The Morse Index of Wente Tori

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Abstract

We find various lower and upper bounds for the index of Wente tori that contain a continuous family of planar principal curves. We then prove a result that gives an algorithm for computing the index sharply.

1 Introduction

Hopf conjectured that every constant mean curvature (CMC) closed (closed = compact, without boundary) surface in $\mathbb{R}^3$ is a round sphere. This is true if the surface is assumed to be either genus 0 or embedded [H], but in general it is false, and the first counterexamples (with genus 1) were found by Wente [We]. Abresch [A] then gave a more explicit representation for those Wente tori which contain a continuous family of planar principal curves, referred to here as symmetric Wente tori. Walter [Wa] later found an even more explicit representation for symmetric Wente tori, by noticing that if one family of principal curves are all planar, then the perpendicular principal curves each lie in a sphere. Finally, Spruck [Sp] showed that these symmetric Wente tori are exactly the same surfaces that Wente originally found. Pinkall and Sterling [PS] and Bobenko [B] went on to classify all closed CMC tori in $\mathbb{R}^3$, and Kapouleas constructed closed CMC surfaces for every genus greater than one [K1], [K2].

The index of minimal surfaces in $\mathbb{R}^3$ has been well studied. In this case there are no closed surfaces. Do Carmo and Peng [CP] showed that the only stable (index 0) complete minimal surface is a plane. Fischer-Colbrie [FC] showed that minimal surfaces have finite index if and only if they have finite total curvature. And for many minimal surfaces with finite total curvature the index has been found or has known bounds (for example, see [FC], [N], [MR], [T], [Cho]).

However, less is known about the index of nonminimal CMC surfaces. It is known that the surface is stable if and only if it is a round sphere [BC] (analogous to the do Carmo-Peng result). It is also known that CMC surfaces without boundary in $\mathbb{R}^3$ have finite index if and only if they are compact [LR], [Si] (analogous to Fischer-Colbrie’s result). However, other than the round sphere, there is no closed CMC surface with known index. (Unlike CMC surfaces in general, minimal surfaces have meromorphic Gauss maps when given conformal coordinates. Furthermore, for the index of nonminimal CMC surfaces we must consider a volume constraint that does not appear in the minimal case. This accounts for why more progress has been made in finding the index of minimal surfaces.)

Our purpose here is to expand the collection of nonminimal CMC surfaces for which we have specific estimates of the index (to include surfaces other than just round spheres). We restrict ourselves to symmetric Wente tori, because they are the simplest known examples
of nonspherical closed CMC surfaces in $\mathbb{R}^3$, and can be represented nicely [A], [Wa]. Hence they are the obvious candidates to consider in an initial investigation.

In section 2, we briefly reiterate Walter’s description of symmetric Wente tori, describe the Jacobi operator and the index problem, and discuss the eigenvalues and eigenfunctions of the Laplacian on a torus (which play an essential role in our discussion).

In section 3, we use Courant’s nodal domain theorem to find initial lower bounds for the index, obtaining as a corollary that for every natural number $N$, there exist only finitely many symmetric Wente tori with index less than $N$. In section 4, we use upper and lower bounds of the Jacobi operator’s potential function to determine upper and lower bounds for the index.

In section 5, we use eigenfunctions of the Laplacian to create function spaces on which the Jacobi operator is negative definite, producing much sharper lower bounds for the index. These bounds imply that every symmetric Wente torus has index at least 7. Furthermore, the two simplest Wente tori (the first two surfaces shown in Figure 1) have index at least 9 and 8, respectively.

In section 6, we prove the correctness of an algorithm (a variation of the finite element method [FS]) for computing the index exactly, up to a possible change by 1 from the volume constraint. In this proof, we essentially show that on a flat torus, the spectrum of a Schrodinger operator $\triangle + V$ with $C^\infty$ potential function $V$ can be determined from the restrictions of the operator to function spaces spanned by a finite number of explicitly known eigenfunctions of the Laplacian $\triangle$.

Finally, in section 7, we implement the algorithm to give numerical estimates for the index of 17 different Wente tori (Table 3). These numerical estimates show that the lower bounds of section 5 are quite sharp, and show that it is reasonable to conjecture that every symmetric Wente torus has index at least 9.

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## 2 Preliminaries

### 2.1 Walter’s representation [Wa]

Consider a smooth conformal immersion $\mathcal{X}: \mathcal{M} \to \mathbb{R}^3$, where $\mathcal{M}$ is a compact 2-dimensional torus with the induced metric. If $(x, y)$ are isothermal coordinates on $\mathcal{M}$, then the first and second fundamental forms, Gaussian curvature, mean curvature, and Hopf differential are

\[ ds^2 = E(dx^2 + dy^2), \quad II = Ldx^2 + 2Mdx dy + Ndy^2, \]

\[ K = \frac{LN - M^2}{E^2}, \quad H = \frac{L + N}{2E}, \quad \Phi = \frac{1}{2}(L - N) - iM. \]

The Gauss and Codazzi equations imply that $\Phi$ is holomorphic with respect to $z = x + iy$ when $H$ is constant.

Since $\mathcal{M}$ is a torus, we may assume that $\mathcal{M} = \mathbb{C}/\Gamma$, where $\Gamma$ is a lattice in the plane $\mathbb{C}$. Since $dz$ is a global 1-form on $\mathbb{C}/\Gamma$, and since we will assume $H$ is constant, we see that $\Phi dz^2$ is a constant multiple of $dz^2$.

We define a function $F$ by

\[ HE = e^F. \]
By a linear transformation of \( \mathbb{C} \), we may arrange that \( \Phi dz^2 = dz^2 \), hence \( M = 0, L = e^F + 1, \) and \( N = e^F - 1 \). Thus \( \mathcal{X} \) has no umbilic points, and \((x, y)\) become curvature line coordinates.

Denoting the oriented unit normal by \( \bar{N} \), we may compute that

\[
\begin{align*}
\mathcal{X}_{xx} &= \frac{1}{2} F_x \mathcal{X}_x - \frac{1}{2} F_y \mathcal{X}_y - (e^F + 1) \bar{N}, \quad \mathcal{X}_{xy} = \frac{1}{2} F_y \mathcal{X}_x + \frac{1}{2} F_x \mathcal{X}_y, \\
\mathcal{X}_{yy} &= -\frac{1}{2} F_x \mathcal{X}_x + \frac{1}{2} F_y \mathcal{X}_y - (e^F - 1) \bar{N}, \quad \bar{N}_x = H(1 + e^{-F}) \mathcal{X}_x, \quad \bar{N}_y = H(1 - e^{-F}) \mathcal{X}_y, \\
\Delta F + 4H \sinh F &= 0, \quad (2.1)
\end{align*}
\]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \). Thus finding the immersion \( \mathcal{X} : \mathbb{C}/\Gamma \to \mathbb{R}^3 \) is reduced to solving for \( F \) in equation \((2.1)\). In the case of symmetric Wente tori, Walter determined \( F \) and the immersion \( \mathcal{X} \) explicitly. In the process of doing so, he proved the following lemmas.

**Lemma 2.1** The set of all symmetric Wente tori are in a one-to-one correspondence with the set of reduced fractions \( \ell/n \in (1, 2) \).

For each \( \ell/n \), we call the corresponding symmetric Wente torus \( \mathcal{W}_{\ell/n} \). Each \( \mathcal{W}_{\ell/n} \) has either one or two planar geodesic loops in the central symmetry plane: two loops if \( \ell \) is odd, and one loop if \( \ell \) is even. Each loop can be partitioned into \( 2n \) congruent curve segments, and \( \ell \) is the total winding order of the Gauss map along each loop. (The central planar geodesic loops are contained in the boundaries of the right-hand side figures in Figure 1.)

**Lemma 2.2** The function \( F \) for the surface \( \mathcal{W}_{\ell/n} \) is

\[
F = 4 \arctanh(f(x)g(y)), \quad \text{with}
\]

\[
f(x) = \gamma \operatorname{cn}_k(\alpha x), \quad g(y) = \bar{\gamma} \operatorname{cn}_{\bar{k}}(\bar{\alpha} y),
\]

where \( \operatorname{cn}_k \) (resp. \( \operatorname{cn}_{\bar{k}} \)) is the amplitude sinus of Jacobi with modulus \( k \) (resp. \( \bar{k} \)), and

\[
k = \sin \theta, \quad \bar{k} = \sin \bar{\theta}, \quad \text{for } \theta, \bar{\theta} \in (0, \pi/2), \quad \text{and } \theta + \bar{\theta} < \frac{\pi}{2},
\]

\[
\gamma = \sqrt{\tan \theta} > 0, \quad \bar{\gamma} = \sqrt{\tan \bar{\theta}} > 0, \quad \alpha = \sqrt{4H \frac{\sin 2\theta}{\sin 2(\theta + \bar{\theta})}}, \quad \bar{\alpha} = \sqrt{4H \frac{\sin 2\bar{\theta}}{\sin 2(\theta + \bar{\theta})}}.
\]

There are two period problems for the immersion. One is a translational period problem in the direction of the rotation axis of the surface, and the other is a rotational period problem around the rotation axis. Using elliptic function theory, Walter determined that for all \( \ell/n \), to solve the translational period problem of the surface, one needs \( \bar{\theta} = 65.354955354^\circ \). To then solve the rotational period problem, there is a unique choice of \( \theta \in (0^\circ, 24.645044646^\circ) \). The value of \( \theta \) (but not \( \bar{\theta} \)) will depend on \( \ell/n \) (see Table 2).

Let \( x_{\ell n} \) (resp. \( y_{\ell n} \)) be the length of the period of \( f(x) \) (resp. \( g(y) \)).

**Lemma 2.3** \( \mathcal{X} : \mathbb{C}/\Gamma \to \mathcal{W}_{\ell/n} \) is a conformal diffeomorphism, where

\[
\Gamma = \operatorname{span}_\mathbb{Z}\{(nx_{\ell n}, 0), (0, y_{\ell n})\} \quad \text{when } \ell \text{ is odd, and}
\]

\[
\Gamma = \operatorname{span}_\mathbb{Z}\{(nx_{\ell n}/2, y_{\ell n}/2), (0, y_{\ell n})\} \quad \text{when } \ell \text{ is even.}
\]

The curves \([x_0, y] \mid x_0 = \text{constant}\) are mapped by \( \mathcal{X} \) to planar curvature lines of \( \mathcal{W}_{\ell/n} \).
Figure 1: The surfaces $W_{4/3}$, $W_{3/2}$, and $W_{6/5}$, with cut-aways to expose the inner parts. (Graphics by Katsunori Sato of Tokyo Institute of Technology.)
The reason why $\Gamma$ is different for $\ell$ odd or even is best explained in the introduction of [A]. (However, the notation "$m/n$" in [A] is only half the value of the notation "$\ell/n$" used here and in [Wa]. One can easily check that the characterization of $\Gamma$ depends on the denominator’s parity in [A]’s notation, and depends on the numerator’s parity in [Wa]’s notation.)

$X$ maps the domain $\{(x, y) \mid 0 \leq x \leq x_{\ell n}/2, 0 \leq y \leq y_{\ell n}/2\}$ to a fundamental piece of $W_{\ell/n}$, bounded by four planar geodesics. The entire surface is composed of pieces congruent to the fundamental piece, in the sense that if one continues the fundamental piece by reflecting across planes containing planar geodesics, one arrives at the entire $W_{\ell/n}$. When $\ell$ is odd (resp. $\ell$ is even), $W_{\ell/n}$ consists of the union of $4n$ (resp. $2n$) fundamental pieces.

### 2.2 Index and the associated eigenvalue problem

Let $\vec{v}_t$ be the variation vector field at time $t$ of a $C^\infty$ variation $M(t) \subset \mathbb{R}^3$ of the surface $W_{\ell/n}$ so that $M(0) = W_{\ell/n}$. Let $A(t)$ be the induced area of $M(t)$, and let

$$u_t := \langle \vec{v}_t, \vec{N} \rangle_{\mathbb{R}^3}, \quad u := u_0.$$  

It is natural to consider only those variations for which $\int_{C/\Gamma} u_t dA_t = 0$ ($dA_t$ is the induced area form on $M(t)$, $dA := dA_0$) for all $t$ close to 0, corresponding to the fact that the volume inside a "soap bubble" is preserved by any physically natural variation of the bubble. Such a variation is volume preserving [BC]. So, the first variation formula implies that for volume preserving variations,

$$A'(0) = H \int_{C/\Gamma} u dA = 0.$$  

The second variation formula for volume preserving variations ([BC]) is

$$A''(0) = \int_{C/\Gamma} |\nabla u|^2 + (2K - 4H^2)u^2 dA = \int_{C/\Gamma} u(-\Delta + 2KE - 4H^2E)(u) dxdy,$$

where

$$\mathcal{L}(u) = -\Delta u - V \cdot u, \quad V := 4H \cosh F$$  

is the Jacobi operator with potential function $V$. (This operator is also computed in [PS], but with notation that differs by constant factors.)

Note that in equation (2.2), we are integrating with respect to the flat metric on $C/\Gamma$, not with respect to the metric induced by the immersion.

**Definition 2.1** The index $\text{Ind}(W_{\ell/n})$ of $W_{\ell/n}$ is the maximum possible dimension of a subspace $\mathcal{V} \subset C^\infty(C/\Gamma)$ such that $\int_{C/\Gamma} u dA = 0$ and $\int_{C/\Gamma} u \mathcal{L} u dxdy < 0$ for all nonzero $u \in \mathcal{V}$.

By Lemma 2.4 of [BC], for every $u$ satisfying $\int_{C/\Gamma} u dA = 0$, there exists a volume preserving variation of $W_{\ell/n}$ with variation vector field $u\vec{N}$ on $W_{\ell/n}$. Thus, loosely speaking, $\text{Ind}(W_{\ell/n})$ is the maximum dimension of a space of variation vector fields for area-decreasing volume-preserving smooth variations, that is, a space on which $A'(0) = 0$ and $A''(0) < 0$ for all its nonzero variation vector fields.
Let $L^2 = L^2(\mathbb{C}/\Gamma)$ denote the measurable functions $u$ on $\mathbb{C}/\Gamma$ (or equivalently on $W_{\ell/n}$) satisfying $\int_{\mathbb{C}/\Gamma} u^2 \, dx \, dy < \infty$, and define the $L^2$ inner product $\langle u, v \rangle_{L^2} := \int_{\mathbb{C}/\Gamma} uv \, dx \, dy$. If a sequence of functions $u_i \in L^2$ converges strongly in the $L^2$ norm to a function $u$, we denote this by $u_i \to_{L^2} u$ as $i \to \infty$. Note that we define this $L^2$ space with respect to the flat metric on $\mathbb{C}/\Gamma$, not the metric induced by the immersion. The following theorem is well known (see, for example, [U]):

**Theorem 2.1** The operator $\mathcal{L} = -\Delta - V$ on $\mathbb{C}/\Gamma$ has a discrete spectrum of eigenvalues

$$\beta_1 \leq \beta_2 \leq \ldots \nearrow +\infty$$

(each considered with multiplicity 1), and has corresponding eigenfunctions

$$v_1, v_2, \ldots \in C^\infty(\mathbb{C}/\Gamma)$$

which form an orthonormal basis for $L^2$. Furthermore, we have the following variational characterization for the eigenvalues:

$$\beta_j = \inf_{V_j} \left( \sup_{\phi \in V_j, ||\phi||_{L^2}=1} \int_{\mathbb{C}/\Gamma} \phi \mathcal{L} \phi \, dx \, dy \right), \quad (2.3)$$

where $V_j$ runs through all $j$ dimensional subspaces of $C^\infty(\mathbb{C}/\Gamma)$.

**Lemma 2.4** If $\mathcal{L}$ has $k$ negative eigenvalues, then $\text{Ind}(W_{\ell/n})$ is either $k$ or $k - 1$.

**Proof.** Since $\beta_{k+1} \geq 0$, from characterization (2.3) with $j = k + 1$, we see there is no $k + 1$ dimensional subspace of $C^\infty(\mathbb{C}/\Gamma)$ on which $\int_{\mathbb{C}/\Gamma} u\mathcal{L}u \, dx \, dy < 0$ for all nonzero functions $u$ in the $k + 1$ dimensional subspace. Thus $\text{Ind}(W_{\ell/n}) \leq k$.

Now let $V_k = \text{span}\{v_1, \ldots, v_k\}$. Since $\beta_j < 0$ for all $j \leq k$, $V_k$ is a $k$ dimensional space satisfying $\int_{\mathbb{C}/\Gamma} u\mathcal{L}u \, dx \, dy < 0$ for all nonzero $u \in V_k$. Define a linear map $F : V_k \to \mathbb{R}$ by $F(u) = \int_{\mathbb{C}/\Gamma} u \, dx \, dy$. The kernel $\text{Ker}(F) = \{u \in V_k \mid \int_{\mathbb{C}/\Gamma} u \, dx \, dy = 0\}$ is a subspace of $V_k$ of dimension at least $k - 1$. Then, by definition, $\text{Ind}(W_{\ell/n}) \geq \text{dim}(\text{Ker}(F)) \geq k - 1$. □

**Remark.** It also follows from Lemma 2.4’s proof that if there exists a $j$ dimensional space $V_j \subset C^\infty(\mathbb{C}/\Gamma)$ with $\int_{\mathbb{C}/\Gamma} u\mathcal{L}u \, dx \, dy < 0$ for all nonzero $u \in V_j$, then $\text{Ind}(W_{\ell/n}) \geq j - 1$. □

We define the nullity $\text{Null}(\mathcal{L})$ of $\mathcal{L}$ to be the multiplicity of the eigenvalue 0 of $\mathcal{L}$.

**Lemma 2.5** $\text{Null}(\mathcal{L}) \geq 6$ for every $W_{\ell/n}$.

**Proof.** For any vector field $\vec{v}$ on $\mathbb{R}^3$ generated by a rigid motion of $\mathbb{R}^3$, the normal part $\langle \vec{v}, \vec{N} \rangle_{\mathbb{R}^3}$ on $W_{\ell/n}$ is a Jacobi field on $W_{\ell/n}$, that is, $\mathcal{L}(\langle \vec{v}, \vec{N} \rangle_{\mathbb{R}^3}) = 0$. Suppose $\text{Null}(\mathcal{L}) \leq 5$. Then there is some such $\vec{v}$ entirely tangent to $W_{\ell/n}$, since the rigid motions of $\mathbb{R}^3$ form a 6 dimensional group. Choose such a $\vec{v}$. For each $p \in W_{\ell/n}$, let $\phi_p(t)$ be the integral curve of $\vec{v}$ in $W_{\ell/n}$ such that $\phi_p(0) = p$. Since $d\phi_p(t)/dt = \vec{v}_{\phi_p(t)}$, $\phi_p(t)$ is also an integral curve of $\vec{v}$ in $\mathbb{R}^3$, hence $W_{\ell/n}$ is invariant under the rigid motion that generates $\vec{v}$. But $W_{\ell/n}$ is not invariant under any rigid motion, a contradiction, proving the lemma. □
2.3 Eigenvalues of the Laplacian

By Lemma 2.4, our goal becomes to compute the number of negative eigenvalues of $\mathcal{L}$. In order to do this, we use a convenient fact: we know explicitly the complete set of eigenvalues $\alpha_i$ and eigenfunctions $u_i$ of $-\Delta$. For $\mathbb{C}/\Gamma$, with $\Gamma = \text{span}_{\mathbb{Z}}\{(a_1, a_2), (b_1, b_2)\}$, the complete set of eigenvalues of $-\Delta u_i = \alpha_i u_i$ are

$$\frac{4\pi^2}{(a_1b_2 - a_2b_1)^2} \left((m_2b_2 - m_1a_2)^2 + (m_1a_1 - m_2b_1)^2\right),$$

with corresponding eigenfunctions

$$c_{m_1, m_2} \cdot (\sin or \cos) \left(\frac{2\pi}{a_1b_2 - a_2b_1}((m_2b_2 - m_1a_2)x + (m_1a_1 - m_2b_1)y)\right),$$

for $m_1, m_2 \in \mathbb{Z}$. We want these eigenfunctions to have $L^2$-norm equal to 1, so we choose $c_{m_1, m_2} = \sqrt{2/((a_1b_2 - a_2b_1)^2)}$ if $|m_1| + |m_2| > 0$ and $a_{0, 0} = \sqrt{1/((a_1b_2 - a_2b_1)^2)}$.

When $\ell$ is even, by Lemma 2.3, we have $a_1 = nx_{\ell n}$, $a_2 = b_1 = 0$, $b_2 = y_{\ell n}$, and we can list these eigenvalues and eigenfunctions by choosing $m_1$ and $m_2$ in the following order: $(m_2, m_1) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (1, -1), (0, 2), (3, 0), (2, 1), (2, -1), (1, 2), (1, -2), (0, 3), ...$, always choosing sine first and cosine second. When $\ell$ is odd, by Lemma 2.3, we have $a_1 = nx_{\ell n}/2$, $a_2 = y_{\ell n}/2$, $b_1 = 0$, $b_2 = y_{\ell n}$, and again we can list the eigenvalues and eigenfunctions by choosing $m_1$ and $m_2$ so that $(2m_2 - m_1, m_1)$ has the following order: $(2m_2 - m_1, m_1) = (0, 0), (2, 0), (1, 1), (1, -1), (0, 2), (4, 0), (3, 1), (3, -1), (2, 2), (2, -2), (1, 3), (1, -3), (0, 4), ...$, always choosing sine first and cosine second. Again the eigenfunctions $u_i$ will form an orthonormal basis for $L^2$.

Note that, with these orderings, we do not necessarily have $\alpha_i \leq \alpha_{i'}$ for $i < i'$. However, we still have $\lim_{i \to \infty} \alpha_i = +\infty$.

Since we later refer to the $\alpha_i$ and $u_i$, without restating their definitions, an unambiguous understanding of their ordering is needed, so for clarity we list the first 13 $\alpha_i, u_i$ in Table 1.

3 Application of Courant’s nodal domain theorem

In this section, we consider a Jacobi function on each surface $\mathcal{W}_{\ell/n}$ produced from rotation about the surface’s central axis. Using the geometry of the surface, we can estimate the number of nodal domains of this function, and then the Courant nodal domain theorem gives the following lower bound for the index:

**Lemma 3.1** $\text{Ind}(\mathcal{W}_{\ell/n}) \geq 2n - 2$ if $\ell$ is odd, and $\text{Ind}(\mathcal{W}_{\ell/n}) \geq n - 2$ if $\ell$ is even.

**Proof.** Let $\vec{v}$ be the variational vector field of $\mathbb{R}^3$ associated to rotation about the axis of rotational symmetry of $\mathcal{W}_{\ell/n}$. The normal part $\langle \vec{v}, \vec{N} \rangle_{\mathbb{R}^3} \vec{N}$ is a Jacobi field on $\mathcal{W}_{\ell/n}$, that is, $v := \langle \vec{v}, \vec{N} \rangle_{\mathbb{R}^3}$ satisfies $\mathcal{L}(v) = 0$. Letting $k$ be the number of negative eigenvalues of $\mathcal{L}$, we may assume $v = v_{k+1}$.

For odd $\ell$ (resp. even $\ell$), the set $\{[x, y] \in \mathbb{C}/\Gamma \mid 2x/x_{\ell n} \in \mathbb{Z}\}$ is mapped by $\mathcal{X}$ to $2n$ (resp. $n$) closed planar geodesics in $\mathcal{W}_{\ell/n}$, each lying in a plane containing the axis of rotational
eigenfunctions

| α_j for ℓ odd | u_i for ℓ odd | eigenvalues | eigenfunctions | u_i for ℓ even |
|--------------|---------------|-------------|----------------|---------------|
| α_1 = 0      | u_1 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_1 = 0 | u_1 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_1 = 0 |
| α_2 = \frac{4 \pi^2}{n^2 x_n^2} | u_2 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_2 = \frac{4 \pi^2}{n^2 x_n^2} | u_2 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_2 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_3 = \frac{4 \pi^2}{n^2 x_n^2} | u_3 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_3 = \frac{4 \pi^2}{n^2 x_n^2} | u_3 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_3 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_4 = \frac{4 \pi^2}{n^2 x_n^2} | u_4 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_4 = \frac{4 \pi^2}{n^2 x_n^2} | u_4 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_4 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_5 = \frac{4 \pi^2}{n^2 x_n^2} | u_5 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_5 = \frac{4 \pi^2}{n^2 x_n^2} | u_5 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_5 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_6 = \frac{10 \pi^2}{n^2 x_n^2} | u_6 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_6 = \frac{10 \pi^2}{n^2 x_n^2} | u_6 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_6 = \frac{10 \pi^2}{n^2 x_n^2} |
| α_7 = \frac{16 \pi^2}{n^2 x_n^2} | u_7 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_7 = \frac{16 \pi^2}{n^2 x_n^2} | u_7 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_7 = \frac{16 \pi^2}{n^2 x_n^2} |
| α_8 = \frac{4 \pi^2}{n^2 x_n^2} | u_8 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_8 = \frac{4 \pi^2}{n^2 x_n^2} | u_8 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_8 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_9 = \frac{4 \pi^2}{n^2 x_n^2} | u_9 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_9 = \frac{4 \pi^2}{n^2 x_n^2} | u_9 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_9 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_10 = \frac{4 \pi^2}{n^2 x_n^2} | u_10 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_10 = \frac{4 \pi^2}{n^2 x_n^2} | u_10 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_10 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_11 = \frac{4 \pi^2}{n^2 x_n^2} | u_11 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_11 = \frac{4 \pi^2}{n^2 x_n^2} | u_11 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_11 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_12 = \frac{4 \pi^2}{n^2 x_n^2} | u_12 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_12 = \frac{4 \pi^2}{n^2 x_n^2} | u_12 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_12 = \frac{4 \pi^2}{n^2 x_n^2} |
| α_13 = \frac{4 \pi^2}{n^2 x_n^2} | u_13 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_13 = \frac{4 \pi^2}{n^2 x_n^2} | u_13 = \frac{1}{\sqrt{n x_n y_n}} \sin(\pi x/(nx_y)) \sqrt{n x_n y_n/2} | α_13 = \frac{4 \pi^2}{n^2 x_n^2} |

Table 1: The first 13 eigenvalues and eigenfunctions of \(-\Delta u_i = \alpha_i u_i\).
4 Bounds for index derived from bounds for \( V \)

As in the previous section, we again search for rough bounds for the index of the surfaces \( \mathcal{W}_{\ell/n} \). But instead of using a Jacobi function, this time we will use upper and lower bounds for the potential function, giving us both upper and lower bounds for the index. When the potential function is replaced by a constant upper or lower bound, we know the eigenvalues of the new resulting operator explicitly, leading to the lemma below. (This idea can be found in [FS], equation (15) of Chapter 6.)

Let \( \alpha_{\rho(1)}, \alpha_{\rho(2)}, \ldots \) be the complete set of eigenvalues with multiplicity 1 of the operator \(-\Delta \) on the flat torus \( \mathbb{C}/\Gamma \), defined as in section 2.3, but reordered by the permutation \( \rho(i) \) so that \( \alpha_{\rho(1)} \leq \alpha_{\rho(2)} \leq \cdots \rightarrow +\infty \). (We use the reordering \( \rho(i) \) only in this section.) Define the constants

\[
V_{\min} = \min_{[x,y] \in \mathbb{C}/\Gamma} V(x,y), \quad V_{\max} = \max_{[x,y] \in \mathbb{C}/\Gamma} V(x,y).
\]

Consider the operators \( L_{\min} = -\Delta - V_{\min} \) and \( L_{\max} = -\Delta - V_{\max} \). The complete set of eigenvalues with multiplicity 1 of \( L_{\min} \) (resp. \( L_{\max} \)) is then \( \alpha_{\rho(i)} = \alpha_{\rho(i)} - V_{\min} \) (resp. \( \alpha_{\rho(i)} = \alpha_{\rho(i)} - V_{\max} \)).

Recall that \( \beta_i \) is a complete set of eigenvalues with multiplicity 1 of \( L \). Comparing (2.3) with similar variational characterizations for the eigenvalues \( \alpha_{\rho(i)} \) and \( \beta_i \), we have

\[
\alpha_{\rho(i)} = \alpha_{\rho(i)} - V_{\min} \geq \beta_i \geq \alpha_{\rho(i)} - V_{\max} \quad \forall i.
\]

By Lemma 2.4, we conclude:

**Lemma 4.1** Choose \( \mu \in \mathbb{Z}^+ \) (resp. \( \nu \in \mathbb{Z}^+ \)) so that \( \alpha_{\rho(\mu)} < V_{\min} \) (resp. \( \alpha_{\rho(\nu)} < V_{\max} \)) and \( \alpha_{\rho(\mu+1)} \geq V_{\min} \) (resp. \( \alpha_{\rho(\nu+1)} \geq V_{\max} \)). Then

\[
\mu - 1 \leq \text{Ind}(\mathcal{W}_{\ell/n}) \leq \nu.
\]

Lemma 4.1 gives explicit upper and lower bounds for the index, since we know the \( \alpha_{\rho(i)} \) explicitly (see Table 2). Since \( V_{\max} \) is generally much larger than \( V_{\min} \), these bounds are very rough. However, when \( \ell/n \) is close to 1, the bounds get somewhat sharp, in the sense that the ratio of the lower and upper bounds is close to 1 (see, for example, the bounds for \( \mathcal{W}_{21/20} \) and \( \mathcal{W}_{71/72} \) in Table 2). We produce much sharper estimates in the next sections.

5 Specific spaces on which \( L \) is negative definite

We now begin considering the Jacobi operator restricted to finite dimensional subspaces. The essential point is that if the operator restricted to some \( N \)-dimensional subspace is negative definite, then the index of \( \mathcal{W}_{\ell/n} \) is at least \( N - 1 \). We will use subspaces generated by a finite number of eigenfunctions of the Laplacian. This will lead us to stronger lower bounds for the index.

Consider a finite subset \( \{ \bar{u}_1 = u_i_1, \ldots, \bar{u}_N = u_i_N \} \) of the eigenfunctions \( u_i \) of \(-\Delta \) on \( \mathbb{C}/\Gamma \) defined in section 2.3 with corresponding eigenvalues \( \tilde{\alpha}_j = \alpha_{i_j} \), \( j = 1, \ldots, N \). Consider any \( u = \sum_{i=1}^N a_i \bar{u}_i \in \text{span}\{ \bar{u}_1, \ldots, \bar{u}_N \} \), \( a_1, \ldots, a_N \in \mathbb{R} \).

\[
\int_{\mathbb{C}/\Gamma} u L u dxdy = \sum_{i,j=1}^N a_i a_j \int_{\mathbb{C}/\Gamma} \bar{u}_i (-\Delta \bar{u}_j - V \bar{u}_j) dxdy = \sum_{i,j=1}^N a_i (\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij}) a_j,
\]
where $b_{ij} := \int_{\mathbb{C}/\Gamma} V \tilde{u}_i \tilde{u}_j dx dy$. So $\int_{\mathbb{C}/\Gamma} u L u dx dy < 0$ for all nonzero $u \in \text{span}\{\tilde{u}_1, ..., \tilde{u}_N\}$ if and only if the matrix $\tilde{A}_N = (\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij})_{i,j=1,...,N}$ is negative definite. The remark following Lemma 2.4 then implies the following result:

**Theorem 5.1** If the $N \times N$ matrix $(\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij})_{i,j=1,...,N}$ is negative definite, then

$$\text{Ind}(W_{\ell/n}) \geq N - 1.$$  

So, for example, consider the symmetric Wente torus $W_{3/2}$. Choose $N = 9$ and choose $\tilde{u}_1 = u_1, \tilde{u}_2 = u_2, \tilde{u}_3 = u_3, \tilde{u}_4 = u_4, \tilde{u}_5 = u_5, \tilde{u}_6 = u_7, \tilde{u}_7 = u_8, \tilde{u}_8 = u_9,$ and $\tilde{u}_9 = u_{17}$. Then, with $H = 1/2$, the matrix $\tilde{A}_N$ is the following $9 \times 9$ negative definite matrix:

$$\tilde{A}_N \approx \begin{pmatrix}
-9.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -7.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -7.99 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.36 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -13.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8.70 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5.76 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.76 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.50 \\
\end{pmatrix},$$

Hence, by Theorem 5.1,

$$\text{Ind}(W_{3/2}) \geq 8.$$  

This is the best we can do for the $\ell/n = 3/2$ case; any $10 \times 10$ matrix of the form $(\tilde{\alpha}_j \delta_{ij} - \tilde{b}_{ij})_{i,j=1,...,10}$ will not be negative definite.

For $W_{4/3}$ we use $N = 10$ and the functions $u_1,...,u_9,u_{13}$ for $\tilde{u}_1,...,\tilde{u}_{10}$, producing the negative definite matrix (with $H = 1/2$)

$$\tilde{A}_N \approx \begin{pmatrix}
-5.17 & 0 & 0 & 0 & 0 & 0 & 0 & -3.23 & 0 \\
0 & -3.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3.53 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3.78 & 0 & -2.29 & 0 & 0 & 0 \\
0 & 0 & 0 & -3.78 & -2.29 & -2.29 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.91 & -2.29 & -3.78 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2.29 & -3.78 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3.78 & 0 & 0 & 0 \\
-3.23 & 0 & 0 & 0 & 0 & 0 & -2.21 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.97 \\
\end{pmatrix},$$

Thus

$$\text{Ind}(W_{4/3}) \geq 9.$$  

For other surfaces we can use the same technique. Without showing the resulting matrices, we provide here some other examples:

- For $W_{5/3}$ we can use $N = 12$ and the functions $u_1,u_2,u_3,u_5,...,u_9,u_{15},u_{16},u_{17},u_{29}$.

- For $W_{5/4}$ we can use $N = 33$ and $u_1,...,u_{23},u_{27},...,u_{35},u_{45}$.

- For $W_{7/4}$ we can use $N = 16$ and $u_1,u_2,u_3,u_5,...,u_9,u_{14},...,u_{17},u_{27},u_{28},u_{29},u_{45}$.

- For $W_{6/5}$ we can use $N = 20$ and $u_1,...,u_{19},u_{29}$.
Table 2: Rough upper and lower bounds for $\text{Ind}(W_{t/n})$. ($x_{tn}$, $y_{tn}$, $V_{min}$, and $V_{max}$ are computed using the value $H = 1/2$.)

For $W_{7/5}$ we can use $N = 27$ and $u_1, \ldots, u_{11}, u_{14}, \ldots, u_9, u_{26}, \ldots, u_{31}, u_{43}, u_{44}, u_{45}, u_{65}$.

For $W_{8/5}$ we can use $N = 12$ and $u_1, \ldots, u_7, u_{10}, \ldots, u_{13}, u_{29}$.

For $W_{9/5}$ we can use $N = 20$ and $u_1, u_2, u_3, u_5, \ldots, u_9, u_{14}, \ldots, u_{17}, u_{26}, \ldots, u_{29}, u_{43}, u_{44}, u_{45}, u_{65}$.

In each case, $A_N$ will be negative definite. $A_N$ is a diagonal matrix for $W_{3/2}, W_{5/3}, W_{7/4}$, and $W_{9/5}$, and is not diagonal for $W_{4/3}, W_{5/4}, W_{6/5}, W_{7/5}$, and $W_{8/5}$. We conclude that

$$\text{Ind}(W_{5/3}) \geq 11, \quad \text{Ind}(W_{5/4}) \geq 32, \quad \text{Ind}(W_{7/4}) \geq 15, \quad \text{Ind}(W_{6/5}) \geq 19,$$

$$\text{Ind}(W_{7/5}) \geq 26, \quad \text{Ind}(W_{8/5}) \geq 11, \quad \text{Ind}(W_{9/5}) \geq 19.$$

We could continue in this way for more $W_{t/n}$, but many computations are required to find sharp lower bounds. However, if we are interested in rougher lower bounds, then we may continue without so many computations. For example, for each of the surfaces $W_{8/7}, W_{10/7}$, and $W_{12/7}$, we can use $N = 9$ and the functions $u_1, \ldots, u_5, u_{10}, \ldots, u_{13}$ to create a negative definite diagonal matrix $A_N$, thus

$$\text{Ind}(W_{8/7}) \geq 8, \quad \text{Ind}(W_{10/7}) \geq 8, \quad \text{Ind}(W_{12/7}) \geq 8.$$
From Lemma 3.1 and the above applications of Theorem 5.1, we conclude:

\[ \text{Ind}(W_{l/n}) \leq 7 \text{ for all } W_{l/n}. \]

**Remark.** In fact, numerical results in section 7 suggest that \( \text{Ind}(W_{l/n}) \geq 9 \) for all \( W_{l/n}. \)

Since the symmetric Wente tori are possible candidates for minimizers of index amongst all closed unstable CMC surfaces, one may conjecture that, except for the round sphere, there does not exist a closed CMC surface in \( \mathbb{R}^3 \) with index less than 9.

**6 A method for computing the index sharply**

In the previous section, we found finite dimensional subspaces on which the restricted operator is negative definite. Now, we will still use finite dimensional subspaces, but we will no longer confine ourselves to subspaces for which the operator becomes negative definite. Instead we will use the number of negative eigenvalues of the restricted operators on the subspaces to estimate the index of the surfaces from below. Again, as they are convenient to work with, we will use subspaces generated by eigenfunctions of the Laplacian. Furthermore, we will show that if we choose a sequence of subspaces whose union is dense in the full space, then the eigenvalues of the restricted operators converge to the eigenvalues of the original operator, providing us with a numerical algorithm for estimating the eigenvalues sharply.

Let \( P_m \) be orthogonal projection of \( L^2 \) to \( \mathcal{V}_m := \text{span}\{u_1, ..., u_m\} \), the span of the first \( m \) eigenfunctions \( u_i \) (as ordered in section 2.3) of \( -\Delta \). Hence

\[ P_m(u) = \sum_{i=1}^{\infty} a_i u_i = \sum_{i=1}^{m} \langle u, u_i \rangle_{L^2} u_i. \]

Note that \( P_m v_j \to_{L^2} v_j \) as \( m \to \infty \) for any eigenfunction \( v_j \) of \( \mathcal{L} \).

Let \( \mathcal{L}_m = P_m \circ \mathcal{L} \circ P_m \). Then

\[ \mathcal{L}_m \left( u = \sum_{i=1}^{\infty} a_i u_i \right) = P_m \circ \left( \sum_{i=1}^{m} a_i \mathcal{L} u_i \right) = \sum_{i,j=1}^{m} a_i (\alpha_{ij} - b_{ij}) u_j, \]

where

\[ b_{ij} := \int_{\mathbb{C}/\Gamma} Vu_i u_j dxdy. \]

So, on \( \mathcal{V}_m, \mathcal{L}_m \) is the linear transformation \( A_m = (\alpha_{ij} - b_{ij})_{1 \leq i,j \leq m} \) with respect to the basis \( \{u_i\}_{i=1}^{m} \). The next lemma states that the linear map \( A_m \) is the restriction of \( \mathcal{L} \) to \( \mathcal{V}_m: \)

**Lemma 5.1** Let \( \dot{x} = (x_1, ..., x_m), \dot{y} = (y_1, ..., y_m) \in \mathbb{R}^m \). Let \( f = \sum_{i=1}^{m} x_i u_i, g = \sum_{i=1}^{m} y_i u_i. \) Then

\[ \langle \dot{x}, A_m \dot{y} \rangle_{\mathbb{R}^m} = \int_{\mathbb{C}/\Gamma} f \mathcal{L} g dxdy \quad \text{and} \quad \langle \dot{x}, \dot{y} \rangle_{\mathbb{R}^m} = \int_{\mathbb{C}/\Gamma} f g dxdy. \]

**Proof.** \( \int_{\mathbb{C}/\Gamma} f \mathcal{L} g dxdy = \sum_{i,j=1}^{m} x_i y_j (\alpha_{ij} - b_{ij}) = \langle \dot{x}, A_m \dot{y} \rangle_{\mathbb{R}^m}. \) Similarly, \( \langle \dot{x}, \dot{y} \rangle_{\mathbb{R}^m} = \int_{\mathbb{C}/\Gamma} f g dxdy. \)

Let \( \lambda_1^{(m)} \leq ... \leq \lambda_{m}^{(m)} \) be the eigenvalues of \( A_m \) with corresponding orthonormal eigenvectors \( u_1^{(m)}, ..., u_{m}^{(m)} \). Note that since \( A_m \) is a symmetric matrix, \( \lambda_i^{(m)} \in \mathbb{R} \) for all \( i \). The following lemma is known ([RS], Theorem VIII.3, or [FS], equation (23) in Chapter 6), but we include a brief proof here.
Lemma 6.2 $\lambda_j^{(m)} \geq \lambda_j^{(m')} \geq \beta_j$ for all $j$, $m$, $m'$ such that $j \leq m \leq m'$.

Proof. We have the following variational characterization:

$$
\lambda_j^{(m)} = \sup_{\phi_1, \ldots, \phi_{j-1} \in V_m} \left( \inf_{\psi \in V_m \cap \text{span}\{\phi_1, \ldots, \phi_{j-1}\}} \int_{\Gamma} \psi L_m \psi \, dx \, dy \right)
$$

$$
= \sup_{\phi_1, \ldots, \phi_{j-1} \in L^2} \left( \inf_{\psi \in V_m \cap \text{span}\{\phi_1, \ldots, \phi_{j-1}\}} \int_{\Gamma} \psi (P_m \circ L \circ P_m(\psi)) \, dx \, dy \right).
$$

However, for $\psi \in V_m$ we have

$$
\int_{\Gamma} \psi (P_m \circ L \circ P_m(\psi)) \, dx \, dy = \int_{\Gamma} \psi L \psi \, dx \, dy,
$$

so

$$
\lambda_j^{(m)} = \sup_{\phi_1, \ldots, \phi_{j-1} \in L^2} \left( \inf_{\psi \in L^2 \cap \text{span}\{\phi_1, \ldots, \phi_{j-1}\}} \int_{\Gamma} \psi L \psi \, dx \, dy \right).
$$

So $\lambda_j^{(m)}$ is nonincreasing as $m$ increases. Also, $\beta_j$ has the following characterization [U]:

$$
\beta_j = \sup_{\phi_1, \ldots, \phi_{j-1} \in L^2} \left( \inf_{\psi \in L^2 \cap \text{span}\{\phi_1, \ldots, \phi_{j-1}\}} \int_{\Gamma} \psi L \psi \, dx \, dy \right).
$$

Hence $\lambda_j^{(m)} \geq \beta_j$ for all $m$. \hfill \Box

We define

$$
\gamma_j := \lim_{m \to \infty} \lambda_j^{(m)}.
$$

This limit clearly exists, since $\{\lambda_j^{(m)}\}_{m=j}^\infty$ is nonincreasing and bounded below by $\beta_j$.

Theorem 6.1 $\gamma_j = \beta_j$ for all $j$.

Remark. The motivation for this theorem is that it gives us a method for estimating the eigenvalues $\beta_j$ of $L$, since one can estimate $\gamma_j$ numerically. Thus one can estimate the number of negative eigenvalues $\beta_j < 0$, and hence the index (see section 7). \hfill \Box

Before proving Theorem 6.1, we prove a crucial preliminary lemma, which essentially says that the Rayleigh quotient

$$
\frac{\int_{\Gamma} (P_m v_j)(P_m v_j) \, dx \, dy}{\int_{\Gamma} (P_m v_j)^2 \, dx \, dy}
$$

of $P_m v_j$ converges to the Rayleigh quotient

$$
\frac{\int_{\Gamma} v_j L v_j \, dx \, dy}{\int_{\Gamma} (v_j)^2 \, dx \, dy} = \beta_j
$$

of $v_j$ as $m \to \infty$, for all $j$.

Lemma 6.3 $\int_{\Gamma} (P_m v_j)(P_m v_j) \, dx \, dy \to \int_{\Gamma} v_j L v_j \, dx \, dy = \beta_j$ as $m \to \infty$ for all $j$. 

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Proof. \((-\Delta v_j, v_j)_{L^2} = \sum_{l=1}^{\infty} \langle -\Delta ((v_j, u_i)L^2u_i), \langle v_j, u_i\rangle_{L^2L^2} \rangle_{L^2} = \sum_{i=1}^{\infty} \langle \nu, u_i \rangle_{L^2L^2} \alpha_i \leq \infty.\)
Similarly, \((-\Delta P_m v_j, P_m v_j)_{L^2} = \sum_{i=1}^{\infty} \langle v_j, u_i \rangle_{L^2L^2} \alpha_i, \) hence

\[
\int_{\mathbb{C}/\Gamma} (P_m v_j)(-\Delta P_m v_j) dx dy = \sum_{i=1}^{m} \langle v_j, u_i \rangle_{L^2L^2} \alpha_i \to \sum_{i=1}^{\infty} \langle v_j, u_i \rangle_{L^2L^2} \alpha_i = \int_{\mathbb{C}/\Gamma} v_j(-\Delta v_j) dx dy
\]
as \(m \to \infty.\)

We complete the proof by showing \(\int_{\mathbb{C}/\Gamma} V(P_m v_j)^2 dx dy \to \int_{\mathbb{C}/\Gamma} Vv_j^2 dx dy\) as \(m \to \infty.\) Let \(M^+ = \{[x, y] \in \mathbb{C}/\Gamma \mid (P_m v_j)^2 - v_j^2 \geq 0\},\) and let \(M^- = \{[x, y] \in \mathbb{C}/\Gamma \mid (P_m v_j)^2 - v_j^2 < 0\}.\)

\[
\int_{\mathbb{C}/\Gamma} (P_m v_j - v_j)^2 dx dy \to 0 \Rightarrow \int_{M^+} (P_m v_j - v_j)^2 dx dy \to 0 \text{ and } \int_{M^-} (P_m v_j - v_j)^2 dx dy \to 0\]
as \(m \to \infty,\) so for \(M^*\) equal to either \(M^+\) or \(M^-;\) we have

\[
\int_{M^*} ((P_m v_j)^2 - v_j^2) dx dy = \int_{M^*} ((P_m v_j)^2 - v_j^2) dx dy = \int_{M^*} (P_m v_j - v_j)^2 + 2v_j(P_m v_j - v_j) dx dy
\]

by the Cauchy-Schwarz inequality. Hence

\[
\int_{\mathbb{C}/\Gamma} V((P_m v_j)^2 - v_j^2) dx dy \leq V_{\text{max}} \int_{\mathbb{C}/\Gamma} ((P_m v_j)^2 - v_j^2) dx dy = V_{\text{max}} \int_{M^*} ((P_m v_j)^2 - v_j^2) dx dy \to 0 \text{ as } m \to \infty.
\]

Proof of Theorem 6.1. Since \(w_i^{(m)}\) is a linear combination of \(u_j\) for \(j \leq m,\) we may consider \(w_i^{(m)}\) as a function in \(C^\infty(\mathbb{C}/\Gamma).\) Then \(\int_{\mathbb{C}/\Gamma} u_{\mu}^{(m)} \mathcal{L} u_{\nu}^{(m)} dx dy = \langle u_{\mu}^{(m)}, A_{\nu} u_{\nu}^{(m)} \rangle_{\mathbb{R}^m} = \lambda_{\mu}^{(m)} \delta_{\mu\nu} \) and \(\int_{\mathbb{C}/\Gamma} u_{\mu}^{(m)} w_i^{(m)} dx dy = \langle u_{\mu}^{(m)}, w_i^{(m)} \rangle_{\mathbb{R}^m} = \delta_{\mu i},\) by Lemma 6.1. Choose \(a_{i,j}^{(m)} \in \mathbb{R}\) so that \(P_m v_j = \sum_{i=1}^{m} a_{i,j}^{(m)} w_i^{(m)}\) for all \(m\) and \(j,\) then

\[
\int_{\mathbb{C}/\Gamma} (P_m v_j) \mathcal{L}(P_m v_j) dx dy = \sum_{i=1}^{m} (a_{i,j}^{(m)})^2 \lambda_i^{(m)} \to \beta_j \quad (6.4)
\]
by Lemma 6.3, and

\[
\langle P_m v_j, P_m v_j \rangle = \sum_{i=1}^{m} (a_{i,j}^{(m)})^2 \to 1 \quad (6.5)
\]
for all \(j \in \mathbb{Z}^+\) as \(m \to \infty.\)

Let \(P_l^m\) be the orthogonal projection from \(L^2\) to \(\text{span}\{w_1^{(m)}, ..., w_l^{(m)}\}.\)

Choose \(k_1, k_2, ...\) such that \(\beta_1 = ... = \beta_{k_1} < \beta_{k_1+1} = ... = \beta_{k_2} < \beta_{k_2+1} = ... = \beta_{k_3} < \beta_{k_3+1} = ...\). Note that \(k_1 = 1,\) since Courant’s nodal domain theorem implies that any function in the eigenspace of the first eigenvalue has only one nodal domain, hence this eigenspace cannot contain two \(L^2\)-orthogonal functions.
We will prove the result by induction on \(i\), the subscript of \(k_i\), for \(i \in \mathbb{Z}^+\).

**Base case:** \(\gamma_1 = \beta_1\), and \(P^m v_1 \to L^2 v_1\) as \(m \to \infty\).

We have

\[
0 \leq (a_{1,1}^{(m)})^2 (\gamma_1 - \beta_1) + \left( \sum_{\rho=2}^{m} (a_{\rho,1}^{(m)})^2 \right) (\beta_2 - \beta_1)
\]

\[
\leq (a_{1,1}^{(m)})^2 (\gamma_1 - \beta_1) + \left( \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \right) - 1 \beta_1 + \left| \left( \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \right) - 1 \right| \cdot |\beta_1| + \left( \sum_{\rho=2}^{m} (a_{\rho,1}^{(m)})^2 \right) (\beta_2 - \beta_1)
\]

\[
= (a_{1,1}^{(m)})^2 \gamma_1 - \beta_1 + \left( \sum_{\rho=2}^{m} (a_{\rho,1}^{(m)})^2 \right) \beta_2 + \left| \left( \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \right) - 1 \right| \cdot |\beta_1|
\]

\[
\leq \left( \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \lambda_\rho^{(m)} \right) - \beta_1 + \left| \left( \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \right) - 1 \right| \cdot |\beta_1| \to 0
\]

as \(m \to \infty\), by equations (6.4) and (6.5). Therefore, since \(\beta_2 - \beta_1 > 0\), we have

\[
||P^m v_1 - P_m v_1||^2_{L^2} = \sum_{\rho=1}^{m} (a_{\rho,1}^{(m)})^2 \to 0,
\]

\[
(a_{1,1}^{(m)})^2 \to 1, \text{ and } (a_{1,1}^{(m)})^2 (\gamma_1 - \beta_1) \to 0 \text{ as } m \to \infty.
\]

Thus \(\gamma_1 = \beta_1\), and \(||P^m v_1 - v_1||_{L^2} \leq ||P^m v_1 - P_m v_1||_{L^2} + ||P_m v_1 - v_1||_{L^2} \to 0 \text{ as } m \to \infty\).

**Inductive step:** If \(\gamma_j = \beta_j\) and \(P^m v_j \to L^2 v_j\) as \(m \to \infty\) for all \(j \leq k_i\), then \(\gamma_j = \beta_j\) and \(P^m v_j \to L^2 v_j\) as \(m \to \infty\) for all \(j \leq k_{i+1}\).

Since \(P^m v_j \to L^2 v_j\) as \(m \to \infty\) for all \(j \leq k_i\), we have \(P^m v_j \to L^2 0\) for all \(j \in [k_i+1, k_{i+1}]\) as \(m \to \infty\) (since the \(v_j\) are orthonormal), so

\[
a_{i,j}^{(m)} \to 0 \text{ for all } l \leq k_i \text{ and all } j \in [k_i + 1, k_{i+1}].
\]

Then for all \(j \in [k_i + 1, k_{i+1}]\), we have

\[
0 \leq \left( \sum_{\rho=k_i+1}^{k_i+1} (a_{\rho,j}^{(m)})^2 (\gamma_\rho - \beta_\rho) \right) + \left( \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \right) (\beta_{k_{i+1}+1} - \beta_{k_i+1})
\]

\[
\leq \left( \sum_{\rho=1}^{k_i} (a_{\rho,j}^{(m)})^2 (\gamma_\rho - \beta_\rho) \right) + \left( \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \right) (\beta_{k_{i+1}+1} - \beta_{k_i+1}) +
\]

\[
\left( \sum_{\rho=1}^{k_i} (a_{\rho,j}^{(m)})^2 (\gamma_\rho - \beta_\rho) \right) + \left( \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \right) - 1 \beta_{k_{i+1}+1} + \left( \sum_{\rho=1}^{k_i} (a_{\rho,j}^{(m)})^2 |\beta_\rho| \right) +
\]

\[
\left| \left( \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \right) - 1 \right| \cdot |\beta_{k_{i+1}+1}| = \sum_{\rho=1}^{k_i} (a_{\rho,j}^{(m)})^2 (\gamma_\rho - \beta_{k_{i+1}+1}) +
\]

\[
\left( \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \right) \beta_{k_{i+1}+1} + \left( \sum_{\rho=1}^{k_i} (a_{\rho,j}^{(m)})^2 |\beta_\rho| \right) +
\]

\[
15
\]
\[
|\left(\sum_{\rho=k_i+1}^{k_{i+1}} (a_{\rho,j}^{(m)})^2\right) - 1| \cdot |\beta_{k_i+1}| \leq \left(\sum_{\rho=1}^{k_{i+1}} (a_{\rho,j}^{(m)})^2 + \lambda^{(m)}\right) - \beta_{k_i+1} + \\
\left(\sum_{\rho=1}^{k_{i+1}} (a_{\rho,j}^{(m)})^2 |\beta_{\rho}|\right) + \left(\sum_{\rho=k_i+1}^{k_{i+1}} (a_{\rho,j}^{(m)})^2\right) - 1| \cdot |\beta_{k_i+1}| \to 0
\]
as \(m \to \infty\), by (6.4), (6.5), and (6.6). Therefore, since \(\beta_{k_i+1} - \beta_{k_i+1} > 0\), we have

\[\sum_{\rho=k_i+1}^{k_{i+1}} (a_{\rho,j}^{(m)})^2 (\gamma_\rho - \beta_\rho) \to 0 \quad (6.7)\]
and

\[\|P_{k_{i+1}} v_j - P_m v_j\|_2^2 = \sum_{\rho=k_{i+1}+1}^{m} (a_{\rho,j}^{(m)})^2 \to 0 \quad (6.8)\]
as \(m \to \infty\).

By (6.8) we have \(\|P_{k_{i+1}} v_j - v_j\|_2 \leq \|P_{k_{i+1}} v_j - P_m v_j\|_2 + \|P_m v_j - v_j\|_2 \to 0\) as \(m \to \infty\), hence \(P_{k_{i+1}}^m v_j \to v_j\) as \(m \to \infty\) for all \(j \in [k_i+1, k_{i+1}]\). Since, by the inductive assumption, \(P_{k_{i+1}}^m v_j \to v_j\) as \(m \to \infty\) for all \(j \leq k_i\), it follows that also \(P_{k_{i+1}}^m v_j \to v_j\) as \(m \to \infty\) for all \(j \leq k_i\). Hence \(P_{k_{i+1}}^m v_j \to v_j\) as \(m \to \infty\) for all \(j \leq k_{i+1}\).

Since \(P_{k_{i+1}}^m v_j \to v_j\) as \(m \to \infty\) for all \(j \leq k_{i+1}\), we have \(\langle P_{k_{i+1}}^m v_\mu, P_{k_{i+1}}^m v_\nu \rangle_{L_2} \to \delta_{\mu\nu}\) as \(m \to \infty\) for all \(\mu, \nu \leq k_{i+1}\). By (6.6), \(\sum_{k_{i+1}=k_i+1}^{k_{i+1}} a_{\mu,\mu}^{(m)} a_{\nu,\nu}^{(m)} = \langle P_{k_{i+1}}^m v_\mu, P_{k_{i+1}}^m v_\nu \rangle_{L_2} - \sum_{l=1}^{k_i} a_{\mu,l}^{(m)} a_{\nu,l}^{(m)} \to \delta_{\mu\nu}\) as \(m \to \infty\) when \(\mu, \nu \in [k_i+1, k_{i+1}]\). So the matrix \(a_{\mu,\nu}^{(m)}\) is almost orthogonal for large \(m\). This implies, as (6.7) holds for all \(j \in [k_i+1, k_{i+1}]\), that \(\gamma_{k_i+1} = \cdots = \gamma_{k_{i+1}} = \beta_{k_{i+1}}\).

\[\square\]

7 Numerical computation of the index

At first glance, the numerical computation of \(A_m\) seems to involve a separate computation for each \(b_{ij}\) for \(1 \leq i \leq j \leq m\). Each \(b_{ij}\) is essentially a triple integral, since \(V\) is defined via the Jacobi \(cn\) function, and \(cn\) is defined via an integral. Numerically estimating \(m(m+1)/2\) triple integrals \(b_{ij}\) would require much computer time. However, most of the \(b_{ij}\) are 0, and there are clear relationships between many of the nonzero \(b_{ij}\). So one wishes to find a minimal set of integrals that will determine all \(b_{ij}\), hopefully having much less than \(m(m+1)/2\) integrals, thus reducing the amount of required numerical computation. This is the purpose of this section.

When \(\ell\) is odd, we have

\[b_{11} = \frac{1}{n x_{\ell n} y_{\ell n}} \int_0^{n x_{\ell n}} \int_0^{y_{\ell n}} V(x,y)dydx\ ,\]
and

\[b_{ij} = \frac{\sqrt{2}}{n x_{\ell n} y_{\ell n}} \int_0^{n x_{\ell n}} \int_0^{y_{\ell n}} V(x,y)(\sin or \cos)\left(\frac{2\pi ax}{n x_{\ell n}} + \frac{2\pi by}{y_{\ell n}}\right)dydx\ ,\]
for some \(a, b \in \mathbb{Z}\) when \(j \geq 2\), and

\[b_{ij} = \frac{2}{n x_{\ell n} y_{\ell n}} \int_0^{n x_{\ell n}} \int_0^{y_{\ell n}} V(x,y)(\sin or \cos)\left(\frac{2\pi ax}{n x_{\ell n}} + \frac{2\pi by}{y_{\ell n}}\right)(\sin or \cos)\left(\frac{2\pi cx}{n x_{\ell n}} + \frac{2\pi dy}{y_{\ell n}}\right)dydx\ ,\]
for some \(a, b, c, d \in \mathbb{Z}\) when \(i, j \geq 2\). When \(\ell\) is even, we have
\[
b_{11} = \frac{2}{n x_{\ell} y_{\ell}} \int_0^{n x_{\ell}/2} \int_0^{y_{\ell}} V(x, y) dy dx,
\]
and
\[
b_{1j} = \frac{2\sqrt{2}}{n x_{\ell} y_{\ell}} \int_0^{n x_{\ell}/2} \int_0^{y_{\ell}} V(x, y) (\sin \text{ or } \cos) \left(\frac{2\pi ax}{n x_{\ell}} + \frac{2\pi by}{y_{\ell}}\right) dy dx,
\]
for some \(a, b \in \mathbb{Z}\) with \(a + b\) even when \(j \geq 2\), and
\[
b_{ij} = \frac{4}{n x_{\ell} y_{\ell}} \int_0^{n x_{\ell}/2} \int_0^{y_{\ell}} V(x, y) (\sin \text{ or } \cos) \left(\frac{2\pi ax}{n x_{\ell}} + \frac{2\pi by}{y_{\ell}}\right) dy dx \left(\sin \text{ or } \cos\right) \left(\frac{2\pi cx}{n x_{\ell}} + \frac{2\pi dy}{y_{\ell}}\right) dx dy,
\]
for some \(a, b, c, d \in \mathbb{Z}\) with \(a + b\) and \(c + d\) even when \(i, j \geq 2\). When \(\ell\) is even, we have changed the domain of integration from \(\{(x, y) \mid 0 \leq x \leq n x_{\ell}/2, y_{\ell}/n x_{\ell} \leq y \leq y_{\ell}/n x_{\ell} + y_{\ell}\}\) to \(\{(x, y) \mid 0 \leq x \leq n x_{\ell}/2, 0 \leq y \leq y_{\ell}\}\), as this will not affect the value of the integral.

We list the following facts without proof, as they follow easily from the symmetries \(V(x, y) = V(-x, y) = V(x, -y) = V((x_{\ell}/2) - x, y) = V(x, (y_{\ell}/2) - y)\) of \(V\).

- If \(i + j\) is odd, then \(b_{ij} = b_{ji} = 0\).
- If \(j\) is odd, and if \(a\) and \(b\) are not both even, then \(b_{ij} = b_{ji} = 0\). 

Table 3: Sharper lower bounds and numerical estimates of \(\text{Ind}(W_{\ell/n})\). (The values in the final two columns are computed using \(H = 1/2\).)
For $b_{ij} \neq 0$,

\[
b_{ij} = b_{ji} = \sum_{0 \leq A, B \leq |a|+|b|+|c|+|d|, \ A, B \ \text{even}} c_{i,j,A,B} \cdot I(A, B),
\]

where $c_{i,j,A,B} \in \mathbb{Z}$ if $i = j = 1$ or $i, j \geq 2$, and $c_{i,j,A,B}/\sqrt{2} \in \mathbb{Z}$ if precisely one of $i$ or $j$ is 1, and

\[
I(A, B) := \frac{1}{nx_{tn}y_{tn}} \int_0^{nx_{tn}} \int_0^{y_{tn}} V(x, y)(\cos \frac{2\pi x}{nx_{tn}})^A(\cos \frac{2\pi y}{y_{tn}})^B \, dx \, dy.
\]

The $c_{i,j,A,B}$ are easily determined by repeated use of the identities $\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$, $\sin(\alpha \pm \beta) = \cos(\alpha) \sin(\beta) \pm \cos(\beta) \sin(\alpha)$, and $\sin^2(\alpha) = 1 - \cos^2(\alpha)$.

So, for $\ell$ odd (resp. $\ell$ even), if we compute $A_m = 2\ell^2 - 2\ell + 1$ (resp. $A_m = 4\ell^2 - 4\ell + 1$) for some $l \in \mathbb{Z}^+$, then we need compute only $l(l + 1)/2$ (resp. $2\ell^2 - \ell$) integrals of the form $I(A, B)$. 

**Remark.** We can compute the number of negative eigenvalues of $A_m$. By Theorem 6.1 and Lemma 2.4 this number or one less than this number estimates $\text{Ind}(W_{\ell/n})$. These estimates are listed in the third column of Table 3. The choice of $A_m$ is listed in this table’s fourth column. For example, for $W_{3/2}$, we chose $A_m = A_{181}$, and computed numerically that $A_{181}$ has 11 negative eigenvalues. Thus $\text{Ind}(W_{3/2})$ is estimated to be either 10 or 11, as written in the third column. And those 11 negative eigenvalues range from $\lambda_{11}^{(181)} \approx -35.4$ to $\lambda_{11}^{(181)} \approx -1.47$ when $H = 1/2$, as written in the fifth column. (Changing $H$ changes the eigenvalues by a factor of $H$, so the number of negative eigenvalues is independent of $H$.)

The size of $A_m$ varies with the choice of $\ell/n$, since we choose $A_m$ as large as possible in each case without creating significant numerical errors. (Note that as the size of $A_m$ increases, the $c_{i,j,A,B}$ quickly become very large, hence even very small numerical errors in the estimates of $I(A, B)$ can result in large numerical errors, for large $m$.)

By Lemma 2.5, $\text{Null}(W_{\ell/n}) \geq 6$, hence the first six nonnegative eigenvalues of $A_m$ will converge to zero as $m \to \infty$. This provides some indication of how close the eigenvalues of $A_m$ are to the first $m$ eigenvalues of $\mathcal{L}$. Hence, in Table 3’s final column, we include the range of the first six positive eigenvalues of $A_m$. For example, the first six positive eigenvalues of the matrix $A_{181}$ for $W_{3/2}$ range from $\lambda_{12}^{(181)} \approx 0.059$ to $\lambda_{17}^{(181)} \approx 0.65$ when $H = 1/2$. \hfill $\Box$

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