Glivenko’s theorem, finite height, and local finiteness

Ilya B. Shapirovsky
September 5, 2018

Abstract

The well-known Glivenko’s theorem states that a formula is derivable in the classical propositional logic \( \mathbf{CL} \) iff under the double negation it is derivable in the intuitionistic propositional logic \( \mathbf{IL} \): \( \mathbf{CL} \vdash \varphi \) iff \( \mathbf{IL} \vdash \neg \neg \varphi \). Its analog for the modal logics \( \mathbf{S5} \) into \( \mathbf{S4} \) states that \( \mathbf{S5} \vdash \varphi \) iff \( \mathbf{S4} \vdash \neg \Box \neg \Box \varphi \). We provide generalizations of this translation for a wide family of modal logics. We consider modal logics in which the master modality \( \Box^* \) is expressible. For a logic \( L \) and a finite \( h \), the logic \( L[h] \) is the extension of \( L \) with the formula of height \( h \). The first result states that for all pretransitive logics \( L \), we have \( L[1] \vdash \varphi \) iff \( L \vdash \neg \Box^* \neg \Box^* \varphi \). Then we consider logics of arbitrary finite height. For formulas in a given finite set of variables, we describe translations from \( L[h + 1] \) to \( L \) in the case when \( L[h] \) is locally tabular.

Keywords: Glivenko’s translation, modal logic, intermediate logic, finite height, pretransitive logic, local tabularity, local finiteness, top-heavy frame

1 Introduction

In Kripke semantics, the intuitionistic propositional logic \( \mathbf{IL} \) is the logic of partial orders. At the same time, the classical propositional logic \( \mathbf{CL} \) is the logic of a singleton, or equivalently, is the logic of partial orders of height 1. The well-known Glivenko’s theorem [Gli29] states that \( \mathbf{CL} \vdash \varphi \) iff \( \mathbf{IL} \vdash \neg \neg \varphi \). The modal analog of this translation embeds the logic \( \mathbf{S5} \) into \( \mathbf{S4} \): namely, \( \mathbf{S5} \vdash \varphi \) iff \( \mathbf{S4} \vdash \Diamond \Box \varphi \) [Mat55]. Recall that \( \mathbf{S4} \) is the logic of preorders, and \( \mathbf{S5} \) is the logic of equivalence relations, which can be considered as preorders of height 1.
We provide two generalizations of these facts. We consider propositional polymodal logics, in which the master modality $\Box^*$ is expressible (pretransitive logics). In the pretransitive case, we can write formulas restricting the height of frames. Let $L[h]$ be the extension of $L$ with the formula of height $h$.

In the complete analogy with the transitive case, for all pretransitive logics we have $L[1] \vdash \varphi$ iff $L \vdash \Diamond^* \Box^* \varphi$. This result for the unimodal case was obtained in [KS17]; here we provide its polymodal version. The explanation of this embedding is rather simple: in a pretransitive canonical frame, the set of maximal (with respect to the preorder induced by the the accessibility relations) points is cofinal. (This property of transitive canonical frames is well-known [Fin85]; it holds in the pretransitive case as well.) It remains to observe that maximal elements of a canonical frame of $L$ forms a canonical frame of $L[1]$.

To describe translations for logics of arbitrary finite height, we need more efforts. $L$ is said to be $k$-tabular if, up to the equivalence in $L$, there exist only finitely many $k$-formulas (i.e., formulas in variables $p_0, \ldots, p_{k-1}$); equivalently, $L$ is $k$-tabular if its $k$-generated free algebra is finite. $L$ is locally tabular if it is $k$-tabular for all $k < \omega$; equivalently, $L$ is locally tabular if the variety of $L$-algebras is locally finite, i.e., every finitely generated $L$-algebra is finite. The main result (Theorem 4) describes the translation from $L[h+1]$ to $L$ for $k$-formulas in the case when $L[h]$ is $k$-tabular.

Again, the proof is based on the structure of upper parts of pretransitive canonical frames. The upper part of the $k$-canonical frame $F$ of $S4$ (which is built from maximal consistent sets of $k$-formulas) is known to have the following (top-heavy) property: for all $h < \omega$, every point $x$ is either in the $k$-canonical frame $F[h]$ of $S4[h]$, or below a point in $F[h]$ [She78, Fin85, Bel85]. Without transitivity the picture is much more complicated. However, the following fact (Theorem 3) holds for every pretransitive $L$: if $L[h]$ is $k$-tabular, then every point $x$ in the $k$-canonical frame of $L$ is either in the $k$-canonical frame of $L[h+1]$, or below (with respect to the preorder induced by the the accessibility relations) one of its points.

2 Preliminaries

Fix a finite $n > 0$. $ML_n$ denotes the set of all $n$-modal formulas; they are built from a countable set $\{p_1, p_2, \ldots\}$ of proposition letters, the classical connectives $\to, \bot$, and the modal connectives $\Diamond_i$, $i < n$; the other Boolean connectives are defined as standard abbreviations; $\Box_i$ abbreviates $\neg \Diamond_i \neg$. By a logic we mean a propositional $n$-modal normal logic, that is a subset of $ML_n$. 2
containing all classical tautologies, formulas ♦_i(p ∨ q) → ♦_i p ∨ ♦_i q and ¬♦_i⊥ for all i < n, and closed under the rules of Modus Ponens, Substitution, and Monotonicity (if ϕ → ψ is in the logic, then so is ♦ϕ → ♦ψ). For a logic L and a set of formulas Ψ, the least logic containing L ∪ Ψ is denoted by L + Ψ. For a formula ϕ, the notation L + ϕ abbreviates L + {ϕ}. L ⊩ ϕ is a synonym for ϕ ∈ L.

The truth and the validity of modal formulas in Kripke frames and models are defined as usual, see, e.g., [BdRV02]. By a frame we always mean a Kripke frame (W; (R_i)_{i<n}), W ≠ ∅, R_i ⊆ W × W. Put R_F = ∪_{i<n} R_i. The transitive reflexive closure of a relation R is denoted by R^*; the notation R(x) is used for the set {y | xRy}. The restriction of F onto its subset V, F|V in symbols, is (V, (R_i ∩ (V × V))_{i<n}). In particular, F⟨x⟩ = F|R^*_F(x).

For k ≤ ω, a k-formula is a formula in proposition letters p_i, i < k.

Let L be a consistent logic. For k ≤ ω, the k-canonical model of L is built from maximal L-consistent sets of k-formulas; the relations and the valuation are defined in the standard way, see e.g. [CZ97]. L is k-canonical if it is valid in its k-canonical frame. Recall the following fact.

**Proposition 1** (Canonical model theorem). Suppose k ≤ ω, M is the k-canonical model of a logic L. Then for all k-formulas ϕ,

- for all x in M, we have M, x ⊩ ϕ ⇔ ϕ ∈ x, and
- M ⊩ x ⇔ L ⊩ ϕ.

**Pretransitive logics and frames**

Here we quote some basic facts about logics with expressible transitive reflexive closure modality (‘master modality’); for more details on this topic, see, e.g., [Kra99].

For a binary relation R on a set W, put R^{≤m} = ∪_{i≤m} R^i, where R^0 = Id(W), R^{i+1} = R ∘ R^i. R is called m-transitive, if R^{≤m} = R^*. R is pretransitive if it is m-transitive for some m. A frame F is m-transitive if R_F is m-transitive.

Let 0^ϕ = ϕ, 1^i+1ϕ = 0^i(0_0ϕ ∨ ... ∨ 0_n−1ϕ), 0^≤mϕ = ∀_{i≤m}0^iϕ, 0^≤mϕ = ¬0^≤m−nϕ.

**Proposition 2.** Let F be a frame. The following are equivalent:

- R^*_F = R^{≤m}_F,
- R^{m+1}_F ⊆ R^{≤m}_F,
- F ⊩ ♦^{m+1}p → ♦^{≤m}p.
The proof is straightforward, details can be found, e.g., in [Kra99].

**Definition 1.** A logic $L$ is said to be $m$-transitive if $L \vdash \Diamond^{m+1}p \to \Diamond^{\leq m}p$. $L$ is pretransitive if it is $m$-transitive for some $m \geq 0$.

If $L$ is pretransitive, there exists the least $m$ such that $L$ is $m$-transitive; in this case we write $\Diamond^*\varphi$ for $\Diamond^{\leq m}\varphi$, and $\Box^*\varphi$ for $\Box^{\leq m}\varphi$.

For a unimodal formula $\varphi$, $\varphi^{[\ast]}$ denotes the formula obtained from $\varphi$ by replacing $\Diamond$ with $\Diamond^*$ and $\Box$ with $\Box^*$.

**Proposition 3.** For a pretransitive logic $L$, the set $\{\varphi \mid L \vdash \varphi^{[\ast]}\}$ is a logic containing $S4$.

*Proof. Follows from [GSS09, Lemma 1.3.45].

**Proposition 4.** Let $F$ be the $k$-canonical frame of a pretransitive logic $L$ ($k \leq \omega$). For all $x, y$ in $F$, we have

$$xR^*_F y \text{ iff } \forall \varphi (\varphi \in y \Rightarrow \Diamond^*\varphi \in x).$$

The proof can be found, e.g., in [CZ97], [Kra99].

**Formulas and frames of finite height**

A poset is of height $h < \omega$ if it contains a chain of $h$ elements and no chains of cardinality $> h$.

A *cluster* in a frame $F$ is an equivalence class with respect to the relation $\sim_F = R^*_F \cap R^{-1}_F$. For clusters $C, D$, put $C \leq_F D$ iff $xR^*_F y$ for some $x \in C, y \in D$. The poset $(W/\sim_F, \leq_F)$ is called the *skeleton* of $F$.

The height of a frame $F$, in symbols, $ht(F)$, is the height of its skeleton. Put

$$B_0 = \bot, \quad B_{i+1} = p_{i+1} \to \Box^*(\Diamond^*p_{i+1} \lor B^*_i).$$

In the unimodal transitive case, the formula $B_h$ expresses the fact that the height of a frame $\leq h$ [Seg71]. In the case of a pretransitive frame $F = (W, (R_i)_{i<\omega})$, the operators $\Diamond^*$ and $\Box^*$ relate to $R^*_F$ (again, $\Diamond^*$ abbreviates $\Diamond^{\leq m}$ for the least $m$ such that $R^*_F = R^{=m}_F$). Since the height of $F$ is the height of the preorder $(W, R^*_F)$, we have

**Proposition 5.** For a pretransitive frame $F$, $F \models B_h$ iff $ht(F) \leq h$.

---

1Logics with master modality sometimes are said to be *weakly transitive*. However, it the other terminology, the term ‘weakly transitive’ is used for logics containing the formula $\Diamond\Diamond p \to \Diamond p \lor p$. 

---

4
Definition 2. A pretransitive logic is of finite height if it contains $B_h$ for some $h < \omega$.

For a pretransitive $L$, put

$$L[h] = L + B_h.$$ 

Example 1. If $L \vdash p \rightarrow \Box \Diamond p$ for a pretransitive logic $L$, then $L \vdash p \rightarrow \Box^* \Diamond^* p$. In particular, the logic $KTB + \Diamond^3 p \rightarrow \Diamond^2 p$ is a 2-transitive logic of height 1 ($KTB$ is the least unimodal logic containing \{p \rightarrow \Diamond p, p \rightarrow \Box \Diamond p\}).

If $L$ is the least unimodal $m$-pretransitive logic, $m \geq 2$, then $L[h] \subseteq KTB + \Diamond^3 p \rightarrow \Diamond^2 p$ for all $h < \omega$.

Example 2. If $L_1$ and $L_2$ are Kripke complete logics, $L_i$ is $m_i$-transitive of height $h_i$ ($i = 1, 2$), then the product $L_1 \times L_2$ is $m_1 + m_2$-transitive of height $h_1 + h_2 - 1$. In particular, the product of two transitive Kripke complete logics is 2-transitive; $S5 \times S5$ is 2-transitive of height 1.

Example 3 (V. Shehtman, private communications). $K5 = K + \Diamond p \rightarrow \Box \Diamond p$ is a 2-transitive logic of height 2. To show this, recall that $K5$ is Kripke complete and its frames are those which validate the sentence $\forall x \forall y \forall z (xRy \land xRz \rightarrow yRz)$. Every $K5$-frame is 2-transitive. Indeed, suppose that $aRb Rc Rd$ for some elements of an $K5$-frame. Then $bRb$; we also have $bRc$, thus $cRb$; from $cRb$ and $cRd$ we infer that $bRd$. Thus $aR^2d$. If a $K5$-frame has an irreflexive serial point, then its height is 2; otherwise it is a disjoint sum of $S5$-frames and irreflexive singletons, so its height is 1.

Proposition 6. A pretransitive logics $L$ is consistent iff $L[1]$ is consistent.

Proof. Easily follows from Proposition 3 and the fact that if a logic containing $S4$ is consistent, then its extension with the formula $p \rightarrow \Box \Diamond p$ is consistent.

Since $L[1] \supseteq L[2] \supseteq L[3] \supseteq \ldots$, it follows that if $L$ is consistent, then $L[h]$ is consistent for any $h > 0$.

For a frame $F$, let $F[h]$ be the restriction of $F$ onto the set of its points $x$ such that $F\langle x \rangle \leq h$.

Proposition 7. Let $F$ be the $k$-canonical frame of a pretransitive logic $L$, $k \leq \omega$.

1. For all $x$ in $F$, $0 \leq h < \omega$,

$$ht(F\langle x \rangle) \leq h \text{ iff } B_h(\psi_1, \ldots, \psi_h) \in x \text{ for all } k\text{-formulas } \psi_1, \ldots, \psi_h.$$
2. For \(0 < h < \omega\), the frame \(F[h]\) is the canonical frame of \(L[h]\).

Proof. 1. If \(ht(F(x)) \leq h\), then \(B_h\) is valid at \(x\) in \(F\); by the Canonical model theorem, \(B_h(\psi_1, \ldots, \psi_h) \in x\) for all \(k\)-formulas \(\psi_1, \ldots, \psi_h\).

By induction on \(h\), let us show that if \(ht(F(x)) > h\), then \(B_h(\psi_1, \ldots, \psi_h) \notin x\) for some \(\psi_1, \ldots, \psi_h\). The basis is trivial, since there are no points containing \(B_0 = \bot\) in \(F\). Suppose \(ht(F(x)) > h + 1\). Then there exists \(y\) such that \(ht(F(y)) > h\), \((x, y) \in R^*_F\), and \((y, x) \notin R^*_F\). By induction hypothesis, \(\neg B_h(\psi_1, \ldots, \psi_h) \in y\) for some \(\psi_1, \ldots, \psi_h\). By Proposition 4, for some \(\psi_{h+1}\) we have \(\psi_{h+1} \in x\) and \(\varphi \notin y\). It follows that \(B_{h+1}(\psi_1, \ldots, \psi_{h+1}) \notin x\).

2. Since \(L \subseteq L[h]\), the \(k\)-canonical frame of \(L[h]\) is a generated subframe of \(F\). Now the statement follows from the first statement of the proposition.

**Proposition 8.** If a pretransitive \(L\) is \(k\)-canonical \((k \leq \omega)\), then \(L[h]\) is \(k\)-canonical for all \(0 < h < \omega\).

Proof. Follows from Proposition 4.

### 3 Translation for logics of height 1

For a pretransitive logics \(L\), \(L[1] = L + B_1\), that is \(L[1]\) is the least logic containing \(L \cup \{p \to \Box^* \Diamond^* p\}\). It is known that \(S4 \vdash \Diamond \Box \Diamond \varphi\) iff \(S4[1] \vdash \varphi\), and \(S5 \vdash \Box \psi \rightarrow \Box \varphi\) iff \(S4[1] \vdash \Diamond \Box \psi \rightarrow \Diamond \Box \varphi\) [Mat55], [Ryb92]. In [KSI17], it was shown that in the pretransitive unimodal case we have \(L[1] \vdash \varphi\) iff \(L \vdash \Diamond^* \Box^* \varphi\). We are going to generalize these facts for the polymodal case.

Consider a frame \(F\) and its subset \(V\). We say that \(x \in V\) is a maximal element of \(V\), if for all \(y \in V\), \(x R^*_F y\) implies \(y R^*_F x\).

It is known that in canonical transitive frames every non-empty definable subset has a maximal element [Fin85]; the next proposition shows that this property holds in the pretransitive case as well.

**Proposition 9** (Maximality lemma). Suppose that \(F\) is the \(k\)-canonical frame of a pretransitive \(L\), \(k \leq \omega\). Let \(\varphi \in x\) for some \(x\) in \(F\) and some \(\varphi\). Then \(R^*_F(x) \cap \{y \mid \varphi \in y\}\) has a maximal element.

Proof. For a formula \(\alpha\), put \(\|\alpha\| = \{y \mid \alpha \in y\}\). Since \(\varphi \in x\), \(R^*_F(x) \cap \|\varphi\|\) is non-empty.

Let \(\Sigma\) be an \(R^*_F\)-chain in \(R^*_F(x) \cap \|\varphi\|\). The family \(\{R^*_F(y) \cap \|\varphi\| \mid y \in \Sigma\}\) has the finite intersection property (indeed, if \(\Sigma_0\) is a non-empty finite subset of \(\Sigma\), then for some \(y_0 \in \Sigma_0\) we have \(y R^*_F y_0\) for all \(y \in \Sigma_0\); so \(y_0 \in R^*_F(y) \cap \|\varphi\|\) for all \(y \in \Sigma_0\)). By Proposition 4, \(R^*_F(y) = \bigcap\{\|\varphi\| \mid \Box^* \Box \varphi \in y\}\). It follows that all the sets \(R^*_F(y) \cap \|\varphi\|\) are closed in the Stone topology on \(F\) (see, e.g.,...
It follows by the compactness that \( \bigcap \{ R^*_F(y) \cap \| \varphi \| \mid y \in \Sigma \} \) is non-empty. Thus \( \Sigma \) has an upper bound in \( \| \varphi \| \). By Zorn’s lemma, \( R^*_F(x) \cap \| \varphi \| \) contains a maximal element.

**Theorem 1.** Suppose that \( L \) is a pretransitive logic. For all formulas \( \varphi, \psi \) we have

\[
L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \iff L \vdash \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi.
\]

**Proof.** By Proposition 3 we may assume that both \( L \) and \( L[1] \) are consistent. Let \( F \) be the canonical frame of \( L \), and \( G \) the canonical frame of \( L[1] \).

Suppose \( L \vdash \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi \). Consider an element \( x \) of \( G \). By Proposition 3, \( \{ \varphi \mid L[1] \vdash \varphi \} \) is a logic containing \( S5 \). Thus \( x \) contains formulas \( \Box^* \psi \rightarrow \Diamond^* \Box^* \psi \) and \( \Diamond^* \Box^* \varphi \rightarrow \Box^* \varphi \). Since \( L \subseteq L[1] \), \( x \) also contains \( \Diamond^* \Box^* \psi \rightarrow \Diamond^* \Box^* \varphi \). It follows that if \( x \) contains \( \Box^* \psi \), then \( x \) contains \( \Box^* \varphi \). By the Canonical model theorem, \( L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \).

Now suppose \( L[1] \vdash \Box^* \psi \rightarrow \Box^* \varphi \). Assume that \( \Diamond^* \Box^* \psi \in x \) for some element \( x \) of \( F \). Then for some \( y \) we have \( \Box^* \psi \in y \) and \( x R_F^* y \). The set \( R_F^*(y) \) has a maximal element \( z \) by Proposition 9. It follows that if \( x \) contains \( \Box^* \psi \), then \( x \) contains \( \Box^* \varphi \).

**Theorem 2.** Let \( L \) be a pretransitive logic.

1. For all \( \varphi \), we have \( L[1] \vdash \varphi \) iff \( L \vdash \Diamond^* \Box^* \varphi \).

2. If \( L \) is decidable, then so is \( L[1] \).

3. If \( L \) has the finite model property, then so does \( L[1] \).

**Proof.** By the above theorem, we have

\[
L[1] \vdash \Box^* \top \rightarrow \Box^* \varphi \iff L \vdash \Diamond^* \Box^* \top \rightarrow \Diamond^* \Box^* \varphi.
\]

By Proposition 3 we have \( \top \leftrightarrow \Box^* \top \) and \( \top \leftrightarrow \Diamond^* \Box^* \top \) in every pretransitive logic; also we have \( \Box^* \varphi \in L[1] \) iff \( \varphi \in L[1] \). Now the first statement follows.

The second statement is an immediate consequence of the first one.

Suppose \( L \) has the finite model property. Consider a formula \( \varphi \notin L[1] \). Then \( \Diamond^* \Box^* \varphi \notin L \). Then \( \Diamond^* \Box^* \varphi \) is refuted in some finite \( L \)-frame \( F \). If follows that \( \varphi \) is refuted in \( F \) at some point in a maximal cluster \( C \). The restriction \( F|C \) is a generated subframe of \( F \). Thus \( F|C \) refutes \( \varphi \) and validates \( L \). The height of this restriction is 1, so \( F|C = L[1] \). Thus \( L[1] \) has the finite model property.
Example 4. A natural family of pretransitive frames are birelational frames \((W, \leq, R)\) with transitive \(R\). Recall that \((W, \leq, R)\) is a birelational frame, if \(\leq\) is a partial order on \(W\), \(R \subseteq W^2\), and
\[
(R \circ \leq) \subseteq (\leq \circ R), \quad (R^{-1} \circ \leq) \subseteq (\leq \circ R^{-1}).
\]
Consider the class of all birelational frames \((W, \leq, R)\) with transitive reflexive \(R\). Its modal logic \(L\) is the least bimodal logic containing the axioms of \(S4\) for modalities \(\Box_0, \Box_1\), and the formulas \(\Diamond_1 \Diamond_0 p \rightarrow \Diamond_0 \Diamond_1 p\) and \(\Diamond_0 \Box_1 p \rightarrow \Box_1 \Diamond_0 p\) \([GKWZ03]\) (recall that in the semantics of modal intuitionistic logic, the logic of this class is known to be \(IS4\), one of the “most prominent logics for which decidability is left open” \([Sim94]\)). In this case, \(\Diamond_0 \Diamond_1\) plays the role of the master modality, and the formula \(B_1\) says that \(\leq \circ R\) is an equivalence. The decidability and the finite model property of the logic \(L\), as well as of the logic \(IS4\), is an open question. By the above theorem, we have \(L \vdash \Diamond_0 \Diamond_1 \Box_0 \Box_1 \phi\) iff \(L[1] \vdash \phi\).

Question. Is the logic \(L[1]\) decidable? Does it have the fmp?

4 Translation for logics of arbitrary finite height

For a pretransitive \(L\), Theorem \(\|$ provides the translation from \(L[1]\) to \(L\). Now our aim is to describe its analogs for logics of arbitrary finite height.

In the proof of Theorem \(\|$ we used the following property of a canonical frame \(F\) of \(L\): every point in \(F\) is below (w.r.t to the preorder \(R^*_F\)) a maximal point; maximal points form \(F[1]\), the canonical frame of \(L[1]\). To describe translations from \(L[h]\) to \(L\) for \(h > 1\), we shall use the following analogs of this property.

Definition 3. Let \(0 < h < \omega\). A frame \(F\) is said to be \(h\)-heavy if for every its element \(x\) which is not in \(F[h]\) there exists \(y\) such that \(x R^*_F y\) and \(ht(F(y)) = h\).

\(F\) is said to be top-heavy if it is \(h\)-heavy for all \(0 < h < \omega\).

Proposition 10. The \(k\)-canonical frame of a consistent pretransitive logic is 1-heavy for every \(k \leq \omega\).

Proof. In the Maximality lemma (Proposition\(\|$), put \(\phi = \top\).

It is known that \(k\)-canonical frames of unimodal transitive logics are top-heavy for all finite \(k\) \([She78], [Fin85]\). To generalize Theorem \(\|$ we need to tame the top-heavy property in the pretransitive case.
L is said to be $k$-tabular if there exist only finitely many $k$-formulas up to L-equivalence (equivalently, L is $k$-tabular if its $k$-generated free algebra is finite). L is locally tabular if it is $k$-tabular for all $k < \omega$ (equivalently, L is locally tabular if the variety of L-algebras is locally finite, i.e., every finitely generated L-algebra is finite).

**Theorem 3.** Consider a consistent pretransitive logic L and $h, k < \omega$. If $L[h]$ is $k$-tabular, then the $k$-canonical frame of L is $(h + 1)$-heavy.

**Proof.** The case $h = 0$ follows from Proposition 10. Suppose $h > 0$.

Let $F = (W, (R_i)_{i < n})$ be the $k$-canonical frame of L. By proposition 7, the frame $F[h] = (\overline{W}, (\overline{R_i})_{i < n})$ is the $k$-canonical frame of $L[h]$. Since $L[h]$ is $k$-tabular, it follows that $\overline{W}$ is finite and for every $a \in \overline{W}$ there exists a $k$-formula $\alpha(a)$ such that

$$\forall b \in \overline{W} (\alpha(a) \in b \iff b = a).$$

Without loss of generality we may assume that $\alpha(a)$ is of the form

$$p_0^+ \land \ldots \land p_{k-1}^+ \land \varphi,$$

where $p_i^+ \in \{p_i, \neg p_i\}$.

For $a \in \overline{W}$ let $\beta(a)$ be the following Jankov-Fine formula:

$$\beta(a) = \alpha(a) \land \gamma,$$

where $\gamma$ is the conjunction of the formulas

$$\Box^* \land \{ \alpha(b_1) \rightarrow \lozenge_i \alpha(b_2) \mid (b_1, b_2) \in \overline{R_i}, i < n \}$$

$$\Box^* \land \{ \alpha(b_1) \rightarrow \neg \lozenge_i \alpha(b_2) \mid (b_1, b_2) \in \overline{W}^2 \setminus \overline{R_i}, i < n \}$$

$$\Box^* \lor \{ \alpha(b) \mid b \in \overline{W} \}$$

For all $x, y \in W, i < n$ we have

if $\gamma \in x$ and $xR_iy$, then $\gamma \in y$.

We claim that

$$\forall a \in \overline{W} \forall x \in W (\beta(a) \in x \iff x = a).$$

To prove this, by induction on the length of formulas let us show that for all $k$-formulas $\varphi$, all $a \in \overline{W}$, and all $x \in W$,

if $\beta(a) \in x$, then $\varphi \in a \iff \varphi \in x$. (9)
The basis of induction follows from 2. The Boolean cases are trivial.

Assume that \( \varphi = \Diamond_i \psi \).

First, suppose \( \Diamond_i \psi \in a \). We have \( \psi \in b \) for some \( b \) with \( aR_ib \). Since \( \beta(a) \in x \), by (14) we have \( \Diamond_i \alpha(b) \in x \). Then we have \( \alpha(b) \in y \) for some \( y \) with \( xR_i y \); by (7), \( \beta(b) \in y \). Hence \( \psi \in y \) by induction hypothesis. Thus \( \Diamond_i \psi \in x \).

Now let us show that \( \Diamond_i \psi \in a \) whenever \( \Diamond_i \psi \in x \). In this case we have \( \psi \in y \) for some \( y \) with \( xR_i y \). By (6) we infer that \( \alpha(b) \in y \) for some \( b \in W \). Thus \( \Diamond_i \alpha(b) \in x \). Since \( \alpha(a) \in x \), it follows from (5) that \( aR_ib \). By (7) we have \( \gamma \in y \), thus \( \beta \in y \); by induction hypothesis \( \psi \in b \). Hence \( \Diamond_i \psi \in a \), as required.

Thus (9) is proved and (8) follows.

It follows that \( W \setminus \overline{W} \) is definable in the canonical model of \( L \):

\[
x \in W \setminus \overline{W} \iff \neg \bigvee \{ \beta(a) \mid a \in \overline{W} \} \in x.
\]

Now by Proposition 9 we have that if \( x \) is not in \( \overline{W} \), then there exists a maximal \( y \in R^*_F(x) \setminus \overline{W} \). Hence if \( (y, z) \in R^*_F \) and \( (z, y) \notin R^*_F \) for some \( z \), then \( z \) belongs to \( \overline{W} \), which means \( ht(F(z)) \leq h \). Thus \( ht(F(y)) = h + 1 \). On the other hand, \( y \notin \overline{W} \). It follows that \( ht(F(y)) = h + 1 \), as required.

The logic \( L[0] \) is inconsistent, so it is \( k \)-tabular. Hence Proposition 10 can be considered as a particular case of the above theorem.

Using the formulas (3), for \( i \leq h \) we can define the formulas \( B^{(L,k)}_i \) such that

\[
B^{(L,k)}_i \in x \text{ iff } x \in F[i].
\]

for all \( x \) in \( F \); for this, put

\[
B^{(L,k)}_i = \bigvee \{ \beta(a) \mid ht(F(a)) \leq i \}
\]

Finally, for a formula \( \varphi \), put

\[
C^{(L,k)}_i(\varphi) = \neg B^{(L,k)}_i \rightarrow \Diamond^*(\neg B^{(L,k)}_i \land \Box^* \varphi).
\]

Note that \( B^{(L,k)}_i \) is \( \perp \); hence \( C^{(L,k)}_0(\varphi) \) is equivalent to \( \Diamond^* \Box^* \varphi \). The following theorem generalizes the translation described in Theorem 2.

**Theorem 4.** Consider a pretransitive logic \( L \) and \( h, k < \omega \). If \( L[h] \) is \( k \)-tabular, then for all \( k \)-formulas \( \varphi \) we have

\[
L[h + 1] \vdash \varphi \text{ iff } L \vdash C^{(L,k)}_0(\varphi) \land \ldots \land C^{(L,k)}_h(\varphi).
\]

10
Proof. We may assume that both L and L[h + 1] are consistent (see Proposition 6). Let F be the k-canonical frame of L.

Suppose L[h + 1] ⊨ ϕ. Let us show that for all i ≤ h, C_i^{(L,k)}(ϕ) is true at every point x in the k-canonical model of L. Let ¬B_i^{(L,k)} be in x. Let us show that ¬B_i^{(L,k)} ∧ □*ϕ ∈ y for some y with xR_F y. First, assume that x is in F[h + 1]. By Proposition 7, x contains L[h + 1]. Since ϕ ∈ L[h + 1], we have □*ϕ ∈ x. Since R_F is reflexive, in this case we can put y = x. Second, suppose x is not in F[h + 1]. By Theorem 3, there exists y such that xR_F y and ht[F⟨y⟩] = h + 1. We have □*ϕ ∈ y and B_i^{(L,k)} ∉ y. This proves the “only if” part.

Now suppose that L ⊨ C_i^{(L,k)}(ϕ) for all i ≤ h. Assume ht(F⟨x⟩) = i ≤ h + 1. In this case B_i^{(L,k)} ∉ x. Since C_i^{(L,k)}(ϕ) ∈ x, it follows that ¬B_i^{(L,k)} ∧ □*ϕ ∈ y for some y with xR_F y. The first conjunct says that y is not in F[i − 1]. Since y is in F[i], it follows that ht(F⟨y⟩) = i. Hence y and x belong to the same cluster. Since □*ϕ ∈ y, we obtain ϕ ∈ x. It follows that ϕ ∈ x for all x in F[h + 1]. By Proposition 7, L[h + 1] ⊨ ϕ. □

5 Corollaries and examples

The translation (13) holds for all finite h, k in the case when L is a transitive unimodal logic. This is due to the well-known result by Segerberg [Seg71] and Maksimova [Mak75]: a unimodal logic containing K4 is locally tabular iff it is of finite height.

The Segerberg – Maksimova criterion was recently generalized to a wide family of pretransitive logics [SS16]. In particular, if a unimodal L contains ♦^{m+1}p → ♦p ∨ p for some m > 0, then L is locally tabular iff it is of finite height. Thus (13) holds for all finite h, k in this case too.

In the non-transitive case the situation is much more complicated. In [SS16], it was shown that every locally tabular unimodal logic is a pretransitive logic of finite height; in fact, the proof yields the following stronger formulation.

Theorem 5. If a logic is 1-tabular, then it is a pretransitive logic of finite height.

Proof. Let L be 1-tabular. Then its 1-canonical frame is finite. Every finite frame is m-transitive for some m. Thus L is m-transitive.

By Proposition 3, the set *L = {ϕ | L ⊨ ϕ[*]} is a logic containing S4. Since L is 1-tabular, *L is 1-tabular. In [Mak75], it was shown that for transitive logics 1-tabularity implies local tabularity. Thus *L is of finite height. It follows that L is of finite height too. □
Thus all locally tabular logics are pretransitive of finite height. However, unlike the transitive case, the converse is not true in general even for the unimodal logics. Let $\mathbf{Tr}_m$ be the least $m$-transitive unimodal logic. For $m \geq 2$, $h \geq 1$, none of the logics $\mathbf{Tr}_m[h]$ are locally tabular; moreover, for $k \geq 1$, they are not $k$-tabular: for $k \geq 2$, this result was strengthened for $k \geq 1$ in [Mak81].

In the non-transitive case, local tabularity of $L[h]$ depends on $h$.

**Example 5.** Consider the least reflexive 2-transitive logic $K + \{p \to \Diamond p, \Diamond^3 p \to \Diamond^2 p\}$ and its extension $L$ with the McKinsey formula for the master modality $\Box^2 \Diamond^2 p \to \Diamond^3 \Box^2 p$. Maximal clusters in the canonical frames of $L$ are reflexive singletons, so $L[1] = K + p \leftrightarrow \Box p$ by Proposition 7. Clearly, $L[1]$ is locally tabular. It follows that we have the translation (13) from $L[2]$ to $L$ for all finite $k$.

However, $L[2]$ is not even 1-tabular. To see this, consider the frame $F_0 = (\omega, R_0)$, where

$$xR_0y \quad \text{iff} \quad x \neq y + 1 \text{ and } y \neq x + 1.$$ 

Let $F = (\omega + 1, R)$, where $xRy$ iff $xR_0y$ or $y = \omega$. Clearly, $F \models L[2]$. Consider a model $M$ on $F$ such that $x \models p_0$ iff $x = 0$ or $x = \omega$. Put $\alpha_0 = p_0 \land \Diamond \neg p_0$, $\alpha_1 = \neg \Diamond \alpha_0 \land \neg p_0$, and $\alpha_{i+1} = (\neg \Diamond \alpha_i \lor \alpha_{i-1}) \land \neg p_0$ for $i > 0$. By an easy induction, in $M$ we have for all $i$: $x \models \alpha_i$ iff $x = i$. Thus if $i \neq j$, then $\alpha_i \leftrightarrow \alpha_j \notin L$.

With the parameter $k$, the situation is much more intriguing. The following result was proved in [Mak75]:

a unimodal transitive logic is locally tabular iff it is 1-tabular.

The recent results [SS16] show that this equivalence also holds for a wide class of non-transitive logics; in particular, if $L$ contains $\Diamond^{m+1} p \to \Diamond p \lor p$ for some $m > 0$, then, again, it is locally tabular iff it is 1-tabular. This situation is typical, and the following question has been open since 1970s: does this equivalence hold for every modal logic?

Up to now, all known examples of 1-tabular logics were locally tabular. Let us announce the first counterexample.

**Theorem 6.** There exists a unimodal 1-tabular logic $L$ which is not locally tabular.

**Proof (sketch).** Let $L$ be the logic of the frame $(\omega + 1, R)$, where

$$xRy \quad \text{iff} \quad x \leq y \text{ or } x = \omega.$$
First, we claim that $L$ is not locally tabular.

The following fact follows from Theorem 4.3 and Lemma 5.9 in [SS16]: if the logic of a frame $(W, R)$ is locally tabular, then the logic of the restriction $(V, R \cap (V \times V))$ is locally tabular for any non-empty $V \subseteq W$.

The restriction of $(\omega + 1, R)$ onto $\omega$ is the frame $(\omega, \leq)$, which is not locally tabular (it is of infinite height). Thus $L$ is not locally tabular.

To show that $L$ is 1-tabular, we need the following observation. If every $k$-generated subalgebra of an algebra $A$ contains at most $m$ elements for some fixed $m < \omega$, then the free $k$-generated algebra in the variety generated by $A$ is finite; see [Bez01b, Mal73].

Consider the complex algebra $A$ of the frame $(\omega + 1, R)$. It can be readily checked that its every 1-generated subalgebra contains at most 8 elements. By the above observation, $L$ is 1-tabular.

\textbf{Question.} Does $k$-tabularity imply local tabularity, for some fixed $k$ for all modal logics?

The similar question relates to intermediate logics: all known examples of 2-tabular intermediate logics are locally tabular. Whether it is true in general is another old open question.

In the intuitionistic case, the logics of finite height are known to be locally tabular [Kuz71, Kom75]. The top parts of finitely-canonical frames in the intuitionistic case were explicitly described in [She85, Bel85]. Note that the formula $C^{(L(k))}_i(\varphi)$ is equivalent to $\Box^*(\Box^*\varphi \rightarrow B^{(L(k))}_i \rightarrow B^{(L(k))}_i)$, and its intuitionistic analogs can easily be constructed. Thus the analog of Theorem 4 holds for intermediate logics of finite height.

Finite height is not a necessary condition for local tabularity of intermediate logics. What can an analog of Gliveko’s translation be in the case of a locally tabular intermediate logic with no finite height axioms?

Another generalizations can probably be found in the area of modal intuitionistic logics. In [Bez01a], Glivenko type theorems were proved for intuitionistic modal logics above $\text{MIPC}$, the intuitionistic variant of $\text{S5}$ (the equational theory of monadic Heyting algebras). In particular, in [Bez01a] it was shown that for all $L$ between $\text{MIPC}$ and $\text{WS5} = \text{MIPC} + \Diamond p \leftrightarrow \neg \Box \neg p$, we have $\text{WS5} \vdash \varphi$ iff $L \vdash \neg \neg \Box \varphi$. The paper [BG98] considered local tabularity of extensions of $\text{MIPC}$. What can be an analog of Theorem 4 for modal intuitionistic logics?
References

[BdRV02] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*. Cambridge University Press, 2002.

[Bel85] F. Bellissima. An effective representation for finitely generated free interior algebras. *Algebra Universalis*, 20(3):302–317, Oct 1985.

[Bez01a] G. Bezhanishvili. Glivenko type theorems for intuitionistic modal logics. *Studia Logica*, 67(1):89–109, 2001.

[Bez01b] G. Bezhanishvili. Locally finite varieties. *Algebra universalis*, 46(4):531–548, 2001.

[BG98] G. Bezhanishvili and R. Grigolia. Locally tabular extensions of MIPC. In *Advances in Modal Logic*, pages 101–120, 1998.

[Byr78] M. Byrd. On the addition of weakened L-reduction axioms to the Brouwer system. *Mathematical Logic Quarterly*, 24(25-30):405–408, 1978.

[CZ97] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[Fin85] K. Fine. Logics containing K4. part ii. *Journal of Symbolic Logic*, 50(3):619–651, 09 1985.

[GKWZ03] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*. Studies in Logic and the Foundations of Mathematics. Elsevier, 2003.

[Gli29] V. Glivenko. Sur quelques points de la logique de M. brouwer. *Bulletins de la classe des sciences*, 15:183–188, 1929.

[GoL93] R. Goldblatt. *Mathematics of modality*. CSLI Publications, 1993.

[GSS09] D. Gabbay, V. Shehtman, and D. Skvortsov. *Quantification in nonclassical logic*. Elsevier, 2009.

[Kom75] Y. Komori. The finite model property of the intermediate propositional logics on finite slices. *Journal of the Faculty of Science, the University of Tokyo*, 22(2):117–120, 1975.

[Kra99] M. Kracht. *Tools and techniques in modal logic*. Elsevier, 1999.
A. Kudinov and I. Shapirovsky. Partitioning kripke frames of finite height. *Izvestiya: Mathematics*, 81(3):592, 2017.

A. Kuznetsov. Some properties of the structure of varieties of pseudo-boolean algebras. In *Proceedings of the Xth USSR Algebraic Colloquium*, pages 255–256, 1971.

L. Maksimova. Modal logics of finite slices. *Algebra and Logic*, 14(3):304–319, 1975.

D. Makinson. Non-equivalent formulae in one variable in A strong omnitemporal modal logic. *Math. Log. Q.*, 27(7):111–112, 1981.

A. Malcev. *Algebraic systems*. Springer, 1973.

K. Matsumoto. Reduction theorem in Lewis’s sentential calculi. *Mathematica Japonicae*, 3:133–135, 1955.

V. Rybakov. A modal analog for glivenko’s theorem and its applications. *Notre Dame J. Formal Logic*, 33(2):244–248, 03 1992.

K. Segerberg. *An essay in classical modal logic*. Filosofska Studier, vol.13. Uppsala Universitet, 1971.

V. Shehtman. Rieger-nishimura lattices. *Doklady Mathematics*, 19:1014–1018, 1978.

V. Shehtman. *Applying Kripke models to the investigation of superintuitionistic and modal logics*. PhD thesis, Moscow State University, 1985. In Russian.

A. Simpson. *The proof theory and semantics of intuitionistic modal logic*. PhD thesis, University of Edinburgh, 1994.

I. Shapirovsky and V. Shehtman. Local tabularity without transitivity. In *Advances in Modal Logic*, volume 11, pages 520–534. College Publications, 2016.