ON THE NUMBER OF EVEN VALUES OF AN ETA-QUOTIENT

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Abstract. The goal of this note is to provide a general lower bound on the number of even values of the Fourier coefficients of an arbitrary eta-quotient $F$, over any arithmetic progression. Namely, if $g_{a,b}(x)$ denotes the number of even coefficients of $F$ in degrees $n \equiv b \pmod{a}$ such that $n \leq x$, then we show that $g_{a,b}(x)/\sqrt{x}$ is unbounded for $x$ large.

Note that our result is very close to the best bound currently known even in the special case of the partition function $p(n)$ (namely, $\sqrt{x \log \log x}$, proven by Bellaïche and Nicolas in 2016). Our argument substantially relies upon, and generalizes, Serre’s classical theorem on the number of even values of $p(n)$, combined with a recent modular-form result by Cotron et al. on the lacunarity modulo 2 of certain eta-quotients.

Interestingly, even in the case of $p(n)$ first shown by Serre, no elementary proof is known of this bound. At the end, we propose an elegant problem on quadratic representations, whose solution would finally yield a modular form-free proof of Serre’s theorem.

1. Introduction and preliminaries

The goal of this brief note is to present a general result on the longstanding problem of estimating the number of even values of the Fourier coefficients of arbitrary eta-quotients (see below for the relevant definitions). In fact, we will do so over any arithmetic progression. Namely, denoting by $g_{a,b}(x)$ the number of even coefficients of an eta-quotient $F$ in degrees $n \equiv b \pmod{a}$ such that $n \leq x$, in Theorem 2 we show that

$$\frac{g_{a,b}(x)}{\sqrt{x}}$$

is unbounded for $x$ large.

Our paper was originally motivated by the preprint [17], which asked whether a specific eta-quotient assumes infinitely many even, and infinitely many odd values over any arithmetic progression. (See Conjecture 2 of the arXiv version v3 of [17], which has since been updated because of a mistake in one of the proofs, unrelated to our own paper. We thank the author for informing us.) Our Theorem 2 positively answers the even part of that conjecture as a...
very special case; for the odd part, see Question 5 at the end of this note, again in the much broader framework of arbitrary eta-quotients.

The proof of Theorem 2 is substantially based upon, and generalizes, Serre’s classical theorem [16] on the parity of the ordinary partition function \( p(n) \) (in fact, of a broader class of functions) over any arithmetic progression, combined with a recent result by Cotron et al. [5] on the lacunarity modulo 2 of eta-quotients satisfying a suitable technical assumption. The use of the latter result, which implicitly requires the theory of modular forms, will constitute the only non-elementary portion of our argument.

We note that even for the special case of \( p(n) \), Serre’s proof also required modular forms in an essential fashion. In fact, while it is easy to see that the number of even values of the partition function for \( n \leq x \) has order \( \sqrt{x} \), no elementary proof is known to date that this number grows faster than \( \sqrt{x} \). At the end of this paper, we propose a problem, phrased entirely in terms of quadratic representations, whose solution would finally yield a modular form-free proof of Serre’s theorem for \( p(n) \).

We first briefly recall the main definitions. We refer the reader to, e.g., [8] and its references for any unexplained terminology. Set \( f_j = f_j(q) = \prod_{i \geq 1} (1 - q^{2i}) \). Then an eta-quotient is a quotient of the form

\[
F(q) = \frac{\prod_{i=1}^{u} f_{\alpha_i}^{r_i}}{\prod_{i=1}^{t} f_{\gamma_i}^{s_i}},
\]

for integers \( \alpha_i \) and \( \gamma_i \) positive and distinct, \( r_i, s_i > 0 \), and \( u, t \geq 0 \). (Note that, for simplicity, here we omit the extra factor of \( q^{(1/24)(\sum \alpha_i r_i - \sum \gamma_i s_i)} \) that appears in some definitions of \( F \), since this factor is irrelevant for the asymptotic estimates of this paper.)

We say that \( F(q) = \sum_{n \geq 0} a_n q^n \) is odd with density \( \delta \) if the number of odd coefficients \( a_n \) with \( n \leq x \) is asymptotic to \( \delta x \), for \( x \) large. Further, \( F \) is lacunary modulo 2 if it is odd with density zero (equivalently, if its number of odd coefficients is \( o(x) \)).

One of the best-known instances of an eta-quotient (1) is arguably

\[
\frac{1}{f_1} = \sum_{n \geq 0} p(n) q^n.
\]

Understanding the parity of \( p(n) \) is a horrendously difficult and truly fascinating problem, which has historically attracted the interest of the best mathematical minds. A classical conjecture by Parkin-Shanks [4, 13] predicts that \( p(n) \) is odd with density \( 1/2 \) (see [6, 7, 8] for generalizations of this conjecture). However, the best bounds available today, obtained after a number of incremental results, only guarantee that the even values of \( p(n) \) are of order at least \( \sqrt{x} \log \log x \) [3], and the odd values at least \( \sqrt{x}/ \log \log x \) [2].
Thanks to theorems by Ono [12] and Radu [14], we also know that \( p(n) \) assumes infinitely many odd, and infinitely many even values over any arithmetic progression. In fact, as we mentioned earlier, Serre’s result [16] established that the number of even values of \( p(n) \) for \( n \leq x, n \equiv b \pmod{a} \) grows faster than \( \sqrt{x} \), for any choice of \( a \) and \( b \). However, it is reasonable to believe, as a generalization of the Parkin-Shanks conjecture (see [10]), that \( p(n) \) is even with density \( 1/2 \) over any arithmetic progression. For a broader set of conjectures on the parity of eta-quotients, including their behavior over arithmetic progressions, see our recent paper with Keith ([8], Conjecture 4).

In view of the above, it appears that our bound of Theorem 2 — which holds in full generality for any eta-quotient, and is very close to the best known result even in the case of \( p(n) \) [3] — might be hard to improve significantly with the existing technology.

2. The bound

We first need a recent theorem by Cotron et al., which we restate in the following terms:

**Lemma 1** ([5], Theorem 1.1). Let \( F(q) = \prod_{i=1}^{u} f_{r_i}^{r_i} \prod_{i=1}^{t} f_{s_i}^{s_i} \) be an eta-quotient as in (1), and assume that

\[
\sum_{i=1}^{u} r_i \alpha_i \geq \sum_{i=1}^{t} s_i \gamma_i.
\]

Then \( F \) is lacunary modulo 2.

We are now ready for the main result of this note. In what follows, given two power series \( A(q) = \sum_{n \geq 0} a(n)q^n \) and \( B(q) = \sum_{n \geq 0} b(n)q^n \), if we write \( A(q) \equiv B(q) \) we always mean that \( a(n) \equiv b(n) \pmod{2} \), for all \( n \).

**Theorem 2.** Let \( F(q) = \sum_{n \geq 0} a_nq^n \) be an eta-quotient as in (1), and denote by \( g_{a,b}(x) \) the number of even values of \( a_n \) over the arithmetic progression \( n \equiv b \pmod{a} \), for \( n \leq x \). Then

\[
\lim_{x \to \infty} \frac{g_{a,b}(x)}{\sqrt{x}} = \infty.
\]

**Proof.** Let

\[
F(q) = \frac{\prod_{i=1}^{u} f_{r_i}^{r_i}}{\prod_{i=1}^{t} f_{s_i}^{s_i}} = \sum_{n \geq 0} a_nq^n
\]

be as in the statement. Since

\[
\frac{q^b}{1-q^a} = \sum_{j \geq 0} q^{b+ja},
\]

it is clear that the coefficients of the series \( G \) defined by

\[
G(q) = \frac{q^b}{1-q^a} + F(q)
\]
coincide with those of $F$ except precisely in degrees $n \equiv b \pmod{a}$, where they differ by 1. In particular, $F$ is even in any degree $n \equiv b \pmod{a}$ if and only if $G$ is odd in that degree.

Now fix a positive integer $d$. By definition of $G$, we have the identity:

$$G(q) \cdot f_{a}^{2d} = q^b \cdot \frac{f_{a}^{2d}}{1-q^a} + f_{a}^{2d} \cdot \prod_{i=1}^{u} f_{\alpha_{i}}^{r_{i}} \prod_{i=1}^{t} f_{\gamma_{i}}^{s_{i}}.$$ 

Using the reduction modulo 2 of Euler’s Pentagonal Number Theorem (see, e.g., [1]),

$$f_{1} \equiv \sum_{n \in \mathbb{Z}} q^{n(3n-1)/2},$$

we obtain:

$$f_{a}^{2d} = \left( \sum_{n \in \mathbb{Z}} q^{an(3n-1)/2} \right)^{2d} \equiv \sum_{n \in \mathbb{Z}} q^{2d \cdot an(3n-1)/2}.$$

It follows by standard computations that, for $x$ large, the number of odd coefficients of $f_{a}^{2d}$ in degrees $n \leq x$ is asymptotic to

$$\frac{c_{0} \sqrt{x}}{2^{d}},$$

where

$$c_{0} = \frac{2 \sqrt{2}}{a \sqrt{3}}.$$ 

Further, all odd coefficients appear in degrees $n \equiv 0 \pmod{a}$.

From the Pentagonal Number Theorem, we also deduce the modulo 2 identity:

$$\frac{f_{1}}{1-q} \equiv (1 + q + q^2 + q^5 + q^7 + q^{12} + q^{15} + q^{22} + \ldots)(1 + q + q^2 + q^3 + \ldots)$$

$$\equiv 1 + (q^2 + q^3 + q^4) + (q^7 + \ldots + q^{11}) + (q^{15} + \ldots + q^{21}) + \ldots.$$ 

(3)

Note that the last series alternates strings of consecutive powers with coefficient 1 to strings (omitted) of consecutive powers with coefficient 0, where a new string begins any time a degree is a generalized pentagonal number.

Given this, it is easy to see that, for $x$ large, the number of odd coefficients of $f_{1}/(1-q)$ (or equivalently, the number of 1s appearing in (3)) in degrees $n \leq x$ is asymptotic to $2x/3$.

Moreover, since $f_{a}^{2d} \equiv f_{2d} \pmod{2}$, an entirely similar argument gives that the corresponding asymptotic value for the odd coefficients of

$$\frac{f_{a}^{2d}}{1-q^a}$$

is again $2x/3$. Thus, if we replace $q$ with $q^a$, it is clear that the number of odd coefficients of

$$\frac{f_{a}^{2d}}{1-q^a}$$

in degrees $n \leq x$ is asymptotic to $c_{1} x$, with $c_{1} = 2/(3a)$. 

It follows that the number of odd coefficients of the first term on the right side of (2),

$$q^b \cdot \frac{f_a^{2d}}{1 - q^d},$$

is asymptotic to $c_1 x$. Note that these coefficients all appear in degrees $n \equiv b \pmod{a}$.

Now consider the second term on the right side of (2), namely

$$f_a^{2d} \cdot \prod_{t=1}^u f_{\alpha_t} \prod_{i=1}^t f_{\gamma_i}.$$

By Lemma 1, we obtain that (4) is lacunary modulo 2 whenever

$$\frac{2d}{a} + \sum_{i=1}^u \frac{r_i}{\alpha_i} \geq \sum_{i=1}^t s_i \gamma_i,$$

or equivalently,

$$2^d \geq a \left( \sum_{i=1}^t s_i \gamma_i - \sum_{i=1}^u \frac{r_i}{\alpha_i} \right).$$

Thus, the lacunarity of (4) is guaranteed for all integers $d$ large enough.

Putting the above together, for any large integer $d$, the number of odd coefficients on the right side of (2) in degrees $n \leq x$ is asymptotic to

$$c_1 x + o(x),$$

or simply to $c_1 x$. Asymptotically, again $c_1 x$ of these odd coefficients are in degrees $n \equiv b \pmod{a}$.

Recall that the number of odd coefficients of $f_a^{2d}$, which is the second factor on the left side of (2), was shown to be asymptotic to $c_0 \sqrt{x}/2^d$ (where the constant $c_0$ is independent of $d$). Also, all such coefficients appear in degrees $n \equiv 0 \pmod{a}$.

We conclude that, for $n \equiv b \pmod{a}$, $n \leq x$, the number of odd coefficients of the first factor, $G$, on the left side of (2) — or equivalently, the number $g_{a,b}(x)$ of even coefficients of the original eta-quotient $F$ — must satisfy:

$$g_{a,b}(x) \geq c_2 \cdot \frac{c_1 x}{(c_0 \sqrt{x})/2^d} = c_3 \cdot 2^d \sqrt{x},$$

for a suitable positive constant $c_2$ and for $x$ large, where

$$c_3 = \frac{c_2 c_1}{c_0} > 0$$

is independent of $d$. Thus,

$$\frac{g_{a,b}(x) \sqrt{x}}{2^d} \geq c_3 \cdot 2^d,$$

for $x$ large. Since this is true for all $d$ large, the theorem follows. $\square$
3. Questions for future research

As we mentioned earlier, Lemma 1, which was a key ingredient in the proof of Theorem 2, made an essential use of modular forms. We are not aware of a modular form-free proof of our result, even for \( p(n) \). In fact, Serre’s argument \[16\] that the even values of \( p(n) \), for \( n \leq x \), grow faster than \( \sqrt{x} \) also relied on modular forms; to be precise, it can be seen that their use may essentially be limited to showing that \( f_1^{2d-1} \) is lacunary modulo 2 for infinitely many values of \( d \).

We now propose a problem, elegantly stated in terms of quadratic representations, which is equivalent to the lacunarity modulo 2 of \( f_1^{2d-1} \). (For brevity’s sake, we omit the proof of this equivalence, which employs the Jacobi triple product identity and other elementary tools.) Thus, a direct proof of Problem 3 would lead, as a byproduct, to the first modular form-free proof of Serre’s theorem in the case of \( p(n) \).

**Problem 3.** For a positive integer \( d \), consider the polynomial

\[
T_d(x_1, \ldots, x_{2d-1}) = \sum_{i=1}^{2d-1} 2^d x_i^2 - (2i - 1)x_i,
\]

and let

\[
R_d(n) = \# \{(x_1, \ldots, x_{2d-1}) \in \mathbb{Z}^{2d-1} : T_d(x_1, \ldots, x_{2d-1}) = n \}.
\]

Show that, for any \( d \geq 1 \),

\[
\# \{n \leq x : R_d(n) \text{ is odd} \} = o(x).
\]

**Remark 4.**

1. As we saw earlier, the lacunarity modulo 2 of \( f_1^{2d-1} \), which is equivalent to the statement of Problem 3 for \( d \), is already known via modular forms \[16\]. In fact, when \( d \geq 2 \), with more work (and more modular forms) one can estimate exactly that, for \( x \) large, \( \# \{n \leq x : R_d(n) \text{ is odd} \} \) is asymptotic to

\[
c_1 x (\log \log x)^{c_2} / \log x,
\]

for suitable constants \( c_1 \) and \( c_2 \) depending on \( d \) (see \[11, 15\] for details).

2. In order to reprove Serre’s theorem on the even values of \( p(n) \), one can show that, in fact, it suffices to solve Problem 3 only for infinitely many values of \( d \), and over any arithmetic progression \( cn + r \) in lieu of \( n \), provided that \( c \) grow slower than \( 4^d \).

3. As an illustration, when \( d = 1 \), it is easy to see that \( R_1(n) \) is odd (it equals 1) if and only if \( n = \binom{a+1}{2} \). Thus, the estimate \( o(x) \) is trivial. When \( d = 2 \), \( R_2(n) \) is odd precisely when the number of representations of \( n \) as \( 4\binom{a+1}{2} + \binom{b+1}{2} \) is odd. Since by a classical result of Landau \[9, 15\], a binary quadratic form is lacunary over the integers, this is a fortiori true modulo 2, and the result again follows.
We conclude by posing a general question on the number of odd values of arbitrary eta-quotients over arithmetic progressions.

**Question 5.** Let \( F(q) = \sum_{n \geq 0} a_n q^n \) be an eta-quotient as in (7), and assume \( F \) is not constant modulo 2 over the arithmetic progression \( n \equiv b \pmod{a} \). Is it true that the number of odd values of \( a_n \) for \( n \equiv b \pmod{a} \), \( n \leq x \), has always order at least \( \sqrt{x} \)?

Note that the lower bound of Question 5 in general cannot be improved (\( \sqrt{x} \) is well known to be the correct order for the odd values of, e.g., \( f_1 \) and \( f_3 \)). However, the question is still open in many important instances; among others, for the generating function \( 1/f_1 \) of \( p(n) \), and for all of its positive powers \( 1/f_1^t \), which define the \( t \)-multipartition functions \( p_t(n) \) 2 6 7.

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**References**

[1] G. Andrews: “The Theory of Partitions,” Encyclopedia of Mathematics and its Applications, Vol. II. Addison-Wesley, Reading, Mass.-London-Amsterdam (1976).

[2] J. Bellaïche, B. Green, and K. Soundararajan: *Non-zero Coefficients of Half-Integral Weight Modular Forms Mod \( \ell \)*, Res. Math. Sci. 5 (2018), no. 1, Paper no. 6, 10 pp.

[3] J. Bellaïche and J.-L. Nicolas: *Parité des coefficients de formes modulaires*, Ramanujan J. 40 (2016), no. 1, 1–44.

[4] N. Calkin, J. Davis, K. James, E. Perez, and C. Swannack: *Computing the integer partition function*, Math. Comp. 76 (2007), 1619–1638.

[5] T. Cotron, A. Michaelsen, E. Stamm, and W. Zhu: *Lacunary Eta-quotients Modulo Powers of Primes*, Ramanujan J. 53 (2020), 269–284.

[6] S. Judge, W.J. Keith, and F. Zanello: *On the Density of the Odd Values of the Partition Function*, Ann. Comb. 22 (2018), no. 3, 583–600.

[7] S. Judge and F. Zanello: *On the density of the odd values of the partition function, II: An infinite conjectural framework*, J. Number Theory 188 (2018), 357–370.

[8] W.J. Keith and F. Zanello: *Parity of the coefficients of certain eta-quotients*, J. Number Theory 235 (2022), 275–304.

[9] E. Landau: *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. Math. Phys. (3) 13 (1908), 305–312.

[10] J.-L. Nicolas and A. Sárközy: *On the parity of partition functions*, Illinois J. Math. 39 (1995), no. 4, 586–597.

[11] J.-L. Nicolas and J.-P. Serre: *Formes modulaires modulo 2: l’ordre de nilpotence des opérateurs de Hecke*, C.R. Acad. Sci. Paris, Ser. I 350 (2012), 343–348.
[12] K. Ono: *On the parity of the partition function in arithmetic progressions*, J. Reine Angew. Math. 472 (1996), 1–15.

[13] T.R. Parkin and D. Shanks: *On the distribution of parity in the partition function*, Math. Comp. 21 (1967), 466–480.

[14] C.-S. Radu: *A proof of Subbarao’s conjecture*, J. Reine Angew. Math. 672 (2012), 161–175.

[15] J.-P. Serre: *Divisibilité de certaines fonctions arithmétiques*, L’Enseignement Math. 22 (1976), 227–260.

[16] J.-P. Serre: Appendix to: J.-L. Nicolas, I.Z. Ruzsa, and A. Sárközy: *On the parity of additive representation functions*, J. Number Theory 73 (1998), no. 2, 292–317.

[17] Q.-Y. Zheng: *Distribution of partitions of n in which no part appears exactly once*, preprint (arXiv:2205.03191; version v3 of May 12, 2022).

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