SPECTRAL DIMENSION FOR 
\( \beta \)-ALMOST PERIODIC SINGULAR JACOBI OPERATORS 
AND THE EXTENDED HARPER’S MODEL

By

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Abstract. We study fractal dimension properties of singular Jacobi operators. We prove quantitative lower spectral/quantum dynamical bounds for general operators with strong repetition properties and controlled singularities. For analytic quasiperiodic Jacobi operators in the positive Lyapunov exponent regime, we obtain a sharp arithmetic criterion of full spectral dimensionality. The applications include the extended Harper’s model where we obtain arithmetic results on spectral dimensions and quantum dynamical exponents.

1 Introduction

In this paper, we study self-adjoint Jacobi operators on \( \ell^2(\mathbb{Z}) \) given by:

\[(Hu)_n = w_n u_{n+1} + \overline{w}_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z},\]

where \( w_n \in \mathbb{C} \setminus \{0\} \) and \( v_n \in \mathbb{R} \) are bounded sequences in \( n \). If \( w_n \equiv 1 \), \( H \)

is a discrete Schrödinger operator. We will focus on singular Jacobi operators, where the off-diagonal sequence \( w_n \) has an accumulation point 0 at \( \pm \infty \). A prime example of such operator, both in math and in physics literature, is the extended Harper’s model (EHM); see (2.13).

We are interested in the fractal decomposition of the spectral measure and quantitative spectral/quantum dynamical bounds. In a recent work of Jitomirskaya and Zhang [32], many quantitative criteria of fractal dimensions of spectral measure for lattice Schrödinger operators were obtained. Their criteria have various applications to quasiperiodic Schrödinger operators, e.g., the almost Mathieu operator or the Sturmian Hamiltonians. However, whether their results could be applied to the Jacobi case, in particular the singular Jacobi case, has remained a question. Indeed, many generalizations of the spectral theory from the Schrödinger case to the singular Jacobi case have been shown to be highly non-trivial, e.g., [23, 28, 36, 22, 21, 3, 40].
In this paper we give general sufficient conditions for spectral continuity in the singular Jacobi case; see Theorem 2.1. We show spectral continuity follows if the parameters \( w_n, v_n \) of \( H \) satisfy: (i) strong repetition properties; (ii) control of the averaged closeness between \( w_n \) and 0. In particular, condition (ii) is imposed to control the strength of singularity. We also show such operators exist widely in the general context of quasiperiodic setting. In the positive Lyapunov exponent regime of analytic quasiperiodic Jacobi operators, the general statement leads to the first arithmetic if-and-only-if criterion for full spectral dimensionality. Notably, our results have applications to the extended Harper’s model for both spectral and quantum dynamical properties.

Our proof is based on a general dynamical system approach, which has recently shown to be extremely powerful in the study of spectral properties, e.g., [34, 5, 33, 16, 12, 24, 35]. The eigenvalue equation of (1.1) is associated to a linear cocycle system; see (3.3). In the Schrödinger case the cocycles are SL(2, \( \mathbb{R} \))-valued, whereas in the singular Jacobi case the cocycles are GL(2, \( \mathbb{C} \))-valued with determinants approaching zero along a subsequence. This presents the main obstruction in [23, 28, 36, 22, 21, 3, 40] and in our paper.

It was shown in [32] that the fractal dimensions of spectral measures depend on the competition between the quality of repetitions and the growth of the Schrödinger cocycles. Such competition was resolved in the SL(2, \( \mathbb{R} \)) setting involving delicate algebraic arguments, which are difficult to carry out directly in the GL(2, \( \mathbb{C} \)) setting due to the presence of a singularity. To reduce to the SL(2, \( \mathbb{R} \)) case, we employ a family of conjugacies which were first introduced in a recent work of Avila–Jitomirskaya–Marx [3]. Such regularization moves the singularity into the conjugate matrices. The main technical accomplishment of our work is to develop general quantitative estimates (see Lemmas 5.3, 5.4 and 5.5) of the conjugacy under assumptions (i) and (ii). The successful combination of these estimates with the mechanism in [32] proves the quantitative spectral continuity results for the singular Jacobi case.

We show the assumptions (i) and (ii) hold for singular Jacobi operators over a quasiperiodic base. In particular, the proof of (ii) is close in spirit to the characterization of singularity in [22] (see also [27, 31]). Here we need to study the finer decomposition of the singular spectral measure, thus a strengthened characterization is developed. Moreover, our estimates hold for general \( C^k \) sampling functions with finitely many non-degenerate zeros, which reduces the analytic regularity requirements in [27, 31, 22]. This part is also of independent interest in the study of uniform upper semi-continuity of the growth of the Lyapunov exponent.
The rest of this paper is organized in the following way. In Section 2, we give all the definitions and state our main results. After giving the preliminaries in Section 3, we proceed to discuss the $(\Lambda, \beta)$ bound in the quasiperiodic case in Section 4. In Section 5, we prove the general spectral continuity results. In Section 6, we focus on the analytic quasiperiodic Jacobi operator and prove an arithmetic if-and-only-if criterion for full spectral dimensionality. In the last section, we discuss the explicit parameter partitions for the extended Harper’s model.

2 Main results

To formulate the main results, we introduce the following definitions.

**Definition 2.1.** A sequence $\{a_n\}_{n \in \mathbb{Z}}$ is said to be $\beta$-$q$ almost periodic if there exist $\delta > 0$, $\beta > 0$, $q \in \mathbb{N}$, such that the following holds:

$$\max_{|m| \leq e^{\delta \beta q}} |a_m - a_{m \pm q}| \leq e^{-\beta q}.$$  

We say $\{a_n\}_{n \in \mathbb{Z}}$ is $\beta$-almost periodic (about $q_n$) if there exists a sequence of positive integers $q_n \to \infty$, such that $\{a_n\}$ is $\beta$-$q_n$ almost periodic.

**Remark 2.1.** The $\beta$-almost periodicity was first introduced in [32] to study quantitative spectral bounds in the Schrödinger case. Note that the $\beta$-almost periodicity does not imply the almost periodicity in the usual sense. A typical example is the sequence generated by skew-shift map

$$(x, y) \mapsto (x + y, y + 2\alpha)$$

with a smooth sampling function $f(x, y)$ on $\mathbb{T}^2$. The sequence

$$v_n = f(x + ny + n(n - 1)a, y + 2na)$$

is $\beta$-almost periodic for typical $\alpha$, but not almost periodic for any irrational $\alpha$.

**Definition 2.2.** We say $\{w_n\}_{n \in \mathbb{Z}}$ is $(\Lambda, \beta)$-$q$ bounded if there exist $\Lambda > 0$, $\beta > 0$, $\delta > 0$, and $q \in \mathbb{N}$, such that

$$\min_{|m| \leq e^{\delta \beta q}} \prod_{j=m}^{m+q-1} |w_j| > e^{-\Lambda q}.$$  

We say $w_n$ is $(\Lambda, \beta)$ bounded (about $q_n$) if there exists a sequence of positive integers $q_n \to \infty$, such that $w_n$ is $(\Lambda, \beta)$-$q_n$ bounded.
Remark 2.2. If we only consider the maximum of (2.1) and (2.2) over $|m| \leq 2q_n$, then the standard Gordon-type argument will be enough to show the absence of a point spectrum for the associated Jacobi operator, provided $\beta \gtrsim \Lambda$. Assume further the Lyapunov exponent is positive, then the operator has purely singular continuous spectrum by Kotani theory; see, e.g., [8, 6]. See more discussion on the Gordon-type argument and purely singular continuous spectrum in [9] and references therein.

Let $\mu$ be the spectral measure of the Jacobi operator given as in (1.1). The fractal properties of $\mu$ are closely related to the boundary behavior of its Borel transforms; see, e.g., [13]. Let

\begin{equation}
M(E + i\epsilon) = \int \frac{d\mu(E')}{E' - (E + i\epsilon)}
\end{equation}

be the (whole line) Weyl–Titchmarsh $m$-function of $H$. We are interested in the following fractal dimension of $\mu$:

**Definition 2.3.** We say $\mu$ is (upper) $\gamma$-spectral continuous if for some $\gamma \in (0, 1)$ and $\mu$ a.e. $E$, we have

\begin{equation}
\liminf_{\epsilon \downarrow 0} \epsilon^{1-\gamma} |M(E + i\epsilon)| < \infty.
\end{equation}

Define the (upper) spectral dimension of $\mu$ to be

\begin{equation}
\dim_{\text{spe}}(\mu) = \sup\{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral continuous} \}.
\end{equation}

Our first result is about spectral continuity and the lower bound on the spectral dimension.

**Theorem 2.1.** Let $H$ be given as in (1.1) and let $\mu$ be the spectral measure of $H$. Assume that there are positive constants $\Lambda$, $\beta$, $\delta$ and a sequence of positive integers $q_k \to \infty$ such that $w_n$, $v_n$ are $\beta$-almost periodic and $w_n$ is $(\Lambda, \beta)$ bounded about $q_k$. There exists an explicit constant $C = C(\delta, \Lambda, \|w\|_{\infty}, \|v\|_{\infty}) > 0$, such that if $\beta > C$ and $\gamma < 1 - \frac{C}{\beta}$, then $\mu$ is $\gamma$-spectral continuous. Consequently, we have the following lower bound on the spectral dimension of $\mu$:

\begin{equation}
\dim_{\text{spe}}(\mu) \geq 1 - \frac{C}{\beta}.
\end{equation}

We will formulate a more precise lower bound (specifying the dependence of $C$ on $\delta, \Lambda, \|w\|_{\infty}, \|v\|_{\infty}$) in Theorem 5.2.

It is well known that periodicity implies absolute continuity. We actually prove a quantitative weakening version of this result: $\beta$-almost periodicity implies $\gamma$-spectral continuity. On the other hand, it is well known that the Gordon condition implies absence of a point spectrum, which predicts a purely singular continuous spectrum in many situations; see, e.g., [9, 6]. Our result distinguishes the
singular continuous spectrum further according to their spectral dimensions. This can be viewed as a quantitative strengthening of Gordon-type results. Quantitative results directly linking easily formulated properties of the potential to dimensional/quantum dynamical results were first proved in [32] for the Schrödinger case. Theorem 2.1 was a further generalization of this type of estimates to more general singular Jacobi operators.

An important context where we have generic $\beta$-almost periodicity and $(\Lambda, \beta)$ bound is the quasiperiodic Jacobi operators with smooth sampling functions defined as follows. Consider real and complex valued sampling functions $v: \mathbb{T} \mapsto \mathbb{R}$ and $c : \mathbb{T} \mapsto \mathbb{C}$. We also assume $\ln |c| \in L^1(\mathbb{T})$, which is the minimum requirement for the Lyapunov exponent to exist. Let $H_{\alpha,\theta} = H_{\alpha,\theta,c,v}$ be the Jacobi operator on $\ell^2(\mathbb{Z})$ given by

$$ (H_{\alpha,\theta} u)_n = c(\theta + n\alpha) u_{n+1} + \overline{c}(\theta + (n-1)\alpha) u_{n-1} + v(\theta + n\alpha) u_n, \quad n \in \mathbb{Z}, $$

where $\theta \in \mathbb{T} := [0, 1]$ is the phase, $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $\overline{c}(\theta)$ is the complex conjugate of $c(\theta)$ in the usual sense.

Given $\alpha$, let $p_n/q_n$ be the continued fraction approximants to $\alpha$. Define

$$ \beta(\alpha) := \limsup_n \frac{\ln q_{n+1}}{q_n} \in [0, \infty]. $$

It is easy to check that for any Lipschitz continuous sampling functions $v$ and $c$, the sequences $v(\theta + n\alpha), c(\theta + n\alpha)$ are $\beta$-almost periodic as defined in (2.1) for any $\theta \in \mathbb{T}$ and any $\beta < \beta(\alpha)/2$. See a proof about this simple fact in Section 4. Furthermore, if we require some non-degenerate regularity near the zeros of $c$, then $c(\theta + na)$ will be $(\Lambda, \beta)$ bounded for a.e. $\theta$. As a consequence of Theorem 2.1, we have spectral continuity for a.e. $\theta$ for (2.7). More precisely, let $\mu_{\alpha,\theta}$ be the spectral measure of $H_{\alpha,\theta}$ (2.7), then

**Corollary 2.2.** Assume $v(\theta)$ is Lipschitz continuous on $\mathbb{T}$ and $c(\theta)$ is $C^k$ continuous on $\mathbb{T}$ with finitely many non-degenerate zeros. For all $k = 1, 2, \ldots$, there exists an explicit constant $C = C(c, v, k) > 0$ and a full measure set $\Theta = \Theta(\alpha, c) \subseteq \mathbb{T}$, only depending on $\alpha$ and the zeros of $c(\theta)$ with the following properties: suppose $\beta(\alpha) > C$, then for any $\theta \in \Theta$,

(a) $H_{\alpha,\theta}$ has no eigenvalues in the spectrum;
(b) the spectral dimension of $\mu_{\alpha,\theta}$ is bounded from below as

$$ \dim_{\text{spec}}(\mu_{\alpha,\theta}) \geq 1 - \frac{C}{\beta(\alpha)}. $$

In particular, if $\beta(\alpha) = \infty$, then for a.e. $\theta$, $\dim_{\text{spec}}(\mu_{\alpha,\theta}) = 1$.\hspace{1cm}^{1}

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1We say $\theta_0 \in \mathbb{T}$ is a non-degenerate zero of $f \in C^k(\mathbb{T}, \mathbb{C})$ if $f(\theta_0) = 0$ and $f^{(k)}(\theta_0) \neq 0$. 
We will prove the \((\Lambda, \beta)\) bound of \(c(\theta + n\alpha)\) for a.e. \(\theta\) in Section 4 and then part (b) follows directly from Theorem 2.1. The main ingredient is one fundamental estimate (see Lemma 4.1) about the trigonometric product over irrational rotation in [1]. Similar arguments have been used in [27, 22, 31] to study the arithmetic criterion of a purely singular continuous spectrum. In those papers, the authors considered periodic approximation based on a Gordon-type argument. The growth of the transfer matrix only need be controlled within at most two periods. In our case, the quantitative spectral continuity relies on the \((\Lambda, \beta)\) bound over exponentially many periods. The use of Lemma 4.1 is more delicate and involved. See more details in Lemma 4.2.

As mentioned before, the absence of a point spectrum in part (a) is a direct consequence of the \((\Lambda, \beta)\) boundedness of \(c(\theta + n\alpha)\) and the standard Gordon-type argument. In view of Definition 2.3, it is easy to check that a point measure has zero (spectral/Hausdorff/packing) dimension; (2.9) implies that the spectral measure \(\mu_{a,\theta}\) has positive spectral dimension for \(\beta(\alpha) > C\). Part (a) can also be derived as a corollary of (2.9). An interesting question that remained here is whether the assumption on \(c(\theta)\) can be weakened: For example, could any Lipschitz continuous function with finitely many zeros generate a \((\Lambda, \beta)\) bounded sequence? Will the associated Jacobi operator have absence of a point spectrum and full spectral dimension? We will not go further in this direction in the current paper. Some of these questions are partially answered in a recent preprint by the second and the third authors, [41].

It is clear that our general results (2.6) and (2.9) only go in one direction, as even absolute continuity of the spectral measures does not imply \(\beta\)-almost periodicity for \(\beta > 0\). However, in the important context of analytic quasiperiodic operators (e.g., EHM) this leads to a sharp if and only if result in the positive Lyapunov exponent regime.

Let \(H_{a,\theta}\) be the Jacobi operator on \(\ell^2(\mathbb{Z})\) defined as in (2.7). The Lyapunov exponent of \(H_{a,\theta}\) at energy \(E\) is defined through the associated skew-product over irrational rotations (quasiperiodic cocycles). For any irrational \(\alpha\), the Lyapunov exponent is only a function of \(E, \alpha\) and is independent of \(\theta\), therefore, denoted as \(L(E, \alpha)\). See more basic properties and discussions about Lyapunov exponent in Section 3.

Assume further \(v, c\) of \(H_{a,\theta}\) are analytic on \(\mathbb{T}\) with real and complex values, respectively. Let \(L(E, \alpha)\) be the Lyapunov exponent, \(\sigma(H_{a,\theta})\) be the spectrum of \(H_{a,\theta}\), and let \(\mu_{a,\theta,\Sigma}\) be the restriction of the spectral measure \(\mu_{a,\theta}\) of \(H_{a,\theta}\) on \(\Sigma := \{ E \in \sigma(H_{a,\theta}) : L(E, \alpha) > 0 \}\). We have the following sharp estimate on the spectral dimension of \(\mu_{a,\theta,\Sigma}\):
Theorem 2.3. For any $\alpha \in [0, 1]$, let $\beta(\alpha)$ be defined as in (2.8). For any analytic sampling functions $v$ and $c$, there is a full Lebesgue measure set $\Theta = \Theta(\alpha, c) \subseteq \mathbb{T}$ that explicitly depends on $\alpha$ and $c$ such that for any $\theta \in \Theta$, $\dim_{\text{spe}}(\mu, \alpha, \theta, \Sigma) = 1$ if and only if $\beta(\alpha) = \infty$.

The proof of Theorem 2.3 contains two parts. Clearly, the ‘if’ part of Theorem 2.3 is a direct consequence of spectral continuity and follows from Corollary 2.2. The ‘only if’ part is usually referred to as the so-called spectral singularity, defined through the singular boundary behavior of the $m$ function. More precisely, we say the spectral measure $\mu$ is (upper) $\gamma$-spectral singular if for some $\gamma \in (0, 1)$ and $\mu$ a.e. $E$,

\[
\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| = +\infty.
\]

Define

\[
\dim_{\text{sp}}(\mu) = \inf\{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral singular} \}.
\]

Obviously, $\dim_{\text{spe}}(\mu) \leq \dim_{\text{sp}}(\mu)$. Theorem 2.3 also holds for $\dim_{\text{sp}}(\mu)$. We actually can prove the following local quantitative upper bound of the spectral dimension which completes the sufficient part of Theorem 2.3.

Theorem 2.4. Consider the quasiperiodic Jacobi operator defined in (2.7) with analytic sampling functions $v, c$. Let $L(E)$ be the associated Lyapunov exponent defined in (3.12). Assume $L(E) \geq a > 0$ on a compact set $S$. Consider the spectral measure $\mu_{\alpha, \theta}$ restricted on $S$, denoted by $\mu_{\alpha, \theta, S}$. Suppose $\beta(\alpha) < \infty$, then there is a $C = C(a, v, c, S) > 0$ and a full Lebesgue measure set $\Theta = \Theta(\alpha, c)$, such that for any $\theta \in \Theta$, $\mu_{\alpha, \theta}$ is $\gamma$-spectral singular for any $\gamma \geq \frac{1}{1 + \frac{C}{\beta(\alpha)}}$. Consequently,

\[
\dim_{\text{spe}}(\mu, \alpha, \theta, S) \leq \dim_{\text{sp}}(\mu, \alpha, \theta, S) \leq \frac{1}{1 + \frac{C}{\beta(\alpha)}} < 1.
\]

The spectral singularity can be viewed as a “weak-type of localization”. It involves the decay of the Green’s function in a finite box with a low density (see Lemma 6.4). Such decay/localization density was previously known either with a strong non-resonance condition on $\alpha$ (e.g., $\beta(\alpha) = 0$, see [12]), or for a concrete example with $\beta(\alpha) \lesssim L$ (see [4, 26]). Such a phenomenon was first found in [32] for general analytic quasiperiodic Schrödinger operators with extremely large $\beta$.

Two crucial ingredients for the quantitative spectral singularity are:

1. quantitative subordinacy theory (Jitomirskaya–Last inequality, Lemma 3.2);
2. existence of generalized eigenfunctions with sub-linear growth by the Last–Simon estimate (Lemma 3.3).
Theorem 2.4 generalizes the result for Schrödinger operators in [32] to singular Jacobi operators. The techniques to deal with the singular Jacobi case are more involved and very delicate in view of the quantitative estimates (2.12). In Section 6, we reduce the proof of Theorem 2.4 to a quantitative result (see Lemma 6.1) obtained in [32]. One key observation in [32] is that the norm of the analytic transfer matrix can be approximated by trigonometric polynomials with uniform linear degree. The generalization of this result to the meromorphic transfer matrix in our case (see Lemma 6.2) becomes an important part of the proof of Theorem 2.4.

2.1 Applications to the extended Harper’s model. Quasiperiodic Jacobi operators arise naturally from the study of tight-binding electrons on a two-dimensional lattice exposed to a perpendicular magnetic field. A more general model is the extended Harper’s model (EHM), defined as follows:

\[
(H_{\lambda,\alpha,\theta})_n = c_\lambda(\theta + n\alpha)u_{n+1} + \bar{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}.
\]

Here,

\[
c_\lambda(\theta) = \lambda_1 e^{-2\pi i (\theta + \frac{3\theta}{2})} + \lambda_2 + \lambda_3 e^{2\pi i (\theta + \frac{\theta}{2})},
\]

\(\bar{c}_\lambda(\theta)\) is the complex conjugate of \(c_\lambda(\theta)\) in the usual sense and \(\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\) are real coupling constants. EHM was introduced by D. J. Thouless in 1983 [39], which includes the AMO as a special case.

The extended Harper’s model is a prime example of a quasiperiodic Jacobi matrix. It has attracted great attention from both mathematics and physics (see, e.g., [7, 10, 20]) literature in the past several decades. Recent developments on the spectral theory of the AMO and EHM include: a pure point spectrum for Diophantine frequencies in the positive Lyapunov exponent regime in [23]; an explicit formula for the Lyapunov exponent \(L(E, \lambda)\) (see (7.3)) on the spectrum throughout all the three regions [28]; the dry ten Martini problem for Diophantine frequencies in the self-dual regions [21]; complete spectral decomposition for all \(\alpha\) and a.e. \(\theta\) in the zero Lyapunov exponent regimes [3]; and arithmetic spectral transition in \(\alpha\) in the positive Lyapunov exponent regime [22].

As a central example of the analytic quasiperiodic singular Jacobi operators, Theorem 2.3 can be applied to the extended Harper’s model \(H_{\lambda,\alpha,\theta}\) defined in (2.13). As a consequence of the Lyapunov exponent formula of EHM in terms of the coupling constants \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\), we have more explicit conclusions on the full spectral dimensionality of EHM. Moreover, our lower bounds in Theorem 2.3
are effective for $\beta(\alpha) > \max\{ C \sup_{E \in \sigma(H)} L(E), 0 \}$, where $\sigma(H)$ is the spectrum of $H$, by some simple scaling arguments (see Lemma 4.2 and Section 7). Thus the range of $\beta(\alpha)$ is increased for smaller Lyapunov exponents. In particular, we obtain full spectral dimensionality as long as $\beta(\alpha) > 0$, when Lyapunov exponents are zero on the spectrum. This applies, in particular, to the parameter region $R_3$ (see below) of EHM.

For the parameter partitions of the EHM, we consider the following three parameter regions of $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$:

$$R_1 = \{ \lambda \in \mathbb{R}^3 : 0 < \lambda_1 + \lambda_3 < 1, 0 < \lambda_2 < 1 \}.$$  

$$R_2 = \left\{ \lambda \in \mathbb{R}^3 : \begin{array}{c} \lambda_2 > \max\{ \lambda_1 + \lambda_3, 1 \}, \lambda_1 + \lambda_3 \geq 0 \\
\text{or } \lambda_1 + \lambda_3 > \max\{ \lambda_2, 1 \}, \lambda_1 \neq \lambda_3, \lambda_2 > 0 \end{array} \right\}.$$  

$$R_3 = \left\{ \lambda \in \mathbb{R}^3 : \begin{array}{c} 0 \leq \lambda_1 + \lambda_3 \leq 1, \lambda_2 = 1 \\
\text{or } \lambda_1 + \lambda_3 \geq \max\{ \lambda_2, 1 \}, \lambda_1 = \lambda_3, \lambda_2 > 0 \end{array} \right\}.$$  

**Corollary 2.5.** Let $\mu_{\lambda, \alpha, \theta}$ be the spectral measure of EHM: $H_{\lambda, \alpha, \theta}$. For any $\alpha \in [0, 1]$, there is a full measure set $\Theta = \Theta(\alpha) \subset \mathbb{T}$ such that for all $\theta \in \Theta$, the following hold:

1. For $\lambda \in R_1$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$ if and only if $\beta(\alpha) = \infty$.
2. For $\lambda \in R_2$ and for all $\alpha \in [0, 1]$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$.
3. For $\lambda \in R_3$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$ if $\beta(\alpha) > 0$.

We will see the explicit formula of the Lyapunov exponent and the spectral decomposition of EHM, in Section 7. In region $R_1$, EHM has positive Lyapunov exponent for all $\alpha$. Part (1) then follows from Theorem 2.3 directly. Region $R$ is actually where EHM has purely absolutely continuous measure for all $\alpha$ and a.e. $\theta$, see [3] and Theorem 7.2 in Section 7. In view of Definition 2.3, it is well known that if a measure is absolutely continuous w.r.t. Lebesgue measure, then it has full spectral dimension. Part (2) is then a direct consequence of the a.c. spectrum and this fact. We list part (2) here for completeness only. $R_3$ is the region where EHM has zero Lyapunov exponent and purely singular continuous spectrum for almost all $(\theta, \alpha)$, part (3) follows from Theorem 2.1 and some technical improvements of the $(\Lambda, \beta)$ bound for analytic sampling functions. We will discuss in detail about these three parts in Section 7.

Full spectral dimensionality is defined through the boundary behavior of the Borel transform of the spectral measure. It implies a range of properties, in particular, maximal packing dimension and quasiballistic quantum dynamics. Thus our criterion links a purely analytic property of the spectral measure to the arithmetic
property of the frequency in a sharp way. In particular, consider $H_{\lambda,\alpha,\theta}$, the extended Harper’s model (EHM) given in (2.13). In this part, we will focus on EHM and discuss the consequences of the full spectral dimensionality in terms of these explicit parameters.

Recall that the Hausdorff/packing dimension of a (Borel) measure $\mu$, namely, $\dim_H(\mu)/\dim_P(\mu)$ is defined through the lim sup / lim inf ($\mu$ almost everywhere) of its local scaling exponents:

$$\lim_{\epsilon \downarrow 0} \frac{\ln \mu(E-\epsilon, E+\epsilon)}{\ln \epsilon}.$$ 

If the lim inf is replaced by lim sup in the Definition 2.3, we can define correspondingly the lower spectral dimension $\dim_{\text{spe}}(\mu)$. It is well known (see, e.g., [10, 14, 32]) that the relation between these fractal dimensions

$$\dim_H(\mu) = \dim_{\text{spe}}(\mu) \leq \dim_{\text{spe}}(\mu) \leq \dim_P(\mu).$$

Therefore, lower bounds on spectral dimension lead to lower bounds on packing dimension, thus also for the packing/upper box counting dimensions of the spectrum as a set. We obtain corresponding non-trivial results for all the above quantities. The lower bounds also provide explicit examples where the spectral measure has different Hausdorff and packing dimension.

Lower bounds on spectral dimension also have immediate applications to the lower bounds on quantum dynamics. Let $\delta_j \in \ell^2(\mathbb{Z})$ be the delta vector in the usual sense. For $p > 0$, define

$$\langle |X|_{\delta_0}^p \rangle(T) = \frac{2}{T} \int_0^T e^{-2it/T} \sum_n |n|^p \langle e^{-itH} \delta_0, \delta_n \rangle|^2.$$ 

The power law of $\langle |X|_{\delta_0}^p \rangle(T)$ characterizes the propagation rate of $e^{-itH} \delta_0$. Define the upper/lower transport exponents to be

$$\beta^+(\delta_0)(p) = \limsup_{T \to \infty} \frac{\ln \langle |X|_{\delta_0}^p \rangle(T)}{p \ln T}, \quad \beta^-(\delta_0)(p) = \liminf_{T \to \infty} \frac{\ln \langle |X|_{\delta_0}^p \rangle(T)}{p \ln T};$$

$\beta^+(\delta_0)(p) = 1$ for all $p > 0$ corresponds to ballistic motion, and $\beta^+(\delta_0)(p) = 1$ for all $p > 0$ corresponds to quasiballistic motion. $\beta^-(\delta_0)(p) = 0$ is sometimes called quasilocalized motion. It was proved in [19] that $\beta^+(\delta_0)(p) \geq \dim_P(\mu), \forall p > 0$. In view of Corollary 2.5, we have:

\[\text{2In contrast to the Hausdorff dimension, the relation for the packing dimension only goes in one direction.}\]
Corollary 2.6. Let \( \mu_{\lambda,a,\theta} \) be the spectral measure of EHM: \( H_{\lambda,a,\theta} \) defined in (2.13). For any \( a \in [0,1] \), there is a full measure set \( \Theta = \Theta(a) \subset \mathbb{T} \) such that for any \( \theta \in \Theta \), \( H_{\lambda,a,\theta} \) has full packing dimension of \( \mu_{\lambda,a,\theta} \) and quasiballistic motion if
1. \( \lambda \in \mathbb{R}_1 \) and \( \beta(a) = \infty \).
2. \( \lambda \in \mathbb{R}_2 \) and for all \( a \in [0,1] \).
3. \( \lambda \in \mathbb{R}_3 \) and \( \beta(a) > 0 \).

Hausdorff dimension of the spectral measure is always equal to zero for a.e. phase for any ergodic operator [38] in the regime of positive Lyapunov exponents. Combining the Lyapunov exponent formula of \( H_{\lambda,a,\theta} \) (see (7.3)) with the result of Simon in [38], we have
\[
\dim_H(\mu_{\lambda,a,\theta}) = 0
\]
for \( \lambda \in \mathbb{R}_1 \), a.e. \( \theta \) and any \( a \). In view of part (1) of Corollary 2.6, for \( \lambda \in \mathbb{R}_1 \), a.e. \( \theta \) and \( \beta(a) = \infty \), we have
\[
0 = \dim_H(\mu_{\lambda,a,\theta}) < \dim_P(\mu_{\lambda,a,\theta}) = 1.
\]

We are also interested in the fractal dimensional properties of the density of states measure and the dimension of the spectrum as a set. Let \( dN_{\lambda,a} \) be the density of states measure and \( \Sigma_{\lambda,a} \) be the spectrum of \( H_{\lambda,a,\theta} \). For irrational \( a \), they are both \( \theta \) independent. It is well known that
\[
dN_{\lambda,a} = \mathbb{E}_\theta(\mu_{\lambda,a,\theta})
\]
and \( \Sigma_{\lambda,a} = \text{supp}(dN_{\lambda,a}) \). By these relations and the general properties of the packing dimension of a measure and its (topological) support (see, e.g., [14]), Corollary 2.6 implies that
\[
\dim_P(dN_{\lambda,a}) = \dim_P(\Sigma_{\lambda,a}) = 1
\]
in the corresponding parameter regions where the spectral measure of EHM has full packing dimension.

For the dynamical transport part, Last in [34] proved that the almost Mathieu operator with an appropriate Liouville frequency has quasiballistic motion for the first time. In general, the quasiballistic property is a \( G_\delta \) in any regular space (see, e.g., [37, 16]), thus this was known for (unspecified) topologically generic frequencies. In [32], the authors gave a precise arithmetic condition on \( a \) for the quasiballistic motion depending on whether or not the Lyapunov exponent vanishes in the quasiperiodic Schrödinger setting. Here, we provide the parametric conditions for the EHM. The conclusions can also be extended directly to more general singular Jacobi operators with analytic quasiperiodic potentials.
3 Preliminaries

We recall some commonly used notations for the reader’s convenience. We denote $L^\infty(T, \mathbb{R})$ and $L^\infty(T, \mathbb{C})$ to be the space of all 1-periodic bounded functions, taking values in $\mathbb{R}$ and $\mathbb{C}$ respectively. Denote the usual $L^\infty$ norm in both spaces by
\[ \|f\|_\infty := \sup_{x \in T} |f(x)|. \]

Note that we only require the diagonal potential function $v$ to be a real valued function; all the other sampling functions are allowed to take values in $\mathbb{C}$. We do not emphasize the real/complex value anymore unless necessary. Denote by $L^1(T, \mathbb{C})$ the usual Lebesgue space with the 1-norm
\[ \|f\|_1 := \int_T |f(\theta)| d\theta. \]

Denote by $C^\omega(T, \mathbb{C})$ the space of all 1-periodic analytic functions and by $C^k(T, \mathbb{C})$ the space of all functions with continuous $k$-th order derivatives for all $k = 0, 1, \ldots, \infty$. We denote by Lip$(T, \mathbb{C})$ the space of all 1-periodic Lipschitz continuous functions, induced with the Lipschitz norm given by
\[ \|f\|_{\text{Lip}} := \|f\|_\infty + \sup_{x, y \in T} \frac{|f(x) - f(y)|}{|x - y|}. \]

We identify the sequence $u = \{u_n\}_{n \in \mathbb{Z}}$ with $u_n$ whenever it is clear that $n$ is the index. Denote the $\ell^\infty$ norm of $u \in \ell^\infty(\mathbb{Z}, \mathbb{C})$ by $\|u\|_\infty := \sup_{n \in \mathbb{Z}} |u_n|$. We will denote the distance on $T_1$ by
\[ \|\theta\|_T := \inf_{n \in \mathbb{Z}} |\theta - n| \]
and may drop the subindex $\| \cdot \| = \| \cdot \|_T$ whenever it is clear.

3.1 Transfer matrices and Lyapunov exponents. Let $H$ be given as in (1.1):
\[ (Hu)_n = w_n u_{n+1} + \overline{w}_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z}. \]

The eigenvalue equation $Hu = Eu$ can be rewritten via the following skew product:
\[ \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(E) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \]

where
\[ A_n(E) = \frac{1}{\overline{w}_n} D_n(E), \quad D_n(E) = \begin{pmatrix} E - v_n & -\overline{w}_{n-1} \\ w_n & 0 \end{pmatrix}. \]
For \( n \in \mathbb{N}^+ \) and \( m \in \mathbb{Z} \), define the \( n \)-step transfer matrix\(^3\) at position \( m \) to be

\[
A(n, m; E) = \prod_{j=n+m-1}^{m} A_j(E), \tag{3.5}
\]

\[
D(n, m; E) = \prod_{j=n+m-1}^{m} D_j(E). \tag{3.6}
\]

We denote the scalar product of \( w_n \) by the similar notation

\[
w(n, m) = \prod_{j=m}^{n+m-1} w_j, \quad m \in \mathbb{Z}, \ n \in \mathbb{N}^+
\]

and denote

\[
A(n; E) = A(n, 1; E), \ n > 0; \quad A(0; E) = \text{Id}; \tag{3.8}
\]

\[
A(n; E) = A^{-1}(-n, n + 1; E), \ n < 0, \tag{3.9}
\]

\[
D(n; E) = D(n, 1; E), \ n > 0; \tag{3.10}
\]

\[
D(0; E) = \text{Id}; \quad D(n; E) = D^{-1}(-n, n + 1; E), \ n < 0,
\]

for simplicity.

The (upper) Lyapunov exponent characterizes the grow(decay) rate of the norm of the transfer matrix \( \|A(n, m)\| \); it will be convenient to introduce the Lyapunov exponent by using the dynamical notations. We refer readers to [32, 2] and references therein for the general definition of the Lyapunov exponent of the linear skew product. In this part, we will restrict ourselves to the quasiperiodic cocycles. Let \( \alpha \in \mathbb{R}\backslash\mathbb{Q} \) and \( A : \mathbb{T} \to \text{GL}(2, \mathbb{C}) \). We call \((\alpha, A)\) a (complex) cocycle. In view of (3.5) and (3.8), denote the transfer matrix in the quasiperiodic cocycle case by

\[
A(n; \theta, \alpha) = \prod_{j=n}^{1} A(\theta + (j - 1)\alpha), \quad \theta \in \mathbb{T}, \ n \in \mathbb{N}^+. \tag{3.11}
\]

The Lyapunov exponent is given by the formula:

\[
L(A, \alpha) = \lim_{n \to +\infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|A(n; \theta, \alpha)\|\,d\theta = \inf_{n>0} \frac{1}{n} \int_{\mathbb{T}} \ln \|A(n; \theta, \alpha)\|\,d\theta. \tag{3.12}
\]

\(^{3}\)Throughout the paper, the matrix product notation \( \prod_{j=n+m-1}^{m} A_j \) starts from the left \( j = n+m-1 \), and ends at the right \( j = m \). This definition meets the iteration of the orbit in (3.3), from the right \( j = m \) to the left \( j = n+m-1 \). The order of the product is only important for matrices and will not affect scalar products.
For irrational $\alpha$, the point-wise limit

$$L(A, \alpha) = \lim_{n \to +\infty} \frac{1}{n} \ln \|A(n; \theta, \alpha)\|$$

also holds true for a.e. $\theta \in \mathbb{T}$ by subadditive ergodic theory. When $A$ is the one-parameter family of $E$ as in \((3.4)\), we denote the associated Lyapunov exponent by $L(E, \alpha)$ to emphasize the dependence on $E$. We also frequently write $L(E) = L(E, \alpha)$ for simplicity whenever it is clear.

By unique ergodicity of the irrational rotations we have the following uniform upper bound (in $\theta$) for both matrix and scalar cases:

**Lemma 3.1** (e.g., [15, 30]). If $A \in C^0(\mathbb{T}, \text{GL}(2, \mathbb{C}))$, then

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \|A(n; \theta, \alpha)\| \leq L(A, \alpha)$$

uniformly in $\theta \in \mathbb{T}$.

If $a \in C^0(\mathbb{T}, \mathbb{C})$ and $\ln |a(\theta)| \in L^1(\mathbb{T})$, then

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \left| \prod_{j=1}^{n} a(\theta + (j-1)\alpha) \right| \leq \int_{\mathbb{T}} \ln |a(\theta)| d\theta$$

uniformly in $\theta \in \mathbb{T}$.

**Remark 3.1.** If $a \in C^0(\mathbb{T}, \mathbb{C})$ has no zeros, then $\frac{1}{a(\theta)}$ is also continuous. By \((3.14)\), we have

$$\frac{1}{n} \ln \left| \prod_{j=1}^{n} \frac{1}{a(\theta + (j-1)\alpha)} \right| \leq \int_{\mathbb{T}} \ln \left| \frac{1}{a(\theta)} \right| d\theta + \epsilon$$

$$\Longleftrightarrow \prod_{j=1}^{n} |a(\theta + (j-1)\alpha)| \geq e^{\int_{\mathbb{T}} \ln |a(\theta)| d\theta - \epsilon}$$

for $n > n_0(\epsilon)$ (uniform in $\theta$). This immediately gives the desired lower bound in \((2.2)\) in a uniform way. If $a(\theta)$ has zeros, there is no such uniform lower bound for the scalar product anymore. One technical achievement in the paper is, with some mild assumptions on the non-degeneracy of the zeros, we are able to get a weakened version of \((3.15)\) (see Lemma 4.2), which will be sufficient for the spectral continuity.

### 3.2 The Weyl–Titchmarsh $m$-function and subordinacy theory

The boundary behavior of the $m$ function is linked to the power law of the half-line solution and the growth of the transfer matrix norm $A(n, m; E)$ via the well known Gilbert–Pearson subordinacy theory [17, 18]. We give a brief review on the $m$-function and the subordinacy theory. More details can be found, e.g., in [7].
Let $H$ be as in (1.1) and $z = E + i\varepsilon \in \mathbb{C}$. Consider equation

$$Hu = zu,$$

with the family of normalized phase boundary conditions

$$u_0^\phi \cos \varphi + u_1^\phi \sin \varphi = 0, \quad -\pi/2 < \varphi < \pi/2, \quad |u_0^\phi|^2 + |u_1^\phi|^2 = 1.$$  

Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^- = \{\ldots, -2, -1, 0\}$. Denote by $u^\phi = \{u_j^\phi\}_{j \geq 0}$ the right half-line solution on $\mathbb{Z}^+$ of (3.17) with boundary condition (3.18) and by $u^{\phi,-} = \{u_j^{\phi,-}\}_{j \leq 0}$ the left half-line solution on $\mathbb{Z}^-$ of the same equation. Also denote by $v^\phi$ and $v^{\phi,-}$ the right and left half-line solutions of (3.17) with the orthogonal boundary conditions to $u^\phi$ and $u^{\phi,-}$, i.e., $v^\phi = u^{\phi+\pi/2}$, $v^{\phi,-} = u^{\phi+\pi/2,-}$.

For any function $u : \mathbb{Z}^+ \to \mathbb{C}$ we denote by $\|u\|_\ell$ the $\ell$-norm of $u$ over a lattice interval of length $\ell$, that is,

$$\|u\|_\ell = \left[ \sum_{n=1}^{[\ell]} |u_n|^2 + (\ell - [\ell])|u_{[\ell]+1}|^2 \right]^{1/2}. \tag{3.19}$$

Similarly, for $u : \mathbb{Z}^- \to \mathbb{C}$, we define

$$\|u\|_\ell = \left[ \sum_{n=1}^{[\ell]-1} |u_{-n}|^2 + (\ell - [\ell])|u_{-[\ell]}|^2 \right]^{1/2}. \tag{3.20}$$

For any $\varepsilon > 0$, let $\ell = \ell(\varphi, \varepsilon, E)$ be\(^\text{4}\) such that

$$\|u^\phi\|_{\ell(\varphi, \varepsilon)} \|v^\phi\|_{\ell(\varphi, \varepsilon)} = \frac{1}{2E}. \tag{3.21}$$

Now $\ell^-(\varphi)$ is defined through the same equation by $u^{\phi,-}$, $v^{\phi,-}$. The solutions $u^\phi$ and $v^\phi$ have orthogonal boundary conditions, and therefore their Wronskian is (constant and) equal to one,

$$w_n(u^\phi(n+1)v^\phi(n) - u^\phi(n)v^\phi(n+1)) = 1, \quad n \geq 0.$$  

Direct computation by the Cauchy–Schwarz inequality implies that

$$\|u^\phi\|_\ell \cdot \|v^\phi\|_\ell \geq \sum_{n=1}^{[\ell]-1} |w_n|^{-1} \geq \frac{1}{2\|w\|_\infty}(\ell - 1). \tag{3.22}$$

Let $m_\varphi(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ and $m_-^\varphi(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ be the right and left Weyl–Titchmarsh $m$-functions (half line) associated with the boundary condition (3.18). Let $m = m_0$ and $m^- = m_0^-$ be the half line $m$-functions corresponding to the Dirichlet boundary conditions. The following quantitative subordinacy theory was proved in [24], well known as the Jitomirskaya–Last inequality.

\(^{4}\)Such $\ell$ exists for all $\varepsilon > 0$ and $\varphi$ due to the monotonicity of the product $\|u^\phi\|_\ell \|v^\phi\|_\ell$. See more details about this part in [24, 32].
Lemma 3.2 (Jitomirskaya–Last inequality, Theorem 1.1 in [24]). For $E \in \mathbb{R}$ and $\varepsilon > 0$, the following inequality holds for any $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\frac{5 - \sqrt{24}}{|m_\varphi(E + i\varepsilon)|} < \frac{\|u_\varphi\|_{\ell(\varphi, \varepsilon)}}{\|v_\varphi\|_{\ell(\varphi, \varepsilon)}} < \frac{5 + \sqrt{24}}{|m_\varphi(E + i\varepsilon)|}$$

(3.23)

There is also one general statement about the existence of generalized eigenfunctions with sub-linear growth in its $\ell$-norm:

Lemma 3.3 ([35]). For $\mu_\theta$-a.e. $E$, there exists $\varphi \in (-\pi/2, \pi/2]$ such that $u_\varphi$ and $u_{\varphi,-}$ both obey

$$\limsup_{\ell \to \infty} \frac{\|u\|_\ell}{\ell^{1/2} \ln \ell} < \infty.$$  

(3.24)

This inequality provides us an upper bound for the $\ell$-norm of the solution, which is crucial in the proof of the spectral singularity.

The next proposition relates the whole-line $m$-function $M$ and half-line $m$-function $m_\varphi$, which can be found in [11].

Proposition 3.4 (Corollary 21 in [11]). Fix $E \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$|M(E + i\varepsilon)| \leq \sup_{\varphi} |m_\varphi(E + i\varepsilon)|.$$  

(3.25)

By this proposition, to bound $M$ from above and get spectral continuity as in (2.4), it is enough to obtain uniform upper bounds of $m_\varphi$ in boundary condition $\varphi$ for the right half-line problem.

For spectral singularity, we need to consider both $m_\varphi(z)$ and $m^-_\varphi(z)$. Let

$$(U\psi)_n = \psi_{-n+1}, \quad n \in \mathbb{Z}$$

be a unitary operator on $\ell^2(\mathbb{Z})$. Let $\tilde{H} = UHU^{-1}$. Denote by $\tilde{m}$, $\tilde{m}_\varphi$, $\tilde{u}^\varphi$ and $\tilde{\ell}(\varphi)$, correspondingly, $m$, $m_\varphi$, $u^\varphi$ and $\ell(\varphi)$ of the operator $\tilde{H}$. The following facts are well known in the past literatures (see, e.g., Section 3, [25]). For any $\varphi \in (-\pi/2, \pi/2)$,

$$M(z) = \frac{m_\varphi(z)\tilde{m}_{\pi/2-\varphi} - 1}{m_\varphi(z) + \tilde{m}_{\pi/2-\varphi}}$$

(3.26)

and

$$\tilde{\ell}(\pi/2 - \varphi) = \ell^-(\varphi), \quad \|u\|_\ell = \|Uu\|_\ell.$$  

(3.27)
In view of (2.10), a direct consequence of (3.26) is (e.g., Lemma 5 in [25]):

**Lemma 3.5.** For any $0 < \gamma < 1$, suppose that there exists a $\varphi \in (-\pi/2, \pi/2)$ such that for $\mu$-a.e. $E$ in some Borel set $S$, we have that

$$
\liminf_{\epsilon \to 0} \epsilon^{1-\gamma}|m_\varphi(E + i\epsilon)| = \infty \quad \text{and} \quad \liminf_{\epsilon \to 0} \epsilon^{1-\gamma}|\tilde{m}_{\pi/2-\varphi}(E + i\epsilon)| = \infty.
$$

Then for $\mu$-a.e. $E$ in $S$,

$$
\liminf_{\epsilon \to 0} \epsilon^{1-\gamma}|M(E + i\epsilon)| = \infty,
$$

namely, the restriction $\mu(S \cap \cdot)$ is $\gamma$-spectral singular.

### 3.3 Continued fraction.

An important tool in the study of a quasiperiodic sequence is the continued fraction expansion of irrational numbers. Let $\alpha \in [0, 1] \setminus \mathbb{Q}$; $\alpha$ has the following unique expression with $a_n \in \mathbb{N}$:

$$
\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.
$$

Let

$$
\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
$$

be the continued fraction approximants of $\alpha$. Let

$$
\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.
$$

Now $\beta(\alpha)$ being large means $\alpha$ can be approximated very well by a sequence of rational numbers. Let us mention that $\{\alpha : \beta(\alpha) = 0\}$ is a full measure set.

The following properties about the continued fraction expansion are well-known:

$$
\frac{1}{2q_{n+1}} \leq \|q_n \alpha\|_T \leq \frac{1}{q_{n+1}}.
$$

For any $q_n \leq |k| < q_{n+1}$,

$$
\|q_n \alpha\|_T \leq |k\alpha|_T.
$$

Combining definition of $\beta(\alpha)$ (2.8) with (3.31), we have: If $\beta(\alpha) = 0$, then for any $\delta > 0$, for $|k|$ large, the following inequality holds:

$$
|k\alpha|_T > e^{-\delta |k|}.
$$
3.4 More about $\beta$-almost periodicity and the $(\Lambda, \beta)$ bound. In this paper, we consider bounded sequences $v_n, w_n$ (for example, $v_n, w_n$ are both generated by some smooth sampling functions). Let
\[ N(H) := [-2\|v\|_\infty - \|v\|_\infty, \|v\|_\infty + 2\|w\|_\infty] \]
be the numerical range of $H$. It is enough to consider $E \in N$ since the spectrum of $H$ is contained in $N$. Let $D(n, m)$ and $w(n, m)$ be defined as in (3.6) and (3.7). The mild assumption on $v_n, w_n$ yields the following trivial upper bound for $D(n, m)$ and $w(n, m)$: there is $\Lambda_0 > 0$ such that for any $n \in \mathbb{N}$, and any $E \in N(H),$
\begin{align}
\sup_{m \in \mathbb{Z}} \|D(n, m; E)\| &\leq e^{\Lambda_0 n}, \\
\sup_{m \in \mathbb{Z}} |w(n, m)| &\leq e^{\Lambda_0 n}. 
\end{align}
Suppose $w_n$ has $(\Lambda, \beta)$-bound as in (2.2). Without loss of generality, we assume $\Lambda_0 = \Lambda$ for simplicity. For $1 \leq r \leq q$ and $m \in \mathbb{Z},$ write
\[ w(q, m) = w(q - r, m + r)w(r, m). \]
Combine (2.2) with the upper bound (3.34) to obtain
\begin{equation}
\min_{|m| \leq e^{\beta q}} |w(r, m)| \geq e^{-2\Lambda q}, \quad 1 \leq r < q. 
\end{equation}
In particular, $r = 1$ gives
\begin{equation}
\min_{|m| \leq e^{\beta q}} |w_m| \geq e^{-2\Lambda q}. 
\end{equation}
Assume further $w_n$ has $\beta$-q almost periodicity as in (2.1). By (3.34), $\beta$-q periodicity can be strengthened:
\begin{equation}
\max_{|m| \leq e^{\beta q}} \left| \frac{w_{m+q}}{w_m} - 1 \right| < e^{-(\beta - 2\Lambda)q}. 
\end{equation}
We also abuse the notation frequently by saying the operator $H$ or the transfer matrix $A(n, E)$ has $\beta$-almost periodicity and $(\Lambda, \beta)$ boundedness if the corresponding $v_n, w_n$ has $\beta$ almost periodicity and $(\Lambda, \beta)$ boundedness .

The lower bound on $w(n, m)$ and upper bound on $D(n, m)$ also imply that for any $E \in N$, and $|m| \leq e^{\beta q},$
\begin{equation}
\|A(q, m)\| < e^{2\Lambda q}, \quad \max_{0 \leq r < q} \|A(r, m)\| < e^{3\Lambda q}. 
\end{equation}
Assume now $\beta$-almost periodicity and $(\Lambda, \beta)$ bound hold true for the sequence $q_n \to \infty$. We will use these induced bounds (3.35)–(3.38) for the sequence $q_n$ frequently.
4 \((\Lambda, \beta)\) bound for quasiperiodic smooth sequence and the proof of Corollary 2.2

Assume we have a quasiperiodic sequence \(v(\theta + na)\) generated by a Lipschitz sampling function \(v\). Let \(q_n\) be given as in (2.8). By (2.8), for any \(0 < \beta < \beta(\alpha)/2\), there is a subsequence \(q_{n_k}\) such that \(\ln q_{n_k + 1} > 2\beta q_{n_k}\). Then by (3.30), for any \(\theta, m\) and \(q_{n_k}\),

\[
|v(\theta + ma) - v(\theta + (m \pm q_{n_k})\alpha)| \leq \|v\|_{\text{Lip}} \cdot \|q_{n_k}\| \leq \|v\|_{\text{Lip}} \cdot \frac{1}{q_{n_k + 1}} \leq \|v\|_{\text{Lip}} \cdot e^{-2\beta q_{n_k}} \leq e^{-\beta q_{n_k}},
\]

provided \(q_{n_k}\) is large. Same computation works for \(c\). Therefore, \(v(\theta + na)\) and \(c(\theta + na)\) are \(\beta\)-almost periodic for Lipschitz continuous functions \(v\) and \(c\).

The more challenging part is the \(\|\Lambda_1, \beta\) bound on \(c(\theta + na)\), where we need some further assumption on \(c\). We will focus on this throughout the rest of this section.

The key ingredient for the proof of the \((\Lambda, \beta)\) bound is the following lemma in [1]:

**Lemma 4.1.** Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}, \theta \in \mathbb{R}\) and \(0 \leq j_0 \leq q_n - 1\) be such that

\[
|\sin \pi(\theta + j_0\alpha)| = \inf_{0 \leq j \leq q_n - 1} |\sin \pi(\theta + j\alpha)|.
\]

Then for some absolute constant \(C > 0\),

\[
-C \ln q_n \leq \sum_{j=0, j \neq j_0}^{q_n-1} \ln |\sin \pi(\theta + j\alpha)| + (q_n - 1) \ln 2 \leq C \ln q_n.
\]

This lemma was used in [31] to prove some optimal singular continuous spectrum results. By extending the argument in [31] to exponentially many periods, we are able to prove the \((\Lambda, \beta)\) bound for any analytic sampling function. Actually, we can deal with more general sampling functions with much weaker regularities.

Define

\[
\mathcal{F}(\mathbb{T}, \mathbb{C}) := \left\{ c \in L^\infty(\mathbb{T}, \mathbb{C}) : \begin{array}{l}
\exists m \in \mathbb{N}^+, \theta_\ell \in \mathbb{T}, \tau_\ell \in (0, 1], \ell = 1, \ldots, m, \\
\text{such that } g(\theta) := \frac{c(\theta)}{\prod_{\ell=1}^{m} |\sin \pi(\theta - \theta_\ell)|^{\tau_\ell}} \in L^\infty(\mathbb{T}, \mathbb{C}) \\
\text{and } \inf_{\theta} |g(\theta)| > 0
\end{array} \right\}.
\]
Assume $c(\theta) \in \mathcal{F}(\mathbb{T}, \mathbb{C})$ with $\theta_\ell$ and $g(\theta)$ given as in (4.2) such that

$$
(4.3) \quad c(\theta) = g(\theta) \prod_{\ell=1}^{m} |\sin \pi (\theta - \theta_\ell)|^{\tau_\ell}.
$$

Clearly, $\ln |g(\theta)| \in L^1(\mathbb{T})$. By the well known integral $\int_{\mathbb{T}} \ln |\sin \pi \theta| d\theta = -\ln 2$, it is easy to check that $\ln |c(\theta)| \in L^1(\mathbb{T})$ and is linked to $\ln |g|$ by

$$
(4.4) \quad \int_{\mathbb{T}} \ln |c(\theta)| d\theta = \int_{\mathbb{T}} \ln |g(\theta)| d\theta - 2 \sum_{\ell=1}^{m} \tau_\ell.
$$

The following technical lemma shows that any sampling function in $\mathcal{F}(\mathbb{T}, \mathbb{C})$ with an irrational force can generate a $(\Lambda, \beta)$ bounded sequence.

**Lemma 4.2.** Assume that there exists $m \in \mathbb{N}^+, \theta_\ell \in \mathbb{T}, \tau_\ell \in (0, 1], \ell = 1, \ldots, m,$ $g(\theta) \in L^\infty(\mathbb{T}, \mathbb{C})$ such that $\inf_{\mathbb{T}} |g(\tilde{\theta})| > 0$ and

$$
(4.5) \quad c(\theta) = g(\theta) \prod_{\ell=1}^{m} |\sin \pi (\theta - \theta_\ell)|^{\tau_\ell}.
$$

Then for any $\alpha$ with $0 < 2\beta < \beta(\alpha)$ and $0 < \delta < \frac{1}{2} \sum_{k=1}^{m} \tau_\ell$, there is a sequence $q_n \to \infty$ and a full Lebesgue measure set $\Theta = \Theta(\alpha, \alpha_1, \ldots, \alpha_m)$ such that for any $\theta \in \Theta$ and $q_n$ large enough,\(^5\) $c(\theta + n\alpha)$ satisfies

$$
(4.6) \quad \min_{|k| \leq q_n} \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-\Lambda_1 q_n},
$$

where

$$
(4.7) \quad \Lambda_1 := \Lambda_1(\tau, g, \delta\beta) = \ln 2 \sum_{\ell=1}^{m} \tau_\ell - \ln(\inf_{\mathbb{T}} |g(\tilde{\theta})|) + \delta^2 \min \{\beta, 1\}.
$$

Assume further $g(\theta) \in C^0(\mathbb{T}, \mathbb{C})$ and $\ln |g(\theta)| \in L^1(\mathbb{T})$. Then $\Lambda_1$ in (4.7) can be replaced by

$$
(4.8) \quad \Lambda_1 = -\int_{\mathbb{T}} \ln |c(\theta)| d\theta + 2\delta^2 \min \{\beta, 1\}.
$$

Moreover, $c(\theta + n\alpha)$ is $(\Lambda, \beta)$ bounded as defined in (2.2) such that

$$
(4.9) \quad \min_{|m| \leq e^\beta q_n} \prod_{j=m}^{m+q_n-1} |c(\theta + j\alpha)| > e^{-\Lambda q_n}, \quad \theta \in \Theta,
$$

where

$$
(4.10) \quad \Lambda = -\int_{\mathbb{T}} \ln |c(\theta)| d\theta + 6\delta^2 \min \{\beta, 1\}.
$$

\(^5\)The sequence itself only depends on $\beta(\alpha)$, while the largeness depends on $\theta, \alpha, \beta, \delta, \tau$.\}
**Remark 4.1.** The above $\Lambda_1$ and $\Lambda$ can be negative in general, which makes (4.6) and (4.9) actually grow exponentially (instead of decay). This is natural since there is actually a ‘large’ scaling of size $\int \ln |g| \sim \int \ln |c|$ for the product in these cases. We are more interested in the case $\int \ln |c| \leq 0$ where $\Lambda_1$ and $\Lambda$ are indeed positive. In particular, it is always possible to re-scale $c(\theta)$ to make the logarithm average zero. This will lead to an arbitrarily small $\Lambda$ (positive) in (4.10).

**Proof.** Let $0 < 2\beta < \beta(\alpha)$ and $q_n$ be defined as in (2.8). For any $\delta > 0$, let $q_{n_k}$ be the subsequence such that $\ln q_{n_k+1} \geq 2\beta q_{n_k}$. For simplicity, drop the subindex $n_k$ and denote the subsequence still by $q_n$, i.e.,

$$q_{n+1} > e^{2\beta q_n}.$$  

We also write $\tilde{\beta} = \min\{\beta, 1\}$. It is obvious that for any $m \in \mathbb{Z}$ and $\theta \in \mathbb{T}$,

$$\prod_{j=\tilde{\beta}q_n}^{(k+1)q_n-1} |g(\theta + j\alpha)| > (\inf_{\mathbb{T}} |g(\tilde{\theta})|)^{q_n} = e^{q_n \ln(\inf_{\mathbb{T}} |g(\tilde{\theta})|)}.$$  

In view of (4.5), it is enough to study the lower bound for each $\|\theta - \theta_\ell\|_\mathbb{T}$. For any $\alpha \in [0, 1] \setminus \mathbb{Q}$ and any $\theta_\ell$, $\ell = 1, \ldots, m$, let

$$\Theta_\ell := \bigcup_{\gamma > 0} \{\theta \in \mathbb{T} : \|\theta - \theta_\ell + n\alpha\|_\mathbb{T} \geq \gamma |n|^{-2}, \forall n \in \mathbb{Z}\setminus\{0\} \}.$$  

**Remark 4.2.** It is an easy exercise that if $c(\theta)$ is $C^k$ continuous on $\mathbb{T}$ with finitely many zeros with non-degenerate $k$-th order derivatives, then

$$c(\theta) \in \mathcal{F}(\mathbb{T}, \mathbb{C}) \bigcap \text{Lip}(\mathbb{T}, \mathbb{C})$$

with a continuous $g(\theta)$ for all $k \geq 1$. By Lemma 4.2, there is $\Lambda = \Lambda(c)$ such that $c(\theta + n\alpha)$ is $(\Lambda, \beta)$ bounded for any $0 < 2\beta < \beta(\alpha)$ and a.e. $\theta \in \mathbb{T}$. From the proof of Lemma 4.2, see (4.11), the subsequence $q_{n_k}$ can be taken to be the same as in the $\beta$-almost periodicity (4.1). Therefore, Corollary 2.2 follows from Theorem 2.1. We omit the details here. An interesting question is whether the non-degenerate condition on $c$ can be weakened and what is the appropriate ‘non-degenerate’ condition on any Lipschitz function such that (4.2) holds.

**Proof.** Let $0 < 2\beta < \beta(\alpha)$ and $q_n$ be defined as in (2.8). For any $\delta > 0$, let $q_{n_k}$ be the subsequence such that $\ln q_{n_k+1} \geq 2\beta q_{n_k}$. For simplicity, drop the subindex $n_k$ and denote the subsequence still by $q_n$, i.e.,

$$q_{n+1} > e^{2\beta q_n}.$$  

We also write $\tilde{\beta} = \min\{\beta, 1\}$. It is obvious that for any $m \in \mathbb{Z}$ and $\theta \in \mathbb{T}$,

$$\prod_{j=\tilde{\beta}q_n}^{(k+1)q_n-1} |g(\theta + j\alpha)| > (\inf_{\mathbb{T}} |g(\tilde{\theta})|)^{q_n} = e^{q_n \ln(\inf_{\mathbb{T}} |g(\tilde{\theta})|)}.$$  

In view of (4.5), it is enough to study the lower bound for each $\|\theta - \theta_\ell\|_\mathbb{T}$. For any $\alpha \in [0, 1] \setminus \mathbb{Q}$ and any $\theta_\ell$, $\ell = 1, \ldots, m$, let

$$\Theta_\ell := \bigcup_{\gamma > 0} \{\theta \in \mathbb{T} : \|\theta - \theta_\ell + n\alpha\|_\mathbb{T} \geq \gamma |n|^{-2}, \forall n \in \mathbb{Z}\setminus\{0\} \}.$$
It is well known that $\Theta_\ell$ is a full measure set. Let
\begin{equation}
\Theta := \bigcap_{\ell=1}^m \Theta_\ell.
\end{equation}

For any $\theta \in \Theta$ and $1 \leq \ell \leq m$, there is $\gamma_\ell = \gamma(\theta, \theta_\ell, \alpha) > 0$ such that
\begin{equation}
\|\theta - \theta_\ell + n\alpha\|_T \geq \frac{\gamma_\ell}{|n|^2}, \quad \forall n \in \mathbb{Z}\setminus\{0\}.
\end{equation}

For all $1 \leq \ell \leq m$ and $|k| < q_n^{-1}e^{\delta \beta q_n}$, let $j_{\ell,k} \in [0, q_n)$ be such that the following holds:

\begin{equation}
|\sin \pi(\theta - \theta_\ell + k\alpha + j_{\ell,k}\alpha)| = \inf_{0 \leq j < q_n} |\sin \pi(\theta - \theta_\ell + k\alpha + j\alpha)|.
\end{equation}

By (4.15), for all $j_{\ell,k}$,
\begin{equation}
\|\theta - \theta_\ell + j_{\ell,k}\alpha\|_T \geq \frac{\gamma_\ell}{|j_{\ell,k}|^2} \geq \frac{\gamma_\ell}{q_n^2}.
\end{equation}

Let $\tau = \sum_{\ell=1}^m \tau_\ell$. For all $|k| < q_n^{-1}e^{\delta \beta q_n} < e^{\delta \beta q_n}$, we have that
\begin{align}
|\sin \pi(\theta - \theta_\ell + k\alpha + j_{\ell,k}\alpha)| &\geq \|\theta - \theta_\ell + k\alpha + j_{\ell,k}\alpha\|_T \\
&\geq \|\theta - \theta_\ell + k\alpha\|_T - \|k\alpha\|_T \\
&\geq \frac{\gamma_\ell}{q_n^2} - |k| \frac{1}{q_n^{n+1}} \\
&\geq \frac{\gamma_\ell}{q_n^2} - e^{\delta \beta q_n}e^{-2 \beta q_n} \\
&\geq 2e^{-\tau^{-1}\delta \tilde{\beta} q_n} - e^{-(2-\delta)\tilde{\beta} q_n} \\
&\geq e^{-\tau^{-1}\delta \tilde{\beta} q_n}
\end{align}

provided $q_n^{-2}e^{\tau^{-1}\delta \tilde{\beta} q_n} \geq 2\gamma_\ell^{-1}$ and $2 - \delta > \tau^{-1}\delta > \tau^{-1}\delta^2$. The latter gives the restriction on $\delta$ such that $\delta < \frac{2}{2+\tau^{-1}}$.

By Lemma 4.1,
\begin{equation}
\prod_{j=0, j \neq j_{\ell,k}}^{q_n-1} |\sin \pi(\theta - \theta_\ell + k\alpha + j\alpha)| \geq e^{-(q_n-1)\ln 2 - C\ln q_n} \geq e^{-q_n\ln 2 - \tau^{-1}\delta^2 \tilde{\beta} q_n}
\end{equation}

provided $C\ln q_n < \tau^{-1}\delta^2 \tilde{\beta} q_n$, where $C$ is the absolute constant in Lemma 4.1 and $\tau$ is the same in (4.17).
Now putting (4.17) and (4.18) together, we have

\[
\prod_{j=kq_n}^{(k+1)q_n-1} \left( \prod_{\ell=1}^{m} | \sin \pi (\theta + j\alpha - \theta_{\ell}) | \tau_{\ell} \right)
\]

\[
= \prod_{\ell=1}^{m} \left( \prod_{j=0, j \neq j_{1, k}}^{q_n-1} | \sin \pi (\theta_{\ell} + kq_n\alpha + j\alpha) | \right) \tau_{\ell}
\]

\[
= \left( \prod_{\ell=1}^{m} \left( \prod_{j=0, j \neq j_{1, k}}^{q_n-1} | \sin (\theta_{\ell} + kq_n\alpha + j\alpha) | \right) \right) \tau_{\ell}
\]

\[
\times \left( \prod_{\ell=1}^{m} | \sin (\theta_{\ell} + kq_n\alpha + j_{1, k}\alpha) | \right)
\]

\[
\geq \left( \prod_{\ell=1}^{m} (e^{-q_n \ln 2 - \tau_{\ell} \partial^{2} \tilde{\theta} q_n}) \right) \cdot \left( \prod_{\ell=1}^{m} (e^{-\tau_{\ell} \partial^{2} \tilde{\theta} q_n}) \right)
\]

\[
= e^{-q_n (\ln 2 \sum_{1}^{m} \tau_{\ell}) - \partial^{2} \tilde{\theta} q_n}.
\]

Combined with (4.12), we have that for all \(|k| \leq q_n^{-1} e^{\beta_{n} q_n} \),

\[
(4.19) \quad \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-(\ln 2 \sum_{\ell=1}^{m} \tau_{\ell} - \ln \inf_{\mathbb{T}} |g(\theta)| + \partial^{2} \tilde{\theta} q_n)}
\]

provided \(q_n > \tilde{q} = \tilde{q}(\max \gamma_{\ell}^{-1}, \partial, \alpha, \sum_{\ell=1}^{m} \tau_{\ell})\).

Assume further \(g(\theta), c(\theta) \in C^{0}(\mathbb{T}, \mathbb{C})\). Since \(\inf |g(\theta)| > 0\), \(\ln |g(\theta)|^{-1}\) is also continuous. By Lemma 3.1, there is \(n_0 = n_0(\partial^{2} \tilde{\theta})\) such that the following upper bound holds uniformly in \(\theta \in \mathbb{T}\) for \(n > n_0\):

\[
(4.20) \quad \frac{1}{n} \sum_{j=1}^{n} \ln |g(\theta + j\alpha)|^{-1} \leq \int_{\mathbb{T}} \ln |g(\theta)|^{-1} d\theta + \partial^{2} \tilde{\theta}.
\]

In particular, for all \(q_n \geq n_0\) and any \(k \in \mathbb{Z}\), we have

\[
\left( \prod_{j=0}^{q_n-1} |g(\theta + kq_n\alpha + j\alpha)|^{-1} \right)^{-1} \leq e^{-\int_{\mathbb{T}} \ln |g(\theta)| d\theta + \partial^{2} \tilde{\theta}}
\]

\[
\Rightarrow \prod_{j=kq_n}^{(k+1)q_n-1} |g(\theta + j\alpha)| \geq e^{q_n (\int_{\mathbb{T}} \ln |g(\theta)| d\theta - \partial^{2} \tilde{\theta})}.
\]

Therefore, we can replace \(\ln(\inf_{\mathbb{T}} |g(\tilde{\theta})|)\) in (4.19) by

\[
\int_{\mathbb{T}} \ln |g(\theta)| d\theta - \partial^{2} \min\{ \beta, 1 \}.
\]
In view of (4.4), we have

\[
\Lambda_1 = \ln 2 \sum_{\ell=1}^m \tau_\ell - \int_T \ln |g(\theta)| d\theta + 2\delta^2 \min\{\beta, 1\} \\
\tag{4.21}
\]

\[
= - \int_T \ln |c(\theta)| d\theta + 2\delta^2 \min\{\beta, 1\},
\]

which gives the desired expression of \(\Lambda_1\) in (4.8).

Let \(\bar{c}(\theta) = c(\theta) e^{-\int_T \ln |c(\theta)| d\theta}\). It is easy to check that \(\int_T \ln |\bar{c}(\theta)| d\theta = 0\). By (4.21), we have for all \(|k| \leq q_n^{-1} e^{\delta^2 \bar{\beta}}\),

\[
\prod_{j=kq_n}^{(k+1)q_n-1} |\bar{c}(\theta + j\alpha)| > e^{-2\delta^2 \bar{\beta}} q_n. \tag{4.22}
\]

By Lemma 3.1, there is \(r_0 = r_0(\delta^2 \bar{\beta}) \in \mathbb{N}\) such that for any \(m \in \mathbb{Z}\) and \(r \geq r_0\),

\[
\prod_{j=m}^{m+r-1} |\bar{c}(\theta + j\alpha)| \leq e^{r(\int_T |\bar{c}(\theta)| d\theta + \delta^2 \bar{\beta})} = e^{\delta^2 \bar{\beta} r}. \tag{4.23}
\]

For \(0 \leq r < r_0\), we have the trivial upper bound

\[
\prod_{j=m}^{m+r-1} |\bar{c}(\theta + j\alpha)| \leq e^{r(\|\bar{c}\|_\infty)} \leq e^{r_0 \ln(\|\bar{c}\|_\infty + 1)}. \tag{4.24}
\]

Thus, for any \(m \in \mathbb{Z}\) and \(1 \leq r \leq q_n\),

\[
\prod_{j=m}^{m+r-1} |\bar{c}(\theta + j\alpha)| \leq e^{\delta^2 \bar{\beta} q_n}
\]

provided \(q_n \geq \delta^{-1} \bar{\beta}^{-1} r_0 \ln(\|\bar{c}\|_\infty + 1)\).

For any \(|m| < e^{\delta^2 \bar{\beta} q_n}\), there is a \(k\) such that \(kq_n \in (m, m + q_n - 1)\). Then we have that

\[
\prod_{j=m}^{m+q_n-1} |\bar{c}(\theta + j\alpha)| = \prod_{j=m}^{kq_n-1} |\bar{c}(\theta + j\alpha)| \cdot \prod_{j=kq_n}^{m+q_n-1} |\bar{c}(\theta + j\alpha)|
\]

\[
\tag{4.25}
= \frac{\prod_{j=(k-1)q_n}^{kq_n-1} |\bar{c}(\theta + j\alpha)|}{\prod_{j=(k-1)q_n}^{m-1} |\bar{c}(\theta + j\alpha)|} \cdot \frac{\prod_{j=kq_n}^{(k+1)q_n-1} |\bar{c}(\theta + j\alpha)|}{\prod_{j=m+q_n}^{(k+1)q_n-1} |\bar{c}(\theta + j\alpha)|}
\]

\[
> e^{-2\delta^2 \bar{\beta} q_n} \cdot e^{-2\delta^2 \bar{\beta} q_n} = e^{-6\delta^2 \bar{\beta} q_n}.
\]
Therefore,

\[
\prod_{j=m}^{m+q_n-1} |c(\theta + j\alpha)| = e^{q_n \int_T \ln |c(\theta)|d\theta} \prod_{j=m}^{m+q_n-1} |\tilde{c}(\theta + j\alpha)| \\
\geq e^{-(\int_T \ln |c(\theta)|d\theta + 6\delta^2 \beta) q_n} =: e^{-\Lambda q_n},
\]

as claimed. This completes the proof of Lemma 4.2.

As an explicit example, we have the following arbitrarily small lower bound for the analytic case with zero logarithm mean.

**Corollary 4.3.** Assume that \(c(\theta) \in C^\infty(\mathbb{T}, \mathbb{C})\) and \(\int_T \ln |c(\theta)|d\theta = 0\). Denote all the zeros\(^6\) of \(c(\theta)\) on \(\mathbb{T}\) by \(c^{-1}(0) = \{\theta_1, \ldots, \theta_m\}\). For any \(\beta\) with \(0 < 2\beta < \beta(\alpha)\) and \(0 < \delta < 1\), there is a sequence \(q_n \to \infty\) and a full Lebesgue measure set \(\Theta = \Theta(\alpha, c^{-1}(0))\) such that for any \(\theta \in \Theta\), \(c(\theta + n\alpha)\) satisfies

\[
\min_{|k| \leq e^{\theta q_n}} \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-2\delta^2 \min\{\beta, 1\} q_n},
\]

\[
\min_{|m| \leq e^{\theta q_n}} \prod_{j=mq_n}^{m+q_n-1} |c(\theta + j\alpha)| > e^{-6\delta^2 \min\{\beta, 1\} q_n}.
\]

**Proof.** Clearly, there is an analytic function \(\tilde{g}(\theta)\) such that

\[
c(\theta) = \tilde{g}(\theta) \prod_{\ell=1}^{m} (e^{2\pi i \theta} - e^{2\pi i \theta_{\ell}}), \quad \inf_{\mathbb{T}} \tilde{g}(\theta) > 0.
\]

Direct computation shows \(\int_{\mathbb{T}} \ln |\tilde{g}(\theta)|d\theta = \int_{\mathbb{T}} \ln |c(\theta)|d\theta = 0\) and

\[
c(\theta) = \tilde{g}(\theta) \prod_{\ell=1}^{m} (e^{2\pi i \theta} - e^{2\pi i \theta_{\ell}}) = \tilde{g}(\theta)(2i)^{m} \prod_{\ell=1}^{m} e^{i\pi(\theta + \theta_{\ell})} \sin(\pi(\theta - \theta_{\ell})).
\]

Therefore,

\[
\prod_{\ell=1}^{m} \left| \frac{|c(\theta)|}{\sin(\pi(\theta - \theta_{\ell}))} \right| = 2^m |\tilde{g}(\theta)|.
\]

Applying Lemma 4.2 to (4.30) where \(\tau_1 = \cdots = \tau_m = 1\) and \(|g(\theta)| = 2^m |\tilde{g}(\theta)|\), we have (4.6) and (4.9) hold with \(\Lambda_1 = 2\delta^2 \min\{\beta, 1\}\) and \(\Lambda = 6\delta^2 \min\{\beta, 1\}\). \(\square\)

\(^6\) Clearly, analytic function \(c(\theta)\) only has finitely many zeros on \(\mathbb{T}\).
5 Spectral continuity: Proof of Theorem 2.1

Following the notations and assumptions in Theorem 2.1, consider
\[(5.1) \quad (Hu)_n = w_n u_{n+1} + w_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z}.
\]
Assume that there are positive constants \(\beta, \delta, \Lambda > 0\) and a sequence of positive integers \(q_n \to \infty\) such that \(w_n, v_n\) has \(\beta\)-\(q_n\) almost periodicity and \(w_n\) has \((\Lambda, \beta)\)-\(q_n\) bound.

The key observation is: if \(H\) has \(\beta\)-\(q_n\) almost periodicity, then it can be approximated by a \(q_n\) periodic operator exponentially fast in a finite (exponentially large) lattice. The estimates on the \(q_n\) periodic operator eventually lead to the quantitative upper bound for the \(m\)-function as in (2.4) through the subordinacy theory Lemma 3.2.

In view of Lemma 3.2, let \(v^\varphi\) be the right half-line solution to \(Hu = Eu\) with initial condition \(\varphi\) and \(\ell = \ell(\varphi, \varepsilon, E)\) is defined as in (3.21). As a direct consequence of Lemma 3.2 and Proposition 3.4, the following relation between the power law of \(\|v^\varphi\|_\ell\) and the spectral continuity was proved in [32] (see Lemma 2.1 and the proof of Theorem 6 there).

**Lemma 5.1.** Fix \(0 < \gamma < 1\). Suppose for \(\mu\)-a.e. \(E\), there is a sequence of positive numbers \(\eta_k \to 0\) and \(L_k = \ell_k(\varphi, \eta_k, E) \to \infty\) such that for any \(\varphi\)
\[(5.2) \quad \frac{1}{16}(L_k)^\gamma \leq \|v^\varphi\|_{L_k}^2 \leq (L_k)^{2-\gamma}.
\]
Then the spectral measure \(\mu\) is \(\gamma\)-spectral continuous.

Let \(A(n; E)\) be defined as in (3.8). Denote by \(\text{Tr} A\) the trace of any matrix \(A \in \text{GL}(2, \mathbb{C})\). The following estimate on \(\text{Tr} A(q_n; E)\) is the key to prove the above power law and spectral continuity.

**Theorem 5.2.** Let \(H, \beta, \delta, \Lambda\) and \(q_n\) be given as in (5.1). Suppose
\[\beta > 260\left(1 + \frac{1}{\delta}\right)\Lambda.
\]
Then for \(\mu\) a.e. \(E\), there exists \(K(E) \in \mathbb{N}\), for \(k \geq K(E)\), so we have
\[(5.3) \quad |\text{Tr} A(q_k; E)| < 2 - 2e^{-60\Lambda q_k}.
\]
For any \(0 < \gamma < 1\), assume further that
\[(5.4) \quad \beta > 300\left(1 + \frac{1}{\delta}\right) \frac{\Lambda}{1 - \gamma},
\]
we have the power law required in (5.2).
Let
\begin{equation}
C = C(\delta, \Lambda) = 300 \left( 1 + \frac{1}{\delta} \right) \Lambda.
\end{equation}

Combining Lemma 5.1 and (5.4) in Theorem 5.2, if $\beta > C$, then $\mu$ is $\gamma$-spectral continuous for any $\gamma < 1 - \frac{C}{\beta} < 1$ and therefore
\[
\text{dim}_{\text{spec}}(\mu) \geq 1 - \frac{C}{\beta}.
\]

This proves Theorem 2.1.

The trace estimate (5.3) shows that spectrally almost everywhere, $A(q_k; E)$ is strictly elliptic eventually. The quantitative estimate (5.3) allows us to iterate the transfer matrix up to the length scale $e^{\Lambda q_k}$, which gives a good control on the norm of $A(q_k; E)$. The norm estimate eventually leads to the power law as required in (5.2) through (3.3).

The proof of (5.4) and the required power law follows the outline of the Schrödinger case (see [32], Lemma 2.1). The main difference is now that the transfer matrix $A(n; E)$ is in $GL(2, \mathbb{C})$. We need to consider some transformations introduced in [22] which conjugate $A(n; E)$ to some $SL(2, \mathbb{R})$ matrix. Then many important techniques developed in [32] for $SL(2, \mathbb{R})$ cocycles are now applicable. The trace estimate (5.3) leads to a norm estimate of $A(q_k; E)$ and eventually leads to the estimate (5.2) for the truncated $\ell^2$ norm of the eigenfunction $v \psi$ by (3.3).

We will omit the details here and focus on the proof of (5.4) and the power law (5.2) in the Appendix A.1 for the reader’s convenience.

The rest of the section is organized as follows: In Section 5.1, we introduce the transformation we will use to conjugate $GL(2, \mathbb{C})$ to $SL(2, \mathbb{R})$ and develop all the useful lemmas about the conjugate. In Section 5.2, we study the case where the trace of the transfer matrix is greater than 2. In Section 5.3, we study the case where the trace of the transfer matrix is close to 2.

Throughout this section, we assume $v_n, w_n$ have $\beta$-$q$ almost periodicity and $w_n$ has $(\Lambda, \beta)$-$q$ bound for some $q$ large enough such that $e^{-(\beta - 2\Lambda)q} < 1/10$. We also use the induced estimates (3.35)–(3.38) discussed in Section 3.4 directly, referred to also as $\beta$-$q$ almost periodicity and $(\Lambda, \beta)$-$q$ bound.

### 5.1 Conjugate between $SL(2, \mathbb{R})$ and $GL(2, \mathbb{C})$ matrices.

The trace estimate (5.3) was first proved in [32] for $SL(2, \mathbb{R})$ cocycles. The generalization to the $GL(2, \mathbb{C})$ case is very delicate. We need to consider the following transformation:
Let
\begin{equation}
T_n = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{w_n}{w_{n+1}}} \end{pmatrix}
\end{equation}
and
\begin{equation}
r_n = \frac{w_{n+1}}{\sqrt{|w_{n+1}w_n|}}.
\end{equation}

Let \( A_n(E) \) be given as in (3.4). Define
\begin{equation}
\tilde{A}_n(E) := r_n - 1 T_n^{-1} A_n T_n - 1 = \frac{1}{\sqrt{|w_n w_{n-1}|}} \begin{pmatrix} E - v_n & -w_{n-1} \\ w_n & 0 \end{pmatrix}.
\end{equation}

The \( n \)-transfer matrix \( \tilde{A}(n, m; E) \) and \( \tilde{A}(n; E) \) for \( \tilde{A}_n \) will be defined in the same way as in (3.5):
\begin{equation}
\tilde{A}(n, m; E) = \prod_{j=m}^{n} \tilde{A}_j(E), \quad n \in \mathbb{N}^+, \ m \in \mathbb{Z}
\end{equation}
and
\begin{equation}
\tilde{A}(n; E) = \tilde{A}(n, 1; E), \ n > 0; \quad \tilde{A}(0; E) = \text{Id};
\end{equation}
\begin{equation}
\tilde{A}(n; E) = \tilde{A}^{-1}(-n, n+1; E), \ n < 0.
\end{equation}

We also denote the scalar product of \( r_n \) in the same way as \( w(n, m) \) in (3.10) for \( n \in \mathbb{N}^+ \),
\begin{equation}
r(n, m) = \prod_{j=m}^{n+m-1} r_j, \quad m \in \mathbb{Z},
\end{equation}
and \( r^{-1}(n, m) = \frac{1}{r(n, m)} \) is the inverse of a complex number in the usual sense.

Notice that \( \tilde{A}_n \) and \( r_{n-1} A_n \) are conjugated by \( T_n \) and the conjugation preserves the skew product due to the telescoping structure. Direct computation by (5.8) shows that
\begin{equation}
\tilde{A}(n, m; E) = r(n, m - 1) T_{n+m-1}^{-1} A(n, m; E) T_{m-1}.
\end{equation}

In view of (5.6) and (5.8), it is easy to check that
\[ ||T_n|| = 1 \quad \text{and} \quad \tilde{A}_n, \tilde{A}(n, m) \in \text{SL}(2, \mathbb{R}) \]
for any \( n, m \). By (5.8) and (5.12), we are able to apply the techniques developed in [32] for the SL(2, \( \mathbb{R} \)) matrix. We are going to switch between the singular GL(2, \( \mathbb{C} \)) case and the SL(2, \( \mathbb{R} \)) case.

The \( \beta \)-almost periodicity and the \((\Lambda, \beta)\) boundedness of \( w_n \) imply the \( \beta \)-almost periodicity of \( r \) and \( T \) in the following sense:
Lemma 5.3. If \( \beta > 2\Lambda \), then for all \( m \in \mathbb{Z} \) such that \( |m| < e^{\beta q} \),

\[
(5.13) \quad |r(q, m)| - 1| < e^{-(\beta - 2\Lambda)q}; \quad |r^{-1}(q, m)| - 1| < e^{-(\beta - 2\Lambda)q};
\]

\[
(5.14) \quad \| T_{m+q}^{-1} \cdot T_m - I \| = \| T_m \cdot T_{m+q}^{-1} - I \| < 4e^{-(\beta - 2\Lambda)q}.
\]

Assume further that \( N \in \mathbb{N}^+ \), \( Nq \leq e^{\beta q} \). Then

\[
(5.15) \quad |r(Nq, 0)| - 1| < 2Ne^{-(\beta - 2\Lambda)q}; \quad |r^{-1}(Nq, 0)| - 1| < 2Ne^{-(\beta - 2\Lambda)q};
\]

\[
(5.16) \quad \| T_{Nq}^{-1} \cdot T_0 - I \| = \| T_0 \cdot T_{Nq}^{-1} - I \| < 4Ne^{-(\beta - 2\Lambda)q}.
\]

Note that \( r(n, m) \) and \( T_n \) are essentially scalar products. The proof is based on the following direct computation:

**Proof.** Set \( z_m = \frac{w_m}{w_{m+q}} \). By (3.37), for \( |m| \leq e^{\beta q} \) and \( q \) large,

\[
|z_m|^{\pm} - 1| \leq |z^{\pm} - 1| < e^{-(\beta - 2\Lambda)q} < \frac{1}{2}.
\]

Therefore, \( |z_m|^{\pm} > \frac{1}{2} \), and

\[
|\sqrt{|z_m|^{\pm}} - 1| = \left| \frac{|z_m|^{\pm} - 1}{\sqrt{|z_m|^{\pm} + 1}} \right| \leq |z^{\pm} - 1| < e^{-(\beta - 2\Lambda)q}.
\]

We will sketch the proof for \( r(q, m) \), \( T_{m+q}^{-1} \cdot T_m \), \( r(Nq, 0) \) and \( T_{Nq}^{-1} \cdot T_0 \). The proof for \( r^{-1}(q, m) \) and \( r^{-1}(Nq, 0) \) will be exactly the same. Clearly, \( |r_n| = \sqrt{\frac{|w_{m+1}|}{|w_m|}} \). In view of (5.11) and (5.6), we have

\[
(5.17) \quad |r(q, m)| - 1| = \left| \sqrt{\frac{|w_{m+1}|}{|w_m|}} \cdots \frac{|w_{m+q}|}{|w_{m+q-1}|} - 1 \right| = |\sqrt{|z_m|^{1-1}} - 1| < e^{-(\beta - 2\Lambda)q},
\]

and

\[
T_{m+q}^{-1} \cdot T_m = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{|w_{m+q}|w_{m+q}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{|w_m|w_m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{|w_{m+1}|}{|w_m|} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{z_m}{|z_m|} \end{pmatrix}.
\]

The triangle inequality implies that \( |z_m - |z_m|| \leq |z_m - 1| + 1 - |z_m|| \leq 2|z_m - 1| \). Therefore,

\[
(5.18) \quad \| T_{m+q}^{-1} \cdot T_m - I \| \leq |z_m - 1| = \frac{|z_m - |z_m||}{|z_m|} \leq 2|z_m - 1| = 4e^{-(\beta - 2\Lambda)q},
\]

since \( |z_m - 1| < e^{-(\beta - 2\Lambda)q} \) and \( |z_m| > \frac{1}{2} \).

In particular, let \( m = 0, q, 2q, \ldots, (N - 1)q \) for \( Nq < e^{\beta q} \) in (5.17); we have that \( |r(q, m)| - 1| < e^{-(\beta - 2\Lambda)q} < \frac{1}{2} \). Notice that \( |r(q, mq)| - 1 \) are all scalars; direct computation shows that for all \( 0 \leq j < N \),

\[
(5.19) \quad \prod_{k=j}^{N-1} |r(q, kq)| \leq (1 + e^{-(\beta - 2\Lambda)q})^{N-j} \leq 1 + 2Ne^{-(\beta - 2\Lambda)q} \leq 2,
\]

provided \( q \) is large.
A telescoping argument implies that
\begin{equation}
|r(Nq, 0)| - 1 = \prod_{k=0}^{N-1} |r(q, kq)| - 1 \leq \sum_{j=0}^{N-1} |r(q, jq)| - 1 \prod_{k=j+1}^{N-1} |r(q, kq)| \leq 2 \sum_{j=0}^{N-1} |r(q, jq)| - 1 \leq 2Ne^{-(\beta-2\Lambda)q}.
\end{equation}

As obtained in (5.18), we have that for 0 \leq k < N,
\begin{equation}
\left| \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq 4e^{-(\beta-2\Lambda)q}.
\end{equation}

Notice that \( \prod_{k=j+1}^{N-1} |\frac{z_{kq}}{|z_{kq}|}| = 1 \) for all \( j \). By the same telescoping argument in (5.20), we have that
\begin{equation}
\left| \prod_{k=0}^{N-1} \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq \prod_{j=0}^{N-1} \left| \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq \prod_{j=k+1}^{N-1} \left| \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq 4Ne^{-(\beta-2\Lambda)q}.
\end{equation}

Direct computation shows that
\begin{equation}
T_{Nq}^{-1} T_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{w_{Nq}^{-1}}{w_0 w_{Nq}^{-1}}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{w_0^{-1}}{w_0 w_{Nq}^{-1}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_0 w_{Nq}^{-1}}{|w_0 w_{Nq}^{-1}|} \end{pmatrix},
\end{equation}
and therefore
\begin{equation}
\|T_{Nq}^{-1} T_0 - I\| \leq \left| \frac{w_0 w_{Nq}^{-1}}{|w_0 w_{Nq}^{-1}|} - 1 \right| \leq \prod_{k=0}^{N-1} \left| \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq 4Ne^{-(\beta-2\Lambda)q}.
\end{equation}

Equation (5.12) only implies \( \|\tilde{A}(q; E)\| \approx \|A(q; E)\| \), while \( \|A(q; E) - \tilde{A}(q; E)\| \) is not necessarily small. Lemma 5.3 actually shows that \( \tilde{A}(q; E) \) and \( A(q; E) \) are close to each other up to a conjugate. This will be enough to control the difference between their traces. Fix \( E \); we write \( A(n, m) = A(n, m; E) \) for short.

**Lemma 5.4.** For all \( m \in \mathbb{Z} \) such that \( |m| < e^{3\beta q} \), let \( \Phi = \text{Arg} r(q, m) \) be the principal value of \( r(q, m) \in \mathbb{C} \). For \( \beta > 4\Lambda \),
\begin{align}
\frac{1}{2} \|A(q, m)\| &\leq \|\tilde{A}(q, m)\| \leq 2\|A(q, m)\| < 2e^{2\Lambda q} \tag{5.23} \\
\|\tilde{A}(q, m + 1) - e^{i\Phi} T_m^{-1} A(q, m + 1) T_m\| &< 12e^{-(\beta-4\Lambda)q} \tag{5.24}
\end{align}
and consequently
\begin{equation}
\|\text{Tr} \tilde{A}(q, m) - |\text{Tr} A(q, m)|\| < 12 e^{-(\beta-4\Lambda)q}. \tag{5.25}
\end{equation}
**Proof.** By (5.13), we have \(|r(q, m - 1)|, |r^{-1}(q, m - 1)| \leq 2\). Now (5.23) follows from (3.38) and (5.12) since

\[
\|A(q, m)\| \leq |r(q, m - 1)| \cdot \|T_{q+m-1}^{-1}\| \cdot \|A(q, m)\| \cdot \|T_{m-1}\| \leq 2\|A(q, m)\|;
\]

\[
\|A(q, m)\| \leq |r^{-1}(q, m - 1)| \cdot \|T_{q+m-1}^{-1}\| \cdot \|A(q, m)\| \cdot \|T_{m-1}^{-1}\| \leq 2\|\tilde{A}(q, m)\|,
\]

where we use the fact that \(\|T_m^{\pm}\| = 1\) for all \(m \in \mathbb{Z}\).

By (5.12) and \(r(q, m) = |r(q, m)|e^{i\Phi}\), we have that

\[
\tilde{A}(q, m + 1) = r(q, m)T_{m+q}^{-1}A(q, m + 1)T_m
\]

(5.26)

\[
=r(q, m)|e^{i\Phi} T_{m+q}^{-1} T_m T_m^{-1} A(q, m + 1)T_m
\]

\[
= (|r(q, m)|T_{m+q}^{-1} T_m) e^{i\Phi} T_m^{-1} A(q, m + 1)T_m.
\]

Therefore,

\[
\|\tilde{A}(q, m + 1) - e^{i\Phi} T_m^{-1} A(q, m + 1)T_m\|
\]

\[
= \|(r(q, m)|T_{m+q}^{-1} T_m - I) e^{i\Phi} T_m^{-1} A(q, m + 1)T_m\|
\]

(5.27)

\[
\leq \|(r(q, m)|T_{m+q}^{-1} T_m - I)\| \cdot \|e^{i\Phi} T_m^{-1} A(q, m + 1)T_m\|
\]

\[
\leq 6e^{-(\beta-2\Lambda)q}\|A(q, m + 1)\|
\]

\[
\leq 12e^{-(\beta-4\Lambda)q}.
\]

The penultimate (one before the last) inequality follows from (5.13) and (5.14) since

\[
\|(|r(q, m)|T_{m+q}^{-1} T_m - I)\|
\]

\[
\leq \|r(q, m)| - 1\| \cdot \|T_{m+q}^{-1} T_m - I\| + \|r(q, m)| - 1\| + \|T_{m+q}^{-1} T_m - I\|.
\]

The conjugation by the unimodular matrix \(T_m\) preserves the trace of \(A(q, m + 1)\), \(\text{Tr} A(q, m + 1) = \text{Tr}(T_m^{-1} A(q, m + 1) T_m)\), and the scalar rotation \(e^{i\Phi}\) preserves the modulus of the trace,

\[
|\text{Tr}(T_m^{-1} A(q, m + 1) T_m)| = |\text{Tr}(e^{i\Phi} T_m^{-1} A(q, m + 1) T_m)|.
\]

Inequality (5.25) follows directly from (5.24) since

\[
|\text{Tr} A(q, m + 1)| = |\text{Tr}(e^{i\Phi} T_m^{-1} A(q, m + 1) T_m)|
\]

and

\[
|\text{Tr}\tilde{A}(q, m + 1)| - |\text{Tr}(e^{i\Phi} T_m^{-1} A(q, m + 1) T_m)|
\]

\[
\leq |\text{Tr}\tilde{A}(q, m + 1) - \text{Tr}(e^{i\Phi} T_m^{-1} A(q, m + 1) T_m)|
\]

\[
\leq \|\tilde{A}(q, m + 1) - e^{i\Phi} T_m^{-1} A(q, m + 1) T_m\|.
\]

□
A standard telescoping argument allows us to pass the $\beta$-almost periodicity from the sequences $w_n, v_n$ to the matrices $A(n, m), A(n, m)$, up to product a of length $q$:

**Lemma 5.5.** For all $m \in \mathbb{Z}$ such that $|m| < e^{6\beta q}$, $\beta > 6\Lambda$,

\begin{equation}
\|A(q, m; E) - A(q, m + q; E)\| \leq e^{(-\beta + 6\Lambda)q}
\end{equation}

and

\begin{equation}
\|\tilde{A}(q, m; E) - \tilde{A}(q, m + q; E)\| \leq e^{(-\beta + 6\Lambda)q}.
\end{equation}

**Proof.** Write $m' = m + q$ for short. Then

$$A(q, m) - A(q, m') = \sum_{j=0}^{q-1} A(q - j - 1, m + j + 1)(A_{m+j} - A_{m'+j})A(j, m')$$

$$= \sum_{j=0}^{q-1} D(q - j - 1, m + j + 1)(D_{m+j}\frac{D_{m'+j}}{w_{m+j}})D(j, m') \cdot \frac{D(j, m')}{w(j, m')}.$$  

By the trivial upper bound (3.33) for $D(n, m)$ and the lower bound (3.35) for $w(n, m)$, we have

$$\|A(q, m) - A(q, m')\|$$

$$\leq \sum_{j=0}^{q-1} \frac{e^{(q-j-1)\Lambda}}{|w(q - j - 1, m + j)|}|w_{m'+j}D_{m+j} - w_{m+j}D_{m'+j}|e^{\Lambda j} \frac{e^{\Lambda}}{w(j + 1, m')}$$

$$\leq \sum_{j=0}^{q-1} \frac{e^{(q-j-1)\Lambda}}{e^{2\Lambda q}}|w_{m'+j}D_{m+j} - w_{m+j}D_{m'+j}|e^{\Lambda} \frac{e^{\Lambda}}{e^{2\Lambda q}}$$

$$\leq q e^{5\Lambda q} \max_{|m| \leq e^{6\beta q}} |w_{m+q}D_m - w_mD_{m+q}|$$

$$\leq q e^{5\Lambda q} \max_{|m| \leq e^{6\beta q}} (|w_{m+q} - w_m|D_m + |w_mD_{m+q} - D_m|)$$

$$\leq 2q e^{5\Lambda q} e^{\Lambda} e^{-\beta q}$$

$$\leq e^{-(\beta - 6\Lambda)q},$$

provided $\sup_n |w_n|, \sup_{n,E} \|D_n\| \leq e^{\Lambda}$ and $q$ is large such that $2qe^{\Lambda} \leq e^{\Lambda q}$.

Let

$$\tilde{w}_n = \sqrt{|w_n w_{n-1}|}, \quad \tilde{D}_n = \begin{pmatrix} E - v_n & -|w_{n-1}| \\ |w_n| & 0 \end{pmatrix}.$$
Define \( \tilde{w}(n, m), \tilde{D}(n, m; E) \) exactly in the same as for \( w, D \). It is easy to check that the \( \Lambda \) bounds of \( w_n \) and \( D_n \) hold true for \( \tilde{w}_n, \tilde{D}_n \):

\[
|\tilde{w}(n, m)| \leq e^{\Lambda n}, \quad \sup_E \|\tilde{D}(n, m; E)\| \leq e^{\Lambda n}, \quad \forall n \geq 0, \quad m \in \mathbb{Z}
\]

and

\[
|\tilde{w}(r, m)| \geq e^{-2\Lambda q}, \quad 0 \leq r \leq q, \quad |m| \leq e^{\delta \beta q}.
\]

The \( \beta \)-almost periodicity of \( w_n \) and \( v_n \) is also passed directly to \( \tilde{w}_n, \tilde{D}_n \):

\[
\max_{|m| \leq e^{\delta \beta q}} |\tilde{w}_m - \tilde{w}_{m \pm q}| \leq 2e^{\Lambda} e^{-\beta q}, \quad \max_{|m| \leq e^{\delta \beta q}} \|\tilde{D}_m - \tilde{D}_{m \pm q}\| \leq e^{-\beta q}.
\]

By the definition of \( \tilde{A}_n \) in (5.8) we have \( \tilde{A}_n = \frac{1}{w_n} \tilde{D}_n \). Exactly the same computation proves that

\[
\|\tilde{A}(q, m) - \tilde{A}(q, m')\| \leq q(2e^{2\Lambda} + e^{\Lambda})e^{5\Lambda q} e^{-\beta q} \leq e^{-(\beta-6\Lambda)q}.
\]

The above telescoping argument can not be extended to exponential scale \( e^{\Lambda q} \) as for \( r \) and \( T \) in (5.15) and (5.16) directly. One main reason is that we lose control of the matrix norm super-exponentially as \( \|D(e^{\Lambda q})\| \lesssim e^{\Lambda e^{\Lambda q}} \). Such growth can not be controlled by a condition such as \( \beta \gtrsim \Lambda \). The key to prove the trace estimate (5.3) is to avoid using such a rough bound for the matrix norm at an exponential scale. This is one breakthrough in [32]. By all the above estimates of the conjugate \( r, T \) and some simple linear algebra facts of the \( \text{SL}(2, \mathbb{R}) \) matrix found in [32], we are able to prove this extension for the \( \text{GL}(2, \mathbb{C}) \) case. We will see more details in the next two subsections.

Similar to [32], we consider the following two cases where \( |\text{Tr}A(q)| \) is away from 2 and close to 2.

### 5.2 The case where the trace is away from 2.

We start with the hyperbolic case in the following sense: let

\[
S^1_q = \{ E : |\text{Tr}A(q; E)| > 2 + 2e^{-60\Lambda q}\}.
\]

We may fix \( E \) and write \( A(q) = A(q; E) \) for simplicity whenever it is clear.

**Lemma 5.6.** Let \( q_n \) be given as in Theorem 5.2. If \( \beta > (260 + \delta)\Lambda \), then the set

\[
\limsup_{n \to \infty} S^1_{q_n} = \{ E : E \text{ belongs to infinitely many } S^1_{q_n} \}
\]

has spectral measure zero.
Lemma 5.6 will be proved later in this subsection. We need some technical lemmas first. Lemma 5.4 implies for large $\beta$ that $\text{Tr}\tilde{A}(q; E)$ and $\text{Tr}A(q; E)$ lie in the same region, i.e., if $E \in S^1_q$, then

$$|\text{Tr}\tilde{A}(q; E)| > 2 + 2e^{-60\Lambda q} - 12e^{-(\beta - 4\Lambda)q} > 2 + e^{-60\Lambda q},$$

provided $e^{(\beta - 64\Lambda)q} > 12$.

The following linear algebra facts were proved in [32]:

**Lemma 5.7.** Suppose $G \in \text{SL}(2, \mathbb{R})$ with $2 < |\text{Tr} G| \leq 6$. There exists invertible matrix $B$ such that

$$G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1},$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of $G$; $B$ satisfies $|\text{det} B| = 1$ and

$$\|B\| = \|B^{-1}\| < \frac{\sqrt{|G|}}{|\text{Tr} G| - 2}.$$ 

If $|\text{Tr} G| > 6$, then $\|B\| \leq \frac{2\sqrt{|G|}}{\sqrt{|\text{Tr} G| - 2}}$.

Apply the above lemma to $\tilde{A}(q; E) \in \text{SL}(2, \mathbb{R})$ satisfying (5.23) and (5.36). We have the following decomposition:

$$\tilde{A}(q) = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1},$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of $\tilde{A}(q)$ with

$$|\rho| > |\text{Tr}\tilde{A}(q)| - 1 > 1 + e^{-60\Lambda q} \quad \text{and} \quad |\text{det} B| = 1.$$

By (5.38), (5.23) and (5.36), $B$ satisfies

$$\|B\| = \|B^{-1}\| < \frac{2|\tilde{A}(q)|}{|\text{Tr}\tilde{A}(q)| - 2} < \frac{4e^{2\Lambda q}}{\sqrt{e^{-60\Lambda q}}} < e^{33\Lambda q}.$$ 

By (5.39) and (5.40), we have that for any $N \in \mathbb{N}^+$,

$$\tilde{A}(q)^N := [\tilde{A}(q)]^N = B \begin{pmatrix} \rho^N & 0 \\ 0 & \rho^{-N} \end{pmatrix} B^{-1}, \quad \|\tilde{A}(q)^N\| \leq e^{66\Lambda q}|\rho|^N.$$ 

In the rest of this section, consider $\beta > \frac{61}{\delta} \Lambda$ and set

$$N = [e^{61\Lambda q}] < e^{6\beta q}.$$
The above decomposition now turns the matrix product $[\tilde{A}(q)]^N$ into a scalar product of $\rho^N$ with a uniformly controlled conjugate $B$ (independent of $N$). This is one key algebra ingredient observed in [32]. This technique now allows us to extend the orbit of $\tilde{A}(q)$ to the exponentially long scale $N = e^{61\Lambda q}$.

The following technical lemma was proved in [32]:

**Lemma 5.8** (Lemma A.1, [32]). Suppose $G$ is a two by two matrix satisfying

\begin{equation}
\|G^j\| \leq M < \infty, \quad \text{for all } 0 < j \leq N \in \mathbb{N}^+,
\end{equation}

where $M \geq 1$ only depends on $N$. Let $G_j = G + \Delta_j$, $j = 1, \ldots, N$, be a sequence of two by two matrices with

\begin{equation}
\delta = \max_{1 \leq j \leq N} \|\Delta_j\|.
\end{equation}

If

\begin{equation}
NM\delta < 1/2,
\end{equation}

then for any $1 \leq n \leq N$

\begin{equation}
\left\| \prod_{j=n}^1 G_j - G^n \right\| \leq 2NM^2\delta.
\end{equation}

Let $N = [e^{61\Lambda q}]$, $G = \frac{1}{\rho} \tilde{A}(q)$ and $G_j = \frac{1}{\rho} \tilde{A}(jq, jq+1)$, $|j| = 0, 1, \ldots, N$. By (5.41), we have that for all $0 \leq j \leq N$,

$$
\|G^j\| = \frac{1}{|\rho|^j}\|\tilde{A}^j\| \leq e^{66\Lambda q},
$$

and by (5.29), we have that for all $1 \leq j \leq N$,

$$
\|G_j - G\| \leq \frac{1}{|\rho|} \sum_{k=0}^{j-1} \|\tilde{A}(q, (k+1)q+1) - \tilde{A}(q, kq + 1)\| \leq Ne^{(-\beta + 66\Lambda)q} \leq e^{(-\beta + 67\Lambda)q}.
$$

The above lemma is applicable provided $\beta > (260 + \frac{64}{\Lambda})\Lambda$. One can prove that

\begin{equation}
\|\tilde{A}(Nq) - \tilde{A}^N(q)\| \leq |\rho|^N e^{(-\beta + 260\Lambda)q},
\end{equation}

\begin{equation}
\|\tilde{A}(-Nq) - \tilde{A}^{-1}(Nq)\| \leq 2|\rho|^N e^{(-\beta + 260\Lambda)q}.
\end{equation}

The proof (5.47) and (5.48) is a direct application of Lemma 5.8 and resembles the proof of Claim 3 in [32]. We omit the details here.

Similar to (5.24), we can prove $A(\pm Nq)$ and $\tilde{A}(\pm Nq)$ are close to each other up to size $|\rho|^N$. 
Lemma 5.9. Let $\eta = r^{-1}(Nq, 0)$, $\zeta = r(Nq, -Nq)$ and $\phi = \text{Arg } \eta$, $\psi = \text{Arg } \zeta$ be the principal values of $\eta$ and $\zeta$ accordingly. For $\beta > (260 + \frac{64}{\rho})\Lambda$,

\begin{align}
(5.49) \quad & \|A^\pm(Nq) - e^{\pm \phi} T_0 \tilde{A}^\pm(Nq) T_0^{-1}\| < e^{(-\beta + 132\Lambda)q} |\rho|^N, \\
(5.50) \quad & \|A(-Nq) - e^{i\psi} T_0 \tilde{A}(-Nq) T_0^{-1}\| < e^{(-\beta + 132\Lambda)q} |\rho|^N,
\end{align}

and consequently,

\begin{equation}
(5.51) \quad \|A^{-1}(Nq) - e^{-i(\phi + \psi)} A(-Nq)\| < 4e^{(-\beta + 260\Lambda)q} |\rho|^N.
\end{equation}

Proof. By (5.8),

\begin{equation}
(5.52) \quad A(Nq) = \eta T_{Nq} \tilde{A}(Nq) T_0^{-1} = (|\eta| T_{Nq} T_0^{-1}) e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1}.
\end{equation}

By (5.15) and (5.16),

\begin{equation}
(5.53) \quad \| |\eta| T_{Nq} T_0^{-1} - I\| \leq |\eta| \cdot \| T_{Nq} T_0^{-1} - I\| + |\eta| - 1 \leq 10Ne^{(-\beta + 2\Lambda)q} \leq e^{(-\beta + 6\Lambda)q},
\end{equation}

provided $N < e^{6\Lambda q}$ and $e^{\Lambda q} > 10$.

By (5.41) and (5.47),

\begin{equation}
\|\tilde{A}(Nq)\| \leq \|\tilde{A}^N(q)\| + e^{(-\beta + 260\Lambda)q} |\rho|^N \leq e^{6\Lambda q} |\rho|^N + e^{(-\beta + 260\Lambda)q} |\rho|^N \\
\leq e^{67\Lambda q} |\rho|^N,
\end{equation}

provided $e^{(-\beta + 260\Lambda)q} < e^{67\Lambda q} - e^{6\Lambda q}$.

Therefore,

\begin{equation}
\|A(Nq) - e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1}\| \leq \| |\eta| T_{Nq} T_0^{-1} - I\| \cdot \| e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1}\| \leq e^{(-\beta + 131\Lambda)q} |\rho|^N,
\end{equation}

provided $\beta$ and $q$ are large.

Note that $\tilde{A}(Nq) \in \text{SL}(2, \mathbb{R})$; then $\|\tilde{A}^{-1}(Nq)\| = \|\tilde{A}(Nq)\| \leq e^{67\Lambda q} |\rho|^N$. Therefore, (5.48) implies that

\begin{equation}
\|\tilde{A}(-Nq)\| \leq \|\tilde{A}^{-1}(Nq)\| + 2|\rho|^N e^{(-\beta + 260\Lambda)q} \leq e^{68\Lambda q} |\rho|^N,
\end{equation}

provided $e^{67\Lambda q} + 2e^{(-\beta + 260\Lambda)q} < e^{68\Lambda q}$.

The proof for $A^{-1}(Nq)$ is exactly the same as the proof for $A(Nq)$ since

\begin{equation}
(5.54) \quad A^{-1}(Nq) = \eta^{-1} T_0 \tilde{A}^{-1}(Nq) T_{Nq}^{-1} = e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1} (|\eta|^{-1} T_0 T_{Nq}^{-1}).
\end{equation}
Now (5.8) and (5.10) imply that
\[ A(-Nq) = A^{-1}(Nq, -Nq + 1) = \left[ r^{-1}(Nq, -Nq)T_0 \tilde{A}(Nq, -Nq + 1)T_{-Nq}^{-1} \right]^{-1} = r(Nq, -Nq)T_{-Nq} \tilde{A}(-Nq)T_0^{-1}. \]

By (5.15) and (5.16), exactly the same argument for (5.49) proves (5.50) provided $\beta$ and $q$ large.

The proof of (5.51) follows directly from (5.48)), (5.49) and (5.50) since
\[
\|A^{-1}(Nq) - e^{-i(\phi + \psi)} A(-Nq)\| \leq \|A^{-1}(Nq) - e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1}\|
+ \|e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1} - e^{-i\phi} T_0 \tilde{A}(-Nq) T_0^{-1}\|
+ \|e^{-i\phi} T_0 \tilde{A}(-Nq) T_0^{-1} - e^{-i(\phi + \psi)} A(-Nq)\|.
\]
\[
\leq \|A^{-1}(Nq) - e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1}\|
+ \|\tilde{A}^{-1}(Nq) - \tilde{A}(-Nq)\|
+ \|e^{i\psi} T_0 \tilde{A}(-Nq) T_0^{-1} - A(-Nq)\|
\leq 2e^{(-\beta + 132\Lambda)q} |\rho|^N + 2e^{(-\beta + 260\Lambda)q} |\rho|^N
\leq 4e^{(-\beta + 260\Lambda)q} |\rho|^N. \]

With the above preparation, we are in place to prove Lemma 5.6. It is easy to see that all the estimates from (5.47) to (5.51) preserve errors between the traces. Now combine (5.47) with (5.49), to obtain
\[
\text{det}A(Nq) = \text{det} \tilde{A}(q) \leq 2e^{(-\beta + 260\Lambda)q} |\rho|^N \leq \frac{1}{2} |\rho|^N, \tag{5.55}
\]
provided $e^{(\beta - 260\Lambda)q} > 4$. Therefore, by (5.41),
\[
\text{Tr}A(Nq) = \text{Tr} \tilde{A}(q) \geq |\text{Tr} \tilde{A}(q)| - \frac{1}{2} |\rho|^N \geq \frac{1}{2} |\rho|^N. \tag{5.56}
\]
Inequality (5.51) implies that for any vector $X \in \mathbb{C}^2$,
\[
\|A^{-1}(Nq)X\| \leq \|A(-Nq)X\| + 4e^{(-\beta + 260\Lambda)q} |\rho|^N \|X\|
\leq \|A(-Nq)X\| + \frac{1}{8} |\rho|^N \|X\|, \tag{5.57}
\]
provided $e^{(\beta - 260\Lambda)q} > 32$.

By (5.8) and (5.12), it is easy to check that $|\text{det}A(Nq)| = |r^{-1}(Nq, 0)|$. Therefore, (5.15) implies that
\[
|\text{det}A(Nq)| < 1 + 2e^{(-\beta - 63\Lambda)q} < 2, \tag{5.58}
\]
provided $N < e^{61\Lambda q}$ and $e^{(\beta - 63\Lambda)q} > 2$. 

\[ \]
Consider the generalized eigenvalue equation $Hu = Eu$ with normalized initial value $X = \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix}$, $\|X\| = 1$. By (3.3) and the Cayley–Hamilton theorem for the $GL(2, \mathbb{C})$ matrix $A(Nq)$, we have

$$A(Nq) X = \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix},$$

and

$$A(-Nq) X = \begin{pmatrix} u_{-Nq+1} \\ u_{-Nq} \end{pmatrix},$$

and

$$A(Nq) X + (\det A) \cdot A^{-1}(Nq) X = -(\text{Tr} A(Nq)) X.$$

Combine (5.56), (5.57) and (5.58) with (5.60) to yield

$$\|A(Nq) X\| + \|A(-Nq) X\| \geq \frac{1}{8} |\rho|^N.$$

Now by the choice of $\rho$ and $N$, for $q$ large, we have

$$\|A(Nq) X\| + \|A(-Nq) X\| \geq \frac{1}{8}(1 + e^{-60\Lambda q})[e^{61\Lambda q}] \geq 4e^q$$

which implies

$$\max\{ |u_{Nq+1}|, |u_{Nq}|, |u_{-Nq+1}|, |u_{-Nq}| \} \geq e^q.$$

In conclusion, we can claim the existence of a subsequence of $u_n$ at energy $E$ with the following exponential growth:

Claim 5.10. Assume $v_n, w_n$ have $\beta$-q almost periodicity as in (2.1) and $w_n$ has $(\Lambda, \beta)$-q bound Definition 2.2, (2.2) for $q > q_0(\Lambda, \delta, \beta)$. Suppose $E \in S^1_q$ and $\beta > (260 + \frac{61}{\delta})\Lambda$. Then there are integer sequences $x^{i_1}_q, x^{i_2}_q, x^{i_3}_q, x^{i_4}_q \in \mathbb{Z}$ independent of $E$, such that $\min_i |x^{i_j}_q| \to \infty$ as $q \to \infty$ and

$$\max_i |u^{E}_{x^{i_j}_q}| > e^q,$$

where $u^{E}_{x^{i_j}_q}$ solves the half-line problem $Hu = Eu$ with normalized boundary condition $|u_0|^2 + |u_1|^2 = 1$.

Now Lemma 5.6 follows from Claim 5.10 and the following theorem.

Theorem 5.11 (Extended Schonl’s Theorem, Lemma 2.4, [32]). Fix any $y > 1/2$. For any sequence $|x_k| \to \infty$ (where the sequence is independent of $E$), for spectrally a.e. $E$, there is a generalized eigenvector $u^E$ of $Hu = Eu$, such that

$$|u^E_{x^{i_j}_q}| < C(1 + |k|)^y.$$
Proof of Lemma 5.6. Let \( q_n \) be given as in Theorem 5.2. Now apply Claim 5.10 to those \( E \) belonging to infinitely many \( E \in S_{q_k} \). Let \( x_{q_k}^i, i = 1, \ldots, 4 \) be given as in (5.64). For all \( k \in \mathbb{N} \), let \( \bar{x}_k \) be the \( x_{q_k}^i \) where the maximum in (5.64) is obtained. Clearly, \( \lim_k |x_k| = \infty \). By (5.64) and the choice \( \bar{x}_k, |u_E^{x_k}| > e^{q_k} \gg 1 + |k| \) for all boundary conditions. Therefore, the collection of such \( E \) has spectral measure zero because of (5.65).

\[ \square \]

5.3 The case where the trace is close to 2. In this part, we consider those energy \( E \) where the trace of \( A(q;E) \) is close to 2. Let

\[ S_q^2 = \{ E : |\text{Tr} A(q;E) - 2| < 2e^{-60\Lambda q} \}. \]

Again we assume that \( q \) is large and \( v_n, w_n \) satisfy \( \beta \)-almost periodicity (2.1) and \( \Lambda-q \) bound in (2.2) with positive finite parameters \( \beta, \Lambda, \delta \). We have

Lemma 5.12. If \( \beta > (130 + \frac{29}{\delta})\Lambda \), then

\[ \mu(S_q^2) < e^{-\frac{10}{\Lambda q}}, \]

where \( \mu \) is the spectral measure of \( H \).

We will prove Lemma 5.12 later. Before that, we can first finish

Proof of Theorem 5.2. Assume now \( \beta > 260(1 + \frac{1}{q})\Lambda \). Let \( q_n \) be given as in Theorem 5.2. Lemma 5.6 implies that for spectrally a.e. \( E \), there is \( K_1(E) \) such that

\[ |\text{Tr} A(q;E)| < 2 + 2e^{-60\Lambda q_k}, \quad \forall k \geq K_1(E). \]

For those \( E \) such that \( \text{Tr} A(q_n;E) \) is close to 2, Lemma 5.12 implies that \( \mu(S_{q_n}^2) < e^{-\frac{10}{\Lambda q_n}} \). Therefore, \( \sum_n \mu(S_{q_n}^2) < \infty \). By the Borel–Cantelli lemma, we have that \( \mu(\limsup_{n} S_{q_n}^2) = 0 \), where

\[ \limsup_{n} S_{q_n}^2 = \bigcap_{n \geq 1} \bigcup_{k \geq n} S_{q_k}^2. \]

Therefore, for spectrally a.e. \( E \), there is \( K_2(E) \) such that

\[ |\text{Tr} A(q;E)| - 2 | > 2e^{-60\Lambda q_k}, \quad \forall k \geq K_2(E). \]

Clearly, (5.68) and (5.69) complete the proof of Theorem 5.2 by taking \( K = \max\{K_1, K_2\}. \)

In the rest of the section, we focus on proving (5.67). Similar to the hyperbolic case, \( \text{Tr} \tilde{A}(q;E) \) and \( \text{Tr} A(q;E) \) are close up to an exponential error by Lemma 5.4. More precisely, let

\[ \tilde{S}_q^2 := \{ E : |\text{Tr} \tilde{A}(q;E) - 2| < 3e^{-60\Lambda q} \}. \]

Clearly, Lemma 5.4 implies that for \( \beta > 6\Lambda, \tilde{S}_q^2 \subset S_q^2 \).
The following elementary linear algebra facts were proved in [32]

**Lemma 5.13** (Lemma 2.9, Lemma 2.10 [32]). Suppose $A \in \text{SL}(2, \mathbb{R})$ has eigenvalues $\rho \pm 1$, $\rho > 1$. For any $k \in \mathbb{N}$, if $\text{Tr}A \neq 2$, then

\[
A^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left( A - \frac{\text{Tr}A}{2} \cdot I \right) + \frac{\rho^k + \rho^{-k}}{2} \cdot I.
\]

Otherwise, $A^k = k(A - I) + I$.

Assume further that $|\text{Tr}A - 2| < \tau < 1$. Then there are universal constants $1 < C_1 < \infty$, $c_1 > 1/3$ such that for $1 \leq k \leq \tau^{-1}$, we have

\[
c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1, \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k.
\]

Now fix $E \in \tilde{S}_2^q$. The above lemma actually shows that the $k$-th power of $\tilde{A}(q; E)$ grows almost linearly in $k$ as

\[
\tilde{A}^k(q) \sim k(\tilde{A}(q) - \frac{1}{2} \text{Tr} \tilde{A}(q)) + I, \quad 1 \leq k \leq N.
\]

This simple observation will be an important part of our quantitative estimates in the case where the transfer matrix is close to a parabolic one. The arguments to derive (5.67) from (5.73) follow the outline in [32]. We need some slight modifications concerning all the estimates of the conjugacy in Section 5.1. We sketch the proof below for the reader’s convenience.

**Proof of Lemma 5.12.** First, Lemma 5.13 provides the following norm estimates: there is an absolute constant $C_1 > 0$ such that for any

\[
1 \leq j < N = [e^{29\Lambda q}] < e^{29\Lambda q} < e^{\delta q},
\]

we have

\[
\|\tilde{A}^j(q; E)\| < 3C_1 j \cdot \|\tilde{A}(q)\|.
\]

By (5.23) and the choice of $N$, we have

\[
\|\tilde{A}^j(q; E)\| < 6C_1 j e^{2\Lambda q} < j e^{3\Lambda q} < e^{32\Lambda q}.
\]

Similar to the proof of (5.47) and (5.48), for any $1 \leq k \leq N$, combining (5.75) with (5.29), we can apply Lemma 5.8 to obtain

\[
\|\tilde{A}(kq) - \tilde{A}^k(q)\| \leq e^{(-\beta + 130\Lambda)q} < 1,
\]

provided $\beta > (130 + \frac{29}{\delta})\Lambda$ and $q$ is large.
In view of (5.71), (5.73) and (5.76), it is clear that \( \tilde{A}(kq) \) has the same linear expansion as in (5.73). By (5.71), (5.76), and the conjugate relation

\[
A(kq)X = r^{-1}(kq, 0) T(kq)\tilde{A}(kq)T^{-1}(0) X, \quad X \in \mathbb{C}^2,
\]

we can prove that:

**Claim 5.14.** For any \( \varepsilon > 0 \), \( E \) and \( \varphi \in [0, 2\pi) \), let \( \ell = \ell(\varphi, \varepsilon, E) \), \( u^\varphi, v^\varphi \) be given as in (3.21). Suppose \( E \in S^2_q, \varepsilon < e^{-2\Lambda q} \) and \( \beta > (130 + \frac{29}{3})\Lambda \). Then

\[
\|u^\varphi\|_\ell^2 > e^{\frac{1}{\ell}\Lambda q}.
\]

The proof of Claim 5.14 follows from the proof of Claim 5 in [32]. The key is to use the linear expression (5.73) to control both the upper and lower bounds of \( \|A(n)X\| \). The main difference is that we need to consider the conjugacy (5.77) and switch between the orbits of \( A(n)X \) and \( \tilde{A}(n)\tilde{X} \). We omit the details here. For the sake of completeness, we include the proof in Appendix A.2.

We proceed to prove Lemma 5.12 by Claim 5.14. Here \( \text{Tr} A(q; E) \) and \( \text{Tr} \tilde{A}(q; E) \) are both polynomials in \( E \) of degree \( q \). Therefore, \( S^2_q \) can be written as a union of at most \( q \) bands: \( S^2_q = \bigcup_{j=1}^q I_j \). We want to estimate each \( |I_j| \) by \( |\tilde{S}^2_q| \) from above, where \( |\cdot| \) stands for the Lebesgue measure of a set on the real line. Note that \( \tilde{S}^2_q \) is the preimage of \( \text{Tr} \tilde{A}(q; E) \) which has real coefficients. By Proposition A.3, we have

\[
|\tilde{S}^2_q| \leq C_2 \sqrt{6e^{-60\Lambda q}},
\]

where \( C_2 \) only depends on \( \|w\|_\infty, \|v\|_\infty \). Then this gives us a uniform control on the width of each band \( I_j \):

\[
S^2_q = \bigcup_{j=1}^q I_j, \quad \varepsilon^j_q := |I_j| \leq |S^2_q| \leq |\tilde{S}^2_q| \leq e^{-29\Lambda q}.
\]

Now pick \( E_j \in I_j \cap \sigma(H) \neq \emptyset \) to be the center in the sense that

\[
I_j \subset (E_j - \varepsilon^j_q, E_j + \varepsilon^j_q).
\]

For any \( \varphi \), let \( u^\varphi(E_j) \) be the right half line solution associated with the energy \( E_j \). By Claim 5.14, we have

\[
\|u^\varphi(E_j)\|_{\ell^2(j)}^2 \geq e^{\frac{1}{\ell^j_q}\Lambda q}, \quad j = 1, \ldots, q,
\]

where \( \ell_q(j) = \ell(\varphi, E_j, \varepsilon^j_q) \) is given as in (3.21).

A direct consequence of (5.80) and the subordinacy theory Lemma 3.2 is

\[
\varepsilon^j_q \cdot |m_q(E_j + i\varepsilon^j_q)| < \frac{5 + \sqrt{24}}{2} e^{-\frac{1}{\ell^j_q}\Lambda q}, \quad j = 1, \ldots, q.
\]
Then by (2.3) and (3.25), we have

\[
\mu(I_j) \leq \sup_{\phi} 2e^{|\phi|} |m_\phi(E_j + i\epsilon_j)| < (5 + \sqrt{24})e^{-\frac{1}{6}\Lambda q}, \quad j = 1, \ldots, q.
\]

Clearly, (5.82) completes the proof of Lemma 5.12 provided

\[q(5 + \sqrt{24})e^{-\frac{1}{6}\Lambda q} \leq e^{-\frac{1}{10}\Lambda q}.\]

□

6 Spectral Singularity for analytic quasiperiodic Jacobi operator

In this Section, we focus on an analytic quasiperiodic potential given by

\[v_n = v(\theta + n\alpha), \quad w_n = c(\theta + n\alpha), \quad n \in \mathbb{Z}, \quad \theta \in \mathbb{T},\]

where \(v \in C^0(\mathbb{T}, \mathbb{R})\) and \(c \in C^0(\mathbb{T}, \mathbb{C})\) are analytic functions on \(\mathbb{T}\) taking values in \(\mathbb{R}\) and \(\mathbb{C}\) respectively. Both \(v(\theta)\) and \(c(\theta)\) have bounded analytic extensions to the strip

\[\mathbb{T}_\rho := \{z : |\text{Im}z| < \rho\}.
\]

We denote the supremum norm \(\|f\|_\rho := \sup_{\mathbb{T}_\rho} |f(z)|\) for functions on \(\mathbb{T}_\rho\).

Follow the notations in Section 3.1. We list the corresponding quasiperiodic versions here again for the reader’s convenience. The analytic quasiperiodic Jacobi operator on \(\ell^2(\mathbb{Z})\) is given by

\[
(H_{v,c}u)_n = c(\theta + n\alpha)u_{n+1} + \bar{c}(\theta + (n - 1)\alpha)u_{n-1} + v(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}.
\]

The transfer matrix is given by

\[
A(\theta, E, \alpha) = \frac{1}{c(\theta)} \begin{pmatrix} E - v(\theta) & -\bar{c}(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix}
\]

and

\[
A(n; \theta, E, \alpha) = \prod_{j=n}^{1} A(\theta + (j - 1)\alpha, E, \alpha), \quad n > 0.
\]

Let \(L(E)\) be the Lyapunov exponent defined in (3.12), where we omit the dependence on \(v, c\) and \(\alpha\). Let \(S\) be the compact set where \(L(E)|_S \geq a > 0\) as in Theorem 2.4. The analytic function \(c(\theta)\) only has finitely many zeros \(\theta_1, \theta_2, \ldots, \theta_m\) on the torus. Fix \(\alpha \in \mathbb{T}\setminus \mathbb{Q}\); let

\[
\Theta := \{\theta \in \mathbb{T} : \theta \neq \theta_j + k\alpha, \quad \text{for all } j = 1, \ldots, m, \ k \in \mathbb{Z}\}.
\]

Clearly, its complement \(\hat{\Theta}\) are the orbits of the zeros of \(c(\theta)\) under the shift \(a\mathbb{Z}\), which only contains countably many points. The above transfer matrix is well defined on \(\hat{\Theta}\) which has full measure in \(\mathbb{T}\). Let \(\Theta\) be the full measure subset
defined in (4.14) and Corollary 4.3, where the spectral continuity holds true. It is also easy to check that \( \Theta \subset \tilde{\Theta} \).

The spectral singularity in Theorem 2.4 is reduced to the following lemma about the norm of the transfer matrices, which was proved in [32]:

**Lemma 6.1** ([32], Lemma 3.1). Let \( S \) be the compact set where \( L(E) \mid _{S} \geq a > 0 \) for some \( a > 0 \). Fix \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \beta = \beta(\alpha) < \infty \) and \( \theta \in \tilde{\Theta} \). Suppose there is a constant \( c > 0 \) such that for any \( E \in S \), there is \( \ell_{0} = \ell(E, \beta, \rho, \theta) \) such that for any \( \ell > \ell_{0} \), the following two estimates hold:

\[
\sum_{k=1}^{\ell} \| A(k; \theta, E, \alpha) \|^{2} \geq \ell^{1+\frac{2}{\beta}} \tag{6.3}
\]

and

\[
\sum_{k=1}^{\ell} \| A(k; \theta - \alpha, E, -\alpha) \|^{2} \geq \ell^{1+\frac{2}{\beta}}. \tag{6.4}
\]

Then we have the following upper bound for the spectral dimension defined in (2.3) of the spectral measure \( \mu = \mu_{\alpha, \theta} \):

\[
\text{dim}_{\text{spe}}(\mu) \leq \gamma_{0} := \frac{1}{1 + c/\beta} < 1. \tag{6.5}
\]

This is a direct consequence of the subordinacy theory (3.23) and Last–Simon upper bound on the generalized eigenfunction (3.24). Actually, in view of Lemma 3.5, it is enough to find a \( \varphi \) such that both \( m_\varphi \) and \( \tilde{m}_{\pi/2-\varphi} \) are \( \gamma \)-spectral singular, where \( m_\varphi \) and \( \tilde{m}_{\pi/2-\varphi} \) are half-line \( m \)-functions defined in Section 3.2. The estimate on the half-line \( m \)-functions relies on the subordinacy theory Lemma 3.2. The quantitative estimates need both an upper bound and a lower bound on the \( \ell \)-norm of \( u^\varphi, v^\varphi \). Lemma 3.3 provides two eigenfunctions \( u^\varphi \) and \( u^{\varphi,-} \), both obeying the sub-linear growth as in (3.24). Estimates (6.3) and (6.4) provide the lower bound as required in the subordinacy theory for \( m_\varphi \) and \( \tilde{m}_{\pi/2-\varphi} \) respectively, which eventually leads to the spectral singularity. In the rest of this section, we will focus on the proof of (6.3) and (6.4). We refer readers to [32], Section 3 for more details about this lemma and spectral singularity.

For a \( GL(2, \mathbb{C}) \) matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

we denote by \( \| \cdot \|_{HS} \) the Hilbert–Smith norm of \( A \):

\[
\| A \|_{HS} = \sqrt{|a|^{2} + |b|^{2} + |c|^{2} + |d|^{2}}. \tag{6.6}
\]
In the rest of this section, we write \( \| \cdot \| = \| \cdot \|_{HS} \) for simplicity whenever it is clear. We also denote by \( \text{Leb}(\cdot) \) the usual Lebesgue measure on the torus.

The key to prove (6.3) and (6.4) is the following lemma:

**Lemma 6.2.** Let \( a \) and \( S \) be as in Lemma 6.1. There are \( c_2 = c_2(a, S, \rho) > 0 \), \( n_0 = n_0(a, \rho) > 0 \) and a positive integer \( d = d(S, \rho, \|v\|_\rho, \|c\|_\rho) \in \mathbb{N}^+ \) such that for \( E \in S \) and \( n > n_0 \), there exists an interval \( \Delta_n \subset \mathbb{T} \) satisfying the following properties:

\[
\text{Leb}(\Delta_n) \geq \frac{c_2}{4dn},
\]

and for any \( \theta \in \Delta_n \setminus \tilde{\Theta}^c \)

\[
\| A(n; \theta, E, \alpha) \|^2_{HS} > e^{nL(E)/8}.
\]

Lemma 6.2 will be the key ingredient to the proof of spectral singularity; we will return to its proof at the end of this section. We will derive (6.3) and (6.4) from Lemma 6.2 and finish the proof of Theorem 2.4 first.

Let \( q_n \) be given as in the continued fraction approximants to \( \alpha \); see (2.8). The following lemma about the ergodicity of an irrational rotation was proved in [25].

**Lemma 6.3** (Lemma 9, [25]). Let \( \Delta \subset [0, 1] \) be an arbitrary segment. If \( |\Delta| > \frac{1}{q_n} \), then for any \( \theta \) there exists a \( j \) in \( \{0, 1, \ldots, q_n + q_{n-1} - 1\} \) such that \( \theta + ja \in \Delta \).

Combining Lemma 6.2 with Lemma 6.3, we immediately have the following localization density result:

**Lemma 6.4.** Fix \( E \in S \), \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), and \( \theta \in \tilde{\Theta} \). There is \( n_1 = n_1(E, \rho, \alpha, \theta) \) such that for any \( q_n \geq n_1 \) and any \( m \in \mathbb{N} \), there is \( j_m = j_m(\theta) \in [2mq_n, (2m+2)q_n) \) such that

\[
\| A(j_m; \theta, E, \alpha) \| > e^{c_0 q_n L(E)},
\]

where \( c_0 = c_0(a, \rho) \) explicitly depends on \( c_2 \) and \( d \) given in Lemma 6.2.

**Proof.** We fix \( E, \alpha \) and write \( A(n; \theta) = A(n; \theta, E, \alpha) \) for simplicity. Let \( n_0 \) be given as in Lemma 6.2. Given \( q_n \), let

\[
k_n = \left\lfloor \frac{c_2 q_n}{4d} \right\rfloor - 1 \geq \frac{c_2}{5d} q_n \geq n_0,
\]

provided \( q_n \) is large, where \( c_2 \) and \( d \) are given as in Lemma 6.2. By Lemma 6.2, there is an interval \( \Delta_{k_n} \subset \mathbb{T} \) such that the following hold:

\[
\text{Leb}(\Delta_{k_n}) \geq \frac{c_2}{4dk_n} > \frac{1}{q_n}
\]
and
\[(6.12) \quad \|A(k_n; \theta)\|^2 > e^{k_n L(E)/8} > e^{\frac{c_0}{c_2} q_n L(E)}, \quad \forall \theta \in \Delta_{k_n} \Theta^c.\]

Fix \(\theta\) and \(m \in \mathbb{N}\), apply Lemma 6.3 to \(\Delta_{k_n}\) and \(\theta + 2m q_n\); we have that there exists a \(j\) in \([0, 1, \ldots, q_n + q_{n-1} - 1]\) such that \((\theta + 2m q_n \alpha) + j \alpha \in \Delta_{k_n}\). By (6.12), we have
\[(6.13) \quad \|A(k_n; \theta + 2m q_n \alpha + j \alpha)\| > e^{4c_0 q_n L(E)},\]
where \(c_0 = \frac{c_2}{20 a}\).

It is easy to check that
\[(6.14) \quad A(2m q_n + j + k_n; \theta) = A(k_n; \theta + 2m q_n \alpha + j \alpha) A(2m q_n + j; \theta).\]
By (6.13), we have that either
\[(6.15) \quad \|A^{-1}(2m q_n + j; \theta)\| \geq e^{2c_0 q_n L(E)}\]
or
\[(6.16) \quad \|A(2m q_n + j + k_n; \theta)\| \geq e^{2c_0 q_n L(E)}.\]

Direct computation shows that
\[(6.17) \quad \|A^{-1}(2m q_n + j; \theta)\| = \frac{1}{|\det A(2m q_n + j; \theta)|} \|A(2m q_n + j; \theta)\| \leq \frac{|c(\theta + (2m q_n + j) \alpha)|}{|c(\theta)|} \|A(2m q_n + j; \theta)\| \leq \frac{\|c\|_\infty}{|c(\theta)|} \|A(2m q_n + j; \theta)\|.\]
Suppose (6.15) holds. Then
\[(6.20) \quad \|A(2m q_n + j; \theta)\| \geq \frac{|c(\theta)|}{\|c\|_\infty} e^{2c_0 q_n L(E)} \geq e^{c_0 q_n L(E)}\]
provided
\[(6.21) \quad e^{c_0 q_n L(E)} \geq \frac{\|c\|_\infty}{|c(\theta)|}.\]

Let \(j_m\) be \(2m q_n + j\) or \(2m q_n + j + k_n\), for which \(j_m\) satisfies (6.9). Clearly, by the choice of \(j, k_n, j_m(\theta) \in [2m q_n, (2m + 2) q_n]\) for all \(m \in \mathbb{N}\) and
\[(6.22) \quad q_n \geq n_1 := \max \left\{ \frac{5d_{n_0}}{c_2}, \frac{\ln \left| \frac{\|c\|_\infty}{|c(\theta)|} \right|}{c_0 L(E)} \right\} \cdot\]
Note that if \(m = j = 0\) in (6.13), we pick \(j_0 = k_n \geq 1\). So \(j_0 \in [1, 2q_n]\). \(\square\)
With the above localization density lemma, we can complete the proof of Theorem 2.4 by checking (6.3) and (6.4) in Lemma 6.1 for a.e. \( \theta \in \mathbb{T} \).

**Proof of Theorem 2.4.** For any \( \ell \in \mathbb{N} \), there is \( q_n \) such that \( \ell \in [2q_n, 2q_{n+1}) \). Let \( \ell = 2Nq_n + r \), where \( 0 \leq r < 2q_n \), \( 1 \leq N < \frac{q_{n+1}}{q_n} \). Let \( n_1 \) be given as in (6.22). Note that \( \ell < 2q_{n+1} < 2e^{2\beta(\alpha)q_n} \) since \( 0 < \beta(\alpha) < \infty \). It is easy to check that \( q_n \geq n_1 \) provided

\[
\ell \geq 2e^{2n_1\beta(\alpha)}.
\]

Now apply Lemma 6.4 to \( q_n \) and \( 0 \leq m \leq N - 1 \). There are \( j_m \in [2m q_n, (2m + 2) q_n) \subset [0, \ell] \) such that \( \|A(j_m; \theta, E, \alpha)\| > e^{c_0 q_n L(E)} \) for \( m = 0, 1, \ldots, N - 1 \). Therefore, the following sum can be bounded from below over the subsequence at \( j_m \):

\[
\sum_{k=1}^{\ell} \|A(k; \theta, E, \alpha)\|^2 \geq \sum_{m=0}^{N-1} \|A(j_m; \theta, E, \alpha)\|^2 \geq N e^{2c_0 q_n L(E)}.
\]

Clearly, \( \ell = 2Nq_n + r < 4Nq_n \). By (6.24), we have

\[
\sum_{k=1}^{\ell} \|A_k(\theta)\|^2 \geq \frac{\ell}{4q_n} e^{2c_0 q_n L(E)} \geq \ell e^{c_0 q_n L(E)} \geq \ell e^{c_0 a q_n}
\]

provided \( e^{c_0 q_n L(E)} \geq 4q_n \). Then for sufficiently large \( \ell \in [2q_n, 2q_{n+1}) \) such that \( \frac{\ln q_{n+1}}{q_n} < 2\beta \), we have

\[
\sum_{k=1}^{\ell} \|A_k(\theta)\|^2 \geq \ell q_n^{c_0} \geq \ell \cdot \left( \frac{\ell}{2} \right)^{c_0} \geq \ell \cdot \ell^{\frac{c_0 a}{2}} = \ell^{1 + \frac{c_0 a}{2}}
\]

provided \( \ell \geq 4 \), where \( c = \frac{1}{8} c_0 a \). This proves (6.3).

For the same \( \theta \) and \( E \), repeat the above procedure for \( A(n; \theta - \alpha, E, -\alpha) \). We have a sequence of positive integers \( j_n = j_n(\theta - \alpha) \in [2mq_n, 2(m + 1)q_n) \) for any \( N \in \mathbb{N} \) and \( q_n \geq n_1(E, \rho, -\alpha, \theta - \alpha) \) such that

\[
\|A(j_n; \theta - \alpha, E - \alpha, E)\| > e^{c_0 q_n L(E)}.
\]

Note that \( c_0 = c_0(\alpha, \rho) \) does not depend on \( \theta - \alpha \) and is the same as in (6.9) and (6.25). The same reasoning proves (6.4).

---

7We also assume that \( \beta(\alpha) > 0 \) in the proof of Theorem 2.4. Otherwise we have \( q_{n+1} < e^{q_n} \) for arbitrary small \( \epsilon > 0 \). Lemma 6.1 and a limiting argument imply that the upper spectral dimension is 0 in (6.5), which is the case for \( \beta(\alpha) = 0 \).
Then by Lemma 6.1, we have for all $\theta \in \Theta$ and $\beta(\alpha) < \infty$,
\[
\dim_{\text{spec}}(\mu_{\alpha, \theta}) < \frac{1}{1 + c/\beta} < 1,
\]
which completes the proof of Theorem 2.4. $\Box$

In the rest of the section, we focus on the proof of Lemma 6.2. In [32], the authors proved the analytic $SL(2, \mathbb{R})$ version of this lemma. One advantage for the Schrödinger case is that the $HS$ norm $\|A(n; \theta)\|_{HS}^2$ is a real analytic function which can be approximated by trigonometric functions in some uniform sense. For the $GL(2, \mathbb{C})$ case, the $HS$ norm of the transfer matrices is a meromorphic function. We need a finer decomposition to deal with the poles.

Fix $E, \alpha$, for $n \in \mathbb{N}^+, \theta \in \tilde{\Theta}$; let
\[
F_n(\theta) = \|A(n; \theta, E, \alpha)\|_{HS}^2
\]
be defined as in (6.6). We have the following decomposition of $F_n(\theta)$:

**Lemma 6.5.** For any $E$ and $n \in \mathbb{N}^+$, there are positive functions $f_n(\theta)$ and $g_n(\theta)$ such that

\[
F_n(\theta) = \frac{f_n(\theta)}{g_n(\theta)},
\]
\[
\inf_n \frac{1}{n} \int \ln g_n(\theta) \, d\theta = 0, \quad \inf_n \frac{1}{n} \int \ln f_n(\theta) \, d\theta = 2L(E).
\]

For any $\varepsilon > 0$, there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that for any $n > n_1$ and any $\theta \in \mathbb{T}$,
\[
0 < g_n(\theta) < e^{\varepsilon n}.
\]

Furthermore, for $E$ in a compact set $S$, there are positive integers $n_2 = n_2(\rho)$ and $d = d(S, \rho, \|v\|_\rho, \|c\|_\rho)$ such that for any $n > n_2$, there are two functions $P_n(\theta), R_n(\theta)$ satisfying the following decomposition:

\[
f_n(\theta) = P_n(\theta) + R_n(\theta),
\]
\[
|R_n(\theta)| < 1,
\]
\[
P_n(\theta) = \sum_{|k| \leq d \cdot n} \hat{f}_n(k) e^{2\pi i k \theta},
\]
where $\hat{f}_n(k)$ is the $k$-th Fourier coefficient of $f_n(\theta)$.
Proof. Follow the notations in (3.4), let
\[ A(\theta, E) = \frac{1}{c(\theta)} D(\theta, E), \quad D(\theta, E) = \begin{pmatrix} E - v(\theta) & -\bar{c}(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix} \]
and
\[ A(n; \theta, E) = \frac{1}{c(n; \theta)} D(n; \theta, E), \]
where
\[ c(n; \theta) = \prod_{j=1}^{n} c(\theta + (j - 1)\alpha), \]
\[ D(n; \theta, E) = \prod_{j=n}^{1} D(\theta + (j - 1)\alpha) = \begin{pmatrix} D^1(\theta) & D^2(\theta) \\ D^3(\theta) & D^4(\theta) \end{pmatrix}. \]

Without loss of generality, we assume
\[ \int_{\mathbb{T}} \ln |c(\theta)| \, d\theta = 0. \]
Otherwise, the argument simply differs by a constant factor. See Remark 6.1 after the proof.

Let \( g_n(\theta) := |c(n; \theta)|^2 \) and \( f_n(\theta) := \|D(n; \theta, E)\|_{HS}^2 \). Clearly,
\[ F_n(\theta) = \|A(n; \theta, E, \alpha)\|_{HS}^2 = \frac{f_n(\theta)}{g_n(\theta)}, \]
\[ f_n(\theta) = \|D(n; \theta, E)\|_{HS}^2 = |D^1(\theta)|^2 + |D^2(\theta)|^2 + |D^3(\theta)|^2 + |D^4(\theta)|^2. \]

Birkhoff ergodic theory implies that for any irrational \( \alpha \),
\[ \lim_{n} \int_{\mathbb{T}} \frac{1}{n} \ln g_n(\theta) \, d\theta = \inf_{n} \frac{1}{n} \int_{\mathbb{T}} \ln g_n(\theta) \, d\theta \]
\[ = \lim_{n} \int_{\mathbb{T}} \frac{1}{n} \sum_{j=1}^{n} \ln |c(\theta + (j - 1)\alpha)|^2 \, d\theta \]
\[ = \int_{\mathbb{T}} \ln |c(\theta)|^2 \, d\theta = 0. \]

In view of (6.36) and the definition of Lyapunov exponent (3.12), we have
\[ \inf_{n} \int_{\mathbb{T}} \frac{1}{n} \ln f_n(\theta) \, d\theta = \inf_{n} \int_{\mathbb{T}} \frac{1}{n} \ln \left( g_n(\theta) \|A(n; \theta)\|_{HS}^2 \right) \, d\theta \]
\[ = \inf_{n} \int_{\mathbb{T}} \frac{1}{n} \ln g_n(\theta) \, d\theta + \inf_{n} \int_{\mathbb{T}} \frac{1}{n} \ln \|A(n; \theta)\|_{HS}^2 \, d\theta \]
\[ = 2L(E). \]
Note that $c(\theta)$ is continuous in $\theta$. By (3.14), for any $\varepsilon > 0$, there is $n_1 = n_1(\varepsilon)$ such that for any $n > n_1$ and any $\theta \in \mathbb{T}$, we have the following upper semi-continuity (uniform in $\theta$):

\[
\frac{1}{n} \ln g_n(\theta) \leq \int_{\mathbb{T}} \ln |c(\theta)|^2 \, d\theta + \varepsilon = \varepsilon.
\]

This gives $g_n(\theta) \leq e^{\varepsilon n}$ and finishes the proof of (6.28)--(6.30).

The further decomposition of $f_n(\theta)$ into $P_n$ and $R_n$ follows the strategy in [32].

Note that $v(\theta)$ and $c(\theta)$ are both analytic with bounded extension to the strip $\{ z : |\text{Im} z | < \rho \}$. In view of (6.35), all $D^j(\theta)$, $i = 1, 2, 3, 4$ have analytic extension to the strip $\{ z : |\text{Im} z | < \rho \}$. For compact $S$, there is $C_2 = C_2(S, \rho, \| v \|_\rho, \| c \|_\rho)$ such that

\[
\| D^i \|_\rho := \sup_{|\text{Im} z | < \rho} | D^i(z) | < \sup_{|\text{Im} z | < \rho} \| D_n(z) \|^2_{HS} < e^{C_2 n},
\]

(6.41) $E \in S$, $i = 1, 2, 3, 4$.

Consider the Fourier expansion of the one-periodic functions $D^j(\theta)$:

\[
D^j(\theta) = \sum_{k \in \mathbb{Z}} \hat{D}^j(k) e^{2\pi i k \theta}, \quad i = 1, 2, 3, 4.
\]

The Fourier coefficients of $D^j(\theta)$ have exponential decay as

\[
|\hat{D}^j(k)| < \| D^j \|_\rho \cdot e^{-2\pi \rho |k|} < e^{C_2 n} \cdot e^{-2\pi \rho |k|}, \quad \forall k \in \mathbb{Z}, \ i = 1, 2, 3, 4.
\]

(6.43)

Combine

\[
|D^j(\theta)|^2 = \left( \sum_{k \in \mathbb{Z}} \hat{D}^j(k) e^{2\pi i k \theta} \right) \left( \sum_{k \in \mathbb{Z}} \overline{\hat{D}^j(k)} e^{-2\pi i k \theta} \right)
\]

with (6.43). It is easy to check that the Fourier coefficients of $|D^j(\theta)|^2$ have exponential decay as

\[
|\hat{D}^j(\cdot)(k)| < e^{C_2 n} \cdot e^{-\pi \rho |k|}, \quad \forall k \in \mathbb{Z}, \ i = 1, 2, 3, 4.
\]

(6.45)

Let $f_n(\theta)$ be given as in (6.37). Consider the Fourier expansion of $f_n(\theta)$:

\[
f_n(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_n(k) e^{2\pi i k \theta}.
\]

(6.46)

By (6.37) and (6.45), clearly $\hat{f}_n(k)$ has the same exponential decay in $|k|$

\[
|\hat{f}_n(k)| < 4 e^{C_2 n} \cdot e^{-\pi \rho |k|}, \quad \forall k \in \mathbb{Z}.
\]

(6.47)
Pick
\begin{equation}
(6.48) \quad d = \left\lfloor \frac{C_2}{\pi \rho} \right\rfloor + 2.
\end{equation}
We split \( f_n(\theta) \) into two parts:
\begin{equation*}
 f_n(\theta) = P_n(\theta) + R_n(\theta), \quad P_n(\theta) = \sum_{|k| \leq d-n} \hat{f}_n(k)e^{2\pi ik\theta}, \quad R_n(\theta) = \sum_{|k| > d-n} \hat{f}_n(k)e^{2\pi ik\theta}.
\end{equation*}
For any \( \theta \in \mathbb{T} \),
\begin{equation*}
|R_n(\theta)| \leq \sum_{|k| > d-n} |\hat{f}_n(k)| \leq \sum_{|k| > d-n} 4e^{C_2n}e^{-\pi \rho |k|} \leq \frac{8}{1 - e^{-\pi \rho}} e^{C_2n}e^{-\pi \rho dn} \leq \frac{8}{1 - e^{-\pi \rho}} e^{-(\pi \rho d - C_2)n}.
\end{equation*}
By the choice of \( d \) in (6.48), we have \( \pi \rho d > C_2 + \pi \rho \). Then for any \( \theta \in \mathbb{T} \),
\begin{equation}
(6.49) \quad |R_n(\theta)| \leq \frac{8}{1 - e^{-\pi \rho}} e^{-\pi \rho n} < 1,
\end{equation}
provided \( n > n_2(\rho) := (\pi \rho)^{-1} \ln(\frac{8}{1 - e^{-\pi \rho}}) \). This finishes the proof of (6.31)–(6.33).

\textbf{Remark 6.1.} Suppose \( b = \int_\mathbb{T} \ln |c(\theta)|d\theta \neq 0 \). In (6), we set
\begin{equation}
A(\theta, E) = \frac{1}{\tilde{c}(\theta)} \tilde{D}(\theta, E), \quad \text{where} \quad \tilde{c}(\theta) = e^{-b}c(\theta), \quad \tilde{D}(\theta, E) = e^{-b}D(\theta, E).
\end{equation}
Clearly,
\begin{equation}
(6.50) \quad \int_\mathbb{T} \ln |\tilde{c}(\theta)|d\theta = 0, \quad \lim_n \int_\mathbb{T} \frac{1}{n} \ln \| \tilde{D}(n; \theta, E) \| d\theta = L(E).
\end{equation}
Let \( g_n(\theta) := |\tilde{c}(n; \theta)|^2 \) and \( f_n(\theta) := \| \tilde{D}(n; \theta, E) \|^2_{HS} \). The rest of the decomposition is exactly the same.

Combine Lemma 6.5 with the positive assumption on the Lyapunov exponent. We can now finish:

\textbf{Proof of Lemma 6.2.} Assume that the Lyapunov exponent \( L(E) \geq a > 0 \) for \( E \in S \). Pick \( \varepsilon = a/8 \). Let \( n_1 = n_1(\varepsilon) \) and \( n_2 = n_2(\rho) \) be given as in Lemma 6.5. Then for all \( n > \max\{n_1, n_2\} \), we have \( g_n(\theta), f_n(\theta), P_n(\theta) \) and \( R_n(\theta) \) as in Lemma 6.5, satisfying (6.28)–(6.33). Denote
\begin{align*}
\Theta_1 &= \{ \theta \in \mathbb{T} : F_n(\theta) > e^{nL(E)/8} \}, \\
\Theta_2 &= \{ \theta \in \mathbb{T} : P_n(\theta) > e^{nL(E)/3} \}, \\
\Theta_3 &= \{ \theta \in \mathbb{T} : f_n(\theta) > e^{nL(E)/2} \}.
\end{align*}
Let \( n_3 := 4a^{-1} \). Then for all \( n > n_3 \), we have \( e^{nL(E)} > e^{n\theta} > e^4 > 50 \). By using the fact that \( x^{1/2} - x^{1/3} > x^{1/3} - x^{1/4} > 1 \) for all \( x > 50 \), it is easy to check that for \( n > n_3 \),

\[
e^{nL(E)/2} - e^{nL(E)/3} > e^{nL(E)/3} - e^{nL(E)/4} > 1.
\]

Assume that \( f_n(\theta) > e^{nL(E)/2} \). By (6.31) and (6.52) we have, for \( n > n_3 \),

\[
P_n(\theta) > f_n(\theta) - |R_n(\theta)| > e^{nL(E)/2} - 1 > e^{nL(E)/3}.
\]

Then

\[
f_n(\theta) > P_n(\theta) - |R_n(\theta)| > e^{nL(E)/3} - 1 > e^{nL(E)/4}.
\]

In view of (6.28) and (6.30), we then have for \( n > \max\{n_1, n_3\} \)

\[
F_n(\theta) = \frac{f_n(\theta)}{g_n(\theta)} > \frac{e^{nL(E)/4}}{e^{n\epsilon}} > \frac{e^{nL(E)/4}}{e^{nL(E)/8}} = e^{nL(E)/8}.
\]

Therefore, for \( n > n_0 := \max\{n_1, n_2, n_3\} \),

\[
\Theta_n^3 \subseteq \Theta_n^2 \subseteq \Theta_n^1,
\]

except for countably many points in \( \Theta^c \).

Let \( C_2 \) be as in (6.41). By (6.29),

\[
2nL(E) \leq \int_{\mathbb{T}} \ln f_n(\theta) d\theta \leq \text{Leb}(\Theta_n^3) \ln \|f_n\|_\rho + (1 - \text{Leb}(\Theta_n^3)) \ln e^{nL(E)/2}
\]

\[
\leq \text{Leb}(\Theta_n^3) \cdot C_2 n + (1 - \text{Leb}(\Theta_n^3)) \cdot nL(E)/2,
\]

where on \( \Theta_n^3 \) we use the trivial upper bound \( f_n(\theta) < \|f_n\|_\rho < e^{C_2 n} \) in (6.41). This implies

\[
\text{Leb}(\Theta_n^3) \geq \frac{3L(E)}{2C_2 - L(E)}.
\]

Note that \( C_2 > L(E) \geq a > 0 \) for all \( E \in S \). We have that

\[
\text{Leb}(\Theta_n^3) \geq \frac{3a}{2C_2 - a} =: c_2(a, S, \rho) > 0.
\]

In view of (6.53) we have, for \( n > n_0 \),

\[
\text{Leb}(\Theta_n^2) \geq c_2(a, S, \rho) > 0.
\]

By (6.33), \( P_n(\theta) \) is a trigonometric polynomial of degree (at most) \( 2dn \), where \( d \) is given by (6.48) in Lemma 6.5. The set \( \Theta_n^2 \) consists of no more than \( 4dn \) intervals.
Therefore, there exists a segment, \( \Delta_n \subset \Theta_n^2 \subset \Theta_n^1 \cup \Theta_n^c \), with \( \text{Leb}(\Delta_n) > \frac{C}{4dn} \). For any \( n > n_0 \) and \( \theta \in \Delta_n \setminus \Theta_n^c \subset \Theta_n^1 \),

\[
\| A_n(\theta) \|_{HS}^2 = F_n(\theta) > e^{nL(E)/8}
\]

and

\[
\text{Leb}(\Delta_n) > \frac{C_2}{4dn},
\]
as claimed. \( \square \)

7 The extended Harper’s model: Proof of Corollary 2.5

Recall the extended Harper’s model (EHM) defined in (2.13) as

\[
(H_{\lambda, \alpha, \theta} u)_n = c_{\lambda}(\theta + n\alpha)u_{n+1} + \bar{c}_{\lambda}(\theta + (n - 1)\alpha)u_{n-1} + 2 \cos 2\pi(\theta + n\alpha)u_n,
\]

where

\[
c_{\lambda}(\theta) = \lambda_1 e^{-2\pi i(\theta + \frac{\pi}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta + \frac{\pi}{4})}.
\]

By some earlier work [28], we consider the following partitioning of the parameter space into the following three regions:

**Region I** \( 0 \leq \lambda_1 + \lambda_3 \leq 1, \ 0 < \lambda_2 \leq 1 \).

**Region II** \( \max\{\lambda_1 + \lambda_3, 1\} \leq \lambda_2, \ \lambda_1 + \lambda_3 > 0 \).

**Region III** \( \max\{1, \lambda_2\} \leq \lambda_1 + \lambda_3, \ \lambda_2 > 0 \).
Let $L(E, \lambda)$ be the Lyapunov exponent of the extended Harper’s model, defined as in (3.12). The main achievement of [28] is to prove the following explicit formula of $L(E, \lambda)$, valid for all $\lambda$ and all irrational $\alpha$:

**Theorem 7.1** ([28]). Fix an irrational frequency $\alpha$. Then $L(E, \lambda)$ restricted to the spectrum is zero within both region II and III. In region I it is given by the formula on the spectrum,

$$L(E, \lambda) = \begin{cases} 
\ln\left(\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_3}\right), & \text{if } \lambda_1 \geq \lambda_3, \lambda_2 \leq \lambda_3 + \lambda_1, \\
\ln\left(\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_3}\right), & \text{if } \lambda_3 \geq \lambda_1, \lambda_2 \leq \lambda_3 + \lambda_1, \\
\ln\left(\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}\right), & \text{if } \lambda_2 \geq \lambda_3 + \lambda_1.
\end{cases}$$

Denote by Region $I^\circ$, Region $II^\circ$, Region $III^\circ$ the interior of Region I, II, III respectively. A complete understanding of the spectral properties of the extended Harper’s model for a.e. $\theta$ has been established in [23, 22, 3, 20]. We collect the spectral decomposition results in these papers as the follow theorem for the reader’s convenience. Follow the notations in Corollary 2.5; denote the three parameter regions of $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ by:

$$\mathcal{R}_1 = \{ \lambda \in \mathbb{R}^3 : 0 < \lambda_1 + \lambda_3 < 1, 0 < \lambda_2 < 1 \}.$$  

$$\mathcal{R}_2 = \left\{ \lambda \in \mathbb{R}^3 : \begin{array}{l}
\lambda_2 > \max\{\lambda_1 + \lambda_3, 1\}, \lambda_1 + \lambda_3 \geq 0 \\
\lambda_1 + \lambda_3 > \max\{\lambda_2, 1\}, \lambda_1 \neq \lambda_3, \lambda_2 > 0,
\end{array} \right\}.$$  

$$\mathcal{R}_3 = \left\{ \lambda \in \mathbb{R}^3 : \begin{array}{l}
0 \leq \lambda_1 + \lambda_3 \leq 1, \lambda_2 = 1 \\
\lambda_1 + \lambda_3 \geq \max\{\lambda_2, 1\}, \lambda_1 = \lambda_3, \lambda_2 > 0
\end{array} \right\}.$$  

**Theorem 7.2** ([23, 22, 3, 20]). The following Lebesgue decomposition of the spectrum of $H_{\lambda,a,\theta}$ holds for a.e. $\theta$.

- For $\lambda \in \mathcal{R}$, if $\beta(a) < L(E, \lambda)$, then $H_{\lambda,a,\theta}$ has a pure point spectrum. If $\beta(a) > L(E, \lambda)$, then $H_{\lambda,a,\theta}$ has a purely singular continuous spectrum.

- For $\lambda \in \mathcal{R}$ and all irrational $a$, $H_{\lambda,a,\theta}$ has a purely absolutely continuous spectrum.

- For $\lambda \in \mathcal{R}$ and all irrational $a$, $H_{\lambda,a,\theta}$ has a purely singular continuous spectrum.

Now we are in place to analyze the spectral dimension of EHM in each region. Clearly, Region $I^\circ = \mathcal{R}_1$. In view of (7.3), it is easy to check that $L(E, \lambda) > 0$ on $\mathcal{R}_1$ for all $a$ and $E$. Therefore, by Theorem 2.3, we have part (1) of Corollary 2.5.
Next, consider Region $\mathcal{R}_2$. Theorem 7.2 shows that $H_{\lambda,\alpha,\theta}$ has a purely a.c. spectrum in region $\mathcal{R}_2$ for all $\alpha$ and a.e. $\theta$. In view of Definition 2.3, absolutely continuous measure has full spectral dimension.\(^8\) This gives part (2) of Corollary 2.5.

Part (3) (region $\mathcal{R}_3$) is the only place that requires extra work. By Theorem 7.1 and Theorem 7.2, in region $\mathcal{R}_3$, $L(E,\lambda) = 0$ on the spectrum and $H_{\lambda,\alpha,\theta}$ does not have an a.c. spectrum. Due to the lack of positivity of the Lyapunov exponent, we do not have the spectral singularity and the upper bound provided by Theorem 2.4, while the lower bound from Theorem 2.1 still holds. Moreover, in view of Lemma 3.1 and Corollary 4.3, we can obtain arbitrarily small exponential growth of the transfer matrix. This allows us to obtain the increased range of $\beta(\alpha)$ in the critical region in part (3).

Recall the notations of the transfer matrix in (3.6) and (3.7) for EHM: let

$$A^j(\theta, E, \alpha) = \frac{1}{c_\lambda(\theta)}D^j(\theta, E, \alpha), \quad D^j(\theta, E, \alpha) = \begin{pmatrix} E - v(\theta) & -\tilde{c}_\lambda(\theta - \alpha) \\ \tilde{c}_\lambda(\theta) & 0 \end{pmatrix}. \quad (7.4)$$

For $n > 0, m \in \mathbb{Z}$,

$$A^n(n, m; \theta) = \prod_{j=m+n-1}^m A^j(\theta + j\alpha), \quad (7.5)$$

$$D^n(n, m; \theta) = \prod_{j=m+n-1}^m D^j(\theta + j\alpha), \quad c^n(n, m; \theta) = \prod_{j=m}^{m+n-1} c_\lambda(\theta + j\alpha).$$

It is easy to check that

$$L(E, \lambda) = L(D^n) - \int_T \ln |c^n(\theta)| d\theta. \quad (7.6)$$

Note that

$$b_\lambda := \int_T \ln |c_\lambda(\theta)| d\theta \quad (7.7)$$

is not necessarily zero in region $\mathcal{R}_3$. Suppose not, consider the rescaling trick in Remark 6.1. Set

$$\tilde{c}_\lambda(\theta) = e^{-b_\lambda}c_\lambda(\theta), \quad \tilde{D}^j(\theta, E) = e^{-b_\lambda}D^j(\theta, E). \quad (7.8)$$

Now in Region $\mathcal{R}_3$, we have

$$\int_T \ln |\tilde{c}_\lambda(\theta)| = 0, \quad L(\tilde{D}^j) = L(D^j) - b_\lambda = L(E, \lambda) = 0. \quad (7.9)$$

---

\(^8\)Actually, it is well known that a.c. measure has full dimension for most commonly used fractal dimensions, e.g., Hausdorff/packing dimension etc. See more background knowledge about fractal dimensions in, e.g., [14].
Let $\tilde{D}^\lambda(n, m; \theta)$, $\tilde{c}_\lambda(n, m; \theta)$ be defined the same way as in (7.4). For irrational $\alpha$, let $\beta(\alpha)$ and $q_n$\footnote{We still denote the subsequence reaching the lim sup by $q_n$.} be defined as in (2.8). Now assume $\beta(\alpha) > 0$, let $\tilde{\beta} = \min\{\beta(\alpha)/3, 1\}$. It was proved in [28] that $L(E, \alpha)$ is continuous in $E$ for irrational $\alpha$. In view of Lemma 3.1, the lim sup is uniform in both $\theta$ and $E$. Therefore, for any $\delta > 0$, there is $n_0 = n_0(\delta, \tilde{\beta})$ such that for any $n > n_0$, $m \in \mathbb{Z}$, $\theta \in \mathbb{T}$ and $E \in \sigma(H_\lambda, \alpha, \theta)$,

$$(7.10) \quad \|\tilde{D}^\lambda(n, m; \theta)\| \leq e^{\delta \tilde{\beta} n},$$

$$(7.11) \quad |\tilde{c}_\lambda(n, m; \theta)| \leq e^{\delta \tilde{\beta} n}.$$  

Note that in the proof of Theorem 5.2, we only need to consider the above upper bound for $E$ restricted in the spectrum. By Corollary 4.3, for a.e. $\theta$ and $q_n$ large,

$$(7.12) \quad \min_{|m| \leq e^{\delta \beta q_n}} |\tilde{c}_\lambda(q_n, m; \theta)| > e^{-6\delta \tilde{\beta} q_n}.$$  

By (7.10), (7.11) and (7.12), exactly the same computation in Section 3.4 shows that for a.e. $\theta$, $0 < \delta < \frac{1}{\sqrt{7}}$ and $q_n$ large,

$$(7.13) \quad \max_{|m| \leq e^{\delta \beta q_n}} |\tilde{c}_\lambda(r, m; \theta)| \geq e^{-7\delta \tilde{\beta} q_n}, \quad 1 \leq r \leq q_n,$$

$$(7.14) \quad \min_{|m| \leq e^{\delta \beta q_n}} |\tilde{c}_\lambda(n, m; \theta)| \geq e^{-7\delta \tilde{\beta} q_n},$$

$$(7.15) \quad \sup_{E \in \sigma(H_\lambda, \alpha, \theta)} \|A^\lambda(r, m; \theta)\| < e^{8 \delta \tilde{\beta} q_n}, \quad 0 \leq r \leq q_n, \quad |m| \leq e^{\delta \beta q_n}.$$  

Therefore, we can replace every $\Lambda$ in the proof Theorem 5.2 by $10\delta^2 \tilde{\beta}$. Then for any $\beta(\alpha) > 0$ and $0 < \gamma < 1$, (5.4) holds true provided

$$(7.16) \quad \delta < \frac{1}{6000}(1 - \gamma).$$  

Therefore, by Lemma 5.1 and Theorem 5.2, for any $\beta(\alpha) > 0$, $\gamma < 1$ and a.e. $\theta$, $\mu_\lambda, \alpha, \theta$ is $\gamma$-spectral continuous. By (2.5), $\dim_{\text{spec}}(\mu_\lambda, \alpha, \theta) = 1$, which completes the proof of part (3) of Corollary 2.5. 

\section*{Appendix A}

\subsection*{A.1 Proof of (5.4) in Theorem 5.2.}

We have shown in the first part of Theorem 5.2 that if $\beta > 260(1 + \frac{1}{\delta})\Lambda$, then for $\mu$-a.e. $E$, there exists $K(E) \in \mathbb{N}$, for $k \geq K(E)$, and we have

$$(A.1) \quad |\text{Tr} A(q_k; E)| < 2 - 2e^{-60\Lambda q_k}.$$
Now by (5.25), we have
\begin{equation}
|\text{Tr} \tilde{A}(q_k; E)| < 2 - 2e^{-60\Lambda q_k} + 12e^{(-\beta+4\Lambda)q_k} < 2 - e^{-60\Lambda q_k},
\end{equation}
provided \(e^{(\beta-64\Lambda)q_k} > 12\). Fix \(E\) and \(q = q_k\) and write \(\tilde{A}(q_k; E) = \tilde{A}(q)\). Now apply Lemma 5.13 to these \(\tilde{A}(q)\) satisfying (A.2). Note that \(\tilde{A}(q) \in \text{SL}(2, \mathbb{R})\) and \(|\text{Tr} \tilde{A}(q)| < 2\); the eigenvalue \(\rho\) of \(\tilde{A}(q)\) is purely imaginary with modulus 1, i.e., \(\rho = e^{i\psi}\), for some \(\psi \in (-\pi, \pi)\). By (5.71) we have, for any \(j\),
\begin{equation}
\tilde{A}(q) = \frac{\sin j\psi}{\sin \psi} \cdot (\tilde{A}(q) - \frac{\text{Tr} \tilde{A}(q)}{2} \cdot I) + \frac{\cos j\psi}{2} \cdot I, \quad \psi \in (-\pi, \pi).
\end{equation}
Then \(|2 \cos \psi| = |\text{Tr} \tilde{A}(q)| < 2 - e^{-60\Lambda q}\) implies
\[|\sin \psi| > \sqrt{1 - \left(1 - \frac{1}{2}e^{-60\Lambda q}\right)^2} > e^{-40\Lambda q}.
\]
By (A.3) and (5.23),
\begin{equation}
\|\tilde{A}(q)\| \leq 2e^{40\Lambda q} \|\tilde{A}(q)\| + 1 \leq e^{43\Lambda q},
\end{equation}
provided \(q > q(\Lambda)\).

Now for any \(0 < \gamma < 1\), let \(\bar{\epsilon} = \frac{95}{1-\gamma} < e^{\delta \beta q}\) and
\begin{equation}
N = [e^{\bar{\epsilon} \lambda q}].
\end{equation}
Apply Lemma 5.8 to \(G = \tilde{A}(q), G_j = \tilde{A}(q, jq + 1), j = 0, \ldots, N\); by (5.29) and (A.4), for all \(j \leq N\) we have
\begin{equation}
\|\tilde{A}(jq) - \tilde{A}(q)\| < e^{(-\beta+93\lambda q+2\epsilon \lambda)q} < e^{-\lambda q} < 1,
\end{equation}
provided \(\beta > (94 + 2\epsilon)\lambda\). Therefore, by (A.4),
\[\|\tilde{A}(jq)\| \leq \|\tilde{A}(q)\| + 1 \leq 2e^{40\lambda q} \|\tilde{A}(q)\| + 2 \leq e^{43\lambda q}.
\]
By (5.15),
\[|r^{-1}(jq, 0)| \leq 1 + Ne^{(-\beta+2\lambda)q} \leq 1 + e^{(\beta+2\lambda)q} < 2
\]
provided \(\beta > 3\lambda + \bar{\epsilon} \lambda\). Then by (5.12) and (5.23), for all \(0 \leq j \leq N\) and \(1 \leq r \leq q\),
\begin{equation}
\|A(jq)\| \leq |r^{-1}(jq, 0)| \cdot \|T_{jq}\| \cdot \|\tilde{A}(jq)\| \cdot \|T_{0}^{-1}\| \leq 2e^{43\lambda q},
\end{equation}
\begin{equation}
\|A(jq + r)\| \leq \|A(r, jq + 1)\| \cdot \|A(jq)\| \leq e^{46\lambda q}.
\end{equation}
Therefore,

\begin{equation}
\sum_{n=1}^{Nq} \|A(n; E)\|^2 \leq \sum_{k=0}^{N-1} \sum_{r=1}^{q} \|A(kq + r; E)\|^2 \leq Nq e^{92\Lambda q} \leq e^{(\xi + 93)\Lambda q}, \tag{A.9}
\end{equation}

\begin{equation}
\frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A(n; E)\|^2 \leq e^{(-(1-\gamma)\xi + 94)\Lambda q} = e^{-\Lambda q} < 1. \tag{A.10}
\end{equation}

In conclusion, for any $0 < \gamma < 1$ and $\mu$ a.e. $E$, we have a sequence $q_k \to \infty$ and $\ell_k = [e^{95(1-\gamma)^{-1}\Lambda q}]q_k$ such that

\begin{equation}
\sum_{n=1}^{\ell_k} \|A(n; E)\|^2 \leq \ell_k^{2-\gamma} \tag{A.11}
\end{equation}

provided

\begin{equation}
\beta > (3\xi + \xi/\delta)\Lambda = \left(285 + \frac{95}{\delta}\right) \frac{\Lambda}{1-\gamma} > (94 + 2\xi + \xi/\delta)\Lambda. \tag{A.12}
\end{equation}

It was proved in [32] that (A.11) implies (5.2) directly from the relation (3.3) and (3.21). We omit the proof for this part here. See more details about this direct computation in the proof of Lemma 2.1 in [32].

\section{Proof of Claim 5.14.}

For any $0 < \varepsilon < e^{-29\Lambda q}$, let $\ell = \ell(\varphi, \varepsilon, E)$, $u^\varphi$, $v^\varphi$ be given as in (3.21). Write $\ell(\varepsilon) = [\ell] + \ell - [\ell]$, and $[\ell] = K(\varepsilon) \cdot q + r(\varepsilon)$, where $0 \leq r = [\ell] \text{mod} q < q$ and $0 \leq \ell - [\ell] < 1$. Let

\[ X = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} \quad \text{and} \quad \tilde{X} = T_0^{-1}X. \]

Clearly, $\|X\| = \|\tilde{X}\| = 1$.

We need to show first that if $K < N_q = [e^{29\Lambda q}]$, then for any $\varepsilon < e^{-29\Lambda q}$

\begin{equation}
K > e^{\Lambda q}. \tag{A.13}
\end{equation}

For any $n \leq [\ell] + 1$, write $n = kq + r$, where $0 \leq k \leq K$, $0 \leq r \leq q$. By (5.15), (5.75) and (5.76), we have

\[ \|A(kq)\| \leq |r^{-1}(kq, 0)| \cdot (\|\tilde{A}(q)\| + 1) \leq 2 (6C_1 k e^{2\Lambda q} + 1) < k e^{3\Lambda q}. \]

Then by (5.23),

\[ \|A(kq + r)X\| \leq \|A(r, kq + 1)\| \cdot \|A(kq)\| \cdot \|X\| \leq k e^{5\Lambda q}. \]
Direct computation shows that
\[
\|u^\theta\|_{\ell}^2 \leq \sum_{n=1}^{[\ell]+1} \|A(n) \cdot X\|^2 \leq \sum_{r=1}^{q} \|A(r) \cdot X\|^2 + \sum_{k=1}^{K} \sum_{r=1}^{q} \|A(kq + r) \cdot X\|^2 \\
\leq q \cdot e^{Aq} + \sum_{k=1}^{K} \sum_{r=1}^{q} k^2 e^{10q} \\
\leq q \cdot e^{Aq} + K^3 q e^{10q} \\
\leq K^3 e^{11q}.
\]

Since \(\phi\) is arbitrary, we have \(\|v^\theta\|_{\ell}^2 \leq K^3 e^{11q}\) in the same way. By the definition of \(\ell\) in (3.21), we have
\[
(A.14) \quad K^6 e^{22q} \geq \|u^\theta\|_{\ell(\epsilon)} \|v^\theta(\epsilon)\|_{\ell} = \frac{1}{2} + e^{28q}.
\]

Therefore, \(K > e^{Aq}\) as claimed in (A.13).

To bound \(\|u^\theta\|_{\ell}^2\) from below, we need to consider two cases of initial the value \(\phi\).

**Case I.** Assume \(\phi\) satisfies
\[
(A.15) \quad \left\| \left( \tilde{A}(q) - \frac{\text{Tr} \tilde{A}(q)}{2} \cdot I \right) \cdot \tilde{X} \right\| \geq e^{-\frac{1}{2}Aq}.
\]

By (5.71), for any \(e^{\frac{1}{4}Aq} \leq k \leq K \leq N_q\), we have
\[
\|\tilde{A}^k(q) \cdot \tilde{X}\| \geq \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left( \tilde{A}(q) - \frac{\text{Tr} \tilde{A}(q)}{2} \cdot I \right) \tilde{X} \right\| - \frac{\rho^k + \rho^{-k}}{2} \cdot \|\tilde{X}\| \\
\geq \frac{1}{3} k \cdot e^{-\frac{1}{4}Aq} - C_1 \geq 3,
\]

provided \(e^{\frac{1}{4}Aq} > 3(C_1 + 3)\).

By (5.76), we then have
\[
\|\tilde{A}(kq) \cdot \tilde{X}\| \geq \|\tilde{A}^k(q) \cdot \tilde{X}\| - \|\tilde{A}(kq) - \tilde{A}^k(q)\) \cdot \tilde{X}\| \geq 2.
\]

By (5.12) and (5.15), for \(e^{\frac{1}{4}Aq} \leq k \leq K\), we have
\[
(A.16) \quad \|A(kq)X\| = |r^{-1}(kq, 0)| \cdot \|T_{kq} \tilde{A}(kq) T_{kq}^{-1} X\| \\
= |r^{-1}(kq, 0)| \cdot \|\tilde{A}(kq) \tilde{X}\| \geq 1.
\]

Therefore,
\[
\|u^\theta\|_{\ell}^2 \geq \frac{1}{2} \sum_{n=1}^{[\ell]-1} \|A_n \cdot X\|^2 \geq \frac{1}{2} \sum_{e^{\frac{1}{4}Aq} \leq k \leq K} \|A(kq) \cdot X\|^2 \\
\geq \frac{1}{2} (K - e^{\frac{1}{4}Aq}) > e^{\frac{1}{2}Aq}.
\]
Case II. Assume $\varphi$ satisfies
\begin{equation}
(A.17) \quad \| (\tilde{A}(q) - \frac{\text{Tr}\tilde{A}(q)}{2} I) \cdot \tilde{X} \| < e^{-\frac{1}{4}\Lambda q}.
\end{equation}

By (5.71), for any $1 \leq k \leq e^{\frac{1}{8}\Lambda q} < N_q$, we get
\begin{align*}
\| \tilde{A}^k(q) \cdot \tilde{X} \| &\geq \frac{\rho^k + \rho^{-k}}{2} \cdot \| \tilde{X} \| - \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \| (\tilde{A}(q) - \frac{\text{Tr}\tilde{A}(q)}{2} I) \tilde{X} \|
\geq \frac{1}{2} - C_1 k \cdot e^{-\frac{1}{4}\Lambda q} \geq \frac{1}{3},
\end{align*}
provided $e^{\frac{1}{8}\Lambda q} > 6C_1$.

By (5.76), we have
\begin{align*}
\| \tilde{A}(kq) \cdot \tilde{X} \| &\geq \| \tilde{A}^k(q) \cdot \tilde{X} \| - \| (\tilde{A}(kq) - \tilde{A}^k(q)) \cdot \tilde{X} \| \geq \frac{1}{4}.
\end{align*}

By (5.12) and (5.15), for any $1 \leq k \leq e^{\frac{1}{8}\Lambda q} < N_q$, we have
\begin{align*}
(A.18) \quad \| A(kq)X \| &= |r^{-1}(kq, 0)| \cdot \| T_{kq}\tilde{A}(kq)T_0^{-1}X \|
\geq |r^{-1}(kq, 0)| \cdot \| \tilde{A}(kq)\tilde{X} \| \geq \frac{1}{5},
\end{align*}
Therefore,
\begin{align*}
\| u^\varphi \|_\ell^2 &\geq \frac{1}{2} \sum_{n=1}^{[\ell]-1} \| A_n \cdot X \|^2 \geq \frac{1}{2} \sum_{1 \leq k \leq e^{\frac{1}{8}\Lambda q}} \| A(kq) \cdot X \|^2 \\
&\geq \frac{1}{50} e^{\frac{1}{8}\Lambda q} \geq e^{\frac{1}{4}\Lambda q}. \quad \square
\end{align*}

A.3 The refined estimate on the preimage of $\mathcal{P}_n(\mathbb{R})$. Let $\mathcal{P}_n(\mathbb{R})$ denote the polynomials over $\mathbb{R}$ of exact degree $n$. Let the class $\mathcal{P}_{n,\varnothing}(\mathbb{R})$ be elements in $\mathcal{P}_n(\mathbb{R})$ with $n$ distinct real zeros. The following proposition was proved in Theorem 6.1 in [29]:

**Proposition A.1.** Let $p \in \mathcal{P}_{n,\varnothing}(\mathbb{R})$ with $y_1 < \cdots < y_{n-1}$ the local extrema of $p$. Let
\begin{equation}
(A.19) \quad \zeta(p) := \min_{1 \leq j \leq n-1} |p(y_j)|
\end{equation}
and $0 \leq a < b$. Then,
\begin{equation}
(A.20) \quad |p^{-1}(a, b)| \leq 2\text{diam}(z(p - a)) \max \left\{ \frac{b - a}{\zeta(p) + a}, \left( \frac{b - a}{\zeta(p) + a} \right)^2 \right\},
\end{equation}
where $z(p)$ is the zero set of $p$ and $| \cdot |$ denotes the Lebesgue measure.
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