On Coretractable Module

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Abstract The main aim of this paper is to discuss the structure of coretractable and mono-coretractable modules. We have provided the characterizations of coretractable modules in terms of Kasch ring, CS modules, mono-coretractable modules, projective modules, epi-retractable modules, and retractable modules. We have also discussed some equivalent condition related with coretractable and mono-coretractable modules.

Keywords Retractable Module · Coretractable Module · Mono coretractable Module · Kasch Ring · Kasch Module

1 Introduction

In this paper, all rings are associative and modules are unitary right modules. Any terminology not defined here may be found in Goodearl and Warfield [1] and McConnell and Robson [2]. Zelmanowitz considered compressible modules in detail in a series of papers [11]. Following [12] the right R-module M is called compressible if for each non-zero zero submodule N of M, there exists a monomorphism \( f : N \rightarrow M \). McConnell and Robson considered the epi-retractable modules as the dual of compressible module and defined it as follows a module M is called epi-retractable if every submodule of M is a homomorphic image of M [2]. An R-module M is said to be retractable if \( \text{Hom}(M, N) \neq 0 \) for any non-zero submodule N of M. Retractable modules have been discussed by some authors in series of papers [5], [6], [7], [8], [9], [10]. Dual concept to retractable modules was investigated in paper [13]. B.
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Amini et al. [13] proved that the right CC rings are two sided perfect and provided condition under which free modules are coretractable. In [9], P.F. Smith introduced the concept of essentially compressible modules as a generalization of compressible modules. Follows Vedadi [9], an R-module M is said to be essentially retractable if \( \text{Hom}(M, N) \neq 0 \), for any non-zero essential submodule of M which is a natural generalization of retractable modules. In [7], Abhay K. Singh discussed some results related with essentially slightly compressible modules and rings. Some structure of completely coretractable rings have been discussed by J. Zemlicka [14].

If N is a submodule of M, we write \( N \leq M \) and if N is an essential submodule of M then we write \( N \subseteq M \). A partial endomorphism of a module M is a homomorphism from a submodule of M into M. We say that N is a dense submodule of M (written \( N \subseteq_d M \)) if, for any \( yeM \) and \( xeM/\{0\}, xy^{-1}N \neq 0 \) (i.e., there exists \( reR \) such that \( xr \neq 0 \) and \( yreN \)). If \( N \subseteq_d M \), we also say that M is a rational extension of N. Let \( I = E(M) \), and let \( H = \text{End} \ (I_R) \), operating on the left of I. We define Rational Hull of a Module \( \hat{E}(M) \) as \( \hat{E}(M) = \{icI : \forall h \in H, h(M) = 0, h(i) = 0\} \). Clearly, this is an R-submodule of I containing M. An R-module MR is said to be rationally complete if it has no proper rational extensions, or equivalently \( \hat{E}(M) = M \).

A module is called mono coretractable if each of its factors can be embedded in it. That is, a module M is mono-coretractable if for each submodule N of M there exists a monomorphism \( g : M/N \rightarrow M \). Mono coretractable module is coretractable. Every simple and semi simple module is mono coretractable \( Z_p^\infty \) as a \( Z \)-module is mono coretractable and \( Z \) as \( Z \)-module is not mono coretractable. Some characterizations of mono-coretractable and coretractable modules discussed in [15]. In section 2 we study some properties of coretractable modules. We developed some relation between epi-retractable modules, coretractable modules, semiprimitive modules, and Kasch ring. In section 3, we discussed the characterizations mono-retractable modules in terms of non-singular uniform modules and kasch ring.

2 Structure of Coretractable Modules

A module M is called coretractable (see [13]) if \( \text{Hom}_R(M/K, M) \neq 0 \) with \( f(K) = 0 \) for any proper submodule K of M. Module M is said to be torsion free module if \( \forall xeR \) and \( meM, rm = 0 \Rightarrow r = 0 \) or \( m = 0 \). Before we prove some general module theoretic correction of coretractability, we investigate role of free, dense and rational submodules in coretractable modules. In our first result we discuss a coretractable module M in terms of torsion free module.

Proposition 2.1. Let M be coretractable module over a commutative ring R then M is torsion free.

Proof. Since M coretractable thus we have a homomorphism s.t. \( \text{Hom}_R(M/K, M) \neq 0 \).
If possible let $M$ is not torsion free then $\forall r \in \mathbb{R}$ and $m \in M$ \Rightarrow $rm = 0$.

Thus we have $f(rM/K) = fr(m + K) = f(rm) + f(rK) = 0$, since $f(K) = 0$. A contradiction to the fact that $f \neq 0$. Hence $M$ is torsion free.

An $R$-module $M_R$ is said to be rationally complete if it has no proper rational extensions, or equivalently $\hat{E}(M) = M$.

Theorem 2.2. For any module $M_R$, the followings are equivalent:

1. $M$ is rationally complete.
2. For any right $R$-modules $A \leq B$ such that $\text{Hom}_R(B/A, E(M)) = 0$, any $R$-homomorphism $f : A \rightarrow M$ can be extended to $B$.

If $M$ is Rationally complete then $M$ is not coretractable. We know that an $R$-module $M$ is called monoform if every submodule is dense. Now we are ready to establish following proposition.

Proposition 2.3. Let $I \neq R$ be an ideal in a commutative ring $R$ and let $M_R = R/I$, then followings are equivalent in $M$:

1. Every submodule in $M$ is dense.
2. $M$ is monoform.
3. $M$ is not coretractable.
4. Every ideal in $R/I$ is prime.

Proof. (1) $\Rightarrow$ (2).
(1) $\Rightarrow$ (3).
(1) $\Leftrightarrow$ (4). Let $M_R$ is a module and $I \neq R$ is an ideal of $R$. Let us suppose that any submodule $N$ of $M$ is dense in $M$. Then $\text{Hom}_R(N/I, M) = 0$ $\Leftrightarrow$ $f(I) = 0$ $\Leftrightarrow$ $f(ab) = 0$ $\Leftrightarrow$ $ab = 0$ $\Leftrightarrow$ either $a = 0$ or $b = 0$. $\Leftrightarrow$ $I$ is prime ideal.

We say that $M$ is a epi-retractable module if for every submodule $N$ of $M$, we have $\text{Hom}(M, N) \neq 0$. Our next result gives relation between epi-retractable module and coretractable module.

Proposition 2.4. The following statements are equivalent for a module $M$:

1. $M$ is epi-retractable.
2. There exist surjective homomorphisms $M \rightarrow N$ and $N \rightarrow M$ for some epi-retractive module $N$.
3. There exists a surjective homomorphism $M/K \rightarrow M$ for some epi-retractive factor module $M/K$.
4. $M$ is coretractable.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) By Lemma 2.7 \[10\].
(3) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (1) Let $N$ be any submodule of $M$. Since $M$ is coretractable we have $f : M/K \rightarrow M \neq 0$. So we have $f_{|_{N/K}} : N/K \rightarrow N$. Consider a projection map $\pi : M/K \rightarrow N/K$ and a canonical epimorphism $g : M \rightarrow M/K$. The
composition \( f|_{N/K} \pi g : M \to N \) implies that \( M \) is epi-retractable modules.

Proposition 2.5. Let \( M_R \) be an epi-retractable module, then \( M_R \) is coretractable if and only if \( M \) is semiprimitive.

Proof. Let \( M \) is coretractable then \( M \) is semisimple (see 2.3[13]). So \( M \) is semiprimitive. For the sufficient part let \( M \) is semiprimitive. Now since \( M \) is epi-retractable module, so we have a surjective mapping s.t. \( f(M, N) \neq 0 \) and \( g(N, M) \neq 0 \) by lemma 2.7[10]. Clearly we have a non zero homomorphism \( h(M/N, N) \neq 0 \). Thus the composition \( gh \neq 0 \). So \( M \) is coretractable.

Proposition 2.6. Let \( M \) be a projective \( R \)-module, then every epi-retractable module is coretractable.

Proof. Let \( M \) is a projective \( R \)-module and \( N \) is a submodule of \( M \). Consider a non zero map \( f : M/N \to N \). \( R \) is a right perfect ring so we have a surjective map \( \theta : P \to N \); \( P \) is projective cover of \( N \). Since \( M \) is projective so \( f = \theta h \neq 0 \) where \( h : M/N \to P \) is a non zero map. Since \( M \) is epi-retractable module so we have a surjective mapping from \( \pi : M \to N \) and \( g : N \to M \). (Prop. 2.1[10]). Clearly the mapping \( gf : M/K \to M \neq 0 \). Hence the proof.

An \( R \)-module \( M \) is satisfies (***) property if every \( f \in \text{End}(M_R) \), \( f \) is surjective mapping from \( M \) to \( M \).

We say that \( R \) is a right Kasch ring if every simple right \( R \)-module \( M \) can be embedded in \( R_R \). "Left Kasch ring" is defined similarly. \( R \) is called a Kasch ring if it is both right and left Kasch. A module \( M \) is called kasch module if it contains a copy of every simple module in \( \sigma(M) \).

Proposition 2.7. Let \( M_R \) be a epi-retractable, then

1. \( M_R \) is right kasch.
2. \( M_R \) satisfies (*** ) property.
3. \( M_R \) is coretractable.

Proof. (1) \( \Rightarrow \) (2) Let \( f \in \text{End}(M_R) \) and \( f : M \to K \) be a non zero homomorphism where \( K \) is a proper submodule of \( M \). Now \( M \) is right kasch so it contains copy of simple module. Thus we have \( f(M) = K = M \). \( f \) is surjective. \( M \) satisfies (***) property.

(2) \( \Rightarrow \) (3) Let \( M \) satisfies (*** ) property. Also \( M \) is epi-retractable module so we have non-zero homomorphism from \( f : M/K \to N/K \) where \( K \leq N \leq M \). Now we have a natural epimorphism \( f : N/K \to N \) and let \( ge\text{End}_R(M) \neq 0 \) where \( g \) is an onto map from \( N \) to \( M \). So we have a onto composition \( ghf : M/K \to M \neq 0 \).

(3) \( \Rightarrow \) (1) (Prop.2.14 [13]).

The following two proposition shows that coretractable modules satisfies (***) property and \( M \) is also a direct summand of a free epi-retractable \( R \)-modules.

Proposition 2.8. A Coretractable module \( M \) satisfies (*** ) property.
Proof. Let $M$ be a coretractable Module so we have a non zero homomorphism $f : (M/K, M) \neq 0$ for some proper submodule $K$. Consider a nonzero $g \in \text{End}(M_R)$. We have $gf : (M/K, M) \neq 0$ and also the mapping $gf$ is epimorphism. So $M$ satisfies (**) property.

Proposition 2.9. Let $R$ be a ring and $\alpha$ be an infinite ordinal $\geq |R|$. Suppose that $M = K \bigoplus N$ where $K$ is a free epiretractable $R$-module with a basic set of cardinality $\alpha$ and $N$ is a $\beta$-generated $R$-module with $\alpha \geq \beta$. Then, $M_R$ is coretractable.

Proof. $K$ is epi-retractable so $N$ is a homomorphic image of $K$. Since $K \simeq K \bigoplus K$, then there exist surjective homomorphisms $f : M \rightarrow K$ and $g : K \rightarrow M$. Let $L = \text{Ker}f$, then there exist natural homomorphism $h : M/L \rightarrow M$. Hence $M$ is coretractable.

A module $M$ is said to be fully retractable if for every nonzero submodule $N$ of $M$ and every nonzero element $g \in \text{Hom}_R(N, M)$, we have $\text{Hom}_R(M, N)g \neq 0$. This definition motivated us to give our next result.

Proposition 2.10. Every fully retractable module is coretractable if $\text{Hom}_R(M, N)g$ is epimorphism.

Proof. clear by prop. 2.4.

Proposition 2.11. Let $R$ be a ring. If $M_R$ is cyclic CS module then $M$ is coretractable.

Proof. since $M$ is cyclic CS module then $M$ is direct sum of uniform modules i.e. $M = \bigoplus_{i=1}^{n} N_i$ where each $N_i$ is essential in $M$. Let $K$ any submodule of $M$ then $K$ be essential in direct summand of $M$. Now exploiting the fact that every quotient module of a cyclic module is cyclic then we get for $\text{Hom}_R(M/K, M) \neq 0$ giving us that $M$ is coretractable.

Proposition 2.12. Let $R$ be a commutative artinian ring, then

1. If $p$ be any prime ideal of $R$ then $p$ is maximal.
2. $R$ is local ring.
3. $R_R$ is semisimple.
4. $\text{Hom}_R(R/p, R) \neq 0$.
5. $R$ is a kasch ring.
6. $R_R$ is coretractable.

Proof. Let $p$ be any prime ideal of $R$, then $R/p$ become an integral domain. i.e. $R/p$ is a field consequently $p$ is maximal ideal of $R$.

(1) $\Rightarrow$ (2) Since any commutative ring $R$ with prime ideal $p$ has got unique maximal ideal. Thus $R$ should be a local ring.

Since $R$ is artinian, hence (3) is obvious.

(1) $\Rightarrow$ (4) Let $p$ be a prime ideal of an Artinian ring $R$. Then $R/p$ is a prime Artinian ring, but such rings are simple. Hence $R$ is kasch.
We know that direct sum of coretractable module is coretractable but in the case of direct product of coretractable module the result may not be same. Next result will validate our supposition.

**Proposition 2.13.** The infinite direct sum of coretractable module is not coretractable.

**Proof.** Let \( K = K_1 \oplus K_2 \oplus \ldots \subset M \) be the infinite direct product of coretractable module where \( K_i, (i = 1, 2, \ldots) \) are projective. Consider a map \( f : (M/K, K) \) with \( f(K) = 0 \). Also suppose that \( M = M_1 \oplus M_2 \). Since \( f(K) = 0 \) so \( f(M_1) = 0 \) and \( f(M_2) = 0 \) \( \Rightarrow f(M) = 0 \) \( \Rightarrow f : (M/K, M) = 0 \). Thus infinite direct sum of coretractable module is may not be coretractable.

**Proposition 2.14.** Let \( R \) be hereditary ring, then \( M_R \) is coretractable.

**Proof.** Let \( N \) be a injective submodule of \( M \). Then there exist a submodule \( K \) of \( M \) such that \( M = K \oplus N \) by lemma 2.8. Let \( f(M/N, N) \neq 0 \). Consider a injection map \( g : N \to M \). Clearly the composition \( gf(M/N, M) \neq 0 \). So \( M_R \) is coretractable.

### 3 Mono coretractable Module

Now we introduce Mono-coretractable module and we will discuss various propeties of mono-coretractable modules.

An R-Module \( M \) is said to be Mono-coretractable if \( \text{Hom}_R(M/K, M) \neq 0 \) is monomorphism where \( K \) is a submodule of \( M \). A coretractable modules are not mono-coretractable in general for example it can be easily seen that \( \mathbb{Z}_4 \) over \( \mathbb{Z} \) is coretractable but not mono-coretractable.

Let us discuss in this section various properties of mono-coretractable modules.

**Proposition 3.1.** For a module \( M_R \) following is equivqlent:

1. \( M \) is mono coretractable.
2. Every non zero partial endomorphism of \( M/K \) is monomorphism.

**Proof.** (1) \( \Rightarrow \) (2) Let \( M \) is mono coretractable. We have \( \text{HOM}_R(M/K, M) \neq 0 \). Take \( N/K \leq M/K \) where \( N \leq K \leq M \) and let \( f : N/K \to M/K \) be a nonzero partial endomorphism of \( M/K \). Then we have \( f : N/K \to N.kerf \cong \text{im} f \). Also \( f_1(N/kkerf) \to M/K \) is monomorphism.

The composition \( N/K \to M/K \to M \) is monomorphism. Thus \( kerf = K \). Thus Every non zero partial endomorphism of \( M/K \) is monomorphism.

(1) \( \Rightarrow \) (2) let Every non zero partial \( \text{End}(M/K) \) is monomorphism and let \( M \) be module s.t. \( f : M/K \to M \neq 0 \). Now \( f : N/K \to M/K \) is monomorphism. Consider a projection map \( h : N/K \to M \) define as \( f(x+K) = x, x \in N \). Clearly \( h \) is monomorphism.

Since \( h \) is projection map, so \( h = fg \) and the composition \( N/K \to M/K \to M \).
should be monomorphism ⇒ f is monomorphism ⇒ M is mono coretractable.

Proposition 3.2. Direct sum of mono coretractable modules is mono coretractable.

Proof. Obvious.

Proposition 3.3. Let \( M_R \) be a non singular uniform module in R then \( M_R \) is not mono coretractable.

Proof. take any \( N \leq M \) then \( M/N \) is singular. Hence \( \text{Hom}_R(M/N, M) = 0 \). So M is not mono coretractable.

Now we discuss some condition for mono-coretractability.

Proposition 3.4. For a ring R, Following holds:
1. R is a right Kasch ring.
2. \( R_R \) is a mono- coretractable module.
3. Every finitely generated free right R-module is mono-coretractable.
4. \( R_R \) has no proper dense submodules.
5. \( R_R \) not monoform.

Proof:- (1) ⇒ (2) For any maximal ideal I in R, \( \text{Hom}_R(R/I, R) \neq 0 \). Thus \( R_R \) is monocoretractable.
(2) ⇒ (3). Obvious
(2) ⇒ (4) Take a R-module M and Consider \( K \leq N \leq M \). Due to Mono coretractability \( \text{Hom}_R(M/K, M) \neq 0 \Rightarrow \text{Hom}_R(M/K, N) \neq 0 \). So M has no proper dense submodules.
(4) ⇒ (5). Obvious.

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