Monotonic core solutions: beyond Young’s theorem

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Abstract  Young’s theorem implies that every core concept violates monotonicity. In this paper, we investigate when such a violation of monotonicity by a given core concept is justified. We introduce a new monotonicity property for core concepts. We pose several open questions for this new property. The open questions arise because the most important core concepts (the nucleolus and the per capita nucleolus) do not satisfy the property even in the class of convex games.

Keywords  Monotonicity · Core · TU games · Per capita Nucleolus

1 Introduction

Young (1985) formulates an impossibility result for the problem of finding core concepts satisfying monotonicity in the domain of balanced TU games. This result opens up two paths of research: One is to restrict the search for monotonic core concepts to certain classes of games. The other is to define new properties of monotonicity (weaker than the one formulated by Young) and to deal with the class of all balanced games. This paper discusses this second path.

In considering new monotonicity properties we first analyze the compatibility between monotonicity and core stability. Only when such a compatibility exists do we require a core concept to be monotonic. Following this approach we introduce a new property: core monotonicity. Once we have checked that core concepts such as nucleolus, per capita nucleolus and egalitarian core concepts do not satisfy the property the existence of a core concept satisfying this property arises as an open question. We

1 See Sects. 2 and 3 for formal definitions of core concept and monotonicity.

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also discuss the existence of core concepts that are monotonic on the class of convex games, on the class of veto balanced games and on the class of games with a large core.

2 Preliminaries

2.1 TU games

A cooperative $n$-person game in characteristic function form is a pair $(N, v)$, where $N$ is a finite set of $n$ elements and $v : 2^N \rightarrow \mathbb{R}$ is a real-valued function in the family $2^N$ of all subsets of $N$ with $v(\emptyset) = 0$. Elements of $N$ are called players and the real-valued function $v$ the characteristic function of the game. Any subset $S$ of $N$ is called a coalition. The number of players in $S$ is denoted by $|S|$. Given $S \subset N$ we denote by $N \setminus S$ the set of players of $N$ that are not in $S$. Let $N$ be a set of players and $\Gamma_0$ a class of games. If there is no confusion, we write $v \in \Gamma_0$ instead of $(N, v) \in \Gamma_0$.

A distribution of $v(N)$ among the players is a real-valued vector $x \in \mathbb{R}^N$ where $x_i$ is the payoff assigned by $x$ to player $i$. A distribution satisfying $x_i \geq v(\{i\})$ for all $i \in N$ is called an imputation and the set of imputations is denoted by $I(v)$. We denote $\sum i \in S x_i$ by $x(S)$. The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$
C(v) = \{ x \in I(v) : x(S) \geq v(S) \text{ for all } S \subset N \}.
$$

A game with a nonempty core is called a balanced game. Player $i$ is a veto player if $v(S) = 0$ for all $S$ where player $i$ is not present. A balanced game with at least one veto player is called a veto balanced game. We denote by $\Gamma_B$ the class of balanced games and by $\Gamma_{VB}$ the class of veto balanced games.

We say that a game $(N, v)$ is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. We denote by $\Gamma_C$ the class of convex games.

A game $(N, v)$ has a large core if for every $y \in \mathbb{R}^N$ that satisfies $y(S) \geq v(S)$ for all $S \subseteq N$ there exists $x \in C(v)$ such that $x_i \leq y_i$ for all $i \in N$. We denote by $\Gamma_L$ the class of games with a large core.

Games with a large core are balanced games. A convex game has a large core. In this paper we only consider balanced games.

A solution $\phi$ on a class of games $\Gamma_0$ is a correspondence that associates a set $\phi(N, v)$ in $\mathbb{R}^N$ with every game $(N, v)$ in $\Gamma_0$ such that $x(N) \leq v(N)$ for all $x \in \phi(N, v)$. This solution is efficient if this inequality holds with equality. The solution is single-valued if the set contains a unique element for each game on the class. In this paper, we study solution concepts that select precisely one core allocation for each balanced game. We call such concepts core concepts.

For any vector $z \in \mathbb{R}^d$ we denote by $\theta(z)$ the vector that results from $z$ by permuting the coordinates in such a way that $\theta_1(z) \leq \theta_2(z) \leq \cdots \leq \theta_d(z)$. Let $x, y \in \mathbb{R}^d$.

We say that the vector $x$ Lorenz dominates the vector $y$ if $\sum_{i=1}^{k} \theta_i(x) \geq \sum_{i=1}^{k} \theta_i(y)$ for all $k \in \{1, 2, \ldots, d\}$ and at least one of these inequalities is strict.

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We say that the vector $x$ weakly lexicographically dominates the vector $y$ (denoted by $x \leq_L y$) if either $\theta(x) = \theta(y)$ or there exists $k$ such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, \ldots, k - 1\}$ and $\theta_k(x) > \theta_k(y)$.

We say that the vector $x$ weakly lexmax dominates the vector $y$ (denoted by $x \leq_{lm} y$) if either $\theta(x) = \theta(y)$ or there exists $k$ such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, \ldots, k - 1\}$ and $\theta_k(x) < \theta_k(y)$.

A core allocation is Lorenz undominated if there does not exist a core allocation that Lorenz dominates it. An egalitarian core concept is a core concept that selects Lorenz undominated core allocations. The Lexmax solution of a balanced game $v$, denoted by $L_{\text{max}}(v)$, as the unique core allocation that weakly lexmax dominates any core allocation of the game $v$. In formula:

$$\{ L_{\text{max}}(v) \} = \{ x \in C(v) \mid x \leq_{lm} y \text{ for all } y \in C(v) \} .$$

Given $x \in \mathbb{R}^N$ the excess of a coalition $S$ with respect to $x$ in a game $v$ is defined as $e(S, x) := v(S) - x(S)$.

Schmeidler (1969) introduced the nucleolus of a game $v$, denoted by $\eta(v)$, as the unique imputation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of imputations. In formula:

$$\{ \eta(v) \} = \{ x \in I(v) \mid \theta(x) \leq_L \theta(y) \text{ for all } y \in I(v) \} .$$

For any game $v$ with a nonempty imputation set, the nucleolus is a single-valued solution and lies in the core provided that the core is nonempty.

The per capita nucleolus of a game $v$ (Grotte 1970) is defined analogously by using the concept of per capita excess instead of excess. Given $S$ and $x$ the per capita excess of $S$ at $x$ is:

$$e_{\text{pc}}(S, x) := \frac{v(S) - x(S)}{|S|} .$$

The per capita nucleolus of a game $v$ is denoted by $pc\eta(v)$.

2.2 On monotonicity properties

We present some monotonicity properties for single-valued solution concepts.

Let $v, w \in \Gamma_0$ be such that for all $T$ containing player $i$, $v(T) \leq w(T)$, and for all $S \subseteq N \setminus \{i\}$, $v(S) = w(S)$. We say that game $w$ is a monotonic transformation of game $v$ with respect to $i$. We also say that player $i$ is a benefited player from $v$ to $w$.

Let $\phi$ be a single-valued solution on a class of games $\Gamma_0$.

We say that solution $\phi$ satisfies coalitional monotonicity (Young 1985) if the following condition is satisfied: if $v, w \in \Gamma_0$ are such that $w$ is monotonic transformation of $v$ with respect to $i$, then $\phi_i(w) \geq \phi_i(v)$.

We say that solution $\phi$ satisfies aggregate-monotonicity (Meggido 1974) if the following condition is satisfied: if $v, w \in \Gamma_0$ are such that for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i \in N$, $\phi_i(v) \leq \phi_i(w)$.
We say that solution \( \phi \) satisfies **strong aggregate-monotonicity** if the following condition is satisfied: if \( v, w \in \Gamma_0 \) are such that for all \( S \neq N \), \( v(S) = w(S) \) and \( v(N) < w(N) \), then for all \( i, j \in N \), \( \phi_i(w) - \phi_i(v) = \phi_j(w) - \phi_j(v) > 0 \).

Clearly, coalitional monotonicity implies aggregate-monotonicity. In the following, for simplicity we use the term monotonicity instead of coalitional monotonicity.

The following two concepts are also used in this paper.

We say that game \( w \) is a **monotonic transformation** of game \( v \) if there exists \( i \in N \) such that for all \( T \) containing player \( i \), \( v(T) \leq w(T) \), and \( v(S) = w(S) \) for all \( S \subseteq N \setminus \{i\} \).

We say that a core concept \( \phi \) violates monotonicity on \( \Gamma_0 \) if there exist \( v, w \in \Gamma_0 \) such that \( \phi \) violates monotonicity with respect to \( v \) and \( w \).

Young (1985) shows that on the class of balanced games there is no core concept that satisfies monotonicity. The result is presented by means of two balanced games with only one core allocation each. Therefore in these games a core concept must choose the allocation contained in the core. To some extent, we can say that in this case the core restrictions do not allow a core concept to be monotonic. This observation motivates the following note. We investigate new monotonicity properties that require some compatibility between monotonicity and core stability. In other terms, we investigate when a violation of monotonicity by a core concept is justified. Certainly, in the example of Young the violation of monotonicity is justified. Formally, we may say that a violation of monotonicity by a core concept \( \phi \) with respect to games \( v \) and \( w \) is justified if all core concepts violate monotonicity with respect to \( v \) and \( w \). That is, the core concept \( \phi \) does not have any possibility of avoiding the violation of monotonicity.

On the class of convex games the Shapley value\(^2\) is a core concept that satisfies monotonicity. Arin and Feltkamp (2011) show that on the class of veto balanced games there are several core concepts that satisfy monotonicity. It is also shown that the nucleolus and the per capita nucleolus do not satisfy the property. Therefore, it seems that the violation of monotonicity by the nucleolus and the per capita nucleolus on the class of veto balanced games is not justified.

Those facts open the following question: When a violation of monotonicity by a core concept is justified?

### 3 On core monotonicity

We introduce a new monotonicity property\(^3\). The idea of the new property is that a core concept **should be monotonic** whenever core restrictions are compatible with this monotonicity. Our analysis will mainly discuss variations of coalitional monotonicity, and moreover, will relate our property with the classic notion of aggregate-monotonicity.

With the following example we argue that although there exist core concepts satisfying monotonicity with respect to two games \( v \) and \( w \) the violation of monotonicity with respect to the games \( v \) and \( w \) by a core concept may be justified.

\(^2\) See Shapley (1953). In this paper we denote by \( Sh(v) \) the Shapley value of a game \((N, v)\).

\(^3\) We define the property for core concepts.
Example 3.1 Let $N = \{1, 2, 3, 4\}$ and consider the following 4-person balanced games:

$$v(S) = \begin{cases} 
12 & \text{if } S \in \left\{ \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \right\} \\
24 & \text{if } S = N \\
0 & \text{otherwise}
\end{cases}$$

and

$$w^i(S) = \begin{cases} 
v(S) & \text{if } S \neq N \setminus \{i\} \\
24 & \text{if } S = N \setminus \{i\}
\end{cases}$$

where $i \in N$.

The games $w^i$ are monotonic transformations of game $v$. The core of games $w^1$ and $w^2$ contains only the allocation $z^1 = (0, 0, 12, 12)$ and the core of games $w^3$ and $w^4$ only contains the allocation $z^3 = (12, 12, 0, 0)$.

Let $\phi$ a core concept satisfying monotonicity. By applying monotonicity with respect to $w^1$ and $v$ we conclude that $\phi(v) = z^1$. By applying monotonicity with respect to $w^3$ and $v$ we conclude that $\phi(v) = z^3$. Therefore a core concept must violate monotonicity with respect to $v$ and $w^3$ or with respect to $v$ and $w^1$.

Housman and Clark (1998) use this example to prove that Young’s impossibility theorem holds also on the class of 4-person balanced games. In this example, core restrictions do not allow a core concept to behave monotonically from game $v$ to games $w^1$ and $w^3$.

Therefore, we may say that a violation of monotonicity by a core concept $\phi$ with respect to games $v$ and $w$ is justified if there exists a nonempty set of monotonic transformations of $v$, $A(v)$, such that for any core concept there exists a game in the set $A(v) \cup \{w\}$ such that the core concept violates monotonicity with respect to $v$ and this game.

This first approach is not completely satisfactory as the following example illustrates.

Example 3.2 Let $N = \{1, 2, 3, 4\}$ and consider the following 4-person balanced games:

$$v(S) = \begin{cases} 
12 & \text{if } S \in \left\{ \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \right\} \\
24 & \text{if } S = N \\
0 & \text{otherwise}
\end{cases}$$

and

$$q(S) = \begin{cases} 
v(S) & \text{if } S \neq N \\
28 & \text{if } S = N.
\end{cases}$$

It seems intuitively clear that a core concept $\phi$ such that $\phi_i(v) > \phi_i(q)$ for a player $i$ cannot be considered as a monotonic core concept. But since there exists a nonempty set of monotonic transformations of $v$, $A(v) = \{w^1, w^3\}$, such that for any core concept there exists a game in the set $A(v) \cup \{q\}$ such that this core concept violates monotonicity with respect to $v$ and this game, we may assert that the
violation of aggregate-monotonicity with respect to \( v \) and \( q \) is justified. However, the fact that there exist core concepts satisfying aggregate-monotonicity indicates that the violation of aggregate-monotonicity is not a consequence of the necessity of satisfying core requirements.

We introduce the definition of justified violation of monotonicity by a core concept.

**Definition 3.1** Let \( v, w \in \Gamma_0 \) and let \( \phi \) be a core concept defined on \( \Gamma_0 \). A violation of monotonicity by \( \phi \) with respect to \( v \) and \( w \) is justified if there exists a nonempty set of monotonic transformations of \( v \), \( A(v) \subseteq \Gamma_0 \setminus \{v, w\} \), such that:

1. There exist core concepts satisfying monotonicity with respect to \( v \) and any game in \( A(v) \).
2. For any core concept we can find a game in the set \( A(v) \cup \{w\} \) such that this core concept violates monotonicity with respect to \( v \) and this game.

With this definition we introduce the concept of core monotonicity.

**Definition 3.2 Core monotonicity**: A core concept \( \phi \) on \( \Gamma_0 \) is core monotonic if any violation of monotonicity on \( \Gamma_0 \) is justified.

If a violation of monotonicity occurs with respect to \( v \) and \( w \), then game \( w \) should be a necessary member of a set of monotonic transformations of \( v \) with the following property: For any core concept we can find a game in the set such that this core concept violates monotonicity with respect to \( v \) and this game.

Next proposition relates core monotonicity and aggregate-monotonicity.

**Proposition 3.1 Core monotonicity implies aggregate-monotonicity.**

*Proof* Assume that there exists a core concept \( \phi \) that is core monotonic but not aggregate-monotonic. Then there exist two balanced games \((N, v)\) and \((N, w)\) such that:

1. \( w(N) > v(N) \) and \( w(S) = v(S) \) for any other coalition \( S \).
2. The violation of monotonicity by \( \phi \) with respect to games \( v \) and \( w \) is justified.

The second fact implies that there exists a nonempty set of monotonic transformations of \( v \), \( A(v) \), such that:

1. There exist core concepts satisfying monotonicity with respect to \( v \) and any game in \( A(v) \).
2. For any core concept we can find a game in the set \( A(v) \cup \{w\} \) such that this core concept violates monotonicity with respect to \( v \) and this game.

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4 Other definitions may be considered. For example:

A violation of monotonicity by \( \phi \) with respect to games \( v \) and \( w \) is justified if there exists a nonempty set of monotonic transformations of \( v \), \( A(v) \subseteq \Gamma_0 \setminus \{v, w\} \), such that:

1. There exist monotonic core concepts in \( A(v) \cup \{v\} \).
2. There does not exist monotonic core concept in \( A(v) \cup \{v, w\} \).
Let $\tau$ be a core concept satisfying monotonicity with respect to $v$ and any game in $A(v)$. Consider a core concept $\varepsilon$ defined in $A(v) \cup \{v, w\}$ as follows:

$$
\varepsilon(q) = \begin{cases} 
\tau(q) & \text{if } q \in A(v) \cup \{v\} \\
\tau(v) + \left( \frac{w(N) - v(N)}{|N|}, \ldots, \frac{w(N) - v(N)}{|N|} \right) & \text{if } q = w.
\end{cases}
$$

It is clear that there is not game in $A(v) \cup \{w\}$ such that $\varepsilon$ violates monotonicity with respect to $v$ and this game. \qed

It is well-known that on the classes of balanced games, convex games and veto balanced games the nucleolus does not satisfy aggregate-monotonicity. (See Hokari 2000 for the case of convex games and Arin and Feltkamp (2005) for the case of veto balanced games.)

Therefore, on the class of balanced games, convex games and veto balanced games the nucleolus is not core monotonic.

The per capita nucleolus satisfies aggregate-monotonicity on the class of all balanced games, which is why some authors consider it a good candidate when seeking to select core allocations. The following result shows that the per capita nucleolus violates core monotonicity and therefore, even if it does satisfy aggregate-monotonicity, it can hardly be seen as a monotonic core concept (at least on the class of all balanced games and in the subclass of veto balanced games).

**Proposition 3.2** On the class of balanced games the per capita nucleolus is not core monotonic.

**Proof** Let $N = \{1, 2, 3, 4, 5, 6\}$ a set of players and consider the following 6-person veto balanced games:

$$
v(S) = \begin{cases} 
4 & \text{if } 1 \in S \text{ and } |S| = 5 \\
10 & \text{if } S = N \\
0 & \text{otherwise}
\end{cases}
$$

$$
w(S) = \begin{cases} 
1 & \text{if } 1 \in S \text{ and } |S| = 4 \\
8 & \text{if } S = N \setminus \{6\} \\
12 & \text{if } S = N \\
v(S) & \text{otherwise}
\end{cases}
$$

The per capita nucleolus selects the allocation $(5, 1, 1, 1, 1, 1)$ in the first game and in the second game selects $(\frac{35}{27}, \frac{38}{27}, \frac{38}{27}, \frac{38}{27}, \frac{14}{21})$. Therefore, the per capita nucleolus is not monotonic since player 1, the only benefited player from $v$ to $w$, receives a lower payoff in the second game. In order to be core monotonic this violation of monotonicity must be justified. Therefore, there exists a nonempty set of monotonic transformations of $v$, $A(v)$, such that:

1. There exist core concepts satisfying monotonicity with respect to $v$ and any game in $A(v)$.
2. For any core concept we can find a game in the set $A(v) \cup \{w\}$ such that this core concept violates monotonicity with respect to $v$ and this game.
Let \( \tau \) be a core concept satisfying monotonicity with respect to \( v \) and any game in \( A(v) \). Consider a core concept \( \varepsilon \) defined on \( A(v) \cup \{v, w\} \) as follows:

\[
\varepsilon(q) = \begin{cases} 
\tau(q) & \text{if } q \in A(v) \cup \{v\} \\
(12, 0, \ldots, 0) & \text{if } q = w.
\end{cases}
\]

Clearly, there is no game in the set \( A(v) \cup \{w\} \) such that \( \varepsilon \) violates monotonicity with respect to \( v \) and this game. \( \square \)

Note that the proof is constructed using veto balanced convex games and therefore we have proved that on the class of veto balanced games and on the class of convex games the per capita nucleolus is not monotonic.

4 Results and open questions

The foregoing analysis suggests the following general question:

Are there any core concept satisfying core monotonicity on the class of balanced games?

It also suggests several other questions concerning classes of balanced games. On the class of convex games, the Shapley value is a core concept and therefore satisfies core monotonicity. But, in general, the Shapley value is not a core concept. This prompts the following question: if there are core concepts that are core monotonic on the class of balanced games do those core concepts coincide with the Shapley value on the class of convex games? The answer is negative, as the following example shows.

Example 4.1 Let \( N = \{1, 2, 3, 4\} \) and consider the following 4-person balanced games:

\[
v(S) = \begin{cases} 
4 & \text{if } S \in \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, N\} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
w(S) = \begin{cases} 
v(S) & \text{if } S \neq N \\
8 & \text{if } S = N.
\end{cases}
\]

The game \( w \) is convex and therefore its Shapley value, \((3, \frac{5}{7}, \frac{5}{7}, \frac{5}{7})\) is a core allocation. The only core allocation of the game \((N, v)\) is \((4, 0, 0, 0)\). Note that game \( w \) is a monotonic transformation of \( v \) with respect to any player in \( N \). Consider a core concept \( \phi \) that is monotonic on the class of convex games and on the class of veto balanced games. Clearly, \( \phi(v) = (4, 0, 0, 0) \) and by core monotonicity of \( \phi \) on \( \Gamma_{VB} \) we conclude that \( \phi_1(w) \geq 4 \) and therefore \( \phi(w) \neq Sh(w) \).

On the class of veto balanced games there are core concepts that are monotonic (see Arin and Feltkamp 2012). The solution concepts analyzed in this paper\(^5\) are not

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\(^5\) To be precise this paper deals with the class of non negative veto balanced games. Non negativity implies that the worth of any coalition is non negative. Therefore, in this section, by \( \Gamma_{VB} \) we refer to the class of non negative veto balanced games.
defined on the class of balanced games and therefore a similar question can be asked. That is, given a monotonic concept defined on the class of veto balanced games, say \( \phi \), are there any core concepts that are defined on the class of balanced games coinciding with \( \phi \) on the class of veto balanced games?

An immediate positive answer to the questions is the following. Let \( AVS \) (All for veto players solution) be a monotonic core concept defined on the class of non negative veto balanced games as follows:

\[
AVS_i(v) = \begin{cases} 
\frac{v(N)}{|T|} & \text{if } i \in T \text{ where } T \text{ is the set of veto players} \\
0 & \text{otherwise}
\end{cases}
\]

and let \( \sigma \) be a core concept defined on the class of balanced games (\( \Gamma_B \)) as follows:

\[
\sigma(v) = \begin{cases} 
AVS(v) & \text{if } v \in \Gamma_{VB} \\
Sh(v) & \text{if } v \in \Gamma_C \text{ and } v \notin \Gamma_{VB} \\
p\epsilon(v) & \text{otherwise.}
\end{cases}
\]

It is immediately apparent that \( \sigma \) is monotonic in the class of convex games and in the class of veto balanced games. This answer is not entirely satisfactory since it is clear that \( \sigma \) does not satisfy continuity. Therefore, the problem is still unsolved if we ask for continuous single-valued core concepts.

Calleja et al. (2009) introduce single-valued solutions that satisfy core stability and aggregate-monotonicity. They do not study whether their solutions satisfy other desirable properties such as continuity, equal treatment of equal players or consistency.

In fact, more open questions can be added to the list. On the class of large core games (Sharkey 1982) the Shapley value is not a core concept (see Hoffmann and Sudholter 2007). The following proposition relates core monotonicity and large core games.

**Proposition 4.1** Let \( v, w \in \Gamma_L \) be such that game \( w \) is a monotonic transformation of game \( v \). Then a violation of monotonicity with respect to games \( v \) and \( w \) is not justified.

**Proof** Assume that the proposition is not true. Then there exist two large core games \((N, v)\) and \((N, w)\) such that:

1. Game \( w \) is a monotonic transformation of game \( v \).
2. There exists a nonempty set of monotonic transformations\(^6\) of \( v \), \( A(v) \subseteq \Gamma_B \setminus \{v, w\} \), such that:
   1. There exist core concepts satisfying monotonicity with respect to \( v \) and any game in \( A(v) \).
   2. For any core concept we can find a game in the set \( A(v) \cup \{v, w\} \) such that this core concept violates monotonicity with respect to \( v \) and this game.

Let \( P \) be the set of benefited players from \( v \) to \( w \). Let \( \tau \) be a core concept satisfying monotonicity with respect to \( v \) and any game in \( A(v) \). Consider a core concept \( \epsilon \) defined on \( A(v) \cup \{v, w\} \) as follows:

\(^6\) Not necessarily games with a large core.
\[ \varepsilon(q) = \begin{cases} \tau(q) & \text{if } q \in A \cup \{v\} \\ z & \text{if } q = w \end{cases} \]

where \( z \) is defined as follows:

If \( \tau(v) \in C(w) \) then \( z = \tau(v) \). If \( \tau(v) \notin C(w) \) then it must occur that for some coalition \( S \) containing the set of benefited players holds that \( \tau(v)(S) < w(S) \). Consider a vector \( y \) obtained from \( \tau(v) \) by increasing the payoff of players in \( P \) equally until \( y(S) \geq w(S) \) for all \( S \) containing the set \( P \) and \( y(S) = w(S) \) for at least one of those coalitions. Clearly, any core allocation, \( x \), obtained from \( y \) satisfying \( x_i \leq y_i \) for any \( l \in N \), needs to satisfy \( x_i = y_i > \tau_i(v) \) for all \( i \in P \). Since game \( w \) has a large core the existence of such core allocation, \( x \), is guaranteed. Let \( z = x \). Clearly, there is no game in the set \( A(v) \cup \{w\} \) such that \( \varepsilon \) violates monotonicity with respect to \( v \) and this game. \( \square \)

A consequence of the proposition above is that if a core concept satisfies core monotonicity then it satisfies monotonicity on the class of large core games (that contains the class of convex games).

Arin, Kuipers and Vermeulen (2002) characterize an egalitarian core concept, the Lmax solution, on the class of large core games. In the characterization aggregate-monotonicity is used and it is not difficult to prove that the solution also satisfies coalitional monotonicity, but not strong aggregate-monotonic. Egalitarian core concepts are defined only for balanced games and do not satisfy covariance. On the class of veto balanced games egalitarian core concepts do not satisfy aggregate-monotonicity\(^7\) and therefore are not core monotonic.

The result adds one more question to the list, concerning the existence of monotonic core concepts on the class of large core games. Note that we do not consider a solution \( \sigma \) defined on the class of balanced games (\( \Gamma_B \)) as follows to be entirely satisfactory:

\[ \sigma(v) = \begin{cases} AVS(v) & \text{if } v \in \Gamma_{VB} \\ L \max(v) & \text{if } v \in \Gamma_L \text{ and } v \notin \Gamma_{VB} \\ pcone(v) & \text{otherwise.} \end{cases} \]

Finally, we show that if we only consider monotonic changes in coalitions of two players then there is no incompatibility between core monotonicity and therefore 2-monotonic core concepts may exist\(^8\).

We say that solution \( \phi \) satisfies k-monotonicity if for all \( v, w \in \Gamma_0 \), such that for \( S \neq T, |T| = k, v(S) = w(S) \) and \( v(T) < w(T) \), then \( \phi_i(w) \geq \phi_i(v) \) for all \( i \in T \).

\(^7\) Let \( N = \{1, 2, 3\} \) and consider the following 3-person veto balanced games:

\[
v(S) = \begin{cases} 4 & \text{if } S \in \{1, 2, 3\}, \{1, 2\}, \{1, 3\} \\ 0 & \text{otherwise} \end{cases}
\]

\[
w(S) = \begin{cases} v(S) & \text{if } S \neq N \\ 6 & \text{if } S = N. \end{cases}
\]

Any egalitarian concept selects \((4, 0, 0)\) in game \( v \) and \((2, 2, 2)\) in game \( w \).

\(^8\) Zhou (1991) shows that the nucleolus satisfies 1-monotonicity. Our approach is different and we show that any violation of 2-monotonicity is not justified.
Proposition 4.2 Let $v, w \in \Gamma_B$, such that for all $S \neq T$, $|T| = 2$, $v(S) = w(S)$ and $v(T) < w(T)$. Then for any $\alpha \in C(v)$ there exists $\alpha \in C(w)$ such that for all $i \in T$, $y_i \geq x_i$.

Proof Consider a game $q \in \Gamma_B$ such that for all $S \neq T$, $|T| = 2$, $v(S) = q(S)$ and $v(T) < w(T) \leq q(T)$. Let $q(T)$ be the maximal value of coalition $T$ compatible with nonemptiness of the core of $q$. Assume that $T = \{1, 2\}$ and let $f_{ij}(x, v) = \min_{i \in S \subseteq N \setminus \{j\}} (x(S) - v(S))$. We consider two cases:

(a) $q(\{1, 2\}) + q(N \setminus \{1\}) + q(N \setminus \{2\}) = 2q(N)$. In this case, any core allocation, say $x$, implies that $x_i = q(N) - q(N \setminus \{i\}) \geq z_i$ for $i \in T$. The last inequality is a consequence of $z \in C(v)$. Since $x \in C(w)$ the result holds.

(b) $q(\{1, 2\}) + q(N \setminus \{1\}) + q(N \setminus \{2\}) < 2q(N)$. We consider two cases:

(b1) Assume there exists a core allocation $y \in C(v)$ such that $z_i = y_i$ for all $i \in T$ and there exists player $l \in N \setminus T$ such that $f_{j1}(y, v) \geq f_{j2}(y, v) \geq 0$ (with one of the two inequalities strict). Construct a new allocation $y^1$ where $y^1_2 = z_2 + \alpha$, $y^1_j = z_l - \alpha$ and $y^1_1 = y_i$ otherwise. The amount $\alpha$ is the maximal amount compatible with $f_{j1}(y^1, v) \geq 0$ and $f_{j2}(y^1, v) \geq 0$. If $y^1 \in C(q)$ the proof is completed. If $y^1 \notin C(q)$ then find player $m \in N \setminus T$ such that $f_{m1}(y^1, v) \geq f_{m2}(y^2, v) \geq 0$ (with one of the two inequalities strict). Construct a new allocation $y^2$ where $y^2_1 = y^1_1 + \alpha$, $y^2_m = y^1_m - \alpha$ and $y^2_2 = y^1_2$ otherwise. The amount $\alpha$ is the maximal amount compatible with $f_{m1}(y^2, v) \geq 0$ and $f_{m2}(y^2, v) \geq 0$. If $y^2 \in C(q)$ the proof is completed. Otherwise follow the procedure until a core allocation is achieved or go to case b2) whenever is impossible to continue with the procedure. (b2) Assume that for any allocation $y \in C(v)$ such that $z_i = y_i$ for all $i \in T$ and $y(T) < q(T)$ there is no player $l \in N \setminus T$ such that $f_{j1}(y, v) \geq f_{j2}(y, v) \geq 0$ (with one of the two inequalities strict). Consider $y$ such that there exist players $l, m \in N \setminus T$ such that

$$\min_{l \in S \subseteq N \setminus T} (y(S) - v(S)) > 0 \quad \text{and} \quad \min_{m \in S \subseteq N \setminus T} (y(S) - v(S)) > 0.$$ 

If this condition does not hold we need to conclude that $q$ is not balanced since it is impossible to transfer any amount from the set $N \setminus T$ to the set $T$. Construct a new allocation $y^1$ where $y^1_1 = y_1 + \alpha$, $y^1_2 = y_2 + \beta$, $y^1_1 = y_1 - \alpha$, $y^1_m = y_m - \beta$ and $y^1_i = y_i$ for all $i \in N \setminus (T \cup \{l, m\})$. The amounts $\alpha$ and $\beta$ are the maximal amounts compatible with

$$\min_{l \in S \subseteq N \setminus T} (y^1(S) - v(S)) \geq 0 \quad \text{and} \quad \min_{m \in S \subseteq N \setminus T} (y^1(S) - v(S)) \geq 0$$

and $f_{j1}(y^1, v) = f_{j2}(y^1, v) = f_{m1}(y^1, v) = f_{m2}(y^1, v) = 0$. If $y^2 \in C(q)$ the proof is completed. Otherwise continue with the procedure until a core allocation is achieved. \(\square\)

9 In case that $f_{j2}(y, v) \geq f_{j1}(y, v) \geq 0$ the bilateral transfer is from player $l$ to player $1$. 

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The proof is a procedure for constructing a core allocation that respects the monotonicity requirement. In the procedure it is important to identify which players transfer amounts to players in the set $T$. The case b2) implies that two players transfer simultaneously to the set $T$.

With this result it is immediately apparent that on the class of balanced games core monotonicity implies 2-monotonicity.

Note that Example 1 shows the incompatibility between 3-monotonicity and core stability.

5 Concluding remarks

This paper investigates the compatibility between two very desirable properties: core stability and monotonicity. The approach introduced in the paper is different from the classic one, which consists of checking whether well-known solution concepts satisfy the properties or not in domains where the solutions are applied. In our approach we try to identify incompatibilities by checking the games themselves. Our main idea is that a solution may be monotonic whenever any violation of monotonicity is justified. To justify such a violation is to show that there was an incompatibility between the two properties. In fact, the paper suggests the following path of research. First, find a single-valued continuous core concept that is monotonic on the class of large core games and on the class of veto balanced games. Second, use this solution to find the largest class of games where there is compatibility between core stability and monotonicity. If this research is satisfactorily completed it can be determined why a solution concept does not behave monotonically and therefore we can state whether the solution is monotonic or not. Note that we have argued that the existing core concepts are not monotonic on the class of balanced games.

A related but slightly different approach is taken in Tauman and Zapechelnyuk (2010). In their paper they use an example to argue that in some cases even when the possibility of the two properties being compatible exists the violation of aggregate-monotonicity may be reasonable. Therefore the open question posed by this paper is the following: when is it reasonable to expect a violation of monotonicity by a core concept in circumstances where this violation is not justified by an incompatibility between core stability and monotonicity?

10 For example, consider the following case. Let $N = \{1, 2, 3, 4\}$ be a set of players and consider the following two games:

$$v(S) = \begin{cases} 2 & \text{if } |S| = 2 \text{ and } S \notin \{\{1, 2\}, \{3, 4\}\} \\ 4 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}$$

$$w(S) = \begin{cases} 2 & \text{if } S = \{1, 2\} \\ v(S) & \text{otherwise.} \end{cases}$$

Consider the allocation $(0, 0, 2, 2)$. Only simultaneous transfers from player 3 and 4 to players 1 and 2 allow the construction of allocation $(1, 1, 1, 1)$. 
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