A Survey on the Theory of Bonds

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Abstract

Many researchers tried to understand/explain the geometric reasons for paradoxical mobility of a mechanical linkage, i.e. the situation when a linkage allows more motions than expected from counting parameters and constraints. Bond theory is a method that aims at understanding paradoxical mobility from an algebraic point of view. Here we give a self-contained introduction of this theory and discuss its results on closed linkages with revolute or prismatic joints.

Introduction

By definition, the mobility of a mechanical linkage is the dimension of its configuration space. We say that a mechanical linkage moves paradoxically if the mobility is positive, but one does not expect this by counting parameters and constraints. We are especially interested in the case when the expected mobility is zero, but the linkage is still mobile. Examples are closed linkages with 6 revolute joints: for a generic choice of parameters, the closure equations have 16 complex solutions. But there are many families of special cases with mobility 1, such as Hooke’s linkage [3], Bricard’s line symmetric linkage[5], or Wohlhart’s partially symmetric linkage [21].

The theory of bonds was introduced in [12] as a tool for systematically explaining and analyzing paradoxical mobility of closed loops with only revolute joints. That paper contains a simplified proof of Karger’s classification of mobile closed 5R linkages (Karger’s original proof [13] uses computer algebra). In [10], the theory is used to prove that the genus of the configuration curve of a mobile 6R linkage is at most 5, and to classify all cases where the maximum is attained. In [17], the theory is used to obtain equations in the Denavit/Hartenberg parameters of a 6R linkage that are necessary for mobility. The paper [1] introduces

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bonds for prismatic joints and the paper \cite{18} introduces bonds for Stewart platforms. The theory can also be used if the mobility is bigger than one; however, in this paper we will focus on mobility one linkages.

The main purpose of this paper is to make the theory more accessible, because we think that there is still potential to derive new results on paradoxically moving linkages. The paper is therefore a survey on bond theory with a tutorial ambition.

In Section 1, we recall a well-known isomorphism between the Euclidean group \( SE_3 \) of direct isometries from \( \mathbb{R}^3 \) into itself and the quotient group of dual quaternions with nonzero real norm by the subgroup of nonzero real scalars. We use the language of dual quaternions to formulate configuration spaces of linkages and the closure equations. In Section 3, we recall a well-known description of all paradoxically moving closed \( nR \)-loops with revolute joints for \( n = 3, 4, 5 \); the classification of paradoxically moving closed \( 6R \)-loops is an open problem which will be the main question addressed in the subsequent sections. Bonds for \( R \)- and \( P \)-joints are introduced in Section 3; this section also contains properties that translate into geometric conditions on the Denavit/Hartenberg parameters of the linkage under consideration and a proposition illustrating the use of bond theory for showing non-trivial (but known) geometric conditions for paradoxically moving loops of type PRRRR. Section 4 introduces the bond diagram of a linkage, which is useful for “reading off” the degree of various coupler motions. The last section summarizes the current knowledge on paradoxically moving \( 6R \)-loops and their bond diagrams and points out open questions. It also contains the single new result of this paper (Example 5.2): by specializing a line symmetric linkage, one may obtain a linkage with three additional rotations in its configuration set.

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1 Dual Quaternions and the Closure Equations

The algebra \( \mathbb{D} \mathbb{H} \) of dual quaternions is defined as the 8-dimensional vector space over \( \mathbb{R} \) with basis \((1, i, j, k, \epsilon, \epsilon i, \epsilon j, \epsilon k)\). The multiplication is defined in the usual way for quaternions; the symbol \( \epsilon \) is supposed to commute with all quaternions and to fulfill the equation \( \epsilon^2 = 0 \). Every dual quaternion \( h \) can be written as \( h = p + \epsilon q \) with quaternions \( p, q \) called the primal and dual part of \( h \). Alternatively, we may write \( h \) as an expression \( h = a_0 + a_1 i + a_2 j + a_3 k \) with coefficients \( a_0, a_1, a_2, a_3 \in \mathbb{D} := \mathbb{R} \oplus \epsilon \mathbb{R} \) in the ring of dual numbers. The conjugate of \( h \) is defined as \( \overline{h} := a_0 - a_1 i - a_2 j - a_3 k \); conjugation is an anti-automorphism of \( \mathbb{D} \mathbb{H} \).

The norm of the dual quaternion is defined as \( N(h) := hh^\dagger \). Norm is a homomorphism of the semigroup \((\mathbb{D} \mathbb{H}, \cdot)\) to the semigroup \((\mathbb{D}, \cdot)\). Let \( \mathbb{S} := \{ h \in \mathbb{D} \mathbb{H} \mid N(h) \in \mathbb{R} \} \) and \( \mathbb{S}^* := \{ h \in \mathbb{D} \mathbb{H} \mid N(h) \in \mathbb{R}^* \} \), where \( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). Then \( \mathbb{S}^* \) is a group and \( \mathbb{R}^* \) is a normal subgroup. The quotient group \( \mathbb{S}^*/\mathbb{R}^* \) is isomorphic to the group \( SE_3 \), see \cite{19} Section 9.3; we may consider it as a
locally closed subset \( G \) in \( \mathbb{P}^7 \). Its closure is the Study quadric \( S \), represented by all dual quaternions in \( S \). We also introduce the null cone \( Y \) defined by the quadratic form \( p + \epsilon q \mapsto p\overrightarrow{p} \). For instance, if \( t \in \mathbb{R} \), then \([t - i]\) corresponds to a rotation around the first axis by an angle \( 2\arccot(t) \), and \([1 - \epsilon it]\) corresponds to a translation by a distance \( 2t \) in the direction of the first axis.

**Remark 1.1.** In order to parametrize the full rotation group around a fixed axis, we choose the parameter \( t \) in \( \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \). For any dual quaternion \( h \), the element \((\infty - h)\) is not finite but we still can consistently say that the class \([\infty - h]\) is equal to \([1]\). The corresponding group element is the identity.

A **linkage** is a collection of rigid bodies, called **links**, where two links may be connected by a **joint**. A joint restricts the relative position of the joined links. We consider two types of joints:

1. (R) revolute joints: allow rotations around a fixed axis;
2. (P) prismatic joints: allow translations in a fixed direction;

The **link graph** of a linkage is defined by putting a vertex for each link, and an edge whenever two links are joined by a joint. In order to specify the linkage completely, it suffices to specify the allowed subset of \( \text{SE}^3 \) for each joint.

Relative positions can be composed by the group operation: the relative position of link 1 with respect to link 2 times the relative position of link 2 with respect to link 3 is equal to the relative position of link 1 with respect to link 3. By multiplying relative positions in a cycle in the link graph, we get the **closure equations**. The solution set of the closure equations is the **configuration set** of the linkage.

**Example 1.2.** Let \( n \geq 3 \) be an integer. The link graph of a closed \( nR \) linkage is an \( n \)-cycle. For \( r = 1, \ldots, n \), the set of allowed relative position can be written as \( \{[t_r - h_r] \mid t_r \in \mathbb{R} \cup \{\infty\}\} = \{[t_r - h_r] \mid t_r \in \mathbb{R}\} \cup \{[1]\} \), where \( h_r \) is a dual quaternion such that \( h_r^2 = -1 \) specifying the rotation axis in an initial position.

This gives the closure equation

\[
[(t_1 - h_1)(t_2 - h_2) \ldots (t_n - h_n)] = [1].
\]

The class on the left hand side is \([1]\) if and only if 7 of the 8 coordinates of the product are zero, hence we have 7 polynomial equations in \( t_1, \ldots, t_n \). Actually, one of the 7 equations is redundant, because it is clear that the product is contained in the Study quadric \( S \); and if \( h = a_0 + a_1i + a_2j + a_3k \), \( a_0, \ldots, a_3 \in \mathbb{D} \), is a dual quaternion with norm in \( \mathbb{R}^* \) such that \( a_1 = a_2 = a_3 = 0 \), then it follows that \( h \in \mathbb{R}^* \).

Assume \( n = 6 \). For generic choice of \( h_1, \ldots, h_6 \in \mathbb{D} \mathbb{H} \) such that \( h_r^2 = -1 \) for \( r = 1, \ldots, 6 \), one gets at most 16 real solutions for these 6 equations in \( t_1, \ldots, t_6 \), including the solution \( t_r = \infty \) for \( r = 1, \ldots, 6 \). The number of complex solutions is infinite, but if one excludes solutions contained in the null cone \( Y \), then one generically gets 16 complex solutions \([19\text{, pp. 262–264}]\).
**Example 1.3.** In the case of a closed nR linkage (or more general simply closed linkages), it is possible to use a more invariant specification which does not depend on the choice of an initial position. In essence, this is the method of Denavit/Hartenberg \[7\]. For \( r = 1, \ldots, n \), let \( \phi_r \) be the angle between the \( r \)-th and \((r + 1)\)-th rotation axis, with indices modulo \( n \); let \( d_r \) be the normal distance between these axes; let \( s_r \) be the signed distance of the intersections of the common normals of neighboring axes on the \( r \)-th rotation axis. For each linkage, we introduce an internal frame of reference in which the first joint is the first coordinate axes, the intersection with the common normal with the previous axes is the origin, and the second axis lies in the first coordinate plane. When we express relative positions in this frame of reference, then the allowed positions are composed by a translation in direction of the first axes by \( s_r \), rotation around first axis by \( \phi_r \), translation in direction of the second axis by \( d_r \), and rotation around the second axis by an arbitrary angle. So the closure equation is

\[
[(t_1 - i) g_1 (t_2 - i) g_2 \cdots (t_n - i) g_n] = [1],
\]

where

\[
g_r = \left( 1 - \frac{s_r^2}{2} \epsilon_i \right) (w_r - k) \left( 1 - \frac{d_r}{2} \epsilon_k \right)
\]

and \( w_r = \cot \left( \frac{\phi_r}{2} \right) \) for \( r = 1, \ldots, n \). If \( \phi_r \) is a multiple of \( \pi \) for some \( r \), then we set \( w_r = \infty \) and \( g_r = \left( 1 - \frac{\phi_r}{2} \epsilon_i \right) (1 - \frac{d_r}{2} \epsilon_k) \).

Similar as in the previous example, we get 6 equations in \( n \) parameters, and if \( n = 6 \), then for generic choice of the parameters \( s_1, w_1, d_1, \ldots, s_6, w_6, d_6 \) we get 16 isolated complex solutions. (But now there is no trivial solution at infinity.)

**Remark 1.4.** The invariant parameters for a closed nR linkage above do depend on a choice of orientation of the rotation axis. Generically, there are \( 2^n \) choices leading to different parameters. If we change the orientation of the \( k \)-th axis, then \( w_k \) gets replaced by \(-1/w_k\) and \( s_k \) gets replaced by \(-s_k\), and all other parameters stay the same. Also, the parameter \( d_k \) can be replaced by \(-d_k\) without changing the linkage, one just needs to reparametrize the sets of allowed positions (replacing \( t_k \) by \(-1/t_k\)).

If \( w_k = 0 \) for some \( k \), then one can add a constant to \( s_k \) and \( s_{k-1} \) without changing the linkage, because then the axes are parallel and the common normal is not unique. Similarly, if \( w_k = \infty \), then one can add a constant to \( s_k \) and subtract the constant from \( s_{k-1} \).

**Example 1.5.** With the notation from Example [1.3] we set \( n = 4 \) and

\[
(w_1, w_2, w_3, w_4) = (1, 2, 1, 2),
\]

\[
(d_1, d_2, d_3, d_4) = (4, 5, 4, 5),
\]

\[
(s_1, s_2, s_3, s_4) = (0, 0, 0, 0).
\]
The closure equation (2) for \((t_1, t_2, t_3, t_4)\) can be simplified using computer algebra (we used Maple). The simplified system is

\[
3t_1t_2 + 1 = t_1 + t_3 = t_2 + t_4 = 0, \quad t_1^2 + 1 \neq 0, \quad t_2^2 + 1 \neq 0.
\]

Its solution set is a curve that can be easily parametrized; it is

\[
(t_1, t_2, t_3, t_4) = \left(t, -\frac{1}{3t}, -t, \frac{1}{3t}\right).
\]

This linkage is therefore mobile. It is an example of a Bennett linkage [4] (Figure 1). The general description of Bennett linkages is given by the equations

\[
s_1 = s_2 = s_3 = s_4 = 0, \quad w_1 = w_3 \neq 0, \infty, \quad w_2 = w_4 \neq 0, \infty
\]

\[
d_1 = d_3 \neq 0, \quad d_2 = d_4 \neq 0, \quad \frac{2d_1w_1}{w_1^2 + 1} = \frac{2d_2w_2}{w_2^2 + 1},
\]

which lead to a similar one-dimensional solution set.

2 Mobile Closed 4R and 5R Linkages

For \(n = 3, 4, 5\), and for generic choice of parameters, the closure equation (1) has only the trivial solution, and the closure equation (2) has no solution at all (disregarding the complex solutions with factors in the null cone \(Y\)). Nevertheless, there are special cases of mobile linkages of these types. Our main question is what are the implications of the assumption of mobility for the structure of a linkage. The hope is to find enough conditions that allow a classification of mobile loops with 4, 5 (and later 6) links.
Example 2.1. If \( h_1 = h_s \) for some \( s, 2 \leq s \leq n \), then the closure equation (1) has the solution

\[
    t_s = -t_1, t_r = \infty \quad \text{for } r \neq 1, s.
\]

These are configurations where two of the axes coincide, and one part of the linkage just rotates about the coinciding axes.

It is easy to show that every mobile 3R linkage is of this type. In case all 3 axes coincide, the mobility is 2.

Example 2.2. Assume that \( n = 4 \). Assume that the dual parts of \( h_1, \ldots, h_4 \) are all zero. Then the dual part of the left hand side of equation (1) is automatically zero and the closure equation boils down to three polynomial equations in \( t_1, t_2, t_3, t_4 \). In general, the solution has complex dimension 1.

Geometrically, the vanishing of the dual part means that all four axes pass through the origin. This type of linkage is known as spherical 4R linkage. It can also be characterized by the conditions

\[
    s_1 = s_2 = s_3 = s_4 = d_1 = d_2 = d_3 = d_4 = 0.
\]

A similar case (which may actually be considered as limiting case of a spherical linkage) is the planar 4R linkage, where all 4 axes are parallel.

It is well-known [6] that every mobile 4R linkage either has two coinciding axes or is spherical, planar, or Bennett.

Example 2.3. Let \( n = 4 \). Let \( h_1, h_2, h_3, h_4 \) be the dual quaternions defining the rotation axes of a mobile 4R linkage (e.g. spherical). Choose an arbitrary dual quaternion \( h_5 \) such that \( h_2 h_5 = -1 \). Then any configuration of the mobile 4R linkage can be extended to a configuration of the 5R linkage by setting \( t_5 := \infty \). Hence the 5R linkage is again movable.

The 5th joint in this linkage remains frozen during this particular motion.

Example 2.4. Consider the triply closed linkage with 6 links that are connected according to the link graph in Figure 2. Its mobility is one: the motion of the joint corresponding to the vertical edge in the middle determines the motion of the two Bennett 4R linkages, and then the motion of the 3R linkage below is also determined. If we remove the link in the center, then we get a mobile closed 5R linkage. This construction is due to Goldberg [9].

It is well-known that every mobile closed 5R linkage either has coinciding axes, or a frozen joint, or is planar, spherical, or a Goldberg linkage. The original proof [13] uses computer algebra; a simpler proof without computers is based on bond theory [12].

Let \( \mathcal{L}_n := \mathbb{R}^{2n} \times (\mathbb{P}^1)^n \) denote the parameter space of closed nR-linkages. Let \( \mathcal{D}_n := \mathcal{L}_n \times (\mathbb{P}^1)^n \) be the Zariski closure of the set of all solutions of equation (2). The projection \( \mathcal{D}_n \to \mathcal{L}_n \) is a proper morphism, hence the dimension of its fiber is upper semicontinuous in the Zariski topology as a function in \( \mathcal{L}_n \). The subset \( \mathcal{M}_n \subset \mathcal{L}_n \) of all parameters of mobile linkages is then also
Zariski closed, i.e., it is a subset defined by algebraic equations in the parameters $s_1, w_1, d_1, \ldots, s_n, w_n, d_n$. The linkages with parameters in $\mathcal{M}_n$ have an infinite solution set over the complex numbers. It is possible that none of these solutions are real, for instance for a planar linkage where $d_1 > d_2 + d_3 + d_4$.

For $n = 3, 4, 5$, these equations are known (for $n = 3, 4$, we essentially gave them above; for $n = 5$, see [8]). The dimension of $\mathcal{M}_n$ is determined by the largest components, corresponding to linkages with coinciding/parallel axes ($\dim(\mathcal{M}_3) = 3$, $\dim(\mathcal{M}_4) = 7$, $\dim(\mathcal{M}_5) = 10$); but the more interesting components have smaller dimension (Bennett linkages form a 3-dimensional component of $\mathcal{M}_4$, and Goldberg linkages form a 5-dimensional component of $\mathcal{M}_5$). For $\mathcal{M}_6$, we do know the dimension: it is 14, again because of components with coinciding axes. Several other components are known and will be discussed in the following. Even more algebraic subsets of $\mathcal{M}_6$ are known which are not contained in any known component, but for which it is not clear whether they form a component or they are properly contained in some yet unknown component. It is an open problem to determine all components and to give equations for them. A broad discussion of partial results can be found in [15].

3 Definition and First Properties of Bonds

Assume that we have a linkage with $e$ joints. If the $k$-th joint is of type R, then the set of allowed motions can be parametrized by $t_k \mapsto m_k := (t_k - h_k)g_k$ for some $h_k, g_k \in \mathbb{D}\mathbb{H}$ of norm 1 with $h_k^2 = -1$, with $t_k \in \mathbb{R}$. If we pass to classes, we may even allow $t_k \in \mathbb{P}_k^1$, see Remark [1.1]. The parametric dual quaternion $m_k$ also appears as a factor in closure equations. Note that $N(m_k) = t_k^2 + 1$.

If the $k$-th joint is of type P, then the set of allowed motions can be parametrized by $t_k \mapsto m_k := (t_k - \epsilon p_k)g_k$ for some purely vectorial $p_k \in \mathbb{H}$ (that is, $p_k + \overline{p}_k = 0$) and $g_k \in \mathbb{D}\mathbb{H}$, both of norm 1, with $t_k \in \mathbb{R} \setminus \{0\}$. Here, we have $N(m_k) = t_k^2$. 

Figure 2: The link graph of a triply closed linkage with 6 links and 8 joints (three joints with coinciding axes) of mobility 1. If the link in the center is removed, one obtains a Goldberg 5R linkage.
Recall that the configuration set $K$ is the set of all $(t_1, \ldots, t_e)$ such that $m_{i_1} \ldots m_{i_e} \in \mathbb{R}^*$ for all loops with edges $i_1, \ldots, i_e$. In order to define bonds, we have to allow also complex parameters $t_1, \ldots, t_e$. Recall that the Zariski closure of any set $X$ is defined as the set of all points, maybe with complex coordinates, which satisfy all polynomial equations that are satisfied by all points in $X$. The Zariski closure of $K$ in the product of complex projective lines is denoted by $\overline{K}$. The set $B$ of bonds is defined as the set of all elements $(t_1, \ldots, t_e) \in K$ such that at least one of the $m_k(t_k)$ has norm zero. For a fixed bond, the subset of joints $k$ such that $N(m_k(t_k)) = 0$ are called the joints attached to the bond. The bond also induces a partition of all links into the connected components of the subgraph which is obtained by deleting all attached joints.

The following proposition guarantees that the set of bonds of a mobile linkage is non-empty.

**Proposition 3.1.** A joint is frozen in a linkage if and only if it is not attached to any bond.

**Proof.** If the $k$-th joint is frozen, then $t_k = c$ for some constant $c$ for all configurations in $K$. Hence we also have $t_k = c$ for all points in $\overline{K}$, which includes all bonds. Hence $N(m_k) \neq 0$.

Conversely, if the $k$-th joint is not attached to any bond, then the projection from $\overline{K}$ to the coordinate $t_k$ is not surjective. On the other hand, the image is a closed subvariety. It follows that the image is finite and therefore the joint is frozen.

**Example 3.2.** The Zariski closure of the configuration set of the Bennett linkage in Example 1.5 is

$$\{(t_1, t_2, t_3, t_4) \mid 3t_1t_2 + 1 = t_1 + t_3 = t_2 + t_4 = 0\}.$$  

It has 4 bonds:

$$\beta_1 := (i, -i/3, -i, i/3), \quad \beta_2 := (-i, i/3, i, -i/3),$$

$$\beta_3 := (i/3, -i, -i/3, i), \quad \beta_4 := (-i/3, i, i/3, -i).$$

The joints 1 and 3 are attached to $\beta_1$ and $\beta_2$. The joints 2 and 4 are attached to $\beta_3$ and $\beta_4$.

**Proposition 3.3.** Let $i_1, \ldots, i_k$ be a set of joints forming a path in the link graph such that starting point and ending point are in the same subset of the partition induced by the bond $\beta = (t_1, \ldots, t_e)$. If at least one of the joints is attached to the bond $\beta$, then $m_{i_1}(t_{i_1}) \cdots m_{i_k}(t_{i_k}) = 0$.

**Proof.** Since the starting and ending point are in the same subset of the partition, there exists a path $(j_1, \ldots, j_l)$ of joints not attached to the bond $\beta$, with the same starting and ending point. The closure equation

$$[m_{i_1} \cdots m_{i_k}] = [m_{j_1} \cdots m_{j_l}]$$
is valid for all configurations, but it is not valid for $\beta$ because the left side has norm zero and the right side has norm different from zero. Since $\beta$ is in the closure of the configuration set, there is only one possibility: the left side is not defined, because the product is zero.

**Example 3.4.** In our running example of the Bennett linkage (Example 1.5 and Example 3.2), the following equations and their conjugate counterparts obtained by replacing $i$ by $-i$ are valid:

\[
(i - i)g_1(-i/3 - i)g_2(-i - i)g_3 = 0,
\]
\[
(i - i)g_2(i/3 - i)g_3(-i - i)g_4 = 0,
\]
\[
(i - i)g_3(-i/3 - i)g_4(-i - i)g_1 = 0,
\]
\[
(i - i)g_4(i/3 - i)g_1(-i - i)g_2 = 0.
\]

Assume that we have a minimal chain $i_1, \ldots, i_k$ such that $m_{i_1}(t_{i_1}) \cdots m_{i_k}(t_{i_k}) = 0$ for some fixed bond $(t_1, \ldots, t_k)$. Then $N(m_{i_1}(t_{i_1})) = N(m_{i_k}(t_{i_k})) = 0$ - otherwise we could multiply by $m_{i_1}(t_{i_1})$ from the left or by $m_{i_k}(t_{i_k})$ from the right and make the chain shorter. The condition

\[
m_{i_1}(t_{i_1}) \cdots m_{i_k}(t_{i_k}) = 0, \quad N(m_{i_1}(t_{i_1})) = N(m_{i_k}(t_{i_k})) = 0 \quad (5)
\]

is called bond condition. The validity of a bond condition for some chain in the link graph has some interesting geometric consequences on the geometric parameters of the linkage.

**Lemma 3.5.** 1. If joint 1 is of type $R$ and joint 2 is of type $R$ or $P$, then the bond condition 1-2 is never valid.

2. If joint 1 is of type $P$ and joints 2 and 3 of type $R$, and the bond condition 1-2-3 is valid, then the axes of joints 2 and 3 are parallel.

**Proof.** (1): Assume that joints 1 and 2 are of type $R$, and the axis have distance $d$ and twist angle $2\arccot(w)$. Assume, without loss of generality, that the bond coordinates at joints 1 and 2 are both $i$ and not $-i$ (this can always be achieved by a change of orientation of the axes). With

\[
g = (w - k) \left(1 - \frac{d}{2}e_k\right),
\]

the bond condition reduces to the equation

\[
(i - i)g(i - i) = (2wi - edi)(i - i) = 0,
\]
hence $w = d = 0$ and the axes are equal.

Assume now that joint 2 is of type $P$. Then the bond coordinate at the second joint is 0, and the bond condition has the form $(t_1 - i)e_p = 0$ for some quaternion $p$ specifying the direction of the $P$-joint. Since $p$ is invertible, it follows that $t_1 - i = 0$, which is impossible.

The statement (2) reduces to a similar short and straightforward calculations. \qed
For any chain $c := (i_1, \ldots, i_k)$, the coupling space $L_c$ is defined as the linear subspace of $DH$ generated by all products $m_{i_1}(t_{i_1}) \cdots m_{i_k}(t_{i_k})$, where $t_{i_1}, \ldots, t_{i_k}$ range over the full parameter space. If all joints are of type R or P, then $L_c$ has a generating set of cardinality $2^k$ which can be obtained by expanding the product and taking all coefficients with respect to $t_{i_1}, \ldots, t_{i_k}$. The projectivization of the coupling spaces contains the coupling varieties, consisting of all relative positions of the two links that are connected by the chain.

Lemma 3.6. Let $c := (i_1, \ldots, i_k)$ be a chain of joints.

1. If the joint $i_1$ or the joint $i_k$ is of type R, then $\dim(L_c)$ is even.

2. If $\dim(L_c) = 2$, then all joints are of the same type and have the same axis (for R-joints) resp. directions (for P-joints).

3. If all joints are of type R and $\dim(L_c) = 4$, then all axes are parallel or pass through a common point.

4. If $k = 3$ and there is a bond such that the bond condition for $c$ is valid, then $\dim(L_c) < 8$.

Proof. This is [12, Theorem 1]. The proofs of (1) and (4) do give some insight, so we include them here.

(1): assume that $i_1$ is an R-joint. Then $L_c$ is closed under multiplication by $h_1$ from the left. Since left multiplication by $h_1$ is a linear map whose square is negative identity, it follows that $L_c$ may be considered as a vector space over $\mathbb{C}$. Its real dimension is two times its dimension over $\mathbb{C}$, which proves the claim.

(4): Expanding the bond condition, we obtain a nontrivial linear equation between the products generating $L_c$. Hence these products cannot be linearly independent.

Assume that we have a chain 1-2-3 of three joints of type R. Then we say that the chain satisfies the Bennett condition if $\dim(L_c) = 6$. The following proposition expresses the condition in terms of the Denavit/Hartenberg parameters. The proof is straightforward.

Lemma 3.7. Let $d_1, d_2, w_1, w_2, s_2$ be the distances, angles, and offset of a 3-chain of R-joints. Then the Bennett condition is equivalent to

$$s_2 = 0, \quad \frac{2d_1w_1}{w_1^2 + 1} = \frac{2d_2w_2}{w_2^2 + 1}$$

(compare with Equation 4).

The lemmas 3.5, 3.6, and 3.7 above give necessary conditions on the geometric parameters of a linkage for the existence of bonds. The following proposition, which is taken from [1, Theorem 6], demonstrates how these lemmas are applied to classify linkages with a given link diagram and types of joints.

Proposition 3.8. Consider a mobile closed PRRRR linkage. Then one of the following conditions must be satisfied.
1. The P-joint is frozen.

2. Two of the axes of the rotational joints coincide.

3. Three of the rotational axes are parallel, and the fourth axes is frozen.

4. The axes of joints 2 and 3 are parallel, and the axes of joints 4 and 5 are parallel.

Proof. By Lemma 3.5 we get that either joints 2 and 3 are parallel or joints 4 and 5 are parallel. In order to show that actually both are necessary, we consider the closure equation modulo $\epsilon$. If, say, joints 2 and 3 are parallel, and joints 4 and 5 are not, then there are three different rotation axes, since parallel axes only differ in their dual part. Modulo $\epsilon$, we get a closure equation of a 3R loop with three different axes, but such a link is never movable.

For chains of four R-joints, the bond condition does not imply geometric conditions on the four axes. Indeed, given four generic dual quaternions $h_1, h_2, h_3, h_4$, there are two solutions of the equation

$$(i - h_1)(t_2 - h_2)(t_3 - h_3)(i - h_4) = 0$$

in the unknowns $t_2, t_3$. This gives room for two bonds with $t_1 = t_4 = i$. Varying the signs of $t_1, t_4$, one has up to 8 choices for the coordinates $(t_1, t_2, t_3, t_4)$ of a bond in such a chain.

In a 6R loop with joints 1-2-3-4-5-6-1, one can get a geometric condition by comparing the possible solutions of bond coordinates in the chains 1-2-3-4 and 4-5-6-1. The idea is to take into account the equality

$$[(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)] = [(t_6 + h_6)(t_5 + h_5)(t_4 + h_4)],$$

which holds for all points in the configuration set, and therefore also for the bonds in case both sides are defined. The lengthy calculations have been done in [17] for 6R linkages given in terms of their Denavit/Hartenberg parameters $d_1, w_1, s_1, \ldots, d_6, w_6, s_6$. For $i = 1, \ldots, 6$, we define $c_i := \frac{w_i^2 - 1}{w_i^2 + 1}$ and $b_i := \frac{2d_i w_i}{w_i^2 + 1}$. Then we define the quad polynomial as the quadratic polynomial in a variable $x$

$$Q_i^+(x) = \left(x + \frac{b_3 c_3 - b_1 c_1}{2} - \frac{s_1 i}{2}\right)^2 +$$

$$\frac{i}{2} \left(b_1 s_2 + b_3 s_3 + s_2 b_3 c_2 + s_3 b_1 c_2\right) -$$

$$b_1 b_3 c_2 - s_2 s_3 c_2 + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2 c_2^2}{4}.$$

For $i = 2, \ldots, 6$, we define the quad polynomial $Q_i^+(x)$ by a cyclic shift of indices that shifts 1 to $i$. Finally, we define $Q_i^-(x)$ by replacing the parameters
c_1, \ldots, c_6, b_1, \ldots, b_6 \text{ and } s_2, s_4, s_6 \text{ by their negatives, and leaving } s_1, s_3, s_5 \text{ as they are. For instance,}
\begin{align*}
Q_1^-(x) &= \left( x + \frac{b_3 c_3 - b_1 c_1}{2} - \frac{s_1}{2} \right)^2 + \\
&\quad \frac{i}{2} \left( b_1 s_2 - b_3 s_3 - s_2 b_3 c_2 + s_3 b_1 c_2 \right) - \\
&\quad \frac{-b_1 b_3 c_2 - s_2 s_3 c_2}{2} + s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2 c_2^2.
\end{align*}

**Theorem 3.9.** Let \( k \) be the number of bond connections of 1 and 4. Then
\[
k \leq \text{deg}(\text{gcd}(Q_1^+, Q_4^+)) + \text{deg}(\text{gcd}(Q_1^-, Q_4^-)).
\]

One can use this to derive a necessary condition for the existence of a bond connecting 1 and 4, since two polynomials have a nontrivial gcd if and only if their resultant is zero. The equations get simpler if one assumes that some of the gcd’s have degree 2, because then the two quad polynomials need to be equal.

Similar conditions should be possible for 6-loops with P-joints, but the equations have not yet been derived, and so its consequences are not yet known.

### 4 Bond Diagrams

The combinatorial structure of the bonds – how many bonds are attached to which joints – can be visualized in a diagram. This diagram can be used to read off the degree of coupling curves.

Consider a linkage of mobility 1 with \( e \) joints. Let \( K \) be its configuration set and let \( \beta = (\beta_1, \ldots, \beta_e) \in K \) be a bond (\( \overline{K} \) denotes the Zariski closure of \( K \)). For any two links \( i, j \), define the **coupling map** \( f_{i,j} : K \to SE3 \) as the map that computes the relative position of link \( j \) with respect to the link \( i \). We define the **local distance** \( d_{\beta}(i, j) \) as the order of the Taylor expansion of the analytic function \( N \circ f_{i,j} \circ \lambda \) divided by 2, where \( \lambda \) is a local parametrization of \( \overline{K} \) around \( \beta \).

**Example 4.1.** In the Bennett linkage of Example 1.5 whose bonds are given in Example 3.2, we consider the bond \( \beta_1 = (i, -i/3, -i, i/3) \). A local parametrization \( \lambda \) of \( \overline{K} \) is \( (t - i, (t + i)/3, -t + i, (t - i)/3) \). We label the links cyclically, the link attached to joints 1 and 2 is indexed by 2. Then \( f_{4,2} \circ \lambda \) is \( (t - i - 1)g_1((-t + i)/3)g_2 \), and its norm has Taylor expansion
\[
N((-t + i)N(g_1)N((-t + i)/3 - i)N(g_2) = -160/9it + 40/9t^2 + O(t^3),
\]
which is of order 1. Hence \( d_{\beta_1}(4, 2) = 1/2 \).

For the other links, one gets \( d_{\beta_1}(4, 1) = d_{\beta_1}(1, 4) = d_{\beta_1}(2, 3) = d_{\beta_1}(3, 2) = 1/2 \), and all other values of \( d_{\beta_i} \) are zero.
Lemma 4.2. The local distance is a pseudo-metric on the set of links, i.e. for any three links \(i, j, k\) we have
\[
\begin{align*}
d_{\beta}(i, i) &= 0, \quad d_{\beta}(i, j) = d_{\beta}(j, i), \\
d_{\beta}(i, k) &\leq d_{\beta}(i, j) + d_{\beta}(j, k).
\end{align*}
\]
Moreover, the perimeter \(d_{\beta}(i, j) + d_{\beta}(j, k) + d_{\beta}(k, i)\) of the triangle \((i, j, k)\) is even.

Lemma 4.3. Assume that the link graph contains a chain from \(i\) to \(k\) passing \(j\), for which the bond condition imposed by \(\beta\) is not valid. Then
\[
d_{\beta}(i, k) = d_{\beta}(i, j) + d_{\beta}(j, k).
\]

Proof. Both Lemmas are consequences of [12, Theorem 3]. The necessary adaptations to include prismatic joints are easy.

The vast majority of bonds we studied so far have a very simple local distance: the link graph is partitioned into two subsets; \(d_{\beta}(i, j) = \frac{1}{2}\) if \(i\) and \(j\) lie in different subsets, and \(d_{\beta}(i, j) = 0\) otherwise (see e.g. Example 4.1 above). We visualize linkages with these bonds by adding to the link diagram additional lines connecting the edges, one for each conjugated pair of bonds, that separate the vertices in the same way. Figure 3 shows the bond diagrams of the Bennett 4R linkage and of the Goldberg 5R linkage.

For any two links \(i, j\), the algebraic degree of the coupling curve \(C_{i,j}\) is defined as the number of all points \(p \in \mathbb{P}^7\) such that \(f_{i,j}(p)\) lies in a fixed generic hyperplane of \(\mathbb{P}^7\). If \(f_{i,j} : \mathbb{P}^7 \rightarrow C_{i,j}\) is birational, then this is simply the degree of \(C_{i,j}\) as a curve in \(\mathbb{P}^7\). In general, it is equal to the degree of \(C_{i,j}\) multiplied with the mapping degree of \(f_{i,j}\), i.e. the number of preimages of a generic point of \(C_{i,j}\).
Figure 4: The bond diagrams of Dietmaier’s 6R linkage and of Wohlhart’s double Goldberg linkage. The degree of the coupler curve $C_{25}$ is 8 in Dietmaier’s linkage and 4 in Wohlhart’s linkage (see also Example 4.6 and Example 4.7).

**Theorem 4.4.** For any two links, the algebraic degree of $C_{i,j}$ is equal to the sum of all local distances $d_\beta(i,j)$ over all bonds $\beta$.

**Proof.** For R joints, this is [12, Theorem 4]. The adaptions to make it work for P joints are easy to make. The idea of the proof is the following: instead of computing the algebraic degree by intersection with a hyperplane, we can also intersect with a quadric not containing any coupler curve. We take the null cone $Y$. Then we do the counting taking multiplicities into account, and finally divide by 2.

In the bond diagram, the local distance of a bond visualized by a single line separating the link diagram into two is 0 for any two vertices in the same subset and it is 1 2 for any two vertices in different subsets. Summing up over all bonds, and taking into account that conjugate bonds separate the same links, we obtain that the algebraic degree of a coupling curve is equal to the number of lines that must be crossed when one draws a line between the two vertices.

**Example 4.5.** Figure 3 shows the bond diagram of Goldberg’s 5R linkage. The coupling curves $C_{13}$, $C_{35}$, and $C_{24}$ are conics (the two first actually appear in the Bennett linkage in the construction). The coupling curves $C_{14}$ and $C_{25}$ are cubics. The remaining 5 coupling curves are lines, parametrizing rotations around the joint axes. The coupling map $f_{15}$ is 2:1, all other coupling maps are birational.

**Example 4.6.** The mobile 6R linkage found by Dietmaier [8] can be characterized by the following condition: the coupling spaces of two disjoint chains of length 3 both have dimension 6, and intersect in a space of dimension 5. Then the coupling curve with respect to the two links connected by the two chains is contained in the projectivization of the intersection, which is a $\mathbb{P}^4$. It is defined by three quadratic equations and therefore it has degree 8.
Figure 5: Bond diagrams of multiply closed linkages consisting of Bennett 4R linkages and 3R linkages with the same axis ("s.a.") in all three joints, including bond diagrams. If one leaves away the links in the interior, then one obtains Waldron’s double Bennett linkage, the Goldberg L-form 6R linkage, Goldberg’s serial 6R linkage, Goldberg’s second serial 6R linkage, the cube linkage, and Wohlhart’s double Goldberg linkage.

The bond diagram is shown in Figure 4. One can see the algebraic degree of $C_{25}$. Also, the Bennett conditions need to hold for two triples of axes because of the bond connections 1-5 and 2-4.

**Example 4.7.** In [22], Wohlhart constructed a movable 6R linkage by combing four Bennett linkages (see Example 4.8 for the construction). Its bond diagram is resembling the bond diagram of Dietmair’s linkage. The dimensions of coupling spaces are the same, and the number of bonds attached to any pair of joints is exactly half as in Example 4.6. Consequently, the degree of each coupler curve is exactly half of the degree of the corresponding coupler curve in Dietmaier’s linkage.

**Example 4.8.** By combining Bennett linkages and 3R linkages such that all 3 axes coincide, one can construct various multiply closed linkages. Leaving away the linkages with vertices drawn in the interior of the planar representation of the link graph, we obtain the following 6R linkages: Waldron’s double Bennett linkage [20], the Goldberg L-form 6R linkage, Goldberg’s serial 6R linkage, Goldberg’s second serial 6R linkage (all in [9, 2]), the cube linkage [11], and Wohlhart’s double Goldberg linkage [22]. Figure 5 shows the bond diagrams of the multiply closed linkages.

A complete list of all linkages with three or four conjugate pairs of bonds can be found in [15].
5 Open questions for 6R linkages

Bond theory has helped to discover new families of 6R linkages (see [12, 14, 17]), sometimes containing known families (see [10]). The possible list of bond diagrams is finite. For some of these diagrams, we know all linkages, for other diagrams we do have examples but no proof of completeness, and for other possible diagrams we do not if they appear as the diagrams of any linkage. In this section, we give a summary of the open cases.

In this overview we exclude degenerate cases where one of the joints is frozen or where two consecutive axes coincide. Also, we exclude cases where three consecutive axes are incident to a single point; then the three joints can be replaced by a spherical joint, and the mobile SRRR linkages are well understood. We also exclude cases with three consecutive axes being parallel, since they can be seen as limit cases of three consecutive axes that are incident to a single point. By Lemma 3.6, the dimension of the coupling spaces $L_{i,i+1,i+2}$ is either 6 or 8. We label the links and joints cyclically modulo 6, the link attached to joint 1 and 2 has index 1.

We start by restricting the possible candidates of bond diagrams of such 6R linkages. We may distinguish “long bond connections” connecting joints 1-4, 2-5, or 3-6, and “short bond connections” connecting $i-i+2$ for $i = 1, \ldots, 6$ modulo 6. By Lemma 3.5, consecutive joints are not connected.

Theorem 5.1. For any mobile 6R linkage, the following conditions on the bonds are known.

1. Any joint is attached to at most 4 conjugate pairs of bonds (counted with multiplicities).

2. If $\dim(L_{i,i+1,i+2}) = \dim(L_{i+1,i+2,i+3}) = 6$, then there is no bond connecting joints $i$ and $i + 3$.

3. If $\dim(L_{i,i+1,i+2}) = \dim(L_{i+3,i+4,i+5}) = 6$, and the algebraic degree of $C_{i+2,i+5}$ is bigger than 4, then the linkage is a Dietmaier linkage (see Example 4.6).

4. If $\dim(L_{i,i+1,i+2}) = 6$ and $\dim(L_{i+3,i+4,i+5}) = 8$, and the algebraic degree of $C_{i+2,i+5}$ is bigger than 6, then it is 8.

Proof. (1): by Theorem 4.4, the number $k$ of pairs of bonds attached to joint $i$ is equal to the algebraic degree of $C_{i-1,i}$. Since $C_{i-1,i}$ is a line parametrizing rotations around a fixed axis, $k$ is the number of configurations with fixed joint parameter $t_i$. Since the maximal number of configurations of a non-mobile 5R linkage with no three consecutive axes incident to a single point is 4, the claim follows.

For (2), we refer to [15] Lemma 5.6. For (3) and (4), we refer to [10]. □

When the dimensions of all coupling spaces $L_{i,i+1,i+2}$ is 8, then there are no short connections. Let us assume that the number of bonds connecting joints
1 and 4 is $k_{14}$, the number of bonds connecting joints 2 and 5 is $k_{25}$, and the number of bonds connecting joints 3 and 6 is $k_{36}$. Without loss of generality, we may assume that $k_{14} \leq k_{25} \leq k_{36}$; also, we have $k_{15} \geq 1$ by Proposition 3.1 and $k_{36} \leq 4$ by Theorem 5.1. Here is a summary about what is known for these cases.

- All linkages with $k_{14} = k_{25} = 1$ are known: there is one family [11] with $k_{36} = 1$ and another family [14] with $k_{36} = 2$. Both families are maximal, i.e. they are irreducible components of the variety $M_6$ of movable 6R loops.

- All linkages with $k_{25} = k_{36} = 4$ are known: there is one family with $k_{14} = 2$ [17] and two families with $k_{14} = 4$ [10]. All three families are maximal.

- The family of line symmetric linkages has $k_{14} = k_{25} = k_{36} = 2$. It is maximal. Another family with the same bond diagram can be found in [16]. But we do not know if this family is maximal, or if there are other families with $k_{14} = k_{25} = k_{36} = 2$.

- There are examples with linkages with $k_{14} = k_{25} = k_{36} = 3$ with reducible configuration space (see below). But we do not know if there are other examples.

- For any triple $(k_{14}, k_{25}, k_{36})$ not covered by the above cases, we do not know if there are any linkages.

Example 5.2. Let $h_1, h_2, h_3$ be three dual quaternions corresponding to three random lines. Set

$$h_4 := h_1, \ h_5 := h_2, \ h_6 := h_3.$$

The configuration space of the closed linkage specified by the axes corresponding to $h_1, \ldots, h_6$ is a reducible curve with four components: three lines parametrizing rotations around a coincident axis, and one curve parametrizing the motion of a line symmetric linkage. The first three lines intersect in a common point, corresponding to the initial configuration with three coincident lines. The fourth component does not meet the other three.

The question whether a known family of linkages is maximal is more difficult when short connections are present. In this case, maximality is known only for just one family, namely Dietmaier’s linkage [8]. The situation is more complicated because one has to exclude that the family under consideration is a specialization of another family yet unknown for which the dimension of the coupling spaces is 8. In the case of Dietmaier’s linkage, it is possible to prove maximality by semicontinuity of the genus of the configuration curve (see [10]).

Let us consider only bond diagrams with at least one short connection. The thesis [15] contains a complete list of linkages with three or four conjugated pair of bonds (i.e. the bond diagram contains three or four lines; some of these diagrams are depicted in Figure 5). For most diagrams compatible with
Figure 6: The bond diagrams of the Wohlhart's partially symmetric linkage and its third isomerization. For a generic partially symmetric linkage, three consecutive triples of axes satisfy the Bennett condition and three do not, which one can be seen from the short connections in the diagram. Isomerization works only if all six triples satisfy the Bennett condition.

Theorem 5.1 and Lemmas 3.5, 3.6 we do not know if they are bond diagrams of movable linkages, and for many others we have examples but we do not know if the known examples are all linkages with the bond diagram under consideration. There are just two diagrams with more than four lines and at least one short connection for which all linkages are known, namely the diagram of Wohlhart's partially symmetric linkage [21] and its third isomerization shown in Figure 6.

Isomerization is a technique introduced in [23] that allows to construct new families from known ones by interchanging two links; it is possible only if their affected joints satisfy the Bennett condition.

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