Decay Of Correlation For Expanding Toral Endomorphisms

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(In memory of Professor Liao Shantao)

Abstract

Let \( A \) be an expanding endomorphism on the torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) with its smallest eigenvalue \( \lambda > 1 \). Consider the ergodic system \( (\mathbb{T}^d, A, \mu) \) where \( \mu \) is Haar measure. We prove that the correlation \( \rho_{f,g}(n) \) of a pair of functions \( f, g \in L^2(\mu) \) is controlled by the modulus of \( L^2 \)-continuity \( \Omega_{f,2}(\lambda^{-n}) \) and that the estimate is to some extent optimal. We also prove the central limit theorem for the stationary process \( f(A^n x) \) defined by a function \( f \) satisfying \( \sum_n \Omega_{f,2}(\lambda^{-n}) < \infty \). An application is given to the Ulam-von Neumann system.

1. Introduction and Main Results

There has been much interest in the study of rates of correlation decay for various kinds of systems [1, 2, 9, 13] (see the references therein). However few rates for a given class of test functions are known to be optimal. We intend in this note to study the simple dynamics of expanding torus endomorphisms and we shall see that optimal rates may be obtained in this case. Another motivation is to better understand a method introduced in [1, 2] to study decay of correlations in a general case, which provides rather precise decay rates but which seems not be able to cover a classical result that there is an exponential decay rate for Hölder functions.

We consider the dynamical system \( (\mathbb{T}^d, A) \) where \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) \((d \geq 1)\) is the \( d \)-dimensional torus and \( A \) is an endomorphism on \( \mathbb{T}^d \). We suppose that \( A \) is expanding, that is, all of its eigenvalues have absolute value strictly larger than 1. A central theme of the ergodic theory for such a system is to consider the behavior of \( A^n \) as \( n \to \infty \). A natural way is to describe the behavior of \( A^n \) through an invariant measure. We take the Haar-Lebesgue measure \( \mu = dx \) on...
The system $(\mathbb{T}^d, A, \mu)$ is strong mixing. That means the correlation
\[ \rho(n) = \rho_f, g(n) := \int f \cdot g \circ A^n d\mu - \int f d\mu \cdot \int g d\mu \]
tends to zero, as $n \to \infty$, for any $f \in L^2(\mu)$ and any $g \in L^2(\mu)$ (called test functions). Our purpose is to study the rate of correlation decay for a given pair of test functions $f, g \in L^2(\mu)$. It is well known that if $g$ and $f$ are Hölder functions, the correlation decays exponentially fast. We shall show that for less regular functions, the correlations decay more slowly and that different kinds of decay rates are possible.

In order to state our results, we recall here the modulus of continuity and the modulus of $L^r$-continuity ($1 \leq r < \infty$) of a function $f$ defined on the torus:
\[
\Omega_f(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|
\]
\[
\Omega_{f,r}(\delta) = \sup_{|v| \leq \delta} \|f(\cdot + v) - f(\cdot)\|_r
\]
where $\|f\|_r$ denotes the norm of $f \in L^r(\mu)$. Formally, we may write $\Omega_f(\delta) = \Omega_{f,\infty}(\delta)$.

**Theorem 1** Let $A$ be an expanding integral matrix with the least eigenvalue (in absolute value) $\lambda > 1$. For $f, g \in L^2(\mu)$, we have
\[
|\rho_f, g(n)| \leq C\|g\|_2 \Omega_{f,2}(\lambda^{-n}) \quad (\forall n \geq 1)
\]
where $C > 0$ is a constant depending only on $A$.

This is actually a consequence of an estimate on a transfer operator that we state as the following theorem. Let $D (\subset \mathbb{Z}^d)$ be a set of representatives of cosets in $\mathbb{Z}^d/A\mathbb{Z}^d$ (one and only one for each coset). We call $D$ a set of digits. The cardinal of $D$ is equal to $q = |\det A|$. For any function $f$ on $\mathbb{T}^d$, define
\[
\mathcal{L}f(x) = \frac{1}{q} \sum_{\gamma \in D} f(A^{-1}(x + \gamma)).
\]
The operator $\mathcal{L}$ is called a transfer operator. Note that $\mathcal{L}$ doesn’t depend on the choice of $D$.

**Theorem 2** Let $A$ be an expanding integral matrix with the least eigenvalue (in absolute value) $\lambda > 1$. Suppose $\int f d\mu = 0$. If $f \in L^r(\mu)$ ($1 \leq r \leq \infty$), we have
\[
\|\mathcal{L}^n f\|_r \leq C \Omega_{f,r}(\lambda^{-n}) \quad (\forall n \geq 1).
\]
Exponential decays are obtained for Hölder continuous functions and functions of bounded variation (not necessary continuous). However, few results are known to be optimal and few work has been carried out for less regular test functions. As far as we know, an optimal decay is obtained for some systems studied in [4] and less regular test functions are discussed for expanding systems in [1, 2, 11]. The method introduced in [1, 2] gives rather precise estimate on the correlations for functions having Dini continuity \( \int_0^1 \Omega_f(t)/dt < \infty \). For the endomorphisms discussed here, it seems that the modulus of \( L^2 \)-continuity is a good tool to describe the decay of correlation. It gives a decay rate for every pair of test functions in \( L^2 \) and the decay rate is optimal to some extent, as we shall see by considering lacunary trigonometric series.

Another consequence of the above theorem is the following CLT (Central Limit theorem).

**Theorem 3** Let \( A \) be an expanding integral matrix and let \( f \in L^2(\mathbb{T}) \) with \( \int f = 0 \). Suppose

\[
\int_0^1 \frac{\Omega_{f,2}(t)}{t} dt < \infty.
\]

Then

\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f(A^n x)
\]

converges in law to a Gaussian variable of zero mean and finite variance

\[
\sigma^2 = -\int f^2(x) dx + 2 \sum_{n=0}^{\infty} \int f(x)f(A^n x) dx.
\]

In the one-dimensional case, M. Kac [6] first proved the CLT for a function of the class \( \text{Lip}_\alpha \) with \( \alpha > 1/2 \). I.A. Ibragimov [5] weakened the \( \text{Lip}_\alpha \) condition to that the modulus of \( L^r \)-continuity \( \Omega_{f,r}(\delta) \) of \( f \) for some \( r > 2 \) is of order \( \delta^\beta \) \( (\beta > 0) \). The above Theorem 3 improves significantly these results. (But it should be pointed out that a convergence speed for CLT was obtained in [5]). For the higher dimensional case, there were no similar satisfactory results.

2. Tiling

We refer to [3, 8] for the facts recalled here and for further information about tilings. Given a measurable set \( T \subset \mathbb{R}^d \), we use \( 1_T \) to denote its characteristic function and \( |T| \) to denote its Lebesgue measure. Given two measurable sets \( T \) and \( S \), the notation \( T \simeq S \) means that \( T \) and \( S \) are equal up to a set of null Lebesgue measure.
An endomorphism of the torus is represented by an integral matrix. Suppose $A$ is a $d \times d$ integral matrix which is expanding, that is, all of its eigenvalues $\lambda_i$ have $|\lambda_i| > 1$. Denote $\lambda = \inf |\lambda_i|$ and $q = |\det A|$ ($q \geq 2$ and is an integer).

Take a digit set $D$. Recall that it consists of representatives of cosets in $\mathbb{Z}^d/\mathbb{A}^d$ (one and only one for a coset). For each $\gamma \in D$, define $S_\gamma : \mathbb{R}^d \to \mathbb{R}^d$ by

$$S_\gamma x = A^{-1}(x + \gamma).$$

As for hyperbolic iterated function systems, it can be proved that there exists a unique compact set $T$ having the self-affinity

$$T = \bigcup_{\gamma \in D} S_\gamma(T).$$

Actually, $|S_\gamma(T) \cap S_{\gamma'}(T)| = 0$ when $\gamma' \neq \gamma''$. Therefore the self-affinity implies

$$\sum_{k \in \mathbb{D}} 1_T(Ax - k) = 1_T(x) \quad \text{a.e.} \quad (1)$$

It is also known that the compact set $T$ has the tiling property

$$\sum_{k \in \mathbb{D}} 1_T(x - k) = 1 \quad \text{a.e.} \quad (2)$$

Since $T$ satisfies (1) and (2), we say it generates an integral self-affine tiling.

The compact set $T$ is called a self-affine tiling tile. We have $|T| = 1$. The tiling property allows us to identify $\mathbb{T}^d$ with $T$ up to a null measure set. The self-affinity allows us to decompose $T$ into $q$ disjoint (up to a null measure set) self-affine parts. If $A$ is a similarity, we say $T$ is self-similar tiling tile.

### 3. Proofs of Theorems

**Notation:** For $\gamma = (\gamma_1, \cdots, \gamma_n) \in D^n$, write

$$S_\gamma x = S_{\gamma_n} \circ S_{\gamma_{n-1}} \circ \cdots \circ S_{\gamma_1} x, \quad T_\gamma = S_\gamma(T).$$

Clearly

$$S_\gamma x = A^{-n}x + A^{-n}\gamma_1 + \cdots + A^{-2}\gamma_{n-1} + A^{-1}\gamma_n.$$

Denoting $b_\gamma = S_\gamma 0$, we get

$$\mathcal{L}^n f(x) = \frac{1}{q^n} \sum_{\gamma \in D^n} f(A^{-n}x + b_\gamma).$$
Proof of Theorem 2 For $\gamma \in D^n$, write

$$f_\gamma = \frac{1}{|T_\gamma|} \int_{T_\gamma} f(x) dx.$$ 

Note that $|T_\gamma| = q^{-n}|T| = q^{-n}$ and that

$$\sum_{\gamma \in D^n} f_\gamma = q^n \int_T f = q^n \int_{\mathbb{R}^d} f = 0.$$ 

Then

$$L^n f(x) = \frac{1}{q^n} \sum_{\gamma \in D^n} f(A^{-n}x + b_\gamma)$$

$$= \frac{1}{q^n} \sum_{\gamma \in D^n} [f(A^{-n}x + b_\gamma) - f_\gamma].$$

Since $T_\gamma = A^{-n}T + b_\gamma$, we have immediately

$$|L^n f(x)| \leq \frac{1}{q^n} \sum_{\gamma \in D^n} \Omega_f(\text{diam } A^{-n}T) \leq C\Omega_f(\lambda^{-n})$$

where diam$B$ denotes the diameter of a set $B$. We used the fact that diam$A^{-n}T \leq a\lambda^{-n}$ for some $a > 0$ and the fact that $\Omega_f(2\delta) \leq 2\Omega_f(\delta)$. The estimate on $\|L^n f\|_\infty$ is thus proved.

For the estimate on $\|L^n f\|_r$, we first use the Hölder inequality to get

$$|L^n f(x)|^r \leq \frac{1}{q^n} \sum_{\gamma \in D^n} |f(A^{-n}x + b_\gamma) - f_\gamma|^r.$$ 

By making the change of variables $y = A^{-n}x + b_\gamma$, we get

$$\frac{1}{q^n} \int_T |f(A^{-n}x + b_\gamma) - f_\gamma|^r dx = \int_{T_\gamma} |f(y) - f_\gamma|^r dy.$$ 

Then

$$\|L^n f\|_r \leq \sum_{\gamma \in D^n} \int_{T_\gamma} |f(y) - f_\gamma|^r dy.$$ 

However

$$\int_{T_\gamma} |f(y) - f_\gamma|^r dy = \int_{T_\gamma} \left| \int_{T_\gamma} (f(y) - f(x)) \frac{dx}{|T_\gamma|} \right|^r dy$$
\[
\begin{align*}
\leq \int_{T_n} dy \int_{T_n} |f(y) - f(x)|^r \frac{dx}{|T|} \\
\leq q^n \int_{T_n} dy \int_{1_{T_n}(y-x)} |f(y) - f(x)|^r dx \\
= q^n \int_{T_n} dy \int_{1_{A^n(T-T)}} (y-x) |f(y) - f(x)|^r dx \\
\end{align*}
\]

Then, if we make a change of variables 
\[ u = y - x \quad \text{and} \quad v = x, \]
we get
\[
\|L^n f\|_r^r \leq q^n \int_{T} dy \int_{A^n(T-T)} (y-x) |f(y) - f(x)|^r dx \\
\leq q^n \int_{T} dv \int_{A^n(T-T)} |f(u + v) - f(v)|^r du \\
= q^n \int_{A^n(T-T)} du \int_{T} |f(u + v) - f(v)|^r dv \\
\leq q^n |A^{-n}(T - T)| \cdot \left[ \Omega_{f,r}(\text{diam}A^{-n}(T - T)) \right]^r.
\]

Suppose \(|T - T| \leq C_1|T|\) with some sufficiently large constant \(C_1\) (which may depend on the digit set \(D\)), we have
\[
|A^{-n}(T - T)| \leq C_1 q^n.
\]

There is another constant \(C_2\) such that
\[
\text{diam}A^{-n}(T - T) \leq C_2 \lambda^{-n}.
\]

Therefore for some constant \(C_3 > 0\), we have
\[
\|L^n f\|_r^r \leq C_3 (\Omega_{f,r}(\lambda^{-n}))^r.
\]

\[ \square \]

**Proof of Theorem 1** Assume \(\int f = 0\), without loss of generality. Consider the operator \(V : L^2 \to L^2\) defined by \(V f = f \circ A\). It can be verified that \(\mathcal{L}\) is just the adjoint operator of \(V\). Therefore
\[
|\rho_{f,g}(n)| = \left| \int g L^n f \right| \leq \|g\|_2 \|L^n f\|_2.
\]

Then Theorem 1 follows from Theorem 2. \[ \square \]

**Proof of Theorem 3** By Theorem 1.1 in [10], it suffices to verify
\[
\sum_{n=0}^\infty \left| \int f \cdot f \circ A^n \right| < \infty, \quad \sum_{n=0}^\infty |L^n f| < \infty.
\]
However, both sums are finite whenever $\sum_{n=0}^\infty \Omega_{f,2}(\lambda^{-n}) < \infty$. This last condition is equivalent to $\int_0^1 \frac{\Omega_{f,2}(s)}{s} ds < \infty$. □

4. Transfer operator, Fourier series and Modulus of continuity

Proposition 4 For $f \in L^1$, we have

$$L^n f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(A^n k) e^{2\pi i \langle k, x \rangle} \quad (\forall n \geq 1).$$

Proof It suffices to prove the expression for $n = 1$. Take a digit set $D^*$ representing $\mathbb{Z}^d/A^* \mathbb{Z}^d$. Assume that $0 \in D^*$. Write

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{2\pi i \langle k, x \rangle} = \sum_{k \in \mathbb{Z}^d} \sum_{\beta \in D^*} \hat{f}(A^* k + \beta) e^{2\pi i \langle A^* k + \beta, x \rangle}.$$ 

We have

$$f(A^{-1}(x + \gamma)) = \sum_{k \in \mathbb{Z}^d} \sum_{\beta \in D^*} \hat{f}(A^* k + \beta) e^{2\pi i \langle A^* k + \beta, A^{-1}(x + \gamma) \rangle}$$

then

$$L f(x) = \frac{1}{q} \sum_{k \in \mathbb{Z}^d} \sum_{\beta \in D^*} \sum_{\gamma \in D} \hat{f}(A^* k + \beta) e^{2\pi i \langle k + A^{-1} \beta, x + \gamma \rangle}$$

$$= \frac{1}{q} \sum_{k \in \mathbb{Z}^d} e^{2\pi i \langle k, x \rangle} \sum_{\beta \in D^*} \hat{f}(A^* k + \beta) e^{2\pi i \langle A^{-1} \beta, x \rangle} \sum_{\gamma \in D} e^{2\pi i \langle \gamma \rangle}$$

So, in order to get

$$L f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(A^* k) e^{2\pi i \langle k, x \rangle}$$

It suffices to note that

$$\frac{1}{q} \sum_{\gamma \in D} e^{2\pi i \langle A^{-1} \beta, \gamma \rangle} = \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta \neq 0. \end{cases}$$

In fact, suppose the above sum is not zero. We have only to show that $\beta = 0$. Since the group $D$ is a product of cyclic groups and $m^{-1} \sum_{j=0}^{m-1} e^{2\pi ijx} = 1$ or 0 for a real number $x$ according to $x \in \mathbb{Z}$ or not, we must have $\langle A^{-1} \beta, \gamma \rangle$ is
an integer for any cyclic group generator γ'. Then \( \langle A^{*-1} \beta, \gamma \rangle \) is an integer for any \( \gamma \in D \). Let \( z = \gamma + Ak \) with \( \gamma \in D \) and \( k \in \mathbb{Z}^d \). Then
\[
\langle A^{*-1} \beta, z \rangle = \langle A^{*-1} \beta, \gamma \rangle + \langle \beta, k \rangle \equiv 0 \pmod{\mathbb{Z}}.
\]
It follows that \( \beta \equiv 0 \pmod{A^* \mathbb{Z}^d} \). So \( \beta = 0 \).

\[\Box\]

**Notation:** "\( a_n \approx b_n \)" means there are constants \( C_1 > 0, C_2 > 0 \) such that
\[
C_1 a_n \leq b_n \leq C_2 a_n \forall n \geq 1.
\]

From Theorem 2 and Proposition 1, we get
\[
\|L^n f\|_2 \approx \left( \sum_{k \in \mathbb{Z}^d} |\hat{f}(A^* k)|^2 \right)^{1/2} \leq C \Omega_{f, 2}(\lambda^{-n}).
\]

\[
\|L^n f\|_\infty \leq \sum_{k \in \mathbb{Z}^d} |\hat{f}(A^* k)| \leq C \Omega_{f, 2}(\lambda^{-n}).
\]

These inequalities may become "\( \approx \)" for some functions \( f \). Let us consider the class of functions defined by lacunary trigonometric series
\[
H(x) = \sum_{k=1}^\infty a_n e^{2\pi i (A^* h, x)} \quad (h \in \mathbb{Z}^d \setminus \{0\}).
\]

Since \( \{A^* h\}_{k \geq 1} \) is a Sidon set [7], we have
\[
\|L^n H\|_r \approx \|L^n H\|_2 \quad (1 \leq r < \infty)
\]
and
\[
\|L^n H\|_\infty \approx \sum_{k=n+1}^\infty |a_k| \leq C \Omega_{H}(\lambda^{-n}) \quad (3)
\]
\[
\|L^n H\|_2 \approx \left( \sum_{k=n+1}^\infty |a_k|^2 \right)^{1/2} \leq C \Omega_{H, 2}(\lambda^{-n}). \quad (4)
\]

Now let us estimate the modulus of continuity of \( H \) by its Fourier coefficients. The following proposition is immediate, just because
\[
|H(x + y) - H(x)| \leq \sum_{k=1}^n |a_k| \left| e^{2\pi i (A^* h, y)} - 1 \right| + \sum_{k=n+1}^\infty |a_k|
\]
\[
\int |H(x + y) - H(x)|^2 dx \leq \sum_{k=1}^n |a_k|^2 \left| e^{2\pi i (A^* h, y)} - 1 \right|^2 + \sum_{k=n+1}^\infty |a_k|^2.
\]
Proposition 5 Let $A$ be an expanding integral similarity matrix with spectral radius $\lambda > 1$. Let $H$ be the function defined by the above lacunary trigonometric series. We have

$$\Omega_H(\lambda^{-n}) \leq C \lambda^{-n} \sum_{k=1}^{n} |a_k| \lambda^k + \sum_{k=n+1}^{\infty} |a_k|$$

$$\Omega_{H,2}(\lambda^{-n})^2 \leq C \lambda^{-2n} \sum_{k=1}^{n} |a_k|^2 \lambda^{2k} + \sum_{k=n+1}^{\infty} |a_k|^2$$

where $C > 0$ is a constant.

If $a_n = \frac{1}{n^\alpha}$ with $\alpha > 1$, then

$$\|L^n H\|_\infty \approx \Omega_H(\lambda^{-n}) \approx \frac{1}{n^{\alpha-1}}$$

$$\|L^n H\|_2 \approx \Omega_{H,2}(\lambda^{-n}) \approx \frac{1}{n^{\alpha-1/2}}.$$ 

If $a_n = \frac{1}{n \log n}$ with $\beta > 1$, then

$$\|L^n H\|_\infty \approx \Omega_H(\lambda^{-n}) \approx \frac{1}{(\log n)^{\beta-1}}$$

$$\|L^n H\|_2 \approx \Omega_{H,2}(\lambda^{-n}) \approx \frac{1}{(\log n)^{\beta-1/2}}.$$ 

If $a_n = \theta^n$ with $\frac{1}{\lambda} < \theta < 1$, then

$$\|L^n H\|_\infty \approx \|L^n H\|_2 \approx \Omega_H(\lambda^{-n}) \approx \Omega_{H,2}(\lambda^{-n}) \approx \theta^n.$$ 

In fact, we have only to use (3) and (4) for getting lower bounds and to use Proposition 2 for getting upper bounds. When $a_n = \frac{1}{n^{\alpha}}$, it suffices to remark that (for any $\lambda > 1$ and any $\alpha > 1$)

$$\sum_{k=1}^{n} \frac{\lambda^k}{k^n} \approx \sum_{k=1}^{[n/2]} \frac{\lambda^k}{k^n} + \sum_{k=[n/2]+1}^{n} \frac{\lambda^k}{k^n} \leq C \left( \lambda^{n/2} + \frac{\lambda^n}{n^{\alpha}} \right)$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k^n} \approx \frac{1}{n^{\alpha-1}}.$$
where \( [n/2] \) denotes the integral part of \( n/2 \). In the same way, we treat the case \( a_n = \frac{1}{n \log n} \). The case \( a_n = \theta^n \) is simpler. More generally, suppose \( |a_n| \) is a decreasing sequence such that

\[
\limsup_{n \to \infty} \frac{|a_{[\delta n]}|}{|a_n|} < \infty, \quad \limsup_{n \to \infty} \frac{\lambda^{-(1-\delta)n}}{|a_n|} < \infty
\]

for some \( 0 < \delta < 1 \). Then we have

\[
\Omega_H(\lambda^{-n}) \approx \sum_{k=n+1}^{\infty} |a_k|, \quad \Omega_{H,2}(\lambda^{-n}) \approx \left( \sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1/2}.
\]

Let us finish our discussion by making some remarks:

1. In all three cases of \( H \) discussed above, the estimate provided by Theorem 2 is optimal. Let us point out that \( H \) belongs to the class of functions with \( \Omega_f(\delta) = O(1/|\log \delta|^{\alpha-1}) \) when \( a_n = \frac{1}{n^{\alpha}} \); \( H \) belongs to the class of functions with \( \Omega_f(\delta) = O(1/(\log |\log \delta|)^{\beta-1}) \) when \( a_n = \frac{1}{n \log^{\beta} n} \); \( H \) belongs to the class of functions with \( \Omega_f(\delta) = O(\delta^{\log_{\lambda} \theta}) \) when \( a_n = \theta^n \).

2. For every \( f \in L^r (1 \leq r < \infty) \), \( \|f(\cdot + y) - f(\cdot)\|_r \) is continuous as a function of \( y \). It follows that \( \lim_{\delta \to 0} \Omega_{f,\tau}(\delta) = 0 \). Then, by Theorem 2, \( \|L^n f - \int f \|_r \) tends to zero for any \( f \in L^r \).

3. Let \( 0 < \delta_n < 1 \) be an arbitrary decreasing sequence tending to zero. There is a function \( H \) such that \( \|L^n H\|_\infty \approx \delta_n \). In fact, it suffices to take the function \( H \) defined by \( a_n = \delta_{n-1} - \delta_n \) (with \( \delta_0 = 1 \)). Also, if we take the function \( H \) defined by \( a_n = \sqrt{\delta_{n-1}^2 - \delta_n^2} \), we have \( \|L^n H\|_2 \approx \delta_n \).

4. The function \( H \) defined above with \( a_n = \frac{1}{n^{\alpha}} \) (\( 1 < \alpha \leq 2 \)) or \( a_n = \frac{1}{n \log^{\beta} n} \) (\( \beta > 1 \)) is a continuous function but not of summable variation. However, \( \|L^n f\|_\infty \) tends to zero with a precisely known convergence speed. Such a situation was not seen before.

5. Ulam-von Neumann map

The map \( Uy = 1 - 2y^2 \) from \( I = [-1,1] \) into itself was studied by Ulam and von Neumann [12]. This Ulam-von Neumann map \( U \) is conjugate to the tent map \( Tx = 1 - 2|x| \). More precisely, we have \( U \circ h = h \circ T \) where \( h \) is the conjugacy defined by \( h(x) = \sin \frac{\pi}{2} x \). The Lebesgue measure \( dx \) (normalized so that \( I \) has measure 1) is \( T \)-invariant and its image under \( h, d\mu = \frac{2}{\sqrt{1-y^2}} dy \), is
U-invariant. Consider now the system \((I, U, \mu)\). For this system, the transfer operator is defined by

\[
L_U f(y) = \sum_{z \in U^{-1}y} \frac{f(z)}{|U'(z)|}.
\]

**Theorem 6** For any \(f \in \text{Lip}_\alpha (0 < \alpha < 1)\) with \(\int f \, d\mu = 0\), we have \(\|L_U^n f\|_2 \leq C 2^{-\alpha n}\). For \(g(y) = \log|y| + \log 2\), we have \(\int g \, d\mu = 0\) and \(\|L_U^n g\|_2 \approx 2^{-n}\).

**Proof.** First observe that Theorem 2 remains true for the system \((I, T, dx)\) because \(I\) can be decomposed into \(I = S_0(I) \cup S_1(I)\), where \(S_0x = \frac{x-1}{2}, S_1x = \frac{1-x}{2}\) are inverses of \(T\). Let \(L_T\) be the transfer operator associated to \(T\), which is defined in the same way as \(L_U\). By using the fact that \(L_U\) is the adjoint operator of \(f \rightarrow f \circ U\) acting on \(L^2(\mu)\) and the similar fact about \(L_T\), we get the relation between \(L_U\) and \(L_T\): for any \(g, f \in L^2(\mu)\)

\[
\int g \cdot L_U^n f \, d\mu = \int g \circ U^n \cdot f \, d\mu
\]

\[
= \int g \circ U^n \circ h \cdot f \circ h \, dx = \int g \circ h \circ T^n \cdot f \circ h \, dx
\]

\[
= \int g \circ h \cdot L_T^n(f \circ h) \, dx = \int g \cdot L_T^n(f \circ h) \circ h^{-1} \, d\mu.
\]

It follows that \(\|L_U^n f\|_2 = \|L_T^n(f \circ h)\|_2\). In fact,

\[
\int L_U^n f \cdot \overline{L_U^n f} \, d\mu = \int L_T^n f \cdot \overline{L_T^n f} \circ h^{-1} \, d\mu
\]

\[
= \int L_T^n(f \circ h) \circ h^{-1} \cdot L_T^n(f \circ h) \circ h^{-1} \, d\mu
\]

\[
= \int L_T^n(f \circ h) \cdot L_T^n(\overline{f} \circ h) \, dx.
\]

So, we have only to estimate \(\|L_T^n(f \circ h)\|_2\). To this end, apply Theorem 2 (its variant mentioned above). We are then led to estimate the modulus of continuity of \(f \circ h\). Since \(h\) is Lipschitz, we have \(\Omega_{f \circ h, 2}(\delta) \leq C \Omega_f(\delta) \leq C\Omega_f(\delta)\).

However, \(\log|y|\) is neither continuous nor of bounded variation. But it is in \(L^2(\mu)\), equivalently \(\log|h(x)| \in L^2(dx)\). Note that

\[
\log|h(x)| = \log|\sin \frac{\pi}{2}x| = -\log 2 - \sum_{n=1}^{\infty} \cos \frac{\pi n x}{n}.
\]

It follows that \(\int \log|y| \, d\mu = \int \log|\sin \frac{\pi}{2}x| \, dx = -\log 2\). From the above series and Proposition 1, we get \(\|L_T^n g\|_2 = \|L_T^n g \circ h\|_2 \approx 2^{-n}\). □
Remark that we can also get $\Omega_{gh,2}(\delta) \leq C\sqrt{\delta}$. In fact, for any $u \neq 0$, take $N$ to be the integral part of $1/|u|$. According to Parseval’s equality, we have

$$\int [\log |h(x + u)| - \log |h(x)||^2 dx = 2 \sum_{n=1}^{\infty} \frac{\sin^2 \frac{\pi}{2} nu}{n^2} \leq \frac{\pi^2}{2} N|u|^2 + 2 \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq C\sqrt{|u|}.$$ 

It is known that the Liapunov exponent of $(I, U, \mu)$ is equal to

$$\int \log |U'(y)|d\mu(y) = \log 2.$$ 

That means for $\mu$ almost all points $y$, $\frac{1}{n} \log |(U^n)'(y)|$ converges to $\log 2$. As a consequence of the last theorem, we get that $\frac{1}{n^n}[\log |(U^n)'(y)| - n \log 2]$ converges in law to a centered gaussian variable (following the arguments in the proof of Theorem 3).

The above discussion on Ulam-von Neumann map reveals a possibility to reduce the study of a general system to that of an endomorphism on torus or a system like the tent map. It is the case when there is a conjugacy between the two systems and when the conjugacy has some smoothness so that the modulus of continuity of $f \circ h$ is small.

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