The volume growth of hyperkähler manifolds of type $A_\infty$

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Abstract

We study the volume growth of hyperkähler manifolds of type $A_\infty$ constructed by Anderson-Kronheimer-LeBrun [11] and Goto [10]. These are noncompact complete 4-dimensional hyperkähler manifolds of infinite topological type. These manifolds have the same topology but the hyperkähler metrics are depend on the choice of parameters. By taking a certain parameter, we show that there exists a hyperkähler manifold of type $A_\infty$ whose volume growth is $r^\alpha$ for each $3 < \alpha < 4$.

1 Introduction

A hyperkähler manifold is, by definition, a Riemannian manifold $(X, g)$ of real dimension $4n$ equipped with three complex structures $I_1, I_2, I_3$ satisfying the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1I_2I_3 = -\text{id}$ with respect to all of which the metric $g$ is Kählerian. Then the holonomy group of $g$ is a subgroup of $Sp(n)$ and $g$ is Ricci-flat.

In this paper we focus on the volume growth of hyperkähler metrics. The notion of the volume growth are considered for a Riemannian manifold $(X, g)$. We denote by $V_g(p_0, r)$ the volume of the ball $B_g(p_0, r) \subset X$ of radius $r$ centered at $p_0 \in X$. Then we say that the volume growth of $g$ is $f(r)$ if the condition

$$0 < \liminf_{r \to +\infty} \frac{V_g(p_0, r)}{f(r)} \leq \limsup_{r \to +\infty} \frac{V_g(p_0, r)}{f(r)} < +\infty.$$ 

holds for some $p_0 \in X$. From Bishop-Gromov comparison theorem [6] [12], the above condition is independent of $p_0 \in X$ if $g$ has the nonnegative Ricci
curvature.

For instance, the volume growth of Euclidean space $\mathbb{R}^4$ is $r^4$ and the volume growth of $\mathbb{R}^3 \times S^1$ with the flat metric is $r^3$. These are trivial examples and there are also nontrivial examples such as ALE hyperkähler metrics constructed in [8][9][15] whose volume growth is $r^4$, and multi-Taub-NUT metrics [13][16][18] whose volume growth is $r^3$.

Thus there are several examples of complete hyperkähler manifolds whose volume growth is $r^k$ for positive integers $k$. On the other hand, there exist complete Ricci-flat Kähler manifolds of complex dimension $n$ whose volume growth are $r^{(2n)/(n+1)}$ [2][3][19].

In this paper we will show that there is a family of complete hyperkähler manifolds of real dimension 4 whose volume growth is less than $r^4$ and more than $r^3$.

Anderson, Kronheimer and LeBrun constructed the hyperkähler manifolds of type $A_\infty$ by Gibbons Hawking ansatz in [1], which are 4-dimensional non-compact complete hyperkähler manifolds of infinite topological type. Here infinite topological type means that the homology groups are infinitely generated. The same metrics were constructed by hyperkähler quotient method due to Goto in [10]. Each of the metrics in [1] is constructed from an element of

$$(\text{Im}\mathbb{H})_0^\mathbb{Z} := \{\lambda = (\lambda_n)_{n \in \mathbb{Z}} \in (\text{Im}\mathbb{H})^\mathbb{Z}; \sum_{n \in \mathbb{Z}} \frac{1}{1 + |\lambda_n|} < +\infty\},$$

where $\mathbb{H}$ is the quaternions and $\text{Im}\mathbb{H}$ the 3-dimensional subspace formed by the purely imaginary quaternions. We denote by $(X_\lambda, g_\lambda)$ the hyperkähler metric of type $A_\infty$ constructed from $\lambda = (\lambda_n)_{n \in \mathbb{Z}} \in (\text{Im}\mathbb{H})_0^\mathbb{Z}$. The purpose of this paper is studying the asymptotic behavior of $V_{g_\lambda}(p_0, r)$ for some $p_0 \in X_\lambda$, and observe how the volume growth of $g_\lambda$ depends on the choice of $\lambda$. The main result is described as follows.

**Theorem 1.1.** For each $\lambda \in (\text{Im}\mathbb{H})_0^\mathbb{Z}$ and $p_0 \in X_\lambda$, the function $V_{g_\lambda}(p_0, r)$ satisfies

$$0 < \liminf_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_{\lambda}^{-1}(r^2)} \leq \limsup_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_{\lambda}^{-1}(r^2)} < +\infty,$$

where the function $\tau_{\lambda} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\tau_{\lambda}(R) := \sum_{n \in \mathbb{Z}} \frac{R^2}{R + |\lambda_n|}$$

for $R \geq 0$. 

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Applying Theorem 1.1 to some $\lambda \in (\text{Im} \mathbb{H})_0^Z$, we can find hyperkähler manifolds whose volume growth is given as follows.

**Theorem 1.2.** (1) There is a hyperkähler manifold $(X_\lambda, g_\lambda)$ for each $3 < \alpha < 4$ which satisfies

$$0 < \liminf_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} \leq \limsup_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} < +\infty.$$  

(2) There is a hyperkähler manifold $(X_\lambda, g_\lambda)$ which satisfies

$$\lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^\alpha} = +\infty$$  

for any $\alpha < 4$.

Next we denote by $g_\lambda^{(s)}$ the Taub-NUT deformation of $g_\lambda$ where $s > 0$ is the parameter of deformations. Then the volume growth of $g_\lambda^{(s)}$ is given by the following.

**Theorem 1.3.** Let $\lambda \in (\text{Im} \mathbb{H})_0^Z$ and $s > 0$. Then the volume growth of hyperkähler metric $g_\lambda^{(s)}$ satisfies

$$\lim_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(r)}{r^3} = \frac{8\pi^2}{3\sqrt{s}}$$

In Section 2, we review the construction and the basic properties of hyperkähler manifolds of type $A_\infty$. Although there are two constructions by Gibbons-Hawking ansatz and hyperkähler quotient method, we adopt the latter way according to [10] since the argument in Section 3 is due to Goto’s construction. Then we obtain the hyperkähler manifold $(X_\lambda, g_\lambda)$ as a hyperkähler quotient, and there is an $S^1$-action on $X_\lambda$ preserving the hyperkähler structure and the hyperkähler moment map $\mu_\lambda : X_\lambda \to \text{Im} \mathbb{H}$.

For the proof of Theorem 1.1, we need upper and lower bounds for the function $V_{g_\lambda}(p_0, r)$, which are discussed in Sections 3 and 4, respectively. In Section 3 the upper bound for $V_{g_\lambda}(p_0, r)$ will be obtained as follows. We fix $p_0 \in X_\lambda$ to be $\mu_\lambda(p_0) = 0 \in \text{Im} \mathbb{H}$. Then we obtain two inequalities

$$\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) \leq P_+ R^2 \varphi_\lambda(R),$$

$$d_{g_\lambda}(p_0, p)^2 \geq Q_- |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|),$$

for each $p \in X_\lambda$, where $\text{vol}_{g_\lambda}$ is the Riemannian measure, $d_{g_\lambda}$ is the Riemannian distance, $B_R := \{\zeta \in \text{Im} \mathbb{H}; |\zeta| < R\}$, and $P_+, Q_-$ are positive constants.
The former inequality is obtained from the hyperkähler construction, and the latter one is obtained from Gibbons Hawking ansatz. By combining these inequalities we have the upper bound for $V_{g_\lambda}(p_0, r)$. Next we discuss the lower bound for $V_{g_\lambda}(p_0, r)$ in Section 4. If we try to bound the function $V_{g_\lambda}(p_0, r)$ from below in a similar method as in Section 3, then we need to show the inequality

$$d_{g_\lambda}(p_0, p)^2 \leq Q_+|\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|),$$

which seems to be hard to show for any $p \in X_\lambda$. But it does not mean that the author can give a counterexample to the inequality. In Section 4, we take an open subset of $B_{g_\lambda}(p_0, r)$ which consists of points $p \in B_{g_\lambda}(p_0, r)$ satisfying the inequality $d_{g_\lambda}(p_0, p)^2 \leq Q_+|\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|)$, and compute the volume of the subset, which is the lower bound for $V_{g_\lambda}(p_0, r)$. From two types of estimates obtained in Sections 3 and 4, we prove Theorem 1.1 in Section 5.

We compute the volume growth of some examples concretely in Section 6, and obtain Theorem 1.2. Section 7 is devoted to studying the volume growth of the Taub-NUT deformations of $(X_\lambda, g_\lambda)$.

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## 2 Hyperkähler manifolds of type $A_\infty$

In this section, we review the construction of hyperkähler manifolds of type $A_\infty$ according to [10]. Although they can be constructed by Gibbons-Hawking ansatz [1], we need hyperkähler quotient construction in [10] for arguments in Section 3. Before the construction, we start with some basic definitions.

**Definition 2.1.** Let $(X, g)$ be a Riemannian manifold of dimension $4n$ and $I_1, I_2, I_3$ be complex structures on $X$ compatible with $g$. Then $(X, g, I_1, I_2, I_3)$ is a hyperkähler manifold if $(I_1, I_2, I_3)$ satisfies the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\text{id}$ and each $\omega_i := g(I_i \cdot, \cdot)$ is closed for $i = 1, 2, 3$.

Let $H = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ be the quaternions and $\text{Im}\ H = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ the 3-dimensional subspace formed by the purely imaginary quaternions. We can combine three fundamental 2-forms into a single $\text{Im}\ H$-valued 2-form $\omega := \omega_1 i + \omega_2 j + \omega_3 k$. Then it turns out that three complex
structures and the metric are determined by the 2-form $\omega$. Hence we call $\omega$ the hyperkähler structure on $X$ instead of $(g, I_1, I_2, I_3)$.

Let $G$ be a Lie group acting on a manifold $X$. Then each element $\xi$ of the Lie algebra of $G$ generates a vector field $\xi^* \in X(X)$ defined by $\xi^*_x := \frac{d}{dt}|_{t=0} x \exp(t\xi) \in T_xX$ for $x \in X$.

**Definition 2.2.** Suppose that the $n$-dimensional torus $T^n$ with Lie algebra $t^n$ acts smoothly on a hyperkähler manifold $(X, \omega)$ preserving the hyperkähler structure $\omega$. Then the hyperkähler moment map $\mu : X \to \text{Im} \mathbb{H} \otimes (t^n)^*$ for the action of $T^n$ on $X$ is defined by two properties (i) $\mu$ is $T^n$-invariant, (ii) $\langle d\mu_x(V), \xi \rangle = \omega(\xi^*_x, V_x) \in \text{Im} \mathbb{H}$ for all $x \in X$ and $V \in T_xX$. Here $\langle \cdot, \cdot \rangle : \text{Im} \mathbb{H} \otimes (t^n)^* \otimes t^n \to \text{Im} \mathbb{H}$ is the natural pairing induced from the contraction.

Next we review the construction of hyperkähler manifolds of type $A_\infty$. Let $S^Z$ be the set of maps from the integers $\mathbb{Z}$ to a set $S$. Then each $x \in S^Z$ is expressed in the sequence $x = (x_n)_{n \in \mathbb{Z}}$ where $x_n \in S$. Then we define a Hilbert space $M$ in the infinite sequences of the quaternions by

$$M := \{ v \in \mathbb{H}^Z; \|v\|^2_M < +\infty \},$$

where the Hilbert metric is given by

$$\langle u, v \rangle_M := \sum_{n \in \mathbb{Z}} u_n \overline{v}_n, \quad \|v\|^2_M := \langle v, v \rangle_M$$

for $u, v \in \mathbb{H}^Z$.

Put $\mathbb{H}^Z_0 := \{ \Lambda \in \mathbb{H}^Z; \sum_{n \in \mathbb{Z}} (1 + |\Lambda_n|^2)^{-1} < +\infty \}$. Then for each $\Lambda \in \mathbb{H}^Z_0$, we have the following Hilbert manifolds

$$M_\Lambda := \Lambda + M = \{ \Lambda + v; \ v \in M \},$$

$$U_\Lambda := \{ g = (g_n)_{n \in \mathbb{Z}} \in (S^1)^Z; \|1_n - g\|^2_\Lambda < +\infty \},$$

$$u_\Lambda := \{ \xi = (\xi_n)_{n \in \mathbb{Z}} \in \mathbb{R}^Z; \|\xi\|^2_\Lambda < +\infty \},$$

$$G_\Lambda := \{ g = (g_n)_{n \in \mathbb{Z}} \in U_\Lambda; \prod_{n \in \mathbb{Z}} g_n = 1 \},$$

$$g_\Lambda := \{ \xi = (\xi_n)_{n \in \mathbb{Z}} \in u_\Lambda; \sum_{n \in \mathbb{Z}} \xi_n = 0 \},$$

where

$$\langle \xi, \eta \rangle_\Lambda := \sum_{n \in \mathbb{Z}} (1 + |\Lambda_n|^2) \xi_n \overline{\eta}_n, \quad \|\xi\|^2_\Lambda := \langle \xi, \xi \rangle_\Lambda$$

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for $\xi, \eta \in \mathbb{C}^Z$. Here, $1_Z \in (S^1)^Z$ is the constant map $(1_Z)_n := 1$. The convergence of $\prod_{n \in \mathbb{Z}} g_n$ and $\sum_{n \in \mathbb{Z}} \xi_n$ follows from the condition $\sum_{n \in \mathbb{Z}} (1 + \|\Lambda_n\|^2)^{-1} < +\infty$. Then $G_A$ is a Hilbert Lie group whose Lie algebra is $\mathfrak{g}_A$.

We can define a right action of $H$ of $M_a$ on $\Lambda$ for $\langle, \rangle$ and inner product structures induced by the left multiplication by $i, j, k$.

We obtain the quotient space $\hat{\mu}$ for $\Lambda$. This $\hat{\mu}$ is a hyperkähler manifold and the action of $G_A$ on $M_A$ preserves the hyperkähler structure. Then we define a map $\hat{\mu}_A : M_A \rightarrow \text{Im} \mathbb{H} \otimes \mathfrak{g}_A$ by

$$\langle \hat{\mu}_A(x), \xi \rangle := \sum_{n \in \mathbb{Z}} (x_n i\bar{x}_n - \Lambda_n i\bar{\Lambda}_n)\xi_n \in \text{Im} \mathbb{H}$$

for $x \in M_A, \xi \in \mathfrak{g}_A$. This $\hat{\mu}_A$ is the hyperkähler moment map for the action of infinite dimensional torus $G_A$ on $M_A$.

Since $\hat{\mu}_A$ is $G_A$-invariant, then $G_A$ acts on the inverse image $\hat{\mu}_A^{-1}(0)$. Hence we obtain the quotient space $\hat{\mu}_A^{-1}(0)/G_A$ which is called hyperkähler quotient. In general, there is no guarantee that $\hat{\mu}_A^{-1}(0)/G_A$ is a smooth manifold for every $\Lambda \in \mathbb{H}_0^Z$. For the smoothness of $\hat{\mu}_A^{-1}(0)/G_A$ we need the following condition.

**Definition 2.3.** An element $\Lambda \in \mathbb{H}_0^Z$ is generic if $\Lambda_n i\bar{\Lambda}_n - \Lambda_m i\bar{\Lambda}_m \neq 0$ for all distinct $n, m \in \mathbb{Z}$.

**Proposition 2.4.** For a generic $\Lambda \in \mathbb{H}_0^Z$, the Lie group $G_A$ acts freely on $\hat{\mu}_A^{-1}(0)$.

**Proof.** Let $e_l \in \mathfrak{g}_A$ be defined by $e_l := (e_{l, n})_{n \in \mathbb{Z}}$, where

$$e_{l, n} = \begin{cases} 1 & (n = l), \\ 0 & (n \neq l). \end{cases}$$

Since $\mathfrak{g}_A$ is generated by elements $e_l - e_m$, then $x \in M_A$ satisfies $\hat{\mu}_A(x) = 0$ if and only if $\langle \hat{\mu}_A(x), e_l - e_m \rangle = 0$ for all $l, m \in \mathbb{Z}$. Hence the value of $x_n i\bar{x}_n - \Lambda_n i\bar{\Lambda}_n$ is independent of $n \in \mathbb{Z}$ for $x \in \hat{\mu}_A^{-1}(0)$.

Assume that there is a pair of $x \in \hat{\mu}_A^{-1}(0)$ and $g \in G_A$ satisfies $xg = x$. If $x_n \neq 0$ for any $n \in \mathbb{Z}$, then $g = 1$. Therefore we may assume $x_s = 0$ for some $s \in \mathbb{Z}$. Then we have $x_n i\bar{x}_n - \Lambda_n i\bar{\Lambda}_n = -\Lambda_s i\bar{\Lambda}_s$ for all $n \in \mathbb{Z}$, which implies

$$x_n i\bar{x}_n = \Lambda_n i\bar{\Lambda}_n - \Lambda_s i\bar{\Lambda}_s \neq 0$$
for \( n \neq s \). Since \( x_n = 0 \) if and only if \( x_n\bar{x}_n = 0 \), we have \( x_n \neq 0 \) if \( n \neq s \). Thus we have shown \( g_n = 1 \) if \( n \neq s \), and also \( g_s \) should be 1 from the condition \( \prod_{n \in \mathbb{Z}} g_n = 1 \). \( \square \)

**Theorem 2.5** (\cite{10}). If \( \Lambda \in \mathbb{H}_0^\mathbb{Z} \) is generic, then \( \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \) is a smooth manifold of dimension 4, and the hyperkähler structure on \( M_\Lambda \) induces a hyperkähler structure \( \hat{\omega}_\Lambda \) on \( \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \).

**Remark 2.1.** In \cite{10}, the above theorem is proved in the case of

\[
\Lambda_n = \begin{cases} 
ni & (n \geq 0), \\
 nk & (n < 0).
\end{cases}
\]

But it is easy to show the theorem for the case of any generic \( \Lambda \in \mathbb{H}_0^\mathbb{Z} \).

Take \( \Lambda \in \mathbb{H}_0^\mathbb{Z} \) and \( (e^{i\theta_n})_{n \in \mathbb{Z}} \in (S^1)^\mathbb{Z} \) and put \( \Lambda' = (\Lambda_n e^{i\theta_n})_{n \in \mathbb{Z}} \). Then there is a canonical isomorphism of hyperkähler structure between \( \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \) and \( \hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'} \) as follows. Define a map \( \hat{F} : \hat{\mu}_\Lambda^{-1}(0) \to \hat{\mu}_\Lambda^{-1}(0) \) by

\[
\hat{F}((x_n)_{n \in \mathbb{Z}}) := (x_n e^{i\theta_n})_{n \in \mathbb{Z}}.
\]

Since \( \hat{F} \) is equivariant with respect to \( G_\Lambda \) and \( G_{\Lambda'} \)-actions, we obtain \( F : \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \to \hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'} \), which preserves hyperkähler structures \( \hat{\omega}_\Lambda \) and \( \hat{\omega}_{\Lambda'} \). In this case we have \( \Lambda_n i\Lambda_n = \Lambda'_n i\Lambda'_n \) for all \( n \in \mathbb{Z} \). Conversely, if we take \( \Lambda, \Lambda' \in \mathbb{H}_0^\mathbb{Z} \) to be \( (\Lambda_n i\Lambda_n)_{n \in \mathbb{Z}} = (\Lambda'_n i\Lambda'_n)_{n \in \mathbb{Z}} \), then there is \( (e^{i\theta_n})_{n \in \mathbb{Z}} \in (S^1)^\mathbb{Z} \) such that \( \Lambda'_n = \Lambda_n e^{i\theta_n} \). Thus \( \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \) and \( \hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'} \) are isomorphic as hyperkähler manifolds if \( (\Lambda_n i\Lambda_n)_{n \in \mathbb{Z}} = (\Lambda'_n i\Lambda'_n)_{n \in \mathbb{Z}} \). Therefore we may put

\[
(X_\lambda, \omega_\lambda) := (\hat{\mu}_\lambda^{-1}(0)/G_\lambda, \hat{\omega}_\lambda)
\]

for

\[
\lambda \in (\text{Im} \mathbb{H})_0^\mathbb{Z} := \{(\Lambda_n i\Lambda_n)_{n \in \mathbb{Z}} \in (\text{Im} \mathbb{H})^\mathbb{Z}; \ \Lambda \in \mathbb{H}_0^\mathbb{Z}\}
\]

\[
= \{\lambda \in (\text{Im} \mathbb{H})^\mathbb{Z}; \ \sum_{n \in \mathbb{Z}} \frac{1}{1 + |\lambda_n|} < +\infty\}
\]

and \( \Lambda \in \hat{\nu}^{-1}(\lambda) \). Then the condition for \( X_\lambda \) to be smooth is written as follows.

**Definition 2.6.** An element \( \lambda \in (\text{Im} \mathbb{H})_0^\mathbb{Z} \) is generic if \( \lambda_n - \lambda_m \neq 0 \) for all distinct \( n, m \in \mathbb{Z} \).

Then Theorem 2.5 implies that \((X_\lambda, \omega_\lambda) \) is a smooth hyperkähler manifold for each generic \( \lambda \in (\text{Im} \mathbb{H})_0^\mathbb{Z} \).

We denote by \( g_\lambda \) the hyperkähler metric induced from \( \omega_\lambda \).

**Theorem 2.7** (\cite{11}). Let \((X_\lambda, g_\lambda) \) be as above. Then the Riemannian metric \( g_\lambda \) is complete.

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We denote by \([x] \in \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda\) the equivalence class represented by \(x \in \hat{\mu}_\Lambda^{-1}(0)\). Then by putting
\[
[x]g := [(\cdots, x_{-1}, x_0g, x_1, \cdots, x_n, \cdots)]
\]
for \(x = (x_n)_{n \in \mathbb{Z}} = (\cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots) \in \hat{\mu}_\Lambda^{-1}(0)\) and \(g \in S^1\), we have the action of \(S^1\) on \(X_\Lambda\). Then the hyperkähler moment map \(\mu_\Lambda : X_\Lambda \to \text{Im} \mathbb{H} = \mathbb{R}^3\) is given by
\[
\mu_\Lambda([x]) := x_0i \bar{x}_0 - \lambda_0 \in \text{Im} \mathbb{H}.
\]
Since \(x \in \hat{\mu}_\Lambda^{-1}(0)\), we have \(x_0i \bar{x}_0 - \lambda_0 = x_ni \bar{x}_n - \lambda_n\), where we identify \((t^1)^*\) with \(\mathbb{R}\) by taking a generator of \(t^1\).

If we put \(\text{Stab}([x]) := \{g \in S^1; [x]g = [x]\}\) for \([x] \in X_\Lambda\), then it is obvious that \(\text{Stab}([x]) = \{1\}\) if and only if \(x_n \neq 0\) for any \(n \in \mathbb{Z}\), otherwise \(\text{Stab}([x]) = S^1\). Hence we have a principal \(S^1\)-bundle \(\mu_\Lambda|_{X_\Lambda^*} : X_\Lambda^* \to Y_\Lambda\) where
\[
X_\Lambda^* := \{[x] \in X_\Lambda; x_n \neq 0 \text{ for all } n \in \mathbb{Z}\},
\]
\[
Y_\Lambda := \text{Im} \mathbb{H}\{\lambda_n; n \in \mathbb{Z}\}.
\]

**Definition 2.8.** An \(S^1\)-action on a 4-dimensional hyperkähler manifold \((X, \omega)\) is tri-Hamiltonian if the action preserves \(\omega\) and there exists a hyperkähler moment map \(\mu : X \to \text{Im} \mathbb{H}\) for the action of \(S^1\).

**Theorem 2.9 (\cite{H}).** There exists a canonical one-to-one correspondence between the followings; (i) a hyperkähler manifold of real dimension 4 with free tri-Hamiltonian \(S^1\)-action, (ii) a principal \(S^1\)-bundle \(\mu : X \to Y\) where \(Y\) is an open subset of \(\mathbb{R}^3\), and an \(S^1\)-connection \(A\) on \(X\) and a positive valued harmonic function \(\Phi\) on \(Y\) such that \(\frac{dA}{2\sqrt{-1}} = \mu^*(-d\Phi)\). Here \(*\) is the Hodge star operator with respect to the Euclidean metric on \(\mathbb{R}^3\).

Here we see a sketch of the proof, and the details can be seen in \cite{H}. Let \((X, \omega)\) be a hyperkähler manifold of dimension 4 with a free \(S^1\)-action preserving \(\omega\), and \(\mu : X \to \text{Im} \mathbb{H}\) the hyperkähler moment map. Then \((Y, \Phi, A)\) is given by the followings. Let \(Y\) be the image \(\mu(X)\). Then the function \(\Phi : Y \to \mathbb{R}\) is defined by
\[
\frac{1}{\Phi(\mu(x))} := 4g_x(\xi^*_x, \xi^*_x)
\]
for \(x \in X\), where \(g\) is the hyperkähler metric and \(\xi := \sqrt{-1}\) is a generator of Lie algebra of the Lie group \(S^1\). For each \(x \in X\), we denote by \(V_x \subset T_xX\) the subspace spanned by \(\xi^*_x\). Then the \(S^1\)-connection \(A\) on \(X\) is defined
by the horizontal distribution \((H_x)_{x \in X}\) where \(H_x \subset T_x X\) is the orthogonal complement of \(V_x\).

Conversely, let \(\mu = (\mu_1, \mu_2, \mu_3) : X \to Y \subset \text{Im} \mathbb{H}\) be a principal \(S^1\)-bundle and \((\Phi, A)\) a pair consisting of a positive valued harmonic function and a connection of the \(S^1\)-bundle with \(\frac{dA}{2\sqrt{-1}} = \mu^*(\ast d\Phi)\). Then the hyperkähler structure \(\omega = (\omega_1, \omega_2, \omega_3) \in \Omega^2(X) \otimes \text{Im} \mathbb{H}\) is defined by

\[
\begin{align*}
\omega_1 &= d\mu_1 \wedge \frac{A}{2\sqrt{-1}} + \mu^* \Phi d\mu_2 \wedge d\mu_3, \\
\omega_2 &= d\mu_2 \wedge \frac{A}{2\sqrt{-1}} + \mu^* \Phi d\mu_3 \wedge d\mu_1, \\
\omega_3 &= d\mu_3 \wedge \frac{A}{2\sqrt{-1}} + \mu^* \Phi d\mu_1 \wedge d\mu_2.
\end{align*}
\]

Then it follows from the condition \(\frac{dA}{2\sqrt{-1}} = \mu^*(\ast d\Phi)\) that \(\omega\) is closed.

Theorem 2.9 gives the positive valued harmonic function \(\Phi_\lambda\) on \(Y_\lambda\) and \(S^1\)-connection \(A_\lambda\) on \(X^*\). It is known in [10] that \(\Phi_\lambda\) is given by

\[
\Phi_\lambda(\zeta) = \frac{1}{4} \sum_{n \in \mathbb{Z}} \frac{1}{|\zeta + \lambda n|}
\]

for \(\zeta \in Y_\lambda\).

Let \((X, \omega)\) be a hyperkähler manifold satisfying the condition (i) of Theorem 2.9, and \((Y, \Phi, A)\) be what corresponds to \((X, \omega)\) satisfying the condition (ii).

Denote by \(g\) the hyperkähler metric of \(\omega\) and let \(\text{vol}_g(B)\) be the volume of a subset \(B \subset X\).

**Lemma 2.10.** Let \(U \subset \text{Im} \mathbb{H}\) be an open set. Then we have the following formula

\[
\text{vol}_g(\mu^{-1}(U)) = \pi \int_{\zeta \in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3,
\]

where \(\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \text{Im} \mathbb{H}\) is the Cartesian coordinate.

**Proof.** It suffices to show the assertion for all open set \(U \subset Y\).

First of all we suppose that the principal \(S^1\)-bundle \(\mu : \mu^{-1}(U) \to U\) is trivial. Then we can take a \(C^\infty\) trivialization \(\sigma : U \to \mu^{-1}(U)\) and define \(C^\infty\) map \(t : \mu^{-1}(U) \to \mathbb{R}/2\pi \mathbb{Z}\) by \(t(\sigma(\zeta)e^{i\theta}) := \theta\) for \(\zeta \in U\) and \(\theta \in \mathbb{R}/2\pi \mathbb{Z}\) and obtain a local coordinate \((t, \mu_1, \mu_2, \mu_3)\) on \(\mu^{-1}(U)\). Since we may write \(dt = -\sqrt{-1}A + \sum_{i=1}^3 a_i d\mu_i\) for some \(a_1, a_2, a_3 \in C^\infty(\mu^{-1}(U))\), the volume
form $vol_g$ is given by

$$vol_g = \frac{\mu^*\Phi(-\sqrt{-1})}{2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge A$$

$$= \frac{\mu^*\Phi}{2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge dt.$$

Thus we have

$$vol_g(\mu^{-1}(U)) = \int_{\mu^{-1}(U)} vol_g$$

$$= \int_{\mu^{-1}(U)} \frac{\mu^*\Phi}{2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge dt$$

$$= \pi \int_{\zeta \in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3.$$

For a general $U \subset Y$, we take open sets $U_1, U_2, \ldots, U_m \subset Y$ such that each principal $S^1$-bundle $\mu^{-1}(U_\alpha) \to U_\alpha$ is trivial and $\coprod_{\alpha=1}^m U_\alpha \subset U \subset \coprod_{\alpha=1}^m \overline{U_\alpha}$. Then we have

$$vol_g(\mu^{-1}(U)) = \sum_{\alpha=1}^m vol_g(\mu^{-1}(U_\alpha))$$

$$= \sum_{\alpha=1}^m \pi \int_{\zeta \in U_\alpha} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3$$

$$= \pi \int_{\zeta \in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3.$$

\[\square\]

### 3 The upper bound for the volume growth

For a Riemannian manifold $(X,g)$ and a point $p_0 \in X$, we denote by $V_g(p_0,r)$ the volume of a ball $B_g(p_0,r) := \{ p \in X ; d_g(p_0,p) < r \}$ with respect to the Riemannian distance $d_g$. In this section, we will evaluate the upper bound for $V_{g_\lambda}(p_0,r)$ for the hyperkähler manifold $(X_\lambda,g_\lambda)$.

In Section 2, we supposed $\lambda \in (\text{Im}H)^Z \subset Z_0$ to be generic for the smoothness of $(X_\lambda,g_\lambda)$. But the function $V_{g_\lambda}(p_0,r)$ is determined only by the Riemannian measure $vol_{g_\lambda}$ and the Riemannian distance $d_{g_\lambda}$. Even if $\lambda \in (\text{Im}H)^Z$ is taken not to be generic, $vol_{g_\lambda}$ and $d_{g_\lambda}$ can be extended to $X_\lambda$ naturally. Hence we take $\lambda \in (\text{Im}H)^Z$ not necessary to be generic from now on.
Fix $p_0 \in X_\lambda$ to be $\mu_\lambda(p_0) = -\lambda_0$. We may assume $\lambda_0 = 0$ since the hyperkähler quotient constructed from $(\lambda_n - \lambda_0)_{n \in \mathbb{Z}}$ is isometric to $(X_\lambda, g_\lambda)$. For each $R > 0$, put

$$\varphi_\lambda(R) := \sum_{n \in \mathbb{Z}} \frac{R}{R + |\lambda_n|},$$

and $\varphi_\lambda(0) := \lim_{R \to 0} \varphi_\lambda(R)$.

**Proposition 3.1.** We have the following inequality

$$d_{g_\lambda}(p_0, p)^2 \geq Q_- |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|)$$

for any $p \in X_\lambda$, where $Q_- = 1/8$.

**Proof.** Take $\Lambda = (\Lambda_n)_{n \in \mathbb{Z}} \in \mathbb{H}_0^\mathbb{Z}$ to be $\Lambda_n i\Lambda_n = \lambda_n$ for all $n \in \mathbb{Z}$. We fix $x \in \hat{\mu_\lambda}^{-1}(0)$ such that $[x] = p$. If we regard $\hat{\mu_\lambda}^{-1}(0)$ as the infinite dimensional Riemannian submanifold of $M_\lambda$, then the quotient map $\hat{\mu_\lambda}^{-1}(0) \to X_\lambda = \hat{\mu_\lambda}^{-1}(0)/G_\lambda$ is a Riemannian submersion and $G_\lambda$ acts on $\hat{\mu_\lambda}^{-1}(0)$ as an isometry. Then the horizontal lift of the geodesic from $p_0$ to $p$ has the same length as $d_{g_\lambda}(p_0, p)$. Since the Riemannian distance between $\Lambda$ and $x' \in \hat{\mu_\lambda}^{-1}(0)$ is larger than $\|\Lambda - x'\|_M$, then we have

$$d_{g_\lambda}(p_0, p) \geq \inf_{\sigma \in G_\lambda} \|\Lambda - x\sigma\|_M.$$

If we put $\Lambda = (\alpha_n + \beta_n j)_{n \in \mathbb{Z}}$, $x = (z_n + w_n j)_{n \in \mathbb{Z}}$ and $\sigma = (e^{i\theta_n})_{n \in \mathbb{Z}}$, then

$$\|\Lambda - x\sigma\|_M^2 = \sum_{n \in \mathbb{Z}} (|\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{i\theta_n}|^2).$$

If the function $f_n(t) := |\alpha_n - z_n e^{it}|^2 + |\beta_n - w_n e^{-it}|^2$ attains its minimum at $\theta_n$ for each $n \in \mathbb{Z}$, then $\sigma = (e^{i\theta_n})_{n \in \mathbb{Z}}$ satisfies $\|\Lambda - x\sigma\|_M \leq \inf_{\tau \in G_\lambda} \|\Lambda - x\tau\|_M$.

From the equations

$$\lambda_n = \Lambda_n i\Lambda_n = i(|\alpha_n|^2 - |\beta_n|^2) - 2\alpha_n \beta_n k,$$

$$\zeta + \lambda_n = x_n i\sigma_n = i(|z_n|^2 - |w_n|^2) - 2z_n w_n k,$$

we have

$$f_n(t) = |\lambda_n| + |\zeta + \lambda_n| - 2 Re \{ (\alpha_n \bar{z}_n + \beta_n w_n) e^{-it} \},$$

$$|\alpha_n \bar{z}_n + \beta_n w_n|^2 = \frac{1}{2} (|\lambda_n| |\zeta + \lambda_n| + \langle \lambda_n, \zeta + \lambda_n \rangle_{\mathbb{R}^3}),$$

where $\langle \cdot , \cdot \rangle_{\mathbb{R}^3}$ is the standard inner product on $\text{Im} \mathbb{H} = \mathbb{R}^3$.

Suppose $n \neq 0$. If $\alpha_n \bar{z}_n + \beta_n w_n \neq 0$, then we put $e^{i\theta_n} := \frac{\alpha_n \bar{z}_n + \beta_n w_n}{|\alpha_n \bar{z}_n + \beta_n w_n|}$. If
\[ \alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0, \]
we may put \( \theta_n := 0 \) since \( f_n(t) \) is constant.
Let \( S := \{ n \in \mathbb{Z}; \ \alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0 \} \). For each \( n \notin S \), we have
\[
|\alpha_n - z_ne^{i\theta_n}|^2 + |\beta_n - w_ne^{-i\theta_n}|^2 = \frac{|\zeta|^2}{|\lambda_n| + |\zeta + \lambda_n| + 2|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|} \leq \frac{|\zeta|^2}{|\lambda_n|}.
\]
Hence we deduce
\[
\sum_{n \in \mathbb{Z}} |\Lambda_n - x_ne^{i\theta_n}|^2 \leq \sum_{n \in S} |\Lambda_n - x_n|^2 + \sum_{n \notin S} \frac{|\zeta|^2}{|\lambda_n|} < +\infty.
\]
Thus we obtain \( \sum_{n \in \mathbb{Z} \setminus \{0\}} |\Lambda_n|^2 |1 - e^{i\theta_n}|^2 < +\infty \), which ensures the convergence of \( \Pi_{n \in \mathbb{Z} \setminus \{0\}} e^{i\theta_n} \).

It follows from \( \lambda_0 = 0 \) that \( \alpha_0 = \beta_0 = 0 \). Then \( f_0(t) \) is constant. Since \( f_0(t) \) attains its minimum at any \( \theta_0 \), then we may put \( e^{i\theta_0} := \Pi_{n \in \mathbb{Z} \setminus \{0\}} e^{-i\theta_n} \). Thus the function \( \|\Lambda - x\tau\| \) attains its minimum at \( \tau = \sigma = (e^{i\theta_n})_{n \in \mathbb{Z}} \in G_\Lambda \). For this \( \sigma \), we have
\[
\|\Lambda - x\sigma\|^2 = \sum_{n \in \mathbb{Z}} |\lambda_n| + |\zeta + \lambda_n| - \sqrt{2} \sqrt{|\lambda_n| |\zeta + \lambda_n| + \langle \lambda_n, \zeta + \lambda_n \rangle} \geq \sum_{n \in \mathbb{Z}} \frac{|\zeta|^2}{8|\lambda_n| + 2|\zeta|} \geq \frac{1}{8} |\zeta| \cdot \varphi_\Lambda(|\zeta|).
\]
Then the assertion is obtained by putting \( Q_- = \frac{1}{8} \).

Put \( B_R := \{ \zeta \in \text{Im}\mathbb{H}; \ |\zeta| < R \} \). Next we discuss the upper bound for the volume of \( \mu^{-1}_\Lambda(B_R) \).

**Lemma 3.2.** We define two functions \( N_\lambda(R) \) and \( \psi_\lambda(R) \) by
\[
N_\lambda(R) := \{ n \in \mathbb{Z}; |\lambda_n| \leq R \},
\]
\[
\psi_\lambda(R) := \#N_\lambda(R) + \sum_{n \notin N_\lambda(R)} \frac{1}{|\lambda_n|} R
\]
for \( R \geq 0 \). Then we have
\[
\varphi_\lambda(R) \leq \psi_\lambda(R) \leq 2\varphi_\lambda(R).
\]
Proof. We define two functions \( p(x) \) and \( q(x) \) by
\[
p(x) := \frac{x}{1 + x}, \quad q(x) := \min\{1, x\}
\]
for \( x \geq 0 \). Then the inequalities
\[
p(x) \leq q(x) \leq 2p(x)
\]
hold for each \( x \geq 0 \). Therefore the assertion follows from
\[
\varphi_\lambda(R) = \sum_{n \in \mathbb{Z}} p\left(\frac{R}{|\lambda_n|}\right), \quad \psi_\lambda(R) = \sum_{n \in \mathbb{Z}} q\left(\frac{R}{|\lambda_n|}\right).
\]

Lemma 3.3. For \( \alpha \geq 1 \) and \( R \geq 0 \), we have \( \varphi_\lambda(\alpha R) \leq \alpha \varphi_\lambda(R) \) and \( \psi_\lambda(\alpha R) \leq \alpha \psi_\lambda(R) \).

Proof. Since \( \frac{\varphi_\lambda(R)}{R} \) is strictly decreasing for \( R \), we have
\[
\varphi_\lambda(\alpha R) = \frac{\varphi_\lambda(\alpha R)}{\alpha R} \alpha R \leq \frac{\varphi_\lambda(R)}{R} \alpha R = \alpha \varphi_\lambda(R)
\]
for \( \alpha \geq 1 \) and \( R \geq 0 \). It follows from the same argument that \( \psi_\lambda(\alpha R) \leq \alpha \psi_\lambda(R) \).

Proposition 3.4. There is a constant \( P_+ > 0 \) independent of \( \lambda \) and \( R \) such that
\[
\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) \leq P_+ R^2 \varphi_\lambda(R)
\]
for all \( R \geq 0 \).

Proof. From Lemma 2.10, the volume of \( \mu_\lambda^{-1}(B_R) \) is given by
\[
\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) = \pi \int_{\zeta \in B_R} \Phi_\lambda(\zeta) d\zeta_1 d\zeta_2 d\zeta_3 \\
= \pi \sum_{n \in \mathbb{Z}} \int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3.
\]
Let \((r \geq 0, \Theta)\) be the polar coordinate over \( \text{ImH} = \mathbb{R}^3 \), where \( \Theta \) is a coordinate on \( S^2 = \partial B_1 \). If \( n \in N_\lambda(R) \), then the change of variables \( \zeta' = \zeta + \lambda_n \) gives
\[
\int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 \leq \int_{\zeta' \in B_{R+|\lambda_n|}} \frac{1}{|\zeta'|} d\zeta_1' d\zeta_2' d\zeta_3' \\
= 4\pi \int_0^{R+|\lambda_n|} rdr \\
= 2\pi (R + |\lambda_n|)^2 \leq 8\pi R^2.
\]
If \( n \not\in N_\lambda(R) \), the mean value property of harmonic functions gives

\[
\int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda n|} d\zeta_1 d\zeta_2 d\zeta_3 = \frac{4\pi R^3}{3 |\lambda n|},
\]

since \( \frac{1}{|\zeta + \lambda n|} \) is harmonic on \( B_R \). Hence the upper bound for \( \text{vol}_{g_\lambda}(\mu^{-1}_\lambda(B_R)) \) is given by

\[
\text{vol}_{g_\lambda}(\mu^{-1}_\lambda(B_R)) \leq 2\pi^2 |N_\lambda(R)| \cdot R^2 + \frac{\pi^2}{3} \sum_{n \not\in N_\lambda(R)} \frac{R}{|\lambda n|} \cdot R^2 
\]

\[
\leq 2\pi^2 \varphi_\lambda(R) R^2 \leq 4\pi^2 \varphi_\lambda(R) R^2.
\]

Then we have the assertion by putting \( P_+ := 4\pi^2 \).

Let \( \theta_{\lambda,C}(R) := C\varphi_\lambda(R) R^2 \), \( \tau_{\lambda,C}(R) := C\varphi_\lambda(R) R \) for \( C > 0 \), and \( \tau^{-1}_{\lambda,C} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be the inverse function of \( \tau_{\lambda,C} \).

**Proposition 3.5.** Let \( P_+ \) and \( Q_- \) be as in Proposition 3.1 and 3.4. Then the inequality

\[
V_{g_\lambda}(p_0, r) \leq \theta_{\lambda,P_+} \circ \tau^{-1}_{\lambda,Q_-}(r^2)
\]

holds for all \( r \geq 0 \).

**Proof.** Since \( p \in B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)}) \) satisfies \( d_{g_\lambda}(p_0, p)^2 \leq \tau_{\lambda,Q_-}(R) \), then from Proposition 3.1 we obtain

\[
\tau_{\lambda,Q_-}(|\mu_\lambda(p)|) = Q_-|\mu_\lambda(p)|\varphi(|\mu_\lambda(p)|) \leq \tau_{\lambda,Q_-}(R).
\]

Since the function \( \tau_{\lambda,Q_-}(R) \) is strictly increasing, we have \( |\mu_\lambda(p)| \leq R \). Then we have \( B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)}) \subset \mu^{-1}_\lambda(B_R) \). Then it follows from Proposition 3.4 that we have

\[
\text{vol}_{g_\lambda}(B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)})) \leq \text{vol}_{g_\lambda}(\mu^{-1}_\lambda(B_R)) \leq P_+ R^2 \varphi(R) = \theta_{\lambda,P_+}(R).
\]

Set \( r = \sqrt{\tau_{\lambda,Q_-}(R)} \). Then substituting for \( R \) the inverse function \( \tau^{-1}_{\lambda,Q_-}(r^2) \), we obtain the result.

**4 The lower bound for the volume growth**

In the previous section we obtained the upper bound for \( V_{g_\lambda}(p_0, r) \). The purpose of this section is to obtain the inequality

\[
\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2) \leq V_{g_\lambda}(p_0, r),
\]
where $P_-, Q_+ > 0$ are constants independent of $\lambda$ and $r$. In similar consideration as in Section 3, it seems that the following two inequalities should be shown,

$$
\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(BR)) \geq P_- R^2 \varphi_\lambda(R),
$$

(1)

$$
d_{g_\lambda}(p_0, p)^2 \leq Q_+ |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|).
$$

(2)

But it seems to be hard to show the inequality (2) for the author. Accordingly we shall show two modified inequalities; one is a stronger inequality than (1), and the other is weaker than (2). First of all we consider the former estimate.

Let $U \subset S^2$ be a measurable set and put

$$
BR,U := \{ t\zeta \in \text{Im}H; 0 \leq t \leq R, \zeta \in U \}.
$$

We denote by $m_{S^2}$ the measure induced from the Riemannian metric with constant curvature over $S^2$ whose total measure is given by $m_{S^2}(S^2) = 4\pi$.

**Proposition 4.1.** There is a constant $C_- > 0$ independent of $\lambda$, $R$ and $U \subset S^2$, which satisfies

$$
\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(BR,U)) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R).
$$

**Proof.** From the triangle inequality, we have

$$
\text{vol}_{g_\lambda}(\mu_\lambda^{-1}(BR,U)) \geq \frac{\pi}{4} \sum_{n \in \mathbb{Z}} \int_{\zeta \in BR,U} \frac{1}{|\zeta| + |\lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3
$$

$$
= \frac{\pi}{4} \sum_{n \in \mathbb{Z}} \int_{r \in [0,R]} \int_{\Theta \in U} \frac{1}{r + |\lambda_n|} r^2 dr dm_{S^2}
$$

$$
= \frac{\pi m_{S^2}(U)}{4} \sum_{n \in \mathbb{Z}} \int_{r \in [0,R]} \frac{1}{r + |\lambda_n|} r^2 dr.
$$

If $n \in N_\lambda(R)$, then

$$
\int_{r \in [0,R]} \frac{1}{r + |\lambda_n|} r^2 dr \geq \int_{r \in [0,R]} \frac{1}{r + R} r^2 dr = (\log 2 - \frac{1}{2})R^2.
$$

If $n \notin N_\lambda(R)$, then

$$
\int_{r \in [0,R]} \frac{1}{r + |\lambda_n|} r^2 dr = \int_{r \in [0,R]} \frac{1}{r + |\lambda_n|} r^2 dr
$$

$$
= |\lambda_n|^2 \left\{ - \frac{R}{|\lambda_n|} + \frac{1}{2} \left( \frac{R}{|\lambda_n|} \right)^2 + \log \left( 1 + \frac{R}{|\lambda_n|} \right) \right\}.
$$
Since the inequality
\[
\log(1 + x) \geq x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4
\]
holds for \(x \geq 0\), then we have
\[
\int_{r \in [0, R]} \frac{1}{r + |\lambda_n|} r^2 dr \geq |\lambda_n|^2 \left\{ \frac{1}{3} \left( \frac{R}{|\lambda_n|} \right)^3 - \frac{1}{4} \left( \frac{R}{|\lambda_n|} \right)^4 \right\}
\]
\[
\geq \frac{1}{12} \frac{R^3}{|\lambda_n|}.
\]
By taking \(\frac{4C}{\pi} := \min\{\log 2 - \frac{1}{2}, \frac{1}{12}\} > 0\), we have
\[
\text{vol}_{g_\lambda}(\mu^{-1}_\lambda(B_{R, U})) \geq C m_{S^2}(U) R^2 \cdot \psi_\lambda(R) \geq C m_{S^2}(U) R^2 \cdot \varphi_\lambda(R).
\]

Next we need the upper bound for \(d_{g_\lambda}(p_0, p)\). If we can calculate the length of a piecewise smooth path from \(p_0\) to \(p\), then the length is larger than \(d_{g_\lambda}(p_0, p)\). Here we take the path as follows.

Put \(\zeta = \mu_\lambda(p)\). For \(|\zeta| \leq 1\), we define \(\gamma_p\) to be the geodesic from \(p_0\) to \(p\). If \(|\zeta| > 1\), we define a function \(\delta : [1, |\zeta|] \to \text{Im} H\) by \(\delta(t) := t \frac{\zeta}{|\zeta|}\) and take the horizontal lift \(\tilde{\delta} : [1, |\zeta|] \to X_\lambda\) of \(\delta\) with respect to the connection \(A_\lambda\) such that \(\tilde{\delta}(|\zeta|) = p\). If there exists a point \(t \in [1, |\zeta|]\) such that \(\delta(t) = -\lambda_n\) for some \(n \in \mathbb{Z}\), then \(\tilde{\delta}\) is not always smooth but continuous and piecewise smooth. Then we obtain a path \(\gamma_p\) by connecting the geodesic from \(p_0\) to \(\tilde{\delta}(1)\) and \(\tilde{\delta}\).

The length of \(\gamma_p\) is given by \(d_{g_\lambda}(p_0, \tilde{\delta}(1)) + l_\lambda(\mu_\lambda(p))\) where \(l_\lambda : \text{Im} H \to \mathbb{R}\) is defined by
\[
l_\lambda(\zeta) := \int_1^{||\zeta||} \sqrt{\Phi_\lambda\left( t \frac{\zeta}{|\zeta|} \right)} dt
\]
for \(|\zeta| \geq 1\), and \(l_\lambda(\zeta) := 0\) for \(|\zeta| \leq 1\).

If there exists a point \(t \in [1, |\zeta|]\) such that \(\delta(t) = -\lambda_n\), then \(\sqrt{\Phi_\lambda\left( t \frac{\zeta}{|\zeta|} \right)}\) is not continuous at \(t = |\lambda_n|\). But the integral \(\int_1^{||\zeta||} \sqrt{\Phi_\lambda\left( t \frac{\zeta}{|\zeta|} \right)} dt\) is well-defined since the integral
\[
\int_{b-\varepsilon}^{b+\varepsilon} \sqrt{\frac{1}{a|t-b| + h(t)}} dt
\]

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is a finite value for any constant $a > 0$, $b \in \mathbb{R}$ and smooth function $h(t)$.

Now we have $d_{g_{\lambda}}(p_0, p) \leq L_{\lambda} + l_{\lambda}(\mu_{\lambda}(p))$, where

$$L_{\lambda} := \sup_{p \in \mu_{\lambda}^{-1}(B_1)} d_{g_{\lambda}}(p_0, p) < +\infty.$$ 

**Proposition 4.2.** There is a constant $C_+ > 0$ independent of $\lambda$ and $R$ which satisfies

$$\int_{\Theta \in \partial B_1} l_{\lambda}(R\Theta)dm_{S^2} \leq 4\pi \sqrt{C_+ R \varphi_{\lambda}(R)}.$$

**Proof.** We may suppose $R \geq 1$, since the left hand side of the assertion is equal to 0 if $R < 1$. The definition of $l_{\lambda}$ gives

$$\int_{\Theta \in \partial B_1} l_{\lambda}(R\Theta)dm_{S^2} \leq \int_{\Theta \in \partial B_1} \int_0^R \sqrt{\Phi_{\lambda}(t\Theta)}dt dm_{S^2}$$

$$= \int_{\zeta \in B_R \setminus B_1} \frac{\sqrt{\Phi_{\lambda}(\zeta)}}{|\zeta|^2} d\zeta,$$

where $d\zeta = d\zeta_1 d\zeta_2 d\zeta_3$.

Take $m \in \mathbb{Z}_{\geq 0}$ to be $2^m \leq R < 2^{m+1}$. Then the Cauchy-Schwarz inequality gives

$$\int_{\zeta \in B_{R\setminus B_1}} \frac{\Phi_{\lambda}(\zeta)}{|\zeta|^2} d\zeta \leq \left( \sum_{l=0}^{m-1} \int_{\zeta \in B_{2^{l+1}} \setminus B_{2^l}} \frac{\sqrt{\Phi_{\lambda}(\zeta)}}{|\zeta|^2} d\zeta \right)^{1/2} \left( \int_{\zeta \in B_{R\setminus B_1}} \Phi_{\lambda}(\zeta) d\zeta \right)^{1/2}.$$

From Proposition 3.4, the inequalities

$$\int_{\zeta \in B_{R\setminus B_1}} \Phi_{\lambda}(\zeta) d\zeta \leq \int_{\zeta \in B_t} \Phi_{\lambda}(\zeta) d\zeta \leq \frac{P_+}{\pi} t^2 \varphi_{\lambda}(t)$$

hold for any $0 \leq t' \leq t$. Therefore the assertion follows from

$$\int_{\zeta \in B_{R\setminus B_1}} \Phi_{\lambda}(\zeta) d\zeta \leq \sum_{l=0}^{m-1} \int_{\zeta \in B_{2^{l+1}} \setminus B_{2^l}} \frac{1}{2^{l+1}} \int_{\zeta \in B_{R\setminus B_1}} \Phi_{\lambda} d\zeta \leq \frac{P_+}{\pi} 2^{2(l+1)} \varphi_{\lambda}(2^{l+1})^{1/2}.$$
\[
+ \sqrt{4\pi} \sqrt{\frac{1}{2^m} - \frac{1}{R}} \sqrt{\frac{P_+}{\pi} R^2 \varphi_\lambda(R)}
\]
\[
\leq 2 \sum_{l=0}^{m-1} \sqrt{2^{l+1}} \sqrt{P_+ \varphi_\lambda(R)} + 2 \sqrt{P_+ R \varphi_\lambda(R)}
\]
\[
\leq 2(3 + \sqrt{2}) \sqrt{P_+ R \varphi_\lambda(R)}
\]
by putting \( C_+ = (\frac{3+\sqrt{2}}{2\pi})^2 P_+ \).

Since \( R \varphi_\lambda(R) \) diverges to \(+\infty\) for \( R \to +\infty \), there is a constant \( R_0 > 0 \) which satisfies
\[
4\pi L_\lambda + \int_{\Theta \in S^2} l_\lambda(R\Theta) \, dm_{S^2} \leq 4\pi \sqrt{2C_+ R \varphi_\lambda(R)}
\]
for all \( R \geq R_0 \). Now we put
\[
U_{R,T} := \{ \Theta \in S^2; \, L_\lambda + l_\lambda(R\Theta) \leq \sqrt{TR \varphi_\lambda(R)} \}
\]
for \( R,T > 0 \). For a fixed \( R > 0 \), we introduce a function \( F(\Theta) \) on \( S^2 \) by
\[
F(\Theta) := L_\lambda + l_\lambda(R\Theta).
\]
Then \( U_{R,T} \) is the inverse image \( F^{-1}([0, \sqrt{TR \varphi_\lambda(R)}]) \), hence \( U_{R,T} \) is a measurable set.

**Lemma 4.3.** There exists a sufficiently large \( R_0 > 0 \) and we have
\[
m_{S^2}(U_{R,T}) \geq 4\pi \sqrt{\frac{T - \sqrt{2C_+}}{T - \sqrt{Q_+}}} \]
for \( R \geq R_0 \) and \( T > 2C_+ \).

**Proof.** From (4), we have the upper bound
\[
4\pi \sqrt{2C_+ R \varphi_\lambda(R)} \geq \int_{\Theta \in S^2} F(\Theta) \, dm_{S^2}.
\]
Since \( F(\Theta) := L_\lambda + l_\lambda(R\Theta) \geq d_{\rho_\lambda}(p_0, p) \) and \( \mu_\lambda(p) = \zeta = R\Theta \), Lemma 3.1 yields the lower bound for \( F(\Theta) \),
\[
F(\Theta) \geq \sqrt{Q_+ R \varphi_\lambda(R)}.
\]
On the complement $U_{\bar{R},T}$ of $U_{R,T}$ in $S^2$, we have $F(\Theta) \geq \sqrt{TR\varphi_\lambda(R)}$. Then we have
\[
\int_{\Theta \in S^2} F(\Theta)dm_{S^2} = \int_{\Theta \in U_{R,T}} F(\Theta)dm_{S^2} + \int_{\Theta \in U_{\bar{R},T}} F(\Theta)dm_{S^2} \\
\geq m_{S^2}(U_{R,T}) \sqrt{Q_- R\varphi_\lambda(R)} + (4\pi - m_{S^2}(U_{R,T})) \sqrt{TR\varphi_\lambda(R)}.
\]
(6)

From inequalities (5) and (6), we have
\[
4\pi \sqrt{2C_+ R\varphi_\lambda(R)} \geq m_{S^2}(U_{R,T}) \sqrt{Q_- R\varphi_\lambda(R)} + (4\pi - m_{S^2}(U_{R,T})) \sqrt{TR\varphi_\lambda(R)}.
\]

Thus we have
\[
(\sqrt{T} - \sqrt{Q_-})m_{S^2}(U_{R,T}) \sqrt{R\varphi_\lambda(R)} \geq 4\pi(\sqrt{T} - \sqrt{2C_+}) \sqrt{R\varphi_\lambda(R)}.
\]

Since $T > 2C_+ > Q_-$, we obtain
\[
m_{S^2}(U_{R,T}) \geq 4\pi \frac{\sqrt{T} - \sqrt{2C_+}}{\sqrt{T} - \sqrt{Q_-}}.
\]

\[\square\]

**Lemma 4.4.** For each $R \geq 0$ and $T > 0$, $\mu_\lambda^{-1}(B_{R,U_{R,T}})$ is a subset of $B_{\lambda}(p_0, \sqrt{\tau_\lambda(T)}(R))$.

**Proof.** First of all we take $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ such that $|\mu_\lambda(p)| > 1$. Since
\[
\frac{\mu_\lambda(p)}{|\mu_\lambda(p)|} \text{ is an element of } U_{R,T}, \text{ we have}
\]
\[
L_\lambda + l_\lambda(\mu_\lambda(p)) \leq L_\lambda + l_\lambda(R \frac{\mu_\lambda(p)}{|\mu_\lambda(p)|}) \leq \sqrt{TR\varphi_\lambda(R)}.
\]

from $R \geq |\mu_\lambda(p)| > 1$. Then we obtain
\[
d_{g_\lambda}(p_0, p) \leq \sqrt{TR\varphi_\lambda(R)}
\]

from $d_{g_\lambda}(p_0, p) \leq L_\lambda + l_\lambda(\mu_\lambda(p))$.

If $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ is taken to be $|\mu_\lambda(p)| \leq 1$, then we have the same conclusion as above since $l_\lambda(\mu_\lambda(p)) = 0$ in this case. \[\square\]

Now we fix a constant $Q_+$ to be $Q_+ > 2C_+$ and put
\[
m_0 := 4\pi \frac{\sqrt{Q_+ - \sqrt{2C_+}}}{\sqrt{Q_+ - \sqrt{Q_-}}}
\]

and $P_- := m_0C_-$. 19
Proposition 4.5. Let $P_-, Q_+ > 0$ be as above. Then we have

$$\liminf_{r \to \infty} \frac{V_{g_\lambda}(p_0, r)}{\theta_{\lambda, P_-} \circ \tau_{\lambda, Q_+}^{-1}(r^2)} > 0.$$ 

Proof. Let $R \geq 0$. From Lemma 4.4, we have

$$V_{g_\lambda}(p_0, \sqrt[2]{\tau_{\lambda, Q_+}(R)}) \geq \operatorname{vol}_{g_\lambda}(\mu_{\lambda}^{-1}(B_{R,U_{R,Q_+}})).$$

Then Proposition 4.1 gives

$$V_{g_\lambda}(p_0, \sqrt[2]{\tau_{\lambda, Q_+}(R)}) \geq m_S^2(U_{R,Q_+})C_-R^2\varphi_\lambda(R) \geq m_0C_-R^2\varphi_\lambda(R) = \theta_{\lambda, P_-}(R)$$

for $R \geq R_0$. Thus we have the assertion by putting $R = \tau_{\lambda, Q_+}^{-1}(r^2)$. \qed

5 The volume growth

In Sections 3 and 4, we estimate $V_{g_\lambda}(p_0, r)$ from the above by $\theta_{\lambda, P_-} \circ \tau_{\lambda, Q_+}^{-1}(r^2)$ and from the bottom by $\theta_{\lambda, P_-} \circ \tau_{\lambda, Q_+}^{-1}(r^2)$. In this section we show that the asymptotic behavior of the functions $\theta_{\lambda, P_-} \circ \tau_{\lambda, Q_+}^{-1}(r^2)$ and $\theta_{\lambda, P_-} \circ \tau_{\lambda, Q_+}^{-1}(r^2)$ are equal up to constant, and prove the main results.

The asymptotic behavior of $V_{g_\lambda}(p, r)$ is independent of the choice of $p \in X_{\lambda}$ from the next well-known fact.

Proposition 5.1. Let $(X, g)$ be a connected Riemannian manifold of dimension $n$, whose Ricci curvature is nonnegative. Then we have

$$\lim_{r \to +\infty} \frac{V_g(p_1, r)}{V_g(p_0, r)} = 1$$

for any $p_0, p_1 \in X$.

Proof. From the Bishop-Gromov comparison inequality $\frac{V_g(p_0, r)}{r^n}$ is nonincreasing with respect to $r$. If we put $r_0 := d_g(p_0, p_1)$, then we have

$$\frac{V_g(p_0, r_0)}{r^n} \leq \frac{V_g(p_0, r + r_0)}{V_g(p_0, r)} \frac{V_g(p_0, r + r_0)}{r^n} \frac{(r + r_0)^n}{r^n} \leq \frac{1}{r} \to 1 \quad (r \to +\infty).$$
By considering the same argument after exchanging \( p_0 \) and \( p_1 \), we have the assertion. \( \square \)

We denote by \( C^0_+ \) the set of the nondecreasing continuous functions from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R}_{\geq 0} \).

**Definition 5.2.** For \( f_0, f_1 \in C^0_+ \), we define \( f_0(r) \preceq_r f_1(r) \) if
\[
\limsup_{r \to +\infty} \frac{f_0(r)}{f_1(r)} < +\infty.
\]

**Definition 5.3.** For \( f_0, f_1 \in C^0_+ \), we define \( f_0(r) \simeq_r f_1(r) \) if \( f_0(r) \preceq_r f_1(r) \) and \( f_1(r) \preceq_r f_0(r) \).

For a Riemannian manifold \((X, g)\) and a point \( p_0 \in X \), the function \( V_g(p_0, r) \) is an element of \( C^0_+ \). If \((X, g)\) satisfies the assumption of Proposition 5.1, then the equivalence class of \( V_g(p_0, r) \) with respect to \( \simeq_r \) is independent of the choice of \( p_0 \in X \). Therefore we denoted by \( V_g(r) \) the equivalence class of \( V_g(p_0, r) \). For example, if we write \( V_g(r) \preceq_r r^n \), then it implies that
\[
\limsup_{r \to +\infty} \frac{V_g(p_0, r)}{r^n} < +\infty
\]
for some (hence all) \( p_0 \in X \).

The main purpose of this section is to look for the function in \( C^0_+ \) which is equivalent to \( V_g(r) \).

**Lemma 5.4.** Let \( S_+, S_-, T_+, T_- > 0 \). Then \( \theta_{\lambda, S_+} \circ \tau_{\lambda, T_+}^{-1} \) and \( \theta_{\lambda, S_-} \circ \tau_{\lambda, T_+}^{-1} \) are elements of \( C^0_+ \) and we have \( \theta_{\lambda, S_+} \circ \tau_{\lambda, T_+}^{-1}(r^2) \simeq_r \theta_{\lambda, S_-} \circ \tau_{\lambda, T_+}^{-1}(r^2) \).

**Proof.** We may suppose \( T_- \leq T_+ \) without loss of generality. Since \( \tau_{\lambda, T_+} \) and \( \tau_{\lambda, T_-} \) are continuous, strictly increasing and satisfy \( \tau_{\lambda, T_+}(0) = \tau_{\lambda, T_-}(0) = 0 \), then \( \tau_{\lambda, T_+}^{-1} \) and \( \tau_{\lambda, T_-}^{-1} \) are also the elements of \( C^0_+ \). Hence the composite functions \( \theta_{\lambda, S_+} \circ \tau_{\lambda, T_+}^{-1} \) and \( \theta_{\lambda, S_-} \circ \tau_{\lambda, T_+}^{-1} \) are also the elements of \( C^0_+ \). Next we show (i) \( \theta_{\lambda, S_+} \circ \tau_{\lambda, T_+}^{-1}(r^2) \preceq_r \theta_{\lambda, S_+} \circ \tau_{\lambda, T_-}^{-1}(r^2) \) and (ii) \( \theta_{\lambda, S_-} \circ \tau_{\lambda, T_-}^{-1}(r^2) \preceq_r \theta_{\lambda, S_-} \circ \tau_{\lambda, T_+}^{-1}(r^2) \).
(i) From \( \tau_{\lambda, T_+}(R) \geq \tau_{\lambda, T_-}(R) \) and strictly increasingness of \( \tau_{\lambda, T_\pm} \), then we have \( \tau_{\lambda, T_+}^{-1}(r^2) \leq \tau_{\lambda, T_-}^{-1}(r^2) \). Hence we obtain
\[
\frac{\theta_{\lambda, S_+} \circ \tau_{\lambda, T_+}^{-1}(r^2)}{\theta_{\lambda, S_+} \circ \tau_{\lambda, T_-}^{-1}(r^2)} \leq \frac{\theta_{\lambda, S_-} \circ \tau_{\lambda, T_-}^{-1}(r^2)}{\theta_{\lambda, S_-} \circ \tau_{\lambda, T_+}^{-1}(r^2)} \leq \frac{S_-}{S_+}.
\]
(ii) Put \( R_\pm := \tau_{\lambda, T_\pm}^{-1}(r^2) \). Then we have
\[
r^2 = T_+ R_+ \varphi_{\lambda}(R_+) = T_- R_- \varphi_{\lambda}(R_-).
\]

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Since $\varphi_\lambda$ is nondecreasing, it holds
\[ R_- \varphi_\lambda(R_-) = \frac{T_+}{T_-} R_+ \varphi_\lambda(R_+) \leq \frac{T_+}{T_-} R_+ \varphi_\lambda \left( \frac{T_+}{T_-} R_+ \right) \] (7)

Since the function $R \varphi_\lambda(R)$ is strictly increasing with respect to $R$, the expression (7) gives $R_- \leq \frac{T_+}{T_-} R_+$. Recall that $\varphi_\lambda$ satisfies $\varphi_\lambda(\alpha R) \leq \alpha \varphi_\lambda(R)$ for $\alpha \geq 1$, which implies $\theta_{\lambda,S_+}(\alpha R) \leq \alpha^3 \theta_{\lambda,S_+}(R)$. Thus we have
\[
\frac{\theta_{\lambda,S_+} \circ \tau_{\lambda,T_-}^{-1}(r^2)}{\theta_{\lambda,S_-} \circ \tau_{\lambda,T_+}^{-1}(r^2)} = \frac{\theta_{\lambda,S_+}(R_-)}{\theta_{\lambda,S_-}(R_+)} \leq \left( \frac{T_+}{T_-} \right)^3 \frac{\varphi_\lambda(R_+)}{\varphi_\lambda(R_-)} = \left( \frac{T_+}{T_-} \right)^3 \frac{S_+}{S_-}.
\]

\[ \square \]

Put $\theta_\lambda := \theta_{\lambda,1}$, $\tau_\lambda := \tau_{\lambda,1}$. Then the main result in this paper is described as follows.

**Theorem 5.5.** For each $\lambda \in (\text{Im} \mathbb{H}_0^\perp)$ and $p \in X_\lambda$, the function $V_{g_\lambda}(p,r)$ satisfies
\[ 0 < \liminf_{r \to +\infty} \frac{V_{g_\lambda}(p,r)}{r^2 \tau_\lambda^{-1}(r^2)} \leq \limsup_{r \to +\infty} \frac{V_{g_\lambda}(p,r)}{r^2 \tau_\lambda^{-1}(r^2)} < +\infty. \]

**Proof.** We have shown that
\[ \theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \preceq_r V_{g_\lambda}(r) \preceq_r \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2) \]
in Sections 3 and 4, and
\[ \theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \simeq_r \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2) \simeq_r \theta_\lambda \circ \tau_\lambda^{-1}(r^2) \]
in Lemma 5.4. Thus we have the assertion from
\[ \theta_\lambda \circ \tau_\lambda^{-1}(r^2) = \tau_\lambda^{-1}(r^2) \cdot \tau_\lambda(\tau_\lambda^{-1}(r^2)) = r^2 \tau_\lambda^{-1}(r^2). \]

\[ \square \]

**Corollary 5.6.** For $\lambda, \lambda' \in (\text{Im} \mathbb{H}_0^\perp)$, the condition $V_{g_\lambda}(r) \preceq_r V_{g_{\lambda'}}(r)$ is equivalent to $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$. 

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Proof. The condition \(V_{g_R}(r) \leq \tau_{\lambda}^{-1}(r^2) \leq \tau_{\lambda}^{-1}(r^2)\) from Theorem 5.5.
If we assume \(\tau_{\lambda}^{-1}(r^2) \leq \tau_{\lambda}^{-1}(r^2)\) then there are constants \(r_0 \geq 0\) and \(C \geq 1\) which satisfy \(\frac{\tau_{\lambda}^{-1}(r^2)}{\tau_{\lambda}^{-1}(r^2)} \leq C\) for all \(r \geq r_0\). Now we put \(r^2 := \tau_{\lambda}(R)\) and \((r')^2 := \tau_{\lambda}(R)\) for \(R \geq \tau_{\lambda}^{-1}(r^2)\). Since we have
\[
R = \tau_{\lambda}^{-1}((r')^2) = \tau_{\lambda}^{-1}(r^2) \leq C \tau_{\lambda}^{-1}(r^2),
\]
then
\[
(r')^2 \leq \tau_{\lambda}(C \tau_{\lambda}^{-1}(r^2)) \leq C^2 \tau_{\lambda}(\tau_{\lambda}^{-1}(r^2)) = C^2 r^2
\]
is given by the monotonicity of \(\tau_{\lambda}\). Thus we obtain \(\frac{\tau_{\lambda}(R)}{\tau_{\lambda}(R)} \leq C^2\) for all \(R \geq \tau_{\lambda}^{-1}(r^2)\), which implies \(\tau_{\lambda}(R) \leq C \tau_{\lambda}(R)\).
On the other hand, if we assume \(\tau_{\lambda}(R) \leq C \tau_{\lambda}(R)\) then \(\tau_{\lambda}^{-1}(r^2) \leq \tau_{\lambda}^{-1}(r^2)\) is obtained in the same way. Thus we have the assertion since the condition \(\tau_{\lambda}(R) \leq C \tau_{\lambda}(R)\) is equivalent to \(\varphi_{\lambda}(R) \leq C \varphi_{\lambda}(R)\). \(\square\)

Lemma 5.7. For all \(\lambda \in \Im H_0^Z\), we have
\[
\lim_{R \to +\infty} \varphi_{\lambda}(R) = +\infty.
\]
Proof. From Lemma 3.2, it is enough to show \(\lim_{R \to +\infty} \psi_{\lambda}(R) = +\infty\), which follows directly from \(\lim_{R \to +\infty} 2N_{\lambda}(R) = +\infty\). \(\square\)

Lemma 5.8. For all \(\lambda \in \Im H_0^Z\), we have
\[
\lim_{R \to +\infty} \frac{\varphi_{\lambda}(R)}{R} = 0.
\]
Proof. For each \(\varepsilon > 0\) there exists a sufficiently large \(n_\varepsilon \in \mathbb{Z}_{>0}\) such that \(\sum_{|n| > n_\varepsilon} \frac{1}{|\lambda_n|} < \frac{\varepsilon}{2}\). Hence we have
\[
\frac{\varphi_{\lambda}(R)}{R} = \sum_{|n| \leq n_\varepsilon} \frac{1}{R + |\lambda_n|} + \sum_{|n| > n_\varepsilon} \frac{1}{R + |\lambda_n|}
\leq \sum_{|n| \leq n_\varepsilon} \frac{1}{R} + \sum_{|n| > n_\varepsilon} \frac{1}{|\lambda_n|}
\leq \frac{2n_\varepsilon + 1}{R} + \frac{\varepsilon}{2}.
\]
Then the inequality \(\frac{\varphi_{\lambda}(R)}{R} \leq \varepsilon\) holds for any \(R \geq \frac{2(2n_\varepsilon + 1)}{\varepsilon}\), which implies \(\lim_{R \to +\infty} \frac{\varphi_{\lambda}(R)}{R} \leq \varepsilon\). The assertion follows by taking \(\varepsilon \to 0\). \(\square\)
Corollary 5.9. For all $\lambda \in (\text{Im} H)_0^Z$, we have
\[
\lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^3} = +\infty.
\]

Proof. It suffices to show that
\[
\lim_{r \to +\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = 0, \quad \lim_{r \to +\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = +\infty.
\]
We put $R = \tau_\lambda^{-1}(r^2)$ and consider the limit of $R \to +\infty$. Then we have
\[
\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^2} = \frac{1}{\varphi_\lambda(R)},
\]
and
\[
\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^2} = \sqrt{\frac{R}{\varphi_\lambda(R)}}.
\]
Hence we obtain
\[
\lim_{r \to +\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = \lim_{R \to +\infty} \frac{1}{\varphi_\lambda(R)} = 0,
\]
\[
\lim_{r \to +\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = \lim_{R \to +\infty} \sqrt{\frac{R}{\varphi_\lambda(R)}} = +\infty.
\]
from Lemmas 5.7 and 5.8. \hfill \Box

The condition $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$ is rather difficult to check. But we can describe the sufficient condition for $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$ easier as follows.

**Proposition 5.10.** Let $\lambda, \lambda' \in (\text{Im} H)_0^Z$. Suppose that there are some $\alpha > 0$ and $R_0 > 0$ such that $\sharp N_\lambda(R) \leq \sharp N_{\alpha \lambda'}(R)$ for $R \geq R_0$, where $\alpha \lambda'$ := $(\alpha \lambda'_n)_{n \in \mathbb{Z}}$. Then we have $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$.

Proof. Take bijections $a, b : \mathbb{Z}_{>0} \to \mathbb{Z}$ to be $|\lambda_{a(n)}| \leq |\lambda_{a(n+1)}|, |\lambda_{b(n)}| \leq |\lambda_{b(n+1)}|$ for each $n \in \mathbb{Z}_{>0}$. Then the condition $\sharp N_\lambda(R) \leq \sharp N_{\alpha \lambda'}(R)$ is equivalent to $a^{-1}(N_\lambda(R)) \subset b^{-1}(N_{\alpha \lambda'}(R))$.

Take $n_0 \in \mathbb{Z}_{>0}$ sufficiently large such that $|\lambda_{a(n_0)}| \geq R_0$. Then we have $|\alpha \lambda'_{b(n)}| \leq |\lambda_{a(n)}|$ for each $n \geq n_0$ from
\[
a^{-1}(N_\lambda(|\lambda_{a(n_0)}|)) \subset b^{-1}(N_{\alpha \lambda'}(|\lambda_{a(n_0)}|)).
\]
If \( n \) is an element of \( b^{-1}(N_{a\lambda'}(R))\setminus a^{-1}(N_{\lambda}(R)) \) for \( R \geq R_0 \), then we have

\[
\frac{1}{|\lambda_{a(n)}|} \leq \frac{1}{R}, \quad \frac{1}{|\lambda_{a(n)}|} \leq \frac{1}{\alpha|\lambda_{b(n)}'|}.
\]

Thus we have

\[
\psi_{\lambda}(R) = \frac{n}{\tau}N_{\lambda}(R) + \sum_{n\notin a^{-1}(N_{\lambda}(R))} \frac{R}{|\lambda_{a(n)}|}.
\]

We fix \( \lambda \) to \( R \in \mathbb{Z} \) such that \( |\lambda_{a(n)}| = |\lambda_{b(n)}| \). Then \( \psi_{\lambda}(R) \leq \psi_{\lambda'}(R) \). Hence the volume growth of \( g_{\lambda} \) is equivalent to \( \varphi_{\lambda}(R) \geq \varphi_{\lambda'}(R) \).

**6 Examples**

In this section we evaluate \( V_{g_{\lambda}}(r) \) concretely for some \( \lambda \in (\text{Im}\mathbb{H})_{0}^{\mathbb{Z}} \). We fix a bijection \( a : \mathbb{Z}_{>0} \to \mathbb{Z} \) and identify \( \mathbb{Z} \) with \( \mathbb{Z}_{>0} \) throughout this section.

Now we fix \( \lambda \in (\text{Im}\mathbb{H})_{0}^{\mathbb{Z}} \). Assume that \( |\lambda_{a(n)}| \) is nondecreasing with respect to \( n \in \mathbb{Z}_{>0} \) and there exists a continuous nondecreasing function \( \lambda_{b} : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) which satisfies \( |\lambda_{a(n)}| = \lambda_{b}(n) \). Then a function \( \varphi_{\lambda_{a}}(R) := \int_{0}^{\infty} \frac{Rdx}{R+\lambda_{a}(x)} \) is strictly increasing and satisfies \( \varphi_{\lambda_{a}}(R) \simeq R \varphi_{\lambda}(R) \). In this case we have \( V_{g_{\lambda}}(r) \simeq r^{2}\varphi_{\lambda_{a}}^{-1}(r^{2}) \) where \( \varphi_{\lambda_{a}} \) is defined by \( \varphi_{\lambda_{a}}(R) := R\varphi_{\lambda_{a}}(R) \). Now we compute the volume growth of \( g_{\lambda} \) in the following two cases.

1. Fix \( \alpha > 1 \) and put \( \lambda_{a}(x) = x^{\alpha} \), \( \lambda_{a}(n) = \lambda_{b}(n)i \). Then \( \varphi_{\lambda_{a}} \) is given by

\[
\varphi_{\lambda_{a}}(R) = \int_{0}^{\infty} \frac{Rdx}{R+x^{\alpha}} = \int_{0}^{\infty} \frac{dy}{1+y^{\alpha}},
\]

where we put \( y = \frac{x}{R}^{\alpha} \). Since \( \int_{0}^{\infty} \frac{dy}{1+y^{\alpha}} \) is a constant, we have \( \varphi_{\lambda_{a}}(R) \simeq R^{\frac{1}{\alpha}} \), which gives \( \varphi_{\lambda_{a}}(R) \simeq R^{1+\frac{1}{\alpha}} \) and \( \varphi_{\lambda_{a}}^{-1}(r^{2}) \simeq r^{\frac{2\alpha}{\alpha+1}} \). Hence the volume growth is given by

\[
V_{g_{\lambda}}(r) \simeq r^{4-\frac{2\alpha}{\alpha+1}}.
\]

Thus we obtain the following result.

2.5
Theorem 6.1. There exists a complete 4-dimensional hyperkähler manifold $(X_\lambda, g_\lambda)$ for each $3 < \alpha < 4$ whose volume growth is given by

$$V_{g_\lambda}(r) \simeq_r r^\alpha.$$  

2. Fix $\alpha > 0$ and put $\lambda_R(x) = e^{\alpha x}$, $\lambda_{a(n)} = \lambda_R(n)i$. Then $\varphi_{\lambda_R}$ is given by

$$\varphi_{\lambda_R}(r) = \int_0^\infty \frac{Rdx}{R + e^{\alpha x}}.$$  

By putting $y = e^{\alpha x}$, we have

$$\varphi_{\lambda_R}(R) = \int_1^\infty \frac{Rdy}{\alpha y(y + R)} = \frac{1}{\alpha} \log(R + 1).$$  

Hence we have $\varphi_{\lambda_R}(R) = \frac{1}{\alpha} \log(R + 1)$ and $\hat{\tau}_{\lambda_R}(R) = \frac{1}{\alpha} R \log(R + 1)$.  

Proposition 6.2. Let $\lambda \in (\text{Im}\mathbb{H})_0^\mathbb{Z}$ be as above. Then the volume growth of $g_\lambda$ satisfies $V_{g_\lambda}(r) \simeq_r \frac{r^4}{\log r}$.

Proof. It is enough to see the behavior of $\frac{r^2\hat{\tau}_{\lambda_R}^{-1}(r^2) \log r}{r^4}$ at $r \to +\infty$. Put $R = \hat{\tau}_{\lambda_R}^{-1}(r^2)$. Then we have $r^2 = \frac{1}{\alpha} R \log(R + 1)$ and $\log r = \frac{1}{2} (\log R + \log \log(R + 1) - \log \alpha)$. Thus we have

$$\lim_{r \to +\infty} \frac{r^2\hat{\tau}_{\lambda_R}^{-1}(r^2) \log r}{r^4} = \lim_{R \to +\infty} \frac{\alpha}{2} \frac{R(\log R + \log \log(R + 1) - \log \alpha)}{R \log(R + 1)} = \frac{\alpha}{2}.$$  

Thus we have the following theorem.

Theorem 6.3. There exists a complete 4-dimensional hyperkähler manifold $(X_\lambda, g_\lambda)$ whose volume growth satisfies

$$\lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^\alpha} = +\infty$$  

for any $\alpha < 4$.  

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7 Taub-NUT deformations

We consider the volume growth of the Taub-NUT deformations of $(X_\lambda, g_\lambda)$ in this section. First of all, we define the Taub-NUT deformations of hyperkähler manifolds with tri-Hamiltonian $S^1$-actions. Let $(X, \omega)$ be a hyperkähler manifold of dimension 4 with tri-Hamiltonian $S^1$-action, and $\mu : X \to \text{Im} \mathbb{H}$ be the hyperkähler moment map. An action of $\mathbb{R}$ on $\mathbb{H}$ is defined by $x \to x + \sqrt{s}^{-1} t$ for $x \in \mathbb{H}$ and $t \in \mathbb{R}$ by fixing a constant $s > 0$. This action preserves the standard hyperkähler structure on $\mathbb{H}$ and the hyperkähler moment map is given by $\sqrt{s}^{-1} : \text{Im} \mathbb{H} \to \text{Im} \mathbb{H}$. Then we obtain the hyperkähler quotient with respect to the action of $\mathbb{R}$ on $X \times \mathbb{H}$, that is, the quotient $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ where $\zeta$ is an element of $\text{Im} \mathbb{H}$ and $\mu^{(s)} : X \times \mathbb{H} \to \text{Im} \mathbb{H}$ is defined by $\mu^{(s)}(x, y) := \mu(x) + 2\sqrt{s}^{-1}\text{Im}(y)$. The hyperkähler structure on $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ is independent of $\zeta$.

For each $\zeta \in \text{Im} \mathbb{H}$ we have an embedding $\tilde{\iota}_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)$ defined by $\tilde{\iota}_{s,\zeta}(x) := (x, \sqrt{s}^{-2}(-\mu(x) + \zeta))$ which induces a diffeomorphism $\iota_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$. Then we have another hyperkähler structure on $X$ independent of $\zeta$ by the pull-back, which is called Taub-NUT deformation of $\omega$ denoted by $\omega^{(s)}$. If we denote by $g$ the hyperkähler metric of $(X, \omega)$, then we denote by $g^{(s)}$ the hyperkähler metric of $(X, \omega^{(s)})$.

There is the pair of a harmonic function and an $S^1$-connection $(\Phi, A)$ corresponding to the hyperkähler structure $\omega$ by Theorem 2.9. Then the corresponding pair to $\omega^{(s)}$ is given by $(\Phi + \frac{\delta}{s}, A)$.

The $S^1$-action on $X$ also preserves $\omega^{(s)}$ and $\mu : X \to \text{Im} \mathbb{H}$ is also the hyperkähler moment map with respect to $\omega^{(s)}$.

**Lemma 7.1.** Let $(X, g)$ be as above and take $p_0, p \in X$. We suppose that $p_0$ is a fixed point by the $S^1$-action. Then we have the inequality

$$d_{g^{(s)}}(p_0, p)^2 \geq d_g(p_0, p)^2 + \frac{s}{4}|\mu(p) - \mu(p_0)|^2.$$  

**Proof.** We apply the same argument as Proposition 3.1. Then the assertion follows from

$$d_{g^{(s)}}(p_0, p)^2 \geq \inf_{t \in \mathbb{R}} d_{g \times g_{\lambda}}(\tilde{\iota}_{s,\zeta}(p_0) \cdot t, \tilde{\iota}_{s,\zeta}(p))^2$$

$$= \inf_{t \in \mathbb{R}} (d_g(p_0 e^{it}, p)^2 + \frac{\sqrt{s}}{2} (\mu(p_0) - \mu(p)) + \frac{1}{\sqrt{s}} t^2)$$

$$= \inf_{t \in \mathbb{R}} (d_g(p_0, p)^2 + \frac{s}{4} |\mu(p_0) - \mu(p)|^2 + \frac{t^2}{s})$$

$$= d_g(p_0, p)^2 + \frac{s}{4} |\mu(p_0) - \mu(p)|^2,$$
where \( g_{\mathbb{H}} \) is the Euclidean metric on \( \mathbb{H} \) and \( g \times g_{\mathbb{H}} \) is the direct product metric.

**Lemma 7.2.** Let \((X, g)\) be as above and \(B \subset \text{Im} \mathbb{H}\) be a measurable set. Then we have

\[
\text{vol}_{g}(\mu^{-1}(B)) = \text{vol}_{g}(\mu^{-1}(B)) + \frac{\pi s}{4} m_{\text{Im} \mathbb{H}}(B),
\]

where \( m_{\text{Im} \mathbb{H}} \) is the Lebesgue measure of \( \text{Im} \mathbb{H} \).

**Proof.** It follows directly from Lemma 2.10 and that \( \omega^{(s)} \) corresponds to \((\Phi + \frac{s}{4}, A)\).

For \( s, C > 0 \) and \( \lambda \in (\text{Im} \mathbb{H})_{0}^{\mathbb{Z}} \), put

\[
\theta^{(s)}_{\lambda, C}(R) := CR^2 \varphi_{\lambda}(R) + \frac{\pi^2 s}{3} R^3, \quad \tau^{(s)}_{\lambda, C}(R) := CR \varphi_{\lambda}(R) + \frac{s}{4} R^2.
\]

**Proposition 7.3.** For \( \lambda \in (\text{Im} \mathbb{H})_{0}^{\mathbb{Z}} \) and \( s > 0 \), we have

\[
\limsup_{r \to +\infty} \frac{V_{g^{(s)}}(p_0, r)}{r^3} \leq \frac{8 \pi^2}{3 \sqrt{s}}.
\]

**Proof.** From Lemma 7.1 and 7.2, we have

\[
V_{g^{(s)}}(p_0, r) \leq \theta^{(s)}_{\lambda, P_+} \circ (\tau^{(s)}_{\lambda, Q_-})^{-1}(r^2)
\]

for \( r > 0 \). Then it suffices to show

\[
\limsup_{r \to +\infty} \theta^{(s)}_{\lambda, P_+} \circ (\tau^{(s)}_{\lambda, Q_-})^{-1}(r^2) \leq \frac{8 \pi^2}{3 \sqrt{s}}.
\]

Put \( R = (\tau^{(s)}_{\lambda, Q_-})^{-1}(r^2) \). Then we have

\[
r^2 = Q_- R \varphi_{\lambda}(R) + \frac{s}{4} R^2 \geq \frac{s}{4} R^2.
\]

Therefore Lemma 5.8 gives

\[
\limsup_{r \to +\infty} \frac{\theta^{(s)}_{\lambda, P_+} \circ (\tau^{(s)}_{\lambda, Q_-})^{-1}(r^2)}{r^3} \leq \limsup_{R \to +\infty} \frac{8(P + R^2 \varphi_{\lambda}(R) + \frac{s^2}{3} R^3)}{\sqrt{s} R^3} = \frac{8 \pi^2}{3 \sqrt{s}}.
\]

\]
Next we consider the lower bound for $V_{g(t)}(s)$. We apply the same way as Section 4. Put $l^s_\lambda : \text{Im} \mathbb{H} \to \mathbb{R}$ as

$$l^s_\lambda (\zeta) := \int_1^{\vert \zeta \vert} \sqrt{\Phi_\lambda \left( \frac{t \zeta}{|\zeta|} \right) + \frac{s}{4}} \, dt$$
onumber

on $|\zeta| > 1$, and $l^s_\lambda (\zeta) := 0$ on $|\zeta| \leq 1$. Then the inequality $d_{g(t)}(p_0, p) \leq L^s_\lambda + l^s_\lambda (\mu(t))$ holds where $p_0 \in X$ is taken as in Sections 3 and 4, and we put $L^s_\lambda := \sup_{p \in \mu^{-1}(B_1)} d_{g(t)}(p_0, p)$.

**Lemma 7.4.** Let $C_+ > 0$ be as in Proposition 4.2. Then we have

$$\int_{\Theta \in S^2} l^s_\lambda (R \Theta) \, dm_{S^2} \leq 4\pi \left( \sqrt{\tau_{\lambda, C_+}}(R) + \sqrt{\frac{sR^2}{4}} \right).$$

**Proof.** The assertion follows from

$$\int_{\Theta \in S^2} \left( \int_1^R \sqrt{\Phi_\lambda (t \Theta) + \frac{s}{4}} \, dt \right) \, dm_{S^2} \leq \int_{\Theta \in S^2} \left( \int_1^R \sqrt{\frac{s}{4}} \, dt \right) \, dm_{S^2} + \int_{\Theta \in S^2} l_\lambda (R \Theta) \, dm_{S^2} \leq 4\pi \sqrt{\frac{sR^2}{4}} + 4\pi \sqrt{\tau_{\lambda, C_+}(R)}.$$



Put $U^{(s)}_{R, T} := \{ \Theta \in S^2; L^s_\lambda + l^s_\lambda (R \Theta) \leq \sqrt{\tau_{\lambda, T}(R)} + \sqrt{\frac{sR^2}{4}} \}.$

**Lemma 7.5.** There is a constant $R_0 > 0$ such that

$$m_{S^2}(U^{(s)}_{R, T}) \geq \frac{4\pi \left( \sqrt{T} - \sqrt{2C_+} \right)}{\sqrt{T}}$$

for any $R \geq R_0$ and $T > 2C_+$.

**Proof.** We consider the same argument as in Lemma 4.3. First of all we remark that there exists sufficiently large $R_0 > 0$ such that

$$\int_{\Theta \in S^2} (L^s_\lambda + l^s_\lambda (R \Theta)) \, dm_{S^2} \leq 4\pi \left( \sqrt{\tau_{\lambda, 2C_+}(R)} + \sqrt{\frac{sR^2}{4}} \right).$$
for any $R \geq R_0$. Then we have

$$\int_{\Theta \in S^2} (L^{(s)}_{\lambda} + i^{(s)}_{\lambda}(R\Theta)) dm_{S^2} = \int_{\Theta \in U^{(s)}_{R,T}} (L^{(s)}_{\lambda} + i^{(s)}_{\lambda}(R\Theta)) dm_{S^2}$$

$$+ \int_{\Theta \in S^2 \setminus U^{(s)}_{R,T}} (L^{(s)}_{\lambda} + i^{(s)}_{\lambda}(R\Theta)) dm_{S^2}$$

$$\geq m_{S^2}(U^{(s)}_{R,T}) \sqrt{\tau_{\lambda,Q_-}(R)} + \frac{sR^2}{4}$$

$$+ (4\pi - m_{S^2}(U^{(s)}_{R,T}))(\sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}),$$

where the constant $Q_- > 0$ is as in Proposition 3.1. Then an inequality

$$\sqrt{\tau_{\lambda,Q_-}(R)} + \frac{sR^2}{4} \geq \sqrt{\frac{sR^2}{4}}$$

gives

$$\int_{\Theta \in S^2} (L^{(s)}_{\lambda} + i^{(s)}_{\lambda}(R\Theta)) dm_{S^2} \geq 4\pi \sqrt{\frac{sR^2}{4}}$$

$$+ (4\pi - m_{S^2}(U^{(s)}_{R,T}))(\sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}).$$

Thus we have the conclusion. \qed

**Lemma 7.6.** For each $R \geq 0$ and $T > 0$, $\mu^{-1}_{\lambda}(B_{R,U^{(s)}_{R,T}})$ is a subset of $B_{g^{(s)}_{\lambda}}(p_0, \sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}})$.

**Proof.** It is shown by the same argument as Lemma 4.4. \qed

**Proposition 7.7.** For each $\lambda \in (\Im H^Z_0)$ and $s > 0$ we have

$$\liminf_{r \to +\infty} \frac{V_{g^{(s)}_{\lambda}}(p_0, r)}{r^3} \geq \frac{8\pi^2}{3\sqrt{s}}.$$

**Proof.** For each sufficiently small $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$m_{S^2}(U^{(s)}_{R,T_\varepsilon}) > 4\pi(1 - \varepsilon)$$

from Lemma 7.5. Since $\frac{\varphi_{\lambda}(R)}{R}$ converges to 0, there exists $R_\varepsilon > 0$ such that $T_\varepsilon \varphi_{\lambda}(R) < \varepsilon^2 R$ for any $R > R_\varepsilon$. Then it holds

$$\text{vol}_{g^{(s)}_{\lambda}}(\mu^{-1}_{\lambda}(B_{R,U^{(s)}_{R,T_\varepsilon}})) \leq V_{g^{(s)}_{\lambda}}(p_0, \sqrt{\tau_{\lambda,T_\varepsilon}(R)} + \sqrt{\frac{sR^2}{4}})$$

$$\leq V_{g^{(s)}_{\lambda}}(p_0, (\varepsilon + \sqrt{\frac{s}{4}})R) \quad (8)$$
for $R > R_\varepsilon$. On the other hand, we have the following inequality

$$\text{vol}_{g_\lambda^{(s)}}(\mu_\lambda^{-1}(B_{R,T})) \geq C_m s^2(U) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m s^2(U)}{12} s R^3 \quad (9)$$

for $U \subset \text{Im} \mathbb{H}$ from Proposition 4.1. Thus we obtain

$$V_{g_\lambda^{(s)}} \left( p_0, \left( \varepsilon + \sqrt{\frac{s}{4}} \right) R \right) \geq C_m s^2(U_{R,T}^{(s)}) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m s^2(U_{R,T}^{(s)})}{12} s R^3$$

from (8) and (9). Hence substituting $r = (\varepsilon + \sqrt{\frac{s}{4}}) R$ gives

$$\liminf_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} = \liminf_{R \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, (\varepsilon + \sqrt{\frac{s}{4}}) R)}{(\varepsilon + \sqrt{\frac{s}{4}})^3 R^3} \geq \frac{1}{3} \frac{\pi^2 (1 - \varepsilon) s}{(\varepsilon + \sqrt{\frac{s}{4}})^3}$$

for any sufficiently small $\varepsilon > 0$. Therefore the conclusion follows by taking the limit for $\varepsilon \to 0$. \hfill \Box

From Proposition 7.3 and 7.7, we obtain the followings.

**Theorem 7.8.** Let $\lambda \in (\text{Im} \mathbb{H})_0^\mathbb{Z}$ and $s > 0$. Then the volume growth of hyperkähler metric $g_\lambda^{(s)}$ is given by

$$\lim_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(r)}{r^3} = \frac{8 \pi^2}{3 \sqrt{s}}$$

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