ITERATION OF POLYNOMIAL PAIR UNDER THUE-MORSE DYNAMIC

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ABSTRACT. We study the behavior of a polynomial sequence which is defined by iterating a polynomial pair under Thue-Morse dynamic. We show that in suitable sense, the sequence will behave like \( \{2 \cos 2^n x : n \geq 1\} \). Basing on this property we can show that the Hausdorff dimension of the spectrum of the Thue-Morse Hamiltonian has a common positive lower bound for all coupling.

1. Introduction

The trace polynomials related to Thue-Morse sequence has been studied since 1980s. See especially the early works \( \[1, 2, 3\] \). Let us recall the definitions. Consider the Thue-Morse substitution

\[
\begin{align*}
\sigma(a) &= ab \\
\sigma(b) &= ba.
\end{align*}
\]

We denote the free group generated by \( a, b \) as \( \text{FG}(a, b) \). Given \( \lambda, x \in \mathbb{R} \), we can define a homomorphism \( \tau : \text{FG}(a, b) \to \text{SL}(2, \mathbb{R}) \) as

\[
\tau(a) = \begin{bmatrix} x - \lambda & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tau(b) = \begin{bmatrix} x + \lambda & -1 \\ 1 & 0 \end{bmatrix}
\]

and \( \tau(a_1 \cdots a_n) = \tau(a_n) \cdots \tau(a_1) \). Define \( h_n(x) := \text{tr}(\tau(\sigma^n(a))) \) (where \( \text{tr}(A) \) denotes the trace of the matrix \( A \) ), by a direct computation we have

\[
\begin{align*}
h_1(x) &= x^2 - \lambda^2 - 2; \\
h_2(x) &= (x^2 - \lambda^2)^2 - 4x^2 + 2; \\
h_{n+1}(x) &= h_{n-1}(x)(h_n(x) - 2) + 2 \quad (n \geq 2).
\end{align*}
\]

\( \{h_n : n \geq 1\} \) is called the sequence of trace polynomials related to Thue-Morse sequence.

\( \[1\] \) motivates the following definition. Define \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) as

\[
\Phi(x, y) = (y^2(x - 2) + 2, x).
\]

Then the recurrence relation \( \[1\] \) is equivalent to \( (h_{n+1}, h_n) = \Phi(h_n, h_{n-1}) \).

We thus call \( \Phi \) the Thue-Morse dynamic.
In general, starting from a polynomial pair \((P_{-1}, P_0)\), we can define
\[(P_n, P_{n-1}) := \Phi^n(P_0, P_{-1}).\]

The main goal of this paper is to understand the behavior of the polynomial sequence \(\{P_n : n \geq -1\}\). Let us start with one simple situation.

1.1. A special sequence.

We take \(\lambda = 0\) and consider the sequence \(\{h_n : n \geq 1\}\). In this case
\[h_1(x) = x^2 - 2\]
and
\[h_2(x) = x^4 - 4x^2 + 2 = h_1^2(x) - 2 = h_1 \circ h_1(x).\]

By induction it is easy to show that
\[h_n(x) = h_1 \circ \cdots \circ h_1(x).\]

Thus in this special case, the iterations of \((h_1, h_2)\) are quite clear.

Observe that if \(\{P_n : n \geq -1\}\) is defined by the Thue-Morse dynamic, then for any change of variable \(x = \varphi(y)\), the sequence \(\{Q_n = P_n \circ \varphi : n \geq -1\}\) still satisfies the Thue-Morse dynamic. Of course, now \(Q_n\) need not to be a polynomial. If we do the change of variable \(x = 2 \cos y\), then by a simple computation we get
\[g_n(y) := h_n(2 \cos y) = 2 \cos(2^n y).\]

Thus we conclude that \(\{2 \cos 2^n x : n \geq 1\}\) satisfies the Thue-Morse dynamic.

1.2. General pictures.

A natural question is that what does \(\{h_n : n \geq 1\}\) look like when \(\lambda \neq 0\)? More generally, starting from any polynomial pair \((P_{-1}, P_0)\), and defining \(P_n\) according to Thue-Morse dynamic, what is the behavior of \(P_n\)?

In this paper we will answer this question partly. Roughly speaking, let \(Z := \{x : P_k(x) = 0 \text{ for some } k\}\), then we will show that under some mild condition, around each \(x \in Z\), after suitable renormalization, the sequence \(\{P_n : n \geq -1\}\) will behave like \(\{2 \cos 2^n x : n \geq 1\}\). We will make this precise in Section 2.

1.3. Application to Thue-Morse Hamiltonian.

This general result can be applied to the spectral problem of Thue-Morse Hamiltonian and give a uniform lower bound for the Hausdorff dimension of the spectrum.
Let us recall the definition of discrete Schrödinger operator. Given a bounded real sequence \( v = \{v(n)\}_{n \in \mathbb{Z}} \) and \( \lambda \in \mathbb{R} \), we can define an operator \( H_{\lambda,v} \) act on \( l^2(\mathbb{Z}) \) as
\[
(H_{\lambda,v} \psi)(n) = \psi(n + 1) + \psi(n - 1) + \lambda v(n) \psi(n), \quad \forall n \in \mathbb{Z}.
\]
\( H_{\lambda,v} \) is called an discrete Schrödinger operator with potential \( \lambda v \); \( \lambda \) is called the coupling constant. We denote the spectrum of \( H_{\lambda,v} \) by \( \sigma(H_{\lambda,v}) \).

We define the two-sided Thue-Morse sequence \( v \) as follows: Let \( \sigma \) be the Thue-Morse substitution, let \( u = u_1 u_2 \cdots := \sigma^\infty(a) \). For \( n \geq 1 \), let \( v(n) = 1 \) if \( u_n = a \); let \( v(n) = -1 \) if \( u_n = b \); let \( v(1-n) = v(n) \) for \( n \geq 1 \). The operator \( H_{\lambda,v} \) with Thue-Morse sequence \( v \) is called Thue-Morse Hamiltonian. We will prove the following theorem.

**Theorem 1.1.** There exists an absolute constant \( C > 0 \) such that for Thue-Morse sequence \( v \) and any \( \lambda \in \mathbb{R} \),
\[
\dim_H \sigma(H_{\lambda,v}) \geq C.
\]

**Remark 1.2.** Axel and Peyrière ([1, 2]) study the spectrum, and prove numerically that its Lesbesgue measure is zero, and the Box dimension is strictly less than 1. Then it is rigorously proven that the spectrum is a Cantor set of Lebesgue measure zero (see for example [3, 4, 11, 12]). By our best knowledge, no rigorous results about the Hausdorff dimension of the spectrum has been proven before.

**Remark 1.3.** By our result, the dimension property of Thue-Morse Hamiltonian is quite different from another heavily studied model – the Fibonacci Hamiltonian. Recall that the Fibonacci sequence \( w \) is defined by
\[
w(n) = \chi_{[1-\alpha,1)}(n\alpha \mod 1), \quad \forall n \in \mathbb{Z}
\]
with \( \alpha = (\sqrt{5} + 1)/2 \). The Fibonacci Hamiltonian \( H_{\lambda,w} \) is a central model in discrete Schrödinger operator. Its dimensional properties has been extensively studied, see for example [14, 10, 13, 6, 5, 7, 8]. In particular the following property is shown in [3]:
\[
\lim_{|\lambda| \to \infty} \dim_H \sigma(H_{\lambda,w}) \ln |\lambda| = \ln(1 + \sqrt{2}).
\]
This implies that \( \dim_H \sigma(H_{\lambda,w}) \to 0 \) with the speed \( 1/\ln |\lambda| \) when \( |\lambda| \to \infty \). However by our result, the dimension of spectrum for Thue-Morse potential (i.e., \( \sigma(H_{\lambda,v}) \)) has a uniform positive lower bound.
Remark 1.4. The rough idea of proving Theorem 1.1 is the following. Recall that \( \{h_n : n \geq 1\} \) is the trace polynomial sequence related to the Thue-Morse sequence. Define
\[
\Sigma = \{ x \in \mathbb{R} : \exists n > 0, h_n(x) = 0 \}.
\]
It is shown in [2, 3] that \( \Sigma \subset \sigma(H_{\lambda,v}) \). Basing on the general picture described in Section 1.2 it is possible to construct a Cantor subset \( \mathcal{C} \) of \( \sigma(H_{\lambda,v}) \) in a controllable fashion, then we can estimate the Hausdorff dimension of \( \mathcal{C} \), which in turn offer a lower bound for the dimension of the spectrum.

The rest of the paper is organized as follows. In Section 2 we show that the polynomial sequence will behave like \( \{2 \cos 2^n x : n \geq 1\} \) near a base point of a germ. In Section 3 we prepare the proof of the lower bound of the spectrum. In Section 4 we prove Theorem 1.1.

2. Convergence towards \( \{2 \cos 2^n x : n \geq 1\} \)

Given polynomial pair \((f_{-1}, f_0)\). We recall that defining \( \{f_n : n \geq 1\} \) according to \( (f_n, f_{n-1}) := \Phi^n(f_0, f_{-1}) \) for \( n \geq 0 \) is equivalent to define it according to the recurrence relation \( f_{n+1} = f_{n-1}^2(f_n - 2) + 2 \).

2.1. Germ of a polynomial pair.

Given polynomial pair \((f_{-1}, f_0)\). Define \( f_{n+1} = f_{n-1}^2(f_n - 2) + 2 \) for \( n \geq 0 \). Assume \( f_0(x_0) = 0 \), at first we study the local behavior of \( f_n \) at \( x_0 \). Write
\[
f_0(x) = f'(x_0)(x - x_0) + O((x - x_0)^2) \quad \text{and} \quad f_1(x) = f_1(x_0) + O((x - x_0)).
\]
By the recurrence relation we have
\[
f_2(x) = 2 - (2 - f_1(x_0)) f_0(x_0)(x - x_0)^2 + O((x - x_0)^3)
\]
\[
f_k(x) = 2 - 4^{k-3} (2 - f_1(x_0)) (f_0'(x_0) f_1(x_0))^2 (x - x_0)^2 + O((x - x_0)^3) \quad (k \geq 3).
\]
If moreover \( f_1(x_0) < 2 \) and \( f_0'(x_0), f_1(x_0) \neq 0 \), then
\[
\rho := \sqrt{2 - f_1(x_0)}|f_0'(x_0) f_1(x_0)| > 0
\]
and for \( k \geq 3 \)
\[
f_k(x) = 2 - 4^{k-3} \rho^2 (x - x_0)^2 + O((x - x_0)^3).
\]
(2)
If we define \( \tilde{f}_k(x) = f_{k+3}(x/\rho + x_0) \), then for \( k \geq 0 \)
\[
\tilde{f}_k(x) = 2 - (2^k x)^2 + O(x^3).
\]

Notice that we also have

\[ 2 \cos 2^k x = 2 - (2^k x)^2 + O(x^3). \]

Thus \( \{ \tilde{f}_k(x) : k \geq 1 \} \) is a good candidate of polynomial sequence which converge to \( \{ 2 \cos 2^k x : k \geq 1 \} \). This also motivates the following definition.

Given a polynomial pair \((P_{-1}, P_0)\). Assume at \( x_0 \in \mathbb{R} \), there exists \( \rho > 0 \) such that

\[
\begin{align*}
P_{-1}(x) &= 2 - \rho^2 (x - x_0)^2 + O((x - x_0)^3); \\
P_0(x) &= 2 - \rho^2 (x - x_0)^2 + O((x - x_0)^3)
\end{align*}
\]

Then we say \((P_{-1}, P_0)\) has a \( \rho \)-germ at \( x_0 \). \( x_0 \) is called the base point of the germ.

Assume \((P_{-1}, P_0)\) has a \( \rho \)-germ at \( x_0 \). For \( k \geq 1 \), define

\[ P_k = P_{k-2}(P_{k-1} - 2) + 2. \] (3)

For \( k \geq -1 \) define

\[ Q_k(x) = P_k\left(\frac{x}{2^k \rho} + x_0\right). \] (4)

It is ready to show that \( Q_k(x) = 2 - x^2 + O(x^3) \). Since \( 2 \cos x = 2 - x^2 + O(x^3) \), we conclude that \( Q_k(x) = 2 \cos x + O(x^3) \). Write \( \Delta_k(x) = Q_k(x) - 2 \cos x \), then

\[ \Delta_k(x) = Q_k(x) - 2 \cos x = \sum_{k \geq 3} \Delta_{k,n} x^n. \] (5)

### 2.2. Regular germ of polynomial pair.

Our goal is to show that \( \Delta_k(x) \to 0 \) for \( x \) in any bounded interval, to achieve this we need to impose some condition on the initial pair \((P_{-1}, P_0)\), or equivalently on \((Q_{-1}, Q_0)\). Our condition is about the coefficients of \( \Delta_{-1} \) and \( \Delta_0 \). Let us do some preparation.

Given two formal series with real coefficients

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n. \]

If we only concern about their coefficients, we can define the following partial order:

\[ f \preceq g \iff a_n \leq b_n \quad (\forall n \geq 0). \]

We further define \( |f(x)|^* := \sum_{n=0}^{\infty} |a_n| x^n \). Then it is easy to check that

\[ |fg|^* \leq |f|^* |g|^* \quad \text{and} \quad |f + g|^* \leq |f|^* + |g|^*. \]
Moreover if $|f| \leq \tilde{f}$ and $|g| \leq \tilde{g}$, then it is seen that $|fg| \leq \tilde{f}\tilde{g}$. Later we will use these properties repeatedly to estimate the coefficients of certain series.

Let us go back to $\{P_k : k \geq -1\}$ discussed above. If moreover there exist $\delta > 0$ and $\beta \geq 1$ such that

$$|\Delta_{-1}|^* \leq \delta \sum_{n=3}^{\infty} \frac{x^n}{\beta^n}$$

Then we say that $(P_{-1}, P_0)$ has a $(\delta, \beta)$-regular $\rho$-germ at $x_0$. We also say that $(P_{-1}, P_0)$ is $(\delta, \beta)$-regular at $x_0$ with renormalization factor $\rho$, or simply as $(\delta, \beta)$-regular at $x_0$.

With this definition, now we can state our main convergence theorem.

**Theorem 2.1.** Assume $(P_{-1}, P_0)$ has a $(1,1)$-regular $\rho$-germ at $x_0$. Define $(P_k)_{k \geq -1}$ and $\{Q_k\}_{k \geq -1}$ according to (3) and (4) respectively. Let $\Delta_k(x) = Q_k(x) - 2 \cos x$. Then for any $m \geq 0$, there exists an absolute constant $C_m > 0$ such that for any $k \geq 2m + 1$ and any $x \in [-2^{m-1}\pi, 2^{m-1}\pi]$,

$$|\Delta_k(x)| \leq \tilde{C}_m \alpha_k |x|^3 \leq C_m \alpha_k.$$

**Remark 2.2.** This theorem says that in any bounded interval, the polynomial sequence $Q_k(x)$ will converge to $2 \cos x$ uniformly with exponential speed. If we define $\tilde{P}_k(x) := Q_k(2^k x)$, then the sequence $\{\tilde{P}_k(x) : k \geq 1\}$ will behave like $\{2 \cos 2^k x : k \geq 1\}$ locally. On the other hand, notice that by (4), we have

$$\tilde{P}_k(x) = Q_k(2^k x) = P_k(\frac{x}{\rho} + x_0).$$

Thus $\tilde{P}_k$ is a renormalization of $P_k$. As a conclusion, the renormalization of the sequence $\{P_k : k \geq 1\}$ behave like $\{2 \cos 2^k x : k \geq 1\}$ locally, which gives a precise version of Section 1.2.

### 2.3. Convergence properties.

We prove Theorem 2.1 at the end of this subsection. By the recurrence relation (3), we have for $k \geq 1$

$$Q_k(x) = Q_{k-2}(x/4)(Q_{k-1}(x/2) - 2) + 2$$

$$= \left(2 \cos x/4 + \Delta_{k-2}(x/4)\right)^2 \left(2 \cos x/2 - 2 + \Delta_{k-1}(x/2)\right) + 2$$

$$= 2 \cos x + (2 + 2 \cos \frac{x}{2}) \cdot \Delta_{k-1}(\frac{x}{2}) +$$

$$\Delta_{k-2}(\frac{x}{4}) \left(4 \cos \frac{x}{4} + \Delta_{k-2}(\frac{x}{4})\right) \left(2 \cos \frac{x}{2} - 2 + \Delta_{k-1}(\frac{x}{2})\right).$$
Thus we conclude that for \( k \geq 1 \)

\[
\Delta_k(x) = (2 + 2 \cos \frac{x}{2}) \cdot \Delta_{k-1}(\frac{x}{2}) + \Delta_{k-2}(\frac{x}{4}) \left( 4 \cos \frac{x}{4} + \Delta_{k-2}(\frac{x}{4}) \right) \left( 2 \cos \frac{x}{2} - 2 + \Delta_{k-1}(\frac{x}{2}) \right).
\]

**Proposition 2.3.** Assume \( \Delta_k(x) = \sum_{k \geq 3} \Delta_{k,n} x^n \) and \( (\Delta_k(x))_{k \geq -1} \) satisfy (6). Fix \( k \geq 0 \). If there exists \( 0 < \delta \leq 1 \) such that \( |\Delta_{k-1,n}|, |\Delta_{k,n}| \leq \delta \) for any \( n \geq 3 \), then

\[
|\Delta_{k+1}(x)|^* \leq \delta \left( 4 \frac{x^3}{2^5} + 4 \frac{x^4}{2^4} + 9 \sum_{n \geq 5} \frac{x^n}{2^n} \right).
\]

**Proof.** We only need to prove the case \( k = 1 \). By (6) we write

\[
\Delta_1(x) = (2 + 2 \cos \frac{x}{2}) \cdot \Delta_0(\frac{x}{2}) + \Delta_{-1}(\frac{x}{4}) \left( 4 \cos \frac{x}{4} + \Delta_{-1}(\frac{x}{4}) \right) \left( 2 \cos \frac{x}{2} - 2 + \Delta_0(\frac{x}{2}) \right)
\]

\[
= (I) + \Delta_{-1}(\frac{x}{4})(II)(III),
\]

where

\[
\begin{aligned}
(I) &= (2 + 2 \cos \frac{x}{2}) \cdot \Delta_0(\frac{x}{2}), \\
(II) &= 4 \cos \frac{x}{4} + \Delta_{-1}(\frac{x}{4}), \\
(III) &= 2 \cos \frac{x}{2} - 2 + \Delta_0(\frac{x}{2}).
\end{aligned}
\]

Since \( \delta \leq 1 \), we have

\[
|I|^* \leq \left( 4 + 2 \sum_{n \geq 2} \frac{x^n}{n!2^n} \right) \sum_{n \geq 3} \frac{\delta x^n}{2^n} = 4 \delta \sum_{n \geq 3} \frac{x^n}{2^n} + 2 \delta \sum_{n \geq 5} \frac{x^n}{n!} (e - 2) \leq 4 \delta \sum_{n \geq 3} \frac{x^n}{2^n} + 4 \delta \sum_{n \geq 4} \frac{x^n}{n!} + 6 \delta \sum_{n \geq 5} \frac{x^n}{n!}
\]

\[
|II|^* \leq 4 + 4 \sum_{n \geq 2} \frac{x^n}{n!4^n} + \delta \sum_{n \geq 3} \frac{x^n}{n!} \leq 4 + 2 \sum_{n \geq 2} \frac{x^n}{4^n}
\]

\[
|III|^* \leq \frac{x^2}{2^2} + \sum_{n \geq 3} \frac{x^n}{n\cdot 2^n} (\frac{2}{n!} + \delta) \leq \frac{x^2}{2^2} + 2 \sum_{n \geq 3} \frac{x^n}{2^n}.
\]
Thus

\[ |\Delta_{-1}(\frac{x}{4}) \times (II)|^* \lesssim \delta \sum_{n \geq 3} \frac{x^n}{n!} \left( 4 + 2 \sum_{n \geq 2} \frac{x^n}{n!} \right) = 4\delta \sum_{n \geq 3} \frac{x^n}{n!} + 2\delta \sum_{n \geq 5} \frac{x^n}{n!} (n - 4) = 4\delta \frac{x^3}{4^3} + 2\delta \sum_{n \geq 4} \frac{x^n}{n!} (n - 2) \]

\[ |\Delta_{-1}(\frac{x}{4}) \times (II) \times (III)|^* \lesssim \frac{x^5}{2!} + \sum_{n \geq 6} \frac{x^n}{n!} (n - 4) + \delta \sum_{n \geq 6} \frac{x^n}{n!} \sum_{k=4}^{n-3} \frac{k-2}{k-2} \]

\[ \leq \frac{x^5}{2!} + \frac{x^6}{3!} + 3\delta \sum_{n \geq 7} \frac{x^n}{n!}. \]

This proves the proposition.

To prepare the proof for the dimension of the spectrum, we also need to study a variant of (6), i.e., for \( \tilde{\Delta}_k(x) = \sum_{n \geq 0} \tilde{\Delta}_{k,n} x^n, (k \geq -1) \) there exists a constant \( t_0 \) such that for any \( k \geq 1 \),

\[
\tilde{\Delta}_k(x) = (2 + 2 \cos \frac{x + t_0}{2}) \cdot \tilde{\Delta}_{k-1}(\frac{x}{2}) + \tilde{\Delta}_{k-2}(\frac{x}{4}) \cdot \left( 4 \cos \frac{x + t_0}{4} - 2 + \tilde{\Delta}_{k-1}(\frac{x}{2}) \right) \tag{7}
\]

**Proposition 2.4.** Assume \( (\tilde{\Delta}_k(x))_{k \geq -1} \) satisfy (7). Fix \( k \geq 0 \). If there exist \( 0 < \delta, \beta \leq 1 \) such that \( |\tilde{\Delta}_{k-1,n}|, |\tilde{\Delta}_{k,n}| \leq \delta \beta^{-n} \) for any \( n \geq 0 \), then

\[ |\tilde{\Delta}_{k+1}(x)|^* \leq 152\delta \sum_{n \geq 0} \frac{x^n}{(2\beta)^n}. \]

**Proof.** We only need to prove the case \( k = 1 \). By (7), we can write

\[
\tilde{\Delta}_1(x) = (I) + \tilde{\Delta}_{-1}(\frac{x}{4})(II)(III). \]

where

\[
(I) = \left( 2 + 2 \cos \left( \frac{x}{4} + \frac{t_0}{4} \right) \right) \cdot \tilde{\Delta}_0(\frac{x}{2}), \]

\[
(II) = 4 \cos \left( \frac{x}{4} + \frac{t_0}{4} \right) + \tilde{\Delta}_{-1}(\frac{x}{2}), \]

\[
(III) = 2 \cos \left( \frac{x}{2} + \frac{t_0}{2} \right) - 2 + \tilde{\Delta}_0(\frac{x}{2}). \]

Note that

\[ |\cos(x + x_0)|^* = \left| \sum_{n \geq 0} \frac{\cos(x_0 + n\pi/2)}{n!} x^n \right|^* \leq \sum_{n \geq 0} \frac{x^n}{n!}. \tag{8} \]
By Proposition 2.3, \\
\\n\[
(\Delta_k, \beta, \delta, \alpha, x) = (x, 2, 3, \delta, \alpha) \\
\]

So for any \( n \geq 3 \),
\\n\[
|\Delta_k| \leq 9\alpha^{k-2} \sum_{n \geq 3} x^n. \\
\]

Consequently \( (P_{k-1}, P_k) \) is \( (9\alpha^{k-3}, 2) \)-regular at \( x_0 \) for any \( k \geq 2 \).

**Proof.** By the condition we have
\\n\[
|\Delta_{-1,n}| \leq 1, \quad |\Delta_{0,n}| \leq 1, \quad \forall n \geq 3. \\
\]

By Proposition 2.3
\\n\[
|\Delta_1| \leq 4\alpha^3 + 4\alpha^4 + 9 \sum_{n \geq 5} x^n. \\
\]

So for any \( n \geq 3 \),
\\n\[
|\Delta_{1,n}| \leq \min\{1/2, 9/2^n\}. \\
\]

For any \( n \geq 3 \), \( \Delta_{0,n} \leq 1, \Delta_{1,n} \leq 1/2 < 1 \), then by a same argument as above,
\\n\[
|\Delta_{2,n}| \leq \min\{1/2, 9/2^n\}. \\
\]
Continue this process, by Proposition 2.3 and induction, for any $k \geq 0$ and $n \geq 3$,

$$\Delta_{2k+1,n} \leq \min \{2^{-k-1}, (9 \times 2^{-k}) \times 2^{-n}\}$$
$$\Delta_{2k+2,n} \leq \min \{2^{-k-1}, (9 \times 2^{-k}) \times 2^{-n}\}.$$ 

Recall that $\alpha = \sqrt{2}/2$. Thus for any $n \geq 3$,

$$|\Delta_{k,n}| \leq 9\alpha^{k-2} \times 2^{-n},$$

which implies (9). □

Now we are ready for the proof of Theorem 2.1. At first we define the absolute constants mentioned in that theorem. Let $\tilde{C}_0 = 9/(4 - \pi)$ and for any $m \geq 0$ define

$$\begin{align*}
C_m &= \tilde{C}_m (2^{m-1}\pi)^3 \\
\tilde{C}_{m+1} &= \tilde{C}_m \left(\frac{1}{2\alpha} + \left(\frac{4 + C_{n-1}}{64\alpha^2}\right)^2\right).
\end{align*}$$

**Proof of Theorem 2.1** We prove it by induction on $m$.

At first consider $m = 0$. Fix $x \in [-\pi/2, \pi/2]$. By Proposition 2.5, for $k \geq 1$,

$$|\Delta_k(x)| \leq 9\alpha^{k-2} \sum_{n=3}^{\infty} \frac{|x|^n}{2^n} = \frac{9\alpha^{k-2}|x|^3}{8 - 4|x|} \leq \frac{9\alpha^k|x|^3}{4 - \pi} = \tilde{C}_0 \alpha^k|x|^3 \leq C_0 \alpha^k.$$

Next we assume the result holds for $m < n$. Take $x \in [-2^{n-1}\pi, 2^{n-1}\pi]$, then $x/2, x/4 \in [-2^{n-2}\pi, 2^{n-2}\pi]$. For any $k \geq 2n+1$, we have $k-1, k-2 \geq 2n-1$. By (6) and induction,

$$\begin{align*}
|\Delta_k(x)| &= |4 \cos^2 \frac{x}{4} \cdot \Delta_{k-1}(\frac{x}{2}) + \\
&\quad \Delta_{k-2}(\frac{x}{4}) \left(4 \cos \frac{x}{4} + \Delta_{k-2}(\frac{x}{4})\right) \left(2(\cos \frac{x}{2} - 1) + \Delta_{k-1}(\frac{x}{2})\right)| \\
&\leq 4|\Delta_{k-1}(\frac{x}{2})| + |\Delta_{k-2}(\frac{x}{4})||(4 + |\Delta_{k-2}(\frac{x}{4})|)(4 + |\Delta_{k-1}(\frac{x}{2})|) \\
&\leq \tilde{C}_{n-1} \alpha^{k-1} \frac{C_{n-1}}{2} |x|^3 + \tilde{C}_{n-1} \alpha^{k-2} \frac{C_{n-1}}{64} |x|^3 (4 + C_{n-1})^2 \\
&\leq \tilde{C}_{n-1} \left(\frac{1}{2\alpha} + \left(\frac{4 + C_{n-1}}{64\alpha^2}\right)^2\right) \alpha^k |x|^3 \\
&= \tilde{C}_n \alpha^k |x|^3 \leq C_n \alpha^k.
\end{align*}$$

By induction the proof is finished. □
3. Generate new germs and control the distance of base points

In this section we prepare the proof of Theorem 1.1. Especially we will show that under some condition, new germs will appear and we can control the distance of the base points of the germs.

3.1. Birth of new germs.

In this subsection, we exhibit that how a regular germ of one pair can give birth to some new regular germs for the iterated pairs. At first we prove several preliminary results. Define $\delta_0 = 0.01$, $\delta_1 = 0.0005$ and $\delta_2 = 10^{-10}$.

**Lemma 3.1.** Fix $\delta \in (0, \delta_0)$ and $k \in \mathbb{Z}$. Assume $\varphi$ is a polynomial satisfying

$$|\varphi(x) - 2 \cos x| \leq \delta, \quad \forall x \in 2k\pi + [0, \pi] \ (\text{resp. } 2k\pi + [-\pi, 0]).$$

We further assume $x_+$ (resp. $x_-$) is the minimal $x \in 2k\pi + [0, \pi]$ (resp. maximal $x \in 2k\pi + [-\pi, 0]$) such that $\varphi(x) = 0$. Then

$$|x_+ - (2k\pi + \pi/2)| \leq \delta \quad \text{(resp. } |x_- - (2k\pi - \pi/2)| \leq \delta).$$

**Proof.** We only prove the result for $x_+$ when $k = 0$ since the other cases can be proven similarly.

By the assumption we have

$$|2 \cos x_+| = |\varphi(x_+) - 2 \cos x_+| \leq \delta.$$

On the other hand since $|\sin x| \geq 2|x|/\pi$ for $|x| \leq \pi/2$ and $|2k\pi + \pi/2 - x_+| \leq \pi/2$

$$|2 \cos x_+| = 2|\sin(2k\pi + \pi/2 - x_+)| \geq 4/\pi|x_+ - \pi/2 - 2k\pi|.$$

This prove the lemma. \qed

**Corollary 3.2.** Fix $\delta \in (0, \delta_0)$, $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. Assume $\varphi$ is a polynomial such that for any $x \in 2^{1-m}k\pi + [0, 2^{-m}\pi]$ (resp. $2^{1-m}k\pi + [-2^{-m}\pi, 0]$)

$$|\varphi(x) - 2 \cos 2^mx| \leq \delta.$$

We further assume $x_+$ (resp. $x_-$) is the minimal $x \in 2^{1-m}k\pi + [0, 2^{-m}\pi]$ (resp. maximal $x \in 2^{1-m}k\pi + [-2^{-m}\pi, 0]$) such that $\varphi(x) = 0$. Then

$$|x_+ - 2^{-m}(2k\pi + \pi/2)| \leq 2^{-m}\delta \quad \text{(resp. } |x_- - 2^{-m}(2k\pi - \pi/2)| \leq 2^{-m}\delta).$$

**Proof.** Define $\tilde{\varphi}(x) = \varphi(2^mx)$, then apply Lemma 3.1, the result follows. \qed
Corollary 3.3. Let $\delta \in (0, \delta_0)$. Assume $\varphi, \psi$ are two polynomials satisfies 
\[ |\varphi(x) - 2 \cos x|, \quad |\psi(x) - 2 \cos 8x| \leq \delta \quad \forall x \in [0, \pi]. \]
Let $x^*$ be the minimal $x \in [0, \pi]$ such that $\varphi(x) = 0$ and $x_*$ be the maximal $x \in (0, x^*)$ such that $\psi(x) = 0$. Then 
\[ |(x^* - x_*) - \pi/16| \leq 2\delta. \]

Proof. By Corollary 3.2 we have 
\[ |x^* - \pi/2| \leq \delta \quad \text{and} \quad |x_* - 7\pi/16| \leq 2^{-3}\delta. \]
Thus the result follows. \hfill \square

Proposition 3.4. Take any $\delta \leq \delta_1$. Assume polynomial pair $(P_{-1}, P_0)$ has a $(\delta, 2)$-regular $\rho$-germ at $x_0$. Then there exist $y_{-1} < y_0^* < x_0 < y_0^+ < y_{-1}^+$ such that 
\[ P_k(y_k^+) = P_k(y_k^+)-y_0 = 0; \quad P_k(x) > 0 \quad (x \in I^+_k \cup I^-_k), \quad (k = -1, 0). \]
where $I^+_k = (y_k^-, x_0], I^-_k = [x_0, y_k^+)$. Moreover 
\[ \frac{|I^+_k|}{|I^-_k|}, \quad \frac{|I^-_k|}{|I^+_k|} \leq 2.1. \]

Proof. For $k = -1, 0$, define $Q_k(x)$ as in (1). By the assumption $(P_{-1}, P_0)$ is $(\delta, 2)$-regular at $x_0$, for $|x| < 2$, 
\[ |Q_{-1}(x) - 2 \cos x|, \quad |Q_0(x) - 2 \cos x| \leq \frac{|x|^3}{8 - 4|x|}. \]
Especially for $|x| \leq 1.9$, we have 
\[ |Q_{-1}(x) - 2 \cos x|, \quad |Q_0(x) - 2 \cos x| \leq \frac{1.93\delta}{8 - 4 \cdot 1.9} < 20\delta \leq 0.01 = \delta_0. \]
Let $t_0$ be the minimal $t \in (0, 2)$ such that $Q_0(t) = 0$, $t_{-1}$ be the minimal $t \in (0, 2)$ such that $Q_{-1}(t) = 0$. Then by Lemma 3.1 
\[ |t_0 - \frac{\pi}{2}| \leq \delta_0, \quad |t_{-1} - \frac{\pi}{2}| \leq \delta_0, \]
Define 
\[ y_0^+ = \frac{t_0}{a} + x_0, \quad y_{-1}^+ = \frac{t_{-1}}{a} + x_0, \]
we see $P_0(y_0^+) = P_{-1}(y_{-1}^+) = 0$ and 
\[ \frac{|I^+_0|}{|I^-_0|} = \frac{|y_0^+-x_0|}{|y_0^+-x_0|} = \frac{2t_{-1}}{t_0} \leq \frac{2(\frac{\pi}{2} + 0.01)}{\frac{\pi}{2} - 0.01} < 2.1. \]
The proof of the other part of the proposition is analogous. \hfill \square

Now we can state the main result in this subsection.
Proposition 3.5. Take any $\delta \leq \delta_2$. Assume $(P_{-1}, P_0)$ is $(\delta, 2)$-regular at $x_0$ with renormalization factor $a$, define $P_n = P_{n-2}^2(P_{n-1} - 2) + 2$ for $n \geq 1$. Let $y^+_0, y^-_0$ be defined as in Proposition 3.4 then $(P_3, P_4)$ is $(1, 1)$-regular at both $y^+_0$ and $y^-_0$.

Proof. Recall that we have defined

$$Q_k(x) = P_k\left(\frac{x}{2^k a} + x_0\right) = 2 \cos x + \Delta_k(x), \quad k = -1, 0. \quad (10)$$

Then

$$|\Delta_{-1}(x)|^* \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n}, \quad |\Delta_0(x)|^* \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n}. \quad (11)$$

Consequently for $x \in (0, 2)$ it is ready to show that

$$\begin{cases} |\Delta^{(n)}_0(x)|^* \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n}, & n = 0 \\ |\Delta^{(n)}_{-1}(x)| \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n} = \delta \frac{x^3}{n} & n = 1 \\ |\Delta^{(n)}_{-2}(x)| \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n} = \frac{2\delta}{(2-x)^2} & n = 2 \\ |\Delta^{(n)}_{-3}(x)| \leq \delta \frac{\sum_{n=3}^{\infty} x^n}{2n} = \frac{2\delta}{(2-x)^{n+1}} & n \geq 3. \end{cases}$$

Let $t_0$ be the minimal $t \in (0, 2)$ such that $Q_0(t) = 0$. Then

$$P_0(t_0/a + x_0) = 0, \quad |t_0 - \frac{\pi}{2}| \leq 0.01. \quad (12)$$

Consequently $y^+_0 = t_0/a + x_0$. We have

$$P_{-1}(y^+_0) = Q_{-1}(\frac{t_0}{2}) = 2 \cos \frac{t_0}{2} + \Delta_{-1}(\frac{t_0}{2}).$$

By (11) and (12) we get $|P_{-1}(y^+_0) - \sqrt{2}| \leq 0.03$. Since

$$P_1(y^+_0) = P_{-1}(y^+_0)(P_0(y^+_0) - 2) + 2,$$

we conclude that

$$|P_1(y^+_0) + 2| \leq 0.2. \quad (13)$$

By (10) we have $P_0'(y^+_0) = aQ_0'(t_0) = a(-2 \sin t_0 + \Delta_0'(t_0))$. By (11) and (12) we get

$$\left|\frac{P_0'(y^+_0)}{a} + 2\right| \leq 0.2. \quad (14)$$

By (2) and (13), (14), for $k = 3, 4$ we have

$$P_k(x) = 2 - (2^{k-3})^2(x - y^+_0)^2 + O((x - y^+_0)^3)$$

with

$$\rho = \sqrt{2 - P_1(y^+_0)|P_0'(y^+_0)P_1(y^+_0)| \geq 6a.}$$
For $k \geq 3$, if we define $\tilde{Q}_k(x) := P_k(\frac{x}{\sqrt{1+\rho^2}} + y_0^+)$, then

$$\tilde{Q}_k(x) = 2 - x^2 + O(x^3) = 2 \cos x + O(x^3) =: 2 \cos x + \Delta_k(x).$$

(15)

We need to show that

$$|\Delta_3(x)|^*, |\Delta_4(x)|^* \leq \sum_{n \geq 3} x^n.$$

For $k \geq -1$ define $\Delta_k(2^k x) := \Delta_k(2^k(x + t_0))$. Then

$$P_k(\frac{x}{a} + y_0^+) = P_k(\frac{x + t_0}{a} + x_0) =: 2 \cos(2^k(x + t_0)) + \Delta_k(2^k x).$$

(16)

By the recurrence relation of $P_k$, it is ready to show that $(\Delta_k(x))_{k \geq -1}$ satisfies (7). We have

$$\begin{cases}
\tilde{\Delta}_0(x) & = \Delta_0(x + t_0) = \sum_{n=0}^{\infty} \Delta_0^{(n)}(t_0) x^n \\
\tilde{\Delta}_{-1}(x) & = \Delta_{-1}(x + t_0/2) = \sum_{n=0}^{\infty} \Delta_{-1}^{(n)}(t_0/2) x^n.
\end{cases}$$

Write $\beta = 2 - \pi/2 - 0.01 = 0.419 \cdots$ and $M_0 = 10$. By (11) and (12) we get

$$|\tilde{\Delta}_0(x)|^*, |\tilde{\Delta}_{-1}(x)|^* \leq M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{\beta^n}.$$ 

Since $\delta \leq \delta_2 = 10^{-10}$, we have $152^2 M_0 \delta < 1$. By Proposition 2.4

$$|\tilde{\Delta}_1(x)|^* \leq 152 M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{(2\beta)^n}.$$ 

Consequently we have

$$|\tilde{\Delta}_0(x)|^* \leq 152 M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{\beta^n}, \quad |\tilde{\Delta}_1(x)|^* \leq 152 M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{\beta^n}.$$ 

Again by Proposition 2.4 we get

$$|\tilde{\Delta}_2(x)|^* \leq 152^2 M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{(2\beta)^n}.$$ 

Continue this process, since $152^3 M_0 \delta < 1$ and $2\beta < 1$, for $k = 3, 4$ we get

$$|\tilde{\Delta}_k(x)|^* \leq 152^k M_0 \delta \sum_{n=0}^{\infty} \frac{x^n}{(4\beta)^n}.$$ 

By these inequalities and (11), (12), we have, for $k = 3, 4$,

$$|P_k(\frac{x}{a} + y_0^+)|^* \leq \sum_{n \geq 0} \left( \frac{(2^k x)^n}{(4\beta)^n} \left( \frac{(2(4\beta))^n}{n!} + 152^k M_0 \delta \right) \right).$$
Notice that $\tilde{Q}_k(x) = P_k\left(\frac{ax/2^{k-3}ρ}{a} + y_0^+\right)$ and $ρ \geq 6a$, we get

$$|\tilde{Q}_k(x)|^* \leq \sum_{n \geq 0} \frac{(4x/3)^n}{(4β)^n} \left(\frac{2(4β)^n}{n!} + 152^k M_0δ\right).$$

Combine with (15) we have

$$|\tilde{Q}_k(x) - 2 + x^2|^* \leq \sum_{n \geq 3} \frac{(4x/3)^n}{(4β)^n} \left(\frac{2(4β)^n}{n!} + 152^k M_0δ\right) + \sum_{n \geq 4} \frac{2x^n}{n!},$$

Now by a direct computation, for $k = 3, 4,$

$$|\bar{Δ}_k(x)|^* = |\tilde{Q}_k(x) - 2 \cos x|^* \leq \sum_{n \geq 3} x^n.$$

This prove the result for $y_0^+$. The proof for $y_0^-$ is the same. $\square$

3.2 Control the distance between the base points of germs.

In this subsection we will show that besides the birth of new germs, we can even control the distance between the base points of the old and new germs once we iterate sufficient long time.

At first we define a big absolute integer. Let $n_α = 40$, then $9α^{n_α-3} \leq δ_1$. Define $δ_3 := 30^{-n_α}/4000$. Choose an absolute constant $K \in \mathbb{N}$ such that

$$K \geq n_α + 4, \quad 9α^{K-7} < δ_2, \quad C_6α^{K-4} \leq δ_3, \quad (17)$$

where $C_6$ is an absolute constant defined in Theorem 2.1.

We need one more lemma.

**Lemma 3.6.** Fix $N \geq 2$ and $δ > 0$. Assume $φ_0(x), φ_1(x)$ are two polynomials satisfying

$$\begin{cases}
|φ_0(x) - 2 \cos 8x| \leq δ & x \in [-π, π] \\
|φ_1(x) - 2 \cos 16x| \leq δ & x \in [-π/2, π/2].
\end{cases}$$

Define $φ_n = φ_{n-2}^2(φ_{n-1} - 2) + 2$ for $n \geq 2$. Then for any $2 \leq n \leq N$ and $x \in [-π/2^n, π/2^n]$

$$|φ_n(x) - 2 \cos 2^{n+3}x| \leq 30^{n-1}δ. \quad (18)$$

**Proof.** Write $φ_n(x) = 2 \cos 2^{n+3}x + Δ_n(2^{n+3}x)$ for $n \geq 0$. It is equivalent to prove that for any $n \geq 2$

$$|Δ_n(x)| \leq 30^{n-1}δ \quad (\forall x \in [-8π, 8π]). \quad (19)$$

We prove it by induction. The condition implies that

$$|Δ_0(x)|, |Δ_1(x)| \leq δ \quad (x \in [-8π, 8π]). \quad (20)$$
By the recurrence relation, similar with (9), for \( n \geq 2 \) we have

\[
\Delta_n(2^{n+3}x) = (2 + 2\cos 2^{n+2}x) \cdot \Delta_{n-1}(2^{n+2}x) + \Delta_{n-2}(2^{n+1}x) \cdot \\
(4\cos 2^{n+1}x + \Delta_{n-2}(2^{n+1}x))(2\cos 2^{n+2}x - 2 + \Delta_{n-1}(2^{n+2}x)).
\]

For any \( x \in [-\pi/2^2, \pi/2^2] \), we have \( 2^3x, 2^4x \in [-4\pi, 4\pi] \). Thus by (20)

\[
|\Delta_2(2^5x)| \leq 4\delta + \delta(4 + \delta)^2 \leq 4\delta + 25\delta < 30\delta,
\]
i.e., (19) holds for \( n = 2 \).

Now assume (18) holds for \( n < m \leq N \). For any \( x \in [-\pi/2^m, \pi/2^m] \), we have \( 2^{m+1}x, 2^{m+2}x \in [-4\pi, 4\pi] \). Then

\[
|\Delta_m(2^{m+3}x)| \leq 4|\Delta_{m-1}(2^{m+2}x)| + |\Delta_{m-2}(2^{m+1}x)| \cdot (4 + |\Delta_{m-2}(2^{m+1}x)}|(4 + |\Delta_{m-1}(2^{m+2}x)|)
\leq 4 \cdot 30^{m-2}\delta + 30^{m-3}\delta(4 + 30^{m-3}\delta)(4 + 30^{m-2}\delta)
\leq 4 \cdot 30^{m-2}\delta + 25 \cdot 30^{m-2}\delta \leq 30^{m-1}\delta.
\]

Thus (19) holds for \( n = m \). This proves the lemma. \( \square \)

**Proposition 3.7.** Assume \((P_-, P_0)\) is \((1,1)\)-regular at \( \theta \). Let \( \theta^+ > \theta \) be the minimal zero of \( P_{K-4} \) in \([\theta, \infty)\). Then \((P_{K-1}, P_K)\) is \((1,1)\)-regular at \( \theta^+ \). Moreover assume \( \theta^+ \in (\theta, \theta^+) \) is the maximal zero of \( P_{2K-4} \), then

\[
\frac{\theta^+ - \theta^+}{\theta^+ - \theta} \geq 2.1^{-K}.
\]

Similarly let \( \theta^- < \theta \) be the maximal zero of \( P_{K-4} \) in \([-\infty, \theta) \). Then \((P_{K-1}, P_K)\) is \((1,1)\)-regular at \( \theta^- \). Moreover assume \( \theta^- \in (\theta^-, \theta) \) is the minimal zero of \( P_{2K-4} \), then

\[
\frac{\theta^- - \theta^-}{\theta^- - \theta} \geq 2.1^{-K}.
\]

**Proof.** We only prove the first result, since the second one is the same.

Since \((P_-, P_0)\) is \((1,1)\)-regular at \( \theta \), by Proposition 2.3 \((P_{K-5}, K-4)\) is \((9\alpha^{K-7}, 2)\)-regular at \( \theta \). By (17), \((P_{K-5}, K-4)\) is \((\delta_2, 2)\)-regular at \( \theta \). Then by Proposition 3.3 \((P_{K-1}, P_K)\) is \((1,1)\)-regular at \( \theta^+ \).

For any \(-1 \leq k \leq K-4\) define \( \theta_k \) to be the largest \( y \) in \([\theta, \theta^+]\) such that \( P_{K+k}(y) = 0 \), then \( \theta_{K-4} = \theta^+ \). We have

\[
\frac{\theta^+ - \theta^+}{\theta^+ - \theta} = \frac{\theta^+ - \theta_{-1}}{\theta^+ - \theta} \cdot \prod_{k=0}^{K-4} \frac{\theta^+ - \theta_k}{\theta^+ - \theta_{k-1}}.
\]

At first we estimate \((\theta^+ - \theta_{-1})/(\theta^+ - \theta)\).
Assume \((P_{-1}, P_0)\) has renormalization factor \(a\) at \(\theta\). let \(L_k(x) = \frac{x}{2^k a} + \theta\).

By Theorem 2.1 for any \(k \geq 13\) and any \(x \in [-32\pi, 32\pi]\) we have
\[
|P_k \circ L_k(x) - 2\cos x| \leq C_0 \alpha^k. \tag{22}
\]

By (22) and (17), for \(x \in [-4\pi, 4\pi]\),
\[
|P_{K-4} \circ L_{K-4}(x) - 2\cos x| \leq \delta_3, \quad |P_{K-1} \circ L_{K-4}(x) - 2\cos 8x| \leq \delta_3.
\]

By Lemma 3.1 and Corollary 3.3
\[
|L_{K-4}^{-1}(\theta^+ - \pi/2)| \leq \delta_3 \leq \delta_0, \quad |(L_{K-4}^{-1}(\theta^+) - L_{K-4}^{-1}(\theta_-)) - \pi/16| \leq 2\delta_3 \leq \delta_0. \tag{23}
\]

Since \(L_k^{-1} x = 2^k a(x - \theta)\) we conclude that
\[
\frac{\theta^+ - \theta}{\theta^+ - \theta_1} \geq 2.1^{-3}. \tag{24}
\]

Next we we estimate \((\theta^+ - \theta_k)/(\theta^+ - \theta_{k-1})\) for \(k = 0, \ldots, K-4\). We will discuss two cases. We have shown that \((P_{K-1}, P_K)\) is \((1,1)\)-regular at \(\theta^+\).

By Proposition 2.5 \((P_{K+k-1}, P_{K+k})\) is \((9\alpha^{k-3}, 2)\)-regular.

In the following, we prove \(\theta^+ - \theta_k \geq 2.1^{-1}\) in two cases:
\[
n_\alpha \leq k \leq K - 4 \quad \text{and} \quad 0 \leq k < n_\alpha.
\]

Case i: \(n_\alpha \leq k \leq K - 4\).

Recall that \(n_\alpha\) is such that \(9\alpha^{n_\alpha-3} \leq \delta_1\). In this case \((P_{K+k-1}, P_{K+k})\) is \((\delta_1, 2)\)-regular, then by Proposition 3.4
\[
\frac{\theta^+ - \theta_k}{\theta^+ - \theta_{k-1}} \geq 2.1^{-1}. \tag{25}
\]

Case ii: \(0 \leq k < n_\alpha\). By (22) and (17),
\[
\begin{aligned}
|P_{K-1} \circ L_{K-4}(x) - 2\cos 8x| &\leq \delta_3 \quad x \in [-4\pi, 4\pi] \\
|P_K \circ L_{K-4}(x) - 2\cos 16x| &\leq \delta_3 \quad x \in [-2\pi, 2\pi].
\end{aligned}
\]

Let \(x^* = L_{K-4}^{-1}(\theta^+)\), define \(\xi_k(x) := P_{K+k-1} \circ L_{K-4}(x + x^*)\) for \(k \geq 0\).

For \(x \in [-\pi, \pi]\),
\[
|\xi_0(x) - 2\cos 8x| \\
= |P_{K-1} \circ L_{K-4}(x + x^*) - 2\cos 8x| \\
\leq |P_{K-1} \circ L_{K-4}(x + x^*) - 2\cos 8(x + x^*)| + |2\cos 8(x + x^*) - 2\cos 8x| \\
\leq \delta_3 + 4|\sin 4x^*| = \delta_3 + 4|\sin 4(x^* - \frac{\pi}{2})| \leq 17\delta_3 \leq 30^{-n_\alpha}/100,
\]

where the third inequality is due to (23), and the last inequality is by definition of \(\delta_3\).
Similarly we have for \( x \in [-\pi/2, \pi/2] \)
\[
|\xi_1(x) - 2 \cos 16x| \\
= |P_K \circ L_{K-4}(x + x^*) - 2 \cos 16x| \\
\leq |P_K \circ L_{K-4}(x + x^*) - 2 \cos 16(x + x^*)| + |2 \cos 16(x + x^*) - 2 \cos 16x| \\
\leq \delta_3 + 4|\sin 8x^*| \leq \delta_3 + 4|\sin (x^* - \frac{\pi}{2})| \leq 33\delta_3 \leq 30^{-n_\alpha}/100.
\]

By Lemma 3.6, for any \( 2 \leq k \leq n_\alpha \) and \( x \in [-\pi/2^k, \pi/2^k] \),
\[
|\xi_k(x) - 2 \cos 2^{k+3}x| \leq 30^{k-1} \cdot 30^{-n_\alpha}/100 \leq 1/100 = \delta_0.
\]

Notice that for all \( -1 \leq k < n_\alpha \), \( L_{K-4}^{-1}(\theta_k) - x^* = 2^{K-4}a(\theta_k - \theta^+) \) is the largest zero point of \( \xi_{k+1} \) in \( [-\frac{\pi}{2^{k+1}}, 0] \). Then by Corollary 3.2,
\[
|2^{K-4}a(\theta_k - \theta^+) + \frac{\pi}{2^{k+5}}| \leq \frac{1}{100 \cdot 2^{k+4}} \leq \frac{1}{100} \frac{\pi}{2^{k+5}}.
\]
Consequently, for any integer \( 0 \leq k < n_\alpha \),
\[
\frac{\theta^+ - \theta_k}{\theta^+ - \theta_{k-1}} \geq 2.1^{-1}.
\]
Combining (24), (25) and (26) we get (21). \( \square \)

4. Lower Bound for the Hausdorff Dimension of the Spectrum

Now we go back to the spectrum of the Thue-Morse Hamiltonian. Recall that \( \{ h_n(x) : n \geq 1 \} \) is the related trace polynomials and \( h_1(x) = x^2 - \lambda^2 - 2 \). Let \( a_\emptyset = \sqrt{2 + \lambda^2} \), then \( h_1(a_\emptyset) = 0 \).

At first we will show that \( (h_4, h_5) \) is \((1,1)\)-regular at \( a_\emptyset \), then by Proposition 3.7 we will construct a Cantor subset of the spectrum around \( a_\emptyset \) and estimate the dimension of the Cantor set. Consequently we get a lower bound for the Hausdorff dimension of the spectrum.

4.1. \((h_4, h_5)\) is \((1,1)\)-regular at \( a_\emptyset \).

Recall that \( h_2(x) = (x^2 - \lambda^2)^2 - 4x^2 + 2 \), then \( h_2(a_\emptyset) = -2 - 4\lambda^2 \). Thus
\[
\begin{align*}
\{ h_1(x) &= h'_1(a_\emptyset)(x - a_\emptyset) + O((x - a_\emptyset)^2) = 2a_\emptyset(x - a_\emptyset) + O((x - a_\emptyset)^2) \\
\{ h_2(x) &= h_2(a_\emptyset) + O((x - a_\emptyset)) = -2(1 + 2\lambda^2) + O((x - a_\emptyset)).
\end{align*}
\]

Write \( \rho := (1 + 2\lambda^2)^{\frac{1}{2}}(1 + \lambda^2)(2 + \lambda^2) \). By using the recurrent relation we get
\[
h_k(x) = 2 - 4^{k-1} \rho^2(x - a_\emptyset)^2 + O((x - a_\emptyset)^3) \quad (k \geq 4).
\]

**Lemma 4.1.** \((h_4, h_5)\) is \((1,1)\)-regular at \( a_\emptyset \) with renormalization factor \( 2^4 \rho \).
Proof. Write $t := 2a_\emptyset$. Then $t \geq 2\sqrt{2}$. Define

$$g_n(x) = h_n(x + a_\emptyset),$$

we have

$$g_1(x) = x^2 + tx \quad \text{and} \quad g_2(x) = x^4 + 2tx^3 + t^2x^2 + 6 - t^2.$$

Then we can compute that for $n \geq 4$,

$$g_n(x) = 2 - 4^{n-4}t^2(t^2 - 6)^2(t^2 - 4)x^2 + O(x^3).$$

Write $\tau := t(t^2 - 6)\sqrt{t^2 - 4}$. Define $f_n(x) = g_n(x/\tau)$. Then for $n \geq 4$ we have

$$f_n(x) = 2 - 4^{n-4}x^2 + O(x^3).$$

We also have

$$f_1(x) = t\tau^{-1}x + \tau^{-2}x^2$$
$$f_2(x) = (6 - t^2) + (t/\tau)^2x^2 + (2t/\tau^3)x^3 + x^4/\tau^4$$
$$f_3(x) = 2 - x^2/(t^2 - 4) - O(x^3).$$

By the fact that $t \geq 2\sqrt{2}$ and $\tau = t(t^2 - 6)\sqrt{t^2 - 4}$, it is direct to verify that

$$|f_1(x)|^* \leq \frac{t}{\tau}xe^{x^3/32}, \quad |f_2(x)|^* \leq (t^2 - 6)e^{x^3/4}, \quad |f_2(x) - 2|^* \leq (t^2 - 4)e^{x^3/4}.$$

Then, by $f_n = 2 + f_{n-2}^2(f_{n-1} - 2)$, we have,

$$|f_3(x)|^* \leq 2 + 4x^2e^{13x/16}$$
$$|f_4(x)|^* \leq 2 + x^2e^{13x/16}$$
$$|f_5(x)|^* \leq 2 + 4x^2e^{13x/16} + 4\frac{x^4}{(t^2 - 6)^2}e^{18x/16} + \frac{x^6}{(t^2 - 6)^4}e^{23x/16}.$$

Now, it is ready to verify that

$$|f_4(x) - 2\cos x|^* \leq \sum_{n \geq 3} x^n, \quad |f_5(x/2) - 2\cos x|^* \leq \sum_{n \geq 3} x^n.$$

This implies that $(h_4, h_5)$ is $(1, 1)$-regular at $a_\emptyset$ with renormalization factor $2\tau = 2^4\rho$. □
4.2. Lower bound of the spectrum.

Now we will construct the desired Cantor set. To simplify the notation we write \( P_k(x) := h_{k+5}(x) \) for \( k \geq -1 \). Then we have shown that \((P_{-1},P_0)\) is \((1,1)\)-regular at \( a_\emptyset \).

Fix an absolute constant \( K \in \mathbb{N} \) according to (17). Assume \( b_\emptyset > a_\emptyset \) is the first zero of \( P_{K-4} \) to the right of \( a_\emptyset \). Define \( I_\emptyset := [a_\emptyset, b_\emptyset] \). Assume \( b_0 \) is the smallest zero of \( P_{2K-4} \) in \( I_\emptyset \) and \( a_1 \) is the biggest zero of \( P_{2K-4} \) in \( I_\emptyset \). Define

\[
I_0 := [a_\emptyset, b_0] = [a_0, b_0] \quad \text{and} \quad I_1 := [a_1, b_0] = [a_1, b_1].
\]

Take any \( w \in \{0,1\}^k \), suppose \( I_w = [a_w, b_w] \) is defined. Assume \( b_{w0} \) is the smallest zero of \( h_{(k+2)K-4} \) in \( I_w \) and assume \( a_{w1} \) is the biggest zero of \( h_{(k+2)K-4} \) in \( I_w \). Write \( a_{w0} = a_w, b_{w1} = b_w \) and define

\[
I_{w0} = [a_w, b_{w0}] = [a_{w0}, b_{w0}] \quad \text{and} \quad I_{w1} = [a_{w1}, b_w] = [a_{w1}, b_{w1}].
\]

Define a Cantor set as

\[
\mathcal{C} := \bigcap_{n \geq 1} \bigcup_{|w|=n} I_w
\]

Proposition 4.2.

\[
\dim_H \mathcal{C} \geq \frac{\ln 2}{K \ln 2.1}.
\]

Proof. Given \( w \in \{0,1\}^k \). By induction it is easy to show that

\[
\{P_{(k+1)K-4}(a_w), P_{(k+1)K-4}(b_w)\} = \{0,2\}.
\]

For definiteness let \( \alpha_w \) be the endpoint of \( I_w \) such that \( P_{(k+1)K-4}(\alpha_w) = 2 \) and \( \beta_w \) be another endpoint of \( I_w \), thus \( P_{(k+1)K-4}(\beta_w) = 0 \).

Claim: \((P_{kK-1}, P_{kK})\) is \((1,1)\)-regular at \( \alpha_w \).

\(<\) We show it by induction on \( k \). When \( k = 0, w = \emptyset \). By Lemma 4.1 \((P_{-1}, P_0)\) is \((1,1)\) regular at \( \alpha_w = a_w = a_\emptyset \).

Assume the result holds for any \( w \in \{0,1\}^l \) with \( l < k \). Now fix \( w \in \{0,1\}^k \), then \( w = \tilde{w}i \) with \( \tilde{w} \in \{0,1\}^{k-1} \) and \( i \in \{0,1\} \). Then \( \alpha_w \) must be one of the endpoints of \( I_{\tilde{w}} \) since \( P_{(k+1)K-4}(\alpha_w) = 2 \). By the induction assumption \((P_{(k-1)K-1}, P_{(k-1)K})\) is \((1,1)\)-regular at \( \alpha_{\tilde{w}} \). Consequently \((P_{kK-1}, P_{kK})\) is \((9\alpha_{K-3}, 2)\)-regular at \( \alpha_{\tilde{w}} \). By (17), \((P_{kK-1}, P_{kK})\) is \((1,1)\)-regular at \( \alpha_{\tilde{w}} \). On the other hand by Proposition 3.7 we have \((P_{kK-1}, P_{kK})\) is \((1,1)\)-regular at \( \beta_{\tilde{w}} \). Since \( \alpha_w = \alpha_{\tilde{w}} \) or \( \beta_{\tilde{w}} \), the result follows. \(\rangle\)

Now fix any \( w \in \{0,1\}^k \), let us estimate \(|Iwi|/|Iw|\) for \( i = 0,1 \). Without loss of generality we assume \( \alpha_w = a_w \) and \( \beta_w = b_w \). By the claim above \((P_{kK-1}, P_{kK})\) is \((1,1)\)-regular at \( a_w \). By Proposition 2.5 \((P_{kK+l-1}, P_{kK+l})\)
\(9\alpha_l - 3\), \(2)\)-regular for any \(l \geq 2\). By (17), we have \(9\alpha_l - 3 < \delta_1\) for any \(l \geq K - 4\). Let \(y_l\) be the smallest zero of \(P_{K+1}\) in \([a_w, \infty)\), then \(y_{K-4} = b_w\) and \(y_{2K-4} = b_{w0}\). By Proposition 3.4, for any \(l \geq K - 4\)

\[
\frac{y_{l+1} - a_w}{y_l - a_w} \geq 2.1^{-1}.
\]

Consequently we have

\[
\frac{|I_{w0}|}{|I_w|} = \frac{b_{w0} - a_w}{b_w - a_w} = \prod_{l=K-4}^{2K-5} \frac{y_{l+1} - a_w}{y_l - a_w} \geq 2.1^{-K}.
\]

On the other hand, since \((P_{K-1}, P_{K})\) is \((1, 1)\)-regular at \(a_w\), by Proposition 3.7 \((P_{(k+1)K-1}, P_{kK})\) is \((1, 1)\)-regular at \(b_w\). Moreover since \(a_{w1}\) is the maximal zero of \(P_{(k+2)K-4}\) in \(I_w\),

\[
\frac{|I_{w1}|}{|I_w|} = \frac{b_w - a_{w1}}{b_w - a_w} \geq 2.1^{-K}.
\]

Now it is well known that (see for example [9])

\[
\dim_H C \geq \frac{\ln 2}{-\ln 2.1^{-K}} = \frac{\ln 2}{K \ln 2.1}.
\]

\(\square\)

**Proof of Theorem 1.1.** Recall that \(\Sigma\) is defined in Remark 1.4 thus for any \(w \in \{0,1\}^*\), \(a_w, b_w \in \Sigma \subset \sigma(H_{\lambda,v})\). Since \(C = \{a_w, b_w : w \in \{0,1\}^*\}\), we conclude that \(C \subset \sigma(H_{\lambda,v})\). Consequently

\[
\dim_H \sigma(H_{\lambda,v}) \geq \dim_H C \geq \frac{\ln 2}{K \ln 2.1}.
\]

Since \(K\) is an absolute positive constant, the result follows. \(\square\)

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**References**

[1] Axel, F., Peyrière, J.: Extended states in a chain with controlled disorder. C. R. Acad. Sci. Paris Sr. II Mc. Phys. Chim. Sci. Univers Sci. Terre 306, 179-182 (1988)

[2] Axel, F., Peyrière, J.: Spectrum and extended states in a harmonic chain with controlled disorder: Effects of the Thue-Morse symmetry. J. Statist. Phys. 57, 1013-1047 (1989)

[3] Bellissard, J.: Spectral properties of Schrödinger operator with a Thue-Morse potential. In: Number Theory and Physics (Les Houches, 1989), Springer Proc. Phys. 47, pp. 140-150. Springer, Berlin (1990)
[4] Bovier, A., Ghez, J. M.: Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions. Commun. Math. Phys. 158, 45-66 (1993)

[5] Cantat, S.: Bers and Hénon, Painlevé and Schrödinger. Duke Math. J. 149, 411-460 (2009)

[6] Damanik, D., Embree, M., Gorodetski, A., Tcheremchantsev, S.: The fractal dimension of the spectrum of the Fibonacci Hamiltonian. Commun. Math. Phys. 280, 499-516 (2008)

[7] Damanik, D., Gorodetski, A.: Hyperbolicity of the trace map for the weakly coupled Fibonacci Hamiltonian. Nonlinearity 22, 123-143 (2009)

[8] Damanik, D., Gorodetski, A.: Spectral and quantum dynamical properties of the weakly coupled Fibonacci Hamiltonian. Commun. Math. Phys. 305, 221-277 (2011)

[9] Falconer, K.: Fractal geometry. Mathematical foundations and applications. John Wiley & Sons, Ltd., Chichester (1990)

[10] Jitomirskaya, S., Last, Y.: Power-law subordinacy and singular spectra. II. Line operators. Commun. Math. Phys. 211, 643-658 (2000)

[11] Lenz, D.: Uniform ergodic theorems on subshifts over a finite alphabet. Ergodic Theory Dynam. Systems 22, 245-255 (2002)

[12] Liu, Q. H., Tan, B., Wen, Z. X., Wu, J.: Measure zero spectrum of a class of Schrödinger operators. J. Statist. Phys. 106, 681-691 (2002)

[13] Liu, Q. H., Wen, Z. Y.: Hausdorff dimension of spectrum of one-dimensional Schrödinger operator with Sturmian potentials. Potential Analysis 20, 33-59 (2004)

[14] Raymond, L.: A constructive gap labelling for the discrete schrödinger operator on a quasiperiodic chain. (Preprint, 1997)

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