ON TANGENTS TO CURVES

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ABSTRACT: In this paper, we give a simple definition of tangents to a curve in elementary geometry. From which, we characterize the existence of the tangent to a curve at a point.

1 Introduction

It has long been known that the notion of tangents to a curve is one of most important notions of analytic geometry and classical analytic. The first definition of tangents was "a right line which touches a curve, but which when produced, does not cut it" [7]. This old definition prevents inflection points from having any tangent. It has been dismissed and the modern definitions are equivalent to those of Leibniz. Pierre de Fermat developed a general technique for determining tangents of a curve by using his method of adequality in the 1630s. Leibniz defined a tangent line as a line through a pair of infinitely close points on the curve (see e.g. [8]).

The notion of tangents to an arbitrary curve can be traced back to the work of Archimedes in the third century B.C, when he solved the problem of finding tangents to spirals. The geometric idea of tangent lines as the limit of secant lines serves as the motivation for analytical methods that are used to find tangent lines explicitly (see e.g. [8]). The question of finding tangent lines to a graph, or the tangent line problem, was one of the central questions leading to the development of calculus in the 17th century. In the second book of his Geometry [1], René Descartes said of the problem of constructing tangents to a curve. "And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know" [6].

Up to now, in analytic geometry and classical analytic, one use the following definition of tangents (see e.g. [2, 3, 4, 5]).

Definition 1.1. Consider the sequence of straight lines (secant lines) passing through two points, A and M, those that lie on C. The tangent to C at A is defined as the limiting position of the secant line AM as M tends to A along the curve C.

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However, in elementary geometry, this definition is still very hard to explain, because what is "the limiting position of secant lines"? And how to overcome this problem?

In this paper, we first would like to give a simple definition of tangents to a curve in elementary geometry. Next, we characterize the existence of tangents to a curve in the space and the relationship between the existence of tangents and the derivative.

2 The definition of tangents

In elementary geometry, it is very difficult to make the notion about "the limiting position of secant lines" in Definition 1.1 accurate. Because one need to put lines in a metric space or a topological space to consider the limit of them.

Overcoming this problem is based on the idea of the definition of tangents in [2] by Flett and remark that: each line passing through a fixed point is determined by its direction vectors. This view helps us move considering "the limiting position of secant lines" to considering "the limiting position of direction vectors". But the direction vectors of a line must be not zero.

The remark above leads us to considering "the limiting position of direction vectors of a stable length".

That is why we give the following definition.

Definition 2.1. Let $L$ be a curve in the space and a point $A \in L$. Consider an arc of $L$ containing $A$ which is divided into two parts $L_1$ and $L_2$ by $A$ such that $L_1$ and $L_2$ intersect at only $A$, and in each part there always exist points that are not $A$. For each point $M \in L$ and $M \neq A$, the secant line $AM$ has a unit vector $\frac{\overrightarrow{AM}}{|\overrightarrow{AM}|}$. Then if the limits

$$\lim_{M \rightarrow A} \frac{\overrightarrow{AM}}{|\overrightarrow{AM}|} \quad \text{and} \quad \lim_{M \rightarrow A} \frac{\overrightarrow{AM}}{|\overrightarrow{AM}|}$$

exist and they are collinear vectors, we call that $L$ has the tangent at $A$, and both these limits are direction vectors of this tangent.
3 The condition for the existence of tangents

Suppose that a curve $L$ has the parametric equation
\[
\begin{align*}
  x &= x(t) \\
  y &= y(t) \\
  z &= z(t)
\end{align*}
\]
and $A(x(t_0), y(t_0), z(t_0)) \in L$. Let $M(x(t), y(t), z(t)) \in L$, where $t \neq t_0$. Set
\[
\Delta t = t - t_0; \Delta x = x(t) - x(t_0); \Delta y = y(t) - y(t_0); \Delta z = z(t) - z(t_0).
\]
Then we have
\[
\frac{AM}{|AM|} = \frac{\Delta t}{|\Delta t|} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right).
\]
From this equation it follows that $L$ has the tangent at $A$ if and only if the following limits
\[
\lim_{\Delta t \to 0^+} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)
\]
and
\[
\lim_{\Delta t \to 0^-} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)
\]
exist and are collinear vectors.

Hence we obtain the following theorem.

**Theorem 3.1.** Let $L$ be a curve having the parametric equation
\[
\begin{align*}
  x &= x(t) \\
  y &= y(t) \\
  z &= z(t)
\end{align*}
\]
and $A(x(t_0), y(t_0), z(t_0)) \in L$. Set
\[
\Delta t = t - t_0; \Delta x = x(t) - x(t_0); \Delta y = y(t) - y(t_0); \Delta z = z(t) - z(t_0).
\]
Then $L$ has the tangent at $A$ if and only if the following limits
\[
\lim_{\Delta t \to 0^+} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)
\]
and
\[
\lim_{\Delta t \to 0^-} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)
\]
exist and these vectors are collinear.

From this theorem, we immediately get the following result.

**Corollary 3.2.** Let $L$ be a curve having the parametric equation
\[
\begin{align*}
  x &= x(t) \\
  y &= y(t) \\
  z &= z(t)
\end{align*}
\]
$A(x(t_0), y(t_0), x(t_0)) \in L$. Set
\[
\Delta t = t - t_0; \Delta x = x(t) - x(t_0); \Delta y = y(t) - y(t_0); \Delta z = z(t) - z(t_0).
\]
Then $L$ has the tangent at $A$ if the following limit exists
\[
\lim_{\Delta t \to 0} \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right).
\]

Particularly, if $x(t), y(t), z(t)$ are differentiable at $t_0$ and $x'(t_0), y'(t_0), z'(t_0)$ are not all zero, then $L$ has the tangent at $A$ and $(x'(t_0), y'(t_0), z'(t_0))$ is a direction vector of this tangent. In the two-dimensions space, if $L$ is the graph of the function $y = f(x)$ and $A \in L$ has the abscissa $x_0$ and $f(x)$ is differentiable at $x_0$, then from above result, it implies that $L$ has the tangent at $A$ and $(1, f'(x_0))$ is a direction vector of this tangent, i.e., the slope of this tangent is $f'(x_0)$.

**Example 3.3.** Let $(C)$ be a circle with the center $I(a, b)$ and the radius $R$. Then it has the parametric equation
\[
\begin{align*}
  x &= a + R\cos t \\
  y &= b + R\sin t.
\end{align*}
\]
Now assume that
\[
A(a + R\cos t_0, b + R\sin t_0) \in (C).
\]
Then the direction vector of the tangent at $A$ is $\vec{u} = (-\sin t_0, \cos t_0)$. It is easily to check that $\vec{u}$ is perpendicular to $\vec{IA} = (R\cos t_0, R\sin t_0)$. Therefore the tangent to $(C)$ at $A$ is the line passing through $A$ and perpendicular to the radius $IA$. We obtain the classical method of determining tangents to circles.
Example 3.4. The curve $y = \sqrt[3]{x}$ is not differentiable at 0, however since

$$\lim_{\Delta x \to 0} \frac{1}{\sqrt{1 + (\frac{\Delta \sqrt[3]{x}}{\Delta x})^2}} \left(1, \frac{\Delta \sqrt[3]{x}}{\Delta x}\right) = (0, 1),$$

it follows that the vertical axis is the tangent to this curve at the point $O(0, 0)$.

This example shows that the problem of the existence of tangents and the problem of differentiability are not equivalent.

Return to consider closely the curve $L$ which is the graph of the function $y = f(x)$. Let a point $A \in L$ have the abscissa $x_0$. Then $L$ has the tangent at $A$ if and only if the following limits

$$\lim_{\Delta x \to 0^+} \frac{1}{\sqrt{1 + (\frac{\Delta y}{\Delta x})^2}} \left(1, \frac{\Delta y}{\Delta x}\right)$$

and

$$\lim_{\Delta x \to 0^-} \frac{1}{\sqrt{1 + (\frac{\Delta y}{\Delta x})^2}} \left(1, \frac{\Delta y}{\Delta x}\right)$$

exist and these vectors are collinear vectors. But it is clear that this condition is equivalent to that $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is finite or infinite. Concretely, if

$$\lim_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} = \pm \infty \quad \text{and} \quad \lim_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} = \pm \infty,$$

then the tangent to $L$ at $A$ is parallel to the vertical axis, i.e., the slope of the tangent is infinite. If $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is finite, i.e., $f(x)$ is differentiable at $x_0$ and $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$, then

$$\lim_{\Delta x \to 0} \frac{1}{\sqrt{1 + (\frac{\Delta y}{\Delta x})^2}} \left(1, \frac{\Delta y}{\Delta x}\right) = \frac{1}{\sqrt{1 + (f'(x_0))^2}} \left(1, f'(x_0)\right),$$

and the slope of the tangent at $A$ is $f'(x_0)$.

To express the relationship between the existence of tangents and the derivative, we consider $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \infty$ as the derivative of the function at $x_0$ and write $f'(x_0) = \infty$. And we call extended derivative of the function $y = f(x)$ at $x_0$ the finite or infinite limit of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Then we obtain the following corollary.

Corollary 3.5. The graph of the function $y = f(x)$ has the tangent at $A$ with the abscissa $x_0$ if and only if $f(x)$ has the extended derivative at $x_0$, and in this case the slope of the tangent is $f'(x_0)$.

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