Multihorizon spherically symmetric spacetimes with several scales of vacuum energy

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Abstract

We present a family of spherically symmetric multihorizon spacetimes with a vacuum dark fluid, associated with a time-dependent and spatially inhomogeneous cosmological term. The vacuum dark fluid is defined in a model-independent way by the symmetry of its stress–energy tensor, i.e. its invariance under Lorentz boosts in a distinguished spatial direction ($p_r = -\rho$ for the spherically symmetric fluid), which makes dark fluid essentially anisotropic and allows its density to evolve. The related cosmological models belong to the Lemaître class of models with anisotropic fluids and describe evolution of a universe with several scales of vacuum energy related to phase transitions during its evolution. The typical behavior of solutions and the number of spacetime horizons are determined by the number of vacuum scales. We study in detail the model with three vacuum scales: GUT, QCD and that responsible for the present accelerated expansion. The model parameters are fixed by the observational data and by conditions of analyticity and causality. We find that our Universe has three horizons. During the first inflation, the Universe enters a $T$-region, which makes expansion irreversible. After second phase transition at the QCD scale, the Universe enters $R$-region, where for a long time its geometry remains almost pseudo-Euclidean. After crossing the third horizon related to the present vacuum density, the Universe should have to enter the next $T$-region with the inevitable expansion.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Astronomical observations give a compelling evidence for the existence of a dark energy dominating our Universe at above 73% of its density and responsible for its accelerated expansion due to negative pressure, \( p = w \rho, w < -1/3 \) [1–9] with the best fit \( w = -1 \) [10–15], which corresponds to the Einstein cosmological term \( \lambda g_{\mu \nu} \) related to the de Sitter vacuum \( T_{\mu \nu} = 8\pi G \rho_{\text{vac}} g_{\mu \nu}/(\lambda = 8\pi G \rho_{\text{vac}}) \).

The well-known cosmological constant problem has two aspects. (i) Quantum field theory predicts for \( \rho_{\text{vac}} \) the Planckian scale \( \rho_{\text{vac}} = \rho_{\text{Pl}} = 5 \times 10^{93} \text{ g cm}^{-3} \). Confronting this with the observational value \( \rho_{\text{vac}} \simeq 3.6 \times 10^{-30} \text{ g cm}^{-3} = 1.4 \times 10^{-123} \rho_{\text{Pl}} \) creates the fine-tuning problem. (ii) The inflationary paradigm needs a large value of \( \rho_{\text{vac}} \) at the beginning of the Universe evolution, typically of the GUT scale \( \rho_{\text{vac}} \simeq \rho_{\text{GUT}} \simeq 5 \times 10^{77} \text{ g cm}^{-3} \); the observations indicate its much smaller value, and the Einstein equations require \( \rho_{\text{vac}} = \text{const} \).

A typical solution, which can be found in the literature, is to put \( \rho_{\text{vac}} = 0 \) for some reason and to introduce a dark energy of non-vacuum origin, which mimics the cosmological constant \( \lambda \) when necessary. A lot of theories and models have been developed to describe a dynamical dark energy (for a comprehensive review, see [9, 10]). The alternative to the cosmological constant provided by quintessence assumes the existence of a hypothetical component of matter content with \( w_Q \neq -1 \) [16]. Q-models based on a scalar field, rolling down its self-interaction potential [17], were tested using different methods with WMAP–CMB data. This gave the constraint \( w_Q \leq -0.7 \), with the best fit \( w_Q = -1 \) [18], which evidently corresponds to the cosmological constant \( \lambda \).

The quartessence models describe a transition from a dust-dominated stage to a late-time inflationary stage. The Chaplygin gas (CG) model, with the postulated equation of state \( p = -A/\rho \) gives the FRW flat model interpolating between \( p = 0 \) and a negative pressure at late times [19]. It can be obtained in the model of a superfluid CG with the potential \( V(\phi\phi) = M(\phi^*\phi/\mu + \mu/\phi^*\phi) \), which gives \( p = -4M^2/\rho \) [20, 21]. In the holographic dark energy approach (for a recent review see [22]) with an interaction between dark matter with \( p = 0 \) and dark energy [23], quartessence can be recovered as an isotropic perfect fluid [24], but the perfect fluid for holographic dark energy was found to be classically unstable [25]. The generalized CG (GCG) model, \( p = -A/\rho^\alpha \) [26], was introduced to overcome difficulties with satisfying the CMB constraints [27]. The observational constraint on the parameter \( \alpha \), \( 0 < \alpha < 0.2 \), implies a little difference between the GCG and \( \lambda = \text{const} \) [28].

Quintom cosmology describes dark energy in the framework of the braneworld cosmology by introducing two scalar fields, one being quintessence and the other phantom (for a review see [29]). The multiple-lambda cosmology [30] describes the Universe as a kind of multiverse [31] filled with a phantom energy; it may describe the evolution as a sequence of transitions between different inflationary stages. It is based on a perfect isotropic fluid with a time-dependent energy approach (for a recent review see [22]) with an interaction between dark matter with satisfying the CMB constraints [27]. The observational constraint on the parameter \( \alpha \), \( 0 < \alpha < 0.2 \), implies a little difference between the GCG and \( \lambda = \text{const} \) [28].

Quintom cosmology describes dark energy in the framework of the braneworld cosmology by introducing two scalar fields, one being quintessence and the other phantom (for a review see [29]). The multiple-lambda cosmology [30] describes the Universe as a kind of multiverse [31] filled with a phantom energy; it may describe the evolution as a sequence of transitions between different inflationary stages. It is based on a perfect isotropic fluid with a time-dependent equation of state [32], introduced phenomenologically [30] and describing phantom–nonphantom transitions [33]. Another approach to creating different effective scales of vacuum energy density rests on curvature-nonlinear multidimensional gravity with at least two extra factor spaces whose scale factors behave as scalar fields in four-dimensional spacetime [34].

According to observational data, dark energy is described by the inflationary equation of state, \( p = -\rho \). The stress–energy tensor (SET) has the form

\[
T_{\mu \nu}^\rho = \rho_{\text{vac}} \delta_{\mu \nu}, \quad \rho_{\text{vac}} = \text{const}.
\]

It represents de Sitter vacuum that generates the de Sitter geometry responsible for accelerated expansion independent of specific properties of particular models for \( \rho_{\text{vac}} \).
The standard models of cosmology and particle physics suggest a series of phase transitions that occurred in the course of the expansion and cooling history of the Universe [35]. A connection between particle physics and cosmology predicts the first inflation related to a phase transition at the grand unification (GUT) scale. The first inflation solves the key problems of the standard Big Bang model (see [36] and references therein) and has been confirmed by CMB observations [35]. The standard model of particle physics predicts another phase transition at the quantum chromodynamics (QCD) scale of 100–200 MeV (see [35] and references therein), which can be related to a second inflationary stage [37]. A quasi-stable QCD vacuum state can lead to a short period of inflation (7 e-foldings) consequently diluting the net baryon to photon ratio to its currently observed value. The second inflationary stage is considered in the model of a thermal inflation [38], with a duration of about 10-foldings. Arguments for a second inflation at the QCD scale exist also in an effective model of a QCD phase transition, which displays a high degree of supercooling at a critical temperature of the order of 100 MeV, so that the Universe increases exponentially during the quark–hadron transition [39].

The aim of this paper is to present and study a family of cosmological solutions to the Einstein equations describing a vacuum-dominated universe with several scales of vacuum energy related to phase transitions in the course of its evolution.

The gauge non-invariance of quantum cosmology leads to connection between the gauge and the quantum spectrum of a certain physical quantity, which can be specified in the framework of the minisuperspace model. There exists such a particular gauge in which the cosmological constant $\Lambda_1$ is quantized. Transitions between quantum levels of the operator $\Lambda_1$ can be related to several stages in the universe evolution with different values of vacuum density $\rho_{\text{vac}}$ [40].

At the classical level, the key point is the algebraic structure of a source term in the Einstein equations. In the model-independent approach, a vacuum is defined by the symmetry of its SET [41–43], as suggested by the Petrov classification for SETs. The Einstein cosmological term corresponds to the maximally symmetric de Sitter vacuum with $\rho_{\text{vac}} = \text{const}.$ The Petrov classification of SETs implies the existence of vacua whose symmetry is reduced as compared with (1), which allows the vacuum energy to become time dependent and spatially inhomogeneous [43, 46, 47]. A relevant class of SETs describes vacuum dark fluid [48] specified by the inflationary equation of state $p_\alpha = -\rho$ in only one or two distinguished spatial directions so that vacuum dark fluid is intrinsically anisotropic.

In the spherically symmetric case, a cosmological vacuum is specified by $T^t_t = T^r_r (p_r = -\rho)$ [42, 43]. The radial direction is distinguished by cosmological expansion. Regular solutions with source terms specified by $T^t_t = T^{\mu\nu}$ have an obligatory de Sitter center [46]. In the case of two vacuum scales, at the center and at infinity, source terms evolve from $\Lambda_1 g_{\mu\nu}$ to $\lambda g_{\mu\nu}$ with $\lambda < \Lambda$ [43, 46]. Cosmological models belong to the Lemaître class models with anisotropic pressures. They are asymptotically de Sitter at the early-time and late-time stages [49, 50].

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5 Quantum field theory in curved spacetime does not contain a unique specification for the quantum state of a system, and the symmetry of the vacuum expectation value of a SET does not always coincide with the symmetry of the background spacetime [44]. In the case of de Sitter space, the renormalized expectation value of $(T_{\mu\nu})$ for a scalar field with an arbitrary mass $m$ and curvature coupling $\xi$ is proved to have a fixed point attractor behavior at late times (see [44] and references therein), approaching, depending on $m$ and $\xi$, either the Bunch–Davies de Sitter-invariant vacuum or, in the massless minimally coupled case ($m = \xi = 0$), the de Sitter-invariant Allen–Folacci vacuum. The latter case is peculiar since the de Sitter-invariant two-point function is infrared divergent, and the vacuum states free of this divergence are $O(4)$-invariant Fock vacua; the vacuum energy density in the $O(4)$-invariant case is not the same (larger) as in the de Sitter-invariant case [45].
In this paper, we study in general setting spherically symmetric spacetimes with several vacuum scales. The relevant Lemaître models involve several de Sitter (inflationary) stages in the universe evolution. We study in detail the cosmological model with three basic vacuum scales, the GUT and QCD scales and that responsible for the currently observed accelerated expansion, $\rho_\Lambda = (8\pi G)^{-1}\lambda$. We introduce a phenomenological density profile describing vacuum decay at each stage by the exponential function as typical for a decay and fix the rate of decay by the conditions of analyticity and causality. This approach allows us to reveal certain general features of our Universe including the number of its spacetime horizons.

This paper is organized as follows. In section 2, we introduce spherically symmetric vacuum spacetimes. In section 3, we show how the number of vacuum scales determines general features of a spacetime including the number of horizons. In section 4, we describe a transition from the static reference frame to geodesic reference frames representing the Lemaître cosmologies. Section 5 presents a Lemaître cosmological model with the vacuum dark fluid $T^\nu_\mu = T^\tau_\tau$. Section 6 describes the model with three vacuum scales, GUT, QCD and present-day vacuum density, with the parameters fixed by the observational data. In section 7, we summarize and discuss the results.

2. Vacuum energy in general spherically symmetric spacetimes

A general model-independent approach based on the Petrov classification of SETs defines a vacuum by the symmetry properties of its SET [41–43]. The Einstein cosmological term corresponds to the de Sitter vacuum with the SET

$$T^\nu_\mu = \rho \delta^\nu_\mu, \quad p = -\rho.$$  \hspace{1cm} (2)

The medium specified by (2) is interpreted as a vacuum due to the algebraic structure of its SET (2). It has an infinite set of comoving reference frames so that an observer cannot in principle measure his/her velocity with respect to it [41], which is the intrinsic property of a vacuum [51]. The Einstein equations imply $\nabla_\nu T^\nu_\mu = 0$, which leads to $\rho = \text{const}$ for the de Sitter vacuum (2). The maximum symmetry of the vacuum SET (2) can be reduced while keeping its vacuum identity [42], and this inevitably leads (due to $\nabla_\nu T^\nu_\mu = 0$) to a dynamical vacuum energy [43]. The vacuum SET with a reduced symmetry, such that only one or two of its spatial eigenvalues coincide with the temporal eigenvalue, represents a vacuum dark fluid with the equation of state $p = -\rho$ in the distinguished direction(s) [48].

The general time-dependent spherically symmetric spacetime is described by the metric

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} dr^2 - r^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$  \hspace{1cm} (3)

where $\alpha$, $\gamma$ and $r$ are the functions of $R$ and $t$. The Einstein equations with source terms whose algebraic structure is specified by [42]

$$T^\nu_\mu = T^\tau_\tau (p_r = -\rho)$$  \hspace{1cm} (4)

admit a class of regular solutions with a de Sitter center, $T^\nu_\mu = \Lambda \delta^\nu_\mu$ as $r = 0$, where $\Lambda = 8\pi G \rho_0$ corresponds to a certain fundamental scale of symmetry breaking $\rho_0 = \rho_{(\text{vac})}$ at $r = 0$ [46, 47].

In a comoving reference frame, a vacuum dark fluid specified by (4) is presented by the SET

$$T^\nu_\mu = \text{diag}(\rho, \rho, -p_\perp, -p_\perp),$$  \hspace{1cm} (5)

where $p_r$ and $p_\perp$ are the radial and transversal pressures, respectively.

It can be easily shown that, under these general conditions, we necessarily have $\rho = \rho (r)$ and hence also $p_\perp = p_\perp (r)$. To begin with, the tensor (5) is invariant under any coordinate
transformations in the \((t,R)\) 2D subspace, which is just a definitive property of a vacuum. Moreover, if there is no material source of gravity other than \((5)\), the system satisfies all conditions of the generalized Birkhoff theorem \([52, 53]\), whence it follows that there exists a coordinate frame \((t,R)\) in which the metric \((3)\) is \(t\)-independent, and consequently, \(\rho \) and \(p_{\perp}\) are the functions of \(R\) alone. Let us show, however, that it is unnecessary to assume that \((5)\) is the only source of gravity: it is sufficient to require that it does not interact with other kinds of matter, and thus, the conservation law \(\nabla_\mu T^\mu\) = 0 holds.

Indeed, in this case, we have (dots and primes stand for \(\partial/\partial t\) and \(\partial/\partial R\), respectively)
\[
\dot{\rho} + 2\frac{\dot{r}}{r}(\rho + p_{\perp}) = 0, \tag{6}
\]
\[
\rho' + 2\frac{\dot{r}}{r}(\rho + p_{\perp}) = 0. \tag{7}
\]

If \(r = r(R)\), and hence \(\dot{r} = 0\), equation \((6)\) immediately gives \(\rho = \rho(R)\). If, on the contrary, \(r\) is \(t\)-dependent so that in the most general case we can suppose \(\rho = \rho(R, r)\), then \((6)\) and \((7)\) combined lead to \(\partial \rho/\partial R = 0\), as was asserted, and then, from \((6)\) we have the following:
\[
p_{\perp} = p_{\perp}(r) = -\rho r \frac{\partial \rho}{\partial r}. \tag{8}
\]

From \((4)\), it follows that \(G_{00}^0 = G_1^1\) for the Einstein equations. In the static reference frame, with using the Schwarzschild coordinate \(R = r\), this gives \(\alpha' + \gamma' = 0\) in \((3)\), whence, choosing the appropriate time scaling, we obtain \(\alpha + \gamma = 0\), and the metric \((3)\) takes the form
\[
dx^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2; \quad A(r) = e^{2 \varphi(r)}, \tag{9}
\]
where the metric function \(A(r)\) is given by
\[
A(r) = 1 - \frac{2M(r)}{r}, \tag{10}
\]
and \(M(r)\) is the mass function:
\[
M(r) = 4\pi \int_0^r \rho(x) x^2 \dx. \tag{11}
\]
We adopt the weak energy condition, i.e. a non-negative density for any observer on a time-like curve, which is natural for cosmological models describing the evolution of our Universe. This condition requires \(p_{\perp} + \rho \geq 0\), which leads, by \((8)\), to a monotonically decreasing density profile \(\rho(r)\) \([46]\). This fact, together with the number of vacuum scales for which \(p_{\perp}\) satisfies \(p_{\perp} = -\rho\), determines the generic behavior of the metric function \(A(r)\), and in consequence the maximum number of spacetime horizons. The actual number of horizons is determined by the specific form of the profile \(\rho(r)\). In section 6, we shall introduce a density profile appropriate for three vacuum scales and show how the number of horizons in our Universe follows from the observational constraints.

Equations \((4)\) and \((8)\) give the \(r\)-dependent equation of state. It is evident that an anisotropic fluid would need two different equation-of-state parameters \(w_a = p_a/\rho\). In the considered case, \(w_r = p_r/\rho = -1\) due to equation \((4)\), and \(w_{\perp} = p_{\perp}/\rho = -1 - (r/2)d(\ln \rho)/dr\) due to equation \((8)\). The parameter \(w_{\perp}\) satisfies \(w_{\perp} \geq -1\) since \(\rho(r)\) is monotonically decreasing function. The parameter \(w_{\perp}\) approaches \(w_{\perp} = -1\) for \(r \to 0\), for \(r \to \infty\), where \(\rho\) approaches the present-day vacuum density \(\rho_\lambda\), as well as for each intermediate inflationary stage with \(\rho_{(\text{vac})} = \text{const}\).

The mass function \((11)\) can be related to the Schwarzschild mass if we separate in \((11)\) the currently observed vacuum density \(\rho_\lambda\) as the background density. This is possible because \(\rho(r)\) is monotonically decreasing function, and \(\rho_\lambda\) is its minimal value. If we introduce
\[ \rho(r) = \rho_d + \rho_s \], where \( \rho_d \) is dynamical density decreasing smoothly from the value at the center \( \rho_0 = (8\pi G)^{-1}\Lambda \) to zero at infinity, then the mass function (11) takes the form

\[ M(r) = 4\pi \int_0^r \rho_d(x)x^2 dx + r^3\lambda/6 \]

and contains, in the limit \( r \to \infty \), the Schwarzschild mass \( M = 4\pi \int_0^\infty \rho_d(r)r^2 dr \) (in the Schwarzschild geometry, it is measured by the Kepler law in the Newton limit of the Schwarzschild solution in its asymptotically flat infinity). The mass function (11) differs from the proper mass in a curved spacetime, which is obtained by integration with the proper volume element \( dV = \sqrt{g^3}dx \), where \( g^3 \) refers to the determinant of the spatial metric. The difference represents the binding energy [54], also called the gravitational mass defect [51]. Dynamics of the Lemaître class models is determined by the mass function (11); therefore, we do not consider here the proper mass. Let us note however that in the case of cosmology asymptotically de Sitter at infinity, the notion of the total conserved (proper) mass of the universe can be introduced, because any asymptotically de Sitter spacetime must have an asymptotic isometry generated by the Killing vector \( \partial/\partial t \), and there exists the notion of a conserved total mass for the spacetime as computed at the future infinity [55].

3. The number of horizons and the number of scales

Typical behavior of the metric function \( A(r) \) in (9) is dictated by dynamics of the transversal pressure \( p_\perp \) in its source (5), which determines the maximal number and character of its extrema, which in turn determines the maximal number of spacetime horizons.

According to the Einstein equations, the transversal pressure \( p_\perp \) can be expressed in terms of the metric function \( A(r) \) as

\[ 8\pi G p_\perp = \frac{1}{2} A'' + \frac{A'}{r}. \]  

(12)

At an extremum of \( A(r) \), \( A' = 0 \), and hence, if \( p_\perp > 0 \), this extremum is a minimum, and this minimum of \( A \) is unique in the domain where \( p_\perp > 0 \) (otherwise there would be a maximum between two minima). Assuming that \( p_\perp \) is normally positive (so that the strong energy condition holds) and becomes negative only at distinguished spatial domains related to particular stages with a de Sitter vacuum behavior, we fix the number of zeros of \( p_\perp \) and restrict the maximum number of zeros of \( A(r) \), i.e. of spacetime horizons. One vacuum scale is related to the de Sitter center, where \( p_\perp \) is negative and \( A(r) \) has the maximum. If there is no other such scale, then transversal pressure changes its sign once, and \( A(r) \) has one minimum in which \( A(r) \) can be negative. In the asymptotically flat case, \( A(r) \to 1 \) as \( r \to \infty \), and hence, one zero of \( p_\perp \) can result in two zeros of \( A(r) \) and spacetime can have maximum two horizons [46]. If there is the de Sitter asymptotic also at infinity, this gives another domain where \( p_\perp \) is negative. Then, \( A(r) \) has two maxima and can have one minimum in between where \( p_\perp \) is positive. In this case, \( p_\perp \) changes its sign twice, and the single minimum of \( A(r) \) leads to at most three horizons [49].

Each intermediate vacuum scale produces two more zeros of \( p_\perp \) and at most two horizons. Hence, for \( n \) vacuum scales with negative pressure, we find at most \( 2n \) horizons in asymptotically flat spaces and at most \( 2n - 1 \) horizons for asymptotically de Sitter spaces. Note that it is the maximum number of horizons, and it will be smaller if, under the same behavior of \( p_\perp \), the metric function \( A(r) > 0 \) at some of its minima or \( A(r) < 0 \) at some of its maxima.

Dynamics of the transversal pressure determines also a typical behavior of the equation-of-state parameter \( w_\perp \). For example, in the case of three vacuum scales of interest here, \( \rho_{QCD} \) at the center, \( \rho_{QCD} \) and \( \rho_s \) appearing successively, there can exist two domains where \( p_\perp \) is
positive. The parameter \( w_\perp = p_\perp/\rho \) takes the value \( w_\perp = -1 \) during each inflationary stage, and it can be positive during the transitions \( \rho_{\text{GUT}} \rightarrow \rho_{\text{QCD}} \) and \( \rho_{\text{QCD}} \rightarrow \rho_\perp \).

The requirements of regularity at the center and the dominant energy condition do not restrict the total number of horizons. This can be seen from the following example.

Let \( \rho > 0 \) and consider two extreme equations of state compatible with the dominant energy condition: (a) \( p_\perp = -\rho \) and (b) \( p_\perp = \rho \). In case (a) the most general static, spherically symmetric solution is Schwarzschild–de Sitter; in case (b) the SET structure is the same as for a radial electromagnetic field, so that we arrive at the Reissner–Nordström metric. The metric function \( A \) in these two cases is

\[
\begin{align*}
(a) & \quad A(r) = 1 - \frac{2m}{r} - H^2 r^2, \\
(b) & \quad A(r) = 1 - \frac{2\mu}{r} + \frac{q^2}{r^2},
\end{align*}
\]

with constant parameters \( m, \mu, H \) and \( q \). (Note that here \( q \) is not a charge, and the notation is adopted for convenience).

Now, suppose that in the spacetime with a regular de Sitter center, the equation of state (a), leading to the exact de Sitter metric, holds in a finite interval of \( r \) until \( A \) reaches zero (i.e. for \( r < h_1 = 1/H \)). Beyond this horizon, let us take the equation of state (b) so that the metric function \( A \) has the form (14). It matches to the solution in the previous interval of \( r \) if the constants \( \mu \) and \( q \) are found from the continuity conditions for \( A \) and \( A' \). Since \( A'(h_1) < 0 \), this necessarily means that \( h_1 \) is the inner horizon of the metric (14). With growing \( r \), it will eventually reach the outer horizon \( r = h_2 \) with \( A' > 0 \). At this point, we again switch the equation of state to (a), so that the next interval will be described by \( A(r) \) given in equation (13), with \( m \) and \( H \) determined from the continuity of \( A \) and \( A' \). At the inevitable next horizon \( h_3 \), we will again join the Reissner–Nordström metric in the same way and so on.

The process can be continued as long as one wishes. Its feasibility is guaranteed by the following facts verified by a direct inspection.

(i) Given any \( h > 0 \) and \( C < 0 \), one can always find such \( \mu \) and \( q \) that the function (14) satisfies the conditions \( A(h) = 0 \) and \( A'(h) = C \). Thus, a next Reissner–Nordström segment can always be joined at the points \( h_1, h_3, \) etc.

(ii) Given any \( h > 0 \) and \( C > 0 \), such that \( hC < 1 \), one can always find such \( m \) and \( H \) that the function (13) satisfies the conditions \( A(h) = 0 \) and \( A'(h) = C \); moreover, the condition \( hC < 1 \) always holds if \( h \) is the greater of two zeros of the function (14) and \( C = A'(h) \) for the same function. Thus, a next Schwarzschild–de Sitter segment can always be joined at the points \( h_2, h_4, \) etc.

The process can be stopped at any stage. If the last equation of state is (a), we obtain an asymptotically de Sitter model with an odd number of horizons; on the contrary, (b) leads to an asymptotically flat model with an even number of horizons.

The density profile is continuous but contains fractures (jumps of the derivative \( \rho' \)). When \( p_\perp = -\rho \), we have \( \rho = \text{const} \), while when \( p_\perp = \rho \), the function \( \rho(r) \) behaves as \( 1/r^4 \). The fractures can, however, be smoothed by arbitrarily small additions to \( \rho(r) \) without changing the whole qualitative picture, which will then correspond to an entirely smooth density distribution.

Each plateau in the density profile must, from a physical viewpoint, manifest an intermediate energy scale, eventually connected with some phase transition. We can anticipate that the existence of such scales can appreciably complicate the set of possible spacetime structures.
4. Transition to Lemaître reference frames

Consider the general static metric (9) that solves the Einstein equations with the SET (5). A transition to the geodesic coordinates \((R, \tau)\), where \(\tau\) is the proper time along a geodesic and the radial coordinate \(R\) is the congruence parameter, different for different geodesics, can be described in a general form. A radial timelike geodesic in the metric (9) satisfies the equations

\[
\left(\frac{dr}{d\tau}\right)^2 = E^2 - A(r), \quad \frac{dt}{d\tau} = \frac{E}{A(r)},
\]

where the constant \(E\) is connected with the initial velocity of a particle moving along this particular geodesic at a given value of the congruence parameter \(R\). In general, \(E = E(R)\), i.e. it is different for different geodesics.

Equations (15) give two of the four components of the transition matrix \(\|\partial(t, r)/\partial(\tau, R)\|\), namely \(\dot{r}\) and \(i\) (dots and primes stand for \(\partial/\partial\tau\) and \(\partial/\partial R\), respectively), since this partial differentiation occurs along the geodesics:

\[
\dot{r} = \pm \sqrt{E^2(R) - A(r)}, \quad i = E(R)/A(r).
\]

A relation between the other two components, \(\dot{t}'\) and \(i'\), can be found from the condition \(g_{tR} = 0\) when we substitute \(dr = idr + \dot{t}'dR\) and \(dr = itr + \dot{t}'dR\) into the metric (9):

\[
\dot{t}' = \frac{\sqrt{E^2(R) - A(r)}}{E(R)A(r)} \dot{t}.
\]

It remains to determine \(\dot{t}'(R, \tau)\), which can be done by using the integrability condition \((\partial_\tau \partial_R - \partial_R \partial_\tau) r = 0\). The latter takes the form of a linear first-order differential equation with respect to \(\dot{r}'\) = \(y(R, r)\):

\[
\partial_\tau y = -\frac{y \partial_r A}{2(E^2 - A)} + \frac{EE'}{E^2 - A}.
\]

Solving it, we obtain

\[
y = \dot{t}'(R, \tau) = \sqrt{E^2 - A} \left\{ f_0(R) + EE' \int \frac{dr}{(E^2 - A(r))^{3/2}} \right\}.
\]

The other integrability condition \((\partial_\tau \partial_R - \partial_R \partial_\tau) t = 0\) holds automatically if (18) holds. The functions \(f(R, \tau)\) and \(r(R, \tau)\) can now be found by further integration of equations (16)–(19).

The resulting metric can be written as follows:

\[
\begin{align*}
d\bar{s}^2 &= d\tau^2 - \frac{\dot{r}'(R, \tau)^2}{E^2(R)} dR^2 - r^2(R, \tau) \ d\Omega^2, \\
\end{align*}
\]

One can see how this procedure works using the de Sitter space as an example. Its static form is (9) with \(A(r) = 1 - H^2r^2\), \(H = \text{const}\). We will choose three different families of geodesics such that

\[
E(R) = \sqrt{1 - KR^2}, \quad K = 0, \pm 1,
\]

and show that their corresponding reference frames represent the three well-known forms of the de Sitter metric as isotropic cosmologies with different signs of spatial curvature (see, e.g., [56]). Indeed, integrating the first relation in (16) as an equation for \(\tau = \tau(r, R)\) and properly choosing the arbitrary function of \(R\) that appears as an integration constant, we obtain the following expressions for \(r\):

\[
r(R, \tau) = \frac{(R/H)}{\sinh(H\tau)} \times \{ \cosh(H\tau), \ e^{H\tau}, \ \sin(H\tau) \}.
\]
where the expressions in the curly brackets are ordered according to $K = 1, 0, -1$. Substituting them into (20), we obtain the metric in the form

$$d\tau^2 = a^2(\tau) \left( \frac{dR^2}{1 - KR^2} + R^2 d\Omega^2 \right),$$

$$a(\tau) = (1/H) \times [\cosh(H\tau), e^{H\tau}, \sinh(H\tau)],$$

as was intended. One can also verify that expression (19) with $E(R)$ given by (21) (provided the function $f_0(R)$ is chosen properly) coincides with the expression for $r'$ obtained directly from (22) in all three variants. So the transition has been completed.

5. Lemaître cosmology with the vacuum dark fluid $T_{\tau\tau} = T_r$

We have shown above that the behavior of the static metric function $A(r)$ is dictated by the number of vacuum scales and that the static spherically symmetric metric (9) can always be transformed to the Lemaître form. Hence, the cosmological evolution in this case can be described by a model from the Lemaître class satisfying the condition $p_r = -\rho$ that specifies a vacuum dark fluid, with the appropriate choice of the density profile $\rho(r)$ modeling smoothed jumps between different values of $\rho_{\text{vac}}$, which will be discussed in the next section.

A Lemaître class model is described by the line element [51]

$$d\tau^2 = e^{2v} (dR^2 - \tau^2 R^2 + r^2 \frac{d\Omega^2}{r^2}).$$

(24)

Coordinates $R$ and $\tau$ are the Lagrange (comoving) coordinates. The coordinate $\tau$ measures the proper time along the world lines of a fluid. The function $r(R, \tau)$ corresponds to the Euler coordinate, which is called the luminosity distance.

For the metric (24), the Einstein equations with the SET (5) read [51]

$$8\pi G p_r = \frac{1}{r^2} \left( e^{-2v} r^2 - 2 r \tau - \tau^2 - 1 \right),$$

(25)

$$8\pi G p_\perp = \frac{e^{-2v}}{r^2} \left( \tau'' - r' \tau' - \tau^2 \right) - \tau - \tau^2 - \frac{\tau}{r},$$

(26)

$$8\pi G \rho = - \frac{e^{-2v}}{r^2} (2 r \tau'' + r^2 - 2 r \tau'') + \frac{1}{r^2} \left( 2 r \tau' + r^2 + 1 \right),$$

(27)

$$8\pi G T'_\tau = \frac{2 e^{-2v}}{r} (r' - \tau' \tau) = 0,$$

(28)

where dots and primes stand for $\partial/\partial \tau$ and $\partial/\partial R$, respectively.

The component $T'_\tau$ of the SET vanishes in the comoving reference frame since there is no momentum in the radial direction, and equation (28) is integrated giving [51, 57]

$$e^{2v} = \frac{\tau^2}{1 + f(R)},$$

(29)

where $f(R)$ is an arbitrary function. Putting (29) into (25), we obtain the equation of motion

$$\dot{r}^2 + 2r\dot{r} + 8\pi G p_r r^2 = f(R).$$

(30)

Taking into account that $p_r + \rho = 0$, the first integration of (30) gives

$$\dot{r}^2 = \frac{2GM(r)}{r} + f(R) + \frac{F(R)}{r},$$

(31)

where the mass function $M(r)$ is defined by

$$M(r) = 4\pi \int_0^r \rho(x) x^2 \, dx.$$
An arbitrary function \( F(R) \) (‘integration constant’ parametrized by \( R \)) should be chosen equal to zero for models regular at \( r = 0 \) since \( M(r) \to 0 \) as \( r \to 0 \), where \( \rho(r) \to \rho_0 < \infty \).

The second integration of equation (30) gives

\[
\tau - \tau_0(R) = \int \frac{dr}{r \sqrt{2GM(r)/r + f(R)}}.
\]  

(33)

The new arbitrary function \( \tau_0(R) \) due to this integration is called the bang-time function [58]. For example, in the case of the Tolman–Bondi model for dust (\( p_r = p_\perp = 0 \)), the evolution is described by \( r(R, \tau) = [9GM(R)/2]^{1/3}[\tau - \tau_0(R)]^{2/3} \), where \( \tau_0(R) \) is an arbitrary function of \( R \) representing the Big Bang singularity surface at which \( r(R, \tau) = 0 \) [59].

In the regular case asymptotically de Sitter in the \( R \)-region near \( r = 0 \) considered here, evolution starts from the timelike regular surface \( r(R, \tau) = r_b \). For \( f(R) \geq 0 \), a bang surface is \( r(R, \tau) = 0 \), and the solution (33) near this surface reduces to

\[
\tau - \tau_0(R) = \int \frac{dr}{\sqrt{r^2/r_0^2 + f(R)}}.
\]  

(34)

where

\[
r_0 = \sqrt{\frac{3}{8\pi G\rho_0}}
\]  

(35)

is the curvature radius at \( r = 0 \), and \( \rho_0 \) is the density at \( r = 0 \). In the case \( f(R) < 0 \), the bang surface is \( r(R, \tau) = r_b \), where \( r_b \) satisfies \( 2GM(r)/r + f(R) = 0 \). For small values of \( f(R) \), we can apply (34), which gives \( r = r_0\sqrt{-f(R)} \cosh [(\tau - \tau_0(R))/r_0] \). For \( f(R) > 0 \), we obtain \( r = r_0\sqrt{f(R)} \sinh [(\tau - \tau_0(R))/r_0] \), and \( r = r_0 \exp[(\tau - \tau_0(R))/r_0] \) for \( f(R) = 0 \).

Different points of the regular timelike bang surfaces start at different moments of the synchronous time \( \tau \) so that bangs are non-singular and non-simultaneous.

For \( f(R) = 0 \) (parabolic motion), equation (34) gives at small \( r \) the expansion law

\[
r = r_0 e^{(\tau - \tau_0(R))/r_0}
\]  

(36)

gives

\[
e^{2\nu} = r^2 \left[ \frac{d\tau_0(R)}{dr_0} \right]^2.
\]  

(37)

The metric takes the FRW form with the de Sitter scale factor

\[
dx^2 = dr^2 - r_0^2 e^{2\tau_0(r_0)}(d\Omega^2 + q^2)
\]  

(38)

where the variable \( q = e^{\tau_0(r_0)} \) is introduced to transform the metric to the FRW form. In accordance with (36), it describes a non-singular non-simultaneous de Sitter bang from the surface \( r(\tau - \tau_0(R) \to -\infty) = 0 \) [60].

The inflationary stage is followed by an anisotropic Kasner-like stage: one scale factor, corresponding to the transversal direction, is given by \( r(R, \tau) \), and the other, corresponding to the radial direction, is proportional to \( r' \) according to (29), and its particular form depends on the density profile \( \rho(r) \) and the choice of arbitrary functions of \( R \). If we choose \( f(R) = 0 \) and \( \tau(R) = R \), the metric can be approximated by \( [60, 49] \)

\[
dx^2 = dr^2 - (\tau + R)^{-2/3}K(R) dr^2 - L(\tau + R)^{4/3} d\Omega^2,
\]  

(39)

where \( K(R) \) is a smooth regular function and \( L \) is a constant.

A similar behavior can be found for a density profile, which approximates phase transitions with several scales of vacuum energy (see below). At each transition an inflationary stage is followed by an anisotropic Kasner-like stage.
6. Lemaître cosmology with GUT and QCD phase transitions

6.1. Basic features

According to the conventional scenario, the first inflationary stage corresponding to the GUT phase transition occurred at the GUT scale $E_{\text{GUT}} \sim 10^{15}$ GeV, the relevant density being $\rho_{\text{GUT}} \simeq 2.3 \times 10^{17}$ g cm$^{-3}$. It was followed by decay of vacuum energy resulting ultimately in a radiation-dominated stage. The next phase transition that could drive the second inflation [37–39] occurred at the QCD scale $E_{\text{QCD}} \sim (100–200)$ MeV, at about $10^{-5}$ s after the Big Bang, when the Hubble radius, $d_H = c/H$, was about $10$ km [35]. The density $\rho_{\text{QCD}}$ is smaller by a factor of $(E_{\text{QCD}}/E_{\text{GUT}})^4$ than the GUT density $\rho_{\text{GUT}} = \rho_0$, i.e. of the order of the nuclear matter density. The last inflationary stage corresponds to the currently observed dark energy density $\rho_e$, which is about $10^7$ orders of magnitude smaller than the GUT density.

This situation can be modeled by the density profile

$$\rho = \rho_0\left[1 - (1 - B_1)\exp\left(-r^n/r^n_0\right) - (B_1 - B_3)\exp\left(-r^n_3/r^n_0\right)\right], \quad (40)$$

where $n$, $B_1$, $B_3$, $r_1$, $r_3$ are constants, for which we adopt

$$B_1 = \rho_{\text{QCD}}/\rho_0 \approx 10^{-64}, \quad B_3 = \rho_e/\rho_0 \approx 10^{-107}; \quad r_0 < r_1 \ll r_3. \quad (41)$$

The exponential function in (40) is chosen as typical for processes of decay. The parameter $n$ characterizing the decay rate will be fixed below by conditions of analyticity and causality.

We choose $f(R) \equiv 0$ in equations (29)–(33), because in this case each 3-hypersurface $\tau = \text{const}$ is flat, with zero curvature [61], which guarantees fulfillment of the spatial flatness condition $\Omega = 1$ required by the observational data.

Under the above choice, the model undergoes the following stages.

(a) $r \ll r_1$: the first inflation, $\rho \approx \rho_0$; the mass function (32) is approximated by $M(r) = \frac{4\pi}{3} r^3$; and equation (33) yields, in agreement with (36),

$$\tau - \tau_0(R) \simeq \tau_0 \ln \frac{r}{r_0}. \quad (42)$$

(b) $r \sim r_1$: end of the first inflation since the second term in (40) becomes significant.

(c) $r_1 \ll r \ll r_3$ so that

$$\rho \approx \rho_0(B_1 + r^n_1/r^n_0). \quad (43)$$

This stage in turn splits into two periods. As long as $r$ is sufficiently small,

$$r < r_2, \quad r_2 = r_1 B_1^{1/n} = 10^{64/n} r_1, \quad (44)$$

the second term in (43) is dominant so that $\rho(r)$ rapidly decreases. It is an intermediate period between the first and the second inflation. At $r = r_2$, the two terms coincide, and at $r > r_2$, we have $\rho \approx B_1 \rho_0 = \rho_{\text{QCD}} = \text{const}$, which corresponds to the second inflation.

(d) $r \sim r_3$: end of the second inflation since the third term in (40) becomes significant.

(e) $r \gg r_3$: the density is

$$\rho \approx \rho_0(B_3 + r^n_3/r^n_0). \quad (45)$$

Similarly to stage (c), at some value of $r$, namely, at $r = r_4$ defined by

$$r_4 = r_3 (B_1/B_3)^{1/n} = 10^{64/n} r_3 = 10^{107/n} r_1, \quad (46)$$

$$\rho = \rho_0\left[1 - (1 - B_1)\exp\left(-r^n_1/r^n_0\right) - (B_1 - B_3)\exp\left(-r^n_3/r^n_0\right)\right]. \quad (40)$$

where $n$, $B_1$, $B_3$, $r_1$, $r_3$ are constants, for which we adopt

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We choose $f(R) \equiv 0$ in equations (29)–(33), because in this case each 3-hypersurface $\tau = \text{const}$ is flat, with zero curvature [61], which guarantees fulfillment of the spatial flatness condition $\Omega = 1$ required by the observational data.

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Similarly to stage (c), at some value of $r$, namely, at $r = r_4$ defined by

$$r_4 = r_3 (B_1/B_3)^{1/n} = 10^{64/n} r_3 = 10^{107/n} r_1, \quad (46)$$
the two terms in (45) are equal. At \( r_2 < r < r_4 \), we have one more intermediate period, where \( \rho(r) \) rapidly decreases, while at \( r > r_4 \), it approaches a constant corresponding to the present-day dark energy density.

The time elapsed between the first and the second inflation \( \tau_2 - \tau_1 \) (we denote \( \tau_i = \tau(r_i) \)) is estimated by integrating between \( r_1 \) and \( r_2 \) in equation (33). To this end, we find the mass function in the same interval:

\[
M(r) = 4\pi \left( \int_{r_1}^r + \int_r^{r_2} \right) \rho(r) r^2 \, dr.
\]

In the first term, we take \( \rho \approx \rho_0 \), while in the second one, in accord with (43), we approximate \( \rho(r) \) by \( \rho_0(1/r)^n \). Hence,

\[
M(r) = \frac{4}{3} \pi \rho_0 r_1^3 + \frac{4\pi \rho_0}{n-3} \left( \frac{1}{r_1^{n-3}} - \frac{1}{r^{n-3}} \right).
\]  

In the last term, almost in the whole interval of interest, \( r \gg r_1 \), and therefore, for our estimation purpose, we can neglect the last term thus obtaining a constant value of \( M \):

\[
M(r) \approx M_2 = \frac{4\pi n}{3(n-3)} \rho_0 r_1^3 = \text{const.}
\]  

Substituting it into (33), we obtain

\[
\tau_2 - \tau_1 \approx \sqrt{\frac{4(n-3)}{9n}} \rho_0 \left( \frac{r_2}{r_1} \right)^{3/2}.
\]  

The first restriction on the parameter \( n \) is evident: \( n > 3 \). The second constraint follows directly from the dominant energy condition, which requires \( p_\perp \leq \rho \) and guarantees that the speed of sound never exceeds the speed of light thus maintaining causality in the course of evolution. A simple analysis of equation (8) for several vacuum scales shows that the difference \( \rho - p_\perp \) as a function of \( r \) decreases at each transition starting from 2\( \rho \) with \( \rho = \text{const.} \). Let us introduce the function \( f_{\text{DEC}} = \rho - p_\perp \) characterizing the dominant energy condition. According to (8), \( f_{\text{DEC}} = 2\rho + r\rho' / 2 \). It should be a decreasing function since \( \rho(r) \) is monotonically decreasing function and its derivative \( \rho' \) is negative. During the first transition, this function should decrease from 2\( \rho_0 \) to 2\( \rho_{\text{GUT}} \) to 2\( \rho_{\text{QCD}} \). For the density profile (40), we have

\[
f(r) = f_{\text{DEC}}(r_1)^{-1} = 2 \left[ 1 - (1 - B_1) e^{-r_1/r} \left( 1 + n \left( \frac{r_1}{r} \right)^n \right) \right].
\]  

It should be non-negative and decreasing from 2 to 2\( B_1 \), where \( B_1 = \rho_{\text{QCD}} / \rho_{\text{GUT}} \) is given by (41). For \( r \gg r_1 \), the exponent in (50) can be presented as a series in \( (r_1/r)^n \), which gives

\[
f = 2B_1 + 2(1 - B_1) \left[ \left( 1 - \frac{n}{4} \right) \left( \frac{r_1}{r} \right)^n + \frac{n - 1}{2} \left( \frac{r_1}{r} \right)^{2n} + O \left( \frac{r_1}{r} \right)^{3n} \right].
\]

The condition of non-negativity of this function is \( n \leq 4 \). The derivative is given by

\[
f' = \left( 1 - B_1 \right) \frac{n}{2} \left( \frac{r_1}{r} \right)^n e^{-r_1/r} \left[ n - 4 - n \left( \frac{r_1}{r} \right)^n \right].
\]  

The condition \( n \leq 4 \) guarantees the monotonic decrease of the function \( \rho - p_\perp \) during the first transition. It is easily to show that this also concerns the second transition at which the function \( f_{\text{DEC}}(r) \) decreases from 2\( \rho_{\text{QCD}} \) to 2\( \rho_0 \).

The condition \( n \leq 4 \) provides thus non-negativity of the function \( f_{\text{DEC}}(r) = \rho - p_\perp \) during the cosmological evolution with the density profile (40). Therefore, we fix \( n = 4 \) as the only integer compatible with analyticity and causality.

Let us note that at the end of both transitions, for \( r \gg r_1 \) and \( r \gg r_3 \), the density in (40) behaves like \( \rho \propto r^{-3} \), in a way typical for radiation in the FRW cosmology, where \( \rho a^3 = \text{const.} \), and also agrees with our qualitative analysis at the second part of section 3.
With \( n = 4 \), we obtain from (47)

\[
M(r) = \frac{4}{3} \pi \rho_0 r_1^3 + 4 \pi \rho_0 r_1^4 \left( \frac{1}{r_1} - \frac{1}{r} \right).
\]

The value of \( M_2 \) in (48) is now

\[
M_2 = \frac{16}{3} \pi \rho_0 r_1^3.
\]

Substituting it into (49), we obtain

\[
\tau_2 - \tau_1 \approx \frac{r_0}{3} \left( \frac{r_2}{r_1} \right)^{3/2} \sim 10^{24} r_0.
\]

Recalling that \( r_0 \sim 10^8 l_{pl} \), we find \( \tau_2 - \tau_1 \sim 10^{32} l_{pl}/c \sim 10^{-11} \) s. It is of interest that this estimate does not depend on the particular choice of the free parameter \( r_1 \) or, equivalently, the number of e-foldings \( N_e := \ln(r_1/r_0) \).

Furthermore, if the second inflation contains seven e-foldings [37], it means that \( r_3 \sim 10^3 r_2 \). It is then also easy to find \( r_4 \), the value of \( r \) at which the DE density has reached its modern value, from (46): \( r_4 \sim 10^{43/4} r_3 \). To estimate the duration of the second inflation \( \tau_3 - \tau_2 \) and the time \( \tau_4 \) of the onset of the latest \( \lambda \)-dominated stage, we should integrate in (33) from \( r_2 \) to \( r_3 \) and then to \( r_4 \). Acting in the same manner as in finding \( \tau_2 \), we see that the main contribution to the mass function comes from the range \( r < r_2 \) and is given by (54), and hence, the duration of the second inflation is

\[
\tau_3 - \tau_2 \approx \sqrt{\frac{1}{12 \pi G \rho_0} \left( \frac{r_3}{r_1} \right)^{3/2}} \approx 10^{-7} \text{ s}.
\]

Finally, for \( \tau_4 \), we obtain the same relation as (56) but with \( r_3 \) replaced by \( r_4 \). It results in

\[
\tau_4 - \tau_3 \sim 10^{10} \text{ s} \sim 1000 \text{ years}.
\]

Let us note that all these times are practically independent of the number of e-foldings \( N_e \) during the first inflation. However, the duration of the later period up to the present epoch does depend on \( N_e \). Namely, since at \( r \gg r_4 \), we have \( \rho \approx B_3 \rho_0 = \text{const} \), integration in (32) at large enough \( r \) gives

\[
M(r) \approx \frac{4\pi}{3} B_3 r_0 r^3.
\]

and integration in (33) yields immediately

\[
\tau - \tau_0(R) \approx 10^{18} \text{ s} \cdot \ln(r/r_*) ,
\]

where \( r_* \approx 10^9 r_2 \approx 10^{36} r_1 \) is the value of \( r \) at which the contribution of \( r^3 \) to the mass function begins to exceed \( M_2 \).

The qualitative behavior of the vacuum density profile (40) is shown schematically in figure 1.

Equation (59) gives the expansion law \( r = r_* e^{(\tau - \tau_0(R))/r_*} \), and equation (29) gives for the second scale factor in (24) \( e^{2\nu} = \frac{\dot{r}^2}{(\ddot{r}/r)^2} \). Introducing the variable \( q = e^{(\tau - \tau_0(R))/r_*} \), we transform the metric (24) to the FRW form

\[
d\hat{s}^2 = dr^2 - r_*^2 e^{2\nu/r_*} (dq^2 + q^2 d\Omega^2),
\]

at the stage when the vacuum density achieves its present value.
As we have seen, the only free parameter of the model is the number of e-foldings at first inflation \( N_e = \ln (r_1 / r_0) \), where the characteristic de Sitter radius for the GUT scale vacuum is \( r_0 \approx 2.4 \times 10^{-25} \) cm. The time interval corresponding to \( r_1 \) is, according to (42), approximately \( 10^{-35} N_e \) s. In the next section, we evaluate an admissible interval for the parameter \( N_e \) from the requirement of the late-time homogeneity and isotropy.

For the density profile (40), the transversal pressure is given by

\[
p_{\perp} = \rho_0 \left[ -1 + (1 - B_1) \left( 1 + \frac{2 r_1^4}{r^4} \right) e^{-r_1^4/r^4} + (B_1 - B_3) \left( 1 + \frac{2 r_3^4}{r^4} \right) e^{-r_3^4/r^4} \right].
\]  

(61)

It satisfies equation \( p_{\perp} = -\rho \) during each inflationary stage. It has two maxima, \( p_{\perp} \approx 0.213 \rho_0 \) at \( r_{m1} \approx 1.2 r_1 \) and \( p_{\perp} \approx 0.213 B_1 \rho_0 = 0.213 \rho_{QCD} \) at \( r_{m2} \approx 1.2 r_2 \), and ultimately quickly achieves \( p_{\perp} = -B_3 \rho_0 = -\rho_\lambda \). Behavior of the transversal pressure (61) is shown schematically in figure 2.

The parameter \( w_{\perp} = -1 \) during the first inflation then rapidly increases to \( w_{\perp} \approx 0.213 \) at \( r_{m1} \approx 1.2 r_1 \), decreases to \( w_{\perp} = -1 \), quickly increases to \( w_{\perp} \approx 0.213 \) at \( r_{m2} \approx 1.2 r_2 \), and finally approaches \( w_{\perp} = -1 \) as \( \rho \) approaches \( \rho_{\lambda} \).
6.2. Late-time homogeneity and isotropy

At $r > r_4$, the model evolution is governed by the small effective cosmological constant $\lambda = B_3 \rho_0 / (8\pi G)$ and tends to a de Sitter regime, i.e. becomes homogeneous and isotropic.

The degree of inhomogeneity can be characterized by the dimensionless parameter $(r/\rho) d\rho/dr$ showing how the density $\rho$ changes at a distance $\sim r$. By (45) with $n = 4$, at $r \gg r_4$, this parameter is approximately equal to $4r_4^2/r^4$ and rapidly decreases with growing $r$.

Thus, at $r \gg r_4$, one can estimate the degree of anisotropy of our model as is conventionally done for homogeneous models, e.g., using the anisotropy parameter [62, 63]

\[ A = \frac{1}{3H^2} \sum_{i=1}^{3} H_i^2, \]

where $H_i = \dot{a}_i/a_i$ are the directional Hubble parameters corresponding to the three scale factors $a_i(\tau)$, the dot stands for $d/d\tau$, and $H = (H_1 + H_2 + H_3)/3$ is the mean Hubble parameter (see [63] for a discussion of different anisotropy characteristics). In our model with the metric (24), where $e^{2\nu} = r^2$ and $f(R) = 0$, these scale factors are $a_1 = |r|$ and $a_2 = a_3 = r$. The expression for $r'$ is found from (33):

\[ r' = -\sqrt{2GM(r)/r_0^2(R)}. \]

Using this, one obtains for the anisotropy parameter

\[ A = 2 \left( \frac{M/M - 3r/r^2}{M/M + 3r/r^2} \right)^2 = 2 \left[ \frac{3M_4 - 4\pi \rho_4 r_4^3}{3M_4 + 4\pi \rho_4 (2r_4^3 - r_4^3)} \right]^2. \]

(64)

where $M_4 = M(r_4)$ and $\rho_4 = \rho(r_4) = 2B_3 \rho_0$ according to (45). At large $r$, the parameter $A \sim r^{-6}$, but, as can be directly verified, this rapid decrease does not begin from $r_4$ but only from much larger values of $r$ because at $r \sim r_4$ both the numerator and the denominator of (64) are dominated by the constant $M_4$.

To agree with CMB observations, the vacuum contribution must be already highly isotropic ($A < 10^{-6}$) when $r$ reaches the value of the scale factor $r = r_5 \sim 10^{35}$ cm corresponding to the recombination epoch with redshifts $z \sim 1000$. This requirement constrains the possible value of the free parameter $N_e = \ln(r_1/r_0)$. Indeed, the condition $A(r = r_5) < 10^{-6}$ gives

\[ 3M_4 < 2\sqrt{2\pi G \rho_0 r_4^3 \cdot 10^{-3}} \]

(65)

(taking into account that $M_4 \gg \pi \rho_4 r_4^3$). In turn, $GM_4$ is expressed in terms of $r_1$. From (54), we know the value of $M_2 = M(r_2) = (16/3)\pi \rho_0 r_4^3$. To find $M_4$, we must integrate in (32) from $r_2$ to $r_4$; it turns out, however, that this integration contributes only a relative correction of the order $10^{-7}$ to $M_2$. Thus,

\[ M_4 \approx \frac{16}{3} \pi \rho_0 r_4^3. \]

(66)

Comparing (66) with (65), we obtain the constraint

\[ r_1 < B_3^{1/3} r_5/10 \approx 10^{-37} r_5 = 3 \times 10^{-12} \text{ cm}, \quad N_e = \ln(r_1/r_0) < 30. \]

(67)

Hence, after the recombination time corresponding to $r = r_5$, the Lemaître model (24) practically behaves as a homogeneous and isotropic FRW model (60).

6.3. Evolution of the scale factors

Now, having established the constraint (67), we can discuss the model evolution at all stages.

For the spatially flat model, $f(R) = 0$, the line element (24) takes the form

\[ ds^2 = d\tau^2 - B^2(\tau, R) dR^2 - r^2(\tau, R) d\Omega^2, \]

(68)
where we have introduced explicitly two scale factors: \( r(\tau, R) \) in accordance with (24) and \( b(\tau, R) \equiv r'(\tau, R) \) in accordance with (29). For the integration ‘constant’ in (34), we choose \( \tau_0(R) = -R \) to make the de Sitter asymptotics familiar. It is easily seen that in this case, 
\[
\dot{r}(\tau + R) = \frac{dr}{d(\tau + R)} \frac{\partial(\tau + R)}{\partial \tau} = \frac{dr}{d(\tau + R)}.
\]

Numerical integration of the Lemaître equations during the first transition \( \rho_{\text{GUT}} \to \rho_{\text{QCD}} \) shows an exponential growth of both scale factors at the beginning when \( p_{\perp} \simeq p_r = -\rho \), followed by an anisotropic Kasner-like stage when the anisotropy of the pressures leads to an anisotropic expansion. The behavior of two scale factors \( r(\tau + R), b(\tau + R) \) and their evolution during first phase transition are shown in figures 3 and 4.

The behavior of the scale factors at the second phase transition is qualitatively quite the same for the density profile (40).

The evolution of the scale factors during the whole Universe history is shown schematically in figure 5 plotted on the basis of figure 3 extended to the third inflationary stage (the hypersurface \( \tau + R = t_4 \) in figure 5). The first inflation ends at the hypersurface \( \tau + R = t_1 \), and the second inflation occurs between \( \tau + R = t_2 \) and \( \tau + R = t_3 \). The sharp maximum in \( b(\tau + R) \) in figure 4 and two maxima in the second picture of figure 5 are related to maximum
of $p_\perp$, which is shown in figure 2. At late times, due to isotropy, the evolution of the two scale factors is common and conforms to the standard flat de Sitter cosmology.

6.4. Horizons

We have described the Universe evolution from the viewpoint of the Lemaître cosmological reference frame. Now, we can find the number of horizons and present the global structure of spacetime. The proof in section 3 concerned only the maximum number of horizons, and their smaller number is certainly possible dependent on the dispositions and durations of the phase transitions on the $r$ scale.
Let us find how the whole scenario looks from the static reference frame. The metric function $A(r)$ calculated with the above fixed parameters is shown in figure 6, where the characteristic scales designated on the $r$ axis are

$$r_1 = 10^{-12} \text{ cm}, N_e = 29 \text{ at the end of the first inflation},$$
$$r_2 = 1.2 \times 10^{9} \text{ cm at the beginning of the second inflation},$$
$$r_3 = 1.2 \times 10^{7} \text{ cm at the end of the second inflation},$$
$$r_4 = 6.7 \times 10^{17} \text{ cm at achieving the present-day vacuum (dark energy) density $\rho_\lambda$.}$$

The values of $r_5 = 10^{25}$ and $r_6 = 10^{28}$ cm approximately correspond to the recombination and to the beginning of the third (currently observed) inflation, respectively. The value $r_7 = 7.5 \times 10^{29}$ cm corresponds to the cosmological horizon due to de Sitter vacuum with the density $\rho_\lambda$. The behavior of the metric function $A(r)$, see figure 6, testifies for the existence of three horizons for the case of the above fixed parameters corresponding to those of our Universe.

The first horizon in our Universe $r_-$ is close to the de Sitter radius $r_0$ characteristic for the GUT scale of the first phase transition. The second horizon $r_+ \simeq 7.4 \times 10^{13}$ cm distinguishes the additional essential length scale: during a long time our Universe evolves as the $T_+$-region $r_- < r < r_+$, i.e. the expansion is inevitable for all observers as dictated by the causal structure of spacetime. The present epoch gets into the the $R$-region between two $T_+$-regions. Near the point of achieving the present vacuum density, the geometry generated by the vacuum fluid background becomes almost pseudo-Euclidean ($A(r) \simeq 1 - 10^{-6}$).

The global structure of spherically symmetric spacetime with three horizons asymptotically de Sitter in the origin and at infinity is shown in figure 7 [49, 64]. This picture shows how the manifold of events is seen by different observers. Let us note that the Carter–Penrose diagram figure 7 covers the whole plane except the squares bounded by the lines $r = 0$ and $r = \infty$. The lines $r_-$ correspond to cosmological horizons for static observers (observers in hats) in the $R$-regions $0 \leq r < r_-$ denoted as $RC$; $r_+$ are the black (white) hole horizons for static observers in the $R$-regions $r_- < r < r_+$ denoted as $U$, and $r_{++}$ are their cosmological horizons. $T_+$-regions $r_- < r < r_+$ denoted as $CC$ correspond to regular homogeneous anisotropic cosmological $T$-models of the Kantowski–Sachs type [49, 50]. The Lemaître cosmological model shown in figure 7 starts evolution in the $R$-region $RC_1$ and goes consequently through the $T_+$-region $r_- < r < r_+$, the $R$-region $r_+ < r < r_{++}$ and the $T_+$-region $r_{++}, < r < \infty$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The metric function $A(r)$ plotted with parameters corresponding to our Universe.}
\end{figure}
7. Summary and discussion

We presented the general analysis for the case of several scales of vacuum energy $\rho_{\text{vac}}$ corresponding to phase transitions involving inflationary stages. In our approach, vacuum dark energy is described by vacuum dark fluid defined by symmetry of its SET. In the spherically symmetric case, it is invariant under the radial Lorentz boosts acquiring the maximal symmetry of de Sitter vacuum only at inflationary stages. This makes the vacuum density $\rho_{\text{vac}}$ time dependent and spatially inhomogeneous. Cosmological solutions generated by vacuum dark fluid belong to the Lemaître class models with anisotropic pressures (their anisotropy follows directly from the variability of $\rho_{\text{vac}}$).

Intrinsic properties of the de Sitter spacetime are responsible for accelerated expansion independent of particular properties of particular models for vacuum density associated with the cosmological constant. In a similar way, intrinsic properties of geometries generated by a vacuum dark fluid can be in principle responsible for variable vacuum density, and make it possible to describe on the common ground the first inflationary expansion, the currently observed accelerated expansion, as well as predicted by the standard model inflationary stages related to phase transitions in the universe evolution.

To our knowledge, such an approach is applied for the first time for the analysis of cosmological evolution in the case of several vacuum scales.

Dynamics of cosmological models with the vacuum dark fluid is dictated by the number of vacuum scales: their number determines behavior of the transversal pressure $p_\perp$ that in turn

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Figure 7. Global structure of spherical spacetime with three horizons.
determines the maximal number of the spacetime horizons; their actual number depends on a model parameters.

We studied in detail the cosmological model for the case of three vacuum scales: GUT, QCD and that responsible for the currently observed accelerated expansion. We used the phenomenological density profile with typical behavior for a cosmological scenario with inflationary stages followed by decay of vacuum energy, which we describe by the exponential function typical for decay processes. The parameter characterizing the rate of the decay is tightly fixed by the requirements of analyticity and causality. Other parameters of the model are fixed by the values $\rho_{\text{GUT}}$, $\rho_{\text{QCD}}$, $\rho_\lambda$ and $\Omega = 1$. The only free parameter is the e-folding number for the first inflation, which we estimate with using the observational constraint on the CMB anisotropy.

This model reveals the following features of our Universe.

(i) Spacetime has three horizons.
(ii) The cosmological evolution starts with the non-simultaneous timelike de Sitter bang followed by the short stage of inhomogeneous and anisotropic expansion.
(iii) During the first inflationary stage, the Universe enters quickly the $T^+$-region that makes the expansion irreversible. The second phase transition occurs during this period.
(iv) The Universe enters the $R$-region (in which we actually live) when the second inflation already terminated. Soon, a vacuum density achieves its present value (it occurs at $r \approx r_4 \sim 10^{18}$ cm and $\tau \approx \tau_4 \sim 1000$ years). For a long time, the Universe geometry remains almost pseudo-Euclidean up to the scale factor $r$ of approximately $3 \times 10^{27}$ cm that corresponds to the age of about $3 \times 10^9$ years, when, according to the observational data, the present vacuum density $\rho_\lambda$ begins to dominate and the third inflation starts. Let us note that in a model taking into account matter and radiation, the vacuum density could achieve its present value later.
(v) After crossing the third horizon related to the present vacuum density ($r_h \approx 7.5 \times 10^{29}$ cm), the Universe enters the second $T_3$-region with the inevitable expansion.

We have seen that even our purely vacuum model can fairly well conform to the basic observational features of our Universe, which proves the viability of our approach. We hope that inclusion of matter and radiation in a further development of this approach will result in the models more completely describing the observed cosmological picture.

A general exact solution for homogeneous $T$-models has been found for a mixture of vacuum dark fluid with $T_t = T_r$ and dust-like matter [50]. It presents the class of $T$-models specified by the density profile of a vacuum fluid. The solution contains one arbitrary integration constant related to the dust density. Numerical estimates for a particular model illustrate the ability of such models to satisfy the observational constraints [50]. A similar solution for the Lemaître class models is now under consideration.

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