CHARACTER FACTORIZATIONS FOR REPRESENTATIONS OF $\text{GL}(n, \mathbb{C})$

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ABSTRACT. The aim of this paper is to give another proof of a theorem of D. Prasad, Theorem 2 of [DP], which calculates the character of an irreducible representation of $\text{GL}(mn, \mathbb{C})$ at the diagonal elements of the form $t \cdot c_n$, where $t = (t_1, t_2, \cdots, t_m) \in (\mathbb{C}^*)^m$ and $c_n = (1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1})$, where $\omega_n = e^{\frac{2\pi i}{n}}$, and expresses it as a product of certain characters for $\text{GL}(m, \mathbb{C})$ at $(t^n)$.

June 14, 2023

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1. INTRODUCTION

In the work [DP] of Dipendra Prasad, he discovered a certain factorization theorem for characters of $\text{GL}(mn, \mathbb{C})$ at certain special elements of the diagonal torus, those of the form

$$t \cdot c_n = \begin{pmatrix} t \cdot \cdots \cdot \omega_n^{n-1} \end{pmatrix},$$

where $t = (t_1, t_2, \cdots, t_m)$ and $\omega_n$ is a primitive $n$th root of unity. D. Prasad proved that the character of a finite dimensional highest weight representation $\pi_\lambda$ of $\text{GL}(mn, \mathbb{C})$ of highest weight $\lambda$ at such elements $t \cdot c_n$ is the product of characters of certain highest weight representations of $\text{GL}(m, \mathbb{C})$ at the element $t^n = (t_1^n, t_2^n, \cdots, t_m^n)$. This work of D. Prasad was recently generalized for all classical groups in [AK]. Both the works [DP] and [AK] are achieved via direct manipulation with the Weyl character formula expressed as a determinantal identity. The present work aims at giving another proof of Prasad’s factorisation theorem in which we manipulate directly with the Weyl numerator which is an alternating sum over the Weyl group. Assume that for each $k$, $0 \leq k \leq n - 1$, there are exactly $m$ integers in $\lambda + \rho_{mn}$ that are congruent to $k$ modulo $n$; this is a necessary condition for the character of the associated representation of $\text{GL}(mn, \mathbb{C})$ to be nonzero at some element of the form $t \cdot c_n$ (see Proposition 1 below for a proof). If
this necessary condition on $\lambda + \rho_{mn}$ is satisfied, we can replace $\lambda + \rho_{mn}$ by its conjugate $w_0(\lambda + \rho_{mn}) = \mu$, $w_0 \in S_m$, which affects the numerator in the Weyl character formula only by a sign, such that the first $m$ entries in $\mu$ are a set of (distinct) integers that are congruent to 0 modulo $n$, the next $m$ entries are congruent to 1 modulo $n$ and so on. Then we sum the Weyl numerator over various left cosets of a particularly chosen subgroup $R$ which is isomorphic to $S_m^n$. We find that there is a subgroup $C$ such that those cosets of $R$ which have a representative from $C$ contribute a nonzero term whereas sum over cosets not contained in $CR$ contribute zero. Adding the summations obtained for each of these distinct cosets, we get the Weyl numerator of a representation of $GL(m, \mathbb{C})$ at the diagonal element $t^n$ in each of $GL(m, \mathbb{C})$ in $GL(m, \mathbb{C})^n$. This gives a proof of the factorization theorem of Prasad, hopefully a more conceptual one; this is Theorem 4.1 of this paper which we refer to for a more precise statement of the Theorem.

The paper is written in the hope that manipulations with the Weyl group may be available in many more situations, yielding character identities of the kind obtained in [DP] as well as [AK].

2. Calculation of the Weyl Denominator $A_\rho$

Our work involves factorizing the Weyl numerator and the denominator. Since the Weyl denominator is a special case of the Weyl numerator (for $\lambda = 0$), in particular if we prove a factorization theorem for the numerator, it proves a factorization theorem for the Weyl denominator. However, the Weyl denominator is much a simpler expression, and its factorization is easy enough, so we begin with the factorization of the Weyl denominator $A_{\rho_{mn}}(t \cdot c_n)$ in this section.

\begin{align*}
A_{\rho_{mn}}(t \cdot c_n) &= (1) \prod_{k<l, i \geq j} (\omega_n^k t_i - \omega_n^l t_j) \times \prod_{i<j, k \leq l} (\omega_n^k t_i - \omega_n^l t_j), \\
&= (2) \prod_{k<l, i=j} (\omega_n^k t_i - \omega_n^l t_i) \times \prod_{k<l, i>j} (\omega_n^k t_i - \omega_n^l t_j) \times \prod_{i<j, k \leq l} (\omega_n^k t_i - \omega_n^l t_j), \\
&= A \times B \times C.
\end{align*}

We have

\begin{align*}
(1) \quad A &= \prod_{i<j} (\omega_n^i - \omega_n^j)^m \cdot \left( \prod_{s=1}^m t_s \right)^{\frac{m(m-1)}{2}}, \\
(2) \quad B \times C &= \prod_{\substack{k<l, i=j \\
i > j}} (\omega_n^k t_i - \omega_n^l t_j) \times \prod_{\substack{i<j, k \leq l \\
k \leq l}} (\omega_n^k t_i - \omega_n^l t_j), \\
&= \prod_{k=0}^{n-1} P_k,
\end{align*}

where $P_k = \prod_{i<j} \prod_{l \geq k} (\omega_n^k t_i - \omega_n^l t_j) \times \prod_{i>j} \prod_{l \leq k} (\omega_n^s t_i - \omega_n^k t_j)$. Let us evaluate $P_k$. Now
\[ P_k = \prod_{i<j} (\omega_{i}^k t_i - \omega_{j}^k t_j) \times \prod_{\{0 \leq s < k\}} (\omega_{n}^s t_i - \omega_{n}^s t_j), \]

\[ = \prod_{i<j} (\omega_{n}^n) \prod_{l \geq k} (t_i - \omega_{n}^{l-k} t_j) \times \prod_{\{0 \leq s < k\}} (-1)^k \prod_{\{0 \leq s < k\}} (\omega_{n}^s t_i - \omega_{n}^s t_j), \]

\[ = \prod_{i<j} (\omega_{n}^n) \prod_{l \geq k} (t_i - \omega_{n}^{l-k} t_j) \times \prod_{\{0 \leq s < k\}} (-1)^k (\omega_{n}^k) \prod_{\{0 \leq s < k\}} (t_i - \omega_{n}^{s-k} t_j), \]

\[ = \prod_{i<j} (-1)^k \prod_{l \geq k} (t_i - \omega_{n}^{l-k} t_j) \times \prod_{\{0 \leq s < k\}} (t_i - \omega_{n}^{s-k} t_j). \]

Now for each \( k \), we have,

\[ \prod_{l \geq k} (t_i - \omega_{n}^{l-k} t_j) \times \prod_{\{0 \leq s < k\}} (t_i - \omega_{n}^{s-k} t_j) = (t_i^n - t_j^n). \]

So from equation (3) we conclude that

\[ P_k = (-1)^{\frac{km(m-1)}{2}} \prod_{i,j} (t_i^n - t_j^n). \]

Hence putting the values of \( P_k \), we get

\[ A_{\rho_{mn}}(t \cdot c_n) = \prod_{i<j} (\omega_{n}^i - \omega_{n}^j)^m \cdot \prod_{s=1}^{m} t_s^{\frac{n(n-1)}{2}} \times \prod_{k=0}^{n-1} (-1)^{\frac{km(m-1)}{2}} \prod_{i<j} (t_i^n - t_j^n), \]

\[ = (-1)^{\frac{mn(m-1)(n-1)}{4}} \times \prod_{i<j} (\omega_{n}^i - \omega_{n}^j)^m \times \prod_{i<j} (t_i^n - t_j^n)^m \times \prod_{s=1}^{m} t_s^{\frac{n(n-1)}{2}}. \]

3. Calculation of the Weyl numerator

We will consider \( G = \text{GL}(mn, \mathbb{C}) \) in the standard basis \( \mathcal{A} = \{e_1, e_2, \ldots, e_{mn}\} \) of \( \mathbb{C}^{mn} \). Define \( B_k = \{e_{mk+s} \mid 1 \leq s \leq m\} \) for each \( 0 \leq k \leq n-1 \). Let us define \( V_k = \text{span}\{B_k\} \). This will give rise to a decomposition \( \mathbb{C}^{mn} = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1} \). Define the subgroup \( H \cong \text{GL}(m, \mathbb{C})^n \) to be the stabilizer of this decomposition of \( \mathbb{C}^{mn} \). The Weyl group of \( G \) is \( S_{mn} \), the symmetric group of \( mn \) integers from 1 to \( mn \). Now the Weyl group of \( H \) is \( \prod_{k=0}^{n-1} S(I_k) \), where we define \( I_k = \{mk+1, mk+2, \ldots, (k+1)m\} \), for each \( 0 \leq k \leq n-1 \) and \( S(I_k) \) is the group of permutations of the set \( I_k \). We will denote \((-1)^n\) to be the sign character on \( S_n \) and also on any of its subgroups. As discussed in the Introduction, we will take the Weyl numerator \( A_{\Sigma, \rho_{mn}} \) to be \( \sum_{\mu \in S_{mn}} (-1)^{\mu} e^{\text{w} \mu} \), where \( \mu \) is a conjugate of \( \lambda + \rho_{mn} \) by an element of the Weyl group \( S_{mn} \) so that it satisfies:

\[ \text{property #} = \begin{cases} \text{the first } m \text{ numbers of } \mu \text{ are congruent to } 0 \text{ modulo } n, \\
\text{next } m \text{ numbers are congruent to } 1 \text{ modulo } n \text{ etc.} \end{cases} \]

We begin this section proving the necessary condition on \( \mu \) for the Weyl numerator \( A_{\Sigma} \) to be nonzero.
Proposition 1. If the integers in $\mu$ do not satisfy the property that each residue class modulo $n$ is represented by exactly $m$ integers, then the Weyl numerator $A_\mu\left(\mathbf{t}, c_n\right)$ is 0.

Proof. Let us rewrite $\mu$ as a $n \times m$ matrix whose $i$-th row consists of the integers
\[ \{\mu(i-1)+1, \mu(i-1)+2, \ldots, \mu i m\} \] in decreasing order. Under the assumption on $\mu$ in the Proposition, at least one column of $\mu$ say the $v$-th column contains a pair of integers say $\mu k m + v$ and $\mu k 2 m v$ which are congruent modulo $n$. $\sigma \in S_{mn}$ acts on $\mu$ by $\sigma \cdot \mu = (\mu_{\sigma^{-1}(k)})_{1 \leq k \leq mn}$. Let us consider the transposition $\eta_0 = (k_1 m + v, k_2 m + v)$. Now for each $\sigma \in S_{mn}$ the $v$-th column of $\sigma \cdot \mu$ contains integers that are the powers of $\omega_{n^k} v$ in $e^{\sigma / t\left(\mathbf{t}, c_n\right)}$. For $\sigma \in S_{mn}$ the columns of $\sigma \cdot \mu$ and $\eta_0 \sigma \cdot \mu$ are same except the $v$-th column where the integers $\mu_{\sigma^{-1}(m k 1 + v)}$ and $\mu_{\sigma^{-1}(m k 2 + v)}$ are interchanged. As $\mu_{\sigma^{-1}(m k 1 + v)}$ and $\mu_{\sigma^{-1}(m k 2 + v)}$ are congruent modulo $n$, we have
\[ e^{\eta_0 \sigma / \mu\left(\mathbf{t}, c_n\right)} = e^{\sigma / t\left(\mathbf{t}, c_n\right)}, \]
for all $\sigma \in S_{mn}$. Now we have
\[
\sum_{\sigma \in S_{mn}} (-1)^{\sigma} e^{\sigma \cdot \mu\left(\mathbf{t}, c_n\right)} = \frac{1}{2} \left( \sum_{\sigma \in S_{mn}} (-1)^{\sigma} e^{\sigma \cdot \mu\left(\mathbf{t}, c_n\right)} + \sum_{\sigma \in S_{mn}} (-1)^{\eta_0 \sigma} e^{\eta_0 \sigma \cdot \mu\left(\mathbf{t}, c_n\right)} \right) = 0.
\]
This completes the proof of the Proposition.

The proof of the main theorem of this paper will go through several lemmas and propositions. We first start with a lemma about the symmetric group which may be of independent interest.

Lemma 1. Consider the symmetric group $S_{mn}$. Suppose that we rewrite $\mu$ as a $n \times m$ matrix in such a way that the indices of the $\mu_i$ along each row increase by 1 and the indices along each columns increase by $m$. The symmetric Group $S_{mn}$ acts on $\mu$ as follows: $\sigma \cdot \mu = (\mu_{\sigma^{-1}(1)}, \mu_{\sigma^{-1}(2)}, \ldots, \mu_{\sigma^{-1}(mn)})$, for all $\sigma \in S_{mn}$. Let $R$ be the subgroup of $S_{mn}$ which fixes the rows of $\mu$ and $C$ be the subgroup of $S_{mn}$ which fixes the columns. We clearly have $R \cap C = \{1 \}$.

Now a permutation $\tau \in CR$ if and only if $\tau$ takes no two elements of the same row to elements of one column.

Proof. We have
\[ \mu = (\mu_{ij}) = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{m-1} & \mu_m \\ \mu_{m+1} & \mu_{m+2} & \mu_{m+3} & \cdots & \mu_{2m-1} & \mu_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{rn+1} & \mu_{rn+2} & \mu_{rn+3} & \cdots & \mu_{rn+m-1} & \mu_{rn+m} \end{pmatrix}, \]
where $r = n - 1$. If $\tau \in CR$, $\tau$ takes no two elements of the same row to a column. Now we will calculate the number of permutations having this property. We will start...
with the first row. The number of ways the integers of the first row can be put in different columns is given by $mn \times (mn - n) \times \cdots \times (mn - (m - 1)n) = m!n^m$. After the first step we have remaining $mn - m$ integers. Now the number of ways the integers of the second row can be put in different columns is given by $m!(n - 1)^m$. Proceeding similarly we get that the number of ways in which the integers of the $k$th row can be put in different columns is $m! \times (n - k + 1)^m$. Therefore the total number of ways in which we can permute the integers of $\mu$ such that no two integers of a row goes to a column is $\prod_{k=1}^{n} m! \times (n - k + 1)^m = (m!)^n \times (n!)^m = |CR|$. So that the set of permutations in $CR$ is the only set satisfying the property that it takes no two elements of a row of $\mu$ to some column. This completes the proof of the Lemma. \hfill \Box

The next lemma will determine the conditions on the left cosets $\tau R$ for which the summation $\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n)$ is zero.

Lemma 2. Suppose $\mu$ satisfies property # (defined in the beginning of this section), then with the notation as in Lemma 1,

$$\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n) = 0,$$

if $\tau \not\in CR$.

Proof. Let us consider $\mu$ as a $n \times m$ matrix as mentioned in Lemma 1. The subgroup $C$ will act along the columns of $\mu$ and the subgroup $R$ acts along the rows (see Lemma 1 for the definition of $C$ and $R$). We have $C \cap R = \{id_{mn}\}$. Now from Lemma 1 it follows that if $\tau \not\in CR$ then $\tau$ takes two elements say $\mu_{mu+s_1}$ and $\mu_{mu+s_2}$ of the $u$th row of $\mu$ to some $v$th column. Observe that $\mu_{mu+s_1}$ and $\mu_{mu+s_2}$ are congruent modulo $n$. (This follows from the fact that $\mu$ satisfies property # (defined in the beginning of this section)). Take $\sigma_0 = (mu + s_1 mu + s_2) \in R$. Now for each $\sigma \in S_{mn}$ the $v$-th column of $\sigma \cdot \mu$ contains integers that are the powers of $\omega_{n}^{k_{v}} \in e^{\sigma_{0}(\ell \cdot c_n)}$. For $\sigma \in S_{mn}$ the columns of $\sigma \cdot \mu$ and $\sigma_{0} \sigma \cdot \mu$ are same except the $v$-th column where the integers $\mu_{\sigma^{-1}(mu+s_1)}$ and $\mu_{\sigma^{-1}(mu+s_2)}$ are interchanged. As $\mu_{\sigma^{-1}(mu+s_1)}$ and $\mu_{\sigma^{-1}(mk_2 v)}$ are congruent modulo $n (\sigma \in R)$, we have

$$e^{\tau \sigma \mu}(\ell \cdot c_n) = e^{\tau \sigma_{0} \mu}(\ell \cdot c_n),$$

for all $\sigma \in R$. Now the summation

$$\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n) = \sum_{\sigma \in R} (-1)^{\tau \sigma_{0}} e^{\tau \sigma_{0} \mu}(\ell \cdot c_n),$$

$$= (-1)^{\sigma_0} \sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n),$$

$$= -\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n).$$

Therefore the summation $\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n)$ is zero. Hence the Lemma is proved. \hfill \Box

Next we will calculate the summation $\sum_{\sigma \in R} (-1)^{\tau \sigma} e^{\tau \sigma \mu}(\ell \cdot c_n)$, where $\tau \in CR$, in the following two lemmas.
Let us recall $\mu$ written as a $n \times m$ matrix (see Lemma 1). By the property # (defined in the beginning of this section) of $\mu$, we have:

$$\mu \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \cdots & n-1 \end{pmatrix} \mod n.$$ 

Now rewrite

$$L \cdot c_n = \begin{pmatrix} t_1 \\ \omega_n t_1 \\ \vdots \\ \omega^{n-1} t_1 \\ \omega_n t_2 \\ \vdots \\ \omega^{n-1} t_2 \\ \omega_n t_m \\ \vdots \\ \omega^{n-1} t_m \end{pmatrix} = (t_{ij}).$$

The subgroup $C$ will permute the columns of $\mu$ and the subgroup $R$, the rows of $\mu$. Now we have

$$\mu(L \cdot c_n) = \prod_{i,j} t_{ij}^{\mu_{ij}} = \prod_i (\prod_j t_{ij}^{\mu_{ij}}),$$

from which we easily deduce the following Lemma.

**Lemma 3.** For each $\eta \in C$, $\mu$ a character of the maximal torus $(\mathbb{C}^*)^{mn}$ of $GL(mn, \mathbb{C})$ with property # (defined in the beginning of this section), there exists constants $C_{\eta}(\mu) \in \mathbb{C}^*$ such that

1. $\mu^\eta(L \cdot c_n) = C_{\eta}(\mu)\mu(L \cdot c_n)$.

2. If $\eta \in C$ and $\sigma \in R$, then $C_{\eta}(\mu) = C_{\eta}(\mu^\sigma)$.

**Proof.** The proof follows from the fact that the elements of $C$ will permute the columns of $\mu$ and the elements of $R$ permute the rows of $\mu$ (hence the value of $\mu$ remains the same modulo $n$).

For each $0 \leq k \leq n - 1$, define $L_k = \{\mu_j \mid \mu_j \equiv k \pmod{n}\}$. From now on we will assume that $L_k = \{\mu_{(k-1)m+1}, \mu_{(k-1)m+2}, \ldots, \mu_{km}\}$, for each $k, 0 \leq k \leq n - 1$. Define $L_k^1 = \{\frac{\mu_j - k}{n} \mid \mu_j \in L_k\}$. Observe that $I_k$ is the set of all indices of the elements of $L_k$.

**Lemma 4.** Let $\eta \in C$ then for $\mu$ with property # (defined in the beginning of this section),

$$\sum_{\sigma \in R} (-1)^{\eta\sigma} e^{i\theta_{\sigma}} \mu(L \cdot c_n) = (-1)^{\eta} C_{\eta}(\mu) D \cdot \prod_{k=0}^{n-1} Q_k \cdot \prod_{s=1}^{m} t_s^{n^{(n-1)}},$$

where $Q_k = \sum_{I_k \in S(I_k)} (-1)^{\tau_{k+1}} e^{i\tau_k \cdot L_k^1} \mu(I_k)$ and $D$ is a constant.

**Proof.** Let $\sigma \in R$ be arbitrary. We have $\mu = \bigcup_{k=0}^{m-1} L_k$ and $R$ is the direct product of the groups $S(I_k)$. We can write $\sigma = \prod_{k=0}^{n-1} \sigma_k$, where $\sigma_k \in S(I_k)$. From Lemma 3 we have

$$e^{i\theta_{\sigma}} (L \cdot c_n) = C_{\eta}(\mu) e^{i\theta_{\sigma}} (L \cdot c_n),$$
for each \( \eta \in C \). Therefore we get

\[
\sum_{\sigma \in R} (-1)^{\eta} C_\eta(\mu)(-1)^{\eta \sigma} e^{(\eta \sigma)} \mu(t \cdot c_n) = \sum_{\sigma \in R} (-1)^{\eta} C_\eta(\mu) \sum_{e = R} (-1)^\sigma e^{\sigma \cdot L_k}(\mu(t \cdot c_n),
\]

\[
= (-1)^\eta C_\eta(\mu) \sum_{e = R} \prod_{k = 0}^{n-1} (-1)^{\sigma_k} e^{\sigma_k \cdot L_k}(\omega_{\eta}^k),
\]

\[
= (-1)^\eta C_\eta(\mu) \sum_{e = R} \prod_{k = 0}^{n-1} (-1)^{\sigma_k} \omega_n^{k \mu k-1(mk+s)} e^{\sigma k \cdot L_k}(t),
\]

\[
\overset{(*)}{=} (-1)^\eta C_\eta(\mu) \sum_{e = R} \prod_{k = 0}^{n-1} (-1)^{\sigma_k} \omega_n^{k \mu k-1(sR)} e^{\sigma k \cdot L_k}(t),
\]

\[
= (-1)^\eta C_\eta(\mu) D \cdot \sum_{e = R} \prod_{k = 0}^{n-1} (-1)^{\sigma_k} e^{\sigma k \cdot L_k}(t),
\]

\[
= (-1)^\eta C_\eta(\mu) D \sum_{e = R} \sum_{\sigma_0} \sigma_1 \cdots \sigma_{n-1} \prod_{k = 0}^{n-1} (-1)^{\sigma_k} e^{\sigma k \cdot L_k}(t),
\]

\[
= (-1)^\eta C_\eta(\mu) D \prod_{k = 0}^{n-1} \sum_{e = R} (-1)^{\sigma_k} e^{\sigma k \cdot L_k}(t),
\]

\[
(5)
\]

The equality (*) follows from the fact that for each \( k \) and \( s \), the integers \( \mu k-1(mk+s) \) and \( \mu mk+s \) are congruent modulo \( n \). For \( 1 \leq j, s \leq m \), we can write

\[
t_j^{\mu_{mk+s}} = (t_j^n)^{\frac{\mu_{mk+s}}{m}} \times t_j^k.
\]

This implies that we have

\[
e^{\sigma k \cdot L_k}(t) = e^{\sigma k \cdot L_k}(t^n) \times \prod_{s = 1}^{m} t_s^k,
\]

\[
(6)
\]

for each \( 0 \leq k \leq n - 1 \). Now from the equations (5) and (6) we have

\[
\sum_{e = R} (-1)^\eta e^{\eta \cdot \mu}(t \cdot c_n) = (-1)^\eta C_\eta(\mu) D \prod_{k = 0}^{n-1} Q_k \cdot \prod_{s = 1}^{m} t_s^{\frac{n(n-1)}{2}},
\]

where \( Q_k = \sum_{\tau_k \in S(I_k)} (-1)^{\tau_k \cdot L_k}(t^n) \). Hence the Lemma is proved. \( \square \)

The next Proposition will give the Weyl numerators at \( t \cdot c_n \).

**Proposition 2.** With the notation as above (assuming \( \mu \) satisfies property # (defined in the beginning of this section)), we have

\[
A_{\mu}(t \cdot c_n) = \sum_{s} \sum_{e = R} (-1)^\sigma e^{\sigma \cdot \mu}(t \cdot c_n) = E \cdot \prod_{s = 1}^{m} t_s^{\frac{n(n-1)}{2}} \cdot \prod_{i = 0}^{n-1} S_i,
\]
where \( s \in S_{mn}/R \), \( E \) is a constant and \( S_j \) are Weyl numerators of the highest weight representations \( \pi_{\eta_j} \) of \( GL(m, \mathbb{C}) \) with the highest weight \( \eta_k \) satisfies the condition:

\[
\eta_k + \rho_m = L^1_k \text{(defined in the beginning of Lemma 4)}.
\]

**Proof.** We get the Weyl numerator by adding the summations of the forms \( \sum_{\sigma \in R} (-1)^{\tau} e^{\tau(\eta \cdot c_n)} \), where \( \tau \in S_{mn}/R \). Notice that from Lemma 2 and Lemma 4, for each \( s \not\in CR \),

\[
\sum_{\sigma \in R} (-1)^{\sigma} e^{\sigma(\eta \cdot t \cdot c_n)} = 0
\]

and the summation is nonzero when \( \tau \in CR \). Therefore

\[
A_{\eta}(t \cdot c_n) = \sum_{s \in S_{mn}/R} \sum_{\sigma \in R} (-1)^{\sigma} e^{\sigma(\eta \cdot t \cdot c_n)},
\]

\[
(\ast) = \sum_{s \in C} \sum_{\sigma \in R} (-1)^{\sigma} e^{\sigma(\mu \cdot t \cdot c_n)},
\]

\[
(\ast\ast) \sum_{s \in C} (-1)^{s} C_s(\mu) D \prod_{k=0}^{n-1} Q_k \cdot \left( \prod_{s=1}^{m} t_s \right)^{\frac{n(n-1)}{2}},
\]

\[
(8) = E \cdot \prod_{k=0}^{n-1} Q_k \cdot \left( \prod_{s=1}^{m} t_s \right)^{\frac{n(n-1)}{2}},
\]

where \( E \) is a constant and \( Q_k \)'s are defined in Lemma 4. The equality \((\ast)\) follows from the fact that every left coset of the form \( \eta R \), where \( \eta \in CR \) has a representative in \( C \) and the equality \((\ast\ast)\) follows from Lemma 4.

For each \( 0 \leq k \leq n - 1 \), let \( \pi_{\eta_k} \) be the highest weight representation of \( GL(m, \mathbb{C}) \) with highest weight \( \eta_k \) satisfying the relation \( \eta_k + \rho_m = L^1_k \), where \( L^1_k = \{ \mu_j - k \} \) \( \mu_j \in L_k \}. \)

Then we have

\[
Q_k = A_{\eta_k + \rho_m}(t^n).
\]

Now from equation (8) and (9) we have

\[
A_{L}(t \cdot c_n) = E \cdot \prod_{k=0}^{n-1} A_{\eta_k + \rho_m}(t^n) \cdot \left( \prod_{s=1}^{m} t_s \right)^{\frac{n(n-1)}{2}}.
\]

This completes the proof of the Proposition. \(\Box\)

Now using proposition 2 and equation 4 we deduce that

\[
\Theta_{L}(t \cdot c_n) = B \cdot \prod_{k=0}^{n-1} \Theta_{\eta_k}(t^n),
\]

where \( B \) is a constant. We will show that \( B = \pm 1 \) in the next theorem which is the main result of this paper.
Theorem 3.1. For the irreducible highest weight representation $\pi_\lambda$ of $GL(mn, \mathbb{C})$, such that for each $k$, $0 \leq k \leq n-1$, there are exactly $m$ integers in $\lambda + \rho_{mn}$ that are congruent to $k$ modulo $n$, we have

$$\Theta_\Delta(t \cdot c_n) = \pm \prod_{k=0}^{n-1} \Theta_{\eta_k}(t^n),$$

where $\Theta_{\eta_k}$ is the character of the highest weight representation of $GL(m, \mathbb{C})$ (see Proposition 2).

Proof. We will show that if for some constant $B$, we have,

$$\Theta_\Delta(t \cdot c_n) = B \cdot \prod_{k=0}^{n-1} \Theta_{\eta_k}(t^n), \quad \forall t \in (\mathbb{C}^*)^m,$$

then $B = \pm 1$, thus proposition 2 implies Theorem 3.1.

We have $t \cdot c_n = (t, \omega_n t, \cdots, \omega_n^{n-1} t)$, where $t = (t_1, t_2, \cdots, t_m)$. Let us choose $t_j = \alpha^{j-1}$, where $\alpha = e^{2\pi i/m}$. We have $\omega_n = \alpha^m$ and $t \cdot c_n = (1, \alpha, \cdots, \alpha^{m-1})$. By the choice of $t_j$ the element $t \cdot c_n$ is a Coxeter element of $GL(mn, \mathbb{C})$. Now by using a theorem of Kostant, Theorem 1 of [KO] we have $\Theta_\Delta(t \cdot c_n) = \pm 1$ (see [DP2] for another proof). The element $t^n = (1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1})$ is also a Coxeter element of $GL(m, \mathbb{C})$. Again by Kostant’s theorem, we have $\Theta_{\eta_k}(t^n) = \pm 1$. Now from equation (10) we conclude that $B = \pm 1$. This completes the proof of the Lemma.

Acknowledgement: The author thanks Prof. Dipendra Prasad for suggesting the question and for his help in the writing of this paper.

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