Operator approach to analytical evaluation of Feynman diagrams

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Abstract. The operator approach to analytical evaluation of multi-loop Feynman diagrams is proposed. We show that the known analytical methods of evaluation of massless Feynman integrals, such as the integration by parts method and the method of ”uniqueness” (which is based on the star-triangle relation), can be drastically simplified by using this operator approach. To demonstrate the advantages of the operator method of analytical evaluation of multi-loop Feynman diagrams, we calculate ladder diagrams for the massless $\phi^3$ theory (analytical results for these diagrams are expressed in terms of multiple polylogarithms). It is shown how operator formalism can be applied to calculation of certain massive Feynman diagrams and investigation of Lipatov integrable chain model.

1 Introduction

Elaboration of methods of analytical evaluation of multiple integrals visualized as multi-loop Feynman diagrams is motivated from the point of view of physical applications and the mathematical point of view.

From the physical point of view such investigations are important, since in calculations of the physical characteristics in quantum field theories the number of Feynman diagrams, in higher order of the perturbation theory, grows so quickly [1] that numerical calculations are not sufficient to obtain the desirable precision for the corresponding sums of perturbative integrals.

From the point of view of mathematics and mathematical physics such investigations are interesting since they reveal some structures specific for the theory of quantum integrable systems [2], [3], [4] (see also [5] and references therein). Besides, as a rule, analytical results are expressed in terms of multiple zeta-values and polylogarithms – special functions which are extremely interesting and important objects of investigations in modern mathematics and mathematical physics (see, e.g., [6], [7], [8]).

Finally, as we will see in this paper, the analytical results in calculations of Feynman diagrams (FDs) give explicit expressions for Green functions of certain special quantum mechanical models. And vice versa, the expansion of such Green functions over coupling constants gives explicit expressions for multi-loop FDs. This connection of perturbative Feynman integrals and Green functions for integrable

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quantum mechanical models is one of the substantial implications of the proposed algebraic approach.

The main idea of our algebraic method for evaluation of FDs is that we replace manipulations with multiple integrals by manipulations with the corresponding algebraic expressions. In other words, identical transformations of multiple perturbative integrals are substituted with transformations of generators of special infinite dimensional algebra. This drastically simplifies all calculations. We stress that here we mostly consider the massless case of FDs (see, however, Subsection 4.3).

2 Feynman diagrams in configuration space

The Feynman diagrams, which will be considered in this paper, are graphs with vertices connected by edges (propagators). To each edge we assign a complex number (the index of the propagator). With each vertex we associate the point in the D-dimensional space $\mathbb{R}^D$ while the edges of the graph (with index $\alpha$) are associated with the propagator of massless particle

$$x \quad \alpha \quad y = 1/(x - y)^{2\alpha}$$

where $x, y \in \mathbb{R}^D$, $(x - y)^{2\alpha} := (\sum_{i=1}^{D} (x_i - y_i) (x_i - y_i))^{\alpha}$, and $\alpha \in \mathbb{C}$. Moreover, we consider the graphs with two types of vertices: boldface vertices $\bullet$ denote that the corresponding points are integrated over $\mathbb{R}^D$. These FDs are called FDs in the configuration space. Note that more commonly known the momentum space FDs are dual graphs with respect to the graphs which correspond to the FD in the configuration space. That is, loops in the momentum space FD are replaced by boldface vertices in FD in the configuration space, and vice versa, vertices in the momentum space FD are replaced by loops in FD in the configuration space.

Consider examples of FDs in the configuration space and present the corresponding multiple integrals.

1. Graph with 5 vertices and 5 edges (3-point function):

$$\int \frac{d^Dx \ d^Du}{(x - y)^{2\alpha_1} (x - x_2)^{2\alpha_2} (x - x_3)^{2\alpha_3}}$$

2. ”Star” graph:

$$\int \frac{d^Dx}{(x - x_1)^{2\alpha_1} (x - x_2)^{2\alpha_2} (x - x_3)^{2\alpha_3}}$$

3. Propagator type graph (2-point function):
The problem (which we need to solve when analytically calculate multiple integrals corresponding to FDs) consists in searching for special transformations of graphs (FDs) such that the number of boldface vertices (the number of integrations) decreases at each step. In the next section, we discuss these special transformations and describe the corresponding operator formalism, which gives us a possibility to represent these transformations using a more compact algebraic language.

3 Operator formalism

3.1 Algebraic manipulations with perturbative integrals

Consider the $D$-dimensional Euclidean space $\mathbb{R}^D$ with the coordinates $x_i$, ($i = 1, 2, \ldots, D$). We use the notation: $\hat{x}^{2\alpha} = (\sum_{i=1}^{D} x^2_i)^\alpha$. Let the operators of coordinates $\hat{q}_i = \hat{q}_i^\dagger$ and momenta $\hat{p}_i = \hat{p}_i^\dagger$ be generators of the Heisenberg algebra

$$[\hat{q}_k, \hat{p}_j] = i \delta_{kj}.$$ 

Introduce the vectors $|x\rangle \equiv |\{x_i\}\rangle$, $|k\rangle \equiv |\{k_i\}\rangle$ that are eigenstates of the operators of coordinates and momenta, respectively: $\hat{q}_i |x\rangle = x_i |x\rangle$, $\hat{p}_i |k\rangle = k_i |k\rangle$. We normalize these states as follows

$$\langle x|k \rangle = \frac{1}{(2\pi)^{D/2}} \exp(i k_j x_j) , \quad \int d^D k \langle k|k \rangle = 1 = \int d^D x |x \rangle \langle x| .$$

Consider the heat-kernels ("matrix representations") of the operators $\hat{p}^{-2\beta}$:

$$\langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle = a(\beta) \frac{1}{(x-y)^{2\beta}} , \quad a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2} 2^{2\beta} \Gamma(\beta)} . \quad (3.1)$$

where $\beta' = D/2 - \beta$ and $\Gamma(\beta)$ is the Euler gamma-function. Formula (3.1) relates the propagators for massless particles and pseudo-differential operators $\hat{p}^{-2\beta}$. For the operators $\hat{q}^{2\alpha}$ the "matrix representations" have the form:

$$\langle x| \hat{q}^{2\alpha} |y\rangle = x^{2\alpha} \delta^D(x-y) . \quad (3.2)$$

Below we consider three $(a,b,c)$ algebraic relations which are operator analogs of relations used for the analytical evaluation of multi-loop perturbative integrals for FDs. Recall that these relations give us a possibility to reconstruct FD in such a way that the number of integrations (the number of boldface vertices in the graph) decreases to zero (this will indicate that a given FD is calculated analytically).
**a. Group relation.** Consider a convolution product of two propagators:
\[
\int \frac{d^Dz}{(x-z)^{2\alpha}(z-y)^{2\beta}} = \frac{G(\alpha', \beta')}{(x-y)^{2(\alpha+\beta-D/2)}}, \quad \left(P(\alpha, \beta) = \frac{a(\alpha + \beta)}{a(\alpha) a(\beta)}\right).
\] (3.3)

We graphically represent this convolution product as
\[
\begin{align*}
\text{We graphically represent this convolution product as} \\
x & \quad \alpha \quad \bullet \quad \beta \quad \gamma \\
\end{align*}
\]

Thus, relation (3.3) describes such a reconstruction of the graph in which the number of integrations (boldface vertices) decreases by one. This relation is the "matrix representation" of the operator identity (group relation)
\[
\hat{\mathcal{P}}^{-2\alpha'} \hat{\mathcal{P}}^{-2\beta'} = \hat{\mathcal{P}}^{-2(\alpha'+\beta')},
\] (3.4)

Indeed, by using (3.1) and (3.2) we easily demonstrate that the "matrix" analog of (3.4)
\[
\int d^Dz \langle x|\hat{\mathcal{P}}^{-2\alpha'}|z\rangle \langle z|\hat{\mathcal{P}}^{-2\beta'}|y\rangle = \langle x|\hat{\mathcal{P}}^{-2(\alpha'+\beta')}|y\rangle
\]

coincides with relation (3.3). Note that in the operator relation (3.4) the tedious coefficient \(G(\alpha', \beta')\) is vanished.

**b. Star-triangle relation.** This relation is in the basis of the so-called "method of uniqueness" [10] (see also [11]) – an efficient method of analytical evaluation of FD. In fact, this relation is a special case of the Yang-Baxter equation [2], [13]. The star-triangle relation (STR) has the form
\[
\int \frac{d^Dz}{(x-z)^{2\alpha'} z^{2(\alpha+\beta)} (z-y)^{2\beta'}} = \frac{G(\alpha, \beta)}{(x)^{2\beta} (x-y)^{2\frac{D}{2} - \alpha - \beta} (y)^{2\alpha}},
\] (3.5)

and was initially used in the framework of investigations of multi-dimensional conformal field theories [12]. The identity (3.5) can be graphically represented as
\[
\begin{align*}
\text{Thus, STR (3.5) describes such a reconstruction of the graph for which the number of integrations (boldface vertices) decreases by one. The operator version of this relation was proposed in [14] and is written in the form} \\
\text{Again we note the absence of the coefficient } G(\alpha, \beta). \\
\end{align*}
\]

\[
\hat{\mathcal{P}}^{-2\alpha} \hat{\mathcal{Q}}^{-2(\alpha+\beta)} \hat{\mathcal{P}}^{-2\beta} = \hat{\mathcal{Q}}^{-2(\alpha+\beta)} \hat{\mathcal{P}}^{-2\alpha} \hat{\mathcal{Q}}^{-2\beta} \quad (\forall \alpha, \beta),
\] (3.6)

Again we note the absence of the coefficient \(G(\alpha, \beta)\). To demonstrate the equivalence of (3.5) and (3.6) we act on (3.6) by the vectors \(\langle x|\) and \(|y\rangle\) from the left and right, respectively, and use the representations (3.1), (3.2). Identity (3.6) can be written in the form of the Yang-Baxter equation:
\[
R_{12}(\alpha) R_{23}(\alpha + \beta) R_{12}(\beta) = R_{23}(\beta) R_{12}(\alpha + \beta) R_{23}(\alpha),
\]
where $R_{ab}(\alpha) = (\hat{q}^{(a)} - \hat{p}^{(b)})^{2\alpha}$ and $[\hat{q}_i^{(a)}, \hat{p}_j^{(b)}] = i \delta_{ij} \delta_{ab},$ ($i, j = 1, \ldots, D$). Now a few remarks about STR (3.6) are in order.

**Remark 1.** The algebraic version of STR is equivalent to the commutativity for the infinite set of operators $H(\alpha) = \hat{p}^{2\alpha} \hat{q}^{2\alpha}$:

$$H(\alpha) H^{-1}(-\beta) = H^{-1}(-\beta) H(\alpha) \Rightarrow \hat{p}^{2\alpha} \hat{q}^{2(\alpha + \beta)} \hat{p}^{2\beta} = \hat{q}^{2\beta} \hat{p}^{2(\alpha + \beta)} \hat{q}^{2\alpha}.$$

**Remark 2.** Here we present the algebraic proof of STR (3.6). Introduce an inversion operator $\mathcal{R}_\Delta$ which obeys the conditions

$$\mathcal{R}_\Delta^2 = 1, \quad \mathcal{R}_\Delta \hat{q}_i \mathcal{R}_\Delta = \hat{q}_i / \hat{q}^2, \quad \langle x_i | \mathcal{R}_\Delta = \left( \frac{x_i}{x^2} \right)^{(\Delta - D)/2}, \quad (3.7)$$

$$\mathcal{R}_\Delta \hat{p}_j \mathcal{R}_\Delta = \hat{q}^2 \hat{p}_j - 2 \hat{q}_j (\hat{q} \hat{p}) + i(D - \Delta) \hat{q}_j, \quad (3.8)$$

We note that $\mathcal{R}_\Delta = \mathcal{R} q^{-2\hat{\Delta}}$, where $\mathcal{R} \equiv \mathcal{R}_0$. Using (3.7), (3.8) the algebraic version of STR is proved as follows:

$$\mathcal{R} \hat{p}^{2\alpha} \hat{p}^{2\beta} \mathcal{R} = \mathcal{R} \hat{p}^{2(\alpha + \beta)} \mathcal{R} \Rightarrow \hat{p}^{2\alpha} \hat{q}^{2(\alpha + \beta)} \hat{p}^{2\beta} = \hat{q}^{2\beta} \hat{p}^{2(\alpha + \beta)} \hat{q}^{2\alpha}.$$

**Remark 3.** From operator identity (3.6) one can immediately obtain new STR

$$(\hat{p}^{2\alpha} + a \hat{q}^{-2\alpha})(\hat{q}^{2(\alpha + \beta)} + b \hat{p}^{-2(\alpha + \beta)})(\hat{p}^{2\beta} + c \hat{q}^{-2\beta}) =$$

$$= (\hat{q}^{2\beta} + c \hat{p}^{-2\beta})(\hat{p}^{2(\alpha + \beta)} + b \hat{q}^{-2(\alpha + \beta)})(\hat{q}^{2\alpha} + a \hat{p}^{-2\alpha}), \quad (3.9)$$

where $a, b, c$ are arbitrary constants.

**Remark 4.** One can introduce one more ”local” STR [15] which is related to the $\alpha$-representation of perturbative Feynman integrals

$$W(q^2| \alpha_1) W(p^2| \frac{1}{\alpha_2}) W(q^2| \alpha_3) = W(p^2| \frac{1}{\beta_3}) W(q^2| \beta_2) W(p^2| \frac{1}{\beta_1}) ,$$

where $W(x^2| \alpha) = \exp (-x^2/(2\alpha))$, and the parameters $\alpha_i$ and $\beta_i$ are related by the identity $\alpha_i = 3\beta_i^2 + \beta_i^2\alpha_i$, which is known as star-triangle transformation for resistances in electric networks.

**c. Integration by parts rule** [9].

First, we present the graphical version of this rule

$$\text{Fig. 1}$$
With the help of this rule we obtain the reconstruction of graphs in which the number of integrations (boldface vertices) does not decrease. However, this rule is extremely useful, since the corresponding reconstruction of the graphs leads to variations of the indices on the lines, which further permits one to apply previous relations \textbf{a}, \textbf{b} and decrease the number of integrations.

The operator version of the integration by parts rule (Fig. 1) has the form

\[
(2\gamma - \alpha - \beta) \hat{p}^{2\alpha} \hat{q}^{2\beta} \hat{p}^{2\beta} = \frac{\hat{q}^2 - \hat{p}^{2(\alpha+1)}}{4(\alpha+1)} \hat{q}^{2\gamma} \hat{p}^{2\beta} - \hat{p}^{2\alpha} \hat{q}^{2\gamma} \frac{\hat{q}^2 - \hat{p}^{2(\beta+1)}}{4(\beta+1)}
\]  

(3.10)

where \(\alpha = -\alpha', \gamma = -\alpha_2\) and \(\beta = -\alpha_3\). Identity (3.10) can be directly proved by using the relations for the Heisenberg algebra:

\[
\left[\hat{q}^2, \hat{p}^{2(\alpha+1)}\right] = 4(\alpha + 1) (H + \alpha) \hat{p}^{2\alpha},
\]

\[
H \hat{q}^{2\alpha} = \hat{q}^{2\alpha} (H + 2\alpha), \quad H \hat{p}^{2\alpha} = \hat{p}^{2\alpha} (H - 2\alpha),
\]  

(3.11)

where \(H := \frac{1}{4} (\hat{p} \hat{q} \hat{i} + \hat{q} \hat{i} \hat{p})\) is the dilatation operator. It follows from (3.11) that the operators \(\hat{q}^2, \hat{p}^2, H\) generate the algebra \(sl(2)\):

\[
\left[\hat{q}^2, \hat{p}^2\right] = 4 H, \quad [H, \hat{q}^2] = 2 \hat{q}^2, \quad H \hat{p}^2 = -2 \hat{p}^2.
\]  

(3.12)

### 3.2 Group \(\mathcal{H}\) and characters on \(\mathcal{H}\).

Note that all operator identities \textbf{a}, \textbf{b}, \textbf{c} (3.4), (3.6), (3.10) are relations on the operators \(\hat{p}^{2\alpha}, \hat{q}^{2\beta}\) and their products. So it is natural to introduce a group \(\mathcal{H}\) which is generated by the operators \(\{\hat{p}^{2\alpha}, \hat{q}^{2\beta}\} (\forall \alpha, \beta \in \mathbb{C})\). The dilatation operator \(H\) belongs to the group algebra of \(\mathcal{H}\) in view of the first relation in (3.12). Consider any element of the group \(\mathcal{H}\)

\[
\Psi(\alpha_i) = \hat{p}^{-2\alpha_i} \hat{q}^{-2\alpha_2} \hat{p}^{-2\alpha_3} \hat{q}^{-2\alpha_4} \hat{p}^{-2\alpha_5} \ldots \hat{q}^{-2\alpha_{2k}} \hat{p}^{-2\alpha_{2k+1}}.
\]  

(3.13)

For \(\sum_{m=1}^{k} \alpha_{2m} = \sum_{m=0}^{k} \alpha_{2m+1}\) and \(\forall k \in \mathbb{Z}_+\), such elements form a commutative subgroup \(\mathcal{H}_0\) in \(\mathcal{H}\) (see Remark 1 in Sect. 3). The element (3.13) can be interpreted as an operator version of the 3-point function:

\[
\langle x|\Psi(\alpha_i)|y\rangle \sim \int d^D z_1 |z_1| d^D z_2 |z_2| \ldots d^D z_k |z_k| \int d^D z_{k+1} |z_{k+1}|
\]  

(3.14)

Indeed, the corresponding multiple integral is obtained from the representation

\[
\langle x|\Psi(\alpha_i)|y\rangle = \langle x|\hat{p}^{-2\alpha_i} \hat{q}^{-2\alpha_2} \hat{p}^{-2\alpha_3} \hat{q}^{-2\alpha_4} \hat{p}^{-2\alpha_5} \ldots \hat{q}^{-2\alpha_{2k}} \hat{p}^{-2\alpha_{2k+1}}|y\rangle
\]

\[
\int d^D z_1 |z_1| \int d^D z_2 |z_2| \ldots \int d^D z_k |z_k| \int d^D z_{k+1} |z_{k+1}|
\]

if we take into account eqs. (3.1), (3.2). Note that the expression

\[
\langle x|\Psi(\alpha_i)|x\rangle = \chi_D(\alpha_i) \frac{1}{x^2 \sum_{i=1}^{2k}}
\]  

(3.15)
where \( \chi_D(\alpha_i) \) is a coefficient function, gives the representation for the 2-point function, or for the propagator-type FD. The advantage of the operator approach to the evaluation of the 3-point function \((3.14)\) consists in that we can apply relations \(a,b,c\) \((3.4), (3.6), (3.10)\) to the element \((3.13)\) directly at the operator level, i.e. use the commutation relations in the group algebra of \(\mathcal{H}\) instead of making the corresponding manipulations with multiple integrals.

A remarkable fact is that one can define a trace for the elements of the group \(\mathcal{H}\) (generated by the operators \(\{\hat{p}^{2\alpha}, \hat{q}^{2\beta}\}\)). First, recall that the dimension regularization scheme requires the identity \([17]\)

\[
\int \frac{d^D x}{x^{2(D/2+\alpha)}} = 0 \quad \forall \alpha \neq 0 .
\]  

The extension of the definition for the integral \((3.16)\) at the point \(\alpha = 0\) was proposed \([18]\) and has the form

\[
\int \frac{d^D x}{x^{2(D/2+\alpha)}} = \pi \Omega_D \delta(|\alpha|) ,
\]  

where \(\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\) is the area of the unit hypersphere in \(\mathbb{R}^D\), \(\alpha = |\alpha|e^{i\arg(\alpha)}\) and \(\delta(.)\) is the one-dimensional delta-function. Consider the formal integral for the 2-point function \((3.15)\):

\[
\int d^D x \frac{d^D x}{x^{2(\beta+\gamma)}} = \int d^D x |\Psi(\alpha_i)\hat{q}^{-\gamma}|x \rangle = \text{Tr} \left( \Psi(\alpha_i)\hat{q}^{-\gamma} \right) \pi \Omega_D \delta(|\beta + \gamma - D/2|) .
\]  

Here \(\beta = \sum_i \alpha_i - k D / 2\) and \(\chi_D(\alpha_i)\) is a coefficient function for propagator type diagram which now can be interpreted as a character for the group element

\[
\hat{p}^{-2\alpha_1} \hat{q}^{-2\alpha_2} \cdots \hat{p}^{-2\alpha_{k+1}} \hat{q}^{-2\gamma} \in \mathcal{H}_0 ,
\]

where \(\beta + \gamma = D / 2\) (or \(\sum_{m=1}^k \alpha_{2m} + \gamma = \sum_{m=0}^k \alpha'_{2m+1}\)). The cyclic property of the trace \((3.18)\) \(\text{Tr}(AB) = \text{Tr}(BA)\) can be checked directly. The definition of the trace \((3.18)\) permits one to reduce the evaluation of propagator-type FDs (and searching for their symmetries) to the evaluation (and searching symmetries) of vacuum FDs (for details see \([18]\)).

### 4 Applications

#### 4.1 Ladder FDs for \(\phi^3\) theory in \(D = 4\); relation to conformal quantum mechanics

Consider dimensionally and analytically regularized massless perturbative integrals

\[
D_L(p_0, p_{L+1}; \alpha, \beta, \gamma) = \left[ \prod_{k=1}^L \int \frac{d^D p_k}{p_k^{2\alpha_k} (p_k + p)^{2\beta}} \right] \prod_{m=0}^L \frac{1}{(p_{m+1} - p_m)^{2\gamma}} ,
\]  

\(L\)
which correspond to FDs \((x_1 = p_0, x_2 = p_{L+1}, x_3 = p)\)

\[\begin{align*}
\text{Fig. 2}
\end{align*}\]

Perturbative integral (4.1) can be graphically represented in two ways – as diagrams in configuration and momentum spaces, as it is shown in Fig. 2, where \(\alpha, \beta, \gamma\) are indices on the lines in the left diagram (in configuration space), while in the right diagram (in momentum space) the indices \(p_i\) indicate momenta flowing over the lines. As we have mentioned above, these diagrams are dual to each other (boldface vertices in the left diagram correspond to the loops in the right diagram). The operator version of the integral (4.1) follows from the representation for the left diagram

\[D_L(x_a; \alpha, \beta, \gamma) = (a(\gamma'))^{-L-1} \langle x_1 | \hat{p}^{2\gamma'} \left( \prod_{k=1}^{L} \hat{q}^{-2\alpha}(\hat{q} - x_3)^{-2\beta} \hat{p}^{2\gamma'} \right) | x_2 \rangle .\]

It is convenient to consider the generating function for the integrals \(D_L\) (4.1)

\[D_g(x_a; \alpha, \beta, \gamma) = a(\gamma') \sum_{L=0}^{\infty} g^L D_L(x_a; \alpha, \beta, \gamma) = \langle x_1 | \left( \hat{p}^{2\gamma'} - \frac{g/a(\gamma')}{\hat{q}^{2\alpha}(\hat{q} - x_3)^{2\beta}} \right)^{-1} | x_2 \rangle .\]

(4.2)

If the indices on the lines are related by the condition \(\alpha + \beta = 2\gamma'\), then by using properties (3.7), (3.8) of the inversion operator \(\mathcal{R}\) we obtain (for details see [14]):

\[D_g(x_a; 2\gamma' - \beta, \beta, \gamma) = \frac{1}{(x_1^2 x_2^2)^{\gamma'}} \langle u | \left( \hat{p}^{2\gamma'} - \frac{g_{\gamma',\beta}}{\hat{q}^{2\beta}} \right)^{-1} | v \rangle ,\]

(4.3)

where \(g_{\gamma,\beta} = \frac{g}{(x_1^2 x_2^2)^{2\gamma}}\), \(u_i = \frac{(x_1)}{(x_3)^2} - \frac{(x_2)}{(x_3)^2}\), \(v_i = \frac{(x_2)}{(x_3)^2} - \frac{(x_2)}{(x_3)^2}\) \((i = 1, \ldots, D)\). In the case when we fix the indices on the lines as \(\gamma' = \beta\), the generating function \(D_g\) (4.3) is related to the conformal function

\[D_g(x_a; \beta, \beta, D/2 - \beta) = \frac{1}{(x_1^2 x_2^2)^{D/2-\beta}} \langle u | \left( \hat{p}^{2\beta} - \frac{g_{\beta,\beta}}{\hat{q}^{2\beta}} \right)^{-1} | v \rangle ,\]

(4.4)

and then taking \(\beta = 1\) we obtain the Green function for \(D\)-dimensional conformal mechanics

\[D_g(x_a; 1, 1, D/2 - 1) = \frac{1}{(x_1^2 x_2^2)^{(D/2-1)}} \langle u | \left( \hat{p}^{2} - \frac{g_{1,1}}{\hat{q}^{2}} \right)^{-1} | v \rangle =\]

\[a(1) \sum_{L=0}^{\infty} g^L D_L(x_a; 1, 1, D/2 - 1) .\]

(4.5)

Thus, we have shown that with a special choice of indices on the lines \(\alpha = \beta = 1, \gamma = \frac{D}{2} - 1 = 1 - \epsilon\) the ladder diagrams (in momentum space):
are related to the Green function for $D$-dimensional conformal mechanics. Moreover, according to the definition of the generating function $D_g$, the expression $D_L$ for the ladder diagram with $L$ loops ($L$ boldface vertices for FD in the configuration space) is obtained in the expansion of the Green function (4.5) over the coupling constant $g$ (coefficient in order $g^L$).

The operator method of evaluation of Green function (4.5) is based on the remarkable identity [14]

$$\frac{1}{\hat{p}^2 - g/\hat{q}^2} = \sum_{L=0}^{\infty} \left( -\frac{g}{4} \right)^L \left[ \hat{q}^{2\alpha} \frac{(H-1)}{(H-1+\alpha)^L+1} \frac{1}{\hat{p}^2} \hat{q}^{-2\alpha} \right]_{\alpha=L}, \quad (4.6)$$

where we have used the notation $[\ldots]_{\alpha,L} = \frac{1}{L!} \left( \partial_{\alpha}^{\alpha} [\ldots] \right)_{\alpha=0}$. The leading terms in the expansion of (4.6) over $g$ give the identities

$$\hat{p}^{-2} \hat{q}^{-2} \hat{p}^{-2} = -\frac{1}{4 (H-1)} [\log(\hat{q}^2), \hat{p}^{-2}]_L,$$

$$\hat{p}^{-2} \hat{q}^{-2} \hat{p}^{-2} \hat{q}^{-2} \hat{p}^{-2} = \frac{(H-1)^{-2}}{32} [\log(\hat{q}^2), [\log(\hat{q}^2), \hat{p}^{-2}]] + \frac{(H-1)^{-3}}{16} [\log(\hat{q}^2), \hat{p}^{-2}]_L,$$

which can be proved directly and immediately lead to analytical expressions for the ladder (box) diagrams with one and two loops ($L = 1, 2$). In general, taking into account the integral representation for the rational function of $H$ (in the right-hand side of (4.6))

$$\frac{(H-1)}{(H-1+\alpha)^L+1} = \frac{(-1)^L}{L!} \int_0^\infty dt \, t^L \, e^{tH} \partial_t (e^{t(H-1)})_L,$$

and using obvious properties of the operator $e^{tH}$: $e^{t(H+\frac{D}{2})}|x\rangle = |e^{-t}x\rangle$, we can rewrite Green function appearing in (4.5) in the form

$$\langle u | \frac{1}{(\hat{p}^2 - g_{1,1}/\hat{q}^2)} | v \rangle = \sum_{L=0}^{\infty} \frac{1}{L!} \left( \frac{g_{1,1}}{4} \right)^L \Phi_L(u, v), \quad (4.7)$$

where

$$\Phi_L(u, v) = -a(1) \int_0^\infty dt \, t^L \left[ \frac{u^2}{v^2} \right]^{\alpha} \partial_t \left( e^{-t} \right)^{(D-1)} \left( \frac{e^{-t}}{u - e^{-t}v} \right)^{(D-1)} =$$

$$= \frac{a(1)}{u^{2(D/2-1)}} \left( \frac{v^2}{2} \right) \left( \frac{2(uv)}{a^2} \right)^{(D-1)} \Psi_L \left( \frac{v^2}{u^2} \right).$$

(4.8)
Thus, the result for the evaluation of the $L$-loop ladder diagram (Fig. 3) is

$$D_L(x_1, x_2, x_3; 1, 1, \frac{D}{2} - 1) = \frac{1}{L!4^L a^L} \frac{x_2^{2(D/2-L-1)}}{(x_1^2 x_2^2)^{D/2-1}} \Psi_L(\frac{v^2}{u^2}, 2 \frac{(uv)}{u^2}),$$

(4.9)

where $u^2 = \frac{x_1^2}{x_1^2 + x_2^2}$, $v^2 = \frac{x_2^2}{x_1^2 + x_2^2}$, $(u - v)^2 = \frac{x_2^2}{x_1^2 + x_2^2}$ and $x_{ab} = x_a - x_b$.

For $D = 4 - 2\epsilon$ the function $\Psi_L(\frac{v^2}{u^2}, 2 \frac{(uv)}{u^2})$ (4.8) is expanded over $\epsilon$

$$\Psi_L(\frac{v^2}{u^2}, 2 \frac{(uv)}{u^2}) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \Phi_L^{(k)}(z_1, z_2),$$

where $z_1 + z_2 = 2(\frac{uv}{u^2}) + u^2$ and $z_1 z_2 = \frac{v^2}{u^2}$. The coefficient functions $\Phi_L^{(l)}$ are

$$\Phi_L^{(l)} = \sum_{f=0}^{L} \frac{(- \ln(z_1 z_2))^f (2L - f)!}{f! (L - f)!} \sum_{m=0}^{l} (-)^m C_l^m Z_m (z_1, z_2; 2L + l - f),$$

(4.10)

where $C_l^m = \frac{\binom{L}{m} \ln(l - m)!}{m!}$ is a binomial coefficient and

$$Z_m(z_1, z_2; k) = \frac{\Gamma(k - m)}{(z_1 - z_2)} \sum_{n_0, \ldots, n_m = 1}^{\infty} \frac{(z_1^{n_0} - z_2^{n_0})}{(\sum_{i=0}^{m} n_i)^{k-m}} \left(\prod_{i=1}^{m} \frac{z_i^{n_i} + z_2^{n_i}}{n_i}\right),$$

is expressed in terms of multiple polylogarithms

$$\text{Li}_{m_0, m_1, \ldots, m_r}(w_0, w_1, \ldots, w_r) = \sum_{n_0 > n_1 > \ldots > n_r > 0} \frac{w_0^{n_0} w_1^{n_1} \ldots w_r^{n_r}}{n_0^{m_0} n_1^{m_1} \ldots n_r^{m_r}}.$$  (4.11)

The first coefficient (for $D = 4$ or $\epsilon = 0$) has the form [19], [20]

$$\Phi_L^{(0)}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} \sum_{f=0}^{L} \frac{(-)^f (2L - f)!}{f! (L - f)!} \ln^f(z_1 z_2) \left[\text{Li}_{2L-f}(z_1) - \text{Li}_{2L-f}(z_2)\right].$$

and is expressed via the standard polylogarithms $\text{Li}_m(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^m}$. The following coefficient was calculated in [14]:

$$\Phi_L^{(1)}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} \sum_{n=L}^{2L} n! \ln^{2L-n}(z_1 z_2) \left[(n \text{Li}_{n+1}(z_1) - \text{Li}_{n+1}(z_1, 1) - \text{Li}_{n+1}(z_1, z_2)) - (z_1 \leftrightarrow z_2)\right],$$

where the functions $\text{Li}_{n+1}(w_0, w_1)$ have been defined in (4.11).

**Remark.** Using conformal and scaling properties of the Green function (4.3) one can deduce the representation (cf. (4.8)):

$$\langle u \mid \left(\hat{p}^{2\gamma} - \frac{g(u^2 v^2)^{\frac{\beta - \gamma}{2}}}{q^{2\beta}}\right)^{-1} \mid v \rangle = \frac{1}{u^{2(D/2-\gamma)}} \Psi_{\gamma, \beta}(\frac{u^2}{u^2}, 2 \frac{(uv)}{u^2}),$$

(4.12)
where the conformal symmetry requires \( \Psi^{(\gamma,\beta)}(u_1, u_2) = \Psi^{(\gamma,2\gamma-\beta)}(u_1, u_2) \). From the representation (4.12) we obtain the identity

\[
\begin{align*}
    u^{2(D/2-\gamma)} \langle u | \left( \hat{p}^{2\gamma} - g \frac{u \cdot v}{q^2} \right) \frac{1}{2} | v \rangle = \\
    = (u')^{2(D/2-\gamma)} \langle u' | \left( \hat{p}^{2\gamma} - g \frac{u' \cdot v'}{q'^2} \right) \frac{1}{2} | v' \rangle,
\end{align*}
\]

(4.13)

where \( u = \frac{1}{x_1} - \frac{1}{x_3}, v = \frac{1}{x_2} - \frac{1}{x_3}, u' = \frac{1}{x_1} - \frac{1}{x_{12}}, v' = \frac{1}{x_2} - \frac{1}{x_{12}} \) and we have introduced the concise notation \( \langle \frac{1}{x_a} \rangle_i = \frac{\langle x_a \rangle_i}{x_a} \). To prove eq. (4.13) it is only needed to note the cross-ratio identities

\[
\frac{v^2}{u^2} = \frac{(v')^2}{(u')^2} = \frac{x_{23}^2}{x_{12}^2 x_{13}^2}, \quad \frac{(u-v)^2}{u^2} = \frac{(u'-v')^2}{(u')^2} = \frac{x_{12}^2 x_3^2}{x_{13}^2 x_2^2}.
\]

Now for both sides of eq. (4.13) we make the inverse transformation with respect to that used in passing from eq. (4.2) to eq. (4.3). As a result, we rewrite (4.13) in the form

\[
\begin{align*}
    x_{12}^{2(\gamma-D/2)} \langle x_1 | \left( \hat{p}^{2\gamma} - g \frac{x_3^{2\gamma}}{q^{2(2\gamma-\beta)}(q - x_3)^{2\beta}} \right) \frac{1}{2} | x_2 \rangle = \\
    = x_{12}^{2(\gamma-D/2)} \langle x_1 | \left( \hat{p}^{2\gamma} - g \frac{x_{12}^{2\gamma}}{q^{2(2\gamma-\beta)}(\hat{q} - x_{12})^{2\beta}} \right) \frac{1}{2} | x_{13} \rangle.
\end{align*}
\]

where \( \tilde{u} = \frac{x_{12}^2 x_3^2}{x_1^2 x_2} \) and \( \tilde{v} = \frac{x_{12}^2 x_3^2}{x_1^2 x_{13}} \). Note that this identity is nothing but the relation on the generating functions for the ladder diagrams (4.2). Expanding both the sides over the coupling constant \( g \), we obtain \( D \)-dimensional identities for the \( L \)-loop ladder momentum diagrams in the order \( g^L \):

\[
\begin{align*}
    &x_3^{2(\gamma\gamma - D/2)} \tilde{u}^{L(\beta-\gamma)} x_{12}^{L(\beta-\gamma)} \times \gamma' \cdots \gamma' \frac{x_1 - x_3}{x_1} \frac{x_2 - x_3}{x_2} = \\
    &x_{12}^{2(\gamma\gamma - D/2)} \tilde{v}^{L(\beta-\gamma)} x_{13}^{L(\beta-\gamma)} \times \gamma' \cdots \gamma' \frac{x_1 - x_3}{x_1} \frac{x_2 - x_3}{x_2}.
\end{align*}
\]

where \( \beta, \beta = 2\gamma - \beta \) and \( \gamma' = D/2 - \gamma \) are special indices on the lines and \( x_1, x_2, x_3 \) parametrize external momenta. These identities, in the special case \( D = 4 (\epsilon = 0) \) and \( \beta = \gamma' = \beta = 1 \), were obtained in [21] and used there for deriving many remarkable relations for various planar FDs.
4.2 Application to Lipatov chain model

It was shown in [3] that a wave function for bound states of gluons at high energies has the property of the holomorphic factorization. The Lipatov Hamiltonian for each of these two holomorphic subsystems has the form: $H = \sum_{i=1}^{n} H_{ii+1}$, where

$$H_{ik} = \hat{p}_i \ln(\rho_{ik}) \hat{p}_i^{-1} + \hat{p}_k \ln(\rho_{ik}) \hat{p}_k^{-1} + \ln(\hat{p}_i \hat{p}_k) + 2\gamma = 2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k) \rho_{ik}^{-1} + 2\gamma .$$ (4.15)

Here $\gamma = -\Gamma'(1)$ is the Euler constant, $\rho_{ik} = q_i - q_k$, where $q_i$ are complex coordinates, and momentum operators $\hat{p}_i = -i\partial_{q_i}$ are complex derivatives. In this subsection we show how one can easily demonstrate the equality of two expressions (4.14) and (4.15) for the Lipatov pair Hamiltonian by using the operator technique discussed above.

Note that the expression (4.15) (up to the constant $2\gamma$) appears in the expansion over $\epsilon$ of the following operator:

$$R_{ik}(\epsilon) := \rho_{ik}^{1+\epsilon}(\hat{p}_i \hat{p}_k)^{\epsilon} \rho_{ik}^{-1+\epsilon} = 1 + \epsilon \left( 2 \ln(\rho_{ik}) + \rho_{ik} \ln(\hat{p}_i \hat{p}_k) \rho_{ik}^{-1} \right) + \epsilon^2 \ldots .$$ (4.16)

Now we use the one-dimensional analog of the operator ”star-triangle” identity (3.6)

$$\rho_{ik}^{\alpha_\beta} \hat{p}_i^{\alpha+\beta} \rho_{ik} = \hat{p}_i^\beta \rho_{ik}^{\alpha_\beta} \hat{p}_i^\alpha \Leftrightarrow \rho_{ki}^{\beta_\alpha} \hat{p}_i^{\alpha+\beta} \rho_{ki} = \hat{p}_i^\alpha \rho_{ki}^{\beta_\alpha} \hat{p}_i^\beta .$$ (4.17)

Then, we have

$$\rho_{ik}^{1+\epsilon}(\hat{p}_i \hat{p}_k)^{\epsilon} \rho_{ik}^{-1+\epsilon} = \rho_{ik}^{1+\epsilon}(\hat{p}_i) \hat{p}_k^{1+\epsilon} \rho_{ik}^{-1+\epsilon} \hat{p}_i \rho_{ik}^{1+\epsilon} \rho_{ik}^{-1+\epsilon} = \hat{p}_i \rho_{ik}^{1+\epsilon} \rho_{ik}^{-1+\epsilon} \hat{p}_k \rho_{ik}^{1+\epsilon} \rho_{ik}^{-1+\epsilon} = 1 + \epsilon \left( \hat{p}_i \ln(\rho_{ik}) \hat{p}_i + \hat{p}_k \ln(\rho_{ik}) \hat{p}_k \right) + \epsilon^2 \ldots ,$$

and this proves the equivalence of expressions (4.14) and (4.15).

We stress that from the point of view of the integrability of the Lipatov model the form of the pair Hamiltonian (4.14) and its relation to the operator (4.16) is not accidental. Indeed, the $R$-operator (4.16) satisfies the Yang-Baxter equation

$$R_{i,i+1}(\epsilon) R_{i+1,i+2}(\epsilon + \epsilon') R_{i+1,i+1}(\epsilon') = R_{i+1,i+2}(\epsilon') R_{i,i+1}(\epsilon + \epsilon') R_{i+1,i+2}(\epsilon) ,$$

which can be easily proved by using the operator ”star-triangle” relation (4.17). Then the complete holomorphic Hamiltonian $H = \sum_{i=1}^{n} H_{ii+1}$ appears in the expansion over $\epsilon$ of the monodromy matrix (in the order $\epsilon^3$)

$$T(1,2,\ldots,n+1)(\epsilon) = R_{12}(\epsilon) R_{23}(\epsilon) R_{34}(\epsilon) \cdots R_{n,n+1}(\epsilon) .$$

Finally, we should like to note that recently the solutions of the Yang-Baxter equation for principal series of infinite dimensional representations of $SL(N)$ have been considered in [22]. The results of [22] generalize the discussion presented in this subsection.
4.3 Diagrams with massive propagators

In this subsection we consider an example of the operator approach to analytical evaluation of the 1-loop 3-point function with one massive propagator.

First, we use the automorphism \( \hat{p}^2 \leftrightarrow \hat{q}^2, H \leftrightarrow -H \) of the \( sl(2) \)-algebra (3.12) to write the first relation in (3.11) as

\[
[q^{2\beta}, \hat{p}^2] = 4\beta (H - \beta + 1)q^{2(\beta - 1)},
\]

and, then, generalize it by introducing the massive parameter \( m \) as follows:

\[
[(\hat{q}^2 + m^2)^\beta, \hat{p}^2] = 4\beta (H + m^2 \partial_m - \beta + 1)(\hat{q}^2 + m^2)^{\beta - 1}.
\]

This identity can be converted into the integral form

\[
\frac{1}{p^2}(\hat{q}^2 + m^2)^{\beta - 1} \frac{1}{p^2} = \frac{1}{4\beta} \int_0^\infty dt e^{(H - \beta + m^2 \partial_m)} \frac{1}{p^2} (\hat{q}^2 + m^2)^\beta
\]

from which the representation for the 3-point function is deduced

\[
\langle x_1 | \frac{1}{p^2}(\hat{q}^2 + m^2)^{\beta - 1} \frac{1}{p^2} | x_2 \rangle = \frac{a(1)}{4\beta} \int_0^\infty dt e^{(D/2-1)\left( (e^{-t}x_1^2 + m^2)^\beta - (e^t x_1^2 + m^2)^\beta \right)}
\]

Here the left-hand side is represented in the form of the perturbative integral and we obtain the equality

\[
\int \frac{d^D k}{(k-x_1)^2(D/2-1)(k^2 + m^2)^{1-\beta}(k-x_2)^2(D/2-1)} =
\]

\[
= \frac{1}{4a(1)} \int_0^\infty dt e^{(D/2-1)\left( (e^{-t}x_2^2 + m^2)^\beta - (x_2^2 e^t + m^2)^\beta \right)}
\]

Finally, we consider the limit \( D \to 4, \beta \to 0 \) for this relation and deduce the identity

\[
\int \frac{d^4 k}{(k-x_1)^2(k^2 + m^2)(k-x_2)^2} = \pi^2 \int_0^\infty dt \frac{e^{-t}}{(x_1 - e^{-t} x_2)^2} \log \left( \frac{e^{-t} x_2^2 + m^2}{e^t x_1^2 + m^2} \right),
\]

which is important for physical applications and corresponds to the evaluation of the 3-point one-loop FD (in the momentum space) with one massive line:
Conclusion

A large enough class of perturbative integrals (which are graphically represented as FDs) generates a commutative algebra of functions $\mathcal{M}$. For some subset $\mathcal{M}_0 \subset \mathcal{M}$ of generators of this algebra, the explicit analytical expressions in terms of special functions (of the type of multiple polylogarithms) are known. The problem of analytical evaluation of a specific perturbative integral $I \in \mathcal{M}$ is now reduced to searching for the representation of $I$ (e.g., by means of transformations $(a,b,c)$ from Section 3) in terms of the elements of $\mathcal{M}_0$. Unfortunately, one cannot always succeed in finding such a representation, since the subset $\mathcal{M}_0$ which is known nowadays, as well as the collection of transformations (of the type $a,b,c$), do not give the possibility to speak about $\mathcal{M}_0$ as a complete system in $\mathcal{M}$. In this case, by using the transformations $(a,b,c)$, one can obtain certain relations between the elements $I_\alpha \in \mathcal{M}$, which are considered as a set of functional equations for $I_\alpha$. Thus, the problem of analytical evaluation is reduced to the extraction of an independent set of equations and then to searching of their solution.

Now let us make a few remarks about the results presented above.

1. It should be noted that the coefficient functions $\Phi_L(u, v)$ (4.7) appear in the calculations of the 4-point functions in the $N = 4$ supersymmetric Yang-Mills theory [16].

2. The proposed operator relations (3.6), (4.17) not only clarify the structure of the integrable Lipatov model (see subsection 4.2), but also help to investigate certain generalizations of this model (see [22]).

3. The important problem is the search for generalizations of the described algebraic formalism in the cases of supersymmetric quantum mechanics and for massive propagators. In the last case we have succeeded in calculating the special 3-point FD with one massive propagator (see subsection 4.3). However, this calculation is particular. From this point of view it would be important to calculate the coefficients $\Phi_L(u, v; m^2)$ in the expansion over $g$ of the spectral Green function for conformal mechanics:

$$
\langle u | \frac{1}{(\hat{p}^2 - g/\hat{q}^2 + m^2)} | v \rangle = \sum_{L=0}^{\infty} g^L \Phi_L(u, v; m^2).
$$

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