An upper bound on the number of Killing-Yano tensors

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Abstract. We discuss a simple method for computing an upper bound on the number of Killing-Yano tensors for a given metric. The method is applied to some metrics in four and five dimensions.

1. Introduction
Spacetime symmetry has played an important role in general relativity. When our spacetime admits isometry described by Killing vector fields, if one calculates a physical quantity at a point of spacetime one can obtain the quantity at another point of spacetime without solving equations of motion. Similarly, Killing-Yano symmetry described by Killing-Yano tensors [1] is called hidden symmetry of spacetimes because due to the symmetry spacetimes can possess remarkable properties, e.g., separability of Hamilton-Jacobi for geodesics, Klein-Gordon and Dirac equations. However, it is not always easy to find Killing-Yano tensors for a given metric as we have to partial differential equations obtained from the Killing equations. In this article, we discuss a simple method to find Killing-Yano tensors, as well as Killing vector fields, for a given metric. The detailed calculation can be found in [2].

2. Killing vector fields
Let us consider Killing vector fields on a spacetime \((M, g_{\mu\nu})\) in \(n\) dimensions. The equation defining a Killing vector field \(\xi^\mu\) is given by
\[
\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0,
\]
where \(\nabla\) is the Levi-Civita connection. Introducing a 2-form \(L_{\mu\nu} = L_{[\mu\nu]}\), the equation can be decomposed into two equations
\[
\nabla_{\mu}\xi_{\nu} = L_{\mu\nu}, \tag{2}
\]
\[
L_{\mu\nu} = \nabla_{[\mu} \xi_{\nu]} \tag{3}.
\]
Covariantly differentiating eq. (1), we also obtain the equation
\[
\nabla_{\mu} L_{\nu\rho} = -R_{\nu\rho\mu} \sigma \xi_{\sigma} \tag{4}.
\]
The point here is that if we start with eqs. (2) and (4), we obtain eq. (3). Since eqs. (2) and (4) are packaged into the form
\[
D_{\mu} \left( \begin{array}{c} \xi_{\nu} \\ L_{\nu\rho} \end{array} \right) \equiv \left( \begin{array}{c} \nabla_{\mu}\xi_{\nu} - L_{\mu\nu} \\ \nabla_{\mu} L_{\nu\rho} + R_{\nu\rho\mu} \sigma \xi_{\sigma} \end{array} \right) = 0, \tag{5}
\]
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we find that eq. (5) is equivalent to the Killing equation (1). Eq. (5) implies that one can introduce a connection $\mathcal{D}$ on the vector bundle $E^1 \equiv T^*M \oplus \Lambda^2 T^*M$ whose parallel sections $\xi_A = (\xi_\mu, L_{\mu\nu})$ are one-to-one corresponding to Killing vector fields,

$$\mathcal{D}_\mu \xi_A = 0. \quad (6)$$

Since the number of parallel sections is bound by the rank of $E^1$, the maximum number of Killing vector fields is given by $n(n+1)/2$.

Calculating the curvature of the Killing connection, we obtain some conditions for the parallel sections. Since we have eq. (6), the parallel sections $\xi_A$ satisfy the curvature condition

$$R_{\mu\nu A}^B \xi_B \equiv (\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) \xi_A = 0. \quad (7)$$

Since parallel sections satisfy the curvature condition (7), the number of the parallel sections is bound by the number of solutions of the curvature condition. The condition is no longer partial differential equations but linear algebraic equations for $\xi_A$, which enables us to solve the curvature conditions. Thus, we can easily compute an upper bound on the number of Killing vector fields for any metric.

3. Killing-Yano tensors

Let $(M, g)$ be an $n$-dimensional spacetime and $\nabla$ be the Levi-Civita connection. We work in a local orthonormal frame of $TM$ denoted by $\{X_a\}$ and its dual frame of $T^*M$ denoted by $\{e^a\}$. They satisfy $X_a \cdot e^b = \delta^b_a$ where $\cdot$ is the inner product. The Latin indices $a, b, \ldots$ range from 1 to $n$.

A rank-$p$ KY tensor $k$ is defined as a $p$-form satisfying the equation

$$\nabla_X k = \frac{1}{p+1} X \cdot dk, \quad (8)$$

for any vector field $X$. Covariantly differentiating (8), we obtain

$$\nabla_X (dk) = \frac{p+1}{p} R^+(X) k, \quad (9)$$

where

$$R^+(X) \equiv e^a \wedge R(X, X_a), \quad (10)$$

with the Riemann curvature

$$R(X, Y) \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$ 

Combining eqs. (8) and (9), one can introduce a connection $\mathcal{D}$ on the vector bundle $E^p(M) \equiv \Lambda^p T^*M \oplus \Lambda^{p+1} T^*M$ [3],

$$\mathcal{D}_X \begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \nabla_X - \left( \begin{array}{ccc} 0 & 0 \\ \frac{p+1}{p} & \frac{1}{p+1} \end{array} \right) \frac{1}{p+1} X \cdot \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}, \quad (12)$$

where $\omega$ is a $p$-form and $\eta$ is a $(p+1)$-form. If a section $\hat{\omega} = (\omega, \eta)$ of $E^p(M)$ is given by a KY $p$-form $\omega = k$ and its exterior derivative $\eta = dk$, then it satisfies the parallel equation

$$\mathcal{D}_X \hat{\omega} = 0. \quad (13)$$
Conversely, if $\tilde{\omega} = (\omega, \eta)$ is a parallel section of $E^p(M)$, $\omega$ is a KY p-form and $\eta$ is its exterior derivative, $\eta = d\omega$. It follows that KY p-forms on $M$ are in one-to-one correspondence with parallel sections of $E^p(M)$. Hence, the maximum number of KY p-forms is bound by the rank of $E^p(M)$ [3], which is given by

$$\text{rank } E^p(M) = \binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}. \quad (14)$$

The equality is attained if a spacetime is maximally symmetric. Note that when we take $p = 1$, eqs. (8) and (9) are equivalent to eqs. (2) and (3). Eqs. (12) and (13) correspond to eqs. (5) and (6), respectively. The maximum number (14) becomes $n(n+1)/2$ for $p = 1$.

We calculate the curvature of the Killing connection (12) by

$$R(X, Y) \equiv \mathcal{D}_X \mathcal{D}_Y - \mathcal{D}_Y \mathcal{D}_X - \mathcal{D}_{[X,Y]}, \quad (15)$$

on the vector bundle $E^p(M)$. Then, the Killing curvature is given by

$$R(X, Y) = \begin{pmatrix} N_{11}(X, Y) & 0 \\ N_{21}(X, Y) & N_{22}(X, Y) \end{pmatrix}, \quad (16)$$

The entries are given by

$$N_{11}(X, Y) = R(X, Y) + \frac{1}{p} \left\{ X \rhd R^+(Y) - Y \rhd R^+(X) \right\},$$

$$N_{21}(X, Y) = -\frac{p+1}{p} \left\{ (\nabla_X R)^+(Y) - (\nabla_Y R)^+(X) \right\}, \quad (17)$$

$$N_{22}(X, Y) = R(X, Y) + \frac{1}{p} \left\{ R^+(X)(Y \rhd \bullet) - R^+(Y)(X \rhd \bullet) \right\},$$

where

$$(\nabla_X R)^+(Y) = e^a \wedge (\nabla_X R)(Y, X_a). \quad (18)$$
Table 2. The numbers of rank-p KY tensors for Myers-Perry, Emparan-Reall and Kerr string metrics in five dimensions.

|                | dim $K^p(M)$ |
|----------------|--------------|
|                | $p = 1$      | $p = 2$ | $p = 3$ | $p = 4$ |
| Maximally symmetric space | 15         | 20      | 15      | 6       |
| Myers-Perry    | 3           | 0       | 1       | 0       |
| Emparan-Reall  | 3           | 0       | 0       | 0       |
| Kerr string    | 3           | 1       | 0       | 1       |

4. Applications

By use of the method, we can compute the upper bounds on the number of Killing-Yano tensors (including Killing vector fields as rank-1 Killing-Yano tensors) for various metrics. The point is that since the curvature conditions are linear algebraic equations, one can easily solve them for any metric. Another point is that the method can be applied to any metric in any dimensions written in any coordinates. Furthermore, by use of the solutions of the curvature conditions, it is possible to integrate the Killing equations. As a result, we can obtain the precise number of Killing-Yano tensors for a given metric. The results for various metrics in four and five dimensions are listed in Tables 1 & 2.

5. Summary

This article has illustrated an idea for counting the number of Killing-Yano tensors. Using the fact that rank-p Killing-Yano tensors are in one-to-one correspondence with parallel sections of the vector bundle $E^p \equiv \Lambda^p T^* M \oplus \Lambda^{p+1} T^* M$, we have seen that the maximum number of rank-p Killing-Yano tensors is given by the rank of $E^p$. We have also obtained curvature conditions. Using the curvature conditions, we have actually computed the precise numbers of Killing-Yano tensors for various metrics in four and five dimensions.

One possible extension is to see curvature conditions for conformal Killing-Yano tensors. It was shown [3] that one can introduce a Killing connection on the vector bundle $E^p(M) \equiv \Lambda^p T^* M \oplus \Lambda^{p+1} T^* M \oplus \Lambda^{p-1} T^* M \oplus \Lambda^p T^* M$ whose parallel sections are one-to-one corresponding to rank-p conformal Killing-Yano tensors. Similar to this article, one could calculate the curvature of the Killing connection.

Finally, I would like to draw the reader’s attention to a package of the computational software, Mathematica, which automatically solves the curvature conditions and returns the upper bound on the number of Killing-Yano tensors for a given metric [4].

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References

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