Bounds and Constructions of Singleton-Optimal Locally Repairable Codes With Small Localities

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Abstract—An $(n, k, d; r_j)$-locally repairable code (LRC) is called a Singleton-optimal LRC if it achieves the Singleton-type bound. Analogous to the classical MDS conjecture, the maximal length problem of Singleton-optimal LRCs has attracted a lot of attention in recent years. In this paper, we give an improved upper bound for the length of $q$-ary Singleton-optimal LRCs with disjoint repair groups such that $(r_j + 1) | n$ based on the parity-check matrix approach. In particular, for any Singleton-optimal $(n, k, d; r_j)$-LRCs, we show that: 1) $n \leq q + d - 4$, when $r = 2$ and $d = 3$; 2) $n \leq (r + 1) \left\lceil \frac{2(2q^2 + q + 1)}{d} \right\rceil + e + 1$, when $d \geq 8$ and $\max\{3, \frac{d - e - 6}{d - e - 3}\} \leq r^*$ for any $0 \leq e \leq \left\lceil \frac{d - 6}{d - 5} \right\rceil$. Furthermore, we establish equivalent connections between the existence of Singleton-optimal $(n, k, d; r_j)$-LRCs for $d = 6, r = 3$ and $d = 7, r = 2$ with disjoint repair groups and some subsets of lines in finite projective space with certain properties. Consequently, we prove that the length of $q$-ary Singleton-optimal LRCs with minimum distance $d = 6$ and locality $r = 3$ is upper bounded by $O(q^{1.5})$. We construct Singleton-optimal $(8 \leq n \leq q + 1, k, d = 6, r = 3)$-LRC with disjoint repair groups such that $4 \leq n$ and determine the exact value of the maximum code length for some specific $q$. We also prove the existence of $(n, k, d = 7; r = 2)$-Singleton-optimal LRCs for $n \approx \sqrt{2q}$.

Index Terms—Locally repairable codes, singleton-type bound, singleton-optimal LRCs, finite geometry, constant weight codes.

I. INTRODUCTION

In a large-scale distributed storage system (DSS), node failures frequently occur and it is critical to recover a failed node in time. Due to the large storage overhead of the replication strategy, erasure codes were introduced to increase storage efficiency and reduce the cost of repairing a failed node. To minimize the number of storage nodes involved in a repairing process, Gopalan et al. [13] introduced locally repairable codes to recover a failed node by accessing a few available nodes. Suppose that $C$ is an $(n, k, d, r_j)$ linear code. The $i$-th symbol of $C$ has locality $r_j$ if there exists a subset $R_i \subset [n] = \{1, 2, \cdots, n\}$ with $i \in R_i$ and $|R_i| \leq r_j + 1$ such that the $i$-th symbol $c_i$ can be represented as a linear combination of $\{c_j\}_{j \in R_i \setminus \{i\}}$. The subset $R_i$ is called the repair group of the $i$-th symbol. The linear code $C$ is called a locally repairable code (LRC) with locality $r_j$ if each symbol of $C$ has locality $r_j$. We denote such a code as an $(n, k, d; r_j)_q$-LRC. An LRC is said to have disjoint local repair groups if the set of coordinates $[n]$ is exactly the disjoint union of some repair groups, i.e., $[n] = \bigcup_i R_i$, and $|R_i| = r_j + 1$. In this paper, we mainly focus on LRCs with disjoint local repair groups, which are widely adopted in constructing LRCs in the previous work. Any $(n, k, d; r_j)_q$-LRCs have to satisfy the well-known Singleton-type bound in [13], i.e.,

$$d \leq n - k - \left\lceil \frac{k}{r_j} \right\rceil + 2. \quad (1)$$

A $q$-ary $(n, k, d)$ linear code $C$ with locality $r_j$ is called a Singleton-optimal $(n, k, d, r_j)_q$-LRC if it achieves the bound (1) with equality. Constructing Singleton-optimal LRCs has been extensively investigated in the literature ([5], [6], [7], [10], [13], [15], [16], [17], [24], [27], [28], [29]). When the locality $r_j = k$, the Singleton-type bound reduces to the classical Singleton bound. Note that from the Singleton-type bound (1), we have

$$\frac{k}{n} \leq \frac{r_j}{r_j + 1} - \frac{d - 2}{n},$$

thus given $d$ and $r_j$, a Singleton-optimal LRC with longer length $n$ would have larger code rate $k/n$. So it is interesting to construct Singleton-optimal LRCs with long lengths and deriving upper bounds on the maximal code length for given $q, d$, and $r_j$. In [29], Tamo and Barg first gave a breakthrough construction of Singleton-optimal LRCs via subcodes of Reed-Solomon codes whose code length can go up to the alphabet size. Chen et al. [5], [7] have constructed the $q$-ary Singleton-optimal LRCs of length $n | (q + 1)$ for all possible parameters via cyclic and constacyclic codes. In [21], Jin et al. also constructed a family of $q$-ary Singleton-optimal $r$-LRCs with length up to $q + 1$ by using the automorphism groups of rational function fields. Analogous to the well-known MDS
codes, for some particular \( r = 2, 3, 5, 7, 11 \) or 23, Singleton-optimal LRCs with \((r+1)d\) and code lengths up to \(q+2\sqrt{q}\) are constructed via elliptic curves [22]. For more flexible locality, Ma and Xing [25] constructed Singleton-optimal LRCs with \((r+1)d\) and length \( n < q + 2\sqrt{q} \) by utilizing subgroups of the automorphism groups of elliptic curves over finite fields, which involve both translations and automorphisms that fix the point at infinity. When the minimum distance \( d = 3 \) or 4, Luo et al. [24] proposed a construction of cyclic Singleton-optimal LRCs whose length can be arbitrarily large. Analogous to the well-known MDS conjecture, the problem of the maximal code lengths of Singleton-optimal LRCs has attracted the attention of many scholars. For minimum distance \( d \geq 5 \), Guruswami et al. [15] proved that the code length of a \( q \)-ary Singleton-optimal LRC must be at most roughly \( O(dq^3) \). Jin [20] presented an explicit construction of \( q \)-ary optimal LRCs of length \( \Omega_r(q^2) \) via binary constant weight codes for minimum distance \( d = 5, 6 \) and the locality \( r > d - 2 \). In [30], Xing and Yuan generalized Jin’s results and presented a construction of \( q \)-ary Singleton-optimal LRCs for general \( d \geq 7 \) via hypergraph theory. In [8], Chen et al. constructed an explicit family of optimal LRCs with length \( n = q(r+1) \) for given \( 1 \leq r < q + 1 \) and \( 3 \leq d \leq \min\{r+1, q+1-r\} \). In our prior work [6], [10], we considered the case of \( d = 6 \) and \( r = 2 \) and provided some new upper bounds on code length and new constructions with long code length.

To repair multiple failure nodes, LRCs with \((r, \delta)\) locality were introduced by Prakash et al. [27]. Let \( C \) be a \( q \)-ary \([n, k]\) linear code. The \( i \)-th symbol \( c_i \) of \( C \) is said to have \((r, \delta)\)-locality \((\delta \geq 2)\) if there exists a subset \( S_i \subseteq [n] \) such that \( i \in S_i \) and \( |S_i| \leq r + \delta - 1 \) and \( d(C|_{S_i}) \geq \delta \), where \( |S_i| \) denotes the size of \( S_i \) and \( d(C|_{S_i}) \) denotes the minimum distance of the punctured code of \( C \) on \([n] \setminus S_i \). If each \( i \)-th symbol \( c_i \) of \( C \) has \((r, \delta)\) locality for \( i = 1, 2, \ldots, n \), then we call \( C \) has \((r, \delta)\) locality or an \((r, \delta)\)-LRC. Recently, Cai et al. [1], [2] improved these results and generalized them to the \((r, \delta)\)-LRCs.

In [20] and [30], some \( q \)-ary Singleton-optimal LRCs with super-linear code length have been constructed. However, these LRCs have locality \( r \geq d - 2 \). Smaller locality does lead to lower storage efficiency, but it significantly reduces the number of helper nodes required during the repair process. In some application scenarios, it may be necessary to consider LRCs with a smaller locality \( r \). Therefore, we mainly focus on new bounds and constructions of Singleton-optimal LRCs with \( r \leq d - 2 \) in this paper. Firstly, we give an improved upper bound on the length of \( q \)-ary Singleton-optimal LRCs with disjoint repair groups such that \((r+1) \mid n\) based on the parity-check matrix approach. Secondly, we explore the Singleton-optimal LRCs with minimum distance \( d = 6 \), locality \( r = 3 \) and minimum distance \( d = 7 \), locality \( r = 2 \), respectively. We establish equivalent connections between the existence of these two families of LRCs and the existence of some subsets of lines in projective spaces with special structures. New improved upper bounds and optimal constructions of LRCs with long lengths are then obtained.

In this paper, we consider the LRCs with disjoint repair groups such that \((r+1) \mid n\) and list our main results as follows.

- We give an improved bound for the length of \( q \)-ary Singleton-optimal LRCs with minimum distance \( d \geq 7 \) (see Theorem 2). In particular, for \( d = 7 \) and \( 2 \leq r \leq 4 \), we prove that \( n \leq (r+1) \lfloor \frac{2q^2+q+1}{r(r+1)} \rfloor \) (see Corollary 1); for \( r = 2 \) and \( d = 3e+8 \) with \( e \geq 0 \), we show that \( n \leq q+d-4 \) (see Corollary 2). Specifically, we have \( n \leq q+4 \) when \( d = 8 \) and \( r = 2 \); when \( d \geq 8 \) and \( \max(3, \frac{d-e-6}{e+1}) \leq d \leq \frac{d-e-4}{e+1} \) for any \( 0 \leq e \leq \frac{d-6}{4} \), we show that \( n \leq (r+1) \lfloor \frac{2q^2+q+1}{r(r+1)} \rfloor + e + 1 \) (see Corollary 3). Specifically, we have \( n \leq \frac{2}{3}q^2 + o(q^2) \) when \( d = 8 \) with \( 3 \leq r \leq 5 \), \( d = 9 \) with \( 4 \leq r \leq 6 \) and \( d \geq 10 \) with \( 6 \leq r \leq d-3 \) (see Corollary 3 and Remark 4), where \( o(q^2) \) means that \( o(q^2)/q^2 \to 0 \) as \( q \to \infty \).

- We provide sufficient and necessary conditions on the existence of Singleton-optimal \((n, k, d; r)\)-LRCs for \( d = 6, r = 3 \) and \( d = 7, r = 2 \) with disjoint repair groups, respectively, which establish equivalent connections with some subsets of lines in finite projective space with certain properties (see Theorems 3 and 9);

- For any Singleton-optimal \((n, k, d = 6; r = 3)\)-LRCs with disjoint repair groups, we prove that \( n \approx O(q^{1.5}) \) (see Theorem 5);

- We prove the existence of Singleton-optimal \((8 \leq n \leq q+1, k; d = 6; r = 3)\)-LRC with disjoint repair groups such that \( 4 \mid n \) and present the detail construction from the subsets of \((q+1)\)-arc and \((q+2)\)-arc (see Theorem 7). We also give an explicit construction of Singleton-optimal \((n, k, d = 6; r = 3)\)-LRCs with maximal code length for \( q = 7, 8, 9 \) respectively (see Example 1);

- We show the existence of Singleton-optimal \((n, k, d = 7; r = 2)\)-LRCs with \( n \approx \sqrt{2}q \) (see Theorems 10 and 11).

In Table I, we present a list of some known constructions as well as our new constructions of \((n, k, d; r)\)-Singleton-optimal LRCs. Table II includes both some known and our new upper bounds on the code length. We give a brief comparison as follows.

- Based on the last four rows of Table I, we provide some new constructions of Singleton-optimal LRCs for the cases where \( d = 6, r = 3 \) and \( d = 7, r = 2 \). To the best of our knowledge, these code lengths are the longest reported so far.

- From rows 1-4 of Table II, our new bounds on the code length \( n \) of \((n, k, d = 6, r = 3)\)-Singleton-optimal LRCs improve the results in [6] and [23].

- By comparing rows 5-8 of Table II, our new bounds improve the results in [6] for \((n, k, d = 7, r = 2)\)-Singleton-optimal LRCs and the results in [23] for \((n, k, d = 7, r = 3)\) with \( k \not\equiv 0 \pmod{3} \) and \((n, k, d = 7, r = 4)\)-Singleton-optimal LRCs, respectively.

- Due to rows 9 and 10 of Table II, our new bounds on the code length \( n \) of Singleton-optimal LRCs are better than the results in [23] when \( d \equiv 5 \pmod{r+1} \) and \( r = 2 \).
TABLE I

| Construction | Length $n$ | Minimum distance $d$ | Locality $r$ | References |
|--------------|------------|----------------------|--------------|------------|
| $n = q$      | $d \leq n$ | $r \leq k$           | [29]         |
| $n = q + 1$  | $d \leq n$ | $r \leq k$           |              |
| $n = q + 2\sqrt{q}$ | $(r + 1) \mid d$ | $r = 2, 3, 5, 7, 11, 23$ | [22]         |
| Unbounded    | $d = 3, 4$ | $r \geq 2$ and $r + 1 \mid q + 1$ | [24]         |
| $n = 3(2q - 4)$ | $d = 6$   | $r = 2$              | [6], [10]    |
| $n = \Omega_2(q^q)$ | $d = 5, 6$ | $r \geq 4$, $r$ is a prime power | [20], [30]  |
| $n = q^{2\ell - \alpha(1)}$ | $d = 7, 8$ | $r \geq d - 2$       | [30]         |
| $n = q^{3/2 - \alpha(1)}$ | $d = 9, 10$ | $r \geq d - 2$       | [30]         |
| $n = \Omega_2(q(q \log q)^{\frac{3(q - 1)}{q + 1}})$ | $d \geq 11$ | $r \geq d - 2$       | [30]         |
| $8 \leq n \leq q + 1$, 4 $\mid n$ | $d = 6$ | $r = 3$              | Theorem 7    |
| $q = 7, n = 12$ | $d = 6$ | $r = 3$              | Example 1    |
| $q = 8, 9, n = 16$ | $d = 6$ | $r = 3$              | Example 1    |
| $n \approx \sqrt{2q}$ | $d = 7$ | $r = 2$              | Theorems 10, 11 |

TABLE II

| Case | Minimum distance $d$ | Locality $r$ | Length $n$ | References |
|------|----------------------|--------------|------------|------------|
| 1    | $d = 6$              | $r = 3$      | $n \leq \frac{4}{3}(q^2 + q) + 26$ | [23]         |
| 2    | $d = 6$              | $r = 3$      | $n \leq \frac{4}{3}(q^2 + 2q + 1)$ | [6]          |
| 3    | $d = 6$              | $r = 3$      | $n \leq \frac{4}{3}(q^2 - q + 1)$ | Theorem 4    |
| 4    | $d = 6$              | $r = 3$      | $n \leq O(q^1.5)$ | Theorem 5    |
| 5    | $d = 6$              | $r = 2$      | $n \leq \frac{3}{2}(q^{1/3} + 1)$ | [6]          |
| 6    | $d = 7$              | $r = 3, k \neq 0$ (mod 3) | $n \leq \frac{3}{2}(q^2 + q) + 31$ | [23]         |
| 7    | $d = 7$              | $r = 4$      | $n \leq \frac{3}{2}(q^2 + q) + 37$ | [23]         |
| 8    | $d = 7$              | $2 \leq r \leq 4$ | $n \leq (r + 1)\frac{2(q^r + 2)}{r(r + 1)} + 1$ | Corollary 1  |
| 9    | $d \equiv 5$ (mod $r + 1$) | $r = 2$ | $n \leq \frac{3}{2}q + d + 12$ | [23]         |
| 10   | $d \equiv 5$ (mod $r + 1$) | $r = 2$ | $n \leq q + d - 4$ | Corollary 2  |

- Additionally, Liu et al. [23] showed that $n \leq \frac{r + 1}{r}q^2 + o(q^2)$ (here $o(q^2)$ means that $o(q^2)/q^2 \to 0$ as $q \to \infty$) when $r = 3, d \equiv 0, 2$ (mod 4); or $r = 3, d \equiv 3$ (mod 4) with $3 \nmid k$; or $r \geq 4$, $d \equiv 1, 2, 3, 4, 5$ (mod $r + 1$). In this paper, for $d = 8$ with $3 \leq r \leq 5$, or $d = 9$ with $4 \leq r \leq 6$ or $d \geq 10$ with $d - 6 \leq r \leq d - 3$, we show that $n \leq \frac{3}{2}q^2 + o(q^2)$ (see Corollary 3 and Remark 4), which improve their results.

The rest of this paper is organized as follows. In Section II, we introduce some basic results on LRCs, finite geometry and the Johnson bound for binary constant weight codes. In Section III, we give an improved bound of the length of Singleton-optimal LRCs. In Section IV, we consider the new constructions and bounds of Singleton-optimal LRCs with $d \geq 7$. In Section V, we conclude this paper in Section V.

II. PRELIMINARIES

In this section, we introduce some basic notations and results about LRCs, finite geometry and binary constant weight codes.

A. LRCs With Disjoint Repair Groups

Let $q$ be a prime power and $\mathbb{F}_q$ be a finite field with $q$ elements. Denote $[n] \triangleq \{1, 2, \ldots, n\}$. The support of a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n$ is defined as $\text{supp}(v) \triangleq \{i \in [n] : v_i \neq 0\}$. The Hamming weight $\text{wt}(v)$ of $v$ is defined as the size of $\text{supp}(v)$, i.e., $\text{wt}(v) = |\text{supp}(v)|$. A linear code $C$ of length $n$ and dimension $k$ is just a $k$-dimensional subspace of $\mathbb{F}_q^n$. The minimum distance $d$ of $C$ is the minimum weight of the nonzero codewords of $C$, i.e., $d = \min_{c \in C, c \neq 0} \text{wt}(c)$. Let $C^\perp$ be the dual code of $C$ with respect to the Euclidean inner product.

Next, we recall the parity-check matrix approach to LRCs proposed in [6] and [18]. Let $C$ be an $[n, k, d]_q$-linear code and $(r + 1) \mid n$. Then $C$ is an LRC with locality $r$ if and only if for each $i \in [n]$, there exists a codeword $h_i$ of the dual code $C^\perp$ such that $i \in \text{supp}(h_i)$ and $\text{wt}(h_i) \leq r + 1$. Throughout this paper, we assume that the LRCs have disjoint local repair groups. Thus there exist $\ell \triangleq \frac{n}{r + 1}$ vectors $h_1, h_2, \ldots, h_\ell$ of $C^\perp$, called locality vectors, such that $\text{wt}(h_i) = r + 1$ and $\text{supp}(h_i) \cap \text{supp}(h_j) = \emptyset$ for any $1 \leq i \neq j \leq \ell$. Two equivalent linear codes have the same code length, dimension, minimum distance.
and locality. According to [18], we have the following lemma.

**Lemma 1** [18]: Suppose that $C$ is a $q$-ary $(n, k, d, r)$-LRC with disjoint local repair groups such that $(r + 1) \mid n$. Then $C$ has an equivalent parity-check matrix $H$ as the following form:

$$H = \begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
& & \ddots & & & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\
\end{pmatrix}.$$  \tag{2}

where the upper part of $H$ contains $\ell$ locality rows and the lower part of $H$ contains $u \triangleq n - k - \ell$ rows. The bold-type letters $v_j^{(i)}$, $i \in [\ell]$, $j \in [r]$ are column vectors in $\mathbb{F}_q^n$, and $0$ represents the all-zero column vector in $\mathbb{F}_q^n$. For $i \in [\ell]$, we take $v_0^{(i)} = 0$.

For convenience, we denote $H$ as follows:

$$H = (h_{1,0}, h_{1,1}, \ldots, h_{n,1}, x_{2,0}, h_{2,1}, h_{2,2}, \ldots, h_{\ell,0}, h_{\ell,1}, \ldots, h_{\ell,\ell(r)},$$

where for each $i \in [\ell]$, $j \in [0] \cup [r], \quad h_{i,j} = \left(0, 0, \ldots, 0, 1, 0, \ldots, 0, (v_{(i)}^{(j)})^T \right)^T$$

is a column vector of $\mathbb{F}_q^{n-k}$ and $v_0^{(i)} = 0 \in \mathbb{F}_q^n$.

**B. Singleton-Optimal LRCs**

The Singleton-optimal LRCs could be equivalently characterized by an alternative equality as follows.

**Lemma 2** [15, Lemma II.2.]: Alternative Singleton-type bound) Let $n, k, d, r$ be positive integers with $(r + 1) \mid n$. If the Singleton-type bound (1) is achieved, then

$$n - k = \ell + d - 2 - \frac{d - 2}{r + 1}. \tag{4}$$

Conversely, if $d - 2 \not\equiv r \pmod{r + 1}$ and Eq. (4) is satisfied, then the Singleton-type bound (1) is achieved.

**Remark 1:** If $d - 2 \equiv r \pmod{r + 1}$ and $(r + 1) \mid n$, then there are no $(n, k, d; r)_q$ linear codes such that $d = n - k - \left\lceil \frac{k}{r} \right\rceil + 2$. Otherwise, by setting $\ell = \frac{n}{r + 1}$ and $u = n - k - \ell$, we deduce that $n = \ell (r + 1)$, $k = n - \ell - u = \ell r - u$ and

$$d = n - k - \left\lfloor \frac{k}{r} \right\rfloor + 2 = u + \ell - \left\lceil \frac{\ell r - u}{r} \right\rceil + 2 = u + \ell - \left(\ell - \frac{u}{r} \right) + 2 = u + \ell - \frac{u}{r} + 2.$$

Let $u = ar + b$ with $0 \leq b < r$. Then $d = ar + b + a = a(r + 1) + b + 2$. We deduce that $d - 2 \equiv b \pmod{r + 1}$ with $0 \leq b < r$, which contradicts with the condition that $d - 2 \equiv r \pmod{r + 1}$.

Therefore, when $d - 2 \equiv r \pmod{r + 1}$ and $(r + 1) \mid n$, we must have $d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 1.$ It can be similarly verified that $d = n - k - \left\lceil \frac{k}{r} \right\rceil + 1$ is equivalent to Eq. (4) in this case. Therefore, in this paper, an LRC is called a Singleton-optimal LRC if it satisfies Eq. (4).

**C. Finite Geometry**

In this subsection, we introduce some basic notions and results of finite geometry over finite fields, which will be used in Sections III and IV. And we refer the readers to [3], [4], [9], [19], and [26] for more details about finite geometry.

Let $\mathbb{F}_q^{N+1}$ be the $(N + 1)$-dimensional vector space over $\mathbb{F}_q$ with origin $0$. We consider the equivalence relation on the vectors of $\mathbb{F}_q^{N+1} \setminus \{0\}$ whose equivalence classes are the one-dimensional subspaces of $\mathbb{F}_q^{N+1}$ without the origin. Precisely, for two nonzero vectors $u = (u_1, u_2, \ldots, u_{N+1}), v = (v_1, v_2, \ldots, v_{N+1}) \in \mathbb{F}_q^{N+1} \setminus \{0\}$, we say $u$ is equivalent to $v$ if there exists $t \in \mathbb{F}_q^r$ such that $u = tv$.

**Definition 1** [19]: The $N$-dimensional projective space over $\mathbb{F}_q$ is defined as the set of equivalence classes and is denoted by $PG(N, q)$. The elements of $PG(N, q)$ are called points. In particular, when $N = 2$, $PG(N, q)$ is called the projective plane.

If the point $P(u)$ is the equivalence class of the vector $u \in \mathbb{F}_q^{N+1} \setminus \{0\}$, we say that $u$ is a vector representing $P(u)$. A subspace of dimension $m$, or an $m$-subspace of $PG(N, q)$ is a set of points all of whose representing vectors form (together with the origin) a subspace of dimension $m + 1$ of $\mathbb{F}_q^{N+1}$. A subspace of dimension zero in $PG(N, q)$ has already been called a point. A subspace of dimension one and two in $PG(N, q)$ are called a line and a plane, respectively. We say that the points $P(u_1), P(u_2), \ldots, P(u_s)$ are collinear if they lie on a common line, and the lines $L_1, L_2, \ldots, L_t$ are concurrent if they pass through a common point.

Some basic properties of $PG(N, q)$ are summarized in the following.

**Lemma 3** [19]:

(i) Each line in $PG(N, q)$ has $q + 1$ points;

(ii) The number of lines in $PG(N, q)$ passing through a fixed point is $\frac{q^N - 1}{q - 1};$

(iii) Any two distinct lines in the projective plane $PG(2, q)$ meet at exactly one point;

(iv) The number of points in the projective plane $PG(2, q)$ is $q^2 + q + 1$.

The following theorem is the well-known principle of duality in the projective plane $PG(2, q)$.

**Theorem 1** [3, Theorem 2.1, The Principle of Duality]: If $T$ is a theorem valid in the projective plane $PG(2, q)$, and $T'$ is the statement obtained from $T$ by making the following changes:

- point $\leftrightarrow$ line,
- collinear $\leftrightarrow$ concurrent,
- join $\leftrightarrow$ intersection,

and whatever grammatical adjustments are necessary, then $T'$ (called the Dual Theorem) is a valid theorem in the $PG(2, q)$.

Finally, we introduce the definitions of spreads and sunflowers in projective spaces, respectively.

**Definition 2** (Spreads in Projective Spaces [4]): Suppose $t \leq N$. A $t$-spread in an $N$-dimensional projective space $PG(N, q)$ is a set $S$ of $t$-dimensional subspaces such that any point of $PG(N, q)$ is on exactly one element of $S$. 
Lemma 4 [4]: A finite projective space $PG(N, q)$ contains a t-spread if and only if $(t + 1) | (N + 1)$. In particular, there exists a spread of lines in $PG(3, q)$ of size $q^2 + 1$.

Definition 3 (Sunflowers in Projective Spaces [9]): Suppose $t \leq s \leq N$. A sunflower $SF_q(t, s, N)$ is a set $S$ of $s$-subspaces of $PG(N, q)$ such that they meet at a common $t$-subspace. The common $t$-subspace is called the center of $SF_q(t, s, N)$ and the $s$-subspaces are called petals of $SF_q(t, s, N)$. A sunflower $SF_q(t, s, N)$ is called maximal if it has the largest possible size for fixed $q, t, s, N$.

When $s = t + 1$, we can calculate the size of a maximal sunflower $SF_q(t, t + 1, N)$.

Lemma 5 [11, Lemma 4]: The size of a maximal sunflower $SF_q(t, t + 1, N)$ is equal to $q^{N-t-1}$.

Let $m \geq 3$ be an integer. An $m$-arc $A$ of $PG(2, q)$ is a set of $m$ points such that no three of them are collinear. It is known that $m \leq q + 2$.

When $q$ is odd, a $(q+1)$-arc is called an oval. Let

$$\mathcal{O}_1 = \{(1, x, x^2) : x \in \mathbb{F}_q\} \cup \{(0,0,1)\}. \quad (5)$$

The set $\mathcal{O}$ is called a oval, which is exactly an oval in $PG(2, q)$.

When $q$ is even, a $(q+2)$-arc is called a hyperoval. Let $q > 2$ be a power of 2. A well-known construction of hyperoval in $PG(2, q)$ is as follows.

$$\mathcal{O}_2 = \{(1, x, x^2) : x \in \mathbb{F}_q\} \cup \{(0,0,1)\} \cup \{(0,1,0)\}. \quad (6)$$

For more details on arcs, please refer to [19].

D. Binary Constant Weight Codes

A binary $(n, M, d; w)$ constant weight code is a set of binary vectors of length $n$ with $M$ vectors, such that each vector contains $w$ ones and $n - w$ zeros, and any two distinct vectors differ in at least $d$ positions. The following is the well-known Johnson bound for binary constant weight codes, which is important for our subsequent derivation.

Lemma 6 [26, Johnson Bound]: Let $C$ be a binary $(n, M, d = 2\delta; w)$ constant weight code, then

$$M(w^2 - wn + \delta n) \leq \delta n.$$ 

III. IMPROVED UPPER BOUNDS FOR SINGLETON-OPTIMAL LRCs WITH $d \geq 7$

In the remaining part of this paper, we assume that $(r + 1) | n$ and $C$ is an $(n, k, d; r, q)$ LRC with $\ell$ disjoint repair groups. Then we can assume that the parity-check matrix of $C$ is $H$ as in Eq. (2).

Lemma 7: Let $s$ be a positive integer such that $s \leq d - 2$, then for each $i \in [\ell]$, the vectors $v^{(i)}_{1}, \ldots, v^{(i)}_{s}$ in $H$ are linearly independent.

Proof: Suppose

$$\sum_{j=1}^{s} a_j v^{(i)}_{j} = 0,$$

for some $a_1, \ldots, a_s \in \mathbb{F}_q$. By the expression of $H$ in Eq. (2) and $h_{i,j}$ in Eq. (3), we have

$$-\left(\sum_{j=1}^{s} a_j\right) h_{i,0} + a_1 h_{i,1} + \ldots + a_s h_{i,s} = 0.$$ 

Note that any $d - 1$ columns of $H$ are linearly independent and $s \leq d - 2$, from the above equation, we have $a_1 = \ldots = a_s = 0$. Thus $v^{(i)}_{1}, \ldots, v^{(i)}_{s}$ are linearly independent.

Now, we give our improved upper bound on the length of q-ary Singleton-optimal LRCs with minimum distance $d \geq 7$.

Theorem 2: Suppose that $(r + 1) | n$. Let $C$ be a q-ary Singleton-optimal LRC of length $n$, minimum distance $d \geq 7$ and locality $r$ with disjoint repair groups. For each integer $e$ such that $0 \leq e \leq \min\{\ell - 2, \left\lfloor \frac{d-6}{r+1}\right\rfloor\}$, let $t_0 = \min\{r, d - er - e - 6\}$ and $t = er + t_0$. Then

$$n \leq (r + 1) \left[\frac{2(q^{t-\ell} - 1)}{r(r + 1)(q - 1)} + e + 1\right], \quad (7)$$

where $u = d - 2 - \left\lfloor \frac{d-2}{r+1}\right\rfloor$.

Proof: Suppose $C$ has a parity-check matrix $H$ as in Eq. (2). By Lemma 2, we have $u = n - k - \ell - d - 2 - \left\lfloor \frac{d-6}{r+1}\right\rfloor$. We split the proof into two cases according to $e = 0$ or not.

Case 1: If $e = 0$, then $t = t_0 = \min\{r, d - 6\} \leq r$.

Let $V_0 = \langle v^{(1)}_{1}, \ldots, v^{(1)}_{\ell}\rangle$ be the subspace of $\mathbb{F}_q^n$ spanned by $v^{(1)}_{1}, \ldots, v^{(1)}_{\ell}$. By Lemma 7 and $t \leq d - 6 < d - 2$, the dimension of $V_0$ is $t$.

For $i \in \{2, 3, \ldots, \ell\}$ and $j_1, j_2 \in \{0, 1, 2, \ldots, r\}$ with $j_1 > j_2$, define

$$V_{i,j_1,j_2} = \langle v^{(i)}_{1}, \ldots, v^{(i)}_{1}, v^{(i)}_{j_1} - v^{(i)}_{j_2}\rangle. \quad (8)$$

We claim that the dimension of $V_{i,j_1,j_2}$ is $t + 1$. Otherwise, there exist not all zero elements in $\mathbb{F}_q^t, a_1, a_2, \ldots, a_t$ such that

$$v^{(i)}_{j_1} - v^{(i)}_{j_2} = \sum_{j=1}^{t} a_j v^{(i)}_{j}.$$ 

Then, we have

$$h_{i,j_1} - h_{i,j_2} = \sum_{j=1}^{t} a_j (h_{1,j_1} - h_{1,0}) = \sum_{j=1}^{t} a_j h_{1,j_1} - (\sum_{j=1}^{t} a_j) h_{1,0}.$$ 

It means that there are $t + 3$ columns of $H$ linearly dependent, hence $d \leq t + 3$, which contradicts to $d \geq 7 - 6$. Define

$$\mathcal{V} = \{V_{i,j_1,j_2} : i \in \{2, 3, \ldots, \ell\}, j_1, j_2 \in \{0, 1, 2, \ldots, r\}, j_1 > j_2\}. \quad (9)$$

We now show that the $(t+1)$-dimensional subspaces in $\mathcal{V}$ are pairwise distinct. Otherwise, suppose that $V_{i_1,j_1,j_2} = V_{i_2,j_3,j_4}$ for some $i_1, i_2 \in \{2, 3, \ldots, \ell\}, j_1, j_2, j_3, j_4 \in \{0, 1, 2, \ldots, r\}$ such that $j_1 > j_2, j_3 > j_4$ and $(i_1, j_1, j_2) \neq (i_2, j_3, j_4)$. Recall that $V_0 = \langle v^{(1)}_{1}, \ldots, v^{(1)}_{\ell}\rangle$. Then

$$v^{(i_1)}_{j_1} - v^{(i_2)}_{j_2} \in \langle V_0, v^{(i_2)}_{j_3} - v^{(i_2)}_{j_4}\rangle.$$ 

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There exist not all zero elements in $\mathbb{F}_q$, $a_1, a_2, \ldots, a_t, a_{t+1}$ such that
\[
\mathbf{v}_{j_1} - \mathbf{v}_{j_2} = \sum_{j=1}^{t} a_j (\mathbf{v}_{j_1} - \mathbf{v}_{j_2}).
\]

Then, we have
\[
h_{i_1,j_1} - h_{i_1,j_2} = h_{i_1,j_2} - h_{i_1,j_0} + a_{i_1} (h_{i_2,j_0} - h_{i_2,j_2}).
\]

It means that there are $t+5$ columns of $H$ linearly dependent, hence $d \leq t+5$, which contradicts with the condition $t \leq d-6$. Thus, the size of $\mathcal{V}$ is $(\ell - e + 1) \binom{r}{2}$. Note that there are totally $\frac{q^n-1}{q-1}$ distinct $(t+1)$-dimensional subspaces of $\mathbb{F}_q^n$ containing $V_0$ by Lemma 5. Therefore, we have
\[
(\ell - 1) \binom{r+1}{2} \leq \frac{q^{n-t} - 1}{q - 1},
\]
that is,
\[
n \leq (r + 1) \left\lfloor \frac{2(q^{n-t} - 1)}{r(r+1)(q-1)} + 1 \right\rfloor.
\]
We complete the proof for the case that $e = 0$.

**Case 2:** If $1 \leq e \leq \ell - 2$, then $t_0 = \min\{r, d - er - e - 6\}$ and $t = er + t_0$.

Let
\[
\mathcal{V}_{t,i,j_1,j_2} = \{\mathbf{v}_{k}^{(i)}, \mathbf{v}_{k}^{(j_1)}, \mathbf{v}_{k}^{(j_2)}\},
\]
be a subspace of $\mathbb{F}_q^n$ spanned by $\mathbf{v}_{1}^{(i)}, \mathbf{v}_{1}^{(j_1)}, \mathbf{v}_{1}^{(j_2)}, \ldots, \mathbf{v}_{t_0}^{(i)}, \mathbf{v}_{t_0}^{(j_1)}, \mathbf{v}_{t_0}^{(j_2)}$.

By the similar arguments as in Case 1, we can show that $\mathcal{V}_{t,i,j_1,j_2}$ is a $t$-dimensional subspace of $\mathbb{F}_q^n$. For simplicity, we omit the proof. For $i \in \{e+2, \ldots, \ell\}$, $j_1, j_2 \in \{0, 1, 2, \ldots, r\}$ such that $j_1 > j_2$, define
\[
V_{t,i,j_1,j_2} = \langle \mathcal{V}_{t,i,j_1,j_2} \rangle.
\]
We claim that the dimension of $V_{t,i,j_1,j_2}$ is $t+1$. Otherwise, we have $\mathbf{v}_{j_1}^{(i)} - \mathbf{v}_{j_2}^{(i)} \in V_{t,i,j_1,j_2}$ and there exist not all zero elements $a_{i,j} \in \mathbb{F}_q$, $1 \leq i \leq e+1$, $1 \leq j \leq r$ such that
\[
\sum_{i=1}^{e} \sum_{j=1}^{t_0} a_{i,j} (\mathbf{v}_{j}^{(i)} - \mathbf{v}_{j}^{(j_1)}) + \sum_{j=1}^{r} a_{e+1,j} (\mathbf{v}_{j}^{(e+1)} - \mathbf{v}_{j}^{(j_1)}) = \mathbf{v}_{j_1}^{(i)} - \mathbf{v}_{j_2}^{(i)}.
\]
It follows that
\[
\sum_{i=1}^{e} \sum_{j=1}^{t_0} a_{i,j} (h_{i,j} - h_{i,0}) + \sum_{j=1}^{r} a_{e+1,j} (h_{e+1,j} - h_{e+1,0}) = h_{i_1,j_1} - h_{i_1,j_2}.
\]
It means that there are $er + e + t_0 + 3$ columns of $H$ linearly dependent, which contradicts to $t_0 \leq d - er - e - 6$.

Define
\[
\mathcal{V}_t = \{V_{t,i,j_1,j_2} : i \in \{e+2, \ldots, \ell\}, j_1, j_2 \in \{0, 1, 2, \ldots, r\}, j_1 > j_2\},
\]
and
\[
\mathcal{V}_t = \{V_{t,i,j_1,j_2} : i \in \{e+2, \ldots, \ell\}, j_1, j_2 \in \{0, 1, 2, \ldots, r\}, j_1 > j_2\},
\]
We now show that the $(t+1)$-dimensional subspaces in $\mathcal{V}_t$ are pairwise distinct. Suppose that $V_{t,i_1,j_1,j_2} = V_{t,i_2,j_1,j_2}$ for some $i_1, i_2 \in \{e+2, \ldots, \ell\}$, $j_1, j_2, j_3, j_4 \in \{0, 1, 2, \ldots, r\}$ such that $j_1 > j_2, j_3 > j_4$ and $(i_1, j_1, j_2) \neq (i_2, j_3, j_4)$. Then
\[
\mathbf{v}_{j_1}^{(i_1)} - \mathbf{v}_{j_2}^{(i_1)} \in \langle V_{t,i_2,j_1,j_2} \rangle - \langle V_{t,i_2,j_1,j_2} \rangle.
\]
There exist not all zero elements $a_{i,j}, b \in \mathbb{F}_q$, $1 \leq i \leq e+1$, $1 \leq j \leq r$ such that
\[
\mathbf{v}_{j_1}^{(i)} - \mathbf{v}_{j_2}^{(i)} = \sum_{i=1}^{e} \sum_{j=1}^{t_0} a_{i,j} (\mathbf{v}_{j}^{(i)} - \mathbf{v}_{j}^{(j_1)}) + \sum_{j=1}^{r} a_{e+1,j} (\mathbf{v}_{j}^{(e+1)} - \mathbf{v}_{j}^{(j_1)}) + b (\mathbf{v}_{j_2}^{(i)} - \mathbf{v}_{j_2}^{(i)}).
\]
It follows that
\[
h_{i_1,j_1} - h_{i_1,j_2} = \sum_{i=1}^{e} \sum_{j=1}^{t_0} a_{i,j} (h_{i,j} - h_{i,0}) + \sum_{j=1}^{r} a_{e+1,j} (h_{e+1,j} - h_{e+1,0}) + b (h_{i_2,j_3} - h_{i_2,j_4}).
\]
It means that there are $er + e + t_0 + 3$ columns of $H$ linearly dependent, which contradicts to $t_0 \leq d - er - e - 6$. Thus, $\mathcal{V}_t$ is a set of $(t+1)$-dimensional subspaces in $\mathbb{F}_q^n$ containing $V_t$ with size $(\ell - (e+1)) \binom{r}{2}$. There are in total $\frac{q^{n-t} - 1}{q-1}$ distinct $(t+1)$-dimensional subspaces of $\mathbb{F}_q^n$ containing $V_t$ by Lemma 5. Therefore, we have
\[
(\ell - (e+1)) \binom{r+1}{2} \leq \frac{q^{n-t} - 1}{q - 1},
\]
that is,
\[
n \leq (r + 1) \left\lfloor \frac{2(q^{n-t} - 1)}{r(r+1)(q-1)} + e + 1 \right\rfloor.
\]
We complete the proof.

**Corollary 1:** Suppose that $(r+1) \mid n$. Let $C$ be a $q$-ary Singleton-optimal LRC of length $n$, minimum distance $d = 7$ and locality $r$ with disjoint repair groups. For $2 \leq r \leq 4$, we have $n \leq (r + 1) \left\lfloor \frac{2(q^2 + q)}{r(r+1)} + 1 \right\rfloor$.

**Proof:** For $d = 7$ and $2 \leq r \leq 4$, we have $u = d - 2 - \left\lfloor \frac{d-2}{r+1} \right\rfloor = 4$. In Theorem 2, we take $e = 0$. Then $t_0 = \min\{r, d - 6\} = 1$ and $V_0 = \{\mathbf{v}_1^{(1)}\}$. By the proof of Case 1 in Theorem 2, $V$ is a set with size $(\ell - 1) \binom{r}{2}$. It is easy to check that $\langle \mathbf{v}_1^{(1)}, \mathbf{v}_2^{(2)} \rangle \notin V$. By the same argument as the proof of Case 1 in Theorem 2, we have
\[
1 + (\ell - 1) \binom{r+1}{2} \leq \frac{q^3 - 1}{q - 1},
\]
that is,
\[
n \leq (r + 1) \left\lfloor \frac{2(q^2 + q)}{r(r+1)} + 1 \right\rfloor.
\]

**Remark 2:** In particular, for $r = 2$ and $d = 7$, we have $n \leq \left\lfloor \frac{q^2 + q + 3}{3} \right\rfloor$, which improves the bound on the code length of $q$-ary $(n, k, d = 7; r = 2)$ Singleton-optimal LRCs in [6] from the fifth row in Table II; for $r = 3$ and $d = 7$, we have $n \leq \left\lfloor \frac{2(q+4)}{3} \right\rfloor$, which improves the bound on the code length of $q$-ary $(n, k, d = 7; r = 3)$ Singleton-optimal...
LRCs in [23] when $k \not\equiv 0 \pmod{3}$ by the sixth row in Table II. For $r = 4$ and $d = 7$, we have $n \leq 5 \left\lfloor \frac{2^2 + 4 + 10}{10} \right\rfloor$, which also improves the bound on the code length of q-ary $(n, k, d; r = 4)$ Singleton-optimal LRCs in [23] from the seventh row in Table II.

Corollary 2: Suppose $3 \mid n$, $0 \leq e \leq \ell - 2$ and $C$ is a q-ary $(n, k, d = 3e + 8; r = 2)$-Singleton-optimal LRC with disjoint repair groups. Then

$$n \leq 3 \left\lfloor \frac{q + 1}{3} + e + 1 \right\rfloor \leq q + d - 4.$$  

Proof: Since $r = 2$ and $d = 3e + 8$, then $d = d - e - 6$, $t_0 = \min\{r, d - e - 6\} = d - e - 6$ and $t = e + t_0 = d - e - 6$ in Theorem 2. By Lemma 2, $u = d - 2 - \left\lfloor \frac{d - 2}{r + 1} \right\rfloor$. Then

$$u - t = d - 2 - (e + 2) - (d - e - 6) = 2.$$  

By (7) in Theorem 2, we have

$$n \leq 3 \left\lfloor \frac{q + 1}{3} + e + 1 \right\rfloor \leq q + 1 + 3(e + 1) \leq q + d - 4.$$  

Remark 3: In Theorem 3.1 (ii), Theorem 4.4 and Theorem 4.5 of [23], the authors showed that $n \leq \frac{2}{3} q + d + 12$ when $r = 2$, $d = 5 \pmod{r + 1}$. In Corollary 2, we improve this bound to $n \leq q + d - 4$ for $r = 2$, $d = 5 \pmod{r + 1}$ and $(r + 1) \mid n$.

Corollary 3: Suppose that $(r + 1) \mid n$. Let $C$ be a q-ary Singleton-optimal LRC of length $n$, minimum distance $d \geq 8$ and locality $r$ with disjoint repair groups. For any integer $e$ with $0 \leq e \leq \left\lfloor \frac{d - 2}{r + 1} \right\rfloor$ and $\max\{3, \frac{d - 2}{r + 1} - 1\} \leq r \leq \frac{d - 6}{e + 1}$, then we have

$$n \leq (r + 1) \left\lceil \frac{2q^2 + q + 1}{r(r + 1)} \right\rceil + e + 1.$$  

Proof: By conditions, we have $t_0 = \min\{r, d - e - 6\} = d - e - 6$ and $t = e + t_0 = d - e - 6$ in Theorem 2. By Lemma 2, $u = d - 2 - \left\lfloor \frac{d - 2}{r + 1} \right\rfloor$. It follows that

$$u - t = d - 2 - \frac{d - 2}{r + 1} - (d - e - 6) = 4 + e - \frac{d - 2}{r + 1}.$$  

By $\frac{d - 6}{e + 1} \leq r \leq \frac{d - 6}{e + 1} - 1$, we have

$$e + 1 \leq \frac{d - 2}{r + 1} \leq e + 1 + \frac{3}{r + 1}.$$  

Since $r \geq 3$, then $\left\lfloor \frac{d - 2}{r + 1} \right\rfloor = e + 1$ and so $u - t = 3$. By (7) in Theorem 2, we have

$$n \leq (r + 1) \left\lceil \frac{2q^2 + q + 1}{r(r + 1)} \right\rceil + e + 1.$$  

The proof is completed.

Remark 4: In Theorems 4.5 and 4.6 of [23], the authors showed that $n \leq \frac{r + 1}{r + 1} q^2 + o(q^2)$ when $r = 3$, $d = 0 \pmod{4}$; $r = 3$, $d = 3 \pmod{4}$ with $3 \nmid k$; and $r \geq 4$, $d = 1, 2, 3, 4, 5 \pmod{r + 1}$, where $o(q^2)$ means that $o(q^2)/q^2 \to 0$ as $q \to \infty$. In Corollary 3, we consider the case that $d \geq 8$, $r \geq 3$ with $(r + 1) \mid n$. We show that

$$n \leq (r + 1) \left\lceil \frac{2q^2 + q + 1}{r(r + 1)} \right\rceil + e + 1$$  

for any integer $e$ with $0 \leq e \leq \frac{d - 6}{4}$ and $\max\{3, \frac{d - 6}{r + 1} - 1\} \leq r \leq \frac{d - 6}{3}$. In particular, by taking $e = 0$ in Corollary 3, the upper bounds in Corollary 3 improve the results in Theorems 4.5 and 4.6 of [23] when $d = 8$ with $3 \leq r \leq 5$, $d = 9$ with $4 \leq r \leq 6$ and $d \geq 10$ with $d - 6 \leq d - 3$. In Theorem 10 of [15] and Remark 1 of [1], the authors showed that $n = O(dq^3)$ for $d \geq 10$ respectively. Corollary 3 indicates that the code length $n$ of Singleton-optimal LRCs with disjoint repair groups such that $(r + 1) \mid n$ is strictly less than $O(dq^3)$ in many cases.

Corollary 4: Suppose that $(r + 1) \mid n$. Let $C$ be a q-ary Singleton-optimal LRC of length $n$, minimum distance $d \geq 7$ and locality $r$ with disjoint repair groups. If $r \geq 2$ and $e = 0$, or $\frac{d - 3 - e}{e + 1} < r \leq \frac{d - e - 2}{e + 1}$ and $1 \leq e \leq \ell - 2$, then

$$n \leq \frac{r + 1}{r + 1} \left\lceil \frac{2q^2 + q + 1}{r(r + 1)} \right\rceil + e + 1.$$  

Proof: When $e = 0$ and $r \geq d - 2 > d - 6$, we have $t = t_0 = \min\{r, d - 6\} = d - 6$. Then

$$u - t = d - 2 - \frac{d - 2}{r + 1} - (d - e - 6) = 4 + e - \frac{d - 2}{r + 1}.$$  

By Theorem 2, the claim follows.

When $1 \leq e \leq \ell - 2$ and $\frac{d - 3 - e}{e + 1} < r \leq \frac{d - e - 2}{e + 1}$, we have

$$u - t = d - 2 - \frac{d - 2}{r + 1} - (d - e - 6) = 4 + e - \frac{d - 2}{r + 1}.$$  

Thus, $\frac{d - 2}{r + 1} = e$ and

$$u - t = d - 2 - \frac{d - 2}{r + 1} - (d - e - 6) = 4 + e - e = 4.$$  

By Theorem 2, the claim follows.

IV. SINGLETON-OPTIMAL LRCs WITH SMALL LOCALITIES

A. Singleton-Optimal LRCs With $d = 6$ and $r = 3$

In this subsection, we consider Singleton-optimal LRCs with minimum distance $d = 6$ and locality $r = 3$. Throughout this subsection, we assume that $4 \mid n$ and $\ell = \frac{n}{4}$.

Suppose $C$ is an LRC of length $n$, dimension $k$, minimum distance $d = 6$ and locality $r = 3$ with disjoint repair groups. From Section II-A, we assume that $C$ has a parity-check matrix $H$ as the following form:

$$H = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & u_1 & v_1 & w_1 & 0 & u_2 & v_2 & w_2 & \cdots & 0 & u_\ell & v_\ell & w_\ell
\end{pmatrix},$$  

where the column vectors $u_i, v_i, w_i \in \mathbb{F}_q^u$, $i \in [\ell]$, $u = n - k - \ell$.

For convenience, we also denote

$$H = \langle h_{1,0}, h_{1,1}, h_{1,2}, h_{1,3}, \cdots, h_{\ell,0}, h_{\ell,1}, h_{\ell,2}, h_{\ell,3} \rangle,$$

where the column vectors $h_{i,j} \in \mathbb{F}_q^{u-k}$.

Let

$$L_{1,i} = \langle u_i, v_i \rangle \subseteq \mathbb{F}_q^u,$$

$$L_{2,i} = \langle v_i, w_i \rangle \subseteq \mathbb{F}_q^u,$$

$$L_{3,i} = \langle w_i, u_i \rangle \subseteq \mathbb{F}_q^u,$$
and
\[ L_{4,i} \triangleq \{ u_i - v_i, v_i - w_i \} = \{ au_i + bw_i + cw_i : a, b, c \in \mathbb{F}_q, \text{ and } a + b + c = 0 \} \subseteq \mathbb{F}_q. \]

From Lemma 7, for any \( 1 \leq i \neq i' \leq 4 \) and \( 1 \leq j \leq \ell \), we have \( \dim(L_{i,j}) = 2 \) and \( L_{i,j} \neq L_{i',j} \).

For convenience, we will use the language of projective geometry. Each \( L_{i,j} \) can be regarded as a line in \( PG(u-1, q) \). Moreover, we will use \( u_i \) to represent the point \( P(u_i) \) in \( PG(u-1, q) \). In this sense, \( u_i \) and \( \lambda u_i \) represent the same point, for any \( \lambda \in \mathbb{F}_q^* \).

**Proposition 1:** Let \( C \) be an LRC of length \( n \), dimension \( k \), minimum distance \( d = 6 \) and locality \( r = 3 \) with a parity-check matrix \( H \) given in Eq. (12). Let \( L_{1,i}, L_{2,i}, L_{3,i}, L_{4,i} \) be 4 lines in \( PG(u-1, q) \) defined as above. Denote \( B_i \triangleq \{ L_{1,i}, L_{2,i}, L_{3,i}, L_{4,i} \} \). Then for any \( 1 \leq i \neq j \leq \ell \),

(i) \( B_i \cap B_j = \emptyset \);

(ii) there are no 3 lines in \( B_i \cup B_j \) pass through a common point.

**Proof:** (i): If \( B_i \cap B_j \neq \emptyset \), without loss of generality, we assume that \( L_{1,1} = L_{1,3} \). Then \( u_i \in L_{1,3} \), suppose \( u_i = au_j + bw_j \), for some \( a, b \in \mathbb{F}_q^* \). From the form of \( H \) (see Eq. (12)), we know that
\[ -h_{i,0} + h_{i,1} = -(a+b)h_{j,0} + ah_{j,1} + bh_{j,2}, \]
where \( h_{i,0}, h_{i,1}, h_{i,2}, h_{j,0}, h_{j,1}, h_{j,2} \) are linearly dependent, which leads to \( d \leq 5 \).

(ii) Firstly, note that the 4 lines in \( B_i \) produce 6 distinct intersection points: \( u_i, v_i, w_i, u_i - v_i, v_i - w_i, w_i - u_i \). Thus there are no 3 lines in \( B_i \) pass through a common point.

By contradiction, we suppose there are three lines of \( B_i \cup B_j \) pass through a common point. Without loss of generality, we assume that two of them are from \( B_i \) and the other one from \( B_j \). There are essentially 4 cases that need to be considered.

1) Suppose \( L_{1,1}, L_{2,1}, L_{3,1} \) pass through a common point. Note that \( L_{1,1} \) and \( L_{2,1} \) meet at the point \( v_i \). Thus \( v_i \in L_{1,1} \) and \( v_i = au_j + bw_j \), for some \( a, b \in \mathbb{F}_q^* \). From the form of \( H \) (see Eq. (12)), we know that
\[ -h_{i,0} + h_{i,1} = -(a+b)h_{j,0} + ah_{j,1} + bh_{j,2}, \]
where \( h_{i,0}, h_{i,1}, h_{i,2}, h_{j,0}, h_{j,1}, h_{j,2} \) are linearly dependent, which leads to \( d \leq 5 \);

2) Suppose \( L_{1,1}, L_{2,2}, L_{3,3} \) pass through a common point. We can similarly verify that \( h_{i,0}, h_{i,2}, h_{j,0}, h_{j,2}, h_{3,0}, h_{3,2} \) and \( h_{j,3} \) are linearly dependent, which leads to \( d \leq 5 \);

3) Suppose \( L_{1,1}, L_{4,1}, L_{3,1} \) pass through a common point. Note that \( L_{1,1} \) and \( L_{4,1} \) meet at the point \( u_i - v_i \). We can verify that \( h_{i,1}, h_{i,2}, h_{j,0}, h_{j,1} \) and \( h_{j,2} \) are linearly dependent, which also leads to \( d \leq 5 \);

4) Suppose \( L_{1,1}, L_{4,1}, L_{4,1} \) pass through a common point. We can similarly verify that \( h_{i,1}, h_{i,2}, h_{j,1}, h_{j,2} \) and \( h_{j,3} \) are linearly dependent, which leads to \( d \leq 5 \).

The proposition is proved.

Now, we are ready to present a sufficient and necessary condition of the existence of \( q \)-ary Singleton-optimal LRCs of length \( n \), minimum distance \( d = 6 \) and locality \( r = 3 \) with disjoint repair groups.

**Theorem 3:** Suppose \( 4 \mid n \). Then, there exists a \( q \)-ary Singleton-optimal LRC of length \( n \), minimum distance \( d = 6 \) and locality \( r = 3 \) with disjoint repair groups if and only if there exist \( \ell \) sets \( B_1, B_2, \ldots, B_\ell \), each of which consists of 4 lines in \( PG(2, q) \), such that for any \( 1 \leq i \neq j \leq \ell \),

(i) \( B_i \cap B_j = \emptyset \);

(ii) there are no 3 lines in \( B_i \cup B_j \) pass through a common point.

**Proof:** **Necessity:** Assume that there exists a \( q \)-ary Singleton-optimal LRC of length \( n \), minimum distance \( d = 6 \) and locality \( r = 3 \) with disjoint repair groups. From Eq. (4), we have \( u = n - k = \ell = d - 2 - \left[ \frac{d-2}{r+1} \right] = 3 \). Then the conclusion follows from Proposition 1 by taking \( u = 3 \).

**Sufficiency:** Assume that there exist \( \ell \) sets \( B_1, B_2, \ldots, B_\ell \) satisfying the conditions (i) and (ii). Suppose \( B_i = \{ L_{i,1}, L_{i,2}, L_{i,3}, L_{i,4} \} \), then any three lines of \( B_i \) do not pass through a common point. Suppose \( u_i = L_{i,1} \cap L_{i,3} \), \( v_i = L_{i,1} \cap L_{i,2} \), \( w_i = L_{i,1} \cap L_{i,4} \). Then \( u_i, v_i, w_i \) are linearly independent. Thus \( L_{i,1} = \langle u_i, v_i, w_i \rangle \), \( L_{i,2} = \langle v_i, w_i \rangle \) and \( L_{i,4} = \langle w_i, u_i \rangle \). Suppose \( L_{i,1} \cap L_{i,3} = \langle u_i, v_i, w_i \rangle \), and \( L_{i,2} \cap L_{i,4} = \langle w_i, u_i \rangle \). Without loss of generality, we may assume that \( \beta = \gamma \). Moreover, by replacing \( u_i, v_i, w_i \) by \( \alpha u_i, \beta v_i, \gamma w_i \) respectively, we may assume that \( L_{i,1} \cap L_{i,4} = u_i - v_i \) and \( L_{i,2} \cap L_{i,3} = v_i - w_i \). Then we can deduce that
\[ L_{i,1} = \langle u_i - v_i, v_i - w_i \rangle = \langle au_i + bw_i + cw_i : a, b, c \in \mathbb{F}_q, a + b + c = 0 \rangle. \]

Now we let \( C \) be the linear code with parity-check \( H \) given as Eq. (12), where \( u_i, v_i, w_i \) are given as above. Then \( C \) has locality \( r = 3 \) and the dimension \( k = n - 3 = \frac{3}{2} n - 3 \). By the Singleton-type bound,
\[ d \leq n - k - 2 - \left[ \frac{k}{3} \right] \leq 6. \]

From condition (ii) and similar discussions in the proof of Proposition 1, we can verify that any 5 columns of \( H \) are linearly independent. Thus \( d \geq 6 \), hence \( d = 6 \) and \( C \) is a \( q \)-ary Singleton-optimal LRC of length \( n \), minimum distance \( d = 6 \) and locality \( r = 3 \) with disjoint repair groups.

The proof is completed.

According to Theorem 3, we can immediately obtain the first bound on the code length of Singleton-optimal LRCs with \( d = 6 \) and \( r = 3 \).

**Theorem 4:** Suppose \( 4 \mid n \). Let \( C \) be a \( q \)-ary Singleton-optimal LRC of length \( n \), minimum distance \( d = 6 \) and locality \( r = 3 \). Then
\[ n \leq 4 \left( \frac{q^2 - 3q + 9}{6} \right). \]

**Proof:** Let \( B_i \) be the set of lines in \( PG(2, q) \) given by Theorem 3, \( i = 1, 2, \ldots, \ell \). Note that the 4 lines in the set \( B_i \) produce 6 distinct intersection points. Let \( E_i \) be the set of these 6 points. If \( E_i \cap E_j \neq \emptyset \) for some \( 1 \leq i \neq j \leq \ell \), we suppose \( P \in E_i \cap E_j \), then there are 4 lines from \( B_i \) and \( B_j \) passing through the point \( P \), which contradicts with condition
(ii) of Theorem 3. Thus all these 6ℓ intersection points are mutually distinct. Moreover, each line of B_1 does not pass through any points in E_j (2 ≤ j ≤ ℓ). Note that there are 4(q+1)−6 points in the 4 lines of B_1. Thus 6(ℓ−1)+4(q+1)−6 ≤ q^2 + q + 1, i.e., ℓ ≤ \frac{(2q^2+3q+1)}{6}. Hence
\[ n = 4ℓ ≤ 4 \left[ \frac{q^2 − 3q + 9}{6} \right]. \]

**Remark 5:** In [6], the authors showed that a q-ary Singleton-optimal (n, k, d = 6; r = 3) LRC has to satisfy that \( n ≤ 4 \left[ \frac{2q^2+3q+1}{6} \right] \). Thus Theorem 4 improves their result.

Furthermore, by using the techniques of incidence matrix and Johnson bound on constant weight codes, we can deduce a tighter upper bound.

**Theorem 5:** Suppose 4 \mid n. Let C be a q-ary Singleton-optimal LRC of length n, minimum distance d = 6 and locality r = 3 with disjoint repair groups. Then
\[ n ≤ 4 \left[ \frac{7q + 3 + \sqrt{24q^2 + q^2 - 6q - 63}}{24} \right] = O(q^{1.5}). \]

**Proof:** Let B_i be the set of lines in PG(2, q) given by Theorem 3, i = 1, 2, ···, ℓ. Let E_i be the set of 6 points produced by the 4 lines in B_i. From the proof of Theorem 4, \( E_i \cap E_j = \emptyset \) for any 1 ≤ i ≠ j ≤ ℓ. Denote \( E = \bigcup_{i=1}^{ℓ} E_i \), then \( |E| = 6ℓ. \)

We consider a 4ℓ \times (q^2 + q + 1) binary matrix A = (a_{ij}) defined by Eq. (13), shown at the bottom of the next page, whose rows are indexed by the 4ℓ lines in B_1, B_2, ···, B_ℓ, and columns are indexed by the points of PG(2, q). The \((i, j)\)-th entry a_{ij} = 1 if and only if the “i-th line” passes through the “j-th point”. Note that each line in PG(2, q) has q + 1 points (see Lemma 3), thus each row of A has weight q + 1.

Let G be the sub-matrix of A consisting of the 6ℓ columns indexed by E. According to Theorem 3, for each line L_i of B_i, there are exactly three points of E_i lie on L_i and no points of E \setminus E_i lie on L_i. So we can deduce that each row of G has weight 3. Let B be the sub-matrix obtained by deleting G in A (see below). Then B is a 4ℓ \times (q^2 + q + 1 − 6ℓ) binary matrix and each row of B has weight q + 3 = q − 2.

Let D be the binary code of length q^2 + q + 1 − 6ℓ whose codewords are the rows of B. Note that any two lines in PG(2, q) meet at exactly one point (see Lemma 3). Thus any two distinct codewords of D have Hamming distance at least 2(q^2 − 2) − 2q − 6. So D is a binary \( (q^2 + q + 1 − 6ℓ; M = 4ℓ; d = 2q − 6; w = q − 2)\)-constant weight code. By Johnson bound (see Lemma 6),
\[ 4ℓ((q^2 − 2) − (q^2 + q + 1 − 6ℓ)) ≤ (q^2 + q + 1 − 6ℓ)(q − 3), \]
\[ ⇒ 24ℓ^2 − (14q + 6ℓ)(q^2 + q + 1)(q − 3) \leq 0, \]
\[ ⇒ ℓ ≤ \frac{7q + 3 + \sqrt{24q^2 + q^2 + 1}(q^2 + q + 1)(q − 3)}{24} \leq \ell \leq \frac{7q + 3 + \sqrt{24q^2 + q^2 + 1}(q^2 + q + 1)(q − 3)}{24} \]
\[ ⇒ n = 4ℓ ≤ 4 \left[ \frac{7q + 3 + \sqrt{24q^2 + q^2 + 1}(q^2 + q + 1)(q − 3)}{24} \right] = O(q^{1.5}). \]

**Remark 6:** From Table II, to the best of our knowledge, Theorem 5 is the best bound on the code length of \( (n, k, d = 6; r = 3) \)-Singleton-optimal LRCs until now.

Next, we will propose a new construction of Singleton-optimal \( (n, k, d = 6; r = 3) \)-LRCs. Firstly, by the duality of the projective plane (see Theorem 1), we present the dual version of Theorem 3 as follows.

**Theorem 6 (Dual version of Theorem 3):** Suppose 4 \mid n. Then, there exists a q-ary Singleton-optimal LRC of length n, minimum distance d = 6 and locality r = 3 with disjoint repair groups if and only if there exist ℓ sets \( S_1, S_2, \ldots, S_ℓ \), each of which consisting of 4 points in PG(2, q), such that for any 1 ≤ i ≠ j ≤ ℓ,
(i) \( S_i \cap S_j = \emptyset \);
(ii) there are no 3 points in \( S_i \cup S_j \) lie on a common line.

**Theorem 7:** Suppose q ≥ 7. For any integer n such that 4 \mid n and 8 ≤ n ≤ q+1, there exists a q-ary Singleton-optimal LRC of length n, minimum distance d = 6 and locality r = 3 with disjoint repair groups.

**Proof:** Suppose q ≥ 7. Let \( O \) be defined by \( O_1 \) as Eq. (5) when q is odd and be defined by \( O_2 \) as Eq. (6) when q is even. Let n be a positive integer such that 4 \mid n and 8 ≤ n ≤ q+1. We take n points from \( O \) and divide every 4 points into one point set. Then we get \( \frac{n}{4} \) point sets, denoted as \( S_1, S_2, \ldots, S_\frac{n}{4} \). It is easy to check that \( S_1, S_2, \ldots, S_\frac{n}{4} \) satisfy the conditions (i) and (ii) in Theorem 6 from the properties of oval and hyperoval. By Theorem 6, the claim follows.

**Remark 7:** In [5] and [6], Chen et al. constructed \( (n, k, d, r) \)-q Singleton-optimal LRCs with disjoint repair groups such that \( (r + 1) \mid n \) and \( n \mid q + 1 \). In [21], Jin et al. constructed \( (n, k, d, r) \)-q Singleton-optimal LRCs via automorphism groups of rational function fields. In Theorem 7, we prove the existence of Singleton-optimal \( (n, k, d = 6, r = 3) \)-LRC with disjoint repair groups for all integers n such that 8 ≤ n ≤ q+1 and 4 \mid n. We also present the detail constructions from the subsets of \( (q + 1) \)-arc and \( (q + 2) \)-arc.

In the following, we will use Theorem 6 to determine the maximum code length of q-ary Singleton-optimal LRCs with d = 6 and r = 3 for some specific q. Firstly, we aim to show that we can choose \( S_1 = \{ 1, 0, 0 \} \), \( \{ 0, 1, 0 \} \), \( \{ 1, 0, 0 \} \) in the Theorem 6.

Let GL(3, q) and PGL(3, q) denote general linear group and projective general linear group over \( \mathbb{F}_q \) of dimension 3 respectively. Let \( Z(GL(3, q)) \) denote the center of GL(3, q). For more details, please refer to Chapter 1 of [14].

**Lemma 8:** Let
\[ S_1 = \{ 1, 0, 0 \}, \{ 0, 1, 0 \}, \{ 0, 0, 1 \}, \{ 1, 1, 1 \} \]
be a subset of PG(2, q). Then, for any subset \( S = \{ P_1, P_2, P_3, P_4 \} \subseteq PG(2, q) \) in which no three of these four points are collinear, there exists \( g \in PGL(3, q) \) such that \( g S_i = S \).

**Proof:** Let \( P_1 = \langle \alpha_1 \rangle \), where \( \alpha_1 = (\alpha_{11}, \alpha_{12}, \alpha_{13}) \), \( i = 1, 2, 3, 4 \). Since no three of \( P_1, P_2, P_3, P_4 \) in PG(2, q) are
collinear, then there exist \( \lambda, \mu, \delta \in \mathbb{F}_q^* \) such that \( a_4 = \lambda a_0 + \mu a_2 + \delta a_3 \). Let

\[
M = \begin{pmatrix}
\lambda a_0 \\
\mu a_2 \\
\delta a_3 \\
a_4
\end{pmatrix}_{3 \times 3}.
\]

It is clear that \( M \in GL(3, q) \). Then we have

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix} M = \begin{pmatrix}
\lambda a_0 \\
\mu a_2 \\
\delta a_3 \\
a_4
\end{pmatrix}_{4 \times 3}.
\]

Take \( g = M/Z(GL(3, q)) \). Then we have \( S_i^q = S \).

By Lemma 8, we have that

\[
S = \{ S_i^q : g \in PGL(3, q) \}
\]

is the set of all point sets of \( PG(2, q) \) with 4 points such that no three of these 4 points are collinear.

**Theorem 8:** Let

\[
S_1 = \{ \langle (1, 0, 0) \rangle, \langle (0, 1, 0) \rangle, \langle (0, 0, 1) \rangle, \langle (1, 1, 1) \rangle \}
\]

and \( m \) be the maximal integer such that \( S_1, S_2, \ldots, S_m \) satisfy Conditions (i) and (ii) in Theorem 6. Then \( m \) is the maximal number of point sets \( T_i, 1 \leq i \leq m \) such that \( T_1, T_2, \ldots, T_m \) satisfy Conditions (i) and (ii) in Theorem 6.

**Proof:** Suppose that there exist point sets \( T_1, T_2, \ldots, T_m, T_{m+1} \) satisfy Conditions (i) and (ii) in Theorem 6. By Lemma 8, there exist \( g \in PGL(3, q) \) such that \( S_1 = T_1^g \). Set \( S_i = T_i^g \) for \( 2 \leq i \leq m+1 \). Since \( T_i \cap T_j = \emptyset \) for \( 1 \leq i \neq j \leq m+1 \), then \( S_i \cap S_j = \emptyset \) for \( 1 \leq i \neq j \leq m+1 \).

If there exist \( P_k = \langle b_k \rangle \) in \( S_1 \cup S_j, k = 1, 2, 3 \) for some \( 1 \leq i \neq j \leq m+1 \) such that \( P_1, P_2, P_3 \) are collinear. That is, there exist \( \lambda_1, \lambda_2 \in \mathbb{F}_q^* \) such that \( b_k = \lambda_1 b_1 + \lambda_2 b_2 + b_3 \). Then \( c_k = (b_k + b_3) \) in \( T_{i} \cup T_j \), \( k = 1, 2, 3 \), and \( Q_1, Q_2, Q_3 \) are collinear, which is a contradiction. Therefore, \( S_1, S_2, \ldots, S_m \) satisfy the Conditions (i) and (ii) in Theorem 6, which is contradictory to the maximum of \( m \).

Denote

\[
n_{max}(q) = \max \{ n | \text{there exists a Singleton-optimal (}n, k, 6; 3)_{q} \text{-LRC with disjoint repair groups} \}.
\]

By Theorems 6 and 8, given \( q \), to determine the exact value of \( n_{max}(q, d, r) \) is equivalent to find the maximum number \( m \) such that \( S_1 = \{ \langle (1, 0, 0) \rangle, \langle (0, 1, 0) \rangle, \langle (0, 0, 1) \rangle, \langle (1, 1, 1) \rangle \}, S_2, \ldots, S_m \) satisfy Conditions (i) and (ii) in Theorem 6.

Let \( \mathcal{P} \) be the set of all points in \( PG(2, q) \) and \( S \) be as Eq. (14). We write \( S_i = \{ P_{1}, P_2, P_3, P_4 \} \), where \( P_{i,j} = \langle a_{i,j} \rangle \) and \( a_{i,j} \in \mathbb{F}_q^* \), \( 1 \leq j \leq 4 \). For each \( S_i \), define

\[
\mathcal{R}_i = \{ \langle a \rangle : a = \langle a_{i,j}, a_{i,j} \rangle, 1 \leq j < j_2 \leq 4 \}, \text{ which is the set of all lines formed by joining any two points from } S_i. \text{ If we have found } S_1, S_2, \ldots, S_i, \text{ by Condition (ii) in Theorem 6, to find } S_{i+1} \text{, we only need to search } S_{i+1} \text{ from } S \text{ such that } S_{i+1} \cap (\cup_{j=1}^{i} \mathcal{R}_i) = \emptyset. \text{ Thus we have the following Algorithm 1 to find the maximal number } m.
\]

**Remark 8:** In Algorithm 1, we assume that \( m \leq 5 \). For \( q = 7, 8, 9 \), we use Algorithm 1 to go through all 4-subsets of \( SRP \cap S, SRP \cap S, \ldots, SRP_{m-1} \cap S \) step by step to search \( S_2, \ldots, S_m \). The returned results show that \( m = 3, 4, 4 \) for \( q = 7, 8, 9 \) respectively. All of them are less than 5, which shows that it is enough to set \( m \leq 5 \) for \( q \leq 9 \). When \( q \geq 11 \), we just need to repeat the search process to compute \( S_5, S_5, \ldots \) by point-line incidence of the projective plane (cf. the proof of Theorem 3). By Eq. (12), we get the parity-check matrix and compute the minimum distance of \( C \). It is worth mentioning that the coefficients \( a_1, b_1, c_1, \) and \( d_i \) in Algorithm 2 are all non-zero, as shown in the proof of Theorem 3. We implemented Algorithm 2 and conducted experiments with \( q = 7, 8, 9 \) in Example 1 using Magma. This yielded the parity-check matrix and confirmed that \( d(C) = 6 \) for each of the three cases.

In the following example, according to Algorithm 1 and Algorithm 2, we determine the exact value of \( n_{max}(q) \) for \( q = 7, 8, 9 \), respectively.

**Example 1:** 1) For \( q = 7 \), by Algorithm 1, we have \( m = 3 \) and \( n_{max}(7) = m(r + 1) = 12 \). One example for \( S_1, S_2, \ldots, S_m \) is

\[
S_1 = \{ \langle (1, 0, 0) \rangle, \langle (0, 1, 0) \rangle, \langle (0, 0, 1) \rangle, \langle (1, 1, 1) \rangle \}, S_2 = \{ \langle (1, 4, 6) \rangle, \langle (1, 2, 4) \rangle, \langle (1, 6, 3) \rangle, \langle (1, 5, 2) \rangle \}, S_3 = \{ \langle (1, 6, 4) \rangle, \langle (1, 4, 2) \rangle, \langle (1, 3, 6) \rangle, \langle (1, 2, 5) \rangle \}.
\]

The corresponding parity-check matrix is

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 4 & 0 & 6 & 5 & 3 \\
0 & 1 & 0 & 0 & 0 & 4 & 5 & 3 & 0 & 5 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 3 & 6 & 1 & 0 & 1 & 4 & 6
\end{pmatrix}.
\]

\[(13)\]
Algorithm 1 Scheduled Algorithm for Searching Singleton-Optimal LRCs With Largest Code Lengths When $d = 6$ and $r = 3$ With Small $q$

**Input:** $P$, $S_1$, $S$, $R_1$ and $RP_1$; **Output:** $m$, $S_1$, $S_2$, $S_3$.

1. Let $SRP_1 = \text{Subsets}(RP_1, 4)$, $tm = [1, 1, 1, 1]$ and $TS = \{\{S_1\}, \{S_1\}, \{S_1\}, \{S_1\}\}$.
2. for $t_2$ in $SRP_1 \cap S$ do
   3. $S_2 \leftarrow t_2$
   4. compute $R_2$
   5. if $R_2 \cap S_1 = \emptyset$ then
   6. $RP_2 \leftarrow RP_1 \setminus R_2$, $tm[2] \leftarrow 2$, $TS[2] \leftarrow \{S_1, S_2\}$
   7. if $\#RP_2 < 4$ then continue $t_2$
   8. else
   9. $SRP_2 = \text{Subsets}(RP_2, 4)$
  10. for $t_3$ in $SRP_2 \cap S$ do
   11. $S_3 \leftarrow t_3$
   12. compute $R_3$
   13. if $R_3 \cap (S_1 \cup S_2) = \emptyset$ then
   14. $RP_3 \leftarrow RP_2 \setminus R_3$, $tm[3] \leftarrow 3$, $TS[3] \leftarrow \{S_1, S_2, S_3\}$
   15. if $\#RP_3 < 4$ then continue $t_3$
  16. else
   17. $SRP_3 = \text{Subsets}(RP_3, 4)$
  18. for $t_4$ in $SRP_3 \cap S$ do
   19. $S_4 \leftarrow t_4$
   20. compute $R_4$
   21. if $R_4 \cap (S_1 \cup S_2 \cup S_3) = \emptyset$ then
   22. $RP_4 \leftarrow RP_3 \setminus R_4$, $tm[4] \leftarrow 4$, $TS[4] \leftarrow \{S_1, S_2, S_3, S_4\}$
   23. if $\#RP_4 < 4$ then continue $t_4$
  24. else
   25. $SRP_4 = \text{Subsets}(RP_4, 4)$
  26. for $t_5$ in $SRP_4 \cap S$ do
   27. $S_5 \leftarrow t_5$
   28. compute $R_5$
   29. if $R_5 \cap (S_1 \cup S_2 \cup S_3 \cup S_4) = \emptyset$ then
   30. $tm[5] \leftarrow 5$, $TS[5] \leftarrow \{S_1, S_2, S_3, S_4, S_5\}$
  31. end if
  32. end for
  33. end if
  34. end if
  35. end for
  36. end if
  37. end if
  38. end for
  39. end if
  40. end if
  41. end for
  42. $m \leftarrow \max\{tm[i] : i = 1, 2, 3, 4, 5\}$
  43. print $m$, $TS[m]$

Algorithm 2 Scheduled Algorithm for Computing Parity-Check Matrix When Point Sets $S_1, S_2, \ldots, S_t$ in Theorem 6 Are Given.

**Input:** $S_1 = \{P_{1,1}, P_{1,2}, P_{1,3}, P_{1,4}\}$, $S_2 = \{P_{2,1}, P_{2,2}, P_{2,3}, P_{2,4}\}$, $\ldots$, $S_t = \{P_{t,1}, P_{t,2}, P_{t,3}, P_{t,4}\}$; **Output:** $H$, $d(C)$;

1. for $\ell = 1$ to $\ell$ do
2. for $j = 1$ to $4$ do
3. compute $P_{i,j}^+$: (the dual subspace of $P_{i,j}$ in $F_q^2$)
4. Let $L_{i,j} = P_{i,j}$
5. end for
6. Let $u_i = L_{i,1} \cap L_{i,3}$, $v_i = L_{i,1} \cap L_{i,2}$, $w_i = L_{i,2} \cap L_{i,3}$
7. Let $t_{i,j} = L_{i,1} \cap L_{i,4}$, $t_{i,j} = L_{i,2} \cap L_{i,4}$
8. Compute $[a_i, b_i]$ such that $t_{i,1} = a_i u_i - b_i v_i$
9. Compute $[c_i, d_i]$ such that $t_{i,2} = c_i v_i - d_i w_i$
10. $u_i \leftarrow a_i u_i$, $v_i \leftarrow b_i v_i$, $w_i \leftarrow c_i^{-1} d_i w_i$
11. end for
12. Compute $H$ by (12)
13. Compute $d(C)$ (minimum distance of $C$)
14. print $H$, $d(C)$

$S_2 = \{(1, \omega^2, \omega^4), (1, \omega^4, \omega^2), (1, \omega, \omega^2), (1, \omega^5, \omega^3)\}$
$S_3 = \{(1, \omega^2, \omega), (1, \omega^3, \omega), (1, \omega^4, \omega^3), (1, \omega^5, \omega^2)\}$
$S_4 = \{(1, \omega^5, \omega), (1, \omega^2, \omega^3), (1, \omega^3, \omega^4), (1, \omega^4, \omega^2)\}$

The corresponding parity-check matrix is given in Eq. (15), shown at the bottom of the next page.

3) For $q = 9$, by Algorithm 1, we have $m = 4$ and $n_{\text{max}}(9) = m(r+1) = 16$. Let $\delta$ be a primitive element of $F_9$ such that $\delta^2 + 2\delta + 2 = 0$. One example for $S_1, S_2, \ldots, S_m$ is

$S_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$
$S_2 = \{(1, \delta, \delta^2), (1, 2, \delta^2), (1, \delta^2, \delta), (1, \delta, \delta^2)\}$
$S_3 = \{(1, \delta^2, \delta^2), (1, \delta, \delta^3), (1, \delta^3, \delta), (1, \delta^2, \delta^3)\}$
$S_4 = \{(1, \delta^3, \delta), (1, \delta^3, \delta^2), (1, \delta^2, \delta^3), (1, \delta^3, \delta^3)\}$

The corresponding parity-check matrix is given in Eq. (16), shown at the bottom of the next page.

B. Singleton-Optimal LRCs With $d = 7$ and $r = 2$

In this subsection, we consider Singleton-optimal LRCs with minimum distance $d = 7$ and locality $r = 2$. Throughout this subsection, we assume $3 \mid n$, $\ell = \frac{2}{3}$. Firstly, note that in this case, $d - 2 \equiv r(\text{mod } r + 1)$. Thus from Lemma 2 and Remark 1, a Singleton-optimal $(n, k, d = 7; r = 2)$-LRC means $n - k = \ell + d - 2 - \lceil \frac{2d}{r+1} \rceil = \ell + 4$.

Suppose $C$ is an LRC of length $n$, dimension $k$, minimum distance $d = 7$ and locality $r = 2$ with disjoint repair groups. From Section II-A, we assume that $C$ has a parity-check matrix $H$ in Eq. (17).

$$H = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\
0 & u_1 & v_1 & 0 & u_2 & v_2 & \cdots & 0 & u_\ell & v_\ell
\end{pmatrix}$$ (17)
In Eq. (17), the column vectors \( \mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^n \), \( i \in [\ell] \), \( u = n - k - \ell \). We present a sufficient and necessary condition of the existence of \( q \)-ary Singleton-optimal LRCs of length \( n \), minimum distance \( d = 7 \) and locality \( r = 2 \) with disjoint repair groups as follows.

**Theorem 9:** Suppose \( 3 | n \). Then, there exists a \( q \)-ary Singleton-optimal LRC of length \( n \), minimum distance \( 7 \leq d \leq 8 \) and locality \( r = 2 \) with disjoint repair groups if and only if there exist \( \ell \) lines \( L_1, L_2, \cdots, L_\ell \) in \( PG(3, q) \), such that

(i) for any \( 1 \leq i \neq j \leq \ell \), \( L_i \cap L_j = \emptyset \);

(ii) there exist three distinct points \( P_{i,1}, P_{i,2}, P_{i,3} \) in \( L_i \) (\( i = 1, 2, \cdots, \ell \)) satisfying that \( P_{i,j}, P_{i,k}, P_{i,\ell} \) do not lie on a common line for any \( 1 \leq s < t < m \leq \ell \) and \( \mu, \nu, \omega \in \{1, 2, 3\} \).

**Proof:** **Necessity:** Suppose \( C \) is a \( q \)-ary Singleton-optimal LRC with length \( n \), minimum distance \( 7 \leq d \leq 8 \) and locality \( r = 2 \). Let \( H \) be a parity-check matrix of \( C \) given by Eq. (17). From Remark 1 and Eq. (4), we have \( u = 4 \) when \( d = 7 \) or \( d = 8 \). Let

\[
\mathcal{L}_i \triangleq \langle \mathbf{u}_i, \mathbf{v}_i \rangle \subseteq \mathbb{F}_q^4.
\]

Similar to Lemma 7, we can show that \( \dim(\mathcal{L}_i) = 2 \). Suppose \( x \in \mathcal{L}_i \cap \mathcal{L}_j \) (\( i \neq j \)), then

\[
x = a_i \mathbf{u}_i + b_i \mathbf{v}_i = c_j \mathbf{u}_j + d_j \mathbf{v}_j,
\]

for some \( a_i, b_i, c_j, d_j \in \mathbb{F}_q \). We take the notation as in Eq. (3) and denote the columns of \( H \) as \( h_{1,0}, h_{1,1}, h_{1,2}, \cdots, h_{\ell,0}, h_{\ell,1}, h_{\ell,2} \), then

\[
(h_i + b_i)h_{i,0} - a_i h_{i,1} - b_i h_{i,2} = (c_j + d_j)h_{j,0} - c_j h_{j,1} - d_j h_{j,2}.
\]

Since \( d \geq 7 \), we have \( a_i = b_i = c_j = d_j = 0 \), hence \( \mathcal{L}_i \cap \mathcal{L}_j = \emptyset \). Let \( L_1 \) be the line corresponding to \( PG(3, q) \) corresponds to the 2-dimensional subspace \( \mathcal{L}_i \) of \( \mathbb{F}_q^4 \). Then for any \( 1 \leq i \neq j \leq \ell \), \( L_i \cap L_j = \emptyset \).

Denote \( P_{i,1}, P_{i,2} \) and \( P_{i,3} \) the points of \( L_i \) represented by \( \mathbf{u}_i, \mathbf{v}_i \) and \( \mathbf{u}_i - \mathbf{v}_i \), respectively. We will prove that these points satisfy condition (ii). By contradiction, w.l.o.g., we suppose \( P_{1,1} \in L_1, P_{2,1} \in L_2, P_{3,1} \in L_3 \) lie on a common line. Then there exist \( a, b, c \in \mathbb{F}_q^* \) such that

\[
a \mathbf{u}_1 + b \mathbf{u}_2 + c \mathbf{u}_3 = 0
\]

\[
\Rightarrow a(h_{1,1} - h_{1,0}) + b(h_{2,1} - h_{2,0}) + c(h_{3,1} - h_{3,0}) = 0,
\]

which leads to \( d \leq 6 \). The necessity is proved.

**Sufficiency:** Suppose there exist \( \ell \) lines \( L_1, L_2, \cdots, L_\ell \) in \( PG(3, q) \) satisfying the conditions (i) and (ii). Assume that the three points \( P_{i,1}, P_{i,2}, P_{i,3} \) in \( L_i \) corresponding to the nonzero vectors \( \mathbf{u}_i, \mathbf{v}_i, w_i \in \mathbb{F}_q^3 \). Thus \( w_i = a_i \mathbf{u}_i - b_i \mathbf{v}_i \), for some \( a, b \in \mathbb{F}_q \). By replacing \( \mathbf{u}_i \) and \( \mathbf{v}_i \) with \( a_i \mathbf{u}_i \) and \( b_i \mathbf{v}_i \), respectively, we can assume that \( w_i = \mathbf{u}_i - \mathbf{v}_i \). Now we let \( C \) be the linear code with parity-check \( H \) given as Eq. (17). Obviously, \( C \) is a \( q \)-ary LRC of length \( n = 3 \ell \), locality \( r = 2 \), dimension \( k \geq n - \ell - 4 = 2 \ell - 4 \). From Lemma 2, we know that \( d \leq n - k - 2 - \left[ \frac{\ell}{2} \right] \leq 8 \). Thus we only need to prove that \( d \geq 7 \), i.e., any 6 columns of \( H \) are linearly independent.

Note that a column from a repair group can not be a linear combination of columns from other repair groups. Thus we divide our discussions into 2 cases.

**Case 1:** These 6 columns come from two repair groups. W.l.o.g., we suppose \( a_1 h_{1,0} + b_1 h_{1,1} + c_1 h_{1,2} + a_2 h_{2,0} + b_2 h_{2,1} + c_2 h_{2,2} = 0 \). Then \( a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = 0 \) and \( b_1 \mathbf{u}_1 + c_1 \mathbf{v}_1 + b_2 \mathbf{u}_2 + c_2 \mathbf{v}_2 = 0 \). Note that \( b_1 \mathbf{u}_1 + c_1 \mathbf{v}_1 \in \mathcal{L}_1 \) and \( b_2 \mathbf{u}_2 + c_2 \mathbf{v}_2 \in \mathcal{L}_2 \). Since \( L_1 \cap L_2 = \emptyset \), we have \( b_1 \mathbf{u}_1 + c_1 \mathbf{v}_1 = b_2 \mathbf{u}_2 + c_2 \mathbf{v}_2 = 0 \), hence \( b_1 = c_1 = b_2 = c_2 = 0 \) and \( a_1 = a_2 = 0 \). Thus \( h_{1,0}, h_{1,1}, h_{1,2}, h_{2,0}, h_{2,1}, h_{2,2} \) are linearly independent.

**Case 2:** These 6 columns come from three repair groups, and each group contains exactly two of these columns. W.l.o.g., we suppose \( a_1 h_{1,1} + b_1 h_{1,2} + a_2 h_{2,1} + b_2 h_{2,2} + a_3 h_{3,1} + b_3 h_{3,2} = 0 \). Then \( a_1 + b_1 + a_2 + b_2 + a_3 + b_3 = 0 \) and \( a_1 (\mathbf{u}_1 - \mathbf{v}_1) + a_2 (\mathbf{u}_2 - \mathbf{v}_2) + a_3 (\mathbf{u}_3 - \mathbf{v}_3) = 0 \). Note \( P_{1,3}, P_{2,3}, P_{3,3} \) do not lie on a common line, thus \( a_1 = a_2 = a_3 = 0 \) and \( b_1 = b_2 = b_3 = 0 \). Thus \( h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}, h_{3,1}, h_{3,2} \) are linearly independent.

The sufficiency is proved and the theorem follows.

In the following, we use the spread of lines in \( PG(3, q) \) to show the existence of \( q \)-ary Singleton-optimal LRCs with

\[
H = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \omega^3 & \omega^4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \omega^3 & \omega^6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \omega & \omega^2 & \omega^6 & \omega & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
length \( n = 3 \left\lceil \frac{7 + \sqrt{72q^2 + 121}}{18} \right\rceil \), minimum distance \( d = 7 \) and locality \( r = 2 \).

**Theorem 10:** For any \( q \geq 4 \), there exists a \( q \)-ary Singleton-optimal LRC with length \( n = 3 \left\lceil \frac{7 + \sqrt{72q^2 + 121}}{18} \right\rceil \), minimum distance \( 7 \) and locality \( 2 \).

**Proof:** By Lemma 4, suppose \( \mathcal{L} \) is a spread of lines in \( PG(3, q) \), that is
\[
\mathcal{L} = \{ L_1, L_2, \ldots, L_{q^2 + 1} \},
\]
where each \( L_i \) is a line of \( PG(3, q) \) with \( L_i \cap L_j = \emptyset \) for any \( 1 \leq i \neq j \leq q^2 + 1 \). Suppose \( m \) is the maximal number of lines in \( \mathcal{L} \) satisfying the condition (ii) of Theorem 9. Obviously, \( m \geq 2 \). W.l.o.g., we assume that \( L_1, L_2, \ldots, L_m \) satisfy the condition (ii) of Theorem 9. By condition (ii), we claim that the lines \( P_{i,\mu}P_{j,\nu} \), \( 1 \leq i < j \leq m \) and \( \mu, \nu \in \{1, 2, 3\} \), are mutually distinct. Indeed, suppose \( P_{i,\mu}P_{j,\nu} = P_{i',\mu'}P_{j',\nu'} \), \( 1 \leq i < j \leq m, 1 \leq i' < j' \leq m \) and \( \mu, \nu, \mu', \nu' \in \{1, 2, 3\} \), if \( i, j, i', j' \) are mutually distinct, then \( P_{i,\mu}, P_{j,\nu}, P_{i',\mu'}, P_{j',\nu'} \) lie on a common line, which contradicts with the condition (ii) of Theorem 9. So we assume that \( j = j' \). Then \( P_{j,\nu}, P_{j',\nu}' \) lie in \( L_j \), hence \( P_{j,\nu} = P_{j',\nu}' \) and \( P_{i,\mu}, P_{j,\nu}, P_{j',\nu}' \) lie on a common line, which also contradicts with the condition (ii) of Theorem 9. The claim is proved. Thus
\[
|\{ P_{i,\mu}P_{j,\nu} : 1 \leq i < j \leq m \text{ and } \mu, \nu \in \{1, 2, 3\} \}| = 3m(3m - 3)/2 = 9m(m - 1)/2.
\]
For \( m + 1 \leq \tau \leq q^2 + 1 \), if there exist three points \( P_{r,1}, P_{r,2}, P_{r,3} \) on \( L_\tau \) which do not lie on any lines \( P_{i,\mu}P_{j,\nu} \) \( (1 \leq i < j \leq m \) and \( \mu, \nu \in \{1, 2, 3\} \). Then we can obtain \( m + 1 \) lines \( L_1, L_2, \ldots, L_{m}, L_\tau \) satisfying the condition (ii) of Theorem 9. Thus from the maximality of \( m \), for each \( \tau = m + 1, \ldots, q^2 + 1 \), the number of points on \( L_\tau \), which also lie on \( P_{i,\mu}P_{j,\nu} \) for some \( 1 \leq i < j \leq m \) and \( \mu, \nu \in \{1, 2, 3\} \), is no less than \( q - 1 \). For a set \( S \), we use \( \mathcal{T}_S \) to denote the number of points in \( S \). Then for any \( 1 \leq \tau \leq q^2 + 1 \), and \( \tau = m + 1, \ldots, q^2 + 1 \), we have
\[
\bigcup_{\tau = m + 1}^{q^2 + 1} \left( L_\tau \cap P_{i,\mu}P_{j,\nu} \right) \geq q - 1.
\]
Hence
\[
\# \left\{ \bigcup_{1 \leq i < j \leq m}^{\tau = m + 1} \left( L_\tau \cap P_{i,\mu}P_{j,\nu} \right) \right\} \geq (q - 1)(q^2 + 1 - m).
\]
On the other hand, \( P_{i,\mu}, P_{j,\nu} \in P_{i,\mu}P_{j,\nu} \cap \bigcup_{s=1}^{m} L_s \) \((1 \leq i < j \leq m \) and \( \mu, \nu \in \{1, 2, 3\} \). Since \( L_s \cap L_\tau = \emptyset \) \((s = 1, 2, \ldots, m \)
and \( \tau = m + 1, \ldots, q^2 + 1 \), we have
\[
\bigcup_{\tau = m + 1}^{q^2 + 1} \left( L_\tau \cap P_{i,\mu}P_{j,\nu} \right) \subseteq P_{i,\mu}P_{j,\nu} \setminus \{ P_{i,\mu}, P_{j,\nu} \},
\]
and hence
\[
\# \left\{ \bigcup_{1 \leq i < j \leq m}^{\tau = m + 1} \left( L_\tau \cap P_{i,\mu}P_{j,\nu} \right) \right\} = \# \left\{ \bigcup_{1 \leq i < j \leq m}^{\tau = m + 1} \left( L_\tau \cap P_{i,\mu}P_{j,\nu} \right) \right\} \leq \sum_{1 \leq i < j \leq m}^{\tau = m + 1} (q + 1 - 2) \leq (q - 1)(9m(m - 1)/2).
\]
Combining with (18), we have
\[
(q - 1)(9m(m - 1)/2) \geq (q - 1)(q^2 + 1 - m),
\]
i.e., \( 9m^2 - 7m - 2(q^2 + 1) \geq 0 \), which leads to \( m \geq \left\lceil \frac{7 + \sqrt{72q^2 + 121}}{18} \right\rceil \). Thus there exist at least \( \left\lceil \frac{7 + \sqrt{72q^2 + 121}}{18} \right\rceil \) lines satisfy the conditions (i) and (ii) of Theorem 9. It is clear that \( 3 \left\lceil \frac{7 + \sqrt{72q^2 + 121}}{18} \right\rceil > q + 4 \) when \( q \geq 4 \). We deduce that \( d = 7 \) from Theorem 9 and Corollary 2. The theorem then follows.

**Remark 10:** Near the completion of the original manuscript, we become aware of the construction of Singleton-optimal LRCs in Theorem 10 was obtained independently in the recent paper [12], which uses a slightly different method.

Finally, we give another construction of \( q \)-ary Singleton-optimal LRCs with length \( n = 3 \left\lceil \frac{7 + \sqrt{8q^2 - 16q - 7}}{6} \right\rceil \), minimum distance \( d = 7 \) and locality \( r = 2 \) by the sunflowers in \( PG(3, q) \).

**Theorem 11:** For any \( q \geq 7 \), there exists a \( q \)-ary Singleton-optimal LRC with length \( n = 3 \left\lceil \frac{7 + \sqrt{8q^2 - 16q - 7}}{6} \right\rceil \), minimum distance \( d = 7 \) and locality \( r = 2 \).

**Proof:** We fix a line \( L \) of \( PG(3, q) \) and label the points in \( L \) as \( A_1, A_2, \ldots, A_{q+1} \). Consider the maximal sunflower \( SF(1, 2, 3) \) in \( PG(3, q) \) with center \( L \). From Lemma 5, \( |SF(1, 2, 3)| = q + 1 \). Each petal \( \pi_i \) \((i = 1, 2, \ldots, q + 1) \) of \( SF(1, 2, 3) \) is a plane containing \( L \). For \( i = 1, 2, \ldots, q + 1 \), let \( B_i \) be any point in \( \pi_i \) with \( B_i \neq L \), we claim that
\[
\overline{A_1B_1} \cap \overline{A_2B_2} = \emptyset, \text{ for any } 1 \leq i \neq j \leq q + 1.
\]
Otherwise, suppose \( P = A_1B_1 \cap A_2B_2 \). Since \( A_1B_1 \subseteq F_1B_1 \cup F_2B_2 \), then \( P \in F_1 \cap F_2 = L \). Thus \( A_1B_1 \cap F_2B_2 \subseteq L \), hence \( P = A_1 = A_2 \), which is a contradiction. The claim is proved.

Now, let \( L_1(\neq L) \) and \( L_2(\neq L) \) be two lines in \( \pi_1, \pi_2 \) which through \( P_{1,1} \), \( P_{2,1} \), respectively. Choose any points \( P_{1,2}, P_{1,3} \in L_1 \setminus \{P_{1,1}\} \) and \( P_{2,2}, P_{2,3} \in L_2 \setminus \{P_{2,1}\} \) with \( P_{1,2} \neq P_{1,3} \) and \( P_{2,2} \neq P_{2,3} \). Suppose \( m \) is the maximal number such that there exist \( m \) lines \( L_1, L_2, \cdots, L_m \), with \( L_i \subseteq \pi_1, L_i \neq L, A_i \in L_i \) \((i = 1, 2, \cdots, m)\), and satisfying the conditions (i) and (ii) of Theorem 9. Firstly, by the above claim, these \( m \) lines satisfy condition (i) of Theorem 9. Obviously, \( m \geq 2 \). Let \( L_{1,1}, L_{1,2}, L_{1,3} \) be three points in \( L_i \) \((i = 1, 2, \cdots, m)\) satisfying the condition (ii) of Theorem 9.

We consider the set of lines \( S = \{P_{i,\mu}P_{j,\nu} : 1 \leq i < j \leq m, \mu, \nu \in \{1, 2, 3\}\} \).

Then similar to the proof of Theorem 10, we know these lines are mutually distinct, i.e.,

\[
|S| = \frac{9m(m - 1)}{2}.
\]

If \( P_{i,\mu}P_{j,\nu} \subseteq \pi_{m_1} \), then \( P_{i,\mu} \subseteq \pi_1 \cap \pi_{m_1} = L \) and \( P_{j,\nu} \subseteq \pi_2 \cap \pi_{m+1} = L \), thus \( P_{i,\mu}P_{j,\nu} \subseteq L \). So except for the line \( P_{1,1}P_{2,1}(= L) \), the other lines of \( S \) do not lie on the plane \( \pi_{m+1} \), hence each line of \( S \setminus \{L\} \) meets \( \pi_{m+1} \) at exactly one point.

Denote \( S_1 = \{l \in S : P_{1,1} \in l \text{ or } P_{2,1} \in l\} \), the subset of lines in \( S \) which through \( P_{1,1} \) or \( P_{2,1} \), then \( |S_1| = 2 \times (3(m - 1) - 1) = 6m - 7 \). Consider the \( q + 1 \) lines \( l_1, l_2, \cdots, l_{q+1} \) in \( \pi_{m+1} \) which through the common point \( A_{m+1} \). Then

\[
\# \left\{ \bigcup_{j=2}^{q+1} (l_j \cap l) \right\} = \# \left\{ \bigcup_{l \in S \setminus S_1} \bigcup_{j=2}^{q+1} (l_j \cap l) \right\} \leq \sum_{l \in S \setminus S_1} \# (l \cap \bigcup_{j=2}^{q+1} l_j) \leq \sum_{l \in S \setminus S_1} 1 = |S \setminus S_1| = \frac{9m(m - 1)}{2} - (6m - 7).
\]

On the other hand,

\[
\# \left\{ \bigcup_{j=2}^{q+1} (l_j \cap l) \right\} = \sum_{j=2}^{q+1} \# \left\{ \bigcup_{l \in S \setminus S_1} (l_j \cap l) \right\} = \sum_{j=2}^{q+1} \# (l_j \cap \bigcup_{l \in S \setminus S_1} l) = \sum_{j=2}^{q+1} \# \left\{ l_j \cap \left( \bigcup_{l \in S \setminus S_1} l \right) \right\}.
\]

Thus

\[
\sum_{j=2}^{q+1} \# \left\{ l_j \cap \left( \bigcup_{l \in S \setminus S_1} l \right) \right\} \leq \frac{9m(m - 1)}{2} - (6m - 7).
\]

If \( \frac{9m(m - 1)}{2} - (6m - 7) < q(q - 2) \), then there exists some \( j \) with \( 2 \leq j \leq q + 1 \) such that \( \# \left\{ l_j \cap \left( \bigcup_{l \in S \setminus S_1} l \right) \right\} < q - 2 \), i.e., \( \# \left\{ l_j \cap \left( \bigcup_{l \in S \setminus S_1} l \right) \right\} \leq q - 3 \). Note that \( A_{m+1} \not\subseteq \bigcup_{l \in S \setminus S_1} l \), thus we can find three points \( A, B, C \) in \( l_j \setminus \{A_{m+1}\} \) such that \( A, B, C \not\subseteq \bigcup_{l \in S \setminus S_1} l \). For any \( l \in S_1 \), if \( l \neq L \) then \( \cap l_{m+1} = P_{1,1} \) or \( P_{2,1} \), hence \( \cap l_j = \emptyset \). Thus \( A, B, C \not\subseteq \bigcup_{l \in S_1} l \). In summary, the three points \( A, B, C \) do not lie on any lines of \( S \). At this moment, the \( m+1 \) lines \( L_1, \cdots, L_m, l_j \) satisfy the conditions of Theorem 9, which contradicts with the maximality of \( m \) and thus

\[
\frac{9m(m - 1)}{2} - (6m - 7) \geq q(q - 2),
\]

\[
\Rightarrow \frac{9m^2}{2} - 21m - 2(q^2 - 2q - 7) \geq 0,
\]

\[
\Rightarrow m \geq \left( \frac{7 + \sqrt{8q^2 - 16q - 7}}{6} \right).
\]

Thus there exist at least \( \left( \frac{7 + \sqrt{8q^2 - 16q - 7}}{6} \right) \) lines satisfy the conditions (i) and (ii) of Theorem 9. It is clear that \( 7 + \sqrt{8q^2 - 16q - 7} \geq q + 4 \) when \( q \geq 7 \). We deduce that \( d = 7 \) from Theorem 9 and Corollary 2. The theorem then follows.

Remark 11: Both of Theorems 10 and 11 show the existence of \( q \)-ary \((n, k, d; r = 2)\) Singleton-optimal LRCs with \( n \approx \sqrt{2q} \).

V. CONCLUSION

In this paper, we investigate the new bounds and constructions of Singleton-optimal LRCs with disjoint repair groups. Firstly, we present an improved upper bound for the length of Singleton-optimal LRCs with \( d \geq 7 \). In particular, for some specific values of \( d \) and \( r \), we obtain some new bounds, which are better than the previous work. Secondly, we give a further study on the \((n, k, d; r)_q\) Singleton-optimal LRCs for \( d = 6, r = 3 \) and \( d = 7, r = 2 \), respectively. We reduce the existence of these optimal LRCs to the existence of some lines in finite projective space with certain properties. By some techniques of finite field and finite geometry, some new constructions and bounds of Singleton-optimal LRCs are obtained. To the best of our knowledge, our new bounds on the code length are tighter than previous known results. However, there still exists a gap between the upper bound and the known best construction of the length of Singleton-optimal LRCs. Thus it will be a challenge to obtain an asymptotically optimal bound, and even determine the exact value of the maximal length of Singleton-optimal LRCs.

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