Bisected theta series, least $r$-gaps in partitions, and polygonal numbers

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Abstract
The least $r$-gap, $g_r(\lambda)$, of a partition $\lambda$ is the smallest part of $\lambda$ appearing less than $r$ times. In this article we introduce two new partition functions involving least $r$-gaps. We consider a bisection of a classical theta identity and prove new identities relating Euler’s partition function $p(n)$, polygonal numbers, and the new partition functions. To prove the results we use an interplay of combinatorial and $q$-series methods.

We also give a combinatorial interpretation for
\[ \sum_{n=0}^{\infty} (\pm 1)^{k(k+1)/2} p(n - r \cdot k(k+1)/2). \]

Keywords: partitions, least gap, polygonal numbers, theta series

MSC 2010: 05A17, 11P83

*This work was partially supported by a grant from the Simons Foundation (#245997 to Cristina Ballantine).
1 Introduction

In [6], the second author considered a bisection of Euler’s pentagonal number theorem

\( (q; q)_{\infty} = \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} q^{G_k} \)

based on the parity of the \( k \)-th generalized pentagonal number

\[ G_k = \frac{1}{2} \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{3k+1}{2} \right\rfloor, \]

and obtained the following result:

\[ \sum_{k=0}^{\infty} 1 + (-1)^{G_k} (-1)^{\lceil k/2 \rceil} q^{G_k} = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{30}, q^{32}; q^{32})_{\infty}, \quad (1) \]

where

\[ (a_1, a_2 \ldots, a_n; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_n; q)_{\infty}. \]

Because the infinite product

\[ (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) \]

diverges when \( a \neq 0 \) and \( |q| \geq 1 \), whenever \( (a; q)_{\infty} \) appears in a formula, we shall assume that \( |q| < 1 \).

The following identity for Euler’s partition function \( p(n) \) was obtained in [6] as a combinatorial interpretation of (1):

\[ \sum_{k=0}^{\infty} \frac{1 + (-1)^{G_k}}{2} (-1)^{\lceil k/2 \rceil} p(n - G_k) = L(n), \quad (2) \]

where \( L(n) \) is the number of partitions of \( n \) into parts not congruent to 0, 2, 12, 14, 16, 18, 20 or 30 mod 32. This identity is a bisection of Euler’s well-known recurrence relation for the partition function \( p(n) \):

\[ \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} p(n - G_k) = \delta_{0,n}, \quad (3) \]

where \( \delta_{i,j} \) is the Kronecker delta function. For details on (3) see Andrews’s book [1].

In this paper, motivated by these results, we consider a bisection of another classical theta identity [1, eq. 2.2.13]

\[ \frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = \sum_{k=0}^{\infty} (-q)^{k(k+1)/2} \quad (4) \]
Table 1: The partition functions $g_r$ for $\lambda \vdash 5$

| $\lambda$ | $\vdash 5$ | $4+1$ | $3+2$ | $3+1+1$ | $2+2+1$ | $2+1+1+1$ | $1+1+1+1+1$ |
|-----------|----------|-------|-------|---------|---------|-----------|--------------|
| $g_1(\lambda)$ | 1        | 2     | 1     | 2       | 3       | 3         | 2           |
| $g_2(\lambda)$ | 1        | 1     | 1     | 2       | 1       | 2         | 2           |
| $g_3(\lambda)$ | 1        | 1     | 1     | 1       | 1       | 2         | 2           |
| $g_4(\lambda)$ | 1        | 1     | 1     | 1       | 1       | 1         | 2           |
| $g_5(\lambda)$ | 1        | 1     | 1     | 1       | 1       | 1         | 2           |
| $g_6(\lambda)$ | 1        | 1     | 1     | 1       | 1       | 1         | 1           |

in order to derive new identities for Euler’s partition function. These identities involve new partition functions which we define below.

For what follows, we denote by $g_r(l)$ the smallest part of the partition $l$ appearing less than $r$ times. The limit distribution of $g_r(l)$ has been studied in [9]. In the literature, $g_1(l)$ is referred to as the least gap of $l$. By analogy, we refer to $g_r(\lambda)$ as the least $r$-gap of $\lambda$. To make formulas more concise, we set $g_0(n) = \infty$. We denote by $S_r(n)$ the sum of the least $r$-gaps in all partitions of $\lambda$, i.e.,

$$S_r(n) = \sum_{l \vdash n} g_r(l).$$

Thus, $S_1(n)$ is the sum of the least gaps in all partitions of $n$. By Table 1 we see, for example, that

$$S_1(5) = 1 + 2 + 1 + 2 + 3 + 3 + 2 = 14$$

and

$$S_4(n) = 1 + 1 + 1 + 1 + 1 + 1 + 2 = 8.$$

When $r \geq 2$, for each partition $l$ we have $g_r(l) \leq g_{r-1}(l)$. Let $G_r(n)$ be the number of partitions $l$ of $n$ satisfying $g_r(l) < g_{r-1}(l)$. It is clear that $G_1(n) = p(n)$ and $G_r(n) = 0$ for $r \geq n + 2$.

To our knowledge, the functions $S_r(n)$ and $G_r(n)$ have not been considered previously in the literature.

It is known [8, A022567] that the sum of the least gaps in all partitions of $n$ can be expressed in terms of the Euler’s partition function $p(n)$:

$$\sum_{k=0}^{\infty} p(n - T_k) = S_1(n),$$

where $T_n = n(n + 1)/2$ is the $n$-th triangular number. Upon reflection, one expects that there might be infinite families of such identities where [8] is the first entry. As far as we know, the following identity has not been remarked before.
Theorem 1.1. For \( n \geq 0 \) and \( r \geq 1 \),

\[
\sum_{k=0}^{\infty} p(n - rT_k) = S_r(n). \tag{6}
\]

In section 2 we provide a combinatorial proof of Theorem 1.1. Then, theta identity (4) and Theorem 1.1 allow us to find the generating function for \( S_r(n) \) and to prove the following result that also involves the partitions of \( n \) into parts not congruent to 0, \( r \) or \( 3r \mod 4r \). We denote by \( U_r(n) \) the number of these partitions.

Theorem 1.2. For \( n \geq 0 \) and \( r \geq 1 \),

\[
(i) \sum_{k=0}^{\infty} (p(n - rT_{4k}) + p(n - rT_{4k+3})) = \frac{S_r(n)}{2} + \frac{U_r(n)}{2};
\]

\[
(ii) \sum_{k=0}^{\infty} (p(n - rT_{4k+1}) + p(n - rT_{4k+2})) = \frac{S_r(n)}{2} - \frac{U_r(n)}{2}.
\]

By this theorem, we see that \( S_r(n) \) and \( U_r(n) \) have the same parity.

Corollary 1.3. For \( n \geq 0 \) and \( r \geq 1 \), the sum of the least \( r \)-gaps in all partitions of \( n \) and the number of partitions of \( n \) into parts not congruent to 0, \( r \) or \( 3r \mod 4r \) have the same parity.

In addition, we have the following identity.

Corollary 1.4. For \( n \geq 0 \) and \( r \geq 1 \),

\[
\sum_{k=0}^{\infty} (-1)^T_k p(n - rT_k) = U_r(n).
\]

Replacing \( r \) by 1 in Corollary 1.4, we obtain another known identity (see [4, the proof of Theorem 2.3]).

Corollary 1.5. For \( n \geq 0 \),

\[
\sum_{k=0}^{\infty} (-1)^T_k p(n - T_k) = \begin{cases} q \left( \frac{n}{2} \right), & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases}
\]

where \( q(n) \) is the number of partitions of \( n \) into distinct parts.

It is shown in [7, Corollary 4.7] that \( q(n) \) is odd if and only if \( n \) is a generalized pentagonal number. Thus, we deduce the following result related to the parity of \( S_1(n) \).

Corollary 1.6. For \( n \geq 0 \), the sum of the least gaps in all partitions of \( n \) is even except when \( n \) is twice a generalized pentagonal number.
If \( s \geq 3 \) is the number of sides of a polygon, the \( n \)-th \( s \)-polygonal number (or \( s \)-gonal number) is
\[
P(s, n) = \frac{n^2(s-2) - n(s-4)}{2}.
\]
If we allow \( n \in \mathbb{Z} \), we obtain generalized \( s \)-gonal numbers. Note that, for \( n > 0 \), we have \( P(3, -n) = P(3, n-1) \) and for all \( n \) we have \( P(4, -n) = P(4, n) \). For \( s \geq 5 \) and \( n > 0 \), \( P(s, -n) \) is not an ordinary \( s \)-gonal number. We remark that the \( n \)-th \( s \)-gonal number can be expressed in term of the triangular numbers \( T_n \) as follows:
\[
P(s, n) = (s-3)T_{n-1} + T_n.
\]

Beside Theorem 1.1, there is another infinite family of identities involving Euler’s partition function \( p(n) \) for which (5) is the special case \( r = 1 \).

**Theorem 1.7.** For \( n \geq 0 \) and \( r \geq 1 \)
\[
\sum_{k=0}^{\infty} p(n - P(r+2, -k)) = S_r(n) + G_r(n).
\]

In this paper, we provide a purely combinatorial proof of this result and some applications involving partitions into even numbers of parts, partitions with nonnegative rank, and partitions with nonnegative crank.

### 2 Combinatorial proof of Theorem 1.1

Fix \( r \geq 1 \) and, for each \( k \geq 0 \) consider the fat staircase partition (written in exponential notation)
\[
\delta_r(k) = (1^r, 2^r, \ldots, (k-1)^r, k^r).
\]
This is the staircase partition with largest part \( k \) in which each part is repeated \( r \) times. Its size is equal to \( rT_k \).

As before, fix \( r \geq 1 \) and also fix \( n \geq 0 \). For each \( k \geq 0 \) we create an injection from the set of partitions of \( n - rT_k \) into the set of partitions of \( n \)
\[
\varphi_{r,n,k} : \{ \mu \vdash n - rT_k \} \leftrightarrow \{ l \vdash n \}
\]
where \( \varphi_{r,n,k}(\mu) \) is the partition obtained from \( \mu \) by inserting the parts of the staircase \( \delta_r(k) \). Denote by \( A_{r,n,k} \) the image of \( \{ \mu \vdash n - rT_k \} \) under \( \varphi_{r,n,k} \). Thus, \( p(n - rT_k) = |A_{r,n,k}| \) and \( A_{r,n,k} \) consists precisely of the partitions \( l \) of \( n \) satisfying \( g_r(l) > k \).

Consider an arbitrary partition \( l \) of \( n \) with \( g_r(l) = k \). Then \( l \in A_{r,n,i}, i = 0, 1, \ldots, k-1 \) and \( l \notin A_{r,n,j} \) with \( j \geq k \). Therefore, each partition of \( n \) with \( g_r(l) = k \) is counted by the left hand side of (6) exactly \( k \) times.
3 Proof of Theorem 1.2

We rewrite the identity (4) as
\[ \sum_{k=0}^{\infty} (-q)^{T_k} = \frac{(q; q)_{\infty}}{(q^2; q^4)_{\infty}}. \]
(8)

Applying bisection on (8), we obtain:
\[ \frac{1}{2} \sum_{k=0}^{\infty} (q^{T_k} \pm (-q)^{T_k}) = \frac{1}{2} \frac{(-q; -q)_{\infty} \pm (q; q)_{\infty}}{(q^2; q^4)_{\infty}}. \]
(9)

Multiplying both sides of (9) by the reciprocal of \((q; q)_{\infty}\), we give
\[ \frac{1}{2} \frac{(q; q)_{\infty}}{(q^2; q^4)_{\infty}} \sum_{k=0}^{\infty} (q^{T_k} \pm (-q)^{T_k}) = \frac{1}{2} \frac{(-q^2; q^2)_{\infty} \pm (q^2; q^2)_{\infty}}{(q; q)_{\infty}}. \]

By this identity, with \(q\) replaced by \(q^r\), we obtain the relation
\[ \frac{1}{2} \frac{(q^r; q^r)_{\infty}}{(q^{2r}; q^{2r})_{\infty}} \sum_{k=0}^{\infty} (q^{T_k} \pm (-q^r)^{T_k}) = \frac{1}{2} \frac{(-q^r; q^r)_{\infty} \pm (q^r; q^r)_{\infty}}{(q^2; q^2)_{\infty}}. \]
that can be rewritten as
\[ \frac{1}{2} \frac{(q; q)_{\infty}}{(q^2; q^4)_{\infty}} \sum_{k=0}^{\infty} (q^{T_k} \pm (-q^r)^{T_k}) \]
\[ = \frac{1}{2} \left( \frac{(-q^r; q^r)_{\infty} \pm (q^r; q^r)_{\infty}}{(q; q)_{\infty}} \right) \frac{(q^2; q^2)_{\infty}}{(q^2; q^4)_{\infty}} \]
\[ = \frac{1}{2} \left( \frac{(-q^2; q^2r)_{\infty} \pm (q^2r; q^2r)_{\infty}}{(q; q)_{\infty}} \right) \frac{(q^2; q^4r)_{\infty}}{(q; q)_{\infty}}. \]
(10)

Considering the generating function for \(p(n)\), i.e.,
\[ \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}} \]
and the theta identity (4), by Theorem 1.1 we deduce that
\[ \sum_{k=0}^{\infty} S_r(k)q^k = \frac{(q^{2r}; q^{2r})_{\infty}}{(q; q)_{\infty} (q^r; q^{2r})_{\infty}}. \]
On the other hand, we have
\[ \sum_{k=0}^{\infty} U_r(k)q^k = \frac{(q^r, q^{3r}, q^{5r}; q^{4r})_\infty}{(q; q)_\infty}. \]

Taking into account the well-known Cauchy multiplication of two power series, we deduce our identities as combinatorial interpretations of (10).

### 4 Combinatorial proof of Theorem 1.7

The proof of Theorem 1.7 is analogous to the proof of Theorem 1.1. For fixed \( r \geq 1 \) and, for each \( k \geq 0 \) we denote by \( \delta_r(k) = (1^r, 2^r, \ldots, (k-1)^r, k^{r-1}) \) the staircase partition in which the largest part is \( k \) and is repeated \( r-1 \) times and all other parts are repeated \( r \) times. Its size is equal to \( P(r+2, -k) \).

As before, fix \( r \geq 1 \) and also fix \( n \geq 0 \). For each \( k \geq 0 \) we create an injection from the set of partitions of \( n - P(r+2, -k) \) into the set of partitions of \( n \)
\[ \varphi_{r,n,k} : \{ \mu \vdash n - P(r+2, -k) \} \leftrightarrow \{ l \vdash n \} \]
where \( \varphi_{r,n,k}(\mu) \) is the partition obtained from \( \mu \) by inserting the parts of the staircase \( \delta_r(k) \). If \( A'_{r,n,k} \) consists precisely of the partitions \( l \) of \( n \) satisfying \( g_r(l) = k \) and \( g_{r-1}(l) > k \).

If \( l \vdash n \) has \( g_r(l) = k \), then \( l \in A'_{r,n,i} \) if \( i = 0, 1, \ldots, k - 1 \). If \( g_{r-1}(l) = k \), then \( l \notin A'_{r,n,j} \) with \( j \geq k \). If \( g_{r-1} > k \), then \( l \in A'_{r,n,k} \) but \( l \notin A'_{r,n,j} \) with \( j > k \). Therefore, each partition of \( n \) with \( g_r(l) = k \) is counted by the left hand side of (7) exactly \( k \) times if \( g_r(l) = g_{r-1}(l) \) and exactly \( k + 1 \) times if \( g_r(l) < g_{r-1}(l) \).

### 5 Applications of Theorem 1.7

In this section we consider some special cases of Theorem 1.7 in order to discover and prove new identities involving Euler’s partition function \( p(n) \).

#### 5.1 Partitions into even numbers of parts

Now we consider the following classical theta identity [1, eq. 2.2.12]
\[ \frac{(q; q)_\infty}{(-q; q)_\infty} = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{2k}. \]

(11)

Elementary techniques in the theory of partition [1] allow us to derive a known combinatorial interpretation of this identity, namely
\[ p(n) + 2 \sum_{j=k}^{n} (-1)^j p(n - k^2) = p_e(n) - p_o(n), \]

(12)
where \( p_e(n) \) is the number of partitions of \( n \) into even number of parts and \( p_o(n) \) is the number of partitions of \( n \) into odd number of parts. Moreover, it is known that

\[
p_e(n) = p(n) + \sum_{k=1}^{n} (-1)^k p(n-k^2) \quad \text{(13)}
\]

and

\[
p_o(n) = -\sum_{k=1}^{n} (-1)^k p(n-k^2).
\]

These relations can be considered a bisection of the identity (12). Combining identity (13) with the case \( r = 2 \) of Theorem 1.7, we derive the following result.

**Corollary 5.1.** For \( n \geq 0 \),

\[
(i) \sum_{k=0}^{\infty} p(n-(2k)^2) = \frac{S_2(n) + G_2(n) + p_e(n)}{2};
\]

\[
(ii) \sum_{k=0}^{\infty} p(n-(2k+1)^2) = \frac{S_2(n) + G_2(n) - p_e(n)}{2}.
\]

### 5.2 Partitions with nonnegative rank

In 1944, Dyson [5] defined the rank of a partition as the difference between its largest part and the number of its parts. Then he observed empirically that the partitions of \( 5n + 4 \) (respectively \( 7n + 5 \)) form 5 (respectively 7) groups of equal size when sorted by their ranks modulo 5 (respectively 7). This interesting conjecture of Dyson was proved ten years later by Atkin and Swinnerton-Dyer [3]. In this section, we denote by \( R(n) \) the number of partitions of \( n \) with nonnegative rank.

It is known [8, A064174] that the number of partitions of \( n \) with nonnegative rank can be expressed in terms of Euler’s partition function as follows:

\[
R(n) = \sum_{k=0}^{n} (-1)^k p(n-k(3k+1)/2). \quad \text{(14)}
\]

Considering the case \( r = 3 \) of Theorem 1.7, we obtain the following result.

**Corollary 5.2.** For \( n \geq 0 \),

\[
(i) \sum_{k=0}^{\infty} p(n-k(6k+1)) = \frac{S_3(n) + G_3(n) + R(n)}{2};
\]

\[
(ii) \sum_{k=0}^{\infty} p(n-(2k+1)(3k+2)) = \frac{S_3(n) + G_3(n) - R(n)}{2}.
\]
5.3 Partitions with nonnegative crank

Dyson [5] conjectured the existence of a crank function for partitions that would provide a combinatorial proof of Ramanujan’s congruence modulo 11. Forty-four years later, Andrews and Garvan [2] successfully found such a function which yields a combinatorial explanation of Ramanujan congruences modulo 5, 7, and 11. For a partition $\lambda$, let $l(\lambda)$ denote the largest part of $\lambda$, $\omega(\lambda)$ denote the number of 1’s in $\lambda$, and $\mu(\lambda)$ denote the number of parts of $\lambda$ greater than $\omega(\lambda)$. The crank $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} 
    l(\lambda), & \text{for } \omega(\lambda) = 0, \\
    \mu(\lambda) - \omega(\lambda), & \text{for } \omega(\lambda) > 0.
\end{cases}$$

In this section, we denote by $C(n)$ the number of partitions of $n$ with nonnegative crank.

We known [8, A064428] that the number of partitions of $n$ with nonnegative crank can be expressed in terms of Euler’s partition function $p(n)$:

$$C(n) = \sum_{k=0}^{\infty} (-1)^k p(n - T_k).$$

We have the following result related to the parity of $C(n)$.

**Corollary 5.3.** For $n \geq 0$, the number of partitions of $n$ with nonnegative crank is even except when $n$ is twice a generalized pentagonal number.

**Proof.** Considering the case $r = 1$ of Theorem 1.7 and the identity (15), we obtain

$$\sum_{k=0}^{\infty} p(n - T_{2k}) = \frac{C(n) + S_1(n)}{2}.$$

We see that the number of partitions of $n$ with nonnegative crank and the sum of the least gaps in all partitions of $n$ have the same parity. According to Corollary 1.6 the proof is finished. \qed

By the identity (15) and Corollary 1.4, we easily get two identities.

**Corollary 5.4.** For $n \geq 0$,

\begin{align*}
(i) \quad & \sum_{k=0}^{\infty} (-1)^k p(n - T_{k+2\lfloor k/2 \rfloor}) = \frac{C(n) + U_1(n)}{2}; \\
(ii) \quad & \sum_{k=0}^{\infty} (-1)^k p(n - T_{k+2\lfloor k/2 \rfloor} + 2) = \frac{C(n) - U_1(n)}{2}.
\end{align*}
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