GENERIC VANISHING INDEX AND THE BIRATIONALITY OF THE BICANONICAL MAP OF IRREGULAR VARIETIES

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ABSTRACT. We prove that any smooth complex projective variety with generic vanishing index bigger or equal than 2 has birational bicanonical map. Therefore, if $X$ is a smooth complex projective variety with maximal Albanese dimension and non-birational bicanonical map, then the Albanese image of $X$ is fibred by subvarieties of codimension at most 1 of an abelian subvariety of Alb$X$.

1. INTRODUCTION

In the study of smooth complex algebraic varieties, the natural maps provided by the holomorphic forms defined in the variety, have a special importance. For example, the invertible sheaf $\omega_X$ of differential $n$-forms (where $n$ is the dimension of $X$) produces a map to a projective space, known as the canonical map. The multiples of this canonical sheaf $\omega_X^{m}$ produce in this way the pluricanonical maps

$$\varphi_m : X \dashrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \omega_X^{\otimes m})^\vee).$$

When $\varphi_m$ gives a birational equivalence between $X$ and its image, we will simply say that $\varphi_m$ is birational. We say that $X$ is of general type if for some $m > 0$ the rational map $\varphi_m$ is birational.

For example, the curves of general type are those of genus $g \geq 2$. The tricanonical map $\varphi_3$ is always birational for such curves and the bicanonical $\varphi_2$ is also birational once that $g \geq 3$. Moreover, the canonical map is birational as soon as the curve is non-hyperelliptic.

For surfaces, Bombieri [Bo] has given sharp numerical conditions for the birationality of $\varphi_m$ for $m \geq 3$. The bicanonical map has revealed to be more complicated and has been studied by many algebraic geometers. In fact, the surfaces with irregularity $q(S) \leq 1$ and $\chi(S, \omega_S) = 1$ are not completely understood and there is no classification about which ones have birational $\varphi_2$. For a modern review of the state of the art in the surface case, we refer to [BCP, Theorem 8].

For higher dimensions not many results are known in general. Nevertheless, the example of the bicanonical map on surfaces shows that for small irregularity $q(X) = h^0(X, \Omega_X)$, the classification becomes more difficult. For complex varieties, recall that the differential 1-forms

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give rise to the Albanese map

\[ \text{alb} : X \rightarrow \text{Alb} X = H^0(X, \Omega^1_X)^{\vee}/H_1(X, \mathbb{Z}). \]

from \( X \) to an abelian variety of dimension \( q(X) = h^0(X, \Omega^1_X) \). We say that \( X \) is \textit{irregular} if, and only if, \( \text{Alb} X \) is not trivial, i.e. \( q(X) > 0 \). And we say that \( X \) is of \textit{maximal Albanese dimension} (m.A.d) if, and only if, the Albanese map \( \text{alb} : X \rightarrow \text{Alb} X \) is generically finite onto its image.

It turns out that some properties of m.A.d varieties seem to behave independently of the dimension and, indeed, Chen-Hacon showed that this is the case for their pluricanonical maps.

**Theorem** (Chen-Hacon. [CH2]).

(a) \( X \) m.A.d and \( \chi(\omega_X) > 0 \) \( \Rightarrow \) \( X \) is of general type, furthermore, \( \varphi_3 \) is birational.

(b) \( X \) m.A.d \( \Rightarrow \) \( \varphi_6 \) is the stable pluricanonical map.

For \( \varphi_2 \), we cannot expect to use \( \chi(\omega_X) \) to control directly its birationality. For example, if \( C \) is a curve of genus 2, then the bicanonical map of the product \( C \times Y \) is never birational. In fact, it is clear that any variety that admits a fibration whose general fibre has non-birational \( \varphi_2 \) will have a non-birational bicanonical map. This should be considered, at least at first glance, as the standard case for higher dimensional varieties.

The following theorem provides geometric constraints for the non-birationality of the bicanonical map (see Theorem 5.2).

**Theorem A.** Let \( X \) be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then, the Albanese image of \( X \) is fibred by subvarieties of codimension at most 1 of an abelian subvariety of \( \text{Alb} X \). The base of the fibration is also of maximal Albanese dimension.

That is, \( X \) admits a fibration onto a normal projective variety \( Y \) with \( 0 \leq \dim Y < \dim X \), such that any smooth model \( \tilde{Y} \) of \( Y \) is of maximal Albanese dimension and

\[ q(X) - \dim X \leq q(\tilde{Y}) - \dim Y + 1. \]

Hence, if \( q(X) > \dim X + 1 \), the inequality implies the existence of an actual fibration, i.e. \( \dim Y > 0 \), whose general fibre is mapped generically finite through the Albanese map of \( X \) either onto a fixed abelian subvariety of \( \text{Alb} X \), or onto a divisor of this fixed abelian subvariety. When \( \dim Y = 0 \) the theorem simply says that the image of \( X \) in \( \text{Alb} X \) has codimension at most 1.

In particular, when \( X \) does not admit any fibration and \( q(X) > \dim X \), there is only one possible case, i.e. \( X \) is birationally equivalent to a theta-divisor of an indecomposable principally polarized abelian variety (see [BLNP, Theorem A]). When \( X \) does not admit any fibration and \( q(X) = \dim X \), there is only one known case of variety of general type and non-birational
bicanonical map: a double cover of a principally polarized abelian variety $(A, \Theta)$ branched along a reduced divisor $B \in |2\Theta|$. Is this the only case? The answer is affirmative in the case of surfaces due to Ciliberto-Mendes Lopes [CM, Theorem 1.1].

To deduce Theorem A it is useful to consider the generic vanishing index introduced by Pareschi–Popa in [PP3, Definition 3.1]

$$\text{gv}(\omega_X) = \min_{i > 0} \{ \text{codim}_{\text{Pic}^0 X} V^i(\omega_X) - i \},$$

where $V^i(\omega_X) = \{ \alpha \in \text{Pic}^0 X \mid h^i(X, \omega_X \otimes \alpha) > 0 \}$. As a consequence of Generic Vanishing Theorem of Green–Lazarsfeld [GL1, Theorem 1], we have that for any irregular variety $1 - \dim X \leq \text{gv}(\omega_X) \leq q(X) - \dim X$.

Moreover, the negative values of $\text{gv}(\omega_X)$ can be interpreted in terms of the dimension of the generic fibre of the Albanese map (see Theorem 3.7) and $X$ is a m.A.d variety if, and only if, $\text{gv}(\omega_X) \geq 0$. Due to the work of Pareschi–Popa [PP3] we can interpret the positive values of $\text{gv}(\omega_X)$ in terms of the local properties of the Fourier-Mukai transform of the structural sheaf (see Theorem 3.3). They have also proved that the positive values of $\text{gv}(\omega_X)$ give a lower bound for the Euler characteristic $\chi(\omega_X)$ (see Theorem 3.4).

Using the generic vanishing index we have the following more synthetic result.

**Theorem B.** Let $X$ be a smooth projective complex variety such that $\text{gv}(\omega_X) \geq 2$. Then, the rational map associated to $\omega_X^2 \otimes \alpha$ is birational onto its image for every $\alpha \in \text{Pic}^0 X$.

Theorem A is deduced from this result by an argument of Pareschi-Popa. On the other hand, this result (see Theorem 5.1) is proved using a birationality criterion (see Lemma 4.2) that is a slight modification of [BLNP, Theorem 4.13].

For curves, $\text{gv}(\omega_C) \geq 2$ is equivalent to $g(C) \geq 3$. For surfaces, $\text{gv}(\omega_S) \geq 2$ is equivalent to suppose that $q(S) \geq 4$ and does not admit an irregular fibration to a curve of genus $\leq q(S) - 3$ (see Example 5.3).

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2. **Generalized Fourier-Mukai transform**

$X$ will be a smooth projective variety over an algebraically closed field $k$ (from section 3.3 on, we will restrict to $k = \mathbb{C}$). It will be equipped with a morphism $a : X \to A$ to a non-trivial abelian variety $A$, in particular, $X$ will be irregular. Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \text{Pic}^0 A$. We will denote

$$P_\alpha = (a \times \text{id}_{\text{Pic}^0 X})^* \mathcal{P}, \quad (1)$$
the induced Poincaré line bundle in $X \times \Pic^0 A$. When $a = \text{alb}$, the Albanese map of $X$, then the map $\text{alb}^*$ identifies $\Pic^0 (\text{Alb} X)$ to $\Pic^0 X$ and the line bundle $P_{\text{alb}}$ will be simply denoted by $P$.

Letting $p$ and $q$ the two projections of $X \times \Pic^0 A$, we consider the left exact functor $\Phi_{P_a} F = q_*(p^* F \otimes P_a)$, and its right derived functors

\[ R^i \Phi_{P_a} F = R^i q_*(p^* F \otimes P_a). \]

Sometimes we will have to consider the analogous derived functor $R^i \Phi_{P_{-1}} F$ as well. By the Seesaw Theorem \cite[Corollary 6, p. 54]{Mu}, $P_{-1} = (1_A \times \Pic^0 A)^* P$, so

\[ R^i \Phi_{P_{-1}} F = (-1_{\Pic^0 A})^* R^i \Phi_{P_a} F \quad \text{for any } i. \]

Given a coherent sheaf $F$ on $X$, its $i$-th cohomological support locus with respect to $a$ is

\[ V^i_a(F) = \{ \alpha \in \Pic^0 A \mid h^i(F \otimes a^* \alpha) > 0 \} \]

Again, when $a$ is the Albanese map of $X$, we will omit the subscript, simply writing $V^i(F)$. By base change, these loci contain the set-theoretical support of $R^i \Phi_{P_a} F$, i.e. $\text{supp} R^i \Phi_{P_a} F \subseteq V^i_a(F)$.

A way to measure the size of all the $V^i_a(F)$’s is provided by the following invariant introduced by Pareschi–Popa.

**Definition 2.1** (\cite[Definition 3.1]{PP3}). Given a coherent sheaf $F$ on $X$, the generic vanishing index of $F$ (with respect to $a$) is

\[ \text{gv}_a(F) := \min_{i>0} \{ \text{codim}_{\Pic^0 A} V^i_a(F) - i \}. \]

By convention we define $\text{gv}_a(F) = \infty$, when $V^i_a(F) = \emptyset$ for every $i > 0$. When $a$ is the Albanese map of $X$, we will omit the subscript, simply writing $\text{gv}(F)$.

By base change (see \cite[Lemma 2.1]{PP3}) it is easy to see that $\text{gv}_a(F)$ can be also defined as the $\min_{i>0} \{ \text{codim}_{\Pic^0 A} \text{supp} R^i \Phi_{P_a} F - i \}$.

## 3. Generic vanishing index of the canonical sheaf

### 3.1. Relations between $\text{gv}(\omega_X)$ and the Fourier-Mukai transform of $O_X$. Here we specialize some general results of Pareschi–Popa \cite{PP3, PP4} to the canonical sheaf of a smooth projective variety of dimension $d$. Some of these results were previously obtained by Hacon (see \cite{Ha}).

The negative values of the $\text{gv}$-index are related with the vanishing of the lowest cohomologies of the Fourier-Mukai transform of its Grothendieck dual. In the case of $\omega_X$ this can be stressed simply as:

**Theorem 3.1** (\cite[Theorem 2.2]{PP3}). The following are equivalent,
(a) \( gv_a(\omega_X) \geq -e \) for \( e \geq 0 \);
(b) \( R^d\Phi_P a \mathcal{O}_X = 0 \) for all \( i \neq d - e, \ldots, d \).

Hence, when \( gv_a(\omega_X) \geq 0 \), \( R^i\Phi_P a \mathcal{O}_X = 0 \) for all \( i \neq d \), and we usually denote 
\[
\mathcal{O}_X = R^d\Phi_P a \mathcal{O}_X.
\]

Note that, in this case, \( H^i(X, \omega_X \otimes a^*\alpha) = 0 \) for all \( i > 0 \) and general \( \alpha \in \text{Pic}^0 A \). Therefore, by deformation-invariance of \( \chi \), the generic value of \( h^0(X, \omega_X \otimes a^*\alpha) \) equals \( \chi(\omega_X) \), in particular \( \chi(\omega_X) \geq 0 \). Since, by base-change, the fibre of \( \mathcal{O}_X \) at a general point \( \alpha \in \text{Pic}^0 A \) is isomorphic to \( H^d(X, a^*\alpha) \cong H^0(X, \omega_X \otimes a^*\alpha^{-1})^* \), the (generic) rank of \( \mathcal{O}_X \) is \( \text{rk} \mathcal{O}_X = \chi(\omega_X) \).

From Grothendieck-Verdier duality [Co, Theorem 4.3.1] and Theorem 3.1 it follows that,

**Corollary 3.2 ([PP4, Remark 3.13]).** If \( gv_a(\omega_X) \geq 0 \) then \( \text{Ext}_i^{\mathcal{O}_\text{Pic}^0 A}((\mathcal{O}^\mathcal{O}_\mathcal{O}^\text{Pic}^0 A)^*, \mathcal{O}_{\text{Pic}^0 A}) \cong R^d\Phi_P a \omega_X \).

The following result of Pareschi–Popa gives a dictionary between the positive values of \( gv_a(\omega_X) \) and the local properties of the Fourier-Mukai transform of \( \mathcal{O}_X \).

**Theorem 3.3 ([PP3, Corollary 3.2]).** Assume that \( gv_a(\omega_X) \geq 0 \). Then,

(4) \( gv_a(\omega_X) \geq m \) if, and only if, \( \mathcal{O}_X \) is a \( m \)-syzygy sheaf.

In particular, \( gv_a(\omega_X) \geq 1 \) is equivalent to \( \mathcal{O}_X \) being torsion-free and \( gv_a(\omega_X) \geq 2 \) to \( \mathcal{O}_X \) being reflexive.

Using the Evans–Griffith Syzygy Theorem and the previous theorem, Pareschi–Popa obtain the following bound on the Euler holomorphic characteristic that generalizes to higher dimensions the Castelnuovo-de Franchis inequality.

**Theorem 3.4 ([PP3, Theorem 3.3]).** Assume that \( gv_a(\omega_X) \geq 0 \). Then, \( \chi(\omega_X) \geq gv_a(\omega_X) \).

**Remark 3.5.** In fact, the theorem of Pareschi–Popa is more general, namely that for any coherent sheaf \( \mathcal{F} \) if \( \infty > gv_a(\mathcal{F}) \geq 0 \), then \( \chi(\mathcal{F}) \geq gv_a(\mathcal{F}) \). As a consequence, we easily obtain that for any non-zero coherent sheaf \( \mathcal{F} \), \( gv_a(\mathcal{F}) \geq 1 \Rightarrow \chi(\mathcal{F}) \geq 1 \). Observe also that if \( a \) is non-trivial, we always have \( gv_a(\omega_X) < \infty \).

### 3.2. Top Fourier-Mukai transform of the canonical sheaf

In the case of abelian varieties (or complex torus) the following result is well-known and crucial in the proof of the Mukai Equivalence Theorem [M, Theorem 2.2]. We will need it in the proof of Theorem 5.1.

**Proposition 3.6 ([BLNP, Proposition 6.1]).** If \( a^* : \text{Pic}^0 A \to \text{Pic}^0 X \) is an embedding, then 
\[
R^d\Phi_P a \omega_X \cong k(\hat{0}).
\]
3.3. Generic vanishing theorem of Green–Lazarsfeld. The name of the \( \text{gv} \)-index comes from the well-known Generic Vanishing Theorem of Green–Lazarsfeld. As other general vanishing theorems, it requires \( \text{char } k = 0 \) so from now on we will restrict ourselves to the case \( k = \mathbb{C} \). Basically, the following theorem is [GL1, Theorem 1]. The converse implication was proven independently in [LP, Theorem B] and [BLNP, Proposition 2.9].

**Theorem 3.7.** For any \( \epsilon > 0 \), the following are equivalent:

(a) the generic fibre of \( a : X \to A \) has dimension \( \epsilon \),
(b) \( \text{gv}_a(\omega_X) = -\epsilon \).

Moreover \( \text{gv}_a(\omega_X) \geq 0 \) if, and only if, \( a : X \to A \) is generically finite onto its image.

In particular, observe that for any irregular variety \( 1 - \dim X \leq \text{gv}(\omega_X) \leq q(X) - \dim X \).

**Remark 3.8.** If \( \text{gv}_a(\omega_X) \geq 0 \) and \( \chi(\omega_X) > 0 \), then \( X \) is a variety of general type. Indeed, by the previous result \( a : X \to A \) is generically finite and since \( \chi(\omega_X) > 0 \), we have that \( V^0_\alpha(\omega_X) = \text{Pic}^0 A \), so by [CH1, Corollary 2.4], \( \kappa(X) = \dim X \). In particular, if \( \text{gv}_a(\omega_X) \geq 1 \), then \( X \) is of general type.

3.4. Subtorus theorem of Green–Lazarsfeld and Simpson. The following theorem is due to Green and Lazarsfeld [GL2, Theorem 0.1] with an important addition due to Simpson [S, Sections 4,6, and 7].

**Theorem 3.9.** Let \( W \) an irreducible component of \( V^i(\omega_X) \) for some \( i \). Then,

(a) There exists a torsion point \( \beta \in \text{Pic}^0 X \) and a subtorus \( B \) of \( \text{Pic}^0 X \) such that \( W = \beta + B \).
(b) There exists a normal variety \( Y \) of dimension \( \leq d - i \), such that any smooth model of \( Y \) has maximal Albanese dimension and a morphism with connected fibres \( f : X \to Y \) such that \( B \) is contained in \( f^* \text{Pic}^0 Y \).

**Remark 3.10.** It is useful to recall that the morphism \( f : X \to Y \) in the second part of the previous theorem, arises as the Stein factorization of the morphism \( \pi : \text{Alb} X \to \text{Pic}^0 W \), where \( \pi : \text{Alb} X \to \text{Pic}^0 W \) is the dual map of the inclusion \( W \subseteq \text{Pic}^0 X \). Hence, the key point of the second part of the theorem is the dimensional bound for \( Y \).

4. Birationality criterion for maximal Albanese dimension varieties

In this section, we will assume that \( a : X \to A \) is a generically finite morphism onto its image, where \( A \) is an abelian variety. We introduce another piece of notation.

**Notation 4.1.** Let \( F \) be a subsheaf of a line bundle and suppose that \( \text{gv}_a(F) \geq 1 \).

(a) We denote \( U_F \), the open subset where \( h^0(F \otimes a^*\alpha) \) has the minimal value, i.e. \( \chi(F) \).
(b) Let \( Z \) be the exceptional locus of \( a : X \to A \), that is \( Z = a^{-1}(T) \), where \( T \) is the locus of points in \( A \) over which the fibre of \( a \) has positive dimension.
(c) We define
\[ B_a^\mathcal{F}(x) = \{ \alpha \in U_F \mid x \text{ is a base point of } |\mathcal{F} \otimes a^*\alpha| \}. \]

By Remark 3.5, \( \chi(\mathcal{F}) \geq 1 \). So, by semicontinuity, it makes sense to speak of the base locus of \( \mathcal{F} \otimes a^*\alpha \) for all \( \alpha \in \text{Pic}^0 A \).

The following lemma is a slight modification of [BLNP, Theorem 4.13] and it is based on [PP1, Proposition 2.12 and 2.13].

**Lemma 4.2.** Suppose that \( a : X \to A \) is a generically finite morphism onto its image and let \( \mathcal{F} \) be a subsheaf of a line bundle such that \( \text{gv}_a(\mathcal{F}) \geq 1 \) and \( R^i a_* \mathcal{F} = 0 \) for all \( i > 0 \). Suppose that for a general \( x \in X \),
\[ \text{codim}_{U_F} B_a^\mathcal{F}(x) \geq 2. \]

Then, the rational map associated to the linear system \( |\mathcal{F} \otimes L| \) is birational for every line bundle \( L \) such that \( \text{gv}_a(L) \geq 1 \).

**Proof.** We first compare the Fourier-Mukai transform of \( \mathcal{F} \otimes I_x \) and \( \mathcal{F} \).

**Claim.** Let \( x \in X \) be a closed point out of \( Z \). Then \( R^i a_*(\mathcal{F} \otimes I_x \otimes a^*\alpha) = 0 \) for \( i > 0 \). This follows immediately from the exact sequence
\[ 0 \to \mathcal{F} \otimes I_x \to \mathcal{F} \to k(x) \to 0 \]
and the hypotheses that \( R^i a_* \mathcal{F} = 0 \), \( a \) is generically finite, and \( x \notin Z \). Hence, the degeneration of the Leray spectral sequence yields to
\[ V^i_a(\mathcal{F} \otimes I_x) = V^i(a_*(\mathcal{F} \otimes I_x)). \]

By sequence (5), tensored by \( a^*\alpha \), it follows that
\[ V^i_a(\mathcal{F} \otimes I_x) = V^i(\mathcal{F}) \quad \text{for all } i \geq 2. \]

For \( i = 1 \) we have the surjection \( H^1(\mathcal{F} \otimes I_x \otimes a^*\alpha) \to H^1(\mathcal{F} \otimes a^*\alpha) \), that is an isomorphism if, and only if, \( x \) is not a base point of \( |\mathcal{F} \otimes a^*\alpha| \). In other words \( V^1_a(\mathcal{F} \otimes I_x) \subseteq B_a^\mathcal{F}(x) \cup V^1_a(\mathcal{F}) \).

Since \( \text{gv}_a(\mathcal{F}) \geq 1 \), the hypothesis on \( B_a^\mathcal{F}(x) \) guarantees that
\[ \text{codim} V^1_a(\mathcal{F} \otimes I_x) \geq 2, \]
for a general \( x \in X \setminus Z \). Hence by (6), (7) and (8), \( \text{gv}(a_*(\mathcal{F} \otimes I_x)) \geq 1 \). By [PP1, Proposition 2.13], \( a_*(\mathcal{F} \otimes I_x) \) is continuously globally generated (CGG, see [PP1]). Therefore \( \mathcal{F} \otimes I_x \) itself is CGG outside \( Z \) (with respect to \( a \)). Since the same is true for \( L \), it follows from [PP1, Proposition 2.12] that for all \( \alpha \in \text{Pic}^0 A \), \( \mathcal{F} \otimes L \otimes I_x \) is globally generated outside \( Z \). So the rational map associated to \( |\mathcal{F} \otimes L| \) is birational. \( \square \)

**Remark 4.3.** From the proof we see that if \( \text{codim}_{U_F} B_a^\mathcal{F}(x) \geq 2 \) for every \( x \in X \setminus Z \), then \( \mathcal{F} \otimes L \) is very ample out of \( Z \), the exceptional locus of \( a \).
4.1. Adjoint line bundles. When $F = \omega_X$ we will call $U_F$ simply $U_0$ and $B^\omega_X(a)(x)$ simply by
\begin{equation}
B_a(x) = \{ \alpha \in U_0 \mid x \text{ is a base point of } \omega_X \otimes a^*\alpha \}.
\end{equation}
Throughout subsections §4.1 and §4.2, we will assume that $\text{gv}_a(\omega_X) \geq 1$.

**Proposition-Definition 4.4.** Let $X$ be a variety such that $\text{gv}_a(\omega_X) \geq 1$ and let $L$ be any line bundle on $X$ such that $\text{gv}_a(L) \geq 1$. Suppose that there exists $\alpha \in \text{Pic}^0 A$ such that $\omega_X \otimes L \otimes a^*\alpha$ is not birational. Then,
\begin{equation}
\text{codim}_{X \times U_0} \{(x, \alpha) \in X \times U_0 \mid x \text{ is a base point of } \omega_X \otimes a^*\alpha \} = 1,
\end{equation}
and its divisorial part is dominant on $X$ and surjects on $U_0$ via the projections $p$ and $q$. We endow this set with the natural subscheme structure given by the image of the relative evaluation map $q^*(q_*L) \otimes L^{-1} \to \mathcal{O}_{X \times U_0}$, where $L = (p^*\omega_X) \otimes P_a)|_{X \times U_0}$ and we call $\mathcal{Y}$ the union of its divisorial components that dominate $U_0$. Let $\overline{\mathcal{Y}}$ be its closure in $X \times \text{Pic}^0 X$. Then
\begin{itemize}
  \item[(a)] $X$ is covered by the scheme-theoretic fibres of the projection $\overline{\mathcal{Y}} \to U_0$, that we will call $F_\alpha$, for $\alpha$ varying in $U_0$. By definition, at a general point $\alpha \in U_0$, $F_\alpha$ is the fixed divisor of $\omega_X \otimes a^*\alpha$.
  \item[(b)] For a general $x \in X$, the fibre of the projection $\overline{\mathcal{Y}} \to X$ is a divisor, that we will call $D_x$. By definition, $D_x$ is the closure of the union of the divisorial components of the locus of $\alpha \in U_0$ such that $x \in \text{Bs}(\omega_X \otimes a^*\alpha)$.
\end{itemize}

**Proof.** Everything follows from taking $F = \omega_X$ in Lemma 4.2. The surjectivity of the projection to $U_0$ is consequence of the Castelnuovo-de Franchis inequality 3.4, i.e. $\chi(\omega_X) \geq \text{gv}_a(\omega_X) \geq 1$. \hfill $\Box$

4.2. Decomposition. In the sequel we will need $a^* : \text{Pic}^0 A \to \text{Pic}^0 X$ to be an embedding. However, for simplicity we will go one step further and we will simply suppose that $A = \text{Alb} X$. Suppose that we are under the hypotheses of the previous Proposition-Definition and consider a fixed point $\alpha_0 \in U_0$, and the map
\begin{equation}
f_{\alpha_0} : U_0 \to \text{Pic}^0 X \quad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0}),
\end{equation}
where $F_\alpha$ is the divisor defined in Proposition-Definition 4.4(a). For $\alpha \in U_0$, all the $F_\alpha$ are algebraically equivalent since they are the fibres of $\overline{\mathcal{Y}} \to U_0$, so the map is well-defined.

The following lemma shows that this map induces a decomposition of $\text{Pic}^0 X$ and that the divisors $F_\alpha$ move algebraically along a non-trivial factor of $\text{Pic}^0 X$. Although the proof is basically the same as [BLNP, Lemma 5.1], we do not require $V^1(\omega_X)$ to be a finite set, but only a proper subvariety.

**Lemma 4.5.** The map defined in (10), induces an homomorphism $f : \text{Pic}^0 X \to \text{Pic}^0 X$ such that,
Lemma 4.6 ([BLNP, Lemmas 5.1 & 5.3])

(a) \( f^2 = f \) and \( \text{Pic}^0 X \) decomposes as \( \text{Pic}^0 X \cong \ker f \times \ker (\text{id} - f) \). Moreover \( \dim \ker (\text{id} - f) > 0 \).

(b) Fix \( \beta \in \ker f \) such that \( U_0 \cap (\{\beta\} \times \ker (\text{id} - f)) \) is non-empty. Then, for \( \gamma \in U_0 \cap \ker (\text{id} - f) \) the line bundle \( \mathcal{O}_X(F_{\beta \otimes \gamma}) \otimes \gamma^{-1} \) does not depend on \( \gamma \). Since it is effective by semicontinuity, we call it \( \mathcal{O}_X(F) \).

(c) For all \( (\beta, \gamma) \in \ker f \times \ker (\text{id} - f) \) \( \cong \text{Pic}^0 X \) such that \( \beta \otimes \gamma \in U_0 \), \( |\mathcal{O}_X(F) \otimes \gamma| \) is contained in the fixed divisor of \( \omega_X \otimes \beta \otimes \gamma \).

Proof. Let \( \mathcal{O}_X(M_a) = \omega_X \otimes a^* \alpha \otimes \mathcal{O}_X(-F_a) \). Then, the proof of (a) is the same as [BLNP, Lemma 5.1](a). Item (b) follows directly from the definition of \( f \). To prove (c), let \( (\beta, \gamma) \in \ker f \times \ker (\text{id} - f) \) such that \( \beta \otimes \gamma \in U_0 \) and \( E \in \mathcal{O}_X(F) \otimes |\gamma| \). Then \( \mathcal{O}_X(F_{\beta \otimes \gamma} - E) \cong \mathcal{O}_X(F_{\beta \otimes \gamma} - F_{\beta \otimes \gamma}) = f(\beta \otimes \gamma^{-1}) = \mathcal{O}_X \). Since \( F_{\beta \otimes \gamma} \) is a fixed divisor of \( |\omega_X \otimes \beta \otimes \gamma| \), also \( E = F_{\beta \otimes \gamma} \) is a fixed divisor in \( \mathcal{O}_X \). \( \square \)

Using the decomposition given by the previous Lemma we give an explicit description of the “half” Poincaré line bundle.

Lemma 4.6 ([BLNP, Lemmas 5.1 & 5.3]). We call \( B = \text{Pic}^0(\ker f) \) and \( C = \text{Pic}^0(\ker (\text{id} - f)) \) so that

\[
\text{Alb } X \cong B \times C \quad \text{and} \quad \text{Pic}^0 X \cong \text{Pic}^0 B \times \text{Pic}^0 C,
\]

with \( \dim C > 0 \). Then we have the following description of the “half” Poincaré line bundle.

\[
(\text{alb} \times \text{id}_{\text{Pic}^0 X})^*(\mathcal{O}_{B \times \text{Pic}^0 B} \boxtimes \mathcal{P}_C) \cong \mathcal{O}_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* \mathcal{O}_X(-F) \otimes q^* \mathcal{O}_{\text{Pic}^0 X}(-D_\bar{x}),
\]

where \( \bar{x} \) is such that \( \text{alb}(\bar{x}) = 0 \) in \( \text{Alb } X \) and \( \mathcal{P}_C \) is the Poincaré line bundle in \( C \times \text{Pic}^0 C \).

Proof. The decomposition of \( \text{Pic}^0 X \) comes directly from Lemma 4.5(a). By the definition of \( \overline{\mathcal{Y}} \) (see Proposition-Definition 4.4) and the definition of \( F \) (see Lemma 4.5(b)) we have that the line bundle

\[
\mathcal{O}_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* \mathcal{O}_X(-F) \otimes q^* \mathcal{O}_{\text{Pic}^0 X}(-D_\bar{x}),
\]

- restricted to \( X \times \{\beta \otimes \gamma\} \) is isomorphic to \( \mathcal{O}_X(F_{\beta \otimes \gamma} - F) = \gamma \), for all \( (\beta, \gamma) \in U_0 \subseteq \ker f \times \ker (\text{id} - f) \);
- restricted to \( \{\bar{x}\} \times \text{Pic}^0 X \) is isomorphic to \( \mathcal{O}_{\text{Pic}^0 X}(D_\bar{x}) \otimes \mathcal{O}_{\text{Pic}^0 X}(-D_\bar{x}) \), i.e. trivial.

On the other hand, \( (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*(\mathcal{O}_{B \times \text{Pic}^0 B} \boxtimes \mathcal{P}_C) \),

- restricted to \( X \times \{\beta \otimes \gamma\} \) is isomorphic to \( \gamma \), for all \( (\beta, \gamma) \in \ker f \times \ker (\text{id} - f) \);
- restricted to \( \{\bar{x}\} \times \text{Pic}^0 X \) is isomorphic to \( \mathcal{O}_{\text{Pic}^0 X} \), i.e. trivial.

Then, the Lemma follows from the seesaw principle. \( \square \)
5. The bicanonical map of irregular varieties

The next theorem gives a sufficient numerical condition for the birationality of the bicanonical map, analogous to Pareschi–Popa Theorem [PP2, Theorem 6.1] for the tricanonical map.

**Theorem 5.1.** Let \( X \) be a smooth projective complex variety such that \( \text{gv}(\omega_X) \geq 2 \). Then, the rational map associated to \( \omega_X^2 \otimes \alpha \) is birational onto its image for every \( \alpha \in \text{Pic}^0 X \).

As a first corollary we have the following geometric interpretation.

**Theorem 5.2.** Let \( X \) be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then \( 0 \leq \text{gv}(\omega_X) \leq 1 \). Moreover, it admits a fibration onto a normal projective variety \( Y \) with \( 0 \leq \dim Y < \dim X \), any smooth model \( \tilde{Y} \) of \( Y \) is of maximal Albanese dimension and

\[
q(X) - \dim X \leq q(\tilde{Y}) - \dim Y + \text{gv}(\omega_X).
\]

**Proof.** By Theorems 3.7 and 5.1, it is clear that \( 0 \leq \text{gv}(\omega_X) \leq 1 \). Now, the proof is the same as the proof of [PP3, Theorem B]. \( \square \)

**Example 5.3.** We would like to show examples of varieties with \( \text{gv}(\omega_X) \geq 2 \). For curves \( C \), this is equivalent to \( g(C) \geq 3 \). For surfaces \( S \), is equivalent to suppose that \( q(S) \geq 4 \) and \( S \) does not admit an irregular fibration to a curve of genus \( \leq q(S) - 3 \) (see [Be, Corollary 2.3]).

On the other hand, if \( A \) is a simple abelian variety, then every subvariety \( X \) of codimension \( \geq 2 \) has \( \text{gv}(\omega_X) \geq 2 \). Moreover, the property of having \( \text{gv}(\omega_X) \geq 2 \) is closed under taking products and cyclic coverings induced by a torsion point \( \alpha \in \text{Pic}^0 X - V^1(\omega_X) \).

The rest of the paper is devoted to the proof of Theorem 5.1.

**Proof.** Assume that \( \text{gv}(\omega_X) \geq 1 \) and there exists \( \alpha \in \text{Pic}^0 X \) such that \( \omega_X^2 \otimes \alpha \) is non-birational. Then, we want to see that \( \text{gv}(\omega_X) = 1 \). Under these hypotheses we can apply Proposition-Definition 4.4 and Lemma 4.6, so \( \text{Alb} X \cong B \times C \), where \( B = \text{Pic}^0 \ker(\text{id}-f) \) and \( C = \text{Pic}^0 \ker f \). We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Pic}^0 X & \xrightarrow{q} & X \times \text{Pic}^0 X \\
\downarrow{p_b} & & \downarrow{\text{id} \times p_b} \\
\text{Pic}^0 B & \xrightarrow{q} & X \times \text{Pic}^0 B \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Alb} X \times \text{Pic}^0 X & \xrightarrow{\text{alb} \times \text{id}} & \text{Alb} X \times \text{Pic}^0 X \\
\downarrow{p_b \times p_b} & & \downarrow{p_b \times p_b} \\
B \times \text{Pic}^0 B & \xrightarrow{b \times \text{id}} & B \times \text{Pic}^0 B
\end{array}
\]

where

- \( p_b : \text{Alb} X \to B \) and \( p_b : \text{Pic}^0 X \to \text{Pic}^0 B \) are the corresponding projections,
- \( b \) is the composition by \( b : X \text{alb} \to \text{Alb} X \overset{p_b}{\to} B \), and
- abusing notation we also call \( q \) either the projection \( X \times \text{Pic}^0 X \to \text{Pic}^0 X \) or \( X \times \text{Pic}^0 B \to \text{Pic}^0 B \) and \( p \) the projections \( X \times \text{Pic}^0 X \to X \) or \( X \times \text{Pic}^0 B \to X \).
The effectiveness of $\mathcal{Y}$ give us the following short exact sequence on $X \times \text{Pic}^0 X$

$$0 \to (\text{alb} \times \text{id})^*\left((\mathcal{O}_{B \times \text{Pic}^0 X} \boxtimes \mathcal{P}_C)^{-1}\right) \xrightarrow{\mathcal{Y}} p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(D_Z) \to (p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(D_Z))|_{\mathcal{Y}} \to 0.$$

Recall that $P = (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B \boxtimes \mathcal{P}_C)\right)$ since the Poincaré line bundle $\mathcal{P}$ in $\text{Alb} X \times \text{Pic}^0 X$ is isomorphic to $\mathcal{P}_B \boxtimes \mathcal{P}_C$. We apply the functor $R^d q_*\left(\cdot \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right)$, that is, we tensor by the other “half” Poincaré line bundle and we consider the top direct image. We get

$$\cdots \to R^d \Phi_{P^{-1}}(\mathcal{O}_X) \to R^d q_*\left(p^*\mathcal{O}_X(F) \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right) \otimes \mathcal{O}_{\text{Pic}^0 X}(D_Z) \to$$

$$R^d q_*\left(p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}_{\text{Pic}^0 X}(D_Z))|_{\mathcal{Y}} \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right) \to 0.$$

Using that $R^d \Phi_{P^{-1}} \cong (\text{Pic}^0 X)^n R^d \Phi_P$ (see (3)), we have the following short exact sequence,

$$(12) \quad 0 \to (\text{Pic}^0 X)\mathcal{O}_X \xrightarrow{\mu} \mathcal{E}(D_Z) \to \mathcal{T} \to 0$$

where:

(a) By base change, $\mathcal{E} = R^d q_*\left(p^*\mathcal{O}_X(F) \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right)$ is a coherent sheaf of rank $\text{rk}(\mathcal{O}_X(F) \otimes \beta^{-1})$ by a general $\beta \in \ker f$, i.e. $h^0(\mathcal{O}_X(-F) \otimes \beta) = \chi(\mathcal{O}_X)$ by Lemma 4.5 (c). Then,

$$\mathcal{E} = R^d q_*\left(p^*\mathcal{O}_X(F) \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right)$$

right square of (11)

$$= R^d q_*\left(\text{id} \times p_b)^*\left((\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right)\right)$$

abuse of notation on $p$

$$= p_b^* R^d q_*\left(\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right)$$

flat base change

$$= p_b^* R^d \Phi_{P^{-1}}(\mathcal{O}_X(F)),$$

following the notation of (1) and (2).

(b) $\mathcal{T} = R^d q_*\left(\left((\text{alb} \times \text{id}_{\text{Pic}^0 X})^*\left((\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \text{Pic}^0 C})\right)\right))$ is supported at the locus of the $\alpha \in \text{Pic}^0 X$ such that the fibre of the projection $q: \mathcal{Y} \to \text{Pic}^0 X$ has dimension $d$, i.e. it coincides with $X$. Such locus is contained in $V^1(\mathcal{O}_X)$, therefore, since $\text{gv}(\mathcal{O}_X) \geq 1$, codim $\mathcal{T} \geq 2$.

(c) The map $\mu$ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\text{rk}(\mathcal{O}_X) = \chi(\mathcal{O}_X)$), and, as $\text{gv}(\mathcal{O}_X) \geq 1$, the source $\mathcal{O}_X$ is torsion-free (Theorem 3.3).

(d) $\mu$ is $R^d q_* (m_s)$, where $m_s$ is the multiplication by the section defining $\mathcal{Y}$. By base change $[\text{Mu, Corollary 3, p. 53}], R^d q_* (m_s) \otimes \mathbb{C}(\alpha) = H^d(m_s)_{q^{-1}(\alpha)}$ where $q$ is the projection $q: \mathcal{Y} \to \text{Pic}^0 X$. When $q^{-1}(\alpha) = X$, $m_s|_{q^{-1}(\alpha)} = 0$, so in these points $R^d q_* (m_s) \otimes \mathbb{C}(\alpha) = 0.$

Claim 5.4. $\mathcal{T} \neq 0.$
Proof of the Claim. Suppose that $\mathcal{T} = 0$, so $\mu$ is an isomorphism. Taking $\mathcal{E}xt^d(\cdot, \mathcal{O}_{\text{Pic}^0 X})$ we get

\[
\begin{align*}
k(\hat{0}) &= R^d\Phi P\omega_X \\
&= \mathcal{E}xt^d(\mathcal{E}, \mathcal{O}_{\text{Pic}^0 X}) \otimes \mathcal{O}(-\mathcal{D}_x) \\
&= p_b^! \mathcal{E}xt^d(\mathcal{R}^d\Phi P_{\lambda}(\mathcal{O}_X(F)), \mathcal{O}_{\text{Pic}^0 B}) \otimes \mathcal{O}(-\mathcal{D}_x)
\end{align*}
\]

which implies that $\text{codim}_{\text{Alb} X} B = \dim \ker(id - f) = 0$ contradicting Lemma 4.6.

Let $\tau(\mathcal{E}(\mathcal{D}_x))$ be the torsion part of $\mathcal{E}(\mathcal{D}_x)$ and $\mathcal{E}(\mathcal{D}_x)$ the quotient of $\mathcal{E}(\mathcal{D}_x)$ by its torsion part. Hence $\mathcal{E}(\mathcal{D}_x)$ is torsion-free. Now consider the following composition

\[
\begin{array}{ccc}
(-1)^*_{\text{Pic}^0 X} \mathcal{O}_X & \xrightarrow{\mu} & \mathcal{E}(\mathcal{D}_x) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{E}}(\mathcal{D}_x) & & \\
\end{array}
\]

Since $\mu$ is generically surjective and $(-1)^*_{\text{Pic}^0 X} \mathcal{O}_X$ is torsion-free (recall that $\operatorname{gv}(\omega_X) \geq 1$), we have that $\tilde{\mu}$ is injective. Completing the diagram we get,

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\tau(\mathcal{E}(\mathcal{D}_x)) & \tau(\mathcal{E}(\mathcal{D}_x)) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & (-1)^*_{\text{Pic}^0 X} \mathcal{O}_X & \mathcal{E}(\mathcal{D}_x) \\
\downarrow & \downarrow & \downarrow \\
0 & (-1)^*_{\text{Pic}^0 X} \mathcal{O}_X & \mathcal{E}(\mathcal{D}_x) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

If $\tilde{\mathcal{T}} = 0$, then the middle horizontal short exact sequence splits. But, for $\alpha$ a closed point in the support of $\mathcal{T}$ (by the previous claim we know that $\mathcal{T} \neq 0$), $\mu \otimes \mathcal{C}(\alpha) = 0$ by item (d), so $\mu$ cannot split. Therefore $\tilde{\mathcal{T}} \neq 0$.

Let $e = \text{codim}_{\text{Pic}^0 X} \supp \tilde{\mathcal{T}} \geq 2$ (see item (c)). Then $\text{codim}_{\text{Pic}^0 X} \supp \mathcal{E}xt^e(\tilde{\mathcal{T}}, \mathcal{O}_{\text{Pic}^0 X}) = e$. Now, we apply the functor $\mathcal{E}xt^i(\cdot, \mathcal{O}_{\text{Pic}^0 X})$ to the bottom row of (13) using Corollary 3.2

\[
\ldots \rightarrow R^{e-1}\Phi P\omega_X \rightarrow \mathcal{E}xt^e(\tilde{\mathcal{T}}, \mathcal{O}_{\text{Pic}^0 X}) \rightarrow \mathcal{E}xt^e(\mathcal{E}(\mathcal{D}_x), \mathcal{O}_{\text{Pic}^0 X}) \rightarrow \ldots
\]

Since $\mathcal{E}(\mathcal{D}_x)$ is torsion-free, $\text{codim}_{\text{Pic}^0 X} \supp \mathcal{E}xt^e(\mathcal{E}(\mathcal{D}_x), \mathcal{O}_{\text{Pic}^0 X}) > e$. Therefore, we must have $\text{codim}_{\text{Pic}^0 X} \supp R^{e-1}\Phi P\omega_X = e$ and $\operatorname{gv}(\omega_X) \leq 1$. 
\qed
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