STRAIGHT RULED SURFACES IN THE HEISENBERG GROUP

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Abstract. We generalise a result of Garofalo and Pauls: a horizontally minimal smooth surface embedded in the Heisenberg group is locally a straight ruled surface, i.e. it consists of straight lines tangent to a horizontal vector field along a smooth curve. We show additionally that any horizontally minimal surface is locally contactomorphic to the complex plane.

1. Introduction

A ruled surface in the Heisenberg group $\mathcal{H}$ is a surface which is foliated by geodesics of the Carnot–Caratheodory metric $d_{cc}$ in $\mathcal{H}$. These geodesics are the rulings of the surface, and when they are Euclidean straight lines we call the ruled surface straight. The class of Heisenberg ruled surfaces is the analogue of its Euclidean counterpart; it is a classical theorem of elementary differential geometry of surfaces that ruled surfaces embedded in $\mathbb{R}^3$ have vanishing Gaussian curvature $K$ and are locally isometric to the plane. Moreover, every sufficient small portion of a surface which is locally isometric to the plane is a generalised cylinder, or a generalised cone or a tangent developable, see for instance [3].

A smooth surface $S$ embedded in $\mathcal{H} = \mathbb{R}^3$ inherits a sub–Riemannian structure from the one of $(\mathcal{H}, d_{cc})$; this is described by the horizontal normal vector field $\nu_S$ on the surface. Points of the surface where $\nu_S$ can not be defined are called characteristic and the set of these points form the characteristic locus of $S$. The pull–back of the contact form $\omega$ of $\mathcal{H}$ defines a 1–form $\omega_S$ in $S$ and two surfaces $S$ and $\tilde{S}$ are called locally contactomorphic if there exists a local diffeomorphism $f: S \to \tilde{S}$ away from the characteristic loci so that $f^* \omega_{\tilde{S}} = \lambda \omega_S$. Such contactomorphisms between surfaces are the sub–Riemannian analogues of local isometries in the Euclidean case. A notion of mean curvature, the horizontal mean curvature $H^h$, is defined in non characteristic points of $S$ in terms of the derivatives of the components of $\nu_S$: if $X$ and $Y$ are the horizontal vector fields of $\mathcal{H}$ and $\nu_S = \nu_1 X + \nu_2 Y$ then

$$H^h = X \nu_1 + Y \nu_2.$$ 

Surfaces with vanishing horizontal mean curvature are called $H$–minimal. In [7], Garofalo and Pauls proved the following result concerning surfaces in $\mathcal{H}$ which are graphs of functions over the $xy$–plane (Corollary 5.3):

Theorem. If $S$ is a portion of a $C^2$– surface $S$ which is a graph of a function over the $xy$–plane in $\mathcal{H}$ with non characteristic points, then it is $H$–minimal if and only if it is a piece of a ruled surface whose rulings are straight lines (i.e. a straight ruled surface).

In this article, we consider arbitrary smooth surfaces (not necessarily graphs) embedded in $\mathcal{H}$, see Section 3.1 for details. Our main theorem is the following version of Garofalo–Pauls’ result:

Theorem 1.1. Straight ruled surfaces have zero horizontal mean curvature and are all locally contactomorphic to the complex plane. Moreover, if a surface $S$ has everywhere zero horizontal mean curvature, then every sufficiently small portion of $S$ comprising only of non characteristic points is a straight ruled surface.

1991 Mathematics Subject Classification. 53C17, 49Q05.
Key words and phrases. Heisenberg geometry, horizontally minimal surfaces.
The method of proof is elementary and may be summarised as follows. The kernel of the induced contact form $\omega_S$ of $S$ is the vector field $J_\nu S \in T(S)$, where $J$ is the natural complex structure acting naturally on the horizontal bundle of $\mathcal{H}$. Therefore, away from characteristic points there is a foliation of $S$ by horizontal surface curves (the horizontal flow of $S$). It is proved (see Proposition 3.13 and also [1]) that the horizontal mean curvature $H_h(p)$ at a non characteristic point of $S$ is equal to $\kappa_s(p')$, where $p'$ is $pr_C(p)$ and $\kappa_s(p')$ is the signed curvature of the plane curve which is the projection to $C$ of the leaf of the horizontal flow passing from $p$. Now, a surface in $\mathcal{H}$ which is locally contactomorphic to the complex plane must have $H^h = 0$ (see Proposition 3.15) and straight ruled surfaces share this property (Proposition 4.1). On the other hand, in an $H-$minimal surface all projected curves of the horizontal flow are straight lines and the only option for a sufficiently small portion of $S$ containing an arbitrary non characteristic point $p$ is to be a straight ruled surface, (see proof of Theorem 1.1 in Section 4).

Next, we give two examples of classes of smooth surfaces with empty characteristic locus. The first one is that of the horizontal tangent developables, see Section 5.1. The second is that of surfaces which besides empty characteristic locus, also have closed induced contact form. We prove the following:

**Proposition 5.3** Smooth surfaces in $\mathcal{H}$ with empty characteristic locus and closed induced contact form are exactly the planes which are perpendicular to the complex plane $C$.

This result is comparable to the next Bernstein type Theorem, see [6]:

**Theorem.** The only stable $C^2-$minimal entire graphs in $\mathcal{H}$ with empty characteristic locus, are the vertical planes

$$\Pi = \{(x, y, t) \in \mathcal{H} \mid ax + by = c, \ a, b, c \in \mathbb{R}\}.$$  

Stability here is in the sense that every compact subset of a surface $S$ minimises the horizontal area (or perimeter) up to the second order; for details see [5] and [6].

There is a quite large bibliography in $H-$minimal surfaces. Illustratively, a characterisation of minimal surfaces in terms of a subelliptic PDE may be found in [11]; Benstein type problems are addressed (and solved) in [1], [5], [6], [7], [9]. More general results may be also found in [2].

This paper is organised as follows. In Section 2 we discuss in brief the Heisenberg group and its sub–Riemannian geometry. In Section 3 we set up the environment of our work, discussing regular surfaces in $\mathcal{H}$ and elements of their horizontal geometry. In Section 4 we discuss straight ruled surfaces and prove our main theorem and finally, surfaces with empty characteristic locus are presented in Section 5.

2. The Heisenberg Group

The material of this section is standard; we refer the reader for instance to [1], [8] and [10]. The Heisenberg group $\mathcal{H}$ is the set $\mathbb{R}^2 \times \mathbb{R}$ with the group law

$$(x, y, t) \ast (x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy')),$$

and it is a two–step nilpotent Lie group with underlying manifold $\mathbb{R}^2 \times \mathbb{R}$. Consider the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$  

The Lie algebra of left invariant vector fields of $\mathcal{H}$ has a grading $\mathfrak{h} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ with

$$\mathfrak{v}_1 = \text{span}_\mathbb{R}\{X, Y\} \quad \text{and} \quad \mathfrak{v}_2 = \text{span}_\mathbb{R}\{T\}.$$  

The contact form $\omega$ of $\mathcal{H}$ is defined as the unique 1–form satisfying $X, Y \in \ker \omega$, $\omega(T) = 1$. Uniqueness here is modulo change of coordinates as it follows by the Darboux Theorem. The
distribution in $\mathcal{H}$ defined by the first layer $v_1$ is called the horizontal distribution. In Heisenberg coordinates $x, y, t$, the contact form $\omega$ is given by

$$\omega = dt + 2(\text{d}xy - ydx).$$

There are two natural metrics defined on $\mathcal{H}$; the first arises from the Korányi gauge which is given by

$$| (x, y, t) |_\mathcal{H} = | x + iy |_{t}^{1/2}.$$

The Korányi–Cygan metric $d_\mathcal{H}$ is derived from it on $\mathcal{H}$, and is defined by the relation

$$d_\mathcal{H}((x_1, y_1, t_1), (x_2, y_2, t_2)) = | (x_1, y_1, t_1)^{-1} \ast (x_2, y_2, t_2) |.$$

The sub-Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is given in the horizontal subbundle by the following relations:

$$\langle X, X \rangle = \langle Y, Y \rangle = 1, \quad \langle X, Y \rangle = \langle Y, X \rangle = 0,$$

and the induced norm shall be denoted by $\| \cdot \|$. The geodesics of this metric form the Legendrian foliation of $\mathcal{H}$ i.e. the foliation of $\mathcal{H}$ by horizontal curves. An (in general) absolutely continuous curve $\gamma : [a, b] \to \mathcal{H}$ (in the Euclidean sense) with

$$\gamma(\tau) = (x(\tau), y(\tau), t(\tau)) \in \mathcal{H}$$

is called horizontal if

$$\dot{\gamma}(\tau) \in H_{\gamma(\tau)}(\mathcal{H}) \quad \text{for almost every } \tau \in [a, b],$$

or equivalently if

$$\dot{t}(\tau) = 2(y(\tau)\dot{x}(\tau) - x(\tau)\dot{y}(\tau)).$$

For a horizontal curve $\gamma$,

$$\ell(\gamma) = \int_{a}^{b} \| \dot{\gamma}(\tau) \| \, d\tau = \int_{a}^{b} \sqrt{\langle \dot{\gamma}(\tau), X_{\gamma(\tau)} \rangle^2 + \langle \dot{\gamma}(\tau), Y_{\gamma(\tau)} \rangle^2} \, d\tau$$

and the Carnot–Caratheodory distance of two arbitrary points $p, q \in \mathcal{H}$ is

$$d_{cc}(p, q) = \inf_{\gamma} \ell(\gamma)$$

where $\gamma$ is a horizontal curve joining $p$ and $q$. It is proved that the Korányi–Cygan and Carnot–Caratheodory metrics generate the same infinitesimal structure and moreover, the isometry groups of $(\mathcal{H}, d_\mathcal{H})$ and $(\mathcal{H}, d_{cc})$ are the same.

3. Elements of Horizontal Geometry of Surfaces in $\mathcal{H}$

In this section we define regular surfaces in the Heisenberg group $\mathcal{H}$ and their horizontal normal vector field (Section 3.1). Regular surfaces induce a contact structure from $\mathcal{H}$; we study this structure in Section 3.2 in fact we comment on (local) contactomorphisms between surfaces and the horizontal flow of a regular surface (that is the foliation of the surface by horizontal surface curves). Finally, in Section 3.3 we define the horizontal mean curvature of a regular surface and prove that $H$–minimal regular surfaces are locally contactomorphic to the plane.
3.1. **Regular Surfaces—Horizontal Normal Vector Field.** By a regular surface $S$ embedded in the Heisenberg group $\mathcal{H}$ we shall always mean an oriented regular surface of $\mathbb{R}^3$, see [3], i.e. a countable collection of surface patches $\sigma_\alpha : U_\alpha \rightarrow V_\alpha \cap \mathbb{R}^3$ where $U_\alpha$ and $V_\alpha$ are open sets of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, such that

1. each $\sigma_\alpha$ is a smooth homeomorphism, and
2. the differential $(\sigma_\alpha)_* : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is of rank 2 everywhere.

Let $\mathcal{S} : U \rightarrow \mathbb{R}^3$ be a regular surface and suppose that a surface patch $\sigma$ is defined in an open domain $U \subset \mathbb{R}^2$ by

$$\sigma(u,v) = (x(u,v), y(u,v), t(u,v))$$

so that its differential $\sigma_*$ is of rank 2. The tangent plane $T_\sigma(S)$ of $\mathcal{S}$ at $\sigma$ is defined by the normal vector

$$\nu_\sigma \in T_\sigma(S) = \text{span} \left\{ \sigma_u = \sigma_* \frac{\partial}{\partial u}, \sigma_v = \sigma_* \frac{\partial}{\partial v} \right\}$$

which is also defined by the normal vector

$$N_\sigma = \sigma_u \wedge \sigma_v = \frac{\partial(y,t)}{\partial(u,v)} \frac{\partial}{\partial x} + \frac{\partial(t,x)}{\partial(u,v)} \frac{\partial}{\partial y} + \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial}{\partial t},$$

where $\wedge$ is the exterior product in $\mathbb{R}^3$. That is

$$T_\sigma(S) = \left\{ V_\sigma \in T_\sigma(\mathbb{R}^3) : N_\sigma \cdot V_\sigma = 0 \right\}$$

where the dot is the usual Euclidean product in $\mathbb{R}^3$. The unit normal vector field of $\nu_\mathcal{S}$ of $\mathcal{S}$ is uniquely defined at each local chart by the relation

$$\nu_\sigma = \frac{\sigma_u \wedge \sigma_v}{|\sigma_u \wedge \sigma_v|}$$

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^3$.

**Definition 3.1.** Let $\mathcal{S}$ be a regular surface and $p \in \mathcal{S}$. The horizontal plane $\mathbb{H}_p(S)$ of $\mathcal{S}$ at $p$ is the horizontal plane $\mathbb{H}_p(\mathcal{S})$.

For arbitrary $p \in \mathcal{S}$, we wish to find the relation between the horizontal plane $\mathbb{H}_p(S)$ and the tangent plane $T_p(S)$. We begin by defining a suitable wedge product.

**Definition 3.2.** For $p \in \mathcal{S}$, the Heisenberg wedge product $\wedge^\mathcal{H}$ is a mapping $T_p(\mathcal{S}) \times T_p(\mathcal{S}) \rightarrow T_p(\mathcal{S})$ which assigns to each two vectors

$$a = a_1 X + a_2 Y + a_3 T, \quad \text{and} \quad b = b_1 X + b_2 Y + b_3 T$$

of $T_p(\mathcal{S})$ the vector $a \wedge^\mathcal{H} b \in T_p(\mathcal{S})$ which is given by the formal determinant

$$a \wedge^\mathcal{H} b = \begin{vmatrix} X & Y & T \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} X + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} Y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} T.$$

Obviously $a \wedge^\mathcal{H} b = -b \wedge^\mathcal{H} a$ and the following clock rule holds.

$$X \wedge^\mathcal{H} Y = T, \quad Y \wedge^\mathcal{H} T = X, \quad T \wedge^\mathcal{H} X = Y.$$

Thus defined, this wedge product leads to the following.
**Definition 3.3.** If $\sigma : U \to \mathbb{R}^3$ is a surface patch of a regular surface $S$, the horizontal normal $N^h_\sigma$ to $\sigma$ is the horizontal part of

$$\sigma_u \wedge \delta \sigma_v = \sigma_u \frac{\partial}{\partial u} \wedge \delta \sigma_v \frac{\partial}{\partial v},$$

that is

$$N^h_\sigma = (\sigma_u \wedge \delta \sigma_v)^h = \sigma_u \wedge \delta \sigma_v - \omega (\sigma_u \wedge \delta \sigma_v) T.$$

The unit horizontal normal $\nu^h_\sigma$ to $\sigma$ is

$$\nu^h_\sigma = \frac{N^h_\sigma}{\|N^h_\sigma\|}$$

where $\| \cdot \|$ denotes the norm of the product $\langle \cdot, \cdot \rangle$ in $\mathcal{H}$ (recall that $\|X\| = \|Y\| = 1$ and $\langle X, Y \rangle = 0$).

We have

$$\nu^h_\sigma = \frac{(\sigma_u \wedge \delta \sigma_v)^h}{\| (\sigma_u \wedge \delta \sigma_v)^h \|}.$$ 

Observe that $N^h_p$ is not the horizontal part of $N_\sigma$. Simple calculations induce the following explicit formula:

$$N^h_\sigma = \left( \frac{\partial(y, t)}{\partial(u, v)} + 2y \frac{\partial(x, y)}{\partial(u, v)} \right) X + \left( \frac{\partial(t, x)}{\partial(u, v)} - 2x \frac{\partial(x, y)}{\partial(u, v)} \right) Y.$$

From its very definition, it is immediately derived that the horizontal normal $N^h_p$ at a point $p \in S$ depends on the choice of the surface patch in the following way: suppose that $(U, \sigma)$ and $(\tilde{U}, \tilde{\sigma})$ are two overlapping patches at $p$. Then if $\Phi = \sigma^{-1} \circ \tilde{\sigma}$ is the transition mapping, we may find from (3.1) that around $p$ we have

$$N^h_{\tilde{\sigma}} = \det(\Phi) N^h_\sigma,$$

where $\det(\Phi) > 0$ since we have already presupposed that $S$ is oriented. At this point, we would have been ready to define the unit horizontal normal vector field in $S$ in accordance with the unit normal vector field which is defined everywhere in a regular surface embedded into Euclidean space, but there is no assurance that a) $N^h_p \neq 0$ at all $p \in S$ and b) $\nu^h_p$ is not in $T_p(S)$. To this end we give the following definition.

**Definition 3.4.** Let $S$ be a regular surface. A point $p \in S$ is called *non characteristic* if $N^h_p \neq 0$. The set of characteristic points

$$\mathcal{C}(S) = \{ p \in S \mid N^h_p = 0 \}$$

is called the characteristic locus of $S$.

By definition, the points of $\mathcal{C}(S)$ are given in a local chart $(U, \sigma)$ by the equations

$$\frac{\partial(y, t)}{\partial(u, v)} + 2y \frac{\partial(x, y)}{\partial(u, v)} = 0 \quad \text{and} \quad \frac{\partial(t, x)}{\partial(u, v)} - 2x \frac{\partial(x, y)}{\partial(u, v)} = 0,$$

and therefore the Lebesgue measure of $\mathcal{C}(S)$ is 0 or 1. An equivalent, but not depending on coordinates definition of the characteristic locus will be given in the next section. It remains to show that at non characteristic points of $S$, $\nu^h_p$ is not in $T_p(S)$:
Proposition 3.5. A point \( p = (x, y, t) \in S \) is non characteristic if and only if \( N_p \cdot N_p^h \neq 0 \), (where the dot denotes the Euclidean product in \( \mathbb{R}^3 \)). Moreover, \( N_p = N_p^h \) as vectors in \( \mathbb{R}^3 \) if and only if \( x = y = 0 \) or \( \partial(x, y) = 0 \). In this case,
\[
N_p = (\partial(y, t), \partial(t, x), 0), \quad N_p^h = \partial(y, t)X + \partial(t, x)Y, \quad \text{and} \quad |N_p| = |N_p^h|
\]
and the surface at \( p \) is tangent to a plane passing through \( p \) which is orthogonal to the complex plane.

Proof. If \( p = \sigma(u, v) \), then \( N_p^h \) may be written as a vector of \( \mathbb{R}^3 \) as follows
\[
N_p^h = ((\partial(y, t) + 2y\partial(x, y)), (\partial(t, x) - 2x\partial(x, y)), 4(x^2 + y^2)\partial(x, y)),
\]
where we have denoted \( \partial(y, t)/\partial(u, v) \) by \( \partial(y, t) \) etc. By taking the Euclidean dot product we find
\[
N_p \cdot N_p^h = (\partial(y, t) + 2y\partial(x, y))^2 + (\partial(t, x) - 2x\partial(x, y))^2,
\]
and this vanishes if and only if \( p \) is characteristic. Our second claim is immediate. \( \square \)

Corollary 3.6. Let \( S \) be a regular surface of \( \mathcal{H} \). Then away from the characteristic locus, \( \text{3.5} \) defines a nowhere vanishing vector field \( \nu^h_S \in \mathbb{H}(S) \), such that \( \|\nu^h_S\| = 1 \).

Denote by \( \mathcal{J} \) the complex operator acting in \( \mathbb{H}(\mathcal{H}) \) by the relations
\[
\mathcal{J}X = Y, \quad \mathcal{J}Y = -X.
\]
The operator \( \mathcal{J} \) acts in the horizontal space of a regular surface \( S \), and if \( \nu^h_S = \nu_1X + \nu_2Y \) then
\[
\mathcal{J}\nu^h_S = -\nu_2X + \nu_1Y.
\]

3.2. The Induced 1–Form. Contactomorphisms. Horizontal Flow. Let \( S \) be a regular surface in \( \mathcal{H} \) and denote by \( \iota_S \) the inclusion map \( \iota_S : S \hookrightarrow \mathcal{H} \), given locally by a parametrisation \( \sigma(u, v) = (x(u, v), y(u, v), t(u, v)) \). Let \( \omega = dt + 2xyd\bar{y} - 2yd\bar{x} \) be the contact form of \( \mathcal{H} \); the pullback \( \omega_S = \iota^*\omega \) defines a 1–form on \( S \) which, in the local parametrisation is given by
\[
\omega_S = \sigma^*\omega = (t_u + 2xy_u - 2yx_u)du + (t_v + 2xy_v - 2yx_v)dv.
\]

Proposition 3.7. The characteristic locus \( \mathcal{C}(S) \) is the (closed) set of points of \( S \) at which \( \omega_S = 0 \).

Proof. We have:
\[
\omega_S(p) = 0 \quad \text{for some} \quad p \in S
\]
\[
\iff \omega_p(\sigma_u) = \omega_p(\sigma_v) = 0 \quad \text{for each chart} \quad (U, \sigma) \text{ containing } p
\]
\[
\iff \sigma_u \text{ and } \sigma_v \in \mathbb{P}_p(S) \quad \text{for each chart} \quad (U, \sigma) \text{ containing } p
\]
\[
\iff (\sigma_u \times \sigma_v)^h = 0 \quad \text{for each chart} \quad (U, \sigma) \text{ containing } p
\]
\[
\iff p \in \mathcal{C}(S).
\]
\( \square \)

Regular surfaces in \( \mathcal{H} \) with empty characteristic locus and will be treated separately in Section \( \text{5} \), where we will see some consequences of Proposition \( \text{3.7} \).
Definition 3.8. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \tilde{\mathcal{S}}$ be a smooth diffeomorphism. We may assume a weaker condition, that is we will require $f$ to be a local diffeomorphism outside the characteristic loci of $\mathcal{S}$ and $\tilde{\mathcal{S}}$. The mapping $f$ is called a local contactomorphism of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ if there exists a smooth function $\lambda$ so that

$$f^* \omega_{\tilde{\mathcal{S}}} = \lambda \omega_{\mathcal{S}}.$$

Since $f$ is a local diffeomorphism, if $\sigma : U \to \mathbb{R}^3$ is a surface patch for $\mathcal{S}$ then $\tilde{\sigma} = f \circ \sigma$ is a surface patch for $\tilde{\mathcal{S}}$ (with the possible exception of characteristic points). It follows that $f : \mathcal{S} \to \tilde{\mathcal{S}}$ is a contactomorphism if and only if

$$\omega_{\tilde{\mathcal{S}}} (u, v) = \lambda(u, v) \omega_{\mathcal{S}} (u, v), \quad \text{for almost all } (u, v) \in U.$$

By a surface curve on a regular surface $\mathcal{S}$ we shall always mean a smooth mapping $\gamma : I \to \mathcal{S}$ where $I$ is an open interval of $\mathbb{R}$. We wish to find conditions so that a surface curve is horizontal, i.e. its horizontal tangent $\dot{\gamma}_h(s) \in H_{\gamma(s)}(\mathcal{S})$.

Proposition 3.9. Suppose that $\sigma : U \to \mathbb{R}^3$ is a surface patch, and $\gamma(s) = \sigma(u(s), v(s))$, $s \in I$ is a smooth surface curve (that is, $\dot{\gamma}(s) = (u(s), v(s))$ a smooth curve in $U$). Then away from the characteristic locus $\gamma$ is horizontal if and only if

$$\dot{\gamma}_h \in \ker \omega_{\mathcal{S}},$$

or in other words,

$$(tu + 2xy - 2yxu)\ddot{u} + (tv + 2xyv - 2yvx)\dot{v} = 0$$

where the dot denotes $d/ds$. In this case,

$$\dot{\gamma} = (x\ddot{u} + x\dot{v})X + (y\ddot{u} + y\dot{v})Y.$$

Proof. We only prove the first statement; the other two are derived immediately. We have

$$\gamma \text{ horizontal } \iff \omega(\dot{\gamma}_h) = 0 \iff \omega(\sigma^* \dot{\gamma}) = 0 \iff (\sigma^* \omega)(\dot{\gamma}) = 0 \iff \dot{\gamma} \in \ker \omega_{\mathcal{S}}.$$

The following Proposition indicates the importance of the unit horizontal normal vector field $\mathbb{J}v_{\mathcal{S}}$.

Proposition 3.10. The 1–form $\omega_{\mathcal{S}}$ defines an integrable foliation of $\mathcal{S}$ (with singularities at characteristic points) by horizontal surface curves. These curves are tangent to $\mathbb{J}v_h^\mathcal{S}$.

Proof. Integrability is obvious: $\omega_{\mathcal{S}}$ is a 1–form defined in a two–dimensional manifold. For the second statement, we set

$$\alpha = \frac{1}{\|N^h\|} (tu - 2yxu + 2xyu), \quad \beta = \frac{1}{\|N^h\|} (tv - 2yxv + 2xyv),$$

where $\|N^h\| = \|(\sigma_u \wedge \delta \sigma_v)^h\|$, and consider

$$JY = \beta \frac{\partial}{\partial u} - \alpha \frac{\partial}{\partial v} \in \ker \omega_{\mathcal{S}}.$$
By observing that
\[ \beta y_u - \alpha y_v = \frac{\partial(y, t) + 2y\partial(x, y)}{\|N^h\|} = \nu_1, \]
\[ \beta x_u - \alpha x_v = -\frac{\partial(t, x) - 2x\partial(x, y)}{\|N^h\|} = -\nu_2, \]
we obtain
\[ \sigma_*(J\mathcal{V}) = \beta \sigma_u - \alpha \sigma_v \]
\[ = \beta(x_u X + y_u Y + \|N^h\|\alpha T) - \alpha(x_v X + y_v Y + \|N^h\|\beta T) \]
\[ = (\beta x_u - \alpha x_v)X + (\beta y_u - \alpha y_v)Y \]
\[ = -\nu_2X + \nu_1Y = \mathcal{J}\nu_S. \]

Note finally that by 3.7 the integral curves of \( \mathcal{J}\nu_S \) are the solutions of the system of differential equations
\[ \dot{u} = \beta, \quad \dot{v} = -\alpha. \]

□

We remark for later use that when \( D = \partial(x, y) \neq 0 \) we also have the following expressions for \( \alpha \) and \( \beta \):
\[ (3.8) \quad \alpha = -\frac{\nu_1 x_u + \nu_2 y_u}{D}, \quad \beta = -\frac{\nu_1 x_v + \nu_2 y_v}{D}. \]

**Definition 3.11.** The foliation of \( S \) by the integrable curves of \( \mathcal{J}\nu_S \) is called the horizontal flow of \( S \).

3.3. **Horizontal Mean Curvature.** Horizontal mean curvature is defined as follows.

**Definition 3.12.** Let \( S \) be a non characteristic point of a regular surface \( S \) and let also \( \nu^h_p = \nu_1X + \nu_2Y \) be the unit horizontal normal of \( S \) at \( p \). The horizontal mean curvature \( H^h(p) \) of \( S \) at \( p \) is given by
\[ H^h(p) = X_p\nu_1 + Y_p\nu_2. \]

A more geometric but equivalent definition is following by the next proposition according to which, the horizontal mean curvature at non characteristic points of a regular surface may be defined as the signed curvature of the projection to \( \mathbb{C} \) of the leaf of the horizontal flow passing from \( p \) (see also Proposition 4.24 of [1]).

**Proposition 3.13.** Let \( S \) be a regular surface and \( p \in S \) a non characteristic point. Let \( \nu^h_S = \nu_1X + \nu_2Y \) be the unit horizontal normal vector field of \( S \), and \( \gamma \) the unique unit speed surface curve passing from \( p \) which is tangent to \( \mathcal{J}\nu^h_S \) at \( p \). If \( \pi = \text{pr}_C\gamma \) is the projection of \( \gamma \) on \( \mathbb{C} \), \( p' \) is the projection of \( p \) and \( \kappa_s \) is the signed curvature of \( \pi \), then
\[ \kappa_s(p') = X_p\nu_1 + Y_p\nu_2. \]

**Proof.** Let \( p \in S \) and \( \gamma(s) \) the unit speed horizontal surface curve passing from \( p \). Let \( \pi(s) \) be the projection of \( \gamma(s) \) in \( \mathbb{C} = \mathbb{R}^2 \); then its tangent is
\[ \dot{\pi}(s) = (-\nu_2(s), \nu_1(s)) \]
and its of unit speed. We have by applying the chain rule that
\[ \dot{v}_1 = (\nu_1)_x \dot{x} + (\nu_1)_y \dot{y} + (\nu_1)_t \]
\[ = (Xv_1 - 2yTv_1)(x_u \dot{u} + x_v \dot{v}) + (Yv_1 + 2xTv_1)(y_u \dot{u} + y_v \dot{v}) + Tv_1(t_u \dot{u} + t_v \dot{v}) \]
\[ = -(Xv_1 - 2yTv_1)v_2 + (Yv_1 + 2xTv_1)v_1 + Tv_1(-2yv_2 - 2xv_1) \]
\[ = -\nu_2(Xv_1 + Yv_2), \]
where we have used
\[ \dot{\gamma}(s) = (x_u \dot{u} + x_v \dot{v})X + (y_u \dot{u} + y_v \dot{v})Y = -\nu_2X + \nu_1Y, \]
and the relation \[ \nu_1Yv_1 = -\nu_2Yv_2 \] which follow from \[ \nu_1^2 + \nu_2^2 = 1 \]. Working analogously for \( \dot{v}_2 \) (using \( \nu_1Xv_1 = -\nu_1Xv_2 \) this time), we have \( \dot{v}_2 = \nu_1(Xv_1 + Yv_2) \), hence
\[ \pi = (-\dot{v}_2, \dot{v}_1) = (-\nu_1(Xv_1 + Yv_2), -\nu_2(Xv_1 + Yv_2)) \]
\[ = \kappa_s(-\nu_1, -\nu_2) \]
where \( \kappa_s \) is the signed curvature of the curve \( \pi \). This yields \( \kappa_s = Xv_1 + Yv_2. \)

A local expression for \( H^h \) is in order:

**Proposition 3.14.** Let \( S \) be a regular surface in \( \mathcal{S} \). In every surface patch \( \sigma = (x, y, t) \) with \( \partial(x, y) \neq 0 \) and sufficiently away from the characteristic locus, the horizontal mean curvature is given by

\[ H^h(\sigma) = \frac{\partial(\nu_1, y) + \partial(x, \nu_2)}{\partial(x, y)}, \]

where \( \nu_i, i = 1, 2 \) are the components of the unit horizontal normal vector \( \nu \) of \( S \). If \( \partial(x, y) = 0 \), then \( H^h(\sigma) = 0 \).

**Proof.** Suppose first that \( \partial(x, y) \neq 0 \). Using the chain rule we write
\[ (\nu_1)_u = (\nu_1)_x x_u + (\nu_1)_y y_u + (\nu_1)_t t_u = (Xv_1)_x x_u + (Yv_1)_y y_u + (t_u - 2yv_2 + 2xy_2)Tv_1, \]
\[ (\nu_2)_u = (\nu_2)_x x_u + (\nu_2)_y y_u + (\nu_2)_t t_u = (Xv_2)_x x_u + (Yv_2)_y y_u + (t_u - 2yv_2 + 2xy_2)Tv_2, \]
\[ (\nu_1)_v = (\nu_1)_x x_v + (\nu_1)_y y_v + (\nu_1)_t t_v = (Xv_1)_x x_v + (Yv_1)_y y_v + (t_v - 2yv_2 + 2xy_2)Tv_1, \]
\[ (\nu_2)_v = (\nu_2)_x x_v + (\nu_2)_y y_v + (\nu_2)_t t_v = (Xv_2)_x x_v + (Yv_2)_y y_v + (t_v - 2yv_2 + 2xy_2)Tv_2. \]
The first and the third equation are written as
\[ (Xv_1)_x x_u + (Yv_1)_y y_u = (\nu_1)_u - \alpha \|N^h\|Tv_1, \]
\[ (Xv_1)_x x_v + (Yv_1)_y y_v = (\nu_1)_v - \beta \|N^h\|Tv_1, \]
where we have used Equations 3.6. Solving the system we obtain
\[ Xv_1 = \frac{\partial(\nu_1, y) + \|N^h\|Tv_1}{\partial(x, y)}, \quad Yv_1 = \frac{\partial(x, \nu_1) + \|N^h\|Tv_1}{\partial(x, y)}, \]
where we have used Equations 3.8. In an analogous manner, we obtain the following from the second and the fourth equations:
\[ Xv_2 = \frac{\partial(\nu_2, y) + \|N^h\|Tv_2}{\partial(x, y)}, \quad Yv_2 = \frac{\partial(x, \nu_2) + \|N^h\|Tv_2}{\partial(x, y)}. \]
Therefore
\[ X\nu_1 + Y\nu_2 = \frac{\partial(\nu_1, y) + \partial(x, \nu_2)}{\partial(x, y)} + \|N^h\| (\nu_1 T\nu_1 + \nu_2 T\nu_2) \]
\[ = \frac{\partial(\nu_1, y) + \partial(x, \nu_2)}{\partial(x, y)}, \]
since \( \nu_1^2 + \nu_2^2 = 1 \) and hence \( \nu_1 T\nu_1 + \nu_2 T\nu_2 = 0 \).

Finally if \( \partial(x, y) = 0 \), then from Proposition 3.14 it is deduced that the horizontal normal vector field \( \nu^h_\sigma \) is orthogonal to a plane vertical to the complex plane and the image of \( \sigma \) belongs to that plane. Thus \( \mathbb{J}\nu^h_\sigma \) is tangent to the plane and the horizontal flow comprises of straight lines. The proof is complete. \( \square \)

**Proposition 3.15.** If a regular surface \( S \) in \( \mathbb{H} \) is locally contactomorphic to the complex plane, then it is \( H \)-minimal.

**Proof.** First we prove the statement for graphs \( G_f \) of smooth functions \( t = f(x, y) \) over \( \mathbb{C} \). Here \( (x, y) \) lie in an open subset of the plane. Let
\[ \sigma(x, y) = (x, y, f(x, y)), \quad (x, y) \in U. \]
The induced 1-form is \( \omega_{G_f} = (f_x - 2y)dx + (f_y + 2x)dy \). From the contactomorphism condition we also have
\[ f_x - 2y = -2\lambda y, \quad \text{and} \quad f_y + 2x = 2\lambda x \]
for some non zero function \( \lambda \). Moreover,
\[ N^h = (-f_x + 2y)X + (-f_y - 2x)Y = 2\lambda(yX - xY), \]
and therefore
\[ \nu_{G_f} = \nu_1 X + \nu_2 Y = \pm \frac{yX - xY}{(x^2 + y^2)^{1/2}}. \]

Using Proposition 3.14 we have for the positive sign case (the other case is treated analogously):
\[ H^h = \partial(\nu_1, y) + \partial(x, \nu_2) \quad (\partial(x, y) = 1), \]
\[ = \partial \left( \frac{y}{(x^2 + y^2)^{1/2}}, y \right) + \partial \left( \frac{x}{(x^2 + y^2)^{1/2}}, x \right) \]
\[ = y\partial_x \left( \frac{1}{(x^2 + y^2)^{1/2}} \right) - x\partial_y \left( \frac{1}{(x^2 + y^2)^{1/2}} \right) \]
\[ = y \cdot \frac{-x}{(x^2 + y^2)^{3/2}} - x \cdot \frac{-y}{(x^2 + y^2)^{3/2}} \]
\[ = 0. \]

Next we show that all coordinate planes are locally contactomorphic; we will treat the case of the planes \( x = 0 \) and \( t = 0 \) and leave the other cases as an exercise. We parametrise the plane \( x = 0 \) by \( \sigma(u, v) = (0, u, v) \) and consider the map \( f : \{x = 0\} \to \{t = 0\} \) given by
\[ (0, u, v) \mapsto (uv, v, 0). \]
Denote by \( \tilde{\sigma} \) the surface patch \( f \circ \sigma \). Then
\[ \omega_\sigma = dv \quad \text{and} \quad \omega_{\tilde{\sigma}} = -2u^2 dv \]
which by the contact condition \[\text{[5.5]}\] proves our assertion.

If now \(\sigma(u,v) = (x(u,v),y(u,v),t(u,v))\) is an arbitrary surface patch for \(S\), from regularity we have that at least one of \(\partial(x,y),\partial(y,t)\) and \(\partial(t,x)\) is different from zero. We may now assume that \(\partial(x,y) \neq 0\) and reparametrize if necessary by

\[
\tilde{u} = x(u,v), \quad \tilde{v} = y(u,v)
\]

to obtain the regular surface patch \(\sigma(\tilde{u},\tilde{v},t(\tilde{u},\tilde{v}))\) which is a local graph of a function over the complex plane.

\(\square\)

4. Straight Ruled Surfaces

In this section we define straight ruled surfaces in \(\mathcal{H}\) and prove Theorem \[\text{[4.1]}\]. For the proof, we use two different ways to show that straight ruled surfaces are \(H\)-minimal; the first one is by showing that they are locally contactomorphic to the complex plane and the second is straightforward.

A straight ruled surface in \(\mathcal{H}\) is a surface which is formed by a union of straight lines (the rulings of the surface), in the following manner. Suppose that \(\gamma = \gamma(s)\), where \(s\) lies in an open interval \(I\) of \(\mathbb{R}\), is a (not necessarily horizontal) smooth curve and \(V = V(s)\) is a unit horizontal vector field along \(\gamma\); i.e. \(V(s) \in \mathbb{H}_{\gamma(s)}(\mathcal{H})\). For reasons that will be justified below, we assume that the projected curve \(pr_C(\gamma)\) is not a straight line. At any point \(q \in \gamma\), say \(q = \gamma(s)\) we consider the straight line passing from \(q\) in the direction of \(V(s)\). Then a point \(p\) on the straight line satisfies \(p = \gamma(s) + vV(s)\) for some \(v\). The straight ruled surface \(\mathcal{R}(\gamma)\) is the union of all such straight lines, therefore it admits a parametrisation by the (single) surface patch \(\sigma : I_s \times \mathbb{R} \to \mathbb{R}^3\) where \(I_s\) is an open interval of \(\mathbb{R}\) and

\[
\sigma(s,v) = \gamma(s) + vV(s).
\]

If \(\gamma = (x,y,t)\) and \(V = aX + bY, \ a^2 + b^2 = 1\), we write

\[
\sigma(s,v) = (\tilde{x}(s,v),\tilde{y}(s,v),\tilde{t}(s,v))
\]

\[
= (x(s) + va(s), y(s) + vb(s), t(s) + 2v(y(s)a(s) - x(s)b(s))),
\]

and calculate (denoting \(d/ds\) by dot)

\[
\begin{align*}
\tilde{x}_s &= \dot{x} + v\dot{a}, & \tilde{y}_s &= \dot{y} + v\dot{b}, & \tilde{t}_s &= \dot{t} + 2v(\dot{y}a + y\dot{a} - \dot{x}b - x\dot{b}) \\
\tilde{x}_v &= a, & \tilde{y}_v &= b, & \tilde{t}_v &= 2(ya - xb).
\end{align*}
\]

Regularity: \(\sigma\) has to be a regular surface patch. Set

\[
\delta(s) = (a(s),b(s),2(y(s)a(s) - x(s)b(s))).
\]

Since \(\sigma_s = \dot{\gamma} + v\dot{\delta}\) and \(\sigma_v = \delta\), \(\sigma\) is regular if \(\dot{\gamma} + v\dot{\delta}\) and \(\delta\) are linearly independent. For example, this happens if

\[
(\dot{x}(s),\dot{y}(s),\dot{t}(s)) \quad \text{and} \quad \delta(s)
\]

are linearly independent and \(v\) is sufficiently small. Thus regularity is assured if \(V(s)\) is never tangent to \(\gamma\).

\[
\begin{align*}
\sigma_s &= (\dot{x} + v\dot{a})X + (\dot{y} + v\dot{b})Y + (\dot{t} + 4v(a\dot{y} - b\dot{x}) + 2(x\dot{y} - y\dot{x}) + 2v^2(ab - b\dot{a})\big)T, \\
\sigma_v &= aX + bY = V,
\end{align*}
\]

and

\[
\left(\sigma_s \wedge^h \sigma_v\right)^h = \eta(-bX + aY) = \eta V,
\]

where

\[
\eta = \eta(s,v) = \dot{t}(s) + 2(x(s)\dot{y}(s) - y(s)\dot{x}(s)) + 4v(a(s)\dot{y}(s) - b(s)\dot{x}(s)) + 2v^2(a(s)b(s) - b(s)\dot{a}(s)).
\]
Thus the characteristic locus is
\[ \mathcal{C}(\mathcal{R}(\gamma)) = \{(s, v) \in I_s \times I_v : \eta(s, v) = 0 \}, \]
where \(I_v\) is an appropriately small open interval of \(\mathbb{R}\). The exceptional case when \(\eta\) vanishes identically occurs when the projection \(pr_C(\gamma)\) is a straight line. This can be seen as follows. The function \(\eta\) is a quadratic polynomial in \(v\) therefore it vanishes identically if and only if the following relations hold simultaneously:
\[
\begin{align*}
\dot{s}(s) &= 2(y(s)\dot{x}(s) - x(s)\dot{y}(s)), \\
a(s)\dot{y}(s) - b(s)\dot{x}(s) &= 0, \\
a(s)\dot{b}(s) - b(s)\dot{a}(s) &= 0.
\end{align*}
\]
From the first relation it follows that \(\gamma\) has to be horizontal; from the second we have that \(V\) is parallel to the horizontal tangent \(\dot{\gamma} = \dot{x}X + \dot{y}Y\) and since \(V\) has been supposed to be unit, we have \(V = \pm \dot{\gamma}\). Then the third relation reads
\[
\pm (\dot{x}\ddot{y} - \dot{y}\ddot{x}) = 0.
\]
But the left hand side is (up to sign) equal to the signed curvature of the projected curve \(pr_C(\gamma)\). Hence \(pr_C(\gamma)\) has to be a straight line, which contradicts our assumptions for \(\mathcal{R}(\gamma)\). (Note that in this case there is no surface defined). Another special case occurs when \(\gamma\) is horizontal; then \(\dot{t} + 2(xy - y\dot{x}) = 0\) and thus the characteristic locus includes all points of \(\gamma\).

**Proposition 4.1.** Any straight ruled surface \(\mathcal{R}(\gamma)\) is locally contactomorphic to the complex plane \(\mathbb{C}\) and thus is \(H\)–minimal.

**Proof.** We only have to prove our first statement; the second follows from Proposition 3.15. Let \(\gamma = (x, y, t)\) and \(V = aX + bY\) as before and also
\[
\sigma(s, v) = (\bar{x}(s, v), \bar{y}(s, v), \bar{t}(s, v))
\]
be the surface patch for \(\mathcal{R}(\gamma)\). Then
\[
\omega_{\mathcal{R}(\gamma)} = \sigma^* \omega = (\bar{t}_s + 2\bar{x}\ddot{y}_s - 2\bar{y}\ddot{x}_s)ds + (\bar{t}_v + 2\bar{x}\ddot{y}_v - 2\bar{y}\ddot{x}_v)dv = \eta ds.
\]
We now consider the following local parametrisation for \(\mathbb{C}\):
\[
\bar{\sigma}(s, v) = (a(s)v, b(s)v, 0).
\]
Under this parametrisation, \(\mathbb{C}\) is trivially a straight ruled surface; the curve \(\gamma\) is the single point \((0, 0, 0)\), and the horizontal flow comprises of the straight lines passing through the origin. Then
\[
\omega_{\mathbb{C}} = \bar{\sigma}^* \omega = 2v^2(ab - b\dot{a})ds = (2v^2(ab - b\dot{a})/\eta)\omega_{\mathcal{R}(\gamma)}.
\]

**Remark 4.2.** Here is a straightforward proof of \(H\)–minimality of straight ruled surfaces. The unit horizontal vector field is
\[
\nu = \nu_1 X + \nu_2 Y = \pm (bX - aY).
\]
We suppose first that \(\nu_1 = b\) and \(\nu_2 = -a\); the other case is treated similarly. We find
\[
(\nu_1)_s = \dot{b}, \quad (\nu_1)_v = 0, \quad (\nu_2)_s = -\dot{a}, \quad (\nu_2)_v = 0.
\]
Using Proposition 3.14 we have at non characteristic points
\[ H^h = \frac{\partial (v_1, \bar{y}) + \partial (\bar{x}, v_2)}{\partial (\bar{x}, \bar{y})} = \frac{b\bar{a} + a\bar{b}}{b\bar{x} - a\bar{y} + v(b\bar{a} - ab)} = 0, \]
since \( a^2 + b^2 = 1 \).

We also stress here that it is geometrically clear that the parametric lines \( s = \text{const} \) are the horizontal flow of \( R(\gamma) \). This can also be seen by solving the system of equations 3.8 and 3.8 to obtain \( \beta = 0 \).

**Proof of the Main Theorem 1.1.** The first statement of the Theorem follows from Proposition 4.1. For the second statement, let first \( S \) be a regular surface and \( p \in S \) be a non characteristic point. Since the horizontal flow foliates \( S \) by horizontal surface curves \( \gamma_s \) of unit horizontal speed tangent to \( \nu_S \), \( s \in I \), consider the integral curve \( \gamma_{s_0}(v) \) passing from \( p \), where \( v \) lies in a sufficiently small interval: \( \gamma_{s_0}(v_0) = p \) for some \( v_0 \) in that interval. There exists an open subset \( U \) of \( \mathbb{R}^2 \), with \((s_0, v_0) \in U \) and a smooth mapping \( \sigma : U \to S \) so that
\[ \sigma(s, v) = \gamma_s(v), \quad (s, v) \in U \]
and we may shrink \( U \) so that it does not contain any characteristic points. Suppose now that \( S \) has zero horizontal mean curvature; by Proposition 3.13 the curves \( pr_C(\gamma_s) \) have zero signed curvature, therefore they are pieces of straight lines. It follows that if
\[ \sigma(s, v) = (x_s(v), y_s(v), t_s(v)), \]
then we have \( d^2 x_s/dv^2 = d^2 y_s/dv^2 = 0 \) and thus
\[ x_s(v) = a(s)v + x(s), \quad y_s(v) = b(s)v + y(s), \]
for some smooth functions \( x, y, a, b \). Since \( \gamma_s \) has unit horizontal speed, we have \( a^2 + b^2 = 1 \) and since it is horizontal, we also have
\[ \frac{dt_s}{dv} = 2 \left( yv \frac{dx_s}{dv} - xv \frac{dy_s}{dv} \right) = 2((y + b)a - (x + a)b) = 2(ya - xb), \]
and therefore \( t_s(v) = 2v(y(s)a(s) - x(s)b(s)) + t(s) \), where \( t(s) \) is a smooth function of \( s \). Therefore the patch \( \sigma \) above is a patch of a piece of a straight ruled surface. Since our point \( p \) is arbitrary, we conclude the Theorem.

5. **Regular Surfaces in \( \mathcal{H} \) with Empty Characteristic Locus**

In this section we give two examples of regular surfaces \( S \) with empty \( \mathcal{C}(S) \). First, we examine horizontal tangent developables which comprise of the counterparts of tangent developables in the Euclidean case. Secondly, we show that surfaces with empty characteristic locus and closed induced 1-form can be only generalised cylinders which have constant horizontal mean curvature. An arbitrary generalised cylinder is not a straight ruled surface; this happend only in the case of a plane orthogonal to \( C \). Two indicative examples of surfaces are given in the end of this section. The first, that of the hyperbolic paraboloid shows that there exists a developable Euclidean surface with negative Gaussian curvature which is also a straight ruled surface. The second, that of the cone, shows that a Euclidean cone, although having empty characteristic locus and zero Gaussian curvature, can not be a straight ruled surface.

We start with the following proposition which is a direct consequence of Proposition 3.5.

**Proposition 5.1.** Let \( S \) be an oriented regular surface curve. Then the following are equivalent:
(1) The characteristic locus \( C(S) \) of \( S \) is the null set.
(2) The induced 1–form \( \omega_S \) is nowhere zero.
(3) The characteristic locus \( C(S) \) of \( S \) is the null set, if and only if the horizontal flow has no singularities.

5.1. Horizontal tangent developables. Let \( \gamma \) be a horizontal curve parametrised so that it is of unit horizontal speed, that is
\[
\gamma(s) = (x(s), y(s), t(s)), \quad \dot{i}(s) = 2(y(s)\dot{x}(s) - x(s)\dot{y}(s)),
\]
and \( \dot{x}(s)^2 + \dot{y}(s)^2 = 1 \) for \( s \) in an open interval \( I \) of \( \mathbb{R} \). We also suppose that \( \gamma \) is not a straight line; for \( \gamma \) horizontal \( \text{pr}_C(\gamma) \) is a straight line if and only if \( \gamma \) is a straight line. The surface \( T(\gamma) \) of horizontal tangent developables of \( \gamma \) is defined by the single surface patch
\[
\sigma(s, v) = \gamma(s) + v\dot{\gamma}(s).
\]
Regularity: Since
\[
\sigma_s = (\dot{x} + v\ddot{x}, \dot{y} + v\ddot{y}, \dot{t} + v\ddot{t}), \quad \sigma_v = (\dot{x}, \dot{y}, \dot{t}),
\]
we have \( \sigma_s \wedge \sigma_v = v\dddot{x} \wedge \dot{\gamma} \). Hence, in the first place, the (usual) curvature \( \kappa(\gamma) \) of \( \gamma \) has to be positive everywhere. Since \( \gamma \) is horizontal,
\[
\kappa(\gamma) = |(\dot{x}, \dot{y}, 2(y\dot{x} - x\dot{y}))|,
\]
which vanishes only if \( \dot{x} = \dot{y} = 0 \), i.e. only if \( \gamma \) is a straight line. Moreover, we have to exclude the points of \( \gamma \) since at these points \( v = 0 \).

Thus defined, \( T(\gamma) \) is a special case of a straight ruled surface (here \( V(s) = \dot{\gamma}(s) \) the unit horizontal tangent of \( \gamma \)) and therefore it is locally contactomorphic to the plane \( \mathbb{C} \) and has vanishing horizontal mean curvature. Note that the characteristic locus of \( T(\gamma) \) is empty, since we have assumed regularity for \( T(\gamma) \).

5.2. Surfaces with empty characteristic locus and closed induced form. Below we trace all regular oriented surfaces \( S \) in \( \mathcal{F} \) with empty characteristic locus and with the additional property that \( \omega_S \) is closed.

Proposition 5.2. Regular surfaces \( S \) in \( \mathcal{F} \) with empty characteristic locus and closed induced 1–form \( \omega_S \) are exactly the Euclidean generalised cylinders which are obtained by translating a regular curve lying in the complex plane \( \mathbb{C} \) along the vector field \( T \).

Proof. If \( \sigma : U \to S, \sigma = (x, y, t) \) is an arbitrary surface patch for \( S \), then \( d\omega_S = 0 \) induces \( \partial(x, y) = 0 \). From Proposition 5.1 we see that if \( S \) is such a surface, then for every parametrisation \( \sigma \) we have
\[
\sigma_u \times \sigma_v \perp T = \frac{\partial}{\partial t}
\]
as vectors in \( \mathbb{R}^3 \). But this is equivalent to say that either \( \sigma_u = \rho(u, v)\partial_t \) or \( \sigma_v = \rho^*(u, v)\partial_t \) where \( \rho \) and \( \rho^* \) are smooth functions of \((u, v)\). Suppose the first holds; the second case is treated analogously. We obtain
\[
\sigma(u, v) = \left( x(v), y(v), \int_{u_0}^u \rho(\xi, v)d\xi \right),
\]
and we may reparametrise by
\[
\tilde{u} = \int_{u_0}^u \rho(\xi, v)d\xi, \quad \tilde{v} = v,
\]

5.3. Straight ruled surfaces.
to obtain
\[ \sigma(u, v) = (x(v), y(v), u). \]
Since \( S \) is regular, condition \( \sigma_u \land \sigma_v = (\dot{y} \rho, \dot{x} \rho, 0) \neq 0 \), where the dot stands for \( d/dv \), is equivalent to that the curve \( \gamma(v) = (x(v), y(v), 0) \) is regular. Thus
\[ \sigma(u, v) = \gamma(v) + u \partial_t. \]
The proof is complete.

**Proposition 5.3.** The only regular surfaces \( S \) in \( \mathcal{F} \) with empty characteristic locus, closed induced 1–form \( \omega_S \) and constant horizontal mean curvature are

1. the planes which are perpendicular to \( \mathbb{C} \); these have \( H^h \equiv 0 \) and
2. the right cylinders whose profile curve is a circle of radius \( R \); these have \( H^h \equiv 1/R \).

**Proof.** Let
\[ \sigma(u, v) = (x(v), y(v), u) \]
a generalised cylinder. Since \( \gamma(v) \) is regular we may reparametrise so that it has unit speed, \( \dot{x}(v)^2 + \dot{y}(v)^2 = 1 \). Then,
\[ \nu^h = \dot{y} X - \dot{x} Y, \quad \| \nu^h \| = \dot{x} X + \dot{y} Y, \]
and therefore the horizontal flow is comprising of all horizontal lifts of \( \gamma \). Thus \( H^h = \kappa_s(\gamma) \), where \( \kappa_s(\gamma) \) is the signed curvature of \( \gamma \). \( \square \)

### 5.3. Two examples: Euclidean Ruled Surfaces vs. Straight Ruled Surfaces.

**Hyperbolic paraboloid.** The hyperbolic paraboloid \( z = y^2 - x^2 \) is a doubly ruled surface in the usual sense and a straight ruled surface \( \mathcal{R}(\gamma) \) where \( \gamma \) is the parabola \( z = y^2 \):
\[ \sigma(s, v) = (0, s, s^2) + \frac{1}{\sqrt{2}} v(X + Y). \]
Its characteristic locus is the plane \( x + y = 0 \). Recall that as a surface in \( \mathbb{R}^3 \) it has negative Gaussian curvature; however, since it is a straight ruled surface in \( \mathcal{F} \) it has zero horizontal mean curvature.

**Cone.** The cone \( x^2 + y^2 = z^2 \) is a Euclidean ruled surface with zero Gaussian curvature. On the other hand, as a regular surface in \( \mathcal{F} \) it has empty characteristic locus (the origin is not a regular point for the cone) and non zero (actually non constant) horizontal mean curvature. To see this, parametrise the lower part of the cone by
\[ \sigma(u, v) = (u \cos v, u \sin v, u), \quad u < 0, \ v \in (0, 2\pi). \]
One finds
\[ \nu^h = \frac{(\cos v - 2u \sin v)X + (\sin v + 2u \cos v)}{(1 + 4u^2)^{1/2}} \]
and
\[ H^h = \frac{1}{u(1 + 4u^2)^{3/2}}. \]
Thus it is not a straight ruled surface in \( \mathcal{F} \).
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