The theory of pseudo-differential operators on the noncommutative n-torus

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The theory of pseudo-differential operators on the noncommutative n-torus

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Abstract. The methods of spectral geometry are useful for investigating the metric aspects of noncommutative geometry and in these contexts require extensive use of pseudo-differential operators. In a foundational paper, Connes showed that, by direct analogy with the theory of pseudo-differential operators on finite-dimensional real vector spaces, one may derive a similar pseudo-differential calculus on noncommutative n-tori, and with the development of this calculus came many results concerning the local differential geometry of noncommutative tori for \( n = 2, 4 \), as shown in the groundbreaking paper in which the Gauss–Bonnet theorem on the noncommutative two-torus is proved and later papers. Certain details of the proofs in the original derivation of the calculus were omitted, such as the evaluation of oscillatory integrals, so we make it the objective of this paper to fill in all the details. After reproving in more detail the formula for the symbol of the adjoint of a pseudo-differential operator and the formula for the symbol of a product of two pseudo-differential operators, we extend these results to finitely generated projective right modules over the noncommutative n-torus. Then we define the corresponding analog of Sobolev spaces and prove equivalents of the Sobolev and Rellich lemmas.

1. Introduction

The methods of spectral geometry are useful for investigating the metric aspects of noncommutative geometry \([1–4]\) and in these contexts require extensive use of pseudo-differential operators. In the foundational paper \([5]\), Connes showed that, by direct analogy with the theory \([6–8]\) of pseudo-differential operators on \( \mathbb{R}^n \), one may derive a similar pseudo-differential calculus on noncommutative n tori \( \mathbb{T}_\theta^n \). With the development of this calculus came many results concerning the local differential geometry of noncommutative tori for \( n = 2, 4 \), as shown in the groundbreaking paper \([9]\) in which the Gauss–Bonnet theorem on \( \mathbb{T}_\theta^2 \) is proved and later papers \([10–14]\). In these papers, the flat geometry of \( \mathbb{T}_\theta^n \) which was studied in \([5]\) is conformally perturbed using a Weyl factor given by a positive invertible smooth element in \( C^\infty(\mathbb{T}_\theta^n) \). Connes’ pseudo-differential calculus is critically used to apply heat kernel techniques to geometric operators on \( \mathbb{T}_\theta^n \) to derive small time heat kernel expansions that encode local geometric information such as scalar curvature. As discovered in \([10, 12, 13]\), a purely noncommutative feature that appears in the computations and in the final formula for the curvature is the modular automorphism of the state implementing the conformal perturbation of the metric.

Certain details of the proofs in the derivation of the calculus in \([5]\) were omitted, such as the evaluation of oscillatory integrals, so we make it the objective of this paper to fill in all the details.
After reproving in more detail the formula for the symbol of the adjoint of a pseudo-differential operator and the formula for the symbol of a product of two pseudo-differential operators, we extend these results to finitely generated projective right modules over the noncommutative n torus. Then we define the corresponding analog of Sobolev spaces for which we prove the Sobolev and Rellich lemmas.

We list these results below.

**Theorem 1.1.** Suppose that $P$ is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order $M$. Then the symbol of the adjoint $P^*$ is of order $M$ and satisfies $\sigma(P^*) \sim \sum_{\ell \in \mathbb{Z}_{\geq 0}^n} \frac{\partial^\ell [\rho(\xi)]}{\ell_1! \cdots \ell_n!}$. 

**Theorem 1.2.** Suppose that $P$ is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order $M_1$, and $Q$ is a pseudo-differential operator with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order $M_2$. Then the symbol of the product $QP$ is of order $M_1 + M_2$ and satisfies $\sigma(QP) \sim \sum_{\ell \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\ell_1! \cdots \ell_n!} \phi(\xi) \delta^\ell(\xi)$, where $\delta^\ell := \prod_j \delta_j^{\ell_j}$ and $\delta_j := \delta_j^{\ell_j}$.

**Theorem 1.3.** (a) For a pseudo-differential operator $P$ with $r \times r$ matrix valued symbol $\sigma(P) = \rho = \rho(\xi)$, the symbol of the adjoint $P^*$ satisfies 

$$\sigma(P^*) \sim \sum_{(\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geq 0}^n} \frac{\partial^{\ell_1} \cdots \partial^{\ell_n} \delta(\xi) \delta^{\ell_1} \cdots \delta^{\ell_n}(\rho(\xi))^*}{\ell_1! \cdots \ell_n!}.$$ 

(b) If $Q$ is a pseudo-differential operator with $r \times r$ matrix valued symbol $\sigma(Q) = \rho' = \rho'(\xi)$, then the product $PQ$ is also a pseudo-differential operator and has symbol 

$$\sigma(PQ) \sim \sum_{(\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geq 0}^n} \frac{\partial^{\ell_1} \cdots \partial^{\ell_n} \delta(\xi) \delta^{\ell_1} \cdots \delta^{\ell_n}(\rho'(\xi))}{\ell_1! \cdots \ell_n!}.$$ 

**Theorem 1.4.** For $s > k + 1$, $H^s \subseteq (A_0^\infty)'e$.

**Theorem 1.5.** Let $\{\tilde{a}_N\} \in (A_0^\infty)'e$ be a sequence. Suppose that there is a constant $C$ so that $||\tilde{a}_N||_s \leq C$ for all $N$. Let $s > t$. Then there is a subsequence $\{\tilde{a}_{N_i}\}$ that converges in $H^t$.

2. Preliminaries

Fix some skew symmetric $n \times n$ matrix $\theta$ with upper triangular entries in $\mathbb{R} \setminus \mathbb{Q}$ that are linearly independent over $\mathbb{Q}$. Consider the irrational rotation $C^\ast$-algebra $A_\theta$ with $n$ unitary generators $U_1, \ldots, U_n$ which satisfy $U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k$ and $U_j^* = U_j^{-1}$. Let $\{\alpha_s\}_{s \in \mathbb{R}}$ be a $n$-parameter group of automorphisms given by $\prod_j U_j^{m_j} \mapsto e^{is \cdot \prod_j U_j^{m_j}}$. We define the subset $A_\theta^\infty$ of $C^\ast$ elements to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is $C^k$, and we define the subalgebra $A_\theta^\infty$ of smooth elements of $A_\theta$ to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is smooth. An alternative definition of the subalgebra $A_\theta^\infty$ of smooth elements is the elements in $A_\theta$ that can be expressed by an expansion of the form $\sum_{m \in \mathbb{Z}^n} a_{m} \prod_j U_j^{m_j}$, where the sequence $\{a_m\}_{m \in \mathbb{Z}^n}$ is in the Schwartz space $S(\mathbb{Z}^n)$ in the sense that, for all $\alpha \in \mathbb{Z}^n$, $\sup_{m \in \mathbb{Z}^n} (\prod_j |m_j|^{\alpha_j} |a_{m}|) < \infty$. Define the trace $\tau : A_\theta \to \mathbb{C}$ by $\tau(\prod_j U_j^{m_j}) = 0$ for $m_j$ not all zero and $\tau(1) = 1$ and define an inner product $\langle \cdot, \cdot \rangle : A_\theta \times A_\theta \to \mathbb{C}$ by $\langle a, b \rangle = \tau(b^* a)$ with induced norm $|| \cdot || : A_\theta \to \mathbb{R}_{\geq 0}$. Let $D_j = -i \partial_j$ and define derivations $\delta_j$ by the relations $\delta_j(U_j) = U_j$ and $\delta_j(U_k) = 0$ for $j \neq k$. For convenience, denote $\delta^\ell := \prod_j \delta_j^{\ell_j}$, $\delta^\ell := \prod_j \delta_j^{\ell_j}$, and $\ell! := \prod_j \ell_j!$ for multi-indices $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$. We define a map $\psi : \rho \mapsto P_\rho$ assigning a pseudo-differential operator on $A_\theta^\infty$ to a symbol $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$.

**Definition 2.1.** For $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$, let $P_\rho$ be the pseudo-differential operator sending arbitrary $a \in A_\theta^\infty$ to $P_\rho(a) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i \xi \cdot \rho(\xi)} \alpha_s(a) \, ds \, d\xi$. 

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The integral above does not converge absolutely; it is an oscillatory integral. We define oscillatory integrals below as in [7].

**Definition 2.2.** Let $q$ be a nondegenerate real quadratic form on $\mathbb{R}^n$, $a$ be a $C^\infty$ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1 + |x|^2)^{-m/2} \partial^a a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$, and $\varphi$ be a Schwartz function, i.e. the functions $x^\alpha \partial^\beta \varphi(x)$ are bounded on $\mathbb{R}^n$ for all pairs $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. Suppose further that $\varphi(0) = 1$. Then the limit $\lim_{x \to 0} \int e^{iq(x)} a(x) \varphi(\mathbf{cx}) \, dx$ exists, is independent of $\varphi$ (as long as $\varphi(0) = 1$), and is equal to $\int e^{iq(x)} a(x) \, dx$ when $a \in L^1$. When $a \not\in L^1$, we continue to denote this limit by $\int e^{iq(x)} a(x) \, dx$, and have an estimate $|\int e^{iq(x)} a(x) \, dx| \leq C_{q,m} \max_{|\alpha| \leq m+n+1} \inf \{U \in \mathbb{R} : |(1 + |x|^2)^{-m/2} \partial^\alpha a| \leq U \text{ almost everywhere}\}$ where $C_{q,m}$ depends only on the quadratic form $q$ and the order $m$.

As shown in [7], oscillatory integrals behave essentially like absolutely convergent integrals in that one can still make changes of variables, integrate by parts, differentiate under the integral sign, and interverse integral signs. Given certain conditions on $\rho$, $P_\rho(a)$ satisfies the conditions of the oscillatory integral, and we can evaluate $P_\rho(a)$.

**Definition 2.3.** An element $\rho = \rho(\xi) = \rho(\xi_1, \ldots, \xi_n)$ of $C^\infty(\mathbb{R}^n, A^{\infty}_\theta)$ is a symbol of order $m$ if and only if for all non-negative integers $i_1, i_2, j_1, \ldots, j_n$, $||\delta_{i_1}^\xi \cdots \delta_{i_n}^\xi (\partial_{j_1}^\xi \cdots \partial_{j_n}^\xi \rho)(\xi)|| \leq C_\rho(1 + |\xi|)^{m-|\beta|}$.

Example 2.6(i) of [7] gives a convenient formula for evaluating the oscillatory integrals that appear in our calculation of $P_\rho(a)$.

**Proposition 2.4.** Suppose that, for some $m$, $a$ is a $C^\infty$ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1 + |x|^2)^{-m/2} \partial^a a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. Then $\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} a(y) \, dy \, d\eta = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy \cdot \eta} a(y) \, dy \, d\eta = a(0)$.

We apply Proposition 2.4 to get a basic result.

**Lemma 2.5.** Let $a = \sum_{m \in \mathbb{Z}^n} a_m \prod_j U_j^{m_j}$ be an arbitrary element of $A^{\infty}_\theta$ and let $\rho \in C^\infty(\mathbb{R}^n, A^{\infty}_\theta)$ be a symbol of order $M$. Then $P_\rho(a) = \sum_{m \in \mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j}$.

**Proof.** First consider the case $a = \prod_j U_j^{m_j}$. We get

$$P_\rho \left( \prod_{j=1}^n U_j^{m_j} \right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \rho(\xi) \alpha_s \left( \prod_{j=1}^n U_j^{m_j} \right) \, ds \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \rho(\xi) e^{is \cdot m} \prod_{j=1}^n U_j^{m_j} \, ds \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi - m} \rho(\xi) \, ds \, d\xi \prod_{j=1}^n U_j^{m_j}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \rho(\xi + m) \, ds \, d\xi \prod_{j=1}^n U_j^{m_j} = \rho(m) \prod_{j=1}^n U_j^{m_j},$$

as desired, having substituted $\eta = \xi - m$ and applied the result of Proposition 2.4. Now consider the general case $a = \sum_{m \in \mathbb{Z}^n} a_m \prod_j U_j^{m_j}$. Since $\alpha_s$ is an automorphism on $A_\theta$, we get $P_\rho(a) = \sum_{m \in \mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j}$, and we are done. 

\[ \square \]
3. Asymptotic formula for the symbol of the adjoint of a pseudo-differential operator

Here we prove the formula for the symbol of the adjoint for the noncommutative n torus, adapting the proof of Lemma 1.2.3 of [6] to the noncommutative n torus.

**Theorem 3.1.** Suppose $P$ is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order $M$. Then the symbol of the adjoint $P^*$ is of order $M$ and satisfies $\sigma(P^*) \sim \sum_{\ell \in \mathbb{Z}_{\geq 0}} \frac{\partial^\ell \delta[(\rho(\xi))^*]}{\ell!}$.

**Proof.** Let $a, b \in A^\infty_\theta$. We have

$$\langle P_\rho(a), b \rangle = \tau \left( b^* \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\xi} \rho(\xi) \alpha_s(a) \, ds \, d\xi \right)$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\xi} \tau(b^* \rho(\xi) \alpha_s(a)) \, ds \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\xi} \tau(\alpha_s(\rho(\xi)^*b)a) \, ds \, d\xi$$

$$= \tau \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\xi} \alpha_s(\rho(\xi)^*b) \, ds \, d\xi \right)^* a \rangle \langle a, P_\rho^*(b) \rangle$$

where

$$P_\rho^*(b) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\xi} \alpha_s(\rho(\xi)^*b) \, ds \, d\xi$$

so

$$\sigma(P_\rho^*)(\xi) = \left[ \sum_m \rho_m(\xi - m) \prod_{j=1}^n U_j^{m_j} \right]^* \alpha_x \left[ \prod_{j=1}^n U_j^{m_j} \right] \frac{d\xi \, dy}{\mathbb{R}^n}$$

$$= \left[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\xi \alpha_s(\rho(\xi)^*b)} \, ds \, d\xi \right]^*.$$

We have $\rho(\xi - y) = \sum_{|\ell| < N_1} \frac{(-y)^\ell}{\ell!} (\partial^\ell \rho)(\xi) + R_{N_1}(\xi, y)$ and

$$\alpha_x(\rho(\xi - y)) = \sum_{|\ell| < N_1} \sum_m \frac{(-y)^\ell}{\ell!} (\partial^\ell \rho_m)(\xi) e^{ix \cdot m} \left[ \prod_{j=1}^n U_j^{m_j} \right] + \alpha_x(R_{N_1}(\xi, y)).$$
where \( R_{N_1}(\xi, y) = N_1 \sum_{|\ell| = N_1} x_{\ell} \left( \frac{(-\partial^{|\ell|}}{D^{|\ell|}} \right) (1 - \gamma)^{N_1-1}(\partial^{|\ell|}\rho)(\xi - y \gamma) \right) \) dy with corresponding symbol

\[
\sigma(P^{\ast}_{\rho})(\xi) = \left[ \sum_{|\ell| < N_1} \sum_m (-m)^{\ell}! (\partial^{|\ell|}\rho_m)(\xi) \prod_{j=1}^m U_j^{m_j} + T_{N_1}(\xi, y) \right]^* = \sum_{|\ell| < N_1} \frac{\partial^{|\ell|} \delta^{|\ell|}(\rho(\xi))^*}{\ell !} + T_{N_1}(\xi, y)^* \]

where \( T_{N_1}(\xi, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \alpha_x(R_{N_1}(\xi, y)) \) dy. It remains to show that this symbol is of order \( M \). Obviously, \( \sum_{N \leq \ell < N_1} \frac{\partial^{|\ell|} \delta^{|\ell|}(\rho(\xi))^*}{\ell !} \in S^{M-N} \), so we need to show that the remainder is of order \( M - N_1 \). Note that \( \sigma(P^{\ast}_{\rho})(\xi) - \sum_{|\ell| < N_1} \frac{\partial^{|\ell|} \delta^{|\ell|}(\rho(\xi))^*}{\ell !} = T_{N_1}(\xi, y)^* \). Integrating by parts,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \left( \frac{(-y)^{\ell}}{\ell !} \right) \alpha_x \left( \int_0^1 (1 - \gamma)^{N_1-1}(\partial^{|\ell|}\rho)(\xi - y \gamma) \right) \) \ dx \ dy
\]

\[
= \frac{1}{\ell !} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \left( \frac{(-y)^{\ell}}{\ell !} \right) \alpha_x \left( \int_0^1 (1 - \gamma)^{N_1-1}(\partial^{|\ell|}\rho)(\xi - y \gamma) \right) \) \ dx \ dy
\]

\[
= \frac{1}{\ell !} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \left( \frac{(-y)^{\ell}}{\ell !} \right) \alpha_x \left( \int_0^1 (1 - \gamma)^{N_1-1}(\partial^{|\ell|}\rho)(\xi - y \gamma) \right) \) \ dx \ dy
\]

\[
= \frac{1}{\ell !} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \left( \frac{(-y)^{\ell}}{\ell !} \right) \alpha_x \left( \int_0^1 (1 - \gamma)^{N_1-1}(\partial^{|\ell|}\rho)(\xi - y \gamma) \right) \) \ dx \ dy
\]

where, for arbitrary \( a = \sum_m a_m \prod_j U_j^{m_j} \),

\[
(-D_x)^{\ell} \alpha_x(a) = (-D_x)^{\ell} \alpha_x \left( \sum_m a_m \prod_j U_j^{m_j} \right) = (-D_x)^{\ell} \sum_m (-m)^{\ell} \alpha_x \left( \prod_j U_j^{m_j} \right) = \sum_m (-m)^{\ell} \alpha_x \left( \prod_j U_j^{m_j} \right)
\]

Since \( \rho \in S^M \) and \( |\ell| = N_1 \) we have \( \delta^{|\ell|} \partial^{|\ell|} \rho \in S^{M-N_1} \). We get the boundedness of \( \delta^{N_1-M}(\xi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \alpha_x(R_{N_1}(\xi, y)) \) dy because \( \partial^{|\ell|} R_{N_1}(y, \xi) \) is the rest of index \( N_1 \) in the Taylor expansion of \( \partial^{|\ell|}(\xi - y \gamma) \) for which one has \( \partial^{|\ell|} \rho \in S^{M-N_1} \). □

4. Asymptotic formula for the symbol of a product of two pseudo-differential operators

Next we prove the formula for the product or composition of symbols for the noncommutative \( n \) torus, adapting the proof of Theorem 7.1 of [8].

**Theorem 4.1.** Suppose that \( P \) is a pseudo-differential operator with symbol \( \sigma(P) = \rho = \rho_0(\xi) \) of order \( M_1 \), and \( Q \) is a pseudo-differential operator with symbol \( \sigma(Q) = \phi = \phi_0(\xi) \) of order \( M_2 \). Then the symbol of the product \( PQ \) is of order \( M_1 + M_2 \) and satisfies \( \sigma(PQ) \sim \sum_{|\ell| < N_1} \frac{1}{\ell !} \partial^{|\ell|} \phi(\xi) \delta^{|\ell|} \rho(\xi) \), where \( \partial^{|\ell|} := \prod_j \partial^{|\ell_j|}_j \) and \( \delta^{|\ell|} := \prod_j \delta^{|\ell_j|}_j \).

**Proof.** We want to show that if \( \rho : \mathbb{R}^n \to A^\infty_0 \) is of order \( M_1 \) and \( \phi : \mathbb{R}^n \to A^\infty_0 \) is of order \( M_2 \), \( P_\phi \circ P_\rho = P_{\mu} \) where \( \mu \) is of order \( M_1 + M_2 \) and has asymptotic expansion \( \mu \sim \sum_{|\ell| < N_1} \frac{1}{\ell !} \partial^{|\ell|} \phi_0(\xi) \delta^{|\ell|} \rho_0(\xi) \). Let \( \{ \phi_k \} \) be the partition of unity constructed in Theorem 6.1 of [8] and define \( \phi_k(\xi) := \phi(\xi) \varphi_k(\xi) \). We have \( P_\phi(a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \sum_{k=0}^{\infty} \phi_k(\xi) \alpha_s(a) \) d\xi. Summing over \( k \) from zero to infinity and applying Fubini’s Theorem, we get \( \sum_{k=0}^{\infty} P_\phi(a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \sum_{k=0}^{\infty} \phi_k(\xi) \alpha_s(a) \) d\xi =
\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi(\xi) \alpha_s(a) \, ds \, d\xi, \] so \( P_\phi(a) = \sum_{k=0}^\infty P_{\phi_k}(a) \) and the convergence of the series is absolute and uniform for all \( a \in A^\infty_\theta \). We want to compute the symbol of \( P_\phi \circ P_\rho \), but issues with convergence of integrals make it so we need to compute the symbol of \( P_{\phi_k} \circ P_\rho \). Let \( a \in A^\infty_\theta \) be arbitrary. Applying \( P_{\phi_k} \circ P_\rho \) we get

\[
P_{\phi_k}(P_\rho(a)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is \cdot \xi} \phi_k(\xi) \alpha_s(P_\rho(a)) \, ds \, d\xi
\]

where \( \mu_k(\tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi_k(\sigma + \tau) \alpha_x(\rho(\tau)) \, dx \, d\sigma \), having done the changes of variables \((x, y) = (s, s + t)\) and \((\sigma, \tau) = (\xi - \eta, \eta)\) and applied Proposition 2.4. This suggests that \( P_{\phi_k}(P_\rho(a)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy \cdot \eta} \mu_k(\eta) \alpha_y(a) \, d\eta \), where \( \mu_k(\tau) = \sum_{k=0}^\infty \mu_k(\tau) \). We need to show that \( \mu \) is a symbol in \( S^{M_1+M_2} \) and has our desired asymptotic expression. Define \( \mu_k(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} \phi_k(\xi + \eta) \alpha_x(\rho(\xi)) \, dx \) for all \( \xi \in \mathbb{R}^n \). By Taylor’s formula with integral remainder given in Theorem 6.3 of [8], we get \( \phi_k(\xi + y) = \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) + R_{N_1}(y, \xi) \) where \( R_{N_1}(y, \xi) = N_1 \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} \int_0^1 (1 - \gamma)^N_1 - 1 (\partial^\ell \phi_k)(\xi + \gamma y) \, d\gamma \) for all \( y, \xi \in \mathbb{R}^n \). Substituting back into our expression for \( \mu_k(\xi) \) we get \( \mu_k(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) \alpha_x(\rho(\xi)) \, dx \, dy + T^{(k)}_{N_1}(\xi) \) where \( T^{(k)}_{N_1}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} R_{N_1}(y, \xi) \alpha_x(\rho(\xi)) \, dx \, dy \). Expressing \( \rho(\xi) \) as \( \rho(\xi) = \sum_m \rho_m(\xi) \Pi_{j=1}^m U_{ij}^{m_j} \), we see that \( \alpha_x(\rho(\xi)) = \sum_m \rho_m(\xi) e^{ix \cdot m} \Pi_{j=1}^m U_{ij}^{m_j} \) so

\[
\mu_k(\xi) - T^{(k)}_{N_1}(\xi) = \sum_{|\ell| < N_1} \sum_m \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) \rho_m(\xi) e^{ix \cdot m} \Pi_{j=1}^m U_{ij}^{m_j} \, dx \, dy
\]

where \( \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) (\partial^\ell \rho)(\xi) \in S^{M_1+M_2-|\ell|} \), so we just need to show that the remainder is of order \( M_1 + M_2 - N_1 \). Note that \( \mu_k(\xi) - \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi)(\xi) (\partial^\ell \rho)(\xi) = \sum_{k=0}^\infty T^{(k)}_{N_1}(\xi) \) where \( T^{(k)}_{N_1}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot \eta} R_{N_1}(y, \xi) \alpha_x(\rho(\xi)) \, dx \, dy \) and \( R_{N_1}(y, \xi) = N_1 \sum_{|\ell| = N_1} \frac{y^\ell}{\ell!} \int_0^1 (1 - \gamma)^N_1 - 1 (\partial^\ell \phi_1)(\xi + \gamma y) \, d\gamma \).
γ)^{N_1-1}(∂^ℓ φ_k)(ξ + γ y) dγ. Integrating by parts,
\[
\int_{\mathbb{R}^n} e^{-ix y} \frac{y^ℓ}{ℓ!} \int_0^1 (1 - γ)^{N_1-1}(∂^ℓ φ_k)(ξ + γ y) dγ α_x(ρ(ξ)) dx dy
\]
\[
= \frac{1}{ℓ!} \int_{\mathbb{R}^n} \int_0^1 (1 - γ)^{N_1-1}(∂^ℓ φ_k)(ξ + γ y) dγ D^ℓ_x α_x(ρ(ξ)) dx dy
\]
where for arbitrary a = \( \sum a_m \prod_j U_j^{m_j} \) we have
\[
(D_x)^ℓ α_x(a) = (D_x)^ℓ α_x \left( \sum m \prod_{j=1}^n U_j^{m_j} \right) = (D_x)^ℓ \sum m e^{ix m} \prod_{j=1}^n U_j^{m_j}
\]
\[
= \sum m a_m m^ℓ e^{ix m} \prod_{j=1}^n U_j^{m_j} = δ^ℓ α_x(a).
\]

Since |ℓ| = N_1, we have \( ∂^ℓ φ_k \in S^{M_2-N_1} \) and \( δ^ℓ ρ \in S^{M_1} \). We get the boundedness of \( μ_{N_1-M_1-M_2}(ξ) \int_{\mathbb{R}^n} e^{-ix y} R_{N_1}(y,ξ) α_x(ρ(ξ)) dx dy \) since \( δ^ℓ ρ \in S^{M_1} \) and \( ∂^ℓ R_{N_1}(y,ξ) \) is the rest of index N_1 in Taylor’s expansion of \( ∂^ℓ φ_k(ξ + γ y) \) for which on has \( ∂^ℓ φ_k \in S^{M_2-N_1} \).

5. The pseudo-differential calculus on finitely generated projective modules over the noncommutative n torus

We can generalize these results to arbitrary finitely generated projective right modules over the noncommutative n torus following p. 553 of [2], which considers finitely generated projective modules over an arbitrary unital *-algebra. Let E be a finitely generated projective right \( A^\infty_θ \)-module. Since E is a finitely generated projective right \( A^\infty_θ \)-module, we can write E as a direct summand \( E = (A^\infty_θ)^r e \) of a free module \( (A^\infty_θ)^r \) with direct complement \( F = (A^\infty_θ)^r (id - e) \), where the idempotent \( e \in M_r(A^\infty_θ) \) is self-adjoint. Consider an \( r \times r \) matrix valued symbol \( ρ = (ρ_{j,k}) \) where \( ρ_{j,k} : \mathbb{R}^n \to A^\infty_θ \) are scalar symbols and \( ρ_{j,k} \in S^d \). Define the operator \( P_ρ : E \to E \) as follows: \( P_ρ(\tilde{a}) := (2π)^{-n} \int_{\mathbb{R}^n} e^{-isξ} ρ(ξ) α_s(\tilde{a}) ds dξ \). Define the inner product \( \langle \tilde{a}, \tilde{b} \rangle : E \times E \to \mathbb{C} \) sending \( (\tilde{a}, \tilde{b}) \to ρ(\tilde{b}^* \tilde{a}) \). Since \( P_ρ(\tilde{a})_j = (2π)^{-n} \int_{\mathbb{R}^n} e^{-isξ} \sum_{k=1}^r ρ_{j,k}(ξ) α_s(\tilde{a}_k) ds dξ \), Lemma 2.5 generalizes to E as follows after applying it to each component: \( P_ρ(\tilde{a}) = \sum m ρ(m)\tilde{a}_m \prod_{j=1}^n U_j^{m_j} \). Theorems 3.1 and 4.1 generalize as follows.

**Theorem 5.1.** (a) For a pseudo-differential operator \( P \) with \( r \times r \) matrix valued symbol \( σ(P) = ρ = ρ(ξ) \), the symbol of the adjoint \( P^* \) satisfies
\[
σ(P^*) \sim \sum_{(ℓ_1, ..., ℓ_n) \in (\mathbb{Z}_{≥0})^n} \frac{∂_1^{ℓ_1} \cdots ∂_n^{ℓ_n} δ_1^{ℓ_1} \cdots δ_n^{ℓ_n} (ρ(ξ))^*}{ℓ_1! \cdots ℓ_n!}.
\]
(b) If \( Q \) is a pseudo-differential operator with \( r \times r \) matrix valued symbol \( σ(Q) = ρ' = ρ'(ξ) \), then the product \( PQ \) is also a pseudo-differential operator and has symbol
\[
σ(PQ) \sim \sum_{(ℓ_1, ..., ℓ_n) \in (\mathbb{Z}_{≥0})^n} \frac{∂_1^{ℓ_1} \cdots ∂_n^{ℓ_n} (ρ(ξ)) δ_1^{ℓ_1} \cdots δ_n^{ℓ_n} (ρ'(ξ))}{ℓ_1! \cdots ℓ_n!}.
\]
Proof. First let’s prove part (a). Let ρ be an \( r \times r \) matrix valued symbol of order \( M \) and \( \vec{a}, \vec{b} \in E \). We have

\[
\langle P_\rho(\vec{a}), \vec{b} \rangle = \tau \left( \vec{b}^* \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \rho(\xi) \alpha_s(\vec{a}) \, ds \, d\xi \right)
= \tau \left( \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \alpha_{-s}(\rho(\xi)^\ast \vec{b}) \, d\xi \right)^* \vec{a} \right) = \langle \vec{a}, P_\rho^\ast(\vec{b}) \rangle
\]

where

\[
P_\rho^\ast(\vec{b})_j = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \sum_{k=1}^r \alpha_{-s}(\rho(\xi))_j b_k \, ds \, d\xi
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \sum_{k=1}^r \alpha_{-s}(\rho(\xi))_j \alpha_{-s}(b_k) \, ds \, d\xi
= \sum_{m,p} \left[ \rho_m(p - m) \prod_{h=1}^n U_{m_h} \right]^* \left( b_p \prod_{h=1}^n U_{p_h} \right)
\]

so

\[
\sigma(P_\rho^\ast)(\xi) = \left[ \sum_m \rho_m(\xi - m) \prod_{h=1}^n U_{m_h} \right]^* = \left[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\cdot\cdot\cdot} \sum_m \rho_m(\xi - y) \alpha_x \left( \prod_{h=1}^n U_{m_h} \right) \, dx \, dy \right]^*
= \left[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\cdot\cdot\cdot} \alpha_x(\rho(\xi - y)) \, dx \, dy \right]^*.
\]

The rest of the proof reduces to the \( r = 1 \) case, applying it to each entry in \( \rho = (\rho_{j,k}) \).

We proceed to part (b). Let \( \rho \) be an \( r \times r \) matrix valued symbol of order \( m_1 \) and \( \phi \) be an \( r \times r \) matrix valued symbol of order \( m_2 \). Let \( \{ \varphi_k \} \) be a partition of unity and define \( \phi_k(\xi) := \phi(\xi) \varphi_k(\xi) \). Let \( \vec{a} \in E \). We have

\[
P_{\phi_k}(P_\rho(\vec{a})) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \phi_k(\xi) \alpha_x(P_\rho(\vec{a})) \, ds \, d\xi
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \phi_k(\xi) \alpha_x \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-it\cdot\eta} \rho(\eta) \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, dt \, d\eta \right) \, ds \, d\xi
= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi - it\cdot\eta} \phi_k(\xi) \alpha_x(\rho(\eta)) \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, dt \, d\eta \right\} \, ds \, d\xi
= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi - iy\cdot\eta} \phi_k(\xi) \alpha_x(\rho(\eta)) \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, dx \, dy \right\} \, ds \, d\xi
= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\cdot\cdot - iy\cdot\cdot\cdot} \phi_k(\sigma + \tau) \alpha_x(\rho(\tau)) \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, dx \, dy \right\} \, ds \, d\sigma
= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\tau} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\cdot\cdot - iy\cdot\cdot\cdot} \phi_k(\sigma + \tau) \alpha_x(\rho(\tau)) \, dx \, d\sigma \right\} \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, d\tau
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\tau} \lambda_k(\tau) \alpha_{\cdot\cdot\cdot} \eta \alpha_{\cdot\cdot\cdot} \eta \, d\tau
where \( \lambda_k(\tau) = \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} e^{-ix\cdot\sigma} \phi_k(\sigma + \tau) d\sigma \) so

\[
P_\phi(P_\rho(\vec{a})) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\tau} \lambda(\tau) d\gamma d\tau
\]

where \( \lambda(\tau) = \sum_{k=0}^\infty \lambda_k(\tau) \). Let \( \lambda_k(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \phi_k(\xi + \tau) d\gamma d\tau \). Since

\[
\lambda_k(\xi)_{r,\gamma} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \sum_{\beta=1}^r \phi_k(\xi + \tau)_{\alpha,\beta} \alpha_x(\rho(\xi)_{\beta,\gamma}) d\gamma d\tau
\]

the rest of the proof reduces to the \( r = 1 \) case, applying it to each summand in the above sum.

\[\square\]

6. Sobolev spaces on finitely generated projective modules over the noncommutative n torus

Let \( \lambda(\xi) = (1 + \ell^2 + \cdots + \ell^2_n)^{1/2} d\xi \). Consider the following inner product on \( E \).

**Definition 6.1.** Define the Sobolev inner product \( \langle \cdot, \cdot \rangle_s : E \times E \to \mathbb{C} \) by \( \langle \vec{a}, \vec{b} \rangle_s := \langle P_{\lambda^s}(\vec{a}), P_{\lambda^s}(\vec{b}) \rangle = \sum_{j,m} (1 + |m|^2 + \cdots + |m|^2) b_{j,m} a_{j,m} \).

Note that for \( s = 0 \) this agrees with \( \langle \cdot, \cdot \rangle \). This inner product induces the following norm.

**Definition 6.2.** Define the Sobolev norm \( || \cdot ||_s : E \to \mathbb{R}_{\geq 0} \) by \( ||\vec{a}||_s^2 := \langle P_{\lambda}(\vec{a}), P_{\lambda}(\vec{a}) \rangle = \sum_{j,m} (1 + |m|^2 + \cdots + |m|^2) b_{j,m} a_{j,m}^2 \).

Using this norm, we can define the analog of Sobolev spaces on \( E \).

**Definition 6.3.** Define the Sobolev space \( H^s \) to be the completion of \( E \) with respect to \( || \cdot ||_s \).

We can prove that a pseudo-differential operator of order \( d \in \mathbb{R} \) continuously maps \( H^s \) into \( H^{s-d} \). However we must first prove the case where \( s = d \).

**Theorem 6.4.** Suppose \( \rho \) is a matrix valued symbol of order \( d \). Then, for any \( \vec{a} \in E \), \( ||P_\rho(\vec{a})||_0 \leq C||\vec{a}||_d \) for some constant \( C > 0 \) and \( P_\rho \) defines a bounded operator \( P_\rho : H^d \to H^0 \).

**Proof.** Let \( F \) be an orthogonal eigenbasis of \( e \) normalized with respect to \( \langle \cdot, \cdot \rangle \), and let \( F_1 := \{ F_h : 1 \leq h \leq r_1 \} \) be the subset of eigenvectors with eigenvalue 1. Note that \( \{ \Pi_g U_{g}^m f_j \}_{j=1}^{r_1} \) is an orthogonal basis of \( E \) considered as a \( \mathbb{C} \)-vector space, with respect to \( \langle \cdot, \cdot \rangle_s \). We have \( \| \Pi_g U_{g}^m f_j \|^2 = 1 \), \( \| \rho_m(\xi) h_j \| \Pi_g U_{g}^m f_j \|^2 = \| \rho_m(\xi) h_j \|^2 \), and \( \| \rho(\xi) h_j f_j \|^2 = \sum_m | \rho_m(\xi) h_j |^2 \).

Since \( \rho_h j \) is of order \( d \), we have \( \| \rho(\xi) h_j f_j \|^2 \leq C_\rho (1 + |\xi|^2)^d \), and since \( (1 + |\xi|^2)^d \geq 0 \) gives us \( (1 + |\xi|^2)^d \leq 2(1 + |\xi|^2) \), we have \( \| \rho(\xi) h_j f_j \|^2 \leq C_\rho^2 (1 + |\xi|^2)^d \leq C_\rho^2 (1 + |\xi|^2)^d \). Let \( k_\rho := C_\rho^2 \).

Then we have \( \sum_m | \rho_m(\xi) h_j |^2 \geq k_\rho (1 + |\xi|^2)^d \). Let \( e_{s,m} := (1 + |m|^2 + \cdots + |m|^2)^{-s/2} \Pi_g U_{g}^m \) and \( E_s := \{ e_{s,m} \}_{m \in \mathbb{Z}} \). By definition we have \( E_s F_1 \) orthonormal with respect to \( \langle \cdot, \cdot \rangle_s \).

It suffices to prove this theorem for the case \( \vec{a} = e_{d,m} f_j \) by the orthonormality of \( E_s F_1 \) since \( ||P_\rho(\vec{a})||_0^2 = \sum_m |a_{j,m}|^2 ||P_\rho(e_{d,m} f_j)||_0^2 \) and \( ||\vec{a}||_0^2 = \sum_m |a_{j,m}|^2 ||e_{d,m} f_j||_2^2 \). Since \( ||e_{d,m} f_j||_2^2 = 1 \), it suffices to show that \( ||P_\rho(e_{d,m} f_j)||_2^2 \leq K \) for some constant \( K > 0 \).

\[
||P_\rho(e_{d,m} f_j)||_2^2 = ||\rho(m)e_{d,m} f_j||_0^2 = \rho(m)(1 + |m|^2 + \cdots + |m|^2)^{-d/2} \Pi_g U_{g}^m f_j \leq K
\]

for each \( k \) since \( ||\rho(m) \Pi g U_{g}^m f_j||_0^2 = \sum_k \rho_k(m) \Pi g U_{g}^m f_j ||^2 \leq K \) for some constant \( K > 0 \).
\[
(1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left| \sum_k \rho_k(m) \prod_g U_g^{k_g} \prod_j U_j^{m_j} f_j \right|^2_0
\]

\[
= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left| \sum_k \rho_k(m)w(m, k) \prod_g U_g^{k_g+m_g} f_j \right|^2_0
\]

\[
= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left| \sum_k \rho_{k-m}(m)w(m, k-m) \prod_g U_g^{k_g} f_j \right|^2_0
\]

\[
= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left| \sum_{h,k} \rho_{k-m}(m)_{h,j}w(m, k-m) \prod_g U_g^{k_g} f_j \right|^2_0
\]

\[
= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \sum_{h,k} |\rho_{k-m}(m)_{h,j}|^2
\]

\[
\leq (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} r_1 k_{\rho}(1 + |m_1|^2 + \cdots + |m_n|^2)^d = r_1 k_{\rho}
\]

where \( w(k, m) := \prod_{j=1}^n U_j^{m_j} \prod_{j=1}^n U_j^{k_j} \left( \prod_{j=1}^n U_j^{m_j+k_j} \right)^{-1} \in S^1 \subset C \) so our desired constant is \( K = r_1 k_{\rho} = r_1 C_{g,2}^d \) and we are done. \( \square \)

For the general case \( s \neq d \) we need to prove a lemma saying that \( || \cdot ||_s = || \cdot ||_{s-t} \circ P_{\lambda^t} \).

**Lemma 6.5.** For any \( \bar{a} \in E \) and \( s, t \in R, \bar{a} \in H^s \) if and only if \( P_{\lambda^t}(\bar{a}) \in H^{s-t} \) with \( ||\bar{a}||_s = ||P_{\lambda^t}(\bar{a})||_{s-t} \).

**Proof.** Suppose that \( \bar{a} \in H^s \) or \( P_{\lambda^t}(\bar{a}) \in H^{s-t} \). Then

\[
||P_{\lambda^t}(\bar{a})||_{s-t}^2 = \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^{s-t} \lambda^{2t}(m)|a_{j,m}|^2
\]

\[
\leq \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^{s-t}(1 + |m_1|^2 + \cdots + |m_n|^2)^t|a_{j,m}|^2
\]

\[
\leq (1 + |m_1|^2 + \cdots + |m_n|^2)^t|a_{j,m}|^2 = ||\bar{a}||_s
\]

so we know that \( \bar{a} \in H^s \) and \( P_{\lambda^t}(\bar{a}) \in H^{s-t} \). \( \square \)

Then the general case follows quite easily.

**Corollary 6.6.** Suppose \( \rho \) is a matrix valued symbol of order \( d \). Then \( ||P_{\lambda^s}(\bar{a})||_{s-d} \leq C ||\bar{a}||_s \) for some constant \( C > 0 \) and \( P_{\lambda^s} \) defines a bounded operator \( P_{\lambda^s}: H^s \rightarrow H^{s-d} \).

**Proof.** By Lemma 6.5, we have \( ||P_{\lambda^s}(\bar{a})||_{s-d} = ||P_{\lambda^s-\lambda^t}(P_{\lambda^t}(\bar{a}))||_0 \). By Proposition 5.1(b), the matrix valued symbol \( \sigma(P_{\lambda^s-\lambda^t} \circ P_{\lambda^t}) \) is of order \( d + (s - d) = s \), so Theorem 6.4 gives us \( ||P_{\lambda^s-\lambda^t}(P_{\lambda^t}(\bar{a}))||_0 \leq C ||\bar{a}||_s \) for some constant \( C > 0 \). \( \square \)

We can also define an analog of the \( C^k \) norm on \( E \).

**Definition 6.7.** Define the \( C^k \) norm \( || \cdot ||_{\infty,k} : E \rightarrow \mathbb{R}_{\geq 0} \) as follows: \( ||\bar{a}||_{\infty,k} := \sum_{|\ell| \leq k} ||\delta^\ell(\bar{a})||_{C^*} \) where the \( C^* \) norm \( || \cdot ||_{C^*} \) is given by \( ||\bar{a}||_{C^*}^2 := \sup \{|\lambda| : \bar{a}^* \bar{a} - \lambda \cdot 1 \text{ not invertible}\} \).
Thus we get $C$ for $\{1 + \vec{a}\}$. For $H$ we have $\langle \vec{a} \rangle_0 = |\vec{a}|_\infty$ and $H^s \subseteq (A_0^s)^\ast e$. We can easily prove an analog of the Sobolev lemma on $E$ as follows.

**Theorem 6.8.** For $s > k + 1$, $H^s \subseteq (A_0^s)^\ast e$.

**Proof.** First consider the case $k = 0$. Note that $|\cdot|_{\infty,0} = |\cdot|_{C^\infty}$ so for arbitrary $a_{j,m}$ we have $|a_{j,m} \prod_{g=1}^n U_g f_j|^2_{\infty,0} = \sup \{ |\lambda| : |a_{j,m}|^2 - \lambda \cdot 1 \text{ not invertible} \} = |a_{j,m}|^2$ and for arbitrary $\tilde{\vec{a}} = \sum_{j,m} a_{j,m} \prod_{g=1}^n U_g f_j$ we have $|\tilde{\vec{a}}_s|^2_{\infty,0} \leq \sum_{j,m} |a_{j,m}|^2 \prod_{g=1}^n U_g f_j = \prod_{g=1}^n U_g f_j = \prod_{g=1}^n U_g f_j$ so by the Cauchy-Schwarz inequality we get $|\tilde{\vec{a}}_s|^2_0 \leq |\tilde{\vec{a}}|^2_0 \prod_{g=1}^n (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}$. Since $2s > 2$, $(1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}$ is summable over $m \in \mathbb{Z}^n$ and $j \in \{1, \ldots, r\}$ so $|\tilde{\vec{a}}_s|^2 \leq C|\tilde{\vec{a}}|^s$. Thus we get $|\tilde{\vec{a}}_s|_{\infty,0} \leq |\tilde{\vec{a}}|_0 \leq C|\tilde{\vec{a}}|^s$ and $H^s \subseteq (A_0^s)^\ast e$.

Now suppose $k > 0$. Using what we’ve proven for the previous case, we have

$$
|\delta^\ell (\tilde{\vec{a}})|_{\infty,0} \leq C |\delta^\ell (\tilde{\vec{a}})|_{s-|\ell|} = C \left| \sum_{j,m} m_\ell a_{j,m} \prod_{g=1}^n U_g f_j \right|_{s-|\ell|} \leq C \left| \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^{|\ell|} a_{j,m} \prod_{g=1}^n U_g f_j \right|_{s-|\ell|} = C |P_{\lambda^{|\ell|}} (\tilde{\vec{a}})|_{s-|\ell|} = C |\tilde{\vec{a}}|^s
$$

for $|\ell| \leq k$ since $s - |\ell| \geq s - k > 1$. Therefore, $|\tilde{\vec{a}}|^s_{\infty,k} = \sum_{|\ell| \leq k} |\delta^\ell (\tilde{\vec{a}})|_{\infty,0} \leq \sum_{|\ell| \leq k} C |\tilde{\vec{a}}|^s \leq C |\tilde{\vec{a}}|^s_{s-(k+1)(k+2)/2}$ and we get $H^s \subseteq (A_0^s)^\ast e$.

We get the following corollary.

**Corollary 6.9.** $\cap_{s \in \mathbb{R}} H^s = (A_0^\infty)^\ast e$.

**Proof.** Suppose $a \in \cap_{s \in \mathbb{R}} H^s$. Then for any $k \in \mathbb{Z}_{\geq 0}$, $\tilde{\vec{a}} \in H^{k+2}$, so by the theorem we just proved, $\tilde{\vec{a}} \in (A_0^\infty)^\ast e$. Consequently $\tilde{\vec{a}} \in (A_0^\infty)^\ast e$, so $\cap_{s \in \mathbb{R}} H^s \subseteq (A_0^\infty)^\ast e$.

Suppose $a \in (A_0^\infty)^\ast e$. Then since $H^s$ is the completion of $(A_0^\infty)^\ast e$ with respect to $|\cdot|_s$, $(A_0^\infty)^\ast e \subseteq H^s$ for all $s \in \mathbb{R}$, and $(A_0^\infty)^\ast e \subseteq \cap_{s \in \mathbb{R}} H^s$.

We can also prove an analog of the Rellich lemma on $E$.

**Theorem 6.10.** Let $\{\tilde{a}_N\} \in (A_0^\infty)^\ast e$ be a sequence. Suppose that there is a constant $C$ so that $|\tilde{a}_N|^s \leq C$ for all $N$. Let $s > t$. Then there is a subsequence $\{\tilde{a}_{N_j}\}$ that converges in $H^t$.

**Proof.** Let $F$ be an orthogonal eigenbasis of $e$ normalized with respect to $\langle \cdot, \cdot \rangle$, and let $F_1 := \{f_h : 1 \leq h \leq r_1\}$ be the subset of eigenvectors with eigenvalue 1. Let $e_{s,m} := (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s/2} \prod_{g=1}^n U_g f_j$ and $E_s := \{e_{s,m} \mid m \in \mathbb{Z}^n\}$. $E_s F_1$ is an orthogonal basis of $E$ considered as a $C$-vector space normalized with respect to $\langle \cdot, \cdot \rangle$, so we can write $\tilde{a}_N := \sum_{h,k} a_{N,h,k} e_{s,m} f_h$. Then $|a_{N,h,k}|^2 \leq \sum_{h,k} |a_{N,h,k}|^2 \leq C^2$ and $|a_{N,h,k}|^2 \leq C$. Applying the Arzela-Ascoli theorem to $\{a_{N,h,k}\}$ for some fixed $(h,k)$, we can get a subsequence $\{a_{N_j,h,k}\}$ of
\[ a_{N,h,k} \] such that for any \( \epsilon > 0 \) there exists \( M(\epsilon) \in \mathbb{N} \) such that \( |a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon \) whenever \( i,j \geq M(\epsilon) \). Do this for all \( 1 \leq h \leq r_1 \) and \( |k_1|^2 + \cdots + |k_n|^2 \leq R^2 \), replacing \( \{a_N\} \) with \( \{a_{N_i}\} \) each time. Then we get a subsequence \( \{a_{N_i}\} \) of \( \{a_N\} \) such that for any \( \epsilon > 0 \) there exists \( M(\epsilon) \in \mathbb{N} \) such that, for all \( 1 \leq h \leq r_1 \) and \( |k_1|^2 + \cdots + |k_n|^2 \leq R^2 \), \( |a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon \) whenever \( i,j \geq M(\epsilon) \). Now consider the sum \( ||a_{N_i} - a_N||^2 = \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} \). Decompose it into two parts: one where \( |k_1|^2 + \cdots + |k_n|^2 > R^2 \) and one where \( |k_1|^2 + \cdots + |k_n|^2 \leq R^2 \). On \( |k_1|^2 + \cdots + |k_n|^2 > R^2 \) we estimate \( (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s} \) so that \( \sum_h \sum_{j,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s} \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 \leq 2r_1^2 1 + R^2)^{t-s} \). If \( \epsilon > 0 \) is given, we choose \( R \) so that \( 2r_1^2 (1 + R^2)^{t-s} < \epsilon \). The remaining part of the sum is over \( |k_1|^2 + \cdots + |k_n|^2 \leq R^2 \) and can be bounded above by \( \epsilon' = \epsilon - 2r_1^2 (1 + R^2)^{t-s} \) if \( i,j \geq M(\sqrt{\epsilon}/|r_1(2R+1)^n|) \) because an \( n \)-ball of radius \( R \) centered at the origin is contained in an \( n \)-cube of side length \( 2R \) that has \( (2R+1)^n \) lattice points. Then the total sum is bounded above by \( \epsilon \), and we are done.

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