Completely geometric theory I: gravitation

S V Siparov
State University of Civil Aviation, St. Petersburg, 196210, RF
E-mail: sergey@siparov.ru

Abstract. In order to describe motions in arbitrary physical systems, a geometric approach is proposed, in which the goal is not to find the Lagrangian, but to find the geometry of space that models the experimental space. The requirement that the observed trajectory coincides with the geodesic makes it possible to use geometric identity to find the equation for the metric. As a result, it is possible to give an interpretation of a number of observations that do not have such in the existing theories.

When constructing the dynamics, Newton relied on the observations of Kepler, who discovered the elliptical form of the trajectories of the planets. Along with Galileo's principle of relativity, this led to the formulation of the equation of motion in the form of the second law of dynamics, \( m \frac{d\vec{v}}{dt} = \vec{F} \). Having determined the force with the help of trajectories, Newton considered his task completed. If the force of gravity is considered to be the gradient of some potential, then the field equation corresponding to the observed trajectories is the Poisson equation. Therefore, Newton's theory of gravity is reduced to the following equations

\[
\rho \Delta \varphi = m \frac{d\vec{v}}{ds}, \quad \Delta \varphi = \rho(r),
\]

where \( \vec{v} = \frac{d\vec{r}}{dt} \), and \( \rho(r) \) is the source distribution density, setting which one can solve the direct problem and find the trajectory.

Along with the well-known successes, Newton's theory of gravity incorrectly describes i) the precession of the Mercury orbit (missing 43'' in a century) and ii) the deflection of a light beam by a massive body (the angle differs by half).

2. If the field potential is known, then the trajectory can be found using Hamilton's principle. The variation method leads to the equation of motion in the form of the Euler-Lagrange equation containing this potential. Turning to geometrodynamics, we see that a trajectory which is a solution to the Euler-Lagrange equation coincides with the geodesic equation on a surface with a given curvature. This equation has the form

\[
\frac{d^2 x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0,
\]

where \( \Gamma^i_{kl} \) are the connectivity coefficients defined by the choice of metric (i.e. of geometry), \( s \) is a natural parameter along the trajectory. In order to find the metric, and hence the curvature at each point, the field equation is needed. Einstein obtained it by equating the zero divergence of the energy-
momentum tensor (conservation law) to the zero divergence of a combination of geometric parameters that had the desired structure. The resulting equation of the gravitational field has the form

$$ R^a - \frac{1}{2} g^a R = -\kappa T^{ak} $$

(4)

By specifying the energy-momentum tensor, within the framework of the direct problem, one can solve equation (4), substitute the found metric into equation (3), and find the trajectory. And vice versa, by specifying the trajectory, one can solve the inverse problem and find the components of the energy-momentum tensor. Thus, Einstein’s theory of gravity (GRT) is reduced to equations (3-4).

The use of geometrodynamics made it possible to cope with the problems i)-ii) of Newton’s theory, as well as predict the gravitational redshift, subsequently found in experiment. However, on the galactic scale, GRT ran into problems: iii) in spiral galaxies, the rotation curves turned out to be flat, and iv) in some gravitational lenses, the detected light refraction turned out to be significantly larger than the calculated one.

3. In electrodynamics, the equation of motion contains the Lorentz force,

$$ m \frac{d\vec{v}}{dt} = q \left( \vec{E} + [\vec{v}, \vec{H}] \right), $$

(5)

which includes two types of measured forces: \( \vec{E} = -\frac{\partial \phi_e}{\partial \vec{r}} \) and \( \vec{H} = \text{rot}\vec{A} \), where \( \phi_e \) and \( \vec{A} \) are scalar and vector potentials. The electromagnetic field equations (Maxwell’s equations) in convenient units have the form

$$ \frac{\partial \vec{H}}{\partial t} + \text{rot}\vec{E} = 0 $$

(6)

$$ \text{div}\vec{H} = 0 $$

$$ \text{rot}\vec{H} - \frac{\partial \vec{E}}{\partial t} = \vec{j} $$

(7)

$$ \text{div}\vec{E} = \rho $$

They were obtained as a result of generalization of observations related to measurements of the strengths of electric and magnetic fields and the introduction of the concept of "displacement current", which follows from the continuity equation. This introduction became possible due to the identical equality of two zeros: zero, which is on the right in the continuity equation, and zero, which is equal to the divergence of the rotor. Equations (6-7) can be transformed into d'Alembert equations for the corresponding potentials:

$$ \Delta \phi_e - \frac{\partial^2 \phi_e}{\partial t^2} = -\rho $$

(8)

$$ \Delta \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -\vec{j} $$

or

$$ \Delta A^i - \frac{\partial^2 A^i}{\partial t^2} = -I^i, $$

(9)

where \( A_i \) are the components of the 4-potential (\( \phi_e, \vec{A} \)), \( I^0 = \rho; I^i \equiv \vec{j} \). Thus, equation (9) is the electromagnetic field equation.

It is important to note that in Maxwell's theory based on Faraday's ideas about lines of force, the sources on the right-hand sides of equations (7-9) are auxiliary quantities and are actually functions of strengths [1]. However, due to the complexity of solving the complete direct problem, the goal of which is to find the trajectory, the sources’ distribution is usually assumed to be known. If the sources are
given, equations (9) allow you to find the potentials, then calculate the strengths and find the trajectory with equation (5).

1. **Construction of a completely geometric theory**

In the middle of the 19th century, Clifford pointed out that there is no experimental possibility of distinguishing the effect of physical fields from the manifestation of the geometry with which the experimental space is described. Therefore, a variety of physical ideas and results that made it possible to obtain the above equations of motion and field equations can be applied to the construction of a completely geometric theory that is not tied to a specific physical problem. For this, it is necessary to obtain analogs of the equation of motion and of the field equation. The first, of course, is the geodesic equation for a space with a given geometry. Taking into account the possibility of the existence of two types of forces, we will consider space not only curved, but also anisotropic. Then the geodesic equation (3) should be generalized, and we should take into account the possible dependence of metric on the derivative with respect to the natural parameter, i.e. put \( g_{ik} = g_{ik}(x, y) \), where \( y' = \frac{dx'}{ds} \). This greatly complicates the geometric aspect of the matter. For the sake of simplicity, let us set \( g_{ik}(x, y) = \eta_{ik} + \varepsilon_{ik}(x, y) ; \eta_{ik} = diag(-1,1,1) \), where \( \varepsilon_{ik} = \sigma \varepsilon_{ik} \). Then the generalized geodesic takes the form

\[
\frac{dy^i}{ds} + \left( \Gamma^i_{jk} + \frac{1}{2} \eta^{ij} \frac{\partial^2 \varepsilon_{kl}}{\partial x^j \partial y^l} y^j \right) y^j = 0 .
\]

Then we use the well-known [2,3] simplifying assumptions for the geodesic and obtain

\[
\frac{dy^i}{ds} + \Gamma^i_{00} + \frac{1}{2} \eta^{ij} \frac{\partial^2 \varepsilon_{00}}{\partial x^j} y^j = 0 .
\]

Thus, the only significant component of the metric tensor that remains in the equations for the considered case of small curvature and small anisotropy is \( \varepsilon_{00} \). Let us transform the third term so as to select the “potential” and “solenoid” parts. Then the spatial 3D-section of equation (11) takes the form

\[
\frac{d\vec{v}}{dt} = -\nabla \varepsilon_{00} + \nabla (\vec{v} \cdot \frac{\partial \varepsilon_{00}}{\partial \vec{v}}) + [\vec{\nabla}, \tilde{\nabla}] \nabla \varepsilon_{00} .
\]

Now let us find the field equation. The well-known expression \( F_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \) for the tensor of electromagnetic field, where \( A_i \) is the 4-potential, satisfies the identity \( \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0 \). Now we note that this identity has not a specific, but a universal geometric character and holds for any covariant anti-symmetric tensor of the second rank. The consequences of this fact are as follows. Let \( F_{ik} \) be an arbitrary anti-symmetric tensor, \( g_{ik} \) be the metric (fundamental tensor) of the modeling space, so that \( F^{ij} = g^{ik} g^{jm} F_{mk} \). If we introduce the notation \( \vec{F}^{(v)} = (F_{12}, F_{31}, F_{10}) \) and \( \vec{F}^{(w)} = (-F_{30}, F_{20}, -F_{23}) \), as well as \( I^i = \frac{\partial F^{ij}}{\partial x^j} \) and \( I^0 \equiv \rho ; I^1 \equiv j_x ; I^2 \equiv j_y ; I^3 \equiv j_z \), then the two geometric identities

\[
\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0 ; \frac{\partial F^{ij}}{\partial x^k} + \frac{\partial F^{jk}}{\partial x^i} + \frac{\partial F^{ki}}{\partial x^j} = 0 ,
\]

lead to the equations

\[
\frac{\partial \vec{F}^{(v)}}{\partial t} + \nabla \times \vec{F}^{(v)} = 0 \quad \text{and} \quad \nabla \cdot \vec{F}^{(w)} = 0 .
\]
\[ \text{rot}\vec{F}^{(\ast)} - \frac{\partial \vec{F}^{(\ast)}}{\partial t} = \vec{j}, \]  
\[ \text{div}\vec{F}^{(\ast)} = \rho \]  
\[ (15) \]

that coincide in form with equations (6-7), but do not have any pre-assigned physical meaning. This allows to remain within the framework of exclusively geometric constructions in all further reasoning.

Instead of electromagnetic field tensor, we choose a kinematical tensor in the form
\[ F_{ik} = B_{ik} - B_{ik}, \]
where \( B_{ik} = \frac{\partial y_j}{\partial x^i} \). It is an anti-symmetric 4x4 matrix, the lower right corner of which is occupied by the components of the curl of the velocity, and the first row and column are the acceleration components.

We introduce
\[ gF \vec{\phi} - \nabla = (\ast) \rho \]
\[ gF \vec{\phi} - \Delta g = (\ast) \rho \]
and
\[ gU_{rot} F \rho = (\ast) \rho, \]
where \( \vec{\phi} \) and \( \vec{U} \) are “metric potentials”, determined exclusively by the metric. Then, similarly to equation (9), we obtain the equation of the "metric field"
\[ \Delta D^i - \frac{\partial^2 D^i}{\partial x_0^2} = -I^i, \]  
\[ (16) \]
where \( D_i \) is the metrical 4-potential.

The equation of motion (10) (geodesic) and the equation of the metric field (16) represent a closed system of geometric equations based on identity and suitable for solving both direct and inverse problems. The theory based on them is completely geometric, because no additional physical quantities are initially included into it. The equations of the form (14-16) can be solved with the help of efficient mathematical apparatus developed in electrodynamics.

2. **Geometric theory and classical physics**

Consider the applications of the constructed geometric theory to physics.

a) We use the notation from electrodynamics and put \( \varepsilon_{00} \equiv \varphi_e, \frac{\partial \varepsilon_{00}}{\partial \vec{v}} \equiv \vec{A}, \ (\text{rot}\vec{A} \equiv \vec{H}). \) Then the geodesic equation (12) takes the form of the equation of motion of the test (charged) particle
\[ \frac{d}{dt} \vec{v} = -\nabla \varphi_e + (\vec{v} \cdot \nabla) \vec{A} + [\vec{v}, \text{rot}\vec{A}]. \]  
\[ (17) \]
Choose \( \vec{A} = \text{Re}\{\vec{A}_0 e^{(i\vec{k} - \omega t)}\} \) and make replacements \( \vec{E} = i\vec{k} \vec{A} \) and \( \vec{H} = i[\vec{k}, \vec{A}]. \) Then, for the quantity \( \Psi_{(a)} = E_{(a)} + iH_{(a)}, \) the pair of homogeneous Maxwell equations (14) obtained from the geometric identity can be transformed into the equation
\[ i \frac{\partial \Psi_{(a)}}{\partial t} = \text{rot}\Psi_{(a)}. \]  
\[ (18) \]
Solving equation (18) we can find the fields in the region free from sources, and then use equation (17) to obtain the trajectory of a probe body. The stationary equation corresponding to equation (18)
\[ \text{rot}\Psi_{(a)} = \lambda \Psi_{(a)} \]  
\[ (19) \]
has eigenvalues of the following form [4]
\[ \lambda_{n,m} = \rho_{m,n} R_A^{-1}, \]  
\[ (20) \]
where \( R_A \) is the radius of the (spherical) domain of definition, and \( \rho_{m,n} \) are the zeros of the Bessel function of half-integer order. Their arrangement is similar to the arrangement of the zeros of a sinusoid. The eigenfunctions of equation (19) are expressed in terms of spherical functions. In plasma physics, they are called force-free fields, and this fully corresponds to the geometric approach under consideration. The appearance of Bessel functions is typical for various wave problems. In this case, the
Bessel beam (wave field) does neither refract nor diffract, and, being partially obscured by an obstacle, self-regenerates. Thus, the geometric approach applied to electrodynamics leads to physically significant consequences.

b) Now use the notation from hydrodynamics and put \( \frac{\partial \varepsilon_{\omega}}{\partial y} \equiv u_i, \) \( \text{rot} u_i \equiv \Omega, \) \( \varphi = \varepsilon_{\omega 0} - (\vec{v}, \vec{u}). \) Then equation (12), after rearranging and multiplying and dividing the right-hand side of the equation by a constant value \( \rho_0, \) takes the form

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{u} + 2[\vec{\Omega}, \vec{v}] = -\frac{1}{\rho_0} \nabla \{ \rho_0 \varepsilon_{\omega 0} - (\vec{v}, \vec{u})\rho_0 \} \tag{21}
\]

If we consider vector \( \vec{u} \) characterizing the anisotropy of the modeling space as the flow velocity of an incompressible fluid, then, assuming the gradient of the two terms on the right-hand side to be the effective pressure, we obtain the Euler equation for a (sufficiently large) fluid region rotating as a whole with angular velocity \( \vec{\Omega}. \) Accordingly, equations (21) have both potential and wave solutions. Indeed, assuming the velocity of a liquid particle to be equal to the velocity of its environment, \( \vec{v} \equiv \vec{u}, \) we will seek solution of the equation (21) in the form

\[
\vec{v} = \vec{V} e^{i(k, \vec{r}) - \alpha t}. \tag{22}
\]

The amplitude \( \vec{V} \) will be considered small so that the term \( (\vec{v}, \nabla)\vec{u} \to 0 \) can be neglected. Then

\[
\frac{\partial \vec{v}}{\partial t} + 2[\vec{\Omega}, \vec{v}] = -\nabla \varphi. \tag{23}
\]

Let us choose the local \( Oz \) axis parallel to the vector \( \vec{\Omega}, \) and apply the \( \text{rot} \) operator to both sides of equation (23), taking into account that \( \text{rot}[\vec{\Omega}, \vec{v}] = \vec{\Omega} \vec{d}\vec{v} - (\vec{\Omega}, \nabla)\vec{v} = -((\vec{\Omega}, \nabla)\vec{v}). \) We get [5]

\[
\frac{\partial}{\partial t} \text{rot}\vec{v} = 2\Omega \frac{\partial \vec{v}}{\partial \z}. \tag{24}
\]

In such a wave, the velocity vector \( \vec{v} \) of a liquid particle retains its value and changes only in direction. In classical hydrodynamics, such waves are generated by the "Coriolis force" and are called inertial waves. However, in geometric theory there are no Coriolis forces, but only a metric tensor that describes the anisotropic space in which the motion occurs. In this space, the zero component of the metric is a wave in accordance with the definition \( \vec{v} \equiv \vec{u}, \) i.e.

\[
\varepsilon_{\omega 0}(\vec{v}) = \frac{1}{2} \nu^2 = \frac{V^2}{2c^2} e^{2i(k, \vec{r}) - \alpha t}. \tag{25}
\]

Thus, the desired small correction to the metric has a wave character.

3. Geometric theory and gravitation: interpretation of observations

Applying a completely geometric approach to the theory of gravity, it is natural, as in Newton's theory, to start with the inverse problem, i.e. "Do not invent hypotheses" and, knowing the trajectory, find a suitable metric that satisfies equation (12). For illustration, consider two limiting cases. In the first, only the potential contribution of the anisotropy of the modeling space is taken into account

\[
\frac{dv}{dt} = -\nabla \{ \varepsilon_{\omega 0} - \vec{v}, \frac{\partial \varepsilon_{\omega 0}}{\partial \vec{v}} \}. \tag{26}
\]

and in the second, only the solenoid contribution of the anisotropy is taken into account

\[
\frac{dv}{dt} = -\nabla \varepsilon_{\omega 0} + \{ \vec{v}, \text{rot} \frac{\partial \varepsilon_{\omega 0}}{\partial \vec{v}} \}. \tag{27}
\]

Let us compare equations (26, 27) with the motion of a test body in a spiral galaxy. The simplest model of a spiral galaxy is the "center + current" (C+C) system [6-10], in which all the motion is concentrated.
along a circle, in the vicinity of which there is also a test body. So, we will assume that on the periphery of the galaxy, the motion occurs in a circle. Then in both left-hand sides of equation (26) and equation (27) there is a centripetal acceleration. The first terms on the right-hand sides correspond to the approximate equations of general relativity, leading to Newton’s equation. In order to save this result, choose \( \varepsilon_{\omega 0}(r) \sim \frac{C_1}{r} \). (Under physical interpretation, \( C_1 \) will become the Schwarzschild radius, \( C_1 = r_s \)).

The second terms in equations (26, 27) arise from the geometric approach and are associated with the dependence of the acceleration on the speed of both the test body itself and other surrounding bodies, which can also be selected as test ones. We will talk about the "mass current density" in the vicinity of the test body and denote \( \frac{\partial \varepsilon_{\omega 0}}{\partial v} \equiv \ddot{u} \). We emphasize that this terminology is formal in nature, and no substance is initially considered here.

If it turns out that the first and second terms on the right-hand side of equation (26) make the same contribution, then the classical GRT will not be enough for an adequate description of the motion. Assuming \( \ddot{u} \sim \ddot{v} \), we find the condition under which this will happen

\[
\frac{GM}{r} \sim v^2. \tag{28}
\]

Then if the galaxy contains \( 10^{11} \) stars with a mass of the Sun of \( 10^{30} \) kg, and the radius of the galaxy is \( 10^{20} \) m, then according to (28), both terms will give the same contribution if the velocity of the test body is \( 10^5 \) m/s. It is just this value that corresponds to observations, therefore, GRT is not enough to describe galactic phenomena.

Let us now estimate the second term on the right-hand side of equation (27). To do this, we introduce formal notation \( \text{rot} \frac{\partial \varepsilon_{\omega 0}}{\partial v} = \text{rot} \ddot{u} (\equiv \text{rot} \dddot{A} \equiv \dddot{H}) \) and use the geometric analogue of the Biot-Savart-Laplace law. Then we get 

\[
||[\dddot{v}, \text{rot} \frac{\partial \varepsilon_{\omega 0}}{\partial v}]|| \sim v \frac{C_2}{r},
\]

and from equation (27) it follows

\[
v^2 = \frac{C_1}{r} + vC_2. \tag{29}
\]

With \( r \to \infty \), this equation has two roots: \( v_1 = 0 \), which corresponds to both GRT and Newton’s theory, and \( v_2 = C_2 = \text{const} \), which corresponds to observations. Thus, in a completely geometric theory which in this case becomes anisotropic geometrodynamics (AGD), a spiral galaxy has flat rotation curves, and problem iii) does not arise.

When studying gravitational lensing within the framework of the geometric theory, it turns out that if a spiral galaxy-lens has a certain orientation, namely, with its plane parallel to the direction of the beam from the source to the observer, Figure 1a, then the refraction will also depend on the rotation speed of this galaxy, which leads to a significant increase in the angle of refraction, Figure 1b, [11]. This removes problem iv).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{a) Orientations of a spiral galaxy-lens; b) Refraction on a Coulomb centre and on a C+C system.}
\end{figure}
Thus, the current situation with (gravitational) dark matter resembles the situation with (electromagnetic) ether at the turn of the 19th and 20th centuries. Now observations that contradict GRT are explained using a completely geometric theory, and the introduction of dark matter turns out to be superfluous. All circumstances considered as proof of its existence (the Bullet cluster, possibility of the primary nucleosynthesis, etc.) require not matter as such, but additional interaction, which manifests itself in the geometric theory in the form of additional acceleration.

Let us find the value $C_2$ arising in geometric theory. Simple estimates give: $C_2 \sim M/T$, with $M \sim R^2$, where $R$ is the apparent radius of a spiral galaxy, the mass $M$ of which is concentrated in the plane of the disk, and $T$ is the orbital period corresponding to the periphery. Kepler's law implies $T \sim R^{3/2}$, therefore $C_2 \sim R^{1/2}$. On the other hand, the luminosity $L$ is also proportional to the area of the galaxy, $L \sim R^2$. Therefore, $R \sim L^{1/2}$, and finally

$$v \sim L^{1/4}. \quad (30)$$

This corresponds to the Tully-Fisher's law known from observations, and it also has no theoretical explanation in GRT. Exponents other than $\frac{1}{4}$ found in Faber-Jackson's law for elliptical galaxies can be obtained from a more accurate calculation using numerical methods.

Thus, the correction to the Minkowski metric obtained for the C+C system within the framework of the geometric approach can be chosen in the form

$$\mathcal{E}_{00}(r,v) = \frac{GM}{c^2 r} + \frac{V^2}{2c^2} e^{2i((\vec{k},\vec{x})-\omega t)}. \quad (31)$$

The corresponding trajectory of the test body is not flat and has the form of a helix wound on a circle along which the motion occurs.

In addition to the results mentioned above, the constructed geometric theory or AGD makes it possible to obtain trajectories of motion corresponding to the strange shape of the distribution of cold gas in the center of the Milky Way (in the form of the $\infty$ sign), obtained by the observatory of the Herschel mission [12], Figure 2. The equations of geometric theory also allow us to propose a new interpretation of the linear Hubble's law. Since now inertia and gravity are completely indistinguishable, the motion of distant stars across the line of sight of an arbitrary observer imparts to them a centripetal acceleration, which is linearly dependent on the distance to them. This acceleration corresponds to the "centrifugal" force of inertia, and thus the detected redshift is not cosmological, but gravitational one.

![Figure 2](image_url)

Figure 2. (a) Numerically calculated trajectories of the equal parts explosion in the C+C system; (b) Galaxy NHG-1365 (Hubble telescope, NASA) (free access); (c) Distribution of the cold gas in the Milky Way center (Herschel observ.) (free access).
References

[1] Tamm I E 1976 *Foundations of the theory of electricity* (Moscow: Nauka)
[2] Einstein A 1979 *Foundations of the general theory of relativity* In: *Albert Einstein and the Theory of Gravity* (Moscow: Mir)
[3] Siparov S 2011 *Introduction to the Anisotropic Geometrodynamics* (London-New Jersey-Singapore: World Scientific)
[4] Sax R S 2013 *Ufa mathematical journal* 5 63
[5] Landau L D and Lifshits E M 1986 *Hydrodynamics* (Moscow Nauka)
[6] Siparov S V 2008 *HNGPh* 10 64 (Preprint gr-qc/0809.1817v3).
[7] Siparov S V 2009 HNGPh 12 140 (Preprint gen-ph/1001.1501)
[8] Siparov S V 2009 *AIP Conf. Proc. Proc. 4-th Int. Gamow Conf.* Odessa 2010 1206 Ed Chakrabarti S K, Bisnovatyi-Kogan G S and Zhuk A I (New York: Melville) p 152
[9] Siparov S V 2010 BSG *Proc. Proc. Int. Conf. DGDS-2009* (Geom.Balkan Press) p 190
[10] Siparov S V 2010 *AIP Conf. Proc. Proc.Int. Conf. “Mathematics and Astronomy”* (New York: Melville) p 222
[11] Siparov S V 2010 *Proc. 10th Int. Gamov Conf.* Odessa-2010 p 71
[12] Siparov S V 2012 *Space, time and fundamental interactions* 1 99