Environmental Effects on the Geometric Phase

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Abstract

The behavior of the geometric phase gained by a single spin-1/2 nucleus immersed into a thermal or a squeezed environment is investigated. Both the time dependence of the phase and its value at infinity are examined against several physical parameters. It is observed that for some intermediate ranges of the temperature and the coupling strength, the presence of squeezing enhances the geometric phase.

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I. INTRODUCTION

The geometry of the Hilbert space of a quantum system is registered to the memory of the system as a geometric phase factor\[1, 2, 3, 4, 5, 6, 7\]. It is inherently fault tolerant because of its geometric nature\[8\]. For this reason, it is used in the implementation of controlled gates for quantum computers\[9, 10\]. These gates are also of crucial importance for a universal set of quantum logic gates\[9, 10, 11\]. For this reason, a full understanding of the nature of quantum geometric phase in different environments is vital for quantum computations.

Various aspects of the effects of the environment on the GP of open quantum systems have been studied. Rezakhani and Zanardi analyzed the temperature effects on mixed-state GP for a single and two coupled spin-1/2 particles\[12\]. Wang et al. analyzed the effects of a squeezed vacuum reservoir on GP of a two-level atom in an electromagnetic field by a formulation entirely in terms of geometric structures\[13\]. Carollo et al. showed that GP can be induced by cyclic evolution in an adiabatically manipulated squeezed vacuum reservoir\[14\]. Banerjee and Srikanth studied the effects of a squeezed-thermal environment on the GP of a two-level system\[15\] for dissipative and non-dissipative cases, and analyzed the initial-state and temperature dependence of GP for the system. The purpose of this article is to examine the effect of various physical parameters on the time-dependence of GP.

The system analyzed in this article is a two-level nucleus inside a magnetic field. The time evolution is non-unitary due to the interaction with the environment. The environment is first taken as a thermal bath. Then, this bath is considered to be driven by an electromagnetic field in a squeezed state in order to see whether the GP can be enhanced. The dependence of the GP on temperature, external magnetic field, coupling strength, squeezing and initial state of the system are then analyzed. The GP is computed by using the kinematic definition given by Tong et al.\[16\]. In their approach, when the system’s density matrix in the Schrödinger picture is \( \dot{\rho}(t) \), the GP gained in the time interval \([0, \tau]\) is given by the expression

\[
\Phi(\tau) = \text{arg} \left( \sum_k \sqrt{\lambda_k(0)\lambda_k(\tau)} \langle \varphi_k(0) | \varphi_k(\tau) \rangle e^{-\int_0^\tau \langle \varphi_k(t) | \dot{\varphi}_k(t) \rangle dt} \right),
\]

where \( \lambda_k(t) \) and \( | \varphi_k(t) \rangle \) are the eigenvalues and the eigenvectors of \( \dot{\rho}(t) \), respectively.

The content of the article is as follows. In section II, the system and its interaction with
the environment is described. This section also includes the derivation of the non-unitary dynamics of the reduced density matrix by using Markov approximation. After that, the density matrix of the system is computed analytically. In section III, the dependence of the GP on various physical parameters is analyzed. Finally, section IV contains brief conclusions.

II. THE NUCLEUS INSIDE A BATH

Our specific system is a single spin-1/2 nucleus in an external static magnetic field $\vec{B}$ which is taken to be in the $z$ direction. The Hamiltonian of the nucleus is

$$\hat{H}_N = -\hbar \omega_N \hat{I}^z ,$$

(2)

where $\hbar$ is the Planck constant, $\omega_N = \gamma_N |\vec{B}|$, $\gamma_N$ is the gyromagnetic ratio of the nucleus, $\hat{I}^z = \hat{\sigma}^z/2$ and $\hat{\sigma}^z$ is the $z-$component of the Pauli spin operator.

The environment (reservoir) is assumed to be a bath of harmonic oscillators (such as electromagnetic radiation) where the frequency spectrum forms a continuum. The annihilation (creation) operator for the mode at frequency $\omega$ is denoted by $\hat{b}(\omega)$ ($\hat{b}^\dagger(\omega)$) and the Hamiltonian of the environment is

$$\hat{H}_R = \hbar \int_0^{\infty} \omega \hat{b}^\dagger(\omega) \hat{b}(\omega) d\omega .$$

(3)

The interaction of nucleus with the oscillators is described by the interaction Hamiltonian,

$$\hat{H}_{NR} = \hbar \int_0^{\infty} g_N(\omega) \hat{I}^+ \hat{b}^\dagger(\omega) d\omega + h.c .$$

(4)

where $g_N(\omega)$ is the coupling coefficient between the nucleus and the $\omega$-mode of the reservoir, $\hat{I}^\pm = \hat{I}^x \pm i\hat{I}^y$ are the spin ladder operators, and $h.c.$ indicates the hermitian conjugate term.

The total Hamiltonian $\hat{H} = \hat{H}_N + \hat{H}_R + \hat{H}_{NR}$, is transformed into the interaction picture as

$$\hat{V}(t) = \hbar \int_0^{\infty} g_N(\omega) e^{i(\omega - \omega_N)t} \hat{I}^+ \hat{b}^\dagger(\omega) d\omega + h.c .$$

(5)

Let $\hat{\rho}_{NR}(t)$ denote the state of the combined system of the nucleus and the environment at time $t$, and let $\hat{\rho}_{NR}^I(t)$, $\hat{\rho}_R^I(t)$ and $\hat{\rho}_N^I(t)$ denote the interaction picture states of the combined system, the environment and the nucleus, respectively. We study this system in the regime in which the Markov approximation can be applied. In this approximation, it is assumed that the environment remains in its initial state during the evolution, $\hat{\rho}_R^I(t) = \hat{\rho}_R(0)$, and
the state of the combined system is taken to be in product form, \( \dot{\rho}_{NR}(t) = \dot{\rho}_N(t) \otimes \dot{\rho}_R(0) \). As a result, the equation of motion of the state of the nucleus can be obtained as

\[
\dot{\rho}_N(t) = -\frac{i}{\hbar} \text{tr}_R[\dot{V}(t), \rho_N(0) \otimes \dot{\rho}_R(0)] - \frac{1}{\hbar^2} \text{tr}_R \int_0^t [\dot{V}(t), [\dot{V}(t'), \rho_N(t') \otimes \dot{\rho}_R(0)]] \, dt',
\]

where the dot denotes the time derivative and \( \text{tr}_R \) represents trace over the degrees of freedom of the environment\[^{17, 18}\].

For the thermal environment, the state is given by \( \dot{\rho}_R(0) = \dot{\rho}_{\text{th}} = \exp(-\dot{H}_R/k_B T)/\text{tr} \exp(-\dot{H}_R/k_B T) \). If the bath is driven by a squeezed field, the state of the squeezed-thermal environment is given by

\[
\dot{\rho}_R(0) = \hat{S}\dot{\rho}_{\text{th}}\hat{S}^\dagger,
\]

where

\[
\hat{S} = \hat{S}[\xi(\omega)] = \exp \left( -\frac{1}{2} \int_\omega^{2\Omega} \, d\omega [\xi(\omega)\hat{b}^\dagger(\omega)\hat{b}(2\Omega - \omega) - \xi^*(\omega)\hat{b}(2\Omega - \omega)\hat{b}(\omega)] \right),
\]

where \( 2\Omega \) is the squeezing-carrier frequency, \( \xi(\omega) = r(\omega)e^{i\phi(\omega)} \), \( r(\omega) \) and \( \phi(\omega) \) are real numbers characterizing the squeezing which satisfy \( \xi(\omega) = \xi(2\Omega - \omega) \).

For both types of environments, we have \( \langle b(\omega) \rangle = 0 \) which makes the first term on the right-hand side of Eq. (6) vanish. The second term can be expressed in the form

\[
\dot{\rho}_N(t) = -\frac{i}{\hbar} [\Delta \hat{H}_N, \rho_N(t)] + C_- \left( 2\hat{I}^- \rho_N(t) \hat{I}^+ - \rho_N(t) \hat{I}^+ \hat{I}^- - \hat{I}^+ \hat{I}^- \rho_N(t) \right) + C_+ \left( 2\hat{I}^+ \rho_N(t) \hat{I}^- - \rho_N(t) \hat{I}^- \hat{I}^+ - \hat{I}^- \hat{I}^+ \rho_N(t) \right) - D \hat{I}^+ \rho_N(t) \hat{I}^- - D^* \hat{I}^- \rho_N(t) \hat{I}^+,
\]

where \( \Delta \hat{H}_N = -\hbar\Delta \omega \hat{\sigma}_z / 2 \) represents a re-normalization of the frequency \( \omega_N \) of the nucleus, and \( C_\pm \) and \( D \) are numbers that capture the collective effect of all of the oscillators in the reservoir satisfying

\[
C_+ = C_- + \pi |g_N(\omega_N)|^2.
\]

Each of these constants can be expressed as an integral over the frequency \( \omega \), which can be decomposed into a principal-value integral, which depends on the precise frequency dependence of \( g_N(\omega) \), and a Dirac-delta integral term, which depends only on \( g_N(\omega_N) \). The coupling function \( g_N(\omega) \) is freely adjustable; by changing that function suitably, the values of all principal-value integrals and therefore all final constants \( C_\pm, D \) and \( \Delta \omega_N \) can be adjusted to any desirable value. Here, for simplicity, all of these principal-value integrals are
taken to be zero and the resultant constants are used. With this choice, the re-normalized Hamiltonian $\Delta\hat{H}_N$ becomes zero.

For the purely thermal environment, the constants in Eq. (9) can be computed as $D = 0$ and $C_- = \pi|g_N(\omega_N)|^2n(\omega_N)$, where

$$n(\omega) = \frac{1}{\exp(h\omega/k_BT) - 1}$$

(11)

denotes the average occupation number of the mode at frequency $\omega$. In this case, the matrix elements of the interaction picture density matrix can be integrated to

$$\rho_{N11}^I(t) = \rho_{N11}^I(\infty) + (\rho_{N11}^I(0) - \rho_{N11}^I(\infty))e^{-2(C_+ + C_-)t},$$

(12)

$$\rho_{N12}^I(t) = \rho_{N11}^I(0)e^{-(C_+ + C_-)t},$$

(13)

where

$$\rho_{N11}^I(\infty) = \rho_{N11}(\infty) = \frac{C_+}{C_+ + C_-}.$$  

(14)

For the squeezed-thermal environment given in Eq. (7), the constants can be found as

$$C_- = \pi|g_N(\omega_N)|^2\left\{ n(\omega_N) + [n(\omega_N) + n(2\Omega - \omega_N) + 1] \sinh^2 r(\omega_N) \right\},$$

(15)

$$D = \pi g_N(\omega_N)g_N(2\Omega - \omega_N) [n(\omega_N) + n(2\Omega - \omega_N) + 1] \sinh(2r(\omega_N))e^{2i(\Omega - \omega_N)t - i\phi(\omega_N)}.$$  

(16)

For this case too, the time-dependent density matrix can be found analytically. The diagonal entry $\rho_{N11}^I(t)$ is still given by Eq. (12) and the off-diagonal entry can be expressed in general as

$$\rho_{N12}^I(t) = (A_1e^{st} + A_2e^{-st}) e^{-(C_+ + C_-)t + i(\Omega - \omega_N)t},$$

(17)

where $s$ is the purely real or purely imaginary constant

$$s = \sqrt{|D|^2 - (\Omega - \omega_N)^2},$$

(18)

and $A_1$ and $A_2$ are constants that should be determined from the initial state $\hat{\rho}_N(0)$. The squeezing changes the time-dependence of the density matrix as follows. First, it changes the long-time limit of the density matrix $\hat{\rho}_N(\infty)$. It also makes the diagonal relaxation time, $(2(C_+ + C_-))^{-1}$, shorter. Apart from those, it changes the time-dependence of the off-diagonal entry; in particular it produces new oscillatory behavior for sufficiently large squeezing.
The geometric phase \( \Phi(\tau) \) given in Eq. (1) can be computed analytically in the thermal case. The argument of the exponential in this expression is given by

\[
-\int_0^\tau \langle \varphi_\pm(t) | \dot{\varphi}_\pm(t) \rangle \, dt = \mp i \frac{\omega_N}{2} \left( \tau + \frac{1}{C_+ + C_-} (F(\tau) - F(0)) \right)
\]

where

\[
F(\tau) = \ln(A_\tau - \delta_\tau + 2\delta_\infty) - \ln \left( 2(\delta_0 - \delta_\infty) + \frac{|\rho_{12}(0)|^2}{A_\tau + \delta_\tau} \right),
\]

\[
A_\tau = \sqrt{\frac{1}{4} - \det \hat{\rho}_N(\tau)},
\]

\[
\delta_\tau = \frac{1}{2} (\rho_{11}(\tau) - \rho_{22}(\tau))
\]

and \( |\varphi_\pm(\tau)\rangle \) are eigenvectors with corresponding eigenvalues \( \lambda_\pm(\tau) = 1/2 \pm A_\tau \).

It should be noted that when \( \tau \) goes to infinity, the density-matrix \( \hat{\rho}_N(\tau) \) goes to a diagonal time-independent state. Hence \( F(\tau) \) and consequently the phase \( \Phi(\tau) \) tend to constant limits. For the squeezed-thermal environment, the argument of the exponential needs to be computed numerically but it can be shown that the same behavior holds at infinity, i.e., the GP settles down to a well-defined finite limit.

III. THE DEPENDENCE OF THE GEOMETRIC PHASE ON PHYSICAL PARAMETERS

For investigating the behavior of the GP under different physical conditions, the principal-value integrals are taken to be zero as explained in the previous section, and both of the coupling coefficients \( g_N(\omega_N) \) and \( g_N(2\Omega - \omega_N) \) are taken to be equal to a real positive value \( g \). The phase parameter \( \phi(\omega_N) \) of the squeezed state is taken to be 0. The relevant parameter \( r(\omega_N) \) that gives the amount of squeezing is simply denoted by \( r \) below.

In the problem, there are three parameters having the dimension of energy: the excitation energy \( \hbar \omega_N \) (which is proportional to the external magnetic field), the thermal energy \( k_B T \) and an energy related to the coupling strength \( \hbar g^2 \). The behavior of the GP is invariant if all three of these parameters are scaled by the same amount. For this reason, only the ratios of these parameters need to be specified as done below.
A. Time Dependence of the Geometric Phase

The dependence of the geometric phase $\Phi(\tau)$ on time $\tau$ has periodic structures with period equal to $2\pi/\omega_N$. In each of the oscillations over one period, $\Phi(\tau)$ also changes by an amount depending crucially on the values of all physical parameters. If there is squeezing, an additional periodic structure coming from the time-dependence of the off-diagonal entry given in Eq. (17) may appear.

The effects of changing the temperature, the coupling strength and the magnetic field on the time-dependence of the GP are shown in the Figures 1, 2 and 3, respectively. As it can be seen, increasing the temperature or the coupling strength, or decreasing the magnetic field have regular effects when there is no squeezing (the figures on the left of Figs. 1, 2 and 3). With these changes, the oscillations of the GP are destroyed and the limiting value of GP is reached at earlier times.

But when there is squeezing, that regular dependence is lost for the case of increasing the temperature or the coupling strength (Fig.s 1 and 2). For sufficiently large values of the temperature or the coupling constant, the limiting value of the GP is higher in comparison to those with lower temperatures or coupling strengths. Moreover the GP reaches the limiting values at a shorter time for higher temperatures and at a longer time for higher coupling strengths. For low temperatures and coupling strengths, presence of squeezing in general decreases the GP.

![Graphs showing the GP vs time for different temperatures for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The coupling strength is $g^2/\omega_N = 0.01$ and the initial state is a pure spin up state along $x$. The numbers on the graph indicate the value of $k_B T/h\omega_N$.](image)

FIG. 1: The GP vs time for different temperatures for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The coupling strength is $g^2/\omega_N = 0.01$ and the initial state is a pure spin up state along $x$. The numbers on the graph indicate the value of $k_B T/h\omega_N$. 
FIG. 2: The GP vs time for different coupling strengths for a thermal environment (left) and a squeezed-thermal environment with $r = 2$ (right). The magnetic field is $\hbar \omega_N/k_B T = 1$ and the initial state is a pure spin up state along $x$. The values of the ratio $g^2/\omega_N$ are shown on the graphs. As can be seen, squeezing enhances the GP for a particular case with large coupling. Note also small oscillations which are produced by the effect of squeezed field on the spin state.

For high magnetic fields, the squeezing has almost no effect (Fig. 3). At lower fields, the squeezing tends to decrease the GP, destroys the oscillations and make the limiting values reached at earlier times. And for small enough field strengths it is possible to get higher values for the GP with increasing the squeezing.

FIG. 3: The GP vs time for different magnetic fields for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The coupling coefficient is taken as $\hbar g^2/k_B T = 0.01$ and the initial state is a pure spin up state along $x$. The values of the ratio $\hbar \omega_N/k_B T$ are shown on the graph. We have taken $\Omega = 3\omega_N$ for all data points.
Figure 4 shows the dependence of the GP on the initial state. The initial state is taken as a pure state along a direction on the \(xz\)-plane having the spherical angle \(\theta\). When \(\theta\) is increased above \(\pi/2\), the GP changes significantly. Especially, the GP tends to decrease at the initial moments for \(\theta > \pi/2\). If the initial state is mixed, then the GP is suppressed in magnitude but its overall behavior does not change.

FIG. 4: The GP vs time for different initial states for a thermal environment (left) and a squeezed-thermal environment with \(r = 1\) (right). The magnetic field is \(\hbar \omega_N/k_B T = 1\) and the coupling strength is \(g^2/\omega_N = 0.01\). The initial state is chosen as a pure state and the value of the spherical angle \(\theta\) of the initial orientation is shown on the graph.

B. The Dependence of the Phase at Infinity on Physical Parameters

The behavior of the GP at infinity is also a quantity of interest. Its magnitude gives an indication of the overall magnitudes of the GP that can be obtained at any time. This value can also be used for showing the effect of the squeezing on the GP. In figure 5, the long time limit of the GP, i.e., \(\Phi(\infty)\), is shown as a function of temperature. When there is no squeezing, increasing the temperature always suppresses the GP. With the presence of squeezing, three different temperature regimes appear. For very low and very high temperatures, the GP is still suppressed with increasing \(T\). However, there is now an intermediate range of temperatures over which the opposite trend appears. Increasing the squeezing parameter \(r\) moves this intermediate range to lower temperatures. Fig. 6 shows the dependence of the limiting value of the GP on the coupling strength. The behavior is similar to the case of
changing the temperature parameter.

FIG. 5: The GP at infinity as a function of temperature for different values of the squeezing parameter \( r \). The coupling strength satisfies \( g^2/\omega_N = 0.01 \).

FIG. 6: The GP at infinity as a function of the coupling strength for different values of the squeezing parameter \( r \) for the case \( \hbar \omega_N/k_B T = 1 \).
Figure 7 shows the dependence of the limiting value of the GP on the magnetic field. It can be seen that there are two regions for this case. For high enough magnetic fields this dependence is almost linear. Squeezing essentially decreases the slope. But at lower magnetic fields, GP has a different behavior under squeezing. In this regime, squeezing enhances the GP which is now a nonlinear function of the magnetic field. The limiting GP also reaches to a maximum value at an optimum value of the magnetic field. Increasing the squeezing expands that region of nonlinearity to higher magnetic fields.

![Graph showing the dependence of \( \Phi(\infty) \) on the magnetic field for different values of the squeezing parameter \( r \). The coupling strength satisfies \( \hbar g^2/k_B T = 0.1 \).](image)

FIG. 7: The GP at infinity as a function of the magnetic field for different values of the squeezing parameter \( r \). The coupling strength satisfies \( \hbar g^2/k_B T = 0.1 \).

The dependence of \( \Phi(\infty) \) on the initial state is shown in Fig. III B. As can be seen, for low squeezing \( \Phi(\infty) \) reaches its largest values for \( \theta \) near \( \pi \), i.e., a spin orientation which is almost spin down. Small squeezing suppresses the GP but does not change the behavior. For large squeezing however, \( \Phi(\infty) \) can be enhanced significantly and its largest values are reached when \( \theta \) is around \( \pi/2 \).
IV. CONCLUSION

The availability of easy manipulation with the current technology NMR techniques makes nuclear systems good candidates as carrier systems of the entities that are necessary for quantum information processing[9, 19, 20, 21, 22, 23]. With this motivation, the effects of the coupling between a nucleus and a dissipative environment on the GP that the state of the nucleus gains during its time evolution are studied. Although it is affected by the physical parameters, it is always possible to get a finite GP. Its dependence on these parameters are almost as expected; it decreases with increasing temperature or coupling strength, because both increase the decoherence caused by the environment, and it increases with increasing magnetic field, since increasing $B$ has the same effect as decreasing both the temperature and the coupling constant.

If the GP needs to be increased further, the environment could be driven by an electromagnetic field in a squeezed state. However, in this case, one should be very careful because the squeezing may have a destructive or an enhancing effect on the GP depending on the parameters. In order to enhance the GP, the temperature should be held at some appropriate
intermediate value; it should not be too small or too large. The coupling strength should be
taken at a relatively high value so that the squeezed field can interact with the spin more;
but it should not be too large lest the spin relaxes to its equilibrium state quickly. And
larger magnetic fields do not always enhance the GP; it is true only up to an upper bound
for the field strength. A geometric phase that can be described by the model used in this
article can promote the robustness in quantum computations.

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