Establishment of thermodynamic equilibrium in cosmological model with an arbitrary acceleration
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Abstract
It has been built a strict mathematical model of the thermodynamic equilibrium’s establishment in the originally nonequilibrium cosmological ultrarelativistic plasma for the arbitrary accelerated Universe under the assumption that scaling of interactions of elementary particles is restored at energies above the unitary limit. It is shown, that in case of Universe positive acceleration, thermodynamic equilibrium never can be completely restored.

1 Introduction
In 1986 a hypothesis that particles interaction scaling is restored at energies above the unitary limit [3] has been put forward by the author; it was based on the results of the analysis of axiomatic S-matrix theory and experimental data about extra-high energy particles’ cross-section of scattering. Author also proposed a formula for the asymptotic cross-section of scattering above the unitary limit which had scaling behavior and was composed of 3 fundamental constants - [1]. A restoration of the elementary particles scaling in range of an extra-high energies is equivalent to the field equations’ conformal invariance restoration in the short-wave, quasi-classical limit. The conformal invariance of the relativistic kinetic theory in the ultra-relativistic limit is also restored at that [2]. Basic regulations of theory of thermodynamic equilibrium’s restoration in ultrarelativistic cosmological plasma were formulated in [1]: based on these regulations the numerical model of this process at ultrarelativistic stage of the Universe expansion was built and main results of the theory were formulated. Developed in mentioned articles theory revealed a number of interesting peculiarities of the process of thermodynamic equilibrium restoration in cosmological plasma and allowed to impose constrains on the parameters of the initial distribution of particles. In particular, it was shown that particles possessing energies more than $10^{12} \div 10^{13}$ Gev in modern Universe can be relict ones carrying information about cosmological singularity. In articles [1], [7] –[9] it was formulated a diffused model and investigated an evolution model of extra-high energy particles’ spectrum in ultrarelativistic Universe. On the seminar “Gracos” (Yalchik, 2009) professor V.N.Melnikov put a question to the author, which corrections
to the Universe nonequilibrium models can bring the factor of Universe acceleration that was revealed in 1998 and became an acknowledged fact nowadays. An attempt to answer this question stretched for three years and unexpectedly brought to the significant generalization and the development of theory of Universe thermodynamic restoration as well as detection of series of rather interesting facts. This paper is devoted to that question’s answer, for which author is greatly thankful to professor V.N.Melnikov.

2 Common principles of energy-balance equation’s construction and solution

2.1 Matter model

As is known, Einstein equations in case of isotropic homogeneous cosmological model with zero three-dimensional curvature are reduced to the system of two ordinary first-order differential equations:

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\varepsilon; \] (1)

\[ \dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p(\varepsilon)) = 0. \] (2)

Then:

\[ \varepsilon = \varepsilon_p + \varepsilon_s; \quad p = p_p + p_s, \] (3)

where \( \varepsilon_p, p_p \) are energy density and pressure of the cosmological plasma, \( \varepsilon_s, p_s \) are energy density and pressure of various fundamental fields, probably, scalar ones, leading to the Universe acceleration.

Invariant acceleration of the Universe

\[ \Omega = \frac{a\ddot{a}}{\dot{a}^2} \] (4)

is associated with an effective barotropic factor of matter, \( \kappa \equiv p/\varepsilon \), by relation:

\[ \Omega = -\frac{1}{2}(1 + 3\kappa). \] (5)

Thereby, during the acceleration stage \( t > t_1 \):

\[ 0 < \Omega < 1, \Rightarrow -1 < \kappa < -\frac{1}{3}. \] (6)

Until this moment \( \kappa \in [1/3, -1/3] \). According to (1) - (2) scale factor and total energy density at given constant barotropic factor are changing by rule:

\[ a = a_1t^{2/3(\kappa + 1)}; \quad \varepsilon = \frac{1}{6\pi(\kappa + 1)^{2/3}t^2}, \quad \kappa + 1 \neq 0 \] (7)
Let us rewrite relations (7) in more convenient form using relation (5):

\[ a = a_1 t^{1/(1-\Omega)}; \quad \varepsilon = \frac{3}{8\pi(1 - \Omega)^{2/2}}, \quad \Omega < 1. \]  

(8)

Let us note that at any values of invariant acceleration \( \Omega = \text{Const} \in [-1, 1) \) energy density is proportional to \( t^{-2} \).

2.2 General assumptions of the model

Let us formulate general assumptions, which are laid in the basis of the presented below mathematical model of thermodynamic equilibrium’s restoration in the expanding Universe:

1°. An asymptotic character of elementary particles interaction in range of extra-high energies:

1°a. A restoration of elementary particles’ scaling of interactions at energies above the unitary limit

\[ \sigma|_{s > s_u} \sim \frac{1}{s}, \]  

(9)

where \( s \) is a first kinematic invariant – the relativistic square of a moment of two colliding particles, “a” and “b”, (see, e.g. [10]):

\[ s = (p_a + p_b)^2 \equiv g_{ik}(p^i_a + p^i_b)(p^k_a + p^k_b); \]

1°b. unification of all interparticle interactions on basis of fundamental constants \( G, h, c \) at energies above the unitary limit, which subject to (9) leads to formula of universal asymptotic cross-section of scattering (details see in [4]):

\[ \sigma_0(s) = \frac{8\pi}{s\Lambda(s)}, \]  

(10)

\[ \Lambda(s) = \ln^2 \left( 1 + \frac{s_0}{s} \right), \]  

(11)

\( s_0 = 4 \) is a squared total energy of two colliding Planck masses\(^2\) in the ordinary units.

2°. Minimality of fundamental fields’ coupling with a cosmological plasma. This automatically means that “energy conservation law” (2) is held separately for fundamental fields and plasma:

\[ \dot{\varepsilon}_s + 3\frac{\dot{a}}{a}(\varepsilon_s + p_s(\varepsilon_s)) = 0; \]  

(12)

\[ \dot{\varepsilon}_p + 3\frac{\dot{a}}{a}(\varepsilon_p + p_p(\varepsilon_p)) = 0. \]  

(13)

\(^1s_u \simeq m^2_u, \text{ where } m_u \text{ is a mass of the heaviest , “latest”, intermediate X-boson type particle participating in the 4-particle interactions}\)

\(^2\text{in the universal system of units, which is used in this article } G = h = c = 1; \text{ } s_0 = 4\hbar c^5/G\)
Ultrarelativistic state equation of cosmological plasma at the stage of expansion that is being researched:

\[ p_p = \frac{1}{3} \varepsilon_p, \quad (13) \Rightarrow \varepsilon_p a^4 = \text{Const}. \]

Ultrarelativistic start of the Universe:

\[ \lim_{t \to 0} x = \frac{1}{3}. \]

In the works quoted above [1, 5, 6] weak violation of thermodynamical equilibrium was supposed in the Universe, when the greater part of particles, \( n_{ne} \), is in a thermal equilibrium state, and only for a minor part of particles, \( n_{ne} \), the thermal equilibrium is violated. As thermodynamic equilibrium at early stages of the Universe can be broken only behind a unitary limit \( E(p) > E_u \), it leads to a following kind of function of distribution of particles [1, 5]:

\[
 f(p) \approx \begin{cases} 
 f_0(p) = \left[ \exp \left( -\mu + E_a(p) \right) + 1 \right]^{-1}, & p < p_u; \\
 \Delta f(p); f_0(p) \ll \Delta f(p) \ll 1, & p > p_u,
\end{cases}
\]

where \( \mu(t) = \mu_a(t) \) are chemical potentials and \( T(t) \) is the temperature of the equilibrium component of the plasma. Thus in the range \( E > E_u \rightarrow s > s_u \) one can observe an anomalously great number of particles as compared with the equilibrium one, but still it will be small as compared with the total number of equilibrium particles. However, thus energy nonequilibrium components can be more energy equilibrium components.

It allows to consider plasma, as the system consisting of two subsystems: equilibrium with some temperature \( T(t) \) and nonequilibrium, gradually losing energy at the expense of a transmission of energy equilibrium a component at collisions with equilibrium particles [1, 5]. Thus, in our model equilibrium a component it is warmed up by superthermal particles nonequilibrium components, that as a result leads to slower cooling of equilibrium plasma, than in standard model. This process is described by the power balance equation which the plasma component (13) is a direct consequence of the law of preservation of total energy) and will be formulated more low.

2.3 Energy balance of the cosmological plasma

From (13) subject to (14) right away follows:

\[ \varepsilon_p a^4 \equiv \tilde{\varepsilon}_p = \text{Const}, \]

where \( \tilde{\varepsilon}_p \) is a conformal energy density of the cosmological plasma;

\[ \varepsilon_p = \varepsilon_e + \varepsilon_{ne}, \]
\( \varepsilon_e \) is a energy density of the equilibrium components of plasma, and \( \varepsilon_{ne} \) a energy density of the nonequilibrium components of plasma. As both components of plasma, equilibrium and nonequilibrium, are ultrarelativistics, for them conditions are separately satisfied:

\[
p_e = \frac{1}{3} \varepsilon_e; \quad p_{ne} = \frac{1}{3} \varepsilon_{ne}.
\]

(19)

Let us determine this constant, setting according to (8) on the initial ultrarelativistic stage of expansion:

\[ a(t)|_{t \to 0} = \sqrt{t}. \]

(20)

Then for the conformal energy density of plasma we obtain, suggesting that cosmological plasma if the single ultrarelativistic component of matter, —

\[ \tilde{\varepsilon}_p = \frac{3}{32\pi}. \]

(21)

Let us then introduce temperature \( T_0(t) \) of the cosmological plasma in the ideal Universe, in which at this point in cosmological time \( t \) the whole plasma is locally balanced. Thus, plasma’s energy density is equal to

\[ \varepsilon^0_p = \frac{N_0 \pi^2}{15} T^4_0(t), \]

(22)

where \( N_0 = N^0_B + 1/2 N^0_F \) is an effective number of equilibrium particles’ types (bosons and fermions) in plasma with temperature \( T_0 \).

Hence subject to (21) we obtain the evolution law of plasma’s temperature in the equilibrium Universe:

\[ T_0(t) = \frac{1}{a(t)} \left( \frac{45}{32\pi^3 N_0} \right)^{\frac{1}{4}}. \]

(23)

With respect to \( N_0 \) — effective number of types of thermodynamically equilibrium particles, we suggest that \( N_0(t) \) is a slowly changing function of the cosmological time:

\[ N_0 \dot{t} \ll 1. \]

(24)

Let then \( T(t) \) is a real temperature of equilibrium component of cosmological plasma, and \( \Delta f_a(p,t) \) is a distribution function of “a”-sort nonequilibrium plasma particles. Then we find energy densities of the equilibrium, \( \varepsilon_e \), and nonequilibrium, \( \varepsilon_{ne} \), components:

\[ \varepsilon_e = \frac{N \pi^2}{15} T^4(t); \]

(25)

\[ \varepsilon_{ne} = \frac{1}{2\pi^2} \sum_a (2S + 1) \int_0^\infty p^3 \Delta f_a(p,t) dp. \]

(26)
where $N(t) = N_B + 1/2N_F$ is an effective number of equilibrium particles’ (bosons and fermions) types in plasma with temperature $T(t)$. Expressing further with a use of (23) the scale factor via temperature $T_0(t)$ and introducing new dimensionless conformal momentum variable $\tilde{\rho}$:

$$p = \left(\frac{45}{32\pi^3}\right)^{\frac{1}{4}} \frac{\tilde{\rho}}{a(t)} = T_0(t)N_0^{\frac{1}{4}} \tilde{\rho},$$  \hspace{1cm} (27)

we obtain for (26):

$$\tilde{\epsilon}_{ne} = \frac{45}{64\pi^3} \sum_{a} (2S + 1) \int_{0}^{\infty} \tilde{\rho}^3 \Delta f_a(\tilde{\rho}, t) d\tilde{\rho}. \hspace{1cm} (28)$$

Next, from (23) and (25) we obtain for the conformal energy density of the equilibrium component of plasma:

$$\tilde{\epsilon}_e = \frac{3}{32\pi} y^4,$$  \hspace{1cm} (29)

where the dimensionless function is introduced, $y(t)$ is a – relative temperature [5]:

$$y(t) = \frac{T(t)}{T_0(t)} \leq 1. \hspace{1cm} (30)$$

From (29) it is possible to obtain the following relation:

$$\sigma(t) \equiv y^4(t) \equiv \frac{\tilde{\epsilon}_e}{\tilde{\epsilon}_p} \equiv \frac{\tilde{\epsilon}_e}{\tilde{\epsilon}_e + \tilde{\epsilon}_{ne}}. \hspace{1cm} (31)$$

Thus, cosmological plasma’s energy conservation law (21) with a use of (26) and (29) can be rewritten in form:

$$y^4 + \frac{15}{2\pi^4} \sum_{a} (2S + 1) \int_{0}^{\infty} \tilde{\rho}^3 \Delta f_a(\tilde{\rho}, t) d\tilde{\rho} = 1. \hspace{1cm} (32)$$

Relation (32) is called plasma energy-balance equation. It is obtained with a usage of three model suggestions — $2^o$, $3^o$, $5^o$. Let us note that in previous articles this basic relation of mathematical model of thermodynamical equilibrium’s restoration has been obtained under more special suggestions. At given dependency of nonequilibrium particles’ distribution function on the temperature of plasma’s equilibrium component and cosmological time the energy-balance equation becomes a nonlinear integral equation relative to equilibrium component’s temperature. Therefore, to obtain this equation in the explicit form it is necessary to solve the kinetic equation for the nonequilibrium particles.
3 Kinetic equation for nonequilibrium particles

3.1 Solution of kinetic equation

As was shown in [5], kinetic equation for superthermal particles in ultrarelativistic plasma has a form:

\[ p \frac{\partial \Delta f_a}{\partial t} = -\frac{4\pi N}{3} \frac{T^2(t)}{\Lambda(pT/2)} \Delta f_a, \quad (33) \]

where \( \Delta f_a = \Delta f_a(\tilde{p}, t) \), \( \Lambda = \Lambda(s) \). Using here relation (27), we reduce (33) to form:

\[ \frac{\partial \Delta f_a}{\partial t} = -\frac{8\pi N}{3\tilde{p}\Lambda(\frac{1}{2} \tilde{p} T/2)} \left( \frac{2\pi^3}{45} \right)^{1/4} T^2(t)a(t) \Delta f_a. \quad (34) \]

Solving (34), obtain:

\[ \Delta f_a(t, \tilde{p}) = \Delta f^0_a(\tilde{p}) \times \exp \left[ -\frac{8\pi}{3\tilde{p}} \left( \frac{2\pi^3}{45} \right)^{1/4} \int_0^t NaT^2 dt \right] \Lambda(\frac{1}{2} \tilde{p} T/2)N^{1/4}, \quad (35) \]

where

\[ \Delta f^0_a(\tilde{p}) \equiv \Delta f_a(0, \tilde{p}). \quad (36) \]

3.2 Transition to dimensionless normalized variables

Let us introduce an average conformal energy of ultrarelativistic particles’ nonequilibrium component at the initial point in time, \( \langle \tilde{p} \rangle_0 \), –

\[ \langle \tilde{p} \rangle_0 = \frac{\tilde{E}(0)}{\tilde{n}(0)} \equiv \sum_a (2S + 1) \int_0^\infty \Delta f^0_a(\tilde{p})\tilde{p}^3 d\tilde{p} \]

\[ \sum_a (2S + 1) \int_0^\infty \Delta f^0_a(\tilde{p})\tilde{p}^2 d\tilde{p} \quad (37) \]

and a dimensionless normalized momentum variable, \( \rho \), –

\[ \rho \equiv \frac{\tilde{p}}{\langle \tilde{p} \rangle_0}, \quad (38) \]

so that

\[ \langle \rho \rangle_0 = \frac{\tilde{E}(0)}{\langle \tilde{p} \rangle_0 \tilde{n}(0)} \equiv 1 \Rightarrow \]

\[ \langle \rho \rangle_0 = \frac{\sum_a (2S + 1) \int_0^\infty \Delta f^0_a(\rho)\rho^3 d\rho}{\sum_a (2S + 1) \int_0^\infty \Delta f^0_a(\rho)\rho^2 d\rho} = 1. \quad (39) \]
According to nonequilibrium plasma’s mathematical model particles’ average energy in the initial nonequilibrium distribution must be greater and even much greater than particles’ thermal energy, thereby according to (27), (37) in the model being considered:

\[ \langle \tilde{p} \rangle_0 \gg 1. \]  

(40)

Value \( \langle \tilde{p} \rangle_0 \) is in fact an independent parameter of the model that is under consideration here and its physical meaning is a relation of an average energy of nonequilibrium distribution particles to plasma temperature in the equilibrium Universe in the initial point in time. As opposite to the conformal momentum variable \( \tilde{p} \) the average value of the dimensionless conformal momentum variable \( \rho \) in the initial distribution is identically equal to 1.

Let us transform an expression in the exponent (35), by transition to dimensionless variables \( y, \rho \). Taking into account a weak dependency of the logarithmic factor \( \Lambda \) on its arguments and the decreasing character of the integrand in (35), we receive the following evaluation of the logarithmic factor:

\[ \Lambda \left( \frac{1}{2} \tilde{p} T_0 T N_0^{1/4} \right) \simeq \Lambda(\langle \tilde{p} \rangle_0 T_0^2) \equiv \Lambda_0(t). \]  

(41)

Thus, with a logarithmic accuracy we represent a solution (35) in compact form:

\[ \Delta f_a(t, \rho) = \Delta f_a^0(\rho) \exp \left( -\frac{2}{\rho} \int_0^t \xi y^2 \frac{2}{a} dt \right), \]  

(42)

where it is introduced the denotation:

\[ \xi = \xi(t) = \left( \frac{5\pi}{18} \right)^{1/4} \frac{N}{N_0^{1/2} \Lambda_0(t)} \approx 0.967 \frac{N}{N_0^{1/2} \Lambda_0(t)}. \]  

(43)

Introducing then a new dimensionless time variable, \( \tau \),

\[ \tau = 2 \int_0^t \frac{\xi}{a} dt, \]  

(44)

so that:

\[ \frac{d\tau}{dt} = \frac{2\xi}{a} > 0, \]  

(45)

and a new dimensionless function, \( Z(\tau) \),

\[ Z(\tau) = \int_0^\tau y^2(\tau) d\tau, \]  

(46)

\(^5\)These values themselves are infinite but their ratio is finite.
we reduce the kinetic equation’s solution to form:

\[ \Delta f_a(\tau, \rho) = \Delta f^0_a(\rho) \cdot e^{-\frac{Z(\tau)}{\rho}}. \] (47)

Let us investigate in details the equation of coupling between dimensionless variable \( \tau \) and cosmological time \( t \). Setting in a power dependence of the scale factor \( a(t) \) on the cosmological time and taking into account a weak dependency of \( \xi \) factor on time, we obtain:

\[ a \sim t^\alpha, \quad (\alpha \neq 1, 0) \Rightarrow \tau \sim t^{1-\alpha}; \quad \alpha = 1 \Rightarrow \tau \sim \ln t. \] (48)

Hence it follows that at \( \alpha \leq 1 \rightarrow \tau(\infty) = \infty \), and at \( \alpha < 1 \rightarrow \tau(\infty) = \tau_\infty < \infty \).

Comparing relation (48) to relations (5) – (8), we find:

\[ \kappa \geq -\frac{1}{3} (\Omega \leq 0) \Rightarrow \tau(\infty) = +\infty; \] (49)

\[ \kappa < -\frac{1}{3} (\Omega > 0) \Rightarrow \tau(\infty) = \tau_\infty < +\infty. \] (50)

Since the distribution function of cosmological plasma’s nonequilibrium component depends on time just by means of monotonously increasing function \( Z(\tau) \) of the dimensionless time variable, relations (49) – (50) mean that in ultrarelativistic cosmological plasma in the Universe with a negative acceleration thermodynamical equilibrium is reached asymptotically while in the accelerating Universe thermodynamical equilibrium is never strictly reached.

### 3.3 Conformal energy density of the nonequilibrium component

Substituting the solution of the kinetic equation in form (47) in the expression for nonequilibrium particles’ conformal energy density, we obtain

\[ \tilde{\varepsilon}_{ne} = \frac{45}{64\pi^5} \sum_a (2S + 1) \int_0^\infty r^3 \Delta f^0_a(r)e^{-\frac{Z(\tau)}{r}}. \] (51)

Let us perform an identity substitution on the given expression, accounting that subject to the definition and energy-balance equation (52):

\[ \tilde{\varepsilon}_{ne}^0 = (1 - \sigma_0) \frac{3}{32\pi}; \] (52)

\[ \tilde{\varepsilon}_{ne} \equiv \frac{\tilde{\varepsilon}_{ne}}{\tilde{\varepsilon}_{ne}^0} = (1 - \sigma_0)\Phi(Z) \frac{3}{32\pi}, \] (53)

where we introduced new dimensionless function \( \Phi(Z) \) with a use of transformation to the dimensionless momentum variable \( \rho \) (53):

\[ \Phi(Z) \equiv \frac{\sum_a (2S + 1) \int_0^\infty r^3 \Delta f^0_a(r)e^{-\frac{Z(\tau)}{r}}}{\sum_a (2S + 1) \int_0^\infty r^3 \Delta f^0_a(r)}. \] (54)
3.4 Energy-balance equation solution and analysis

As a result of definition (46) function \( Z(\tau) \) satisfies following conditions:

\[
Z'(\tau) = y^2(\tau) \Rightarrow Z'' = \sigma(\tau); \quad (55)
\]

\[
Z(0) = 0; \quad Z'(0) = y^2(0) = \sqrt{\sigma_0}, \quad (56)
\]

where

\[
Z' \equiv \frac{dZ}{d\tau} > 0. \quad (57)
\]

Thus, subject to (53) – (55) energy-balance equation (32) can be rewritten in form of the differential equation relative to function \( Z(\tau) \):

\[
y^2 + (1 - \sigma_0) \Phi(Z) = 1 \Rightarrow
\]

\[
Z'' + (1 - \sigma_0) \Phi(Z) = 1, \quad (58)
\]

solving which subject to relations (56) – (57), we find a formal solution in the implicit form:

\[
\int_0^Z \frac{du}{\sqrt{1 - (1 - \sigma_0) \Phi(u)}} = \tau. \quad (59)
\]

According to definition \( \Phi(Z) \) is a nonnegative one:

\[
\Phi(Z) > 0, \quad (Z \in [0, +\infty)), \quad (60)
\]

and

\[
\Phi(0) = 1; \quad \lim_{Z \to +\infty} \Phi(Z) = 0. \quad (61)
\]

Calculating the first and the second derivatives of function \( \Phi(Z) \), by \( Z \) and differentiating the relation (54) by \( Z \), obtain:

\[
\Phi(Z)' < 0, \quad (Z \in [0, +\infty)); \quad (62)
\]

\[
\Phi(Z)'' > 0, \quad (Z \in [0, +\infty)). \quad (63)
\]

In consequence of (62) function \( \Phi(Z) \) is strictly monotonously decreasing one but then as a result of the relations (61) this function is limited on the interval:

\[
\Phi(Z) \in [0, 1]; \quad (Z \in [0, +\infty)), \quad (64)
\]

and function \( \Phi(Z) \) has a concave graph. As a result of these properties of function \( \Phi(Z) \) equation \( \Phi(Z) = \Phi_0 \) within the limit being researched, always has a single and just only one solution \( Z = Z_0 \), i.e. mapping \( Y = \Phi(Z) \) on the set of nonnegative numbers is bijective.

Next, from (55) it follows that function \( Z(\tau) \) monotonously increases over the interval \( \tau \in [0, \tau_\infty] \). Differentiating relation (58) by \( \tau \) like a composite function, we find:

\[
Z'[2Z'' + (1 - \sigma_0) \Phi'_Z] = 0. \quad (65)
\]
Hence as a result of $Z'$ positivity \cite{58} let us find the second derivative:

$$Z'' = -\frac{1}{2}(1 - \sigma_0)\Phi' Z.$$  \hfill (66)

Therefore in consequence of \cite{62} and \cite{30} - \cite{31} we obtain from (66):

$$Z'' > 0,$$  \hfill (67)

i.e. graph of $Z(\tau)$ function is also concave. Then, differentiating (IV.53), subject to (67) find:

$$y' > 0,$$  \hfill (68)

— i.e. function $y(\tau)$ (and function $\sigma(\tau)$ together with it) is a monotonously increasing one. From the other hand it is limited from below by the initial value $y_0 (\sigma_0)$, and from above by value 1:

$$y' > 0, y \in [y_0, 1); \quad \sigma' > 0, \sigma \in [\sigma_0, 1).$$ \hfill (69)

Listed properties of functions $y(\tau)$, $Z(\tau)$ and $\Phi(Z)$ assert the bijectivity of chain of mappings $\tau \leftrightarrow y$, $y \leftrightarrow Z$, $Z \leftrightarrow \Phi$. Finally, each value $\Phi$ has a one and only one corresponding value $Z$ and one and only one value $\tau$: $\tau \leftrightarrow \Phi$. To close this chain it is enough to determine functions $y(\tau)$ and $Z(\tau)$ coupling by means of the energy-balance equation \cite{69}:

$$y = [1 - (1 - \sigma_0)\Phi(Z)]^{1/4}.$$ \hfill (70)

Equations \cite{59} and \cite{70} are the parametric solution of the energy-balance equation \cite{58}, and above mentioned properties of functions $\Phi(Z)$ and $Z(\tau)$ assert the uniqueness of the solution. According to \cite{54} function $\Phi(Z)$ is fully determined by the initial distribution of nonequilibrium particles $\Delta f_{\rho}^{\text{eq}}$. Therefore from the mathematical point of view the problem of thermodynamical equilibrium recreation in Universe with arbitrary acceleration is completely solved. Concrete models are determined by dark matter model and model of initial nonequilibrium distribution of particles.

Let us differentiate now the relation (66) by $\tau$ and take account of the association (55) between $y(\tau)$ and $Z(\tau)$:

$$Z'''' = -\frac{1}{2}(1 - \sigma_0)\Phi'' ZZ'.$$

$$\Rightarrow y'' y = -y^2 - \frac{1}{4}(1 - \sigma_0)\Phi'' ZZ y^2.$$  

Thus, as a result of (63):

$$y'' < 0,$$ \hfill (71)

— i.e. graph of function $y(\tau)$, and $\sigma(\tau)$ graph together with it, are concave. Then since $\Phi_Z(Z \rightarrow \infty) = 0$, from (66) it follows:

$$\lim_{\tau \rightarrow \infty} y'(\tau) = 0 \Rightarrow \lim_{\tau \rightarrow \infty} \sigma'(\tau) = 0,$$ \hfill (72)
— i.e., value $\sigma = 1$ is reached asymptotically at $\tau \to \infty$. This allows to draw a qualitative graph of functions $y(\tau)$ (Fig. 1). Finiteness of the dimensionless time $\tau_\infty$ conducts to the establishment of the limiting value of function $y(\tau)$:

$$y(\tau_\infty) = y_\infty < 1 \Rightarrow \lim_{t \to \infty} y(t) = y_\infty < 1.$$  \hspace{1cm} (73)

In consequence of that the certain part of cosmological plasma energy is forever conserved in the nonequilibrium superthermal component:

$$\lim_{t \to \infty} \frac{\varepsilon_{ne}(t)}{\varepsilon_p(t)} = 1 - \sigma_\infty = \begin{cases} = 0, & \tau_\infty = \infty \\ > 0, & \tau_\infty < \infty \end{cases}.$$  \hspace{1cm} (74)

According to (49) – (50) it is possible just for the accelerated expanding Universe.

4 Exact model of transition from the ultrarelativistic stage to the inflationary one

Let us consider a simple model of matter consisting of 2 components – minimally coupled massive scalar field (cosmological member) with a state equation:

$$p_s = -\varepsilon_s,$$  \hspace{1cm} (75)

and a ultrarelativistic plasma with a state equation [14]. Then an overall barotropic factor and invariant acceleration can be written in form:

$$\kappa(t) = \frac{1}{3} \frac{1 - 3\delta}{1 + \delta}, \quad \Omega(t) = -\frac{1 - \delta}{1 + \delta},$$  \hspace{1cm} (76)

where

$$\delta = \delta(t) = \frac{\varepsilon_s}{\varepsilon_p}.$$  \hspace{1cm} (77)
Thus at $\delta = \text{Const}$ formulas (7) can be written in the following convenient form:

$$a = a_1 (1 + \delta)^{1/2}; \quad \varepsilon = \frac{3}{32\pi} \frac{(1 + \delta)^2}{t^2}, \quad \Omega < 1.$$  \hfill (78)

Energy conservation laws (12) – (13) take form:

$$\varepsilon_s = \text{Const} = \frac{3\Xi^2}{8\pi}; \quad \varepsilon_\rho a^4 \equiv \tilde{\varepsilon}_\rho = \text{Const} \simeq \frac{3}{32\pi}.$$  \hfill (79)

(80)

Substituting (79)-(80) into equation (1) and integrating it, we obtain:

$$a(t) = \frac{1}{\sqrt{2}} \left[ \left( t_0 + \sqrt{t_0^2 + b^2} \right) e^{(t-t_0)/2\Xi} - \frac{b^2}{t_0 + \sqrt{t_0^2 + b^2}} e^{-(t-t_0)/2\Xi} \right]^{1/2},$$  \hfill (81)

where:

$$b^2 = \frac{3}{32\pi \Xi^2}.$$  \hfill (82)

Hence we have, in particular, for the scale factor at $t_0 = 0$:

$$a(t) = \frac{1}{\Xi} \sqrt{\frac{3}{32\pi} \text{sh} \frac{t}{2\Xi}}.$$  \hfill (83)

Calculating according to (77), (79), (80) and (83) relation $\delta$, we find:

$$\delta(t) = \left( \frac{3}{16\pi \Xi} \text{sh} \frac{t}{2\Xi} \right)^2.$$  \hfill (84)

Next according to (77) it is possible to calculate an effective barotropic factor and an invariant acceleration (see Fig. 2).

**Fig. 2.** Evolution of the effective barotropic factor $\kappa(t)$ (thin line) and invariant acceleration $\Omega(t)$ (heavy line) relative to the exact solution (83) at $\Xi = 1$. Asymptotes $-1; \ 1/3; \ 1$ are denoted by the dotted lines.
Following figure shows that by means of parameter $\Xi$ it is easy to control the time of transition to the inflationary acceleration regime $\kappa \rightarrow -1$. Let us recall that cosmological time $t$ is measured in Planck units.

Thus according to (44) let us determine new dimensionless time variable, $\tau$:

$$\tau = \frac{2\Xi \langle \xi \rangle}{\langle \rho \rangle_0} F(\varphi, 1/\sqrt{2}),$$

(85)

where:

$$\varphi = \arccos \frac{1 - \text{sh} \, t / 2\Xi}{1 + \text{sh} \, t / 2\Xi};$$

(86)

$F(\varphi, k)$ is an elliptic integral of the first type (see e.g. [11]):

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}; \quad (k^2 < 1).$$

(87)

Thus:

$$\frac{d\tau}{dt} = \frac{1}{2\Xi} \frac{1}{\text{sh} \, \varphi} \frac{\text{ch} \, \varphi}{1 + \text{sh} \, \varphi} > 0; \quad \tau \in [0, \tau_\infty),$$

(88)

where

$$\tau_\infty = \lim_{t \rightarrow +\infty} \tau(t) = \frac{2\Xi \langle \xi \rangle}{\langle \rho \rangle_0} F(1, 1/\sqrt{2}).$$

(89)

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Fig. 3. Evolution of the effective barotropic factor $\kappa(t)$ at $\Xi = 1; 10; 100; 1000$ (left to right) relative to the exact solution $\mathbf{[11]}$. Dotted lines denote asymptotes $\kappa = -1; \kappa = 1/3$. Along the abscissa axis values log$_{10}$ $t$ are put.

$F(1, 1/\sqrt{2}) \approx 1.083216773.$

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