Slowly Rotating Non-Abelian Black Holes

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Abstract

It is shown that the well-known non-Abelian static SU(2) black hole solutions have rotating generalizations, provided that the hypothesis of linearization stability is accepted. Surprisingly, this rotating branch has an asymptotically Abelian gauge field with an electric charge that cannot vanish, although the non-rotating limit is uncharged. We argue that this may be related to our second finding, namely that there are no globally regular slowly rotating excitations of the particle-like Bartnik-McKinnon solutions.
Introduction.– Ever since the discovery of a discrete family of particle-like solutions of the Einstein-Yang-Mills (EYM) equations by Bartnik and McKinnon [1], as well as of their black hole analogues by several workers [2], [3], it has been a challenge to find rotating generalizations of these remarkable objects. This is an exceedingly difficult task. It is not even clear how to parametrize the metric, because the coupled field equations do not imply the Frobenius integrability conditions for the Killing fields [4]. The usual Papapetrou ansatz [5] may therefore be too narrow. Even if this ansatz is adopted, the resulting system of partial differential equations becomes considerably more involved than in the Abelian case.

In view of this situation it is reasonable to pursue in a first step a more modest goal, based on the assumption of linearization stability [6]. Suppose that there is (at least) a one-parameter family of stationary black hole or regular solutions of the EYM equations, approaching the static solutions mentioned above for angular momentum $J = 0$, then the tangent to this family at $J = 0$ satisfies the linearized EYM equations. Conversely, it is reasonable to expect that for a well-behaved solution of the linearized equations around the static configurations there exists an exact one-parameter family of stationary solutions.

In this paper we show that, under the above assumption of linearization stability, there are slowly rotating non-Abelian black holes branching off from the (numerically known) static ones. To our surprise these black holes are “charged up”, in that they possess an asymptotically Abelian gauge field with a non-vanishing electric charge. A second result, which may also be surprising to many workers in the field, concerns slowly rotating globally regular solutions: It turns out that there are no acceptable rotational modes in this case; all nontrivial ones develop singularities at the origin. At the end of the paper we shall give a heuristic interpretation of how this may be related to the charging up of the black holes. We can, of course, not exclude that there is a disconnected rapidly rotating branch above some non-vanishing angular momentum.

Recently, there have been a number of investigations [7, 8] on static axially symmetric solutions for matter models with YM fields which do not exhibit spherical symmetry, but, to our knowledge, we present here the first results on rotational deformations.

Perturbation equations.– The EYM action for the SU(2) gauge group reads in standard notation

\[ S = \int \left( -\frac{1}{4} R + \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} \, d^4x, \]

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. In the decomposition $A_\mu = A^a_\mu T_a$ of the gauge potential we choose the gauge group generators $T_a = \tau^a/2i$ with $\tau^a$ being the Pauli matrices.

An intensively studied family of non-Abelian black hole solutions [2], [3] is described by static, spherically symmetric metrics parametrized as

\[ ds^2 = \left( 1 - \frac{2m(r)}{r} \right) \sigma(r)^2 dt^2 - \frac{dr^2}{1 - 2m(r)/r} - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2), \]

and purely magnetic gauge fields of the form

\[ A = w(r) (-T_2 \, d\theta + T_1 \sin \theta \, d\varphi) + T_3 \cos \theta \, d\varphi. \]

The functions $w(r)$, $m(r)$, and $\sigma(r)$ are subject to the EYM field equations. The solutions are characterized by the event horizon radius $r_h$ and an integer $n$ counting the number
of nodes of \( w \). We shall need the asymptotic behavior of the solutions,

\[
 w = w(r_h) + O(x), \quad m = \frac{r_h}{2} + O(x); \quad w = -1 + \frac{a}{r} + O\left(\frac{1}{r^2}\right), \quad m = M + O\left(\frac{1}{r^3}\right),
\]

at the horizon and infinity, respectively; here \( x = r - r_h \), and \( w(r_h) \), \( a \), and \( M \) are fixed by the solution parameters \( r_h \) and \( n \); the behavior of \( \sigma \) is determined by that for \( w \).

Consider perturbations of a given background equilibrium solution \((g_{\mu\nu}, A_\mu)\),

\[
g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad A_\nu \to A_\nu + \delta A_\nu.
\]

The perturbation equations are obtained by linearization of the field equations:

\[
 \delta R_{\mu\nu} = 2 \delta T_{\mu\nu}, \quad \delta (D_\sigma F_{\mu\nu}) = 0,
\]

where \( D_\nu \equiv \nabla_\nu + [A_\nu, \cdot] \). With the notations \( h_{\mu\nu} \equiv \delta g_{\mu\nu} \) and \( \psi_\mu \equiv \delta A_\mu \) the perturbation equations take the form

\[
 -\nabla_\sigma \nabla^\sigma h_{\mu\nu} - 2 R_{\mu\nu\beta\gamma} h^{\alpha\beta} + R^\sigma_{\mu} h_{\sigma\nu} + R^\sigma_{\nu} h_{\sigma\mu} = 4 \delta T_{\mu\nu},
\]

\[
 - D_\sigma D^\sigma \psi_\nu + R^\sigma_\nu \psi_\sigma - 2 [F_{\nu\sigma}, \psi_\sigma] + h^{\alpha\beta} D_\alpha F_{\beta\nu} + F^{\alpha\beta} \nabla_\alpha h_{\beta\nu} = 0,
\]

provided that the following gauge conditions are imposed:

\[
 \nabla_\sigma h_\mu^\sigma = h_\mu^\sigma = D_\sigma \psi^\sigma = 0.
\]

The quantity \( \delta T_{\mu\nu} \) in Eqs. (4) is obtained by varying the energy-momentum tensor

\[
 T_{\mu\nu} = \frac{1}{2} \text{tr} \left( F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} g^{\alpha\beta} \right)
\]

with respect to the metric and the gauge field.

**Rotational perturbations.** To identify the most general rotational degrees of freedom, we determine those amplitudes in the partial wave decomposition which can give a non-vanishing contribution to the ADM flux integral for the total angular momentum

\[
 J^i = \frac{1}{32\pi} \int_{S^2} \epsilon_{i kn} (x^k \partial_j h^{0n} + \delta_j^n h^{0k}) d^2 S^j.
\]

We do this by expanding \( h_{\mu\nu} \) and \( \psi_\mu \) with respect to an appropriately chosen complex null tetrad. (This approach has proved to be extremely efficient for solving wave equations.) The complete separation of the angular variables in Eqs. (4) is achieved, provided that each of the tetrad projections (for \( \psi_\mu \) we project in addition onto \( T_{\pm} = T_1 \pm i T_2 \), and \( T_3 \)) is chosen as the product of a radial amplitude, depending only on \( t \) and \( r \), and a spin-weighted spherical harmonic \( \chi Y_{lm}(\theta, \phi) \). Here the spin weight is \( s = 0, \pm 1, \pm 2 \), depending on the projection under consideration. Passing back to the coordinate basis, we obtain the full mode decomposition for perturbations with respect to the angular momentum quantum numbers. (We do not present here the explicit expressions in view of their complexity.)

Since the background solutions are spherically symmetric, group theoretical arguments imply that modes with different pairs \((l, m)\) decouple, and for a given \( l \) (an irreducible representation of the rotation group) there is a \((2l + 1)\)-fold degeneracy labeled by \( m \). The angular order \( m \) does not occur in the equations for the radial amplitudes,
and we can thus choose \( m = 0 \). Next, passing to cartesian coordinates \( x^i \), we compute the ADM angular momentum by integrating over a two-sphere at finite radius \( r \) in Eq. (10) and then taking the limit \( r \to \infty \). The angular dependence of the integrand implies then that the integral vanishes for any \( r \) unless \( l = 1 \), and that only the \( h_{0\varphi} \) perturbation component can give a non-vanishing contribution. Choosing \( l = 1 \) and suppressing the time-dependence, the tensor modes with the spin weight \( s = \pm 2 \) vanish and Eqs. (7) reduce to a system of 18 coupled equations.

Now, the transformation behavior of the angular momentum under space and time reflections (P,T) implies that only those perturbation amplitudes are relevant which are even under P and odd under T. For \( l = 1 \) these appear only in \( h_{0\varphi} \) and in two isotopic component of \( \psi_0 \). They decouple from the remaining modes because the background solutions are P and T symmetric. This analysis leads finally to the following most general ansatz (up to global coordinate rotations) for the stationary rotational modes:

\[
\begin{align*}
    h &= 2S(r) \sin^2 \theta \, dt \, d\varphi, \\
    \psi &= \left( T_1 \frac{\chi(r)}{r} \sin \theta + T_3 \frac{\eta(r)}{r} \cos \theta \right) dt.
\end{align*}
\]

(11)

(The invariance of \( \psi \) and the background gauge field (3) under P becomes manifest after a suitable gauge transformation). Conditions (8) for the ansatz are fulfilled identically and the perturbation equations (7) reduce to the following coupled system for the radial amplitudes \( S, \chi, \) and \( \eta \) in (11):

\[
\begin{align*}
    -r^2 N \sigma \left( \frac{S'}{\sigma} \right)' &+ \left( 2N + 4 \left( \frac{w^2 - 1)^2}{r^2} \right) S + 4Nr^2 w' \left( \frac{\chi}{r} \right)' + \frac{4(w^2 - 1)}{r} (w \chi - \eta) = 0, \\
    -r^2 N \sigma \left( \frac{\chi'}{\sigma} \right)' &+ \left( 1 + w^2 - 2w^2 N \right) \chi - 2w \eta - rN (w'S)' + \left( 2Nw^2 + \frac{w(w^2 - 1)}{r} \right) S = 0, \\
    -r^2 N \sigma \left( \frac{\eta'}{\sigma} \right)' &+ 2 \left( 1 + w^2 - w^2 N \right) \eta - 4w \chi + \frac{2(1 - w^2)}{r} S = 0.
\end{align*}
\]

(12)

Here \( w, m, \) and \( \sigma \) refer to the background solutions, \( N \equiv 1 - 2m/r \). For a solution of these equations, the ADM angular momentum is

\[
J^i = \delta^i_z \lim_{r \to \infty} \frac{1}{6} r^4 \left( \frac{S'}{r^2} \right)'
\]

(13)

**Qualitative considerations.**– Before turning to the numerical analysis of the radial equations, we make some qualitative remarks. Consider first a Kerr-Newman black hole with mass \( M \), angular momentum \( J \) and electric charge \( Q \). Let \( A \) denote the electromagnetic potential of the solution. Next, consider the embedding of this Abelian solution into the SU(2) gauge theory, such that the gauge field potential is \( A = AT_3 \). Suppose that \( |J| \) and \( |Q| \) are small and linearize the solution with respect to \( J \) and \( Q \). The result can be viewed as a Schwarzschild black hole with two linear hairs. The axial gravitational perturbation can be represented in the form of Eq. (11)

\[
S(r) = -\frac{2JM}{r},
\]

(14)

while the Yang-Mills hair is described by \( A = T_3(Q/r)dt \). After a gauge transformation with the SU(2)-valued function \( U = \exp((\pi - \theta)T_2) \exp(\varphi T_3) \) this gauge field becomes

\[
A = \frac{Q}{r} UT_3U^{-1} + UdU^{-1} = \psi + A_{\text{pure}},
\]

(15)
where $A_{\text{pure}}$ is a pure gauge potential of the form (3) with $w(r) = -1$, and the perturbation $\psi$ is given by

$$\psi = \frac{Q}{r} (T_1 \sin \theta - T_3 \cos \theta) dt. \quad (16)$$

Now let us return to the perturbation equations (12) and concentrate on their solutions in the asymptotic region $r \gg r_h$. In this limit, the background geometry is approximately Schwarzschild and $w(r) \approx -1$, i.e., the background gauge field is almost pure gauge. Eqs. (12) then split into the two independent groups – one equation for the metric perturbation $S$, and two coupled equations for the Yang-Mills amplitudes $\chi$ and $\eta$.

The acceptable solution of the first equation is given by Eq. (14). The equations for $\chi$ and $\eta$ admit the solution $\chi(r) = -\eta(r) = Q$ with constant $Q$, which, together with the background pure gauge field, exactly corresponds to the gauge field specified by Eqs. (15), (16). We therefore conclude that the solution of the perturbation equations in the asymptotic region is close to the linearized Kerr-Newman solution with charge $Q$ and angular momentum $J$.

So far the parameters $J$ and $Q$ are independent. However, Eqs. (12) split into the two independent groups only asymptotically. It is to be expected that the regularity of the solution in the entire domain $r \geq r_h$ will imply relations for the parameters $J$ and $Q$ describing the asymptotics. We shall indeed show below that the charge $Q$ of the black hole is uniquely fixed by the value of its angular momentum $J$.

**Numerical solutions.**– We now describe briefly the numerical analysis of the full problem. Guided by the Abelian example, we are looking for global solutions of Eqs. (12) which are everywhere regular. Note that they do not have to be normalizable, as the Kerr-Newman example already shows. Eqs. (12) have regular singular points at the horizon and at infinity. At the horizon, the formal power-series solution is found to be (using Eq. (4)):

$$S = c_0 + c_1 x + O(x^2), \quad \chi = -\frac{c_0 w(r_h)}{r_h} x + c_2 x + O(x^2), \quad \eta = -\frac{c_0}{r_h} x + c_3 x + O(x^2), \quad (17)$$

where $c_0$, $c_1$, $c_2$, and $c_3$ are four independent integration constants. The solution at infinity contains three independent parameters, $J$, $Q$, and $c_4$:

$$S = -\frac{2JM}{r} + \frac{aQ}{r^2} + O\left(\frac{1}{r^3}\right), \quad \chi = Q \left(1 + \frac{a^2 \ln r}{15 r^2}\right) + \frac{c_4}{r^2} + O\left(\frac{\ln r}{r^3}\right),$$

$$\eta = Q \left(-1 + \frac{2a^2 (\ln r - 5)}{15 r^2}\right) + \frac{2c_4}{r^2} + O\left(\frac{\ln r}{r^3}\right). \quad (18)$$

Clearly, $J$ and $Q$ are, respectively, the angular momentum and the charge of the black hole.

Now, we extend these asymptotic solutions from both sides to the intermediate region and impose the matching conditions for the functions $S$, $\chi$, and $\eta$, and their derivatives. This gives six linear algebraic equations for the seven free parameters in (17), (18), and hence the matching can generically be fulfilled. Thus there exists globally a solution whose asymptotic behavior is specified by Eqs. (17) and (18). For a given $J$ the values of the remaining six coefficients will be proportional to $J$. In particular, we have a relation of the type

$$Q = \Omega_n(r_h) J, \quad (19)$$
Figure 1: The radial behavior of the rotational excitations for the $n = 1$, $r_h = 1$ non-Abelian black hole solution.

where $\Omega_n(r_h)$ is a function determined by the background non-Abelian black hole solution.

The numerical integration of Eqs. (12) (see Fig.1) with the boundary conditions specified by Eqs.(17), (18) reveals that in general none of the seven coefficients in (17), (18) vanishes. We therefore conclude that, under the assumption of linearization stability, each non-Abelian black hole solution admits stationary rotational generalizations with electric charge given by Eq. (19). The behavior of the function $\Omega_n(r_h)$ for the lowest $n$'s and for $0 < r_h < \infty$ is shown in Fig. 2; numerical values are given in Table 1.

Table 1. Numerical data for $\Omega_n(r_h)$.

| $n$ | $r_h = 0$ | $r_h = 0.5$ | $r_h = 1$ | $r_h = 5$ | $1/\alpha \equiv r_h \gg 1$ |
|-----|-----------|-------------|-----------|-----------|------------------|
| 1   | -0.4396   | -0.2416     | -0.1117   | -0.0108   | -0.0052 $\alpha + O(\alpha^2)$ |
| 2   | +0.0092   | -0.0091     | -0.0096   | -0.0003   | -0.0014 $\alpha + O(\alpha^2)$ |
| 3   | -0.0145   | -0.0084     | -0.0010   | -8 $\times$ 10^{-6} | -3 $\times$ 10^{-5} $\alpha + O(\alpha^2)$ |

Discussion.— To obtain the asymptotic behavior of $\Omega_n(r_h)$ for $r_h \gg 1$ we make use of the fact that, in this limit, the spacetime geometry of the non-Abelian black holes is Schwarzschild, up to corrections of order $1/r_h^2$. Then, introducing the new radial coordinate $\xi = \alpha r$ with $\alpha \equiv 1/r_h$, in terms of which the metric functions are $N = 1 - 1/\xi + O(\alpha^2)$, $\sigma = 1 + O(\alpha^2)$, we expand Eqs. (12) with respect to $\alpha$. The solution in zeroth order of the expansion is $S^{(0)}(\xi) = -J/\xi$, and $\eta^{(0)} = \chi^{(0)} = 0$. In first order, we obtain two linear equations for $\eta^{(1)}$ and $\chi^{(1)}$ with the source terms constructed from $S^{(0)}$. These equations, as Eqs. (12), admit solutions which become constant at infinity, thus giving rise to an electric charge of order $\alpha$.

In the opposite limit, $r_h \to 0$, the functions $\Omega_n(r_h)$ assume finite values. The non-Abelian black hole solutions converge for $r_h \to 0$ in the region $r > 0$ pointwise to the regular Bartnik-McKinnon solutions [1], which suggests that the same holds for their
perturbations. In other words, one can expect that the particle-like solutions admit rotational excitations as well. We have to investigate, however, what happens at the origin. There, the behavior of the unperturbed solutions is \( w = \pm (1 - \beta r^2 + O(r^4)) \), \( m = 2\beta^2 r^3 + O(r^5) \), \( \beta \) being a parameter. Eqs. (12) then give for the perturbations

\[
S = -4a_0 \beta \left( r - \frac{\beta}{5} (1 + 4\beta^2) r^3 \right) + a_2 r^2 + O(r^4),
\]

\[
\chi = a_0 \left( 1 - 10\beta^2 r^2 \right) + a_1 r + \left( a_2 \beta - \frac{a_3}{2} \right) r^3 + O(r^4),
\]

\[
\eta = a_0 \left( 1 - 8\beta^2 r^2 \right) + a_1 r + a_3 r^3 + O(r^4),
\]

(20)

where \( a_0, a_1, a_2, \) and \( a_3 \) are four integration constants. Notice that the number of free parameters, together with those in the asymptotic solution (18), is again large enough for matching in the intermediate region. The numerical integration of Eqs. (12) for the regular backgrounds with the boundary conditions given by (18) and (20) shows that none of the seven parameters in (18), (20) vanishes, and the solutions in the region \( r > 0 \) are indeed very close to the black hole rotational modes obtained in the limit \( r_h \to 0 \). In particular, the ratio \( Q/J \) for the particle-like solutions is given by \( \Omega_n(0) \). However, although the functions \( S, \chi \) and \( \eta \) are regular, the corresponding perturbations (11) nevertheless become singular at the origin, since the field ansatz (11) involves the factor \( 1/r \), while the coefficient \( a_0 \) in Eq. (20) does not vanish. We therefore conclude that the regular solutions do not admit rotational excitations.

The appearance of a nonvanishing electric charge induced by the rotation is a somewhat surprising feature of the solutions under consideration – the static non-Abelian black holes being neutral. It is natural to wonder where this charge originates from and where it resides. These questions cannot be answered in a gauge invariant manner, but...
it may be instructive to compute the flux

\[ I = \oint_{S^2} * \delta F \]

through a two-sphere of finite radius in the distinguished gauge where the field \( \delta F \) becomes asymptotically Abelian. This gives uniquely

\[ I = -T_3 \oint_{S^2} \left\{ \left( -\left( \frac{\chi}{r} \right) \right)' \sin^2 \theta + \left( \frac{\eta}{r} \right)' \cos^2 \theta \right\} dt \wedge dr, \]

which motivates us to introduce the charge function

\[ Q(r) = -2 \text{tr} (T_3 I) = r^2 \left( \frac{\eta(r) - 2\chi(r)}{3r} \right)', \]

normalized such that \( Q(\infty) = Q \). The behavior of \( Q(r) \), shown in Fig. 3, suggests that part of the total charge \( Q \) is hidden behind the horizon, and the rest is distributed between the horizon and infinity. If \( r_h \) approaches zero, the hidden charge assumes a finite value, such that its distribution becomes \( \delta \)-like, making it plausible that there are no acceptable rotational perturbations for the solitons.

To summarize, we have found that there is a branch of slowly rotating colored black holes with an asymptotically Abelian gauge field, whose electric flux is proportional to the angular momentum \( J \). Mathematically the appearance of the charge is understood as follows: The static black holes, similarly to the Schwarzschild solution, admit in the far zone two independent hairs, describing rotational and charged excitations. However, unlike the situation in the vacuum case, these two hairs become coupled to each other in the near zone via the background gauge field, and regularity enforces a relation between \( Q \) and \( J \). Physically, on the other hand, this rotational “charging up” may seem rather unusual, since one normally expects that rotation induces only dipole corrections to the static field. – Our second main result is that the particle-like static Bartnik-McKinnon
solutions do not admit continuously connected stationary rotational excitations. Despite this fact, we do not see any reason why solitons in other matter models should not have rotational states. Rotating magnetic monopoles or Skyrmions might be examples of such objects.

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