Dynamical system analysis at background and perturbation levels: Quintessence in severe disadvantage comparing to ΛCDM

Spyros Basilakos,1,2 Genly Leon,3 G. Papagiannopoulos,4 and Emmanuel N. Saridakis5,6

1Academy of Athens, Research Center for Astronomy and Applied Mathematics, Soranou Efesiou 4, 11527, Athens, Greece
2National Observatory of Athens, V. Paulou and I. Metaxa 15236, Penteli, Greece
3Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280 Antofagasta, Chile
4Physics Department, University of Athens, Panepistimiopolis, Athens 157 83, Greece
5Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece
6Department of Astronomy, School of Physical Sciences, University of Science and Technology of China, Hefei 230026, China

We perform for the first time a dynamical system analysis of both the background and perturbation equations, of ΛCDM and quintessence scenarios. In the former case the perturbations do not change the stability of the late-time attractor of the background equations, and the system still results in the dark-energy dominated, de Sitter solution, having passed from the correct dark-matter era with $\gamma \approx 6/11$. However, in the case of quintessence the incorporation of perturbations ruins the stability of the background evolution, and the only conditionally stable point corresponds to a dark-matter dominated universe with a dust dark-energy sector, not favored by observations. This result is a severe disadvantage of quintessence cosmology comparing to ΛCDM paradigm.

Introduction – Dynamical system approach is a powerful tool that allows to extract information on the evolution of a cosmological model, independently of the initial conditions or its specific behavior at intermediate times [1]. In particular, although a general cosmological scenario may exhibit an infinite number of possible evolutions, its asymptotic behavior, namely its behavior at late times, can be classified in a few different classes, which correspond to the stable critical points of the autonomous-form transformed cosmological equations. Thus, through such an analysis one obtains information of the late-time universe, bypassing the complications of the cosmological equations, which prevent complete analytical treatments, as well as the ambiguity of the initial conditions.

The dynamical system approach has been applied to numerous cosmological scenarios since the late 90’s (see [2] and references therein), nevertheless up to now it remained only at the background level, namely examining the behavior of the background equations and calculating at the critical points the values of background-related quantities such as the density parameters, the equation-of-state parameter etc. Although this analysis was important and adequate for the earlier cosmology advance, the significantly advancing cosmological progresses and especially the huge amount of data related to perturbations (such as the growth index and the Large Scale Structure), leads to the need to extend the dynamical system approach in order to investigate cosmological scenarios at both the background and perturbation levels.

Dynamical analysis at the background level – Let us briefly review the phase space analysis of ΛCDM paradigm, as well as of the basic dynamical dark energy scenario, namely the quintessence one. Considering a flat Friedmann-Robertson-Walker (FRW) metric $ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j$, the equations of a general cosmological scenario read as

\begin{align}
H^2 &= \frac{\kappa^2}{3} (\rho_m + \rho_d), \\
\dot{H} &= -\frac{\kappa^2}{2} (\rho_m + p_m + \rho_d + p_d),
\end{align}

with $\kappa^2 = 8\pi G$, and where $\rho_m$, $p_m$ are respectively the energy density and pressure of the matter fluid, while $\rho_d$, $p_d$ are the energy density and pressure of the (effective) dark energy fluid. Finally, assuming that interactions do not take place among the cosmic fluid components, the system of equations closes with the conservation equations

\begin{align}
\dot{\rho}_m + 3H(1 + w_m)\rho_m &= 0, \\
\dot{\rho}_d + 3H(1 + w_d)\rho_d &= 0,
\end{align}

where we have introduced the equation-of-state parameters $w_i \equiv p_i/\rho_i$. Note that only three out of four equations (1)-(4) are independent.

The above framework provides ΛCDM cosmology for $\rho_d = -\rho_d = \Lambda/\kappa^2$, with $\Lambda$ the cosmological constant, and in this case Eq. (4) becomes trivial. Additionally, for the case of the basic quintessence scenario, in which a scalar field $\phi$ is introduced, we have $\rho_d = \dot{\phi}^2/2 + V$ and $p_d = \dot{\phi}^2/2 - V$, with $V(\phi)$ its potential, and then Eq. (4) becomes the Klein-Gordon equation $\ddot{\phi} + 3H\dot{\phi} + V' = 0$, with $V'(\phi) \equiv \partial V/\partial \phi$.

The essence of the dynamical system approach is to transform the equations into an autonomous system, using $\tau \equiv \ln a$ as the dynamical variable, extract its critical points, perturbing around them, and investigate their stability by examining the eigenvalues of the involved perturbation matrix [1, 2].

For ΛCDM cosmology the cosmological equations can be transformed into an autonomous form by simply using...
the matter density parameter $\Omega_m \equiv \kappa^2 \rho_m/(3H^2)$ as the auxiliary variable. Thus, Eqs. (1) and (3) give rise to the one-dimensional system

$$\Omega_m' = 3(\Omega_m - 1)\Omega_m,$$

where primes denote derivatives with respect to $\tau$. The system has two critical points, characterized by $\Omega_m = 1$ and $\Omega_m = 0$, and one can see that the former is unstable while the latter stable. Therefore, for $\Lambda$CDM cosmology, the cosmological-constant dominated ($\Omega_m = 0$ according to (1) implies that $\Omega_d \equiv (\kappa^2 \rho_d/3H^2) = 1$), de-Sitter solution is the stable late-time attractor, and thus the universe will result to it independently of the initial conditions and its evolutions at intermediate times. We mention that actually the dynamical system analysis is not needed in this scenario, since the equations are integrable, with the solution

$$\Omega_m = \Omega_{m0} e^{3(1+w_m)\tau(1-\Omega_{m0})} + \Omega_{m0},$$

with $\Omega_{m0}$ the value of $\Omega_m$ at $a = 1$. Hence, we can immediately see that at late times the system always reaches the de-Sitter solution (for matter sectors that do not violate the null energy condition).

In the case of quintessence scenario, and focusing on the basic model where an exponential potential $V = V_0 e^{\lambda \phi}$ for the scalar field is imposed, introducing the auxiliary variables [4]

$$x \equiv \frac{\kappa \phi}{\sqrt{6} H}, \quad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3} H},$$

we result to the dynamical system

$$x' = \frac{3}{2} x (x^2 - y^2 + 1) - 3x + \sqrt{\frac{3}{2}} y^2,$$

$$y' = \frac{3}{2} y (x^2 - y^2 + 1) - \sqrt{\frac{3}{2}} \lambda xy,$$

in terms of which the various density parameters are expressed as $\Omega_d = x^2 + y^2$, $\Omega_m = 1 - \Omega_d$, while $w_d = x^2 - y^2/\sqrt{x^2 + y^2}$.

The critical points of the system (8)-(9), along with their stability conditions and the corresponding values of $\Omega_d$ and $w_d$, for the quintessence scenario, with $\gamma_m \equiv w_m + 1$.

| C.P. | $x$ | $y$ | Existence | Stability | $\Omega_d$ | $w_d$ |
|------|-----|-----|-----------|-----------|----------|------|
| A    | 0   | 0   | Always    | Saddle for $0 < \gamma_m < 2$ | 0        | Undefined |
| B    | 1   | 0   | Always    | Unstable node for $\lambda < \sqrt{6}$ | 1        | 1    |
| C    | -1  | 0   | Always    | Unstable node for $\lambda > -\sqrt{6}$ | 1        | 1    |
| D    | $\lambda/\sqrt{6}$ | $[1 - \lambda^2/6]^{1/2}$ | $\lambda^2 < 6$ | Stable node for $\lambda^2 < 3\gamma_m$ | 1        | $\frac{3\gamma_m}{\lambda^2} - 1$ |
| E    | $(3/2)^{1/2} \gamma_m / \lambda$ | $[3(2 - \gamma_m)(\gamma_m/2\lambda^2)]^{1/2}$ | $\lambda^2 > 3\gamma_m$ | Stable node for $3\gamma_m < \lambda^2 < 24\gamma_m/(9\gamma_m - 2)$ | $3\gamma_m/\lambda^2$ | $w_m$ |

TABLE I: The critical points, their stability conditions (the corresponding eigenvalues are given in [4]), and the values of $\Omega_d$ and $w_d$, for the quintessence scenario, with $\gamma_m = w_m + 1$.
above perturbation equations must be considered alongside the background evolution equations (1)-(4).

Let us first investigate the case of ΛCDM paradigm, which is obtained by the above general framework for \( \delta_d = 0 \) and \( \theta_d = 0 \), alongside \( w_d = -1 \) and \( p_d = \Lambda/\kappa^2 \).

As auxiliary variables we introduce \( \Omega_m \), as well as the variable

\[
U_m = \frac{\delta'_m}{\delta_m}.
\]

Hence, in terms of \( \Omega_m, U_m \), the equations (1)-(4) and (10)-(13) become

\[
\Omega'_m = 3(\Omega_m - 1)\Omega_m, \quad (16)
\]

\[
U'_m = \frac{3}{2}(U_m + 1)\Omega_m - U_m(U_m + 2). \quad (17)
\]

The critical points of the system (16)-(17), along with the corresponding eigenvalues and their stability conditions are presented in Table II. The system admits four critical points, with \( P_3 \) being the stable one. It corresponds to the cosmological-constant dominated, de Sitter solution, which moreover has \( \delta_m = \text{const.} \) (since \( U_m = 0 \)). Similarly, one can observe the saddle point \( P_4 \), which is a matter dominated universe in which the perturbations increase as \( \delta_m \propto e^{g'\tau} = a \) exactly at the critical point. Thus, for ΛCDM cosmology the incorporation of perturbations does not change the late-time attractor of the background evolution.

For completeness we must examine the possibility of critical points that exist at “infinity” and hence that are missed through the above basic analysis. Introducing the transformation \( \{\Omega_m, U_m\} \rightarrow \{\Omega_m, U_m\} \) with \( U_m = \frac{2}{\pi} \arctan(U_m) \), we find that such critical points at infinity do not exist, since \( U'_m|_{C_m=\pm 1} = -2/\pi \neq 0 \).

Finally, we note that in the literature it is standard to consider that in the matter-dominated phase, in which the large scale structure builds up due to the increase of matter perturbations, we have the relation \( d\ln(\delta_m)/d\ln a \simeq \Omega_m \), where \( \gamma \) is the growth index [5], which in our notation becomes just \( U_m \simeq \Omega_m^2 \). Inserting it into (17) we obtain

\[
3\gamma(\Omega_m - 1)\Omega_m^2 + (\Omega_m^2 + 2)\Omega_m^3 - \frac{3}{2}\Omega_m(\Omega_m^3 + 1) = 0, \quad (18)
\]

which expanded around \( \Omega_m = 1 \) leads to

\[
- \left( \frac{11\gamma}{2} - 3 \right)(1 - \Omega_m) + O((1 - \Omega_m)^2) = 0. \quad (19)
\]

As expected the asymptotic value of the growth index is \( \gamma = \frac{6}{11} \). The curve (18) is depicted in Fig. 1 with a thick (brown) line, and as we observe it coincides with the unstable manifold of the matter dominated solution \( P_4 \).

![FIG. 1: The phase-space diagram for ΛCDM cosmology, at both background and perturbation levels. At late times the system is attracted by the de-Sitter point \( P_3 \). The thick line is the curve (18), which coincides with the unstable manifold of the matter dominated solution \( P_4 \), and which for \( \Omega_m \) close to 1 gives analytically \( \gamma = \frac{6}{11} \) as expected (see text).](image)

We mention here that the dynamical system analysis is not needed for ΛCDM cosmology, since even including the perturbations the system remains integrable. In particular, the general solution reads

\[
\Omega_m(\tau) = \frac{\Omega_{m0}}{e^{3\tau}(1 - \Omega_{m0}) + \Omega_{m0}}, \quad (20)
\]

\[
U_m(\tau) = \left\{ 2\Omega_m(2U_m + 3\Omega_m^3)\left(\Omega_{m0}^2 - \Omega_m^2g\right) + 8(1 - \Omega_{m0})^{5/6}\Omega_m^{2/3} \right\}^{-1}
\]

\[
\left\{ 3\Omega_m(2U_m + 3\Omega_m^3)\left(\Omega_m^{2/3} - \Omega_{m0}^2g\right) + 4\Omega_m^{2/3}\left[(2U_m + 3\Omega_m^3)(1 - \Omega_m)^{5/6}ight. \right.
\]

\[
\left. - 3\Omega_m^{2/3}\Omega_{m0}^2(1 - \Omega_{m0})^{5/6} \right\} \right\}, \quad (21)
\]

where \( g(\tau) = 2F_1 \left[ \frac{1}{3}, \frac{2}{3}; \frac{5}{3}; \Omega_m(\tau) \right] \) and \( \gamma_0 = 2F_1 \left[ \frac{1}{3}, \frac{2}{3}; \frac{5}{3}; \Omega_{m0} \right] \), with \( \Omega_{m0} \) and \( U_{m0} \) the values of \( \Omega_m \) and \( U_m \) at \( \tau = 0 \) (i.e. at \( a = 1 \)). From the analytical solutions (20),(21) we can easily see that for \( \tau \rightarrow \infty \) we have \( \Omega_m \rightarrow 0 \) and \( U_m \rightarrow 0 \), i.e. the system results to the de Sitter point \( P_3 \).

We now proceed to the investigation of perturbations in quintessence. Having in mind the \( x, y \) given in (7), we will use the auxiliary variables

\[
\gamma_\phi = \frac{2x^2}{x^2 + y^2}, \quad Q = \frac{\delta_d}{\delta_m}, \quad U_m = \frac{\delta'_m}{\delta_m}, \quad V_d = \frac{d\ln \delta_d}{d\ln \delta_m}, \quad (22)
\]

and we introduce the time derivative \( f' = \gamma_\phi \frac{df}{d\tau} \), where \( \tau = \ln a \), in terms of which the cosmological equations (1)-(4) and (10)-(13) lead to the five-dimensional au-

| C.P. | \( \Omega_m \) | \( U_m \) | Existence | Eigenvalues | Stability |
|------|-------------|------------|-----------|-------------|-----------|
| \( P_3 \) | 0 | -2 | Always | \(-3, 2\) | Saddle |
| \( P_2 \) | 1 | -\frac{2}{3} | Always | \(3, \frac{2}{3}\) | Unstable |
| \( P_3 \) | 0 | 0 | Always | \(-3, -2\) | Stable |
| \( P_4 \) | 1 | 1 | Always | \(3, -\frac{3}{2}\) | Saddle |
TABLE III: The physical (real with $0 \leq \Omega_m \leq 1$) critical points ($O$ is a curve of critical points), their stability conditions, and their properties, of both background and perturbation equations, in the case of quintessence. The functions $\{F_i(\lambda), G_i(\lambda), J_i(\lambda)\}$ are lengthy expressions not presented explicitly for simplicity. The stability conditions arise from the examination of the sign of the eigenvalues (not presented explicitly for simplicity) of the involved perturbation matrix, which for points $D_1,D_2,E_2$ is performed semi-analytically.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{C.P. } \{x, \gamma_\phi\} & \{Q,U_m,V_d\} & \Omega_m & w_d & \text{Stability} \\
\hline
O & \{0,0\} & \{Q,U_m,V_d\} & 0 \leq \Omega_m \leq 1 & -1 & \text{non-stable} \\
A_1 & \{0,2\} & \{1,-1,-1\} & 1 & \text{Undefined} & \text{non-stable} \\
A_2 & \{0,2\} & \{-\frac{5}{3},-1,\frac{2}{3}\} & 1 & \text{Undefined} & \text{non-stable} \\
A_2^+ & \{0,2\} & \{0,0,\frac{2}{3}\pm\frac{2}{3}\} & 1 & \text{Undefined} & \text{non-stable} \\
B_+ & \{1,2\} & \{-\frac{11}{12}, \frac{5}{6} \{2 \pm \sqrt{6}, 2 \pm \sqrt{6} \{3 \pm \sqrt{6}\}\} & 0 & 1 & \text{non-stable} \\
C^\pm & \{-1,2\} & \{-\frac{11}{12}, \frac{5}{6} \{2 \pm \sqrt{6}, 2 \pm \sqrt{6} \{3 \pm \sqrt{6}\}\} & 0 & 1 & \text{non-stable} \\
D_1 & \{\frac{1}{\sqrt{2}}, 1\} & \{F_1(\lambda), G_1(\lambda), J_1(\lambda)\} & 0 & -1 + \frac{x^2}{\Omega} & \text{non-stable} \\
D_2 & \{\frac{1}{\sqrt{2}}, 1\} & \{F_3(\lambda), G_3(\lambda), J_3(\lambda)\} & 0 & -1 + \frac{x^2}{\Omega} & \text{non-stable} \\
E_1 & \{\frac{1}{\sqrt{2}}, 1\} & \{-\frac{3}{\Omega^{1/2}}, 0, 0\} & 1 - \frac{3}{\Omega^{1/2}} & 0 & \text{non-stable} \\
E_2 & \{\frac{1}{\sqrt{2}}, 1\} & \{F_3(\lambda), G_3(\lambda), J_3(\lambda)\} & 1 - \frac{3}{\Omega^{1/2}} & 0 & \text{stable for } |\lambda| > 3.99631 \text{ (See Fig. 2)} \\
\hline
\end{array}
\]

Note that since $\gamma_\phi = w_d + 1$ and $\Omega_m = 1 - \frac{2x^2}{\gamma_\phi}$, the first two equations, namely (23),(24), which are decoupled from the rest, coincide with Eqs. (8)-(9) of the background analysis. Finally, in terms of the above auxiliary variables the growth index becomes just $\gamma = \frac{\ln U_m}{\ln(\Omega_m)}$.

The physical (real with $0 \leq \Omega_m \leq 1$) critical points and their stability conditions are presented in Table III. Since Eqs. (23),(24) are decoupled from the rest, the scenario as hand admits the four critical points of the background level presented in Table I, expressed in the current auxiliary variables, each of which is now split into more points due to the remaining three variables, plus the extra curve of critical points $O$.

FIG. 2: The real part of the five eigenvalues $\mu$ corresponding to $E_2$ versus $\lambda$, for $-3.99631 < \lambda < 3.99631$ (upper graph), for $\lambda < -3.99631$ (middle graph), and for $\lambda > 3.99631$ (lower graph). $\text{Re}(\mu) < 0$ for all eigenvalues in the case $|\lambda| > 3.99631$.

\[
x' = \frac{1}{2}x \left[-3\gamma_\phi + 6(\gamma_\phi - 2)x^2 - \sqrt{6}(\gamma_\phi - 2)x\right], \quad (23)
\]

\[
\gamma_\phi' = (\gamma_\phi - 2)\gamma_\phi \left(3\gamma_\phi - \sqrt{6}\lambda x\right), \quad (24)
\]

\[
Q' = -\gamma_\phi QU_m(1 - V_d), \quad (25)
\]

\[
U'_m = \frac{1}{2} \left[6x^2(Q - \gamma_\phi U_m + U_m - 1) - \gamma_\phi (2U_m^2 - U_m - 3)\right], \quad (26)
\]

\[
V'_d = \frac{x}{U_m} \left[2\sqrt{6}\lambda(U_m V_d + 1) - 3x(2U_m V_d - V_d + 1)\right]
+ \frac{\gamma_\phi}{2U_m} \left\{18x^2 + 12U_m V_d x^2 - 2\sqrt{6}\lambda x(U_m V_d + 1) - V_d\{2U_m[U_m(V_d - 1) + 10] + 3\} - 13\right\}
- \frac{3QV_d x^2}{U_m} - \frac{3\gamma_\phi (2x^2 - \gamma_\phi)}{2QU_m}. \quad (27)
\]

tononomous dynamical system: $\quad x' = \frac{1}{2}x \left[-3\gamma_\phi + 6(\gamma_\phi - 2)x^2 - \sqrt{6}(\gamma_\phi - 2)x\right], \quad (23)$
exists for $|\lambda| > \sqrt{\frac{3}{2}}$, a semi-analytical elaboration of its eigenvalues, using the characteristic polynomial of the linearization matrix and the Descartes rule, shows that it is stable if $|\lambda| > 3.99631$ (see Fig. 2). Finally, the analysis at infinity shows that stable critical points do not exist, too.

Hence, as we observe, the incorporation of perturbation ruins the stability of the dark-energy dominated, de-Sitter solution, which is the physically interesting late-time attractor of the background equations, and leaves only one conditionally stable point which corresponds to dark matter domination with dark energy behaving like dust and therefore not favored by observations. To make things worse, even this point does not exist for $|\lambda| < \sqrt{\frac{3}{2}}$ which (although consistent with swampland bounds) is the region favored by the data for a successful quintessence.

**Conclusions** – We performed for the first time a dynamical system analysis of both the background and perturbation equations, of $\Lambda$CDM and quintessence scenarios. In the former case, the incorporation of perturbations does not change the stability of the late-time attractor of the background equations, and the system still results in the dark-energy dominated, de Sitter solution, having passed from the correct dark-matter era with $\gamma \approx \frac{6}{11}$ (actually in this scenario one extracts analytical solutions). However, in the case of quintessence, the incorporation of perturbation ruins the stability of the background evolution, and the only conditionally stable point corresponds to a dark-matter dominated universe with a dust dark-energy sector, not favored by observation. This can only become worse if one extends the analysis in the non-standard cases of a general $w_m$ or $c^2_{\text{eff}}$. In summary, the above results are a severe disadvantage of quintessence cosmology comparing to $\Lambda$CDM paradigm.

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