H-function extension of the NBD: further applications

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(March 26, 2022)

The H-function extension of the Negative Binomial Distribution is investigated for scaling exponents \( \mu < 0 \). Its analytic form is derived via a convolution property of the H-function. Applications are provided using multihadron and galaxy count data for \( P_n \).

In a series of recent papers the present author developed a generalization of the Negative Binomial Distribution \([1-3]\). \( P_n \) was obtained by means of the Poisson transform \([4]\)

\[
P_n = \int_0^\infty f(x) \frac{x^n}{n!} e^{-x} \, dx
\]

of the continuous probability density \( f(x) \) which asymptotically yields the KNO scaling form of \( P_n \) (provided that \( f(x) \) belongs to the family of scale parameter generating density functions). Choosing \( f(x) \) to be the generalized gamma distribution \([5]\)

\[
f(x) = \frac{|\mu|}{\Gamma(k)} \lambda^k x^{\mu k - 1} \exp(-\lambda x^\mu)
\]

with moments of the form

\[
\langle x^q \rangle = \int_0^\infty x^q f(x) \, dx = \begin{cases} \frac{\Gamma(k + q/\mu)}{\Gamma(k)} \frac{1}{\lambda^{q/\mu}} & \text{for } q/\mu > -k \\ \infty & \text{otherwise} \end{cases}
\]

one obtains the above cited generalization of the NBD. In Eqs. (2-3) we have parameters \( k > 0 \) (shape parameter), \( \lambda > 0 \) (scale parameter), and \( \mu \neq 0 \) (scaling exponent). In the limit \( \mu \to 0 \) and \( k \to \infty \) the generalized gamma density converges to the log-normal law. The \( k = 1, \mu > 0 \) special case of Eq. (2) provides the Weibull distribution whereas the \( \mu = 1 \) case is, of course, the ordinary gamma distribution whose Poisson transform yields the NBD.

In refs. [1-3] the Poisson transform of the generalized gamma density Eq. (2) was investigated only for scaling exponents \( \mu > 0 \). The analytic form of \( P_n \) was shown to be expressible in terms of Fox’s H-function (see refs. [1,3] for a summary and ref. [6] for details). Therefore we will call this particular extension of the Negative Binomial Distribution as HNBD for short. It takes the form

\[
P_n = \frac{1}{n!} \frac{\Gamma(k + n)}{\Gamma(k)} \theta^n (1 + \theta)^{-k - n} \quad \text{for } \mu = 1
\]

\[
P_n = \frac{1}{n! \Gamma(k)} H_{1,1}^{1,1} \left[ \frac{1}{\theta} \right] \frac{(1 - k, 1/\mu)}{(n, 1)} \quad \text{for } \mu > 1
\]

\[
P_n = \frac{1}{n! \Gamma(k)} H_{1,1}^{1,1} \left[ \frac{1}{\theta} \right] \frac{(1 - n, 1)}{(k, 1/\mu)} \quad \text{for } 0 < \mu < 1
\]

where \( H(\cdot) \) denotes the H-function of Fox and \( \theta = \lambda^{1/\mu} \) can be expressed in terms of the average multiplicity \( \langle n \rangle \) according to \( \theta = \langle n \rangle \Gamma(k)/\Gamma(k + 1/\mu) \). The pure NBD given by Eq. (4) is the \( \mu = 1 \) marginal case of Eq. (5) for \( \langle n \rangle < k \) and of Eq. (6) for \( \langle n \rangle > k \) [3]. The \( \mu \to 0, k \to \infty \) Poisson transformed log-normal limit of the HNBD lacks a representation in terms of known functions.

The main goal of the present Letter is to complete the derivation of the Poisson transformed generalized gamma distribution by extending its validity to negative values of the scaling exponent. For \( \mu < 0 \) we shall consider the Poisson transform of the probability density
\[ f(x) = \frac{\mu}{\Gamma(k)} \lambda^k x^{-\mu k - 1} \exp(-\lambda x^{1/\mu}) \]  
Eq. (7)

with \( \mu > 0 \). In the derivation of \( P_n \) two particular cases of \( H(x) \) will be utilized,

\[ H_{0,1}^{1,0} \left[ ax \right]_{0}^{(0, 1)} = e^{-ax} \]  
Eq. (8)

and

\[ \frac{1}{\Gamma(k)} H_{0,1}^{1,0} \left[ \frac{\theta}{x} \right]_{(k, 1/\mu)} = \frac{\mu}{\Gamma(k)} (\theta/x)^{\mu k} \exp\left(-\frac{\theta}{x}\right), \]  
Eq. (9)

the latter provides \( xf(x) \) with \( \theta = \lambda^{1/\mu} \) as before. We shall also make use of the following two properties of \( H(x) \):

\[ x^p H_{p, q}^{m, n} \left[ x \right]_{(a_1, \alpha_1), \ldots, (a_p, \alpha_p)}_{(b_1, \beta_1), \ldots, (b_q, \beta_q)} = H_{p, q}^{m, n} \left[ x \right]_{(a_1 + r \alpha_1, \alpha_1), \ldots, (a_p + r \alpha_p, \alpha_p)}_{(b_1 + r \beta_1, \beta_1), \ldots, (b_q + r \beta_q, \beta_q)} \]  
Eq. (10)

and

\[ \int_{0}^{\infty} x^{-1} H_{p, q}^{m, n} \left[ ax \right]_{(a_1, \alpha_1), \ldots, (a_p, \alpha_p)}_{(b_1, \beta_1), \ldots, (b_q, \beta_q)} H_{p, q}^{m, n} \left[ b \right]_{(c_1, \gamma_1), \ldots, (c_p, \gamma_p)}_{(d_1, \delta_1), \ldots, (d_q, \delta_q)} dx = H_{p+p+\beta+\gamma+\delta}^{m+n+M+N+P} \left[ ab \right]_{(e_1, E_1), \ldots, (e_{p+P}, \epsilon_{p+P})}_{(f_1, F_1), \ldots, (f_{q+Q}, \epsilon_{q+Q})} \]  
Eq. (11)

with parameters

\{ (e_j, E_j) \} = (a_1, \alpha_1), \ldots, (a_n, \alpha_n), (c_1, \gamma_1), \ldots, (c_p, \gamma_p), (a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p), \]

\{ (f_j, F_j) \} = (b_1, \beta_1), \ldots, (b_{m+1}, \beta_{m+1}), (d_1, \delta_1), \ldots, (d_q, \delta_q), (b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q). \]  
Eq. (12)

For the conditions of validity of various properties of \( H(x) \), see ref. [6]. Absorbing \( x^n \) into the lhs. of Eq. (9) via identity (10) one obtains for \( P_n \) defined by Eq. (1) with \( f(x) \) given by Eq. (7) the following integral:

\[ P_n = \theta^n \frac{1}{n! \Gamma(k)} \int_{0}^{\infty} x^{-1} H_{0,1}^{1,0} \left[ \frac{\theta}{x} \right]_{(k - n/\mu, 1/\mu)} H_{0,1}^{1,0} \left[ x \right]_{(0, 1)} dx. \]  
Eq. (13)

Comparison with the convolution property Eq. (11) yields

\[ P_n = \theta^n \frac{1}{n! \Gamma(k)} H_{0,2}^{2,0} \left[ \theta \right]_{(0, 1), (k - n/\mu, 1/\mu)} \]  
Eq. (14)

and using identity (10) we get

\[ P_n = \frac{1}{n! \Gamma(k)} H_{0,2}^{2,0} \left[ \theta \right]_{(n, 1), (k, 1/\mu)} \]  
Eq. (15)

for the Poisson transform of the probability density Eq. (7). Above, \( \theta \) is expressible in terms of \( \langle n \rangle \) according to \( \theta = \langle n \rangle \Gamma(k)/\Gamma(k - 1/\mu) \) if \( k > 1/\mu \). This can be seen from Eq. (3) recalling that \( \langle x^n \rangle = \langle n(n - 1) \ldots (n - q + 1) \rangle \) for the Poisson transform Eq. (1). Accordingly, the factorial moments of Eq. (15) may diverge. The generating function

\[ G(u) = \sum_{n=0}^{\infty} (1 - u)^n P_n = \int_{0}^{\infty} e^{-ux} f(x) \]  
Eq. (16)

can also be derived via the convolution property Eq. (11), it turns out to be

\[ G(u) = \frac{1}{\Gamma(k)} H_{0,2}^{2,0} \left[ u \theta \right]_{(0, 1), (k, 1/\mu)} \]  
Eq. (17)

Changing \( u \) to \(-it\) in Eq. (17) the characteristic function \( \varphi(t) = \int_{0}^{\infty} e^{itx} f(x) \) dx is obtained for Eq. (7). The generating function of the HNBD corresponding to Eqs. (5-6) was determined in [1].
With Eqs. (15,17) we have completed the derivation of the Poisson transformed generalized gamma distribution. According to a theorem of Bondesson, for $|\mu| > 1$ the characteristic function of Eq. (2) is an entire analytic function of finite order and therefore it must have complex zeroes which is not permitted for infinitely divisible entire characteristic functions [7]. Since the infinite divisibility of $f(x)$ is preserved by $P_n$ for Eq. (1) one can deduce that this important feature holds for the HNBD if $0 \leq |\mu| \leq 1$. In ref. [1] it has already been demonstrated that the factorial cumulant moments of the HNBD exhibit nontrivial (not alternating) sign-changing oscillations for $\mu > 1$ in accordance with the violation of infinite divisibility of $P_n$. The effect is illustrated in Fig. 1 for shape parameter $k = 1$ (Weibull case) which describes the inelastic $pp$ and deep-inelastic $e^+p$ multiplicity data very well [2,3]. Bondesson’s theorem tells us that sign-changing oscillations of the factorial cumulants may arise also for $\mu < -1$. But some preliminary results indicate that this is not the case and the factorial cumulants of the HNBD, when exist, remain positive for $\mu < -1$.

Analysing the experimental data for $P_n$ in different collision processes it was found that the HNBD shape with $\mu < 0$ is, although rare, not completely absent. The best example is the full phase-space TASSO data in $e^+e^-$ annihilations at $\sqrt{s} = 34.8$ GeV [8]. Most of the theoretical models struggle in the description of this high statistics data set. One of the few exceptions is ref. [9] reporting $\chi^2$/d.o.f. = 9.1/14. It corresponds to the NBD fitted in two matched domains of multiplicity $n$ with 3 fit parameters altogether. The HNBD analysis was carried out with fixed shape parameter $k = 1$ (details of the fitting procedure can be found in [1,3]). This particular case of the HNBD produces $\chi^2$/d.o.f. = 7.9/15 with parameters $\langle n \rangle = 13.595 \pm 0.050$ and $\mu = -10.053 \pm 0.388$. The fit is illustrated in Fig. 2. At $\sqrt{s} = 22$ GeV we obtained $\chi^2$/d.o.f. = 4.9/11, $\langle n \rangle = 11.313 \pm 0.096$ and $\mu = -12.745 \pm 1.449$. Despite of the relatively small statistics available at lowest PETRA energy $\sqrt{s} = 14$ GeV, the HNBD fit to the TASSO data fails with $\chi^2$/d.o.f. = 20.4/10. At top PETRA energy $\sqrt{s} = 43.6$ GeV the $\mu \to 0$, $k \to \infty$ log-normal limit of the HNBD produces the best quality fit ($\chi^2$/d.o.f. = 6.4/16) similarly to previous results obtained at the Z$^0$ peak [1,3]. This change in the parametrization of the best-fit HNBD indicates a slight narrowing of the KNO functions with increasing $s$ in the energy range investigated. It is worth mentioning that the value of $\mu$ was found in the fitting procedures to be extremely sensitive to tiny details of the experimental $P_n$. Since the $e^+e^-$ multiplicity data exhibit approximate KNO scaling and dominantly follow the log-normal limit of the HNBD. It is known also as Shane-Wirtanen survey after the name of the authors of the original catalogue [11]. The galaxy counts are contained in 1246 photographic plates covering $\sim 70\%$ of the sky, essentially the same region as the Zwicky sample. Each plate covers $6^\circ \times 6^\circ$ and consists of a 36 $\times$ 36 array of counts in $10^3 \times 10^3$ cells. The catalogue contains $\sim 1.25 \times 10^6$ galaxies of which cca. 34\% are double counts because each plate overlaps its neighbour by at least 1\%. Most of the double counts can be eliminated by using only the center $5^\circ \times 5^\circ$ of each plate. In ref. [12] several multiplicative correction factors were determined to the raw counts of Shane and Wirtanen. The gray-scale map of the corrected galaxy counts presented in [12] shows an impressive network of galaxies with lots of clumps and filaments. An early computer model universe designed to reproduce this fine structure can be found in [13] where the frequency distribution of galaxy counts in the $10^3 \times 10^3$ cells is also displayed.

The count distribution of galaxies analysed here was extracted from the Lick catalogue in ref. [14]. The plate selection criteria, applied in statistical analyses of the catalogue to exclude plates near the galactic plane, resulted 467 plates with 420300 cells in total. The average count in cells is $\langle n \rangle \sim 1.4$. Despite of the small mean of counts, cells with $n > 20$ galaxies are not rare in the Lick survey and the count distribution has a very long tail extending well over $n/\langle n \rangle = 10$. This is a sure sign of highly non-Poissonian behaviour of fluctuations. In [13] a comparison of the Shane-Wirtanen map and a completely random distribution of galaxies is shown — the difference between the visual appearance of the two patterns is huge. Another way of demonstrating this strong departure is fitting the Lick galaxy count distribution in the $10^3 \times 10^3$ cells by the $\mu = 1$, $k \to \infty$ Poisson limit of the HNBD. The fit is displayed in the top left plot in Fig. 3. The best-fit Negative Binomial with $k \sim 1$ improves on the Poissonian but the deviation remains substantial, see the top right plot of the same figure. We mention that the analysis of the Zwicky catalogue in [10] also signals a similar problem with the $\mu = 1$ NBD/gamma model since the good agreement mentioned above was obtained after truncating the high-$n$ tail of the data. In ref. [3] it was shown that long tailed distributions such as the $P_n$ in $pp$ collisions at $\sqrt{s} = 900$ GeV can be successfully described by the $\mu \to 0$, $k \to \infty$ limit of the HNBD. Fitting the Poisson transformed log-normal law to our galaxy data one obtains further improvement, see the bottom left plot in Fig. 3, but the agreement in the high-$n$ tail is, again, unsatisfactory. The main difficulty of reproducing the Lick counts lies in the fact that the distribution decays according to an inverse power-law. The bottom right plot of Fig. 3 illustrates a HNBD fit with negative scaling exponent $\mu$. As is seen this attempt finally succeeds in describing the high-$n$ tail too. The best-fit parameters are $\mu = -0.571 \pm 0.070$ and $k = 8.512 \pm 0.447$. 

3
Summarizing our results, we have investigated the Poisson transform of the generalized gamma distribution Eq. (2) for negative values of the parameter \(\mu\). The analytic form of \(P_n\) was derived by considering the Poisson transform of the probability density Eq. (7) with \(\mu > 0\). Making use of the convolution property Eq. (11) of the H-function we have obtained Eqs. (15) and (17) for the analytic form of \(P_n\) and \(\mathcal{G}(u)\). Our result completes the specification of the Poisson transformed generalized gamma distribution introduced in [1] and developed in [2,3]. Its application to multihadron and galaxy count data revealed \(\mu < 0\) type behaviour in both cases. Nevertheless, it seems to be unlikely that the HNBD shape with negative \(\mu\) will be frequently encountered for the full phase-space multiplicity distributions in particle and nuclear collisions. This feature is more probable for galaxy count data which often decay according to an inverse power-law reflecting the scale-invariant distribution of galaxies in the universe. Concerning multiparticle dynamics, similar behaviour of \(P_n\) should be searched in restricted domains of phase-space.

ACKNOWLEDGEMENTS

I am indebted to A.S. Szalay and I. Szapudi for the discussions concerning the Lick catalogue and for providing me their data. T. Csörgő and G. Jancsó are acknowledged for the many useful conversations on the subject. This work was supported by the Hungarian Science Foundation under Grant No. OTKA-T024094/1997.

[1] S. Hegyi, Phys. Lett. B387, 642 (1996).
[2] S. Hegyi, Phys. Lett. B388, 837 (1996).
[3] S. Hegyi, “H-function extension of the NBD in the light of experimental data”, hep-ph/9707322.
[4] P. Carruthers and C.C. Shih, Int. J. Mod. Phys. A2, 1447 (1987).
[5] N.L. Johnson and S. Kotz, Distributions in Statistics Vol. 2., Continuous Univariate Distributions (Wiley, 1970).
[6] A.M. Mathai and R.K. Saxena, The H-Function with Applications in Statistics and Other Disciplines (Wiley Eastern, 1978).
[7] L. Bondesson, Scand. Actuar. J. 48 (1978).
[8] W. Braunshweig et al., Z. Phys. C45, 193 (1989).
[9] S. Barshay and P. Heiliger, Z. Phys. C51, 399 (1991).
[10] P. Carruthers and Minh Duong-Van, Phys. Lett. B131, 116 (1983).
[11] C.D. Shane and C.A. Wirtanen, Publ. Lick Obs. XXII, Pt. 1 (1967).
[12] M. Seldner, B. Siebers, E.J. Groth and P.J.E. Peebles, Astron. J. 82, 249 (1977).
[13] R.M. Soneira and P.J.E. Peebles, Astron. J. 83, 845 (1978); see also E.J. Groth, P.J.E. Peebles, M. Seldner and R.M. Soneira, Sci. Amer. 237, 5 (1977).
[14] I. Szapudi, A.S. Szalay and P. Boschán, Astrophys. J. 390, 350 (1992).
[15] I.M. Dremin, Mod. Phys. Lett. A8, 2747 (1993); Physics Uspekhi 37, 715 (1994). I.M. Dremin and R.C. Hwa, Phys. Rev. D49, 5805 (1994).
FIG. 1. Sign-changing oscillations of the factorial cumulant-to-moment ratios $H_q$ [15] for the $k = 1$ special case of HNBD. Considering slices with fixed rank $q$, the neighbouring bumps along the $\mu$-axis correspond to $H_q$ having opposite sign. For $\mu \leq 1$ the moment ratios are always positive. The $\mu$-scale is logarithmic and for clarity $\log |H_q|$ is displayed only for odd ranks $q$.

FIG. 2. The best-fit HNBD to the full phase-space TASSO data at $\sqrt{s} = 34.8$ GeV [8]. The fit parameters are: $k = 1$ fixed, $\langle n \rangle = 13.595 \pm 0.050$ and $\mu = -10.053 \pm 0.388$. 
FIG. 3. Various special and limiting cases of HNBD fitted to the Lick galaxy count distribution. Top left: best-fit Poisson ($\mu = 1, k \to \infty$); top right: best-fit Negative Binomial ($\mu = 1, k \sim 2$); bottom left: best-fit Poisson transformed log-normal ($\mu \to 0, k \to \infty$); bottom right: best-fit HNBD with parameters $\mu = -0.571 \pm 0.070$ and $k = 8.512 \pm 0.447$. The average count in cells is $\langle n \rangle \sim 1.4$. The errorbars represent Poisson errors.