Multiple multicontrol unitary operations: Implementation and applications
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The efficient implementation of computational tasks is critical to quantum computations. In quantum circuits, multicontrol unitary operations are important components. Here, we present an extremely efficient and direct approach to multiple multicontrol unitary operations without decomposition to CNOT and single-photon gates. With the proposed approach, the necessary two-photon operations could be reduced from $O(n^3)$ with the traditional decomposition approach to $O(n)$, which will greatly relax the requirements and make large-scale quantum computation feasible. Moreover, we propose the potential application to the $(n-k)$-uniform hypergraph state.

multicontrol unitary operation, cross phase modulation, c-path gate, merging gate, hypergraph state

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1 Introduction

Quantum computations have attracted attention during the last two decades because they are more powerful and extremely faster than classical computational methods [1]. Significant efforts have been devoted for many years in quantum information science and the attention has now turned to large-scale quantum computations. In 2009, 14-qubit entanglement was reported in a trapped ion system [2]. In 2012, based on a spontaneous parametric down-conversion, eight-photon GHz states were discussed by Huang et al. [3] and Yao et al. [4]. Subsequently, Wang et al. [5] reported the development of ten-photon entanglement. However, the probability problem is inevitable in linear optical quantum computations [6]. The photons themselves do not interact with each other; therefore, one has to achieve the necessary interaction in two-photon operations, such as CNOT, through the postselection technique, which will result in nondeterministic two-photon operations with only linear optical elements. For example, the success probability of a CNOT gate is only 1/4 [7, 8]. With an increasing number of photons and operations, the probability of success decreases exponentially. Therefore, the operational efficiency is critical to optical quantum computations.

On the other hand, the recently developed cross phase modulation (XPM) technique is a useful supplement to linear optical technologies for quantum computation. Based on the XPM technique, the analysis of the Bell state [9], CNOT gate [10, 11], and even Toffoli gate [12, 13] is nearly deterministic and is an alternative route to quantum computations [14-18]. Especially, with the element gates, called controlled-path (c-path) and the merging gates proposed in refs. [11-13], the implementation of quantum computation could be direct...
without any decomposition of the CNOT and single-photon gates, and use less resources. For example, the necessary resources for multicontrol gates, such as the general Toffoli gate, Fredkin gate, etc., increase linearly \( (O(n)) \) as a function of the involved photon number \([12, 13]\), compared with the polynomial \( (O(n^2)) \) in other studies \([19]\). Even for general unitary operations, the approach with c-path and merging gates (called CPM approach below) could exponentially relax the requirements on the resources, including the number of operations, the ancilla photons and coherent beams, etc. \([18]\).

Previous works based on the XPM technique either focused on specific gates, such as CNOT \([10,11]\), Toffoli, Fredkin gates \([12,13,17,18]\), among others or the general operations, such as the general unitary operations \([18]\). The problem is whether one can further reduce the complexity or not if the quantum circuit includes special structures. In this study, we consider a quantum circuit constructed by a series of multicontrol operations. As mentioned above, a multicontrol \((n-1)\)-control operation requires \( O(n) \) c-path and merging gates \([12,13]\). We will show below that, with the newly designed c-path gate, even for multiple \( n \) multicontrol operations, \( O(n) \) c-path and merging gates are sufficient. In other words, the increasing photon number is linear. Moreover, this special operation can generate hypergraph states \([20-24]\), which is important to the measurement-based quantum computation.

In sect. 2, we introduce a new c-path gate, which will be used as the element gate to implement multichannel gates, associated with the original c-path and merging gates. As an example, we discuss in sect. 3 the implementation of four triple-control unitary operations that can be generalized to multiple \( (n-1) \)-control unitary operations in sect. 4. To show the efficiency of the proposed approach, we compare the implementation complexity with former approaches in sect. 5. In addition, the proposed approach is generalized in sect. 6. The implementations of the multiple \( (n-k) \)-control unitary operations and the corresponding potential applications are discussed in sect. 7. Finally, we discuss the concluding remarks.

### 2 The controlled-path gate and merging gate

The original controlled-path (c-path) gate was firstly introduced in ref. \([11]\) and was further developed in refs. \([12,13,17,18]\). This gate can be implemented using only linear optics and had been demonstrated in experiment \([25]\). Moreover, it can be used in experiments to calculate unknown eigenvalues \([26]\), compress quantum data \([27]\), or implement a quantum Fredkin gate \([28]\), and widely used in theoretical schemes to generate graph states \([29,30]\), W states \([31]\), Dicke states \([32,33]\), etc.

The control photon of the original c-path gate is encoded by its polarization information as \( |0\rangle \equiv |H\rangle \) and \( |1\rangle \equiv |V\rangle \). However, when we discuss the implementation of multiple multicontrol unitary operations below, the control photon has some spatial modes, e.g., 8, but we only use part of them as the control signal. This means that the control signal sometimes contains a single photon and sometimes it is in vacuum state. In other words, the control signal is encoded by the photon number, i.e., the control signal \( |0\rangle \) denotes a vacuum state and \( |1\rangle \) denotes the quantum state containing a single photon. The desired c-path gate is required to implement the following transformation:

\[
|\Psi_1\rangle |0\rangle^C |\Phi_1\rangle^T + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle^T
\rightarrow |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle^T + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle^T,
\]

where the states \( |\Psi_{1(2)}\rangle \) denote the other components of the multiphoton state. The target single photon denoted by superscript T is separated into one spatial mode when the single photon appears on the control spatial modes \(|1\rangle^C\), whereas it will be separated into another one spatial mode when the single photon appears on other spatial modes that haven’t been used as control modes \(|0\rangle^C\).

Clearly, the original c-path gate cannot be used to implement the above operation. We now design a new c-path gate, shown in Figure 1(a), to implement the above operation. First, the target photon is injected into a 50:50 beam splitter (BS). Subsequently, the control and target photons interact with the two coherent beams \(|\alpha\rangle_{cs} |\alpha\rangle_{cs}\) via the XPM processes in Figure 1(a). The initial state then evolves to the

![Figure 1](image-url) (Color online) (a) The new c-path gate: the target single photon is transformed to the spatial mode 2, when the control mode contains a single photon; otherwise, it is transformed to the spatial mode 1. (b) The original c-path and merging gate: the detailed design of these two gates is found in Appendix. For comparison, the control mode of the original c-path gate is denoted by superscripts \((H, V)\), whereas that of the new c-path gate is denoted by superscripts \((0, 1)\).
following state:
\[
\frac{1}{\sqrt{2}} \left( |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha e^{i\theta/2}\rangle_{cs} |\alpha\rangle_{cs} + |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha\rangle_{cs} |\alpha\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha e^{i\theta/2}\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha\rangle_{cs} |\alpha\rangle_{cs} \right),
\]
where subscripts 1, 2 outside the bracket denote the spatial modes of the target photon, respectively. Two phase shifts \(-\theta/2\) are applied to the two coherent beams, respectively, yielding the following state:
\[
\frac{1}{\sqrt{2}} \left( |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha e^{i\theta/2}\rangle_{cs} |\alpha\rangle_{cs} + |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha\rangle_{cs} |\alpha\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha e^{i\theta/2}\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha\rangle_{cs} |\alpha\rangle_{cs} \right).
\]
(3)
Let the two coherent beams interfere on the 50:50 BS and then obtain the following state,
\[
\frac{1}{\sqrt{2}} \left( |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha^+\rangle_{cs} |\alpha^-\rangle_{cs} + |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha^-\rangle_{cs} |\alpha^+\rangle_{cs} \right.
\]
\[
\left. \sqrt{2} |\alpha e^{-i\theta/2}\rangle_{cs} |0\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha e^{i\theta/2}\rangle_{cs} |0\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha^-\rangle_{cs} |\alpha^+\rangle_{cs} \right),
\]
(4)
where \(|\alpha^-\rangle = |\sqrt{2} \alpha \sin(\theta/2)\rangle\) and \(|\alpha^+\rangle = |\sqrt{2} \alpha \cos(\theta/2)\rangle\).

Next, we detect the second coherent beam with a photon number-resolving detector (PND). The PND is implemented by using the XPM technique, as previously proposed [12, 13]. If \(k\) single photons are registered, the following state is obtained,
\[
e^{ik\pi/2} |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T |\alpha^+\rangle_{cs} + e^{-ik\pi/2} |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\alpha^+\rangle_{cs}.
\]
(5)
In this case, the first coherent state in the above two components is the same, and \(\sqrt{2} \alpha \cos(\theta/2) \sim \sqrt{2} \alpha\) owing to \(\theta \ll 1\), e.g., \(\theta \sim 10^{-5}\) for weak Kerr nonlinearity; therefore, it can be recycled. The unwanted phase shifts \(e^{ik\pi/2}\) and \(e^{-ik\pi/2}\) can be removed by applying additional phase shifts to the first spatial mode controlled by the detection of the PND through classical feedforward. Subsequently, the following desired state is achieved,
\[
|\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T,
\]
(6)
i.e., the target photon will be transformed to the first spatial mode 1 when the controlled state is a vacuum state; otherwise, the target photon will be transformed to the second spatial mode 2.

On the other hand, if no single photons are registered by the PND, the state described by eq. (4) will collapse to the following state:
\[
|\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T \left( |\sqrt{2} \alpha e^{-i\theta/2}\rangle_{cs} + |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T |\sqrt{2} \alpha e^{i\theta/2}\rangle_{cs} .
\]
\]
(7)
Since the rest first coherent beam containing opposite phase shifts, then it should be removed further. We detect the first coherent beam with an additional PND. If \(m\) single photons are registered, one can obtain the following state:
\[
e^{-im\theta/2} |\Psi_1\rangle |0\rangle^C |\Phi_1\rangle_T^T + e^{im\theta/2} |\Psi_2\rangle |1\rangle^C |\Phi_2\rangle_T^T.
\]
(8)
The unwanted phase shifts \(e^{-im\theta/2}\) and \(e^{im\theta/2}\) can be removed and the above state can be transformed to the desired state in eq. (6) by the switch that is controlled by the detection of the first PND through the classical feedforward as well.

Because the original c-path and merging gate are used below, we describe what works without any details given; the design of the c-path and merging gate are given in Appendix. The original c-path gate is implemented to the following transformation,
\[
|\Psi_1\rangle |H\rangle^C |\psi\rangle_T + |\Psi_2\rangle |V\rangle^C |\psi\rangle_T
\]
\[
\rightarrow |\Psi_1\rangle |H\rangle^C |\psi\rangle_T + |\Psi_2\rangle |V\rangle^C |\psi\rangle_T ,
\]
(9)
i.e., the target photon is separated into two spatial modes 1, 2 that depend on the polarizations of the control single photon but not on the photon number in the new c-path gate in eq. (1). Moreover, to complete the quantum computation, we need the inverse transformation of the original or new c-path gate. As shown in Appendix, the necessary inverse transformation can be completed by the same merging gate [12, 13, 17, 18], which can merge the two spatial modes of the target single photon back to one without changing anything else, i.e., by implementing the inverse transformation of eq. (1) or eq. (9). Furthermore, we note here that, in the c-path gate (including the original and new c-path gate) or merging gate, the spatial modes of the target single photon could be more than 2. The c-path gate separates the \(n\) spatial modes of the target photon into \(2n\) spatial modes, whereas the merging gate merges the \(2n\) spatial modes back to the \(n\) spatial modes. Clearly, the c-path and merging gate work well in the case of more than 2 spatial modes, which is a flexible way to use them.

3 Implementation of four triple-control unitary operations

With the new c-path gates and the original c-path and merging gates, the implementation of multiple multicontrol unitary operations becomes more efficient than the one using only the original c-path and merging gates. The discussion regarding efficiency is all in sect. 5. Without losing generality, we
use the implementation of the four triple-control unitary operations in Figure 2, as an example. The general four-photon state can be described as follows:

\[
\begin{align*}
A_1 |HHHH\rangle + A_2 |HHHV\rangle + A_3 |HHVH\rangle + A_4 |HHVV\rangle \\
+ A_5 |HVHH\rangle + A_6 |HVHV\rangle + A_7 |HVVH\rangle + A_8 |HVVV\rangle \\
+ A_9 |VHHH\rangle + A_{10} |VHHV\rangle + A_{11} |VHVH\rangle + A_{12} |VVVV\rangle \\
+ A_{13} |VVHH\rangle + A_{14} |VVHV\rangle + A_{15} |VVVV\rangle + A_{16} |VVVV\rangle ,
\end{align*}
\]

where the coefficients satisfy the normalized condition \(\sum_{i=1}^{16}|A_i|^2 = 1\). To describe the processes clearly, we separate them into the following five steps.

### 3.1 Step 1: adding the polarization information of control single photos to the ancilla single photon

First, we introduce a single photon (\(|H\rangle^a\)) as ancilla and let the first single photon of the above state to control the ancilla single photon through the original c-path gate and thus the following state is achieved,

\[
\begin{align*}
(A_1 |HHHH\rangle + A_2 |HHHV\rangle + A_3 |HHVH\rangle + A_4 |HHVV\rangle \\
+ A_5 |HVHH\rangle + A_6 |HVHV\rangle + A_7 |HVVH\rangle \\
+ A_8 |HVVV\rangle |H\rangle_1^a + (A_9 |VHHH\rangle + A_{10} |VHHV\rangle \\
+ A_{11} |VHVH\rangle + A_{12} |VVHV\rangle + A_{13} |VVHH\rangle + A_{14} |VVVV\rangle \\
+ A_{15} |VVHV\rangle + A_{16} |VVVV\rangle) |H\rangle_1^a .
\end{align*}
\]

Subsequently, the second and third single photons control the ancilla single photon in return by the two original c-path gates, respectively. The ancilla single photon is separated into 8 spatial modes that depend on the polarizations of the three single photons except of the fourth single photon, as indicated in the following state:

\[
\begin{align*}
A_1 |HHHH\rangle |H\rangle_1^a + A_2 |HHHV\rangle |H\rangle_1^a + A_3 |HHVH\rangle |H\rangle_1^a \\
+ A_4 |HHVV\rangle |H\rangle_1^a + A_5 |HVHH\rangle |H\rangle_2^a + A_6 |HVHV\rangle |H\rangle_3^a \\
+ A_7 |HVVH\rangle |H\rangle_4^a + A_8 |HVVV\rangle |H\rangle_5^a + A_9 |VHHH\rangle |H\rangle_6^a \\
+ A_{10} |VHHV\rangle |H\rangle_7^a + A_{11} |VHVH\rangle |H\rangle_8^a + A_{12} |VVHV\rangle |H\rangle_9^a \\
+ A_{13} |VVHH\rangle |H\rangle_1^a + A_{14} |VVHV\rangle |H\rangle_2^a \\
+ A_{15} |VVVV\rangle |H\rangle_3^a + A_{16} |VVVV\rangle |H\rangle_4^a .
\end{align*}
\]

### 3.2 Step 2: the first triple-control unitary operation

Next, the whole polarization information of the three single photons is encoded to the spatial modes of the ancilla single photon. Thus, we use the ancilla single photon to control the fourth single photon and implement the first triple-control unitary operation. We only use the spatial mode 8 of the ancilla single photon to control the fourth single photon by the new c-path gate. According to eq. (1), the fourth single photon is separated into spatial mode 2 when the spatial mode 8 of the ancilla single photon involves a single photon; otherwise, it will be separated into spatial mode 1, i.e., the

Figure 2 (Color online) The implementation of four triple-control unitary operations. The processes are separated into five steps: (1) an ancilla single photon is introduced, which will be separated into 8 spatial modes controlled by the first three single photons \(p_1, p_2, p_3\) through three original c-path gates, respectively; (2) using the spatial mode 8 of the ancilla single photon to control the first target single photon \(p_4\) through three original c-path gates, the first control unitary operation associated with the single-photon unitary operation \(U_1\); applied to the spatial mode 2 of the target single photon and merging gate will be implemented; (3) to implement the second control unitary operation, the information of the single photon \(p_3\) should be removed from the ancilla single photon by the merging gate, whereas the information of the single photon \(p_2\) should be added to the ancilla single photon by the original c-path gate; (4) following similar processes (operations 2, 3, and 4) the desired four triple-control unitary operations can be implemented; and (5) the ancilla single photon is disentangled from the four-photon state by the three merging gates.
following state is obtained,
\[
A_1 |HHHHH⟩|H⟩_1^a + A_2 |HHHVV⟩|H⟩_1^a + A_3 |HHVHV⟩|H⟩_1^a \\
+ A_4 |HHVHV⟩|H⟩_1^a + A_5 |HVHHH⟩|H⟩_2^a + A_6 |HVHVV⟩|H⟩_2^a \\
+ A_7 |HVHVH⟩|H⟩_2^a + A_8 |HHVVV⟩|H⟩_3^a + A_9 |VVHHH⟩|H⟩_3^a \\
+ A_{10} |VVHVV⟩|H⟩_4^a + A_{11} |VVVHV⟩|H⟩_4^a + A_{12} |VVVHH⟩|H⟩_4^a \\
+ A_{13} |VHHHH⟩|H⟩_5^a + A_{14} |VHVHV⟩|H⟩_5^a \\
+ A_{15} |VHVHH⟩|H⟩_6^a + A_{16} |VVVVV⟩|H⟩_6^a .
\] (13)

Only in the last two components of the above equation, the fourth single photon appears in the spatial mode 2, in which the following desired single-photon unitary operation (could be arbitrary)
\[
U_1 = \begin{pmatrix}
  e^{iδ_1} \cos{β_1} & -e^{iδ_1} \sin{β_1} \\
  -e^{-iδ_1} \sin{β_1} & e^{-iδ_1} \cos{β_1}
\end{pmatrix}
\] (14)
is applied and the merging gate is applied to the fourth single photon using the ancilla single photon as control signal. The above state then evolves to
\[
A_1 |HHHHH⟩|H⟩_1^a + A_2 |HHHVV⟩|H⟩_1^a + A_3 |HHVHV⟩|H⟩_1^a \\
+ A_4 |HHVHV⟩|H⟩_1^a + A_5 |HVHHH⟩|H⟩_2^a + A_6 |HVHVV⟩|H⟩_2^a \\
+ A_7 |HVHVH⟩|H⟩_2^a + A_8 |HHVVV⟩|H⟩_3^a + A_9 |VVHHH⟩|H⟩_3^a \\
+ A_{10} |VVHVV⟩|H⟩_4^a + A_{11} |VVVHV⟩|H⟩_4^a + A_{12} |VVVHH⟩|H⟩_4^a \\
+ A_{13} |VHHHH⟩|H⟩_5^a + A_{14} |VHVHV⟩|H⟩_5^a \\
+ (A_{15} e^{iδ_1} \cos{β_1} - A_{16} e^{-iδ_1} \sin{β_1}) |VVVV⟩|H⟩_6^a \\
+ (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) |VVVV⟩|H⟩_6^a ,
\] i.e., the first triple-control unitary operation to the fourth single photon is now complete.

3.3 Step 3: removing and adding process

The second triple-control unitary operation is implemented as above by introducing another ancilla single photon; however, the implementation is not sufficiently efficient. The second triple-control unitary operation requires the polarization information of the first, second, and fourth single photons, whereas the ancilla single photon after the first triple-control unitary operation includes the polarization information of the first, second, and third single photons. Therefore, we only need to remove the polarization information of the third single photon and add that of the fourth single photon to the ancilla single photon and then use the ancilla single photon to control the third single photon. The removing process is completed by the merging gate with the third single photon as control signal, and the spatial modes merge as follows:
\[(1, 2) \rightarrow (1, 3); (3, 4) \rightarrow (2, 3); (5, 6) \rightarrow (3, 5); (7, 8) \rightarrow (4, 6).
\] (16)

The state in eq. (15) evolves back to the following state:
\[
A_1 |HHHHH⟩|H⟩_1^a + A_2 |HHHVV⟩|H⟩_1^a + A_3 |HHVHV⟩|H⟩_1^a \\
+ A_4 |HHVHV⟩|H⟩_1^a + A_5 |HVHHH⟩|H⟩_2^a + A_6 |HVHVV⟩|H⟩_2^a \\
+ A_7 |HVHVH⟩|H⟩_2^a + A_8 |HHVVV⟩|H⟩_3^a + A_9 |VVHHH⟩|H⟩_3^a \\
+ A_{10} |VVHVV⟩|H⟩_4^a + A_{11} |VVVHV⟩|H⟩_4^a + A_{12} |VVVHH⟩|H⟩_4^a \\
+ A_{13} |VHHHH⟩|H⟩_5^a + A_{14} |VHVHV⟩|H⟩_5^a \\
+ (A_{15} e^{iδ_1} \cos{β_1} - A_{16} e^{-iδ_1} \sin{β_1}) |VVVV⟩|H⟩_6^a \\
+ (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) |VVVV⟩|H⟩_6^a .
\] (17)

Subsequently, the addition is implemented by the original c-path gate with the fourth single photon as the control signal. Thus, the following state is achieved,
\[
A_1 |HHHHH⟩|H⟩_1^a + A_2 |HHHVV⟩|H⟩_1^a + A_3 |HHVHV⟩|H⟩_1^a \\
+ A_4 |HHVHV⟩|H⟩_1^a + A_5 |HVHHH⟩|H⟩_2^a + A_6 |HVHVV⟩|H⟩_2^a \\
+ A_7 |HVHVH⟩|H⟩_2^a + A_8 |HHVVV⟩|H⟩_3^a + A_9 |VVHHH⟩|H⟩_3^a \\
+ A_{10} |VVHVV⟩|H⟩_4^a + A_{11} |VVVHV⟩|H⟩_4^a + A_{12} |VVVHH⟩|H⟩_4^a \\
+ A_{13} |VHHHH⟩|H⟩_5^a + A_{14} |VHVHV⟩|H⟩_5^a \\
+ (A_{15} e^{iδ_1} \cos{β_1} - A_{16} e^{-iδ_1} \sin{β_1}) |VVVV⟩|H⟩_6^a \\
+ (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) |VVVV⟩|H⟩_6^a .
\] (18)

3.4 Step 4: the other three triple-control unitary operations

Obviously, the ancilla single photon in eq. (18) contains the entire polarization information of the first, second, and fourth single photons and the second triple-control unitary operation is applied by using the spatial mode 8 of the ancilla single photon as control signal, associated with the new c-path gate and arbitrary single photon unitary operation
\[
U_2 = \begin{pmatrix}
  e^{iδ_2} \cos{β_2} & -e^{iδ_2} \sin{β_2} \\
  -e^{-iδ_2} \sin{β_2} & e^{-iδ_2} \cos{β_2}
\end{pmatrix}
\] (19)

applied to the spatial mode 2 of the third single photon after the c-path gate and the merging gate. The following state is then obtained,
\[
A_1 |HHHHH⟩|H⟩_1^a + A_2 |HHHVV⟩|H⟩_1^a + A_3 |HHVHV⟩|H⟩_1^a \\
+ A_4 |HHVHV⟩|H⟩_1^a + A_5 |HVHHH⟩|H⟩_2^a + A_6 |HVHVV⟩|H⟩_2^a \\
+ A_7 |HVHVH⟩|H⟩_2^a + A_8 |HHVVV⟩|H⟩_3^a + A_9 |VVHHH⟩|H⟩_3^a \\
+ A_{10} |VVHVV⟩|H⟩_4^a + A_{11} |VVVHV⟩|H⟩_4^a + A_{12} |VVVHH⟩|H⟩_4^a \\
+ A_{13} |VHHHH⟩|H⟩_5^a + A_{14} |VHVHV⟩|H⟩_5^a \\
+ (A_{15} e^{iδ_1} \cos{β_1} - A_{16} e^{-iδ_1} \sin{β_1}) |VVVV⟩|H⟩_6^a \\
+ (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) |VVVV⟩|H⟩_6^a \\
+ A_{15} e^{iδ_2} \cos{β_2} \\
- (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) e^{iδ_2} \sin{β_2}] |VVVV⟩|H⟩_6^a \\
+ (A_{15} e^{iδ_1} \cos{β_1} - A_{16} e^{-iδ_1} \sin{β_1}) |VVVV⟩|H⟩_6^a + A_{14} e^{-iδ_2} \sin{β_2} \\
+ (A_{15} e^{-iδ_1} \sin{β_1} + A_{16} e^{iδ_1} \cos{β_1}) e^{-iδ_2} \cos{β_2}] |VVVV⟩|H⟩_6^a .
\] (20)
which is the desired state after two triple-control unitary operations.

Similarly, to implement the third triple-control unitary operation, we first merge the spatial modes as follows:

\[(1, 3) \rightarrow 1; (2, 4) \rightarrow 2; (5, 7) \rightarrow 3; (6, 8) \rightarrow 4, \quad (21)\]

by using the merge gate and the second single photon as control signal to remove the polarization information of the second single photon from the ancilla single photon. Subsequently, following the process described by eq. (18) to eq. (20), the third triple-control unitary operation is completed. The final triple-control unitary operation requires the following merging operation,

\[(1, 5) \rightarrow 1; (3, 7) \rightarrow 2; (2, 6) \rightarrow 3; (4, 8) \rightarrow 4, \quad (22)\]

and similar operations as discussed above.

3.5 Step 5: removing the ancilla single photon

After the desired four controlled unitary operations are completed, the ancilla single photon is disentangled from the four-photon state; otherwise, it may affect the additional operations of the four-photon state. Because the ancilla single photon contains the polarization information of the second, third, and fourth single photons, we can use the second, third, and fourth single photons, as control signals, to merge the ancilla single photon by the three merging gates, as shown in Figure 3. After the spatial mode of the ancilla single photon is back to one, the ancilla single photon is disentangled from the four-photon state.

4 Implementation of multiple \((n - 1)\)-control unitary operation

The above approach is straightforward, and one can implement multiple \((n)\) m tic control \((\text{(n-1)-control})\) unitary operations more efficiently than past methods. The initial state can be described by the following equation:

\[
\sum_{i=1}^{2^n} |\theta_i^{(n)}\rangle = A_1 |H \cdots HH\rangle_{1, \ldots, n} + A_2 |H \cdots HV\rangle_{1, \ldots, n} + A_3 |H \cdots VH\rangle_{1, \ldots, n} + A_4 |H \cdots VV\rangle_{1, \ldots, n} + \cdots, \quad (23)
\]

where \(\sum_{i=1}^{2^n} |A_i|^2 = 1\). First, we use the first \(n - 1\) single photons except of the final single photon as control signals to control the ancilla single photon \(|H\rangle^n\) by \(n - 1\) original c-path gates respectively, yielding the following state:

\[
\sum_{i=1}^{2^{n-1}} A_{2i-1} |2i-1\rangle^{(n)} |H\rangle^n_x + \sum_{i=1}^{2^{n-1}} A_{2i} |2i\rangle^{(n)} |H\rangle^n_y. \quad (24)
\]

To implement the first \((n - 1)\)-control unitary operation, we use the final spatial mode \(2^{n-1}\) of the ancilla single photon as control signal to control the last single photon by a new c-path gate and the last single photon is separated into 2 spatial modes. Applying the desired single photon unitary operation \(U_1\) to the spatial mode 2 of the last single photon and merging the last single photon by the merging gate, the desired \((n - 1)\)-control unitary operation is implemented.

Similarly, to implement the other \((n-1)\)-control unitary operations, the polarization information of the target single photon is removed by merging the corresponding spatial modes through the merging gate. Subsequently, the polarization information of the single photon, which was the target photon in the last control unitary operation, is added to the ancilla single photon by the original c-path gate. The pair of new c-path and original merging gates, associated with the single photon unitary operation will complete the desired control unitary operations as well.

5 Comparison with the former approaches

Next we compare the complexity of the proposed approach with past approaches and the number of the required two-qubit gates is shown in Table 1. In traditional quantum computation, the multicontrol unitary operation is decomposed into two-photon (e.g., CNOT) gates and single-photon unitary operations. Because the single-photon unitary operation can be implemented by a half-wave plate and a quarter-wave plate in an optical system [34], the number of single-photon unitary operations is not considered in this study. It has been demonstrated that a general \((n-1)\)-control gate requires \(O(n^2)\) two-qubit gates [19]. In this case, \(O(n^3)\) two-qubit gates are necessary to implement \(n\) multicontrol gates, i.e., the complexity is about \(O(n^3)\) at least.

The \((n - 1)\)-control gate can be implemented more efficiently than the decomposition approach with the original c-path and merging gate (denoted by the original CPM in Table 1). It has been demonstrated in refs. [12, 13] that only \(n - 1\) pairs of c-path and merging gates are sufficient for a general \((n - 1)\)-control gate; hence, the complexity of \(n\) multicontrol gates with the original c-path and merging gates is about \(O(n^2)\).

Figure 3 (Color online) The three-uniform four-photon hypergraph state. This state is generated by the four 2-control phase gates in the right-hand side.
In the proposed approach (denoted by the new CPM in Table 1), \( n - 1 \) pairs of the original c-path and merging gates are needed to add and remove the polarization information to the ancilla single photon first. For implementing each control operation, only one pair of the new c-path gate and the original merging gate associated with one pair of the original c-path and merging gates are enough. Therefore, only \( 3n - 1 \) pairs of the original or new c-path and merging gates are required in the proposed approach, i.e., the complexity is preserved to be \( O(n) \) and linearly increasing with the number of multicontrol unitary operations. Obviously, the proposed approach is more efficient than former approaches, especially in large-scale quantum computations.

### Table 1

| CNOT \([19]\) | Original CPM \([12,13]\) | New CPM |
|---|---|---|
| \( O(n^2) \) | \( O(n^2) \) | \( O(n) \) |

In the proposed approach, \( n - 1 \) pairs of the original c-path and merging gates are needed to implement multicontrol unitary operations. A direct application is to use the decomposition of the CNOT and single-photon gates. “Original CPM” denotes the approach with the original c-path and merging gate in past works and “new CPM” denotes the present approach.

The implementation of multiple \((n-k)\)-control unitary operations is more complicated than that of multiple \((n-1)\)-control unitary operations. More operations are required because more than the information of a single photon is removed before applying the next control gate, if \( k > 1 \). In other words, the complexity is beyond the linearly increasing regime, i.e., \( O(n) \), and gradually close to \( O(n^2) \) if \( k \) is large. On the other hand, the complexity is reduced with increasing \( k \), if we choose the original CPM approach. For a particular \( k \), the complexity of the present approach is higher than the original CPM approach. In this case, one may naturally choose the better one. In any case, the proposed approach is an important addition to the original CPM approach and one can combine the two approaches to implement a quantum circuit more efficiently.

### 7 Potential applications of multiple multicontrol unitary operations

We discuss the potential applications of the above multiple multicontrol unitary operations. A direct application is to achieve more efficient large-scale quantum computations owing to the fact that the operations can be implemented extremely efficiently with the proposed approach. If a quantum circuit includes the structure built by series of multicontrol unitary operations, the present approach is a better choice. The linear increase with the number of photons is especially suitable to large-scale quantum computations.

Another important application is to generate the hypergraph states [20-24], which are regarded as the special case of locally maximally entangleable (LME) states [35]. Different than ordinary graph states, where edges connect two vertices, the edges in hypergraph states connect more than two vertices; hence, the hypergraph states are the generalized form of graph states. This means that the structure of hypergraph states is more complex than that of the graph states, leading to the more difficult generation of hypergraph states. We discuss the generation of the so-called \((n-k)\)-uniform \( n \)-photon hypergraph state \(|g_{n-k}\rangle\), which can be expressed as follows:

\[
|g_{n-k}\rangle = \prod_{[i_1, \ldots, i_{n-k}] \in E} C_{i_1, \ldots, i_{n-k}} |\uparrow\rangle^{\otimes n},
\]

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where \(\{l_1, \cdots, l_{p-1}\} \in E\) means that the \(k\) vertices are connected by a \(k\)-hyperedge and \(C_{l_i - l_{i+1}}\) denotes the \((n-k)\)-control phase gate. For example, in Figure 3, we show the 3-uniform 4-photograph state, which can be generated with the four two-control phase gates in the right-hand side of Figure 3. Obviously, the desired \((n-k)\)-control phase gates are only the special cases of \((n-k)\)-control unitary operations, which can be efficiently implemented by the proposed approach, as discussed in sect. 6. This implies that the \((n-k)\)-uniform \(n\)-photograph state is efficiently generated with the proposed approach.

8 Discussion and conclusions

The core element in our approach is the XPM technique, which allows for efficient quantum computation. This technique recently has been used widely in quantum logic gates [9-18, 36-39], cluster or graph state generations [40-44], entanglement concentration [45-52], entangled states generation [53-68], etc. Its feasibility had been demonstrated in theory [69-73] and even the multimode has been considered. More recently, a realistic experiment of an efficient XPM based on the closed-loop double-A system has been reported [74]. In addition, a similar technology has been used in other physical systems to implement quantum computational task, such as cavity QED [75-107], etc. Therefore the XPM technique and this approach are feasible with the current experimental technology.

In conclusion, with the new design of the c-path gate, the multiple multicontrol unitary operations can be implemented directly and efficiently by combining the original c-path, merging gates, and the new c-path gates. The linear increase with the number of involved photons makes this approach suitable for large-scale quantum computations.

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Appendix

In this section, we describe the original c-path gate in refs. [11-13, 17, 18]. The initial state is expressed as follows:
\[
|\Psi_1\rangle|H\rangle^c|\phi\rangle_T^+ + |\Psi_2\rangle|V\rangle^c|\psi\rangle_T^T.
\]

(a1)

First, the control single photon is injected into a polarized beamsplitter (PBS), which lets the |H\rangle mode to be transmitted and the |V\rangle mode to be reflected. At the same time, the target single photon is injected into the 50:50 beamsplitter (BS) and these two spatial modes associated with the two modes of the control single photon interact with the two qubus beams |\alpha\rangle_{cs} |\alpha\rangle_{cs}, as shown in Figure a1. Subsequently, the following state is achieved,
\[
\frac{1}{\sqrt{2}} \left( |\Psi_1\rangle|H\rangle^c |\phi\rangle_T^+ |\alpha\rangle_{cs}^i |\alpha\rangle_{cs}^+ \right)_{cs} + |\Psi_2\rangle|V\rangle^c |\psi\rangle_T^T |\alpha\rangle_{cs}^i |\alpha\rangle_{cs}^2 \right)_{cs} + |\Psi_2\rangle|V\rangle^c |\psi\rangle_T^T |\alpha\rangle_{cs}^+ |\alpha\rangle_{cs}^{-} \right)_{cs} + |\Psi_2\rangle|V\rangle^c |\psi\rangle_T^T |\alpha\rangle_{cs}^{-} |\alpha\rangle_{cs}^{-} \right)_{cs}.
\]

(a2)

A two-phase shift $-\theta$ is applied to the two qubus beams and the two qubus beams interfere on the 50:50 BS, yielding the following state:
\[
\frac{1}{\sqrt{2}} \left( |\Psi_1\rangle|H\rangle^c |\phi\rangle_T^+ \sqrt{2} |\alpha\rangle_{cs} \right)_{cs} + |\Psi_2\rangle|V\rangle^c |\psi\rangle_T^T |\beta\rangle_{cs}^{-} + |\Psi_2\rangle|V\rangle^c |\psi\rangle_T^T |\beta\rangle_{cs}^{-} \right)_{cs}.
\]

(a3)
Let the two qubit beams interfere on the 50:50 BS, then, we obtain the following state:

$$\frac{1}{\sqrt{2}} \left( |\Phi_1^C| \left| \Phi'_1 \right|^T \mathcal{C} |\sqrt{2} \alpha \cos \theta \right|_c s |0\rangle_{c s} + |\Psi_1^C| \left| \Psi'_1 \right|^T \mathcal{C} |\sqrt{2} \alpha \cos \theta \right|_c s |0\rangle_{c s} + |\Psi_2^C| \left| \Psi'_2 \right|^T \mathcal{C} |\sqrt{2} \alpha \cos \theta \right|_c s |0\rangle_{c s} \right)$$

where $|\beta_-\rangle_{c s} = \frac{\alpha e^{i \theta}}{\sqrt{2}}$ and $|\beta_+\rangle_{c s} = -\frac{\alpha e^{i \theta}}{\sqrt{2}}$. Clearly, only the vacuum state $|0\rangle_{c s}$ should be distinguished from the state $|\beta_-\rangle$; therefore the photon number non-resolving detector (PNND) is necessary to complete the discrimination. If no photon is registered by the PNND, the desired state is achieved,

$$|\Psi_1^C| \left| \Psi'_1 \right|^T + |\Psi_2^C| \left| \Psi'_2 \right|^T$$

otherwise, by the single-photon operation $\sigma_z$ applied to the control photon and the switch applied to the two spatial modes of the target photon, controlled by the detection through the classical feedforward, the above state is transformed to the desired one in eq. (a9). In this case, the qubus beams are recycled as well because the induced phase shift is also small.

If the initial state in eq. (a6) is replaced by the following state,

$$|\Psi_1^C| \left| 0 \right|^C \mathcal{C} |\Phi_1 \rangle |\Phi'_1 \rangle + |\Psi_2^C| \left| 1 \right|^C \mathcal{C} |\Phi_2 \rangle |\Phi'_2 \rangle$$

i.e., the state in eq. (6) of the main text, the following desired state

$$|\Psi_1^C| \left| 0 \right|^C \mathcal{C} |\Phi_1 \rangle |\Phi'_1 \rangle + |\Psi_2^C| \left| 1 \right|^C \mathcal{C} |\Phi_2 \rangle |\Phi'_2 \rangle$$

is obtained from eq. (a6) to eq. (a9) by following the same processes. This means that the original merging gate can work well as the inverse gate for the new c-path gate without any new design.