Global Solutions for the Kuramoto-Sivashinsky Equation Posed on Unbounded 3D Grooves

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Abstract: Initial boundary value problems for the three-dimensional Kuramoto-Sivashinsky equation posed on unbounded 3D grooves (that may serve as mathematical models for wildfires) were considered. The existence and uniqueness of global strong solutions as well as their exponential decay have been established.

Keywords: kuramoto-sivashinsky equation, global solutions, decay in unbounded domains

MSC: 35B35, 35K91, 35Q53

1. Introduction

This work concerns the existence and uniqueness of global strong solutions as well as exponential decay rates of solutions to initial boundary value problems for the three-dimensional Kuramoto-Sivashinsky equation (K-S):

\[ \phi_t + \Delta^2 \phi + \Delta \phi + \frac{1}{2} |\nabla \phi|^2 = 0. \] (1)

Here \( \Delta \) and \( \nabla \) are the Laplacian and the gradient in \( \mathbb{R}^3 \). In [1], Kuramoto studied the turbulent phase waves, and Sivashinsky in [2] obtained an asymptotic equation which modeled the evolution of a disturbed plane flame front. See also [3]. Moreover, K-S equation has applications in the study of dynamics of plasma and flow of liquid films on inclined planes, [4]. In [5-8], mathematical results on initial and initial boundary value problems for one-dimensional (1) are presented, see references there for more information. Multi-dimensional problems for various types of (1) can be found in [9-15] with some results on the existence, regularity and, nonlinear stability of solutions. In [10-13], stability of global solutions for the K-S equation posed on a rectangle with the width \( L_2 \) and the length \( L_1 \), where \( L_2 \) is a decreasing function of \( L_1 \) and initial data depend on \( L_1 \), was studied. This did not allow to study initial boundary value problems posed on strips. Absence of published results on three-dimensional problems, motivated us to study the existence, uniqueness, and stability of global solutions for the three-dimensional K-S equation. Our approach does not include dependence of \( L_2 \) on \( L_1 \). This allows to prove the existence, uniqueness, and exponential decay of global solutions in two and three dimensions making use of the same approach with the unique restriction on the width of a domain. The
shortage of this approach is smallness conditions for the initial data.

For three dimensions, (1) can be rewritten in the form of the following system:

\[
(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2} \sum_{i=1}^{3} (u_i^2)_{x_j} = 0, \tag{2}
\]

\[
(u_j)_{x_j} = (u_i)_{x_j}, \quad i \neq j; \quad i, j = 1, 2, 3, \tag{3}
\]

where \(u_j = \phi_{x_j}\). The first essential problem that arises while one studies either (1) or (2)-(3) is a destabilizing effect of \(\Delta u_j\) that may be damped by a dissipative term \(\Delta^2 u_j\) provided a domain has some specific properties, see condition (13) of Theorem 3.1. Naturally, the so-called “thin domain” appears where some dimensions are small while the others may be arbitrarily large. The second essential problem is the presence of semi linear interconnected terms in (2). This does not allow to obtain first estimate independent of solutions and leads to a connection between geometric properties of a domain and initial data, see (14) of Theorem 3.1.

Our work has the following structure: Section 1 is Introduction. Section 2 contains notations and auxiliary facts. In Section 3, formulation of an initial boundary value problem for (2)-(3) posed on unbounded grooves is given. The existence and uniqueness of a global strong solution, exponential decay of the \(L^2\)-norm have been established. Section 4 contains ideas of applications results obtained in our paper to some practical problems. Section 5 contains conclusions.

2. Notations and auxiliary facts

Let \(\Omega\) be a domain in \(\mathbb{R}^3\) and \(x = (x_1, x_2, x_3) \in \Omega\). We use the standard notations of Sobolev spaces \(W^{k,p}, L^p\) and \(H^k\) for functions and the following notations for the norms [16]: for scalar functions \(f(x, t)\)

\[
\mathbb{R}^+ = \{ t \in \mathbb{R}; t > 0 \},
\]

\[
\| f \|_p^2 = \int_{\Omega} |f|^2 d\Omega,
\]

\[
\| f \|_{L^p(\Omega)}^p = \int_{\Omega} |f|^p d\Omega,
\]

\[
\| f \|_{W^{k,p}(\Omega)}^p = \sum_{0 \leq \alpha \leq k} \| D^\alpha f \|_{L^p(\Omega)}^p;
\]

\[
\| f \|_{H^k(\Omega)} = \| f \|_{W^{k,2}(\Omega)}.
\]

When \(p = 2\), \(W^{k,p}(\Omega) = H^k(\Omega)\) is a Hilbert space with the scalar product

\[
((u,v))_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v), \| u \|_{L^p(\Omega)} = \text{ess sup}_\Omega |u(x)|.
\]

We use a notation \(H^k_0(\Omega)\) to represent the closure of \(C^\infty_c(\Omega)\), the set of all \(C^\infty\) functions with compact support in \(\Omega\), with respect to the norm of \(H^k(\Omega)\).

**Lemma 2.1** (Steklov’s Inequality [17]). Let \(v \in H^1_0(0, L)\). Then
Lemma 2.2 (Differential form of the Gronwall Inequality. See [18], Lemma 2.2.). Let $I = [t_0, t_1]$. Suppose that functions $a, b : I \to \mathbb{R}$ are integrable and a function $a(t)$ may be of any sign. Let $u : I \to \mathbb{R}$ be a differentiable function satisfying

$$u_t(t) \leq a(t)u(t) + b(t), \quad \text{for } t \in I \text{ and } u(t_0) = u_0,$$

then

$$u(t) \leq u_0 e^{\int_{t_0}^{t} a(r)dr} + \int_{t_0}^{t} e^{\int_{t_0}^{s} a(r)dr} b(s) ds.$$

The next Lemmas will be used in estimates:

Lemma 2.3 (See [19], Lemma 3.5.). Let $v \in H_0^1(\Omega)$ and $n = 3$, then

$$\|v\|_{L^2(\Omega)} \leq \frac{1}{2^2} \|v\|^4 \|\nabla v\|^4.$$

3. K-S system posed on a groove

Define a groove

$$D = \{x \in \mathbb{R}^3; x_1 \in \mathbb{R}^1, x_2 \in (0, B), (B > 0), x_3 \in \mathbb{R}^+ \}, Q_t = (0, t) \times D.$$

Lemma 3.1. Let $f \in H_0^1(D)$. Then

$$a \|f\|^2 \leq \|\nabla f\|^2, \quad a^2 \|f\|^2 \leq \|\Delta f\|^2, \quad a \|\nabla f\|^2 \leq \|\Delta f\|^2,$$

where $a = \frac{\pi^2}{B^2}$.

Proof. By definition,

$$\|\nabla f\|^2 = \sum_{i=1}^{3} \|f_{x_i}\|^2.$$

Making use of Steklov’s inequalities, we get

$$\|\nabla f\|^2 \geq \|f_{x_2}\|^2 \geq \frac{\pi^2}{B^2} \|f\|^2 = a \|f\|^2.$$

On the other hand,
\[ a \| f \|^2 \leq \| \nabla f \|^2 = -\int_D f \Delta f dx \leq \| \Delta f \| \| f \|. \]

This implies
\[ a \| f \| \| \Delta f \| \leq a^2 \| f \|^2 \leq \| \Delta f \|^2. \]

Consequently, \( a \| \nabla f \|^2 \leq \| \Delta f \|^2 \). Proof of Lemma 3.1 is completed.

In \( Q \), consider the following initial boundary value problem:

\begin{align*}
(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2} \sum_{i=1}^{3} (u_i^2)_{x_j} &= 0, \\
(u_j)_{x_j} &= (u_j)_{x_i}, j \neq i, i, j = 1, \ldots, 3; \\
\frac{\partial u_j}{\partial N} |_{\partial D} &= 0, t > 0, \\
\frac{\partial u_j}{\partial N} |_{\partial D} &= 0, t > 0, \\
u_j(x, 0) &= u_{j0}(x), j = 1, \ldots, 3, x \in D, \\
\end{align*}

where \( \frac{\partial}{\partial N} \) is an exterior normal derivative on \( \partial D \).

**Theorem 3.1.** Let

\[ B < \pi, a = \frac{\pi^2}{B^2}, \theta = 1 - \frac{1}{a} > 0. \]

Given \( u_{j0} \in H^4(D) \cap H_0^2(D), j = 1, 2, 3 \) such that

\[ \theta - \frac{48}{3} \sum_{i=1}^{3} \| u_{i0} \|^2 > 0. \]

Then the problem (3.3)-(3.6) has a unique strong solution

\[ u_j \in L^\infty(\mathbb{R}^+; H_0^2(D)), \Delta^2 u_j \in L^\infty(\mathbb{R}^+; L^2(D)); \]

\[ u_{j\tau} \in L^\infty(\mathbb{R}^+; L^2(D)) \cap L^2(\mathbb{R}^+; H_0^2(D)), j = 1, 2, 3. \]

Moreover, \( u_j \) satisfy the following inequality

\[ \sum_{j=1}^{3} \| u_j \|^2(t) + \frac{\theta}{2} \int_0^t \left( \sum_{j=1}^{3} \| \Delta u_j \|^2(\tau) \right) d\tau \leq \sum_{j=1}^{3} \| u_{j0} \|^2, \]
\[
\sum_{j=1}^{3} \| u_j \|^2(t) \leq \left( \sum_{j=1}^{3} \| u_{j0} \|^2 \right) \exp \left(-\frac{a^2 \theta t}{2} \right).
\] (16)

\[
\sum_{j=1}^{3} \| u_{jt} \|^2(t) \leq \left( \sum_{j=1}^{3} \| u_{jt} \|^2(0) \right) \exp \left(-\frac{a^2 \theta t}{2} \right),
\] (17)

\[
\sum_{j=1}^{3} \| u_{jt} \|^2(t) + \frac{\theta}{2} \int_{0}^{t} \left( \sum_{j=1}^{3} \| \Delta u_{jt} \|^2(\tau) \right) d\tau \leq \sum_{j=1}^{3} \| u_{jt} \|^2(0),
\] (18)

where \( t > 0, \ \| u_{jt} \|^2(0) \leq C(\| u_{0j} \|_{W}), \ j = 1,2,3. \)

**Proof.** Define the space \( W = H^4(D) \cap H_0^2(D) \) and let \( \{w_i(x), i \in \mathbb{N}\} \) be a countable dense set in \( W \). We can construct approximate solutions to (9)-(12) in the form

\[
u_{j}^N(x,t) = \sum_{i=1}^{N} g_i^j(t) w_i(x); j = 1,2,3,
\]

Unknown functions \( g_i^j(t) \) satisfy the following initial problems:

\[
\frac{d}{dt} (u_j^N(w_j^N)(t) + \Delta u_j^N(x) \Delta w_j(x) - (\nabla u_j^N \nabla w_j)(t) - \frac{1}{2} \sum_{i=1}^{3} (u_i^N)^2 (w_j)_x)(t) = 0,
\] (19)

\[
g_i^j(0) = g_{0j}^i, j = 1,2,3; i = 1,2,....
\] (20)

By Caratheodory’s existence theorem, [20], Theorems 1.2 of Chapter 1, there exist solutions of (19)-(20) at least locally in \( t \). All the estimates we will prove will be done on smooth solutions of (9)-(12). Naturally, the same estimates are true also for approximate solutions \( u_j^N \).

**Estimate 1** Multiply (9) by \( 2u_j \) to obtain

\[
\frac{d}{dt} \| u_j \|^2(t) + 2 \| \Delta u_j \|^2(t) - 2 \| \nabla u_j \|^2(t) - \sum_{i=1}^{3} (u_i^2)(u_j)_x)(t) = 0, j = 1,2,3.
\] (21)

Since \( a |\nabla u_j|^2(t) \leq |\Delta u_j|^2(t) \), taking into account (13), we get

\[
\frac{d}{dt} \| u_j \|^2(t) + \theta \| \Delta u_j \|^2(t) + \theta \| \nabla u_j \|^2(t) - \sum_{i=1}^{3} (u_i^2)(u_j)_x)(t) \leq 0.
\] (22)

Making use of Lemmas 2.3, 3.1, we estimate
\[ I = \sum_{i=1}^{3} (u_i^2, (u_j)_x) \leq \sum_{i=1}^{3} \| u_i \| \| u_i \|_{L^2(D)} \| (u_j)_x \|_{L^2(D)} \]

\[ \leq \sum_{i=1}^{3} 2 \| u_i \| \| u_j \| \| \nabla u_j \| \| \nabla u_j \| \| \Delta u_j \| \| \Delta u_j \| \]

\[ \leq 2 \sum_{i=1}^{3} 2^{1/4} \| u_i \| \| \nabla u_j \| \| \Delta u_j \| \leq \epsilon \| \Delta u_j \|^2 + \frac{1}{\epsilon} \left( \sum_{i=1}^{3} 2^{1/4} \| u_i \| \| \nabla u_j \| \right)^2 \]

\[ \leq \epsilon \| \Delta u_j \|^2 + \frac{4}{\epsilon} \sum_{i=1}^{3} \| u_i \|^2 \| \nabla u_j \|^2 \]

\[ \leq \epsilon \| \Delta u_j \|^2 + \frac{4}{\epsilon} \sum_{i=1}^{3} \| u_i \|^2 \| \nabla u_j \|^2, \]

where \( \epsilon \) is an arbitrary positive number. Taking \( 2\epsilon = \theta \), substituting \( I \) into (22) and summing up over \( j = 1, 2, 3 \), we get

\[ \frac{d}{dt} \sum_{j=1}^{3} \| u_j \|^2(t) + \frac{\theta}{2} \sum_{j=1}^{3} \| \Delta u_j \|^2(t) + \left[ \theta - \frac{24}{\theta^2 a^{3/2}} \sum_{i=1}^{3} \| u_i \|^2(t) \right] \sum_{j=1}^{3} \| \Delta u_j \|^2(t) \leq 0. \tag{23} \]

Condition (14), positivity of the second term in (23) and standard arguments guarantee that

\[ \left[ \theta - \frac{24}{\theta^2 a^{3/2}} \sum_{i=1}^{3} \| u_i \|^2(t) \right] > 0, t > 0. \tag{24} \]

This transforms (23) into the following inequality:

\[ \frac{d}{dt} \sum_{j=1}^{3} \| u_j \|^2(t) + \frac{\theta}{2} \sum_{j=1}^{3} \| \Delta u_j \|^2(t) \leq 0. \tag{25} \]

Integrating (25), we obtain

\[ \sum_{j=1}^{3} \| u_j \|^2(t) + \int_{0}^{t} \frac{\theta}{2} \sum_{j=1}^{3} \| \Delta u_j \|^2(\tau) d\tau \leq \sum_{j=1}^{3} \| u_j(0) \|^2. \tag{26} \]

On the other hand, Lemma 3.1 allows us to rewrite (25) in the form

\[ \frac{d}{dt} \sum_{j=1}^{3} \| u_j \|^2(t) + a \sum_{j=1}^{3} \| u_j \|^2(t) \leq 0. \tag{27} \]

From this, (16) follows.

**Estimate II** Differentiate (9) with respect to \( t \), then multiply the results respectively by \( 2(u_j)_t \), to get
\[
\frac{d}{dt} \left\| u_{j,t} \right\|^2 (t) + 2 \left\| \Delta u_{j,t} \right\|^2 (t) - 2 \left\| \nabla u_{j,t} \right\|^2 (t) = 2 \sum_{i=1}^{3} (u_{i,j}, (u_j)_{x_{j,i}})(t), \quad j = 1, 2, 3. \tag{28}
\]

Making use of Lemmas 2.4 and 3.1, we estimate

\[
I = 2 \sum_{i=1}^{3} (u_{i,j}, (u_j)_{x_{j,i}})(t) \leq 2 \sum_{i=1}^{3} \| u_i \| \left\| \nabla u_{i,t} \right\|^2 \| \Delta u_{i,t} \|^2 \| \nabla u_{i,t} \|^2 \| u_{i,t} \|^2 \| \Delta u_{i,t} \|^2 \| u_{i,t} \|^2 \leq \epsilon \left\| \Delta u_{i,t} \right\|^2 + \frac{4}{\epsilon} \left( \sum_{i=1}^{3} \| u_i \| \left\| \nabla u_{i,t} \right\|^2 \right)^2 \]

\[
\leq \epsilon \left\| \Delta u_{i,t} \right\|^2 + \frac{16}{a^{1/2}} \sum_{i=1}^{3} \| u_i \|^2 \left\| \nabla u_{i,t} \right\|^2 \]

\[
\leq \epsilon \left\| \Delta u_{i,t} \right\|^2 + \frac{16}{a^{1/2}} \sum_{i=1}^{3} \| u_i \|^2 \left\| \Delta u_{i,t} \right\|^2.
\]

Taking \( 2\epsilon = \theta \), substituting \( I \) into (28) and summing up over \( j = 1, 2, 3 \), we get

\[
\frac{d}{dt} \sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (t) + \frac{\theta}{2} \sum_{j=1}^{3} \left\| \Delta u_{j,t} \right\|^2 (t) + \left[ \theta - \frac{480}{3^{3/2}} \sum_{i=1}^{3} \| u_i \|^2 (t) \right] \sum_{j=1}^{3} \left\| \Delta u_{j,t} \right\|^2 (t) \leq 0. \tag{29}
\]

Taking into account (14), rewrite (29) in the form

\[
\frac{d}{dt} \sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (t) + \frac{\theta}{2} \sum_{j=1}^{3} \left\| \Delta u_{j,t} \right\|^2 (t) \leq 0. \tag{30}
\]

This implies

\[
\sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (t) + \frac{\theta}{2} \int_{0}^{t} \left( \sum_{j=1}^{3} \left\| \Delta u_{j,t} \right\|^2 (r) \right) dr \leq \sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (0), \tag{31}
\]

where \( \left\| u_{j,t} \right\|^2 (0) \leq C(\| u_0 \|_W) \). Making use of Lemma 3.1, rewrite (30) as

\[
\frac{d}{dt} \sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (t) + \frac{\alpha^2 \theta}{2} \sum_{j=1}^{3} \left\| u_{j,t} \right\|^2 (t) \leq 0. \tag{32}
\]

Integrating this, we find
\[
\sum_{j=1}^{3} \| u_{jt} \|^2(t) \leq \left( \sum_{j=1}^{3} \| u_{tt} \|^2(0) \right) \exp \left\{ -\frac{a^2 \theta}{2} t \right\}. \quad (33)
\]

Estimates (3.20) and (3.25) imply that \( u_j \in L^\infty(\mathbb{R}^+; H_0^2(D)), \)
\[
u_{jt} \in L^\infty(\mathbb{R}^+; L^2(D)) \cap L^2(\mathbb{R}^+; H_0^2(D)), \quad j = 1, 2, 3.
\]

These inequalities guarantee the existence of strong solutions \{\( u_j(x, t) \)\} to (9)-(12) satisfying the following identity:
\[
(u_j, \phi)(t) + (\Delta u_j, \Delta \phi)(t) + (\Delta u_j, \phi)(t) - \frac{1}{2} \sum_{i=1}^{3} (u_i)^2, \phi_{x_j})(t) = 0, \quad t > 0,
\]
where \( \phi(x, y) \) is an arbitrary function from \( H_0^2(D) \).

We can rewrite (34) in the form
\[
(\Delta u_j, \Delta \phi)(t) = -((u_j)_t + \Delta u_j + \sum_{i=1}^{3} u_i(x_i, x_j), \phi)(t).
\]

Since \( u_j \in L^\infty(\mathbb{R}^+; H_0^2(D)) \), it follows from here and estimates (26), (31) that
\[
[(u_j)_t + \Delta u_j + \sum_{i=1}^{3} u_i(x_i, x_j)] \in L^\infty(\mathbb{R}^+; L^2(D)), \quad j = 1, 2, 3,
\]

hence
\[
\Delta^2 u_j \in L^\infty(\mathbb{R}^+; L^2(D)), \quad j = 1, 2, 3.
\]

This proves the existence part of Theorem 3.1.

**Lemma 3.2.** The strong solutions of (9)-(12) is unique.

**Proof.** Let \( u_j \) and \( v_j, \ j = 1, 2, 3, \) be two distinct solutions to (9)-(12). Denoting \( w = u_j - v_j \), we come to the following system:
\[
\frac{d}{dt} \| w_j \|^2(t) + 2 \| \Delta w_j \|^2(t) - 2 \| \nabla w_j \|^2(t) = \sum_{i=1}^{4} (u_i + v_j) w_i, (w_j)_x_j)(t), \quad (35)
\]
\[
(w_i)_x_j = (w_j)_x_j, \quad i \neq j, \quad (36)
\]
\[
w_j|_{\partial D} = \frac{\partial}{\partial N} w_j|_{\partial D} = 0, \quad t > 0, \quad (37)
\]
\[
w_j(x, 0) = 0, \quad j = 1, 2, 3. \quad (38)
\]

Making use of Lemmas 2.3, 3.1, we estimate
\[ I = (u_i + v_j)w_j, (w_j)_{x_j} \leq \|w_j\|_{L^4(D)} \|u_i + v_j\|_{L^4(D)} \]
\[ \leq 2 \|w_j\|_{L^4(D)} \|w_j\|_{L^4(D)} \|u_i + v_j\|_{L^4(D)} \]
\[ \leq \frac{2}{a^{1/4}} \|w_j\|_{L^4(D)} \| \Delta(w_j) \|_{L^4(D)} \| \nabla(u_i + v_j) \|_{L^4(D)} \]
\[ \leq \|w_j\|_{L^4(D)} \| \Delta(w_j) \|_{L^4(D)} + \frac{1}{\epsilon} \|w_j\|_{L^4(D)} \| \Delta(u_i + v_j) \|_{L^4(D)} \].

Substituting \( I \) into (35), taking \( 2\epsilon = \theta \) and summing over \( j = 1, 2, 3 \), we get
\[ \frac{d}{dt} \sum_{j=1}^{3} \|w_j\|_{L^4(D)}^2(t) + \theta \sum_{j=1}^{3} \|\Delta w_j\|_{L^4(D)}^2(t) \leq C \left( \sum_{i=1}^{3} \|\Delta u_i\|_{L^4(D)}^2(t) + \|\Delta v_j\|_{L^4(D)}^2(t) \right) \sum_{j=1}^{3} \|w_j\|_{L^4(D)}^2(t)\].

Due to (26), \( \|\Delta u_i\|^2(t) + \|\Delta v_j\|^2(t) \in L^1(\mathbb{R}^+) \), \( i = 1, 2, 3 \).
Applying Lemma 2.2, we obtain that
\[ \|w_j\|_{L^4(D)}(t) = 0, t > 0, j = 1, 2, 3. \]

This proves Lemma 3.2 and consequently Theorem 3.1.

**Remark 3.1.** We can define other grooves:
\[ D_{x_3} = \{ x = (x_1, x_2, x_3); x_1 \in \mathbb{R}^1, x_2 \in \mathbb{R}^+, x_3 \in (0, L_3), (L_3 > 0) \}, \]
\[ D_{x_1} = \{ x = (x_1, x_2, x_3); x_1 \in (0, L_1), (L_1 > 0), x_2 \in \mathbb{R}^1, x_3 \in \mathbb{R}^+ \} \]
and obtain results similar to ones of Theorem 3.1.

**4. Possible applications of theorem 3.1**

Since by [2], the Kuramoto-Sivashinsky equation describes instabilities in laminar flame fronts, it can be used to create mathematical models of wildfires. On the other hand, the Kuramoto-Sivashinsky equation describes also the dynamics of turbulent phase waves and flow of liquid films on an inclined plane [4]. This can be used to create mathematical models for the stability of processes in river flows.

**5. Data availability statement**

I declare that all the data that have been used in my manuscript are mathematical theorems published in open mathematical journals and presented in References.
6. Conclusions

In this work, we studied the stability of global solutions for initial boundary value problems to the three-dimensional Kuramoto-Sivashinsky system (1) posed on unbounded grooves. We defined a set of admissible domains which eliminate destabilizing effects of terms $\Delta u_j$ by dissipativity of $\Delta^2 u_j$. Since these problems do not admit the first a priori estimate independent of $t$ and solutions, in order to prove the existence of global solutions, we put conditions connecting geometrical properties of domains with initial data. We proved the existence and uniqueness of a strong solution as well as exponential decay of $L^2$-norms.

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Conflict of interests

The author declares that there is no conflict of interest regarding the publication of this paper.

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