The power spectrum of cosmological number densities

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Abstract This paper studies the cosmological power spectrum (PS) of the differential and integral galaxy volume number densities, respectively, \( \gamma_i \) and \( \gamma_i^* \), constructed with the cosmological distances \( d_i \) \((i = a, g, t, z) \), where \( d_a \) is the angular diameter distance, \( d_g \) is the galaxy area distance, \( d_t \) is the luminosity distance and \( d_z \) is the redshift distance. Theoretical and observational quantities were obtained in the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime with a non-vanishing cosmological constant. The radial correlation \( \Xi_i \), a quantity defined in the context of these densities, is also discussed in the wave number domain. All observational quantities were computed using luminosity function (LF) data obtained from the FORS Deep Field (FDF) galaxy survey. The theoretical and observational power spectra of \( \gamma_i \), \( \gamma_i^* \), \( \Xi_i \) and the ratio \( \gamma_i / \gamma_i^* \) were calculated by performing Fourier transforms on values of these densities which were previously derived by Iribarrem (Astron. Astrophys. 539:A112, 2012) from observational values \([\gamma_i]_{\text{obs}}\) and \([\gamma_i^*]_{\text{obs}}\) obtained by using the galactic absolute magnitudes and Schechter’s parameters of the galaxy LF presented in Gabasch (Astron. Astrophys. 421:41–58, 2004; Astron. Astrophys. 448:101, 2006). These parameters were evaluated from a I-band selected dataset of the FDF in the redshift range \( 0.5 \leq z \leq 5.0 \) for its blue bands and \( 0.75 \leq z \leq 3.0 \) for its red ones. The results show similar behavior of the power spectra obtained from \( \gamma \) and \( \gamma^* \) using \( d_a \), \( d_t \) and \( d_z \) as distance measures. The PS of the densities defined with \( d_t \) have a different and inconclusive behavior because this cosmological distance reaches a maximum at \( z \approx 1.6 \) in the adopted cosmological model. For the other distances, our results suggest that the PS of \([\gamma_i]_{\text{obs}}, [\gamma_i^*]_{\text{obs}} \) and \([\gamma_i / \gamma_i^*]_{\text{obs}}\), has a general behavior approximately similar to the power spectra obtained with the galaxy two-point correlation function and, by being sample size independent, they may be considered as alternative analytical tools to study the galaxy distribution.

1 Introduction

The galaxy volume number densities are important measures in cosmology as they give information about the evolving universe galaxy content and allow us to test cosmological models. This is so because they are able to connect theory and observation once different types of number densities are evaluated, both theoretically and observationally. Theoretically, these densities can be obtained once one chooses a cosmological spacetime geometry, usually the standard model given by the spatially homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology or, in a recent trend, the spatially inhomogeneous Lemaître-Tolman-Bondi cosmology as well [1, 2]. Observationally, inasmuch as the galaxy luminosity function (LF) is a number density per unit luminosity, several authors have been systematically determining galaxy volume number densities using the LF derived from galaxy redshift surveys.

Comparing cosmological models with observations using number densities determined from LF parameters derived from observed data requires, however, some model connecting relativistic cosmology number counts theory to the LF astronomical data and practice. One way of pursuing this theoretical link was advanced by [3, hereafter RS03] who used general relativistic theory to obtain specialized expressions capable of being applied to LF data. Albani et al. [4, hereafter A07] and [5, hereafter Ir12] further extended such a link and applied the connecting expressions to the data of the Second Canadian Network for Observational Cosmology (COC2; see [6]) and FORS Deep Field (FDF; see [7, 8]; hereafter G04 and G06 respectively) galaxy redshift surveys. Following the methodology advanced by [9] see also [10] for number density definitions, where the adoption of cosmological distance measures is made explicit, A07 and Ir12 calculated the differential number density \( \gamma \) and its integral counterpart \( \gamma^* \) for various cosmological distances and used them to study the observational inhomogeneities in the relativistic radial distribution of galaxies [11] belonging...
to both datasets in the FLRW universe model. Their results indicated that observational inhomogeneities in the relativistic galaxy distribution can arise due to geometrical, past-light cone, effects even in the context of spatially homogeneous cosmologies.

Analyses of the large-scale structure of the universe use, however, a complementary tool to examine such structures, the power spectrum (PS), which is basically a Fourier transform of the quantity under study. If galaxies form the basic units of a cosmological fluid described by a volume number density, this galactic fluid can also be considered as being composed of wave densities with different wave numbers. Then, the PS will give us the intensity, or power, in each component. This is essentially the PS analysis of the galaxy 2-point correlation function, as the information given by the PS is complementary to the correlation function. Various authors performed such an analysis (e.g.,[12–16]) and showed that, in general, the PS behavior is a power-law. Since galaxy volume number densities were used by RS03, A07 and Ir12 to analyze the universe galaxy content and evolution, it is natural to expand these studies to include a PS analysis.

The goal of this work is to study the differential and integral number densities $\gamma$ and $\gamma^*$ in the wave number domain through the PS. This is achieved by performing Fourier transforms of both quantities. Using the relativistic analysis and results presented by Ir12, we computed the theoretical power spectra of $\gamma_i$ and $\gamma_i^*$ obtained using the cosmological distance $d_i$. Here, the index $i$ stands for the adopted distance measure, namely the angular diameter distance $dA_i$, also known as area distance, galaxy area distance $dG_i$, luminosity distance $dL_i$, and redshift distance $d_z$ ($i = A, G, L, z$). Observational values of the differential and integral densities computed with these distance measures, $[\gamma_i]_{\text{obs}}$ and $[\gamma_i^*]_{\text{obs}}$, were obtained using the results of Ir12, which employed the LF parameters derived by G04 and G06 from the FDF survey data. The analysis is also extended to include the power spectra of the ratio $\gamma/\gamma^*$ and of another quantity, named here as radial correlation $\Xi$, which is basically a radial function whose behavior in terms of $d_i$ bears some similarities to the 2-point correlation function [17].

The distance measures $d_i(z)$ were obtained in the cosmology adopted here: the FLRW model with nonzero cosmological constant $\Lambda$ and parameter values equal to $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. The dataset studied here is the I-band selected luminosity functions of the FDF survey. Ir12 calculated observed differential number counts and then $[\gamma(z)]_{\text{obs}}$ and $[\gamma^*(z)]_{\text{obs}}$ using the absolute magnitudes, LF Schechter parameters and their respective redshift parametrization, all previously derived by G04 and G06 directly from the FDF data in eight bandwidths with a redshift range $0.5 \leq z \leq 5.0$, for the blue bands (G04), and $0.45 \leq z \leq 3.75$ for the red bands (G06). Those results allowed us to calculate the ratio $[\gamma/\gamma^*]_{\text{obs}}$ and the radial correlation $[\Xi(d_i)]_{\text{obs}}$. Fourier transforms on these quantities provided their respective power spectra, $P_k[\gamma/\gamma^*]_{\text{obs}}$, $P_k[\Xi]_{\text{obs}}$, and $P_k[\Xi(d_i)]_{\text{obs}}$.

The results show that power spectra analyses with the angular diameter distance $dA_i$ are problematic, because each $dA_i$ corresponds to two different values for $z$, since this distance starts from zero, grows to maximum and then tends to zero as $z \to \infty$. This maximum $dA_i$ value is the only one with unique redshift. Due to these features, the results obtained with $dA_i$ turn out to be problematic in terms of interpretation.

We were also able to reproduce the preliminary results of Ribeiro [17] regarding the radial correlation, confirming its dependency with the sample size and finding out that its respective PS is also sample size dependent. The best results, however, were obtained with the power spectra of the differential and integral densities using $dA_i$ and $dG_i$, which show a behavior bearing similarities, even at wave number values, with the 2-point correlation function PS. This suggests that PS analysis of number densities can also be a complementary tool for large-scale structure studies.

The plan of the paper is as follows. In Sect. 2, the basic quantities used in this paper are presented, as well as the expressions required in their calculation. Section 3 specializes the previously discussed general quantities to the FLRW cosmological model and shows how the luminosity function parameters of the FDF survey are used to calculate the desired observational expressions. Section 4 presents both the theoretical and observational results of all quantities discussed here, as well as their respective power spectra, and Sect. 5 presents our conclusions.

2 Basic definitions

This section defines the basic quantities and expressions necessary for the PS analysis discussed here. Although they have already been introduced in previous papers ([9, 10, 17]; A07; Ir12), one can find below a few additional clarifying remarks.

2.1 Differential and integral densities

Let us start by defining the two key quantities used in this study. The differential density $\gamma_i$ gives the rate of growth in number counts, or more precisely in their density, as one moves along the observational distance $d_i$. It is defined by the following expression [9],

$$\gamma_i = \frac{1}{4\pi (d_i)^2} \frac{dN}{d(d_i)}. \tag{1}$$
where \( N \) is the **cumulative number counts**. The integrated differential density, or simply integral density, \( \gamma_i^* \) gives the number of sources per unit of observational volume located inside the observer’s past light cone down to a distance \( d_i \). It is written as follows:

\[
\gamma_i^* = \frac{1}{V_i} \int \gamma_i \, dV_i, \tag{2}
\]

where \( V_i \) is the **observational volume**,\n
\[
V_i = \frac{4}{3} \pi (d_i)^3. \tag{3}
\]

As it has been previously discussed, these quantities are useful in determining whether or not, and at what ranges, a spatially homogeneous cosmological model exhibits observational inhomogeneity because these densities behave very differently depending on the distance measure used in their definitions. In other words, they show a clear dependence on the adopted cosmological distance ([9, 10]; A07; Ir12).

It is useful to write the two number densities above in terms of the redshift. Thus, the differential density (1) may be written as

\[
\gamma_i = \frac{dN}{dz} \left\{ 4\pi (d_i)^2 \frac{d(d_i)}{dz} \right\}^{-1}, \tag{4}
\]

where \( dN/dz \) is the **differential number counts**. In addition, the expressions above allow us to conclude that the following expression holds

\[
\gamma_i^* = \frac{N}{V_i}. \tag{5}
\]

2.2 Radial correlation

Studies of the large-scale galaxy structure have traditionally employed the 2-point correlation function as a tool to characterize the galaxy distribution, including the possible depth at which this distribution reached homogeneity. This latter application was, however, criticized by Pietronero [18] who argued that the correlation function actually presupposes homogeneity and, therefore, it is a tool unsuited to find its possible presence in the galaxy structure. In order to remove such a built-in hypothesis, Pietronero [18] proposed a different correlation methodology by advancing the conditional density \( \Gamma(d) \), where \( d \) is the distance, as the proper tool capable of detecting the possible homogeneity in the galaxy distribution. He also showed that \( \Gamma \) has a simple functional relationship to the 2-point correlation function \( \xi \), given by the following expression (see also [12, 14])

\[
\xi(d) = \frac{\Gamma(d)}{\bar{n}(R)} - 1, \tag{6}
\]

where \( \bar{n}(R) \) is the mean density of the sample whose radius \( R \) defines the sample volume. This equation shows explicitly the correlation function dependence to the sample depth given by \( R \).

Ribeiro [17] followed these ideas and defined a radial quantity exhibiting the same sample dependence property, dubbed here as **radial correlation** \( \Xi \). In the present context, this quantity may be written as follows:

\[
\Xi_i(d_i) = \frac{\gamma_i(d_i)}{\gamma_i^*(R_i)} - 1. \tag{7}
\]

As the expressions above shows, \( \Xi \) is not a proper statistical correlation, but simply a dimensionless radial function that can be used in the context of number density distributions originated from relativistic cosmology theory, and whose behavior is also dependent on the sample depth, defined here at the radial distance \( d_i = R_i \). Ribeiro [17] studied this quantity in the context of the Einstein-de Sitter model using its theoretical distribution obtained in terms of the luminosity distance and redshift. Here, however, we shall employ the FLRW cosmology as well as the other distance measures discussed above in order to derive observational values for \( \Xi_i \) using the FDF galaxy survey.

2.3 Power spectrum

If one assumes spherical symmetry, the PS of the 2-point correlation function is given by

\[
P_k(\xi) = 4\pi \int_0^\infty \sin(kx) \frac{1}{kx} \xi(x) x^2 \, dx. \tag{8}
\]

This function describes clustering in terms of the wave numbers \( k \) and separates the clustering effects in different scales. To study the differential densities in the wave number domain, we shall employ a similar definition as above, which may be written as follows:

\[
P_k(\gamma_i) = \int_0^\infty \sin(kx) \frac{1}{kx} \gamma_i(x) x^2 \, dx. \tag{9}
\]
For astronomical observations contained in galaxy survey samples, one needs, however, to consider the finiteness of the sample size and divide it in redshift bins so that the function above can actually be plotted from real data. This means rewriting the equation above as follows:

\[ P_k[\gamma_i]_{obs} = \frac{1}{4\pi} \int_0^{d_i(z_j)} \frac{\sin(k_j d_i)}{k_j d_i} \frac{dN}{d(d_i)} \, d(d_i). \]  

(10)

Here, \( d_i(z) \) is the distance-redshift relation given by the adopted cosmological model in a certain distance measure,

\[ k_j = \frac{2\pi}{d_i(z_j)}, \quad \text{for} \ (j = 1, \ldots, m), \]  

(11)

where \( m \) is the total number of redshift bins dividing the sample and \( z_j \) is the central redshift value of \( j \)-th bin. Replacing \( \gamma_i \) with its definition given in Eq. (1) leads to the following expression:

\[ P_k[\gamma_i]_{obs} = \frac{1}{4\pi} \int_0^{d_i(z_j)} \frac{\sin(k_j d_i)}{k_j d_i} \frac{dN}{d(d_i)} \, d(d_i). \]  

(12)

Similarly, the PS of the integral densities yields

\[ P_k[\gamma_i^*]_{obs} = \frac{3}{4\pi} \int_0^{d_i(z_j)} \frac{\sin(k_j d_i)}{k_j d_i^2} N \, d(d_i). \]  

(13)

or, using Eqs. (3) and (5), it can also be written as

\[ P_k[\gamma_i^*]_{obs} = \frac{3}{4\pi} \int_0^{d_i(z_j)} \frac{\sin(k_j d_i)}{k_j d_i^2} N \, d(d_i). \]  

(14)

The angular diameter distance \( d_A \) is the quantity required in theory to obtain the theoretical number counts according to the relativistic cosmology theory developed by Ellis [19]. However, observationally measuring cosmological distances depends on circumstances, that is, they depend on the available observations. If one is only able to measure the redshift and intrinsic luminosity, then the luminosity distance is the distance measure of choice to calculate from observations. On the other hand, if one has observed intrinsic areas and redshift one is able to derive only the angular diameter distance (see also [9]). Hence, using different distance measures and deriving their respective power spectra open the way of applying the theory developed here to a wider range of observational possibilities.

### 3 Cosmologic and sample specific expressions

The quantities defined above are generally, that is, valid for any cosmological model. In this section, we shall adopt the FLRW cosmology and calculate them in this specific universe model, as well as using galaxy redshift survey observations.

As a simple examination of the equations above shows, we only require cosmologic specific expressions for \( d_i(z), d(d_i)/dz, N(z), dN/dz, \) \( [N]_{obs} \) and \( [dN(z)/dz]_{obs} \). The first four quantities are theoretical and come from the chosen spacetime geometry, whereas the last two ones come from a combination of theoretical and observational results, as we shall see next.

#### 3.1 FLRW cosmological model

In this cosmology, we need to obtain the required quantities along the past null cone, a task that is carried out numerically. As Ir12 discusses such a procedure in great details, what we shall present next is just a summary of the results needed here.

The scale factor of the FLRW cosmology with nonzero cosmological constant \( \Lambda \) can be written in terms of the radial coordinate \( r \) as follows:

\[ \frac{dS}{dr} = -H_0 \left[ \frac{(\Omega_{m_0}) S^4 - S_0^2 (\Omega_0 - 1) S^2 + (\Omega_{m_0} S_0^3) S}{c^2 - H_0^2 S_0^2 (\Omega_0 - 1) r^2} \right]^2, \]  

(15)

where \( \Omega_{m_0} \) and \( \Omega_0 \) are, respectively, the matter and cosmological density parameters, \( H_0 \) is the Hubble constant, \( S_0 \) is the current value of the scale factor, assumed to be equal to unity, and

\[ \Omega_0 \equiv \Omega_{m_0} + \Omega_{\Lambda_0}. \]  

(16)

To find solutions for \( S(r) \), we shall use the following numerical values for the parameters above \( \Omega_{m_0} = 0.3, \ \Omega_{\Lambda_0} = 0.7 \) and \( H_0 = 70 \ \text{km s}^{-1} \text{ Mpc}^{-1} \). The redshift \( z \) can be written as

\[ 1 + z = \frac{S_0}{S}, \]  

(17)
where it is clear that a numerical solution of the scale factor also produces numerical solutions for \( z(r) \). The differential number counts yields

\[
\frac{dN}{dz} = \left( \frac{3 \Omega_{m0} H_0 S_0^2}{2G M_g} \right) \times \frac{r^2 S^2}{\sqrt{(\Omega_{\Lambda0})S^4 - S_0^2 (\Omega_0 - 1)S^2 + (\Omega_{m0} S_0^3)S}}.
\]

where \( G \) is the gravitational constant, \( c \) is the light speed and \( M_g \) is the average galactic rest mass, dark matter included.

As pointed out by A07 and Ir12, the details of the galaxy mass function and how it evolves with the redshift are imprinted in the LF itself and will be included in any observationally derived functions stemming from the LF. For the theoretical quantities, we shall assume a constant average galaxy rest mass with the working value of the LF itself and will be included in any observationally derived functions stemming from the LF. For the theoretical quantities, \( \approx 10^{11} M_\odot \) based on the estimate by [20]. To actually extract from the LF, the implicit galaxy mass function so that the function \( M_g(z) \) can be estimated and used in the theoretical quantities is a problem already dealt with in Lopes et al. [21].

The angular diameter distance \( d_A \) is obtained by means of a relation between the intrinsically measured cross-sectional area element \( d\sigma \) of the source and the observed solid angle \( d\Omega_0 \). In a spherically symmetric spacetime, it may be written as follows (Ellis [19, 22]; Ribeiro [9]; Plebanski and Krasinski [23]),

\[
(d_A)^2 = \frac{d\sigma}{d\Omega_0} = (Sr)^2.
\]

The redshift distance is defined by the following expression:

\[
d_z = \frac{cz}{H_0},
\]

and the remaining distance measures can be obtained from the area distance by means of the Etherington [24] reciprocity law ([19, 22])

\[
d_L = (1+z)^2d_A = (1+z) d_G.
\]

The particular case of this law relating only \( d_A \) to \( d_G \) is known as the distance duality relation and can be determined observationally [25–27]. Thus, considering Eqs. (17), (19) and (21), the other cosmological distances are straightforwardly written in terms of the scale factor as follows:

\[
d_k = S_0^2 \left( \frac{r}{S} \right),
\]

\[
d_G = S_0 r,
\]

\[
d_c = \frac{c}{H_0} \left( \frac{S_0}{S} - 1 \right).
\]

Fig. 1 shows the redshift evolution of these distances in the adopted FLRW spacetime.

The derivatives of each distance measure in terms of the redshift are also necessary. Remembering Eqs. (15), (17) and (19), one can easily conclude that

\[
\frac{d(d_A)}{dz} = \frac{dS}{dz} \frac{dr}{dS} \frac{d(d_k)}{dz} = -\frac{S^2}{S_0} \left[ r + S \left( \frac{dS}{dr} \right)^{-1} \right].
\]

Similarly, both \( d(d_L)/dz \) and \( d(d_c)/dz \) are easily obtained, whereas \( d(d_G)/dz \) comes directly from Eq. (20).

Finally, the cumulative number counts \( N \) is derived as follows. The number of cosmological sources per proper volume unit \( n \) is related to the matter density \( \rho_m \) by means of

\[
n = \frac{N}{V_p} = \frac{\rho_m}{M_g},
\]

where \( V_p \) is the proper volume. As shown by Ir12, in the FLRW cosmology, the right-hand side of the Einstein equations allows us to write the number density in terms of the proper volume as follows:

\[
n = \left( \frac{3 \Omega_{m0} H_0^2 S_0^3}{8\pi G M_g} \right) \frac{1}{S^3}.
\]

In what follows, we shall need to write the number density in terms of the comoving volume \( V_c \) and, therefore, a conversion between \( V_p \) and \( V_c \) is needed. In the FLRW model, this is given by \( V_p/V_c = S^3 \), since \( V_c = (4/3)\pi r^3 \) and \( V_p = (4/3)\pi r^3 S^3 \).
Fig. 1 Redshift evolution of the four FLRW cosmological distance measures adopted here assuming \( \Omega_m = 0.3, \Omega_\Lambda = 0.7 \) and \( H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \). Symbols are as in the legend.

3.2 Observational quantities

The next step is to specialize the expressions above to obtain observational counterparts based on the LF data of a specific galaxy catalog. To do so, we need to use the methodology advanced by RS03 connecting theoretical expressions to astronomically derived quantities.

In summary, assuming that an observational quantity \([T]_{\text{obs}}\) can be related to its theoretical counterpart \(T\) by means of a consistency function \(J\) such that \( [T]_{\text{obs}} = J T \), then the relationship between the selection function \(\psi\), which gives the volume number density of galaxies with luminosity above a given threshold, to the comoving volume number density \(n_C(z)\) can be written as follows:

\[
\psi(z) = J(z) n_C(z).
\]

(28)

The redefinition of the number density in terms of comoving volume is a consequence of the fact that in observational cosmology, this is the most adopted volume definition, rather than the proper volume commonly found in theoretical calculations. Hence,

\[
n_C = \frac{N}{V_C} = \frac{3N}{4\pi r^3}.
\]

(29)

What Eq. (28) tells us is that the consistency function basically represents the undetected mass fraction in relation to the one predicted by theory. It also includes the galaxy redshift survey data, since the selection function (SF) is written in terms of the LF \(\phi\) as follows:

\[
\psi(z) = \int_{l_{\text{lim}(z)}}^{\infty} \phi(l) \, dl,
\]

(30)

where \(l_{\text{lim}}(z)\) is the lower luminosity threshold below which cosmological sources are not observed.

The galaxy LF is a number density per unit of luminosity, obtained by fitting the galaxy distribution in a given dataset to some function and determining its parameter values. The most common analytical form used to fit galaxy survey data is the [28] function, given as below,

\[
\phi(l) = \phi^* \, l^\alpha \, e^{-l}.
\]

(31)

Here, \(l \equiv L/L_*\), \(L\) is the observed luminosity, \(L_*\) is the luminosity scale parameter, \(\phi^*\) is a normalization parameter and \(\alpha\) gives the faint-end slope parameter. It was found by several works that the LF evolves with the redshift ([1, 2, 5] and references therein).

The observational quantities of interest studied in this paper require previous knowledge of the observed differential number counts \(\frac{dN}{dz}_{\text{obs}}\). Therefore, linking this quantity to its theoretical counterpart is an essential step, yielding (Ir12),

\[
\left[ \frac{dN}{dz} \right]_{\text{obs}} = J(z) \frac{dN}{dz} = \frac{\psi(z)}{n_C(z)} \frac{dN}{dz}.
\]

(32)

This is our key equation relating the relativistic theory to the observations. Thus, the observed cumulative number counts can be calculated as

\[
[N(z)]_{\text{obs}} = \int_0^z \left[ \frac{dN}{dz} \right]_{\text{obs}} \, dz',
\]

(33)

and, as discussed in Ir12, the uncertainty in \([N]_{\text{obs}}\) can be estimated from the uncertainty in \([dN/dz]_{\text{obs}}\).

Note that, the approach summarized above indicates that all observational quantities obtained by applying \(J(z)\) to their theoretical counterparts will inherit the same empirical number count redshift evolution encoded in the parametrization of the LF. In addition,
Fig. 2 These plots show the theoretical and observational relativistic differential densities for the four FLRW cosmological distances adopted here. The observational densities are shown in the combined UV, optical and red bands of the FDF datasets of G04 and G06. The difference between the theoretical curves and observational ones comes from the fact that the SF is limited by the observations themselves, which reflects on $J(z)$. So, such difference indicates large quantities of dark matter, as already pointed out by RS03 (see Sect. 5.1). Subsequent plots will also reflect the same behavior. Symbols are as in the legend.

Eq. (28) shows that the consistency function is independent of volume units and, hence, if an observational quantity is obtained using $J(z)$, its original volume unit dependence is preserved.

One must also point out that the consistency function $J(z)$ is characterized by the luminosity limits of the SF, but not only that because besides representing its integration limit, the SF also represents the LF parameters themselves, which were obtained for a specific dataset, so much so that they vary depending on the filters used during observation and according to the specific galaxy types. The LF is estimated for a given filter and for distinct types of galaxies in specific environments, hence the SF inherits the same characteristics and limitations of the LF. Therefore, when one estimates $J(z)$ from a given SF, its value will reflect not only the luminosity limit but also all other observational limitations.

Finally, as discussed at length in RS03, the consistency function reflects the differences between observational results and theoretical predictions. Hence, its value should always be in the interval $0 \leq J \leq 1$, where the unit means perfect agreement between theory and observations. The case of $J > 1$ indicates that the model must be revised, because such a result means that the theory cannot describe the observations. The case $J < 1$ is to be expected since our observational techniques are limited. From Eq. (28), one can see that $J$ represents the undetected, observational wise, fraction of objects in relation to what is predicted by theory.

3.3 FORS deep field galaxy redshift survey

As seen above, the consistency function is required for the calculation of the observational quantities, but to do so, one needs first to compute the SF in terms of the redshift in a given galaxy redshift survey. That was done by Ir12, where one can find a very thorough discussion of the procedure adopted in the calculation of the FDF SFs, as well as details of the numerical scheme. Hence, what we shall present below is a summary of Ir12’s methodology on this respect.
Fig. 3 Plots showing the theoretical and observational relativistic integral densities for the four FLRW cosmological distances adopted here. The observational points are shown in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Symbols are as in the legend.

Fig. 4 This graph shows the theoretical and observational relativistic integral densities obtained with the angular diameter distance $d_A$ against the redshift. The observational points are in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Note the difference between this plot ($\gamma_*^A$ vs. $z$) and the one in the lower right panel of Fig. 3 ($\gamma_*^A$ vs. $d_A$), although the Y-axis scale is basically the same. Symbols are as in the legend.

G04 and G06 fitted the LF Schechter parameters over the redshifts of 5558 I-band selected galaxies in the FORS Deep Field dataset, photometrically measured down to an apparent magnitude limit of $I_{AB} = 26.8$. All galaxies were selected in the I-band and then had their magnitudes for each of the five blue bands (1500 Å, 2800 Å, $u'$, $g'$ and $B$) and the three red ones ($r'$, $i'$ and $z'$) computed using the best fitting SED given by the photometric redshift code developed by the authors convolved with the associated filter function. G04 and G06 determined the photometric redshifts by fitting template spectra to the measured fluxes on the optical and near infrared images of the galaxies. The redshift ranges $0.75 \leq z \leq 3.0$ for the red bands and $0.5 \leq z \leq 5.0$ for the blue bands.
This figure shows the theoretical and observational relativistic integral densities computed with the angular diameter distance versus $d_A$ and the redshift. These graphs combine the information of Figs. 1 and 4, basically splitting the lower right plot of Fig. 3 in four different plots and adding a redshift scale at the right side of each graph. Note that, for the same redshift scale on the right side of each plot one needs a different scale for $\gamma^*_A$ on the left side.

To calculate the actual values of the SFs at each waveband, we need the result obtained in RS03 for the limited bandwidth version of the SF of a given LF, fitted by a Schechter's analytical profile (Eqs. 30 and 31). In terms of the absolute magnitudes $M(z)$, this expression reads,

$$
\psi^W(z) = 0.4 \ln 10 \int_{-\infty}^{M_{\text{lim}}^W(z)} \phi^*(z) 10^{0.4[M^*(z) - M^W]} \times \exp[-10^{0.4[M^*(z) - M^W]}] \, dM^W,
$$

(34)

where the index $W$ indicates the bandwidth filter. The expressions for the redshift evolution of the LF parameters adopted by G04 and G06 are

$$
\phi^*(z) = \phi^*_0 (1 + z)^{B^W},
$$

$$
M^*(z) = M^*_0 + A^W \ln(1 + z),
$$

$$
\alpha(z) = \alpha_0,
$$

where $A^W$ and $B^W$ are the evolution parameters fitted for the different $W$ bands and $M^*_0$, $\phi^*_0$ and $\alpha_0$ the local ($z \approx 0$) values of the Schechter parameters. Since all galaxies were detected and selected in the I-band, we can write

$$
M_{\text{lim}}^W(z) = M_{\text{lim}}^I(z) = I_{\text{lim}} - 5 \log[d_L(z)] - 25 + A^I,
$$

(35)

for a luminosity distance $d_L$ given in Mpc. $I_{\text{lim}} = 26.8$ is the limiting apparent magnitude of the I-band in the FDF survey. Its reddening correction is $A^I = 0.035 [29]$.

The SFs for all eight bands of the dataset were calculated by Ir12 using simple numerical integrations at equally spaced values spanning the whole redshift interval using parameter values as given by G04 and G06. The UV bands, 1500 Å, 2800Å, and $u'$,
Fig. 6 This figure shows a panel with the power spectra of the theoretical and observational relativistic differential densities computed with the four FLRW cosmological distances adopted here. The observational points are presented in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Symbols are as in the legend.

evolve tightly with redshift, having values that are consistent with each other within the uncertainties, while assuming values outside the uncertainties of the ones calculated in the blue optical bands $g'$, and $B$. Due to this, Ir12 chose to use the SFs of the combined UV bands and those of the combined blue optical bands separately. The SFs in the red-band dataset of G06, $r'$, $i'$ and $z'$ were also combined. Once the combined SFs were obtained, Ir12 is calculated \(\frac{dN}{dz}\) by means of Eq. (32).

4 Power spectrum analysis of the FDF survey

The observational values of the differential number counts \(\frac{dN}{dz}\) for all filters in the dataset allowed us to evaluate the observational differential number densities \(\gamma\) for the FLRW cosmology distance measures discussed above by simply replacing \(\frac{dN}{dz}\) with \(\frac{dN}{dz}\) in Eq. (4). The dependence of these densities on the distance definitions is shown in Fig. 2. Similarly, the observational values of the integral densities \(\gamma^*\) were obtained by replacing \(N\) with \(N\) in Eq. (5). The results are shown in Fig. 3.

The differential density constructed with the area distance \(d_A\) becomes discontinuous at \(z \approx 1.6\) because \(d_A\) reaches a maximum at that redshift and, therefore, its derivative with respect to \(z\) vanishes, rendering \(\gamma^*_A\) undefined. In addition, due to this, maximum \(\gamma^*_A\) has a curious turn around in \(d_A\) at \(z \approx 1.6\), as shown in Fig. 3. Nevertheless, Fig. 4 shows that such a curious behavior does not occur when the integral density \(\gamma^*_A\) is plotted in terms of \(z\), which in this case keeps on increasing. This is due to the fact that each \(d_A\) corresponds to two redshift values (see Fig. 1), but since each \(z\) produces only one \(\gamma^*_A\), each \(d_A\) generates then two values for \(z\) and \(\gamma^*_A\). The single point where this behavior is absent is when \(d_A\) reaches its maximum, which corresponds to single values for \(d_A\), \(z\) and \(\gamma^*_A\). This effect is better visualized in Fig. 5 where a redshift scale was added to the right side of each plot.
Panel showing the power spectrum of theoretical and observational relativistic integral densities. The observational points are presented in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Symbols are as in the legend.

Graph of the radial correlation for the luminosity distance in the combined UV waveband for different sample sizes. Symbols are as in the legend.

Figure 6 shows the PS of the FDF observational differential densities $P_k[\gamma|\lambda]_{\text{obs}}$ and their theoretical counterparts plotted against the wave number $k_i$ and the redshift. The observational points were determined by means of the combined UV, optical and red bands in the G04 and G06 dataset. It is clear from the graphs that $P_k[\gamma|\lambda]_{\text{obs}}$ possesses an odd behavior due to its discontinuity at $z \approx 1.6$, bearing no resemblance to the PS of the two-point correlation function, whereas the other power spectra $P_k[\gamma|\gamma, \lambda]_{\text{obs}}$ show a similar behavior to the two-point correlation function PS as presented by Tegmark et al. [15, Fig. 38]. We note that these...
The authors calculated the two-point correlation function $PS$ using different types of data, such as cosmic background radiation, galaxies, clusters of galaxies, lensing and Lyman alpha forest. Nevertheless, their general $PS$ behavior is basically similar to ours, apart from the one calculated with $dA$.

The $PS$ of the observational integral densities in the UV, optical and red combined bands versus the wave number and the redshift are shown in Fig. 7. We note that $P_k[γ_i equal G, L, Z]_{obs}$ differs from $P_k[γ_i equal G, L, Z]_{obs}$ shown in the previous figure only at small wave number values or, equivalently, at higher redshifts. Such difference occurs because the differential densities measure the rate of growth in number counts since $γ_i \propto dN/dz$, whereas $γ_i^* \propto N$ (see Eqs. 4 and 5). As $N$ is a cumulative quantity, it can only increase or remain constant, while $dN/dz$ increases, reaches a maximum and then decreases at scales dependent on the ratio of increase in $N$. Clearly, the declining behavior of $dN/dz$ will occur at higher values of $z$, beyond its maximum, leading the decline in $γ$ to become even more pronounced at those scales. In addition, by measuring the rate of growth in number counts, $γ$ is more sensitive to local fluctuations due to noisy data. Thus, the steeper decline detected in the slopes of $P_k[γ_i equal G, L, Z]_{obs}$ at small wave numbers as compared to those at $P_k[γ_i^* equal G, L, Z]_{obs}$ can be attributed to these distortion effects at the redshift limits of the sample.

As discussed above, the radial correlation $Ξ(d_z)$ proposed by Ribeiro [17] was then studied in the Einstein-de Sitter model, but only with the luminosity distance. Here, we extend the analysis of Eq. (7) to the galaxy area distance and redshift distance in the FLRW model with $Ω_m = 0.3$ and $Ω_\Lambda = 0.7$. These results are shown in Fig. 8 where one can see that, similarly to the Einstein-de Sitter cosmology, the amplitude of $[Ξ_{z}]_{obs}$ in the FLRW model also depends on the sample size. The extension of these results to $d_z$ and $dG$ is presented in Fig. 9, where different sample sizes were used, confirming thus such a dependence. Due to the pathological behavior of $dA$, $[Ξ_{z}]_{obs}$ was not evaluated.

Once in possession of the $[Ξ_{i equal G, L, Z}]_{obs}$ computations for the FDF dataset, it is natural to look at the behavior of their respective power spectrum. Fig. 10 shows such PS results, where one can see a distortion at small wave numbers probably caused by the subtraction of one in Eq. (7). To see if that is actually the reason behind this distortion, we evaluated the PS of the ratio $γ/γ^*$ for

![Fig. 9 Radial correlation for the luminosity distance, redshift distance and galaxy area distance in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Symbols are as in the legend.](image-url)
Fig. 10 PS of Radial correlation for the luminosity distance, redshift distance and galaxy area distance in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Values of $R_i$ are as in Fig. 9. Symbols are as in the legend.

$P_k[d_L]_{\text{obs}}$, $P_k[d_z]_{\text{obs}}$ and $P_k[d_G]_{\text{obs}}$. Graphs of these quantities are shown in Fig. 11 where we can clearly see the absence of the previous distortion. This means that as far the observed PS for the radial correlation $P_k[\Xi_i]_{\text{obs}}$ is concerned, this is better calculated if one redefines $\Xi_i$ in Eq. (7) as follows:

$$P_k[\Xi_i(d_i)]_{\text{obs}} = P_k\left[\frac{\gamma_i(d_i)}{\gamma_i^*(R_i)}\right]_{\text{obs}}.$$ (36)

Finally, the PS of the ratio $\gamma/\gamma^*$ for the area distance are shown in Fig. 12 where the discontinuity in $\gamma_i$ and $[\gamma_i]_{\text{obs}}$ translates themselves to discontinuous behavior for $P_k[\gamma/\gamma^*]_{\text{obs}}$.

5 Conclusion

In this paper, we have studied the theoretical and observational power spectrum of the differential and integral galaxy volume number densities $\gamma$ and $\gamma^*$ using various relativistic cosmology distance measures $d_i$, namely the angular diameter distance $d_A$, galaxy area distance $d_G$, luminosity distance $d_L$, and redshift distance $d_z$ ($i = A, G, L, Z$). The cosmology adopted here is the FLRW model with nonzero cosmological constant $\Lambda$ having $\Omega_{m0} = 0.3$, $\Omega_{\Lambda0} = 0.7$ and $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. Observational results were obtained by following the framework that connects the relativistic cosmology number counts theory with the astronomical data extracted from the galaxy luminosity function (LF), as proposed by Ribeiro and Stoeger [3] and further developed by Albani [4] and [5]. This framework uses the LF parameters of the FDF galaxy survey provided by Gabasch et al. [7, 8] in order to calculate the observed differential number counts $d[N/dz]_{\text{obs}}$ required to build the observational densities $[\gamma_i]_{\text{obs}}$ and $[\gamma_i^*]_{\text{obs}}$. The observational results were produced in the combined red, blue optical and UV bands in the redshift range $0.5 \leq z \leq 5.0$. 

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Fig. 11 PS of the ratio $\gamma/\gamma^*$ for the luminosity distance, redshift distance and galaxy area distance in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Values of $[R_i]_{\text{obs}}$ are as in Fig. 9. Symbols are as in the legend.

Fig. 12 PS of the ratio $\gamma/\gamma^*$ for the area distance in the combined UV, optical and red bands of the FDF datasets of G04 and G06. Symbols are as in the legend.
The PS of the galaxy volume number densities were obtained directly from both \([\gamma_i]_{\text{obs}}\) and \([\gamma^*_i]_{\text{obs}}\), and two other combinations of these quantities, namely the ratio \([\gamma_i/\gamma^*_i]_{\text{obs}}\) and the radial correlation \([\Xi_i]_{\text{obs}}\) which mix up these two densities in a way that they can be discussed in terms of the sample size and each distance measure used above. Therefore, our analysis was able to compute \(P_k[\gamma_i]_{\text{obs}}, P_k[\gamma^*_i]_{\text{obs}}, P_k[\gamma_1/\gamma^*_1]_{\text{obs}}\) and \(P_k[\Xi_1]_{\text{obs}}\).

The results show that the graphs for \(P_k[\gamma_i]_{\text{obs}, G, l, z}\), \(P_k[\gamma^*_i]_{\text{obs}, G, l, z}\) and \(P_k[\gamma_1/\gamma^*_1]_{\text{obs}, G, l, z}\) have similar behavior for larger wave numbers, but the first differs from the other two at small wave numbers. Both \(P_k[\gamma_i]_{\text{obs}, G, l, z}\) and \(P_k[\gamma_1/\gamma^*_1]_{\text{obs}, G, l, z}\) have general power-law behavior, whereas \(P_k[\gamma^*_i]_{\text{obs}, G, l, z}\) behaves more similarly to the PS derived from the 2-point correlation function (see Fig. 38 of [15]) having a more pronounced decline at very small wave numbers.

The PS analysis using \(d_4\) is problematic because this distance measure starts increasing, reaches a maximum and then decreases, resulting in a discontinuous \([\gamma_i]_{\text{obs}}\) at \(z \approx 1.6\). Therefore, both \(P_k[\gamma_i]_{\text{obs}}\) and \(P_k[\gamma_1/\gamma^*_1]_{\text{obs}}\) also present discontinuities. \(P_k[\gamma^*_i]_{\text{obs}}\) is, however, continuous, but has a minimum wave number value just as \(d_4\) has a maximum.

The radial correlation \(\Xi_1\), a quantity defined to possibly exhibit sample dependence in observational values, actually showed such a dependence for \([\Xi_1]_{\text{obs}, G, l, z}\), confirming a previous theoretical prediction made by using only the luminosity distance \(d_L\) [17]. Nevertheless, their respective PS \(P_k[\Xi_i]_{\text{obs}, G, l, z}\) presents some strong distortions.

In summary, the study presented here suggests that the PS analysis of the galaxy number densities can be considered as a complementary tools for studies of the large-scale galaxy distribution. That may be particularly the case of models describing the galaxy distribution as a fractal system, as advanced by Conde-Saavedra et al. [30] and Teles et al. [31], because these works showed that both \(\gamma_i\) and \(\gamma^*_i\) are essential tools for the fractal characterization of the galaxy distribution. This is particularly the case because these densities present decaying power-law behavior for \(l = (g, l, z)\) at increasing distances, as shown in Figs. 2 and 3. Since their respective power spectra also behave mostly as decaying power-laws for larger wave numbers, as seen in Figs. 6 and 7, as well as with the radial correlations in Fig. 9 and their redefined power spectra (Eq. 36) as shown in Fig. 11, these quantities provide additional tools for probing the large-scale galaxy distribution in the universe.

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