Vector Multiplets and the
Phases of $N = 2$ Theories in 2D
Through the Looking Glass

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ABSTRACT

We extend Witten’s discussion of actions related to the Landau-Ginzburg description of Calabi-Yau hypersurfaces in weighted projective spaces to cover the mirror class of models that include twisted chiral matter multiplets and a newly discovered 2D, $N = 2$ twisted vector multiplet. Certain integrability obstructions are observed that constrain the most general constructions containing both matter and twisted matter simultaneously. It is conjectured that knot invariants will ultimately play a role in describing the most general such model.

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I. Introduction

Over a decade ago, it was noted that 2D supersymmetry representations included some unusual (from the view of 4D) representations, the twisted chiral multiplets \[ \text{[1, 2]} \]. Our discovery began precisely with the problem of the toroidal compactification of 4D, N = 2 vector multiplets to the 2D, N = 4 and N = 2 theories. The output of those investigations led to the first appearance in the literature of special Kähler geometries, Kählerian N = 2 vector multiplets, the use of duality as a tool for finding new supersymmetric representations and the appearance of torsion in 2D nonlinear sigma-models. All of these topics have reappeared with a vengeance in compactified heterotic and superstring theory where they play important roles.

Although twisted chiral multiplets were discovered prior to the advent of compactified heterotic string theory, nevertheless they encode the local part of the “mirror symmetry transformation.” They allow Lagrangian field theories to represent both (c,c) and (a,c) rings in the language of superconformal field theory. The local part of mirror symmetry, in its simplest form, is just the statement that given one action written in terms of chiral matter scalars, there exist another theory where all of the chiral scalars are replaced by twisted chiral scalars.

In more recent times Witten [4], has used similar techniques applied to supersymmetric world sheet actions to show how the Calabi-Yau hypersurfaces in weighted projective spaces occur as a phase of 2D, N = 2 supersymmetric Landau-Ginzburg type models. The matter field sectors of his models included only chiral scalar multiplets. One of the questions left open in this investigation was how the extension of such techniques might apply to models where the matter fields included twisted chiral multiplets.

It is the purpose of this paper to research this open question. In the following we will gaze “through the looking glass” at these models. In other words, we will perform a mirror transformation on the models described in reference one to obtain their “mirror images.” This will be a new class of models where the matter multiplets are described by twisted chiral superfields. In order to achieve this, it will be necessary to introduce a previously unknown representation of 2D, N = 2 supersymmetry; the twisted vector multiplet. Since the ordinary vector multiplet is described by a twisted chiral scalar field strength, the twisted vector multiplet is described by an ordinary chiral scalar field strength.

\[^{4}\text{Recently [3], 4D Kählerian vector multiplet theories have been used to make progress in understanding a number of issues.}\]
We first describe the usual 2D, N = 2 vector multiplet (VM-I) in 2D, N = 2 superspace. For this purpose we introduce a superspace Yang-Mills covariant derivative \( \nabla_A \equiv D_A + ig\Gamma_A t \) (in this expression \( t \) denotes a U(1) Lie algebra generator) that has a supercommutator algebra given by,

\[
[\nabla_\alpha, \nabla_\beta] = 0 , \quad [\nabla_\alpha, \nabla_\beta] = 2\gamma_{\alpha\beta}\nabla_c W = i2(\gamma^c)_{a\beta}\nabla_c + 2g[C_{\alpha\beta}S - i(\gamma^3)_{a\beta}P]t ,
\]

\[
[\nabla_\alpha, \nabla_\beta] = i\epsilon_{ab}\nabla_b W = -ig\epsilon_{ab}\nabla_b W = 0 , \quad [\nabla_\alpha, \nabla_\beta] = -ig\epsilon_{ab}\nabla_b W = 0
\]

and

\[
\nabla_\alpha S = -iW = \nabla_\alpha P = -(\gamma^3)_{a\beta}\nabla_c W = \nabla_\alpha W = 0 , \quad \nabla_\alpha d = (\gamma^c)_{a\beta}\nabla_c W = \nabla_\alpha W = 0
\]

This is the vector multiplet that comes down via dimensional reduction from higher dimensions. The standard kinetic action for a U(1) gauge group for this multiplet forms a supersymmetric invariant,

\[
S_{VM-I} = \int d^2\sigma d^2\bar{\zeta}d^2\zeta [ -\frac{1}{4}S^2 ] = \int d^2\sigma d^2\bar{\zeta}d^2\zeta [ -\frac{1}{4}P^2 ] =
\]

\[
= \int d^2\sigma [ -\frac{1}{4}(F_{ab}(A))^2 + \frac{1}{2}(\nabla_\alpha S)^2 + \frac{1}{2}(\nabla_\alpha P)^2 - i\lambda_{(\gamma^c)_{a\beta}\nabla_c W = \nabla_\alpha W = 0} + \frac{i}{2}\lambda_{(\gamma^3)_{a\beta}\nabla_c W = \nabla_\alpha W = 0} + \frac{i}{2}\lambda_{(\gamma^3)_{a\beta}\nabla_c W = \nabla_\alpha W = 0} ] ,
\]

where we have used the facts that \( W| = \frac{1}{2}\epsilon^{ab}F_{ab}(A) \) and \( W_a| = \lambda_a \). (It is also of interest to note that we utilize a definition \( \int d^2\bar{\zeta}d^2\zeta \equiv \frac{1}{8}[\nabla^\alpha\nabla^\beta\nabla^\gamma\nabla^\delta + \nabla^\alpha\nabla^\beta\nabla^\gamma\nabla^\delta\nabla^\beta\nabla^\beta|].\)

One of the interesting properties of the vector multiplet is that it can be coupled to chiral superfields but cannot be coupled to twisted chiral superfields. The proof of this statement can be seen as follows. We can use the following notations for \( \nabla_+ , \nabla_- , \nabla_+ , \) and \( \nabla_- .\)

\[
\nabla_+ \equiv \frac{1}{2}(1 + \gamma^3)_{a\beta}\nabla_+ + \nabla_+ \equiv \frac{1}{2}(1 - \gamma^3)_{a\beta}\nabla_+ + \nabla_+ \equiv \frac{1}{2}(1 - \gamma^3)_{a\beta}\nabla_+ + \nabla_-
\]

Defining the linear combination \( \Psi \equiv S + iP \), we see that \( \nabla_+ \Psi = 0 \) and \( \nabla_- \Psi = 0 \). This identifies \( \Psi \) as a twisted chiral superfield. We now introduce a matter scalar multiplet \( \tilde{\chi} \) that is also a twisted chiral superfield. We thus have,

\[
\nabla_+ \tilde{\chi} = 0 \rightarrow \nabla_- \nabla_+ \tilde{\chi} = 0 , \quad \nabla_- \tilde{\chi} = 0 \rightarrow \nabla_+ \nabla_- \tilde{\chi} = 0 ,
\]

\[
\nabla_+ \tilde{\chi} = 0 \rightarrow \nabla_- \nabla_+ \tilde{\chi} = 0 , \quad \nabla_- \tilde{\chi} = 0 \rightarrow \nabla_+ \nabla_- \tilde{\chi} = 0 ,
\]

\[
\nabla_+ \tilde{\chi} = 0 \rightarrow \nabla_- \nabla_+ \tilde{\chi} = 0 , \quad \nabla_- \tilde{\chi} = 0 \rightarrow \nabla_+ \nabla_- \tilde{\chi} = 0 .
\]
and adding these two results together yields
\[ i2g\bar{\Psi} [ t, \bar{\chi}] = 0 \tag{7} \]

after taking the appropriate chiral projections of the second result in equation (1.).

This is the standard integrability-type argument that often occurs in superspace. The only reasonable way to satisfy this condition is to demand that \( \chi \) lie in the trivial representation of the gauge group of the vector multiplet or equivalently the matter field \( \chi \) does not carry a charge to which the gauge field couples.

We now come to the "mirror image" of the usual vector multiplet. Below we will see that there is actually a second such multiplet! Initially this may be extremely surprising to the reader. This should not be the case. Mirror symmetry is apparently a fundamental part of 2D, N = 2 supersymmetry. Mirror symmetry is connected with the definition of a 2D parity operator. For example, the most important difference between a 2D chiral multiplet and a 2D twisted chiral multiplet is that the former includes two scalar spin-0 fields in its spectrum while the latter includes one scalar and one pseudoscalar in its spectrum. We have a precedent for this behavior in 3D, N = 4 theories \[ 3 \] as well as 2D, N = 2 supergravity theories \[ 6 \].

The second vector multiplet (VM-II) can be introduced in the following manner. For this purpose we introduce a superspace Yang-Mills covariant derivative \( \nabla_A \equiv D_A + ig' \tilde{\Gamma}_A t' \) that has a supercommutator algebra given by,

\[
\begin{align*}
[\nabla_A , \nabla_B] &= i4g'(\gamma^3)_{\alpha\beta} \tilde{\rho} t' , \quad [\nabla_A , \nabla_B] = i2(\gamma^c)_{\alpha\beta} \nabla_c , \\
[\nabla_A , \nabla_B] &= -g'(\gamma^3)_{\alpha\beta} \tilde{\Omega}_t , \quad [\nabla_A , \nabla_B] = -ig'\epsilon_{ab} \mathcal{U} t' ,
\end{align*}
\tag{8}
\]

and

\[
\begin{align*}
\nabla_A \tilde{\rho} &= 0 , \quad \nabla_A \tilde{\Omega}_t = \tilde{\omega}_\beta , \quad \nabla_A \tilde{\Omega}_t = i2(\gamma^a)_{\alpha\beta}(\nabla_A \tilde{\rho}) , \\
\nabla_A \omega_\beta &= C_{\alpha\beta} [ \mathcal{U} + i\tilde{d} ] , \quad \nabla_A \tilde{d} = (\gamma^c)_{\alpha\beta} \nabla_c \tilde{\Omega}_t .
\end{align*}
\tag{9}
\]

This vector multiplet cannot be obtained via dimensional reduction from higher dimensions. The kinetic action for a U(1) gauge group for the multiplet is the mirror image of the standard one above.

\[
\mathcal{S}_{VM-II} = \int d^2\sigma d^2\bar{z} d^2\bar{\zeta} \left[ \frac{1}{2} \bar{\mathcal{P}} \mathcal{P} \right] =
\]

\[
= \int d^2\sigma \left[ -\frac{1}{4}(F_{ab}(B))^2 + 2|\nabla_A \mathcal{P}|^2 - i\tilde{\rho}_\alpha (\gamma^c)_{\alpha\beta} \nabla_c \tilde{\rho}_\beta + \frac{1}{2} \tilde{d}^2 \right] ,
\tag{10}
\]

where \( \mathcal{U} = \frac{1}{2} \epsilon^{ab} F_{ab}(B) \) and \( \Omega_\alpha = \rho_\alpha \).
It has the mirror image property to that of the VM-I theory in that it can be
 coupled to twisted chiral superfields but cannot be coupled to chiral superfields. The
 argument is just the mirror image of that given in equations (4-6). We note that the
definition of a chiral matter scalar $\bar{\Phi}$ implies the following integrability argument.
\[ \nabla_{\alpha} \bar{\Phi} = 0 \rightarrow \nabla_{\beta} \nabla_{\alpha} \bar{\Phi} = 0 \rightarrow \{ \nabla_{\beta}, \nabla_{\alpha} \} \bar{\Phi} = 0 \rightarrow i4g' (\gamma^{3})_{\alpha\beta} \bar{P} [ t', \bar{\Phi} ] = 0 \quad (11) \]

We end this section by considering the unconstrained prepotential formulation of
the twisted vector multiplet. For simplicity we will only consider abelian theories.
It is well known that a chiral (anti-chiral) superfield can be obtained from a general
complex superfield $U$ via the equations
\[ \Phi \equiv \frac{1}{2} C^{\alpha\beta} \bar{D}_{\alpha}D_{\beta}U \quad , \quad \bar{\Phi} \equiv \frac{1}{2} C^{\alpha\beta} D_{\alpha}\bar{D}_{\beta}\bar{U} \quad . \quad (12) \]
What is less well known is that similar equations apply to twisted chiral scalar mul-
tiple also.
\[ S \equiv \frac{1}{4} [ C^{\alpha\beta} D_{\alpha}D_{\beta}( U + \bar{U} ) + (\gamma^{3})^{\alpha\beta} D_{\alpha}D_{\beta}( U - \bar{U} ) ] \quad , \]
\[ P \equiv -i\frac{1}{4} [ (\gamma^{3})^{\alpha\beta} D_{\alpha}\bar{D}_{\beta}( U + \bar{U} ) + C^{\alpha\beta} D_{\alpha}\bar{D}_{\beta}( U - \bar{U} ) ] \quad , \]
\[ \chi = \frac{1}{2} ( 1 + \gamma^{3} )^{\alpha\beta} D_{\alpha}\bar{D}_{\beta}U \quad , \quad \bar{\chi} = \frac{1}{2} ( 1 - \gamma^{3} )^{\alpha\beta} D_{\alpha}\bar{D}_{\beta}\bar{U} \quad . \quad (13) \]
These last two equations are extremely useful because they determine the structure
of the superpropagators for twisted chiral superfields. Finally we come to the vector
multiplets.

For the usual vector multiplet the covariant derivative ($\nabla_{A} \equiv D_{A} + ig\Gamma_{A}t$) can be
explicitly expressed in terms of a prepotential superfield $V$ that is the fundamental
gauge superfield of any SUSY YM-type theory. In particular the components of the
superconnection are defined by,
\[ \Gamma_{\alpha} = iD_{\alpha}V \quad , \quad \bar{\Gamma}_{\alpha} = -i\bar{D}_{\alpha}V \quad , \]
\[ \Gamma_{a} = \frac{1}{4}(\gamma_{a})^{\alpha\beta} [ (D_{a}\bar{D}_{\beta} - \bar{D}_{\beta}D_{a})V ] \quad , \quad (14) \]
and their gauge variations follow from that of the prepotential $\delta_{G}V = -i(\Lambda - \bar{\Lambda})$
where $\Lambda$ is a chiral superfield. For the twisted vector multiplet the covariant deriva-
tive ($\tilde{\nabla}_{A} \equiv D_{A} + ig\bar{\Gamma}_{A}t$) can also explicitly expressed in terms of a prepotential
superfield $\tilde{V}$ that is a fundamental gauge superfield. In particular the components of
the superconnection now are defined by,
\[ \tilde{\Gamma}_{\alpha} = i(\gamma^{3})_{\alpha}^{\beta} D_{\beta}\tilde{V} \quad , \quad \tilde{\Gamma}_{\alpha} = -i(\gamma^{3})_{\alpha}^{\beta} D_{\beta}\bar{V} \quad , \]
\[ 5 \]
\[ \tilde{\Gamma}_a = \frac{1}{4} (\gamma^3 \gamma_a)^{\alpha \beta} \left[ (D_\alpha \bar{D}_\beta - \bar{D}_\beta D_\alpha) \tilde{V} \right], \]  

and their gauge variations follow from that of the prepotential \( \delta_G \tilde{V} = -i(\Lambda - \bar{\Lambda}) \) where \( \Lambda \) is a twisted chiral superfield.

A final point of interest to note is that these superspace results allow a simple generalization of a well known result for 2D gauge fields. Any such field \( (v_a) \) has a natural decomposition of the form \( v_a = \partial_a \lambda + \epsilon_{ab} \partial_b \tilde{\lambda} \) plus a harmonic piece. This same result holds for a supergauge prepotential \( V \) in the form \( V = -i(\Lambda - \bar{\Lambda}) - i(\tilde{\Lambda} - \bar{\tilde{\Lambda}}) \) plus a superharmonic piece.

### III. CY-LG Model Actions Through the Looking Glass

The most general action for chiral matter fields \((\Phi)\), twisted chiral matter \((\chi)\), vector multiplets \((\Psi)\) and twisted vector multiplets \((P)\) takes the form of a typical \( N = 2 \) nonlinear \( \sigma \)-model with superpotential and twisted superpotential terms form,

\[ S_K(\Phi, \chi : \Psi, P) = S_K + S_W + S_{\tilde{W}}, \]  

where

\[ S_K = \int d^2 \sigma d^2 \bar{\zeta} d^2 \zeta \ K(\Phi, \chi : \Psi, P), \]  

\[ S_W = \int d^2 \sigma d^2 \zeta \ W(\Phi : P) + \text{h.c.}, \]  

\[ S_{\tilde{W}} = \int d^2 \sigma d^2 \bar{\zeta} d^2 \zeta^+ \ \tilde{W}(\chi : \Psi) + \text{h.c.}. \]  

The work of Witten [4] specialized to the case \( S_K(\Phi, 0 : \Psi, 0) \) and the simple mirror reflection of this would be \( S_K(0, \chi : 0, P) \). For example, the (local part of) the mirror transformation acting on the Kähler-like potential term consists of the operations

\[ \Phi \rightarrow \frac{1}{2} \chi, \quad \chi \rightarrow 2\Phi, \quad P \rightarrow \frac{1}{2} \Psi, \quad \Psi \rightarrow 2P, \quad K \rightarrow -K. \]  

The whole philosophy of the CY-LG models is to restrict the class of kinetic energy terms to be flat. So the Kähler-like potential \((K)\) takes the form of being purely quadratic in the superfields. Thus, we have for the most general Kähler-like potential

\[ K(\Phi, \chi : \Psi, P) = -\frac{1}{8} \bar{\psi} \Psi + \frac{1}{2} \bar{\rho} \rho + \frac{1}{2} \bar{\Phi} \Phi - \frac{1}{8} \bar{\chi} \chi. \]  

It is important that we say a few words about some of the terms in the potential above. There is the interesting change of sign in comparing the the kinetic energy of a chiral multiplet and a twisted chiral multiplet. This sign difference is no accident.
If we think of the derivatives with respect to the chiral superfields as providing the natural basis of a tangent bundle. Then the derivatives with respect to the twisted chiral superfields provide the basis for the cotangent bundle. The curvature of a manifold taken with respect to the two different bases differs by a sign. The tangent cotangent interpretation is inherent in the presence of mirror symmetry. Also it may look as though we have not coupled the matter multiplets to the vector multiplets. In fact, we have. The matter multiplets in this expression have their chiral and twisted chirality conditions defined with respect to the $U(1)$ $\times$ $U'(1)$ supercovariant derivative $\nabla_A \equiv D_A + ig\Gamma_A t + ig'\bar{\Gamma}_A t'$ not the “bare” supercovariant derivatives. This automatically insures minimal coupling. The $U(1)$ and $U'(1)$ generators are still restricted to satisfy equations (7.) and (11.). Additionally, we have $[t, \Phi] = iQ\Phi$ and $[t', \chi] = iQ'\chi$ for charges $Q$ and $Q'$. This leaves the most interesting part of the actions to reside in the superpotential and twisted superpotential terms. Expanding out the action in (18) in terms of components yields,

$$S_W = \int d^2\sigma \, \frac{1}{2} \{ [ W''_\alpha \psi_\alpha + 2W'F + \dot{W}_\alpha \rho_\alpha + 2\dot{W}'\rho^\alpha \psi_\alpha + h.c. ]
- (\dot{W} + \dot{W}^*) \epsilon^{ab} F_{ab}(B) - 2i(\dot{W} - \dot{W}^*) \tilde{d} \} \quad (22)$$

where $(A, \psi_\alpha, F)$ are the components of the chiral multiplet $(\Phi)$ defined by $\Phi| = A, \nabla_\alpha \Phi| = \psi_\alpha$ and $\frac{1}{2} \nabla^\alpha \nabla_\alpha \Phi| = F$. We have also used the notations

$$W = W(A : P) \quad , \quad W' \equiv \frac{\partial W}{\partial \Phi} \quad , \quad \dot{W} \equiv \frac{\partial W}{\partial P} . \quad (23)$$

In a similar fashion equation (20) yields

$$S_{\tilde{W}} = \int d^2\sigma \{ [ 4\tilde{W}''_+ \varphi_+ + 2\tilde{W}' h + 4\tilde{W}_- \lambda_- + 4\tilde{W}'(\lambda_- \varphi_+ + \varphi_- + \bar{\lambda}_+) + h.c. ]
+ (\tilde{W} + \tilde{W}^*) \epsilon^{ab} F_{ab}(A) + i2(\tilde{W} - \tilde{W}^*) d \} \quad (24)$$

where $(a, \varphi_+, h)$ are the components of the twisted chiral multiplet $(\chi)$ defined by $\chi| = a, \nabla_+ \chi| = -i2\bar{\varphi}_+, \nabla_- \chi| = i2\varphi_-$ and $\nabla_- \nabla_+ \chi| = 2h$. Acting on $\tilde{W}$ we use the notations

$$\tilde{W} = \tilde{W}(a : \Psi) \quad , \quad \tilde{W}' \equiv \frac{\partial \tilde{W}}{\partial a} \quad , \quad \tilde{\dot{W}} \equiv \frac{\partial \tilde{W}}{\partial \Psi} . \quad (25)$$

It is here that the manifestation of having all the multiplets and their full complement of mirror images produces something new. In particular, there is a possibility to introduce non-minimal couplings between chiral matter and twisted vector multiplets.
as well as the mirror image coupling between twisted chiral matter and vector multiplets. These new couplings have an exceedingly interesting interpretation if we view them as arising from the dimensional reductions from a 3D model. Namely, we see that the presence of the most general terms in the two types of superpotentials lead to 2D, $N = 2$ Chern-Simons terms making their appearance! In (22) the quantity $(\dot{W} + \dot{W}^*)$ plays the role of the “third” component of the gauge field in a 3D CS action and a similar role can be seen for $(\dot{\tilde{W}} + \dot{\tilde{W}}^*)$ in (24). It is also obvious now that the parameter $t$ introduced by Witten can be interpreted as the the vacuum value (moduli-like parameter) of the complex “third” gauge component $\dot{\tilde{W}}$.

In the nonlinear $\sigma$-model formulation of ref. [2], it was shown that the introduction of torsion required that the mixed derivative of the Kähler-like potential taken with respect to one chiral superfield and one twisted chiral superfield should be non-vanishing. This suggests that within the context of these CY-LG models, the introduction of torsion is dependent upon the either the chiral superpotential or twisted chiral superpotential depending upon twisted vector or vector multiplets, respectively.

IV. Summary and Conclusion

As we have demonstrated, the extension of the CY Landau-Ginzburg models, as first proposed by Witten, to cover a larger class of models that contain both chiral and twisted chiral matter is possible. This larger class of actions is characterized by formula (16.) containing a new class of terms over and above those that one would expect as simple mirror reflections. These are the non-minimal couplings of the VM-I multiplet to twisted chiral matter and their mirror reflections, i.e. the 2D Chern-Simons terms. The algebraic-geometrical significance of these new interactions is not completely clear at this time. But the interpretation of the new terms as the result of dimensional reduction from the 3D Chern-Simons action suggests the possibility of intersection polynomials [7] playing a powerful and previously unsuspected role.

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APPENDIX A: A Comment on VM + SG Theories

In the work of ref. [4], there was never an introduction of local supersymmetry into the structure of the models. However, in a later work [8], this extension was studied. In this brief appendix, we will comment upon a general aspect that seems not to have been realized previously with regard to the introduction of spin-1 gauge fields in the presence of $N \geq 1$ supergravity. Below, we will show that generically in such theories, the removal of the supergravity spin-0 auxiliary fields by their algebraic equations of motion strongly restricts the appearance of the usual spin-1 field strength in any such local theory!

We shall show this effect within the context of 2D, $N = 1$ SG coupled to matter. Out starting point is the covariant description of SG + SUSY YM (abelian) theory.

$\nabla_\alpha, \nabla_\beta \} = i2(\gamma^a)_{\alpha\beta}\nabla_a + 2(\gamma^3)_{\alpha\beta}[ R\mathcal{M} + igPt ]$ , (A.1)

$\nabla_\alpha, \nabla_b \} = i[ \frac{1}{2}R(\gamma_b)_{\alpha}^{\beta}\nabla_\beta + (\gamma^3)_{\alpha\beta}(\nabla_\beta R)\mathcal{M} - ig(\gamma_b)_{\alpha}^{\beta}W_\beta ]$ , (A.2)

$\nabla_a, \nabla_b \} = -\epsilon_{ab} [ \frac{1}{2}(\nabla^\alpha R)(\gamma^3)_{\alpha}^{\beta}\nabla_\beta - (\nabla^2 R - R^2)\mathcal{M} + ig\mathcal{W}_t ]$ , (A.3)

The quantities $(P, W_\alpha, \mathcal{W})$ are the components of a 2D, $N = 1$ vector multiplet (where $\mathcal{W}_t = \frac{1}{2}e^{ab}F_{ab}(A)$). These must satisfy the relations

$\nabla_\alpha P = (\gamma^3)_{\alpha}^{\beta}W_\beta$ , \quad $\nabla^2 P = -\mathcal{W} + RP$ . (A.4)

so that the Bianchi identities on the vector multiplet are also satisfied. Our goal is to show that any Lagrangian of the form,

$\mathcal{L} = E^{-1} \left[ \frac{1}{4}C^{\alpha\beta}(\nabla_\alpha P)(\nabla_\beta P) + \frac{1}{4}C^{\alpha\beta}\sum_{i=1}^{M}(\nabla_\alpha \Phi_i)(\nabla_\beta \Phi_i) + U(\Phi_i; P) \right]$ , (A.5)

has the property that the elimination of the supergravity auxiliary spin-0 field by its algebraic equation of motion always leads to the unexpected result of an action that is at most linear in $F_{ab}(A)$! The proof is very direct. In order to find the component expression that follows from (A.5) we note the following useful identities.

$\int d^2\sigma d^2\theta E^{-1}\mathcal{L} = \int d^2\sigma e^{-1}[\nabla^2 - i\psi^\beta(\gamma^\alpha)_{\alpha\beta}\nabla_\alpha - B + \epsilon^{bc}\psi^\alpha(\gamma^3)_{\alpha\beta}\psi^\beta]\mathcal{L} | , (A.6)$

$\nabla_\alpha\nabla_\beta = i(\gamma^a)_{\alpha\beta}\nabla_a + (\gamma^3)_{\alpha\beta}R\mathcal{M} + C_{\beta\alpha}\nabla^2$ , (A.7)

$\nabla^2\nabla_\alpha = -i(\gamma^a)_{\alpha\beta}\nabla_a\nabla_\beta + R[ \nabla_\alpha - (\gamma^3)_{\alpha\beta}\mathcal{M}\nabla_\beta ] - 2(\gamma^3)_{\alpha\beta}(\nabla_\beta R)\mathcal{M}$ , (A.8)

where $R| \equiv B$. Since the effect in which we are interested involves the bosonic fields, we will set all the fermionic fields to zero.
\[ S = \int d^2 \sigma e^{-1} \left\{ \frac{1}{2} \eta^{ab} (\hat{\nabla}_a p)(\hat{\nabla}_b p) + \frac{1}{2} \eta^{ab} (\hat{\nabla}_a \phi^i)(\hat{\nabla}_b \phi_i) + \frac{1}{2} (F_{i})^2 
\right. \]
\[ + F_{i} U_{,i}(\phi_i : p) - \left[ \frac{\partial}{\partial p} U(\phi_i : p) \right] (\epsilon^{ab} F_{ab}(A) - pB) \]
\[ + \frac{1}{2} (\epsilon^{ab} F_{ab}(A) - pB)^2 - BU(\phi_i : p) \} , \quad (A.9) \]

where
\[ \Phi_i \equiv \phi_i , \quad P_i \equiv p , \quad \nabla^2 \Phi_i \equiv F_i , \quad (A.10) \]
\[ \hat{\nabla}_a = e_a^m \partial_m - \frac{1}{2} (\epsilon^{bc} C_{bca}) \mathcal{M} + ig A_a t . \quad (A.11) \]

Now eliminating the auxiliary fields by their algebraic equations of motion,
\[ B = p^{-1} \{ \epsilon^{ab} F_{ab}(A) - p^2 \frac{\partial}{\partial p} [p^{-1} U(\phi_i : p) ] \} , \quad (A.12) \]
\[ F_i = -U_{,i}(\phi_i : p) , \quad (A.13) \]

and substituting these back into the action then yields
\[ S = \int d^2 \sigma e^{-1} \left\{ \frac{1}{2} \eta^{ab} (\hat{\nabla}_a p)(\hat{\nabla}_b p) + \frac{1}{2} \eta^{ab} (\hat{\nabla}_a \phi^i)(\hat{\nabla}_b \phi_i) - \frac{1}{2} (U_{,i}(\phi_i : p))^2 
\right. \]
\[ - p^{-1} U(\phi_i : p) \epsilon^{ab} F_{ab}(A) - \frac{1}{2} \left[p^2 \frac{\partial}{\partial p} (p^{-1} U(\phi_i : p)) \right]^2 \} . \quad (A.14) \]

The origin of why such massive cancellations occur is given in (A.4). In any action where the field strength occurs via spinorial differentiation of \( P \), the removal of the supergravity auxiliary scalar field will have this result. The final result shows us that potentials that are linear in \( p \) have a preferred role. For such a potential all of the second line of the action vanishes. Additionally, the penultimate 2D-CS term, under these conditions, is independent of the “prima-photon” \( p \) and the last term in the action also vanishes. Finally, the most unusual feature is that the spin-1 field strength does not appear at all unless there is a non-trivial potential \( U(\Phi_i : P) \)!

Since all \( N > 1 \) theories must have an \( N = 1 \) theory embedded within them, this proves our general assertion.
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