Equitable [[2, 10], [6, 6]]-partitions of the 12-cube∗†

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Abstract

We describe the computer-aided classification of equitable partitions of the 12-cube with quotient matrix [[2, 10], [6, 6]], or, equivalently, simple orthogonal arrays OA(1536, 12, 2, 7), or order-7 correlation-immune Boolean functions in 12 variables with 1536 ones (which completes the classification of unbalanced order-7 correlation-immune Boolean functions in 12 variables). We find that there are 103 equivalence classes of the considered objects, and there are only two almost-OA(1536, 12, 2, 8) among them. Additionally, we find that there are 40 equivalence classes of pairs of disjoint simple OA(1536, 12, 2, 7) (equivalently, equitable partitions of the 12-cube with quotient matrix [[2, 6, 4], [6, 2, 4], [6, 6, 0]]) and discuss the existence of a non-simple OA(1536, 12, 2, 7).

Keywords: orthogonal arrays, correlation-immune Boolean functions, equitable partitions, perfect colorings, intriguing sets.

1 Introduction

The main result of the current work is completing the classification of unbalanced order-7 correlation-immune Boolean functions in 12 variables (equivalently, equitable 2-partitions of the 12-cube with asymmetric quotient matrix having an eigenvalue −4). We solve the case of functions with 1536 ones, finding 103 equivalence classes, while the functions with 1024 ones and with 1792 ones from the considered family were characterized in [9] (16 equivalence classes and 2 equivalence classes, respectively). Computationally, the

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classification considered in the current paper is harder than the one in [9]; it requires the subdivision of the classified family into three subfamilies, each of which is solved with a special modification of the algorithm, and still spends almost thirty CPU years. The classification [9] of functions with 1024 ones was much more easy and straightforward because of smaller number of ones, while the 1792-ones case was solved using the Fourier analysis, which required more analytical than computational work (the crucial fact for the success of the Fourier-based classification was that the average density $1792/2^{12} = 9/16$ of ones of the function was not a multiple of $1/8$, which is not true for the 1536-ones case).

The $n$-cube is a graph $Q_n$ on the set $\{0,1\}^n$ of binary $n$-tuples, where two $n$-tuples are adjacent if and only if they differ in exactly one position. A partition $(C_0, \ldots, C_{k-1})$ of the vertex set of $Q_n$ into two cells is called an equitable $k$-partition (of $Q_n$) with quotient $k \times k$ matrix $S = (S_{ij})$ if for all $i, j \in \{0, \ldots, k-1\}$ any vertex of $C_i$ has exactly $S_{ij}$ neighbors in $C_j$. A (Boolean) function $f : \{0,1\}^n \rightarrow \{0,1\}$ is called unbalanced if the number of its ones is different from $0$, $2^n-1$, and $2^n$. It is called $t$-th order correlation immune if the number of ones (equivalently, zeros) $(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) = 1$ is statistically independent on the values of any $t$ arguments; that is, it is the same in all subgraphs of $Q_n$ isomorphic to $Q_{n-t}$. Correlation immune Boolean functions of order $t$ are known to be equivalent to simple orthogonal arrays $OA(M, n, 2, t)$, where $M$ is the number of ones of the function. Fon-Der-Flaass [1] proved that the correlation-immunity order of an unbalanced Boolean function in $n$ variables cannot exceed $2n/3 - 1$; moreover, any unbalanced Boolean function $f$ of correlation-immunity order $2n/3 - 1$ corresponds to an equitable 2-partition of the $n$-cube $Q_n$ with quotient matrix $[[a, b], [c, d]]$, where $a + b = c + d = n$, $a - c = -n/3$, and the number of ones of $f$ is $2^n c/(b+c)$. Nowadays, there are three known families of quotient matrices corresponding to such functions: $[[0, 3T], [T, 2T]]$, $[[T, 5T], [3T, 3T]]$ (found in [11]), $[[3T, 9T], [7T, 5T]]$ (found in [3]). For each of the matrices $[[0, 3], [1, 2]]$, $[[1, 5], [3, 3]]$, and $[[0, 6], [2, 4]]$, a function is unique up to equivalence. Kirienko [7] found that there are exactly 2 inequivalent unbalanced Boolean functions in 9 variables attaining the bound on the order of correlation immunity (the corresponding quotient matrix is $[[0, 9], [3, 6]]$). Fon-Der-Flaass started the investigation of equitable 2-partitions of $Q_{12}$ attaining the correlation-immunity bound: in [3], it was shown that equitable partitions with quotient matrix $[[1, 11], [5, 7]]$ do not exist, while equitable partitions with quotient matrix $[[3, 9], [7, 5]]$ were built. In [9], combining the theoretical and computational approaches, equitable partitions with quotient matrices $[[0, 12], [4, 8]]$ and $[[3, 9], [7, 5]]$ were classified up to equivalence. The current work is aimed on the remaining case.
This case additionally motivated by the inner order 2 of the first cell of such a partition, which means that this cell induces a 2-factor in $Q_n$, i.e., the union of disjoint cycles at distance at least 2 from each other. As we will see from the result of our classification, the length of cycles can vary over the all family of such partitions, as well as in one partition.

Throughout this paper, we use the notation $S = [[S_{++}, S_{+-}], [S_{-+}, S_{--}]] = [[2, 10], [6, 6]]$ and the term $S$-partition in the meaning “equitable partition of the 12-cube with quotient matrix $S$”.

The results of the present paper can be treated in terms of orthogonal arrays. A binary orthogonal array $OA(M, n, 2, t)$ is a multiset of size $M$ of vertices on the $n$-cube such that every $(n - t)$-subcube contains exactly $M/2^t$ of them. Simple orthogonal arrays (i.e., without multiplicity larger than 1) naturally correspond to $t$-th order correlation immune Boolean functions in $n$ arguments with $M$ ones: the characteristic function of a binary simple orthogonal array of strength $t$ is a $t$th order correlation immune Boolean function, and vice versa.

In Section 2, we describe the computer-aided classification of equitable $[[2, 10], [6, 6]]$-partitions of the 12-cube. The general approach for the classification of equitable 2-partitions of the hypercube is described in Sections 2.1 and 2.2; the approach is the further development of the algorithms used in [9] and [8]. However, straightforward applying this approach to the current parameters requires unreasonable computational resources. In Section 2.3, we discuss the possibility to reduce the amount of computation by dividing the considered class of equitable partitions into subclasses, “square” and “non-square” partitions, “heavy” and “light” partitions. The classification of equitable partitions from different subclasses is described in Sections 2.4, 2.5, and 2.6.

In Section 3, we discuss the results of the classification and properties of the found equitable partitions and orthogonal arrays, including the lengths of the induced cycles (Section 3.1), the order of the automorphism group (Section 3.2), the Fourier coefficients and being so-called “almost orthogonal array” of strength 8 (Section 3.3; spoiler: only 2 of 103 simple $OA(1536, 12, 2, 7)$ and only 1 of 16 $OA(1024, 12, 2, 7)$ satisfy this nice property), relations with known constructions (Section 3.4).

In Section 4, we discuss related objects. In Section 4.1, equitable 3-partitions with quotient matrix $[[2, 6, 4], [6, 2, 4], [6, 6, 0]$ are considered (equivalently, pairs of disjoint simple $OA(1536, 12, 2, 7)$; we find that there are 40 equivalence classes of such pairs), which can also be treated as 3-valued functions attaining the correlation-immunity bound. In Section 4.2, we discuss the existence of non-simple $OA(1536, 12, 2, 7)$. Non-simple orthogonal arrays cannot be treated as Boolean functions; however, the correlation-immunity
bound can be generalized to non-simple binary orthogonal arrays [6], and
the strength 7 of OA(1536, 12, 2, 7) attains that generalised bound, which
motivates their further study.

2 Classification

The general classification approach we use is similar to the one used in [8] for
[[0, 13], [3, 10]]-partitions of $Q_{13}$, which, in its turn, is a development of the
algorithm used in [9] for [[0, 12], [4, 8]]-partitions of $Q_{12}$. The description
of the algorithm (in particular, the definition of local partitions) in the following
two subsections slightly differs from that in [8] because we now have no zeros
in the quotient. However, the main difference and originality of the current
search is described in Subsection 2.3, where the class of all [[2, 10], [6, 6]]-
partitions is divided into three subclasses, according to containing special
configurations in the partition. As shown preliminary experiments, without
this subdivision, the classification would hardly be possible with modern
computational possibilities.

Our goal is to classify the [[2, 10], [6, 6]]-partitions of $Q_{12}$ up to equiv-
alence, so the definition of the equivalence plays a key role. Two subsets
$C$ and $C'$ of $\{0, 1\}^n$ are equivalent if there is an automorphism $\pi$ of the
domain $Q_n$ such that $\pi(C) = C'$. Two 2-partitions $(C_+, C_-)$ and $(C'_+, C'_-)$
of $\{0, 1\}^n$ are equivalent if $C_+$ is equivalent to $C'_+$. (In this paper, we com-
pare only equitable 2-partitions with the same asymmetric quotient matrix,
so the first cell of one partition cannot be equivalent to the second cell of
the other). An automorphism of a subset $C$ of $\{0, 1\}^n$ (and of the partition
$(C, \{0, 1\}^n \setminus C)$) is an automorphism $\pi$ of $Q_n$ such that $\pi(C) = C$. The set
$\text{Aut}(C)$ of all automorphisms of $C$ forms a group with respect to the composi-
tion, the automorphism group of $C$. The number of 2-partitions equivalent to
a given 2-partition $(C_+, C_-)$ is calculated as $|\text{Aut}(\{0, 1\}^n)|/|\text{Aut}(C_+)|$, where
$\text{Aut}(\{0, 1\}^n)$ is the set of automorphisms of $Q_n$, $|\text{Aut}(\{0, 1\}^n)| = 2^n \cdot n!$.

2.1 Local partitions

For classification by exhaustive search, we use an approach based on the
local properties of the equitable partitions. We define objects that satisfy
that properties on the words of small weight (the weight of a binary word is
the number of ones in it). Say that the pair of disjoint sets $P_+, P_-$ of vertices
of $Q_{12}$ is an $(r_0, r_1)$-local partition (sometimes, we will omit the parameters
$(r_0, r_1)$) if
(I) $P_+ \cup P_-$ are the all words starting with 0 and having weight at most $r_0$ or starting with 1 and having weight at most $r_1$;

(II) $P_+$ contains the all-zero word $\overline{0}$;

(III) $P_-$ contains $100000000000$;

(IV) the neighborhood of every vertex $\overline{v} = (v_1, \ldots, v_{12})$ of weight less than $r_v$ satisfies the local condition from the definition of an equitable partition with quotient matrix $[[2, 10], [6, 6]]$ (that is, if $\overline{v} \in P_+$ then $\overline{v}$ has 2 neighbors in $P_+$ and 6 neighbors in $P_+$. if $\overline{v} \in P_-$ then the neighborhood has exactly 6 elements in $P_+$ and 6 in $P_-$);

(V) Every vertex from $P_+$ has at most 2 neighbors in $P_+$.

Two $(r_0, r_1)$-local partitions $(P_+, P_-)$ and $(P'_+, P'_-)$ are equivalent ($(r_0, r_1)$-equivalent) if there is a permutation of coordinates that fixes the first coordinate and sends $P_+$ to $P'_+$. By an $r$-local partition, we mean an $(r, r)$-local partition, but the equivalence for this notion is counted differently: Two $r$-local partitions $(P_+, P_-)$ and $(P'_+, P'_-)$ are equivalent $(r$-equivalent) if there is a permutation of coordinates that sends $P_+$ to $P'_+$. So, equivalent $(r, r)$-local partitions are necessarily $r$-equivalent (equivalent as $r$-local partitions), but not vice versa.

The general approach is to classify all inequivalent $(r_0, r_1)$-local partitions subsequently for $(r_0, r_1)$ equal $(2, 2), (2, 3), (3, 3), (3, 4), (4, 4), (12, 12)$, where $(12, 12)$ corresponds to the complete equitable partitions. In an obvious way, every equitable partition $(C, \overline{C})$ such that $\overline{0} \in C$ includes a $(4, 4)$-local partition $(P_+^{(4,4)}, P_-^{(4,4)})$, $P_+^{(4,4)} \subseteq C$ and $P_-^{(4,4)} \subseteq \overline{C}$, every $(4, 4)$-local partition $(P_+^{(4,4)}, P_-^{(4,4)})$ includes a $(3, 4)$-local partition $(P_+^{(3,4)}, P_-^{(3,4)})$, $P_+^{(3,4)} \subseteq P_+^{(4,4)}$ and $P_-^{(3,4)} \subseteq P_-^{(4,4)}$, and so on. So, the strategy is to reconstruct, in all possible ways, a $(r_0, r_1)$-local partition from each of the inequivalent $(r_0 - 1, r_1)$-local or $(r_0, r_1 - 1)$-local partitions, and then to choose and keep only inequivalent solutions, one representative for each equivalence class found. The details of the reconstruction are described in the next subsection (except the reconstruction from $(4, 4)$ to $(12, 12)$, which is straightforward from the correlation immunity), followed by Section 2.3, where we discuss how to reduce the amount of computations for the concrete quotient matrix $[[2, 12], [6, 6]]$.

2.2 A reconstruction step

Assume that we have a $(r_0 - 1, r_1)$-local partition $(P_+, P_-)$ (the case $(r_0, r_1 - 1)$ is considered similarly). To find all possible $(r_0, r_1)$-local partitions $(R_+, R_-)$
that include it, we construct the following instance of the exact cover problem.

- Among all words $\bar{x}$ of weight $r_0$ and starting from 0, we choose only those whose inclusion in the first cell does not contradict (V), for $\bar{x}$ itself and for all its neighbors. Call them candidates (for the inclusion in $R_+ \setminus P_+$).

- Next, for each words $\bar{v}$ of weight $r_0 - 1$ and starting from 0, we count the number $\alpha_{\bar{v}}$ of its neighbors to add to the first cell for (IV) to be satisfied. If $\bar{v} \in P_i$, $i \in \{+,-\}$, then $\alpha_{\bar{v}} = S_i + \beta_{\bar{v}}$, where $\beta_{\bar{v}}$ is the number of neighbors of $\bar{v}$ that are already in $P_+$.

- We construct a 0,1-matrix $M = (m_{\bar{x},\bar{v}})$ whose rows are indexed by candidates, columns are indexed by the words of weight $r_0 - 1$ starting from 0, and $m_{\bar{x},\bar{v}} = 1$ if and only if $\bar{x}$, $\bar{v}$ are neighbors.

- Now, finding $(r_0, r_1)$-local continuation of $(P_+, P_-)$ is equivalent to finding a set of rows of $M$ whose sum is the row $(\alpha_{\bar{v}})$. This is an instance of the so-called exact multiple cover problem (multiple, because the coefficients $\alpha_{\bar{v}}$ can be larger than 1), and can be solved with an appropriate software; we use libexact [5].

- For each exact cover found, we form $R_+$ by adding the indices of the chosen rows to $P_+$, while $R_-$ is found as the complement of $R_+$.

An important step, traditionally called the isomorph rejection, is choosing nonequivalent representatives from the set of all found solutions, intermediate or final. A standard technique to deal with equivalence, see [4, Sect. 3.3], is to represent sets (in our case, $R_+$) by graphs and use a software [10] recognizing the isomorphism of graphs. At this point, it is possible to make some partial validation of the intermediate results by double-counting the total number of found continuations $(R_+, R_-)$ of $(P_+, P_-)$ with the help of the orbit-stabilizer theorem, see [4, Sect. 10.2] for the general strategy. Here, we do not stop on details of the isomorph rejection and the validation, because they are the same as in [8], and focus on the improvements specific for the current parameters of equitable partitions.

### 2.3 How to reduce the amount of computation

In the case when $r_0 = r_1$, it is sufficient to continue calculation for a representative of every equivalence class of $r$-local partitions, $r := r_0 = r_1$ (we apply this strategy for $r = 2$). However, the choice of the representative
becomes important if we want to make the experiment repeatable with the same intermediate numerical results: for different representatives of the same equivalence class \( E \) of \( r \)-local partitions, the number of \((r, r+1)\)-local continuations can be different, but it is the same for equivalent \((r, r)\)-local partitions. So, for the subdivision \( \{E_1, \ldots, E_s\} \) of \( E \) into \((r, r)\)-equivalence classes, we need to choose one \((r, r)\)-equivalence class explicitly. For this reason, we require that the representative chosen for further calculation must have the minimum number of inequivalent \((r, r+1)\)-local continuations. If, with this condition, there is still more than one candidate \((r, r)\)-equivalence class, then we choose the one with the minimum number of inequivalent \((r+1, r+1)\)-local continuations, then \((r+1, r+2)\), and so on. This approach implies that we produce some redundant computation (to make the choice, we need to find the \((r, r+1)\)-local continuations for each subclass \( E_1, \ldots, E_s \), and sometimes need to find the \((r+1, r+1)\)-local continuations for more than one of them, while only one of them is used for the classification), but makes the choice explicit and the experiment completely repeatable. It reduces the total number of calculations when \( s \) is large (up to 10); we essentially use this approach in Section 2.4.

The general algorithm described above works faster than the similar the algorithm based on \( r \)-local partitions only, but still very heavy for the considered parameters, without taking into account the specific properties of the classified partitions. To make the classification doable, we distinguish the following subfamilies of the family of all \([2, 10], [6, 6]\)-equitable partitions of the 12-cube.

We say that a \([2, 10], [6, 6]\)-equitable (local) partition \((C_+, C_-)\) of \( Q_{12} \) is square if there are four vertices of \( C_+ \) that induce a square subgraph of \( Q_{12} \) (for square local partitions, we additionally require one of these vertices to be the all-zero word). A \([2, 10], [6, 6]\)-equitable (local) partition \((C_+, C_-)\) is square-free if there are no four vertices of \( C_+ \) inducing a square subgraph of \( Q_{12} \).

We say that a \([2, 10], [6, 6]\)-equitable partition \((C_+, C_-)\) of \( Q_{12} \) is heavy (light) if there is (there is no) a subgraph of \( Q_{12} \) isomorphic to the cube \( Q_3 \) and having at least 5 five vertices from \( C_+ \).

The classification is divided into the following three stages; each stage is considered in a separate subsection below. At stage one, we classify the square partitions. Stage two deals with heavy partitions, and its main result is that all heavy \([2, 10], [6, 6]\)-equitable partitions of \( Q_{12} \) are square. Stage three is the classification of the square-free partitions; by the results of the second stage they are all light.
2.4 The square partitions

Up to equivalence, any square $S$-partition or square local partition contains the four words $\bar{0} = 0...000$, $\bar{e}_{12} = 0...001$, $\bar{e}_{11} = 0...010$, $\bar{e}_{11} + \bar{e}_{12} = 0...011$ in its first cell.

**Lemma 1.** There are 66462606 2-local partitions $(C_+, C_-)$ with $\bar{0}$, $\bar{e}_{12}$, $\bar{e}_{11}$, $\bar{e}_{11} + \bar{e}_{12} \in C_+$, which are partitioned into 60 2-equivalence classes or 286 $(2, 2)$-equivalence classes.

**Proof.** Assume that we have a 2-local partition $(C_+, C_-)$ with $\bar{0}$, $\bar{e}_{12}$, $\bar{e}_{11}$, $\bar{e}_{11} + \bar{e}_{12} \in C_+$. Every weight-2 word from $C_+$ is adjacent to exactly two words from $\bar{e}_1, \ldots, \bar{e}_{10}$. On the other hand, every word from $\bar{e}_1, \ldots, \bar{e}_{10}$ has exactly 5 weight-2 neighbors in $C_+$ (totally $S_{-,+} = 6$ neighbors in $C_+$, but one of them is $\bar{0}$). Clearly, this incidence structure corresponds to a 5-regular graph on the 10 vertices $\bar{e}_1, \ldots, \bar{e}_{10}$. It is easy to see that this correspondence with the class of 5-regular graphs of order 10 is one-to-one; moreover, two graphs are isomorphic if and only if the corresponding 2-local partitions are equivalent. The number of 5-regular graphs on 10 labelled vertices is 66462606, see [http://oeis.org/A059441](http://oeis.org/A059441), and the number of their isomorphism classes is 60, see [http://oeis.org/A006821](http://oeis.org/A006821). The number of $(2, 2)$-equivalence classes is counted computationally; it equals the number of vertex orbits summed over the 60 non-isomorphic 5-regular graphs of order 10 (equivalently, the number of non-isomorphic 5-regular graphs of order 10 with one marked vertex, which corresponds to the first coordinate of the $(2, 2)$-local partition).

Next, realizing the strategy described in Section 2.3, we choose 60 representatives of $(2, 2)$-local partitions, and for each of them find all non-equivalent $(2, 3)$-local, $(3, 3)$-local, $(3, 4)$-local, $(4, 4)$-local, and finally $S$-partitions. The local partitions that continue the chosen 60 representatives will be called *leading*, in contrast to the local partitions that continue the rest 286 − 60 $(2, 2)$-equivalence classes. Up to equivalence, we have the following numbers of square partitions:

- there are 37141023 (few CPU days) square $(2, 3)$-local partitions, 4979729 of them are leading;
- there are 659276500 square $(3, 3)$-local partitions, 94275707 (736 CPU days) of them are leading; there are 65945212 3-local partitions (comparing 94275707 and 65945212, we see that applying the strategy described in Section 2.3 can reduce the number of further calculation by factor about 1.4; it was decided not to implement this improvement);
• there are 16535880038 (1795 CPU days) leading square (3, 4)-local partitions;

• there are 3111 square (4, 4)-local partitions, 667 (2322 CPU days) of them are leading; there are 429 square 4-local partitions, and all of them continue to $S$-partitions;

• there are 77 square $S$-partitions.

2.5 The heavy partitions

Define the $(1, 2)$-local partition $(B_+, B_-)$ where $B_+ = \{\bar{e}_1 + \bar{e}_2, \bar{e}_2, 0, \bar{e}_3, \bar{e}_3 + \bar{e}_1, \bar{e}_1 + \bar{e}_4, \bar{e}_1 + \bar{e}_5, \bar{e}_1 + \bar{e}_6\}$.

Lemma 2. Every heavy $S$-partition is equivalent to a partition continuing the $(1, 2)$-local partition $(B_+, B_-)$.

Proof. (i) It is easy to observe that if $C$ is a vertex set of $Q_3$ of cardinality at least 5 and such that the induced subgraph has no vertices of degree more than two, then $C$ induces either a 5-path or a 6-cycle, being equivalent to either

\{110, 010, 000, 001, 101\} or \{110, 010, 000, 001, 101, 111\},

respectively.

(ii) Let $(C_+, C_-)$ be a heavy $S$-partition. Without loss of generality, according to the definition of heavy $S$-partition, we can assume without loss of generality that the 8 vertices ending by 9 zeros include at least 5 vertices from $C_+$. From (i) we see that these vertices are 11000000000, 01000000000, 00000000000, 00100000000, 10100000000, and may be 11100000000. It remains to note that the first cell of the corresponding $(1, 2)$-local partition contains the first 5 of these vertices and 3 more neighbors of 100000000000, which are 10010000000, 10001000000, and 10000100000, up to coordinate permutation.

In contrast to the other subcases, in this section we choose the $(1, 3)$-local partitions for the next step of the classification instead of $(2, 2)$-local partitions. Up to equivalence, we have the following number of partitions containing $\bar{e}_1 + \bar{e}_2, \bar{e}_2, 0, \bar{e}_3, \bar{e}_3 + \bar{e}_1$:

• there are 178 (2, 2)-local partitions (2-local partitions);

• there are 953730 (1 CPU minute) (2, 3)-local partitions;
• there are 815364 (30 CPU days) (3,3)-local partitions (3-local partitions); up to this step, all local partitions are square-free;
• there are 2325257827 (3,4)-local partitions, 560234770 (14 CPU days) of them are square-free;
• there are 54 (4,4)-local partitions (4-local partitions), all of them continue to \( S \)-partitions, 0 (7 CPU days) of them are square-free;
• there are 17 heavy \( S \)-partitions, none of them is square-free (17 CPU days).

Remark 1. In the considered case, the \( r \)-equivalence coincides with the \((r, r)\)-equivalence because the first coordinate is special and cannot be permuted with any other.

Finally, we have established the following fact, which essentially simplifies the next-section classification of the square-free \( S \)-partitions.

**Lemma 3.** All square-free \( S \)-partitions are light.

Remark 2. To validate Lemma 3, it was sufficient to consider only square-free partitions. The numbers of partitions without the square-free condition provided above are based on early computations, which occur to be redundant. The CPU time was not kept; it is essentially smaller than that for the step in the previous section, and it is surely possible to save some time if consider only light square partitions in Section 2.4. In any case, the largest part of computation is described in the next section.

### 2.6 The square-free partitions

**Lemma 4.** Every square-free \( S \)-partition is equivalent to a partition containing \( \bar{\epsilon}_1 + \bar{\epsilon}_2, \bar{\epsilon}_2, 0, \bar{\epsilon}_3, \) and \( \bar{\epsilon}_3 + \bar{\epsilon}_4 \), in its first cell \( C_+ \).

**Proof.** Without loss of generality, we can assume that \( \bar{0} \) belongs to \( C_+ \), and its two neighbors in \( C_+ \) are \( \bar{\epsilon}_2 \) and \( \bar{\epsilon}_3 \). The word \( \bar{\epsilon}_2 + \bar{\epsilon}_3 \) is not in \( C_0 \) because the partition is square-free. Hence, the second \( C_+ \)-neighbor of \( \bar{\epsilon}_2 \) is \( \bar{\epsilon}_i + \bar{\epsilon}_2 \), where \( i \neq 2, 3 \). We can assume \( i = 1 \). Similarly, the second \( C_+ \)-neighbor of \( \bar{\epsilon}_2 \) is \( \bar{\epsilon}_3 + \bar{\epsilon}_j \), where \( j \) is not 2 or 3. Additionally, \( j \) is not 1, because the partition is not light otherwise, contradicting Lemma 3. So, we can assume \( j = 4 \).

Based on the lemma above, we will search for only local partitions having \( \bar{\epsilon}_1 + \bar{\epsilon}_2, \bar{\epsilon}_2, 0, \bar{\epsilon}_3, \) and \( \bar{\epsilon}_3 + \bar{\epsilon}_4 \), in the first cell. Among such partitions,
• there are 1786 equivalence classes of \((2, 2)\)-local partitions, 1010 equivalence classes of \(2\)-local partitions.

Realizing the strategy described in Section 2.3, we can choose 1010 representatives of \((2, 2)\)-local partitions, and for each of them find all nonequivalent \((2, 3)\)-local, \((3, 3)\)-local, \((3, 4)\)-local, \((4, 4)\)-local, and finally \(S\)-partitions. The local partitions that continue the chosen 1010 representatives are called leading. Up to equivalence, we have the following numbers of square-free partitions:

• there are 226841305 (few CPU days) square-free \((2, 3)\)-local partitions, 77868835 of them are light, 38039061 of them are leading;

• there are 166399852 light square-free \((3, 3)\)-local partitions, 96211116 (1500 CPU days) of them are leading; there are 83210833 3-local partitions (again, comparing 96211116 and 83210833, we find unreasonable applying the strategy of Section 2.3 for \(r = 3\));

• there are 23752571733 light square-free \((3, 4)\)-local partitions 14937782031 of them (2328 CPU days) are leading;

• there are 923 light square-free \((4, 4)\)-local partitions, 663 (1306 CPU days) of them are leading; there are 490 light square-free 4-local partitions, 312 of them are completable to square-free \(S\)-partitions (the other are also completable to \(S\)-partitions, but with squares);

• there are 26 square-free \(S\)-partitions.

3 Results and properties

As we see in Sections 2.4–2.6, there are 103 nonequivalent \([2, 10], [6, 6]\)-partitions, 77 of them have squares in the first cell, while the remaining 26 are square-free. In this section, we discuss properties of the found equitable partitions.

3.1 Cycle length

The first cell of a \([2, 10], [6, 6]\)-partition induce a disjoint union of cycles in \(Q_n\). These cycles can be of different lengths, the possible values are 4, 8, 10, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 44, 48, 52, 60, 88, 120. The following partitions have only one cycle length:

\#1: \(4^{384}\) (i.e., 384 cycles of length 4);
3.2 Automorphisms

The order of the automorphism group of a partition from the considered class can possess the values 8 (#64, #65), 16, 32, 64, 128, 160 (#69, #71), 256, 384 (#19), 512, 640 (#73, #74), 768 (#95), 1024, 2048, 2560 (#33), 3072, 4096, 6192 (#15, #102), 12288 (#103), 16384 (#8, #12), 24576 (#101), 32768 (#3), 983040 (#1). The number of translational automorphisms (i.e., the number of periods $\bar{x}: C_+ + \bar{x} = C_+$) can be 4, 8, 16, 32 (#2, #12, #13, #15, #28), 64 (#3, #8, #101), 128 (#1). The only partitions that have odd-weight (with odd, 5 or 7, number of ones) periods are #41, #43, #74. The cell $C_+$ is partitioned into orbits under the action of $\text{Aut}(C_+)$; the following partitions have small number of the orbits: #1, #101 (1 orbit), #16, #95, #103 (2 orbits), #3, #8, #12, #32, #88, #102 (3 orbits), #15, #20, #28, #30, #33, #96, #97 (4 orbits). A notable partition is #95: the first cell is divided into only two orbits, while the induced cycles are of length 24.

3.3 Good orthogonal arrays and Fourier coefficients

A multiset $C$ of tuples from $\{0, 1\}^n$ is called a (binary) orthogonal array of strength $t$, $\text{OA}(|C|, n, 2, t)$, if every $(n - t)$-subcube of $Q_n$ has exactly $|C|/2^t$ elements of $C$, taking into account the multiplicities. If there are no elements of multiplicity more than one (i.e., $C$ can be treated as an ordinary set), the orthogonal array is called simple. In other words, a simple $\text{OA}(M, n, 2, t)$ is a subset of $\{0, 1\}^n$ of size $M$ whose characteristic function is correlation immune of order $t$. In particular, $\text{[2,10], [6.6]}$-partitions correspond to simple $\text{OA}(1536,12,2,7)$. By the bound in [6] (see [1] for the simple case), an array of cardinality less than $2^{n-1}$ has strength at most $2n/3 - 1$ (for $n = 12$, at most 7). So, among orthogonal arrays of strength 7
and size less than 2048, there are no orthogonal arrays of strength 8. It is natural to ask which of them better approximate orthogonal arrays of strength 8. We will say that $C$ is an almost orthogonal array of strength $t + 1$, OA$(|C|, n, 2, t)$, if it is OA$(|C|, n, 2, t)$ and every $(n - t - 1)$-subcube of $Q_n$ has exactly $|C|/2^{t+1} - 1$, $|C|/2^{t+1}$, or $|C|/2^{t+1} + 1$ elements of $C$. From [9], we know that there are two simple OA$(1792, 12, 2, 7)$ and both of them are OA$(1792, 12, 2, 7+)$. From current computations, we find that there are 103 simple OA$(1536, 12, 2, 7)$ and only two of them, #81 and #82 (with 32 cycles of length 48), are OA$(1536, 12, 2, 7+)$. Also, from data in [9], we find that among the 16 OA$(1024, 12, 2, 7)$, there is only one OA$(1024, 12, 2, 7)$, number 16.

The Fourier decomposition of a real-valued function $f : \{0, 1\}^n \to \mathbb{R}$ is the representation of $f$ in the form

$$f(\bar{x}) = \sum_{\bar{y} \in \{0,1\}^n} \hat{f}(\bar{y})(-1)^{\bar{y} \cdot \bar{x}},$$

where $\bar{y} \cdot \bar{x}$ is the inner product defined as $(y_1, ..., y_n) \cdot (x_1, ..., x_n) = y_1x_1 + ... + y_nx_n$, and $\hat{f}(\bar{y})$ are the Fourier coefficients defined uniquely for each $f$ (indeed, the functions $\chi_{\bar{y}}(\bar{x}) = (-1)^{\bar{y} \cdot \bar{x}}$, $\bar{y} \in \{0,1\}^n$, form an orthogonal basis of the space of all real-valued functions on $\{0,1\}^n$). The function $\hat{f}$ is called the Fourier transform of $f$. For an equitable $S$-partition $(C_+, C_-)$, it is convenient to consider the Fourier transform $\hat{f}_{C_+, C_-}$ of

$$f_{C_+, C_-}(\bar{x}) = \begin{cases} \quad S_+ & \text{if } \bar{x} \in C_+ \\ -S_- & \text{if } \bar{x} \in C_- \end{cases}.$$

It is known that

(a) $\hat{f}_{C_+, C_-}(\bar{y}) = 0$ unless $\bar{y}$ has exactly $(b + c)/2$ ones;

(b) if $\bar{y}$ has exactly $(b + c)/2$ ones, then

$$\hat{f}_{C_+, C_-}(\bar{y}) = 2^{(S_++S_-)/2-n} \sum_{\bar{x} \perp \bar{y}} f_{C_+, C_-}(\bar{x}),$$

where $\bar{x} \perp \bar{y}$ means that $\bar{x}$ and $\bar{y}$ have no ones in the same position;

(c) in particular, every Fourier coefficient is a multiple of $2^{(S_++S_-)/2-n}(S_++S_-)$;

(d) $\sum_{\bar{y} \in \{0,1\}^n}(\hat{f}_{C_+, C_-}(\bar{y}))^2 = S_+S_-$. 

13
It follows from (c) and (d) that the Fourier transform of every $[2, 10], [6, 6]$-partition is integer and the sum of the squares of all coefficients is 60. In fact, the possible values of coefficients are 0, ±1, ±2. It can be seen from (b) that the partitions with all Fourier coefficients in $\{0, 1, -1\}$ (60 non-zero coefficients in total) correspond to OA(1536, 12, 2, 7+) (equivalence classes #81 and #82); i.e., for these partitions, every 4-subcube has 11, 12, or 13 elements of the first cell. There are four inequivalent partitions with the Fourier coefficients in $\{0, 2, -2\}$ (15 non-zero coefficients), #1, #3, #8, and #103; for these partitions, every 4-subcube has 10, 12, or 14 elements of the first cell. For the partitions from the other 98 equivalence classes, each value from 10 to 14 is realized in some 4-subcube.

3.4 Known constructions

The partitions #1, #3, #15, and #103 (cycle formulas $4^{384}, 4^{256}8^{64}, 4^{128}8^{128}, 8^{192}$, respectively) are obtained from the unique $[[1, 5], [3, 3]]$-partition $(P_+, P_-)$ by the constructions

$$C_{i} = \{(\bar{x}, \bar{y}) \mid \bar{x} + \bar{y} \in P_{i}\},$$

where the addition is coordinatewise over $\mathbb{Z}_2$ for #1, by pairs of coordinates over $\mathbb{Z}_4$ for #103 (binary pairs are treated as elements of $\mathbb{Z}_4$, via the map $00 \rightarrow 0, 01 \rightarrow 1, 11 \rightarrow 2, 10 \rightarrow 3$), and of mixed $\mathbb{Z}_2\mathbb{Z}_4$ type for #3 and #15 (one or two pairs of coordinates are treated as elements of $\mathbb{Z}_4$, while the remaining 4 or 2 coordinates are treated as elements of $\mathbb{Z}_2$).

The next construction is described here without some technical details, which can be found in [2] and [12]. $[[2, 10], [6, 6]]$-partitions can be constructed in the following way. At first, we take a $[[2, 1], [3, 0]]$-partition $(A_+, A_-)$, say $A_- = \{000, 111\}$; note that $A_+$ can be partitioned into 3 edges of the 3-cube. Defining $B_+ = \{(\bar{x}, \bar{y}) \mid \bar{x} + \bar{y} \in A_+\}$ (with the coordinatewise binary addition), we get a $[[4, 2], [6, 0]]$-partition $(B_+, B_-)$, where $B_+$ can be partitioned into twelve quadruples $Q_1, \ldots, Q_{12}$ inducing 2-subcubes of the 6-cube. The next step is a special case of the construction in [2, Sect. 3], with the additional possibility to switch some subsets observed in [12]. Applying the doubling construction again results in a $[[8, 4], [12, 0]]$-partition $(C_+, C_-)$, where $C_+$ can be split into $C'_+$ and $C''_+$ to form a $[[2, 6, 4], [6, 2, 4], [6, 6, 0]]$-partition $(C'_+, C''_+, C_-)$. There are many ways to split: each set $\{(\bar{x}, \bar{y}) \mid \bar{x} + \bar{y} \in Q_{i}\}$, is divided into two subsets $Q'_i$ and $Q''_i$ according to the formula in [2], but, as noted in [12], for each $i$ we are free to choose which of $Q'_i, Q''_i$ is included in $C'_+$ and which in $C''_+$. So, we can get $2^{12}$ different $[[2, 6, 4], [6, 2, 4], [6, 6, 0]]$-partitions $(C'_+, C''_+, C_-)$. It is straightforward from the quotient matrix that for each of them, $(C'_+, C''_+ \cup C_-)$ is a $[[2, 10], [6, 6]]$-partition. In such a way,
we get 8 equivalence classes of \([2, 10], [6, 6]\)-partitions, \#1, \#2, \#3, \#5, \#7, \#8, \#16, and \#101. With some accuracy, this switching approach can be combined with varying the addition as in the previous paragraph, but this possibility was not yet developed.

For \#1, \#73 (cycle formula \(4^{41}20^{16}60^{16}\)), \#74 (\(4^{64}10^{16}20^{8}30^{16}60^{8}\)), the first cell of a partition can be obtained as the projection (puncturing) of the first cell of the unique \([0, 13], [3, 10]\)-partition (the construction is given in [2], the uniqueness is established in [8]) in one of the directions (one direction corresponds to \#1, six directions to \#73, and six to \#74).

4 Related structures

4.1 Splitting into equitable 3-partitions

Here, we discuss pairs of disjoint orthogonal arrays \(\text{OA}(1536, 12, 2, 7)\). The complement of the union of two disjoint \(\text{OA}(1536, 12, 2, 7)\) is necessarily an \(\text{OA}(1024, 12, 2, 7)\), and these three arrays necessarily form an equitable partition with quotient matrix \([2, 6, 4], [6, 2, 4], [6, 6, 0]\). The easiest way to classify such partitions is, starting from \(\text{OA}(1536, 12, 2, 7)\), to find all \(\text{OA}(1024, 12, 2, 7)\) disjoint to it. This can be done with the same approach as the classification of \(\text{OA}(1024, 12, 2, 7)\) in [9]. It can be considered as a simplified version of the computational approach considered in the present paper, and we do not discuss the details here; the only thing we note is that the number of solutions at each step is relatively small, and isomorph rejection is not necessary until the final step. As a result, we find that for only 36 of 103 inequivalent \(\text{OA}(1536, 12, 2, 7)\) the complement can be split into \(\text{OA}(1536, 12, 2, 7)\) and \(\text{OA}(1024, 12, 2, 7)\). For 6 of 103 inequivalent \(\text{OA}(1536, 12, 2, 7)\) (No 1, 3, 8, 15, 101, 103), the complement can be split in 5 ways, (No 2, 4–7, 9–14, 16–21, 27–30, 32, 88–89, 96–100, 102) for the rest 67 \(\text{OA}(1536, 12, 2, 7)\), the complement in unsplittable. Totally, there are 40 inequivalent pairs of disjoint \(\text{OA}(1536, 12, 2, 7)\) (essentially, equitable \([2, 6, 4], [6, 2, 4], [6, 6, 0]\)-partitions). In 38 of them, the two \(\text{OA}(1536, 12, 2, 7)\) are equivalent to each other; for 2 pairs, they are not equivalent. Only 5 (No 1, 2, 3, 15, and 16, according to [9]) of the 16 inequivalent \(\text{OA}(1024, 12, 2, 7)\) can occur as the complement of two disjoint \(\text{OA}(1536, 12, 2, 7)\). See more details in the appendix.

4.2 Non-simple \(\text{OA}(1536, 12, 2, 7)\)

In this section, we discuss the existence on non-simple orthogonal arrays with parameters \(\text{OA}(1536, 12, 2, 7)\). One such array can be constructed by (1) from
the following non-simple OA(24, 6, 2, 3): the 20 weight-3 words of length 6 are taken with multiplicity 1 and two words 000000 and 111111, with multiplicity 2. It is not difficult to observe that different meanings of “+” in (1) result in equivalent arrays. A nice property of this OA(24, 6, 2, 3) and the corresponding OA(1536, 12, 2, 7) is that they are also related to equitable partitions: the non-simple OA(24, 6, 2, 3) and the corresponding OA(1536, 12, 2, 7) are obtained from equitable partitions \((C_0, C_1, C_2, C_3)\) of \(Q_6\) and \(Q_{12}\) with quotient matrices

\[
\begin{pmatrix}
0 & 6 & 0 & 0 \\
1 & 0 & 5 & 0 \\
0 & 2 & 0 & 4 \\
0 & 0 & 6 & 0
\end{pmatrix}
\quad \quad \text{and} \quad \quad
\begin{pmatrix}
0 & 12 & 0 & 0 \\
2 & 0 & 10 & 0 \\
0 & 4 & 0 & 8 \\
0 & 0 & 12 & 0
\end{pmatrix}
\]

respectively, by taking \(C_0\) with multiplicity 2 and \(C_3\) with multiplicity 1. The next theorem shows that we cannot construct inequivalent non-simple OA(1536, 12, 2, 7) related to such equitable partitions.

**Theorem 1.** *Up to equivalence, there is only one equitable partition of \(Q_{12}\) with quotient matrix (2).*

**Proof.** Without loss of generality, we assume that the all-zero word \(\bar{0}\) is in \(C_0\). It follows that all weight-1 words are in \(C_1\). Since every vertex of \(C_1\) has exactly 2 neighbors in \(C_0\), we see that there are 6 weight-2 words in \(C_0\) and no two of them have 1 in a common position. W.l.o.g., they are of form \((x, x)\), where \(x\) is a weight-1 word of length 6. They have \(10 \cdot 6 = 60\) neighbors of weight 3 in total, which are necessarily in \(C_1\). The 60 other weight-2 words (that are not of form \((x, x)\)) are in \(C_2\). Each of them has exactly 8 neighbors in \(C_3\), which are necessarily of weight-3, while every weight-3 word in \(C_3\) has exactly 3 weight-2 neighbors in \(C_2\). We find that there are exactly \(60 \cdot 8/3 = 160\) weight-3 words in \(C_3\). Since \(\binom{12}{3} = 60 + 160\), we see that all the weight-3 words that are not neighbors of weight-2 words of \(C_0\) (that is, the weight-3 words of form \((x, y)\) where \(x\) and \(y\) have no 1 in a common position) are in \(C_3\). Now, consider the words of weight 4 in \(C_0\). They can only be of form \((x, x)\), because every other weight-4 word has neighbors in \(C_3\) (of weight 3). On the other hand, by counting the number of \(C_0\)-neighbors for the \(C_1\)-neighbors of weight 3, we conclude that all words of form \((x, x)\) and weight 4 (as well as weight 0 and 2) are in \(C_0\). The rest of the proof consists of simple numbered claims.

Claim 1: if \(x, y, \text{ and } z\) belong to \(C_0\) and \(y, z\) are at distance 2 from \(x\), then \(x + y + z \in C_0\). Proof: from the consideration above, we see that (*) holds for \(x = 0\). Similarly, it holds for every other \(x\) from \(C_0\).
Denote by \( C^{(i)} \) the set of all length-12 binary words of form \((x, y)\) such that \( x + y \) has weight \( i \).

Claim 2: \( C^{(0)} \) is a subset of \( C_0 \). Proof: straightforwardly from Claim 1.

Claim 3: \( C^{(i)} \) has exactly \( 2i \) neighbors in \( C^{(i-1)} \) and \( 12 - 2i \) neighbors in \( C^{(i+1)} \), \( i = 0, \ldots, 6 \). Proof: straightforwardly from the definition.

Claim 4: \( C^{(1)} \subset C_1, C^{(2)} \subset C_2, C^{(3)} \subset C_3, C^{(4)} \subset C_2, C^{(5)} \subset C_1, C^{(6)} \subset C_0 \). Proof: straightforwardly from Claim 2, Claim 3, and the quotient matrix.

Claim 5: \( C_0 = C^{(0)} \cup C^{(6)}, C_1 = C^{(1)} \cup C^{(5)}, C_2 = C^{(2)} \cup C^{(4)}, C_3 = C^{(3)} \).

Finally, we see that the partition is reconstructed up to equivalence.

\[ \square \]

**Problem 1.** Are there inequivalent non-simple \( \text{OA}(1536, 12, 2, 7) \)? Is there a non-simple \( \text{OA}(1792, 12, 2, 7) \)? Are there orthogonal arrays \( \text{OA}(M, 12, 2, 7) \) with \( M < 2048, M \not\in \{1024, 1536, 1792\} \) (such arrays cannot be simple)?

**A List of partitions**

Below, we list representatives of all 103 equivalence classes of equitable partitions with quotient matrix \( [[2, 10], [6, 6]] \), which correspond to simple \( \text{OA}(1536, 12, 2, 7) \), and 40 classes of equitable partitions with quotient matrix \( [[2, 6, 4], [6, 2, 4], [6, 6, 0]] \) (partitions into two \( \text{OA}(1536, 12, 2, 7) \) and one \( \text{OA}(1024, 12, 2, 7) \)).

We first describe the list of equitable \( [[2, 10], [6, 6]] \)-partitions. It would take unreasonably much space to list each partition completely. We use some easy-to-recover form, representing each partition \((C_+, C_-)\) by the characteristic function \( \chi_{C_+} \) of the first cell, restricted by the weight-4 words only. The values of this function on the words of weight 3 (similarly, weight 2, weight 1, and 0) can be easily reconstructed by the following rule: if a weight-3 word has at most 2 weight-4 neighbors in \( C_+ \), then it belongs to \( C_+ \), otherwise it is in \( C_- \). The values on each word \( x \) of weight \( w = 5 \) (then, 6, 7, \ldots, 12) can be found using the correlation immunity of \( \chi_{C_+} \): the number of ones in the set \( \{y \mid y \preceq x\} \) is \( \frac{6}{10+6} \cdot 2^w \). The list of values on the 495 weight-4 words, listed in the lexicographic ordering \( 000000001111, 000000010111, \ldots, 111100000000 \), is represented in the hexadecimal form, each symbol corresponding to 4 binary values, except the first symbol of the sequence, which corresponds to 3 binary values (note that the first value can be 0, representatives No 75, 76, 77); each list is given in two lines following the class number with period (1. \ldots, 2. \ldots, 103. \ldots).

The representatives are lexicographically ordered in the following manner: for each partition, we order the first cell lexicographically, forming a
tuple from 1536 words; then, such lists are compared in the lexicographical way. Each equivalence class is represented by the lexicographically first partition, and all 103 representatives of different classes are ordered as well. In particular, the first 77 contain 0...0, 0...01, 0...010, 0...011, forming a square, and the last 26, square-free, contain 0...0, 0...01, 0...010, 0...0101, 0...01010. Note that the lexicographic order between functions on the whole 12-cube is not kept after restricting by the weight-4 words only; that is why the list of sequences below does not look ordered.

If in an equitable \([2, 10], [6, 6]\)-partition \((C_+, C_-)\) the cell \(C_-\) is splittable into cells \(C_*, C_\bullet\) of an equitable \([2, 6, 4], [6, 2, 4], [6, 6, 0]\)-partition, then we also represent the smallest cell \(C_\bullet\) over all inequivalent ways to split. The algorithm to reconstruct all vertices of \(C_\bullet\) from the weight-4 vertices, which are listed, is similar to the one for \(C_+\): if a weight-3 (similarly, for weight-2 and weight-1) word has no weight-4 neighbors in \(C_\bullet\), then then it belongs to \(C_\bullet\), otherwise it is in \(C_+\) or \(C_*\). For words of weight \(w = 5\) (then, 6, 7, \ldots, 12), the words of \(C_\bullet\) are determined from the orthogonal array property. Each list representing \(C_\bullet\) is given in two lines following the class number (of equitable \([2, 6, 4], [6, 2, 4], [6, 6, 0]\)-partitions) in parentheses. The numbers in parentheses can repeat because inequivalent equitable \([2, 10], [6, 6]\)-partition (representatives No 8 and No 15, No 101 and No 103) can be split into equivalent \([2, 6, 4], [6, 2, 4], [6, 6, 0]\)-partitions. So, there are 40 inequivalent equitable \([2, 6, 4], [6, 2, 4], [6, 6, 0]\)-partitions, while our list contains 42 representatives. The number in the brackets in the right of the list indicates the equivalence class of the equitable \([0, 12], [4, 8]\)-partition \((C_\bullet, C_+ \cup C_*)\), according to the classification in [9]; the following number is the number of ways a code equivalent to \(C_\bullet\) can be embedded in the complement of \(C_+\).
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