Horizontal cohomology of a local Lie group

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Abstract

We define and study the invariant linear and nonlinear horizontal double complexes of a local Lie group.

1 Introduction

Let $M$ be a smooth manifold with $\dim M = n$ and $\mathcal{F} \to M$ be fiber bundle with $k$th jet extension $J_k \mathcal{F} \to M$. A fibered submanifold $\mathcal{E} \subset J_k \mathcal{F} \to M$ defines a $k$th order PDE on $M$. The horizontal cohomology of $\mathcal{R}$ is defined in [17] as a part of the general formalism of Vinogradov spectral sequence and studied further in [18], [14], [15], [16] and by various authors (the references in [14] contain an extensive list on horizontal cohomology). In the presence of a group structure as in this work, it is possible to define also the invariant horizontal complex as the “edge sequence” of the invariant variational bicomplex (see [2], [9]).

The study of local Lie groups is initiated in [11], [1], [8], [13]. As shown in [11], the theory of local Lie groups is not a simple consequence of the global theory but has its own set of interesting and delicate geometric structures. Slightly modifying the definition of a local Lie group in [11], we showed in [1] that a Lie group can be defined as a globalizable local Lie group, hence reinstating the paradigm of local to global to its historical record. As we indicated in [12], local Lie groups are particular pre-homogeneous geometric structures with vanishing curvatures (see also [3] for a similar approach to geometric structures based on Cartan algebroids).

A local Lie group is defined by a first order nonlinear PDE on $J_1(M \times M)$ and its Lie algebra as a first order linear PDE on $J_1(T)$ where $T \to M$ is the tangent bundle. The elementary nature of these PDE’s allows one to make a concrete study of their invariant horizontal cohomologies which is the purpose of this work (Sections 1, 2). We show that the group structure enables one to realize these horizontal complexes as the second rows of two double complexes (Section 4). The linearization map determines a homomorphism from the nonlinear double complex to the linear one (Sections 3, 4). In the nonlinear case, it
turns out that the first row of this invariant double complex computes the Lie algebra cohomology and the first column computes the Lie group cohomology in analogy with the Van Est spectral sequence. In particular, the nonlinear double complex assigns an infinite number of seemingly new cohomology groups to a Lie group and uncovers some direct links between Lie group cohomology as initiated in [6], [7] and later generalized to groupoids in [19], [5] and the horizontal cohomology mentioned in the above works.

2 Local Lie groups

In this section we shortly recall the theory of local Lie groups. We refer to [1] for more details for some points (see also Section 2 of [13]).

Let \((M, \bar{\varepsilon})\) be a manifold with a splitting \(\bar{\varepsilon}\) of \(J_1(M \times M) \to M \times M\). So \(\bar{\varepsilon}\) assigns to any ordered pair \((p, q)\) a 1-arrow from \(p\) to \(q\) and this assignment preserves the composition and inversion of arrows. Such a splitting exists if and only if \(M\) is parallelizable. We define the components

\[
\bar{\Gamma}_{kj}^i(x) \overset{\text{def}}{=} \left. \frac{\partial \bar{\varepsilon}_j^i(x, y)}{\partial y^k} \right|_{y=x} = - \left. \frac{\partial \bar{\varepsilon}_j^i(x, y)}{\partial x^k} \right|_{x=y} \overset{\text{def}}{=} \hat{\Gamma}_{kj}^i(x) \quad (1)
\]

For a vector field \(\xi = (\xi^i)\), we define \(\bar{\nabla}_j \xi^i \overset{\text{def}}{=} \frac{\partial \xi^i}{\partial x^j} - \bar{\Gamma}_{ja}^i \xi^a\) and \(\hat{\nabla}_j \xi^i \overset{\text{def}}{=} \frac{\partial \xi^i}{\partial x^j} - \hat{\Gamma}_{ja}^i \xi^a\). The actions of the covariant differentiation operators \(\bar{\nabla}, \hat{\nabla}\) extend naturally from vector fields to all tensor fields. A tensor field \(t\) is called \(\bar{\varepsilon}\)-invariant if \(\bar{\varepsilon}(p, q) \circ t(p) = t(q)\), \(\bar{\nabla}\)-invariant if \(\bar{\nabla}t = 0\) and \(\hat{\nabla}\)-invariant if \(\hat{\nabla}t = 0\). It is easy to show that \(t\) is \(\bar{\varepsilon}\)-invariant if and only if it is \(\bar{\nabla}\)-invariant (Proposition 1 in [13], see also Proposition 5 below). So \(\bar{\nabla}\)-invariance is defined without the object \(\bar{\varepsilon}\) whose definition needs a further assumption (see (5) below). Since the linear PDE \(\bar{\nabla} \xi = 0\) admits \(\bar{\varepsilon}\)-invariant vector fields as solutions, its integrability condition \(\bar{\mathcal{R}} = 0\) is satisfied. Let \(\mathcal{R} = 0\) denote the integrability condition of \(\bar{\nabla} \xi = 0\). We define the torsion tensors \(\bar{T}, \hat{T}\) by \(\bar{T}_{jk}^i \overset{\text{def}}{=} \bar{\Gamma}_{jk}^i - \bar{\Gamma}_{kj}^i\) and \(\hat{T}_{jk}^i \overset{\text{def}}{=} \hat{\Gamma}_{jk}^i - \hat{\Gamma}_{kj}^i\). Clearly \(\bar{T} = - \hat{T}\). Straightforward computations using the definitions prove the fundamental formulas

\[
\bar{\nabla}_i \bar{T}_{kl}^j = \bar{\mathcal{R}}_{kl,i} \quad (2)
\]

\[
\hat{\nabla} \hat{T} = 0 \quad \text{if} \quad \hat{\mathcal{R}} = 0
\]

The splitting \(\bar{\varepsilon}\) determines the nonlinear PDE

\[
\frac{\partial f^i(x)}{\partial x^j} = \bar{\varepsilon}_j^i(x, f(x)) \quad (3)
\]

with the integrability condition \(\bar{\mathcal{R}} = 0\).

**Definition 1** \((M, \bar{\varepsilon})\) is a local Lie group if \(\bar{\mathcal{R}} = 0\).
In this case the local solutions of (3) are uniquely determined by their initial conditions \( f(p) = q \) and they form a simply transitive pseudogroup on \( M \) denoted by \( \tilde{\mathcal{G}} \). If all \( f \in \tilde{\mathcal{G}} \) extend (necessarily uniquely) to global diffeomorphisms of \( M \), then \((M, \tilde{\mathcal{E}})\) is called globalizable. In this case \( \tilde{\mathcal{G}} \) becomes a global transformation group of \( M \) which acts simply transitively. It is a fundamental fact that \( \mathcal{R} = 0 \Leftrightarrow \tilde{\mathcal{R}} = 0 \), the implication \( \Leftarrow \) being the Lie's third fundamental theorem. For a local Lie group \((M, \tilde{\mathcal{E}})\), the solutions \( \Theta \) of \( \nabla \xi = 0 \) becomes a Lie algebra of vector fields on \( M \) which can be localized at any point \( p \in M \). As a crucial fact, it is not \( \Theta \) that integrates to \( \tilde{\mathcal{G}} \) but \( \Theta \) to be defined below.

For a local Lie group \((M, \tilde{\mathcal{E}})\), let \( g(a, b, z) \) denote the unique local solution of (3) in the variable \( z \) satisfying the initial condition \( a \to b \). We fix some \( p, q \in (U, x^i) \) and define

\[
\tilde{\varepsilon}^i_j(p, q) \overset{\text{def}}{=} \left[ \frac{\partial g^i(p, x, q)}{\partial x^j} \right]_{x=p} \tag{4}
\]

Choosing \( f \in \tilde{\mathcal{G}} \) with \( f(p) = q \) and replacing \( p, x \) with \( p', x' \) close to \( p, x \), (4) shows that the local diffeomorphism \( h : x \to g(p, x, q) \) satisfies \( h(p) = q \) and is the unique local solution of

\[
\frac{\partial h^i(x)}{\partial x^j} = \tilde{\varepsilon}^i_j(x, h(x)) \tag{5}
\]

satisfying the initial condition \( h(p) = q \). In particular, the integrability condition \( \mathcal{R} = 0 \) of (5) is satisfied. Note that \( \tilde{\varepsilon}(p, q) \) is defined for sufficiently close \( p, q \) unless \((M, \tilde{\mathcal{E}})\) is globalizable. We now check that \( \tilde{\varepsilon} \) is a (local) splitting. In analogy with (3) we also check \( \left[ \frac{\partial g^i(x, y)}{\partial y^e} \right]_{y=x} = \tilde{\Gamma}^i_{kj}(x) \). The local solutions of (5) define the locally transitive pseudogroup \( \tilde{\mathcal{G}} \). If \((M, \tilde{\mathcal{E}})\) is globalizable, that is, if the local transformations of \( \tilde{\mathcal{G}} \) globalize, then so do the local transformations of \( \tilde{\mathcal{G}} \). In this case we have the map \( \Psi : \tilde{\mathcal{G}} \to \mathcal{G} \) defined as follows: let \( f \in \tilde{\mathcal{G}} \) and fix some \( p \in M \). Then \( \Psi(f) \) is the unique transformation of \( \mathcal{G} \) whose 1-arrow from \( p \) to \( q \) is \( \tilde{\varepsilon}(p, q) \). This definition does not depend on \( p \) and \( \Psi(f \circ g) = \Psi(f) \circ \Psi(g) \), \( \Psi(f^{-1}) = \Psi(f)^{-1} \). As expected, a tensor field \( t \) is \( \tilde{\varepsilon} \)-invariant if and only if it is \( \nabla \)-invariant. Now \( \hat{\Theta} \) integrates to \( \tilde{\mathcal{G}} \) and the Lie algebra \( \hat{\Theta} \) of solutions of \( \nabla \xi = 0 \) integrates to \( \tilde{\mathcal{G}} \), that is, \( \hat{\Theta} \), \( \hat{\Theta} \) are the "Lie algebras" of the transformation groups \( \tilde{\mathcal{G}}, \mathcal{G} \) respectively. This corresponds to the well known fact that on a Lie group left (right) invariant vector fields integrate to right (left) translations. However, observe that there is no canonical choice of left and right for a local Lie group even if it is globalizable. It is for this reason that we avoid the notation \( \mathcal{G}_L \) (or \( \mathcal{G}_R \)) for \( \tilde{\mathcal{G}} \). However, observe that the roles of \( \sim \) and \( \tilde{\sim} \) are not symmetric unless \( \mathcal{R} = 0 \) and \((M, \tilde{\mathcal{E}})\) is globalizable. Some contemplation reveals that the static concepts of left/right on a Lie group are replaced with the dynamic concept of "time" in a local Lie group. Now the isomorphism \( \Psi : \tilde{\mathcal{G}} \to \mathcal{G} \) determines the isomorphism \( d\Psi : \hat{\Theta} \to \Theta \) as follows: let \( \xi \in \hat{\Theta} \) and fix some \( p \in M \). We
define \( d\Psi(\xi) \) as the unique \( \eta \in \tilde{\Theta} \) satisfying \( \eta(p) = \xi(p) \). This definition is again independent of \( p \).

### 3 The linear horizontal complex

Let \((M, \bar{\varepsilon})\) be a local Lie group and \( \pi : T \to M \) be the tangent bundle.

**Definition 2** A horizontal \( k \)-form on \( \pi : T \to M \) is a function which assigns to any \( \xi \in T \) an ordinary \( k \)-form at \( \pi(\xi) \in M \).

For the moment, we do not assume that a horizontal form is linear on the fibers \( \pi^{-1}(x), x \in M \). We denote the vector space of the horizontal \( k \)-forms by \( \Lambda^k_{hor}(T) \). Choosing a coordinate patch \((U, x^r)\), some \( \Omega \in \Lambda^k_{hor}(T) \) over \( \pi^{-1}(U) \)

\[
\text{is of the form } \Omega_{i_1i_2...i_k}(x, \xi_x) \text{ where } \xi_x \text{ denotes an arbitrary point in the fiber } \pi^{-1}(x) \text{ so that } x \text{ and } \xi_x \text{ are independent variables. For simplicity of notation, we write } \xi \text{ for } \xi_x. \text{ We call } \Omega \text{ smooth if its components are smooth functions. Henceforth we always assume that our forms are smooth.}
\]

We can express \( \Omega \) also as

\[
\frac{1}{k!} \sum \Omega_{i_1i_2...i_k}(x, \xi) \; dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k} = \Omega(x, \xi)
\]

(6)

Using Einstein summation convention, our shorthand notation for (6) will be \( \Omega_I(x, \xi) \) or just \( \Omega_I \) with the obvious meaning of the index symbol \( I \).

We define the total derivative \( \tilde{D}_r \) with respect to the variable \( x^r \) of a horizontal \( 0 \)-form by the formula

\[
\tilde{D}_r \Omega(x, \xi) \overset{def}{=} \frac{\partial \Omega(x, \xi)}{\partial x^r} + \frac{\partial \Omega(x, \xi)}{\partial \xi^a} \hat{\Gamma}^{a}_{rb}(x) \xi(x)^b
\]

(7)

In other words, we pretend that \( \xi \) depends on \( x \), differentiate \( \Omega(x, \xi(x)) \) with respect to \( x^r \) and formally substitute \( \frac{\partial \xi^i}{\partial x^r} \) from the equation \( \tilde{\nabla} \xi = 0 \). Note that a horizontal form on \( J^\infty T \) is of the form \( \Omega_s(x^i, \xi, \xi^s, \xi^j, \xi^2, ..., \xi^{j_2}, ..., \xi^{j_s}) \) for arbitrary \( s \) but \( \tilde{\nabla} \xi = 0 \) (or \( \tilde{\nabla} \xi = 0 \)) implies that all derivatives of \( \xi \) are determined by \( \xi \). Henceforth we sometimes omit the notation for the variables \( x, \xi \) from our formulas and write, for instance, (7) as \( \tilde{D}_r \Omega \overset{def}{=} \frac{\partial \Omega}{\partial x^r} + \frac{\partial \Omega}{\partial \xi^a} \hat{\Gamma}^{a}_{rb} \xi^b \).

Now we define an operator \( \tilde{d} : \Lambda^k_{hor}(T) \to \Lambda^{k+1}_{hor}(T) \) by the formula

\[
\left( \tilde{d}\Omega \right)_{i_1i_2...i_k} \overset{def}{=} \left[ \tilde{D}_r \Omega_{i_1i_2...i_k} \right]_{[i_1i_2...i_k]} \quad (8)
\]

\[
= \tilde{D}_r \Omega_{i_1i_2...i_k} - \tilde{D}_{i_1} \Omega_{i_2...i_k} - \tilde{D}_{i_2} \Omega_{i_1...i_k} ... - \tilde{D}_{i_k} \Omega_{i_1i_2...r}
\]

Equivalently, \( \tilde{d}\Omega \overset{def}{=} \tilde{D}_r \Omega \; dx^r \wedge dx^r \).

**Proposition 3** \( \tilde{d} \circ \tilde{d} = 0 \)
Proof: It suffices to show that \( \hat{D}_s \hat{D}_r \Omega \) is symmetric in \( s, r \) for a horizontal 0-form \( \Omega \). Applying \( \hat{D}_s \) to (7) gives

\[
\begin{aligned}
\frac{\partial \Omega}{\partial x^s \partial x^r} &+ \frac{\partial \Omega}{\partial \xi^a \partial x^r} \hat{\Gamma}^{cb}_{ab} \xi^b + \frac{\partial \Omega}{\partial \xi^a \partial x^r} \hat{\Gamma}^{cb}_{ra} \xi^b + \frac{\partial \Omega}{\partial \xi^c \partial x^r} \hat{\Gamma}^{cb}_{rc} \xi^d \\
+ \frac{\partial f}{\partial \xi^a} \left( \frac{\partial \hat{\Gamma}^{ra}}{\partial x^s} + \hat{\Gamma}^{sb}_{ra} \hat{\Gamma}^{ca}_{rc} \right) \xi^b
\end{aligned}
\]  

(9)

The sum of the first four terms is clearly symmetric in \( s, r \). The last term is also symmetric in view of \( \hat{R}^{a}_{sr,b} = 0 \). □

Thus we get the complex

\[
\Lambda^0_{\text{hor}}(T) \xrightarrow{\hat{d}} \Lambda^1_{\text{hor}}(T) \xrightarrow{\hat{d}} \Lambda^2_{\text{hor}}(T) \xrightarrow{\hat{d}} \ldots \xrightarrow{\hat{d}} \Lambda^n_{\text{hor}}(T)
\]  

(10)

Note that (10) can be constructed on any parallelizable manifold using \( \tilde{D} \) instead of \( \hat{D} \) because we always have \( \tilde{R} = 0 \).

Unfortunately, the cohomology groups of (10) turn out to be infinite dimensional. To see this, we consider now the kernel of the first operator in (10). First, note that \( \hat{\varepsilon}(p, q) \) (or \( \tilde{\varepsilon}(p, q) \)) defines an isomorphism of tensor spaces

\[
\hat{\varepsilon}^*(p, q) : T^r_s(p) \rightarrow T^r_s(q).
\]

**Definition 4** \( \Omega \in \Lambda^0_{\text{hor}}(T) \) is \( \hat{\varepsilon} \)-invariant if \( \Omega(q, \hat{\varepsilon}(p, q), \xi) = \Omega(p, \xi), \ p, q \in U, \ \xi \in \pi^{-1}(p) \)

Unlike \( \tilde{\varepsilon} \)-invariance, \( \hat{\varepsilon} \)-invariance is a local concept unless \( (M, \hat{\varepsilon}) \) is globalizable. Equivalently, we may fix \( p \) arbitrarily and let \( q \) vary in the condition of Definition 4. Therefore, choosing coordinates around \( p, q \), we write this invariance condition as

\[
\Omega(x, \eta) = \Omega(p, \xi) \quad \eta^i = \hat{\varepsilon}_i^a(p, x) \xi^a
\]  

(11)

We denote the vector space of \( \hat{\varepsilon} \)-invariant 0-forms by \( \Lambda^0_{\text{hor}}(T) \).

**Proposition 5** The kernel of the first operator \( \hat{d} \) in (10) coincides with \( \Lambda^0_{\text{hor}}(T) \).

Proof: Differentiation of (11) with respect to \( x^r \) at \( x = p \) gives \( \left( \hat{D}, \Omega \right)(p) = 0 \). Since \( p \) is arbitrary, we conclude \( \hat{D}, \Omega = 0 \). Conversely assume \( \hat{D}, \Omega = 0 \), fix \( p, \xi, x \) and consider the LHS of (11) defined by the condition in (11). We want to show the equality in (11). Now \( \hat{D}_r, \Omega = 0 \) implies \( \frac{\partial \Omega(x, \eta)}{\partial x^r} = 0 \) so that \( \Omega(x, \eta) \) is independent of \( x \). Setting \( x = p \) we get (11). □

We will use the principle in the proof of Proposition 5 several times later on without giving further details. Now since some \( \Omega \in \Lambda^0_{\text{hor}}(T) \) is globally determined by its values on some fiber \( \pi^{-1}(p) \) and the vector space of smooth functions on \( \pi^{-1}(p) \) is infinite dimensional, we conclude \( \dim \Lambda^0_{\text{hor}}(T) = \infty \). This
deficiency of (10) forces us to assume the linearity of our horizontal forms on the fibers. Surprisingly, if \((M, \tilde{\varepsilon})\) is globalizable and \(M\) is compact, this assumption makes the cohomology of (10) finite dimensional and even computable as we will see shortly.

**Definition 6**  A horizontal \(k\)-form is linear if it is a linear function on the fibers \(\pi^{-1}(x)\) of \(\pi: T \to M\).

A horizontal linear \(k\)-form \(\omega\) is locally of the form \(\omega(x, \xi) = \omega_{a,i_1i_2...i_k}(x)\xi^a\) where \(\omega_{a,i_1i_2...i_k}\) is a tensor alternating in the indices \(i_1i_2...i_k\). Therefore a horizontal linear \(k\)-form on \(T \to M\) is simply a section of \(T^* \otimes \Lambda^k(M) \to M\), whose total space (and also the space of its sections) will be denoted simply by \(T^* \otimes \Lambda^k\).

Thus we get the subcomplex

\[
0 \longrightarrow \hat{\Lambda}^1 \longrightarrow \Lambda^1 = T^* \longrightarrow \Lambda^2 \longrightarrow \Lambda^3 \longrightarrow \Lambda^n \quad (13)
\]
of (10) and clearly \(\dim \hat{\Lambda}^1 = \dim M\).

**Definition 7** (13) is the horizontal linear complex (LHC) of the local Lie group \((M, \tilde{\varepsilon})\).

A horizontal \(k\)-form \(\Omega\) (not necessarily linear) is \(\tilde{\varepsilon}\)-invariant if \(\tilde{\varepsilon}(p, x)\Omega(x, \eta) = \Omega(p, \tilde{\varepsilon}(p, x)\eta)\). In coordinates this condition is

\[
\tilde{\varepsilon}_{i_1}^a(p, x)\Omega_{a_1...a_k}(x, \eta) = \Omega_{i_1...i_k}(p, \eta)\quad \eta^i = \tilde{\varepsilon}_a^i(p, x)\xi^a \quad (14)
\]

Differentiation of (14) at \(x = p\) gives

\[
\Gamma_{i_1i_2...i_k}^a \Omega_{a_1...a_k} + \ldots + \tilde{\Gamma}_{i_1i_2...i_k}^a \Omega_{a_1...a_k} + \frac{\partial \Omega_{i_1...i_k}}{\partial x^r} + \frac{\partial \Omega_{i_1...i_k}}{\partial \eta^a} \tilde{\Gamma}_{r}^a \xi^b = 0 \quad (15)
\]

We denote the expression on the LHS of (15) by \(\tilde{\Box}_r \Omega\) and call \(\tilde{\Box}_r\) the \(\tilde{\varepsilon}\)-covariant derivative of \(\Omega\) with respect to \(x^r\). Since \(p, \xi\) are arbitrary in (15), if \(\Omega\) is \(\tilde{\varepsilon}\)-invariant then \(\tilde{\Box} \Omega = 0\). Converse also holds and the proof is identical with the proof of Proposition 5. It is crucial to observe that \(\tilde{D}_r \Omega\) is not a linear object like \(\Box \Omega\) unless \(\Omega\) is a 0-form and \(\tilde{D}_r \Omega = \Box \Omega\) for a 0-form \(\Omega\). This is the reason why we alternate as in (8) to get the linear object \((d\Omega)_{i_1i_2...i_k}\) from \(\tilde{D}_r \Omega_{i_1i_2...i_k}\). As another crucial fact, if we replace, for instance, \(\tilde{D}_r\) by \(\tilde{\Box}_r\) in (8), we get a quite different linear object unless \(\Omega\) is a 0-form but the new
operator obtained this way does not give a complex like (10) due to the presence of torsion.

To summarize, the space of $\bar{\varepsilon}$-invariant (horizontal) linear $k$-forms is $T^* \otimes \Lambda^k$. For $\omega = (\omega_{a,i_1i_2...i_k}) \in T^* \otimes \Lambda^k$ (12) and (15) give

$$\bar{\Delta}_a \left( \omega_{b,i_1i_2...i_k} \xi^b \right) = \bar{\Gamma}^a_{r_1i_1} \omega_{b,a_1i_2...i_k} \xi^b + ... + \bar{\Gamma}^a_{r_ki_k} \omega_{b,i_1i_2...a_k} \xi^b$$

$$+ \frac{\partial \omega_{b,i_1...i_k}}{\partial x^r} \xi^b + \bar{\Gamma}^a_{rb} \omega_{a_1i_2...a_k} \xi^b$$

$$= \left( \bar{\nabla}_r \omega_{a,i_1...i_k} \right) \xi^b \quad (16)$$

So $\bar{\Delta}_a \omega = \bar{\nabla} \omega$ where we interpret $\omega$ as a horizontal linear $k$-form in $\bar{\Delta}_a \omega$ and as a section of $T^* \otimes \Lambda^k \to M$ in $\bar{\nabla} \omega$. Now if $\bar{\Delta}_a \omega = 0$, then (16) gives

$$\bar{\Delta}_a \omega_{a,i_1i_2...a_k} = -\bar{\Gamma}^a_{r_1i_1} \omega_{b,a_1i_2...i_k} - ... - \bar{\Gamma}^a_{r_ki_k} \omega_{b,i_1i_2...a_k} - \omega_{a,i_1i_2...a_k} + \bar{\Gamma}^a_{rb} \omega_{a_1i_2...a_k} \quad (17)$$

The proof of the next Proposition is almost identical with the proof of Proposition 7 in [13]. This is not surprising for if we replace $T^*$ in (13) by $T$, then (13) becomes (30) in [13].

**Proposition 8** If $\bar{\Delta}_a \omega = 0$, then $\bar{\Delta}_a d \omega = 0$. Therefore $\hat{d} : T^* \otimes \Lambda^k \to T^* \otimes \Lambda^{k+1}$

Proof (sketch): We observe that each term in the alternation $\left[ \omega_{a,i_1i_2...a_k} \hat{\Gamma}^a_{rb} \right]_{r_1...i_k}$ of the last term of (17) is a tensor. Applying $\bar{\nabla}_a$ to each such term gives zero by (2). The alternation $\left[ \hat{\Gamma}^a_{r_1i_1} \omega_{b,a_1i_2...i_k} + ... + \hat{\Gamma}^a_{r_ki_k} \omega_{b,i_1i_2...a_k} \right]_{r_1...i_k}$ is a sum of terms of the form $\hat{T}^a_{rb} \omega_{b,a_1i_2...a_k}$ and we argue as before. $\square$

Let $\hat{\Lambda}^1$ denote $\bar{\varepsilon}$-invariant 1-forms which are also $\bar{\varepsilon}$-invariant. Clearly $\hat{\Lambda}^1 = \hat{\Lambda}^1 = \hat{\Lambda}^1 \cap \hat{\Lambda}^1$. Now Proposition 8 gives the subcomplex

$$0 \to \hat{\Lambda}^1 \to \hat{\Lambda}^1 \xrightarrow{\hat{d}} T^* \otimes \hat{\Lambda}^1 \xrightarrow{\hat{d}} ... \xrightarrow{\hat{d}} T^* \otimes \hat{\Lambda}^n \quad (18)$$

of (13) which localizes at any point $p \in M$ and can therefore reduces to algebra.

**Definition 9** (18) is the invariant linear horizontal complex (ILHC) of $(M, \bar{\varepsilon})$.

If $(M, \bar{\varepsilon})$ is globalizable and $M$ compact, then $M$ admits a measure invariant under both the global transformation groups $\hat{G}$ and $\hat{G}$ and the standard averaging process over $M$ proves that the inclusion of (18) in (13) induces isomorphism in cohomology in positive degrees. However, (18) computes the cohomology of the Lie algebra $\hat{g} \overset{def}{=} \hat{\Theta}$ with coefficients $g^* = \hat{\Lambda}^1$ in the same way as (30) in [13] computes the cohomology of $g$ with coefficients $g$. Thus we conclude
**Proposition 10** If the local Lie group \((M, \tilde{\varepsilon})\) is globalizable and \(M\) is compact, then the \(k'\)th cohomology groups of (18) and (19) are both isomorphic to \(H^k(\mathfrak{g}, \mathfrak{g}^*)\) in positive degrees where \(\mathfrak{g} \overset{\text{def}}{=} \tilde{\Theta}\) and \(\mathfrak{g}^* = \tilde{\Lambda}\).

Finally, we remark that our construction in this section works if we replace \(T^* \rightarrow M\) by the \((r, s)\)-tensor bundle \(T^*_r \rightarrow M\) and in fact by any natural vector bundle \(E \rightarrow M\) of order one (see [10] and the references there for natural bundles). For \(T^*_r \rightarrow M\), (18) computes \(H^*(\mathfrak{g}, T^*_r(\mathfrak{g}))\).

4 The nonlinear horizontal complex

**Definition 11** A nonlinear horizontal \(k\)-form \(\omega\) on \(M \times M\) assigns to \((p, q) \in M \times M\) an ordinary \(k\)-form at \(p\).

Note that \(\omega\) can be defined also as a function \(\omega: M \times M \rightarrow \Lambda^k(\tilde{M})\) since elements of \(\Lambda^k(\tilde{M})\) are globally determined by their values at any point. We denote the space of (nonlinear) horizontal \(k\)-forms by \(\Lambda^k_{\text{hor}}(M \times M)\). Choosing coordinates \(p \in (U, x^i), q \in (V, y^j)\), we write \(\omega\) as \(\omega_{i_1i_2...i_k}(x, y)\). There is an ambiguity with this notation: it does not specify the coordinates to which the \(k\)-form indices \(i_1, i_2, ..., i_k\) refer to. Except in the proof of Proposition 22, we agree that they refer to the coordinates around the source point \(p\). Note that a choice of coordinates around some point canonically defines coordinates around all points if \((M, \tilde{\varepsilon})\) is a local Lie group.

In view of (3), we define the total differentiation operator \(\tilde{D}: \Lambda^0_{\text{hor}}(M \times M) \rightarrow \Lambda^1_{\text{hor}}(M \times M)\) by

\[
(\tilde{D}\theta)_r(x, y) \overset{\text{def}}{=} \frac{\partial \theta(x, y)}{\partial x^r} + \frac{\partial \theta(x, y)}{\partial y^a} \varepsilon^a_r(x, y) \quad (19)
\]

Now we define \(\tilde{d}: \Lambda^k_{\text{hor}}(M \times M) \rightarrow \Lambda^{k+1}_{\text{hor}}(M \times M)\) by

\[
\tilde{d}\omega \overset{\text{def}}{=} \left(\tilde{D} \omega_1\right)_r dx^r \wedge dx^l \quad (20)
\]

**Proposition 12** If \((M, \varepsilon)\) is a local Lie group, then \(\tilde{d} \circ \tilde{d} = 0\).

Proof: Writing \((\tilde{D} f)\), as \(\tilde{D}_r f\) and applying \(\tilde{D}_s\) to (19) gives

\[
\frac{\partial^2 \theta(x, y)}{\partial x^s \partial x^r} + \frac{\partial^2 \theta(x, y)}{\partial y^a \partial x^r} \varepsilon^a_s(x, y) + \frac{\partial \theta(x, y)}{\partial x^s} \frac{\partial \varepsilon^a_r(x, y)}{\partial y^a} \varepsilon^a_s(x, y) + \frac{\partial \theta(x, y)}{\partial y^b} \left( \frac{\partial \varepsilon^a_r(x, y)}{\partial x^s} + \frac{\partial \varepsilon^a_r(x, y)}{\partial y^a} \varepsilon^b_s(x, y) \right) \quad (21)
\]

which is symmetric in \(s, r\) since \(\tilde{\mathcal{R}}^a_{sr}(x, y) = 0\). □

Thus we get the complex

\[
\Lambda^0_{\text{hor}}(M \times M) \overset{\tilde{d}}{\rightarrow} \Lambda^1_{\text{hor}}(M \times M) \overset{\tilde{d}}{\rightarrow} \cdots \overset{\tilde{d}}{\rightarrow} \Lambda^n_{\text{hor}}(M \times M) \quad (22)
\]
Definition 13 (22) is the nonlinear horizontal complex (NHC) of the local Lie group \((M, \overline{\varepsilon})\).

The construction of (22) needs the parallelizable manifold \((M, \overline{\varepsilon})\) together with the assumption \(R = 0\). The restriction \((U, \overline{\varepsilon}|_U)\) of \((M, \overline{\varepsilon})\) to some \(U \subset M\) satisfies both these conditions. Hence we can meaningfully speak of the restriction of (22) to \(U\). Now the following question arises naturally

Q : Is (22) locally exact?

Let \(f \in \hat{\mathcal{G}}\) be the unique local solution of (3) with the initial condition \(f(p) = q\), \(p \in (U, x')\) and \(\{(x, f(x))\}, x \in U\) be the local graph of \(f\). For \(\theta \in \Lambda^0_{\text{hor}}(M \times M)\) we consider the restriction \(\theta(x, f(x))\) of \(\theta\) to the graph of \(f \in \hat{\mathcal{G}}\).

The proof of the next proposition follows easily from the definitions.

Proposition 14 The following are equivalent

i) \(\theta \in \Lambda^0_{\text{hor}}(M \times M)\) belongs to the kernel of the first operator in (22)

ii) The restriction of \(\theta\) to the graph of \(f\) is constant for all \(f \in \hat{\mathcal{G}}\).

To understand this kernel better, it is useful to assume that \((M, \overline{\varepsilon})\) is globalizable so that \(\hat{\mathcal{G}}\) is a global transformation group of \(M\) which acts simply transitively. So we may identify \(f \in \hat{\mathcal{G}}\) with its graph \(\{(p, f(p)), p \in M\} \subset M \times M\).

Since \(\theta\) is constant on this graph (we always assume that \(M\) is connected), we interpret this constant value as the value of \(\theta\) on \(f\). This identifies the kernel with the functions \(\theta : \hat{\mathcal{G}} \to \mathbb{R}\).

We recall that \(g(a, b, x)\) is the unique solution of (3) in the variable \(x\) satisfying the initial condition \(a \to b\). Let \(\theta \in \Lambda^0_{\text{hor}}(M \times M)\). We call \(\theta\) \(\overline{\varepsilon}\)-invariant if \(\theta(x, g(p, x, q)) = \theta(p, q)\) for \(p, q, x \in M\). Since \(g(a, b, x)\) is defined for sufficiently close \(a, x\) and \(p, q\) are arbitrary in our definition, we make the flat assumption of globalizability henceforth so that \(\hat{\mathcal{G}}\) is another global transformation group of \(M\) which acts simply transitively. Differentiating \(\theta(x, g(p, x, q)) = \theta(p, q)\) with respect to \(x\) at \(x = p\) gives

\[
\hat{D}_x(p, q) = \frac{\partial \theta}{\partial x^r}(p, q) + \frac{\partial \theta}{\partial y^a}(p, q)\overline{\varepsilon}(p, q) = 0
\]  

(23)

Since \(p, q\) are arbitrary in (23), we deduce \(\hat{D}_x = 0\) and we easily show as before that conversely \(\hat{D}_x = 0\) implies the \(\overline{\varepsilon}\)- invariance of \(\theta\). Let \(\Lambda^0_{\text{hor}}(M \times M)\) denote the space of \(\overline{\varepsilon}\)-invariant functions and let \(\theta \in \Lambda^0_{\text{hor}}(M \times M)\) satisfy \(\hat{D}_x = 0\). The proof of Proposition 14 shows that \(\theta\) is constant on the graphs of \(h \in \hat{\mathcal{G}}\) and therefore may be interpreted as a function \(\theta : \hat{\mathcal{G}} \to \mathbb{R}\). Now fix some \(f \in \hat{\mathcal{G}}\), some \(p \in M\) and suppose \(f(p) = q\). Let \(x \in M\). There is a unique \(k_x \in \hat{\mathcal{G}}\) with \(k_x(p) = x\). Therefore \(g(p, x, q) = k_x(q) = (k_x \circ f \circ k_x^{-1})(x)\) and \(h\) in (5) is the transformation \(x \to k_x \circ f \circ k_x^{-1}(x)\). Hence we conclude that this transformation belongs to \(\hat{\mathcal{G}}\). Now since \(\theta \in \Lambda^0_{\text{hor}}(M \times M), \theta(k_x \circ f \circ k_x^{-1})\) has the same value \(\theta(p, q)\) independent of \(x\). However, since also \(\hat{D}_x = 0\), we have \(\theta(p, q) = \theta(f)\). Whence
\[ \theta(f) = \theta(k \circ f \circ k^{-1}) \quad f, k \in \tilde{G} \]  

(24)

Recall that a function on a Lie group which is constant on the conjugacy classes is called a character function. The trace of a representation is a character function and these functions play a fundamental role in representation theory. Let \( \mathcal{C}(\tilde{G}) \) denote the space of character functions defined by (24). Thus we proved

**Proposition 15** The sequence

\[ 0 \to \mathcal{C}(\tilde{G}) \to \Lambda^0_{\text{hor}}(M \times M) \to \Lambda^1_{\text{hor}}(M \times M) \to \cdots \]  

(25)

is exact.

**Definition 16** \( \omega \in \Lambda^k_{\text{hor}}(M \times M) \) is \( \tilde{\varepsilon} \)-invariant if

\[ \tilde{\varepsilon}(p, x)_\ast \omega(x, g(p, x, q)) = \omega(p, q) \]  

(26)

In coordinates the condition of invariance is

\[ \tilde{\varepsilon}_{i_1}^{a_1}(p, x) \cdots \tilde{\varepsilon}_{i_k}^{a_k}(p, x) \omega_{a_1 \cdots a_k}(x, g(p, x, q)) = \omega_{i_1 \cdots i_k}(p, q) \]  

(27)

Differentiation of (26) with respect to \( x^r \) at \( x = p \) gives (omitting \( p, q \) from our notation and using the same symbol \( \hat{\Box} \) as before)

\[ \hat{\Box}_r \omega_{a_1 \cdots i_k} = 0 \]  

where

\[ \hat{\Box}_r \omega_{a_1 \cdots i_k} = \Gamma^a_{r1} \omega_{a_1 \cdots i_k} + \Gamma^a_{r2} \omega_{i_1 a_2 \cdots i_k} + \cdots + \Gamma^a_{rk} \omega_{i_1 \cdots i_k} \]  

(28)

As before, some \( \omega \in \Lambda^k_{\text{hor}}(M \times M) \) is \( \tilde{\varepsilon} \)-invariant if and only if \( \hat{\Box} \omega = 0 \). Using (27), an argument similar to the proof of Proposition 8 gives

**Proposition 17** We have the complex

\[ 0 \to \mathcal{C}(\tilde{G}) \to \Lambda^0_{\text{hor}}(M \times M) \to \Lambda^1_{\text{hor}}(M \times M) \to \cdots \to \Lambda^n_{\text{hor}}(M \times M) \]  

(28)

**Definition 18** (28) is the invariant nonlinear horizontal complex (INHC) of the local Lie group \( (M, \tilde{\varepsilon}) \).
Observe that (28) does not restrict to $U \subset M$ since $(U, \tilde{\varepsilon}|_U)$ need not be globalizable even if $(M, \tilde{\varepsilon})$ is. We are unable to express the cohomology of (28) in positive degrees in terms of some known cohomology groups. We also do not know any sufficient condition which makes the cohomologies of (28) and (22) isomorphic. However it is worthwhile to note that elements of $\Lambda^n_{\text{hor}}(M \times M)$ may be viewed as functionals on the diffeomorphism group $Diff(M)$ of $M$ if $M$ is compact. Indeed, if $\omega \in \Lambda^n_{\text{hor}}(M \times M)$ and $f \in Diff(M)$, then $\omega(x, f(x))$ defines a volume form as $x$ ranges over $M$ and therefore can be integrated over $M$ giving the functional $\omega : f \mapsto \int_M \omega(x, f(x))$. This suggests to continue (22) one step to the right by the Euler-Lagrange operator $EL$ but we will not enter this issue here.

5 The linearization map

Our purpose in this section is to define a chain map from (22) to (13) which restricts to the invariant subcomplexes (28), (18).

Let $\omega = \omega_I(x, y)$ be a nonlinear horizontal $k$-form and $\xi$ be a tangent vector at $x$. The idea is to let $y$ approach $x$ along the tangent vector $\xi$. Since $\omega_I(x, x + t\xi)$ and $\omega_I(x, x)$ are two ordinary $k$-forms at the same point $x$, $\left[ \frac{d\omega_I(x, x + t\xi)}{dt} \right]_{t=0}$ is well defined and is an ordinary $k$-form at $x$ which depends on $\xi$, that is, an element of $\Lambda^k_{\text{hor}}(T)$. It depends linearly on $\xi$ because

$$\left[ \frac{d\omega_I(x, x + t\xi)}{dt} \right]_{t=0} = \left[ \frac{\partial\omega_I(x, y)}{\partial y^a} \right]_{y=x} \xi^a = \frac{\partial\omega_I(x, x)}{\partial y^a} \xi^a.$$ (29)

Now (29) defines a map $L : \Lambda^k_{\text{hor}}(M \times M) \to T^* \otimes \Lambda^k$.

**Proposition 19** The following diagram commutes

$$\begin{array}{cccc}
\Lambda^0_{\text{hor}}(M \times M) & \overset{\hat{d}}{\longrightarrow} & \Lambda^1_{\text{hor}}(M \times M) & \overset{\hat{d}}{\longrightarrow} & \cdots & \overset{\hat{d}}{\longrightarrow} & \Lambda^n_{\text{hor}}(M \times M) \\
\downarrow L & & \downarrow L & & \cdots & & \downarrow L \\
T^* = \Lambda^1 & \overset{\hat{d}}{\longrightarrow} & T^* \otimes \Lambda^1 & \overset{\hat{d}}{\longrightarrow} & \cdots & \overset{\hat{d}}{\longrightarrow} & T^* \otimes \Lambda^n.
\end{array}$$ (30)

Proof: Let $\omega \in \Lambda^k_{\text{hor}}(M \times M)$. To compute $(\hat{D}_r \circ L) \omega$, we apply $\hat{D}_r$ to (29) which gives

$$\frac{\partial^2 \omega_I(x, x)}{\partial x^a \partial y^a} \xi^a + \frac{\partial^2 \omega_I(x, x)}{\partial y^a \partial y^a} \xi^a + \frac{\partial \omega_I(x, x)}{\partial y^a} \hat{\Gamma}^a_{rb} \xi^b.$$ (31)

To compute $(L \circ \hat{D}_r) \omega$ we apply $L$ to (19) which gives
\[
\frac{d}{dt} \left[ \frac{\partial \omega_I(x, x + t\xi)}{\partial x^a} + \frac{\partial \omega_I(x, x + t\xi)}{\partial y^a} \tilde{\varepsilon}_r(x, x + t\xi) \right]_{t=0} = 0 \tag{32}
\]

and (1) shows that (31) and (32) are equal. □

Unfortunately, we do not have \( L : \hat{\Lambda}_k \longrightarrow T^* \otimes \Lambda^k \). For instance, let \( k = 2 \) and \( \omega \in \hat{\Lambda}_2 \). By (27), the condition \( \hat{\Box}_r \omega_{ij} = 0 \) is

\[
\Gamma_{ri}(x) \omega_{aj}(x, y) + \Gamma_{rj}(x) \omega_{ia}(x, y) + \frac{\partial \omega_{ij}(x, y)}{\partial x^a} + \frac{\partial \omega_{ij}(x, y)}{\partial y^a} \tilde{\varepsilon}_r(x, y) = 0 \tag{33}
\]

Setting \( y = x + t\xi \) in (33) and differentiating at \( t = 0 \) gives

\[
0 = \Gamma_{ri}(x) \frac{\partial \omega_{aj}(x, x)}{\partial y^b} \xi^b + \Gamma_{rj}(x) \frac{\partial \omega_{ia}(x, x)}{\partial y^b} \xi^b + \frac{\partial \omega_{ij}(x, x)}{\partial y^a} \tilde{\varepsilon}_r(x) \xi^b \\
+ \frac{\partial \omega_{ij}(x, x)}{\partial y^a} \Gamma_{rb}(x) \xi^b \\
= \Gamma_{ri}(x) \frac{\partial \omega_{aj}(x, x)}{\partial y^b} \xi^b + \Gamma_{rj}(x) \frac{\partial \omega_{ia}(x, x)}{\partial y^b} \xi^b + \frac{\partial \omega_{ij}(x, x)}{\partial y^a} \left[ \frac{\partial \omega_{ij}(x, x)}{\partial y^a} \xi^a \right] \\
+ \frac{\partial \omega_{ij}(x, x)}{\partial y^a} \Gamma_{rb}(x) \xi^b \\
= \Gamma_{ri}(x) (L\omega_{aj}) + \Gamma_{rj}(x) (L\omega_{ia}) + \tilde{D}_r (L\omega)_{ij} \\
= \hat{\Box}_r (L\omega)_{ij} \tag{34}
\]

whereas what we want is \( \hat{\Box}_r (L\omega)_{ij} = 0 \). Clearly we can replace \( \hat{\Box}_r \) with \( \tilde{\Box}_r \) in (34). This makes it necessary to consider forms which are both \( \tilde{\varepsilon} \) and \( \hat{\varepsilon} \) invariant.

Now the proof of Proposition 8 shows that other than (18) we also have the subcomplex

\[
\Lambda^1 \xrightarrow{\hat{d}} T^* \otimes \Lambda^1 \xrightarrow{\hat{d}} \ldots \xrightarrow{\hat{d}} T^* \otimes \Lambda^n \tag{35}
\]

of (13). Observe that the first operator in (35) vanishes on \( \hat{\Lambda}^1 \). The interpretations of (18) and (35) in the modern formalism are somewhat intriguing: (35) computes the cohomology of \( g = \check{\Theta} \) with coefficients \( \hat{\Lambda}^1 \) but the representation is trivial. So (35) computes \( n \)-copies of the cohomology of \( \check{\Theta} \) with trivial coefficients \( \mathbb{R} \). However the representation in (18) is “honest” (which comes, of course, from the Lie derivative \( \hat{L}_\xi \), see (49) below). Now (18) and (35) give the subcomplex

\[
\tilde{\Lambda}^1 \xrightarrow{\tilde{d}} T^* \otimes \tilde{\Lambda}^1 \xrightarrow{\tilde{d}} \ldots \xrightarrow{\tilde{d}} T^* \otimes \tilde{\Lambda}^n \tag{36}
\]

(note that a biinvariant form need not be closed in the presence of a “representation” as can easily be seen from the second formula in (16) which the reader
may compare to 19 in [4]). We call (36) the biinvariant subcomplex. Similarly we construct the biinvariant nonlinear complex. The computation in (34) now gives the following

**Proposition 20** $L$ restricts to biinvariant subcomplexes.

### 6 Double complexes

The main idea of the nonlinear double complex is quite simple: we let the number of copies of $M$ be arbitrary, modify $d$ accordingly and define the vertical operator $\delta$ by the well known formula from topology and group cohomology. Some formulas look quite complicated in coordinates even though they are straightforward generalizations of our previous formulas and state some facts which are evident at this stage. For this reason our treatment will be short.

**Definition 21** A nonlinear horizontal $k$-form $\omega$ on $M \times M \times \ldots \times M$ ($m$ copies, $m \geq 1$) assigns to $(p, q, \ldots, t) \in M \times M \times \ldots \times M$ an ordinary $k$-form at $p$.

Let $M^{(m)}$ denote $M \times M \times \ldots \times M$ ($m$ copies, $m \geq 1$). So $\omega$ is a function $\omega : M^{(m)} \to \Lambda^k(M)$. According to the formalism of groupoids, the groupoid $\mathcal{E}(M \times M) \subset J_1(M \times M)$ has a representation on the vector bundle $\Lambda^k(M) \to M$ and a (nonlinear) horizontal $k$-form is an $(m - 1)$-composable string. Thus we can define the differentiable cohomology of this groupoid with coefficients $\Lambda^k(M)$ (see [5] for details).

We denote the space of horizontal $k$-forms on $M^{(m)}$ by $\Lambda^m_{\text{hor}}(M^{(m)})$. Choosing coordinates $(U, x^i)$, $(V, y^i)$, ..., $(W, z^i)$ around $p, q, \ldots, t$, we express $\omega$ as $\omega_I(x, y, \ldots, z)$ where the index $I$ refers to $(x^i)$ as before. We define

$$\tilde{D}_I \omega_i \ldots i_k (x, y, \ldots, z) \overset{\text{def}}{=} \frac{\partial \omega_{i_1} \ldots i_k (x, y, \ldots, z)}{\partial x^j}$$

(37)

Using $\tilde{D}$ we now define $\tilde{d} : \Lambda^m_{\text{hor}}(M^{(m)}) \to \Lambda^{m,k+1}_{\text{hor}}(M^{(m)})$. Since $\tilde{R} = 0$ identically on $M \times M$, we get the complex

$$\Lambda^m_{\text{hor}}(M^{(m)}) \xrightarrow{\tilde{d}} \Lambda^{m+1}_{\text{hor}}(M^{(m)}) \xrightarrow{\tilde{d}} \ldots \xrightarrow{\tilde{d}} \Lambda^{m,n}_{\text{hor}}(M^{(m)})$$

(38)

For $m = 1$, (38) reduces to the ordinary de Rham complex on $M$ and for $m = 2$ it reduces to (22). If $(M, \mathcal{E})$ is globalizable, which we assume henceforth, the kernel of the first operator in (38) can be identified with functions $\theta : \mathcal{G}^{(m-1)} \to \mathbb{R}$ where we set $\mathcal{G}^{(0)} = \mathbb{R}$.

Now we define $\delta : \Lambda^m_{\text{hor}}(M^{(m)}) \to \Lambda^{m+1,k}_{\text{hor}}(M^{(m)})$ by the well known formula

$$(\delta \omega)_{(p_0, p_1, \ldots, p_m)} \overset{\text{def}}{=} \sum_{0 \leq i \leq m} (-1)^i \omega_{(p_0, p_1, \ldots, p_i, \ldots, p_m)}$$

(39)
where \((p_i)\) indicates the omission of the point \(p_i\). In (39) we interpret \(\omega\) as a function \(\omega : M^{(m)} \to \Lambda^k(M)\). Clearly \(\delta^2 = 0\). Thus we get the diagram

\[
\begin{array}{cccccccc}
\uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \\
\vdots & \quad & \quad & \quad & \quad & \quad & \quad & \\
\Lambda_{\text{hor}}(M^{(3)}) & \quad & \Lambda_{\text{hor}}(M^{(3)}) & \quad & \Lambda_{\text{hor}}(M^{(3)}) & \quad & \Lambda_{\text{hor}}(M^{(3)}) & \\
\uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \\
\Lambda_{\text{hor}}(M^{(2)}) & \quad & \Lambda_{\text{hor}}(M^{(2)}) & \quad & \Lambda_{\text{hor}}(M^{(2)}) & \quad & \Lambda_{\text{hor}}(M^{(2)}) & \\
\uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \quad & \uparrow \delta & \\
\Lambda_{\text{hor}}(M) & \quad & \Lambda_{\text{hor}}(M) & \quad & \Lambda_{\text{hor}}(M) & \quad & \Lambda_{\text{hor}}(M) & \\
\end{array}
\]

(40)

**Proposition 22** The diagram (40) commutes

Proof: We check the commutativity of the square

\[
\begin{array}{ccc}
\Lambda_{\text{hor}}^{2,k}(M) & \quad & \Lambda_{\text{hor}}^{2,k}(M) \\
\uparrow \delta & \quad & \uparrow \delta \\
\Lambda_{\text{hor}}^{3,k}(M^3) & \quad & \Lambda_{\text{hor}}^{3,k}(M^3) \\
\uparrow \delta & \quad & \uparrow \delta
\end{array}
\]

and the general case is similar. For \(\omega_I \in \Lambda_{\text{hor}}^{2,k}(M)\), we have

\[
(\delta\omega)_I(x, y, z) = \omega_I(y, z) - \omega_I(x, z) + \omega_I(x, y)
\]

(42)

We should be careful with (42): \(I\) refers to \((x^i)\) and \(\omega_I(y, z)\) denotes the value of \(\omega(y, z) \in \Lambda^k(M)\) at \(x\). Now we assume \(y = y(x)\) and \(z = z(y) = z(y(x))\) belong to \(\tilde{G}\) with \(y(p) = q, z(q) = o\), substitute \(y(x), z(x)\) into (42) and differentiate (42) with respect to \(x^r\) at \(x = p\). The result is

\[
\tilde{D}_r(\delta\omega)_I(p, q, o) = (\tilde{D}_r\omega_I)(q, o) - (\tilde{D}_r\omega_I)(p, o) + (\tilde{D}_r\omega_I)(p, q)
\]

\[
= \left(\delta \left(\tilde{D}_r\omega_I\right)\right)(p, q, o)
\]

(43)

and (43) implies \(\tilde{D} \circ \delta = \delta \circ \tilde{D}\). \(\square\)

**Definition 23** The diagram (40) is the nonlinear horizontal double complex (NHDC) of the local Lie group.

Some \(\omega \in \Lambda_{\text{hor}}^{m,k}(M^{(m)})\) is \(\tilde{e}\)-invariant if \(\tilde{e}(p, x), \omega(x, g(p, x, q), ..., g(p, x, t)) = \omega(p, q, ..., t)\). This condition is (26) in coordinates except that we should take also the other components into account. Differentiation of this formula at \(x = p\) gives (27) where \(\tilde{D}_r\omega_{i_1i_2...i_k}\) is defined by (35). In this way we get the subcomplex
For $m = 1$, $\Lambda^{k,1}_{\text{hor}}(M^1) = \hat{\Lambda}^k$, $\tilde{d}$ is the ordinary exterior derivative and (44) computes the cohomology of the Lie algebra $\hat{\Theta} \simeq \tilde{\Theta}$. The kernel of the first operator in (44) is the space of funtions $\tilde{G}^{(m-1)} \rightarrow \mathbb{R}$ which are invariant with respect to the conjugation by the elements of $\tilde{G}$.

**Proposition 24** $\delta : \Lambda^{m,k}_{\text{hor}}(M^m) \rightarrow \Lambda^{m+1,k}_{\text{hor}}(M^{m+1})$

Proof: Follows easily from the definitions. □

So (44) restricts to $\hat{\Lambda}$-invariant subspaces. We call the resulting diagram invariant nonlinear horizontal double complex (INHDC) of the localizable local Lie group $(M, \varepsilon)$. Proposition 24 implies the following

**Corollary 25** The restriction of $\delta$ to the kernels of the first horizontal operators in INHDC computes the Lie group cohomology $H^*(\tilde{G}, \mathbb{R})$ of $\tilde{G}$.

In an attempt to generalize Corollary 25 to higher cohomology groups, we now observe some further facts about local Lie groups. We recall that if $(M, \tilde{\varepsilon})$ is a local Lie group, that is, if $\tilde{\varepsilon} = 0$, then $\tilde{\Theta}$ is a Lie algebra. So we have the representation $\mathcal{L} : \tilde{\Theta} \rightarrow gl(\tilde{\Theta})$ defined by $\mathcal{L}_\xi \eta = [\xi, \eta]$, where $\mathcal{L}$ denotes Lie derivative. More generally, let $T^r_s(M)$ denote the space of $\tilde{\varepsilon}$-invariant $(r, s)$-tensor fields. We have the representation of $\mathfrak{g} = \tilde{\Theta}$ on $V = T^r_s(M)$ defined by

$$\mathcal{L}_\xi : T^r_s(M) \rightarrow T^r_s(M) \quad \xi \in \tilde{\Theta}$$

(45)

Observe that for $\xi \in \tilde{\Theta}$, $\mathcal{L}_\xi = 0$ as an operator on $T^r_s(M)$ since $\tilde{\Theta}$ integrates to $\tilde{G}$ and elements of $T^r_s(M)$ are $\tilde{\varepsilon}$-invariant by definition. Now assuming globalizability, $G = \tilde{G}$ has a representation $I\mathcal{L}$ ($I$ denotes integration) on $V = T^r_s(M)$ defined by

$$(I\mathcal{L}f)(\xi)(p) \overset{\text{def}}{=} \tilde{\varepsilon}(f^{-1}(p), p)_* \xi(f^{-1}(p)) \quad f \in \tilde{G}, \; \xi \in T^r_s(M), \; p \in M$$

(46)

and the derivative of the representation (46) is (45), that is, $d(I\mathcal{L}) = \mathcal{L}$.

Now let $H^{r,s}_{\text{sd}}$ denote the cohomology group of INHDC at $(r, s)$ taken first in the horizontal, then in the vertical directions. Motivated by Corollary 25 and the above general facts, we make (assuming globalizability) the following conjecture

$$\mathbf{C} : H^{r,k}_{\text{sd}} \simeq H^k(G, V) \text{ where } G = \tilde{G} \text{ and } V = \Lambda^k(M).$$

Therefore, if $M$ is compact, $\mathbf{C}$ implies the vanishing of $H^{r,k}_{\text{sd}}$ for $k \geq 1$. As we indicated above, the vertical complexes of (40) coincide with the the complex of the composable cochains in the sense of groupoids with representations as
defined in [5]. It is therefore not surprising that for compact $M$ they vanish too by Proposition 1 in Section 2.1 of [5].

To linearize $\omega \in \Lambda^{m,k}_r(M)$, we choose $(m - 1)$-tangent vectors $\xi, \eta, ..., \zeta$ at $x$ and let the target variables $y, z, ..., w$ in $\omega(x, y, z, ..., w)$ approach $x$ along these directions independently so that

$$L\omega_I(x, \xi, \eta, ..., \zeta) \overset{\text{def}}{=} \frac{\partial \omega_I(x, x, ..., x)}{\partial y^a \partial z^b \partial w^c} \xi^a \eta^b ... \zeta^c \tag{47}$$

Let $\pi : E_s \to M$ be the vector bundle over $M$ whose fiber $\pi^{-1}(p)$ is the space of $s$-linear maps $T_p \times ... \times T_p \to \mathbb{R}$, that is, $E_s = \otimes^s T^*$. Now (47) defines a map

$$L : \Lambda^{m,k}_r(M) \to (\otimes^{m-1} T^*) \otimes \Lambda^k \tag{48}$$

Using (48) we define the linear horizontal double complex (LHDC) and its invariantization (ILHDC) in such a way that $L$ becomes a homomorphism of these two biinvariant double complexes. The the $m'$th row of ILHDC computes $H^*(g, \otimes_{m-1} \mathfrak{g}^*) = \bar{\Theta}$ and we can show that its vertical cohomology vanishes for compact $M$.

Finally, consider the covariant differentiation operator $\hat{\nabla}_X : T^*_r(M) \to T^*_r(M)$, $X \in \mathfrak{X}(M)$ = the Lie algebra of smooth vector fields on $M$. A fundamental fact is expressed by the formulas

$$\hat{\nabla}_\xi = L_\eta \xi \in \bar{\Theta}, \eta = d\Psi(\xi) \in \bar{\Theta} \tag{49}$$
$$\hat{\nabla}_\xi = L_\eta \xi \in \bar{\Theta}, \eta = d\Psi^{-1}(\xi) \in \bar{\Theta}$$

where $\Psi$ and $d\Psi$ are defined in Section 2. The formula (49) continues to be valid if we replace $T^*_r(M)$ by more general geometric object bundles as in this note and (49) underlies Propositions 8, 17 and Proposition 7 in [13]. So in a sense everything in this note and in [13] reduces to a duality between $\hat{\nabla}$ and $\bar{\nabla}$ together with the concept of invariance on a local Lie group (which, we believe, is the origin of the concept of torsion), the theory of Lie derivative on form valued geometric objects (as in [20]), and the relation between exterior derivative and Lie derivative.

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