ABSTRACT
This paper describes systematic design process toward third-order autonomous deterministic hyperchaotic oscillators with two coupled generative two-terminal elements. These active devices represent alternative to conventional dissipative accumulation elements such as non-ideal capacitors and inductors. Analyzed network structures contain generalized bipolar transistor as only active element. Three different networks are studied, depending on common-electrode configurations. In each case, transistor is modelled as two-port using admittance parameters with the non-zero linear backward and polynomial forward transconductance. As proved in paper, scalar nonlinearity caused by the amplification property of transistor can push circuit into chaotic and, more interestingly, hyperchaotic steady states. Existence of parameter spaces leading to robust chaotic and hyperchaotic solution is documented by using concept of one-dimensional Lyapunov exponents and colored high-resolution surface-contour plots of two largest numbers. Geometrical structural stability of generated strange attractors is proved via construction of the flow-equivalent chaotic oscillator and real measurement. Plane projections of carefully selected strange attractors were captured by oscilloscope and visualized.

INDEX TERMS
Admittance parameters, bipolar transistor, frequency dependent negative resistor, chaos, chaotic oscillator, hyperchaos, Lyapunov exponents, strange attractors.

I. INTRODUCTION
It is known that autonomous deterministic dynamical system can exhibit chaotic (hyperchaotic) state attractors under two necessary conditions: at least three (four) degrees of freedom and presence of nonlinearity. Such kind of dynamic motion is extremely sensitive to small deviations of initial conditions, exhibits increased entropy and contains many sinusoidal waveforms. Due to this, chaotic signal has broad-band noise-like frequency spectrum. Common types of lumped chaotic oscillators contain basic harmonic oscillators as a core part. The opposite statement is reasonable as well: second-order sinusoidal oscillator can be forced to chaotic steady state if additional accumulation element is connected in proper place in circuit and nonlinear nature of active device is considered (usually in a form of input-output characteristics). Of course, required nonlinearity can be included into circuit artificially, to create robust generator of chaotic signals. This approach is used in brief paper [1], where authors proposed few very simple network structures having a single current-feedback operational amplifier and diode as two-terminal nonlinearity. Unfortunately, chaos is confirmed only by computer-aided simulations. More mature work on simple chaotic oscillators can be found in paper [2]. Authors provide design process toward chaotic electronic systems that utilize composite of accumulation element and single two-terminal device having nonlinear ampere-voltage characteristics. Both numerical and measurement results are provided. Paper [3] introduces chaotic network with two operational amplifiers (opamps) and one diode. Work [4] gives inductorless chaotic oscillator where passive two-terminal device having only resistors and capacitors in ladder interconnection is loaded by
a piece-wise-linear (PWL) active resistor. Nice collection of several Wien-bridge oscillators simply modified to generate spiral chaotic attractors is provided in paper [5]. Despite ancient date of its discovery, the so-called Chua’s circuit [6, 7] still belongs to the worldwide most famous chaotic oscillators. It is mainly because its richness of behavior, circuit simplicity, cheap and easy construction. It can be considered as parallel connection of a third-order passive immittance and three-segment PWL resistor. To illustrate wealth of options how we can generate chaos, paper [8] discuss coupling of the same PWL resistor with Wien-bridge oscillator. Likely the most frequently used sinusoidal generator suitable for radiofrequency applications is Colpitts oscillator. It contains two capacitors, inductor, and one transistor as active element. Pioneering paper [9] admits the existence of a single voltage-controlled nonlinear resistor, leading to evolution of interesting, robust strange attractors. Bilateral interchange of inductors and capacitors results into network topology widely known as Hartley oscillator. This is also good candidate for generator of settled chaotic signals, as demonstrated in work [10]. Chaotic systems presented in this paper can be understood as the higher-order alternatives to Colpitts oscillator. Of course, selected papers mentioned above by no means represent comprehensive review focused on the synthesis of chaotic oscillators. There are hundreds of works that describe interesting generators of chaos, including those having the mem-elements [11], fractional-order (FO) accumulation elements [12], FO mem-elements [13], etc.

This paper is organized as follows. Next section describes mathematical models associated with different connection of bipolar transistor. Obviously, there is no transformation of coordinates between any two cases. Third section contains numerical analysis of investigated systems. As a part of this, sets of parameters that represent the biasing point of bipolar transistor and giving chaotic self-oscillations of each circuit are provided. Fourth paper section answers the problem of long-term structural stability of observed strange attractors. Common-emitter structure (CES) is practically implemented as the chaotic oscillator by using two grounded Antoniou’s immittance converters. Both common-collector (CCS) and common base (CBS) structures are realized by following the concept known as analog computer (integrometer-based block diagram). It is because of simplicity of final circuits, i.e., to reach minimum number of opamps and multipliers needed. Next section discusses similarities and discrepancies between simulated and measured results. Paper ends providing some concluding remarks and suggestions for future research.

II. MATHEMATICAL MODELS OF SIMPLIFIED SINGLE TRANSISTOR BASED OSCILLATORS, THREE CASES

It is well known that three two-port configurations of bipolar transistor can be derived; based on the nature of terminal that is common for input and output port of the bipolar transistor. All fundamental topologies are provided by means of Fig. 1. Two-terminal black-boxes marked as $Z_1$ and $Z_2$ are defined by intrinsic voltage-current relations

\[ iZ_1 = D_1 \cdot \frac{d^2}{dt^2} vZ_1, \quad iZ_2 = D_2 \cdot \frac{d^2}{dt^2} vZ_2, \quad \text{(1)} \]

where $iZ_1$, $vZ_1$ and $iZ_2$, $vZ_2$ is current and voltage across first and second two-terminal element respectively. Coefficients $D_1$ and $D_2$ are positive real numbers expressed using $F \cdot s$ unit. Note that if a sinusoidal voltage $v_x(t)$ is applied across such two-terminal element, current response $i_x(t)$ will be

\[ v_x = A \cdot \sin(\omega \cdot t), \quad i_x = -A \cdot \omega^2 \cdot \sin(\omega \cdot t), \quad \text{(2)} \]

where $A$ is amplitude and $\omega$ is arbitrary angular frequency.

It means that phase of current is shifted by $180^\circ$ and two-port represents frequency dependent negative resistance (FDNR). Since this element represents ideal energy source, it belongs to a large group of the so-called generative circuit elements. Since potentially hyperchaotic systems need to be at least of fourth order, two FDNRs and single linear resistor represent minimal feedback configuration. If speaking about “the only chaotic” electronic circuits, FDNR was already successfully utilized in several cases. For example, authors in paper [14] presents chaos generator with single FDNR as the only active device and passive nonlinear circuit element.

Recent papers [15], [16], [17] admits non-ideal circuit model of bipolar transistor working as two-port described by following admittance parameters: linear input admittance, linear back-ward trans-conductance and nonlinear (with a saturation-type shape) forward trans-conductance of the form

\[ y_{21}(v) = a \cdot v^3 + b \cdot v \Rightarrow y_{21}(-v) = -y_{21}(v). \quad \text{(3)} \]

Absence of offset and quadratic term leads to odd symmetry of nonlinearity and symmetry of vector field with respect to origin simultaneously. Zero output admittance suggests that active element works as the ideal current source.

Calculation schematic for AC signal and CES is provided in Fig. 2a. Dynamical behavior of this circuit (let denote it as case I) is uniquely determined as solution of the following ordinary differential equations

\[ \frac{d}{dt} x = y, \quad \frac{d}{dt} y = \frac{1}{D_2} \left[ \frac{z-x}{R} - y_{12} \cdot z - y_{11} \cdot x \right], \quad \text{(4a)} \]

\[ \frac{d}{dt} z = w, \quad \frac{d}{dt} w = \frac{1}{D_1} \left[ \frac{x-z}{R} - a \cdot x^3 - b \cdot x \right], \quad \text{(4b)} \]
where state vector is

$$x = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^T = \begin{pmatrix} v_1 \\ \frac{d}{dt}v_1 \\ v_2 \\ \frac{d}{dt}v_2 \end{pmatrix}^T.$$  \hspace{1cm} (5)$$

Fundamental CCS topology is given in Fig. 2b. Behavior of this network is uniquely determined by solution of system of ordinary differential equations

$$\frac{d}{dt}x = y, \quad \frac{d}{dt}y = \frac{1}{D_2} \left[ y_{12} \cdot z - y_{11} \cdot x - \frac{x + z}{R} \right],$$

$$\frac{d}{dt}z = w, \quad \frac{d}{dt}w = \frac{1}{D_1} \left[ a \cdot x^3 + b \cdot x - \frac{x + z}{R} \right],$$  \hspace{1cm} (6)$$

where second FDNR is floating such that first state variable is not nodal voltage directly but a voltage difference between $v_1$ and $v_2$. Finally, third schematic provided in Fig. 2c is valid for system CBS, and a describing set of ordinary differential equations becomes

$$\frac{d}{dt}x = y, \quad \frac{d}{dt}y = \frac{1}{D_1} \left[ x + z - \frac{x + z}{R} \right],$$

$$\frac{d}{dt}z = w, \quad \frac{d}{dt}w = \frac{1}{D_2} \left[ \frac{x + z}{R} + y_{12} \cdot z - y_{11} \cdot z - a \cdot z^3 - b \cdot z \right].$$  \hspace{1cm} (7)$$

In set of equations (7), first state variable represents voltage difference between two independent circuit nodes rather than voltage between node and ground potential. At this moment, math models (4), (6), and (7) are normalized with respect to impedance and time. In the upcoming numerical analysis, parameters $R = 10 \ \Omega$ and $D_1 = D_2 = 1 \ F \cdot s$ will be fixed, i.e., not affected by a changing bias point of bipolar transistor. This idea greatly speeds up convergence rate of the searching-for-chaos optimization routine.

**III. NUMERICAL RESULTS**

Numerical procedures used throughout this work are based on fourth order Runge-Kutta integration method with a fixed step size. Although individual routines were implemented in Matlab program (with emphasize on the parallel processing based on the multiple CPU cores), graphical visualizations of achieved results are built up using Mathcad software. Details concerning all numerical scenarios and values of procedure’s input parameters are given in proper places in following text.

**A. SEARCHING FOR CHAOS**

By definition, a closed-form analytic solution generally valid for chaotic system cannot be derived. However, there are still possibilities how distinguish chaotic from regular (periodic) dynamical movement [18]. The accurate flow-quantifiers are always based on the numerical integration process and can be used for both optimization purposes [19], [20] and searching-for-chaos algorithms [21]. Procedures that utilize sequence of measured/observed data as input parameters for determining properties of the reconstructed state attractor are less precise. Within very interesting research paper [21], authors describe searching three-step automatized procedure, and demonstrate that chaos localization can be considered as optimization task as well, with suitable choice of objective function.

Fitness function or, more likely, several fitness functions, should be chosen carefully and with respect to the algebraic properties of investigated mathematical model and geometric structure of vector field. Let’s focus on self-excited attractor, namely strange attractor excited by the unstable fixed point located at origin of the state space. Searching routine should contain two steps. First step removes parameter combinations that stabilize the equilibrium point. Second step, significantly time consuming, is calculation of suitable flow-quantifier, for example Kaplan-Yorke dimension (KYD) of state attractor. Another possibility for fitness function is to watch directly on two largest one-dimensional Lyapunov exponents (LE) [22]. To experience chaos, first LE should be positive and second zero. Fingerprints of hyperchaos occur if two LE are positive. Simultaneously, sum of all LEs determine dissipation level of dynamical flow [23] which, for all cases of investigated math models, tends to zero. It turns out that it is not necessary to search for numerical values of unknown system parameters characterized by two (or more) decimal places. Moreover, it is not necessary to encode ± sign of any parameter.

**B. DISCOVERED CHAOTIC SYSTEMS**

Using numerical algorithm described in the previous section following set of values leads to the robust chaotic attractor for
c) subspace

\[ y_{11} = 18 S, \quad y_{12} = 1 S, \quad a = -10 A/V^3, \quad b = 5 S. \quad (8) \]

Selected calculation results are provided in Fig. 3. Besides all available 3D projections of typical strange attractor generated by system (4) with set (8), this figure also illustrates extreme sensitivity of the system solution to small deviations in initial conditions. Suppose group of \(10^4\) initial conditions uniformly distributed around origin of the state space, forming centered cube with edge size \(10^{-2}\) (black points, Fig. 3e,f). After short-time evolution (1 s), final states are stored (red points). Then, final states after average-time (10 s) and long-time (100 s) evolution are stored as green and blue points respectively. Typical chaotic attractor (Fig. 3a-d) generated for initial conditions \(x_0 = (0, 0, 0, 0)^T\) final time 1000 s and time step 10 ms. Note that two solutions of this system in time domain (see subplot Fig. 3g) diverges very quickly, although initial difference between voltages \(v_1\) is only 10 mV and the rest of the state variables remains the same.

Extensive search performed for CCS of dynamical system reveals following values

\[ y_{11} = 1 S, \quad y_{12} = 0 S, \quad a = -1 A/V^3, \quad b = 3.2 S. \quad (9) \]

For these parameters, typical strange attractor is depicted in Fig. 4, namely using all 3D projections in Fig. 4a,b,c,d).

Individual parts of Fig. 4 stand for CCS system and have the same meaning as corresponding parts provided in frame of Fig. 3 for CES dynamical system. There is one exception, Fig. 4g); this subplot illustrates coexistence of four strange attractors within dynamics of system (6) considering set (9). Initial conditions were adopted as \(x_0 = (0.1, 0, 0, 0)^T\) for blue curve, \(x_0 = (-0.1, 0, 0, 0)^T\) for red trajectory, \(x_0 = (0, 0, 0.1, 0)^T\) for green trajectory, and choice \(x_0 = (0, 0, 0, -0.1)^T\) leads to black orbit. It is worth to be noted (Fig. 4f) that, starting in a close neighborhood of origin, four-dimensional state space volume is, during relatively long evolution time, expanded only in two directions (plane expansion).

For the third analyzed dynamical system, denoted as CBS, following numerical values can be adopted to generate robust chaotic attractor

\[ y_{11} = 1 S, \quad y_{12} = 0.8 S, \quad a = 1 A/V^3, \quad b = -1 S. \quad (10) \]

For these parameters, double scroll like chaotic attractor can be generated, see Fig. 5 for details. Opposite signs associated with cubic polynomial function suggests inverse conductivity of bipolar transistor. Due to the much slower rate of volume expansion near state space origin, final times for sensitivity demonstration were moved to 40 s (green points) and 400 s (blue states). Of course, fragments of the hyperspace of the internal system parameters that do not contain physically reasonable values, such as the negative input admittance of the bipolar transistor, are forbidden for searching procedure. All analyzed systems (CES, CCS, CBS) possess mirrored attractor associated with any kind of attractor observed.

For all calculations performed in this section, parameters for numerical integration process were chosen accordingly to the advice given in paper [24]. Since eigenvalues associated with equilibrium point located at origin are calculated for each step of the searching routine (one point in the parameter hyperspace), we also have comprehensive idea about optimal time step of integration.

Figure 6 illustrates the changes in dynamical behavior near biasing point for CES, defined by set of numerical values (8).
FIGURE 4. Selected numerical results associated with analyzed chaotic system, CCS: a) projection $v_1 - \frac{dv_1}{dt} - v_2$, b) projection $v_1 - \frac{dv_1}{dt} - \frac{dv_2}{dt}$, c) subspace $v_1 - v_2 - \frac{dv_2}{dt}$, d) projection $\frac{dv_1}{dt} - v_2 - \frac{dv_2}{dt}$, e) sensitivity to changes in initial conditions, f) zoomed group of the initial conditions, g) coexistence of four strange attractors in dynamics of one system.

Provided grid is calculated with uniform parameter step 0.01 such that each plot contains 10201 points. Decreasing value of backward trans-conductance $y_{12}$ leads to the reduction of region where chaotic motion can be observed. Instead of this, various limit cycles are generated by CES circuit. Obviously, hyperchaotic and chaotic regimes are surrounded by limit cycle and unbounded solution. The value of maximal LE in investigated parameter area is 0.126.

Figure 7 shows similar plot for CCS system. Two largest LE are visualized here; with respect to both parameters of nonlinear function. Note that wider parameter area and full grid calculation is considered. In these plots, $y_{11}$ as well as $y_{12}$ was fixed on the values (9), and both shaping coefficient was swept with step 0.01. Note that dynamical behavior can be marked as chaotic only if parameter $b$ is close to value 3.2. Moreover, areas that represent a truly chaotic behavior are narrow, surrounded by limit cycles.

Figure 8 presents the numerical analysis results associated with system CBS, i.e., differential equations (7) with nominal biasing point given by set (10). Note that within investigated area $a \in (0.1, 1)$ A/V and $b \in (-4, 0)$ S we can observe either chaotic or hyperchaotic dynamic motion. Initial conditions for black orbits in subplots Fig. 8e-j) are $x_0 = (0.1, 0, 0, 0)^T$, while for orange trajectories in Fig. 8h,i,j) initial states were chosen as $x_0 = (1, 0, 0, 0)^T$. Obviously, there are at least four coexisting strange attractors (two visualized and two mirrors) and this number is probably not final.
In this place, some explanatory relating to Fig. 6, Fig. 7 and Fig. 8 needs to be given. Three graphs mentioned above are composed by contour plot of the first LE and surface plot of the second largest LE. In regions where both plots exhibit distinguishable positive value, dynamic behavior can be marked as hyperchaotic. In areas where contour plot is red and surface is close to zero, strong chaotic movement is corresponding solution. If contour color changes to yellow or green and surface still stands on zero, dynamical behavior changes into chaotic and weakly chaotic, with increased time predictability of the flow. Note that the largest LE calculated for CBS system is quite high, but second LE tends to stay close to zero. This means that CBS system likes working in strong chaotic regime, but there are some regions of weakly hyperchaotic states. Cubic polynomial term, influenced by value \( a \), can serve as bifurcation parameter.

Chaotic attractors discovered in CES, CCS, and CBS cases belongs to self-excited; with basins of attraction neighboring with a state space origin. Process of uncovering the so-called hidden attractors [25] is possible topic for further research. Hidden strange attractors displayed by means of Fig. 8h,i,j) represent accidental discoveries.

C. CHAOS-HYPERCHAOS TRANSITIONS

This brief section brings a note about transition between chaotic and hyperchaotic steady state. These considerations are based on the knowledge summarized in paper [26] where smooth polynomial vector fields were also analyzed. Authors suppose that the chaos-hyperchaos (or vice versa) transitions with continuous change of some internal system parameter is allowed due to change of eigenvalues, stability indexes, birth, or death of equilibrium points. However, dynamical systems investigated in this paper exhibit these transitions without any of the changes mentioned above. Unexpected transitions are not exceptional phenomenon among other known and published chaotic/hyperchaotic systems.

IV. DESIGN OF CHAOTIC OSCILLATORS

Having the knowledge of complete mathematical description of dynamical system synthesis of lumped electronic circuit
represents easy and straightforward task. Since mathematical models are normalized, we have two degrees of freedom to setup numerical values of individual circuit elements. There are many papers where comprehensive review dealing with lumped circuit synthesis of nonlinear dynamical systems can be consulted.

Author in paper [27] nicely and systematically explains the process of circuit design based on set of ordinary differential equations. Many examples can be found here, and readers are guided from mathematical expressions to interconnections of circuit elements. Author focuses on the voltage-mode system, i.e., where individual state variables are the nodal voltages. Interesting practical design examples are demonstrated also in papers [28], [29]. The main drawback of this approach based on the analog computers is the necessity to use many active devices. This number can be reduced if circuit is designed as mixed-mode [30] or current-mode oscillator [31]. In the latter case of realization, all summations are performed simply by node and subtraction by adding current inverter. For current-mode circuits, few relatively modern active devices having low input and high output impedances are available, such as current conveyors, current multipliers, and distributors. More details about current-processing devices and many examples can be found in comprehensive review paper [32].

Specific forms of mathematical model can lead to a very simple chaotic oscillator. This is demonstrated in paper [33] where third order differential equation is implemented using closed loop of a low-pass passive ladder filter and active two-port feedback with polynomial input-output characteristics. Some authors [11], [34] focus on a full on-chip implementation of chaotic systems and solving associated problems: with the state attractor rescaling, restrictions regarding the component values, temperature, and process variations. Other engineers focus on different kind of realizations of the chaotic systems, using FPGA development kit [35]. However, discretization of differential equations and its solving via modern FPGA platform belongs to computer-aided design, control software usually provides a graphical user interface where suitable functional blocks can be placed, set up and interconnected. Therefore, such approach cannot be considered as the truly experimental; it is more likely a bridge between numerical integration and a practical breadboard design followed by laboratory measurement.

In upcoming parts of this section, three different structures of chaotic systems will be designed as the lumped electronic circuits. Individual circuits will relate to mathematical model of CES, CCS and CBS cells.

### A. CES-BASED OSCILLATOR

Since both generative second-order two-terminal devices are grounded, each FDNR can be realized using two opamps and suitable choice of five simple impedances. Now suppose a choice visualized in Fig. 9. Then, describing set of ordinary differential equations can be expressed as

\[
\frac{d^2}{dt^2}v_1 = \frac{R_1}{C_1 \cdot C_2 \cdot R_2 \cdot R_3} \left( \frac{v_2 - v_1}{R_5} - \frac{v_1 - K \cdot v_a}{R_a} \right),
\]

\[
\frac{d^2}{dt^2}v_2 = \frac{R_6}{C_3 \cdot C_4 \cdot R_7 \cdot R_8} \left( \frac{v_1 - v_2}{R_5} - \frac{K \cdot v_b}{R_b} - \frac{K \cdot v_c^3}{R_b} \right),
\]

where \( K = 0.1 \) is internally trimmed transfer constant of analog multiplier. Note that dynamical system (11) is formally the same as the mathematical model of CES (4). First derivatives (with respect to time) of the state variables \( v_1 \) and \( v_2 \) are represented by the linear combinations of these voltages and voltages measured at nodes A and C respectively, namely

\[
\frac{d}{dt}v_1 = \frac{v_A - v_1}{C_2 \cdot R_3}, \quad \frac{d}{dt}v_2 = \frac{v_C - v_2}{C_4 \cdot R_8}.
\]

Simultaneously, voltage observed at node \( v_B \) and \( v_D \) can be written as linear combinations

\[
v_B = v_1 - C_1 \cdot C_2 \cdot R_2 \cdot R_3 \frac{d^2}{dt^2}v_1,
\]

\[
v_D = v_2 - C_3 \cdot C_4 \cdot R_7 \cdot R_8 \frac{d^2}{dt^2}v_2.
\]

Formulas (12) and (13) hold only if opamps behave close to ideal and its parasitic properties can be neglected. Resistors \( R_x = 10 \, \text{k} \Omega \) and \( R_y = 90 \, \text{k} \Omega \) are chosen to compensate transfer constant of first multiplier in cascade. Therefore, term \( K^2 \) is missing in group of differential equations (11).

Note that parameter \( v_{11} \) of original mathematical model can be adjusted via resistor \( R_4 \) while parameter \( v_{12} \) relates to the external DC voltage \( V_d \). In accordance with value set (8), by choosing frequency norm as \( 10^4 \) and impedance scaling factor as \( 10^3 \), numerical values of individual passive

![Figure 9. Circuitry realization of chaotic oscillator based on CES using four opamps and three analog multipliers.](image-url)
network components are \( C_1 = C_2 = C_3 = C_4 = 100 \, \text{nF}, \)
\( R_1 = R_2 = R_3 = R_6 = R_7 = = R_8 = 1 \, \text{k} \Omega, \)
\( R_5 = 10 \, \text{k} \Omega, \quad R_a = 33 \, \text{\Omega}, \quad R_b = 100 \, \text{\Omega}, \) and DC voltages \( V_a = 6 \, \text{V}, \)
\( V_b = 5 \, \text{V}. \)

**B. CCS-BASED OSCILLATOR**

As depicted in Fig. 1b, CCS-based chaotic oscillator needs an ideal floating FDNR. By sticking on Antoniou’s structure, four opamps are required to make it. Final network topology will contain huge number of the active devices. Therefore, to implement CCS-based lumped chaotic system, we are going to follow different approach known as the analog computer concept. We will need only three analog building blocks: the lossless integrators (inverting or non-inverting), differential (summing) amplifiers, and two-port with desired polynomial input-output characteristics.

Final circuit is provided in Fig. 10 and can be described by following set of ordinary differential equations

\[
\begin{align*}
C_1 \frac{d}{dt} v_1 &= \frac{v_2}{R_1}, \quad C_2 \frac{d}{dt} v_2 = -\frac{v_1}{R_3} - \frac{v_3}{R_5}, \quad C_3 \frac{d}{dt} v_3 = \frac{v_4}{R_2} \\
C_4 \frac{d}{dt} v_4 &= -\frac{v_3}{R_4} - \frac{K \cdot V_a}{R_6} v_1 + \frac{K}{R_6} v_1,
\end{align*}
\]

(14)

By comparing system (14) with mathematical model (6) with parameters (9) we can establish numerical values of passive circuit components as \( C_1 = C_2 = C_3 = C_4 = 100 \, \text{nF}, \)
\( R_1 = R_2 = 1 \, \text{k} \Omega, \quad R_3 = 900 \, \text{\Omega}, \quad R_4 = R_5 = 10 \, \text{k} \Omega, \quad R_6 = 100 \, \text{\Omega}, \) together with external DC voltage \( V_a = 3.1 \, \text{V}. \)

Note that final circuit is very simple, contains both inverting (TL084) and non-inverting (AD844) integrators. Resistors \( R_1 = 10 \, \text{k} \Omega \) and \( R_3 = 90 \, \text{k} \Omega \) are dedicated to compensating transfer coefficient \( K \) of the first multiplier, luckily avoiding term \( K^2 \) in describing set of equations (14). The input admittance \( y_{11} \) can be adjusted via resistor \( R_3 \) while backward trans-conductance \( y_{12} \) is affected by resistor \( R_5 \).

**C. CBS-BASED OSCILLATOR**

Due to the necessity of floating FDNR, circuit synthesis based on integrator block schematic will be adopted again. Circuit realization is provided by means of Fig. 11. Straight-forward analysis yields following set of differential equations that describe its behavior

\[
\begin{align*}
C_1 \frac{d}{dt} v_1 &= -\frac{1}{R_1} v_2, \\
C_2 \frac{d}{dt} v_2 &= -\frac{R_{11}}{R_3 \cdot R_{12}} v_1 - \frac{R_{11}}{R_4 \cdot R_{12}} v_3 - \frac{K}{R_5} v_3^2 + \frac{K \cdot V_a}{R_5} v_3, \\
C_3 \frac{d}{dt} v_3 &= -\frac{1}{R_2} v_4, \quad C_4 \frac{d}{dt} v_4 = \left( \frac{1}{R_{10}} - \frac{R_{11}}{R_3 \cdot R_{13}} \right) v_1 + \\
&\quad + \left( \frac{R_8}{R_7 \cdot R_9} - \frac{R_{11}}{R_4 \cdot R_{13}} + \frac{K \cdot V_a}{R_6} \right) v_3 - \frac{K}{R_6} v_3^2, \\
&\quad (15)
\end{align*}
\]

where transfer constant \( K = 0.1 \) and resistors \( R_3 = 9 \cdot R_5 = 90 \, \text{k} \Omega \). Note that this dynamical system is the same as mathematical model given by equations (7) after a linear transformation of coordinates \((-x, y, -z, w)^T \rightarrow (v_1, v_2, v_3, v_4)^T\). This design allows us to change parameter \( y_{11} \) via variable resistor \( R_9 \), vary parameter \( y_{12} \) via resistor \( R_{10} \), and continuously change shaping coefficient \( a, b \) through the value of resistor \( R_5 \) and DC voltage source \( V_a \). Note that system of equations (15) can be directly put into the context

**FIGURE 10.** Simple circuit implementation of chaotic oscillator based on CCS using two opamps, two CFOAs and two analog multipliers.

**FIGURE 11.** Circuitry realization of chaotic oscillator based on CBS using six opamps and two analog multipliers.
with mathematical model (7). Numerical values (10) and a chosen time constant $\tau = 100 \mu s$ led to the passive circuit components $C_1 = C_2 = C_3 = C_4 = 100 \text{ nF}$, $R_1 = R_2 = R_9 = R_{12} = R_{13} = 1 \text{ k}\Omega$, $R_3 = R_5 = R_6 = 100 \text{ \Omega}$, $R_{10} = 1250 \text{ \Omega}$, and $R_3 = R_4 = R_7 = R_8 = R_{11} = 10 \text{ k}\Omega$. External DC voltage is $V_o = 1 \text{ V}$.

As already mentioned, time constant chosen for all chaotic oscillators described above is $\tau = 100 \mu s$. It means that wide frequency spectrum associated with generated chaotic and/or hyperchaotic signals fits audio frequency range and ends at hundreds of kHz. Let recall that integrated circuits needed for our designs are TL084, AD844, and AD633. Because good frequency responses of these items, namely transfer functions with flat characteristics with roll-offs about units of MHz, there is no need to solve frequency limitations mentioned in paper [36].

Average power dissipation reported for individual system cases (simulated CES, measured CCS and CBS) relates to the state attractor generated, from hundreds of mW for small limit cycles up to several W for expected strange attractors. The most power demanding is unbounded solution, that is represented by trajectory collapsed into large limit cycle.

Presence of ideal integrators can cause problems if desired strange attractor (or any other kind of a state attractor) is not centered around the state space origin. This means that one or several chaotic waveforms have a nonzero DC frequency component causing saturation of a lossless active integrator. Mentioned problem is avoided in CCS-based and CBS-based configuration of the chaotic oscillator, at least for discovered strange attractor. However, it could be very difficult to reveal the so-called hidden attractors [37], [38], [39] located far away from origin using practical experiment. In frame of this paper, no such attractors were discovered.

**D. FRACTIONAL-ORDER CHAOTIC OSCILLATORS**

The reason for transformation of an integer-order chaotic system into FO equivalent is that the latter case sometimes offers the larger scale of complex dynamics. Each structure of chaotic oscillator described in the previous section of this work can be easily modified into FO chaotic system. Let’s briefly discuss how to do this.

In the case of CES, total order of first FDNR is given by a product of orders of capacitors $C_3$ and $C_4$. Analogically, total order of second FDNR is given by orders of capacitors $C_1$ and $C_2$. Thus, substitution of any conventional capacitor by FO capacitor that is approximated by suitable two-terminal device has immediate effect of decreasing order of chaotic oscillator. For example, by using popular RC passive ladder networks characterized by the decimal orders (starting with 0.1 and ending with 0.9) we can decrease mathematical order of the chaotic oscillator to 0.4. Of course, total order of the circuit (number of describing differential equations) will be much higher. Numerical analysis of such chaotic systems is also possible since individual RC approximants of fractional-order capacitors can be included directly into set of ordinary differential equations of whole analyzed dynamical system. However, RC approximants are designed using wide range of capacitors, i.e., dynamics of system contains both fast and slow processes. Therefore, values of the input parameters for numerical integration need to be chosen carefully.

Proposed circuitry realizations of CCS and CBS contain ideal integer-order inverting and/or non-inverting integrators. First approach that leads to FO nature of chaotic oscillator is replacement of a conventional capacitor by RC approximant. As a result, corresponding differential equation will be of non-integer order (having order less than one). Alternatively, approximation of FO two-port in frequency domain can be done by cascade of bilinear filters. Each filter implements a single pair of transfer function’s zero and pole. If these pairs are precisely located on the frequency axis, we create desired phase shift rippled around theoretical constant value $-90 \cdot \alpha ^\circ$, where $\alpha$ is order of FO integrator. Number of zero-pole pairs (and required number of filters) is a compromise between accuracy of approximation and the frequency range where approximation is valid. More details can be found in many papers, curious readers are referred to works of prof. Elwakil, prof. Radwan, prof. Psychalinos, prof. Petras, and others.

**E. APPLICATION POTENTIAL OF DESIGNED CIRCUITS**

As mentioned before, the chaotic waveforms have several properties that can be of a key importance in some practical applications. Many research papers were already devoted to study utilization of chaos inside the communication systems. Continuous-time chaotic signals can be used for masking,
modulation, cryptography, compression, frequency testing of network functions, and many other areas where wideband signals are needed. Thus, introduced chaotic systems can be utilized as the steady state sources of strongly chaotic signals, that is the generators of analog signals with very short time predictability. If we want to be more detailed, two state orbits starting near state space origin and being separated by 10 mV diverges rapidly after a few tens of seconds of time evolution (considering normalized time). Of course, specific value also depends on system case studied (CES, CCS, CBS).

V. CIRCUIT SIMULATION AND MEASUREMENT
This section provides proof that strange attractors observed in the numerical part are neither numerical artifacts nor long chaotic transients. CCS and CBS cases of single transistor-based systems was implemented on the breadboard and fed by a symmetrical ±15 V supply voltage. Since neighborhood of origin (zero output voltages of all integrators) is unstable fixed point, there is no need to impose specific set of initial condition into these circuits (nodes).

Figure 12 shows oscilloscope screenshots captured during experimental verification of CCS-based chaotic oscillator. Similarly, Fig. 13 provides the same for CBS-based circuit.

Since all state variables (CCS, CBS) are easily accessible and measurable (in contrast to proposed CES-based system) watched plane projections can be picked freely. Note that in both analyzed cases (CCS, CBS) good agreement between theory and measurement can be concluded.

VI. CONCLUSION
In this paper, very simple single transistor-based structures with two generative two-terminal devices are investigated. Addressed mathematical models can be roughly understood as nonlinearly coupled second order jerk dynamical systems. In first part of this paper, numerical procedure capable to discover solution of dynamical system sensitive to the initial conditions was successfully applied on models of oscillators with two FDNRs. If sought hyperspace is a low-dimensional, full-grid calculation can be utilized. However, this is not the case of three systems addressed in this work. Any kind of problem simplification, the reduction of dimension or size of hyperspace scanned by the routine is helpful and welcomed. Note that feedback of the very first and the most fundamental circuit, CES, represents in fact Colpitts-like trans-resistance feedback two-port under Bruton’s transformation.

Design approach that leads to mathematical models with chaotic behavior presented in this paper is not as systematic as illustrated, for example, in research paper [2]. It is product of numerical search procedure that contains the random-like principles. In addition, here is a right place to emphasize that numerical analysis provided in this paper does not represents complete investigation of three dynamical systems. It is more likely a carefully selected pack of the procedures required for fundamental numerical proof of chaos and/or hyperchaos. Therefore, question asking about importance of discoveries presented in this paper is still up to date. Answer can be summarized in few points mentioned below.

Discovered dynamical systems have three properties that are unique among other chaotic systems: a low dissipation, hyperchaotic solution, and multi-stability [40], [41]. The latter case means that different sets of initial conditions can induce differently shaped and located strange attractors. Also, there is no need to make rescaling of the individual state variables to reach undistorted shape of desired strange state attractors. Even in the case of large, chaotic attractor associated with CES, parameters of final circuit can be adjusted such that the observed chaotic waveforms are not cropped.

Each fundamental topology of chaotic oscillator proposed in this paper contains circuit elements described by a second-order dynamics. In the case of CES system, these FDNRs are the only accumulation elements. This could be a key factor while studying effect of FO nature of differential equations to a global behavior of chaotic system. Assuming that we can change mathematical order of two-terminal element between zero and two, we are able to change total order of chaotic system arbitrarily.

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