GLOBAL EXISTENCE FOR NULL-FORM WAVE EQUATIONS 
WITH DATA IN A SOBOLEV SPACE OF LOWER REGULARITY 
AND WEIGHT

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Abstract. Assuming initial data have small weighted $H^4 \times H^3$ norm, we prove global existence of solutions to the Cauchy problem for systems of quasi-linear wave equations in three space dimensions satisfying the null condition of Klainerman. Compared with the work of Christodoulou, our result assumes smallness of data with respect to $H^4 \times H^3$ norm having a lower weight. Our proof uses the space-time $L^2$ estimate due to Alinhac for some special derivatives of solutions to variable-coefficient wave equations. It also uses the conformal energy estimate for inhomogeneous wave equation $\Box u = F$. A new observation made in this paper is that, in comparison with the proofs of Klainerman and Hörmander, we can limit the number of occurrences of the generators of hyperbolic rotations or dilations in the bootstrap argument. This limitation allows us to obtain global solutions for radially symmetric data, when a certain norm with considerably lower weight is small enough.

1. Introduction

This paper is concerned with the Cauchy problem for systems of quasi-linear wave equations

$$(1.1) \quad \Box u_i + F_i(\partial u, \partial^2 u) + C_i(u, \partial u, \partial^2 u) = 0, \quad t > 0, \quad x \in \mathbb{R}^3$$

$(i = 1, \ldots, N$ for some $N \in \mathbb{N})$ with initial data

$$(1.2) \quad u_i(0) = f_i, \quad \partial_t u_i(0) = g_i.$$ 

Here, $\Box = \partial_t^2 - \Delta$, $u = (u_1, \ldots, u_N)$, $\partial u = (\partial u_1, \ldots, \partial u_N)$, $\partial u_i = (\partial_0 u_i, \ldots, \partial_3 u_i)$, $\partial^2 u = (\partial^2 u_1, \ldots, \partial^2 u_N)$, $\partial^2 u_i = (\partial_0^2 u_i, \partial_0 \partial_1 u_i, \ldots, \partial_3^2 u_i)$, $\partial_a = \partial / \partial x_a$ $(a = 1, 2, 3)$. As in the seminal papers [3] and [12], we will discuss the diagonal system, and we suppose that the quadratic nonlinear term $F_i(\partial u, \partial^2 u)$ has the form

$$(1.3) \quad F_i(\partial u, \partial^2 u) = F_i^{j, \alpha \beta \gamma}(\partial_j u_j)(\partial_{\alpha \beta}^2 u_i) + F_i^{j k, \alpha \beta}(\partial_\alpha u_j)(\partial_\beta u_k), \quad i = 1, \ldots, N$$

for real constants $F_i^{j, \alpha \beta \gamma}$ and $F_i^{j k, \alpha \beta}$. Here, and in the following, we use the summation convention, that is, if lowered and raised, repeated indices of Greek letters and Roman letters are summed from 0 to 3 and 1 to $N$, respectively. As for the higher-order term $C_i(u, \partial u, \partial^2 u)$, we may suppose without loss of generality that it

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is cubic because this paper is concerned only with small solutions. We thus suppose it has the form

\[ C_i(u, \partial u, \partial^2 u) = G^\alpha_\beta(u, \partial u) \partial^2_{\alpha\beta} u_i + H_i(u, \partial u), \quad i = 1, \ldots, N, \]

where \( G^\alpha_\beta(u, v) \) is a homogeneous polynomial of degree 2, and \( H_i(u, v) \) is a homogeneous polynomial of degree 3 in \( u \) and \( v \). Since we consider \( C^2 \)-solutions, we may suppose without loss of generality

\[ F^\alpha_\beta = F^\beta_\alpha, \quad G^\alpha_\beta(u, \partial u) = G^\alpha_\beta(u, \partial u). \]

For given \( i, j \), we say that the set of the coefficients \( \{ F^\alpha_\beta, \alpha, \beta = 0, \ldots, 3 \} \) satisfies the null condition if we have

\[ F^\alpha_\beta X_\alpha X_\beta X_\gamma = 0 \]

for any \( X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \) satisfying \( X_0^2 = X_1^2 + X_2^2 + X_3^2 \). Also, for given \( i, j, k \), we say that the set of the coefficients \( \{ F^{jk}_i, \alpha, \beta = 0, \ldots, 3 \} \) satisfies the null condition if we have

\[ F^{jk}_i X_\alpha X_\beta = 0 \]

for any \( X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \) satisfying \( X_0^2 = X_1^2 + X_2^2 + X_3^2 \). We say that the system (1.1) satisfies the null condition if the sets of the coefficients \( \{ F^\alpha_\beta \} \) and \( \{ F^{jk}_i \} \) satisfy the null condition for all given \( i, j \) and \( i, j, k \), respectively.

For the scalar wave equations, thus \( N = 1 \) in (1.1), with a quadratic nonlinear term \( \Box u = (\partial_t u)^2 \) and \( \Box u = |\nabla u|^2 \), nonexistence of global smooth solutions was shown even for small data by John [9] and Sideris [19], respectively. Actually, in these two papers, nonexistence of global solutions was shown also for some types of quasi-linear wave equations. On the other hand, if the system (1.1) satisfies the null condition and the initial data is sufficiently small, smooth, and compactly supported, then the Cauchy problem (1.1)–(1.2) admits a unique global smooth solution. This was shown by Klainerman [12] with use of the generators of the Lorentz transformations and the dilations, in addition to the standard partial differential operators \( \partial_\alpha \) (the generators of translations). The conformal energy, the Klainerman inequality (see (1.20) below), and his \( L^1 - L^\infty \) weighted estimate for inhomogeneous wave equations [14], which are all written in terms of these generators, played an important role in his proof. (To be precise, an earlier version of (1.20) was employed in [12].) Later, Hörmander [8] refined the \( L^1 - L^\infty \) weighted estimate of Klainerman and showed

\[ (1 + t + |x|)|u(t, x)| \leq C \sum_{j+|a|+|b|+|c|+d \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\partial_a \partial_b \partial_y^c \Omega^b L^d S^t F(s, y)|}{1 + s + |y|} \, dy \, ds \]

for the equation \( \Box u = F \) with zero data. Making an effective use of (1.8), he gave a more precise assumption on smallness of data. Namely, Hörmander gave an alternative proof of global existence under the weaker assumption that the quantities
related with the given initial data

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} (1 + |x|) |\partial^j_t \partial_x^\alpha \Omega^b \Lambda^d u_i(0,x)|, \]

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \| \partial^j_t \partial_x^\alpha \Omega^b \Lambda^d S^d u_i(0) \|_{L^2(\mathbb{R}^3)} \]

are small enough. (The definition of $\Omega^b \Lambda^d S^d$ is given below. We remark that for $j = 2, \ldots, 9$, we can calculate $\partial^j_t u_i(0,x)$ with the help of the equation (1.1), thus these two quantities are determined by the given small data. We also remark that, by virtue of the Sobolev type inequality (2.24), the smallness of (1.10) actually ensures that of (1.9).) On the other hand, in [3], Christodoulou assumed smallness of data with respect to the weighted $H^4 \times H^3$ norm

\[ \sum_{i=1}^{N} \left( \sum_{|\alpha| \leq 4} \| \langle x \rangle^{3+|\alpha|} \partial_x^\alpha f_i \|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 3} \| \langle x \rangle^{4+|\alpha|} \partial_x^\alpha g_i \|_{L^2(\mathbb{R}^3)} \right) < \varepsilon \]

and proved global existence result under the null condition by the conformal mapping method. (Here, and in the following as well, we employ the standard notation $\langle x \rangle := \sqrt{1 + |x|^2}.$) In comparison with this Christodoulou’s size condition, a question naturally arises: does the method of using the generators yield the proof of global existence of solutions to (1.1)–(1.2) under the null condition when some weighted $H^4 \times H^3$ norm of data is small enough? Exploiting a new way of handling the null-form quadratic nonlinear terms with use of the weighted $L^2$ estimate for some special derivatives, Alinhac proved his truly remarkable energy inequality and gave an affirmative answer to this long-standing problem, in the special case where all the cubic terms $C_i(u, \partial u, \partial^2 u)$ are absent. See pages 92–94 in [2]. In this connection, we cite Theorem 1.4 of [7] here. (We remark that the theorem of Alinhac on page 94 in [2] was slightly improved in [6]. See Theorem 1.5 there. This result in [6] was then slightly improved in [7].)

**Theorem 1.1.** Suppose that (1.11) satisfies the null condition and that in (1.11),

\[ G_{i}^{\alpha \beta}(u, v) \equiv 0, \quad H_i(u, v) \equiv 0 \]

for every $i$, $\alpha$, and $\beta$. Then there exists $\varepsilon > 0$ such that if $f_i \in L^6(\mathbb{R}^3)$ ($i = 1, \ldots, N$) and

\[ \sum_{i=1}^{N} \sum_{|\alpha| + |\beta| + d \leq 3} \left( \| \partial_x^\alpha \Omega^b \Lambda^d f_i \|_{L^2(\mathbb{R}^3)} + \| \partial_x^\alpha \Omega^b \Lambda^d g_i \|_{L^2(\mathbb{R}^3)} \right) < \varepsilon \]

then the unique local (in time) solution to (1.11) can be continued globally in time.

We remark that Theorem 1.4 in [7] is concerned with the scalar equation, i.e., (1.1) with $N = 1$, but obviously the method there is general enough to prove Theorem
above. The proof of this theorem is carried out by the combination of the ghost weight energy method of Alinhac [2] with the Klainerman-Sideris method [14]. We must enhance the discussions in [2, 6, and 7] concerning the special case (1.12), because the system (1.1) contains $u$ itself in the nonlinear terms. It is a natural attempt to inject into the argument in [2, 6, and 7] such key elements of the proof due to Klainerman [12] and Hörmander [8] as the conformal energy and the $L^1$–$L^\infty$ weighted estimate for inhomogeneous wave equations (1.8). Combining these elements with the ghost weight energy method in [2 and 6], we can indeed obtain the following:

**Proposition 1.2.** Suppose that the system (1.1) satisfies the null condition. There exist positive constants $C$ and $\varepsilon$ such that if

$$
\sum_{i=1}^{N} \left( \sum_{|a| \leq 4} \| \langle x \rangle^{|a|} \partial_x^a f_i \|_{L^2(\mathbb{R}^3)} + \sum_{|a| \leq 3} \| \langle x \rangle^{|a|+1} \partial_x^a g_i \|_{L^2(\mathbb{R}^3)} \right) < \varepsilon,
$$

then the Cauchy problem (1.1)–(1.2) admits a unique global solution satisfying

$$
\sum_{i=1}^{N} \left\{ \| (1 + t + |x|) u_i \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)} + \sum_{|a| \leq 3} \| \partial^a u_i \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^3))} 
\right. 
\left. + \sum_{|a| \leq 2} \| (1 + t)^{-\delta} \Gamma^a u_i \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^3))} 
\right.
\left. + \sum_{|a| = 3} \| (1 + t)^{-2\delta} \Gamma^a u_i(t) \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^3))} \right\} \leq C \varepsilon.
$$

Here $\mathbb{R}^+ := (0, \infty)$, $\delta$ and $\eta$ are sufficiently small positive constants and $T_j = \partial_j + (x_j/|x|)\partial_t$, $j = 1, 2, 3$.

Concerning the definition of the commonly used operators $\Gamma^a$, see, e.g., [10] p. 46, [12] p. 301. Namely,

$$
\sum_{|a| \leq 3} \| \partial^a u_i \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^3))} = \sum_{j+|a|+|b|+|c|+d \leq 3} \| \partial^j x^a \partial_t^b \partial_x^c \Omega^d u_i \|_{L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R}^3))}
$$

and so on. In comparison with the Christodoulou’s size condition (1.11), the above one (1.14) has an advantage; it obviously assumes less decay on the data. Compared with (1.14), however, the size condition (1.13), which though applies to the special case (1.12), has an advantage that if $f_i$ and $g_i$ are radially symmetric (hence $\Omega f_i = \Omega g_i = 0$) and the norm with the low weight

$$
\sum_{i=1}^{N} \left( \| \partial_x f_i \|_{L^2(\mathbb{R}^3)} + \| g_i \|_{L^2(\mathbb{R}^3)} \right)
$$

would be more convenient to use.
is small enough, then (1.1)–(1.2) admits global solutions. In view of the current state of the art, the purpose of this paper is to show global existence of small solutions to (1.1)–(1.2) under the null condition when initial data have lower regularity than was assumed in [12] and [8], and have weaker decay than was assumed in [3] and Proposition 1.2. In particular, taking into account Theorem 1.1 which holds for the special case (1.12), we would naturally like to obtain global solutions for radially symmetric data when a low weight norm of data is small enough. Recall \( \Lambda := x \cdot \nabla \).

We define

\[
\sum_{i=1}^{N} \sum_{1 \leq |a| \leq 3} \left( \| \langle x \rangle \partial_x^a \Omega^b f_i \|_{L^2(\mathbb{R}^3)} + \| \langle x \rangle \partial_x^a g_i \|_{L^2(\mathbb{R}^3)} \right)
\]

Now we are in a position to state our main theorem.

**Theorem 1.3.** Suppose that the system (1.1) satisfies the null condition. There exist positive constants \( C, \varepsilon \) such that if \( D(f, g) < \varepsilon \), then the Cauchy problem (1.1)–(1.2) admits a unique global solution satisfying

\[
\sum_{i=1}^{N} \sum_{|a|+|b|+|c|+d \leq 3} \| \partial_x^a \Omega^b L^c S^d u_i \|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} + \sum_{|b| \leq 2} \| \Omega^b u_i \|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} 
\]

\[
+ \sum_{|b| \leq 2} \left( \| \langle t \rangle^{-\delta} (S + 2) \Omega^b u_i \|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} + \| \langle t \rangle^{-\delta} \Omega^b u_i \|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} \right) 
\]

\[
+ \sum_{k=1}^{3} \left( \sum_{|a|+|b|+|c|+d \leq 3} \| \langle t - r \rangle^{-(1/2)-\eta} T_k \partial_x^a \Omega^b L^c S^d u_i \|_{L^2((0, \infty) \times \mathbb{R}^3)} 
\]

\[
+ \sum_{|a|+|b|+|c|+d \leq 2} \| \langle t - r \rangle^{-(1/2)-\eta} T_k \partial_x^a \Omega^b L^c S^d u_i \|_{L^2((0, \infty) \times \mathbb{R}^3)} \right)
\]
Here, $\delta$, and $\eta$ are positive constants satisfying $\delta < 1/6$, $\eta < 1/3$.

Let $\varepsilon$ be sufficiently small. We easily see that such an oscillating and decaying data as $u(0, x) = \varepsilon(\sin x_1)|x|^{-d}$ can be allowed in the theorem of Christodoulou if $d > 17/2$, while Theorem 1.3 above allows the smaller values of $d$, that is, $d > 9/2$. Also, we benefit from the size condition (1.17) and obtain global solutions when $f_i$ and $g_i$ are radially symmetric and the low weight norm

$$\sum_{i=1}^{N} \left( \|f_i\|_{L^2(\mathbb{R}^3)} + \sum_{1 \leq |a| \leq 4} \|\langle x \rangle \partial_{x}^a f_i\|_{L^2(\mathbb{R}^3)} + \sum_{|a| \leq 3} \|\langle x \rangle \partial_{x}^a g_i\|_{L^2(\mathbb{R}^3)} \right)$$

is small enough. It means that such an oscillating and more slowly decaying radially symmetric data as $u(0, x) = \varepsilon(\sin(x))|x|^{-d}$ with $d > 5/2$ is allowed in Theorem 1.3.

The new size condition (1.17), where the number of occurrences of $\Lambda$ is limited at most to 1 in the norms there, is a direct consequence of the limitation of that of occurrences of $S$ in the norms (1.18). Also, in (1.17) we are allowed to employ the low weight norms to measure the size of data, which results from the limitation of the number of occurrences of $L_j$ in the norms (1.18). While the $L^1-L^\infty$ estimate (1.8) and the Klainerman inequality (1.3)

$$\sum_{j + |a| + |b| + |c| + d \leq 2} \| \partial_{x}^j \partial_{t}^a \partial_{x}^b \partial_{t}^c \partial_{x}^d u(t) \|_{L^2(\mathbb{R}^3)}$$

play an important role in the proof of Proposition 1.2, we encounter $L^c S^d$ with $|c| + d = 2$ in (1.8) and (1.20), and therefore must refrain from using these two well-known inequalities in the proof of Theorem 1.3. To get over this difficulty, we will exploit the effective idea of estimating nonlinear terms over the set $\{x \in \mathbb{R}^3 : |x| < (t + 1)/2\}$ and its complement (for any fixed $t > 0$) separately. (See, e.g., [15], [21], and [4] for earlier papers using this simple but important idea.) As a consequence, some simple Sobolev-type or trace-type inequalities (2.20)-(2.24) and (2.26), combined with the weighted space-time $L^2$ estimate (2.23) and the Li-Yu estimate (2.23), play a role as the good substitute for (1.8) and (1.20). Using the ghost weight energy inequality for variable-coefficient wave equations and the conformal energy estimate for the standard wave equation $\Box u = F$ together with these good substitutes, we will prove Theorem 1.3.

We end this section with setting the notation in this paper and giving attention to how to handle the cubic terms $C_i(u, \partial u, \partial^2 u)$ at the stage of carrying out the
is used repeatedly. We set \( \Lambda := (c_1 \Omega b_3) \), \( L := (c_1 \Omega b_3) \), \( \Omega^b := \Omega_{12}^b \Omega_{13}^b \Omega_{23}^b \), and \( L^c := L_1^c L_2^c L_3^c \) for \( a = (a_1, a_2, a_3) \), \( b = (b_1, b_2, b_3) \), and \( c = (c_1, c_2, c_3) \), respectively. In this paper, we denote \( \partial_1, \partial_2, \partial_3 \), \( \Omega_{12}, \Omega_{23}, \Omega_{13} \), \( L_1, L_2, L_3 \), and \( S \) by \( Z_1, Z_2, \ldots, Z_{10} \). The notation

\[
(1.21) \quad \tilde{Z}^a := \partial_1^a \partial_2^a \partial_3^a \Omega_{12}^a \Omega_{13}^a \Omega_{23}^a, \quad a = (a_1, \ldots, a_6)
\]

is used repeatedly. We set \( \Lambda := x \cdot \nabla = r \partial_r \).

We define the energy and its associated quantity

\[
(1.22) \quad E_1(v(t)) := \frac{1}{2} \int_{\mathbb{R}^3} \left( (\partial_t v(t, x))^2 + |\nabla v(t, x)|^2 \right) dx, \quad N_1(v(t)) := \sqrt{E_1(v(t))},
\]

\[
(1.23) \quad N_{j+1}(v(t)) := \left( \sum_{|a| + |b| + |c| + d \leq j} E_1(\partial_x^a \Omega^b v(t)) \right)^{1/2}, \quad j = 1, 2, 3.
\]

Moreover, we use the conformal energy

\[
(1.24) \quad Q(v(t)) := \frac{1}{2} \int_{\mathbb{R}^3} \left( (S + 2)v(t, x))^2 + \sum_{|b|=1} (\Omega^b v(t, x))^2 + \sum_{|c|=1} (L^c v(t, x))^2 \right) dx.
\]

We mention the important fact that the inequality

\[
(1.25) \quad \|v(t)\|_{L^2(\mathbb{R}^3)}^2 \leq C Q(v(t))
\]

holds for a positive constant \( C \). For the proof, see [12, pp. 311–322]. See also [8, pp. 101–102] or [11, pp. 98–101], where a different proof can be found. By (1.25), it is easy to verify the equivalence of \( Q(v(t)) \) and

\[
\tilde{Q}(v(t)) := \|v(t)\|_{L^2(\mathbb{R}^3)}^2 + \sum_{|b|=1} \|\Omega^b v(t)\|_{L^2(\mathbb{R}^3)}^2 + \sum_{|c|=1} \|L^c v(t)\|_{L^2(\mathbb{R}^3)}^2 + \|S v(t)\|_{L^2(\mathbb{R}^3)}^2.
\]

We set

\[
(1.26) \quad M_{j+1}(v(t)) := \left( \sum_{|a| + |b| \leq j} \tilde{Q}(\partial_x^a \Omega^b v(t)) \right)^{1/2}, \quad j = 0, 1, 2.
\]

We also need

\[
(1.27) \quad X_j(v(t)) := \left( \sum_{|a| + |b| \leq j} \|\partial_x^a \Omega^b v(t)\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2}, \quad j = 0, 1, 2.
\]

For \( \mathbb{R}^N \)-valued functions \( w(t, x) = (w_1(t, x), \ldots, w_N(t, x)) \), we set

\[
(1.28) \quad E_1(w(t)) = \sum_{i=1}^N E_1(w_i(t)), \quad \tilde{Q}(w(t)) = \sum_{i=1}^N \tilde{Q}(w_i(t)),
\]
we employed $N_j(w(t))$ and $X_j(w(t))$ ($j = 0, 1, 2$) are defined similarly. We obviously have $X_0(w(t)) \leq M_1(w(t))$, $X_j(w(t)) \leq N_j(w(t)) + M_j(w(t))$, $j = 1, 2$. Let us recall that careful attention should be paid not only on the quadratic null-form terms but also on the cubic terms $C_i(u, \partial u, \partial^2 u)$, especially at the stage where the estimate of the conformal energy is carried out. Since the weighted norm of the forcing term $F$ appears on the right-hand side of (2.27) below, the cubic terms as well as the quadratic null-form terms are regarded as “critical” ones, and therefore a mildly growing (in time) bound for the conformal energy is the most that one can obtain in general. The bootstrap argument of [12], [8] successfully employed such a weak bound for the conformal energy in combination with a sharp point-wise decay estimate obtained with the use of linear estimates such as (1.8), when handling the cubic terms and closing the estimates. See, e.g., [8, p. 141], especially the sentences: “In each term we can estimate all factors except one using (6.6.30). For the third order terms $f_i^2$ and $f_i^4$, this gives a factor $\leq C\tilde{x}^2(1 + x_0 + |\tilde{x}|)^2$ in addition to a factor with norm square $O(E_k(u; x_0))$”. In place of such a sharp point-wise decay estimate, our bootstrap argument employs a sharp estimate for local solutions in the $X_2$ norm obtained with the use of the Li-Yu estimate (see (2.28) below). Actually, the use of the $X_2$ norm is one of the crucial ingredients in order to limit the number of occurrences of the generators of hyperbolic rotations or dilations in the bootstrap argument. (Another key ingredient is to use the weighted space-time $L^2$ norm for the estimate of the energy-type norm $N_4(u(t))$. See (3.27).) The $X_2$ norm is effectively used, especially at the stage where the cubic terms are handled in the course of carrying out the conformal-energy type estimate for local solutions (1.1). See, in particular, the terms $\int_0^\tau \langle \tau \rangle^{-1} X_2(u(\tau))^2 N_4(u(\tau))d\tau$ and $\int_0^\tau \langle \tau \rangle^{-1} X_2(u(\tau))^3 d\tau$ on the right-hand side of (1.1) below, where the $X_2$ norm plays a crucial role. Indeed, if we employed $N_2(u(\tau)) + M_2(u(\tau))$ there in place of $X_2(u(\tau))$ (recall that one always has $X_2(u(\tau)) \leq N_2(u(\tau)) + M_2(u(\tau))$), we could not close the estimates.

This paper is organized as follows. In the next section, we first recall some special properties that the null-form nonlinear terms enjoy, and then we recall several key inequalities that play an important role in our arguments. In Section 3 we carry out the energy estimate, following the ghost weight energy method of Alinhac. Sections 4 and 5 are devoted to obtaining bounds for $M_3(u(t))$ and $X_2(u(t))$, respectively. In Section 6 we carry out the $L^2$ weighted space-time estimate, using the Keel-Smith-Sogge type estimate. In the final section, we complete the proof of Theorem 1.3.

2. Preliminaries

The proof of our theorems builds on several lemmas. Let $[\cdot, \cdot]$ stand for the commutator: $[A, B] := AB - BA$. 

\[
N_{j+1}(w(t)) = \left( \sum_{i=1}^{N} N_{j+1}(w_i(t))^2 \right)^{1/2}, \quad j = 1, 2, 3.
\]
Lemma 2.1. The following commutation relations hold for $1 \leq j < k \leq 3$, $l = 1, 2, 3$, and $\alpha = 0, \ldots, 3$:

\begin{align}
&[\Omega_{jk}, \square] = 0, \quad [L_l, \square] = 0, \quad [S, \square] = -2\square, \\
&[S, \Omega_{jk}] = 0, \quad [S, L_l] = 0, \quad [S, \partial_\alpha] = -\partial_\alpha, \\
&[L_l, \Omega_{jk}] = \delta_{lj}L_k - \delta_{lk}L_j.
\end{align}

We also have for $l, j = 1, 2, 3$

\begin{align}
&[L_l, \partial_i] = -\partial_i, \quad [L_l, \partial_j] = -\delta_{lj}\partial_l.
\end{align}

Furthermore, we have for $1 \leq j < k \leq 3$, $1 \leq l < m \leq 3$

\begin{align}
&[\Omega_{jk}, \Omega_{lm}] = \delta_{kl}\Omega_{jm} + \delta_{km}\Omega_{lj} + \delta_{jl}\Omega_{mk} + \delta_{jm}\Omega_{kl}
\end{align}

and for $1 \leq j < k \leq 3$, $l = 1, 2, 3$

\begin{align}
&[\Omega_{jk}, \partial_l] = -\delta_{lj}\partial_k + \delta_{lk}\partial_j.
\end{align}

Recall that in this paper, we denote $\partial_1, \partial_2, \partial_3, \Omega_{12}, \Omega_{23}, \Omega_{13}, L_1, L_2, L_3$, and $S$ by $Z_1, Z_2, \ldots, Z_{10}$. The next lemma states that the null condition is preserved under the differentiation.

Lemma 2.2. Suppose that for given $j$, the coefficients $F^{j,\alpha\beta\gamma}$ satisfy the null condition. Also, suppose that for given $j, k$ the coefficients $F^{jk,\alpha\beta}$ satisfy the null condition. Then, for any $Z_l$ ($l = 1, \ldots, 10$) we have

\begin{align}
Z_l F^{j,\alpha\beta\gamma}(\partial_\gamma v)(\partial^2_{\alpha\beta} w) &= F^{j,\alpha\beta\gamma}(\partial_\gamma Z_l)(\partial^2_{\alpha\beta} w) + F^{j,\alpha\beta\gamma}(\partial_\gamma v)(\partial^2_{\alpha\beta} Z_l w) + \tilde{F}_l^{j,\alpha\beta\gamma}(\partial_\gamma v)(\partial^2_{\alpha\beta} w)
\end{align}

holds with the new coefficients $\tilde{F}_l^{j,\alpha\beta\gamma}$ also satisfying the null condition. Also, the equality

\begin{align}
Z_l F^{jk,\alpha\beta}(\partial_\alpha v)(\partial_\beta w) &= F^{jk,\alpha\beta}(\partial_\alpha Z_l)(\partial_\beta w) + F^{jk,\alpha\beta}(\partial_\alpha v)(\partial_\beta Z_l w) + \tilde{F}_l^{jk,\alpha\beta}(\partial_\alpha v)(\partial_\beta w)
\end{align}

holds with the new coefficients $\tilde{F}_l^{jk,\alpha\beta}$ also satisfying the null condition.

For the proof, see, e.g., [2] p. 91.

The next lemma can be shown essentially in the same way as in [2] pp. 90–91. Together with it, we will later exploit the fact that for local solutions $u$, the special derivatives $T_i u$ have better space-time $L^2$ integrability and improved time decay property of their $L^\infty(\mathbb{R}^3)$ norms.

Lemma 2.3. Suppose that for every $i, j$, and $k$, the coefficients $F_i^{j,\alpha\beta\gamma}$ and $F_i^{jk,\alpha\beta}$ satisfy the null condition. Then, we have for smooth functions $w_i(t, x)$ ($i = 1, 2, 3$)

\begin{align}
&|F_i^{j,\alpha\beta\gamma}(\partial_\gamma w_1)(\partial^2_{\alpha\beta} w_2)| \leq C(|T w_1| |\partial^2 w_2| + |\partial w_1| |T \partial w_2|), \\
&|F_i^{jk,\alpha\beta}(\partial_\alpha w_1)(\partial_\beta w_2)| \leq C(|T \partial w_1| |\partial w_2| + |\partial^2 w_1| |T w_2|),
\end{align}
(2.11) \[ |F_{i}^{j, \alpha \beta \gamma} (\partial_{\alpha} w_{1})(\partial_{\beta} w_{2})(\partial_{\gamma} w_{3})|, |F_{i}^{j, \alpha \beta \gamma} (\partial_{\alpha} w_{1})(\partial_{\beta} w_{2})(-\omega_{\alpha})(\partial_{\gamma} w_{3})| \leq C (|Tw_{1}| |\partial w_{2}| |\partial w_{3}| + |\partial w_{1}| |Tw_{2}| |\partial w_{3}| + |\partial w_{1}| |\partial w_{2}| |Tw_{3}|), \]

(2.12) \[ |F_{i}^{j, \alpha \beta}(\partial_{\alpha} v)(\partial_{\beta} w)| \leq C (|Tv||\partial w| + |\partial v||Tw|). \]

Here, and in the following, we use the notation \( \omega_{0} = -1, \omega_{k} = x_{k}/|x|, k = 1, 2, 3. \) Also, for \( v \) and \( \partial v = (\partial v, \ldots, \partial_3 v) \), we use

\[
|Tv| := \left( \sum_{k=1}^{3} |T_k v|^2 \right)^{1/2}, \quad |T\partial v| := \left( \sum_{k=1}^{3} \sum_{\gamma=0}^{3} |T_k \partial_{\gamma} v|^2 \right)^{1/2},
\]

where \( T_k = \partial_k + \omega_k \partial_t \), as before.

Inspired by [2], we also use the remarkable improvement of point-wise decay of the special derivatives \( T_k v(t, x) \).

**Lemma 2.4** ([2], pp. 90–91). The inequalities

(2.14) \[ |Tv(t, x)| \leq \frac{C}{|x|} \left( \sum_{|b|=1} |\Omega^b v(t, x)| + \sum_{|c|=1} |L^c v(t, x)| + |Sv(t, x)| \right), \]

(2.15) \[ |Tv(t, x)| \leq \frac{C}{t} \left( \sum_{|b|=1} |\Omega^b v(t, x)| + \sum_{|c|=1} |L^c v(t, x)| + |Sv(t, x)| \right) \]

hold for smooth functions \( v(t, x) \).

This is a direct consequence of

(2.16) \[ T_k = \frac{1}{r} (L_k + (r-t) \partial_k) = \frac{1}{t} (L_k + (t-r) \omega_k \partial_t), \]

(2.17) \[ tL_k + \sum_{j=1}^{3} x_j \Omega_{kj} - x_k S = \frac{tS - \sum_{j=1}^{3} x_j L_j}{t^2 - r^2}, \quad \partial_k = \frac{t^2 - r^2}{t^2 - r^2}. \]

We thus omit the proof of (2.14).

Next, let us show some Sobolev-type or trace-type inequalities. In the following, we use the notation \( \partial_r := (x/|x|) \cdot \nabla \), and for \( p \in [1, \infty] \) and \( q \in [1, \infty) \)

(2.18) \[ \|v\|_{L^q \cap L^p(S^3)} := \sup_{r > 0} \|v(r \cdot)\|_{L^p(S^2)}, \]

(2.19) \[ \|v\|_{L^q \cap L^p(S^3)} := \left( \int_{0}^{\infty} \|v(r \cdot)\|^q_{L^p(S^2)} r^2 dr \right)^{1/q}. \]

**Lemma 2.5.** Suppose that \( v \) decays sufficiently fast as \( |x| \to \infty \). Then, we have

(2.20) \[ \|\langle t-r \rangle v(t)\|_{L^p(S^3)} \leq C \sum_{|a|+|b|+|c|+d \leq 1} \|\partial^{a}_x \Omega^{b} L^c S^d v(t)\|_{L^2(S^3)} \]

(2.21) \[ \langle t-r \rangle |v(t, x)| \leq C \|v(t)\|_{H^2(S^3)} + \sum_{|a| \leq 1} \sum_{|b|+|c|+d = 1} \|\partial^{a}_x \Omega^{b} L^c S^d v(t)\|_{L^2(S^3)}. \]
Moreover, we have
\begin{equation}
\|r^{1/2}v(t)\|_{L^\infty_t L^2_x} \leq C \|\nabla v(t)\|_{L^2(\mathbb{R}^3)},
\end{equation}
(2.22)
\begin{equation}
\|rv(t)\|_{L^\infty_t L^2_x} \leq C \|\partial_r v\|_{L^2(\mathbb{R}^3)}^{1/2}\left(\sum_{|b| \leq 1} \|\Omega^b v(t)\|_{L^2(\mathbb{R}^3)}\right)^{1/2},
\end{equation}
(2.23)
\begin{equation}
r|v(t,x)| \leq C\left(\sum_{|b| \leq 1} \|\partial_r \Omega^b v(t)\|_{L^2(\mathbb{R}^3)}\right)^{1/2}\left(\sum_{|b| \leq 2} \|\Omega^b v(t)\|_{L^2(\mathbb{R}^3)}\right)^{1/2}.
\end{equation}
(2.24)

**Proof.** For the proof of (2.20), we first employ the well-known inequality \(\|w\|_{L^6(\mathbb{R}^3)} \leq C\|w\|_{L^2(\mathbb{R}^3)}\) for \(w = (t-r)v(t,x)\) and then use the first equality in (2.17) to obtain
\begin{equation}
|t-r|\|\partial_k v(t,x)| \leq |L_k v(t,x)| + \sum_{j=1}^{3} |\Omega_{kj} v(t,x)| + |S v(t,x)|.
\end{equation}
(2.25)

For the proof of (2.21), we apply the Sobolev embedding \(W^{1.6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)\) to the function \((t-r)v(t,x)\) and then use (2.20). For the proof of the first trace-type inequality (2.22), see, e.g., [20] (3.16). For the proof of the second trace-type inequality (2.23), see, e.g., [20] (3.19)]. The Sobolev embedding \(W^{1.4}(S^2) \hookrightarrow L^\infty(S^2)\) together with (2.20) immediately yields (2.24).

**Lemma 2.6.** Suppose that \(v\) decays sufficiently fast \(|x| \to \infty\). For any \(\theta \in [0,1/2]\), there exists a constant \(C > 0\) such that the inequality
\begin{equation}
r^{(1/2)+\theta}(t-r)^{1-\theta}\|v(t,r)\|_{L^4(S^2)} \leq C \sum_{|a|+|b|} \|\partial^a_x \Omega^b L^c S^d v(t)\|_{L^2(\mathbb{R}^3)}
\end{equation}
holds.

**Proof.** For \(\theta = 1/2\), we first follow the proof of [20] (3.19) with \(\beta = 0\) and then use (2.24) above. For \(\theta = 0\), we first apply (2.22) above to the function \(w(t,x) = (t-r)v(t,x)\) and then use (2.25). We follow the idea in Section 2 of [18] and obtain (2.20) for \(\theta \in (0,1/2)\) by interpolation.

To bound local solutions in the \(M_3\) norm, we employ the conformal energy estimate (see, e.g., [11] Theorem 6.11]) together with the equivalence of \(Q(v(t))\) and \(\tilde{Q}(v(t))\).

**Lemma 2.7.** The solution \(u\) to the inhomogeneous wave equation \(\partial_t^2 u - \Delta u = F\) in \(\mathbb{R}^3 \times (0,\infty)\) with data \((f,g)\) at \(t = 0\) satisfies the conformal energy estimate:
\begin{equation}
\sum_{|b| = 1} \|\Omega^b u(t)\|_{L^2(\mathbb{R}^3)} + \sum_{|c| = 1} \|L^c u(t)\|_{L^2(\mathbb{R}^3)} + \|(S + 2)u(t)\|_{L^2(\mathbb{R}^3)} \leq C(\|f\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)})
\end{equation}
\begin{equation}
+ C \int_0^t \|((\tau + |x|)F(\tau))\|_{L^2(\mathbb{R}^3)} d\tau.
\end{equation}
(2.27)
The following estimate is essentially due to Li and Yu \cite{15}, and we employ it to bound local solutions in the $X_2$ norm.

**Lemma 2.8.** The solution $u$ to the inhomogeneous wave equation $\partial_t^2 u - \Delta u = F$ in $\mathbb{R}^3 \times (0, \infty)$ with data $(f, g)$ at $t = 0$ satisfies

\begin{equation}
\|u(t)\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)} + \|D^{-1}g\|_{L^2(\mathbb{R}^3)} + C \int_0^t \|\chi_1 F(\tau)\|_{L^{6/5}(\mathbb{R}^3)} d\tau
\end{equation}

\begin{equation}
+ C \int_0^t \langle \tau \rangle^{-1/2} \|\chi_2 F(\tau)\|_{L^1_x L^{4/3}(\mathbb{R}^3)} d\tau.
\end{equation}

The functions $\chi_1$ and $\chi_2$ denote the characteristic functions of $\{x \in \mathbb{R}^3 : |x| < (1 + \tau)/2\}$ and $\{x \in \mathbb{R}^3 : |x| > (1 + \tau)/2\}$, respectively.

**Proof.** This is a consequence of the Sobolev inequality $\|v\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}$, the trace inequality \eqref{2.22} above, and the duality argument. \hfill \Box

We also need the space-time $L^2$ estimates for the variable-coefficient operator $P$ defined as

\begin{equation}
P := \partial_t^2 - \Delta + h^{\alpha\beta}(t, x)\partial_{\alpha\beta}^2 = (-m^{\alpha\beta} + h^{\alpha\beta}(t, x))\partial_{\alpha\beta}^2.
\end{equation}

Here, $(m^{\alpha\beta}) = \text{diag}(-1, 1, 1, 1)$, and the variable coefficients $h^{\alpha\beta} \in C^\infty((0, T) \times \mathbb{R}^3)$ $(\alpha, \beta = 0, 1, 2, 3)$ satisfy the symmetry condition $h^{\alpha\beta} = h^{\beta\alpha}$. Define the (modified) energy-momentum tensor as

\begin{equation}
T^{\alpha\beta} := m^{\alpha\mu}(m^{\beta\nu} - h^{\beta\nu})(\partial_\mu v)(\partial_\nu v) - \frac{1}{2} m^{\alpha\beta}(m^{\mu\nu} - h^{\mu\nu})(\partial_\mu v)(\partial_\nu v).
\end{equation}

A straightforward computation yields:

**Lemma 2.9.** Let $g = g(\rho) \in C^1(\mathbb{R})$. The equality

\begin{equation}
\partial_\beta (e^{g(t-r)} T^{0\beta})
\end{equation}

\begin{equation}
= - e^{g(t-r)} \left( (\partial_\nu v)(P v) + (\partial_\beta h^{\beta\nu})(\partial_\nu v)(\partial_\nu v) - \frac{1}{2} (\partial_\nu h^{\mu\nu})(\partial_\mu v)(\partial_\nu v) \right)
\end{equation}

\begin{equation}
- e^{g(t-r)} g'(t-r)(-\omega_\beta) T^{0\beta} = 0
\end{equation}

holds. Here, as in Lemma 2.3, $\omega_0 = -1$, $\omega_k = x_k/|x|$, $k = 1, 2, 3$.

In the next section, we employ Lemma 2.9 together with

\begin{equation}
T^{00} = \frac{1}{2} |\partial_\nu v|^2 + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} h^{0\nu}(\partial_\nu v)(\partial_\nu v) - \frac{1}{2} h^{\mu\nu}(\partial_\mu v)(\partial_\nu v)
\end{equation}

and

\begin{equation}
(-\omega_\beta) T^{0\beta} = \frac{1}{2} \sum_{j=1}^3 (T_j v)^2 - \omega_\beta h^{0\nu}(\partial_\nu v)(\partial_\nu v) - \frac{1}{2} h^{\mu\nu}(\partial_\mu v)(\partial_\nu v)
\end{equation}

to obtain the Alinhac type $L^2$ weighted space-time estimate for the special derivatives $T_k u_i$ of local solutions $u = (u_1, \ldots, u_N)$. Though employing the ghost weight
energy method of Alinhac, we mention that it is possible to get essentially the same $L^2$ weighted space-time estimate by following the idea of Lindblad and Rodnianski [17], Lindblad, Nakamura, and Sogge [16, Lemma A.1].

We also need the Keel-Smith-Sogge type $L^2$ weighted space-time estimate for the standard derivatives. In addition to the symmetry condition, we further suppose the size condition $\sum |h^{\alpha\beta}(t, x)| \leq 1/2$. Owing to the method in [23, Appendix], we have the following:

**Lemma 2.10** (Theorem 2.1 of [3]). For $0 < \mu < 1/2$, there exists a positive constant $C$ such that the inequality

\[(2.34) \quad (1 + T)^{-2\mu} \left( \|r^{-(3/2)+\mu} u\|^2_{L^2((0,T) \times \mathbb{R}^3)} + \|r^{-(1/2)+\mu} \partial u\|^2_{L^2((0,T) \times \mathbb{R}^3)} \right) \leq C \|\partial u(0, \cdot)\|^2_{L^2(\mathbb{R}^3)} + C \int_0^T \int_{\mathbb{R}^3} \left( |\partial u||Pu| + \frac{|u||Pu|}{r^{1-2\mu}(r)^{2\mu}} + |\partial h||\partial u|^2 \right. \]

\[+ \left. \frac{|\partial h||u\partial u|}{r^{1-2\mu}(r)^{2\mu}} + \frac{|h||\partial u|^2}{r^{1-2\mu}(r)^{2\mu}} + \frac{|h||u\partial u|}{r^{2-2\mu}(r)^{2\mu}} \right) \, dx \, dt \]

holds for smooth and compactly supported (for any fixed time) functions $u(t, x)$.

3. **Bound for $N_4(u(t))$**

The proof of local (in time) existence due to Hörmander for scalar wave equations (see [8, Theorem 6.4.11]) is obviously valid for the systems (1.1) under consideration. In what follows, we always assume that for a small constant $\hat{\varepsilon}$, the initial data $(f, g)$ (see (1.2)) satisfies

\[(3.1) \quad \|f\|_{H^3} + \|g\|_{H^2} < \hat{\varepsilon} \]

so that for all $i = 1, \ldots, N$ and $\alpha, \beta = 0, 1, 2, 3$ the quantities $|F_{i,\alpha\beta}^j g_j + F_{i,\alpha\beta\gamma} \partial_k f_j|$ and $|G_{i,\alpha\beta}(f, g, \partial_x f)|$ (see (1.3)–(1.4)) are also small by the Sobolev embedding and we can therefore rely upon the local existence theorem mentioned above.

We may focus on a priori estimates for the local solutions. Let us first consider the case where initial data are smooth and compactly supported so that there exists a constant $R > 0$ such that $\text{supp} \{f, g\} \subset \{x \in \mathbb{R}^3 : |x| < R\}$. Moreover, we know that the local solution is smooth and satisfies

\[(3.2) \quad u_i(t, x) = \partial_t u_i(t, x) = 0, \quad |x| > t + R, \quad i = 1, \ldots, N. \]

John proved (3.2) for $C^2$ solutions for scalar wave equations (see [9], [10]), and his proof is obviously valid for the systems (1.1) under consideration. We temporarily assume the regularity and support conditions on the data because the proof of Theorem 1.3 becomes easier. Note, however, that all the constants $C$ appearing below will never depend upon this constant $R$, and these conditions can be finally removed by the standard argument.
Recall the definition of $D(f, g)$ (see (1.17)). To prove the global existence, we must assume that the initial data $(f, g)$ is smaller than we have done in (3.1). That is, using some appropriate constants which will appear later in our discussion, we assume

\begin{equation}
D(f, g) \leq \varepsilon_0
\end{equation}

where $0 < \varepsilon_0 < \min \left\{ 1, \frac{\varepsilon_1^*}{6C^*C_0}, \frac{1}{2C_0(3C_{12} + 4C_0C_{14})}, \frac{C_{11}}{C_{15}}, \frac{C_{21}}{2C_{22}}, \frac{C_{31}}{2C_{32}} \right\}$.

Using the equality $\partial_t L_j v = \partial_j v + x_j \partial^2_t v$ at $t = 0$, the equation (1.1), and the Sobolev type inequality (2.24), we see that there exists a numerical constant $C_d > 0$ such that the local solution initially satisfies

\begin{equation}
N_4(u(0)), M_3(u(0)), X_2(u(0)) \leq C_d D(f, g),
\end{equation}

due to the size condition (3.3). We remark that the equality

\begin{equation}
r \partial_k = \omega_k \Lambda + \sum_{j \neq k} \omega_j \Omega_{jk}
\end{equation}

plays a role in showing (3.4). (Here, we have used the notation $\Omega_{ij} := x_i \partial_j - x_j \partial_i$ not only for $i < j$ but also for $i > j$.)

Before starting a priori estimates for the local solutions, we must mention some point-wise estimates as in [4], [7]. These inequalities compensate for the absence of $\partial_i t w(t, x)$ ($i = 2, 3, 4$) in the definition of the norms (1.22), (1.23), (1.26), (1.27), (3.24)–(3.25). For the local solutions $u$, we use the notation

\begin{equation}
\langle \langle u(t) \rangle \rangle := \sum_{i=1}^{N} \left\{ \sum_{|\ell|+|b| \leq 1} \langle t \rangle^{|\ell|} \| \partial^\ell_x \Omega^b u_i(t) \|_{L^\infty(\mathbb{R}^3)} + \| \partial L^c u_i(t) \|_{L^\infty(\mathbb{R}^3)} + \| \partial S u_i(t) \|_{L^\infty(\mathbb{R}^3)} + \| \Omega^b u_i(t) \|_{L^\infty(\mathbb{R}^3)} + \| \Omega^b S^d u_i(t) \|_{L^\infty(\mathbb{R}^3)} \right\}.
\end{equation}

Recall the definition of the notation $Z_k$.

**Lemma 3.1.** There exists a small constant $\varepsilon_1^* > 0$ with the following property: whenever smooth solutions to (1.11) satisfy

\begin{equation}
\langle \langle u(t) \rangle \rangle \leq \varepsilon_1^*,
\end{equation}
the point-wise inequalities hold for \( i = 1, 2, \ldots, N, \ j = 1, 2, 3, \ k = 1, 2, \ldots, 10, \) and \( l = 1, 2, \ldots, 6 : \)

\[
(3.7) \quad |\partial_t^2 u_i(t, x)| \leq C \sum_{|a| \leq 1} |\partial \partial_x^a u(t, x)| + C|u(t, x)|^3,
\]

\[
(3.8) \quad |\partial_t^3 u_i(t, x)| \leq C \sum_{|a| \leq 2} |\partial \partial_x^a u(t, x)| + C|u(t, x)|^3,
\]

\[
(3.9) \quad |\partial_t^2 \partial_x Z_k u_i(t, x)| \leq C \sum_{|a| \leq 1} (|\partial \partial_x^a Z_k u(t, x)| + |\partial \partial_x^a u(t, x)|)
+ C|u(t, x)|^2 |Z_k u(t, x)| + C|u(t, x)|^3,
\]

\[
(3.10) \quad |\partial_t^2 Z_t Z_k u_i(t, x)|
\leq C \sum_{|a| \leq 1} (|\partial \partial_x^a Z_t Z_k u(t, x)| + |\partial \partial_x^a Z_t u(t, x)| + |\partial \partial_x^a Z_k u(t, x)|)
+ C|u(t, x)||Z_t u(t, x)||Z_k u(t, x)| + C|u(t, x)|^2 |Z_k u(t, x)|
+ C|u(t, x)|^2 |Z_t u(t, x)| + C|u(t, x)|^3,
\]

\[
(3.11) \quad |\partial_t^3 Z_k u_i(t, x)| \leq C \sum_{|a| \leq 2} |\partial \partial_x^a Z_k u(t, x)| + C \sum_{|a| \leq 2} |\partial \partial_x^a u(t, x)|
+ C|u(t, x)|^2 |Z_k u(t, x)| + C|u(t, x)|^3,
\]

\[
(3.12) \quad |T_j \partial_t^2 u_i(t, x)| \leq C|T_j \partial_x u(t, x)| + C|T_j \partial u(t, x)| + C|T_j u(t, x)|,
\]

and

\[
(3.13) \quad |T_j \partial_t^2 Z_k u_i(t, x)|
\leq C|T_j \partial_x Z_k u(t, x)|
+ C(|T_j \partial u(t, x)| + |T_j \partial Z_k u(t, x)|) + C(|T_j u(t, x)| + |T_j Z_k u(t, x)|)
+ C|T_j \partial u(t, x)||\partial_x Z_k u(t, x)| + C|T_j u(t, x)||\partial_x Z_k u(t, x)|
+ C|Z_k u(t, x)||(T_j \partial u(t, x)| + |T_j \partial u(t, x)| + |T_j \partial_x u(t, x)|).
\]

The proof is based on straightforward computations. Note that we have not pursued the best possible. The above inequalities suffice for our purpose.

We may obviously focus on the energy of the highest order. Moreover, we may focus on the bound for \( E_1(Z^a S_1 u_i(t)) \) \((i = 1, \ldots, N, |a| = 2)\) because we can obtain a similar bound for \( E_1(Z^a L^a u_i(t)) \) \(|a| = 2, |c| = 1\) in the same way and the bound for \( E_1(Z^a u_i(t)) \) \(|a| = 3\) is easier to get.

Using Lemma 2.1, we get \( \square Z^a S_1 u_i = Z^a S \Box u_i + 2Z^a \Box u_i. \) By Lemma 2.2, we therefore obtain for \(|a| = 2\)

\[
(3.14) \quad \square Z^a S_1 u_i
\]
we obtain for every $i, j$

we have omitted the dependence of $\hat{\alpha}$.

Using Lemma 2.9, (2.32) and (2.33) for

$$|\nabla \cdot \{ \cdots \}^{\alpha} = e^g(u, \partial u)(\partial^2_{\alpha\beta}Z^a Su_i)$$

$$+ \sum \bar{\nabla}^a S(G_i^{\alpha\beta}(u, \partial u)(\partial^2_{\alpha\beta}Z^a S u_i)$$

$$+ \sum \bar{\nabla}^a S(F_i^{\alpha\beta}(u, \partial u)(\partial^2_{\alpha\beta}Z^a S u_i))$$

$$+ \left( Z^a (G_i^{\alpha\beta}(u, \partial u)(\partial^2_{\alpha\beta}Z^a S u_i) - G_i^{\alpha\beta}(u, \partial u)(\partial^2_{\alpha\beta}Z^a S u_i)$$

$$+ \bar{Z}^a S H_i(u, \partial u) - 2\bar{Z}^a C_i(u, \partial u, \partial^2 u) = 0. $$

Here, for given $i, j$ the new coefficients $\hat{F}_i^{\alpha\gamma}$ satisfy the null condition. Also, for given $i, j$, and $k$ the new coefficients $\hat{F}_i^{jk,\alpha\beta}$ satisfy the null condition. By $\sum$ we mean the summation over $|a'| + |a''| \leq 2$, $d' + d'' \leq 1$, $|a''| + d'' \leq 2$. Also, by $\sum''$ we mean the summation over $|a'| + |a''| \leq 2$, $d' + d'' \leq 1$. Just for simplicity of notation, we have omitted the dependence of $\hat{F}_i^{\alpha\gamma} F_i^{\alpha\beta}$ and $\hat{F}_i^{jk,\alpha\beta}$ on $a, a', a'', d, d'$, and $d''$. Using Lemma 2.3.9. (2.32) and (2.33) for $v = Z^a S u_i$, $h^{\alpha\beta} = F_i^{\alpha\beta}(u, \partial u)$, we obtain for every $i = 1, \ldots, N$ and the function $g = g(t - r)$ chosen below (see (3.16))

$$1/2 \partial_t \left\{ e^g((\partial_t Z^a S u_i)^2 + |\nabla Z^a S u_i|^2$$

$$+ 2F_i^{\alpha\beta}(\partial_t Su_i)(\partial_\alpha Z^a S u_i)$$

$$+ 2G_i^{\alpha\beta}(u, \partial u)(\partial_\alpha Z^a S u_i)$$

$$- F_i^{\alpha\beta}(\partial_t Su_i)(\partial_\alpha Z^a S u_i)$$

$$- G_i^{\alpha\beta}(u, \partial u)(\partial_\alpha Z^a S u_i)) \right\}$$

$$+ \nabla \cdot \{ \cdots \}^{\alpha} + e^g q + e^g (J_i, 1 + J_i, 2 + J_i, 3 + J_i, 4 + J_i, 5) = 0,$$

Here,

$$q = q_1 = -\frac{1}{2} g'(t - r) \sum_{j=1}^3 (T_j Z^a S u_i)^2 - g'(t - r) q_2,$$

$$q_1 = - F_i^{\alpha\beta}(\partial_t^2 u_i)(\partial_\alpha Z^a S u_i)$$

$$- (\partial_\alpha G_i^{\alpha\beta}(u, \partial u))(\partial_\alpha Z^a S u_i)$$

$$+ \frac{1}{2} F_i^{\alpha\beta}(\partial_t^2 u_i)(\partial_\alpha Z^a S u_i)$$

$$+ \frac{1}{2} (\partial_\alpha G_i^{\alpha\beta}(u, \partial u))(\partial_\alpha Z^a S u_i),$$

$$q_2 = -\omega_\alpha F_i^{\alpha\beta}(\partial_t Z^a S u_i)(\partial_\beta Z^a S u_i)$$

$$- \omega_\alpha G_i^{\alpha\beta}(u, \partial u)(\partial_\beta Z^a S u_i)(\partial_t Z^a S u_i)$$
and suppose that the local solution \( u \) satisfies (3.24)

Proposition 3.2. Suppose that initial data (1.2) is smooth and compactly supported, and suppose that the local solution \( u \) satisfies (3.6) in some interval \( (0, T) \). Then

\[
- \frac{1}{2} F_{ij}^{\alpha \beta \gamma} (\partial_\gamma u_j) (\partial_\beta \bar{Z}^a S u_i) (\partial_\alpha \bar{Z}^a S u_i)
- \frac{1}{2} G_{i}^{\alpha \beta} (u, \partial u) (\partial_\alpha \bar{Z}^a S u_i) (\partial_\beta \bar{Z}^a S u_i),
\]

where \( \omega_0 = -1 \), \( \omega_k = x_k / |x| \), \( k = 1, 2, 3 \). Also, (see (3.14) above for \( \sum', \sum'' \))

(3.19) \( J_{i,1} = \sum' F_{ij}^{\alpha \beta \gamma} (\partial_\gamma u_j) (\partial_\beta \bar{Z}^a S^d u_i) (\partial_\alpha \bar{Z}^a S^d u_i), \)

(3.20) \( J_{i,2} = \sum' F_{ij}^{\alpha \beta \gamma} (\partial_\gamma \bar{Z}^a S^d u_j) (\partial_\beta \bar{Z}^a S^d u_i) (\partial_\alpha \bar{Z}^a S^d u_i), \)

(3.21) \( J_{i,3} = (\bar{Z}^a S (G_{i}^{\alpha \beta} (u, \partial u) \partial_\alpha \bar{Z}^a S^d u_i) - G_{i}^{\alpha \beta} (u, \partial u) \partial_\alpha \bar{Z}^a S^d u_i) (\partial_\alpha \bar{Z}^a S^d u_i), \)

(3.22) \( J_{i,4} = (\bar{Z}^a S H_i (u, \partial u)) (\partial_\alpha \bar{Z}^a S^d u_i), \)

(3.23) \( J_{i,5} = (-2 \bar{Z}^a C_i (u, \partial u, \partial^2 u)) (\partial_\alpha \bar{Z}^a S^d u_i). \)

As in [7], we use the following quantities \( G(v(t)) \) and \( L(v(t)) \) which are related to the ghost energy and the localized energy, respectively:

(3.24) \( G(v(t)) := \left\{ \sum_{j=1}^{3} \left( \sum_{|a| + |c| + d \leq 3, |c| + d \leq 1} \| (t - r)^{-(1/2) - \eta} T_j \bar{Z}^a L^c S^d v(t) \|_{L^2(\mathbb{R}^3)}^2 \right) + \sum_{|a| + |c| + d \leq 2, |c| + d \leq 1} \| (t - r)^{-(1/2) - \eta} T_j \partial_\alpha \bar{Z}^a L^c S^d v(t) \|_{L^2(\mathbb{R}^3)}^2 \right\}^{1/2}, \)

(3.25) \( L(v(t)) := \left\{ \sum_{|a| + |c| + d \leq 3} \left( \| r^{-1/4} \bar{Z}^a L^c S^d v(t) \|_{L^2(\mathbb{R}^3)}^2 \right) \right\}^{1/2}. \)

We remark that the norm \( \| (t - r)^{-(1/2) - \eta} T_j \partial_\alpha \bar{Z}^a L^c S^d v(t) \|_{L^2(\mathbb{R}^3)} \) \( (|a| + |c| + d \leq 2, |c| + d \leq 1) \), which requires a separate treatment, naturally comes up later. See, e.g., [3,66] below. For \( w(t, x) = (w_1(t, x), \ldots, w_N(t, x)) \), we set

(3.26) \( G(w(t)) := \left( \sum_{i=1}^{N} G(w_i(t))^2 \right)^{1/2}, \quad L(w(t)) := \left( \sum_{i=1}^{N} L(w_i(t))^2 \right)^{1/2}. \)

Recall the definition of \( D(f, g) \) (see (1.17)). By \( \eta \), we mean a sufficiently small positive constant satisfying \( 0 < \eta < 1/3 \). The main purpose of this section is to prove:

**Proposition 3.2.** Suppose that initial data (1.2) is smooth and compactly supported, and suppose that the local solution \( u \) satisfies (3.6) in some interval \( (0, T) \). Then
the following inequality holds for all $t \in (0, T)$:

\begin{align}
N_4(u(t))^2 + \int_0^t G(u(\tau))^2 d\tau \\
\leq CN_4(u(0))^2 + CD(f, g)^6 \\
+ C \int_0^t (\tau)^{-1} (M_3(u(\tau)) + N_4(u(\tau))) L(u(\tau))^2 d\tau \\
+ C \int_0^t (\tau)^{-3/2} (M_3(u(\tau)) + N_4(u(\tau))) N_4(u(\tau))^2 d\tau \\
+ C \int_0^t (\tau)^{-1+\eta} (M_3(u(\tau)) + N_4(u(\tau))) N_4(u(\tau)) G(u(\tau)) d\tau \\
+ C \int_0^t (\tau)^{-1} (M_3(u(\tau))^2 + N_4(u(\tau))^2) L(u(\tau))^2 d\tau \\
+ C \int_0^t (\tau)^{-2} (M_3(u(\tau))^3 + N_4(u(\tau))^3) N_4(u(\tau)) d\tau \\
+ C \int_0^t (\tau)^{-2} (M_3(u(\tau))^2 + N_4(u(\tau))^2) N_4(u(\tau))^2 d\tau.
\end{align}

Proof. Due to (3.6), we may use Lemma 3.1 repeatedly. We also remark that in view of (1.23) and (1.26), we have the Sobolev-type inequalities

\begin{align}
\|t-r\|L^6(\mathbb{R}^3) & \leq CM_1(v(t)) + CN_1(v(t)), \\
\|t-r\|L_6(\mathbb{R}^3) & \leq CN_2(v(t)), \\
\|t-r\|L^6(\mathbb{R}^3) & \leq CM_1(v(t)) + CN_2(v(t)), \\
\|t-r\|L_6(\mathbb{R}^3) & \leq CN_3(v(t)), \\
\|r^{(1/2)+\theta} (t-r)^{1-\theta} v(t, x)\|L^{p}\mathbb{L}^q & \leq CM_1(u(t)) + CN_1(u(t)), \\
\|r^{(1/2)+\theta} (t-r)^{1-\theta} \partial v(t, x)\|L^{p}\mathbb{L}^q & \leq CN_2(u(t))
\end{align}

(see Lemma 2.5 and Lemma 2.6) which will be frequently employed in the following discussion.

The estimate of the $L^1(\mathbb{R}^3)$-norm of each term in (3.15) is carried out over the set \{ $x \in \mathbb{R}^3: |x| < (1 + t)/2$ \} and its complement set, separately, for any fixed time $t \in (0, T)$. It is therefore useful to introduce the characteristic function $\chi_1(x)$ of the former set, and we set $\chi_2(x) := 1 - \chi_1(x)$.

3.1. Estimate over the set \{ $x \in \mathbb{R}^3: |x| < (1 + t)/2$ \}.

- Estimate of $\chi_1 q$. Recall the definition of $q$, $q_1$, and $q_2$ (see (3.16), (3.17), and (3.18)). Due to (3.6), we easily obtain the elementary bound

\begin{align}
|G_i^{\alpha\beta}(u, \partial u)| & \leq C(|u| + |\partial u|), \\
|\partial G_i^{\alpha\beta}(u, \partial u)| & \leq C(|\partial u| + |\partial^2 u|).
\end{align}
We employ (3.7), (3.9) and (3.10) to deal with \( \partial_t^2 u_j(t, x) \) and using (3.6) to get the simple inequality \( |u(t, x)|^3 \leq C|u(t, x)| \), we obtain
\[
\| \chi_1 q_1 \|_{L^1(\mathbb{R}^3)} 
\leq C \sum_{|b| \leq 1} \| \chi_1 |\partial_x^b u(t)| \|_{L^1(\mathbb{R}^3)} + \| \chi_1 |u(t)| \|_{L^1(\mathbb{R}^3)}
\leq C \langle t \rangle^{-1} \left( \sum_{|b| \leq 1} \| r^{1/2} (t - r) |\partial_x^b u(t)| \|_{L^\infty(\mathbb{R}^3)} \right) \| r^{-1/4} \partial^a S u_i(t) \|_{L^2(\mathbb{R}^3)}^2
\leq C \langle t \rangle^{-1} \left( M_2(\rho(u)) + N_3(\rho(u)) \right) \langle \rho(u) \rangle^2.
\]

Using (3.7) to handle \( \partial_t^2 u_j(t, x) \) and using (3.6) to get the simple inequality \( |u(t, x)|^3 \leq C|u(t, x)| \),

\[
\| \chi_1 g'(t - r) q_2 \|_{L^1(\mathbb{R}^3)} 
\leq C \langle t \rangle^{-2 - 2\eta} \left( \| \chi_1 (t - r) u(t) \|_{L^\infty(\mathbb{R}^3)} + \| \chi_1 (t - r) \partial u(t) \|_{L^\infty(\mathbb{R}^3)} \right)
\times \| \partial^a S u_i(t) \|_{L^2(\mathbb{R}^3)}^2
\leq C \langle t \rangle^{-2 - 2\eta} \left( M_1(\rho(u)) + N_3(\rho(u)) \right) N_4(\rho(u))^2.
\]

The estimate of \( \chi_1 g \) has been finished.

- **Estimate of** \( \chi_1 J_{i,1} \). We next estimate \( \chi_1 J_{i,1} \) by basically following [7] and paying attention on the number of occurrences of \( S \).

**Case 1.** \( d' = 1, d'' = 0 \).

**Case 1-1.** \( |a'| = 0, |a''| \leq 2 \). We employ (3.7), (3.9) and (3.10) to deal with \( \partial_{\nu x}^a u_j \), and we then use (3.6) to get \( |u| |\bar{Z}^b u|^2, |u^2| |\bar{Z}^b u| \leq C|u| \) for \( |b| \leq 1 \) and \( |b'| \leq 2 \). In this way, we get
\[
\| \chi_1 (\partial S u_j(t))(\partial^2 \bar{Z}^a u_i(t))(\partial_t \bar{Z}^a S u_i(t)) \|_{L^1(\mathbb{R}^3)}
\leq C \sum_{|b| \leq |a''|} \| \chi_1 (\partial S u_j(t))(\partial_x \bar{Z}^b u(t))(\partial_t \bar{Z}^a S u_i(t)) \|_{L^1(\mathbb{R}^3)}
+ C \sum_{|b| \leq |a''|} \| \chi_1 (\partial S u_j(t))(\partial \bar{Z}^b u(t))(\partial_t \bar{Z}^a S u_i(t)) \|_{L^1(\mathbb{R}^3)}
+ C \| \chi_1 (\partial S u_j(t)) u(t)(\partial_t \bar{Z}^a S u_i(t)) \|_{L^1(\mathbb{R}^3)}
=: J_{i,1}^{(1)} + J_{i,1}^{(2)} + J_{i,1}^{(3)}.
\]
Using (3.22) and (3.25), we obtain
\[
J_{i,1}^{(1)} \leq C \sum_{|b| \leq 2} \langle t \rangle^{-1} \| r^{1/2} \partial Su_j(t) \|_{L^\infty(\mathbb{R}^3)} \| r^{-1/4} (t-r) \partial_x \partial Z^b u(t) \|_{L^2(\mathbb{R}^3)}
\]
\[
\times \| r^{-1/4} \partial_t \tilde{Z}^a Su_i(t) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \langle t \rangle^{-1} N_4(u(t)) L(u(t))^2.
\]
We use (3.33) with \(\theta = 0\) to get
\[
J_{i,1}^{(2)} \leq C \sum_{|b| \leq 2} \langle t \rangle^{-1} \| r^{-1/4} \partial Su_j(t) \|_{L^2 L^2_2} \| r^{1/2} (t-r) \partial \tilde{Z}^b u(t) \|_{L^\infty L^2_2}
\]
\[
\times \| r^{-1/4} \partial_t \tilde{Z}^a Su_i(t) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \langle t \rangle^{-1} N_4(u(t)) L(u(t))^2.
\]
Similarly, we obtain by (3.32)
\[
J_{i,1}^{(3)} \leq C \langle t \rangle^{-1} (M_1(u(t)) + N_1(u(t))) L(u(t))^2.
\]

Case 1-2. \(|a'| \leq 1, |a''| \leq 1\). Using (3.36), (3.37), and (3.39), we get
\[
J_{i,1}^{(4)} \leq C \sum_{|b| \leq |a'|} \sum_{|b'| \leq 1} \| \chi_1(\partial Z^{a'} Su_j(t)) (\partial^2 Z^{a''} u_i(t)) (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)}
\]
\[
\leq C \| \chi_1(\partial Z^{a'} Su_j(t)) (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)} + C \| \chi_1(\partial Z^{a'} Su_j(t)) |u(t)| (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)}
\]
\[
=: J_{i,1}^{(4)} + J_{i,1}^{(5)}.
\]
We have only to handle \(J_{i,1}^{(4)}\) and \(J_{i,1}^{(5)}\) in the same way as in (3.40) and (3.41), respectively.

Case 1-3. \(|a'| \leq 2, |a''| = 0\). Proceeding as in (3.35), we can obtain
\[
J_{i,1}^{(6)} \leq C \sum_{|b|, |b'| \leq 1} \| \chi_1(\partial Z^{a'} u_j(t)) (\partial^2 Z^{a''} Su_i(t)) (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)}
\]
\[
\leq C \| \chi_1(\partial Z^{a'} u_j(t)) (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)} + C \| \chi_1(\partial Z^{a'} u_j(t)) |u(t)| (\partial_t \tilde{Z}^a Su_i(t)) \|_{L^1(\mathbb{R}^3)}
\]
\[
=: J_{i,1}^{(6)} + J_{i,1}^{(7)}.
\]

Using (3.46), (3.9) and (3.10), we get
\[
J_{i,1}^{(6)} \leq C \langle t \rangle^{-1} (M_2(u(t)) + N_1(u(t))) L(u(t))^2.
\]

Case 2. \(d' = 0, d'' = 1\). In this case, we know \(|a''| \leq 1\).

Case 2-1. \(|a'| \leq 1, |a''| \leq 1\).
Here, the term \( J_{i,1}^{(7)} \) has appeared because we have used (3.6) to get
\begin{equation}
|u(t, x)||\bar{Z}^b u(t, x)||Su(t, x)|, |u(t, x)|^2|\bar{Z}^b Su(t, x)|
\leq C\langle t \rangle^{-1 + 2\theta}(\bar{\varepsilon}_1^2)|u(t, x)| \leq C|u(t, x)|, \ |b| \leq 1.
\end{equation}
(Recall that we are assuming \( \delta \leq 1/2 \).) In order to bound \( \|r^{1/2}(t-r)\partial \bar{Z}^a u_j(t)\|_{L^\infty(\mathbb{R}^3)} \) and \( \|r^{1/2}(t-r)\partial u(t)\|_{L^\infty(\mathbb{R}^3)} \), we employ (3.33) and (3.32) with \( \theta = 0 \) together with the Sobolev embedding \( W^{1,4}(S^2) \hookrightarrow L^\infty(S^2) \), as in (3.35). We thus get
\begin{equation}
J_{i,1}^{(6)} + J_{i,1}^{(7)} \leq C\langle t \rangle^{-1} (M_2(u(t)) + N_4(u(t))) L(u(t))^2.
\end{equation}
Case 2-2. \( |a'| \leq 2, |a''| = 0 \). We suitably modify the argument in (3.46) by employing the \( L^\infty_t L^2_x \) and the \( L^\infty_t L^4_x \) norms. Using (3.6) and (3.9), we get
\begin{equation}
\|\chi_1(\partial \bar{Z}^a u_j(t))(\partial^2 Su_i(t))(\partial_t \bar{Z}^a Su_i(t))\|_{L^1(\mathbb{R}^3)}
\leq C\langle t \rangle^{-1} (M_2(u(t)) + N_4(u(t))) L(u(t))^2.
\end{equation}
**Estimate of \( \chi_1 J_{i,2} \).** It suffices to explain how to bound
\( \|\chi_1(\partial \bar{Z}^a u_j(t))(\partial \bar{Z}^a u_k(t))(\partial_t \bar{Z}^a u_i(t))\|_{L^1(\mathbb{R}^3)} \) for \( |a'| + |a''| \leq 2 \). We employ \( \|\chi_1 r^{1/2}(t-r)\partial \bar{Z}^a u_j(t)\|_{L^\infty(\mathbb{R}^3)} \) if \( |a'| \leq 1 \), \( \|\chi_1 r^{1/2}(t-r)\partial \bar{Z}^a u_j(t)\|_{L^\infty_t L^2_x(\mathbb{R}^3)} \) if \( |a'| = 2 \). Then we obtain
\begin{equation}
\|\chi_1 J_{i,2}\|_{L^1(\mathbb{R}^3)} \leq C\langle t \rangle^{-1} N_4(u(t)) L(u(t))^2.
\end{equation}
**Estimate of \( \chi_1 J_{i,3} \).** We recall that \( G_i^{\alpha \beta}(u, v) \) is a homogeneous polynomial of degree 2, and therefore \( G_i^{\alpha \beta}(u, \partial u) \) has the form of sum of constant multiples of \( u_j u_k, u_j \partial_\gamma u_k, \) and \( (\partial_\gamma u_j)(\partial_\delta u_k) \). For the estimate of \( \chi_1 J_{i,3} \), it suffices to repeat the same argument as we have done above and obtain
\begin{equation}
\|\chi_1 J_{i,3}\|_{L^1(\mathbb{R}^3)} \leq C\langle t \rangle^{-1} (M_3(u(t))^2 + N_4(u(t))^2) L(u(t))^2
+ C\langle t \rangle^{-2} (M_2(u(t)) + N_3(u(t)))^3 N_4(u(t)).
\end{equation}
As for the proof of this bound, it suffices to mention how to deal with such a typical term as \( (S(u_j u_k))(\bar{Z}^a \partial_\alpha u_i)(\partial_t \bar{Z}^a Su_i(t)) \) \(|a| = 2\). We use (3.10) to deal with \( \bar{Z}^a \partial_\alpha u_i \), and then we use (3.6) to get \( |u(\bar{Z}^a u)(\bar{Z}^a u)| \leq C|u| \) for \( |a'|, |a''| \leq 1 \) and \( |u^2 \bar{Z}^a u| \leq C|u| \) for \( |a| \leq 2 \). We thus obtain by (3.32) with \( \theta = 0 \) and (3.28)
\begin{equation}
\|\chi_1 (S(u_j u_k))(\bar{Z}^a \partial_\alpha u_i)(\partial_t \bar{Z}^a Su_i(t))\|_{L^1(\mathbb{R}^3)}
\leq C\langle t \rangle^{-1} r^{1/2}(t-r)u(t)\|_{L^\infty(\mathbb{R}^3)}|Su(t)|_{L^\infty(\mathbb{R}^3)} L(u(t))^2
+ C\langle t \rangle^{-2} (t-r)u(t)\|_{L^p(\mathbb{R}^3)}|Su(t)|_{L^p(\mathbb{R}^3)} N_4(u(t))
\leq C\langle t \rangle^{-1} (M_2(u(t)) + N_2(u(t))) M_3(u(t)) L(u(t))^2
+ C\langle t \rangle^{-2} (M_1(u(t)) + N_1(u(t)))^2 N_2(u(t)) N_4(u(t)).
\end{equation}
Here, we have used the standard Sobolev inequalities to handle \( |Su_j(t)|_{L^p(\mathbb{R}^3)} \), \( p = \infty, 6 \). All the other terms can be handled in a similar way.
• **Estimate of \(\chi_1 J_{1,4}\).** We recall that \(H_1(u, v)\) is a homogeneous polynomial of degree 3 in \(u\) and \(v\), and therefore \(H(u, \partial u)\) has the form of sum of constant multiples of \(u_j u_k u_l, u_j u_k \partial_r u_l, u_j (\partial_j u_k) (\partial_r u_l)\), and \((\partial_\alpha u_j) (\partial_\beta u_k) (\partial_r u_l)\). It is possible to obtain

\[
\|\chi_1 J_{1,4}\|_{L^1(\mathbb{R}^3)} \leq C(t)^{-2} \left( M_3(u(t))^3 + N_4(u(t))^3 \right) N_4(u(t))
\]

by using (3.28) together with the Hölder-type inequality

\[
\|\partial q_k\|_{L^1} \leq C(t)^{-2} \|\langle t-r\rangle v_1\|_{L^5} \|\langle t-r\rangle v_2\|_{L^6} \|v_3\|_{L^8} \|v_4\|_{L^2}
\]

(see (3.50)) or (3.30), (3.31) together with the Hölder-type inequality

\[
\|\chi_1 v_1 \cdots v_4\|_{L^1} \leq C(t)^{-2} \|\langle t-r\rangle v_1\|_{L^\infty} \|\langle t-r\rangle v_2\|_{L^\infty} \|v_3\|_{L^2} \|v_4\|_{L^2}.
\]

• **Estimate of \(\chi_1 J_{1,5}\).** Recall that \(\bar{Z}^a\) does not contain the operator \(S\). Therefore, using (3.7), (3.9), and (3.11) to deal with \(\partial_\beta Z^a u\) \((|a| \leq 2)\) and proceeding as we have done in dealing with \(\chi_1 J_{1,4}\) just above, we easily obtain

\[
\|\chi_1 J_{1,5}\|_{L^1(\mathbb{R}^3)} \leq C(t)^{-2} \left( M_3(u(t))^3 + N_4(u(t))^3 \right) N_4(u(t)).
\]

### 3.2. **Estimate over the set** \(\{x \in \mathbb{R}^3 : |x| > (1+t)/2\}\). In contrast with the former subsection, we fully exploit the null condition. We start with the estimate of \(\chi_2 q_1\). As for the third term on the right-hand side of (3.17), we basically follow the argument in [7]. Namely, we first employ (2.11) and then (2.14), (3.6)–(3.7), (2.24), and (3.32)–(3.33) with \(\theta = (1/2) - \eta\) to get

\[
\|\chi_2 F_{i}^{j,\beta}(\partial_\alpha u_j(t)) (\partial_\beta Z^a S u_i(t)) (\partial_\alpha Z^a S u_i(t))\|_{L^1(\mathbb{R}^3)}
\leq C \sum_{k,j} \|\chi_2 (T_k \partial_t u_j(t)) (\partial_\beta Z^a S u_i(t))^2\|_{L^1(\mathbb{R}^3)}
+C \sum_{k,j} \|\chi_2 (\partial_\nu \partial_t u_j(t)) (T_k \partial_\beta Z^a S u_i(t)) (\partial_\alpha Z^a S u_i(t))\|_{L^1(\mathbb{R}^3)}
\leq C(t)^{-2} \sum_{\frac{\theta}{2} + \gamma \leq 1} \frac{\|\partial_\nu S^d \partial_t u_j(t)\|_{L^\infty(\mathbb{R}^3)} \|\partial_\nu \partial_\beta S^d u_i(t)\|_{L^2(\mathbb{R}^3)}}{r^{\gamma \theta}(t-r)^{(1/2)+\eta}(|\partial_\nu u(t)| + |\partial_t u(t)| + |u(t)|)}
\leq C(t)^{-2} N_4(u(t))^3 + C(t)^{-1+\eta} (M_2(u(t)) + N_4(u(t))) G(u(t)) N_4(u(t)).
\]

Using (2.10) in place of (2.11) and repeating the same argument as above, we have a similar bound for the first term on the right-hand side of (3.17).

As for the second and the fourth terms on the right-hand side of (3.17), we use the elementary bound

\[
|\partial G(u, \partial u)| \leq C(|u| + |\partial u|) (|\partial u| + |\partial^2 u|)
\]

and employ (3.6)–(3.7) to handle \(\partial_\beta^2 u\). We get by (2.24)

\[
\|\chi_2 (\partial G(u, \partial u)) \|_{L^1(\mathbb{R}^3)}
\]
Obviously, it suffices to handle only the case

\[ |a'| + |a''| \leq 2, \quad d' + d'' \leq 1, \quad \text{and} \quad |a''| + d'' \leq 2. \]

It suffices to discuss only the case \( d' + d'' = 1 \); the argument becomes easier otherwise.

Case 1. \( d' = 1, d'' = 0 \).

Case 1-1. \( |a'| = 0, |a''| \leq 2 \). Obviously, it suffices to handle only the case \( |a''| = 2 \).

Using (3.6), (3.10), (2.25), and (3.32)–(3.33) with \( \theta = (1/2) - \eta \), we obtain

\[
K_1 \leq C(t)^{-1} \| r(t-r)^{-1} T_k S u_j(t) \|_{L^\infty(\mathbb{R}^3)} \| r(t-r)^{-1/2-n} T_k S u_j(t) \|_{L^2 L^\infty_t L^3_x} \\
+ C(t)^{-1+n} \| r(t-r)^{-1/2-n} T_k S u_j(t) \|_{L^2 L^\infty_t L^3_x} \\
\quad \times \sum_{|b| \leq 2} \| r^{1-\eta} (t-r)^{(1/2)+\eta} \partial \tilde{Z}^b u(t) \|_{L^\infty L^3_x} \\
+ C(t)^{-1+n} \| r(t-r)^{-1/2-n} T_k S u_j(t) \|_{L^2(\mathbb{R}^3)} \| r^{1-\eta} (t-r)^{(1/2)+\eta} u(t) \|_{L^\infty(\mathbb{R}^3)} \\
\leq C(t)^{-1} G(u(t)) N_4(u(t)) + C(t)^{-1+n} G(u(t)) (N_4(u(t)) + M_2(u(t))).
\]

Here, to handle \( \| r(t-r)^{-1} T_k S u_j(t) \|_{L^\infty(\mathbb{R}^3)} \), we have used (see, e.g., (27), (28) in [24])

\[
[\Omega_{ij}, T_k] = \delta_{kj} T_i - \delta_{ki} T_j, \quad \partial_t T_i = \sum_{k=1}^3 \frac{x_k}{r} T_i \partial_k
\]
together with (2.24). It is easy to get by (2.24) and (2.14)
\[
(3.62) \quad K_2 \leq C(t)^{-2} \| r \partial S u_j(t) \|_{L^\infty(R^3)} \| r T_k \partial \bar{Z}^{a''} u_i(t) \|_{L^2(R^3)} \\
\leq C(t)^{-2} N_4(u(t))^2.
\]

Case 1-2. \( |a'| \leq 1 \) and \( |a''| \leq 1 \). Employing \( \| r(t-r)^{-1} T_k \bar{Z}^{a'} S u_j(t) \|_{L^\infty L^2} \) and \( \| (t-r) \partial_x \partial \bar{Z}^{a''} u_i(t) \|_{L^2 L^2} \), and naturally modifying the argument in Case 1-1, we get the same bound for \( K_1 \) as in Case 1-1. Also, employing \( \| r \partial \bar{Z}^{a'} S u_j(t) \|_{L^\infty L^4} \) and \( \| r T_k \partial \bar{Z}^{a''} u_i(t) \|_{L^2 L^4} \), we get the same bound for \( K_2 \) as in Case 1-1.

Case 1-3. \( |a'| \leq 2 \) and \( |a''| = 0 \). Using (3.6), (3.7) and (3.32) with \( \theta = (1/2) - \eta \), we easily obtain
\[
(3.63) \quad K_1 \leq C(t)^{-1+\eta} G(u(t)) N_4(u(t)) + C(t)^{-1+\eta} G(u(t)) M_2(u(t)).
\]

Also, using (2.11) first and then (2.22), we easily get
\[
(3.64) \quad K_2 \leq C(t)^{-2} N_4(u(t))^2 \| r (r T_k \partial u_i(t)) \|_{L^\infty(R^3)} \leq C(t)^{-2} N_4(u(t))^2.
\]

Case 2. \( d' = 0 \) and \( d'' = 1 \).

Case 2-1. \( |a'| \leq 1 \) and \( |a''| \leq 1 \). We employ (3.9), (3.10) together with (3.45) to get
\[
(3.65) \quad K_1 \leq \| \chi_2 (T_k \bar{Z}^{a'} u_j(t)) (\partial^2 \bar{Z}^{a''} S u_i(t)) \|_{L^2(R^3)} \\
\leq C(t)^{-3/2} \| r^{1/2} (r T_k \bar{Z}^{a'} u_j(t)) \|_{L^\infty(R^3)} (N_4(u(t)) + M_1(u(t))) \\
\leq C(t)^{-3/2} N_4(u(t)) (M_1(u(t)) + N_4(u(t))).
\]

Here, we have used (2.11), (2.22), (1.26). As for \( K_2 \), we employ (3.38) with \( \theta = (1/2) - \eta \) and easily get
\[
(3.66) \quad K_2 \leq C(t)^{-1+\eta} N_4(u(t)) G(u(t)).
\]

It should be noted that this is the one of the places where we encounter the norm
\[ \| (t-r)^{-(1/2)-\eta} T_k \partial_x \partial \bar{Z}^{a'} L^c S u_i(t) \|_{L^2(R^3)} \] \( \| \langle t \rangle + |c| + d \leq 2, |c| + d \leq 1 \).

Case 2-2. \( |a'| \leq 2 \) and \( |a''| = 0 \). Using the \( L^\infty L^4 \)-norm and the \( L^2 L^4 \)-norm, we naturally modify the argument in the above case to get the same bound for \( K_1 \) and \( K_2 \) as in Case 2-1. We have finished the estimate of \( \chi_2 J_{i,2} \).

\textbf{Estimate of} \( \chi_2 J_{i,2} \). We need to bound \( \| \chi_2 \bar{F}^{jk,\alpha\beta}_i (\partial_\alpha \bar{Z}^{a'} S d' u_j) (\partial_\beta \bar{Z}^{a''} S d'' u_k) \|_{L^2(R^3)} \) for \( |a'| + |a''| \leq 2, d' + d'' \leq 1 \). Obviously, we may focus on the case \( d' + d'' = 1 \). It follows from (2.12) that
\[
(3.67) \quad \| \chi_2 \bar{F}^{jk,\alpha\beta}_i (\partial_\alpha \bar{Z}^{a'} S d' u_j) (\partial_\beta \bar{Z}^{a''} S d'' u_k) \|_{L^2(R^3)} \\
\leq C \sum_{j,k,l} (\| \chi_2 (T_l \bar{Z}^{a'} S d' u_j) (\partial \bar{Z}^{a''} S d'' u_k) \|_{L^2(R^3)} \\
+ \| \chi_2 (\partial \bar{Z}^{a'} S d' u_j) (T_l \bar{Z}^{a''} S d'' u_k) \|_{L^2(R^3)}).
\]

Due to symmetry, we may suppose \( d' = 1 \) and \( d'' = 0 \). When \( |a'| = 0 \) and \( |a''| \leq 2 \) or \( |a'| \leq 1 \) and \( |a''| \leq 1 \), we employ the \( L^2 L^4 \)-norm and the \( L^\infty L^4 \)-norm. We get by
with \( \theta = (1/2) - \eta \)

\[
\| \chi_2(T_t \bar{Z}^{a''} S u_j(t)) (\partial \bar{Z}^{a''} u_k(t)) \|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1+\eta} G(u(t)) N_4(u(t)).
\]

Also, we get by (2.22), (2.14)

\[
\| \chi_2(\partial \bar{Z}^{a''} S u_j(t)) (T_t \bar{Z}^{a''} u_k(t)) \|_{L^2(\mathbb{R}^3)} \leq C(t)^{-3/2} N_4(u(t))^2.
\]

When \( |a'| \leq 2 \) and \( |a''| = 0 \), we have only to modify the argument just above and employ the \( L^2(\mathbb{R}^3) \)-norm and the \( L^\infty(\mathbb{R}^3) \)-norm. We have finished the estimate of \( \chi_2 J_{i,2} \).

**Estimate of \( \chi_2 J_{i,3}, \chi_2 J_{i,4}, \) and \( \chi_2 J_{i,5} \).** Using (2.23) and (2.24), we obtain

\[
\sum_{k=3}^5 \| \chi_2 J_{i,k} \|_{L^1(\mathbb{R}^3)} \leq C \sum_{m=0}^4 \| \langle \tau - r \rangle^{-(1/2) - \eta} T_j \bar{Z}^a S u_i(\tau) \| _{L^1(\mathbb{R}^3)}^2 N_4(u(t))^2.
\]

The proof is direct and is therefore omitted.

Now we are in a position to complete the proof of Proposition 3.2. We first note that the function \( g = g(\rho) (\rho \in \mathbb{R}) \) is bounded (see (3.36)), and hence there exists a positive constant \( c \) such that \( c \leq c^0 \leq c^{-1} (\rho \in \mathbb{R}) \). We also note that \( g' \) is a negative function, and it therefore follows from (3.15), (3.35)–(3.70) that for \( i = 1, \ldots, N \) and \( |a| = 2 \),

\[
E_1(\bar{Z}^a S u_i(t)) + \sum_{j=1}^3 \int_0^t \| \langle \tau - r \rangle^{-(1/2) - \eta} T_j \bar{Z}^a S u_i(\tau) \| _{L^2(\mathbb{R}^3)}^2 d\tau
\]

is estimated from above by the right-hand side of (3.27). (Strictly speaking, the term \( CD(f, g)^6 \) there plays no role at this moment.) We should mention how to estimate \( \| \langle \tau - r \rangle^{-(1/2) - \eta} T_j \bar{Z}^a S u_i(\tau) \| _{L^2(\mathbb{R}^3)} \) for \( |a| = 1 \). In (3.14)–(3.22), we replace \( \bar{Z}^a S \) \( (|a| = 2) \) with \( \partial_i \bar{Z}^a S \) \( (|a| = 1) \), accordingly modifying \( \bar{Z}^a S^d, \bar{Z}^{a''} S^{d''} \) suitably. Also, at the last term on the left-hand side of (3.14) and in (3.23), we replace \( \bar{Z}^a \) \( (|a| = 2) \) with \( \partial_i \bar{Z} \) \( (|a| = 1) \). Though we then encounter such a little troublesome terms as \( \partial_i^2 \bar{Z}^a S^d u_i \) \( (|a| + d \leq 1) \), we can rely upon (3.3) and (3.11) to handle such terms in the same way as we have done above. Also, note that the term \( CD(f, g)^3 \) naturally comes up from \( \| \partial_i^2 \bar{Z}^a S u_i(0) \| _{L^2(\mathbb{R}^3)} \) \( (|a| = 1) \), and this is the reason why we need \( CD(f, g)^6 \) on the right-hand side of (3.27). Finally, we also mention that another troublesome term \( T_j \partial_i^2 \bar{Z}^a S^d u_i \) \( (|a| + d \leq 1) \) comes up when we rely upon Lemma 2.3. We can get over this difficulty by employing (3.12) and (3.13). We have finished the estimate of

\[
\sum_{|a|=2} \sum_{j=1}^3 \int_0^t \| \langle \tau - r \rangle^{-(1/2) - \eta} T_j \bar{Z}^a S u_i(\tau) \| _{L^2(\mathbb{R}^3)}^2 d\tau
\]

also we get by (2.22), (2.14)

\[
\| \chi_2(\partial \bar{Z}^{a''} S u_j(t)) (T_t \bar{Z}^{a''} u_k(t)) \|_{L^2(\mathbb{R}^3)} \leq C(t)^{-3/2} N_4(u(t))^2.
\]
The other terms appearing on the left-hand side of (3.27) can be estimated in a similar way, and we have therefore completed the proof of Proposition 3.2.

4. Bound for $M_3(u(t))$

The main purpose of this section is to prove:

**Proposition 4.1.** Suppose that initial data (1.2) is smooth and compactly supported, and suppose that the local solution $u$ satisfies (3.10) in some interval $(0, T)$. Then the following inequality holds for all $t \in (0, T)$:

\[
M_3(u(t)) \leq C \sum_{|a| \leq 2} \left( \| \bar{Z}^a f \|_{L^2(\mathbb{R}^3)} + \| x | \partial_x \bar{Z}^a f \|_{L^2(\mathbb{R}^3)} + \| x | \bar{Z}^a g \|_{L^2(\mathbb{R}^3)} \right)
\]

\[
+ C \int_0^t \langle \tau \rangle^{-1} \left( M_3(u(\tau)) + N_3(u(\tau)) \right) N_4(u(\tau)) d\tau
\]

\[
+ C \int_0^t \langle \tau \rangle^{-3/2} M_3(u(\tau)) N_2(u(\tau))^3 d\tau
\]

\[
+ C \int_0^t \langle \tau \rangle^{-1} (N_3(u(\tau))^2 + X_2(u(\tau))^2) N_4(u(\tau)) d\tau
\]

\[
+ C \int_0^t \langle \tau \rangle^{-2} (M_3(u(\tau))^2 + N_3(u(\tau))^2) N_4(u(\tau)) d\tau
\]

\[
+ C \int_0^t \langle \tau \rangle^{-2} M_3(u(\tau))^3 d\tau + C \int_0^t \langle \tau \rangle^{-1} X_2(u(\tau))^3 d\tau.
\]

In view of the conformal energy estimate (2.27), it amounts to bounding

\[
(t + |x|) \bar{Z}^a F_{i}^{j,\alpha\beta}(\partial_\gamma u_j)(\partial_{\alpha\beta} u_i) \|_{L^2(\mathbb{R}^3)},
\]

\[
(t + |x|) \bar{Z}^a F_{i}^{jk,\alpha\beta}(\partial_\alpha u_j)(\partial_{\alpha\beta} u_k) \|_{L^2(\mathbb{R}^3)},
\]

\[
(t + |x|) \bar{Z}^a \left( G^\alpha_{i\alpha}(u, \partial u) \partial_{\alpha\beta}^2 u_i \right) \|_{L^2(\mathbb{R}^3)},
\]

\[
(t + |x|) \bar{Z}^a H_i(u, \partial u) \|_{L^2(\mathbb{R}^3)}
\]

(see (1.1), (1.3), (1.4)) for $|a| \leq 2$. As in the previous section, we treat them by considering the $L^2$ norm over the set $\{ x \in \mathbb{R}^3 : |x| < (1 + t)/2 \}$ and its complement set, separately. Furthermore, when considering the $L^2$ norm over the former set, we deal with the case $t < 3$ and $t > 3$, separately.

- **$L^2$ norm over the set $\{ x \in \mathbb{R}^3 : |x| < (1 + t)/2 \}$ with $t < 3$.** Obviously, it suffices to discuss how to bound the $L^2(\mathbb{R}^3)$ norm of $\partial_\gamma F_{i}^{j,\alpha\beta}(\partial_\gamma u_j)(\partial_{\alpha\beta}^2 u_i)$, $\partial_{x} F_{i}^{jk,\alpha\beta}(\partial_\alpha u_j)(\partial_{\alpha\beta} u_k)$, $\partial_{x} \left( G^\alpha_{i\alpha}(u, \partial u) \partial_{\alpha\beta}^2 u_i \right)$, $\partial_{x} H_i(u, \partial u)$ for $|a| \leq 2$. Using (3.6), (3.7), (3.9), and (3.10), we easily get

\[
\sum_{|a| \leq 2} \left( \| \partial_\gamma F_{i}^{j,\alpha\beta}(\partial_\gamma u_j)(\partial_{\alpha\beta}^2 u_i) \|_{L^2(\mathbb{R}^3)} + \| \partial_{x} F_{i}^{jk,\alpha\beta}(\partial_\alpha u_j)(\partial_{\alpha\beta} u_k) \|_{L^2(\mathbb{R}^3)} \right)
\]
Moreover, using not only (3.6), (3.7), (3.9), and (3.10) but also the Hölder inequality (4.7) we carry out the estimate of $P_3$ and the one $t > 3$ which holds on the set $\{x \in \mathbb{R}^3 : |x| < (1+t)/2\}$ together with the Sobolev-type inequality $\|v\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}$, we easily get

$$\sum_{|\alpha| \leq 2} (\|\partial_x^2 (g_{ij}^\alpha (u, \partial u) \partial_\alpha u_i)\|_{L^2(\mathbb{R}^3)} + \|\partial_x^2 H_i (u, \partial u)\|_{L^2(\mathbb{R}^3)}) \leq C (M_1 (u(t)) + N_3 (u(t))) N_4 (u(t)).$$

Next let us bound the $L^2$ norm of (4.2), (4.5) over the set $\{x \in \mathbb{R}^3 : |x| > (1 + t)/2\}$ with $t > 3$ and the one $\{x \in \mathbb{R}^3 : |x| > (1 + t)/2\}$ with $t > 0$.

**Estimate of (4.2).** On account of (2.7), we need to estimate

$$(t + |x|) \hat{F}_i^{j, \alpha \beta \gamma} (\partial_\alpha \bar{Z}^a u_j) (\partial_\alpha \bar{Z}^{a''} u_i)$$

$(|\alpha'| + |\alpha''| \leq 2)$, where the new coefficients $\hat{F}_i^{j, \alpha \beta \gamma}$, which in fact may depend on $\alpha'$ and $\alpha''$, satisfy the null condition. We first note that due to the null condition, (2.9), and (2.14) (2.15) we have

$$\sum_{|\beta| + |\alpha| + |\alpha''| = 1} |\alpha''| \leq |\alpha''| = 2$$

$$\sum_{|\beta| + |\alpha| + |\alpha''| = 1} |\alpha''| \leq 2$$

We carry out the estimate of $P_1$, $P_2$ over the set $\{x \in \mathbb{R}^3 : |x| < (1 + t)/2\}$ with $t > 3$ and the one $\{x \in \mathbb{R}^3 : |x| > (1 + t)/2\}$ with $t > 0$, separately.

When $|\alpha''| = 0$ (and hence $|\alpha'| \leq 2$), we get by (2.17) and the Sobolev embedding

$$\|\chi_1 P_1\|_{L^2(\mathbb{R}^3)} \leq \|t^{-1} M_3 (u(t)) \|_1 - r \|\partial (\bar{u}(t))\|_{L^\infty(\mathbb{R}^3)} \leq C \langle t \rangle^{-1} M_3 (u(t)) \sum_{|\beta| + |\alpha| + |\alpha''| = 1} \|\Omega^b L^c S^d (\partial \bar{u}(t))\|_{L^\infty(\mathbb{R}^3)} \leq C \langle t \rangle^{-1} M_3 (u(t)) N_4 (u(t)).$$

Here we have used the inequality

$$t - r \geq t - \frac{1 + t}{2} = \frac{t}{4} + \frac{t}{4} - \frac{1}{2} \geq \frac{t}{4} + \frac{1}{4},$$

which holds on the set $\{x \in \mathbb{R}^3 : |x| < (1 + t)/2\}$ with $t > 3$. When $|\alpha''| = 1$ (and hence $|\alpha'| \leq 1$) or $|\alpha''| = 2$ (and hence $|\alpha'| = 0$), we get by (4.10), (2.17), and the Sobolev embedding

$$\|\chi_1 P_1\|_{L^2(\mathbb{R}^3)} \|$$
As for $P$, we use (3.29) to get for $|a''| = 0$ (and hence $|a'| \leq 2$)

$$\|\chi_1 P_2\|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1}\|\langle t - r \rangle \partial Z^a u(t)\|_{L^6(\mathbb{R}^3)} \left( \sum_{|b| + |c| + d = 1} \|\Omega^b L^c S^d Z^a u(t)\|_{L^\infty(\mathbb{R}^3)} \right) \leq C(t)^{-1}N_3(u(t))N_4(u(t)).$$

For $|a''| = 1$ (and hence $|a'| \leq 1$) or $|a''| = 2$ (and hence $|a'| = 0$), we apply (3.31) to $\|\langle t - r \rangle \partial Z^a u(t)\|_{L^\infty(\mathbb{R}^3)}$ and obtain the same bound for $\|\chi_1 P_2\|_{L^2(\mathbb{R}^3)}$ as in (4.12).

Turning our attention to the estimate of $\chi_2 P_1$ and $\chi_2 P_2$, we get by (3.7) and (2.22), for $|a''| = 0$ (and hence $|a'| \leq 2$)

$$\|\chi_2 P_1\|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1}M_5(u(t)) \sum_{|a| \leq 1} \|r \partial Z^a u(t)\|_{L^\infty(\mathbb{R}^3)} + C(t)^{-3/2}M_5(u(t))\|r^{1/2} u(t)\|_{L^\infty(\mathbb{R}^3)}^3 \leq C(t)^{-1}M_5(u(t))N_4(u(t)) + C(t)^{-3/2}M_3(u(t))N_2(u(t))^3.$$  

For $|a''| = 1$ (and hence $|a'| \leq 1$), we get by (3.9) and (2.22)

$$\|\chi_2 P_1\|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1}\left( \sum_{|b| + |c| + d = 1} \|r \Omega^b L^c S^d Z^a u(t)\|_{L^\infty L^2} \right) \left( \sum_{|a| \leq 2} \|\partial Z^a u(t)\|_{L^2 L^2} \right) + C(t)^{-3/2}\left( \sum_{|b| + |c| + d = 1} \|\Omega^b L^c S^d Z^a u(t)\|_{L^2 L^2} \right) \|r^{1/2} u(t)\|_{L^2 L^2}^2 \times \left( \sum_{|a| \leq 1} \|r^{1/2} Z^a u(t)\|_{L^\infty L^2} \right) \leq C(t)^{-1}\left( N_3(u(t))^{1/2}M_3(u(t))^{1/2}N_4(u(t)) + C(t)^{-3/2}M_3(u(t))N_2(u(t))^3. \right.$$  

For $|a''| = 2$ (and hence $|a'| = 0$), we employ (3.10) and modify the argument above to get the same estimate as in (4.14). As for $\chi_2 P_2$, it is easy to get

$$\|\chi_2 P_2\|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1}N_3(u(t))N_4(u(t)).$$

**Estimate of (4.3).** In view of (2.8), we need to deal with

$$(t + |x|)\tilde{F}_i^j k^j \alpha \beta (\partial_\alpha Z^a u_j)(\partial_\beta Z^{a''} u_k), \quad |a'| + |a''| \leq 2,$$
where the new coefficients $\hat{F}^j_{ik \alpha \beta}$, which may depend on $a'$ and $a''$, satisfy the null condition. We then have, as in (4.3)
\begin{equation}
(4.16) \quad (t + |x|)|\hat{F}^j_{ik \alpha \beta}(\partial \tilde{Z}^{a'} u_j(\partial \tilde{Z}^{a''} u_k)|
\leq \left( \sum_{|b| + |c| + |d| = 1} |\Omega^{b} L^c S^d \tilde{Z}^{a'}| \right) |\partial \tilde{Z}^{a''}| u + C |\partial \tilde{Z}^{a''} u| \left( \sum_{|b| + |c| + |d| = 1} |\Omega^{b} L^c S^d \tilde{Z}^{a''}| u \right)
=: P_3 + P_4.
\end{equation}

By symmetry, we have only to deal with $P_3$. When $|a''| = 0$ (and hence $|a'| \leq 2$) or $|a''| = 1$ (and hence $|a'| \leq 1$), we get by (3.31)
\begin{equation}
(4.17) \quad \|\chi_1 P_3\|_{L^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-1} M_3(u(t)) \|\langle t - r \rangle \partial \tilde{Z}^{a''} u(t)\|_{L^\infty(\mathbb{R}^3)}
\leq C \langle t \rangle^{-1} M_3(u(t)) N_4(u(t)).
\end{equation}

When $|a''| = 2$ (and hence $|a'| = 0$), we employ (3.23) and obtain
\begin{equation}
(4.18) \quad \|\chi_1 P_3\|_{L^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-1} \left( \sum_{|b| + |c| + |d| = 1} \|\Omega^{b} L^c S^d u(t)\|_{L^3(\mathbb{R}^3)} \right) \|\langle t - r \rangle \partial \tilde{Z}^{a''} u(t)\|_{L^6(\mathbb{R}^3)}
\leq C \langle t \rangle^{-1} M_3(u(t)) N_4(u(t)).
\end{equation}

As for $\chi_2 P_3$, we get in a way similar to (4.13)–(4.14)
\begin{equation}
(4.19) \quad \|\chi_2 P_3\|_{L^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-1} M_3(u(t)) N_4(u(t))
+ C \langle t \rangle^{-1} \left( N_3(u(t))^{1/2} M_3(u(t))^{1/2} \right) N_3(u(t)).
\end{equation}

- Estimate of (4.4). We can obtain for $|a| \leq 2$, $i = 1, \ldots, N$
\begin{equation}
(4.20) \quad \|\chi_1 (t + |x|) \tilde{Z}^a (G_i^{\alpha \beta}(u, \partial u) \partial^2 u_i(t))\|_{L^2(\mathbb{R}^3)}
\leq C \langle t \rangle^{-2} (M_3(u(t)) + N_3(u(t)))^2 N_4(u(t)),
\end{equation}
\begin{equation}
(4.21) \quad \|\chi_2 (t + |x|) \tilde{Z}^a (G_i^{\alpha \beta}(u, \partial u) \partial^2 u_i(t))\|_{L^2(\mathbb{R}^3)}
\leq C \langle t \rangle^{-1} \left( N_3(u(t))^2 + \chi_3(u(t))^2 \right) N_4(u(t)).
\end{equation}

For the proof of (4.20), it suffices to explain how to handle such typical terms as $u_j(\tilde{Z}^{a} u_k) \partial^2 u_i$ and $u_j u_k \tilde{Z}^a \partial^2 u_i$ for $|a| = 2$ because the other terms can be treated in a similar way.

Recall that we are assuming $t \geq 3$ when considering the estimate over the set $\{x \in \mathbb{R}^3 : |x| < (1 + t)/2\}$. Using (3.28) and (2.17), we obtain
\begin{equation}
(4.22) \quad \|\chi_1 (t + |x|) u_j(t) (\tilde{Z}^{a} u_k(t)) \partial^2 u_i(t)\|_{L^2(\mathbb{R}^3)}
\leq C \langle t \rangle^{-2} \|\langle t - r \rangle u_j(t)\|_{L^6(\mathbb{R}^3)} \|\langle t - r \rangle \tilde{Z}^{a} u_k(t)\|_{L^6(\mathbb{R}^3)} \|\langle t - r \rangle \partial(u_i(t))\|_{L^6(\mathbb{R}^3)}
\leq C \langle t \rangle^{-2} (M_3(u(t)) + N_3(u(t)))^2 N_3(u(t)).
\end{equation}

Also, using (3.30) and (2.17), we get
\begin{equation}
(4.23) \quad \|\chi_1 (t + |x|) u_j(t) u_k(t) \tilde{Z}^a \partial^2 u_i(t)\|_{L^2(\mathbb{R}^3)}
\end{equation}
\[
\leq C(t)^{-2} \| (t - r) u(t) \|_{L^\infty_c(\mathbb{R}^3)}^2 \sum_{|b| \leq 2} \| t - r | \partial (\partial Z^b u(t)) |_{L^2(\mathbb{R}^3)}
\]
\[
\leq C(t)^{-2} (M_1(u(t)) + N_2(u(t)))^2 N_4(u(t)).
\]

For the proof of (4.21), it suffices to explain how to treat \((\bar{Z}^{a'}) j u_j (\bar{Z}^{a''} u_k) \partial^2 u_i\) with \(|a'| = |a''| = 1\) and \(u_j (\bar{Z}^a u_k) \partial^2 u_i\) with \(|a| = 2\); the other terms can be handled in a similar manner. Using (2.24) and (3.6)–(3.7), we get
\[
\| \chi_2 (t + |x|) (\bar{Z}^{a'} u_j (t)) (\bar{Z}^{a''} u_k (t)) \partial^2 u_i (t) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C(t)^{-1} \| r \bar{Z}^{a'} u_j (t) \|_{L^\infty L^1} \| r \bar{Z}^{a''} u_k (t) \|_{L^\infty L^1} \| \partial^2 u_i (t) \|_{L^2 L^\infty}
\]
\[
\leq C(t)^{-1} (N_2(u(t)))^{1/2} X_2(u(t))^{1/2} (N_4(u(t)) + X_2(u(t)))
\]
\[
\leq C(t)^{-1} (N_2(u(t))^2 + X_2(u(t))^2) N_4(u(t)),
\]
where we have used the Young inequality. Also, using (2.24), (3.6)–(3.7), we obtain
\[
\| \chi_2 (t + |x|) j u_j (t) (\bar{Z}^a u_k (t)) \partial^2 u_i (t) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C(t)^{-1} N_2(u(t))^{1/2} X_2(u(t))^{1/2} (N_4(u(t)) + N_2(u(t))^{1/2} X_2(u(t))^{1/2})
\]
\[
\leq C(t)^{-1} (N_2(u(t))^2 + X_2(u(t))^2) N_4(u(t)),
\]
where we have used the Young inequality again.

- **Estimate of (4.5)**. Repeating essentially the same argument as above, we can obtain for \(|a| \leq 2\)
\[
\| (t + |x|) \bar{Z}^a H_4(u, \partial u) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C(t)^{-2} (M_3(u(t))^3 + N_3(u(t))^3) + C(t)^{-1} (N_5(u(t))^3 + X_2(u(t))^3).
\]

It is now obvious that (4.1) is an immediate consequence of the estimates we have obtained above. The proof of Proposition 4.1 has been finished.

5. **Bound for \(X_2(u(t))\)**

The purpose of this section is to prove

**Proposition 5.1.** Suppose that initial data \([1.2]\) is smooth and compactly supported, and suppose that the local solution \(u\) satisfies \([3.6]\) in some interval \((0, T)\). Then the following inequality holds for all \(t \in (0, T)\):
\[
(5.1) \quad X_2(u(t))
\]
\[ \leq C \left( \sum_{|a| \leq 2} \| \tilde{Z}^a f \|_{L^2(\mathbb{R}^3)} + \sum_{|a| \leq 2} \| x | \tilde{Z}^a g \|_{L^2(\mathbb{R}^3)} \right) + C \int_0^t \langle \tau \rangle^{-2} (M_3(u(\tau)) + N_3(u(\tau))) (M_3(u(\tau)) + N_4(u(\tau))) d\tau \\
+ C \int_0^t \langle \tau \rangle^{-2} (M_3(u(\tau)) + N_3(u(\tau)))^2 (N_4(u(\tau)) + X_2(u(\tau))) d\tau \\
+ C \int_0^t \langle \tau \rangle^{-3/2} (M_3(u(\tau)) + N_3(u(\tau))) (N_4(u(\tau)) + X_1(u(\tau))) d\tau \\
+ C \int_0^t \langle \tau \rangle^{-3/2} (N_3(u(\tau)) + X_2(u(\tau)))^2 (N_4(u(\tau)) + X_2(u(\tau))) d\tau. \]

In view of the Li-Yu estimate (2.28) together with the well-known inequality \( \| D^{-1}v \|_{L^2(\mathbb{R}^n)} \leq C \| x | v \|_{L^2(\mathbb{R}^n)} \) \( (n \geq 3) \), the proof of this proposition amounts to showing decay estimates of the following norms for \( |a| \leq 2 \):

\begin{align*}
(5.2) & \quad \| \chi_1 \tilde{Z}^a F_{i}^{j, \alpha \beta \gamma} (\partial_i u_j) (\partial^2_{\alpha \beta} u_i) \|_{L^{6/5}(\mathbb{R}^3)} , \quad \| \chi_2 \tilde{Z}^a F_{i}^{j, \alpha \beta \gamma} (\partial_i u_j) (\partial^2_{\alpha \beta} u_i) \|_{L^{4/3}(\mathbb{R}^3)} ,
(5.3) & \quad \| \chi_1 \tilde{Z}^a F_{i}^{j, \alpha \beta} (\partial_i u_j) (\partial^2_{\alpha \beta} u_i) \|_{L^{6/5}(\mathbb{R}^3)} , \quad \| \chi_2 \tilde{Z}^a F_{i}^{j, \alpha \beta} (\partial_i u_j) (\partial^2_{\alpha \beta} u_i) \|_{L^{4/3}(\mathbb{R}^3)} ,
(5.4) & \quad \| \chi_1 \tilde{Z}^a (G_{i}^{\alpha \beta} (u, \partial_u) \partial^2_{\alpha \beta} u_i) \|_{L^{6/5}(\mathbb{R}^3)} , \quad \| \chi_2 \tilde{Z}^a (G_{i}^{\alpha \beta} (u, \partial_u) \partial^2_{\alpha \beta} u_i) \|_{L^{4/3}(\mathbb{R}^3)} ,
(5.5) & \quad \| \chi_1 \tilde{Z}^a H_i (u, \partial_u) \|_{L^{6/5}(\mathbb{R}^3)} , \quad \| \chi_2 \tilde{Z}^a H_i (u, \partial_u) \|_{L^{4/3}(\mathbb{R}^3)} .
\end{align*}

\textbf{Estimate of (5.2).} We need to handle the \( L^{6/5} \) norm for \( t < 3 \) and \( t > 3 \), separately. It is easy to get for \( |a| \leq 2 \)

\begin{align*}
(5.6) & \quad \| \chi_1 \tilde{Z}^a F_{i}^{j, \alpha \beta \gamma} (\partial_i u_j) (\partial^2_{\alpha \beta} u_i) \|_{L^{6/5}(\mathbb{R}^3)} \\
& \leq C N_3(u(t))(M_1(u(t)) + N_4(u(t))) , \quad 0 < t < 3.
\end{align*}

On the other hand, for \( t > 3 \), we need to handle \( (t + |x|)^{-1} P_i \) \( (i = 1, 2) \), as in (1.8). When \( |a''| = 0 \) (and hence \( |a'| \leq 2 \)) or \( |a''| = 1 \) (and hence \( |a'| \leq 1 \)), we get owing to (1.10) and (2.17)

\begin{align*}
(5.7) & \quad \| \chi_1 (t + |x|)^{-1} P_i \|_{L^{6/5}(\mathbb{R}^3)} \\
& \leq C \langle t \rangle^{-2} \left( \sum_{|\alpha| + |\beta| + |\gamma| \leq 1} \| \Omega^b L^c S^d \tilde{Z}^{a' \gamma} u(t) \|_{L^2(\mathbb{R}^3)} \right) \| t - r | \partial (\partial \tilde{Z}^{a''} u(t)) \|_{L^3(\mathbb{R}^3)} \\
& \leq C \langle t \rangle^{-2} M_3(u(t)) N_4(u(t)) , \quad 3 < t < T.
\end{align*}

When \( |a''| = 2 \) (and hence \( |a'| = 0 \)), we get in a similar way

\begin{align*}
(5.8) & \quad \| \chi_1 (t + |x|)^{-1} P_i \|_{L^{6/5}(\mathbb{R}^3)} \\
& \leq C \langle t \rangle^{-2} \left( \sum_{|\alpha| + |\beta| + |\gamma| \leq 1} \| \Omega^b L^c S^d u(t) \|_{L^3(\mathbb{R}^3)} \right) \| t - r | \partial (\partial \tilde{Z}^{a''} u(t)) \|_{L^2(\mathbb{R}^3)}
\end{align*}
\[ \leq C(t)^{-2} M_3(u(t)) N_4(u(t)), \quad 3 < t < T. \]

As for \( P_2 \), we get for \(|a''| = 0\) (hence \(|a'| \leq 2\)) or \(|a''| = 1\) (hence \(|a'| \leq 1\))

\begin{equation}
\| \chi_1(t + |x|)^{-1} P_2 \|_{L^3(\mathbb{R}^3)} \leq C(t)^{-2} \| t - r |\partial Z' u(t) \|_{L^3(\mathbb{R}^3)} \left\langle \sum_{\{b+c\} + d = 1} \| \Omega^b L^c S^d \partial Z'' u(t) \|_{L^3(\mathbb{R}^3)} \right\rangle \end{equation}

\[ \leq C(t)^{-2} M_3(u(t)) N_4(u(t)), \quad 3 < t < T. \]

When \(|a''| = 2\) (hence \(|a'| = 0\)) we obtain

\begin{equation}
\| \chi_1(t + |x|)^{-1} P_2 \|_{L^3(\mathbb{R}^3)} \leq C(t)^{-2} \| t - r |\partial u(t) \|_{L^3(\mathbb{R}^3)} \left\langle \sum_{\{b+c\} + d = 1} \| \Omega^b L^c S^d \partial Z'' u(t) \|_{L^3(\mathbb{R}^3)} \right\rangle \end{equation}

\[ \leq C(t)^{-2} M_3(u(t)) N_4(u(t)), \quad 3 < t < T. \]

Let us turn our attention to the estimate of \( \chi_2(t + |x|)^{-1} P_i \) \((i = 1, 2)\) for \( t > 0 \).
When \(|a''| = 0\) (hence \(|a'| \leq 2\)) or \(|a''| = 1\) (hence \(|a'| \leq 1\)) we get by using \((3.7)\), \((3.8)\), and \((3.9)\)

\begin{equation}
\| \chi_2(t + |x|)^{-1} P_1 \|_{L^3(\mathbb{R}^3)} \leq C(t)^{-1} \left( \sum_{\{b+c\} + d = 1} \| \Omega^b L^c S^d \partial Z' u(t) \|_{L^3(\mathbb{R}^3)} \right) \| \partial^2 \partial Z'' u(t) \|_{L^3(\mathbb{R}^3)} \end{equation}

\[ \leq C(t)^{-1} M_3(u(t)) \left( \sum_{|a| \leq 2} \| \partial Z' u(t) \|_{L^3(\mathbb{R}^3)} + \| u(t) \|_{L^3(\mathbb{R}^3)} \right) \]

\[ \leq C(t)^{-1} M_3(u(t)) \left( N_4(u(t)) + X_1(u(t)) \right). \]

When \(|a''| = 2\) (and hence \(|a'| = 0\)) we get \((3.10)\)

\begin{equation}
\| \chi_2(t + |x|)^{-1} P_1 \|_{L^3(\mathbb{R}^3)} \leq C(t)^{-1} \left( \sum_{\{b+c\} + d = 1} \| \Omega^b L^c S^d u(t) \|_{L^3(\mathbb{R}^3)} \right) \| \partial^2 Z'' u(t) \|_{L^3(\mathbb{R}^3)} \end{equation}

\[ \leq C(t)^{-1} M_2(u(t)) \left( N_4(u(t)) + \| u(t) \|_{L^3(\mathbb{R}^3)} \right) \]

\[ \leq C(t)^{-1} M_2(u(t)) \left( N_4(u(t)) + X_0(u(t)) \right). \]

As for \( P_2\), we get

\begin{equation}
\| \chi_2(t + |x|)^{-1} P_2 \|_{L^3(\mathbb{R}^3)} \leq C(t)^{-1} \| \partial Z' u(t) \|_{L^3(\mathbb{R}^3)} \sum_{\{b+c\} + d = 1} \| \Omega^b L^c S^d \partial Z'' u(t) \|_{L^3(\mathbb{R}^3)} \end{equation}
\[ \leq C(t)^{-1}N_3(u(t))N_4(u(t)). \]

**Estimate of (5.3).** As in the estimate of (5.2), we need to handle the $L^{6/5}$ norm for $t < 3$ and $t > 3$, separately. It is easy to get for $|a| \leq 2$

\[ \| \chi_1 Z^a F_i^{j,k,\alpha\beta}(\partial_a u_j)(\partial_{\beta} u_k) \|_{L^{6/5}(\mathbb{R}^3)} \leq CN_2(u(t))N_3(u(t)), \quad 0 < t < 3 \]

On the other hand, for $t > 3$ we need to bound $(t + |x|)^{-1} P_i (i = 3, 4)$. By symmetry, it suffices to treat only $(t + |x|)^{-1} P_3$. See (4.16).

When $|a''| = 0$ (and hence $|a'| \leq 2$) or $|a''| = 1$ (and hence $|a'| \leq 1$), we get by (2.17)

\[ \| \chi_1(t + |x|)^{-1} P_3 \|_{L^{6/5}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} \left( \sum_{|b| + |c| + d = 1} \| \Omega^b L^c S^d \tilde{Z}^{a'} u(t) \|_{L^2(\mathbb{R}^3)} \right) \| t - r | \tilde{Z}^{a''} u(t) \|_{L^2(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} M_2(u(t)) M_3(u(t)), \quad 3 < t < T. \]

When $|a''| = 2$, we get

\[ \| \chi_1(t + |x|)^{-1} P_3 \|_{L^{6/5}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} \left( \sum_{|b| + |c| + d = 1} \| \Omega^b L^c S^d u(t) \|_{L^2(\mathbb{R}^3)} \right) \| t - r | \tilde{Z}^{a''} u(t) \|_{L^2(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} M_2(u(t)) M_3(u(t)), \quad 3 < t < T. \]

Moreover, we easily obtain for $t > 0$

\[ \| \chi_2(t + |x|)^{-1} P_3 \|_{L^1 L^{4/3}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-1} \left( \sum_{|b| + |c| + d = 1} \| \Omega^b L^c S^d \tilde{Z}^{a''} u(t) \|_{L^2(\mathbb{R}^3)} \right) \| \partial \tilde{Z}^{a''} u(t) \|_{L^2 L^2(\mathbb{R}^3)} \]

\[ \leq C(t)^{-1} M_3(u(t)) N_4(u(t)). \]

**Estimate of (5.4).** We obtain the following for $|a| \leq 2$:

\[ \| \chi_1 \tilde{Z}^a (G_i^{\alpha\beta}(u, \partial u) \partial^2_{\alpha\beta} u_i) \|_{L^{6/5}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} \left( M_3(u(t))^2 + N_3(u(t))^2 \right) \left( N_4(u(t)) + X_0(u(t)) \right). \]

\[ \| \chi_2 \tilde{Z}^a (G_i^{\alpha\beta}(u, \partial u) \partial^2_{\alpha\beta} u_i) \|_{L^1 L^{4/3}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-1} \left( N_3(u(t))^2 + X_2(u(t))^2 \right) \left( N_4(u(t)) + X_2(u(t)) \right). \]

For the proof of (5.18)–(5.19), it suffices to explain how to deal with such a typical term as $u_j(\tilde{Z}^a u_k) \partial^2 u_i$ ($|a| = 2$). Using (3.6)–(3.7) and (3.28), we get

\[ \| \chi_1 u_j(t)(\tilde{Z}^a u_k(t)) \partial^2 u_i(t) \|_{L^{6/5}(\mathbb{R}^3)} \]

\[ \leq C(t)^{-2} \left( \langle t \rangle - r \right) u_j(t) \|_{L^6(\mathbb{R}^3)} \left( \langle t \rangle - r \right) \tilde{Z}^a u_k(t) \|_{L^6(\mathbb{R}^3)} \| \partial^2 u_i(t) \|_{L^2(\mathbb{R}^3)} \]
\[ \leq C(t)^{-2} \left( M_3(u(t)) + N_3(u(t)) \right)^2 \left( N_2(u(t)) + X_0(u(t)) \right). \]

We also obtain by (2.23), (3.6)–(3.7)

\begin{align}
(5.21) & \quad \| \chi_2 u_j(t)(Z^n u_k(t)) \partial^2 u_i(t) \|_{L^1_t L^{4/3}_x(\mathbb{R}^3)} \\
& \leq C(t)^{-1} \| r u_j(t) \|_{L^\infty_t L^4_x(\mathbb{R}^3)} \| \tilde{Z}^a u_k(t) \|_{L^2_t L^2_x(\mathbb{R}^3)} \| \partial^2 u_i(t) \|_{L^2_t L^\infty_x(\mathbb{R}^3)} \\
& \leq C(t)^{-1} \left( N_1(u(t)) + X_1(u(t)) \right) \left( N_2(u(t)) + X_2(u(t)) \right). 
\end{align}

**Estimate of (5.5).** We can prove for \(|a| \leq 2\)

\begin{align}
(5.22) & \quad \| \chi_1 \tilde{Z}^a H_i(u, \partial u) \|_{L^6(\mathbb{R}^3)} \\
& \leq C(t)^{-2} \left( M_2(u(t)) + N_2(u(t)) \right)^2 \left( N_3(u(t)) + X_2(u(t)) \right), \\
(5.23) & \quad \| \chi_2 \tilde{Z}^a H_i(u, \partial u) \|_{L^1_t L^3_x(\mathbb{R}^3)} \leq C(t)^{-1} \left( N_3(u(t)) + X_2(u(t)) \right)^3.
\end{align}

The proof is similar to what we have done above. We may therefore omit it.

Obviously, the estimate (5.1) follows from what we have just obtained above. The proof of Proposition 5.1 has been finished.

### 6. Space-time \(L^2\) estimate

Recall the definition of \(L(v(t))\) (see (5.25)). The purpose of this section is to prove the following proposition.

**Proposition 6.1.** Suppose that initial data (1.2) is smooth and compactly supported, and suppose that the local solution \(u\) satisfies (3.6) in some interval \((0, T)\). Then the following inequality holds for all \(t \in (0, T)\):

\begin{align}
(6.1) & \quad (1 + t)^{-1/2} \int_0^t L(u_i(\tau))^2 d\tau \\
& \leq C \sum_{a + |i| + d \leq 3} \| (\partial \tilde{Z}^a L^c S^d u_i)(0) \|_{L^2(\mathbb{R}^3)}^2 \\
& \quad + C \int_0^t \langle \tau \rangle^{-1} \left( M_3(u(\tau)) + N_4(u(\tau)) \right) L(u(\tau))^2 d\tau \\
& \quad + C \int_0^t \langle \tau \rangle^{-3/2} \left( M_3(u(\tau)) + N_4(u(\tau)) \right) N_4(u(\tau))^2 d\tau \\
& \quad + C \int_0^t \langle \tau \rangle^{-1+\eta} \left( M_3(u(\tau)) + N_4(u(\tau)) \right) N_4(u(\tau)) G(u(\tau)) d\tau \\
& \quad + C \int_0^t \langle \tau \rangle^{-1} \left( M_3(u(\tau))^2 + N_4(u(\tau))^2 \right) L(u(\tau))^2 d\tau \\
& \quad + C \int_0^t \langle \tau \rangle^{-2} \left( M_3(u(\tau))^3 + N_4(u(\tau))^3 \right) N_4(u(\tau)) d\tau
\end{align}
Next, we consider $L^1$ norm $|\tau|^{-1}$.

We start with $6.2$. We thus see that it is bounded from above by the sum of the third, fourth, ..., and the eighth terms on the right-hand side of (3.27).

6.2. Estimate of $\int |x|^{-1/2} \, dx$.

Next, we consider $\int |x|^{-1/2} \, dx$, that is, $\|\partial Z^a S u_i|_{L^1(\mathbb{R}^3)}$. Actually, this norm has been already estimated in Section 3. See (3.14), (3.19)–(3.23), (3.38)–(3.54), and (3.59)–(3.70). We thus see that it is bounded from above by the sum of the third, fourth, ..., and the eighth terms on the right-hand side of (3.27).

6.3. Estimate of $\int |\partial h| \, du$.

Our next concern is to bound $\int |\partial h| \, du$, that is, $\|\partial h(\partial Z^a S u_i)|_{L^1(\mathbb{R}^3)}$, with $h = F^j_{\alpha \beta \gamma} \partial_i u_j + G^a_{\alpha \beta} (u, \partial u)$. Using (3.6) and (3.7), we easily obtain

\begin{align*}
(6.3) \quad & \|\partial h(t)(\partial Z^a S u_i(t))\|_{L^1(\mathbb{R}^3)} \\
& \leq C \sum_{|\beta| \leq 1} \|\partial Z^a S u_i(t)\|_{L^1(\mathbb{R}^3)} + C \|u(t)(\partial Z^a S u_i(t))\|_{L^1(\mathbb{R}^3)} \\
& \leq C(t)^{-1} \left( \sum_{|\beta| \leq 1} \langle t-r \rangle \partial Z^a S u_i(t) \right)_{L^\infty(\mathbb{R}^3)} + C(t)^{-1} \|ru(t)\|_{L^\infty(\mathbb{R}^3)} \\
& \leq C(t)^{-1} \left( N_4(u(t)) + M_1(u(t)) + N_2(u(t))^{1/2} X_2(u(t))^{1/2} \right) N_4(u(t))^2 \\
& \leq C(t)^{-1} \left( N_4(u(t)) + M_1(u(t)) + X_2(u(t)) \right) N_4(u(t))^2.
\end{align*}
6.4. Estimate of $\int |x|^{-1/2}(x)^{-1/2} |\partial h| |u\partial u| dx$.
We next consider $\int |x|^{-1/2}(x)^{-1/2} |\partial h| |u\partial u| dx$, which is $\|x^{-1/2}(x)^{-1/2}(\partial h)(Z^a S u_i)\partial Z^a S u_i\|_{L^1(\mathbb{R}^3)}$. We rely upon the Hardy inequality to get

$$\|x^{-1/2}(x)^{-1/2}(\partial h(t))(Z^a S u_i(t))\partial Z^a S u_i(t)\|_{L^1(\mathbb{R}^3)} \leq C\|\partial h(t)\|_{L^\infty(\mathbb{R}^3)}\|x^{-1}(Z^a S u_i(t))\|_{L^2(\mathbb{R}^3)}\|\partial Z^a S u_i(t)\|_{L^2(\mathbb{R}^3)} \leq C\|\partial h(t)\|_{L^\infty(\mathbb{R}^3)} N_4(u(t))^2,$$

which implies that we have only to repeat the same argument as in Subsection 6.3.

6.5. Estimate of $\int |x|^{-1/2}(x)^{-1/2}|h| |\partial u|^2 dx$.
Next, let us consider $\int |x|^{-1/2}(x)^{-1/2}|h| |\partial u|^2 dx$, that is, $\|x^{-1/2}(x)^{-1/2} h(\partial Z^a S u_i)^2\|_{L^1(\mathbb{R}^3)}$.
Recalling that $h = F_i^{\lambda \beta \gamma} \partial \gamma u_j + G_i^\beta(u, \partial u)$, we get by (3.30), (3.31), and (2.24)

$$\|x^{-1/2}(x)^{-1/2}|h(t)|(\partial Z^a S u_i(t))^2\|_{L^1(\mathbb{R}^3)} \leq C(t)^{-1}\|\partial h(t)\|_{L^\infty(\mathbb{R}^3)} + \|t^{-1}|u(t)\|_{L^\infty(\mathbb{R}^3)}$$

$$+ \|r\partial u(t)\|_{L^\infty(\mathbb{R}^3)} + \|r u(t)\|_{L^\infty(\mathbb{R}^3)}\|x^{-1/4} \partial Z^a S u_i(t)\|_{L^2(\mathbb{R}^3)}^2 \leq C(t)^{-1}(N_3(u(t)) + M_1(u(t)) + X_2(u(t))) (L(u(t))^2).$$

6.6. Estimate of $\int |x|^{-3/2}(x)^{-1/2}|h| |\partial u|^2 dx$.
Finally, we consider $\int |x|^{-3/2}(x)^{-1/2}|h| |\partial u|^2 dx$, that is,

$$\|x^{-3/2}(x)^{-1/2}|h|(Z^a S u_i(t))(\partial Z^a S u_i(t))\|_{L^1(\mathbb{R}^3)}.$$
Recalling the definition of $L(u(t))$ (see (3.25)) and using the norm $\|x^{-5/4} Z^a S u_i(t)\|_{L^2(\mathbb{R}^3)}$, we can bound it in the same way as in Subsection 6.5.

Now we are in a position to complete the proof of Proposition 6.1. Obviously, the estimate (6.1) is a direct consequence of what we have just obtained above. The proof has been finished.

7. Proof of Theorem 1.3

We are in a position to prove Theorem 1.3, firstly for smooth data with compact support. Our proof of global existence uses the method of continuity, and the important property (7.14) is easier to show when smooth data have compact support.

We use the notation

$$\mathcal{N}_T(w) := \sup_{0 < t < T} N_4(w(t)), \quad (7.1)$$
$$\mathcal{M}_T(w) := \sup_{0 < t < T} \langle t \rangle^{-\delta} M_3(w(t)), \quad (7.2)$$
$$\mathcal{X}_T(w) := \sup_{0 < t < T} X_2(w(t)), \quad (7.3)$$
$$\mathcal{G}_T(w) := \left( \int_0^T G(w(t))^2 dt \right)^{1/2}, \quad (7.4)$$
$$\mathcal{L}_T(w) := \sup_{0 < t < T} \langle t \rangle^{-(1/4) - \delta/2} \left( \int_0^T L(w(\tau))^2 d\tau \right)^{1/2}. \quad (7.5)$$
Here, $0 < \delta < 1/6$. The proof of global existence basically consists of two steps. We firstly show that the estimate $\max\{N_T(u), M_T(u), X_T(u)\} \leq 2C_0 D(f, g)$ implies $\sup\{\|u(t)\| : t \in (0, T^*)\} \leq \varepsilon_1^*$, and we secondly show that the latter implies the improved estimate $\max\{N_T(u), M_T(u), X_T(u)\} \leq \sqrt{3C_0} D(f, g)$. See (3.3) for $\|u(t)\|$, and see (1.17) for $D(f, g)$. See (7.6) and (125) for $T^*$ and $\varepsilon_1^*$. We will set

the constant $C_0$ below (see (7.13)). The last estimate, together with the standard local existence theorem, implies existence of global solutions.

We first note that, using the idea of decomposing the time interval $(1, T)$ dyadically as in Sogge [22, p. 363] (see also [7], (161) and (125)), we get

(7.6) \[ \int_0^T \langle \tau \rangle^{-1+2\delta} L(w(\tau))^2 d\tau \leq C\mathcal{L}_T(w)^2 \]

owing to $\delta < 1/6$,

(7.7) \[ \int_0^T \langle \tau \rangle^{-1+\eta+\delta} G(w(\tau)) d\tau \leq C\mathcal{G}_T(w) \]

owing to $\eta < 1/3$ (and hence $\eta + \delta < 1/2$). We also note that it follows from (2.21),

(2.24) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$ that

(7.8) \[ \sup_{t \in (0, T)} \|w(t)\| \leq C^*(N_T(w) + M_T(w) + X_T(w)) \]

for a constant $C^* > 0$ independent of $T$. Therefore, using (3.4), Propositions 3.2, 4.1, 5.1 and 6.1 together with (7.6)–(7.8) and the Young inequality, we see that, if the local solution defined for $(t, x) \in [0, T) \times \mathbb{R}^3$ satisfies

(7.9) \[ C^*(N_T(u) + M_T(u) + X_T(u)) \leq \varepsilon_1^*, \]

then we have

(7.10) \[ N_T(u)^2 + G_T(u)^2 + L_T(u)^2 \]

\[ \leq C_{11} D(f, g)^2 (1 + D(f, g)^4) \]

\[ + C(N_T(u)^3 + M_T(u)^3 + X_T(u)^3) + C(N_T(u)^4 + M_T(u)^4) \]

\[ + C_{12} (N_T(u) + M_T(u) + X_T(u)) L_T(u)^2 + C_{13} (N_T(u)^2 + M_T(u)^2) G_T(u) \]

\[ + C_{14} (N_T(u)^2 + M_T(u)^2) L_T(u)^2, \]

(7.11) \[ M_T(u) \leq C_{21} D(f, g) + C(N_T(u)^2 + M_T(u)^2) \]

\[ + C(N_T(u)^3 + M_T(u)^3 + X_T(u)^3) + C M_T(u) N_T(u)^3, \]

(7.12) \[ X_T(u) \leq C_{31} D(f, g) + C(N_T(u)^2 + M_T(u)^2 + X_T(u)^2) \]

\[ + C(N_T(u)^3 + M_T(u)^3 + X_T(u)^3). \]

We remark that in (7.11), the equality $r \partial_r = \Lambda + \omega_j \Omega_j$ has been used. We set

(7.13) \[ C_0 := \max\{C_d, \sqrt{C_{11}}, C_{21}, C_{31}\}. \]
(See (3.4) for \(C_d\).)

Recall that due to the size condition (3.1) (see also (3.3)), we enjoy a unique solution at least for a short time interval, say, \([0, T_\ast]\). By virtue of the finite speed of propagation, it is easy to observe the important property that this local solution satisfies

\[
N_4(u(t)), \quad M_3(u(t)), \quad X_2(u(t)) \in C([0, T_\ast]),
\]

which implies that

\[
\max\{N_4(u(t)), (1 + t)^{-\delta}M_3(u(t)), X_2(u(t))\} \leq 2C_dD(f, g)
\]

at least for a short time interval, say, \([0, \hat{T}]\) with \(\hat{T} \leq T_\ast\). See (3.4). For the given data \((f_i, g_i) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3) (i = 1, \ldots, N)\) satisfying (3.3), we therefore have the non-empty set \(\{T > 0 : \) There exists a unique smooth solution \(u(t, x)\) to (1.1) defined for all \((t, x) \in [0, T) \times \mathbb{R}^3\) satisfying

\[
\max\{N_4(u(t)), (1 + t)^{-\delta}M_3(u(t)), X_2(u(t))\} \leq 2C_0D(f, g)
\]

for all \(t \in [0, T]\). Define \(T^* \in (0, \infty]\) as the supremum of this non-empty set. (Readers are advised not to confuse it with \(T_\ast\).)

Let \(T'\) be an arbitrary number such that \(T' < T^*\). Since \(T'\) is finite and \(u\) is a smooth solution defined for all \((t, x) \in [0, T^*) \times \mathbb{R}^3\) with \(\text{supp} u(t, \cdot) \subset \{x \in \mathbb{R}^3 : |x| < t + R\}\) for some \(R > 0\), we can easily verify \(\mathcal{L}_{T'}(u) < \infty\) by using the Hardy-type inequality. It is also easy to verify \(\mathcal{G}_{T'}(u) < \infty\). We are in a position to show that the inequality (7.16) holds for \(T = T'\) and the last three terms on its right-hand side can be absorbed into its left-hand side. (These terms are allowed to move to the left-hand side, thanks to \(\mathcal{L}_{T'}(u), \mathcal{G}_{T'}(u) < \infty\).) We start with the inequality

\[
\max\{N_{T'}(u), M_{T'}(u), X_{T'}(u)\} \leq 2C_0D(f, g),
\]

which holds owing to (7.16) and the definition of \(T^*\). This inequality (7.17), combined with the size condition (3.3), implies (7.9) with \(T = T'\). Therefore, we see by (7.8) that the inequality (3.6) is true for all \(t \in [0, T']\), which among others implies that the inequality (7.10) holds for \(T = T'\). Due to the size condition (3.3), it is possible to carry out the absorption step indicated above, which yields \(N_T(u)^2 \leq 2C_{11}D(f, g)^2 + C_{15}D(f, g)^3\) for a suitable constant \(C_{15} > 0\). Here, we have used the assumption \(D(f, g) \leq 1\) and the inequality (7.17). Just to be sure, we note that the second last term on the right-hand side of (7.10) with \(T = T'\) has been handled as

\[
C_{13}(N_{T'}(u)^2 + M_{T'}(u)^2)\mathcal{G}_{T'}(u)
\]

\[
\leq 2C_{13}(2C_0D(f, g))^2\mathcal{G}_{T'}(u) \leq C_{13}^2(4C_0^2D(f, g)^3/2)^2 + D(f, g)\mathcal{G}_{T'}(u)^2
\]

and the last term above has moved to the left-hand side of (7.10) with \(T = T'\). Since \(T' < T^*\) is arbitrary, we finally get

\[
N_{T'}(u)^2 \leq 3C_{11}D(f, g)^2
\]

due to the condition (3.3).
As for the estimate of $\mathcal{M}_{T^*}(u)$ and $\mathcal{X}_{T^*}(u)$, we may start with the inequality
\begin{equation}
\max\{N_{T^*}(u), \mathcal{M}_{T^*}(u), \mathcal{X}_{T^*}(u)\} \leq 2C_0D(f, g),
\end{equation}
which is a direct consequence of (7.10) and the definition of $T^*$, and combine it with the size condition (3.3) to see that the condition (7.9) holds for $T = T^*$. It therefore follows directly from (7.11), (7.12) with $T = T^*$ and (7.19) that
\begin{align}
\mathcal{M}_{T^*}(u) &\leq C_{21}D(f, g) + C_{22}D(f, g)^2, \\
\mathcal{X}_{T^*}(u) &\leq C_{31}D(f, g) + C_{32}D(f, g)^2
\end{align}
for suitable constants $C_{22}, C_{32} > 0$. Here, we have naturally used the assumption $D(f, g) \leq 1$. Using the size condition (3.3), we finally obtain
\begin{equation}
\mathcal{M}_{T^*} \leq \frac{3}{2}C_{21}D(f, g), \quad \mathcal{X}_{T^*} \leq \frac{3}{2}C_{31}D(f, g).
\end{equation}
In sum, we have obtained
\begin{equation}
\max\{N_{T^*}(u), \mathcal{M}_{T^*}(u), \mathcal{X}_{T^*}(u)\} \leq \max\{\sqrt{3}C_{11}, \frac{3}{2}C_{21}, \frac{3}{2}C_{31}\}D(f, g)
\leq \sqrt{3}C_0D(f, g).
\end{equation}

If we suppose $T^* < \infty$, then it is possible to show that there exists $\bar{T} > T^*$ such that the unique local solution $u(t, x)$ to (1.1)–(1.2) exists for all $(t, x) \in [0, \bar{T}) \times \mathbb{R}^3$ satisfying (7.16) for all $t \in [0, \bar{T})$. This contradicts the definition of $T^*$. We therefore have $T^* = \infty$, which proves Theorem 1.3 at least for smooth, compactly supported data.

It remains to relax the regularity of data and eliminate compactness of the support of data. We naturally rely upon the standard idea of using the mollifier and the cut-off technique. It is then possible to show that, for any data $(f, g)$ satisfying $D(f, g) \leq \varepsilon_0/2$ (see (3.3) for $\varepsilon_0$), there exists a sequence of pairs of compactly supported smooth $\mathbb{R}^N$-valued functions $(f^{(n)}, g^{(n)})$ ($n = 1, 2, \ldots$) such that $D(f^{(n)}, g^{(n)}) \leq \varepsilon_0$ for sufficiently large $n$, and $D(f - f^{(n)}, g - g^{(n)}) \to 0$ as $n \to \infty$. It then follows from the above argument that, for sufficiently large $n$, we enjoy the unique solution $u^{(n)}$ to (1.1) with data $(f^{(n)}, g^{(n)})$ given at $t = 0$ which satisfies the estimate (7.23) with $T^* = \infty$. Moreover, repeating the arguments that we have done in the previous sections, we obtain for $u_{m,n} := u^{(m)} - u^{(n)}$
\begin{equation}
\max\left\{\sup_{t>0} N_2(u_{m,n}(t)), \sup_{t>0} (t)^{-\delta} M_1(u_{m,n}(t)), \sup_{t>0} X_0(u_{m,n}(t)), \right\}
\leq \left|\frac{3}{2}C_{21} + \frac{3}{2}C_{31}\right|D(f, g)
\leq \sqrt{3}C_0D(f, g).
\end{equation}
\[
\sum_{\substack{|a|+|c|+d\leq 1 \\ T>0}} \sup_{T>0} (1 + T)^{-(1/4)-\delta/2} \| |x|^{-1/4} \partial \bar{Z}^a L^c S^d u_{m,n} \|_{L^2((0,T)\times \mathbb{R}^3)} \}
\leq CD(f^{(m)} - f^{(n)}, g^{(m)} - g^{(n)}).
\]

(Here we are supposed to choose \(\varepsilon_0\) smaller than before, if necessary.) By the standard argument, we see that the sequence \(\{u^{(n)}\}\) has the limit and it is the solution to (1.1) that we have sought for. The proof of Theorem 1.3 has been finished.

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**References**

1. Alinhac, S., Hyperbolic partial differential equations, Universitext, Springer, Dordrecht, 2009.
2. Alinhac, S., Geometric analysis of hyperbolic differential equations: an introduction, London Mathematical Society Lecture Note Series, 374. Cambridge University Press, Cambridge, 2010.
3. Christodoulou, D., Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.*, 39 (2), 1986, 267–282.
4. Hidano, K., Regularity and lifespan of small solutions to systems of quasi-linear wave equations with multiple speeds, I: almost global existence, RIMS Kôkyûroku Bessatsu B65: Harmonic Analysis and Nonlinear Partial Differential Equations (eds. Hideo Kubo and Hideo Takaoka) , Res. Inst. Math. Sci. (RIMS), Kyoto, 2017, 37–61.
5. Hidano, K., Wang, C., and Yokoyama, K., On almost global existence and local well posedness for some 3-D quasi-linear wave equations, *Adv. Differential Equations*, 17 (3-4), 2012, 267–306.
6. Hidano, K. and Yokoyama, K., Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition, *Int. Math. Res. Not. IMRN*, 2020 (1), 39–70.
7. Hidano, K. and Zha, D., Remarks on a system of quasi-linear wave equations in 3D satisfying the weak null condition, *Commun. Pure Appl. Anal.*, 18 (4), 2019, 1735–1767.
8. Hörmander, L., Lectures on nonlinear hyperbolic differential equations, Mathématiques & Applications, 26. Springer-Verlag, Berlin, 1997.
9. John, F., Blow-up for quasilinear wave equations in three space dimensions, *Comm. Pure Appl. Math.*, 34 (1), 1981, 29–51.
10. John, F., Nonlinear wave equations, formation of singularities. Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989, University Lecture Series, 2, American Mathematical Society, Providence, RI, 1990.
11. Klainerman, S., Weighted \(L^\infty-L^1\) estimates for solutions to the classical wave equation in three space dimensions, *Comm. Pure Appl. Math.*, 37 (2), 1984, 269–288.
12. Klainerman, S., The null condition and global existence to nonlinear wave equations, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986, 293–326.
13. Klainerman, S., Remarks on the global Sobolev inequalities in the Minkowski space \(\mathbb{R}^{n+1}\), *Comm. Pure Appl. Math.*, 40 (1), 1987, 111–117.
14. Klainerman, S. and Sideris, T. C., On almost global existence for nonrelativistic wave equations in 3D, *Comm. Pure Appl. Math.*, 49 (3), 1996, 307–321.
15. Li, T. T. and Yu, X., Life-span of classical solutions to fully nonlinear wave equations, *Comm. Partial Differential Equations*, 16 (6-7), 1991, 909–940.

16. Lindblad, H., Nakamura, M., and Sogge, C. D., Remarks on global solutions for nonlinear wave equations under the standard null conditions, *J. Differential Equations*, 254 (3), 2013, 1396–1436.

17. Lindblad, H. and Rodnianski, I., Global existence for the Einstein vacuum equations in wave coordinates, *Comm. Math. Phys.*, 256 (1), 2005, 43–110.

18. Metcalfe, J., Nakamura, M., and Sogge, C. D., Global existence of quasilinear, nonrelativistic wave equations satisfying the null condition, *Japan. J. Math. (N.S.)*, 31 (2), 2005, 391–472.

19. Sideris, T. C., Global behavior of solutions to nonlinear wave equations in three dimensions, *Comm. Partial Differential Equations*, 8 (12), 1983, 1291–1323.

20. Sideris, T. C., Nonresonance and global existence of prestressed nonlinear elastic waves, *Ann. of Math.* (2), 151 (2), 2000, 849–874.

21. Sideris, T. C. and Tu, S. -Y., Global existence for systems of nonlinear wave equations in 3D with multiple speeds, *SIAM J. Math. Anal.*, 33 (2), 2001, 477–488.

22. Sogge, C. D., Global existence for nonlinear wave equations with multiple speeds, Harmonic Analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math., 320, Amer.Math. Soc., Providence, RI, 2003, 353–366.

23. Sterbenz, J., Angular regularity and Strichartz estimates for the wave equation. With an appendix by Igor Rodnianski, *Int. Math. Res. Not.*, 2005 (4), 187–231.

24. Zha, D., Some remarks on quasilinear wave equations with null condition in 3-D, *Math. Methods Appl. Sci.*, 39 (15), 2016, 4484–4495.

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