INTERTWINING OPERATOR ASSOCIATED TO THE COMPLEX DUNKL OPERATOR OF TYPE \( G(m, 1, N) \)

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Abstract. In this work, we consider the Dunkl complex reflection operators related to the group \( G(m, 1, N) \) in the complex plane

\[
T_i = \frac{\partial}{\partial z_i} + k_0 \sum_{j \neq i} \sum_{r=0}^{m-1} \frac{1 - s_r^j s_i^j}{z_i - \epsilon^r z_j} + \sum_{j=1}^{m-1} k_j \sum_{r=0}^{m-1} \frac{\epsilon^{-r} s_i^j}{z_i}, \quad 1 \leq i \leq N.
\]

We first review the theory of Dunkl operators for complex reflection groups we recall some results related to the hyper–Bessel functions, which are solutions of a higher order differential equation. Secondly, we construct a new explicit intertwining operator between the operator \( T_i \) and the partial derivative operator \( \frac{\partial}{\partial z_i} \). As application we given an explicit solution of the system:

\[
T_i f(x) = \kappa \lambda_i f(x), \quad f(0) = 1.
\]

1. Introduction

The idea of intertwining operator \( V \) such that \( VP = QV \) for \( P \) and \( Q \) ordinary differential operators goes back to Gelfand, Levitan, Marchenko, Naimark, Delsarte and Lions (see [11], [12]). It was picked up again by C. F. Dunkl, Rösler and K. Trimèche (see [8], [15] [16]) who established some fundamental ideas related to the class of differential difference operators. In this work we investigate in the rank one case the particular cases of complex reflection Dunkl operator \( T(k) \), associated with complex reflection group \( G(m, 1, 1) \), on the set of radial rays \( U = \bigcup_{j=1}^{m} \epsilon^j \mathbb{R} \), which is given by

\[
T(k)f(x) := \frac{df(x)}{dx} + \sum_{i=1}^{m-1} k_i \sum_{j=0}^{m-1} \epsilon^{-ij} f(\epsilon^j x), \quad \epsilon = e^{\frac{2\pi i}{m}} \quad \text{and} \quad k_i \in \mathbb{C}.
\]

In particular, when \( m = 2 \), \( T(k) \) coincides with the following Dunkl operator on the real line

\[
Tf(x) := \frac{df(x)}{dx} + \frac{\nu + 1/2}{x} (f(x) - f(-x)).
\]

First, we indicate briefly some results involving intertwining operators. In [8], C. F. Dunkl has proved that there exists a linear isomorphism \( V \), called the Dunkl intertwining operator, from the space of polynomials on \( \mathbb{R} \) of degree \( n \) onto itself, satisfying the transmutation relation

\[
T \circ V = V \circ \frac{d}{dx}, \quad V(1) = 1.
\]

In [15], Rösler has obtained an integral representation of \( V \) and K. Trimèche [16] extended it to a topological isomorphism from \( E(\mathbb{R}) \), the space of even \( C^\infty \)-functions on \( \mathbb{R} \), onto itself satisfying the relation \( (1.3) \) and obtained the following form

\[
V(f) := R_\nu(f_e) + \frac{d}{dx} \circ R_\nu \circ I(f_o),
\]

where \( f_e \) and \( f_o \) are respectively the even and odd parts of the function \( f \),

\[
I(f)(x) := \int_0^x f(t) dt
\]

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and $\mathcal{R}_\nu$ is the Riemann-Liouville operator given by

$$
\mathcal{R}_\nu(f)(x) := \frac{\Gamma(\nu + 1)}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t)^{\nu - \frac{1}{2}} t^{\frac{1}{2}} f(x t^{\frac{1}{2}}) dt.
$$

The goal of this paper is to provide a similar construction for an intertwining operator $V_m$ between the complex Dunkl operator $T(k)$ and the derivative operator $\frac{d}{dx}$. Our construction is based on some hyper-Bessel operator and Riemann-Liouville type transform.

The remaining sections of this paper are organized as follows. In Section 2, we first recall notations and some results for Dunkl operator, we establish a new representation for the complex Dunkl operator by using circular matrices. In section 3, we discuss some results satisfied by the hyper-Bessel functions which can be found in the literature. In section 4, we give a new intertwining operator between $T(k)$ and $\frac{d}{dx}$.

2. Complex Dunkl operators of type $G(m, 1, N)$

Let $m \in \mathbb{N}$ ($m \geq 2$). We denote by $G$ the cyclic group generated by $\varepsilon = e^{\frac{2i\pi}{m}}$ and by

$$
U = \bigcup_{j=1}^m \varepsilon^j \mathbb{R}
$$

a set of radial rays in complex plane. For $i = 1, \ldots, m$, we define the operators

$$
p_i(f)(x) = \frac{1}{m} \sum_{j=0}^{m-1} \varepsilon^{-ij} f(\varepsilon^j x).
$$

These obey

$$
id = \sum_{i=1}^m p_i, \quad p_ip_j = \delta_{ij}p_i.
$$

Then, the elements $p_i$ are idempotents which are generalizations of the primitive idempotents $(1 - s)/2$ and $(1 + s)/2$ for a real reflection $s$.

**Definition 2.1.** A function $f : U \to \mathbb{C}$ is called of type $j$ with respect to $G$, if

$$
f(\varepsilon x) = \varepsilon^j f(x),
$$

hold for every $x \in U$.

**Lemma 2.1.** Let $f$ be a function $f : U \to \mathbb{C}$. Then, $f$ can be decomposed uniquely in the form

$$
f = \sum_{j=0}^{m-1} f_j,
$$

where the component function $f_j$ is of type $j$, given by

$$
f_j = p_j(f).
$$

**Example 2.1.** Let $\kappa = e^{\frac{2i\pi}{m}}$. By using the previous Lemma we obtain easily the following decomposition of the exponential function $e^{\kappa x}$

$$
e^{\kappa x} = \cos_m(x) + \sum_{l=1}^{m-1} \kappa^l \sin_{m,l}(x),
$$

where the hyper-trigonometric functions $\cos_m(x)$ and $\sin_{m}(x)$ are given by

$$
\cos_m(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{nm}}{(nm)!}, \quad \text{and} \quad \sin_{m,l}(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{nm+l}}{(nm + l)!}.
$$

The function $y(x) = \cos_m(\lambda x)$ is the unique $C^\infty$-solution of the system

$$
\begin{cases}
    y^{(m)}(x) = -\lambda^m y(x), \\
    y(0) = 1, \quad y^{(1)}(0) = \ldots = y^{(m-1)}(0) = 0.
\end{cases}
$$
We denote by $\mathcal{E}(U)$ the space of $C^\infty$-complex valued functions on $U$ equipped with the topology of uniform convergence on compacts of the functions and all their derivatives, is a Frechet space and we denote by $\mathcal{E}_j(U)$ the subspace of $\mathcal{E}(U)$ of functions of type $j$ with respect to the group $G$. Of course we have

$$\mathcal{E}(U) = \bigoplus_{j=0}^{m-1} \mathcal{E}_j(U).$$

Let $\nu = (\nu_1, ..., \nu_{m-1}, 0) \in \mathbb{C}^m$ and $k = (k_1, ..., k_{m-1}, 0)$, with $k_j = m\nu_j + m - j$.

The complex reflection Dunkl operator associated to cyclic $G$ generated by $\varepsilon = e^{2i\pi m}$ is defined by ([9], [2])

$$T(k)f(x) := \frac{df(x)}{dx} + \sum_{i=1}^{m-1} k_i \sum_{j=0}^{m-1} \varepsilon^{-ij}f(\varepsilon^j x).$$

**Proposition 2.2.** The operator $T(k)$ can be written in the the form

$$T(k) = \frac{d}{dx} + \omega_k(x),$$

where

$$\omega_k(f) = \Omega \Lambda(f),$$

$\Omega$ is the Fourier $m \times m$ matrix, which is given by $\Omega = (\varepsilon^{-(i-1)(j-1)})_{i,j}$ and $\Lambda(f)(z)$ is the vector valued function form $U$ into $\mathbb{C}^m$, given by $\Lambda f(x) = (f(x), f(\varepsilon x), ..., f(\varepsilon^{m-1} x))$.

**Proof.** Put

$$\omega_k := \sum_{i=1}^{m-1} k_i p_i.$$ 

A simple calculation shows that (see [4])

$$\omega_k(f) = \langle \Omega \Lambda(f), k \rangle,$$

and

$$T(k)f = \frac{df}{dx} + \frac{\omega_k(f)}{x}.$$ 

**Lemma 2.3.** 1) If $f \in \mathcal{E}(U)$, then $T(k)(f) \in \mathcal{E}(U)$.

2) For $j = 1, ..., m - 1$, we have

$$p_j \circ \frac{d}{dx} = \frac{d}{dx} \circ p_{j+1}.$$ 

Furthermore, if $f \in \mathcal{E}_j(U)$, then $T(k)(f) \in \mathcal{E}_j(U)$.

**Proof.** This follows immediately from the fact that:

For $i = 1, ..., m - 1,$

$$p_i(f)(x) = \frac{1}{m} \sum_{j=0}^{m-1} \varepsilon^{-ij} f(\varepsilon^j x) = x \int_0^1 p_{i-1}(f^{(1)})(xt) dt.$$
3. The hyper-Bessel functions

Let \( \nu = (\nu_1, ..., \nu_{m-1}) \in \mathbb{R}^{m-1} \), satisfying \( \nu_k \geq -1 + \frac{k}{m} \), we denote by
\[
|\nu| := \nu_1 + ... + \nu_{m-1},
\]
\[
\nu + n := (\nu_1 + n, ..., \nu_{m-1} + n) (n \in \mathbb{N}),
\]
\[
\Gamma(\nu) := \Gamma(\nu_1) ... \Gamma(\nu_{m-1}).
\]
The normalized hyper-Bessel function with vector index \( \nu \)
\[ (3.2) \]
where the hyper-Bessel is given by following differential recurrence relations for the normalized hyper-Bessel functions \( (3.1) \)
of the following problem \[14\]
\[ \text{where } J \]
Here \( B \)
\[ (3.3) \]
second order differential operator of Bessel \( \text{B} \)
\[ (3.4) \]
The simplest higher order hyper-Bessel operator is the operator of \( m \)-fold differentiation
\[ (3.5) \]
where the hyper-Bessel is given by
\[ (3.2) \]
The hyper-Bessel is a unique \( C^\infty \)-solution of the following problem \[14\]
\[
\begin{aligned}
B_m(f)(x) &:= -\lambda^m f(x), \\
f(0) &= 1, \quad f^{(1)}(0) = \ldots = f^{(m-1)}(0) = 0.
\end{aligned}
\]
\[
B_m = \prod_{j=1}^{m-1} \left( \frac{d}{dx} + \frac{m\nu_j + m - j}{x} \right) \frac{d}{dx}.
\]
The hyper-Bessel operator generalizes the well known second order differential operator of Bessel \( B_2 \) given by where
\[ (3.3) \]
\[
B_2 := \frac{d^2}{dx^2} + \frac{2\nu + 1}{x} \frac{d}{dx},
\]
and the corresponding normalized Bessel function is given by
\[
\mathcal{J}_{\nu,2}(x) := \frac{2^\nu \Gamma(\nu + 1)}{x^\nu} J_\nu(x),
\]
where \( J_\nu(x) \) is the classical Bessel function (see, \[17\]). From Corollary 2 in \[14\] we obtain the following differential recurrence relations for the normalized hyper-Bessel functions \( \mathcal{J}_{\nu,m}(x) \)
\[ (3.4) \]
\[
\frac{d}{dx} \mathcal{J}_{\nu,m}(x) = -\frac{(x^m)^{m-1}}{(\nu_1 + 1) ... (\nu_{m-1} + 1)} \mathcal{J}_{\nu+1,m}(x),
\]
\[ (3.5) \]
\[
\left( \frac{d}{dx} + \frac{m\nu_k}{x} \right) \mathcal{J}_{\nu,m}(x) = \frac{m\nu_k}{x} \mathcal{J}_{\nu-e_k,m}(x),
\]
where \( e_k, (1 \leq k \leq m - 1) \) are the standard basis of \( \mathbb{R}^{m-1} \).

4. Intertwining Operator

Let \( \nu = (\nu_1, ..., \nu_{m-1}) \in \mathbb{C}^{m-1} \) such that \( \Re(\nu_j) > 0 \). We define the fractional integrals \( \mathcal{R}_{\nu,m} \) of Riemann-Liouville type for \( f \in \mathcal{E}_m(U) (\mathcal{E}_m(U) \) the subspace of \( \mathcal{E}(U) \) of functions of type \( m \) by
\[ (4.1) \]
\[
\mathcal{R}_{\nu,m}(f)(x) := \frac{m^{3/2} \Gamma(\nu + 1)}{(2\pi)^{(m-1)/2}} \int_0^1 G_{m-1,m-1}^{1,0} \left( \nu_1, \nu_2, ..., \nu_{m-1}, 1 \right) \left( \frac{1}{m}, ..., \frac{m-1}{m} \right) f(x t^{1/m}) dt,
\]
where \( G_{p,q}^{m,n} (a_1, a_2, ..., a_p ; b_1, b_2, ..., b_q | z) \) is the Meijer’s function (see \[10\]). This operator intertwines the hyper-Bessel operator \( B_m \) and the \( m \)-th differential operator \( \frac{d^m}{dx^m} \)
\[ (4.2) \]
\[
B_m \circ \mathcal{R}_{\nu,m} = \mathcal{R}_{\nu,m} \circ \frac{d^m}{dx^m}.
\]
and maps the hyper-cosine function $\cos_m(\lambda x)$ of order $m \geq 2$ into a normalized hyper-Bessel function $J_{\nu,m}$

$$J_{\nu,m}(\lambda x) = R_{\nu,m} (\cos_m(\lambda .))(x).$$

For $m = 2$, $R_{\nu,m}$ is reduced to the so called Riemann-Liouville transform $R_k$ defined in (1.6).

The operator $R_{\nu,m}^m$ can be written also as a product of the Erdélyi-Kober integrals

$$R_{\nu,m}f(x) = \frac{m^{3/2}\Gamma(\nu + 1)}{(2\pi)^{(m-1)/2}} \prod_{k=1}^{m} \int_{m-1} f(x) (\nu_k \gamma_k + \frac{1}{m}) f(x),$$

where the Erdélyi-Kober fractional integrals is defined by

$$I_{\lambda, \gamma} f(x) := \int_{0}^{1} \frac{(1-t)^{\alpha-1}t^\gamma}{\Gamma(\alpha)} f(x) dt, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0.$$

By Theorem 3.5.7 in [13] and by similar argument as [16], we can show that the operator $R_{\nu,m}$ is a topological isomorphism from $E_m(U)$ onto itself and its inverse is given by

$$R_{\nu,m}^{-1} f(x) = \frac{(2\pi)^{(m-1)/2}}{m^{3/2}\Gamma(\nu + 1)} \prod_{k=1}^{m} \prod_{j=1}^{n_k} (-1 + j + \frac{k}{m} + \frac{1}{m} d dx)) I_{m-1}^{(\nu_k \gamma_k + \frac{1}{m} + 1)} f(x),$$

where

$$n_k = \begin{cases} \lfloor \nu_k - \frac{k}{m} \rfloor + 1, & \text{if } \nu_k - \frac{k}{m} \text{ is non integer}, \\ \nu_k - \frac{k}{m} + 1, & \text{if } \nu_k - \frac{k}{m} \text{ is integer}. \end{cases}$$

Let consider the operator $V_m$ defined for $f \in E(U)$ by

$$V_m(f) = \sum_{j=1}^{m} A_j \circ R_{\nu,m} \circ I_{m-j} \circ p_j(f),$$

where the operator $I$ is defined in (1.5) and

$$A_m = 1, \quad A_{m-1} = \frac{d}{dx}, \quad A_j = \prod_{k=j+1}^{m-1} \left( \frac{d}{dx} + \frac{mv_k + m - k}{x} \right) \frac{d}{dx}, \quad 1 \leq j \leq m - 2.$$

The operator $V_m$ is well defined on the space $E(U)$, since for $f \in E(U)$, we have

$$I_{m-j} \circ p_j(f) \in E_m(U).$$

**Theorem 4.1.** The operator $V_m$ satisfy the following intertwining relation on the space $E(U)$

$$T(k) \circ V_m = V_m \circ \frac{d}{dx}.$$

**Proof.** Let $f \in E(U)$. It is clearly that for $j = 1, ..., m$, the function

$$A_j \circ I_{m-j} \circ p_j(f) \in E_j(U).$$

Then,

$$T(k) \circ V_m(f) = \frac{d}{dx} \circ R_{\nu,m} \circ p_m(f) + \sum_{j=1}^{m-1} \left( \frac{d}{dx} + \frac{k_j}{x} \right) \circ A_j \circ R_{\nu,m} \circ I_{m-j} \circ p_j(f)$$

$$= \frac{d}{dx} \circ R_{\nu,m} \circ p_m(f) + B_m \circ R_{\nu,m} \circ I_{m-1} \circ p_1(f) + \sum_{j=2}^{m-1} A_{j-1} \circ R_{\nu,m} \circ I_{m-j} \circ p_j(f).$$

On the other hand from (4.2), we can write

$$B_m \circ R_{\nu,m} \circ I_{m} \circ p_1 = R_{\nu,m} \circ \frac{d^m}{dx^m} \circ I_{m-1} \circ p_1 = R_{\nu,m} \circ \frac{d}{dx} \circ p_1 = A_m \circ R_{\nu,m} \circ p_m \circ \frac{d}{dx}.$$
Similarly
\[ \frac{d}{dx} \circ R_{\nu,m} \circ p_m = A_{m-1} \circ R_{\nu,m} \circ I \circ p_{m-1} \circ \frac{d}{dx}. \]
So that
\[
\sum_{j=2}^{m-1} A_{j-1} \circ R_{\nu,m} \circ I^{m-j} \circ p_j = \sum_{j=1}^{m} A_j \circ R_{\nu,m} \circ I^{m-j-1} \circ p_{j+1}, \\
= \sum_{j=1}^{m-2} A_j \circ R_{\nu,m} \circ I^{-j} \circ \frac{d}{dx} \circ p_{j+1}, \\
= \sum_{j=1}^{m-2} A_j \circ R_{\nu,m} \circ I^{-j} \circ p_j \circ \frac{d}{dx}.
\]
Thus,
\[
T(k) \circ V_m(f) = \sum_{j=1}^{m} A_j \circ R_{\nu,m} \circ I^{-j} \circ p_j \circ \frac{d}{dx}(f) = V_m \circ \frac{d}{dx}(f).
\]

**Theorem 4.2.** Under the condition
\[ k_j = m
\nu_j + m - j \geq 0, \quad j = 1, \ldots, m - 1. \]
The following system
\[ T(k) f(x) = \kappa \lambda f(x), \quad f(0) = 1. \]
has the following solution
\[
D_{\nu}(\lambda, x) = J_{\nu}(\lambda x) + \sum_{j=1}^{m-1} \frac{(\kappa \lambda)^j}{m^j (\nu_1 + 1) \cdots (\nu_{m-j} + 1)} J_{\nu_1+1, \ldots, \nu_j+1, \nu_{j+1}, \ldots, \nu_{m-1}}(\lambda x).
\]
**Proof.** According to Theorem 4.1, \( V_m \) intertwines \( B_m \) and \( \frac{d}{dx} \) in \( E(\mathbb{R}) \). We apply the intertwines operator \( V_m \) to the initial value problem
\[ f'(x) = \kappa \lambda f(x), \quad f(0) = 1. \]
Then if \( f \) is a solution of \( 4.12 \) then \( V_m(f) \) is a solution \( 4.10 \). Therefore \( D(\lambda, x) = V_m(e^{\kappa \lambda \cdot})(x) \) is a solution of the system \( 4.10 \). Using \( 3.4 \) and \( 3.5 \) we can write \( D(\lambda, x) \) in form \( 4.11 \).

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