Bilinear Equation and Additional Symmetries for an Extension of the Kadomtsev–Petviashvili Hierarchy

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Abstract

An extension of the Kadomtsev–Petviashvili (KP) hierarchy defined via scalar pseudo-differential operators was studied in Szablikowski and Blaszak (J. Math. Phys. 49(8), 082701, 20, 2008) and Wu and Zhou (J. Geom. Phys. 106, 327–341, 2016). In this paper, we represent the extended KP hierarchy into the form of bilinear equation of (adjoint) Baker–Akhiezer functions, and construct its additional symmetries. As a byproduct, we derive the Virasoro symmetries for the constrained KP hierarchies.

Keywords Kadomtsev–Petviashvili hierarchy · Baker–Akhiezer function · Additional symmetry

Mathematics Subject Classification (2010) 37K10 · 37K30

1 Introduction

As a fundamental model in the theory of integrable systems, the Kadomtsev–Petviashvili (KP) hierarchy can be defined as follows. Let

$$L_{KP} = \partial + v_1 \partial^{-1} + v_2 \partial^{-2} + \ldots, \quad \partial = \frac{d}{dx}, \quad (1.1)$$

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be a pseudo-differential operator whose coefficients $\psi_i$ are scalar unknown functions of the spatial coordinate $x$, then the KP hierarchy is composed by the following evolutionary equations of $\psi_i$ as

$$\frac{\partial L_{KP}}{\partial t_k} = \left[(L_{KP}^k)^{+}, L_{KP}\right], \quad k = 1, 2, 3, \ldots . \quad (1.2)$$

Here and below the subscript “+” of a pseudo-differential operator means to take its purely differential part, while the subscript “-” means to take its negative part. Suppose that the equations (1.2) are imposed with the constraint $(L_{KP}^n)^{-} = 0$ with some integer $n \geq 2$, then they form the Gelfand–Dickey (or the $(n - 1)$-th Korteweg–de Vries) hierarchy.

It is known that the KP hierarchy (1.2) possesses a series of bi-Hamiltonian structures derived by the $R$-matrix formalism [15], and that it can be represented into the form of bilinear equation of (adjoint) Baker–Akhiezer functions or of a tau function [8]. Such bi-Hamiltonian structures and the bilinear equation can be reduced to that of the Gelfand–Dickey hierarchies. What is more, for the KP hierarchy there is a class of non-isospectral symmetries named as the additional symmetries that can be constructed via the so-called Orlov–Schulman operators [14]. The flows of such additional symmetries commute with all the time flows $\partial/\partial t_k$ but do not commute between themselves; instead, they generate the $\mathfrak{W}_{1+\infty}$ algebra. As an application of the additional symmetries for the KP hierarchy, part of these symmetries can be reduced to the Virasoro symmetries for its subhierarchies such like the Gelfand–Dickey hierarchies. Note that Virasoro symmetries reveal crucial properties of a large amount of integrable hierarchies including the Gelfand–Dickey hierarchies and the Drinfeld–Sokolov hierarchies, see e.g. [1, 9, 10, 19] and references therein.

In the definition of the KP hierarchy (1.2), it is used pseudo-differential operators with only finitely many positive powers in $\partial$. The notion of pseudo-differential operator was generalized in [13] to be over a certain graded differential algebra $\mathcal{A}$ such that these operators may contain infinitely many positive powers in $\partial$ (see Section 2 below for details). By using these operators, in [20] Zhou and one of the authors of the present paper considered an integrable hierarchy, which can be viewed as a subhierarchy of the dispersionful analogue [16] of the universal Whitham hierarchy. More exactly, let

$$P = \partial + \sum_{i \geq 1} u_i (\partial - \varphi)^{-i}, \quad \hat{P} = (\partial - \varphi)^{-1} \hat{u}_{-1} + \sum_{i \geq 0} \hat{u}_i (\partial - \varphi)^i \quad (1.3)$$

with $u_i$, $\hat{u}_i$ and $\varphi$ being certain unknown functions belong to $\mathcal{A}$, then the following evolutionary equations are well defined:

$$\frac{\partial}{\partial t_k} (P, \hat{P}) = \left((P^k)^{+}, P, [((P^k)^{+}, \hat{P})\right], \quad \frac{\partial}{\partial t_k} (P, \hat{P}) = \left([-(\hat{P}^k)^{-}, P, [-(\hat{P}^k)^{-}, \hat{P})\right), \quad (1.4)$$

where $k = 1, 2, 3, \ldots$. These evolutionary equations compose an integrable hierarchy, which is named as the extended KP hierarchy. In fact, this hierarchy is reduced to the KP hierarchy (1.2) whenever $\hat{P} = 0$ (note that the operator $P$ gives an alternative expression of $L_{KP}$). On the other hand, if one lets $\varphi \rightarrow 0$ and impose certain B-type symmetry conditions to the operators $P$ and $\hat{P}$, then the flows in (1.4) with $k \in \mathbb{Z}_{>0}$ give the two-component BKP (2-BKP) hierarchy [7, 13]. With the $R$-matrix method
applied in the cases of the KP and the 2-BKP hierarchies \cite{15, 17}, the extended KP hierarchy (1.4) was shown to possess infinitely many bi-Hamiltonian structures \cite{20}.

In this paper we assume \( \varphi = \partial(f) \) in (1.3) with some homogeneous function \( f \in A \) of degree 0. We will show that the operators \( P \) and \( \hat{P} \) can be represented in a dressing form as

\[
P = \Phi \partial \Phi^{-1}, \quad \hat{P} = \hat{\Phi} \partial \hat{\Phi}^{-1},
\]

where \( \Phi \) and \( \hat{\Phi} \) are pseudo-differential operators of the form

\[
\Phi = 1 + \sum_{i \geq 1} a_i \partial^{-i}, \quad \hat{\Phi} = e^f \left( 1 + \sum_{i \geq 1} b_i \partial^i \right).
\]

With the help of these two dressing operators in the extended KP hierarchy, we will introduce two Baker–Akhiezer functions \( \psi(t, \hat{t}; z), \hat{\psi}(t, \hat{t}; z) \) and their adjoints \( \psi^+(t, \hat{t}; z), \hat{\psi}^+(t, \hat{t}; z) \) that depend on the time variables \( t = (t_1, t_2, t_3, \ldots), \hat{t} = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \ldots) \), and a nonzero parameter \( z \). Our first main result is the following theorem (see Theorem 3.8 for a more precise version).

**Theorem 1.1** The extended KP hierarchy (1.4) can be represented equivalently to a bilinear equation as

\[
\text{res}_z \left( \psi(t, \hat{t}; z) \psi^+(t', \hat{t}'; z) \right) = \text{res}_z \left( \hat{\psi}(t, \hat{t}; z) \hat{\psi}^+(t', \hat{t}'; z) \right)
\]

with arbitrary time variables \( (t, \hat{t}) \) and \( (t', \hat{t}') \).

Our second main result is the construction of additional symmetries for the extended KP hierarchy, with the help of certain Orlov–Schulman operators given by the above dressing operators \( \Phi \) and \( \hat{\Phi} \). Such additional symmetries will be shown to generate a \( W_{1+\infty} \times W_{1+\infty} \) algebra. These results will be applied to study the \((n, 1)\)-constrained KP hierarchy (see, e.g., \cite{2, 4, 6, 11, 12}), which is denoted as cKP\(_{n,1}\) and can be reduced from the extended KP hierarchy (1.4) under the constraint

\[
P^n = \hat{P}
\]

with a given integer \( n \geq 1 \). In fact, the Virasoro symmetries for the cKP\(_{n,1}\) hierarchy (even for more general cases) were proposed by Aratyn, Nissimov and Pacheva \cite{2}, with the method of adding certain “ghost” symmetry flows such that some non-local actions on functions are involved. They also showed that, the cKP\(_{n,1}\) hierarchy subject to the subsidiary condition of invariance under the lowest Virasoro symmetry flow can be applied to compute explicit Wronskian solution for the two-matrix model partition function \cite{3}. As to be seen, the Virasoro symmetries appearing in \cite{2, 3} for the cKP\(_{n,1}\) hierarchy can be obtained in an alternative way, starting from the additional symmetries for the extended KP hierarchy (see Proposition 4.9 below).

This paper is arranged as follows. In the next section, we will recall the pseudo-differential operators of the first and the second types over a graded differential algebra. In Section 3, we will recall the definition of the extended KP hierarchy, and represent it into the form of a bilinear equation. In Section 4, we will construct the...
additional symmetries for the extended KP hierarchy, and then study the Virasoro symmetries for the cKP \(_{n,1}\) hierarchy. The final section is devoted to some remarks.

2 Pseudo-Differential Operators

Let \( \mathcal{A} \) be a commutative associative algebra, and \( \partial : \mathcal{A} \to \mathcal{A} \) be a derivation. We consider the linear space \( \mathcal{D}(\mathcal{A}) = \left\{ \sum_{i \in \mathbb{Z}} f_i \partial^i \mid f_i \in \mathcal{A} \right\} \) and its subsets. For instance, the set of pseudo-differential operators is

\[
\mathcal{A}((\partial^{-1})) = \left\{ \sum_{i \leq k} f_i \partial^i \mid f_i \in \mathcal{A}, k \in \mathbb{Z} \right\},
\]

and it becomes an associative algebra if a product is defined by

\[
f \partial^i \cdot g \partial^j = \sum_{r \geq 0} \binom{i}{r} f \partial^r(g) \partial^{i+j-r}, \quad f, g \in \mathcal{A}.
\]  

(2.1)

For any two pseudo-differential operators \( A \) and \( B \), their commutator means \([A, B] = AB - BA\). Clearly, one has \([\partial, f] = \partial(f)\) for any \( f \in \mathcal{A}\).

In the present paper we assume the algebra \( \mathcal{A} \) to be a graded one. Namely, \( \mathcal{A} = \prod_{i \geq 0} \mathcal{A}_i \), such that

\[
\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}, \quad \partial(\mathcal{A}_i) \subset \mathcal{A}_{i+1}.
\]

Denote \( \mathcal{D}^- = \mathcal{A}((\partial^{-1})) \), which is called the algebra of pseudo-differential operators of the first type over \( \mathcal{A} \). In contrast, by the algebra of pseudo-differential operators of the second type over \( \mathcal{A} \) it means [13]

\[
\mathcal{D}^+ = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j = \max(0, k-i)} a_{i,j} \partial^i \mid a_{i,j} \in \mathcal{A}_j, k \in \mathbb{Z} \right\},
\]  

(2.2)

whose product is also defined by (2.1). One observes that an operator in \( \mathcal{D}^+ \) may contain infinitely many positive powers in \( \partial \).

Given an element \( A = \sum_{i \in \mathbb{Z}} f_i \partial^i \in \mathcal{D}(\mathcal{A}) \), its differential part, negative part and residue are defined respectively as:

\[
A_+ = \sum_{i \geq 0} f_i \partial^i, \quad A_- = \sum_{i < 0} f_i \partial^i, \quad \text{res} \ A = f_{-1}.
\]  

(2.3)

It is easy to see that

\[
(\mathcal{D}^\mp)_\pm \subset \mathcal{D}^- \cap \mathcal{D}^+.
\]

What is more, on each \( \mathcal{D}^\mp \) there is an anti-automorphism defined by

\[
\partial^* = -\partial, \quad f^* = f \text{ with } f \in \mathcal{A}.
\]

Clearly, for any \( A \in \mathcal{D}^\mp \), one has

\[
\text{res} \ A^* = -\text{res} \ A.
\]
In what follows we will use the notation \( A_{\geq r} = \prod_{i \geq r} A_i \) with \( r \in \mathbb{Z}_{\geq 0} \). Given an element \( \varphi \in A_{\geq 1} \), the following two maps are well defined with \( \partial \) replaced by \( \partial - \varphi \), that is,

\[
\mathcal{S}_\varphi : \mathcal{D}^\varphi \rightarrow \mathcal{D}^\varphi, \\
\sum f_i \partial^i \mapsto \sum f_i (\partial - \varphi)^i.
\]  

(2.4)

For instance, we have

\[
\mathcal{S}_\varphi (\partial^{-1}) = (\partial - \varphi)^{-1} = \partial^{-1} (1 - \varphi \partial^{-1})^{-1} = \partial^{-1} + \partial^{-1} \varphi \partial^{-1} + \partial^{-1} \varphi \partial^{-1} + \ldots.
\]  

(2.5)

In [20], it was verified that the maps \( \mathcal{S}_\varphi \) are automorphisms on each \( \mathcal{D}^\varphi \). Accordingly, the algebras \( \mathcal{D}^\varphi \) can be represented as follows:

\[
\mathcal{D}^- = \left\{ \sum_{i \leq k} g_i (\partial - \varphi)^i \mid g_i \in A, k \in \mathbb{Z} \right\},
\]  

(2.6)

\[
\mathcal{D}^+ = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j \geq \max(0, k-i)} b_{i,j} (\partial - \varphi)^i \mid b_{i,j} \in A_j, k \in \mathbb{Z} \right\},
\]  

(2.7)

and their product can be defined equivalently by

\[
f (\partial - \varphi)^i \cdot g (\partial - \varphi)^j = \sum_{r \geq 0} \begin{pmatrix} i \cr r \end{pmatrix} f \partial^r (g) (\partial - \varphi)^{i+j-r}, \quad f, g \in A.
\]

Moreover, it is easy to verify the following properties: for any \( A \in \mathcal{D}^\varphi \),

\[
(S_\varphi A)_{\pm} = S_\varphi (A_{\pm}), \quad \text{res} (S_\varphi A) = \text{res} A, \quad (S_\varphi A)^* = S_{-\varphi} A^*.
\]  

(2.8)

### 3 The Extended KP Hierarchy and its Bilinear Equation

We proceed to recall the definition of the extended KP hierarchy, and then capsule it into a bilinear equation of certain (adjoint) Baker–Akhiezer functions.

#### 3.1 The Extended KP Hierarchy

Let \( M \) be an infinite-dimensional manifold with coordinate

\[
a = (a_1, a_2, a_3, \ldots; f, b_1, b_2, b_3, \ldots).
\]

We consider the following graded algebra of formal differential polynomials:

\[
A = C^\infty (S^1 \rightarrow M)[[\partial^r (a) \mid r \geq 1]],
\]  

(3.1)

in which the derivation is \( \partial = d/dx \) with \( x \) being the coordinate of the loop \( S^1 \), and

\[
\text{deg} a = 0, \quad \text{deg} \partial^r (a) = r.
\]

Over the graded differential algebra \( A \), it is defined the algebras \( \mathcal{D}^- \) and \( \mathcal{D}^+ \) of pseudo-differential operators of the first type and of the second type, respectively.
Let us consider two pseudo-differential operators as follows:

\begin{equation}
\Phi = 1 + \sum_{i \geq 1} a_i \partial^{-i} \in \mathcal{D}^-,
\end{equation}

\begin{equation}
\hat{\Phi} = e^{f \left( 1 + \sum_{i \geq 1} b_i \partial^i \right)} \in \mathcal{D}^+.
\end{equation}

These two operators have inverses of the form

\begin{equation}
\Phi^{-1} = 1 + \sum_{i \geq 1} \hat{a}_i \partial^{-i} \in \mathcal{D}^-,
\end{equation}

\begin{equation}
\hat{\Phi}^{-1} = \left( 1 + \sum_{i \geq 1} \hat{b}_i \partial^i \right) e^{-f} \in \mathcal{D}^+.
\end{equation}

In fact, by expanding \( \Phi^{-1} \Phi = 1 \) and \( \hat{\Phi}^{-1} \hat{\Phi} = 1 \), one sees that the coefficients \( \hat{a}_i \) and \( \hat{b}_i \) are represented as

\[ \hat{a}_i = -a_i + g_i(a_1, a_2, \ldots, a_{i-1}), \quad \hat{b}_i = -b_i + h_i(b_1, b_2, b_3, \ldots) \]

with \( g_i, h_i \in \mathcal{A}_{\geq 1} \). For instance, one has \( \hat{a}_1 = -a_1 \), and that \( \hat{b}_1 \) is given recursively by

\[ \hat{b}_1 \big|_{\mathcal{A}_0} = -b_1, \quad \hat{b}_1 \big|_{\mathcal{A}_j} = -\sum_{r=1}^{j} b_r \left( \hat{b}_1 \big|_{\mathcal{A}_{j-r}} \right)^{(r)} \text{ for } j \geq 1. \]

Here \( \hat{b}_1 \big|_{\mathcal{A}_j} \) means the degree- \( j \) component of \( \hat{b}_1 \).

Now let us introduce two pseudo-differential operators as follows:

\begin{equation}
P = \Phi \partial \Phi^{-1} \in \mathcal{D}^-,
\end{equation}

\begin{equation}
\hat{P} = \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} \in \mathcal{D}^+.
\end{equation}

**Proposition 3.1** The operators \( P \) and \( \hat{P} \) given above can be represented in the form:

\begin{equation}
P = \partial + \sum_{i \geq 1} v_i \partial^{-i}, \quad \hat{P} = (\partial - f')^{-1} \rho + \sum_{i \geq 0} \hat{v}_i \partial^i,
\end{equation}

where \( v_{i+1}, \hat{v}_i \in \mathcal{A}_{\geq 1} \) for \( i \in \mathbb{Z}_{\geq 0} \), and

\begin{equation}
\rho = e^{f \left( \Phi^{-1} \right)^*} (1).
\end{equation}

**Proof** The representation of \( P \) is well known, so let us verify the case of \( \hat{P} \). To this end, firstly let us check that \( \hat{P} \) given in \( (3.6) \) takes the form

\begin{equation}
\hat{P} = (\partial - f')^{-1} \left( \sum_{i \geq 0} \hat{v}_i (\partial - f')^i \right), \quad \hat{v}_i - \delta_{i0} \in \mathcal{A}_{\geq 1}.
\end{equation}

We substitute this expansion and \( (3.3) \) into an equivalent version of \( \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} = \hat{P} \), say,

\[ e^{-f} (\partial - f') \hat{\Phi} \partial^{-1} = e^{-f} (\partial - f') \hat{P} \hat{\Phi}. \]
Note $e^{-f}(\partial - f')e^f = \partial$, then we have
\[
1 + b_i' + \sum_{i \geq 1} (b_i + b_{i+1}) \partial^i = \left( \sum_{i \geq 0} \tilde{v}_i \partial^i \right) \left( 1 + \sum_{i \geq 1} b_i \partial^i \right),
\]
in which the coefficients of $\partial^i$ lead to
\[
i = 0 : \quad 1 + b_0' = \tilde{v}_0, \tag{3.11}
\]
\[
i \geq 1 : \quad b_i + b_{i+1} = \tilde{v}_i + \sum_{r=1}^{i} \sum_{s \geq 0} \binom{i-r+s}{s} \tilde{v}_{i-r+s} b_r^{(s)}. \tag{3.12}
\]

The equations (3.12) yield
\[
b_i = \tilde{v}_i |_{A_0} + \sum_{r=1}^{i} \tilde{v}_{i-r} |_{A_0} b_r, \quad i \geq 1,
\]
which together with (3.11) gives
\[
\tilde{v}_i |_{A_0} = \delta_{i0}, \quad i \geq 0.
\]
The equations (3.12) also yield
\[
b_i' + 1 = \tilde{v}_i |_{A_1} + \sum_{r=1}^{i} \tilde{v}_{i-r} |_{A_1} b_r, \quad i \geq 1,
\]
which together with (3.11) determine $\tilde{v}_i |_{A_1}$ recursively. In fact, we can obtain a generating function for $\tilde{v}_i |_{A_1}$ of a parameter $z$ as
\[
\sum_{i \geq 0} \tilde{v}_i |_{A_1} z^{i+1} = \partial \log \left( 1 + \sum_{i \geq 1} b_i z^i \right).
\]

For $j \geq 2$, it follows from (3.12) that
\[
\tilde{v}_i |_{A_j} + \sum_{r=1}^{i} \sum_{s \geq 0} \binom{i-r+s}{s} \tilde{v}_{i-r+s} |_{A_{j-s}} b_r^{(s)} = 0, \quad i \geq 1,
\]
hence $\tilde{v}_i |_{A_j}$ are determined recursively. By using (2.4), we derive that the operator $\hat{\Phi}$ takes the form (3.7), in which
\[
\rho = \text{res} \left( \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} \right) = \text{res} \left( e^f \partial^{-1} \hat{\Phi}^{-1} \right) = e^f \left( \hat{\Phi}^{-1} \right)^* (1).
\]

Therefore the proposition is proved. \hfill \Box

**Remark 3.2** From (3.8) and (3.5) it follows that
\[
\rho = e^f \left( \hat{\Phi}^{-1} \right)^* (1) = 1 + \sum_{i \geq 1} (-1)^i \tilde{b}_i^{(i)}, \tag{3.14}
\]
which implies $\rho - 1 \in \mathcal{A}_{\geq 1}$. One also sees that, the coefficients of $P$ and $\hat{P}$ have different degrees from that in [20], since now the graded differential algebra $\mathcal{A}$ is chosen differently.

With the help of the operators $P$ and $\hat{P}$, let us define a class of evolutionary equations on $\mathcal{M}$: for $k \in \mathbb{Z}_{>0}$,

$$\frac{\partial \Phi}{\partial t_k} = -(P^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial t_k} = ((P^k)_+ - \delta_{k1}\hat{P}^{-1})\hat{\Phi}, \quad (3.15)$$

$$\frac{\partial \Phi}{\partial \hat{t}_k} = -(\hat{P}^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{t}_k} = (\hat{P}^k)_+ \hat{\Phi}, \quad (3.16)$$

Here we note that the right hand sides make sense since the operators $(P^k)_+, (\hat{P}^k)_- \in \mathcal{D}^- \cap \mathcal{D}^+$, and we assume that the flows $\partial/\partial t_k$ and $\partial/\partial \hat{t}_k$ commute with $\partial/\partial x$. In particular, it can be seen $\partial/\partial t_1 = \partial/\partial x$, so in what follows we will just take $t_1 = x$.

**Proposition 3.3** The flows (3.15), (3.16) satisfy, for $k \in \mathbb{Z}_{>0}$,

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \quad (3.17)$$

$$\frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}], \quad (3.18)$$

Moreover, these flows commute with each other.

**Proof** The proposition can be verified case by case. For instance, we have

\[
\frac{\partial \hat{P}}{\partial t_k} \left[ \frac{\partial \Phi}{\partial t_k} \right] = \frac{\partial \Phi}{\partial t_k} \left[ (P^k)_+ - \delta_{k1}\hat{P}^{-1} \right] = \left[ [(P^k)_+, \hat{P}] \right],
\]

\[
\left[ \frac{\partial}{\partial \hat{t}_k}, \frac{\partial}{\partial t_k} \right] \Phi = \frac{\partial}{\partial t_k} \left[ (\hat{P}^l)_+ \hat{\Phi} \right] - \frac{\partial}{\partial \hat{t}_l} \left[ ((P^k)_+ - \delta_{k1}\hat{P}^{-1})\hat{\Phi} \right]
\]

\[
= \left[ (\hat{P}^l)_+, (P^k)_+ - \delta_{k1}\hat{P}^{-1} \right] \hat{\Phi} + \left[ (P^k)_+, \hat{P}^l \right] \hat{\Phi}
\]

\[
- \left[ [-(\hat{P}^l)_-, P^k]_+ - \delta_{k1} [-(\hat{P}^l)_-, \hat{P}^{-1}] \right] \hat{\Phi}
\]

\[
= \left[ (\hat{P}^l)_+, (P^k)_+ \right] \hat{\Phi} - \left[ \hat{P}^l, (P^k)_+ \right] \hat{\Phi}
\]

\[
+ \left[ [-(\hat{P}^l)_-, (P^k)_+]_+ \right] \hat{\Phi} - \delta_{k1} \left[ \hat{P}^l, \hat{P}^{-1} \right] \hat{\Phi}
\]

\[
= 0.
\]

The other cases are almost the same. So we complete the proof. $\square$

The system of equations (3.17), (3.18) was studied in [20] by Zhou and one of the authors (cf. [16]), with the set of unknown functions as

$$\{u_{i+1}, \hat{v}_i \mid i \in \mathbb{Z}_{\geq 0}\} \cup \{\rho, \varphi = f'\}.$$ 

This system is called the extended KP hierarchy for the reason that the flows $\partial P/\partial t_k$ in (3.17) with $k \in \mathbb{Z}_{>0}$ compose the well-known KP hierarchy. By virtue of the above
proposition, we will also call the system of equations (3.15), (3.16) the extended KP hierarchy.

**Example 3.4** From the extended KP hierarchy one can write down some equations explicitly as follows:

\[
\frac{\partial^2 v_1}{\partial t_2^2} = \left( \frac{4}{3} \frac{\partial v_1}{\partial t_3} - 4v_1 v_1' - \frac{1}{3} v_1'' \right)', \quad \frac{\partial f}{\partial t_2} = 2v_1 + (f')^2 + f'',
\]
\[
\frac{\partial \rho}{\partial t_2} = (2\rho f' - \rho)'.
\] (3.19)

**Example 3.5** By using (3.3) and (3.15) we have, for \( k \in \mathbb{Z}_{>0} \),

\[
\frac{\partial f}{\partial t_k} = e^{-f} \frac{\partial \hat{\Phi}}{\partial t_k}(1) = e^{-f} ((P^k)_+ - \delta_{k1} \hat{P}^{-1}) \hat{\Phi}(1) = e^{-f}(P^k)_+ e^f(1)
\]
\[
= \text{res} \left( e^{-f} p^k e^f \partial^{-1} \right) = \text{res} \mathcal{J}_{-f}' \left( p^k e^f \partial^{-1} e^{-f} \right)
\]
\[
= \text{res} \left( p^k e^f \partial^{-1} e^{-f} \right) = \text{res} \left( p^k (\partial - f')^{-1} \right).
\] (3.20)

Here in the fifth equality we have used (2.8). With the same method, we obtain

\[
\frac{\partial f}{\partial t_k} = \text{res} \left( \hat{p}^k (\partial - f')^{-1} \right), \quad k \in \mathbb{Z}_{>0}.
\] (3.21)

Moreover, for \( \hat{t}_k = t_k \) or \( \hat{t}_k \) (correspondingly \( \hat{P} = P \) or \( \hat{P} \)), we have

\[
\frac{\partial \rho}{\partial t_k} = \sum_{i \geq 1} (-1)^{i-1} \partial^i \left( \rho \text{ res} \hat{p}^k (\partial - f')^{-i-1} \right).
\] (3.22)

By letting \( \varphi = f' \), we recover the equations (2.23) and (2.24) in [20].

### 3.2 Baker–Akhiezer Functions and the Bilinear Equation

With \( z \) being a nonzero parameter, we assign \( \partial^i (e^{xz}) = z^i e^{xz} \) for any \( i \in \mathbb{Z} \), and more generally,

\[
\partial^i (ge^{xz}) = (\partial^i g) (e^{xz}), \quad g \in A, \quad i \in \mathbb{Z}.
\]

Namely, it is the usual action of a differential operator on a function whenever \( i \geq 0 \), and the integral constants are fixed in a special way whenever \( i < 0 \). In order to simplify notations, for any \( A \in D^\pm \) and exponential functions of the form \( e^{\pm xz} \), we will just write \( Ae^{\pm xz} \) instead of \( A(e^{\pm xz}) \).

Denote \( t = (t_1 = x, t_2, t_3, \ldots) \) and \( \hat{t} = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \ldots) \), and let \( \xi \) be given by

\[
\xi(t; z) = \sum_{k \in \mathbb{Z}_{>0}} t_k z^k.
\]

Given a solution of the extended KP hierarchy (3.15), (3.16), let us introduce two Baker–Akhiezer functions:

\[
\psi(t, \hat{t}; z) = \Phi e^{\xi(t; z)}, \quad \hat{\psi}(t, \hat{t}; z) = \hat{\Phi} e^{e^{xz} - \xi(\hat{t}; z^{-1})},
\] (3.23)
and two adjoint Baker–Akhiezer functions:

$$
\psi^+(t, \hat{t}; z) = \left( \Phi^{-1} \right)^* e^{-\xi(t; z)}, \quad \hat{\psi}^+(t, \hat{t}; z) = \left( \Phi^{-1} \right)^* e^{xz+\xi(\hat{t}; z^{-1})}.
$$

(3.24)

When there is no confusion, we will just write $\eta(z) = \eta(t, \hat{t}; z)$ with $\eta \in \{\psi, \hat{\psi}, \psi^+, \hat{\psi}^+\}$. Based on (3.15) and (3.16), it is straightforward to verify the following Lemma.

**Lemma 3.6** The (adjoint) Baker–Akhiezer functions given above satisfy:

(i) $P\psi(z) = z\psi(z), \quad P^*\psi^+(z) = z\psi^+(z),$ 

(ii) $\hat{P}\hat{\psi}(z) = z^{-1}\hat{\psi}(z), \quad \hat{P}^*\hat{\psi}^+(z) = z^{-1}\hat{\psi}^+(z);$

where $\psi \in \{\psi(z), \hat{\psi}(z)\}$ and $\psi^+ \in \{\psi^+(z), \hat{\psi}^+(z)\}$.

For any formal series $\sum_{i \in \mathbb{Z}} g_i z^i$, its residue is defined by

$$
\text{res}_z \sum_{i \in \mathbb{Z}} g_i z^i = g_{-1}.
$$

The following lemma is useful.

**Lemma 3.7** (see, for example, [8]) For any pseudo-differential operators $Q, R \in \mathcal{D}^\pm$, the following equality holds true whenever both sides make sense:

$$
\text{res}_z \left( Q e^{xz} \cdot R^* e^{-xz} \right) = \text{res} (QR).
$$

(3.29)

**Theorem 3.8** The (adjoint) Baker–Akhiezer functions of the extended KP hierarchy satisfy the following bilinear equation

$$
\text{res}_z \left( \psi(t, \hat{t}; z) \psi^+(t', \hat{t}'); z \right) = \text{res}_z \left( \hat{\psi}(t, \hat{t}; z) \hat{\psi}^+(t', \hat{t}'); z \right),
$$

(3.30)

for arbitrary time variables $(t, \hat{t})$ and $(t', \hat{t}')$. Conversely, suppose that four functions of the form

$$
\psi(t, \hat{t}; z) = \left( 1 + \sum_{i \geq 1} a_i (t, \hat{t}) z^{-i} \right) e^{\xi(t; z)},
$$

(3.31)

$$
\hat{\psi}(t, \hat{t}; z) = e^{f(t, \hat{t})} \left( 1 + \sum_{i \geq 1} b_i (t, \hat{t}) z^i \right) e^{xz-\xi(\hat{t}; z^{-1})},
$$

(3.32)
\[\psi^+ (t, \hat{t}; z) = \left( 1 + \sum_{i \geq 1} a_i^+ (t, \hat{t}) z^{-i} \right) e^{-\xi (t; z)}, \quad (3.33)\]

\[\hat{\psi}^+ (t, \hat{t}; z) = e^{-f(t, \hat{t})} \left( 1 + \sum_{i \geq 1} b_i^+ (t, \hat{t}) z^i \right) e^{-xz + \xi (\hat{t}; z^{-1})}. \quad (3.34)\]

satisfy the bilinear (3.30), then they are the Baker–Akhiezer functions and the adjoint Baker–Akhiezer functions of the extended KP hierarchy.

**Proof** As a preparation, we introduce the set of indices as

\[I = \{(m_1, m_2, m_3, \ldots) \mid m_i \in \mathbb{Z}_{\geq 0} \text{ such that } m_i = 0 \text{ for } i \gg 0\}.\]

For \(m = (m_1, m_2, m_3, \ldots) \in I\), denote

\[\partial_t^m = \prod_{k \geq 1} \left( \frac{\partial}{\partial t_k} \right)^{m_k}, \quad \partial_{\hat{t}}^m = \prod_{k \geq 1} \left( \frac{\partial}{\partial \hat{t}_k} \right)^{m_k}.\]

In order to show the equality (3.30), we only need to check that the Baker–Akhiezer functions and the adjoint Baker–Akhiezer functions satisfy

\[\text{res}_z \left( \partial_t^m \partial_{\hat{t}}^n \psi (t, \hat{t}; z) \cdot \psi^+ (t, \hat{t}; z) \right) = \text{res}_z \left( \partial_t^m \partial_{\hat{t}}^n \hat{\psi} (t, \hat{t}; z) \cdot \hat{\psi}^+ (t, \hat{t}; z) \right) \quad (3.35)\]

for any indices \(m, n \in I\). In fact, given any \(m, n \in I\), according to Lemma 3.6 and Proposition 3.3 there is a pseudo-differential operator \(A^{m,n} \in D^- \cap D^+\) such that the following two equalities hold simultaneously:

\[\partial_t^m \partial_{\hat{t}}^n \psi (z) = A^{m,n} \psi (z), \quad \partial_{\hat{t}}^m \partial_t^n \hat{\psi} (z) = A^{m,n} \hat{\psi} (z).\]

Then, by using (3.24) and Lemma 3.7, the equality (3.35) is recast to

\[\text{res} \left( A^{m,n} \Phi \Phi^{-1} \right) = \text{res} \left( A^{m,n} \hat{\Phi} \hat{\Phi}^{-1} \right),\]

which is clearly valid. Hence the equality (3.35) holds true, and the first assertion is verified.

For the second assertion, one sees that there are uniquely pseudo-differential operators \(\Phi, \hat{\Phi}, \Psi\) and \(\hat{\Psi}\) such that the functions (3.31)–(3.34) are represented as

\[\psi (t, \hat{t}; z) = \Phi e^{\xi (t; z)}, \quad \hat{\psi} (t, \hat{t}; z) = \hat{\Phi} e^{xz - \xi (\hat{t}; z^{-1})},\]

\[\psi^+ (t, \hat{t}; z) = \Psi^* e^{-\xi (t; z)}, \quad \hat{\psi}^+ (t, \hat{t}; z) = \hat{\Psi}^* e^{-xz + \xi (\hat{t}; z^{-1})}.\]

Moreover, the operators \(\Phi\) and \(\hat{\Phi}\) take the form (3.2) and (3.3) respectively, while \(\Psi\) and \(\hat{\Psi}\) take the form (3.4) and (3.5) respectively. The bilinear equation (3.30) leads to the following facts.

(i) For \(i \in \mathbb{Z}\), we have (note that when \(i < 0\) the integral constants on both sides are fixed in the same way)

\[\text{res}_z \left( \partial^i \psi (t, \hat{t}; z) \psi^+ (t', \hat{t}; z) \right) = \text{res}_z \left( \partial^i \hat{\psi} (t, \hat{t}; z) \hat{\psi}^+ (t', \hat{t}; z) \right).\]
Let \((t', \hat{t}) = (t, \hat{t})\), and then with the help of (3.29) we obtain
\[
\text{res } \partial_i \Phi \psi = \text{res } \partial_i \hat{\Phi} \hat{\psi} = 0, \quad i \geq 0;
\]
\[
\text{res } \partial_i \hat{\Phi} \hat{\psi} = \text{res } \partial_i \Phi \psi = \delta_{i,-1}, \quad i < 0.
\]
which implies \(\Phi \psi = \hat{\Phi} \hat{\psi} = 1\). So we derive
\[
\psi = \Phi^{-1}, \quad \hat{\psi} = \hat{\Phi}^{-1}.
\]

(ii) Denote
\[
X_k = \frac{\partial \Phi}{\partial t_k} \Phi^{-1}, \quad \hat{X}_k = \frac{\partial \hat{\Phi}}{\partial t_k} \hat{\Phi}^{-1}.
\]
Clearly, one has \((X_k)_+ = (\hat{X}_k)_+ = 0\), and that
\[
\frac{\partial \psi(z)}{\partial t_k} = \left( X_k \Phi + \Phi \partial^k \right) e^{\xi(t, z)} = \left( X_k + \Phi \partial^k \Phi^{-1} \right) \psi(z),
\]
\[
\frac{\partial \hat{\psi}(z)}{\partial t_k} = \left( \hat{X}_k \hat{\Phi} + \delta_{k1} \hat{\Phi} \partial \right) e^{x z - \xi(\hat{t}, z^{-1})} = \left( \hat{X}_k + \delta_{k1} \hat{\Phi} \partial \hat{\Phi}^{-1} \right) \hat{\psi}(z).
\]
For any \(i \in \mathbb{Z}\), we let \(\partial^i\) act on the derivative of (3.30) with respect to \(t_k\), and let \((t', \hat{t}') = (t, \hat{t})\), then by using (3.29) again we obtain
\[
\text{res } \partial^i \left( X_k + \Phi \partial^k \Phi^{-1} \right) \Phi \Phi^{-1} = \text{res } \partial^i \left( \hat{X}_k + \delta_{k1} \hat{\Phi} \partial \hat{\Phi}^{-1} \right) \hat{\Phi} \hat{\Phi}^{-1}.
\]
Hence \(X_k + \Phi \partial^k \Phi^{-1} = \hat{X}_k + \delta_{k1} \hat{\Phi} \partial \hat{\Phi}^{-1}\), and we arrive at
\[
X_k = -\left( \Phi \partial^k \Phi^{-1} \right)_-, \quad \hat{X}_k = \left( \Phi \partial^k \Phi^{-1} \right)_+ - \delta_{k1} \hat{\Phi} \partial \hat{\Phi}^{-1}.
\]
Similarly, we can derive
\[
\frac{\partial \Phi}{\partial t_k} \Phi^{-1} = -\left( \hat{\Phi} \partial^{-k} \hat{\Phi}^{-1} \right)_-, \quad \frac{\partial \hat{\Phi}}{\partial t_k} \hat{\Phi}^{-1} = \left( \hat{\Phi} \partial^{-k} \hat{\Phi}^{-1} \right)_+.
\]
Taking (i) and (ii) together we achieve the second assertion. The theorem is proved.

### 4 Additional Symmetries Versus Virasoro Symmetries

In this section, we want to construct a class of additional symmetries for the extended KP hierarchy, following the approach of [14, 18], and then study the Virasoro symmetries for the constrained KP hierarchies.

#### 4.1 Additional Symmetries for the Extended KP Hierarchy

Suppose that the operators \(\Phi\) and \(\hat{\Phi}\) solve the hierarchy (3.15), (3.16). Let us introduce two Orlov–Schulman operators as follows:
\[
M = \Phi \Xi \Phi^{-1}, \quad \hat{M} = \hat{\Phi} \hat{\Xi} \hat{\Phi}^{-1},
\]
where

\[ E = \sum_{k \in \mathbb{Z}_{>0}} k t_k \partial_k^{k-1}, \quad \hat{E} = x + \sum_{k \in \mathbb{Z}_{>0}} k \hat{t}_k \partial_k^{-k-1}. \]

Here we assume that all \( t_k \) and \( \hat{t}_k \) vanish except finitely many of them, such that the operators \( M \) and \( \hat{M} \) are well defined.

**Remark 4.1** There is another way to ensure the Orlov–Schulman operators \( M \) and \( \hat{M} \) (even with infinitely many time variables) to be well defined. Indeed, as what was done in [18], one can extend the graded algebra \( \mathcal{A} \) to include also \( \{ t_k, \hat{t}_k \mid k \in \mathbb{Z}_{>0} \} \) with \( \deg t_k = \deg \hat{t}_k = k \), so that a new graded algebra \( \hat{\mathcal{A}} \) is obtained. Then, the set \( \mathcal{D}^- \) of pseudo-differential operators of the first type can be extended to

\[ \hat{\mathcal{D}}^- = \left\{ \sum_{i \in \mathbb{Z}} \sum_{j \geq \max\{0, i-k\}} a_{i,j} \partial^i | a_{i,j} \in \hat{\mathcal{A}}_j, k \in \mathbb{Z} \right\}. \]

So, the operators \( M \) and \( \hat{M} \) are elements of \( \hat{\mathcal{D}}^- \) and \( \mathcal{D}^+ \) (with \( \mathcal{A} \) replaced by \( \hat{\mathcal{A}} \)) respectively.

**Lemma 4.2** The Orlov–Schulman operators \( M \) and \( \hat{M} \) satisfy:

\[ [P, M] = 1, \quad [\hat{P}^{-1}, \hat{M}] = 1, \]

\[ M \psi(z) = \frac{\partial \psi(z)}{\partial z}, \quad \hat{M} \hat{\psi}(z) = \frac{\partial \hat{\psi}(z)}{\partial z}, \]

\[ \frac{\partial \hat{M}}{\partial t_k} = [(P^k)_+, \hat{M}], \quad \frac{\partial \hat{M}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{M}], \]

where \( \hat{M} = M \) or \( \hat{M} \).

**Proof** Based on the definition of \( M \) and \( \hat{M} \) in (4.1), the first line (4.2) follows from (3.6), the second line (4.3) follows from (3.23), while the third line (4.4) follows from (3.15) and (3.16). The lemma is proved.

For any pair of integers \( (m, p) \) with \( m \geq 0 \), let

\[ B_{mp} = M^m P^p, \quad \hat{B}_{mp} = \hat{M}^m \hat{P}^{-p}, \]

and we introduce the following evolutionary equations:

\[ \frac{\partial \Phi}{\partial \hat{p}_{mp}} = -(B_{mp})_+ \Phi, \quad \frac{\partial \Phi}{\partial p_{mp}} = (B_{mp})_+ \Phi, \]

\[ \frac{\partial \Phi}{\partial \hat{p}_{mp}} = -(\hat{B}_{mp})_+ \Phi, \quad \frac{\partial \Phi}{\partial \hat{p}_{mp}} = (\hat{B}_{mp})_+ \Phi. \]

As before, such flows are assumed to commute with \( \partial / \partial x \).
Lemma 4.3 For any $m, m' \in \mathbb{Z}_{\geq 0}$ and $p, p' \in \mathbb{Z}$, the following equalities hold:

\[
\begin{align*}
\frac{\partial \hat{\psi}(z)}{\partial \hat{\beta}_{mp}} &= - (\hat{\mathcal{B}}_{mp})_+ \psi(z), & \frac{\partial \hat{\psi}(z)}{\partial \hat{\beta}_{mp}} &= (\hat{\mathcal{B}}_{mp})_+ \hat{\psi}(z), \\
\frac{\partial \hat{P}}{\partial \hat{\beta}_{mp}} &= - (\hat{\mathcal{B}}_{mp})_- P, & \frac{\partial \hat{P}}{\partial \hat{\beta}_{mp}} &= [(\hat{\mathcal{B}}_{mp})_+, \hat{P}], \\
\frac{\partial \hat{M}}{\partial \hat{\beta}_{mp}} &= - (\hat{\mathcal{B}}_{mp})_- M, & \frac{\partial \hat{M}}{\partial \hat{\beta}_{mp}} &= [(\hat{\mathcal{B}}_{mp})_+, \hat{M}], \\
\frac{\partial \hat{B}_{m'p'}}{\partial \hat{\beta}_{mp}} &= - (\hat{\mathcal{B}}_{mp})_-, \hat{B}_{m'p'}, & \frac{\partial \hat{B}_{m'p'}}{\partial \hat{\beta}_{mp}} &= [(\hat{\mathcal{B}}_{mp})_+, \hat{B}_{m'p'}],
\end{align*}
\]

where $\hat{\beta}_{mp} = \beta_{mp}$, $\hat{\beta}_{mp}$ correspond to $\hat{B}_{mp} = B_{mp}$, $\hat{B}_{mp}$ respectively.

Proof The equalities (4.8)–(4.10) follow from the definition (4.6), (4.7). Subsequently, the equalities (4.11) follow from (4.9) and (4.10). The lemma is proved.

Theorem 4.4 The flows defined by (4.6), (4.7) commute with those in (3.15), (3.16) that compose the extended KP hierarchy. More exactly, for any $\hat{\beta}_{mp} = \beta_{mp}$, $\hat{\beta}_{mp}$ and $\hat{\beta}_k = \beta_k$, $\hat{\beta}_k$ it holds that

\[
\frac{\partial}{\partial \hat{\beta}_{mp}} \frac{\partial}{\partial \hat{\beta}_k} \Phi = 0, \quad m \in \mathbb{Z}_{\geq 0}, \quad p \in \mathbb{Z}, \quad k \in \mathbb{Z}_{> 0},
\]

where $\Phi = \Phi$ or $\hat{\Phi}$.

Proof Firstly, from (3.17), (3.18) and (4.4) it follows that

\[
\begin{align*}
\frac{\partial \hat{B}_{mp}}{\partial \hat{\beta}_k} &= [(P^k)_+, \hat{B}_{mp}], & \frac{\partial \hat{B}_{mp}}{\partial \hat{\beta}_k} &= -[(\hat{P}^k)_-, \hat{B}_{mp}],
\end{align*}
\]

with $\hat{\beta}_{mp} = B_{mp}$, $\hat{\beta}_{mp}$. Then the proposition is checked case by case with the help of Lemmas 4.2 and 4.3. For instance,

\[
\begin{align*}
\hat{\Phi} &= \frac{\partial}{\partial \hat{\beta}_{mp}} \left( (P^k)_+ - \delta_k \hat{P}^{-1} \hat{\Phi} \right) - \frac{\partial}{\partial \hat{\beta}_{mp}} \left( (\hat{B}_{mp})_+ \hat{\Phi} \right) \\
&= [(P^k)_+ - \delta_k \hat{P}^{-1}, (\hat{B}_{mp})_+] \hat{\Phi} \\
&\quad + \left( [-(\hat{B}_{mp})_-, (P^k)_+] - \delta_k [(\hat{B}_{mp})_+, \hat{P}^{-1}] \right) \hat{\Phi} - [(P^k)_+, (\hat{B}_{mp})_+] \hat{\Phi} \\
&= \left( [(P^k)_+, (\hat{B}_{mp})_+] + [(P^k)_+, (\hat{B}_{mp})_-] - [(P^k)_+, (\hat{B}_{mp})_+] \right) \hat{\Phi} = 0.
\end{align*}
\]

The other cases are similar. Thus the proposition is proved.

Proposition 4.4 means that the flows (4.6), (4.7) give a class of symmetries for the extended KP hierarchy, which are called the additional symmetries.
Let us study the commutation relation between the additional symmetries themselves. Observe that each commutator $[B_{mp}, B_{m'p'}]$ is a polynomial in $M$ and $P^{\pm 1}$, hence, by virtue of (4.2), there exist certain constants $c^{nq}_{mp, m'p'}$ such that

$$[B_{mp}, B_{m'p'}] = \sum_{n, q} c^{nq}_{mp, m'p'} B_{nq}.$$  

In fact, one has $c^{nq}_{mp, m'p'} = 0$ whenever $n \geq m + m'$ or $|q - (p + p')| > \max(m, m')$, which implies that all but finitely many structure constants on the right hand side of (4.14) vanish. For instance, when $m + m' \leq 2$ one has

$$c^{0q}_{0p, 0p'} = 0, \quad c^{0q}_{0p, 1p'} = p\delta_{00}\delta_{q, p + p' - 1}, \quad c^{0q}_{1p, 1p'} = (p - p')\delta_{11}\delta_{q, p + p' - 1},$$

$$c^{0q}_{0p, 2p'} = p(p - 1)\delta_{00}\delta_{q, p + p' - 2} + 2p\delta_{10}\delta_{q, p + p' - 1}.$$  

By virtue of (4.2), it holds for the same structure constants that

$$[\hat{B}_{mp}, \hat{B}_{m'p'}] = \sum_{n, q} c^{nq}_{mp, m'p'} \hat{B}_{nq}.$$  

**Proposition 4.5** For the extended KP hierarchy (3.15), (3.16), its additional symmetries defined by (4.6), (4.7) satisfy:

$$\left[ \frac{\partial}{\partial \beta_{mp}}, \frac{\partial}{\partial \beta_{m'p'}} \right] \Phi = -\sum_{n, q} c^{nq}_{mp, m'p'} \frac{\partial \hat{\Phi}}{\partial \beta_{nq}},$$  

$$\left[ \frac{\partial}{\partial \hat{\beta}_{mp}}, \frac{\partial}{\partial \hat{\beta}_{m'p'}} \right] \hat{\Phi} = \sum_{n, q} c^{nq}_{mp, m'p'} \frac{\partial \hat{\Phi}}{\partial \beta_{nq}},$$  

$$\left[ \frac{\partial}{\partial \beta_{mp}}, \frac{\partial}{\partial \hat{\beta}_{m'p'}} \right] \hat{\Phi} = 0,$$

where $\hat{\Phi} = \Phi$ or $\hat{\Phi}$.

**Proof** The conclusion can be checked case by case with the help of Lemma 4.3. For instance,

$$\left[ \frac{\partial}{\partial \beta_{mp}}, \frac{\partial}{\partial \beta_{m'p'}} \right] \hat{\Phi} = [(B_{m'p'})_+, (B_{mp})_+] \hat{\Phi} + [-(B_{mp})_-, B_{m'p'}]_+ \hat{\Phi} - [-(B_{m'p'})_-, B_{mp}]_+ \hat{\Phi}$$

$$= -[(B_{mp})_+, (B_{m'p'})_+] \hat{\Phi} - [(B_{mp})_-, (B_{m'p'})_+]_+ \hat{\Phi} - [B_{mp}, (B_{m'p'})_-]_+ \hat{\Phi}$$

$$= -[B_{mp}, B_{m'p'}]_+ \hat{\Phi} = -\sum_{n, q} c^{nq}_{mp, m'p'} (B_{nq})_+ \hat{\Phi}$$

$$= -\sum_{n, q} c^{nq}_{mp, m'p'} \frac{\partial \hat{\Phi}}{\partial \beta_{nq}},$$  

(4.19)
\[
\left[ \frac{\partial}{\partial \hat{\beta}_{mp}}, \frac{\partial}{\partial \hat{\beta}_{m'p'}} \right] \hat{\Phi} = \left[ (\hat{B}_{m'p'})_+, (\hat{B}_{mp})_+ \right] \hat{\Phi} + \left[ (\hat{B}_{mp})_+, (\hat{B}_{m'p'})_+ \right] \hat{\Phi} - \left[ -(\hat{B}_{m'p'})_-, B_{mp} \right] \hat{\Phi} \\
= \left[ (\hat{B}_{mp})_+, (\hat{B}_{m'p'})_+ \right] \hat{\Phi} + [\hat{B}_{mp} (\hat{B}_{m'p'})_+] \hat{\Phi} \\
= \left[ \hat{B}_{mp} (\hat{B}_{m'p'})_+ \right] \hat{\Phi} = \sum_{n,q} c_{mp,m'p'}^{nq} (\hat{B}_{nq} + \hat{\Phi}) \\
= \sum_{n,q} c_{mp,m'p'}^{nq} \frac{\partial \hat{\Phi}}{\partial \beta_{nq}},
\]
(4.20)

\[
\left[ \frac{\partial}{\partial \beta_{mp}}, \frac{\partial}{\partial \beta_{m'p'}} \right] \hat{\Phi} = \left[ (\hat{B}_{m'p'})_+, (B_{mp})_+ \right] \hat{\Phi} + \left[ (B_{mp})_+, (\hat{B}_{m'p'})_+ \right] \hat{\Phi} - \left[ -(\hat{B}_{m'p'})_-, B_{mp} \right] \hat{\Phi} \\
= \left[ (B_{mp})_+, (\hat{B}_{m'p'})_+ \right] \hat{\Phi} + \left[ (\hat{B}_{m'p'})_-, (B_{mp})_+ \right] \hat{\Phi} = 0.
\]
(4.21)

The other cases are checked in the same way. Thus the proposition is proved. \( \square \)

This proposition means that the additional symmetries (4.6), (4.7) for the extended KP hierarchy generate a \( W_{1+\infty} \times W_{1+\infty} \) algebra.

### 4.2 Virasoro Symmetries for the Constrained KP Hierarchies

Given an arbitrary positive integer \( n \), let us consider the extended KP hierarchy (3.15), (3.16) imposed with the following constraint

\[
\Phi \partial^n \Phi^{-1} = \hat{\Phi} \partial^{-1} \Phi^{-1}.
\]
(4.22)

Under this constraint, one has \( P^n = \hat{P} \) and hence \( \partial / \partial t_{nk} = \partial / \partial \hat{t}_k \) for \( k \geq 1 \). Consequently, the extended KP hierarchy is reduced to

\[
\frac{\partial L}{\partial t_k} = [(P^k)_+, L], \quad k = 1, 2, 3, \ldots,
\]
(4.23)

where \( L := P^n = \hat{P} \) takes the form

\[
L = \partial^n + u_1 \partial^{n-2} + u_2 \partial^{n-3} + \cdots + u_{n-2} \partial + u_{n-1} + (\partial - f')^{-1} \rho.
\]
(4.24)

The system of equations (4.23) is called the constrained KP hierarchy, denoted by cKP\(_n\), (see [2, 4, 5, 11]). In fact, if we write \( v = e^f \) and \( w = e^{-f} \), then

\[
L_- = v \partial^{-1} w
\]
(4.25)

and it is yielded by the equations (4.23) that

\[
\frac{\partial v}{\partial t_k} = (P^k)_+ (v), \quad \frac{\partial w}{\partial t_k} = - (P^k)_+^* (w).
\]
(4.26)
Example 4.6 The hierarchy (4.23) is known as the nonlinear Shrödinger hierarchy (see, e.g., [5]) when \( n = 1 \), and it is the Yajima–Oikawa hierarchy [21] when \( n = 2 \). For instance, when \( n = 2 \) the operator \( L \) takes the form

\[
L = \partial^2 + u + (\partial - f')^{-1} \partial = \partial^2 + u + v\partial^{-1}w,
\]

then the first nontrivial equations defined by (4.23) read (cf. (3.19))

\[
\frac{\partial u}{\partial t_2} = 2\rho', \quad \frac{\partial \rho}{\partial t_2} = (2\rho f' - \rho')', \quad \frac{\partial f}{\partial t_2} = u + (f')^2 + f'',
\]

(4.27)
or equivalently,

\[
\frac{\partial u}{\partial t_2} = 2(vw)', \quad \frac{\partial v}{\partial t_2} = v'' + uv, \quad \frac{\partial w}{\partial t_2} = -w'' - uw,
\]

(4.28)
which is called the Yajima–Oikawa system.

We proceed to construct a series of Virasoro symmetries for the cKP\(_{n,1}\) hierarchy (4.23) with the help of the extended KP hierarchy (3.15), (3.16). To this end, let us introduce a class of operators as

\[
S_p = \frac{1}{n} M P^{n_p+1} + \hat{M} \hat{P}^{p-1}, \quad p \in \mathbb{Z},
\]

(4.29)
where

\[
M = \Phi \left( \sum_{k \in \mathbb{Z}_{>0}} k t_k \partial^k \right) \Phi^{-1}, \quad \hat{M} = \hat{\Phi} \hat{\Phi}^{-1}.
\]

Here the operator \( M \) takes the same form as in (4.1), while for convenience \( \hat{M} \) does not contain the time variables \( \hat{t}_k \) as before. One observes that the operators \( S_p \) belong to the space \( \mathcal{D}^- + \mathcal{D}^+ \), hence \( (S_p)_- \in \mathcal{D}^- \) and \( (S_p)_+ \in \mathcal{D}^+ \). The following evolutionary equations are well defined:

\[
\frac{\partial \Phi}{\partial s_p} = -(S_p)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial s_p} = (S_p)_+ \hat{\Phi}, \quad p \geq -1.
\]

(4.30)

Proposition 4.7 The flows defined by (4.30) are consistent with the constraint (4.22), and they are reduced to

\[
\frac{\partial L}{\partial s_p} = [(S_p)_-, L] = [(S_p)_+, L], \quad p \geq -1,
\]

(4.31)
Moreover, the reduced flows defined by (4.30) and by (3.15) under the constraint (4.22) satisfy:

\[
(i) \quad \left[ \frac{\partial}{\partial t_k}, \frac{\partial}{\partial s_p} \right] L = 0,
\]

(4.32)

\[
(ii) \quad \left[ \frac{\partial}{\partial s_p}, \frac{\partial}{\partial s_q} \right] L = (q - p) \frac{\partial L}{\partial s_{p+q}},
\]

(4.33)
where \( k \in \mathbb{Z}_{>0} \) and \( p, q \in \mathbb{Z}_{\geq -1} \).
Proof Since

\[ L = \Phi \partial^n \Phi^{-1} = \hat{\Phi} \partial^{-1} \Phi^{-1}. \]  

(4.34)

then the equations in (4.30) lead respectively to

\[ \frac{\partial L}{\partial s_p} = \left[ -(S_p)_-, L \right], \quad \frac{\partial L}{\partial s_p} = \left[ (S_p)_+, L \right]. \]  

(4.35)

When \( p \geq -1 \), it is straightforward to verify

\[ [S_p, L] = \frac{1}{n} \left[ MP^{n+1}, P^n \right] \left[ \hat{M} \hat{P}^{p-1}, \hat{P} \right] = -P^{n(p+1)} + \hat{P}^{p+1} \]

\[ = -L^{p+1} + P^{p+1} = 0, \]  

(4.36)

which implies that the equations in (4.35) coincide with each other. Thus the first assertion is obtained.

Let us show the second assertion. On the one hand, by using (3.15) we have, for \( p \geq -1 \) and \( k \geq 1 \),

\[ \frac{\partial S_p}{\partial t_k} = \frac{1}{n} \left[ -(P^k)_-, MP^{n+1} \right] + \frac{k}{n} P^{k+n} \]

\[ + \left[ (P^k)_+ - \delta_{k1} \hat{P}^{-1}, \hat{M} \hat{P}^{p-1} \right] + \delta_{k1} \hat{P}^{p-1} \]

\[ = \frac{1}{n} \left[ -(P^k)_+ + P^k, MP^{n+1} \right] \left[ (P^k)_+, \hat{M} \hat{P}^{p-1} \right] \]

\[ = \left[ (P^k)_+, S_p \right], \]  

(4.37)

On the other hand, by using (4.30) we have, for \( p, q \geq -1 \),

\[ \frac{\partial (MP^p)}{\partial t_q} = \left[ -(S_q)_-, MP^p \right] \quad \frac{\partial (\hat{M} \hat{P}^p)}{\partial t_q} = \left[ (S_q)_+, \hat{M} \hat{P}^p \right]. \]  

(4.38)

Then the first item is checked as

\[ \left[ \frac{\partial}{\partial t_k}, \frac{\partial}{\partial s_p} \right] L = \frac{\partial}{\partial t_k} \left[ (S_p)_+, L \right] - \frac{\partial}{\partial s_p} \left[ (P^k)_+, L \right] \]

\[ = \left[ \left[ (P^k)_+, S_p \right]_+, L \right] \left[ (S_p)_+, \left[ (P^k)_+, L \right] \right] \]

\[ - \left[ \left[ -(S_p)_-, P^k \right]_+, L \right] - \left[ (P^k)_+, \left[ (S_p)_+, L \right] \right] \]

\[ = \left[ \left[ (P^k)_+, (S_p)_+ \right]_+, L \right] - \left[ \left[ (P^k)_+, L \right], (S_p)_+ \right] \]

\[ - \left[ (P^k)_+, \left[ (S_p)_+, L \right] \right] = 0. \]
For the second item, it is straightforward to calculate

\[
\begin{align*}
\left[ \frac{\partial}{\partial s_p}, \frac{\partial}{\partial s_q} \right] L & = \frac{\partial}{\partial s_p} \left[ (S_q)_+, L \right] - \frac{\partial}{\partial s_q} \left[ (S_p)_+, L \right] \\
& = \left[ \left( - (S_q)_-, \frac{1}{n} MP^{nq+1} \right)_+ + \left( (S_p)_+, \hat{M} \hat{P}^{q-1} \right)_+, L \right] \\
& \quad + \left[ (S_q)_+, [(S_p)_+, L] \right] - (p \leftrightarrow q) \\
& = \left[ \left( - (S_q)_-, \frac{1}{n} MP^{nq+1} \right)_+ + \left( (S_p)_+, \hat{M} \hat{P}^{q-1} \right)_+ \\
& \quad - \left[ (S_q)_-, \frac{1}{n} MP^{np+1} \right]_+ \\
& \quad - \left[ (S_q)_+, \hat{M} \hat{P}^{p-1} \right]_+ + \left[ (S_q)_+, (S_p)_+ \right], L \right] \\
& = \left[ \frac{1}{n^2} X + \frac{1}{n} Y + Z, L \right]
\end{align*}
\]

where, with (4.29) substituted,

\[
X = \left[ - (MP^{np+1})_-, MP^{nq+1} \right]_+ - \left[ - (MP^{nq+1})_-, MP^{np+1} \right]_+ \\
+ \left[ (MP^{nq+1})_+, (MP^{np+1})_+ \right]
\]

\[
Y = \left[ - (\hat{M} \hat{P}^{p-1})_-, MP^{nq+1} \right]_+ + \left[ (MP^{np+1})_+, \hat{M} \hat{P}^{q-1} \right]_+ \\
- \left[ - (\hat{M} \hat{P}^{q-1})_-, MP^{np+1} \right]_+ - \left[ (MP^{nq+1})_+, \hat{M} \hat{P}^{p-1} \right]_+ \\
+ \left[ (MP^{nq+1})_+, (\hat{M} \hat{P}^{p-1})_+ \right] + \left[ (\hat{M} \hat{P}^{q-1})_+, (MP^{np+1})_+ \right]
\]

\[
Z = \left[ (\hat{M} \hat{P}^{p-1})_+, \hat{M} \hat{P}^{q-1} \right]_+ + \left[ - (\hat{M} \hat{P}^{q-1})_+, \hat{M} \hat{P}^{p-1} \right]_+ \\
+ \left[ (\hat{M} \hat{P}^{q-1})_+, (\hat{M} \hat{P}^{p-1})_+ \right]
\]

\[
= \left[ (\hat{M} \hat{P}^{p-1})_+, \hat{M} \hat{P}^{q-1} \right]_+ - \left[ (\hat{M} \hat{P}^{q-1})_+, (\hat{M} \hat{P}^{p-1})_- \right]_+ \\
= \left[ \hat{M} \hat{P}^{p-1}, \hat{M} \hat{P}^{q-1} \right]_+ = (q - p)(\hat{M} \hat{P}^{p+q-1})_+.
\]

So we obtain

\[
\left[ \frac{\partial}{\partial s_p}, \frac{\partial}{\partial s_q} \right] L = (q - p) \left[ (S_{p+q})_+, L \right] = (q - p) \frac{\partial L}{\partial s_{p+q}}.
\]

The proposition is proved. \(\square\)
According to this proposition, the flows (4.31) give a series of Virasoro symmetries for the cKP\(_{n,1}\) hierarchy (4.23). It is worthwhile to indicate that, here the condition \(p \geq -1\) is necessary, otherwise (4.36) may not vanish since \(L\) has different inverse as an operator in \(D^-\) or in \(D^+\).

One observes that the flows \(\partial / \partial s_p\) in (4.30) can be considered as reductions of the linear combinations
\[
\frac{1}{n} \frac{\partial}{\partial \beta_{1,np+1}} + \frac{\partial}{\partial \hat{\beta}_{1,-p+1}}
\]
of the additional symmetries (4.6), (4.7) for the extended KP hierarchy constrained by (4.34). This observation motivates us to consider whether the symmetries like \(\partial / \partial \beta_0p\) or \(\partial / \partial \hat{\beta}_0p\) for the extended KP hierarchy could be reduced to that for the cKP\(_{n,1}\) hierarchy.

Similar as in (4.30), for arbitrary constants \(\kappa\) and \(\lambda\) we can define the following equations:

\[
\frac{\partial \Phi}{\partial s_p} = - \left( S_p + (\kappa p + \lambda) \hat{P}^p \right) \Phi, \quad \frac{\partial \hat{\Phi}}{\partial s'_p} = \left( S_p + (\kappa p + \lambda) \hat{P}^p \right) \hat{\Phi}, \quad p \geq -1.
\]

In the same way as above, by doing the replacement \(S_p \mapsto S_p + (\kappa p + \lambda) \hat{P}^p\), we achieve that the flows \(\partial / \partial s'_p\) are consistent with the constraint (4.22), and they are reduced to

\[
\frac{\partial L}{\partial s'_p} = \left[ -(S_p + (\kappa p + \lambda) \hat{P}^p) \right] L, \quad p \geq -1.
\]

Moreover, such reduced flows satisfy:

\[
\begin{align*}
(i) & \quad \left[ \frac{\partial L}{\partial k}, \frac{\partial}{\partial s'_p} \right] L = 0, \\
(ii) & \quad \left[ \frac{\partial}{\partial s'_p}, \frac{\partial}{\partial s_q'} \right] L = (q - p) \frac{\partial L}{\partial s'_{p+q}},
\end{align*}
\]

where \(k \in \mathbb{Z}_{>0}\) and \(p, q \in \mathbb{Z}_{\geq -1}\).

**Remark 4.8** The reduced flows defined by (4.30) and by (4.39) under the constraint (4.22) satisfy

\[
\left( \frac{\partial}{\partial s'_p} - \frac{\partial}{\partial s_p} \right) L = \begin{cases} 
0, & p = -1, 0; \\
-(\kappa p + \lambda) \left[ (L^p)_{-} \right] L, & p \geq 1.
\end{cases}
\]

It is known that, for the cKP\(_{n,1}\) hierarchy (4.23), a series of the Virasoro symmetries was constructed by Aratyn, Nissimov and Pacheva in [2] by adding certain “ghost” symmetry flows related to the eigenfunctions characteristic for the operators \(L\) and \(L^*\) (in fact more general cases were studied there). More exactly, in Proposition 2 of [2] the following Virasoro symmetries were given:

\[
\frac{\partial L}{\partial s_p} = \left[ -\frac{1}{n} (MP^{np+1})_{-} + Y_p, L \right], \quad p \geq -1,
\]
where \( Y_p = 0 \) for \( p = -1, 0, 1, \) and
\[
Y_p = \sum_{j=0}^{p-1} \frac{2j - p + 1}{2} L^{p-j-1}(v) \partial^{-1}(L^*)^j(w), \quad p \geq 2.
\] (4.45)

Here we repeat
\[
v = e^f = \hat{\Phi}(1), \quad w = \rho e^{-f} = \left( \hat{\Phi}^{-1} \right)^*(1),
\]
and note that there are nonlocal-action terms in \( Y_p \) whenever \( p \geq 2 \).

**Proposition 4.9** The flows defined by (4.44) satisfy
\[
\frac{\partial L}{\partial s_p} = \frac{\partial L}{\partial s_p'}, \quad p \geq -1,
\]
where \( \partial / \partial s_p' \) are given in (4.40) with \( \kappa = -\frac{1}{2} \) and \( \lambda = \frac{1}{2} \).

**Proof** Let us fix \( \kappa = -\frac{1}{2} \) and \( \lambda = \frac{1}{2} \) in (4.40), then
\[
\left( \frac{\partial}{\partial s_p} - \frac{\partial}{\partial s_p'} \right) L = \left[ -\frac{1}{n} \left( MP^{np+1} \right)_- + Y_p + \left( S_p - \frac{p-1}{2} \hat{\rho}^p \right)_- , L \right] = [Y_p + Z_p, L],
\]
where
\[
Z_p := \left( \hat{\mathcal{M}} \hat{\rho}^{p-1} - \frac{p-1}{2} \hat{\rho}^p \right)_- = \left( \hat{\Phi} \partial^{-p+1} \hat{\Phi}^{-1} \right)_- - \frac{p-1}{2} \left( \hat{\Phi} \partial^{-p} \hat{\Phi}^{-1} \right)_-.
\] (4.46)

When \( p = -1, 0, 1 \), clearly \( Z_p = 0 = Y_p \), then the flows \( \partial / \partial s_p \) and \( \partial / \partial s_p' \) acting on \( L \) coincide. When \( p = 2 \), on one hand from (4.45) we have
\[
Y_2 = \frac{1}{2} \left( -L(v) \partial^{-1}w + v \partial^{-1}L^*(w) \right)
= -L(v) \partial^{-1}w + \frac{1}{2} \left( L(v) \partial^{-1}w + v \partial^{-1}L^*(w) \right)
= -L(v) \partial^{-1}w + \frac{1}{2} \left( L^2 \right)_- ,
\]
where the last equality is due to the formula (61) in the appendix of [2], say, in the present case
\[
(L^k)_- = \sum_{j=0}^{k-1} L^{k-j-1}(v) \partial^{-1}(L^*)^j(w), \quad k \geq 1.
\]

On the other hand, since
\[
\left( \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} \right)_- = \hat{\Phi}(x) \cdot \partial^{-1} \cdot \left( \hat{\Phi}^{-1} \right)^*(1) = \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} \hat{\Phi} \partial(x) \cdot \partial^{-1} \cdot \left( \hat{\Phi}^{-1} \right)^*(1) = L \hat{\Phi}(1) \cdot \partial^{-1} \cdot w = L(v) \partial^{-1}w,
\]
\[ \hat{\Phi}(x) \cdot \partial^{-1} \cdot (\hat{\Phi}^{-1})^*(1) = \hat{\Phi} \partial^{-1} \hat{\Phi}^{-1} \hat{\Phi} \partial(x) \cdot \partial^{-1} \cdot (\hat{\Phi}^{-1})^*(1) = L \hat{\Phi}(1) \cdot \partial^{-1} \cdot w = L(v) \partial^{-1}w, \]
then from (4.46) we have

\[ Z_2 = L(v)\partial^{-1}w - \frac{1}{2} \left( L^2 \right)_- . \]

Clearly \( Y_2 + Z_2 = 0 \), then we obtain \( \partial L/\partial \delta_2^\prime = \partial L/\partial \delta_2^\prime \). When \( p \geq 3 \), the flows \( \partial/\partial \delta_p^\prime \) and \( \partial/\partial \delta_p^\prime \) can be determined by those flows with \( p = -1, 0, 1, 2 \) via the Virasoro commutation relations. Therefore we complete the proof.

5 Concluding Remarks

In this paper we have represented the extended KP hierarchy into a bilinear equation satisfied by the (adjoint) Baker–Akhiezer functions. What is more, we have constructed a class of additional symmetries for this hierarchy, and studied their reduction properties. Such results, which are analogue to those for the KP and the 2-BKP hierarchies, are expected to extend our understanding to the theory of integrable systems.

At the end of [20], a tau function of the extended KP hierarchy was introduced by using the densities of Hamiltonian functionals. Similar to the KP hierarchy, one can derive a Sato formula that links the tau function to the (adjoint) Baker–Akhiezer functions \( \psi(z) \) and \( \psi^\dagger(z) \). However, the relationship between the tau function and \( \hat{\psi}(z) \) (or \( \hat{\psi}^\dagger(z) \)) still needs to be clarified. It is why a bilinear equation of tau function is still missing. We also hope that the tau function of the extended KP hierarchy could be applied in the two-matrix model such as in the context of constrained KP hierarchies [3]. This will be studied elsewhere.

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References

1. Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional \( W_p \)-gravity. Comm. Math. Phys. 147(1), 25–56 (1992)
2. Aratyn, H., Nissimov, E., Pacheva, S.: Virasoro symmetry of constrained KP hierarchies. Phys. Lett. A 228(3), 164–175 (1997)
3. Aratyn, H., Nissimov, E., Pacheva, S.: Constrained KP hierarchies: additional symmetries, Darboux-Bäcklund solutions and relations to multi-matrix models. Internat. J. Modern Phys. A 12(7), 1265–1340 (1997)
4. Bonora, L., Xiong, C.S.: The \( (N, M) \)th Korteweg-de Vries hierarchy and the associated \( W \)-algebra. J. Math. Phys. 35(11), 5781–5819 (1994)
5. Cheng, Y.: Constraints of the Kadomtsev-Petviashvili hierarchy. J. Math. Phys. 33(11), 3774–3782 (1992)
6. Cheng, Y.: Modifying the KP, the \( n^{th} \) constrained KP hierarchies and their Hamiltonian structures. Comm. Math. Phys. 171(3), 661–682 (1995)
7. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type. Phys. D 4(3), 343–365 (1981/82)
8. Date, E., Kashiwara, M., Jimbo, M., Miwa, T.: Transformation Groups for Soliton Equations. Nonlinear Integrable Systems—Classical Theory and Quantum Theory (Kyoto, 1981), 39–119. World Sci Publishing, Singapore (1983)
9. Dickey, L.A.: Additional symmetries of KP, Grassmannian, and the string equation. Modern Phys. Lett. A 8(13), 1259–1272 (1993)
10. Dickey, L.A.: Additional symmetries of KP, Grassmannian, and the string equation. II. Modern Phys. Lett. A 8(14), 1357–1377 (1993)
11. Dickey, L.A.: On the constrained KP hierarchy. II. Lett. Math. Phys. 35 3, 229–236 (1995)
12. Krichever, I.: Linear operators with self-consistent coefficients and rational reductions of KP hierarchy. The nonlinear Schrödinger equation (Chernogolovka, 1994). Phys. D 87(1–4), 14–19 (1995)
13. Liu, S.-Q., Wu, C.-Z., Zhang, Y.: On the Drinfeld-Sokolov hierarchies of $D$ type. Intern. Math. Res. Notices 8, 1952–1996 (2011)
14. Orlov, A.Y., Schulman, E.I.: Additional symmetries for integrable equations and conformal algebra representation. Lett. Math. Phys. 12(3), 171–179 (1986)
15. Semenov-Tyan-Shanskii, M.A.: What a classical $r$-matrix is. (Russian) Funktsional. Anal. i Prilozhen. 17(4), 17–33 (1983). English translation: Functional Anal. Appl. 17 (1983), 259–272
16. Szablikowski, B., Blaszak, M.: Dispersionful analog of the Whitham hierarchy. J. Math. Phys. 49(8), 082701, 20 (2008)
17. Wu, C.-Z.: $R$-matrices and Hamiltonian structures for certain Lax equations. Rev. Math. Phys. 25(3), 1350005, 27 (2013)
18. Wu, C.-Z.: From additional symmetries to linearization of Virasoro symmetries. Physica D 249, 25–37 (2013)
19. Wu, C.-Z.: Tau functions and Virasoro symmetries for Drinfeld-Sokolov hierarchies. Adv. Math. 306, 603–652 (2017)
20. Wu, C.-Z., Zhou, X.: An extension of the Kadomtsev-Petviashvili hierarchy and its hamiltonian structures. J. Geom. Phys. 106, 327–341 (2016)
21. Yajima, N., Oikawa, M.: Formation and interaction of Sonic-Langmuir solitons. Prog. Theor. Phys. 56(6), 1719–1739 (1976)

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