Planck scale inflationary spectra from quantum gravity

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We derive the semiclassical evolution of massless minimally coupled scalar matter in the de Sitter space-time from the Born-Oppenheimer reduction of the Wheeler-DeWitt equation. We show that the dynamics of trans-Planckian modes can be cast in the form of an effective modified dispersion relation and that high energy corrections in the power spectrum of the cosmic microwave background radiation produced during inflation remain very small if the initial state is the Bunch-Davies vacuum.

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I. INTRODUCTION

It is commonly accepted that inflation [1] is a viable paradigm for the early Universe which solves some of the problems of the standard big-bang scenario and allows one to make testable predictions about the spectrum of the cosmic microwave background radiation (CMBR). However, it has also been realized that inflation provides a window towards trans-Planckian physics [2] as it magnifies all quantum fluctuations and red-shifts originally trans-Planckian frequencies down to the range of low energy physics currently observed. This has raised two related issues: a) how to describe the original quantum fluctuations in such a high energy regime, given that there is currently no universally accepted theory of quantum gravity, and b) whether the effect of now red-shifted trans-Planckian frequencies can be observed with the precision of present and future experiments.

One can take the pragmatic view of the renormalization group approach to high energy physics and use quantum field theory on a classical metric background to describe quantum fluctuations after their frequencies have been red-shifted below the scale of quantum gravity. However, the higher energy dynamics cannot be a priori neglected as it could enter (at least) in the form of initial conditions for the quantum state of the field theory and, in turn, affect the CMBR spectrum (to an extent which is currently being debated; see, e.g., Refs. [3, 4, 5, 6, 7]).

As a consequence of the above observations, one expects that the dispersion relations of matter fields change on approaching the Planckian regime, so that the frequency \( \omega \) of massless modes will depend on their wavenumber \( k \) according to an expression of the general form

\[
\omega = \frac{k}{a} \left[ 1 + f(k, \ell_p/a, \dot{a}, \ldots) \right],
\]

in which \( \ell_p \) is the Planck length and \( a = a(t) \) the scale factor of the flat Robertson-Walker metric \( g \),

\[
ds^2 = -N^2 dt^2 + a^2 (d\vec{x})^2.
\]

Most attempts have tested the effects of functions \( f \) chosen ad hoc in Eq. (1). We shall instead derive the dispersion relation for a minimally coupled massless scalar field from the action principle previously employed to study the semiclassical dynamics of non-minimally coupled scalar fields in Ref. [2].

In that paper, the principle of time-reparameterization invariance was lifted to a quantum symmetry. We then obtained an Hamiltonian constraint from which the Born-Oppenheimer (BO) reduction allowed to properly and unambiguously recover the semiclassical limit of quantum field theory on a curved background starting from the Wheeler-DeWitt (WDW) equation. This required that certain “quantum fluctuation” terms be negligible with respect to the usual matter contributions. Precisely such terms in the matter equation will now be studied and their effect on the power spectrum derived for the simplest model of inflation.

In the next Section, we briefly introduce the model and its classical and Wheeler-DeWitt dynamics in the BO decomposition (for more details, see Ref. [3]). In Section IV we then focus on the quantum equation for matter which we solve perturbatively in order to determine a dispersion relation and the corresponding CMBR spectrum. Finally, we comment on our findings in Section V. We shall use units with \( c = \hbar = 1 \).

II. SCALAR FIELD IN DE SITTER SPACE

As the simplest model of inflation [10], we shall consider the de Sitter metric [2] with

\[
a = a_0 \exp(\mathcal{H} t)
\]

and cosmological constant \( \Lambda = 3 \mathcal{H}^2 \). The massless minimally coupled scalar field \( \phi \) will be treated as a perturbation with respect to \( \Lambda \) in the interval \( t_i \leq t \leq t_f \), where the relevant modes \( k \) were first produced at \( t = t_i \) and

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inflation ends at \( t = t_f \). Upon varying the corresponding action (\( \tilde{f} \equiv \partial_t f \)),

\[
S = -\frac{1}{2\ell_p^2} \int_{t_i}^{t_f} a^3 N \, dt \left( \frac{\dot{a}^2}{N^2 a^2} + \frac{\Lambda}{3} \right) + \frac{1}{2} \int_{t_i}^{t_f} a^3 N \, dt \int d^3 x \left[ \frac{\dot{\phi}^2}{N^2} - \frac{(\partial \phi)^2}{a^2} \right],
\]

one obtains the Euler-Lagrange equations of motion in the form of the Hamiltonian constraint

\[
\delta_N S \equiv -H = 0,
\]

and the dynamical equations

\[
\delta_k S = \delta_\phi S = 0.
\]

The former allows one to work in the gauge \( N(t) = 1 \), provided the initial conditions are such that \( H(t_i) = 0 \), and Eqs. (5a) will then evolve the initial data consistently, since the Hamiltonian constraint (5a) is then preserved,

\[
\dot{H} = -\dot{a} \delta_a S - \dot{\phi} \delta_\phi S = 0.
\]

However, since

\[
H(t) = \delta_\phi S = 0 \Rightarrow \delta_a S = 0,
\]

for our purposes it is more convenient to impose the Hamiltonian constraint together with the Klein-Gordon equation at all times.

### A. WDW equation

At the quantum level, Eq. (5a) becomes the WDW constraint which reads

\[
H |\Psi\rangle = \left( \hat{H}_G + \sum_k \hat{H}_k \right) |\Psi\rangle = 0,
\]

where the gravitational Hamiltonian is

\[
\hat{H}_G = \frac{1}{2} \left( \ell_p^2 \frac{\partial^2}{a^2} + \frac{a^3 \Lambda}{3 \ell_p^2} \right),
\]

and the matter Hamiltonian is given by the sum of the contributions for each mode \( \phi_k \) of the scalar field,

\[
\hat{H}_k = \frac{1}{2} \left( \frac{\pi_k^2}{\dot{a}^2} + \ddot{a} k^2 \dot{\phi}_k^2 \right),
\]

with \([\dot{\phi}_k, \ddot{\phi}_k]\) = \( i \delta_{k,k'} \). This equation is rather involved, but the following treatment allows us to obtain more tractable equations.

### B. BO decomposition

We can decompose the wavefunction of the Universe according to the BO prescription

\[
|\Psi(a, \phi)\rangle = \psi(a) |\chi(a, \phi)\rangle \equiv \psi \prod_k |\chi_k\rangle,
\]

where \( |\chi_k\rangle \) is the wavefunction for the mode \( k \). Eq. (5) can then be shown to be equivalent to a system of decoupled Schwinger-Tomonaga equations (one for each mode) and a Friedmann-like equation expressing the back-reaction of matter on the expansion of the Universe. In fact, starting from the ansatz one can reduce Eq. (5) to the two coupled equations

\[
\left( \frac{\ell_p^2}{2} D_+^2 + \frac{a^4 \Lambda}{6 \ell_p^2} + a \langle \hat{H}_\phi \rangle \right) \psi = -\frac{\ell_p^2}{2} \langle D_+^2 \rangle \psi
\]

\[
\frac{\ell_p^2}{a} \left( D_- \psi \right) D_+ |\chi\rangle + \psi \left( \hat{H}_\phi - \langle \hat{H}_\phi \rangle \right) |\chi\rangle = -\frac{\ell_p^2}{2a} \psi \left( D_+^2 - \langle D_+^2 \rangle \right) |\chi\rangle,
\]

where \( \langle \hat{O} \rangle \equiv \langle \chi | \hat{O} | \chi \rangle \) and the action of the “covariant” derivatives \( D_\pm \equiv \partial_\pm \pm \langle \partial_\pm \rangle \) on \( \psi \) and \( |\chi\rangle \) reduces to that of \( \partial_\pm \) after rescaling by a geometric phase \[3\].

\[
|\chi\rangle \rightarrow |\tilde{\chi}\rangle \equiv e^{\pm i \int \langle \partial_{\pm} \rangle \, dt} |\chi\rangle
\]

\[
\psi \rightarrow \tilde{\psi} \equiv e^{-i \int \langle \partial_{\pm} \rangle \, dt} \psi.
\]

Eq. (12b) is still rather involved since it contains the full matter wavefunction. Upon projecting onto \( \prod_{n \neq k} |\tilde{\chi}_n\rangle \), one can obtain independent equations for each mode,

\[
\frac{\ell_p^2}{a} \left( \partial_a \tilde{\psi} \right) \partial_a \langle \tilde{\chi}_k \rangle = \left( \hat{H}_k - \langle \hat{H}_k \rangle \right) \langle \tilde{\chi}_k \rangle
\]

\[
= -\frac{\ell_p^2}{2a} \left( \partial_a^2 - \langle \partial_a^2 \rangle \right) \langle \tilde{\chi}_k \rangle,
\]

where \( \langle \hat{O}_k \rangle \equiv \langle \tilde{\chi}_k | \hat{O} | \tilde{\chi}_k \rangle \) and the terms in the right hand side represent the kind of “quantum gravitational fluctuations” already mentioned in the Introduction. Finally, one can also write the gravitational Eq. (12a) in terms of the solutions to the matter Eq. (12b) as

\[
\left( \frac{\ell_p^2}{2a} \partial_a^2 + \frac{a^3 \Lambda}{6 \ell_p^2} + \sum_k \langle \hat{H}_k \rangle \right) \tilde{\psi} = -\frac{\ell_p^2}{2a} \sum_k \langle \partial_a^2 \rangle \langle \tilde{\chi}_k \rangle \tilde{\psi}.
\]

Again, “quantum gravitational fluctuations” appear in the right hand side which will affect the way matter back-reacts on the evolution of the scale factor. However, we shall not consider the latter equation any further in the present paper.
### III. MATTER EQUATION

Our aim is to solve Eq. (14a) for modes with wavelengths $a/k \lesssim \ell_p \ll \mathcal{H}^{-1}$ at the time $t \sim t_1$ and evolve them to a later time $t = t_1$ when $a/k \gg \mathcal{H}^{-1}$ in order to determine the power spectrum. The choice of the state $\langle \text{vac} \rangle$ is very important and will be discussed later. First we take the semiclassical limit for gravity which allows us to introduce the time as $t \sim \mathcal{H}^2 \ell_p^2 (\partial_a \ln \psi) \partial_a \sim i a \partial_t$.

In the infrared sector, $a/k \gg \ell_p$, the right hand side (RHS) of Eq. (14a) can be discarded, since

$$\text{RHS} \sim (\ell_p^2 / a^2) (\partial_a^2 \tilde{\chi}_{k}) \ell_p^2 (k^2 / a^3) (\tilde{\chi}_{k})$$

is suppressed by a factor of order $k \ell_p^2 / a^2 \ll 1$ with respect to $\langle \tilde{H}_k \rangle \sim k/a$. Eq. (14a) therefore coincides with the Schwinger-Tomonaga equation for $\langle \chi_k \rangle$ for the same reason, however, one expects that the RHS of Eq. (14a) significantly affects the scalar field dynamics in the ultraviolet sector $a/k \lesssim \ell_p$. If we define

$$\langle \chi_s \rangle = e^{-i \int (H_{s,k}) dt'} |\tilde{\chi}_k\rangle,$$

Eq. (14a) then becomes

$$\left( 1 - \frac{3 i \delta^2}{2 a^3 \mathcal{H}^3} \right) \left(i \partial_t - \tilde{H}_k \right) |\chi_s\rangle = \frac{\delta^2}{2 a^3 \mathcal{H}^3} \Delta |\chi_s\rangle,$$

where $\delta \equiv \mathcal{H} \ell_p$ and $\Delta = \sum_1^3 (\tilde{O}_i - (\chi_s \otimes \tilde{O}_i) \chi_s)$ with

$$\tilde{O}_1 = 2 \mathcal{H}^{-1} (\chi_s \otimes \tilde{H}_k) i \partial_t$$
$$\tilde{O}_2 = 3 i \tilde{H}_k$$
$$\tilde{O}_3 = \mathcal{H}^{-1} \partial_t^2.$$

Eq. (18) is not linear and the superposition principle no longer holds. However, we note that in a (semi)classical Universe the dimensionless parameter $\delta^2 \ll 1$, and we can therefore identify two regimes: I) in the very early stages $a^3 \mathcal{H}^3 \lesssim \delta^2$ matter evolves according to

$$\left(i \partial_t - \tilde{H}_k \right) |\chi_s\rangle \simeq i \frac{3}{\delta^2} \sum_1^3 (\Delta_i \chi_s) \equiv \tilde{W}_1 |\chi_s\rangle; \quad (20a)$$

II) after $a$ has become sufficiently large ($a \gg \delta^3 / \mathcal{H}^{-1}$), Eq. (18) can be expanded to leading order in $\delta$ as

$$\left(i \partial_t - \tilde{H}_k \right) |\chi_s\rangle \simeq \frac{\delta^2}{2 a^3 \mathcal{H}^3} \sum_1^3 (\Delta_i \chi_s) \equiv \delta^2 \tilde{W}_1 |\chi_s\rangle; \quad (20b)$$

These expressions represent our main qualitative result: Eq. (20a) shows that, at late stages of the cosmological evolution, corrections coming from $\tilde{W}_1$ are of order $\delta^2 \ll 1$ and thus very small; further, although the RHS of Eq. (14a) seems to produce corrections of order $\delta^0 \sim 1$ in the very early stages, we shall see that the effect of $\tilde{W}_1$ is actually negligible in Eq. (20a).

### A. Perturbative analysis

On neglecting $\tilde{W}$ ($\equiv \tilde{W}_1$ or $\tilde{W}_{11}$), the matter equations (20a) and (20b) simply read

$$i \partial_t |\chi_s\rangle = \tilde{H}_k |\chi_s\rangle = \frac{k}{a} \left( a^\dagger a + \frac{1}{2} \right) |\chi_s\rangle,$$

where

$$\tilde{a} \equiv \sqrt{\frac{k a^2}{2}} \left( \hat{\phi}_k + i \frac{\hat{p}_k}{k a^2} \right),$$

and $\hat{a}^\dagger$ and $\hat{a}$ are the usual annihilation and creation operators such that $[\hat{a}, \hat{a}^\dagger] = 1$ and can be used to evaluate the energy of the matter state.

Because of the time dependent $a = a(t)$, Hamiltonian eigenstates defined by $\hat{a}^\dagger \hat{a} |n_E\rangle = n |n_E\rangle$ do not satisfy Eq. (21). A basis of exact solutions of the time-dependent problem is instead given by eigenstates of the invariant number operator $\hat{b}^\dagger \hat{b} |n\rangle = n |n\rangle$, where

$$\hat{b} \equiv \frac{1}{\sqrt{2}} \left[ \hat{\phi}_k \rho + i \left( \rho \hat{\pi}_k - a^3 \hat{a} \hat{\phi}_k \right) \right],$$

and $\rho = \rho(t)$ must satisfy

$$\ddot{\rho} + 3 \mathcal{H} \dot{\rho} + \frac{k^2}{a^2} \rho = \frac{1}{a^2 \rho}.$$ 

One then has $[\hat{b}, \hat{b}^\dagger] = 1$ and exact solutions are given by superpositions of the base vectors

$$|n\rangle = \frac{e^{i n \theta}}{n!} (\hat{b}^\dagger)^n |0\rangle,$$

with $\hat{b} |0\rangle = 0$ and the phase

$$\Theta(t) = \int_0^t \frac{dt'}{a^3(t')} \rho^2(t').$$

Note that the invariant operators $\hat{b}$ and $\hat{b}^\dagger$ (as well as the states $|n\rangle$) are a mathematical tool to determine the solutions and do not in general have a physical meaning. They are however related to $\hat{a}$ and $\hat{a}^\dagger$ by the Bogoliubov transformation

$$\hat{a} = B^* \hat{b} + A^* \hat{b}^\dagger$$
$$\hat{b} = \frac{1}{2} \left[ a \sqrt{k} \rho + \frac{1}{a \sqrt{k} \rho} + i \frac{a^2 \rho}{\sqrt{k}} \right] \hat{b}$$
$$+ \frac{1}{2} \left[ a \sqrt{k} \rho - \frac{1}{a \sqrt{k} \rho} + i \frac{a^2 \rho}{\sqrt{k}} \right] \hat{b}^\dagger,$$

which also relates Hamiltonian eigenstates to exact solutions. In terms of such operators, one finds

$$\tilde{W} |n\rangle = \left[ \alpha \hat{b}^2 - \alpha^* \left( \hat{b}^\dagger \right)^2 + \beta \hat{b}^4 + \beta^* \left( \hat{b}^\dagger \right)^4 \right] |n\rangle.$$
where terms proportional to $\beta$ will be omitted from now on since they do not affect the final power spectrum [to order $\delta^2$, see Eq. (14)], and

$$\alpha_{11} = \frac{3\alpha_1}{2i a^3 \mathcal{H}^3} = \frac{k^2}{a^3 \mathcal{H}^4} \left\{ 2A B^* \left[ \left( |A|^2 + \frac{1}{2} \right) + \frac{i \mathcal{H}}{k/a} \right] - 2 \frac{i \mathcal{H}}{k/a} \left( (B^*)^2 + A^2 \right) \right\} \quad (28b)$$

Since $\alpha_{11}/(\dot{\mathcal{H}}_k) \sim a \mathcal{H}/k \lesssim |\delta|/k \ll 1$, it appears that $\dot{W}_1$ is actually negligible with respect to $\dot{\mathcal{H}}_k$. In the regime I, one can therefore choose suitable initial conditions $\rho(t_i)$ and $\dot{\rho}(t_i)$ for which

$$A(t_i) = 0 \quad \text{and} \quad B(t_i) = 1 , \quad (29)$$

so that $|n_E \rangle = |n \rangle$ and $|\chi_s \rangle = |n \rangle$ at $t = t_i$. This state will not change significantly until it enters the regime II and starts to be acted upon by $\dot{W}_1$. We can then assume a perturbative expansion of the form

$$|\chi_s \rangle = |n_s \rangle \simeq |n \rangle + \delta^2 |n^{(1)} \rangle = (\hat{1} + \delta^2 \hat{R}_n) |n \rangle . \quad (30)$$

From this it then follows that

$$(i \partial_t \hat{R}_n - [\hat{H}_k, \hat{R}_n]) |n \rangle = \left[ \alpha \hat{b}^2 - \alpha^* \left( \hat{b}^\dagger \right)^2 \right] |n \rangle , \quad (31)$$

which admits the solution

$$\hat{R}_n |n \rangle = \left[ r \hat{b}^2 + r^* \left( \hat{b}^\dagger \right)^2 \right] |n \rangle , \quad (32a)$$

provided $r$ satisfies the differential equation

$$i \dot{r} + 2 \Theta \dot{r} = \alpha . \quad (32b)$$

Although $\hat{R}_n$ contains Hermitian parts, matter evolution is unitary in general due to the form of Eq. (18) and in the case at hand one can in fact show that $\langle \chi_s | \chi_s \rangle = 1$ to order $\delta^2$ at all times for the states (30).

### B. Dispersion relations

Two quantum states $|\chi_s \rangle$ and $|\bar{\chi}_s \rangle$ are physically indistinguishable when they share the same expectation values for all the measurable observables $\hat{X}$,

$$\langle \chi_s | \hat{X} | \chi_s \rangle = \langle \bar{\chi}_s | \hat{X} | \bar{\chi}_s \rangle . \quad (33)$$

In this sense, since we are interested in observable quantities, such as the power-spectrum, quadratic in $\phi$ and $\pi_\phi$, we cannot distinguish between $|n_s \rangle$ and the state

$$|\bar{n}_s \rangle = (\hat{1} + i \delta^2 \hat{H}_n) |n \rangle , \quad (34a)$$

where

$$\hat{H}_n = i \frac{n^2 + n + 1}{2n + 1} \left[ r \hat{b}^2 - r^* \left( \hat{b}^\dagger \right)^2 \right] , \quad (34b)$$

which evolves in an explicitly unitary way [12].

One can for instance consider the invariant vacuum, $|n_s \rangle = |0 \rangle$ at $t = t_i$, corresponding to an initial state devoid of particles (of wavenumber $k$). The Hamiltonian associated with its effective evolution (34a) is given by

$$\hat{H}_{\text{eff}} = \hat{H}_k - \delta^2 \left[ \alpha \hat{b}^2 + \alpha^* \left( \hat{b}^\dagger \right)^2 \right]$$

$$= \frac{1}{2} \left[ \frac{\hat{a}_k^2}{\mu} + \mu \omega^2 \hat{\phi}^2_k \right] + \gamma \left( \hat{\phi}_k \hat{\pi}_k + \hat{\pi}_k \hat{\phi}_k \right) , \quad (35)$$

where $\omega$ is the effective frequency and, for $\ell_p^2/(a^3 \mathcal{H}) \ll 1$ and $a \mathcal{H}/k \ll 1$, the effective mass is

$$\mu \simeq \frac{a^3}{2} \left[ 1 - \frac{\ell_p^2}{a^3 \mathcal{H}} \left( \frac{a}{k} \right) \right] \quad (36)$$

and

$$\gamma \simeq - \frac{\ell_p^2}{a^3} \left( \frac{k}{a \mathcal{H}} \right) . \quad (37)$$

Apart from the squeezing term proportional to $\gamma$, the perturbed dynamics can be described by a modified dispersion relation (11). In fact, Eq. (24) with the initial conditions (29) at $t_i \to -\infty$ admits the solution

$$\rho(t) = \frac{1}{a k^{1/2}} \sqrt{1 + \left( \frac{a \mathcal{H}}{k} \right)^2} , \quad (38)$$

from which one obtains

$$\omega \simeq \frac{k}{a} \left[ 1 + \frac{321 \mathcal{H}^2 \ell_p^5}{64 a^3} \left( \frac{a}{k \ell_p} \right)^3 \right] , \quad (39)$$

valid for trans-Planckian modes with $a/k \ell_p \lesssim 1$. The correction inside the square brackets is proportional to the factor $(\ell_p/a)^3$. In a classical Robertson-Walker Universe there is no fundamental length scale and one would not expect any dependence on the numerical value of $a = a(t)$, since only its ratio at two different times has a physical meaning. However, in the quantum theory $\ell_p$ plays the role of the fundamental length which breaks scale invariance and in terms of which all quantities with length dimension must be measured. In fact, “quantum gravitational fluctuations” in the right hand side of the matter Eq. (34a) are precisely proportional to $(\ell_p/a)^3$ and $(\mathcal{H} \ell_p)^2$.

### C. CMBR spectrum

We can also compute the power spectrum

$$\mathcal{P}_\phi(k) \equiv k^3 \langle \text{vac} | \hat{\phi}_k^2 | \text{vac} \rangle , \quad (40)$$

and compare to the well-known case $\delta = 0$. The result will in general depend on the initial quantum state.
we choose at $t = t_1$ (see, e.g., Ref. [3]). As we did for the dispersion relation, we choose the adiabatic invariant vacuum
\[ |\text{vac}⟩ = |0_s⟩ , \]
which initially coincides with the Bunch-Davies vacuum [17] and is known to give a flat, scale invariant spectrum for modes well outside the horizon, $a/k \gg \mathcal{H}^{-1}$. Adding the leading order corrections from Eq. (28a) yields
\[ \mathcal{P}_\phi(k) \simeq k^3 \left( \langle 0_s | \dot{\phi}_k^2 | 0_s⟩ + 2 \delta^2 \Re \langle 0_s | \dot{\phi}_k \dot{R} | 0_s⟩ \right) , \] where
\[ \langle 0_s | \dot{\phi}_k \dot{R} | 0_s⟩ = \frac{r^*}{k a^2} \left[ (B^*)^2 + A^2 + 2 B^* A \right] , \]
and $r = r(t)$ is a solution of Eq. (32b). We again employ Eq. (38) for trans-Planckian modes with $k \gtrsim a(t)/\ell_p$ to evaluate the expression [12] at $t = t_1$ such that $a(t_1) \gg k/\mathcal{H}$, and obtain
\[ \mathcal{P}_\phi(k) \simeq \frac{3}{8} \mathcal{H}^2 \left[ 1 + \delta^2 \left( u + \frac{v}{k^3} \right) \right] , \]
where $u$ is an integration constant related to the initial conditions for $r = r(t)$ and $v \simeq 1.22$. The zero order is precisely the expected flat spectrum, and $k$-dependent deviations are of order $\delta^2$, as we had already anticipated from the general form of Eq. (28a).

### IV. CONCLUSIONS

We have derived an effective modified dispersion relation [38] for the trans-Planckian modes of a minimally coupled massless scalar field in the de Sitter space-time from the action principle [14] and the semiclassical treatment of the WDW equation [8, 10]. The effects induced by such modified dynamics on the power spectrum [14] during inflation appear as corrections of order $\delta^2 = (\mathcal{H} \ell_p)^2$ and thus quite difficult to observe. However, in order to obtain the expression in Eq. (44), we used the condition that the initial state be the Bunch-Davies vacuum [11]. Different results would follow from other choices, such as the one considered in Ref. [3] which produces corrections of order $\mathcal{H} \ell_p$ and thus possibly observable [3, 10], and the non-linearity of the matter equation [12] may then lead to more non-trivial effects. Our result (44) can thus be considered as a lower bound for the size of quantum gravitational corrections.

With regard to the two main issues mentioned in the Introduction, we can therefore say that our work has shown: a) that the semiclassical approach to the WDW equation of Refs. [10, 14] can consistently describe the evolution of trans-Planckian matter modes starting from the very early stages and b) that the initial quantum matter state can be more relevant than the corrections produced during its subsequent semiclassical evolution (see also Ref. [3] in this respect). It appears then that initial conditions other than [11] could only be justified by appealing to new physics beyond the semiclassical level of quantum gravity. Quite remarkably, earlier computations in the low energy limit of some string and M-theories also yielded corrections of order $(\mathcal{H} \ell_p)^2$ for the CMBR spectrum [3], thus in agreement with our (lower bound) estimate.

Let us end by mentioning that our present conclusions hold for minimal coupling between matter and gravity and may change in more general cases [15]. Moreover, the backreaction of the quantum fluctuations in the gravitational equation [14], which was presently neglected, is also worth analyzing.

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