TOEPLITZ OPERATORS ON THE FOCK SPACE VIA THE FOURIER TRANSFORM

SHENGKUN WU¹ AND DECHAO ZHENG²

Abstract. Inspired by Berger-Coburn theorems and their conjecture in [5], we use the Fourier transform to decompose $T_g$ as an infinite sum of Toeplitz operators with symbols which have compact support in the frequency domain. As a consequence, we obtain a sufficient condition for $T_g$ to be bounded in terms of the Carleson measure conditions defined by the heat transform of the symbol $g$. Moreover the decomposition of a Toeplitz operator leads us to get easily understanding that for a bounded function $g$, if its Berezin transform vanishes at infinity, then the Toeplitz operator $T_g$ is compact [10] and the Toeplitz algebra generated by Toeplitz operators with symbols in $L^\infty$ is indeed generated by Toeplitz operators with symbols which on uniformly continuous are $C^\infty[2]$. For any positive parameter $\alpha$, let
\[
d\lambda_{\alpha}(x) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2}d\lambda(x)\]
denote the Gaussian measure on $\mathbb{C}^n$. Let $L^2(\mathbb{C}^n, d\lambda_{\alpha})$ be the Hilbert space consisting of all square integrable functions with respect to the Gaussian measure. The Fock space $F^2_n$ is a subspace of $L^2(\mathbb{C}^n, d\lambda_{\alpha})$ which consists all holomorphic functions $f$ on $\mathbb{C}^n$. Let $P$ be the projection from $L^2(\mathbb{C}^n, d\lambda_{\alpha})$ onto the Fock space. The inner product of two functions $f, g \in F^2_n$ is given by
\[
\langle f, g \rangle = \int_{\mathbb{C}^n} f\overline{g} d\lambda_{\alpha}.
\]
For $z, w \in \mathbb{C}^n$, on one hand the inner product $\langle z, w \rangle$ is defined by
\[
\langle z, w \rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}
\]
where $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$. On the other hand, $z$ and $w$ can be viewed as two vectors in $\mathbb{R}^{2n}$. We denote the inner product of $z$ and $w$ in $\mathbb{R}^{2n}$ as
\[
z \cdot w = \Re z \cdot \Re w + \Im z \cdot \Im w = \Re z_1\Re w_1 + \cdots + \Re z_n\Re w_n + \Im z_1\Im w_1 + \cdots + \Im z_n\Im w_n
\]
where
\[
\Re z = (\Re z_1, \Re z_2, \ldots, \Re z_n), \quad \text{and} \quad \Im z = (\Im z_1, \Im z_2, \ldots, \Im z_n),
\]
$\Re z_i$ and $\Im z_i$ denote the real part and the imagery part of the complex number $z_i$ respectively. Thus
\[
z \cdot w = \Re \langle z, w \rangle.
\]
For a multi-index $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^n_+$, the order $|\sigma|$ of the multi-index $\sigma$ is defined by
\[
|\sigma| = \sigma_1 + \cdots + \sigma_n.
\]
The monomial $z^\sigma$ is defined by
\[
z^\sigma = z_1^{\sigma_1}z_2^{\sigma_2} \cdots z_n^{\sigma_n}.
\]
Similarly, we define the differential operators $\partial^\sigma_{\Re z}$ and $\partial^\sigma_{\Im z}$ by
\[
\partial^\sigma_{\Re z} = \partial^{\sigma_1}_{\Re z_1} \cdots \partial^{\sigma_n}_{\Re z_n} \quad \text{and} \quad \partial^\sigma_{\Im z} = \partial^{\sigma_1}_{\Im z_1} \cdots \partial^{\sigma_n}_{\Im z_n}.
\]

Key words and phrases. Toeplitz operators, Fock space, boundedness.

The first author is supported by CSC201906050022. This work is partially supported by NFSC.
The Fock space is a reproducing kernel Hilbert space and for any \( w, z \in \mathbb{C}^n \) with the reproducing kernel \( K_w(z) \) given by

\[
K_w(z) = e^{\alpha(z,w)}.
\]

The normalized reproducing kernel \( k_w(z) \) is given by

\[
k_w(z) = e^{\alpha(z,w) - \frac{1}{2}|w|^2}.
\]

Since the linear expansion of \( \{k_w\} \) is dense in the Fock space, for a measurable function \( g \) on \( \mathbb{C}^n \), if for any \( w \in \mathbb{C}^n \) we have \( g k_w \in L^2(\mathbb{C}^n, d\lambda_n) \), then we define the Toeplitz operator \( T_g \) with symbol \( g \) on the linear combinations of \( \{k_w\} \) by

\[
T_g \sum_{j=1}^{n} c_j k_{w_j} = P \left( g \sum_{j=1}^{n} c_j k_{w_j} \right).
\]

A natural problem is what necessary and sufficient conditions are for \( T_g \) to be bounded. In [5], Berger and Coburn use some trace formula to obtain some necessary conditions for \( T_g \) to be bounded and use the Bargmann transform and Calderon-Vaillancourt pseudo-differential estimates to obtain some sufficient conditions for \( T_g \) to be bounded in terms of the heat transform of the symbol \( g \). These conditions lead them to make conjecture on the problem [5].

Inspired by Berger-Coburn theorems and their conjecture in [5], we use the Fourier transform to decompose \( T_g \) as an infinite sum of Toeplitz operators with symbols which have compact support in the frequency domain.

To state our results precisely we need to introduce notations. For any positive parameter \( t \), the heat transform of \( g \) is defined by

\[
\mathcal{H}_t g(z) = \left( \frac{1}{t\pi} \right)^n \int_{\mathbb{C}^n} g(w) e^{-\frac{|z-w|^2}{t}} dv(w), \quad \forall z \in \mathbb{C}^n.
\]

In fact, if \( t = \frac{1}{\alpha} \), a simple calculation gives that the heat transform of \( g \) is equal to the Berezin transform of the Toeplitz operator \( T_g \):

\[
\mathcal{H}_t g(z) = \langle T_g k_z, k_z \rangle.
\]

For two functions \( f \) and \( g \) in \( L^2(\mathbb{C}^n, dv) \), the convolution \( f \ast g \) of \( f \) and \( g \) is defined by

\[
f \ast g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)dv(w)
\]

for any \( z \in \mathbb{C}^n \).

For any \( y \in \mathbb{C}^n \), \( \tau_y \) is a translation operator:

\[
\tau_y f(x) = f(x - y).
\]

For a Lebesgue integrable function \( f \) on \( \mathbb{C}^n \), the Fourier transform \( \mathcal{F}f \) of \( f \) is defined by

\[
\mathcal{F}f(z) = \int_{\mathbb{R}^{2n}} f(w)e^{-2\pi i z \cdot w}dv(w).
\]

Let \( \gamma_t(z) \) be \( 2n \)-dimension heat kernel \( \left( \frac{1}{t\pi} \right)^n e^{-\frac{|z|^2}{t}} \) and let \( a_t(z) \) be the Gaussian function \( e^{-\pi^2 |z|^2} \).

Then the heat transform of \( g \) is given by

\[
\mathcal{H}_t g(z) = g \ast \gamma_t(z)
\]

and

\[
\mathcal{F}^{-1}(a_t(z)) = \mathcal{F}(\gamma_t(z)) = a_t(z)
\]

as the Gaussian function \( e^{-\pi^2 |z|^2} \) is fixed by the Fourier transform.

In the case of \( \alpha = \frac{1}{2} \), in [5], Berger and Coburn proved that if \( \mathcal{H}_t g \) is bounded for some \( t \in (0, \frac{1}{2\alpha}) \), then \( T_g \) is bounded. In fact, they showed that if

\[
\partial_{\bar{y}} \partial_t \mathcal{H}_{\frac{1}{2\alpha}} g(y)
\]
is bounded for any $a,b \in \mathbb{Z}^n$ with $|a| + |b| \leq 2n + 1$, then $T_g$ is bounded. In this paper, we will improve the above conditions replacing the boundedness by the Carleson measure condition on the Fock space.

Let $d\mu$ be a positive measure on $\mathbb{C}^n$. For any $r > 0$ and $x \in \mathbb{C}^n$, let $B(x,r)$ denote the ball in $\mathbb{C}^n$ with center $x$ and radius $r$. If there is a positive constant $C_r$ such that

$$ \sup_{x \in \mathbb{C}^n} \mu(B(x,r)) < C_r, $$

then we say that $d\mu$ is a Carleson measure on the Fock space. It is shown in [19, Theorem 3.29] that $d\mu$ is a Carleson measure on the Fock space if and only if

$$ \sup_{w \in \mathbb{C}^n} \int_{\mathbb{C}^n} e^{-\frac{\pi}{2}|z-w|^2} d\mu(z) < \infty, $$

for some $0 < p < \infty$. Thus, for a nonnegative function $f$ on $\mathbb{C}^n$, $\mathcal{H}_w(f) \in L^\infty$ if and only if $f(y)dv(y)$ is a Carleson measure on the Fock space. Moreover, for any $r > 0$, there is a $C_r$ such that

$$ \frac{1}{C_r} \|\mathcal{H}_w(f)\|_\infty \leq \sup_{x \in B(x,r)} f(y)dv(y) \leq C_r \|\mathcal{H}_w(f)\|_\infty. \quad (1.1) $$

We will obtain a sufficient condition for a Toeplitz operator to be bounded and show that how the heat transformation of the symbol of a Toeplitz operator is related to the operator itself. In fact, a Toeplitz operator $T_g$ can be represented by the weighted integral of "translates" of $T_{H_{\frac{1}{2}x_0 + y}}$. To do so, we introduce the partition of unity.

Let $Q_0 = \{x + iy : x,y \in (-\frac{1}{2}, \frac{1}{2})^n\}$ be a rectangle in $\mathbb{C}^n$ and $x_0 = (0, \ldots, 0)$ be the center of $Q_0$. Let $\Gamma$ be the lattice $\mathbb{Z}^{2n}$, i.e.,

$$ \Gamma = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \Re z_1, \Im z_1, \ldots, \Re z_n, \Im z_n \in \mathbb{Z}\}. $$

For any $x \in \Gamma$, let $Q_x$ denote the rectangle with center $x$, which is a translate of $Q_0$, i.e.,

$$ Q_x = Q_0 + x. $$

Then the collection $\{Q_x : x \in \Gamma\}$ of rectangles tiles $\mathbb{C}^n$.

Let $\phi$ be a smooth function such that

$$ \phi(z) = \begin{cases} 1 & z \in Q_0 \\ 0 & z \notin 2Q_0. \end{cases} $$

where $2Q_0 = (-1,1)^{2n}$.

For any $x \in \Gamma$ and $z \in \mathbb{C}^n$, let $\phi_x(z) = \phi(z-x)$. Since

$$ \sum_{x \in \Gamma} \frac{1}{(1 + |\Re x| + |\Im x|)^{2n+1}} \leq \sum_{x \in \Gamma} \frac{1}{(1 + |x|)^{2n+1}} < \infty, \quad (1.2) $$

the series $\sum_{x \in \Gamma} \phi_x$ of smooth functions converges uniformly on compact subsets of $\mathbb{C}^n$ and satisfies

$$ \frac{1}{c} \leq |\sum_{x \in \Gamma} \phi_x(z)| \leq c $$

for $z$ in $\mathbb{C}^n$ and for some positive constant $c$. Letting

$$ \varphi_x = \frac{\phi_x}{\sum_{x \in \Gamma} \phi_x}, $$

we obtain

$$ \varphi_x(z + x) = \varphi_{x_0}(z), $$

and

$$ \sum_{x \in \Gamma} \varphi_x = 1. $$

The following theorem is our main result.
**Theorem 1.1.** Let \( g \) be a measurable function on \( \mathbb{C}^n \) such that \( gk_w \in L^2(\mathbb{C}^n, d\lambda_n) \) for any \( w \in \mathbb{C}^n \). For each \( x \) in \( \Gamma = \mathbb{Z}^{2n} \), let

\[
g_x = (\mathcal{H}_{\frac{\partial}{\partial x}} g) * \mathcal{F}(\varphi_x a^{-1}).
\]

If for any \( a, b \in \mathbb{Z}^n_+ \) with \( |a| + |b| \leq 2n + 1 \), \( |\partial^{a}_{\mathfrak{R}g} \partial^{b}_{\mathfrak{I}g} \mathcal{H}_{\frac{\partial}{\partial x}} g(y)| dv(y) \) is a Carleson measure on the Fock space, then the Toeplitz operator \( T_g \) is decomposed as a sum of Toeplitz operators with symbols which have compact support in the frequency domain:

\[
T_g = \sum_{x \in \Gamma} T_{g_x}
\]

and for any \( t \geq 0 \),

\[
T_{g_x} = \int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a^{-1}) (y) T_{\tau_y \mathcal{H}_{\frac{\partial}{\partial x}} + t} g dv(y),
\]

where the summation and the integral are both convergent in the operator norm topology and hence \( T_g \) is bounded. Moreover

\[
\|T_g\| \leq C_{n, \alpha} \sum_{|a| + |b| \leq 2n + 1} \|\mathcal{H}_{\frac{\partial}{\partial x}} ((\partial^{a}_{\mathfrak{R}g} \partial^{b}_{\mathfrak{I}g} \mathcal{H}_{\frac{\partial}{\partial x}} g(y)))\|_{\infty},
\]

where \( C_{n, \alpha} \) is a positive constant depending only on \( n \) and \( \alpha \).

**Remarks.** Let us make some remarks about the above theorem. First as

\[
g_x = (\mathcal{H}_{\frac{\partial}{\partial x}} g) * \mathcal{F}(\varphi_x a^{-1}),
\]

its Fourier inverse transform

\[
\mathcal{F}^{-1}(g_x)(z) = \mathcal{F}^{-1}(\mathcal{H}_{\frac{\partial}{\partial x}} g)(z) \varphi_x(z) a^{-1}(z)
\]

has support in \( 2Q_x = 2Q_0 + x \) and hence \( g_x \) has compact support in the frequency domain.

Since

\[
\sum_{|a| + |b| \leq 2n + 1} \|\mathcal{H}_{\frac{\partial}{\partial x}} ((\partial^{a}_{\mathfrak{R}g} \partial^{b}_{\mathfrak{I}g} \mathcal{H}_{\frac{\partial}{\partial x}} g(y)))\|_{\infty} \lesssim \sum_{|a| + |b| \leq 2n + 1} \|\partial^{a}_{\mathfrak{R}g} \partial^{b}_{\mathfrak{I}g} \mathcal{H}_{\frac{\partial}{\partial x}} g(y)\|_{\infty} \lesssim \|H_{\mathcal{F}} g\|_{\infty}, \quad (1.3)
\]

where \( t \in (0, \frac{1}{2\alpha}) \), the above theorem improves [19, Theorem 6.18].

The decomposition of a Toeplitz operator in the above theorem is useful for us to understand two results on Toeplitz operators and Toeplitz algebras in [2] and [10]. If \( \mathcal{A} \) is a set of functions that satisfies the condition in Theorem 1.1 let \( \mathfrak{T}(\mathcal{A}) \) denote the \( C^* \)-algebra generated by \( \{T_u : u \in \mathcal{A}\} \) and \( \mathfrak{T}^1(\mathcal{A}) \) denote the closed space which is generated by \( \{T_u : u \in \mathcal{A}\} \) in the norm topology.

**Corollary 1.2.** Let \( g \) be a measurable function on \( \mathbb{C}^n \) such that \( gk_w \in L^2(\mathbb{C}^n, d\lambda_n) \) for any \( w \in \mathbb{C}^n \). Suppose that for any \( a, b \in \mathbb{Z}^n_+ \) with \( |a| + |b| \leq 2n + 1 \), \( |\partial^{a}_{\mathfrak{R}g} \partial^{b}_{\mathfrak{I}g} \mathcal{H}_{\frac{\partial}{\partial x}} g(y)| dv(y) \) is a Carleson measure on the Fock space. If for a fixed \( t \geq 0 \), \( \tau_y \mathcal{H}_{\frac{\partial}{\partial x}} + t g \) is in \( \mathcal{A} \) for any \( y \in \mathbb{C}^n \), then \( T_g \in \mathfrak{T}^1(\mathcal{A}) \).

**Proof.** Suppose that for a fixed \( t \geq 0 \), \( \tau_y \mathcal{H}_{\frac{\partial}{\partial x}} + t g \) is in \( \mathcal{A} \) for any \( y \in \mathbb{C}^n \). Thus \( T_{\tau_y \mathcal{H}_{\frac{\partial}{\partial x}} + t} g \) is in \( \mathfrak{T}^1(\mathcal{A}) \). By Theorem 1.1 we have

\[
T_{g_x} = \int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a^{-1}) (y) T_{\tau_y \mathcal{H}_{\frac{\partial}{\partial x}} + t} g dv(y).
\]

Since the above integral converges to \( T_{g_x} \) in the operator norm topology, \( T_{g_x} \) is in \( \mathfrak{T}^1(\mathcal{A}) \). By Theorem 1.1 again, we have

\[
T_g = \sum_{x \in \Gamma} T_{g_x}
\]

to conclude that \( T_g \) is in \( \mathfrak{T}^1(\mathcal{A}) \).
Corollary 1.2 generalizes [13, Proposition 7.4.1] in the case \( p = 2 \) and immediately leads to the two results on the Toeplitz algebras and compact Toeplitz operators in [2] and [10].

Let \( C_{buc}(\mathbb{C}^n) \) be the set of bounded uniformly continuous function ( [2 page 1345]). In [2], on Toeplitz algebras, Bauer and Isralowitz showed that
\[
\mathcal{F}(L^\infty) = \mathcal{F}(C_{buc}(\mathbb{C}^n)).
\]

We will show how one gets easily
\[
\mathcal{F}^1(L^\infty) = \mathcal{F}^1(C_{buc}(\mathbb{C}^n)) \text{ and } \mathcal{F}(L^\infty) = \mathcal{F}(C_{buc}(\mathbb{C}^n)). \tag{1.4}
\]

To do so, we notice that for any \( g \in L^\infty \) and \( y, \tau_y \mathcal{H}_y g \) is in \( C_{buc}(\mathbb{C}^n) \). Thus \( T_g \) is in \( \mathcal{F}^1(C_{buc}(\mathbb{C}^n)) \), and so the above corollary gives (1.4). By [18], we have
\[
\mathcal{F}^1(L^\infty) = \mathcal{F}(L^\infty).
\]

Thus, we actually get
\[
\mathcal{F}^1(L^\infty) = \mathcal{F}^1(C_{buc}(\mathbb{C}^n)) = \mathcal{F}(L^\infty) = \mathcal{F}(C_{buc}(\mathbb{C}^n)).
\]

On compact Toeplitz operators, in [10], Engliš showed that for \( g \in L^\infty \), if the Berezin transform of \( T_g \) vanishes at infinity, that is,
\[
\lim_{|z| \to \infty} \mathcal{H}_y g(z) = 0,
\]
then the Toeplitz operator \( T_g \) is compact.

To get the above result, we observe that if for \( g \in L^\infty \),
\[
\lim_{|z| \to \infty} \mathcal{H}_y g(z) = 0,
\]
then for each \( y \) in \( \mathbb{C}^n \),
\[
\lim_{|z| \to \infty} \tau_y \mathcal{H}_y g(z) = \lim_{|z| \to \infty} \mathcal{H}_y g(z - y) = \lim_{|w| \to \infty} \mathcal{H}_y g(w) = 0.
\]

Let
\[
\mathcal{A} = \{ g \in L^\infty(\mathbb{C}^n) : \lim_{|z| \to \infty} g(z) = 0 \}.
\]

Then \( \tau_y \mathcal{H}_y g \) is in \( \mathcal{A} \) for any \( y \). So we have that \( T_g \in \mathcal{F}^1(\mathcal{A}) \). Noting that it is not difficult to show that \( \mathcal{F}^1(\mathcal{A}) \subset \mathcal{K} \), we conclude that \( T_g \) is compact.

This paper is organized as follows. In Section 2, using the partition of unity to decompose the symbol \( g \) of the Toeplitz operator \( T_g \), we will show that each part in the decomposition is bounded and has compact support in frequency domain. In Section 3, we will obtain the norm estimation of the Toeplitz operator with symbol equal to each part of the decomposition to establish a norm estimation of \( T_g \). In Section 4, we will apply our decomposition theory for a Toeplitz operator to estimate the Schatten \( p \)-norm of the product of two Toeplitz operators.

2. DECOMPOSITION

In this section, first we will establish some decomposition of a symbol \( g \) of a Toeplitz operator \( T_g \). Even if \( g \) is a measurable function on \( \mathbb{C}^n \) such that \( g k_w \in L^2(\mathbb{C}^n, d\lambda) \) for any \( w \in \mathbb{C}^n \), the Fourier transform \( \mathcal{F}(f) \) may be a tempered distribution. So we need to recall some facts on tempered distributions and the Fourier transform. Next using the decomposition of the symbol we will obtain a decomposition of the symbol \( g \) of the Toeplitz operator \( T_g \).

We now introduce the Schwartz space \( \mathcal{S}(\mathbb{R}^{2n}) \). A smooth complex-valued function \( f \) on \( \mathbb{R}^{2n} \) is called a Schwartz function if for every pair of multi-indices \( \alpha \) and \( \beta \) there exists a positive constant \( C_{\alpha,\beta} \) such that
\[
\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^{2n}} |\partial^\alpha(x^\beta f(x))| = C_{\alpha,\beta} < \infty.
\]

The set of all Schwartz functions on \( \mathbb{R}^{2n} \) is called the Schwartz space and denoted by \( \mathcal{S}(\mathbb{R}^{2n}) \). The Schwartz space \( \mathcal{S}(\mathbb{R}^{2n}) \) is a locally convex topological vector space equipped with the family of seminorms \( \rho_{\alpha,\beta} \).
Elements of the dual space $\mathcal{S}'(\mathbb{R}^{2n})$ of the Schwartz space are called tempered distributions. A function $g$ is said to be a tempered distribution if for any $f$ in the Schwartz space, the pair $(f, g)$ defined by

$$(f, g) = \int_{\mathbb{R}^{2n}} f(x)g(x)dx$$

gives a continuous linear functional on the Schwartz space.

Next, we need to recall some facts about the tempered distribution. If $f, h \in \mathcal{S}(\mathbb{R}^{2n})$ and $G \in \mathcal{S}'(\mathbb{R}^{2n})$. The Fourier transformation can be extend to the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ such that $\mathcal{F}(G)$ is a tempered distribution with

$$(\mathcal{F}(G), f) = (G, \mathcal{F}(f)).$$

$Gh$ is a tempered distribution given by

$$(Gh, f) = (G, hf),$$

and $G * h$ is a tempered distribution such that

$$(G * h, f) = (G, \hat{h} * f),$$

where $\hat{h}(x) = h(-x)$ for any $x \in \mathbb{R}^{2n}$. $\check{G}$ is also a tempered distribution such that

$$(\check{G}, h) = (G, \hat{h}).$$

For any $y \in \mathbb{R}^{2n}$, $\tau_y$ is a translation operator on $\mathcal{S}(\mathbb{R}^{2n})$ such that $\tau_y f(x) = f(x - y)$. $\tau_y$ can be extended on $\mathcal{S}'(\mathbb{R}^{2n})$ such that

$$(\tau_y G, f) = (G, \tau_y^{-1} f).$$

The following lemma says that a Carleson measure on the Fock space induces a tempered distribution.

**Lemma 2.1.** Let $h$ be a positive function on $\mathbb{C}^n$ such that $h(y)dv(y)$ is a Carleson measure on the Fock space, then $h$ is a tempered distribution. Moreover, if $f$ is a Schwartz function, then $h * f$ is in $L^\infty(\mathbb{C}^n)$.

*Proof.* Since $h(y)dv(y)$ is a Carleson measure on the Fock space, we have

$$\sup_{x \in \mathbb{C}^n} \int_{B(x, 2)} |h(y)|dv(y) = C_2 < \infty.$$ 

To show that $h$ is a tempered distribution, by [13, Proposition 2.3.4], it is sufficient to show

$$|\int_{\mathbb{C}^n} h(y)f(y)dv(y)| \leq C_3 \sup_y (1 + |\Re y| + |\Im y|)^{2n+1}|f(y)|$$

for all $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ and for some positive constant $C_3$. As we pointed out in the introduction that $\{Q_x\}_{x \in \Gamma}$ tiles $\mathbb{C}^n$, for any $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we have

$$|\int_{\mathbb{C}^n} h(y)f(y)dv(y)| = |\int_{\bigcup_{x \in \Gamma} Q_x} h(y)f(y)dv(y)|$$

$$\leq \sum_{x \in \Gamma} |\int_{Q_x} \frac{1}{(1 + |\Re y| + |\Im y|)^{2n+1}} h(y)(1 + |\Re y| + |\Im y|)^{2n+1} f(y)dv(y)|.$$ 

Since for any $y \in Q_x$ with $x \in \Gamma$, there is a positive constant $C$ such that

$$\frac{1}{(1 + |\Re y| + |\Im y|)^{2n+1}} \leq C \frac{1}{(1 + |\Re x| + |\Im x|)^{2n+1}}.$$
Thus
\[
\left| \int_{\mathbb{C}^n} h(y)f(y)dv(y) \right| \\
\leq \sum_{x \in \Gamma} \left| \int_{Q_x} \frac{1}{(1 + |\Re y| + |\Im y|)^{2n+1}} h(y)(1 + |\Re y| + |\Im y|)^{2n+1} f(y)dv(y) \right|
\]
\[
\leq \sum_{x \in \Gamma} C \left( \frac{1}{(1 + |\Re x| + |\Im x|)^{2n+1}} \right) \int_{Q_x} |h(y)(1 + |\Re y| + |\Im y|)^{2n+1} f(y)|dv(y)
\]
\[
\leq \sum_{x \in \Gamma} C \left( \frac{1}{(1 + |\Re x| + |\Im x|)^{2n+1}} \right) \int_{Q_x} h(y)dv(y) \sup_{y}(1 + |\Re y| + |\Im y|)^{2n+1} |f(y)|
\]
\[
\leq \sum_{x \in \Gamma} C \left( \frac{1}{(1 + |\Re x| + |\Im x|)^{2n+1}} \right) \int_{B(x,2)} h(y)dv(y) \sup_{y}(1 + |\Re y| + |\Im y|)^{2n+1} |f(y)|
\]
\[
\leq C_3 \sup_{y}(1 + |\Re y| + |\Im y|)^{2n+1} |f(y)|
\]
for some positive constant $C_3$. The last inequality follows from (1.2). Thus $h$ is a tempered distribution.

In fact, the above argument gives that for any $z \in \mathbb{C}^n$
\[
\left| \int_{\mathbb{C}^n} h(z-y)f(y)dv(y) \right| \leq C_3 \sup_{y}(1 + |\Re y| + |\Im y|)^{2n+1} |f(y)|.
\]
This implies that $h \ast f$ is in $L^\infty(\mathbb{C}^n)$.

\[\square\]

**Lemma 2.2.** Let $g$ be a measurable function on $\mathbb{C}^n$ such that $g_{\omega \in L^2(\mathbb{C}^n, d\lambda_\alpha)}$ for any $w \in \mathbb{C}^n$. If $|H_{2^{-n}}g(y)|dv(y)$ is a Carleson measure on the Fock space, then $(H_{2^{-n}}g) \ast F(\varphi \alpha^{-1}_{2_{-n}}) \in L^\infty(\mathbb{C}^n)$ for any $x \in \Gamma$ and
\[
H_{2^{-n}}g(z) = \sum_{x \in \Gamma} H_{2^{-n}} \left[ (H_{2^{-n}}g) \ast F(\varphi x^{-1}_{2^{-n-n}}) \right](z),
\]
for any $z \in \mathbb{C}^n$.

**Proof.** Since for each $x \in \Gamma$, $\varphi x^{-1}_{2^{-n-n}}$ is a smooth function with compact support, $\varphi x^{-1}_{2^{-n-n}}$ is in the Schwartz space. Thus $F(\varphi x^{-1}_{2^{-n-n}})$ is a Schwartz function as the Fourier transform $F$ maps the Schwartz space onto the Schwartz space. So Lemma 2.1 gives that
\[
(H_{2^{-n}}g) \ast F(\varphi x^{-1}_{2^{-n-n}}) \in L^\infty(\mathbb{C}^n)
\]
as $|H_{2^{-n}}g(y)|dv(y)$ is a Carleson measure on the Fock space.

Since $g_{\omega \in L^2(\mathbb{C}^n, d\lambda_\alpha)}$ for any $w \in \mathbb{C}^n$, we have
\[
H_{2^{-n}}g(z) = \langle g_{\omega}, k_z \rangle = g \ast \gamma_{2^{-n-n}}(z)
\]
is finite for any $z \in \mathbb{C}^n$. Then by the Fubini theorem, we have
\[
H_{2^{-n}}g = (g \ast \gamma_{2^{-n-n}}) \ast \gamma_{2^{-n}} = (H_{2^{-n}}g) \ast \gamma_{2^{-n}} = (H_{2^{-n}}g) \ast F(a_{2^{-n}}).
\]

By Lemma 2.1 $H_{2^{-n}}g$ is a tempered distribution, thus $\tau_y H_{2^{-n}}g$ is also a tempered distribution. So we have
\[
H_{2^{-n}}g(z) = \int_{\mathbb{C}^n} H_{2^{-n}}g(z-x)F(a_{2^{-n}})(x)dv(x)
\]
\[
= (\tau_y H_{2^{-n}}g, F(a_{2^{-n}}))
\]
\[
= (F(\tau_y H_{2^{-n}}g), a_{2^{-n}}).
\]
On the other hand, using properties of the convolution, the Fourier transform and the heat transform, we have

\[ \mathcal{H}_{\frac{1}{n}} \left[ (\mathcal{H}_{\frac{1}{n}} g) * \mathcal{F}(\varphi_x a_{\frac{1}{2n}}) \right](z) = (\mathcal{H}_{\frac{1}{n}} g) * \mathcal{F}(\varphi_x a_{\frac{1}{2n}}) * \gamma_\alpha(z) \]

\[ = (\mathcal{H}_{\frac{1}{n}} g) * [\mathcal{F}(\varphi_x a_{\frac{1}{2n}}) * \gamma_\alpha](z) \]

\[ = (\mathcal{H}_{\frac{1}{n}} g) * [\mathcal{F}[\mathcal{F}^{-1}(\varphi_x a_{\frac{1}{2n}}) * \gamma_\alpha]](z) \]

\[ = (\mathcal{H}_{\frac{1}{n}} g) * \mathcal{F}[\gamma_\alpha(\varphi_x a_{\frac{1}{2n}})](z) \]

\[ = (\mathcal{H}_{\frac{1}{n}} g) * \mathcal{F}[\varphi_x a_{\frac{1}{2n}}](z) \]

\[ = (\tau_z \mathcal{H}_{\frac{1}{2n}} g, \mathcal{F}[\varphi_x a_{\frac{1}{2n}}]) \]

\[ = (\mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g], \varphi_x a_{\frac{1}{2n}}). \]

Since \( \mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g] \) is a tempered distribution and \( \sum_{x \in \Gamma} \varphi_x a_{\frac{1}{2n}} \) converges to \( a_{\frac{1}{2n}} \) in the Schwartz space, we have

\[ \sum_{x \in \Gamma} (\mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g], \varphi_x a_{\frac{1}{2n}}) = (\mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g], a_{\frac{1}{2n}}). \]

Thus we conclude

\[ \mathcal{H}_{\frac{1}{n}} g(z) = (\mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g], a_{\frac{1}{2n}}) \]

\[ = \sum_{x \in \Gamma} (\mathcal{F}[\tau_z \mathcal{H}_{\frac{1}{2n}} g], \varphi_x a_{\frac{1}{2n}}) \]

\[ = \sum_{x \in \Gamma} \mathcal{H}_{\frac{1}{n}} \left[ (\mathcal{H}_{\frac{1}{n}} g) * \mathcal{F}(\varphi_x a_{\frac{1}{2n}}) \right](z), \]

to complete the proof. \( \square \)

For \( x \) in \( \Gamma \), let

\[ g_x = (\mathcal{H}_{\frac{1}{2n}} g) * \mathcal{F}(\varphi_x a_{\frac{1}{2n}}). \]

The above lemma tells us that the summation of the Berezin transform of \( T_{g_x} \) is equal to the Berezin transform of \( T_g \). In fact, we will show that \( \sum_{x \in \Gamma} T_{g_x} \) converges to \( T_g \) in operator norm topology in the last section.

For any \( z \in \mathbb{C}^n \), we define a unitary operator \( W_z \) on the Fock space such that

\[ W_z f(w) = f(w - z) k_z(w) \]

for any \( f \) in the Fock space. We have

\[ W_z^* = W_{-z}. \]

Let

\[ b_z(w) = e^{2\pi i w \cdot x} \]

for any \( w, x \in \mathbb{C}^n \).

**Lemma 2.3.** Let \( g \) be a measurable function on \( \mathbb{C}^n \) such that \( g_{k_w} \in L^2(\mathbb{C}^n, d\lambda) \) for any \( w \in \mathbb{C}^n \). If \( |\mathcal{H}_{\frac{1}{2n}} g(y)|dy(y) \) is a Carleson measure on the Fock space, then

\[ W_{-i\pi x} T_{g_x} W_{-i\pi x} = T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) * \mathcal{F}[\varphi_{x_{\cdot0}}]} \]

Further, we have

\[ T_{g_x} W_{-i\pi x} = T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) * \tau_{-i\pi x} \mathcal{F}[\varphi_{x_{\cdot0}}]} \]

and

\[ W_{-i\pi x} T_{g_x} = T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) * \tau_{i\pi x} \mathcal{F}[\varphi_{x_{\cdot0}}]}. \]
Proof. Since \( a^{-1} \phi_{x_0} \) is a smooth function with compact support, we have
\[
F[a^{-1} \phi_{x_0}] \in \mathcal{F}(\mathbb{R}^2).
\]
Since \(|b_z(z)| = 1\), by Lemma [2.1], we have
\[
(b_z \mathcal{H} \phi_z) * F[a^{-1} \phi_{x_0}] \in L^\infty(\mathbb{C}^n).
\]
By Lemma [2.2] we have \( g_x \in L^\infty(\mathbb{C}^n) \) and hence \( T_g \) is a bounded operator. Because the Berezin transform is injective, to get
\[
W \frac{1}{2\pi} T_g \frac{1}{2\pi} = T(b_z \mathcal{H} \phi_z) * F[a^{-1} \phi_{x_0}],
\]
we only need to show that
\[
\langle W \frac{1}{2\pi} T_g \frac{1}{2\pi} k_z, k_z \rangle = \langle T(b_z \mathcal{H} \phi_z) * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle.
\]
On one hand, since \( \mathcal{H} \phi_z \) is the Berezin transform of the Toeplitz operator \( T_g \), we have
\[
\langle T(b_z \mathcal{H} \phi_z) * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle = \langle b_z \mathcal{H} \phi_z * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle = \langle b_z \mathcal{H} \phi_z * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle.
\]
By properties of the Fourier transform and noting that \( F \tau = b_x F \), we have
\[
F[a \frac{1}{2\pi} \tau x \alpha] = F[a \frac{1}{2\pi} \tau x \alpha] = \mathcal{F}(F(a \frac{1}{2\pi} \tau x \alpha)) = \mathcal{F}[\tau (\varphi, a \frac{1}{2\pi} \tau x \alpha)] = \mathcal{F}[\tau (\varphi, a \frac{1}{2\pi} \tau x \alpha)] = b_x f a \frac{1}{2\pi} \tau x \alpha.
\]
For any \( w \in \mathbb{C}^n \), a simple calculation gives
\[
F[a \frac{1}{2\pi} \tau x \alpha](w) = \int_{\mathbb{C}^n} e^{\frac{-2\pi x}{2\pi} |z|^{2} \frac{1}{2\pi} e^{\frac{-2\pi x}{2\pi} |z|^{2} \frac{1}{2\pi} e^{-2\pi i z \cdot \omega}} dv(z) = e^{\frac{-2\pi x}{2\pi} |w|^{2} \frac{1}{2\pi} e^{-2\pi i z \cdot \omega}} dv(z) = e^{\frac{-2\pi x}{2\pi} |w|^{2} \frac{1}{2\pi} e^{-2\pi i z \cdot \omega}} dv(z) = e^{\frac{-2\pi x}{2\pi} |w|^{2} \frac{1}{2\pi} e^{-2\pi i z \cdot \omega}} dv(z) = e^{\frac{-2\pi x}{2\pi} |w|^{2} \frac{1}{2\pi} e^{-2\pi i z \cdot \omega}} dv(z).
\]
Thus, we have
\[
\langle T(b_z \mathcal{H} \phi_z) * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle = \langle b_z \mathcal{H} \phi_z * F[a^{-1} \phi_{x_0}] k_z, k_z \rangle = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} b_z(y) \mathcal{H} \phi_z g(y) b_z(y) (F \varphi, a \frac{1}{2\pi} \tau x \alpha) (w - y) dv(y) b_z(z - w) F(a \frac{1}{2\pi} \tau x \alpha) (w) (z - w) dv(w) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \mathcal{H} \phi_z g(y) (F \varphi, a \frac{1}{2\pi} \tau x \alpha) (w - y) dv(y) e^{\frac{2\pi i z \cdot \omega}{2\pi} F(a \frac{1}{2\pi} \tau x \alpha) (w)} (z - w) dv(w) = \int_{\mathbb{C}^n} g_z(w) e^{\frac{-2\pi x}{2\pi} |w|^{2} e^{\frac{-2\pi x}{2\pi} |w|^{2} e^{-2\pi i z \cdot \omega}} dv(w).
On the other hand, we have
\[
\langle W_{\frac{i\pi x}{2\alpha}} T_{g_x} W_{\frac{i\pi x}{2\alpha}} | k_z, k_z \rangle
= (T_{g_x} W_{\frac{i\pi x}{2\alpha}} k_z, W_{\frac{i\pi x}{2\alpha}} k_z)
= \int_{\mathbb{C}^n} g_x(k_z) \overline{k_z} \left( w + \frac{i\pi x}{2\alpha} \right) \overline{k_z} \left( w - \frac{i\pi x}{2\alpha} \right) d\lambda_\alpha(w)
= \int_{\mathbb{C}^n} g_x e^{\alpha(w + \frac{i\pi x}{2\alpha} , z) - \alpha(w, \frac{i\pi x}{2\alpha})} - \alpha(z, w - \frac{i\pi x}{2\alpha}) + \alpha(w, \frac{i\pi x}{2\alpha}) - \alpha(z, w) \left( \frac{\alpha}{n} \right)^n e^{-|z-w|^2} dv(w).
\]
Since
\[
\alpha \left( \frac{i\pi x}{2\alpha} , z \right) - \alpha \left( w, \frac{i\pi x}{2\alpha} \right) + \alpha \left( z, \frac{i\pi x}{2\alpha} \right) + \alpha \left( w, \frac{i\pi x}{2\alpha} \right) = \frac{i\pi}{2} \left( x, z \right) + \frac{i\pi}{2} \left( w, x \right) + \frac{i\pi}{2} \left( z, x \right) + \frac{i\pi}{2} \left( w, x \right)
= i\pi \Re \left( x, z + w \right)
= i\pi x \cdot (z + w),
\]
we have
\[
\langle W_{\frac{i\pi x}{2\alpha}} T_{g_x} W_{\frac{i\pi x}{2\alpha}} | k_z, k_z \rangle
= \int_{\mathbb{C}^n} g_x(w) e^{\frac{i\pi^2 x^2}{4\alpha^2}} e^{i\pi x \cdot (z+w)} \left( \frac{\alpha}{n} \right)^n e^{-|z-w|^2} dv(w)
= (T_{(b_x H_{\frac{1}{2\alpha}} g) \ast F[a^{-1} \varphi_{x_o}]} k_z, k_z).
\]
Further, we have
\[
T_{g_x} W_{\frac{i\pi x}{2\alpha}} = W_{\frac{i\pi x}{2\alpha}} T_{g_x} W_{\frac{i\pi x}{2\alpha}} W_{\frac{i\pi x}{2\alpha}}
= W_{\frac{i\pi x}{2\alpha}} T_{(b_x H_{\frac{1}{2\alpha}} g) \ast F[a^{-1} \varphi_{x_o}]} W_{\frac{i\pi x}{2\alpha}}
= T_{(b_x H_{\frac{1}{2\alpha}} g) \ast \tau_{\frac{i\pi x}{2\alpha}} F[a^{-1} \varphi_{x_o}]}.
\]
Similarly, we have
\[
W_{\frac{i\pi x}{2\alpha}} T_{g_x} = W_{\frac{i\pi x}{2\alpha}} W_{\frac{i\pi x}{2\alpha}} T_{g_x} W_{\frac{i\pi x}{2\alpha}}
= W_{\frac{i\pi x}{2\alpha}} T_{(b_x H_{\frac{1}{2\alpha}} g) \ast F[a^{-1} \varphi_{x_o}]} W_{\frac{i\pi x}{2\alpha}}
= T_{(b_x H_{\frac{1}{2\alpha}} g) \ast \tau_{\frac{i\pi x}{2\alpha}} F[a^{-1} \varphi_{x_o}]}.
\]
to complete the proof.

\[
\square
\]

3. Estimation

In this section, we will present the proof of our main theorem. In Section 2, we have obtained that $T_g$ is decomposed as a sum $\sum_{z \in \Gamma} T_g$, of the Toeplitz operators via the Berezin transform (Lemma 2.2). To show that $T_g = \sum_{z \in \Gamma} T_g$, we need to estimate the norm of a Toeplitz operator on the Fock space and show that the series of operators converges in operator norm topology. The following lemma is a generalization of [3, Lemma 4.9].

Lemma 3.1. Let $T$ be a densely defined operator on the Fock space and the linear span of reproducing kernels is contained in the domain of $T$ and $T^*$. If
\[
\sup_z \int_{\mathbb{C}^n} |\langle T k_z, k_w \rangle| dv(w) \sup_w \int_{\mathbb{C}^n} |\langle T k_z, k_w \rangle| dv(z) < \infty,
\]
then $T$ is bounded and
\[
\|T\| \leq \left( \left( \frac{\alpha}{n} \right)^n \sup_z \int_{\mathbb{C}^n} |\langle T k_z, k_w \rangle| dv(w) \sup_w \int_{\mathbb{C}^n} |\langle T k_z, k_w \rangle| dv(z) \right)^{1/2}.
\]
Proof. Let \( f \) and \( h \) be in the linear span of reproducing kernels, we have
\[
\langle T f, h \rangle = \int_{\mathbb{C}^n} (T f)(w) h(w) d\lambda_\alpha(w)
\]
\[
= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \langle T f, k_w \rangle \langle k_w, h \rangle d\nu(w)
\]
\[
= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \langle f, T^* k_w \rangle \langle k_w, h \rangle d\nu(w)
\]
\[
= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle k_z, T^* k_w \rangle d\nu(z) \langle k_w, h \rangle d\nu(w)
\]
\[
= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle Tk_z, k_w \rangle d\nu(z) \langle k_w, h \rangle d\nu(w)
\]
Thus applying the Cauchy-Schwarz inequality gives
\[
|\langle T f, h \rangle| \leq (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| |\langle f, k_z \rangle| d\nu(z) \langle k_w, h \rangle d\nu(w)
\]
\[
\leq (\frac{\alpha}{\pi})^n \left[ \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle|^2 d\nu(w) \right]^{1/2} \left[ \int_{\mathbb{C}^n} \langle k_w, h \rangle|^2 d\nu(w) \right]^{1/2}
\]
\[
= (\frac{\alpha}{\pi})^n \left[ \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle|^2 d\nu(w) \right]^{1/2} \left[ \int_{\mathbb{C}^n} |h(w)|^2 d\lambda_\alpha(w) \right]^{1/2}
\]
Since the Cauchy-Schwarz inequality gives
\[
\int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| |\langle f, k_z \rangle| d\nu(z) \right)^2 d\nu(w)
\]
\[
= \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle|^{1/2} |\langle f, k_z \rangle| d\nu(z) \right)^2 d\nu(w)
\]
\[
\leq \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle|^{1/2} |\langle f, k_z \rangle| d\nu(z) d\nu(w)
\]
\[
\leq \sup_w \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle|^{1/2} |\langle f, k_z \rangle| d\nu(z) d\nu(w)
\]
\[
\leq \sup_w \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \sup_z \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(w) \int_{\mathbb{C}^n} |f(z)|^2 d\lambda_\alpha(z)
\]
\[
= \sup_w \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \sup_z \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(w) \int_{\mathbb{C}^n} |f(z)|^2 d\lambda_\alpha(z),
\]
we obtain
\[
|\langle T f, h \rangle| \leq (\frac{\alpha}{\pi})^n \left[ \sup_w \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \right] \left[ \sup_z \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(w) \right] \int_{\mathbb{C}^n} |f(z)|^2 d\lambda_\alpha(z)
\]
\[
= (\frac{\alpha}{\pi})^n \left[ \sup_w \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(z) \right] \left[ \sup_z \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle| d\nu(w) \right] \|f\|_{L^2}^2 \|h\|_{L^2}^2.
\]
This completes the proof. \(\square\)

We introduce two-variable Berezin transform. Let \( g \) be a measurable function on \( \mathbb{C}^n \) such that \( g k_w \in L^2(\mathbb{C}^n, d\lambda_\alpha) \) for any \( w \in \mathbb{C}^n \). The two-variable Berezin transform \( \hat{g}(z, w) \) of \( g \) is defined by
\[
\hat{g}(z, w) = \langle g k_z, k_w \rangle
\]
for \( z, w \) in \( \mathbb{C}^n \). By Lemma 5.1 we immediately obtain the following proposition which will give an estimation of the norm of the Toeplitz operator \( T_g \) in terms of the two-variable Berezin transform \( \hat{g}(z, w) \) of \( g \).
Proposition 3.2. Let $g$ be a measurable function on $\mathbb{C}^n$ such that $g_{xw} \in L^2(\mathbb{C}^n, d\lambda_{\alpha})$ for any $w \in \mathbb{C}^n$. If
\[
\left| \sup_z \int_{\mathbb{C}^n} |\hat{g}(z, w)| dv(w) \right| \left( \sup_w \int_{\mathbb{C}^n} |\hat{g}(z, w)| dv(z) \right) < \infty,
\]
then $T_g$ is bounded and
\[
\|T_g\| \leq \left( \left( \frac{\alpha}{\pi} \right)^n \sup_z \int_{\mathbb{C}^n} |\hat{g}(z, w)| dv(w) \| \sup_w \int_{\mathbb{C}^n} |\hat{g}(z, w)| dv(z) \| \right)^{1/2}.
\]

For any $x \in \Gamma$, Lemma 2.2 tells us that $g_x = (\mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}(\varphi_x a_{-\frac{1}{2n}})$ is in $L^\infty$. Thus $T_{g_x}$ is a bounded operator. Next, we will obtain a better estimation on the norm of $T_{g_x}$ than $\|g_x\|_\infty$ to guarantee that the series $\sum_{x \in \Gamma} T_{g_x}$ converges in the operator norm topology.

Proposition 3.3. Let $g$ be a measurable function on $\mathbb{C}^n$ such that $g_{xw} \in L^2(\mathbb{C}^n, d\lambda_{\alpha})$ for any $w \in \mathbb{C}^n$. If for any $a, b \in \mathbb{Z}_+^n$ with $|a| + |b| \leq 2n + 1$, $|\partial_{\mathbb{R}^n \partial_{\mathbb{C}^n}} \mathcal{H}_{\frac{1}{2n}} g(y)dv(y)$ is a Carleson measure on the Fock space. Then there is a constant $C_{n, \alpha}$ such that
\[
\|T_{g_x}\| \leq C_{n, \alpha} \sum_{|a|_1 + |b|_1 \leq 2n + 1} \left\| \mathcal{H}_{\frac{1}{2n}} \left( \partial_{\mathbb{R}^n} \partial_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}} g(y) \right) \right\|_{\infty}.
\]

Proof. Since $W_{-\frac{ixz}{2n}}$ is an unitary operator and $g_x \in L^\infty$, we have
\[
\|T_{g_x}\| = \|T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}]}\|.
\]

By Lemma 2.3, we have
\[
\|T_{g_x}\| = \|T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}]}\|.
\]

Let $\psi_x = (b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}]$. Proposition 3.2 gives
\[
\|T_{g_x}\| = \|T_{\psi_x}\| \leq \left( \sup_z \int_{\mathbb{C}^n} |\tilde{\psi}_x(z, w)| dv(w) \right) \left( \sup_w \int_{\mathbb{C}^n} |\tilde{\psi}_x(z, w)| dv(z) \right)^{1/2}.
\]

Let $M_{z,w}(y) = k_z(y) \overline{k_w(y)} (\frac{\alpha}{\pi})^n e^{-\alpha |y|^2}$. For any $a, b \in \mathbb{Z}_+^n$ with $|a| + |b| \leq 2n + 1$, we have
\[
\left| (\mathbb{R}^n)^a (\mathbb{C}^n)^b \tilde{\psi}_x(z, w) \right| = \left| (\mathbb{R}^n)^a (\mathbb{C}^n)^b ((b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}]) k_z(k_w) \right|
\]
\[
= \int_{\mathbb{C}^n} (\mathbb{R}^n)^a (\mathbb{C}^n)^b (b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] |y| k_z(y) \overline{k_w(y)} (\frac{\alpha}{\pi})^n e^{-\alpha |y|^2} dv(y)
\]
\[
= \int_{\mathbb{C}^n} (\mathbb{R}^n)^a (\mathbb{C}^n)^b (b_x \mathcal{H}_{\frac{1}{2n}} g) \ast \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] |y| M_{z,w}(y) dv(y)
\]
\[
= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (\mathbb{R}^n)^a (\mathbb{C}^n)^b e^{-2\pi i x \cdot (y - \eta)} H_{\frac{1}{2n}} g(y - \eta) \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta) dv(\eta) M_{z,w}(y) dv(y)
\]
\[
= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (\mathbb{R}^n)^a (\mathbb{C}^n)^b e^{-2\pi i x \cdot \eta} H_{\frac{1}{2n}} g(y - \eta) \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta) dv(\eta) e^{2\pi i x \cdot \eta} M_{z,w}(y) dv(y)
\]

Since for any $a, b \in \mathbb{Z}_+^n$ with $|a| + |b| \leq 2n + 1$, $\partial_{\mathbb{R}^n \partial_{\mathbb{C}^n}} \mathcal{H}_{\frac{1}{2n}} g(y)dv(y)$ is a Carleson measure on the Fock space and $\mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}]$ is in the Schwartz space, by the proof of Lemma 2.1 we have $\partial_{\mathbb{R}^n} \partial_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}} g(y - \eta) \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta)$ is integrable with respect to $\eta$. By the properties of Fourier transform [18, page 12], we have
\[
\int_{\mathbb{C}^n} (\mathbb{R}^n)^a (\mathbb{C}^n)^b e^{-2\pi i x \cdot \eta} H_{\frac{1}{2n}} g(y - \eta) \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta) dv(\eta)
\]
\[
= e^{-\frac{1}{2} \pi |a + b|} \int_{\mathbb{C}^n} e^{-2\pi i x \cdot \eta} \partial_{\mathbb{R}^n} \partial_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}} g(y - \eta) \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta) dv(\eta)
\]
\[
= \sum_{a' \leq a, b' \leq b} C_{a, b, a', b'} [\partial_{\mathbb{R}^n} \partial_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}} g(y - \eta)] \partial_{\mathbb{R}^n}^{-a'} \partial_{\mathbb{C}^n}^{-b'} \mathcal{F}[a_{-\frac{1}{2n}} \varphi_{x_0}] (\eta) dv(\eta),
\]
where \( \{C_{a,b,a',b'}\} \) are constants and \( a' \leq a, b' \leq b \) means for any \( j \leq n \) we have \( a'_j \leq a_j, b'_j \leq b_j \). Since \( |M_{z,w}(y)| = \left( \frac{\pi}{n} \right)^n e^{-\frac{\pi}{2} |y-z|^2 - \frac{\pi}{2} |y-w|^2} \) and
\[
\int_{\mathbb{C}^n} |M_{z,w}(y)| dv(w) \approx e^{-\frac{\pi}{2} |y-z|^2},
\]
we have
\[
|\langle Rx \rangle^a (3x)^b| \sup_z \int_{\mathbb{C}^n} |\tilde{\psi}_z(z, w)| dv(w)
= \sup_z \int_{\mathbb{C}^n} |\int_{\mathbb{C}^n} e^{-\frac{\pi}{2} i x \cdot \eta} \sum_{a', b' \leq b} C_{a,b,a',b'} [\partial_{\mathbb{R}_y} \partial_{\mathbb{R}_z} \mathcal{H}_{\frac{1}{2n}} g(y - \eta)]
\times \partial_{\mathbb{R}_\eta}^{a-a'} \partial_{\mathbb{R}_\eta}^{b-b'} \mathcal{F}[a^{-1} \varphi_{x_0}(\eta)] dv(\eta) e^{\frac{\pi i}{2} y \cdot z} M_{z,w}(y) dv(y) dv(w)
\leq C \sum_{a' \leq a, b' \leq b} \sup_z \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g(y - \eta)|
\times |\partial_{\mathbb{R}_\eta}^{a-a'} \partial_{\mathbb{R}_\eta}^{b-b'} \mathcal{F}[a^{-1} \varphi_{x_0}(\eta)]| dv(\eta) M_{z,w}(y) dv(y) dv(w)
\leq C \sum_{a' \leq a, b' \leq b} \sup_z \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g(y - \eta) e^{-\frac{\pi}{2} |y-z|^2} dv(y) \partial_{\mathbb{R}_\eta}^{a-a'} \partial_{\mathbb{R}_\eta}^{b-b'} \mathcal{F}[a^{-1} \varphi_{x_0}(\eta)]| dv(\eta).
\]
Since for any \( \eta \),
\[
\sup_z \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g(y - \eta) e^{-\frac{\pi}{2} |y-z|^2} dv(y)
= \sup_z \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g(y - \eta) e^{-\frac{\pi}{2} |y-z|^2} dv(y)
= \sup_z \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g(y) e^{-\frac{\pi}{2} |y+\eta-z|^2} dv(y)
= \left( \frac{\pi}{\alpha} \right)^n \| \mathcal{H}_{\frac{1}{2n}} ((\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g)) \|_{\infty},
\]
we obtain
\[
|\langle Rx \rangle^a (3x)^b| \sup_z \int_{\mathbb{C}^n} |\tilde{\psi}_z(z, w)| dv(w)
\leq C \sum_{a' \leq a, b' \leq b} \left( \frac{\pi}{\alpha} \right)^n \| \mathcal{H}_{\frac{1}{2n}} ((\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g)) \|_{\infty} \int_{\mathbb{C}^n} |\partial_{\mathbb{R}_\eta}^{a-a'} \partial_{\mathbb{R}_\eta}^{b-b'} \mathcal{F}[a^{-1} \varphi_{x_0}(\eta)]| dv(\eta)
\leq C' \sum_{a' \leq a, b' \leq b} \| \mathcal{H}_{\frac{1}{2n}} ((\partial_{\mathbb{R}_y}^{a'} \partial_{\mathbb{R}_z}^{b'} \mathcal{H}_{\frac{1}{2n}} g)) \|_{\infty},
\]
where \( C' \) is a constant and the last inequality follows from the fact that \( \mathcal{F}[a^{-1} \varphi_{x_0}(\eta)] \) is a Schwartz function. The binomial expansion gives that there is a constant \( c \) such that
\[
(1 + |Rx| + |3x|)^{2n+1} \leq c \sum_{|a|+|b| \leq 2n+1} |\langle Rx \rangle^a||\langle 3x \rangle^b|.
\]
Thus we have
\[
\sup_z \int_{C^n} |\psi_x(z, w)| dv(w) \\
\leq c \sum_{|a|+|b|\leq 2n+1} (\|R^a\|^4 (\|3z\|^2) \sup_z \int_{C^n} |\langle b_x H_{3z}^+ g \rangle \ast F[a_{-\frac{1}{2n}} \varphi_x, k_z, k_w]| dv(w) \\
\leq C_{n,\alpha} \sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty \\
(1 + |Rz| + |3z|)^{2n+1}
\]

Similarly, we have
\[
\sup_w \int_{C^n} |\tilde{\psi}_x(z, w)| dv(z) \\
\leq C_{n,\alpha} \sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty \\
(1 + |Rz| + |3z|)^{2n+1}
\]

Thus we conclude
\[
\|T_{gx}\| = \|T_{\psi_x}\| \\
\leq C_{n,\alpha} \sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty \\
(1 + |Rz| + |3z|)^{2n+1}
\]

to complete the proof. \(\square\)

Now we are ready to present the proof of the main theorem.

**Proof of Theorem 1.1.** Since \(|\partial_y^a \partial_y^b H_{3z}^+ g(y)| dv(y)|\) is a Carleson measure, it means
\[
\sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty < \infty.
\]

For \(x \in \Gamma\), let \(g_x = (H_{3z}^\pm g) \ast F(\varphi_x a_{-\frac{1}{2n}})\). By Proposition 4.3, we have
\[
\|T_{gx}\| \leq C_{n,\alpha} \sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty \\
(1 + |Rz| + |3z|)^{2n+1}
\]

Since
\[
\sum_{x \in \Gamma} \frac{1}{(1 + |Rz| + |3z|)^{2n+1}} < C < \infty,
\]

the series \(\sum_{x \in \Gamma} T_{gx}\) of bounded operators converges to a bounded operator \(X\) in the operator norm topology and
\[
\|X\| \leq C_{n,\alpha} C \sum_{|a|+|b|\leq 2n+1} \|H_{3z}^\pm (|\partial_y^a \partial_y^b H_{3z}^+ g(y))\| \infty.
\]

We will show that \(X\) is the bounded extension of \(T_g\). To do so, by Lemma 2.2, for any \(z \in C^n\), we have
\[
\langle T_g k_z, k_z \rangle = H_{3z}^\pm g(z) \\
= \sum_{x \in \Gamma} \langle H_{3z}^\pm g \rangle \ast F(\varphi_x a_{-\frac{1}{2n}}) (z) \\
= \sum_{x \in \Gamma} \langle T(\varphi_x a_{-\frac{1}{2n}}) k_z, k_z \rangle \\
= \sum_{x \in \Gamma} \langle T_g k_z, k_z \rangle \\
= \langle X k_z, k_z \rangle.
\]

This implies
\[
\langle T_g K_z, K_z \rangle = \langle X K_z, K_z \rangle.
\]
Since $\langle T_g K_z, K_w \rangle$ and $\langle X K_z, K_w \rangle$ are both analytic with respect to $w$ and anti-analytic with respect to $z$, we have

$$\langle T_g K_z, K_w \rangle = \langle X K_z, K_w \rangle$$

for any $z, w \in \mathbb{C}^n$. Thus this implies

$$T_g K_z = X K_z$$

for any $z \in \mathbb{C}^n$. So $X$ is the extension of $T_g$ and hence $\sum_{x \in \Gamma} T_{g_x}$ converges to $T_g$ in the operator norm topology.

To finish the proof of the main theorem, we need only show that $T_{g_x}$ has the following integral representation: for any $t \geq 0$,

$$T_{g_x} = \int_{\mathbb{C}^n} F(\varphi_{x,a^{-1}}^t)(y) T_{\gamma_{\mathcal{H}^\infty_{\frac{1}{2n}+t}}(g)} dv(y),$$

To do so, first we show that the map

$$y \to W_y T_{\mathcal{H}^\infty_{\frac{1}{2n}+t}}(g) W_y^* = T_{\gamma_{\mathcal{H}^\infty_{\frac{1}{2n}+t}}(g)}$$

is uniformly continuous from $\mathbb{C}^n$ to the space of bounded linear operator with respect to the operator norm topology.

Since $|\mathcal{H}_{\frac{1}{2n}}(g)|dv(y)$ is a Carleson measure, by Lemma 3 we have

$$\mathcal{H}_{\frac{1}{2n}}(\mathcal{H}_{\frac{1}{2n}+t}(g)) = \gamma_{\frac{1}{2n}+t} * \mathcal{H}_{\frac{1}{2n}}(g)$$

is bounded for any $t \geq 0$. Thus 13 and the norm estimation above to replace $g$ by $\mathcal{H}_{\frac{1}{2n}+t}g$ give that

$$\|T_{\mathcal{H}_{\frac{1}{2n}+t}g}\| \lesssim \sum \|\mathcal{H}_{\frac{1}{2n}}(g)|dv(y)\|_{\infty} \lesssim \|\mathcal{H}_{\frac{1}{2n}}(\mathcal{H}_{\frac{1}{2n}+t}g)\|_{\infty} < \infty.$$

So $T_{\mathcal{H}_{\frac{1}{2n}+t}g}$ is bounded. Direct calculation shows

$$W_y^* W_{y'} = e^{-i\alpha 3(-y,y')} W_{y'-y}$$

for any $y, y' \in \mathbb{C}^n$. As $W_y$ is a unitary operator, the following equalities hold:

$$\|W_y T_{\mathcal{H}_{\frac{1}{2n}+t}g} W_{y'} - W_y' T_{\mathcal{H}_{\frac{1}{2n}+t}g} W_{y'}^*\| = \|T_{\mathcal{H}_{\frac{1}{2n}+t}g} - W_{y'-y} T_{\mathcal{H}_{\frac{1}{2n}+t}g} W_{y'-y}\|$$

$$\quad = \|T_{\mathcal{H}_{\frac{1}{2n}+t}g} - T_{\mathcal{H}_{\frac{1}{2n}+t}g} W_{y'-y}\|$$

$$\quad = \|T_{\mathcal{H}_{\frac{1}{2n}+t}g} - \gamma_{\frac{1}{2n}+t}(g)\|.$$

By 13 and the norm estimation above, for some $s \in (0, \frac{1}{2n})$, we have

$$\|T_{\mathcal{H}_{\frac{1}{2n}+t}g} - \gamma_{\frac{1}{2n}+t}(g)\| \lesssim \sum z \|\mathcal{H}_{\frac{1}{2n}+t}g(w)\| |\gamma_s(z-w) - \gamma_s(z-w+y-y')|dv(w)$$

$$\quad \leq \frac{1}{s^n} \sum z \int_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}+t}g(w)|e^{-\frac{|z-w|^2}{s}} - e^{-\frac{|z-w+y-y'|^2}{s}}|dv(w)$$

$$\quad = \frac{1}{s^n} \sum z \int_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}+t}g(w)e^{-\frac{|z-w|^2}{s}}||1 - e^{-\frac{|z-w|^2}{s} - \frac{|z-w+y-y'|^2}{s}}|dv(w)$$

$$\quad = \frac{1}{s^n} \sum z \int_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}+t}g(w)e^{-\frac{|z-w|^2}{s}}||1 - e^{-\frac{2R(z-w,y-y') + |y-y'|^2}{s}}|dv(w)$$

$$\quad \leq \frac{1}{s^n} \sum z \int_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}+t}g(w)e^{-\frac{|z-w|^2}{s}}\sum_{n=1}^\infty \frac{1}{n!}\left(-\frac{2R(z-w,y-y') + |y-y'|^2}{s}\right)^n|dv(w)$$

$$\quad \leq \frac{1}{s^n} \sum z \int_{\mathbb{C}^n} \mathcal{H}_{\frac{1}{2n}+t}g(w)e^{-\frac{|z-w|^2}{s}}\sum_{n=1}^\infty \frac{1}{n!}\left(\frac{2R(z-w,y-y') + |y-y'|^2}{s}\right)^n|dv(w).$$
Without loss of generality, we can suppose $|y - y'| \leq 1$. The above estimations give

$$
\|T(\mathcal{H}^{1+(t-g)} - T_{y'}^y(\mathcal{H}^{1+(t-g)})\| \\
\leq \frac{1}{8\pi} n \sup_{z} \int_{\mathbb{C}^n} |\mathcal{H}^{1+(t-g)} e^{-\frac{|z-w|^2}{s}} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{2|z-w| + |y-y'|^2}{s} |^{n} |y-y'|^n dv(w)
$$

$$
\leq |y - y'| \frac{1}{8\pi} n \sup_{z} \int_{\mathbb{C}^n} |\mathcal{H}^{1+(t-g)} e^{-\frac{|z-w|^2}{s}} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{2|z-w| + 1}{s} |^{n} dv(w)
$$

$$
\leq |y - y'| \frac{1}{8\pi} n \sup_{z} \int_{\mathbb{C}^n} |\mathcal{H}^{1+(t-g)} e^{-\frac{|z-w|^2 + |y-y'|^2}{s}} dv(w)
$$

$$
= |y - y'| \frac{1}{8\pi} n \||\mathcal{H}^{1+(t-g)}| * f|_\infty,
$$

where $f(w) = e^{-\frac{2|w|^2}{s} + \frac{2r}{s}}$ is a Schwartz function. As Lemma 2.1 gives

$$
(\frac{1}{8\pi}) n \||\mathcal{H}^{1+(t-g)}| * f|_\infty < \infty,
$$

the above estimations imply

$$
\|W_y T_{\mathcal{H}^{1+(t-g)}} W_y^* - W_{y'} T_{\mathcal{H}^{1+(t-g)}} W_{y'}^*\| \lesssim |y - y'|.
$$

Thus the map

$$
y \to W_y T_{\mathcal{H}^{1+(t-g)}} W_y^*
$$

is uniformly continuous from $\mathbb{C}^n$ to the space of bounded linear operator with respect to the norm topology. Since $\mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})$ is a Schwartz function and

$$
\|W_y T_{\mathcal{H}^{1+(t-g)}} W_y^*\| \leq \|T_{\mathcal{H}^{1+(t-g)}}\|,
$$

the integral

$$
\int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) W_y T_{\mathcal{H}^{1+(t-g)}} W_y^* dv(y),
$$

converges in the operator norm topology.

To establish the integral representation of $T_{g_z}$, as the Berezin transform is injective, next we calculate the Berezin transform of $\int_{\mathbb{C}^n} W_y T_{\mathcal{H}^{1+(t-g)}} W_y^* \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) dv(y)$ and $T_{g_z}$ to get

$$
\langle \int_{\mathbb{C}^n} W_y T_{\mathcal{H}^{1+(t-g)}} W_y^* \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) dv(y) | k_z, k_z \rangle
$$

$$
= \int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) \langle [W_y T_{\mathcal{H}^{1+(t-g)}} W_y^*] k_z, k_z \rangle dv(y)
$$

$$
= \int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) \langle [T_{g_y} H_{\frac{1}{2n} + t}] k_z, k_z \rangle dv(y)
$$

$$
= \int_{\mathbb{C}^n} \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(y) \tau_{g_y} H_{\frac{1}{2n} + t} \gamma_{\frac{1}{n}}(z) dv(y)
$$

$$
= \mathcal{H}_{\frac{1}{2n} + t} g * \gamma_{\frac{1}{n}} * \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(z)
$$

$$
= \mathcal{H}_{\frac{1}{2n} + t} g * \gamma_{\frac{1}{n}} * \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(z)
$$

$$
= \mathcal{H}_{\frac{1}{2n} + t} g * \mathcal{F}(a_t) * \gamma_{\frac{1}{n}} * \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(z)
$$

$$
= \mathcal{H}_{\frac{1}{2n} + t} g * \mathcal{F}(\varphi_x a_{\frac{1}{2n} + t})(z)
$$

$$
= T_{g_z} k_z, k_z \rangle}
where the second equality follows from
\[ W_y T_{\frac{1}{2n} + g} W_y^* = T_{\tau_y} T_{\frac{1}{2n} + g}; \]
the third equality follows from the relation between the Berezin transform and the heat transform; the fourth and fifth equalities follow from the semigroup property of the heat transform. As the Berezin transform is injective, we conclude
\[
T_{\phi_x} = \int_{\mathbb{C}^n} F(\varphi_x a^{-1/2n}) (y) W_y T_{\frac{1}{2n} + g} W_y^* dv(y)
\]
\[
= \int_{\mathbb{C}^n} F(\varphi_x a^{-1/2n}) (y) T_{\tau_y} T_{\frac{1}{2n} + g} dv(y)
\]
to complete the proof. \qed

4. **Schatten \( p \)-class**

In this section, we will apply our decomposition theory for a Toeplitz operator to estimate the Schatten \( p \)-norm of the product of two Toeplitz operators.

Let \( S_p \) denote the Schatten \( p \)-class on \( F^2_\alpha \). A compact operator \( A \) on \( F^2_\alpha \) is in \( S_p \) if
\[
\| A \|_{S_p} = \sup \left\{ \left( \sum \| A e_n \|^p \right)^{1/p} : \{e_n\} \text{ is an orthonormal set} \right\} < \infty.
\]
For a Toeplitz operator \( T_f \), we have
\[
\| T_f \|_{S_p} \lesssim \| f \|_{L^p(\mathbb{C}^n, dv)}, \tag{4.1}
\]
see [19] Lemma 6.30].

**Lemma 4.1.** Let \( \{X, M, m\} \) be a measure space and \( \mathcal{H} \) be a separable Hilbert space. Let \( S_p \) denote the Schatten \( p \)-class on \( \mathcal{H} \). Suppose that \( F : X \to S_p \) is a weakly \( M \)-measurable map. If
\[
\int_X \| F(x) \|_{S_p} dm(x) < \infty,
\]
then
\[
K = \int_X F(x) dm(x) \in S_p, \text{ and } \| K \|_{S_p} \leq \int_X \| F(x) \|_{S_p} dm(x)
\]
where the integral is taken in the weak sense.

**Proof.** Let \( \{e_i : i = 1, \cdots, n, \cdots\} \) denote the orthogonal basis in \( \mathcal{H} \). Let \( P_n \) be the project on to the space generated by \( \{e_1, \cdots, e_n\} \). We have
\[
K - KP_n = \int_X F(x) - F(x) P_n dm(x).
\]
By the definition of the \( S_p \) norm, we have
\[
\| K - KP_n \|_{S_p} \leq \int_X \| F(x) - F(x) P_n \|_{S_p} dm(x) \text{ and } \lim_{n \to \infty} \| F(x) - F(x) P_n \|_{S_p} = 0.
\]
Since
\[
\| F(x) - F(x) P_n \|_{S_p} \leq 2 \| F(x) \|_{S_p},
\]
which is integrable, by the dominated convergence theorem we have
\[
\lim_{n \to \infty} \| K - KP_n \|_{S_p} = \int_X \lim_{n \to \infty} \| F(x) - F(x) P_n \|_{S_p} dm(x) = 0.
\]
Since \( S_p \) is closed with respect to the Schatten \( p \)-norm, we have \( K \in S_p \). The norm estimation follows directly. \qed
Recall that the Weyl operator $W_z$ on $F_α^2$ is defined by
\[ W_z f(w) = k_z f(w - z). \]

One can check that
\[ W_w W_z = e^{-i(\alpha w - \beta z)}/\pi} W_{w+z}, \quad W_w K_z = e^{-i(\alpha w - \beta z)} k_{w+z} \]
and
\[ \|W_z f\|_{F_α^2} = \|f\|_{F_α^2}. \quad (4.2) \]

**Theorem 4.2.** Let $A$ be a bounded operator on $F_α^2$, if
\[ \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle A_k, k_{z+w} \rangle|^p dv(z) \right)^{1/p} dv(w) < \infty, \]
then $A$ is in the Schatten $p$-class and
\[ \|A\|_{S_p} \leq \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |\langle A_k, k_{z+w} \rangle|^p dv(z) \right)^{1/p} dv(w). \]

**Proof.** For any $f, g \in F_α^2$ we have
\[ \langle Af, g \rangle = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \langle Af, k_w \rangle \langle k_w, g \rangle dv(w) = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \langle f, A^* k_w \rangle \langle k_w, g \rangle dv(w) = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle k_z, A^* k_w \rangle \langle k_w, g \rangle dv(z) dv(w) \]
\[ = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle A_k, k_{w+z} \rangle \langle k_{w+z}, g \rangle dv(z) dv(w) \]
\[ = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle A_k, W_w k_z \rangle \langle W_w k_z, g \rangle dv(z) dv(w) \]
\[ = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle f, k_z \rangle \langle A_k, W_w k_z \rangle \langle k_z, W_w^* g \rangle dv(z) dv(w). \]

Let
\[ S_w = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \langle A_k, W_w k_z \rangle k_z \otimes k_z dv(z), \]
where the integral is taken in the weak sense. $S_w$ is actually a Toeplitz operator, we have
\[ \langle Af, g \rangle = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \langle S_w f, W_w^* g \rangle dv(w) = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} \langle W_w S_w f, g \rangle dv(w). \]

Thus
\[ A = \frac{\alpha^n}{\pi^n} \int_{\mathbb{C}^n} W_w S_w dv(w), \]
where the integral is taken in the weak sense. Since $S_w$ is a Toeplitz operator, we have
\[ \|W_w S_w\|_{S_p} \leq \|S_w\|_{S_{p'}} \leq \left( \int_{\mathbb{C}^n} \left| \langle A_k, W_w k_z \rangle \right|^p dv(z) \right)^{1/p} = \left( \int_{\mathbb{C}^n} \left| \langle A_k, k_{z+w} \rangle \right|^p dv(z) \right)^{1/p}. \]

By the hypothesis and Lemma 11 we obtain $A \in S_p$. The norm estimation follows directly. \qed

For any $a, b \in \mathbb{Z}_+^n$, let
\[ J^{a, b} g(y) = \partial_{\eta_y} a \partial_{\eta_y} b \mathcal{H} \frac{1}{\pi} g(y). \]

In the proof of [17, Theorem 1], the authors get the Schatten $p$-norm estimation for one Toeplitz operator through the result about the pseudo-differential operator in [16]. That is
\[ \|T_g\|_{S_p} \lesssim \sum_{|a| + |b| \leq 2n+1} \left( \int_{\mathbb{C}^n} |J^{a, b} g(\eta)|^p dv(\eta) \right)^{1/p}. \]

We will show an estimation for the product of Toeplitz operators and our proof has nothing to do with the pseudo-differential operator. Moreover, we will show that our result implies their result.
Theorem 4.3. Let \( g \) and \( f \) be two measurable functions on \( \mathbb{C}^n \) such that \( g_{k,w}, f_{k,w} \in L^2(\mathbb{C}^n, d\lambda_{\alpha}) \) for any \( w \in \mathbb{C}^n \). If for any \( a, b \in \mathbb{Z}^n \) with \(|a| + |b| \leq 2n + 1\), \( |J^{n,b}g(y)|dv(y) \) and \( |J^{n,b}f(y)|dv(y) \) are Carleson measures on the Fock space, we have

\[
\|T_f T_g\|_{S_p} \lesssim \sum_{|a| + |b| \leq 2n + 1} \sup_{w \in \mathbb{C}^n} \left( \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |J^{n,b}f(\xi)|^p |J^{n,b'}g(\eta)|^p e^{-\frac{\omega}{\omega' + 1} |\xi - \eta| + |w|^2} dv(\xi)dv(\eta) \right)^{1/p}.
\]

Proof. By Theorem 1.1 we know that \( T_f \) and \( T_g \) are bounded and we have two decompositions

\[
T_f = \sum_{y \in \mathcal{F}} T_{f_y} \quad \text{and} \quad T_g = \sum_{z \in \mathcal{F}} T_{g_z},
\]

where \( f_y = (\mathcal{H}_{\frac{1}{2n}}f) * \mathcal{F}(\varphi_ya_{\frac{1}{2n}}) \) and \( g_x = (\mathcal{H}_{\frac{1}{2n}}g) * \mathcal{F}(\varphi_xa_{\frac{1}{2n}}) \). Thus we have

\[
\|T_f T_g\|_{S_p} \leq \sum_{y \in \mathcal{F}} \|T_{f_y}\|_{S_p} \|T_{g_x}\|_{S_p} = \sum_{y \in \mathcal{F}} \sum_{x \in \mathcal{F}} \|T_{f_y} T_{g_x}\|_{S_p}
\]

\[
\lesssim \sum_{y \in \mathcal{F}} \sum_{x \in \mathcal{F}} \left( \|\mathcal{F}(\varphi_ya_{\frac{1}{2n}})\|^p \|\mathcal{F}(\varphi_xa_{\frac{1}{2n}})\|^p \right) \|T_{f_y} T_{g_x}\|_{S_p}
\]

By Lemma 2.3 we have

\[
W_{-\frac{1}{2n}} T_{f_y} = T_{(b_y \mathcal{H}_{\frac{1}{2n}} f) * \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y]} \quad \text{and} \quad T_{g_x} W_{-\frac{1}{2n}} = T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) * \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y]}.
\]

where \( y_0 = 0 \). Then we get

\[
\|T_{f_y} T_{g_x}\|_{S_p} = \|W_{-\frac{1}{2n}} T_{f_y} T_{g_x} W_{-\frac{1}{2n}}\|_{S_p} = \|T_{(b_y \mathcal{H}_{\frac{1}{2n}} f) * \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y]} T_{(b_x \mathcal{H}_{\frac{1}{2n}} g) * \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y]}\|_{S_p}.
\]

By (3.1), we have

\[
(\mathcal{F}(\varphi_ya_{\frac{1}{2n}}))^p(\mathcal{F}(\varphi_xa_{\frac{1}{2n}}))^p \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y] = \sum_{c < a, d < b} (b_y J^{c,d} f) * h_{y,c,d} \quad \text{and} \quad (\mathcal{F}(\varphi_xa_{\frac{1}{2n}}))^p \tau_{-\frac{1}{2n}} \mathcal{F}[a_{\frac{1}{2n}} \varphi_y] = \sum_{c < a', d < b'} (b_x J^{c,d} f) * h_{x,c,d}
\]

where \( h_{y,c,d} \) and \( h_{x,c,d} \) are Schwartz functions and

\[
\int_{\mathbb{C}^n} |h_{x,c,d}(z)|dv(z) \quad \text{and} \quad \int_{\mathbb{C}^n} |h_{y,c,d}(z)|dv(z)
\]

are independent of \( x \) and \( y \). Thus

\[
\sum_{|a| + |b| \leq 2n + 1} \|\mathcal{F}(\varphi_ya_{\frac{1}{2n}})\|^p \|\mathcal{F}(\varphi_xa_{\frac{1}{2n}})\|^p \|T_{f_y} T_{g_x}\|_{S_p}
\]

\[
\lesssim \sum_{|a| + |b| \leq 2n + 1} \|T_{(b_y J^{n,b} f) * h_{y,a,b}} T_{(b_x J^{n,b'} g) * h_{x,a',b'}}\|_{S_p}.
\]

Thus, we have

\[
\|T_f T_g\|_{S_p} \lesssim \sum_{|a| + |b| \leq 2n + 1} \|T_{(b_y J^{n,b} f) * h_{y,a,b}} T_{(b_x J^{n,b'} g) * h_{x,a',b'}}\|_{S_p}.
\]
By Lemma 3.2, we have \( T_{(b_y, J^{a,b} f) \ast h_{y,a,b}} T_{(b_z, J^{a',b'} g) \ast h_{x,a',b'}} \) is a product of Toeplitz operators with bounded symbols, we can apply Theorem 4.2. We have
\[
|T_{(b_y, J^{a,b} f) \ast h_{y,a,b}} T_{(b_z, J^{a',b'} g) \ast h_{x,a',b'}} k_z, k_z + w)| \\
\leq \int_{C^n} J^{a,b} f \ast h_{y,a,b}(\xi) \int_{C^n} J^{a',b'} g \ast h_{x,a',b'}(\eta) k_z(\eta) \hat{K}(\eta) d\lambda_{\alpha}(\eta) k_{z + w}(\xi) d\lambda_{\alpha}(\eta)
\leq \int_{C^n} \int_{C^n} |J^{a,b} f| \ast |h_{y,a,b}(\xi)||J^{a',b'} g| \ast |h_{x,a',b'}|(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} dv(\xi) dv(\eta).
\]
Denote
\[
L^{a,b}_{y} f(\xi) = |J^{a,b} f| \ast |h_{y,a,b}(\xi)| \text{ and } L^{a',b'}_{x} g(\eta) = |J^{a',b'} g| \ast |h_{x,a',b'}|(\eta).
\]
By Theorem 4.2 we have
\[
\|T_{(b_y, J^{a,b} f) \ast h_{y,a,b}} T_{(b_z, J^{a',b'} g) \ast h_{x,a',b'}}\|_{S_p} \\
\leq \int_{C^n} \left( \int_{C^n} |J^{a,b} f| L^{a,b}_{x} g(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} dv(\eta)\right)^{\frac{1}{p}} dv(\xi)
\]
For simplicity, we will denote \( dv(w) \) by \( dw \). If \( p = 1 \), we have
\[
\|T_{(b_y, J^{a,b} f) \ast h_{y,a,b}} T_{(b_z, J^{a',b'} g) \ast h_{x,a',b'}}\|_{S_p} \\
\leq \int_{C^n} \left( \int_{C^n} \left( \int_{C^n} L^{a,b}_{y} f(\xi) L^{a,b}_{x} g(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} d\xi d\eta dv(z) \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} dv(w).
\]
We have completed the proof when \( p = 1 \). If \( p > 1 \), let \( q > 1 \) such that \( 1/p + 1/q = 1 \). We have
\[
\|T_{(b_y, J^{a,b} f) \ast h_{y,a,b}} T_{(b_z, J^{a',b'} g) \ast h_{x,a',b'}}\|_{S_p} \\
\leq \int_{C^n} \left( \int_{C^n} \left( \int_{C^n} L^{a,b}_{y} f(\xi) L^{a,b}_{x} g(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} d\xi d\eta\right)^{p} \right)^{\frac{1}{p}} dv(w).
\]
Using Hölder inequality, we get
\[
\int_{C^n} \left( \int_{C^n} L^{a,b}_{y} f(\xi) L^{a,b}_{x} g(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} d\xi d\eta\right)^{p} \\
\leq \int_{C^n} \left( \int_{C^n} L^{a,b}_{y} f(\xi) L^{a,b}_{x} g(\eta) e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} d\xi d\eta\right)^{\frac{p}{q}} \int_{C^n} \left( \int_{C^n} e^{\frac{a|x|^2}{2}} e^{\frac{a|z|^2}{2}} e^{\frac{a|z - w|^2}{2}} d\xi d\eta\right)^{\frac{q}{p}}
\]
Thus
\[ \left\| T_{g} \right\|_{S_{p}} \lesssim \sum_{|a'| + |b'| \leq 2n+1} \left( \int_{\mathbb{C}^{n}} |J^{a',b'} g(\eta)|^{p} d\nu(\eta) \right)^{\frac{1}{p}}. \]

**Corollary 4.4.** Let \( g \) be a measurable function on \( \mathbb{C}^{n} \) such that \( g k_{w} \in L^{2}(\mathbb{C}^{n}, d\lambda_{n}) \) for any \( w \in \mathbb{C}^{n} \). We have

\[ \left\| T_{g} \right\|_{S_{p}} \lesssim \sum_{|a'| + |b'| \leq 2n+1} \left( \int_{\mathbb{C}^{n}} |J^{a',b'} g(\eta)|^{p} d\nu(\eta) \right)^{\frac{1}{p}}. \]

**Proof.** If \( f = 1 \), then \( T_{f} T_{g} = T_{g} \). By Theorem 4.3, we have

\[ \left\| T_{g} \right\|_{S_{p}} = \left\| T_{f} T_{g} \right\|_{S_{p}} \lesssim \sum_{|a'| + |b'| \leq 2n+1} \sup_{w \in \mathbb{C}^{n}} \left( \int_{\mathbb{C}^{n}} |J^{a',b'} g(\eta)|^{p} e^{-\frac{1}{p}|\xi - \eta + w|^{2}} d\omega(\xi) d\nu(\eta) \right)^{\frac{1}{p}} \]

\[ \lesssim \sum_{|a'| + |b'| \leq 2n+1} \left( \int_{\mathbb{C}^{n}} |J^{a',b'} g(\eta)|^{p} d\nu(\eta) \right)^{\frac{1}{p}}. \]

\[ \square \]

**Acknowledgement**

It is our pleasure to thank Robert Fulsche for useful comments.

**References**

[1] W. Bauer, L.A. Coburn and J. Isralowitz, Heat flow, BMO, and the compactness of Toeplitz operators, J. Funct. Anal. 250 (2010), 57–78.

[2] W. Bauer and J. Isralowitz, Compactness characterization of operators in the Toeplitz algebra of the Fock space \( F_{p}^{\infty} \), J. Funct. Anal. 263 (2012), 1323–1355.

[3] W. Bauer and R. Fulsche, Berger-Coburn Theorem, Localized Operators, and the Toeplitz Algebra, Operator Algebras: Toeplitz Operators and Related Topics. 279 (2020), 53–77.

[4] C. Berger and L.A. Coburn, Toeplitz operators and quantum mechanics, J. Funct. Anal. 68 (1986), 273–299.

[5] C. Berger and L.A. Coburn, Heat flow and Berezin-Toeplitz Estimates, American Journal of Mathematics. 116 (1994), 563–590.

[6] C. Berger and L.A. Coburn, Toeplitz operators on the Segal-Bargmann space, Trans. Amer. Math. Soc. 310 (1987), 813–829.

[7] W. Bauer, L.A. Coburn and J. Isralowitz, Heat flow, BMO, and the compactness of Toeplitz operators, J. Funct. Anal. 250 (2010), 57–78.

[8] A. Boukhemair, \( L^{2} \) estimates for Weyl Quantization, J. Funct. Anal. 165 (1999), 173–204.

[9] Javier Duoandikoetxea, Fourier Analysis. Graduate Studies in Mathematics, vol. 29, American Mathematical Society.

[10] M. Engliš, Compact Toeplitz operators via the Berezin transform on bounded symmetric domains, Integral Equations Operator Theory 33 (1999), 426-455.

[11] Gerald B. Folland, Harmonic analysis in phase space. Princeton University Press, Princeton, New Jersey, 1989.

[12] R. Fulsche, Correspondence theory on \( p \)-Fock spaces with applications to Toeplitz algebras, J. Funct. Anal. 279 (2020), no. 7.
[13] R. Fulsche, *Toeplitz operators and generated algebras on non-Hilbertian spaces*, PhD thesis, Leibniz University Hannover (2020).

[14] Loukas Grafakos, *Classical Fourier Analysis*. Graduate Texts in Mathematics, **249**, Second Edition, Springer.

[15] J. Isralowitz, M. Mitkovski and B. D. Wick, *Localization and compactness in Bergman and Fock spaces*, Indiana Univ. Math. J. **64** (2015), 1553-1573.

[16] C. Rondeaux, *Classes de Schatten d’opérateurs pseudo-différentiels*, Ann. Sci. École Norm. Sup. **4** (1984), 67-81.

[17] Shengkun Wu and Xianfeng Zhao, *Toeplitz algebras over Fock and Bergman spaces*, arXiv:2105.03950

[18] J. Xia, *Localization and the Toeplitz algebra on the Bergman space*, J. Funct. Anal. **269** (2015), 781–814.

[19] K. Zhu, *Analysis on Fock Spaces*. Graduate Texts in Mathematics, vol. **263**, Springer, New York, 2012.

---

1 **College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, PR China**

*Email address: shengkunwu@foxmail.com*

2 **Department of Mathematics, Vanderbilt University, Nashville, TN 37240 and Center of Mathematics, Chongqing University, Chongqing, 401331, PR China**

*Email address: dechao.zheng@vanderbilt.edu*