A Geometric Approach to Aggregate Flexibility Modeling of Thermostatically Controlled Loads
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Abstract—Coordinated aggregation of a large population of thermostatically controlled loads (TCLs) presents a great potential to provide various ancillary services to the grid. One of the key challenges of integrating TCLs into system level operation and control is developing a simple and portable model to accurately capture their aggregate flexibility. In this paper, we propose a geometric approach to model the aggregate flexibility of TCLs. We show that the set of admissible power profiles of an individual TCL is a polytope, and their aggregate flexibility is the Minkowski sum of the individual polytopes. In order to represent their aggregate flexibility in an intuitive way and achieve a tractable approximation, we develop optimization-based algorithms to approximate the polytopes by the homothets of a given convex set. As a special application, this set is chosen as a virtual battery model and the corresponding optimal approximations are solved efficiently by equivalent linear programming problems. Numerical results show that our algorithms yield significant improvement in characterizing the aggregate flexibility over existing modeling methods. We also conduct case studies to demonstrate the efficacy of our approaches by coordinating TCLs to track a frequency regulation signal from the Pennsylvania-New Jersey-Maryland (PJM) Interconnection.

NOMENCLATURE

\( \theta \) Temperature of a TCL system.
\( \theta_c \) User specified temperature set-point.
\( \delta \) Sampling time of a TCL system.
\( \Delta \) Half of the temperature deadband.
\( C_{th} \) Thermal capacitance.
\( R_{th} \) Thermal resistance.
\( P_0 \) Nominal power (baseline power) that maintains the set-point temperature.
\( P_m \) Rated power of a TCL system.
\( q(k) \) Operating state "OFF/ON" at time instant \( k \).
\( \eta \) Coefficient of performance of the power consumption.
\( \Omega^i \) The set of the \( i \)th TCL’s parameters.
\( \Omega_o \) The set of mean TCL parameters.
\( T \) Time horizon set \{1, 2, \ldots m\}.
\( A^i, B^i \) \( m \)-dimensional system/input matrix associated with the vector representation of the \( i \)th TCL’s discrete dynamics.

\( C^i \) The vector representation of the \( i \)th TCL’s initial condition.
\( A^i \) The inverse of the matrix \( A^i \).
\( \alpha \) Energy dissipation rate.
\( X(k) \) Energy state of a virtual battery at time \( k \).
\( U(k) \) Power supply/draw of a virtual battery at time \( k \).
\( C \) Initial condition vector of a virtual battery, \( C = [\alpha X(0), 0, \ldots , 0]^T \).
\( D, \overline{D} \) Lower/Upper power limits of a virtual battery.
\( \bar{E}, \overline{E} \) Lower/Upper energy capacity limits of a virtual battery.
\( \phi \) The set of the virtual battery parameters, \( \phi = \{C, D, \overline{D}, E, \overline{E}\} \).
\( B_o \) Prototype virtual battery model.
\( B_s, B_n \) Sufficient/Necessary virtual battery model.
\( B^*_o, B^*_n \) Optimal sufficient/necessary virtual battery model.
\( B^1_s, B^1_n \) Suboptimal sufficient/necessary virtual battery model.
\( B^2_s, B^2_n \) Sufficient/Necessary virtual battery model obtained using the algorithms proposed in [1], [2].
\( \mathcal{P} \) Exact aggregate flexibility.
\( \mathcal{P}^i \) Individual flexibility of the \( i \)th TCL.
\( \mathcal{P}_o \) Prototype set.
\( \beta, \beta^i, \beta^i_* \) Scaling factors.
\( t, t^i, t^i_* \) Translation vectors.
\( I_m \) Identity diagonal matrix of dimension \( m \).
\( \text{diag}(s; -1) \) Lower subdiagonal matrix consisting of element \( s \) with appropriate dimension.
\( \bigcup \) Operator for Minkowski sum.

I. INTRODUCTION

Renewable energy resources such as wind and solar have a high degree of variability. Recent studies show that deep penetration of variable generations into the power grid requires substantial reserves from the generation side and flexible consumption via demand response [3]–[5]. Thermostatically controlled loads (TCLs) such as air conditioners, heat pumps, water heaters, and refrigerators are an important class of demand response assets due to their resource size and inherent flexibility. It is well recognized that coordination of TCLs presents a huge potential to provide various services to the grid, such as frequency regulation, energy arbitrage, renewable integration, and peak shaving, etc. [1], [3]–[10]. However, to integrate a large population of TCLs into system level operation and control, a fundamental challenge is to construct...
a simple and user-friendly model to manage them. This model
should be able to accurately capture their aggregate flexibility,
while being amenable to system level optimization and control.

The existing literature on aggregate modeling of TCLs
can be generally divided into two categories: modeling the
population dynamics of the loads, and characterizing the set
of admissible aggregate power profiles. In the first category,
the studies focus on establishing dynamical equations that
describe the probability density evolution of a population
of TCLs. These include partial differential equations [7], [11]–
[13] and Markov chains [6], [8], [14], [15]. However, in
order to reproduce the population dynamics accurately, these
models often require fine gridding of the state space, which
is computationally expensive [8], [14]. Moreover, the above
methods do not explicitly characterize the ex-ante flexibility
that TCLs can offer to the grid.

To address the above issue, the second category of aggregate
modeling aims to characterize the set of admissible aggregate
power profiles that can be consumed by the TCL population
without violating any comfort or operational constraint [1], [2],
[9], [16], [17]. The set of admissible power profiles represents
the aggregate flexibility of the TCL population. Such models
are for ex-ante planning, which can be easily incorporated into
solving various problems, such as multi-period optimal power
flow problems [13], [19], or unit commitment problems [20],
among others. The aggregate flexibility model characterizes
the power capacity of the load population, and thus can assist
their provision of ancillary services under the system level
coordination [21].

In the literature, the aggregate flexibility is often modeled
as a virtual battery model [1], [2], [9], [19], [22]. The virtual
battery model is a scalar linear system that resembles a simpli-
fied battery dynamics parameterized by charge and discharge
power limits, energy capacity limits, and self-discharge rate.
However, the existing virtual battery models for characterizing
the aggregate flexibility are very conservative [1], [2], espe-
cially for a TCL population with heterogeneous model param-
eters. The authors in [16] proposed several ways to improve
the flexibility characterization, but only certain specific battery
parameters were optimized independently under special cases
of limited TCL population heterogeneities.

This paper proposes a novel geometric approach which is
able to characterize the aggregate flexibility of heterogeneous
TCLs more accurately. We show that the power flexibility of
an individual TCL can be represented by a polytope, and the aggregate flexibility is the Minkowski sum of these polytopes.
However, an exact computation of this Minkowski sum is
numerically intractable when the number of TCLs is large.
Therefore, we estimate the aggregate flexibility by a subset
of the Minkowski sum of the individual flexibilities. Specifically, we first approximate each individual flexibility
polytope by its subset and superset, respectively, and then
calculate the Minkowski sum of the resulting approximations
accordingly. The key to facilitating the second step is to restrict
the approximation sets to be the homothets (i.e., the dilation
and translation) of a given convex set, the latter of which will
be referred to as the prototype set. Hence, for each TCL, we
calculate the maximum inner approximation and the minimum
outer approximation of its flexibility polytope with respect to
the homothets of the prototype set. Moreover, we show if the
prototype set is chosen as a polytope, then the optimization
problems can be formulated as linear programming problems,
and therefore can be solved very efficiently.

The above proposed geometric approach provides a general
framework for aggregating a large number of constrained linear dynamical systems when the summation of individual
quantities is of interest. In particular, the virtual battery modeling
in [1], [2], [9] can be viewed as a special case by choosing
the prototype set as the virtual battery model. Compared to the
optimization methods proposed in [1], [2], [9], our approach
takes advantage of the geometric information of the flexibility
polytopes and optimizes over additional decision variables
which represent the translation vector. These features improve
the modeling accuracy significantly and can deal with much
stronger parameter heterogeneity. We show that with 10%,
20%, and 30% TCL parameter heterogeneities, our approach
can improve the flexibility characterization accuracy by as
much as 129%, 141%, and 156% respectively. Moreover, we
demonstrate the efficacy of our geometric approach through
an example of providing frequency regulation service to the
grid, where we control the aggregate power of a population
of TCLs to track a regulation signal from the PJM Interconnec-
tion [23]. We show that the proposed approach substantially
increases the regulation capacity that TCLs can provide to the
ancillary service market, and the dispatched regulation signal
can be followed successfully without violating any comfort or
operational constraint of TCLs.

Other closely-related works on aggregate flexibility model-
ing of power system demand-side resources include [17],
[24]–[26]. These methods are based on the general idea
of computing the exact or approximate Minkowski sum of
different types of polytopes. An outer approximation of the
Minkowski sum of general polytopes was proposed in [17].
However, the number of inequality constraints resulted from
this method is non-deterministic and is very large in general.
Besides, the outer approximation cannot guarantee the fea-
sibility of the aggregate power profile. The references [24]–
[26] deals with the so-called resource polytopes, which arise
from the flexibility modeling of deferrable loads, such as plug-
in electric vehicles (PEVs), dishwashers, among many others.
In particular, the exact Minkowski sum of resource polytopes
was considered in [24]. However, it cannot be applied to the
flexibility polytopes resulted from TCL systems. In addition,
its cannot deal with high dimensional polytopes since the
number of inequality constraints increases exponentially with
the system dimension. Moreover, an inner approximation
of the resource polytopes using Zonotopes was proposed in [25],
whereas using general polytopes was investigated in [26]
via a lift and projection method.

The rest of the paper is organized as follows. In Section III
we present the problem statement. The geometric approach
to aggregate flexibility characterization is proposed in Section IV.
In Section V we apply the geometric approach to obtain the
virtual battery models. We demonstrate its efficacy through
numerical examples and case studies in Section VI. Finally,
we summarize our research and discuss the future work in
II. MODELING OF TCL AND FLEXIBILITY

In this section, we first present a nonlinear switching model that governs the temperature dynamics of a TCL. To facilitate aggregate flexibility modeling, we adopt a constrained linear system model to approximate the power consumption of the switching model. Based on this linear system model, we define the individual and aggregate flexibility of TCLs. It is worth mentioning that the linear system model is only employed for analysis purpose, and the nonlinear switching model is used in all the simulation studies presented in Section VI.

A. Nonlinear Switching Model of TCLs

The temperature evolution of a TCL can be described by a discrete-time switching model \([1], [7], [9]\):

\[
\theta(k) = a\theta(k-1) + (1-a)(\theta_a - bq(k)P_m),
\]

where \(\theta(k)\) is the TCL temperature at time step \(k\), \(\theta_a\) is the ambient temperature whose dynamics are much slower than \(\theta(k)\), \(P_m\) is the rated power, and \(q(k) \in \{0,1\}\) is a binary variable representing the operating state “ON/ OFF” of the system. The model parameters \(a\) and \(b\) are related to the thermal capacitance \(C_{th}\), thermal resistance \(R_{th}\), and coefficient of performance \(\eta\) of the system by \(a = e^{-\Delta T/(R_{th}C_{th})} \approx 1 - \Delta T/(R_{th}C_{th})\) and \(b = R_{th}\eta\), where \(\Delta T\) is the sampling time. Without loss of generality, we assume each TCL is a cooling device with \(P_m > 0\). The TCL switches between “ON” and “OFF” subject to the following local control rules,

\[
q(k) = \begin{cases} 
1, & \theta(k-1) \geq \theta_r + \Delta, \\
0, & \theta(k-1) \leq \theta_r - \Delta, \\
q(k-1), & \text{otherwise}, 
\end{cases}
\]

where \(\theta_r\) is the user-specified temperature set-point and \(\Delta > 0\) is half of the deadband.

B. Linear System Model of TCLs

To aggregate the flexibility of TCLs, the above switching model \([1], [2], [7]\) is very challenging for analysis due to its nonlinearity. Therefore, we consider a linear system model to approximate it,

\[
\theta(k) = a\theta(k-1) + (1-a)(\theta_a - bP(k)),
\]

where \(P(k) \in [0,P_m]\) is a continuous variable instead of a binary input of \(\{0,P_m\}\). It is shown in \([1], [2], [27]\) that the aggregate behavior of a large population of TCLs with model \([1]-[2]\) can be accurately approximated by model \([3]\). The continuous power input \(P(k)\) can be considered as the average of the binary power input of model \([1]-[3]\) over time. Additionally, for a large population of TCLs, the aggregate power of the linear system models can match that of the nonlinear switching models closely. After a change of variables, \(x(k) = C_{th}(\theta(k)-\theta_r)/\eta\), and \(u(k) = P(k) - P_0(k)\), where \(P_0(k) = (\theta_a - \theta_r)/b\) is the nominal power that keeps the temperature of model \([3]\) at its set-point, we can rewrite the above model as,

\[
x(k) = ax(k-1) + u(k)\delta,
\]

where \(\delta = (1-a)R_{th}C_{th} \approx \Delta T\). Additionally, the model has energy constraint \(x(k) \in [-x_-, x_+]\) with \(x_+ = x_- = C_{th}\Delta/\eta\), and input constraint \(u(k) \in [-u_-(k), u_+(k)]\) with \(u_-(k) = P_0(k)\) and \(u_+(k) = P_m - P_0(k)\).

C. Modeling of Flexibility

We consider a heterogeneous population of \(N\) TCLs modeled by \([4]\). Each TCL is parameterized by \(\mathbf{Q}^i := \{R_{th}^i, C_{th}^i, \theta_r^i, \Delta^i, \eta^i, \theta^i(0), P_m^i\}\). The aggregate power consumption of a population of TCLs has many feasible solutions that respect all the temperature and power constraints of TCLs. The key of non-disruptive control of TCLs for demand response is to accurately characterize their aggregate power flexibility over a considered time horizon \(\mathcal{T} := \{1,2,\ldots,m\}\). Before we proceed, we first define the individual and the aggregate flexibilities of TCLs.

**Definition 1.** For each TCL \(i = 1, \ldots, N\), its individual flexibility is defined as the set of all admissible power profiles

\[
\mathcal{P}^i = \left\{ [u^i(k)] \in \mathbb{R}^m \left| \begin{array}{c} x^i(k) = a x^i(k-1) + u^i(k)\delta, \quad \forall k \in \mathcal{T} \\ -u^i_- (k) \leq u^i(k) \leq u^i_+ (k), \quad \forall k \in \mathcal{T} \\ -x^i_- \leq x^i(k) \leq x^i_+, \quad \forall k \in \mathcal{T} \end{array} \right. \right\},
\]

where \([u^i(k)]\) denotes a vector whose \(k\)th element is \(u^i(k)\). The aggregate flexibility of a population of TCLs is a set of power profiles satisfying

\[
\mathcal{P} = \bigcup_{i=1}^N \mathcal{P}^i,
\]

where \(\bigcup\) denotes the Minkowski sum.

The set \(\mathcal{P}\) contains all the aggregate power profiles that are admissible to the population of TCLs. However, the expression of set \(\mathcal{P}\) is very abstract, and it is challenging to integrate it into the power system level operation and control. To represent the aggregate flexibility in an intuitive way, we define a simple and portable virtual battery model which will be used to describe the aggregate flexibility of TCLs.

**Definition 2.** An \(m\)-horizon discrete-time virtual battery model is a set of power profiles satisfying

\[
\left\{ [U(k)] \in \mathbb{R}^m \left| \begin{array}{c} X(k) = aX(k-1) + U(k)\delta, \quad \forall k \in \mathcal{T} \\ -D_- (k) \leq U(k) \leq D_+ (k), \quad \forall k \in \mathcal{T} \\ -E_- (k) \leq X(k) \leq E_+ (k), \quad \forall k \in \mathcal{T} \end{array} \right. \right\},
\]

The virtual battery model is specified by parameters \(\phi := (a, X(0), D_- (k), D_+ (k), E_- (k), E_+ (k), \forall k \in \mathcal{T})\), and we write it compactly as \(\mathcal{B}(\phi)\). In addition, \(\mathcal{B}(\phi)\) is called sufficient if \(\mathcal{B}(\phi) \subseteq \mathcal{P}\), and it is called necessary if \(\mathcal{B}(\phi) \supseteq \mathcal{P}\).

Note that the definition of the virtual battery model comes naturally from the definition of the individual flexibility. We can regard \(U(k)\) as the power draw of the battery and \(X(k)\) as its charging state which indicates the level of the energy stored in the battery. The quantities \(D_+ (k)\) and \(D_- (k)\) represent the
time-varying charging/discharging rate limits, and $E_+$ and $E_-$ represent upper/lower energy capacity limits relative to the nominal energy level of the battery. In addition, the parameter $a$ represents the self-discharge rate of the battery, which is due to the thermal exchange between the inner air and the ambient environment. We will show in the next section that the virtual battery model offers us great convenience in describing and characterizing the aggregate flexibility of TCLs.

III. GEOMETRIC APPROACH TO FLEXIBILITY CHARACTERIZATION

In this section, we present a geometric interpretation of the aggregate flexibility. Additionally, we show that it is generally intractable to compute the exact aggregate flexibility. Therefore, optimal approximations of the aggregate flexibility are proposed. They are further formulated as linear programming problems which can be solved very efficiently.

A. Polytope Interpretation of Flexibility

A polytope $Q$ is a solution set of a system of finite linear inequalities: $Q := \{U \in \mathbb{R}^m \mid FU \leq H\}$, where $\leq$ denotes elementwise inequality. A polytope $Q \subset \mathbb{R}^m$ is called full dimensional if it contains an interior point in $\mathbb{R}^m$. In this subsection, we show that the aggregate flexibility of TCLs can be represented by a polytope. Denoting the state and input vectors by $X^i = [x^i(k)]$ and $U^i = [u^i(k)]$, we rewrite (3) as,

\[
A^iX^i = B^iU^i + C^i,
\]

where $A^i = I_m + \text{diag}(-a^i; -1)$ is a lower bidiagonal matrix with 1’s on the main diagonal, and $-a^i$’s on the lower subdiagonal, $B^i = \delta I_m$, in which $I_m$ denotes the $m$-dimensional identity matrix, and $C^i = [a^i x^i(0), 0, \cdots, 0]^T$. The inverse of $A^i$ can be derived in an explicit form with polynomials of $a^i$ as its entries. It will be denoted by $\Lambda^i$ in the sequel. Additionally, the constraint sets for $X^i$ and $U^i$ are

\[
[\Lambda^i]U^i \leq U^i \leq \bar{U}^i, \quad [\Lambda^i]X^i \leq X^i \leq \bar{X}^i,
\]

where $U^i = [u^i_1(k), u^i_2(k)], U^i = [\bar{u}^i_1(k)], X^i = 1_m x^i, \text{ and } \bar{X}^i = 1_m \bar{x}^i$, in which $1_m$ is the $m$-dimensional column vector of all ones.

Using (5) and (6), the individual flexibility $P^i$ of the $i$th TCL can be expressed as

\[
P^i = U^i \cap X^i,
\]

where

\[
U^i = \{U^i \in \mathbb{R}^m \mid -\bar{U}^i \leq U^i \leq \bar{U}^i\},
\]

\[
X^i = \{U^i \in \mathbb{R}^m \mid -X^i \leq A^i B^i U^i + A^i C^i \leq \bar{X}^i\}.
\]

Since $-\infty < -\bar{u}^i_1 < \bar{u}^i_1 < +\infty$, it is straightforward to show that $U^i$ is a full dimensional hyper-rectangular (and thus a polytope). Similarly, because $A^i$ and $B^i$ are invertible and $-\infty < -\bar{x}^i_1 < \bar{x}^i_1 < +\infty$, we can show that $X^i$ is also a full dimensional polytope. It then follows from (8) that their intersection $P^i = U^i \cap X^i$ is a polytope if $P^i \neq \emptyset$. Moreover, it can be proven that the Minkowski sum of polytopes $P = \bigcup_{i=1}^N P^i$ is also a polytope [29].

For each TCL, its individual flexibility $P^i$ can be determined by [5]-[7]. However, the numerical complexity of calculating their Minkowski sum is prohibitively expensive when the number of TCLs is large. In fact, calculating the Minkowski sum of two sets $Q_1$ and $Q_2$, when they are polytopes specified by facets is NP-hard since the facets of the obtained polytope can grow exponentially with the number of the facets of $Q_1$ and $Q_2$ [30], [31]. Therefore, we take an alternative route and find its maximum inner (subset) approximation and minimum outer (superset) approximation instead.

B. Optimal Approximations of the Aggregate Flexibility

In this subsection, we aim to find sets $P_o$ and $P_n$ such that $P_o \subset P \subset P_n$. Any such sets $P_o$ and $P_n$ will be referred to as the sufficient approximation and the necessary approximation, respectively. Given a power profile $U$, if $U \notin P_o$, we can conclude that $U$ is not an admissible aggregate power profile for TCLs. On the other hand, if $U \in P_o$, then there exists a decomposition of $U$ such that $U = \sum_{i=1}^N U^i$, and $U^i \in P^i$ for all $i = 1, \cdots, N$.

Given a compact convex set $P_o$, we call $\beta^i P_o + t^i := \{U^i | U^i = \beta^i \xi + t^i, \forall \xi \in P_o\}$ a homothet of $P_o$, that is, the dilation and translation of $P_o$, where $\beta^i > 0$ is a scaling factor and $t^i \in \mathbb{R}^m$ is a translation factor. Since all $P^i$’s have the same structure (7), we conduct the inner and outer approximations of each $P^i$ with respect to a given set, $P_o$. The set $P_o$ will be referred to as the prototype set hereafter. Specifically, we will find within the homothets of $P_o$ the optimal approximations of each $P^i$’s. Fig. 1 illustrates this idea using a 2-dimensional example (i.e., the time horizon is taken as $m = 2$), where $\beta^i P_o + t^i_o$ is a minimum outer approximation of $P^i$ and $\beta^i P_o + t^i_i$ is a maximum inner approximation of $P^i$.

The benefit of using homothets to approximate $P^i$’s is that it admits an efficient calculation of their Minkowski sum. It was shown in [29] that given a convex set $Q$, non-negative scalars $\beta^i$ and $\beta^o$, and any scalars $t^i$ and $t^o$, the calculation of their Minkowski sum is simply as follows,

\[
(\beta^i Q + t^i) \bigcup (\beta^o Q + t^o) = (\beta^i + \beta^o) Q + (t^i + t^o),
\]

which says that the Minkowski sum of homothets of a convex set reduces to the sum of the scaling factors and the sum of the translation factors. Therefore, once we have the $P_o$-homothetic approximations of the individual flexibility polytopes, the
approximation of the aggregate flexibility can be calculated very easily.

Our focus now becomes how to choose the \( \mathcal{P}_o \)-homothet that optimally approximates \( \mathcal{P}_i \). Specifically, for each \( i = 1, \cdots, N \), we are interested in finding the maximal \( \mathcal{P}_o \)-homothet that is contained in \( \mathcal{P}_i \), which can be mathematically expressed as,

\[
\begin{align*}
\text{maximize} & \quad \beta^i \\
\text{subject to} & \quad \beta^i \mathcal{P}_o + t^i \subset \mathcal{P}_i, \\
& \quad \beta^i > 0,
\end{align*}
\]

and the minimal \( \mathcal{P}_o \)-homothet that contains \( \mathcal{P}_i \),

\[
\begin{align*}
\text{minimize} & \quad \beta^i \\
\text{subject to} & \quad \beta^i \mathcal{P}_o + t^i \supset \mathcal{P}_i, \\
& \quad \beta^i > 0.
\end{align*}
\]

In the above optimization problems, the optimality is in the sense of inclusion, i.e., if \( (\beta^i, t^i) \) is an optimal solution, then there is no other homothet of \( \mathcal{P}_o \) contained in between \( \beta^i \mathcal{P}_o + t^i \) and \( \mathcal{P}_i \). We will refer to the optimal solution of (9) and (10) as the Maximum Inner Approximation (MIA) and Minimum Outer Approximation (MOA), respectively. For convenience, we denote the solutions of problems (9) and (10) by \( \beta^i, t^i \) = MIA(\( \mathcal{P}_i, \mathcal{P}_o \)) and \( \beta^i, t^i \) = MOA(\( \mathcal{P}_i, \mathcal{P}_o \)), respectively.

In order to have tractable solutions of the MIA and MOA problems, we have to specify \( \mathcal{P}_o \) beforehand. Since each \( \mathcal{P}_i \) is a polytope, to achieve a good approximation of it, it is natural to choose \( \mathcal{P}_o \) as a polytope too. Furthermore, we show that the MIA and MOA problems under such choice can be solved efficiently by equivalent linear programming problems (32). The specific optimization algorithms to solve for \( (\beta^i, t^i) = \text{MIA}(\mathcal{P}_i, \mathcal{P}_o) \) and \( (\beta^i, t^i) = \text{MOA}(\mathcal{P}_i, \mathcal{P}_o) \) are derived as follows. In view of (5)-(7), the individual flexibility polytope can be written as \( \mathcal{P}_i^\star = \{U \in \mathbb{R}^m : FU \leq H^i, \} \), where \( F^i = (I_m, -I_m, A^iB^i, -A^iB^i), \)
\( H^i = (U^i, U^i, \bar{X}^i - N^iC^i, \bar{X}^i + N^iC^i), \)
where \((x, y)\) denotes the matrix \([x^T, y^T]^T\) for two matrices \(x\) and \(y\) with the same number of columns. Moreover, if \( \mathcal{P}_o \) is chosen to have the form \( \{U \in \mathbb{R}^m : FU \leq H \} \), then we have the following theorem:

**Theorem 1.** The optimal solution of MIA(\( \mathcal{P}_i, \mathcal{P}_o \)) is given by
\( \beta^i_s = 1/s^i \) and \( t^i_s = -t^i/s^i \), if \( (s^i, t^i, G_s) \) is an optimal solution of the following linear programming problem,

\[
\begin{align*}
\text{minimize} & \quad s^i \\
\text{subject to} & \quad GF = F^i, \\
& \quad GH \leq s^iH^i + F^i t^i. 
\end{align*}
\]

Similarly, the optimal solution \( \beta^i, t^i = \text{MOA}(\mathcal{P}_i, \mathcal{P}_o) \) is solved by

\[
\begin{align*}
\text{minimize} & \quad \beta^i \\
\text{subject to} & \quad GF^i = F, \\
& \quad GH^i \leq \beta^i H + F t^i. 
\end{align*}
\]

**Proof:** See Appendix A

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**Figure 2.** Diagram of the suboptimal sufficient virtual battery characterization

The above theorem provides algorithms for solving the optimal inner/outer \( \mathcal{P}_o \)-homothetic approximations of individual flexibility polytopes. Furthermore, it is easy to see that if \( \mathcal{P}_i \subset \mathcal{Q}^i \) for some polytope \( \mathcal{Q}^i \) for each \( i \), then we also have \( \cup_i \mathcal{P}_i \subset \cup_i \mathcal{Q}^i \). Therefore, the inner/outer approximation of the aggregate flexibility can be obtained by the Minkowski sum of the obtained \( \mathcal{P}_o \)-homothets. By the formula (8), we see that \( \|y_i|(\beta^i \mathcal{P}_o + t^i) = (\sum \beta^i) \mathcal{P}_o + \sum t^i \) is a sufficient approximation of \( \mathcal{P} \) if \( (\beta^i, t^i) = \text{MIA}(\mathcal{P}_i, \mathcal{P}_o) \), and is a necessary approximation of \( \mathcal{P} \) if \( (\beta^i, t^i) = \text{MOA}(\mathcal{P}_i, \mathcal{P}_o) \). Note that the obtained aggregate flexibility approximations are also \( \mathcal{P}_o \)-homothets. Therefore, some desired properties of the aggregate flexibility model can be achieved through specifying \( \mathcal{P}_o \)’s structure. In the next section, we will choose \( \mathcal{P}_o \) as virtual battery models and derive the battery parameters for the approximated aggregate flexibility. We further propose computationally more efficient virtual battery modeling methods by exploiting their special structures.

### IV. Virtual Battery based Flexibility Characterization

We now consider a special case of our geometric approach proposed in the last section, where the prototype set \( \mathcal{P}_o \) is chosen as the virtual battery model (see Definition 2). In the sequel, we assume that the battery parameter \( \alpha \) is predetermined and fixed (e.g., taking the mean of all TCL parameters \( \alpha^i \’s \)), and focus on estimating its power limits, energy capacity, and initial energy state. Hence, we denote \( \phi = (C, D, E, F, \bar{E}) \) as the parameter of the virtual battery in the sequel, where the vector notations are given by \( C = [aX(0), 0, \cdots, 0]^T, \ D = [D_{-}(k)], \bar{E} = [E_{-}(k)], \bar{E} = [E_{+}(k)] \). We will use \( \mathbb{B}_\alpha \) as the short notation for the virtual battery \( \mathbb{B}(\phi, \alpha) \), where \( \alpha \) denotes the subscript \( \alpha \), \( s \), and \( n \) meaning prototype, sufficient, and necessary virtual battery, respectively.

#### A. Optimal Virtual Battery

In this subsection, we choose \( \mathcal{P}_o \) to be in the same form as \( \mathcal{P}_i \’s \) with parameter \( \Omega \), being the mean of all TCL parameters \( \Omega^i \’s \). Clearly, the \( \mathcal{P}_o \) of this choice is also a virtual battery model. To emphasize it, we will denote this prototype virtual battery by \( \mathbb{B}_o \), with parameters \( \phi_0 = (C_o, D_o, E_o, \bar{E}_o) \) obtained from the average of the TCL parameters. In view of (5)-(7), the prototype virtual battery can be expressed as \( \mathbb{B}_o = \).
\( U \in \mathbb{R}^m : F U \leq H \), where \( F = (I_m, -I_m, \Lambda B, -\Lambda B) \) and \( H = (D_o, D_o, E_o - \Lambda C_o, E_o + \Lambda C_o) \). As discussed in the previous section, we can approximate each \( P^i \) by \( \beta^i \mathbb{B}_o + t^i \), where \( \beta^i \) and \( t^i \) are the optimal solutions of (11) or (12). Now let \( \beta = \sum_{i=1}^N \beta^i \) and \( t = \sum_{i=1}^N t^i \). We will show that \( \beta \mathbb{B}_o + t \) is a sufficient or necessary battery depending on whether \( \{\beta^i, t^i\} \)’s are obtained from (11) or (12).

**Theorem 2.** \( B_s := \beta \mathbb{B}_o + t \) is a sufficient battery if \( (\beta^i, t^i) = \text{MIA}(P^i, \mathbb{B}_o) \), and is a necessary battery if \( (\beta^i, t^i) = \text{MOA}(P^i, \mathbb{B}_o) \). The battery parameters \( \phi_s = (C, D, \bar{D}, E, \bar{E}) \) is given by

\[
\begin{align*}
C &= \beta C_o, \\
D &= \beta D_o - \bar{D} = \beta D_o + t, \\
\bar{D} &= \beta D_o - \Lambda B t, \\
E &= \beta E_o - \Lambda B t, \\
\bar{E} &= \beta E_o + \Lambda B t.
\end{align*}
\]

In addition, \( \forall U \in \mathbb{B}_s \), the individual admissible power profile is given by

\[
U^i = \frac{\beta^i}{\beta}(U - t) + t^i, \quad \forall i.
\]

**Proof:** See Appendix [B].

This theorem is a direct result of the proposed general aggregate flexibility modeling method by choosing \( P_o \) as the virtual battery model. The advantage of the virtual battery model lies in that the resulted aggregate flexibility model has a very simple form, which is very desirable in practice for system level optimization. In addition, as will be shown later through simulations, it can give a very accurate approximation of the exact aggregate flexibility. Moreover, we emphasize that \( \beta^i \)’s and \( t^i \)’s for different \( i \)’s can be computed in parallel. This makes our algorithm scalable and it can be executed efficiently even with a large number of TCLs.

### B. Suboptimal Virtual Battery

To further reduce the computational complexity, we will propose a suboptimal method. In the above optimal method, we approximate \( P^i = U^i \cap \mathcal{X}^i \) with respect to \( \mathbb{B}_o \) as a whole. Alternatively, we can approximate its components \( U^i \) and \( \mathcal{X}^i \) separately. Although this is generally more conservative than the optimal method, it substantially reduces the numerical complexities since the corresponding linear programming problem has a smaller dimension.

We first consider the sufficient battery characterization. For notational convenience, we denote \( \mathbb{B}_o = \cup_{U^i} \cap \mathcal{X}^i \) with the same structure of (7). There are mainly 3 steps of this method, which are illustrated in Fig. 2 by a 2-dimensional example (i.e., the time horizons \( m = 2 \)). Assuming \( \mathcal{X}_o \) is given as one of the components of \( \mathbb{B}_o \), in the first step each \( \mathcal{X}^i \) is approximated with respect to \( \mathcal{X}_o \), which yields \( (\beta^i, t^i) = \text{MIA}(\mathcal{X}^i, \mathcal{X}_o) \). Next, to guarantee the homothetic transformation between \( \mathbb{B}_o \) and \( P^i \), the \( \mathcal{X}_o \)-component of \( \mathbb{B}_o \) can be determined by enforcing \( \beta^i U_o + t^i \subset U^i \) for all \( i \), or equivalently by enforcing

\[
U_o \subset \cap_i (U^i - t^i)/\beta^i.
\]

Since \( U^i \)’s are all hyper rectangulars, the right hand side of (15) can be calculated exactly which is given below in (16).

Clearly, this is the largest \( U_o \) that satisfies (15). The last step is to obtain \( \mathbb{B}_o \) as \( \mathcal{X}_o \cap U_o \), and then \( \mathbb{B}_s \) is obtained by \( \beta \mathbb{B}_o + t \).

We next prove that with \( \mathbb{B}_o \) obtained in this way, \( \beta \mathbb{B}_o + t \) is a sufficient battery.

**Theorem 3.** Let \( (\beta^i, t^i) = \text{MIA}(\mathcal{X}^i, \mathcal{X}_o) \) and \( U_o = \{ U \in \mathbb{R}^m | -U \leq U \leq \bar{U} \} \), where

\[
U = \min_i \bar{U}^i + t^i, \quad \bar{U} = \min_i \bar{U}^i - t^i,
\]

and \( \min \) is element-wise. Then \( \mathbb{B}_s := \beta \mathbb{B}_o + t \) is a sufficient battery. The battery parameters \( \phi_s \) are given by (14).

**Proof:** See Appendix [C].

It is worth mentioning that Theorem 3 can be easily adapted to obtain a suboptimal necessary battery [33]. However, its estimation of the power limits can be very inaccurate. In the following theorem, we develop a suboptimal necessary battery modeling method that generalizes the methods proposed in [1], [2] to improve the power limits characterization. The main idea is that the two components of the necessary battery can be obtained independently. In particular, its power limits can be obtained by simply adding up the individual power limits, while the energy capacity limits are obtained through the \( \mathcal{X}_o \)-homothetic approximation of each \( \mathcal{X}^i \).

**Theorem 4.** Let \( (\beta, t) = \text{MOA}(\mathcal{X}^i, \mathcal{X}_o) \) and \( U_o = \cup_{i} U^i \).

Then \( \mathbb{B}_n := \beta \mathcal{X}_o + t \cap U_o \) is a necessary battery. The battery parameters are given by

\[
\begin{align*}
C &= \beta C_o, \\
D &= \sum_{i=1}^N U^i, \\
\bar{D} &= \sum_{i=1}^N \bar{U}^i, \\
E &= \beta E_o - \Lambda B t, \\
\bar{E} &= \beta E_o + \Lambda B t.
\end{align*}
\]

**Proof:** See Appendix [D].

Our approach (Theorems 2-4) includes the approach proposed in [1], [2] as a special case. As an example, it can be seen from (17) that the suboptimal necessary battery calculates the power bounds in the same way as [1], [2], i.e., by summing over the individual power bounds. In addition, it optimizes the approximation of \( \mathcal{X}^i \) using \( \mathcal{X}_o \)-homothetic. In fact, if we further drop these optimization schemes (i.e., \( \text{MIA}(\mathcal{X}^i, \mathcal{X}_o) \) and \( \text{MOA}(\mathcal{X}^i, \mathcal{X}_o) \) in Theorems 3-4) and instead obtain the energy bounds from (5) using matrix norm inequalities, then we will get the counterpart of the modeling method proposed in [1], [2] in the discrete-time finite-horizon case. Moreover, the optimal method proposed in Theorem 2 approximates \( P^i \) directly, and therefore offers the best performance among all these methods.

The major computation of the suboptimal methods only involves solving \( \text{MIA}(\mathcal{X}^i, \mathcal{X}_o) \) or \( \text{MOA}(\mathcal{X}^i, \mathcal{X}_o) \), which involves about 50\% fewer constraints and 75\% fewer decision variables as compared to the optimal method. Therefore, it can be solved much faster than the optimal case.

### V. Case Studies

In this section, we first compare the characterized flexibilities using our geometric method and the method in [1], [2].
We next demonstrate the efficacy of our approach through an example of providing frequency regulation service. We then show that the population of TCLs using our flexibility characterization approach can provide more regulation capacity to the grid while achieving excellent tracking of the regulation signal and guaranteeing the thermal comfort of the end users.

We consider a population of 1000 heterogeneous TCLs. Their model parameters \( \Omega \)'s are assumed to be uniformly distributed \([1, 7, 15] \), e.g., the thermal capacities \( C_{th} \)'s are uniformly distributed within \([ (1-\epsilon)C_{th}, (1+\epsilon)C_{th} ] \), where \( \bar{C}_{th} \) is the mean value, and \( \epsilon \) models the degree of heterogeneity. Additionally, the ambient temperature profile is picked as a typical hot summer day in Columbus, OH \([34] \).

For the comparison of the virtual batteries, note that if a sufficient battery has both larger power and energy limits than another one, we claim that the former extracts more flexibility than the latter and the latter is more conservative. On the other hand, if a necessary battery has both larger power and energy limits than another one, we claim that the former overestimates more flexibility than the latter, and the latter is more accurate.

### A. Performance Comparison

We first calculate the optimal sufficient battery using Theorem 2. The MIA and MOA problems are solved using the GLPK linear programming solver \([35] \) interfaced with YALMIP \([36] \). The blue dash-dot lines in Figs. 3 (a) (respectively, (b)) represent the lower and upper power (respectively, energy) limits of the optimal sufficient battery (denoted by \( B_s^* \)). If a given power profile \( U \in \mathbb{R}^m \) belongs to \( B_s^* \), then it must lie between the two blue dash-dot lines in Fig. 3(a), and the energy state resulting from this power profile through the battery dynamical equation \( \dot{X}(k) = aX(k-1) + bU(k) \) (see Definition 2) must lie between the blue dash-dot lines in Fig. 3(b). In other words, given a power profile \( U \) which lies between the two blue dash-dot lines in Figs. 3 (a), if the associated energy state vector \( X \) also lies between the blue dash-dot lines in Figs. 3 (b), then \( U \) is a feasible power trajectory of the TCL population.

Our first observation is that the ambient temperature has a significant impact on the aggregate flexibility. For example, when the temperature is the lowest around 6:00 AM (the temperature profile is not shown here due to the space limit), the baseline aggregate power consumption is the lowest at this time. As a result, the lower power limit (which corresponds to the largest possible down regulation capacity) is the smallest (see Fig. 3), while the upper power limit (which corresponds to the largest possible up regulation capacity) is the largest. The situation is inverted when the temperature reaches the highest around 3 PM, since the nominal aggregate power consumption is the highest.

Next, we use Theorem 3 to obtain the suboptimal sufficient battery (denoted by \( B_s^1 \)) and the red solid lines in Fig. 3 (a) (respectively, (b)) are respectively its lower and upper power (respectively, energy) limits of the suboptimal sufficient battery. Roughly speaking, the optimal sufficient battery \( B_s^* \) extracts more flexibility than the suboptimal battery \( B_s^1 \), since \( B_s^* \) has larger power limits than \( B_s^1 \) and their energy limits are similar. Moreover, we compare our geometric approach with the characterization methods in \([1], [2] \), where the sufficient battery (denoted by \( B_s^\infty \)) is obtained by solving the optimization problem in \([2] \) Theorem 3]. The orange dashed lines in Figs. 3 (a) and (b) represent \( B_s^\infty \). It can be seen that both \( B_s^* \) and \( B_s^1 \) extracts more flexibility than \( B_s^\infty \), i.e., \( B_s^* \supset B_s^\infty \) and \( B_s^1 \supset B_s^\infty \), since both their power and energy limits are larger than those of \( B_s^\infty \).

Furthermore, we compare in Fig. 4 the power and energy limits of the optimal necessary battery \( B_n^* \), suboptimal necessary battery \( B_n^1 \), and necessary battery \( B_n^\infty \) obtained in \([1] \).
Table I

| Heterogeneity | $B^*_n$ | $B^*_m$ | $B^*_n$ | $B^*_m$ |
|---------------|--------|--------|--------|--------|
| $\epsilon = 10\%$ | 129.16% | 86.13% | 0.34% | 0.34% |
| $\epsilon = 20\%$ | 141.05% | 87.86% | 0.61% | 0.61% |
| $\epsilon = 30\%$ | 155.94% | 88.43% | 0.82% | 0.82% |

[2]. In obtaining the optimal necessary battery, we choose the suboptimal necessary battery as its prototype battery model. It can be seen all the necessary batteries have similar estimations since their power and energy limits are similar. Even though no strict inclusion relationship is present in our numerical comparison, $B^*_n$ and $B^*_m$ are generally more accurate than $B^*\hat{n}$, since their energy limits are slightly tighter than $B^*\hat{n}$ most of the time, as shown in Fig. 4 (b).

Compared to the method in [1], [2], our approach takes advantage of the geometric information of each individual flexibility set, and thus improves the approximation of the aggregate flexibility. In addition, the flexibility characterization method in [1], [2] requires the power and energy bounds (e.g., $u_-(k)$, $u_+(k)$, $x_-$, and $x_+$) to be non-negative for each TCL. This non-negativity requirement restricts the allowable degree of parameter heterogeneity. In contrast, the proposed geometric approach removes such a restriction, and allows us to characterize the aggregate flexibility of a population of TCLs where their model parameters are strongly heterogeneous. We show the performance improvement of the proposed approach at different heterogeneity degrees $\epsilon$ in Table I. For the convenience of comparison, we assume only the thermal parameters $C_{th}$ and $R_{th}$ are heterogeneous. The numbers in the table represent the average percentage improvement of the power and energy limits as compared to the battery $B^*\hat{m}$ and $B^*\hat{n}$ in [1], [2].

$$0.5 | \Gamma(D, D^o) + \Gamma(E, E^o) | ,$$

where $\Gamma(D, D^o) := \frac{1}{m} \sum_{k=1}^{m} \frac{D^o_{-}(k) + D^o_{+}(k) - D_{-}(k) - D_{+}(k)}{D^o_{-}(k) + D^o_{+}(k)}$, $D_{-}(k), D_{+}(k), \forall k \in T$ denote the power limits of the optimal or suboptimal batteries, and the ones with superscript $\diamond$ represent the power limits of the virtual batteries proposed in [1], [2]. The other term $\Gamma(E, E^o)$ is defined similarly for the energy bounds. It can be seen from Table I that the stronger the heterogeneity is, the larger the improvement can be achieved by the proposed approach.

### B. Providing Frequency Regulation Service

In order to provide frequency regulation service, each service provider needs to bid its regulation capacities into the day-ahead or hour-ahead ancillary service market. After the market is cleared, each awarded regulating resource will be dispatched a regulation signal $r(t)$ in real-time. The regulation signal will be within the bidded capacity, and it is generally broadcast every 4 seconds depending on the independent system operators. Each regulation resource is obliged to follow this regulation signal accurately since the tracking accuracy will be reflected in the financial settlement.
the regulation signal \( r(t) \), the aggregate power deviation from baseline \( U(t) \), and the energy state of the virtual battery \( X(t) \) are plotted in Fig. 5. It can be seen that as long as \( r(t) \in \mathbb{B}^o \) (i.e., both the regulation signal \( r(t) \) and the resulted charging state are within the power limits and the energy limits of \( \mathbb{B}^o \); respectively), \( U(t) \) can track \( r(t) \) successfully, even when \( r(t) \) violates the power and energy limits of the sufficient battery \( \mathbb{B}^\infty \) obtained in \( \mathbb{L} \). This shows that our characterization method is more accurate in estimating the aggregate flexibility of TCLs. Moreover, we observe that \( U(t) \) fails to track \( r(t) \) shortly after the energy state exceeds the upper energy limit of optimal sufficient battery model \( \mathbb{B}^o \) at around 3000 seconds. This again implies that our geometric approach makes a very accurate approximation to the aggregate flexibility of TCLs.

Moreover, Fig. 6 shows the temperature evolutions of several randomly chosen TCLs, where the black dashed lines represent the corresponding allowable temperature bands for each TCLs. We observe that these TCLs are well regulated within the user-specified temperature bands, which means the thermal comfort of end users is strictly respected.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a novel geometric approach to characterize the aggregate flexibility of a population of TCLs. We showed that the power flexibility of an individual TCL could be modeled as a polytope, and their aggregate flexibility was represented by the Minkowski sum of the aforementioned polytopes. However, an exact computation of the Minkowski sum was numerically expensive. We thus developed two optimization-based algorithms to approximate the aggregate flexibility using the maximum inner approximation and minimum outer approximation with respect to the homothets of a prototype set. Additionally, we showed that if the prototype was chosen to be a virtual battery model, our geometric approach extracted more flexibility than existing algorithms in the literature. Moreover, we demonstrated the efficacy of our method through a case study of controlling TCLs to provide regulation service to the grid. We showed that our method could enable TCLs to bid more regulation capacities to the ancillary service market, while achieving excellent tracking of the regulation signal and respecting the thermal comfort requirement of end users. In the future, we are interested in examining the impact of no-short-cycling or minimum off-time constraint on the aggregate flexibility and the aggregate power ramping rate of TCLs.

APPENDIX

A. Proof of Theorem 7

To prove the theorem, we first state the following version of Farkas’s lemma \([32, 37]\), which assists in deriving the algorithms for solving the MIA and MOA problems.

**Lemma 1** (Farkas’ lemma). Suppose that the system of inequalities \( Lx \leq b, L \in \mathbb{R}^{m \times n} \) has a solution and that every solution satisfies \( Mx \leq d, M \in \mathbb{R}^{k \times n} \). Then there exists \( G \in \mathbb{R}^{k \times m} \), \( G \geq 0 \), such that \( GL = M \) and \( Gb \leq d \). The converse is also true.

After a change of variables \( \beta^i = 1/s^i \), and \( t^i = -r^i/s^i \), where \( s^i > 0 \), the optimization problem (9) is equivalent to

\[
\begin{align*}
\text{minimize} & \quad s^i > 0, r^i \\
\text{subject to} & \quad \mathcal{P}_o \subset s^i \mathcal{P}^i + r^i.
\end{align*}
\]

Moreover, we see that \( s^i \mathcal{P}^i + r^i \) is the solution set of \( F^i U^i \leq s^i H^i + F^i r^i \) with respect to \( U^i \in \mathbb{R}^m \). Therefore, by Lemma 1, there exists a matrix \( G \) such that \( GF = F^i \), and \( GH \leq s^i H^i + F^i r^i \). As such, we showed that the optimization problem (18) or (19) is equivalent to the linear programming problem (11). Similarly, we can prove that the optimization problem (10) is equivalent to the linear programming problem (12).

B. Proof of Theorem 2

In this section, we only prove the results for the sufficient battery characterization. The necessary battery characterization can be proved analogously. Suppose \( \{ \beta^i, t^i \mid \forall i = 1, \cdots, N \} \) are the solutions of the corresponding MIA problems and let \( \mathbb{B}_o := \bigcup_{i=1}^N (\beta^i \mathbb{B}_o + t^i) \). It can be shown that

\[
\mathbb{B}_o = \bigcup_{i=1}^N \beta^i \mathbb{B}_o + t = \beta \mathbb{B}_o + t,
\]

where the last equality is the Minkowski sum of the homothets of \( \mathbb{B}_o \) (recall the formula (6)). Furthermore, since \( \beta^i \mathbb{B}_o + t^i \subset \mathcal{P}^i \), we have

\[
\bigcup_{i=1}^N (\beta^i \mathbb{B}_o + t^i) \subset \bigcup_{i=1}^N \mathcal{P}^i = \mathcal{P}.
\]

This implies \( \mathbb{B}_o \) is a sufficient battery. Its battery parameters \( \phi_o \) can be obtained by formulating the constraint sets on \( U \) from \( (U - t)/\beta \in \mathbb{B}_o \). Now \( \forall U \in \mathbb{B}_o \), we obtain from (19) that \( (U - t)/\beta \in \mathbb{B}_o \). It follows by the scaling and translating of \( \mathbb{B}_o \) that \( \beta(U - t)/\beta + t^i \in \mathcal{P}^i \). This completes the proof.

C. Proof of Theorem 3

From the solution of the MIA problem, we have \( \forall i = 1, \cdots, N, X^i \supseteq \beta^i X_o + t^i \). We next show that \( U^i \supseteq \beta^i U_o + t^i, \forall i = 1, \cdots, N \). Since \( U^i \)'s are hyper rectangles, the largest \( U_o \) we can obtain is \( U_o = \cap_i (U^i - t^i)/\beta^i \). This is
equivalent to having the power limits of \( B_o \) in (16). The above two inclusion relationships yield
\[
P = \bigcup_{i=1}^{N} \mathcal{X}^i \cap \mathcal{U}^i \supseteq \bigcup_{i=1}^{N} (\beta_i \mathcal{X}^i_o + t_i) \cap (\beta_i \mathcal{U}^i + t_i) = \bigcup_{i=1}^{N} \beta_i (\mathcal{X}^i_o \cup \mathcal{U}^i) + t_i = \beta \mathcal{B}^+_o + t,
\]
where \( \mathcal{B}^+_o := \mathcal{X}^i_o \cup \mathcal{U}^i_o \) and the last equality is the Minkowski sum of the homothets of \( \mathcal{X}^i \cup \mathcal{U}^i \). It is straightforward to see that \( P \supset \mathcal{B}^+_o := \beta \mathcal{B}^+_o + t \), and thus \( \mathcal{B}^+_o \) is sufficient. The rest of the proof regarding the derivation of the parameter \( \phi \) and the power profile decomposition is the same as those in the proof of Theorem 2.

\section*{D. Proof of Theorem 2}

For arbitrary subsets \( Q_i \) of \( \mathbb{R}^m \), it is easy to verify that the following holds,
\[
\bigcup_{i=1}^{N} (Q_i \cap S_i) \subseteq \left( \bigcup_{i=1}^{N} Q_i \right) \cap \left( \bigcup_{i=1}^{N} S_i \right).
\]
Therefore, we have
\[
\bigcup_{i=1}^{N} (\beta_i \mathcal{X}^i_o + t_i) \cap \mathcal{U}^i \subseteq \bigcup_{i=1}^{N} (\beta_i \mathcal{X}^i_o + t_i) \cap \left( \bigcup_{i=1}^{N} \mathcal{U}^i \right) = (\beta \mathcal{X}^i + t) \cap \mathcal{U}^i = \mathcal{B}^+_o.
\]
Combining with (20), we have \( P^+ \subseteq \mathcal{B}^+_o \). Hence, \( \mathcal{B}^+_o \) is a necessary battery. The battery parameters \( C, D, \tilde{D} \) in (17) can be obtained from formulating the constraint sets on \( U \) from \((U - t)/\beta \in \mathcal{X}^i_o\), while \( E, \tilde{E} \) can be obtained by noticing that the Minkowski sum of hyper-rectangulars can be simply calculated by adding the individual bounds.

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