Maximal function characterizations for Hardy spaces associated with nonnegative self-adjoint operators on spaces of homogeneous type

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Abstract. Let $X$ be a metric measure space with a doubling measure and $L$ be a nonnegative self-adjoint operator acting on $L^2(X)$. Assume that $L$ generates an analytic semigroup $e^{-tL}$ whose kernels $p_t(x,y)$ satisfy Gaussian upper bounds but without any assumptions on the regularity of space variables $x$ and $y$. In this article, we continue a study in Song and Yan (Adv Math 287:463–484, 2016) to give an atomic decomposition for the Hardy spaces $H^p_{L,\text{max}}(X)$ in terms of the nontangential maximal function associated with the heat semigroup of $L$, and hence, we establish characterizations of Hardy spaces associated with an operator $L$, via an atomic decomposition or the nontangential maximal function. We also obtain an equivalence of $H^p_{L,\text{max}}(X)$ in terms of the radial maximal function.

1. Introduction

Our goal in this paper is to continue a study in [23] to establish the equivalence of the maximal and atomic Hardy spaces on spaces of homogeneous type, associated with nonnegative self-adjoint operators whose heat kernel has Gaussian upper bounds. The theory of Hardy spaces associated with operators has attracted a lot of attention in the last decades and has been a very active research topic in harmonic analysis—see, for example, [1–3, 7, 10–13, 15–18, 23, 25, 26].

Let $(X, d, \mu)$ be a metric measure space endowed with a distance $d$ and a nonnegative Borel doubling measure $\mu$ on $X$. Recall that a measure $\mu$ is doubling, provided that there exists a constant $C > 0$ such that for all $x \in X$ and for all $r > 0$,

$$V(x, 2r) \leq CV(x, r)$$  \hspace{1cm} (1.1)

where $V(x, r) = \mu(B(x, r))$, the volume of the open ball $B = B(x, r) := \{ y \in X : d(y, x) < r \}$.

Note that the doubling property implies the following strong homogeneity property,

$$V(x, \lambda r) \leq C\lambda^n V(x, r)$$  \hspace{1cm} (1.2)

Mathematics Subject Classification: Primary 42B30; Secondary 42B35, 47B38

Keywords: Hardy space, Nonnegative self-adjoint operator, Atomic decomposition, The nontangential and radial maximal functions, Spaces of homogeneous type.
for some $C, n > 0$ uniformly for all $\lambda \geq 1, r > 0$ and $x \in X$. In Euclidean space $\mathbb{R}^n$ with Lebesgue measure, the parameter $n$ corresponds to the dimension of the space, but in our more abstract setting, the optimal $n$ need not even be an integer. There also exists $C > 0$ so that

$$V(y, r) \leq C \left(1 + \frac{d(x, y)}{r}\right)^n V(x, r)$$

(1.3)

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (1.3) is a direct consequence of the triangle inequality for the metric $d$ and the strong homogeneity property (1.2).

The following will be assumed throughout the article unless otherwise specified:

**(H1)** $L$ is a nonnegative self-adjoint operator on $L^2(X)$;

**(H2)** Assume that the semigroup $e^{-tL}$ has a kernel $p_t(x, y)$, i.e., a continuous function $X \times X \to \mathbb{C}$ such that

$$e^{-tL} f(x) = \int_X p_t(x, y) f(y) \, d\mu(y), \text{ for every } f \in L^2(X), \text{ and a.e. } x \in X,$$

where $p_t(x, y)$ satisfies a Gaussian upper bound, that is

$$|p_t(x, y)| \leq C \frac{1}{V(x, \sqrt{t})} \exp \left(-\frac{d(x, y)^2}{ct}\right)$$

(GE)

for all $t > 0$, and $x, y \in X$, where $C$ and $c$ are positive constants.

We now recall the notion of a $(p, q, M)$-atom associated with an operator $L$ ([2, 11, 15]).

**DEFINITION 1.1.** Given $0 < p \leq 1 \leq q \leq \infty$, $p < q$ and $M \in \mathbb{N}$, a function $a \in L^2(X)$ is called a $(p, q, M)$-atom associated with the operator $L$ if there exists a function $b \in \mathcal{D}(L^M)$ and a ball $B \subset X$ such that

(i) $a = L^M b$;

(ii) $\text{supp } L^k b \subset B$, $k = 0, 1, \ldots, M$;

(iii) $\| (\frac{r}{B})^k L \|_{L^q(X)} \leq r^{2M} V(B)^{1/q - 1/p}$, $k = 0, 1, \ldots, M$.

The atomic Hardy space $H^p_{L, \text{at}, q, M}(X)$ is defined as follows.

**DEFINITION 1.2.** For a function $f \in L^2(X)$, we will say that $f = \sum \lambda_j a_j$ is an atomic $(p, q, M)$-representation (of $f$) if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^p$, each $a_j$ is a $(p, q, M)$-atom, and the sum converges to $f$ in $L^2(X)$. Set

$$\mathbb{H}^p_{L, \text{at}, q, M}(X) := \left\{ f \in L^2(X) : f \text{ has an atomic } (p, q, M)\text{-representation} \right\},$$

with the norm $\| f \|_{\mathbb{H}^p_{L, \text{at}, q, M}(X)}$ given by

$$\inf \left\{ \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is an atomic } (p, q, M)\text{-representation} \right\}.$$
Given a function \( f \in L^2(X) \), consider the following nontangential maximal function associated with the heat semigroup generated by the operator \( L \),

\[
f^*_L(x) = \sup_{d(x,y) < t} |e^{-r^2L} f(y)|. \tag{1.4}
\]

We may define the spaces \( H^p_{L,\text{max}}(X) \), \( 0 < p \leq 1 \) as the completion of \( \{L^2(X) : \|f^*_L\|_{L^p(X)} < \infty \} \) with respect to \( L^p \)-norm of the nontangential maximal function; i.e.,

\[
\|f\|_{H^p_{L,\text{max}}(X)} := \|f^*_L\|_{L^p(X)} \tag{1.5}
\]

It can be verified (see [11,15,18]) that for all \( q > p \) with \( 1 \leq q \leq \infty \) and every number \( M > \frac{q}{2} \left( \frac{1}{p} - 1 \right) \), any \((p, q, M)\)-atom \( a \) is in \( H^p_{L,\text{max}}(X) \) and so the following continuous inclusion holds:

\[
H^p_{L,\text{at},q,M}(X) \subseteq H^p_{L,\text{max}}(X). \tag{1.6}
\]

A natural question is to show the following continuous inclusion \( H^p_{L,\text{max}}(X) \subseteq H^p_{L,\text{at},q,M}(X) \). In the case of \( X = \mathbb{R}^n \), it is known that the inclusion \( H^p_{L,\text{max}}(\mathbb{R}^n) \subseteq H^p_{L,\text{at},q,M}(\mathbb{R}^n) \) holds for certain operators including Schrödinger operators with non-negative potentials and second-order divergence form elliptic operators via particular PDE techniques (see, for example, [13–16]). Very recently, the authors of this article have made a reformulation and modification of a technique due to Calderón [5] to obtain an atomic decomposition directly from \( H^p_{L,\text{max}}(\mathbb{R}^n) \). Precisely, under the assumptions (H1) and (H2) of the operator \( L \) but without any assumptions on the regularity of \( p_t(x, y) \) of space variables \( x \) and \( y \), we have that \( H^p_{L,\text{max}}(\mathbb{R}^n) \subseteq H^p_{L,\text{at},q,M}(\mathbb{R}^n) \) for \( 0 < p \leq 1 \leq q \leq \infty \) with \( q > p \), and all integers \( M > \frac{q}{2} \left( \frac{1}{p} - 1 \right) \), and hence by (1.6),

\[
H^p_{L,\text{max}}(\mathbb{R}^n) \simeq H^p_{L,\text{at},q,M}(\mathbb{R}^n). \]

That is, the spaces \( H^p_{L,\text{max}}(\mathbb{R}^n) \) and \( H^p_{L,\text{at},q,M}(\mathbb{R}^n) \) coincide, and their norms are equivalent.

We point out that in [5], a decomposition of the function \( F(x, t) = f \ast \varphi_t(x) \) associated with the distribution \( f \) was given, and convolution operation in the function \( F \) played an important role in the proof. In [23, Theorem 1.4], there is no analogue of convolution operation in the function \( t^2 Le^{-t^2L} f(x) \). Moreover, the proof depends critically on the geometry of \( \mathbb{R}^n \) to use oblique cylinders of \( \mathbb{R}^n+1 \); for every cube \( Q \) of \( \mathbb{R}^n \) and for \( \tilde{e} = (1, \ldots, 1) \in \mathbb{R}^n \)

\[
\tilde{Q} := \{(y, t) \in \mathbb{R}^n+1 : y + 3t\tilde{e} \in Q\},
\]

in place of vertical cylinders in Calderón’s construction in [5]. However, “oblique cylinders” do not exist on spaces of homogeneous type, and hence, it is not trivial to generalize the method in [23] to the case of spaces of homogeneous type. So, we may ask the following question:
QUESTION 1. Is it possible to show an inclusion $H_{L,\text{max}}^p(X) \subseteq H_{L,\text{at},q,M}^p(X)$ on spaces of homogeneous type $X$?

Next we consider the Hardy spaces $H_{L,\text{max}}^p(X)$ in terms of the radial maximal function. Given an operator $L$ satisfying (H1)–(H2), we may define the spaces $H_{L,\text{rad}}^p(X)$, $0 < p \leq 1$ as the completion of $\{ f \in L^2(X) : \| f_L^+ \|_{L^p(X)} < \infty \}$ with respect to the $L^p$-norm of the radial maximal function; i.e.,

$$
\| f \|_{H_{L,\text{rad}}^p(X)} := \| f_L^+ \|_{L^p(X)} := \sup_{t > 0} \| e^{-t^2 L} f \|_{L^p(X)}.
$$

(1.7)

Fix $0 < p \leq 1$. For all $q > p$ with $1 \leq q \leq \infty$ and for all integers $M > \frac{q}{2}(\frac{1}{p} - 1)$, the following continuous inclusion holds:

$$
H_{L,\text{at},q,M}^p(X) \subseteq H_{L,\text{max}}^p(X) \subseteq H_{L,\text{rad}}^p(X)
$$

(1.8)

by (1.4)–(1.7). In the case of $X = \mathbb{R}^n$, it is known that the inclusion $H_{L,\text{rad}}^p(\mathbb{R}^n) \subseteq H_{L,\text{max}}^p(\mathbb{R}^n)$ holds for certain operators including Schrödinger operators with non-negative potentials and second-order divergence form elliptic operators via particular PDE techniques (see, for example, [13–15]). We may ask the following question:

QUESTION 2. Is it possible to show an inclusion $H_{L,\text{rad}}^p(X) \subseteq H_{L,\text{max}}^p(X)$ assuming merely that an operator $L$ satisfies (H1)–(H2)?

The aim of this article gives an affirmative answer to Questions 1 and 2 to establish the equivalent characterizations of Hardy spaces associated with an operator $L$ satisfying (H1) and (H2) on spaces of homogeneous type $X$, including an atomic decomposition, the nontangential maximal functions and the radial maximal functions. Throughout the article, we always assume that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$.

THEOREM 1.3. Let $(X, d, \mu)$ be as in (1.1) and (1.2). Suppose that an operator $L$ satisfies (H1) and (H2). Fix $0 < p \leq 1$. For all $q > p$ with $1 \leq q \leq \infty$ and for all integers $M > \frac{q}{2}(\frac{1}{p} - 1)$, we have that

(i) $H_{L,\text{rad}}^p(X) \subseteq H_{L,\text{max}}^p(X)$;

(ii) $H_{L,\text{max}}^p(X) \subseteq H_{L,\text{at},q,M}^p(X)$.

Hence by (1.8),

$$
H_{L,\text{at},q,M}^p(X) \simeq H_{L,\text{max}}^p(X) \simeq H_{L,\text{rad}}^p(X).
$$

We would like to mention that Dekel et al. [10] have established the equivalence of the maximal and atomic Hardy spaces associated with an operator $L$ on the space of homogeneous type $X$, under the four assumptions that $L$ satisfies (H1), (H2), and (H3) the kernel $p_t(x, y)$ of the semigroup $e^{-tL}$ satisfies the Hölder continuity: There exists a constant $\alpha > 0$ such that
\[|p_t(x, y) - p_t(x, y')| \leq C \left( \frac{d(y, y')}{\sqrt{t}} \right)^\alpha \]

for \(x, y, y' \in X\) and \(t > 0\), whenever \(d(y, y') \leq \sqrt{t}\); and

(H4) Markov property:

\[\int_X p_t(x, y) d\mu(y) = 1\]

for \(x \in X\) and \(t > 0\).

Note that in our Theorem 1.3, we remove the need to assume conditions (H3)–(H4). There are numbers of operators which satisfy conditions (H1)–(H2), among them, there exist many for which (H3) or (H4) fails. This happens, e.g., for the harmonic oscillator, Schrödinger operators with rough potentials and second-order elliptic operators with rough lower order terms [2,12,13,19,22,23]. This is indeed one of the main obstacles in this article and makes the theory quite subtle and delicate.

The layout of the article is as follows: In Sect. 2, we recall some basic properties of heat kernels and finite propagation speed for the wave equation and build the necessary kernel estimates for functions of an operator, which is useful in the proof of Theorem 1.3. In Sect. 3, we show (i) of Theorem 1.3 to obtain an equivalence of Hardy spaces on spaces of homogeneous type, in terms of the nontangential and radial maximal functions. In Sect. 4, we will show our main result (ii) of Theorem 1.3. A crucial idea in the proof is to make a modification of [5,23] to decompose \(X \times (0, \infty)\) into certain “cones” in place of “vertical cylinders” of \(\mathbb{R}^{n+1}_+\) in [5] or “oblique cylinders” of \(\mathbb{R}^{n+1}_+\) in [23] (see (4.6)). This leads us to obtain characterizations of Hardy spaces associated with an operator \(L\) on spaces of homogeneous type, via an atomic decomposition or the nontangential maximal function.

Throughout, the letters “\(c\)” and “\(C\)” will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

Recall that, if \(L\) is a nonnegative, self-adjoint operator on \(L^2(X)\), and \(E_L(\lambda)\) denotes a spectral decomposition associated with \(L\), then for every bounded Borel function \(F : [0, \infty) \to \mathbb{C}\), one defines the operator \(F(L) : L^2(X) \to L^2(X)\) by the formula

\[F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda). \tag{2.1}\]

In particular, the operator \(\cos(t \sqrt{L})\) is then well defined on \(L^2(X)\). Moreover, it follows from Theorem 3.4 of [8] that the integral kernel \(K_{\cos(t \sqrt{L})}\) of \(\cos(t \sqrt{L})\) satisfies

\[\text{supp} K_{\cos(t \sqrt{L})} \subseteq \{(x, y) \in X \times X : d(x, y) \leq t\}. \tag{2.2}\]
By the Fourier inversion formula, whenever \( F \) is an even bounded Borel function with the Fourier transform of \( F \), \( \hat{F} \in L^1(\mathbb{R}) \), we can write \( F(\sqrt{L}) \) in terms of \( \cos(t \sqrt{L}) \). Concretely, by recalling (2.1) we have

\[
F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t \sqrt{L}) \, dt,
\]

which, when combined with (2.2), gives

\[
K_F(\sqrt{L})(x, y) = (2\pi)^{-1} \int_{|t| \geq d(x, y)} \hat{F}(t) K_{\cos(t \sqrt{L})}(x, y) \, dt. \tag{2.3}
\]

This property leads us to the following result (see [15, Lemma 3.5]): for every even function \( \varphi \in \mathcal{C}_0^\infty(\mathbb{R}) \) with \( \text{supp} \varphi \subset (-1, 1) \),

\[
\text{supp} K_{\Phi(\sqrt{L})} \subseteq \{ (x, y) \in X \times X : d(x, y) \leq t \}, \tag{2.4}
\]

where \( \Phi \) denotes the Fourier transform of \( \varphi \).

Next, for \( s > 0 \), we define

\[
\mathbb{F}(s) := \left\{ \psi : \mathbb{C} \rightarrow \mathbb{C} \text{ measurable} : |\psi(z)| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right\}.
\]

Then for any nonzero function \( \psi \in \mathbb{F}(s) \), we have that \( \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} < \infty \). It follows from the spectral theory in [27] that for any \( f \in L^2(X) \),

\[
\left\{ \int_0^\infty \left\| \psi(t \sqrt{L}) f \right\|^2_{L^2(X)} \frac{dt}{t} \right\}^{1/2} = \left\{ \int_0^\infty \left( \psi(t \sqrt{L}) \psi(t \sqrt{L}) f, f \right) \frac{dt}{t} \right\}^{1/2} \]

\[
= \left\{ \int_0^\infty \left| \psi(t \sqrt{L}) f, f \right|^2 \frac{dt}{t} \right\}^{1/2} \]

\[
= \kappa \| f \|_{L^2(X)}, \tag{2.5}
\]

where \( \kappa = \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} \).

The following Calderón reproducing formula will play an important role in this paper.

**Lemma 2.1.** Let \( \varphi \in \mathcal{C}_0^\infty(\mathbb{R}) \) be even, \( \text{supp} \varphi \subset (-1, 1) \). Let \( \Phi \) denote the Fourier transform of \( \varphi \) and set \( \Psi(x) := x^{2\kappa} \Phi(x) \) and \( \kappa = 1, 2, \ldots \). Let \( \psi \in \mathcal{S}(\mathbb{R}) \) be a Schwartz function with \( C_{\Psi, \psi}^{-1} = \int_0^\infty \Psi(s) \psi(s) \, ds/s \neq 0 \). Then for every \( f \in L^2(X) \),

\[
f = C_{\Psi, \psi_2} \int_0^\infty \Psi(s \sqrt{L}) \psi(s \sqrt{L}) f \frac{ds}{s}
\]

where the integral converges in \( L^2(X) \).

**Proof.** Lemma 2.1 is a consequence of \( L^2 \)-functional calculus (see for example, [21, Section 5]). See also [1,15,27]. □
LEMMA 2.2. Assume that an operator $L$ satisfies (H1)–(H2).

(i) Let $\varphi \in \mathcal{S}(\mathbb{R})$ be an even function. Then for every $\beta > 0$, there exists a positive constant $C = C(n, \beta, \varphi)$ such that the kernel $K_{\varphi(t\sqrt{L})}(x, y)$ of $\varphi(t\sqrt{L})$ satisfies

$$|K_{\varphi(t\sqrt{L})}(x, y)| \leq C \frac{1}{\max(V(x, t), V(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-n-\beta}$$

(2.6)

for all $t > 0$ and $x, y \in X$.

(ii) Let $\psi_i \in \mathcal{S}(\mathbb{R})$ be even functions, $\psi_i(0) = 0$, $i = 1, 2$. Then for every $\beta > 0$, there exists a positive constant $C = C(n, \beta, \psi_1, \psi_2)$ such that the kernel $K_{\psi_1(t\sqrt{L})\psi_2(t\sqrt{L})}(x, y)$ of $\psi_1(t\sqrt{L})\psi_2(t\sqrt{L})$ satisfies

$$|K_{\psi_1(t\sqrt{L})\psi_2(t\sqrt{L})}(x, y)| \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right) \frac{1}{\max(V(x, \max(s, t)), V(y, \max(s, t)))} \left(1 + \frac{d(x, y)}{\max(s, t)}\right)^{-n-\beta}$$

(2.7)

for all $t > 0$ and $x, y \in X$.

Proof. The proof of (i) and (ii) is similar to that of [4, Lemma 2.3] and [23, Lemma 2.3] on the Euclidean spaces $\mathbb{R}^n$, respectively. We omit the detail here. \hfill \Box

Let $F(y, t)$ be a $\mu$-measurable function of $X \times (0, +\infty)$. For $\alpha > 0$, set $F_\alpha^*(x) = \sup_{d(x, y) < \alpha t} |F(y, t)|$. With the notation above, we have the following result.

LEMMA 2.3. For any $p > 0$ and $0 < \alpha_2 \leq \alpha_1$,

$$\|F_{\alpha_1}^*\|_{L^p(X)} \leq C \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{n/p} \|F_{\alpha_2}^*\|_{L^p(X)},$$

where $C = C(p, n)$ is independent of $\alpha_1, \alpha_2$ and $F$.

Proof. The proof of Lemma 2.3 is standard (see, for instance, [6, Theorem 2.3] for the case of $X = \mathbb{R}^n$).

We write

$$\|F_{\alpha_1}^*\|_{L^p(X)}^p = p \int_0^\infty \lambda^{p-1} \mu\{x \in X : F_{\alpha_1}^*(x) > \lambda\} \, d\lambda.$$

Observe that

$$\{x \in X : F_{\alpha_1}^*(x) > \lambda\} \subset \left\{x \in X : \mathcal{M}(\chi_E)(x) > C^{-2}(1 + \frac{2\alpha_1}{\alpha_2})^{-n}\right\},$$

(2.8)

where $E := \{x \in X : F_{\alpha_2}^*(x) > \lambda\}$ and $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. Indeed, if $F_{\alpha_1}^*(x_0) > \lambda$, then there exist $y_0 \in X$ and $t_0 > 0$ such that $d(x_0, y_0) < \alpha_1 t_0$ and $|F(y_0, t_0)| > \lambda$. Hence, $B(y_0, \alpha_2 t_0) \subset E$. It follows that

$$\frac{\mu(B(x_0, (\alpha_1 + \alpha_2) t_0) \cap E)}{V(x_0, (\alpha_1 + \alpha_2) t_0)} \geq \frac{V(y_0, \alpha_2 t_0)}{V(x_0, (\alpha_1 + \alpha_2) t_0)}.$$
By (1.2) and (1.3),
\[
V(x_0, (\alpha_1 + \alpha_2)t_0) \leq CV(y_0, (\alpha_1 + \alpha_2)t_0)\left(1 + \frac{d(x_0, y_0)}{(\alpha_1 + \alpha_2)t_0}\right)^n
\]
\[
\leq C^2V(y_0, \alpha_1\alpha_2t_0)\left(\frac{\alpha_1 + \alpha_2}{\alpha_2}\right)^n\left(1 + \frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^n
\]
\[
\leq C^2V(y_0, \alpha_2t_0)\left(1 + \frac{2\alpha_1}{\alpha_2}\right)^n,
\]
which gives
\[
\frac{\mu(B(x_0, (\alpha_1 + \alpha_2)t_0) \cap E)}{V(x_0, (\alpha_1 + \alpha_2)t_0)} \geq C^{-2}\left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{-n}.
\]
This proves (2.8). By the weak (1,1) boundedness of Hardy–Littlewood maximal function, we obtain the proof of Lemma 2.3. □

Throughout the article, for every even functions \( \varphi \in \mathcal{S}(\mathbb{R}) \), and for every \( f \in L^2(X) \) we define
\[
\varphi^\ast_{L,\alpha}(f)(x) = \sup_{d(x, y) < \alpha t} |\varphi(t \sqrt{L})f(y)|
\]
and
\[
\varphi^+_{L}(f)(x) = \sup_{t > 0} |\varphi(t \sqrt{L})f(x)|.
\]
For simplicity, we will write \( \varphi^\ast_{L,1}(f) \) instead of \( \varphi^\ast_{L,1}(f) \).

**Proposition 2.4.** Let \( 0 < p \leq 1 \). Suppose that an operator \( L \) satisfies (H1) and (H2). Let \( \varphi_i \in \mathcal{S}(\mathbb{R}) \) be even functions with \( \varphi_i(0) = 1 \) and \( \alpha_i > 0, i = 1, 2 \). Then there exists a constant \( C = C(n, \varphi_1, \varphi_2, \alpha_1, \alpha_2) \) such that for every \( f \in L^2(X) \),
\[
\left\| (\varphi_1)^\ast_{L,\alpha_1}(f) \right\|_{L^p(X)} \leq C \left\| (\varphi_2)^\ast_{L,\alpha_2}(f) \right\|_{L^p(X)}.
\]
**Proof:** The argument is similar to that of [23, Proposition 3.1] with minor modifications. We give a brief argument of this proof for completeness and convenience for the reader.

For any \( 0 < \alpha_2 \leq \alpha_1 \), we apply Lemma 2.3 to have that
\[
\left\| \varphi^\ast_{L,\alpha_1}(f) \right\|_{L^p(X)} \leq C(p, n)\left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{n/p} \left\| \varphi^\ast_{L,\alpha_2}(f) \right\|_{L^p(X)}
\]
for any \( \varphi \in \mathcal{S}(\mathbb{R}) \). Now, we let \( \psi(x) := \varphi_1(x) - \varphi_2(x) \), and then, the proof of (2.11) reduces to show that
\[
\left\| \psi^\ast_{L,1}(f) \right\|_{L^p(X)} \leq C \left\| (\varphi_2)^\ast_{L,1}(f) \right\|_{L^p(X)}.
\]
Let us show (2.12). Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be even, \( \text{supp } \varphi \subset (-1, 1) \). Let \( \Phi \) denote the Fourier transform of \( \varphi \) and set \( \Psi(x) := x^{2\kappa} \Phi(x) \) and \( 2\kappa > (n + 1)/p \). By the Calderón reproducing formula (Lemma 2.1), we have

\[
f = C_\varphi \varphi_2 \int_0^\infty \Psi(s \sqrt{L}) \varphi_2(s \sqrt{L}) f \frac{ds}{s}.
\]

Therefore,

\[
\psi(t \sqrt{L}) f(x) = C \int_0^\infty \left( \psi(t \sqrt{L}) \Psi(s \sqrt{L}) \right) \varphi_2(s \sqrt{L}) f(x) \frac{ds}{s}.
\]

Let us denote the kernel of \( \psi(t \sqrt{L}) \Psi(s \sqrt{L}) \) by \( K_{\psi(t \sqrt{L}) \Psi(s \sqrt{L})}(x, y) \). For every \( \lambda \in (\frac{n}{p}, 2\kappa) \), we write

\[
\sup_{d(x, y) < t} |\psi(t \sqrt{L}) f(y)|
\]

\[
= C \sup_{d(x, y) < t} \left| \int_0^\infty \int_X K_{\psi(t \sqrt{L}) \Psi(s \sqrt{L})}(y, z) \varphi_2(s \sqrt{L}) f(z) d\mu(z) \frac{ds}{s} \right|
\]

\[
\leq \sup_{z, s} |\varphi_2(s \sqrt{L}) f(z)| \left( 1 + \frac{d(x, z)}{s} \right)^{-\lambda}
\]

\[
\times \sup_{d(x, y) < t} \int_0^\infty \int_X |K_{\psi(t \sqrt{L}) \Psi(s \sqrt{L})}(y, z)| \left( 1 + \frac{d(x, z)}{s} \right)^\lambda d\mu(z) \frac{ds}{s}.
\]  (2.13)

By (ii) of Lemma 2.2, it follows that for \( \eta \in (\lambda, 2\kappa) \),

\[
|K_{\psi(t \sqrt{L}) \Psi(s \sqrt{L})}(y, z)| \leq C \min \left( \left( \frac{s}{t} \right)^{2\kappa}, \left( \frac{t}{s} \right)^2 \right)
\]

\[
\times \frac{1}{1 + \frac{d(y, z)}{\text{max}(s, t)}} \frac{1}{n + \eta} V(y, \text{max}(s, t)).
\]

For any \( d(x, y) < t \), one can compute

\[
\int_X \frac{1}{V(y, \text{max}(s, t))} \left( 1 + \frac{d(x, z)}{s} \right)^\lambda d\mu(z) \leq C \max \left( 1, \left( \frac{t}{s} \right)^\eta \right),
\]

which implies,

\[
\int_X |K_{\psi(t \sqrt{L}) \Psi(s \sqrt{L})}(y, z)| \left( 1 + \frac{d(x, z)}{s} \right)^\lambda d\mu(z)
\]

\[
\leq C \min \left( \left( \frac{s}{t} \right)^{2\kappa}, \left( \frac{t}{s} \right)^2 \right) \max \left( 1, \left( \frac{t}{s} \right)^\eta \right)
\]

\[
\leq C \min \left( \left( \frac{s}{t} \right)^{2\kappa - \eta}, \left( \frac{t}{s} \right)^2 \right),
\]
for any \( d(x, y) < t \). Hence
\[
\sup_{d(x, y) < t} \int_0^\infty \int_X |K_{\psi(t\sqrt{L})\psi(t\sqrt{L})}(y, z)| \left(1 + \frac{d(x, z)}{s}\right)^\lambda \frac{d\mu(z)}{s} ds \\
\leq C \int_0^\infty \min \left(\frac{s}{t} \right)^{2\kappa - \eta}, \left(\frac{t}{s}\right)^{2\lambda} \right) ds \\
\leq C.
\]
This, in combination with (2.13) and the condition \( \lambda \in \left(\frac{n}{p}, 2\kappa\right) \), implies
\[
\|\psi^*_{L,1}(f)\|_{L^p(X)} = \left\| \sup_{d(x, y) < t} |\psi(t\sqrt{L})f(y)| \right\|_{L^p_t(X)} \\
\leq C \left\| \sup_{s, z} |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{d(x, z)}{s}\right)^{-\lambda} \right\|_{L^p_t(X)} \\
\leq C \left\| \sup_{d(x, y) < t} |\varphi_2(t\sqrt{L})f(y)| \right\|_{L^p_t(X)} \\
= C \left\| (\varphi_2)^*_{L,1}(f) \right\|_{L^p(X)},
\]
where the second inequality above can be proved easily by combining Lemma 2.3 and the argument of [6, Theorem 2.4]. This completes the proof of Proposition 2.4.

COROLLARY 2.5. Let \( 0 < p \leq 1 \). Suppose that an operator \( L \) satisfies (H1) and (H2). Then for any even function \( \varphi \in \mathcal{S}(\mathbb{R}) \) with \( \varphi(0) = 1 \) and \( \alpha > 0 \), there exists a constant \( C = C(n, \varphi, \alpha, p) \) such that for every \( f \in L^2(X) \),
\[
C^{-1} \|f^*_{L}\|_{L^p(X)} \leq \|\varphi^*_{L,\alpha}(f)\|_{L^p(X)} \leq C \|f^*_{L}\|_{L^p(X)}.
\]

Proof. Corollary 2.5 is a direct consequence of Proposition 2.4.

3. Equivalence of Hardy spaces \( H^p_{L,\text{max}}(X) \) and \( H^p_{L,\text{rad}}(X) \)

Assume that the metric measure space \( X \) satisfies the doubling conditions (1.1) and (1.2) with exponent \( n \). In this section, we will show (i) of Theorem 1.3 to obtain an equivalence of Hardy spaces \( H^p_{L,\text{rad}}(X) \) and \( H^p_{L,\text{max}}(X) \). By (1.8), we have
\[
H^p_{L,\text{max}}(X) \subseteq H^p_{L,\text{rad}}(X).
\]
Then we have the following result.

THEOREM 3.1. Let \( (X, d, \mu) \) be as in (1.1) and (1.2). Suppose that an operator \( L \) satisfies (H1) and (H2). For every \( 0 < p \leq 1 \), we have
\[
H^p_{L,\text{max}}(X) \simeq H^p_{L,\text{rad}}(X).
\]
That is, the spaces \( H^p_{L,\text{max}}(\mathbb{R}^n) \) and \( H^p_{L,\text{rad}}(\mathbb{R}^n) \) coincide, and their norms are equivalent.
Proof. By (3.1), the proof of Theorem 3.1 reduces to show that there exists a constant $C = C(p, \phi) > 0$ such that for every $f \in L^2(X)$,

$$\|\varphi^*_L(f)\|_{L^p(X)} \leq C\|\varphi^+_L(f)\|_{L^p(X)}. \tag{3.2}$$

In order to prove (3.2), for every $N > 0$ we define

$$M^{**}_{L,\phi,N}(f)(x) := \sup_{y \in X, s > 0} \frac{|\varphi(s\sqrt{L})f(y)|}{(1 + d(x,y)/s)^N}.$$

By the definition of $\varphi^+_L(f)$ in (2.9), we have

$$\varphi^*_L(f)(x) \leq 2^N M^{**}_{L,\phi,N}(f)(x). \tag{3.3}$$

We now claim that if $0 < \theta < 1$ and $N\theta > 2n$, then there exists $C = C(p, \phi, N, \theta) > 0$ such that for every $f \in L^2(X)$,

$$M^{**}_{L,\phi,N}(f)(x) \leq C \left[ \mathcal{M}((\varphi^+_L(f))^\theta)(x) \right]^{1/\theta} \text{ a.e. } x \in X. \tag{3.4}$$

If the claim is proved, then we can choose $N = 2(n+1)/p$ and $\theta = \frac{(2n+1)p}{2(n+1)}$ and apply the $L^r$ ($r > 1$) boundedness of Hardy–Littlewood maximal operator to obtain that for any $f \in L^2(X)$

$$\|M^{**}_{L,\phi,N}(f)\|_{L^p(X)} \leq C \left\| \mathcal{M}((\varphi^+_L(f))^\theta)(x) \right\|_{L^p(X)}^{1/\theta} \leq C\|\varphi^+_L(f)\|_{L^p(X)},$$

which, together with (3.3), yields (3.2).

It remains to prove (3.4). Let $\phi \in C_0^\infty(\mathbb{R})$ be even, supp $\phi \subset (-1, 1)$. Let $\Phi$ denote the Fourier transform of $\phi$ and set $\Psi(x) := x^{2\kappa} \Phi(x)$, $x \in \mathbb{R}$ and $\kappa > N/2$. For every $f \in L^2(X)$, by the Calderón reproducing formula (Lemma 2.1), one can write

$$f = \lim_{\epsilon \to 0} c_{\psi, \phi} \int_1^1 \Psi(t\sqrt{L})\varphi(t\sqrt{L})f(t) \frac{dt}{t} \tag{3.5}$$

with the integral converging in $L^2(X)$.

Set

$$\eta(x) := c_{\psi, \phi} \int_1^\infty \Psi(tx)\varphi(tx) \frac{dt}{t} = c_{\psi, \phi} \int_x^\infty \Psi(y)\varphi(y) \frac{dy}{y}, \quad x \in \mathbb{R} \setminus \{0\}$$

with $\eta(0) = 1$. It follows that $\eta \in \mathscr{S}(\mathbb{R})$ is an even function. By the spectral theory ([27]) again, one can write, for any $s > 0$,

$$\eta(s\sqrt{L})f = c_{\psi, \phi} \int_s^\infty \Psi(t\sqrt{L})\varphi(t\sqrt{L})f(t) \frac{dt}{t}. \tag{3.6}$$
which, together with (3.5), yields that for any \( f \in L^2(X), \)
\[
f = \eta(s\sqrt{L})f + c_{\psi, \varphi}\int_0^s \Psi(t\sqrt{L})\varphi(t\sqrt{L})f \frac{dt}{t}. \tag{3.7}
\]
Let \( 0 < \theta < 1, N\theta > 2n. \) By (3.7), there holds
\[
\left| \varphi(s\sqrt{L})f(y) \right| \leq \frac{\left| \eta(s\sqrt{L})\varphi(s\sqrt{L})f(y) \right|}{(1 + \frac{d(x,y)}{s})^N} + \frac{c_{\psi, \varphi}}{(1 + \frac{d(x,y)}{s})^N} \left| \int_0^s \varphi(t\sqrt{L})\Psi(t\sqrt{L})\varphi(t\sqrt{L})f(y) \frac{dt}{t} \right| =: I + II.
\]
Now we apply an argument of Strömberg and Torchinsky as [24, Chapter V, Theorem 5] on page 64. For the term \( I, \) we use (i) of Lemma 2.2 to obtain
\[
I \leq C \left( \frac{1}{1 + \frac{d(x,y)}{s}} \right)^N \int_X \frac{1}{V(z, s)} \frac{1}{1 + \frac{d(y,z)}{s}}^N \left| \varphi(s\sqrt{L})f(z) \right| d\mu(z)
\leq C \int_X \frac{1}{V(z, s)} \frac{1}{1 + \frac{d(x,z)}{s}}^N \left| \varphi(s\sqrt{L})f(z) \right| d\mu(z)
\leq C \int_X \frac{1}{V(x, s)} \frac{1}{1 + \frac{d(x,z)}{s}}^N \left| \varphi(s\sqrt{L})f(z) \right| d\mu(z) \left( M_{L, \varphi, N}^*(f)(x) \right)^{1-\theta},
\]
where in the last inequality above we have used (1.3). By noting that \( N\theta > 2n, \) we obtain,
\[
I \leq C M \left( \left| \varphi_L^+ f \right|^\theta \right)(x) \left( M_{L, \varphi, N}^*(f)(x) \right)^{1-\theta}. \tag{3.8}
\]
Let us estimate the term \( II. \) One writes \( \Psi(tx)\psi(sx) = (\frac{t}{s})^{2\kappa}[\Phi(tx)(sx)^{2\kappa}\psi(sx)]. \) We then apply an argument as in (ii) of Lemma 2.2 to show that the kernel \( K_{\Psi(t\sqrt{L})\varphi(s\sqrt{L})(x, y)} \) of \( \Psi(t\sqrt{L})\varphi(s\sqrt{L}) \) satisfies
\[
\left| K_{\Psi(t\sqrt{L})\varphi(s\sqrt{L})(y, z)} \right| \leq C \left( \frac{t}{s} \right)^{2\kappa} \frac{1}{V(z, s)} \frac{1}{1 + \frac{d(y,z)}{s}}^N.
\]
for all $s > t > 0$ and $y, z \in X$. This yields

\[
II \leq \frac{C}{(1 + \frac{d(x,y)}{s})^N} \int_0^s \int_X \left( \frac{t}{s} \right)^{2\kappa} \frac{1}{V(z,s)} \left( \frac{1}{1 + d(y,z)} \right)^N \left| \varphi(t\sqrt{L}) f(z) \right| d\mu(z) \frac{dt}{t}
\]

\[
\leq C \int_0^s \int_X \left( \frac{t}{s} \right)^{2\kappa-N} \frac{1}{V(z,t)} \left( \frac{1}{1 + \frac{d(x,z)}{t}} \right)^{N(\theta-N)} d\mu(z) \frac{dt}{t} \left( M_{**}^{L,\varphi,N}(f)(x) \right)^{1-\theta}
\]

where in the last inequality above we have used (1.3). Since $2\kappa > N$ and $N\theta > 2n$, we have

\[
II \leq C \int_0^s \left( \frac{t}{s} \right)^{2\kappa-N} dt \mathcal{M} \left( |\varphi_L^+ f|^\theta \right)(x) \left( M_{**}^{L,\varphi,N}(f)(x) \right)^{1-\theta}
\]

\[
\leq C \mathcal{M} \left( |\varphi_L^+ f|^\theta \right)(x) \left( M_{**}^{L,\varphi,N}(f)(x) \right)^{1-\theta}.
\]

Combining (3.8) and (3.9), we have proved that

\[
M_{**}^{L,\varphi,N}(f)(x) \leq C \mathcal{M} \left( |\varphi_L^+ f|^\theta \right)(x) \left( M_{**}^{L,\varphi,N}(f)(x) \right)^{1-\theta}.
\]  

Finally, let us verify that for any $f \in L^2(X)$, $M_{**}^{L,\varphi,N}(f)(x) < \infty$, for a.e. $x \in X$. In fact, note that

\[
M_{**}^{L,\varphi,N}(f)(x) \leq \sup_{y \in X, d(x,y) < s} \sum_{k=1}^\infty \left( \frac{1}{1 + d(y,z)} \right)^N |\varphi(s\sqrt{L}) f(y)|
\]

\[
\leq \sup_{y \in X, d(x,y) < s} |\varphi(s\sqrt{L}) f(y)| + \sum_{k=1}^\infty \sup_{d(x,y) < 2^k s} 2^{-(k-1)N} |\varphi(s\sqrt{L}) f(y)|,
\]

which, combined with Lemma 2.3, gives
\[ \| M_{L,\varphi,N}^*(f) \|_{L^2(X)} \leq \| \varphi_L^*(f) \|_{L^2(X)} + \sum_{k=1}^{\infty} 2^{-(k-1)N} \| \varphi_{L,2k}^*(f) \|_{L^2(X)} \]

\[ \leq \sum_{k=1}^{\infty} 2^{-(k-1)N} (1 + 2^{k+1})^{n/2} \| \varphi_L^*(f) \|_{L^2(X)} \]

\[ \leq C \sum_{k=1}^{\infty} 2^{-(k-1)N} (2^{k+2})^{n/2} \| \mathcal{M}(f) \|_{L^2(X)} \]

\[ \leq C \| f \|_{L^2(X)}, \]

since \( N > 2n/\theta > n/2 \).

From (3.10), (3.4) follows readily. The proof of Theorem 3.1 is complete. \( \square \)

4. Proof of Theorem 1.3

In this section, we prove (ii) of Theorem 1.3 to give a \((p, \infty, M)\)-atomic representation for the Hardy spaces \( H_{L,\max}^p(X) \). To do it, we first recall the following Whitney-type covering lemma on space of homogeneous type \( X \).

**Lemma 4.1.** Suppose that \( O \subseteq X \) is an open set with finite measure. There exists a sequence of points \( \{ \xi_k \}_{k=1}^{\infty} \in O \) and a collection of balls \( B(\xi_k, \rho_k) \) where \( \rho_k := d(\xi_k, O^c) \) such that

(i) \( \bigcup_k B(\xi_k, \rho_k/2) = O \);

(ii) \( \{ B(\xi_k, \rho_k/10) \}_{k=1}^{\infty} \) are disjoint.

**Proof.** The proof of this lemma is essentially given in [9, Chapter III, Theorem 1.3] and is omitted. See also [10,20]. \( \square \)

**Proof of (ii) of Theorem 1.3.** It suffices to show that for \( f \in H_{L,\max}^p(X) \cap L^2(X) \), \( f \) has a \((p, \infty, M)\) atomic representation.

We start with a suitable version of the Calderón reproducing formula. Let \( \varphi \in \mathcal{C}^\infty_0(\mathbb{R}) \) be an even function with \( \text{supp} \varphi \subset (-1,1) \). Let \( \Phi \) denote the Fourier transform of \( \varphi \), and set \( \Psi(x) := x^{2M} \Phi(x), \ x \in \mathbb{R} \). By Lemma 2.1, for every \( f \in L^2(X) \) one can write

\[ f = \lim_{\epsilon \to 0} c \varphi \int_{\epsilon}^{1/\epsilon} \Psi(t \sqrt{L}) t^2 L e^{-t^2 L} f \frac{dt}{t} \quad (4.1) \]

with the integral converging in \( L^2(X) \).

Set

\[ \eta(x) := c \varphi \int_1^{\infty} t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t} = c \varphi \int_1^{\infty} y \Psi(y) e^{-y^2} dy, \quad x \neq 0 \]

with \( \eta(0) = 1 \). Then \( \eta \in \mathcal{S}(\mathbb{R}) \) is an even function. By the spectral theory ([27]) again, one has
\[c\psi \int_a^b \Psi(t\sqrt{L})t^2Le^{-t^2L} f \frac{df}{t} = \eta(a\sqrt{L})f(x) - \eta(b\sqrt{L})f(x).\] (4.2)

Define,
\[\mathcal{M}_L f(x) := \sup_{d(x,y)<5t} \left(|t^2Le^{-t^2L} f(y)| + |\eta(t\sqrt{L}) f(y)|\right).\]

By Proposition 2.4, it follows that
\[\|\mathcal{M}_L f\|_{L^p(X)} \leq C \|f\|_{H^p_{L,\max}(X)}, \quad 0 < p \leq 1.\]

In the sequel, if \(O\) is an open subset of \(\mathbb{R}^n\), then the “tent” over \(O\), denoted by \(\hat{O}\), is given as
\[\hat{O} := \left\{(x, t) \in X \times (0, +\infty) : B(x, 4t) \subseteq O\right\}.\]

For \(i \in \mathbb{Z}\), we define the family of sets \(O_i := \{x \in X : \mathcal{M}_L f(x) > 2^i\}\). Up to sets of measure zero, we obtain a decomposition for \(X \times (0, +\infty)\) as follows:
\[X \times (0, +\infty) = \bigcup_{i \in \mathbb{Z}} \hat{O}_i = \bigcup_{i \in \mathbb{Z}} (\hat{O}_i \setminus \hat{O}_{i+1}) = \bigcup_{i \in \mathbb{Z}} T_i,\] (4.3)

where
\[T_i := \hat{O}_i \setminus \hat{O}_{i+1}.\]

Note that for each \(i \in \mathbb{Z}\), \(O_i\) is open set with \(\mu(O_i) < \infty\). By Lemma 4.1, we can further decomposition \(O_i\) into “balls” of \(X\). More precisely, for each \(i \in \mathbb{Z}\), there exists a sequence of points \(\{\xi^k_i\}_{k=1}^{\infty} \in O_i\), such that
1. \(O_i = \bigcup_{k=1}^{\infty} B^k_i;\)
2. \(\{\frac{1}{2}B^k_i\}_{k=1}^{\infty}\) are disjoint, where \(B^k_i := B(\xi^k_i, \rho^k_i / 2)\) and \(\rho^k_i := d(\xi^k_i, O_i^c)\).

For any \(E \subseteq X\), we define the “cone” of \(E\) by
\[\mathcal{R}(E) := \{(y, t) : d(y, E) < 2t\}.\] (4.4)

For every \(k = 0, 1, 2, \ldots\), we set
\[\mathcal{R}(B^0_i) := \emptyset, \quad T^k_i := T_i \cap (\mathcal{R}(B^k_i) \setminus \bigcup_{j=0}^{k-1} \mathcal{R}(B^j_i)).\] (4.5)

It is easy to see that \(\hat{O}_i \subseteq \bigcup_{j \in \mathbb{N}} \mathcal{R}(B^j_i)\) and \(T^k_i \cap T^{k'}_{i'} = \emptyset\) if \(k \neq k'\) or \(i \neq i'\). By (4.3) and (4.5), up to sets of measure zero, we can obtain a further decomposition for \(X \times (0, +\infty)\) as follows:
By Hölder’s inequality and the fact that
\[ T_k = \bigcup_{i \in \mathbb{Z}, k \in \mathbb{N}} \left( T_i \cap \mathcal{R}(B_i^k) \right) = \bigcup_{i \in \mathbb{Z}, k \in \mathbb{N}} \left( T_i \cap (\mathcal{R}(B_i^k) \setminus \bigcup_{j=0}^{k-1} \mathcal{R}(B_i^j)) \right) = \bigcup_{i \in \mathbb{Z}, k \in \mathbb{N}} T_i^k. \]  
(4.6)

By (4.1), this leads us to write
\[
\begin{align*}
f &= \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} c_\Psi \int_0^\infty \Phi(t \sqrt{L}) \left( \chi_{T_i^k} t^2 L e^{-t^2 L} f \right) \frac{d\mu(y)dt}{t} \\
&=: \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} \lambda_i^k a_i^k,
\end{align*}
\]  
(4.7)

where \( \lambda_i^k := 2^i \mu(B_i^k)^{1/p} \), \( a_i^k := L^M b_i^k \), and
\[
b_i^k := (\lambda_i^k)^{-1} c_\Psi \int_0^\infty t^{2M} \Phi(t \sqrt{L}) \left( \chi_{T_i^k} t^2 L e^{-t^2 L} f \right) \frac{d\mu(y)dt}{t}.
\]

We see that the sum (4.7) converges in \( L^2(X) \). Indeed, since for each \( f \in L^2(X) \),
\[
\left( \int_{X \times (0, +\infty)} |t^2 L e^{-t \sqrt{L}} f(y)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(X)}.
\]

By (4.7),
\[
\sum_{|i| > N_1, k > N_2} |a_i^k|_{L^2(X)}
\leq c_\Psi \left( \sum_{|i| > N_1, k > N_2} \int_{X \times (0, +\infty)} K(t^2 L)^M \Phi(t \sqrt{L}) \chi_{T_i^k} (y,t) t^2 L e^{-t \sqrt{L}} f(y) \frac{d\mu(y)dt}{t} \right)_{L^2(X)}
\leq \sup_{\|g\|_{L^2(X)} \leq 1} \sum_{|i| > N_1, k > N_2} \int_{T_i^k} \left| (t^2 L)^M \Phi(t \sqrt{L}) g(y) t^2 L e^{-t \sqrt{L}} f(y) \right| \frac{d\mu(y)dt}{t}.
\]

By Hölder’s inequality and the fact that \( T_i^k \) are disjoint for different \( i \) or \( k \), we have
\[
\sum_{|i| > N_1, k > N_2} \int_{T_i^k} \left| (t^2 L)^M \Phi(t \sqrt{L}) g(y) t^2 L e^{-t \sqrt{L}} f(y) \right| \frac{d\mu(y)dt}{t}
\leq \left( \int_{X \times (0, +\infty)} \left| (t^2 L)^M \Phi(t \sqrt{L}) g(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2}
\times \left( \sum_{|i| > N_1, k > N_2} \int_{T_i^k} \left| t^2 L e^{-t \sqrt{L}} f(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2}
\leq C \|g\|_{L^2(X)} \left( \sum_{|i| > N_1, k > N_2} \int_{T_i^k} \left| t^2 L e^{-t \sqrt{L}} f(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2}.
\]
in the last inequality above we have used (2.5). Hence, we obtain that
\[
\left\| \sum_{|i|>N_1, k>N_2} \lambda_i^k a_i^k \right\|_{L^2(X)} \leq C \left( \sum_{|i|>N_1, k>N_2} \int_{T_i^k} |t^2 Le^{-t\sqrt{L}} f(\gamma)|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} \to 0
\]
as \(N_1 \to \infty, N_2 \to \infty\).

Next, we will show that, up to a normalization by a multiplicative constant, the \(a_i^k\) are \((p, \infty, M)\)-atoms. Once the claim is established, we have
\[
\sum_{i \in \mathbb{Z}, k \in \mathbb{N}} |\lambda_i^k|^p = \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} 2^{ip} \mu(B_i^k) \leq 5^n \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} 2^{ip} \mu\left(\frac{1}{5}B_i^k\right)
\]
\[
\leq C \sum_{i \in \mathbb{Z}} 2^{ip} \mu(O_i)
\]
\[
\leq C \|f\|_{H^p_{\lambda\text{max}}(X)}^p
\]
as desired.

Let us now prove that for every \(i \in \mathbb{Z}\) and \(k \in \mathbb{N}\), the function \(C^{-1}a_i^k \) is a \((p, \infty, M)\)-atom associated with the ball \(B(\xi_i^k, 5\rho_i^k)\) for some constant \(C\). Observe that if \((y, t) \in T_i^k\), then \(B(y, 4t) \subseteq O_i\). It implies that \(\rho_i \leq \rho(y, (O_i)^c)\). Note that \(d(y, B_i^k) < 2t\), then \(d(y, (O_i)^c) \leq d(y, B_i^k) + 2\rho_i^k < 2t + 2\rho_i^k\). Hence, we have that \(t < \rho_i^k\). It follows from the finite propagation speed property and Lemma 3.5 of [15] that the integral kernel \(K_{(t^2 L)^k \Phi(t \sqrt{L})}\) of the operator \((t^2 L)^k \Phi(t \sqrt{L})\) satisfies
\[
\text{supp} \ K_{(t^2 L)^k \Phi(t \sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : d(x, y) \leq t\}.
\]
It follows that \(d(x, B_i^k) \leq d(x, y) + d(y, B_i^k) < 3t \leq 3\rho_i^k\) and \(d(x, \xi_i^k) < 4\rho_i^k\).
Hence, for every \(j = 0, 1, \ldots, M\)
\[
\text{supp} \ \left( L^j b_i^k \right) \subseteq B(\xi_i^k, 4\rho_i^k).
\]
It remains to show that \(\| (\rho_i^k)^2 L \| L^j b_i^k \|_{L^\infty(X)} \leq C (\rho_i^k)^{2M} \mu(B_i^k)^{-1/p}, j = 0, 1, \ldots, M\).

In this case \(j = 0, 1, \ldots, M - 1\), it reduces to show
\[
\left| \int_0^\infty \int_{\mathbb{R}^n} K_{(t^2 L)^j \Phi(t \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) t^2 Le^{-t^2 L} f(\gamma) d\mu(y) \frac{dt}{t} \right| \leq C 2^j (\rho_i^k)^{2(M-j)}.
\]
Indeed, if \(\chi_{T_i^k}(y, t) = 1\), then \((y, t) \in (O_i + 1)^c\), and so \(B(y, 4t) \cap (O_i + 1)^c \neq \emptyset\). Let \(\bar{x} \in B(y, 4t) \cap (O_i + 1)^c\). We have that \(|t^2 Le^{-t^2 L} f(\gamma)| \leq M_L f(\bar{x}) \leq 2^{i+1}\). By (i) of Lemma 2.2,
Recall that for any set $E$ there exist intervals $T_k$ such that $\eta(T_k) = \mu(E)$ and $\h(t) = \int_{T_k} f(y) \, d\mu(y)$. We have

$$
\int_0^\infty \int_{\mathbb{R}^n} K_{\rho^2 L} \Phi(t, \sqrt{L}) (x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) \, d\mu(y) \, dt 
\leq C 2^j \int_0^\infty \int_{\mathbb{R}^n} K_{\rho^2 L} \Phi(t, \sqrt{L}) (x, y) \, d\mu(y) \, dt
\leq C 2^j \rho^2 L \int_0^\infty \int_{\mathbb{R}^n} f(y) \, d\mu(y) \, dt
\leq C 2^j \rho^2 L \int_0^\infty \int_{\mathbb{R}^n} f(y) \, d\mu(y) \, dt
$$

since $j = 0, 1, \ldots, M - 1$.

Next, let us consider this case $j = M$. We will show that for every $i \in \mathbb{Z}, k \in \mathbb{N}$,

$$(4.9) \quad \int_0^\infty \int_X K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) \, d\mu(y) \, dt \leq C 2^j.$$ 

To prove (4.9), we need the following result, which plays a crucial role in the proof of (ii) of Theorem 1.3.

**Lemma 4.2.** Fix $x \in O_i$, the properties of the set defining $\chi_{T_i^k}(y, t)$ imply that there exist intervals $(0, b_0), (a_1, b_1), \ldots, (a_N, +\infty)$, where $0 < b_0 \leq a_1 < b_1 \leq \ldots \leq a_N < +\infty$, $1 \leq N \leq 4$ such that for $j = 0, 1, \ldots, N - 1$, there hold $a_{j+1} \leq 3^j b_j$ and

(a) $K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) = 0$ for $t > a_N$;
(b) either $K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) = 0$ or $K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) = K_{\Psi(t, \sqrt{L})} (x, y)$ for all $t \in (a_j, b_j)$;
(c) either $K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) = 0$ or $K_{\Psi(t, \sqrt{L})} (x, y) \chi_{T_i^k}(y, t) = K_{\Psi(t, \sqrt{L})} (x, y)$ for all $t \in (0, b_0)$.

**Proof.** Recall that for any set $E \subset X$, $\mathcal{R}(E)$ is given in (4.4). It is easy to see that for every set $E_1 \subset X$ and $E_2 \subset X$, there holds $\mathcal{R}(E_1) \cup \mathcal{R}(E_2) = \mathcal{R}(E_1 \cup E_2)$. One can write

$$
\mathcal{R}(B_i^j) \cup \mathcal{R}(B_i^1) = \bigcup_{j=0}^{k-1} \mathcal{R}(B_i^j) \bigcup_{j=0}^{k-1} \mathcal{R}(B_i^1)
= \mathcal{R}\left(\bigcup_{j=0}^{k-1} B_i^j\right) \bigcup \mathcal{R}\left(\bigcup_{j=0}^{k-1} B_i^1\right)
= \mathcal{R}(E_i^k) \setminus \mathcal{R}(E_i^{k-1}),
$$

which, in combination with $T_i^k = T_i \cap \mathcal{R}(B_i^k) \setminus \bigcup_{j=1}^{k-1} \mathcal{R}(B_i^j))(\text{see (4.5)), gives}

$$
\chi_{T_i^k}(y, t) = \chi_{\sigma_i}(y, t) \cdot \chi_{\sigma_{\mathcal{R}(E_i^k)}}(y, t) \cdot \chi_{\mathcal{R}(E_i^1)}(y, t) \cdot \chi_{\mathcal{R}(E_i^{k-1})}(y, t)
= \prod_{\ell=1}^{4} \chi_{\ell}(y, t).
$$
From (4.10), we know that if $\chi_{T^k_\ell}(y, t) = 1$, then $\chi_{\ell}(y, t) = 1$ for all $\ell = 1, 2, 3, 4$. That is, if either of $\chi_{\ell}(y, t) = 0$, then $\chi_{T^k_\ell}(y, t) = 0$.

To prove Lemma 4.2, we claim that for $\ell = 1, 2, 3, 4$, there exist numbers $b^{(\ell)}$ and $a^{(\ell+1)}$, $0 < b^{(\ell)} \leq a^{(\ell+1)}$, $a^{(\ell+1)} \leq 3b^{(\ell)}$ such that either $K_{\psi(t\sqrt{L})} (x, y) \chi_{\ell}(y, t) = 0$ or $K_{\psi(t\sqrt{L})} (x, y) \chi_{\ell}(y, t) = K_{\psi(t\sqrt{L})} (x, y)$ for all $t$ in each of the intervals complementary to $(b^{(\ell)}, a^{(\ell+1)})$. And for at least one of $\chi_{\ell}(y, t)$, $K_{\psi(t\sqrt{L})} (x, y) \chi_{\ell}(y, t) = 0$ for $\ell > a^{(\ell+1)}$. By (4.10), we see that

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{T^k_\ell}(y, t) = \prod_{\ell=1}^{4} \chi_{\ell}(y, t) K_{\psi(t\sqrt{L})} (x, y)$$

equals $K_{\psi(t\sqrt{L})} (x, y)$ or 0 when $t$ is in each of the intervals complementary to $\bigcup_{\ell=1}^{4} (b^{(\ell)}, a^{(\ell+1)})$. From this, Lemma 4.2 follows readily.

We now prove our claim. Fix $x \in O_i$ and $d(x, y) < t$. Let us consider the following four cases.

**Case 1** $\chi_1(y, t) = \chi_{\sigma_i^c}(y, t) = 1$.

In this case, we choose $b^{(1)} = \frac{1}{2} d(x, O_i^c)$ and $a^{(2)} = \frac{1}{2} d(x, O_i^c)$, and so $a^{(2)} \leq 3b^{(1)}$. If $t < b^{(1)}$, then $d(y, O_i^c) \geq d(x, O_i^c) - d(x, y) > 5t - t = 4t$. This tells us

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{\sigma_i^c}(y, t) = K_{\psi(t\sqrt{L})} (x, y), \quad \text{for } t < b^{(1)}.$$ 

On the other hand, if $t > a^{(2)}$, then $d(y, O_i^c) \leq d(x, O_i^c) + d(x, y) < 4t$. From this, we have

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{\sigma_i^c}(y, t) = 0, \quad \text{for } t > a^{(2)}.$$ 

**Case 2** $\chi_2(y, t) = \chi_{\overline{O_i}(\sigma_i^c)}(y, t) = 1$.

In this case, we consider two cases: $d(x, O_{i+1}^c) = 0$ and $d(x, O_{i+1}^c) > 0$.

**Subcase 2.1** $d(x, O_{i+1}^c) = 0$.

It follows that $d(y, O_{i+1}^c) \leq d(x, O_{i+1}^c) + d(x, y) < t < 4t$. Hence,

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{\sigma_i^c}(y, t) = K_{\psi(t\sqrt{L})} (x, y), \quad \text{for } t > 0.$$ 

So we can choose $b^{(2)}$ and $a^{(3)}$ to be any positive number. For example, we let $b^{(2)} = b^{(1)}$ and $a^{(3)} = a^{(2)}$.

**Subcase 2.2** $d(x, O_{i+1}^c) > 0$.

Let us choose $b^{(2)} = \frac{1}{2} d(x, O_{i+1}^c)$ and $a^{(3)} = \frac{1}{2} d(x, O_{i+1}^c)$. If $t < b^{(2)}$, then $d(y, O_{i+1}^c) \geq d(x, O_{i+1}^c) - d(x, y) > 5t - t = 4t$, which gives

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{\sigma_i^c}(y, t) = 0 \quad \text{for } t < b^{(2)},$$ 

If $t > a^{(3)}$, then $d(y, O_{i+1}^c) \leq d(x, O_{i+1}^c) + d(x, y) < 4t$. Therefore,

$$K_{\psi(t\sqrt{L})} (x, y) \chi_{\sigma_i^c}(y, t) = K_{\psi(t\sqrt{L})} (x, y), \quad \text{for } t > a^{(3)}.$$
Case 3 \( \chi_3(y, t) = \chi_{\mathcal{R}(E^k_i)}(y, t) = 1 \).

In this case, we consider two cases: \( d(x, E^k_i) = 0 \) and \( d(x, E^k_i) > 0 \).

Subcase 3.1 \( d(x, E^k_i) = 0 \).

It follows that \( d(y, E^k_i) \leq d(x, y) < t < 2t \). Hence,

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^k_i)}(y, t) = K_{\psi(t, \sqrt{tL})}(x, y), \quad \text{for} \quad t > 0.
\]

In this case, we can choose \( b^{(3)} \) and \( a^{(4)} \) to be any positive number. For example, we let \( b^{(3)} = b^{(1)} \) and \( a^{(4)} = a^{(2)} \).

Subcase 3.2 \( d(x, E^k_i) > 0 \).

We choose \( b^{(3)} = d(x, E^k_i)/3 \) and \( a^{(4)} = d(x, E^k_i) \). If \( t < b^{(3)} \), then \( d(y, E^k_i) \geq d(x, E^k_i) - d(x, y) > 3t - t = 2t \). Hence,

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^k_i)}(y, t) = 0 \quad \text{for} \quad t < b^{(3)}.
\]

If \( t > a^{(4)} \), then \( d(y, E^k_i) \leq d(x, E^k_i) + d(x, y) < 2t \). This tells us that for \( t > a^{(4)} \),

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^k_i)}(y, t) = K_{\psi(t, \sqrt{tL})}(x, y).
\]

Case 4 \( \chi_4(y, t) = \chi_{\mathcal{R}(E^{k-1}_i)}(y, t) = 1 \).

In this case, we consider two cases: \( d(x, E^{k-1}_i) = 0 \) and \( d(x, E^{k-1}_i) > 0 \).

Subcase 4.1 \( d(x, E^{k-1}_i) = 0 \).

It follows that \( d(y, E^{k-1}_i) \leq d(x, y) < t < 2t \). Hence,

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^{k-1}_i)}(y, t) = 0, \quad \text{for} \quad t > 0.
\]

We let \( b^{(4)} \) and \( a^{(5)} \) be any positive number. For example, we choose \( b^{(4)} = b^{(1)} \) and \( a^{(5)} = a^{(2)} \).

Subcase 4.2 \( d(x, E^{k-1}_i) > 0 \).

We choose \( b^{(4)} = d(x, E^{k-1}_i)/3 \) and \( a^{(5)} = d(x, E^{k-1}_i) \). If \( t < b^{(4)} \), then \( d(y, E^{k-1}_i) \geq d(x, E^{k-1}_i) - d(x, y) > 3t - t = 2t \). This tells us

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^{k-1}_i)}(y, t) = K_{\psi(t, \sqrt{tL})}(x, y) \quad \text{for} \quad t < b^{(4)}.
\]

If \( t > d(x, E^{k-1}_i) \), then \( d(y, E^{k-1}_i) \leq d(x, E^{k-1}_i) + d(x, y) < 2t \). Therefore,

\[
K_{\psi(t, \sqrt{tL})}(x, y)\chi_{\mathcal{R}(E^{k-1}_i)}(y, t) = 0, \quad \text{for} \quad t > a^{(5)}.
\]

From Cases 1, 2, 3 and 4, we have obtained our claim, and then, the proof of Lemma 4.2 is complete. \( \square \)
Back to the proof of (ii) of Theorem 1.3. We continue to show (4.9). Note that the conditions $d(x, y) < t$ and $B(y, 4t) \in O_i$ imply that $x \in O_i$. If $x \in O_i$, then

$$
\int_0^\infty \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} = 0.
$$

Fix $x \in O_i$. We apply Lemma 4.2 to write

$$
\int_0^\infty \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t}
= \left\{ \int_0^{b_0} + \sum_{l=1}^{N-1} \int_{a_l}^{b_l} \right\} \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t}
+ \sum_{l=0}^{N-1} \int_{b_l}^{a_{l+1}} \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t}
= I_1(x) + I_2(x).
$$

(4.11)

To estimate $I_1(x)$, we note that if $0 \leq a < b \leq b_1$ or $a_l \leq a < b \leq b_l$, then one has either

$$
\int_a^b \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} = 0,
$$
or by (4.2),

$$
\int_a^b \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t}
= \int_a^b \Psi(t \sqrt{T}) t^2 L e^{-t^2 L} f(x) \frac{dt}{t}
= \eta(a \sqrt{T}) f(x) - \eta(b \sqrt{T}) f(x).
$$

Observe that for each $a \leq t \leq b$, if $d(x, y) < t$, then $\chi_{T_i^k}(y, t) = 1$. This tells us that $(y, t) \in (O_{i+1})^c$, hence $B(y, 4t) \cap (O_{i+1})^c \neq \emptyset$. Assume that $\bar{x} \in B(y, 4t) \cap (O_{i+1})^c$. From this, we have that $d(x, \bar{x}) \leq d(x, y) + d(y, \bar{x}) < 5t$ and $\mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$. It implies that $|\eta(t \sqrt{T}) f(x)| \leq \mathcal{M}_L f(\bar{x}) \leq C 2^{i+1}$ for every $a \leq t \leq b$. Therefore, $|\eta(a \sqrt{T}) f(x)| \leq C 2^{i+1}$ and $|\eta(b \sqrt{T}) f(x)| \leq C 2^{i+1}$, and so $|I_1(x)| \leq C 2^{i+1}$.

Consider $I_2(x)$. If $\chi_{T_i^k}(y, t) = 1$, then $(y, t) \in (O_{i+1})^c$. Thus $B(y, 4t) \cap (O_{i+1})^c \neq \emptyset$. Assume that $\bar{x} \in B(y, 4t) \cap (O_{i+1})^c$. We have that $|t^2 L e^{-t^2 L} f(y)| \leq \mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$. This, together with $a_{l+1} \leq 3^l b_l (l = 0, 1, \ldots, N - 1)$, implies that

$$
\left| \int_{b_l}^{a_{l+1}} \int_X K_{\psi(t, \sqrt{T})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} \right|
\leq 2^{i+1} \left| \int_{b_l}^{a_{l+1}} \int_X |K_{\psi(t, \sqrt{T})}(x, y)| d\mu(y) \frac{dt}{t} \right|
\leq C 2^{i+1} \int_{b_l}^{a_{l+1}} \frac{1}{t} dt \leq C 2^{i+1},
$$

(4.12)
which yields that $|I_2(x)| \leq C2^{i+1}$.

Combining estimates of $I_1(x)$ and $I_2(x)$, we have obtained (4.9). This proves (ii) of Theorem 1.3, and the proof of Theorem 1.3 is complete. \qed

Finally, we assume that an operator $L$ satisfies conditions (H1)–(H2). For $f \in L^2(X)$, we define an area function $S_L f$ associated with the heat semigroup generated by $L$,

$$S_L f(X) := \left( \int_0^\infty \int_{d(x,y) < t} \left| \int_{t^2L} \frac{d\mu(y)dt}{tV(y,t)} \right|^2 \mu(y) \, dy \right)^{1/2}, \quad x \in X. \quad (4.13)$$

Given $0 < p \leq 1$. The Hardy space $H^p_{L,S}(\mathbb{R}^n)$ is defined as the completion of $\{ f \in L^2(X) : \| S_L f \|_{L^p(X)} < \infty \}$ with norm

$$\| f \|_{H^p_{L,S}(X)} := \| S_L f \|_{L^p(X)}.$$

From [11,15] and Theorem 1.3,

$$H^p_{L,at,q,M}(X) \simeq H^p_{L,S}(X) \simeq H^p_{L,max}(X) \simeq H^p_{L,rad}(X)$$

for every $0 < p \leq 1$, and for all $q > p$ with $1 \leq q < \infty$ and all integers $M > \frac{n}{2}(\frac{1}{p}-1)$.

Acknowledgements

The authors would like to thank the referee for carefully reading the manuscript and for offering numerous valuable suggestions. L. Song was supported by NNSF of China (Grant Nos. 11471338 and 11622113) and Guangdong Natural Science Funds for Distinguished Young Scholar (No. 2016A030306040). L. Yan was supported by NNSF of China (Grant Nos. 11371378 and 11521101) and Guangdong Special Support Program.

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