Dispersive Approach to the Trace Anomaly.

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Abstract

In the scalar $\phi^4$ field model the dispersive approach to the trace anomaly is proposed. It is shown that it is impossible to get dispersion representation for all formfactors so that preserve both the translation and dilatation Ward identities. Subtractions which preserve energy-momentum conservation violate the classical trace Ward–Takahashi identity and give rise to an anomalous contribution to the matrix element of stress tensor $\theta_{\mu}$. This contribution coincides with one-loop $\beta$-function in accordance with previous analyses based on RG technique.

Some years ago in Ref. [1] there was proposed a derivation of the axial anomaly [2] using dispersion relations for formfactors. In this approach the anomaly appears in a natural way as (finite) subtractions in dispersion integrals. This formalism was developed in Ref. [3] for different kinematical regimes. Another well known anomaly—that of energy-momentum tensor trace [4]—had been extensively studied in different field theories [5, 6, 8]. In this paper we apply the dispersion technique to the derivation of the trace anomaly following the line of paper [1].

In the case of the axial anomaly it was realized that there must be a 'pair' of symmetries (Ward identities at the quantum level) which cannot be preserved under renormalization simultaneously and thus give rise to anomaly. The trace anomaly, as it will be seen below, in fact is a conflict between the translation and dilatation invariances.

To derive the axial anomaly by dispersion method one starts from the three-point Green’s function of one axial and two vector currents in a gauge theory with a fermion field $\psi$

$$T_{\alpha\mu}(k_1, k_2) = \int dx \, dy \, e^{i(xk_1 + yk_2)} \langle J_5^\alpha(0) J_\mu(x) J_\nu(y) \rangle .$$

Conservation of the vector current leads to the vector Ward identity

$$k_1^\mu T_{\alpha\mu} = k_2^\nu T_{\alpha\nu} = 0 ,$$

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while (partial) conservation of the axial current is followed by the axial Ward identity

\[(k_1 + k_2)\alpha T_{\alpha\mu\nu} = 2mT_{\mu\nu} .\]  

(3)

Here \(T_{\mu\nu}\) is the Green’s function with the axial current replaced by its divergence—chiral density \(\bar{\psi}\gamma_5\psi\) and \(m\) is the mass of the fermion.

It is well known that Ward identities (2) and (3) are incompatible at the one-loop level. However, one can check that for the absorptive part of \(T\) both vector and axial identities hold true (no matter which variable the discontinuity is taken with respect to). Using analytical properties of amplitude \(T\) dispersion relations for formfactors can be written down. Then the anomaly manifests itself as additional subtractions in dispersion integrals \([1, 3]\).

One may think that such the scheme would work in the case of the trace anomaly as well. Below we will show that it does (at the one-loop level). As an example we considered \(\phi^4\) theory in four dimensions but generalizations of this procedure on any scalar field theory is straightforward.

Let us start from the usual \(\phi^4\)-lagrangian of a real scalar field \(\phi\) with a mass \(m\) and a coupling constant \(\lambda\)

\[\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.\]  

(4)

At the classical level translations in the configuration space are associated with the canonical stress tensor \(\theta^{\text{can}}_{\mu\nu}\). However, in our analysis it is more convenient to use the so-called ‘improved’ tensor \([7]\)

\[\theta_{\mu\nu} = \frac{1}{2}(\partial_{\mu}\phi)(\partial_{\nu}\phi) + g_{\mu\nu}m^2\phi^2 + g_{\mu\nu}\frac{\lambda}{4!}\phi^4 + \frac{1}{6}(g_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu})\phi^2,\]  

(5)

which coincides with the canonical one \(\theta^{\text{can}}_{\mu\nu}\) apart from the last term. Both \(\theta_{\mu\nu}\) and \(\theta^{\text{can}}_{\mu\nu}\) generate space-time translations and have the trace \(m^2\phi^2+\text{total divergences}\) (about an ambiguity of energy-momentum tensor definition see e.g. review \([8]\)). Improved tensor (5) has an advantage of having the trace just \(m^2\phi^2\) if equations of motion are applied

\[\theta^\mu_\mu = m^2\phi^2.\]  

(6)

Eq. (6) can be thought about as the ‘classical trace identity’ while in the quantum theory the r.h.s. of (6) changes because of quantum loop corrections. One must consider corresponding Ward identities instead of formula (6). The translation and trace identities
\[ q^\mu T_{\mu \nu}(p_1, \ldots, p_n) = \sum_{i=1}^{n} (p_i + q) \mu G^{(n)}(p_1, \ldots, p_i + q, \ldots, p_n) , \quad (7) \]

\[ g^{\mu \nu} T_{\mu \nu}(q; p_1, \ldots, p_n) = T(q; p_1, \ldots, p_n) + \sum_{i=1}^{n} d G^{(n)}(p_1, \ldots, p_i + q, \ldots, p_n) , \quad (8) \]

Here \( G^{(n)} \) is the \( n \)-point Green’s function having a canonical dimension \( d \) while \( T_{\mu \nu} \) and \( T_{\mu} \) are Green’s functions of the operators \( \theta_{\mu \nu} \) and \( \theta_{\mu} \) respectively, i.e.

\[ G^{(n)}(p_1, \ldots, p_n) = \mathcal{F}\langle T\phi(x_1) \ldots \phi(x_n) \rangle , \quad (9) \]

\[ T_{\mu \nu}(q; p_1, \ldots, p_n) = \mathcal{F}\langle T\theta_{\mu \nu}(0) \phi(x_1) \ldots \phi(x_n) \rangle , \quad (10) \]

\[ T(q; p_1, \ldots, p_n) = \mathcal{F}\langle T\theta_{\mu}(0) \phi(x_1) \ldots \phi(x_n) \rangle , \quad (11) \]

where symbol \( \mathcal{F} \) stands for the Fourier transformation. In the above formulae we reckon that all momenta are incoming.

Consider first 2-point functions. Then \( T_{\mu \nu} \) and \( T_{\mu} \) are operator insertions of \( \theta_{\mu \nu} \) and \( \theta_{\mu} \) into propagator \( G^{(2)} \). It was shown in Ref. \[ 6 \] that there is no anomaly in this case. Namely, one can redefine tensor \( \theta_{\mu \nu} \) in such a way that both identities are fulfilled. In another language it manifests the well known fact that there is no mass renormalization to the first order in \( \phi^4 \) theory.

Let us now turn to the case \( n = 4 \) which we will deal with in the rest of this paper. One-loop corrections to the matrix element \( (10) \) are given by 9 diagrams. But we are interested in those three only that contribute to \( \text{Im} q^2 T_{\mu \nu} \). One of them is shown in Fig.1 while two others are obtained by permutating momenta \( p_i \). Diagrams of other type corresponding to \( \phi^4 \) terms in \( \theta_{\mu \nu} \) are irrelevant when cut along \( q^2 \).

The problem greatly simplifies if we restrict ourselves to the following special kinematics \[^1\]

\[ (p_i + p_j)^2 = p^2 < 0 \quad \text{for } i, j = 1, \ldots, 4, \quad i \neq j. \quad (12) \]

In such a regime each of the three crossed diagrams gives an equal contribution to form-factors depending only on two variables \( p^2 \) and \( q^2 \). Thus one can consider only one graph keeping in mind that the final result is to be multiplied by the factor 3.

\[^1\]This symmetrical kinematics is realized when the momenta \( \vec{p}_i \) in the c.m. frame \( \vec{q} = 0 \) are directed from the centre of the regular tetrahedron to its vertices.
Denoting the amplitude corresponding to the diagram of Fig.1 by $\Delta_{\mu\nu}$ and introducing vectors $p = p_1 + p_2$ and $k = p_3 + p_4$ we write the following decomposition

$$\Delta_{\mu\nu}(p, k) = g_{\mu\nu}F_1 + (p_\mu p_\nu + k_\mu k_\nu)F_2 + (p_\mu k_\nu + k_\mu p_\nu)F_3,$$

$$\Delta_\mu(p, k) = 2m^2F_0,$$

where formfactor $F_0$ corresponds to the triangle graph in Fig.1 with $\theta_{\mu\nu}$ replaced by trace operator $\theta_{\mu\mu}$. The same decomposition is valid for imaginary parts with replacement $F_i \to A_i = \text{Im}F_i$. The Feynman rules for operator vertexes are shown in Fig.2. Calculation of the diagram of Fig.1 is straightforward and yields

$$A_1 = \frac{\lambda^2}{32\pi} \frac{1}{s + \beta} \left\{ \left( \frac{s^2}{3} + \frac{s(2\beta + 3)}{6} + \frac{\beta(\beta + 4)}{8} \right)L + \left( s + \frac{\beta}{2} \right)S \right\},$$

$$A_2 = \frac{\lambda^2}{32\pi} \frac{1}{4m^2(s + \beta)^2} \left\{ \left( \frac{2s^2}{3} + \frac{s(\beta + 6)}{12} - \frac{\beta^2(\beta + 4)}{8s} \right)L + \left( 2s - \frac{\beta^2}{2s} \right)S \right\},$$

$$A_3 = \frac{\lambda^2}{32\pi} \frac{1}{4m^2(s + \beta)^2} \left\{ \left( -\frac{3s^2}{4} - \frac{s(5\beta + 3)}{3} - \frac{\beta(5\beta + 9)}{6} - \frac{\beta^2(\beta + 4)}{8s} \right)Light.$$

$$+ \left( -4s - 3\beta - \frac{\beta^2}{2s} \right)S \right\},$$

$$A_0 = \frac{\lambda^2}{32\pi} \frac{1}{2m^2}L,$$

where we introduce the dimensionless variables

$$s = \frac{q^2}{4m^2} \quad \text{and} \quad \beta = -\frac{p^2}{m^2},$$

and

$$L = \frac{1}{\sqrt{s(s + \beta)}} \log \frac{s + \beta/2 - \sqrt{(s - 1)(s + \beta)}}{s + \beta/2 + \sqrt{(s - 1)(s + \beta)}},$$

$$S = \sqrt{1 - \frac{1}{s}}.$$

Now let us consider Ward identities (7) and (8) again. Substituting (13),(14) into these equations we obtain the Ward identities written in terms of formfactors

$$F_1(q^2, p^2) + \frac{q^2}{2} \left( F_2(q^2, p^2) + F_3(q^2, p^2) \right) = G^{(4)}(p^2),$$

$$4F_1(q^2, p^2) + 2p^2F_2(q^2, p^2) + (q^2 - 2p^2)F_3(q^2, p^2) = 2m^2F_0(q^2, p^2) + 4G^{(4)}(p^2),$$

where $G^{(4)}$ is the usual 4-point Green’s function depending only on $p^2$. Let us emphasize that there are 'pure kinematical' contributions ($G^{(4)}$) in the r.h.s. of Eqs. (22) and (23).
Such terms are absent in the axial–vector Ward identities. To get rid of the auxiliary function $G^{(4)}$ one can introduce a new formfactor $F'_{1} = F_{1} - G^{(4)}$. Then (22) and (23) take the form
\begin{align}
F'_{1}(q^{2}, p^{2}) + \frac{q^{2}}{2} \left( F_{2}(q^{2}, p^{2}) + F_{3}(q^{2}, p^{2}) \right) &= 0, \\
4F'_{1}(q^{2}, p^{2}) + 2p^{2}F_{2}(q^{2}, p^{2}) + (q^{2} - 2p^{2})F_{3}(q^{2}, p^{2}) &= 2m^{2}F_{0}(q^{2}, p^{2}).
\end{align}

Furthermore, in our analysis we do not care much about $G^{(4)}$ for its discontinuity along $q^{2}$ axis vanishes. For imaginary parts equations similar to (22) and (23) exist
\begin{align}
A_{1}(t, p^{2}) + \frac{t}{2} \left( A_{2}(t, p^{2}) + A_{3}(t, p^{2}) \right) &= 0, \\
4A_{1}(t, p^{2}) + 2p^{2}A_{2}(t, p^{2}) + (t - 2p^{2})A_{3}(t, p^{2}) &= 2m^{2}A_{0}(t, p^{2}),
\end{align}

with $A_{i}(q^{2}) = \text{Im} q^{2}F_{i}$. It should be noted also that $A_{1}(t, p^{2}) \equiv A'_{1}(t, p^{2})$.

Another difference from axial anomaly case is that ultraviolet divergences appear. Namely, functions $F_{1}$ and $G^{(4)}$ are logarithmically divergent while $F_{2}, F_{3}, F_{0}$ are finite. One can convince oneself that functions $A_{2,3,0}(q^{2})$ fall like $1/q^{2}$ as $q^{2} \to \infty$. Thus we can postulate dispersion representations for these formfactors without any subtractions
\begin{equation}
F_{2,3,0}(q^{2}) = \frac{1}{\pi} \int_{4m^{2}}^{\infty} \frac{A_{2,3,0}(t)}{t - q^{2}} \, dt.
\end{equation}

The situation is different for $A_{1}$. As one can expect there is a logarithmic divergence and at least a single subtraction is needed. Let us make the subtraction at an euclidean point $q^{2} = \mu^{2} < 0$
\begin{equation}
F_{1}(q^{2}) = \frac{1}{\pi} (q^{2} - \mu^{2}) \int_{4m^{2}}^{\infty} \frac{A_{1}(t)}{(t - \mu^{2})(t - q^{2})} \, dt + c.
\end{equation}

Note that generally $c$ is a function of $p^{2}$ and $\mu^{2}$.

Integrating the Ward identities (26) and (27) we get for the translation identity
\begin{equation}
F_{1}(\mu) + c + \frac{q^{2}}{2}(F_{2} + F_{3}) + \frac{1}{\pi} \int_{4m^{2}}^{\infty} \left( \frac{A_{1}}{t - \mu^{2}} + \frac{1}{2}A_{2} + \frac{1}{2}A_{3} \right) \, dt = 0,
\end{equation}

while the trace identity now reads
\begin{equation}
4F_{1}(\mu) + 4c + 2p^{2}F_{2} + (q^{2} - 2p^{2})F_{3} + \frac{1}{\pi} \int_{4m^{2}}^{\infty} \left( \frac{4A_{1}}{t - \mu^{2}} + A_{3} \right) \, dt = 2m^{2}F_{0}.
\end{equation}
From Eqs. (30) and (31) one can read that it is impossible to satisfy both translation (24) and trace (25) identities having the only auxiliary parameter \( c \). One is free though to choose \( c \) so that to preserve the former or the latter. It is more natural to have the translation identity untouched, i.e. one should require that Eq. (30) takes form (24). This fixes the subtraction \( c(p^2) \):

\[
c(p^2) = -\frac{1}{\pi} \int_{4m^2}^{\infty} \left( \frac{A_1}{t - \mu^2} - \frac{1}{2} A_2 + \frac{1}{2} A_3 \right) \, dt .
\]  

Substituting (32) into the trace identity (31) we obtain

\[
F_1(\mu) + 2p^2 F_2 + (q^2 - 2p^2) F_3 = 2m^2 F_0 + \Delta ,
\]

where \( \Delta \) is, generally speaking, a function of \( \mu^2 \) and \( p^2 \) only.

\[
\Delta(\mu^2, p^2) = -4c(\mu^2, p^2) - \frac{1}{\pi} \int_{4m^2}^{\infty} \left( \frac{4A_1}{t - \mu^2} + A_3 \right) \, dt
\]

\[
= \frac{1}{\pi} \int_{4m^2}^{\infty} (2A_2 + A_3) \, dt .
\]

Let us emphasize that \( \Delta \) in (34) is actually independent on \( \mu \), and by dimensional reasons (as it may depend only on ratio \( \mu^2/p^2 \), on \( p^2 \). All \( \mu \)-dependence is now absorbed in formfactor \( F_1 \) (its finite renormalization actually changes it to \( F' \)). Moreover, expressing \( A_1 \) in terms of \( A_2 \) and \( A_3 \) by use of the translation invariance (26) and assuming the validity of (24), we may keep all the derivation ultraviolate finite on each stage.

Thus we see that the trace identity (33) develops the extra term \( \Delta \) which is in fact the constant 'trace anomaly'. Considering the chiral limit \( m \to 0 \), one may conclude, that because of the constant value of the integral (34),

\[
2A_2(t) + A_3(t) \to \pi \Delta \delta(t),
\]

manifesting the zero mass ”dilaton” singularity, in complete analogy with the axial anomaly case. Trace anomaly is therefore appearing as a purely infrared phenomenon.

One may, in principle, consider the opposite case, when the dilatation invariance is preserved, while the translation invariance is anomalously broken. Expressing \( A_1 \) from (27) and assuming the validity of (23) one immediately get:

\[
F'_{1}(q^2, p^2) + \frac{q^2}{2} (F_2(q^2, p^2) + F_3(q^2, p^2)) = -\frac{1}{4} \Delta ,
\]
the "translational anomaly" being described by the same expression (34) as the trace one. It is leading to the similar massless singularity (35) in the chiral limit, contrary to the case of spontaneous breaking of translation invariance [9].

The integration in (34) can be performed analytically. Evaluation yields that the result is really constant $\lambda^2/16\pi^2$. Taking into account that the contribution to the Ward identity (8) is three times more because of the cross diagrams we obtain the anomalous term in the r.h.s. of (8)

$$T_{\text{anom}} = 3\Delta(\mu^2, p^2) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3).$$

(37)

Thus we have obtained the result which must be compared with that given in Ref. [8]. In terms of renormalized operators the trace operator looks like [8]

$$\theta^\mu = m^2\phi^2 + \beta(\lambda)\frac{1}{4!}\phi^4 + \ldots,$$

(38)

where dots stand for the terms which do not contribute to the matrix element at hand and $\beta(\lambda) = 3\lambda^2/16\pi^2 + O(\lambda^3)$ is the renormalization-group $\beta$-function. Taking matrix element of (38) one can see that $T_{\text{anom}}$ is nothing but one-loop $\beta$-function of the theory. At the same time, the translation anomaly leads to:

$$\partial_\mu \theta^\mu = -\frac{1}{4} \beta(\lambda)\frac{1}{4!}\partial_\nu\phi^4.$$

(39)

The equations (38,39) are representing the two different choices of the anomaly for the pair of the symmetries and may be compared with the similar equations for the case of the another pair [10], namely the general coordinate invariance and Weyl invariance for the massless scalar field interacting with the 2-dimensional gravity. The r.h.s. of the analogies of (38,39) are $R/24\pi$ and $\partial_\nu R/48\pi$, respectively, $R$ being the scalar curvature. Note that the different magnitude of the ratio of the numerical coefficients is the direct consequence of the different dimensionality of the space-time.

When this investigation was finished, we have learned about another efforts in this direction.

The generalization of our method to the more realistic case of QED and QCD appears to be rather lengthy, and the analysis of the Ward identities for the trace itself (assuming the translation invariance), successfully performed recently [11], is more useful. However, we believe that the full analysis of the pair of anomalies, presented here, is also of some interest.
Also, the earlier calculations in the framework of the source theory \[12\] are in fact manifesting many properties of the dispersive derivation and may be considered as its pioneering application to the trace anomaly case.

At the same time, the scalar fields are known to be of particular interest for the quantum cosmology and general relativity, where \[13\] some analog of the dispersive analysis (which seems to have a number of counterparts in our approach) was presented recently.

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    pendix B.
Fig.1 One of the graphs contributing to the matrix element of $\theta_{\mu\nu}$.

\[ i[g_{\mu\nu}(p_1 p_2 + m^2) - (p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu}) + \frac{1}{3}(q_{\mu} q_{\nu} - q^2 g_{\mu\nu})] \]

Fig.2 The Feynman rules for operator $\theta_{\mu\nu}$. 

\[ i\lambda g_{\mu\nu} \]