On the metric of the space of states in a modified QCD

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The form of the resulting Feynman propagators in a proposed local and gauge invariant QCD for massive fermions suggests the existence of indefinite metric associated to quark states, a property that might relate it with the known Lee-Wick theories. Thus, the nature of the asymptotic free quark states in the theory is investigated here by quantizing the quadratic part of the quark action. As opposite to the case in the standard QCD, the free theory does not show Hamiltonian constraints. The propagation modes include a family of massless waves and a complementary set of massive states. The theory can be quantized in a way that the massive modes show positive metric and the massless ones exhibit negative norms. It is remarked that, since QCD is expected to not exhibit gluon or quark asymptotic states, the presence of negative metric massless modes does not constitute a definite drawback of the theory. In addition, the fact that the positive metric quark states are massive, seems to be a good feature of the model, being consistent with the approximate existence of asymptotically free massive states in high energy processes.

PACS numbers: 12.38.Aw;12.38.Bx;12.38.Cy;14.65.Ha

I. INTRODUCTION

In a recent work [1] it had been motivated an alternative for the description of quarks interacting through gluons, which possible equivalence with massless QCD was also argued. The outcome was a theory with an action given by the massless QCD one, which also incorporates additional terms, one for each quark flavour. The theory constitutes a local and gauge invariant completion of the one discussed in Ref. [2–9], and, up to our knowledge, it also furnishes a new local formulation for strongly interacting massive quarks. The new Lagrangian terms appearing could also be interpreted as possible counterterms, within a special renormalization scheme for massless QCD. That is, it was pointed out that such terms could be present in an effective action for massless QCD. These terms do not eliminate the power counting renormalizability of the model, because the new quadratic part of the action modifies the behavior of the free quark propagator, which now gets a $1/p^2$ behavior at large momenta. The expansion is expected to be employed in further studies directed to explore physical questions as: the possibility of a flavour symmetry breaking effect determining a hierarchical quark mass spectrum, and the properties of quark-quark interaction potential in the scheme. However, the new diagrammatic expansion, shows a conceptual feature which is convenient to be studied before further employing it in the above cited studies. This special property is that the new more convergent free propagator is given by the difference between a free massless quark propagator and a massive one. This structure is very similar to the typical ones appearing in the so called Lee-Wick theories, in which usual bosons propagators are modified by substracting massive propagators in order to make the expansion more convergent [10–12]. Such substractions, for the bosonic fields, although making the propagators decreasing more rapidly at large momenta, introduce states of negative norm in the space of states, which mass shells are defined by the poles of the new propagators [13, 14]. Thus, the quark modified propagator appearing in reference [1], which is the difference between a massless and a massive one as described before, might in principle lead to the appearance of an indefinite metric in the quark sector of the state space of the theory. As remarked in Ref. [10], the presence of an indefinite metric could not directly imply a limitation in the physical applicability of the theory. In our case this possibility might be indicated by the fact that for QCD, it is conceived that there are not asymptotic states for the elementary quarks and gluons. Also, the negative metric states in the Lee-Wick theories can be compatible with an unitary scattering matrix by the appearance of decaying rates for these negative norm states due to radiative corrections. Thus, the presence of those modes on the proposed variation of QCD might not automatically define its lack of applicability. However, the existence of such modes, could occasionally lead to drastic modifications of the predictions at very high energy processes, where the perturbative expansion is expected to approximately work due to asymptotic freedom. Thus, to know the type of negative metric states which can appear in the proposed theory becomes an important question. It can be also noted that the gluon free action is already including an indefinite metric associated to the unphysical gluon components. However, since the gluon and quark sectors do not couple in the free part of the action, the gluon free theory quantization is identical as in the usual QCD.

Therefore, in this work, we investigate the quantization of the free part of the quark action of the proposed theory. For this purpose we first consider a reparametrization of the classical action obtained in [1] in order to express the relativistic quark propagator in a form in which the massive states appear as related with poles associated to positive metric states and the massless ones on the contrary, as linked with negative metric ones [10]. Next, the description of
the classical propagation modes of the free Lagrangian equations is considered. Two sets of quark modes arise. One of them is identical to the one corresponding to massless quarks in QCD, and a second one is a set of massive states of polarizations being equivalent to the usual set of modes satisfying the Dirac’s equation. The classical Hamiltonian and the canonical equations are also written. A helpful form of the Hamiltonian, to be further employed for the quantization process is obtained. It was used the specific form of the generalized Hamiltonian procedure presented in Ref. [13]. The consistency of the Hamiltonian and Lagrangian equations were checked. Further the quantization is considered by defining field operators as superpositions of the previously determined wave modes. The Hamiltonian is then expressed in terms of the creation and annihilation operators defining the massive and massless quark fields. The result, as usual was written as a linear combination of the number operators for each quark mode with the coefficient defined by the energy for the mode. This conclusion followed after choosing the form of the anticommutation relation among all the creation and annihilation operators of the fermion modes, that define the quantum states as showing positive norm for the massive quarks and negative ones for the massless ones. Central in determining the chosen selection of the creation and annihilation operators for the single particle states and the metric of the state vector space, was, on one side, the requirement of defining a bounded from below value of the energy of the system. The process to implementing these properties was also basic in defining the positive or negative metric character of the various quantized modes. The satisfaction of the same classical field equations by the field operators and their momenta at all times was shown. The quantum commutators between all the fields and their momenta satisfied the usual quantization rule of being given by the imaginary unit times the results of the classical Poisson brackets.

The presentation proceeds as follows. In Section II, the general form of the action for the theory is presented and the notations employed are defined. Afterwards, the general action and the quark propagators are rewritten, to fully evidence the similarity of the propagator structure with the ones appearing in the Lee-Wick theories. This is done in a way suggesting the positive metric of massive states and negative metric for massless states. Section III, presents the determination of the quark modes associated to the poles of the propagator. The Section IV considers the writing of the classical propagation modes of the free Lagrangian equations is considered. Two sets of quark modes arise. One among all the creation and annihilation operators of the fermion modes, that define the quantum states as showing positive norm for the massive quarks and negative ones for the massless ones. Central in determining the chosen selection of the creation and annihilation operators for the single particle states and the metric of the state vector space, was, on one side, the requirement of defining a bounded from below value of the energy of the system. The process to implementing these properties was also basic in defining the positive or negative metric character of the various quantized modes. The satisfaction of the same classical field equations by the field operators and their momenta at all times was shown. The quantum commutators between all the fields and their momenta satisfied the usual quantization rule of being given by the imaginary unit times the results of the classical Poisson brackets.

The classical action defining the model introduced in Ref. [1] had the explicit form

$$S = \int dx \left( -\frac{1}{4} F_{\mu\nu}^a(x) F^{a\mu\nu}(x) - \frac{1}{2\alpha} \partial_\mu A^a_\mu(x) \partial_\nu A^a_\nu(x) + \Gamma^i(x) \partial_\mu D^{ab}_\mu c^b(x) + \sum_f \bar{\Psi}_f(x) i\gamma^\mu D^\mu_f \Psi_f(x) - \sum_f \chi \int dx \bar{\Psi}_f(x) \gamma_\mu \overline{D}^{ij}_\mu \gamma_\nu D^{ik\nu} \Psi_f^k(x) \right),$$

(1)

in which the conventions for the various quantities coincide with the ones employed in Ref. [17]. In details, the appearing fields are defined as follows

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu,$$

$$\Psi_f^k(x) = \left( \begin{array}{c} \Psi_{f,1}^k(x) \\ \Psi_{f,2}^k(x) \\ \Psi_{f,3}^k(x) \\ \Psi_{f,4}^k(x) \end{array} \right),$$

(2)

$$\Psi_f^{1k}(x) \equiv (\Psi_f^k(x))^T = \left( (\Psi_f^{1k}(x))^* (\Psi_f^{2k}(x))^* (\Psi_f^{3k}(x))^* (\Psi_f^{4k}(x))^* \right),$$

(3)

where \( f = 1, \ldots, 6 \) indicates the flavour index. The expressions for the Dirac conjugate spinors and covariant derivatives are

$$\bar{\Psi}_f(x) = \Psi_f^{1k}(x) \gamma^0,$$

(4)

$$D^{ij}_\mu = \partial_\mu \delta^{ij} - i g A^{a}_\mu T^{ij}_a, \quad \overline{D}^{ij}_\mu = -\partial_\mu \delta^{ij} - i g A^{a}_\mu T^{ij}_a,$$

(5)

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{abc} A^c_\mu.$$  

(6)
in which the Dirac’s matrices, SU(3) generators and the metric tensor are defined in the conventions of Ref. [17], as

\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad [T_a, T_b] = i f^{abc}T_c, \quad \gamma^0 = \beta, \quad \gamma^j = \beta \alpha^j, \quad j = 1, 2, 3, \]

\[ g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \]

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & -1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (7)

Other useful definitions and relations for the coordinates are

\[ x \equiv x^\mu = (x^0, x^2) = (x^0, x^1, x^2, x^3), \quad x_\mu = g_{\mu\nu}x^\nu, \quad x^0 = t. \]

The complex conjugate operation is symbolized by the superscript *, by also assuming that the conjugate operation of a quantity also implies the Hermitian conjugation of any operator entering its definition. For example, in the conjugate transposed fermion field \( \Psi_f^T(x) = (\Psi_f(x))^T \) the operator structure entering the definition of \( \Psi_f^T(x) \) should be substituted by its Hermitian transposed structure. In what follows the color and spinor indices of the fields and other magnitudes will be omitted in order to simplify the writing. The time variable \( t \) will be defined by \( t = x^0 \). The system will be assumed to be enclosed in a large cubic spatial box of volume \( V \), on which periodic boundary conditions are imposed for all the fields. This define the spatial momenta as taken the usual discrete values.

The quadratic in the quark field of flavour \( f \), free action of the theory is given by

\[ S_{0,f} = \int dx \overline{\Psi}_f(x)(i\gamma^\mu \partial_\mu - \kappa \partial^2\nu)\Psi_f(x) \]

\[ = -\int dx \overline{\Psi}_f(x)\Lambda_f(\partial)\Psi_f(x) \] (8)

in which \( \kappa \) is a constant with dimension of length, which the discussion in former works motivating the proposal of the action [11] in Ref. [1], suggested to be related with quark condensation effects. Those analysis were associated to the consideration of a fermion squeezed state as the vacuum leading to modified non local perturbative Wick expansion for massless QCD. They strongly suggested the local action form [11] as possible effective action of the massless QCD [2, 9].

The Feynman propagator \( S_f \) for the considered quark of flavour \( f \), is given by the above defined inverse of the kernel \( \Lambda_f \) and satisfies

\[ -(i\gamma^\mu \partial_\mu - \kappa \partial^2\nu) S_f(x-y) = \delta(x-y), \] (9)

or in terms of its Fourier transform \( S_f(x) = \int dp S_f(p) \exp(-ip.x)) \)

\[ -(\gamma^\mu p_\mu + \kappa p^2)S_f(p) = I. \] (10)

In order to simplify the notation, in what follows we will use the same symbol \( I \) (or simply the unity 1) for the Kronecker delta in the color and spinor indexes, for both cases: the two component and the four component spinors, when no confusion can arise.

Thus, the quark propagator for the quark of flavour \( f \) has the form

\[ S_f(p) = \frac{1}{-\gamma_\nu p^\nu - \kappa p^2} \equiv \frac{(-\gamma_\nu p^\nu - \lambda p^2)^{rr'}\delta^{rv}}{p^2(1-\lambda^2 p^2)} \]

\[ = \frac{1}{\gamma_\nu p^\nu} - \left( \frac{1}{\gamma_\nu p^\nu + m_f} \right), \quad m_f = \frac{1}{\kappa}. \] (11)

The free quark propagator shows poles of mass \( m_f = \frac{1}{\kappa} \), which are inversely proportional to the parameter \( \kappa \). It also present poles of zero mass. This propagator has the structure given in the last line of [11], which expresses it, in the here employed notation, as the usual massless quark propagator minus the Dirac propagator for a massive quark in which the sign of the mass had been reversed. As mentioned in the introduction, this structure closely resembles the Pauli Villars kind of regularization employed in the Lee-Wick theories, which might lead to the appearance of negative metric states in the resulting quantum field theories. The massless component of the propagator shows the
standard form for massless QCD with positive metric, and the massive component has a changed sign for the usual Dirac component of the action (as evidenced by the large momentum limit of this term). This suggests that the quantization of the Lagrangian with the employed field parametrization, will lead to negative metric for the massive fermion sector of the asymptotic states. However, since the theory shows space time inversion invariance, the sign of the time which could effectively correspond with the physical time is not well defined at the current stage of the analysis of the theory. Thus, this freedom can be considered as factor to employed in the process of constructing the theory. After adopting this point of view, it seems possible to redefine the fields in the action by performing the following change of variables and coordinates within the functional integral defining the Green’s functions generating functional

$$x = -x', \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = -\partial'^\mu \equiv \frac{\partial}{\partial x'^\mu},$$

$$\Psi(x) = \Psi_f^i(-x') = \Psi_f^i(x'), \quad \overline{\Psi}(x) = \overline{\Psi}_f^k(-x') = \overline{\Psi}_f^k(x'),$$

$$A^{a\mu}(x) = A^{a\mu}(-x') = -A^{a\mu}(x'),$$

$$c(x) = c(-x') = c'(x'), \quad \overline{c}(x) = \overline{c}(-x') = \overline{c}'(x').$$

After performing this changes in the general expression for the action (1), its expression in terms of the new primed fields and coordinates, for afterwards returning to the previous notation for the coordinates and fields by removing the primes in all the magnitudes, it follows for $S$

$$S = \int dx \left( -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{2\alpha} \partial_\mu A^{a\mu}(x) \partial_\nu A^{a\nu}(x) + \overline{\Psi}^i(x) \partial_\mu D^{a\mu\nu} A^{a\nu}(x) - \sum_f \int dx \overline{\Psi}_f^j(x) i\gamma^\mu D^{a\mu\nu} \Psi_f^j(x) - \int dx \overline{\Psi}_f^j(x) \gamma^\mu D^{a\mu\nu} \gamma_\nu D^{a\nu\rho} \Psi_f^j(x) \right). \tag{12}$$

The new expression for the free quark action takes the form

$$S_{0,f} = \int dx \overline{\Psi}_f(x) \left( -i\gamma^\mu \partial_\mu - \kappa \partial^2 \right) \Psi_f(x) = -\int dx \overline{\Psi}_f(x) \Delta_f(\partial) \Psi_f(x), \tag{13}$$

which defines for the new propagator kernel

$$S_f(p) = \frac{1}{\gamma^\nu p^\nu - \kappa p^2} = \frac{1}{\gamma^\nu p^\nu - \left( \frac{1}{\gamma^\nu p^\nu - m_f} \right)}, \tag{14}$$

where now the massive term has the form of the typical Feynman quark propagator for massive quarks and the massless one has a different sign, which suggests its association to negative metric states. In order to define the precise way in which the theory implements the introduction of a space of states with indefinite metric for quarks, in next sections we perform the explicit quantization of this new form of the action.

### III. CLASSICAL QUARK PROPAGATION MODES

Let us determine in this section the quark propagation modes defined by the free fermion component of the action. These $u(x)$ modes satisfies the Lagrange equations defined by (13), which are

$$(-i\gamma^\mu \partial_\mu - \kappa \partial^2) u(x) = 0, \quad u(x) = \int dq u(q) \exp(-q_\mu x^\mu). \tag{15}$$

They imply for the Fourier components $u(q)$, the matrix equation

$$(-\gamma^\mu q_\mu + \kappa q^2) u(q) = 0. \tag{16}$$

This expression clearly shows that there exist massless as well as massive propagation modes. Let us separately consider in what follows the massive and massless solutions.
A. Massive solutions

The Fourier four momentum \( q \) associated to any of the waves being determined, will be expressed in the form \( q = (q^0, q^i) \). Since we are searching for the massive modes, in the Lorentz frame in which the momentum vanishes, the momentum can be written in the form \( q = (q^0, 0) \), and thus, the equation (16) reduces to

\[
q_0(-\gamma^0 + xq_0) u(q) = 0 \equiv q_0 \begin{pmatrix} (-1 + \kappa q_0)I & 0 \\ 0 & (\kappa q_0 + 1)I \end{pmatrix} \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix},
\]

which clearly allows to easily define in this frame the solutions for the separate cases in which the energy component \( q_0 \) takes positive or negative values. Both cases are considered in what follows in this subsection.

1. Positive energy waves

Below, it will be convenient to define a mass parameter \( m_f \) as

\[
\kappa = \frac{1}{m_f}.
\]

For this case of positive energy waves, in the selected rest frame, the energy \( q_0 \), takes the positive value \( q_0 = m_f \) and the positive energy spinor polarization gets the simple form

\[
u^r(0) = \begin{pmatrix} \beta^r \\ 0 \end{pmatrix}, r = 1, 2
\]

\[
\beta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then, the massive positive energy polarization of the considered theory coincide with the ones associated to the positive energy modes of the usual solutions of the Dirac’s equation for a massive particle. Therefore, the spinor polarizations in any Lorentz frame might be written in the form

\[
u^r(\vec{p}) = \sqrt{\epsilon_m(\vec{p}) + m_f} \begin{pmatrix} \beta^r \\ \frac{\bar{\sigma} \cdot \vec{p}}{\epsilon_m(\vec{p}) + m_f} \beta^r \end{pmatrix}, r = 1, 2,
\]

in which the energies of the modes are given by \( \epsilon_m(\vec{p}) = \sqrt{m_f^2 + \vec{p}^2} \). The polarizations had been chosen to satisfy the normalization conditions

\[
u^r(\vec{p})u^s(\vec{p}) = \delta^{rs}, \quad \bar{u}(\vec{p})u^s(\vec{p}) = \frac{m_f}{\epsilon_m(\vec{p})} \delta^{rs}.
\]

Note that the normalization for the spinors is fixed in a slightly different way as the usual one. This was done in order to assure the normalization to one of the spinors in the large quantization volume for the usual spinor scalar product.

2. Negative energy waves

In a similar way, for the negative energy case \( q_0 = -m_f \) the solutions of the equation (17) leads to the spinor polarization of the form

\[
u^r(\vec{p}) = \sqrt{\epsilon_m(\vec{p}) - m_f} \begin{pmatrix} -\frac{\bar{\sigma} \cdot \vec{p}}{\epsilon_m(\vec{p}) - m_f} \beta^r \\ \beta^r \end{pmatrix}, r = 1, 2,
\]

in which the energies of the modes are given by \( \epsilon_m(\vec{p}) = \sqrt{m_f^2 + \vec{p}^2} \).
in which again the energy parameter $\epsilon_m^{\pm}(p) = \sqrt{m_f^2 + p^2}$ and the polarizations were chosen to satisfy the normalization conditions

$$v^r(\vec{p}) \, v^s(\vec{p}) = \delta^{rs}, \quad \overline{v}^r(\vec{p}) \, v^s(\vec{p}) = -\frac{m_f}{\epsilon_m^{\pm}(\vec{p})} \delta^{rs}.$$ 

The four momenta of the negative energy waves is given by $q = (-\epsilon_m^{\pm}(\vec{p}), \vec{p})$. Note that as usual, in order to assure orthonormality for different values of the spatial momentum $\vec{p}$ of the modes, the spatial momentum index in the Fourier expansion of the original wave $u(x)$, had been changed in sign for the negative energy solutions.

**B. Massless solutions**

These solutions of the general equation (16), since having $q^2 = (q^0)^2 - \vec{q}^2 = 0$, satisfy the massless Dirac equation

$$i \gamma^\mu \partial_\mu u(x) = 0, \quad u(x) = \int dq \, u(q) \exp(-q_\mu x^\mu),$$

which for the Fourier components $u(q)$ take the form

$$\gamma^\mu q_\mu u(q) = 0.$$ 

In terms of the null four momentum $q = (q^0, q^i)$ it can be written as follows

$$(q_0 \gamma^0 - \gamma^i q^i)u(q) = \beta(q_0 - \alpha^i q^i)u(q)$$

$$\equiv \beta \left( \frac{q_0}{-\vec{\sigma} \cdot \vec{q}} I - \vec{\sigma} \cdot \vec{q} \frac{q_0}{q_0 I} \right) \left( \beta_1 \beta_2 \right) = 0.$$ 

It is clear that by considering the two component spinors $\beta_1$ and $\beta_2$ as eigenfunctions of the helicity operator $\vec{\sigma} \cdot \vec{q}$, the positive and negative energy polarizations can be determined. Below we examine the two cases.

1. **Positive energy massless waves**

After defining the $\beta^{+1}(\vec{p}), \beta^{-1}(\vec{p})$ as the eigenfunctions of $\vec{\sigma} \cdot \vec{p}$, given by (18)

$$\beta^{+1}(\vec{p}) = \frac{1}{\sqrt{2(n_3 + 1)}} \begin{pmatrix} n_3 + 1 \\ n_1 + i n_2 \end{pmatrix}, \quad \beta^{-1}(\vec{p}) = \frac{1}{\sqrt{2(n_3 + 1)}} \begin{pmatrix} -n_1 + i n_2 \\ n_3 + 1 \end{pmatrix},$$

$$\vec{\sigma} \cdot \vec{p} \, \beta^l(\vec{p}) = l \, \beta^l(\vec{p}), \quad l = +1, -1, \quad \vec{n}(\vec{p}) = \frac{\vec{p}}{|\vec{p}|} = (n_1, n_2, n_3),$$

the positive energy solutions of momentum $q = (\epsilon_0(\vec{p}), \vec{p})$ can be written in the form

$$u^l_0(\vec{p}) = \sqrt{\frac{1}{2}} \beta^l(\vec{p}), \quad l = +1, -1,$$

$$\epsilon_0(\vec{p}) = |\vec{p}|, \quad q = (\epsilon_0(\vec{p}), \vec{p}).$$

The normalization of these modes satisfy

$$u^d_0(\vec{p}) \, u^d_0(\vec{p}) = \beta^{d \prime} \cdot \beta^{d \prime}, \quad \overline{u}^d_0(\vec{p}) \, u^d_0(\vec{p}) = 0.$$
2. Negative energy massless waves

The negative energy modes, defined by \( q_0 = -|\vec{q}| \), have the polarization

\[
\begin{align*}
\psi_0^i(\vec{p}) &= \sqrt{\frac{1}{2}} \left( \beta^{-1}(\vec{p}) \right), \quad l = +1, -1, \\
\vec{n}^l(\vec{p}) &= \frac{\vec{p}}{|\vec{p}|} = (n_1, n_2, n_3),
\end{align*}
\]

(26)

where, as it is convenient for assuring orthonormality of the modes for different spatial momentum index, the spatial Fourier momentum \( \vec{q} \) was fixed to be equal to minus \( -\vec{p} \). The appearing two component spinors have the same already defined explicit expressions

\[
\beta^+(\vec{p}) = \frac{1}{\sqrt{2(1+n_3)}} \left( \begin{array}{c} 1 + n_3 \\ n_1 + i n_2 \end{array} \right), \beta^-(p) = \frac{1}{\sqrt{2(1+n_3)}} \left( \begin{array}{c} -n_1 + i n_2 \\ 1 + n_3 \end{array} \right),
\]

(27)

and the four Fourier momentum of the mode has the form \( q = (-\epsilon_0(\vec{p}), \vec{p}) \) with \( \epsilon_0(\vec{p}) = |\vec{p}| \). These polarizations obey the orthonormality relations

\[
v_0^i(\vec{p}) v_0^j(\vec{p}) = \delta^i_j, \quad \vec{n}^l(\vec{p}) v_0^i(\vec{p}) = 0.
\]

(28)

IV. FREE QUARK FIELD QUANTIZATION

In this section we firstly present the application of the generalized Hamiltonian formalism to the free part of the quark action of the modified QCD in the form \( [13] \). This discussion will also permit to obtain a formula for the Hamiltonian of the system which is further employed to quantize the theory.

A. Generalized canonical procedure

After performing few integrations by parts, the classical action \( S \) in \( [13] \) can be expressed as a functional of the quark fields and their first time derivatives in a form which determines for the Lagrangian \( L \) of the system

\[
L = \int dt d^3x \Psi^\dagger(x) \left( -\frac{i}{2} \partial^\mu \partial_{\mu} - \kappa \left( \partial_t \partial_{\mu} - \beta \vec{\nabla} \right) \beta \right) \Psi(x),
\]

(29)

\[
\partial_{\mu} \beta = \partial_{\mu} \beta - \partial_{\mu} \beta = (I, \alpha^1, \alpha^2, \alpha^3), \Psi^\dagger(x) = \Psi^T(x),
\]

(30)

where the right or left arrows appearing over the derivatives indicate whether they are acting on the right or the left expressions respectively. It can be helpful to recall that when operators will appear, the complex conjugation operation, designed by the superindex "*" also includes the Hermitian transpose of the operator structure.

The momenta associated to the, up to now, classical fields \( \Psi \) and \( \Psi^\dagger \) are defined in the generalized mechanics with anticommuting variables (See Ref. [15]) in the form

\[
\Pi_\Psi(x) = L \frac{\delta}{\delta(\partial_t \Psi(x))} = -\frac{i}{2} \Psi^\dagger(x) + \kappa \partial_t \Psi^\dagger(x) \beta,
\]

(31)

\[
\Pi_{\Psi^\dagger}(x) = L \frac{\delta}{\delta(\partial_t \Psi^\dagger(x))} = -\frac{i}{2} \Psi(x) - \kappa \beta \partial_t \Psi(x),
\]

(32)

in which the "right" Grassman functional derivative employed is defined for functionals which depend of the field and their derivatives as functions of the spatial variables \( \vec{x} \) at a time slice defined by a instant \( t \). This instant is given by the fixed time slice employed to perform the spatial volume integration defining the Lagrangian in \( [29] \).

It is interesting that this fermion theory is regular, since it does not have Hamiltonian constraints. Then, all the velocities can be explicitly expressed in terms of the coordinates (fields) and their momenta as follows

\[
\partial_t \Psi^\dagger(x) \beta = \frac{1}{\kappa} \frac{i}{2} \Psi^\dagger(x) + \Pi_\Psi(x),
\]

(33)

\[
\beta \partial_t \Psi(x) = \frac{1}{\kappa} (\Pi_{\Psi^\dagger}(x) + \frac{i}{2} \Psi(x)).
\]

(34)
With these expressions, the Hamiltonian at the defined time slice can be expressed as a functional of the fields and momenta in the way

$$H = \int dx^3 (\Pi_\Psi(x) \partial_t \Psi(x) + \Pi_{\Psi^\dagger}(x) \partial_t \Psi^\dagger(x)) - L$$

$$= \int dx^3 \left\{ \frac{1}{\kappa} ( - \Pi_\Psi(x) \beta \Pi_{\Psi^\dagger}(x) + \frac{1}{4} \Psi^\dagger(x) \beta \Psi(x) - \frac{i}{2} \Psi^\dagger(x) \beta \Pi_{\Psi^\dagger}(x) - \frac{i}{2} \Pi_\Psi(x) \beta \Psi(x)) + \frac{i}{2} \Psi^\dagger(x) \alpha^i \nabla_i \Psi(x) + \kappa \Psi^\dagger(x) \nabla \cdot \nabla \beta \Psi(x) \right\}. \quad (35)$$

The expression for the Hamiltonian, allows to write the canonical equations of motion in the form [15]

$$\partial_t \Psi(x) = \frac{\delta}{\delta \Pi_\Psi(x)} H = - \frac{1}{\kappa} \beta (\Pi_{\Psi^\dagger}(x) + \frac{i}{2} \Psi(x)),$$

$$\partial_t \Psi^\dagger(x) = \frac{\delta}{\delta \Pi_{\Psi^\dagger}(x)} H = \frac{1}{\kappa} (\Pi_\Psi(x) + \frac{i}{2} \Psi^\dagger(x)) \beta,$$

$$\partial_t \Pi_\Psi(x) = -H \frac{\delta}{\delta \Psi(x)} = \frac{i}{2 \kappa} \Pi_\Psi(x) \beta - \frac{1}{4 \kappa} \Psi^\dagger(x) \beta + i \Psi^\dagger(x) \alpha^i \nabla_i \Psi(x) + \kappa \nabla^2 \Psi^\dagger(x) \beta,$$

$$\partial_t \Pi_{\Psi^\dagger}(x) = -H \frac{\delta}{\delta \Psi^\dagger(x)} = - \frac{i}{2 \kappa} \beta \Pi_{\Psi^\dagger}(x) + \frac{1}{4 \kappa} \beta \Psi(x) + i \alpha^i \partial_i \Psi(x) - \kappa \beta \nabla^2 \Psi^\dagger(x), \quad (36)$$

which after eliminating the momenta in terms of the fields and their time derivatives lead to the Lagrange equations

$$(-i \gamma^\mu \partial_\mu - \kappa \partial^2) \Psi(x) = 0,$$

in consistency with the starting ones in [15].

### B. Quantization

Let us consider in this section the quantization of the quark fields $\Psi$ and $\Psi^\dagger$. For this purpose it seems convenient to find a formula for the Hamiltonian $H$ in terms of the fields and their time derivatives. In the case of the Dirac’s equation, and thanks to its constrained canonical structure, the Hamiltonian can be expressed in terms of the fields $\Psi$ and $\Psi^\dagger$ and their spatial derivatives. However, the here considered mechanical system is regular and the formula for the energy should also contain the time derivatives of the fields. The expression for $H$ can be rewritten after eliminating the momenta in [35] by employing the canonical equations [36]

$$H = \int dx^3 \{ \kappa \partial_t \Psi^\dagger(x) \beta \partial_i \Psi(x) + i \Psi^\dagger(x) \alpha^i \partial_i \Psi(x) + \kappa \Psi^\dagger(x) \nabla \cdot \nabla \beta \Psi(x) \}$$

$$= \int dx^3 \Psi^\dagger(x) \{ \kappa \partial_t \beta \partial_i \Psi(x) + i \alpha^i \partial_i + \kappa \nabla \cdot \nabla \beta \} \Psi(x)$$

$$= \int dx^3 \Psi^\dagger(x) h(\partial) \Psi(x), \quad (37)$$

in which, as it was remarked before, the time derivatives of the fields enter the definition of the kernel $h$.

Let us now start the quantization procedure by considering the quark fields as superpositions of all the oscillator modes with coefficients defined by operators. They will be identified as creation and annihilation operators for all the modes. For this purpose it will be useful to separate the field as a sum of two components: one assoceate to the massless modes and another defined by the massive ones. The kind of the mode will be represented by the greek
index $\kappa$ taking two values: 0 or $m$. Then the total field will be written as follows

$$
\Psi(x) = \Psi_m(x) + \Psi_0(x) = \sum_{\kappa=0,m} \Psi_\kappa(x), \tag{38}
$$

$$
\Psi_0(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{l=\pm 1} (a_l(\vec{p}) u^\dagger_l(\vec{p}) \exp(-i (\epsilon_0(\vec{p}) t - \vec{p} \cdot \vec{x}))) + c^\dagger_l(\vec{p}) v^\dagger_l(\vec{p}) \exp(i (\epsilon_0(\vec{p}) t + \vec{p} \cdot \vec{x})))
= u_0(x) + v_0(x) \tag{39}
$$

$$
\Psi_m(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{r=1,2} (b_r(\vec{p}) u^\dagger_m(\vec{p}) \exp(-i (\epsilon_{m,r}(\vec{p}) t - \vec{p} \cdot \vec{x}))) + d^\dagger_r(\vec{p}) v^\dagger_m(\vec{p}) \exp(i (\epsilon_{m,r}(\vec{p}) t + \vec{p} \cdot \vec{x})))
= u_m(x) + v_m(x), \tag{40}
$$

where the polarization vectors for all the massless or massive modes were before determined. The assignment of the creation or annihilation nature for each of the eight operators ($a_1(\vec{p}), a^\dagger_1(\vec{p}), c_1(\vec{p}), c^\dagger_1(\vec{p}), b_r(\vec{p}), b^\dagger_r(\vec{p}), d_r(\vec{p}), d^\dagger_r(\vec{p})$) is not yet defined at this level. It will be fixed afterwards, by the requirements to be imposed on the metric of the state space to define a bounded from below energy for the many particle states. Thus, up to now the symbols $\dagger$ over any of these quantities will only mean that it is the Hermitian conjugate of the operator, not that it is a creation one.

One important point to note is that the expression \[37\] is defined by the spatial integral in a given fixed time slice. Thus the time derivatives of the fields entering, will be evaluated using the time dependences of the classical modes. Therefore, for the sake of consistency it should be required that after quantization of this free theory, the expectation of the Hamiltonian $H$ should reproduce the classical time dependency of the fields through the unitary transformation defining the operators in the Heisenberg representation $\Psi(\vec{x}, t) = \exp(iH t) \Psi(\vec{x}, 0) \exp(-iH t)$.

In addition, the field will be also decomposed in the sum of its the terms associated to the positive energy modes $u_\kappa(x)$, plus the terms defined by the negative energy modes $v_\kappa(x)$, for each of the two values of the mass $\kappa$, as follows

$$
\Psi(x) = \sum_{\kappa=0,m} u_\kappa(x) + \sum_{\kappa=0,m} v_\kappa(x). \tag{41}
$$

In terms of this expansion the Hamiltonian has the expression

$$
H = \int dx^3 \Psi^\dagger(x) h(\partial) \Psi(x)
= \sum_{\kappa=0,m} \int dx^3 u^\dagger_\kappa(x) h(\partial) u_\kappa(x) + \int dx^3 v^\dagger_\kappa(x) h(\partial) v_\kappa(x) + \int dx^3 u^\dagger_m(x) h(\partial) v_m(x) + \int dx^3 v^\dagger_m(x) h(\partial) u_m(x). \tag{42}
$$

The procedures for the calculation of the appearing matrix elements of the kernel $h(\partial)$ are illustrated in the Appendix A. It follows that they vanish when both entering modes have different $u$ or $v$ types and also when they have different masses. The calculation of the few non vanishing terms leads to the following formula for the Hamiltonian operator

$$
H = \int dx^3 \Psi^\dagger(x) h(\partial) \Psi(x)
= \int dx^3 u^\dagger_m(x) h(\partial) u_m(x) + \int dx^3 v^\dagger_m(x) h(\partial) v_m(x) + \int dx^3 u^\dagger_0(x) h(\partial) u_0(x) + \int dx^3 v^\dagger_0(x) h(\partial) v_0(x)
= \sum_{\vec{p}} \sum_{l=\pm 1} (-\epsilon_0(\vec{p})) a^\dagger_l(\vec{p}) a_l(\vec{p}) + \epsilon_0(\vec{p}) c^\dagger_l(\vec{p}) c_l(\vec{p}) + (\epsilon_{m,r}(\vec{p})) b^\dagger_r(\vec{p}) b_r(\vec{p}) + \epsilon_{m,r}(\vec{p}) d^\dagger_r(\vec{p}) d_r(\vec{p}). \tag{43}
$$
This expression is central for the purpose of deciding about the assignation of the creation and annihilation qualities to the introduced operators and the type of indefinite metric of the Fock space of states. This formula indicates that for massive modes quantities, the usual assignation \( b_r(\overrightarrow{p}) \) and \( d_r(\overrightarrow{p}) \) to annihilation and \( b^\dagger_r(\overrightarrow{p}) \) and \( d^\dagger_r(\overrightarrow{p}) \) to creation operators, assures that the contribution of their created particles (over a vacuum state annihilated by all the \( b_r(\overrightarrow{p}) \) and \( d_r(\overrightarrow{p}) \) to the energy will be bounded from below. Then, let us adopt the following commutation relations for the massive type operators

\[
\{b^\dagger_r(\overrightarrow{p}), b_r(\overrightarrow{p})\} = s_m \delta_{r,r'}\delta^{(K)}(\overrightarrow{p_1}, \overrightarrow{p}_2),
\]

\[
\{d^\dagger_r(\overrightarrow{p}), d_r(\overrightarrow{p})\} = s_m \delta_{r,r'}\delta^{(K)}(\overrightarrow{p_1}, \overrightarrow{p}_2)
\]

\[
\{A, B\} = AB + BA,
\]

where the number \( s_m = \pm 1 \) yet allows to select positive or negative metric quantization of the associated states for massive fermions \( \text{[13, 14]} \). The function \( \delta^{(K)}(\overrightarrow{p}_1, \overrightarrow{p}_2) \) represent the Kronecker Delta function, equal to one for equal arguments and vanishing when they differ. In what follows, the definition of the Dirac’s Delta function of the coordinates will be employed which is related with the discrete momenta in the quantization box as follows

\[
\delta^{(D)}(\overrightarrow{x}_1 - \overrightarrow{x}_2) = \sum_{\overrightarrow{p}} \frac{1}{V} \exp(i \overrightarrow{p}.(\overrightarrow{x}_1 - \overrightarrow{x}_2)) = \int \frac{dp^3}{(2\pi)^3} \exp(i \overrightarrow{p}.(\overrightarrow{x}_1 - \overrightarrow{x}_2)).
\]

For the case of the massless modes quantities, it is clear that in order to the energy be positive, given the fermion character of the fields, the assignation can be to consider a creation type for the operators \( a_{l}(\overrightarrow{p}), c_{l}(\overrightarrow{p}) \) and an annihilation nature for the operators \( a^\dagger_{l}(\overrightarrow{p}), c^\dagger_{l}(\overrightarrow{p}) \). With this interpretation, and since the particles are fermions we will define the adopted here creation and annihilation operators \( \hat{a}_{l}(\overrightarrow{p}), \hat{c}_{l}(\overrightarrow{p}) \) and their Hermitian conjugates as

\[
\hat{a}_{l}(\overrightarrow{p}) = a^\dagger_{l}(\overrightarrow{p}), \quad \hat{c}_{l}(\overrightarrow{p}) = c^\dagger_{l}(\overrightarrow{p}),
\]

\[
\hat{a}_{l}(\overrightarrow{p}) = a_{l}(\overrightarrow{p}), \quad \hat{c}_{l}(\overrightarrow{p}) = c_{l}(\overrightarrow{p}),
\]

assuming to obey the anticommutation relations

\[
\{\hat{a}^\dagger_{l}(\overrightarrow{p}), \hat{a}_{r}(\overrightarrow{p})\} = s_0 \delta_{l,r}\delta^{(K)}(\overrightarrow{p}_1, \overrightarrow{p}_2), \quad \{\hat{c}^\dagger_{l}(\overrightarrow{p}), \hat{c}_{r}(\overrightarrow{p})\} = s_0 \delta_{l,r}\delta^{(K)}(\overrightarrow{p}_1, \overrightarrow{p}_2),
\]

where in addition, any anticommutator between any two elements pertaining to the set formed by the operators and their Hermitian transposes, vanishes, if the two operators pertain to different classes of the four types "a", "c", "b" and "d". The number \( s_0 = \pm 1 \) in this case, also allows for fixing positive or negative norm for the massless single particle states. Therefore, in what follows, the quantities \( \hat{a}_{l}(\overrightarrow{p}), \hat{c}_{l}(\overrightarrow{p}), b_{r}(\overrightarrow{p}) \) and \( d_{r}(\overrightarrow{p}) \) will be assumed to define the annihilation operators of the massless and massive quarks, and their Hermitian conjugate quantities are associated to the creation operators of the corresponding quarks.

Considering these definitions \( H \) can be expressed in the traditional form

\[
H = \sum_{\overrightarrow{p}} \sum_{l=\pm 1} (\epsilon_0(\overrightarrow{p}) \hat{a}^\dagger_{l}(\overrightarrow{p})\hat{a}_{l}(\overrightarrow{p}) + \epsilon_0(\overrightarrow{p}) \hat{c}^\dagger_{l}(\overrightarrow{p})\hat{c}_{l}(\overrightarrow{p}))+
\]

\[
\sum_{\overrightarrow{p}} \sum_{r=1,2} (\epsilon_{m_r}(\overrightarrow{p}) b^\dagger_{r}(\overrightarrow{p})b_{r}(\overrightarrow{p}) + \epsilon_{m_r}(\overrightarrow{p}) d^\dagger_{r}(\overrightarrow{p})d_{r}(\overrightarrow{p})).
\]

The fields are now written as follows

\[
\Psi(x) = \Psi_{0}(x) + \Psi_{m}(x) = \sum_{\kappa=0, m} \Psi_{\kappa}(x),
\]

\[
\Psi_{0}(x) = \frac{1}{\sqrt{V}} \sum_{\overrightarrow{p}} \sum_{l=\pm 1} (\hat{a}^\dagger_{l}(\overrightarrow{p})u^\dagger_{l}(\overrightarrow{p}) \exp(-i (\epsilon_0(\overrightarrow{p})t - \overrightarrow{p} \cdot \overrightarrow{x})) +
\]

\[
\hat{c}_{l}(\overrightarrow{p})v^\dagger_{l}(\overrightarrow{p}) \exp(i (\epsilon_0(\overrightarrow{p})t + \overrightarrow{p} \cdot \overrightarrow{x}))) = u_0(x) + v_0(x),
\]

\[
\Psi_{m}(x) = \frac{1}{\sqrt{V}} \sum_{\overrightarrow{p}} \sum_{r=1,2} (b_{r}(\overrightarrow{p})u_{m_r}(\overrightarrow{p}) \exp(-i (\epsilon_{m_r}(\overrightarrow{p})t - \overrightarrow{p} \cdot \overrightarrow{x}))+
\]

\[
d^\dagger_{r}(\overrightarrow{p})v_{m_r}(\overrightarrow{p}) \exp(i (\epsilon_{m_r}(\overrightarrow{p})t + \overrightarrow{p} \cdot \overrightarrow{x}))) = u_m(x) + v_m(x).
\]
This indicates that the massless field states should have negative norm. For the massless case, this condition imposes that the time evolution of the operators generated by the usual similarity transformation with the evolution operator of the operators $\hat{a}_i(\vec{p})$ and $\hat{c}_i^+(\vec{p})$ should reproduce the classical time dependence of the field $\Psi_0(x)$. That is

$$\hat{a}_i(\vec{p},t) \equiv \exp(i H t) \hat{a}_i(\vec{p}) \exp(-i H t).$$

$$= \exp(i \epsilon_0(\vec{p}) \hat{a}_i(\vec{p})^\dagger \hat{a}_i(\vec{p}) t) \hat{a}_i(\vec{p}) \exp(-i \epsilon_0(\vec{p}) \hat{a}_i(\vec{p})^\dagger \hat{a}_i(\vec{p}) t).$$

$$= \exp(-s_0 \epsilon_0(\vec{p}) \hat{a}_i(\vec{p})^\dagger \rightarrow \exp(i \epsilon_0(\vec{p}) \hat{a}_i(\vec{p}) \rightarrow \exp(-i \epsilon_0(\vec{p}) \hat{a}_i(\vec{p}) t).$$

(53)

$$\hat{c}_i^+(\vec{p},t) = \exp(i H t) \hat{c}_i^+(\vec{p}) \exp(-i H t).$$

$$= \exp(i \epsilon_0(\vec{p}) \hat{c}_i(\vec{p})^\dagger \hat{c}_i(\vec{p}) t) \hat{c}_i(\vec{p}) \exp(-i \epsilon_0(\vec{p}) \hat{c}_i(\vec{p})^\dagger \hat{c}_i(\vec{p}) t).$$

$$= \exp(i s_0 \epsilon_0(\vec{p}) \hat{c}_i(\vec{p}) \rightarrow \exp(-i \epsilon_0(\vec{p}) \hat{c}_i(\vec{p}), 0).$$

(54)

Thus, the Heisenberg time evolution of the operators will coincide with the classical one, upon the selection $s_0 = -1$. This indicates that the massless field states should have negative norms.

On the contrary, the similar consistency conditions directly lead to the usual selection $s_m = 1$ determining the usual positive norm for the one particle massive states. We thus conclude that, as suggested by the initial form of the free propagator, the Fock space of the free theory show an indefinite metric for fermion states, in which massless one particle states have negative norm and massive states show a positive one.

C. The operator field equations

Let us show that the defined field operator and its Hermitian conjugate solve the quantum version of the classical Hamiltonian and Lagrange equations following by applying the standard quantization rules to the generalized Hamiltonian system.

Since the entering two fields in any of the following expressions, are defined by two sets of creation or annihilation operators with a vanishing anticommutator between any element in one of the sets with any element pertaining to the other set, the following standard vanishing commutator relation at different spatial point and coincident time, follow for the massless and massive components of the fields

$$\{\Psi_0(x), \Psi_0(x')\}|_{x_0=x'_0} = 0,$$

$$\{\Psi_0(x), \Psi_m(x')\}|_{x_0=x'_0} = 0,$$

$$\{\Psi_0(x), \Psi_m(x')\}|_{x_0=x'_0} = 0,$$

$$\{\Psi_m(x), \Psi_m(x')\}|_{x_0=x'_0} = 0.$$  

(55)

Their Hermitian conjugate expressions furnish a similar set of relations including Hermitian conjugate fields. For the relation between the fields and their Hermitian conjugate (h.c.) being of the same massless or massive class, it follows

$$\{\Psi_0(x), \Psi_0^{\dagger}(x')\}|_{x_0=x'_0} = T_1 + T_2,$$

$$T_1 = -\frac{1}{V} \sum_{\vec{p},\vec{l}} \sum_{\vec{p}',\vec{l}'} u_i(p) \otimes u_i^\dagger(\vec{p}') \{\hat{a}_i(\vec{p}), \hat{a}_i(\vec{p}')\} \times$$

$$\exp (i (\epsilon_0(\vec{p}') - \epsilon_0(\vec{p})) x_0 - i(\vec{p}' - \vec{p}, \vec{x} - \vec{x}')) \times$$

$$= -\frac{1}{V} \sum_{\vec{p},\vec{l}} u_i(p) \otimes \sum_{\vec{p}',\vec{l}'} \hat{a}_i(\vec{p}) \exp (\epsilon_0(\vec{p}') \vec{x} - \vec{x}'),$$

$$T_2 = -\frac{1}{V} \sum_{\vec{p},\vec{l}} v_\ell(p) \otimes v_\ell^\dagger(\vec{p}) \exp (\epsilon_0(\vec{p}' \vec{x} - \vec{x}'))$$

$$\{\Psi_0(x), \Psi_0^{\dagger}(x')\}|_{x_0=x'_0} = -\delta^{(D)}(\vec{x} - \vec{x}')) I,$$

(56)

in which the term $T_2$ is following after similar steps. The symbol $\otimes$ will indicate the external product of two spinors leading to a spinor matrix. The negative metric anticommutation relation and spinor completeness relations had been
employed

\[
\{\hat{a}_r^\dagger(\vec{p}'),\hat{a}_l(\vec{p})\} = -\delta_{l,r'}\delta^{(K)}(\vec{p}',\vec{p}) I, \tag{57}
\]

\[
\{\hat{c}_l^\dagger(\vec{p}'),\hat{c}_l(\vec{p})\} = -\delta_{l,l'}\delta^{(K)}(\vec{p}',\vec{p}) I, \tag{58}
\]

\[I = \sum_{l=\pm 1}(u_l(p) \otimes u_l^\dagger(\vec{p}')) + v_l(p) \otimes v_l^\dagger(\vec{p}'), \tag{59}\]

\[
\delta^{(D)}(\vec{x} - \vec{x}') = \sum_{\vec{p}} \frac{1}{V} \exp (-i\vec{p} \cdot (\vec{x}' - \vec{x})) \approx \int \frac{dp^3}{(2\pi)^3} \exp (-i\vec{p} \cdot (\vec{x}' - \vec{x})).
\]

The completeness condition follows from the massless spinor definitions in (24) and (26). Similarly, for the commutation relations between the massive field and its h.c. it is possible to write

\[
\{\Psi_m(x), \Psi_m^\dagger(x')\}|_{x_0 = x_0'} = S_1 + S_2,
\]

\[
S_1 = \frac{1}{\vec{p}}, r, r'=1,2 \sum_{r'=1,2} u_r(p) \otimes u_r^\dagger(\vec{p}') \{'b_r(\vec{p}'), b_r(\vec{p})\} \times \\
\exp (i\epsilon_m(\vec{p}') - \epsilon_m(\vec{p}'))x_0 - i(\vec{p}' \cdot \vec{x}' - \vec{p} \cdot \vec{x})) = \frac{1}{\vec{p}}, r=1,2 \sum_{r'=1,2} u_r(p) \otimes u_r^\dagger(\vec{p}') \exp (-i\vec{p} \cdot (\vec{x}' - \vec{x})),
\]

\[
S_2 = \frac{1}{\vec{p}}, r=1,2 \sum_{r'=1,2} v_r(p) \otimes v_r^\dagger(\vec{p}') \exp (-i\vec{p} \cdot (\vec{x}' - \vec{x})),
\]

\[
\{\Psi_m(x), \Psi_m^\dagger(x')\}|_{x_0 = x_0'} = \delta^{(D)}(\vec{x} - \vec{x}') I. \tag{60}
\]

In this massive case, the relations employed to arrive to the expression of the commutator were

\[
\{b_r^\dagger(\vec{p}'), b_r(\vec{p})\} = \delta_{r,r'}\delta^{(K)}(\vec{p}',\vec{p}) I, \tag{61}
\]

\[
\{d_r(\vec{p}'), d_r(\vec{p})\} = \delta_{r,r'}\delta^{(K)}(\vec{p}',\vec{p}) I, \tag{62}
\]

\[
I = \sum_{r=\pm 1}(u_r(p) \otimes u_r^\dagger(\vec{p}')) + v_r(p) \otimes v_r^\dagger(\vec{p}'). \tag{63}
\]

The obtained commutation relations between the massless and massive components of the fields, then allow to write for the total fields \(\Psi\) and \(\Psi^\dagger\) the commutations relations

\[
\{\Psi(x), \Psi(x')\}|_{x_0 = x_0'} = \{\Psi^\dagger(x), \Psi^\dagger(x')\}|_{x_0 = x_0'} = 0, \tag{64}
\]

\[
\{\Psi(x), \Psi^\dagger(x')\}|_{x_0 = x_0'} = \{\Psi_0(x), \Psi_0^\dagger(x')\}|_{x_0 = x_0'} + \{\Psi_m(x), \Psi_m^\dagger(x')\}|_{x_0 = x_0'} \tag{65}
\]

which reproduce the similar relations between the Poisson brackets between the coordinates in the classical theory.

Let us now represent the momenta operators defined in terms of the operators \(\Psi(x), \Psi^\dagger(x)\) by the same classical expressions

\[
\Pi_{\Psi}(x) = -i\frac{\beta}{2}\Psi^\dagger(x) + \epsilon \partial_t \Psi^\dagger(x)\beta, \tag{66}
\]

\[
\Pi_{\Psi^\dagger}(x) = -i\frac{\beta}{2}\Psi(x) - \epsilon \beta \partial_t \Psi(x), \tag{67}
\]

written in terms of the defined field operators. Then, for the anticommutator of them with the "coordinates" \(\Psi(x)\)
and Ψ(x) it follows

\[ \{ \Psi(x), \Pi_\Psi(x') \} \big|_{x_0=x'_0} = \propto \{ \Psi_0(x), \partial_t \Psi_{01}(x') \} \big|_{x_0=x'_0} + \propto \{ \Psi_m(x), \partial_t \Psi_{m1}(x') \} \big|_{x_0=x'_0} = R_1 + R_2, \]

\[ R_1 = -\frac{i}{\sqrt{\mathcal{V}}} \sum_{p,l} \epsilon_0(p) (u_l(p) \otimes u_l^\dagger(p') - v_l(p) \otimes v_l^\dagger(p')) \exp \left( -i \mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) \right), \]

\[ R_2 = \frac{i}{\sqrt{\mathcal{V}}} \sum_{p,r} \epsilon_m(p) (u_r(p) \otimes u_r^\dagger(p') - v_r(p) \otimes v_r^\dagger(p')) \exp \left( -i \mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) \right). \]  

(68)

But, after employing the spinors definitions (19), (20) and (24), (20) it follows

\[ \sum_{l=\pm 1} (u_l(p) \otimes u_l^\dagger(p') - v_l(p) \otimes v_l^\dagger(p')) = \left( \begin{array}{cc} 0 & \mathbf{\sigma} \cdot \mathbf{p} \\ \mathbf{\sigma} \cdot \mathbf{p} & 0 \end{array} \right), \]

\[ \sum_{r=1,2} (u_r(p) \otimes u_r^\dagger(p') - v_r(p) \otimes v_r^\dagger(p')) = \frac{m_f}{\epsilon_m(p)} \left( \begin{array}{cc} I & \mathbf{\sigma} \cdot \mathbf{p} \\ -\mathbf{\sigma} \cdot \mathbf{p} & -I \end{array} \right), \]  

(69)

(70)

which gives the commutation rule

\[ \{ \Psi(x), \Pi_\Psi(x') \} \big|_{x_0=x'_0} = i \delta^{(D)}(\mathbf{x} - \mathbf{x}') I. \]  

(71)

Therefore, the momentum operators defined in terms of the field operators as following from the classical equations also satisfies the usual quantization condition defining the anticommutator between the momenta and coordinates as given by the classical Poisson bracket between them times the imaginary unit.

Considering that in identical form as it happens for the classical Grassman fields and momenta (\( \Pi_\Psi(x') \dagger = -\Pi_{\Psi\dagger}(x') \)), and after taking the Hermitian transpose of (71) it also follows

\[ \{ \Psi^\dagger(x), \Pi_{\Psi\dagger}(x') \} \big|_{x_0=x'_0} = i \delta^{(D)}(\mathbf{x} - \mathbf{x}') I, \]  

(72)

for the other set of conjugate pair of coordinate and momentum in the system.

The anticommutation between momenta operators associated to the same fields Ψ or Ψ\dagger

\[ \{ \Pi_{\Psi}(x), \Pi_{\Psi}(x') \} \big|_{x_0=x'_0} = \{ \Pi_{\Psi\dagger}(x), \Pi_{\Psi\dagger}(x') \} = 0, \]  

(73)

directly follow because any of the creation or annihilation operators included in the set \( S \) of them defining each of the entering momenta does not has its Hermitian transposed included in this same set \( S \).

Further, after employing the definitions of the momenta in terms of the fields, the anticommutator between momenta operators associated to the fields Ψ and Ψ\dagger can be expressed in the form

\[ \{ \Pi_{\Psi}(x), \Pi_{\Psi\dagger}(x') \} \big|_{x_0=x'_0} = -\mathcal{V}^2 \{ \partial_t \Psi_{01}(x') \beta, \beta, \partial_t \Psi_0(x) \} + \]

\[ \frac{i}{2} \{ \{ \Psi(x'), \beta, \partial_t \Psi(x) \} - \{ \partial_t \Psi_{01}(x') \beta, \Psi_0(x) \} \} - \mathcal{V}^2 \{ \partial_t \Psi_{m1}(x') \beta, \beta, \partial_t \Psi_m(x) \} + \]

\[ -\frac{i}{2} \{ \Psi^\dagger(x'), \Pi_{\Psi\dagger}(x) \} - \frac{i}{2} \{ \Pi_{\Psi}(x'), \Psi(x) \} = \frac{i}{\sqrt{\mathcal{V}}} \sum_{p'} \psi_{p'}(x') \exp \left( i \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') \right), \]  

(74)

But, the direct evaluation leads to

\[ -\mathcal{V}^2 \{ \partial_t \Psi_0^\dagger(x') \beta, \beta, \partial_t \Psi_0(x) \} = \mathcal{V}^2 \sum_{p} \frac{|\mathbf{p}|^2}{m_f^2} \exp(i \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')), \]

\[ -\mathcal{V}^2 \{ \partial_t \Psi_m^\dagger(x') \beta, \beta, \partial_t \Psi_m(x) \} = -\mathcal{V}^2 \sum_{p} \frac{(|\mathbf{p}|^2 + m_f^2)}{m_f^2} \exp(i \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')), \]

\[ -\frac{i}{2} \{ \Psi^\dagger(x'), \Pi_{\Psi\dagger}(x) \} - \frac{i}{2} \{ \Pi_{\Psi}(x'), \Psi(x) \} = \mathcal{V} \sum_{p} \exp(i \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')), \]  

(75)

(76)

(77)
which after substituted show the resting standard anticommutation rules

$$\{\Pi_\Psi(x),\Pi_{\Psi^\dagger}(x')\}|_{x_0=x_0'} = 0. \quad (78)$$

Therefore, the new defined quantum fields \(\Psi(x)\) and \(\Psi^\dagger(x)\) and their momenta \(\Pi_\Psi(x)\) and \(\Pi_{\Psi^\dagger}\) satisfy the standard quantization rules of the classical classical Grassman counterparts. The Hamiltonian of the system \(H\) can be now defined in terms of the classical one by substituting the classical fields and canonical momenta by the above defined operators in a way assuring that \(H\) is hermitian. The same classical expression gives a Hermitian representation in the form

$$
H = \int dx^3(\Pi_\Psi(x)\partial_t\Psi(x) + \Pi_{\Psi^\dagger}(x)\partial_t\Psi^\dagger(x)) - L
$$

$$= \int dx^3(\frac{1}{x} - \Pi_\Psi (x) \beta \Pi_{\Psi^\dagger} (x) + \frac{1}{4}\Psi^\dagger(x) \beta \Psi(x) -
\frac{i}{2}\Psi^\dagger(x) \beta \Pi_{\Psi^\dagger}(x) - \frac{i}{2}\Pi_\Psi(x) \beta \Psi(x) +
\frac{i}{2}\Psi^\dagger(x)\alpha^\dagger \partial_i \Psi(x) + x^\dagger \Psi^\dagger(x) \nabla \nabla \beta \Psi(x)\}
\quad (79)$$

Then, the use of the commutation relations

$$\{\Psi^\dagger(x),\Pi_{\Psi^\dagger}(x')\}|_{x_0=x_0'} = i \delta^{(D)}(x-x') I, \quad (80)$$

$$\{\Psi(x),\Pi_\Psi(x')\}|_{x_0=x_0'} = i \delta^{(D)}(x-x') I, \quad (81)$$

in conjunction with the rest of them among the fields and momenta which vanish, and the Heisenberg equation of motions for arbitrary operators \(A\)

$$\partial_t A(t) = i \left[ H, A(t) \right], \quad (82)$$

permits to check that the fields and momenta obey the equations of motion

$$\partial_t \Psi(x) = -\frac{1}{\kappa} \beta (\Pi_{\Psi^\dagger}(x) + \frac{i}{2}\Psi(x)), \quad (83)$$

$$\partial_t \Psi^\dagger = \frac{1}{\kappa}(\Pi_\Psi(x) + \frac{i}{2}\Psi^\dagger(x)) \beta, \quad (84)$$

$$\partial_t \Pi_\Psi(x) = i \frac{\beta}{2\kappa} \Pi_\Psi(x) \beta - \frac{1}{4\kappa}\Psi^\dagger(x) \beta
+ i \Psi^\dagger \alpha^\dagger \nabla_i + x^\dagger \nabla^2 \Psi^\dagger \beta, \quad (85)$$

$$\partial_t \Pi_{\Psi^\dagger}(x) = -i \frac{\beta}{2\kappa} \Pi_{\Psi^\dagger}(x) \beta + \frac{1}{4\kappa} \beta \Psi(x)
+ i \alpha^\dagger \partial_i \Psi^\dagger - x^\dagger \beta \nabla^2 \Psi. \quad (86)$$

The further elimination of the momenta among these equations check that the "coordinate" fields also satisfy the original Lagrange equations

$$(-i \gamma^\mu \partial_\mu - x^\dagger \partial^2) \Psi(x) = 0, \quad (88)$$

$$\Psi^\dagger(x) \left( i \gamma^\mu \partial_\mu - x^\dagger \partial^2 \right) = 0. \quad (89)$$

### V. SUMMARY

In this work, it has been studied the nature of the free quantization of the quark free Lagrangian associated to the recently proposed local and gauge invariant alternative for QCD for interacting massive quarks. It was argued that the free theory can be quantized in a way in which the massive quarks states can show a positive metric and the massless ones a negative norm. The quantization of the classical theory had been implemented, and an explicit representation of the fields and momenta operators solving the Heisenberg equations of motions was constructed. It is underlined that the accepted absence of asymptotic states for quark and gluons in QCD indicates that the appearance of indefinite metric in the quark sector of the theory does not necessarily present a definite limitation of the the
in a very similar way for the negative energy contributions illustrated for two components. The evaluation for rest of the terms follows very similar algebraical lines. Below, the main algebraic steps in evaluating the general components of the energy operator (37) are explicitly as the superposition of the number operators for the various particles after multiplied by their respective energies.

Appendix A

In this appendix we sketch the derivation of the terms which define the standard formula for the energy operator as the superposition of the number operators for the various particles after multiplied by their respective energies. Below, the main algebraic steps in evaluating the general components of the energy operator are explicitly illustrated for two components. The evaluation for rest of the terms follows very similar algebraical lines.

For the contribution of massless positive energy states it follows

\[
\int dx^3 u_0^\dagger(x)h(\partial)u_0(x) = \sum_{\vec{p}} \sum_{\vec{p}'} \sum_{l=\pm 1} \sum_{l'=\pm 1} \frac{1}{2} \times 
\]

\[
(\beta l' \beta^T(\vec{p}')) (l' \beta^T(\vec{p}')) \times 
\]

\[
\left( \varphi(\epsilon_0(\vec{p}') \epsilon_0(\vec{p})) + \vec{p}' \cdot \vec{p} \right) - \varphi(\epsilon_0(\vec{p}') \epsilon_0(\vec{p}) + \vec{p}' \cdot \vec{p}) \right) \times 
\]

\[
\frac{1}{V} \int dx^3 \exp(i (\vec{p} - \vec{p}').\vec{x}) \hat{a}_{l'}^\dagger(\vec{p}') \hat{a}_l(\vec{p}) 
\]

\[
= \sum_{\vec{p}} \sum_{\vec{p}'} \sum_{l=\pm 1} \sum_{l'=\pm 1} \delta^l l' \delta^K(\vec{p}', \vec{p}) \frac{1}{2} (\beta l' \beta^T(\vec{p})) \frac{1}{2} \beta^T(\vec{p}) \times \epsilon_0(\vec{p}') \epsilon_0(\vec{p}) +
\]

\[
\hat{a}_l(\vec{p}) \hat{a}_{l'}^\dagger(\vec{p}') \delta^l l' \delta^K(\vec{p}', \vec{p}) (1 \mp l' l + l(l + l') |\vec{p}'\rangle |\vec{p}\rangle) \hat{a}_l(\vec{p}') \hat{a}_{l'}^\dagger(\vec{p}) 
\]

\[
= \sum_{\vec{p}} \sum_{l=\pm 1} \frac{1}{2} 2 |\vec{p}'\rangle |\vec{p}\rangle \hat{a}_{l'}^\dagger(\vec{p}) \hat{a}_l(\vec{p}') 
\]

\[
= \sum_{\vec{p}} \sum_{l=\pm 1} \epsilon_0(\vec{p}) \hat{a}_{l'}^\dagger(\vec{p}) \hat{a}_l(\vec{p}') 
\]

(A1)

In a very similar way for the negative energy contributions

\[
\int dx^3 v_0^\dagger(x)h(\partial)v_0(x) = \sum_{\vec{p}} \sum_{l=\pm 1} \epsilon_0(\vec{p}) \hat{c}_l^\dagger(\vec{p}) \hat{c}_l(\vec{p}') 
\]

(A2)
In the case of the terms related with massive quarks, the derivation of the energy term is sketched as follows

\[
\int dx^3 u_m^\dagger(x) h(\partial) u_m(x) = \sum_{\vec{p}, \vec{p}'} \sum_{r=1,2} \sum_{r'=1,2} \sum_{m_f} \sqrt{\frac{\epsilon_{m_f}(\vec{p}')}{-\vec{p} \cdot \vec{p}'} + m_f} \times \sqrt{\frac{\epsilon_{m_f}(\vec{p})}{2\epsilon_{m_f}(\vec{p})}} \times \frac{1}{\sqrt{2}} \int dx^3 \exp(i(\vec{p} - \vec{p}').x) \times b_r(\vec{p}) \\
\times \left( u^{tr}(\vec{p}'), u^{tr'}(\vec{p}') \frac{\sigma \cdot \vec{p}}{\epsilon_m(\vec{p}) + m_f} \right) \times \left( \epsilon_m(\vec{p}) \epsilon_{m_f}(\vec{p}') + \vec{p} \cdot \vec{p}' - \frac{-\sigma \cdot \vec{p}}{\epsilon_{m_f}(\vec{p}) + m_f} \right) \times \left( \frac{-\sigma \cdot \vec{p}}{\epsilon_{m_f}(\vec{p}) + m_f} \right) u^r(\vec{p}) \right).
\]

After evaluating the spinor products, expressing the spatial integral as the Kronecker delta of the two spatial momenta, and using the orthonormality of the two component spinors, this expression can be simplified in the way described below

\[
\int dx^3 u_m^\dagger(x) h(\partial) u_m(x) = \sum_{\vec{p}, \vec{p}'} \sum_{r=1,2} \left( \epsilon_m(\vec{p}) \epsilon_{m_f}(\vec{p}') + \vec{p} \cdot \vec{p}' \right) \left( 1 - \frac{|\vec{p}|^2}{(\epsilon_{m_f}(\vec{p}) + m_f)^2} \right) - \frac{2}{\epsilon_{m_f}(\vec{p}) + m_f} b_r(\vec{p}) \epsilon_{m_f}(\vec{p}) + m_f \\
= \sum_{\vec{p}, \vec{p}'} \sum_{r=1,2} \left( \epsilon_m(\vec{p}) \epsilon_{m_f}(\vec{p}') + \vec{p} \cdot \vec{p}' \right) \left( \frac{|\vec{p}|^2 + m_f^2 - |\vec{p}'|^2 + m_f^2 - m_f^2}{(\epsilon_{m_f}(\vec{p}) + m_f)^2} \right) \\
\times \frac{2}{\epsilon_{m_f}(\vec{p}) + m_f} b_r(\vec{p}) \epsilon_{m_f}(\vec{p}) + m_f \\
= \sum_{\vec{p}, \vec{p}'} \epsilon_m(\vec{p}) \epsilon_{m_f}(\vec{p}') b_r(\vec{p}) \\
= \sum_{\vec{p}, \vec{p}'} \epsilon_m(\vec{p}) \epsilon_{m_f}(\vec{p}') \left( d_r(\vec{p}) + d_r(\vec{p}') \right) \]

Finally, for the negative energy massive term, a closely similar calculation gives

\[
\int dx^3 v_m(\vec{p}) h(\partial) v_m(x) = \sum_{\vec{p}, r=1,2} \epsilon_{m_f}(\vec{p}) \left( d_r(\vec{p}) + d_r(\vec{p}') \right). \quad (A4)
\]

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