THE DIRICHLET PROBLEM FOR MIXED HESSIAN TYPE EQUATIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we derive $C^2$ estimates for a class of mixed Hessian type equations with Dirichlet boundary condition, and obtain the existence theorem of admissible solutions for the classical Dirichlet problem of these mixed Hessian type equations.

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1. Introduction

In this paper, we consider the Dirichlet problem for a class of mixed Hessian type equations with the following form

\begin{equation}
\begin{cases}
\sigma_k(\nabla^2 u + \chi(x, u, \nabla u)) = \sum_{l=0}^{k-1} \alpha_l(x) \sigma_l(\nabla^2 u + \chi(x, u, \nabla u)), & \text{in } M, \\
u = \varphi(x), & \text{on } \partial M,
\end{cases}
\end{equation}

on a Riemannian manifold $(M^n, g)$ of dimension $n \geq 3$ with smooth boundary $\partial M$, where $3 \leq k \leq n$, $\chi(x, u, \nabla u)$ is a $(0, 2)$-tensor on $\overline{M}$, $\nabla u$ and $\nabla^2 u$ are the gradient and Hessian of the function $u$, respectively. Note that $\sigma_k$ is a $k$-Hessian operator defined by

$$\sigma_k(W) := \sigma_k(\lambda(W)),$$

where $\lambda(W)$ are the eigenvalues of a $(0, 2)$-tensor $W$ with respect to the metric $g$. Recall that the Gårding’s cone is defined as

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}.$$

A function $u \in C^2(\overline{M})$ is called admissible if $\lambda(\nabla^2 u + \chi(x, u, \nabla u)) \in \Gamma_{k-1}$ for any $x \in M$. Note that for fixed $x \in \overline{M}$, $z \in \mathbb{R}$ and $p \in T^*_x M$,

$$\chi(x, z, p) : T^*_x M \times T^*_x M \to \mathbb{R}$$

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is a symmetric bilinear map. We shall use the notation
\[
\chi^{\xi\eta}(x,\cdot,\cdot) := \chi(x,\cdot,\cdot)(\xi,\eta), \quad \forall \xi,\eta \in T_x^* M.
\]

The equation in (1.1) with \( \chi = 0 \)
\[
(1.3) \quad \sigma_k(\nabla^2 u) = \sum_{l=0}^{k-1} \alpha_l(x) \sigma_l(\nabla^2 u), \quad \text{in } M
\]
is known to have attracted much research interest and have many applications. Special\-ly, it is Monge-Ampère equation when \( k = n \) and \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \), \( k \)-Hessian equation when \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \), and \( (k,l) \)-Hessian quotient equation when \( \alpha_0 = \cdots = \alpha_{l-1} = \alpha_{l+1} = \cdots = \alpha_{k-1} = 0 \). The corresponding Dirichlet problem was studied extensively, see [1, 19, 2, 25, 10, 12, 13] and so on. In fact, the mixed Hessian equation (1.3) is motivated from the study of many important geometric problems. For example, special Lagrangian equations introduced by Harvey and Lawson [18] can be written as the following form,
\[
\sin \theta \sum_{k=0}^{[n/2]} (-1)^k \sigma_{2k}(\nabla^2 u) + \cos \theta \sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(\nabla^2 u) = 0.
\]

Another important example for the equation (1.3) was the following equation
\[
\sigma_1(\nabla^2 u) + b \sigma_n(\nabla^2 u) = C
\]
for some constants \( b \geq 0, C > 0 \), arising from the study of \( J \)-equation on toric manifolds by Collins-Székelyhidi [3], which was raised as a conjecture by Chen [9] in the study of Mabuchi energy.

As an important example for the applications of the general notion of fully nonlinear elliptic equations developed in [20], Krylov studied Dirichlet problem of the equation (1.3) in a \((k-1)\)-convex domain in \( \mathbb{R}^n \) with \( \alpha_l > 0 (0 \leq l \leq k-1) \). Recently, Guan-Zhang [17] observed that the equation (1.3) is equivalent to the following equation
\[
(1.4) \quad \frac{\sigma_k}{\sigma_{k-1}}(\nabla^2 u) - \sum_{l=0}^{k-2} \alpha_l(x) \frac{\sigma_l}{\sigma_{k-1}}(\nabla^2 u) = \alpha_{k-1},
\]
and the equation is elliptic and concave in \( \Gamma_{k-1} \). Then they obtained the existence of \((k-1)\)-admissible solution for the Dirichlet problem of the equation (1.4) without sign requirement for \( \alpha_{k-1} \). Later the corresponding in Neumann problem, prescribed curvature problem, complex manifolds were also discussed in [4, 5, 31, 6, 29, 30, 26].
The main motivations to our study of the equation (1.1) with the dependence of \( \chi \) come from many interesting geometric problems. These include the Christoffel-Minkowski problem (see [16]) and the Alexandrov problem of prescribed curvature measure (see [15]), which are associated with the equation (1.1) on \( S^n \) for \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \) and \( \chi = uI \). Moreover, Guan-Zhang [17] studied

\[
\sigma_k(\nabla^2 u + uI) = \sum_{l=0}^{k-1} \alpha_l(x) \sigma_l(\nabla^2 u + uI), \quad \text{on } S^n,
\]

which arises in the problem of prescribed convex combination of area measures [23]. Another analogue example for the equation (1.1) with the dependence of \( \chi \) include the Darboux equation, which appears in isometric embedding (see [14, 22]); the Schouten tensor equation, which is connected with a natural fully nonlinear version of the Yamabe problem (see [28]). A natural problem is raised whether we can consider the Dirichlet problem for the equation (1.1) with the dependence of \( \chi \).

In the study of the equation (1.1), a priori \( C^2 \) estimates are crucial to the existence and regularity of solutions. Compared with the equation (1.4) in [17], the equation (1.1) involves a \((0,2)\)-tensor \( \chi \), which is more complicated. Therefore, it is more difficult to obtain a priori estimates, and suitable constraints on \( \chi \) should be needed. Recently, Guan-Jiao [12, 13] considered a fully nonlinear elliptic equation with the general form

\[
f(\lambda(\nabla^2 u + A[u])) = \psi(x, u, \nabla u)
\]

on a Riemannian manifold and derived the estimates under conditions for a \((0,2)\) tensor \( A[u] = A(x, u, \nabla u) \) and \( \psi \) which are close to optimal. Inspired by the Guan-Jiao’s work, we introduce the following conditions:

**Condition 1.1.** For any \( x \in \overline{M}, z \in \mathbb{R}, p \in T_x \overline{M}, \xi \in T_x M, \) \( \chi \) satisfies

\[
(1.5) \quad \chi^{\xi \xi}(x, z, p) \quad \text{is concave in } p,
\]

\[
(1.6) \quad \chi^{\xi \xi}_z \geq 0.
\]

Then the second order estimates for the equation (1.1) are as follows.

**Theorem 1.2.** Let \( 3 \leq k \leq n, \varphi, \alpha_l \) be smooth functions with \( \alpha_l > 0 \) for \( 0 \leq l \leq k-2 \), \( u \) be a smooth admissible solution (i.e. \( \lambda(\nabla^2 u + \chi(x, u, \nabla u)) \in \Gamma_{k-1} \)) for the equation (1.1). Assume that the \((0,2)\)-tensor \( \chi \) satisfies Condition 1.1 and there exists an
admissible subsolution $u \in C^2(M)$ satisfying

$$\begin{cases}
\sigma_k(\nabla^2 u + \chi(x, u, \nabla u)) \geq \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(\nabla^2 u + \chi(x, u, \nabla u)), & \text{in } M, \\
u = \varphi(x), & \text{on } \partial M.
\end{cases}$$

(1.7)

Then there exists $C > 0$ depending on $n, k, l, \|u\|_{C^1}, \|u\|_{C^2}, \|\chi\|_{C^2}, \|\alpha_{k-1}\|_{C^2}, \|\alpha_l\|_{C^2}$ and $\inf \alpha_l$ with $0 \leq l \leq k - 2$ such that

$$\max_M |\nabla^2 u| \leq C.$$

In particular, in order to obtain the gradient estimates for the equation (1.1), we restrict our study in case $\chi = \chi(x, p)$ and add the following conditions:

**Condition 1.3.** For any $x \in \overline{M}, p \in T_x\overline{M}, \xi, \eta \in T_xM$, $\chi$ satisfies

$$\begin{cases}
p \cdot \nabla_x \chi^{\xi\xi}(x, p) \leq \overline{\psi}_1(x)|\xi|^2(1 + |p|^{\gamma_1}), \\
|\chi^{\xi\eta}(x, p)|^2 \leq \overline{\psi}_2(x)|\xi||\eta|(1 + |p|^{\gamma_2}),
\end{cases}$$

(1.8)

with some functions $\overline{\psi}_1, \overline{\psi}_2 > 0$ and constants $\gamma_1, \gamma_2 \in (0, 2)$.

Then we consider the solvability of the Dirichlet problem for the equation (1.1) on Riemannian manifolds.

**Theorem 1.4.** Let $(M, g)$ be a Riemannian manifold with nonnegative sectional curvature, $3 \leq k \leq n$, $\varphi, \alpha_l$ be smooth functions with $\alpha_l > 0$ for $0 \leq l \leq k - 2$. Assume that the $(0, 2)$-tensor $\chi$ satisfies $\chi = \chi(x, p)$, Condition 1.1, Condition 1.3 and there exists an admissible subsolution $u \in C^2(M)$ satisfying (1.7), then there exists an admissible solution $u \in C^\infty(\overline{M})$ for the equation (1.1).

**Remark 1.5.** Following the idea in [12], [13] and [17], we obtain the second order estimates for admissible solutions under Condition 1.1 and establish gradient estimates under Condition 1.2, Condition 1.3. The sub-solution condition is critical in all steps of the a priori estimates. We emphasize that, there is no sign requirement for $\alpha_{k-1}$ in the above theorem.

The organization of the paper is as follows. In Section 2 we start with some preliminaries. Our proof of the estimates heavily depends on results in Section 3 and Section 4. $C^1$ estimates are given in Section 3. In Section 4 we derive the global estimates for the second order derivatives, and finish the proof of Theorem 1.4.
2. Preliminaries

In this section, we give some basic notations and some basic properties of elementary symmetric functions, which could be found in [21], and establish some key lemmas.

2.1. Basic properties of elementary symmetric functions. For \( \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \), the \( k \)-th elementary symmetric function is defined by

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.
\]

We also set \( \sigma_0 = 1 \) and denote \( \sigma_k(\lambda|_{i}) \) the \( k \)-th symmetric function with \( \lambda_i = 0 \).

Recall that the Gårding’s cone is defined as (1.2).

The generalized Newton-MacLaurin inequality is as follows, which will be used all the time.

Proposition 2.1. For \( \lambda \in \Gamma_m \) and \( m > l \geq 0 \), \( r > s \geq 0 \), \( m \geq r \), \( l \geq s \), we have

\[
\left[ \frac{\sigma_m(\lambda) / C_n^m}{\sigma_l(\lambda) / C_n^l} \right]^{\frac{1}{m-l}} \leq \left[ \frac{\sigma_r(\lambda) / C_n^r}{\sigma_s(\lambda) / C_n^s} \right]^{\frac{1}{r-s}}.
\]

Proof. See [24].

2.2. Basic notations and some key lemmas. In this paper, \( \nabla \) denotes the Levi-Civita connection on \((M, g)\) and the curvature tensor is defined by

\[
R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.
\]

Let \( \{e_1, e_2, \cdots, e_n\} \) be local frames on \( M \) and denote \( g_{ij} = g(e_i, e_j) \), \( \{g^{ij}\} = \{g_{ij}\}^{-1} \), while the Christoffel symbols \( \Gamma^k_{ij} \) and curvature coefficients are given respectively by

\[
\nabla_{e_i} e_j = \Gamma^k_{ij} e_k \quad \text{and} \quad R_{ijkl} = g(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mijkl}.
\]

We shall write \( \nabla_i = \nabla_{e_i} \), \( \nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k \), etc. For a differentiable function \( u \) defined on \( M \), we usually identify \( \nabla u \) with its gradient, and use \( \nabla^2 u \) to denote its Hessian which is locally given by \( \nabla_{ij} u = \nabla_{i}(\nabla_j u) - \Gamma^k_{ij} \nabla_k u \). We note that \( \nabla_{ij} u = \nabla_{ji} u \) and

\[
\nabla_{ij_k} u - \nabla_{jik} u = R^l_{kij} \nabla_l u,
\]

(2.1)

\[
\nabla_i (\nabla_k u) = \nabla_{ijk} u + \Gamma^l_{ik} \nabla_j u + \Gamma^l_{jk} \nabla_i u + \nabla_{\nabla_{ij} e_k} u,
\]

(2.2)
(2.3) \[ \nabla_{ijkl} u - \nabla_{ikjl} u = R_{ijkm}^l \nabla_{im} u + \nabla_i R_{ljkm}^l \nabla_{m} u, \]

(2.4) \[ \nabla_{ijkl} u - \nabla_{jikl} u = R_{mijl}^k \nabla_{im} u + R_{mljk}^i \nabla_{m} u. \]

From (2.3) and (2.4), we obtain

\[ \nabla_{ijkl} u - \nabla_{klji} u = R_{mijl}^k \nabla_{im} u + \nabla_i R_{ljkm}^l \nabla_{m} u + R_{mlik}^j \nabla_{jm} u + R_{mjik}^l \nabla_{lm} u + R_{mjlk}^i \nabla_{km} u + \nabla_k R_{jilm}^m \nabla_{m} u. \]

(2.5)

For convenience, we introduce the following notations

\[ U := \nabla^2 u + \chi(x, u, \nabla u), \quad U_{ij} := \nabla_{ij} u + \chi^{ij}(x, u, \nabla u), \]
\[ U_{ijkl} := \nabla_{ijkl} u + \chi^{ijkl}(x, u, \nabla u), \]
\[ G_k(U) := \frac{\sigma_k(U)}{\sigma_{k-1}(U)}, \quad G_l(U) := -\frac{\sigma_l(U)}{\sigma_{k-1}(U)}, \quad 0 \leq l \leq k-2, \]
\[ G(U) := G_k(U) + \sum_{l=0}^{k-2} \alpha_l(x) G_l(U), \]
\[ G^{ij} := \frac{\partial G}{\partial U_{ij}}, \quad G^{ij,rs} := \frac{\partial^2 G}{\partial U_{ij} \partial U_{rs}}, \quad \chi^{ij}_{ps} := \frac{\partial \chi^{ij}}{\partial (\nabla_s u)}, \quad 1 \leq i, j, r, s \leq n, \]

and

\[ (2.6) \quad \mathcal{L} := G^{ij} \nabla_{ij} + G^{ij} \chi^{ij}_{ps} \nabla_s. \]

Let \( u \in C^\infty(M) \) be an admissible solution of the equation (1.1). Under orthonormal local frames \( \{e_1, \cdots, e_n\} \), then the equation (1.1) can be rewritten as the following form:

\[ \begin{cases} G(U) := f(\lambda[U]) = \alpha_{k-1}(x), & \text{in } M, \\ u(x) = \varphi(x), & \text{on } \partial M. \end{cases} \]

(2.7)

For simplicity, we shall still write equation (1.1) in the form (2.7) even if \( \{e_1, \cdots, e_n\} \) are not necessarily orthonormal, although more precisely it should be

\[ G([\gamma^{jk} U_{ki} \gamma^{ij}]) = \alpha_{k-1}(x), \]

where \( \gamma^{ij} \) is the square root of \( g^{ij} : \gamma^{ik} \gamma^{kj} = g^{ij} \). Whenever we differentiate the equation, it will make no difference as long as we use covariant derivatives. Under a local frame \( \{e_1, \cdots, e_n\} \), we have

\[ \nabla_k U_{ij} = \nabla_{kij} u + \nabla_k \chi^{ij}(x, u, \nabla u) + \chi^{ij}_{z}(x, u, \nabla u) \nabla_k u + \chi^{ij}_{ps}(x, u, \nabla u) \nabla_{kl} u. \]

(2.8)
In the establishment of the a priori estimates, the following lemma will play an important role. For more details, see [17].

**Lemma 2.2.** If \( U \in C^2(M) \) with \( \lambda(U) \in \Gamma_{k-1} \) and \( \alpha_l(x) > 0 \) with \( 0 \leq l \leq k - 2 \), then the operator \( G(U) \) is elliptic and concave. Moreover

\[
\sum_{i=1}^{n} G^{ii} \geq \frac{n-k+1}{k} + \sum_{l=0}^{k-2} \frac{(n-k+2)(k-l-1)}{k-1} \alpha_l(x) \frac{\sigma_l \sigma_{k-2}}{\sigma_{k-1}^2}
\]

(2.9)

**Proof.** See [8]. \( \square \)

### 3. A PRIORI ESTIMATES

#### 3.1. \( C^1 \) estimates

In order to estimate the gradient of (1.1), we use a method similar to Theorem 4.2 in [13], but there is no sign requirement for the right hand function \( \alpha_{k-1}(x) \).

**Theorem 3.1.** Let \((M, g)\) be a Riemannian manifold with nonnegative sectional curvature, \( 3 \leq k \leq n \), \( \varphi, \alpha_l \) be smooth functions with \( \alpha_l > 0 \) for \( 0 \leq l \leq k - 2 \). Assume that the \((0,2)\)-tensor \( \chi = \chi(x,p) \) satisfies Condition 1.1 and Condition 1.3, there exists an admissible subsolution \( u \in C^2(M) \) satisfying (1.7). Let \( u \in C^\infty(M) \) be an admissible solution for the equation (1.1), then

\[
\max_M |\nabla u| \leq C
\]

for a constant \( C \) depending on \( n, k, l, \|u\|_{C^0}, \|u\|_{C^2}, \|\alpha_{k-1}\|_{C^1}, \|\alpha_l\|_{C^1} \) and \( \inf \alpha_l \) with \( 0 \leq l \leq k - 2 \).

**Proof.** As in [13], in order to derive the \( C^0 \) estimates and \( C^1 \) estimates, we need to restrict \( \chi = \chi(x,p) \). Since \( \chi \) and \( \alpha_{k-1} \) are assumed to be independent of \( u \), by the comparison principle, it is easy to obtain

\[
\max_M |u| + \max_{\partial M} |\nabla u| \leq C.
\]

Hence, we only need to establish the interior gradient estimates. Let \( \omega = |\nabla u| \), \( \psi = (u - u_0) + \sup_M (u - u) + 1 \). Assume that \( \omega \psi^{-\delta} \) achieves a positive maximum at an interior point \( x_0 \in M \) where \( \delta \in (0, \frac{1}{2}) \) is a constant. We may choose the local orthonormal frame \( \{e_1, e_2, \cdots, e_n\} \) about \( x_0 \) such that \( \nabla_i e_j = 0 \) at \( x_0 \) and \( \{U_{ij}(x_0)\} \) is diagonal.
Thus the function $\log \omega - \delta \log \psi$ attains its maximum at $x_0$ for $i = 1, \cdots, n$, hence at $x_0$,

$$\frac{\nabla \omega}{\omega} - \frac{\delta \nabla \psi}{\psi} = 0, \tag{3.1}$$

$$\frac{\nabla_i \omega}{\omega} + (\delta - \delta^2)\frac{\nabla_i \psi}{\psi} - \frac{\delta \nabla_i \psi}{\psi} \leq 0. \tag{3.2}$$

Since $(M, g)$ has nonnegative sectional curvature, in orthonormal local frame,

$$R^k_{\ iil} \nabla_k u \nabla_l u \geq 0.$$

Recall that $\nabla_{ij} u = \nabla_{ji} u$ and $\nabla_{ijk} u = -\nabla_{jik} u = R^l_{\ ijk} \nabla_l u$. Then

$$\omega \nabla_i \omega = \sum_s \nabla_s u \nabla_{iis} u + \sum_s \nabla_s u \nabla_{is} u - \nabla_i \omega \nabla_i \omega$$

$$= \sum_s (\nabla_{sii} u + R^k_{\ sii} \nabla_k u) \nabla_s u + \sum_s (U_{is} - \chi_{is})^2 - (\nabla_i \omega)^2$$

$$\geq \sum_s \nabla_s u \nabla_s U_{ii} - \sum_s \nabla_s u \nabla_s \chi_{ii} - \sum_s \nabla_s u \chi_{pi} \nabla_{sk} u$$

$$+ \frac{1}{2} U_{ii}^2 - \sum_s (\chi_{is})^2 - (\nabla_i \omega)^2. \tag{3.3}$$

Then by (2.6), (3.1)-(3.3) and condition (1.8), we get at $x_0$,

$$0 \geq \frac{G^{ii} \nabla_s u \nabla_s U_{ii}}{\omega^2} - \frac{G^{ii} \nabla_s u \nabla_s \chi_{ii}}{\omega^2} - \frac{G^{ii} \nabla_s u \chi_{pi} \nabla_{sk} u}{\omega^2}$$

$$+ \frac{(\delta - 2\delta^2)|\nabla_i \psi|^2}{\psi^2} - \frac{\delta G^{ii} \nabla_i \psi}{\psi} + \frac{1}{2\omega^2} G^{ii} U_{ii}^2 - \frac{1}{\omega^2} \sum_s G^{ii} (\chi_{is})^2$$

$$\geq \frac{G^{ii} \nabla_s u \nabla_s U_{ii}}{|\nabla u|^2} + \frac{\delta}{\psi} \mathcal{L}(u - u) + CG^{ii} |\nabla_i \psi|^2 + \frac{1}{2|\nabla u|^2} G^{ii} U_{ii}^2$$

$$- C(|\nabla u|^{-2} + |\nabla u|^{-1} - 2 + |\nabla u|^{-2}) \sum_i G^{ii}. \tag{3.4}$$

Next we need to deal with the term $\frac{\nabla u}{|\nabla u|} G^{ii} \nabla_s U_{ii}$, and we can divide into two cases:
Case 1: If there is a positive constant \( N \) such that \( \frac{\sigma_l}{\sigma_{k-1}} \leq N, \ l = 0, \ldots, k - 2, \) then for some positive constant \( C_0, \)

\[
C_0 \sum_i G^{ii} + \frac{\nabla_s u}{|\nabla u|^2} G^{ii} \nabla_s U_{ii} = C_0 \sum_i G^{ii} + \frac{\nabla_s u}{|\nabla u|^2} \left( \nabla_s \alpha_{k-1} - \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l \right)
\]

\[
\geq \frac{C_0(n-k+1)}{k} - C \frac{\nabla_s u}{|\nabla u|^2} \sum_{l=0}^{k-2} \frac{\sigma_l}{\sigma_{k-1}} - C \frac{C}{|\nabla u|}
\]

\[
\geq \frac{C_0(n-k+1)}{k} - C(N + 1) \frac{C}{|\nabla u|} \geq 0,
\]

by choosing \(|\nabla u|\) large enough.

Case 2: If \( \frac{\sigma_k}{\sigma_{k-1}} > N \), then

\[
\frac{\sigma_{k-2}}{\sigma_{k-1}} \geq \left( \frac{\sigma_l}{\sigma_{k-1}} \right)^{\frac{1}{k-1}} \geq N^{\frac{1}{k-1}}.
\]

Hence by (2.9), for some positive constant \( C_0, \)

\[
C_0 \sum_i G^{ii} + \frac{\nabla_s u}{|\nabla u|^2} G^{ii} \nabla_s U_{ii}
\]

\[
\geq \frac{C_0(n-k+1)}{k} + \sum_{l=0}^{k-2} C_0 C(n, k, l) \alpha_l(x) \frac{\sigma_l \sigma_{k-2}}{\sigma_{k-1}^2} + \frac{\nabla_s u}{|\nabla u|^2} \left( \nabla_s \alpha_{k-1} - \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l \right)
\]

\[
\geq \sum_{l=0}^{k-2} \left( C_0 C(n, k, l) \inf \alpha_l(x) N^{\frac{1}{k-1}} - \frac{C}{|\nabla u|} \right) \frac{\sigma_l}{\sigma_{k-1}} + \frac{C_0(n-k+1)}{k} - C \frac{C}{|\nabla u|} \geq 0,
\]

by choosing \(|\nabla u|\) large enough.

So \( \frac{\nabla_s u}{|\nabla u|^2} G^{ii} \nabla_s U_{ii} \geq -C_0 \sum_i G^{ii} \). Let \( \lambda = \lambda(U), \mu = \lambda(U) \) be the eigenvalues of \( U \) and \( \bar{U} \) respectively, and \( \beta \in (0, \frac{1}{2\sqrt{n}}) \) be a uniform constant such that

\[
\nu_\mu - 2\beta \mathbf{1} \in \Gamma_n, \ \forall x \in \overline{M},
\]

where \( \nu_\lambda := \frac{\partial f(\lambda)}{\partial f(\lambda)} \) is the unit normal vector to the level hypersurface \( \partial \Gamma^{f(\lambda)} \) for \( \lambda \in \Gamma \) and \( \mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n, \Gamma \) is a symmetric open and convex cone in \( \mathbb{R}^n \) with \( \Gamma_n \subset \Gamma. \)

First, we consider the case \( |\nu_\mu - \nu_\lambda| \geq \beta, \) by Lemma 2.1 in [13], we have for some uniform constant \( \varepsilon > 0, \)

\[
G^{ii}(\bar{U}_{ii} - U_{ii}) \geq G(U) - G(U) + \varepsilon(1 + \sum_i G^{ii}).
\]
By condition (1.5), we get
\[ \chi_{ii} \nabla_k (\underline{u} - u) \geq \chi_{ii} (x, \nabla \underline{u}) - \chi_{ii} (x, \nabla u), \]
then
\[ \mathcal{L}(\underline{u} - u) = G^{ij} \nabla_{ij} (\underline{u} - u) + G^{ij} \chi_{ij} \nabla_s (\underline{u} - u) \]
\[ \geq G^{ii} (\underline{U}_{ii} - U_{ii}) \geq \varepsilon (1 + \sum_i G^{ii}). \]
Hence by (3.4) and choosing \( C_0 \leq \inf_M \frac{\varepsilon \delta}{2\psi}, \) \( |\nabla u| \) large enough, we derive
\[ 0 \geq \frac{\varepsilon \delta}{\psi} \sum_i G^{ii} + \frac{\varepsilon \delta}{\psi} - (C_0 + C(|\nabla u|^2 + |\nabla u|^\gamma_1 - 2 + |\nabla u|^\gamma_2 - 2)) \sum_i G^{ii} \]
\[ \geq \left( \frac{\varepsilon \delta}{2\psi} - C(|\nabla u|^2 + |\nabla u|^\gamma_1 - 2 + |\nabla u|^\gamma_2 - 2) \right) \sum_i G^{ii} \]
\[ \geq \left( C_0 - C(|\nabla u|^2 + |\nabla u|^\gamma_1 - 2 + |\nabla u|^\gamma_2 - 2) \right) \sum_i G^{ii}, \]
which implies \( |\nabla u(x_0)| \leq C. \)

Next we consider the case \( |\nu_\mu - \nu_\lambda| < \beta, \) then we have \( G^{ii} \geq \frac{\beta}{\sqrt{n}} \sum_k G^{kk} \) for \( 1 \leq i \leq n \) and \( \mathcal{L}(\underline{u} - u) \geq 0. \) Hence by (3.4) and choosing \( |\nabla u| \) large enough, we have
\[ C|\nabla u|^4 \sum_i G^{ii} \leq (C_0 |\nabla u|^2 + C(1 + |\nabla u|^\gamma_1 + |\nabla u|^\gamma_2)) \sum_i G^{ii}. \]
Hence we derive \( |\nabla u(x_0)| \leq C \) and the proof is completed. \( \Box \)

3.2. Interior \( C^2 \) estimates. In this section, we derive the following interior \( C^2 \) estimates. The treatment of this section follows from [13].

**Theorem 3.2.** Let \( u \in C^4(M) \cap C^2(\overline{M}) \) be an admissible solution of the equation (1.1), \( \varphi \in C^\infty(\overline{M}), \) suppose Condition 1.4 holds and there exists an admissible sub-solution \( \underline{u} \in C^2(M) \) satisfying (1.7), then
\[ \max_M |\nabla^2 u| \leq C (1 + \max_{\partial M} |\nabla^2 u|), \]
where \( C \) is a constant depending on \( n, k, l, \|u\|_{C^1}, \|u\|_{C^2}, \|\chi^{ij}\|_{C^2}, \|\alpha_{k-1}\|_{C^2}, \|\alpha_1\|_{C^2} \) and \( \inf \alpha_l \) with \( 0 \leq l \leq k - 2. \)

**Proof.** Consider the auxiliary function
\[ W(x) = \max_{\xi \in T_{2,M},|\xi|=1} (\nabla_{\xi} u + \chi^{i\xi}(x, u, \nabla u)) e^\phi, \]
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where \( \phi = \frac{a}{2} |\nabla u|^2 + b(u - u) \) and \( a \in (0, 1) \), \( b \) are constants to be determined later. Assume that \( W(x) \) attains its maximum at an interior point \( x_0 \in M \), otherwise we are done. Choose a smooth orthonormal local frame \( \{e_1, e_2, \ldots, e_n\} \) about \( x_0 \) such that \( \nabla e_i e_j = 0 \) and \( U_{ij} = \nabla_{ij} u + \chi^{ij}(x, u, \nabla u) \) is diagonal. Suppose

\[
U_{11}(x_0) \geq \cdots \geq U_{nn}(x_0),
\]

so \( W(x_0) = U_{11}(x_0)e^{\phi(x_0)} \). We define a new function \( \tilde{W} = \log U_{11} + \phi \). Then at \( x_0 \),

\[
(3.5) \quad 0 = \tilde{W}_i = \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi,
\]

\[
(3.6) \quad 0 \geq \tilde{W}_{ii} = \frac{U_{11} \nabla_{ii} U_{11} - (\nabla_i U_{11})^2}{U_{11}^2} + \nabla_{ii} \phi.
\]

By (2.1) and (2.5),

\[
(\nabla_i U_{11})^2 \leq (\nabla_i U_{1i})^2 + C U_{11}^2,
\]

\[
(3.7) \quad \nabla_{ii} U_{11} \geq \nabla_{11} U_{ii} - \nabla_{ii} \chi_{11} + \nabla_{ii} \chi_{11} - C U_{11}.
\]

Differentiating the equation (1.1) twice, then we get

\[
(3.8) \quad G^{ij} \nabla_{11} U_{ij} + \sum_{l=0}^{k-2} \nabla_1 \alpha_l G_{1l} = \nabla_1 \alpha_{k-1},
\]

and

\[
(3.9) \quad G^{ii} \nabla_{ii} U_{11} + G^{ij,rs}_{1l} \nabla_{11} U_{ij} \nabla_{11} U_{rs} + \sum_{l=0}^{k-2} \nabla_{11} \alpha_l G_{1l} + 2 \sum_{l=0}^{k-2} \nabla_1 \alpha_l G^{ij}_{1l} \nabla_{11} U_{ij} + \sum_{l=0}^{k-2} \alpha_l G^{ij,rs}_{1l} \nabla_1 U_{ij} \nabla_1 U_{rs} = \nabla_{11} \alpha_{k-1}.
\]

By (3.5) and (3.8),

\[
G^{ii} (\nabla_{ii} \chi_{11} - \nabla_{11} \chi_{ii}) \geq U_{11} G^{ii} \chi_{11} \nabla_{ii} \phi - C U_{11} \sum_i G^{ii} - C \sum_{i \geq 2} G^{ii} U_{11}^2
\]

\[
- \sum_{l=0}^{k-2} \chi_{11} \nabla_{ii} \alpha_l G_{1l} - U_{11}^2 \sum_{i \geq 2} G^{ii} \chi_{p_1p_1}.
\]

(3.10)
Therefore combined with (3.6), (3.7), (3.9), (3.10) and the definition of $L$, 

\[ 0 \geq G^{ii} \nabla_{ii} \phi + \frac{1}{U_{11}} G^{ii} \nabla_{ii} U_{11} - \frac{1}{U_{11}^2} G^{ii}(\nabla_i U_{11})^2 \]

\[ \geq L \phi + \frac{1}{U_{11}} G^{ii} \nabla_{11} U_{ii} - C \sum_i G^{ii} - \frac{C}{U_{11}} \sum_{i \geq 2} G^{ii} U_{ii}^2 \]

\[ - \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{p_s}^{11} \nabla_{s} \alpha_l G_l - U_{11} \sum_{i \geq 2} G^{ii} \chi_{p_{i1}}^{ii} - \frac{1}{U_{11}^2} G^{ii}(\nabla_i U_{11})^2 \]

\[ = L \phi - \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{p_s}^{11} \nabla_{s} \alpha_l G_l - C \sum_i G^{ii} - \frac{C}{U_{11}} \sum_{i \geq 2} G^{ii} U_{ii}^2 \]

\[ + \frac{1}{U_{11}} [\nabla_{11} \alpha_{k-1} - G^{ij,rs}_{G_l} \nabla_i U_{ij} \nabla_1 U_{rs} - \sum_{l=0}^{k-2} \nabla_{11} \alpha_l G_l - 2 \sum_{l=0}^{k-2} \nabla_1 \alpha_l G^{ij}_{G_l} \nabla_1 U_{ij} \]

\[ (3.11) - \sum_{l=0}^{k-2} \alpha_l G^{ij,rs}_{G_l} \nabla_i U_{ij} \nabla_1 U_{rs}] - U_{11} \sum_{i \geq 2} G^{ii} \chi_{p_{i1}}^{ii} - \frac{1}{U_{11}^2} G^{ii}(\nabla_i U_{11})^2. \]

Since $(-\frac{1}{G_l})$ is a concave operator for $l = 0, \ldots, k - 2$, we obtain

(3.12)

\[ - \frac{1}{2} \sum_{l=0}^{k-2} \alpha_l G^{ij,rs}_{G_l} \nabla_i U_{ij} \nabla_1 U_{rs} - 2 \sum_{l=0}^{k-2} \nabla_1 \alpha_l G^{ii}_{G_l} \nabla_1 U_{ii} \]

\[ \geq - \sum_{l=0}^{k-2} \frac{\alpha_l}{2} (1 + \frac{1}{k - 1 - l}) G^{-1}_{G_l} G^{ij}_{G_l} G^{rs}_{G_l} \nabla_i U_{ij} \nabla_1 U_{rs} - 2 \sum_{l=0}^{k-2} \nabla_1 \alpha_l G^{ii}_{G_l} \nabla_1 U_{ii} \]

\[ = - \sum_{l=0}^{k-2} \frac{k - l}{2(k - 1 - l)} \alpha_l G^{-1}_{G_l} [G^{ij}_{G_l} \nabla_i U_{ij} + \frac{2(k - l - 1) \nabla_1 \alpha_l G_l}{(k - l) \alpha_l} G_l]^2 + \sum_{l=0}^{k-2} \frac{2(k - l - 1) (\nabla_1 \alpha_l)^2}{k - l} \alpha_l G_l \]

\[ \geq \sum_{l=0}^{k-2} \frac{2(k - l - 1) (\nabla_1 \alpha_l)^2}{k - l} \alpha_l G_l. \]
Hence by (1.5), (3.11) and (3.12),

\[
\mathcal{L}\phi \leq \frac{C}{U_{11}} \sum_i G^{ii} U_{ii}^2 + C \sum_i G^{ii} + \frac{C}{U_{11}} + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{pl}^{11} \alpha_l G_l + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \frac{2(k - l - 1)}{k - l} \frac{(\nabla_1 \alpha_l)^2}{\alpha_l} G_l - \sum_{l=0}^{k-2} \nabla_{11} \alpha_l G_l
\]

(3.13) \quad -G^{ij,rs}_k \nabla_1 \alpha_i \nabla_1 \alpha_j \nabla_1 \alpha_k \nabla_1 \alpha_l.

In order to estimate the third derivative term, we follow the idea of [27]. Let \( J = \{ i : 3U_{ii} \leq -U_{11} \} \), as the estimates in [11] and [13], we have

\[
\frac{1}{U_{11}} \alpha_i G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{\alpha_i G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs}}{U_{11}^2} \leq \frac{1}{U_{11}} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{1}{U_{11}^2} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \sum_{i \in J} \alpha_i \sum_{i \in J} G^{ii}_l,
\]

and similarly

\[
\frac{\alpha_i}{2U_{11}} G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{\alpha_i G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs}}{U_{11}^2} \leq \frac{\alpha_i}{U_{11}^2} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{\alpha_i}{U_{11}^2} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \sum_{i \in J} \alpha_i \sum_{i \in J} G^{ii}_l.
\]

Then combined with (3.5), we get

\[
\frac{G^{ii}_l \nabla_1 \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \alpha_i G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs}}{U_{11}^2} \leq \frac{1}{U_{11}} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{1}{2U_{11}} \sum_{l=0}^{k-2} \alpha_i G^{ij,rs}_l \nabla_1 \alpha_i \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs}
\]

(3.14) \quad \leq \frac{1}{U_{11}} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{1}{U_{11}} \sum_{i \in J} G^{11}_l \nabla_1 \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \frac{1}{U_{11}^2} \sum_{i \in J} \alpha_i G^{ij,rs}_l \nabla_1 \nabla_1 \nabla_{ij,rs} U_{ij,rs} + \sum_{i \in J} \alpha_i \sum_{i \in J} G^{ii}_l
\]

Recall that \( \phi = \frac{a}{2} |\nabla u|^2 + b(u - u) \), then

\[
\mathcal{L}\phi = G^{ij} \nabla_{ij} \phi + G^{ij} \chi_{lip} \nabla_s \phi
\]

(3.15) \quad \geq \frac{a}{2} G^{ii}_l U_{ii}^2 - Ca \sum_i G^{ii}_l + a G^{ii}_l \nabla_s u \nabla_{si} u + b G^{ii}_l \nabla_i (u - u)
\]

\[+a G^{ij} \nabla_{ps} \chi^{ij} \nabla_s u U_{ss} - a G^{ii}_l \nabla_s u \chi^{si} + b G^{ij} \chi^{ij} \nabla_s (u - u)
\]

\[b \mathcal{L}(u - u) + \frac{a}{2} G^{ii}_l U_{ii}^2 - (Ca + C) \sum_i G^{ii}_l + C - a \nabla_s u \sum_{l=0}^{k-2} \nabla_{s} \alpha_l G_l.
\]
By (3.13)-(3.15), we obtain

\[
 b\mathcal{L}(u - u) \leq \left(\frac{C}{U_1} + Ca^2 - \frac{a}{2}\right) \sum_i G_{ii} U_{1i}^2 + Cb^2 \sum_{i \in J} G_{ii} \\
+ CG^{11}(a^2 U_{11}^2 + b^2) + (Ca + C) \sum_i G_{ii} + C \\
+ \frac{C}{U_{11}} + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{l,s}^{11} \alpha_l G_l + a \nabla_s u \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l \\
- \frac{1}{U_{11}} \left[\sum_{l=0}^{k-2} \frac{2(k - l - 1) (\nabla_1 \alpha_l)^2}{k - l} G_l - \sum_{l=0}^{k-2} \nabla_1 \alpha_l G_l\right].
\]

(3.16)

Then we will study the term

\[
\frac{C}{U_{11}} + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{l,s}^{11} \alpha_l G_l + a \nabla_s u \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l - \frac{1}{U_{11}} \left[\sum_{l=0}^{k-2} \frac{2(k - l - 1) (\nabla_1 \alpha_l)^2}{k - l} G_l - \sum_{l=0}^{k-2} \nabla_1 \alpha_l G_l\right].
\]

When \(\frac{\alpha_l}{\sigma_{k-1}} \leq N, l = 0, \cdots, k - 2,\)

\[
\sum_i G_{ii} + \frac{1}{U_{11}} \left[\sum_{l=0}^{k-2} \frac{2(k - l - 1) (\nabla_1 \alpha_l)^2}{k - l} G_l - \sum_{l=0}^{k-2} \nabla_1 \alpha_l G_l\right] \\
- \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{l,s}^{11} \alpha_l G_l - \frac{C}{U_{11}} - a \nabla_s u \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l \\
\geq \frac{n - k + 1}{k} - \left(\frac{1}{U_{11} \inf \alpha_l} + a\right) CN \geq 0,
\]

(3.17)

by choosing \(U_{11}\) large enough and \(a\) small enough.
When \( \frac{a}{\sigma_{k-1}} > N \), combined with (2.9), we have

\[
(3.18) \sum_i G^{ii} + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \frac{2(k-l-1)}{k-l} \frac{(\nabla_i \alpha_l)^2}{\alpha_l} G_l - \sum_{l=0}^{k-2} \nabla_{11} \alpha_l G_l
\]

\[
- \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{p_s}^{11} \nabla_s \alpha_l G_l - \frac{C}{U_{11}} - a \nabla_s u \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l
\]

\[
\geq \frac{n-k+1}{k} + \frac{1}{U_{11}} \sum_{l=0}^{k-2} \frac{2(k-l-1)}{k-l} \frac{(\nabla_i \alpha_l)^2}{\alpha_l} G_l - \sum_{l=0}^{k-2} \nabla_{11} \alpha_l G_l
\]

\[
+ \sum_{l=0}^{k-2} C(n, k, l) \alpha_l(x) \frac{\sigma_l \sigma_{k-2}}{\sigma_{k-1}} - \frac{1}{U_{11}} \sum_{l=0}^{k-2} \chi_{p_s}^{11} \nabla_s \alpha_l G_l - \frac{C}{U_{11}} - a \nabla_s u \sum_{l=0}^{k-2} \nabla_s \alpha_l G_l
\]

\[
\geq \sum_{l=0}^{k-2} [C(n, k, l) \inf \alpha_l(x) N^{-k-1-t} - (\frac{1}{U_{11}} \inf \alpha_l + a) C] \frac{\sigma_l}{\sigma_{k-1}} + \frac{n-k+1}{k} - \frac{C}{U_{11}} \geq 0,
\]

by choosing \( U_{11} \) large enough and \( a \) small enough.

Inserting (3.17) and (3.18) into (3.16), then

\[
b L (x - u) \leq (\frac{C}{U_{11}} + Ca^2 - \frac{a}{2}) \sum_i G^{ii} U_{ii}^2 + C b^2 \sum_{i \in J} G^{ii}
\]

\[
(3.19) + CG^{11}(a^2 U_{11}^2 + b^2) + (Ca + C + 1) \sum_i G^{ii} + C.
\]

In order to deal with (3.19), we can also consider the two cases: \( |\nu_\mu - \nu_\lambda| \geq \beta \) and \( |\nu_\mu - \nu_\lambda| < \beta \).

When \( |\nu_\mu - \nu_\lambda| \geq \beta \), by Condition 1.1, we have

\[
\chi_{p_s}^{11} \nabla_k (x - u) \geq \chi^{11}(x, u, \nabla u) - \chi^{11}(x, u, \nabla u)
\]

\[
\geq \chi^{11}(x, u, \nabla u) - \chi^{11}(x, u, \nabla u),
\]

then

\[
L (x - u) \geq G^{ii} (U_{ii} - U_{ii}) \geq \varepsilon (1 + \sum_i G^{ii}).
\]

By (3.19), when \( b \) large enough, we can obtain

\[
(\frac{C}{U_{11}} + Ca^2 - \frac{a}{2}) \sum_i G^{ii} U_{ii}^2 + C b^2 \sum_{i \in J} G^{ii} + CG^{11}(a^2 U_{11}^2 + b^2) \geq 0,
\]

which implies \( U_{11}(x_0) \leq C \). Otherwise the first term will be negative when \( a \) small enough, and \( |U_{ii}| \geq \frac{1}{3} U_{11} \) for \( i \in J \).
When $|\nu_{\mu} - \nu_{\lambda}| < \beta$, then $\nu_{\lambda} - \beta 1 \in \Gamma_n$ and therefore
\[
G^{ii} \geq \frac{\beta}{\sqrt{n}} \sum_k G^{kk}, \quad \forall \ 1 \leq i \leq n.
\]
By (1.5)-(1.7) and concavity of the operator $G$, we get $L(u - u) \geq 0$. Denote $|\lambda|^2 = \sum_i \lambda_i^2 = \sum_i U_{ii}^2$, by (3.19) we derive
\[
(3.20) \quad \frac{\beta}{\sqrt{n}} |\lambda|^2 \sum_i G^{ii} \leq \sum_i G^{ii} U_{ii}^2 \leq C(1 + \sum_i G^{ii}),
\]
by choosing $U_{11}$ large enough and $a$ small enough.

Combined with (2.9) and (3.20), we have
\[
\frac{\beta}{\sqrt{n}} |\lambda|^2 \sum_i G^{ii} \leq \frac{C(n + 1)}{n - k + 1} \sum_i G^{ii},
\]
which implies $|\lambda| \leq C$ and the proof is completed. \(\square\)

3.3. **Second order derivatives boundary estimates.** For any fixed $x_0 \in \partial M$, we can choose smooth orthonormal local frames $e_1, \ldots, e_n$ around $x_0$ such that when restricted on $\partial M$, $e_n$ is normal to $\partial M$. For $x \in \overline{M}$, let $\rho(x)$ and $d(x)$ denote the distances from $x$ to $x_0$ and $\partial M$ respectively, and set $M_\delta = \{x \in M : \rho(x) < \delta\}$. We may assume $\rho$ and $d$ are smooth in $M_\delta$ by taking $\delta$ small. Then we get the following important lemma, which plays a key role in our boundary estimates.

**Lemma 3.3.** Let $v = u - u + \tau d - \frac{N}{2} d^2$, $L$ is defined as in (2.6), then for a positive constant $\epsilon$, there exist some uniform positive constants $t, \delta$ sufficiently small and $N$ sufficiently large such that
\[
(3.21) \quad \begin{cases} 
L v \leq -\epsilon (1 + \sum_{i=1}^n G^{ii}), & \text{in } M_\delta, \\
v \geq 0, & \text{on } \partial M_\delta.
\end{cases}
\]

**Proof.** The proof is similar to lemma 4.3 in [17]. Although the operator $L$ is more complex than $G^{ij}$, it will make no difference since the extra term can be controlled. We can also consider $|\nu_{\mu} - \nu_{\lambda}| \geq \beta$ and $|\nu_{\mu} - \nu_{\lambda}| < \beta$ the two cases to derive (3.21) by choosing $N$ large enough and $t, \delta$ small enough. \(\square\)

**Proof of Theorem 1.2.** By Theorem 3.2, we only need to derive boundary estimates.

**Case 1:** Estimates of $\nabla_{\alpha\beta} u, \alpha, \beta = 1, \ldots, n - 1$ on $\partial M$. 
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Since \( u - \underline{u} = 0 \) on \( \partial M \), therefore,
\[
\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n (u - \underline{u}) B_{\alpha\beta}, \quad \text{on } \partial M,
\]
where \( B_{\alpha\beta} = \langle \nabla_{\alpha} e_{\beta}, e_n \rangle \) denotes the second fundamental form of \( \partial M \). Therefore,
\begin{equation}
(3.22)
|\nabla_{\alpha\beta} u| \leq C, \quad \text{on } \partial M.
\end{equation}

**Case 2**: Estimates of \( \nabla_{\alpha n} u \), \( \alpha = 1, \cdots, n-1 \) on \( \partial M \).

Consider the following barrier function
\begin{equation}
(3.23)
\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{\beta < n} |\nabla_{\beta}(u - \varphi)|^2.
\end{equation}

Combined with lemma 3.3, we claim that
\begin{equation}
(3.24)
\begin{cases}
\mathcal{L}(\Psi \pm \nabla_{\alpha}(u - \varphi)) \leq 0, & \text{in } M_\delta, \\
\Psi \pm \nabla_{\alpha}(u - \varphi) \geq 0, & \text{on } \partial M_\delta,
\end{cases}
\end{equation}

for suitable chosen positive constants \( A_1, A_2, A_3 \) and \( \mathcal{L}, v \) are defined in lemma 3.3.

By (2.2), (2.8) and (3.8), we get
\[
G^{ij} \nabla_{ij}(\nabla_{\beta}(u - \varphi)) \leq -k - 2 \sum_{l=0}^{k-2} \alpha_l G_{ii} + C \sum_{i} G^{ii} |\lambda_i| + C \sum_{i} G^{ii} + C
\]
\[
\leq C \sum_{l=0}^{k-2} \frac{\sigma_l}{\sigma_{k-1}} + C(1 + \sum_{i} G^{ii} + \sum_{i} G^{ii} |\lambda_i|),
\]
where \( \lambda = (\lambda_1, \cdots, \lambda_n) \) are the eigenvalues of \( \{U_{ij}\} \). Since
\[
\sum_{i} G^{ii} |\lambda_i| \geq \sum_{i} \left[ \nabla_{ii} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) U_{ii} - \sum_{l=0}^{k-2} \alpha_l \nabla_{ii} \left( \frac{\sigma_l}{\sigma_{k-1}} \right) U_{ii} \right]
\]
\[
\geq \alpha_{k-1} + \varepsilon_0 \sum_{l=0}^{k-2} \frac{\sigma_l}{\sigma_{k-1}},
\]
where \( \varepsilon_0 \) is a positive constant depending on \( \inf \alpha_l \) for \( 0 \leq l \leq k - 2 \). We obtain
\[
G^{ij} \nabla_{ij}(\nabla_{\beta}(u - \varphi)) \leq C(1 + \sum_{i} G^{ii} + \sum_{i} G^{ii} |\lambda_i|), \quad \text{in } M_\delta.
\]

Then the key is to derive
\[
\mathcal{L}\Psi \leq -K(1 + \sum_{i} G^{ii} + \sum_{i} G^{ii} |\lambda_i|), \quad \text{in } M_\delta
\]
for any positive constant \( K \) and it is same as lemma 5.2 in [12]. We can choose \( A_1 \gg A_2 \gg A_3 \gg 1 \) to get (3.24), more details see [12] and [13].
By the maximum principle, we have
\[ \Psi \pm \nabla_\alpha (u - \varphi) \geq 0, \quad \text{in } M_{\delta}, \]
and therefore
\[ |\nabla_n \alpha u| \leq \nabla_n \Psi + |\nabla_n \alpha \varphi| \leq C, \quad \text{on } \partial M. \tag{3.25} \]

**Case 3**: Estimates of \( \nabla_n u \) on \( \partial M \).

We only need to show the uniform upper bound
\[ \nabla_n u(x) \leq C, \quad \forall x \in \partial M, \]
since \( \Gamma_{k-1} \subset \Gamma_1 \) and the lower bound for \( \nabla_n u \) follows from the estimates of \( \nabla_{\alpha\beta} u \) and \( \nabla_{\alpha n} u \).

According to the main idea in [11], which was originally due to Trudinger [25], we will show that there are uniform constants \( c_0, R_0 \) such that \( (\lambda'(U_{\alpha\beta}(x)), R) \in \Gamma_{k-1} \) and
\[ G(\lambda'(U_{\alpha\beta}(x)), R) \geq \alpha_{k-1}(x) + c_0, \tag{3.26} \]
for all \( R > R_0 \) and \( x \in \partial M \). Here \( \lambda'(U_{\alpha\beta}) = (\lambda'_1, \ldots, \lambda'_{n-1}) \) denotes the eigenvalues of the \((n-1) \times (n-1)\) matrix \( \{U_{\alpha\beta}\}(1 \leq \alpha, \beta \leq n-1) \). Suppose that we have found such \( c_0 \) and \( R_0 \), by Lemma 1.2 in [2], it follows from estimates (3.22) and (3.25) that we can find \( R_1 \geq R_0 \) such that, if \( U_{nn}(x) > R_1 \), then
\[ G(U(x)) \geq G(\lambda'(U_{\alpha\beta}(x)), U_{nn}(x)) - \frac{c_0}{2} \geq \alpha_{k-1}(x) + \frac{c_0}{2}, \]
which contradicts to \( G(U(x)) = \alpha_{k-1}(x) \). Thus \( U_{nn}(x) \leq R_1 \).

In order to obtain the claim (3.26), we only need to show that
\[ \bar{m} := \min_{x \in \partial M} \left( \lim_{R \to +\infty} G(\lambda'(U_{\alpha\beta}(x)), R) - \alpha_{k-1}(x) \right) \geq c_0. \]

Define
\[ \bar{c} = \min_{x \in \partial M} \left( \lim_{R \to +\infty} G(\lambda'(U_{\alpha\beta}(x)), R) - G(U(x)) \right) > 0, \]
and
\[ \bar{F}[r_{\alpha\beta}] = \lim_{R \to +\infty} G(\lambda'([r_{\alpha\beta}]), R), \]
for a symmetric \((n-1) \times (n-1)\) matrix \([r_{\alpha\beta}]\) with \((\lambda'([r_{\alpha\beta}]), R) \in \Gamma_{k-1} \).
Suppose that \( \tilde{m} \) is achieved at a point \( x_0 \in \partial M \). Choose a local orthonormal frame around \( x_0 \) such that \( U_{\alpha\beta}(x_0)(1 \leq \alpha, \beta \leq n - 1) \) is diagonal, \( e_n \) is normal to \( \partial M \) and assume \( \nabla_{nn}u(x_0) \geq \nabla_{nn}u(x_0) \). Using the concavity, we know that
\[
\tilde{F}_0^{\alpha\beta}(u_{\alpha\beta}(x) - U_{\alpha\beta}(x_0)) \geq \tilde{F}[U_{\alpha\beta}(x)] - \tilde{F}[U_{\alpha\beta}(x_0)] - \sum_{i=0}^{k-2} (\alpha_{k-1}(x) - \alpha_{k-1}(x_0)) \frac{\sigma_{i-1}}{\sigma_{k-2}} (U_{\alpha\beta}(x)),
\]
where \( \tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial u_{\alpha\beta}}(U_{\alpha\beta}(x_0)) \). In particular, this implies,
\[
\tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x) - \alpha_{k-1}(x) - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \alpha_{k-1}(x_0) \geq C \text{dist}(x, x_0), \quad \text{on } \partial M,
\]
for a constant \( C \) depending on \( \|\alpha_i\|_{C^1} \) and \( \|U_{\alpha\beta}\|_{L^\infty} \). Hence
\[
U_{\alpha\beta} = U_{\alpha\beta} - \nabla_n(u - u)B_{\alpha\beta} + \chi^{\alpha\beta}[u] - \chi^{\alpha\beta}[\tilde{u}], \quad \text{on } \partial M,
\]
where \( \chi[u] := \chi(x, u, \nabla u) \) and \( \chi[\tilde{u}] := \chi(x, u, \nabla u) \). Without loss of generality, we assume \( \tilde{m} < \tilde{c}/2 \). By (3.27), we have at \( x_0 \),
\[
\nabla_n(u - u)(x_0)\tilde{F}_0^{\alpha\beta}B_{\alpha\beta}(x_0) \geq \frac{\tilde{c}}{2} + H[u(x_0)] - H[u(x_0)],
\]
where \( H[u] = \tilde{F}_0^{\alpha\beta}\chi^{\alpha\beta}[u] - \alpha_{k-1}(x) \). Define
\[
\Phi = -\eta\nabla_n(u - u) + H[u] + Q,
\]
with \( \eta = \tilde{F}_0^{\alpha\beta}B_{\alpha\beta}(x) \) and
\[
Q = \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x) - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \alpha_{k-1}(x_0) + C \text{dist}(x, x_0).
\]
From (3.27) and (3.28) we see that \( \Phi(x_0) = 0 \) and \( \Phi \geq 0 \) on \( \partial M \) near \( x_0 \). Note that
\[
|\mathcal{L}\nabla_k(u - u)| \leq C(1 + \sum_i G^{ii} + \sum_i G^{ii} |\lambda_i|),
\]
and by (1.5), we have
\[
\mathcal{L}H \leq H\sum [u]\mathcal{L}u + H\sum [u]\mathcal{L}\nabla_ku + G^{ij}H_{ij}P_i[u]\nabla_{ki}u\nabla_{ij}u + C(1 + \sum_i G^{ii} + \sum_i G^{ii} |\lambda_i|)
\]
\[
\leq C(1 + \sum_i G^{ii} + \sum_i G^{ii} |\lambda_i|).
\]
Therefore,
\[
\mathcal{L}\Phi \leq C(1 + \sum_i G^{ii} + \sum_i G^{ii} |\lambda_i|).
\]
Consider the function $\Psi$ defined in (3.23), then for $A_1 \gg A_2 \gg A_3 \gg 1$
\[
\left\{
\begin{array}{l}
\mathcal{L}(\Psi + \Phi) \leq 0, \text{ in } M_\delta, \\
\Psi + \Phi \geq 0, \text{ on } \partial M_\delta.
\end{array}
\right.
\]
By the maximum principle, $\Psi + \Phi \geq 0$ in $M_\delta$. Thus
\[
\nabla_n \Phi(x_0) \geq -\nabla_n \Psi(x_0) \geq -C.
\]
Let $u' = tu + (1 - t)\bar{u}$, then we have
\[
H[u] - H[\bar{u}] = (u - \bar{u}) \int_0^1 H_z[u']dt + \sum_k \nabla_k (u - \bar{u}) \int_0^1 H_{pk}[u']dt.
\]
Therefore,
\[
(3.30) \quad H[u](x_0) - H[\bar{u}](x_0) = \nabla_n (u - \bar{u})(x_0) \int_0^1 H_{pn}[u'](x_0)dt,
\]
and
\[
\nabla_n H[u](x_0) \leq \nabla_{nn} (u - \bar{u})(x_0) \int_0^1 H_{pn}[u'](x_0)dt + C,
\]
since $H_{pn,nn} \leq 0$, $\nabla_{nn} (u - \bar{u})(x_0) \geq 0$ and $\nabla_n (u - \bar{u})(x_0) \geq 0$. It follows that
\[
\nabla_n \Phi(x_0) \leq -\eta(x_0) \nabla_{nn} (u - \bar{u})(x_0) + \nabla_n H[u](x_0) + C
\]
\[
\leq -\eta(x_0) + \int_0^1 H_{pn}[u'](x_0)dt \nabla_{nn} u(x_0) + C.
\]
By (3.29) and (3.30),
\[
\eta(x_0) - \int_0^1 H_{pn}[u'](x_0)dt \geq \frac{\tilde{c}}{2\nabla_n (u - \bar{u})(x_0)} \geq \epsilon_1 \tilde{c} > 0
\]
for some uniform constant $\epsilon_1 > 0$. This gives
\[
\nabla_{nn} u(x_0) \leq \frac{C}{\epsilon_1 \tilde{c}}.
\]
Combined with (3.32) and (3.25) we know all eigenvalues of $U(x_0)$ have a priori bound, which implies that eigenvalues of $U(x_0)$ are contained in $\Gamma_{k-1} \cap M_\delta$. On the other hand, if the eigenvalues can not touch $\partial \Gamma_{k-1}$, then for $R > 0$ large enough,
\[
\tilde{m} = G(\lambda(U_{\alpha\beta}(x_0)), R) - \alpha_{k-1}(x_0) > 0.
\]
So we need to show that $\lambda(U(x_0))$ can not touch $\partial \Gamma_{k-1}$. Recall our equation
\[
G(U) = \frac{\sigma_k(U)}{\sigma_{k-1}(U)} - \sum_{l=0}^{k-2} \alpha_l(x) \frac{\sigma_l(U)}{\sigma_{k-1}(U)} = \alpha_{k-1}(x).
\]
For \( \lambda \in \Gamma_{k-1} \), we have \( \sigma_k(\lambda) \sigma_{k-2}(\lambda) \leq c(n,k) \sigma_{k-1}^2(\lambda) \), which implies

\[
\frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} \leq c(n,k) \frac{\sigma_{k-1}(\lambda)}{\sigma_{k-2}(\lambda)} \leq \bar{c}(n,k) \frac{1}{\sigma_{k-1}(\lambda)}.
\]

Then,

\[
(3.31) \quad \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} \leq 0, \quad \text{as} \quad \lambda \to \partial \Gamma_{k-1}.
\]

By the non-degeneracy assumption \( (\alpha_l(x_0) > 0, 0 \leq l \leq k - 2) \), if \( \lambda(U(x_0)) \to \partial \Gamma_{k-1} \), \( G(U(x_0)) \to -\infty \). This contradicts with the condition that \( \alpha_{k-1} \in C^2(M) \).

\[ \square \]

### 4. The Dirichlet problem

We now turn to the existence of solutions for the Dirichlet problem (1.1). We consider the special case \( \chi = \chi(x,p) \).

**Proof of Theorem 1.4.** By Theorem 3.1 and 3.2 we obtain

\[
(4.1) \quad \|u\|_{C^2(M)} \leq C.
\]

From (3.31), we see that the equation (1.1) becomes uniformly elliptic for admissible solutions satisfying (4.1). Applying Evans-Krylov theorem and Schauder theory, we can obtain the \( C^{2,\alpha} \) and higher order estimates for the admissible solutions of the equation (1.1). Theorem 1.4 may be proved by using the standard continuity method.

\[ \square \]

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