In recent years, there has been growing concern over the widening inequalities in income and wealth distributions in the U.S. and elsewhere \(^1, 3\). The statistics are troubling – for instance, as of 2010, the top 1% of households in the U.S. owned 35.4% of all privately held wealth \(^2\), and it had risen from a low of about 20% in 1976.

An important source of the wealth inequality are similar trends in the income and pay or wage distributions. Income remains highly concentrated, with the top 1% of income earners received 17.2% of all income in 2009, and that’s up from 12.8% in 1982 \(^4, 5\). A related trend of equally great concern is the runaway pay packages for CEOs which are reflected in the extraordinarily high CEO pay ratios in the U.S. \(^6, 7\). There is much discussion both in the academic work and popular press about what all these mean, what the consequences are, and what can or should be done about it \(^8, 12\).

However, before we can formulate policies to address these challenges, it is important to understand, in greater depth, what kinds of pay distributions naturally arise in a free market environment and why. While there exists extensive literature on these topics, and we cite only a sample here \(^6, 8, 9, 13–16\), they are largely empirical in nature. Such market data are indeed very valuable in assessing the current state of affairs, but such a purely empirical view, however, is incomplete and unsatisfactory without a deeper understanding of what economic theories and models could say about pay distributions.

Instead of just relying on empirical data alone, can we predict, at least under ideal conditions, what to expect from a theoretical analysis? A fundamental question one would like answered is: What kinds of pay distributions will arise, under ideal conditions, in a free market environment comprising of utility maximizing employees and profit maximizing companies? If we can answer this question, it will serve as a fundamental benchmark against which we can evaluate the distributions seen in real life. This reference can help us measure and understand the deviations caused by non-idealities under actual conditions, and to develop appropriate policy frameworks and incentive structures to try to correct the inequalities. It can give us a quantitative basis for understanding and developing pay packages for executives, tax policies, etc.

Since there appears to be no satisfactory answer to this central question in conventional economic theories and models, this has stimulated, in the past decade or so, much work in the econophysics community to model income and wealth distributions by applying concepts and techniques from statistical mechanics \(^16–28\). While these models are quite interesting, they haven’t bridged the rather wide conceptual gulf that exists between economics and econophysics \(^29, 30\), particularly in two crucial areas. One, the typical particle model of agent behavior in econophysics assumes agents to have nearly “zero intelligence”, acting at random, with no intent or purpose. This does not sit well with an extensive body of economic literature spanning several decades, where one models, in the ideal case, a perfectly rational agent whose goal is to maximize its utility or profit by acting strategically, not randomly. From the perspective of an economist, it is quite reasonable to ask “How can theories and models based on the collective behavior of purpose-free, random, molecules explain the collective behavior of goal-driven, optimizing, strategizing men and women?”

I. INTRODUCTION

The widening inequality in income distribution in recent years, and the associated excessive pay packages of CEOs in the U.S. and elsewhere, is of growing concern among policy makers as well as the common person. However, there seems to be no satisfactory answer, in conventional economic theories and models, to the fundamental question of what kind of pay distribution we ought to see in a free market environment, at least under ideal conditions. We propose a novel game theoretic framework that addresses this question and shows that the lognormal distribution is the fairest inequality of pay in an organization, achieved at equilibrium, under ideal free market conditions. Our theory also shows the deep and direct connection between potential game theory and statistical mechanics through entropy, which is a measure of fairness in a distribution. This leads us to propose the fair market hypothesis, that the self-organizing dynamics of the ideal free market, i.e., Adam Smith’s “invisible hand”, not only promotes efficiency but also maximizes fairness under the given constraints.

Game theory, statistical mechanics, and income inequality

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Another conceptual stumbling block is the role of entropy in economics. In statistical thermodynamics, equilibrium is reached when entropy, which is a measure of randomness or uncertainty, is maximized. So, an economist wonders, why would maximizing randomness or uncertainty be helpful in economic systems? We all know that markets are stable, and function well, when things are orderly, with less uncertainty, not more. As Amartya Sen observed \[31\], “Given the association of doom with entropy in the context of thermodynamics it may take a little time to get used to entropy as a good thing (‘How grand, entropy is on the increase!’).” Similar objections were raised by Paul Samuelson \[32\]: “As will become apparent, I have limited tolerance for the perpetual attempts to fabricate for economics concepts of ‘entropy’ imported from the physical sciences or constructed by analogy to Clausius-Boltzmann magnitudes.” Thus, we run into major conceptual hurdles in the typical statistical mechanics-based approaches to problems in economics, particularly in the study of pay, income and wealth distributions. Besides these conceptual challenges, there is also a more technical one due to the nature of the datasets in economics. As Ormerod \[30\], and Perline \[33\] discuss, one can easily misinterpret data from lognormal distributions, particularly from truncated datasets, as inverse power law or other distributions. Therefore, empirical verification of econophysics models is still in the early stages.

Addressing one of the two conceptual challenges, Venkatasubramanian proposed an information-theoretic framework \[34, 35\] wherein he identified that the entropy is really a measure of fairness in a distribution, not just randomness or uncertainty, which then makes it a suitable candidate in economics. In this paper, we follow up on this line of enquiry and address the other critical challenge of reconciling the behavior of goal-driven, teleological, agents with that of purpose-free molecules. We start from a familiar ground in economics, namely, game theory, to develop a new conceptual framework to address the pay distribution question we raised earlier. This leads to surprising and useful insights about a deep connection between game theory and statistical mechanics, paving the way for a general theoretical framework that unifies the dynamics of purposeful animate agents with that of purpose-free inanimate ones.

II. PAY DISTRIBUTION IN AN IDEAL FREE MARKET ENVIRONMENT: FORMULATING THE PROBLEM

We follow Venkatasubramanian’s \[35\] approach in formulating the problem and restate it here for the convenience of the reader. Consider a competitive, dynamic, free market environment comprising of a large number of utility maximizing rational agents as employees and profit maximizing rational agents as corporations. We assume an ideal environment where the market is perfectly competitive, transaction costs are negligible, and no externalities are present. In this ideal free market, employees are free to switch jobs and move between companies in search of better utilities. Similarly, companies are free to fire and hire employees in order to maximize their profits. We do not consider the effect of taxes.

We also assume that a company needs to retain all its employees in order to survive in this competitive market environment. Thus, a company will take whatever steps necessary, allowed by its constraints, to retain all its employees. Similarly, all employees need a utility to survive and that they will do whatever is necessary, allowed by certain norms, to stay employed. We assume that neither the companies nor the employees engage in illegal practices such as fraud, collusion, and so on.

In this ideal free market, consider a company $A$ with $N$ employees and a salary budget of $M$, with an average salary of $S_{ave} = M/N$. Let us assume that there are $n$ categories of employees – ranging from secretaries to the CEO, contributing in different ways towards the company’s overall success and value creation. All employees in category $i$ contribute value $V_i$, $i \in \{1, 2, \ldots, n\}$, such that $V_1 < V_2 < \cdots < V_n$. Let the corresponding value at $S_{ave}$ be $V_{ave}$, occurring at category $s$. Since all employees are contributing unequally, some more some less, they all need to be compensated differently, communally with their relative contributions towards the overall value created by the company. Instead, $A$ has an egalitarian policy that all employees are equal and therefore pays all of them the same salary, $S_{ave}$, irrespective of their contributions. The salary of the CEO is the same as that of an administrative assistant in the mailroom. This salary distribution is a sharp vertical line at $S_{ave}$, as seen in figure \[1\](a), the Kronecker delta function of magnitude $N$, given by:

\[
\text{Prob}(S) = N\delta_{ls},
\]

where $\delta_{ls} = 1$, if $i = s$ and $\delta_{ls} = 0$, if $i \neq s$.

As noted, while this may seem fair in a social or moral justice sense, clearly it is not in an economic sense. If this were to be the only company in the economic system, or if $A$ is completely isolated from other companies in the economic environment, the employees will be forced to continue to work under these conditions as there is no other choice.

However, in an ideal free market system there are other choices. Therefore, all those employees who contribute more than the average - i.e., those in value categories $V_i$ such that $V_i > V_{ave}$ (e.g., senior engineers, vice presidents, CEO), who feel that their contributions are not fairly valued and compensated for by $A$, will therefore be motivated to leave for other companies where they are offered higher salaries. Hence, in order to survive $A$ will be forced to match the salaries offered by others to retain these employees, thereby forcing the distribution to spread to the right of $S_{ave}$, as seen in figure \[1\](b).

At the same time, the generous compensation paid to all employees in categories $V_i$ such that $V_i < V_{ave}$, will
motivate candidates with the relevant skill sets (e.g., low-level administration, sales and marketing staff) from other companies to compete for these higher paying positions in A. This competition will eventually drive the compensation down for these overpaid employees forcing the distribution to spread to the left of $S_{\text{ave}}$, as seen in Figure II(c). Eventually, we will have a distribution that is not a delta function, but a broader one where different employees earn different salaries depending on the values of their contributions. The funds for the higher salaries now paid to the formerly underpaid employees (i.e., those who satisfy $V_i > V_{\text{ave}}$) come out of the savings resulting from the reduced salaries of the formerly overpaid group (i.e., those who satisfy $V_i < V_{\text{ave}}$), thereby conserving the total salary budget $M$.

Thus, we see that concerns about fairness in pay cause the emergence of a more equitable salary distribution in a free market environment through its self-organizing, adaptive, evolutionary dynamics and that its spread is closely related to fairness in relative compensation. The point of this analysis is not to model the exact details of the free market dynamics but to show that the notion of fairness plays a central role in driving the emergence and spread of the salary (in general, utility) distribution through the free market mechanisms.

Even though an individual employee cares only about her utility and no one else’s, the collective actions of all the employees, combined with the profit maximizing survival actions of all the companies, in an ideal free market environment of supply and demand for talent, under resource constraints, lead towards a more fair allocation of wages, guided by Adam Smith’s “invisible hand” of self-organization.

We have used salary as a proxy for utility in this example to motivate the problem. In general, utility for an employee is a complicated aggregate that depends on a host of factors, some measurable, some not. Obviously, pay (i.e., total compensation including base salary, bonus, options etc.) is an important component of the utility. Other components include, quantity and quality of the work or effort, authority and power of the position, job security, competition, career and personal growth opportunities, work schedule, retirement and health benefits, peer appreciation and recognition, company culture and work environment, job location, and so on, not necessarily in that order.

Given this free market dynamics scenario, two important questions arise: (i) Will this self-organizing dynamics lead to an equilibrium distribution or will the distribution continually evolve without ever settling down? (ii) If there exists an equilibrium distribution, what is it?

Our knowledge of the free market dynamics is incomplete, in an important way, without an answer to these fundamental questions. This requires a theoretical understanding of the free market dynamics, at a reasonable level of depth and detail, particularly from the bottom up, agents-based, perspective as described above. Given the obvious complexity of this dynamics, it is unrealistic to expect to develop a theory, and the associated models, that will address all the details and nuances. Therefore, our goal is to develop a theoretical, quantitative, framework that identifies the key concepts and general principles, helps us model and analyze free market dynamics under ideal conditions, and address these central questions. We propose such a framework in the following sections.

III. A GAME THEORETIC FRAMEWORK: “RESTLESS” AGENTS MODEL

A. Formulating the payoff function

We believe these questions are best addressed using the game theoretic framework. Continuing with the scenario described above, we assume that all employee agents are generally “dissatisfied” in their current positions, due to aforementioned unfairness considerations. In our model, every employee feels that she is unfairly undervalued compared to others in their peer group. Every employee feels they could be doing better, they should be doing better, given their talents and experience, in their company or elsewhere. As a result, they all are constantly on the lookout for job opportunities to improve their utilities. That is, these utility-maximizing, fairness-seeking, teleological agents are always restless, itching to move.

Even though the utility for an employee is a complex aggregate of several factors, we propose that it is broadly composed of three dominant elements: (i) utility derived from salary, (ii) disutility from effort, and (iii) utility from fairness. The first two are rather straightforward to see, but the third requires some more discussion along the lines of the scenario described above. The first two help us model the tendency of an employee to maximize one’s utility from salary while minimizing the effort put into receiving it. As for the third, consider the following.

At any job level, an agent is looking to improve her utility only in the jobs space that is accessible to her based on her education, experience, and other such qualifications. That is, a receptionist is not eyeing the job elsewhere. As a result, they all are constantly on the lookout for job opportunities to improve their utilities. That is, these utility-maximizing, fairness-seeking, teleological agents are always restless, itching to move.
value. For instance, a vice president is not necessarily very happy that she is enjoying much more utility than her receptionist, but is extremely unhappy that her peer, another vice president with comparable (or perhaps even less) skills and contributions, has been better recognized in the organization with a higher salary, better work assignments, more perks etc., thereby enjoying a higher utility than her. Further, she believes she should really be at a higher job category, perhaps as the CEO. Thus, in her own assessment, she should be doing better, a lot better. If her current employer does not offer her the desired utility, she will then go elsewhere where she can get that. Therefore, as far as this “unhappy” agent is concerned, the metric that matters to her is whether she is one of the chosen few or one of the many in her peer level. Her preference is to be one of the few and possibly the only one enjoying a lot of utility. That is, the payoff is better if the agent is one of the select few in her or his job level that is commensurate with her or his qualifications and effort.

Combining all three, we have

\[ h_i(S_i, E_i, N_i) = u(S_i) - v(E_i) + f(N_i), \quad (2) \]

where \( h_i \) is the total utility of an employee earning a salary \( S_i \) by expending an effort \( E_i \), while competing with \((N_i - 1)\) other agents in the same job category \( i \) for a fair recognition of one’s contributions. \( u(\cdot) \) is the utility derived from salary, \( v(\cdot) \) the disutility from effort, and \( f(\cdot) \) is the utility from fairness that depends only on the number of agents in any given category \( i \). We propose the following functional forms for these three elements:

\[ u(S_i) = \alpha \ln S_i \quad (3) \]
\[ v(E_i) = \beta (\ln S_i)^2 \quad (4) \]
\[ f(N_i) = -\gamma \ln N_i \quad (5) \]

where \( \alpha, \beta, \gamma > 0 \). The first one is easy to see, it is the commonly used logarithmic utility function. As for the second, since effort (comprising of both quality and quantity of effort or work) is hard to observe and quantify, we use the salary itself as a proxy to capture it. Generally, salary is a good indicator of the effort required to earn it. By the way, by effort we not only mean the quantity and quality of work involved, but also the necessary qualifications such as education, skills, experience, etc., needed to do the required work. These are naturally embedded in effort, automatically implied. For example, someone with no medical training, obviously, will not be able to perform the work of a cardiac surgeon successfully, no matter how hard he or she tries. Thus, the requisite qualifications to do the job well are implied when we model effort. We assume the commonly used quadratic form for effort \([36, 37]\). For the utility derived from fairness, we believe the negative logarithmic form captures the agents’ preferences and behavior correctly as explained below.

Since \( N_i \in [0, \infty], f_i \in [\infty, -\infty] \). This payoff function may be intuitively interpreted as capturing the following:

(i) When \( N_i \to \infty, f_i \to -\infty \), i.e., an agent in level \( i \) feels “unfairly” treated and undervalued as there are so many other agents at the same job category, thereby reducing her utility.

(ii) When \( N_i = 0, f_i = \infty \). This is the state where the agent can be potentially the “happiest”, most valued and appreciated, the state all the agents strive for. But since this is an elusive state (for when the agent arrives at this level, \( N_i \) is no longer 0 but 1, and \( f_i = 0 \) and not \( \infty \)), the agents are constantly in motion, restless, chasing after this dream of the “perfect” job. While this is clearly a simplification of what really happens in the market place, we believe, this stylized model nevertheless captures an essential, and dominating, aspect of the dynamical behavior of fairness-seeking, utility-maximizing, teleological agents, namely, restlessness, that most employees feel in the real world.

In general, \( \alpha, \beta \) and \( \gamma \), which model the relative importance an agent assigns to these three elements, can vary from agent to agent. However, for the sake of simplicity, we assume that all agents have the same preferences and hence treat these as constant parameters. Further, presumably, there are other expressions one could use to model these three elements, but the choices we have made have interesting properties, revealing important insights and connections as we shall see shortly.

In order to move to a job with better utility, an agent needs job offers. So, the employee agents constantly gather information and scout the market, and their own companies, for job openings that are commensurate with their skill sets, experiences and career and personal goals. Similarly, the company agents (through their human resources department, for example) are also conducting similar searches looking for opportunities to fire and hire employees so that their profits may be improved.

At any given time, an employee agent is faced with one of five job options: (i) no new job offer is available, (ii) new offer has the same utility as the current one, (iii) new offer has less utility than the current one, (iv) new offer has more utility, or (v) is let go from the current job (i.e., zero utility). The agent’s best strategies for the five options are: for (i), (ii) and (iii), the agent stays put in the current position at the current utility, for (iv) accept the new offer, and (v) leave the company and look for a new position. Each agent’s strategy is independent of what the other agents are doing.

We are now ready to answer the first question.

B. Is there an equilibrium distribution?

In a potential game framework, payoff is the gradient of potential \( \phi(x) \) i.e.,

\[ h_i(x) \equiv \partial \phi(x)/\partial x_i, \quad (6) \]
where \( x_i = N_i/N \) and \( x \) is the population vector. Therefore, by integration (we replace partial derivative with total derivative because \( h_i(x) \) can be reduced to \( h_i(x_i) \) expressed in Equations (2)-(5)),
\[
\phi(x) = \sum_{i=1}^{n} h_i(x_i) dx_i,
\]
we obtain the potential of the game:
\[
\phi(x) = \phi_u + \phi_v + \phi_f + \text{constant},
\]
where
\[
\begin{align*}
\phi_u &= \alpha \sum_{i=1}^{n} x_i \ln S_i \\
\phi_v &= -\beta \sum_{i=1}^{n} x_i (\ln S_i)^2 \\
\phi_f &= \frac{\beta}{N} \ln \Pi_{i=1}^{N} (N x_i).
\end{align*}
\]
We can show that \( \phi(x) \) is strictly concave:
\[
\partial^2 \phi(x)/\partial x_i^2 = -\gamma/x_i < 0.
\]
Therefore, a unique Nash Equilibrium for this game exists, as per the well-known theorem [38, p. 60].

It is important to note that this is a stable equilibrium as long as the evolutionary dynamics satisfies positive correlation (e.g., replicator dynamics, Smith dynamics, best response dynamics, etc.), for the potential is a Lyapunov function under such condition, with a guarantee of global convergence [38, p. 223].

This answers our first question.

C. Connection with statistical mechanics

Readers familiar with statistical mechanics will recognize the potential component \( \phi_f \) as entropy, and that maximizing the payoff potential in game theoretic equilibrium would correspond to maximizing entropy in statistical mechanical equilibrium, revealing a deep and useful connection between these seemingly different conceptual frameworks. This connection suggests that one may view the statistical mechanics approach to molecular behavior, also called statistical thermodynamics, from a potential game perspective. In this approach, one may view the molecules as restless agents in a game (let’s call it the thermodynamic game), continually jumping from one energy state to another through intermolecular collisions. However, unlike employees who are continually driven to switch jobs in search of better utilities they desire, molecules are not teleological, i.e., not goal-driven, in their constant search. As prisoners of Newton’s Laws, that are constantly subjected to intermolecular collisions, their search and dynamical evolution is the result of thermal agitation.

D. What is the equilibrium distribution?

This connection to statistical thermodynamics, and the insight that \( \phi_f \) is entropy in this context, helps us in answering the second question: What is the equilibrium distribution?

We first answer this question for the thermodynamic game. Approaching the thermodynamic game from potential game perspective, we have the following “utility” for molecules in state \( i \):
\[
h_i(E_i, N_i) = -\beta E_i - \ln N_i,
\]
where \( E_i \) is the energy of molecule in state \( i \), \( \beta = 1/kT \), \( k = 1.3806488 \times 10^{-23} \text{ JK}^{-1} \) is Boltzmann constant; and \( T \) is temperature. By integrating the utility, we can obtain the potential of the thermodynamic game:
\[
\phi(x) = -\beta E + \frac{1}{N} \ln \Pi_{i=1}^{n} (N x_i)!
\]
where \( E = N \sum_{i=1}^{n} x_i E_i \) is the total energy that is conserved.

We use the method of Lagrange multipliers with \( L \) as the Lagrangian and \( \lambda \) as the Lagrange multiplier for the constraint \( \sum_{i=1}^{n} x_i = 1 \):
\[
L = \phi + \lambda (1 - \sum_{i=1}^{n} x_i).
\]
Solving \( \partial L/\partial x_i = 0 \) and substituting the results back to \( \sum_{i=1}^{n} x_i = 1 \), we obtain the well-known Gibbs-Boltzmann exponential distribution at equilibrium:
\[
x_i = \frac{\exp(-\beta E_i)}{\sum_{j=1}^{n} \exp(-\beta E_j)}.
\]
What we just now did is the standard procedure followed in maximum entropy methods in statistical mechanics and information theory to identify the distribution that maximizes entropy under the given constraints [39-41].

Once again, readers familiar with statistical thermodynamics will recognize that from [14], we have:
\[
\phi = -\frac{1}{NkT} (E - TS) = -\frac{\beta}{N} A,
\]
where \( A \) is the Helmholtz free energy, \( S \) is entropy, and \( T \) is temperature.

For the teledynamic game, i.e., the pay distribution game, we carry out the same procedure to maximize \( \phi(x) \) in Equations (5)-(11) to obtain the following lognormal distribution at equilibrium:
\[
x_i = \frac{1}{S_i Z} \exp \left[ -\frac{\left( \ln S_i - \frac{\alpha + \gamma}{2\beta} \right)^2}{\gamma/\beta} \right],
\]
where \( Z = N \exp \left[ \lambda/\gamma - (\alpha + \gamma)^2/4\beta \gamma \right] \) and \( \lambda \) is the Lagrange multiplier.
1. Replicator Dynamics

Alternatively, we can approach this question from the replicator dynamics point of view in game theory \[38\]. In this approach, an agent revises its strategy based on

$$
\rho_{ij} \propto x_i(h_j - h_i^+).
$$

(19)

Under this protocol, an agent in the job category \(i\) who receives a revision opportunity, i.e., a new job offer in category \(j\), switches from \(i\) to \(j\) with probability \(\rho_{ij}\). Therefore the dynamics becomes:

$$
\dot{x}_i \propto x_i(h_i - \sum_{j=1}^{n} x_j h_j).
$$

(20)

The equilibrium is reached (i.e., \(\dot{x} = 0\)) when individual payoff equals the average payoff of the system:

$$
h_i^* = \sum_{j=1}^{n} x_j h_j^* = h^*.
$$

(21)

We ignore the trivial solution of \(x_i = 0\). Substituting this equation back in our utility function (Equation (2)), we solve to find the equilibrium distribution to be

$$
x_i = \frac{1}{S_i Z} \exp \left[ - \frac{\left( \ln S_i - \frac{a + \gamma}{2\beta} \right)^2}{\gamma/\beta} \right],
$$

(22)

where \(Z = N \exp \left[ h^* / \gamma - (\alpha + \gamma)^2 / 4\alpha\gamma \right]\). This result agrees with \[18\].

This result is also in agreement with what Venkatasubramanian \[34, 35\] derived using an information theoretic framework. In that approach, the constraints are determined by information typically known about the distribution \(a priori\). They are: (i) total number of employees \(N\), (ii) total amount of money \(M\) budgeted to pay all these employees, (iii) minimum salary, \(S_{\text{min}}\), received by the lowest paid employee, often fixed by the minimum wage law or a reservation wage, and (iv) the maximum salary, \(S_{\text{max}}\), cannot exceed \(M\). As Venkatasubramanian has shown, maximizing entropy under these constraints leads to a lognormal distribution at equilibrium given by:

$$
f(S; \mu, \sigma) = \frac{1}{S\sigma\sqrt{2\pi}} \exp \left[ - \frac{(\ln S - \mu)^2}{2\sigma^2} \right],
$$

(23)

where \(\mu = \ln(M/N); \sigma = (\ln M - \ln S_{\text{min}})/2a;\) and \(a\) is a parameter chosen using the Chebychev inequality given by:

$$
\text{Prob}(-a\sigma < X - \mu < a\sigma) \geq 1 - \frac{1}{a^2},
$$

(24)

to the level of confidence desired in the estimate for \(\sigma\) (e.g. for \(a = 10, P \geq 0.99\). Equation (23) is same as \[18\] or \[22\] with the following identities:

$$
\begin{align*}
\mu &= \frac{\alpha + \gamma}{2\beta} \\
\sigma &= \left( \frac{\alpha^2}{2\beta} \right)^{1/2}.
\end{align*}
$$

(25)

For the thermodynamic game, it is easy to show from Equations (19) through (21), and (13), a similar replicator dynamics analysis produces the same Gibbs-Boltzmann exponential distribution in \[10\] at equilibrium.

Thus, we see that, intuitively, maximizing the game theoretic potential (Equation \[8\] or \[14\]) is the same as maximizing entropy subject to the constraints. In the statistical mechanical or information theoretic formulations, these constraints are separately imposed on entropy whereas in the game theoretic formulation (Equation \[8\] or \[14\]) the constraints are already an integral part of the equation (the only additional constraint imposed is the total number of agents, \(N\)). Therefore, the resulting Lagrangian (e.g., Equation (15)) is the same, thereby leading to the same distribution. These demonstrate the internal consistency among the three different approaches, namely, potential game theory, replicator dynamics, and statistical mechanics, which is reassuring.

IV. DISCUSSION AND CONCLUSION

There has been some work in the past that has explored the general connection between game theory and statistical mechanics \[42–44\]. What is new about our contribution is that it shows a deep and direct connection between the dynamics of animate, fairness-driven, utility-maximizing, rational teleological agents and inanimate, purpose-free, thermally-driven molecular entities. Our result reveals the surprising and important connection between entropy and payoff (or utility) potential, demonstrating that the statistical thermodynamic equilibrium reached by molecules is indeed a Nash Equilibrium. We believe that this is a significant insight, for it suggests that statistical thermodynamics can be seen as a special case of potential game theory. Alternatively, one may view this insight as the generalization of the laws of statistical thermodynamics to teleological systems, such as economic systems, yielding a new conceptual framework which we call statistical teleodynamics, that unifies statistical thermodynamics and population game theory.

This framework bridges the conceptual gulf mentioned in the introduction, as our ideal teleological agents are rational, fairness-seeking, utility maximizing strategists, with a natural connection to statistical thermodynamics.

As noted, one could presumably choose other expressions to model the three elements (Equations (3), (4), (5)) in (2), but it is not clear whether they will necessarily lead to the Gibbs-Boltzmann distribution, Helmholtz free energy or entropy in the limiting case of the thermodynamic game involving molecules. We find this correspondence to be particularly appealing, in fact comforting, that statistical teleodynamics properly reduces to well-known results in statistical thermodynamics as a limiting case. This universality has a nice ring to it.

Another important observation is that, in statistical thermodynamics, the claim about the equilibrium state
is a probabilistic one – it is the most probable outcome, one where entropy is maximum. However, our game theoretic result shows that the Nash Equilibrium state reached by the molecules, the one that maximizes the potential $\phi(x)$, is a deterministic outcome, not a probabilistic one. This observation has potentially important implications concerning the philosophical foundations of statistical thermodynamics, and that of information theory, such as ergodicity and metric transitivity, but we are not addressing them here. As we have shown, the deep connection between game theory and statistical mechanics, and with information theory, occurs via entropy, a concept that is often misunderstood and much maligned [31, 32, 34, 35]. The crucial insight is that entropy is a measure of fairness in a distribution, an insight that has not been explicitly recognized and particularly stressed in prior work in statistical thermodynamics, information theory, or economics. Even though there have been attempts such as the Thiele Index in the past, entropy has played, by and large, only a marginal role in economics, even that with strong objections from leading practitioners. It’s pivotal role in economics and free market dynamics has never been recognized. This is mainly because entropy’s essence as fairness appears as different facets in different contexts. In thermodynamics, being fair to all accessible phase space cells at equilibrium under the given constraints – i.e., assigning equal probabilities to all the allowed microstates – projects entropy as a measure of randomness or disorder [45]. This is the appropriate interpretation in this particular context, but it obscures the essential meaning of entropy as a measure of fairness. In information theory, being fair to all messages that could potentially be transmitted in a communication channel – i.e., assigning equal probabilities to all the messages – shows entropy as a measure of uncertainty [39, 40]. Again, while this is the appropriate interpretation for this application, this, too, conceals the real nature of entropy. In the design of teleological systems, being fair to all potential operating environments, entropy emerges as a measure of robustness i.e., maximizing system safety or minimizing risk [40]. Once again, this is the right interpretation for this domain, but this also hides its true meaning.

Thus, the common theme across all these different contexts is the essence of entropy as a measure of fairness, which stems from the notion of equality expressed mathematically. If there are $N$ possible candidates among whom a resource is to be distributed, and if no particular candidate is to be preferred over another, then the fairest distribution of the resource is one of equal allocation among all of them. This quantitative mathematical relationship is at the core of the concept of fairness. Bernoulli and Laplace expressed this notion in probability theory as the Principle of Insufficient Reason. The generalization of this principle is the Principle of Maximum Entropy [54], which addresses the question: “What is the fairest assignment of probabilities of several alternates given a set of constraints?” Thus, the roots of entropy as a fairness measure can be traced all the way back to the Principle of Insufficient Reason [55].

The wide spread misunderstanding that entropy stands for randomness or uncertainty has led to much confusion and the denial of entropy’s long overdue recognition and proper use as an essential concept in social sciences including economics. We believe that by properly recognizing entropy as a measure of fairness, which is a fundamental economic and social principle, and showing how it is naturally and intimately connected to the dynamics of the free market economic environment, our theory makes a significant conceptual advance in revealing the deep and direct connections between game theory, statistical thermodynamics, information theory, and economics. This revelation also sheds new light on a decades-old fundamental question in economics, as Samuelson [17] posed in his Nobel Lecture, “what it is that Adam Smith’s ‘invisible hand’ is supposed to be maximizing”, or as Monderer and Shapley [18] stated regarding the potential function $P^*$ in game theory, “This raises the natural question about the economic content (or interpretation) of $P^*$: What do the firms try to jointly maximize? We do not have an answer to this question.”

Our theory suggests that whatever all the agents in a free market environment are jointly maximizing, i.e., what the “invisible hand” is maximizing, is fairness. Maximizing entropy, or game theoretic potential, is the same as maximizing fairness collectively in economic systems, i.e., being fair to everyone under the given constraints. In other words, economic equilibrium is reached when every agent feels she or he has been fairly compensated for her or his efforts. As we all know, fairness is a fundamental economic principle that lies at the foundation of the free market system. It is so vital to the proper functioning of the markets that we have regulations and watchdog agencies that breakup and punish unfair practices such as monopolies, collusion, and insider trading. Thus, it is eminently reasonable, indeed particularly reassuring, to find that maximizing fairness collectively, i.e., maximizing entropy, is the condition for achieving economic equilibrium. We call this result the fair market hypothesis. We claim that the free market, in addition to being efficient, also promotes fairness to the maximum level allowed by the constraints imposed on it.

A key prediction of this theory is that the lognormal distribution is the fairest inequality of pay in organizations under ideal conditions at equilibrium. While this prediction has support from empirical data, which suggest that the bottom 90-95% of pay distribution is lognormally distributed [16, 24, 33], with the remainder following an inverse power law, further careful analyses are needed, as noted above, to avoid the truncated datasets problem.

One may view our result (i.e., lognormal distribution) as an “economic law” in the statistical thermodynamics sense. The ideal free market, guided by the “invisible hand”, will self-organize to “discover” and obey this eco-
nomic law if allowed to function freely without collusion like practices or other such unfair interferences. This result is the economic equivalent of the Gibbs-Boltzmann exponential distribution in thermodynamics.

Turning our questions around and looking at the problem from the empirical perspective, we can ask the following: What kind of utility and disutility functions will lead to the empirically observed lognormal distribution of pay? The answer, of course, is in Equations (3) and (4). Thus, one may consider this as an experimental validation of the logarithmic utility and quadratic disutility assumptions commonly made in economics. It will be interesting to see whether we have any other such empirical validation of these two assumptions in economics.

There are obvious limitations to our model – we have assumed perfectly rational agents, no externalities, ideal free market conditions, and so on, which are clearly not valid in real life. However, our objective was to develop a general game theoretic framework, identify key principles, and make predictions that are not restricted by market specific details and nuances. Nevertheless, despite such simplifying assumptions, it is encouraging that the prediction of a lognormal distribution in pay is supported by empirical data for the bottom 90-95%. Clearly, the next steps are to conduct more comprehensive studies of pay distributions in various organizations in order to understand in greater detail the deviations from ideality in the market place. Agencies such as the Bureau of Labor Statistics and National Bureau of Economic Research could organize task forces to gather pay data from various companies and organizations. The data should be so grouped to analyze pay distribution patterns across several dimensions such as: (i) organization size - small, medium, large, and very large number of employees, (ii) different industrial sectors, (iii) different types such as private corporations, governments (state and federal), non-profit organizations, etc. Similar studies should be conducted in other countries as well so that we can better understand global patterns.

Further work is also needed to examine whether there are other payoff functions which can explain and predict better than what we have proposed. Another obvious line of research is combining this approach with behavioral models, by endowing the ideal agents with non-idealities such as trending and copying behaviors.

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