OPTIMAL CONTROL OF A NONLOCAL THERMISTOR
PROBLEM WITH ABC FRACTIONAL TIME DERIVATIVES

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Abstract. We study an optimal control problem associated to a fractional nonlocal thermistor problem involving the ABC (Atangana–Baleanu–Caputo) fractional time derivative. We first prove the existence and uniqueness of solution. Then, we show that an optimal control exists. Moreover, we obtain the optimality system that characterizes the control.

1. Introduction

Fractional calculus is a powerful mathematical tool to describe real-world phenomena with memory effects, being used in many scientific fields. Many published works in fractional calculus put emphasis on the Riemann–Liouville power-law differential operator; others suggest different fractional approaches of mathematical modeling to represent physical problems, calling attention that a singularity on the power law leads to models that are singular, which is not convenient for those with no sign of singularity. In particular, several applications of the exponential kernel suggested by Caputo and Fabrizio can be found in chemical reactions, electrostatics, fluid dynamics, geophysics and heat transfer [6, 13].

If an object at one temperature is exposed to a medium with another temperature, the temperature difference between the object and the medium follows an exponential decay, according with Newton’s law of cooling. Other examples may be found in luminescence, pharmacology and toxicology, physical optics, radioactivity and thermo-electricity, where there is a decline in resistance of a negative temperature coefficient thermistor, as the temperature, vibrations, finance or some other aspect is increased. The generalized Mittag–Leffler function, considered as a generalization of the exponential decay and as power-law asymptotic for a very large time, occurs to handle non-locality and avoid singularity [12]. According to Rudolf Gorenflo (1930–2017) [12], one can say that the Mittag–Leffler function is a practical memory function in several physical problems. It can be used as a waiting-time distribution, as well as a first-passage-time distribution for renewal processes [12]. Recently, such considerations lead to the introduction of ABC (Atangana–Baleanu–Caputo) fractional operators [2, 5].
The Riemann–Liouville fractional derivative seems not the most appropriate to describe diffusion at different scales. Thanks to the non-obedience of commutativity and associativity criteria, and due to Mittag–Leffler memory, the ABC fractional derivative promises to be a powerful mathematical tool, allowing to describe heterogeneity and diffusion at different scales, distinguishing between dynamical systems taking place at different scales without steady state. Here, we are interested to study an optimal control problem to the following nonlocal parabolic boundary value problem:

\[
\begin{align*}
\alpha_0^b D_t^\alpha u - \Delta u &= \frac{\lambda f(u)}{\left(\int_{\Omega} f(u) \, dx\right)^2} \quad \text{in } Q_T = \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} &= -\beta u \quad \text{on } S_T = \partial \Omega \times (0, T), \\
u(0) &= u_0 \quad \text{in } \Omega,
\end{align*}
\]

where \(\alpha_0^b D_t^\alpha\), \(\alpha \in (0, 1)\), is the Atangana–Baleanu fractional derivative of order \(\alpha\) in the sense of Caputo with respect to time \(t\), \(\Delta\) is the Laplacian with respect to the spacial variables, defined on \(H^2(\Omega) \cap H_0^1(\Omega)\), \(f\) is a smooth function prescribed below, and \(T\) is a fixed positive real. The domain \(\Omega\) is bounded in \(\mathbb{R}^N\), \(N \geq 1\), with a sufficiently smooth boundary \(\partial \Omega\) and \(Q_T = \Omega \times (0, T)\). Here, \(\nu\) denotes the outward unit normal and \(\frac{\partial}{\partial \nu} = \nu \cdot \nabla\) is the normal derivative on \(\partial \Omega\). Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity \(f\) is strongly dependent on the temperature \(u\) (thermistors). When \(\alpha = 1\), equation (1) describes the diffusion of the temperature \(u\) generated by the electric current with the presence of a nonlocal term \([8,16,21,23,25,29]\). Constant \(\lambda\) is a dimensionless parameter while function \(\beta\) is the positive thermal transfer coefficient. The given value \(u_0\) is the initial condition for the temperature. Mixed boundary conditions of Robin’s type are considered, which are derived from Newton’s cooling law.

Optimal control of problems governed by partial differential equations occurs more and more frequently in different research areas \([1,20,24,28]\). Researchers are interested, essentially, to existence, regularity, and uniqueness of the optimal control problem, as well as necessary optimality conditions. The optimal control theory for systems of thermistor problems with integer-order derivatives on time \(\partial_t\) has been developed in \([7,14,15,18]\). Works on control theory applied to fractional differential equations, where the fractional time derivative is considered in Riemann–Liouville and Caputo senses, have been already studied \([27]\). However, to the best of our knowledge, the use of the Atangana–Baleanu derivative is underdeveloped in this area. Particularly, we are not aware of any paper investigating the optimal control of (1). In our work, we choose the heat transfer coefficient \(\beta\) as a control, because it plays a crucial role in the temperature variations of a thermistor \([11,22,30]\).

Our manuscript is organized as follows. In Section 2, we briefly collect definitions and preliminary results about fractional derivatives. Section 3 is devoted to the existence and uniqueness results for (1), while in Section 4 we investigate the corresponding control problem. Main results characterize, explicitly, the optimal control, extending those of \([14,26]\).
2. Preliminary results

Our main goal consists to find a control $\beta$ belonging to the set

$$U_M = \{ \beta \in L^\infty(\Omega \times (0, T)), 0 < m \leq \beta \leq M \}$$

of admissible controls, which minimizes the cost functional

$$J(\beta) = \int_{Q_T} u dx dt + \int_{S_T} \beta^2 ds dt$$

defined in terms of $u(\beta)$ and $\beta$. Precisely, we purpose to find $\beta \in U_M$ such that

$$J(\beta) = \min_{\beta \in U_M} J(\beta). \quad (2)$$

We now recall some properties on the Mittag–Leffler function and the definition of ABC fractional time derivative. First, we define the two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$, as the family of entire functions of $z$ given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C},$$

where $\Gamma(\cdot)$ denotes the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} dt, \quad Re(z) > 0.$$  

Observe that the exponential function is a particular case of the Mittag–Leffler function: $E_{1,1}(z) = e^z$. Follows the definition of fractional derivative in the sense of Atangana–Baleanu \[3, 9\].

**Definition 1.** For a given function $u \in H^1(a, T), T > a$, the Atangana–Baleanu fractional derivative in Caputo sense, shortly called the ABC fractional derivative, of $u$ of order $\alpha$ with base point $a$, is defined at a point $t \in (a, T)$ by

$$\frac{a^b}{a^t} D_\alpha^a g(t) = \frac{B(\alpha)}{1 - \alpha} \int_a^t u'(\tau)E_{\alpha,\alpha}[-\gamma(t - \tau)\alpha]d\tau, \quad (3)$$

where $\gamma = \frac{\alpha}{\Gamma(\alpha)}$, $E_{\alpha,\alpha}$ stands for the Mittag–Leffler function, and $B(\alpha) = (1 - \alpha) + \frac{\alpha}{\Gamma(\alpha)}$. Furthermore, the Atangana–Baleanu fractional integral of order $\alpha$ with base point $a$ is defined as

$$I_\alpha^a g(t) = \frac{1 - \alpha}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t g(t)(t - \tau)^{-\alpha-1} d\tau. \quad (4)$$

**Remark 2.** For $\alpha = 1$ in (3), we obtain the usual ordinary derivative $\partial_t$. If $\alpha = 0, 1$ in (4), then we get the initial function and the classical integral, respectively.

Roughly speaking, the following result asserts that going backwards in time with the fractional time derivative with nonsingular Mittag–Leffler kernel at the based point $T$ is equivalent as going forward in time with the fractional time derivative operator with nonsingular Mittag–Leffler kernel.

**Lemma 3.** Let $\eta: [0, T] \to \mathbb{R}$. Then, for all $\alpha \in (0, 1)$, the equivalence relation

$$\frac{a^b}{a^t} D_\alpha^a \eta(T - t) = \frac{a^b}{a^t} D_\alpha^a \eta(t)$$

holds.

**Proof.** Follows directly from definition by change of variables. \qed
Along the paper, we always assume that the integrals exist. Moreover, we consider the following assumptions:

(H1) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a positive Lipschitzian continuous function;
(H2) there exist positive constants \( c_1 \) and \( c_2 \) such that \( c_1 \leq f(\xi) \leq c_2 \ \forall \ \xi \in \mathbb{R} \);
(H3) \( u_0 \in L^2(\Omega) \).

**Definition 4.** We say that \( u \) is a weak solution to (1) if

\[
\int_{\Omega} (ab D_t^\alpha u) v dx + \int_{\Omega} \nabla u \nabla v dx + \int_{\partial \Omega} \beta uv ds = \lambda \left( \frac{1}{\int_{\Omega} f(u) dx} \right)^2 \int_{\Omega} f(u) v dx
\]

for all \( v \in H^1(\Omega) \).

**Proposition 5.** Let \( u, v \in C_\infty(Q_T) \). Then,

\[
\int_{\Omega} \int_0^T (ab D_t^\alpha u - \Delta u) v dx dt
\]

\[
= \int_0^T \int_{\partial \Omega} u \frac{\partial v}{\partial \nu} ds dt - \int_0^T \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} ds dt
\]

\[
- \frac{B(\alpha)}{1 - \alpha} \int_{\Omega} \int_0^T u(x,0) E_{\alpha,\alpha}[-\gamma t^\alpha] v dx dt
\]

\[
+ \int_0^T \int_{\Omega} u \left( -ab D_t^\alpha v - \Delta v \right) dx dt
\]

\[
+ \frac{B(\alpha)}{1 - \alpha} \int_{\Omega} \int_0^T v(x,T) \int_0^T u E_{\alpha,\alpha}[-\gamma(T-t)^\alpha] dt dx.
\]

**Proof.** From integration by parts involving the ABC fractional-time derivative (see [10]), a straightforward calculation gives that

\[
\int_0^T ab D_t^\alpha u \cdot v dt = - \int_0^T ab D_t^\alpha v \cdot u dt + \frac{B(\alpha)}{1 - \alpha} v(x,T) \int_0^T u E_{\alpha,\alpha}[-\gamma(T-t)^\alpha] dt
\]

\[
- \frac{B(\alpha)}{1 - \alpha} u(x,0) \int_0^T E_{\alpha}[-\gamma t^\alpha] v dt
\]

and

\[
- \int_{\Omega} \int_0^T \Delta u \cdot v dx dt = \int_{\Omega} \int_0^T u \frac{\partial v}{\partial \nu} ds dt - \int_{\Omega} \int_0^T v \frac{\partial u}{\partial \nu} ds dt
\]

\[
- \int_{\Omega} \int_0^T \Delta v \cdot u dx dt.
\]

Combining (6) and (7), we get the desired result. \( \square \)

Using the boundary conditions of problem (1), we immediately get the following corollary.
Corollary 6. Let \( u, v \in C^\infty(Q_T) \). Then,
\[
\int_\Omega \int_0^T (\frac{\partial}{\partial t} D_t^\alpha u - \Delta u) \, v \, dx \, dt = \int_0^T \int_{\partial \Omega} \beta u v \, ds \, dt + \int_0^T \int_{\partial \Omega} \frac{\partial u}{\partial t} \, v \, ds \, dt - \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega u(x,0) E_{\alpha,\alpha}[-\gamma t^\alpha] + \int_0^T \int_\Omega u (-\frac{\partial}{\partial t} D_t^\alpha v - \Delta v) \, dx \, dt + \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_{\Omega} v(x,T) \, \psi E_{\alpha,\alpha}[-\gamma(T-t)^\alpha] \, dx \, dt.
\]

Along the text, constants \( c \) are generic, and may change at each occurrence.

3. Existence and uniqueness for \([5]\)

We proceed similarly as in \([10]\). Let \( V_m \) define a subspace of \( H^1(\Omega) \) generated by \( w_1, w_2, \ldots, w_m \), space vectors of orthogonal eigenfunctions of the operator \( \Delta \). We seek \( u_m : t \in (0, T] \to u_m(t) \in V_m \), solution of the fractional differential equation
\[
\begin{cases}
\int_0^T \frac{\partial}{\partial t} D_t^\alpha u_m \, v \, dx + \int_\Omega \nabla u_m \cdot \nabla v \, dx + \int_\Omega \beta u_m v \, dx = (g(u_m), v) \quad &\text{for all } v \in V_m, \\
u_m(x,0) = u_{0m} \quad &\text{for } x \in \Omega,
\end{cases}
\]
with \( g(u) = \frac{\lambda_f(u)}{\int_\Omega f(u) \, dx} \).

Theorem 7. Let \( \alpha \in (0, 1) \). Assume that \( f \in L^2(Q_T), u_0 \in L^2(\Omega) \). Let \( \langle \cdot, \cdot \rangle \) be the scalar product in \( L^2(\Omega) \) and \( a(\cdot, \cdot) \) be the bilinear form in \( H_0^1(\Omega) \) defined by
\[
a(\phi, \psi) = \int_\Omega \nabla \phi(x) \cdot \nabla \psi(x) \, dx \quad \forall \phi, \psi \in H^1(\Omega).
\]
Then the problem
\[
\begin{cases}
\frac{\partial}{\partial t} D_t^\alpha u + a(u(t), v) = (f(t), v), \quad &\text{for all } t \in (0, T), \\
u(x,0) = u_0, \quad &\text{for } x \in \Omega,
\end{cases}
\]
has a unique solution \( u \in L^2(0,T, H_0^1(\Omega)) \cap C(0,T, H_0^1(\Omega)) \) given by
\[
u(x,t) = \sum_{i=1}^{+\infty} \left( \zeta_i E_{\alpha,-\gamma_i t^\alpha} u_{i0} + \frac{(1-\alpha)\zeta_i}{B(\alpha)} f_i(t) + K_i \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} [-\gamma_i(t-s)^\alpha] f_i(s) \, ds \right) w_i, \quad \text{for } x \in \Omega,
\]
where \( \gamma_i \) and \( \zeta_i \) are constants. Moreover, provided \( u_0 \in L^2(\Omega) \), \( u \) satisfies the inequalities
\[
\|u\|_{L^2(0,T,H_0^1(\Omega))} \leq \mu_1 (\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q_T)}) \quad \text{(9)}
\]
and
\[
\|u\|_{L^2(\Omega)} \leq \mu_2 (\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}), \quad \text{(10)}
\]
where \( \mu_1 \) and \( \mu_2 \) are positive constants.

Proof. Because \( u_m(t) \in V_m \), one has
\[
u_m(t) = \sum_{i=1}^{m} (u(t), w_i) w_i = \sum_{i=1}^{m} u_i(t) w_i.
\]
The fact that \( g(u) \in L^2(Q_T) \) implies that \( u_m \) can be written in explicit form (see (8)). Arguing exactly as in [10], we can prove that \( u_m(t) \) is a Cauchy sequence in the space \( L^2(0, T, H^1(\Omega)) \) and \( C(0, T, L^2(\Omega)). \) Using the estimates (9)–(10) of Theorem 7, we have that
\[
\begin{align*}
\frac{\partial u_m}{\partial t} &\to \frac{\partial u}{\partial t} \text{ weakly in } L^2(0, T, H^{-1}(\Omega)), \\
u_m &\to u \text{ weakly in } \mathcal{C}(0, T, L^2(\Omega)), \\
u_m &\to u \text{ strongly in } L^2(Q_T), \\
u_m &\to u \text{ a.e. in } L^2(Q_T).
\end{align*}
\]
By standard techniques of Lebesgue’s theorem and some compactness arguments of Lions [19], one gets that \( u(t) \) is a solution of problem (1). Then, the existence and uniqueness result follows. □

4. Existence of an optimal control

We prove existence of an optimal control by using minimizing sequences.

**Theorem 8.** Assume that assumptions (H1)–(H3) are satisfied. Then, there exists at least an optimal solution \( \beta \in L^\infty(Q_T) \) such that (2) holds true.

**Proof.** Let \((\beta_n)_n\) be a minimizing sequence of \( J(\beta) \) in \( U_M \) such that
\[
\lim_{n \to +\infty} J(\beta_n) = \inf_{\beta \in U_M} J(\beta).
\]
Then, \( u_n = u_n(x, t, \beta_n) \), the corresponding solutions to (1), satisfy
\[
\begin{align*}
\frac{\partial^b D_t^\alpha u_n}{\partial t} - \Delta u_n &= \frac{\lambda f(u_n)}{\int_{\Omega} f(u_n) \, dx}^2, \quad \text{in } Q_T = \Omega \times (0, T), \\
\frac{\partial u_n}{\partial \nu} &= -\beta_n u_n, \quad \text{on } S_T = \partial \Omega \times (0, T), \\
u_n(0, x) &= u_0(x), \quad \text{in } \Omega.
\end{align*}
\]
By Theorem 7 we have that \((u_n)_n\) is bounded, independently of \( n \) in \( L^2(0, T, H^1(\Omega)) \). Moreover, for a positive constant independent of \( n \), we have
\[
\|\frac{\partial^b D_t^\alpha u_n}{\partial t} - \Delta u_n\|_{L^2(Q_T)} \leq c.
\]
Therefore, there exists \( u, \) for extracted sequences of \((u_n)_n\), still denoted by \((u_n)_n\), and there exists \( \beta \in U_M \) such that
\[
\begin{align*}
\frac{\partial^b D_t^\alpha u_n}{\partial t} - \Delta u_n &\to \delta \text{ weakly in } L^2(Q_T), \\
\frac{\partial u_n}{\partial t} &\to \frac{\partial u}{\partial t} \text{ weakly in } \mathcal{D}'(Q_T) \text{ and } L^2(0, T, H^{-1}(\Omega)), \\
u_n &\to u \text{ weakly in } L^\infty(0, T, L^2(\Omega)) \text{ and in } L^2(Q_T), \\
u_n &\to u \text{ strongly in } L^2(Q_T), \\
u_n &\to u \text{ a.e. in } L^2(Q_T), \\
\beta_n &\to \beta \text{ weakly in } L^2(\partial \Omega), \\
\beta_n &\to \beta \text{ weakly star in } L^\infty(\partial \Omega),
\end{align*}
\]
where $\mathbb{D}'(Q_T)$ is the dual of $\mathbb{D}(Q_T)$, the set of $C^\infty$ functions on $Q_T$ with compact support. One can prove that

$$a^b_0 D^a_{t} u_n - \triangle u_n \rightarrow a^b_0 D^a_{t} u - \triangle u \text{ weakly in } \mathbb{D}'(Q_T).$$

Indeed, we have

$$\int_0^T \int_{\Omega} a^b_0 D^a_{t} v - \triangle v \, dx \, dt \rightarrow \int_0^T \int_{\Omega} u (-a^b_0 D^a_{t} v - \triangle v) \, dx \, dt, \forall v \in \mathbb{D}(Q_T)$$

and

$$\int_{\Omega} v(x, T) \int_0^T u_n E_{a, \alpha} [-\gamma(T-t)\alpha] \, dt \, dx \rightarrow \int_{\Omega} v(x, T) \int_0^T u E_{a, \alpha} [-\gamma(T-t)\alpha] \, dt \, dx.$$

We now prove that for all $v \in H^1(\Omega)$ and $n \rightarrow \infty$ one has

$$\int_{\partial \Omega} \beta_n u_n v \, ds \rightarrow \int_{\partial \Omega} \beta u v \, ds.$$

In fact,

$$\beta_n u_n v - \beta u v = \beta_n (u_n - u) v + (\beta_n - \beta) u v. \quad (12)$$

By using that $\beta_n$ is essentially bounded, Schwartz’s inequality and the trace inequality $\|u\|_{L^2(\partial \Omega)} \leq c \|u\|_{H^1(\Omega)}$, it leads from limits (11) that the right-hand side of (12) goes to 0 when $n \rightarrow \infty$. Thus,

$$a^b_0 D^a_{t} u_n - \triangle u_n \rightarrow a^b_0 D^a_{t} u - \triangle u \text{ weakly in } \mathbb{D}'(Q_T).$$

From the uniqueness of the limit, we have

$$a^b_0 D^a_{t} u - \triangle u = \delta.$$

Since $u \in L^2(Q_T)$ and $a^b_0 D^a_{t} u - \triangle u \in L^2(Q_T)$, we know that $u/\partial \Omega$ and $\partial u/\partial v$ exist and belong to $H^{-\frac{1}{2}}(\partial \Omega)$ and $H^{-\frac{1}{2}}(\partial \Omega)$, respectively. It follows that

$$\int_{\partial \Omega} u_n \frac{\partial v}{\partial n} \rightarrow \int_{\partial \Omega} u \frac{\partial v}{\partial n} \forall v \in \mathbb{D}(Q_T).$$

On the other hand, we have $u_n \rightarrow u$ a.e. in $\Omega \times (0, T)$. Since $f$ is continuous, $f(u_n) \rightarrow f(u)$ a.e. in $L^2(\Omega)$. It yields that

$$\int_{\Omega} f(u_n) \, dx \rightarrow \int_{\Omega} f(u) \, dx$$

and

$$\int_{\Omega} f(u_n) v \, dx \rightarrow \int_{\Omega} f(u) v \, dx, \forall v \in H^1(\Omega).$$

By passing to the limit in the equation fulfilled by $u_n$, and using Corollary 6, we deduce that $u$ is a solution of (11). Finally, function $\beta \rightarrow J(\beta)$ is lower semi-continuous. Therefore,

$$J(\beta) \leq \lim \inf_{n \rightarrow \infty} J(\beta_n),$$

which implies that $J(\beta) = \inf_{\beta \in U_M} J(\beta)$. The uniqueness of $\beta$ comes from the strict convexity of functional $J$. \hfill \blackbox
5. Optimality conditions

In this section, our aim is to obtain optimality conditions. As we shall see, our necessary optimality conditions involve an adjoint system defined by means of the backward Atangana–Baleanu fractional-time derivative. To prove them, we assume, in addition to hypotheses (H1)–(H3), that

(H4) \( f \) is of class \( C^1 \).

Due to its dependence on \( u \), the objective functional is differentiated with respect to the minimizing control. We calculate the Gâteaux derivative of \( J \) with respect to the control \( \beta \) in the direction \( l \) at \( \beta \). We also need to differentiate \( u \) with respect to the control \( \beta \). The difference quotient \( (u(\beta + \varepsilon l) - u(\beta)) / \varepsilon \) is expected to converge weakly in \( H^1(\Omega) \) to a function \( \psi \) satisfying a linear PDE, which leads to the adjoint system.

**Theorem 9.** Assume hypotheses (H1)–(H4). Then \( \beta \mapsto u(\beta) \) is differentiable in the sense that as \( \varepsilon \to 0 \) one has

\[
\frac{u(\beta + \varepsilon l) - u(\beta)}{\varepsilon} \to \psi \text{ weakly in } H^1(\Omega)
\]

for any \( \beta, l \in U_M \) such that \( (\beta + \varepsilon l) \in U_M \) for small \( \varepsilon \). Moreover, \( \psi \) fulfills the following system:

\[
\begin{align*}
&_{0}^a D_t^{\alpha} \psi - \Delta \psi = \frac{-2\lambda f(u)}{(\int_{\Omega} f(u) \, dx)^3} \int_{\Omega} f'(u) \psi \, dx + \frac{\lambda f'(u) \psi}{(\int_{\Omega} f(u) \, dx)^2} \quad \text{in } \Omega, \\
&\frac{\partial \psi}{\partial \nu} + \beta \psi + lu = 0 \text{ on } \partial \Omega.
\end{align*}
\]

**Proof.** Denote \( u = u(\beta) \) and \( u_\varepsilon = u(\beta_\varepsilon) \), where \( \beta_\varepsilon = \beta + \varepsilon l \). Subtracting equation (11) from the corresponding equation of \( u_\varepsilon \), we have

\[
_{0}^a D_t^{\alpha} \left( \frac{u_\varepsilon - u}{\varepsilon} \right) - \Delta \left( \frac{u_\varepsilon - u}{\varepsilon} \right) = g(u_\varepsilon) - g(u)
\]

with

\[
g(u_\varepsilon) - g(u) = \frac{\lambda (f(u_\varepsilon) - f(u))}{\varepsilon (\int_{\Omega} f(u_\varepsilon) \, dx)^2} + \frac{\lambda f(u)}{\varepsilon} \left( \frac{1}{(\int_{\Omega} f(u) \, dx)^2} - \frac{1}{(\int_{\Omega} f(u) \, dx)^2} \right).\]

As in the first section, since \( g(u_\varepsilon) - g(u) \in L^\infty(\Omega) \subseteq L^2(\Omega) \), by using the energy estimates of Theorem 7, we get that

\[
\frac{u_\varepsilon - u}{\varepsilon} \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)).
\]
We conclude that $\frac{u_\varepsilon - u}{\varepsilon} \to \psi$ weakly in $L^\infty(0,T,L^2(\Omega))$, $\frac{u_\varepsilon - u}{\varepsilon} \to \psi$ weakly in $L^2(0,T, H^1(\Omega))$, $\frac{\partial}{\partial t} \left( \frac{u_\varepsilon - u}{\varepsilon} \right) \to \frac{\partial \psi}{\partial t}$ weakly in $L^2(0,T, H^{-1}(\Omega))$, and

$$\frac{u_\varepsilon - u}{\varepsilon} \to \psi$$ weakly in $L^\infty(0,T,L^2(\partial\Omega))$.

Using (14) and passing to the limit as $\varepsilon \to 0$, we get that

$$\int_0^T \int_\Omega \frac{\partial}{\partial t} D_t^\alpha \left( \frac{u_\varepsilon - u}{\varepsilon} \right) \nabla vdxdt + \int_0^T \int_\Omega \nabla \left( \frac{u_\varepsilon - u}{\varepsilon} \right) \nabla vdxdt + \int_0^T \int_{\partial\Omega} \frac{u_\varepsilon - u}{\varepsilon} vdsdt + \int_0^T \int_\Omega l\varepsilon vdsdt = g(u_\varepsilon) - g(u).$$

By Green’s formula, it follows that

$$\int_0^T \int_\Omega \left( g^b D_t^\alpha \psi - \Delta \psi \right) vdxdt \quad + \quad \int_0^T \int_{\partial\Omega} \left( \frac{\partial \psi}{\partial \nu} + \beta \psi + lu \right) vdsdt = \lim_{\varepsilon \to 0} (g(u_\varepsilon) - g(u)).$$

We conclude that $\psi$ satisfies the system

$$g^b D_t^\alpha \psi - \Delta \psi = \lim_{\varepsilon \to 0} (g(u_\varepsilon) - g(u)),$$

$$\frac{\partial \psi}{\partial \nu} + \beta \psi + lu = 0 \quad \text{on } \partial\Omega.$$

Set $g(u_\varepsilon) - g(u) = (I) + (II)$ with

$$(I) := \frac{\lambda}{\left( \int_\Omega f(u_\varepsilon) \, dx \right)^2} \int_\Omega f(u_\varepsilon) - f(u) \frac{1}{\varepsilon} \cdot vdx$$

and

$$(II) := \frac{\lambda}{\varepsilon} \left( \frac{1}{\left( \int_\Omega f(u_\varepsilon) \, dx \right)^2} - \frac{1}{\left( \int_\Omega f(u) \, dx \right)^2} \right) \int_\Omega f(u) v \, dx.$$
Using the weak convergence (14), we can prove that

\[(II) \to -2\lambda \int_\Omega f(u)\psi dx \quad \text{as } \varepsilon \to 0.\]

Similarly,

\[(I) \to \lambda \int_\Omega f(u)\psi dx \quad \text{as } \varepsilon \to 0.\]

We conclude that \(\psi\) verifies

\[ab D_\alpha^\alpha \psi - \Delta \psi = -2\lambda \int_\Omega f'(u)\psi dx + \lambda f'(u)\psi \quad \text{in } \Omega,\]

\[\frac{\partial \psi}{\partial \nu} + \beta \psi + lu = 0 \quad \text{on } \partial \Omega.\]

This ends the proof of Theorem 9. \(\square\)

5.1. **Derivation of the adjoint system.** To get the optimality system, we need first to derive the adjoint operator associated with \(\psi\). Let \(v\) be an enough smooth function defined in \(Q_T\). By the first equation of (13), we have

\[\int_\Omega \int_0^T \left(ab D_\alpha^\alpha v - \Delta v\right)\psi dx dt = -2\lambda \int_\Omega f'(u)\psi dx \int_\Omega f(u) v dx dt + \int_{Q_T} \lambda f'(u)\psi \phi dx dt \quad \text{in } \Omega.\]

Integrating by parts, one has

\[\int_\Omega \int_0^T \left(ab D_\alpha^\alpha v - \Delta v\right)\psi dx dt = -2\lambda \int_\Omega f'(u)\psi dx \int_\Omega f(u) v dx dt + \int_{Q_T} \lambda f'(u)\psi \phi dx dt \quad \text{in } \Omega.\]

Introducing the boundary and initial conditions

\[\frac{\partial v}{\partial \nu} + \beta v = 0 \quad \text{on } \partial \Omega \times (0, T), \quad v(x, T) = 0,\]

then function \(v\) satisfies the adjoint system given by

\[-ab D_\alpha^\alpha v - \Delta v = -2\lambda \int_\Omega f(u)\phi dx \int_\Omega f(u) v dx dt + \int_{Q_T} \lambda f'(u)\phi dx dt + 1 \quad \text{in } Q_T,\]

\[\frac{\partial v}{\partial \nu} + \beta v = 0 \quad \text{on } \partial \Omega \times (0, T),\]

\[v(T) = 0,\]

where the 1 appears from differentiation of the integrand of \(J(\beta)\) with respect to the state \(u\).
Remark 10. Given an optimal control $\beta \in U_M$ and the corresponding state $u$, the existence of solution to the adjoint system can be established by imposing additional regularity conditions on the electrical conductivity and following the same procedure we have followed for the existence results of (1).

5.2. Derivation of the optimality system. Gathering equation (1) and the adjoint system (15), we obtain the following optimality system:

$$
\begin{align*}
\frac{u_t}{\Delta} - (u) &= \frac{\lambda f(u)}{\left(\int_{\Omega} f(u) \, dx\right)^2}, \\
-\frac{a^b}{T}D_t^\alpha v - \Delta v &= \frac{-2\lambda \int_{\Omega} f(u)v \, dx}{\left(\int_{\Omega} f(u) \, dx\right)^3} f'(u) + \frac{\lambda f''(u)v}{\left(\int_{\Omega} f(u) \, dx\right)^2} + 1 \text{ in } Q_T, \\
\frac{\partial u}{\partial \nu} + \beta u &= 0 \text{ on } \partial \Omega \times (0,T), \\
\frac{\partial v}{\partial \nu} + \beta v &= 0 \text{ on } \partial \Omega \times (0,T), \\
v(T) &= 0, \quad u(0) = u_0.
\end{align*}
$$

(16)

Remark 11. The existence of solution to the optimality system (16) follows from the existence of solution to the state system (1) and the adjoint system (15), combined with the existence of optimal control.

6. Conclusion

In this paper we investigated an optimal control problem for a nonlocal thermistor problem with a fractional time derivative with nonlocal nonsingular Mittag–Leffler kernel. We proved existence and uniqueness of the control. The optimality system describing the optimal control was discussed.

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