Combinatorialization of spaces of nondegenerate spherical curves

Victor Goulart  
goulart@mat.puc-rio.br  
Nicolau C. Saldanha  
saldanha@puc-rio.br

Mathematics Department  
PUC-Rio, Brazil

October 23, 2018

Abstract

A parametric curve \(\gamma\) of class \(C^n\) on the \(n\)-sphere is said to be nondegenerate (or locally convex) when \(\det\left(\gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t)\right) > 0\) for all values of the parameter \(t\). We orthogonalize this ordered basis to obtain the Frenet frame \(F_{\gamma}\) of \(\gamma\) assuming values in the orthogonal group \(SO_{n+1}\) (or its universal double cover, \(Spin_{n+1}\)), which we decompose into Schubert or Bruhat cells. To each nondegenerate curve \(\gamma\) we assign its itinerary: a word \(w\) in the alphabet \(S_{n+1}\setminus\{e\}\) that encodes the succession of non open Schubert cells pierced by the complete flag of \(\mathbb{R}^{n+1}\) spanned by the columns of \(F_{\gamma}\). Without loss of generality, we can focus on nondegenerate curves with initial and final flags both fixed at the (non oriented) standard complete flag. For such curves, given a word \(w\), the subspace of curves following the itinerary \(w\) is a contractible globally collared topological submanifold of finite codimension. By a construction reminiscent of Poincaré duality, we define abstract cell complexes mapped into the original space of curves by weak homotopy equivalences. The gluing instructions come from a partial order in the set of words. The main aim of this construction is to attempt to determine the homotopy type of spaces of nondegenerate curves for \(n > 2\). The reader may want to contrast the present paper’s combinatorial approach with the geometry-flavoured methods of previous works.

1 Introduction

For a fixed positive integer \(n \geq 2\), consider the \(n\)-sphere \(S^n \subset \mathbb{R}^{n+1}\) with respect to the usual Euclidean metric of \(\mathbb{R}^{n+1}\). A parametric curve \(\gamma : [0, 1] \to S^n\) of class \(C^n\) is said to be nondegenerate \([32, 33, 34, 40, 59, 65]\) or locally convex \([2, 51, 52, 53, 54]\) if and only if its derivatives up to \(n^{th}\) order \(\gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t)\) span a complete flag \(\langle \gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t) \rangle\) of \(\mathbb{R}^{n+1}\) for all \(t \in [0, 1]\). Without loss of generality, we shall consider only positive nondegenerate curves, i.e., those satisfying \(\det(\gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t)) > 0\).

Consider the Frenet frame of a nondegenerate curve \(\gamma : [0, 1] \to S^n\) at time \(t \in [0, 1]\) to be the orthogonal matrix \(F_{\gamma}(t) \in SO_{n+1}\) obtained by Gram-Schmidt
orthonormalization of the ordered basis \((\gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t))\). We denote by the same symbol \(\mathcal{F}_\gamma\) the Frenet curve just defined and its lift to the universal double cover of the special orthogonal group, the group Spin\(_{n+1}\). We denote by \(\mathcal{L}_n(\cdot) = \mathcal{L}_n(1, \cdot)\) the resulting space of nondegenerate curves \(\gamma\) satisfying \(\mathcal{F}_\gamma(0) = 1\) with the subspace topology induced by the standard \(C^n\)-norm of \(C^n([0, 1], \mathbb{R}^{n+1})\). It is easily seen that \(\mathcal{L}_n(\cdot)\) is contractible; fixing the final frame \(\mathcal{F}_\gamma(1)\) at a definite element \(z \in \text{Spin}_{n+1}\) produces subspaces \(\mathcal{L}_n(z) = \mathcal{L}_n(1, z)\). Our main problem is to determine the homotopy type of such subspaces. The present paper provides a recipe for the construction of an abstract cell complex weak homotopy equivalent to \(\mathcal{L}_n(z)\).

Consider the subgroup \(\text{Diag}^+_{n+1} \subset \text{SO}_{n+1}\) of diagonal matrices and its lift \(\text{CG}^+_{n+1} \subset \text{Spin}_{n+1}\). The group \(\text{CG}^+_{n+1}\) is often called the Clifford group (see [38]) and \(\text{CG}_3 = \{\pm 1, \pm i, \pm j, \pm k\} \subset S^3 \subset \mathbb{H}\) is the classical quaternion group. Define our working space of nondegenerate curves as

\[
\mathcal{L}_n = \bigcup_{q \in \text{CG}_{n+1}} \mathcal{L}_n(1, q),
\]

the space of nondegenerate curves whose flags \(\langle \gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t) \rangle\) begin and end at the standard complete flag \(\langle e_1, e_2, \cdots, e_{n+1} \rangle\) of \(\mathbb{R}^{n+1}\). It is shown in [54] that for each \(z \in \text{Spin}_{n+1}\) we can determine explicitly an element \(q \in \text{CG}_{n+1}\) such that the spaces \(\mathcal{L}_n(z)\) and \(\mathcal{L}_n(q)\) are homeomorphic. Therefore, in order to understand all the spaces \(\mathcal{L}_n(z)\), one may restrict attention to the disjoint union of \(2^{n+1}\) spaces in Equation 1. It is worth pointing out that some but not all of the remaining spaces exhibit the same homotopy type (see historical remarks below and final remarks on Section 18).

For a number of technical reasons, we shift to a slightly relaxed notion of nondegeneracy that replaces each space \(\mathcal{L}_n(z)\) with a homotopy equivalent smooth Hilbert manifold (see Appendix B). We deliberately blur the distinction between the original space and its Hilbert manifold version by using the same notation for both throughout. Since we are mainly interested in describing homotopy types, this slight abuse is mostly harmless.

Let \(S_{n+1}\) be the group of permutations of \([n+1] = \{1, 2, \ldots, n+1\}\). For \(\sigma \in S_{n+1}\), let \(\text{inv}(\sigma)\) be the number of inversions of \(\sigma\). We denote by \(W_n\) the set of finite words in the alphabet \(S_{n+1} \setminus \{e\}\). For \(w = (\sigma_1, \sigma_2, \cdots, \sigma_\ell) \in W_n\), set \(\dim(w) = \dim(\sigma_1) + \cdots + \dim(\sigma_\ell)\), where \(\dim(\sigma) = \text{inv}(\sigma) - 1\). For \(\gamma \in \mathcal{L}_n\), we define below its itinerary \(w = \text{iti}(\gamma) \in W_n\). Let \(\mathcal{L}_n[w]\) be the set of curves with itinerary \(w\): this yields a stratification

\[
\mathcal{L}_n = \bigcup_{w \in W_n} \mathcal{L}_n[w]
\]

into topological submanifolds \(\mathcal{L}_n[w]\).
Theorem 1. For each $w \in W_n$, there is a unique $q = q_w \in CG_{n+1}$ such that $L_n[w]$ is a contractible globally collared submanifold of $L_n(q)$ of codimension $\dim(w)$.

The map $w \mapsto q_w$ is given a simple formula in Lemma 10.5.

In order to define the itinerary $iti(\gamma)$, recall the Bruhat decomposition

$$GL_{n+1} = \bigcup_{\sigma \in S_{n+1}} UP_{n+1} P_{\sigma} UP_{n+1}$$

of the real general linear group $GL_{n+1} = GL(n + 1, \mathbb{R})$ into double cosets of the subgroup $UP_{n+1}$ of invertible real upper triangular matrices, indexed by the group $S_{n+1}$. The permutation matrix $P_{\sigma} \in GL_{n+1}$ is defined by $e_j^T P_{\sigma} = e_{j^\sigma}$.

We shall often regard $S_{n+1}$ as a Coxeter-Weyl group of type $A_n$ with transpositions $a_1 = (12), a_2 = (23), \ldots, a_n = (n \ n+1)$ as generators. The Coxeter element $\eta$ is the unique element with $inv(\eta) = m = n(n + 1)/2$. The Bruhat cell $Bru_\eta$ is an open dense subset of $Spin_{n+1}$. We show in Lemma 8.2 that for each $\gamma \in L_n$ there are only finitely many instants $0 = t_0 < t_1 < \cdots < t_\ell < t_{\ell+1} = 1$ at which $\gamma(t_j) \not\in Bru_\eta$. Let $\sigma_j$ be such that $\gamma(t_j) \in Bru_{\sigma_j}$. Set $iti(\gamma) = (\sigma_1, \ldots, \sigma_\ell) \in W_n$.

By a construction similar to Poincaré duality, we obtain from the stratification in Equation 2 a CW complex $D_n$ and a weak homotopy equivalence $c : D_n \to L_n$.

Theorem 2. There exists a CW complex $D_n$ with one cell $i_w : D^{\dim(w)} \to D_n$ for each word $w \in W_n$ and a compatible family of continuous maps $c_w : D^{\dim(w)} \to L_n$. Compatibility means that there exists a continuous map $c : D_n \to L_n$ defined by $c \circ i_w = c_w$ (for each $w \in W_n$). The map $c : D_n \to L_n$ is a weak homotopy equivalence.

The sense in which this complex is a dual to the stratification is explained in Section 15, particularly in the concept of valid complexes. The maps $c_w$ are
constructed rather explicitly in Sections 11 and 15. The map $c_w$ turns out to intersect $L_n[w]$ only at $c_w(0)$, in a topologically transversal manner. One important concept in this construction is a partial order $\preceq$ in the set of words $W_n$. We define $w_1 \preceq w_2$ if and only if $L_n[w_2] \subseteq L_n[w_1]$. It turns out that the image $D_{\dim(w)} \setminus \{0\}$ under $c_w$ is contained in strata $L_n[w']$ for $w' \preceq w$. 

This combinatorial approach has already allowed further progress on the subject, which we expect to cover in a forthcoming paper [3]. A sample may be found in the conjectures stated in our final remarks (Section 18).

The space $L_2(I) = L_2(-1) \sqcup L_2(1)$ of closed nondegenerate curves was originally studied by J. Little in the seventies [10], and shown to have three connected components: $L_2(+1)$, containing curves with an odd number of transversal self-intersections; $L_{2,\text{convex}}(-1)$, the subspace of simple curves; and $L_{2,\text{non-convex}}(-1)$, containing curves with positive even number of self-intersections (we will make sense of this notation in due course). The works of B. Khesin, B. Shapiro and M. Shapiro in the nineties [34, 59, 65] extended this result for $n$ and $z \in \text{Spin}_{n+1}$ arbitrary, showing that $L_n(z)$ has one or two connected components: one if and only if it does not contain convex curves (defined in Appendix A) and two otherwise, one of them being the contractible subspace of the said convex curves. The former results are reobtained in the present paper within our combinatorial framework. In [54] it is shown that each space $L_n(z)$ is homeomorphic to one of the spaces $L_n(q)$, $q \in \text{CG}_{n+1}$; also, several among the latter are homeomorphic. In [53] the spaces $L_2(z)$ were completely classified into three homotopy types explicitly described. The analogous problem of describing the homotopy types of the spaces $L_n(z)$ for $n > 2$ is currently open. We hope to contribute to this problem, in particular solving it for $n = 3$, in a subsequent paper [3] using the combinatorial framework proposed in the present work (see conjectures in our final remarks, Section 18). Some partial results for $n = 3$ were obtained in [2] by different methods, more akin to those of [53].

One common motivation for the study of such problems is the study of linear ordinary differential operators [47]. This point of view was the original motivation of B. Khesin and B. Shapiro for considering the problem in the early nineties [32, 33, 34, 61]. The second named author was first led to consider this problem while studying the topology and geometry of critical sets of nonlinear differential operators with periodic coefficients, in a series of works with D. Burghelea and C. Tomei [13, 14, 15, 55]

In Section 2 we review some basics of the symmetric group: useful representations of a permutation and the poset structures given by the strong and weak Bruhat orders. We also introduce the multiplicities of a permutation. In Section 3 we study the hyperoctahedral group $B_{n+1}$ and the double cover $\tilde{B}_{n+1}$ of $B_{n+1}^+ = B_{n+1} \cap \text{SO}_{n+1}$. We are particularly interested in the maps acute : $S_{n+1} \rightarrow \tilde{B}_{n+1}^+$ and hat : $S_{n+1} \rightarrow \text{CG}_{n+1}$, which will come up constantly
in our discussion. In Section 4 we introduce triangular systems of coordinates in large open subsets $U_\varepsilon$ of the group $\text{Spin}_{n+1}$ and study locally convex curves in the nilpotent lower triangular group $L_{n+1}^1$. In Section 5 we recall the important concept of total positivity. More generally, we define the subsets $\text{Pos}_\sigma, \text{Neg}_\sigma \subset L_{n+1}^1$ for $\sigma \in S_{n+1}$. We review some classical results and prove some useful facts. In Section 6 we consider the related concept of accessibility in the triangular group and prove the contractibility of certain sets which will come up later. In Section 7 we prove several useful results about the well known stratification of the group $\text{Spin}_{n+1}$ in Bruhat cells. We also recall the useful notion of projective transformations. In Section 8 we review the chopping operation defined in [54] and introduce the dual notion of advancing. We also study the singular set of a locally convex curve $\Gamma$: the set $\{t_1 < t_2 < \cdots < t_\ell \} \subset (0,1)$ of values of $t$ for which $\Gamma(t) \not\in \text{Bru}_\eta$. In Section 9 we revisit the concept of accessibility, now in the spin group. We prove the contractibility of several subsets of $\text{Spin}_{n+1}$. Section 10 contains the proof of Theorem 1 divided in a series of lemmas. In Section 11 we construct the restriction to an open ball around the origin of the cell $c_w$ in the complex $D_n$. In Section 12 we define the multiplicity vector for a word and study a few examples of the restrictions constructed in Section 11. In Section 13 we prove a few basic necessary facts about the partial order $\preceq$ defined in $W_n$. Section 14 contains a few remarks about lower and upper subsets of the poset $(W_n, \preceq)$. In Section 15 we prove Theorem 2. In Sections 16 and 17 we explicitly construct the 1 and 2-skeletons of the complex $D_n$. As a corollary, we obtain the well known classification of connected components of $L_n$. The 2-skeleton is of course a key ingredient towards proving the simple connectivity of connected components of $L_n$ (see Conjecture 18.3). Section 18 contain our final remarks, with an emphasis on results which we expect to prove in a forthcoming paper [3] and which are stated here as conjectures. Appendix A reviews the notion of convexity of spherical curves and studies its relation with the notion of itinerary. In Appendix B we define precisely the smooth Hilbert manifold homeomorphic to $L_n$ to which Theorems 1 and 2 apply.

This paper is an extended version of the Ph.D. thesis [26] of the first author, supervised by the second. The first named author is grateful to his co-advisor Boris Khesin for the warm hospitality during his time as a visiting graduate student at the Department of Mathematics of the University of Toronto. The second author thanks the kind hospitality of Stockholm University during his visits. Both authors would like to thank Emília Alves, Boris Khesin, Ricardo Leite, Carlos Gustavo Moreira, Paul Schweitzer, Boris Shapiro, Carlos Tomei, David Torres, Cong Zhou and Pedro Zülkhe for helpful conversations and gratefully acknowledge the financial support of PUC-Rio, CAPES, CNPq and FAPERJ (Brazil).
2 The symmetric group

Consider the symmetric group $S_{n+1}$ (acting on $[n+1] = \{1, 2, \ldots, n+1\}$) as the Coxeter-Weyl group $A_n$, i.e., use the $n$ generators $a = a_1 = (12), b = a_2 = (23), \ldots, a_n = (n,n+1)$. For $\sigma \in S_{n+1}$ and $k \in [n+1]$ we use the notation $k^\sigma$ (rather than $\sigma(k)$) so that $(k^\sigma)^{\sigma_2} = k^{(\sigma_1,\sigma_2)}$. A permutation can be denoted in many ways: one common notation is as a list of values $[1^{\sigma_1} 2^{\sigma_2} \cdots n^{\sigma_n} (n+1)^{\sigma_{n+1}}]$, the so called complete notation; another is as a product of the generators above; for instance, $[ab] = [312] \in S_3$. Notice that we enclose between brackets a expression of a permutation in terms of Coxeter generators. We adopt this device to avoid confusion between the single permutation with that expression and a string of generators with no product intended. Strings of permutations are to appear prominently in this paper, encoding the so called itinerary of a nondegenerate curve.

For $\sigma \in S_{n+1}$, let $P_\sigma$ be the permutation matrix defined by $e_k^\top P_\sigma = e_{k^\sigma}^\top$; for instance, for $n = 2$ we have:

\[
P_a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
P_{[ab]} = P_a P_b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{[ab]} = P_b P_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

For $\sigma \in S_{n+1}$, let $\text{inv}(\sigma)$ be the length of $\sigma$ with the generators $a_i, 1 \leq i \leq n$ (we reserve the symbol $\ell$ for lengths of the itineraries mentioned above). Equivalently, $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$ is the number of inversions of $\sigma$; the set of inversions is

\[
\text{Inv}(\sigma) = \{(i, j) \in [n+1]^2 | 1 \leq i < j \leq n+1, i^\sigma > j^\sigma\}.
\]

A set $I \subseteq \{(i,j) \in [n+1]^2 | i < j\}$ is the set of inversions of a permutation $\sigma \in S_{n+1}$ if and only if

\[
\forall i < j < k, \quad ((i, j), (j, k)) \in I \rightarrow (i, k) \in I \land ((i, j), (j, k) \notin I \rightarrow (i, k) \notin I).
\]

Also, if $\rho = \sigma \eta$ then $\text{Inv}(\sigma) \sqcup \text{Inv}(\eta) = \text{Inv}(\eta)$.

Let $U_{n+1,1}^1, L_{n+1,1}^1 \subset \text{GL}_{n+1}$ be the nilpotent triangular groups of real upper and lower triangular matrices with all diagonal entries equal to 1. For $\sigma \in S_{n+1}$ consider the subgroups

\[
U_\sigma = U_{n+1}^1 \cap (P_\sigma L_{n+1}^1 P_\sigma^{-1}) = \{U \in U_{n+1}^1 | \forall i, j \in [n+1], ((i < j, U_{ij} \neq 0) \rightarrow ((i, j) \in \text{Inv}(\sigma))\}, \quad (3)
\]

\[
L_\sigma = L_{n+1}^1 \cap (P_\sigma U_{n+1}^1 P_\sigma^{-1}) = (U_\sigma)^\top = P_\sigma U_{P_\sigma^{-1}} P_\sigma^{-1},
\]

\[
6
\]
affine subspaces of dimension \( \text{inv}(\sigma) \). If \( \rho = \sigma \eta \) then any \( L \in \mathcal{L}_{n+1}^L \) can be written uniquely as \( L = L_1L_2, L_1 \in \mathcal{L}_\rho, L_2 \in \mathcal{L}_\rho \).

A reduced word for \( \sigma \) is an identity
\[
\sigma = a_{i_1}a_{i_2} \cdots a_{i_k}, \quad k = \text{inv}(\sigma),
\]
or, more formally, it is a finite sequence of indices \((i_1, i_2, \ldots, i_k) \in [n]^k\) satisfying the identity above. Two reduced words for the same permutation \( \sigma \) are connected by a finite sequence of local moves of two kinds:
\[
\begin{align*}
(\cdots, i, j, \cdots) &\leftrightarrow (\cdots, j, i, \cdots), & |i - j| &\neq 1; \quad (4) \\
(\cdots, i, i + 1, \cdots) &\leftrightarrow (\cdots, i + 1, i, \cdots); \quad (5)
\end{align*}
\]

(corresponding to the identities \(a_i a_j = a_j a_i\) for \(|i - j| \neq 1\) and \(a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}\), respectively. Recall that there exists a unique \( \eta \in S_{n+1} \) with \( \text{inv}(\eta) = n(n + 1)/2 \), the Coxeter element (elsewhere usually denoted by \( w_0 \); we spare the letter \( w \) for itineraries, as mentioned above); write
\[
P_\eta = \begin{pmatrix} 1 & \cdots & 1 \\
& \ddots & \\
& & 1 \end{pmatrix}.
\]

Recall that the (strong) Bruhat order in the symmetric group \( S_{n+1} \) can be defined as follows: \( \sigma_0 \leq \sigma_1 \) if some reduced word for \( \sigma_0 \) is a substring of some reduced word for \( \sigma_1 \) (a substring here need not have consecutive letters).

Write \( \sigma_0 \triangleleft \sigma_1 \) if \( \sigma_0 \) is an immediate predecessor of \( \sigma_1 \) (in the Bruhat order). Recall that \( \sigma_0 \triangleleft \sigma_1 \) if \( \text{inv}(\sigma_1) = \text{inv}(\sigma_0) + 1 \) and \( \sigma_1 = \sigma_0(i_0j_0) = (i_0j_0)\sigma_0 \); here \( i_0 < i_1, j_0 < j_1, i_0^{\sigma_0} = j_0, i_1^{\sigma_0} = j_1, i_0^{\sigma_1} = j_1, i_1^{\sigma_1} = j_0 \). We have \( \sigma_0 \triangleleft \sigma_k \) (with \( k = \text{inv}(\sigma_k) - \text{inv}(\sigma_0) \)) if and only if there exist \( \sigma_1, \ldots, \sigma_{k-1} \) with \( \sigma_0 \triangleleft \sigma_1 \triangleleft \cdots \triangleleft \sigma_{k-1} \triangleleft \sigma_k \).

If \( \sigma_1 \) is written as \([1^{a_1} \cdots (n+1)^{a_n}] \), it is easy to find its immediate predecessors: look for integers \( j_1 > j_0 \) appearing in the list \([1^{a_1} \cdots (n+1)^{a_n}] \), \( j_1 \) to the left of \( j_0 \), such that the integers which appear in the list between \( j_1 \) and \( j_0 \) are either larger than \( j_1 \) or smaller than \( j_0 \); the permutation \( \sigma_0 \triangleleft \sigma_1 \) is then obtained by switching the entries \( j_1 \) and \( j_0 \). In the matrix \( P_{\sigma_1} \), we must look for positive entries \((i_0, j_1), (i_1, j_0)\) such that the interior of the rectangle with these vertices includes no positive entry. Then \( P_{\sigma_0} \) is obtained by flipping these entries to the diagonal position while leaving the complement of the rectangle unchanged.

The strong Bruhat order must not be confused with the left and right weak Bruhat orders. Define the weak left Bruhat order by taking the transitive closure of: \( \sigma_1 \triangleleft_L \sigma_0 \) if \( \sigma_1 \triangleleft \sigma_0 \) and \( \sigma_0 = a_i \sigma_1 \) (for some \( i \)). Equivalently, \( \sigma_1 \leq_L \sigma_0 \) if \( \text{Inv}(\sigma_1^{-1}) \subseteq \text{Inv}(\sigma_0^{-1}) \). Similarly, \( \sigma_1 \triangleleft_R \sigma_0 \) if \( \sigma_1 \triangleleft \sigma_0 \) and \( \sigma_0 = \sigma_1 a_i \) (for some \( i \)); the transitive closure \( \sigma_1 \leq_R \sigma_0 \) is characterized by \( \text{Inv}(\sigma_1) \subseteq \text{Inv}(\sigma_0) \). Notice
that either $\sigma_1 \triangleleft_L \sigma_0$ or $\sigma_1 \triangleleft_R \sigma_0$ imply $\sigma_1 \triangleleft \sigma_0$; on the other hand, $\sigma_1 = [2143] = [a_1 a_3] \triangleleft \sigma_0 = [4123] = [a_1 a_2 a_3]$, but $\sigma_1 \not\triangleleft_L \sigma_0$ and $\sigma_1 \not\triangleleft_R \sigma_0$. For more on Coxeter groups and Bruhat orders, see [30].

**Lemma 2.1.** Consider $\sigma \in S_{n+1}$ and $i, j \in [n]$ such that $|i - j| > 1$. Then $\sigma \triangleleft \sigma a_i$ if and only if $\sigma a_j \triangleleft \sigma a_j a_i = \sigma a_j a_i$. Similarly, $\sigma \triangleleft \sigma a_j$ if and only if $\sigma a_i \triangleleft \sigma a_i a_j$.

*Proof.* The condition $\sigma \triangleleft \sigma a_i$ is equivalent to $i^\sigma < (i + 1)^\sigma$. But $i^\sigma = i^{(\sigma a_i)}$ and $(i + 1)^\sigma = (i + 1)^{(\sigma a_i)}$, proving the first equivalence. The second one is similar. □

Define $$\sigma_0 \lor e = \sigma_0, \quad \sigma_0 \lor a_i = \begin{cases} \sigma_0, & \text{if } \sigma_0 a_i \triangleleft \sigma_0; \\ \sigma_0 a_i, & \text{if } \sigma_0 \triangleleft \sigma_0 a_i. \end{cases}$$

A simple computation verifies that

$$
|i - j| \neq 1 \quad \implies \quad (\sigma_0 \lor a_i) \lor a_j = (\sigma_0 \lor a_j) \lor a_i;
$$

$$
((\sigma_0 \lor a_i) \lor a_{i+1}) \lor a_i = ((\sigma_0 \lor a_{i+1}) \lor a_i) \lor a_{i+1}.
$$

We may therefore recursively define

$$
\sigma_1 \triangleleft \sigma_1 a_i \quad \implies \quad \sigma_0 \lor (\sigma_1 a_i) = (\sigma_0 \lor \sigma_1) \lor a_i;
$$

the previous remarks, together with the connectivity of reduced words under the moves in Equations 4 and 5 show that this is well defined. Equivalently, $\sigma_0 \lor \sigma_1$ is the smallest $\sigma$ (in the strong Bruhat order) satisfying both $\sigma_0 \leq_R \sigma$ and $\sigma_1 \leq_L \sigma$. Notice that $S_{n+1}$ is not a lattice with the strong Bruhat order; the $\lor$ operation above uses more than one partial order. In general, we may have $\sigma_0 \lor \sigma_1 \neq \sigma_1 \lor \sigma_0$ and $\sigma_0 \lor \sigma_0 \neq \sigma_0$. We do have associativity: $(\sigma_0 \lor \sigma_1) \lor \sigma_2 = (\sigma_0 \lor \sigma_2) \lor (\sigma_1 \lor \sigma_2)$.

**Example 2.2.** Take $n = 3$, $\sigma_0 = [2413]$, $\sigma_1 = [2431] = a_3 a_2 a_3 a_1$. We have $\sigma_0 \lor \sigma_1 = (((\sigma_0 \lor a_3) \lor a_2) \lor a_3) \lor a_1 = ((\sigma_0 \lor a_2) \lor a_3) \lor a_1 = (\sigma_0 a_2 \lor a_3) \lor a_1 = \sigma_0 a_2 a_3 \lor a_1 = \sigma_0 a_2 a_3 a_1 = \eta$.

Another useful representation of a permutation is in terms of its multiplicities, which we now define. For $\sigma \in S_{n+1}$ and $1 \leq k \leq n$, let

$$
\text{mult}_k(\sigma) = \sum_{j \in [k]} (j^\sigma - j), \quad \text{mult}(\sigma) = (\text{mult}_1(\sigma), \text{mult}_2(\sigma), \ldots, \text{mult}_n(\sigma)).
$$

With the convention $\text{mult}_0(\sigma) = \text{mult}_{n+1}(\sigma) = 0$, we have $k^\sigma = k + \text{mult}_k(\sigma) - \text{mult}_{k-1}(\sigma)$, so that the multiplicity vector $\text{mult}(\sigma)$ easily determines $\sigma$. The reason for calling $\text{mult}_k(\sigma)$ a multiplicity will become clear in section 12.

If $d, \bar{d} \in \mathbb{N}^*$ we write $d \leq \bar{d}$ if, for all $k$, $d_k \leq \bar{d}_k$. If $\sigma_0 \leq \sigma_1$ (in the Bruhat order) then $\text{mult}(\sigma_0) \leq \text{mult}(\sigma_1)$ and $\text{inv}(\sigma_0) \leq \text{inv}(\sigma_1)$. 

8
Example 2.3. For $n = 5$, let $\sigma_0 = [432156]$ and $\sigma_1 = [612345]$. We have $\text{mult}(\sigma_0) = (3, 4, 3, 0, 0, 0) \leq \text{mult}(\sigma_1) = (5, 4, 3, 2, 1)$ but $\text{inv}(\sigma_0) = 6 > \text{inv}(\sigma_1) = 5$. For $n = 6$, let $\sigma_2 = [4321567]$ and $\sigma_3 = [7123456]$. We have $\text{inv}(\sigma_2) = \text{inv}(\sigma_3) = 6$ and $\text{mult}(\sigma_2) = (3, 4, 3, 0, 0, 0, 0) < \text{mult}(\sigma_3) = (6, 5, 4, 3, 2, 1)$.

Lemma 2.4. Let $\sigma_0 \triangleleft \sigma_1$ with $\sigma_1 = (i_0 i_1) \sigma_0 = \sigma_0 (j_0 j_1)$. Then

$$\text{mult}_k(\sigma_1) = \text{mult}_k(\sigma_0) + (j_1 - j_0) [i_0 \leq k < i_1].$$

Here we use Iverson notation:

$$[i_0 \leq k < i_1] = \begin{cases} 1, & i_0 \leq k < i_1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. This is an easy computation.

The notion of multiplicity is closely related to a beautiful 1-1 correspondence, discovered by S. Elnitsky [19], between commutation classes of reduced words for a permutation $\sigma \in S_{n+1}$ and the rhombic tilings of a certain (possibly degenerate) $2(n+1)$-gon associated to $\sigma$. An equivalent (if somewhat deformed) version of this construction is obtained by considering tessellations by parallelograms of the plane region $P_\sigma$ between the graphs of $k \mapsto (2 \text{mult}_k(\sigma_0) - \text{mult}_k(\eta))$ and $k \mapsto (- \text{mult}_k(\eta))$. These decompositions can be performed directly on the graph of $k \mapsto \text{mult}_k(\sigma)$, as shown in figure 1 below.

![Figure 1: Conjugate tilings of the graph of $k \mapsto \text{mult}_k \sigma$ and of the Elnitsky’s polygon $P_\sigma$ for $\sigma = [429681735] \in S_9$ corresponding to the commutation class of the reduced word $\sigma_0 = a_1 a_3 a_4 a_5 a_3 a_2 a_1 a_6 a_5 a_7 a_6 a_5 a_4 a_3 a_8 a_7 a_6 a_5$.](image)

3 Signed permutations

Let $B_{n+1}$ be the hyperoctahedral group of signed permutation matrices, i.e., orthogonal matrices $P$ such that there exists a permutation $\sigma$ with $e_k^\top P = \pm e_{k\sigma}$.
for all \( k \). The group \( B_{n+1} \) is a Coxeter group (whence the notation) but we shall not use this presentation. Let \( B_{n+1}^+ = B_{n+1} \cap \text{SO}_{n+1} \). Let \( \text{Diag}^+_{n+1} \subset B_{n+1}^+ \) be the normal subgroup of diagonal matrices, isomorphic to \( \{ \pm 1 \}^n \). We have \( B_{n+1}^+ / \text{Diag}^+_{n+1} = S_{n+1} \), the quotient map being denoted by \( P \mapsto \sigma_P \).

Consider the universal double covering \( 1 \to \{ \pm 1 \} \to \text{Spin}_{n+1} \to \Pi \to \text{SO}_{n+1} \to 1 \) of the special orthogonal group and let \( \text{CG}_{n+1} = \Pi^{-1}[\text{Diag}^+_{n+1}] \subset \text{Spin}_{n+1} \) and \( \tilde{B}_{n+1}^+ = \Pi^{-1}[B_{n+1}^+] \). The group \( \text{CG}_{n+1} \) has \( 2^{(n+1)} \) elements and is a normal subgroup of \( \tilde{B}_{n+1}^+ \); the quotient is again the symmetric group \( S_{n+1} \); In other words, we have the exact sequences

\[
1 \to \text{CG}_{n+1} \to \tilde{B}_{n+1}^+ \to S_{n+1} \to 1; \quad 1 \to \{ \pm 1 \} \to \text{CG}_{n+1} \to \text{Diag}^+_{n+1} \to 1.
\]

Recall that \( \text{Spin}_3 \approx S^3 \). The Clifford group \( \text{CG}_{n+1} \) is a natural generalization of the classical quaternion group \( \text{Quat}_3 = \{ \pm 1, \pm i, \pm j, \pm k \} \subset S^3 \subset \mathbb{H} \).

Let \( a_i \in \mathfrak{so}_{n+1} = \mathfrak{spin}_{n+1} \) be the matrix with only two nonzero entries:

\[
(a_i)_{i+1,i} = 1, \quad (a_i)_{i,i+1} = -1. \tag{6}
\]

Consider \( \alpha_i : \mathbb{R} \to \text{Spin}_{n+1} \) given by \( \alpha_i(t) = \exp(ta_i) \) and let \( \hat{a}_i = \alpha_i \left( \frac{\pi}{2} \right) \in \tilde{B}_{n+1}^+ \) and \( \hat{a}_i = a_i^2 \in \text{CG}_{n+1} \), so that \( (\hat{a}_i)^2 = -1 \). The matrix \( P = \Pi(\hat{a}_i) \in B_{n+1}^+ \) has nonzero entries

\[
P_{i+1,i} = 1, \quad P_{i,i+1} = -1, \quad P_{j,j} = 1, \quad j \notin \{ i, i+1 \}.
\]

\( P \) is a rotation of \( \frac{\pi}{2} \) in the plane spanned by \( e_i \) and \( e_{i+1} \). Notice that \( \sigma_{\Pi(\hat{a}_i)} = a_i \).

**Lemma 3.1.** The following identities hold:

\[
|i - j| \neq 1 \quad \Rightarrow \quad \hat{a}_j \hat{a}_i = \hat{a}_i \hat{a}_j, \quad \hat{a}_j \hat{a}_i = \hat{a}_i \hat{a}_j, \quad \hat{a}_j \hat{a}_i = \hat{a}_i \hat{a}_j;
\]

\[
\hat{a}_i \hat{a}_{i+1} \hat{a}_i = \hat{a}_{i+1} \hat{a}_i \hat{a}_i; \quad (\hat{a}_i)^{-1} \hat{a}_{i+1} (\hat{a}_i)^{-1} = \hat{a}_{i+1} (\hat{a}_i)^{-1} \hat{a}_{i+1};
\]

\[
|i - j| = 1 \quad \Rightarrow \quad \hat{a}_j \hat{a}_i = (\hat{a}_i)^{-1} \hat{a}_j, \quad \hat{a}_j \hat{a}_i = -\hat{a}_i \hat{a}_j.
\]

**Proof.** These are simple computations. \( \Box \)

Each element \( q \in \text{CG}_{n+1} \) can be written uniquely as

\[
q = \pm \hat{a}_{i_1}^{\varepsilon_1} \hat{a}_{i_2}^{\varepsilon_2} \cdots \hat{a}_{i_n}^{\varepsilon_n}, \quad \varepsilon_i \in \{ 0, 1 \}.
\]

In particular, the elements \( \hat{a}_i, 1 \leq i \leq n \), generate \( \text{CG}_{n+1} \). Furthermore, if \( z \in \tilde{B}_{n+1}^+ \) and \( \sigma_{\Pi(z)} = a_{i_1} \cdots a_{i_k} \in S_{n+1} \), take \( z_1 = \hat{a}_{i_1} \cdots \hat{a}_{i_k} \in \tilde{B}_{n+1}^+ \): we have \( \sigma_{\Pi(z)} = \sigma_{\Pi(z_1)} \) and therefore \( z = qz_1 \) with \( q \in \text{CG}_{n+1} \). In particular, the elements \( \hat{a}_i, 1 \leq i \leq n \), generate \( \tilde{B}_{n+1}^+ \). We make this construction more systematic.
Lemma 3.2. If $\sigma \in S_{n+1}$ is expressed by two reduced words $\sigma = a_{i_1} \cdots a_{i_k} = a_{j_1} \cdots a_{j_k}$ then $\hat{\sigma}_i = \hat{a}_{i_1} \cdots \hat{a}_{i_k}$.

Proof. Both moves (as in Equations 4 and 5) are taken care of by Lemma 3.1.

Let $\hat{a}_i = (\hat{a}_i)^{-1}$. For $\sigma \in S_{n+1}$, take a reduced word $\sigma = a_{i_1} \cdots a_{i_k}$: set

\[
\text{acute}(\sigma) = \hat{\sigma} = \hat{a}_{i_1} \cdots \hat{a}_{i_k}; \quad \text{grave}(\sigma) = \hat{\sigma} = \hat{a}_{i_1} \cdots \hat{a}_{i_k}.
\]

Lemma 3.2 shows that the maps acute, grave : $S_{n+1} \to \tilde{B}_{n+1}^+$ are well defined. Notice that these maps are not homomorphisms. Similarly, non-reduced words do not work in the above formulas for $\hat{\sigma}$ and $\hat{\sigma}$; $a_1$ has order 2 but $\hat{a}_1$ has order 8. Also, define

\[
\text{hat}(\sigma) = \hat{\sigma} = \hat{\sigma}(\hat{\sigma})^{-1} = \hat{a}_{i_1} \cdots \hat{a}_{i_k} \hat{a}_{i_k} \cdots \hat{a}_{i_1},
\]

so that $\hat{\sigma} \in CG_{n+1}$ for all $\sigma \in S_{n+1}$. Notice that these notations are consistent with the previously introduced special cases $\hat{a}_i$ and $\hat{a}_i$.

Let $\text{inv}_i(\sigma) = |\text{Inv}_i(\sigma)|$ where

\[
\text{Inv}_i(\sigma) = \{j \mid i < j, i^\sigma > j^\sigma\} = \{j \mid (i, j) \in \text{Inv}(\sigma)\};
\]

notice that $\text{Inv}(\sigma) = \bigsqcup_i (\{i\} \times \text{Inv}_i(\sigma))$ and therefore $\text{inv}(\sigma) = \sum_i \text{inv}_i(\sigma)$.

Lemma 3.3. For any $\sigma \in S_{n+1}$ and for any $i \in \llbracket n + 1 \rrbracket$ we have

\[
\text{inv}_i(\sigma) - \text{inv}_{i^\sigma}(\sigma^{-1}) = i^\sigma - i = \text{mult}_{i+1}(\sigma) - \text{mult}_i(\sigma).
\]

Proof. The permutation $\sigma$ restricts to a bijection between the two sets:

\[
\{i + 1, \ldots, n + 1\} \setminus \text{Inv}_i(\sigma) = \{j \mid i < j, i^\sigma < j^\sigma\},
\]

\[
\{i^\sigma + 1, \ldots, n + 1\} \setminus \text{Inv}_{i^\sigma}(\sigma^{-1}) = \{j' \mid i^\sigma < j', i < (j')^{\sigma^{-1}}\},
\]

with cardinalities $n + 1 - i - \text{inv}_i(\sigma)$ and $n + 1 - i^\sigma - \text{inv}_{i^\sigma}(\sigma^{-1})$.

Lemma 3.4. Consider $\sigma \in S_{n+1}$ and set $\hat{\sigma} = \Pi(\hat{\sigma}) \in B_{n+1}^+$. We have

\[
e_i^\top \hat{\sigma} = (-1)^{\text{inv}_i(\sigma)} e_i^\top, \quad \hat{\sigma} e_j = (-1)^{\text{inv}_{j^\sigma}(\sigma^{-1})} e_j^{\sigma^{-1}}
\]

and therefore $\hat{\sigma}_{ij} = e_i^\top \hat{\sigma} e_j = (-1)^{\text{inv}_i(\sigma)} \delta_{ji^\sigma}$.

The nonzero entries of $\hat{\sigma} = \Pi(\hat{\sigma}) \in \text{Diag}_{n+1}^+$ are

\[
(\hat{\sigma})_{ii} = (-1)^{\text{inv}_i(\sigma) + \text{inv}_{i^\sigma}(\sigma^{-1})} = (-1)^{i + i^\sigma} = (-1)^{\text{mult}_i(\sigma) + \text{mult}_{i+1}(\sigma)}.
\]

We also have $\hat{\sigma} = \pm \hat{a}_1^{\text{mult}_1(\sigma)} \cdots \hat{a}_i^{\text{mult}_i(\sigma)} \cdots \hat{a}_n^{\text{mult}_n(\sigma)}$. 

11
Proof. The first expression for the diagonal entries of $\hat{P} = \Pi(\sigma)\Pi(\acute{\sigma})$ follows directly from the first two formulae, which we now prove by induction on $\text{inv}(\sigma)$. The base cases $\text{inv}(\sigma) \leq 1$ are easy. Assume $\sigma_1 < \sigma = a_k\sigma_1$, so that $\hat{P} = \Pi(\acute{\sigma})\hat{P}_1$, where $\hat{P}_1 = \Pi(\acute{\sigma}_1)$. By the induction hypotheses we have

$$e_i^T\hat{P} = e_i^T\Pi(\acute{\sigma})\hat{P}_1 = (-1)^{\text{inv}(\sigma_k)}e_i^T\hat{P}_1 = (-1)^{\text{inv}(\sigma_k) + \text{inv}(\sigma_1)}e_i^T.$$ 

To see that $\text{inv}(\sigma) = \text{inv}(\sigma_k) + \text{inv}(\sigma_1)$ for all values of $i \in [n+1]$, consider separately the cases $i < k$, $i = k$, $i = k + 1$ and $i > k + 1$. The second formula is similar. The alternate expressions for $\hat{P}_i$ are obtained via Lemma 3.5.

Finally, setting $q = \hat{a}_k\sigma\hat{a}_k\cdots\hat{a}_{i+1}\sigma\hat{a}_{i+1}\cdots\hat{a}_n\sigma$ and $E = \Pi(\sigma)$ we have $E_{i,i} = (-1)^{\text{mult}_i(\sigma) + \text{mult}_{i+1}(\sigma)}$ and therefore $E = \hat{P}$, hence $\hat{\sigma} = \pm q$. \hfill \Box

If $\sigma_1 < \sigma_0 = a_i\sigma_1$ then, by definition,

$$\hat{\sigma}_0 = \hat{a}_i\hat{\sigma}_1, \quad \hat{\sigma}_0 = \hat{a}_i\hat{\sigma}_1\hat{a}_i.$$

We show how to obtain a different recursive formula for $\hat{\sigma}_0$.

Lemma 3.5. Let $q \in \text{CG}_n$ and $E = \Pi(q) \in \text{Diag}_n$; write

$$q = \pm \hat{a}_1^e \cdots \hat{a}_n^e, \quad \varepsilon_j \in \mathbb{Z}.$$

With the convention $\varepsilon_0 = \varepsilon_{n+1} = 0$, we have:

1. If $\varepsilon_{i-1} + \varepsilon_{i+1}$ is odd then $q\hat{a}_i = (\hat{a}_i)^{-1}q, q\hat{a}_i = -\hat{a}_i q, E_{i+1,i+1} = -E_{i,i}$.

2. If $\varepsilon_{i-1} + \varepsilon_{i+1}$ is even then $q\hat{a}_i = \hat{a}_i q, q\hat{a}_i = \hat{a}_i q, E_{i+1,i+1} = E_{i,i}$.

Proof. From Lemma 3.1

$$\hat{a}_i\hat{a}_j = \begin{cases} \hat{a}_j\hat{a}_i, & |i - j| \neq 1, \\ (\hat{a}_i)^{-1}\hat{a}_j, & |i - j| = 1; \end{cases} \quad \hat{a}_j(\hat{a}_i)^{-1} = \begin{cases} (\hat{a}_i)^{-1}\hat{a}_j, & |i - j| \neq 1, \\ \hat{a}_i\hat{a}_j, & |i - j| = 1; \end{cases}$$

these imply the formulas for $q\hat{a}_i$. We then have, for $\varepsilon_{i-1} + \varepsilon_{i+1}$ even,

$$q\hat{a}_i = q\hat{a}_i\hat{a}_i = \hat{a}_iq\hat{a}_i = \hat{a}_i\hat{a}_iq = \hat{a}_i q$$

and, for $\varepsilon_{i-1} + \varepsilon_{i+1}$ odd,

$$q\hat{a}_i = q\hat{a}_i\hat{a}_i = (\hat{a}_i)^{-1}q\hat{a}_i = (\hat{a}_i)^{-1}(\hat{a}_i)^{-1}q = -\hat{a}_i q.$$

Finally, notice that $\varepsilon_{i-1} + \varepsilon_{i+1}$ even implies $E\Pi(\hat{a}_i) = \Pi(\hat{a}_i)E$ and therefore $E_{i+1,i+1} = E_{i,i}$; conversely, $\varepsilon_{i-1} + \varepsilon_{i+1}$ odd implies $E\Pi(\hat{a}_i) = \Pi((\hat{a}_i)^{-1})E = (\Pi(\hat{a}_i))^{-1}E$ and therefore $E_{i+1,i+1} = -E_{i,i}$. \hfill \Box
Lemma 3.6. Let $\sigma_1 \triangleleft \sigma_0 = a_i \sigma_1 = \sigma_1(j_0 j_1)$, $\delta = j_1 - j_0$.

1. If $\delta$ is odd then $\hat{\sigma}_1 \hat{a}_i = \hat{a}_i \hat{\sigma}_1$ and $\hat{\sigma}_0 = \hat{a}_i \hat{\sigma}_1 = \hat{\sigma}_1 \hat{a}_i$.

2. If $\delta$ is even then $\hat{\sigma}_1 \hat{a}_i = -\hat{a}_i \hat{\sigma}_1$ and $\hat{\sigma}_0 = \hat{\sigma}_1 = \hat{a}_i \hat{\sigma}_1 \hat{a}_i$.

Proof. We know (by definition) that $\hat{\sigma}_0 = \hat{a}_i \hat{\sigma}_1$. As in Lemma 3.5, write $\hat{\sigma}_1 = \pm \hat{a}_1^{\varepsilon_1} \cdots \hat{a}_n^{\varepsilon_n}$. We know from Lemma 3.4 that we can take $\varepsilon_j = \text{mult}_j(\sigma_1)$. Thus

$$\varepsilon_{i+1} - \varepsilon_{i-1} = (i + 1)\varepsilon_1 + \varepsilon_1 - (i + 1) - i = j_1 + j_0 - 2i - 1 \equiv \delta + 1 \pmod{2}.$$  

If $\delta$ is odd then $\hat{\sigma}_1 \hat{a}_i = \hat{a}_i \hat{\sigma}_1$ and therefore $\hat{\sigma}_1 \hat{a}_i = \hat{a}_i \hat{\sigma}_1$ and $\hat{\sigma}_0 = \hat{a}_i \hat{\sigma}_1 = \hat{\sigma}_1 \hat{a}_i$. If $\delta$ is even then $\hat{\sigma}_1 \hat{a}_i = (\hat{a}_i)^{-1} \hat{\sigma}_1$ and therefore $\hat{\sigma}_1 \hat{a}_i = -\hat{a}_i \hat{\sigma}_1$ and $\hat{\sigma}_0 = \hat{\sigma}_1 = \hat{a}_i \hat{\sigma}_1 \hat{a}_i$. 

Example 3.7. Using this result it is easy to compute $\hat{\sigma}_0$ given $\sigma_0$. Take, say, $\sigma_0 = [7245136] = [a_1 a_2 a_3 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_6]$. Take

$$\sigma_0 = a_1 \sigma_1 \triangledown \sigma_1 = [2745136], \quad \sigma_1 = a_2 \sigma_2 \triangledown \sigma_2 = [2475136],$$

$$\sigma_2 = a_3 \sigma_3 \triangledown \sigma_3 = [2457136], \quad \sigma_3 = a_4 \sigma_4 \triangledown \sigma_4 = [2451736],$$

$$\sigma_4 = a_5 \sigma_5 \triangledown \sigma_5 = [2415736], \quad \sigma_5 = a_6 \sigma_6 \triangledown \sigma_6 = [2145736],$$

$$\sigma_6 = a_7 \sigma_7 \triangledown \sigma_7 = [1245376], \quad \sigma_7 = a_8 \sigma_8 \triangledown \sigma_8 = [1245376],$$

$$\sigma_8 = a_9 \sigma_9 \triangledown \sigma_9 = [1243576], \quad \sigma_9 = a_{10} \sigma_{10} \triangledown \sigma_{10} = [1234576] = a_6.$$ 

We therefore have

$$\hat{\sigma}_0 = \hat{a}_1 \hat{\sigma}_1 = \hat{a}_1 \hat{a}_2 \hat{\sigma}_2 = \hat{a}_1 \hat{a}_2 \hat{\sigma}_3 = \hat{a}_1 \hat{a}_2 \hat{\sigma}_4 = \hat{a}_1 \hat{a}_2 \hat{\sigma}_5 =$$

$$= \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{\sigma}_6 = \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{\sigma}_7 = \hat{\sigma}_7 = \hat{\sigma}_8 = \hat{\sigma}_9 = \hat{a}_3 \hat{\sigma}_{10} = \hat{a}_3 \hat{a}_6.$$ 

Example 3.8. We have that $\eta = a_1 a_2 a_3 a_4 a_3 a_2 a_1 \cdots a_n a_{n-1} \cdots a_3 a_1$ is a reduced word so that $\hat{\eta} = \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_3 \hat{a}_2 \hat{a}_1 \cdots \hat{a}_n \hat{a}_{n-1} \cdots \hat{a}_2 \hat{a}_1$ and $\hat{\eta} = (\hat{\eta})^2$. From Lemma 3.4 we have $\Pi(\hat{\eta}) = (-1)^n I$ and

$$\Pi(\hat{\eta}) = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 1 \\ -1 & \cdots & 1 \end{pmatrix}, \quad \hat{\eta} = \begin{cases} 1, & n \equiv 0, 6 \pmod{8}, \\ -1, & n \equiv 2, 4 \pmod{8}, \\ \hat{a}_1 \hat{a}_3 \cdots \hat{a}_n, & n \equiv 1, 7 \pmod{8}, \\ -\hat{a}_1 \hat{a}_3 \cdots \hat{a}_n, & n \equiv 3, 5 \pmod{8}. \end{cases}$$

Notice that, for $n$ odd, we have $\hat{a}_1 \hat{a}_3 \cdots \hat{a}_n = [2143 \cdots]$ is in the center of $CG_{n+1}$.

Lemma 3.9.

$$n \leq 3 \implies \forall \sigma \in S_{n+1}, (\hat{\sigma} = 1 \rightarrow \sigma = e) \land (\hat{\sigma} = \hat{\eta} \rightarrow \sigma = \eta).$$

$$n \geq 4 \implies (\exists \sigma \in S_{n+1}, (\hat{\sigma} = 1 \land \sigma \neq e)) \land (\exists \sigma \in S_{n+1}, (\hat{\sigma} = \hat{\eta} \land \sigma \neq \eta)).$$
Proof. For \( n = 2 \) we have \( \hat{\sigma} = \pm 1 \) if and only if \( \sigma \in \{ \varepsilon, \eta \} \), with \( \hat{\varepsilon} = 1 \) and \( \hat{\eta} = -1 \). For \( n = 3 \) we have \( \hat{\sigma} = \pm 1 \) if and only if \( \sigma \in \{ \varepsilon, [1432], [3214], [3412] \} \) with \( \hat{\varepsilon} = 1 \), \( \text{hat}([1432]) = \text{hat}([3214]) = \text{hat}([3412]) = -1 \). Also, \( \hat{\sigma} = \pm \hat{\eta} \) if and only if \( \sigma \in \{ [2143], [1423], [2341], \eta \} \), with \( \text{hat}([2143]) = \text{hat}([4123]) = \text{hat}([2341]) = \hat{\alpha}_1 \hat{\alpha}_3 \), \( \hat{\eta} = -\hat{\alpha}_1 \hat{\alpha}_3 \). For \( n \geq 4 \), take \( \sigma_0 = [52341 \cdots] = [a_1 a_2 a_3 a_4 a_3 a_2 a_1] \); we have \( \hat{\sigma}_0 = 1 \), \( \text{hat}(\eta \sigma_0) = \text{hat}(\sigma_0 \eta) = \hat{\eta} \).

\[ \square \]

### 4 Triangular coordinates

Let \( \text{Lo}^1_{n+1} \) be the nilpotent group of real lower triangular matrices with all diagonal entries equal to 1; let \( \text{Up}^+_{n+1} \) be the group of real upper triangular matrices with all diagonal entries strictly positive.

Recall the \( LU \) decomposition: a matrix \( A \in \text{GL}^+_{n+1} \) can be (uniquely) written as \( A = LU \), \( L \in \text{Lo}^1_{n+1} \) and \( U \in \text{Up}^+_{n+1} \) provided each of its northwest minors has positive determinant. This condition holds in a contractible open neighborhood of the identity matrix \( I \); for \( A \) in this set, \( L \) and \( U \) are smoothly and uniquely defined. We shall be more interested in \( \mathcal{U}_I \subset \text{SO}_{n+1} \), the intersection of this neighborhood with \( \text{SO}_{n+1} \), which is also a contractible open subset. Let \( \mathbf{L} : \mathcal{U}_I \to \text{Lo}^1_{n+1} \) take \( Q \in \mathcal{U}_I \) to the unique \( L = \mathbf{L}(Q) \in \text{Lo}^1_{n+1} \) such that there exists \( U \in \text{Up}^+_{n+1} \) with \( Q = LU \): the map \( \mathbf{L} \) is a diffeomorphism. Indeed, its inverse \( \mathbf{Q} : \text{Lo}^1_{n+1} \to \mathcal{U}_I \) is given by the QR decomposition: given \( L \in \text{Lo}^1_{n+1} \) let \( Q = \mathbf{Q}(L) \in \text{SO}_{n+1} \) be the unique matrix for which there exists \( R \in \text{Up}^+_{n+1} \) with \( L = QR \).

For \( Q_0 \in \text{SO}_{n+1} \), let \( \mathcal{U}_{Q_0} = Q_0 \mathcal{U}_I \); let \( \phi_{Q_0} : \mathcal{U}_{Q_0} \to \text{Lo}^1_{n+1} \) take \( Q \) to \( L = \mathbf{L}(Q_0^{-1}Q) \) so that we can write \( Q = Q_0LU \), \( U \in \text{Up}^+_{n+1} \). Conversely, let \( \psi_{Q_0} : \text{Lo}^1_{n+1} \to \mathcal{U}_{Q_0} \), \( \psi_{Q_0}(L) = Q_0Q(L) \). We have \( \psi_{Q_0} = \phi_{Q_0}^{-1} \); the family \( (\mathcal{U}_P, \phi_P)_{P \in \text{P}^+_{n+1}} \) is an atlas for \( \text{SO}_{n+1} \).

For each \( Q_0 \in \text{SO}_{n+1} \), the set \( \Pi^{-1}[\mathcal{U}_{Q_0}] \subset \text{Spin}_{n+1} \) has two contractible connected components: we call them \( \mathcal{U}_{z_0} \) and \( \mathcal{U}_{-z_0} \) where \( z_0 \in \text{Spin}_{n+1} \), \( \Pi(z_0) = Q_0 \), \( z_0 \in \mathcal{U}_{z_0} \) and \( -z_0 \in \mathcal{U}_{-z_0} \). Let \( \phi_{z_0} : \mathcal{U}_{z_0} \to \text{Lo}^1_{n+1} \) be \( \phi_{z_0} = \phi_{Q_0} \circ \Pi \); the family \( (\mathcal{U}_z, \phi_z)_{z \in \text{P}_{n+1}} \) is an atlas for \( \text{Spin}_{n+1} \). In particular, the preimage of the set \( \mathcal{U}_I \) under \( \Pi : \text{Spin}_{n+1} \to \text{SO}_{n+1} \) has two contractible components \( \mathcal{U}_{\pm 1} \). We abuse notation and write \( \mathbf{L} : \mathcal{U}_I \to \text{Lo}^1_{n+1} \) and \( \mathbf{Q} : \text{Lo}^1_{n+1} \to \mathcal{U}_I \) for the diffeomorphisms obtained by composition.

Recall that \( a_i \in \text{so}_{n+1} \) is the matrix with only two nonzero entries given in Equation 6. Let \( l_i \in \text{lo}_{n+1} \) be the matrix with only one nonzero entry \( (l_i)_{i+1,i} = 1 \). Let \( X_{a_i} \) and \( X_{l_i} \) be the left-invariant vector fields in \( \text{SO}_{n+1} \) and \( \text{Lo}^1_{n+1} \) generated by \( a_i \) and \( l_i \), respectively:

\[ X_{a_i}(Q) = Qa_i, \quad X_{l_i}(L) = Ll_i. \]
Lemma 4.1. The diffeomorphisms \( L : \mathcal{U}_t \to \text{Lo}^1_{n+1} \) and \( Q : \text{Lo}^1_{n+1} \to \mathcal{U}_t \) take the vector fields \( X_{a_i} \) and \( X_{l_i} \) to smooth positive multiples of each other.

Proof. Given \( Q_0 \in \mathcal{U}_t \), take a short arc of the integral line of \( X_{a_i} \) through \( Q_0 \): let \( \epsilon > 0 \) be sufficiently small so that \( Q(t) = Q_0 \exp(\epsilon a_i) \in \mathcal{U}_t \) for \( -\epsilon < t < \epsilon \). Also write \( L(Q(t)) = L(t) \in \text{Lo}^1_{n+1} \), so that \( L(t) = Q(t)R(t) \) for a smooth path \( R : (-\epsilon, \epsilon) \to \text{Up}^+_n \). Differentiating the last equation, we have

\[
(L(t))^{-1}L'(t) = (R(t))^{-1}a_i R(t) + (R(t))^{-1}R'(t).
\]

Since the left hand side is in \( \mathfrak{lo}^1_{n+1} \) and the rightmost summand of the right hand side is in \( \text{up}^+_n \), it is readily seen that \( L'(t) = (R(t))_{i+1} R(t)_{i+1} L(t)_{j_i} \). \( \square \)

A smooth curve \( \gamma : J \to \text{SO}_{n+1} \) or \( \text{Spin}_{n+1} \) is locally convex if, for all \( t \in J \), the logarithmic derivative \( (\Gamma(t))^{-1} \Gamma'(t) \) is tridiagonal with strictly positive sub-diagonal entries, i.e., a positive linear combination of \( a_i \), \( 1 \leq i \leq n \). Notice that in this case \( \gamma = \Gamma e_1 : J \to \mathbb{S}^n \) is locally convex in the sense of the Introduction; conversely, if \( \gamma : J \to \mathbb{S}^n \) is smooth and locally convex (in the sense of the Introduction), then \( \Gamma = \mathcal{F}_\gamma \) is locally convex (in the sense we just introduced). In other works \([54, 53]\), the alternate name holonomic was also used to designate this kind of curves. The reason for calling \( \gamma = \Gamma e_1 \) locally convex in the first place is given in Appendix A.

Similarly, a smooth curve \( \Gamma : J \to \text{Lo}^1_{n+1} \) is locally convex if, for all \( t \in J \), the logarithmic derivative \( (\Gamma(t))^{-1} \Gamma'(t) \) is a positive linear combination of \( l_i \), \( 1 \leq i \leq n \).

Example 4.2. Set \( \mathfrak{h} = \mathfrak{h}_L - \mathfrak{h}_L^T \in \mathfrak{so}_{n+1} \), where

\[
\mathfrak{h}_L = \sum_{k \in [n]} \sqrt{k(n+1-k)} \mathfrak{l}_k \in \mathfrak{lo}_{n+1}, \quad \mathfrak{l} = \sum_{k \in [n]} \mathfrak{l}_k \in \mathfrak{lo}_{n+1}.
\]

We have

\[
[\mathfrak{h}_L, \mathfrak{h}_L^T] = \begin{pmatrix}
-n & \cdots & 0 \\
& \ddots & \vdots \\
0 & \cdots & n - 2
\end{pmatrix}, \quad [\mathfrak{l}_t, [\mathfrak{h}_L, \mathfrak{h}_L^T]] = -2\mathfrak{h}_L, \quad [\mathfrak{h}_L^T, [\mathfrak{h}_L, \mathfrak{h}_L^T]] = 2\mathfrak{h}_L.
\]

It is not hard to verify that, for \( n = 1 \) and \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \),

\[
\exp(\theta(\mathfrak{h}_L - \mathfrak{h}_L^T)) = \exp(\tan(\theta)\mathfrak{h}_L) \exp(\log(\sec(\theta)) [\mathfrak{h}_L, \mathfrak{h}_L^T]) \exp(-\tan(\theta)\mathfrak{h}_L^T).
\]

The symmetric product induces a Lie algebra homomorphism \( S : \mathfrak{sl}_2 \to \mathfrak{sl}_{n+1} \) with \( S(\mathfrak{h}_L) = \mathfrak{h}_L \) (that is, taking \( \mathfrak{h}_L \in \mathbb{R}^{2 \times 2} \) to \( \mathfrak{h}_L \in \mathbb{R}^{(n+1) \times (n+1)} \) and \( S(\mathfrak{h}_L^T) = \mathfrak{h}_L^T \). We
therefore also have a Lie group homomorphism $S : \overline{SL_2} \to \overline{SL_{n+1}}$, $S(\exp(tu)) = \exp(tS(u))$ for all $u \in \mathfrak{sl}_2$, $t \in \mathbb{R}$. Equation 6 therefore holds for any value of $n$. We therefore have $\exp(\theta h) \in U$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with

$$L(\exp(\theta h)) = \exp(\tan(\theta) h_L); \quad Q(\exp(t h_L)) = \exp(\arctan(t) h).$$

Also, the equation $\exp\left(\frac{\pi}{2} h\right) = h'$, which is trivially true for $n = 1$, can be obtained for arbitrary $n$ using the Lie group homomorphism $S$ above and noticing that $S(h') = h$.

For $z_0 \in \text{Spin}_{n+1}$, the curve $\Gamma_{z_0,h}(t) = z_0 \exp(t h)$ is locally convex and satisfies $\Gamma_{z_0,h}(\frac{\pi}{2}) = z_0 h'$, $\Gamma_{z_0,h}(\pi) = z_0 h$. For $L_0 \in L_{n+1}^1$, the curves $\Gamma_{L_0,h_L}(t) = L_0 \exp(t h_L)$ and $\Gamma_{L_0,n}(t) = L_0 \exp(n t)$ are locally convex. Notice that the $(i, j)$ entry of either $\Gamma_{L_0,h_L}(t)$ or $\Gamma_{L_0,n}(t)$ is a polynomial of degree $(i - j)$ in the variable $t$.

One advantage of working with triangular coordinates is that there is then a simple integration formula. Indeed, let $\Gamma : [t_0, t_1] \to L_{n+1}^1$ satisfy $(\Gamma(t))^{-1} \Gamma'(t) = \sum_i \beta_i(t) L_i$. Then

$$(\Gamma(t))_{i+1,i} = (\Gamma(t_0))_{i+1,i} + \int_{t_0}^{t_1} \beta_i(\tau) d\tau.$$  

More generally,

$$((\Gamma(t_0))^{-1} \Gamma(t))_{i+t,i} = \int_{t_0 \leq \tau_1 \leq \cdots \leq \tau_{i+t} \leq t} \beta_{i+t-1}(\tau_1) \cdots \beta_i(\tau_t) d\tau_1 \cdots d\tau_t. \quad (9)$$  

The set of smooth locally convex curves then has an easy generalization. A curve $\Gamma$ is \textit{locally convex} if it is absolutely continuous and its logarithmic derivative $(\Gamma(t))^{-1} \Gamma'(t)$ is almost everywhere a positive linear combination of $t_i$, $1 \leq i \leq n$. In this case the functions $\beta_i : J \to (0, +\infty)$ belong to $L^1(J; \mathbb{R})$ and Equation 9 above makes perfect sense; notice that if each $\beta_i$ is in $L^1(J; \mathbb{R})$ then $(\beta_{i_1}, \ldots, \beta_{i_t})$ is in $L^1(J; \mathbb{R}^t)$. Equivalently, a curve $\Gamma : [t_0, t_1] \to L_{n+1}^1$ is locally convex if and only if there exist finite absolutely continuous (positive) Borel measures $\mu_1, \ldots, \mu_n$ on $J = [t_0, t_1]$ such that $\mu_i(\tilde{J}) > 0$ for any index $i$ and for any nondegenerate interval $\tilde{J} \subseteq J$ such that

$$((\Gamma(t_0))^{-1} \Gamma(t))_{i+t,i} = (\mu_{i+t-1} \times \cdots \times \mu_i)(\Delta), \quad (10)$$  

$$\Delta = \{ (\tau_1, \ldots, \tau_t) \in [t_0, t]^k \mid t_0 \leq \tau_1 \leq \cdots \leq \tau_t \leq t \}.$$  

Similarly, a curve $\Gamma : J \to \text{Spin}_{n+1}$ is \textit{locally convex} if it is absolutely continuous and its logarithmic derivative $(\Gamma(t))^{-1} \Gamma'(t)$ is almost everywhere a positive linear combination of the vectors $a_i$, $i \in [n]$. Equivalently, $\Gamma : J \to \text{Spin}_{n+1}$ is locally convex if and only if, near any point $t_\bullet \in J$, there is a system of triangular coordinates $\Gamma_L(t) = L(z_0^{-1} \Gamma(t))$ with $\Gamma_L$ locally convex in the previous sense.
Let \([n+1]^{(k)}\) be the set of subsets \(i \subseteq [n+1]\) with \(|i| = k\); let \(\sum(i)\) be the sum of the elements of the set \(i\). The \(k\)-th exterior (or alternating) power \(\Lambda^k(V)\) of \(V = \mathbb{R}^{n+1}\) has a basis indexed by \(i \in [n+1]^{(k)}\). For \(i_0, i_1 \in [n+1]^{(k)}\), write:

\[i_0 \xrightarrow{j} i_1 \iff j \in i_1, \ j+1 \notin i_1, \ i_0 = (i_1 \setminus \{j\}) \cup \{j+1\}.\]

Notice that \(i_0 \xrightarrow{j} i_1\) implies \(\sum(i_0) = 1 + \sum(i_1)\). With respect to the basis above, the matrix of the linear endomorphism \(\Lambda^k(i) \in \mathfrak{gl}(\Lambda^k(V))\) given by

\[
\Lambda^k(i)(v_1 \wedge \cdots \wedge v_k) = \sum_{j \in [k]} v_1 \wedge \cdots \wedge i_j(v_j) \wedge \cdots \wedge v_k
\]

has nonzero entries all equal 1 and in positions \((i_0, i_1)\) such that \(i_0 \xrightarrow{j} i_1\). Write \(i_1 < i_0\) if there exists \(j\) such that \(i_0 \xrightarrow{j} i_1\) and define a partial order in \([n+1]^{(k)}\) by taking the transitive closure. Equivalently, for \(i_j = \{i_{j1} < i_{j2} < \cdots < i_{jk}\}\) we have

\[i_1 \leq i_0 \iff i_{11} \leq i_{01}, i_{12} \leq i_{02}, \ldots, i_{1k} \leq i_{0k}.\]

If \(i_0 \geq i_1\), \(\sum(i_0) = l + \sum(i_1)\), write

\[i_0 \xrightarrow{(j_1, \ldots, j_l)} i_1 \iff \exists j_0, \ldots, j_l, \ i_0 = j_0 \xrightarrow{j_1} j_1 \rightarrow \cdots \rightarrow j_{l-1} \xrightarrow{j_l} j_l = i_1;\]

notice that given \(i_0 \) and \(i_1\) there may exist many such \(l\)-tuples \((j_1, \ldots, j_l)\). Order the indices \(i\) consistently with the partial order introduced above (or, more directly, order the subsets \(i\) increasing in the sum of their elements). The matrix \(\Lambda^k(i)\) is then strictly lower triangular.

If \(L \in \text{Lo}_0^{l+1}\) and \(i_0, i_1 \in [n+1]^{(k)}\), define \(L_{i_0,i_1}\) to be the \(k \times k\) submatrix of \(L\) obtained by selecting the rows in \(i_0\) and the columns in \(i_1\). The \((i_0, i_1)\) entry of \(\Lambda^k(L)\) is \(\det(L_{i_0,i_1})\). Clearly, \(i_0 \geq i_1\) implies \(\det(L_{i_0,i_1}) = 0\); also, \(L_{i_1}\) is lower triangular with diagonal entries equal to 1 and therefore \(\det(L_{i_1}) = 1\). The matrix \(\Lambda^k(L)\) is therefore lower triangular with diagonal entries equal to 1. Furthermore, the map \(\Lambda^k : \text{Lo}_0^{l+1} \rightarrow \text{Lo}_0^{l+1}\) is a group homomorphism. The following result generalizes Equations \([3]\) and \([10]\) above.

**Lemma 4.3.** Let \(\Gamma : [t_0, t_1] \rightarrow \text{Lo}_0^{l+1}\) be locally convex with \(\Gamma(t_0) = L_0\) and let \(\beta_i(t) = ((\Gamma(t))^{-1}\Gamma'(t))_{i+1,i}, \ \mu_i(J) = \int_J \beta_i(t)dt\). Let \(i_0, i_1 \in [n+1]^{(k)}\) with \(i_0 \geq i_1\) and \(l = (\sum i_0) - (\sum i_1)\). Then

\[
\det((L_0^{-1}\Gamma(t))_{i_0,i_1}) = (\Lambda^k(L_0^{-1}\Gamma(t)))_{i_0,i_1} = \sum_{i_0 \rightarrow (j_1, \ldots, j_l)_{i_1}} (\mu_{j_1} \times \cdots \times \mu_{j_l})(\Delta);
\]

\[
\Delta = \{(\tau_1, \ldots, \tau_l) \in [t_0, t]^l \mid t_0 \leq \tau_1 \leq \cdots \leq \tau_l \leq t\}.
\]

**Proof.** These are straightforward computations. \(\square\)
5 Totally positive matrices

A matrix $L \in \mathbb{L}_{n+1}^1$ is totally positive if for all $k \in [n+1]$ and for all indices $i_0, i_1 \in [n+1]^{(k)}$, \[ i_0 \geq i_1 \implies \det(L_{i_0, i_1}) > 0. \]

Let $\text{Pos}_\eta \subset \mathbb{L}_{n+1}^1$ be the set of totally positive matrices.

Totally positive matrices were introduced independently in [23] and [58] and have since found widespread applications [4, 9, 31, 43]. The concept of totally positive elements has been generalized to a reductive group $G$ and its flag manifold by G. Lusztig [41, 42, 43] and to Grassmannians by A. Postnikov [48, 49]. The relation of the subject with intersections of Bruhat (Schubert) cells has been studied in [21, 50, 63, 64]. See also [22]. Our particular definition is analogous to that of [6], where $G = N = \text{U}_n^1$ and $N > 0 = \text{Pos}_\eta^\top$: this one is a good reference for facts mentioned without proof in the present section.

Let $\lambda_i(t) = \exp(tL)$: for any reduced word $\eta = a_{i_1}a_{i_2} \cdots a_{i_m}$, $m = \text{inv}(\eta) = n(n + 1)/2$, the map \[ (0, +\infty)^m \to \text{Pos}_\eta, \quad (t_1, t_2, \ldots, t_m) \mapsto \lambda_{i_1}(t_1)\lambda_{i_2}(t_2) \cdots \lambda_{i_m}(t_m) \] is a diffeomorphism. Moreover, there exists a stratification of its closure $\overline{\text{Pos}_\eta}$:

\[ \overline{\text{Pos}_\eta} = \{ L \in \mathbb{L}_{n+1}^1 \mid \forall i_0, i_1, ((i_0 \geq i_1) \implies (\det(L_{i_0, i_1}) \geq 0)) \} = \bigsqcup_{\sigma \in S_{n+1}} \text{Pos}_\sigma; \]

$\text{Pos}_\sigma \subset \mathbb{L}_{n+1}^1$ is a smooth manifold of dimension $\text{inv}(\sigma)$, and if $\sigma = a_{i_1} \cdots a_{i_k}$ is a reduced word (so that $k = \text{inv}(\sigma)$) then the map \[ (0, +\infty)^k \to \text{Pos}_\sigma, \quad (t_1, t_2, \ldots, t_k) \mapsto \lambda_{i_1}(t_1)\lambda_{i_2}(t_2) \cdots \lambda_{i_k}(t_k) \] is a diffeomorphism. Equivalently, if $\sigma_1 \triangleleft \sigma_0 = \sigma_1 a_{i_k}$ then the map \[ \text{Pos}_{\sigma_1} \times (0, +\infty) \to \text{Pos}_{\sigma_0}, \quad (L, t_k) \mapsto L\lambda_{i_k}(t_k) \] is a diffeomorphism.

Different reduced words yield different diffeomorphisms but the same set $\text{Pos}_\sigma$: the equation

\[ \lambda_i(t_1)\lambda_{i+1}(t_2)\lambda_i(t_3) = \lambda_{i+1} \left( \frac{t_2t_3}{t_1 + t_3} \right) \lambda_i(t_1 + t_3)\lambda_{i+1} \left( \frac{t_1t_2}{t_1 + t_3} \right) \] (11)

provides the transition between adjacent parameterizations (i.e., between reduced words connected by the local move in Equation 5; the local move in Equation 4 corresponds to a mere relabeling).
In general, the sets \( \text{Pos}_\sigma \subset \text{Lo}_{a+1}^1 \) are neither subgroups nor semigroups and should not be confused with the subgroups \( \text{Lo}_\sigma = \text{Lo}_{a+1}^1 \cap (P_{\sigma}^{-1} \cup \text{Lo}_{a+1}^1 P_{\sigma}) \). For instance, \( \text{Pos}_\sigma = \{1\} \) consists of a single point and \( \text{Pos}_{a_i} = \{\lambda_i(t), t > 0\} \) is an open half line; in this case, \( \text{Pos}_{a_i} \subset \text{Lo}_{a_i} \). For \( n = 2 \) and

\[
L(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}
\]

we have

\[
\text{Pos}_{[a_1 a_2]} = \{L(x, y, 0) \mid x, y > 0\}; \quad \text{Pos}_{[a_2 a_1]} = \{L(x, y, xy) \mid x, y > 0\}; \\
\text{Pos}_{[a_1 a_2 a_1]} = \{L(x, y, z) \mid x, y > 0; 0 < z < xy\}.
\]

On the other hand,

\[
\text{Lo}_{[a_1 a_2]} = \{L(x, 0, z) \mid x, z \in \mathbb{R}\}; \quad \text{Lo}_{[a_2 a_1]} = \{L(0, y, z) \mid y, z \in \mathbb{R}\}.
\]

If \( L \in \text{Pos}_\sigma \) then there exist matrices \( U_1, U_2 \in \text{Up}_{a+1} \) such that \( L = U_1 P_{\sigma} U_2 \), but the converse is not at all true, not even if we pay attention to signs of diagonal entries of the matrices \( U_i \). In \([63, 64]\) it is shown that the set of matrices which admit such a decomposition is almost always disconnected; each cell \( \text{Pos}_\sigma \) is contractible, and so is its closure \( \text{Pos}_\sigma \); see also Lemma \( 7.6 \) below.

**Lemma 5.1.** Consider \( \sigma \in S_{a+1}, k \in [n+1] \) and indices \( i_0, i_1 \in [n+1]^{(k)} \).

If there exists \( \sigma_1 = a_j_1 \cdots a_j_{k_1} \leq \sigma \), \( \text{inv} (\sigma_1) = l_1 \), such that \( i_0 \xrightarrow{\bullet} (j_0, \cdots, j_{k_1}) \rightarrow i_1 \) then, for all \( L \in \text{Pos}_\sigma \), \( (\Lambda^k (L))_{i_0, i_1} > 0 \). Conversely, if no such \( \sigma_1 \) exists then, for all \( L \in \text{Pos}_\sigma \), \( (\Lambda^k (L))_{i_0, i_1} = 0 \).

**Proof.** Write a reduced word \( \sigma = a_{i_1} \cdots a_{i_l} \). Assume first that such \( \sigma_1 \) exists and that \( j_1 = i_{x_1}, \ldots, j_{l_1} = i_{x_{l_1}} \) (where of course \( 1 \leq x_1 < \cdots < x_{l_1} \leq l \)). Set

\[
i_0 = j_0 \rightarrow j_1 \rightarrow \cdots \rightarrow j_{l_1-1} \rightarrow j_{l_1} = i_1; \quad L = \lambda_{i_1}(t_1) \cdots \lambda_{i_l}(t_l) \in \text{Pos}_\sigma.
\]

We have \( (\Lambda^k (L))_{i_0, i_1} \geq t_{x_1} \cdots t_{x_{l_1}} > 0 \), as desired.

Conversely, assume that \( L = \lambda_{i_1}(t_1) \cdots \lambda_{i_l}(t_l) \), \( (\Lambda^k (L))_{i_0, i_1} > 0 \). We have

\[
(\Lambda^k (L))_{i_0, i_1} = \sum_{i_0 = j_0 \geq \cdots \geq j_{k_1} = i_1} \left( (\Lambda^k (\lambda_{i_1} (t_1)))_{j_0, j_1} \cdots (\Lambda^k (\lambda_{i_l} (t_l)))_{j_{l-1}, j_l} \right).
\]

Consider \( (j_0, \ldots, j_k) \) such that the above product is positive. Let \( x_1, \ldots, x_{l_1} \) be such that \( j_{x_{1-1}} > j_{x_1}, \ldots, j_{x_{l_1-1}} > j_{x_{l_1}} \); this obtains a reduced word for \( \sigma_1 \). \( \square \)
Lemma 5.2. Consider $L_0, L_1 \in \text{Lo}^1_{0,0+1}$. If $L_0 \in \text{Pos}_{\sigma_0}$ and $L_1 \in \text{Pos}_{\sigma_1}$ then $L_0 L_1 \in \text{Pos}_{\sigma_0 \lor \sigma_1}$. Thus, $\text{Pos}_{\sigma_0} \cap \text{Pos}_{\sigma_1} = \text{Pos}_{\sigma_0 \lor \sigma_1}$.

In particular, if $L_0 \in \text{Pos}_\eta$ and $L_1 \in \overline{\text{Pos}_\eta}$ then $L_0 L_1, L_1 L_0 \in \text{Pos}_\eta$. If $L_0 \in \text{Pos}_\eta$ and $L_1 \in \overline{\text{Pos}_\eta}$ then $L_0 L_1 \in \text{Pos}_\eta$.

The operation $\lor$ is the one in Example 2.2.

Proof. The first claim can be proved by induction on $l = \text{inv}(\sigma_1)$; the case $l = 0$ is trivial. For the case $l = 1$, consider $\sigma_1 = a_i$ and two cases. If $\sigma_0 \lor a_i = \sigma_0$, we take a reduced word $\sigma_0 = a_{i_1} \cdots a_{i_k}$ with $i_k = i$. Then

$$L_0 L_1 = (\lambda_{i_1}(t_1) \cdots \lambda_{i_k}(t_k)) \lambda_i(t) = \lambda_{i_1}(t_1) \cdots \lambda_{i_k}(t_k + t) \in \text{Pos}_{\sigma_0}.$$  

The case $\sigma_0 \lor a_i \neq \sigma_0$ is even more direct. The induction step is now easy.

The other claims follow from the first, but a direct proof may be instructive: consider $i_0, i_1 \in [n + 1]^{(k)}$, $i_0 \geq i_1$. If $L_0 \in \text{Pos}_\eta$ and $L_1 \in \overline{\text{Pos}_\eta}$ we have

$$(\Lambda^k(L_0 L_1))_{i_0 i_1} = (\Lambda^k(L_0))_{i_0 i_1} (\Lambda^k(L_1))_{i_1 i_1} + \sum_{i_0 \geq 1 > i_1} (\Lambda^k(L_0))_{i_0 i_1} (\Lambda^k(L_1))_{i_1 i_1} > 0,$$

as desired; the other cases are similar.

Write $L_0 \leq L_1$ if $L_0^{-1} L_1 \in \overline{\text{Pos}_\eta}$ and $L_0 \ll L_1$ if $L_0^{-1} L_1 \in \text{Pos}_\eta$; notice that $L_0^{-1} L_1 \in \text{Pos}_\eta$ is in general not equivalent to $L_1 L_0^{-1} \in \text{Pos}_\eta$. Lemma 5.2 implies that these are partial orders:

$$L_0 \leq L_1 \leq L_2 \implies L_0 \leq L_2; \quad L_0 \leq L_1 \ll L_2 \implies L_0 \ll L_2.$$  \hspace{1cm} (13)

Lemma 5.3. Consider $L_0, L_1 \in \text{Lo}^1_{0,0+1}$. We have that $L_0 \ll L_1$ if and only if there exists a locally convex curve $\Gamma : [0,1] \to \text{Lo}^1_{0,0+1}$ with $\Gamma(0) = L_0$ and $\Gamma(1) = L_1$.

Proof. We first prove that the existence of $\Gamma$ implies $L_0 \ll L_1$. Given $\Gamma$ and $i_0, i_1 \in [n + 1]^{(k)}$ with $i_0 > i_1$, Lemma 4.3 gives us a formula for $(L_0^{-1} L_1)_{i_0 i_1} > 0$.

Conversely, let $\mathfrak{h}_L = \sum_i c_i I_i \in \text{Lo}^1_{0,0+1}$ for fixed positive $c_i$. Consider a small closed ball of radius $r > 0$ centered at $L_0^{-1} L_1$ and contained in $\text{Pos}_\eta$, the image of a continuous map $h : \mathbb{B}^m \to \text{Pos}_\eta \subset \text{Lo}^1_{0,0+1}$ with $h(0) = L_0^{-1} L_1$ such that the topological degree of $h|_{S^{m-1}}$ around $L_0^{-1} L_1$ equals $+1$ (here $m = \dim(\text{Lo}^1_{0,0+1})$). Consider a fixed reduced word $\eta = a_{i_1} \cdots a_{i_m}$. Define continuous functions $\tau_i : \mathbb{B}^m \to (0, +\infty)$ such that $h(s) = \lambda_{i_1}(\tau_1(s)) \cdots \lambda_{i_m}(\tau_m(s))$. For $\epsilon \geq 0$, let

$$\Lambda_{\epsilon}(s)(t) = m \tau_j(s) I_j + \epsilon \mathfrak{h}_L, \quad t \in \left(\frac{j - 1}{m}, \frac{j}{m}\right).$$
Integrate to obtain functions

\[ \Gamma_\epsilon(s) : [0, 1] \to \text{Lo}^1_{\text{n+1}}, \quad \Gamma_\epsilon(s)(0) = L_0, \quad (\Gamma_\epsilon(s)(t))^{-1}(\Gamma_\epsilon(s))'(t) = \Lambda_\epsilon(s)(t). \]

Notice that \( \Gamma_\epsilon(s) \) is a locally convex curve if \( \epsilon > 0 \). Define \( h_\epsilon(s) = L_0^{-1}\Gamma_\epsilon(s)(1) \): clearly \( h_0 = h \), i.e., \( \Gamma_0(s)(1) = L_0h(s) \). By continuity, there exists \( \epsilon > 0 \) such that for all \( s \in \mathbb{R}^m \) we have \( |h_\epsilon(s) - h_0(s)| < r/2 \). The topological degree of \( h_\epsilon|_{\mathbb{R}^{m-1}} \) around \( L_0^{-1}L_1 \) equals \( +1 \). There exists therefore \( s_\epsilon \in \mathbb{R}^m \) with \( h_\epsilon(s_\epsilon) = L_0^{-1}L_1 \).

We have that \( \Gamma = \Gamma_\epsilon(s_\epsilon) \). Let \( \Gamma : [0, 1] \to \text{Lo}^1_{\text{n+1}} \) is a locally convex curve with \( \Gamma(0) = L_0, \Gamma(1) = L_1 \).

We know by now that if \( L_0 \in \text{Pos}_\sigma \) for \( \sigma \neq \eta \) and \( \Gamma : [0, 1] \to \text{Lo}^1_{\text{n+1}} \) is a locally convex curve with \( \Gamma(0) = L_0 \) then \( \Gamma(t) \in \text{Pos}_\eta \) for all \( t > 0 \). The following lemma shows that, at least from the point of view of certain entries, the curve \( \Gamma \) goes in with positive speed.

**Lemma 5.4.** Given \( \sigma \in S_{\text{n+1}}, \ \sigma \neq \eta \), there exist \( k \in [n + 1] \) and indices \( i_0, i_1, i_2 \in [n + 1]^{(k)} \) and \( j \in [n] \) such that \( i_0 \geq i_1 > i_2, i_1 \overset{j}{\rightarrow} i_2 \) and, for all locally convex curves \( \Gamma : [0, 1] \to \text{Lo}^1_{\text{n+1}} \) with \( \Gamma(0) \in \text{Pos}_\sigma \) and \( \Gamma'(0) \neq 0 \) (and well defined), if \( g(t) = (\Lambda^k(\Gamma(t)))_{i_0,i_2} \) then \( g(0) = 0 \) and \( g'(0) > 0 \).

**Proof.** Consider \( k \) and a pair of indices \( (i_0, i_1), i_0 \geq i_3, i_3 \in [n + 1]^{(k)} \) such that \( (\Lambda^k(L))_{i_0,i_1} = 0 \) for \( L \in \text{Pos}_\sigma \) (see Lemma 5.1). Keep \( k \) and \( i_0 \) fixed and search for \( i_2 \leq i_0 \) maximal such that \( (\Lambda^k(L))_{i_2,i_3} = 0 \) for \( L \in \text{Pos}_\sigma \). Maximal implies that there exists \( i_1, i_0 \geq i_1 > i_2 \) and an index \( j \) such that \( i_1 \overset{j}{\rightarrow} i_2 \) and \( (\Lambda^k(L))_{i_0,i_1} > 0 \) for \( L \in \text{Pos}_\sigma \).

Let \( L_0 = \Gamma(0) \), \( c_0 = (\Lambda^k(L_0))_{i_0,i_1} > 0 \). Write \( h_i(t) = (L_0^{-1}\Gamma(t))_{i+1,i} \) so that \( h_i(0) = 0 \) and \( h'_i(0) = c_i > 0 \). Clearly, \( g(0) = 0 \) and, for all \( t > 0 \),

\[ g(t) \geq (\Lambda^k(L_0))_{i_0,i_2}(\Lambda^k(L_0^{-1}\Gamma(t)))_{i_1,i_2} = c_0 h_j(t) = c_0 c_j(t + o(t)) \]

so that \( g'(0) \geq c_0 c_j > 0 \), as desired. \( \square \)

**Remark 5.5.** We now present an explicit construction. Given \( \sigma \neq \eta \), take \( k \) minimal such that \( (n-k+2)^\sigma \neq k \). Set then \( j = (n-k+2)^\sigma - 1 \). Equivalently, \( k \) is minimal such that \( \Lambda^k(L)_{i_0,i_3} = 0 \) for \( L \in \text{Pos}_\sigma \), \( i_0 = \{n-k+2, \ldots, n+1\} \) and \( i_3 = \{1, \ldots, k\} \). If we follow the proof of Lemma 5.4, we have \( i_1 = \{1, \ldots, k-1, j+1\} \) and \( i_2 = \{1, \ldots, k-1, j\} \).

Let

\[ \text{Neg}_\sigma = X \text{Pos}_\sigma X = \{L^{-1}, L \in \text{Pos}_{\sigma-1}\} \]

\[ = \{\lambda_{i_1}(t_1)\lambda_{i_2}(t_2) \cdots \lambda_{i_k}(t_k); \ t_1, t_2, \ldots, t_k \in (-\infty, 0)\} \]

---

21
where $X = \text{diag}(1,-1,1,-1,\ldots)$ and $\sigma = a_{i_1}a_{i_2}\cdots a_{i_k}$ is any reduced word (therefore $k = \text{inv}(\sigma)$). Of course, each cell $\text{Neg}_\sigma \subset \text{Lo}_{n+1}^1$ is a contractible submanifold of dimension $\text{inv}(\sigma)$, forming the stratification

$$\overline{\text{Neg}}_\eta = \bigcup_{\sigma \in S_{n+1}} \text{Neg}_\sigma.$$ 

Notice that $\overline{\text{Pos}}_\eta \cap \overline{\text{Neg}}_\eta = \{1\}$.

**Lemma 5.6.** Consider an interval $J$ and a locally convex curve $\Gamma : J \to \text{Lo}_{n+1}^1$.

1. If $t_{-1} < t_0 < t_1$ and $\Gamma(t_0) \in \text{Pos}_\eta \subset \partial \text{Pos}_\eta$ for some $\sigma \neq \eta$ then $\Gamma(t_1) \in \text{Pos}_\eta$ and $\Gamma(t_{-1}) \notin \text{Pos}_\eta$.

2. If $t_{-1} < t_0 < t_1$ and $\Gamma(t_0) \in \text{Neg}_\sigma \subset \partial \text{Neg}_\eta$ for some $\sigma \neq \eta$ then $\Gamma(t_{-1}) \in \text{Neg}_\eta$ and $\Gamma(t_1) \notin \text{Neg}_\eta$.

3. If $t_0 < t < t_1$ then $\Gamma(t) \in (\Gamma(t_0) \text{Pos}_\eta) \cap (\Gamma(t_1) \text{Neg}_\eta)$.

**Proof.** As in the first item, assume $\Gamma(t_0) \in \text{Pos}_\sigma$, $\sigma \neq \eta$. From Lemma 5.3 $\Gamma(t_0) \ll \Gamma(t_1)$ and, by definition, $\Gamma(t_0)^{-1}\Gamma(t_1) \in \text{Pos}_\eta$. By Lemma 5.2 $\Gamma(t_1) = \Gamma(t_0)\Gamma(t_0)^{-1}\Gamma(t_1) \in \text{Pos}_\eta$, proving the first claim. Assume by contradiction that $\Gamma(t_{-1}) \in \overline{\text{Pos}}_\eta$; from the claim just proved, $\Gamma(t_0) \in \text{Pos}_\eta$, a contradiction. The second item is analogous. The third item follows from the previous ones.

**Lemma 5.7.** Consider a reduced word $a_{i_1}\cdots a_{i_m} = \eta$; consider

$$L = \lambda_{i_1}(t_1)\cdots \lambda_{i_m}(t_m) \in \text{Pos}_\eta,$$

$t_1,\ldots,t_m > 0$.

Then $\lambda_{i_1}(t) \ll L$ if and only if $t < t_1$ and $\lambda_{i_1}(t) \leq L$ if and only if $t \leq t_1$.

**Proof.** Let $\sigma_1 = a_{i_1}\eta = a_{i_2}\cdots a_{i_m} \lambda \eta$; let

$$L_1 = \lambda_{i_1}(-t_1)L = \lambda_{i_2}(t_2)\cdots \lambda_{i_m}(t_m) \in \text{Pos}_{\sigma_1} \subset \overline{\text{Pos}}_\eta.$$ 

By definition, $\lambda_{i_1}(t) \ll L$ if and only if $\lambda_{i_1}(t_1-t)L_1 \in \text{Pos}_\eta$: this clearly holds for $t < t_1$. For $t = t_1$, we have $\lambda_{i_1}(t_1-t)L_1 = L_1 \in \text{Pos}_{\sigma_1}$ and therefore $\lambda_{i_1}(t) \leq L$, $\lambda_{i_1}(t) \ll L$.

Finally, assume by contradiction that for $t > t_1$ we have $\lambda_{i_1}(t_1-t)L_1 \in \text{Pos}_\sigma \subset \overline{\text{Pos}}_\eta$. If $a_{i_1} \sigma < \sigma$ consider a reduced word $\sigma = a_{i_1}a_{i_2}\cdots a_{i_k}$ and write

$$\lambda_{i_1}(t_1-t)L_1 = \lambda_{i_1}(t_1)\lambda_{i_2}(t_2)\cdots \lambda_{i_k}(t_k)$$

so that

$$L_1 = \lambda_{i_1}(t-t_1+\tau_1)\lambda_{i_2}(\tau_2)\cdots \lambda_{i_k}(\tau_k) \in \text{Pos}_\sigma,$$

22
which implies $\sigma = \sigma_1$, contradicting $a_i\sigma < \sigma$. We thus have $a_i\sigma > \sigma$: consider a reduced word $\sigma = a_j \cdots a_k$ and write
\[
\lambda_i(t_1 - t)L_1 = \lambda_j(\tau_1)\lambda_j(\tau_2) \cdots \lambda_j(\tau_k)
\]
so that
\[
L_1 = \lambda_i(t - t_1)\lambda_j(\tau_1)\lambda_j(\tau_2) \cdots \lambda_j(\tau_k) \in \text{Pos}_{a_i\sigma},
\]
which implies $a_i\sigma = \sigma_1$, contradicting $a_i\sigma > \sigma$. \hfill $\Box$

6 Accessibility in triangular coordinates

For $L_x \in \text{Pos}_\eta \subset \text{Lo}_n$ we shall be interested in the interval
\[
[I, L_x) = \overline{\text{Pos}_\eta} \cap (L_x \text{ Neg}_\eta) = \{ L \in \text{Lo}_n \mid I \leq L \ll L_x \} = \bigcup_{\sigma \in S_n} \text{Ac}_\sigma(L_x)
\]
where the strata $\text{Ac}_\sigma(L_x) \subset \text{Pos}_\sigma$ are
\[
\text{Ac}_\sigma(L_x) = [I, L_x) \cap \text{Pos}_\sigma = \{ L \in \text{Pos}_\sigma \mid L \ll L_x \}.
\]
The sets $\text{Ac}_\sigma(L_x)$ will be called accessibility sets, suggesting that for $L \in \text{Pos}_\sigma$, $L \in \text{Ac}_\sigma(L_x)$ if and only if there exists a locally convex curve $\Gamma : [0, 1] \to \text{Lo}_n$ with $\Gamma(0) = L$ and $\Gamma(1) = L_x$.

**Example 6.1.** Take $n = 2$ and write $L(x, y, z)$ as in Equation 12
\[
L_x = L(x, y, z) = \lambda_1(c_1)\lambda_2(c_2)\lambda_1(c_3) = \lambda_2(\tilde{c}_1)\lambda_1(\tilde{c}_2)\lambda_2(\tilde{c}_3),
\]
\[
c_1 = x - \frac{z}{y}, \quad c_2 = y, \quad c_3 = \frac{z}{y}, \quad \tilde{c}_1 = y - \frac{z}{x}, \quad \tilde{c}_2 = x, \quad \tilde{c}_3 = \frac{z}{x}.
\]
We can now explicitly describe the strata $\text{Ac}_\sigma = \text{Ac}_\sigma(L_x)$. The first stratum is a point: $\text{Ac}_{\sigma} = \{ I \}$. Next we have line segments:
\[
\text{Ac}_{a_1} = \{ \lambda_1(t_1) \mid t_1 \in (0, c_1) \}, \quad \text{Ac}_{a_2} = \{ \lambda_2(\tilde{t}_1) \mid t_2 \in (0, \tilde{c}_1) \}.
\]
The next strata are surfaces:
\[
\text{Ac}_{a_1a_2} = \{ \lambda_1(t_1)\lambda_2(t_2) \mid t_1 \in (0, c_1), t_2 \in (0, g_2(t_1)) \}, \quad g_2(t_1) = \frac{c_2c_3}{c_1 + c_3 - t_1},
\]
\[
\text{Ac}_{a_2a_1} = \{ \lambda_2(\tilde{t}_1)\lambda_1(\tilde{t}_2) \mid \tilde{t}_1 \in (0, \tilde{c}_1), \tilde{t}_2 \in (0, \tilde{g}_2(\tilde{t}_1)) \}, \quad \tilde{g}_2(\tilde{t}_1) = \frac{\tilde{c}_2\tilde{c}_3}{\tilde{c}_1 + \tilde{c}_3 - \tilde{t}_1}.
\]
Translating this parametrization back to $(x, y, z)$ coordinates shows that $\text{Ac}_{a_1a_2}$ is contained in the plane $z = 0$ and $\text{Ac}_{a_2a_1}$ is contained in the hyperbolic paraboloid $z = xy$. Finally, the open stratum $\text{Ac}_\eta$ can be described as
\[
\text{Ac}_\eta = \{ \lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3) \mid t_1 \in (0, c_1), t_2 \in (0, g_2(t_1)), t_3 \in (0, g_3(t_1, t_2)) \}
\]
\[
= \{ \lambda_2(\tilde{t}_1)\lambda_3(\tilde{t}_2)\lambda_2(\tilde{t}_3) \mid \tilde{t}_1 \in (0, \tilde{c}_1), \tilde{t}_2 \in (0, \tilde{g}_2(\tilde{t}_1)), \tilde{t}_3 \in (0, \tilde{g}_3(\tilde{t}_1, \tilde{t}_2)) \}, \quad g_3(t_1, t_2) = \frac{c_2(c_1 - t_1)}{c_2 - t_2}, \quad g_3(\tilde{t}_1, \tilde{t}_2) = \frac{\tilde{c}_2(\tilde{c}_1 - \tilde{t}_1)}{\tilde{c}_2 - \tilde{t}_2}.
\]
A *quasiproduct* is a finite sequence \((X_j)_{1 \leq j \leq k}\) of open sets \(X_j \subset (0, +\infty)^j\) such that there exist a constant \(c_1 \in (0, +\infty)\) and continuous functions \(g_j : X_{j-1} \to (0, +\infty)\) for \(2 \leq j \leq k\) such that \(X_1 = (0, c_1)\) and

\[
X_j = \{(t_1, \ldots, t_{j-1}, t_j) \in X_{j-1} \times (0, +\infty) \mid t_j < g_j(t_1, \ldots, t_{j-1})\}, \quad 2 \leq j \leq k.
\]

Notice that \(X_k\) is homeomorphic to \(\mathbb{R}^k\).

**Lemma 6.2.** If \(L_x = \text{Pos}_\eta\), each stratum \(\text{Ac}_\sigma(L_x)\) is an open, bounded and contractible subset of \(\text{Pos}_\sigma\). Moreover, if \(\sigma = \sigma_k = a_{i_1} \cdots a_{i_k}\) is a reduced word and \(\sigma_j = a_{i_1} \cdots a_{i_j}, j \leq k\), then

\[
\text{Ac}_{\sigma_j}(L_x) = \{\lambda_{i_1}(t_1) \cdots \lambda_{i_j}(t_j) \mid (t_1, \ldots, t_j) \in X_j\}
\]

where the sequence \((X_j)_{1 \leq j \leq k}\) is a quasiproduct; the functions \(g_i : X_{i-1} \to (0, +\infty)\) are rational and bounded.

Example 6.1 above illustrates this claim for \(n = 2\).

**Proof.** Notice that \(I \leq L \ll L_x\) implies that \(L \in \text{Pos}_\eta\) and that there exists \(\tilde{L} \in \text{Pos}_\eta\) with \(L \tilde{L} = L_x\). Computing \((L_x)_{ij}\) in this product yields \(0 \leq (L)_{ij} \leq (L_x)_{ij}\); it follows that the interval \([I, L_x]\) is bounded.

The proof is by induction on \(k = \text{inv}(\sigma)\); the case \(k = 1\) is easy. Write

\[
X_j = \{(t_1, \ldots, t_j) \in (0, +\infty)^j \mid \lambda_{i_1}(t_1) \cdots \lambda_{i_j}(t_j) \ll L_x\}.
\]

We assume by induction that \((X_j)_{1 \leq j \leq k-1}\) is a quasiproduct; we need to construct the function \(g_k : X_{k-1} \to (0, +\infty)\) that obtains \(X_k\).

Let \(\eta = a_{j_1} \cdots a_{j_m}\) be a reduced word with \(j_1 = i_k\). Given \((t_1, \ldots, t_{k-1}) \in X_{k-1}\), let

\[
L_{\sigma_{k-1}} = \lambda_{i_1}(t_1) \cdots \lambda_{i_{k-1}}(t_{k-1}) \in \text{Ac}_{\sigma_{k-1}}(L_x) \subset \text{Pos}_{\sigma_{k-1}}
\]

and write

\[
L_{\sigma_{k-1}}^{-1} L_x = \lambda_{j_1}(\tau_1) \cdots \lambda_{j_m}(\tau_m) \in \text{Pos}_\eta
\]

so that \(\tau_1 > 0\) is a function of \((t_1, \ldots, t_{k-1})\): define \(g_k(t_1, \ldots, t_{k-1}) = \tau_1\). It follows from Equation 11 that \(g\) is a rational function. As in Lemma 5.7, \(\lambda_{i_k}(t) \ll L_{\sigma_{k-1}}^{-1} L_x\) if and only if \(t < g_k(t_1, \ldots, t_{k-1})\), as claimed. \(\square\)
7 Bruhat cells

For \( \sigma \in S_{n+1} \), let \( \text{Bru}_\sigma \subset \text{SO}_{n+1} \) be the unsigned Bruhat cell

\[
\text{Bru}_\sigma = \{ Q \in \text{SO}_{n+1} \mid \exists U_0, U_1 \in \text{Up}_{n+1}, Q = U_0 P_\sigma U_1 \};
\]

notice that this set is not connected. The set \( \text{Bru}_\sigma \subset \text{SO}_{n+1} \) is the lift of the corresponding Schubert cell \( \mathcal{C}_\sigma \subset \text{GL}_{n+1} / \text{Up}_{n+1} \) in the complete flag manifold under the inclusion map. These cells, particularly the intersection of translated Bruhat cells, have been extensively studied \([16, 17, 21, 37, 50, 63, 64]\). As in \([26, 53, 54]\), the signed Bruhat cell \( \text{Bru}_\sigma \) have been extensively studied \([16, 17, 21, 37, 50, 63, 64]\). As in \([26, 53, 54]\), the signed Bruhat cell \( \text{Bru}_\sigma \subset \text{SO}_{n+1} \) for \( Q_0 \in B_{n+1}^+ \) is

\[
\text{Bru}_Q = \{ Q \in \text{SO}_{n+1} \mid \exists U_0, U_1 \in \text{Up}_{n+1}, Q = U_0 Q_0 U_1 \}
\]

where \( \text{Up}_{n+1}^+ \) is the group of upper triangular matrices with positive diagonal. The signed Bruhat cell \( \text{Bru}_Q \) is homeomorphic to the Schubert cell \( \mathcal{C}_{\sigma Q_0} \): the signed Bruhat cells are therefore contractible and disjoint. Each unsigned Bruhat cell is a disjoint union of \( 2^n \) signed Bruhat cells. The preimage of each cell by \( \Pi : \text{Spin}_{n+1} \to \text{SO}_{n+1} \) is a disjoint union of two contractible components: we call these connected components the (lifted) Bruhat cells in \( \text{Spin}_{n+1} \): let \( \text{Bru}_z \) be the connected component of \( \Pi^{-1}[\text{Bru}_z] \) containing \( z \).

The group \( \text{Up}_{n+1}^+ \) of upper triangular matrices with positive diagonal entries acts on \( \text{SO}_{n+1} \): define \( Q^U = Q(U^{-1}Q) \). This action preserves Bruhat cells and may be lifted to an action on the spin group \( \text{Spin}_{n+1} \): we write \( z^U = Q(U^{-1}z) \). Also, if \( U \in \text{Up}_{n+1}^+ \) and \( \Gamma : [0, 1] \to \text{Spin}_{n+1} \) is a locally convex curve then \( \Gamma^U : [0, 1] \to \text{Spin}_{n+1} \), \( \Gamma^U(t) = Q(U^{-1}\Gamma(t)) \), is also a locally convex curve. Also, the nilpotent subgroup \( \text{Up}_{n+1}^1 \) acts simply transitively on open Bruhat cells \( \text{Bru}_\eta \) and transitively on any Bruhat cell \( \text{Bru}_z \). The map \( \text{Spin}_{n+1} \to \text{Spin}_{n+1}, z \mapsto z^U \), can be considered to be induced from the projective transformation

\[
\mathbb{S}^n \to \mathbb{S}^n, \quad v \mapsto \frac{U^{-1}v}{|U^{-1}v|};
\]

we thus say that \( \text{Up}_{n+1}^+ \) acts on \( \text{Spin}_{n+1} \) (or \( \text{Bru}_\sigma \) or \( \text{Bru}_z \)) and on spaces of locally convex curves by projective transformations.

Remark 7.1. Consider \( z_0, z_1, \tilde{z}_0, \tilde{z}_1 \in \text{Spin}_{n+1} \) such that \( \tilde{z}_0^{-1} z_1, \tilde{z}_0^{-1} \tilde{z}_1 \in \text{Bru}_q \). Let \( U \in \text{Up}_{n+1}^1 \) be the only such matrix for which \( (z_0^{-1} z_1)^U = \tilde{z}_0^{-1} \tilde{z}_1 \). We define a projective transformation, a homeomorphism from \( \mathcal{L}_n(z_0; z_1) \) to \( \mathcal{L}_n(\tilde{z}_0; \tilde{z}_1) \), taking \( \Gamma \in \mathcal{L}_n(z_0; z_1) \) to \( \tilde{z}_0(z_0^{-1} \Gamma)^U \in \mathcal{L}_n(\tilde{z}_0; \tilde{z}_1) \). We are particularly interested in the restriction \( \mathcal{L}_{n, \text{convex}}(z_0; z_1) \to \mathcal{L}_{n, \text{convex}}(\tilde{z}_0; \tilde{z}_1) \).

The following result is a simple corollary of these observations; compare with Lemma 5.3.
**Lemma 7.2.** For any \( z \in \text{Bru}_q \) there exists a locally convex curve \( \Gamma : [0, 1] \rightarrow \text{Spin}_{n+1} \), \( \Gamma(0) = 1 \), \( \Gamma(\frac{1}{2}) = z \), \( \Gamma(1) = \hat{q} \) and \( \Gamma(t) \in \text{Bru}_q \) for all \( t \in (0, 1) \).

Moreover, if \( h : K \rightarrow \text{Bru}_q \) is a continuous function then there exists a continuous function \( H : K \times [0, 1] \rightarrow \text{Spin}_{n+1} \) such that for any \( s \in K \) the locally convex curve \( \Gamma_s : [0, 1] \rightarrow \text{Spin}_{n+1} \), \( \Gamma_s(t) = H(s, t) \), satisfies \( \Gamma_s(0) = 1 \), \( \Gamma_s(\frac{1}{2}) = h(s) \), \( \Gamma_s(1) = \hat{q} \) and \( \Gamma_s(t) \in \text{Bru}_q \) for all \( t \in (0, 1) \).

**Proof.** As in Example 4.2 take

\[
\mathfrak{h} = \sum_{i \in [n]} \sqrt{i(n+1-i)} \mathfrak{a}_i \in \mathfrak{so}_{n+1}, \quad \Gamma_0(t) = \exp(\pi t \mathfrak{h}).
\]

Recall that \( \Gamma_0(0) = 1 \), \( \Gamma_0(\frac{1}{2}) = \hat{q} \), \( \Gamma_0(1) = \hat{q} \). Equation \( \Phi \) implies that, for \( t \in (0, 1) \), \( \Gamma_0(t) = \exp(\pi(t - \frac{1}{2}) \mathfrak{h}) = U_1(t)\hat{q}U_2(t) \in \text{Bru}_q \), where

\[
U_1(t) = \hat{q} \exp(-\cot(\pi t)\mathfrak{h}_L) \hat{q}^{-1} \in \text{Up}_n^{+},
\]

\[
U_2(t) = \exp(-\log(\sin(\pi t))[\mathfrak{h}_L, \mathfrak{h}_L]) \exp(\cot(\pi t)\mathfrak{h}_L) \in \text{Up}_n^{+}.
\]

Define \( h_U : K \rightarrow \text{Up}_n^{+} \) by \( \hat{q}^{h_U(s)} = h(s) \); define \( H(s, t) = (\Gamma_0(t))^{h_U(s)} \) and \( \Gamma_s = \Gamma_0^{h_U(s)} \).

If \( q \in \text{CG}_{n+1} \) then \( \text{Bru}_q = \{q\} \). If \( z = q\hat{q} \in \bar{B}_{n+1}^{+}, q \in \text{CG}_{n+1} \), then \( \text{Bru}_z = \mathcal{U}_z \).

If \( z = q\alpha_i, q \in \text{CG}_{n+1} \), then \( \text{Bru}_z = \{qa_i(\theta), \theta \in (0, \pi)\} \) where \( \alpha_i(\theta) = \exp(\theta \mathfrak{a}_i) \) and \( \mathfrak{a}_i \in \mathfrak{so}_{n+1} \) is given by Equation 7 (recall that \( \alpha_i(\frac{\pi}{2}) = \hat{a}_i \)). We generalize this observation below. The reader will notice the similarities between these results and the discussion above concerning total positivity in the triangular group.

**Lemma 7.3.** Let \( q \in \text{CG}_{n+1} \), \( \sigma_0, \sigma_1 \in S_{n+1} \) with \( \sigma_1 < \sigma_0 = \sigma_1a_i, \ z_0 = q\sigma_0, \ z_1 = q\sigma_1 \). Then the map

\[
\Phi : \text{Bru}_{z_1} \times (0, \pi) \rightarrow \text{Bru}_{z_0}, \quad \Phi(z, \theta) = z\alpha_i(\theta)
\]

is a diffeomorphism. Similarly, if \( \sigma_1 < \sigma_0 = a_i\sigma_1 \) then the map

\[
\Phi : (0, \pi) \times \text{Bru}_{z_1} \rightarrow \text{Bru}_{z_0}, \quad \Phi(\theta, z) = \alpha_i(\theta)z
\]

is a diffeomorphism.

Before presenting the proof, we see a few applications.

**Corollary 7.4.** Let \( \sigma = a_{i_1} \cdots a_{i_k} \) be a reduced word (so that \( k = \text{inv}(\sigma) \)). Let \( q \in \text{CG}_{n+1} \). Then the map

\[
\Psi_{(q;i_1,\ldots,i_k)} : (0, \pi)^k \rightarrow \text{Bru}_q, \quad (\theta_1, \ldots, \theta_k) \mapsto q\alpha_{i_1}(\theta_1) \cdots \alpha_{i_k}(\theta_k)
\]

is a diffeomorphism.
Proof. The proof is by induction on \( k \). The cases \( k \leq 1 \) are easy and have been discussed above. The induction step is provided by Lemma \( \text{7.}\)3. \( \square \)

A crucial difference between this case and the triangular case is that \( (0, +\infty) \) and \( \text{Pos}_\eta \) are semigroups (i.e., closed under sums and products, respectively) but \( (0, \pi) \) and \( \text{Bru}_\eta \) are not.

**Corollary 7.5.** Consider \( \sigma_0, \sigma_1 \in S_{n+1} \), \( \sigma = \sigma_0 \sigma_1 \). If \( \text{inv}(\sigma) = \text{inv}(\sigma_0) + \text{inv}(\sigma_1) \) then \( \text{Bru}_{\sigma_0} \text{Bru}_{\sigma_1} = \text{Bru}_\sigma \); moreover, the map

\[
\text{Bru}_{\sigma_0} \times \text{Bru}_{\sigma_1} \rightarrow \text{Bru}_\sigma, \quad (z_0, z_1) \mapsto z_0 z_1
\]

is a diffeomorphism.

**Proof.** This follows directly from Corollary \( \text{7.}\)4. \( \square \)

**Lemma 7.6.** Consider \( \sigma \in S_{n+1} \). Then \( \text{Q}[\text{Pos}_\sigma] \subset \text{Bru}_\sigma \). Furthermore, if \( \sigma \neq e \) then \( \dot{\sigma} \) does not belong to \( \text{Q}[\text{Pos}_\sigma] \). Similarly, \( \text{Q}[\text{Neg}_\sigma] \subset \text{Bru}_\sigma \); if \( \sigma \neq e \) then \( \dot{\sigma} \) does not belong to \( \text{Q}[\text{Neg}_\sigma] \).

**Proof.** The case \( \sigma = e \) is trivial; for \( \sigma = a_i \) we have \( \text{Pos}_\sigma = \{ \lambda_i(t), t \geq 0 \} \) and \( Q(\lambda_i(t)) = (e, \pi) \) (where \( \lambda_i(t) = \alpha_i(\text{arctan}(t)) \) (where \( \alpha_i(\theta) = \exp(\theta a_i) \) and \( \lambda_i(t) = \exp(t a_i) \)). We thus have

\[
\lim_{t \to +\infty} Q(\lambda_i(t)) = \alpha_i \left( \frac{\pi}{2} \right) = \dot{\alpha}_i,
\]

as desired.

We proceed to the induction step. Assume \( \sigma_k = a_{i_1} \cdots a_{i_k} \) (a reduced word) and \( \sigma_{k-1} = a_{i_1} \cdots a_{i_{k-1}} \subset \sigma_k = \sigma_{k-1} a_{i_k} \). Consider \( L_k \in \text{Pos}_{\sigma_k} \); write \( L_k = L_{k-1} - a_{i_k} L_k = L_{k-1} \lambda_{i_k}(t_k), t_k \in (0, +\infty), L_{k-1} \in \text{Pos}_{\sigma_{k-1}} \). By induction, we have \( Q(L_{k-1}) = z_{k-1} \in \text{Bru}_{\sigma_{k-1}} \). Consider the curves \( \Gamma_L : [0, t_k] \rightarrow L_{n+1}^1 \) and \( \Gamma : [0, t_k] \rightarrow \text{Spin}_{n+1} \) defined by \( L_L(t) = L_{k-1} - a_{i_k} L(t) \) and \( \Gamma = Q \circ \Gamma_L \). In particular, \( \Gamma(0) = z_{k-1} \). The curve \( \Gamma_L \) is tangent to the vector field \( X_{a_{i_k}} \) and therefore, from Lemma \( \text{4.1} \) the curve \( \Gamma \) is tangent to the vector field \( X_{a_{i_k}} \). We thus have \( \Gamma(t) = z_{k-1} \alpha_{i_k}(\theta(t)) \) for some smooth increasing function \( \theta : [0, +\infty) \rightarrow [0, +\infty) \). But \( z_{k-1} \in U_1 \) implies \( z_{k-1} \alpha_{i_k}(\pi) = z_{k-1} \alpha_i \in U_0 \) and therefore \( z_{k-1} \alpha_{i_k}(\pi) \notin U_1 \). Thus, we have \( \theta : [0, +\infty) \rightarrow [0, \pi) \). From Lemma \( \text{7.3} \), \( z_k = Q(L_k) \in \text{Bru}_{\sigma_k} \), as desired.

Clearly, for \( \sigma \neq e \) we have \( \dot{\sigma} \notin U_1 \), implying \( \dot{\sigma} \notin Q[\text{Pos}_\sigma] \). The claims concerning \( \text{Neg}_\sigma \) follow from the claims for \( \text{Pos}_\sigma \) either by taking inverses or by similar arguments. \( \square \)

**Corollary 7.7.** Consider \( \sigma_{k-1} \subset \sigma_k = \sigma_{k-1} a_{i_k} \in S_{n+1} \). Consider \( z_{k-1} \in \text{Bru}_{\sigma_{k-1}} \) and \( z_k \in \text{Bru}_{\sigma_k} \), \( z_k = z_{k-1} \alpha_{i_k}(\theta_k), \theta_k \in (0, \pi) \). If \( z_k \in Q[\text{Pos}_\sigma] \) then \( z_{k-1} \in Q[\text{Pos}_{\sigma_{k-1}}] \) and \( z_{k-1} \alpha_{i_k}(\theta) \in Q[\text{Pos}_{\sigma_{k-1}}] \) for all \( \theta \in (0, \theta_k) \).
Proof. Let $\sigma_{k-1} = a_{i_1} \cdots a_{i_{k-1}}$ be a reduced word. Let
\[ L_k = L(z_k) = \lambda_{i_1}(t_1) \cdots \lambda_{i_{k-1}}(t_{k-1})\lambda_{i_k}(t_k). \]

Define $\tilde{L}_{k-1} = \lambda_{i_1}(t_1) \cdots \lambda_{i_{k-1}}(t_{k-1})$ and $\tilde{z}_{k-1} = Q(\tilde{L}_{k-1}).$ The curve $\Gamma_L : [0,t_k] \to L_0^{1,1}, \Gamma_L(t) = \tilde{L}_{k-1}\lambda_{i_k}(t)$ is taken to $\Gamma = Q \circ \Gamma_L$ with $\Gamma(t_k) = z_k$ and $\Gamma(t) = \tilde{z}_{k-1}\alpha_{i_k}(\theta(t))$ for some strictly increasing function $\theta$. Invertibility of the map $\Phi$ in Lemma 7.3 implies that $\tilde{z}_{k-1} = z_{k-1}$. Furthermore, $z_{k-1}\alpha_{i_k}(\theta) = \Gamma(t)$ for some $t \in (0,t_k).$ □

Proof of Lemma 7.3. We prove the first claim; the second one is similar. Notice that $\Phi(z_1, q) = z_0.$

We first prove that for all $z \in \text{Bru}_{z_1}$ and $\theta \in (0,\pi)$ we have $\Phi(z, \theta) \in \text{Bru}_{z_0}$. Given the initial remark and connectivity, it suffices to prove that $\Pi(\Phi(z, \theta)) \in \text{Bru}_{\sigma_0}$ (the unsigned Bruhat cell). Abusing the distinction between $z \in \text{Spin}_{n+1}$ and $\Pi(z) \in \text{SO}_{n+1},$ consider $z = U_1q\sigma_1U_2 \in \text{Bru}_{z_1}$ and $\theta \in (0,\pi).$ We have $\Phi(z, \theta) = U_1q\sigma_1U_2\alpha_i(\theta).$ We have $U_2\alpha_i(\theta) = \hat{a}_i\lambda_i(t)U_3$ for some $t \in \mathbb{R}$ and $U_3 \in U_{P_{n+1}^+}$ and therefore $\Phi(z, \theta) = U_1q\sigma_0\lambda_i(t)U_3.$ But since $\sigma_1 \equiv \sigma_0,$ we have $\delta_0\lambda_i(t) = U_4\delta_0$ where $U_4 \in U_{P_{n+1}^+}$ has at most a single nonzero nondiagonal entry at position $(\sigma^{-1}_0, (i + 1)\sigma^{-1}_0) = ((i + 1)\sigma^{-1}_0, i\sigma^{-1}_0).$ We have $\Phi(z, \theta) = U_1qU_4\delta_0U_3,$ as desired.

At this point we know that $\Phi : \text{Bru}_{z_1} \times (0,\pi) \to \text{Bru}_{z_0}$ is a smooth function. It is also injective. Indeed, assume $z\alpha_i(\theta) = \tilde{z}\alpha_i(\tilde{\theta}).$ If $\theta < \tilde{\theta}$ we have both $z \in \text{Bru}_{z_1}$ and $z = \tilde{z}\alpha_i(\tilde{\theta} - \theta) \in \text{Bru}_{z_0},$ contradicting the disjointness of the cells. The case $\theta > \tilde{\theta}$ is similar and the case $\theta = \tilde{\theta}$ is trivial.

Given $U_2 \in U_{P_{n+1}^+},$ the matrix $\hat{a}_iU_2$ is almost upper, with a positive entry in position $(i + 1,i).$ There exist unique $r > 0$ and $\theta \in (0,\pi)$ such that $(\hat{a}_iU_2)_{i+1,i} = r\sin(\theta),$ $(\hat{a}_iU_2)_{i+1,i+1} = r\cos(\theta).$ The matrix $U_3 = \hat{a}_iU_2\alpha_i(-\theta)$ therefore also belongs to $U_{P_{n+1}^+}.$ Let $\theta : U_{P_{n+1}^+} \to (0,\pi),$ $U_2 \mapsto \theta,$ be the smooth function defined by the above argument.

Given $z \in \text{Bru}_{z_0},$ write $z = qU_1\sigma_0U_2,$ $U_1 \in U_{P_{n+1}^+}$. Notice that
\[ z\alpha_i(-\theta_i(U_2)) = qU_1\sigma_1(\hat{a}_iU_2\alpha_i(-\theta_i(U_2))) = qU_1\sigma_1U_3 \in \text{Bru}_{z_1}. \]

Thus, $\Phi(z\alpha_i(-\theta_i(U_2)), \theta_i(U_2)) = z$, proving surjectivity of $\Phi$. Injectivity implies that even though $U_2$ is not well defined (as a function of $z$), $\theta_i(U_2)$ is well defined (and smooth, again as a function of $z$): this gives a formula for $\Phi^{-1}$ and proves its smoothness. □

Remark 7.8. The following function constructed in the proof will be used later. For $z_0 = q\sigma_0,$ $q \in \text{CG}_{n+1},$ $\sigma_0 \in S_{n+1},$ $a_i \leq L \sigma_0$ there is a real analytic function $\Theta_i : \text{Bru}_{z_0} \to (0,\pi).$ Set $\sigma_1 \equiv \sigma_0 = \sigma_1a_i$ and $z_1 = q\sigma_1;$ we can define $\Theta_i(z) = \theta \in (0,\pi)$ if and only if $z\alpha_i(-\theta) \in \text{Bru}_{z_1}.$
Lemma 7.9. Consider $z_0 = q \tilde{q} \in \tilde{B}^+_{n+1} \subset \text{Spin}_{n+1}$, $q \in CG_{n+1}$, $\sigma \in S_{n+1}$, $\sigma \neq \eta$, $k = \text{inv}(\eta) - \text{inv}(\sigma) > 0$. There exists a smooth function $f = (f_1, \ldots, f_k) : U_{q_0} \to \mathbb{R}^k$ with the following properties. For all $z \in U_{q_0}$, the derivative $Df(z)$ is surjective. For all $z \in U_{q_0}$, $z \in \text{Bru}_{z_0}$ if and only if $f(z) = 0$. For any smooth locally convex curve $\Gamma : (-\epsilon, \epsilon) \to U_{q_0}$ we have $(f_k \circ \Gamma)'(t) > 0$ for all $t \in (-\epsilon, \epsilon)$.

Proof. We present an explicit construction of the coordinate functions $f_i$. Write $L_{n+1} = \text{Lo}_{\sigma-1} \text{Lo}_{\sigma-1}^{\sigma}$, i.e., write $L \in L_{n+1}$ as $L = L_1 L_2$, $L_1 \in \text{Lo}_{\sigma-1}$, $L_2 \in \text{Lo}_{\sigma-1}^{\sigma}$ (see Equation 3 in Section 2 for the subgroups $L_1 \subseteq L_{p+1}$). Notice that if $L_1 \in \text{Lo}_{\sigma-1}$ then $z_0 L_1 = U_1 z_0$ for $U_1 = z_0 L_1 z_0^{-1} \in U_\sigma$. Thus, every $z \in U_{q_0}$ can be uniquely written as $z = Q(U_1 z_0 L_2)$, $U_1 \in U_\sigma$, $L_2 \in \text{Lo}_{\sigma-1}^{\sigma}$. Notice that if $U_1, U_2 \in U_\sigma$ and $L_2 \in \text{Lo}_{\sigma-1}^{\sigma}$ then $Q(U_1 z_0 L_2)$ and $Q(U_2 z_0 L_2)$ belong to the same Bruhat cell. Also, $z = Q(U_1 z_0 L_2) \in \text{Bru}_{z_0}$ if and only if $L_2 = I$. The function $f$ can be defined in terms of $z_0 L_2 \in z_0 \text{Lo}_{\sigma-1}^{\sigma}$; in other words, we define an affine function $f_I : z_0 \text{Lo}_{\sigma-1}^{\sigma} \to \mathbb{R}^k$ and set $f(Q(U_1 z_0 L_2)) = f_I(z_0 L_2)$.

We describe a generic element of the set $z_0 \text{Lo}_{\sigma-1}^{\sigma}$, or, more concretely, of $\Pi(z_0) \text{Lo}_{\sigma-1}^{\sigma} \subset \text{GL}_{n+1}$. Start with the matrix $Q_0 = \Pi(z_0)$. In order to obtain $M \in Q_0 \text{Lo}_{\sigma-1}^{\sigma} = z_0 \text{Lo}_{\sigma-1}^{\sigma}$, we are free to change the empty entries of $Q_0$ which are below and to the left of nonzero entries of $Q_0$. Call these entries $x_1, \ldots, x_k$, where we number them as we read, top to bottom and left to right. Apply the sign of the entry of $Q_0$ on the same row. Thus, for instance, the matrix $Q_0$ below yields to following $M$:

$$Q_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -x_1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ x_2 & x_3 & 1 & 0 \end{pmatrix} \in z_0 \text{Lo}_{\sigma-1}^{\sigma}.$$

Finally, set $f_I(M) = (x_1, \ldots, x_k)$. If $x_k$ is in position $(i, j)$ set $\bar{k} = n - i + 2$, $i_0 = \{i, \ldots, n + 1\}$ and $i_2 = \{1, \ldots, \bar{k} - 1, j\}$ (see Lemma 5.4). The desired property of $f_k = \pm(\Lambda^j(\tilde{M}))_{i_0,i_2}$ follows from Remark 5.5. Equivalently, if $\Gamma(t) = z_0 Q(\Gamma(t))$ then $f_k(\Gamma(t)) = (\Gamma(t))_{j+1,j}$, which is clearly strictly increasing with positive derivative. 

Remark 7.10. The explicit construction of the function $f$ will be used again. We shall also consider the smooth projection $\Pi_{z_0} : U_{z_0} \to \text{Bru}_{z_0} \subset U_{q_0}$ defined by $\Pi_{z_0}(Q(U_1 z_0 L_2)) = Q(U_1 z_0)$.

8 Chop, advance and the singular set

The maps chop, adv : $\text{Spin}_{n+1} \to \tilde{\eta} CG_{n+1} \subset \tilde{B}^+_{n+1}$ are defined by

$$\text{adv}(z) = q_0 \eta, \quad \text{chop}(z) = q_0 \eta, \quad z \in \text{Bru}_{z_0} \subset \text{Bru}_{\sigma_0}, \quad z_0 = q_0 \sigma_0 = q_0 \eta_0.$$
For $\sigma_1 = \eta \sigma_0$, we have $\text{adv}(z) = z_0 \ \text{acute}(\sigma_1^{-1}) = z_0(\hat{\sigma}_1)^{-1}$ and $\text{chop}(z)\hat{\sigma}_1 = z_0$. In particular, $\text{adv}(z) = \text{chop}(z)\hat{\sigma}_1$.

**Lemma 8.1.** For $z \in \text{Spin}_{n+1}$, let $\Gamma : (-\epsilon, \epsilon) \to \text{Spin}_{n+1}$ be a locally convex curve such that $\Gamma(0) = z$. There exists $\epsilon_a \in (0, \epsilon)$ such that for all $t \in (0, \epsilon_a)$, $\Gamma(t) \in \text{Bru}_{\text{adv}(z)}$. There exists $\epsilon_c \in (0, \epsilon)$ such that for all $t \in [-\epsilon_c, 0)$, $\Gamma(t) \in \text{Bru}_{\text{chop}(z)}$.

**Proof.** If necessary, apply a projective transformation so that $z = q_a Q(L_0)$, $L_0 \in \text{Pos}_{\sigma_0}$. For any locally convex curve $\Gamma$ as in the statement, there exists $\epsilon_a \in (0, \epsilon)$ such that the restriction $\Gamma|_{[-\epsilon_a, \epsilon_a]}$ can be written in triangular coordinates: $\Gamma(t) = q_a Q(\Gamma_L(t))$, $\Gamma_L(0) = L_0$. It follows from Lemma 5.6 that $\Gamma_L(t) \in \text{Pos}_{\sigma_0}$ for any $t \in (0, \epsilon_a]$. Thus, $\Gamma(t) \in \text{Bru}_{\text{adv}(z)}$ for all $t \in (0, \epsilon_a]$. The proof for chop is similar. $\square$

The chopping map was introduced in [54], where a different combinatorial description is given, together with the topological characterization in Lemma 8.1. The notations $a = \eta = \text{chop}(1)$ and $A = \Pi(a)$ are used there; $A$ is called the Arnold matrix.

For $z_0, z_1 \in \text{Spin}_{n+1}$, let $\mathcal{L}_n(z_0; z_1)$ be the space of locally convex curves $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ such that $\Gamma(0) = z_0$ and $\Gamma(1) = z_1$. There are many possible choices for the exact definition and topology of these spaces. As a small example, one can admit only smooth curves and consider the (Fréchet) topology induced by the family of $C^k$-seminorms in $C^\infty([0, 1], \mathbb{S}^n)$. As a large example, one can admit all the absolutely continuous curves (as in Section 4, next to Equation 10) with the required condition on the logarithmic derivative. For most arguments, the exact choice is immaterial. This issue is discussed in detail in Appendix B.

Clearly, $\mathcal{L}_n(z_0; z_1)$ is homeomorphic to $\mathcal{L}_n(1; z_0^{-1} z_1)$; let $\mathcal{L}_n(z) = \mathcal{L}_n(1; z)$. If $z \in \text{Bru}_{q_0}$ then a projective transformation yields a homeomorphism between $\mathcal{L}_n(z)$ and $\mathcal{L}_n(z_0)$. In [54] (Prop. 6.4) it is proved that if $\text{chop}(z_0) = \text{chop}(z_1)$ then $\mathcal{L}_n(z_0)$ and $\mathcal{L}_n(z_1)$ are homeomorphic (for nice topologies).

Define
\[
\mathcal{L}_n = \bigsqcup_{q \in CG_{n+1}} \mathcal{L}_n(q).
\]
Thus, understanding $\mathcal{L}_n$ is in a sense sufficient to understand all spaces $\mathcal{L}_n(z_0; z_1)$. Moreover, the particularly interesting cases $\mathcal{L}_n(I)$ and, for $n$ odd, $\mathcal{L}_n(-I)$, are explicitly contained in $\mathcal{L}_n$. It follows from Proposition A.1 in Appendix A that $q = \eta$ is the unique $q \in CG_{n+1}$ for which $\mathcal{L}_n(q)$ contains convex curves: the space $\mathcal{L}_n$ has precisely $1 + 2(n+1)$ connected components.

Recall that $\text{Bru}_z \subset \text{Spin}_{n+1}$ is open if and only if $z \in \eta CG_{n+1}$. Define the singular set $\text{Sing}_{n+1} \subset \text{Spin}_{n+1}$ as the complement of the open cells:
\[
\text{Sing}_{n+1} = \text{Spin}_{n+1} \setminus \text{Bru}_\eta = \bigsqcup_{z \in \tilde{E}_{n+1}^+ \setminus (\eta CG_{n+1})} \text{Bru}_z.
\]
The **singular set** of a locally convex curve \( \Gamma : [t_0, t_1] \rightarrow \text{Spin}_{n+1} \) is

\[
\text{sing}(\Gamma) = \Gamma^{-1}[\text{Sing}_{n+1}] \setminus \{t_0, t_1\}.
\]

**Lemma 8.2.** Given a locally convex curve \( \Gamma : [t_0, t_1] \rightarrow \text{Spin}_{n+1} \), the set \( \text{sing}(\Gamma) \) is finite.

**Proof.** We know from Lemma 8.1 that for each \( \tau \in [t_0, t_1] \) there exists \( \epsilon > 0 \) such that \((\tau - \epsilon, \tau + \epsilon) \cap \text{sing}(\Gamma) \subseteq \{\tau\}\). Take a finite subcover of \([t_0, t_1]\).

A curve \( \Gamma \in \mathcal{L}_n(z_0; z_1) \) is **convex** if and only if \( \text{sing}(z_0^{-1} \Gamma) = \emptyset \). We prove in Appendix A the equivalence between this notion of convexity and the geometric, more classical one used for instance in \([34, 53, 59, 65]\). In particular, if \( \Gamma : J_0 \rightarrow \text{Spin}_{n+1} \) is convex and \( J_1 \subset J_0 \) then \( \Gamma|_{J_1} \) is convex. Also, if \( \Gamma : J_0 \rightarrow \text{Spin}_{n+1} \) is locally convex with \( J_0 = [t_0, t_1] \) and the restrictions \( \Gamma|_{[t_0+\epsilon, t_1-\epsilon]} \) are convex (for all \( \epsilon > 0 \)), then \( \Gamma \) is also convex. It then follows that \( \text{sing}(\Gamma) = \emptyset \) implies that \( \Gamma \) is convex. The reciprocal is not true. Indeed, for any \( \sigma \in S_{n+1}, \sigma \neq \eta \), take \( \Gamma : [-\epsilon, \epsilon] \rightarrow \text{Spin}_{n+1}, \Gamma(t) = \sigma e^{\epsilon t} \)). For small \( \epsilon > 0 \), \( \Gamma \) is convex but \( 0 \in \text{sing}(\Gamma) \) (see also Example 9.2 below).

The following lemma is closely related to the known fact that convex curves form a connected component of \( \mathcal{L}_n(\hat{\eta}) \): it shows more generally that, when a curve is deformed, points in the singular set may join or split but never vanish or appear out of nowhere.

**Lemma 8.3.** Let \( K \) be a compact set; let \( H : K \rightarrow \mathcal{L}_n(z_0; z_1) \) be a continuous function. Let

\[
K_1 = \bigsqcup_{s \in K} (\{s\} \times \text{sing}(H(s))) = \{(s, t) \in K \times (0, 1) \mid H(s)(t) \in \text{Sing}_{n+1}\}.
\]

Then \( K_1 \) is a compact set and satisfies the following condition:

\[
\forall (s_0, t_0) \in K_1, \forall \epsilon > 0, \exists \delta > 0, \forall s \in K, |s - s_0| < \delta \rightarrow (\exists t \in (0, 1), (s, t) \in K_1, |t - t_0| < \epsilon).
\]

We comment a little before the proof. Recall that the Hausdorff distance between two compact nonempty sets \( X, Y \subset (0, 1) \) is

\[
d_H(X, Y) = \max \left( \sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y| \right).
\]

Let \( \mathcal{H}((0, 1)) \subset \mathcal{P}((0, 1)) \) be the set of compact subsets of \((0, 1)\); this is a metric space with the Hausdorff distance where the empty set is an isolated point.

**Corollary 8.4.** The map \( \text{sing} : \mathcal{L}_n(z_0; z_1) \rightarrow \mathcal{H}((0, 1)) \) is continuous.
Proof. It follows from the condition in Lemma 8.3 that the composite $\text{sing} \circ H$ is continuous. Since $\mathcal{L}_n(z_0; z_1)$ is metrizable and $K$ is arbitrary, this implies the continuity of the map $\text{sing} : \mathcal{L}_n(z_0; z_1) \to \mathcal{H}((0, 1))$. \hfill $\square$

Let $\mathcal{L}_{n, \text{convex}}(z_0; z_1) \subset \mathcal{L}_n(z_0; z_1)$ be the subset of convex curves. In particular, we write $\mathcal{L}_{n, \text{convex}}(z) = \mathcal{L}_{n, \text{convex}}(1; z)$. The following result is well known \cite{5, 59, 65} and is presented here for completeness and as an example of an application.

**Lemma 8.5.** If $z \in \bar{B}^+_{n+1}$ then the subset $\mathcal{L}_{n, \text{convex}}(z) \subset \mathcal{L}_n(z)$ is either empty or a contractible connected component. It is nonempty if and only if $\text{chop}(z) = \eta$.

Proof. From Corollary 8.4 and the fact that $\emptyset \in \mathcal{H}((0, 1))$ is an isolated point it follows that $\mathcal{L}_{n, \text{convex}}(z)$ is a union of connected components. It suffices to prove that it is contractible.

Consider first the case $z \in \text{Bru}_q$. By applying a projective transformation we may assume $z = \eta = \exp(\frac{n}{2} \eta)$. Take $\Gamma_0 \in \mathcal{L}_{n, \text{convex}}(\eta)$, $\Gamma_0(t) = \exp(\frac{n}{2} \eta t)$. For $s \in (0, 1)$, let $U_s \in \text{Up}^1_{n+1}$ be such that $\eta U_s = \Gamma_0(s)$. For $\Gamma_1 \in \mathcal{L}_{n, \text{convex}}(\eta)$ and $s \in (0, 1]$ define $\Gamma_s \in \mathcal{L}_{n, \text{convex}}(\eta)$ by:

$$
\Gamma_s(t) = \begin{cases} 
(\Gamma_1(t))^U_s, & t \in [0, s], \\
\Gamma_0(t), & t \in [s, 1].
\end{cases}
$$

The map $[0, 1] \to \mathcal{L}_{n, \text{convex}}(\eta), s \mapsto \Gamma_s$ is continuous (even at $s = 0$).

The general case follows from the chopping lemma (Prop. 6.4 in \cite{521}); see also Remark 9.6 \hfill $\square$

**Proof of Lemma 8.3.** Write $\tilde{H}(s, t) = \tilde{z}_{H(s)}(t) \in \text{Spin}_{n+1}$ so that $\tilde{H} : K \times [0, 1] \to \text{Spin}_{n+1}$ is continuous. We assume without loss of generality that $H(s, 1) = z_1 \in \text{CG}_{n+1}$ is fixed for all $s \in K$. Notice that $K_2 \subseteq K \times [0, 1]$ defined by

$$K_2 = K_1 \cup (K \times \{0, 1\}) = \tilde{H}^{-1}[X], \quad X = \sqcup_{\sigma \neq \eta} \text{Bru}_\sigma,$$

is closed and therefore compact. Furthermore, the sets $A_0 = \tilde{H}^{-1}(\text{Bru}_{\text{adv}(1)})$ and $A_1 = \tilde{H}^{-1}(\text{Bru}_{\text{chop}(z_1)})$ are open and disjoint from $K_2$. From Lemma 8.1, for each $s \in K$ there exists $\epsilon_s > 0$ such that $\{s\} \times (0, \epsilon_s) \subset A_0$ and $\{s\} \times (1 - \epsilon_s, 1) \subset A_1$. By compactness of $K$ there exists $\epsilon_s > 0$ such that $K \times (0, \epsilon_s) \subset A_0$ and $K \times (1 - \epsilon_s, 1) \subset A_1$, implying the compactness of $K_1 = K_2 \setminus (K \times \{0, 1\})$.

Consider now $(s_0, t_0) \in K_1$. Take $t_1 \in (t_0, 1]$ (resp. $t_{-1} \in [0, t_0)$) minimal (resp. maximal) such that $(s_0, t_1) \in K_2$ (resp. $(s_0, t_{-1}) \in K_2$); recall that $(s_0, t) \in K_2$ if and only if $\sigma(H(s_0, t)) \neq e$. Take $\epsilon_1 < \min(\epsilon_1, |t_0 - t_1|, |t_0 - t_{-1}|)$ and $\tilde{z} \in \text{CG}_{n+1}$ such that $\tilde{H}(s_0, t_0 - \epsilon_1) \in \text{Bru}_{\text{chop}(\tilde{z})}$. Applying a projective transformation, we may assume that $\tilde{H}(s_0, t_0 - \epsilon_1) \in \tilde{z} Q[\text{Neg}_\eta] \subset U_{z_1}$. Define $\Gamma_{s_0}(t) = L(\tilde{z}^{-1}\tilde{H}(s_0, t))$ in the maximal interval around $t_0 - \epsilon_1$. The curve $\Gamma_{s_0}$ is
locally convex and satisfies $\Gamma_s(t_0 - \epsilon_1) \in \text{Neg}_\eta$; it can not go to infinity without first leaving $\text{Neg}_\eta$, and it can not leave $\text{Neg}_\eta$ before $t_0$. It follows that there exists $\epsilon_2 < \epsilon_1$ such that $\Gamma_s$ is defined in $[t_0 - \epsilon_2, t_0 + \epsilon_2]$, $\Gamma_s(t_0) \in \partial \text{Neg}_\eta$. Take $r > 0$ such that the balls of radius $r$ centered at $\Gamma_s(t_0 - \epsilon_2)$ are contained in $\text{Neg}_\eta$, and $\text{Lo} \{n+1\} \setminus \text{Neg}_\eta$, respectively. Take $\delta > 0$ such that $|s - s_0| < \delta$ implies $\tilde{H}(s, t) \in \mathcal{U}_\delta$ for all $t \in [t_0 - \epsilon_2, t_0 + \epsilon_2]$, $|\Gamma_s(t_0 - \epsilon_2) - \Gamma_{s_0}(t_0 - \epsilon_2)| < r$ and $|\Gamma_s(t_0 + \epsilon_2) - \Gamma_{s_0}(t_0 + \epsilon_2)| < r$ (where $\Gamma_s(t) = \mathbf{L}(z^{-1} \tilde{H}(s, t))$). The locally convex curve $\Gamma_s$ must cross $\partial \text{Neg}_\eta$ at some $t \in (t_0 - \epsilon_2, t_0 + \epsilon_2)$: we have $(s, t) \in \mathcal{K}_1$, as desired.

9 Accessibility in the spin group

For $z_\chi \in Q[\text{Pos}_\eta] \subseteq \mathcal{U}_1 \cap \text{Bru}_\eta$ and $\sigma \in S_{n+1}$ we define

$$Ac_\sigma(z_\chi) = Q[\text{Ac}_\sigma(L(z_\chi))] \subset Q[\text{Pos}_\sigma] \subset \text{Bru}_\sigma.$$

For each $z \in Ac_\sigma(z_\chi)$ there exists a locally convex curve $\Gamma : [0, 1] \to \mathcal{U}_1 \subset \text{Spin}_{n+1}$ with $\Gamma(0) = z$ and $\Gamma(1) = z_\chi$. Indeed, just take a locally convex curve $\Gamma_L : [0, 1] \to \text{Lo} \{n+1\}$ with $\Gamma_L(0) = L(z)$ and $\Gamma_L(1) = L(z_\chi)$ and define $\Gamma = Q \circ \Gamma_L$. Similarly, for $z \in Q[\text{Pos}_\sigma] \setminus Ac_\sigma(z_\chi)$ no such curve exists.

For $z_\chi \in \text{Bru}_\eta$, choose $U \in \text{Up}_{n+1}^\mathcal{U}$ such that $z_\chi = z_0^U$, $z_0 \in Q[\text{Pos}_\eta]$. For $\sigma \in S_{n+1}$ define $Ac_\sigma(z_\chi) = (Ac_\sigma(z_0))^U$; this turns out to be well defined and the properties above still hold. We want to define $Ac_\sigma(z_\chi)$ for any $z_\chi \in \text{chop}^{-1}([\eta])$. This will require a certain detour. We shall first present a topological construction (using curves), then an algebraic one (using coordinates) and then finally prove their equivalence.

For $q \in \text{CG}_{n+1}$, set

$$\text{Bru}_q^0 = \text{adv}^{-1}([\{q\eta\}]) = \bigcup_{\sigma \in S_{n+1}} \text{Bru}_{q\sigma},$$

$$\text{Bru}_q^1 = \text{chop}^{-1}([\{q\eta\}]) = \bigcup_{\sigma \in S_{n+1}} \text{Bru}_{q\sigma}.$$

A locally convex curve $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ satisfying $\Gamma(t) \in \text{Bru}_\eta$ for $t \in (0, 1)$ will necessarily satisfy $\Gamma(0) \in \text{Bru}_q^0$ and $\Gamma(1) \in \text{Bru}_q^1$. Notice that $\text{Bru}_\eta \subseteq \text{Bru}_q^0 \cap \text{Bru}_q^1$. Given $\sigma \in S_{n+1}$, we have

$$\text{Bru}_\sigma \cap \text{Bru}_q^0 = \text{Bru}_\sigma,$$

$$\text{Bru}_\sigma \cap \text{Bru}_q^1 = \text{Bru}_q^{(\sigma)^{-1} \sigma}$$

and therefore

$$\text{Bru}_q^0 \cap \text{Bru}_q^1 = \bigcup_{\sigma \in S_{n+1}, \sigma = \eta} \text{Bru}_\sigma.$$

33
Recall that, from Lemma 3.9, $\text{Bru}^0_{\eta} \cap \text{Bru}^1_{\eta} = \text{Bru}_{\eta}$ precisely for $n \leq 3$. In order to extend locally convex curves in $\text{Bru}_{\eta}$ to the boundary and not mix up entry points with exit points we define a new larger space:

$$\text{Bru}^0_{\eta} = ((\text{adv}^{-1}[[\eta]]) \times \{0\}) \sqcup (\text{chop}^{-1}[[\eta]]) \times \{1\})/\sim$$

where $(z,0) \sim (z,1)$ for $z \in \text{Bru}_{\eta}$ (and only there). We abuse notation by writing

$$\text{Bru}^0_{\eta} \subset \text{Bru}^0_{\eta}, \quad \text{Bru}^1_{\eta} \subset \text{Bru}^0_{\eta};$$

in this context, $\text{Bru}^0_{\eta} \cap \text{Bru}^1_{\eta} = \text{Bru}_{\eta}$. A locally convex curve $\Gamma : [0,1] \to \text{Bru}^0_{\eta}$ corresponds to a locally convex curve $\Gamma : [0,1] \to \text{Spin}_{n+1}$ satisfying $\Gamma(t) \in \text{Bru}_{\eta}$ for $t \in (0,1)$ with $\Gamma(0) = (\Gamma(0),0)$, $\Gamma(1) = (\Gamma(1),1)$.

For $z_0 \in \text{Bru}^0_{\eta}$ and $z_1 \in \text{Bru}^1_{\eta}$ let $L^0_n(z_0; z_1) \subset L_n(z_0; z_1)$ be the set of locally convex curves $\Gamma : [0,1] \to \text{Spin}_{n+1}$ such that $\Gamma(0) = z_0$, $\Gamma(1) = z_1$ and $\Gamma(t) \in \text{Bru}_{\eta}$ for all $t \in (0,1)$. For $z_0, z_1 \in \text{Bru}^0_{\eta}$, write $z_0 \ll z_1$ if and only if $z_0 \in \text{Bru}^0_{\eta}$, $z_1 \in \text{Bru}^1_{\eta}$, and $L^0_n(z_0; z_1) \neq \emptyset$ (compare with Lemma 5.3).

**Lemma 9.1.** Consider $z_0 \in \text{Bru}^0_{\eta}$ and $z_1 \in \text{Bru}^1_{\eta}$. The set $L^0_n(z_0; z_1)$ is either empty or contractible. If $z_0 \ll z_1$ then $z_0^{-1}z_1 \in \text{Bru}^1_{\eta}$ and $L^0_n(z_0; z_1) = L_{n,\text{convex}}(z_0; z_1)$.

**Example 9.2.** Recall from Lemma 8.5 that $L_{n,\text{convex}}(z_0; z_1) \subset L_n(z_0; z_1)$ is a contractible connected component if $z_0^{-1}z_1 \in \text{Bru}^1_{\eta}$ and is empty otherwise. It is entirely possible to have $z_0 \in \text{Bru}^0_{\eta}$, $z_1 \in \text{Bru}^1_{\eta}$, $z_0^{-1}z_1 \in \text{Bru}^1_{\eta}$ and $z_0 \not\ll z_1$ so that $L^0_n(z_0; z_1) = \emptyset$. In this case, there are convex curves in $L(z_0; z_1)$ but they never belong to $L^0_n(z_0; z_1)$.

A simple case is $n = 6$, $\hbar$ as in Example 4.2, $z_0 = \exp(\frac{3\pi}{4}\hbar)$ and $z_1 = \exp(\frac{\pi}{2}\hbar)$. Recall that for $n = 6$ we have $\dot{\eta} = \exp(\pi\hbar) = 1$. We have $z_0^{-1}z_1 = \exp(\frac{\pi}{2}\hbar) = \dot{\eta}$ and the curve $\Gamma(t) = z_0 \exp\left(\frac{\pi}{2}t\hbar\right)$ is convex.

**Proof of Lemma 9.1.** By Corollary 8.4, the function $\text{sing} : L_n(z_0; z_1) \to \mathcal{H}((0,1))$ is continuous. By definition, $L^0_n(z_0; z_1) = \text{sing}^{-1}[[\emptyset]]$. Since $\emptyset$ is an isolated point in $\mathcal{H}((0,1))$, the set $L^0_n(z_0; z_1)$ is a union of connected components of $L_n(z_0; z_1)$.

We know from [52] that if $\Gamma$ is not convex then $\Gamma$ is in the same connected component as $\Gamma$ with added loops, which clearly does not have empty singular set. Thus the only connected component of $L_n(z_0; z_1)$ which may be contained in $L^0_n(z_0; z_1)$ is $L_{n,\text{convex}}(z_0; z_1)$. 

Given $z_x \in \text{Bru}^1_{\eta}$ and $\sigma \in S_{n+1}$, consider $\text{Bru}_{\sigma} \subset \text{Bru}^0_{\eta}$: let

$$\text{Ac}_{\sigma}(z_x) = \{ z \in \text{Bru}_{\sigma} \subset \text{Bru}^0_{\eta} \mid z \ll z_x \}.$$
Lemma 9.3. Consider $z_x \in \text{Bru}_{\tilde{h}}$. Consider $\sigma_{k-1} \prec \sigma_k = \sigma_{k-1}a_{i_k} \in S_{n+1}$, $\text{inv}(\sigma_k) = k$. Consider $z_{k-1} \in \text{Bru}_{\tilde{h}_{k-1}}$ and $z_k = z_{k-1}a_{i_k}(\theta_k) \in \text{Bru}_{\tilde{h}_k}$, $\theta_k \in (0, \pi)$.

If $z_k \ll z_x$ then $z_{k-1}a_{i_k}(\theta_k) \ll z_x$ for all $\theta \in [0, \theta_k]$.

Proof. From Corollary 7.7 if $z_k \in \mathbb{Q}[\text{Pos}_{\sigma_k}]$ then $z_{k-1}a_{i_k}(\theta_k) \in \mathbb{Q}[\text{Pos}_{\sigma_{k-1}} \cup \text{Pos}_{\sigma_k}]$.

In this case, take $L_k = \mathbb{L}(z_k)$ and $L_{\theta} = \mathbb{L}(z_{k-1}a_{i_k}(\theta_k))$. Consider a locally convex curve $\Gamma \in \mathcal{L}_{\mathbb{R}}(z_k)$ and $\Gamma_\theta \in \text{Pos}_\eta$. We have $L_\theta \leq L_k \ll L_\epsilon$ and therefore $L_{\theta} \ll L_\epsilon$ by Lemma 5.3. By Lemma 5.3, there exists a locally convex curve $\Gamma_\epsilon : [0, \epsilon] \rightarrow \text{Lo}_n$, $\Gamma_\epsilon(0) = L_\theta$ and $\Gamma_\epsilon(\epsilon) = L_\epsilon$. Define

$$\Gamma_1(t) = \begin{cases} \mathbb{Q}(\Gamma_\epsilon(t)), & t \in [0, \epsilon], \\ \Gamma(t), & t \in [\epsilon, 1]. \end{cases}$$

Notice that for $t \in (0, \epsilon)$ we have $\Gamma_1(t) \in \text{Pos}_\eta$ and therefore $\Gamma_1(t) \in \text{Bru}_{\tilde{h}}$. The curve $\Gamma_1 : [0, 1] \rightarrow \text{Spin}_{n+1}$ is locally convex and satisfies $\Gamma_1(0) = z_{k-1}a_{i_k}(\theta_k)$, $\Gamma(1) = z_x$ and $\Gamma(t) \in \text{Bru}_{\tilde{h}}$ for all $t \in (0, 1)$. By definition, $z_{k-1}a_{i_k}(\theta_k) \ll z_x$.

In general, there is an upper matrix $U \in \text{Up}_{n+1}^1$ such that the corresponding projective transformation takes $z_k$ to $z_k' = \mathbb{Q}(U^{-1}z_k) \in \mathbb{Q}[\text{Pos}_{\sigma_k}]$, reducing to the previous case. \hfill \square

We now present an algebraic definition. Consider $z_x \in \text{Bru}_{\tilde{h}}$. Consider $\rho_0 \in S_{n+1}$ such that $z_x \in \text{Bru}_{\tilde{h}(\rho_0)^{-1}}$, $y_0 = z_x^{-1}\tilde{h} \in \text{Bru}_{\tilde{h}_0}$. We first define sets $\text{Ac}_{(i_1, \ldots, i_k)}(z_x) \subseteq \text{Bru}_{\tilde{h}}$ where $\sigma = a_{i_1} \cdots a_{i_k}$ is a reduced word. When $z_x$ is fixed (and thus so are $y_0$ and $\rho_0$) we write for simplicity $\text{Ac}_{(i_1, \ldots, i_k)} = \text{Ac}_{(i_1, \ldots, i_k)}(z_x)$.

Set $\sigma_j = a_{i_1} \cdots a_{i_j}$, $\sigma_{j-1} \prec \sigma_j = \sigma_{j-1}a_{i_j}$; set $\rho_j = \rho_0 \lor \sigma_j$ so that either $\rho_{j-1} = \rho_j$ or $\rho_{j-1} \ll \rho_j = \rho_{j-1}a_{i_j}$. Set $\text{Ac}_j = \text{Bru}_{\tilde{h}_j} = \{1\}$. We assume $\text{Ac}_{(i_1, \ldots, i_{k-1})}$ defined and proceed to construct $\text{Ac}_{(i_1, \ldots, i_k)}$:

$$\text{Ac}_{(i_1, \ldots, i_k)} = \{z_{k-1}a_{i_k}(\theta_k) \mid z_{k-1} \in \text{Ac}_{(i_1, \ldots, i_{k-1})}, \theta_k \in (0, \vartheta_{i_k}(z_{k-1}))\};$$

$$\vartheta_{i_k} : \text{Ac}_{(i_1, \ldots, i_{k-1})} \rightarrow (0, \pi]; \quad \vartheta_{i_k}(z_{k-1}) = \begin{cases} \pi, & \rho_{k-1} \ll \rho_k, \\ \pi - \Theta_{i_k}(y_0z_{k-1}), & \rho_{k-1} = \rho_k; \end{cases}$$

notice here that $\Theta_{i_k} : \text{Bru}_{\tilde{h}_k} \rightarrow (0, \pi]$ is well defined if $\rho_{k-1} = \rho_k$ (see Remark 7.8 for the definition of $\Theta_{i_k}$).

Lemma 9.4. The sets $\text{Ac}_{(i_1, \ldots, i_j)}$, $1 \leq j \leq k$, defined above satisfy

$$\text{Ac}_{(i_1, \ldots, i_j)} = \{a_{i_1}(\theta_1) \cdots a_{i_j}(\theta_j) \mid (\theta_1, \ldots, \theta_j) \in X_j \subseteq \text{Bru}_{\tilde{h}_j} \cap (y_0^{-1}\text{Bru}_{\tilde{h}_j})$$

where $(X_j)_{1 \leq j \leq k}$ is a quasiproduct, $\text{Ac}_{(i_1, \ldots, i_j)}$ is diffeomorphic to $\mathbb{R}^j$. \hfill 35
Notice that the inclusion in the statement is necessary to make sense of the definition of $\vartheta_{t_k}$. The reader should compare this result with Lemma 6.2.

**Proof.** The proof is by induction on $k$; the case $k = 0$ is trivial. Take $z_k = z_{k-1}a_{i_k}(\theta_k) \in Ac_{(i_1, \ldots, i_k)}$, $z_{k-1} \in Ac_{(i_1, \ldots, i_{k-1})}$, $\theta_k \in (0, \vartheta_{i_k}(z_{k-1}))$. We assume by induction hypothesis that $Ac_{(i_1, \ldots, i_{k-1})} \subseteq \Bru_{\delta_{i_{k-1}}}$. We therefore have $z_k \in \Bru_{\delta_{k-1}}\Bru_{\delta_i} = \Bru_{\delta_k}$ (the last equation follows from Lemma 7.3). We also assume by induction hypothesis that $y_{k-1} = y_0z_{k-1} \in \Bru_{\rho_{k-1}}$. If $\rho_{k-1} < \rho_k$, Lemma 7.3 implies $y_k = y_0z_k = y_{k-1}a_{i_k}(\theta_k) \in \Bru_{\rho_{k-1}}\Bru_{\delta_i} = \Bru_{\rho_k}$. If $\rho_{k-1} = \rho_k$, let $\tilde{\rho} \prec \rho_k = \rho_{i_k}$. The map $\Theta_{i_k} : \Bru_{\rho_{k-1}} \to (0, \pi)$ is well defined: take $\theta = \Theta_{i_k}(y_{k-1})$ and $\tilde{y} \in \Bru_{\tilde{\rho}}$ such that $y_{k-1} = \tilde{y}a_{i_k}(\theta)$. By our recursive definition, $\theta + \theta_k < \pi$; by Lemma 7.3 $y_k = \tilde{y}a_{i_k}(\theta + \theta_k) \in \Bru_{\rho_k}$.

We now prove that the two definitions are equivalent.

**Lemma 9.5.** Consider $z_x \in \Bru^1_\eta$ and $\sigma_k = a_{i_1} \cdots a_{i_k}$ a reduced word in $S_{n+1}$. Then $Ac_{\sigma_k}(z_x) = Ac_{(i_1, \ldots, i_k)}(z_x)$.

**Proof.** The proof is by induction on $k$; the case $k = 0$ is trivial. Assume therefore $Ac_{\sigma_{k-1}}(z_x) = Ac_{(i_1, \ldots, i_{k-1})}(z_x)$ for $\sigma_{k-1} = a_{i_1} \cdots a_{i_{k-1}}$.

Consider $z_k = z_{k-1}a_{i_k}(\theta_k)$, $z_{k-1} \in \Bru_{\delta_{k-1}}$, $z_k \in \Bru_{\delta_k}$, $\theta_k \in (0, \pi)$. It follows from Lemma 9.3 that $z_k \in Ac_{\sigma_k}$ implies $z_{k-1} \in Ac_{\sigma_{k-1}}$ and therefore

$$Ac_{\sigma_k} \subseteq Ac_{\sigma_{k-1}} \Bru_{\delta_{i_k}}; \quad Ac_{(i_1, \ldots, i_k)} \subseteq Ac_{\sigma_{k-1}} \Bru_{\delta_{i_k}};$$

we have to prove that these two sets are equal.

Given $z_{k-1} \in Ac_{\sigma_{k-1}}$, let $J_{z_{k-1}} \subseteq (0, \pi)$ be the set such that, for all $\theta_k \in (0, \pi)$,

$$\theta_k \in J_{z_{k-1}} \iff z_{k-1}a_{i_k}(\theta_k) \in Ac_{\sigma_k}.$$ 

It follows from Lemma 9.3 that $J_{z_{k-1}}$ is either empty or an initial interval.

We claim that $J_{z_{k-1}}$ is not empty. By applying a projective transformation, we may assume that $z_{k-1} \in Q[\Pos_{\sigma_{k-1}}]$ so that $z_{k-1} = Q(L_{k-1})$, $L_{k-1} \in \Pos_{\sigma_{k-1}}$. Take $\Gamma \in L_{i_{k-1}}(z_{k-1}; z_x)$. Define $\Gamma_L = L \circ \Gamma$, with maximal connected domain containing $t = 0$. Consider $t^* > 0$ in this domain and $L_\bullet = \Gamma_L(t^*)$, $L_\bullet \in \Pos_{\eta}$, $L_{k-1} \ll L_\bullet$. Take $t_k > 0$ such that $L_{k-1}a_{i_k}(t_k) \ll L_\bullet$; define $\theta_k > 0$ by $Q(L_{k-1}a_{i_k}(t_k)) = z_{k-1}a_{i_k}(\theta_k)$. Take $\Gamma_{L_{i_{k-1}}} : [0, t^*) \to L_{i_{k-1}}$ locally convex such that $\Gamma_{L_{i_{k-1}}}(0) = L_{k-1}a_{i_k}(t_k)$ and $\Gamma_{L_{i_{k-1}}}(t^*) = L_\bullet$. Finally, take $\Gamma_1 : [0, 1] \to \Spin_{n+1}$,

$$\Gamma_1(t) = \begin{cases} Q(\Gamma_{L_{i_{k-1}}}(t)), & t \in [0, t^*], \\ \Gamma(t), & t \in [t^*, 1]. \end{cases}$$

This curve is locally convex so that $z_{k-1}a_{i_k}(\theta_k) \in Ac_{\sigma_k}$ and $\theta_k \in J_{z_{k-1}}$, as claimed.
We claim that \( J_{z_{k-1}} \) is open. Assume by contradiction \( \theta_k' = \text{max}(J_{z_{k-1}}), \)
\( z_k' = z_{k-1} \alpha_{i_k}(\theta_k') \). By applying a projective transformation, we may assume that
\( z_k' \in \mathbb{Q}[\text{Pos}_{\sigma_k}] \). As in the previous paragraph, take a curve \( \Gamma \) going from \( z_k' \)
to \( x \), use \( L \) to take its initial segment to \( L_{\theta_k}^{0+1} \) and slightly perturb it to obtain
\( \theta_k \in J_{z_{k-1}}, \theta_k > \theta_k' \). The argument is so similar that we feel a repetition is pointless.

At this point we know that there exists a function \( \check{\theta}_{i_k} : \mathbb{A}_{\sigma_{k-1}} \to (0, \pi) \) such that
\( J_{z_{k-1}} = (0, \check{\theta}_{i_k}(z_{k-1})) \). We are left with proving that \( \theta_k = \check{\theta}_{i_k} \).

We first prove that \( \check{\theta}_{i_k}(z_{k-1}) \leq \theta_{i_k}(z_{k-1}) \) for all \( z_{k-1} \). If \( \rho_{k-1} < \rho_k \) then
\( \theta_{i_k}(z_{k-1}) = \pi \) and we are done. If \( \rho_{k-1} = \rho_k \), take \( \theta_k' = \theta_{i_k}(z_{k-1}), z_k' = z_{k-1} \alpha_{i_k}(\theta_k') \)
and \( y_k' = y_0 z_k \). Recall that in this case there exists \( \rho_k \in S_{n+1}, \theta_k \in \rho_k \).
By definition of \( \theta_{i_k}, y_k' \in \text{Bru}_k \) so that \( \text{adv}(y_k') = q' \eta \) for \( q' \in CG_{n+1}, q' \neq 1 \). Thus any locally convex curve starting at \( y_k' \) immediately enters \( \text{Bru}_n \).
Thus there is no convex curve going from \( y_k' \) to \( \eta \) and therefore no convex curve going from \( z_k' \) to \( x \). It follows that \( z_k' \notin \mathbb{A}_{\sigma_k}(x) \) and therefore \( \theta_k \geq \theta_{i_k}(z_{k-1}) \), proving our claim.

We finally prove that \( \check{\theta}_{i_k}(z_{k-1}) \geq \theta_{i_k}(z_{k-1}) \). Consider \( \theta_k < \theta_{i_k}(z_{k-1}), z_k = z_{k-1} \alpha_{i_k}(\theta_k) \) and \( y_k = y_{k-1} \alpha_{i_k}(\theta_k) = y_0 z_k \in \text{Bru}_k \). Notice that \( z_{k-1} \alpha_{i_k}(\theta_k) \in \text{Bru}_{\theta_k} \)
and \( y_{k-1} \alpha_{i_k}(\theta_k) \in \text{Bru}_{\theta_k} \) for all \( \theta \in [0, \theta_k] \). By compactness and Lemma 8.1 there exists \( c > 0 \) such that for all \( \theta \in [0, \theta_k] \) we have both
\( z_{k-1} \alpha_{i_k}(\theta) \exp(t h) \in \text{Bru}_{\theta} \) and \( y_{k-1} \alpha_{i_k}(\theta) \exp(t h) \in \text{Bru}_{\theta} \). Apply Lemma 7.2 to construct a family \( \Theta : [0, \theta_k] \times [\frac{1}{2}, 1] \to \text{Spin}_{n+1} \) of convex curves \( \Theta(\theta) : [\frac{1}{2}, 1] \to \text{Spin}_{n+1} \)
going from \( y_{k-1} \alpha_{i_k}(\theta) \exp(t h) \) to \( \eta \). Extend this to \( \Theta : [0, \theta_k] \times [0, 1] \to \text{Spin}_{n+1} \) by defining \( \Theta(\theta)(t) = y_{k-1} \alpha_{i_k}(\theta) \exp(2c t h) \in \text{Bru}_k \) for \( t \in [0, \frac{1}{2}] \). For \( \text{each } \theta \in [0, \theta_k] \) the arc \( \Theta(\theta) : [0, 1] \to \text{Spin}_{n+1} \) is convex: indeed, for \( t \in (0, 1) \) we have that \( \Theta(\theta)(t) \in \text{Bru}_k \); the claim follows from Proposition 8.1.

Multiply by \( y_0^{-1} \) to obtain the family \( y_0^{-1} \text{Hovf curves } \Gamma_\theta = y_0^{-1} \Theta(\theta) : [0, 1] \to \text{Spin}_{n+1} \) going from \( z_{k-1} \alpha_{i_k}(\theta) \) to \( x \). We prove that for all \( \theta \) we have
\( \Gamma_\theta \in \mathcal{L}_{n}^*_{\sigma_k}(z_{k-1} \alpha_{i_k}(\theta); x) \), i.e., that \( \Gamma_\theta(0) = \text{Bru}_{\theta} \) for all \( t \in (0, 1) \). We know that
\( \Gamma_0 \) is convex and that \( z_{k-1} \in \mathbb{A}_{\sigma_{k-1}}(x) \) and therefore from Lemma 9.1 that
\( \Gamma_0 \in \mathcal{L}_{n}^*(z_{k-1}; x) \). We know by construction that \( \Gamma_\theta(t) \in \text{Bru}_{\theta} \) for all \( t \in (0, \frac{1}{2}) \). Apply again Lemma 7.2 to construct convex arcs \( \tilde{\Gamma}_\theta : [0, \frac{1}{2}, 1] \to \text{Spin}_{n+1} \) from
\( \tilde{\Gamma}_\theta(0) = 1 \) to \( \tilde{\Gamma}_\theta(\frac{1}{2}) = \tilde{\Gamma}_\theta(\frac{1}{2}) \). Extend \( \tilde{\Gamma}_\theta \) to \( [0, 1] \) by \( \tilde{\Gamma}_\theta(t) = \Gamma_\theta(t) \) for \( t \in [\frac{1}{2}, 1] \). We have \( \text{sing}(\tilde{\Gamma}_\theta) = 0 \). Also, from Corollary 8.4 \( \text{sing}(\tilde{\Gamma}_\theta) \) is a continuous function of \( \theta \). Since \( \emptyset \in \mathcal{H}((0, 1)) \) is an isolated point, we have \( \text{sing}(\tilde{\Gamma}_\theta) = 0 \) for all \( \theta \), as desired. This implies that \( z_k \ll x \) and therefore \( \theta_k < \check{\theta}_{i_k}(z_{k-1}) \). Since this holds for any \( \theta_k < \text{check}(z_{k-1}) \) we have \( \check{\theta}_{i_k}(z_{k-1}) \geq \check{\theta}_{i_k}(z_{k-1}) \), completing our proof.

\[ \square \]

**Remark 9.6.** We saw in Lemmas 8.5 and 9.1 that, given \( z_0 \in \text{Bru}_k \) and \( z_1 \in \text{Bru}_k \), the set \( \mathcal{L}_{n}^*(z_0; z_1) \) is either empty or equal to \( \mathcal{L}_{n,\text{convex}}(z_0; z_1) \) and contractible. In
Lemma 9.1 we saw an explicit contraction if \( z_0^{-1}z_1 \in \text{Bru}_q \) but otherwise used the chopping lemma. We now present a more explicit contraction in general.

For any \( \Gamma \in \mathcal{L}_n(z_0; z_1) \), we have \((\Gamma(0))^{-1}\Gamma(\frac{1}{2}) \in \text{Bru}_q \) and \((\Gamma(\frac{1}{2}))^{-1}\Gamma(1) \in \text{Bru}_q \). Apply the contraction in the proof of Lemma 8.5 to each arc, leaving \( \Gamma(\frac{1}{2}) \) fixed. This takes us to a set of curves parametrized by \( \Gamma(\frac{1}{2}) \in z_0 \text{Ac}_q(z_0^{-1}z_1) \).

We now know that \( \text{Ac}_q(z_0^{-1}z_1) \) is diffeomorphic to \( \mathbb{R}^m \) (with a rather explicit diffeomorphism).

10 Itineraries and paths of curves

Given \( \Gamma \in \mathcal{L}_n(z_0; z_•) \), \( \text{sing}(\Gamma) = \{ t_1 < \cdots < t_\ell \} \), let its itinerary and path be

\[
\text{iti}(\Gamma) = (\sigma_1, \ldots, \sigma_\ell), \quad \text{path}(\Gamma) = (z_1, \ldots, z_\ell), \quad z_j = \Gamma(t_j) \in \text{Bru}_{q\sigma_j}.
\]

Thus, \( \text{iti}(\Gamma) \) is a word in \( W_n = (S_{n+1} \setminus \{e\})^* \); here \( S_{n+1} \setminus \{e\} \) is the alphabet. For \( w = (\sigma_1, \ldots, \sigma_\ell) \in W_n \), let its length be \( \ell(w) = \ell = |\text{sing}(\Gamma)| \in \mathbb{N} \). From Lemma 9.1, \( \Gamma \in \mathcal{L}_n(\hat{q}) \) is convex if and only if \( \text{iti}(\Gamma) \) is the empty word of length 0. Define the stratification

\[
\mathcal{L}_n = \bigsqcup_{w \in W_n} \mathcal{L}_n[w]; \quad \mathcal{L}_n[w] = \{ \Gamma \in \mathcal{L}_n \mid \text{iti}(\Gamma) = w \}.
\]

One of our aims is to describe these strata.

Example 10.1. Consider again the case \( n = 2 \). We draw the locally convex curves \( \gamma : J \to \mathbb{S}^2 \) instead of the corresponding locally convex curves \( \Gamma = \hat{\mathfrak{F}}_\gamma : J \to \mathbb{S}^2 \). A letter \( a = a_1 \) corresponds to the curve \( \gamma \) transversally crossing the equator (i.e., the great circle \( x_3 = 0 \)) at a point different from \( \pm e_1 \). A letter \( b = a_2 \) occurs when the tangent geodesic (great circle) to \( \gamma \) at \( t \) includes the points \( \pm e_1 \) but the \( x_3 \)-coordinate of \( \gamma(t) \) is non-zero. A letter \([ab]\) indicates that the curve is tangent to the equator, but not at \( \pm e_1 \). A letter \([ba]\) declares that the curve crosses the equator transversally and not at \( \pm e_1 \). Finally, \([aba]\) proclaims that the curve is tangent to the equator at \( \pm e_1 \). Figure 2 shows a two-parameter family of (portions of) curves in \( \mathcal{L}_2 \) illustrating all these cases. Notice that we also write a word as a string of letters. For instance, \( baba = (b, a, b, a) \) and \( b[ab] = (b, ab) \). Square brackets are used to avoid confusion between, say, \( a[ba] = (a, ba) \), \([aba] = (aba) \) and \( aba = (a, b, a) \), of respective lengths 2, 1 and 3.

Given the path \( (z_1, \ldots, z_\ell) \) of some \( \Gamma \in \mathcal{L}_n \), it is easy to determine the corresponding itinerary \( w = (\sigma_1, \ldots, \sigma_\ell) \). Conversely, given an itinerary \( w = \)
Figure 2: A family of curves in $\mathcal{L}_2$. The equator is dashed and the fat dot indicates $e_1$. The vector $e_2$ is at the right.

$$(\sigma_1, \ldots, \sigma_\ell) \in \mathbf{W}_n,$$ define $B(w, j) \in \tilde{B}_{n+1}^+$ for $j \in \mathbb{Z}, 0 \leq j \leq \ell + 1$ and $B(w, j + \frac{1}{2}) \in \tilde{B}_{n+1}^+$ for $j \in \mathbb{Z}, 0 \leq j \leq \ell$ by

$$B(w, 0) = 1, \quad B\left(w, \frac{1}{2}\right) = \hat{\eta}, \quad B(w, j) = B\left(w, j - \frac{1}{2}\right) \hat{\sigma}_j,$$

$$B\left(w, j + \frac{1}{2}\right) = B\left(w, j - \frac{1}{2}\right) \hat{\sigma}_j, \quad B(w, \ell + 1) = B\left(w, \ell + \frac{1}{2}\right) \hat{\eta}. \quad (14)$$

In particular, we have $B(w, \ell + 1) = \hat{\eta} \hat{w} \hat{\eta} \in \text{CG}_{n+1}$, where we define the hat of a word by $\hat{w} = \hat{\sigma}_1 \cdots \hat{\sigma}_\ell \in \text{CG}_{n+1}$. We adopt here the conventions $t_0 = 0, t_{\ell+1} = 1,$ $z_0 = 1, z_{\ell+1} = B(w, \ell + 1), \sigma_0 = \sigma_{\ell+1} = \eta$. It follows from Lemma 8.1 that if $\Gamma \in \mathcal{L}_n[w]$ and $\text{sing}(\Gamma) = \{t_1 < \cdots < t_\ell\}$ then

$$\Gamma \in \mathcal{L}_n(\hat{\eta} \hat{w} \hat{\eta}), \quad \Gamma(t_j) \in \text{Bru}_{B(w, j)}, \quad \forall t \in (t_j, t_{j+1}), \quad \Gamma(t) \in \text{Bru}_{B(w, j + \frac{1}{2})}.$$

Thus, if $\Gamma \in \mathcal{L}_n[w]$ then $\text{path}(\Gamma) \in \text{Bru}_{B(w, 1)} \times \cdots \times \text{Bru}_{B(w, \ell)}$. 

39
Given $w = (\sigma_1, \ldots, \sigma_\ell) \in W_n$ and $j \in \mathbb{Z}$, $0 \leq j \leq \ell$, define $q_j \in CG_{n+1}$ by

$$B(w, j) = q_j \text{ acute}(\eta \sigma_j) \in q_j \text{ Bru}_q^0, \quad B\left(w, j + \frac{1}{2}\right) = q_j \acute{\eta},$$

$$B(w, j + 1) = q_j \acute{\eta} \text{ grave}(\eta \sigma_{j+1}) \in q_j \text{ Bru}_q^1.$$

A sequence $(z_1, \ldots, z_\ell) \in \text{Bru}_{B(w,1)} \times \cdots \times \text{Bru}_{B(w,\ell)}$ is an accessible path for $w$ if

$$\forall j \in [\ell], \ q_j^{-1} z_j \in \text{Ac}_{\eta \sigma_j}(q_j^{-1} z_{j+1}).$$

Let $\text{Path}(w) \subseteq \text{Bru}_{B(w,1)} \times \cdots \times \text{Bru}_{B(w,\ell)}$ be the set of accessible paths for $w$.

**Lemma 10.2.** Consider $w \in W_n$. For any $\Gamma \in L_n[w]$, $\text{path}(\Gamma)$ is accessible, i.e., belongs to $\text{Path}(w)$.

*Proof.* Consider $t_j < t_{j+1}$ and the arc $q_j^{-1} \Gamma|_{[t_j, t_{j+1}]}$. Except for the modified domain, this arc belongs to $L_n^\eta(q_j^{-1} \Gamma(t_j); q_j^{-1} \Gamma(t_{j+1}))$ and therefore $q_j^{-1} \Gamma(t_j) \in \text{Ac}_{\eta \sigma_j}(q_j^{-1} \Gamma(t_{j+1}))$, as desired. □

**Lemma 10.3.** Consider $w \in W_n$. For any accessible path $(z_1, \ldots, z_\ell) \in \text{Path}(w)$, the set $\{\Gamma \in L_n[w] \mid \text{path}(\Gamma) = (z_1, \ldots, z_\ell)\}$ is contractible (and nonempty).

*Proof.* The set of sets $\{t_1 < \cdots < t_\ell\} \in \mathcal{H}((0, 1))$ is contractible. At this point, the values of $q_j \in CG_{n+1}$, of $t_j < t_{j+1}$, of $z_j \in q_j \text{ Bru}_{\text{acute}(\eta \sigma_j)}$ and of $z_{j+1} \in q_j \text{ Bru}_q^1$ with $q_j^{-1} z_j \in \text{Ac}_{\eta \sigma_j}(q_j^{-1} z_{j+1})$ are all given. The set of arcs $\Gamma : [t_j, t_{j+1}] \rightarrow \text{Spin}_{n+1}$ with $\Gamma(t_j) = z_j$, $\Gamma(t_{j+1}) = z_{j+1}$ and $\Gamma(t) \in q_j \text{ Bru}_q$ for all $t \in (t_j, t_{j+1})$ is homeomorphic to $L_n^\eta(q_j^{-1} z_j; q_j^{-1} z_{j+1})$; by Lemma 9.1, this set is contractible (with an explicit contraction given by Remark 9.6). Concatenate the above arcs to construct $\Gamma$; this yields the desired result. □

**Lemma 10.4.** Consider $w \in W_n$; the set $\text{Path}(w) \subseteq \text{Bru}_{B(w,1)} \times \cdots \times \text{Bru}_{B(w,\ell)}$ is diffeomorphic to $\mathbb{R}^d$, $d = \text{inv}(\eta \sigma_1) + \cdots + \text{inv}(\eta \sigma_\ell)$. In particular, $\text{Path}(w)$ is contractible (and nonempty).

*Proof.* Start constructing the set from the $\ell$-th coordinate $\text{Bru}_{B(w,\ell)}$ and proceed backwards. Use Lemma 9.4 for the inductive step. The set $\text{Path}(w)$ is parametrized by a quasiproduct. □

**Lemma 10.5.** Consider $w \in W_n$; the set $L_n[w] \subset L_n(\acute{\eta} \hat{\omega} \acute{\eta})$ is contractible (and nonempty).

*Proof.* Let $\text{Path}_1(w) \subset L_n[w]$ be the set of paths $\Gamma$ such that the arcs $\Gamma|_{[t_{i-1}, t_i]}$ are the base points of the contractible sets $L_{\eta, \text{convex}}(\Gamma(t_{i-1}); \Gamma(t_i))$. Here we assume that $\text{sing}(\Gamma) = \{t_1 < \cdots < t_\ell\}$; we may use the construction in Remark 9.6 to select a basepoint.
Lemma 10.3 gives us a deformation retract from $\mathcal{L}_n[w]$ to $\text{Path}_1(w)$, a homotopy $H_0 : [0, 1] \times \mathcal{L}_n[w] \to \mathcal{L}_n[w]$ which starts with an arbitrary curve $\Gamma_0 \in \mathcal{L}_n[w]$ and deforms it: $\Gamma_s = H_0(s, \Gamma_0)$ for $s \in [0, 1]$. The homotopy satisfies $\text{sing}(\Gamma_s) = \text{sing}(\Gamma_0) = \{ t_1 < \cdots < t_\ell \}$ and $\text{path}(\Gamma_s) = \text{path}(\Gamma_0)$ for all $s \in [0, 1]$. We have $\Gamma_1 \in \text{Path}_1(w)$, i.e., the arcs $\Gamma_1|[t_{i-1}, t_i]$ are the base points of the contractible sets $\mathcal{L}_{n,\text{convex}}(\Gamma_0(t_{i-1}); \Gamma_0(t_i))$. Also, if $\Gamma_0 \in \text{Path}_1(w)$ then $\Gamma_s = \Gamma_0$ for all $s \in [0, 1]$.

Let $\text{Path}_2(w) \subset \text{Path}_1(w)$ be the set of paths $\Gamma \in \text{Path}_1(w)$ such that $\text{sing}(\Gamma) = \{ \frac{1}{t+1} < \cdots < \frac{\ell}{t+1} \}$. There is an easy deformation retract $H_1 : [1, 2] \times \text{Path}_1(w) \to \text{Path}_1(w)$ from $\text{Path}_1(w)$ to $\text{Path}_2(w)$: affinely reparametrize each interval $[t_{i-1}, t_i]$.

Lemma 10.2 shows that $\text{Path}_2(w)$ is homeomorphic to $\text{Path}(w)$: the homeomorphism takes $\Gamma$ to $\text{path}(\Gamma)$. Lemma 10.4 shows us how to construct a homotopy $H_2 : [2, 3] \times \text{Path}(w) \to \text{Path}(w)$ with $H_2(2, p) = p$ and $H_2(3, p) = p_0$ where $p_0 \in \text{Path}(w)$ is a base point. Compose with the homeomorphism above to define a deformation retract from $\text{Path}_2(w)$ to a point. Concatenate $H_0, H_1, H_2$ to construct the desired contraction. \qed

In Appendix B we endow $\mathcal{L}_n(z_0; z_1)$ and $\mathcal{L}_n$ with smooth Hilbert manifold structures. The construction is similar to that given in 30 for a different but similar space of curves. Our structure allows for curves $\Gamma$ which are, say, piecewise $C^1$ with logarithmic derivative $(\Gamma(t))^{-1}\Gamma'(t)$ a positive linear combination of the vectors $a_i$. Here by piecewise $C^1$ we mean that there exists a finite family of compact intervals $[0, t_1], [t_1, t_2], \ldots, [t_k, 1]$ covering $[0, 1]$ such that $\Gamma$ is of class $C^1$ in each interval $[t_i, t_{i+1}]$.

The proof of Lemma 10.5 above obtains a rather explicit contraction. A slightly shorter proof is possible using the metrizable topological manifold structure provided by Lemma 10.6 below, Theorem 15 of 46 and the long exact sequence of homotopy groups for the fibration $\pi : \mathcal{L}_n[w] \to \text{Path}(w)$, via Lemmas 10.2, 10.3 and 10.4. We prefer to be more self-contained. For everything that follows, however, we cannot postpone any longer the adoption of a manifold structure for the spaces $\mathcal{L}_n(z_0; z_1)$ (see Appendix B).

Let $M_0$ be a (finite or infinite dimensional) manifold and $M_1 \subset M_0$: the subset $M_1$ is a (globally) collared topological submanifold of codimension $d$ if and only if there exists an open set $A_0$, $M_1 \subset A_0 \subset M_0$, which is a tubular neighborhood of $M_1$ (based on 10). We say that $A_0$ as above is a tubular neighborhood if there exist an open neighborhood $B \subset \mathbb{R}^d$, $0 \in B$, a continuous projection $\Pi : A_0 \to M_1 \subset A_0$ and a continuous map $\hat{F} : A_0 \to B$ such that the map $(\Pi, \hat{F}) : A_0 \to M_1 \times B$ is a homeomorphism. Recall that $\Pi$ being a projection implies $\Pi \circ \Pi = \Pi$. Compact oriented surfaces of class $C^2$ contained in $M_0 = \mathbb{R}^3$ are examples of collared topological submanifolds: in this case $\Pi$ can be taken to be the normal projection.
For \( \sigma \in S_{n+1} \setminus \{e\} \) set \( \dim(\sigma) = \text{inv}(\sigma) - 1 \); for \( w = (\sigma_1, \ldots, \sigma_\ell) \in W_n \) set \( \dim(w) = \dim(\sigma_1) + \cdots + \dim(\sigma_\ell) \). In Section 13 we shall construct a CW complex \( D_n \) with one cell \( c_w \) for each word \( w \in W_n \); we shall have \( \dim(c_w) = \dim(w) \).

**Lemma 10.6.** Consider \( w \in W_n \); The set \( L_n[w] \subset L_n(\eta\wedge) \) is a collared topological submanifold of codimension \( \dim(w) \).

It turns out that \( L_n[w] \) is not a submanifold of class \( C^1 \); see Remark 10.9 below. Before proving Lemma 10.6 we state and prove a preliminary result.

**Lemma 10.7.** Consider \( \sigma \in S_{n+1}, \sigma \neq \eta \) and \( z_0 = q_0 \in B_{n+1}^+, q \in CG_{n+1} \). If \( (\lambda, \ldots, \lambda, \eta, \ldots, \eta) \) is in Lemma 5.6 then \( i(\lambda) \) is convex and there exists \( t_1 \in [-\epsilon, \epsilon] \) with \( \Gamma(t_1) \in \text{Brus}_0 \) then \( \text{sing}(\Gamma) = \{t_1\} \).

**Proof.** Consider a projective transformation \( \phi \) for which \( \phi(z_0) \in qQ[\text{Pos}_\sigma] \subset U_q \). By continuity, there exists an open set \( A \subset U_{s_0}, z_0 \in A \), such that \( \phi[A] \subset U_q \). We may furthermore assume that \( z \in \phi[A] \cap \text{Brus} \) implies \( z \in qQ[\text{Pos}_\sigma] \).

Consider \( \Gamma \) as in the statement. Apply triangular coordinates to define the curve \( \Gamma_L : [-\epsilon, \epsilon] \to L_{n+1}^1, \Gamma(t) = z_0Q[\Gamma_L(t)] \). For \( \lambda \in [1, +\infty) \) define

\[
\Gamma_L^\lambda(t) = \text{diag}(1, \lambda^{-1}, \ldots, \lambda^{-(n+1)})\Gamma_L(t) \text{ diag}(1, \lambda, \ldots, \lambda^{n+1})
\]

and the curve \( \Gamma_L^\lambda(t) = z_0Q[\Gamma_L^\lambda(t)] \). Notice that \( \Gamma_L^\lambda : [-\epsilon, \epsilon] \to U_{s_0} \) is locally convex and satisfies \( \text{sing}(\Gamma_L^\lambda) = \text{sing}(\Gamma) \) and \( i(t_1) = \text{iti}(\Gamma_L^\lambda) \). Given \( t_0 \in [-\epsilon, \epsilon] \) we have \( \lim_{\lambda \to +\infty} \Gamma_L^\lambda(t_0) = z_0 \); by compactness, there exists \( \lambda_0 \) such that \( \Gamma_L^{\lambda_0}([-\epsilon, \epsilon]) \subset A \).

The curve \( \tilde{\Gamma} = \phi \circ \Gamma_L^{\lambda_0} \) therefore admits triangular coordinates \( \tilde{\Gamma}_L : [-\epsilon, \epsilon] \to L_{n+1}^1, \tilde{\Gamma}(t) = qQ[\tilde{\Gamma}_L(t)] \). We have \( \tilde{\Gamma}(t_1) \in \phi[A] \cap \text{Brus} \) and therefore \( \tilde{\Gamma}_L(t_1) \in \text{Pos}_\sigma \). From Lemma 5.6 \( t > t_1 \) implies \( \tilde{\Gamma}_L(t) \in \text{Pos}_\sigma \), i.e., \( \text{sing}(\tilde{\Gamma}) \cap (t_1, \epsilon) = \emptyset \). Thus \( t_1 \) is the last element of \( \text{sing}(\Gamma) \).

A similar argument using the sets \( \text{Neg}_\sigma \) instead of \( \text{Pos}_\sigma \) proves that \( t_1 \) is also the first element of \( \text{sing}(\Gamma) \).

**Proof of Lemma 10.6.** For \( w = \sigma_1 \cdots \sigma_\ell = (\sigma_1, \ldots, \sigma_\ell) \) and \( 2j \in \mathbb{Z} \cap [0, 2\ell + 2] \), set \( B(w, j) \in B_{n+1}^+ \) as in Equation 14 above; in particular, \( B(w, \ell + 1) = q_0 = \eta\wedge \in \text{CG}_{n+1} \). We first define an open subset \( A_w^\ell \subset L_n(q_\ell) \times (0, 1) \). A pair \( (\Gamma, \tilde{e}) \) belongs to \( A_w^\ell \) if there exist \( \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_\ell < \tilde{t}_{\ell+1} = 1 \) such that:

1. For each \( i, \tilde{t}_{i+1} - \tilde{t}_i < 2\tilde{e} \).
2. Each arc \( \Gamma|_{[\tilde{t}_i - 2\tilde{e}, \tilde{t}_i + 2\tilde{e}]} \) is convex, with image in \( U_B(w, i) \). In particular, for \( \tilde{z}_i = \Gamma(\tilde{t}_i) \) we have \( \tilde{z}_i \in U_B(w, i) \).
3. Each arc \( \Gamma|_{[\tilde{t}_i - \tilde{e}, \tilde{t}_i + \tilde{e}]} \) is convex, with image in \( U_B(w, i + 1/2) = \text{Brus}_B(w, i + 1/2) \).
4. For $f_{i,k_i} : U_{B(w,i)} \to \mathbb{R}$ and $k_i$ as in Lemma 7.9, we have $f_{i,k_i}(\tilde{z}_i) = 0$.

5. Let $\Pi_{B(w,i)} : U_{B(w,i)} \to \text{Bru}_{B(w,i)} \subset U_{B(w,i)}$ be the smooth projection defined in Remark 7.10. Set $\tilde{z}_i = \Pi_{B(w,i)}(\tilde{z}_i)$. There exist convex arcs in $U_{B(w,i)}$ from $\Gamma(\tilde{t}_i - \frac{\epsilon}{2})$ to $\tilde{z}_i$ and from $\tilde{z}_i$ to $\Gamma(\tilde{t}_i + \frac{\epsilon}{2})$.

For $\Gamma \in \mathcal{L}_n(q_w)$, set $J_{\Gamma} = \{ \tilde{\epsilon} \in (0,1) \mid (\Gamma, \tilde{\epsilon}) \in \mathcal{A}_w \}$; clearly, $J_{\Gamma}$ is either an open interval or empty; for $\Gamma \in \mathcal{L}_n[w]$, $J_{\Gamma}$ is an interval of the form $J_{\Gamma} = (0, \epsilon)$ for some $\epsilon > 0$. Set

$$A_w = \{ \Gamma \in \mathcal{L}_n(q_w) \mid J_{\Gamma} \neq \emptyset \} \subseteq \mathcal{L}_n(q_w),$$

an open subset. For $\Gamma \in A_w$ the times $\tilde{t}_i$ are well defined and, from Lemma 7.9, the functions $\Gamma \mapsto \tilde{t}_i$ are also continuous. For $\Gamma \in A_w$, set $\epsilon = \epsilon_{\Gamma} = \sup J_{\Gamma}$; from the continuity of $\tilde{t}_i$ and several uses of Lemma 5.4, the function $\Gamma \mapsto \epsilon$ is also continuous. For instance, checking that the pair $(\Gamma, \tilde{\epsilon})$ satisfies item 2 above involves looking for the smallest $t > \tilde{t}_i$ such that, for $\Gamma_L$ defined by $\Gamma_L(t) = B(w,i + \frac{1}{2}) \mathcal{Q}(\Gamma_L(t))$, we have $\Gamma_L(t) \in \partial \text{Neg}_\eta$. Lemmas 5.4 and 5.6 imply that this $t$ is a continuous function of $\Gamma$. We have $\tilde{z}_i = \Gamma(\tilde{t}_i) \in U_{B(w,i)}$; we define $\bar{z}_i = \Gamma(\tilde{t}_i - \frac{\epsilon}{2})$, $\bar{z}_i^+ = \Gamma(\tilde{t}_i + \frac{\epsilon}{2})$, and $\bar{z}_i = \Pi_{B(w,i)}(\tilde{z}_i)$. Also, the map $\Gamma \mapsto (\bar{z}_i, \bar{z}_i^-, \bar{z}_i^+, \bar{z}_i)$ is continuous. For $\Gamma \in A_w$, $\epsilon = \epsilon_{\Gamma}$, $\tilde{t}_i$, $\bar{z}_i$ and $\bar{z}_i$ as above we therefore have the following properties:

1. For each $i$, $\tilde{t}_{i+1} - \tilde{t}_i \leq 8\epsilon$.

2. Each arc $\Gamma_{\tilde{t}_i - \epsilon, \tilde{t}_i + \epsilon}$ is convex, with image in $U_{B(w,i)}$; also, $\bar{z}_i \in U_{B(w,i)}$.

3. Each arc $\Gamma_{\tilde{t}_i, \tilde{t}_i + \frac{\epsilon}{2}}$ is convex, with image in $U_{B(w,i + \frac{1}{2})}$.

4. For $f_{i,k_i} : U_{B(w,i)} \to \mathbb{R}$, we have $f_{i,k_i}(\bar{z}_i) = 0$.

5. There exist convex arcs in $U_{B(w,i)}$ from $\bar{z}_i^-$ to $\bar{z}_i$ and from $\bar{z}_i$ to $\bar{z}_i^+$.

Define $F_i : A_w \to \mathbb{R}^{d_i}$, $d_i = k_i - 1$, $F_i(\Gamma) = (f_{i,1}(\bar{z}_i), \ldots, f_{i,d_i}(\bar{z}_i))$ and $F : A_w \to \mathbb{R}^d$, $F(\Gamma) = (F_1(\Gamma), \ldots, F_\ell(\Gamma))$, $d = \dim(w) = d_1 + \cdots + d_\ell$ (the functions $f_{i,*}$ are constructed in Lemma 7.9). We claim that, for $\Gamma \in A_w$, $F(\Gamma) = 0$ if and only if $\Gamma \in \mathcal{L}_n[w]$.

Indeed, if $\Gamma \in A_w$ and $F(\Gamma) = 0$ we have $\bar{z}_i = \Gamma(\tilde{t}_i) \in \text{Bru}_{\eta_{\sigma_\ell}}$. We already know that $\{ \tilde{t}_1 < \cdots < \tilde{t}_\ell \} \subseteq \text{sing}(\Gamma) \subset \bigcup_i(\tilde{t}_i - \epsilon, \tilde{t}_i + \epsilon)$. By Lemma 10.7 we have $\text{sing}(\Gamma) = \{ \tilde{t}_1 < \cdots < \tilde{t}_\ell \}$ and therefore $\text{iti}(\Gamma) = w$.

We now construct a projection $\Pi : A_w \to \mathcal{L}_n[w] \subset A_w$. Given $\Gamma \in A_w$, the curve $\Gamma = \Pi(\Gamma)$, $\Gamma : [0,1] \to \text{Spin}_{\eta_{\sigma_\ell}}$, will coincide with $\Gamma$ except in the intervals $[\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i + \frac{\epsilon}{2}]$ and will satisfy $\Gamma(\tilde{t}_i) = \bar{z}_i$.

The restrictions $\Gamma_{\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i}$ and $\Gamma_{\tilde{t}_i, \tilde{t}_i + \frac{\epsilon}{2}}$ will be convex arcs contained in $U_{B(w,i)}$ joining $\bar{z}_i^-$ to $\bar{z}_i$ and $\bar{z}_i$ to $\bar{z}_i^+$, respectively. These two convex arcs are obtained from the convex arcs $\Gamma_{\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i}$ and $\Gamma_{\tilde{t}_i, \tilde{t}_i + \frac{\epsilon}{2}}$.
by projective transformations as in Remark 7.1. Notice that \( \epsilon_\Gamma = \epsilon_B \). If \( \Gamma \in \mathcal{L}_n[w] \) we have \( \tilde{z}_i = \hat{z}_i \) and therefore \( \hat{\Gamma} = \Gamma \).

We now have a continuous map \( (\Pi, F): \mathcal{A}_w \to \mathcal{L}_n[w] \times \mathbb{R}^d \); let \( \mathcal{B}_w \subseteq \mathcal{L}_n[w] \times \mathbb{R}^d \) be its image. We construct the inverse map \( \Phi: \mathcal{B}_w \to \mathcal{A}_w; \) in the process we see that the set \( \mathcal{B}_w \) is an open neighborhood of \( \mathcal{L}_n[w] \times \{0\} \). Indeed, given \( \hat{\Gamma} \in \mathcal{L}_n[w] \) construct \( \epsilon, \tilde{t}_i, \hat{z}_i = \hat{\Gamma}(\tilde{t}_i) \) and \( \hat{z}_i^+ = \hat{\Gamma}(\tilde{t}_i \pm \frac{\epsilon}{2}) \) as above. Given \( \mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{R}^{d_1 + \cdots + d_r} \), there exist unique \( \hat{z}_i \in \mathcal{U}_{B(w,i)} \) with \( \Pi_{B(w,i)}(\hat{z}_i) = \hat{z}_i \) and \( F(\hat{z}_i) = \mathbf{x} \). If there exist convex arcs contained in \( \mathcal{U}_{B(w,i)} \) from \( \hat{z}_i^\pm \) then \( (\hat{\Gamma}, \mathbf{x}) \in \mathcal{B}_w \) and the curve \( \hat{\Gamma} = \Phi(\hat{\Gamma}, \mathbf{x}) \) is constructed as before. More precisely, \( \hat{\Gamma} \) coincides with \( \hat{\Gamma} \) except in the intervals \( [\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i + \frac{\epsilon}{2}] \).

The convex arcs \( \hat{\Gamma}|[\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i] \) and \( \hat{\Gamma}|[\tilde{t}_i, \tilde{t}_i + \frac{\epsilon}{2}] \) are obtained from the arcs \( \Gamma|[\tilde{t}_i - \frac{\epsilon}{2}, \tilde{t}_i] \) and \( \Gamma|[\tilde{t}_i, \tilde{t}_i + \frac{\epsilon}{2}] \) by projective transformations.

Recall that there exists a natural diffeomorphism from the product of triangular groups \( U_{\sigma_i} \times L_{\sigma_i} \) to \( \mathcal{U}_{B(w,i)} \), taking \( (U_1, L_2) \) to \( Q(U_1)B(w,i)\Gamma L_2 \) (see the proof of Lemma 7.9). Endow \( U_{\sigma_i} \) and \( L_{\sigma_i} \) with the euclidean metrics coming from the usual sets of coordinates (i.e., the entries) and use the above diffeomorphism to endow \( \mathcal{U}_{B(w,i)} \) with a flat euclidean metric. Similarly, endow the cartesian product \( \mathcal{U}_{B(w,i)}^3 = \mathcal{U}_{B(w,i)} \times \mathcal{U}_{B(w,i)} \times \mathcal{U}_{B(w,i)} \) with a flat euclidean metric. Let \( \mathcal{W}_i \subset \mathcal{U}_{B(w,i)}^3 \) be the open set of triples \((z_i^-, z_i, z_i^+)_i\) such that there exist convex arcs contained in \( \mathcal{U}_{B(w,i)} \) from \( z_i^- \) to \( z_i \) and from \( z_i \) to \( z_i^+ \). Let \( \delta_i : \mathcal{W}_i \to (0, +\infty) \) be the continuous function taking a triple \((z_i^-, z_i, z_i^+)\) to \( \mathcal{W}_i \) to one half of the distance (in the flat euclidean metric constructed above) from the complement \( \mathcal{U}_{B(w,i)}^3 \setminus \mathcal{W}_i \); i.e., \( \delta_i(z_i^-, z_i, z_i^+) = \frac{1}{2}d((z_i^-, z_i, z_i^+), \mathcal{U}_{B(w,i)}^3 \setminus \mathcal{W}_i) \). Given \( \Gamma \in \mathcal{L}_n[w] \), define \( \delta(\Gamma) = \min \{ \delta(z_i^- : z_i, z_i^+) \} \) where, as above, \( z_i^\pm = \Gamma(t_i \pm \frac{\epsilon}{2}) \). Notice that \( \delta : \mathcal{L}_n[w] \to (0, +\infty) \) is continuous (see for instance the proof of Lemma A.4) and that if \( |x_i| \leq \delta(\Gamma) \) then \( (\Gamma, \mathbf{x}) \in \mathcal{B}_w \) (by construction).

Let \( \mathbb{B}^d \subset \mathbb{D}^d \subset \mathbb{R}^d \) be the open and closed balls of radius 1, respectively. Define \( \hat{\Phi} : \mathcal{L}_n[w] \times \mathbb{D}^d \to \mathcal{A}_w \) by \( \hat{\Phi}(\Gamma, \mathbf{x}) = \hat{\Phi}(\Gamma, \delta(\Gamma)\mathbf{x}) \). Let \( \hat{\mathcal{A}}_w = \hat{\Phi}[\mathcal{L}_n[w] \times \mathbb{D}^d] \subset \mathcal{A}_w \) and \( \hat{F} : \hat{\mathcal{A}}_w \to \mathbb{B}^d \) so that \( (\Pi, \hat{F}) = \hat{\Phi}^{-1}: \hat{\mathcal{A}}_w \to \mathcal{L}_n[w] \times \mathbb{D}^d \). This completes the construction of the tubular neighborhood of \( \mathcal{L}_n[w] \).

**Remark 10.8.** The maps \( F_w = F : \mathcal{A}_w \to \mathbb{R}^d \) and \( \hat{F}_w = \hat{F} : \hat{\mathcal{A}}_w \to \mathbb{B}^d \) constructed in the proof of Lemma 10.6 will be used again in the future. More precisely, in Section 11 below we follow the constructions of \( F \) above and of the map \( f \) in Remark 7.10 in order to describe examples of strata \( \mathcal{L}_n[w] \).

**Remark 10.9.** In the proof of Lemma 10.6 above, the auxiliary times \( \tilde{t}_i = \tilde{t}_{i,\Gamma} \) are continuous functions of \( \Gamma \) but it is easy to construct finite dimensional smooth families of curves \( \Gamma_s \) (with each curve not of class \( C^1 \)) such that \( \tilde{t}_{i,\Gamma_s} \) is a continuous function of the variable \( s \), but is not a function of class \( C^1 \) of \( s \). Similarly, the subset \( \mathcal{L}_n[w] \subset \mathcal{L}_n \) is not a submanifold of class \( C^1 \).
A more restrictive topology on \( L_n \) for which all curves are necessarily of class \( C^k \) (for some \( k \geq 1 \)) solves this difficulty but creates others. More precisely, the subset \( L_n[w] \subset L_n \) is then a submanifold of class \( C^k \). On the other hand, several proofs as given above no longer apply: some examples are the proof of Lemma 10.3 and the construction of \( \Pi \) and \( \Phi \) in the proof of Lemma 10.6. These difficulties are surmountable, but imply longer proofs.

Well known results from the homotopy theory of infinite-dimensional manifolds (see [11, 12, 13, 28, 44]) combine to show that different versions of the space \( L_n \) with different manifold structures, are pairwise homotopy equivalent and therefore homeomorphic. This is discussed in Appendix B; in particular we cite Theorem 2 of [13] (Fact B.2) and Theorem 0.1 of [11] (Fact B.3).

11 Transversal sections

The proof that \( L_n[w] \subset L_n \) is a topological submanifold implicitly gives us (topologically) transversal sections. We now construct an explicit transversal section, starting with the case \( w = (\sigma) \); in this case we write \( L_n[\sigma] = L_n[w] \). The construction roughly corresponds to going back to Lemma 10.6, then to Lemma 7.9 and Remark 7.10, then to Lemma 5.4 and Remark 5.5, and following the steps. A key difference is that strictly following Lemma 10.6 gives curves which fail to be smooth precisely at the times \( \tilde{t}_i \); the curves produced by our construction in this section are smooth (indeed algebraic) in a neighborhood of \( \tilde{t}_i \), even though they still violate smoothness elsewhere. We first present the construction as an algorithm, then provide examples.

Consider \( \sigma \in S_{n+1}, \sigma \neq e, \rho = \eta \sigma \) and \( d = \dim(\sigma) = \text{inv}(\sigma) - 1 \). Consider \( z_0 = q \hat{\eta} \hat{\sigma} \in B^+_n, q \in CG_{n+1} \), so that \( \text{chop}(z_0) = q \hat{\eta} \) and \( \text{adv}(z_0) = q \hat{\eta} \hat{\sigma} \). Let \( Q_0 = \Pi(z_0) \in B^+_n \). We first construct an explicit transversal section \( \psi : \mathbb{R}^{d+1} \to \text{SO}_{n+1} \) to the Bruhat cell \( \text{Bru}_{Q_0} \subset \text{SO}_{n+1} \) passing through \( Q_0 = \psi(0) \) (compare with Remarks 5.5 and 7.10). First we define a matrix \( M \in (\mathbb{R}[x_1, \ldots, x_{d+1}])^{(n+1) \times (n+1)} \) where \( x_l, 1 \leq l \leq d+1, \) are new variables. For \( i \in [n+1] \), set \( (M)_{i,\rho} = (Q_0)_{i,\rho} = \pm 1 \). There are \( d+1 \) zero entries in \( Q_0 \) which are simultaneously below a nonzero entry and to the left of a nonzero entry: these are the pairs \((i, j)\) for which \( j < i^\rho \) and \( j^{\rho^{-1}} < i \). Number the positions from 1 to \( d+1 \) in the same order you would read or write them on a page (top to bottom and left to right). For each such position \((i, j)\), set \((M)_{i,\rho} = (Q_0)_{i,\rho} x_l \). The other entries of \( M \) are set to 0: this defines the desired matrix \( M \in (\mathbb{R}[x_1, \ldots, x_{d+1}])^{(n+1) \times (n+1)} \) or, equivalently, a function \( \psi_L : \mathbb{R}^{d+1} \to \text{GL}^+_{n+1} \) where \( \psi_L(x) \) is obtained by evaluating \( M \) at \( x \in \mathbb{R}^{d+1} \). As an example, the two matrices below correspond to
n = 2 and \( \sigma_0 = [321] \) (so that \( d = 2 \)) and \( n = 3 \) and \( \sigma_1 = [3142] \) (so that \( d = 2 \)):

\[
\tilde{M}_0 = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}; \quad \tilde{M}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & x_1 & 0 \\ 1 & 0 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}.
\]

Notice that the map \( \psi_L \) is a homeomorphism from \( \mathbb{R}^{d+1} \) to \( Q_0 L_{\sigma_0} \subset \text{GL}^+_{n+1} \). The function \( \psi_A = Q \circ \psi_L : \mathbb{R}^{d+1} \to \text{SO}_{n+1} \) is the desired transversal section to the Bruhat cell \( \text{Bru}_{Q_0} \). In order to define \( \psi : \mathbb{R}^{d+1} \to \text{Spin}_{n+1} \), \( \psi_A = \Pi \circ \psi \), lift the map \( \psi_A \) starting at \( \psi(0) = z_0 \).

Consider \( \mathbb{R}^d \subset \mathbb{R}^{d+1} \) defined by \( x_{d+1} = 0 \). Let \( n = \sum_i t_i \) be the lower triangular nilpotent matrix whose only nonzero entries are \( n_{i+1,j} = 1 \) (see Example 4.2). For each \( x \in \mathbb{R}^d \) define a curve \( \phi_L(x; \cdot) : \mathbb{R} \to Q_0 L_{\sigma_1} \subset \text{GL}^+_{n+1} \) by the IVP

\[
\frac{d}{dt} \phi_L(x; t) = \phi_L(x; t)n, \quad \phi_L(x; 0) = \psi_L(x),
\]

so that \( \phi_L(x; t) = \psi_L(x) \exp(tn) \). Since entries of \( \phi_L(x; t) \) are polynomials in \( x \) and \( t \), we may equivalently consider the matrix \( M \in (\mathbb{R}[x; t])^{(n+1) \times (n+1)} \), \( M(x, t) = \phi_L(x; t) \), whose entries are polynomials in \( x \) and \( t \), of degree at most \( n \) in the variable \( t \) and satisfying

\[
(M)_{i,j+1} = \frac{d}{dt} (M)_{i,j}.
\]

As an example, the two matrices below again correspond to \( n = 2 \), \( \sigma_0 = [321] \) and \( n = 3 \), \( \sigma_1 = [3142] \):

\[
M_0 = \begin{pmatrix} 1 & 0 & 0 \\ t + x_1 & 1 & 0 \\ \frac{t^2}{2} + x_2 & t & 1 \end{pmatrix}; \quad M_1 = \begin{pmatrix} -t & -1 & 0 & 0 \\ \frac{t^3}{6} + x_1 t & \frac{t^2}{2} + x_1 & t & 1 \\ 1 & 0 & 0 & 0 \\ \frac{t^2}{2} + x_2 & t & 1 \end{pmatrix}.
\]

Notice that, given \( x \in \mathbb{R}^d \), the curve \( Q_0^{-1} \phi_L(x; \cdot) : \mathbb{R} \to L_{\sigma_1}^1 \) is locally convex. Let \( \Gamma_x : \mathbb{R} \to \text{Spin}_{n+1} \) be the locally convex curve defined by \( \Gamma(t) = \text{Q}(\phi_L(x, t)) \), \( \Gamma_x(0) = \psi(x) \). Clearly, \( \Gamma_0(0) = z_0 \), \( \Gamma_0(t) \in \text{Bru}_{\text{adv}(z_0)} \) for \( t < 0 \) and \( \Gamma_0(t) \in \text{Bru}_{\text{adv}(z_0)} \) for \( t > 0 \).

We now construct the desired transversal surface \( \phi : \mathbb{D}^d \to \mathcal{L}_n \). Choose \( z_0 \) above such that \( \text{adv}(z_0) = \hat{\eta} \) and \( \text{adv}(z_0) = \hat{\eta} \hat{\sigma} \); let \( q_1 = \hat{\eta} \hat{\sigma} \hat{\eta} \in \text{CG}_{n+1} \). For sufficiently small \( r \in (0, \frac{\pi}{4}) \), there exists a convex arc contained in \( \text{Bru}_{\text{adv}(z_0)} \) going from \( \exp(r \mathbf{h}) \) to \( \Gamma_0(-r) \). Similarly, for sufficiently small \( r \in (0, \frac{\pi}{4}) \), there exists a convex arc contained in \( \text{Bru}_{\text{adv}(z_0)} \) going from \( \Gamma_0(r) \) to \( q_1 \exp(-r \mathbf{h}) \). Fix such a small \( r \in (0, \frac{\pi}{4}) \). By continuity, there exists a small \( \hat{s} > 0 \) such that, if
\[ |x| \leq \tilde{s} \text{ then there exists a convex arc contained in } \text{Bru}_{\text{chop}(z_0)} \text{ going from } \exp(r\hbar) \text{ to } \Gamma_x^{\epsilon}(-r). \text{ Similarly, for sufficiently small } \tilde{s} > 0 \text{ if } |x| \leq \tilde{s} \text{ then there exists a convex arc in contained in } \text{Bru}_{\text{adv}(z_0)} \text{ going from } \Gamma_x(r) \text{ to } q_1 \exp(-r\hbar). \text{ Fix such a small } \tilde{s} > 0. \text{ Use Lemma 7.2 to define such convex arcs } \tilde{\phi}(x)|_{\tilde{s}/2} \text{ going from } \exp(r\hbar) \text{ to } \Gamma_{\tilde{s}x}(-r) \text{ and } \tilde{\phi}(x)|_{\tilde{s}/2} \text{ going from } \Gamma_{\tilde{s}x}(r) \text{ to } q_1 \exp(-r\hbar). \text{ For } t \in [0, 1], \text{ set } \tilde{\phi}(x)(t) = \exp(8rt\hbar); \text{ for } t \in \left[\frac{7}{8}, 1\right], \text{ set } \tilde{\phi}(x)(t) = q_1 \exp(8r(t - 1)\hbar); \text{ for } t \in \left[\frac{8}{9}, \frac{7}{8}\right], \text{ set } \tilde{\phi}(x)(t) = \Gamma_{s\infty}(8r(t - 1/2)). \text{ Consider now } s \in (0, \tilde{s}) \text{ sufficiently small so that, for all } x \in D^d \text{ with } |x| \leq \frac{\tilde{s}}{2}, \text{ we have } \tilde{\phi}(x) \in \tilde{A}_\sigma \text{ (where } \tilde{A}_\sigma \text{ is the open neighborhood of } L_n[\sigma] \text{ constructed in Lemma 10.6). Define } \phi : D^d \to \tilde{A}_\sigma \subset L_n \text{ by } \phi(x) = \tilde{\phi}(\frac{\tilde{s}}{2}x).

**Lemma 11.1.** Consider } \sigma \in S_{n+1}, \sigma \neq e, \dim(\sigma) = d \text{ and construct the map } \phi : D^d \to L_n \text{ as above. This map is topologically transversal to } L_n[\sigma], \text{ with a unique intersection at } x = 0 \in D^d.

**Proof.** Uniqueness of intersection follows from Lemma 7.9. Topological transversality follows from taking the composition } \hat{F} \circ \phi, \text{ where } \hat{F} : \tilde{A}_\sigma \to D^d \subset R^d \text{ is constructed in Lemma 10.6. The map } \hat{F} \circ \phi : D^d \to D^d \text{ is a positive multiple of the identity.} \square

Notice that the maps } \hat{F} : \tilde{A}_\sigma \to R^d \text{ and } \phi : D^d \to \tilde{A}_\sigma \subset L_n \text{ consistently provide us with a transversal orientation to } L_n[\sigma].

This completes the process of constructing a transversal section to } L_n(\sigma_1) \text{ at the path } (z_1), \text{ } z_1 = q\sigma, q \in \text{CG}_{n+1}. \text{ By applying affine transformations in the interval and projective transformations in the group } \text{Spin}_{n+1}, \text{ this defines a map } \phi_1 \text{ from } x \in D^d (d_1 = \dim(\sigma_1)) \text{ to convex arcs } \Gamma_x : [t_1 - \epsilon, t_1 + \epsilon] \to \text{Spin}_{n+1} \text{ with } \text{sing}(\Gamma_x) \neq \emptyset, \text{ sing}(\Gamma_x) \subset (t_1 - \frac{\epsilon}{2}, t_1 + \frac{\epsilon}{2}) \text{ and } \text{iti}(\Gamma_x) = (\sigma_1) \text{ if and only if } x = 0. \text{ We may furthermore assume that } \Gamma_x(t_1 + \epsilon) = z_1 \exp(\pm \hbar) \text{ for all } x \in D^d \text{ and that } \Gamma_x(t) = z_1 \exp((t - t_1)\hbar) \text{ for } x = 0 \in D^d.

More generally, for any } w = (\sigma_1 \cdots \sigma_k) = (\sigma_1, \ldots, \sigma_k) \in W_n, \text{ for any path } (z_1, \ldots, z_k) \in \text{Path}(w) \text{ and for any set } \{t_1 < \cdots < t_k\} \subset (0, 1) \text{ we show how to construct a smooth map } \phi : D^d \to L_n, d = \dim(w), \text{ transversal to } L_n[w] \text{ at } \phi(0) \in L_n[w], \text{ path}(\phi(0)) = (z_1, \ldots, z_k), \text{ sing}(\phi(0)) = \{t_1 < \cdots < t_k\}. \text{ Make the convention } t_0 = 0, z_0 = 1, t_{k+1} = 1 \text{ and } z_{k+1} = \eta \sigma_1 \cdots \sigma_k \eta. \text{ Define } q_i \in \text{CG}_{n+1} \text{ such that } q_i \eta = \text{adv}(z_{i-1}) = \text{chop}(z_i). \text{ First, choose } \epsilon > 0 \text{ such that for all } i, 0 < i \leq k + 1, t_{i-1} + \epsilon < t_i - \epsilon, z_{i-1} \exp(\epsilon\hbar) \in \text{Bru}_{q_i \eta} \text{ and } z_i \exp(-\epsilon\hbar) \in \text{Bru}_{q_i \eta} \text{. Define } L_{i-, L_i, L_i, L_{i+}} \in L_{n+1}^1 \text{ by } z_i \exp(\epsilon\hbar) = q_i \eta Q(L_{i-}) z_i \exp(-\epsilon\hbar) = q_i \eta Q(L_{i+}): \text{ by taking } \epsilon \text{ sufficiently small we may assume that } L_{i-} \ll L_{i+, \ Choose fixed convex arcs } \Gamma_{i-\frac{1}{2}} : [t_{i-1} + \epsilon, t_i - \epsilon] \to \text{Bru}_{q_i \eta}, \text{ } \Gamma_{i-\frac{1}{2}}(t_{i-1} + \epsilon) = z_{i-1} \exp(\epsilon\hbar), \text{ } \Gamma_{i-\frac{1}{2}}(t_i - \epsilon) = z_i \exp(-\epsilon\hbar). \text{ In each interval } [t_i - \epsilon, t_i + \epsilon], \text{ define as above a map } \phi_i \text{ associating to each } x_i \in D^d \text{ a convex arc } \Gamma_{i,x_i} : [t_i - \epsilon, t_i + \epsilon] \to \text{Spin}_{n+1} \text{ with }
\[ \Gamma_{i,x}(t_i - \epsilon) = z_i \exp(-\epsilon h), \Gamma_{i,x}(t_i + \epsilon) = z_i \exp(\epsilon h). \] Set \( \Gamma_0 : [0, \epsilon] \to \text{Spin}_{n+1}, \)
\( \Gamma_0(t) = \exp(\epsilon h) \) and \( \Gamma_{k+1} : [1 - \epsilon, 1] \to \text{Spin}_{n+1}, \)
\( \Gamma_{k+1}(t) = z_{k+1} \exp((t - 1)h). \) Finally, for \( x = (x_1, \ldots, x_k), \) concatenate these arcs to define \( \phi(x) = \Gamma_x \in L_n; \) the map \( \phi \) is the desired transversal section.

In the next section we continue the description of these examples. We first discuss the concept of multiplicity.

### 12 Multiplicities

Recall that the multiplicity of \( \sigma \in S_{n+1} \) is the vector
\[ \text{mult}(\sigma) = (\text{mult}_1(\sigma), \ldots, \text{mult}_n(\sigma)), \quad \text{mult}_j(\sigma) = (1^\sigma + \cdots + j^\sigma) - (1 + \cdots + j). \]

Consider a matrix \( Q \in \text{SO}_{n+1} \) and \( j \leq n. \) Let \( \text{swminor}(Q, j) \in \mathbb{R}^{j \times j} \) be the southwest \( j \times j \) minor. For a locally convex curve \( \Gamma \in L_n, \) let \( m_j = m_{\Gamma,j} : [0, 1] \to \mathbb{R}, 1 \leq j \leq n, \) be the function defined by the determinant of the southwest \( k \times k \) minor of \( \Gamma: \)
\[ m_{\Gamma,j}(t) = \det(\text{swminor}(\Gamma(t), j)). \] (16)

Write \( \text{mult}_j(\Gamma; t_\bullet) = \mu \) if \( t_\bullet \) is a zero of multiplicity \( \mu \) of the function \( m_{\Gamma,j}, \) that is, if \( (t - t_\bullet)^{(-\mu)} m_{\Gamma,j}(t) \) is continuous and non-zero at \( t = t_\bullet. \) Notice that for a general locally convex curve \( \Gamma, \) \( \text{mult}_j(\Gamma; t_\bullet) \) as above is not always well defined; if \( \Gamma \) is smooth near \( t_\bullet, \) however, \( \text{mult}_j(\Gamma; t_\bullet) = \mu \in \mathbb{N} \) is well defined. Let the multiplicity vector be
\[ \text{mult}(\Gamma; t_\bullet) = (\text{mult}_1(\Gamma; t_\bullet), \text{mult}_2(\Gamma; t_\bullet), \ldots, \text{mult}_n(\Gamma; t_\bullet)). \]

Recall that \( \Gamma(t_\bullet) \in \text{Bru}_\eta \) if and only if there exist upper triangular matrices \( U_1 \) and \( U_2 \) such that \( \Gamma(t_\bullet) = U_1 \eta U_2. \) It is a basic fact of linear algebra that this happens if and only if \( m_{\Gamma;j}(t_\bullet) \neq 0 \) for all \( j. \) In other words, for \( t_\bullet \in (0, 1), \) \( \text{mult}(\Gamma; t_\bullet) = 0 \) if and only if \( t_\bullet \notin \text{sing}(\Gamma); \) roots of the functions \( m_j \) indicate times \( t \) for which \( \Gamma(t) \notin \text{Bru}_\eta. \) The next lemma generalizes this remark and shows how to define the multiplicity vector in general. It also justifies the notation \( \text{mult}(\sigma). \)

**Lemma 12.1.** For a locally convex curve \( \Gamma : J \to \text{Spin}_{n+1} \) and \( t_\bullet \in J, \) if \( \Gamma \) is smooth in an open interval containing \( t_\bullet \) and \( \Gamma(t_\bullet) \in \text{Bru}_\eta \) then
\[ \text{mult}(\Gamma; t_\bullet) = \text{mult}(\sigma). \]

We use this formula to define \( \text{mult}(\Gamma; t_\bullet) \) even if \( \Gamma \) is not smooth. First, however, an easy result in linear algebra.
Lemma 12.2. Let \( k_1, k_2, \ldots, k_n \) be non-negative integers. Let \( M \) be the \( n \times n \) matrix with entries
\[
M_{i,1} = t^{k_i}, \quad M_{i,j+1} = \frac{d}{dt} M_{i,j}.
\]
Then
\[
det(M) = C t^\mu; \quad C = \prod_{i_0 < i_1} (k_{i_1} - k_{i_0}); \quad \mu = -\frac{n(n-1)}{2} + \sum_i k_i.
\]
If \( \tilde{M} \) is obtained from \( M \) by substituting 1 for \( t \) then \( det(\tilde{M}) = C \neq 0 \).

Proof. We have \( M_{i,j} = \tilde{M}_{i,j} t^{(k_i+1-j)} \). All monomials in the expansion of \( det(M) \) have therefore degree \( \mu \). The first column of \( \tilde{M} \) consists of ones; the second column has \( i \)-th entry equal to \( k_i \). The third column has \( i \)-th entry equal to \( k_i(k_i-1) = k_i^2 - k_i \); an operation on columns leaves the determinant unchanged but now makes the third column have entries \( k_i^2 \). Perform similar operations on columns to obtain a Vandermonde matrix, implying \( det(\tilde{M}) = C \), as desired. \( \square \)

Proof of Lemma 12.1. Assume without loss of generality that \( t_0 = 0, J = (-\epsilon, \epsilon) \). Notice that projective transformations have the effect of multiplying the functions \( m_j \) by a positive multiple and therefore do not affect the multiplicity vector. We may therefore assume that \( \Gamma(0) = Q_0 \in B_{n+1}^+ \cap \text{Bru}_{\sigma_0} \). We thus have \( (Q_0)_{i_0}^{\sigma_0} = \varepsilon_i \in \{-1, 1\} \) and \( (Q_0)_{i,j} = 0 \) otherwise. As in Section 11 we use generalized triangular coordinates: \( \Gamma_L : (-\epsilon, \epsilon) \to Q_0 L_0^{1} \cap \text{Bru}_{\sigma_0} \), \( \Gamma(t) = Q(\Gamma_L(t)) \). Notice that \( det(\text{swminor}(\Gamma_L(t), k)) \) is a positive multiple of \( det(\text{swminor}(\Gamma(t), k)) \), so that we may work with \( \Gamma_L \). Let \( A_0 = (\Gamma_L(0))^{-1} \Gamma_L'(0) = \sum_i c_i \xi_i, c_i > 0; \) let \( k_i = \prod_{j<i} c_j \).

For given \( i_0 \in [n+1] \), set \( j_0 = (n+2-i_0)^{\sigma_0} = i_0^{\sigma_0} \). For \( j > j_0 \) we have \( (\Gamma_L(t))_{i_0,j} = 0; \) also, \( (\Gamma_L(t))_{i_0,j_0} = \varepsilon_{i_0} = \pm 1 \). For \( j = j_0 - 1 \), we have that the derivative of the function \( (\Gamma_L(t))_{i_0,j} \) is a smooth positive multiple of \( (\Gamma_L(t))_{i_0,j+1} \); we thus have \( (\Gamma_L(t))_{i_0,j} = t^{c_j} \varepsilon_{i_0} u_{i_0,j}(t) \) where \( u_{i_0,j} \) is smooth and \( u_{i_0,j}(0) = 1 \). Similarly, for \( j = j_0 - \mu, \mu \geq 0 \), we have
\[
(\Gamma_L(t))_{i_0,j} = \frac{\mu}{k_{j_0}} \varepsilon_{i_0} u_{i_0,j}(t), \quad u_{i_0,j}(0) = 1
\]
or, equivalently,
\[
(\Gamma_L(t))_{i,j} = \frac{1}{(\varepsilon_{i_0} - j)!} \frac{\varepsilon_i k_{j_0}^{i_0} t^{i_0}}{k_{j_0}^{i_0} t^j} u_{i,j}(t),
\]
where we follow the convention that \( \frac{1}{\mu!} = 0 \) for \( \mu < 0 \).

Consider now \( det(\text{swminor}(\Gamma_L(t), k)) \) as a function of \( t \). Write the entries as above. The powers of \( t \) can be taken out of the determinant, yielding a factor
The terms $\varepsilon_*$ and $k_*$ can be taken out, giving us a nonzero constant multiplicative factor. Multiply the $i$-th row by $(i^{\sigma_0} - 1)! \neq 0$: the remaining matrix $M(t)$ has entries

$$M_{i,j}(t) = \frac{(i^{\sigma_0} - 1)!}{(i^{\sigma_0} - j)!} u_{i,j}(t).$$

The matrix $\text{swminor}(M(0), k)$ is just like the matrix $\tilde{M}$ in Lemma 12.2 and therefore, $\det(\text{swminor}(M(0), k)) \neq 0$. By continuity, $\det(\text{swminor}(M(t), k))$ is nonzero near $t = 0$. 

For $w = \sigma_1 \cdots \sigma_k = (\sigma_1, \ldots, \sigma_k) \in W_n$, set its multiplicity to be

$$\text{mult}(w) = \sum_j \text{mult}(\sigma_j) \in \mathbb{N}^n.$$

**Example 12.3.** We shall write $a = a_1$, $b = a_2$, $c = a_3$, ... As in Section 11, let $\Gamma_\mathbf{x}$ be a family of convex arcs and let $M$ be a matrix with polynomial entries. Let $m_j(t)$ be the determinant of the southwest $j \times j$ minor of $M$, so that $m_j$ is an explicit polynomial in rational coefficients in $t$ and $\mathbf{x}$ (or $x_i$ for $1 \leq i \leq d$).

In our first example ($n = 2$, $\sigma = [321] = [aba]$; see the matrices in Equation 15), we have

$$m_1(t) = \frac{t^2}{2} + x_2, \quad m_2(t) = \frac{t^2}{2} + x_1 t - x_2.$$ 

Thus, $m_1$ has two real roots $t = \pm \sqrt{-2x_2}$ if $x_2 < 0$ and $m_2$ has two real roots $t = -x_1 \pm \sqrt{x_1^2 + 2x_2}$ if $x_2 > -\frac{x_1^2}{2}$. Thus, if $x_2 > 0$ the itinerary is $bb = (a_2, a_2)$ and if $x_2 < -\frac{x_1^2}{2}$ the itinerary is $aa = (a_1, a_1)$. If $x_1 < 0$ (resp. $x_1 > 0$) and $-\frac{x_1^2}{2} < x_2 < 0$ the itinerary is $abab = (a_1, a_2, a_1, a_2)$ (resp. $baba = (a_2, a_1, a_2, a_1)$); the reader should compare these results with Figures 2 and 6.

![Figure 3: A transversal section to $\mathcal{L}_3[[acb]]$.](image)
In our second example \((n = 3, \sigma = [3142] = [acb])\), we have

\[
m_1(t) = \frac{t^2}{2} + x_2, \quad m_2(t) = t, \quad m_3(t) = \frac{t^2}{2} - x_1.
\]

Thus, \(m_2\) has a simple root at \(t = 0\). If \(x_2 > 0\), \(m_1\) has no real roots; if \(x_2 < 0\), \(m_1\) has roots \(t = \pm \sqrt{-x_2}\). Similarly, for \(x_1 < 0\), \(m_3\) has no real roots and for \(x_1 > 0\), \(m_3\) has roots \(t = \pm \sqrt{x_1}\). It is now easy to verify the itineraries in Figure 3. Notice that this section is transversal to \(L_3[[3142]] = L_3[[acb]]\) (as promised) but is not transversal to \(L_3[[a_1a_3], a_2, [a_1a_3]] = L_3[[ac]b[ac]]\) (which was never promised).

### 13 The poset \(W_n\)

Let \(\sigma_0 \in S_{n+1}, \sigma_0 \neq e\); let \(\sigma_1 = \eta \sigma_0\). Let \(z_1 = q\tilde{\sigma}_1 \in \tilde{B}_{n+1}^+, q \in CG_{n+1}\). Recall from Example 4.2 that \(L(\exp(\theta h)) = \exp(\tan(\theta) h_L)\) for \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Let \(\theta_0 \in (0, \frac{\pi}{2})\). The curve \(\Gamma_0 \in L_n(z_1 \exp(-\theta_0 h); z_1 \exp(\theta_0 h)), \Gamma_0 : [-\theta_0, +\theta_0] \rightarrow \text{Spin}_{n+1}, \Gamma_0(\theta) = \tau_1 \exp(\theta h), \text{ is locally convex with image contained in } U_{z_1}\). Moreover, it can be expressed in triangular coordinates and is therefore convex (see Appendix A): \(\Gamma_0(\theta) = z_1 Q(\Gamma_0, L_{\theta}(\theta))\) where \(\Gamma_0, L_{\theta}(\theta) : [-\theta_0, 0; \theta_0] \rightarrow \text{Lo}^1_{n+1}, \Gamma_0, L_{\theta}(\theta) = \exp(\tan(\theta) h_{L_{\theta}(\theta)})\). The itinerary of \(\Gamma_0\) is \(\text{iti}(\Gamma_0) = (\sigma_0)\).

For \(w \in W_n\), define \(w \preceq \sigma_0\) (or, more precisely, \(w \preceq (\sigma_0)\)) if there exists a convex curve \(\Gamma_1 \in L_{n, \text{convex}}(z_1 \exp(-\theta_0 h); z_1 \exp(\theta_0 h))\) with \(\text{iti}(\Gamma_1) = w\). Our first remark is that this condition does not depend on the choice of \(\theta_0 \in (0, \frac{\pi}{2})\).

**Lemma 13.1.** Consider \(z_1 \in \tilde{B}_{n+1}, \theta_1, \theta_2 \in (0, \frac{\pi}{2})\). Consider \(w \in W_n\). Then there exists \(\Gamma_1 \in L_{n, \text{convex}}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h))\) with \(\text{iti}(\Gamma_1) = w\) if and only if there exists \(\Gamma_2 \in L_{n, \text{convex}}(z_1 \exp(-\theta_2 h); z_1 \exp(\theta_2 h))\) with \(\text{iti}(\Gamma_2) = w\).

Furthermore, if there exists \(\Gamma_1 \in L_{n, \text{convex}}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h))\) such that \(\text{iti}(\Gamma_1) = w\) then there exists a homotopy

\[
H : [0, 1] \rightarrow L_{n, \text{convex}}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h)), \quad H(0) = \Gamma_0, \quad H(1) = \Gamma_1,
\]

\[
\Gamma_0(\theta) = z_1 \exp(\theta h), \quad H(s)|_{[-\theta_1, -s\theta_1]} |_{[s\theta_1, \theta_1]} = \Gamma_0|_{[-\theta_1, -s\theta_1]} \bigcup [s\theta_1, \theta_1]
\]

such that \(\text{iti}(H(s)) = w\) for all \(s \in (0, 1]\).

**Proof.** We start with the first claim. Assume without loss of generality that \(\theta_1 < \theta_2\). Given \(\Gamma_1\) as above, \(\Gamma_2\) can be constructed by attaching arcs: set

\[
\Gamma_2(\theta) = \begin{cases} \Gamma_1(\theta), & \theta \in [-\theta_1, \theta_1], \\ z_1 \exp(\theta h), & \theta \in [-\theta_2, -\theta_1] \cup [\theta_1, \theta_2]. \end{cases}
\]
Conversely, given \( \Gamma_2 \) we apply a projective transformation to obtain \( \Gamma_1 \). More precisely, set \( \Gamma_{2,L} : [-\theta_2, \theta_2] \to L_{0,n+1}^1 \), \( \Gamma_{2,L}(\theta) = L(z_1^{-1} \Gamma_2(\theta)) \). Notice that a diagonal projective transformation takes \( \exp(\pm \tan(\theta_2)h_L) \) to \( \exp(\pm \tan(\theta_1)h_L) \):

\[
\exp(\pm \tan(\theta_1)h_L) = \text{diag}(1, \lambda, \ldots, \lambda^n) \exp(\pm \tan(\theta_2)h_L) \text{diag}(1, \lambda^{-1}, \ldots, \lambda^{-n})
\]

for \( \lambda = \tan(\theta_1) / \tan(\theta_2) \); apply this projective transformation and reparametrize the domain to obtain \( \Gamma_{1,L} \) and therefore \( \Gamma_1 \in L_n, \text{convex}(z_1 \exp(-\theta_1h_1) ; z_1 \exp(\theta_1h_1)) \) with \( \text{iti}(\Gamma_1) = \text{iti}(\Gamma_2) \).

For the second claim, given \( \Gamma_1 \), apply a projective transformation as above to define \( H(s) \) satisfying the conditions in the statement (compare with the construction of the homotopy in the proof of Lemma 8.5). \( \square \)

**Lemma 13.2.** Consider \( w \in W_n \), \( \sigma_0 \in S_{n+1} \), \( \sigma_0 \neq e \), and \( \Gamma_0 \in L_n[[\sigma_0]] \). If there exists a sequence \( \Gamma_k \in L_n \) with \( \text{iti}(\Gamma_k) = w \) for all \( k \in \mathbb{N}^* \) and \( \lim_{k \to \infty} \Gamma_k = \Gamma_0 \) then \( w \preceq \sigma_0 \).

Notice that the converse is already known: if \( w \preceq \sigma_0 \) we constructed in Lemma 13.1 a path \( H \) of curves of itinerary \( w \) tending to \( \Gamma_0 \).

**Proof.** By reparametrizing the domain and applying a projective transformation, we may assume \( \text{sing}(\Gamma_0) = \{ \frac{1}{2} \} \) and \( \Gamma_0(\frac{1}{2}) = z_1 = q\hat{\sigma}_1 \in \tilde{B}_{n+1}^+ \) where \( q \in C \Gamma_{n+1} \) and \( \sigma_1 = \eta \sigma_0 \). Consider \( \epsilon_0 > 0 \) such that \( |t - \frac{1}{2}| \leq \epsilon_0 \) implies \( \Gamma_0(t) \in U_{\epsilon_0} \). For \( t \in [\frac{1}{2} - \epsilon_0, \frac{1}{2} + \epsilon_0] \), define \( \Gamma_{0,L}(t) = L(z_1^{-1} \Gamma_0(t)) \). Notice that \( \Gamma_{0,L}(\frac{1}{2} - \epsilon_0) \ll I \ll \Gamma_{0,L}(\frac{1}{2} + \epsilon_0) \). Take \( \epsilon_1 \in (0, \frac{\pi}{2}) \) such that \( \Gamma_{0,L}(\frac{1}{2} - \epsilon_0) \ll \exp(-\arctan(\epsilon_1)h_L) \) and \( \exp(\arctan(\epsilon_1)h_L) \ll \Gamma_{0,L}(\frac{1}{2} + \epsilon_0) \). Set \( L_{1,-} = \exp(-\arctan(\epsilon_1)h_L) \) and \( L_{1,=} = \exp(\arctan(\epsilon_1)h_L) \). Take \( \epsilon_2 \in (0, \frac{\arctan(\epsilon_1)}{2}) \) such that \( L_{1,-} \ll \Gamma_{0,L}(\frac{1}{2} - \epsilon_2) \ll I \) and \( I \ll \Gamma_{0,L}(\frac{1}{2} + \epsilon_2) \ll L_{1,=} \).

Take open neighborhoods \( A_{0,-} \), \( A_{2,-} \), \( A_{2,=} \) and \( A_{0,=} \subset L_{0,n+1}^1 \) of \( \Gamma_{0,L}(\frac{1}{2} - \epsilon_0) \), \( \Gamma_{0,L}(\frac{1}{2} - \epsilon_2) \), \( \Gamma_{0,L}(\frac{1}{2} + \epsilon_2) \) and \( \Gamma_{0,L}(\frac{1}{2} + \epsilon_0) \), respectively, such that, for all \( \epsilon_i, \epsilon_j \in A_{i,=} \), \( i \in \{0, 2\} \), we have \( \Gamma_{0,-} \ll L_{1,-} \ll L_{2,-} \ll I \ll L_{2,+} \ll L_{1,+} \ll L_{0,+} \). Let \( B_{i,=} = z_1 Q[A_{i,=}] \subset U_{z_1}, \ i \in \{0, 2\} \); notice that \( \Gamma_0(\frac{1}{2} \pm \epsilon_i) \in B_{i,=} \), \( i \in \{0, 2\} \).

For sufficiently large \( k \) we have \( \Gamma_k(\frac{1}{2} \pm \epsilon_i) \in B_{i,=} \), \( i \in \{0, 2\} \). By Corollary 8.4, for sufficiently large \( k \) we also have \( \text{sing}(\Gamma_k) \subset (\frac{1}{2} - \epsilon_0, \frac{1}{2} - \epsilon_2) \). For such large \( k \), define a locally convex curve \( \tilde{\Gamma}_k \) which coincides with \( \Gamma_k \) except in the intervals \( [\frac{1}{2} - \epsilon_0, \frac{1}{2} - \epsilon_2] \) and \( [\frac{1}{2} + \epsilon_2, \frac{1}{2} + \epsilon_0] \). In these arcs, \( \tilde{\Gamma}_k \) is defined so that \( \tilde{\Gamma}_k(\frac{1}{2} - \epsilon_1) = z_1 Q(L_{1,-}) = z_1 \exp(-\epsilon_1h_1) \) and \( \tilde{\Gamma}_k(\frac{1}{2} + \epsilon_1) = z_1 Q(L_{1,+}) = z_1 \exp(\epsilon_1h_1) \): the above conditions guarantee that this is possible. The restriction of any such curve \( \tilde{\Gamma}_k \) to the interval \( [\frac{1}{2} - \epsilon_1, \frac{1}{2} + \epsilon_1] \) yields, by definition, \( w \preceq \sigma_0 \). \( \square \)
We are ready to define a poset structure in \( \mathbf{W}_n \). Recall that the Bruhat order in the symmetric group \( S_{n+1} \) can be defined as follows: \( \sigma_0 \leq \sigma_1 \) if and only if \( \text{Bru}_{\sigma_0} \subseteq \text{Bru}_{\sigma_1} \). For \( w_0, w_1 \in \mathbf{W}_n \), set

\[
w_0 \preceq w_1 \iff \mathcal{L}_n[w_1] \subseteq \mathcal{L}_n[w_0];
\]

notice the reversion. From the previous results, this coincides with our first definition for \( w_1 = \sigma \in S_{n+1} \setminus \{e\} \).

**Lemma 13.3.** For \( \sigma_0 \in S_{n+1} \setminus \{e\} \) and \( w \in \mathbf{W}_n \), the following conditions are equivalent:

(i) \( w \preceq \sigma_0 \);

(ii) given \( \Gamma_1 \in \mathcal{L}_n[(\sigma_0)] \) and an open neighborhood \( U \subset \mathcal{L}_n \) of \( \Gamma_1 \) there exists \( \Gamma \in U \cap \mathcal{L}_n[w] \);

(iii) given \( \Gamma_1 \in \mathcal{L}_n[(\sigma_0)], \epsilon > 0, \text{sing}(\Gamma_1) = \{t_1\} \) and an open neighborhood \( U \subset \mathcal{L}_n \) of \( \Gamma_1 \) there exists \( \Gamma \in U \cap \mathcal{L}_n[w] \) with \( \Gamma \) and \( \Gamma_1 \) coinciding outside \( (t_1 - \epsilon, t_1 + \epsilon) \).

**Proof.** Condition (iii) clearly implies (ii); (ii) is a rewording of the second definition of (i); the old definition of (i) implies (iii). \( \Box \)

The set \( \mathbb{N}[X] \) of polynomials with coefficients in \( \mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\} \) is well ordered by

\[
p_0 < p_1 \iff \exists C \in \mathbb{R}, \forall x \in \mathbb{R}, ((x > C) \rightarrow (p_0(x) < p_1(x))).
\]

For \( w = (\sigma_1, \ldots, \sigma_\ell) \in \mathbf{W}_n \), define

\[
\text{Xdim}(w) = \sum_{1 \leq j \leq \ell} X^{\text{dim}(\sigma_j)} \in \mathbb{N}[X].
\]

In particular, \( w \in \mathbf{W}_n \) implies \( \text{Xdim}(w) < X^m, m = \frac{n(n+1)}{2} \).

Clearly, given \( p \in \mathbb{N}[X] \) there exist only finitely many words \( w \in \mathbf{W}_n \) with \( p = \text{Xdim}(w) \). As usual in set theory, identify the ordinal \( \omega^m \) with the set of ordinals smaller than \( \omega^m \), i.e.,

\[
\omega^m = \{\omega^{m-1}c_{m-1} + \cdots + \omega c_1 + c_0; c_0, c_1, \ldots, c_{m-1} \in \mathbb{N}\}.
\]

There exists therefore a bijection \( \text{ord} : \mathbf{W}_n \rightarrow \omega^m \) such that if \( w_0, w_1 \in \mathbf{W}_n \) and \( \text{Xdim}(w_0) < \text{Xdim}(w_1) \) then \( \text{ord}(w_0) < \text{ord}(w_1) \).
Lemma 13.4. Consider $\sigma_0 \in S_{n+1} \setminus \{e\}$, $w = (\sigma_1, \sigma_2, \ldots, \sigma_\ell) \in W_n$, $w \preceq (\sigma_0)$, $w \neq (\sigma_0)$. Then $\ell > 0$ and $\sigma_i < \sigma_0$ (in the Bruhat order), $\dim(\sigma_i) < \dim(\sigma_0)$ and $\text{mult}(\sigma_i) < \text{mult}(\sigma_0)$ for $1 \leq i \leq \ell$. Furthermore, $X\dim(w) < X\dim(\sigma_0)$, $\text{ord}(w) < \text{ord}(\sigma_0)$.

Proof. Notice first of all that the empty word is known to be isolated in the poset $W_n$: this follows either from Lemma 8.3 above or from the known fact that convex curves form a connected component of $L_n$ (as in Lemma 8.5). This implies $k > 0$. By definition of the Bruhat order, $\sigma_i \leq \sigma_0$. Also, from Lemma 10.7 and $w \neq (\sigma_0)$ we have $\sigma_i < \sigma_0$. It then follows that $\dim(\sigma_i) < \dim(\sigma_0)$ and $\text{mult}(\sigma_i) < \text{mult}(\sigma_0)$; the rest is easy.

Conjecture 13.5. If $\sigma \in S_{n+1}$, $w \in W_n$ and $w \preceq (\sigma)$ then $\text{mult}(w) \leq \text{mult}(\sigma)$.

Remark 13.6. Conjecture 13.5 above implies Conjecture 2.4 in [61]. The cases $n = 2$ and $\text{inv}(\sigma) \leq 1$ are easy. The case $n = 3$ and $\sigma = \eta$ is essentially answered in [62] (with different notation). Notice that for the words which appear in the transversal section constructed in Sections 11 and 12 the claim is correct. Conjecture 13.5 would simplify some arguments below but it will be neither proved nor used.

We might also ask whether $w \preceq (\sigma)$ implies $\dim(w) \leq \dim(\sigma)$. We do not have a counterexample, but lean towards disbelieving the implication. The example $[a_1a_3][a_1a_2] \preceq [a_1a_3a_2]$ shows that $w \preceq (\sigma)$ and $w \neq (\sigma)$ do not imply $\dim(w) < \dim(\sigma)$.

We sum up some of our conclusions.

Lemma 13.7. The set $W_n$ is a poset. Consider $w_0, w_1 \in W_n$, $w_1 = (\sigma_1, \ldots, \sigma_\ell_1)$.

(i) $w_0 \preceq w_1$ if and only if there exist nonempty words $\hat{w}_1, \ldots, \hat{w}_{\ell_1} \in W_n$ such that $w_0$ is the concatenation of $\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{\ell_1}$ and, for all $j$, $\hat{w}_j \preceq \sigma_j$;

(ii) if $w_0 \preceq w_1$ then $\hat{w}_0 = \hat{w}_1$, $\ell(w_0) \geq \ell(w_1)$ and $\text{ord}(w_0) \leq \text{ord}(w_1)$;

(iii) $W_n$ is well-founded;

(iv) $W_n$ is converse well-founded, and given $w_0$ there are only finitely many $w \in W_n$ such that $w_0 \preceq w$.

Proof. By definition, the relation $\preceq$ is reflexive and transitive. Item (i) follows from Lemmas 13.3 and 13.4. It now follows that $w_0 \preceq w_1 \preceq w_0$ implies $w_0 = w_1$, completing the proof the $W_n$ is a poset. Item (ii) follows from Lemma 13.4. Item (iii) is a corollary of (ii); (iv) follows from (ii) and the finiteness of $S_{n+1}$.  

54
Example 13.8. Consider \( \sigma \in S_{n+1} \setminus \{e\} \), \( \dim(\sigma) = k \), and the map \( \phi : \mathbb{D}^k \to \mathcal{L}_n \) transversal to \( \mathcal{L}_n[\sigma] \) constructed in Lemma [11.1]. By construction of \( \phi \), if \( x \in \mathbb{D}^k \setminus \{0\} \) and \( \text{iti}(\phi(x)) = w \) there exists a continuous map \( h : [0,1] \to \mathbb{D}^k \) such that \( h(0) = 0 \), \( h(1) = x \) and \( \text{iti}(\phi(h(s))) = w \) for all \( s \in (0,1] \). In particular, \( w \preceq \sigma \). As we shall see, the reciprocal does not hold.

Thus, for instance, again with \( a = a_1, b = a_2 \) and \( c = a_3 \), it follows from the transversal sections constructed in Example [12.3] that

\[
[a|b], [b|a]a, b|a[b], [a|b]b \preceq [a|b]; \quad [a|b], [b|a], a|b|a[c|b|a], [a|b|a] \preceq [a|b].
\]

Notice that \( \dim([a|b][b|a]) = \dim([a|b]) = 2 \). Also, by transitivity we have

\[
acb|a|c, cab|c|a, [a|b|a][c|b|a], [a|b|a|c|b|a] \preceq [a|b|a]
\]

but none of the itineraries on the left hand side appear in the transversal section constructed in Example [12.3].

It is easy, on the other hand, to perturb the map \( \phi \) to obtain other maps transversal to \( \mathcal{L}_n[[a|b][b|a]] \). Take

\[
M = \begin{pmatrix}
-t & -1 & 0 & 0 \\
\frac{t^3}{6} + xt & t^2 + x & t & 0 \\
\frac{ut}{2} + 1 & u & 0 & 0 \\
\frac{t^2}{2} + y & t & 1 & 0
\end{pmatrix}
\]

where \( u \) is to be thought of as a fixed real number of small absolute value: say, \( |u| < 1/4 \). The previous construction corresponds of course to \( u = 0 \). Figure 4 shows the resulting sections.

![Figure 4: Two other transversal sections to \( \mathcal{L}_3[[a|b]] \) (\( u < 0 \) and \( u > 0 \)).](image)

Notice that the two diagrams differ combinatorially. For \( u < 0 \), the itinerary \( acbac \) appears and \( cabca \) does not; for \( u > 0 \) it is the other way round.
14 Lower and upper sets

A subset $I \subseteq W_n$ is a lower set if, for all $w_1 \in I$ and $w_0 \in W_n$, $w_0 \preceq w_1$ implies $w_0 \in I$. In particular, $\emptyset$ and $W_n$ are lower sets. If $I$ is a lower set then

$$L_n[I] = \bigsqcup_{w \in I} L_n[w] \subseteq L_n$$

is an open subset. Given $w_0 \in W_n$, let $I(w_0) = \{w \in W_n, w \preceq w_0\}$ and $I^*(w_0) = I(w_0) \setminus \{w_0\}$; both are lower sets. The map $\phi : D_k \to L_n$ transversal section to $L_n[I] \subseteq L_n$ is of the form $\phi : D_k \to L_n[I(\omega)] \subseteq L_n$. Figure 5 is a diagram of $I([aba]) = \{aa, abab, bb, baba, [ba]a, a[ba], [ab]b, b[ab], [aba]\}$; compare with Figure 2.

![Figure 5: The lower set $I([aba])$.](image)

For $\alpha \in \omega^m$, let $I(\alpha) = \{w \in W_n \mid \text{ord}(w) < \alpha\}$, a lower set. For instance, $I(\omega)$ is the set of words of degree 0, i.e., words whose letters are generators $a_k$. The set $L_n[I(\omega)]$ is a disjoint union of contractible open sets $L_n[w], w \in I(\omega)$.

Similarly, $U \subseteq W_n$ is an upper set if, for all $w_0 \in U$ and $w_1 \in W_n$, $w_0 \preceq w_1$ implies $w_1 \in U$. If $U$ is an upper set then

$$L_n[U] = \bigsqcup_{w \in U} L_n[w] \subseteq L_n$$

is a closed subset.

A few examples are in order.

**Example 14.1.** The following are examples of upper sets:

\[
\begin{align*}
U_{2,2} &= \{[aba]\} \subset W_2, \quad U_{2,3} = \{[aba], [bacb], [cb]\} \subset W_3, \\
&\quad \{[abc], [ac], [cba]\} \subset W_3, \quad U_{x,3} = \{[cba]\} \subset W_3, \\
U_{2,4} &= \{[aba], [bacb], [cb], [cbdc], [cde]\} \subset W_4, \\
U_{4,4} &= \{[34521], [32541], [52341], [52143], [54123]\} \subset W_4.
\end{align*}
\]
The example $U_{2,2}$ follows directly from Lemma \ref{lem:examples}. There is no letter in $S_3$ of greater dimension than $[aba]$. For $U_{2,3}$, verify that there are no other letters $\sigma \in S_4$ with $\hat{\sigma} = \text{hat}([aba]) = -1$ (notice also that $[aba] \preceq [bacb]$, $[bcb] \preceq [bacb]$).

Also, the only letters $\sigma \in S_4$ with $\hat{\sigma} = \hat{ac}$ are $[abc]$, $[ac]$ and $[cba]$ (notice that $[ac] \preceq [abc]$, $[ac] \preceq [cba]$; see also Lemma \ref{lem:3.9}.

The only letters $\sigma \in S_5$ with $\hat{\sigma} = -1$ are the elements of $U_{2,4}$ and $\eta$. We do not have $\sigma \preceq \eta$ for $\sigma \neq \eta$, however. Indeed, the only point in $\text{Pos}_\eta \cap \text{Neg}_\eta$ is the identity. Consider a locally convex curve $\Gamma_L : [-1, 1] \to L_0^1$ with $\Gamma_L(-1) \in \text{Neg}_\eta$ and $\Gamma_L(1) \in \text{Pos}_\eta$. Set $\Gamma(t) = qQ(\Gamma_L(t))$, $q \in CG_{n+1}$. If $\Gamma_L(t) = I$ for some $t$ then $\text{iti}(\Gamma) = \eta$. Otherwise, $\Gamma_L$ must cross $\partial \text{Neg}_\eta$ and $\partial \text{Pos}_\eta$ at two distinct times $t_- < t_+$, $t_-, t_+ \in \text{sing}(\Gamma)$: thus $\text{iti}(\Gamma) \neq \sigma$. The only letters $\sigma \in S_5 \setminus \{e\}$ with $\hat{\sigma} = 1$ are the elements of $U_{4,4}$.

### 15 Valid complexes $D_n[I]$

Here $D^k$ is the closed disk of dimension $k$. Let $I \subseteq W_n$ be a lower set. A valid (CW) complex $D_n[I]$ (for $I$) has the following ingredients and properties:

(i) A CW complex $D_n[I]$ with one cell of dimension $\dim(w)$ for each $w \in I$.

The continuous gluing maps $g_w : \partial D_{\dim(w)} \to D_n[I]$ have image contained in $\mathcal{D}_n[I]$. Let $i_w : D_{\dim(w)} \to D_n[I]$ be the inclusion (with quotient at the boundary).

(ii) For each $w \in I$, we have a continuous map $c_w : D_{\dim(w)} \to L_n[I_w]$ (which will be called the $w$ cell).

(iii) The maps $c_w$ are compatible: if $i_{w_0}(p_0) = i_{w_1}(p_1)$ then $c_{w_0}(p_0) = c_{w_1}(p_1)$. In particular, if $w_1 \preceq w_0$, $p_0 \in \partial D_{\dim(w_0)}$, $p_1 \in D_{\dim(w_1)}$ and $g_{w_0}(p_0) = i_{w_1}(p_1)$ then $c_{w_0}(p_0) = c_{w_1}(p_1)$. The union of the cells $c_w$ defines a continuous map $\alpha : \mathcal{D}_n[I] \to \mathcal{L}_n[I]$. 

(iv) For any $w \in I$, the cell $c_w$ is a topological embedding in the interior of $D_{\dim(w)}$, smooth in the ball of radius $\frac{1}{2}$, and intersects some tubular neighborhood $\mathcal{A}_w \supset L_n[w]$ transversally around $c_w(0) \in L_n[w]$. More precisely, for $\mathcal{A}_w \supset L_n[w]$ as in the characterization of such tubular neighborhoods, the map $\hat{F}_w : \mathcal{A}_w \to D_{\dim(w)}$ is a homeomorphism from an open neighborhood of 0 to an open neighborhood of 0.

(v) If $s \in D_{\dim(w)}$, $s \neq 0$, then $c_w(s) \in L_n[\Gamma^*(w)]$ (and therefore $c_w(s) \notin L_n[w]$).

We abuse notation and identify a valid complex with the corresponding family of cells and write $D_n[I] = (c_w)_{w \in I}$ (notice that this family yields all the necessary
information). A valid complex is good if for every lower set $\tilde{I} \subseteq I$ the map $c_I : D_n[\tilde{I}] \to \mathcal{L}_n[I]$ is a weak homotopy equivalence.

Remark 15.1. Given a valid complex $D_n[I] = (c_w)_{w \in I}$ and a lower set $\tilde{I} \subseteq I$, the subcomplex $(c_w)_{w \in \tilde{I}}$ is also valid (for $\tilde{I}$). We call this subcomplex $D_n[\tilde{I}]$ the restriction of $D_n[I]$ to $I$. A restriction of a good complex is also good.

Let $J$ be a totally ordered set. Let $(I_j)_{j \in J}$ be a family of lower sets such that $j_0 < j_1$ implies $I_{j_0} \subseteq I_{j_1}$. Let $I = \bigcup_j I_j$, which is therefore also a lower set. For each $j \in J$, let $D_n[I_j]$ be a valid complex. Assume that if $j_0 < j_1$ then $D_n[I_{j_0}]$ is the restriction of $D_n[I_{j_1}]$ to $I_{j_0}$. Then the union $D_n[I]$ of all the valid complexes $D_n[I_j]$ (for all $j \in J$) is a valid complex for $I$. A family $(D_n[I_j])_{j \in J}$ of valid complexes is a chain if it satisfies the conditions above.

Lemma 15.2. Let $(D[I_j])_{j \in J}$ be a chain of good complexes. Let $I = \bigcup_j I_j$ and $D_n[I]$ be the union of the complexes $D_n[I_j]$. Then $D_n[I]$ is a good complex.

Proof. Let $\alpha : S^k \to D_n[I]$ be a continuous map which is homotopically trivial in $\mathcal{L}_n[I]$, i.e., there exists a map $A_0 : D^{k+1} \to \mathcal{L}_n[I]$, $A_0|_{S^k} = c_I \circ \alpha$. By compactness, there exists $j \in J$ such that the image of $A_0$ is contained in $\mathcal{L}_n[I_j]$ so that we may write $A_0 : D^{k+1} \to \mathcal{L}_n[I_j]$ Since $D_n[I_j]$ is good, there exists $A_1 : D^{k+1} \to D_n[I_j]$ such that $A_1|_{S^k} = \alpha$.

Let $\alpha_0 : S^k \to \mathcal{L}_n[I]$ be a continuous map. By compactness, there exists $j \in J$ such that the image of $\alpha_0$ is contained in $\mathcal{L}_n[I_j]$. Since $D_n[I_j]$ is good, there exists $\alpha_1 : S^k \to D_n[I_j] \subset D_n[I]$ such that $c_{I_j} \circ \alpha_1$ is homotopic to $\alpha_0$ in $\mathcal{L}_n[I_j]$ (and therefore also in $\mathcal{L}_n[I]$).

This completes the proof that $c_I : D_n[I] \to \mathcal{L}_n[I]$ is a weak homotopy equivalence. The proof for $\tilde{I} \subseteq I$ is similar. □

Lemma 15.3. Let $\tilde{I} \subset I \subseteq W_n$ be lower sets with $I \setminus \tilde{I} = \{w_0\}$. Let $D_n[\tilde{I}]$ be a good complex (for $\tilde{I}$). Then this complex can be extended to a valid complex $D_n[I]$ (for $I$). Moreover, all such valid complexes are good.

Before proving Lemma 15.3 we present the main conclusion for this section.

Lemma 15.4. For any lower set $I \subseteq W_n$ there exists a valid complex $D_n[I] \subset \mathcal{L}_n[I]$. All such valid complexes are good.

Proof. We prove by transfinite induction on $\alpha \leq \omega^m$ that there exists a chain of good complexes $(D_n[I \cap L(\beta)])_{\beta < \alpha}$. For $\alpha = 0$ there is nothing to do. For $\alpha = \bar{\alpha} + 1$ apply Lemma 15.3. For $\alpha$ a limit ordinal apply Lemma 15.2.

The fact that all valid complexes are good is likewise proved by transfinite induction. We again use Lemma 15.3 for successor ordinals and Lemma 15.2 for limit ordinals. □
Proof of Lemma 15.3. Assume $D_{n}[\hat{I}]$ given (and good): we construct the cell $c_{w_{0}}$ so that $D_{n}[\hat{I}]$ is also good. Let $k = \dim(w_{0})$. For a small ball $B \subset \mathbb{R}^{k}$ around the origin, construct as in Section 11 a smooth map $\tilde{c} : B \to L_{n}[\hat{I}] \subseteq L_{n}$ with $\tilde{c}(0) = \Gamma_{0} \in L_{n}[w_{0}]$ and topologically transversal to $L_{n}[w_{0}]$ at this point. By taking $r > 0$ sufficiently small, the image under $\tilde{c}$ of a ball $B(2r) \subset B$ of radius $2r$ around the origin satisfies the following condition: $s \in B(2r)$ and $s \neq 0$ imply $\tilde{c}(s) \in L_{n}[I_{w_{0}}^{*}] \subseteq L_{n}[\hat{I}]$.

Let $S(r) \subset B(2r)$ be the sphere of radius $r$ around the origin so that $\tilde{c}|_{S(r)} : S(r) \to L_{n}[I_{w_{0}}^{*}]$. Since $D_{n}[\hat{I}]$ is good, the map $c_{I_{w_{0}}} : D_{n}[I_{w_{0}}] \to L_{n}[I_{w_{0}}^{*}]$ is a weak homotopy equivalence. There exists therefore a map $g_{w_{0}} : S^{k-1} \to D_{n}[I_{w_{0}}^{*}]$ such that $c_{I_{w_{0}}} \circ g$ is homotopic (in $L_{n}[I_{w_{0}}^{*}]$) to $\tilde{c}|_{S(r)}$. In other words, there exists $c_{w_{0}} : \mathbb{D}^{k} \to L_{n}[I_{w_{0}}]$ coinciding with $\tilde{c}$ in $B(r)$, assuming values in $L_{n}[I_{w_{0}}]$ outside 0 and with boundary $c_{I_{w_{0}}} \circ g_{w_{0}}$. This completes the construction of a valid complex $D_{n}[\hat{I}]$ and the proof of the first claim.

We now prove the second claim. More precisely, let $c_{w_{0}}$ be such that extending the good complex $D_{n}[\hat{I}]$ by $c_{w_{0}}$ is a valid complex (for $\hat{I}$): we prove that it is a good complex. In other words, we prove that the map $c_{\hat{I}} : D_{n}[\hat{I}] \to L_{n}[\hat{I}]$ is a weak homotopy equivalence for any lower set $\hat{I} \subseteq I$. Again, we present the proof for $\hat{I} = I$: the general case is similar. The construction is similar to that of certain classical results involving CW complexes; a minor difference here is that $L_{n}[w_{0}] \subseteq L_{n}[\hat{I}]$ is not a smooth submanifold. Recall that $L_{n}[w_{0}] \subseteq L_{n}[\hat{I}]$ is a contractible closed subset (of $L_{n}[\hat{I}]$) and a (globally) collared topological submanifold of codimension $d = \dim(w_{0})$. Indeed, in Lemma 10.6 we constructed a tubular neighborhood of $\hat{A}_{w_{0}} \supset L_{n}[w_{0}]$, a projection $\Pi : \hat{A}_{w_{0}} \to L_{n}[w_{0}]$ and a map $\hat{F} = \hat{F}_{w_{0}} : \hat{A}_{w_{0}} \to \mathbb{B}^{d}$ (where $\mathbb{B}^{d}$ is the unit open ball) such that $(\Pi, \hat{F}) : \hat{A}_{w_{0}} \to (L_{n}[w_{0}], \mathbb{B}^{d})$ is a homeomorphism (see Remark 10.8). By construction, the map $c_{w_{0}}$ intersects $\hat{A}_{w_{0}} \supset L_{n}[w_{0}]$ transversally so that $c_{w_{0}}^{-1}[\hat{A}_{w_{0}}]$ is an open neighborhood of 0 in $\mathbb{B}^{d}$. The map $\hat{F} \circ c_{w_{0}}$ (where defined) is a homeomorphism from a neighborhood of 0 to a neighborhood of 0; indeed, in the example constructed in the first part of the proof, it is the multiplication by a positive constant. As we shall see, the tubular neighborhood (and not smoothness) is the essential ingredient in the proof.

Consider a compact manifold $M_{0}$ of dimension $k$ and continuous map $\alpha_{0} : M_{0} \to L_{n}[\hat{I}]$. We prove that there exists $\alpha_{1} : M_{0} \to D_{n}[\hat{I}]$ such that $\alpha_{0}$ is homotopic to $c_{1} \circ \alpha_{1}$. First deform $\alpha_{0}$ to obtain a map $\alpha_{1}$ which intersects $\hat{A}_{w_{0}}$ transversally; this is similar to the more familiar construction for smooth maps. By using the contractibility of $L_{n}[w_{0}]$, we may deform $\alpha_{1}$ to obtain a map $\alpha_{1} \circ \alpha_{1}$ which intersects $\hat{A}_{w_{0}}$ only along the image of $c_{w_{0}}$. More precisely, take $\Gamma_{0} = c_{w_{0}}(0) \in L_{n}[w_{0}]$ as a base point; let $H : [0, 1] \times L_{n}[w_{0}] \to L_{n}[w_{0}]$ be a homotopy such that, for all $\Gamma \in L_{n}[w_{0}]$, $H(0, \Gamma) = \Gamma$ and $H(1, \Gamma) = \Gamma_{0}$; assume without loss of generality that $\Pi(c_{w}(p)) = \Gamma_{0}$ for $p$ is an open neighborhood.
$B_0 \subset \mathbb{B}^d$, $0 \in B_0$; let bump : $\mathbb{B}^d \to [0,1]$ be a bump function with support contained in the neighborhood above and constant equal to 1 in a smaller open ball $B_1 \subset B_0$, $0 \in B_1$. For $s \in [\frac{1}{3}, \frac{2}{3}]$, we define $\alpha_s$ to coincide with $\alpha_{\frac{1}{3}}$ outside $\hat{A}_{w_0}$; if $\alpha_{\frac{1}{3}}(p) \in \hat{A}_{w_0}$ define $\alpha_s(p) \in \hat{A}_{w_0}$ to satisfy

\[
\Pi(\alpha_s(p)) = H((3s-1) \text{bump}(\hat{F}_{w_0}(\alpha_{\frac{1}{3}}(p)))) , \Pi(\alpha_{\frac{1}{3}}(p)) , \quad \hat{F}_{w_0}(\alpha_s(p)) = \hat{F}_{w_0}(\alpha_{\frac{1}{3}}(p)).
\]

Thus, the set of points $p \in M_0$ such that $\alpha_{\frac{1}{3}}(p) \in \hat{A}_{w_0}$ and $\hat{F}_{w_0}(\alpha_{\frac{1}{3}}(p)) \in B_1$ consists of open tubes taken to the image of the cell $c_{w_0}$. By removing these open tubes we have $M_1 \subseteq M_0$, a manifold with boundary, also of dimension $k$. Let $D^*_n[1] = D_n[1] \setminus i_{w_0}[B_1]$ so that both the map $\alpha_1|D^*_n[1] : D^*_n[1] \to \mathcal{L}_n[1]$ and the inclusion $D_n[1] \subset D^*_n[1]$ are weak homotopy equivalences (since $D_n[1]$ is assumed to be good). The restriction $\alpha_{\frac{1}{3}}|M_1 : M_1 \to \mathcal{L}_n[1]$ has boundary $\alpha_{\frac{1}{3}}|_{\partial M_1} = c_1 \circ \beta_1$ for $\beta_1 : \partial M_1 \to D^*_n[1]$. There exists $\beta : M_1 \to D^*_n[1]$ with $\beta_1 = \beta|_{\partial M_1}$ and such that $c_1 \circ \beta : M_1 \to \mathcal{L}_n[1]$ is homotopic to $\alpha_{\frac{1}{3}}|M_1 : M_1 \to \mathcal{L}_n[1]$ (in $\mathcal{L}_n[1]$). Define $\alpha_1$ to coincide with $\beta$ in $M_1$ and such that $c_{w_0} \circ \alpha_1$ coincides with $\alpha_{\frac{1}{3}}$ in the tubes $M_0 \setminus M_1$. This is our desired map.

Conversely, consider a compact manifold with boundary $M$ and its boundary $\partial M = N$. Consider a maps $\alpha_0 : M \to \mathcal{L}_n[1]$ and $\beta_0 : N \to D_n[1]$ such that $c_1 \circ \beta_0 = \alpha_0|_N$. We prove that there exists a map $\alpha_1 : M \to D_n[1]$ with $\alpha_1|_N = \beta_0$. Furthermore, $\alpha_0$ and $c_1 \circ \alpha_1$ are homotopic in $\mathcal{L}_n[1]$, with the homotopy constant in the boundary. Again, we may assume without loss of generality that $\alpha_0$ is transversal to $\hat{A}_{w_0}$. As in the previous paragraph, use the contractibility of $\mathcal{L}_n[w_0]$ to deform $\alpha_0$ in $\hat{A}_{w_0}$ towards $c_{w_0}$, thus defining $\alpha_{\frac{1}{3}}$; notice that this keeps the boundary fixed, as required. Again remove tubes around points taken to $c_{w_0}$, thus defining $M_1 \subset M$, also a manifold with boundary. By hypothesys, $\alpha_{\frac{1}{3}}$ can be deformed in $M_1$ to obtain $\alpha_1 : M_1 \to D_n[1]$; more precisely, $\alpha_{\frac{1}{3}} : M_1 \to \mathcal{L}_n[1]$ is homotopic to $c_1 \circ \alpha_1$. As in the previous paragraph we fill in the tubes to define $\alpha_1 : M \to D_n[1]$, the desired map.

\[
\begin{array}{c}
\begin{array}{c}
\text{abab} \quad \{ab\}b \\
\text{aba} \quad [ab]b \\
\text{a[ba]} \quad b[ab] \\
\text{aa} \quad [ba]a \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{cc} \quad [bc][c] \\
\text{bc}c \quad [bc]c \\
\text{bb} \quad [cb]b \\
\end{array}
\end{array}
\]

Figure 6: The cells $c_{[aba]}$ and $c_{[bc]}$.  

60
Example 15.5. Following the above construction for $I_{[aba]}$ we have the cell shown in Figure 6, the transversal map $\tilde{c}$ in the proof above is shown in Figure 2. Notice that $c_{[bc]}$ is very similar. The immediate predecessors of $[aba]$ are $[ba]a$, $b[ab]$, $[ab]b$ and $a[ba]$ (see Figure 5), and we have

$$\partial[aba] = [ba]a + b[ab] - [ab]b - a[ba], \quad \partial[bc] = [cb][b + c][b] - [bc][c - b][cb].$$

Remark 15.6. As we shall see, for cells of very low dimension the complex can be taken to be polyhedric; it is not clear whether this is true in general.

It seems to be the case that the complex can be constructed so that cells are injective and with disjoint images except for the gluing at the boundaries. We shall not attempt to clarify this issue here.

16 The complexes $D_n[I_1(\omega)] \subset D_n[I_1(\omega^2)] \subset D_n[I_1(\omega^2)]$

In this section we consider a few simple examples of valid complexes. From now on, when $n \leq 4$ we write $a = a_1$, $b = a_2$, $c = a_3$ and $d = a_4$. Recall that $I_1(\omega) \subset W_n$ is the lower set of words of dimension 0. For each $w \in I_1(\omega)$, let $c_w \in L_n[w]$ be an arbitrary curve: we think of $c_w$ as a vertex of the complex $D_n$. In other words, the 0-skeleton of $D_n$ is the infinite countable set of vertices $D_n[I_1(\omega)] = \{c_w, w \in W^{(0)}\}$; the inclusion $D_n[I_1(\omega)] \subset L_n[I_1(\omega)]$ is a homotopy equivalence.

The subset of $I_1(\omega^2) \subset W_n$ of words of dimension at most 1 is also a lower set. A word of dimension 1 is of the form $w = w_0[a_ka_l]w_1$ where $w_0$, $w_1 \in I_1(\omega) \subset W_n$ are (possibly empty) words of dimension 0 and $k \neq l$ so that $[a_ka_l] \in S_{n+1}$ is an element of dimension 1 (i.e., inv($a_ka_l$) = 2). There are three cases: case (i) is $l = k - 1$, case (ii) is $l = k + 1$ and case (iii) is $l > k + 1$ if $l < k - 1$ we write $a_k$ instead; notice that in this case the permutations $a_k$ and $a_l$ commute). In each case the stratum $L_n[w]$ is a hypersurface with an open stratum on either side: let us call them $L_n[w^+]$ and $L_n[w^-]$. The words $w^+, w^- \in W_n$ have dimension 0; their values according to case are:

(i) $w^- = w_0a_lw_1, w^+ = w_0a_ka_lw_1$.
(ii) $w^- = w_0a_ka_lw_1, w^+ = w_0a_lw_1$.
(iii) $w^- = w_0a_ka_lw_1, w^+ = w_0a_lw_1$.

From Lemma 13.4 it follows easily that the only two words $\tilde{w} \in W_n$ such that $\tilde{w} \leq w$ are $\tilde{w} = w^\pm$. In order to construct the 1-skeleton $D_n[I_1(\omega^2)]$ of $D_n$, we add an oriented edge $c_w$ from $c_{w^-}$ to $c_{w^+}$. The edge may be assumed to be contained in $L_n[w^-] \cup L_n[w] \cup L_n[w^+]$ and to cross $L_n[w]$ (topologically) transversally.
and precisely once; edges are also assumed to be simple, and disjoint except at endpoints. Again, the inclusion \( \mathcal{D}_n[I(ω_2)] \subset \mathcal{L}_n[I(ω_2)] \) is a homotopy equivalence.

The four sides of Figure 2 above provide examples of edges of cases (i) and (ii). Figure 7 shows the edges of \( \mathcal{D}_n \). Here we omit the initial and final words \( w_0 \) and \( w_1 \) (since they are not involved anyway). We also prefer to present examples (instead of spelling out conditions as above; a more formal discussion is given in Sections 15 and 11). Thus, cases (i), (ii) and (iii) are represented by \([ba]\), \([ab]\) and \([ac]\), respectively.

![Figure 7: Examples of edges of \( \mathcal{D}_n \).](image)

In a homological language, we write

\[
\partial [ba] = bab - a, \quad \partial [ab] = b - aba, \quad \partial [ac] = ca - ac.
\]

Notice that in all cases \( \hat{w} = \hat{w}^+ = \hat{w}^- \in CG_{n+1} \), consistently with the fact that we are constructing disjoint complexes \( \mathcal{D}_n(z) \), \( z \in CG_{n+1} \).

As a simple application of our methods, we compute the connected components of \( \mathcal{D}_n(z) \) (i.e., we are reproving in our notation the main result of [59]). We need only consider words of dimension 0 (the vertices of \( \mathcal{D}_n \)), i.e., words in the generators \( a_k \), \( 1 \leq k \leq n \). Write \( w_0 \sim w_1 \) if \( c_{w_0} \) and \( c_{w_1} \) are in the same connected component; thus, if \( w_0 \preceq w_1 \) then \( w_0 \sim w_1 \) and if \( w_0 \sim w_1 \) then \( \hat{w}_0 = \hat{w}_1 \).

For \( w \) the empty word, the vertex \( c_w \) is not attached to any edge and thus forms a contractible connected component in the complex \( \mathcal{D}_n \). This of course corresponds to \( \mathcal{L}_n[w] \subset \mathcal{L}_n \), the component of convex curves; notice that \( \hat{a} = \hat{\hat{a}} = \hat{w} = 1 \) but \( aaaa \not\sim w \).

**Proposition 16.1.** Consider two non-empty words \( w_0, w_1 \in \mathcal{D}_n \) of dimension 0. Then \( w_0 \sim w_1 \) if and only if \( \hat{w}_0 = \hat{w}_1 \in CG_{n+1} \).

**Proof.** We have already seen that \( w_0 \sim w_1 \) implies \( \hat{w}_0 = \hat{w}_1 \). We prove the other implication. A **basic word** is either:

(i) a non-empty word \( a_{k_1}a_{k_2}\cdots a_{k_l} \) with \( k_1 < k_2 < \cdots < k_l \);

(ii) of the form \( aaaa_k a_{k_2} \cdots a_{k_l} \) with \( k_1 < k_2 < \cdots < k_l \) (here the word \( aa \) corresponding to \( l = 0 \) is allowed);

(iii) \( aaaa \).

62
In particular, words of length 1 are basic. Clearly, for each \( z \in \mathbb{C}G_{n+1} \) there exists a unique basic word \( w \) with \( z = \hat{w} \).

Using the edges above, first notice that

\[
\begin{align*}
   aa & \sim abab \sim bb \sim bcbc \sim cc \sim \cdots \sim a_k a_k; \\
   a_{k+1} a_k & \sim a_{k+1} a_k + a_k a_{k+1} \sim a a a_k a_{k+1}; \\
   baa & \sim babab \sim aab \text{ and, for } k > 2, a_k a a \sim a a a_k.
\end{align*}
\]

Furthermore, \( a_k a_k + 1 a_k a_k \sim a_k a_k + a_k a_k \sim a_k a_k \).

Thus \( aa \) commutes with all generators \( a_k \) and can be brought to the beginning of the word; other generators either commute (\( a_k a_l \sim a_l a_k \) if \( |k - l| \neq 1 \)) or anticommute (\( a_{k+1} a_k \sim a a a_k a_{k+1} \)). Thus, for an arbitrary non-empty word \( w \), generators can be arranged in increasing order of index, at the price of creating copies of \( aa \) which are taken to the beginning of the word. Duplicate generators can also be transformed into further copies of \( aa \). Finally, if there are more than 4 copies of \( a \), they can be removed 4 by 4 thus arriving at a basic word. \( \square \)

We now construct the complex \( D_n[\mathbf{I}_{(\omega^2)}] \). Notice that \( w \in \mathbf{I}_{(\omega^2)} \) if and only if \( w \) is of the form \( w = w_0 \sigma_1 w_1 \cdots \sigma_l w_l \) where \( \dim(w_j) = 0 \) and \( \dim(\sigma_j) = 1 \) (some of the \( w_j \) may equal the empty word). Set \( c_w \) to be a product cell of dimension \( l \), i.e., the \( l \)-th dimensional cube

\[
c_w = c_{w_0} \times c_{\sigma_1} \times c_{w_1} \times \cdots \times c_{\sigma_l} \times c_{w_l};
\]

the gluing instructions are legitimate. See Figure 8 for the following examples:

\[
\begin{align*}
   \partial([ba][ab]) &= [ba]aba + bab[ab] - [ba]b - a[ab], \\
   \partial([ac]b[ac]) &= [ac]bac + cab[ac] - [ac]bca - acb[ac].
\end{align*}
\]

Figure 8: The cells \([ba][ab]\) and \([ac]b[ac]\).

The construction of product cells works in greater generality. If \( w \in W_n \) is a word of length greater than 1, possibly containing more than one letter of positive
dimension, define $c_w$ as a product cell. As before, write $w = w_0\sigma_1 w_1 \cdots \sigma_l w_l$ where \( \dim(w_j) = 0 \) and \( \dim(\sigma_j) > 0 \) (some of the $w_j$ may equal the empty word). Set

$$c_w = c_{w_0} \times c_{\sigma_1} \times c_{w_1} \times \cdots \times c_{\sigma_l} \times c_{w_l}.$$  

Here we assume the cells $c_{\sigma_j}$ to have been previously constructed.

Also, let $\sigma \in S_{n+1}$ be a letter of dimension $k$ such that $1^\sigma = 1$. Write $\sigma = [a_{n_1} \cdots a_{n_{k+1}}]$ and $s = -1 + \min n_j > 0$. Set $\bar{\sigma} = [a_{n_1-s} \cdots a_{n_{k+1}-s}]$: the cell $c_{\sigma}$ is assumed to be already constructed. Define $c_{\sigma}$ from $c_{\bar{\sigma}}$ by adding $s$ to the index of every generator of every letter. Notice that this fits with our construction of the 1-skeleton; see also Figure 6 for $c_{[bcb]}$.

17 Cells of dimension 2

Given $n$ and the simplifications in the previous section, there is a finite (and short) list of possibilities of letters of dimension 2. In $S_3$, the only letter of dimension 2 is $[aba]$; Figure 5 shows the elements of $W_2$ (or $W_n$) below $[aba]$, i.e., the smallest lower set containing $[aba]$. See also Figures 2 and 6 for a transversal surface to the submanifold $L_2[[aba]]$ and for the cell $c_{[aba]}$.

In $S_4$ we also have $[bcb]$ (which is of course similar to $[aba]$, as in Figure 6) and $[acb]$, which we already discussed and for which valid cells are shown in Figure 4 (see also Figure 3). The three remaining cells are and $[abc]$, $[bac]$ and $[cba]$, for which transversal sections and valid cells are shown in Figure 9; in a homological notation:

$$\partial[abc] = abc[ab] + a[bc] + [ac] - [bc]a - [ab]cba + ab[ac]ba,$$

$$\partial[bac] = [ac]b - [bc]ab - bc[ba] - b[ac] - ba[bc] - [ba]cb,$$

$$\partial[cba] = [ac] + c[ba] + cba[cb] + cb[ac]bc - [cb]abc - [ba]c.$$  

![Figure 9: The cells $c_{[abc]}$, $c_{[bac]}$ and $c_{[cba]}$.](image-url)
In each figure, we consider the family of paths constructed in Section 11 so that the functions $m_j$ are polynomials in the real parameters $x_1$ and $x_2$ and in the variable $t$. The curves in the figure indicate regions for which the itinerary has positive dimension and therefore separate open regions for which the itinerary has dimension 0. These curves are therefore semialgebraic. For instance, for $[abc]$ we have

$$m_1(t) = \frac{t^3}{6} + x_2 t + x_1, \quad m_2(t) = \frac{t^2}{2} + x_2, \quad m_3(t) = -t.$$ 

The vertical line, corresponding to $[ac]$, has equation

$$\text{resultant}(m_1, m_3; t) = x_1 = 0.$$ 

More precisely, the positive semiaxis corresponds to itinerary $[ac]$ and the negative semiaxis corresponds to itinerary $ab[ac]ba$. The horizontal line, corresponding to $[bc]$, has equation

$$\text{discriminant}(m_2; t) = -2x_2 = 0;$$

the positive semiaxis corresponds to itinerary $a[bc]$ and the negative semiaxis to $[bc]a$. Finally, the cusp-like curve in the figure, corresponding to $[ab]$, has equation

$$\text{resultant}(m_1, m_2; t) = \frac{x_1^2}{9} + \frac{x_2^2}{8} = 0;$$

in the third quadrant the itinerary is $[ab]cba$ and in the fourth quadrant it is $abc[ab]$.

In $D_4$ there exist faces (i.e., cells of dimension 2) similar to those above (such as $[bcd]$, which is similar to $[abc]$) but also a few genuinely new ones: $[abd]$, $[acd]$, $[adc]$ and $[bad]$, all shown in Figure 10; more generally, we have

$$\partial[aba_k] = ab[aa_k] + a[ba_k]a + [aa_k]ba + a_k[ab] - [ba_k] - [ab]a_k, \quad k \geq 4;$$

$$\partial[aa_k a_{k+1}] = a[a_k a_{k+1}] + [aa_{k+1}] - [a_k a_{k+1}]a$$

$$- a_k a_{k+1}[aa_k] - a_k[a a_{k+1}]a_k - [aa_k]a_{k+1}a_k, \quad k \geq 3;$$

$$\partial[aa_{k+1} a_k] = a[a_{k+1} a_k] + [aa_{k+1}]a_k a_{k+1} + a_{k+1}[aa_k] a_{k+1}$$

$$+ a_{k+1} a_k [aa_{k+1}] - [a_{k+1} a_k]a - [aa_k], \quad k \geq 3;$$

$$\partial[baa_k] = [aa_k] + a_k[ba] - [ba_k]ab - b[aa_k]b - ba[ba_k] - [ba]a_k, \quad k \geq 4.$$

The only genuinely new face in $D_5$ is $[a_1 a_3 a_5]$, shown in Figure 11; we have

$$\partial[aa_k a_l] = a[a_k a_l] + [aa_l]a_k + a_l[aa_k] - [a_k a_l]a - a_k[aa_l] - [aa_l]a_k, \quad 3 \leq k < l - 1.$$
18 Final Remarks

The present paper casts the foundations of a combinatorial approach. We expect to cover in a forthcoming paper [3] some of the results stated in this section as conjectures.

One important construction in [59] and [53] is the add-loop procedure, which, in certain cases, is used to loosen up compact families of nondegenerate curves through a homotopy in $L_n$. The resulting families of curly curves are then maleable: if a homotopy exists in the space of immersions, another homotopy exists in the space of locally convex curves. In [53], for instance, open dense subsets $\mathcal{Y}_\pm \subset L_2(\pm 1)$ are shown to be homotopy equivalent to the space of loops $\Omega S^3$. Thus, certain questions regarding homotopies can be transferred to the space of continuous paths in the group $\text{Spin}_{n+1}$. This approach is reminiscent of classical constructions such as Thurston’s eversion of the sphere by corrugations.
and the proof of Hirsch-Smale Theorem [29, 66]. It can be considered as an elementary instance of the h-principle of Eliashberg and Gromov [18, 27].

We expect to give a combinatorial version of this method which generalizes the result above to higher dimensions. This would imply the following results.

Recall that the center $Z(SO_{n+1})$ of $SO_{n+1}$ equals $\{I\}$ if $n$ is even and $\{\pm I\}$ if $n$ is odd (here $I$ is the identity matrix). We have $Z(CG_{n+1}) = \Pi^{-1}[Z(SO_{n+1})]$.

**Conjecture 18.1.** If $q \in CG_{n+1} \setminus Z(CG_{n+1})$ then the inclusion $i_q : \mathcal{L}_n(q) \to \Omega Spin_{n+1}$ is a weak homotopy equivalence.

In particular, there are at most two or four homotopy types distinct from that of $\Omega Spin_{n+1}$, depending on the parity of $n$.

Recall that $CG_3 = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and $CG_4 = Q_8 \times Q_8$. The main result in [33] classifies the spaces $\mathcal{L}_2(q)$, $q \in CG_3$, into the following three weak homotopy types: for $q \neq \pm 1$, consistently with Conjecture 18.1 we have $\mathcal{L}_2(q) \approx \Omega S^3$; otherwise,

$$
\mathcal{L}_2(-1) \approx \Omega S^3 \vee S^0 \vee S^2 \vee S^8 \vee \cdots, \quad \mathcal{L}_2(1) \approx \Omega S^3 \vee S^2 \vee S^6 \vee S^{10} \vee \cdots. \quad (17)
$$

The following conjecture promises a similar result for $n = 3$ (see [2] for more on these spaces).

**Conjecture 18.2.** We have the following weak homotopy equivalences:

\begin{align*}
\mathcal{L}_3((+1, +1)) & \approx \Omega(S^3 \times S^3) \vee S^4 \vee S^8 \vee S^8 \vee S^{12} \vee S^{12} \vee S^{12} \vee \cdots, \\
\mathcal{L}_3((-1, -1)) & \approx \Omega(S^3 \times S^3) \vee S^2 \vee S^6 \vee S^6 \vee S^{10} \vee S^{10} \vee S^{10} \vee \cdots, \\
\mathcal{L}_3((+1, -1)) & \approx \Omega(S^3 \times S^3) \vee S^0 \vee S^4 \vee S^4 \vee S^8 \vee S^8 \vee S^8 \vee \cdots, \\
\mathcal{L}_3((-1, +1)) & \approx \Omega(S^3 \times S^3) \vee S^2 \vee S^6 \vee S^6 \vee S^{10} \vee S^{10} \vee S^{10} \vee \cdots.
\end{align*}

The above bouquets include one copy of $S^k$, two copies of $S^{(k+4)}$, \ldots, $j+1$ copies of $S^{(k+4j)}$, \ldots, and so on.

From the 1 and 2-skeletons of $\mathcal{D}_n$, constructed in Sections 16 and 17, we also expect to prove the following conjecture.

**Conjecture 18.3.** Let $z \in CG_{n+1}$. Every connected component of $\mathcal{L}_n(z)$ is simply connected.

Another interesting aspect of the subject is its relation with the theory of differential operators, mentioned in the Introduction. A linear differential operator can be canonically associated to a nondegenerate curve $\gamma : [0, 1] \to S^n$ (see [32]). There is a Poisson structure in the space of the differential operators, given by the Adler-Gelfand-Dickey bracket [18, 24, 25]. The identification above relates the spaces $\mathcal{L}_n(z)$ with symplectic leaves of this structure [32, 33]. Notice that the spheres appearing in the bouquets in Equation 17 and Conjecture 18.2 are all even-dimensional. We wonder whether this is fortuitous or a manifestation of this symplectic structure; this question is maybe worth clarification.
A Convex curves

A smooth parametric curve $\gamma : J \to S^n$ defined on a compact interval $J \subset \mathbb{R}$ is said to be strictly convex if for each nonzero linear functional $\omega \in (\mathbb{R}^{n+1})^* \setminus \{0\}$ the function $\omega \gamma : J \to \mathbb{R}$ has at most $n$ zeroes counted with multiplicities (zeroes at endpoints taken into account). It is said to be convex if its restriction to any proper compact subinterval of $J$ is strictly convex.

In other words, a convex curve is one that (possibly neglecting one endpoint at a time) intersects each $n$-dimensional vector subspace $H^n \subset \mathbb{R}^{n+1}$ at most $n$ times with multiplicities taken into account. Thus, for instance, a transversal intersection counts as 1; a tangency counts as 2; an osculation counts as 3. Other terms used for the same or closely related concepts are non-oscillatory curves and disconjugate curves.

The goal of this appendix is to show that a smooth nondegenerate curve $\gamma : [0, 1] \to S^n$ with initial frame $\mathfrak{F}_\gamma(0) = 1$ is convex if and only if its itinerary is the empty word, i.e., that the notion of convexity introduced in Section 8 and given in terms of the singular set of $\mathfrak{F}_\gamma$ coincides with this geometric definition. These results are essentially present in [65] but we feel that it may help the reader to have a mostly self-contained presentation.

Clearly, convexity implies nondegeneracy. Conversely, as we shall see in Proposition A.1, (smooth) nondegeneracy implies local convexity: this is why the terms nondegenerate and locally convex are used interchangeably.

Now we present the main result of this Appendix. For $J$ a compact interval, we say that a locally convex curve $\Gamma : J \to \text{Spin}_{n+1}$ is short if there exists $z \in \text{Spin}_{n+1}$ such that $\Gamma[J] \subset U_z$. Recall that $U_z \subset \text{Spin}_{n+1}$ is the domain of a triangular system of coordinates (see Section 4).

**Proposition A.1.** Let $\gamma : J \to S^n$ be a smooth nondegenerate curve defined on a compact interval $J \subset \mathbb{R}$. The following conditions are equivalent:

1. $\gamma$ is strictly convex;
2. $\mathfrak{F}_\gamma$ is short;
3. $\forall t_0, t_+ \in J ((t_0 < t_+) \Rightarrow (\mathfrak{F}_\gamma(t_+) \in \mathfrak{F}_\gamma(t_0) \text{Bru}))$;
4. $\forall t_0, t_- \in J ((t_- < t_0) \Rightarrow (\mathfrak{F}_\gamma(t_-) \in \mathfrak{F}_\gamma(t_0) \text{Bru}))$.

Closely related sufficient conditions for convexity may be found in [45, 65].
Example A.2. Given \( z \in \text{Bru}_d \), consider the curve \( \Gamma = \Gamma_z : [0, 1] \to \text{Spin}_{n+1} \) passing through \( \Gamma_z(\frac{1}{2}) = z \) given by Lemma 7.2. It follows immediately from Proposition A.1 that \( \gamma_z = \Gamma_z e_1 \) is a convex curve (though not strictly convex). For an alternative proof, recall that \( \Gamma_z \) is obtained from \( \Gamma_{\eta}(t) = \exp(\pi t h) \) through a projective transformation and see Lemma 2.2 of [54] for a direct proof of the convexity of \( \Gamma_{\eta} \). For \( n = 2 \), \( \gamma_{\eta}(t) = \frac{1}{2}(1 + \cos(2\pi t)), \sqrt{2} \sin(2\pi t), 1 - \cos(2\pi t) \) is the circle of diameter \( e_1 e_3 \) in \( \mathbb{S}^2 \). Notice that \( \gamma_{\eta} \) is closed if and only if \( n \) is even (as usual, a curve \( \gamma : [0, 1] \to \mathbb{S}^n \) is closed if \( \text{F}_\gamma(0) = \text{F}_\gamma(1) \)).

The well known fact that there are no closed convex curves in \( \mathbb{S}^n \) for \( n \) odd also follows as an easy consequence of Proposition A.1. Of course, the projectivization \([\gamma_{\eta}] : [0, 1] \to \mathbb{RP}^n \) (where \([\gamma_{\eta}](t) = \mathbb{R}\gamma_{\eta}(t) \in \mathbb{RP}^n \)) is a closed convex curve for arbitrary \( n > 1 \). In [5] it is shown that the space of closed convex curves in \( \mathbb{RP}^n \) is a contractible connected component of the space of closed, locally convex curves in \( \mathbb{RP}^n \). In the same spirit, we have Lemma 8.5.

In order to prove proposition A.1, we shall need a couple of technical results.

In this section, \( \text{Lo}_{n+1} \) (respectively, \( \text{Up}_{n+1} \)) stands for the group of invertible real lower (resp., upper) triangular matrices of order \( n + 1 \). Given a real square matrix \( M \) of order \( n + 1 \), we call a factorization of the form \( M = LU \), \( L \in \text{Lo}_{n+1} \), \( U \in \text{Up}_{n+1} \) an \( LU \) decomposition of \( M \). Recall that a necessary and sufficient condition for \( M = (M_{ij}) \) having an \( LU \) decomposition is the nonvanishing of all of its northwest minor determinants

\[
\Lambda^k(M)_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}} = \det (M_{ij})_{1 \leq i,j \leq k}.
\]

Lemma A.3. Let \( J \subseteq \mathbb{R} \) be an interval and \( \phi : J \to \mathbb{R}^{n+1} \) be a smooth map. If the matrix \( W_\phi(t) = (\phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) \) admits an \( LU \) decomposition for all \( t \in J \), then, given sequences \( t \) of real numbers \( \{t_0 < t_1 < \cdots < t_k\} \subseteq J \) and \( m \) of positive integers \( m_0, m_1, \ldots, m_k \) whose sum is \( n + 1 \), the matrix

\[
W_\phi^{t,m} = (\phi(t_0), \phi'(t_0), \ldots, \phi^{(m_0-1)}(t_0), \ldots, \phi(t_k), \phi'(t_k), \ldots, \phi^{(m_k-1)}(t_k))
\]

admits an \( LU \) decomposition as well and, in particular, \( \det W_\phi^{t,m} \neq 0 \).

In what follows, we only need the nonvanishing of \( \det W_\phi^{t,m} \) when \( W_\phi \) has an \( LU \) decomposition. In fact, a smooth curve \( \gamma : J \to \mathbb{R}^{n+1} \) is strictly convex if and only if \( \det W_\gamma^{t,m} \neq 0 \) for all increasing sequences \( t_0 < t_1 < \cdots < t_k \) in the compact interval \( J \) and all positive integers \( m_0, m_1, \ldots, m_k \) whose sum is \( n + 1 \). The stronger way the conclusion of Lemma A.3 is stated is convenient for the induction argument. Lemma A.3 can be considered a reformulation of Theorem V of [17]; see also [61], [65] and [45].
Proof. Write \( \phi(t) = \sum_{0 \leq j \leq n} \phi_j(t) e_{j+1} \) and \( W_{\phi}(t) = L(t)U(t) \), with \( L(t) \in L_{n+1} \) and \( U(t) \in U_{n+1} \). We proceed by induction in \( n \). The cases \( n \in \{0, 1\} \) are trivial. There is no loss of generality in assuming \( \phi_0 \) constant equal to 1, since \( \phi_0(t) \neq 0 \) for all \( t \) and we can solve \( e_1^\top W_{\phi}(t)V(t) = e_1^\top \) explicitly for \( V(t) \in U_{n+1} \) obtaining a matrix whose entries are rational functions of \( \phi_0(t), \phi_0'(t), \ldots, \phi_0^{(n)}(t) \) with powers of \( \phi_0(t) \) as denominators. Besides, given a sequence of instants \( t_0 < t_1 < \cdots < t_k \) and positive integers \( m_0, m_1, \ldots, m_k \) as above, we only need to prove that \( \det W_{\phi}^{t,m} \neq 0 \), since the northwest minor determinants of order \( k \leq n \) of \( W_{\phi}^{t,m} \) are all nonvanishing by the induction hypothesis. Suppose, to the contrary, that there is \( \omega \in (\mathbb{R}^{n+1})^* \setminus \{0\} \) such that for all \( j \in \{0, 1, \ldots, k\} \) one has \( \omega \cdot \phi(t_j) = \omega \cdot \phi'(t_j) = \cdots = \omega \cdot \phi^{(m_j-1)}(t_j) = 0 \), and define \( \hat{\phi} : \mathbb{R} \rightarrow \mathbb{R}^n \) by \( \hat{\phi}(t) = \sum_{1 \leq j \leq n} \phi_j(t)e_j \). Note that \( W_{\hat{\phi}}(t) \) is the \( n \times n \) southeast block of \( W_{\phi}(t) \) and therefore admits an \( LU \) decomposition as well. Consider now the restriction \( \hat{\omega} = \omega|_{\mathbb{R}^n} \), where we make the identification \( \mathbb{R}^n = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \). By Rolle’s Theorem, there is an instant \( t_{j+1/2} \in (t_j, t_{j+1}) \) for each \( j \in \{0, 1, \ldots, k-1\} \) when \( \hat{\omega} \cdot \hat{\phi}(t_{j+1/2}) = \omega \cdot \phi'(t_{j+1/2}) = 0 \). Besides, for all \( j \in \{0, 1, \ldots, k\} \), \( \hat{\omega} \) inherits from \( \omega \) the zeroes \( \hat{\omega} \cdot \hat{\phi}(t_j) = \hat{\omega} \cdot \hat{\phi}'(t_j) = \cdots = \hat{\omega} \cdot \hat{\phi}^{(m_j-2)}(t_j) = 0 \). Now, take \( \hat{k} = 2k \) and define the refined sequence \( \hat{t}_0 < \hat{t}_1 < \cdots < \hat{t}_k \) by \( \hat{t}_j = t_{j/2} \) with associated multiplicities

\[
\hat{m}_j = \begin{cases} 
    m_{j/2} - 1, & \text{if } j \text{ is even} \\
    1, & \text{if } j \text{ is odd}
\end{cases}
\]

Then, \( \det W_{\hat{\phi}}^{\hat{t},\hat{m}} = 0 \), what contradicts our induction hypothesis. \( \square \)

The following result amounts to the intuitive fact that the distance between two continuously moving compact subsets of a metric space is a continuous function of time.

**Lemma A.4.** Let \( (M,d) \) be a metric space and \( K,L \) be compact topological spaces. Let \( F : \mathbb{R} \times K \rightarrow M \) and \( G : \mathbb{R} \times L \rightarrow M \) be continuous maps. The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(s) = d(F_s[K], G_s[L]) \) is continuous.

**Proof.** Let \( X,Y \) be topological spaces, \( Y \) compact, and \( \Phi : X \times Y \rightarrow \mathbb{R} \) a continuous map. It is an easy exercise in point-set topology to prove that the map \( \phi : X \rightarrow \mathbb{R} \) given by \( \phi(x) = \inf\{\Phi(x,y) \mid y \in Y\} \) is well-defined and continuous. Our result then follows. \( \square \)

**Proof of Proposition A.7** In this proof we write \( \mathfrak{F}_\gamma(t_0; t_1) = (\mathfrak{F}_\gamma(t_0))^{-1} \mathfrak{F}_\gamma(t_1) \). Also, for each \( Q \in SO_{n+1} \), we will denote by \( B_Q \in B^+_{n+1} \) the unique signed permutation matrix such that \( Q \in \text{Bru}_{H_M} \).

\[ (2 \rightarrow 1) \] Rotations preserve both nondegeneracy and strict convexity, i.e., for all \( Q \in SO_{n+1} \), if the smooth curve \( \gamma : J \rightarrow \mathbb{R}^{n+1} \) is strictly convex (respectively,
nondegenerate) then so is $Q^T \gamma$. Take $Q \in SO_{n+1}$ such that $\mathcal{F}_\gamma[J] \subset \mathcal{U}_Q$ and apply Lemma A.3 to $\phi = Q^T \gamma$.

(1) Suppose that $\gamma$ violates condition (3) for certain $t_0 < t_+$. We know from Lemma 8.1 that $\mathcal{F}_\gamma(t_0; t) \in \text{Bru}_q$ for sufficiently small $t - t_0 > 0$. Assume without loss that we have taken $t_+ > t_0$ minimal such that $\mathcal{F}_\gamma(t_0; t_+) \notin \text{Bru}_q$. Thus, $\mathcal{F}_\gamma(t_0; t_+)$ will exhibit a first non-invertible southwest block of order $k + 1$ for some $k \in \{0, 1, \ldots, n - 1\}$ and the corresponding block in the signed permutation matrix $B(t_0; t_+) = B_{\mathcal{F}_\gamma(t_0; t_+)}$ will have the form

$$\begin{pmatrix}
(-1)^{k+1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}.$$ 

This means that the $(k + 1)$th columns of $B(t_0; t_+)$ is one of the previous vectors of the standard basis of $\mathbb{R}^{n+1}$, i.e., some $e_j$ with $j \in \{1, 2, \ldots, n - k\}$. Explicitly, one has $B(t_0; t_+) e_{k+1} = e_j$, which boils down to the following relation between the Wronskian matrices of $\gamma$ in $t_0$ and $t_+$:

$$W_\gamma(t_+) U_{+, k+1} = W_\gamma(t_0) U_0 e_j$$

for some $U_0, U_+ \in \text{Up}_{n+1}^+$ and $j \in \{1, 2, \ldots, n - k\}$. That means a linear dependence between the first $k + 1$ columns of $W_\gamma(t_+)$ and the first $n - k$ columns of $W_\gamma(t_0)$:

$$\sum_{1 \leq i_0 \leq n - k} (U_0)_{i_0, j} \gamma^{(i_0 - 1)}(t_0) + \sum_{1 \leq i_+ \leq k+1} (U_+)_{i_+, k+1} \gamma^{(i_+ - 1)}(t_+) = 0.$$ 

Notice that the coefficient of $\gamma^{(k)}(t_+)$ is $(U_+)_{k+1, k+1} > 0$. In the notation of lemma A.3 we have $\det W_\gamma^{t, m} = 0$ for $t = (t_0, t_+)$ and $m = (n - k, k + 1)$, thus ruling out strict convexity for the proper subarc $\gamma|[t_0, t_+]$.

(4) Notice that for all $t_a, t_b \in J$ and all $U_0, U_1 \in \text{Up}_{n+1}^+$ we have $\mathcal{F}_\gamma(t_a; t_b) = U_0 \Pi(\eta) U_1^{-1}$ if $\mathcal{F}_\gamma(t_b; t_a) = U_1^{-1} \Pi(\eta) U_0^{-1} = U_1^{-1} \Pi(\eta) U_0^{-1}$.

(3) Take $\Gamma = \mathcal{F}_\gamma$ and $J = [t_0, t_1]$. From (3), $\Gamma[[t_0, t_1]] \subset \Gamma(t_0) \text{Bru}_\eta = \mathcal{U}_{\Gamma(t_0)\eta}$. Take $\epsilon > 0$ and a smooth nondegenerate extension $\tilde{\Gamma} : [t_0 - \epsilon, t_1] \to \mathcal{S}^n$ of $\Gamma$ (i.e., $\tilde{\Gamma}|_J = \Gamma$). We claim that $\Gamma[[t_0, t_1]] \subset \mathcal{U}_{\tilde{\Gamma}^{(s)\eta}}$ provided $s \in [t_0 - \epsilon, t_0]$ is sufficiently near $t_0$. By taking $\epsilon$ sufficiently small we may assume that $\Gamma|[t_0 - \epsilon, t_0 + \epsilon]$ is short. By implications (2) $\rightarrow$ (1) $\rightarrow$ (3) we have $\tilde{\Gamma}[[t_0, t_0 + \epsilon]] \subset \mathcal{U}_{\tilde{\Gamma}^{(s)\eta}}$ for all $s \in [t_0 - \epsilon, t_0]$. Now define $f : [t_0 - \epsilon, t_0] \to \mathbb{R}$ by $f(s) = d(\Gamma|[t_0 + \epsilon, t_1]), \text{Spin}_{n+1} \sim \mathcal{U}_{\tilde{\Gamma}^{(s)\eta}}$ (where $d$ is the distance in Spin$_{n+1}$). By Lemma A.4 $f$ is a continuous function and $f(t_0) > 0$. Thus, $f(s) > 0$ for $s < t_0$ sufficiently near $t_0$ and we have $\Gamma[[t_0 + \epsilon, t_1]] \subset \mathcal{U}_{\tilde{\Gamma}^{(s)\eta}}$, completing the proof. $\square$
B  Hilbert manifolds of curves

In this appendix we introduce a definition of the spaces \( \mathcal{L}_n(z_0; z_1) \) that has some technical advantages as compared to more straightforward ones. Among its nice features is a smooth Hilbert manifold structure, which has enabled some developments reminiscent of standard differential topology ones. Also, in many arguments we have allowed for discontinuities in the derivatives of our locally convex curves. More explicitly, we have many times concatenated locally convex arcs \( \Gamma_1 \) and \( \Gamma_2 \) on \( \text{Spin}_{n+1} \) regardless of the differentiability of the resulting path \( \Gamma_1 \ast \Gamma_2 \) at the welding point, and considered it nonetheless as a locally convex curve, omitting thereby a tedious smoothening out argument. This appendix therefore plays the same role as Section 2 in [53], Section 1 in either [56] or [57].

Recall from Section 4 that a map \( \Gamma : J \to \text{Spin}_{n+1} \) defined on an interval \( J \subseteq \mathbb{R} \) is said to be locally convex if it is absolutely continuous with logarithmic derivative \( \Lambda_\Gamma(t) = (\Gamma(t))^{-1}\Gamma'(t) \) given almost everywhere by a positive linear combination of the skew-symmetric matrices \( a_k \in \mathfrak{so}_{n+1} \) (defined in Equation 6):

\[
\Lambda_\Gamma(t) = \sum_{k \in [n]} \kappa_k(t)a_k, \quad (a_k)_{i,j} = [k = j = (i-1)] - [k = i = (j-1)], \quad \kappa_k(t) > 0.
\]

We shall call the functions (defined on \( J \) up to a set of measure zero) \( \kappa_1, \ldots, \kappa_n \) the generalized curvatures of \( \Gamma \). In fact, the classical Frenet-Serret formulae say that a smooth parametric curve \( \gamma : J \to \mathbb{S}^n \subset \mathbb{R}^{n+1} \) is nondegenerate (in the sense of the Introduction) if and only if the logarithmic derivative \( \Lambda_\gamma = \Lambda_{\tilde{\gamma}} \) of its Frenet lift \( \tilde{\gamma} \) is of the form above with smooth functions \( \kappa_k : J \to (0, +\infty) \) (defined everywhere). Notice that in the smooth case we have \( \kappa_1 = v_\gamma = |\gamma'| \), the velocity of \( \gamma \); \( \kappa_2 = v_\gamma \kappa_1 \), where \( \kappa_1 \) is the geodesic curvature of \( \gamma \); \( \kappa_3 = v_\gamma \kappa_2 \), where \( \kappa_2 \) is the geodesic torsion of \( \gamma \), and so on (see [35, 45]). A natural relaxation of the notion of local convexity is therefore to allow for curves with less regular generalized curvatures. The definition of Section 4 is too wide, though: we stick to a compromise that will allow enough freedom in the construction of locally convex curves, plus a Hilbert manifold structure. This manifold structure is the key to showing that different spaces are equivalent (see Proposition B.6 below). Our approach is reminiscent of the construction of a Hilbert manifold \( \mathcal{H}^1(J, M) \) of absolutely continuous curves in a compact Riemannian manifold \( M \) (see [36]).

Let \( J \subseteq \mathbb{R} \) be an interval. A measurable positive function \( \kappa : J \to (0, +\infty) \) will be called \( L^2 \)-admissible if it satisfies \( \int_{t_0}^{t_1} (\kappa(t))^2 dt < \infty \) and \( \int_{t_0}^{t_1} (\kappa(t))^{-2} dt < \infty \) for all \( t_0 < t_1 \) in \( J \). Let us denote by \( \mathcal{K}_J \) the set of admissible functions defined on the interval \( J \). A locally convex curve \( \Gamma : J \to \text{Spin}_{n+1} \) (in the sense of Section 4) is said to be \( L^2 \)-admissible if its generalized curvatures \( \kappa_1, \ldots, \kappa_n \) are all \( L^2 \)-admissible functions (i.e., \( \kappa_1, \ldots, \kappa_n \in \mathcal{K}_J \)). Also, an absolutely continuous curve...
\( \gamma : J \to \mathbb{S}^n \) is said to be \( L^2\)-nondegenerate if there exists an \( L^2\)-admissible locally convex curve \( \Gamma : J \to \text{Spin}_{n+1} \) such that \( \gamma = \Gamma e_1 \). In this case, we set \( \mathfrak{g}_n = \Gamma \) and call the admissible functions \( \kappa_1, \ldots, \kappa_n \in \mathcal{K}_J \) the generalized curvatures of \( \gamma \).

A theorem of Carathéodory’s guarantees existence and uniqueness for the solution (in an extended sense) of the Frenet-Serret IVP

\[
\Gamma'(t) = \Gamma(t) \sum_{k \in [n]} \kappa_k(t) a_k, \quad \Gamma(t_0) = z_0 \in \text{Spin}_{n+1}
\]

as long as the functions \( \kappa_k \) are all Lebesgue integrable on each compact subinterval of their common domain (see [20]). In this case, the coordinate-functions \( (\Gamma'(t))_{ij} \) of the unique solution \( \Gamma \) are all absolutely continuous and therefore differentiable almost everywhere with derivatives integrable on compact intervals. Also, given \( t \in J \), the element \( \Gamma(t) \in \text{Spin}_{n+1} \) depends continuously on the functions \( \kappa_k \) with respect to the \( L^1 \)-norm in any compact interval containing \( t_0 \) and \( t \). The somewhat more stringent \( L^2 \) hypotheses are meant to produce Hilbert manifolds of curves below.

Notice that \( \mathfrak{g}_n \) and \( \kappa_1, \ldots, \kappa_n \) are well-defined for each given \( L^2 \)-nondegenerate spherical curve \( \gamma \). In fact, for all \( t_0, t \in J \) and all \( j \in [n+1] \), an \( L^2 \)-admissible curve \( \Gamma : J \to \text{Spin}_{n+1} \) with logarithmic derivative \( \Lambda = \sum_k \kappa_k a_k \) satisfies

\[
\Gamma_j(t) = \Gamma_j(t_0) + \int_{t_0}^{t} (\kappa_j(s) \Gamma_{j+1}(s) - \kappa_{j-1}(s) \Gamma_{j-1}(s)) ds,
\]

where \( \Gamma_j = \Gamma e_j \) for \( j \in [n+1] \) and \( \Gamma_j \equiv 0 \) and \( \kappa_j \equiv 0 \) otherwise. Therefore, if \( \Gamma, \hat{\Gamma} : J \to \text{Spin}_{n+1} \) are both \( L^2 \)-admissible satisfying \( \Gamma_j = \hat{\Gamma}_j \), \( \Gamma_{j-1} = \hat{\Gamma}_{j-1} \) and \( \kappa_{j-1} = \hat{\kappa}_{j-1} \) a.e. for some \( j \), then \( \Gamma_j' = \hat{\Gamma}_j' \) a.e. tells us that \( \kappa_j \Gamma_{j+1} = \hat{\kappa}_j \hat{\Gamma}_{j+1} \) a.e., and, since \( \kappa_j, \hat{\kappa}_j > 0 \) a.e. and \( |\Gamma_{j+1}| = |\hat{\Gamma}_{j+1}| \equiv 1 \), we have \( \kappa_j = \hat{\kappa}_j \) a.e. and \( \Gamma_{j+1} = \hat{\Gamma}_{j+1} \). By induction we see that if \( \gamma = \Gamma_1 = \hat{\Gamma}_1 \) with \( \Gamma \) and \( \hat{\Gamma} \) \( L^2 \)-admissible then \( \Gamma = \hat{\Gamma} \) and \( \Lambda = \hat{\Lambda} \) a.e.. In other words, once we fix the interval \( J \subseteq \mathbb{R} \) and the initial conditions \( \mathfrak{g}_n(t_0) = \Gamma(t_0) = z_0 \in \text{Spin}_{n+1} \), there are natural and mutually compatible bijections \( (\kappa_1, \ldots, \kappa_n) \mapsto \mathfrak{g}_n \), \( \gamma \mapsto \gamma \) between the set \( \mathcal{K}_J \) of \( n \)-tuples of admissible functions, the set of \( L^2 \)-admissible curves and the set of \( L^2 \)-nondegenerate spherical curves. We make the identifications \( \gamma \approx \mathfrak{g}_n \approx (\kappa_1, \ldots, \kappa_n) \) without further clarification.

**Remark B.1.** Notice that \( \gamma \) of class \( C^n \) implies \( \kappa_1 \in C^{n-1}, \kappa_2 \in C^{n-2}, \ldots, \kappa_n \in C^0 \). The less obvious converse follows from mixing up Equations [19] and noticing that \( \Gamma = \mathfrak{g}_n \) is necessarily of class \( C^1 \). Slightly more explicitly: we already know that \( \Gamma_j \in C^1 \) for all \( j \in [n+1] \). Use Equations [19] to show recursively that \( \Gamma_j \in C^2 \) for \( j \in [n] \). Use again Equations [19] to show recursively that \( \Gamma_j \in C^3 \) for \( j \in [n-1] \); repeat the procedure to obtain the desired result. We spare the reader the rather tedious formalization of this argument.
Henceforth, we fix $z_0 \in \text{Spin}_{n+1}$ and consider the set of $L^2$-nondegenerate curves $\gamma : [0, 1] \to \mathbb{S}^n$ satisfying $\mathfrak{F}^n(0) = z_0$. We can identify this set with $\mathcal{K}^n_{[0,1]}$. In order to turn it into a topological Hilbert manifold modeled on the separable Hilbert space $\mathbf{H} := (L^2([0,1], \mathbb{R}))^n$, we identify $(\kappa_1, \ldots, \kappa_n) \in \mathcal{K}^n_{[0,1]}$ with $(\xi_1, \ldots, \xi_n) \in \mathbf{H}$ via

$$
\xi_j = \kappa_j - \frac{1}{\kappa_j}, \quad \kappa_j = \frac{\xi_j + \sqrt{\xi_j^2 + 4}}{2}
$$

(20)

This choice of topological chart is rather arbitrary, but is simple enough and does the job: notice that $\int_0^1 \xi_j^2(t)dt < \infty$ iff $\kappa_j \in \mathcal{K}_{[0,1]}$. Throughout the remaining of this appendix, we denote by $\mathcal{L}^{[L^2]}_n(z_0; \cdot)$ the topological Hilbert manifold thus obtained (homeomorphic to Hilbert space $\mathbf{H}$ by construction). By contrast, we denote by $\mathcal{L}^{[C^m]}_n(z_0; \cdot)$ the space of nondegenerate curves defined in the introduction, as a means of explicit reference to both the $C^n$ regularity required of its elements and its natural $C^m$-metric. Under the identification above, we have $\xi_1 \in C^{n-1}$, $\xi_2 \in C^{n-2}$, $\ldots$, $\xi_n \in C^0$ for $\gamma \in \mathcal{L}^{[C^m]}_n(z_0; \cdot)$, and therefore the proper point-set inclusion $\mathcal{L}^{[C^m]}_n(z_0; \cdot) \subset \mathcal{L}^{[L^2]}_n(z_0; \cdot)$. Alternate metrics for $\mathcal{L}^{[C^m]}_n(z_0; \cdot)$ are the metric induced by this inclusion map, and the product metric of the respective $C^r$-metrics in $\mathbf{B} = C^{n-1}(0,1, \mathbb{R}) \times \cdots \times C^0(0,1, \mathbb{R})$, both under the identification $\Gamma \approx (\xi_1, \ldots, \xi_n)$. The former shall not be used in what follows, while the latter is readily seen to yield the same topology as the natural (not complete) $C^n$-metric inherited from $C^n([0,1], \mathbb{R}^{n+1})$. In fact, the continuity of the map $\gamma \in \mathcal{L}^{[C^m]}_n(z_0; \cdot) \mapsto (\xi_1, \ldots, \xi_n) \in \mathbf{B}$ follows from $\kappa_j = (\mathfrak{F}_j, e_{j+1}) \cdot (\mathfrak{F}_j, e_j)$, via Equations (20) while the existence and continuity of its inverse is due to Remark B.4 and the continuous dependence on parameters for the IVP in Equation 18 (alternatively, one could use Equations 19 to obtain the appropriate estimates).

From now on, we identify $\mathcal{L}^{[C^m]}_n(z_0; \cdot)$ with the separable Banach space $\mathbf{B}$, just as we have identified $\mathcal{L}^{[L^2]}_n(z_0; \cdot)$ and the separable Hilbert space $\mathbf{H}$. With respect to these metrics, the inclusion $\iota : \mathbf{B} \hookrightarrow \mathbf{H}$ is clearly continuous with dense image.

For future reference, we now quote the following two general results from the homotopy theory of infinite dimensional manifolds.

**Fact B.2** (Theorem 2 of [13]). Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be infinite dimensional separable Banach spaces. Suppose $\iota : \mathbf{B}_1 \to \mathbf{B}_2$ is a bounded, injective linear map with dense image and $M_2 \subset \mathbf{B}_2$ is a smooth closed Banach submanifold of finite codimension. Then, $M_1 = \iota^{-1}[M_2]$ is a smooth closed Banach submanifold of $\mathbf{B}_1$ and $\iota : (\mathbf{B}_1, M_1) \to (\mathbf{B}_2, M_2)$ is a homotopy equivalence of pairs.

**Fact B.3** (from Theorem 0.1 of [11] and Corollary 1 of [28]). Let $M_1$ and $M_2$ be topological manifolds modeled on infinite dimensional separable Banach spaces. Any homotopy equivalence $\iota : M_1 \to M_2$ is homotopic to a homeomorphism.
We now endow $L^{t,2}_{n}(z_0, \cdot)$ with a smooth differentiable structure. The previous identification $L^{t,2}_{n}(z_0, \cdot) \approx H$ could be used to define a differentiable structure, but the one we are about to introduce is more convenient. They may well be equivalent, but we shall not try to prove this fact. In any case, Corollary 2 of [28] implies that the resulting two smooth Hilbert manifolds are diffeomorphic. Around each $\Gamma_0 \in L^{t,2}_{n}(z_0, \cdot)$ we construct a coordinate system $(U_{t,z}, \phi(t,z))$ with $\phi(t,z) : U_{t,z} \to H^N$, $N \in \mathbb{N}^*$, $t = (t_1 < \cdots < t_N) \in (0, 1)^N$, $z = (z_1, \cdots, z_N) \in \text{Spin}^N_{n+1}$.

Fix $\Gamma_0 \in L^{t,2}_{n}(z_0, \cdot)$ and decompose it into a finite number of strictly convex arcs (equivalently, short arcs; see Proposition [L.1]). Explicitly, choose times $0 = t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1} = 1$ and $\Gamma_0(0) = (z_0, z_1, \ldots, z_N, z_{N+1}) = \Gamma_0(1) \in \text{Spin}^n_{n+1}$ such that $\Gamma_0[[t_r, t_{r+1}]] \subset U_{z_r}$ for each $r \in \{0, 1, \ldots, N\}$. Now, on each open subset $U_{z_r} \subset \text{Spin}^n_{n+1}$ we use the triangular system of coordinates of Section 4 to define

$$\Gamma_{L,r}(t) = z_r^{-1}\Gamma_0(t) R_r(t) \in L_{n+1}^1, \quad R_r(t) \in U_{p_{n+1}^+}, \quad t_r \leq t \leq t_{r+1}.$$  

The curve $\Gamma_{L,r} : [t_r, t_{r+1}] \to L_{n+1}^1$ is locally convex in the sense of Section 4, i.e., its logarithmic derivative is almost everywhere of the form

$$(\Gamma_{L,r}(t))^{-1} \Gamma_{L,r}'(t) = \sum_{k \in [n]} \beta_{r,k}(t) I_k, \quad (I_k)_{i,j} = [k = j = i-1] \in \mathfrak{L}_{n+1}^1, \quad \beta_{r,k}(t) > 0.$$  

In fact, we have (see Lemma 4.1)

$$\beta_{r,k}(t) = \frac{R(t)_{k,k}}{R(t)_{k+1,k+1}} \kappa_k(t). \quad (21)$$  

In particular, the coefficients $\beta_{r,k}$ are all admissible functions. Write as before

$$\chi_{r,k} = \beta_{r,k} - \frac{1}{\beta_{r,k}}, \quad \beta_{r,k} = \frac{\chi_{r,k} + \sqrt{\chi_{r,k}^2 + 4}}{2}. \quad (22)$$  

We therefore have $\chi_r = (\chi_{r,1}, \ldots, \chi_{r,n}) \in H$. For such $t = (t_1, \ldots, t_N) \in (0, 1)^N$ and $z = (z_1, \ldots, z_N) \in \text{Spin}^N_{n+1}$ consider the set $U_{t,z} \subset L^{t,2}_{n}(z_0, \cdot)$ of the locally convex curves $\Gamma$ satisfying $\Gamma[[t_r, t_{r+1}]] \subset U_{z_r}$ for all $r \in \{0, 1, \ldots, N\}$. Now consider the map $\phi_{t,z} : U_{t,z} \to H^N$ defined by $\phi_{t,z}(\Gamma) = (\chi_1, \ldots, \chi_N)$.

We now establish the compatibility conditions for the atlas formed by all the pairs $(U_{t,z}, \phi_{t,z})$, indexed on the set of valid indices $(t, z)$. Let $\Gamma \in U_{t,z} \cap U_{\tilde{t},\tilde{z}}$, where $t = (t_1, \ldots, t_N)$, $z = (z_1, \ldots, z_N)$, $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_N)$ and $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_N)$. It suffices to show that the $\chi$s depend smoothly on the $\chi$s with respect to the standard differentiable structure of the separable Hilbert space $H$. Write
\[ \phi_{t,z}(\Gamma) = (\chi_1, \ldots, \chi_N) \] and \[ \tilde{\phi}_{t,z}(\Gamma) = (\tilde{\chi}_1, \ldots, \tilde{\chi}_N). \] If \( \mathcal{U}_{s_r} \cap \mathcal{U}_z \neq \emptyset \) and \( J_{r,s} = [t_r, t_{r+1}] \cap [\tilde{t}_s, t_{s+1}] \neq \emptyset \), write the LU decompositions of \( \tilde{z}_r^{-1}\Gamma(t) \) and \( \tilde{z}_s^{-1}\Gamma(t) \) as

\[ \Gamma(t) = z_r \Gamma_L(t) R_r^{-1}(t) = \tilde{z}_s \tilde{\Gamma}_L(t) \tilde{R}_s^{-1}(t), \quad t \in J_{r,s}, \]

where \( \Gamma_L(t), \tilde{\Gamma}_L(t) \in \text{Lo}^1_{n+1} \) and \( R_r(t), \tilde{R}_s(t) \in \text{Up}_{n+1}^+. \) Equations \ref{eq:21} imply

\[ \tilde{\beta}_{s,k}(t) = \frac{\tilde{R}(t)_{k,k} R(t)_{k+1,k+1}}{R(t)_{k+1,k}} \beta_{r,k}(t). \]

Now, the entries of both \( \tilde{R}(t) \) and \( R(t) \) are rational functions in the entries of \( \Gamma(t) \), while \( \Gamma(t) \) can be obtained from the functions \( \beta_{r,k} \) using Equations \ref{eq:9}. The desired smoothness of \((\tilde{\chi}_1, \ldots, \tilde{\chi}_N) \in \mathbb{H}^N_r \) with respect to \((\chi_1, \ldots, \chi_N) \in \mathbb{H}^N \) now follows from these considerations plus Equations \ref{eq:22}.

We now consider the the monodromy map \( \mu_{z_0} : \mathcal{L}^{[L^2]}_n(z_0; \cdot) \rightarrow \text{Spin}_{n+1} \) given by \( \mu_{z_0}(\Gamma) = \Gamma(1) \) and, for each element \( z_1 \in \text{Spin}_{n+1} \), the monodromy subspaces \( \mathcal{L}^{[L^2]}_n(z_0; z_1) = \mu_{z_0}^{-1}\{z_1\} \) and \( \mathcal{L}^{[C^n]}_n(z_0; z_1) = \mathcal{L}^{[L^2]}_n(z_0; z_1) \cap \mathcal{L}^{[C^n]}_n(z_0; \cdot) \). The reader might wish to compare \( \mu_{z_0} \) with the monodromy map studied in \cite{14}.

Continuous dependence on parameters for the IVP in Equation \ref{eq:18} implies the continuity of \( \mu_{z_0} \). Smooth dependence on parameters already implies the smoothness of \( \mu_{z_0} \) restricted to \( \mathcal{B} = \mathcal{L}^{[C^n]}_n(z_0; \cdot) \), but we do not have such a general result for IVPs in the Carathéodory sense. The differentiable structure for \( \mathcal{L}^{[L^2]}_n(z_0; \cdot) \) provided by the charts \((\mathcal{U}_{t,z}, \phi_{t,z})\) was adopted precisely to address the smoothness of \( \mu_{z_0} \).

The next result shows that the monodromy subspaces are closed embedded submanifolds of codimension \( m = n(n + 1)/2 \).

**Lemma B.4.** The monodromy map \( \mu_{z_0} : \mathcal{L}^{[L^2]}_n(z_0; \cdot) \rightarrow \text{Spin}_{n+1} \) is a surjective smooth submersion.

**Proof.** The local expression for \( \mu_{z_0} \) with respect to each chart \((\mathcal{U}_{t,z}, \phi_{t,z})\) is a smooth function \( \mu_{z_0} \circ \phi_{t,z}^{-1} : \mathcal{B} \rightarrow \text{Spin}_{n+1} \) of the coordinates \((\chi_1, \ldots, \chi_N)\) by Equations \ref{eq:9} and \ref{eq:22}. To see that the derivative \( D\mu_{z_0}(\Gamma) \) is surjective, assume \( \Gamma \in \mathcal{U}_{t,z} \). By strict convexity and Proposition A.1, we have \( \Gamma([t_N, 1]) \subset \Gamma(t_N) \Bru_q \). Let \( \Gamma_0 \) and \( \Gamma_N \) be the linear reparametrizations on the unit interval \([0, 1]\) of the restrictions \( \Gamma|_{[0,t_N]} \) and \( \Gamma|_{[t_N,1]} \), respectively. Use Remark 7.1 to produce a map \( \psi_N : \Gamma(t_N) \Bru_q \rightarrow \mathcal{L}^{[L^2]}_n(\Gamma(t_N); \cdot) \) (given by a projective transformations of \( \Gamma_N \)) taking \( \Gamma(1) \) to \( \Gamma_N \) and \( z \in \Gamma(t_N) \Bru_q \) to a convex curve \( \psi_N(z) \in \mathcal{L}_n(\Gamma(t_N); z) \). A straightforward computation verifies that \( \psi_N \) is smooth. Now, define the map \( \psi : \Gamma(t_N) \Bru_q \rightarrow \mathcal{L}^{[L^2]}_n(z_0; \cdot) \) by the concatenation \( \psi(z) = \Gamma_0 \ast \psi_N(z) \); The map \( \psi \) is also smooth, by construction. Thus, the composition \( \mu_{z_0} \circ \psi \) is the identity.
map of $\Gamma(t_N)\Bru \eta \subset \Spin_{n+1}$. In particular, $\mu_{z_0}$ is an open map. Surjectivity of the map $\mu_{z_0}$ is well known [54] but we present a short proof. It suffices to prove that the image of $\mu_{z_0}$ is a closed subset of the connected space $\Spin_{n+1}$. In fact, let $z \in \Spin_{n+1}$ and assume $\Gamma_0(1) \in U_z = z\eta \Bru \eta$ for some $\Gamma_0 \in L_n^2(z_0; \cdot)$. Use Lemma 7.2 to obtain a convex arc $\Gamma_1 \in L_n^2(\Gamma_0(1); z\eta)$; let $\Gamma_2 \in L_n^2(z\eta; z)$ $\Gamma_2(t) = z\eta \exp(t \frac{\pi}{2} \hat{h})$. We have $z = \mu_{z_0}(\Gamma_0 \ast \Gamma_1 \ast \Gamma_2)$ (notice the similarity between this proof and the add-loop construction in [54]).

Remark B.5. A natural question would be if $\mu_{z_0}$ qualifies as some sort of fibration. The reader of course knows that the spaces $L_n(z)$ exhibit different homotopy types as $z$ ranges over $\Spin_{n+1}$ [40, 53, 54, 59, 65]. In fact, $\mu_1$ is not even a Serre fibration, since it lacks the homotopy lifting property for polyhedra (see [34, 51]).

**Proposition B.6.** For all $z_0, z_1 \in \Spin_{n+1}$ we have that:

1. the monodromy subspace $L_n^2(z_0; z_1)$ is a closed embedded submanifold of $L_n^1(z_0; \cdot)$ of codimension $m = n(n+1)/2$;

2. the inclusion map $i : (L_n^1(z_0; \cdot), L_n^2(z_0; z_1)) \hookrightarrow (L_n^2(z_0; \cdot), L_n^2(z_0; z_1))$ is a homotopy equivalence of pairs.

3. There are homeomorphisms $L_n^2(z_0; \cdot) \approx L_n^1(z_0; \cdot)$ and $L_n^2(z_0; z_1) \approx L_n^2(z_0; z_1)$ each homotopic to the corresponding inclusion map.

**Proof.** Item 1 follows from Lemma B.4 and the Regular Value Theorem applied to $\mu_{z_0}$. Item 2 follows from Item 1 and Fact B.2. Item 3 follows from Item 2 and Fact B.3.

The previous result warrants us the right to drop the superscripts $[L^2]$ and $[C^m]$ and to adopt either definition of $L_n(z_0; z_1)$ depending on the purpose at hand.

**References**

[1] Adler, M. *On a Trace Functional for Formal Pseudo-Differential Operators and the Symplectic Structure of the Korteweg-Devries Type Equations*, Inventiones Math. 50, 219–248 (1979)

[2] Alves, E. and Saldanha, N. *Results on the homotopy type of the spaces of locally convex curves on $S^3$*, to appear in Annales d’Institut Fourier, available at [https://arxiv.org/pdf/1703.02581.pdf](https://arxiv.org/pdf/1703.02581.pdf) (2017)
[3] Alves, E., Goulart, V., Saldanha, N. and Shapiro, B. *The homotopy type of spaces of locally convex curves in spheres*, in preparation (2018)

[4] Ando, T. *Totally positive matrices*, Linear Algebra Appl. 90, 165–219 (1987)

[5] Anisov, S. *Convex curves in $\mathbb{RP}^n$*, Proc. Steklov Inst. Math. 221, 2, pp. 3–39, Moscow, Russia (1998)

[6] Berenstein, A., Fomin, S., Zelevinsky, A. *Parametrizations of Canonical Bases and Totally Positive Matrices*, Adv. Math. 122, 49–149 (1996)

[7] Bernstein, I., Gelfand, I., Gelfand, S. *Schubert cells and cohomology of the spaces $G/P$*, Russian Math. Surveys, 28 : 3, pp. 1–26 (1973)

[8] Björner, A., Brenti, F. *Combinatorics of Coxeter groups*, Grad. Texts in Math., 231, Springer, New York (2005)

[9] Brenti, F. *Combinatorics and Total Positivity*, Journal of Combinatorial Theory, Series A 71, 175–218 (1995)

[10] Brown, M. *Locally flat imbeddings of topological manifolds*, Annals of Mathematics, Second series, Vol. 75, pp. 331–341 (1962)

[11] Burghelea, D. and Henderson, D. *Smoothings and homeomorphisms for Hilbert manifolds*, Bull. Am. Math. Soc. 76, 1261–1265 (1970)

[12] Burghelea, D. and Kuiper, N. *Hilbert manifolds*, Ann. of Math. (2) 90, 379–417 (1969)

[13] Burghelea, D., Saldanha and N., Tomei, C. *Results on infinite dimensional topology and applications to the structure of the critical set of nonlinear Sturm-Liouville operators*, J. Differential Equations 188, 569–590 (2003)

[14] Burghelea, D., Saldanha, N. and Tomei, C. *The topology of the monodromy map of a second order ODE*, J. Differential Equations 227, 581–597 (2006)

[15] Burghelea, D., Saldanha, N. and Tomei, C. *The geometry of the critical set of nonlinear periodic Sturm-Liouville operators*, J. Differential Equations 246, 3380–3397 (2009)

[16] Chevalley, C. *Sur les décompositions cellulaires des espaces $G/B$* (ca. 1958), Haboush, W. (ed.), Algebraic Groups and their Generalizations: Classical Methods, Proc. Symp. Pure Math., 56:1, Amer. Math. Soc. pp. 1–23 (1994)

[17] Demazure, M. *Désingularization des variétés de Schubert généralisées*, Ann. Sci. cole Norm. Sup. (4), 7, pp. 53–88 (1974)
[18] Eliashberg, Y. and Mishachev, N. Introduction to the h-principle, Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI (2002)

[19] Elnitsky, S. Rhombic tilings of polygons and classes of reduced words in Coxeter groups, Journal of Combinatorial Theory, series A, vol. 77, no. 2 (1997)

[20] Filippov, A. Differential Equations with Discontinuous Right Hand Sides, Kluwer Acad. Publ. (1988).

[21] Fomin, S., Zelevinsky, A. Double Bruhat Cells and Total Positivity, Journal of the AMS, Vol. 12, No. 2, pp. 335-380 (1999)

[22] Fomin, S., Zelevinsky, A. Total Positivity: Tests and Parametrizations, Math. Intelligencer, 22, 23–33 (2000)

[23] Gantmacher, F. and Krein, M. Sur les matrices oscillatoires, C.R.Acad.Sci.Paris, 201 , 577–579 (1935)

[24] Gel’fand, I. and Dikii, L. Fractional powers of operators and Hamiltonian systems, Funkts. Anal. Prilozhen., 10 (4), 13–29 (1976)

[25] Gel’fand, I. and Dikii, L. A family of Hamiltonian structures related to non-linear integrable differential equations, Prepr. Inst. Appl. Mat. 136 (1978)

[26] Goulart, V. Towards a combinatorial approach to the topology of spaces of nondegenerate spherical curves, Ph.D. thesis, PUC-Rio (2016)

[27] Gromov, M. Partial Differential Relations, Springer-Verlag (1986)

[28] Henderson, D. Infinite-dimensional manifolds are open subsets of Hilbert space, Topology 9, 25–33 (1970)

[29] Hirsch, M. Immersions of manifolds, Trans. Am. Math. Soc. 93, 242-276 (1959)

[30] Humphreys, J. Reflection Groups and Coxeter Groups, Cambridge University Press (1990)

[31] Karlin, S. Total Positivity, vol. 1, Stanford University Press (1968)

[32] Khesin, B. and Ovsienko, V. Symplectic leaves of the Gelfand-Dickey brackets and homotopy classes of nondegenerate curves, Funktsional’nyi Analiz i Ego Prilozheniya, Vol. 24, No. 1, pp. 38–47 (1990)

[33] Khesin, B. and Shapiro, B. Nondegenerate curves on $S^2$ and orbit classification of the Zamolodchikov algebra, Commun. Math. Phys. 145, 357–362 (1992)
[34] Khesin, B. and Shapiro, B. *Homotopy classification of nondegenerate quasiperiodic curves on the 2-sphere*, Publ. de l'Institut Mathématique, Nouvelle série, tome 66 (80), 127–156 (1999)

[35] Klingenberg, W. *A course in Differential Geometry*, Springer-Verlag, New York (1978)

[36] Klingenberg, W. *Riemannian Geometry*, De Gruyter, Berlin (1982)

[37] Kostant, B. *Lie algebra cohomology and generalized Schubert cells*, Ann. of Math. (2) 77, 72–144; MR 26 # 266.(1963)

[38] Lawson, H. and Michelsohn, M. *Spin Geometry*, Princeton University Press (1989)

[39] Levy, S., Maxwell, D. and Munzner, T. *Outside In*, AK Peters, Wellesley, MA. Narrated video (21 min) from the Geometry Center. (1994)

[40] Little, J. *Nondegenerate homotopies of curves on the unit 2-sphere*, J. of Differential Geometry 4, 339–348 (1970)

[41] Lusztig, G. *Total positivity in reductive groups* in *Lie Theory and Geometry: in honour of B. Konstant*, Progress in Mathematics, Vol. 123, Birkhäuser, Basel (1994)

[42] Lusztig, G. *Total positivity in partial flag manifolds*, Representation Theory 2, 70–78 (1998)

[43] Lusztig, G. *A Survey of Total Positivity*, Milan Journal of Mathematics, 76, 125–134 (2008)

[44] Moulis, N. *Sur les variétés Hilbertiennes et les fonctions non dégénérées*, Nederl. Akad. Wetensch. Proc., Ser. A 71, Indag. Math. 30, 497–511 (1968)

[45] Novikov, D. and Yakovenko, S. *Integral curvatures, oscillation and rotation of spatial curves around affine subspaces*, Journal of Dynamical and Control Systems, Volume 2, Issue 2, 157–191 (1996)

[46] Palais, R. *Homotopy theory of infinite dimensional manifolds*, Topology 5 1–16 (1966)

[47] Pólya, G. *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Am. Math. Soc., 24, 312–324 (1922)

[48] Postnikov, A. *Total positivity, Grassmannians, and networks*, https://arxiv.org/abs/math/0609764 (2006)

80
[49] Postnikov, A. *Positive Grassmannian and polyhedral subdivisions*, [https://arxiv.org/abs/1806.05307](https://arxiv.org/abs/1806.05307) (2018)

[50] Rietsch, K. *Intersections of Bruhat Cells in Real Flag Varieties*, Int. Math. Res. Notices, No. 13 (1997)

[51] Saldanha, N. *The homotopy and cohomology of spaces of locally convex curves in the sphere — I*, [http://www.arxiv.org/abs/0905.2111](http://www.arxiv.org/abs/0905.2111) (2012)

[52] Saldanha, N. *The homotopy and cohomology of spaces of locally convex curves in the sphere — II*, [http://www.arxiv.org/abs/0905.2116](http://www.arxiv.org/abs/0905.2116) (2012)

[53] Saldanha, N. *The homotopy type of spaces of locally convex curves in the sphere*, Geometry & Topology 19 1155–1203 (2015)

[54] Saldanha, N. and Shapiro, B. *Spaces of locally convex curves in $S^n$ and combinatorics of the group $B_{n+1}^+$*, Journal of Singularities, 4, 1–22 (2012)

[55] Saldanha, N. and Tomei, C. *The topology of critical sets of some ordinary differential operators*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 66, 491–504 (2005)

[56] Saldanha, N. and Zühlke, P. *On the components of spaces of curves on the 2-sphere with geodesic curvature in a prescribed interval*, International Journal of Mathematics, Vol. 24, No. 14, 1350101 (2013).

[57] Saldanha, N. and Zühlke, P. *Components of spaces of curves with constrained curvature on flat surfaces*, Pacific Journal of Mathematics, Vol. 281, No. 1, 185–242 (2016).

[58] Schoenberg, I. *Uber variationsvermindende lineare Transformationen*, Math.Z. 32, 321–328 (1930)

[59] Shapiro, B. and Shapiro, M. *On the number of connected components in the space of closed nondegenerate curves on $S^n$*, Bull. Am. Math. Soc., 25, no.1:75–79 (1991)

[60] Shapiro, B. and Shapiro, M. *On the boundary of totally positive upper triangular matrices*, Linear Algebra Appl. 231, 105–109 (1995)

[61] Shapiro, B. and Shapiro, M. *Linear ordinary differential equations and Schubert calculus*, Proceedings of 13th Gökova Geometry-Topology Conference, pp. 1–9 (2010)

[62] Shapiro, B. and Shapiro, M. *Projective convexity in $\mathbb{P}^3$ implies Grassmann convexity*, Int. J. Math., vol 11, issue 4, 579–588 (2000)
[63] Shapiro, B., Shapiro, M., Vainshtein, A. *On the number of connected components in the intersection of two open opposite Schubert cells in SL_n/B*, Int. Math. Res. Notices, Issue 10, 469–493 (1997)

[64] Shapiro, B., Shapiro, M., Vainshtein, A. *Skew-symmetric vanishing lattices and intersections of Schubert cells*, Int. Math. Res. Notices, Issue 11, 563–588 (1998)

[65] Shapiro, M. *Topology of the space of nondegenerate curves*, Russian Acad. Sci. Izv. Math. 43 291 (1994)

[66] Smale, S. *The classification of immersions of spheres in euclidean spaces*, Ann. of Math. 69, 327–344 (1959)

[67] Verma, D. *Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc., Vol. 74, 1, 160–166 (1968)