Investigation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps

Tomoyuki Mao$^{1,2}$, Hidetoshi Okutomi$^2$ and Ken Umeno$^1$

$^1$ Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
$^2$ Toshiba Information Systems (Japan) Corporation, 1-53 Nissin-cho, Kawasaki-ku, Kawasaki-city, Kanagawa 210-8540, Japan

E-mail mao.tomoyuki.48m@st.kyoto-u.ac.jp

Received April 03, 2019, Accepted June 01, 2019

Abstract

Lyapunov exponent is commonly used as a measure of chaos. Chaos Degree is proposed as another measure of chaos based on information theory. The advantage of Chaos Degree is that it is calculable from data. However, there is a difference between Chaos Degree and Lyapunov exponent and determination of chaos requires caution by using Chaos Degree. In this paper, we find out information-theoretic interpretation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps.

Keywords chaos, Chaos Degree, Lyapunov exponent

Research Activity Group Applied Chaos

1. Introduction

Lyapunov exponent is commonly used as a measure of chaos. However, it is difficult to calculate Lyapunov exponents if equations of dynamical systems are not given. On the other hand, Entropic Chaos Degree (Chaos Degree) [1] is proposed as another measure of chaos based on information theory. The advantage of Chaos Degree is that it is calculable from data. However, there is a difference between Chaos Degree and Lyapunov exponent and determination of chaos requires caution by using Chaos Degree. In this paper, we find out information-theoretic interpretation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps.

Relations among Chaos Degree, Lyapunov exponent and KS entropy are discussed in [2]. Previous research [3] analytically shows that Chaos Degree in the limit that partition number approaches to infinity is greater than Lyapunov exponent, and suggests that the difference between Chaos Degree and Lyapunov exponent depends on the choice of partition. In many cases, Chaos Degree is greater than Lyapunov exponent and we should be careful to determine chaos by using Chaos Degree. For example, preceding studies [4, 5] show that Chaos Degree is positive for quasi-periodic orbits and it is hard to distinguish chaotic orbits from quasi-periodic orbits.

This paper reports on an analytical investigation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps.

2. Definition of Chaos Degree

The definition of Chaos Degree in difference equations is as below.

We assume that the difference equation is determined by a map $f : I \rightarrow I$ ($\equiv [a, b]^d \subset \mathbb{R}^d, a, b \in \mathbb{R}, d \in \mathbb{N}$), i.e., $x_{n+1} = f(x_n)$ ($n = 0, 1, \ldots$). Let $x_0$ be an initial value, and $A = \{A_i\}$ be a finite partition of $I$ such that

$$I = \bigcup_{k=1}^{N} A_k, \quad A_i \cap A_j = \emptyset \quad (i \neq j).$$

The probability distribution $p_i^{(m)}(M)$ at the time $m$ is given as

$$p_i^{(m)}(M) = \frac{1}{M} \sum_{k=m}^{m+M-1} 1_{A_i}(x_k),$$

and the joint probability distribution $p_{i,j,A}^{(m,m+1)}(M)$ between the time $m$ and $m + 1$ is given as

$$p_{i,j,A}^{(m,m+1)}(M) = \frac{1}{M} \sum_{k=m}^{m+M-1} 1_{A_i}(x_k)1_{A_j}(x_{k+1}).$$

Then Chaos Degree $D_{Ch}(M, m)(A, f)$ for the orbit $\{x_k\}$ is defined by

$$D_{Ch}(M, m)(A, f) = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{i,j,A}^{(m,m+1)}(M) \log \frac{p_i^{(m)}(M)}{p_{i,j,A}^{(m,m+1)}(M)}.$$ 

In this paper, we simplify $p_i^{(m)}(M) = p(i)$ and $p_{i,j,A}^{(m,m+1)}(M) = p(i, j)$, then Chaos Degree $H_{CD}$ is calculated as:

$$H_{CD} = \sum_{i=1}^{N} \sum_{j=1}^{N} p(i, j) \log \frac{p(i)}{p(i, j)} = -\sum_{i=1}^{N} p(i) \sum_{j=1}^{N} p(j|i) \log p(j|i), \quad (1)$$

where the conditional probability $p(j|i) = p(i, j)/p(i)$.
3. Asymmetric tent map

Let \( T_k(x) \) be a tent map with the peak at \( x = 1/k \) \((k \in \mathbb{N}, k \geq 2)\) such that

\[
T_k(x) = \begin{cases} 
  \frac{kx}{k-1}(1-x), & (0 \leq x \leq \frac{1}{k}) \\
  \frac{k}{k-1}x, & (\frac{1}{k} \leq x \leq 1)
\end{cases}.
\]

(2)

If \( k = 2 \) then \( T_k(x) \) is a symmetric tent map, else \( T_k(x) \) is an asymmetric tent map.

We try to find out how partitions affect the difference between Chaos Degree and Lyapunov exponent, focusing on asymmetric tent map \( T_k(x) \) because its shape is simple so that it is easy to calculate both theoretical values of Chaos Degree and Lyapunov exponents.

4. Investigation procedure

First, we calculate Chaos Degree of an asymmetric tent map \( T_k(x) \). Next, we calculate a difference between Chaos Degree and Lyapunov Exponent \((H_{CD} - \lambda)\) and try to interpret what the difference means.

5. Calculation of Chaos Degree

Let \( \{A_i\} \) be an equipartition with partition number \( N \) such that \( N = nk \) \((n \in \mathbb{N})\). For example, the case \( k = 4 \) and \( n = 2 \) is shown in Fig. 1.

There are two kinds of repeating patterns in \( 0 \leq x \leq 1/k \) and \( 1/k \leq x \leq 1 \). In Fig. 1, each pattern is colored red or green. Each of these patterns appears \( n \) times.

If \( \{A_i, A_j\} \) such that \( x_m \in A_i \) and \( x_{m+1} \in A_j \) does not belong to any patterns, then \( p(j|i) = 0 \) and Chaos Degree is not affected. Therefore, we consider the two patterns in calculation of Chaos Degree below.

In case \( 0 \leq x \leq 1/k \), as shown in Fig. 2 (a), \( T_k(A_i) \) intersects with just \( k \) components, therefore conditional probability is

\[
p(j|i_1) = \begin{cases} 
  \frac{1}{k} & (j = j_1, j_2, \ldots, j_k) \\
  0 & \text{(otherwise)}
\end{cases}.
\]

(3)

In case \( 1/k \leq x \leq 1 \), as shown in Fig. 2 (b), each of \( T_k(A_i) \) \((i = 1, 2, \ldots, k)\) intersects with just two components. Suppose the index numbers of the two components are \( v_1 \) and \( v_2 \), then

\[
p(j|i_u) = \begin{cases} 
  \frac{u}{k} & (j = j_{v_1}) \\
  \frac{k-u}{k} & (j = j_{v_2}) \\
  0 & \text{(otherwise)}
\end{cases}.
\]

(4)

Further, \( T_k(x) \) has the uniform invariant probability density, therefore

\[
p(i) = \frac{1}{N} = \frac{1}{nk}.
\]

(5)

From equations (1), (3), (4) and (5), we obtain

\[
H_{CD} = \frac{1}{nk} \sum_{i=1}^{k-1} \left( \frac{k-u}{k} \log \frac{u}{k} - \frac{k-u}{k} \log \frac{k-u}{k} \right) + \frac{1}{nk} \sum_{u=1}^{k-1} \left( \frac{u}{k} \log \frac{k-u}{k} - \frac{u}{k} \log \frac{k-u}{k} \right) = \frac{1}{k} \log \frac{1}{k}.
\]

(6)

Note that \( H_{CD} \) does not depend on \( n \).

6. Calculation of the difference between Chaos Degree and Lyapunov Exponent

Lyapunov exponent \( \lambda \) is given as

\[
\lambda = -\frac{1}{k} \log \frac{1}{k} - \frac{k-1}{k} \log \frac{k-1}{k}.
\]

(7)

Chaos Degree \( H_{CD} \) for \( k \geq 2 \) and Lyapunov exponent \( \lambda \) is shown in Fig. 3. The horizontal axis \( a \) is the peak of \( T_k(x) \) i.e. \( a = 1/k \). As can be seen from the figure, when \( k = 2 \) \((a = 1/2)\), \( T_k(x) \) is a symmetric tent map, Chaos Degree equals Lyapunov exponent, and in other cases, Chaos Degree is greater than Lyapunov exponent.

**Theorem 1** Let \( T_k(x) \) be an asymmetric tent map which is given by equation (2) and \( \{A_i\} \) be an \( nk \) equipartition of \( I = [0,1] \), then we have

\[
H_{CD} \geq \lambda,
\]

where \( H_{CD} \) is Chaos Degree and \( \lambda \) is Lyapunov expo-
Thus, we have $H_{CD} - \lambda \geq 0$, i.e. $H_{CD} \geq \lambda$. The equality holds for $k = 2$ (symmetric tent map case).

(QED)

The difference between Chaos Degree and Lyapunov Exponent is shown in Fig. 4. The horizontal axis $a$ is the peak of $T_k(x)$ i.e. $a = 1/k$. When $a = 0$, although Lyapunov exponent $\lambda = 0$, the difference $H_{CD} - \lambda$ is maximum. This shows that it is difficult to determine chaos for weak chaos by using Chaos Degree $H_{CD}$.

Fig. 4 also shows that values of the difference $H_{CD} - \lambda$ are close to the line $-a + 1/2$. We can understand this behavior as below. From equation (8),

$$H_{CD} - \lambda = \frac{1}{k} \sum_{u=1}^{k-1} \left( -\frac{u}{k} \log \frac{u}{k} - \frac{k-u}{k} \log \frac{k-u}{k} \right)$$

$$= 2 \left( 1 - \frac{1}{k} \right)^2 \sum_{u=1}^{k-1} \left( -\frac{u}{k} \log \frac{u}{k} \right) \frac{1}{k-1}.$$

For sufficiently large $k$,

$$H_{CD} - \lambda \sim 2 \left( 1 - \frac{2}{k} \right) \int_0^1 (-x \log x) dx$$

$$= -a + \frac{1}{2}.$$

7. Interpretation of the difference

**Theorem 2** Suppose that $T_k(x)$ is an asymmetric tent map and $\{A_i\}$ is an nk equipartition of $I = [0, 1]$. The difference between Chaos Degree and Lyapunov exponent $H_{CD} - \lambda$ can be calculated as an average of some kind of information $-\log q(i,j)$, where $q(i,j)$ is defined as

$$q(i,j) \overset{\text{def}}{=} \frac{\|T_k(A_i) \cap A_j\|}{\|A_j\|}.$$

**Proof** Since $\{A_i\}$ is an nk equipartition,

$$\|A_i\| = \Delta = \frac{1}{nk} \ (i = 1, 2, \ldots, N).$$

In case $0 \leq x \leq 1/k$,

$$q(i_1, j) = \frac{\|A_j\|}{\|A_j\|} = 1 \quad (j = j_1, j_2, \ldots, j_k).$$

In case $1/k \leq x \leq 1$, as shown in Fig. 5,

$$q(i_u, j) = \begin{cases} \frac{k-u}{k} & (j = j_v) \\ \frac{k}{k} & (j = j_v) \\ 0 & (\text{otherwise}) \end{cases}.$$
8. Conclusion

We try to interpret the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps and find out that the difference $H_{CD} - \lambda$ can be calculated as an average of some kind of information $- \log q(i, j)$ and thus be non-negative. It is assumed that information $- \log q(i, j)$ is related to intersection of mapping $T_k(x)$ and components of partition $\{A_i\}$.

It is left for a future work to generalize the interpretation to other one-dimensional maps.

References

[1] M. Ohya, Complexities and their applications to characterization of chaos, Internat. J. Theoret. Phys., 37 (1998), 495–505.
[2] T. Kamizawa, T. Hara and M. Ohya, On relations among the entropic chaos degree, the Kolmogorov-Sinai entropy and the Lyapunov exponent, J. Math. Phys., 55 (2014), 032702.
[3] H. Okutomi and T. Mao, Mathematical relationship between chaos degree and Lyapunov exponent (in Japanese), IEICE Tech. Rep., 117(288) (2017), 5–10.
[4] K. Inoue, Basic properties of entropic chaos degree in classical systems, Information-An International Interdisciplinary Journal, 16 (2013), 8589–8596.
[5] K. Inoue, On evaluation of quasi-periodic orbits by chaos degree (in Japanese), Trans. JSIAM, 25 (2015), 105–115.