SPLITTING CRITERIA FOR VECTOR BUNDLES
ON MINUSCULE HOMOGENEOUS VARIETIES

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Abstract. I prove that a vector bundle on a minuscule homogeneous variety splits into a direct sum of line bundles if and only if its restriction to the union of two-dimensional Schubert subvarieties splits. A case-by-case analysis is done.

INTRODUCTION

As stated in the abstract, the goal of this note is to prove the following:

Theorem. A vector bundle on a minuscule homogeneous variety $X$ with cyclic Picard group splits if and only if its restriction to the union of 2-dimensional Schubert subvarieties of $X$ splits.

The list of minuscule flag varieties can be found e.g. [11, Ch. 5, §2]. For the groups of type $A_n, B_n, D_n$ one gets respectively the Grassmannians, the even dimensional quadrics, and the spinor varieties. There are two ‘exceptional’ cases, the Cayley plane and the Freudenthal variety, corresponding to the groups $E_6, E_7$ respectively.

A case-by-case analysis shows that the statement above is very down-to-earth. Minuscule homogeneous varieties have only one irreducible 2-dimensional Schubert subvarieties, isomorphic to $\mathbb{CP}^2$, with two exceptions: the Grassmannians $\text{Gr}(k, n)$, $1 < k < n - 1$, and the 4-dimensional quadric, which have two irreducible 2-dimensional Schubert varieties, both isomorphic to $\mathbb{CP}^2$, intersecting along a Schubert line.

The problem of deciding the splitting of vector bundles goes a long way back. One of the first and widely known criterion is that of Horrocks [10], saying that a vector bundle on the projective space splits if and only if it does so along a 2-plane. Cohomological splitting criteria for vector bundles on Grassmannians and quadrics have been obtained in [17, 14]. Recent progress is achieved in [16], where the authors proved a splitting criterion for uniform vector bundles of low rank on more general Fano varieties with cyclic Picard group. (The term ‘uniform’ refers to the fact that the splitting type of the vector bundle is the same along all the rational curves representing a fixed homology class. Earlier articles which investigated uniform vector bundles on projective spaces and Grassmannians are [20, 6, 11, 7].) One of the main applications yields splitting criteria for uniform vector bundles of low rank on Hermitian symmetric spaces, also known as co-minuscule homogeneous varieties. The assumption on the rank of the vector bundles to be sufficiently small is essential: there are easy examples of homogeneous, thus uniform, non-split vector bundles; optimal upper bounds for the rank are given in op. cit.

The theorem above requires to probe the splitting of vector bundles along a single 2-plane or a wedge of two 2-planes; moreover, the rank is arbitrary. Furthermore, there is
an elementary cohomological test (cf. proposition 3.1) to verify whether a vector bundle on the projective plane splits or not. From a ‘computational’ point of view, these facts are an advantage because, unless additional information is available, it is difficult to check whether a vector bundle is uniform or not. Indeed, the generic splitting type of any vector bundle being constant, one has to verify the uniformity on every line (within a suitable homology class).

One of the essential tools used for proving the main result is Pieri’s formula in Schubert calculus, which is recalled in the first section. The main result itself is proved in section 2 and detailed in section 3.

1. Preliminaries

Here we recall from [13 §1], [11, Ch. IV,V] a few basic facts about the Bruhat decomposition of projective homogeneous varieties and their Schubert subvarieties; a thorough description of the cohomology ring can be found in [2, 5]. Let $G$ be a connected, reductive linear algebraic group and $P \subset G$ a parabolic subgroup; denote by $W_G$ and $W_P$ the Weyl groups of $G$ and of the Levi subgroup of $P$ respectively. The Bruhat decomposition states that

\[ X := G/P = \coprod_{w \in W_G/W_P} BwP. \]  

Each $W_P$-orbit in $W_G$ admits a unique representative of shortest length, and the dimension of the corresponding cell $BwP$ equals $\text{length}(w)$; let $W^P$ be the set of shortest representatives. We denote by $w_0 \in W_G$, $w_0 P \in W_P$, $w_0^P \in W^P$ be the longest words; they are related by $w_0 = w_0^P w_0$. By definition, the Schubert subvarieties of $G/P$ are the closures of the Bruhat cells:

\[ \overline{BwP}, \quad \text{dim} \overline{BwP} = \text{length}(w), \quad \forall w \in W^P. \]

Since we are interested in intersection products, it is more convenient to index the Schubert subvarieties by their codimension. The Poincaré duality on $G/P$ acts as:

\[ W^P \to W^P, \quad w \mapsto \check{w} := w_0 w w_0. \]  

(The right-hand-side is not the reduced decomposition; it can be simplified by using the commutation relations in $W_G$.) We denote

\[ X(w) := \overline{BwP}, \quad \text{codim} X(w) = \text{length}(w), \quad \forall w \in W^P. \]  

The (reversed) Bruhat order on $W^P$ is defined by:

\[ w \prec w' \iff X(w') \subsetneq X(w). \]

It defines an oriented graph (the Hasse diagram) with vertices $W^P$, and edges

\[ \{(w, w') \mid w \prec w', \text{ length}(w') = \text{length}(w) + 1\}. \]  

(Notation: $w \to w'$).

The Bruhat cells are isomorphic to affine spaces, so (1.1) is a cell-decomposition of $X$, and therefore $\{X(w)\}_{w \in W^P}$ is a $\mathbb{Z}$-basis of the cohomology ring $H^*(X; \mathbb{Z})$ of $X$; in particular, $H^{\text{odd}}(X; \mathbb{Z}) = 0$. The Picard group is isomorphic to $H^2(X; \mathbb{Z})$, generated by the classes of the divisors $D_\alpha := X(\tau_\alpha)$, where $\alpha$ is a simple root such that $\tau_\alpha \in W^P$. (Here $\tau_\alpha$ stands for the reflection defined by $\alpha$.) The intersection product between $D_\alpha$ and the Schubert variety $X(w)$ is given by Pieri’s formula:

\[ [D_\alpha \cdot X(w)] = \sum_{\beta \in \Delta_+, \ w \to w \beta} (\omega_\alpha, \beta') \cdot X(w \beta). \]  

(1.4)
(Here $(\cdot, \cdot)$ stands for the Weyl-invariant scalar product between the characters of the maximal torus of $G$ with $\mathbb{R}$-coefficients, $\Delta^+$ denotes the positive roots of $G$, $\omega_\alpha$ is the fundamental weight corresponding to the simple root $\alpha$, and $\beta^\vee := 2\beta/(\beta, \beta)$ is the co-root of $\beta$.)

**Minuscule homogeneous varieties.** Now we specialize to the case relevant for us.

**Definition 1.1** (cf. [11, Ch. V, Proposition 2.2]) A fundamental weight $\omega_\alpha$ is called **minuscule** if it satisfies the condition

$$
(\omega_\alpha, \beta^\vee) \in \{0, 1\}, \quad \forall \beta \in \Delta^+.
$$

Let $P_\alpha \subset G$ be the standard parabolic subgroup defined by omitting the simple root $\alpha$. One says that $X = G/P_\alpha$ is a **minuscule flag variety** if $\omega_\alpha$ is minuscule.

Since $P = P_\alpha \subset G$ is maximal, there is a unique Schubert divisor $D = D_\alpha$ and a unique Schubert line $\ell \cong \mathbb{C}P^1$. The line bundle $\mathcal{L} := \mathcal{O}_X(D) \in \text{Pic}(X)$ is ample and $\text{Pic}(X) = \mathbb{Z}\mathcal{L}$.

The essential property of minuscule weights is that all the coefficients in Pieri’s formula (1.4) are either zero or one, thus

$$
[D \cdot X(w)] = \left[ \sum_{w' \in W^P, w \rightarrow w'} X(w') \right].
$$

In other words, the intersection product between $D$ and $X(w)$ is represented by the union of the (reduced) Weyl divisors $X(w')$, with $w \prec w'$. This is the main reason for restricting ourselves in this article to minuscule homogeneous varieties.

**Definition 1.2** For $d \geq 0$, let $W^P_d := \{w \in W^P \mid \text{length}(w) = \dim X - d\}$ and

$$
X_d = \bigcup_{w \in W^P_d} X(w).
$$

be the union of $d$-dimensional Schubert subvarieties of $X$. It is a connected subvariety of $X$ (all the Schubert varieties contain the point $X(w_0^P) = X(1)$). For $w \in W^P_{d+1}$, consider

$$
D(w) := \bigcup_{w \rightarrow w' \in W^P_d} X(w').
$$

Usually $D(w)$ is called the **boundary** of the Schubert variety $X(w)$, and is denoted by $\partial X(w)$. Obviously, $D(w) \subset X_d$; the assumption that $X$ is minuscule implies

$$
\mathcal{L}_{X(w)} \cong \mathcal{O}_{X(w)}(D(w)).
$$

We will use the following important properties of Schubert varieties:

- $X(w), D(w)$ are (equidimensional) Cohen-Macaulay varieties (cf. [13, 11, 3, Corollary 2.2.7]);
- $D(w)$ is Frobenius split, actually its reduction in positive characteristics is so (cf. ibid.).
- For any $w_1, w_2 \in W^P$, the scheme theoretic intersection $X(w_1) \cap X(w_2)$ is reduced (cf. [13, Theorem 3], [3, Proposition 1.2.1]).

**Lemma 1.3** (i) For any $d \geq 1$, $w \in W^P_{d+1}$, the restriction homomorphisms below are bijective:

$$
\mathbb{Z}\mathcal{L} = \text{Pic}(X) \to \text{Pic}(X(w)),
$$

$$
\mathbb{Z}\mathcal{L} = \text{Pic}(X) \to \text{Pic}(D(w)).
$$
(ii) For \( d \geq 2 \) and \( w \) as above holds:
\[
H^t(D(w), \Lambda) = 0, \quad \forall 0 < t < \dim D(w) = d, \quad \forall \Lambda \in \text{Pic}(D(w)).
\]

**Proof.** (i) For the first claim, the restriction homomorphism is always surjective (cf. [15, Ch. 12], [3 Proposition 2.2.8]). The injectivity is trivial: if the restriction of a line bundle to \( X(w) \) is trivial, then its degree along \( \ell \) vanishes. For the second claim, \( D(w) \) is a union of Schubert varieties: pairwise, their intersections are reduced Schubert varieties of dimension \( d - 1 \geq 1 \) (or empty), all containing \( \ell \).

(ii) We have seen that \( \text{Pic}(D(w)) = \mathbb{Z}\mathcal{L}_{D(w)} \). For positive multiples of \( \mathcal{L}_{D(w)} \), the cohomology vanishing is [18 Theorem 2(a)]. For large negative multiples, use that \( D(w) \) is an equidimensional Cohen-Macaulay variety, so Serre duality holds. The vanishing for arbitrary (strictly) negative multiples of \( \mathcal{L}_{D(w)} \) is implied now by the fact that \( D(w) \) is Frobenius-split (cf. [4, Lemma 1.2.7]). \( \square \)

2. The main result

Henceforth, for any locally free sheaf \( \mathcal{V} \), we denote \( \mathcal{E} : = \text{End}(\mathcal{V}) \). We are going to use the following elementary lemma:

**Lemma 2.1** Let \( S \) be a closed subscheme of a scheme \( S' \), such that \( \Gamma(\mathcal{O}_S) = \Gamma(\mathcal{O}_{S'}) = \mathbb{C} \). Let \( \mathcal{V}' \) be a locally free sheaf on \( S' \); we assume that the restriction \( \Gamma(\mathcal{E}'(\mathcal{V})) \to \Gamma(\mathcal{E}(\mathcal{V})) \) is surjective. Then \( \mathcal{V}' \) splits if and only if \( \mathcal{V} \) splits.

**Proof.** See [9, §1]. \( \square \)

**Theorem 2.2** An arbitrary vector bundle \( \mathcal{V} \) on \( X \) splits if and only if \( \mathcal{V}_{X_d} \) splits.

**Proof.** The proof is by induction: we show that the splitting of \( \mathcal{V} \) along \( X_d \), with \( d \geq 2 \), implies its splitting along \( X_{d+1} \).

**Claim 1** For \( d \geq 1, w \in W_{d+1}^D \) holds:
\[
H^1(X(w), \mathcal{E} \otimes \mathcal{L}^{-1}) = 0.
\]
Indeed, since \( X(w) \) is Cohen-Macaulay (cf. (1.10)), the Serre duality implies
\[
a_0 : = \min\{a \geq 1 \mid H^1(X(w), \mathcal{E} \otimes \mathcal{L}^{-a}) = 0\} < +\infty.
\]
We claim that \( a_0 = 1 \). Otherwise, if \( a_0 \geq 2 \), the exact sequence
\[
0 \to \mathcal{L}_{X(w)}^{-a_0} \to \mathcal{L}_{X(w)}^{-(a_0-1)} \to \mathcal{L}_{D(w)}^{-(a_0-1)} \to 0
\]
tensored by \( \mathcal{E} \) yields:
\[
H^1(\mathcal{E} \otimes \mathcal{L}_{X(w)}^{-a_0}) \to H^1(\mathcal{E} \otimes \mathcal{L}_{X(w)}^{-(a_0-1)}) \to H^1(\mathcal{E} \otimes \mathcal{L}_{D(w)}^{-(a_0-1)}).
\]

The left-hand-side vanishes, by definition. The induction hypothesis says that \( \mathcal{E}_{D(w)} \) is a direct sum of line bundles, so lemma [13(ii)] implies that the right-hand-side vanishes too. Therefore the middle term vanishes, with \( a_0 - 1 \geq 1 \), which contradicts the minimality of \( a_0 \).

**Claim 2** \( \mathcal{V}_{X(w)} \) splits, for all \( w \in W_{d+1}^D \).
Indeed, the previous claim and the exact sequence
\[
0 \to (\mathcal{E} \otimes \mathcal{L}^{-1})_{X(w)} \to \mathcal{E}_{X(w)} \to \mathcal{E}_{D(w)} \to 0
\]
yields that \( \text{res}_{D(w)}^X(\mathcal{E}) : \Gamma(X(w), \mathcal{E}) \to \Gamma(D(w), \mathcal{E}) \) is surjective, and we apply the lemma [2.1]
Claim 3 \( \mathcal{V}_{X_{d+1}} \) splits. The splitting of \( \mathcal{V}_{X_d} \) implies that there is \( \varphi_d \in \Gamma(X_d, \mathcal{E}) \) with \( \text{rk}(\mathcal{V}) \) pairwise distinct eigenvalues, whose eigenspaces are the direct summands of \( \mathcal{V}_{X_d} \). As \( D(w) \subset X_d \), for any \( w \in W_{d+1} \), the surjectivity of \( \text{res}_{D(w)}^{X(w)} \) implies that \( \varphi_d|_{D(w)} \) extends to some \( \varphi_d(w) \in \Gamma(X(w), \mathcal{E}) \) which still has the same eigenvalues. Now define \( \varphi_{d+1} \in \Gamma(X_{d+1}, \mathcal{E}) \) by setting

\[
\varphi_{d+1}|_{X(w)} := \varphi_d(w), \ \forall w \in W_{d+1}.
\]

We claim that \( \varphi_{d+1} \) is well-defined: indeed, the intersection of any two distinct components \( X(w_1), X(w_2) \subset X_{d+1} \) is reduced (cf. \( \text{(1.10)} \)) and

\[
X(w_1) \cap X(w_2) \subset D(w_1) \cap D(w_2) \subset X_d.
\]

Moreover, we have \( \varphi_d(w_j)|_{D(w_j)} = \varphi_d|_{D(w_j)} \) and \( \varphi_d \) is defined over \( X_d \supset D(w_1) \cap D(w_2) \). \( \square \)

Remark 2.3 There are stable vector bundles of arbitrary rank on \( \mathbb{P}^3 \) with trivial splitting type along the generic line (the ADHM construction). Thus is not possible to verify the splitting of a vector bundle by restricting it to any finite union of test lines.

3. The list

In this section we perform a case-by-case analysis of the theorem \( \text{(2.2)} \). The minuscule varieties with cyclic Picard group are listed e.g. in \( [11, \S 5.2] \) and \( [13, \S 2.3] \). Some varieties are minuscule for several groups (e.g. \( \mathbb{P}^{2n-1} \) is minuscule for \( G = \text{SL}(2n) \) and \( \text{Sp}(n) \)). Since we are primarily interested in the variety itself, we choose the easier description (\( \text{SL}(2n) \) in the example). As we will see, it turns out that the 2-dimensional Schubert varieties are very simple geometric objects, although their embedding in the ambient homogeneous space can be subtle: they are mostly projective 2-planes.

(i) \( X = \text{Gr}(k; n) \), the Grassmannian of \( k \)-planes in \( \mathbb{C}^n \), is minuscule for \( G = \text{SL}(n) \). Let \( e_1, \ldots, e_n \) be the standard base in \( \mathbb{C}^n \).

(a) \( k = 1, n - 1 \Rightarrow X \cong \mathbb{C}^{n-1} \Rightarrow X_2 \cong \mathbb{C}^2 \).

(b) \( 1 < k < n - 1 \Rightarrow X_2 \cong \mathbb{C}^2 \cup_{\ell} \mathbb{C}^2 \).

So, \( X_2 \) is the union (wedge) of the Schubert 2-planes

\[
\{ \langle e_1, \ldots, e_{k-2}, f_{k-1}, f_k \rangle \mid f_{k-1}, f_k \in \langle e_{k-1}, e_k, e_{k+1} \rangle \},
\]

\[
\{ \langle e_1, \ldots, e_{k-2}, e_{k-1}, f_k \rangle \mid f_k \in \langle e_k, e_{k+1}, e_{k+2} \rangle \}
\]

glued along the line \( \ell = \{ \langle e_1, \ldots, e_{k-2}, e_{k-1}, f_k \rangle \mid f_k \in \langle e_k, e_{k+1} \rangle \} \).

(ii) \( X = S_n \), the Grassmannian of maximal isotropic (Lagrangian) \( n \)-planes in \( \mathbb{C}^{2n} \) with respect to a non-degenerate symmetric bilinear form (known as the spinor variety), is minuscule for \( G = \text{SO}(2n) \); its dimension equals \( \frac{n(n-1)}{2} \).

Let \( C \subset \ldots C^{n-1} \subset C^n \) be a flag of isotropic subspaces in \( \mathbb{C}^{2n} \). The unique 2-dimensional Schubert subvariety of \( X \) is:

\[
X_2 = \{ U \subset \mathbb{C}^{2n} \mid \dim U = n, C^{n-3} \subset U, \dim(U \cap C^n) \geq n - 2 \} \cong \mathbb{P}^2.
\]
(iii) The $2n$-dimensional quadric $Q_{2n} := \{x_0 y_0 + x_1 y_1 + \ldots + x_n y_n = 0\} \subset \mathbb{CP}^{2n+1}$, is minuscule for $G = SO(2n + 2)$. For $n \geq 3$, the 2-dimensional Schubert variety is

$$X_2 = \{x_0 = \ldots = x_n = y_3 = \ldots = y_n = 0\} \cong \mathbb{CP}^2.$$

For $n = 2$, we have $X_2 = \{x_0 = x_1 = x_2 = y_3 = \ldots = y_n = 0\} \cup \{x_0 = y_1 = y_2 = 0\} \cong \mathbb{CP}^2 \cup \mathbb{CP}^1 \mathbb{CP}^2$.

(iv) The Cayley plane $X = \mathbb{OP}^2$ is studied in detail in [12, 19]. It is a 16-dimensional variety, which can be identified with the projective plane over the octonions, and is minuscule for the exceptional group $G = E_6$. We are interested only in the fact that there is only one 2-dimensional Schubert cell, which is isomorphic to $\mathbb{CP}^2$; the embedding is explicitly described in [12, pp. 151].

(v) The Freundenthal variety is a 27-dimensional variety, minuscule for $G = E_7$. The Schubert varieties and the corresponding Hasse diagram can be found in [19, §3.1]. Again, there is only one smooth, 2-dimensional Schubert variety $X_2$. Since $X_2$ is rational and its Picard group is cyclic, we deduce that $X_2 \cong \mathbb{CP}^2$.

The conclusions of our discussion are summarized in the table below: Theorem 2.2 yields the following splitting criteria.

| Variety $X = G/P$ | group $G$ | $X_2 \cong \ldots$ | splitting criterion for $\mathcal{V}$ on $X$ |
|-------------------|-----------|---------------------|------------------------------------------|
| $\mathbb{P}^{n-1}$ | $\text{SL}(n)$, $n \geq 3$ | $\mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2}$ splits (Horrocks’ criterion) |
| $\text{Gr}(k; n)$, $1 < k < n - 1$ | $\text{SL}(n)$, $n \geq 4$ | $\mathbb{P}^2 \cup \mathbb{P}^1 \mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2 \cup \mathbb{P}^1 \mathbb{P}^2}$ splits |
| spinor variety $S_n$ | $\text{SO}(2n)$, $n \geq 3$ | $\mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2}$ splits |
| quadric $Q_4$ | $\text{SO}(6)$ | $\mathbb{P}^2 \cup \mathbb{P}^1 \mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2 \cup \mathbb{P}^1 \mathbb{P}^2}$ splits |
| quadric $Q_{2n}$, $n \geq 3$ | $\text{SO}(2n + 2)$ | $\mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2}$ splits |
| Cayley plane $\mathbb{OP}^2$ | $E_6$ | $\mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2}$ splits |
| Freundenthal variety | $E_7$ | $\mathbb{P}^2$ | $\mathcal{V} \iff \mathcal{V}_{\mathbb{P}^2}$ splits |

Let us observe that this analysis yields splitting criteria for vector bundles on the odd-dimensional quadric $Q_{2n+1}$, $n \geq 2$, as well: $Q_{2n} \subset Q_{2n+1}$ is the hyperplane section, and $\mathcal{V}$ on $Q_{2n+1}$ splits if and only if $\mathcal{V}_{Q_{2n}}$ splits (cf. [21, §2]). We deduce that:

$$\mathcal{V} \text{ on } Q_5 \text{ splits } \iff \mathcal{V}_{\mathbb{P}^2 \cup \mathbb{P}^1 \mathbb{P}^2} \text{ splits},$$

$$\mathcal{V} \text{ on } Q_{2n+1}, n \geq 3, \text{ splits } \iff \mathcal{V}_{\mathbb{P}^2} \text{ splits}. \quad (3.2)$$

**How to decide the splitting along $X_2$?** As the table above shows, the two dimensional Schubert subvarieties are mostly projective planes. So, one would like to have a tool to check the splitting of a vector bundle on $\mathbb{CP}^2$. 


Proposition 3.1 A vector bundle $\mathcal{V}$ on $\mathbb{P}^2$ splits if and only if $H^1(\mathcal{E}(-1)) = 0$.

Proof. The condition is necessary, so let us prove that it is also sufficient. Let $\ell \subset \mathbb{P}^2$ be a straight line. Then $\mathcal{V}_\ell$ decomposes into a direct sum of line bundles, by Grothendieck’s theorem: thus there is $\varphi_\ell \in \Gamma(\mathcal{E})$ with $\text{rk}(\mathcal{V})$ distinct eigenvalues. The hypothesis implies that $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}_\ell)$ is surjective, so $\varphi_\ell$ extends to $\varphi \in \Gamma(\mathcal{E})$ with the same eigenvalues. □

Note that, in the situation (3.1), the natural choice for $\ell$ is the (unique) Schubert line in $X$.

For the cases when $X_2 \cong \mathbb{CP}^2 \cup_\ell \mathbb{CP}^2$, we apply the proposition for both components.

4. Concluding remarks

It is natural to ask if there is a similar splitting criterion for vector bundles on homogeneous varieties which are not necessarily minuscule. This hypothesis was used (cf. (1.6)) to represent the class of the intersection product $D \cdot X(w)$ as a sum of reduced boundary divisors of $X(w)$. In general, Pieri’s formula (1.4) involves multiplicities. For this reason, one should consider in theorem 2.2 a union of thickenings of the Schubert surfaces, instead of the set-theoretical union $X_2$. However, the resulting form does not seem appealing.

Let us remark that in (3.1) and (3.2) there is one notable absence from the list of co-minuscule (also called Hermitian symmetric) varieties: the Lagrangian Grassmannian $LG(n)$.

If $\omega$ is a non-degenerate, skew-symmetric form on $\mathbb{C}^{2n}$, then define

$$LG(n) := \{ U \subset \mathbb{C}^{2n} \mid \dim(U) = n, \omega|_U = 0 \} \quad (4.1)$$

It is homogeneous for the action of the symplectic group $Sp(n)$, of dimension $\frac{n(n+1)}{2}$, and one may wonder whether a similar splitting criterion holds for $LG(n)$. The same approach as in section 2 fails; as explained above, one has to take into account multiplicities (for $LG(n)$, some of the coefficients in (1.4) equal 2). However, one can can still prove the following weaker result.

Theorem 4.1 A vector bundle $\mathcal{V}$ on $LG(n)$ splits if and only if its restriction to a very generally embedded $LG(2) \subset LG(n)$ does so.

Note that $Q_3 \cong LG(2) \subset Gr(2; 4) \cong Q_4$. Although the statement of the theorem is in the same vein as (2.2) the method of its proof is totally different and considerably more difficult (cf. \[8, \S 7.2\]).

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