Question on the Existence of Gravitational Anomalies

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Abstract

The existence of gravitational anomalies claimed by Alvarez-Gaumé and Witten is examined critically. It is pointed out that they were unaware of the essential difference between T-product quantities and T*-product quantities. Field equations and, therefore, the Noether theorem are, in general, violated in the case of T*-product quantities, that is, those directly calculable from Feynman integrals. In the 2-dimensional case, it is explicitly confirmed that the energy-momentum tensor is strictly conserved if the above stated property of the T*-product quantities is correctly taken into account. The non-existence of gravitational anomalies is explicitly demonstrated for the BRS-formulated 2-dimensional quantum gravity in the Heisenberg picture.

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§1. Introduction

In 1984, Alvarez-Gaumé and Witten\textsuperscript{1} (A.-G. and W.) claimed that there exist gravitational anomalies in the \((4k + 2)\)-dimensional spacetime \((k = 0, 1, 2, \ldots)\). In that work, they calculated the 1-loop Feynman integrals for the \((2k + 2)\)-point functions of the energy-momentum tensor, and showed that the results are inconsistent with the conservation law of the energy-momentum tensor. The purpose of the present paper is to point out that their argument is not sufficient for establishing the existence of gravitational anomalies and that, in fact, there exists no gravitational anomaly in the 2-dimensional case.\footnote{A preliminary report was made half a decade ago.}

The flaw in their reasoning results from the fact that they were careless with regard to the essential difference between the T-product and the T*-product. The former is converted into the latter when the Hamiltonian formalism is transcribed into the Lagrangian formalism. The expressions appearing in the covariant perturbation theory and in the path-integral formalism are written as the vacuum expectation values of the T*-product but not of the T-product. Nevertheless, A.-G. and W. regarded the Feynman integrals as quantities written in terms of the T-product. The T-product is a sum over products of local operators multiplied by a product of \(\theta\)-functions of time differences; therefore, in general, it becomes non-covariant when differentiated. By contrast, the T*-product is defined in such a way that time differentiations always act \textit{after} the vacuum expectation value of the T-product of the canonical fields is taken. Hence, T*-product quantities are always covariant. The price paid for this bonus is that the T*-product is no longer a product in the mathematical sense; a T*-product quantity involving a factor 0 is not necessarily equal to 0. Accordingly, Feynman integrals are not necessarily consistent with field equations and, therefore, with the Noether theorem. This violation of the Noether theorem should not be confused with an anomaly, as it can be calculated explicitly from the difference between the T*-product and the T-product.

As is well known, the Adler-Bell-Jackiw anomaly is established by calculating the Feynman integral corresponding to the triangle diagram. The reason that the chiral anomaly is correctly obtained in this case is that the expression for the chiral current involves no differentiation. Contrastingly, the expression for the energy-momentum tensor necessarily contains time differentiations. In the T*-product quantity, those differentiations act from outside the vacuum expectation value, but not directly on the fields in the expression for the energy-momentum tensor. This fact implies a nontrivial difference between the T*-product and the T-product. However no analysis of this difference is given in the work of A.-G. and W.
The main purpose of the present paper is to show explicitly that, at least in the 2-dimensional case, what A.-G. and W. called the gravitational anomaly is nothing more than the contribution from the difference between the \( T^* \)-product and the \( T \)-product. Thus, we establish that the energy-momentum tensor is strictly conserved if correctly calculated. We believe that this fact implies that in fact there exists no gravitational anomaly.

The present paper is organized as follows. In §2, we discuss the apparent violation of the Noether theorem encountered in the path-integral approach. In §3, we consider the 2-dimensional Weyl field and show that what A.-G. and W. interpreted as a gravitational anomaly is nothing more than the apparent violation of the conservation law due to the use of the \( T^* \)-product quantities. In §4, we examine the paper of A.-G. and W. more closely, because some people assert that even if the energy-momentum tensor is conserved, the conclusion of A.-G. and W. would remain valid. With regard to this point, some comments are made about the validity of the Virasoro anomaly. Furthermore, we closely analyze the meaning of the general-coordinate non-invariance that cannot be removed by local counter-terms. In §5, in order to demonstrate that there exists no gravitational anomaly, we consider BRS-formulated 2-dimensional quantum gravity coupled with the Weyl fields. The final section is devoted to discussion. In the Appendix, some formulae for singular-function products are presented.

\section*{§2. \( T^* \)-product and the Noether theorem}

In a previous paper, we investigated the pathological nature of the covariant perturbation theory and the path-integral formalism caused by the \( T^* \)-product.\(^3\) In this section, we first reproduce the general formula for the field-equation-violating contribution due to the \( T^* \)-product in the path integral.

The generating functional, \( Z(J) \), of the Green functions is formally expressed as a path integral:

\[
Z(J) = \int \left( \prod_A \mathcal{D}\varphi_A \right) \exp \int d^N x \left( \mathcal{L} + \sum_A J_A \varphi_A \right).
\]

Here, \( S = \int d^N x \mathcal{L}(x) \) is the action for the fields \( \varphi_A(x) \) in the \( N \)-dimensional spacetime, \( \mathcal{D}\varphi_A \) is the path-integral measure normalized as \( Z(0) = 1 \), and \( J_A(x) \) denotes the source function for \( \varphi_A(x) \). Let \( F(\varphi) \) be an arbitrary function of \( \varphi_{A_1}(y_1), \ldots, \varphi_{A_m}(y_m) \), and let \( \delta \varphi_A \) be a field-independent variation of \( \varphi_A \). The path-integral measure should be invariant under the functional translation \( \varphi_A \rightarrow \varphi_A + \delta \varphi_A \). Accordingly, by considering a variation of a particular field \( \varphi_A \) in \( F(i^{-1}\partial/\partial J)Z|_{J=0} \), we obtain

\[
i \left\langle T^* \frac{\delta}{\delta \varphi_A} S \cdot F(\varphi) \right\rangle + \left\langle T^* \frac{\delta}{\delta \varphi_A} F(\varphi) \right\rangle = 0. \quad (2.2)
\]
This is the T*-product version of the field equation $(\delta/\delta \phi_A)S = 0$. The second term in (2.2) is the field-equation-violating term resulting from the use of the T*-product. It is thus seen that we cannot naively use the field equations, and therefore the Noether theorem, with the T*-product quantities. Hence, we must treat the current conservation law very carefully in the covariant perturbation theory.

Now, we consider the infinitesimal symmetry transformation

$$\delta^\epsilon \phi_A(x) \equiv \phi'_A(x') - \phi_A(x). \quad (2.3)$$

Thus, we define

$$\delta^\epsilon \phi_A(x) \equiv \phi'_A(x) - \phi_A(x), \quad (2.4)$$

so that

$$\delta^\epsilon \phi_A = \delta^\epsilon \phi_A + \delta^\epsilon x^\mu \cdot \partial_\mu \phi_A, \quad (2.5)$$

where $\delta^\epsilon x^\mu \equiv x'^\mu - x^\mu$.

The Noether current $J^\mu$ is defined by

$$\epsilon J^\mu \equiv \sum_A \delta^\epsilon \phi_A \cdot \frac{\partial}{\partial (\partial_\mu \phi_A)} L + \delta^\epsilon x^\mu \cdot L. \quad (2.6)$$

Then, the Noether identity is

$$\epsilon \partial_\mu J^\mu = - \sum_A \delta^\epsilon \phi_A \cdot \frac{\delta}{\delta \phi_A} S + \delta^\epsilon L, \quad (2.7)$$

where the last term on the right-hand side vanishes if the Lagrangian density is invariant under the symmetry transformation. However, the first term cannot be set to zero, because the field equations do not hold for the T*-product quantities, as stated above. In particular, for the energy-momentum tensor $T^{\mu \nu}$, we have

$$\partial_\mu T^{\mu \nu} = - \sum_A \partial^\nu \phi_A \frac{\delta}{\delta \phi_A} S. \quad (2.8)$$

Hence, with the help of (2.2), we obtain

$$\langle T^* \partial_\mu T^{\mu \nu}(x) \cdot F(y) \rangle = -i \left\langle T^* \sum_A \epsilon_A \frac{\delta}{\delta \phi_A(x)} [\partial^\nu \phi_A(x) \cdot F(y)] \right\rangle, \quad (2.9)$$

where $\epsilon_A = 1$ if $A$ is bosonic and $\epsilon_A = -1$ if $A$ is fermionic.

For illustration, we consider a free massive scalar field $\phi(x)$ in $N$ dimensions. Its field equation is

$$(\Box + m^2)\phi = 0. \quad (2.10)$$
The energy-momentum tensor is given by

\[ T_{\mu\nu} = \partial_\mu \phi \cdot \partial_\nu \phi - \frac{1}{2} \eta^{\mu\nu} (\partial_\sigma \phi \cdot \partial_\rho \phi - m^2 \phi^2), \]

which, of course, satisfies the conservation law

\[ \partial_\mu T^{\mu\nu} = (\Box + m^2) \phi \cdot \partial_\nu \phi = 0, \]

owing to the field equation. We should note, however, that the Feynman propagator,

\[ \langle T^* \phi(x) \phi(y) \rangle = \langle T^\phi(x) \phi(y) \rangle = \Delta_F(x - y), \]

does not satisfy the Klein-Gordon equation but, instead, the equation

\[ (\Box + m^2) \Delta_F(x - y) = -i \delta^N(x - y). \]

Accordingly, by straightforward calculation, we find a nonvanishing result for the divergence of the 2-point function of the energy-momentum tensor:

\[ \langle T^* \partial_\mu T^{\mu\nu}(x) \cdot T^{\lambda\rho}(y) \rangle \equiv \partial_\mu \langle T^* T^{\mu\nu}(x) T^{\lambda\rho}(y) \rangle
\]

\[ = -i(\eta^{\lambda\sigma} \partial^\rho + \eta^{\rho\sigma} \partial^\lambda - \eta^{\lambda\rho} \partial^\sigma) \partial_\rho \Delta_F(x - y) \cdot \partial_\sigma \delta^N(x - y) \]

\[ -im^2 \eta^{\lambda\rho} \Delta_F(x - y) \cdot \delta^N(x - y). \]

This is, of course, not a gravitational anomaly. Indeed, the right-hand side of (2.9) becomes

\[ -i \langle T^* \frac{\delta}{\delta \phi(x)} [\partial_\nu \phi(x) \cdot T^{\lambda\rho}(y)] \rangle \]

\[ = -i \langle T^* \partial_\nu \phi(x) [(-\partial_\sigma) \{(\eta^{\lambda\sigma} \partial^\rho + \eta^{\rho\sigma} \partial^\lambda) \phi(y) \cdot \delta^N(x - y)\} \]

\[ -\eta^{\lambda\rho} (-\partial_\sigma)^\rho [\partial^\sigma \phi(y) \cdot \delta^N(x - y) + m^2 \eta^{\lambda\rho} \phi(y) \delta^N(x - y)] \rangle, \]

which is exactly equal to (2.15).

In order to avoid the obstruction caused by the T*-product, it is convenient to define the anomaly in terms of Wightman functions.\(^4\) Let \( j^\mu \) be a symmetry current; then the anomaly for the corresponding symmetry exists if we have

\[ \partial_\mu \langle j^\mu(x) \phi_1(y_1) \cdots \phi_n(y_n) \rangle \neq 0 \]

for some fields \( \phi_1, \ldots, \phi_n \). With regard to the gravitational anomaly of the above model, we have only to calculate the quantity

\[ \partial_\mu \langle T^{\mu\nu}(x) \phi(y_1) \phi(y_2) \rangle. \]

It is readily shown that this quantity vanishes using the Klein-Gordon equation for \( \Delta^{(+)}(x - y_j) \). Thus, we conclude that the gravitational anomaly does not exist.
§3. Energy-momentum conservation in the Weyl theory

We consider the 2-dimensional complex Weyl field $\psi(x)$. Its free-field action is given by

$$ S = i \int d^2x \psi^\dagger \partial_- \psi. \quad (3.1) $$

We employ the light-cone coordinates, $x^\pm = (x^0 \pm x^1)/\sqrt{2}$. The field equation and the 2-dimensional anti-commutation relations are

$$ \partial_- \psi = 0, \quad (3.2) $$
$$ \{\psi(x), \psi(y)\} = 0, \quad \{\psi(x), \psi^\dagger(y)\} = \delta(x^+ - y^+), \quad (3.3) $$

respectively. Accordingly, the 2-point Wightman function and the Feynman propagator are, respectively, as follows:

$$ \langle \psi(x)\psi^\dagger(y) \rangle = \frac{1}{2\pi i} \cdot \frac{1}{x^+ - y^+ - i0^+} \quad (3.4) $$
$$ \langle T^*\psi(x)\psi^\dagger(y) \rangle = \frac{1}{2\pi i} \cdot \frac{1}{x^+ - y^+ - i0(x^- - y^-)} \equiv \frac{1}{2\pi i} \left[ \frac{\theta(x^- - y^-)}{x^+ - y^+ + i0} + \frac{\theta(-x^- + y^-)}{x^+ - y^+ + i0} \right]. \quad (3.5) $$

It is very important to write explicitly the fact that the Feynman propagator depends on $x^- - y^-$. The energy-momentum tensor $T^\mu_\nu$ is given by

$$ T^-_+ = T^+_+ = \frac{i}{2}(\psi^\dagger \partial_+ \psi - \partial_+ \psi^\dagger \cdot \psi), \quad (3.6) $$
$$ T^+_+ = T^-_- = \frac{i}{2}(-\psi^\dagger \partial_- \psi + \partial_- \psi^\dagger \cdot \psi). \quad (3.7) $$

Although $T^-_+$ vanishes owing to the field equation, it cannot be ignored in the calculation of the $T^*$-product quantities.

A straightforward calculation yields

$$ \langle T^* T_+^+(x) T_+^+(y) \rangle = \frac{1}{8\pi^2} \cdot \frac{1}{(x^+ - y^+ - i0(x^- - y^-))^4} $$
$$ = \frac{1}{8\pi^2} \left[ \frac{\theta(x^- - y^-)}{(x^+ - y^+ - i0)^4} + \frac{\theta(-x^- + y^-)}{(x^+ - y^+ + i0)^4} \right]. \quad (3.8) $$

Hence, we have

$$ \partial_+^2 \langle T^* T_+^+(x) T_+^+(y) \rangle = \frac{1}{24\pi i} \delta'''(x^+ - y^+) \delta(x^- - y^-). \quad (3.9) $$
We can rewrite the above results into those in momentum space. With the help of the formula
\[
\int d^2 z \frac{\theta(\pm z^0)}{z^0 + i0} e^{ipz} = -2\pi \cdot \frac{\theta(\mp p_+)}{p_- + i0},
\] (3.10)
we see that the Fourier transform of (3.8) is given by
\[
\frac{i}{24\pi} \cdot \frac{p_+^3}{p_- + i0},
\] (3.11)
It is important to write explicitly the infinitesimal imaginary part of the denominator. Differentiating (3.8) with respect to \(x^-\) is equivalent to multiplying (3.11) by \(-ip_-\), and therefore we obtain \((1/24\pi)p_+^3\), which is, of course, the Fourier transform of (3.9). This quantity is identically that which A.-G. and W. regarded as the gravitational anomaly (see §4).1)

We can show, however, that it is merely the contribution from the difference between the \(T^*\)-product quantity and the \(T\)-product quantity.

It is convenient to work in the spacetime representation. Dropping the terms proportional to \(\epsilon (x^- - y^-)\delta(x^- - y^-) = 0\), we obtain
\[
\langle T^* T_-(x) T_+(y) \rangle = \frac{1}{4\pi i} \left[ \text{Pf} \frac{1}{(x^+ - y^+)^2} \cdot \delta(x^+ - y^+) + \text{Pf} \frac{1}{x^+ - y^+} \cdot \delta'(x^+ - y^+) \right] \delta(x^- - y^-),
\] (3.12)
where Pf denotes the “finite part”, i.e., Pf\((1/z^n) \equiv \Re [1/(z - i0)^n]\). Hence, we find
\[
\partial_+^x \langle T^* T_-(x) T_+(y) \rangle = \frac{1}{4\pi i} \left[ -2 \text{Pf} \frac{1}{(x^+ - y^+)^2} \cdot \delta(x^+ - y^+) + \text{Pf} \frac{1}{x^+ - y^+} \cdot \delta''(x^+ - y^+) \right] \delta(x^- - y^-).
\] (3.13)
From (3.9) and (3.13), we obtain
\[
\langle T^* [\partial_- T_+(x) + \partial_+ T_- (x)] T_+(y) \rangle = \frac{1}{4\pi i} \left[ -\frac{1}{6} \delta''(x^+ - y^+) - 4 \text{Pf} \frac{1}{(x^+ - y^+)^3} \cdot \delta(x^+ - y^+) \right] \delta(x^- - y^-),
\] (3.14)
where we have made use of the identity (A.6) presented in the Appendix.

Next, note that the quantity on the right-hand side of (2.9) becomes
\[
i \langle T^* \frac{\delta}{\delta \psi(x)} [\partial_+ \psi(x) \cdot T_+(y)] \rangle + i \langle T^* \frac{\delta}{\delta \psi^\dagger(x)} [\partial_+ \psi^\dagger(x) \cdot T_+(y)] \rangle \]
\[
= \frac{1}{2\pi i} \left[ -\text{Pf} \frac{1}{(x^+ - y^+)^2} \cdot \delta'(x^+ - y^+) - 2 \text{Pf} \frac{1}{(x^+ - y^+)^3} \cdot \delta(x^+ - y^+) \right] \delta(x^- - y^-).
\] (3.15)
With the help of the identity (A.4), we see that (3.15) coincides with (3.14). Thus, we have shown that what A.-G. and W. called the gravitational anomaly is simply the contribution from the T*-product.

Finally, we make a remark concerning the Majorana Weyl field $\psi$, used by Green, Schwarz and Witten.\textsuperscript{5} This field has only one field degree of freedom. Hence, without introducing an extra field, it cannot be quantized, because there is no canonical conjugate independent of $\psi$. Nevertheless, setting up the 2-dimensional anticommutation relation

$$\{\psi(x), \psi(y)\} = \delta(x^+ - y^+) \quad (3.16)$$

by hand and replacing (3.6) and (3.7) by the same expressions with the daggers removed, we can repeat the above analysis. Doing so, we obtain the same results, except for the appearance of an overall multiplicative factor of 2 on the left-hand side only. Thus, (2.9) does not hold in this model. The reason for this is that the path integral does not exist.\textsuperscript{*)}

§4. Examination of the paper of A.-G. and W.

In §3, we established that the energy-momentum tenser is strictly conserved, at least in the 2-dimensional case, if the difference between the T-product and the T*-product is correctly taken into account. However, some people believe that the claim of A.-G. and W. itself is valid nevertheless. To explicitly show that in fact it is not valid, we more closely examine the reasoning employed in the paper of A.-G. and W. in this section.

We first quote one paragraph, together with a footnote, of their paper.

"With $\gamma_- \psi = \partial_- \psi = 0$, the only non-vanishing component of the energy-momentum tensor is $T_{++} = \frac{1}{2} i \bar{\psi} \gamma_+ \overleftrightarrow{\partial_+} \psi$, and the linearized interaction of fermions with the gravitational field is $\Delta L = -h_{--} \frac{1}{4} i \bar{\psi} \gamma_+ \overleftrightarrow{\partial_+} \psi$. We will study the effective action to second order in the metric perturbation $h$, by studying the two-point function $U(p) = \int d^2 x e^{ipx} \langle \Omega | T(T_{++}(x)T_{++}(0))|\Omega \rangle. \quad (A.-G. and W. 11)\textsuperscript{11)}"

Now, it is possible to see without any computation that there must be an anomaly. The naive conservation law for $T_{++}$ is $\partial_- T_{++} = 0$; it leads to the naive Ward identity $p_- U = 0.\textsuperscript{*})$

If true, this would imply $U = 0$ for all non-zero $p_-$, and hence (by analyticity) for all $p_-$.\textsuperscript{\textsuperscript{*})}

\textsuperscript{*)} To write down the path integral, we expand $\psi$ as $\psi = \sum_i \theta_i \varphi_i$, where the quantities $\theta_i$ are Grassmann numbers and we have $i \int d^2 x \varphi_i \partial_- \varphi_j = \delta_{ij}$. Then the action vanishes because $S = i \int d^2 x \psi \partial_- \psi = \sum_i \theta_i^2 = 0$. We cannot avoid this result by introducing conjugate Grassmann numbers $\theta_i^\dagger$, because $\psi$ satisfies a first-order differential equation. Indeed, if we do so, then the action becomes that of the complex Weyl field.
But $U$, as the two-point function of the hermitian operator $T_{++}$, cannot vanish. So there must be an anomaly.

[footnote]

* Naively there is no equal time commutator term in this Ward identity. If one looks at (A.-G. and W. 11) as a two-point function in flat space, the anomaly we will find can be regarded as an anomalous commutation relation $[T_{++}(x), T_{++}(y)] = (i/48\pi)\delta'''(x - y) + \text{tree level terms}$. It is closely related to the anomaly in the Virasoro algebra in string theories. But we will see that upon coupling $T_{++}$ to the gravitational field, the anomaly is a breakdown of general covariance.

In the succeeding paragraph, they calculate $U(p)$ using the Feynman diagrammatic method and obtain

$$U(p) = \frac{i}{24\pi} \frac{p_+^3}{p_-}. \quad (\text{A.-G. and W. 13})$$

Then they proceed to discussing the “question of the covariance of the effective action”. As the effective action corresponding to (A.-G. and W. 13), they consider

$$-\frac{1}{192\pi} \int d^2p \frac{p_+^3}{p_-} h_-(p) h_-(\bar{p}). \quad (\text{A.-G. and W. 15})$$

Under the general coordinate transformation of the external gravitational field $h_{\mu \nu}(x)$, they show that (A.-G. and W. 15) cannot be made invariant by adding a local functional of fields.

Now, in the paragraph quoted above, A.-G. and W. defined the 2-point function $U(p)$ as a $T$-product quantity. Nevertheless, in the succeeding paragraph, they calculated it by means of a $T^*$-product quantity. Evidently, they were unaware of the essential difference between the T-product and the $T^*$-product. Indeed, while they used the equation $T_{-+} = 0$, which is not valid in the case of the $T^*$-product, they calculated a quantity involving $T_{++}$ with the method for calculating the $T^*$-product quantity. Thus their reasoning is clearly internally inconsistent.

It is quite interesting to analyze the naive argument appearing below (A.-G. and W. 11), in which A.-G. and W. indicated the existence of an anomaly: Why did they reach an invalid conclusion in spite of the fact that they considered T-product quantities in this paragraph?*

The reason can be clarified by examining the footnote presented there. For this purpose, we reproduce the description concerning the Virasoro anomaly in a book of Green, Schwarz and Witten.** It is in essence as follows:**

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* One of the authors (N. N.) would like to thank Prof. T. Kugo for raising this question.

** In their notation, spacetime coordinates are denoted by $(\tau, \sigma)$ and $\sigma^\pm = \tau \pm \sigma$, without the factor of $1/\sqrt{2}$.  

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9
We consider a 2-dimensional massless scalar field \( \phi \), which satisfies
\[
\partial_+ \partial_- \phi = 0. \tag{4.1}
\]
The \( ++ \) component of its energy-momentum tensor is
\[
T_{++} = \partial_+ \phi \cdot \partial_+ \phi, \tag{4.2}
\]
which satisfies the conservation law
\[
\partial_- T_{++} = 0. \tag{4.3}
\]
But, because the T-product is noncommutative with the time differentiation, we obtain
\[
\partial_- \langle T_{++}(\sigma, \tau) T_{++}(\sigma', \tau') \rangle = \frac{1}{2} \delta(\tau - \tau') \langle [T_{++}(\sigma, \tau), T_{++}(\sigma', \tau')] \rangle. \tag{4.4}
\]
Perturbative calculation yields
\[
\langle T_{++}(\sigma, \tau) T_{++}(\sigma', \tau') \rangle = \frac{1}{8} \cdot \frac{1}{(\sigma^+ - \sigma'^+)^4}. \tag{4.5}
\]
Substituting (4.5) into (4.4) and making use of the formula
\[
\partial_- \frac{1}{\sigma^+ - i 0} = i \pi \delta(\sigma) \delta(\tau), \tag{4.6}
\]
we find
\[
\langle [T_{++}(\sigma, \tau), T_{++}(\sigma', \tau')] \rangle = -\frac{i \pi}{24} \delta'''(\sigma - \sigma'), \tag{4.7}
\]
which is the equal-time anomalous commutator.

In the above, it is (4.5) that is mistaken: It is a perturbative result, and therefore it is a \( T^* \)-product quantity. Because differentiation commutes with the \( T^* \)-product, it cannot be substituted into the formula (4.4), which is a formula for the T-product. The curious formula (4.6) is correctly written
\[
\partial_- \frac{1}{\sigma^+ - i 0} = i \pi \delta(\sigma) \delta(\tau). \tag{4.8}
\]
Correspondingly, the correct version of (4.5) is
\[
\langle T^* T_{++}(\sigma, \tau) T_{++}(\sigma', \tau') \rangle = \frac{1}{8} \cdot \frac{1}{[\sigma^+ - \sigma'^+ - i 0(\sigma^- - \sigma'^-)]^4}. \tag{4.9}
\]
It is easy to correctly carry out the calculation if one employs the Wightman functions. Noting the relations

\[
\langle T_{++}(\sigma, \tau) T_{++}(\sigma', \tau') \rangle = \frac{1}{8} \cdot \frac{1}{(\sigma^+ - \sigma'^+ - i0)^4},
\]

(4.10)

\[
\langle T_{++}(\sigma', \tau') T_{++}(\sigma, \tau) \rangle = \frac{1}{8} \cdot \frac{1}{(\sigma^+ - \sigma'^+ + i0)^4},
\]

(4.11)

we have

\[
\langle [T_{++}(\sigma, \tau), T_{++}(\sigma', \tau')] \rangle = \frac{1}{8} \left[ \frac{1}{(\sigma^+ - \sigma'^+ - i0)^4} - \frac{1}{(\sigma^+ - \sigma'^+ + i0)^4} \right] = -\frac{i\pi}{24} \delta'''(\sigma^+ - \sigma'^+)\].

(4.12)

In particular, when \( \tau = \tau' \), (4.12) reduces to (4.7), but (4.12) is not anomalous. Note that \( \delta''' \) can be expressed in terms of \( \delta' \), as is seen in (A.4).

Thus, we see that A.-G. and W. omitted the commutator term because they misunderstood it to be anomalous.

It is also noteworthy that A.-G. and W. regarded the violation of the conservation law of \( T_{\mu\nu} \) as evidence of a gravitational anomaly. Their consideration of the effective action amounts to no more than simply checking whether or not (A.-G. and W. 15) can be made invariant by adding some local terms. However, such an investigation is completely unnecessary, because the quantity in (A.-G. and W. 15) itself does not appear.

There is the assertion\(^*\) that the “gravitational anomaly” should be defined not in terms of the inevitable violation of translational invariance but in terms of the existence of generally-noncovariant terms in the path integral generally-covariantized by introducing an external gravitational field \( g_{\mu\nu}(x) \). The (logarithm of the) path integral \( S_{\text{eff}} \) is expanded into powers of \( h_{\mu\nu}(x) = g_{\mu\nu}(x) - \eta_{\mu\nu} \), and one considers its variation (nontrivial lowest order in \( h_{\mu\nu} \) only) under an infinitesimal general coordinate transformation. That is, what one considers is the infinitesimal variation of

\[
S_{\text{eff}}^{(2)} = \text{const} \int d^N x \int d^N y \ h_{\mu\nu}(x) \langle T^* T_{\mu\nu}^{\mu}(x) T^{\lambda\rho}(y) \rangle h_{\lambda\rho}(y) \]

(4.13)

under the transformation \( \delta h_{\mu\nu}(x) = -\partial_\mu \epsilon_\nu(x) - \partial_\nu \epsilon_\mu(x) \). Integrating by parts, we encounter the quantity \( \langle T^* \partial_{\mu} T_{\mu\nu}^{\mu}(x) \cdot T^{\lambda\rho}(y) \rangle \).

The above formulation does not straightforwardly apply to the case of the Weyl field, because the action for the Weyl field cannot be generally covariantized by using \( g_{\mu\nu} \), as it is necessary to introduce the zweibein. Although A.-G. and W. defined the zweibein as

\(^*\) The authors would like to thank Prof. T. Kugo for detailed discussions about this standpoint.
\( e_{\mu a} = \eta_{\mu a} + (1/2)h_{\mu a} \), this form represents nothing but the unwarranted omission of the antisymmetric part of \( e_{\mu a} \). Indeed, the transformation implied by this definition explicitly violates the general-coordinate invariance of the action. The zweibein transforms not as a tensor but as two vectors under general-coordinate transformations. Therefore, the expansion must be carried out in powers not of \( h_{\mu\nu} \) but of \( \tilde{h}_{\mu a} = e_{\mu a} - \eta_{\mu a} \).

The starting action must be invariant under general-coordinate transformations. The invariant action for the Weyl field is given by

\[
S_W = \int d^2x (e^+ j^- - e^- j^+), \quad (4.14)
\]

with \( j_{\pm} = \pm T_{\pm} \), where \( T_{\pm} \) is given by (3.6) and (3.7). We can explicitly confirm its invariance under the infinitesimal general-coordinate transformation defined by \( \delta e^\pm = -\partial_\pm e^\lambda e_{\lambda -} - e^\lambda \partial_\lambda e_{\pm -} \) and \( \delta \psi = -e^\lambda \partial_\lambda \psi \). [Note that the integrand of (4.14) can be written \( \epsilon_{\mu\nu} e^\mu - j^\nu \).]

Thus, for the Weyl field, (4.13) should be replaced by

\[
S_{\text{eff}}^{(2)} = \text{const} \int d^2x \int d^2y \tilde{h}_{\mu -}(x)\langle T^\mu_+(x)T^\lambda_+(y)\rangle \tilde{h}_{\lambda -}(y), \quad (4.15)
\]

with \( \delta \tilde{h}_{\mu a} = -\partial_\mu \epsilon_a(x) \). Only the term for \( \mu = \lambda = - \) is nontrivial; in momentum space, it yields (A.-G. W. and 15). The other three terms are local terms; they would vanish if the \( T^* \)-product were not taken. In momentum space, they are quadratically divergent. After regularization, they can be written

\[
\text{const} \int d^2p \left[ -p^2 \tilde{h}_{- -}(p)\tilde{h}_{+ -}(-p) - p^2 \tilde{h}_{+ -}(p)\tilde{h}_{- -}(-p) + p_+p_-\tilde{h}_{+ -}(p)\tilde{h}_{- -}(-p) \right]. \quad (4.16)
\]

By choosing the coefficient appropriately, we see that the sum of (A.-G. and W. 15) and (4.16) is invariant under the transformation \( \delta \epsilon_\mu \tilde{h}_{\mu -}(p) = -ip_+\epsilon_- (p) \).

Up to this point, everything is essentially the same as in the 2-dimensional massless scalar field theory, in which \( \langle T^\mu T^\nu(x)T^\lambda_+(y) \rangle \) has a nonlocal term proportional to

\[
\frac{p^\mu p^\nu p^\lambda p^\rho}{p^2 + i0} \quad (4.17)
\]

in momentum space. Its divergence is, of course, a local term. This fact is well known as the “conformal anomaly”. 6)

The qualitative difference between the presently considered case and the scalar-field case arises when one requires local Lorentz invariance for the Weyl theory. While \( S_W \) in (4.14) is invariant, \( S_{\text{eff}}^{(2)} \) in (4.15) is not. Indeed, the infinitesimal local Lorentz transformation of \( e_{\mu -} \) yields \( \delta \tilde{h}_{- -} = 0 \) and \( \delta \tilde{h}_{+ -} = \tilde{\epsilon} \) at lowest order. Hence (A.-G. and W. 15) is already invariant
at lowest order, but any local term depending on $\tilde{h}_{++}$ is not. However, we must note that this non-invariance is caused by the fact that the free action of the Weyl field is not invariant under the local Lorentz transformation. In general, under a symmetry transformation that does not leave the free action invariant, the invariance is not preserved at each order in the perturbation theory.

Shortly after A.-G. and W., Langouche\(^7\) and Leutwyler\(^8\) resolved this problem by exactly carrying out the path integral of the effective action. According to Leutwyler, $S_{\text{eff}}$ is consistent with the general-coordinate invariance but not with the local-Lorentz invariance, and the local-Lorentz non-invariance cannot be transferred into the general-coordinate non-invariance in an admissible way. Thus, although a Lorentz anomaly may exist, it cannot be claimed that a “gravitational anomaly” exists in the exact expression for $S_{\text{eff}}$.

According to Leutwyler, the energy-momentum tensor $T^{\mu\nu}$, defined in a frame-independent manner, does not satisfy the general-covariant conservation law: $\nabla_\mu T^{\mu\nu}$ is a local polynomial, but it cannot be removed by adding a local polynomial to $T^{\mu\nu}$. However, the general-covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$ is important only in classical gravity; it has no physical significance in quantum gravity. Indeed, we emphasize that the existence of a “gravitational anomaly” in this sense has nothing to do with the obstruction to the unitarity of the physical $S$-matrix of quantum gravity.

§5. 2-dimensional gravitational theory

A precise treatment of the gravitational anomaly must be made in the framework of quantum gravity. We emphasize that when the gravitational field is quantized, there is no general-coordinate invariance. The gravitational anomaly is nothing but an anomaly with respect to translational invariance\(^9\) in the framework of quantum gravity.

In this section, in order to explicitly demonstrate that there is no gravitational anomaly, we consider the BRS-formulated conformal-gauge 2-dimensional quantum gravity coupled with $D$ Weyl fields. If it is coupled with $D$ scalar fields instead, the model can be interpreted as a string theory in $D$-dimensional spacetime. Previously, we thoroughly investigated this model and found the complete solution in terms of Wightman functions\(^9\)–\(^12\). Extension to the case of Weyl fields is straightforward.

In the conformal gauge, the gravitational field $g^{\mu\nu}$ is parametrized as $g^{\pm\mp} = \exp(-\theta)$ and

\(^9\) Here, of course, the “translation” is defined as a (pseudo-)unitary transformation of quantum fields, under which the effective action is always non-invariant. One might say that the effective action is translationally invariant under a linear approximation with respect to $h_{\mu\nu}$, but this merely implies that it is “translationally invariant” if any arbitrary function can be regarded as a constant. Genuine translational invariance must be checked for the exact solution in the framework of quantum gravity.
\[ g^{±±} = \exp(-\theta)h^{±} \]. Hence, to first order, the zweibein \( e_{μa} \) is given by \( e_{±a} = \exp(\frac{1}{2} \theta) \) and 
\[ e_{±±} = -\frac{1}{2} \exp(\frac{1}{2} \theta)h_{±} \] in the symmetric gauge \( e_{μa} = e_{aμ} \).

The action for \( D \) Weyl fields coupled to the zweibein is given by \(^\text{)}\)

\[ S_W = \frac{i}{2} \int d^2x \left[ \tilde{ψ}_M^\dagger \left( e_{+-} \partial_- \tilde{ψ}_M - e_{--} \partial_+ \tilde{ψ}_M \right) - \left( e_{+-} \partial_- \tilde{ψ}_M - e_{--} \partial_+ \tilde{ψ}_M \right) \cdot \tilde{ψ}_M \right], \quad (5.1) \]

where the sum over \( M = 1, \ldots, D \) is understood. Setting \( \tilde{ψ}_M = \exp(-\frac{1}{4} \theta)ψ_M \), we have

\[ S_W = \frac{i}{2} \int d^2x \left[ ψ_M^\dagger \left( \partial_- ψ_M + \frac{1}{2} h_+ \partial_+ ψ_M \right) - \left( \partial_- ψ_M^\dagger + \frac{1}{2} h_+ \partial_+ ψ_M^\dagger \right) \cdot ψ_M \right] \quad (5.2) \]
to first order.

Let \( \tilde{b}^μ \), \( c^μ \) and \( \tilde{c}^μ \) be the B field, the FP ghost and the FP anti-ghost, respectively. The BRS transformation, which is, of course, nilpotent, is defined by

\[ \delta_\ast h_± = 2 \partial_± c^± + (\partial_± c^± - \partial_± \tilde{c}^±) \cdot h_± - c^λ \partial_λ h_± - \frac{1}{2} \partial_± c^± \cdot h_±^2, \quad (5.3) \]
\[ \delta_\ast ψ_M = -\frac{1}{2} \partial_+ c_+ \cdot ψ_M - c^λ \partial_λ ψ_M + \frac{1}{4} \partial_+ c_+ \cdot h_+ ψ_M, \quad (5.4) \]
\[ \delta_\ast c^μ = -c^λ \partial_λ c^μ, \quad \delta_\ast \tilde{c}^μ = \tilde{b}^μ, \quad \delta_\ast \tilde{b}^μ = 0. \quad (5.5) \]

The Weyl action \((5.2)\) is exactly BRS invariant, in spite of the fact that we have adopted a non-covariant gauge. Because the BRS transformation \((5.3)\) of \( h_± \) is the same as that given previously, except for the second-order terms, the action proper to the zweibein is essentially the same as in our previous work. \(^9\) Then, from \((5.2)\), we see that the total action is given by

\[ S = \int d^2x (L_0 + L_I), \quad (5.6) \]
with

\[ L_0 = -\frac{1}{2} \tilde{b}^+ h_+ - i \tilde{c}^+ \partial_- c^+ + (+ \leftrightarrow -) + \frac{i}{2} \left[ ψ_M^\dagger \partial_- ψ_M - \partial_- ψ_M^\dagger \cdot ψ_M \right], \quad (5.7) \]
\[ L_I = \frac{1}{2} h_+ \left[ -2i \tilde{c}^+ \partial_+ c^+ - i (\partial_+ c^+ \cdot c^+ + \partial_- c^± \cdot c^-) \right] + (+ \leftrightarrow -) + \frac{1}{2} h_+ T_{++} + O(h^2), \quad (5.8) \]

where

\[ T_{++} \equiv \frac{i}{2} (ψ_M^\dagger \partial_+ ψ_M - \partial_+ ψ_M^\dagger \cdot ψ_M) \quad (5.9) \]
and \((+ \leftrightarrow -)\) indicates to interchange attached letters in the preceding expression.

In the operator formalism, higher-order terms, i.e., \( O(h^2) \), yield no contribution, because of the field equation \( h_± = 0 \), and moreover the left-moving mode and the right-moving mode

\(^9\) For \( D = 1 \), \((5.1)\) reduces to \((4.14)\).
decouple completely, because of the field equations $\partial_{\mp}c_{\pm} = 0$, $\partial_{\mp}\bar{c}_{\pm} = 0$, etc. It should be noted that such simplicity is never realized in the path-integral formalism, because the $T^*$-product violates the field equation. The terms of first order in $h_{\pm}$ yield the B-field equations:

\begin{align}
    T^+ &\equiv -\hat{b}^+ - 2i\bar{c}^+ \partial_+ c^+ - i\partial_+ \bar{c}^+ \cdot c^+ + T_{++} = 0, \\
    T^- &\equiv -\hat{b}^- - 2i\bar{c}^- \partial_- c^- - i\partial_- \bar{c}^- \cdot c^- = 0.
\end{align}

(5.10) (5.11)

From this point, everything is carried out as in our previous work, except for the part in which the Weyl fields are relevant. For this reason, we do not describe our reasoning and proceed directly to the Wightman functions involving the Weyl fields. The nonvanishing $n$-point truncated Wightman functions involving the Weyl fields are only those which consist of one $\psi_M$, one $\psi^l_M$ and $(n-2)$ $\hat{b}^+$ fields ($n \geq 2$). Their explicit expressions are

\[
    \langle \psi_M(x_1)\hat{b}^+(x_2)\cdots\hat{b}^+(x_{n-1})\psi^l_M(x_n) \rangle_T
    = \delta_{MN} \left( \frac{i}{2} \right)^{n-2} \frac{1}{(2\pi i)^{n-1}} \sum_{P(j_2,\ldots,j_{n-1})} (\partial^R_{j_2} - \partial^L_{j_2}) (\partial^R_{j_3} - \partial^L_{j_3}) \cdots (\partial^R_{j_{n-1}} - \partial^L_{j_{n-1}})
    \times \left( \frac{1}{x_{j_2}^+ - x_{j_2}^+ - i0} \right) \cdots \left( \frac{1}{x_{j_{n-2}}^+ - x_{j_{n-1}}^+ - i0} \right) \times \frac{1}{x_{j_{n-1}}^+ - x_{n}^+ - i0},
\]

(5.12)

with similar expressions for other orderings of field operators. Here, $P(j_2,\ldots,j_{n-1})$ denotes a permutation of $(2,\ldots,n-1)$, we have $x_j^+ - x_k^+ \mp i0 = x_j^+ - x_k^+ - i0$ for $j < k$ and $x_j^+ - x_k^+ + i0$ for $j > k$, and $\partial^R/L_j$ acts only on the $x_j^+$ appearing in the right/left factor.\(^*)

All truncated Wightman functions are consistent with translational invariance. The fact that there is no gravitational anomaly is easily seen as follows. The Noether currents for translational invariance are

\begin{align}
    J_{-}^+ &= -i\bar{c}^+ \partial_+ c^+ + T_{++}, \\
    J_{-}^- &= -i\bar{c}^- \partial_- c^-.
\end{align}

(5.13) (5.14)

with $J_{\pm}^\pm = 0$. Because any nonvanishing truncated vacuum expectation value of a product of $J_{-}^\nu(x)$ and fundamental fields $\varphi_1(y_1),\ldots,\varphi_n(y_n)$ depends only on $x^+,y_1^+,\ldots,y_n^+$, but not on $x^-$, we trivially have

\[
    \partial^\nu \langle J_{-}^\nu(x)\varphi_1(y_1)\cdots\varphi_n(y_n) \rangle = 0.
\]

(5.15)

\(^*)\text{For example, } \partial^R_j[f(x_j^+)g(x_j^+)] \equiv f(x_j^+)\partial_j g(x_j^+).\)
We have a similar expression for $J^+_\nu$. Thus, there is no gravitational anomaly (see §2).

However, this model is not free of all anomalies. As in the scalar-field case,9) the B-field equations exhibit a “field-equation anomaly”, namely, a slight violation of field equations at the representation level. Explicitly, we find that

\begin{align}
\langle \tilde{b}^+(x_1)T^+(x_2) \rangle &= -\langle T^+(x_1)\tilde{T}^+(x_2) \rangle = (D - 26)\Phi^{++}(x_1^+ - x_2^+), \\
\langle \tilde{b}^-(x_1)T^-(x_2) \rangle &= -\langle T^-(x_1)\tilde{T}^-(x_2) \rangle = -26\Phi^{++}(x_1^+ - x_2^+),
\end{align}

where

$$\Phi^{++}(z^+) \equiv \frac{1}{8\pi^2} \cdot \frac{1}{(z^+ - i0)^4},$$

just as in the scalar-field case. The perturbation-theoretical counterparts of (5.16) and (5.17) are the unexpected 1-loop contributions to $\langle T^*\tilde{b}^\pm(x_1)b^\pm(x_2) \rangle$, which are easily calculated by using (5.8) according to the Feynman rules; the reason for their presence is that the Feynman propagator $\langle T^*\tilde{h}^\pm(x)h^\pm(y) \rangle$ is nonvanishing, owing to the fact that the $T^*$-product violates the field equation.

The Noether currents for translational invariance contain an anomalous contribution from the above field-equation anomaly at the representation level, but we can define the anomaly-free currents as

$$\tilde{J}^\pm_\pm \equiv J^\pm_\pm - T^\pm - i\partial^\pm(\overline{c}^\pm c^\pm) = \tilde{b}^\pm,$$

so that the anomaly-free translational generators are given by

$$P_\pm \equiv \int dx^\pm \tilde{b}^\pm.$$

We can define the anomaly-free BRS current similarly.9)

§6. Discussion

In the present paper, we have shown that the quantities calculated by A.-G. and W. do not directly show the existence of gravitational anomalies. In order to establish the existence of gravitational anomalies, it is necessary to calculate T-product quantities, not $T^*$-product quantities directly calculable using Feynman integrals. We have explicitly demonstrated that, in the 2-dimensional case, the energy-momentum tensor is strictly conserved if the fact that the $T^*$-product quantities violate the field equation is correctly taken into account. Although it is very difficult to demonstrate this explicitly in higher-dimensional cases, it is clear that the reasoning of A.-G. and W. is insufficient for establishing the existence of gravitational anomalies. Note that A.-G. and W. also presented a naive argument from
which they inferred the existence of gravitational anomalies in higher-dimensional cases from their result in the 2-dimensional case. It is therefore natural to conjecture that the non-existence of a gravitational anomaly in higher-dimensional cases follows from that in the 2-dimensional case. It is interesting to note that the cancellation of the 10-dimensional gravitational anomaly is known as the anomaly-free condition of the superstring theory proposed by Green and Schwarz.

We emphasize the fundamental importance of strictly distinguishing T*-product quantities from T-product quantities. In recent studies, most calculations are carried out using T*-product quantities, but it seems that they are done without being aware of the fact that field equations and the Noether theorem do not hold in the case of T*-product quantities. It is generally quite dangerous to investigate symmetry properties on the basis of Feynman diagrammatic calculations. As it is much more difficult to calculate T-product quantities than T*-product quantities, because the former are, in general, non-covariant, it is preferable to investigate symmetry properties by means of Wightman functions in the Heisenberg picture.

Appendix

Singular Function Identities

In this appendix, we present some identities for products of singular functions.

Taking the imaginary part of the self-evident identity

\[
\left[ \frac{1}{(z - i0)^{n+1}} \right]^2 = \frac{1}{(z - i0)^{2n+2}}, \tag{A.1}
\]

we obtain

\[
Pf \frac{1}{z^{n+1}} \cdot \delta^{(n)}(z) = \frac{(-1)^{n+1}n!}{2 \cdot (2n + 1)!} \delta^{(2n+1)}(z). \tag{A.2}
\]

In particular, for \( n = 0 \) and for \( n = 1 \), this becomes

\[
Pf \frac{1}{z} \cdot \delta(z) = -\frac{1}{2} \delta'(z), \tag{A.3}
\]

and

\[
Pf \frac{1}{z^2} \cdot \delta'(z) = \frac{1}{12} \delta'''(z), \tag{A.4}
\]

respectively.

Differentiating (A.3) twice, we obtain

\[
2Pf \frac{1}{z^3} \cdot \delta(z) - 2Pf \frac{1}{z^2} \cdot \delta'(z) + Pf \frac{1}{z} \cdot \delta''(z) = -\frac{1}{2} \delta'''(z). \tag{A.5}
\]

Adding (A.4) twice to (A.5), we have

\[
Pf \frac{1}{z} \cdot \delta''(z) = -\frac{1}{3} \delta'''(z) - 2Pf \frac{1}{z^3} \cdot \delta(z). \tag{A.6}
\]
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