An Introduction to Wave-Trapping in Supergranulation

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Abstract. This paper is an introduction to modelling waves trapped in a supergranular cell. The supergranular cell is generalized to the form of a hexagon with a cylinder inscribed within its boundaries. A cylindrical wave equation is implemented and solved and we account for the edges of the hexagon through boundary conditions. Plots are created of the solution and will serve as a test as to whether the model reflects actual wave conditions inside a single supergranular cell.

1. Introduction

Over the years many aspects of supergranulation regarding the dimensions, origins, and attributes associated with the convection of flows in relation to local magnetic fields have remained a mystery. Supergranules are large scale convective polygonal structures on the Sun’s surface. Certain characteristics of supergranules that have been observed or estimated include having a maximum mean diameter ranging from 25 to 85 Mm, an estimated maximum depth of 10 Mm, a mean lifetime of about 24 hours, and root mean squared horizontal and vertical flow velocities ranging from .3 to .4 km/s and .1 to .2 km/s respectively. There are an estimated 2500 supergranules covering the Sun’s surface at any one time [1]. Below the Sun’s surface the flow of plasma constituents convect in large part due to temperature and pressure gradients. As the flows rise to the surface much of this flow advects across the surface of the Sun out to supergranule boundaries. More recently it has been speculated that there are not only flows inside supergranular cells, but the fluid inside cells could feature oscillatory wave properties. In this paper we attempt to model these waves using a four-dimensional wave equation featuring independent variables $r, \theta, z,$ and $t$ for a single supergranule cell. We first introduce a single template for the supergranular cell. The template for our model takes the form of a hexagon and this shape was chosen since it is sufficiently complex yet not so complex that the geometry would be too difficult to model. The model is obtained from a four-dimensional wave equation written in terms of independent variables $r, \theta, z,$ and $t$. The solution to the wave equation is found by separation of variables. Various plots of the model will be shown and these plots will serve as a diagnostic test of how the model relates to observations. In the future we hope to compare the oscillation attributes found from the model to actual observations using the HMI (Helioseismic Magnetic Imager) instrument on SDO. The conclusion will summarize our findings and outline goals for future work.

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2. The Model

The template of the cell used to trap waves for the model supergranule is shown below in Figure 1.

![Diagram of the hexagonal template used for wave-trapping. Boundaries are treated as perfectly reflecting and waves generated originate from the centre of the template.](image)

The template shown above features an inscribed circle within the boundaries of the hexagon. The circle is actually a cylinder and therefore there is a depth associated with the model which extends down to 10Mm. We can use the geometry of the template to write the radial component as,

\[
R = \frac{X \sin \theta_R}{\sin \theta_X} = \frac{X \sin \theta_R}{\sin(180^\circ - \theta_R - \theta_C)},
\]

To account for the edges of the hexagon we employ the boundary conditions which yield the eigenvalues:

\[
\lambda_{nm} = \left(\frac{\mu_{nm}}{R}\right) = \frac{\mu_{nm} \sin(180^\circ - \theta_R - \theta_C)}{X \sin \theta_R} \quad \text{and} \quad \beta_{nm}^2 = \lambda_{nm}^2 - \left(\frac{p \pi}{l}\right)^2
\]

where \(\beta_{nm}\) combines both horizontal and vertical boundary conditions, \(\mu_{nm}\) and \(p\) are parameters that arise through the implementation of the boundary conditions, and \(X\) and \(l\) are based upon the horizontal and vertical dimensions of the model supergranule respectively. Cylindrical geometry gives the four dimensional wave equation as,

\[
a^2 \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta_C^2}\right) = \frac{\partial^2 U}{\partial t^2}
\]

The wave equation found in equation (2) will be decomposed through separation of variables into four eigenfunctions and the solution is \(U(r, \theta, z, t) = R'(r)M(\theta)Z(z)T(t)\). Written explicitly, the solution becomes,
\[ U(r,\theta,z,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left( A_{nmp} \cos(n\theta) + A'_{nmp} \sin(n\theta) \right) \cos(a\beta_{nmp}t) + \left( B_{nmp} \cos(n\theta) + B'_{nmp} \sin(n\theta) \right) \sin(\beta_{nmp}z) \right\} J_n(\kappa_{nmp} r) \]  

\text{(3)}

where \( A_{nmp}, A'_{nmp}, B_{nmp}, \) and \( B'_{nmp} \) are coefficients. \( \alpha \) is a parameter having units of velocity, \( J_n(\kappa_{nmp} r) \) is a Bessel’s function of the first kind, and \( \theta = \theta_c \). It is worth mentioning that equation (3) was obtained through the following steps. First determine the boundary conditions from the template in Figure 1. Second use separation of variables to decouple the wave equation into two and later into four eigenfunctions. Boundary conditions yield the eigenvalues \( \kappa_{nmp}, \beta_{nmp} \) and each are used in the construction of the general solution. Finally note that the radial component is modelled so that \( X = 15 \text{Mm} \) and time is sampled at \( t = 35 \text{ h} \).

3. Tests On The Model

The solution to the wave equation given in equation (3) can be used in plotting routines to determine if the model solution reflects some of the actual wave properties and fluid flow characteristics found in actual supergranulation. We show three plots found from the solution given in equation (3). The first are a contour plot and a polar plot shown in Figure 2. Figure 3 illustrates a plot of the wave attributes of the model.

Figure 2. A contour plot shown (left) and a polar plot (right). The contour plot illustrates waves trapped within the template given in Figure 1. Notice that the contour plot is partitioned into six sectors. These sectors show how the waves travel as they are generated from the centre and are reflected by each side due to boundary conditions. The colour scheme shows that the amplitude of the waves are greater near the centre. The polar plot shows various wave modes as generated with an intrinsic time step and the direction of propagation is shown as a function of angle.
Figure 3. The three-dimensional plot above represents the oscillatory flow as a function of $r^2$. As seen in this plot the oscillations diverge from the centre and the magnitude of the oscillatory flow found at the centre is greater than that found at the boundaries.

The left panel in Figure 2 shows the oscillations as wedges between the contours that branch out from the centre of the plot. The waves emerge and reflect as modelled in an isotropic medium from the centre out to each side of the hexagon where they are seen to reflect off of the boundaries used for the template. On the right of Figure 2 the polar plot shows the distribution of the amplitude of the oscillations with angle for a given time step. Physically the polar plot may also be interpreted through the fact that, in each direction, there is a net dissipative effect that each wave front causes on the medium it propagates in with a larger and larger time step. Finally Figure 3 shows the oscillation pattern as oscillatory flow versus $R^2$. The modes are hard to discern because the plot is actually in three dimensions as seen in a two dimensional layout. Finally in Figure 3 note that as one approaches the portion of the plot at $R^2 = 0$ the model blows up at that value.

4. Conclusion

This paper illustrates that a cylindrical wave equation constrained by boundary conditions can be used to model waves trapped within a hexagonal template. Waves are generated at the centre of the template and the model shows that the waves diverge from the centre with higher amplitude at that position and subside at the boundaries. This behaviour is in good agreement with the actual fluid flow found in supergranules modelled with 10 Mm depth and 30 Mm diameter. Future work will involve using dynamic boundary conditions so that the shape of the regular polygon template changes with each time step. We also will extend our wave equation to include the magnetic field, pressure gradients as well as gravitational stratification. Finally we hope to couple the theoretical aspects of the model with observation. This will be done by using the Fourier transform of the solution compared with ring diagram analysis generated with the Helioseismic Magnetic Imager from Dopplergrams. The comparison of the model and observations will help fine-tune the independent variables of the model and may enable us to obtain a better estimate for the depth of supergranulation.

References

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