Multiparticle Entanglement Measure

Chang-shui Yu* and He-shan Song
Department of Physics, Dalian University of Technology,
Dalian 116024, China.
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In this paper, we generalize the residual entanglement to the case of multiparticle states in arbitrary dimensions by making use of a new method. Through the introduction of a special entanglement measure, the residual entanglement of mixed states takes on a form that is more elegant than that in Ref.[7] (Phys.Rev.A 61 (2000) 052306). The result obtained in this paper is different from the previous one given in Ref.[8] (Phys.Rev.A 63 (2000) 044301). Several examples demonstrate that our present result is a good measurement of the multiparticle entanglement. Furthermore, the original residual entanglement is a special case of our result.

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I. INTRODUCTION

Entanglement is the interesting feature which distinguishes the quantum world from the classical one. It has become a useful physical resource in quantum information processing (QIP) that has undergone a rapid development in recent years [1,2]. Because quantum entangled states play an important role in storage and transport of quantum information, a great attention has been focused on the study of their properties. Entanglement has been quantified through the concept of concurrence that was introduced by Wootters in 1998 [3]. However, Wootters’ concurrence is available only for bipartite systems with two levels, which, as known to all, is developed to different extents later [4,5,6]. The quantification of multiparticle entanglement (multi-way entanglement) is still an open problem.

In 2000, Valerie Coffman, Joydip Kundu and William K. Wootters [7] shed new light on the understanding of multiparticle entanglement by the study of distributed entanglement. They discovered an interesting quantity for a tripartite two-level system, referred to as the residual entanglement that is defined by

\[
\tau_{ABC} = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2,
\]

(1)

where \(C_{AB}\) and \(C_{AC}\) are the concurrences of the original pure state \(\rho_{ABC}\) with traces taken over qubits \(C\) and \(B\), respectively. \(C_{A(BC)}\) is the concurrence of \(\rho_{A(BC)}\) with qubits \(B\) and \(C\) regarded as a single object. In Ref.[7], it is shown that the residual entanglement of the pure two-level state \(|\psi\rangle = \sum_{ijk} a_{ijk} |ijk\rangle\) is given by

\[
\tau_{ABC} = 2 \left| \sum_{ijk} a_{ijk} a_{i'j'n} a_{npk} a_{n'p'm} \epsilon_{ii'} \epsilon_{jj'} \epsilon_{mm'} \epsilon_{nn'} \epsilon_{pp'} \right|,
\]

(2)

where the summation over repeated indices is implied and \(\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = \delta_{\alpha\beta}\). Since the residual entanglement is unchanged by permutations of \(A, B\) and \(C\), it can be regarded as representing a collective property of three qubit and can be used to quantify the 3-way entanglement [8].

In 2001, Alexander Wong and Nelson Christensen [8] demonstrated a generalization of the 3-tangle for \(n\) qubits. Starting from the definition of the \(n\)-tangle in a form similar to eq.(2), that is a generalization of the concurrence of pure states with three or an even number of qubits, they obtained a result with elegant form analogous to the result in Ref.[7]. Their result has also the property that \(\tau_{1...n}\) is unchanged by permutations. However, \(\tau_{1...n}\) itself is not a measure of \(n\)-way entanglement.

In this paper, starting from eq.(1) and the generalized bipartite entanglement measure in arbitrary dimensions [5,6,10], we generalize the residual entanglement to multiparticle systems in arbitrary dimensions in an approach similar to Ref.[7]. Our result provides a good measure for \(n\)-way entanglement in arbitrary dimensions. As to the case discussed in Ref.[7], the residual entanglement given in this paper can be easily reduced to the original one. The paper is organized as follows: Starting with the entanglement measure for the bipartite pure state-linear entropy, we generalize it to the case of mixed state; then we prove that the inequality similar to \(C_{A(BC)}^2 \geq C_{AB}^2 + C_{AC}^2\) mentioned in Ref.[7] holds for tripartite pure states in arbitrary dimensions, generalize it to the case of mixed state and define the residual entanglement something different from the original one; and then we generalize the residual entanglement to the case of multiparticle systems; finally we give several examples to demonstrate our generalization can work well for the multiparticle entanglement in its right.

II. THE SPECIAL ENTANGLEMENT MEASURE FOR BIPARTITE SYSTEMS

The linear entropy of a pure state \(\varphi\) is defined by

\[
E(\varphi) = 1 - Tr(\rho_\varphi^2),
\]
where $\rho_\alpha$ denotes the reduced density matrix of a bipartite system. The pure state concurrence in arbitrary dimensions [9] is defined by

$$C(\psi) = \sqrt{2(1 - Tr(\rho^2_{\text{min}}))},$$

which is the same to the concurrence [10], i.e.

$$C = |C| = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2},$$  \quad (3)

where $C_{\alpha\beta} = \langle \psi | \tilde{\psi}_{\alpha\beta} \rangle$, $|\tilde{\psi}_{\alpha\beta}\rangle = (L_\alpha \otimes L_\beta | \psi \rangle$ with $L_\alpha$, $L_\beta$ denoting generators of $SO(N_1)$ and $SO(N_2)$ respectively. Therefore,

$$2E(\varphi) = C^2 = \frac{N_1(N_1-1)/2 \ N_2(N_2-1)/2}{\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2}. \quad (4)$$

If we employ the linear entropy $C'(\varphi) = 2E(\varphi)$ as entanglement measure for pure states, we can generalize it to the case of mixed states.

The mixed states $\rho = \sum_k \omega_k |\psi^k\rangle \langle \psi^k|$ can be written in matrix notation as $\rho = \Psi W \Psi^T$, where $W$ is a diagonal matrix with $W_{kk} = \omega_k$, the columns of the matrix $\Psi$ correspond to the vectors $\psi^k$. Consider the eigenvalue decomposition, $\rho = \Phi M \Phi^T$, where $M$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\rho$, and $\Phi$ is a unitary matrix whose columns are the eigenvectors of $\rho$. From Ref.[5], one can get $\Psi W^{1/2} = \Phi M^{1/2} T$, where $T$ is a Right-unitary matrix. The mixed states are separable iff there exist a decomposition such that $\psi^k$ for every $k$ is separable. The entanglement measure of formation can be defined as the infimum of the average $|C'(\psi^k)|$.

Namely, $C(\rho) = \inf \left\{ \sum_k \omega_k |C'(\psi^k)| \right\}$, if $C(\rho)$ is assigned as the entanglement measure for tripartite mixed states. Therefore, for any a decomposition

$$\rho = \sum_k \omega_k |\psi^k\rangle \langle \psi^k|,$$

considering eq.(4),

$$\sum_k \omega_k C'(\psi^k) = 2 \sum_k \omega_k E(\psi^k)$$

$$= \sum_k \omega_k \left( \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2 \right)^2. \quad (5)$$

According to the convexity property of eq.(4), we have,

$$\sum_k \omega_k C'(\psi^k) \geq \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} \left( \sum_k \omega_k \left| \langle \psi^k | \tilde{\psi}_{\alpha\beta}^k \rangle \right|^2 \right)^2 \quad (6)$$

$$= \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} \left( \sum_k \omega_k \left| \langle \psi^k | \tilde{\psi}_{\alpha\beta}^k \rangle \right|^2 \right)^2$$

$$\geq \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} \left( \sum_k \omega_k \left| \langle \psi^k | \tilde{\psi}_{\alpha\beta}^k \rangle \right|^2 \right)^2$$

with $S_{\alpha\beta} = L_\alpha \otimes L_\beta$. If the inequality $\sum_i |x_i| \geq |\sum_i x_i|$ and the Cauchy-Schwarz inequality

$$\left( \sum_i x_i^2 \right)^{1/2} \left( \sum_i y_i^2 \right)^{1/2} \geq \sum_i x_i y_i,$$

are considered, eq.(6) can be written as

$$\sum_k \omega_k C'(\psi^k) \geq \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} \left( \sum_k |T^T \Phi^* S_{\alpha\beta} \Phi^T M^{1/2} T_{kk} |^2 \right)^2$$

$$\geq \left[ \sum_k \left| T^T \left( \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} z_{\alpha\beta} A_{\alpha\beta} \right) T_{kk} \right| \right]^2 \quad (8)$$

in matrix notation of $\rho$, where $A_{\alpha\beta} = M^{1/2} \Phi T S_{\alpha\beta} \Phi^T M^{1/2}$ and $z_{\alpha\beta} = y_{\alpha\beta} e^{i\theta}$ with $y_{\alpha\beta} > 0$, $\sum_{\alpha\beta} y_{\alpha\beta}^2 = 1$. By virtue of the definition of measure of entanglement of formation, one can get

$$C'(\rho) = \inf \left\{ \sum_k \left| T^T \left( \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} z_{\alpha\beta} A_{\alpha\beta} \right) T_{kk} \right| \right\} \quad (9)$$

for mixed states. The infimum is given by

$$\left[ \max_{z \in C^\alpha \beta} \lambda_1(z) - \sum_{i > 1} \lambda_i(z) \right]$$

with $\lambda_i(z)$ are the singular values of $\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} A_{\alpha\beta}$. Hence we get the lower bound

$$C'(\rho) \geq \left[ \max_{z \in C^\alpha \beta} \lambda_1(z) - \sum_{i > 1} \lambda_i(z) \right] \geq [C(\rho)]^2,$$

which is consistent to the result in Ref.[5,6] neglecting the square.

### III. RESIDUAL ENTANGLEMENT OF TRIPARTITE SYSTEMS

Consider a given tripartite pure state $|\psi\rangle_{ABC}$ in arbitrary dimension. For the subsystem $\rho_{AB}$ made up of $A$
and $B$ by tracing over $C$, similar to ref.[7], we can employ the eq.(10) to write the following inequality

$$C'_{AB}(\rho) = \left[ \max_{z \in C'_{\alpha\beta}} \lambda_1(z) - \sum_{i=1}^{n} \lambda_i(z) \right]^2 \leq Tr(Q_{AB}Q^I_{AB})$$

$$\leq Tr(\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} \left| z_{\alpha\beta} \right|^2 |A_{\alpha\beta}|^2)^{1/2} \times \left( \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |A_{\alpha\beta}|^2 \right)^{1/2}$$

$$= \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} Tr \left( M^{1/2} \Phi^I S_{\alpha\beta} \Phi^* M^{1/2} \right)^2$$

$$= \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} Tr \left( \rho_{AB} (\tilde{\rho}_{AB})_{\alpha\beta} \right),$$

i.e.

$$C'_{AB}(\rho) \leq \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} Tr \left( \rho_{AB} (\tilde{\rho}_{AB})_{\alpha\beta} \right),$$

(11)

where $Q_{AB} = \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} z_{\alpha\beta} A_{\alpha\beta}, \ (\tilde{\rho}_{AB})_{\alpha\beta} = S_{\alpha\beta} \rho_{AB} S_{\alpha\beta}$ and the other parameters are defined the same as above section. For the subsystem $\rho_{AC}$, the inequality analogous to (11) holds. Therefore, we can bound the sum $C'_{AB}(\rho) + C'_{AC}(\rho)$:

$$C'_{AB}(\rho) + C'_{AC}(\rho) \leq \sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} Tr \left( \rho_{AB} (\tilde{\rho}_{AB})_{\alpha\beta} \right)$$

$$+ \sum_{\gamma=1}^{N_1(N_1-1)/2} \sum_{\delta=1}^{N_2(N_2-1)/2} Tr \left( \rho_{AC} (\tilde{\rho}_{AC})_{\alpha\beta} \right).$$

(12)

If we write the pure state $|\psi\rangle_{ABC}$ in the standard basis defined in $n_1 \times n_2 \times n_3$ dimension, i.e.

$$|\psi\rangle_{ABC} = \sum_{ijk} a_{ijk} |i\rangle|j\rangle|k\rangle,$$

where $i = 0, 1, \cdots n_1 - 1$, $j = 0, 1, \cdots n_2 - 1$ and $k = 0, 1, \cdots n_3 - 1$. Considering the coefficients $a_{ijk}$, one can obtain the following three equations

$$\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} Tr \left( \rho_{AB} (\tilde{\rho}_{AB})_{\alpha\beta} \right)$$

$$= \sum_{\alpha=1}^N a_{ijk}^* a_{mkn}\epsilon_{mm'\epsilon_{nn'}} a_{m'n'p}\epsilon_{ii'} \epsilon_{jj'} \epsilon_{kk'}.$$
According to the convex $C^2_A(\rho)$, the left-hand side of the inequality (19) is greater than or equal to $C^2_{AB}(\rho) + C^2_{AC}(\rho)$; according to the definition of entanglement measure of mixed bipartite states, the right-hand of the inequality is equal to $C^2_{A(BC)}(\rho)$. Therefore, we have

$$C^2_{AB}(\rho) + C^2_{AC}(\rho) \leq C^2_{A(BC)}(\rho) = C^2_{A(BC)}(\rho)$$

holds for the mixed state $\rho$.

If particle $A$ in $C^2_{A(BC)}$ is called the focus analogous to Ref.[7], we find that the inequality holds independent on the choice of the focus. If we introduce a quantity $\tau$ for every analogous inequality corresponding every possible focus, then every one of the inequalities can be converted to an equation. Namely, we can obtain

$$\tau_{A(BC)} = C^2_{A(BC)}(\rho).$$

$$\tau_{B(AC)} + C^2_{AB}(\rho) + C^2_{BC}(\rho) = C^2_{B(AC)}(\rho).$$

$$\tau_{C(AB)} + C^2_{CB}(\rho) + C^2_{AC}(\rho) = C^2_{C(AB)}(\rho).$$

Unlike the case of pure states in Ref.[7], it is difficult to tell whether $\tau_{A(BC)}$, $\tau_{B(AC)}$ and $\tau_{C(AB)}$ are equal because of the introduction of $z_{\alpha\beta}$ in optimization, even for the pure state $\rho$ in arbitrary dimension. However, in Ref.[7], for tripartite states with two levels, $\tau$ embodies a kind of global property, which can measure 3-way entanglement, therefore, no matter whether $\tau_\alpha$s are equal or not, it is reasonable to believe that $\tau_\alpha$s, with $\alpha$ corresponding to the different foci $A(BC)$, $B(AC)$ or $C(AB)$, include a common quantity which embodies a kind of collective property, independent of permutations and can be used to measure 3-way entanglement just like $\tau$. What’s more, we know that entanglement measure is relative, hence we can define the minimal $\tau_\alpha$ as the residual entanglement, which measures 3-way entanglement.

**IV. RESIDUAL ENTANGLEMENT OF MULTIPARTITE SYSTEMS**

For an $N$-partite quantum state $\rho_{AB\ldots N}$ in arbitrary dimension, one always regards it as a tripartite quantum state which can be assumed to be $\rho_{AB(C\ldots N)}$, therefore the analogous inequality holds

$$C^2_{AB}(\rho) + C^2_{AC(\ldots N)}(\rho) \leq C^2_{A(BC\ldots N)}(\rho).$$

In the same way, one can get

$$C^2_{AC(\ldots N)}(\rho) + C^2_{A(D\ldots N)}(\rho) \leq C^2_{A(C\ldots N)}(\rho).$$

This iteration of the above inequalities leads to

$$C^2_{AB}(\rho) + C^2_{AC}(\rho) + \ldots + C^2_{AN}(\rho) \leq C^2_{A(BC\ldots N)}(\rho).$$

Analogous to the last section, we have

$$\tau_{A(BC\ldots N)} + C^2_{AB}(\rho) + C^2_{AC}(\rho) + \ldots + C^2_{AN}(\rho) = C^2_{A(BC\ldots N)}(\rho).$$

If changing the focus, one will obtain the other $N-1$ analogous equations. It is worth noting that $AB$, $ABC$ and so on can all be regarded as an object and all can be used as the focus. Therefore, there should be

$$\sum_{i=1}^{N} \tau_{i} \alpha \tau_{\alpha}$$

in all, where $\tau_{i} \alpha \tau_{\alpha}$ includes a common quantity that embodies the collective property of the given quantum state, and independent on the choice of the focus (or permutations). Therefore, analogous to the case of tripartite systems, we can also select the minimal $\tau_\alpha$, where $\alpha$ belongs to the set of all the different foci, as the residual entanglement. Now, the residual entanglement can be written in a general form considering the tripartite case, in a more rigorous way.

**Definition.** The residual entanglement $\tau_{A(BC\ldots N)}$ of an $N$-particle system $\rho_{A(BC\ldots N)}$ is defined as

$$\tau_{A(BC\ldots N)} = \min \{ \tau_{\alpha} | \alpha = 1, 2, \ldots, \sum_{i=1}^{N} C^2_{i}\},$$

where $\alpha$ corresponds to all the possible foci.

**V. EXAMPLES**

For the generalized GHZ states [11]

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |0\ldots0\rangle_n + |1\ldots1\rangle_n \right),$$

no matter which particles are selected as the focus, the concurrence of the state $|\psi\rangle$ which is regarded as a bipartite state corresponding to the focus and the others, is one; and the concurrence of any subsystem by tracing over other $N - focus$ particles are zero. Hence, $\tau_{A(BC\ldots N)} = 1$.

For the generalized $n$-qubit state

$$|\phi\rangle = \alpha_1 |10\ldots0\rangle + \alpha_2 |01\ldots0\rangle + \alpha_3 |001\ldots0\rangle + \ldots + \alpha_n |000\ldots1\rangle,$$

our definition is reduced to the case introduced in Ref.[7], therefore

$$C^2_{12} + C^2_{13} + \ldots + C^2_{1n} = C^2_{A(BC\ldots N)},$$

namely, $\tau_{12\ldots n} = 0$. 
For N-particle product states, \[ \rho = \sum_i \rho^1_i \otimes \rho^2_i, \] where \( \rho^1 \) corresponds to \( N_1 \) particles and \( \rho^2 \) corresponds to the others, if the particles corresponding to \( \rho^1 \) (or \( \rho^2 \)) are selected as the focus, we can obtain that the concurrence of the two product subsystem \( \rho^1 \) and \( \rho^2 \) is zero, i.e. \( C_{N_1,N_2}^{2} (\rho) = 0 \). Since \( \tau \geq 0 \) is the minimum of all the \( \tau^\alpha \)'s, then \( \tau = 0 \) in this case. Especially for the four-particle pure product state \( \Psi_{ABCD} \) of two singlet states mentioned in Ref.[8], one can obtain \( \tau_{ABCD} = 0 \), which shows that the state \( \Psi_{ABCD} \) does not include 4-way entanglement. The result is consistent to the fact and what was implied in Ref.[8].

VI. CONCLUSION AND DISCUSSION

In summary, we introduced a special entanglement measure for bipartite states. We generalize residual entanglement to the case of multipartite states in arbitrary dimensions. Unlike Ref.[8], we generalize the residual entanglement to the case of any \( N \)-partite system, and the examples show that our residual entanglement can well measure the \( n \)-way entanglement. Recalling the original residual entanglement [7], one can find that it is just the special case of ours: it is a tripartite state with two levels (in the case, we can get the same result). However, the special case easily shows that \( \tau^A_{(BC)} \), \( \tau^B_{(AC)} \) and \( \tau^C_{(AB)} \) are equal which have a more elegant form, but it is difficult to tell whether the same relation holds in other cases. For tripartite states, it is impossible to select two particles as a focus, there exist only three ways to select the focus which is enough; but for multipartite states, \( \tau^\alpha \) corresponding to only one particle as the focus is not enough to completely embody the collective property, the other cases are valid and necessary. Furthermore, one can find that the introduction of the special bipartite entanglement measure in section II is just a medium of generalization of residual entanglement from the pure states to the mixed states, which leads to the difference between \( C_{3}^{2} (\rho) \) in this paper and \( (C_{A|BC}^{2})_{\text{min}} (\rho) \) in Ref.[7] for mixed states and provides inestimable conveniences to generalize the residual entanglement from tripartite case to multipartite one.

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