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Curved Flats and Isothermic Surfaces

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Abstract

We show how pairs of isothermic surfaces are given by curved flats in a pseudo Riemannian symmetric space and vice versa. Calapso’s fourth order partial differential equation is derived and, using a solution of this equation, a Möbius invariant frame for an isothermic surface is built.

1 Introduction

These notes grew out of a series of discussions on a recent paper by J. Cieślinski, P. Goldstein and A. Sym [4]: these authors give a characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter. In trying to understand the geometric meaning of this spectral parameter, we observed some analogies with the theory of conformally flat hypersurfaces in a four-dimensional space form: Guichard’s nets may be understood as a kind of analogue of isothermic parametrizations of Riemannian surfaces (cf. [4, no.3.4.1]), and so it seems natural to look for relations between the theory of isothermic surfaces in three-dimensional space forms and the theory of conformally flat hypersurfaces in four-dimensional space forms. Here we would like to present some results we found — especially the possibility of constructing isothermic surfaces using

2 Curved Flats

A curved flat is the natural generalization of a developable surface in Euclidean space: it is a submanifold $M \subset G/K$ of a (pseudo-Riemannian) symmetric space for which the curvature operator of $G/K$ vanishes on $\wedge^2 TM$. Thus, a curved flat may be thought of as the enveloping submanifold of a congruence of flats — totally geodesic submanifolds — of the symmetric space. Taking a regular parametrization $\gamma : M \rightarrow G/K$ of a curved flat and a framing $F : M \rightarrow G$ of this parametrization, the Maurer-Cartan form $\Phi = F^{-1}dF$

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$^1$Thus $M$ is curvature isotropic in the sense of [3]
of the framing has a natural decomposition Φ = Φ_t + Φ_p according to the symmetric decomposition of the Lie algebra g = ℱ ⊕ ℙ of the Lie algebra g. Now the condition for γ to parametrize a curved flat may be formulated as

$$[[Φ_p ∧ Φ_p], p] \equiv 0.$$  

In case that G is semisimple, it is straightforward to see that this is equivalent to

$$[Φ_p ∧ Φ_p] \equiv 0.$$  

To summarise, we have the

**Definition** of a curved flat: An immersion γ : M → G/K is said to parametrize a curved flat, if the p-part in the symmetric decomposition of the Maurer-Cartan form $F^{-1}dF = Φ = Φ_t + Φ_p$ of a framing $F : M → G$ of γ defines a congruence $p ↦ Φ_p|_{p}(T_p M)$ of abelian subalgebras of g.

At this point we should remark that curved flats naturally arise in one parameter families: setting

$$Φ_λ := Φ_t + λΦ_p$$  

the Maurer-Cartan equation $dΦ_λ + \frac{1}{2}[Φ_λ ∧ Φ_λ] = 0$ for the loop $λ ↦ Φ_λ$ of forms splits into the three equations

$$0 = dΦ_t + \frac{1}{2}[Φ_t ∧ Φ_t]$$
$$0 = dΦ_p + [Φ_t ∧ Φ_p]$$
$$0 = [Φ_p ∧ Φ_p],$$

and hence the integrability of the loop $λ ↦ Φ_λ$ is equivalent to the forms $Φ_λ$ being the Maurer-Cartan forms for some framings $F_λ : M → G$ of curved flats $γ_λ : M → G/K$. Thus integrable systems theory may be applied to produce examples.

Now we will consider the case leading to the theory of isothermic surfaces: let

$$G := O_1(5) \quad \text{and} \quad K := O(3) × O_1(2).$$

2Thus ℱ and ℙ are the +1 and −1-eigenspaces, respectively, of the involution fixing ℱ and so satisfy the characteristic conditions

$$[ℱ, ℱ] ⊂ ℱ, [ℱ, ℙ] ⊂ ℙ, [ℙ, ℙ] ⊂ ℱ.$$  

3The product

$$[Φ ∧ Ψ](v, w) := [Φ(v), Ψ(w)] − [Φ(w), Ψ(v)]$$

defines a symmetric product on the space of Lie algebra valued 1-forms with values in the space of Lie algebra valued 2-forms.

4In fact, $ℙ ⊕ ℙ$ is an ideal of g so that we have a decomposition $g = ℱ' ⊕ [ℙ, ℙ] ⊕ ℙ$ where $ℱ'$ is a complementary ideal commuting with $[ℙ, ℙ] ⊕ ℙ$. Thus, if $a ⊂ ℙ$ satisfies $[[a, a], ℙ] = 0$ we deduce that $[a, a]$ lies in the center of g and so vanishes.
The coset space \( G_+ (5, 3) = G / K \) of space-like 3-planes in the Minkowski space \( \mathbb{R}^5_1 \) becomes a six dimensional pseudo-Riemannian symmetric space of signature \((3, 3)\) when endowed with the metric induced by the Killing form. We will consider two-dimensional curved flats

\[
\gamma : M^2 \rightarrow G_+ (5, 3)
\]
satisfying the regularity assumption that the metric on \( M^2 \) induced by \( \gamma \) is non-degenerate.

Fixing a pseudo orthonormal basis \((e_1, \ldots, e_5)\) of the Minkowski space \( \mathbb{R}^5_1 \) with

\[
(\langle e_i, e_j \rangle)_{ij} = \begin{pmatrix}
I_3 & 0 \\
0 & 1
\end{pmatrix}
\]
we get the matrix representations

\[
O_1(5) = \{ A \in GL(5, \mathbb{R}) | A^t E_5 A = E_5 \}
\]
\[
o_1(5) = \{ \mathfrak{X} \in \mathfrak{gl}(5, \mathbb{R}) | (E_5 \mathfrak{X}) + (\mathfrak{E}_5 \mathfrak{X})^t = 0 \}.
\]

The subalgebra \( \mathfrak{k} \) and its complementary linear subspace \( \mathfrak{p} \) in the symmetric decomposition of \( o_1(5) \) are given by the +1- resp. −1-eigenspaces of the involutive automorphism \( \text{Ad}(Q) : o_1(5) \rightarrow o_1(5) \) with \( Q = \begin{pmatrix}
-I_3 & 0 \\
0 & I_2
\end{pmatrix} \). Writing down the Maurer-Cartan form of a framing \( F : M^2 \rightarrow O_1(5) \) of our curved flat \( \gamma : M^2 \rightarrow G_+ (5, 3) \) with this notation we obtain

\[
F^{-1} dF = \Phi = \Phi_t + \Phi_p \quad \text{with}
\]
\[
\Phi_t = \begin{pmatrix}
\Omega & 0 \\
0 & \nu
\end{pmatrix} : TM \rightarrow \mathfrak{o}(3) \times o_1(2)
\]
\[
\Phi_p = \begin{pmatrix}
0 & \eta \\
-E_2 \eta^t & 0
\end{pmatrix} : TM \rightarrow \mathfrak{p}.
\]

The image of \( \Phi_p \) at each \( p \in M^2 \) is a 2-dimensional abelian subspace of \( \mathfrak{p} \) on which the Killing form is non-degenerate. One can show that there are precisely two \( K \)-orbits of maximal abelian subspaces of \( \mathfrak{p} \): one consists of 3-dimensional subspaces which are isotropic for the Killing form while the other consists of 2-dimensional subspaces on which the Killing form has signature \((1, 1)\). We therefore conclude that the images of each \( \Phi_p \) are maximal abelian and \( K \)-conjugate and so we can put \( \eta \) into the standard form

\[
\eta = \begin{pmatrix}
\omega_1 & -\omega_1 \\
\omega_2 & -\omega_2 \\
0 & 0
\end{pmatrix}
\]
by applying a gauge transformation \( M \rightarrow K \).

Calculating the Maurer-Cartan equation using the ansatz

\[
\Omega = \begin{pmatrix}
0 & \omega & -\psi_1 \\
-\omega & 0 & -\psi_2 \\
\psi_1 & \psi_2 & 0
\end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix}
\nu & 0 \\
0 & -\nu
\end{pmatrix}
\]

3
together with \( \eta \) given by (10), we see that
\[
d\omega_1 = d\omega_2 = 0.
\]

So we are given canonical coordinates \((x, y) : M \to \mathbb{R}^2\) by integrating the forms \(\omega_1\) and \(\omega_2\). Moreover, since we also have \(dv = 0\), we may set \(\nu = -du\) for a suitable function \(u \in C^\infty(M)\) — this gives us \(\omega = u_y dx - u_x dy\), where \(u_x\) and \(u_y\) denote the partial derivatives of \(u\) in \(x\)- resp. \(y\)-directions. Finally, the equations \(\psi_1 \wedge \omega_1 = 0\) and \(\psi_2 \wedge \omega_2 = 0\) show that \(\psi_1 = e^u k_1 dx\) and \(\psi_2 = e^u k_2 dy\) for two functions \(k_i \in C^\infty(M)\).

We now perform a final \(O_1(2)\)-gauge \((I_3^0 e^u 0 0 e^{-u}, 0 0 e^{-u}) : M \to O(3) \times O_1(2)\) and insert the spectral parameter \(\lambda\) to obtain the Maurer-Cartan form discussed in (cf.\([4]\)):
\[
\Phi_\lambda = \begin{pmatrix} I_3 & e^u & 0 & 0 \\ 0 & e^{u_x} & 0 & 0 \\ 0 & 0 & e^{u_y} & 0 \\ -e^{u_x} & -e^{u_y} & 0 & 0 \\ -e^{-u_x} & -e^{-u_y} & 0 & 0 \\ e^{-u_x} & e^{-u_y} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We are now lead directly to the theory of

3 Isothermic Surfaces

In the context of Möbius geometry the three sphere \(S^3\) is viewed as the projective light-cone \(IPL^4\) in \(I\mathbb{R}_1\) while the Lorentzian sphere \(\{v \in I\mathbb{R}_1^5 | \langle v, v \rangle = 1\}\) should be interpreted as the space of (oriented) spheres in the three sphere \(I\mathbb{R}_1^5\) (cf.\([1]\)). Now, denoting by
\[
\begin{align*}
n &: F e_3 : M \to S_1^5 = \{v \in I\mathbb{R}_1^5 | \langle v, v \rangle = 1\} \\
f &: F e_4 : M \to L^4 = \{v \in I\mathbb{R}_1^5 | \langle v, v \rangle = 0\} \\
\hat{f} &: F e_5 : M \to L^4
\end{align*}
\]

one of the sphere congruences resp. the two immersions given by our frame \(F\), we see that

**Theorem:** The sphere congruence \(n\) given by our curved flat is a Ribeaucour sphere congruence\([1]\) which is enveloped by two isothermic immersions \(f\) and \(\hat{f}\) (cf.\([1]\) p.362):

Since
\[
\begin{align*}
\langle f, n \rangle &= 0 \quad \text{and} \quad \langle df, n \rangle \equiv 0, \\
\langle \hat{f}, n \rangle &= 0 \quad \text{and} \quad \langle d\hat{f}, n \rangle \equiv 0,
\end{align*}
\]

\(\text{Since our theory is local, all closed forms may be assumed to be exact.}\)

\(\text{Or, equivalently, it may be interpreted as the space of (oriented) spheres and planes in Euclidean three space } \mathbb{R}^3: \text{the polar hyperplane to a vector } v \text{ of the Lorentz sphere intersects the three sphere — thought of as the absolute quadric in projective four space} — \text{in a two sphere. Stereographic projection yields a sphere in } \mathbb{R}^3 \text{ or, if the projection center lies on the sphere, a plane.}\)

\(\text{The curvature lines on the two enveloping immersions correspond.}\)
the immersions \( f \) and \( \hat{f} \) do envelop the sphere congruence \( n \) and, since the bilinear forms

\[
\begin{align*}
\langle df, dn \rangle &= \lambda e^{2u}(k_1 dx^2 + k_2 dy^2), \\
\langle d\hat{f}, dn \rangle &= \lambda (-k_1 dx^2 + k_2 dy^2)
\end{align*}
\]

are diagonal with respect to the induced metrics

\[
\begin{align*}
\langle df, df \rangle &= \lambda^2 e^{2u}(dx^2 + dy^2), \\
\langle d\hat{f}, d\hat{f} \rangle &= \lambda^2 e^{-2u}(dx^2 + dy^2)
\end{align*}
\]

the two immersions \( f \) and \( \hat{f} \) are isothermic.

It is quite difficult to calculate the first and second fundamental forms of these isothermic immersions, when they are projected to \( S^3 \) resp. \( IR^3 \), but applying a (constant) conformal change (constant \( O_1(2) \)-gauge)

\[
\begin{align*}
f &\sim \frac{1}{\lambda^2} f \quad \text{and} \quad \hat{f} \sim \lambda \hat{f} \quad \text{or} \\
f &\sim \lambda f \quad \text{and} \quad \hat{f} \sim \frac{1}{\lambda \hat{f}}
\end{align*}
\]

and sending \( \lambda \to 0 \), \( \hat{f} \)resp. \( f \) becomes a constant vector — \( \Phi_{\lambda=0} e_5 \) resp. \( \Phi_{\lambda=0} e_4 \) vanishes. This constant light-like vector may be interpreted as the point at infinity and we therefore obtain an isothermic immersion \( f : M \to IR^3 \) with first and second fundamental forms

\[
\begin{align*}
I &= e^{2u}(dx^2 + dy^2) \\
II &= e^{2u}(k_1 dx^2 + k_2 dy^2)
\end{align*}
\]

resp. its Euclidean dual surface \( \hat{f} : M \to IR^3 \) with first and second fundamental forms

\[
\begin{align*}
\hat{I} &= e^{-2u}(dx^2 + dy^2) \\
\hat{II} &= -k_1 dx^2 + k_2 dy^2.
\end{align*}
\]

We now recognise the remaining three equations from the Maurer-Cartan equation for \( \Phi_{\lambda} \)

\[
\begin{align*}
0 &= \Delta u + e^{2u} k_1 k_2 \\
0 &= k_{1y} + (k_1 - k_2) u_y \\
0 &= k_{2x} - (k_1 - k_2) u_x
\end{align*}
\]

\(^8\)The bundle defined by \( \text{span}\{n, f, \hat{f}\} \) over \( M \) is flat (cf.(13)) and so the map \( p \mapsto df(T_p M) \) defines a normal congruence of circles \[5\]: for each \( p \in M \)

\[
t \mapsto f_t(p) := \frac{1}{\sqrt{2}} \sin t \cdot n(p) + \frac{1}{2}(1 + \cos t) \cdot f(p) - \frac{1}{2}(1 - \cos t) \cdot \hat{f}(p)
\]

parametrizes the circle \((df(T_p M))^\perp\) meeting the sphere \( n(p) \) in \( f(p) \) and \( \hat{f}(p) \) orthogonal. Since \( n, f \) and \( \hat{f} \) are parallel sections in this bundle, the maps \( p \mapsto f_t(p) \) (which generically are not degenerate) parametrize the surfaces orthogonal to this congruence of circles.

In general the immersions \( f \) and \( \hat{f} = f_\pi \) will be the only isothermic surfaces among the surfaces.
as the Gauß and Codazzi equations of the Euclidean immersion \( f \) resp. its dual \( \hat{f} \) \[3\]. As a consequence, we can invert our construction and build a curved flat from an isothermic surface:

**Theorem.** Given an isothermic surface \( f : M^2 \to \mathbb{IR}^3 \) and its Euclidean dual surface \( \hat{f} : M \to \mathbb{IR}^3 \) we get a curved flat \( \gamma : M \to G_+(5,3) \) integrating the Maurer-Cartan form \((13)\), which we are able to write down knowing the first and second fundamental forms of the immersions \( f \) and \( \hat{f} \) \[4\].

Another way to obtain these two Euclidean immersions is presented in \[4\]. Applying Sym’s formula to the associated family of frames \( F = F(\lambda) \), we obtain a map

\[
(\frac{\partial}{\partial \lambda} F)F^{-1}|_{\lambda=0} : M \to p
\]

interpreting \( p \) as two copies of Euclidean three space \( \mathbb{IR}^3 \) this map gives us the immersion \( f \), and in the other copy of \( \mathbb{IR}^3 \), its dual \( \hat{f} \): this can be seen by looking at the differential

\[
d(\frac{\partial}{\partial \lambda} F)F^{-1}|_{\lambda=0} = F_0 \Phi_p F_0^{-1}
\]

\[
\cong H_3 \begin{pmatrix}
    e^u dx & -e^{-u} dx \\
    e^u dy & e^{-u} dy \\
    0 & 0
\end{pmatrix}.
\]

Here \( F_0 = \begin{pmatrix}
    H_3 & 0 \\
    0 & I_2
\end{pmatrix} \) solves the equation \( F_0^{-1} dF_0 = \Phi_F \) and thus we may view \( H_3 : M \to O(3) \) as a Euclidean framing of \( f \) resp. \( \hat{f} \).

There is another possibility for producing isothermal surfaces in Euclidean space \( \mathbb{IR}^3 \) (or \( S^3 \)): that is, by using a solution of

\[4\] Calapso’s equation

To understand this, we write down the Maurer-Cartan form of a frame \( F : M \to O_1(5) \), which is M"obius-invariantly connected to a given immersion: taking \( f = Fe_4 \) the (unique) isometric lift of the isothermic immersion and \( n = Fe_3 \) the central sphere congruence (conformal Gauß map) of the immersion, the frame is determined by the assumption of

---

9The Euclidean dual of an isothermic surface is obtained by integrating the closed 1-form \( d\hat{f} := e^{-2u}(-f_x dx + f_y dy) \): see for example \[2\] p.14.

When the normal congruence of circles mentioned in footnote 8 is projected to Euclidean three space \( \mathbb{IR}^3 \), we see that, in the limit \( \lambda \to 0 \), the circles become straight lines — circles meeting the collapsed surface \( \hat{f} \) resp. \( f \) in the point at infinity — while the Ribeaucour sphere congruence enveloped by the two surfaces \( f \) and \( \hat{f} \) becomes the congruence of tangent planes of \( f \) resp. \( \hat{f} \).

10Since this construction depends on the conformal rather than the Euclidean geometry of the ambient space, we generally get a whole three parameter family of loops of curved flats from one isothermic surface: when viewing our given isothermic surface as a surface in the three sphere \( S^3 \), we may choose the point at infinity arbitrarily.

11Here the Euclidean metric is induced by the quadratic form \( \frac{1}{2} \text{tr} \Phi_p \Phi_p \) instead of the Killing form.
being an adapted frame (i.e. $F\varepsilon_1 = f_x$ and $F\varepsilon_2 = f_y$). The associated Maurer-Cartan form will be

$$ \Phi = \begin{pmatrix} 0 & 0 & kdx & dx & \chi_1 \\ 0 & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix}, $$

$k^2$ being the conformal factor relating the metric induced by the central sphere congruence to the isometric one, and the 1-forms $\chi_1$, $\chi_2$ and $\tau$ to be determined. From the Maurer-Cartan equation for this form we learn that

$$ \tau = k_x dx - k_y dy $$

$$ \chi_1 = \left( \frac{1}{2} k^2 - u \right) dx - \frac{k_x}{k} dy $$

$$ \chi_2 = -\frac{k_x}{k} dx + \left( \frac{1}{2} k^2 + u \right) dy , $$

where $u \in C^\infty(M)$ is a function satisfying the differential equation

$$ du = -\left( \left( \frac{k_x}{k} \right)_y + \left( k^2 \right)_x \right) dx + \left( \left( \frac{k_x}{k} \right)_x + \left( k^2 \right)_y \right) dy $$

— the integrability condition of this equation is a fourth order partial differential equation closely related to Calapso’s original equation [3]:

$$ 0 = \Delta \left( \frac{k_x}{k} \right) + 2 \left( k^2 \right)_{xy} $$

This shows, that

**Theorem:** Any isothermic surface gives rise to a solution of Calapso’s equation.

Conversely, from a solution $k \in C^\infty(M)$ of Calapso’s equation we can construct a Möbius invariant frame of an isothermic surface by integrating the Maurer-Cartan form (24), where the function $u$ is a solution of (26).

Now, applying a conformal change $f \sim \frac{1}{f'}$ while fixing the central sphere congruence $n \sim n$, the Maurer-Cartan form of the associated frame becomes

$$ \Phi = \begin{pmatrix} 0 & \omega & kdx & dx & \chi_1 \\ -\omega & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -\frac{\omega}{k} dx & -\frac{\omega}{k} dy & 0 & 0 & 0 \end{pmatrix}, $$

where

$$ \omega = -\frac{k_x}{k} dx + \frac{k_x}{k} dy $$

$$ \chi_1 = k \left( \frac{k_x}{k} - \frac{k_x}{2} + \frac{1}{2} k^2 - u \right) dx . $$

$$ \chi_2 = k \left( \frac{k_x}{2} - \frac{k_x}{2} + \frac{1}{2} k^2 + u \right) dy $$

Here we see that the central sphere congruence of an isothermic surface is a Ribaucour sphere congruence, which actually is a characterisation of isothermic surfaces (cf.[4].
and hence it has flat normal bundle as a codimension two surface in the Lorentz sphere $\mathbb{S}^4_1$.

In general, the second enveloping surface of the central sphere congruence of an isothermic surface will not be an isothermic surface and it seems to be difficult to built a curved flat starting with it. But in a quite simple case this is possible:

## 5 Example

Starting with a surface of revolution

$$f(x, y) = (r(x) \cos y, r(x) \sin y, z(x)),$$

the functions $r$ and $z$ satisfying the differential equation

$$r^2 = r'^2 + z'^2,$$

i.e. the curve $(r, z)$ being parametrized by arc length (thought of as a curve in the Poincaré half plane), we may write down the loop of Maurer-Cartan forms

$$\Phi_\lambda = \begin{pmatrix}
0 & -\frac{r'}{\sqrt{r'}} dy & -\frac{r'^{\prime\prime}}{\sqrt{r'}} dz & \lambda r dx & -\frac{\lambda}{\sqrt{r'}} dx \\
\frac{r'}{\sqrt{r'}} dy & 0 & -\frac{r'^{\prime\prime}}{\sqrt{r'}} dz & \lambda r dx & -\frac{\lambda}{\sqrt{r'}} dx \\
\frac{r'}{\sqrt{r'}} dz & -\frac{r'^{\prime\prime}}{\sqrt{r'}} dz & 0 & 0 & 0 \\
\lambda r dx & -\lambda r dy & 0 & 0 & 0
\end{pmatrix},$$

which gives us the immersion $f$ and its dual $\hat{f}$ in the limit $\lambda \to 0$.

On the other hand, denoting by $H = \frac{1}{2}(\frac{r'}{r} + \frac{r'^{\prime\prime}}{r})$ the mean curvature of our surface of rotation, the central sphere congruence of $f$ is $n + Hf$. The metric it induces has conformal factor $k^2$ (relative to the metric induced by $f$) given by

$$k = \frac{1}{2\sqrt{r'}}(rz' + r'^{}z'' + r''z').$$

Since $k_y \equiv 0$, this is obviously a solution of Calapso’s equation and a function $u$ solving (26) is $u = \lambda^2 - k^2$. So the Maurer-Cartan form (24) becomes

$$\Phi_\lambda = \begin{pmatrix}
0 & 0 & k dx & dx & (\frac{1}{2}k^2 - \lambda^2) dx \\
0 & 0 & -kd y & dy & (\frac{1}{2}k^2 + \lambda^2) dy \\
-k dx & k dy & 0 & 0 & 0 \\
-k dx & -dy & 0 & 0 & 0
\end{pmatrix}. $$

A change $n \sim n + kf$ of the sphere congruence, enveloped by $f$, followed by an $O_1(2)$-gauge $f \sim \lambda f$ and $\hat{f} \sim \lambda^{-1} \hat{f}$ gives us the Maurer-Cartan form

$$\Phi_\lambda = \begin{pmatrix}
0 & 0 & 2k dx & \lambda dx & -\lambda dx \\
0 & 0 & 0 & \lambda dy & -\lambda dy \\
-2k dx & 0 & 0 & 0 & 0 \\
\lambda dx & -\lambda dy & 0 & 0 & 0 \\
-\lambda dx & -\lambda dy & 0 & 0 & 0
\end{pmatrix}. $$
of a curved flat, quite different from that coming from (32).

To understand the geometry of the two enveloping immersions \( f = F e_4 \) and \( \hat{f} = F e_5 \), we remark that the sphere congruence \( n = F e_3 \) depends only on one variable and hence the two immersions parametrize a channel surface; moreover all spheres of the congruence are perpendicular to the fixed circle \( e_2 \) given by span\{\( F e_2, F(e_4 + e_5) \)\}, which may be thought as an axis of rotation: the immersions \( f \) and \( \hat{f} \) parametrize pieces of a surface of revolution \( \hat{F} \), \( f \) and \( \hat{f} \) being axisymmetric.\(^{14}\) Taking now the limit \( \lambda \to 0 \), we obtain a cylinder resp. its dual, which is an (axial) reflection of the original cylinder.

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\(^{12}\) We have \( \Phi e_2 = -(e_4 + e_5)dy \) and \( \Phi (e_4 + e_5) = 2e_2dy \).

\(^{13}\) The meridian curve is given by \( \frac{1}{\sqrt{2}}(f - \hat{f}) \) — which only depends on one variable — thought as a curve in the Poincaré half plane; its tangent field is given by \( F e_1 \) and its unit normal field by \( n = F e_3 \).

\(^{14}\) The circles \( \{F(p)e_1, F(p)e_2\}^+ \) intersecting the sphere \( n(p) \) orthogonally in \( f(p) \) and \( \hat{f}(p) \) all meet the axis (cf. footnote 3, page 3).