THE LOCAL-GLOBAL PROPERTY FOR BITANGENTS OF PLANE QUARTICS

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Abstract. We study the arithmetic of bitangents of smooth quartics over global fields. With the aid of computer algebra systems and using Elsenhans–Jahnel’s results on the inverse Galois problem for bitangents, we show that, over any global field of characteristic different from 2, there exist smooth quartics which have bitangents over every local field, but do not have bitangents over the global field. We give an algorithm to find such quartics explicitly, and give an example over \( \mathbb{Q} \). We also discuss a similar problem concerning symmetric determinantal representations. This paper is a summary of the first author’s talk at the JSIAM JANT workshop on algorithmic number theory in March 2019. Details will appear elsewhere.

1. Introduction

Let \( C \subset \mathbb{P}^2 \) be a smooth quartic over a field \( k \) of characteristic different from 2. It is defined by a homogeneous polynomial \( f(X, Y, Z) \) of degree 4:

\[ C = \{ [X : Y : Z] \in \mathbb{P}^2 \mid f(X, Y, Z) = 0 \}. \]

By Bézout’s theorem, the intersection of \( C \) with a line \( L \subset \mathbb{P}^2 \) consists of four points, counted with their multiplicities. A line \( L \subset \mathbb{P}^2 \) is called a bitangent of \( C \) if one of the following conditions is satisfied:

- \( L \) tangents to \( C \) at two distinct points, or
- \( L \) tangents quadruply to \( C \) at one point. (In this case, the line \( L \) is also called a hyperflex line of \( C \).)

It has been known since the 19th century that every smooth quartic has exactly 28 bitangents over an algebraic closure of \( k \) [1, Chapter 6]. When \( k \) is not algebraically closed (e.g. \( k = \mathbb{Q} \)), the number of bitangents defined over \( k \) is usually smaller than 28. Harris and Shioda studied the action of the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the set of the 28 bitangents of a smooth quartic over \( \mathbb{Q} \) [5], [11]. However, the arithmetic of bitangents of smooth quartics are not well understood yet.

In this paper, we shall study the local-global property (or Hasse principle) for bitangents. It is natural to ask the following question. Let \( K \) be a global field of characteristic different from 2. (It is a finite extension of \( \mathbb{Q} \) or \( \mathbb{F}_p(T) \) with \( p \neq 2 \).) Let \( C \subset \mathbb{P}^2 \) be a smooth quartic over \( K \). Assume that \( C \) has a bitangent over the completion \( K_v \) for every place \( v \) of \( K \) (including the infinite places when \( K \) is a number field). Then, does \( C \) have a bitangent over \( K \)?

We shall answer this question negatively. Here is the main theorem of this paper.

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Theorem 1.1. For any global field $K$ of characteristic different from $2$, there exists a smooth quartic $C \subset \mathbb{P}^2$ over $K$ such that

- $C$ has a bitangent over $K_v$ for every place $v$ of $K$, but
- $C$ does not have a bitangent over $K$.

Our proof of Theorem 1.1 is a combination of certain group theoretic results on $\text{Sp}_6(\mathbb{F}_2)$ obtained with the aid of computer algebra systems and the results on the inverse Galois problem for bitangents recently obtained by Elsenhans–Jahnel. We shall also give an algorithm to find quartics failing the local-global property for bitangents explicitly, and give such an example over $\mathbb{Q}$. By a similar method, we also obtain smooth quartics failing the local-global property for symmetric determinantal representations; see Theorem 8.1.

After this work was completed, Jahnel and Loughran told the authors that it is also possible to construct smooth quartics satisfying the conditions in Theorem 1.1 by the results in [9]; see Remark 6.1.

2. BITANGENTS AND QUADRATIC FORMS

Let $k$ be a field of characteristic different from $2$, and $k^\text{sep}$ a separable closure of $k$. Let $C \subset \mathbb{P}^2$ be a smooth quartic over $k$. The Jacobian variety $\text{Jac}(C)$ is an abelian variety of dimension 3 since the genus of $C$ is 3. The group $\text{Jac}(C)[2]$ of $k^\text{sep}$-rational points on $\text{Jac}(C)$ killed by 2 is a 6-dimensional vector space over $\mathbb{F}_2$. It is equipped with the action of $\text{Gal}(k^\text{sep}/k)$ and the Weil pairing

$$\langle \cdot, \cdot \rangle : \text{Jac}(C)[2] \times \text{Jac}(C)[2] \rightarrow \{-1, 1\} \cong \mathbb{F}_2.$$

A map $Q : \text{Jac}(C)[2] \rightarrow \mathbb{F}_2$ is called a quadratic form with polar form $\langle \cdot, \cdot \rangle$ if

$$Q(x + y) - Q(x) - Q(y) = \langle x, y \rangle$$

is satisfied for all $x, y \in \text{Jac}(C)[2]$. Take a symplectic basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ of $\text{Jac}(C)[2]$, i.e. $\langle e_i, e_j \rangle = \delta_{i,j} = 0$ and $\langle e_i, f_j \rangle = \delta_{i,j}$ for all $1 \leq i, j \leq 3$. The Arf invariant of $Q$ is defined by

$$\text{Arf}(Q) := \sum_{1 \leq i < 3} Q(e_i)Q(f_i) \in \mathbb{F}_2,$$

which is independent of the choice of $e_1, e_2, e_3, f_1, f_2, f_3$.

Lemma 2.1 (Mumford). There is a canonical bijection between the set of bitangents of $C$ over $k^\text{sep}$, and the set of quadratic forms on $\text{Jac}(C)[2]$ of Arf invariant 1 whose polar form is the Weil pairing $\langle \cdot, \cdot \rangle$.

See [10, 6] Proposition 6.2 for details. Here we briefly recall the construction of the bijection. Let $L \subset \mathbb{P}^2$ be a bitangent of $C$. It is defined over $k^\text{sep}$ since $\text{char}(k) \neq 2$. We put $C \cap L = \{P_1, P_2\}$ for some $k^\text{sep}$-rational points $P_1, P_2$. (We put $P_1 = P_2$ if $L$ is a hyperflex line.) Then $D_L := P_1 + P_2$ is a divisor on $C$ over $k^\text{sep}$ called an effective theta characteristic. A point on $\text{Jac}(C)[2]$ corresponds to a line bundle $\mathcal{M}$ on $C$ over $k^\text{sep}$ with $\mathcal{M} \otimes \mathcal{O}_C \cong \mathcal{O}_C$. Then the map $\text{Jac}(C)[2] \rightarrow \mathbb{F}_2$ defined by

$$[\mathcal{M}] \mapsto h^0(\mathcal{O}_C(D_L)) + h^0(\mathcal{M}(D_L)) \pmod{2}$$

is a quadratic form of Arf invariant 1 with polar form $\langle \cdot, \cdot \rangle$. Conversely, it can be shown that every quadratic form of Arf invariant 1 with polar form $\langle \cdot, \cdot \rangle$ is obtained from a unique bitangent of $C$. 
A symplectic basis \( \{ e_1, e_2, e_3, f_1, f_2, f_3 \} \) of \( \text{Jac}(C)[2] \) gives an isomorphism \( \text{Jac}(C)[2] \cong \mathbb{F}_2^{3 \times 6} \). The action of \( \text{Gal}(k^{\text{sep}}/k) \) gives a continuous homomorphism

\[
\rho_C: \text{Gal}(k^{\text{sep}}/k) \to \text{Sp}(\text{Jac}(C)[2], \langle \cdot, \cdot \rangle) \cong \text{Sp}_6(\mathbb{F}_2).
\]

By Lemma 2.1, via the map \( \rho \), the action of \( \text{Gal}(k^{\text{sep}}/k) \) on the bitangents is translated into the action of \( \text{Sp}_6(\mathbb{F}_2) \) on the quadratic forms on \( \mathbb{F}_2^{3 \times 6} \) of Arf invariant 1.

**Remark 2.2.** For a ‘generic’ choice of \( C \), Harris and Shioda proved that the map \( \rho_C \) is surjective, at least when \( \text{char}(k) \notin \{3, 5, 7, 11, 29, 1229\} \); see [11, Theorem 7]. For explicit examples of quartics with surjective \( \rho_C \), see [4, p. 26, Corollary 3]. Shioda also constructed smooth quartics over \( \mathbb{Q} \) such that all the 28 bitangents are defined over \( \mathbb{Q} \); see [11 (6.6)].

### 3. Group theoretic results

Let \( \langle \cdot, \cdot \rangle \) be the standard symplectic form on \( \mathbb{F}_2^{3 \times 6} \). Let \( \Omega \) be the set of quadratic forms on \( \mathbb{F}_2^{3 \times 6} \) with polar form \( \langle \cdot, \cdot \rangle \). Let \( \Omega^+ \) (resp. \( \Omega^- \)) be the subset of \( \Omega \) consisting of quadratic forms of Arf invariant 0 (resp. 1), which has 36 (resp. 28) elements. The symplectic group \( \text{Sp}_6(\mathbb{F}_2) \) acts transitively on \( \Omega^+ \) and \( \Omega^- \). Let \( U_{36} \subset \text{Sp}_6(\mathbb{F}_2) \) (resp. \( U_{28} \subset \text{Sp}_6(\mathbb{F}_2) \)) be the stabilizer of an element of \( \Omega^+ \) (resp. \( \Omega^- \)).

The action of \( \text{Sp}_6(\mathbb{F}_2) \) on the set of non-zero vectors \( \mathbb{F}_2^{3 \times 6} \setminus \{0\} \) is transitive, and the stabilizer of a non-zero vector is denoted by \( U_{63} \subset \text{Sp}_6(\mathbb{F}_2) \).

It is known that \( U_{28}, U_{36}, U_{63} \) are maximal subgroups of \( \text{Sp}_6(\mathbb{F}_2) \), and every subgroup of \( \text{Sp}_6(\mathbb{F}_2) \) of index 28, 36, 63 is conjugate to \( U_{28}, U_{36}, U_{63} \), respectively.

We shall consider the following condition.

**Condition 3.1.** Let \( G \subset \text{Sp}_6(\mathbb{F}_2) \) be a subgroup. We say \( G \) satisfies the condition \( (*)^+ \) (resp. \( (*)^- \)) if the following conditions are satisfied:

- No element of \( \Omega^+ \) (resp. \( \Omega^- \)) is fixed by every element of \( G \).
- For every \( g \in G \), the action of \( g \) on \( \Omega^+ \) (resp. \( \Omega^- \)) has a fixed point.

The following results can be confirmed by GAP. (For a sample source code for GAP, see Appendix A.)

**Proposition 3.2.**

1. \( U_{36} \) has 296 subgroups, up to conjugation. Among them, 35 subgroups satisfy the condition \( (*)^- \); all the 35 subgroups are solvable.
2. \( U_{63} \) has 1916 subgroups, up to conjugation. Among them, 548 subgroups satisfy both of the condition \( (*)^+ \) and \( (*^-) \); 536 of the 548 subgroups are solvable.

**Proposition 3.3.** \( \text{Sp}_6(\mathbb{F}_2) \) has 6 subgroups isomorphic to \( \mathbb{F}_2^{3 \times 5} \), up to conjugation. All of them satisfy both of the conditions \( (*^+) \) and \( (*^-) \).

**Remark 3.4.** Any subgroup of \( U_{36} \) does not satisfy the condition \( (*)^+ \) since, by definition, it fixes at least one element of \( \Omega^+ \).

### 4. The inverse Galois problems

The inverse Galois problem asks whether every finite group is realized as the Galois group of a number field over \( \mathbb{Q} \). It is an open problem in general. It was proved by Shafarevich for solvable groups. Sonn observed that Shafarevich’s solution to the inverse Galois problem yields a Galois extension such that every decomposition group is cyclic; see [12, Theorem 2] for details.
**Theorem 4.1** (Shafarevich, Sonn). Let $G$ be a finite solvable group, and $K$ a global field. Then there exists a finite Galois extension $L/K$ such that $\text{Gal}(L/K)$ is isomorphic to $G$, and that, for every place $v$ of $K$, the decomposition group of $L/K$ at $v$ is cyclic.

Recently, Elsenhans–Jahnel studied an analogue of the inverse Galois problem for bitangents of quartics. Let $k$ be a field of characteristic different from 2, and $\rho: \text{Gal}(k^{\text{sep}}/k) \to \text{Sp}_6(\mathbb{F}_2)$ a continuous homomorphism. They asked whether $\rho$ is realized as the map $\rho_C$ associated with a smooth quartic $C \subset \mathbb{P}^2$ over $k$, up to conjugation by an element of $\text{Sp}_6(\mathbb{F}_2)$.

The following result is proved in [2] (resp. [3]) when the image of $\rho$ is contained in a conjugate of $U_{36}$ (resp. $U_{63}$).

**Theorem 4.2** (Elsenhans–Jahnel). If the image of $\rho$ is contained in a conjugate of $U_{36}$ or $U_{63}$, there exists a smooth quartic $C \subset \mathbb{P}^2$ over $k$ such that the maps $\rho, \rho_C$ are conjugate by an element of $\text{Sp}_6(\mathbb{F}_2)$.

**Remark 4.3.** Elsenhans–Jahnel used different geometric methods to construct smooth quartics with prescribed Galois action on bitangents in [2], [3]. Assume that $\rho(\text{Gal}(k^{\text{sep}}/k))$ is contained in a conjugate of $U_{36}$, and it is also contained in a conjugate of $U_{63}$. In this case, the results in [2], [3] give different quartics, in general.

5. **Proof of Theorem 1.1**

Theorem 1.1 is proved by combining Lemma 2.1, Proposition 3.2, Theorem 4.1, and Theorem 4.2.

1. Take a solvable subgroup $G \subset \text{Sp}_6(\mathbb{F}_2)$ contained in $U_{36}$ or $U_{63}$ which satisfies the condition $(-)^-$. Such a subgroup exists by Proposition 3.2.

2. Take a finite Galois extension $L/K$ such that $\text{Gal}(L/K) \cong G$ and, for every place $v$ of $K$, the decomposition group of $L/K$ at $v$ is cyclic. Such an extension exists by Theorem 4.1.

3. Consider the composite of the following maps:

$$\rho: \text{Gal}(k^{\text{sep}}/k) \to \text{Gal}(L/K) \cong G \hookrightarrow \text{Sp}_6(\mathbb{F}_2).$$

Take a smooth quartic $C \subset \mathbb{P}^2$ over $K$ such that the maps $\rho, \rho_C$ are conjugate by an element of $\text{Sp}_6(\mathbb{F}_2)$. Such a quartic exists by Theorem 4.2.

4. Then, the quartic $C$ satisfies all the conditions of Theorem 1.1 by Lemma 2.1.

6. **Description of Algorithm**

Here we shall give an algorithm obtaining smooth quartics failing the local-global property for bitangents.

We need to fix a subgroup $G$ of $\text{Sp}_6(\mathbb{F}_2)$. In this section, we shall take a subgroup $G \subset U_{63} \subset \text{Sp}_6(\mathbb{F}_2)$.
isomorphic to $\mathbb{F}_2^{25}$ since we can use Proposition 3.3 for such a subgroup. Explicitly, the group $U_{63}$ is isomorphic to $(\mathbb{F}_2 \wr S_6) \cap A_{12}$; see [3, Corollary 2.19]. The wreath product $\mathbb{F}_2 \wr S_6$ contains a subgroup isomorphic to $\mathbb{F}_2^{26}$. Let $G$ be the kernel of the sum $\mathbb{F}_2^{26} \to \mathbb{F}_2$. Then $G$ is isomorphic to $\mathbb{F}_2^{35}$, and we have an embedding
\[ G \cong \mathbb{F}_2^{35} \to (\mathbb{F}_2 \wr S_6) \cap A_{12} \cong U_{63}. \]

Our algorithm consists of 3 steps.

**Step 1 (Find a Galois extension $L/K$.)** Take a Galois extension $L/K$ such that
\[ \text{Gal}(L/K) \cong \mathbb{F}_2^{35} \]
and that every decomposition group is cyclic. (The existence of such $L/K$ is guaranteed by Theorem 4.1.1)

Assume that $L$ is described as
\[ L = K (\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}, \sqrt{a_5}) , \]
for some $a_1, a_2, a_3, a_4, a_5 \in K$. We put $a_u = a_1 a_2 a_3 a_4 a_5 u^2$ for some $u \in K^\times$ and set
\[ F(S, T) = (a_1 S - T)(a_2 S - T) \cdots (a_6 S - T). \]
By construction, the splitting field of $F(1, T^2)$ is $L$ and $F(1, 0) = (a_1 a_2 \cdots a_5 u)^2$.

**Step 2 (Construct a conic bundle $B \subset \mathbb{P}^1 \times \mathbb{P}^2$.)** There exists a unique pair of binary quadratic forms $g(S, T), h(S, T)$ satisfying
\[ \det M(S, T) = -F(S, T) \]
and $g(1, 0) = a_1 a_2 a_3 a_4 a_5 u$, where
\[ M(S, T) = \begin{pmatrix} -ST + T^2 & ST & g(S, T) \\ ST & S^2 & T^2 \\ g(S, T) & T^2 & h(S, T) \end{pmatrix}. \]

Let $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ be the hypersurface defined by
\[ (X \quad Y \quad Z) M(S, T) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0. \]
Here $[S : T]$ is the projective coordinate of $\mathbb{P}^1$ and $[X : Y : Z]$ is that of $\mathbb{P}^2$. The first projection $\text{pr}_1 : B \to \mathbb{P}^1$ defines a conic bundle structure with six degenerate fibers with prescribed Galois action; see [3, Proposition 3.5].

**Step 3 (Calculate the branching locus.)** The composite of the embedding
\[ \iota : B \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \]
and the projection
\[ \text{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \]
is a double cover. Its ramification locus is the desired quartic $C \subset \mathbb{P}^2$. (If $C$ is not smooth, then take other parameters $a_1, a_2, \ldots, a_5$ and $u$, and calculate the quartic $C$ again. For a ‘generic’ choice of $a_1, a_2, \ldots, a_5$ and $u$, the quartic $C$ is smooth; see [3, Proposition 3.5].)

Applying the above algorithm, we can construct a smooth quartic $C \subset \mathbb{P}^2$ over $K$ such that the map $\rho_C$ is conjugate to the composite of the following maps:
\[ \text{Gal}(K^{\text{sep}}/K) \to \text{Gal}(L/K) \cong G \hookrightarrow \text{Sp}_6(\mathbb{F}_2). \]
The smooth quartic $C \subset \mathbb{P}^2$ satisfies all the conditions of Theorem 1.1 by Proposition 3.3.

**Remark 6.1 (Jahnel–Loughran).** Here is another construction of smooth quartics satisfying the conditions of Theorem 1.1 by the results in [9]. Take two closed points $P, Q$ of the projective plane $\mathbb{P}^2$ over $K$ (as a $K$-scheme) of degree 3, 4 with splitting fields $\kappa(P), \kappa(Q)$, respectively, such that

- the union $P \sqcup Q$ lies in general position,
- $\kappa(P), \kappa(Q)$ are linearly disjoint Galois extensions of $K$ with $\text{Gal}(\kappa(P)/K) \cong \mathbb{Z}/3\mathbb{Z}$ and $\text{Gal}(\kappa(Q)/K) \cong A_4$, and
- every decomposition group of $\text{Gal}(\kappa(Q)/K) \cong A_4$ is cyclic.

(Such points $P, Q$ exist by Theorem 4.1.) Let $X$ be the blowup of $\mathbb{P}^2$ along $P \sqcup Q$. By [9, Lemma 3.9], it is a del Pezzo surface of degree 2 failing the local-global property for lines. The branching locus $C \subset \mathbb{P}^2$ of the anticanonical map of $X$ is a smooth quartic. We have a 2 : 1 map from the 56 lines on $X$ to the 28 bitangents of $C$. Since $(\mathbb{Z}/3\mathbb{Z}) \times A_4$ has no subgroup of index 2, it is easy to check that $C$ satisfies all the conditions of Theorem 1.1.

7. **An example**

We consider the case $K = \mathbb{Q}$. We put

$$b_1 = -1, \quad b_2 = 17, \quad b_3 = 89, \quad b_4 = 257, \quad b_5 = 769,$$

and put $K_i = \mathbb{Q}(\sqrt{b_i})$ for every $1 \leq i \leq 5$. Then, for every $1 \leq i \leq 5$, only one prime number is ramified in $K_i/\mathbb{Q}$. For every $1 \leq i, j \leq 5$ with $i \neq j$, the prime number $p_i$ splits in $K_j/\mathbb{Q}$. (Here $p_i$ is the unique prime number ramified in $K_i/\mathbb{Q}$.)

We put $L = K_1K_2K_3K_4K_5$, and

$$a_1 = b_1b_5, \quad a_2 = b_2b_4, \quad a_3 = b_3, \quad a_4 = b_4, \quad a_5 = b_5, \quad u = -b_4^{-1}b_5^{-1}.$$

Then $L/K$ satisfies the conditions in Step 1. We have

$$F(S, T) = (-769S - T)(4369S - T)(89S - T)(257S - T)(769S - T)(-1513S - T),$$

$$g(S, T) = \frac{1}{8}(2392149832S^2 + 35008837ST + 12804T^2),$$

$$h(S, T) = -\frac{1}{64}(251582881045706064S^2 + 1084638148302617ST + 594847875240T^2).$$
The quartic is computed as

\[ 4096X^4 - 16384X^3Y - 9869943810048X^3Z \\
+ 143396196352X^2YZ - 52445184XY^2Z - 32768Y^3Z \\
+ 64826445425191482752X^2Z^2 \\
- 277686962456893696XY^2Z^2 \\
+ 1522810595401504768Y^2Z^2 \\
- 577825743806146129742752249Z^4 = 0. \]

It is a smooth quartic. Thus it gives an example of smooth quartics over \( \mathbb{Q} \) failing the local-global property for bitangents.

8. The Local-Global Property for Symmetric Determinantal Representations

Recall that a smooth quartic \( C \subset \mathbb{P}^2 \) over a field \( k \) is said to admit a symmetric determinantal representation over \( k \) if there exist symmetric matrices \( M_1, M_2, M_3 \in \text{Mat}_4(k) \) of size \( 4 \times 4 \) such that the equation

\[ \det(XM_1 + YM_2 + ZM_3) = 0 \]

defines the quartic \( C \subset \mathbb{P}^2 \). (For explicit examples, see [8].)

By the same method as above, it is possible to obtain smooth quartics failing the local-global property for symmetric determinantal representations.

**Theorem 8.1.** For any global field \( K \) of characteristic different from 2, there exists a smooth quartic \( C \subset \mathbb{P}^2 \) over \( K \) such that

- \( C \) admits a symmetric determinantal representation over \( K_v \) for every place \( v \) of \( K \), but
- \( C \) does not admit a symmetric determinantal representation over \( K \).

Here is a sketch of the proof. The point is that a symmetric determinantal representation corresponds to a non-effective theta characteristic.

For a smooth quartic \( C \subset \mathbb{P}^2 \) with a \( K \)-rational point, it admits a symmetric determinantal representation over \( K \) if and only if there exists a \( \text{Gal}(K^{\text{sep}}/K) \)-invariant quadratic form on \( \text{Jac}(C)[2] \) of Arf invariant 0 whose polar form is the Weil pairing; see [6, Theorem 2.2, Corollary 6.3]. (Compare with Lemma 2.1.) Taking a subgroup \( G \subset U_{63} \) satisfying the condition (*)\(^*\), we find smooth quartics over \( K \) satisfying the conditions of Theorem 8.1 by the same way as in the case of bitangents.

**Example 8.2.** Quartics constructed by the algorithm in Section 6 always have the \( \mathbb{Q} \)-rational point \([0 : 1 : 0]\). By Proposition 3.3, the example of quartic in Section 7 satisfies the conditions of Theorem 8.1. (But the quartics constructed by the method described in Remark 6.1 do not satisfy these conditions.)
Remark 8.3. In [6], the first and the second authors proved that, in general, smooth quartics over number fields do not satisfy the local-global property for symmetric determinantal representations. But the quartics constructed in [6] are defined over number fields of large degree. For quartics over $\mathbb{Q}$, this problem was stated in [6, Problem 1.6 (1)], but not answered there.

Remark 8.4. Theorem 8.1 does not hold in characteristic 2. In fact, over a global field of characteristic 2, any smooth plane curve of any degree satisfies the local-global property for symmetric determinantal representations; see [7] for details.

**Appendix A. Sample source codes for GAP and SageMath**

We proved Proposition 3.2 and Proposition 3.3 by GAP. Here is a sample source code for GAP (Version 4.10.2) which performs necessary calculation.

```gap
G := PSp(6,2);
C := List(ConjugacyClassesMaximalSubgroups(G), Representative);
# G = Sp_6(F_2) has 8 maximal subgroups, with order
# 51840 40320 23040 12096 10752 4608 4320 1512
U28 := C[1];;
if Size(G)/Size(U28) = 28 then Print("OK.\nU28 has index 28.\n"); fi;
else Print("Error: U28 does not have index 28.\n"); fi;
U36 := C[2];;
if Size(G)/Size(U36) = 36 then Print("OK.\nU36 has index 36.\n"); fi;
else Print("Error: U36 does not have index 36.\n"); fi;
U63 := C[3];;
if Size(G)/Size(U63) = 63 then Print("OK.\nU63 has index 63.\n"); fi;
else Print("Error: U63 does not have index 63.\n"); fi;
count_solvable_32 := function(GroupList)
local count_solvable, count_32;
count_solvable := 0;
count_32 := 0;
for K in GroupList do
if IsSolvable(K) = true
then count_solvable := count_solvable + 1; fi;
if IsomorphismGroups(K, ElementaryAbelianGroup(32)) <> fail
then count_32 := count_32 + 1; fi;
end;
Print("The number of solvable subgroups: ",count_solvable,"\n");
Print("The number of subgroups isomorphic to (F_2)^5: ",count_32,"\n");
end;
count_cond := function(GroupList)
local count_plus, count_plus_solvable, count_plus_32,
count_minus, count_minus_solvable, count_minus_32,
count_both, count_both_solvable, count_both_32;
count_plus := 0;
count_plus_solvable := 0;
count_plus_32 := 0;
count_minus := 0;
count_minus_solvable := 0;
count_minus_32 := 0;
count_both := 0;
count_both_solvable := 0;
count_both_32 := 0;
end;
```


count_minus_32 := 0;
count_both := 0;
count_both_solvable := 0;
count_both_32 := 0;

for K in GroupList do
  # Condition +
  A := Action(K, RightCosets(G, U36), OnRight);
  cond_plus := 0;
  if NrMovedPoints(A) < 36 then cond_plus := 1;
  else
    for x in A do if NrMovedPoints(x) = 36 then cond_plus := 1; fi; od;
  fi;

  # Condition -
  A := Action(K, RightCosets(G, U28), OnRight);
  cond_minus := 0;
  if NrMovedPoints(A) < 28 then cond_minus := 1;
  else
    for x in A do if NrMovedPoints(x) = 28 then cond_minus := 1; fi; od;
  fi;

  if cond_plus = 0 then
    count_plus := count_plus + 1;
    if IsSolvable(K) = true then
      count_plus_solvable := count_plus_solvable + 1; fi;
    if IsomorphismGroups(K, ElementaryAbelianGroup(32)) <> fail then
      count_plus_32 := count_plus_32 + 1; fi;
  fi;

  if cond_minus = 0 then
    count_minus := count_minus + 1;
    if IsSolvable(K) = true then
      count_minus_solvable := count_minus_solvable + 1; fi;
    if IsomorphismGroups(K, ElementaryAbelianGroup(32)) <> fail then
      count_minus_32 := count_minus_32 + 1; fi;
  fi;

  if cond_plus = 0 and cond_minus = 0 then
    count_both := count_both + 1;
    if IsSolvable(K) = true then
      count_both_solvable := count_both_solvable + 1; fi;
    if IsomorphismGroups(K, ElementaryAbelianGroup(32)) <> fail then
      count_both_32 := count_both_32 + 1; fi;
  fi;

od;

Print("Condition plus: \n", count_plus, "\n");
Print("Condition plus (solvable): \n", count_plus_solvable, "\n");
Print("Condition plus ((F_2)^5): \n", count_plus_32, "\n");
Print("Condition minus: \n", count_minus, "\n");
Print("Condition minus (solvable): \n", count_minus_solvable, "\n");
Print("Condition minus ((F_2)^5): \n", count_minus_32, "\n");
If the above code is executed successfully, it outputs as follows.

1 OK. U28 has index 28.
2 OK. U36 has index 36.
3 OK. U63 has index 63.
4 Sp(6,2) has 1369 subgroups, up to conjugation.
5 The number of solvable subgroups: 1301
6 The number of subgroups isomorphic to \((F_2)^5\): 6
7 Condition plus: 411
8 Condition plus (solvable): 399
9 Condition plus ((F_2)^5): 6
10 Condition minus: 371
11 Condition minus (solvable): 359
12 Condition minus ((F_2)^5): 6
13 Both: 240
14 Both (solvable): 228
15 Both ((F_2)^5): 6
16 U28 has 350 subgroups, up to conjugation.
17 The number of solvable subgroups: 331
18 The number of subgroups isomorphic to \((F_2)^5\): 0
19 Condition plus: 22
20 Condition plus (solvable): 22
21 Condition plus ((F_2)^5): 0
22 Condition minus: 0
23 Condition minus (solvable): 0
24 Condition minus ((F_2)^5): 0
25 Both: 0
26 Both (solvable): 0
27 Both ((F_2)^5): 0
28 U36 has 296 subgroups, up to conjugation.
29 The number of solvable subgroups: 268
The number of subgroups isomorphic to \((F_2)^5\): 0
Condition plus: 0
Condition plus (solvable): 0
Condition plus \((F_2)^5\): 0
Condition minus: 35
Condition minus (solvable): 35
Condition minus \((F_2)^5\): 0
Both: 0
Both (solvable): 0
Both \((F_2)^5\): 0
U63 has 1916 subgroups, up to conjugation.
The number of solvable subgroups: 1880
The number of subgroups isomorphic to \((F_2)^5\): 13
Condition plus: 856
Condition plus (solvable): 844
Condition plus \((F_2)^5\): 13
Condition minus: 711
Condition minus (solvable): 699
Condition minus \((F_2)^5\): 13
Both: 548
Both (solvable): 536
Both \((F_2)^5\): 13

Here is a sample source code for SageMath (Version 8.9) which calculates quartics by the algorithm described in Section 6. It also checks the smoothness of the output. The example in Section 7 is calculated by this code.

```python
P.<X, Y, Z, S, T> = PolynomialRing(QQ)

b1, b2, b3, b4, b5 = -1, 17, 89, 257, 769
a1, a2, a3, a4, a5, u = b1*b5, b2*b4, b3, b4, b5, -b4^(-1)*b5^(-1)

a6 = a1*a2*a3*a4*a5*u^2

F = (a1*S-T)*(a2*S-T)*(a3*S-T)*(a4*S-T)*(a5*S-T)*(a6*S-T)
g0 = (-F.coefficient({S:1, T:5}) -1)/2
g1 = (-F.coefficient({S:2, T:4}) + g0^2)/2
g2 = c
g = g0*T^2 + g1*S*T + g2*S^2
h = (T^5*S - (T^3 - g*S)^2 + F)/(S^3*T)
# print(factor(g))
# print(factor(h))

M = matrix([[[-S*T+T^2, S*T, g], [S*T, S^2, T^2], [g, T^2, h]]])
# print(det(M).factor()) ## check det(M) = -F(S,T)
v = matrix([[X,Y,Z]])
Biquad = (v*M*v.transpose())[0][0]
Biquad = P(Biquad)
# print(factor(Biquad))

q0 = Biquad.coefficient({S:2, T:0})
q1 = Biquad.coefficient({S:1, T:1})
q2 = Biquad.coefficient({S:0, T:2})
Quart = q1^2-4*q0*q2
# print(factor(Quart))
```
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