GENERALISED SOLUTIONS FOR FULLY NONLINEAR PDE SYSTEMS AND EXISTENCE THEOREMS

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ABSTRACT. We introduce a new theory of generalised solutions which applies to fully nonlinear PDE systems of any order and allows the interpretation of merely measurable maps as solutions without any further a priori regularity requirements. This approach bypasses the standard problems arising by the application of distributions to PDEs and is not based on either duality or on integration by parts. Instead, the starting point builds on the probabilistic representation of limits of difference quotients via Young (parameterised) measures over compactifications of the “state space”. After developing some basic theory, as a first application we prove existence of solution to the Dirichlet problem for the $\infty$-Laplace system of vectorial Calculus of Variations in $L^\infty$ and also for fully nonlinear degenerate elliptic 2nd order hessian systems.

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1. INTRODUCTION

It is well known that PDEs, either linear or nonlinear, in general do not possess classical solutions, in the sense that not all derivatives that appear in the

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equation may actually exist. The modern approach to this problem consists of looking for appropriately defined generalised solutions for which the hope is that at least existence can be proved given certain boundary and/or initial conditions. Once existence is settled, subsequent considerations typically include uniqueness, qualitative properties, regularity and of course numerics.

This approach to PDE theory has been enormously successful, but unfortunately so far only equations and systems with fairly special structure have been considered. A standing idea consist of using integration by parts and duality of functional spaces in order to interpret rigorously derivatives which do not exist, by “passing them to test functions”. This approach of Sobolev spaces and Schwartz’s distributions which dates back to the 1930s ([S1, S2, So]) is basically restricted to equations which have divergence structure, like the Euler-Lagrange equation of Calculus of Variations or linear systems with smooth coefficients. A more recent theory discovered in the 1980s is that of viscosity solutions ([CL]) and builds on the idea that the maximum principle allows to “pass derivatives to test functions” without duality. This idea applies mostly to scalar solutions of single equations which support the maximum principle (elliptic or parabolic up to 2nd order), but has been hugely successful because it includes fully nonlinear equations. A relevant notion of solution which bridges the gap between classical and generalised is that of strong solutions, where it is usually assumed that all derivatives appearing exist a.e. but in a weak sense.

In this paper we introduce a new theory of generalised solutions which applies to fully nonlinear PDE systems of any order. Our approach allows for merely measurable maps to be rigorously interpreted and studied as solutions of PDE systems, even fully nonlinear and with discontinuous coefficients. More precisely, let \( p, n, N, M \in \mathbb{N} \), let also \( \Omega \subseteq \mathbb{R}^n \) be an open set and

\[
F : \Omega \times \left( \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2} \times \cdots \times \mathbb{R}^{Nn^p} \right) \longrightarrow \mathbb{R}^M
\]

a Carathéodory map. Here \( \mathbb{R}^{Nn} \) denotes the space of \( N \times n \) matrices and \( \mathbb{R}^{Nn^p}_s \) the space of symmetric tensors

\[
\left\{ X \in \mathbb{R}^{Nn^p} \mid X_{\alpha i_1...i_p} = X_{\alpha \sigma(i_1...i_p)}, \alpha = 1, ..., N, \right. \\
\left. i_k = 1, ..., n, k = 1, ..., p, \sigma \text{ permutation} \right\}
\]

wherein the gradient matrix

\[
Du(x) = \left( D_i u_\alpha(x) \right)_{i=1,...,n}^{\alpha=1,...,N}
\]

and the \( p \)-th order derivative

\[
D^p u(x) = \left( D_{i_1...i_p}^p u_\alpha(x) \right)_{i_1,...,i_p \in \{1,...,n\}}^{\alpha=1,...,N}
\]

of (smooth) maps \( u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N \) are respectively valued. Obviously, \( D_i \equiv \partial/\partial x_i \), \( x = (x_1, ..., x_n)\top \), \( u = (u_1, ..., u_N)\top \) and \( \mathbb{R}^{Nn^p}_s = \mathbb{R}^{Nn} \). The present theory applies to measurable solutions of the system

\[
F(\cdot, u, Du, ..., D^p u) = 0, \quad \text{on} \ \Omega,
\]

without any further restrictions on \( F \). Since we will not assume that the solutions are locally integrable on \( \Omega \), the derivatives \( Du, ..., D^p u \) may not have any classical meaning, not even in the sense of distributions.
The starting point of our approach is not based either on duality or on the maximum principle. Instead, it builds on the probabilistic interpretation of the limits of difference quotients by using the notion of Young measures, also known as parameterised measures. These measures have been introduced by L.C. Young in the late 1930s ([Y]) in order to show existence of “relaxed” solutions to nonconvex variational problems for which the minimum may not be attained at a weakly differentiable minimiser. Since then, they turned out to be exceptionally useful in Calculus of Variations and PDE theory ([E, M, P, FL]) and there is also a modern abstract topological theory for them ([CFV, FG, V]). Their main utility so far has been to study the failure of strong convergence in approximating sequences due to the combination of phenomena of oscillations and/or concentrations ([DPM, KR]) typically occurring in minimisation of functionals or approximation of solutions to an equation.

In the present framework, a version of Young measures is utilised in order to define generalised solutions of (1.2) by applying it to the difference quotients of the candidate solution. The exact definitions are thoroughly motivated later, but roughly the idea restricted to the first order case $p = 1$ of (1.1) is as follows: suppose $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a Lipschitz continuous strong a.e. solution of the system

$$F(\cdot, u, Du) = 0, \quad \text{on } \Omega.$$  

Then, $u$ satisfies

$$F\left(x, u(x), \lim_{h \to 0} D^{1,h}u(x)\right) = 0,$$

for a.e. $x \in \Omega$, where $D^{1,h}$ the first difference quotients operator. Since $F$ is continuous with respect to the gradient variable, this is equivalent to

$$\lim_{h \to 0} F\left(x, u(x), D^{1,h}u(x)\right) = 0,$$

for a.e. $x \in \Omega$. The crucial observation is that the above statement makes sense even if $u$ is not differentiable. In order to represent this limit, we imbed the difference quotients into the space of Young measures over the Alexandroff compactification

$$\mathbb{R}^n : = \mathbb{R}^n \cup \{\infty\}$$

of $\mathbb{R}^n$ (that is the space $\mathcal{Y}(\Omega, \mathbb{R}^n)$ of probability-valued maps $\Omega \rightarrow \mathcal{P}(\mathbb{R}^n)$, see Section 2 for the precise definitions). By the weak* compactness of the space $\mathcal{Y}$, there always exists a probability-valued map

$$Du : \Omega \rightarrow \mathcal{P}(\mathbb{R}^n)$$

such that

$$\delta_{D^{1,h}u} \Rightarrow Du, \quad \text{in } \mathcal{Y}(\Omega, \mathbb{R}^n),$$

as $h \rightarrow 0$ along a sequence (even if $u$ is not differentiable). Then, by using the properties of the space $\mathcal{Y}$, it can be shown that strong solutions of (1.3) satisfy

$$\int_{\mathbb{R}^n} \Phi(P) F(x, u(x), P) d[Du(x)](P) = 0, \quad \text{a.e. } x \in \Omega,$$

for any compactly supported “test” function $\Phi \in C^0_c(\mathbb{R}^n)$. We stress again that this last statement is independent of the Lipschitz regularity of $u$: the only extra piece of information the regularity provides is that $Du$ coincides a.e. with the Dirac mass $\delta_{Du}$ at the pointwise gradient $Du$. 
Despite its appealing elegance, the above property of strong solutions is too simplistic to be taken exactly as it stands as meaningful definition of generalised solution. The problem is that in general existence of solutions may not be possible without considering special adapted difference quotients with respect to appropriate frames which follow the geometry of the level sets of $F$ and should be allowed to vary with the basepoint. This relaxation is necessary and finding the appropriate frame with respect to which there exist difference quotients leading to existence is an essential part of the “weak formulation” of the problem. Notwithstanding, up to this further adaptation of the concept, (1.4) and (1.5) essentially consist the definition of diffuse solutions or just $D$-solutions\footnote{The author is using the letter “$D$-” as a short of either of the modifiers “diffuse” or “dim” or “disintegration” because all of these terms are descriptive of the notion of solution. We leave it to the reader to decide for the interpretation of their preference.} in the special case of the 1st order system (1.3) and will be the central notion of solution in this paper.

Our motivation to introduce and study generalised solutions for nonlinear PDE systems primarily comes from the need to study the recently discovered $\infty$-Laplace system rigorously, which is the fundamental equation of Vectorial Calculus of Variations in the space $L^\infty$. Calculus of Variations in $L^\infty$ has a long history which started in the 1960s by Aronsson ([A1]-[A5]) who was the first to consider variational problems for supremal functionals of the form

$$E_\infty(u, \Omega) := \|H(\cdot, u, Du)\|_{L^\infty(\Omega)}.$$  

Aronsson introduced the appropriate notion of minimisers for such functionals and studied classical solutions of the respective equation which is the $L^\infty$-analogue of the Euler-Lagrange equation. In the simplest case of $H(p) = |p|$ (the Euclidean norm on $\mathbb{R}^n$), the $L^\infty$-equation is called the $\infty$-Laplacian and reads

$$\Delta_\infty u := Du \otimes Du : D^2u = 0.$$  

Since then, the field has undergone huge developement due to both the intrinsic mathematical interest and the important for applications: minimisation of the maximum provides more realistic models when compared to the classical case of integral functionals where the average is minimised instead. A basic difficulty in the study of (1.6) is that (1.7) possesses singular solutions. Aronsson himself exhibited this in [A6, A7] and the field had to wait until the development of viscosity solutions for 2nd order equations in the early 1990s in order to study general solutions (see [C, BEJ, E, E2] and for a pedagogical introduction see [K8]).

Until recently, the study of supremal functionals was restricted exclusively to the scalar case of $N = 1$ and to first order problems. The principal reason for this weakness was the absense of an efficient theory of generalised solutions which would allow the rigorous study of non-divergence PDE systems or higher order equations, including those arising in $L^\infty$. The foundations of the vector case of (1.6), including the discovery of the appropriate system version of (1.7), the correct vectorial minimality notion and the study of classical solutions have been layed in a series of recent papers of the author ([K1]-[K6]). In the simplest case of

$$E_\infty(u, \Omega) = \|Du\|_{L^\infty(\Omega)}$$  

applied to Lipschitz maps $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ (where the $L^\infty$ norm is interpreted as the essential supremum of the Euclidean norm $|Du|$ on $\mathbb{R}^{Nn}$), the respective
The \( \infty \)-Laplace system is
\[
\Delta_{\infty} u := \left( Du \otimes Du + |Du|^2 [Du]^\perp \otimes I \right) : D^2 u = 0.
\]
In the above, \( [Du]^\perp \) denotes the orthogonal projection on the nullspace of the operator \( Du(x)^T : \mathbb{R}^N \to \mathbb{R}^n \) and in index form reads
\[
\sum_{\beta=1}^N \sum_{i,j=1}^n \left( D_i u_\alpha D_j u_\beta + |Du|^2 [Du]^\perp_{\alpha \beta} \delta_{ij} \right) D^2_{ij} u_\beta = 0, \quad \alpha = 1, \ldots, N.
\]
An extra difficulty of (1.9) which is not present in the scalar case is that the nonlinear operator defining the system may have \textit{discontinuous coefficients} even when applied to smooth maps since the new term involving \( [Du]^\perp \) measures the dimension of the tangent spaces of \( u(\Omega) \) ([K1, K6]). Almost simultaneously to [K1], Sheffield and Smart [SS] studied the relevant problem of vectorial Lipschitz extensions and derived a different singular version of “\( \infty \)-Laplacian”, which in the present setting amounts to changing in (1.8) from the Euclidean to the operator norm on \( \mathbb{R}^{Nn} \).

A further motivation to introduce a new theory of generalised solutions stems from the insufficiency of the current PDE approaches to handle even \textit{elliptic linear} systems with rough coefficients: if \( A \) is a \textit{continuous} symmetric 4th order tensor on \( \mathbb{R}^{Nn} \) satisfying the strict Legendre-Hadamard condition (that is, the quadratic form \( P \mapsto A : P \otimes P \) is strictly rank-one convex on \( \mathbb{R}^{Nn} \)), then for the divergence system
\[
\sum_{\beta=1}^N \sum_{i,j=1}^n A_{\alpha i \beta j}(x) D^2_{ij} u_\beta(x) = 0, \quad \alpha = 1, \ldots, N,
\]
“everything” is known: existence-uniqueness of weak solutions, regularity, etc (see e.g. [GM]). On the other hand, for its non-divergence analogue
\[
\sum_{\beta=1}^N \sum_{i,j=1}^n A_{\alpha i \beta j}(x) D^2_{ij} u_\beta(x) = 0, \quad \alpha = 1, \ldots, N,
\]
“nothing” is known, not even what is a meaningful notion of generalised solution, unless \( A \) is \( C^{0,\alpha} \) in which case a priori estimates guarantee that solutions, if they exist, have to be smooth ([GM]).

In the present paper, after motivating, introducing and developing some basic theory of \( D \)-solutions for general nonlinear PDE systems (Section 2), we apply it to two specific problems and prove existence. Accordingly, we first consider the Dirichlet problem for the \( \infty \)-Laplacian
\[
\Delta_{\infty} u = 0, \quad\text{on } \Omega,
\]
\[
u = g, \quad\text{on } \partial \Omega.
\]
when \( \Omega \subseteq \mathbb{R}^n \) is an open domain with finite measure, \( n = N \) and \( g \in W^{1,\infty}(\Omega, \mathbb{R}^n) \). In Section 3 we prove existence of a \( D \)-solution \( u \in W^{1,\infty}_g(\Omega, \mathbb{R}^n) \) to (1.11). The idea of the proof has two main steps. We first apply the analytic counterpart of Gromov’s Convex Integration in the form of the Dacorogna-Marcellini Baire Category method ([DM]) and prove existence of a \( W^{1,\infty} \) solution to a 1st order differential inclusion associated to (1.11). Next, we use the apparatus of \( D \)-solutions to characterise this map as solution to (1.9). A particular technical subtlety is the use of Aumann’s measurable selection theorem of multifunctions, employed in order to construct an
appropriate frame along which convergence of difference quotients in the space of Young measures can be achieved.

The second main question we consider in this paper concerns the existence of $\mathcal{D}$-solutions to the Dirichlet problem for fully nonlinear 2nd order hessian systems

$$
\begin{align*}
F(\cdot, D^2 u) &= f, \quad \text{on } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

when $\Omega \subset \mathbb{R}^n$ is a $C^2$ convex domain, $F : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^N$ is a Carathéodory map and $f \in L^2(\Omega, \mathbb{R}^N)$. The essential hypothesis guaranteeing existence is an appropriate degenerate ellipticity condition. Roughly, we require that $F$ “is not too far away” from a degenerate linear system of the form (1.10) with $A$ constant which satisfies the (weak) Legendre-Hadamard condition. The problem (1.12) has first been considered by Campanato [C1, C2, C3] under a strong uniform ellipticity assumption of Cordes type which roughly requires $F$ to be “near” the Laplacian. Under this condition, it can proved unique solvability of (1.12) in $(H^2 \cap H^1_0)(\Omega, \mathbb{R}^N)$. Very recently, the author ([K9, K10] and also [K7]) has generalised the results of Campanato by proving existence of strong solutions under a new weaker ellipticity notion. These results were stepping stones to the general approach we develop herein for such problems in the context of $\mathcal{D}$-solutions.

In Section 4 we prove existence of a measurable $\mathcal{D}$-solution to (1.12). In particular, we resolve the matter of defining a meaningful notion of generalised solution and proving existence to the Dirichlet problem for the nondivergence systems mentioned above. The proof is rather long and is based on the study of the Dirichlet problem for (1.10) in the $\mathcal{D}$-sense when $A$ is constant, the introduction of apt Sobolev “fibre spaces” of maps possessing weakly differentiable projections along certain rank-one direction and the hypothesis of degenerate ellipticity which acts as “perturbation device”. The latter allows the passage from the linear to the non-linear problem via a fixed point argument in the guises of Campanato’s near operators. Then, we use the machinery of $\mathcal{D}$-solutions to show that there is an appropriate frame depending on $F$ with respect to which we have convergence of the difference quotients in the Young measures.

A particular difficulty in the solvability of (1.12) is the satisfaction of the boundary condition under this low regularity. In the absence of $W^{1,1}_{\text{loc}}$ regularity for the $\mathcal{D}$-solution, there is no trace operator giving boundary values for merely measurable maps. Notwithstanding, by using the structure of $F$ and that of the tailored “fibre spaces”, we extend some part of the classical Sobolev space theory ([E2, T]), including traces, to this new setting.

We conclude this introduction by noting that the table of contents gives an idea of the organisation of the material in this paper, as well as where the reader may find further motivation of the main ideas and proofs. We hope that the systematic theory proposed herein will be the starting point for future developments. In particular, in the companion paper [K11] we consider the relevant problem of existence of $\mathcal{D}$-solutions to the Dirichlet problem of the vectorial equations of Calculus of Variations in $L^\infty$ for (1.6) in one space dimension. Therein we follow the “natural” approach in order to prove existence for the $L^\infty$ system, namely we approximate by the Euler-Lagrange systems of $L^p$ functionals as $p \to \infty$. A central difficulty when one attempts to follow this route is that in the vector case existence is a highly nontrivial matter and a priori estimates are required because $p$-Harmonic limits
are the “good” solutions of (1.9) (see the remarks in Section 3 about uniqueness). The analogue of [K11] for higher space dimensions will be considered in future work.

2. Theory of $D$-solutions for fully nonlinear systems

2.1. Preliminaries. We begin by collecting some introductory material which will be needed in the rest of the paper.

Basics. Let $n, N \in \mathbb{N}$ be fixed, which in this paper will always be the dimensions of domain and range respectively of our candidate solutions $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{ 1, ..., N \}$ and latin indices $i, j, k, ...$ (perhaps indexed $i_1, i_2, ...$) will run in $\{ 1, ..., n \}$, even when the range is not explicitly denoted. The norms $| \cdot |$ will always be the Euclidean and the Euclidean inner products will be denoted by “$\cdot, \cdot$” on $\mathbb{R}^n, \mathbb{R}^N$ and by “$;,$” on tensor spaces, e.g. on $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{N \times N}$ we have

$$|P|^2 = \sum_{\alpha,i} P_{\alpha i} P_{\alpha i} \equiv P : P, \quad |X|^2 = \sum_{\alpha,i,j} X_{\alpha ij} X_{\alpha ij} \equiv X : X,$$

etc. The standard bases on $\mathbb{R}^n, \mathbb{R}^N, \mathbb{R}^{n \times n}$ will be denoted by $\{ e^i \}$, $\{ e^\alpha \}$ and $\{ e^\alpha \otimes e^\beta \}$. By introducing the symmetrised tensor product

$$a \vee b := \frac{1}{2} (a \otimes b + b \otimes a), \quad a, b \in \mathbb{R}^n,$$

we have that the standard basis of the space $\mathbb{R}_s^{N,n}p$ is given by $\{ e^\alpha \otimes e^i \vee ... \vee e^l p \}$. In particular, for $\mathbb{R}_s^{N,n^2}$ the standard basis is $\{ e^\alpha \otimes e^i \vee e^j \},$ etc.

We will follow the convention of denoting vector subspaces of Euclidean spaces as well as the orthogonal projections on them by the same symbol. For example, if $\Sigma \subseteq \mathbb{R}^n$ is a subspace, we denote the projection map $\text{Proj}_\Sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by just $\Sigma$ and we have $\Sigma^2 = \Sigma \cap \Sigma = \Sigma \in \mathbb{R}_s^{N,n^2}$, etc.

Our measure theoretic and function space notation is the standard one as e.g. in [E2, EG]. For example, the Lebesgue measure on $\mathbb{R}^n$ will be denoted by $| \cdot |$ or $\mathcal{L}^n$, the s-Hausdorff measure by $\mathcal{H}^s$, the charasteristic function of the set $A$ by $\chi_A$, the standard Sobolev and $L^p$ spaces of maps $u : \Omega \subseteq \mathbb{R}^n \rightarrow \Sigma \subseteq \mathbb{R}^N$ by $L^p(\Omega, \Sigma)$, $W^{m,p}(\Omega, \Sigma)$ etc.

We will systematically use the Alexandroff 1-point compactification of the space $\mathbb{R}_s^{N,n^p}$. Its topology will be the standard one which makes it homeomorphic to the sphere of the same dimension (via the stereographic projection which identifies $\{ \infty \}$ with the north pole). We will denote it by

$$\mathbb{R}_s^{N,n^p} := \mathbb{R}_s^{N,n^p} \cup \{ \infty \}.$$

General frames, derivative expansions, difference quotients. In the sequel we will also need to consider non-standard orthonormal frames and express derivatives $D^p u$ with respect to such frames of $\mathbb{R}_s^{N,n^p}$. Indeed, let $\{ \eta^\alpha \}$ be an orthonormal frame of $\mathbb{R}^N$ and suppose that for each $\alpha = 1, ..., N$ we have an orthonormal frame $\{ \xi^{(\alpha)1}, ..., \xi^{(\alpha)n} \}$ of $\mathbb{R}^n$. Then, we have

$$\mathbb{R}^{nN} = \text{span}\{ E^{\alpha i} \}, \quad E^{\alpha i} := \eta^\alpha \otimes \xi^{(\alpha)i},
$$

(2.2) $$\mathbb{R}_s^{N^2} = \text{span}\{ E^{\alpha ij} \}, \quad E^{\alpha ij} := \eta^\alpha \otimes \xi^{(\alpha)i} \vee \xi^{(\alpha)j},$$

$$\mathbb{R}_s^{Nnp} = \text{span}\{ E^{\alpha i_1 \ldots i_p} \}, \quad E^{\alpha i_1 \ldots i_p} := \eta^\alpha \otimes \xi^{(\alpha)i_1} \vee \ldots \vee \xi^{(\alpha)i_p}.$$

Given such frames, let
\[
D_{\xi^{(a)}} D_{\xi^{(b)}} := D_{\xi^{(a)}} D_{\xi^{(b)}}, \quad D_{\xi^{(a)}} := D_{\xi^{(a)}} D_{\xi^{(a)}}, \quad D_{\xi^{(a)}} \cdots D_{\xi^{(a)}} := D_{\xi^{(a)}} \cdots D_{\xi^{(a)}},
\]
denote the usual directional derivative operators of 1st, 2nd and \(p\)-th order along \(\xi^{(a)}, \{\xi^{(a)}, \xi^{(b)}\}\) and \(\{\xi^{(a)}, \ldots, \xi^{(b)}\}\) respectively. Then, the derivatives \(Du, D^2u, D^pu\) of smooth maps \(u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N\) can be written as
\[
Du = \sum_{\alpha,i} \left( E^{\alpha i} : Du \right) E^{\alpha i}, \quad D^2u = \sum_{\alpha,i,j} \left( E^{\alpha ij} : D^2u \right) E^{\alpha ij}, \quad D^pu = \sum_{\alpha,i_1,\ldots,i_p} \left( E^{\alpha i_1\ldots i_p} : D^pu \right) E^{\alpha i_1\ldots i_p},
\]
(2.3)

Given \(a \in \mathbb{R}^n\) with \(|a| = 1\) and \(h \in \mathbb{R} \setminus \{0\}\), the 1st difference quotient of \(u\) along the direction \(a\) will be denoted by
\[
D^1_{a,h} u(x) := \frac{u(x + ha) - u(x)}{h}
\]
when \(x, x + ah \in \Omega\). By iteration, the \(p\)-th order difference quotient along \(a_1, \ldots, a_p\) is
\[
D^{p,h_{a_1}\ldots h_{a_p}} u(x) := D^{1,h_{a_1}} \left( \cdots \left( D^{1,h_{a_p}} u \right) \cdots \right)(x),
\]
(2.5)

where \(h_1, \ldots, h_p \neq 0\).

**Borel measurable representatives.** In the sequel we will need to consider compositions of our candidate (Lebesgue) measurable solutions \(u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N\) with measurable maps \(t : \Omega \rightarrow \Omega\). In order to have Lebesgue measurable compositions \(u \circ t\), we will always replace \(u\) by a Borel measurable representative in the equivalence class of maps which are a.e. equal to \(u\). For concreteness, let us indicate how such a representative can be defined:

Let \(\left(u^m\right)_n\) be the standard sequence of simple functions such that \(u^m(x) \rightarrow u(x)\) for every \(x \in \Omega\) (see e.g. [F], p. 47). By replacing each characteristic function \(\chi_A\) of each term of the approximating sequence by the characteristic function \(\chi_{A'}\), where \(A' \subseteq A\) is a Borel set such that \(|A' \setminus A| = 0\), it can be easily shown that there is a Borel nullset \(E \subseteq \Omega\) with \(|E| = 0\) and a Borel measurable map \(u^* : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N\) such that \(u = u^*\) everywhere on \(\Omega \setminus E\) and \(u^* \equiv 0\) on \(E\).

In the sequel we will always replace our Lebesgue measurable map \(u\) by a Borel measurable map (e.g. \(u^*\)) with which it coincides a.e. on \(\Omega\). We stress however that the arguments will not depend on the precise properties of the Borel representative, but only on the fact that it is Borel measurable and a.e. equal to the original version of the map.

**Young Measures.** Let \(E \subseteq \mathbb{R}^n\) be a measurable set and \(K \subseteq \mathbb{R}^d\) a compact set of some Euclidean space, which we will later take to be \(\mathbb{R}^m\). Consider the \(L^1\) space of strongly measurable mappings valued in the (separable Banach) space \(C^0(K)\) of real continuous functions over \(K\), in the standard sense of Bochner integrals:
\[
L^1(E, C^0(K)).
\]
For more details about these spaces we refer e.g. to [FL, F, V] (and references therein). Then, the elements of this space can be identified with the Carathéodory integrands
\[ \Phi : E \times \mathbb{K} \rightarrow \mathbb{R}, \quad (x, Q) \mapsto \Phi(x, Q) \equiv \Phi_x(Q), \]
for which
\[ \|\Phi\|_{L^1(E, C^0(\mathbb{K}))} := \int_{E} \max_{Q \in \mathbb{K}} |\Phi_x(Q)| \, dx < \infty \]
and the identification is given by considering \( \Phi \) as a map \( x \mapsto \Phi_x \in C^0(\mathbb{K}) \). Carathéodory integrands are understood in the standard sense, that is \( x \mapsto \Phi(x, Q) \) is measurable for every \( Q \in \mathbb{K} \) and \( Q \mapsto \Phi(x, Q) \) is continuous for a.e. \( x \in E \). Then, \( L^1(E, C^0(\mathbb{K})) \) is separable and the simple functions in this space (which are norm-dense) have the form
\[ E \ni x \mapsto \sum_{i=1}^{q} \chi_{E_i}(x) \Phi_i \in C^0(\mathbb{K}), \]
where \( E_1, \ldots, E_q \) are measurable disjoint sets and \( \Phi_i \in C^0(\mathbb{K}) \). By using the duality
\[ (C^0(\mathbb{K}))^* = \mathcal{M}(\mathbb{K}) \]
where \( \mathcal{M}(\mathbb{K}) \) denotes the real (signed) Radon measures on \( \mathbb{K} \) endowed with the total variation norm, it can be shown (see e.g. [FL]) that the dual space of \( L^1(E, C^0(\mathbb{K})) \) is given by
\[ (L^1(E, C^0(\mathbb{K})))^* = L^\infty_{w}(E, \mathcal{M}(\mathbb{K})). \]
The Banach space \( L^\infty_{w}(E, \mathcal{M}(\mathbb{K})) \) consists of weakly* measurable maps
\[ E \ni x \mapsto \vartheta(x) \equiv \vartheta_x \in \mathcal{M}(\mathbb{K}) \]
and weak* measurability is meant in the sense that for any Borel set \( B \subseteq \mathbb{K} \), the function \( E \ni x \mapsto \vartheta_x(B) \in \mathbb{R} \) is measurable. The norm of \( L^\infty_{w}(E, \mathcal{M}(\mathbb{K})) \) is
\[ \|\vartheta\|_{L^\infty_{w}(E, \mathcal{M}(\mathbb{K}))} := \text{ess sup}_{x \in E} \|\vartheta_x\|_{(\mathbb{K})} \]
where \( \|\cdot\|_{(\mathbb{K})} \) denotes the total variation. The duality pairing
\[ \langle \cdot, \cdot \rangle : L^\infty_{w}(E, \mathcal{M}(\mathbb{K})) \times L^1(E, C^0(\mathbb{K})) \rightarrow \mathbb{R} \]
is given by
\[ \langle \vartheta, \Phi \rangle := \int_{E} \int_{\mathbb{K}} \Phi(x, Q) \, d\vartheta_x(Q) \, dx. \]
Since \( L^1(E, C^0(\mathbb{K})) \) is separable, the unit ball of \( L^\infty_{w}(E, \mathcal{M}(\mathbb{K})) \) is sequentially weakly* compact. Hence, for any bounded sequence \( (\vartheta^m)_{m=1}^{\infty} \subseteq L^\infty_{w}(E, \mathcal{M}(\mathbb{K})) \), there is a limit map \( \vartheta \) and a subsequence of \( m \)'s along which \( \vartheta^m \rightharpoonup \vartheta \) as \( m \rightarrow \infty \). By definition, we have
\[ \vartheta^m \rightharpoonup \vartheta \iff \langle \vartheta^m - \vartheta, \Phi \rangle \rightarrow 0, \text{ for all } \Phi \in L^1(E, C^0(\mathbb{K})). \]
Further, by the density of simple functions and linearity, for bounded sequences the weak* convergence \( \vartheta^m \rightharpoonup \vartheta \) is equivalent to
\[ \int_{\mathbb{K}} \Phi(Q) \, d[\vartheta^m - \vartheta](Q) \rightharpoonup 0, \text{ in } L^\infty(E), \]
for any fixed \( \Phi \in C^0(\mathbb{K}) \).
**Definition 1** (Young Measures). The space of Young (or Parameterised) Measures is the subset of the unit sphere of $L_w^\infty(E,\mathcal{M}(\mathbb{K}))$ which consists of probability-valued weakly* measurable maps:

$$\mathcal{Y}(E,\mathbb{K}) := \left\{ \vartheta \in L_w^\infty(E,\mathcal{M}(\mathbb{K})) : \vartheta_x \in \mathcal{P}(\mathbb{K}), \text{ for a.e. } x \in E \right\}.$$ 

**Remark 2** (Properties of $\mathcal{Y}(E,\mathbb{K})$, see e.g. [FG]). i) The set of Young measures is convex and by the compactness of $\mathbb{K}$, it follows that it is weakly* compact in $L_w^\infty(E,\mathcal{M}(\mathbb{K}))$. The idea of the proof is the following: boundedness is obvious; the weak* closedness can be seen by using the equivalence between Young measures and the set of positive finite measures $\nu$ on $E \times \mathbb{K}$ for which the first marginal is the Lebesgue measure (which other authors call “disintegrations”, see also [E]) and also that positive functionals over $C^0(\mathbb{K})$ are measures (see e.g. [EG]).

ii) The space of Lebesgue measurable functions $v : E \subseteq \mathbb{R}^n \rightarrow \mathbb{K}$ has weakly* dense image in $\mathcal{Y}(E,\mathbb{K})$ under the imbedding $v \mapsto \delta_v$ given by $\delta_v(x) := \delta_{v(x)}$.

The next result is a minor variant of a classical result in the theory of Young measures which we give together with its short proof because it plays an important role in our setting.

**Lemma 3.** Suppose $v^m, v^\infty : E \subseteq \mathbb{R}^n \rightarrow \mathbb{K}$ are measurable maps, $m \in \mathbb{N}$. Let also $E$ be measurable with $|E| < \infty$. Then, there exist subsequences $(m_k)_k^\infty$, $(m_l)_l^\infty$ such that:

1. $v^m \rightarrow v^\infty$, a.e. on $E$ $\implies$ $\delta_{v^m} \overset{\ast}{\rightharpoonup} \delta_{v^\infty}$, in $\mathcal{Y}(E,\mathbb{K})$,
2. $\delta_{v^m} \overset{\ast}{\rightharpoonup} \delta_{v^\infty}$ in $\mathcal{Y}(E,\mathbb{K})$ $\implies$ $v^m \rightarrow v^\infty$, a.e. on $E$.

**Proof of Lemma 3.** (1) If $v^m \rightarrow v^\infty$ a.e. on $E$, by sequential weak* compactness there is $(v^{m_k})_k^\infty$ such that $\delta_{v^{m_k}} \overset{\ast}{\rightharpoonup} \vartheta$ in $\mathcal{Y}(E,\mathbb{K})$. If $\Phi \in L^1(E, C^0(\mathbb{K}))$, we have

$$\int_E \Phi(x,v^m(x)) \, dx \rightarrow \int_E \int_{\mathbb{K}} \Phi(x,Q) \, d\vartheta^\perp(x) \, dx$$

and also, since $\|\Phi(v^m)\| \leq \chi_E \|\Phi\|_{C^0(\mathbb{K})}$, the Dominated Convergence theorem implies $\Phi\circ v^{m_k} \rightarrow \Phi\circ v^\infty$ in $L^1(E)$. Hence, by uniqueness of limits it follows that $\vartheta = \delta_{v^\infty}$. 

(2) If $\delta_{v^m} \overset{\ast}{\rightharpoonup} \delta_{v^\infty}$ in $\mathcal{Y}(E,\mathbb{K})$, we fix $\varepsilon > 0$ and choose $\Phi(x,Q) := \min\{1 + \varepsilon, |Q - v^\infty(x)|\}$. Then,

$$0 = \int_E \Phi(x,v^\infty(x)) \, dx = \lim_{m \rightarrow \infty} \int_E \Phi(x,v^m(x)) \, dx$$

$$\geq \varepsilon \lim_{m \rightarrow \infty} \left| \left\{ x \in E : \min\{1 + \varepsilon, |v^m(x) - v^\infty(x)|\} > \varepsilon \right\} \right|$$

$$= \varepsilon \lim_{m \rightarrow \infty} \left| \left\{ x \in E : |v^m(x) - v^\infty(x)| > \varepsilon \right\} \right|.$$ 

Hence, $v^m \rightarrow v^\infty$ in measure on $E$ and as a result there is $(v^{m_l})_l^\infty$ such that $v^{m_l} \rightarrow v^\infty$ a.e. on $E$. 

The previous result shows that weak* convergence is actually relatively strong since, if the measures are given by Dirac masses at measurable maps then it is (up to a subsequence) equivalent to pointwise a.e. convergence.
2.2. Motivation of the notions. We seek to find a meaningful notion of generalised solution for fully nonlinear PDE systems which relaxes the notion of strong solutions and does not require any more a priori regularity of the solution apart from measurability. We derive the notion in the special but instructive case of 2nd order systems. Suppose \( F \) is as in (1.1) with \( p = 2 \) and suppose \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) is a \( W^{2,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \) (or \( W^{2,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \) with \( p > n \)) strong a.e. solution of the system
\[
(2.6) \quad F(\cdot, u, Du, D^2u) = 0, \quad \text{on } \Omega.
\]
Then, \( u \) is pointwise twice differentiable a.e. on \( \Omega \) and hence
\[
F\left(x, u(x), \lim_{h \to 0} D^{1,h}u(x), \lim_{h',h'' \to 0} D^{2,h''h'}u(x)\right) = 0,
\]
for a.e. \( x \in \Omega \), where \( D^{1,h}, D^{2,kh} \) stand for the standard difference quotient operators, i.e.
\[
(2.7) \quad D^{1,h} = (D_{e_1}^{1,h}, ..., D_{e_n}^{1,h}), \quad D^{2,kh} = D^{1,k}D^{1,h}
\]
and \( D_{e_i}^{1,h} \) is given by (2.4). Now note that since \( F \) is a Carathéodory map, the limits commute with the nonlinearity:
\[
(2.8) \quad \lim_{h'' \to 0} \lim_{h' \to 0} \lim_{h \to 0} F\left(x, u(x), D^{1,h}u(x), D^{2,h'h''}u(x)\right) = 0,
\]
for a.e. \( x \in \Omega \). The crucial observation is that the above statement is independent of the \( W^{2,\infty} \) regularity of \( u \) and makes sense if \( u \) is merely measurable. Moreover, we may also relax the full convergence along the variables \( h, h', h'' \) (this will turn out to be necessary) and have only sequential convergence. How can we represent this limit and turn it into a handy definition? Going back to (2.6), we observe that \( u \) is a strong solution of (2.6) if and only if it satisfies
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{n \times n^2}} \Phi(P, X) F(\cdot, u, P, X) d[\delta_{(Du, D^2u)}](P, X) = 0, \quad \text{a.e. on } \Omega,
\]
for any compactly supported “test” function \( \Phi \in C^\infty_0(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{n \times n^2}) \). This gives the idea that we can embed the difference quotient maps
\[
D^{1,h}u : \Omega \rightarrow \mathbb{R}^N, \quad D^{2,h'h''}u : \Omega \rightarrow \mathbb{R}^{n \times n^2}
\]
into the spaces of Young measures and consider instead
\[
\delta_{D^{1,h}u} : \Omega \rightarrow \mathcal{P}(\mathbb{R}^N), \quad \delta_{D^{2,h'h''}u} : \Omega \rightarrow \mathcal{P}(\mathbb{R}^{n \times n^2})
\]
over the Alexandroff compactifications \( \bar{\mathbb{R}}^{n \times n^2} \). The reason we need to attach the point at \( \infty \) and compactify the space is to get “tightness” and have weak* compactness for the spaces \( \mathcal{P}(\Omega, \mathbb{R}^N) \), \( \mathcal{P}(\Omega, \mathbb{R}^{n \times n^2}) \). This compensates the possible loss of mass since the difference quotients of measurable maps may not converge in any sense, not even weakly* in the distributions \( \mathcal{D}' \). However, there exist limits \( Du \) and \( D^2u \) in the Young measures such that, along subsequences we have
\[
(2.9) \quad \delta_{(D^{1,h}u, D^{2,h'h''}u)} \rightharpoonup Du \times D^2u, \quad \text{in } \mathcal{D}(\Omega, \mathbb{R}^N \times \mathbb{R}^{n \times n^2}),
\]
as \( h, h', h'' \to 0 \). It will be also more fruitful to take the limits above separately (regardless of order), because the measure \( Du \times D^2u \) is then a (fibre) product.
measure. Then, by employing the imbedding into the Young measures and
the above observation, (2.8) is equivalent to
\[
\int_{\mathbb{R}^{n} \times \mathbb{R}^{2n}} \Phi(P, X) F(\cdot, u, P, X) \, d[\delta_{D^1 h u} \times \delta_{D^2 h' h'' u}](P, X) \to 0,
\]
as \( h, h', h'' \to 0 \) along sequences a.e. on \( \Omega \), for any \( \Phi \in C_0^0(\mathbb{R}^{n} \times \mathbb{R}^{2n}) \). By using
the duality pairing between \( L^1(\Omega, C^0) \) and \( L^\infty(\Omega, \mathcal{M}) \) and a simple argument
involving Egoroff’s theorem (which will be analysed later in the proof of Lemma
20), we obtain the integral formula
\[
\int_{\mathbb{R}^{n} \times \mathbb{R}^{2n}} \Phi(P, X) F(\cdot, u, P, X) \, d[D u \times D^2 u](P, X) = 0, \quad \text{a.e. on } \Omega,
\]
for any \( \Phi \in C_0^0(\mathbb{R}^{n} \times \mathbb{R}^{2n}) \). We note that this statement is independent of the
regularity of the solution of (2.6). If \( u \) is weakly once differentiable a.e. on \( \Omega \) and
its gradient is measurable, by using Lemma 3 we have \( D u = \delta_{D u} \) a.e. on \( \Omega \) and the
above simplifies to
\[
\int_{\mathbb{R}^{n} \times \mathbb{R}^{2n}} \Phi(X) F(\cdot, u, D u, X) \, d[D^2 u](X) = 0, \quad \text{a.e. on } \Omega,
\]
for any \( \Phi \in C_0^0(\mathbb{R}^{2n}) \). Moreover, in this case \( D^2 u \) arises as a sequential limit of
\( \delta_{D^1 h D u} \overset{\ast}{\rightharpoonup} D^2 u \) in \( \mathcal{Y} (\Omega, \mathbb{R}^{2n}) \), as \( h \to 0 \).

If further \( D u \) is also differentiable a.e. on \( \Omega \) and \( D^2 u \) is measurable, we have \( D^2 u = \delta_{D^2 u} \) a.e. on \( \Omega \) and by applying Lemma 3 again we get back to strong solutions of
(2.6).

**Remark 4** (About adaptivity). As we have already explained in the introduction,
we have to relax the above ideas even further in order to obtain a notion which will
allow to prove existence. The problem is that there is no reason why we should
choose the standard difference quotients (2.7). Instead, we need to take difference
quotients with respect to appropriate infinitesimal frames which should be allowed
to vary measurably with the basepoint \( x \). We will call such frames adaptive and
we have borrowed this term from Numerical Analysis in order to describe this
phenomenon.

In its plainest form, adaptivity in our context means that instead of the usual
difference quotient \( D^i h u \) of a map \( u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N \) given by (2.7) and (2.4), we
need to consider instead
\[
D^i_{e_m(x)} u(x) = \frac{u(x + h_m(x) e_i(x)) - u(x)}{h_m(x)}, \quad m \in \mathbb{N},
\]
where for each \( i = 1, \ldots, n \), we have that
\[
\begin{align*}
|e_m(x) - e^i| & \to 0 \\
|h_m(x)| & \to 0
\end{align*}
\]
essentially uniformly in \( x \in \Omega \), as \( m \to \infty \).

Hence, we need to allow the direction fields \( e_m(x) \), \( e_n(x) \) to vary but asymptot-
ically approach the standard frame \( e^1, \ldots, e^n \) as \( m \to \infty \) and we need to allow the
step sizes \( h_m(x), \ldots, h_m(x) \) to vary but also to depend on the direction.

The actual definition that follows is a bit more complicated than this, since
we need to do the above separately for the projections of the map \( u \) on different
mutually orthogonal subspaces of \( \mathbb{R}^N \). Finding the correct frames which follow the geometry of the level sets of \( F \) is an essential part of the “weak” formulation.

2.3. Main definitions and analytic properties. Fix \( n, N, p \in \mathbb{N} \) and suppose \( \{ \eta^1, \ldots, \eta^N \} \) is a given orthonormal frame of \( \mathbb{R}^N \) and for each \( \alpha = 1, \ldots, N \) we have an orthonormal frame \( \{ \xi^{(\alpha)}_1, \ldots, \xi^{(\alpha)}_n \} \) of \( \mathbb{R}^n \). Then, we have the following induced orthonormal frame of \( \mathbb{R}^_{nNp} \):

\[
E^{\alpha_1 \ldots \alpha_p} := \eta^\alpha \otimes \xi^{(\alpha)_i_1} \vee \ldots \vee \xi^{(\alpha)_i_p}
\]

**Definition 5** (adaptive infinitesimal frames). Given the above orthonormal frames \( \{ \eta^\alpha \} \), \( \{ \xi^{(\alpha)_1}, \ldots, \xi^{(\alpha)_n} \} \) and \( \{ E^{\alpha_1 \ldots \alpha_p} \} \) of \( \mathbb{R}^N \), \( \mathbb{R}^n \) and \( \mathbb{R}^{nNp} \) respectively, an adaptive infinitesimal frame \( s \) (with respect to the above frames) is a sequence of maps in \( L^\infty(\Omega, \mathbb{R}^{nN^2}) \)

\[
s \equiv (s_m)_{m=1}^\infty, \quad s_m \equiv \{ s_m^{(\alpha)i} | \alpha, i \},
\]

such that for each pair of indices \( \alpha, i \), the map \( s_m^{(\alpha)i} \in L^\infty(\Omega, \mathbb{R}^n) \) can be written as

\[
s_m^{(\alpha)i} = h_m^{(\alpha)i} h_m^{(\alpha)i}
\]

and the sequences \( (h_m^{(\alpha)i})_1^\infty, (e_m^{(\alpha)i})_1^\infty \) satisfy

\[
h_m^{(\alpha)i} \in L^\infty(\Omega), \quad |h_m^{(\alpha)i}(x)| > 0, \quad \text{a.e. } x \in \Omega, \quad \|h_m^{(\alpha)i}\|_{L^\infty(\Omega)} \to 0,
\]

and

\[
e_m^{(\alpha)i} \in L^\infty(\Omega, \mathbb{R}^n), \quad |e_m^{(\alpha)i}(x)| = 1, \quad \text{a.e. } x \in \Omega, \quad \|e_m^{(\alpha)i} - \xi^{(\alpha)i}\|_{L^\infty(\Omega)} \to 0,
\]

as \( m \to \infty \).

**Remark 6.** In view of Definition 5, an adaptive infinitesimal frame consists of measurable sequences of “step sizes” \( h_m^{(\alpha)i} \) and measurably varying unit directions \( e_m^{(\alpha)i} \) which asymptotically approach the respective elements of the constant frames \( \{ \xi^{(\alpha)_1}, \ldots, \xi^{(\alpha)_n} \} \):

\[
h_m^{(\alpha)i}(x) \to 0, \quad |e_m^{(\alpha)i}(x) - \xi^{(\alpha)i}| \to 0
\]

and both convergences are essentially uniform in \( x \) as \( m \to \infty \).

**Remark 7** (Special infinitesimal frames). Particular cases of the above frames which will be used later in Sections 3, 4 arise when

a) The \( N \)-many frames \( \{ \xi^{(\alpha)_1}, \ldots, \xi^{(\alpha)_n} \} \) on \( \mathbb{R}^n \) are all the same and independent of \( \alpha \), whilst \( h_m^{(\alpha)i}(x), e_m^{(\alpha)i}(x) \) are also independent of \( \alpha \):

\[
\{ \xi^{(\alpha)_1}, \ldots, \xi^{(\alpha)_n} \} = \{ \xi^1, \ldots, \xi^n \}, \quad h_m^{(\alpha)i}(x) \equiv h_m^i(x), \quad e_m^{(\alpha)i}(x) \equiv e_m^i(x),
\]

for all \( \alpha = 1, \ldots, N \). Then, \( s \) actually depends only on the chosen frame of \( \mathbb{R}^n \) and not on the frame of \( \mathbb{R}^N \) (or on the frames of the spaces \( \mathbb{R}^n_{N^p} \)) and we simplify things by writing

\[
s = \{ (s_m^i)_{m=1}^\infty, s_m^i \in L^\infty(\Omega, \mathbb{R}^n) \}
\]

This case will appear for the \( \infty \)-Laplacian. In addition, it will turn out that the frame \( \{ \xi^1, \ldots, \xi^n \} \) can be chosen to be the standard basis of \( \mathbb{R}^n \).
b) The direction vector fields $x \mapsto e_{m}^{(\alpha)i}(x)$ can be chosen independent of $x$ and identically equal to the frame elements, whilst the step sizes can be chosen independent of $x$, $\alpha$ and $i$:

$$e_{m}^{(\alpha)i}(x) \equiv \xi^{(\alpha)i}, \quad h_{m}^{(\alpha)i}(x) \equiv h_{m}, \quad m \in \mathbb{N}.$$ 

This case will appear for the fully nonlinear hessian system, under a degenerate ellipticity assumption.

The next definition introduces difference quotients taken with respect to the above frames. The only difficulty is the complexity in the notation, so for pedagogical reasons we first give the 1st and 2nd order cases before the general $p$-th order case.

**Definition 8** (adaptive difference quotients). Let $u : \Omega \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}^{N}$ be a measurable map (which we have identified with a Borel measurable representative a.e. equal to it) and suppose $\Omega$ is open.

In the setting of Definition 5, given frames $\{\eta^{\alpha}\}, \{\xi^{(\alpha)i}\}$, induced frames $\{E^{\alpha_{1}...\alpha_{p}}\}$ and an adaptive infinitesimal frame $s$, we define the **adaptive difference quotients of $u$ of 1st, 2nd and $p$-th order** as the measurable sequences of maps

$$D^{1,p} u : \Omega \rightarrow \mathbb{R}^{Nn}, \quad m \in \mathbb{N},$$

$$D^{2,p} u : \Omega \rightarrow \mathbb{R}^{Nn^{2}}, \quad m', m'' \in \mathbb{N},$$

$$D^{p,s_{p}...p_{1}} u : \Omega \rightarrow \mathbb{R}^{Nn^{p}}, \quad m_{1}, ..., m_{p} \in \mathbb{N},$$

given for $p \in \{1,2\}$ by

$$(D^{1,s} u)(x) := \sum_{\alpha,i} \left[ \chi_{\Omega^{m}} D^{1,h_{m}^{(\alpha)i}}(x) (\eta^{\alpha} \cdot u)(x) \right] E^{\alpha i},$$

$$(D^{2,s} u)(x) := \sum_{\alpha,i,j} \left[ \chi_{\Omega^{m}} D^{2,h_{m}^{(\alpha)i}}(x) h_{m}^{(\alpha)i}(x) e_{m}^{(\alpha)i}(x) (\eta^{\alpha} \cdot u)(x) \right] E^{\alpha ij},$$

and for general $p \in \mathbb{N}$ by

$$(D^{p,s_{p}...p_{1}} u)(x) := \sum_{\alpha_{i_{1}},...,\alpha_{i_{p}}} \left[ \chi_{\Omega^{m_{1}}} D^{p,h_{m_{p}}^{(\alpha_{p})}}(x)...h_{m_{1}}^{(\alpha_{1})i_{1}}(x) (\eta^{\alpha} \cdot u)(x) \right] E^{\alpha_{1}...\alpha_{p}}.$$

In the above, for each fixed $x$ the notation

$$D^{p,h_{m_{p}}^{(\alpha_{p})}}(x)...h_{m_{1}}^{(\alpha_{1})i_{1}}(x)$$

denotes the standard difference quotient operator given by (2.4), (2.5). Moreover we have used the notation

$$\Omega^{m} := \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \max_{\alpha,i} \left\| h_{m}^{(\alpha)i} \right\|_{L^{\infty}(\Omega)} \right\}$$

$$\Omega^{m'} := \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \max_{\alpha,i} \left\| h_{l}^{(\alpha)i} \right\|_{L^{\infty}(\Omega)}, \ l = \max \{m', m''\} \right\}$$

$$\Omega^{m_{p}...m_{1}} := \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \max_{\alpha,i} \left\| h_{l}^{(\alpha)i} \right\|_{L^{\infty}(\Omega)}, \ l = \max \{m_{1}, ..., m_{p}\} \right\}$$

and we have multiplied by the characteristic functions of these sets.
Definition 9 (diffuse derivatives). In the setting of Definition 8, suppose we have again given frames \( \{ \eta^\alpha \} \), \( \{ \xi^{(\alpha)j} \} \), induced frames \( \{ E^{\alpha_1 \ldots \alpha_p} \} \), an adaptive infinitesimal frame \( \mathfrak{s} \) and a measurable map \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) with adaptive difference quotients maps \( D^{r_1, s_{mp \ldots m_1}} u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_{s}^{Nn_p} \), \( p \in \mathbb{N} \).

Then, we define the **diffuse gradients**, **diffuse hessians** and the **diffuse derivatives of \( p \)-th order** of \( u \) as the subsequential limits \( D\mathfrak{s}, D^2\mathfrak{s}, D^p\mathfrak{s} \) of the adaptive difference quotients in the respective spaces of Young measures over the 1-point compactifications:

\[
\begin{align*}
\delta_{D^1 \cdot m \mathfrak{s}} & \rightarrow D\mathfrak{s}, \quad \text{in } \mathscr{B} (\Omega, \mathbb{R}^{Nn}), \quad \text{as } m \rightarrow \infty, \\
\delta_{D^2 \cdot m \mathfrak{s}'} & \rightarrow D^2\mathfrak{s}, \quad \text{in } \mathscr{B} (\Omega, \mathbb{R}_{s}^{Nn^2}), \quad \text{as } m', m'' \rightarrow \infty \text{ separately,} \\
\delta_{D^p \cdot m \mathfrak{s} \ldots m_1 \mathfrak{s}} & \rightarrow D^p\mathfrak{s}, \quad \text{in } \mathscr{B} (\Omega, \mathbb{R}_{s}^{Nn_p}), \quad \text{as } m_1, \ldots, m_p \rightarrow \infty \text{ separately.}
\end{align*}
\]

We will call the (fibre) product Young measures of the form

\[
D^{[p]} u := D\mathfrak{s} \times \cdots \times D^p\mathfrak{s} \in \mathscr{B} (\Omega, \mathbb{R}_{s}^{Nn} \times \cdots \times \mathbb{R}_{s}^{Nn_p})
\]

the **diffuse jets of \( p \)-th order** of the measurable map \( u \).

The weak* compactness of the spaces of Young measures readily implies the next

Lemma 10 (Existence of diffuse derivatives). Every measurable mapping \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) possesses diffuse derivatives of any order \( p \in \mathbb{N} \).

Moreover, \( u \) has at least one sequence of diffuse derivatives \( D\mathfrak{s}, D^2\mathfrak{s}, \ldots \) for every choice of frames on \( \mathbb{R}^N, \mathbb{R}^n \) as above (and induced frames on \( \mathbb{R}_{s}^{Nn_p}, \mathbb{R}_{s}^{Nn} \), \( p \in \mathbb{N} \) and every adaptive infinitesimal frame \( \mathfrak{s} \)).

Remark 11 (Nonexistence of derivatives in \( \mathscr{D}' \)). We stress again that since \( u \) may not be in \( L^{1}_{\text{loc}} (\Omega, \mathbb{R}^N) \), it may not possess distributional derivatives in \( \mathscr{D}' \).

Obviously, in general diffuse derivatives **may not be unique** for non-smooth mappings. The next result asserts the rather obvious fact that diffuse derivatives are compatible with classical derivatives.

Lemma 12 (Compatibility of classical/strong with diffuse derivatives). Let \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) be a measurable map and \( E \subseteq \Omega \) a measurable set with \( |E| > 0 \). Suppose that we are given frames on \( \mathbb{R}^n, \mathbb{R}^N \) and \( \mathbb{R}_{s}^{Nn_p} (p \in \mathbb{N}) \) and an adaptive infinitesimal frame \( \mathfrak{s} \) as in Definition 9.

If \( u \) is pointwise differentiable a.e. on \( E \) and its gradient \( D\mathfrak{s} \) is measurable (e.g. when \( u \in W^{1,r}_{\text{loc}} (\Omega, \mathbb{R}^N) \) for \( r > n \)), the restriction of the diffuse gradient \( D\mathfrak{s} \) on \( E \) is unique and we have

\[
\delta_{D\mathfrak{s}u} = D\mathfrak{s}u, \quad \text{a.e. on } E.
\]

More generally, if \( u \) is pointwise \( (p-1) \)-times differentiable a.e. on an open set containing \( E \), \( D^{p-1}\mathfrak{s} \) is differentiable a.e. on \( E \) and the \( p \)-th order derivative \( D^p\mathfrak{s} \) is measurable, the restriction of \( D^1 \times \cdots \times D^p\mathfrak{s} \) on \( E \) is unique and we have

\[
\delta_{D^q\mathfrak{s}u} = D^q\mathfrak{s}u, \quad \text{a.e. on } E, \quad q = 1, \ldots, p.
\]

**Proof of Lemma 12.** If \( u \) is pointwise differentiable a.e. on \( E \), for a.e. \( x \in E \) and any \( \alpha \in \{1, \ldots, N\} \), in view of Definition 5 we have

\[
\left| \eta^\alpha \cdot u(x + y) - \eta^\alpha \cdot u(x) - D(\eta^\alpha \cdot u)(x) \cdot y \right| \leq o(|y|),
\]

for all \( \eta^\alpha \), \( \alpha \in \{1, \ldots, N\} \).
as \( y \to 0 \). We fix such an \( x \) and an \( i \in \{1, \ldots, n\} \) and (in view of Definition 8) we set

\[
y := h_m^{(\alpha)i}(x) \varepsilon_m^{(\alpha)i}(x).
\]

Then, we have

\[
\left| D_{\varepsilon_m^{(\alpha)i}(x)}^1 h_m^{(\alpha)i}(\eta^\alpha \cdot u(x)) - D_{\xi^{(\alpha)i}}(\eta^\alpha \cdot u(x)) \right| \leq o(1) + \left| Du(x) \right| \left| \varepsilon_m^{(\alpha)i}(x) - \xi^{(\alpha)i} \right|,
\]

as \( m \to \infty \). By the orthonormal expansion of \( Du \) in (2.3), the above estimate and the definition of \( \mathcal{E} \) we have \( (D^{1,s_m} u)(x) \to Du(x) \) for a.e. \( x \in \Omega \) as \( m \to \infty \). By applying Lemma 3 to \( \delta_{D^{1,s_m} u} \), the conclusion for the 1st order case follows. The general \( p \)-th order case ensues by arguing analogously.

The next notion of solution will be central in this work. For pedagogical reasons we give first the definition in the 2nd order case before the general case.

**Definition 13** (\( \mathcal{D} \)-solutions of 2nd order systems). Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and

\[
F : \Omega \times \left( \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}_s^{Nn^2} \right) \to \mathbb{R}^M
\]

a Carathéodory map. Consider the PDE system

\[
(2.11) \quad F(\cdot, u, Du, D^2 u) = 0, \quad \text{on } \Omega.
\]

We say that the measurable map \( u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N \) is a \( \mathcal{D} \)-solution of (2.11) when there exist orthonormal frames \( \{\eta^\alpha\} \) of \( \mathbb{R}^N \), \( \{\xi^{(\alpha)i}\} \) of \( \mathbb{R}^n \), \( \alpha = 1, \ldots, N \), induced frames on \( \mathbb{R}^{Nn} \times \mathbb{R}_s^{Nn^2} \) and also an adaptive infinitesimal frame \( \mathcal{E} \) (Definition 5) such that, for a diffuse jet of 2nd order

\[
D^{[2]} u = Du \times D^2 u \in \mathcal{E}
\]

arising from this frame \( \mathcal{E} \) (Definitions 8, 9) we have

\[
\int_{\mathbb{R}^{Nn} \times \mathbb{R}_s^{Nn^2}} \Phi(P, X) F(x, u(x), P, X) d\left[ (Du \times D^2 u)(x) \right](P, X) = 0,
\]

for a.e. \( x \in \Omega \) and any \( \Phi \in C^0_c(\mathbb{R}^{Nn} \times \mathbb{R}_s^{Nn^2}) \).

**Remark 14** (Once differentiable \( \mathcal{D} \)-solutions of 2nd order systems). If \( u \) is pointwise once differentiable a.e. on \( \Omega \) and its gradient \( Du \) is measurable, by using Lemma 3 we have \( Du = \delta_{D_u} \) a.e. on \( \Omega \) and Definition 13 simplifies to

\[
\int_{\mathbb{R}_s^{Nn^2}} \Phi(X) F(\cdot, u, Du, X) d|D^2 u|(X) = 0, \quad \text{a.e. on } \Omega,
\]

for any \( \Phi \in C^0_c(\mathbb{R}_s^{Nn^2}) \). Moreover, the diffuse hessian \( D^2 u \) arises as a sequential limit of adaptive 1st order difference quotients of the gradient \( Du \) with respect to \( \mathcal{E} \):

\[
\delta_{D^{1,s_m}} Du \rightharpoonup D^2 u, \quad \text{in } \mathcal{E}(\Omega, \mathbb{R}_s^{Nn^2}), \quad \text{as } m \to \infty.
\]

Now we give the definition in the general \( p \)-th order case.

**Definition 15** (\( \mathcal{D} \)-solutions for \( p \)-th order systems). Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and

\[
F : \Omega \times \left( \mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn^p} \right) \to \mathbb{R}^M
\]

a Carathéodory map. Consider the PDE system

\[
(2.12) \quad F(\cdot, u, Du, ..., D^p u) = 0, \quad \text{on } \Omega.
\]
We say that the measurable map \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) is a \( \mathcal{D} \)-solution of (2.12) when there exist orthonormal frames \( \{ \eta^\alpha \} \) of \( \mathbb{R}^N \), \( \{ \xi^{(\alpha)}_i \} \) of \( \mathbb{R}^n \), \( \alpha = 1, \ldots, N \), induced frames on \( \mathbb{R}^{Nn} \), \( \ldots, \mathbb{R}_s^{Nn} \) and also an adaptive infinitesimal frame \( s \) (Definition 5) such that, for a diffuse jet of \( p \)-th order

\[
\mathcal{D}^{[p]} u = Du \times \cdots \times D^p u \in \mathcal{Y} \left( \Omega, \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn} \right)
\]

arising from this frame \( s \) (Definitions 8, 9) we have

\[
\int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}^p} \Phi(\mathbf{X}) F(x, u(x), \mathbf{X}) \, d[D^p u(x)](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega,
\]

for any \( \Phi \in C_c^0(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}) \), where

\[
\mathbf{X} \equiv (X_1, \ldots, X_p) \in \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}.
\]

The same comments as in Remark 14 apply here as well when \( u \) is differentiable up to some order less than \( p \). The following proposition asserts that \( \mathcal{D} \)- and strong (or classical) solutions are compatible when the \( \mathcal{D} \)-solution is regular.

**Proposition 16** (Compatibility of strong/classical solutions with \( \mathcal{D} \)-solutions). Let \( \Omega \subseteq \mathbb{R}^n \) be open and \( F \) a Carathéodory map as in (1.1). Consider the PDE system

\[
F(\cdot, u, Du, \ldots, D^p u) = 0, \quad \text{on } \Omega.
\]

Then, we have:

A) Suppose \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) is a \( \mathcal{D} \)-solution of the system and \( E \subseteq \Omega \) is a measurable set with \( |E| > 0 \). If \( u \) is pointwise \((p - 1)\)-times differentiable a.e. on an open set containing \( E \) and \( D^{p-1} u \) is differentiable a.e. on \( E \) with \( D^p u \) measurable, then \( u \) is a strong solution on \( E \):

\[
F(x, u(x), Du(x), \ldots, D^p u(x)) = 0, \quad \text{for a.e. } x \in E \subseteq \Omega.
\]

B) Conversely, suppose \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) is a strong solution of the system on \( \Omega \), in the sense that all pointwise derivatives up to \( p \)-th order exist a.e. on \( \Omega \) and are measurable maps. Then, \( u \) is a \( \mathcal{D} \)-solution of the system on \( \Omega \).

**Proof of Proposition 16.** A) Since \( u \) is a \( \mathcal{D} \)-solution of the system on \( \Omega \), for any \( \Phi \in C_c^0(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}) \) we have

\[
\int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}^p} \Phi(\mathbf{X}) F(\cdot, u, \mathbf{X}) \, d[D^p u](\mathbf{X}) = 0,
\]

a.e. on \( \Omega \). By Lemma 12, we have that \( D^q u = \delta_{D^q u} \), a.e. on \( E \) for all \( q \in \{1, \ldots, p\} \). Hence

\[
\int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}^p} \Phi(\mathbf{X}) F(\cdot, u, \mathbf{X}) \, d[\delta_{(Du, \ldots, D^p u)}](\mathbf{X}) = 0,
\]

a.e. on \( E \), which gives that for any \( \Phi \in C_c^0(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}_s^{Nn}) \), we have

\[
\Phi(Du, \ldots, D^p u) F(\cdot, u, Du, \ldots, D^p u) = 0,
\]

a.e. on \( E \). Hence, \( u \) solves the system a.e. on \( E \).

B) If \( u \) is a strong solution of the system, then by Lemma 12 we have

\[
\mathcal{D}^{[p]} u = \delta_{Du} \times \cdots \times \delta_{D^p u}
\]
(for any infinitesimal frame $s$) and by reversing the steps of part A) above we get that $u$ is a $D$-solution of the PDE system. □

**Remark 17** (Absence of concentration measures). The next estimate shows that "... $= 0$ a.e. on $\Omega$" in the integral formulas of Definitions 13, 15 is equivalent to "... $= 0$ in $L^\infty(\Omega)$". Namely, for any fixed $\Phi$ the left hand side is always a measurable function and no lower-dimensional concentration measures can arise which are mutually singular with respect to the Lebesgue measure. For, we have

$$
\left| \int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nnp}} \Phi(X) F(\cdot, u, X) \, d[D^p]u(X) \right|
\leq \int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nnp}} |\Phi(X) F(\cdot, u, X)| \, d[D^p]u(X)
\leq \|\Phi\|_{C^0(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nnp})} \max_{X \in \text{supp}(\Phi)} |F(\cdot, u, X)|.
$$

The next result gives equivalent formulations of the definition of $D$-solutions. For clarity, we state it only for 2nd order systems with hessian dependence. The general case can be stated and proved in a similar fashion.

**Proposition 18** (Equivalent formulations for $D$-solutions). Suppose that $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a measurable map with $\Omega$ open. Let also $F : \Omega \times \mathbb{R}^{Nn^2} \rightarrow \mathcal{M}$ be a Carathéodory map. Further, let $D^2u$ be a diffuse hessian of $u$ arising from an adaptive infinitesimal frame $s$ with respect to given frames of $\mathbb{R}^n$, $\mathbb{R}^N$, $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}$. We define the **reduced diffuse hessian** on $\mathbb{R}^{Nn^2}$ which is the (fibre) restriction measure of $D^2u$ outside $\{\infty\}$:

$$
D^*_2u(x) := D^2u(x) \cap \mathbb{R}^{Nn^2}, \quad D^*_2u \in L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^{Nn^2})).
$$

Then the following are equivalent:

1. The map $u$ is a $D$-solution of the PDE system
   $$
   F(\cdot, D^2u) = 0, \quad \text{on } \Omega,
   $$
   with respect to the given diffuse hessian $D^2u$, that is
   $$
   \int_{\mathbb{R}^{Nn^2}} \Phi(X) F(\cdot, X) \, d[D^2u](X) = 0, \quad \text{a.e. on } \Omega,
   $$
   for any $\Phi \in C^0_c(\mathbb{R}^{Nn^2})$.

2. The reduced diffuse hessian $D^*_2u$ satisfies the **differential inclusion**:
   For a.e. $x \in \Omega$, $\text{supp}(D^*_2u(x)) \subseteq \{F(x, \cdot) = 0\}$.

3. The weakly* measurable measure-valued map $F^u : \Omega \rightarrow \mathcal{M}(\mathbb{R}^{Nn^2})$ defined by $x \mapsto F^*_x$, such that
   $$
   F^u_x(\mathcal{B}) := \int_{\mathbb{R}^{Nn^2}} F(x, X) \, d[D^2u](X), \quad \mathcal{B} \subseteq \mathbb{R}^{Nn^2}, \text{ Borel},
   $$
   vanishes for a.e. $x \in \Omega$.

If further $F$ does not depend on $x$, then (1)-(3) above are equivalent to the following:

4. The map $u$ satisfies
   $$
   \int_{\mathbb{R}^{Nn^2}} \Psi(X) F(X) \, d[D^2u](X) = 0, \quad \text{a.e. on } \Omega,
   $$
   for any $\Psi \in C^0_c(\mathbb{R}^{Nn^2})$. 

for any \( \Psi \in \mathcal{A} \), where

\[
\mathcal{A} := \left\{ \Psi \in C^0(\mathbb{R}^{Nn}_s) \mid \limsup_{|X| \to \infty} |\Psi(X)| (1 + |F(X)|) = 0 \right\}.
\]

**Proof of Proposition 18.** (1)\(\Leftrightarrow\)(3): Since \( \Phi \in C^0_c(\mathbb{R}^{Nn}_s) \), (1) is equivalent to

\[
\int_{\mathbb{R}^{Nn}_s} \Phi(X) F(\cdot, X) \, d[D^2 u](X) = 0, \quad \text{a.e. on } \Omega.
\]

Now it suffices to note that this integral formula is a restatement of the fact that measure-valued map \( F^u \) (which is absolutely continuous with respect to \( D^2 u \)) actually vanishes because for a.e. \( x \in \Omega \), we have

\[
\int_{\mathbb{R}^{Nn}_s} \Phi(X) \, d|F^u_x|(X) = \int_{\mathbb{R}^{Nn}_s} \Phi(X) F(x, X) \, d[D^2 u(x)](X).
\]

(2)\(\Rightarrow\)(1): Suppose (2) holds and fix \( x \in \Omega \) for which the inclusion is true. Then, if \( \Phi \in C^0_c(\mathbb{R}^{Nn}_s) \), we have that the continuous map \( x \mapsto \Phi(X)F(x, X) \) vanishes on the support of the measure \( D^2 u(x) \). Hence, the right hand side of the next identity vanishes

\[
\int_{\mathbb{R}^{Nn}_s} \Phi(X) F(x, X) \, d[D^2 u(x)](X) = \int_{\text{supp}(D^2 u(x))} \Phi(X) F(x, X) \, d[D^2 u(x)](X)
\]

and as a result (1) ensues.

(1)\(\Rightarrow\)(2): Suppose that (1) holds but for the sake of contradiction assume (2) fails. Fix a point \( x \in \Omega \) for which (1) is true but \( \text{supp}(D^2 u(x)) \nsubseteq \{ F(x, \cdot) = 0 \} \). Then, there is \( X_0 \in \mathbb{R}^{Nn}_s \setminus \{ F(x, \cdot) = 0 \} \) such that for all \( R > 0 \), we have

\[
[D^2 u(x)](B_R(X_0)) > 0
\]

where \( B_R(X_0) \) is the open ball of radius \( R \) centred at \( X_0 \). Since \( F(x, \cdot) \) is continuous and \( F(x, X_0) \neq 0 \), there exist \( c_0, R_0 > 0 \) and an index \( \mu \in \{ 1, \ldots, M \} \) such that

\[
|F_\mu(x, \cdot)| \geq c_0 > 0, \quad \text{on } B_{R_0}(X_0).
\]

We now choose \( \Phi \in C^0_c(\mathbb{R}^{Nn}_s) \) such that

\[
\chi_{B_{R_0/2}(X_0)} \leq \Phi \leq \chi_{B_{R_0}(X_0)}.
\]

As a result, by using that \( u \) is a \( D \)-solution, we have

\[
0 = \int_{\mathbb{R}^{Nn}_s} \Phi(X) F_\mu(x, X) \, d[D^2 u](X)
\]

\[
= \int_{\mathbb{R}^{Nn}_s} \Phi(X) |F_\mu(x, X)| \, d[D^2 u](X)
\]

\[
\geq c_0 \int_{B_{R_0}(X_0)} \Phi(X) \, d[D^2 u](X)
\]

\[
\geq c_0 \left[ D^2 u(x) \right](B_{R_0/2}(X_0)) > 0.
\]

The above contradiction establishes (2).

(1)\(\Leftrightarrow\)(4): Since \( C^0_c(\mathbb{R}^{Nn}_s) \subseteq \mathcal{A} \), (4) readily implies (1). Conversely, fix \( \Psi \in \mathcal{A} \)
and $\varepsilon > 0$ and suppose $u$ is a $D$-solution. Then, for any $\Phi \in C_c^0(\mathbb{R}^{Nn^2})$ we have
\[
\left| \int_{\mathbb{R}^{Nn^2}} \Psi(X) F(X) \, d[D^2u(x)](X) \right|
\leq \left[ D^2u(x) \right](\mathbb{R}^{Nn^2}) \sup_{X \in \mathbb{R}^{Nn^2}} \left\{ |\Psi(X) - \Phi(X)| (1 + |F(X)|) \right\}
\leq \sup_{X \in \mathbb{R}^{Nn^2}} \left\{ |\Psi(X) - \Phi(X)| (1 + |F(X)|) \right\},
\]
for a.e. $x \in \Omega$. Since by assumption $\Psi(1 + |F|) \in C_c^0(\mathbb{R}^{Nn^2})$, we can find $\phi \in C_c(\mathbb{R}^{Nn^2})$ such that
\[
\|\Psi(1 + |F|) - \phi\|_{C^0(\mathbb{R}^{Nn^2})} < \varepsilon.
\]
Hence, by choosing $\Phi := \frac{\phi}{1 + |F|} \in C_c^0(\mathbb{R}^{Nn^2})$, we get that
\[
\sup_{X \in \mathbb{R}^{Nn^2}} \left\{ |\Psi(X) - \Phi(X)| (1 + |F(X)|) \right\} < \varepsilon.
\]
Since $\varepsilon$ is arbitrary, (4) ensues and so does the proposition. \qed

The following result establishes that $D$-solutions are very well behaved under weak* convergence of diffuse derivatives in the spaces of Young measures.

**Theorem 19** (Convergence of $D$-solutions). Let $\Omega \subseteq \mathbb{R}^n$ be open with $|\Omega| < \infty$ and let $(u^m)_{m}^{\infty}$ be $D$-solutions of the $p$-th order systems
\[
F^m(\cdot, u^m, D^{[p]}u^m) = 0, \quad \text{on } \Omega,
\]
where $u^m : \Omega \rightarrow \mathbb{R}^N$ are measurable, $F^m$ are Carathéodory maps (with the same dimensions as in (1.1)) and $D^{[p]}u = (Du, D^2u, \ldots, D^pu)$. Suppose that:
\[
F^m(x, \cdot) \rightarrow F^\infty(x, \cdot) \text{ uniformly on compact subsets, for a.e. } x \in \Omega,
\]
\[
u^m \rightarrow \nu^\infty, \quad \text{a.e. on } \Omega,
\]
\[
D^{[p]}\nu^m \rightharpoonup D^{[p]}\nu^\infty, \quad \text{in } \mathcal{Y}_c(\Omega, \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}),
\]
where $D^{[p]}\nu^m$ is a diffuse jet of $p$-th order with respect to which $u^m$ solves the system. Then, $\nu^\infty$ is a $D$-solution of
\[
F^\infty(\cdot, \nu^\infty, D^{[p]}\nu^\infty) = 0, \quad \text{on } \Omega.
\]

The proof is a direct consequence of the following lemma:

**Lemma 20** (Convergence lemma). Let $\Omega \subseteq \mathbb{R}^n$ be open with $|\Omega| < \infty$ and let $u^\infty, (u^m)_1^{\infty}$ be measurable maps with $u^\infty, u^m : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$. Suppose also that we have Carathéodory maps
\[
F^\infty, F^m : \Omega \times (\mathbb{R}^N \times \mathbb{R}^d) \rightarrow \mathbb{R}^M, \quad m \in \mathbb{N},
\]
and Young measures $\vartheta^\infty, (\vartheta^m)_1^{\infty} \in \mathcal{Y}_c(\Omega, \mathbb{R}^d)$, such that
\[
F^m(x, \cdot) \rightarrow F^\infty(x, \cdot) \text{ in } C^0(\mathbb{R}^N \times \mathbb{R}^d), \quad \text{for a.e. } x \in \Omega,
\]
\[
u^m \rightarrow \nu^\infty, \quad \text{a.e. on } \Omega,
\]
\[
\vartheta^m \rightharpoonup \vartheta^\infty, \quad \text{in } \mathcal{Y}_c(\Omega, \mathbb{R}^d).
\]
Then, if for a fixed $\Phi \in C^0_c(\mathbb{R}^d)$ we have
\[
\int_{\mathbb{R}^d} \Phi(X) F^m(\cdot, u^m, X) \, d\vartheta^m(X) = 0, \quad \text{a.e. on } \Omega,
\]
for all $m \in \mathbb{N}$, it follows that
\[
\int_{\mathbb{R}^d} \Phi(X) F^\infty(\cdot, u^\infty, X) \, d\vartheta^\infty(X) = 0, \quad \text{a.e. on } \Omega.
\]

**Proof of Lemma 20.** It suffices to show that for any given fixed $\Phi \in C^0_c(\mathbb{R}^d)$, we have that
\[
\phi^m := \sup_{X \in \mathbb{R}^d} \left| \Phi(X) \left[ F^m(\cdot, u^m, X) - F^\infty(\cdot, u^\infty, X) \right] \right| \to 0, \quad \text{a.e. on } \Omega.
\]
Indeed, if this is the case, select as $\Phi$ the function of the assumption of the lemma. Since $|\Omega| < \infty$, by Egoroff’s theorem, we can find for each $j \in \mathbb{N}$ a measurable set $E_j \subseteq \Omega$ with $E_{j+1} \subseteq E_j$ and $|E_j| \leq 1/j$ such that
\[
\|\phi^m\|_{L^\infty(\Omega \setminus E_j)} \to 0, \quad \text{as } m \to \infty.
\]
Then, by using the weak*–strong continuity of the pairing
\[
L^\infty_w(\Omega \setminus E_j, \mathcal{M}(\mathbb{R}^d)) \times L^1(\Omega \setminus E_j, C^0(\mathbb{R}^d)) \to \mathbb{R}
\]
and that $L^\infty(\Omega \setminus E_j) \subseteq L^1(\Omega \setminus E_j)$, the convergence $\vartheta^m \rightharpoonup \vartheta^\infty$ and our assumption imply
\[
\int_{\mathbb{R}^d} \Phi(X) F^\infty(x, u^\infty(x), X) \, d[\vartheta^\infty(x)](X) = 0,
\]
for a.e. $x \in \Omega \setminus E_j$. Then, we conclude by letting $j \to \infty$. In order to establish that $\phi^m \to 0$ a.e. on $\Omega$, we recall that $u^m \to u^\infty$ a.e. on $\Omega$ and we fix an $x \in \Omega$ such that $u^m(x) \to u^\infty(x)$. Then, we can find compact sets $K \subseteq \mathbb{R}^N$ and $L \subseteq \mathbb{R}^d$ such that $u^m(x), u^\infty(x) \in K$ and $\text{supp}(\Phi) \subseteq L$. By the convergence assumption on the maps $F^m$, we have
\[
\| F^m(x, \cdot) - F^\infty(x, \cdot) \|_{C^0(K \times L)} \to 0, \quad \text{as } m \to \infty.
\]
If $\omega^\infty_x \in C^0[0, \infty)$ denotes the modulus of continuity of $K \ni \eta \mapsto F^\infty(x, \eta, X) \in \mathbb{R}^d$ which can be chosen uniform with respect to $X \in L$, we have
\[
|\phi^m(x)| \leq \sup_{X \in L} |\Phi| \left\{ \sup_{X \in L} \left| F^\infty(x, u^m(x), X) - F^\infty(x, u^\infty(x), X) \right| + \sup_{X \in \mathbb{R}^d} \left| F^m(x, u^m(x), X) - F^\infty(x, u^m(x), X) \right| \right\}
\leq \sup_{X \in \mathbb{R}^d} |\Phi| \left\{ \omega^\infty_x(|u^m(x) - u^\infty(x)|) + \| F^m(x, \cdot) - F^\infty(x, \cdot) \|_{C^0(K \times L)} \right\}
= o(1),
\]
as $m \to \infty$, because $\omega^\infty_x(0^+) = 0$. Since this holds for a set of points $x \in \Omega$ of full measure, the conclusion follows and the lemma ensues. \qed
Remark 21. The reader should note carefully that Theorem 19 is not a stability theorem, in the sense that we do not have compactness of diffuse jets for the approximating sequence as part of the conclusion of the theorem. In fact, such a result would not be possible without some sort of a priori estimates. Derivation of certain a priori estimates for existence appears to be a central difficulty in the vectorial case. Moreover, in view of Lemma 3, the weak* convergence of Young measures is actually relatively strong: when the maps are regular, it is equivalent (up to subsequences) to a.e. convergence.

2.4. Comparison with distributional and weak solutions. As we have already explained, the theory introduced herein allows to interpret merely measurable maps as solutions of fully nonlinear PDE systems. Since we not assume that our solutions are $L^1_{\text{loc}}$, in this sense $D$-solutions comprise a more general theory than distributional solutions (and a fortiori than weak solutions) because the present theory applies to more general systems and under less regularity requirements.

Notwithstanding, a direct comparison of the type that “distributional $\Rightarrow D$-solutions” or “$D$-solutions $\Rightarrow$ distributional” in the common domain of applicability of the two theories, that is for $L^1_{\text{loc}}$ solutions of linear systems with smooth coefficients, is not at all obvious. This happens because diffuse derivatives of maps $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ may be incompatible with distributional and weak $L^p$ derivatives for $p \leq n$ when the map fails to be pointwise differentiable. The problem is that when the adaptive infinitesimal frame $s$ with respect to which we take difference quotients is not independent of the basepoint $x$, the integration-by-parts formula for difference quotients

\begin{equation}
\int_{\Omega} \phi D^{1,h} u = - \int_{\Omega} u D^{1,-h} \phi
\end{equation}

in general fails. The next trivial but sharp estimate exhibits the failure to estimate in $L^p$ the adaptive difference quotients in terms of weak derivatives when the “step size” depends on the basepoint:

Lemma 22. Let $u \in C^1_c(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ with $|a| = 1$ and $s \in C^1(\mathbb{R}^n)$ with $s > 0$ and $\|Ds\|_{C^0(\mathbb{R}^n)} < 1$. Then, for any $p \geq 1$ we have

$$\int_{\mathbb{R}^n} \left| \frac{u(x+s(x)a) - u(x)}{s(x)} \right|^p dx \leq \int_0^1 \int_{\mathbb{R}^n} \frac{|Du(x)\cdot a|^p}{1 + rDs(x) \cdot a} dx dr.$$  

Proof of Lemma 23. We obviously have

$$\left| \frac{u(x+s(x)a) - u(x)}{s(x)} \right|^p \leq \int_0^1 |Du(x+rs(x)a)\cdot a|^p dr.$$  

By integrating on $\mathbb{R}^n$, using the change of variables $y(x) := x + rs(x)a$ and applying Sylvester’s determinant theorem

$$\det \left( I + rDs(x) \otimes a \right) = 1 + rDs(x) \cdot a,$$

the desired estimate follows. \qed

Hence, when the $D$-solution is not pointwise differentiable but instead only weakly differentiable, the relation between weak and diffuse derivatives is not so simple, unless the infinitesimal frame satisfies extra conditions. The next result connects them in a simple instructive case.
Lemma 23 (Compatibility of weak with diffuse derivatives). Let \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) be in \( W_{1,1}^{1,1}(\Omega, \mathbb{R}^N) \). Suppose we are given an adaptive infinitesimal frame \( \mathfrak{s} \) with respect to certain orthonormal frames \( \{ \eta^\alpha \} \) of \( \mathbb{R}^N \) and for each \( \alpha \), \( \{ \xi^{(\alpha)i} \} \) of \( \mathbb{R}^n \). If \( \mathfrak{s} \) satisfies
\[
s_{m}^{(\alpha)i}(x) = h_{m}^{(\alpha)i} \xi^{(\alpha)i}
\]
for all \( m \in \mathbb{N} \) and all indices \( \alpha, i \) (that is, \( s_{m}^{(\alpha)i} \) is independent of \( x \) and in addition its direction is independent of \( m \)), then the diffuse gradient \( D_u \) with respect to this frame \( \mathfrak{s} \) is unique and we have
\[
\delta_{D_u} = D_u, \quad \text{a.e. on } \Omega,
\]
where \( D_u \in L_{1,loc}^1(\Omega, \mathbb{R}^{Nn}) \) is the weak gradient.

Proof of Lemma 23. By the standard equivalence between weak and strong directional derivatives, since \( u \in W_{1,1}^{1,1}(\Omega, \mathbb{R}^N) \) we have that \( D_{1,h}^e u \rightharpoonup D_e u \) in \( L_{1,loc}^1(\Omega, \mathbb{R}^{Nn}) \) as \( h \rightarrow 0 \) for any fixed \( e \in \mathbb{R}^n \). Hence, since the step sizes \( h_{m}^{(\alpha)i} \) and the directions \( \xi^{(\alpha)i} \) are independent of \( x \), we may apply the previous to the projection \( \eta^\alpha \cdot u \) for \( e := \xi^{(\alpha)i} \) and \( h := h_{m}^{(\alpha)i} \). Thus, by (2.3) and Definition 8 we obtain that \( D_{1,s_{m}u} \rightharpoonup D_u \) in \( L_{1,loc}^1(\Omega, \mathbb{R}^{Nn}) \) and a.e. on \( \Omega \) as \( m \rightarrow \infty \) along a subsequence. Application of Lemma 3 completes the proof. \( \square \)

Comparison of distributional with \( D \)-solutions in a special case. The following discussion demonstrates the relation between these two theories in the case of an adaptive infinitesimal frame \( \mathfrak{s} \) satisfying the assumptions of Lemma 23. For simplicity, we may further consider only the case of standard difference quotients, the general case being only slightly more complicated.

Let \( u \in L_{1,loc}^1(\mathbb{R}^n) \). Then, the distributional gradient \( D_u \) exists in \( \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n) \) and it is well known that by (2.13) the standard difference quotients converge in the weak* topology, that is
\[
\int_{\mathbb{R}^n} \phi D^{1,s_{m}u} \rightarrow \langle \phi, Du \rangle, \quad \text{as } m \rightarrow \infty,
\]
for any \( \phi \in C^\infty_c(\mathbb{R}^n) \). By using the identity
\[
\int_{\mathbb{R}^n} \phi D^{1,s_{m}u} = \int_{\mathbb{R}^n} \phi \left( \int_{\mathbb{R}^n} P d[\delta_{D^{1,s_{m}u}}](P) \right)
\]
we obtain that
\[
\overline{\delta_{D^{1,s_{m}u}}} \rightharpoonup Du, \quad \text{in } \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n), \quad \text{as } m \rightarrow \infty,
\]
where the bar denotes the barycentre of the (restriction on \( \mathbb{R}^n \) of the) measure:
\[
\overline{\vartheta} := \int_{\mathbb{R}^n} P d\vartheta(P).
\]
By the definition of diffuse gradients, along perhaps a further subsequence we have
\[
\delta_{D^{1,s_{m}u}} \rightharpoonup Du, \quad \text{in } \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n), \quad \text{as } m \rightarrow \infty.
\]
By juxtaposing (2.14) with (2.15), our formal interpretation is that when the infinitesimal frame is independent of the basepoint, the barycentres of the diffuse derivative measures are unique and coincide with the distributional derivatives:
\[
\overline{D_u} = Du.
\]
More importantly, diffuse derivatives avoid the impossibility to define products of
distributions with nonsmooth functions and bypass the failure of distributions to
apply to even linear equations with rough coefficients. The next considerations are
again formal but elucidating: let $A \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be essentially bounded (or even
continuous) “coefficient” map. Then, we have

$$A : D^{1,s} \rightarrow A : Du, \quad A : D_u, \quad \text{not well defined!}$$

Hence, for example the 1st order PDE system

$$A(x) : Du(x) = 0, \quad x \in \mathbb{R}^n,$$

makes perfect sense in the context of the present theory by interpreting it as

$$A(x) : \int_{\mathbb{R}^n} \Phi(P) \, d[Du(x)](P) = 0, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for $\Phi \in C^0(\mathbb{R}^n)$, while the barycentre of $Du$ is a distribution and hence its product
with a general nonsmooth $A$ not well defined:

$$A(x) : Du(x) = A : Du(x) = A(x) : \int_{\mathbb{R}^n} P \, d[Du(x)](P) = ?$$

In the case that $A$ is smooth, then both approaches apply and the above imply
that when the infinitesimal frame $s$ satisfies the assumption of Lemma 23, then
$D$-solutions is a stronger notion than distributional solutions. In particular, in
view of Proposition 18, we (loosely speaking) have that $u$ is a $D$-solution when for a.e. $x \in \mathbb{R}^n$
the support of the measure $Du(x)$ is contained in the nullspace of $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, while $u$ is a distributional solution when for a.e. $x \in \mathbb{R}^n$ the
barycentre of the measure $Du(x)$ is contained in the nullspace of $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}$.

We conclude this discussion by underlining the simplicity and handiness of the
analytical theory proposed herein, as opposed to the advanced but more cumber-
some algebraic theories of multiplication of distributions and their inconsistencies
when applied to PDEs (see e.g. [Co]).

3. Existence of $D$-solutions to the $\infty$-Laplace system

In this section we establish our first main existence result for $D$-solutions. We
treat the Dirichlet problem for the $\infty$-Laplace system

$$\Delta_\infty u := \left(Du \otimes Du + |Du|^2 |Du|^\perp \otimes I \right) : D^2 u = 0$$

which is the fundamental equation of vectorial Calculus of Variations in the space
$L^\infty$ (see [K5]) and arises in relation to variational problems of the functional

$$E_\infty(u, U) = \|Du\|_{L^\infty(U)}, \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^N), \quad U \subseteq \Omega.$$

**Theorem 24** (Existence of $\infty$-Harmonic maps). Let $\Omega \subseteq \mathbb{R}^n$ be an open set with
$|\Omega| < \infty$ and $n \geq 1$. Then, for any $g \in W^{1,\infty}(\Omega, \mathbb{R}^N)$, the Dirichlet problem

$$\left\{ \begin{array}{ll}
(Du \otimes Du + |Du|^2 |Du|^\perp \otimes I) : D^2 u = 0, & \text{on } \Omega, \\
u = g, & \text{on } \partial \Omega,
\end{array} \right.$$
has a $\mathcal{D}$-solution $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ in $W^{1,\infty}_g(\Omega, \mathbb{R}^n)$. More specifically, $u - g \in W^{1,\infty}_0(\Omega, \mathbb{R}^n)$ and there exists an adaptive infinitesimal frame $\Phi$ with respect to the standard basis of $\mathbb{R}^n$ (Definitions 5, 8, 9) such that

$$\int_{\mathbb{R}^n} \Phi(X) \left( Du \otimes Du + |Du|^2 [Du]^{\perp} \otimes I + X d[D^2u](X) \right) = 0,$$

a.e. on $\Omega$, for any $\Phi \in C^0_c(\mathbb{R}^{nm^2})$, where $D^2u$ is a diffuse hessian:

$$\delta_{[Du]^{\perp} [Du]} \xrightarrow{\ast} D^2u,$$

in $\mathcal{F}(\Omega, \mathbb{R}^{nm^2})$, as $m \rightarrow \infty$.

The case $n = 1$ of the above theorem is completely trivial (see [K1]), so we will focus only on the case $n \geq 2$. The next corollary will be established in the course of the proof of Theorem 24.

**Corollary 25.** In the setting of Theorem 24, if $n \geq 2$ then the problem (3.3) actually has an infinite set of solutions and each one of those in addition satisfies

$$|Du|^2 = nM^2, \quad |\det(Du)| = M^n, \quad \text{a.e. on } \Omega,$$

for some $M > \| (Dg^\top Dg)^{1/2} \|_{L^\infty(\Omega)}$.

### 3.1. The idea of the proof.

Suppose that $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth and recall that (3.1) in index form reads

$$\sum_{i,j,\alpha,\beta=1}^n D_i u_\alpha D_j u_\beta D^2_{ij} u_{\alpha\beta} + \sum_{i,\beta=1}^n |Du|^2[Du]^{\perp}_{i\alpha\beta} D^2_{n\alpha} u_{\beta} = 0, \quad \alpha = 1, \ldots, n,$$

where $[Du(x)]^{\perp}$ denotes the orthogonal projection on the nullspace of the linear map $Du(x)^{\top} : \mathbb{R}^N \rightarrow \mathbb{R}^n$. By contracting derivatives in the first summand, we can rewrite (3.1) as

$$Du \ D\left( \frac{1}{2} |Du|^2 \right) + |Du|^2 [Du]^{\perp} \Delta u = 0.$$

By observing that the first summand of (3.5) is valued in the tangent bundle of the (immersed) manifold $u(\Omega) \subseteq \mathbb{R}^n$ (defined by the projection $[Du]^{\top} := I - [Du]^{\perp}$) and the second summand is valued in the normal bundle, (3.5) decouples to the pair of independent systems

$$Du \ D\left( \frac{1}{2} |Du|^2 \right) = 0, \quad |Du|^2 [Du]^{\perp} \Delta u = 0.$$

Then, we obtain that smooth solutions of the 1st order differential inclusion

$$Du(x) \in \mathcal{K}_c, \quad \text{for } x \in \Omega,$$

where

$$\mathcal{K}_c := \left\{ Q \in \mathbb{R}^{mn} : |Q| = c, \ |\det(Q)| > 0 \right\}, \quad c > 0,$$

actually are $\infty$-Harmonic mappings: indeed, if $Du(\Omega) \subseteq \mathcal{K}_c$, then we have that $|Du|^2 \equiv c^2$ and $\det(Du) \neq 0$ on $\Omega$. Hence, in view of (3.6) we have that the 1st system is satisfied because the modulus of $Du$ is constant ($u$ is Eikonal map) and the 2nd system is satisfied because $u$ is a submersion since $Du$ is locally invertible (the codimension is zero), which forces $[Du]^{\perp} = 0$ on $\Omega$.

However, the preceding arguments make sense only for classical or strong solutions. In the following subsection, we prove Theorem 24 by using the machinery of
$D$-solutions in order to make the previous arguments rigorous for merely Lipschitz maps, which is the natural regularity requirement for the solutions of (3.1).

We remark that our proof is not “variational”, in the sense that we do not use the functional (3.2). Moreover, we do not follow the standard idea of constructing solutions to the $L^\infty$ equations by approximating by the $L^p$ Euler-Lagrange equations as $p \to \infty$. Instead, we use the Dacorogna-Marcellini Baire Category method [DM] to construct solutions of the inclusion above which we subsequently characterize as $\infty$-Harmonic maps in the $D$-sense.

3.2. Proof of the main result. Now we prove our first main existence result.

**Proof of Theorem 24** (and Corollary 25). Assume we are given $\Omega \subseteq \mathbb{R}^n$ with finite measure and $g \in W^{1,\infty}(\Omega, \mathbb{R}^n)$.

**Claim 26.** If $M > \| (Dg^\top Dg) \|_{L^\infty(\Omega)}^{1/2}$, there exists $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ such that

\[
|Du|^2 = nM^2, \quad \text{a.e. on } \Omega, \\
|\det(Du)| = M^n, \quad \text{a.e. on } \Omega.
\]

**Proof of Claim 26.** Given a map $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ in $W^{1,\infty}(\Omega, \mathbb{R}^n)$, let $\lambda_i(Du)$ denote the $i$-th singular value, that is the $i$-th eigenvalue of $(Dg^\top Dg)^{1/2}$:

\[
\sigma((Du^\top Du)^{1/2}) = \{ \lambda_1(Du), \ldots, \lambda_n(Du) \}, \quad \lambda_i \leq \lambda_{i+1}.
\]

Fix an $M > 0$ as in statement and consider the Dirichlet problem:

\[
\begin{aligned}
\lambda_i(Dv) &= 1, \quad \text{a.e. in } \Omega, \quad i = 1, \ldots, n, \\
v &= g/M, \quad \text{on } \partial\Omega.
\end{aligned}
\]

Then, we have the estimate

\[
\|\lambda_n(Dg)\|_{L^\infty(\Omega)} = \| \max_{|e|=1} (Dg^\top Dg)^{1/2} : e \otimes e \|_{L^\infty(\Omega)}
\leq \| (Dg^\top Dg)^{1/2} \|_{L^\infty(\Omega)}< M.
\]

Moreover, the rank-one convex hull of the orthogonal group $O(n)$ coincides with its convex hull (for details we refer to [DM]). In view of the results of [DM], the estimate (3.8) implies that the required compatibility condition for existence of solution to (3.7) is satisfied. Hence there is a strong solution $v$ to the Dirichlet problem (3.7) such that $v - (g/M) \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ for the given $M$ and the boundary data $g$.

Finally, since $\lambda_i(Dv) = 1$ a.e. on $\Omega$, by setting $u := Mv$ we have

\[
|Du|^2 = M^2|Dv|^2 = M^2 \sum_i \lambda_i(Dv)^2 = nM^2, \quad \text{a.e. on } \Omega, \\
|\det(Du)| = M^n|\det(Dv)| = M^n \prod_i \lambda_i(Dv) = M^n, \quad \text{a.e. on } \Omega,
\]

and in addition, $u - g \in W^{1,\infty}_0(\Omega, \mathbb{R}^n)$. The proof of the claim is complete. \qed

Next, we have:
Claim 27. Let \( u \in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \) be the map of Claim 26. Then, there exists an adaptive infinitesimal frame
\[
\mathfrak{s} = \{ (s^i_m) \}_{m=1}^{\infty}, \quad s^i_m \in L^\infty(\Omega, \mathbb{R}^n)
\]
with respect to the standard frame of \( \mathbb{R}^n \) (see Definition 8 and Remark 7) and by decomposing \( s^i_m \) as
\[
s^i_m(x) = h^i_m(x) e^i_m(x),
\]
then for all \( m \in \mathbb{N}, \) all \( i = 1, \ldots, n \) and a.e. \( x \in \Omega, \) we have
\[
0 < |h^i_m(x)| \leq \frac{1}{m}, \quad |e^i_m(x)| = 1, \quad |e^i_m(x) - e^i| \leq O\left(\frac{1}{m}\right).
\]
Moreover,
\[
\left| Du(x + h^i_m(x) e^i_m(x)) - Du(x) \right| \leq \frac{1}{m}
\]
and in addition for all \( \alpha = 1, \ldots, n \) we have
\[
\sum_{i,j,\beta=1}^n \left\{ D_i u_\alpha(x) \left( \frac{D_j u_\beta(x + h^i_m(x) e^i_m(x)) + D_j u_\beta(x)}{2n} \right) \right. \\
\left. \cdot \left[ D_j u_\beta(x + h^i_m(x) e^i_m(x)) - D_j u_\beta(x) \right] h^i_m(x) \right\} = 0.
\]
(3.9)

The idea of the proof is to employ lower density estimates by using the approximate continuity (in the measure-theoretic sense) of \( Du \) and the fact that \( u \) solves the vectorial Eikonal equation on \( \Omega. \) For background on approximate continuity we refer e.g. to [EG]. A tricky point is that we have to select the elements of \( \mathfrak{s} \) varying measurably in \( x \) as a measurable section of a multi-valued set function.

Proof of Claim 27. Let \( f \in L^\infty(\Omega, \mathbb{R}^{nn}), \) which we may assume that equals its standard Lebesgue precise representative. We will apply the next considerations to the map \( f = Du \) where \( u \) is the map of Claim 26, but for the moment we do not need any of its specific structure. By removing a nullset from the Lebesgue set of \( f \) on which we redefine \( f \) to be zero, we may identify \( f \) with a Borel measurable version (see Subsection 2.1). Hence, there is a Borel set \( B(f) \subseteq \Omega \) with \( |B(f)| = 0 \) such that, by the Lebesgue differentiation theorem, for every \( x \in \Omega \setminus B(f), \) we have
\[
\frac{1}{\alpha(n) \rho} \int_{\mathbb{B}^n_{\rho}} |f(z + x) - f(x)| \, dz \to 0, \quad \text{as } \rho \to 0,
\]
(3.10)
and \( f(x) = 0 \) for all \( x \in B(f) \) with \( f \) Borel measurable on \( \Omega. \) Here \( \alpha(n) \) denotes the volume of the unit \( n \)-ball \( \mathbb{B}^n_1 \) (centred at the origin). Now we modify (3.10) in order to show that it is satisfied on certain sectors rather than balls. Fix \( e \in \mathbb{R}^n \) with \( |e| = 1 \) and we set
\[
e^\perp := I - e \otimes e
\]
for the projection on the normal hyperplane normal to \( e. \) Fix also \( R > 0 \) and \( \phi \in (0, \phi_0), \) where
\[
\phi_0 := \frac{1}{2} \min \left\{ \frac{\pi}{2}, \arcsin \left( \frac{n - 1}{n} \right) \right\}.
\]
Consider the spherical solid sector
\[
\mathfrak{S}^{e\perp}_{R,\phi} := \left\{ x \in \mathbb{B}^n_R : \frac{1}{\tan \phi} |e^\perp x| \leq e \cdot x \leq \sqrt{R^2 - |e^\perp x|^2} \right\}
\]
namely the solid compact set of $\mathbb{R}^n$ enclosed between the graphs of the functions
\[ L(y) := \sqrt{R^2 - |y|^2}, \quad K(y) := \frac{1}{\tan \phi} |y|, \]
when $y \in \mathbb{B}^{n-1}_R$ and $\mathbb{R}^{n-1} \times \{0\}$ is identified with $e^\perp$ (see Figure 1).

![Figure 1](image1.png)

Then, the volume of $S^e_{R,\phi}$ is given by
\[ |S^e_{R,\phi}| = \mathcal{H}^{n-2}(\partial \mathbb{B}^{n-1}_1) \int_0^{R \sin \phi} t^{n-2} \left( \sqrt{R^2 - t^2} - \frac{t}{\tan \phi} \right) dt. \]

Elementary upper and lower bound estimates imply that
\[ C'(n) \left( n \sin^{n-2} \phi - (n-1) \sin^{n-1} \phi \right) \leq \frac{|S^e_{R,\phi}|}{R^n} \leq C''(n) \sin^{n-1} \phi \]
for some constants $C'(n), C''(n) > 0$ depending only on $n$. We set (see Figure 2)
\[ S^e_{R,\phi} := S^e_{R,\phi} \setminus S^e_{R/2,\phi}. \]

Then, by the estimate (3.11), it follows that for any $e, \phi$ fixed (3.10) gives
\[ \frac{1}{|S^e_{R,\phi}|} \int_{S^e_{R,\phi}} |f(z + x) - f(x)| \, dz \longrightarrow 0, \quad \text{as } \rho \rightarrow 0, \]
for every $x \in \Omega \setminus B(f)$.

![Figure 2](image2.png)

Now, for $e, \phi$ again fixed we define
\[ F : \Omega \times [0, \infty) \longrightarrow [0, \infty) \]
by setting
\[ d(x, \rho) := \min \{ \rho, \text{dist}(x, \partial \Omega) \} \]
and
\[
F(x, \rho) := \begin{cases} 
\rho + \sup_{0 < r < d(x, \rho), r \in Q} \left( \frac{1}{|S_{r, \phi}^x|} \int_{S_{r, \phi}^x} |f(z + x) - f(x)| \, dz \right), & \text{when } \rho > 0, \; x \in \Omega \setminus B(f), \\
0, & \text{otherwise}.
\end{cases}
\]

Then, \( F(x, \cdot) \) is strictly increasing for all \( x \in \Omega \setminus B(f) \) and \( F(\cdot, \rho) \) is Borel measurable on \( \Omega \) for any \( \rho \geq 0 \). Moreover, \( F(\cdot, 0) \equiv 0 \) on \( \Omega \). We fix an \( \epsilon > 0 \) and define

\[
(3.13) \quad \rho_\epsilon(x) := \begin{cases} 
(F(x, \cdot))^{-1}(\epsilon^2), & x \in \Omega \setminus B(f), \\
0, & x \in B(f).
\end{cases}
\]

We claim now that the function \( \rho_\epsilon \) is Borel measurable on \( \Omega \) and for all \( x \in \Omega \setminus B(f) \) it satisfies \( \rho_\epsilon(x) > 0 \) and also \( F(x, \rho_\epsilon(x)) = \epsilon^2 \). The only property of \( \rho_\epsilon \) that may not be obvious from (3.13) and needs to be established is measurability. This can be seen by defining

\[
F^i(\cdot, \rho) := \frac{\rho}{i} + F(\cdot, \rho) * \eta^{1/i}, \quad \text{on} \quad \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \frac{1}{i} \right\},
\]

where \( \eta^{1/i} \) is the standard mollifying sequence. Then, \( F^i \) is strictly increasing in \( \rho \), smooth in \( x \) and \( F^i(x, \rho) \to F(x, \rho) \) for all \( x \in \Omega \setminus B(f) \) and any \( \rho \geq 0 \). Hence, by using the strict monotonicity of \( F^i \) with respect to \( \rho \) we easily obtain that

\[
\rho_\epsilon(x) = \lim_{i \to \infty} \left[ \chi_{\Omega \setminus B(f)}(x) \left( F^i(x, \cdot) \right)^{-1}(\epsilon^2) \right]
\]

for all points \( x \in \Omega \), which shows that \( \rho_\epsilon \) is the pointwise limit of Borel measurable functions. Now we use the functions \( F \) and \( \rho_\epsilon \equiv \rho_{\epsilon, \phi} \) defined above to infer that for any \( \epsilon > 0 \) and \( \phi \in (0, \phi_0) \), there is a Borel measurable a.e. positive function \( \rho_{\epsilon, \phi} \) such that for every \( x \in \Omega \setminus B(f) \) and all

\[
0 < r < d(x, \rho_{\epsilon, \phi}(x)),
\]

we have

\[
\epsilon^2 = F(x, \rho_{\epsilon, \phi}(x)) \geq \frac{1}{|S_{r, \phi}^x|} \int_{S_{r, \phi}^x} |f(z + x) - f(x)| \, dz \\
\geq \frac{\epsilon}{|S_{r, \phi}^x|} \left| \left\{ z \in S_{r, \phi}^x : |f(z + x) - f(x)| > \epsilon \right\} \right|.
\]

Hence, by setting

\[
R_{\epsilon, \phi}(x) := \min \left\{ \epsilon, \frac{1}{2} d(x, \rho_{\epsilon, \phi}(x)) \right\},
\]

we obtain that for any \( 0 < \epsilon < 1, \; \phi \in (0, \phi_0) \) and \( \phi \in \mathbb{R}^n \) with \( |e| = 1 \), there is a Borel measurable function \( R_{\epsilon, \phi} \) on \( \Omega \) defined by (3.14) and (3.13) such that

\[
0 < |R_{\epsilon, \phi}(x)| \leq \epsilon
\]

and

\[
(3.15) \quad \left| \left\{ z \in S_{R_{\epsilon, \phi}(x), \phi} : |f(z + x) - f(x)| \leq \epsilon \right\} \right| \geq (1 - \epsilon) \left| S_{R_{\epsilon, \phi}(x), \phi} \right| > 0,
\]

for every \( x \in \Omega \setminus B(f) \).

Consider now the following set-valued mapping \((e, \epsilon, \phi)\) fixed:

\[
\Gamma : \Omega \to 2^{\mathbb{R}^n}, \quad x \mapsto \Gamma(x),
\]
defined by
\[ \Gamma(x) := \begin{cases} 
\{ z \in S^n_{R_{\varepsilon,\phi}(x)} : |f(z + x) - f(x)| \leq \varepsilon \}, & x \in \Omega \setminus B(f), \\
\emptyset, & x \in B(f), 
\end{cases} \]

Then, the graph of \( \Gamma \subseteq \Omega \times \mathbb{R}^n \) is given by
\[ \text{Gr}(\Gamma) = \left\{ (x, y) \in \Omega \times \mathbb{R}^n : y \in \Gamma(x) \right\} = \left\{ (x, y) \in \Omega \times \mathbb{R}^n : |f(y + x) - f(x)| \leq \varepsilon, \ y \in S^n_{R_{\varepsilon,\phi}(x)} \right\}. \]

Hence, by assuming as we can that \( f \equiv 0 \) on \( \mathbb{R}^n \setminus \Omega \) and by the definition of the sectors \( S^n_{R_{\varepsilon,\phi}(x)} \), we obtain that
\[ \text{Gr}(\Gamma) = \left\{ (x, y) \in \Omega \times \mathbb{R}^n : |f(y + x) - f(x)| \leq \varepsilon \right\} \]
\[ \cap \left\{ (x, y) \in \Omega \times \mathbb{R}^n : \frac{R_{\varepsilon,\phi}(x)}{2} - 2 \sin \left( \frac{\phi}{2} \right) \leq |y| \leq \frac{R_{\varepsilon,\phi}(x)}{2} \right\}. \]

Since \( f \) and \( R_{\varepsilon,\phi} \) are Borel measurable mappings on \( \Omega \), by (3.16) we obtain that \( \text{Gr}(\Gamma) \) is a Borel measurable subset of \( \Omega \times \mathbb{R}^n \). Moreover, the estimate (3.15) guarantees that
\[ \Gamma(x) \neq \emptyset, \quad \text{for all } x \in \Omega. \]

Hence, by applying Aumann’s selection theorem (see e.g. [FL], [AC]) there is a (Borel) measurable selection of \( \Gamma \):
\[ \Omega \ni x \mapsto s^n_{\varepsilon,\phi}(x) \in \Gamma(x) \subseteq \mathbb{R}^n. \]

Finally, we apply the previous to \( e := e^i \) (the \( i \)-th basic vector), \( \varepsilon = \phi := 1/m, \ m \in \mathbb{N} \) and to \( f := Du \), where \( u \in W^{1,\infty}(\Omega, \mathbb{R}^N) \) is the map of Claim 26. Hence, by the definition of \( \Gamma \) we conclude that for all \( i = 1, \ldots, n \) and all \( x \in \Omega \setminus B(Du) \), there exists \( s^i_m(x) \in \mathbb{R}^n \) such that \( x + s^i_m(x) \in \Omega \) and
\[ 0 < |s^i_m(x)| \leq \frac{1}{m}, \quad |\text{sgn}(s^i_m(x)) - e^i| \leq 2 \sin \left( \frac{1}{2m} \right), \]

and also
\[ |Du(x + s^i_m(x)) - Du(x)| \leq \frac{1}{m}, \]

for all \( x \in \Omega \setminus B(Du) \) and all \( i = 1, \ldots, n \). Further, since \( |B(Du)| = 0 \), the maps
\[ \Omega \ni x \mapsto s^i_m(x) \in \mathbb{R}^n \]

are defined a.e. on \( \Omega \) and vary measurably with respect to \( x \). As a result, \( \mathbf{s} := (s^i_m)_{m=1}^{\infty} \) is an adaptive infinitesimal frame, provided that we define \( h^i_m(x), e^i_m(x) \) by taking
\[ h^i_m(x) := |s^i_m(x)|, \quad e^i_m(x) := \text{sgn}(s^i_m(x)). \]

Finally, we show (3.9) and this will complete the proof of the claim. Since by Claim 26 the map \( u \) satisfies \( |Du|^2 = nM^2 \) a.e. on \( \Omega \), for a.e. \( x \in \Omega \) we have \( x, x + s^i_m(x) \in \Omega \) and also
\[ |Du(x + s^i_m(x))|^2 - |Du(x)|^2 = 0. \]
Consequently,
\[
\sum_{j,\beta=1}^{n} \left( D_j u_{\beta} (x + s^i_m (x)) + D_j u_{\beta} (x) \right) \left( D_j u_{\beta} (x + s^i_m (x)) - D_j u_{\beta} (x) \right) = 0
\]
for a.e. \( x \in \Omega \). By using the splitting \( s^i_m (x) = h^i_m (x) e^i_m (x) \), multiplying the above identity by \( D_i u_{\alpha} (x) / (2n h^i_m (x)) \) and summing in \( i = 1, \ldots, n \), we obtain (3.9) and the claim ensues. \( \square \)

The next claim completes the proof of the theorem.

Claim 28. If \( u \in W^{1,\infty}_g (\Omega, \mathbb{R}^n) \) is the map of Claim 26 and \( s = \{ s^i_m \}_{m=1}^{\infty} \) is the adaptive infinitesimal frame of Claim 27, then the adaptive difference quotients \( D^{1,s} u \) (see Definition 5, Remark 7) converge in \( \mathcal{W} (\Omega, \mathbb{R}^{nn^2}_s) \) to a diffuse hessian of \( u \) with respect to which \( u \) is a \( D \)-solution of the \( \infty \)-Laplace system on \( \Omega \).

Proof of Claim 28. We define the Carathéodory maps
\[
F^m, F^\infty : \Omega \times \mathbb{R}^{nn^2} \to \mathbb{R}^n, \quad m \in \mathbb{N},
\]
given by
\[
F^m (x, X) := \sum_{i,j,\alpha,\beta=1}^{n} \left[ D_i u_{\alpha} (x) \left( \frac{D_j u_{\beta} (x + s^i_m (x)) + D_j u_{\beta} (x)}{2n} \right) \right] X_{\beta j} e^{\alpha},
\]
\[
F^\infty (x, X) := \sum_{i,j,\alpha,\beta=1}^{n} \left[ D_i u_{\alpha} (x) D_j u_{\beta} (x) + |Du(x)|^2 [Du(x)]_\beta^\top \delta_{ij} \right] X_{\beta j} e^{\alpha}.
\]
By Claim 27, we have that \( Du (x + s^i_m (x)) \to Du (x) \) for a.e. \( x \in \Omega \) as \( m \to \infty \). Moreover, by Claim 26 we have \( \det(Du) \neq 0 \) a.e. on \( \Omega \) and as a result \( Du(x) \) has rank equal to \( n \) in \( \mathbb{R}^m \), which implies that the projection \( |Du(x)|^\top \) on its nullspace vanishes for a.e. \( x \in \Omega \). Hence, the previous two facts together imply that \( F^m (x, \cdot) \to F^\infty (x, \cdot) \) uniformly over the compact subsets of \( \mathbb{R}^{nn^2}_s \) as \( m \to \infty \) for a.e. \( x \in \Omega \). By (3.9) and the definition of \( F^m \), for any \( \Phi \in C^0_c (\mathbb{R}^{nn^2}_s) \) we have
\[
\left( \int_{\mathbb{R}^n} \Phi (X) F^m (x, X) d[\delta_{D^{1,s} Du}] (X) \right) (\Omega) = 0,
\]
for a.e. \( x \in \Omega \) and any \( m \in \mathbb{N} \), where \( D^{1,s} Du \) are the adaptive first difference quotients of \( Du \) with respect to \( s \). By the weak* compactness of \( \mathcal{W} (\Omega, \mathbb{R}^{nn^2}_s) \), there exists a weak* limit \( D^2 u \) of \( \delta_{D^{1,s} Du} \) which is a diffuse hessian of \( u \) with respect to the infinitesimal frame \( s \). To conclude, we apply the Convergence Lemma 20 to \( F^m \) and \( F^\infty \) above over \( \Omega \) with
\[
\vartheta^m = \delta_{D^{1,s} Du}, \quad \vartheta^\infty = D^2 u, \quad \mathbb{R}^d = \mathbb{R}^{nn^2}_s.
\]
By letting \( m \to \infty \), (3.17) implies that \( u \) is a \( D \)-solution on the \( \infty \)-Laplace system on \( \Omega \). The claim ensues. \( \square \)

The proof of Theorem 24 is now complete. \( \square \)

In the course of the proof we have established the following result, which we delineate here for the sake of clarity:
Corollary 29 (Partially-$C^1$ Eikonal $\Rightarrow$ tangentially $\infty$-Harmonic). Let $n, N \geq 1$ and let also $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ be a Lipschitz map which is partially $C^1$, that is there is a subdomain $\Omega_0 \subseteq \Omega$ with $|\Omega \setminus \Omega_0| = 0$ and $u \in C^1(\Omega_0, \mathbb{R}^N)$. Suppose that $u$ is a strong solution of the vectorial Eikonal equation on $\Omega$:

$$|Du|^2 = 1, \quad \text{on } \Omega_0.$$  

Then, $u$ is a $D$-solution of the tangential part of the $\infty$-Laplacian on $\Omega$:

$$Du \otimes Du : D^2u = 0, \quad \text{on } \Omega.$$  

Moreover, for any diffuse hessian $D^2u$ arising from subsequential limits with respect to the standard first difference quotients of $Du$

$$\delta_{D^1,m}Du \rightharpoonup D^2u, \quad \text{in } \mathcal{Y}(\Omega, \mathbb{R}^{Nn^2}), \quad \text{as } m \rightarrow \infty,$$

we have that

$$\int_{\mathbb{R}^{Nn^2}} \Phi(X) \; Du \otimes Du : Xd[D^2u](X) = 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c(\mathbb{R}^{Nn^2})$.

**Proof of Corollary 29.** Apply the method of the proof of Claim 28 to the map $u$ of the statement and use that $Du(x + he^i) \rightarrow Du(x)$ as $h \rightarrow 0$ for all $x \in \Omega_0$ and all $i = 1, \ldots, N$. \hfill $\square$

3.3. Remarks on the (non-)uniqueness and relation to viscosity solutions. The notion of $D$-solutions is designed in order to be flexible and to provide existence results primarily for the vectorial and the higher order nonlinear case where the standard notions of solution (distributional, weak, strong, viscosity) fail. However, by Corollary 29, in the scalar case $N = 1$ of $\Delta_\infty$ when both $D$-solutions and viscosity solutions apply, we have that $D$-solutions are not stronger than viscosity solutions. This means that for $\Delta_\infty$ there exist $D$-solutions which are not viscosity solutions and hence “$D$-solutions $\Rightarrow$ viscosity solutions” is excluded. (Most likely the opposite is true, that is “viscosity $\Rightarrow D$-solutions”, but we don’t know how to prove that yet!) In particular, by Corollary 29 the cone functions $C(x) = \pm |x|$ are $D$-solutions of $\Delta_\infty$ on $\mathbb{R}^n$, although it is well known that they are only sub/super solutions in the viscosity sense rather than solutions. On the other hand, $C^1$ solutions of the Eikonal equation do solve $\Delta_\infty$ in the viscosity sense (see e.g. [CC]).

In view of the above remarks, the reader might think of $D$-solutions as an “inappropriate” theory of generalised solutions since it does not seem to support uniqueness (for $\Delta_\infty$). However, **uniqueness in the vectorial case is not an issue of defining a correct notion of generalised solution, since even classical solutions in general may not be unique.** In the paper [K2] we showed that even when $n = N = 2$, $\Omega$ is the punctured disc and $g$ is the identity map, the Dirichlet problem (1.11) has infinitely many $C^\infty$ smooth solutions. Hence, uniqueness appears to be an issue of finding extra conditions that select a “good” solution. Such problems are not unique to the $\infty$-Laplacian: for instance, the Dirichlet problem for the minimal surface single equation is well posed, while for the minimal surface system may have either non-existence or non-uniqueness in positive codimension (see [OL]).

Going back to the example of [K2] mentioned above, we note that among the many smooth solutions that (1.11) has for these data, the (extension in $\Omega$ of the)
boundary condition \( g(x) = x \) is also a solution of \( \Delta_\infty u = 0 \) on \( \Omega \). Moreover, it is the only solution that is a limit of \( p \)-Harmonic maps as \( p \to \infty \). Namely, for each \( p \in (2, \infty) \), the unique solution of the \( p \)-Laplace system on \( \Omega \)

\[
\text{Div} (|Du|^{p-2}Du) = 0,
\]

with boundary data \( g \) on \( \partial \Omega \) is \( g \) itself. On the other hand, it is well-known that in the scalar case all \( \infty \)-Harmonic functions arise as uniform subsequential limits of \( p \)-Harmonic functions (this is a consequence of Jensen’s uniqueness theorem for the \( \infty \)-Laplacian and of the uniqueness for the \( p \)-Laplacian, see e.g. [C, K8] and references therein). Moreover, plenty of other examples seem to exhibit the same behaviour. Hence, we are led to the following conjecture regarding a selection (or “entropy”) principle of “good” solutions for the \( \infty \)-Laplace system:

**Conjecture (Uniqueness for the Dirichlet problem for \( \Delta_\infty \)).** For any bounded domain \( \Omega \subseteq \mathbb{R}^n \) with Lipschitz boundary and for any \( g \in W^{1,\infty}(\Omega, \mathbb{R}^N) \), the Dirichlet problem (1.11) has a unique \( D \)-solution \( u_\infty : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) in the class of uniform subsequential limits of \( p \)-Harmonic mappings \( u_p \) as \( p \to \infty \).

The investigation of the validity of this conjecture is left for future work.

4. **Existence of \( D \)-solutions to fully nonlinear degenerate elliptic systems**

Fix \( n, N \geq 1 \), let \( \Omega \subseteq \mathbb{R}^n \) be an open set and

\[
F : \Omega \times \mathbb{R}^{Nn}^2 \rightarrow \mathbb{R}^N
\]
a Carathéodory map. In this section we establish our second main result, namely the existence of a \( D \)-solution \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) to the Dirichlet problem

\[
\begin{cases}
F(\cdot, D^2u) = f, \text{ in } \Omega, \\
u = 0, \text{ on } \partial \Omega,
\end{cases}
\tag{4.1}
\]

when \( f \in L^2(\Omega, \mathbb{R}^N) \) and \( F \) satisfies a degenerate ellipticity assumption which in general does not guarantee that solutions are even once weakly differentiable. This extends previous results of the author in the class of strong solution for (4.1) ([K9, K10]) under a stronger ellipticity notion than that we consider herein. As we have already explained, the satisfaction of the boundary condition under this low regularity is a certain issue.

4.1. **The idea of the proof.** The solvability of (4.1) in the class of \( D \)-solutions is based on the study of the linearised problem with constant coefficients

\[
\begin{cases}
A : D^2u = f, \text{ in } \Omega, \\
u = 0, \text{ on } \partial \Omega,
\end{cases}
\tag{4.2}
\]

when \( A : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn} \) is a symmetric positive (i.e. non-negative) 4th order tensor and on a perturbation device provided by our ellipticity assumption. This device allows to prove existence for (4.1) by proving existence for (4.2) and using a fixed point argument in the guises of a classical surjectivity theorem of Campanato taken from [C3]. In order to solve (4.2) in the \( D \)-sense (and not just in the distributional sense), we need a technical (but easy to verify) structural condition on the minors of the tensor \( A \). This assumption allows to construct solutions as mappings which
have twice weakly differentiable projections along certain rank-one lines of $\mathbb{R}^{Nn}$. These projections are, roughly speaking, orthogonal to the nullspace of the operator and are “directions of strict ellipticity” of the system $A : D^2u = f$. We formalise this idea by introducing a “fibre” extension of the classical Sobolev spaces of maps possessing only certain partial regularity. Our “fibre” spaces support feeble yet sufficient for our purpose versions of weak compactness, of trace operators and of Poincaré inequalities. The proof of the existence is completed by characterising the “fibre” object we have obtained via our surjectivity result as a $D$-solution of the problem.

4.2. The notion of degenerate ellipticity and the main result. Before stating our existence result for (4.1) (and (4.2)) we need some preparation. We will use the notation

$$A \in \mathbb{R}^{Nn \times Nn}$$

to denote symmetric linear maps $A : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$, i.e. 4th order tensors satisfying $A_{\alpha \beta ij} = A_{\beta \alpha ij}$ for all indices $\alpha, \beta = 1, \ldots, N$ and $i, j = 1, \ldots, n$. The notation

$$N(A : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}) \quad N(A : \mathbb{R}^{Nn^2} \rightarrow \mathbb{R}^N)$$

will be used to denote the nullspaces of $A$ when seen as linear map with domain and range as indicated in the brackets, i.e. when $A$ acts respectively as

$$Q \mapsto AQ := \sum_{\alpha, \beta, i, j} A_{\alpha \beta ij} Q_{\beta j} e^\alpha \otimes e^i, \quad X \mapsto A : X := \sum_{\alpha, \beta, i, j} A_{\alpha \beta ij} X_{\beta i} e^\alpha.$$

We will also follow the analogous notation for the respective ranges

$$R(A : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}) \quad R(A : \mathbb{R}^{Nn^2} \rightarrow \mathbb{R}^N).$$

If $A$ is rank-one positive, namely if the respective quadratic form on $\mathbb{R}^{Nn}$ is rank-one convex

$$A : \eta \otimes a \otimes \eta \otimes a := \sum_{\alpha, \beta, i, j} A_{\alpha \beta ij} \eta_i a_j \eta_j a_i \geq 0, \quad \eta \in \mathbb{R}^N, \ a \in \mathbb{R}^n,$$

we define

$$\Pi := N(A : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn})^\perp \subseteq \mathbb{R}^{Nn},$$

$$\Sigma := \text{span}\{\eta \mid \eta \otimes a \in \Pi\} \subseteq \mathbb{R}^N,$$

$$\Xi := \text{span}\{\eta \otimes a \vee b \mid \eta \otimes a, \ \eta \otimes b \in \Pi\} \subseteq \mathbb{R}^{Nn^2},$$

$$\nu := \min_{|\eta| = |a| = 1, \ \eta \otimes a \in \Pi} \left\{A : \eta \otimes a \otimes \eta \otimes a\right\} > 0.$$

We will call $\nu$ the ellipticity constant of $A$. We also recall that we will use the same letters $\Pi, \Xi, \Sigma$ to denote the subspaces and the orthogonal projections on them. The degenerate ellipticity assumption we need is introduced next.

**Definition 30 (K-Condition).** Let $n, N \geq 1, \ \Omega \subseteq \mathbb{R}^n$ an open set. We will say that the Carathéodory map

$$F : \Omega \times \mathbb{R}^{Nn^2} \rightarrow \mathbb{R}^N$$
(or the PDE system \( F(\cdot, D^2 u) = f \)) is degenerate elliptic when there exists \( A \in \mathbb{R}^{n \times n} \) rank-one positive, constants \( B, C \geq 0 \) with \( B + C < 1 \) and a positive measurable function \( A \) satisfying \( A, 1/A \in L^\infty(\Omega) \) such that

\[
\left| A : Z - A(x) \left( F(x, X + Z) - F(x, X) \right) \right| \leq B \nu |\Xi Z| + C |A : Z|,
\]

for a.e. \( x \in \Omega \) and all \( X, Z \in \mathbb{R}^{n^2} \). We will moreover require that \( F \) is valued in the subspace \( \Sigma \subseteq \mathbb{R}^N \), i.e.

\[
F(x, X) \in \Sigma, \text{ for a.e. } x \in \Omega \text{ and all } X \in \mathbb{R}^{n^2}.
\]

This notion is an extension to the degenerate elliptic realm of the strict ellipticity assumption introduced in [K10]. In the elliptic case we required \( \Sigma = \mathbb{R}^N, \Pi = \mathbb{R}^{nN} \) and \( \Xi = \mathbb{R}^{n^2} \). We refer to [K9] for further material on the elliptic case. The special case of \( A_{\alpha i \beta j} = \delta_{\alpha \beta} \delta_{ij} \) and \( A(x) = \text{const} \) reduces to the classical notion introduced by Campanato ([C1, C2, C3]).

**Example 31.** It is easy to exhibit non-trivial examples of Carathéodory maps satisfying the K-Condition:

- a) Fix \( A \in \mathbb{R}^{n \times n} \) and any \( f \in C^{0,1}(\mathbb{R}^{n^2}, \mathbb{R}^N) \) with Lipschitz constant Lip\((f)\). Then, for any \( A \) with \( A, 1/A \in L^\infty(\Omega) \), the map

\[
F(x, X) := \frac{1}{A(x)} \left( 1 + \gamma \right) A : X + \Sigma f(\Xi X)
\]

satisfies the K-Condition when \( |\gamma| + \frac{1}{\nu} \text{Lip}(f) < 1 \).

- b) A linear example with non-constant coefficients which satisfies the K-Condition is given by \( F(x, X) := A(x) : X \) where \( A : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is measurable and such that

\[
\left| (A - A(x))A(x) : Z \right| \leq B \nu |\Xi Z|, \quad Z \in \mathbb{R}^{n^2},
\]

for some \( 0 < B < 1 \) and \( A, 1/A \in L^\infty(\Omega) \), \( A > 0 \) a.e. on \( \Omega \).

**Remark 32 (Partial monotonicity).** If \( F \) satisfies the K-Condition and the subspace \( \Xi \) associated to \( A \) satisfies

\[
\Xi \supseteq N \left( A : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^N \right) ^\perp
\]

then the following “partial monotonicity” property is satisfied:

\[
(4.4) \quad \begin{cases}
\text{For a.e. } x \in \Omega, \text{ } F(x, \cdot) \text{ is constant along the directions of } \Xi ^\perp:
\end{cases}
\]

\[
F(x, X + Z) = F(x, Z), \quad Z \in \Xi ^\perp, X \in \mathbb{R}^{n^2}.
\]

This means that \( F(x, X) = F(x, \Xi X) \), i.e. \( F(x, \cdot) \) depends only on the projection along \( \Xi \). Indeed, the assumption on \( \Xi \) says \( \Xi ^\perp \subseteq N(A : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^N) \). Thus, for any \( Z \in \Xi ^\perp \) we have \( A : Z = 0 \) and also \( \Xi Z = 0 \). Hence, by the K-Condition we deduce that for any \( Z \in \Xi ^\perp \) and \( X \in \mathbb{R}^{n^2} \) we have

\[
\left| - A(x) \left( F(x, X + Z) - F(x, X) \right) \right| \leq 0.
\]

This assumption on \( \Xi \) will turn out to be satisfied under a hypothesis we will impose on \( A \). We remark that \((4.4)\) is weaker than the monotonicity required in order to define vector-valued viscosity solutions; the latter requires \( F_\alpha(X) = F_\alpha(X_\alpha) \), that is the system must decouple to \( N \) independent equations \( F_\alpha(\cdot, D^2 u_\alpha) = f_\alpha \).
As we have noted, we will need an extra condition on $A$:

**Definition 33** (Structural Hypothesis (SH)). Let $n, N \geq 1$. We will say that the tensor $A \in \mathbb{R}^{N \times N \times N}$ satisfies the Structural Hypothesis when it can be written as

$$A_{\alpha i \beta j} = B_{\alpha \beta}^0 A_{ij} + \cdots + B_{\alpha \beta}^N A_{ij}$$

such that:

i) The sets of matrices $\{B_1, ..., B^N\} \subseteq \mathbb{R}^{N \times N}$ satisfies

$$B^\gamma \geq 0 \quad \text{and} \quad \Sigma^\gamma \perp \Sigma^\delta \text{ if } \gamma \neq \delta,$$

where $\Sigma^\gamma := R(B^\gamma : \mathbb{R}^N \to \mathbb{R}^N)$.

ii) The set of matrices $\{A^1, ..., A^N\} \subseteq \mathbb{R}^{n \times n}$ satisfies

$$A^\gamma \geq 0 \quad \text{and} \quad \dim \left( \bigcap_{\gamma=1}^N N \left( A^\gamma - \lambda_{i_0}^\gamma f : \mathbb{R}^n \to \mathbb{R}^n \right) \right) \geq 1,$$

where $\lambda_{i_0}^\gamma$ is the smallest positive eigenvalue of $A^\gamma$:

$$\sigma(A^\gamma) = \{0, ..., 0, \lambda_{i_0}^\gamma, \lambda_{i_0+1}^\gamma, ..., \lambda_n^\gamma\}, \quad \lambda_i^\gamma \leq \lambda_{i+1}^\gamma.$$

**Remark 34.** We observe that the structural hypothesis above trivialises in the scalar or in the 1-dimensional case, that is when either $N = 1$ or $n = 1$ and any non-negative matrix $A \in \mathbb{R}^{n \times n}$ or $B \in \mathbb{R}^{N \times N}$ satisfies it.

When $\max\{N, n\} \geq 2$, it is non-trivial, but in view of its constructive nature it is obvious how to exhibit non-trivial examples which satisfy it. Moreover, if $A$ satisfies (SH), it must necessarily be non-negative. Indeed, if $(B^\gamma)^{1/2}$, $(A^\gamma)^{1/2}$ denote the square roots of the matrices $B^\gamma$, $A^\gamma$, for any $Q \in \mathbb{R}^{N \times n}$ we have

$$A : Q \otimes Q = \sum_{\gamma, \alpha, \beta = 1}^N \sum_{i, j = 1}^n B_{\alpha \beta}^\gamma A_{ij}^\gamma Q_{\alpha i} Q_{\beta j}$$

$$= \sum_{\gamma, \alpha, \beta, \alpha = 1}^N \sum_{i, j, k = 1}^n (B_{\alpha \beta}^\gamma)^{1/2} Q_{\alpha i} (A_{ij}^\gamma)^{1/2} (B_{\alpha \beta}^\gamma)^{1/2} Q_{\beta j} (A_{jk}^\gamma)^{1/2}$$

and hence

$$A : Q \otimes Q = \sum_{\gamma=1}^N (B^\gamma)^{1/2} Q (A^\gamma)^{1/2} \geq 0.$$

Now we may finally state the main result of this section.

**Theorem 35** (Existence). Let $n, N \geq 1$ and suppose $\Omega \subseteq \mathbb{R}^n$ is a strictly convex bounded domain with $C^2$ boundary. Suppose further that $F : \Omega \times \mathbb{R}^{N \times N} \to \mathbb{R}^n$ is a Carathéodory map which satisfies Definition 30 with respect to a tensor $A$ which satisfies Definition 33. Let also $\Xi, \Pi, \Sigma$ be given by (4.3).

Then, if $|F(\cdot, 0)| \in L^2(\Omega)$, for any $f \in L^2(\Omega, \Sigma)$, the Dirichlet problem

$$\begin{cases}
F(\cdot, D^2 u) = f, \text{ in } \Omega, \\
u = 0, \text{ on } \partial \Omega,
\end{cases}$$

has a $\mathcal{L}^n$-measurable $D$-solution $u : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^N$ which also is $\mathcal{H}^{n-1} \cup \partial U$-measurable on the boundary of any strictly convex domain $U \subseteq \overline{\Omega}$. In addition, $u = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial \Omega$. 
In particular, there exists an adaptive infinitesimal frame $s$ with respect to certain orthonormal frames of $\mathbb{R}^N$, $\mathbb{R}^n$, $\mathbb{R}^{N_n}$ and $\mathbb{R}^{N_n^2}$ (Definition 5) depending on our assumptions such that the respective diffuse hessian of $u$

$$\delta_{D^2,m',m''} u \overset{\circ}{\rightarrow} D^2 u, \quad \text{in } \mathcal{Y}(\Omega, \mathbb{R}^{N_n^2}), \quad \text{as } m', m'' \to \infty,$$

satisfies

$$\int_{\mathbb{R}^{N_n^2}} \Phi(X) \left( F(x, X) - f(x) \right) d[D^2u(x)](X) = 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c(\mathbb{R}^{N_n^2})$.

Moreover, $u$ has certain weakly differentiable projections along rank-one directions: for any $\eta \otimes a \in \Pi$ and $\eta \otimes a \vee b \in \Xi$, we have that $D_a(\eta \cdot u), D^2_{ab}(\eta \cdot u)$ exist weakly and belong to $L^2(\Omega)$.

We note that $f$ has to be valued in the subspace $\Sigma$ and this is a compatibility condition arising from the degenerate nature of the problem. Before giving the proof of the main result, we need to establish some auxiliary results. This is done in the next two subsections.

4.3. A priori degenerate hessian estimates. In this subsection we establish an a priori estimate for strong solutions in $(W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ of a regularisation of the system

$$A : D^2u = f,$$

when $A$ satisfies Definition 33. This estimate is a generalisation of an elliptic estimate established in [K10] (which extends the classical Miranda-Talenti identity) to the case of degenerate elliptic systems.

**Theorem 36** (Degenerate hessian estimate). Let $n, N \geq 1$ with $\Omega \subseteq \mathbb{R}^n$ a convex bounded $C^2$ domain. Suppose further that $A \in \mathbb{R}^{N_n \times N_n}$ satisfies Definition 33.

If $\Xi, \nu$ are as in (4.3), then for any $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ and any $\varepsilon \geq 0$ we have the estimate

$$\|\Xi D^2 u\|_{L^2(\Omega)} \leq \frac{1}{\nu} \|A^{(\varepsilon)} : D^2 u\|_{L^2(\Omega)}$$

and also the property

$$\Xi \supseteq N \left( A : \mathbb{R}^{N_n^2} \rightarrow \mathbb{R}^N \right)^\perp.$$

The tensor $A^{(\varepsilon)}$ is the following (strictly) rank-one positive regularisation of $A$:

$$A^{(\varepsilon)}_{\alpha_i \beta_j} := \sum_{\gamma=0}^{N} B^{(\varepsilon)}_{\alpha_i \gamma} A^{(\varepsilon)}_{\beta_j \gamma},$$

$$B^{(\varepsilon)}_{\gamma} := \begin{cases} B^\gamma, & \gamma = 1, \ldots, N, \\ \varepsilon I - \varepsilon(B^1 + \cdots + B^N), & \gamma = 0, \end{cases}$$

$$A^{(\varepsilon)}_{\gamma} := \begin{cases} A^\gamma + \varepsilon I, & \gamma = 1, \ldots, N, \\ \varepsilon I, & \gamma = 0, \end{cases}$$

and $B^\gamma, A^\gamma$ are the matrices appearing in Definition 33.
Remark 37. We note that in the vectorial case $N \geq 2$ of Theorem 36, the “correct” approximation in not the vanishing viscosity one, although it reduces to a vanishing viscosity one when $N = 1$. Moreover, the property of $\Xi$ in the statement is very important and implies that $A$ “sees” only the projection on $\Xi$, that is

$$A : X = A : (\Xi X), \quad \text{for all } X \in \mathbb{R}^{nN^2}.$$  

Indeed, we have that $\Xi^\perp \subseteq N\left(A : \mathbb{R}^{nN^2} \rightarrow \mathbb{R}^N\right)$ and hence for any $Z \in \Xi^\perp$ it follows that $A : Z = 0$. Therefore, $A : X = A : (\Xi X + \Xi^\perp X) = A : (\Xi X)$.

Proof of Theorem 36. The first step is to prove a weak version on the scalar case of the theorem.

Claim 38. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex $C^2$ domain and $A \in \mathbb{R}^{n^2}$ a non-negative matrix. Then, there exists a subspace $H \subseteq \mathbb{R}^{n^2}$ such that

$$H \supseteq N\left(A : \mathbb{R}^n \rightarrow \mathbb{R}\right)^\perp$$

and for any $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega)$ and any $\varepsilon \geq 0$ we have the estimate

$$\|HD^2u\|_{L^2(\Omega)} \leq \frac{1}{\nu(A)} \|A : D^2u + \varepsilon \Delta u\|_{L^2(\Omega)}$$

where

$$\nu(A) := \min_{|a|=1, a \in T} \{A : a \otimes a\}, \quad T := N\left(A : \mathbb{R}^n \rightarrow \mathbb{R}\right)^\perp.$$  

Proof of Claim 38. By the Spectral theorem, we can find a diagonal matrix $\Lambda$ with entries $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$ and $O \in O(n)$ such that $A = O\Lambda O^\top$ and $\lambda_{i_0}$ is the smallest positive eigenvalue. We also fix $\varepsilon \geq 0$ and set

$$\Theta := (\Lambda + \varepsilon I)^{1/2}, \quad \Gamma := O\Theta.$$  

Then, since $A = O\Lambda O^\top$ and $\Theta$ is symmetric, we have

$$A + \varepsilon I = O\Lambda O^\top + O(\varepsilon I)O^\top = O\Theta (O\Theta)^\top = \Gamma \Gamma^\top$$

and also

$$\nu(A) = \lambda_{i_0}.$$  

We now define the subspaces of $\mathbb{R}_s^{n^2}$

$$H^0 := \left\{ X \in \mathbb{R}_s^{n^2} : X = \begin{bmatrix} 0 \\ \lambda_{i_0} \\ \vdots \\ 0 \end{bmatrix} \right\},$$

(4.7)

$$H := \left\{ X \in \mathbb{R}_s^{n^2} : O^\top XO \in H^0 \right\}.$$  

We begin by establishing the following algebraic inequality:

$$|\Theta X \Theta| \geq \nu(A) \|H^0 X\|, \quad X \in \mathbb{R}_s^{n^2}.$$  

Indeed, since $\Theta_{ij} = 0$ when $i \neq j$ and $\Theta_{ii} = \sqrt{\lambda_i + \varepsilon}$, in view of (4.7) we have
\[
|\Theta X \Theta|^2 = \sum_{i,j,k,l,p,q=1}^n (\Theta_{ik} X_{kl} \Theta_{lj}) (\Theta_{ip} X_{pq} \Theta_{qj})
\]
\[
= \sum_{i,j=1}^n (\Theta_{ii} X_{ij} \Theta_{jj})^2
\]
\[
\geq \sum_{i,j=1}^n (\lambda_i + \varepsilon) (X_{ij})^2 (\lambda_j + \varepsilon)
\]
\[
\geq (\lambda_{i_0})^2 \sum_{i,j=1}^{n} (X_{ij})^2
\]
\[
= \nu(A)^2 |H^2 X|^2.
\]
Hence, (4.8) has been established. In order to conclude, the goal is to reduce to the classical Miranda-Talenti inequality (see [M, T, K10]) which says that for $U \subset \mathbb{R}^n$ convex $C^2$ domain and any $v \in (W^{2,2} \cap W^{1,2}_0)(U)$, we have
\[
\|D^2 v\|_{L^2(U)} \leq \|\Delta v\|_{L^2(U)}.
\]
We suppose as we can that $\varepsilon > 0$ since the case $\varepsilon = 0$ follows by letting $\varepsilon \to 0$. Given a fixed $u \in C^2(\Omega) \cap C^1_0(\Omega)$, we set
\[
U := \Gamma^{-1} \Omega, \quad v(x) := u(\Gamma x), \quad x \in U.
\]
Then, we calculate
\[
D^2_{ij} v(x) = \sum_{p,q=1}^n D^2_{pq} u(\Gamma x) \Gamma_{pi} \Gamma_{qj}
\]
and hence, by (4.5) and (4.6) we obtain
\[
D^2 v(x) = \Gamma^\top D^2 u(\Gamma x) \Gamma = \Theta \left( O^\top D^2 u(\Gamma x) O \right) \Theta,
\]
\[
\Delta v(x) = D^2 u(\Gamma x) : \Gamma \Gamma^\top = D^2 u(\Gamma x) : (A + \varepsilon I).
\]
We now claim that since $\Omega$ is a $C^2$ bounded convex domain, $U$ is a $C^2$ bounded convex domain as well. Indeed, by (4.5) we have $\Gamma^{-1} = \Theta^{-1} O^\top$ and since $O^\top$ is an isometry, it suffices to show that $\Theta^{-1} V$ is convex, where $V := O^\top \Omega$. To see this, note that we can find a convex $F \in C^2(\mathbb{R}^n)$ such that $\{ F < 0 \} = V$. We set
\[
G(x) := F(\Theta x), \quad G \in C^2(\mathbb{R}^n).
\]
Then, we have
\[
D^2_{ij} G(x) = \sum_{p,q=1}^n D^2_{pq} F(\Theta x) \Theta_{pi} \Theta_{qj}
\]
and hence for any $a \in \mathbb{R}^n$, the convexity of $F$ implies
\[
D^2 G(x) : a \otimes a = \sum_{i,j,p,q=1}^n D^2_{pq} F(\Theta x) \Theta_{pi} \Theta_{qj} a_i a_j
\]
\[
= D^2 F(\Theta x) : (\Theta a) \otimes (\Theta a)
\]
\[
\geq 0.
\]
Hence, $G$ is convex too, which means the sublevel set $\{G < 0\}$ is convex. Moreover,

\[
U = \Theta^{-1} V = \{\Theta^{-1} x \in \mathbb{R}^n : F(x) < 0\} \\
= \{y \in \mathbb{R}^n : F(\Theta y) < 0\} \\
= \{y \in \mathbb{R}^n : G(y) < 0\}.
\]

Thus, $U$ is convex as claimed and we may apply the estimate (4.9) to $v$ over $U \subseteq \mathbb{R}^n$. Hence, by (4.9), (4.10) we get

\[
\int_{U} |D^2 u(\Gamma x) : (A + \varepsilon I)\|^2 dx \geq \int_{U} \left| \Theta \left( O^T D^2 u(\Gamma x) O \right) \Theta \right|^2 dx \geq \nu(A)^2 \int_{U} \left| H^0 \left( O^T D^2 u(\Gamma x) O \right) \right|^2 dx.
\]

By the change of variables $y := \Gamma x$ and by using that $O$ is orthogonal, we obtain

(4.11) \hspace{1em} \|D^2 u : (A + \varepsilon I)\|_{L^2(\Omega)} \geq \nu(A) \|O \left( H^0 \left( O^T D^2 u O \right) \right) O^T\|_{L^2(\Omega)}.

Now we claim that the orthogonal projection on the subspace $H \subseteq \mathbb{R}^n_\ast$ is given by

(4.12) \hspace{1em} HX = O \left( H^0 \left( O^T X O\right) \right) O^T.

Once (4.12) has been established, the desired estimate follows from (4.11), (4.12) and a standard density argument in the Sobolev norm. Indeed, if $K$ denotes the linear operator defined by the right hand side of (4.12), for any $X \in \mathbb{R}^n_\ast$ we have

\[
K(KX) = O \left( H^0 \left( O^T O \left( H^0 \left( O^T X O \right) \right) O^T \right) \right) O^T \\
= O \left( H^0 H^0 \left( O^T X O \right) \right) O^T \\
= O \left( H^0 \left( O^T X O \right) \right) O^T \\
= KX.
\]

Hence, $K^2 = K$. Moreover, $K$ is symmetric as a map $\mathbb{R}^n_\ast \rightarrow \mathbb{R}^n_\ast$, since

\[
(KX) : Y = \left( O \left( H^0 \left( O^T X O \right) \right) O^T \right) : Y \\
= H^0 \left( O^T X O \right) : (O^T Y O)
\]

and by using that $H^0$ is symmetric, we get

\[
(KX) : Y = (O^T X O) : H^0 \left( O^T Y O \right) \\
= X : \left( O \left( H^0 \left( O^T Y O \right) \right) O^T \right) \\
= X : (KY),
\]

for any $X, Y \in \mathbb{R}^n_\ast$. Hence, (4.12) follows. It remains to exhibit the claimed property of $H$. To this end, fix $X \perp H$. Then, we have that the projection of $X$ on $H$ vanishes and as a result of (4.12) we obtain $H^0(O^T X O) = 0$. By recalling that $A = O \Lambda O^T$, we have $A : X = \Lambda : (O^T X O)$ and since $\Lambda$ belongs to $H^0$, we conclude that $A : X = 0$. Hence, we have just proved that $H^\perp \subseteq N(A : \mathbb{R}^n_\ast \rightarrow \mathbb{R})$, which is the desired property of the subspace $H$. The claim has been established. \(\square\)

The next step is to characterise the subspace $H \subseteq \mathbb{R}^n_\ast$ of Claim 38 in terms of the range of $A$. 


Claim 39. In the setting of Claim 38, we have the identity
\[ H = \text{span}\{a \vee b \mid a, b \in H(A : \mathbb{R}^n \to \mathbb{R}^n)^\perp\}, \]
which by recalling that \( T = N(A : \mathbb{R}^n \to \mathbb{R}^n)^\perp \), we may also write it as
\[ H = T \vee T. \]

Proof of Claim 39. We begin by observing that in view of (4.7), we have
\[ H = O H^0 O^\top \] where \( O \in O(n) \). Since \( H^0 = \text{span}\{e^i \vee e^j \mid i, j = i_0, \ldots, n\} \) we obtain that \( H \) has a basis consisting of matrices of the form \( O e^i \vee O e^j, i, j = i_0, \ldots, n \). We recall now that \( A = O \lambda \) where \( \Lambda \) is a diagonal matrix with entries the eigenvalues \( 0, \ldots, 0, \lambda_{i_0}, \ldots, \lambda_n \) of \( A \). We define the vectors
\[ a^i := O e^i = (O e_i, \ldots, O e_n)^\top, \quad i = 1, \ldots, n. \]
Then, \( \{a^1, \ldots, a^n\} \) is an orthonormal frame of \( \mathbb{R}^n \) corresponding to the columns of the matrix \( A \) and is a set of eigenvectors of \( A \). Since \( \{a^0, \ldots, a^n\} \) correspond to the nonzero eigenvalues \( \{\lambda_{i_0}, \ldots, \lambda_n\} \), the nullspace \( N(A : \mathbb{R}^n \to \mathbb{R}^n) \) is spanned by \( \{a^1, \ldots, a^{i_0-1}\} \) and hence
\[ N(A : \mathbb{R}^n \to \mathbb{R}^n)^\perp = \text{span}\{a^{i_0}, \ldots, a^n\}. \]
Since \( H \) has a basis of the form \( \{a^i \vee a^j : i, j = i_0, \ldots, n\} \), the claim follows. \( \square \)

Now we begin working towards the vector case \( N \geq 2 \). Let us first verify that \( A^{(c)} \) is strictly rank-one positive. Indeed, if \( 0 < \varepsilon < 1, \eta \in \mathbb{R}^N, a \in \mathbb{R}^n \), we have
\[ A^{(c)} : \eta \otimes a \otimes \eta \otimes a = \sum_{\gamma=0}^{N} \left( B^{(c)\gamma} : \eta \otimes \eta \right) \left( A^{(c)\gamma} : a \otimes a \right) \]
\[ \geq \min_{\delta=0, \ldots, N} \left( A^{(c)\delta} : a \otimes a \right) \left[ \sum_{\gamma=0}^{N} B^{(c)\gamma} : \eta \otimes \eta \right] \]
\[ \geq \varepsilon |a|^2 \left[ \sum_{\gamma=1}^{N} B^{\gamma} + \varepsilon \left( I - \sum_{\delta=1}^{N} B^{\delta} \right) \right] : \eta \otimes \eta \]
and hence
\[ A^{(c)} : \eta \otimes a \otimes \eta \otimes a \geq \varepsilon^2 |\eta|^2 |a|^2, \]
as claimed. The next step is to characterise the range \( \Pi = R(A : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}) \) of \( A \in \mathbb{R}^{Nn \times Nn} \) satisfying (SH) in terms of the matrices \( B^{\gamma}, A^{\gamma} \).

Claim 40. Let \( \Pi \subseteq \mathbb{R}^{Nn} \) be the range of \( A : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn} \) (see (4.3)). Then,
\[ \Pi = \bigoplus_{\gamma=1}^{N} (\Sigma^{(c)} \otimes T^{(c)}), \]
where
\[ \Sigma^{(c)} = R(B^{\gamma} : \mathbb{R}^N \to \mathbb{R}^N) \subseteq \mathbb{R}^N, \]
\[ T^{(c)} = R(A^{\gamma} : \mathbb{R}^n \to \mathbb{R}^n) \subseteq \mathbb{R}^n. \]
Proof of Claim 40. We first observe that by (SH), $\Sigma^\gamma \perp \Sigma^\delta$ if $\gamma \neq \delta$ and this implies that $\Sigma^\gamma \otimes T^\gamma \perp \Sigma^\delta \otimes T^\delta$ if $\gamma \neq \delta$. Let now $Q \in \mathbb{R}^{N_n}$. Then, $A : Q$ is given in index form by

$$\sum_{\beta,j} A_{\alpha i \beta j} Q_{\beta j} = \sum_{\gamma,\beta,j} B^\gamma_{\alpha \beta} Q_{\beta j} A^\gamma_{ji}$$

which by (4.13) shows that $\Pi \subseteq \oplus_{\gamma}(\Sigma^\gamma \otimes T^\gamma)$. Conversely, let $R \in \oplus_{\gamma}(\Sigma^\gamma \otimes T^\gamma)$. Then, $R$ can be written as

$$R = \sum_{\gamma,\kappa} \left( B^\gamma \eta^{\kappa \gamma} \right) \otimes \left( A^\gamma \xi^{\kappa \gamma} \right)$$

for some $\eta^{\kappa \gamma} \in \Sigma^\gamma, \xi^{\kappa \gamma} \in T^\gamma$. We note that

$$(B^\delta \otimes A^\delta) \left( \sum_{\kappa} \eta^{\kappa \gamma} \otimes \xi^{\kappa \gamma} \right) = 0, \quad \text{if } \gamma \neq \delta,$$

because $\eta^{\kappa \gamma} \perp \Sigma^\delta$ if $\gamma \neq \delta$. We now define $Q := \sum_{\gamma,\kappa} \eta^{\kappa \gamma} \otimes \xi^{\kappa \gamma}$ and we claim that $A : Q = R$. Indeed, we have

$$\sum_{\beta,j} A_{\alpha i \beta j} Q_{\beta j} = \sum_{\delta,\beta,j} \left( B^\delta_{\alpha \beta} A^\delta_{ji} \right) \left( \sum_{\kappa} \eta^{\kappa \gamma} \xi^{\kappa \gamma} \right)$$

$$= \sum_{\delta,\beta,j} \left( B^\delta_{\alpha \beta} A^\delta_{ji} \right) \left( \sum_{\kappa} \eta^{\kappa \delta} \xi^{\kappa \delta} \right)$$

$$= \sum_{\kappa,\delta,\beta,j} \left( B^\delta_{\alpha \beta} \eta^{\kappa \delta} \right) \left( A^\delta_{ji} \xi^{\kappa \delta} \right)$$

$$= R_{\alpha i}.$$  

This establishes that $\Pi \supseteq \oplus_{\gamma}(\Sigma^\gamma \otimes T^\gamma)$, therefore completing the proof.

The next step is to find an upper bound of the ellipticity constant $\nu$ of $A$ in terms of the matrices $B^\gamma, A^\gamma$.

Claim 41. Let $\nu$ be given by (4.3). Then, we have the estimate

$$\nu \leq \left( \min_{\gamma} \min_{\eta \in \Sigma^\gamma, |\eta|=1} \{ B^\gamma : \eta \otimes \eta \} \right) \left( \min_{\delta} \min_{a \in T^\delta, |a|=1} \{ A^\delta : a \otimes a \} \right)$$

where $\Sigma^\gamma, T^\gamma$ are as in (4.13).

Proof of Claim 41. We begin by noting that on top of (SH) we may further assume that all the matrices $A^\gamma$ have the same smallest positive eigenvalue $\lambda^\gamma_{\ast}$ equal to 1 for all $\gamma = 1, \ldots, N$ and is realised at a common eigenvector $\bar{a} \in \mathbb{R}^n$. Indeed, existence of $\bar{a}$ follows from (SH) since the eigenspaces $N\left( A^\gamma - \lambda^\gamma_{\ast} I : \mathbb{R}^n \to \mathbb{R}^n \right)$ intersect for all $\gamma$ at least along a common line. Further, by replacing $\{B^1, \ldots, B^N\}, \{A^1, \ldots, A^N\}$ by the rescaled families $\{\tilde{B}^1, \ldots, \tilde{B}^N\}, \{\tilde{A}^1, \ldots, \tilde{A}^N\}$ where $\tilde{B}^\gamma := \lambda^\gamma_{\ast} B^\gamma, \tilde{A}^\gamma := (1/\lambda^\gamma_{\ast}) A^\gamma$, we have that the new families have the same properties as the
original and in addition all the new $A^\gamma$ matrices have the same minimum positive eigenvalue normalised to 1. Hence, we may assume that $A$ satisfies (SH) and also

\[
(4.14) \quad \exists \ \bar{a} \in \partial B^1 \cap \bigcap_{\gamma=1}^N T^\gamma : \quad \lambda_{\bar{a}}^\gamma = \min_{a \in T^\gamma, |a|=1} \{ A^\gamma : a \otimes a \} = A^\gamma : \bar{a} \otimes \bar{a} = 1,
\]

for all $\gamma = 1, \ldots, N$. By using (4.14), Claim 40 and that $\cup_{\gamma} (\Sigma^\gamma \otimes T^\gamma) \subseteq \oplus_{\gamma} (\Sigma^\gamma \otimes T^\gamma)$, we calculate

\[
\nu = \min_{|\eta|=|a|=1, \eta \otimes a \in \mathcal{H}} \sum_{\delta} \left( B^{\delta} : \eta \otimes \eta \right) \left( A^{\delta} : a \otimes a \right)
\]

\[
\leq \min_{|\eta|=|a|=1, \eta \otimes a \in \cup_{\gamma} (\Sigma^\gamma \otimes T^\gamma)} \sum_{\delta} \left( B^{\delta} : \eta \otimes \eta \right) \left( A^{\delta} : a \otimes a \right)
\]

\[
= \min_{\gamma} \left( \min_{|\eta|=|a|=1, \eta \otimes a \in \Sigma^\gamma} \sum_{\delta} \left( B^{\delta} : \eta \otimes \eta \right) \left( A^{\delta} : a \otimes a \right) \right)
\]

\[
\leq \min_{\gamma} \left( \min_{|\eta|=|a|=1, \eta \otimes a \in \Sigma^\gamma} \sum_{\delta} \left( B^{\delta} : \eta \otimes \eta \right) \right)
\]

\[
= \min_{\gamma} \min_{|\eta|=|a|=1, \eta \otimes a \in \Sigma^\gamma} \sum_{\delta} \left( B^{\delta} : \eta \otimes \eta \right).
\]

Since $B^{\delta} : \eta \otimes \eta = 0$ if $\eta \in \Sigma^\gamma$ for $\gamma \neq \delta$, by using (4.14) again we conclude that

\[
\nu \leq \min_{\gamma} \min_{|\eta|=|a|=1} \left\{ B^{\gamma} : \eta \otimes \eta \right\}
\]

\[
= \left( \min_{\gamma} \min_{|\eta|=|a|=1} \left\{ B^{\gamma} : \eta \otimes \eta \right\} \right) \left( \min_{\delta} \min_{a \in T^\gamma, |a|=1} \left\{ A^{\delta} : a \otimes a \right\} \right),
\]

as desired. \qed

Now we complete the proof of the theorem by using the previous claims. We define

\[
(4.15) \quad \Xi := \bigoplus_{\gamma} (\Sigma^\gamma \otimes T^\gamma \vee T^\gamma) \subseteq \mathbb{R}^{N^2},
\]

and for brevity we set

\[
\Xi^\gamma := T^\gamma \vee T^\gamma \subseteq \mathbb{R}^{N^2},
\]

where $\Sigma^\gamma$, $T^\gamma$ are as in (4.13). Fix a map $u \in C^2(\bar{\Omega}, \mathbb{R}^N) \cap C^1_0(\Omega, \mathbb{R}^N)$. Then, for any indices $\gamma, \alpha = 1, \ldots, N$, by the Claims 38, 39 applied to the scalar function $(\Sigma^\gamma u)_\alpha \in C^2(\bar{\Omega}) \cap C^1_0(\Omega)$, we have the estimate

\[
\int_{\Omega} \left| \Xi^\gamma D^2(\Sigma^\gamma u)_\alpha \right|^2 \leq \int_{\Omega} \left| A^{(\epsilon)\gamma} : D^2(\Sigma^\gamma u)_\alpha \right|^2,
\]

where we have used that $A^{(\epsilon)\gamma} = A^\gamma + \epsilon I$ (by the definition of $A^{(\epsilon)}$) and we have employed the normalisation of (4.14) which forces

\[
\lambda_{\bar{a}}^\gamma = \nu(A^\gamma) = 1.
\]

By summing in $\alpha, \gamma$, the above estimate and (4.15) give

\[
(4.16) \quad \int_{\Omega} \left| \Xi D^2u \right|^2 = \int_{\Omega} \sum_{\gamma} \left| \Sigma^\gamma \otimes \Xi^\gamma : D^2u \right|^2 \leq \int_{\Omega} \sum_{\gamma} \left| \Sigma^\gamma (D^2u : A^{(\epsilon)\gamma}) \right|^2.
\]
For brevity, we set
\[ \xi^{(c)\gamma} := \sum_{\gamma=1}^{N} (D^2 u : A^{(c)\gamma}) , \quad \gamma = 1, \ldots, N. \]

Then, (4.16) says
\[ (4.17) \quad \int_{\Omega} |\Xi D^2 u| \leq \int_{\Omega} \sum_{\gamma=1}^{N} |\xi^{(c)\gamma}|^2. \]

By the definition of \( A^{(c)} \), we have that \( B^{(c)\gamma} \perp B^{(c)\delta} \) for \( \gamma \neq \delta \) in \{0, 1, \ldots, N\}. By using this fact, we calculate
\[ |A^{(c)} : D^2 u|^2 = \left( \sum_{\gamma=0}^{N} B^{(c)\gamma} (D^2 u : A^{(c)\gamma}) \right) \cdot \left( \sum_{\delta=0}^{N} B^{(c)\delta} (D^2 u : A^{(c)\delta}) \right) \]
\[ = \sum_{\gamma=0}^{N} \left( B^{(c)\gamma} (D^2 u : A^{(c)\gamma}) \right) \cdot \left( B^{(c)\gamma} (D^2 u : A^{(c)\gamma}) \right) \]
\[ = |B^{(c)0} (D^2 u : A^{(c)0})|^2 + \sum_{\gamma=1}^{N} |B^{(c)\gamma} (D^2 u : A^{(c)\gamma})|^2 \]
\[ \geq \sum_{\gamma=1}^{N} \left| B^{\gamma} (D^2 u : A^{(c)\gamma}) \right|^2. \]

Hence, (4.18) gives (by using that \( B^{\gamma} = B^{\gamma} \Sigma^{\gamma} \))
\[ |A^{(c)} : D^2 u|^2 \geq \sum_{\gamma=1}^{N} |B^{\gamma} \xi^{(c)\gamma}|^2 \]
\[ = \sum_{\gamma=1}^{N} \max_{|\eta|=1} \left( B^{\gamma} : \xi^{(c)\gamma} \otimes \eta \right)^2 \]
\[ \geq \sum_{\gamma=1}^{N} \left( B^{\gamma} : \text{sgn}(\xi^{(c)\gamma}) \otimes \text{sgn}(\xi^{(c)\gamma}) \right)^2 |\xi^{(c)\gamma}|^2 \]
\[ \geq \left( \min_{\delta=1, \ldots, N} \left\{ B^{\delta} : \text{sgn}(\xi^{(c)\delta}) \otimes \text{sgn}(\xi^{(c)\delta}) \right\} \right)^2 \sum_{\gamma=1}^{N} |\xi^{(c)\gamma}|^2 \]
and as a result we obtain
\[ (4.19) \quad |A^{(c)} : D^2 u|^2 \geq \left( \min_{\delta=1, \ldots, N} \min_{|\eta|=1, \eta \in \Sigma^{\delta}} \left\{ B^{\delta} : \eta \otimes \eta \right\} \right)^2 \sum_{\gamma=1}^{N} |\xi^{(c)\gamma}|^2. \]

By using the Claim 41 (and also the normalisation condition (4.14)), (4.19) gives
\[ (4.20) \quad \int_{\Omega} |A^{(c)} : D^2 u|^2 \geq \nu^2 \int_{\Omega} \sum_{\delta=1}^{N} |\xi^{(c)\delta}|^2. \]

Hence, by (4.20) and (4.17) we obtain the desired estimate for smooth \( u \), the general case following by a standard density argument in the Sobolev norm. We complete the proof by showing that the subspace \( \Xi \subseteq \mathbb{R}^{Nn^2} \) satisfies
\[ \Xi \supseteq N \left( A : \mathbb{R}^{Nn^2} \to \mathbb{R}^N \right)^\perp. \]
Indeed, let \(X \perp \Xi\). Then, by (4.15) we have that \(X\) is normal to \(\Sigma^\gamma \otimes H^\gamma\) for any \(\gamma = 1, \ldots, N\), where we have denote \(H^\gamma := T^\gamma \vee T^\gamma\). Hence the projection of \(X\) on \(\Sigma^\gamma \otimes H^\gamma\) vanishes: \((\Sigma^\gamma \otimes H^\gamma)X = 0\). By Claim 38 and Remark 37 we have that \(A^\gamma : X = A^\gamma : (H^\gamma X)\) for any \(X \in \mathbb{R}^{n^2}\). Hence, we get that \(B^\gamma(X : A^\gamma) = 0\) for all \(\gamma\) and by summing in \(\gamma\) we obtain \(A : X = 0\). Thus, we have shown that

\[
\Xi^\perp \subseteq N\left( A : \mathbb{R}^{Nn^2}_+ \rightarrow \mathbb{R}^N \right),
\]

as desired. The theorem has been established.

4.4. The fibre Sobolev spaces and their properties. This subsection has been written independently of the rest of the paper and is a minor extension of standard result in Sobolev spaces (see e.g. [EG, AF, E]) which we describe in some detail for the sake of completeness. In these Sobolev spaces we obtain compactness of \(\mathcal{D}\)-solutions and consist of partially differentiable maps which possess directional derivatives only along certain (rank-one) projections, corresponding to the “direction of ellipticity” of the PDE system. For simplicity, we treat only the \(L^2\) case which is needed in this paper.

Given \(A \in \mathbb{R}^{Nn \times Nn}\) positive, let \(\Sigma, \Pi, \Xi\) be given by (4.3) and we suppose for the rest of this subsection that \(\Pi\) is spanned by rank-one directions. A sufficient condition regarding when this happens is in the case \(A\) satisfies Definition 33. Let \(\Omega \subseteq \mathbb{R}^n\) be open and consider the space \(W^{2,2}(\Omega, \mathbb{R}^N)\), which we identify with its isometric image into a product of \(L^2\) spaces:

\[
W^{2,2}(\Omega, \mathbb{R}^N) \subseteq L^2(\Omega, \mathbb{R}^N) \times L^2(\Omega, \mathbb{R}^{Nn}) \times L^2(\Omega, \mathbb{R}^{Nn^2}).
\]

We define the fibre Sobolev space \(\mathcal{F}^{2,2}(\Omega, \Sigma)\) as the orthogonal projection on \(L^2(\Omega, \Sigma)\times L^2(\Omega, \Pi) \times L^2(\Omega, \Xi)\) of the closure of \(W^{2,2}(\Omega, \mathbb{R}^N)\) with respect to the seminorm

\[
\|u\|_{\mathcal{F}^{2,2}(\Omega, \Sigma)} := \|\Sigma u\|_{L^2(\Omega)} + \|\Pi Du\|_{L^2(\Omega)} + \|\Xi D^2 u\|_{L^2(\Omega)}.
\]

That is, \(\mathcal{F}^{2,2}(\Omega, \Sigma)\) is the Hilbert space

\[
\mathcal{F}^{2,2}(\Omega, \Sigma) := \text{Proj}_{L^2(\Omega, \Sigma) \times L^2(\Omega, \Pi) \times L^2(\Omega, \Xi)} W^{2,2}(\Omega, \mathbb{R}^N)_{\|\cdot\|_{\mathcal{F}^{2,2}(\Omega, \Sigma)}}.
\]

By utilising the Mazur theorem, \(\mathcal{F}^{2,2}(\Omega, \Sigma)\) can also be written as

\[
\mathcal{F}^{2,2}(\Omega, \Sigma) = \left\{ (u, G(u), H(u)) \mid \exists (u^m) \in W^{2,2}(\Omega, \mathbb{R}^N) : \Sigma u^m \rightharpoonup u, \Pi Du^m \rightharpoonup G(u), \Xi D^2 u^m \rightharpoonup H(u), \text{ in } L^2 \right\}.
\]

We will call \(G(u) \in L^2(\Omega, \Pi)\) the fibre gradient of \(u\) and \(H(u) \in L^2(\Omega, \Xi)\) the fibre hessian of \(u\) and it can be easily seen (by using integration by parts and that \(\Sigma, \Pi, \Xi\) are spanned by directions of the form \(\eta, \eta \otimes a, \eta \otimes a \vee b\) respectively) that these measurable mappings depend only on \(u \in L^2(\Omega, \Sigma)\) and not on the approximating sequence. Further, by using the standard properties of equivalence between strong and weak \(L^2\) directional derivatives, we have that \(G(u), H(u)\) can be characterised as “fibre” derivatives of \(u\): for any directions \(\eta \in \Sigma, \eta \otimes a \in \Pi\) and \(\eta \otimes a \vee b \in \Xi\), we have

\[
G(u) : (\eta \otimes a) = D_a(\eta \cdot u), \quad H(u) : (\eta \otimes a \vee b) = D^2_{ab}(\eta \cdot u),
\]
a.e. on $\Omega$, where $D_a$, $D_a^2$ are the usual directional derivative operators. Similarly, we may define

$$F^{1,2}_0(\Omega, \Sigma) := \text{Proj}_{L^2(\Omega, \Sigma)} \cdot L^2(\Omega, \Pi) W^{1,2}(\Omega, \mathbb{R}^N) \| \cdot \|_{F^{1,2}(\Omega, \Sigma)},$$

and

$$\|u\|_{F^{1,2}(\Omega, \Sigma)} := \|\Sigma u\|_{L^2(\Omega)} + \|\Pi D u\|_{L^2(\Omega)}.$$

The next result provides a Poincaré type inequality in $F^{1,2}_0(\Omega, \Sigma)$.

**Lemma 42** (Poincaré inequality). Let $n, N \geq 1$, $\Omega \subseteq \mathbb{R}^n$ a bounded domain and $a \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$ with $|\eta| = |a| = 1$. Then, for any $u \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, we have the estimate

$$\|\eta \cdot u\|_{L^2(\Omega)} \leq \text{diam}(\Omega) \|D_a(\eta \cdot u)\|_{L^2(\Omega)}.$$

**Proof of Lemma 42.** We set $a^\perp := I - a \otimes a$ and also (see Figure 3)

$$\Omega^a := \{ y \in a^\perp | \exists t \in \mathbb{R} : y + ta \in \Omega \}, \quad I^{y,a} := \{ t \in \mathbb{R} | y + ta \in \Omega \}.$$ For any $v \in C^1_0(\Omega)$ and $y + ta \in \Omega$, in the standard way we have

$$|v(y + ta)|^2 \leq |I^{y,a}| \int_{I^{y,a}} |Dv(y + \lambda a) \cdot a|^2 d\lambda$$

which by integration first over $t \in I^{y,a}$, then over $y \in \Omega^a$ and also by application of Fubini theorem gives

$$\int_{\Omega} |v(x)|^2 dx \leq \int_{\Omega^a} \left( |I^{y,a}| \int_{I^{y,a}} |Dv(y + \lambda a) \cdot a|^2 d\lambda \right) d\mathcal{H}^{n-1}(y) \leq \text{diam}(\Omega)^2 \int_{\Omega} |Dv(x) \cdot a|^2 dx.$$

The lemma follows by applying the above to $v = \eta \cdot u$ for $u \in C^1_0(\Omega, \mathbb{R}^N)$.

**Remark 43** (Interpolation & norm equivalence). Let $n, N \geq 1$, $\Omega \subseteq \mathbb{R}^n$ a bounded domain and $a \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$ with $|\eta| = |a| = 1$. Then, for any function $u \in (W^{1,2}_0 \cap W^{2,2}_0)(\Omega, \mathbb{R}^N)$ and $\varepsilon > 0$, we have the interpolation inequality

$$\|Du : \eta \otimes a\|_{L^2(\Omega)} \leq \varepsilon \|\eta \cdot u\|_{L^2(\Omega)} + \frac{4}{\varepsilon} \|D^2 u : \eta \otimes a \otimes a\|_{L^2(\Omega)}.$$
Moreover, by Lemma 42 and by employing the above estimate and standard arguments we obtain that an equivalent norm on the space $\mathcal{F}^{1,2}(\Omega, \Sigma)$ is

$$
\|\Xi D^2 u\|_{L^2(\Omega)} \approx \| u\|_{\mathcal{F}^{2,2}(\Omega)}.
$$

The next result shows that maps in $\mathcal{F}^{1,2}(\Omega, \Sigma)$ have well-defined trace values, at least for strictly convex domains.

**Lemma 44** (Trace operator). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded strictly convex domain, that is $\Omega$ is convex and $\partial \Omega$ contains no nontrivial straight line segments. Fix also $a \in \mathbb{R}^n$, $|a| = 1$. Then, there exists a closed set $E \subseteq \partial \Omega$ with $H^{n-1}(E) = 0$ such that, for any $\Gamma \in \partial \Omega \setminus E$ compactly contained, there is a $C(n, \Gamma) > 0$ such that

$$
\|v\|_{L^2(\Gamma, H^{n-1})} \leq C(n, \Gamma) \left( \|v\|_{L^2(\Omega)} + \|Da v\|_{L^2(\Omega)} \right),
$$

for any $v \in C^1(\overline{\Omega})$. Moreover, the set $E$ is given by

$$
E = \left\{ x \in \partial \Omega \mid (x + \text{span}[a]) \cap \partial \Omega = \{x\} \right\}.
$$

Hence, there exists a trace operator $T : \mathcal{F}^{1,2}(\Omega, \mathbb{R}^N) \rightarrow L^2_{\text{loc}}(\partial \Omega \setminus E, H^{n-1}, \mathbb{R}^N)$. Actually, in the lemma above, it suffices $\Omega$ to be convex and $\partial \Omega$ to contain no straight line segments parallel to the vector $a$.

**Proof of Lemma 44.** Let $E \subseteq \partial \Omega$ be the set defined in the statement and fix $v \in C^1(\overline{\Omega})$. By our assumption on $\Omega$, it easily follows that $E$ is closed and $H^{n-1}(E) = 0$. We cover $\partial \Omega \setminus E$ by countably many open cubes $\{Q_j\}_{j=1}^{\infty}$ with sides orientated parallel to $\{a, a^\perp\}$ (see Figure 4). For each $Q_j$, we set $\Omega_j := Q_j \cap \Omega$, $\Gamma_j := Q_j \cap \partial \Omega$ and we fix a triple $(Q_j, \Omega_j, \Gamma_j)$. We may assume that $a$ is directed towards the interior of $\Omega_j$ (otherwise we replace it by $-a$). We further consider the restrictions of $v$ and $Dv$ on $\Omega_j$, that is we set $v_j := v_{\chi_{\Omega_j}}$, $(Dv)_j := Dv_{\chi_{\Omega_j}}$.

![Figure 4](image-url)

Then, for any $x \in \Gamma_j$, in the standard way we have

$$
|v(x)|^2 \leq C \left( \int_0^\infty |v_j(x + ta)|^2 dt + \int_0^\infty |(Dv)_j(x + ta) \cdot a|^2 dt \right).
$$

Let now $F_j \in W^{1,\infty}(\mathbb{R}^n)$ be a Lipschitz function such that, when restricted on $\Omega_j$, for any $t \geq 0$ its level set $\{F_j = t\}$ coincides with the translate of the boundary $\partial \Omega + ta$. Such an $F_j$ is given for $z \in \Omega_j$ by $F_j(z) := \sup\{t > 0 : z \in (\Omega + ta) \cap \Omega_j\}$ and can be extended on $\mathbb{R}^n \setminus \Omega_j$ by the classical McShane-Whitney formulas.
by the co-area formula (see e.g. [EG]), integration over \( x \in \Gamma_j \) and application of Fubini theorem give

\[
\int_{\Gamma_j} |v|^2 \, d\mathcal{H}^{n-1} \leq C \int_{\mathbb{R}} \int_{\Gamma_j + t\mathbb{a}} \left( |v_j(y)|^2 + |(Dv)_j(y) \cdot \mathbb{a}|^2 \right) \, d\mathcal{H}^{n-1}(y) \, dt
\]

\[
= C \int_{\mathbb{R}} \int_{\{F_j = t\}} \left( |v_j(y)|^2 + |(Dv)_j(y) \cdot \mathbb{a}|^2 \right) \, d\mathcal{H}^{n-1}(y) \, dt
\]

\[
= C \int_{\mathbb{R}^n} |DF_j(x)||\left( |v_j(x)|^2 + |(Dv)_j(x) \cdot \mathbb{a}|^2 \right) \, dx
\]

\[
\leq C\|DF_j\|_{L^\infty(\mathbb{R}^n)} \int_{\Omega_j} \left( |v(x)|^2 + |Dv(x) \cdot \mathbb{a}|^2 \right) \, dx.
\]

The lemma follows by employing a standard argument of partitions of unity. \( \square \)

4.5. **Proof of the main result.** Now we may finally establish our second main result by utilizing the machinery of the previous two subsections.

**Proof of Theorem 35.** The first step is to prove existence of a map in the fibre space \((\mathcal{F}^{2,2} \cap \mathcal{F}^{1,2}_0)(\Omega, \Sigma)\) solving in a certain sense the linear problem.

**Claim 45.** In the setting of Theorem 35 and under the same assumptions, for any \( f \in L^2(\Omega, \Sigma) \), there exists \( u \in (\mathcal{F}^{2,2} \cap \mathcal{F}^{1,2}_0)(\Omega, \Sigma) \) such that

\[
\begin{align*}
\{ A : H(u) &= f, \ L^n \text{-a.e. on } \Omega, \\
u &= 0, \ \mathcal{H}^{n-1} \text{-a.e. on } \partial \Omega,
\end{align*}
\]

where \( H(u) \) is the fibre hessian of \( u \).

**Proof of Claim 45.** The proof is based on the approximation by strictly elliptic systems and relies on the stable degenerate hessian estimate of Theorem 36. Let \( A^{(\varepsilon)} \) be the strictly elliptic approximation of \( A \) appearing in Theorem 36 and consider for a fixed \( f \in L^2(\Omega, \Sigma) \) the problem

\[
\begin{align*}
\{ A^{(\varepsilon)} : D^2u^{\varepsilon} &= f, \ L^n \text{-a.e. on } \Omega, \\
u^{\varepsilon} &= 0, \ \mathcal{H}^{n-1} \text{-a.e. on } \partial \Omega.
\end{align*}
\]

By standard lower semicontinuity and regularity results (see e.g. [D, GM]), the problem has for any \( \varepsilon > 0 \) a unique strong a.e. solution \( u^{\varepsilon} \in (W^{2,2} \cap W_0^{1,2})(\Omega, \mathbb{R}^N) \). By Theorem 36 and Remark 43, we have the uniform estimate

\[
\|\Sigma u^{\varepsilon}\|_{L^2(\Omega)} + \|\Pi Du^{\varepsilon}\|_{L^2(\Omega)} + \|\Xi D^2u^{\varepsilon}\|_{L^2(\Omega)} \leq \frac{C}{\nu} \|f\|_{L^2(\Omega)}
\]

for some universal \( C > 0 \). By the properties of the fibre space \((\mathcal{F}^{2,2} \cap \mathcal{F}^{1,2}_0)(\Omega, \Sigma)\) and the Poincaré inequality of Lemma 42, we have that there exists \( u \) in this space such that \( \Sigma u^{\varepsilon} \rightharpoonup u, \Pi Du^{\varepsilon} \rightharpoonup G(u) \) and \( \Xi D^2u^{\varepsilon} \rightharpoonup H(u) \), along a sequence \( \varepsilon_k \to 0 \), all convergences in \( L^2 \). Now we pass to the weak limit in the equations. By the form of the approximation \( A^{(\varepsilon)} \) and Definition 33, we have

\[
\sum_{\gamma=1}^N B^{(\varepsilon)\gamma}(D^2u^{\varepsilon} : A^{(\varepsilon)\gamma}) = f - B^{(\varepsilon)0}(D^2u^{\varepsilon} : A^{(\varepsilon)0}),
\]
a.e. on $\Omega$. By using that $B^{(\kappa)} = B^{\kappa}$ for $\kappa = 1, \ldots, N$ and that $B^{(\kappa)} \perp B^{1} + \cdots + B^{N}$, we may project the system above on $\Sigma$ which is the range of $B^{1} + \cdots + B^{N}$. Then, since $\Sigma f = f$ and $A^{(\kappa)} = A^{\kappa} + \varepsilon I$, we obtain
\[
\sum_{\kappa=1}^{N} B^{\kappa} \left( D^2 u^{\varepsilon} : A^{\kappa} + \varepsilon \Delta u^{\varepsilon} \right) = f,
\]
a.e. on $\Omega$. Moreover, we have that $\Xi \supseteq N( A : \mathbb{R}^{Nn^2} \to \mathbb{R}^{N} ) \perp$ and by applying Remark 37 we deduce
\[
A : (\Xi D^2 u^{\varepsilon}) - f = -\varepsilon \sum_{\kappa=1}^{N} B^{\kappa} \Delta (\Sigma u^{\varepsilon}),
\]
a.e. on $\Omega$. Then, for any $\phi \in C_{c}^{\infty}(\Omega, \mathbb{R}^{N})$, integration by parts gives
\[
\int_{\Omega} \left( A : (\Xi D^2 u^{\varepsilon}) - f \right) \cdot \phi = -\varepsilon \int_{\Omega} \sum_{\kappa=1}^{N} B^{\kappa} (\Sigma u^{\varepsilon}) : \Delta \phi.
\]
By letting $\varepsilon = \varepsilon_{k} \to 0$, we obtain $A : H(u) = f$, a.e. on $\Omega$. Finally, application of Lemma 44 shows that $u$ is $H^{n-1}(\partial \Omega)$-measurable and $u = 0$ $H^{n-1}$-a.e. on $\partial \Omega$. \hfill $\square$

An essential ingredient in order to pass from the linear to the non-linear problem is the next theorem of Campanato taken from [C3] (see also [K7]) which we recall together with its proof, since the original proof is imprecise about representatives and does not quote the Axiom of Choice which is tacitly used.

**Claim 46** (Campanato’s surjectivity of near operators). Let $\mathcal{X} \neq \emptyset$ be a set and $(\mathcal{X}, \| \cdot \|)$ a Banach space. Let also $\mathcal{F}, \mathcal{A} : \mathcal{X} \to \mathcal{X}$ be two mappings and suppose there is a $K \in (0, 1)$ such that
\[
\| \mathcal{F}(u) - \mathcal{F}(v) - (\mathcal{A}(u) - \mathcal{A}(v)) \| \leq K \| \mathcal{A}(u) - \mathcal{A}(v) \|
\]
for all $u, v \in \mathcal{X}$. Then, if $\mathcal{A}$ is surjective, $\mathcal{F}$ is surjective as well.

**Proof of Claim 46.** Consider the equivalence relation on $\mathcal{X}$ given by $u \sim v$ iff $\mathcal{A}[u] \sim \mathcal{A}[v]$. For each equivalence class $[u] \in \mathcal{X}/\sim$ in the quotient space, we select a representative $\bar{u} \in [u]$ (by the Axiom of Choice) and we define
\[
\mathcal{A} : \mathcal{X}/\sim \to \mathcal{X}, \quad \mathcal{A}([u]) := \mathcal{A}(\bar{u}), \quad \mathcal{F}([u]) := \mathcal{F}(\bar{u}).
\]
Then, since $\mathcal{A}$ is surjective, $\mathcal{A} : \mathcal{X}/\sim \to \mathcal{X}$ is bijective. We now turn $\mathcal{X}/\sim$ into a complete metric space, by defining the metric $d([u], [v]) := \| \mathcal{A}([u]) - \mathcal{A}([v]) \|$. Then, for any fixed $f \in \mathcal{X}$ we define the map $P : \mathcal{X}/\sim \to \mathcal{X}/\sim$ given by
\[
P([u]) := \mathcal{A}^{-1} \left( \mathcal{A}([u]) - (\mathcal{F}([u]) - f) \right).
\]
Then, it can be easily seen by our assumption that $P$ is a contraction on $(\mathcal{X}/\sim, d)$ and hence $P$ has a unique fixed point $[u]$ such that $P([u]) = [u]$. The latter implies that for any $f \in \mathcal{X}$, there is $\bar{u} \in \mathcal{X}$ such that $\mathcal{F}(\bar{u}) = f$. Hence, $\mathcal{F}$ is surjective. \hfill $\square$

Now we employ Claim 46 in order to show existence of a map in the fibre space $(\mathcal{F}^{2,2} \cap \mathcal{F}^{1,2})_{\Omega}(\Sigma)$ solving in a certain sense the nonlinear problem.
Claim 47. In the setting of Theorem 35 and under the same assumptions, for any $f \in L^2(\Omega, \Sigma)$, there exists $u \in (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma)$ such that
\[
\begin{cases}
F(\cdot, H(u)) = f, & L^n\text{-a.e. on } \Omega, \\
u = 0, & \mathcal{H}^{n-1}\text{-a.e. on } \partial \Omega,
\end{cases}
\]
where $H(u)$ is the fibre hessian of $u$.

Proof of Claim 47. For any fixed $u \in (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma)$, we have that $A : H(u) \in L^2(\Omega, \Sigma)$ because $H(u) \in L^2(\Omega, \Sigma)$ and also $A : X \in \Xi \subseteq \mathbb{R}^N$ for any $X \in \Xi \subseteq \mathbb{R}_+^n$. Moreover, by Definition 30, for $X = 0$ and $Z = H(u)$, we have
\[
|F(\cdot, H(u))| \leq \left( \frac{(C + 1)|A| + B \nu \text{ ess inf}_{x \in \Omega}[A(x)]}{H(u)} + |F(\cdot, 0)| \right) \text{ a.e. on } \Omega.
\]
Hence, $F(\cdot, H(u)) \in L^2(\Omega, \Sigma)$ as well. The previous considerations imply that the maps
\[
\mathcal{A} : (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma) \to L^2(\Omega, \Sigma), \quad \mathcal{A}(u) := A : H(u),
\]
\[
\mathcal{F} : (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma) \to L^2(\Omega, \Sigma), \quad \mathcal{F}(u) := F(\cdot, H(u)),
\]
are well defined. By Claim 45, $\mathcal{A}$ is surjective. We complete the claim by showing that $\mathcal{F}$ is near $\mathcal{A}$ in the sense of Claim 46 and then the surjectivity of $\mathcal{F}$ will conclude the proof. For any $u, v \in (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma)$, by Definition 30 and Theorem 36 (with $\varepsilon = 0$) we have
\[
\left\| A(\cdot) \left( F(\cdot, H(u)) - F(\cdot, H(v)) \right) - A : (H(u) - H(v)) \right\|_{L^2(\Omega)} \\
\leq B \nu \| H(u) - H(v) \|_{L^2(\Omega)} + C \| A : (H(u) - H(v)) \|_{L^2(\Omega)} \\
\leq (B + C) \left\| A : (H(u) - H(v)) \right\|_{L^2(\Omega)}.
\]
Hence, $\mathcal{F}(u) := A(\cdot) F(\cdot, H(u))$ is surjective and since $A, 1/A \in L^\infty(\Omega)$, the same is true for $\mathcal{F}$. The claim ensues. \hfill $\Box$

Now we may complete the proof of Theorem 35. By (4.3) and (4.13) we have that there is an orthonormal frame $\{ \eta^1, ..., \eta^N \} \subseteq \mathbb{R}^N$ and for each $\alpha = 1, ..., N$ there is a frame $\{ \xi^{(\alpha)1}, ..., \xi^{(\alpha)n} \} \subseteq \mathbb{R}^n$ such that each of the mutually orthogonal subspaces $\Sigma^\gamma \subseteq \mathbb{R}^N$ is spanned by a subset of vectors $\eta^\alpha$ and for the same index $\gamma, \gamma^\gamma$ is spanned by $\{ \xi^{(\alpha)1}, ..., \xi^{(\alpha)n} \}$ which is a set of eigenvectors of $A^\gamma$. By (4.3) and (4.15) there are also induced orthonormal frames of $\mathbb{R}^{Nn}$ and $\mathbb{R}_+^{n^2}$ consisting of vectors of the form
\[
E^{\alpha i} := \eta^\alpha \otimes \xi^{(\alpha)i}, \quad E^{\alpha ij} := \eta^\alpha \otimes \xi^{(\alpha)i} \vee \xi^{(\alpha)j}
\]
respectively. These frames are such that a subset of the vectors $E^{\alpha ij}$ spans the subspace $\Xi \subseteq \mathbb{R}_+^{n^2}$ and the rest are orthogonal to $\Xi$. Given these frames, we define the adaptive infinitesimal frame $s = (s^{(\alpha)i})_{\alpha = 1}^\infty$ (see Definitions 5, 8, Remark 7) given by
\[
s^{(\alpha)i}(x) := h_m \xi^{(\alpha)i},
\]
where $(h_m)_{\alpha}$ is any constant infinitesimal sequence. Let $u \in (F^{2,2} \cap F_0^{1,2})(\Omega, \Sigma)$ be the map which satisfies $F(\cdot, H(u)) = f$ a.e. on $\Omega$, provided by Claim 47. By our assumptions and application of Lemma 44, we have that $u$ is $\mathcal{H}^{n-1}\cup \partial U$-measurable.
on the boundary of any strictly convex subdomain $U \subseteq \bar{\Omega}$ and also satisfies the boundary condition $H^{n-1}$-a.e. on $\partial \Omega$. By the characterisation of the fibre hessian $H(u) \in L^2(\Omega, \mathbb{E})$ in terms of directional derivatives of projections, we have

$$H(u) = \sum_{\alpha,i,j} \left( H(u) : E^{\alpha ij} \right) E^{\alpha ij}$$

a.e. on $\Omega$, where the projections of $H(u)$ along the orthonormal directions $E^{\alpha ij}$ are non-zero only for the subset of vector $E^{\alpha ij}$ spanning the subspace $\mathbb{E}$. Hence,

$$H(u) = \lim_{m', m'' \to \infty} \sum_{\alpha,i,j : E^{\alpha ij} \in \mathbb{E}} \left( \chi_{\Omega^{m''}} D^2 h_{\alpha(i)\xi(ij)}(\eta^\alpha \cdot u) \right) E^{\alpha ij},$$

a.e. on $\Omega$. Since $F$ is a Carathéodory map, we have

$$\lim_{m', m'' \to \infty} F \left( x, \sum_{\alpha,i,j : E^{\alpha ij} \in \mathbb{E}} \left[ \chi_{\Omega^{m''}} D^2 h_{\alpha(i)\xi(ij)}(\eta^\alpha \cdot u) \right] (x) E^{\alpha ij} \right) = f(x),$$

for a.e. $x \in \Omega$. In view of Remark 32, the latter equality is equivalent to

$$\lim_{m', m'' \to \infty} F \left( x, \sum_{\alpha,i,j} \left[ \chi_{\Omega^{m''}} D^2 h_{\alpha(i)\xi(ij)}(\eta^\alpha \cdot u) \right] (x) E^{\alpha ij} \right) - f(x) = 0,$$

for a.e. $x \in \Omega$. Let $D^2 h_{\alpha(i)\xi(ij)}$ denote the adaptive difference quotients of $u$ with respect to $\alpha$. We set

$$f^{m', m''}(x) := F \left( x, D^2 h_{\alpha(i)\xi(ij)}(x) \right) - f(x)$$

and rewrite the previous equality as

$$f^{m', m''}(x) \to 0, \quad \text{for a.e. } x \in \Omega, \quad \text{as } m', m'' \to \infty.$$

We define

$$F^{m', m''}(x, X) := F(x, X) - f(x) - f^{m', m''}(x),$$

$$F^\infty(x, X) := F(x, X) - f(x).$$

Then, by the above we infer that for a.e. $x \in \Omega$, $F^{m', m''}(x, \cdot) \to F^\infty(x, \cdot)$ uniformly in $X \in \mathbb{R}^N_+$ and also

$$\int_{\mathbb{R}^N_+} \Phi(X) F^{m', m''}(x, X) d[\delta_{D^2 h_{\alpha(i)\xi(ij)}(x)}](X) = 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c(\mathbb{R}^N_+)$. Moreover, by the weak* compactness of the space of Young measures, we have

$$\delta_{D^2 h_{\alpha(i)\xi(ij)}} \rightharpoonup^* D^2 u, \quad \text{in } \mathcal{M}(\Omega, \mathbb{R}^N_+),$$

perhaps along further subsequences as $m', m'' \to \infty$. We now conclude by applying the Convergence Lemma 20: indeed, we obtain

$$\int_{\mathbb{R}^N_+} \Phi(X) F^\infty(x, X) d[D^2 u(x)](X) = 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c(\mathbb{R}^N_+)$. The proof of the theorem is completed.

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