A Bounded Formulation for The School Bus Scheduling

Problem

Liwei Zeng¹, Sunil Chopra², and Karen Smilowitz¹

¹Department of Industrial Engineering and Management Sciences
²Kellogg School of Management
Northwestern University

March 28, 2018

Abstract

This paper proposes a new formulation for the school bus scheduling problem (SBSP) which optimizes starting times for schools and associated bus routes to minimize transportation cost. Specifically, the problem determines the minimum number of buses required to complete all bus routes under the constraint that routes for the same school must arrive within a set time window before that school starts. We present a new integer linear programming (ILP) formulation for this problem which is based on a time-indexed formulation. We develop a randomized rounding algorithm based on the linear relaxation of the ILP that yields near-optimal solutions for large-scale problem instances.

Keywords: school bus scheduling problem; school bus routing problem; time-indexed formulation; randomized rounding algorithm.

1. Introduction

In this paper, we study the school bus scheduling problem, where the goal is to determine starting times for schools and bus routes and assign buses to the routes. This problem, defined in Raff [19] and Fügenschuh [12], merges two subproblems of the broader school bus routing problem (SBRP). The SBRP is a composite of five subproblems: data preparation, bus stop selection, bus route generation, school bell time adjustment, and route scheduling (Park and Kim [18]). Different names for subproblems are used in literature. The school bell time adjustment is called the school
scheduling problem in Desrosiers et al. [9]. We use the term “school bus scheduling problem” from Raff [19] and Fügenschuh [12] to denote the combined problem of school bell time adjustment and route scheduling.

The school bus scheduling problem defined in Raff [19] and Fügenschuh [12] takes as input a set of routes of known length (time duration) for each school. Given the length of each route, the goal of the problem is to determine starting and ending times of schools and routes to increase bus usage and reduce transportation cost. By staggering starting times for schools, one bus may be able to complete routes for different schools, thus reducing the total number of buses needed. Reusing buses reduces cost because the marginal cost of using a bus for a second route is significantly smaller than the cost of adding a new bus. Previous work on this problem applies column generation, cutting plane method and heuristics (Desrosiers et al. [9], Raff [19], Fügenschuh [12]) to obtain solutions. Column generation methods perform well on small instances but can encounter computational challenges as the scale of the problem increases. For large school districts, this can become an issue. The Boston Public School System operates around 3,000 routes with more than 650 buses, at a cost of $120 million per year (Baskin [2]). Heuristic approaches scale well but lack theoretical guarantee. In this paper, we draw on prior work on both the school bus scheduling problem and related machine scheduling problems to design new algorithms with provable performance guarantees that work well for large-scale problem instances.

We present the formal definition of the school bus scheduling problem using the following notation. Consider $N$ schools and $\Gamma_n$ associated routes for each school $n \in [N]$. Let $M$ represent the number of starting time options, given a discretization of the time period over which all schools may start (e.g. 5 or 10 minutes). If school $n \in [N]$ starts at $s_n \in [M]$, all routes for the school must arrive in the interval $[s_n - l_n, s_n]$ where $l_n \in \mathbb{N}$ is a parameter representing the length of the time window over which buses may arrive to school $n$. A time window of $l_n = 20$ minutes indicates that buses may arrive up to 20 minutes before the start of school $n$. We define route length using a finite set of route length types type-1 to type-$K_{\text{max}}$ where a type-$k$ route requires $k$ consecutive time units to complete. $K_{\text{max}}$ represents the maximum allowed length of a bus route. The maximum route length $K_{\text{max}}$ will be less than $M$ since routes longer than $M$ time units must be the last route a bus takes and can be truncated to length of $M$ time units without affecting the total number of buses in the system. In practice, $K_{\text{max}}$ may be significantly smaller than $M$. After assigning a starting time to each route, we assign routes to buses such that routes assigned to the same bus must operate on disjoint time intervals. The goal is to minimize the total number of
buses required to complete all the routes. We define the school bus scheduling problem (SBSP) as follows.

**Problem 1. (School bus scheduling problem (SBSP))** Given $N, M \in \mathbb{N}^+$, assume there are $N$ schools, where the $n^{th}$ school is associated with $\Gamma_n$ routes. Each school $n \in [N]$ chooses a starting time $s_n \in [M]$ such that routes for school $n$ must arrive between $\max\{s_n - l_n, 1\}$ and $s_n$, where $l_n \in \mathbb{N}$ is a given parameter called the time window length for school $n$. After determining a starting time for each route, routes are assigned to buses such that routes assigned to the same bus operate on disjoint time intervals. The objective is to minimize the number of buses to complete all the routes.

As noted in Desrosiers et al. [9] and Raff [19], when all $l_n$ are equal to 0, the SBSP reduces to the school scheduling problem (SSP) defined below.

**Problem 2. (School scheduling problem (SSP))** Given $N, M \in \mathbb{N}^+$, assume there are $N$ schools, where the $n^{th}$ school is associated with $\Gamma_n$ routes. Each school $n \in [N]$ chooses a starting time $s_n \in [M]$ such that routes for school $n$ must arrive at $s_n$. After determining a starting time for each route, routes are assigned to buses such that routes assigned to the same bus must operate on disjoint time intervals. The objective is to minimize the number of buses to complete all the routes.

The SSP is a special case of the SBSP by restricting routes’ arrival times to one specific time, i.e., the school starting time. Both problems have been studied by Desrosiers et al. [9] and Raff [19] where they present an integer linear programming (ILP) formulation that counts the maximum number of routes operating in the same time period. Their formulation creates a binary variable for each feasible schedule of each school and route. They solve small-sized problem instances using column generation approaches.

The SSP is NP-hard, shown from a reduction from the balanced partition problem (Garey and Johnson [13]), indicating that the SBSP is also NP-hard. To solve the SBSP and the SSP, we build an ILP formulation which is based on the time-indexed formulation (Sousa and Wolsey [22]) commonly seen in the machine scheduling literature. Different from the ILP in Desrosiers et al. [9] and Raff [19], the time-window constraints are implied in our ILP so that fewer binary variables are used to represent starting times of routes. We provide provable theoretical bounds using the LP relaxation of the ILP together with a randomized rounding algorithm. The algorithm is presented by taking an optimal fractional solution to the LP relaxation of the ILP and rounding that solution to a feasible integral solution. Detailed analysis on the convex hull of the feasible
region of the ILP provides an error bound for this rounding algorithm, showing its near-optimality for large-scale problem instances.

The remainder of this paper is organized as follows. In Section 2 we review related work on scheduling problems in transportation, and the machine scheduling, and the bin packing problem. In Section 3 we present an ILP formulation for the SBSP and the SSP. In Section 4 we study the SSP and present a randomized rounding algorithm that solves the SSP to near-optimality. This algorithm is modified in Section 5 to solve the SBSP to near-optimality. Section 6 provides numerical results. Finally, we conclude in Section 7 with a summary of results and discussion of future research.

2. Literature Review

We summarize scheduling problems in transportation systems with a special focus on the SBRP and the time-indexed formulation. We also relate our problem to two fundamental combinatorial optimization problems related to our modeling approaches: the bin packing problem and the machine scheduling problem.

Scheduling is a central element of many transportation systems, such as aircraft planning, crane scheduling and the SBRP. In these problems, trips or routes are scheduled on one or multiple vehicles to minimize one or more objectives (see Chen et al. [3], Cheng and Sin [4] for reviews). The aircraft planning problem (Arabeyre et al. [1], Etschmaier and Mathaisel [11]) aims at finding efficient deployment of airline resources to meet customer demands and optimize revenue. The crane scheduling problem (Daganzo [6]) finds the optimal crane allocation scheme that minimizes the total amount of time that ships spend in port. Our work is motivated by the SBRP (e.g., Desrosiers et al. [8], Fügenschuh [12], Newton and Thomas [16], Park and Kim [18], Raff [19]). As a part of the SBRP, the SBSP specifies starting and ending times for each school and associated routes with the objective of minimizing the number of buses required to complete all the routes. Swersey and Ballard [23] present a mixed-integer programming (MIP) formulation for the route scheduling problem where school starting times are fixed and each route is associated with a set time window for its arrival time. The formulation is further simplified to an integer program (IP) by discretizing the time period into small time intervals. Desrosiers et al. [9] solve the school scheduling problem and the route scheduling problem sequentially. They show that the school scheduling problem is equivalent to minimizing the maximum number of routes operating during a single small time interval and formulate the problem as a min max 0-1 program.
which is solved by column generation approaches for small-sized instances. For large-scale instances, they present a heuristic that updates school starting times and route times alternatively. Fügenschuh [12] presents an IP formulation that solves the school scheduling and route scheduling problems simultaneously using cutting plane methods. In this work, we solve the school scheduling and route scheduling problems together with fast algorithms that have provable error bounds. We adopt the time-indexed formulation of Sousa and Wolsey [22] which has been shown to give stronger bounds than other IP or MIP formulations for machine scheduling problems. The formulation is similar to the one used by Desrosiers et al. [9], where binary variables are used to imply starting times of routes. We present an ILP formulation of the SBSP that also takes time window constraints into account. Different from heuristics or column generation approaches in the literature, we leverage the structure of the ILP formulation by studying the LP relaxation of the min max 0-1 program. We provide a randomized rounding algorithm (Raghavan and Tompson [20]) that transfers an optimal fractional solution to the LP relaxation to a feasible integral solution to the ILP. We show that the resulting solution is near-optimal for large-scale problem instances using the Chernoff bound (Chernoff [5]).

Our work is related to machine scheduling problems that consider dependency among jobs, namely job priority. In these problems, jobs with higher priorities must start (or end) earlier than other jobs. Ikura and Gimple [14] studies the batched scheduling problem where jobs in the same batch have the same priority. The SSP can be restated as a similar machine scheduling problem where routes in the same school have the same priority, and therefore, have to start at the same time. Our problem is also similar to the 1-dimensional bin packing (De La Vega and Lueker [7], Scholl et al. [21]) if we view routes as objects and buses as bins. The time window constraint on routes is equivalent to a constraint on objects’ relative location within bins. We design a greedy algorithm for the SSP that is analogous to the first-fit algorithm (Dósa [10]) and prove a constant approximation ratio for the algorithm.

3. An Integer Linear Programming Formulation

In this section, we present an ILP formulation for the SBSP and the SSP with slight changes on problem definitions.

3.1 A Note on Problem Definition for ILP Formulation

When two routes are executed by the same bus, there must be sufficient time to allow the bus to transit from the first route to the second one. We assume a constant time for bus transition. For
convenience, the constant transition time is included in route length (e.g. a 30 minutes route with 10 minutes time transition becomes a 40 minutes route) and we assume there is no transition time in later discussion.

Furthermore, we define the SBSP and the SSP for morning routes where time windows are on ending times. In expanded work for a local school district, we show how one can easily add offsets for afternoon routes where time windows are on starting times and account for varying day length. In Sections 3, 4 and 5 we invert the timeline of scheduling for notational convenience. For the time period $[1, M]$, this inversion transfers time $m$ to $M + 1 - m$ for any $m \in [M]$. Figure 1 (a) illustrates Problems 1 and 2 (morning routes) where each route has an arrival time window based on the school starting time and its time window length. Figure 1 (b) is obtained by inverting Figure 1 (a). A natural way to think of Figure 1 (b) is the afternoon routes. The inversion of timeline converts the arrival time window to a time window on the starting time for each route. The dashed lines represent a time window for ending time and starting time for (a) and (b), respectively.

3.2 ILP Formulation

We present an ILP formulation for the SBSP that includes the SSP as a special case when $l_n = 0, \forall n \in [N]$. The formulation is based on the following proposition from Desrosiers et al. [9] that computes the minimum number of buses to complete routes with known starting and ending times.

**Proposition 1** (Desrosiers et al. [9]). *Given a set of routes with fixed starting and ending times, the minimum number of buses required to complete all the routes is equal to the maximum number of routes in operation in the same time period.*

We note that given the scheduling of buses, the optimal assignment of routes to buses can be found by a greedy algorithm in polynomial time. The algorithm orders routes by starting times and assigns routes to buses sequentially (see Olariu [17] for details). Since all starting times and route lengths are integers in our problem, it suffices to consider unit time intervals $\{[t, t + 1]\}_{t=1}^{M}$. For a given starting time schedule and $t \in [M]$, let $A(t)$ be the number of routes that are operating during time interval $[t, t + 1]$. The objective is to minimize the maximum of $A(t), t \in [M]$. Motivated by
these results, we build a formulation which is based on the time-indexed formulation (Sousa and Wolsey [22]) where decision variables are used to indicate starting times of routes.

For each school \( n \in [N] \), let \( \Gamma_n \) be the number of routes for school \( n \) and let \( r(i, n) \in \mathbb{N}^+ \) be the length of its \( i^{th} \) route. Recall that \( l_n \in \mathbb{N} \) is the time window length for school \( n \). In the following ILP, we introduce binary variables to represent the starting times of routes, i.e., \( x^{(m)}_{i,n} = 1 \) if the \( i^{th} \) route in school \( n \) starts at time \( m \), and 0 otherwise. SBSP can then be formulated as follows:

\[
\begin{align*}
\min & \quad z \\
\text{s.t.} & \quad \sum_{m=1}^{M} x^{(m)}_{i,n} = 1 & \forall n \in [N], i \in [\Gamma_n], \quad (1a) \\
& \quad \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{\min\{m+l_n,M\}} x^{(t)}_{i,n} \leq z & \forall m \in [M], \quad (1b) \\
& \quad x^{(m)}_{i,n} \leq \sum_{t=\max\{m-l_n,1\}}^{\min\{m+l_n,M\}} x^{(t)}_{j,n} & \forall n \in [N], i, j \in [\Gamma_n], m \in [M], \quad (1c) \\
& \quad x^{(m)}_{i,n} \in \{0, 1\} & \forall n \in [N], i \in [\Gamma_n], m \in [M]. \quad (1d)
\end{align*}
\]

Constraints (1a) are the assignment constraints, ensuring that each route is assigned to exactly one starting time. Constraints (1b) determine the objective function value. For any \( m \in [M] \), the left hand of constraints (1b) counts the number of routes that are in operation during time interval \([m, m + 1]\), which then defines the total number of buses needed \( z \) (Proposition 1). Note that the \( i^{th} \) route in school \( n \) is in operation during \([m, m + 1]\) if and only if its starting time is in the time interval \([m - r(i,n) + 1, m]\). Thus, in (1b) we sum \( x^{(t)}_{i,n} \) for all \( t \) in the interval \([\max\{m-r(i,n)+1,1\}, m]\) for all schools and routes to obtain the number of buses operating in time interval \([m, m + 1]\). Constraints (1c) are the school time window constraints. For any school \( n \in [N] \), constraints (1c) force all routes in school \( n \) to start in a time period of length \( l_n \); i.e, if the \( i^{th} \) route starts at \( m \), all other routes for school \( n \) must start between \( m - l_n \) and \( m + l_n \). Finally, constraints (1d) are the binary constraints.

ILP1 is used in Sections 4 and 5 to provide approximate solutions to the SSP and the SBSP, respectively. We show that the optimal fractional solution to the LP relaxation of ILP1 can be used with a randomized rounding algorithm to obtain good solutions for the ILP. A probabilistic
argument shows that the resulting solution is near optimal for large-scale problem instances.

4. Solution Approaches to the SSP

Recall that the SSP is a special case of the SBSP where the time window length equals 0 for each school; i.e., routes in the same school start at exactly the same time. In Section 4.1, we present a 3-approximation algorithm which is similar to the first-fit algorithm (Dósa [10]) for bin packing. In Section 4.2, we introduce a randomized rounding algorithm based on the LP relaxation of ILP1 that achieves near-optimality for large-scale problem instances.

4.1 Greedy Algorithm for the SSP

The greedy algorithm for the SSP is divided into two parts. In Algorithm 1, given a guess of the total number of buses (from Algorithm 2), we assign starting times to schools using a greedy scheduling heuristic. In Algorithm 2, we keep a search interval \([L, U]\) such that the optimal solution to the SSP is lower bounded by a function of \(L\) and upper bounded by a function of \(U\). We iteratively update upper and lower bounds of the search interval (\(L\) and \(U\)) based on results from Algorithm 1, and we use those bounds to update the guess of the optimal solution for the next iteration of Algorithm 1. The algorithm terminates when the gap between \(L\) and \(U\) is at most 1.

Let \(OPT_{SSP}\) be the optimal solution to the SSP and let \(\Gamma_{\text{max}} = \max_{n \in [N]} \Gamma_n\) be the maximum number of routes at any school. In Algorithm 1, two arrays are created to store the number of routes operating in each unit time interval and starting times of all schools. Given \(OPT_{\text{guess}}\), the current guess of \(OPT_{SSP}\), each iteration of Algorithm 1 searches for a feasible solution to the SSP such that the total number of routes in operation during any unit time period is no more than \(OPT_{\text{guess}} + \Gamma_{\text{max}}\). If such a feasible solution is found, we obtain an upper bound for \(OPT_{SSP}\) and the upper bound of the search interval is updated to \(OPT_{\text{guess}}\). If no feasible solution is found, we obtain a lower bound for \(OPT_{SSP}\) and the lower bound of the search interval is updated to \(OPT_{\text{guess}}\). These bounds on the search interval are then used to update \(OPT_{\text{guess}}\) in Algorithm 2. We return to Algorithm 1 if the gap between the current upper and lower bounds of the search interval is greater than 1.

We now analyze Algorithm 1 in greater detail. Each iteration of Algorithm 1 repeats steps 3–6. In each iteration of Algorithm 1, \(s\) represents the first time index for which the total number of routes \(C_s\) in operation during \([s, s+1]\) is less than \(OPT_{\text{guess}}\). If there is a school whose starting time is not assigned, we start this school at time \(s\) (in Step 4) and update the array \(C\). Algorithm 1 stops with an output “Feasible” if all schools have been assigned or an output “Infeasible” if \(s\)
Algorithm 1 (Greedy Scheduling Algorithm)

Input: $OPT_{\text{guess}}$;
Step 1: create an array $C = [C_1, C_2, \ldots, C_M]$ to store the number of routes in each unit time interval $[m, m + 1], m \in [M]$, and an array $[s_1, s_2, \ldots, s_N]$ to store school starting times;
Step 2: initialize $C = [0, 0, \ldots, 0], s = 1$ and counter $n = 1$;
Step 3: if $s > M$, terminate algorithm and return “Infeasible”;
Step 4: while $n < N$, let $s_n = \text{the starting time of school } n$;
Step 5: for each $m \in [M]$, let $\Delta C_m$ be the number of routes in school $n$ that contain time interval $[m, m + 1]$; $C_m = C_m + \Delta C_m$;
Step 6: let $s = \min\{\arg\min\{m : C_m < OPT_{\text{guess}}, M + 1\}, M + 1\}, n = n + 1$, go to Step 3;
Step 7: terminate algorithm, return “Feasible”.

Algorithm 2 (Greedy Algorithm for the SSP)

Step 1: set initial search interval $[L, U]$ where $L = \Gamma_{\text{max}}, U = N \cdot \Gamma_{\text{max}}$;
Step 2: set $OPT_{\text{guess}} = \lfloor L + U \rfloor$;
Step 3: while $U - L > 1$, run Algorithm 1 with $OPT_{\text{guess}}$, otherwise, terminate and output $U$;
Step 4: if Algorithm 1 returns “Feasible”, $U = OPT_{\text{guess}}$, go to Step 2;
Step 5: if Algorithm 1 returns “Infeasible”, $L = OPT_{\text{guess}}$, go to Step 2.

Lemma 1. When Algorithm 1 returns “Feasible”, we have $OPT_{\text{SSP}} \leq OPT_{\text{guess}} + \Gamma_{\text{max}}$; otherwise, we have $OPT_{\text{SSP}} > \frac{OPT_{\text{guess}}}{2}$.

Proof of Lemma 1

We first show that $OPT_{\text{SSP}} \leq OPT_{\text{guess}} + \Gamma_{\text{max}}$ when the output is “Feasible”. Note that if $s_N \leq M$ in Algorithm 1, i.e., all schools and routes start before or at $M$, it suffices to show that for any $t \in [M]$, the number of routes operating in $[t, t + 1]$ is at most $OPT_{\text{guess}} + \Gamma_{\text{max}}$. Let $s_{N+1} = M + 1$ and assume that $s_j \leq t < s_{j+1}$ for some $1 \leq j \leq N$, i.e., routes from the $(j + 1)^{th}$ to the $N^{th}$ school do not operate in $[t, t + 1]$. Therefore, the number of routes that contain $[t, t + 1]$ is at most the number of routes that contain $[s_j, s_j + 1]$. According to the starting time assignment rule (Step 4 in Algorithm 1), at the time we assign starting time $s_j$ to the $j^{th}$ school, the number of routes that contain $[s_j, s_j + 1]$ is at most $OPT_{\text{guess}}$. Since the $j^{th}$ school has no more than $\Gamma_{\text{max}}$ routes, the number of routes that contain $[s_j, s_j + 1]$ is at most $OPT_{\text{guess}} + \Gamma_{\text{max}}$.

We next show that $OPT_{\text{SSP}} > \frac{OPT_{\text{guess}}}{2}$ if Algorithm 1 returns “Infeasible”. In this case, there is at least one school yet to be assigned a starting time but all time periods between 1 and $M$ have at least $OPT_{\text{guess}}$ buses operating. Thus, the total time duration of all routes (across all schools) must be at least $M \cdot OPT_{\text{guess}}$. 

exceeds $M$ at some point. Lemma 1 details how the upper and lower bounds of the search interval provide bounds on $OPT_{\text{SSP}}$. 
Recall that all schools must start by $M$ and no route is longer than $M$. Thus, at optimality all routes must be completed by time $2M$. Thus, the total duration of all routes (across all schools) is at most $2M \cdot \text{OPT}_{SSP}$.

Each time we return to Algorithm 2, either the upper bound of the search interval has decreased or the lower bound has increased. We now show that $U$ obtained at the conclusion of Algorithm 2 (when the gap between the upper and lower bounds of the search interval is at most 1) provides a 3-approximation of $\text{OPT}_{SSP}$.

**Theorem 1 (Approximation ratio of Algorithm 2).** Let $U$ be the output of Algorithm 2. We have

$$\frac{U + \Gamma_{\text{max}}}{3} \leq \text{OPT}_{SSP} \leq U + \Gamma_{\text{max}}.$$

**Proof of Theorem 1.** We start with trivial lower and upper bounds of $\text{OPT}_{SSP}$. Note that routes in the same school must be assigned to different buses, we have $\text{OPT}_{SSP} \geq \Gamma_{\text{max}}$. Besides, $\text{OPT}_{SSP}$ is no larger than the total number of routes, hence $\text{OPT}_{SSP} \leq N \cdot \Gamma_{\text{max}}$.

We also note that Algorithm 1 returns “Feasible” with $\text{OPT}_{\text{guess}} = N \cdot \Gamma_{\text{max}}$ since all schools will be assigned to starting time 1. If Algorithm 1 returns “Feasible” with $\text{OPT}_{\text{guess}} = \Gamma_{\text{max}}$, the lower bound of the search interval $L$ in Algorithm 2 never updates so that the output $U$ is either $\Gamma_{\text{max}}$ or $\Gamma_{\text{max}} + 1$ and the statement above naturally holds.

If Algorithm 1 returns “Infeasible” with $\text{OPT}_{\text{guess}} = \Gamma_{\text{max}}$, from Steps 4 and 5 of Algorithm 2, the pair $(L, U)$ in any iteration of Algorithm 2 satisfies that Algorithm 1 returns “Feasible” with $\text{OPT}_{\text{guess}} = U$ and “Infeasible” with $\text{OPT}_{\text{guess}} = L$. Since Algorithm 2 terminates at $(L, U)$ such that $U - L \leq 1$, from Lemma 1 we have $\text{OPT}_{SSP} \leq U + \Gamma_{\text{max}}$ and $\text{OPT}_{SSP} > \frac{L}{2}$, i.e., $\text{OPT}_{SSP} \geq \frac{L + 1}{2} \geq \frac{U}{2}$. Combining this inequality with the fact that $\text{OPT}_{SSP} \geq \Gamma_{\text{max}}$, we have $\text{OPT}_{SSP} \geq \frac{1}{3}(\frac{U}{2} + \frac{U}{2} + \Gamma_{\text{max}}) = \frac{U + \Gamma_{\text{max}}}{3}$.

Since Algorithm 1 provides a feasible schedule with at most $U + \Gamma_{\text{max}}$ buses, this is a 3-approximation solution to the SSP.

### 4.2 Randomized Rounding Algorithm for the SSP

We present a randomized rounding algorithm based on the optimal solution to the LP relaxation of an adaptation of [ILPI]. We show that the randomized rounding algorithm solves large-scale problem instances to near optimality.
We adapt [ILP1] for the SSP taking advantage of the fact that $l_n = 0$ for all $n \in [N]$. When $l_n = 0$ for all $n \in [N]$, constraints (1c) are equivalent to $x_{i,n}^{(m)} \leq x_{j,n}^{(m)}$ for all $n \in [N], i, j \in [\Gamma_n]$. A similarly argument shows that $x_{i,n}^{(m)} = x_{i,n}^{(m)}$ for all $i, j \in [\Gamma_n]$. This leads to the following formulation for the SSP.

$$\min z$$

s.t.

1. $\sum_{m=1}^{M} x_{i,n}^{(m)} = 1 \quad \forall n \in [N], i \in [\Gamma_n]$, (2a)
2. $\sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} x_{i,n}^{(t)} \leq z \quad \forall m \in [M]$, (2b)
3. $x_{i,n}^{(m)} = x_{i,n}^{(m)} \quad \forall n \in [N], i, j \in [\Gamma_n], m \in [M]$, (2c)
4. $x_{i,n}^{(m)} \in \{0, 1\} \quad \forall n \in [N], i \in [\Gamma_n], m \in [M]$. (2d)

Let LP2 be the natural linear relaxation of [ILP2] and let $(x^{*}, z_{LP2}^{*})$ be an optimal solution to LP2. From (2a), $\sum_{m=1}^{M} x_{i,n}^{(m)} = 1$ for all $n \in [N]$ and $i \in [\Gamma_n]$. Thus, we randomly assign all routes in school $n$ to starting time $m$ with probability $x_{i,n}^{(m)}$ (this is independent of $i$ due to equation (2c)). The starting time assignment for each school is done independently.

Algorithm 3 (Randomized Rounding Algorithm for the SSP)

Step 1: solve the LP relaxation of [ILP2] to obtain an optimal solution $x^{*} = (x_{1}^{*}, \cdots, x_{N}^{*})$ where $x_{n}^{*}$ is a vector corresponding to all route variables for school $n$;

Step 2: $\forall n \in [N]$, assign school $n$ to starting time $m$ with probability $x_{i,n}^{(m)}$, $m = 1, 2, \cdots, M$, independent of all other schools.

Theorem 2 provides an error bound for the randomized rounding procedure summarized in Algorithm 3.

**Theorem 2.** With probability at least $\frac{1}{2}$, the assignment in Algorithm 3 yields at most $z_{rand}$ buses, where

$$z_{rand} = z_{LP2}^{*} + \sqrt{2\Gamma_{\max} \log(2M) z_{LP2}^{*} + \Gamma_{\max} \log(2M)}. \quad (1)$$

We note that Theorem 2 does not provide a constant approximation ratio since $\Gamma_{\max} \log(2M)$ can be much larger than $OPT_{SSP}$. However, in a large-scale instance where $OPT_{SSP} >> \Gamma_{\max} \log(2M)$, $z_{rand}$ is asymptotically near-optimal.

We use the following inequality to prove Theorem 2.
Lemma 2. (Chernoff Bound \cite{5}) Let \( x_1, x_2, \cdots, x_n \) be \( n \) independent random variables in \([0, T]\) and \( X = \sum_{i=1}^{n} x_i \) be the sum of these random variables with mean \( E[X] = \mu \). For any \( \lambda > 0 \),

\[
Pr(X > \mu + \lambda) \leq \exp\left(-\frac{\lambda^2}{(2\mu + \lambda)T}\right). \tag{2}
\]

Proof of Theorem 2. Let \( \hat{x} \) be the resulting integral solution of the rounding procedure in Algorithm 3. Note that the objective value \( z^*_{LP2} \) equals to

\[
\max_{m \in [M]} \left\{ \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} \hat{x}_{i,n}^{(t)} \right\}.
\]

From the definition of \( \Gamma_{max} \), \( \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} \hat{x}_{i,n}^{(t)} \) is a random variable in \([0, \Gamma_{max}]\) for each \( m \in [M] \) and \( n \in [N] \). Therefore, \( \sum_{n=1}^{N} \left( \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} \hat{x}_{i,n}^{(t)} \right) \) is the sum of \( N \) independent random variables in \([0, \Gamma_{max}]\) with mean less than or equal to \( z^*_{LP2} \).

Let \( \lambda^* \) be the positive root of

\[
\exp\left(-\frac{(\lambda^*)^2}{(2z^*_{LP}+\lambda^*)\Gamma_{max}}\right) = \frac{1}{2M}.
\]

Thus,

\[
\lambda^* = \frac{\Gamma_{max} \log(2M) + \sqrt{((\Gamma_{max} \log(2M))^2 + 8z^*_{LP}\Gamma_{max} \log(2M))}}{2} \tag{3}
\]

\[
\leq \sqrt{2z^*_{LP}\Gamma_{max} \log(2M)} + \Gamma_{max} \log(2M). \tag{4}
\]

By Lemma 2, we have \( Pr\left( \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} \hat{x}_{i,n}^{(t)} > z_{rand} \right) < \frac{1}{2M} \) for each \( m \in [M] \). Applying a union bound over all \( m \in [M] \), we have

\[
\max_{m \in [M]} \left\{ \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=\max\{m-r(i,n)+1,1\}}^{m} \hat{x}_{i,n}^{(t)} \right\} \leq z_{rand}
\]

with probability at least \( \frac{1}{2} \).

Note that the probability \( \frac{1}{2} \) in Theorem 2 can be boosted up to \( 1 - \varepsilon \) by repeating the algorithm \( \log(\frac{1}{\varepsilon}) \) times and selecting the best solution.

5. Solution Approaches to the SBSP

With the introduction of time windows, there is no straightforward way to adapt the greedy heuristic for the SSP to the SBSP. Note that one bus can perform multiple routes for the same school when its time window length is larger than route lengths. This suggests that \( \Gamma_{max} \) is no longer a lower bound of the optimal solution to the SBSP, which invalidates the lower bound statement in Theorem 1. It is also important to observe that the randomized rounding algorithm for the SSP in Algorithm 3 cannot be applied directly to the SBSP because the structure of \( ILP1 \) is more complicated than that of \( ILP2 \) due to constraints (1c). In this section, we present a stronger ILP for-
mulation for the SBSP and use its LP relaxation (LP3) to design a randomized rounding algorithm that achieves a bound similar to that in Theorem 2.

5.1 Randomized Rounding Algorithm for the SBSP

In our modified formulation for the SBSP, we replace constraints (1c) in ILP1 with constraints (3c) in LP3. We provide an intuitive explanation of constraints (3c). Consider any value \( \bar{m} \in [M] \), given that the time window length of school \( n \) is \( l_n \), the difference in starting times of any two routes for this school must be at most \( l_n \). Thus, if route \( i \in \Gamma_n \) starts by time \( \bar{m} \) (represented by the left hand side of (3c)), any route \( j \neq i \in \Gamma_n \) must start by \( \bar{m} + l_n \) (represented by the right hand side of (3c)).

\[
\begin{align*}
\min \quad & z \\
\text{s.t.} \quad & \sum_{m=1}^{M} x_{i,n}^{(m)} = 1 \quad \forall n \in [N], i \in \Gamma_n, \quad (3a) \\
& \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \sum_{t=1}^{\max\{m-r(i,n)+1,1\}} x_{i,n}^{(t)} \leq z \quad \forall m \in [M], \quad (3b) \\
& \sum_{t=1}^{\min\{\bar{m}+l_n,M\}} x_{i,n}^{(t)} \leq \sum_{t=1}^{\min\{\bar{m}+l_n,M\}} x_{j,n}^{(t)} \quad \forall n \in [N], i, j \in \Gamma_n, m \in [M], \bar{m} \in [M], \quad (3c) \\
& 0 \leq x_{i,n}^{(m)} \leq 1. \quad \forall n \in [N], i \in \Gamma_n, m \in [M]. \quad (3d)
\end{align*}
\]

Let \( x_n \in \mathbb{R}^{\Gamma_n \times M} \) be the set of decision variables associated with school \( n \), i.e., \( \{x_{i,n}^{(m)}\} \) where \( i \in \Gamma_n, m \in [M] \). Let \( X_n \subseteq \{0,1\}^{\Gamma_n \times M} \) be the set of 0-1 vectors that satisfy constraints (1a), (1c) and (1d) and let \( \text{Conv}(X_n) \) be its convex hull. Let \( P_n = \{x_n \mid x_n \text{ satisfies } (3a), (3c), (3d)\} \).

Theorem 3 shows that \( P_n = \text{Conv}(X_n) \).

**Theorem 3.** \( \text{Conv}(X_n) = P_n = \{x_n \mid x_n \text{ satisfies } (3a), (3c), (3d)\} \).

**Proof of Theorem 3** We first show that \( X_n \) is the set of all integer points in \( P_n \). For any \( x_n \in X_n \), we prove that inequality (3c) holds for any \( i, j \in \Gamma_n, \bar{m} \in [M] \). Let \( t_i, t_j \) be indices such that \( x_{i,n}^{(t_i)} = 1 \) and \( x_{j,n}^{(t_j)} = 1 \). In constraints (1c), let \( m = t_i \). We know that

\[
t_i - l_n \leq t_j \leq t_i + l_n.
\]
Similarly, we have

\[ t_j - l_n \leq t_i \leq t_j + l_n. \]

Therefore, \(|t_i - t_j| \leq l_n\), which naturally implies that \(x_n\) satisfies (3c). Note that constraints (3a) and (3d) hold for all \(x_n \in X_n\). Thus, we have \(X_n \subseteq P_n\).

We now show that for any integer point \(x_n \in P_n\), the inequalities (1c) are true for all \(i, j \in [\Gamma_n], m \in [M]\), and therefore, \(x_n \in X_n\). We discuss two cases based on the value of \(m - l_n\).

If \(m - l_n \leq 1\), we have \(\max\{m - l_n, 1\} = 1\) and

\[
x^{(m)}_{i,n} \leq \sum_{t=1}^{m} x^{(t)}_{i,n} \leq \sum_{t=1}^{\min\{m+l_n,M\}} x^{(t)}_{j,n} = \sum_{t=\max\{m-l_n,1\}}^{\min\{m+l_n,M\}} x^{(t)}_{j,n}.
\]

If \(m - l_n > 1\), \(\max\{m - l_n, 1\} = m - l_n\).

In inequalities (3c), let \(\tilde{m} = m\), we have

\[
\sum_{t=1}^{m} x^{(t)}_{i,n} \leq \sum_{t=1}^{\min\{m+l_n,M\}} x^{(t)}_{j,n}.
\]  \hspace{1cm} (5)

Let \(\tilde{m} = m - l_n - 1\) and swap indices \(i, j\) to get

\[
\sum_{t=1}^{m-l_n-1} x^{(t)}_{j,n} \leq \sum_{t=1}^{m-1} x^{(t)}_{i,n}.
\]  \hspace{1cm} (6)

Summing up (5) and (6), we have

\[
\sum_{t=1}^{m} x^{(t)}_{i,n} + \sum_{t=1}^{m-l_n-1} x^{(t)}_{j,n} \leq \sum_{t=1}^{\min\{m+l_n,M\}} x^{(t)}_{j,n} + \sum_{t=\max\{m-l_n,1\}}^{m-1} x^{(t)}_{i,n},
\]  \hspace{1cm} (7)

which is equivalent to

\[
x^{(m)}_{i,n} \leq \sum_{t=m-l_n}^{\min\{m+l_n,M\}} x^{(t)}_{j,n} = \sum_{t=\max\{m-l_n,1\}}^{\min\{m+l_n,M\}} x^{(t)}_{j,n},
\]  \hspace{1cm} (8)

Thus, (8) implies that the integer point \(x_n\) satisfies inequalities (1c) and belongs to \(X_n\).

We next show that all extreme points of \(P_n\) are integral, implying \(P_n = \text{Conv}(X_n)\). We do so by proving that the matrix corresponding to constraints (3a), (3c) and (3d) is totally unimodular (TU). In order to do so, we first transform the \(x\) variables to a new set of variables.
For any school $n$, let $S_{i,n}^{(m)} = \sum_{t=1}^{m} x_{i,n}^{(t)}, i \in [\Gamma_n], m \in [M]$. As a result of this transformation, for each school $n$, constraints (3a), (3c) and (3d) are transformed to

\[
S_{i,n}^{(\bar{m})} \leq S_{j,n}^{(\min\{\bar{m}+l_n,M\})} \quad \forall i,j \in [\Gamma_n], \bar{m} \in [M], \quad (3e)
\]

\[
S_{i,n}^{(m-1)} \leq S_{i,n}^{(m)} \quad \forall i \in [\Gamma_n], m \in [M] \setminus \{1\}, \quad (3f)
\]

\[
S_{i,n}^{(M)} = 1 \quad \forall i \in [\Gamma_n], \quad (3g)
\]

\[
S_{i,n}^{(m)} \geq 0 \quad \forall i \in [\Gamma_n], m \in [M], \quad (3h)
\]

Define $P_{S,n} = \{ S \mid S \text{ satisfies (3e) – (3h)} \}$. Note that there is a one-to-one mapping between the extreme points of $P_n$ and $P_{S,n}$. As a result, it suffices to show that the matrix corresponding to constraints (3e)–(3h) is TU. We use the following properties of TU matrices (Nemhauser and Wolsey [15]) in our proof.

**Proposition 2.** If the $(0,1,-1)$ matrix $A$ has no more than two nonzero entries in each column, and if $\sum_i a_{ij} = 0$ if column $j$ contains two nonzero coefficients, then $A$ is TU.

**Proposition 3.** If matrix $A$ is TU, $A^T$ is TU.

**Proposition 4.** If matrix $A$ is TU, $(A,I)$ is TU.

Let $M_S$ be the matrix corresponding to constraints (3e) and (3f). Since each row of $M_S$ contains exactly one +1 and one -1, $M_S^T$ satisfies the condition in Proposition 2 and is thus TU. Furthermore, constraints (3g) and (3h) represent an identity matrix. From Propositions 3 and 4, the matrix corresponding to constraints (3e)–(3h) is thus TU. This implies that $P_{S,n}$ and thus $P_n$ have only integer extreme points.

We now use Theorem 3 to derive a randomized rounding algorithm based on the optimal solution to $LP_3$. We first show that $LP_3$ has an error bound similar to that obtained in Theorem 2.

**Theorem 4 (Error Bound of $LP_3$).** Let $OPT$ and $LP_{OPT}$ be the optimal solutions to the SBSP and $LP_3$, respectively. We have

\[
OPT \leq OPT_{LP} + \sqrt{2\Gamma_{\max} \log(2M)OPT_{LP}} + \Gamma_{\max} \log(2M).
\]

An important step in the proof of Theorem 4 is to round an optimal solution of $LP_3$ in a way that feasibility is maintained. Observe that a simple rounding of the $x$ variables as in Algorithm 3
may result in violation of constraints (3c). Thus, rounding must be done in a way that maintains feasibility.

In order to do so, we exploit the equivalence between PS_n (using S variables) and P_n (using x variables). Given an optimal solution \( x^* \) to \( \text{LP3} \) we first construct the equivalent solution \( S^* \) as shown in Algorithm 4. We then randomize over \( S^* \) to decide the x variables on which to round up or down (as shown in Algorithm 4). This method of rounding allows us to maintain feasibility and prove Theorem 4.

**Algorithm 4 (Randomized Rounding Algorithm for the SBSP)**

Step 1: solve \( \text{LP3} \) to yield an optimal solution \( x^* = (x^{(1)}_1, \ldots, x^{(N)}_N) \) where \( x^{(i)}_n \) is a vector corresponding to all variables for school \( n \), let \( S^{(m)}_{i,n} = \sum_{t=1}^{m} x^{(t)}_{i,n} \) for all \( i \in [N], m \in [M] \);

Step 2: \( \forall n \in [N], \) generate a random variable \( \gamma_n \sim U[0, 1] \);

Step 3: \( \forall i \in \Gamma_n, \) let \( t_i = \arg\min\{m : S^{(m)}_{i,n} \geq \gamma_n \} \) and let \( x^{(t_i)}_{i,n} = 1, x^{(m)}_{i,n} = 0 \) if \( m \neq t_i \).

Figure 2 illustrates Algorithm 4 focusing on school \( n \). For the \( i^{th} \) route in this school, we cut a [0, 1] interval into \( M \) pieces with lengths \( x^{(m)}_{i,n}, m = 1, 2, \ldots, M \). Let \( S^{(m)}_{i,n} = \sum_{t=1}^{m} x^{(t)}_{i,n} \) for all \( m \in [M] \). The [0, 1] interval is cut at \( S^{(m)}_{i,n}, m = 1, 2, \ldots, M \) as shown in Figure 2. We then pick a random number \( \gamma_n \sim U[0, 1] \) and assume that \( \gamma_n \) falls into the line segment \( [S^{(t_i-1)}_{i,n}, S^{(t_i)}_{i,n}] \) (see Figure 2), the \( i^{th} \) route is then assigned to starting time \( t_i \), where \( t_i = \arg\min\{m : S^{(m)}_{i,n} \geq \gamma_n \} \).

In other words, \( S^{(t_i)}_{i,n} \) is the first cutting point in the [0, 1] interval that is greater than or equal to \( \gamma_n \).

Let \( x \) be the integral solution generated from Algorithm 4, we prove the following propositions.

**Proposition 5 (Correctness of Algorithm 4).** The solution \( x \) satisfies constraints (3a), (3c) and (3d).

*Proof.* For each \( n \in [N], i \in \Gamma_n \), exactly one of \( \{x^{(m)}_{i,n}\}_{m=1}^{M} \) is 1 while all others are 0. Therefore, \( x \) satisfies constraints (3a) and (3d). To prove \( x \) also satisfies constraints (3c), it suffices to show that \( t_i - t_j \leq l_n \) for any \( i, j \in \Gamma_n, n \in [N] \). If \( t_i - t_j > l_n \), from \( t_i = \arg\min\{m : S^{(m)}_{i,n} \geq \gamma_n \} \) and \( t_j = \arg\min\{m : S^{(m)}_{j,n} \geq \gamma_n \} \), we have \( S^{(t_j)}_{j,n} \geq S^{(t_i-1)}_{i,n} \geq S^{(t_i+l_n)}_{i,n} \). This contradicts to the fact that \( x^* \) satisfies constraints (3c). \( \square \)

**Proposition 6 (Probabilistic Property of Algorithm 4).** \( \Pr(x^{(m)}_{i,n} = 1) = x^{(m)}_{i,n} \) for any \( m \in [M], n \in [N], i \in \Gamma_n \).
Proof. Let $S_{i,n}^{(0)} = 0$ for any $i \in \Gamma_n$, $n \in [N]$.

\[ \Pr(x_{i,n}^{(m)} = 1) = \Pr(S_{i,n}^{(m-1)} < \gamma_n \leq S_{i,n}^{(m)}) = S_{i,n}^{(m)} - S_{i,n}^{(m-1)} = x_{i,n}^{*}(m) . \]

Proposition 5 shows that Algorithm 4 always outputs a feasible starting time schedule and Proposition 6 proves the probabilistic property needed in the proof of Theorem 4. This completes the algorithmic perspective of the solution approach and shows the error bound in Theorem 4 can be reached by a polynomial algorithm.

Proof of Theorem 4 Let $x^* = (x_1^*, \ldots, x_N^*)$ be an optimal solution to LP3 where $x_n^*$ is a vector corresponding to all variables for school $n$. From Theorem 3, each $x_n^* \in \text{Conv}\{X_n\}$ and can be represented as a linear combination of extreme points of $X_n$, i.e.,

\[ x_n^* = \sum_k \lambda_k^n x_k^n, \quad \sum_k \lambda_k^n = 1, \quad x_k^n \in X_n. \]

Since each extreme point $x_k^n$ corresponds to a feasible starting time schedule for school $n$, we round $x_n$ to $x_k^n$ with probability $\lambda_k^n$ for all $n \in [N]$. Under this rounding scheme, the probability of $x_{i,n}^{(m)} = 1$ is exactly $x_{i,n}^{*}(m)$. Following the proof of Theorem 2 leads to the error bound.

As a final remark, we are able to handle constraints that restrict route starting times. Note that the $i^{th}$ route in school $n$ must (or must not) start at $m$ is equivalent to $S_{i,n}^{(m)} - S_{i,n}^{(m-1)} = 1$ ($S_{i,n}^{(m)} - S_{i,n}^{(m-1)} = 0$). These constraints all satisfy the condition in Proposition 2, hence, do not affect the total unimodularity of the corresponding matrix using $S$ variables. Therefore, all results in

![Figure 2: Randomized rounding algorithm for school $n \in [N]$ given $S$ and random number $\gamma_n$](image-url)
6. Numerical Study

We test the strength of the LP relaxation and the randomized rounding algorithm for the SBSP through numerical experiments. All experiments are conducted through Python v2.7 and Gurobi v7.5.

6.1 An Equivalent Formulation of LP3

We use the $S$ variables rather than the $x$ variables with an equivalent LP of LP3 for all experiments. The equivalent LP is based on the $S$ variables defined in the proof of Theorem 3 and has higher sparsity. Recall that $S_{i,n}(m) = \sum_{t=1}^{\max\{m-r(i,n)+1,1\}} x_{i,n}^{(t)}$, $i \in [\Gamma_n], m \in [M]$ and that constraints (3a), (3c) and (3d) are equivalent to (3e)–(3f). Since $\sum_{t=\max\{m-r(i,n)+1,1\}}^{m} x_{i,n}^{(t)} = S_{i,n}(m) - S_{i,n}(\max\{m-r(i,n),0\})$ (here we define $S_{i,n}(0) = 0$), constraints (3b) can be transformed into $S$ variables and LP3 is equivalent to LP3’ shown below:

\[
\begin{align*}
\min \quad & z \\
\text{s.t.} \quad & S_{i,n}(\tilde{m}) \leq S_{i,n}(\min\{\tilde{m}+l_n,M\}) \quad \forall i,j \in [\Gamma_n], \tilde{m} \in [M], \quad (3e) \\
S_{i,n}(m-1) & \leq S_{i,n}(m) \quad \forall i \in [\Gamma_n], m \in [M] \setminus \{1\}, \quad (3f) \\
S_{i,n}(M) & = 1 \quad \forall i \in [\Gamma_n], \quad (3g) \\
S_{i,n}(m) & \geq 0 \quad \forall i \in [\Gamma_n], m \in [M], \quad (3h) \\
\Gamma_n \sum_{n=1}^{N} \sum_{i=1}^{\Gamma_n} \left( S_{i,n}(m) - S_{i,n}(\max\{m-r(i,n),0\}) \right) & \leq z \quad \forall m \in [M]. \quad (3i)
\end{align*}
\]

Note that each $\sum_{t} x_{i,n}^{(t)}$ is replaced with two $S$ variables for fixed $i$ and $n$, the new LP is much sparser than LP3.

6.2 Experimental Design

We design problem instances with different scales and parametric distributions. Recall that $M$ is the number of starting time options, $N$ is the number of schools, $\Gamma_{\text{max}}$ is the maximum number of routes in one school. We test four different problem sizes summarized in Table 1.

Given $M, N$ and $\Gamma_{\text{max}}$, we generate the following parameters in four different ways: $\Gamma_n$ as the number of routes in school $n$, $l_n$ as the time window length of school $n$, $r(i,n)$ as the length of...
| Size | M  | N  | $\Gamma_{\text{max}}$ |
|------|----|----|-----------------------|
| 1    | 10 | 5  | 50                    |
| 2    | 30 | 50 | 50                    |
| 3    | 50 | 50 | 100                   |
| 4    | 50 | 100| 100                   |

Table 1: Problem sizes

the $i^{th}$ route in school $n$. $U(a, b)$ stands for a uniform distribution (rounded to the nearest integer) in interval $[a, b]$.

Type 1 [base model]: $\Gamma_n \sim U(1, \Gamma_{\text{max}})$, $l_n \sim U(1, M)$, $r_{i,n} \sim U(1, M)$.

Type 2 [short time window model]: $l_n \sim U(1, M/3)$, others are same as type 1.

Type 3 [short route length model]: $\Gamma_n \sim U(1, \Gamma_{\text{max}}/3)$, others are same as type 1.

Type 4 [mixed school model]: $\Gamma_n \sim U(1, \Gamma_{\text{max}}/3)$ for $n \leq N/2$, $\Gamma_n \sim U(2\Gamma_{\text{max}}/3, \Gamma_{\text{max}})$ for $n > N/2$, others are same as type 1.

Type 1 is the base model where all parameters follow a uniform distribution from 1 to the maximum possible value; type 2 is the short time window model which restricts scheduling compared to type 1; type 3 is the short route model which provides more feasible schedules compared to type 1; type 4 is the mixed school model where half of the schools have fewer routes and others have more. All four types of parameter distributions capture features of real problem instances and we test all of them to illustrate the robustness of our results.

6.3 Numerical Results and Analysis

For each size/type combination we generate five random problem instances for a total of 80 problem instances ($4 \times 4 \times 5$). The detailed results for each problem instance are reported in the Appendix in Tables 3-6. As shown in Tables 3-6, $\text{LP}_{\text{OPT}}$ is the optimal LP value, Rounding is the objective function of the best solution obtained after repeating the rounding Algorithm 4 a thousand times, and $\text{IP}_{\text{OPT}}$ is the optimal solution to the IP obtained by letting Gurobi solve the integer program to optimality (when it is able to do so within 8 hours). Whereas Gurobi is able to solve problem instances of size 1 and 2 to optimality within 8 hours, it is not able to do so for larger problem instances of size 3 and 4. We obtain the LP Gap, the Rounding Gap, and the Total
Gap as follows:

\[
\text{LP Gap} = \frac{\text{IP}_{\text{OPT}} - \text{LP}_{\text{OPT}}}{\text{LP}_{\text{OPT}}} \quad (9)
\]
\[
\text{Rounding Gap} = \frac{\text{Rounding} - \text{IP}_{\text{OPT}}}{\text{IP}_{\text{OPT}}} \quad (10)
\]
\[
\text{Total Gap} = \frac{\text{Rounding} - \text{LP}_{\text{OPT}}}{\text{LP}_{\text{OPT}}} \quad (11)
\]

A summary of the results from Tables 3–6 is provided in Table 2. The Average LP Gap, Average Rounding Gap, and the Average Total Gap correspond to the average for the five problem instances of each type/size combination. The Max Total Gap corresponds to the maximum total gap across the corresponding five problem instances.

| Type          | Size | Average LP Gap | Average Rounding Gap | Average Total Gap | Max Total Gap |
|---------------|------|----------------|----------------------|-------------------|--------------|
| (1) Base model| 1    | 1.90%          | 11.27%               | 13.39%            | 23.04%       |
|               | 2    | 0.14%          | 0.72%                | 0.86%             | 1.94%        |
|               | 3    | —              | —                    | 0.46%             | 0.92%        |
|               | 4    | —              | —                    | 0.69%             | 1.47%        |
| (2) Short time window | 1 | 4.96%          | 5.52%                | 10.75%            | 17.55%       |
|               | 2    | 0.38%          | 1.85%                | 2.24%             | 3.11%        |
|               | 3    | —              | —                    | 1.99%             | 2.33%        |
|               | 4    | —              | —                    | 1.19%             | 1.53%        |
| (3) Short route | 1 | 1.02%          | 23.49%               | 24.75%            | 50.84%       |
|               | 2    | 0.15%          | 5.51%                | 5.66%             | 11.00%       |
|               | 3    | —              | —                    | 4.89%             | 9.62%        |
|               | 4    | —              | —                    | 4.40%             | 6.26%        |
| (4) Mixed school | 1 | 2.07%          | 22.30%               | 24.83%            | 34.06%       |
|               | 2    | 0.13%          | 0.59%                | 0.72%             | 1.98%        |
|               | 3    | —              | —                    | 1.67%             | 4.24%        |
|               | 4    | —              | —                    | 0.53%             | 0.85%        |

Table 2: Summary of numerical results

The first observation from our numerical results is that the LP relaxation provided by LP3 and LP3 is very good and provides an LP optimal value that is quite close to the integer optimum. This is evident from the small Average LP Gap for problem instances of size 1 and 2 (where we were able to obtain the IP optimum) and the small Average Total Gap for the larger problem instances of size 3 and 4 (where we were unable to obtain the IP optimum within 8 hours). In instances where the Average Total Gap is large (small instances of size 1), most of the gap arises from the inability of the rounding algorithm to find a good solution. The small LP gap can be explained by the fact that constraints (3e)–(3h) define the convex hull of all feasible $S$ vectors and that only constraints (3i) may not be facet-defining inequalities for the $(S, z)$ polytope and create fractional extreme points. This suggests the possibility of finding valid cuts that eliminate...
fractional solutions brought by constraints (3i) to shrink the total gap.

The second observation from our numerical results is that the rounding algorithm performs quite well as problem size increases. Even with short routes where the rounding algorithm is not expected to do very well, we obtain Average Total Gap below five percent for large problem instances (size 3 and 4). Theorem 4 shows that the randomized rounding algorithm is near-optimal when $OPT >> \Gamma_{\max} \log(2M)$, i.e., the relative gap between $IP_{OPT}$ and the rounding value decreases quickly as the problem size increases. Our numerical results support this claim for all four types of problem instances.

In practice, for small instances (sizes 1 and 2), we solve the IP directly to get the exact optimal solution; for large instances where solving the IP may not be tractable (sizes 3 and 4), we are able to get good solution bounds by solving the LP relaxation and performing the randomized rounding algorithm.

7. Discussion

In this paper, we study the school bus scheduling problem which finds optimal starting time schedule to reduce transportation cost. We leverage work from earlier school bus routing studies and the machine scheduling problem to design efficient algorithm with provable performance guarantee. For the school scheduling problem, where all time window lengths equal to zero, we present a 3-approximation greedy algorithm. For the school bus scheduling problem, we develop an ILP formulation based on the time-indexed formulation. Based on the LP relaxation to the ILP, we provide a randomized rounding algorithm that achieves near-optimality for large-scale problems. Numerical study suggests that the performance of the rounding algorithm can be much better than the theoretical guarantee even for large-scale instances. Note that the randomized rounding algorithm naturally provides several options to compare which gives more flexibility for decision makers. We feel that the approaches developed here may also be applied to other scheduling problems with similar structures.

In this work, we assume that all routes are formed and are taken as inputs for the scheduling problem. For future research, we would like to build a unified framework that includes both route design and starting time scheduling where we need to construct routes in the first stage. Another direction is to consider problems with route or location dependent transition times. Both generalizations capture real life problems and we will continue to work with the school district to provide solution approaches that are robust and easy to implement.
Acknowledgement

This project is supported by the National Science Foundation (CMMI-1727744). The authors thank Konstantin Makarychev for useful discussion.
References

[1] JP Arabeyre, J Fearnley, FC Steiger, and W Teather. The airline crew scheduling problem: A survey. *Transportation Science*, 3(2):140–163, 1969.

[2] Kara Baskin. Creating better bus routes with algorithms, 2017. URL [http://mitsloan.mit.edu/newsroom/articles/creating-better-bus-routes-with-algorithms/](http://mitsloan.mit.edu/newsroom/articles/creating-better-bus-routes-with-algorithms/).

[3] Bo Chen, Chris N Potts, and Gerhard J Woeginger. A review of machine scheduling: Complexity, algorithms and approximability. In *Handbook of Combinatorial Optimization*, pages 1493–1641. Springer, 1998.

[4] TCE Cheng and CCS Sin. A state-of-the-art review of parallel-machine scheduling research. *European Journal of Operational Research*, 47(3):271–292, 1990.

[5] Herman Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics*, pages 493–507, 1952.

[6] Carlos F Daganzo. The crane scheduling problem. *Transportation Research Part B: Methodological*, 23(3):159–175, 1989.

[7] W Fernandez De La Vega and George S Lueker. Bin packing can be solved within \(1+\varepsilon\) in linear time. *Combinatorica*, 1(4):349–355, 1981.

[8] Jacques Desrosiers, Université de Montréal. Centre de recherche sur les transports, and Université de Montréal. Département d’informatique et de recherche opérationnelle. *An Overview of School Busing System*. Montréal: Université de Montréal, Centre de recherche sur les transports, 1980.

[9] Jacques Desrosiers, Jacques-A Ferland, Jean-Marc Rousseau, Guy Lapalme, and Luc Chapleau. Transcol: A multi-period school bus routing and scheduling system. *Management Sciences*, 22:47–71, 1986.

[10] György Dósa. The tight bound of first fit decreasing bin-packing algorithm is ffd (i) 11/9 opt (i)+ 6/9. *Combinatorics, Algorithms, Probabilistic and Experimental Methodologies*, pages 1–11, 2007.

[11] Maximilian M Etschmaier and Dennis FX Mathaisel. Airline scheduling: An overview. *Transportation Science*, 19(2):127–138, 1985.

[12] Armin Fügenschuh. Solving a school bus scheduling problem with integer programming. *European Journal of Operational Research*, 193(3):867–884, 2009.

[13] Michael R Garey and David S Johnson. *Computers and intractability*, volume 29. W. H. Freeman, New York, 2002.
[14] Yoshiro Ikura and Mark Gimple. Efficient scheduling algorithms for a single batch processing machine. *Operations Research Letters*, 5(2):61–65, 1986.

[15] George L Nemhauser and Laurence A Wolsey. Integer programming and combinatorial optimization. Wiley, Chichester. GL Nemhauser, MWP Sавelsbergh, GS Sigismondi (1992). Constraint Classification for Mixed Integer Programming Formulations. COAL Bulletin, 20:8–12, 1988.

[16] Rita M Newton and Warren H Thomas. Design of school bus routes by computer. *Socio-Economic Planning Sciences*, 3(1):75–85, 1969.

[17] Stephan Olariu. An optimal greedy heuristic to color interval graphs. *Information Processing Letters*, 37(1):21–25, 1991.

[18] Junhyuk Park and Byung-In Kim. The school bus routing problem: A review. *European Journal of operational research*, 202(2):311–319, 2010.

[19] Samuel Raff. Routing and scheduling of vehicles and crews: The state of the art. *Computers & Operations Research*, 10(2):6369117149195–67115147193211, 1983.

[20] Prabhakar Raghavan and Clark D Tompson. Randomized rounding: a technique for provably good algorithms and algorithmic proofs. *Combinatorica*, 7(4):365–374, 1987.

[21] Armin Scholl, Robert Klein, and Christian Jürgens. Bison: A fast hybrid procedure for exactly solving the one-dimensional bin packing problem. *Computers & Operations Research*, 24(7):627–645, 1997.

[22] Jorge P Sousa and Laurence A Wolsey. A time indexed formulation of non-preemptive single machine scheduling problems. *Mathematical Programming*, 54(1):353–367, 1992.

[23] Arthur J Swersey and Wilson Ballard. Scheduling school buses. *Management Science*, 30(7):844–853, 1984.
8. Appendix

| Size | Instance | $L_P^{OPT}$ | Rounding | $I_P^{OPT}$ | LP Gap | Rounding Gap | Total Gap |
|------|----------|-------------|----------|-------------|--------|--------------|-----------|
| 1    | 1        | 35.6        | 42       | 37          | 3.93%  | 13.51%       | 17.98%    |
|      | 2        | 49.0        | 51       | 49          | 0.00%  | 4.08%        | 4.08%     |
|      | 3        | 31.9        | 36       | 32          | 0.31%  | 12.50%       | 12.85%    |
|      | 4        | 38.2        | 47       | 39          | 2.09%  | 20.51%       | 23.04%    |
|      | 5        | 45.5        | 51       | 47          | 3.30%  | 8.51%        | 12.09%    |
| 2    | 1        | 333.9       | 336      | 334         | 0.03%  | 0.60%        | 0.63%     |
|      | 2        | 347.3       | 349      | 348         | 0.20%  | 0.29%        | 0.49%     |
|      | 3        | 339.7       | 342      | 340         | 0.09%  | 0.59%        | 0.68%     |
|      | 4        | 395.1       | 397      | 396         | 0.23%  | 0.25%        | 0.48%     |
|      | 5        | 392.4       | 400      | 393         | 0.15%  | 1.78%        | 1.94%     |
| 3    | 1        | 675.8       | 682      | —           | —      | —            | 0.92%     |
|      | 2        | 708.5       | 714      | —           | —      | —            | 0.78%     |
|      | 3        | 800.2       | 801      | —           | —      | —            | 0.10%     |
|      | 4        | 779.1       | 780      | —           | —      | —            | 0.12%     |
|      | 5        | 764.3       | 768      | —           | —      | —            | 0.58%     |
| 4    | 1        | 1614.4      | 1621     | —           | —      | —            | 0.41%     |
|      | 2        | 1554.3      | 1564     | —           | —      | —            | 0.62%     |
|      | 3        | 1536.4      | 1559     | —           | —      | —            | 1.47%     |
|      | 4        | 1364.4      | 1374     | —           | —      | —            | 0.70%     |
|      | 5        | 1541.8      | 1546     | —           | —      | —            | 0.27%     |

Table 3: Numerical results for Type 1 (base model)
| Size | Instance | LPOPT | Rounding | IPOPT | LP Gap | Rounding Gap | Total Gap |
|------|----------|-------|----------|-------|--------|--------------|-----------|
| 1    | 1        | 60.5  | 65       | 62    | 2.48%  | 4.84%        | 7.44%     |
|      | 2        | 52.4  | 58       | 55    | 4.96%  | 5.45%        | 10.69%    |
|      | 3        | 62.1  | 73       | 67    | 7.89%  | 8.96%        | 17.55%    |
|      | 4        | 49.2  | 56       | 52    | 5.69%  | 7.69%        | 13.82%    |
|      | 5        | 52.1  | 54       | 54    | 3.65%  | 0.00%        | 3.65%     |
| 2    | 1        | 429.3 | 437      | 430   | 0.16%  | 1.63%        | 1.79%     |
|      | 2        | 412.2 | 425      | 414   | 0.44%  | 2.66%        | 3.11%     |
|      | 3        | 449.5 | 455      | 452   | 0.56%  | 0.66%        | 1.22%     |
|      | 4        | 488.1 | 501      | 491   | 0.59%  | 2.04%        | 2.64%     |
|      | 5        | 376.7 | 386      | 377   | 0.08%  | 2.39%        | 2.47%     |
| 3    | 1        | 767.1 | 785      | —     | —      | —            | 2.33%     |
|      | 2        | 820.4 | 835      | —     | —      | —            | 1.78%     |
|      | 3        | 697.6 | 712      | —     | —      | —            | 2.06%     |
|      | 4        | 791.7 | 806      | —     | —      | —            | 1.81%     |
|      | 5        | 928.5 | 947      | —     | —      | —            | 1.99%     |
| 4    | 1        | 1710.9| 1737     | —     | —      | —            | 1.53%     |
|      | 2        | 1503.2| 1517     | —     | —      | —            | 0.92%     |
|      | 3        | 1616.0| 1635     | —     | —      | —            | 1.21%     |
|      | 4        | 1664.9| 1682     | —     | —      | —            | 1.03%     |
|      | 5        | 1618.6| 1639     | —     | —      | —            | 1.26%     |

Table 4: Numerical results for Type 2 (short time window model)
| Size | Instance | LP_{OPT} | Rounding | IP_{OPT} | LP Gap | Rounding Gap | Total Gap |
|------|----------|----------|----------|----------|--------|--------------|-----------|
| 1    | 1        | 32.3     | 39       | 33       | 2.17%  | 18.18%       | 20.74%    |
|      | 2        | 37.8     | 44       | 38       | 0.53%  | 15.79%       | 16.40%    |
|      | 3        | 28.4     | 40       | 29       | 0.00%  | 37.93%       | 40.85%    |
|      | 4        | 30.5     | 34       | 31       | 1.64%  | 9.68%        | 11.48%    |
|      | 5        | 17.9     | 27       | 18       | 0.56%  | 50.00%       | 50.84%    |
| 2    | 1        | 200.9    | 223      | 201      | 0.05%  | 10.95%       | 11.00%    |
|      | 2        | 211.9    | 223      | 212      | 0.05%  | 5.19%        | 5.24%     |
|      | 3        | 197.8    | 210      | 198      | 0.10%  | 6.06%        | 6.17%     |
|      | 4        | 181.3    | 187      | 182      | 0.39%  | 2.75%        | 3.14%     |
|      | 5        | 205.6    | 211      | 206      | 0.19%  | 2.43%        | 2.63%     |
| 3    | 1        | 348.3    | 353      | —        | —      | —            | 1.35%     |
|      | 2        | 328.1    | 347      | —        | —      | —            | 5.76%     |
|      | 3        | 314.2    | 330      | —        | —      | —            | 5.03%     |
|      | 4        | 325.4    | 329      | —        | —      | —            | 1.11%     |
|      | 5        | 453.4    | 497      | —        | —      | —            | 9.62%     |
| 4    | 1        | 751.0    | 798      | —        | —      | —            | 6.26%     |
|      | 2        | 734.4    | 759      | —        | —      | —            | 3.35%     |
|      | 3        | 722.7    | 753      | —        | —      | —            | 4.19%     |
|      | 4        | 795.4    | 840      | —        | —      | —            | 5.61%     |
|      | 5        | 744.6    | 763      | —        | —      | —            | 2.47%     |

Table 5: Numerical results for Type 3 (short route model)
| Size | Instance | LP<sub>OPT</sub> | Rounding | IP<sub>OPT</sub> | LP Gap | Rounding Gap | Total Gap |
|------|----------|----------------|----------|----------------|--------|--------------|-----------|
| 1    | 1        | 27.6           | 37       | 28             | 1.45%  | 32.14%       | 34.06%    |
|      | 2        | 33.1           | 44       | 34             | 2.72%  | 29.41%       | 32.93%    |
|      | 3        | 31.2           | 37       | 32             | 2.56%  | 15.63%       | 18.59%    |
|      | 4        | 34.3           | 42       | 35             | 2.04%  | 20.00%       | 22.45%    |
|      | 5        | 18.8           | 21       | 19             | 1.06%  | 10.53%       | 11.70%    |
| 2    | 1        | 380.8          | 382      | 381            | 0.05%  | 0.26%        | 0.32%     |
|      | 2        | 368.9          | 370      | 369            | 0.03%  | 0.27%        | 0.30%     |
|      | 3        | 380.7          | 384      | 382            | 0.34%  | 0.52%        | 0.87%     |
|      | 4        | 358.9          | 366      | 359            | 0.03%  | 1.95%        | 1.98%     |
|      | 5        | 369.3          | 370      | 370            | 0.19%  | 0.00%        | 0.19%     |
| 3    | 1        | 730.1          | 745      | —              | —      | —            | 2.04%     |
|      | 2        | 685.9          | 715      | —              | —      | —            | 4.24%     |
|      | 3        | 720.2          | 730      | —              | —      | —            | 1.36%     |
|      | 4        | 739.2          | 742      | —              | —      | —            | 0.38%     |
|      | 5        | 689.1          | 692      | —              | —      | —            | 0.42%     |
| 4    | 1        | 1420.5         | 1427     | —              | —      | —            | 0.46%     |
|      | 2        | 1427.4         | 1432     | —              | —      | —            | 0.32%     |
|      | 3        | 1492.0         | 1495     | —              | —      | —            | 0.19%     |
|      | 4        | 1498.3         | 1511     | —              | —      | —            | 0.85%     |
|      | 5        | 1416.6         | 1428     | —              | —      | —            | 0.80%     |

Table 6: Numerical results for Type 4 (mixed school model)