Characterization of riemann zeta distribution

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Abstract. The characterization of Riemann Zeta distribution is presented based on its characteristic function properties, where the characteristic function of Riemann Zeta distribution is obtained by using Fourier-Stieltjes transform to be a normalized Zeta function. The characteristic function property is exposed to be in the quadratic form as definite non-negative function property. The combination of this property and its continuity provided infinite divisibility of Riemann Zeta distribution where this characteristic function is also obtained as the characteristic function from compound Poisson distribution.

1. Introduction

The Riemann Zeta distribution is constructed by using Zeta function. We set the Zeta function in complex-value function as explained by Edwards [1] or Titchmarsh [2] in the following form

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $$z = \sigma + it$$, $$\sigma > 1$$, $$t \in \mathbb{R}$$. Borwein et. al [3] has introduced the strategy how to compute the Riemann Zeta function for convergent series. However, the definition of Riemann Zeta distribution is referred by Lin and Hu [4] that random variable $$X = - \log n$$ is said to have Riemann Zeta distribution with parameter $$\sigma$$ if its discrete probability distribution can be expressed as

$$f(X = - \log n) = \frac{1}{\zeta(\sigma + it)n^{\sigma}}$$

for any positive integer $$n$$ and $$\sigma > 1$$, $$t \in \mathbb{R}$$ and support in $$\{- \log n \mid n \in \mathbb{N}\}$$. The infinite divisibility of Riemann Zeta distribution has well-explained by Lin and Hu [4] and Sato and Steutel [5], while Saito [6] has introduced the new methods to show infinitely divisible Zeta distribution with recourse to the Euler product of the Riemann Zeta function.

The infinitely divisible distribution is identified by using Fourier-Stieltjes transform that is called as a characteristic function. The characteristic function is defined as transformation of random variable $$X$$ into $$\phi_X(t) = E[\exp(itX)]$$ for real number $$t$$ and $$\exp(itX) = \cos tX + i \sin tX$$ as imaginary unit. Lukacs [7] or Sato [8] has introduced definition of infinitely divisible characteristic function, the random variable $$X$$ with characteristic function $$\phi_X(t)$$ is infinitely divisible if there exists the characteristic function $$\phi_{X_{in}}(t)$$ of random variable $$X_{in}$$ such that $$\phi_X(t) = (\phi_{X_{in}}(t))^m$$ for any positive integer number $$m$$, while the constructing the infinitely divisible distribution from distribution function is also mentioned by Artikis [9], the random variable $$X$$ with distribution function $$F$$ is said to be infinitely divisible if for
every positive integer number \( m \), there exists independent and identically random variables \( X_m \) with distribution function \( F_m \) such that \( F \) is \( m \)-fold convolution of \( F_m \) in the form of \( F = F_m * F_m * \ldots * F_m \). The most property of characteristic function is in the following form

\[
E[\exp(itX)] = \int \exp(itx) f(x) \, dx = 1.
\] (3)

This property is mentioned that characteristic function always exists for every distribution. This reason makes the characterization of distribution by using characteristic function is more widely use than other characterization properties on distribution.

The existence of characteristic function for every distribution makes the characteristic function becomes most powerful methods to describe the characterization of a distribution. Devianto et al. [11] and Devianto et al. [12] have described the property of convoluted exponential and hypoexponential distribution with stabilizer constant, while Devianto et al. [12] has used the \( m \)-fold convolution method to show the sum of distributions from an exponential distribution with stabilizer constant by using the property of characteristic function as Fourier-Stieltjes transform. Furthermore, Gut [13] has given some properties of Riemann Zeta distribution for the characteristic function as a normalized Zeta function, and while Nakamura [14] has explored the infinitely divisible of Riemann Zeta function based on characteristic function properties. The most recent result concerning Riemann Zeta distribution is introduced by Najnudel [15] on the extreme values of Riemann Zeta distribution and its hypothesis and Arguin et al. [16] developed application of theoretical concepts of maxima of a randomized Riemann zeta function. However, the characteristic function properties of Riemann Zeta distribution are described on its infinite divisibility, but it has not completely described such as the characteristic function of the exponential distribution. This paper will give exploration of the new characterizations of the characteristic function of Riemann Zeta distribution on its continuity, definite non-negative function in the quadratic form and infinite divisibility in sense of characteristic functions.

2. The Characteristic Function of Riemann Zeta Distribution

The characteristic function of Riemann Zeta distribution is derived from Fourier-Stieltjes transform for a random variable \( X = -\log n \) as follows

\[
\phi_X(t) = E[\exp(itX)] = E[\exp(it(-\log n))] = E[n^{-it}].
\] (4)

Now, it is used the property of expectation to the Riemann Zeta distribution, then it is obtained the characteristic function in the form

\[
\phi_X(t) = \sum_{n=1}^{\infty} n^{-it} \frac{1}{\zeta(\sigma)n^\sigma} = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} = \frac{\zeta(\sigma+it)}{\zeta(\sigma)}.
\] (5)

This characteristic function of Riemann Zeta distribution is also known as a normalized Zeta function. The property of complex conjugate of characteristic function from Riemann Zeta distribution is mentioned by using the relation

\[
\phi_{-X}(t) = \overline{\phi_X(t)}.
\] (6)

Let us consider the following Fourier-Stieltjes transform

\[
\phi_{-X}(t) = E[\exp(it(-X))] = E[\exp(it \log n)] = E[n^{it}].
\] (7)

Then we have

\[
\phi_{-X}(t) = \sum_{n=1}^{\infty} n^{it} \frac{1}{\zeta(\sigma)n^\sigma} = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-it}} = \frac{\zeta(\sigma-it)}{\zeta(\sigma)}.
\] (8)

This property of characteristic function from the Riemann Zeta distribution has expressed that \( \phi_X(t) \neq \phi_{-X}(t) \), then characteristic function of Riemann Zeta distribution as complex-valued function has
not only real part but also imaginary part. The characteristic function of Riemann Zeta distribution can be described on the complex plane in the term of parametric curves as in Figure 1.

![Parametric curve of characteristic function of Riemann Zeta distribution](image)

**Figure 1.** Parametric curve of characteristic function of Riemann Zeta distribution, (a) various parameter \( \sigma = 1.15, 1.65, 2.95 \) and (b) various parameter \( \sigma = 2, 4, 8 \).

Figure 1 has described the parametric curve of characteristic function of Riemann Zeta distribution. The parametric curves of characteristic function in (a) and (b) describe graphically continuous, never vanish on the complex plane and exhibit positively definite function. In addition, both of Figure 1 in Part (a) and Part (b) have show tendencies that for \( \sigma \rightarrow \infty \) then characteristic function of Riemann Zeta distribution tends to one at the real part of this characteristic function.

### 3. The Properties of Riemann Zeta Distribution

The aim of this section is to present the properties of the characteristic function of Riemann Zeta distribution. This characterization is stated as the main results in the term of definite positive function and its quadratic form properties to provide the property of infinite divisibility.

**Proposition 3.1.** Let random variable \( X = -\log n \) has Riemann Zeta distribution with characteristic function

\[
\phi_X(t) = \frac{\zeta(\sigma + it)}{\zeta(\sigma)},
\]

then this characteristic function is equal to one for \( t = 0 \).

**Proof.** It is easily to have \( \phi_X(0) = 1 \).

**Proposition 3.2.** The characteristic function of Riemann Zeta distribution is continuous.

**Proof.** Let \( \phi_X(t) \) be characteristic function of Riemann Zeta distribution and let us define \( h = s - t \) for \( s > t \). It will show that for every \( \varepsilon > 0 \) there is exist \( \delta \) such that \( |\phi_X(s) - \phi_X(t)| < \varepsilon \) then \( s - t < \delta \). Furthermore, we have the following equation
$$|\phi_X(s) - \phi_X(t)| = \left| \frac{\zeta(\sigma+i s)}{\zeta(\sigma)} - \frac{\zeta(\sigma+i t)}{\zeta(\sigma)} \right| = \left| \sum_{n=1}^{\infty} \frac{1}{n^\sigma} (n^{-i s} - n^{-i t}) \right|$$

It is for $h = s - t$ and $h \to 0$, then we have

$$|\phi_X(s) - \phi_X(t)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^\sigma} (n^{-i(t+h)} - n^{-i t}) \right| \to 0.$$ 

This last condition is to prove the continuity of characteristic function of Riemann Zeta distribution. ■

**Proposition 3.3.** Let random variable $X = -\log n$ has Riemann Zeta distribution with characteristic function

$$\phi_X(t) = \frac{\zeta(\sigma+it)}{\zeta(\sigma)},$$

then it satisfies the quadratic form

$$\sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j \overline{c_l} \phi_X(t_j - t_l) \geq 0$$

for any integer number $r \geq 2$ and any complex numbers $c_1, c_2, \ldots, c_r$ and real numbers $t_1, t_2, \ldots, t_r$.

**Proof.** It will explain that the characteristic function $\phi_X(t)$ satisfy the quadratic form. We have the following condition for characteristic function from Riemann Zeta distribution

$$\sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j \overline{c_l} \phi_X(t_j - t_l) = \sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j \overline{c_l} \frac{\zeta(\sigma+i(t_j - t_l))}{\zeta(\sigma)} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} n^{-i(t_j - t_l)} \sum_{n=1}^{\infty} \frac{1}{n^\sigma}.$$ 

This condition is rewritten on quadratic form by using the property of complex conjugate for characteristic function in the form $\phi_X(t) = \overline{\phi_X(t)}$, and the property of modulus for complex number, such that we have the quadratic form as follows
\[ \sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j \bar{c}_l \phi_X(t_j - t_l) = \sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} \frac{\sum_{n=1}^{\infty} \frac{1}{n^\sigma} c_j n^{-it_j} c_l n^{-it_l}}{\sum_{n=1}^{\infty} \frac{1}{n^\sigma}} \]

\[ = \left( \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \right)^{-1} \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \left( \sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j n^{-it_j} c_l n^{-it_l} \right) \]

\[ = \left( \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \right)^{-1} \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \left| \sum_{1 \leq j \leq r} \sum_{1 \leq l \leq r} c_j n^{-it_j} \right|^2 \geq 0. \] (12)

Then it is proved that characteristic function \( \phi_X(t) \) from Riemann Zeta distribution as positively defined function where the quadratic form has non-negative values. \( \blacksquare \)

The next proposition will explain the property of infinitely divisible characteristic function of Riemann Zeta distribution referred from Lin and Hu [7]. However, on the construction of infinitely divisible Riemann Zeta distribution is introduced the new setting of proof by mentioning there exists a characteristic function of random variable \( Y_i \) such that we have \( X = Y_1 + Y_2 + \ldots + Y_m \), in other word, there exists a characteristic function \( \phi_Y(t) \) such that \( \phi_X(t) = (\phi_Y(t))^m \) where \( \phi_Y(t) \) is characteristic function from Riemann Zeta distribution.

**Proposition 3.4.** The characteristic function of Riemann Zeta distribution is infinitely divisible.

**Proof.** Let \( \phi_X(t) \) be characteristic function of Riemann Zeta distribution and we can rewrite the characteristic function

\[ \phi_X(t) = \zeta(\sigma + it) / \zeta(\sigma) = \exp \left[ \log \zeta(\sigma) \left( \frac{\log \zeta(\sigma + it)}{\log \zeta(\sigma)} - 1 \right) \right]. \] (13)

It is used the Von Mangoldt function \( \Lambda(n) = \log p \) for \( n = p^k \) where \( k = 0, 1, 2, \ldots \) to have the Zeta function identity as follows

\[ \zeta(\sigma + it) = \exp \left[ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-(\sigma + it)} \right]. \]

then we have the form of characteristic function of Riemann Zeta distribution as follows

\[ \phi_X(t) = \exp \left[ \log \zeta(\sigma) \left( \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} \log \zeta(\sigma) \exp[-it \log n] - 1 \right) \right]. \] (14)

Let us consider the form of infinitely divisible characteristic function from compound Poisson distribution, that is

\[ \phi_X(t) = \exp[\lambda (\exp[\ln \phi_Y(t)] - 1)] \] (15)

for \( X = X_1 + X_2 + \ldots + X_N \) where random variable \( N \) has Poisson distribution with parameter \( \lambda \), and arbitrary independent and identically random variables of \( X_1, X_2, \ldots, X_N \). Now, by setting \( \lambda = \log \zeta(\sigma) \) and the function
\[ \phi_{X_i}(t) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n \log \zeta(\sigma)} \exp[-it \log n], \]  
(16)

as the characteristic function of random variable \( X_i \), then this characteristic function of Riemann Zeta distribution can be written in the form of characteristic function of compound Poisson distribution, so that Riemann Zeta distribution is infinitely divisible. ■

The infinitely divisible distribution originally comes from the idea that the random variable \( X \) can be divided into \( m \) random variables which are independent and identically random variables of \( Y_i \) for \( i = 1, 2, ..., m \). The characteristic function of Riemann Zeta distribution can be written as compound Poisson distribution which is infinitely divisible, then we can set the random variable Riemann Zeta distribution \( X = -\log n \) divided into \( m \) random variable of \( Y_i \) which characteristic function as follows
\[ \phi_{X}(t) = (\phi_{Y_i}(t))^m, \]
so that we have
\[ \phi_{Y_i}(t) = \left( \frac{\phi_X(t)}{m} \right)^m = \exp \left[ \frac{\log \zeta(\sigma)}{m} \left( \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n \log \zeta(\sigma)} \exp[-it \log n] - 1 \right) \right]. \]  
(17)

The function \( \phi_{Y_i}(t) \) is the characteristic function of compound Poisson distribution with parameter \( \lambda = \log \zeta(\sigma)/m \) such that we have \( X = Y_1 + Y_2 + ... + Y_m \) where the random variable \( Y_i \) has characteristic function in the form
\[ \phi_{Y_i}(t) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n \log \zeta(\sigma)} \exp[-it \log n]. \]  
(18)

This condition explained infinitely divisible of characteristic function of Riemann Zeta distribution in the sense of \( m \)-fold convolution of probability distribution of random variable \( Y_i \).
Figure 2. Parametric curves of characteristic function of random variable \( Y_i \), (a) Parameter of Riemann Zeta distribution \( \sigma = 1.15 \) with various \( m = 2, 4, 8 \), (b) 2 fold convolution of random variable \( Y_i \) with various parameter of Riemann Zeta distribution \( \sigma = 2, 4, 8 \).

Figure 2 has described the parametric curve of characteristic function \( (\phi(t))^{1/m} \) where \( \phi(t) \) is characteristic function of Riemann Zeta distribution that is infinitely divisible. The parametric curves of characteristic function in (a) and (b) describe graphically the continuity property such as explained on Proposition 3.2, the curves also have explained that for \( t = 0 \) then the characteristic function is equal to one as in Proposition 3.1 and the parametric curves never vanish on the complex plane and exhibit positively definite function such as explained in Proposition 3.3. Figure 2 Part (a) has shown tendencies that for \( m \to \infty \) then characteristic function \( (\phi(t))^{1/m} \) tends to one at real part of this characteristic function, while from Figure 2 Part (b) has shown tendencies for \( \sigma \to \infty \) then characteristic function \( (\phi(t))^{1/m} \) tends to one at real part of this characteristic. This result gave a new exploration of parametric curves for characterization of characteristic function based on parameter on the Riemann Zeta distribution.

4. Conclusion
The Riemann Zeta distribution is discrete distribution in complex-valued function containing Zeta function. The characteristic function of Riemann Zeta distribution exists and it is obtained by using Fourier-Stieltjes transform to be a normalized Zeta function. The characteristic function of Riemann Zeta distribution has properties as continuity and definite positive function property in quadratic form. The combination properties of definite non-negative property and its continuity provided infinite divisibility of Riemann Zeta distribution in the form of compound Poisson distribution in sense of its characteristic function. The characteristic function of Riemann Zeta distribution tends to one on the real part as parameter \( \sigma \to \infty \) and this is also explained the characteristic function is never vanished on the complex plane.

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