Accelerating many-body entanglement generation by dipolar interactions in the Bose-Hubbard model

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The spin squeezing protocols allow the dynamical generation of massively correlated quantum many-body states, which can be utilized in entanglement-enhanced metrology and technologies. We study a quantum simulator generating twisting dynamics realized in a two-component Bose-Hubbard model with dipolar interactions. We show that the interplay of contact and long-range dipolar interactions between atoms in the superfluid phase activates the anisotropic two-axis counter-twisting mechanism, accelerating the spin squeezing dynamics and allowing the Heisenberg-limited accuracy in spectroscopic measurements.

I. INTRODUCTION

The second Quantum Revolution’s main objective lies in multipartite entangled states: their production, storage, certification, and application. Such states, i.e., many-body entangled and many-body Bell correlated states, are essential resources for quantum-based technologies and quantum-enhancement metrology [1–6]. As such, a general protocol allowing the controlled generation of such states is an extensive research direction in modern quantum science. Spin squeezing represents such a protocol paving the way for high-precision measurements, allowing overcoming the shot-noise limit [7, 8].

The lowest value of the squeezing parameter scales with particle numbers, and for OAT it is $\xi_{\text{best}} \propto N^{-1/3}$ at $\chi t_{\text{best}} \approx N^{-2/3}$ [7]. The TACT Hamiltonian reads

$$\hat{H}_{\text{TACT}} = \hbar \chi (\hat{S}_z^2 - \hat{S}_x^2),$$

where the clockwise and counter-clockwise twisting take places around two orthogonal axes $z$ and $x$. The advantage of the TACT is that it gives the Heisenberg limited level of the best squeezing, namely $\xi_{\text{best}} \propto N^{-1/2}$. In addition, the time scale of the best squeezing is accelerated with respect to OAT, and it is given by $\chi t_{\text{best}} \sim N^{-1} \log(2N)$ [25].

Realizing quantum simulators of OAT or TACT dynamics is essential for quantum enhancement metrology. Ultra-cold atoms form a perfect platform for quantum simulators mimicking such twisting dynamics. OAT has been realized with Bose-Einstein condensates utilizing atom-atom collisions [26–31], and atom-light interactions [32, 33]. Another research directions are ultracold platforms simulating the Hubbard and Heisenberg models, which generate twisting dynamics. In the case of bosons, twisting dynamics is induced by atom-atom collisions [34–37], while for spinful fermions a synthetic spin-orbit coupling is necessary [38–44]. Finally, twisting dynamics can be activated with long-range interacting bosons, what provides a platform for spin squeezing simulators by casting original Hamiltonian onto long-range interacting spin-chain [45–51].

In this work, we propose a quantum simulator for the TACT model realized in a one-dimensional two-component Bose-Hubbard model in the superfluid phase, considering both contact and dipolar interactions. With the help of full many-body dynamics and an effective two-mode model (TMM) description, we show that the realized squeezing dynamics capture properties of the anisotropic TACT model where the clockwise and counter-clockwise twisting take place with different rates.

Next, we show that scaling with the system size of the best squeezing parameter and best squeezing time is...
equivalent to the scaling obtained for the TACT model. Our results show the significant acceleration of the spin squeezing dynamics by dipolar interactions, which is an essential effect from the experimental point of view.

The paper is organized as follows. In Sec.II we introduce the considered model. Starting with general many-body description of the system we provide an effective two mode model accounting for both contact and long-range dipolar interactions. Next, in Sec.III we perform analysis of the mean-field phase space of the anisotropic TACT model. In Sec.IV with the help of Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations [52, 53], we analyze the scaling of the best squeezing and the best squeezing time with the system size. We conclude in Sec.V.

II. EXACT AND EFFECTIVE MODELS

We consider $N$ bosonic atoms in the two internal states $|\uparrow\rangle, |\downarrow\rangle$ which corresponds to the ensemble of $N$ spin-1/2 particles (qubits). The atoms are described by the following Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_d,$$

where

$$\hat{H}_0 = \int d^3 r \hat{\Psi}^\dagger(r) \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{latt}}\right) \hat{\Psi}(r),$$

and

$$\hat{H}_d = \int d^3 r_1 \int d^3 r_2 \hat{\Psi}^\dagger(r_1) \hat{\Psi}^\dagger(r_2) V_{12} \hat{\Psi}(r_2) \hat{\Psi}(r_1),$$

where the vector of bosonic field operators is $\hat{\Psi}^\dagger(r) = (\hat{\Psi}_{\uparrow}(r), \hat{\Psi}_{\downarrow}(r))$ with $\hat{\Psi}_\sigma(r)$ describing an atom at the position $r$ in the state $\sigma = \uparrow, \downarrow$. The interaction potential $V_{12}$ is a sum of two terms, $V_{12} = V_c + V_d$, the short range contact interaction

$$V_c = \frac{4\pi \hbar^2 a_s}{m} \delta(r_1 - r_2),$$

and the long range dipolar interaction

$$V_d = \frac{\mu_1 \cdot \mu_2}{|r_1 - r_2|^3} - \frac{3[\mu_1 \cdot (r_1 - r_2)][\mu_2 \cdot (r_1 - r_2)]}{|r_1 - r_2|^5},$$

where $\mu_{1,2}$ is the dipole moment, $m$ is the atomic mass and $a_s$ is the s-wave scattering length.

The atoms are loaded into the one-dimensional optical lattice potential $V_{\text{latt}} = V_0 \sin^2 k x$, where $k = 2\pi/\lambda_{\text{latt}}$ is a wave-vector associated with the lattice wave-length $\lambda_{\text{latt}}$. We consider the unit filling, so the number of lattice sites $M$ equals the total number of atoms $N$ ($M = N$). We assume the atoms are in the superfluid phase and occupy the lowest Bloch band. In the tight-binding approximation, the field operators is conveniently expanded in the basis of the Wannier functions, and the system Hamiltonian (4) reduces to the two-component Bose-Hubbard model (BHM) extended by the dipolar term, namely

$$\hat{H} = \hat{H}_{\text{BH}} + \hat{H}_d \equiv \hat{H}_{\text{dBH}}.$$
The first term, $\hat{H}_{\text{BH},n=0}$, comes from the zero quasi-momentum mode of the Bose-Hubbard Hamiltonian,

$$\hat{H}_{\text{BH},q_n=0} = -2J\hat{N}_{q_n=0} + \Omega_{NN}\hat{N}^2_{q_n=0} + \Omega_{SN}\hat{S}_{z,q_n=0}\hat{N}_0 + \Omega_{SS}\hat{S}^2_{z,q_n=0},$$

where

$$\Omega_{NN} = \frac{U_{\uparrow\uparrow} + U_{\downarrow\downarrow} + 4U_{\uparrow\downarrow}}{8N},$$

$$\Omega_{SN} = \frac{U_{\uparrow\uparrow} - U_{\downarrow\downarrow}}{2N},$$

$$\Omega_{SS} = \frac{U_{\uparrow\uparrow} + U_{\downarrow\downarrow} - 2U_{\uparrow\downarrow}}{2N},$$

and realizes OAT dynamics [36]. The second term, $\hat{H}_{d,q_n=0}$, consists of zero momentum component of the dipolar interaction:

$$\hat{H}_{d,q_n=0} = \frac{2}{N} \sum_{d=1}^{[N/2]} \frac{h^{(3)}_{\hat{S}^2_{z,q_n=0} - 3\hat{S}^2_{z,q_n=0}}}{\gamma^2\hat{h}_{[N/2]}},$$

with $\hat{H}_{\text{BH},q_n=0}$ coming from the zero quasi-momentum mode of the Bose-Hubbard Hamiltonian, $\hat{N}_0$ and $U_{\uparrow\uparrow} = U_{\downarrow\downarrow} = U$. Note here, the zero quasi-momentum component of the spin operators correspond to the collective spin operators in the position representation, namely $\hat{S}_{\beta,q_n=0} = \frac{1}{\sqrt{N}} \sum_j \hat{S}_{\beta,j}$ with $\beta = x, y, z$. As such, we can replace $\hat{S}_{\beta,0}$ by the collective spin operators $\hat{S}_{\beta}$ in (18). Taking this into account, we identify the effective two-mode model (TMM):

$$\hat{H}_{\text{TMM}} = h\chi \left( \hat{S}^2_z - \eta \hat{S}_x^2 \right),$$

which is the anisotropic TACT with

$$h\chi = \frac{U - U_{\uparrow\downarrow}}{N},$$

$$h\chi\eta = \frac{2\gamma^2 h^{(3)}_{[N/2]}}{\gamma^2 \hat{h}_{[N/2]}},$$

where $\eta$ is the anisotropy parameter and $\chi$ sets the energy scale. In the two limit cases $\eta = 0$ and $\eta = 1$ the effective model (19) reduces to the OAT and TACT model, respectively.

In Fig. 1(a) we show spin squeezing parameter $\xi^2$ defined in (1) for different values of anisotropy parameter $\eta = \{0.0, 0.5, 1.0\}$ (lines from right to left). The two limiting cases, i.e., $\eta = 0$ and $\eta = 1$ correspond to OAT and TACT dynamics, respectively. Points correspond to the results from the exact many-body numerical simulation of $\hat{H}_{\text{BH}}$ given by (9), while solid lines to the numerical results from the effective two-mode model (19) when $N = M = 10, U = 0.01, J = 1.0$ and $U_{\uparrow\downarrow} = 0.95U$. (b) Color encoded values of the spin squeezing parameter $\xi^2$ versus $\chi t$ and $\eta$ obtained from the numerical simulations of the two mode model (19) for $N = 10^3$ atoms. The solid red line indicates the best squeezing time.

FIG. 1. (a) Time evolution of the spin squeezing parameter $\xi^2$ defined in (1) for different values of anisotropy parameter $\eta = \{0.0, 0.5, 1.0\}$ (lines from right to left). The two limiting cases, i.e., $\eta = 0$ and $\eta = 1$ correspond to OAT and TACT dynamics, respectively. Points correspond to the results from the exact many-body numerical simulation of $\hat{H}_{\text{BH}}$ given by (9), while solid lines to the numerical results from the effective two-mode model (19) when $N = M = 10, U = 0.01, J = 1.0$ and $U_{\uparrow\downarrow} = 0.95U$. (b) Color encoded values of the spin squeezing parameter $\xi^2$ versus $\chi t$ and $\eta$ obtained from the numerical simulations of the two mode model (19) for $N = 10^3$ atoms. The solid red line indicates the best squeezing time.
Equations of motion for the canonical position operators by complex numbers \cite{57}, \( \hat{a} \rightarrow \sqrt{N} \rho_a e^{i\phi_a} \), \( \hat{b} \rightarrow \sqrt{N} \rho_b e^{i\phi_b} \) what transforms the spin operators to \( \hat{S}_x \rightarrow N \sqrt{\frac{1}{2} - \eta} \cos \phi \), \( \hat{S}_y \rightarrow N \sqrt{\frac{1}{2} + \eta} \sin \phi \), \( \hat{S}_z \rightarrow \frac{N}{2} z \). This allows introducing the new canonical variables \( z = \rho_a - \rho_b \) and \( \phi = \phi_a - \phi_b \). The Hamiltonian (19) takes the form of the energy functional \( \epsilon(\phi, z) \):

\[
\epsilon(\phi, z) = \frac{N}{4} z^2 - \frac{\eta N}{4} (1 - z^2) \cos^2 \phi.
\]

Equations of motion for the canonical position \( \phi \) and the conjugate momentum \( z \) are set by the Hamilton equations:

\[
\dot{\phi} = \frac{\partial \epsilon(\phi, z)}{\partial z} = \frac{N}{2} z + \frac{\eta N}{2} z \cos^2 \phi
\]

\[
\dot{z} = -\frac{\partial \epsilon(\phi, z)}{\partial \phi} = -\frac{\eta N}{2} (1 - z^2) \cos \phi \sin \phi.
\]

III. MEAN-FIELD PHASE PORTRAITS

The activation of the TACT dynamics by dipolar interactions can be intuitively explained by analyzing the structure of the mean-field phase space of the two-mode model (19). It is a good navigator for the dynamical spin structure of the mean-field phase space of the two-mode model (19). It is a good navigator for the dynamical spin structure of the mean-field phase space of the two-mode model (19). In the following we will analyze the topology of phase portraits which are a geometrical representation of trajectories of a dynamical system in the phase space. In our case, trajectories are tangent to the velocity field \( (\phi, \dot{z}) \). The phase portrait consists of fixed points or closed orbits corresponding to a steady state, and satisfies \( (\phi, \dot{z}) = (0, 0) \). Spin squeezing takes place in the vicinity of unstable fixed points. We are interested in the fixed point located at \( z = 0 \) and \( \phi = \frac{\pi}{2} \) according to the location of our initial spin coherent state. The classification of fixed points can be found by analysis of the eigenproblem of the stability matrix \( \mathcal{M} \) which in our case is

\[
\mathcal{M} = \begin{bmatrix}
\frac{\partial^2 \epsilon}{\partial \phi \partial z} & \frac{\partial^2 \epsilon}{\partial z^2} \\
-\frac{\partial^2 \epsilon}{\partial z \partial \phi} & -\frac{\partial^2 \epsilon}{\partial \phi^2}
\end{bmatrix} = \frac{N}{2} \begin{bmatrix}
0 & 1 \\
\eta & 0
\end{bmatrix}.
\]

When \( \eta \neq 0 \) then the matrix \( \mathcal{M} \) has two non-degenerate real eigenvalues of the opposite sign \( (\lambda_1, \lambda_2) = \frac{1}{2} \sqrt{\eta} (-1, 1) \) and two real eigenvectors

\[
v_1 = \begin{bmatrix}
\frac{1}{\sqrt{\eta}} \\
1
\end{bmatrix} , v_2 = \begin{bmatrix}
\frac{1}{\sqrt{\eta}} \\
1
\end{bmatrix}.
\]

Scalar product of the two eigenvectors \( \{v_1, v_2\} \) defines the angle \( \theta \) between in-going and out-going trajectories crossing at the centre of the unstable saddle fixed points \( \theta = \arccos \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|} \).

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the pure OAT dynamics with non-isolated unstable fixed point. For $\eta > 0$ the nature of fixed point changes to the unstable saddle fixed point, see panels (b)-(d), which in the limiting case $\eta = 1$ corresponds to the TACT dynamics, panel (d). Note, the angle $\theta$ is approximately $\pi/2$ when the value of anisotropy parameter $\eta$ is one.

IV. SCALING WITH THE SYSTEM SIZE

In this paragraph we study the scaling of the best squeezing for the anisotropic TACT model (19). We apply the Gaussian approach within the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [52, 53] which was used in [25] to explain the scaling for the TACT model. Here, we generalize the theory taking into account the values of parameter $\eta$ different than one.

We start with equations of motion for expectation values of spin operators $\langle \hat{S}_j \rangle$ which involve terms that depend on the first-order moments $\langle \hat{S}_j \rangle$ and second-order moments $\langle \hat{S}_j \hat{S}_k \rangle$. Subsequently, the time evolution of the second-order moments depends on themselves and on third-order moments, and so on. It leads to the BBGKY hierarchy of equations of motion for expectation values of operator products. The hierarchy is then truncated by keeping the first- and the second-order moments,

$$\langle \hat{S}_j \hat{S}_k \rangle \simeq \langle \hat{S}_j \rangle \langle \hat{S}_k \rangle + \langle \hat{S}_j \hat{S}_k \rangle \langle \hat{S}_j \rangle + \langle \hat{S}_j \rangle \langle \hat{S}_k \rangle - \langle \hat{S}_j \hat{S}_k \rangle \langle \hat{S}_k \rangle. \quad (27)$$

To perform the scaling analysis, we first introduce a small parameter $\epsilon = 1/N$, and transform the spin components into $\hat{J}_j = \sqrt{\eta} \hat{S}_j$ which obey cyclic commutation relations $[\hat{J}_x, \hat{J}_y] = i\sqrt{\eta} \hat{J}_z$. The Hamiltonian (19) then reads $\hat{H} = \frac{\hbar}{2} (\hat{J}_j^2 - \eta \hat{J}_z^2)$. Equations of motion for expectation values of the spin operators $\langle \hat{J}_j \rangle \equiv h_j$, second order moments $\langle \hat{J}_j \hat{J}_k \rangle \equiv \Delta_{j,j}$ and $\langle \hat{J}_j^2 \rangle - \langle \hat{J}_j \rangle^2 \equiv \delta_{j}$ read

$$\dot{h}_j = 2(1 + \eta) \Delta_{j}, \quad (28)$$
$$\Delta_{x,z} = -2(\delta_{z} + \eta \delta_{x}) h_y, \quad (29)$$
$$\dot{\delta}_z = -4\eta \Delta_{x,z} h_y, \quad (30)$$
$$\dot{\delta}_x = -4\Delta_{x,z} h_y, \quad (31)$$

where time is measured in dimensionless unit $\tau = \chi t/\sqrt{\epsilon}$. The initial coherent state at the unstable saddle fixed point, $|\Psi(0)\rangle = |\theta = \pi/2, \varphi = \pi/2\rangle$, gives the following initial conditions: $h_y(0) = (2\sqrt{\eta})^{-1}$, $\delta_{x}(0) = \delta_{x}(0) = 1/4$ and $\Delta_{x,z}(0) = 0$. In order to find the approximate solution we introduce the two quadratures: $X = \delta_{x} + \sqrt{\eta} \Delta_{x,z}$ and $Y = \delta_{z} - \sqrt{\eta} \Delta_{x,z}$ obeying the dynamical equations $\dot{X} = -4\sqrt{\eta} X h_y$ and $\dot{Y} = -4\sqrt{\eta} Y h_y$ which have the following solutions:

$$X(t) = X(0)e^{-4\sqrt{\eta} f(\tau)}, \quad (32)$$
$$Y(t) = Y(0)e^{4\sqrt{\eta} f(\tau)}, \quad (33)$$

where $f(\tau) = \int_{0}^{\tau} h_y(\tau') d\tau'$ for $\eta \neq 0$. This gives

$$\delta_{x}(\tau) = \delta_{x}(0) \cosh [4\sqrt{\eta} f(\tau)], \quad (34)$$
$$\Delta_{x,z}(\tau) = -\frac{\delta_{z}(0)}{\sqrt{\eta}} \sinh [4\sqrt{\eta} f(\tau)], \quad (35)$$
$$h_y(\tau) - h_y(0) = -\frac{\delta_{x}(0)}{\sqrt{\eta}} \int_{0}^{\tau} \sinh [4\sqrt{\eta} f(\tau')] d\tau'. \quad (36)$$

In principle, the solution for $h_y$ can be find in self-consistent way, here however, we approximate it by taking the first iteration, namely $f(\tau) \approx f(0) + f'(0)\tau$, which results in

$$h_y(\tau) = \frac{1}{2\sqrt{\epsilon}} \left[ 1 + \frac{(1 + \eta)\epsilon}{2\eta} \left( 1 - \cosh \left( 2\tau \sqrt{\frac{\eta}{\epsilon}} \right) \right) \right]. \quad (37)$$

Next, one evaluates the evolution of (34) and (35) by taking (37) in $f(\tau)$. Finally, noting that the spin squeezing parameter (1) is determined by the quadrature $X(t)$, namely $\xi^2 \approx X(t)$, when approximating $\langle S \rangle \approx h_y(0)/\sqrt{\epsilon}$, we obtain the scaling of the best squeezing and the best squeezing time with $N$ by keeping the leading order terms in $\epsilon$, what

$$\xi_{\text{best}}^2 \sim 1/N, \quad \chi t_{\text{best}} \sim \frac{\ln(\eta N)}{\sqrt{\eta} N}. \quad (38)$$
when η is of the order of one. We compared the above analytical predictions with the numerically solved set of differential equations (28)-(31) and confirmed the scaling (38) when η ∈ (0.3, 1).

The quantitative illustration of the above results can be provided by analyzing the scaling of the best squeezing with N obtained from the numerical time evolution of the TMM Hamiltonian (19). Fig. 3 (a) presents the best squeezing $\xi_{\text{best}}^2$ and the best squeezing time $\chi_{\text{best}}$ as a function of particle number $N$. Power-law behaviour can be noticed for various η. Therefore, for each value of the anisotropy parameter η we extracted the scaling exponent α by fitting $\xi_{\text{best}}^2 \sim N^{-\alpha}$. Panel (b) of Fig. 3 shows the change of the fitted exponent α as the function of anisotropy parameter η and is compared to the variation of the angle $\theta$. A characteristic feature is a change in the value of α when η ∈ (10^{-3}, 10^{-1}). In the same range $\theta$ diminishes from π to approximately π/2. We conclude, the variation of $\alpha$ is driven by the change in the structure of the unstable fixed point. It is worth to mention here, that $\alpha \approx 1$ when η ≈ 1. Our results show that the Heisenberg limited level of squeezing is possible in the anisotropic TACT model.

V. CONCLUSIONS

In this work, we show how the OAT mechanism, generating many-body entanglement, can be accelerated by the long-range interactions via activation of the anisotropic TACT mechanisms. We explain the activation of the TACT mechanism during competition of contact and dipolar interactions between bosons in a superfluid phase. We propose the feasible experimentally quantum simulator for the anisotropic TACT dynamics based on dipolar two-component Bose-Hubbard in a one-dimensional optical lattice. With the help of the scaling analysis, we show that it is possible to obtain a Heisenberg limited level of squeezing for a weak anisotropy. The anisotropic TACT model accelerates the spin squeezing dynamics compared to OAT with the improvement of the level of squeezing. Our protocol allows for fast generation of many-body entangled states with entanglement depth larger than in a standard OAT scenario.

Our work provides an essential step toward generating many-body entangled states during two-axis counter-twisting protocol in state-of-the-art experimental setups, paving the way for obtaining the Heisenberg limit of spectroscopic measurements in ultracold atoms systems.

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Appendix A: Dipolar interaction

To obtain the lattice version of dipolar interaction one starts with the Hamiltonian (6) and (8), associates the on-site Hamiltonian with the Pauli matrices as $\mathbf{\sigma}_1 = (\sigma_x, \sigma_y, \sigma_z)$, and the same at $\mathbf{r}_2$. Then one obtains

$$\hat{H}_d = \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 \frac{\gamma^2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \times \left[ (1 - 3 \cos^2 \phi_1) \mathbf{j}_1^+ \mathbf{j}_2^- - \frac{J_1^+ J_2^- + J_1^- J_2^+}{4} \right]$$

$$- \frac{3}{4} \sin^2 \phi_1 \left( e^{i\phi_2} J_1^+ J_2^- + \text{h.c.} \right)$$

$$- \frac{3}{4} \sin 2\phi_1 \left( e^{i\phi_2} (J_1^+ J_2^- + J_1^- J_2^+) + \text{h.c.} \right),$$

with $\mathbf{r}_1 \neq \mathbf{r}_2$, and where $\mathbf{j}_i^+ = \hat{\Psi}_i^\dagger (\mathbf{r}_i) \hat{\Psi}_i (\mathbf{r}_i)$, $\mathbf{j}_i^- = \hat{\Psi}_i^\dagger (\mathbf{r}_i) \hat{\Psi}_i (\mathbf{r}_i)$, $\mathbf{j}_i^\dagger = (\hat{\Psi}_i^\dagger (\mathbf{r}_i) \hat{\Psi}_i (\mathbf{r}_i) - \hat{\Psi}_i^\dagger (\mathbf{r}_i) \hat{\Psi}_i (\mathbf{r}_i))/2$, and similarly at $\mathbf{r}_2$. The two angles $\phi_1, \phi_2$ parameterize the normal vector along $\mathbf{r}_1 - \mathbf{r}_2$ direction, namely $\mathbf{n}_{12} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = (\cos \phi_1 \sin \theta_1, \sin \phi_1 \sin \theta_1, \cos \theta_1).$
and we expand $\hat{\Phi}$

Therefore, we consider the following form of the field operator

$$\hat{\Psi}(r) = \hat{\Phi}(x) \phi(y) \phi(z),$$

(A2)

and we expand $\hat{\Phi}(x)$ in the basis of Wannier functions $w(x - x_j)$ localized around lattice sites, where $x_j$ denotes position of the j-th site in the lowest energy band,

$$\hat{\Phi}(x) = \sum_j \hat{a}_{j,\uparrow} w(x - x_j),$$

(A3)

where $\hat{a}_{j,\uparrow}$ annihilates an atom in the single-particle Wannier state $w(x - x_j)$ of the lowest energy band localized on the j-th site, in the internal state $\uparrow$. In (A2) we assume $\phi(y)$ and $\phi(z)$ are the ground state wave-functions of the system in the $y$ and $z$ directions. The same applies for the $\downarrow$ operator.

The geometry of the system, we have chosen, determines the normal vector, $\vec{n} = (1, 0, 0)$, and sets the value of $\theta_{12} = \pi/2$ and $\phi_{12} = 0$. Taking this into account, in the tight-binding limit when the lattice height is larger than the recoil energy $E_R = (2\pi)^2/(2m\lambda_{\text{latt}}^2)$ and the Wannier functions are well localized around each lattice site, the dipolar Hamiltonian reduces to

$$\hat{H}_d = \sum_{j,k \neq j} \frac{\gamma^2}{d^3 |j - k|^3} \left( \hat{S}_{z,j} \hat{S}_{z,k} - 2\hat{S}_{x,j} \hat{S}_{x,k} + \hat{S}_{y,j} \hat{S}_{y,k} \right),$$

(A4)

due to normalization of the wave functions, and were $d = \lambda_{\text{latt}}/2$ will be absorbed in the parameter $\gamma^2$ in the main part of the paper.

It is worth commenting here about the importance of the geometry chosen. There is a symmetry between the $x$ and $y$ axis, i.e., if the lattice would be along the $y$ axis the resulting Hamiltonian (A4) would have the factor minus two in the front of the $\hat{S}_{y,j} \hat{S}_{y,k}$ term. On the other hand, if the lattice would be along $z$ axis, the factor $-2$ appears in the front of $\hat{S}_{j} \hat{S}_{k}$. This has an important consequence in the resulting effective model (19) which would be the OAT one.

Appendix B: Numerical evaluation of spin squeezing parameter

1. Dipolar Bose-Hubbard model

We performed the full many-body numerical simulations of $\hat{H}_{\text{dBH}} = \hat{H}_{\text{BH}} + \hat{H}_d$ with (10) and (11). To this end we constructed the Fock states basis, as described in [36]. We implemented numerically the matrix representations of the Hamiltonian $\hat{H}_{\text{dBH}}$, and the initial spin coherent state is

$$|\Psi(0)\rangle = |\theta, \psi\rangle = e^{-i\hat{S}_x \psi} e^{-i\hat{S}_y \theta} |\Psi_a\rangle,$$

(B1)

where $|\Psi_a\rangle$ is the ground state of the system when all atoms are in the $|\uparrow\rangle$ state. The system evolves according to the unitary operator, namely

$$|\Psi(t)\rangle = e^{-i\hat{H}_{\text{dBH}} t/\hbar} |\Psi(0)\rangle,$$

(B2)

and the spin squeezing parameter (1) is calculated.

2. Two-mode model

In order to find the scaling exponents we perform numerical time evolution of the two-mode model. We express Hamiltonian (19) in the Fock state basis consisting the vectors of the form $|n, N - n\rangle$, where $N$ is the total number of atoms, $n$ is the number of the particles in the $|\uparrow\rangle$ state and $N - n$ is the number of the particles in the $|\downarrow\rangle$ state.

Our initial state is the spin coherent state, which we obtain as a double rotation of the state $|N, 0\rangle$ according to:

$$|\Psi(0)\rangle = |\theta, \psi\rangle = e^{-i\hat{S}_x \psi} e^{-i\hat{S}_y \theta} |N, 0\rangle.$$  

(B3)

Next, we apply the unitary evolution

$$|\Psi(t)\rangle = e^{-i\hat{H}_{\text{TMM}} t/\hbar} |\Psi(0)\rangle,$$

(B4)

and calculate the spin squeezing parameter (1) and find its first minimum $\xi_{\text{best}}^2$, as well as the time at which it occurs $\chi_{\text{best}}$. 

[1] A. Acín, I. Bloch, H. Buhrman, T. Calarco, C. Eichler, J. Eisert, D. Esteve, N. Gisin, S. J. Glaser, F. Jelezko, S. Kuhr, M. Lewenstein, M. F. Riedel, P. O. Schmidt,
(2022).

[51] T. Comparin, F. Mezzacapo, and T. Roscilde, Phys. Rev. A 105, 022625 (2022).

[52] J. R. Anglin and A. Vardi, Phys. Rev. A 64, 013605 (2001).

[53] A. André and M. D. Lukin, Phys. Rev. A 65, 053819 (2002).

[54] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Lühmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, Reports on Progress in Physics 78, 066001 (2015).

[55] F. Trimborn, D. Witthaut, and H. J. Korsch, Phys. Rev. A 79, 013608 (2009).

[56] V. S. Shchesnovich and V. V. Konotop, Phys. Rev. A 77, 013614 (2008).

[57] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997).