The representations of Temperley-Lieb algebras and entanglement in a Yang-Baxter system

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A method of constructing Temperley-Lieb algebras (TLA) representations has been introduced in [Xue et al. arXiv:0903.3711]. Using this method, we can obtain another series of \( n^2 \times n^2 \) matrices \( U \) which satisfy the TLA with the single loop \( d = \sqrt{n} \). Specifically, we present a \( 9 \times 9 \) matrix \( U \) with \( d = \sqrt{3} \). Via Yang-Baxterization approach, we obtain a unitary \( \hat{R}(\theta, \varphi_1, \varphi_2) \)-matrix, a solution of the Yang-Baxter Equation. This \( 9 \times 9 \) Yang-Baxter matrix is universal for quantum computing.

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I. INTRODUCTION

Quantum entanglement, which has been singled out by Schrödinger as “the characteristic trait of quantum mechanics” many decades ago, is the most surprising nonclassical property of composite quantum systems. In recent years, there has been an ongoing effort to characterize qualitatively and quantitatively the entanglement properties, because it implies a nonclassical nature through which we can investigate the conceptual foundations and interpretation of quantum mechanics, and, more importantly, it provides a fundamental resource in realizing quantum information and quantum computers [2], such as quantum teleportation [3], superdense coding [4], quantum key distribution [5], and telecommuting [6]. Besides, in highly correlated states in condensed-matter systems such as fractional quantum Hall liquids [7] and superconductors [8, 9], the entanglement serves as a unique measure of quantum correlations between degrees of freedom.

The Yang-Baxter equation (YBE) was originated in solving the one-dimensional \( \delta \)-function interaction models by Yang [10] and the statistical models on lattices by Baxter [11], and introduced to solve many quantum integrable models by Faddeev and Leningrad Scholars [12]. Very recently, the YBE and braiding operators have been introduced to the field of quantum information and quantum computation, and also provide a novel way to study the quantum entanglement [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. As is known, the Temperley-Lieb algebras (TLA) are intimately connected with braid group. In fact, the TLA is a quotient of the group algebra of the braid group, and any representation of TLA constructs automatically a representation of a corresponding braid group. A unitary solution of YBE can also be constructed from a representation of TLA via the Yang-Baxterization approach, so TLA has been widely used in the construction of YBE solutions [20, 21, 22]. Recently, Ref. [23] show that TLA is found to present a suitable mathematical framework for describing quantum teleportation, entangle swapping, universal quantum computation and quantum computation flow. In a very recent work, a reducible representation of the TLA is constructed on the tensor product of \( n \)-dimensional spaces [30]. Then we expanded Kulish’s method in Ref. [1], and we obtained a series of solutions of YBE. With this method, another series of solutions of TLA can be determined in this paper.

The paper is organized as follows: In Sec[I] we recall the method of constructing some \( n^2 \times n^2 \) matrix solutions of TLA with the single loop \( d = \sqrt{n} \) which is shown in Ref. [1]. In the following, using the same method, another series of solutions are presented via changing some original conditions. In Sec[III] we present a \( 9 \times 9 \) matrix \( U \) which satisfies the TLA with the single loop \( d = \sqrt{3} \). Via Yang-Baxterization approach, we can obtain a \( 9 \times 9 \) unitary \( \hat{R}(\theta, \varphi_1, \varphi_2) \)-matrix, a solution of the YBE. Then we investigate the entanglement. We show that the arbitrary degree of entanglement for two qutrits entangled states can be generated via the unitary \( \hat{R} \)-matrix acting on the standard basis. And it is also shown that all pure entangled states of two 3-dimensional quantum systems (i.e., two qutrits) can be generated from an initial separable state via the universal \( \hat{R} \)-matrix if one is assisted by local unitary transformations. In fact, we can prove that this unitary Yang-Baxter matrix \( \hat{R}(\theta, \varphi_1, \varphi_2) \) is local equivalent to the solution in Ref. [20]. We end with a summary.

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II. THE REPRESENTATIONS OF TLA

In order to keep the paper self-contained, we first briefly review the theory of TLA. It is a unital algebra generated by $U_i (i = 1, 2, \ldots, N - 1)$ which subject to the following relations,

$$U_i^2 = dU_i, \quad U_iU_{i+1}U_i = U_i, \quad U_iU_j = U_jU_i, \quad |i - j| > 1$$

(1)

where $0 \neq d \in \mathbb{C}$ is the single loop in the knot theory which don’t depend on the sites of the lattices. The notation $U_i \equiv U_{i+1}$ is used, $U_{i+1}$ implies $1_1 \otimes \cdots \otimes 1_{i-1} \otimes U_{i+1} \otimes 1_{i+2} \otimes \cdots \otimes 1_n$, and $1_{i}$ represents the unit matrix of the $j$-th particle. The TLA is easily understood in terms of diagrammatics in Ref. [31].

In Ref. [1], a method of constructing some $n^2 \times n^2$ matrices solutions of TLA with $n^3$ matrix elements has been shown. Let us review it briefly. The representation is defined by two invertible $n \times n$ matrices $A \in GL(n, \mathbb{C})$ and $B \in GL(n, \mathbb{C})$, which can also be seen as an $n^2$ dimensional vector $A_{ab} \in (\mathbb{C}^n \otimes \mathbb{C}^n)$ and an $n^2$ dimensional vector $B_{ab} \in (\mathbb{C}^n \otimes \mathbb{C}^n)$, respectively. The generators $U_i$ can be expressed as

$$(U_i)^{ab}_{cd} = A^{a}_{b}B^{b}_{c} \in \text{Mat}(\mathbb{C}^{n} \otimes \mathbb{C}^{n})$$

(2)

where one explicitly writes the indices corresponding to the factors in the tensor product space $\mathcal{H} = \otimes_1^N \mathbb{C}^n$ for any $n = 2, 3, \ldots$. The notation $U^{ab}_{cd} \equiv U_{ab,cd}$, $A^{a}_{b} \equiv A_{ab}$ and $B^{b}_{c} \equiv B_{cd}$ are used. In order to satisfy the second relation of [1], if and only if

$$(BA)^T(AB) = I_{n \times n},$$

(3)

and the first relation of [1] determines the single loop $d$:

$$U_i^2 = U_i tr(A^T B), \quad tr(A^T B) = d,$$

(4)

where $tr(A^T B)$ denotes the trace of matrix $A^T B$, and $A^T$ denotes the transpose of matrix $A$. By means this method Eq. (2), one can construct lots of $n^2 \times n^2$ matrix $U$ with $d = tr(A^T B)$ as long as $n \times n$ matrices $A$ and $B$ satisfy Eq. (3). Especially, when $B = A^{-1}$, the generators $U_i$ can be expressed as $(U_i)^{ab}_{cd} = A^{a}_{b}(A^{-1})^{b}_{c}$, which has been presented in Ref. [30].

For the sake of constructing some useful TLA matrices, Ref. [1] select the nonzero elements’ locations of $n \times n$ matrix $A$ are symmetric and are the same as the $n \times n$ matrix $B$’s, and every row and every column of them have only one nonzero element. In addition, the nonzero elements of matrices $A$ and $B$ satisfy the relation

$$B^{b}_{c} = (A^{a}_{b})^{-1} \quad \text{namely} \quad A^T B = I_{n \times n}.$$

(5)

Under this case, it is easy to see the constraint (3) is automatically satisfied. And one can easily verify that the single loop $d = Tr(I_{n \times n}) = n$. Then we select $n$ matrices $U^{(i)} (i = 1, 2, \ldots, n)$ which can be expressed as $(U^{(i)})^{ab}_{cd} = (A^{(i)})^{a}_{b}(B^{(i)})^{b}_{c}$, where matrices $A^{(i)}$ and $B^{(i)}$ all satisfy above conditions. And all their nonzero elements for $n$ matrices $A^{(i)}$ occupy different locations. Namely, the non-vanishing matrix elements of $A^{(i)}$ are $(A^{(i)})^{0}_{0-1}$, $(A^{(i)})^{1}_{1-2}$, $(A^{(i)})^{2}_{2-3}$, \ldots, $(A^{(i)})^{i-1}_{i-1}$, $(A^{(i)})^{i}_{n-1}$, \ldots, $(A^{(i)})^{n-1}_{n-1}$. Taking the sums of these $n$ matrices $U^{(i)}$ which are $n$ different solutions of TLA, we can construct a $n^2 \times n^2$ matrix solution of TLA with $n^3$ matrix elements. The combined matrix $U$ reads,

$$U = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U^{(i)}.$$

(6)

The limited condition which makes $U$ be a solution of TLA in Ref. [1] reads,

$$\sum_{j=1}^{n} (B^{(i)}A^{(j)})^T(A^{(k)}B^{(j)}) = 0_{n \times n}$$

$$\sum_{j=1}^{n} (A^{(j)}B^{(i)})(B^{(j)}A^{(k)})^T = 0_{n \times n},$$

(7)

where $i \neq k$ and $i, k = 1, 2, \ldots n$. The nonzero matrix elements of matrices $A^{(i)}$ and $B^{(i)}$ are determined by the limited condition Eq. (7) together with their special matrix structures. And the first one relation of [1] determines the single loop $d = \sqrt{n}$. 
Using the same method, we can obtain another series of TLA matrices. In this paper we select n matrices $U^{(i)} (i = 1, 2, \ldots, n)$, which can be expressed as $(U^{(i)})_{\alpha \beta}^{(i)} = (A^{(i)})_{\alpha \beta}^{(i)} (B^{(i)})_{\alpha \beta}^{(i)}$, where matrix $A^{(i)}$ and matrix $B^{(i)}$ all have special matrix structures. The nonzero elements’ locations of $n \times n$ matrix $A^{(i)}$ are antisymmetric and are the same as the $n \times n$ matrix $B^{(i)}$'s, and every row and every column of them have only one nonzero element. In addition, the nonzero elements of matrices $A^{(i)}$ and $B^{(i)}$ satisfy the relation $B_{\alpha \beta}^{(i)} = (A_{\alpha \beta}^{(i)})^{-1}$, which makes the constraint (8) be automatically satisfied. And all their nonzero elements for $n$ matrices $A^{(i)}$ occupy different locations. Namely, the nonzero matrix elements of $A^{(i)}$ are $(A^{(i)})_{0 i1}^{(i)}, (A^{(i)})_{1 i2}^{(i)}, \ldots, (A^{(i)})_{n i1}^{(i)}, (A^{(i)})_{0 i(n-1)}^{(i)}, (A^{(i)})_{1 i(n-2)}^{(i)}, \ldots, (A^{(i)})_{n i2}^{(i)}$. For example, if $n=4$ and $i=3$, the nonzero matrix elements of $A^{(3)}$ are $(A^{(3)})_{0 i1}^{(3)}, (A^{(3)})_{1 i2}^{(3)}, (A^{(3)})_{2 i3}^{(3)}, (A^{(3)})_{3 i4}^{(3)}$. Under these conditions, these $n$ matrices $U^{(i)}$ are $n$ different solutions of TLA with the single loop $d^{(i)} = n$, and all their nonzero matrix elements occupy different locations. By means of the same method as Eq. (6), another combined matrix $U$ reads,

$$U = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U^{(i)}.$$  (8)

We substitute Eq(8) into Eqs(1), the second relation of Eqs(1) determines $A^{(i)}$ and $B^{(i)}$ subject to the same constraints as Eqs. (7), so the nonzero elements of matrices $A^{(i)}$ and $B^{(i)}$ are determined by the limited condition Eq. (7) together with above special matrix structures. And the first one relation of (1) determines the single loop $d = \sqrt{n}$. So with the same method as Eq. (6), via selecting the nonzero elements’ locations of $n \times n$ matrices $A^{(i)}$ and $B^{(i)}$ are antisymmetric in this paper, we can construct another series of $n^2 \times n^2$ matrices $U$ (with $n^3$ matrix elements) which satisfy the TLA with the single loop $d = \sqrt{n}$.

### III. A 9 × 9 Matrix, Unitary $\tilde{R}$ Matrix and Entanglement

In this section, we first construct a $3^2 \times 3^2$ matrix $U$ which satisfies the TLA for $n = 3$. Via the above summation method $U = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} U^{(i)} = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} (A^{(i)})_{\alpha \beta}^{(i)} (B^{(i)})_{\alpha \beta}^{(i)}$, which satisfies the constraints (7) with above special matrix structures, one can have the solution with standard basis (i.e. $|0\rangle$, $|1\rangle$, $|2\rangle$) as follows,

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{q_1 q_2} & 0 \\ 0 & 0 & \frac{1}{q_1 q_2} \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} \\ 0 & \frac{1}{q_1 q_2} & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} \\ q_1 q_2 & 0 & 0 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ q_1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{q_1 q_2} & 0 \end{pmatrix}, \quad B^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{q_1 q_2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} \end{pmatrix}.  \quad (9)$$

In this work, we choose basis $\{ |00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle \}$ as the standard basis. As a result, the 9 × 9 Hermitian matrix $U$ with $d = \sqrt{3}$ is realized as,

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} & 0 & 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} \\ 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{q_1 q_2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} \\ 0 & 0 & q_1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{q_1 q_2} & 0 \\ 0 & \frac{\sqrt{3}}{q_1} & 0 & 0 & 0 & 1 & \frac{q_2}{q_1 q_2} & 0 & 0 \\ 0 & 0 & q_2 & \frac{\sqrt{3}}{q_2} & 0 & 0 & \frac{1}{q_1 q_2} & 0 & 0 \\ 0 & 0 & \frac{q_1}{q_2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{q_1} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\sqrt{3}}{q_2} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (10)$$

where $\omega = e^{i \frac{2\pi}{3}}$ (here and after $\epsilon = \pm$), $q_1 = e^{i \varphi_1}$, and $q_2 = e^{i \varphi_2}$, with the parameters $\varphi_1$ and $\varphi_2$ both are real. The matrix $U$ of (10) can also be rewritten as a form of projectors

$$U = \sqrt{3} \sum_{i=1}^{3} |\Psi_i\rangle\langle\Psi_i|, \quad (11)$$
where

\[ |\Psi_1\rangle = \frac{1}{\sqrt{3}}((00) + \frac{q_1^2}{q_2^2} + (\frac{\omega}{q_2}|11) + \frac{\omega}{q_2}|22)), \]

\[ |\Psi_2\rangle = \frac{1}{\sqrt{3}}((01) + \frac{\omega}{q_2}|12) + \frac{1}{q_2^2}|20)), \]

\[ |\Psi_3\rangle = \frac{1}{\sqrt{3}}((02) + q_1|10) + \frac{q_1}{q_2}|21)). \]

(12)

It is interesting that all \( |\Psi_i\rangle (i=1,2,3) \) are of the \( SU(3) \) entangled states with the maximal degree of entanglement \( \text{32}. \)

Next we derive a unitary matrix \( \hat{R} \) from \( U \) by the Yang-Baxterization approach. Such a matrix \( \hat{R} \) satisfies the YBE

\[ \hat{R}_{i}(u)\hat{R}_{i+1}(\frac{u+v}{1+\beta^2 uv})\hat{R}_{i}(v) = \hat{R}_{i+1}(v)\hat{R}_{i}(\frac{u+v}{1+\beta^2 uv})\hat{R}_{i+1}(u), \]

(13)

where \( u \) and \( v \) are spectral parameters, and \( \beta^{-1} = i c \) (\( c \) is light velocity). The notation \( \hat{R}_{i}(u) \equiv \hat{R}_{i,i+1}(u) \) is used, \( \hat{R}_{i,i+1}(u) \) implies \( 1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes \hat{R}_{i,i+1}(u) \otimes \cdots \otimes 1_n \), and \( 1_j \) represents the unit matrix of the \( j \)-th particle. The physical meaning of \( \hat{R}(u) \) is two-particle scattering matrix depending on the relative rapidity \( \text{tanh}^{-1}(\beta u) \). Let the unitary Yang-Baxter matrix for two qutrits be the form

\[ \hat{R}(u) = \rho(u)[1_i + G(u)U_i], \]

(14)

where \( \rho(u) \) is a normalization factor, and we can choose appropriate \( \rho(u) \) to ensure \( \hat{R}(u) \) is unitary. \( G(u) \) is determined by the associated YBE \( \text{(13)} \), and we easily get

\[ G(u) + G(v) + \sqrt{3}G(u)G(v) = [1 - G(u)G(v)]G(\frac{u+v}{1+\beta^2 uv}). \]

(15)

Equation \( \text{(14)} \) has the solution \( G(u) = \frac{4\alpha^2\beta u}{1+\beta^2 u^2-2\beta^2\alpha^2 u} \). We further introduce the transformation \( \frac{1+\beta^2 u^2+2\beta^2\alpha u}{1+\beta^2 u^2-2\beta^2\alpha u} = e^{-2i\theta}, \rho(u) = e^{i\theta}, \) where \( \theta \) is real. One can easily verify \( G(u) = e^{-2i\theta}\sqrt{3} \). Then we can obtain the following form of the unitary Yang-Baxter matrix for two qutrits as,

\[ \hat{R}(\theta, \varphi_1, \varphi_2) = \frac{1}{3} \begin{pmatrix}
    b & 0 & 0 & 0 & aq_2\omega & 0 & 0 & 0 & aq_2
    0 & b & 0 & 0 & 0 & 0 & 0 & 0 & aq_2
    0 & 0 & b & aq_2 & aq_2 & aq_2 & aq_2 & aq_2 & aq_2
    0 & 0 & aq_2 & b & 0 & 0 & 0 & 0 & aq_2
    0 & aq_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
    0 & 0 & aq_2 & 0 & 0 & 0 & 0 & 0 & 0
    0 & 0 & aq_2 & 0 & 0 & 0 & 0 & 0 & 0
    0 & 0 & aq_2 & 0 & 0 & 0 & 0 & 0 & 0
    0 & 0 & aq_2 & 0 & 0 & 0 & 0 & 0 & 0
    \end{pmatrix}, \]

(16)

where \( a = -2i \sin \theta \) and \( b = 2e^{i\theta} + e^{-i\theta} \).

The Gell-mann matrices, a basis for the Lie algebra \( SU(3) \) \( 33, \lambda_a \) satisfy \( \{I_\lambda, I_\mu\} = i f_{\lambda\mu\nu}I_\nu \), where \( I_\mu = \frac{1}{2}i \lambda_\mu \). To the later convenience, we denote \( I_\lambda \) by, \( I_\pm = I_1 \pm iI_2, V_\pm = V_4 \mp iV_5, U_\pm = U_6 \pm iU_7, Y = \frac{2}{\sqrt{3}}I_8. \)

In this work, we get rise to three sets of \( SU(3) \) realizations as:

\[ I^{(1)}_\pm = I^\pm_1 I^\pm_2, \quad U^{(1)}_\pm = U^\pm_1 U^\pm_2, \quad V^{(1)}_\pm = V^\pm_1 V^\pm_2, \]

(i) : \[ I^{(1)}_3 = \frac{1}{3}(I^3_1 + I^3_2) + \frac{1}{2}(I^3_1 Y_2 + Y_1 I^3_2), \]

\[ Y^{(1)} = \frac{1}{3}(Y_1 + Y_2) + \frac{2}{3}I_1^3 - \frac{1}{3}Y_1 Y_2; \]

\[ I^{(2)}_\pm = U^\pm_1 V^\pm_2, \quad U^{(2)}_\pm = V^\pm_1 U^\pm_2, \quad V^{(2)}_\pm = I^\pm_1 U^\pm_2, \]

(ii) : \[ I^{(2)}_3 = \frac{1}{2}[-\frac{1}{3}(I^3_1 + I^3_2) + \frac{1}{2}(Y_1 - Y_2) + I^3_1 Y_2 + Y_1 I^3_2], \]

\[ Y^{(2)} = -\frac{1}{3}(I^3_1 - I^3_2) - \frac{1}{6}(Y_1 + Y_2) + \frac{2}{3}I_1^3 - \frac{1}{2}Y_1 Y_2; \]

(17)

(18)
\[
\begin{align*}
(iii): \quad &\begin{cases}
I_\pm^{(3)} = V_\pm^{(3)} U_2^\pm, \quad V_\pm^{(3)} U_2^\pm = I_\pm^{(3)} V_2^\pm, \\
I_\pm^{(3)} = \frac{1}{2} [-\frac{1}{3} (I_3^3 + I_2^3) - \frac{1}{2} (Y_1 - Y_2) + I_3^3 Y_2 + Y_1 I_3^3],
\end{cases} \\
Y^{(3)} = \frac{1}{2} (I_1^3 - I_2^3) - \frac{1}{3} (Y_1 + Y_2) + \frac{2}{3} I_1^3 I_2^3 - \frac{1}{3} Y_1 Y_2.
\end{align*}
\]

We denote \(I_{\pm}^{(k)} = I_{1}^{(k)} \pm iI_{2}^{(k)}, V_{\pm}^{(k)} = V_{1}^{(k)} \mp iV_{2}^{(k)} \). For \(k = 1, 2, 3\). These realizations satisfy the commutation relation \([I_{\lambda}^{(i)}, I_{\mu}^{(j)}] = i\delta_{ij} f_{\lambda\mu\nu} I_{\nu}^{(i)} \). These states can be generated from an initial separable two qutrits.

For \(i\)-th and \((i + 1)\)-th lattices, \(\hat{R}\) can be expressed in terms of the above operators,

\[
\begin{align*}
\hat{R}(\theta, \varphi_1, \varphi_2) &= \frac{a q_2 \omega}{3} I_3^{(1)} + \frac{q_3^2 \omega}{q_2^2} I_3^{(2)} + \frac{q_3 \omega}{q_2} V_1^{(2)} + q_3^2 \omega U_1^{(2)} + \frac{1}{q_1} U_1^{(2)} \\
&+ \frac{q_1 q_2 \omega}{q_4} I_3^{(3)} + \frac{q_2^2 \omega}{q_4} I_3^{(2)} + \frac{q_2 \omega}{q_4} V_1^{(2)} + \frac{1}{q_1} U_1^{(2)} + q_1 \omega U_1^{(2)} \\
&+ \frac{q_1^2 q_2 \omega}{q_4} I_3^{(3)} + \frac{q_2^2 \omega}{q_4} I_3^{(2)} + \frac{q_2 \omega}{q_4} V_1^{(2)} + \frac{1}{q_1} U_1^{(2)} + q_1 \omega U_1^{(2)} + \frac{b}{3} (I \otimes I).
\end{align*}
\]

So we can say the whole tensor space \(C^3 \otimes C^3\) is completely decomposed into three subspaces. i.e. \(C^3 \otimes C^3 = C^3 \oplus C^3 \oplus C^3\). In addition, each block of \(\hat{R}\)-matrix can be represented by fundamental representations of SU(3) algebra.

When one acts \(\hat{R}(\theta, \varphi_1, \varphi_2)\) on the separable state \(|mn\rangle\), he yields the following family of states \(|\psi\rangle_{mn} = \sum_{ij=00}^{22} \hat{R}_{ij}^{(i)} (mn) (m,n = 0,1,2)\). For example, if \(m = 1\) and \(n = 1\),

\[
|\psi\rangle_{11} = \frac{1}{3} \left( a q_2 \omega |00\rangle + b |11\rangle + a q_1^2 \omega |22\rangle \right) \tag{21}
\]

By means of concurrence, we study these entangled states. In Ref. [34], the generalized concurrence (or the degree of entanglement [55]) for two qudits is given by,

\[
C = \sqrt{\frac{d}{d-1} (1 - I_1)}, \tag{22}
\]

where \(I_1 = Tr[\rho_A^2] = Tr[\rho_B^2] = |\kappa_0|^4 + |\kappa_1|^4 + \cdots + |\kappa_{d-1}|^4\), with \(\rho_A\) and \(\rho_B\) are the reduced density matrices for the subsystems, and \(\kappa_j'(j = 0,1,\ldots,d-1)\) are the Schmidt coefficients. Then we can obtain the generalized concurrence of the state \(|\psi\rangle_{11}\) as,

\[
C = \frac{2 \sqrt{2} |\sin \theta|}{3} \sqrt{1 + 2 \cos^2 \theta}. \tag{23}
\]

When \(\theta = \frac{\pi}{3}\), the state \(|\psi\rangle_{11}\) becomes the maximally entangled state of two qutrits as \(|\psi\rangle_{11} = \frac{1}{\sqrt{3}} \left( \frac{a \omega}{q_4} |00\rangle + e^{i \frac{2\pi}{3}} |11\rangle + \frac{1}{q_4} |22\rangle \right)\). In general, if one acts the unitary Yang-Baxter matrix \(\hat{R}(\theta, \varphi_1, \varphi_2)\) on the basis \(|00\rangle, |01\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle\), he will obtain the same concurrence as Eq. (23). It is easy to check that the concurrence ranges from 0 to 1 when the parameter \(\theta\) runs from 0 to \(\pi\). But for \(\theta \in [0, \pi]\), the entanglement is not a monotonic function of \(\theta\) And when \(\theta = \frac{\pi}{3}\), he will generate nine complete and orthogonal maximally entangled states of two qutrits. The entanglement doesn’t depend on the parameters \(\varphi_1\) and \(\varphi_2\). So one can verify that parameter \(\varphi_1\) and \(\varphi_2\) may be absorbed into a local operation.

In fact, we can introduce a local unitary transformation \(P = P_1 \otimes P_2\), where \(P_1 = \text{diag} \{a \omega, \omega, q_1\}\) and \(P_2 = \text{diag} \{a \omega, \omega, q_1\}\). By means of this local transformation \(\hat{R}(\theta) = P \hat{R}(\theta, \varphi_1, \varphi_2) P^{-1}\), the unitary \(\hat{R}(\theta, \varphi_1, \varphi_2)\)-matrix \(\text{[16]}\) is local equivalent to the universal \(\hat{R}(\theta)\) matrix for \(n = 3\) in Ref. [20], where the proof of universality for \(n^2 \times n^2\) Yang-Baxter matrix is presented. So the same as the property of \(\hat{R}(\theta)\) matrix in Ref. [20], we can also say all pure entangled states of two 3-dimensional quantum systems (i.e., two qudits) can be generated from an initial separable state via the universal \(\hat{R}(\theta, \varphi_1, \varphi_2)\)-matrix \(\text{[16]}\) if one is assisted by local unitary transformations.

IV. SUMMARY

In this paper, using the method which is shown in Ref. [1], we can obtain another series of \(n^2 \times n^2\) matrices \(U\) which satisfy the TLA via changing some original conditions. The single loop of these matrices \(U\) is \(d = \sqrt{n}\). Then we
present a $9 \times 9$ matrix representation $U$ which satisfies the TLA with the single loop $d = \sqrt{3}$, and we derived a unitary $\tilde{R}(\theta, \varphi_1, \varphi_2)$-matrix via Yang-Baxterization of the $U$-matrix. Finally, we investigate the entanglement properties of $\tilde{R}$-matrix, and it is shown that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary matrix $\tilde{R}$-matrix acting on the standard basis. We also show that all pure two-qurtit entangled states can be generated via the universal $\tilde{R}$-matrix assisted by local unitary transformations.

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