Abstract. Considering finite extensions $K[A] \subseteq K[B]$ of positive affine semigroup rings over a field $K$ we have developed in [1] an algorithm to decompose $K[B]$ as a direct sum of monomial ideals in $K[A]$. By computing the regularity of homogeneous semigroup rings from the decomposition we have confirmed the Eisenbud-Goto conjecture in a range of new cases not tractable by standard methods. Here we first illustrate this technique and its implementation in our Macaulay2 package MonomialAlgebras by computing the decomposition and the regularity step by step for an explicit example. We then focus on ring-theoretic properties of simplicial semigroup rings. From the characterizations given in [1] we develop and prove explicit algorithms testing properties like Buchsbaum, Cohen-Macaulay, Gorenstein, normal, and seminormal, all of which imply the Eisenbud-Goto conjecture. All algorithms are implemented in our Macaulay2 package.

1. Introduction

Let $B$ be a positive affine semigroup, that is, $B$ is a finitely generated subsemigroup of $\mathbb{N}^m$ for some $m$. Let $K$ be a field and $K[B]$ the affine semigroup ring associated to $B$, which can be identified with the subring of $K[t_1, \ldots, t_m]$ generated by monomials $t^u := t_1^{u_1} \cdots t_m^{u_m}$, where $u = (u_1, \ldots, u_m) \in B$. Denote by $C(B)$ and by $G(B)$ the cone and the group generated by $B$. From now on let $A \subseteq B$ be positive affine semigroups with $C(A) = C(B)$. We will now discuss the decomposition of $K[B]$ into a direct sum of monomial ideals in $K[A]$. Observe that

$$K[B] = \bigoplus_{g \in G} K \cdot \left\{ t^b : b \in B \cap g \right\},$$

where $G := G(B)/G(A)$. Note that $C(A) = C(B)$ if and only if $K[B]$ is a finitely generated $K[A]$-module. From this it follows that $G$ is finite, and we can compute the above decomposition since all summands are finitely generated. Moreover, there are shifts $h_g \in G(B)$ such that

$$I_g := K \cdot \left\{ t^{b-h_g} : b \in B \cap g \right\}$$

is a monomial ideal in $K[A]$. Thus,

$$K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$$

as $\mathbb{Z}^m$-graded $K[A]$-modules (with deg $t^b = b$). A detailed formulation of the algorithm computing the ideals $I_g$ and shifts $h_g$ and a more general version of the decomposition in the setup of cancellative abelian semigroup rings over an integral domain can be found in [1] Algorithm 1, Theorem 2.1.

Our original motivation for developing this decomposition was to provide a fast algorithm to compute the Castelnuovo-Mumford regularity $\text{reg} K[B]$ of a homogeneous semigroup ring in order to test the Eisenbud-Goto conjecture [3]. Recall that the Castelnuovo-Mumford regularity

\[ \text{reg} K[B] = \max_{g \in G} \left( \text{deg} I_g + h_g \right) \]

is the smallest integer $r$ such that $K[B]/I^r$ is a finite $K$-vector space. In many cases, it is enough to compute the regularity up to a certain power of $h_g$. The algorithm described above allows to compute the regularity of $K[B]$ efficiently, and it can be used to test the Eisenbud-Goto conjecture in finite time for a wide range of semigroups.
reg \( M \) of a finitely generated graded module \( M \) over a standard graded polynomial ring \( R = K[x_1, \ldots, x_n] \) is defined as the smallest integer \( m \) such that every \( j \)-th syzygy module of \( M \) is generated by elements of degree \( \leq m + j \). Moreover, \( B \) is called a homogeneous semigroup if there exists a group homomorphism \( \deg : G(B) \to \mathbb{Z} \) with \( \deg b_i = 1 \) for \( i = 1, \ldots, n \), where \( \text{Hilb}(B) = \{ b_1, \ldots, b_n \} \) is the minimal generating set of \( B \); by \( \text{reg} B \) we mean its regularity with respect to the \( R \)-module structure which is given by the \( K \)-algebra homomorphism \( R \to K[B], x_i \mapsto t^b_i \).

The toric Eisenbud-Goto conjecture can be formulated as follows: let \( K \) be a field and \( B \) a homogeneous semigroup, then \( \text{reg} B \leq \deg B - \text{codim} B \), where \( \deg B \) denotes the degree and \( \text{codim} B := \dim_K K[B]_1 - \dim K[B] \) the codimension. Even this special case of the Eisenbud-Goto conjecture is largely open, for references on known results see [1, Section 4]. The regularity of \( K[B] \) is usually computed from a minimal graded free resolution. If \( n \) is large this computation is very expensive, and hence it is impossible to test the conjecture systematically in high codimension using this method. However, choosing \( A \) to be generated by minimal generators \( e_1, \ldots, e_d \) of \( C(B) \) of degree 1 the regularity can be computed as

\[
\text{reg} B = \max \{ \text{reg} I_g + \deg h_g \mid g \in G \},
\]

where \( \text{reg} I_g \) denotes the regularity of \( I_g \) with respect to the canonical \( T = K[x_1, \ldots, x_d]\)-module structure given by \( T \to K[A], x_i \mapsto t^e_i \). Since the free resolution of every ideal \( I_g \) appearing has length at most \( d - 1 \), this computation is typically much faster than the traditional approaches. This enabled us to test the conjecture for a large class of homogeneous semigroup rings by using our regularity algorithm. See [1, Section 4] for details.

In Section 2 we illustrate, step by step, decomposition and regularity computation for an explicit example using our \textsc{Macaulay2} [1] package \textsc{MonomialAlgebras} [2]. We say that \( K[B] \) is a simplicial semigroup ring if the cone \( C(B) \) is simplicial. In Section 3 we focus on simplicial semigroup rings \( K[B] \). Based on the characterizations of ring-theoretic properties given in [1, Proposition 3.1] we develop explicit algorithms for testing whether \( K[B] \) is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. We also discuss that, by known results, all these ring-theoretic properties imply the Eisenbud-Goto conjecture. The algorithms mentioned are implemented in our \textsc{Macaulay2} package.

2. Decomposition and regularity

Our \textsc{Macaulay2} package can be loaded by

\textsc{Macaulay2}, version 1.4

with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

\texttt{i1 : needsPackage "MonomialAlgebras";}

We discuss the decomposition at the example of the homogeneous semigroup \( B \subset \mathbb{N}^3 \) specified by a list of generators

\[
i2 : B = \{(4,0,0),(2,2,0),(2,0,2),(0,2,2),(0,3,1),(3,1,0),(1,1,2)\};
\]

As an input for our algorithm we encode this data in a multigraded polynomial ring

\[
i3 : K = \mathbb{Z}/101;
\]

\[
i4 : S = K[x_1 \ldots x_7, \text{Degrees}=>B];
\]

The command

\[
i5 : \text{dc} = \text{decomposeMonomialAlgebra} S
\]

\[
o5 = \text{HashTable}\{ (-1,1,0) \Rightarrow \{ \text{ideal} (x_1,x_3), (-1,1,0) \} , 0 \Rightarrow \{ \text{ideal} 1, 0 \} \}
\]

decomposes \( K[B] \) over \( K[A] \) where \( A \subset B \) is generated by minimal generators of \( C(B) \) with minimal coordinate sum; so in the example \( A = \langle (4,0,0),(2,2,0),(2,0,2),(0,2,2),(0,3,1) \rangle \). The
keys of the hash table represent the elements of $G$ and the values are the tuples $(I_g, h_g)$, hence

\[ K[B] \cong (\langle x_1, x_3 \rangle ) \oplus K[A] \]

as $\mathbb{Z}^3$-graded $K[A]$-modules; here we write $K[A] \cong T/J$ with $T = K[x_1, x_2, x_3, x_4, x_5]$ and $\pi_t$ for the class of $x_t$. Note that the on-screen output of MACAULAY2 does not distinguish between the class and the representative. To compute $\text{reg} I_g$ we will consider the standard grading on $T$:

```
 i6 : KA = ring first first values dc;
 i7 : T = newRing(ring ideal KA,Degrees=>{5:1});
 i8 : J = sub(ideal KA,T);
o8 : ideal of T
i9 : betti res J
```

```
0 1 2  
total: 1 3 2
1 . .  
2 1 2 2
```

```
o9 : BettiTally
```

The usual approach would be to obtain $\text{reg} K[B]$ from a minimal graded free resolution of the toric ideal $I_B$ with respect to the standard grading.

```
i10 : IB = monomialAlgebraIdeal S;
o10 : ideal of S
i11 : R = newRing(ring IB,Degrees=>{7:1});
i12 : betti res sub(IB,R)
```

```
0 1 2 3 4 5  
total: 1 8 15 13 6 1
0 1 . . . .
1 . 6 8 3 .
2 . 2 3 . .
3 . . 4 10 6 1
```

```
o12 : BettiTally
```

We observe that $\text{reg} K[B] = 3$. With $\text{deg} u = (u_1 + u_2 + u_3)/4$ by Equation (2.1) it holds

\[
\text{reg} K[B] = \max \{ \text{reg} (\langle x_1, x_3 \rangle ) + 0, \ \text{reg} K[A] + 0 \}.
\]

By o9 we have $\text{reg} K[A] = 2$. To compute $\text{reg} (\langle x_1, x_3 \rangle)$ do:

```
i13 : I1 = first (values dc)#0
o13 = ideal (x_1, x_3)
i14 : g = matrix entries sub(gens I1, T);
o14 = matrix entries sub(gens I1, T)
i15 : betti res image map(coker gens J, source g, g)
o15 =
```

```
0 1 2 3  
total: 2 5 4 1
1 2 . .
2 . 2 . .
3 . 3 4 1
```

Hence $\text{reg} (\langle x_1, x_3 \rangle) = 3$ and therefore $\text{reg} K[B] = 3$. Observe, that the resolution of $K[B]$ has length 5, whereas the ideals $I_g$ have resolutions of length at most 3. The command

```
i16 : regularityMA S
o16 : {3, {ideal (x_1, x_3), (-1, 1, 0)}}
```
provides an implementation of this approach, also returning the tuples \((I_g, h_g)\) where the maximum is achieved. By [1] Proposition 4.1 we have \(\deg K[B] = \#G \cdot \deg K[A] = 10\) since

\[
i17 : \text{ degree } J
\]
\[
o17 = 5
\]
Moreover, \(\text{codim } K[B] = 4\) since \(\dim K[B] = \dim C(B) = 3\). Hence the ring \(K[B]\) satisfies the Eisenbud-Goto bound.

3. Algorithms for ring theoretic properties

In this section we focus on simplicial semigroup rings \(K[B]\). Based on the characterizations given in [1] Proposition 3.1 we develop and prove explicit algorithms for testing whether \(K[B]\) is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. Note that, in the simplicial case, all these properties are independent of \(K\). As an example, consider the following homogeneous simplicial semigroup \(B\subset\Bbb{N}^3\) specified by the generators

\[
i18 : \ B = \{\{4,0,0\},\{0,4,0\},\{0,0,4\},\{1,0,3\},\{0,2,2\},\{3,0,1\},\{1,2,1\}\};
\]
We compute the decomposition of \(K[B]\) over \(K[A]\), where \(A = (\{4,0,0\},\{0,4,0\},\{0,0,4\})\subset B\) is generated again by minimal generators of \(C(B)\) with minimal coordinate sum.

\[
i19 : \ S = K[x_1 \ldots x_7, \text{Degrees}=>B];
\]
\[
i20 : \ \text{decomposeMonomialAlgebra } S
\]
\[
o20 : \ \text{HashTable} = \{(-1,0,1) \Rightarrow \text{ideal } 1, (3,0,1)\}
\]
\[
(-1,2,-1) \Rightarrow \text{ideal } 1, (3,2,3)\}
\]
\[
(0,2,2) \Rightarrow \text{ideal } 1, (0,2,2)\}
\]
\[
(1,0,-1) \Rightarrow \text{ideal } 1, (1,0,3)\}
\]
\[
(1,2,1) \Rightarrow \text{ideal } 1, (1,2,1)\}
\]
\[
(2,0,2) \Rightarrow \text{ideal } (x_3,x_1,x_2), (2,0,2)\}
\]
\[
(2,2,0) \Rightarrow \text{ideal } 1, (2,2,4)\}
\]
\[
0 \Rightarrow \text{ideal } 1, 0\}
\]
Hence

\[
K[B] \cong K[A] \oplus K[A](-1)^4 \oplus K[A](-2)^2 \oplus \langle x_1, x_2, x_3 \rangle (-1)
\]
with respect to the standard grading induced by \(\deg u = (u_1 + u_2 + u_3)/4\). It follows that depth \(K[B] = 1\), thus, \(K[B]\) is not Cohen-Macaulay. Hence \(K[B]\) is also not normal by [5]. We can test seminormality via the following algorithm:

**Algorithm 1** Seminormality test

**Input:** A simplicial semigroup \(B \subset \Bbb{N}^m\).

**Output:** \text{true} if \(K[B]\) is seminormal, \text{false} otherwise.

1. Let \(e_1, \ldots, e_d \in B\) be minimal generators of \(C(B)\) with minimal coordinate sum, and set \(A := (e_1, \ldots, e_d)\).
2. Compute \(B_A := \{x \in B \mid x \notin B + (A \setminus \{0\})\}\) as described in [1] Algorithm 1, Step 1.
3. for all \(x \in B_A\) do
4. Solve the linear system of equations \(\sum_{i=1}^{d} \lambda_i e_i = x\) for \(\lambda = (\lambda_1, \ldots, \lambda_d) \in \Bbb{Q}^d\).
5. if \(\|\lambda\|_{\infty} > 1\) then return \text{false}
6. return \text{true}

Here, by \(\|\cdot\|_{\infty}\) we denote the maximum norm. Note that all \(\lambda_i\) are non-negative since \(C(B)\) is a simplicial cone. Verifying in Step 5 the condition \(\|\lambda\|_{\infty} \geq 1\) instead results in an algorithm which tests normality. Using our package we observe that \(K[B]\) is not seminormal:

\[
i21 : \ \text{isSeminormalMA } B
\]
\[
o21 : \ \text{false}
\]
The Buchsbaum property can be tested by the following algorithm. We denote by $K[A]_+$ the homogeneous maximal ideal of $K[A]$.

**Algorithm 2** Buchsbaum test

**Input:** A simplicial semigroup $B = \langle b_1, \ldots, b_n \rangle \subseteq \mathbb{N}^m$.

**Output:** true if $K[B]$ is Buchsbaum, false otherwise.

1. Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
2. Using the (minimal) generators $e_1, \ldots, e_d$ of $A$ decompose

$$K[B] \cong \bigoplus_{g \in G} I_g(-h_g),$$

where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [1] Algorithm 1.

3. If $\exists g \in G$ with $I_g \not\subseteq K[A]$ and $I_g \not\subseteq K[A]_+$ then return false

4. $H := \{h_g \mid g \in G \text{ with } I_g = K[A]_+\}$

5. $C := \{b_1, \ldots, b_n\} \setminus \{0, e_1, \ldots, e_d\}$

6. $H + C := \{h_g + b_i \mid h_g \in H, b_i \in C\}$

7. return true if $(H + C) \cap H = \emptyset$ and false otherwise.

**Proof.** By [1] Proposition 3.1 the ring $K[B]$ is Buchsbaum iff each ideal $I_g$ is either equal to $K[A]$, or to $K[A]_+$, and $h_g + b \in B$ for all $b \in \text{Hilb}(B)$. So, Step 3 is correct and we may now assume that $I_g = K[A]$ or $I_g = K[A]_+$ for all $g \in G$. Recall that $I_g = \{t^{v-h_g} \mid v \in \Gamma_g\}K[A]$, where $\Gamma_g = \{x \in B_A \mid x \not\in g\}$. Moreover, note that $\{t^{v-h_g} \mid v \in \Gamma_g\}$ is always a minimal generating set of $I_g$ and $h_g = \sum_{k=1}^d \min\{\lambda^v_k \mid v \in \Gamma_g\} e_k$, where $v = \sum_{k=1}^d \lambda^v_k e_k$ with $\lambda^v_k \in \mathbb{Q}$.

Since $h + e_k \in B_A$ for all $h \in H$ and all $k = 1, \ldots, d$, we have $H \cap B = \emptyset$. In case that $(H + C) \cap H \neq \emptyset$ we obtain $h + b \not\in B$ for some $h \in H$ and some $b \in B \setminus \{0\}$, that is, $h + \text{Hilb}(B) \not\subseteq B$. Hence $K[B]$ is not Buchsbaum.

In case that $K[B]$ is not Buchsbaum, there is an $h \in H$ and some $b \in \text{Hilb}(B)$ such that $h + b \not\in B$. It is now sufficient to show that $b \in C$ and $h + b \in H$. By the above argument, $b \in C$. Let $m_k = h + b + e_k$ for $k = 1, \ldots, d$. Suppose that $m_i \not\in B_A$ for some $i \in \{1, \ldots, d\}$. Since $m_k - e_k \not\in B$ for all $k = 1, \ldots, d$, necessarily $m_i - e_j \in B$ for some $j \neq i$. Consider $y = m_i - \sum_{k=1}^d n_k e_k \in B$ with $n_k \in \mathbb{N}$ such that $\sum_{k=1}^d n_k e_k$ is maximal. By construction $y \in B_A$, moreover, $n_j = 0$ since $m_j - e_j \not\in B$. In the same way if $x = m_i - e_j - \sum_{k=1}^d n_k e_k \in B$ with $\sum_{k=1}^d n_k e_k$ maximal, then $x \in B_A$. Since $m_i, m_j \in g$ for some $g \in G$, we also have $x, y \in g$. Since $e_1, \ldots, e_d$ are linearly independent, we have $\lambda^x_j - \lambda^y_j \geq 2$. Moreover, since $t^{v-h_g}, t^{v-h_g} \in K[A]$ we get that $t^{v-h_g}$ is not a linear form. Hence $I_g \neq K[A]$ and $I_g \neq K[A]_+$, thus, $m_k \in B_A$ for all $k = 1, \ldots, d$. We have $\# \Gamma_g \in \{1, d\}$ by minimality, hence $\Gamma_g = \{m_1, \ldots, m_d\}$. By construction, $h_g = h + b$ and $I_g = K[A]_+$, therefore $h + b \in H$.

Note that in Step 2 the shifts $h_g$ and hence the ideals $I_g$ are uniquely determined since $e_1, \ldots, e_d$ are linearly independent. This is not true for arbitrary generating sets. By

```csharp
i22 : isBuchsbaumMA B
o22 : true
```

we conclude that $K[B]$ satisfies the Eisenbud-Goto conjecture by [2]. Note that we can read off from the decomposition the regularity and the Eisenbud-Goto bound: we have $\text{reg} K[A] = 0$ and $\text{reg} (x_1, x_2, x_3) = 1$, therefore $\text{reg} K[B] = \max\{0, 1, 2, 1 + 1\} = 2$. Moreover, $\deg K[B]$ is the number of ideals which occur in the decomposition, hence $\deg K[B] = \text{codim} K[B] = 8 - 4 = 4$.

Note that, in case $B$ is Buchsbaum, the regularity of $K[B]$ is independent of the field $K$ since all ideals in the decomposition are equal to the homogeneous maximal ideal or to $K[A]$.

We finish this section by providing an algorithm for testing the Gorenstein property.
Algorithm 3 Gorenstein test

Input: A simplicial semigroup $B \subseteq \mathbb{N}^m$.  
Output: true if $K[B]$ is Gorenstein, false otherwise.

1. Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
2. Using the (minimal) generators $e_1, \ldots, e_d$ of $A$ decompose

$$K[B] \cong \bigoplus_{g \in G} I_g(-h_g),$$

where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [1, Algorithm 1].
3. if $\exists g \in G$ with $I_g \neq K[A]$ then return false
4. $H := \{h_g \mid g \in G\}$
5. if $h \in H$ with maximal coordinate sum is not unique then return false
6. Let $h \in H$ with maximal coordinate sum.
7. while $H \neq \emptyset$
8. Let $h_g \in H$
9. if $h - h_g \notin H$ then return false
10. $H := H \setminus \{h_g, h - h_g\}$
11. return true

Proof. By [1, Proposition 3.1] the ring $K[B]$ is Gorenstein iff $I_g = K[A]$ for all $g \in G$ and $H$ has a unique maximal element with respect to $\preceq$ given by $x \leq y$ if there is a $z \in B$ such that $x + z = y$. Note that $H = B_A$ since $I_g = K[A]$ for all $g \in G$. If there is a maximal element $h \in H$, then this element has maximal coordinate sum. If $H$ has more than one element with maximal coordinate sum, then $H$ does not have a unique maximal element. To complete the proof we need to show that an element $h_g \in H$ satisfies $h_g \leq h$ iff $h - h_g \in H$. This follows from the fact that if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$.

\[ \square \]

Note that performing Steps 1–3 of Algorithm 3 (and returning true afterwards) gives a test for the Cohen-Macaulay property.

References

[1] J. Böhm, D. Eisenbud, and M. J. Nitsche, Decomposition of semigroup algebras, to appear in Experim. Math., http://arxiv.org/abs/1110.3053 2011.
[2] J. Böhm, D. Eisenbud, and M. J. Nitsche, MonomialAlgebras, a Macaulay2 package to compute the decomposition of positive affine semigroup rings, available at http://www.math.uni- sb.de/ag/schreyer/jb/Macaulay2/MonomialAlgebras/html/.
[3] D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, J. Algebra 88 (1984), no. 1, 89–133.
[4] D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/
[5] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. Math. 96 (1972), no. 2, 318–337.
[6] M. J. Nitsche, Castelnuovo-Mumford regularity of seminormal simplicial affine semigroup rings, J. Algebra (2012), http://dx.doi.org/10.1016/j.jalgebra.2012.05.004.
[7] J. Stückrad and W. Vogel, Castelnuovo bounds for locally Cohen-Macaulay schemes, Math. Nachr. 136 (1988), 307–320.
[8] R. Treger, On equations defining arithmetically Cohen-Macaulay schemes. I, Math. Ann. 261 (1982), no. 2, 141–153.

Fachbereich Mathematik, TU Kaiserslautern, 67663 Kaiserslautern, Germany
E-mail address: boehm@mathematik.uni-k1.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA
E-mail address: de@msri.org

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTRASSE 22, 04103 LEIPZIG, GERMANY
E-mail address: nitsche@mis.mpg.de