CONFORMALLY QUASI-RECURRENT
PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Conformally quasi-recurrent pseudo-Riemannian manifolds are investigated, with emphasis on 4-dimensional Lorentzian ones. It is shown that the Ricci tensor and the gradient of the fundamental vector are both Weyl compatible tensors (the notion was introduced recently by the authors); topological conditions for the vanishing of the first Pontryagin form are stated. The fundamental vector of conformally quasi-recurrent 4-dimensional Lorentzian manifolds is null and unique up to a scaling; moreover it is an eigenvector of the Ricci tensor and its integral curves are geodesics. Such Lorentzian manifolds are Petrov type-N with respect to the fundamental null vector.

1. Introduction

Recurrent manifolds were investigated by many geometers: Walker [4] studied manifolds with recurrent Riemann curvature, Adati and Miyazawa [1] studied conformally recurrent manifolds, Mc Lenaghan and Leroy [22] and then Mc Lenaghan and Thompson [23] investigated space-times with complex recurrent conformal curvature tensor; they showed that they are Petrov type D and N and obtained the metric form for the case of real recurrence vector. Other results are found in [15], [16] and [31]. Prvanovic introduced the following notion [27], [28]:

Definition 1.1. A n-dimensional pseudo-Riemannian manifold is conformally quasi recurrent, (CQR)ₙ, if it is not conformally flat, and there is a non-zero vector field Aᵢ (the fundamental vector) such that:

\( \nabla_j C_{jklm} = 2A_i C_{jklm} + A_j C_{iklm} + A_k C_{jilm} + A_l C_{jkim} + A_m C_{jkli}. \)

\( C_{jklm} \) is the conformal curvature tensor,

\( R_{jkl} = -R_{mkl} \) is the Ricci tensor and \( R = R^m_m \) is the scalar curvature.

Condition (1) arises in certain studies on conformal transformations of the metric [4]. Manifolds with the same characterisation were named differently, pseudo conformally symmetric (PCS)ₙ, by De and Biswas [5]. De and Mazumdar [6] proved the following:

1) if a (PCS)ₙ manifold admits a proper conformal motion with scalar field \( \sigma \), then it is either conformally flat or \( \nabla_j \sigma \) is a null vector;

2) a (PCS)₄ space-time with a proper conformal motion is type N or O.
A pseudo-Riemannian manifold admits an infinitesimal conformal motion if there are a vector field $\xi_j$ and a scalar field $\sigma$ such that: $\nabla_i \xi_j + \nabla_j \xi_i = 2\sigma g_{ij}$. If $\nabla_i \sigma \neq 0$ the motion is proper, if $\sigma$ is constant the motion is homothetic (see [30] page 564).

In this paper we present new results for (CQR)$_n$ manifolds, in particular for 4-dimensional Lorentzian ones. In Section 2, after a review of their main properties, based on the works [4] [27] [28], we prove that the Ricci tensor and the tensor $\nabla_i A_j$ of (CQR)$_n$ manifolds are Weyl compatible. The notion of Weyl compatibility was recently introduced and investigated by us in [19], [20] and [21]. In Section 3 we study (4-dimensional) (CQR)$_4$ Lorentzian manifolds (space-times): we prove that the fundamental vector $A$ is null and unique up to a scaling; moreover we show that the same vector is eigenvector of the Ricci tensor and its integral curves are geodesics. Finally we show that such space-times are Petrov type-\textit{N} with respect to $A$.

The manifolds are Hausdorff, connected, of dimension $n \geq 3$, with a Levi-Civita connection ($\nabla_j g_{kl} = 0$), and non-degenerate metric of arbitrary signature (pseudo-Riemannian manifolds. For Lorentzian manifolds the signature is 2).

2. (CQR)$_n$ manifolds: general properties

The following proposition collects some basic known facts:

**Proposition 2.1.** In a pseudo-Riemannian (CQR)$_n$ manifold, with fundamental vector $A_i$, the following relations hold:

(3) $A^m C_{jklm} = 0$,

(4) $\nabla_m C_{jkl}^m = 0$.

(5) $(\nabla_i A^m) C_{jklm} + (A^m A_m) C_{jkl} = 0$.

(6) $A^i (\nabla_i A^m) C_{jklm} = 0$.

**Proof.** The first two identities [27] are here derived as follows: write three versions of (1) with indices $ijk$ cyclically permuted, and sum up (some terms cancel by the first Bianchi identity): $\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = 0$. Contraction with $g^m$ gives (2). By transvecting (1) with $g^m$ one gets $\nabla^m C_{jklm} = 3A^m C_{jklm}$, but the left hand side term was just shown to be zero. The third identity was obtained in [27] and reobtained in [4]. The last one follows from (5) and (3). \qed

The condition $\nabla_m C_{jkl}^m = 0$ has an interesting consequence that is now elucidated. We need a differential identity connecting the Weyl, the Riemann and the Ricci tensors, that we proved in [13], [20], extending an identity by Lovelock to curvature tensors other than Riemann’s:

**Lemma 2.2.** In pseudo-Riemannian manifolds the following holds:

(7) $\nabla_i \nabla_m C_{jkl}^m + \nabla_j \nabla_m C_{kilm} + \nabla_k \nabla_m C_{ijlm} = -\frac{n-3}{n-2} (R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m)$.

**Theorem 2.3.** The Ricci tensor of a (CQR)$_n$ pseudo-Riemannian manifold is Riemann compatible:

(8) $R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m = 0$. 

Proof. The left hand side of (7) vanishes because of property (1). □

**Remark 2.4.** Eq. (8), which we proved straightforwardly, was first obtained in [27]. It coincides with the definition that the Ricci tensor is a Riemann compatible tensor [19], [20]. The geometric and topological consequences were studied in [20] and [21], where it was proven that Riemann compatible tensors are also Weyl compatible. For the Ricci tensor this means:

\[ R_{im}C_{jkl}^{\prime m} + R_{jm}C_{kil}^{\prime m} + R_{km}C_{ijl}^{\prime m} = 0. \]

Weyl compatible tensors and vectors were investigated in [21].

Suppose that the manifold has a matter content described by a stress-energy tensor \( T_{ij} \); the Einstein’s equations link it to the Ricci tensor:

\[ R_{kl} - \frac{(R/2)g_{kl}}{8\pi G} = kT_{kl}, \]

where \( k = 8\pi G \) is Einstein’s gravitational constant, [8], [29]. Then eqs (8) and (9) for the Ricci tensor imply analogous statements for the stress energy tensor:

**Corollary 2.5.** In a \((\text{CQR})_n\) Lorentzian manifold the stress-energy tensor is Riemann and Weyl compatible.

The next new theorem is about the fundamental vector \( A_i \) and its gradient (note that \( \nabla_i A_m \) is not necessarily a symmetric tensor. If such, \( A_i \) would be closed):

**Theorem 2.6.** In a \((\text{CQR})_n\) pseudo-Riemannian manifold with fundamental vector \( A_i \), the tensor \( \nabla_i A_m \) is Weyl compatible,

\[ (\nabla_i A_m)C_{jkl}^{\prime m} + (\nabla_j A_m)C_{kil}^{\prime m} + (\nabla_k A_m)C_{ijl}^{\prime m} = 0. \]

and the following identities hold:

\[ A_i R_{im}C_{jkl}^{\prime m} = 0, \]

\[ (\nabla_i C_{jklm})(\nabla^i C_{jklm}) = 8(A_i A^i)C_{jklm}C_{jklm}. \]

Proof. Write three versions of equation (5) with indices \( ijk \) cyclically permuted and sum up. A cancellation occurs by the first Bianchi identity of the Weyl tensor. The relation (11) follows by transvecting with \( A_i \) the equation (9). Finally, (12) is proven by squaring (1); most terms in the right-hand-side vanish because of (9). □

In the geometric literature a deep study is devoted to pseudo-Riemannian manifolds with a vector field \( A_i \) such that (see for example [7], [11],[12]):

\[ A_i C_{jklm} + A_j C_{kim} + A_k C_{ijm} = 0. \]

We need the following [7]:

**Proposition 2.7.** In a \(n\)-dimensional non-conformally flat pseudo-Riemannian manifold, if (13) holds, then:

1) \( A_i A^i = 0 \),
2) \( C_{imj}^k C_{pqk}^j = 0 \).

Proof. By contracting (13) with \( g^{im} \) one obtains \( A^m C_{jklm} = 0 \). Contraction of (13) with \( A^i \) gives \( (A_i A_i)C_{jklm} = 0 \) from which we infer 1); contracting (13) with \( C^{kj}_{pq} \) and using \( A_mC_{jklm} = 0 \) gives \( A_i C_{jklm}C^{kj}_{pq} = 0 \) from which 2) follows. □
The last relation in Prop. 2.7 is relevant in the study of Pontryagin forms. Given orthonormal tangent vectors $X_r$, Pontryagin forms $p_k$ are antisymmetric combinations $p_k(X_1 \ldots X_{4k}) = \sum_p (-1)^p \omega_k(X_{i1} \ldots X_{ik})$ of forms $\omega_k$.

$$\omega_1(X_1 \ldots X_4) = R_{ija} b R_{klb} a (X_i^1 \wedge X_j^2)(X_k^3 \wedge X_l^4),$$

$$\omega_2(X_1 \ldots X_8) = R_{ija} b R_{klb} c R_{mn} d R_{pq} a (X_i^1 \wedge X_j^2)(X_k^3 \wedge X_l^4) \ldots (X_j^p \wedge X_k^q)$$

etc. (see [9], [10], [20], [24], [26] pp 317-318). It was shown by Avez [2] (see also [9]) that forms are unchanged if Riemann’s tensor is replaced by Weyl’s tensor, for example:

$$\omega_1(X_1 \ldots X_4) = C_{ija} b C_{klb} a (X_i^1 \wedge X_j^2)(X_k^3 \wedge X_l^4)$$

Proposition 2.8. In a pseudo-Riemannian manifold of type (13), the first Pontryagin form vanishes.

Proof. By prop.2.7 it is $C_{lmj} b C_{pqk} a = 0$, then $\omega_1 = 0$ and $p_1 = 0$.

Remark 2.9. For a compact orientable 4-dimensional pseudo-Riemannian manifold $M$, the vanishing of the first Pontryagin form $p_1$ has a topological significance. According to Hirzebruch’s theorem ([14], [26] pp 229-230),

$$\tau(M) = \frac{1}{3} \int_M p_1,$$

where Hirzebruch’s signature $\tau(M)$ is related in 4k dimension to Euler’s topological index by the relation $\tau = \chi \mod 2$ [24], page 465.

For (CQR)$_n$ manifolds the additional condition (13) has been investigated, and makes them conformally recurrent [4], [28]:

$$\nabla_i C_{jklm} = 4 A_i C_{jklm}.$$ 

3. (CQR)$_4$ Lorentzian manifolds

We show that (CQR)$_4$ Lorentzian manifolds, besides being conformally harmonic [4], are also conformally recurrent, and various other properties.

Let us quote two useful lemma on 4-dimensional manifolds:

Lemma 3.1. ([17] page 128)
The Weyl tensor of a 4-dimensional pseudo-Riemannian manifold satisfies the following identity:

$$\delta_r^i C_{jkl}^r + \delta_i^r C_{jkl}^r + \delta_{i}^j C_{rkl}^i + \delta_{i}^k C_{rjl}^i + \delta_{i}^l C_{rjk}^i + \delta_{r}^j C_{ikl}^i + \delta_{r}^k C_{ijl}^i + \delta_{r}^l C_{ijk}^i = 0$$

Lemma 3.2. ([8] page 46, [13], page 148)
In a 4-dimensional pseudo-Riemannian manifold let $A$ be a null vector and $B$ a vector orthogonal to $A$, $A \cdot B = 0$. Then $B$ is either space-like, or null and proportional to $A$, i.e. $B_j = \lambda A_j$ for some $\lambda \in R$.

We then prove:

Theorem 3.3. In a 4-dimensional non conformally flat pseudo-Riemannian manifold, let $A$ and $B$ be vector fields such that $A_mC_{jkl}^m = 0$ and $B_mC_{jkl}^m = 0$. Then they are both null ($A^l A_j = 0, B^l B_j = 0$) and $B_j = \lambda A_j$ for some real $\lambda$. 

Proof. On multiplying eq. (16) by $A_j B^s$ we obtain $(A_j B^s)C^{ki}_{tr} = 0$. Similarly, way we obtain $(A^j A_k)C^{ki}_{st} = 0$ and $(B^j B_k)C^{ki}_{tr} = 0$ (see [17] page 128). Then $A$ and $B$ are orthogonal null vectors. By lemma 3.2 they are proportional. □

**Proposition 3.4.** (CQR) Lorentzian manifolds are conformally recurrent.

Proof. On multiplying eq. (16) by $A_j$, and using (3) we obtain:

$$A_r C^{kist} + A_t C^{kirs} + A_s C^{kitr} = 0,$$

i.e. eq. (13) is verified. Then (15) follows by property (1). □

**Theorem 3.3**, eq. (12) and prop. 2.8 imply:

**Corollary 3.5.** In a non conformally flat (CQR) Lorentzian manifold:
1) the fundamental vector $A$ is null and unique up a scaling,
2) the scalar $(\nabla_i C_{jklm})(\nabla^i C^{jklm})$ vanishes identically.
3) the first Pontryagin form vanishes.

The Bel-Debever version of Petrov’s classification of Weyl tensors on 4-dimensional Lorentzian manifolds (see [13] page 196, [3], [25] and [29]) is based on null vectors $k$. By identifying the null vector $k$ with the fundamental vector $A$, we may assert:

**Proposition 3.6.** On a non conformally flat (CQR) Lorentzian manifold, the Weyl tensor is type-N with respect to its fundamental vector.

**Theorem 3.7.** On a non conformally flat (CQR) Lorentzian manifold,
1) the fundamental vector $A$ is an eigenvector of the Ricci tensor;
2) the integral curves of the fundamental vector $A$ are geodesics;
3) $\nabla_i A_i = 0$.

Proof. Consider eqs (3) and (10), i.e. $A^m C_{jklm} = 0$ and $A^i R_{im} C_{jkl} = 0$; the last one defines a vector $B_m = A^i R_{im}$ such that $B^m C_{jklm} = 0$. By theorem 3.1 it is $A^i R_{im} = \lambda A_m$, i.e. $A$ is an eigenvector of the Ricci tensor. Next, consider equations (3) and (6), i.e. $A^m C_{jklm} = 0$ and $(A^i \nabla_i A^m) C_{jklm} = 0$: by the same arguments we obtain $A^i (\nabla_i A_m) = \lambda A_m$. Therefore the integral curves of the vector $A$ are geodesics (see [29] page 41). Multiply (16) by $\nabla^p A^j$ and use (5):

$$(\nabla^p A_r) C^{kisi} + (\nabla^p A_t) C^{kirs} + (\nabla^p A_s) C^{kitr} = 0,$$

Contraction of $s$ with $p$ gives $(\nabla_s A^s) C^{kitr} = 0$. □

From $A^m C_{jklm} = 0$ and $A^i R_{im} = \lambda A_m$, a direct calculation gives $A^m A^i R_{jklm} = (\lambda - R/6) A_k A_l$ (see also Hall’s theorem in [30] and [21]). It follows that $A^m R_{ijkl} A^m A^j = 0$, i.e. the Riemann’s tensor is algebraically special (i.e. type II or D).

In their study of (PCS) manifolds with proper conformal motion, De and Mazumdar [6] obtained the following relations:

$$(\nabla_i \sigma)(\nabla^i \sigma) = 0,$$

$$C^{kimi} \nabla_m \sigma = 0$$

For $n = 4$ this proves that the vector $\nabla_i \sigma$ is null and that the space is type-N with respect to it. If we consider the previous results [17] and theorem 3.5 we have $\nabla_i \sigma = \lambda A_i$, and we may conclude:

**Proposition 3.8.** If a non conformally flat (CQR) Lorentzian manifold admits a proper conformal motion, then $\nabla_i \sigma = \lambda A_i$, i.e. the fundamental vector $A$ is closed.
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