CONTRIBUTION TO OPERATORS BETWEEN BANACH LATTICES

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Abstract. In this paper we introduce and study a new class of operators related to norm bounded sets on Banach Lattice and which brings together several classical classes of operators (as o-weakly compact operators, b-weakly compact operators, M-weakly compact operators, L-weakly compact operators, almost Dunford-Pettis operators). As consequences, we give some new lattice approximation properties of these classes of operators.

1. Introduction

Along this paper E, F mention Banach lattices, X, Y are Banach spaces. The positive cone of E will be denoted by $E^+$. Recall that a net $(x_\alpha) \subset E$ is unbounded absolutely weakly convergent (abb, uaw-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges weakly to zero for every $u \in E^+$, we write $x_\alpha \xrightarrow{uaw} x$. We note that every disjoint sequence of a Banach lattice is uaw-null [7, Lemma 2]. A net $(x'_\alpha)$ is unbounded absolutely weak* convergent (abb, uaw*-convergent) to $x'$ if $(|x'_\alpha - x'| \wedge u')$ converges weak* to zero for every $0 \leq u' \in E'$, we write $x'_\alpha \xrightarrow{uaw*} x$. Recall from [6] that a norm bounded subset $A$ of a Banach lattice $E$ is L-weakly compact if $\lim_{n \to +\infty} \|x_n\| = 0$ for every disjoint sequence $(x_n)$ contained in $sol(A)$, where $sol(A) := \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}$ is the solid hull of the set $A$. Alternatively, $A$ is L-weakly compact if and only if $\|x_n\| \to 0$ for every norm bounded uaw-nul sequence $(x_n)$ of $sol(A)$ [4, Proposition 3.3].

In this paper, we introduce and study a new class of operators attached on a norm bounded subset of the starting space (Definition 3.1) and which groups together several classes of operators, as M-weakly compact operators (Corollary 3.2), order weakly compact operators (Theorem 3.4), b-weakly compact operators (Theorem 3.5), almost Dunford-Pettis operators.
Dunford-Pettis operators (Proposition 3.6) and L-weakly compact operators (Corollary 3.3). As consequences, we obtain new characterizations of L-weakly compact sets (Corollary 3.5), of order continuous Banach lattice (Corollary 3.6), of KB-space (Corollary 3.7) and of positive Schur property (Corollary 3.9).

2. Preliminaries and notations

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space \((E, \| \cdot \|)\) such that \(E\) is a vector lattice and its norm satisfies the following property: for each \(x, y \in E\) such that \(|x| \leq |y|\), we have \(\| x \| \leq \| y \|\). \(E\) is order continuous if for each net \((x_\alpha)\) such that \(x_\alpha \downarrow 0\) in \(E\), the net \((x_\alpha)\) converges to 0 for the norm \(\| \cdot \|\), where the notation \(x_\alpha \downarrow 0\) means that the net \((x_\alpha)\) is decreasing, its infimum exists and \(\inf (x_\alpha) = 0\).

We will use the term operator \(T : E \to F\) from \(E\) to \(F\) to mean a bounded linear mapping. \(T'\) will be the adjoint operator of \(T : E \to F\) defined from \(F'\) into \(E'\) by \(T'(f)(x) = f(T(x))\) for each \(f \in F'\) and each \(x \in E\). An operator \(T : E \to F\) is positive if \(T(x) \in F^+\) whenever \(x \in E^+\). For more information on positive operators see the book of Aliprantis-Burkinshaw [1]

We need to recall definitions of the following operators:

1. An operator \(T : X \to F\) is said to be L-weakly compact, if \(T(B_X)\) is an L-weakly compact subset of \(F\).
2. An operator \(T : E \to X\) is said to be order weakly compact, if \(T([0, x])\) is a relatively weakly compact subset of \(X\) for every \(x \in E^+\).
3. An operator \(T : E \to X\) is said to be b-weakly compact, if \(T([0, x] \cap E)\) is a relatively weakly compact subset of \(X\) for every \(x \in (E'')^+\).
4. An operator \(T : E \to X\) is said to be M-weakly compact, if \(T(x_n) \rightharpoonup 0\), for every norm bounded disjoint sequence \((x_n)\) in \(E\).
5. An operator \(T : E \to X\) is said to be almost Dunford-Pettis, if \(T(x_n) \rightharpoonup 0\) for every disjoint weakly null sequence \((x_n)\) in \(E\).

3. Main results

We start this section by the following definition.
**Definition 3.1.** Let $S$ be a norm bounded subset of $E$. An operator $T : E \rightarrow Y$ is said to be $S$-L-weakly compact (abb, $S$-Lwc) if for every uaw-null sequence $(x_n) \subset sol(S)$, we have $T(x_n) \overrightarrow{||} 0$.

Observing that for a norm bounded subset $S$ of $E$, $I_d_E$ is $S$-Lwc if and only if $S$ is an L-weakly compact subset of $E$ ([4, Proposition 3.3]) and that $T$ is M-weakly compact if and only if $T$ is a $B_E$-Lwc [4, Corollary 3.1], where $B_E$ denotes the closed unit ball of $E$.

For a norm bounded subset $S$ of $E$, we note by $LWC_S(E,Y)$ the space of all $S$-Lwc operators from $E$ into $Y$. It is a norm closed vector subspace of $L(E,Y)$, the space of all operators from $E$ into $Y$, and it is a left ideal in $L(E,Y)$. In particular, if $S$ is an L-weakly compact subset of $E$, then every operator $T$ defined from $E$ to $Y$ is $S$-Lwc. On the other hand, note that if $A, B$ are two norm bounded subsets of $E$ such that $A \subset B$ and $T$ is an operator from $E$ into $Y$, then $T$ is $A$-Lwc whenever $T$ is $B$-Lwc. On the other hand, $T$ is $B$-Lwc if and only if $T$ is $sol(B)$-Lwc.

**Proposition 3.1.** Let $T : E \rightarrow Y$ be an operator and $S$ be a norm bounded subset of $E$. If $T$ is $S$-Lwc, then for every $\lambda \in \mathbb{R}$ the operator $T$ is $\lambda S$-Lwc.

**Proof.** It follows from the fact that for every $\lambda \in \mathbb{R}$, $sol(\lambda S) = \lambda sol(S)$.

**Proposition 3.2.** Let $T : E \rightarrow Y$ be an operator and $A, B$ are norm bounded subsets of $E$. If $T$ is $A$-Lwc and $B$-Lwc, then $T$ is $(A + B)$-Lwc.

**Proof.** Let $(x_n)$ be a uaw-null sequence of $sol(A + B)$, then there exist two sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $x_n^+ \leq |a_n + b_n|$. Therefore, by the Riesz decomposition property [1, Theorem 1.13] there exist tow positive elements $a_n^+$ and $b_n^+$ satisfying $x_n^+ = a_n^+ + b_n^+$, $|a_n^+| \leq |a_n|$ and $|b_n^+| \leq |b_n|$ for each $n$. So, $(a_n^+)$ is a uaw-null sequence of $sol(A)$ and $(b_n^+)$ is a uaw-null sequence of $sol(B)$. As $T$ is $A$-Lwc and $B$-Lwc, then $(T a_n^+) \overrightarrow{||} 0$ and $T(b_n^+) \overrightarrow{||} 0$ and hence $T(x_n^+) \overrightarrow{||} 0$. By the same reason, we found $T(x_n^-) \overrightarrow{||} 0$. Therefore, $T(x_n) \overrightarrow{||} 0$. That is, $T$ is $(A + B)$-Lwc, as claimed.

As immediate consequences of the previous result, we have the following results.

**Corollary 3.1.** Let $T : E \rightarrow Y$ be an operator and $A, B$ are norm bounded subsets of $E$. We have the following statements:
(1) If $T$ is $A$-Lwc and $B$-Lwc, then for every $(\lambda, \mu) \in \mathbb{R}^2$, we have that $T$ is $(\lambda A + \mu B)$-Lwc.

(2) If $T$ is $A$-Lwc and $B$-Lwc, then $T$ is $(A \cup B)$-Lwc.

(3) If $T$ is $A$-Lwc or $B$-Lwc, then $T$ is $(A \cap B)$-Lwc.

Proposition 3.3. Let $T$ be an operator from $E$ to $Y$, $S$ be a norm bounded subset of $E$ and $I$ be the ideal generated by $S$. If $T$ is $S$-Lwc, then for each $x \in I$ we have that $T x \in [-|x|, |x|]$-Lwc.

Proof. Let $S$ be a norm bounded subset of $E$, $I$ be the ideal generated by $S$ and $x \in I$, then there exist $\alpha > 0$ and some vectors $x_1, \ldots, x_n \in S$ with $|x| \leq \alpha \sum_{i=1}^n |x_i|$. By the Riesz decomposition property [1, Theorem 1.13] we have

$$ [-|x|, |x|] \subset \alpha [-|x_1|, |x_1|] + \ldots + \alpha [-|x_n|, |x_n|]. $$

We observe that, for each $i = 1, \ldots, n$ we have $[-|x_i|, |x_i|] \subset \text{sol}(S)$, then

$$ [-|x|, |x|] \subset \alpha \text{sol}(S) + \ldots + \alpha \text{sol}(S). $$

Since $T$ is $S$-Lwc, then $T$ is $\text{sol}(S)$-Lwc and so by Corollary 3.1, we infer that $T$ is $\underbrace{\alpha \text{sol}(S) + \ldots + \alpha \text{sol}(S)}_{\text{n-times}}$-Lwc. Therefore, $T$ is $[-|x|, |x|]$-Lwc.

Theorem 3.1. Let $T: E \to Y$ be an operator and $u \in E^+$. Then, the following statements are equivalent:

(1) $T$ is $[-u, u]$-Lwc.

(2) For each $\varepsilon > 0$, there exists some $g \in (E')^+$ such that

$$ \forall x \in [-u, u], \quad ||T(x)|| \leq g(|x|) + \varepsilon. $$

Proof. 1) $\Rightarrow$ 2) Let $\varepsilon > 0$ and $u \in E^+$. As $T$ is $[-u, u]$-Lwc, we see that every disjoint sequence of $[0, u]$ converges uniformly to zero on $T'(B_{Y^*})$, then by [1, Theorem 4.40] there exists some $g \in (E')^+$ such that

$$ (|T'(f)| - g)^+(u) < \varepsilon $$

holds for all $f \in B_{Y^*}$.

Let $x \in [-u, u]$, then for each $f \in B_{Y^*}$, we have

$$ |f(T(x))| \leq |T'(f)||(|x|) \leq g(|x|) + \varepsilon, $$

so

$$ ||T(x)|| \leq g(|x|) + \varepsilon $$

2) $\Rightarrow$ 1) Let $\varepsilon > 0$ and $(x_n)$ be a uaw-null sequence of $[-u, u]$. We have to show that $T x_n \xrightarrow{||.||} 0$. By our hypothesis, there exists some
$g \in (E')^+$ such that

$$||T(x_n)|| \leq g(|x_n|) + \frac{\varepsilon}{2} \text{ for every } n \in \mathbb{N}.$$ 

As $x_n \xrightarrow{uaw} 0$ in $E$, then $g(|x_n|) = g(|x_n| \wedge u) \rightarrow 0$ and hence there exists some integer $m$ such that $g(|x_n|) \leq \frac{\varepsilon}{2}$ for every $n \geq m$. So, for every $n \geq m$ we have $||T(x_n)|| \leq \varepsilon$ which implies that $T(x_n) \xrightarrow{|||} 0$. Therefore, $T$ is $[-u, u]$-Lwc. $\square$

In the following result, we present some characterizations of $S$-Lwc operators.

**Theorem 3.2.** For an operator $T : E \rightarrow Y$ and a norm bounded subset of $S \subset E$, the following statements are equivalent:

1. $T$ is $S$-Lwc.
2. For each $\varepsilon > 0$, there exist some $g \in (E')^+$ and $u \in E^+$ such that

$$||T(x)|| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in sol(S).$$

3. For every uaw-null net $(x_\alpha)$ of $sol(S)$, we have $T(x_\alpha) \xrightarrow{|||} 0$.

**Proof.** 1) $\Rightarrow$ 2) Let $\varepsilon > 0$; since $T$ is $S$-Lwc, then by [1, Theorem 4.36] there exists some $u \in E^+$ lying in the ideal generated by $sol(S)$ such that

$$||T(|x| - u)^+|| \leq \frac{\varepsilon}{4} \text{ for all } x \in sol(S).$$

This implies that,

$$||T(|x|)|| \leq ||T(|x| \wedge u)|| + \frac{\varepsilon}{4} \text{ for all } x \in sol(S).$$

Hence, for every $x \in sol(S)$, we have

$$||T(x^+)|| \leq ||T(x^+ \wedge u)|| + \frac{\varepsilon}{4}$$

and

$$||T(x^-)|| \leq ||T(x^- \wedge u)|| + \frac{\varepsilon}{4}.$$ 

This implies that for every $x \in sol(S)$, we have

$$||T(x)|| \leq ||T(x^+ \wedge u)|| + ||T(x^- \wedge u)|| + \frac{\varepsilon}{2} \text{ \,(\star).}$$

On the other hand, by Proposition 3.3 $T$ is $[-u, u]$-Lwc, and so by Theorem 3.1, there exists some $g \in (E')^+$ such that for every $x \in sol(S)$, we have

$$||T(x^+ \wedge u)|| \leq g(x^+ \wedge u) + \frac{\varepsilon}{4}$$

and

$$||T(x^- \wedge u)|| \leq g(x^- \wedge u) + \frac{\varepsilon}{4}.$$
and
\[ \|T(x^- \land u)\| \leq g(x^- \land u) + \frac{\varepsilon}{4}. \]

On the other hand, using the fact that \(|x| \land u = x^+ \land u + x^- \land u\), we see that for every \(x \in \text{sol}(\mathcal{S})\) we have
\[ \|T(x^+ \land u)\| + \|T(x^- \land u)\| \leq g(|x| \land u) + \frac{\varepsilon}{2} \quad (**) . \]

Therefore, by combining (*) and (**), we have
\[ \|T(x)\| \leq g(|x| \land u) + \varepsilon \quad \text{for every} \quad x \in \text{sol}(\mathcal{S}). \]

2) \(\Rightarrow\) 3) Let \(\varepsilon > 0\) and \((x_\alpha)\) a net of \(\text{sol}(\mathcal{S})\). By our hypothesis, there exist some \(g \in (E')^+\) and \(u \in E^+\) such that,
\[ \|T(x_\alpha)\| \leq g(|x_\alpha| \land u) + \frac{\varepsilon}{2} \quad \text{for all} \quad \alpha. \]

Since \(x_\alpha \overset{\text{uaw}}{\rightarrow} 0\), then there exists some \(\alpha_0\) such that \(g(|x_\alpha| \land u) < \frac{\varepsilon}{2}\) for every \(\alpha \geq \alpha_0\). Hence, for every \(\alpha \geq \alpha_0\) we have \(\|T(x_\alpha)\| \leq \varepsilon\), which implies that \(T(x_\alpha) \overset{\|\|}{\rightarrow} 0\).

3) \(\Rightarrow\) 1) Obvious. \(\Box\)

As a consequence of the previous theorem, we have the following characterizations of M-weakly compact operators which is exactly the [4, Corollary 3.1].

**Corollary 3.2.** For an operator \(T : E \rightarrow F\) the following statements are equivalent:

1. \(T\) is M-weakly compact.
2. For each \(\varepsilon > 0\), there exist some \(g \in (E')^+\) and \(u \in E^+\) such that
   \[ \|T(x)\| \leq g(|x| \land u) + \varepsilon \quad \text{for all} \quad x \in B_E. \]
3. For every net \((x_\alpha) \subset B_E\) such that \(x_\alpha \overset{\text{uaw}}{\rightarrow} 0\), we have \(T(x_\alpha) \overset{\|\|}{\rightarrow} 0\).
4. For every sequence \((x_n) \subset B_E\) such that \(x_n \overset{\text{uaw}}{\rightarrow} 0\), we have \(T(x_n) \overset{\|\|}{\rightarrow} 0\).

In a similar way, we may prove the following result which present a dual version of the Theorem 3.2. The proof of this theorem is similar to that of the Theorem 3.2, so we will avoid copying it here.

**Theorem 3.3.** Let \(T : E \rightarrow F\) be an operator and \(\mathcal{S}\) a norm bounded subset of \(F^*\). Then, the following statements are equivalent:

1. \(T'\) is \(\mathcal{S}\)-Lwc.
For each \( \varepsilon > 0 \) there exist some \( g \in (F')^+ \) and \( u \in F^+ \) such that
\[
\| T'(f) \| \leq (|f| \wedge g)(u) + \varepsilon \text{ for all } f \in \text{sol}(S).
\]

(3) For every uaw\(^*\)-null net \( (f_\alpha) \subset \text{sol}(S) \), we have \( T'(f_\alpha) \rightharpoonup 0 \).

(4) For every uaw\(^*\)-null sequence \( (f_n) \subset \text{sol}(S) \), we have \( T'(f_n) \rightharpoonup 0 \).

As consequence of the Theorem 3.3, we obtain new characterizations of L-weakly compact operators.

**Corollary 3.3.** Let \( T : E \to F \) be an operator. Then, the following statements are equivalent:

1. \( T \) is L-weakly compact.
2. For each \( \varepsilon > 0 \) there exist some \( g \in (F')^+ \) and \( u \in F^+ \) such that
   \[
   \| T'(f) \| \leq (|f| \wedge g)(u) + \varepsilon \text{ for all } f \in B_{F'}.
   \]
3. For every net \( (f_\alpha) \subset B_{F'} \) such that \( f_\alpha \rightharpoonup^* 0 \), we have \( T'(f_\alpha) \rightharpoonup 0 \).
4. For every sequence \( (f_n) \subset B_{F'} \) such that \( f_n \rightharpoonup^* 0 \), we have \( T'(f_n) \rightharpoonup 0 \).

Recall that a bounded subset \( A \) in Banach lattice \( E \) is said uaw-compact whenever every net \( (x_\alpha) \) in \( E \) has a subnet, which is uaw-convergent. Note that the standard basis \( (e_n) \) of \( \ell_1 \) is uaw-null in \( \ell_1 \), then the set \( A = \{e_n : n \in \mathbb{N}\} \) is relatively uaw-compact; but \( A \) is not relatively compact.

**Proposition 3.4.** Let \( S \) be a norm bounded subset of \( E \). For an \( S \)-Lwc operator \( T : E \to Y \), we have

1. \( T(A) \) is compact for every norm bounded uaw-compact subset \( A \subset \text{sol}(S) \).
2. \( T(A) \) is relatively compact for every norm bounded relatively uaw-compact subset \( A \subset \text{sol}(S) \).

**Proof.** 1) Let \( A \) be a uaw-compact subset of \( \text{sol}(S) \) and let \( (x_\alpha) \) be a net of \( A \), then there exist a subnet of \( (x_\alpha) \) which we denoted by \( (x_{\alpha}) \) and \( x \in A \) such that \( x_{\alpha} \rightharpoonup^* x \). As \( T \) is \( S \)-Lwc, then by Corollary 3.1 and Theorem 3.2, \( T(x_{\alpha}) \rightharpoonup T(x) \). Therefore, \( T(A) \) is a compact subset of \( Y \).

2) Let \( A \) be a relatively uaw-compact subset of \( \text{sol}(S) \) and let \( (x_\alpha) \) be a net of \( A \), then \( (x_{\alpha}) \) has a uaw-convergent subnet which we denoted
by \((x_\alpha)\). Since the subnet \((x_\alpha)\) is norm bounded and uaw-Cauchy, then the double net \((x_\alpha - x_\beta)_{(\alpha,\beta)}\) is uaw-null. As \(T\) is \(S\)-Lwc, then by Theorem 3.2 and Corollary 3.1 the net \((T(x_\alpha))\) is norm Cauchy and hence \((T(x_\alpha))\) is norm convergent. Therefore, \(T(A)\) is a relatively compact subset of \(Y\).

In the following result we give characterizations of \(S\)-Lwc operators which are uaw-continuous.

**Proposition 3.5.** Let \(S\) be a norm bounded subset of \(E\). For a uaw-continuous operator \(T : E \rightarrow F\) the following statements are equivalent:

1. \(T\) is \(S\)-Lwc.
2. \(T(A)\) is relatively compact for every relatively uaw-compact subset \(A \subset \text{sol}(S)\).
3. \(T(A)\) is compact for every uaw-compact subset \(A \subset \text{sol}(S)\).

**Proof.**

1) \(\Rightarrow\) 2) Follows from the Proposition 3.4.

2) \(\Rightarrow\) 3) Obvious.

3) \(\Rightarrow\) 1) Let \((x_n)\) be a uaw-null sequence of \(\text{sol}(S)\), then the set \(A = \{x_n : n \in \mathbb{N}\} \cup \{0\}\) is uaw-compact, and hence it follows from our hypothesis that \(T(A)\) is relatively compact. So, there exist a subsequence \((x_{\phi(n)})\) of \((x_n)\) and \(y \in T(A)\) such that \(T(x_{\phi(n)}) \xrightarrow{\text{uaw}} y\). Since the sequence \((x_n)\) is uaw-null, then \(x_{\phi(n)} \xrightarrow{\text{uaw}} 0\) and since \(T\) is uaw-continuous then \(y = 0\). Therefore, \(T(x_n) \xrightarrow{||\cdot||} 0\) which implies that \(T\) is \(S\)-Lwc.

An immediate consequence of the previous result is the following.

**Corollary 3.4.** For a uaw-continuous operator \(T : E \rightarrow F\), the following statements are equivalent:

1. \(T\) is \(M\)-weakly compact.
2. \(T(A)\) is compact for every uaw-compact subset \(A \subset B_E\).
3. \(T(A)\) is relatively compact for every relatively uaw-compact subset \(A \subset B_E\).

As another consequence of Theorem 3.2 and Proposition 3.5, we have the following characterizations of L-weakly compact sets.

**Corollary 3.5.** For a norm bounded subset \(S\) of \(E\), the following statements are equivalent:

1. \(S\) is L-weakly compact.
(2) For each \( \varepsilon > 0 \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
\[
||x|| \leq g(|x| \wedge u) + \varepsilon \quad \text{for every } x \in sol(S).
\]

(3) For every net \( (x_\alpha) \subset sol(S) \) such that \( x_\alpha \overset{uaw}{\to} 0 \), we have \( x_\alpha \overset{||-||}{\to} 0 \).

(4) Every norm bounded uaw-compact subset \( A \subset sol(S) \) is compact.

(5) Every norm bounded relatively uaw-compact subset \( A \subset sol(S) \) is relatively compact.

Note that from [4, Proposition 3.4], it is easy to see that an operator \( T : E \to Y \) is order weakly compact if and only if for every \( v \in E^+ \), the operator \( T \) is \( \{v\}\)-Lwc.

With the help of Theorem 3.2 and Proposition 3.4, we are now in a position to present new characterizations of the order weakly compact operators.

**Theorem 3.4.** For an operator \( T : E \to Y \), the following statements are equivalent:

1. \( T \) is order weakly compact.
2. For every relatively compact subset \( S \) of \( E \), the operator \( T \) is \( S\)-Lwc.
3. For every \( v \in E^+ \), the operator \( T \) is \( \{v\}\)-Lwc.
4. For every \( \varepsilon > 0 \) and \( v \in E^+ \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
   \[
   ||T(x)|| \leq g(|x| \wedge u) + \varepsilon \quad \text{for every } x \in [-v,v].
   \]
5. For every \( v \in E^+ \) and for every net \( (x_\alpha) \subset [-v,v] \) such that \( x_\alpha \overset{uaw}{\to} 0 \), we have \( T(x_\alpha) \overset{||-||}{\to} 0 \).
6. For every \( v \in E^+ \) and for every sequence \( (x_n) \subset [-v,v] \) such that \( x_n \overset{uaw}{\to} 0 \), we have \( T(x_n) \overset{||-||}{\to} 0 \).
7. For every order bounded uaw-compact subset \( A \) of \( E \), \( T(A) \) is compact.
8. For every order bounded relatively uaw-compact subset \( A \) of \( E \), \( T(A) \) is relatively compact.

**Proof.** It remains to show 1) \( \Rightarrow \) 2) and 8) \( \Rightarrow \) 1), the other implications are already seen before in the preceding results.

1) \( \Rightarrow \) 2) Let \( T \) be an order weakly compact operator, \( S \) a relatively compact set of \( E \) and \( (x_n) \) a uaw-nul sequence of \( sol(S) \) and let \( \varepsilon > 0 \). Since \( (x_n) \subset sol(S) \) and \( S \) a relatively compact set of \( E \), then there
exist \( u \in E^+ \) and a sequence \( (y_n) \subset S \) such that
\[
||(x_n^+ - u)^+|| \leq ||(y_n - u)^+|| \leq \frac{\varepsilon}{4\|T\|} \quad \text{for all } n,
\]
which implies that for all \( n \), we have
\[
||T(x_n^+)|| \leq ||T(x_n^+ \land u)|| + \frac{\varepsilon}{4}.
\]
Since \( T \) is order weakly compact then, \( T \) is \([-u, u]\)-Lwc operator, thus by Theorem 3.1, there exists \( g \in (E')^+ \) such that
\[
||T(x_n^+ \land u)|| \leq g(x_n^+ \land u) + \frac{\varepsilon}{4} \quad \text{for all } n,
\]
So for every \( n \) we have
\[
||T(x_n^+)|| \leq g(x_n^+ \land u) + \frac{\varepsilon}{2} \quad \text{for all } n.
\]
Since \( x_n \overset{\text{uaw}}{\to} 0 \), then there exists \( n_0 \) such that \( g(x_n^+ \land u) < \frac{\varepsilon}{2} \) for all \( n \geq n_0 \). Thus, for each \( n \geq n_0 \) we have \( ||T(x_n^+)|| \leq \varepsilon \), which implies that \( T(x_n^+) \overset{\text{w}}{\longrightarrow} 0 \). By the same reason, we found \( T(x_n^-) \overset{\text{w}}{\longrightarrow} 0 \). Hence, \( T(x_n) \overset{\text{w}}{\longrightarrow} 0 \), as desired.

8) \( \Rightarrow \) 1) Let \( v \in E^+ \) and \( (x_n) \subset [-v, v] \) be a uaw-null sequence. The set \( A = \{x_n : n \in \mathbb{N}\} \) is relatively uaw compact, implies that \((T(x_n))\) is relatively compact. By observing that the sequence \((x_n)\) is weakly null we infer that \( T(x_n) \overset{\text{w}}{\longrightarrow} 0 \).

As a consequence of Theorem 3.4, we obtain the following characterizations of order continuous Banach lattices.

**Corollary 3.6.** The following statements are equivalent:

1. \( E \) is order continuous.
2. Every relatively compact subset \( S \) of \( E \) is \( L \)-weakly compact.
3. For every \( x \in E^+ \), \( \text{Id}_E \) is \( \{x\}\)-Lw.
4. For every \( \varepsilon > 0 \) and \( v \in E^+ \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
   \[
   ||x|| \leq g(|x| \land u) + \varepsilon \quad \text{for all } x \in [-v, v].
   \]
5. For every \( v \in E^+ \) and for every net \( (x_\alpha) \subset [-v, v] \) such that \( x_\alpha \overset{\text{uaw}}{\to} 0 \), we have \( x_\alpha \overset{\text{w}}{\longrightarrow} 0 \).
6. For every \( v \in E^+ \) and for every sequence \( (x_n) \subset [-v, v] \) such that \( x_n \overset{\text{uaw}}{\to} 0 \), we have \( x_n \overset{\text{w}}{\longrightarrow} 0 \).
7. Every order bounded uaw-compact subset \( A \subset E \) is compact.
Every order bounded relatively uaw-compact subset $A \subset E$ is relatively compact.

Recall from [5] that a subset $A$ of a Banach lattice $E$ is said to be $b$-semi compact if it is almost order bounded as a subset of $E''$, that is, for every $\varepsilon > 0$ there exists $u \in E''_1$ such that $A \subset [-u, u] + \varepsilon B_{E''}$.

By repeating the proof of Theorem 3.4, we can prove a similar result for $b$-weakly compact operators as follows.

**Theorem 3.5.** For an operator $T : E \to Y$, the following statements are equivalent:

1. $T$ is $b$-weakly compact.
2. For every $b$-semi compact subset $S$ of $E$, the operator $T$ is $S$-Lwc.
3. For every $v \in (E'')^+$, $T$ is $([-v, v] \cap E)$-Lwc.
4. For each $\varepsilon > 0$ and $v \in (E'')^+$, there exist some $g \in (E')^+$ and $u \in E^+$ such that
   \[
   \|T(x)\| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in [-v, v] \cap E.
   \]
5. For every $v \in (E'')^+$ and for every net $(x_\alpha) \subset [-v, v] \cap E$ such that $x_\alpha \xrightarrow{uaw} 0$, we have $T(x_\alpha) \xrightarrow{\|\|} 0$.
6. For every $v \in (E'')^+$ and for every sequence $(x_n) \subset [-v, v] \cap E$ such that $x_n \xrightarrow{uaw} 0$, we have $T(x_n) \xrightarrow{\|\|} 0$.
7. For every $b$-order bounded uaw-compact subset $A \subset E$, $T(A)$ is compact.
8. For every $b$-order bounded relatively uaw-compact subset $A \subset E$, $T(A)$ is relatively compact.

As a consequence of Theorem 3.5, we have the following characterizations of KB-spaces.

**Corollary 3.7.** The following statements are equivalent:

1. $E$ is a KB space.
2. Every $b$-semi compact subset $S$ of $E$ is $L$-weakly compact.
3. For every $v \in (E'')^+$, $Id_E$ is $([-v, v] \cap E)$-Lwc.
4. For each $\varepsilon > 0$ and $v \in (E'')^+$, there exist some $g \in (E')^+$ and $u \in E^+$ such that
   \[
   \|x\| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in [-v, v] \cap E.
   \]
5. For every $v \in (E'')^+$ and for every net $(x_\alpha) \subset [-v, v] \cap E$ such that $x_\alpha \xrightarrow{uaw} 0$, we have $x_\alpha \xrightarrow{\|\|} 0$.
6. For every $v \in (E'')^+$ and for every sequence $(x_n) \subset [-v, v] \cap E$ such that $x_n \xrightarrow{uaw} 0$, we have $x_n \xrightarrow{\|\|} 0$.  

Every b-order bounded uaw-compact subset $A \subset E$ is compact.

Every b-order bounded relatively uaw-compact subset $A \subset E$ is relatively compact.

In terms of relatively weakly compact sets and $\mathcal{S}$-Lwc operators the almost Dunford-Pettis operators are characterized as follows.

**Proposition 3.6.** For an operator $T : E \rightarrow Y$, the following statements are equivalent:

1. $T$ is almost Dunford-Pettis.
2. For every relatively weakly compact subset $\mathcal{S}$ of $E$, the operator $T$ is $\mathcal{S}$-Lwc.
3. For every relatively weakly compact subset $\mathcal{S}$ of $E$ and for every net $(x_n) \subset \text{sol}(\mathcal{S})$ such that $x_n \text{uaw} \rightarrow 0$, we have $T(x_n) \rightharpoonup 0$.
4. For every relatively weakly compact subset $\mathcal{S}$ of $E$ and for every sequence $(x_n) \subset \text{sol}(\mathcal{S})$ such that $x_n \text{uaw} \rightarrow 0$, we have $T(x_n) \rightharpoonup 0$.

**Proof.**
1) $\Rightarrow$ 2) Let $\mathcal{S}$ be a relatively weakly compact subset of $E$ and $(x_n)$ be a uaw-null sequence of $\text{sol}(\mathcal{S})$, then by [4, Theorem 3.1], we have $x_n^+ \text{w} \rightarrow 0$ and $x_n^- \text{w} \rightarrow 0$. On the other hand, since $T$ is almost Dunford-Pettis, it follows from [3, Theorem 2.2] that $T(x_n^+) \rightharpoonup 0$ and $T(x_n^-) \rightharpoonup 0$, and so $T(x_n) \rightharpoonup 0$. Therefore, $T$ is $\mathcal{S}$-Lwc.

2) $\Rightarrow$ 3) Obvious.

3) $\Rightarrow$ 4) Obvious.

4) $\Rightarrow$ 1) Let $(x_n)$ be a disjoint weakly null sequence of $E$. Put $K = \{x_n : n \in \mathbb{N}\}$, we note that $K$ is relatively weakly compact and the sequence $(x_n)$ is uaw-null, hence by our hypothesis $T(x_n) \rightharpoonup 0$, as desired.

□

As consequences of Proposition 3.5 and Proposition 3.6, we have the following results which present new characterizations of almost Dunford-Pettis operators which are uaw-continuous.

**Corollary 3.8.** For a uaw-continuous operator $T : E \rightarrow F$, the following statements are equivalent:

1. $T$ is almost Dunford-Pettis.
2. For every relatively weakly compact subset $\mathcal{S}$ of $E$ and for every uaw-compact subset $A$ of $\text{sol}(\mathcal{S})$, we have $T(A)$ is compact.
3. For every relatively weakly compact subset $\mathcal{S}$ of $E$ and for every relatively uaw-compact subset $A$ of $\text{sol}(\mathcal{S})$, we have $T(A)$ is relatively compact.
New characterizations of the positive Schur property are obtained as a consequence of Proposition 3.6 and Corollary 3.8.

**Corollary 3.9.** The following statements are equivalent:

1. $E$ has the positive Schur property.
2. Every relatively weakly compact subset of $E$ is $L$-weakly compact.
3. For every relatively weakly compact subset $S$ of $E$ and for every net $(x_\alpha) \subset \text{sol}(S)$ such that $x_\alpha \rightharpoonup 0$, we have $x_\alpha \rightharpoonup 0$.
4. For every relatively weakly compact subset $S$ of $E$ and for every sequence $(x_n) \subset \text{sol}(S)$ such that $x_n \rightharpoonup 0$, we have $x_n \rightharpoonup 0$.
5. For every relatively weakly compact subset $S$ of $E$ and for every uaw-compact subset $A$ of $\text{sol}(S)$, we have $A$ is compact.
6. For every relatively weakly compact subset $S$ of $E$ and for every relatively uaw-compact subset $A$ of $\text{sol}(S)$, we have $A$ is relatively compact.

The last result of this section presents a set characterization of Banach lattices with weakly sequentially continuous lattice operations.

**Proposition 3.7.** The following statements are equivalent:

1. The lattice operations of $E$ are weakly sequentially continuous.
2. Every relatively weakly compact subset of $E$ is relatively uaw-compact.

**Proof.** 1) $\Rightarrow$ 2) It follows from [4, Corollary 3.5].
2) $\Rightarrow$ 1) Let $T : E \to \ell^\infty$ be an almost Dunford-Pettis operator and let $S$ be a relatively weakly compact subset of $E$, by our hypothesis we have $S$ is a relatively uaw-compact set of $E$. Now by Proposition 3.6 we see that $T$ is $S$-Lwc, therefore by Proposition 3.4 the set $T(S)$ is relatively compact. That is, $T$ is a Dunford-Pettis operator, and so by [2, Corollary 2.4] the lattice operations of $E$ are weakly sequentially continuous, and the proof of the theorem is finished. \qed

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