The structure of fixed-point sets of uniformly lipschitzian semigroups

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Abstract In this paper, by asymptotic center techniques, we shown that the set of fixed points of a uniformly $k$-lipschitzian semigroup (one-parameter or left reversible semitopological) in a uniformly convex Banach space is a retract of the domain if $k$ is close to 1. The results presented in this paper includes (among others, in the discrete situation) many known results as special cases.

Keywords One-parameter semigroup · Left reversible semigroup · Uniformly lipschitzian semigroup · Retraction · Asymptotic center · Fixed point · Uniformly convex Banach space

Mathematics Subject Classification (2000) Primary 47H10 · 47H20; Secondary 47H09 · 54C15

1 Introduction

We will consider a Banach spaces $E$ over the real field. Our notation and terminology are standard. Let $C$ be a nonempty bounded closed convex subset of $E$. We say that a mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for every } x, y \in C.$$ 

The result of Bruck [2] asserts that if a nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of $C$ which is invariant under $T$ and if $C$ is convex and weakly compact, then $F(T) = \{x \in C : Tx = x\}$, the set of fixed points, is a nonexpansive retract of $C$ (that is, there exists a nonexpansive mapping $R : C \to F(T)$ such that $R|_{F(T)} = I$). A few years ago, the Bruck result was extended by Domínguez Benavides

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and Lorenzo Ramírez [4] to the case of asymptotically nonexpansive mappings if the space $E$ was sufficiently regular.

On the other hand it is known, the set of fixed points of $k$-lipschitzian mapping can be very irregular for any $k > 1$.

**Example 1** ([11]) Let $F$ be a nonempty closed subset of $C$. Fix $z \in F$, $0 < \varepsilon < 1$ and put

$$Tx = x + \varepsilon \cdot \text{dist}(x, F) \cdot (z - x), \quad x \in C.$$  

It is not difficult to see that $F(T) = F$ and the Lipschitz constant of $T$ tends to $1$ if $\varepsilon \downarrow 0$.

In 1973, Goebel and Kirk [5] introduced the class of uniformly $k$-lipschitzian mappings and stated a relationship between the existence of fixed point for uniformly $k$-lipschitzian mappings and the Clarkson modulus of convexity $\delta_E$. Recall that a mapping $T : C \to C$ is uniformly $k$-lipschitzian, $k \geq 0$, if

$$\|T^nx - T^ny\| \leq k\|x - y\| \quad \text{for every } x, y \in C \quad \text{and } n \in \mathbb{N}.$$  

**Theorem 2** Let $E$ be a uniformly convex Banach space with modulus of convexity $\delta_E$ and let $C$ be a nonempty bounded closed convex subset of $E$. Suppose $T : C \to C$ is uniformly $k$-lipschitzian map and $k < \gamma$, where $\gamma > 1$ is the unique solution of the equation

$$\gamma \left( 1 - \delta_E \left( \frac{1}{\gamma} \right) \right) = 1.$$  

Then $F(T) \neq \emptyset$ (note that in a Hilbert space, $k < \gamma = \frac{1}{2}\sqrt{3}$, in $L^p$-spaces $(2 \leq p < \infty), k < \gamma = (1 + 2^{-p})^{\frac{1}{p}}$), and $F(T)$ is not only connected but even a retract of $C$ (see [11]).

In this paper we establish some results on the structure of fixed point sets for one-parameter uniformly $k$-lipschitzian semigroups and semi-topological uniformly $k$-lipschitzian semigroups in uniformly convex Banach spaces when $k$ is less than a constant bigger than the constant from Theorem 2.

## 2 Uniformly convex Banach spaces

Recall that the **modulus of convexity** $\delta_E$ is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}$$

and that the space $E$ is uniformly convex if $\delta_E(\varepsilon) > 0$ for $\varepsilon > 0$. A Hilbert space $H$ is uniformly convex. This fact is a direct consequence of parallelogram identity. It is well known that $\delta_E$ is continuous on $[0, 2)$ and strictly increasing in uniformly convex Banach spaces [6, Lemma 5.1].

Recall the concept and the notion of **asymptotic center** due to Edelstein, see [1,6]. Let $C$ be a nonempty closed convex subset of a Banach space $E$, and $G$ an unbounded subset of $[0, +\infty)$ such that $t + h \in G$ for all $t, h \in G$ and $t - h \in G$ for all $t, h \in G$ with $t \geq h$ (i.e., $G = [0, +\infty), G = [0, +\infty) \cap \mathbb{Q}$ or $G = \mathbb{N}_0$, the set of nonnegative integers), and let $\{x_t : t \in G\}$ be a bounded family of elements of $E$. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t \in G}$ with respect to $C$ are the number

$$r(C, \{x_t\}) := \inf_{y \in C} \left( \limsup_{G \ni t \to \infty} \|y - x_t\| \right).$$
and the (possibly empty) set
\[ A(C, \{x_t\}) := \left\{ y \in C : \limsup_{G \ni t \to +\infty} \|y - x_t\| = r(C, \{x_t\}) \right\}, \]
respectively. It is well known that if \( E \) is reflexive, then \( A(C, \{x_t\}) \) is bounded closed convex and nonempty, and if \( E \) is uniformly convex, then \( A(C, \{x_t\}) \) consist only a single point, \( \{z\} = A(C, \{x_t\}) \), i.e., other words \( z \in C \) is the unique point which minimizes functional
\[ \limsup_{G \ni t \to +\infty} \|y - x_t\| \]
over \( y \) in \( C \).

Suppose \( \mathcal{F} = \{T_s : s \in G\} \) is a one-parameter uniformly \( k \)-lipschitzian semigroup on \( C \), i.e., a family of self-mappings on \( C \) satisfying the conditions:
1. \( T_{s+h}x = T_s T_h x \) for all \( s, h \in G \) and \( x \in C \),
2. for each \( x \in C \), the mapping \( s \to T_s x \) from \( G \) into \( C \) is continuous when \( G \) has the relative topology of \([0, +\infty)\),
3. for each \( s \in G \), \( \|T_s x - T_s y\| \leq k\|x - y\| \) for all \( x, y \in C \).

Let \( A : C \to C \) denote a mapping which associates with a given \( x \in C \) a unique \( z \in A(C, \{T_t x\}) \), that is, \( z = Ax \). Now we generalize to uniformly \( k \)-lipschitzian semigroups the lemma due to Sędłak and Wiśnicki [11]. This lemma is crucial to our results.

**Lemma 3** Let \( E \) be a uniformly convex Banach space and let \( C \) be a nonempty bounded closed convex subset of \( E \). Then the mapping \( A : C \to C \) is continuous.

**Proof** On the contrary, suppose that there exists \( x_0 \in C \) and \( \varepsilon_0 > 0 \) such that:
for all \( \eta > 0 \) there exists \( x_1 \in C \) such that \( \|x_1 - x_0\| < \eta \) and \( \|z_1 - z_0\| \geq \varepsilon_0 \), where \( \{z_0\} = A(C, \{T_t x_0\}) \), \( \{z_1\} = A(C, \{T_t x_1\}) \).

Fix \( \eta > 0 \) and take \( x_1 \in C \) such that
\[ \|x_1 - x_0\| < \eta \quad \text{and} \quad \|z_1 - z_0\| \geq \varepsilon_0. \]

Let
\[ R_0 = r(C, \{T_t x_0\}) = \inf_{y \in C} \left( \limsup_{G \ni t \to +\infty} \|y - T_t x_0\| \right), \]
\[ R_1 = r(C, \{T_t x_1\}) = \inf_{y \in C} \left( \limsup_{G \ni t \to +\infty} \|y - T_t x_1\| \right) \]
and
\[ R = \limsup_{G \ni t \to +\infty} \|z_1 - T_t x_0\|. \]

Notice that
\[ R_0 < R. \]

Choose \( \varepsilon > 0 \). Then exists \( s(\varepsilon) \in G \) that
\[ \begin{cases} \|z_1 - T_t x_0\| < R + \varepsilon, \\ \|z_0 - T_t x_0\| < R_0 + \varepsilon < R + \varepsilon, \\ \|z_0 - z_1\| \geq \varepsilon_0. \end{cases} \]  
(2)

for all \( t \in G \) and \( t \geq s(\varepsilon) \).
It follows by (2) and the properties of $\delta_E$ that for $G \ni t \geq s(\epsilon)$,

$$\left\| T_t x_0 - \frac{z_1 + z_0}{2} \right\| \leq \left( 1 - \delta_E \left( \frac{\epsilon_0}{R + \epsilon} \right) \right) (R + \epsilon)$$

and hence

$$R_0 < \limsup_{G \ni t \to +\infty} \left\| T_t x_0 - \frac{z_1 + z_0}{2} \right\| \leq \left( 1 - \delta_E \left( \frac{\epsilon_0}{R + \epsilon} \right) \right) (R + \epsilon). \quad (3)$$

Moreover for $t \in G$, from triangle inequality we have

$$\left\| T_t x_0 - z_1 \right\| \leq \left\| T_t x_o - T_t x_1 \right\| + \left\| T_t x_1 - z_1 \right\| \leq k \left\| x_0 - x_1 \right\| + R_1 + \epsilon,$$

and hence

$$R = \limsup_{G \ni t \to +\infty} \left\| T_t x_0 - z_1 \right\| \leq k \eta + R_1 + \epsilon. \quad (4)$$

Similarly,

$$R_1 < \limsup_{G \ni t \to +\infty} \left\| T_t x_1 - z_0 \right\| \leq k \eta + R_1 + \epsilon. \quad (5)$$

From (4) and (5), we have

$$R \leq k \eta + R_1 + \epsilon < 2k \eta + 2 \epsilon + R. \quad (6)$$

Combining (6) with (3) and applying the monotonicity of $\delta_E$, we obtain

$$R_0 < \left( 1 - \delta_E \left( \frac{\epsilon_0}{2k \eta + 3 \epsilon + R_0} \right) \right) (2k \eta + 3 \epsilon + R_0).$$

Letting $\eta, \epsilon \downarrow 0$, and using the continuity of $\delta_E$, we conclude that

$$1 \leq \left( 1 - \delta_E \left( \frac{\epsilon_0}{R_0} \right) \right) < 1.$$

This contradiction proves the continuity of the mapping $A$. \qed

This result can be extend to left reversible semigroups. Now let $J$ be a semi-topological semigroup, i.e., $J$ is a semigroup with a Hausdorff topology such that for each $a \in J$ the mapping $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from $J$ to $J$ are continuous. A semi-topological semigroup $J$ is said to be left reversible if any two closed right ideals have non-void intersection. (This latter is automatically fulfilled if, for example $J$ is commutative, and in particular if $J = [0, +\infty)$.) In this case $(J, \leq)$ is a directed system when the binary relation “$\leq$” on $J$ is defined by $a \leq b$ if and only if $\{a\} \cup \overline{aJ} \supseteq \{b\} \cup \overline{bJ}$.

Let $\{x_a : a \in J\}$ be a bounded net in uniformly convex Banach space $E$ and let $C$ a nonempty closed convex subset of $E$. For a fixed $p > 1$, let us set

$$r(x) = \inf_{b \in J} \sup_{a \geq b} \left\| x_a - x \right\|^p \quad \text{and} \quad r = \inf_{x \in C} r(x).$$

Then we have a unique point $z \in C$ (called the asymptotic center of the net $\{x_a\}$ in $C$) such that $r(z) = r$.

Let $C$ be a nonempty bounded closed convex subset of a Banach space $E$, let $J$ be a left reversible semi-topological semigroup, and let $T = \{T_s : s \in J\}$ be a family of self-mappings of $C$ into itself. Then $T$ is said to be a left reversible semi-topological uniformly $k$-lipschitzian semigroup on $C$ if the following conditions are satisfied:

\[ \text{Springer} \]
1. \( T_{ts}x = T_tT_sx \) for all \( t, s \in J \) and \( x \in C \),

2. the mapping \( (s, x) \to T_sx \) from \( J \times C \) into \( C \) is continuous when \( J \times C \) has the product topology,

3. for each \( s \in J \), \( \|T_sx - T_sy\| \leq k\|x - y\| \) for all \( x, y \in C \).

**Remark 4** For such a family of mappings Lemma 3 remains true.

Normal structure plays essential role in some problems of metric fixed point theory. Let \( C \) be a nonempty bounded set in a Banach space \( E \). We put

\[
r(C) = \inf_{x \in C} \left( \sup_{y \in C} \|x - y\| \right).
\]

This number is called the Chebyshev radius of \( A \).

A Banach space \( E \) is said to have uniformly normal structure (UNS) if for some \( c \in (0, 1) \) and every bounded closed convex subset \( C \subseteq E \) with \( \text{diam} C > 0 \), it has

\[
r(C) \leq c \cdot \text{diam} C.
\]

The *normal structure coefficient* (also called the Jung constant) was defined by Bynum [3] in the following way

\[
N(E) := \inf \left\{ \frac{\text{diam} C}{r(C)} \right\}
\]

where the infimum is taken over all bounded closed convex sets \( C \subseteq E \) with \( \text{diam} C > 0 \). Clearly, the condition \( N(E) > 1 \) characterizes spaces \( E \) with UNS. It is well known that all uniformly convex Banach spaces possess UNS [10, Theorem 5.12]. It is difficult to calculate the normal structure coefficient in an arbitrary Banach space. However, for a Hilbert space, \( N(H) = \sqrt{2} \), and \( N(l^p) = N(L^p) = \min\{2^{1/p}, 2^{1-1/p}\} \) for \( 1 < p < \infty \), see [1,10].

The following lemma can be proved in exactly the same way as Lim [8, Theorem 1] for sequences and the proof is thus omitted here.

**Lemma 5** Let \( E \) be a Banach space with UNS. Then for every bounded family \( \{x_t\}_{t \in G} \) of elements of \( E \) there exists \( y \) in \( \overline{\text{conv}}\{x_t : t \in G\} \) such that

\[
\limsup_{G \ni t \to +\infty} \|y - x_t\| \leq \frac{1}{N(E)} \cdot \lim_{G \ni t \to +\infty} (\sup\{\|x_i - x_j\| : t \leq i, j \in G\}).
\]

Now we improve the fixed point theorem due to Tan and Xu [12, Theorem 3.5].

**Theorem 6** Let \( E \) be a uniformly convex Banach space and let \( C \) be a nonempty bounded closed convex subset of \( E \). Suppose \( \mathcal{F} = \{T_s : s \in G\} \) is a one-parameter uniformly \( k\)-lipschitzian semigroup on \( C \) with \( k < \alpha \), where \( \alpha > 1 \) is the unique solution of the equation

\[
\alpha^2 \cdot \frac{1}{\delta_E(1 - \frac{1}{\alpha})} \cdot \frac{1}{N(E)} = 1
\]

(in a Hilbert space \( \alpha = (\sqrt{3} - 1)^{-\frac{3}{2}} > \frac{1}{2}\sqrt{5} \)). Then

\[
F(\mathcal{F}) = \{x \in C : T_sx = x \text{ for all } s \in G\} \neq \emptyset
\]

and \( F(\mathcal{F}) \) is a retract of \( C \).
Suppose $E$ is uniformly convex Banach space and $\alpha > 1$ is the unique solution of Eq. (7). Then $\gamma < \alpha$, where $\gamma > 1$ is the unique solution of Eq. (1), see [12, Lemma 3.3].

**Proof** The proof of existence $z$ in $C$ such that $T_s z = z$ for all $s \in G$, based on the Lemma 5, is given in [12, Theorem 3.5] or in [7, Theorem 4]. In this proof by induction we define a sequence $\{x_n\}_{n=0,1,2,\ldots}$ in $C$ in the following manner

$$x_0 = x \quad \text{and} \quad x_{n+1} = A(C, \{T_t x_n\}) = A^{n+1} x, \quad n = 0, 1, \ldots$$

($z = \lim_{n \to +\infty} x_n$). Thus by the inequalities

$$d(x_n) = \sup_{t \in G} \|T_t x_n - x_n\| \leq d(x_{n-1}) \leq B^n d(x),$$

$$\|x_{n+1} - x_n\| \leq \left(\frac{k}{N(E)} + 1\right) B^n d(x) \to 0 \quad \text{as} \quad n \to +\infty,$$

where $B = \frac{k^2}{N(E)} \delta_E^{-1} (1 - \frac{1}{k}) < 1$, we have

$$\|A^{n+1} x - A^n x\| \leq \left(\frac{k}{N(E)} + 1\right) B^n d(x) \leq \left(\frac{k}{N(E)} + 1\right) B^n \text{diam} C$$

for $n = 1, 2, \ldots$ So

$$\sup_{x \in C} \|A^i x - A^m x\| \leq \left(\frac{k}{N(E)} + 1\right) \frac{B^m \text{diam} C}{1 - B} \to 0 \quad \text{if} \quad i, m \to +\infty,$$

which implies that the sequence $\{A^m x\}_{m=1,2,\ldots}$ converges uniformly to a function $Rx = \lim_{m \to +\infty} A^m x$, $x \in C$.

It follows from Lemma 3, that $R : C \to C$ is continuous. Moreover,

$$\|Rx - T_s Rx\| \leq \|Rx - A^m x\| + \|A^m x - T_{s+h} A^m x\| + \|T_{s+h} A^m x - T_s A^m x\| + \|T_s A^m x - T_s Rx\|$$

$$\leq (1 + k) \|Rx - A^m x\| + \|A^m x - T_{s+h} A^m x\| + k \|T_h A^m x - A^m x\|$$

$$\leq (1 + k) \|Rx - A^m x\| + (1 + k) \cdot d(A^m x)$$

$$\leq (1 + k) \|Rx - A^m x\| + (1 + k) \cdot B^m \cdot \text{diam} C \to 0 \quad \text{as} \quad m \to +\infty$$

and $Rx = T_s Rx$ for all $s \in G$ and $x \in C$. Thus $R$ is a retraction of $C$ onto $F(F)$. \qed

This result can be sharpened in some uniformly convex Banach spaces, for example in a Hilbert space and in $L^p$-spaces ($1 < p < \infty$).

### 3 $p$-Uniformly convex Banach spaces

Let $p > 1$ be a real number. A Banach space $E$ is said to be $p$-uniformly convex (or $E$ is said to have the modulus of convexity of power type $p$) if there exists a constant $d > 0$ such that the modulus of convexity $\delta_E(\varepsilon) \geq d \cdot \varepsilon^p$ for $0 \leq \varepsilon \leq 2$. We note that a Hilbert space is 2-uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} \geq \frac{1}{8} \varepsilon^2$) and an $L^p$-space ($1 < p < \infty$) is max($p, 2$)-uniformly convex.

In [9, 14] the following result was proved.
Theorem 7 Let \( p > 1 \) be a real number and let \( E \) be a \( p \)-uniformly convex Banach space. Then there exists a constant \( c_p > 0 \) such that
\[
\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p
\]
for all \( x, y \in E \), \( 0 \leq \lambda \leq 1 \), where \( W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda) \).

Let \( H \) be a Hilbert space, then
\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\]
for all \( x, y \in H \), \( 0 \leq \lambda \leq 1 \).

When \( E \) is an \( L^p \)-space, we have the following

Theorem 8 Suppose \( E \) is an \( L^p \)-space.

(a) If \( 1 < p \leq 2 \), then
\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (p - 1) \cdot \lambda \cdot (1 - \lambda) \cdot \|x - y\|^2
\]
for all \( x, y \in E \) and \( 0 \leq \lambda \leq 1 \) (\( c_p = p - 1 \));

(b) If \( 2 < p < \infty \), then
\[
\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p
\]
for all \( x, y \in E \), \( 0 \leq \lambda \leq 1 \), where \( W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda) \) and
\[
c_p = \frac{1 + t_p^{p-1}}{(1 + t_p)^{p-1}} = (p - 1)(1 + t_p)^{2-p}
\]
with \( t_p \) being the unique solution of the equation
\[
(p - 2)t^{p-1} + (p - 1)t^{p-2} - 1 = 0, \quad 0 < t < 1.
\]

All constant appeared in the above inequalities are the best possible.

In the following theorem we improve the fixed point theorem due to Xu [14] from point of view of the structure of the set of fixed points.

Theorem 9 Let \( p > 1 \) be a real number and let \( E \) be a \( p \)-uniformly convex Banach space, \( C \) a nonempty bounded closed convex subset of \( E \). Suppose \( \mathcal{F} = \{T_s : s \in G\} \) is a one-parameter uniformly \( k \)-lipschitzian semigroup on \( C \) with \( k < k_p \), where \( k_p > 1 \) is the unique solution of the equation
\[
(t^p)^2 - t^p - [N(E)]^p \cdot c_p = 0, \quad t \in (0, +\infty),
\]
i.e., \( k_p = \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot c_p \cdot [N(E)]^p} \right) \right]^\frac{1}{p} \). Then \( F(\mathcal{F}) \neq \emptyset \) and \( F(\mathcal{F}) \) is a retract of \( C \).

Proof We may assume that \( k \geq 1 \) since if \( k < 1 \), the well known Banach Contraction Principle guarantees a fixed point of \( \mathcal{F} \).

For an \( x = x_0 \in C \), we can inductively define a sequence \( \{x_m\}_{m=1,2,...} \) in \( C \) in the following way: \( x_{m+1} \) is the asymptotic center of the sequence \( \{T_t x_m\}_{t \in G} \), that is, \( x_{m+1} \) is the unique point in \( C \) that minimizes the functional
\[
\limsup_{G \ni t \rightarrow +\infty} \|y - T_t x_m\|.
\]
over \( y \) in \( C \). For each \( m \geq 0 \), we set
\[
 r_m = \limsup_{G \ni t \to +\infty} \|x_{m+1} - T_t x_m\| \quad \text{and} \quad d(x_m) = \sup_{t \in G} \|x_m - T_t x_m\|.
\]

Then by Lemma 5, we have
\[
 r_m \leq \frac{1}{N(E)} \cdot \lim_{G \ni t \to +\infty} \left( \sup_{t \leq s \in G \quad h \in G} \|T_{s+h} x_m - T_s x_m\| : t \leq s \right)
\leq \frac{k}{N(E)} \cdot \sup_{t \in G} \|T_t x_m - x_m\| = \frac{k}{N(E)} \cdot d(x_m)
\]
for \( m = 0, 1, 2, \ldots \).

Now from Theorem 7 for each fixed \( m \geq 0 \) and \( s, h \in G \), we have
\[
\|\lambda x_{m+1} + (1 - \lambda) T_s x_{m+1} - T_{s+h} x_m\|^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T_s x_{m+1}\|^p
\leq \lambda \|x_{m+1} - T_s x_m\|^p + (1 - \lambda) \|T_{s+h} x_m - T_s x_{m+1}\|^p
\leq \lambda \|x_{m+1} - T_{s+h} x_m\|^p + (1 - \lambda) \cdot k^p \cdot \|T_h x_m - x_{m+1}\|^p.
\]

Taking the limit superior as \( G \ni h \to +\infty \) and nothing that \( x_{m+1} \) is the asymptotic center of the sequence \( \{T_t x_m\}_{t \in G} \), we obtain for each \( s \in G \),
\[
r_m^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T_s x_{m+1}\|^p \leq [\lambda + (1 - \lambda) \cdot k^p] \cdot r_m^p,
\]
and
\[
\|x_{m+1} - T_s x_{m+1}\|^p \leq \frac{(1 - \lambda) (k^p - 1)}{c_p \cdot W_p(\lambda)} \cdot r_m^p.
\]

It then follows that
\[
[d(x_{m+1})]^p \leq \frac{(1 - \lambda) (k^p - 1)}{c_p \cdot W_p(\lambda)} \cdot r_m^p \leq \frac{(1 - \lambda) (k^p - 1)}{c_p \cdot W_p(\lambda)} \cdot \frac{k^p}{[N(E)]^p} \cdot [d(x_m)]^p.
\]

Letting \( \lambda \uparrow 1 \), we get
\[
[d(x_{m+1})]^p \leq \frac{k^p (k^p - 1)}{c_p \cdot [N(E)]^p} \cdot [d(x_m)]^p,
\]
and
\[
d(x_{m+1}) \leq \left( \frac{k^p (k^p - 1)}{c_p \cdot [N(E)]^p} \right)^{\frac{1}{p}} \cdot d(x_m) = B_p \cdot d(x_m), \quad m = 0, 1, 2, \ldots ,
\]
where \( B_p = \left( \frac{k^p (k^p - 1)}{c_p \cdot [N(E)]^p} \right)^{\frac{1}{p}} < 1 \) by assumption of the theorem. In a similar way, we obtain
\[
d(x_{m+1}) \leq B_p \cdot d(x_m) \leq \cdots \leq (B_p)^{m+1} \cdot d(x).
\]
Since
\[
\|x_{m+1} - x_m\| \leq \|x_{m+1} - T_t x_m\| + \|T_t x_m - x_m\|,
\]
so taking the limit superior as \( G \ni t \to +\infty \), we get by (8), (9),
\[
\|x_{m+1} - x_m\| \leq r_m + d(x_m) \leq \left( \frac{k}{N(E)} + 1 \right) d(x_m)
\leq \left( \frac{k}{N(E)} + 1 \right) \cdot (B_p)^{m+1} \cdot d(x) \to 0 \quad \text{as} \quad m \to +\infty,
\]
and we see that \( \{x_m\} \) is norm Cauchy and hence strong convergent. Let \( z = \lim_{m \to \infty} x_m \). Then we have
\[
\|T_s z - z\| \leq \|T_s z - T_s x_m\| + \|T_s x_m - x_m\| + \|x_m - z\|
\leq (1 + k)\|x_m - z\| + d(x_m)
\leq (1 + k)\|x_m - z\| + (B_p)^m \cdot d(x) \to 0 \quad \text{as } m \to +\infty,
\]
and \( T_s z = z \) for all \( s \in G \).

The proof of the retraction \( R : C \to F(\mathcal{F}) \) can be proved in exactly the same way as in the proof of Theorem 6.

**Corollary 10** Let \( H \) be a Hilbert space, \( C \) a nonempty bounded closed convex subset of \( H \) and \( \mathcal{F} = \{T_s : s \in G\} \) be a one-parameter uniformly \( k \)-lipschitzian semigroup on \( C \) with \( k < \sqrt{2} \). Then \( F(\mathcal{F}) \neq \emptyset \) and \( F(\mathcal{F}) \) is a retract of \( C \).

**Corollary 11** Let \( C \) be a nonempty bounded closed convex subset of \( L^p \)-space \( (1 < p < \infty) \) and \( \mathcal{F} = \{T_s : s \in G\} \) be a one-parameter uniformly \( k \)-lipschitzian semigroup on \( C \). Suppose
\[
k < \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + (p - 1)4^{2-\frac{1}{p}}} \quad \text{if } 1 < p \leq 2 \quad \text{(in particular, in } L^2 \text{-space, } k < \sqrt{2}) \quad \text{and}
k < \left(\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8 \cdot c_p}}\right)^{\frac{1}{p}} \quad \text{if } 2 < p < +\infty \quad \text{(here } c_p \text{ is as in Theorem 8(b)). Then } F(\mathcal{F}) \neq \emptyset \text{ and } F(\mathcal{F}) \text{ is a retract of } C.
\]

For left reversible semi-topological semigroup we have the following [13, Lemma 3].

**Lemma 12** Let \( E \) be \( p \)-uniformly convex Banach space for some \( p > 1 \), \( C \) a nonempty bounded closed convex subset of \( E \). Let \( J \) be a left reversible semi-topological semigroup and \( \{x_a : a \in J\} \) be a net in \( C \). Let us set
\[
r(x) = \inf_{b \in J} \sup_{a \geq b} \|x_a - x\|^p \quad \text{and} \quad r = \inf_{x \in C} r(x).
\]

Then we have a unique point \( z \in C \) (called the asymptotic center of the net \( \{x_a\} \) in \( C \)) such that \( r(z) = r \) and
\[
r(z) \leq r(x) - c_p \|x - z\|^p
\]
for all \( x \in C \), where the constant \( c_p \) is as in Theorem 7.

**Theorem 13** Let \( p > 1 \) be a real number and let \( E \) be \( p \)-uniformly convex Banach space, \( C \) a nonempty bounded closed convex subset of \( E \). Suppose \( T = \{T_s : s \in J\} \) is a left reversible semi-topological uniformly \( k \)-lipschitzian semigroup on \( C \) with \( k < (1 + c_p)^{\frac{1}{p}} \), where the constant \( c_p \) is as in Theorem 7.

Then
\[
F(T) = \{x \in C : T_s x = x \quad \text{for all } s \in J\} \neq \emptyset
\]
and \( F(T) \) is a retract of \( C \).

**Proof** We may assume that \( k \geq 1 \) since if \( k < 1 \), the well known Banach Contraction Principle guarantees a fixed point of \( T \).

Define a sequence \( \{x_n\} \subset C \) in the following way: \( x_{n+1} \) is the asymptotic center of the net \( \{T_s x_n\}_{s \in J} \) in \( C \). Then, by Lemma 12, we have for \( x \in C \) and \( n = 1, 2, \ldots \)
\[
c_p \|x - x_{n+1}\|^p \leq \inf_{s \geq s} \sup_{t \geq s} \|T_s x_n - x\|^p - \inf_{s \geq s} \sup_{t \geq s} \|T_s x_n - x_{n+1}\|^p. \tag{10}
\]
Noting the inequality
\[
\inf_s \sup_{t \geq s} \|T_t y - x\|^p \leq \inf_s \sup_{t \geq s} \|T_{at} y - x\|^p
\]
is valid for all \(x, y \in C\) and every \(a \in J\). Putting \(x = T_ax_{n+1}\) into (10) we get
\[
c_p \|T_ax_{n+1} - x_{n+1}\|^p \leq \inf_s \sup_{t \geq s} \|T_tx_n - T_ax_{n+1}\|^p - \inf_s \sup_{t \geq s} \|T_tx_n - x_{n+1}\|^p
\leq \inf_s \sup_{t \geq s} \|T_tx_n - T_ax_{n+1}\|^p - \inf_s \sup_{t \geq s} \|T_tx_n - x_{n+1}\|^p
\leq (k^p - 1) \inf_s \sup_{t \geq s} \|T_tx_n - x_{n+1}\|^p
\leq (k^p - 1) \inf_s \sup_{t \geq s} \|T_tx_n - x\|^p
\]
and hence
\[
\|T_ax_{n+1} - x_{n+1}\|^p \leq \frac{k^p - 1}{c_p} \inf_s \sup_{t \geq s} \|T_tx_n - x\|^p
\leq M^{n+1} \inf_s \sup_{t \geq s} \|T_tx_0 - x_0\|^p, \quad (11)
\]
where \(M = \frac{k^p - 1}{c_p} < 1\) by assumption of the theorem. Inserting \(x = T_ax_{n-1}\) into (10) and in a similar way to above, we obtain
\[
\|T_ax_n - x_{n+1}\|^p \leq \frac{k^p}{c_p} \inf_s \sup_{t \geq s} \|T_tx_n - x_n\|^p. \quad (12)
\]
Combining (11) and (12) it follows that
\[
\|x_{n+1} - x_n\|^p \leq (\|x_{n+1} - T_ax_n\| + \|T_ax_n - x_n\|)^p
\leq 2^{p-1}(\|x_{n+1} - T_ax_n\|^p + \|T_ax_n - x_n\|^p)
\leq 2^{p-1} \cdot M^n \cdot \left(\frac{k^p}{c_p} + 1\right) \inf_s \sup_{t \geq s} \|T_tx_0 - x_0\|^p, \quad (13)
\]
which shows that \(\{x_n\}\) is Cauchy. Let \(z = \lim_{n \to +\infty} x_n\). Then for each \(a \in J\) we have
\[
\|z - T_az\|^p \leq (\|z - x_n\| + \|x_n - T_ax_n\| + \|T_ax_n - T_az\|)^p
\leq ((1 + k)\|z - x_n\| + \|x_n - T_ax_n\|)^p
\leq 2^{p-1}[(1 + k)^p\|z - x_n\|^p + \|x_n - T_ax_n\|^p]
\leq 2^{p-1}[(1 + k)^p\|z - x_n\|^p + M^n \cdot \inf_s \sup_{t \geq s} \|T_tx_0 - x_0\|^p] \to 0
\]
as \(n \to +\infty\). Therefore \(T_az = z\) for all \(a \in J\).
Note that if \(x_0 = x\) is a arbitrary point in \(C\), then \(x_m = A^m x\) for \(m = 1, 2, \ldots\) and by (13)
\[
\|A^{m+1}x - A^m x\| \leq 2^{p-1} \cdot M^m \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot \inf_s \sup_{t \geq s} \|T_tx - x\|^p
\leq 2^{p-1} \cdot M^m \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot (\text{diam}C)^p
\]
for \(m = 1, 2, \ldots\) Thus
\[
\sup_{x \in C} \|A^i x - A^m x\| \leq 2^{p-1} \cdot \frac{M^m}{1 - M} \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot (\text{diam}C)^p \to 0
\]
if $i, m \to +\infty$, which implies that the sequence $\{A^m x\}$ converges uniformly to a function

$$Rx = \lim_{m \to +\infty} A^m x, \quad x \in C.$$ 

It follows from Lemma 3 and Remark 4 that $R : C \to C$ is continuous. Moreover

$$\|Rx - T_a Rx\|^p \leq \left(\|Rx - A^m x\| + \|A^m x - T_a A^m x\| + \|T_a A^m x - T_a Rx\|\right)^p \leq (1 + k)^p \|Rx - A^m x\|^p + \|A^m x - T_a A^m x\|^p$$

$$\leq 2^{p-1} \left(1 + k\right)^p \cdot \|Rx - A^m x\|^p + \|A^m x - T_a A^m x\|^p$$

$$\leq 2^{p-1} \left(1 + k\right)^p \cdot \|Rx - A^m x\|^p + Mm \cdot \inf_{s \geq T} \sup_{t \geq s} \|T_t x - x\|^p \to 0$$

as $m \to +\infty$, and $Rx = T_a Rx$ for all $a \in J$ and $x \in C$. Thus $R$ is a retraction $C$ onto $F(T)$.

**Corollary 14** Let $H$ be a Hilbert space, $C$ a nonempty bounded closed convex subset of $H$ and $T = \{T_s : s \in J\}$ be a left reversible semi-topological uniformly $k$-lipschitzian semigroup on $C$ with $k < \sqrt{2}$. Then $F(T) \neq \emptyset$ and $F(T)$ is a retract of $C$.

**Corollary 15** Let $C$ be a nonempty bounded closed convex subset of $L^p$-space ($1 < p < \infty$) and $T = \{T_s : s \in J\}$ be a left reversible semi-topological uniformly $k$-lipschitzian semigroup on $C$. Suppose $k < \sqrt{p}$ if $1 < p \leq 2$, and $k < \left(1 + c_p\right)^{1-p}$ if $2 < p < +\infty$ (here $c_p$ is as in Theorem 8(b)). Then $F(T) \neq \emptyset$ and $F(T)$ is a retract of $C$.

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