BETTER BOUNDS FOR PLANAR SETS AVOIDING UNIT DISTANCES

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Abstract. A 1-avoiding set is a subset of $\mathbb{R}^n$ that does not contain pairs of points at distance 1. Let $m_1(\mathbb{R}^n)$ denote the maximum fraction of $\mathbb{R}^n$ that can be covered by a measurable 1-avoiding set. We prove two results. First, we show that any 1-avoiding set in $\mathbb{R}^n$ that displays block structure (i.e., is made up of blocks such that the distance between any two points from the same block is less than 1 and points from distinct blocks lie farther than 1 unit of distance apart from each other) has density strictly less than $1/2^n$. For the special case of sets with block structure this proves a conjecture of Erdős asserting that $m_1(\mathbb{R}^2) < 1/4$. Second, we use linear programming and harmonic analysis to show that $m_1(\mathbb{R}^2) \leq 0.258795$.

1. Introduction

The unit-distance graph in $\mathbb{R}^n$ is the graph whose vertex set is $\mathbb{R}^n$ and in which $x$ and $y$ are adjacent if $\|x-y\| = 1$. A well-known problem in geometry, going back to Nelson and Hadwiger (see Soifer [11] for a historical survey), asks for the chromatic number $\chi(\mathbb{R}^n)$ of the unit-distance graph in $\mathbb{R}^n$.

A related problem considers independent sets of the unit-distance graph. Let $G = (V,E)$ be a graph. A set $I \subseteq V$ is independent if it does not contain a pair of adjacent vertices. The independence number of $G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent set.

A set $A \subseteq \mathbb{R}^n$ is independent in the unit-distance graph if it does not contain pairs of points at distance 1, that is, $\|x-y\| \neq 1$ for all $x, y \in A$. We also say that $A$ avoids distance 1 or that it is a 1-avoiding set. Independent sets of the unit-distance graph can have infinite cardinality, so the independence number is infinite. A better measure for the size of an independent set in this case is its density, that is, the fraction of space that it covers (see [11] for a rigorous definition). So we ask the question: what is the maximum fraction of $\mathbb{R}^n$ that can be covered by a measurable set that avoids distance 1?

We denote this maximum fraction by $m_1(\mathbb{R}^n)$. In [3] we show that $m_1(\mathbb{R}^2) \leq 0.258795$. This result is related to a conjecture of Erdős (cf. Székely [13]), stating that $m_1(\mathbb{R}^2) < 1/4$.

Determining $\chi(\mathbb{R}^n)$ is a difficult problem. Even for the Euclidean plane, all that is known is that $4 \leq \chi(\mathbb{R}^2) \leq 7$. The upper bound comes from a simple periodic coloring of $\mathbb{R}^2$, whereas the lower bound comes from a finite subgraph...
of the unit-distance graph, the Moser spindle (cf. Moser and Moser \[8\]), whose chromatic number is 4 (see Figure 1).

Figure 1. On the left, the Moser spindle \[8\]. Each segment has length exactly 1. On the right, the optimal tortoise in Croft’s construction \[2\].

In view of the difficulty of computing $\chi(\mathbb{R}^n)$, Falconer \[3\] introduced the measurable chromatic number, denoted by $\chi_m(\mathbb{R}^n)$, in which the restriction is added that the color classes must be Lebesgue-measurable sets. In other words, one wishes to partition $\mathbb{R}^n$ into the minimum possible number of Lebesgue-measurable 1-avoiding sets. Obviously, $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. Falconer proved that $\chi_m(\mathbb{R}^2) \geq 5$. Since $m_1(\mathbb{R}^n)\chi_m(\mathbb{R}^n) \geq 1$, showing Erdős’ conjecture would give another proof of Falconer’s result.

A simple lower bound for $m_1(\mathbb{R}^2)$ comes from the following construction. Consider the hexagonal lattice with minimal vectors of length 2 and place at each point of the lattice an open disk of radius $1/2$. This is a 1-avoiding set of density $\pi/(8\sqrt{3}) = 0.2267\ldots$. A slight improvement was given by Croft \[2\]. His construction is as follows. Shrink the hexagonal lattice slightly so as to have minimal vectors of length $1 + x$, where $x < 1$, and place at each lattice point the intersection of an open disk of radius 1/2 and an open regular hexagon of height $x$. The disks guarantee that inside each block every distance is less than 1, while the hexagons guarantee that points from different blocks have distance greater than 1. Taking $x = 0.96553\ldots$ maximizes the density of the union of all the blocks, showing that $m_1(\mathbb{R}^2) \geq 0.22936$. This intersection of a disk with a hexagon has often been called a tortoise; Figure 1 shows an optimal tortoise.

Any finite subgraph $G = (V,E)$ of the unit-distance graph in $\mathbb{R}^n$ provides an upper bound for $m_1(\mathbb{R}^n)$, namely $\alpha(G)/|V|$; this can be seen via a simple averaging argument.\footnote{This is related to the following observation: Let $G$ be a subgraph of a finite vertex-transitive graph $H$. Then $\alpha(H)/|V(H)| \leq \alpha(G)/|V(G)|$.}

For the plane, we then immediately have that $m_1(\mathbb{R}^2) \leq 1/3$, as can be seen from the equilateral triangle. A better bound of $2/7$ is provided by the Moser spindle; this is the best upper bound that has been obtained from a finite subgraph.

Using further ideas, Székely \[14\] proved a bound of $\approx 0.279 < 2/7 \approx 0.285$. Oliveira and Vallentin \[9\] gave the currently best known upper bound of $\approx 0.268$. Their method is based on a mix of linear programming and harmonic analysis; it is a strengthening of it that will be used in \[4\] to prove that $m_1(\mathbb{R}^2) \leq 0.258795$.

Finally, Frankl and Wilson \[4\] construct finite subgraphs of the unit-distance graph of $\mathbb{R}^n$ whose chromatic numbers grow exponentially fast in $n$. These same
subgraphs can be used to provide the asymptotic upper bound
\[ m_1(\mathbb{R}^n) \leq (1 + o(1))1.207^{-n}. \]
via the averaging argument. The method of Oliveira and Vallentin [9] also provides an exponential upper bound, \( m_1(\mathbb{R}^n) \leq (1 + o(1))1.1654^{-n} \), which is somewhat weaker than the bound of Frankl and Wilson. A combination of both arguments, requiring detailed analysis, was used by Bachoc, Passuello, and Thiery [1] to obtain the best known asymptotic upper bound
\[ m_1(\mathbb{R}^n) \leq (1 + o(1))1.268^{-n}. \]

The behavior of \( \chi(\mathbb{R}^n) \) under restrictions placed on the color classes has also been studied. We have already mentioned Falconer’s measurable chromatic number. Other restrictions include requiring all classes to be either open or closed sets — in both cases it can be shown that the chromatic number of \( \mathbb{R}^2 \) is either 6 or 7 (cf. Soifer [11]).

We may place similar restrictions on 1-avoiding sets and study the behavior of \( m_1(\mathbb{R}^n) \). A natural idea is to consider sets with block structure. We say that a set \( A \) has block structure if it is a union
\[ A = \bigcup_{i=0}^{\infty} A_i, \]
of blocks \( A_i \), where \( \|x-y\| < 1 \) if \( x \) and \( y \) belong to the same block, and \( \|x-y\| > 1 \) if \( x \) and \( y \) belong to different blocks.

A set with block structure is therefore a 1-avoiding set. All known constructions of 1-avoiding sets of “high density” are actually constructions of sets with block structure, like for instance the hexagonal lattice construction or Croft’s construction. Recall that Erdős conjectured that \( m_1(\mathbb{R}^2) < 1/4 \). Larman and Rogers, and before them Moser (cf. Larman and Rogers [6]), made the following conjecture: the volume of a closed 1-avoiding set inside a ball of radius 1 in \( \mathbb{R}^n \) is less than \( 1/2^n \) of the volume of the ball. A simple argument shows that this conjecture implies \( m_1(\mathbb{R}^n) < 1/2^n \), and it is therefore a generalization of Erdős’ conjecture. In [2] we will show that any subset of \( \mathbb{R}^n \) with block structure has upper density less than \( 1/2^n \).

1.1. Preliminaries and notation. Throughout the paper, \(|A|\) will denote the Lebesgue measure of a set \( A \). This is the same notation used for the cardinality of a set, but the meaning will be clear on each application.

We have so far talked about the density of a set in an informal way. Here is a formal definition. Let \( A \subseteq \mathbb{R}^n \) be a measurable set. We say that its density is \( \delta(A) \) if for all \( p \in \mathbb{R}^n \) we have
\[ \delta(A) = \lim_{r \to \infty} \frac{|A \cap S(p, r)|}{|S(p, r)|}, \]
where \( S(p, r) \) is the \( n \)-dimensional cube of side \( 2r \) centered at \( p \). For a set that has a density, the cube can be substituted by any reasonable body, like the ball, say, without changing the resulting density.

Not every set has a density, but every set has an upper density
\[ \overline{\delta}(A) = \sup_{p \in \mathbb{R}^n} \limsup_{r \to \infty} \frac{|A \cap S(p, r)|}{|S(p, r)|}. \]
So we may define
\[ m_1(\mathbb{R}^n) = \sup \{ \overline{\delta}(A) : A \subseteq \mathbb{R}^n \text{ is 1-avoiding and measurable} \}. \]
We say set \( A \) is periodic if there is a lattice \( L \subseteq \mathbb{R}^n \) that leaves \( A \) invariant, that is, \( x + A = A \) for all \( x \in L \). Then \( L \) is the periodicity lattice of \( A \).

Periodic sets are well-behaved and have densities. Moreover, a simple argument shows that the densities of periodic 1-avoiding sets can come as close as desired to \( m_1(\mathbb{R}^n) \) (cf. Oliveira and Vallentin [13]). So when computing upper bounds for \( m_1(\mathbb{R}^n) \) we may restrict ourselves to periodic sets.

2. Sets with block structure

In any dimension, all known examples of 1-avoiding sets of “high density” are made up of disjoint blocks, i.e., they are sets of block structure as defined in the introduction (however, see the end of this section for an example of a 1-avoiding set of small positive density which does not have block structure).

On the one hand it is natural to consider this class of sets because human imagination of 1-avoiding sets seems to be more or less restricted to this class. On the other hand, it seems very elusive to prove rigorously that a 1-avoiding set of maximum or close to maximum density must be block-structured.

The following theorem, for \( n = 2 \), is a special case of the conjecture of Erdős (cf. Székely [13]) given in the introduction; for \( n \geq 3 \), it is a special case of a conjecture of Larman, Rogers, and Moser (cf. Larman and Rogers [6]), also discussed in the introduction.

**Theorem 2.1.** Let \( n \geq 2 \) and let \( A \subseteq \mathbb{R}^n \) be a measurable 1-avoiding set having block structure. Then \( \overline{\delta}(A) \leq 1/2^n - \varepsilon_n \) for some \( \varepsilon_n > 0 \).

In fact, we will prove the following slightly stronger result.

**Theorem 2.2.** Let \( A_1, A_2, \ldots \subseteq \mathbb{R}^n \) be measurable sets of diameter at most 1 such that the distance of any two of them is at least 1. Then the upper density of \( A = \bigcup_{i=1}^{\infty} A_i \) is at most \( 1/2^n - \varepsilon_n \) for some \( \varepsilon_n > 0 \).

First we present a rather simple proof for the weaker statement when \( \varepsilon_n \) is discarded, that is, we prove first that \( \overline{\delta}(A) \leq 1/2^n \).

Let \( B_r \) denote the open ball of radius \( r \) around the origin and let

\[
C_i = A_i + B_{1/2} = \{ a + b : a \in A_i, b \in B_{1/2} \}.
\]

It is clear from the assumptions that \( C_i \cap C_j = \emptyset \), for all \( i \neq j \). Note that \( \overline{C_i} = \overline{A_i} + \overline{B_{1/2}} \), where \( \overline{X} \) is the closure of \( X \). We claim that \( |C_i| = |\overline{C_i}| \).

To see this, consider any point \( x \) on the the boundary \( \partial C_i \) of \( C_i \) and for any \( 0 < \varepsilon < 1/2 \) consider an open ball \( B(x, \varepsilon) \) around \( x \). As \( x \) is on the boundary of \( C_i \) there exists an open ball \( B = a + B_{1/2} \) with \( a \in A_i \) such that the distance of \( x \) and \( B \) is less than \( \varepsilon/2 \). Then \( |B \cap B(x, \varepsilon)|/|B(x, \varepsilon)| \geq c_n \), where \( c_n = |B \cap (B_{1/2} + (1/2, 0, \ldots, 0))|/|B_{1/2}| > 0 \). The set \( B \cap B(x, \varepsilon) \) fully belongs to \( C_i \), so it is disjoint from \( \partial C_i \). This shows that the density of \( \partial C_i \) at any point \( x \in \partial C_i \) is at most \( 1 - c_n < 1 \). This implies, via the Lebesgue density theorem, that \( |\partial C_i| = 0 \).

Applying the Brunn-Minkowski inequality to the compact sets \( \overline{A_i}, \overline{B_{1/2}} \) we obtain

\[
|C_i|^{1/n} \leq |\overline{A_i}|^{1/n} + |\overline{B_{1/2}}|^{1/n}.
\]

Furthermore the sets \( \overline{A_i} \) still have diameter \( \leq 1 \) so the isodiametric inequality gives \( |\overline{A_i}| \leq |\overline{B_{1/2}}| \). By combining these inequalities we get

\[
\frac{|A_i|^{1/n}}{|C_i|^{1/n}} \leq \frac{|\overline{A_i}|^{1/n}}{|\overline{A_i}|^{1/n} + |\overline{B_{1/2}}|^{1/n}} \leq \frac{1}{1 + \left(\frac{|\overline{B_{1/2}}|}{|\overline{A_i}|}\right)^{1/n}} \leq \frac{1}{1 + 1} = \frac{1}{2}.
\]

Since the sets \( C_i \) are pairwise disjoint, \( A_i \subseteq C_i \), and \( A = \bigcup_i A_i \), this shows that \( \overline{\delta}(A) \leq 1/2^n \).
The plan to show $\delta(A) \leq 1/2^n - \varepsilon_n$ is the following: If $\delta(A)$ is close to $1/2^n$ then in the above argument we must have that for most $i$ the isodiametric inequality is almost an equality and $A_i$ is close to 1. By a stability theorem this implies that each such $A_i$ is very close to a ball of radius $1/2$ and then most of the sets $C_i = A_i + B_{1/2}$ are very close to unit balls. But the density of any unit ball packing is well-separated from 1, and then so is the density of $\bigcup_i C_i$. Since $A$ has density at most $1/2^n$ in $\bigcup_i C_i$, this implies that the density of $A$ is well separated from $1/2^n$.

The stability result we use is the following theorem of Maggi, Ponsiglione, and Pratelli [7].

**Theorem 2.3.** Let $E \subseteq \mathbb{R}^n$ be a measurable set with $|E| > 0$ and $\text{diam}(E) = 2$. Then there exist $x, y \in \mathbb{R}^n$ such that

$$E \subseteq B(x, 1 + r) \quad \text{and} \quad B(y, 1) \subseteq E + B_{r},$$

where $B(z, R)$ denotes the ball centered at $z$ with radius $R$ and

$$r = K_n \left( \frac{|B_1|}{|E|} - 1 \right)^{1/n}$$

for some constant $K_n$ that depends only on $n$.

Note that for sets of diameter 2 the expression $|B_1|/|E| - 1$, which is called *isodiametric deficit* by Maggi, Ponsiglione, and Pratelli, is nonnegative by the isodiametric inequality and expresses the error in that inequality.

We will need the following simple corollary of the above theorem.

**Corollary 2.4.** For any $n \geq 2$ there exists an increasing function $\beta = \beta_n : (0, \infty) \to (0, \infty)$ with $\lim_{\rho \to 0} \beta_n(\rho) = 0$ with the following property. For every measurable $E \subseteq \mathbb{R}^n$ with $|E| > 0$ and $\text{diam} E \leq 1$ there exist $x, y \in \mathbb{R}^n$ such that

$$E \subseteq B(x, 1/2 + \beta(\rho(E))) \quad \text{and} \quad B(y, 1/2) \subseteq E + B_{\beta(\rho(E))},$$

where

$$\rho(E) = \frac{|B_{1/2}|}{|E|} - 1.$$

**Proof.** By rescaling Theorem 2.3 we get that for any measurable $E \subseteq \mathbb{R}^n$ with $|E| > 0$ and $\text{diam} E < \infty$ there exist $x, y \in \mathbb{R}^n$ such that

$$E \subseteq B \left( x, \frac{\text{diam} E}{2} + r \right) \quad \text{and} \quad B \left( y, \frac{\text{diam} E}{2} \right) \subseteq E + B_r,$$

where

$$r = K_n \frac{\text{diam} E}{2} \left( \frac{|B_{(\text{diam} E)/2}|}{|E|} - 1 \right)^{1/n}.$$

If $\text{diam} E \leq 1$ then

$$r = K_n \frac{\text{diam} E}{2} \left( \frac{|B_{(\text{diam} E)/2}|}{|E|} - 1 \right)^{1/n} \leq K_n \frac{\rho(E)}{2} \rho(E)^{1/n},$$

so

$$E \subseteq B \left( x, \frac{\text{diam} E}{2} + r \right) \subseteq B \left( x, \frac{1}{2} + \frac{K_n}{2} \rho(E)^{1/n} \right).$$

By the isodiametric inequality we have $|E| \leq |B_{(\text{diam} E)/2}|$, so

$$\rho(E) = \frac{|B_{1/2}|}{|E|} - 1 \geq \frac{|B_{1/2}|}{|B_{(\text{diam} E)/2}|} - 1 = \frac{1}{(\text{diam} E)^n} - 1,$$

hence

$$\text{diam} E \geq \left( \frac{1}{1 + \rho(E)} \right)^{1/n}.$$
From the second part of (2), using $B(z, a) + B_y = B(z, a + b)$ we get that
\[ B(y, 1/2) \subseteq E + B_{r + (1 - \text{diam } E)/2}. \]

Then (4), (5) and (6) show that the function
\[ \beta_n(\rho) = \frac{K_n}{2} \rho^{1/n} + \frac{1}{2} \left( 1 - \left( \frac{1}{1 + \rho} \right)^{1/n} \right) \]
has all the required properties.

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** We fix the dimension $n$; all constants may depend on it. We will show that if $N$ is large enough then the density of $A$ in $[-N, N]^n$ is at most $1/2^n - \varepsilon_n$ for some positive $\varepsilon_n$.

Suppose without loss of generality that the blocks $A_i$ are enumerated so that $A_1, \ldots, A_m$ are the ones that have nonempty intersection with $[-N, N]^n$ and so
\[ A \cap [-N, N]^n = \bigcup_{i=1}^m A_i \cap [-N, N]^n. \]

Let
\[ \rho_i = \rho(A_i) = \frac{|B_{1/2}|}{|A_i|} - 1. \]

Then using (1) we get
\[ |A_i|^{1/n} \leq \frac{1}{1 + \left( \frac{|B_{1/2}|}{|A_i|} \right)^{1/n}} = \frac{1}{1 + (\rho_i + 1)^{1/n}}. \]

Let $\alpha$ and $\rho$ be two small positive constants that will be specified later. Let
\[ I = \{ i \in \{1, \ldots, n\} : \rho_i \geq \rho \} \quad \text{and} \quad J = \{1, \ldots, n\} \setminus I. \]

By (8), for every $i \in I$ we have
\[ |A_i| \leq |C_i| \left( \frac{1}{1 + (\rho + 1)^{1/n}} \right)^n. \]

By (1), for every $i \in J$ we have $|A_i| \leq |C_i|/2^n$. Therefore we have
\begin{align*}
|A \cap [-N, N]^n| &\leq \sum_{i \in I} |C_i| \left( \frac{1}{1 + (\rho + 1)^{1/n}} \right)^n + \sum_{i \in J} |C_i|/2^n \\
&\leq \sum_{i=1}^m |C_i|/2^n - \sum_{i \in I} |C_i| \left( \frac{1}{2^n - \left( \frac{1}{1 + (\rho + 1)^{1/n}} \right)^n} \right). 
\end{align*}

First we consider the case when
\[ \sum_{i \in I} |C_i| \geq \alpha(2N)^n. \]

Using that the sets $C_i$ are pairwise disjoint and $\bigcup_{i=1}^m C_i \subseteq [-N - 2, N + 2]^n$, we get
\begin{align*}
\frac{|A \cap [-N, N]^n|}{|[-N, N]^n|} &\leq \frac{(2N + 2)^n}{(2N)^n} \frac{1}{2^n} - \alpha \left( \frac{1}{2^n} - \left( \frac{1}{1 + (\rho + 1)^{1/n}} \right)^n \right); \\
\end{align*}
we will come back to this expression later.

Now we consider the case when
\[ \sum_{i \in I} |C_i| < \alpha(2N)^n. \]
By definition for any \( i \in J \) we have \( \rho(\overline{A}_i) < \rho \). By Corollary 2.4 this implies that for any \( i \in J \) there exist \( x_i \) and \( y_i \) such that
\[
\overline{A}_i \subseteq B(x_i, 1/2 + \beta(\rho)) \quad \text{and} \quad B(y_i, 1/2) \subseteq \overline{A}_i + B(\beta(\rho)).
\]
Since \( \lim_{\rho \to 0} \beta(\rho) = 0 \) we can guarantee \( \beta(\rho) < 1/2 \) by taking \( \rho \) small enough.

Note that the first part of (11) implies
\[
C_i = A_i + B_{1/2} \subseteq B(x_i, 1 + \beta(\rho))
\]
and the second part of (11) implies
\[
B(y_i, 1 - \beta(\rho)) \subseteq \overline{A}_i + B_{1/2} = A_i + B_{1/2} = C_i.
\]
Since the sets \( C_i \) are pairwise disjoint this implies that the balls \( B(y_i, 1 - \beta(\rho)) \), for \( i \in J \), are pairwise disjoint.

Let \( \Delta_n \) be an upper bound on the density of the union of disjoint balls of the same size in \( \mathbb{R}^n \). Although the best \( \Delta_n \) is not known for general \( n \), it is known (and not hard to show) that \( \Delta_n < 1 \) for \( n \geq 2 \).

Thus the density of \( \bigcup_{i \in J} B(y_i, 1 - \beta(\rho)) \) in \([-N - 2, N + 2]^n\) is at most \( \Delta_n \), that is,
\[
\frac{1}{(2(N + 2))^n} \sum_{i \in J} |B(y_i, 1 - \beta(\rho))| \leq \Delta_n.
\]

Notice that \( |B(x_i, 1 + \beta(\rho))|/|B(y_i, 1 - \beta(\rho))| = ((1 + \beta(\rho))/(1 - \beta(\rho)))^n \). Let \( A_J = \bigcup_{i \in J} A_i \). Then, using first (11) then (12) and finally (14) we get that
\[
|A_J|/\|[-N, N]^n\| \leq \frac{1}{(2N)^n} \sum_{i \in J} |C_i| \leq \frac{1}{(2N)^n} \sum_{i \in J} |B(x_i, 1 + \beta(\rho))|
\]
\[
\leq \frac{(2(N + 2))^n}{(2N)^n} \cdot \Delta_n \cdot \left(1 + \beta(\rho)/1 - \beta(\rho)\right)^n \cdot \frac{1}{2^n}.
\]

Let \( A_J = \bigcup_{i \in I} A_i \). By (1) and (10) we have
\[
|A_J|/\|[-N, N]^n\| \leq \frac{\alpha}{2^n}.
\]
Therefore combining (15) and (16) we get in this case
\[
|A \cap [-N, N]^n|/\|[-N, N]^n\| \leq \frac{(2(N + 2))^n}{(2N)^n} \cdot \Delta_n \cdot \left(1 + \beta(\rho)/1 - \beta(\rho)\right)^n \cdot \frac{1}{2^n} + \frac{\alpha}{2^n}.
\]

Finally, choose the positive constants \( \alpha, \rho \) and \( \varepsilon_n \) so that \( \beta(\rho) < 1/2 \),
\[
\Delta_n \cdot \left(1 + \beta(\rho)/1 - \beta(\rho)\right)^n + \alpha + 2^n \varepsilon_n < 1
\]
and
\[
\varepsilon_n \leq \alpha \left(\frac{1}{2^n} - \left(\frac{1}{1 + (\rho + 1)^{1/n}}\right)^n\right).
\]
Then by (9) and (17) we get that \( \delta(A) \leq 1/2^n - \varepsilon_n \) in both cases, which completes the proof. \( \square \)

The theorem above suggests that one should try to prove that any 1-avoiding set of “high density” must have block structure. A natural idea is to take any 1-avoiding set \( A \) and try to modify it in some manner to obtain a new 1-avoiding set \( A \) having block structure and at least the same density as \( A \). Unfortunately, we could not prove anything rigorous along this line.

We end this section by presenting an example of a 1-avoiding set of positive (but small) density which does not have block-structure. This example also shows that
not every 1-avoiding set of positive density can be modified in a natural way to obtain a new 1-avoiding set that has block structure and larger or equal density.

Consider the scaled integer lattice \((c\mathbb{Z})^2 \subseteq \mathbb{R}^2\) with \(c = 2\sqrt{2} - 2\), and place an open disk of radius \(r = (3 - 2\sqrt{2})/2\) at each lattice point. It is easy to check that the distance between points of disks around adjacent lattice points is less than 1 and the distance between points of disks around nonadjacent lattice points is bigger than 1. Therefore this is a 1-avoiding set without block structure. (In fact, it has block structure in a more general sense: there is a graph between the blocks and two points have distance less than 1 if and only if they are from the same block or from neighbor blocks. Note that every 1-avoiding set has such a structure.) Simple calculation shows that the the density of this example is \(\delta = r^2 \pi / c^2 \approx 0.0337\).

3. A better upper bound in the plane

We now show how a strengthening of the method of Oliveira and Vallentin [9] can be used to provide the bound

\[ m_1(\mathbb{R}^2) \leq 0.258795. \]

Let \(A \subseteq \mathbb{R}^n\) be a measurable and periodic 1-avoiding set with periodicity lattice \(L \subseteq \mathbb{R}^n\). Its autocorrelation function is the function \(f : \mathbb{R}^n \to \mathbb{R}\) such that

\[ f(x) = \delta(A \cap (A - x)). \]

Determining \(m_1(\mathbb{R}^n)\) is equivalent to the following optimization problem: find a function \(f\) that maximizes \(f(0)\) and is the autocorrelation function of a periodic 1-avoiding set. The difficulty here lies, of course, in the fact that we do not know a complete and useful characterization of autocorrelation functions of 1-avoiding sets. We may, however, look for constraints necessarily satisfied by such functions. If we give up on finding autocorrelation functions, but settle for functions satisfying a few of the constraints that autocorrelation functions satisfy, then we get a relaxation of our original problem and an upper bound for \(m_1(\mathbb{R}^n)\). The following lemma gives some such constraints. Recall from the introduction that by \(\alpha(G)\) we denote the independence number of the graph \(G\).

**Lemma 3.1.** Let \(f\) be the autocorrelation function of a measurable and periodic 1-avoiding set \(A \subseteq \mathbb{R}^n\). Then:

1. \(f(x) = 0\) if \(\|x\| = 1\);
2. if \(G = (V, E)\) is a finite, nonempty subgraph of the unit-distance graph in \(\mathbb{R}^n\), then
   \[ \sum_{x \in V} f(x) \leq f(0)\alpha(G); \]
3. if \(C \subseteq \mathbb{R}^n\) is a finite set of points, then
   \[ \sum_{\{x,y\} \in \binom{C}{2}} f(x - y) \geq |C|f(0) - 1, \]

where \(\binom{C}{2}\) is the set of all pairs of points in \(C\).

**Proof.** For (1) it suffices to observe that, since \(A\) is 1-avoiding, if \(\|x\| = 1\) then \(A \cap (A - x) = \emptyset\).

For (2), we claim that any \(z \in \mathbb{R}^n\) belongs to at most \(\alpha(G)\) of the sets \(A - x\) for \(x \in V\). Indeed, say \(z\) belongs to all sets \(A - x_i\) for \(\{x_1, \ldots, x_k\} \subseteq V\). Then for distinct \(i, j = 1, \ldots, k\) we have that \(z_i - x_i = z = z_j - x_j\) for some \(z_i, z_j \in A\), implying that \(z_i - z_j = x_i - x_j\). Therefore, \(k > \alpha(G)\) would mean that there is a pair of points \(x_i, x_j\) adjacent in \(G\), implying that \(\|z_i - z_j\| = 1\), a contradiction since \(A\) avoids distance 1.
This observation now gives
\[ \delta(A) \geq \delta \left( \bigcup_{x \in V} A \cap (A - x) \right) \geq \alpha(G)^{-1} \sum_{x \in V} \delta(A \cap (A - x)), \]
and with the definition of the autocorrelation function we have (2).

Property (3) is a simple application of inclusion-exclusion. We have
\[ 1 \geq \delta \left( \bigcup_{x \in C} A - x \right) \]
\[ \geq \sum_{x \in C} \delta(A - x) - \sum_{\{x,y\} \in \binom{C}{2}} \delta((A - x) \cap (A - y)) \]
\[ = |C| \delta(A) - \sum_{\{x,y\} \in \binom{C}{2}} \delta(A \cap (A - (x - y))), \]
and together with the definition of \( f \) we are done. \( \square \)

Constraint (2) was observed by Oliveira and Vallentin [9] in the case that the graph \( G \) is a clique. Constraint (3) was used by Székeley [14].

**Remark 3.2.** The autocorrelation function encodes information about pairs of points in the set. We may also consider higher-order correlation functions, that encode information about tuples of points.

For every \( k \geq 0 \), consider the function \( F_k : (\mathbb{R}^n)^k \times \{0,1\}^{k+1} \to \mathbb{R} \) given by
\[ F_k((x_1, \ldots, x_k), (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)) = \delta(A^{\varepsilon_0} \cap (A - x_1)^{\varepsilon_1} \cap \cdots \cap (A - x_k)^{\varepsilon_k}), \]
where \( X^1 = X \) and \( X^0 = \mathbb{R}^n \setminus X \).

These functions are nonnegative and satisfy the following *self-consistency relation*: for all \( k \geq 1 \), \( x_1, \ldots, x_k \) and \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-1} \) we have
\[ F_{k-1}((x_1, \ldots, x_{k-1}, 1), (\varepsilon_0, \ldots, \varepsilon_{k-1})) \]
\[ = F_k((x_1, \ldots, x_{k-1}, x_k), (\varepsilon_0, \ldots, \varepsilon_{k-1}, 0)) \]
\[ + F_k((x_1, \ldots, x_{k-1}, x_k), (\varepsilon_0, \ldots, \varepsilon_{k-1}, 1)). \]

Also, similarly to property (1) in the lemma above, we have
\[ F_k((x_1, \ldots, x_k), (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)) = 0 \]
whenever there exist indices \( i, j \) such that \( \|x_i - x_j\| = 1 \) and \( \varepsilon_i = \varepsilon_j = 1 \). This property, together with the self-consistency relations, imply properties (2) and (3) in the lemma above (we omit the proof of this fact). Therefore, one may regard the self-consistency of higher-order correlation functions as the common source of properties (2) and (3). \( \diamond \)

In order to construct an optimization problem in which an autocorrelation function of a 1-avoiding set is to be found, it is necessary to parametrize such functions, and a convenient way to do so is via their Fourier series. Moreover, this parametrization will also suggest one further constraint satisfied by autocorrelation functions.

All the facts we state from harmonic analysis are quite standard; the reader looking for background is referred to the book by Katznelson [5].

Given measurable functions \( f, g : \mathbb{R}^n \to \mathbb{C} \), write
\[ \langle f, g \rangle = \lim_{T \to \infty} \frac{1}{(2T)^n} \int_{[-T,T]^n} f(x) \overline{g(x)} \, dx, \]
when the limit exists.
Let $L \subseteq \mathbb{R}^n$ be a lattice. A function $f: \mathbb{R}^n/L \to \mathbb{C}$ is periodic. Seen as a function with domain $\mathbb{R}^n$, it is invariant under the action of $L$: we have $f(x + v) = f(x)$ for all $x \in \mathbb{R}^n$ and $v \in L$. Lattice $L$ is the periodicity lattice of $f$.

Now $\mathcal{L}^2(\mathbb{R}^n/L)$ defines an inner product in the space of square-integrable, complex-valued functions with periodicity lattice $L$, denoted by $\mathcal{L}^2(\mathbb{R}^n/L)$. Equipped with this inner product, $\mathcal{L}^2(\mathbb{R}^n/L)$ is a Hilbert space. Functions $\chi_u(x) = e^{iu \cdot x}$ for $u \in 2\pi L^*$, where $L^* = \{u \in \mathbb{R}^n : u \cdot v \in \mathbb{Z} \text{ for all } v \in L\}$ is the dual lattice of $L$, form a complete orthonormal system of $\mathcal{L}^2(\mathbb{R}^n/L)$.

The Fourier coefficient of $f \in 2\pi L^*$ is $\hat{f}(u) = (f, \chi_u)$. Since the $\chi_u$ form a complete orthonormal system, we have the Fourier inversion formula $f(x) = \sum_{u \in 2\pi L^*} \hat{f}(u) e^{iu \cdot x}$, with $L^2$ convergence.

Let $A \subseteq \mathbb{R}^n$ be a measurable set with periodicity lattice $L$. We denote by $1_A \in \mathcal{L}^2(\mathbb{R}^n/L)$ the characteristic function of $A$. The autocorrelation function of $A$ is then given by $f(x) = (1_A, 1_{A-x})$. It is easy to derive an expression for $f(x)$ for all $x \in \mathbb{R}^n$. Indeed, since $f(x)$ is given in terms of the inner product $\mathcal{L}^2(\mathbb{R}^n/L)$, and since $\hat{1}_{A-x}(u) = \hat{1}_A(u) e^{iu \cdot x}$, Parseval’s identity gives

\begin{equation}
(20) f(x) = \sum_{u \in 2\pi L^*} |\hat{1}_A(u)|^2 e^{iu \cdot x}
\end{equation}

with convergence for all $x$. So we see from the inversion formula that $\hat{f}(u) = |\hat{1}_A(u)|^2$, which gives yet another constraint satisfied by an autocorrelation function, namely that its Fourier coefficients are all nonnegative. We also say in this case that $f$ is a function of positive type.

Let $f \in \mathcal{L}^2(\mathbb{R}^n/L)$. We may radialize $f$ by averaging it over the sphere or, equivalently, over the orthogonal group. In other words we set

\begin{equation}
(21) \tilde{f}(x) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} f(\|x\|) \, d\omega(\xi) = \int_{O(\mathbb{R}^n)} f(Ax) \, d\mu(A), \quad \mu(O(\mathbb{R}^n)) = 1,
\end{equation}

where $\omega$ is the surface measure on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ and $\mu$ is the Haar measure on the orthogonal group $O(\mathbb{R}^n)$, normalized so that $\mu(O(\mathbb{R}^n)) = 1$.

Notice that $\tilde{f}$ is radial, i.e., the value of $\tilde{f}(x)$ depends only on $\|x\|$. Let $\Omega_n$ be the function defined over the nonnegative reals which is such that

\begin{equation}
\Omega_n(\|x\|) = \frac{\omega(S^{n-1})}{\omega(S^{n-1})} \int_{S^{n-1}} e^{ix \cdot \xi} \, d\omega(\xi)
\end{equation}

for all $x \in \mathbb{R}^n$. Then, using the inversion formula we get

\begin{equation}
\tilde{f}(x) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \sum_{u \in 2\pi L^*} \hat{f}(u) e^{iu \cdot \xi \|x\|} \, d\omega(\xi) = \sum_{u \in 2\pi L^*} \hat{f}(u) \Omega_n(u \|x\|).
\end{equation}

We can rewrite this as

\begin{equation}
(22) \tilde{f}(x) = \sum_{\ell \geq 0} \alpha(t) \Omega_n(t \|x\|),
\end{equation}

where $\alpha(t)$ is the sum of $\hat{f}(u)$ over all $u$ such that $\|u\| = t$. The sum in (22) should be interpreted keeping in mind that only countably many of the $\alpha(t)$ coefficients are nonzero.
Now, $Ω_n$ has a convenient expression in terms of Bessel functions, namely

$$Ω_n(t) = Γ(\frac{n}{2}) \left(\frac{2}{t}\right)^{(n-2)/2} J_{(n-2)/2}(t)$$

for $t > 0$ and $Ω_n(0) = 1$, where $J_α$ is the Bessel function of the first kind with parameter $α$; this formula was first observed by Schoenberg [10].

Since all constraints in Lemma 3.1 are rotation-invariant, and since the autocorrelation function of a 1-avoiding set satisfies these constraints, then also its radialization satisfies all of them. Then, using expression (22) for $f$ and the constraints in Lemma 3.1, we may prove the following theorem which provides a way to compute upper bounds for $m_1(\mathbb{R}^n)$.

**Theorem 3.3.** Let $S$ be a finite collection of finite subgraphs of the unit-distance graph in $\mathbb{R}^n$ and let $C$ be a finite collection of finite sets of points in $\mathbb{R}^n$. Suppose that numbers $v_0$, $v_1$, $w_G \geq 0$ for $G \in S$, and $z_C \geq 0$ for $C \in C$ are such that

$$v_0 + v_1 + \sum_{G \in S} w_G |V(G)| - \sum_{C \in C} z_C |C|/2 \geq 1,$$

$$v_0 + v_1 Ω_n(t) + \sum_{G \in S} w_G \sum_{x \in V(G)} Ω_n(∥x∥) - \sum_{C \in C} z_C \sum_{(x,y) \in \binom{C}{2}} Ω_n(∥x - y∥) \geq 0$$

for $t > 0$.

Then for every $λ > 0$ we have that $m_1(\mathbb{R}^n) \leq \max\{λ, ζ(λ)\}$, where

$$ζ(λ) = v_0 + \sum_{G \in S} w_G α(G) + \sum_{C \in C} z_C (λ^{-1} - |C|).$$

**Proof.** If $m_1(\mathbb{R}^n) \leq λ$, then we are done. So suppose $m_1(\mathbb{R}^n) > λ$.

Consider now the following linear programming problem with infinitely many variables:

$$\sup \alpha(0) \quad \sum_{t \geq 0} \alpha(t) = 1,$$

$$\sum_{t \geq 0} \alpha(t) Ω_n(t) = 0,$$

$$\sum_{t \geq 0} \alpha(t) \sum_{x \in V(G)} Ω_n(∥x∥) \leq \alpha(G) \quad \text{for } G \in S,$$

$$\sum_{t \geq 0} \alpha(t) \sum_{(x,y) \in \binom{C}{2}} Ω_n(∥x - y∥) \geq |C| - λ^{-1} \quad \text{for } C \in C,$$

$$\alpha(t) \geq 0 \quad \text{for all } t \geq 0.$$
have

\[ \alpha(0) \leq \sum_{t \geq 0} \alpha(t) \left( v_0 + v_1 \Omega_n(t) + \sum_{G \in S} w_G \sum_{x \in V(G)} \Omega_n(t \| x \|) \right. \]
\[ \left. - \sum_{C \in C} z_C \sum_{\{x,y\} \in (\mathcal{G}_C)} \Omega_n(t \| x - y \|) \right) \]
\[ \leq v_0 + \sum_{G \in S} w_G \alpha(G) + \sum_{C \in C} z_C (\lambda^{-1} - |C|) \]
\[ = \zeta(\lambda), \]

where to go from the first to the second line we exchange the summations and use the fact that \( \alpha \) is feasible for (23).

In the remainder of this section we will show how one can look for good collections \( S \) and \( C \). Our approach for this is experimental. We then give explicit values for \( v, w, \) and \( z \) and show how it can be verified that the conditions of Theorem 3.3 are satisfied, so that a bound can be given.

3.1. Applying Theorem 3.3 for the Euclidean plane. The best upper bound that can be achieved with Theorem 3.3 taking \( S = C = \emptyset \) is 0.287..., thus worse than the bound of \( \frac{2}{7} \approx 0.285 \) coming from the Moser spindle. By taking as \( S \) a few equilateral triangles in \( \mathbb{R}^2 \), Oliveira and Vallentin [9] could provide an upper bound of \( \approx 0.268 \), which was better than the previously known upper bound of \( \approx 0.279 \) due to Székely [14].

A further improvement can be obtained if one takes as \( S \) a few congruent copies of the Moser spindle. Let \( G \) be the Moser spindle as shown in Figure 1, with the lower-left vertex placed at the origin. We then generate congruent copies of \( G \) of the form

\[(t, 0) + R(\theta)G,\]

where \( t \in \mathbb{R} \), \( \theta \in [0, 2\pi] \), and

\[ R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \]

is a rotation matrix. We actually discretize the intervals \([-4, 4]\) and \([0, 2\pi]\) to get values for \( t \) and \( \theta \). Then we take as \( S \) all these copies of \( G \).

The next step is to solve the primal problem (23). An issue here is that we have infinitely many variables. The work-around is to pick numbers \( L > 0 \) and \( \epsilon > 0 \) and then only use variables \( \alpha(t) \) for \( t \) of the form \( k\epsilon \leq L \) with \( k \geq 0 \) integer.

By taking \( L = 200 \) and \( \epsilon = 0.01 \) we get a first estimate of the bound, 0.26305. Note this is better than the bound of \( \approx 0.268 \) using triangles. From the solution of the linear program, we can see that only a few of the constraints coming from copies of the Moser spindle are actually used; the rest may be discarded.

Now we can try to add inclusion-exclusion constraints. Let \( \alpha \) be an optimal solution of the discretized linear program and fix \( N > 0 \). Suppose we want to find a set \( C \) of \( N \) points in \( \mathbb{R}^2 \) that minimizes the left-hand side of the inclusion-exclusion constraint in (23), in the hope that for some values of \( \lambda \) the constraint will then be violated. This we can do as follows. Since the inclusion-exclusion constraint is translation-invariant, we can assume that the origin belongs to \( C \). Then we consider the other \( N - 1 \) points as variables in a nonlinear optimization problem, and we try to minimize the left-hand side of the constraint on such variables using some nonlinear optimization method.

This procedure will give us a set \( C \) of points. For every fixed \( \lambda > 0 \), we can add this constraint to (23) and solve the problem. The optimal value of the linear
program will be a function of \( \lambda \), say \( p(\lambda) \). Then we look for a value of \( \lambda \) such that \( \lambda = p(\lambda) \) by finding a root of \( p(\lambda) - \lambda \) using some numerical method. The idea is then that, if the discretization of the linear program is fine enough, this approach will provide an almost valid bound, that we can fix later by sacrificing only very little of the value.

This procedure can be iterated: We consider an optimal solution and find another set \( C \) of points that minimizes the left-hand side of the inclusion-exclusion constraint. And we repeat. After a few iterations, we start to reach the limit of this approach. In our experiments, we could only get results for \( N = 6 \). For other values of \( N \) we could not find violated constraints.

Then it is time to get a dual solution of the resulting linear program (23), which gives us numbers as in Theorem 3.3. These numbers will not satisfy the conditions of the theorem, but they will nearly satisfy them if our discretization was fine enough. Then we have to alter them a bit, verify that they satisfy the conditions, and see what bound they give.

### 3.2. Explicit values and verification

We now present candidate values for \( v \), \( w \), and \( z \) as in Theorem 3.3 and then show how to modify these values slightly so that they satisfy the conditions of the theorem.

For us, \( S \) is composed of three copies of Moser’s spindle, \( G_1 \), \( G_2 \), and \( G_3 \) given as in (24), corresponding to the following pairs \((t, \theta)\):

\[
(0.4, 5.4), (0.6, 5.4), \text{ and } (0.8, 5.4).
\]

The collection \( C \) we use is given in Table 1.

| \( C_1 \)  | \( C_2 \)  | \( C_3 \)  |
|-----------|-----------|-----------|
| (0.781846561681, 0.923983014983) | (0.976422451180, 0.219342709492) | (0.951509148625, 0.29730535201) |
| (-1.493218191370, 0.715876600816) | (-0.896557239530, 0.403173339690) | (-0.856129318724, 0.498561149113) |
| (-0.195640888520, -0.224511807288) | (-0.552919373200, 1.316137405620) | (-0.613074338850, 1.455125134570) |
| (1.079423910680, -0.016405441239) | (0.051861274857, 0.567345039740) | (0.035657780606, 0.706699050884) |
| \( C_4 \)  | \( C_5 \)  |
| (0.533352656963, 0.891484779083) | (0.680196169514, 1.919904450610) |
| (-0.611400296245, 0.779442596608) | (-0.448270339088, 0.880354589520) |
| (-0.285361365856, 1.69921856820) | (-0.943967767036, 0.337966779380) |
| (0.251297714466, 0.991412992863) | (1.397952081510, 2.088390922180) |

Table 1. Collection \( C \) of point-sets used in our application of Theorem 3.3. Each of the sets also contains the origin \((0,0)\), so each set has 6 points.
We set the values of \( v, w, \) and \( z \) to:

\[
\begin{align*}
v_0 &= 2.3022516897351055 \\
v_1 &= 2.2729338571989154 \\
w_1 &= 0.2021538298582705 \\
w_2 &= 0.431184458316473 \\
w_3 &= 1.3855315999360112 \\
z_1 &= 0.2862826361013497 \\
z_2 &= 0.7908579212800153 \\
z_3 &= 0.9616086568833265 \\
z_4 &= 0.2772120180950884 \\
z_5 &= 0.5311904133936868
\end{align*}
\]

(25)

Here, \( w_i \) is associated with graph \( G_i \) in \( \mathcal{S} \), and similarly \( z_i \) is associated with configuration \( C_i \in \mathcal{C} \).

Recall that \( \Omega_2(t) = J_0(t) \) and let

\[
\varphi(t) = v_0 + v_1 J_0(t) + \sum_{i=1}^3 w_i \sum_{x \in V(G_i)} J_0(t \| x \|) - \sum_{i=1}^5 z_i \sum_{(x,y) \in (C_j^i)} J_0(t \| x - y \|).
\]

The conditions required of \( v, w, \) and \( z \) in Theorem 3.3 now translate to \( \varphi(0) \geq 1 \) and \( \varphi(t) \geq 0 \) for all \( t > 0 \).

It is easy to check that \( \varphi(0) \geq 1 \). How can we check that \( \varphi(t) \geq 0 \) for all \( t > 0 \)? The first step is to notice that

\[
\lim_{t \to \infty} J_0(t) = 0.
\]

This follows from the asymptotic formula for \( J_0 \) for \( \alpha \geq 0 \) (cf. Watson [15], equation (1) in §7.21). So we see that

\[
\lim_{t \to \infty} \varphi(t) = v_0,
\]

and so \( \varphi(t) \geq 0 \) for all large enough \( t \).

We need an estimate on how large \( t \) has to be chosen and this we can get as follows. It is well-known that

\[
\frac{dJ_0(t)}{dt} = -J_1(t).
\]

Now, let \( j_1 < j_2 < j_3 < \cdots \) be the positive zeros of \( J_1 \). By the above expression for the derivative, these are the local optima of \( J_0 \). The local optima of \( J_0 \) decrease in absolute value (cf. Watson [15], §15.31), that is

\[
|J_0(j_1)| > |J_0(j_2)| > |J_0(j_3)| > \cdots.
\]

So, if we want to find an upper bound on the absolute value of \( J_0(t) \) for \( t \geq L \), all we need to do is find the rightmost zero of \( J_1 \) in the interval \([0, L]\) and compute \( J_0 \) at this zero. There are procedures to compute the zeros of \( J_1 \) to any desired precision.

Using this idea, we may check that for \( \mathcal{S} \) and \( \mathcal{C} \) as we have the absolute value of

\[
v_1 J_0(t) + \sum_{i=1}^3 w_i \sum_{x \in V(G_i)} J_0(t \| x \|) - \sum_{i=1}^5 z_i \sum_{(x,y) \in (C_j^i)} J_0(t \| x - y \|)
\]

for \( t \geq 779.8998 \ldots \) (this is the 248th positive zero of \( J_1 \)) is at most \( v_0 - 0.05 \approx 2.2522 \). So we see that \( \varphi(t) \geq 0 \) for all \( t \geq L \) with \( L = 780 \).

Now we have to check that \( \varphi(t) \geq 0 \) in \([0, L]\). This will not be the case: since our solution has been found numerically via sampling, it will be negative at some points. But it will only be slightly negative, and then adding a small number to \( v_0 \) will make it nonnegative everywhere.
So we have to lower bound the minimum of \( \varphi(t) \) in \([0, L] \). Recall that the derivative of \( J_0 \) is \( -J_1 \). Since \(|J_1(t)| \leq \frac{1}{\sqrt{2}} \) for all \( t \geq 0 \) (cf. Watson [15], equation (10) in §13.42), we can provide a rough estimate for \(|\varphi'(t)|\), namely

\[
|\varphi'(t)| \leq 75.9547
\]

for all \( t \geq 0 \).

Then the mean-value theorem implies that

\[
|\varphi(t_1) - \varphi(t_2)| \leq 75.9547|t_1 - t_2|
\]

for every \( t_1, t_2 \geq 0 \).

So if for \( \varepsilon > 0 \) we compute \( \varphi(t) \) for all \( t = k\varepsilon/76 \leq L \) with \( k \geq 0 \) integer and get the minimum value computed, we can be sure of having computed the minimum of \( \varphi \) in \([0, L] \) up to an additive error of \( \varepsilon \).

Taking \( \varepsilon = 10^{-4} \), we obtain the conservative estimate that the minimum of \( \varphi \) in \([0, L] \) is at least \(-0.00011\). Adding this to \( v_0 \) we then get numbers \( v, w, \) and \( z \) that satisfy the conditions of Theorem 3.3.

Now we look for \( \lambda \in [0, 1] \) such that \( \zeta(\lambda) = \lambda \) for \( \zeta \) defined as in the statement of Theorem 3.3. We get \( \lambda = \zeta(\lambda) \leq 0.258795 \), an upper bound for \( m_1(\mathbb{R}^2) \).

We have attempted to include more constraints from Moser spindles after the 6-point configurations were added. This improves the bound slightly, but not much. Attempts to add more 6-point configurations have run into numerical trouble, and we could not derive a rigorous bound from such trials. They suggest that better bounds can be achieved, but we never managed to get below 0.257, which is probably the limit of this method.

The verification procedure we just described was implemented in a Sage [12] script that is available together with the arXiv version of this paper. It is a short program that can be easily checked by the reader. Numerical computations are still carried out to compute Bessel functions, but due to the simplicity of the code, one can have a high degree of confidence on the results obtained. A fully rigorous verification procedure would require the use of rational arithmetic and this would require Bessel functions to be computed up to good precision using rationals. This is not hard to implement, but in our view it is not necessary for the present result.

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