Probabilistic Approach to Pattern Selection

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Abstract

The problem of pattern selection arises when the evolution equations have many solutions, whereas observed patterns constitute a much more restricted set. An approach is advanced for treating the problem of pattern selection by defining the probability distribution of patterns. Then the most probable pattern naturally corresponds to the largest probability weight. This approach provides the ordering principle for the multiplicity of solutions explaining why some of them are more preferable than others. The approach is applied to solving the problem of turbulent photon filamentation in resonant media.

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1 Introduction

Nonequilibrium phenomena in real systems are usually described by sets of nonlinear differential or integro-differential equations in partial derivatives. The solutions to such equations are in many cases nonuniform in space exhibiting the formation of various spatial structures. It often happens that a given set of equations possesses several solutions corresponding to different spatial patterns. And, moreover, it is often impossible to distinguish between these solutions being based on stability analysis since all these solutions can be stable. If an ensemble of solutions can be parametrized by a multiparameter $\beta$ from a manifold $B$, so that all solutions corresponding to any $\beta \in B$ are stable, then the manifold $B$ is called the stability balloon [1].

In the case of such a nonuniqueness of solutions one may wander which of the multiplicity of allowed states would actually be found for a given experimental protocol. This is how the problem of pattern selection arises, when the considered equations have many solutions for given external conditions, whereas observed patterns may constitute a more restricted set. And there no doubts are numerous systems and phenomena in real life with such a complicated behaviour [1].

Not only the solutions to equations can be nonunique but real phenomena may also display a variety of patterns for the same given conditions. This means that the multiplicity of solutions is not just an artifact but a feature expressing the intrinsic complexity of natural phenomena. If real patterns show multiplicity, is there any ordering between them, such that one solution is preferred over the other? In equilibrium thermodynamics this is a familiar concept, with the free energy providing the ordering principle. But in nonequilibrium systems, there is no such a general organizing principle to apply [1]. For some particular cases several heuristic recipes have been suggested. For example, one state may be preferred over the other if it has a larger basin of attraction for typical initial conditions. Or one may think that the fastest growing mode dominates the evolution. Or selection is made assuming that the pattern with the fastest spatial decay rate is preferable. The limits of such heuristic arguments are discussed in detail in Cross and Hohenberg [1] where one can find extensive citations to literature on the problem.

In the present paper a general approach is developed for treating the problem of pattern selection. The approach, which is formulated in Sec. 2, is based on defining the probability distribution of patterns. In Sections 4 to 5, this approach is applied to the problem of turbulent photon filamentation. An introduction to the latter problem is given in Section 3. The obtained results are in very good agreement with experiment. The final Section 7 contains conclusions.

2 Probability Distribution of Patterns

Let us consider a system of evolution equations, which displays the multiplicity of solutions describing different spatial structures. Assume that these solutions can be parametrized by a multiparameter $\beta$. All admissible values of $\beta$ form a manifold $B = \{\beta\}$. In many cases, differential equations in partial derivatives can be reduced to a $d$-dimensional system of ordinary differential equations, with a dimension $d$ that may
equal infinity [2]. For the simplicity of notation, we shall keep in mind this possibility of working with a \(d\)-dimensional dynamical system. A generalization of this case will be given at the end of the present Section.

The state of a \(d\)-dimensional dynamical system is the set

\[
y(t) = \{y_i(t) = y_i(\beta, t) | i = 1, 2, \ldots, d\}
\]

of solutions to the system of differential equations, which can always be presented in the normal form

\[
\frac{d}{dt} y(t) = v(y, t) ,
\]

where \(v = \{v_i | i = 1, 2, \ldots, d\}\) is a velocity field. By assumption, the multiplicity of solutions is parametrized by a multiparameter \(\beta\), but in intermediate calculations we shall not, for the compactness of notation, label functions with \(\beta\), restoring this dependence in final formulas.

Our aim is to classify the states (1) labelled by \(\beta\) by defining a probability measure on the manifold \(\mathcal{B}\). To introduce the probability distribution \(p(\beta, t)\) of patterns labelled by a multiparameter \(\beta\), we may resort to the ideas of statistical mechanics [3], where a probability \(p\) can be connected with entropy \(S\) by the relation \(p \sim e^{-S}\). In the nonequilibrium case, the entropy is a function of time, \(S(t)\). Since it is not the entropy itself but rather its change that is measurable, it is convenient to count entropy from its initial value \(S(0)\), thus, considering the entropy variation

\[
\Delta S(t) = S(t) - S(0) ,
\]

which is a kind of relative entropy [4]. The entropy may be defined as the logarithm of an elementary phase volume [3], which for the nonequilibrium case can be expressed as

\[
S(t) = \ln |\delta \Gamma(t)| ,
\]

with the elementary phase volume

\[
\delta \Gamma(t) \equiv \prod_i \delta y_i(t) .
\]

Hence the entropy variation (3) is

\[
\Delta S(t) = \ln \left| \frac{\delta \Gamma(t)}{\delta \Gamma(0)} \right| .
\]

The probability distribution \(p \sim e^{-\Delta S}\), being normalized with respect to \(\beta\), takes the form

\[
p(\beta, t) = \frac{e^{-\Delta S(\beta, t)}}{Z(t)} , \quad Z(t) \equiv \int e^{-\Delta S(\beta, t)} d\beta ,
\]

where the integration over \(\beta\) runs through the manifold \(\mathcal{B}\). If the latter manifold is not continuous, the integration is to be replaced by summation.

The elementary phase volume (5) can be presented as

\[
\delta \Gamma(t) = \prod_i \sum_j M_{ij}(t) \delta y_j(0) ,
\]
where $M_{ij}(t)$ are the elements of the multiplier matrix \[5\] defined through the variational derivatives
\[
M_{ij}(t) \equiv \frac{\delta y_i(t)}{\delta y_j(0)} , \quad M_{ij}(0) = \delta_{ij} . \tag{9}
\]
Then the entropy variation (6) writes
\[
\Delta S(t) = \sum_i \ln |M_{ii}(t)| . \tag{10}
\]
The multiplier matrix $\hat{M}(t) = [M_{ij}(t)]$ satisfies the equation
\[
\frac{d}{dt} \hat{M}(t) = \hat{J}(y, t) \hat{M}(t) , \tag{11}
\]
which follows from the variation of the evolution equation (2) and where $\hat{J}(y, t) = [J_{ij}(y, t)]$ is the Jacobian matrix with the elements
\[
J_{ij}(y, t) = \frac{\delta v_i(y, t)}{\delta y_j(t)} . \tag{12}
\]
The initial condition for Eq. (11), according to the definition (9), is $\hat{M}(0) = [\delta_{ij}]$. Combining Eqs. (7) and (10), we get the probability distribution of patterns,
\[
p(\beta, t) = \frac{1}{Z(t)} \prod_i \frac{1}{|M_{ii}(\beta, t)|} , \tag{13}
\]
with the normalization factor
\[
Z(t) = \int \prod_i \frac{1}{|M_{ii}(\beta, t)|} d\beta .
\]
This is the probability distribution of solutions classified with a multiparameter $\beta$. Since each solution, by definition, represents a particular pattern, expression (13) is the probability distribution of patterns. This expression naturally connects the notion of probability and the notion of stability. Really, the multipliers are smaller by modulus for more stable solutions and, consequently, for more probable patterns.

To calculate the probability distribution of patterns (13), we need to know the multipliers (9) which are defined by the equation (11). Using the latter equation, the pattern distribution (13) can be transformed as follows. Introduce the matrix $\hat{L}(t) = [L_{ij}(t)]$ with the elements
\[
L_{ij}(t) \equiv \ln |M_{ij}(t)| . \tag{14}
\]
Then the entropy variation (10) writes
\[
\Delta S(t) = \text{Tr} \; \hat{L}(t) . \tag{15}
\]

The trace of a matrix does not depend on the matrix representation. Hence, we may perform intermediate calculations using one particular representation, returning at the end to the form independent of a representation. To this end, let us consider a
representation when the multiplier matrix is diagonal. Because of Eq. (11) with the initial condition \( M_{ij}(0) = \delta_{ij} \), the multiplier matrix is diagonal if and only if the Jacobian matrix is diagonal too. In this case, the evolution equation (11) yields

\[ M_{ii}(t) = \exp \left\{ \int_0^t J_{ii}(y(t'), t') \, dt' \right\} . \]

From Eq. (14) we have

\[ L_{ii}(t) = \int_0^t \text{Re} \, J_{ii}(y(t'), t') \, dt' . \]

Introducing the notation

\[ K(t) \equiv \sum_i \text{Re} \, J_{ii}(y, t) , \quad (16) \]

we get

\[ \text{Tr} \, \hat{L}(t) = \int_0^t K(t') \, dt' . \]

Without the loss of generality, we may assume that the state (1) consists of real functions since any complex function can always be treated as a pair of real functions. Hence the velocity field in Eq. (2) can also be considered as real. Then the eigenvalues of the Jacobian matrix (12) are either real or, if complex, come in complex conjugate pairs. Therefore

\[ \sum_i \text{Re} \, J_{ii}(y, t) = \sum_i J_{ii}(y, t) = \text{Tr} \, \hat{J}(y, t) . \]

Thus, Eq. (16) can be written as

\[ K(t) = \text{Tr} \, \hat{J}(y, t) . \quad (17) \]

And for the entropy variation (15), we find

\[ \Delta S(t) = \int_0^t K(t') \, dt' . \quad (18) \]

Expression (17) in dynamical theory is called the contraction rate [6]. All consideration given above can be straightforwardly generalized to the case when the state (1) consists of functions \( y_i(x, t) \), where \( x \) is the set of space variables. This results not more than in a slight complication of notations. Then everywhere the variable \( x \) appears together with the index \( i \) as an additional continuous index, and the sums over \( i \) are to be complimented by the integrals over \( x \). The multiplier matrix (9) becomes a matrix with respect to indices \( i, j \) as well as with respect to \( x, x' \),

\[ M_{ij}(x, x', t) \equiv \frac{\delta y_i(x, t)}{\delta y_j(x', 0)} , \]

with the initial condition

\[ M_{ij}(x, x', 0) = \delta_{ij} \delta(x - x') . \]
Similarly, the Jacobian matrix (12) becomes a matrix with the elements

\[ J_{ij}(x, x', y, t) \equiv \frac{\delta v_i(x, y, t)}{\delta y_j(x', t)} . \]

Employing the matrix notation with respect to continuous variables [5], we may repeat the same steps as above in deriving the probability distribution (13). As is mentioned, the sum over \( x \) is to be understood as the corresponding integral. And the product over a continuous variable can be defined [7] as

\[ \prod_x f(x) \equiv \exp \int \ln f(x) \, dx . \]

As a result, the pattern distribution (13) reads

\[ p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ - \sum_i \int \ln |M_{ii}(x, x', \beta, t)| \, dx \right\} . \]

The contraction rate (17) becomes

\[ K(t) = \sum_i \int J_{ii}(x, x, y, t) \, dx , \quad (19) \]

where \( y = \{y_i(x, t)\} \). In this way, the contraction rate has always the form of \( \text{Tr} \hat{J}(y, t) \), as in Eq. (17), where the trace has to be defined according to the representation of the Jacobian matrix.

Restoring in the contraction rate the dependence on the parameter \( \beta \) labelling different patterns and using the entropy variation (18), we finally obtain the probability distribution of patterns

\[ p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ - \int_0^t K(\beta, t') \, dt' \right\} , \quad (20) \]

in which the normalization factor is

\[ Z(t) = \int \exp \left\{ - \int_0^t K(\beta, t') \, dt' \right\} \, d\beta . \]

Thus, each solution labelled by \( \beta \) is equipped with the probability weight (20). Consequently, that pattern is preferred over the other which has a higher probability weight. This is equivalent, because of the form (20), to saying that one pattern is preferable over others if its local contraction

\[ \Lambda(\beta, t) \equiv \frac{1}{t} \int_0^t K(\beta, t') \, dt' \]

is minimal with respect to \( \beta \). The latter provides the ordering principle for classifying solutions and related patterns. The local contraction plays for nonequilibrium dynamical systems the role analogous to that of the free energy for equilibrium statistical systems.
3 Turbulent Photon Filamentation

To illustrate the probabilistic approach to pattern selection, developed in the previous section, let us consider spatial structures appearing in resonant samples when increasing the Fresnel number. In active media interacting with electromagnetic field, there appears a variety of spatiotemporal structures. For example, electric field in laser cavities can exhibit spatial states, such as solitons and vortices, which bear a close analogy with similar structures in liquids [8,9]. In general, there is a direct correspondence between the Maxwell-Bloch equations for slowly varying field amplitudes and hydrodynamic equations for compressible viscous liquid [1,10]. The Fresnel number for optical systems plays the same role as the Reynolds number for fluids. In the same way as when increasing the Reynolds number, the fluid becomes turbulent, the field dynamics exhibits chaotic behaviour when increasing the Fresnel number. Similarly to hydrodynamics, optical systems display spatial multi-stability with several coexisting distinct stable states for the same values of parameters [1,11]. Spatiotemporal chaos in optical systems is characterized by the same fast decay of correlation functions as it occurs for turbulent fluids. This is why one calls such chaotic optical phenomena optical turbulence [1,10,11].

In active optical systems, having the standard cylindrical shape, the transition from regular behaviour to spatiotemporal chaos occurs with the increasing Fresnel number

\[ F \equiv \frac{\pi R^2}{\lambda L}, \]

where \( R \) is the cylinder radius, \( L \) is the characteristic length, and \( \lambda \) is the radiation wavelength. Physical processes accompanying the route from a regular regime to chaotic one are similar in different active optical media.

At small Fresnel numbers \( F \ll 1 \), there exists the sole transverse central mode practically uniformly filling the medium. When the Fresnel number is around \( F \sim 1 \), the cavity can house several transverse modes seen as a regular arrangement of bright spots in the transverse cross-section. These regular spatial structures emerge from an initially homogeneous state with a break of space-translational symmetry. They are regular in space forming ordered geometric arrays, such a polygons, and they are regular in time being either stationary or periodically oscillating. These transverse structures are imposed by the cavity geometry and correspond to the empty-cavity Gauss-Laguerre modes. Such regular structures are well understood theoretically, their description being based on field expansions over the modal Gauss-Laguerre functions prescribed by the cylindrical geometry, and the theory being in reasonable agreement with experiments for lasers, e.g. for CO\(_2\) and Na\(_2\) lasers [11–17], and for active nonlinear media, as the photorefractive Bi\(_{12}\)SiO\(_{20}\) crystal pumped by a laser [18–20]. Similar structures also arise in many passive nonlinear media, such as a Kerr medium [20]. For Fresnel numbers up to \( F \approx 5 \), the number of bright spots is proportional to \( F^2 \). In the longitudinal cross-section this corresponds to the existence of bright filaments whose number follows the \( F^2 \) law as \( F \) increases.

Around \( F \approx 10 \) there occurs a principal change of properties, from the existence of regular structures to a turbulent-type state [21–23], with the intermittent behaviour in the region \( 5 < F < 15 \). This happens in lasers [21–23] as well as in photorefractive
crystals [18–20]. For Fresnel numbers $F > 15$, the arising spatial structures are very different from those associated with the empty-cavity modes. The modal expansion is no longer relevant and the boundary conditions have no importance. The medium looks as consisting of a large number of parallel independently flashing filaments, whose number is proportional to $F$, contrary to the case of small Fresnel numbers with the number of filaments proportional to $F^2$. The filaments are chaotically distributed in space, are not correlated with each other, and are aperiodically flashing in time. Such a spatio-temporal chaotic behaviour is characteristic of hydrodynamic turbulence, this is why the similar phenomenon in optics is commonly called the optical turbulence [1,10,11,23].

Contrary to the regime of small $F$, when the regular spatial structures are prescribed by the geometry and boundary conditions imposing their symmetry constraints, the turbulent optical filamentation is strictly self-organized, with its organization emerging from intrinsic properties of the medium [20,24]. This kind of optical turbulence has been observed in both photorefractive crystals [18–20] and lasers [21–23,25–30]. Especially accurate and thorough experimental studies for CO$_2$ and Dye lasers have been accomplished in Refs. [25–30]. The independence of turbulent optical filamentation of boundary conditions and, hence, its purely self-organized nature are confirmed by the observation of this effect in the resonatorless discharge-tube superluminescent samples, such as lasers on Ne, Tl, Pb, N$_2$, and N$_2^+$ vapors [31–35]. Since the optical turbulence is characterized by the formation of bright filaments with a high density of photons, this phenomenon may be named the turbulent photon filamentation.

Let us summarize the main features characterizing the regular and turbulent regimes in active optical media. At low Fresnel numbers $F < 10$, the regular regime occurs whose typical characteristics are: (i) Bright filaments are regularly arranged in space forming an ordered structure seen in the transverse cross-section as a polygon made of bright spots. (ii) The spatially ordered structure is either stationary or periodically oscillating in time. (iii) The number of bright filaments is proportional to $F^2$.

At high Fresnel number $F > 10$, the turbulent regime develops whose typical features are: (i) The bright photon filaments are chaotically distributed in space. (ii) The filaments aperiodically flash in time. (iii) The number of these uncorrelated filaments is proportional to $F$.

While for the low-Fresnel-number regime, with regular optical structures, a good theoretical understanding was achieved [11–20], for the high-Fresnel-number regime of the turbulent photon filamentation, there has been no persuasive theory developed. This problem was addressed in Refs. [36–39], where the consideration was based on a simple model, only the stationary case was analysed, and the minimum-energy arguments were employed. This model consideration showed that photon filaments can really be formed in resonant media due to an effective interaction between atoms through photon exchange. The estimates for the filament radius turned out to be in good agreement with experiment. However, the empirical approach of Refs. [36–39] cannot be accepted as completely satisfactory. This is mainly because of the two principal points. One is that a stationary case was addressed, while turbulence is rather a nonstationary phenomenon and must be treated by time-dependent evolution equations. The second point is that the minimal-energy condition was invoked, while there is no such a principle for nonequilibrium systems.

Here we demonstrate that the problem of turbulent photon filamentation can be
successfully treated by the probabilistic approach to pattern selection formulated in Section 2. Our consideration is based on realistic evolution equations for resonant media. Such equations are necessary for calculating the contraction rate $K$ defining the probability distribution of patterns (20).

4 Equations for Resonant Medium

The system of $N$ resonant atoms interacting with electromagnetic field is described by the Hamiltonian

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{af},$$

consisting of the following terms: The Hamiltonian of two-level resonant atoms

$$\hat{H}_a = \frac{1}{2} \sum_{i=1}^{N} \omega_0 (1 + \sigma_i^z),$$

with the transition frequency $\omega_0$, where $\sigma_i^z$ is a Pauli operator. The radiation-field Hamiltonian

$$\hat{H}_f = \frac{1}{8\pi} \int \left( \vec{E}^2 + \vec{H}^2 \right) d\vec{r},$$

with electric field $\vec{E}$ and magnetic field $\vec{H} = \vec{\nabla} \times \vec{A}$, where the vector potential $\vec{A}$ is assumed to satisfy the Coulomb gauge calibration $\vec{\nabla} \cdot \vec{A} = 0$. The atom-field interaction Hamiltonian

$$\hat{H}_{af} = -\sum_{i=1}^{N} \left( \frac{1}{c} \vec{j}_i \cdot \vec{A}_i + \vec{d}_i \cdot \vec{E}_0 i \right)$$

has the standard dipole form, in which $\vec{A}_i \equiv \vec{A}(\vec{r}_i, t)$; the transition-current and transition dipole operators are

$$\vec{j}_i = i\omega_0 \left( \vec{d} \sigma_i^+ - \vec{d}^* \sigma_i^- \right), \quad \vec{d}_i = \vec{d} \sigma_i^+ + \vec{d}^* \sigma_i^-,$$

where $\vec{d}$ is the transition dipole and $\sigma_i^\pm$ are the rising or lowering operators, respectively; and $\vec{E}_0 i$ is a seed field [40].

To consider nonuniform systems with arising spatial structures, it is convenient to invoke the space-time representation of evolution equations, whose general description is given in the book [40] and a detailed analysis in Refs. [41,42]. The equations are written for the average quantities

$$u(\vec{r}, t) \equiv < \sigma^- (\vec{r}, t) >, \quad s(\vec{r}, t) \equiv < \sigma^z (\vec{r}, t) >,$$

using the standard semiclassical and Born approximations. For the compactness of presentation, let us introduce the notation

$$f(\vec{r}, t) \equiv f_0(\vec{r}, t) + f_{\text{rad}}(\vec{r}, t)$$

for an effective field acting on an atom. This field consists of the term

$$f_0(\vec{r}, t) \equiv -i\vec{d} \cdot \vec{E}_0(\vec{r}, t),$$
due to the cavity seed field, and of the term
\[
f_{\text{rad}}(\vec{r}, t) = -\frac{3}{4}i\gamma\rho \int \left[ \varphi(\vec{r} - \vec{r}') \ u(\vec{r}', t) - \vec{e}_d^2 \varphi^*(\vec{r} - \vec{r}') \ u^*(\vec{r}', t) \right] d\vec{r}'
\] (28)
corresponding to the interaction of atoms through the common radiation field, where \(\rho\) is the density of atoms, \(\vec{d} \equiv d_0 \vec{e}_d, \ d_0 \equiv |\vec{d}|\), and
\[
\varphi(\vec{r}) \equiv \frac{\exp(ik_0 |\vec{r}|)}{k_0 |\vec{r}|}, \quad k_0 \equiv \frac{\omega_0}{c}, \quad \gamma \equiv \frac{4}{3}k_0d_0^2.
\]
The seed field
\[
\vec{E}_0(\vec{r}, t) = \vec{E}_1 e^{i(kz - \omega t)} + \vec{E}_1^* e^{-i(kz - \omega t)}
\] (29)
selects a longitudinal mode propagating along the axis \(z\) which is the axis of a cylindrical sample. The resulting equations are
\[
\frac{du}{dt} = -(i\omega_0 + \gamma_2)u + sf,
\]
\[
\frac{ds}{dt} = -2(u^*f + f^*u) - \gamma_1(s - \zeta),
\]
\[
\frac{d|u|^2}{dt} = -2\gamma_2|u|^2 + s(u^*f + f^*u),
\]
where \(\gamma_1\) and \(\gamma_2\) are the level and line widths, respectively, and \(\zeta\) is the pumping parameter defined by the intensity of nonresonant lamp pumping.

The sample has cylindrical shape typical of lasers. The radiation wavelength \(\lambda\), cylinder radius \(R\), and its length \(L\) are related by the inequalities
\[
\frac{\lambda}{R} \ll 1, \quad \frac{R}{L} \ll 1.
\] (31)
We consider the quasiresonance situation when
\[
\frac{|\Delta|}{\omega_0} \ll 1, \quad \Delta = \omega - \omega_0.
\] (32)
As usual, the relaxation parameters are assumed to be small, so that
\[
\frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1.
\] (33)
The solutions to Eqs. (30) are, in general, nonuniform. This is because atoms interact with each other through the radiation field (28) containing the function \(\varphi(\vec{r})\) fastly oscillating in space and diminishing with increasing \(|\vec{r}|\). Hence the atoms, situated far from each other, do not interact and radiate independently, although in the longitudinal direction they may be correlated due to the cylindrical geometry of the sample and the propagation of radiated field along the axis \(z\). This suggests to look for a solution of Eqs. (30) in the form of a bunch of filaments aligned along the cylinder axis. Let us imagine that the radiation field inside the sample is stratified into \(N_f\) photon filaments stretched along the axis \(z\). These filaments can have different radii \(r_n\),
with $n = 1, 2, \ldots, N_f$. The radiation is mainly concentrated inside the bright filaments, fading away outside them, so that at the radial distance $b_n$ the corresponding solutions are an order of magnitude smaller than at the axis of a filament. The relation between the characteristic lengths $b_n$ and $r_n$ can be written down if the profile of solutions inside a filament is known. If the filament profile can be approximated by the normal law $\exp(-r^2/2r_n^2)$, with the filament radius $r_n$ being the standard deviation, then

$$b_n = \sqrt{2 \ln 10} r_n.$$  \hfill (34)

The filaments do not interact with each other, because of which their locations in the radial cross-section are random. This means that the filament axes are located at random points \{x_n, y_n\}, where $n = 1, 2, \ldots, N_f$.

The above discussion explains why it is reasonable to try to find the solutions of Eqs. (30) in the form of expansions

$$u(\vec{r}, t) = \sum_{n=1}^{N_f} u_n(\vec{r}, t) \Theta_n(x, y) e^{ikz}, \quad s(\vec{r}, t) = \sum_{n=1}^{N_f} s_n(\vec{r}, t) \Theta_n(x, y)$$  \hfill (35)

over filaments, where

$$\Theta_n(x, y) \equiv \Theta \left(b_n^2 - (x - x_n)^2 + (y - y_n)^2\right)$$

is a unit-step function. Substituting the presentation (35) in Eqs. (30), we obtain a system of equations for $u_n$, $s_n$, and $|u_n|^2$. These equations can be simplified if we are not interested in the detailed internal structure of each filament but rather wish to find out their characteristic sizes. Then we may employ the mean-field approximation for the averages

$$u(t) \equiv \frac{1}{V_n} \int_{V_n} u_n(\vec{r}, t) \, d\vec{r}, \quad s(t) \equiv \frac{1}{V_n} \int_{V_n} s_n(\vec{r}, t) \, d\vec{r}, \quad |u(t)|^2 \equiv \frac{1}{V_n} \int_{V_n} |u_n(\vec{r}, t)|^2 \, d\vec{r},$$  \hfill (36)

where the integration is over the volume $V_n \equiv \pi b_n^2 L$ and the index $n$ in the left-hand sides, for short, is dropped.

To present the equations for the functions (36) in a compact form, we introduce two effective coupling parameters:

$$g \equiv \frac{3\gamma \rho}{4\gamma_2 V_n} \int_{V_n} \frac{\sin[k_0|\vec{r} - \vec{r}'| - k(z - z')]}{k_0|\vec{r} - \vec{r}'|} \, d\vec{r} \, d\vec{r}'$$  \hfill (37)

and

$$g' \equiv \frac{3\gamma \rho}{4\gamma_2 V_n} \int_{V_n} \frac{\cos[k_0|\vec{r} - \vec{r}'| - k(z - z')]}{k_0|\vec{r} - \vec{r}'|} \, d\vec{r} \, d\vec{r}'. \hfill (38)$$

These parameters enter the definitions of the collective frequency and collective width, respectively,

$$\Omega \equiv \omega_0 + g'\gamma_2 s, \quad \Gamma \equiv \gamma_2(1 - gs).$$  \hfill (39)
Then from Eqs. (30), employing the introduced notation, for the functions (36) we obtain the equations

\[
\frac{du}{dt} = -(i\Omega + \Gamma)u - is\vec{d} \cdot \vec{E}_1 e^{-i\omega t}, \tag{40}
\]

\[
\frac{ds}{dt} = -4g\gamma_2|u|^2 - \gamma_1(s - \zeta) - 4 \text{Im} \left( u^* \vec{d} \cdot \vec{E}_1 e^{-i\omega t} \right),
\]

\[
\frac{d|u|^2}{dt} = -2\Gamma|u|^2 + 2s \text{Im} \left( u^* \vec{d} \cdot \vec{E}_1 e^{-i\omega t} \right),
\]

describing the time evolution of a filament. Such a reduction of initial equations for a bunch of filaments to the set (40) of equations for each of the filaments has become possible due to the absence of interactions between filaments. Different filaments can have different radii whose distribution is yet to be defined.

It is worth emphasizing that the presentation of solutions as the expansions (35) over filaments is rather general and includes as well the case when there are no separate filaments at all. Really, if it turns out that the most probable filament radius is close to the radius of the sample, \( R \), this would actually mean that there are no several filaments, but the whole volume of the sample is filled by one filament.

5 Characteristics of Arising Filaments

To analyze Eqs. (40), we employ the scale separation approach [43,44] which is a generalization of the averaging technique [45] to the case of nonequilibrium statistical systems. To this end, we notice first of all that the functional variables in Eqs. (40) can be classified onto fast and slow in time. Due to the existence of the small parameters (33), the variable \( u \) is fast as compared to the slow variables \( s \) and \( |u|^2 \). The slow variables play the role of quasi-invariants for the fast function \( u \). With \( s \) being a quasi-invariant, the solution for the fast variable \( u \), described by the first of Eqs. (40), writes

\[
u(t) = u_0 e^{-i(\Omega + \Gamma)t} + \frac{sd\vec{d} \cdot \vec{E}_1}{\omega - \Omega + i\Gamma} \left[ e^{-i\omega t} - e^{-i(\Omega + \Gamma)t} \right], \tag{41}
\]

where \( u_0 = u(0) \). This solution is to be substituted in the second and third of Eqs. (40) whose right-hand sides are to be averaged over time explicitly entering the fastly oscillating functions. In this way, we meet the quantity

\[
\alpha \equiv \frac{\text{Im}}{s\Gamma} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau u^*(t) \vec{d} \cdot \vec{E}_1 e^{-i\omega t} dt \tag{42}
\]

describing the influence of the seed field on atoms. Taking into account Eq. (41), we have

\[
\alpha = \frac{|d \cdot \vec{E}_1|^2}{(\omega - \Omega)^2 + \Gamma^2}. \tag{43}
\]

Recall that the seed field is, by definition, a weak cavity field whose role is to fix the axis of the cylindrical sample. This field does not pump the energy into the system which is standardly done by nonresonant lamps and which is described by the pumping parameter \( \zeta \) in the evolution equations (30) or (40). The seed field selects a longitudinal
mode imposing no restrictions on transverse modes. The selection of the latter has to be done by the internal properties of the evolution equations. The weakness of the seed field is to be understood in the sense of the smallness of the parameter \(\alpha\),

\[
\alpha \ll 1 \, . \tag{44}
\]

In analysing further the evolution equations (40), it is convenient to make the transformation of the functional variable \(|u|^2\) to

\[
w \equiv |u|^2 - \alpha s^2 \, . \tag{45}
\]

Finally, following the scale separation approach \([43,44]\), from the last two of Eqs. (40) we obtain the equations

\[
\frac{ds}{dt} = -4g\gamma_2 w - \gamma_1 (s - \zeta) \, , \quad \frac{dw}{dt} = -2\gamma_2 (1 - gs) w \, . \tag{46}
\]

for the slow variables. These equations can be written in the form (2),

\[
\frac{ds}{dt} = v_1 \, , \quad \frac{dw}{dt} = v_2 \, ,
\]

with the velocity field

\[
v_1 = -4g\gamma_2 w - \gamma_1 (s - \zeta) \, , \quad v_2 = -2\gamma_2 (1 - gs) w \, .
\]

For the Jacobian matrix (12), we now have

\[
J_{11} = \frac{\partial v_1}{\partial s} = -\gamma_1 \, , \quad J_{12} = \frac{\partial v_1}{\partial w} = -4g\gamma_2 \, ,
\]

\[
J_{21} = \frac{\partial v_2}{\partial s} = -2\gamma_2 w \, , \quad J_{22} = \frac{\partial v_2}{\partial w} = -2\gamma_2 (1 - gs) \, .
\]

From here, we find the contraction rate (17),

\[
K = \text{Tr} \, \hat{J} = -\gamma_1 - 2\gamma_2 (1 - gs) \, . \tag{47}
\]

The value of Eq. (47) depends on the characteristics of filaments through the coupling parameter (37) containing in its definition the volume \(V_n = \pi b_n^2 L\), which is the volume of a cylinder enveloping a filament. The radius of the enveloping cylinder, \(b_n\), is related to the filament radius, \(r_n\), by relation (34). For what follows, it is convenient to introduce the dimensionless quantity

\[
\beta \equiv \frac{kb_n^2}{2L} = \frac{\pi b_n^2}{\lambda L} \, . \tag{48}
\]

Thus, the contraction rate (47) is a function \(K = K(\beta, t)\) of the variable (48) and also a function of time entering through \(s = s(t)\). For different filaments, the variable \(\beta\) can take different values from the interval

\[
0 < \beta \leq F \, , \tag{49}
\]
where $F$ is the Fresnel number. The distribution of the variable $\beta$ in the interval (49) is given by the probability distribution (20). The maximum of the latter defines the most probable $\beta$ and, respectively, the most probable filament radius.

Let us consider the very beginning of the process, when $t \to +0$, in order to understand what are the most probable filaments to be formed. At this initial stage, the probability distribution (20) can be written as

$$p(\beta, t) \simeq \frac{1}{Z(t)} \exp \{-K(\beta, 0) t\} .$$

Hence the maximum of this distribution corresponds to the minimal contraction rate. The latter, according to Eq. (47), depends on $\beta$ through $g = g(\beta)$. We consider the standard laser-type setup, when the atoms at the initial time are not inverted, i.e. $s_0 < 0$, and the medium is pumped with a long nonresonant pulse whose action is described by the pumping parameter $\zeta > 0$ in the evolution equations (30), (40), and (46). In this case, the minimum of the contraction rate $K(\beta, 0)$ is equivalent to the maximum of the coupling $g(\beta)$, which is defined by the equations

$$\frac{dg}{d\beta} = 0 , \quad \frac{d^2g}{d\beta^2} < 0 . \quad (50)$$

To solve these equations, we need to analyse the integral (37) giving $g = g(\beta)$.

The coupling $g(\beta)$ defined by the integral (37) can be presented in the form

$$g(\beta) = \frac{3\pi \gamma \rho L}{4 \gamma k^2} \left[ \pi \beta - \int_0^{2\beta} \text{Si}(x) \, dx \right] . \quad (51)$$

The procedure of reducing Eq. (37) to the form (51) is described in the Appendix. The extrema of $g(\beta)$ are given by the equation

$$\text{Si}(2\beta) = \pi . \quad (52)$$

From several solutions of Eq. (52) we have to choose the absolute maximum of $g(\beta)$. The latter occurs at $\beta = 0.96$, which, according to Eq. (48), gives $b_n = 0.55\sqrt{\lambda L}$. Using the relation (34), we find the most probable radius of a filament

$$r_f = 0.26\sqrt{\lambda L} . \quad (53)$$

In the typical experiments [21–23,25–35] observing the turbulent photon filamentation, the excitation was achieved by means of the quasistationary nonresonant pumping. Such a pumping can be treated as a quasistationary process if its duration is much longer than the characteristic time $2\pi/\omega$ of fast oscillations, which is always the case. The quasistationary pumping can be characterized by an effective pumping parameter $\zeta$ in the evolution equations. To describe the finite duration time of the pumping procedure, one can consider $\zeta = \zeta(t)$ as a slow function of time, slow in the sense of the inequality

$$\frac{2\pi}{\omega} \left| \frac{d\zeta}{dt} \right| \ll 1 .$$
In this way, for the standard laser setup, the filament radius (53) defines the most probable pattern for the arising bunch of photon filaments.

The most probable number of filaments can be evaluated from the normalization condition

\[
\frac{1}{V} \int s(\vec{r}, t) \, d\vec{r} = \zeta(t),
\]

where the integration runs over the whole volume of the sample, \( V = \pi R^2 L \). To approximately calculate the integral (54), let us consider the moment of time when the filaments have already been formed and the population difference inside each filament of the radius \( r_f \) has reached a value close to the pumping parameter \( \zeta \). Then we may write

\[
\int s(\vec{r}, t) \, d\vec{r} \simeq N_f V_f \zeta + (V - N_f V_f) s_{\text{out}},
\]

where \( V_f \equiv \pi r_f^2 L \) and \( s_{\text{out}} \) is a population difference outside the filaments. In this case, independently of the value \( s_{\text{out}} \), condition (54) yields

\[
N_f = \left( \frac{R}{r_f} \right)^2.
\]

For the filaments of the most probable radius (53), equation (55) gives

\[
N_f = 4.71 F.
\]

The number of filaments is proportional to the Fresnel number, which is in complete agreement with all experiments on the turbulent photon filamentation. The coefficient 4.71 is also in good agreement with experiments [18–20].

6 Dynamics of Filament Flashing

The time evolution of the filaments is described by Eqs. (46). At the initial stage of the process, when \( \gamma_1 t \ll 1 \), we may omit the term containing \( \gamma_1 \). Then the resulting system of nonlinear equations can be solved exactly yielding the solutions

\[
s = -\frac{\gamma_0}{g\gamma_2} \tanh \left( \frac{t - t_0}{\tau_0} \right) + \frac{1}{g}, \quad w = \frac{\gamma_0^2}{4g^2\gamma_2^2} \text{sech}^2 \left( \frac{t - t_0}{\tau_0} \right),
\]

in which \( \gamma_0 \) is the radiation width, \( \tau_0 \) is the radiation time, so that

\[
\gamma_0^2 = \Gamma_0^2 + 4g^2\gamma_2^2 \left( |u_0|^2 - \alpha_0 s_0^2 \right),
\]

where \( u_0 \equiv u(0), s_0 \equiv s(0), \alpha_0 \equiv \alpha(0) \), and

\[
\Gamma_0^2 \equiv \gamma_2 (1 - gs_0), \quad \gamma_0 \tau_0 \equiv 1;
\]

the delay time \( t_0 \) being given by the expression

\[
t_0 = \frac{\tau_0}{2} \ln \left| \frac{\gamma_0 - \Gamma_0}{\gamma_0 + \Gamma_0} \right|.
\]
Invoking the transformation (45), we get

$$|u|^2 = \frac{\gamma_0^2}{4g^2\gamma_2} \sech^2 \left( \frac{t-t_0}{\tau_0} \right) + \alpha s^2.$$  (60)

The found solutions depend on the coupling $g$ both directly as well as through the characteristic parameters $\gamma_0$, $\tau_0$, and $t_0$. This means that the properties of filaments with different radii are essentially different, since $g = g(\beta)$.

At the later stage, the term with $\gamma_1$ cannot be dropped. Then Eqs. (46) cannot be solved exactly. But we may look at the long-time behaviour, when the system approaches a stationary regime. The latter can be achieved if the duration of the pumping pulse is longer than $\gamma_1^{-1}$, as well as $\gamma_2^{-1}$, which is usually the case when pumping a resonant medium.

Before the stationary regime is reached, the oscillation of solutions is described by the eigenvalues of the Jacobian matrix associated with Eqs. (46). These eigenvalues are

$$\lambda^{\pm} = -\frac{1}{2} \left[ \gamma_1 + \gamma_2(1 - gs) \right] \mp \left\{ \left[ \gamma_1 - 2\gamma_2(1 - gs) \right]^2 - 32\gamma_2^2g^2w \right\}^{1/2}. $$  (61)

Because of the dependence of $s(t)$ and $w(t)$ on time, Eq. (61) shows that $\lambda^{\pm}(t)$ is also time dependent, which means that the oscillation of solutions cannot be defined by fixed frequencies but occurs in an aperiodic way.

At long times, such that $\gamma_1 t \gg 1$ and $\gamma_2 t \gg 1$, the system approaches a stationary regime. There are two types of stationary solutions to Eqs. (46), one of them is given by the fixed point

$$s_1^* = \zeta, \quad w_1^* = 0;$$  (62)

and another, by the fixed point

$$s_2^* = \frac{1}{g}, \quad w_2^* = \frac{\gamma_1(g\zeta - 1)}{4\gamma_2g^2}. $$  (63)

The stability of the stationary solutions can be determined from the Lyapunov analysis. To this end, we have to evaluate the Jacobian eigenvalues (61) at the corresponding fixed points (62) and (63), which results in the characteristic exponents

$$\lambda_1^+ = -\gamma_1, \quad \lambda_1^- = -2\gamma_2(1 - g\zeta) $$  (64)

and, respectively,

$$\lambda_2^+ = -\frac{\gamma_1}{2} \left\{ 1 \pm \left[ 1 + 8 \frac{\gamma_2}{\gamma_1} (1 - g\zeta) \right]^{1/2} \right\}. $$  (65)

The real parts of Eqs. (64) and (65) define the Lyapunov exponents whose signs characterize the stability of the related fixed points. As far as the values of the characteristic exponents (64) and (65) depend on the coupling $g = g(\beta)$ which, according to Eq. (51), essentially depends on the parameter $\beta$, that is, on the radius of a filament, the stability properties for the filaments of different radii can be drastically different. For
the parameter $\beta$ in the interval (49), the coupling (51) varies in a wide diapason. Thus, for small $\beta$, we have

$$g(\beta) \simeq \frac{3\gamma \rho \lambda^2 L}{16 \gamma_2} \beta \quad (\beta \ll 1),$$

hence $g(\beta) \to 0$ as $\beta \to 0$. For large $\beta$, Eq. (51) gives

$$g(\beta) \simeq \frac{3\gamma L}{16\pi \gamma_2 \lambda} \rho \lambda^3 \quad (\beta \gg 1).$$

In the case of the wavelength $\lambda \gg a$, much larger than the mean interatomic distance, one has $\rho \lambda^3 \gg 1$. Since usually $\gamma \sim \gamma_2$ and $L \gg \lambda$, the coupling may reach quite large values $g(\beta) \gg 1$. Such a wide variation of the coupling $g = g(\beta)$ results in rather different characteristic exponents (64) and (65) for different filaments and, as a consequence, in essentially different stability properties of the latter.

All filaments can be separated into three types. One group consists of those filaments for which

$$g(\beta) \zeta < 1.$$ \hspace{1cm} (66)

Then the fixed point (62) is a stable node, while that (63) is a saddle point. The filaments with the radii satisfying condition (66) are characterized by the stationary solutions (62), which shows that these filaments after the time $T_1 \equiv \gamma_1^{-1}$ practically stop radiating coherently.

Increasing $g(\beta)$, one reaches the equality $g(\beta) \zeta = 1$, when both fixed points (62) and (63) merge together becoming neutral. In the interval

$$1 < g(\beta) \zeta \leq 1 + \frac{\gamma_1}{8 \gamma_2},$$ \hspace{1cm} (67)

the fixed point (62) is a saddle point and that (63) is a stable node. Hence, the filaments with the radii satisfying condition (67) are described by the stationary solutions (63). Such filaments continue radiating coherently in the stationary regime, although the level of coherence, characterized by the value of $w_2^*$, is not high.

The third group of filaments corresponds to large coupling, such that

$$g(\beta) \zeta > 1 + \frac{\gamma_1}{8 \gamma_2}.$$ \hspace{1cm} (68)

Then the fixed point (62) remains a saddle point while that (63) becomes a stable focus. The latter means that the approach of the related solutions to the stationary point (63) is not monotonic but through a series of pulses. The filaments with the radii satisfying condition (68) radiate by bright flashes interrupted by the intervals of darkness. This intermittent behaviour, for finite times, is not periodic, as is seen from the form of Eq. (61), but as $t \to \infty$, there appears an asymptotic period. This can be noticed from the characteristic exponent (65) which, for the case (68), can be written as

$$\lambda_2^\pm = -\frac{\gamma_1}{2} \mp i \omega_\infty,$$ \hspace{1cm} (69)

where the asymptotic frequency is

$$\omega_\infty = \frac{\gamma_1}{2} \left\{ 8 \frac{\gamma_2}{\gamma_1} [g(\beta) \zeta - 1] - 1 \right\}^{1/2}.$$ \hspace{1cm} (70)
The value of the latter, because of the dependence on $g(\beta)$, is essentially different for different filaments.

7 Conclusion

An approach is developed for treating the problem of pattern selection. The approach is based on defining the probability distribution of patterns. This gives the ordering principle for the multiplicity of admissible solutions to nonlinear differential equations and, respectively, to the corresponding space structures. The maximum of the probability distribution naturally defines the most probable pattern.

This probabilistic approach is applied to the problem of turbulent photon filamentation in laser media. The most probable filament radius is found, and the number of filaments is evaluated. The found values are in very good agreement with experiment. Thus, the filament radii observed in experiments with dye and CO$_2$ lasers [25–30] are $r_f \approx 0.01$ cm for dye lasers and $r_f \approx 0.08$ cm for CO$_2$ lasers, which is in perfect agreement with formula (53). The filament radii measured in experiments with Ne, Tl, Pb, N$_2$, and N$_2^+$ vapor lasers [31–35] are $r_f \approx 0.01$ cm, which again coincides with the result of Eq. (53) for the corresponding wavelengths and laser lengths. The number of filaments, given by Eq. (55) is $N_f \approx 10^2 - 10^3$ for different experiments [31–35]. The found dependence of the filament radius $r_f \sim F^{-1/2}$ and filament number $N_f \sim F$ on the Fresnel number is the same as observed experimentally.

The turbulent photon filamentation provides a good example of how a rather complicated phenomenon can be successfully described by the probabilistic approach to pattern selection. The formulation of this approach in Section 2 is general, which permits one to apply it for different phenomena where the problem of pattern selection arises.

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Appendix. Effective Coupling

The effective coupling $g$ is given by the integral (37) over the enveloping cylinder of radius $b_n$. Assuming that $\lambda \ll b_n$, the integral (37) can be reduced to the form

$$g = \frac{3\pi \gamma \rho}{2\gamma_2^2} \int_0^{b_n} r \, dr \int_{-L/2}^{L/2} \frac{\sin(k_0 \sqrt{r^2 + z^2} - k z)}{k_0 \sqrt{r^2 + z^2}} \, dz,$$

where $r$ is the radial variable. Because of the quasiresonance condition (32), we have $k_0 \approx k$. Then, with the change of the variable $x \equiv k(\sqrt{r^2 + z^2} - z)$, we get

$$g = \frac{3\pi \gamma \rho}{2\gamma_2 k} \int_0^{b_n} r \, dr \int_{k r^2 / L}^{k L} \frac{\sin x}{x} \, dx.$$

Here, the integration limit $k L$, due to the inequality $\lambda \ll L$, can be replaced by $\infty$. As a result, we have

$$g = \frac{3\pi \gamma \rho}{2\gamma_2 k} \int_0^{b_n} \left[ \frac{\pi}{2} - \text{Si} \left( \frac{k r^2}{L} \right) \right] r \, dr,$$

where

$$\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t} \, dt = \frac{\pi}{2} + \int_{\infty}^x \frac{\sin t}{t} \, dt$$

is the integral sine. Introducing notation (48), we come to $g(\beta)$ given in Eq. (51). In the same way, for the coupling (38) we find

$$g'(\beta) = -\frac{3\pi \gamma \rho L}{4\gamma_2 k^2} \int_0^{2\beta} \text{Ci}(x) \, dx,$$

where

$$\text{Ci}(x) \equiv \int_0^x \frac{\cos t}{t} \, dt$$

is the integral cosine. The found expressions for the effective couplings can also be transformed by means of the integrals

$$\int \text{Si}(x) \, dx = x\text{Si}(x) + \cos x, \quad \int \text{Ci}(x) \, dx = x\text{Ci}(x) - \sin x.$$

This yields

$$g(\beta) = \frac{3\pi \gamma \rho L}{4\gamma_2 k^2} \left[ \pi \beta - 2\beta \text{Si}(2\beta) + 1 - \cos(2\beta) \right],$$

$$g'(\beta) = \frac{3\pi \gamma \rho L}{4\gamma_2 k^2} \left[ \sin(2\beta) - 2\beta \text{Ci}(2\beta) \right].$$
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