GEOMETRY OF $Q$-RECURRENT MAPS

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Abstract. Given a critically periodic quadratic map with no secondary renormalizations, we introduce the notion of $Q$-recurrent quadratic polynomials. We show that the pieces of the principal nest of a $Q$-recurrent map $f_c$ converge in shape to the Julia set of $Q$. We use this fact to compute analytic invariants of the nest of $f_c$, to give a complete characterization of complex quadratic Fibonacci maps and to obtain a new auto-similarity result on the Mandelbrot set.

1. Introduction

The principal nest of a quadratic polynomial is a collection of pieces of the puzzle with good shrinking properties (see [L3]). In [P] we proposed a supplementary construction, the frame of the nest, to help classify different nest types. Based on that classification, we introduce in this paper the family of $Q$-recurrent maps. In essence, these are polynomials for which the nest frames are combinatorially modeled on the puzzle of a chosen critically periodic quadratic polynomial $Q$. We will generalize to the family of $Q$-recurrent maps some results of M. Lyubich and L. Wenstrom that concern the real Fibonacci parameter $c_{\text{fib}}$. It is shown in [L1] that the central pieces of the nest of $f_{c_{\text{fib}}}$ are asymptotically similar to the Julia set of $z^2 - 1$. This fact is expanded in [W], where the shape of pieces is exploited to compute the exact rate of growth of the principal nest moduli. Wenstrom translates these results to the parameter plane in order to obtain the corresponding shape of parapieces and the rate of paramoduli growth around $c_{\text{fib}}$ on the Mandelbrot set $M$.

The corresponding results for $Q$-recurrent maps are as follows:

Theorem. Let $c$ be the center of a prime component of $M$ and $Q$ its associated quadratic polynomial. Then the nest pieces of a $Q$-recurrent map converge in shape to the Julia set $K_Q$.

The asymptotic shape of pieces is used to derive analytic information about the nest. In the simplest case, $Q(z) = z^2 - 1$, the family of $(z^2 - 1)$-recurrent maps has an interesting description.

Theorem. A quadratic polynomial is $(z^2 - 1)$-recurrent if and only if it is complex Fibonacci. Moreover, the principal moduli of any map in this family grow linearly with growth rate $\ln 2/3$.

The corresponding result is different when the critical point of $Q$ has period $\geq 3$.

Theorem. For $Q(z) \neq z^2 - 1$ the principal moduli of a $Q$-recurrent map grow exponentially with a rate that depends on the period of $Q$.

When the shape results are translated to the Mandelbrot set, the above statements find parametric counterparts.

Theorem. The paranest pieces around a $Q$-recurrent parameter $c$ converge in shape to $K_Q$. The growth of paramoduli around $c$ is as in the corresponding dynamical plane. The family of $Q$-recurrent parameters forms a dyadic Cantor set of Hausdorff dimension 0.

As an application of parapiece shape, an easy diagonal argument yields a powerful auto-similarity result on $M$.

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Theorem. Let \( c_1, c_2 \in \partial M \) be two parameters such that \( f_{c_2} \) has no indifferent periodic orbits that are rational or linearizable. Then there exists a sequence of parapieces \( \{ \Upsilon_1, \Upsilon_2, \ldots \} \) converging to \( c_1 \) as compact sets, but such that \( \Upsilon_n \) converges to \( K_{c_2} \) in shape.

1.1. Paper structure. The puzzle of Yoccoz, the principal nest and their parametric counterparts are defined in Section 2 as a means to introduce notation. The adjacency graphs introduced in Subsection 2.4 are used in Section 3 to define the frame of a principal nest, and in Section 4 to describe \( Q \)-recurrent behavior.

In Section 5 we present the classification of complex Fibonacci quadratic polynomials and the results on the shape of pieces and growth of moduli for the nest of \( Q \)-recurrent maps.

The corresponding results on parametric pieces of the Mandelbrot set \( M \) are presented in Section 6. Theorem 6.3 introduces a new similarity phenomenon between different locations of \( \partial M \).

An appendix summarizes the tools borrowed from complex variables. This includes an extension of the Grötzsch inequality and brief discussions of Carathéodory topology, Koebe’s distortion lemma and the Teichmüller space of a surface.

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2. Basic notions

2.1. Basic complex dynamics. In order to fix notation, let us start by defining the basic notions of complex dynamics that will be used; the reader is referred to [DHI] and [MII] for details on this introductory material.

We focus attention on the \textit{quadratic family} \( Q := \{ f_c : z \mapsto z^2 + c \mid c \in \mathbb{C} \} \). For every \( c \), the compact sets \( K_c := \{ z \mid \text{the sequence } \{ f_c^n(z) \} \text{ is bounded} \} \) and \( J_c := \partial K_c \) are called the \textit{filled Julia set} and \textit{Julia set} respectively. Depending on whether the orbit of the critical point 0 is bounded or not, \( J_c \) and \( K_c \) are connected or totally disconnected. The \textit{Mandelbrot set} is defined as \( M := \{ c \mid c \in K_c \} \); that is, the set of parameters with bounded critical orbit.

A component of \( \text{int} M \) that contains a parameter with an attracting periodic orbit will be called a \textit{hyperbolic component}\footnote{Though, of course, it is conjectured that all interior components are hyperbolic.}. The boundary of a hyperbolic component can either be real analytic, or fail to be so at one cusp point. The later kind are called \textit{primitive} components. In particular, the hyperbolic component \( \vee \) associated to \( z \mapsto z^2 \) is bounded by a cardioid known as the \textit{main cardioid}.

\( M \) contains infinitely many small homeomorphic copies of itself, accumulating densely around \( \partial M \). In fact, every hyperbolic component \( H \) other than the main one is the base of one such small copy \( M' \). \( H \) is called \textit{prime} if it is not contained in any other small copy. To simplify later statements, prime components are further subdivided in \textit{immediate} (non-primitive components that share a boundary point with \( \vee \)) and \textit{maximal} (primitive components away from \( \partial \vee \)).

2.2. External rays, wakes and limbs. Since \( f_c^{-1}(\infty) = \{ \infty \} \), the point \( \infty \) is a fixed critical point and a classical result of Böttcher yields a change of coordinates that conjugates \( f_c \) to \( z \mapsto z^2 \) in a neighborhood of \( \infty \). With the requirement that the derivative at \( \infty \) is 1, this conjugating map is denoted \( \varphi_c : N_c \rightarrow \overline{\mathbb{C}} \setminus \mathbb{D}_R \), where \( \mathbb{D}_R \) is the disk of radius \( R \geq 1 \) and \( N_c \) is the maximal domain of unimodality for \( \varphi_c \). It can be shown that \( N_c = \overline{\mathbb{C}} \setminus K_c \) and \( R = 1 \) whenever \( c \in M \). Otherwise, \( N_c \) is the exterior of a figure 8 curve that is real analytic and symmetric with respect to 0. In this case, \( R > 1 \) and \( K_c \) is contained in the two bounded regions determined by the 8 curve.

Consider the system of radial lines and concentric circles in \( \mathbb{C} \setminus \mathbb{D}_R \) that characterizes polar coordinates. The pull-back of these curves by \( \varphi_c \), creates a collection of \textit{external rays} \( r_\theta \ (\theta \in [0,1]) \).
and equipotential curves $e_s$ (here $s \in (R, \infty)$ is called the radius of $e_s$) on $N_c$. These form two orthogonal foliations that behave nicely under dynamics: $f_c(r_\theta) = r_{2\theta}$, $f_c(e_\alpha) = e_{2\alpha}$. When $c \in M$, we say that a ray $r_\theta$ lands at $z \in J_c$ if $z$ is the only point of accumulation of $r_\theta$ on $J_c$.

A similar coordinate system exists around the Mandelbrot set. For $c \notin M$, define the map

$$\Phi_M(c) := \varphi_c(c).$$

In [DH1] it is shown that $\Phi_M: \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$ is a conformal homeomorphism tangent to the identity at $\infty$. This yields connectivity of $M$ and allows us to define parametric external rays and parametric equipotentials as in the dynamical case. Since there is little risk of confusion, we will use the same notation $(r_\theta, e_\alpha)$ to denote these curves and say that a parametric ray lands at a point $c \in \partial M$ if $c$ is the only point of accumulation of the ray on $M$.

For the rest of this work, all rays considered, whether in dynamical or parameter plane, will have rational angles. These are enough to work out our combinatorial constructions and satisfy rather neat properties.

**Proposition 2.1.** ([M], ch.18) Both in the parametric and the dynamical situations, if $\theta \in \mathbb{Q}$ the external ray $r_\theta$ lands. In the dynamical case, the landing point is (pre-)periodic with the period and preperiod determined by the binary expansion of $\theta$. A point in $J_c$ (respectively $\partial M$) can be the landing point of at most, a finite number of rays (respectively parametric rays). If this number is larger than 1, each component of the plane split by the landing rays will intersect $J_c$ (respectively $\partial M$).

Unless $c = \frac{1}{2}$, $f_c$ has two distinct fixed points. If $c \in M$, these can be distinguished since one of them is always the landing point of the ray $r_0$. We call this fixed point $\beta$. The second fixed point is called $\alpha$ and can be attracting, indifferent or repelling, depending on whether the parameter $c$ belongs to $\mathcal{H}$, $\partial \mathcal{H}$, or $\mathbb{C} \setminus \mathcal{H}$. The map $\psi_0: \mathcal{H} \to \mathbb{D}$ given by $c \mapsto f_c'(0, c)$ is the Riemann map of $\mathcal{H}$ normalized by $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. Since the cardioid is a real analytic curve except at $\frac{1}{4}$, $\psi_0$ extends to $\partial \mathcal{H}$.

The fixed point $\alpha$ is parabolic exactly at parameters $c_\eta \in \partial \mathcal{H}$ of the form $c_\eta = \psi_0^{-1}(e^{2\pi i \eta})$ where $\eta \in \mathbb{Q} \cap [0, 1]$. If $\eta \neq 0$, $c_\eta$ is the landing point of two parametric rays at angles $t^{-}(\eta)$ and $t^{+}(\eta)$.

**Definition 2.1.** The closure of the component of $\mathbb{C} \setminus \{r_{t^{-}(\eta)} \cup c_\eta \cup r_{t^{+}(\eta)}\}$ that does not contain $\mathcal{H}$ is called the $\eta$-wake of $M$ and is denoted $W_\eta$. The $\eta$-limb is defined as $L_\eta = M \cap W_\eta$.

**Definition 2.2.** Say that $\eta = \frac{p}{q}$ written in lowest terms. Then $\mathcal{P}(\frac{p}{q})$ will denote the unique set of angles whose behavior under doubling is a cyclic permutation with combinatorial rotation number $\frac{p}{q}$.

If $\mathcal{P}(\frac{p}{q}) = \{t_1, \ldots, t_q\}$, then for any parameter $c \in L_{p/q}$ the corresponding point $\alpha$ splits $K_c$ in $q$ parts, separated by the $q$ rays $\{r_{t_1}, \ldots, r_{t_q}\}$ landing at $\alpha$. The two rays whose angles span the shortest arc separate the critical point 0 from the critical value $c$. These two angles turn out to be $t^{-}(\frac{p}{q})$ and $t^{+}(\frac{p}{q})$.

**2.3. Yoccoz puzzles.** The Yoccoz puzzle is well defined for parameters $c \in L_{p/q}$ for any any $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$ with $(p, q) = 1$. If 0 is not a preimage of $\alpha$, the puzzle is defined at infinitely many depths and we will restrict attention to these parameters. Since we describe properties of a general parameter, it is best to omit subscripts and write $f$ instead of $f_c$, $K$ instead of $K_c$ and so on.

Let us fix the neighborhood $U$ of $K$ bounded by the equipotential of radius 2. The rays that land at $\alpha$ determine a partition of $U \setminus \{r_{t_1}, \ldots, r_{t_q}\}$ in $q$ connected components. We will call the closures $Y^{(0)}_0, Y^{(0)}_1, \ldots, Y^{(0)}_{q-1}$ of these components, puzzle pieces of depth 0. At this stage the labeling is chosen so that $0 \in Y^{(0)}_0$ and $f \left( K \cap Y^{(0)}_j \right) = K \cap Y^{(0)}_{j+1}$; where the subindices are understood as
residues modulo \( q \). In particular, \( Y_1^{(0)} \) contains the critical value \( c \) and the angles of its bounding rays turn out to be \( t^-(\ell_0^q), t^+(\ell_0^q) \).

The puzzle pieces \( Y_j^{(n)} \) of higher depths are recursively defined as the closures of every connected component in \( f^{\circ(n)}(\bigcup \text{int } Y_j^{(0)}) \); see Figure 11. At each depth \( n \), there is a unique piece which contains the critical point and we will always choose the indices so that \( 0 \in Y_0^{(n)} \).

Let us denote by \( P_n \) the collection of pieces of level \( n \). The resulting family \( \mathcal{Y}_c := \{P_0, P_1, \ldots \} \) of puzzle pieces of all depths, has the following two properties:

**P1** Any two puzzle pieces either are nested (with the piece of higher depth contained in the piece of lower depth), or have disjoint interiors.

**P2** The image of any piece \( Y_j^{(n)} (n \geq 1) \) is a piece \( Y_i^{(n-1)} \) of the previous depth \( n-1 \). The restricted map \( f : \text{int } Y_j^{(n)} \rightarrow \text{int } Y_i^{(n-1)} \) is a 2 to 1 branched covering or a conformal homeomorphism, depending on whether \( j = 0 \) or not.

These properties characterize \( \mathcal{Y}_c \) as a Markov family and endow the puzzle partition with dynamical meaning.

Note that the collection of ray angles at depth \( n \) consists of all \( n \)-preimages of \( \{r_{t_1}, \ldots, r_{t_q}\} \) under angle doubling. The union of all pieces of depth \( n \) is the region enclosed by the equipotential \( e_{(2^{n-1})} \). Note also that every piece \( Y \) of depth \( n \) is the \( n \)-th preimage of some piece of level 0. By further iteration, \( Y \) will map onto a region determined by the same rays as \( Y_0^{(0)} \) and a possibly larger equipotential. This provides a 1 to 1 correspondence between puzzle pieces and preimages of 0. The distinguished point inside each piece is called the center of the piece.

### 2.4. Adjacency Graphs

Given a set of puzzle pieces \( P \subset P_n \), define the dual graph \( \Gamma(P) \) as a formal graph whose set of vertices is \( P \) and whose edges join pairs of pieces that share an arc of external ray. Due to its finiteness, it is always possible to produce an isomorphic model of \( \Gamma(P) \) sitting in the plane, without intersecting edges and such that it respects the natural immersion of \( \Gamma(P) \) in the plane.

**Definition 2.3.** When \( P = P_n \), we call \( \Gamma_n := \Gamma(P_n) \) the **puzzle graph** of depth \( n \). In this context, the vertices corresponding to the central piece \( Y_0^{(n)} \) and the piece around the critical value \( f_c(0) \) are denoted \( \xi_n \) and \( \eta_n \) respectively.

**Definition 2.4.** The vertices \( \xi_n \) and \( \eta_n \) determine two partial orders on the vertex set of \( \Gamma_n \) as follows: If \( a, b \in V(\Gamma_n) \), we write \( a \succ \eta_n b \) when every path from \( a \) to \( \eta_n \) passes through \( b \). We write \( a \succ \xi_n b \) when every path from \( a \) to \( \xi_n \) passes through \( b \) or through its symmetric image with respect to the origin.

The following are natural consequences of the definitions; see Figure 11 for reference.

**Proposition 2.2.** The puzzle graphs of \( f \) satisfy:

- **G1** \( \Gamma_n \) has 2-fold central symmetry around \( \xi_n \).
- **G2** \( \Gamma_0 \) is a \( q \)-gon whenever \( c \in L_{p/q} \). For \( n \geq 1 \), \( \Gamma_n \) consists of \( 2^n \) \( q \)-gons linked at their vertices in a tree-like structure; i.e. the only cycles on this graph are the \( q \)-gons themselves.
- **G3** For \( n \geq 1 \), removing \( \xi_n \) and its edges splits \( \Gamma_n \) into 2 disjoint (possibly disconnected) isomorphic graphs. Reattaching \( \xi_n \) to each, and adding the corresponding edges defines the connected graphs \( \text{Puzz}_n \) and \( \text{Puzz}_n^+ \) (here, \( \eta_n \in \text{Puzz}_n^+ \)). Then \( \Gamma_n = \text{Puzz}_n \cup \text{Puzz}_n^+ \) and \( \text{Puzz}_n, \text{Puzz}_n^+ \) are isomorphic to \( \Gamma_{n-1} \) with \( \pm \eta_n \) playing the role of \( \xi_{n-1} \) in \( \text{Puzz}_n^+ \).
- **G4** For \( n \geq 1 \) there are two natural maps: \( f^* : \Gamma_n \rightarrow \Gamma_{n-1} \) induced by \( f \), and \( \iota^* : \Gamma_n \rightarrow \Gamma_{n-1} \) induced by the inclusion among pieces of consecutive depths. \( f^* \) is 2 to 1 except at \( \xi_n \) and sends \( \text{Puzz}_n^+ \) onto \( \Gamma_{n-1} \). In turn, \( \iota^* \) collapses the outermost \( q \)-gons into vertices.
**G5**  The map \( f^* : (\Gamma_n, \succ_{\xi_n}) \rightarrow (\Gamma_{n-1}, \succ_{\eta_{n-1}}) \) respects order. That is, if \( a \succ_{\xi_n} b \) then \( f^*(a) \succ_{\eta_{n-1}} f^*(b) \).

**Definition 2.5.** Let \( \Gamma \) be a graph isomorphic to a subgraph of \( \Gamma_n \) and \( \Gamma' \) a graph isomorphic to a subgraph of \( \Gamma_{n-1} \). A map \( E : \Gamma \rightarrow \Gamma' \) that satisfies \( G1 \) and \( G2 \) will be called **admissible** if it also respects order in the sense of \( G5 \).

**Proof of Proposition 2.2.** Property \( G1 \) and the existence of \( f^* \) and \( \iota^* \) are immediate consequences of the structure of quadratic Julia sets. The configuration of \( \Gamma_0 \) is given by the rotation number around \( \alpha \) and then the tree-like structure of \( \Gamma_n (n \geq 1) \) follows from \( G3 \).

Consider a centrally symmetric simple curve \( \gamma \subset Y_0^{(n)} \) connecting two opposite points of the equipotential curve \( e_{(2^n-n)} \) that bounds \( Y_0^{(n)} \). Then \( \gamma \) splits the simply connected region \( \bigcup_{Y \in P_n} Y \) in 2 identical parts. Therefore, \( \Gamma \setminus \xi_n \) is formed by 2 disjoint graphs justifying the existence of \( \text{Puzz}_{2,n}^\gamma \). However, \( \partial Y_0^{(n)} \) may contain several segments of \( e_{(2^n-n)} \); so \( \gamma \), and consequently \( \text{Puzz}_{2,n}^\gamma \), are not uniquely determined. This ambiguity is not consequential; Lemmas \( 3.4 \) and \( 3.5 \) describe the proper method of handling it.

The fact that \( f \) maps the central piece to a non-central one containing the critical value legitimizes the selection of \( \text{Puzz}_{2,n}^\gamma \) as the unique graph containing \( \eta_n \). By symmetry, every piece of \( P_n \) except the central one has a symmetric partner and they both map in a 1 to 1 fashion to the same piece of \( P_{n-1} \). The isomorphisms in \( G3 \) follow.

If two pieces \( A, B \) of depth \( n \) share a boundary ray, their images will too. Moreover, letting \( A', B' \) be the pieces of depth \( n - 1 \) containing \( A \) and \( B \), it is clear that \( \partial A' \) and \( \partial B' \) must share the **same** ray as \( \partial A \) and \( \partial B \). This shows that \( f^* \) and \( \iota^* \) effectively preserve edges and are well define graph maps. Clearly \( f^* \) is 2 to 1, so to complete the proof of \( G4 \) it is only necessary to justify the collapsing property of \( \iota^* \), and by Property \( G3 \) it is sufficient to consider the case \( \iota^* : \Gamma_1 \rightarrow \Gamma_0 \).

Now the non-critical piece \( Y_j^{(0)} \) contains a unique piece \( Y_j \) of \( P_1 \). However, the critical piece \( Y_0^{(0)} \) contains a total of \( q \) different pieces of depth 1: a smaller central piece \( Y_0^{(1)} \) and \( q - 1 \) lateral pieces \( -Y_j \). The resulting graph, \( \Gamma_1 \), consists then of two \( q \)-gons joined at the vertex \( \xi_1 \). Under \( \iota^* \), one of these \( q \)-gons collapses on the critical vertex \( \xi_0 \).

To prove \( G5 \) let us construct the tree \( \Gamma'_n \) with 2 to 1 central symmetry by collapsing every \( q \)-gon into a single vertex. The orders \( \succ_{\xi_n}, \succ_{\eta_n} \) in \( \Gamma'_n \) are induced by the orders in \( \Gamma_n \). Then the corresponding map \( f^{**} : (\Gamma'_n, \succ_{\xi_n}) \rightarrow (\Gamma'_{n-1}, \succ_{\eta_n}) \) is a 2 to 1 map on trees that takes each

**Figure 1.** Puzzle of depth 2 and its corresponding graph. Splitting the graph at \( \xi_2 \) produces the graphs \( \text{Puzz}_{2,0}^\gamma \) and \( \text{Puzz}_{2,1}^\gamma \); both shaped like a bow tie and isomorphic to \( \Gamma_1 \).
half of $\Gamma_{n}'$ injectively into a sub-tree of $\Gamma_{n-1}'$ and respects order. Since vertices in a cycle are not ordered, $f^*$ respects order as well.

2.5. **Parapuzzle.** While the puzzle encodes the combinatorial behavior of the critical orbit for a specific map $f_c$, the **parapuzzle** dissects the parameter plane into regions of parameters that share similar behaviors: In every wake of $M$ we define a partition in pieces of increasing depths, with the property that all parameters inside a given **parapiece** share the same critical orbit pattern up to a specific depth.

**Definition 2.6.** Consider a wake $W_{p/q}$ and let $n \geq 0$ be given. Call $W_n$ the wake $W_{p/q}$ truncated by the equipotential $e(x^2 - n)$ and consider the set of angles $P_n(\overline{a}) = \{t \mid 2^n t \in P(\overline{a})\}$ (compare Subsection 2.2). The **parapieces** of $W_{p/q}$ at depth $n$ are the closures of the components of $W_n \setminus \{r_t \mid t \in P_n(\overline{a})\}$.

**Note.** Even though the critical value $f_c(0)$ is simply $c$, it will be convenient to write $c \in \Delta$ when $\Delta$ is a parapiece and $f_c(0) \in V$ when $V$ is a piece in the dynamical plane of $f_c$. In general, we will use the notation OBJ[$c$] to refer to dynamically defined objects OBJ associated to a specific parameter $c$.

**Definition 2.7.** When the boundary of a dynamical piece $A$ is described by the same equipotential and ray angles as those of a parapiece $B$, this relation is denoted by $\partial A \bowtie \partial B$.

**Definition 2.8.** Let $c \in M$ be a parameter whose puzzle is defined up to depth $n$. Denote by $CV_n[c] \in P_n[c]$ the piece of depth $n$ that contains the critical value: $f_c(0) \in CV_n[c]$.

A consequence of Formula 1 is the well known fact that follows. For a proof of the main statement, refer to [DH2] or [R]. For a proof of the winding number property, refer to [D2] and Proposition 3.3 of [L4].

**Proposition 2.3.** Let $\Delta$ be a parapiece of depth $n$ in some wake $W$. Then $CV_n[c] \cong \Delta$ for every $c \in \Delta$. The family $\{c \mapsto CV_n[c] \mid c \in \Delta\}$ is well defined and it determines a holomorphic motion of the critical value pieces. The holomorphic motion has $\{c \mapsto f_c(0)\}$ as a section with winding number $1$.

The result on winding number can be interpreted as loosely saying that, as $c$ goes once around $\partial \Delta$, the critical value $f_c(0)$ goes once around $\partial CV_n$. However, this description is not entirely accurate since $\partial CV_n[c]$ changes with $c$.

Let us mention the following examples of combinatorial properties that depend on the behavior of the first $n$ iterates of 0. The fact that these entities remain unchanged for $c \in \Delta$ follows from Proposition 2.3 and will be useful in the next sections.

- The isomorphism type of $\Gamma_n[c]$.
- The combinatorial boundary of every piece of depth $\leq n$.
- The location within $P_n[c]$ of the first $n$ iterates of the critical orbit.

From the general results of [L4], we can say more about the geometric objects associated to the above examples.

**Proposition 2.4.** Each of the sets listed below moves holomorphically as $c$ varies in $\Delta$:

- The boundary of every piece of depth $\leq n$.
- The first $n$ iterates of the critical orbit.
- The collection of $j$-fold preimages of $\alpha$ and $\beta$ ($j \leq n$).
2.6. **The principal nest.** The principal nest is well defined for parameters \(c\) that belong neither to \(\mathcal{V}\) nor to an immediate \(M\) copy. The first condition means that both fixed points are repelling (so the puzzle is defined), while the second condition characterizes those polynomials that do not admit an **immediate renormalization** as described below. We restrict further to parameters \(c\) such that the orbit of 0 is recurrent to ensure that the nest is infinite. These necessary conditions will justify themselves as we describe the nest.

In order to explain the construction of the principal nest, a more detailed description of the puzzle partition at depth 1 (use Figure 2 for reference) is necessary. As a note of warning, the pieces of depth 1 will be renamed to reflect certain properties of \(P_1\). That is, we will override the use of the symbols \(Y_j^{(1)}\).

The puzzle depth \(P_1\) consists of \(2q-1\) pieces of which \(q-1\) are the restriction to lower equipotential of the pieces \(Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_{q-1}^{(0)}\). Such pieces cluster around \(\alpha\) and will be denoted \(Y_1, Y_2, \ldots, Y_{q-1}\).

The restriction of \(Y_0^{(0)}\) however, is further divided into the union of the critical piece \(Y_0^{(1)}\) and \(q-1\) pieces \(Z_1, Z_2, \ldots, Z_{q-1}\) which are symmetric to the corresponding \(Y_j\) and cluster around \(-\alpha\). The indices are again determined by the rotation number of \(\alpha\) so that \(f(Z_j)\) is opposite to \(Y_j\) and consequently \(f(Z_j) = Y_j^{(0)}\).

Note that \(f^\nu(0) \in Y_0^{(0)}\), so we face two possibilities. It may happen that \(f^{\nu j q}(0) \in Y_0^{(1)}\) for all \(j\), in which case it is possible to find **thickenings of** \(Y_0^{(1)}\) and \(Y_0^{(0)}\), that yield the **immediate renormalization** \(f^\nu: Y_0^{(1)} \to Y_0^{(0)}\) described by Douady and Hubbard; or else, we can find the least \(k\) for which the orbit of 0 under \(f^\nu\) escapes from \(Y_0^{(1)}\). We will assume that this is the case, so \(f^{\nu k q}(0) \in Z_\nu\) for some \(\nu\) and call \(k q\) the **first escape time**.

The initial nest piece \(V_0^0\) is defined as the \((k q)\)-fold pull-back of \(Z_\nu\) along the critical orbit; that is, the unique piece that satisfies \(0 \in V_0^0\) and \(f^{\nu k q}(V_0^0) = Z_\nu\). In fact, \(V_0^0\) can also be defined as the **largest central piece that is compactly contained in** \(Y_0^{(1)}\): Notice that \(Z_\nu \subseteq Y_0^{(0)}\) so \(V_0^0 \subseteq Y_0^{(1)}\); that is, \((\operatorname{int} Y_0^{(1)}) \setminus V_0^0\) is a non-degenerate annulus.

The higher levels of the principal nest are defined inductively. Suppose that the pieces \(V_0^0, V_1^0, \ldots, V_0^n\) have already been constructed. If the critical orbit never returns to \(V_0^n\) then the nest is finite. Otherwise, there is a first return time \(\ell_n\) such that \(f^{\ell_n}(0) \in V_0^n\); then we define \(V_0^{n+1}\) as the critical piece that maps to \(V_0^n\) under \(f^{\ell_n}\).

**Proposition 2.5.** The principal nest \(V_0^0 \supset V_0^1 \supset \ldots\) is a family of strictly nested pieces centered around 0.

**Proof.** \(V_0^0\) is a piece of depth \(k q\) (the first escape time). Since \(V_0^1\) is a \(f^{\ell_1}\)-pull-back of \(V_0^0\), it is a piece of depth \(k q + \ell_1\) and, in general, \(V_0^n\) will be a piece of depth \(k q + \ell_1 + \cdots + \ell_n\). Since all pieces contain 0, Property [\(\square\)] implies that \(V_0^n \supset V_0^{n+1}\).

Recall that \(V_0^0 \subseteq Y_0^{(1)}\); thus, the \(f^{\ell_1}\)-pull-backs of these 2 pieces satisfy \(V_0^1 \subseteq X\) with \(X\) a central piece of depth \(1 + \ell_1\). Now, \(0 \not\in Z_\nu\), so \(f^{\nu k q}(0)\) requires further iteration to reach a central piece; i.e., \(\ell_1 > k q\). By construction, \(V_0^0\) is a central piece of depth \(1 + k q\), so Property [\(\square\)] implies \(V_1^1 \subseteq X \subseteq V_0^0\). An analogous argument yields the strict nesting property for the nest pieces of higher depth. \(\square\)

**Definition 2.9.** The **principal annulus** \(V_0^{n-1} \setminus V_0^n\) will be denoted \(A_n\).

It may happen that \(\ell_{n+1} = \ell_n\); this means that not only does 0 return to \(V_0^n\) under \(f^{\ell_n}\), but even deeper to \(V_0^{n+1}\) without further iteration. In this case we say that the return is **central** and call a chain of consecutive central returns \(\ell_n = \ell_{n+1} = \ldots = \ell_{n+s}\) a **cascade of central returns**.


An infinite cascade means that the sequence \( \{\ell_n\} \) is eventually constant, so \( f^{\ell_n}(0) \in \bigcap_{j=n}^{\infty} V^j_0 \). By definition, \( f^{\ell_n} : V^{n+1}_0 \to V^n_0 \) is a renormalization of \( f \); that is, a 2 to 1 branched cover of \( V^n_0 \) such that the orbit of the critical point is defined for all iterates.

The return to \( V^n_0 \), however, can be non-central. In fact, it is possible to have several returns to \( V^n_0 \) before the critical orbit hits \( V^{n+1}_0 \) for the first time. When a return is non-central, the description of the nest at that level is completed by the introduction of the lateral pieces \( V^n_k \in V^{n-1}_0 \setminus V^n_0 \).

Let \( \mathcal{O} \subset K \) denote the critical orbit \( \mathcal{O} = \{ f^j(0) | j \geq 0 \} \) and take a point \( z \in \mathcal{O} \cap V^{n-1}_0 \) whose forward orbit returns to \( V^{n-1}_0 \). If we call \( r_{n-1}(z) \) the first return time of \( z \) back to \( V^{n-1}_0 \), we can define \( V^n(z) \) as the unique puzzle piece that satisfies \( z \in V^n(z) \) and \( f^{r_{n-1}(z)} (V^n(z)) = V^{n-1}_0 \). In particular, it is clear that \( V^n(0) \) is just the same as \( V^n_0 \) and that any 2 pieces created by this process are disjoint or equal.

**Definition 2.10.** The collection of all pieces \( V^n(z) \) where \( z \in \mathcal{O} \cap V^{n-1}_0 \) that actually contain a point of \( \mathcal{O} \) is denoted \( V^n \) and referred to as the level \( n \) of the nest.

From this moment on, we will assume that the principal nest is infinite, with \( |V^n| < \infty \) at every level, and that \( f \) is non-renormalizable; thus excluding the possibility of an infinite cascade of central returns. In this situation we say that \( f \) is **combinatorially recurrent**. It follows from [L2] and [Ma] that \( f \) acts minimally on the postcritical set. In this situation, we can name the pieces \( V^n = \{V^n_0, V^n_1, \ldots, V^n_{m_n}\} \) in such a way that the first visit of the critical orbit to \( V^n_i \) occurs before the first visit to \( V^n_j \) whenever \( i < j \). Obviously, the value of \( r_{n-1}(z) \) is independent of \( z \in V^n_k \); thus we will denote it \( r_{n,k} \).

**Definition 2.11.** For every level of the nest, define the map:

\[
g_n : \bigcup_{k=1}^{m_n} V^n_k \to V^{n-1}_0,
\]
Figure 3. Relation between consecutive nest levels. The curved arrow represents the first return map $f^{\circ n}: V_0^n \to V_0^{n-1}$ which is 2 to 1. The dotted arrows show a possible effect of this map on each nest piece of level $n+1$. Each $V_j^{n+1}$ may require a different number of additional iterates to return to this level and map onto $V_0^n$.

given on each $V^n$ by $g_n|_{V^n_k} = f^{\circ n,k}$.

The map $g_n$ satisfies the properties of a generalized quadratic-like (gql) map, i.e.:

**gql-1:** $\bigcup V^n_k \subseteq V_0^{n-1}$ and all the pieces of $V^n$ are pairwise disjoint.

**gql-2:** $g_n|_{V^n_k}: V^n_k \to V_0^{n-1}$ is a 2 to 1 branched cover or a conformal homeomorphism depending on whether $k = 0$ or not.

Note that $g_n$ usually is the result of a different number of iterates of $f$ when restricted to different $V^n_k$. Since we often refer to the map $g_n$ as acting on individual pieces, it is typographically convenient to introduce the notation

**Definition 2.12.** The map $g_n|_{V^n_k} = f^{\circ n,k}$ will be denoted $g_{n,k}$.

Thus, $g_{n,k}(V^n_k) = V_0^{n-1}$ is a 2 to 1 branched cover or a homeomorphism depending on whether $k = 0$ or not.

2.7. Paraneast. The paraneast is well defined around parameters $c$ outside the main cardioid that are neither immediately renormalizable nor postcritically finite.

**Definition 2.13.** If $c$ is a parameter such that $f_c$ has a well defined nest up to level $n$ (for $n \geq 0$), the paraneast piece $\Delta^n[c]$ is defined by the condition $\partial \Delta^n[c] = \partial f_c(V_0^n)$; where $V_0^n$ is the central piece of level $n$ in the principal nest of $f_c$. By the Douady-Hubbard theory, $\Delta^n[c]$ is a well defined region.

The definition of principal nest, together with Proposition 2.8 imply that when $c' \in \Delta^n[c]$, the principal nests of $f_c$ and $f_{c'}$ are identical until the first return $g_n(0)$ to $V_0^{n-1}$ (which creates $V_0^n$). In fact, the relevant pieces move holomorphically as $c'$ varies and $\Delta^n[c]$ is the largest parameter
region over which the initial set of \( \ell_n \) iterates of 0 (recall that \( g_n \equiv f^\circ \ell_n \)) moves holomorphically without crossing piece boundaries.

Following the presentation of [L4], the family \( \{ g_n[c']: V_0^n[c'] \rightarrow V_0^{n-1}[c'] \mid c' \in \Delta^n[c]\} \) is a proper DH quadratic-like family with winding number 1. The last property follows from Proposition 2.3 since \( g_n \) is the first return to a critical piece at this level.

Since the central nest pieces are strictly nested, the above definition implies that the pieces of the paranest are strictly nested as well. It follows that \( (\text{int } \Delta^n) \setminus \Delta^{n-1} \) is a non-degenerate annulus. One of our main concerns is to estimate its modulus or, as it is sometimes called, the paramodulus.

3. Frame system

Let \( f_c \) have an infinite principal nest. We need a description of the combinatorial structure around nest pieces in order to record their positions relative to each other. In this Section we enhance the principal nest with the addition of a frame system. The notion of frame, introduced in [P], provides the necessary language to locate the lateral nest pieces and describe as a consequence, the behavior of the critical orbit. The idea is to split the central nest pieces in smaller regions by a procedure that resembles the construction of the puzzle.

3.1. Frames. Figure 2 provides a useful reference for the construction of the initial frames \( F_0, F_1 \) and \( F_2 \). Some attention is necessary at these levels to ensure that the properties of Proposition 3.3 hold. Starting with level 3, frames are defined recursively.

Consider the puzzle partition at depth 1 and recall that \( kq \) denotes the first escape of the critical orbit to \( Z_\nu \). The initial frame \( F_0 \) is the collection of nest pieces \( F_0 = \{ Y_0^{(1)} \cup \{ \bigcup_{j=1}^g \{ Z_j \} \} \}; \) clearly, \( \Gamma(F_0) \) is a \( q \)-gon. The frame \( F_1 \) is the collection of \( (f^{\circ kq}) \)-pull-backs of cells in \( F_0 \) along the orbit of 0.

From the definition, the central piece \( V_0^0 \) that maps 2 to 1 onto \( Z_\nu \in F_0 \), is one of the cells of \( F_1 \). The pull-back of any other cell \( A \in F_0 \) consists of two symmetrically opposite cells, each mapping univalently onto \( A \). We say that \( F_1 \) is a well defined unimodal pull-back of \( F_0 \).

**Lemma 3.1.** All the cells of \( F_1 \) are contained in \( Y_0^{(1)} \).

**Proof.** Since \( kq > 1 \), \( f^{\circ kq}(Y_0^{(1)}) \) is an extension of \( Y_0^{(0)} \) to a larger equipotential. Thus, \( f^{\circ kq}(Y_0^{(1)}) \) contains all cells of \( F_0 \). \( \square \)

Let \( \lambda \) be the first return time of 0 to a cell of \( F_1 \). By Lemma 3.1, the collection \( F_2 \) of pull-backs of cells in \( F_1 \) along the \( (f^{\circ \lambda}) \)-orbit of 0 is well defined and 2 to 1.

**Lemma 3.2.** The frame \( F_2 \) satisfies:

1. All cells of \( F_2 \) are contained in \( V_0^0 \).
2. \( V_0^1 \) is contained in the central cell of \( F_2 \).

**Proof.** First note that \( \lambda = kq + (g - \nu) \) is the first return of 0 to \( Y_0^{(1)} \) after the first escape to \( Z_\nu \). It follows that \( kq < \lambda \leq \ell_0 \), where the second inequality is true since \( V_0^0 \in F_1 \). Then the first return to \( F_1 \) occurs no later than the first return to \( V_0^0 \). By definition, \( f^{\circ \lambda}(V_0^0) \) is just \( Y_0^{(0)} \) extended to a larger equipotential. Since all cells of \( F_1 \) are inside \( Y_0^{(1)} \subset f^{\circ \lambda}(V_0^0) \), the first assertion follows.

Now, \( V_0^1 \) is central. By the Markov properties of \( Y_0^1 \), either \( V_0^1 \) is contained in the central cell \( C \) of \( F_2 \) or vice versa. However, both \( f^{\circ \ell_0}(V_0^1) \) and \( f^{\circ \lambda}(C) \) belong to \( F_1 \). Since \( \ell_0 \geq \lambda \), the first possibility is the one that holds. This proves property (2). \( \square \)

After introducing the first frames and linking them to the initial levels of the nest, we can give the complete definition of the frame system. The driving idea of this discussion is that the internal
Both of these parameters belong to the left antenna of $L_{1/3}$; they are centers of components of periods 7 and 4. Above we can see that the structures of the frames of levels 0 and 1 coincide between the two examples. Still, the first return to $F_1$ falls in each case on a different cell, producing dissimilar frames of level 2. The pull-back of cells in $F_1$ produces the frame $F_2$, shown in heavy line on the second row.

Definition 3.1. For $n \geq 0$ consider the first return $g_n(0) \in V_0^n$ and define $F_{n+3}$ as the collection of $g_n$-pull-backs of cells in $F_{n+2}$ along the critical orbit. The family $F_c = \{F_0, F_1, \ldots\}$ is called a frame system for the principal nest of $f_c$ and each piece of a frame is called a cell.

The dual graph $\Gamma(F_n)$ (see Subsection 2.4) is called the frame graph. As in the case of the puzzle graph, consider $\Gamma(F_n)$ with its natural embedding in the plane.

Let us mention now some properties of frame systems (refer to [P]).
Proposition 3.3. The frame system satisfies:

1. Frames exist at all levels.
2. The central cell of $F_n$ contains the nest piece $V_0^{n-1}$.
3. Each $F_n$ has 2-fold central symmetry around 0.
4. Suppose there is a non-central return; then, eventually all nest pieces are compactly contained in cells of the corresponding frame.

3.2. Frame labels. Our next objective is to introduce a labeling system for pieces of the frame. This will allow us to describe the relative position of pieces of the nest within a central piece of the previous level. Unlike the case of unimodal maps, where nest pieces are always located left or right of the critical point, the possible labels for vertices of $\Gamma(F_n)$ will depend on the combinatorics of the critical orbit. Only after determining the labeling, it becomes possible to describe the location of nest pieces in a systematic manner.

Observe that the structure of $F_{n+1}$ is determined by the structure of $F_n$ and the location of $g_n(0)$. A graphic way of seeing this is as follows. Say that the first return $g_{n-1}(0)$ to $V_0^{n-2}$ falls in a cell $X \in F_n$. Let $L_n$ and $R_n$ be two copies of $\Gamma(F_n)$ with disjoint embeddings in the plane. Now connect $L_n$ and $R_n$ with a curve $\gamma$ that does not intersect either graph. Suppose that one extreme of $\gamma$ lands at the vertex of $L_n$ that corresponds to $X$ and the other extreme lands at the corresponding vertex of $R_n$ approaching it from the same access.

Figure 5. The curve $\gamma$ joins two copies of the same frame graph approaching the selected vertex from the same direction. The new frame graph is obtained after $\gamma$ is contracted to a point.

Lemma 3.4. If $\gamma$ is collapsed by a homotopy of the whole ensemble, the resulting graph is isomorphic to $\Gamma(F_{n+1})$.

Lemma 3.5. The plane embedding of $\Gamma$ does not depend on the homotopy class of the curve $\gamma$ in lemma 3.4.

Proof. Since we regard $\Gamma = \Gamma(F_n)$ as embedded in the sphere, the exterior of $\Gamma$ is simply connected, so there is a natural cyclic order of accesses to vertices (some vertices can be accessed from more
that contains \( \lambda \) g ical clarity, we will write until the composition of returns of level \( n \) 2 vertices corresponding to symbol of each label.

First, put the labels \( \{Z_0', Z_1', \ldots, Z_{(q-1)'}\} \) on the cells of \( F_0 \), starting at the central piece \( Y_0^{(0)} \) and moving counterclockwise.

Let \( \sigma_0 \) be the label of the cell that holds the first return of 0 to \( F_0 \) and, in general, let \( \sigma_n \) denote the label of the cell in \( \Gamma(F_n) \) that holds the first return of 0. In order to label \( \Gamma(F_{n+1}) \), assume that the number \( q \) of pieces in \( F_0 \) is known, and the label sequence \( (q; \sigma_1, \ldots, \sigma_n) \) that identify the location of first returns of 0 to levels 0, \( \ldots, n-1 \) of the nest. In particular, all frames up to \( \Gamma(F_n) \) have been successfully labeled.

Duplicate in \( L \) the labels of \( \Gamma(F_n) \), but concatenate an extra ' \( ' \) at the beginning. Do a similar labeling on \( R \) by concatenating an extra ' \( R' \) to the duplicated labels. Note that the labels of the two vertices corresponding to \( X \) are ' \( ' \sigma' \) and ' \( R' \sigma_n \). The labels on \( \Gamma(F_{n+1}) \) will be the same as those in the union of \( L \) and \( R \) except that we change the label of the identified vertex, to become ' \( 'Z_0' \sigma_n \). Clearly, \( f \) induces a map \( f_* : \Gamma(F_{n+1}) \rightarrow \Gamma_n \) for \( n \geq 2 \), that acts by forgetting the leftmost symbol of each label.

It is important to mention that the resulting labeling of \( \Gamma(F_n) \) does depend on the access to \( \xi_n \) approached by \( \gamma \). However, the final unlabeled graphs are equivalent as embedded in the plane.

As was just mentioned, some vertices are accessible from \( \infty \) in two or more directions. These are precisely the vertices whose label contains the symbol ' \( 'Z_0' \) (for \( n \geq 1 \)). Since such a vertex represents a frame that maps (eventually) to a central frame cell, the tail of a label with ' \( 'Z_0' \) to position \( j \) must be \( \sigma_j \). On the other hand, for every \( j \) there must be labels with a ' \( 'Z_0' \) in position \( j \). It follows that the set of labels of \( \Gamma(F_n) \) and the sequence \( (q; \sigma_0, \ldots, \sigma_n) \) can be recovered from each other.

### 3.3. Frame system and nest together

The definition of frame system was conceived to satisfy the properties of Proposition 3.3. An extension of the argument used to prove those properties shows that every piece \( V_j^n \) of the nest is contained in a frame cell of level \( n+1 \). Moreover, we would like to extend the definition of frames so that each \( V_j^n \) can be partitioned by a pull-back of an adequate central frame. For this, recall first that \( g_{n,j}(V_j^n) = V_0^{n-1} \cap F_{n+1} \).

**Definition 3.2.** The frame \( F_{n,k} \) is the collection of pieces inside \( V_k^{n-2} \) obtained by the \( g_{n-2,k} \)-pull-back of \( F_{n-1} \). Elements of the frame \( F_{n,k} \) are called **cells** and we will write \( F_{n,0} \) instead of \( F_n \), when there is a need to stress that a property holds in \( F_{n,k} \) for every \( k \).

If a puzzle piece \( A \) is contained in a cell \( B \in F_{n,k} \), denote \( B \) by \( \Phi_{n,k}(A) \).

We have described already how to label \( F_n \). The other frames \( F_{n,k} (k \geq 1) \), mapping univalently onto \( F_{n-1} \), have a natural labeling induced from that of \( F_{n-1} \) by the corresponding \( g_{n-2,k} \)-pull-back.

Let us describe now the itinerary of a piece \( V_j^n \). Since \( V_j^n \subseteq V_0^{n-1} \), the map \( g_{n-1} \) takes \( V_j^n \) inside some piece \( V_{k_1(j)}^{n-1} \subseteq V_0^{n-2} \). Then, \( g_{n-1,k_1(j)} \) takes \( g_{n-1}(V_j^n) \) inside a new piece \( V_{k_2(j)}^{n-1} \) and so on, until the composition of returns of level \( n-1 \)

\[
g_{n-1,k_r(j)} \circ \ldots \circ g_{n-1,k_1(j)} \circ g_{n-1}(V_j^n)_{|_{V_0^{n-1}}}
\]

is exactly \( g_{n,j} : V_j^n \rightarrow V_0^{n-1} \). Of course, \( k_r \) is just 0, and we will write it accordingly.

There is extra information that deems this description more accurate. For the sake of typographical clarity, we will write \( k_i \) instead of \( k_i(j) \). For \( i \leq r \), let \( \Phi_{n+1,k_i} \) be the cell in \( F_{n+1,k_i} \subseteq V_{k_i}^{n-1} \) that contains

\[
g_{n-1,k_r} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n)
\]

and denote by \( \lambda_{n+1,k_i} \) the label of \( \Phi_{n+1,k_i} \).
Definition 3.3. The itinerary of $V^n_j$ is the list of piece-label pairs:

\[ \chi(V^n_j) = \left( [V^n_{k_1}; \lambda_{n+1,k_1}], [V^n_{k_2}; \lambda_{n+1,k_2}], \ldots, [V^n_{k_{r-1}}; \lambda_{n+1,k_{r-1}}], [V^n_{k_r}; \lambda_{n+1,0}] \right) \]

up to the moment when $V^n_j$ maps onto $V^{n-1}_0$.

Note first of all that the last label, $\lambda_{n+1,0}$, will start with 'Z' due to the fact that $V^{n-1}_0$ is in the central cell of $F_n$. More importantly, the conditions

\[ V^n_{k_i} \subset g_{n-1}(\Phi_{n+1,k_i}) \quad 2 \leq i < r \]

must hold since $g_{n-1,k_i} \circ \ldots \circ g_{n-1,k_1}(V^n_j) \subset \Phi_{n+1,k_i}$ and $g_{n-1,k_i} \circ \ldots \circ g_{n-1,k_1}(V^n_j) \subset V^{n-1}_{k_{i+1}}$.

Definition 3.4. When the sequence of frame labelings is specified up to a given level $n$, the locations of the nest pieces and their (admissible) itineraries, we say that we have described the combinatorial type of the map at level $n$.

4. Q-recurrency

Lyubich and Milnor established in [LM] the uniqueness of the real quadratic Fibonacci map $f_{\text{fib}}$ and described in detail its asymptotic geometry. The real parameter $c_{\text{fib}} = -1.8705286321 \ldots$ is determined by either of the following two equivalent conditions:

F1 The closest returns to 0 of the critical orbit occur exactly when the iterates are the Fibonacci numbers.

F2 For $n \geq 2$, each level of the principal nest consists of the central piece $V^n_0$ and a unique lateral piece $V^n_l$. The first return map of previous level $g_{n-1} : V^{n-1}_0 \rightarrow V^{n-2}_0$ interchanges the central and lateral roles:

\[ g_{n-1}(V^n_0) \subset V^{n-1}_1, g_{n-1}(V^n_1) = V^{n-1}_0. \]

Additionally, the first returns to $Y^{(1)}_0$ and $V^n_0$ happen on the third and fifth iterates respectively.

The critical behavior of $f_{\text{fib}}$ is the simplest among maps whose nest has no central returns: Every level of the nest has a unique lateral piece, so in a way, every first return comes as close as possible to being central without actually being central. This means that $f_{\text{fib}}$ is not renormalizable in the classical sense, although its combinatorics can be described as an infinite cascade of Fibonacci renormalizations in the space of gql maps with one lateral piece.

The papers [LB] and [W] analyze an unexpected feature of the Fibonacci map. If the central pieces $V^n_0$ are rescaled to regions $\tilde{V}_n$ of fixed size, each $g_n$ induces a map $G_n : \tilde{V}_n \rightarrow \tilde{V}_n^{-1}$. On increasing levels, the criss-cross behavior that determines $c_{\text{fib}}$ in condition [P2] approximates with exponential accuracy the pattern of the critical orbit of $P_{-1}(z) = z^2 - 1$ (i.e., $0 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto \ldots$). In fact, $G_n \rightarrow P_{-1}$ locally uniformly in the $C^1$ norm. Also, since diam $\tilde{V}_n \asymp 1$, it is shown that the rescaled pieces converge in the Hausdorff metric to the filled Julia set of $P_{-1}$.

In [W], Wenstrom translates this behavior to the Mandelbrot set and obtains pieces of the paranest around $c_{\text{fib}}$ that asymptotically resemble $K_{-1}$; see Figure 1 of [W]. As consequences of this control on shape, he computes the exact rate of linear growth of the principal moduli and proves hairiness around the parameter $c_{\text{fib}}$.

Let $c$ be the center of a prime hyperbolic component and $Q(z)$ its associated polynomial. The critical orbit is periodic (of least period $m$) and $Q^{\circ m}$ is the only renormalization of $Q$. An important consequence of this is that high enough depths of the puzzle of $Q$ will isolate in individual pieces
each point of the critical orbit \( \mathcal{O}(Q) = \{ 0 \mapsto c \mapsto z_2 \mapsto \cdots \mapsto z_{m-1} \} \). Let us assume that the fixed point \( \alpha \) of \( Q \) has combinatorial rotation number \( \frac{p}{q} \). In what follows we will save notation by restricting the use of “\( P_n \)” to refer to the puzzle of \( Q \) and “\( V^n \)”, “\( F_n \)” for the nest and frames of \( Q \)-recurrent maps.

Let us label \( \Gamma(P_0) \), the graph of the puzzle of \( Q \) at depth 0, with symbols ‘\( Z_0 \)’ to ‘\( Z_{q-1} \)’ starting at the critical point piece and moving counterclockwise. Since \( P_{n+1} \) is a 2 to 1 pull-back of \( P_n \), the graph \( \Gamma(P_{n+1}) \) consists of two copies of \( \Gamma(P_n) \) identified at the critical value vertex and we can launch a labeling procedure identical to the frame labeling of Subsection 3.2. Note that \( \Gamma(P_n) \) is symmetric, but a canonical orientation can be specified by dictating that the label on the critical \( P \) the graph \( \Gamma(\cdot) \) at depth 0, with symbols ‘\( Z_0 \)’ to ‘\( Z_{q-1} \)’ starting at the critical point piece and moving counterclockwise. Since \( P_{n+1} \) is a 2 to 1 pull-back of \( P_n \), the graph \( \Gamma(P_{n+1}) \) consists of two copies of \( \Gamma(P_n) \) identified at the critical value vertex and we can launch a labeling procedure identical to the frame labeling of Subsection 3.2. Note that \( \Gamma(P_n) \) is symmetric, but a canonical orientation can be specified by dictating that the label on the critical value vertex begins with the symbol ‘\( L \)’.

Note that \( \Gamma(P_n) \) is symmetric, but a canonical orientation can be specified by dictating that the label on the critical value vertex begins with the symbol ‘\( L \)’.

To see this, remember that the definition of \( Q \)-recurrency requires that the composition of maps \( g_{n-2} \circ \cdots \circ g_{n+m-3} \) falls in the cell of \( F_{n+m-2} \) that corresponds to \( L_1^{n+m-2} \). Next, \( g_{n+m-4} g_{n+m-3} \) falls in the cell of \( F_{n+m-3} \) corresponding to \( L_2^{n+m-3} \). Continue in this manner, with \( g_{n+m-3-j} \circ \cdots \circ g_{n+m-3} \) (where \( 0 \leq j \leq m-2 \)) falling in the cell of level \( n+m-2+2 \) that corresponds to \( L_{j+1}^{n+m-2+2} \). At every step, jump out one nest level and create in the process (by adequate pull-backs) the nest pieces \( V_{1}^{n+m-4}, V_{2}^{n+m-5}, \ldots, V_{m-1}^{n+m-2} \). Note that all these are lateral pieces since they are contained in a frame cell that is not central. In fact, \( V_{j+1}^{n+m-4+j} \) is in the cell of \( F_{n+m-2+j} \) that corresponds to \( L_{j+1}^{n+m-2+j} \); see Figure 6.

The last map in this chain of compositions is \( g_{n-2} \). It brings the critical orbit very nearly to the center, inside \( V_{0}^{n+m-3} \). To see this, remember that the definition of \( Q \)-recurrency requires that the composition of maps \( g_{n-2} \circ \cdots \circ g_{n+m-3} : V_{0}^{n+m-2} \rightarrow V_{0}^{n+m-3} \) is the first return to \( V_{0}^{n+m-3} \), i.e.
$g_{n-2} \circ \ldots \circ g_{n+m-3}$ is exactly the map $g_{n+m-2}$.

In summary, if a point of $\mathcal{O}(f)$ falls in a piece $V_j^n$ (for $j \leq m-1$), the next return falls inside $V_{j+1}^{n-1}$. If it falls on a piece $V_{m-1}^n$, the next return falls $m$ levels deeper, inside $V_0^{n+m-1}$ and is in fact, the first return to this piece. Repeating this procedure at $m-1$ consecutive levels creates various pieces of different levels. Among these, the $m-1$ lateral nest pieces of level $n$, each corresponding to a point $z_j$ ($1 \leq j \leq m-1$) of the critical orbit of $Q$. □

Figure 6. In a $Q$-recurrent map there are $m-1$ lateral pieces at each level, where $m$ is the period of 0 under $Q$. Under successive first return maps, the critical orbit jumps out to lower levels (as $V_{i-1}^{n-i+1}$ goes inside $V_{i+1}^{n-i}$) until at step $m-1$ it returns to the center. See also Figure 8.

A first consequence of Proposition 4.1 is the fact that the itinerary of $V_j^n$ in the nest of level $n-1$ is

$$\chi(V_j^n) = ([V_{j+1}^{n-1}; \lambda_{n+1,j+1}, [V_{0}^{n-1}; \lambda_{n+1,0}]) \text{ when } 0 \leq j < m-1$$

$$\chi(V_{m-1}^n) = ([V_{0}^{n-1}; \lambda_{n+1,0}]).$$

This hints to a similarity between the actions of $Q$ and $g_n$ that will be made precise in the next Section, where we develop the asymptotic properties of $Q$-recurrent maps and their principal nests.

5. Asymptotics of $Q$-recurrency

This Section presents the geometric properties of $Q$-recurrent maps. Their explicit relation to the combinatorics of the map $Q$ gives control over the shapes of nest pieces and this will yield very precise estimates of the analytic invariants of the nest.

For reference, let us state again the fundamental relation between levels of a $Q$-recurrent nest:

$$g_{n+m} = g_n \circ \ldots \circ g_{n+m-1}$$

$$g_{n+m} (V_{n+m}^{n+m}, V_{n+m-1}^{n}) = (V_{n+m-1}^{n}, V_{n}^{n-1})$$
We will also make repeated reference to the following result (see \[L3\]).

**Lyubich’s Theorem.** Let $\kappa(n)$ count the levels of the principal nest up to level $n$, which are non-central. Then the moduli of the principal annuli grow linearly:

$$\mu(n+2) \geq B \cdot \kappa(n) + C.$$

where the constant $B$ depends only on the initial modulus.

5.1. **Complex Fibonacci maps.** Section 4 begins with the definition of the real parameter $c_{\text{fib}}$. We will see that properties $F_1$ and $F_2$ are shared by many complex maps.

**Definition 5.1.** A complex polynomial map, or equivalently its corresponding parameter, is said to be **complex Fibonacci** if the first return to each level of the nest happens exactly when the iterates are the Fibonacci numbers.

**Note.** The first return to a level can be viewed as a close return in a combinatorial sense; that is, a return to a small central piece. Since Lyubich’s Theorem guarantees that central pieces decrease in size, the definition of complex Fibonacci parameter is equivalent to its metric analogue, property $F_1$ at the beginning of Section 4.

Our first result is a classification of the Fibonacci behavior in the complex case.

**Definition 5.2.** The set of all $(z^2 - 1)$-recurrent parameters is denoted $\mathbb{F}_{\text{ibo}}$.

**Theorem 5.1.** A parameter $c$ is complex Fibonacci if and only if $c \in \mathbb{F}_{\text{ibo}}$.

**Proof.** All $(z^2 - 1)$-recurrent maps have the same weak combinatorial type. As was pointed out in the note above, the first returns of high levels are just predetermined compositions of lower level ones. Thus, the number of iterates until the first return to a piece $V_n^0$ is independent of the parameter $c \in \mathbb{F}_{\text{ibo}}$. Since the real parameter $c_{\text{fib}} \in \mathbb{F}_{\text{ibo}}$ is complex Fibonacci, the first direction of the assertion follows.

To show the converse, it is only necessary to observe that the first return times in a Fibonacci nest are strictly increasing, so there are no central returns. Therefore the first return map $g_{n+1}$ must be the composition of at least two first return maps of the two preceding levels. If the nest does not have $(z^2 - 1)$-recurrent type, there must be more than one lateral piece at some level $n$. Then the composition of maps generating $g_{n+1}$ will actually contain more than two maps and the sequence $\{\ell_n\}$ of first return times grows faster than the sequence generated by the recursion $\ell_{n+1} = \ell_n + \ell_{n-1}$. This contradicts the assumption that the map is complex Fibonacci. \[\square\]

Although Yoccoz’s Theorem on rigidity of non-renormalizable maps allows us to characterize $\mathbb{F}_{\text{ibo}}$ as a Cantor set, we must wait until next Section to show that the relevant parapieces shrink exponentially fast, thus allowing us to complete the description of the set $\mathbb{F}_{\text{ibo}}$ as a Cantor set of Hausdorff dimension 0 on which we can impose a natural dyadic decomposition.

5.2. **Shape.** We want to study the shape of nest pieces in the following sense.

**Definition 5.3.** A sequence of compact sets $\{C_j \subset \mathbb{C}\}$ is said to **converge in shape** to a compact $K$ if there exist rescalings $\tilde{C}_j = a_j \cdot C_j$ (with $a_j \in \mathbb{C}$) such that $\{\tilde{C}_j\} \rightarrow K$ in the Hausdorff metric.

The main Theorem of this Section is a vast extension of the result on the shape of central pieces of $f_{c_0}$ found in $[L1]$. In order to give the statement, some notation is needed.

Let $Q = Q(z)$ be the center of a prime hyperbolic component and $c_0$ a $Q$-recurrent parameter. Recall that $f_{c_0}$ is described by a dyadic choice of labels $('L'$ and $'R'$) on every level. These frame orientations determine the sequence of paranest pieces $\{\Delta^n\}$ around $c_0$. If $c \in \Delta^n$ is any nearby parameter, the combinatorics of $f_c$ are identical to those of $f_{c_0}$ including the orientations of the
homeomorphic frames, at all levels \( j \leq n \). In particular, for any \( c \in \Delta^n \) we can find a (unique) point \( s_j \) in \( F_j \) corresponding to the fixed point \( \alpha \) of \( Q \). In what follows, we omit from the notation the fact that the objects described depend on \( c \). Let \( \alpha_j = \frac{c}{s_j} \) and define the complex rescalings 
\[
\tilde{V}^j := \alpha_j \cdot V^j_j \quad \text{of the central nest pieces of} \quad f_{c_j} \quad \text{up to level} \quad n.
\]
Then, the first return maps \( g_j \) induce maps 
\[
G_j : \tilde{V}^j \to \tilde{V}^{j-1}
\]
on the rescaled pieces whose action on the rescaled frame \( \tilde{F}_{j-2} := (\alpha_j \cdot F_{j-2}) \subset \tilde{V}^j \) is isotopic to the action of \( Q \) on its own puzzle.

**Theorem 5.2.** Given \( \varepsilon > 0 \), there is an \( N \) such that for every parameter \( c \in \Delta^n \) and level \( n \geq N \), the maps \( G_N, G_{N+1}, \ldots, G_n \) are all \( \varepsilon \)-close to \( Q \) in the \( C^1 \) topology inside the ball of radius \( \frac{1}{2} \).

**Corollary 5.3.** The sequence of central nest pieces \( \{V^n_j\} \) of \( f_{c_0} \) converges in shape to the filled Julia set \( K_Q \).

**Proof of Corollary 5.3.** The point \( \alpha \in K_Q \) is fixed under \( G_n \) and is surrounded by \( \tilde{V}^n \). This rescaled nest piece also surrounds the critical point \( 0 \) which attracts every point in \( K_Q \setminus J_Q \). Now, \( \tilde{V}^n \) is the pull-back of \( V^{n-1} \) under \( G_n \). By Theorem 5.2, \( G_n \) is a small perturbation of \( Q \); since the rescaled pieces in the sequence \( \{\tilde{V}^{n+1}, \tilde{V}^{n+2}, \ldots\} \) have bounded diameter, they become exponentially close to the regions in the sequence \( \{Q^{n-1}(\tilde{V}^n), Q^{n-2}(\tilde{V}^n), \ldots\} \) which converge to \( K_Q \). This yields the result.

In particular, the central pieces of any quadratic complex Fibonacci map look like \( K_{-1} \), although each one may be tilted at a bizarre angle (recall that the \( \tilde{V}^n \) are rescaled by a complex number). Other examples can be seen in Figure 7, showing puzzle pieces that approximate the behavior of periodic orbits of period 3.

Notice that, since the frames are defined by the same sequence of pull-backs as the central nest pieces, the result of Corollary 5.3 holds also for frames; i.e. the union of cells in \( F_n \) converges in shape to \( K_Q \).

The proof of Theorem 5.2 depends on the convergence of Thurston’s map on an appropriate Teichmüller space (see the Appendix for definitions). Let \( \mathcal{O} \equiv \mathcal{O}(Q) \) and consider the surface \( S \) obtained by puncturing the plane at the critical orbit of \( Q \); that is, \( S = \mathbb{C} \setminus \mathcal{O} \). Since deformations are considered only up to an isotopy that leaves \( \mathcal{O} \) invariant, the structure of a puzzle-like construction does not change. Thus, when \( h \) is a deformation in the class of id, the deformation \( h(P(Q)) \) of the puzzle of \( Q \) can be isotoped back to the puzzle \( P(Q) \) itself without changing its configuration and without moving \( \mathcal{O} \).

Thurston’s map is best described via the alternate description of \( T_S \) in terms of Beltrami differentials. First, normalize every deformation \( h \) by an affine change of coordinates \( \varphi \) so that \( \varphi \circ h \) leaves 0, \( c \in \mathcal{O} \) fixed. The Beltrami coefficient \( \mu = \frac{\partial h}{\partial z} \frac{\overline{\partial h}}{\overline{\partial z}} \) determines a conformal structure associated to \( h \).

**Definition 5.4.** The map \( \tau_Q : T_S \to T_S \) induced on equivalence classes of conformal structures by the pull-back \( \mu \mapsto Q^* \mu \) is called the **Thurston map** associated to \( Q \).

The action of \( \tau_Q \) on a deformation class \( h \) is easy to describe. The class \( \tau_Q([h]) \) is represented by a deformation \( \tilde{h} \) such that the map \( Q_{h} = h \circ Q \circ \tilde{h}^{-1} \) is analytic. Because of conjugacy, \( Q_{h} \) replicates the critical orbit behavior of \( Q \) in a neighborhood of \( \tilde{h}(\mathcal{O}) \). In particular, one can specify a puzzle-like structure around \( \tilde{h}(\mathcal{O}) \) which pulls back according to the same combinatorics as \( Q \). Since \( \mathcal{O} \) is finite, and \( Q^{em} \) is not renormalizable, such a puzzle structure of high enough depth will isolate all the elements of the critical orbit in individual cells. We conclude that the isotopy class of \( \tilde{h} \) relative to punctures consists of those \( Q_h \)-pull-backs of \( \tilde{h}(\mathcal{O}) \) that keep the puzzle structure intact (however deformed).
Consider the maps $Q_1 : z \mapsto z^2 - 1.75487...$ (the “airplane”) and $Q_2 : z \mapsto z^2 - (0.123...) + i(0.745...)$ (the “rabbit”), as displayed in the first row. Both maps have critical orbits of period 3. The pictures in the second row show close-ups near 0 of two other Julia sets. On the left, a $Q_1$-recurrent map ($c_1 = -1.87449300988719...$). On the right, a $Q_2$-recurrent map ($c_2 = -0.023918090959967... + i0.984732550113053...$). In each case, there is a central nest piece approximating the Julia set of the corresponding $Q_i$.

**Proof of Theorem 5.2** Let $X$ be any finite collection of simply connected compact subsets of $\mathbb{C}$. By a multicurve $\Gamma$ around $X$ we mean a system of disjoint isotopy classes of simple closed curves in $\mathbb{C} \setminus X$ such that each curve $\gamma_i \in \Gamma$ splits $\mathbb{C}$ in two regions, each enclosing at least two elements of $X$ (i.e. $\gamma_i$ is non-peripheral). If $f : \mathbb{C} \setminus X \to \mathbb{C} \setminus X$ fixes every element of $X$, denote by $\Gamma_f^{-1}$ the multicurve consisting of the classes of $f$-preimages of elements $\gamma_i \in \Gamma$ that are not peripheral. The multicurve $\Gamma$ is said to be $f$-stable if $\Gamma_f^{-1} \subset \Gamma$.

Given a map $f : \mathbb{C} \to \mathbb{C}$ fixing the critical orbit $O$ of $Q$ and an arbitrary $f$-stable multicurve $\Gamma$ around $O$, we can construct the linear space $\mathbb{R}^{\Gamma}$ generated by the curves of $\Gamma$, and an induced linear map $\hat{f}_\Gamma : \mathbb{R}^{\Gamma} \to \mathbb{R}^{\Gamma}$ given as follows. If $\gamma_i \in \Gamma$, let $\gamma_{i,j,k}$ denote the components of $f^{-1}(\gamma_i)$ that are in the class of $\gamma_j \in \Gamma_f^{-1}$. Then

$$\hat{f}_\Gamma(\gamma_i) = \sum_{j,k} \frac{1}{d_{i,j,k}} \gamma_j,$$

where $d_{i,j,k}$ denotes the degree of $f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \to \gamma_i$.

An obstruction to the convergence of Thurston’s map $\tau_f$ is represented by a $f$-stable multicurve around $O$, for which $\hat{f}_\Gamma$ has an eigenvalue $\lambda \geq 1$. In our case, $Q$ is a polynomial so it represents the
fixed point of its own Thurston map. In particular, there are no obstructions to the convergence of \( \tau_Q \): see [DH3].

Now, since \( Q \) belongs to a prime hyperbolic component of period \( m \), the map \( Q^{\circ m} \) is a renormalization conjugate to \( z \mapsto z^2 \). By hyperbolicity, the central puzzle pieces of \( K_Q \) get arbitrarily close to the immediate basin of 0. In particular, there is a finite depth so that 0 is the only point of \( \mathcal{O} \) inside the central piece. By further iteration, the same will be true of any point in \( \mathcal{O} \).

Let us choose a level \( k \) high enough so that the puzzle \( P_{k-1} \) isolates all the points in the critical orbit of \( Q \). Again, this is possible since \( Q^{\circ m} \) is not renormalizable. Then, any \( Q \)-stable multicurve \( \Gamma \) can be represented with curves that are constructed from segments of the arcs defining \( P_{k-1} \). In this way, \( \Gamma \) is described in terms of the structure of \( P_{k'} \), for any level \( k' \geq k - 1 \). Moreover, \( \Gamma^{-1}_Q \) is a multicurve around \( \mathcal{O} \) that can be described in terms of the combinatorial structure of \( P_{k'+1} \).

Now consider \( f_c \) with \( c \in \Delta^k \). Any \( G_k \)-stable multicurve \( \Gamma' \) around the pieces \( \tilde{V}^{k+1}_j \) can be described with segments of curves in the boundary of the frame \( \tilde{F}_{k-1} \). Since \( \tilde{F}_{k-1} \) is isomorphic to \( P_{k-1} \), there is a correspondence between \( G_k \)-stable multicurves around \( \bigcup_j \tilde{V}^{k+1}_j \) and \( Q \)-stable multicurves around \( \mathcal{O} \). This means that the only possible obstructions for \( \tau_{G_k} \) must form inside one of those pieces; that is, a multicurve realizing such obstruction would intersect at least one of the pieces \( \tilde{V}^{k+1}_j \). Note that such multicurve cannot be represented by curves that are close to the boundary of \( \tilde{F}_{k-1} \).

By [L3], the size of \( \tilde{V}^{k+2}_0 \) with respect to \( \tilde{V}^{k+1}_0 \) decreases exponentially as \( k \to \infty \). Then, Koebe’s Theorem implies that \( G_k \) is exponentially close to being quadratic; that is, it can be decomposed as \( G_k = D_k \circ Q_{h_k} \), where the maps \( D_k \) become linear and the deformations \( h_k \) are given by iteration of the Thurston map \( \tau_Q \). Moreover, both \( Q_{h_k} \) and \( G_k \) fix 0 and send 0 close to itself, so we can conclude that \( D_k \to \text{id} \). It follows that \( G_k \) rapidly approaches \( Q_{h_k} \).

Select any \( Q \)-stable multicurve \( \Gamma' \) around \( \mathcal{O} \). If there is a level \( k \) such that \( \Gamma' \) does not intersect any of the pieces \( \tilde{V}^{k+1}_j \), then \( \Gamma' \) can be pushed to the boundary of \( \tilde{F}_{k-1} \) to represent a \( G_k \)-stable multicurve around the pieces \( \tilde{V}^{k+1}_j \). Since \( \Gamma' \) is not an obstruction for \( Q \), we deduce that, outside the \( \tilde{V}^{k+1}_j \), the map \( G_k \) is isotopic to \( Q \). However, the only possible Thurston obstructions are restricted to extremely small regions, then the distortion of \( h_k \) goes to 0 and the maps \( G_k \) converge to \( Q \) exponentially fast in a neighborhood of \( K_Q \setminus \mathcal{O} \). The Koebe space between \( V^{k+1}_0 \) and \( V^k_0 \) increases without bound, so we can claim convergence of the maps \( G_k \) in arbitrarily big neighborhoods of \( K_Q \).

Theorem 5.2 has broad implications since it provides excellent control of the shapes of nest pieces. In the next Subsection, we use our knowledge on the shape of the central pieces to compute the rate of growth of the principal moduli.

5.3. Growth of annuli. Here we study the moduli of the principal annuli in \( Q \)-recurrent maps. For this family, we can state a more precise version of Lyubich’s Theorem on the linear growth of moduli. The key ingredient in our proof is Theorem 5.2 giving control over the shape of pieces, together with the extended Grötzsch inequality as stated in the Appendix.

As a preparation for Theorem 5.5, we compute first the capacities of \( K_Q \) with respect to 0 and \( \infty \).

**Lemma 5.4.** Let \( Q = Q(z) \) be the center of a hyperbolic component with critical period \( m \). Then \( \text{cap}_\infty(K_Q) = 0 \) and

\[
\text{cap}_0(K_Q) = -(m - 1) \ln 2 - \sum_{j=1}^{m-1} \ln |Q^{\circ j}(0)|.
\]
Proof. $K_Q$ is connected, so the Böttcher coordinate $\varphi : \mathbb{C} \setminus K_Q \to \mathbb{C} \setminus \mathbb{D}$ sending 0 to $\infty$ is the Riemann mapping with derivative 1, so the first equality is obvious.

The capacity of $K_Q$ at 0 is simply $\text{cap}_0(U)$, where $U$ is the immediate basin of attraction of 0. Consider the iterated polynomial $Q^m : U \to U$. It is a 2 to 1 map of a simply connected domain with fixed critical point. Therefore, there is a map $\psi : \mathbb{D} \to U$ such that

$$\psi(z^2) = Q^m \circ \psi(z)$$

and it is clear that $\text{cap}_0(K_Q) = \text{cap}_0(U) = \ln |\psi'(0)|$.

Equation (7) shows that $\psi'(0)$ is the inverse of the quadratic coefficient in the series expansion of $Q^m(z)$. Since the constant term of $Q^m(z)$ is just $Q^m(0)$, it is easy to find recursively that

$$\frac{1}{\psi(0)} = \prod_{j=1}^{m-1} 2Q^{oj}(0)$$

thus,

$$\text{cap}_0(K_Q) = \ln \psi'(0) = -(m-1) \ln 2 - \sum_{j=1}^{m-1} \ln |Q^{oj}(0)|.$$ 

\[\Box\]

Recall that capacity and modulus are invariants that vary continuously with respect to the Carathéodory topology. Thus, given a sequence of topological disks around 0 converging in the Hausdorff topology to a set with pinched points, the sequence of capacities will converge to the capacity of the component of the limit set that contains 0. Similarly, for a sequence of annuli with adequate convergence, the limit of moduli detects only the modulus of the limit component that contains the limit closed geodesic.

**Theorem 5.5.** For any parameter $c \in \text{Fibo}$ the principal moduli grow linearly at the rate

$$\lim_{n \to \infty} \frac{\mu_n}{n} = \frac{\ln 2}{3}.$$ 

If $Q = Q(z)$ is the center of a prime hyperbolic component with critical period $m \geq 3$, the rate of growth is exponential

$$\lim_{n \to \infty} \frac{\mu_n}{\mu_{n+1}} = \kappa_m,$$

depends only on the period $m$ of $Q$ ad nd satisfies $\kappa_m \nearrow \frac{3}{2}$ as $m$ increases.

**Proof.** Fix a level $N$ large enough so that the shape of rescaled nest pieces is already close to the shape of $K_Q$. In particular, $\text{cap}_0(\tilde{V}^n) \sim \text{cap}_0((K_Q)_0)$ for all $n \geq N$, where $(K_Q)_0$ is the Fatou component of $K_Q$ containing 0.

We also require the lateral pieces are small enough to sit in the center of their (almost pinched) regions, far away from the boundary. This is possible since Lyubich’s Theorem on linear growth forces shrinking and Theorem 5.2 locates the nest pieces in positions that resemble the critical orbit of $Q$.

Theorem 5.2 also gives the recursion formula

$$g_{n+m} = g_n \circ \cdots \circ g_{n+m-2} \circ g_{n+m-1}.$$ 

On consecutive levels the first returns of a central piece $V_0^{n+m+1}$ fall inside the pieces $V_1^{n+m}, V_2^{n+m-1}, \ldots, V_m^{n+2}$ and $V_0^{n+m}$. From this, we obtain the annuli relation

$$g_{n+m}^{-1}(V_0^n \setminus V_0^{n+m}) = (V_0^n \setminus V_0^{n+m+1}).$$

In order to estimate the modulus of $(V_0^{n+m} \setminus V_0^{n+m+1})$, let us split the return map $g_{n+m}$ on

in the above mentioned composition of first returns. First, $g_{n+m-1}$ is 2 to 1 on the annulus $(V_0^{n+m} \setminus V_0^{n+m+1})$. Note that the image of $V_0^{n+m+1}$ is deep inside $V_1^{n+m}$; in fact, these two pieces are separated by a nested sequence of preimages of the central pieces $V_0^{n+1}, \ldots, V_0^{n+m-1}$. Due to
the pinching of pieces near repelling points, most of the modulus of \( g_{n+m-1} (V_0^{n+m} \setminus V_0^{n+m+1}) \subset V_0^{n+m-1} \) is concentrated in a region of \( V_0^{n+m-1} \) that resembles the immediate basin of the critical value of \( Q \). On this region, \( g_{n+m-2} \) is injective.

The remaining returns \( g_{n+m-3}, \ldots, g_n \) behave in a similar manner, essentially preserving the modulus on regions around non-central pieces. We can conclude that

\[
\text{mod} \left( V_0^{n+m} \setminus V_0^{n+m+1} \right) \approx \frac{1}{2} \text{mod} \left( V_0^n \setminus V_0^{n+m} \right).
\]

The right hand side can be estimated by applying the Extended Grötzsch Inequality to the annulus \( V_0^n \setminus V_0^{n+m} = (V_0^n \setminus V_0^{n+1}) \cup \ldots \cup (V_0^{n+m-1} \setminus V_0^{n+m}) \), we obtain

\[
\text{mod} \left( V_0^{n+m} \setminus V_0^{n+m+1} \right) \approx \frac{1}{2} \sum_{j=0}^{m-1} \text{mod}(V_0^j \setminus V_0^{j+1}) + \varepsilon_Q
\]

where \( \varepsilon_Q = (m-1) |\text{cap}_0(K_Q)| = (m-1) |(m-1) \ln 2 + \sum_{j=1}^{m-1} \ln |Q^{i_j}(0)|| \).

The recursive formula \( x_{n+m} = \frac{1}{2} (x_{n+m-1} + x_{n+m-2} + \ldots + x_n) + \varepsilon_Q \) has an asymptotic behavior that is ruled by the largest real root of its characteristic polynomial

\[
p(z) = z^m - \frac{1}{2} \left( z^{m-1} + z^{m-2} + \ldots + z + 1 \right)
\]

or, equivalently

\[
p(z) = z^m + \frac{1}{2} \frac{1 - z^m}{z - 1}.
\]

When \( m = 2 \), the largest root of \( z^2 - \frac{1}{2}(z + 1) = 1 \). Consequently, the growth of the moduli is dominated by a linear term \( \mu_n \sim An + B \). The only map with critical orbit of period \( m = 2 \) is \( Q(z) = z^2 - 1 \), for which \( \varepsilon_Q = \ln 2 \). The recursive relation \( \mu_n \sim \frac{2n-1}{2} + \frac{2n-2}{3} + \ln 2 \) gives

\[
A = \lim_{n \to \infty} \frac{\mu_n}{n} = \frac{\ln 2}{3}.
\]

The analysis is different when \( m \geq 3 \). If \( z \geq \frac{3}{2} \), the second term of \( \{S\} \) is \( \frac{1}{2} - z \) \( \leq 1 - z^m \), so the characteristic polynomial \( p \) satisfies \( p(z) \geq 1 \). On the other hand, it follows from \( \{S\} \) that \( p(1) < 0 \) (since \( m \geq 3 \)). Thus the largest root \( \kappa \) of \( p(z) \) is in the interval \( (1, \frac{3}{2}) \) and the exponential growth of \( \mu_n \) follows.

Clearly, \( \kappa \) does not depend asymptotically on the value of \( \varepsilon_Q \). Nevertheless, \( \kappa \) does vary with the period \( m \) of \( Q \). In fact, the sequence \( \{\kappa(m)\} \) converges to \( \frac{3}{2} \) as \( m \) increases. To see this, note from \( \{S\} \) and \( \kappa > 1 \) that \( p(\kappa) = 0 \) implies \( 2\kappa^{m+1} - 3\kappa^m + 1 = 0 \). The polynomial \( \hat{p}(z) = 2z^{m+1} - 3z^m + 1 \) satisfies \( \hat{p}(1) = 0, \hat{p}(\frac{3}{2}) = 1 \) and \( \hat{p}'(1) < 0 \) for all \( m \geq 3 \). Since \( \hat{p}'(z) = 2(m+1)z^m - 3mz^{m-1} \), the interval \( (1, \frac{3}{2}) \) contains one critical point \( z = \frac{3m}{2(m+1)} \) which squeezes \( \kappa(m) \) towards \( \frac{3}{2} \).

**Note.** One should contrast the above result with \( \{AM\} \). There, the authors show that for almost every non-hyperbolic real parameter, the principal moduli grow at least as fast as a tower of exponentials. The “slower” growth displayed by \( Q \)-recurrent polynomials has immediate geometric consequences.

**Definition 5.5.** Say that a compact set \( K \) is **hairy** at a point \( c \in K \) if there is a sequence \( \{\varepsilon_1, \varepsilon_2, \ldots\} \) converging to 0, such that \( \frac{1}{\varepsilon} \cdot (K - c) \cap \mathbb{D} \) becomes dense in \( \mathbb{D} \).

If \( K \) is hairy at \( c \) for any sequence of scaling factors \( \{\varepsilon_j\} \), we say that it satisfies **hairiness at arbitrary scales**.
By an observation of Rivera-Letelier [R-L], the construction of [W] can be extended to prove hairiness of $M$ at any critically recurrent non-renormalizable parameter. The idea is as follows:

Since $K_c$ is connected, it contains a path from 0 to $\beta$. This crosses every principal annuli from one boundary component to the other. Choosing a high enough level, the annulus $A_n$ can be rescaled to constant diameter containing a hair that connects the outer boundary with a small neighborhood of 0. The pull-backs by consecutive first return maps duplicate the number of hairs inside deeper annuli and this collection of hairs is equidistributed around 0 (control of geometry). The hairiness of $K_c$ is then translated to the parameter plane to obtain the result.

Rivera-Letelier has announced a proof that the real quadratic Fibonacci polynomial displays hairiness at arbitrary scales. The proof relies in an essential way on the linear growth of moduli, so it holds true for any parameter in $\mathbb{F}_{\text{Fib}}$. Other $Q$-recurrent polynomials miss this sharper property in account of the exponential growth of their principal moduli. It should be observed that this same property creates a somewhat embarrassing difficulty; since the moduli grow so fast, computer generated pictures fail to exhibit a convincingly hairy picture. In order to do so, it would be necessary to reach deep levels of the nest that may be out of the range of resolution of the software used.

5.4. Meta-Chebyshev. Starting with the chain $L_R \tilde{R} \tilde{L} L$, construct an infinite sequence by the following iterative procedure. At each step, concatenate a second copy of the current chain on which the second to last marked symbol is substituted by its opposite. The result is

$$\Theta : L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\tilde{R}L\tilde{L}L\cdots$$

In order to verify that this is an admissible kneading sequence, we have to describe the sequence $\epsilon_1 \epsilon_2 \ldots$ of accumulated orientation reversals in $\Theta$; that is, $\epsilon_j$ is + or − depending on whether the number of L’s up to position $j$ is even or odd. Then we only have to prove that for any $m$, the least $i$ such that $\epsilon_{m+i} \neq \epsilon_m \cdot \epsilon_i$ satisfies $\epsilon_i = -$. The sequence of $\epsilon_j$ begins:

$$-\cdots - + + + - - - - + + - - + + + - + - - - + + - - + - + - + \cdots$$

and the rule to construct it is as follows: Start with the chain $-\cdots - +$. At each step make a second unchecked copy of the current chain and invert every symbol that appears to the left of the second to last check; then concatenate this copy to the right and put a check on the last symbol.

This sequence starts with $-\cdots - +$ and every mark will be on a $-$ symbol. It follows that there cannot be more that three $+$ in a row so the admissibility condition is satisfied. Thus, the kneading sequence $\Theta$ can be realized by a real polynomial. In fact, by $P_{\text{MCheb}}(z) := z^2 - 1.87450961730020085\ldots$

This map was constructed with the requirement that the graphs $\Gamma(F_n)$ are isomorphic to $\Gamma(P_n(f_{-2}))$, where $f_{-2} : z \mapsto z^2 - 2$ is the Chebyshev polynomial. The motivation for this example is to investigate what properties of $Q$-recurrent polynomials will hold for parameters that imitate the behavior of (non-periodic) postcritically finite maps. Actually, the construction of $P_{\text{MCheb}}$ is very similar to that of $Q$-recurrent polynomials; the main difference being as follows. Since the critical orbit of $f_{-2}$ does not return to the center, the first return to level $n+1$ must be delayed until after the composition of $n$ first return maps $g_1 \circ \ldots \circ g_n$, when the critical orbit falls in $V_1^{(0)}$.

The parameter was chosen so that the first return to level $n+1$ occurs exactly at this moment; that is, $g_{n+1}$ is precisely $g_1 \circ g_2 \circ \ldots \circ g_n$ for all $n$. These are the iterates marked with a check. Moreover, the choice of frame orientations that result in a real parameter imposes the required label sequence $(2; 'Z_1', 'LZ_0', 'LRZ_0', 'LRRZ_0', 'LRRRZ_0', \ldots)$. By analogy with the critical orbit of $f_{-2}$, every nest level of $P_{\text{MCheb}}$ has two lateral pieces and the itinerary of $V_1^n$ includes an infinite number of visits to $V_0^n$ after the first return to $V_0^{n+1}$.

Since this combinatorial type is admissible, the results of [R] guarantee an uncountable set of complex parameters with the same combinatorics. By the rigidity result of Yoccoz, there cannot
be other real polynomials in this class.

The methods used to work with $Q$-recurrent polynomials are not enough to study the nest of $P_{MCheb}$ in a metric sense. In particular, we have relied on the fact that $K_Q$ had a non-empty interior; this is not the case for $K_{-2}$. Nevertheless, the analogy is good enough that it is natural to pose the following.

**Conjecture.** There are suitable rescalings of the nest pieces of $P_{MCheb}$ such that the functions induced from the first return maps converge to $f_{-2}$ and such that properly rescaled pieces converge to the interval $[-2, 2]$ in the Hausdorff topology.

### 6. Parameter space

One of the most amazing attributes of complex quadratic dynamics is the replication of dynamical features in the parameter plane. For instance, the structure of $\Delta$ features in the parameter plane. For instance, the structure of a limb $L_{p/q}$ reflects the initial steps of the critical orbit for any parameter contained in it. In [W], Tan Lei showed that for a strictly preperiodic parameter $c$, the Julia set of $f_c$ and the Mandelbrot set exhibit local asymptotic similarity around $c$.

As it has been mentioned, a result of similar nature appears in [W] where Wenstrom shows that the parapieces around the real Fibonacci parameter $c_{fib}$ are asymptotically similar to the central pieces in the principal nest of $f_{c_{fib}}$. Thus, $\Delta^n(c_{fib}) \rightarrow K_{-1}$ in shape and the author exploits this geometric result to obtain hairiness of $M$ around $c_{fib}$.

This Section discusses a generalization of the above results to the family of all $Q$-recurrent parameters. Note that the maps $Q$ are dense in $\partial M$ and that for each one there is an uncountable set of $Q$-recurrent parameters.

**Theorem 6.1.** Let $Q = Q(z)$ be the center of a prime hyperbolic component with critical period $m$, and let $c_Q$ be a $Q$-recurrent parameter. Then the parapiece around $c_Q$ is infinite and the parapieces $\{\Delta^j(c_Q)\}$ converge in shape to the filled Julia set $K_Q$.

This will require translating the corresponding result obtained in the dynamical plane to the space of parameters. To do this, we need to introduce certain auxiliary parapieces; describe in detail the boundary of $\Delta^j(c_Q)$ and define a map $M_n : \Delta^n(c_Q) \rightarrow \mathbb{C}$ that “rescales” $\Delta^n$ to a compact set close to $K_Q$.

From the above result follows the possibility of computing the rate of growth of the paramoduli. Since the paramoduli increase at least linearly, the set of $Q$-recurrent parameters is a Cantor set of Hausdorff dimension $0$.

For the rest of this Section, unless explicitly mentioned, fix a map $Q = Q(z)$ in the center of a prime hyperbolic component such that the critical orbit has period $m$; also, $c_Q$ will stand for a fixed $Q$-recurrent parameter.

#### 6.1. Auxiliary parapieces

Consider the first return map $g_{n-1} : V_{0}^{n-1} \rightarrow V_{0}^{n-2}$. Given that $V_1^n \subset V_0^{n-1}$, we can study the effect of $g_{n-1}$ on $V_1^n$; there, the condition of $Q$-recurrency gives:

$$g_{n-1}(V_1^n) \subset V_2^{n-1}.$$  

Applying $g_{n-2}$ we obtain

$$g_{n-2} \circ g_{n-1}(V_1^n) \subset g_{n-2}(V_2^{n-1}) \subset V_3^{n-2}.$$  

This procedure can continue further for a total of $m - 2$ steps:

$$g_{n-m+2} \circ \cdots \circ g_{n-1}(V_1^n) \subset V_{m-1}^{n-m+2},$$

where in fact, the last image is contained in a nest of intermediate pieces inside $V_{m-1}^{n-m+2}$; see Figure [S]. Under $g_{n-m+1}$, the piece $V_{m-1}^{n-m+2}$ maps onto $V_0^{n-m+1}$ so the intermediate pieces mentioned will
map onto the central pieces $V_0^{n-m+2}, V_0^{n-m+3}, \ldots, V_0^{n-2}$ and the combined effect on $V_1^n$ will be (recall Definition 2.12):

\[
g_{n-m+1} \circ \ldots g_{n-1}(V_1^n) = g_{n,1}(V_1^n) = V_0^{n-1}.
\]

Since $V_0^n \subseteq V_0^{n-1}$, the following is well defined.

**Definition 6.1.** Denote by $U_1^n \subseteq V_1^n$ and $F^*_{n+2} \subseteq U_1^n$ the $(g_{n,1})$-pull-backs of $V_0^n$ and $F_{n+2} \subseteq V_0^n$, respectively. Compare Figure 8.

Note that $F^*_{n+2}$ is known once the nest structure up to level $n$ is given. However, assuming that the nest of our parameter displays the $Q$-recurrency type up to level $n+1$, it is possible to say more. Since $g_{n+1}(0) = g_{n-m+1} \circ \ldots g_{n-1} \circ g_n(0)$, we must have

\[
g_n(0) \in F^*_{n+2}.
\]

**Figure 8.** Construction of $U^n$ and $F^*_{n+2}$. The domain of $g_{n-1}$ is $V^{n-1}$; in particular, it maps $V_1^n$ inside $V_2^{n-1}$. Further maps $g_{n-2}, g_{n-3}, \ldots$ take the current ensemble inside the next piece of previous level, until $V_{m-1}^{n-m+2}$. Note that $g_{n-m+1}$ takes $V_{m-1}^{n-m+2}$ onto the central piece $V_0^{n-m+1}$, instead of a lateral one. This maps the gray piece (the image of $V_1^n$) onto $V_0^{n-1}$. Then, we can pull $V_0^n$ back all the way to the piece $U^n$ inside $V_1^n$. Also, $U^n$ has a frame $F^*_{n+2}$ which is the corresponding pull-back of $F_{n+2} \subseteq V_0^n$. Neither $U^n$ nor $F^*_{n+2}$ are drawn. See also Figure 6.

Let us pass to the parameter plane. Our initial goal is to obtain a precise control of the combinatorics inside relevant consecutive parapieces. In the first place, $\Delta^n$ is the set of parameters that have the same nest combinatorics as $c_Q$, up to the first return $g_n(0)$ to $V_0^{n-1}$.

**Definition 6.2.** We introduce two new auxiliary parapieces.

- $\Delta_{n+2}^n$ is the set of parameters such that $g_n(0)$ falls inside the frame $F_{n+1} \subseteq V_0^{n-1}$. 

• \( \Xi^{n+1} \) is the set of parameters such that \( g_n(0) \) falls in \( V_1^n \subseteq F_{n+1} \).

Each region \( \Xi^n \) is well defined as a parapiece since it represents the return to an explicit piece of the puzzle. On the other hand, \( \Delta^{n+2}_* \) is actually the union of several parapieces; nevertheless, it is convenient to regard it as a parapiece to avoid longer descriptions. With this in mind, we are interested in the fact that parapieces of consecutive levels can be described in terms of a single first return map. From Formulas \( (6), (9) \) and \( (10) \), we obtain:

\[
\begin{align*}
    c \in \Delta^n & \iff g_n(0) \in V_0^{n-1} \iff g_{n+1}(0) \in V_0^{n-m} \\
    c \in \Delta_{n+2}^* & \iff g_n(0) \in F_{n+1} \iff g_{n+1}(0) \in F_{n-m+2} \\
    c \in \Xi^{n+1} & \iff g_n(0) \in V_1^n \iff g_{n+1}(0) \in V_0^{n-1} \\
    c \in \Delta_{n+1} & \iff g_n(0) \in U^n \iff g_{n+1}(0) \in V_0^n \\
    c \in \Delta_{n+3}^* & \iff g_n(0) \in F_{n+2}^* \iff g_{n+1}(0) \in F_{n+2}.
\end{align*}
\]

(11)

Since \( F_{n+2}^* \subseteq U^n \subseteq V_1^n \subseteq F_{n+1} \subseteq V_0^{n-1} \), we have the following parapiece inclusions:

\[
\Delta_{n+3}^* \subseteq \Delta_{n+1} \subseteq \Xi^{n+1} \subseteq \Delta_{n+2}^* \subseteq \Delta^n.
\]

(12)

### 6.2. Shape and paramoduli.

In order to prove Theorem 6.1, let us introduce the map \( M_n : \Xi^{n-1} \rightarrow \mathbb{C} \), where \( \Xi^{n-1} \) belongs to the parapiece of the fixed parameter \( c_Q \). Recall that \( \alpha_{n-2} = \frac{\alpha_Q}{\alpha_{n-2}} \) is the rescaling factor that defines \( \tilde{F}_n[c_Q] = \alpha_{n-2} \cdot F_n[c_Q] \).

For \( c \in \Xi^{n-1} \), let

\[ M_n(c) = \alpha_{n-2} \cdot g_{n-1}(0)[c]. \]

**Proof of Theorem 6.1.** From Table \((11)\), when \( c \in \Delta_{n+1}^* \) the first return \( g_{n-1}(0)[c] \) is in \( F_n \). Recall that for \( n \) large, \( \tilde{F}_n[c] \) is exponentially close to \( K_Q \).

Fix an \( \varepsilon > 0 \) and find \( n \) big enough so that both rescalings \( \alpha_{n-2}[c] \cdot F_n = \tilde{F}_n \) and \( \alpha_{n-2} \cdot F_n \) are at most \( \frac{\varepsilon}{2} \) close to each other and to \( K_Q \). This means that \( M_n(\Delta^{n+1}_*) \) is a compact set \( \varepsilon \)-close to \( K_Q \).

By definition, the parapiece \( \Xi^{n-1} \) is the set of parameters \( c \) for which \( g_{n-2}(0)[c] \) falls on the lateral piece \( V_0^{n-2} \). Since this map is a first return, Proposition 2.3 implies that the correspondence \( c \mapsto g_{n-2}(0)[c] \) is univalent in \( \Xi^{n-1} \). Moreover, \( V_0^{n-2}[c] \) is at a finite distance away from the central piece \( V_0^{n-2}[c] \) for all \( c \), so the image of \( \Xi^{n-1} \) under \( c \mapsto g_{n-2}(0)[c] \) is uniformly far from \( 0 \) and similarly for all further iterations up to the first return \( g_{n-1}(0)[c] \). Again, we can use Proposition 2.3 to deduce that \( M_n \) is univalent in its entire domain.

Since \( n \) is big, the modulus of the annulus \( \left( \text{int} V_0^{n-3} \setminus F_n \right)[c] \) is large for every \( c \in \Xi^{n-1} \). In particular, since \( \tilde{F}_n[c] \) has bounded diameter, this implies that the distance between the point \( \alpha_{n-2}[c] \cdot g_{n-1}(0)[c] \in \partial V_0^{n-3}[c] \) and the curve \( \partial \tilde{F}_n[c] \) is exponentially big for \( c \in \partial \Xi^{n-1} \). Let \( d_n \) be the minimum of these distances over all \( c \). Then, \( M_n(\partial \Xi^{n-1}) \) and \( M_n(\partial \Delta^{n+1}_*) \) are at least a distance \( d_n - \varepsilon \sim d_n \gg \infty \) apart. We can conclude that the modulus of \( M_n(\text{int} \Xi^{n-1} \setminus \Delta^{n+1}_*) \) is arbitrarily large and so will be the modulus of \( \text{int} \Xi^{n-1} \setminus \Delta^{n+1}_* \).

We have shown that the map \( M_n \) is univalent in its domain and the modulus of \( \text{int} \Xi^{n-1} \setminus \Delta^{n+1}_* \) is big. Then, by the Koebe distortion Theorem, \( M_n \) is asymptotically linear in a neighborhood of \( \Delta^{n+1}_* \). Since \( M_n(\Delta^{n+1}_*) \) is \( \varepsilon \)-close to \( K_Q \), we can conclude the proof.

As an immediate consequence of this control over the shape of parapieces, we can compute the rate of growth of principal paramoduli \( \mu_n \).

**Corollary 6.2.** The annuli of consecutive parapieces in the nest of a \((z^2 - 1)\)-recurrent map grow linearly at the rate

\[
\lim_{n \to \infty} \frac{\mu_n}{n} = \frac{2 \ln 2}{3}.
\]
For any other $Q$-recurrent map (where $Q$ has critical orbit of period $m \geq 3$) the moduli grow exponentially at the rate

$$\lim_{n \to \infty} \frac{\mu_n}{\mu_{n+1}} = \kappa_m,$$

where $\kappa_m$ is the same constant as in Theorem 5.5, converging to $\frac{3}{2}$ as the period $m$ of $Q$ increases.

**Proof.** First note that, although $U^n$ is defined as a pull-back of $V^n_0$, relation 10 shows that this piece is just $g_n(V^n_0 + 1)$.

Now, when $c \in \Delta^n$, the first return $g_n(0)$ falls in $V^n_0 - 1$. For $c \in \Delta^{n+1}$, $g_n(0)$ is in $U^n$. From the previous result, $c \mapsto g_n(0)[c]$ is an almost linear map taking the annulus $(\Delta^n \setminus \Delta^{n+1})$ close to $(V_0^n \setminus U_n)[c]$. Therefore

$$\mod(\Delta^n \setminus \Delta^{n+1}) \sim \mod(V_0^n \setminus U^n) \sim 2 \mod(V_0^n \setminus V_0^{n+1}).$$

Corollary 6.2 now follows from Theorem 5.5. 

6.3. **Auto-similarity in the Mandelbrot set.** The discovery that parapieces around $c_{4b}$ are similar to the Julia set of $-1$ revealed one more level of complexity in the structure of $M$ since it relates the dynamics of two different parameters. In this Subsection we use our results to take one further step. Having at our disposal an infinite collection of superattracting parameters, we reveal an interesting relation between two arbitrary parameters on $\partial M$ whose combinatorics can be completely dissimilar.

The $Q$-recurrency phenomenon is not restricted to the Cantor sets described so far. As part of the proof of the next Theorem, we will show that parapieces whose shape approximates $K_Q$ are dense on $\partial M$. This requires relaxing the definition of $Q$-recurrency which assumes that the correct combinatorics start from level 0. Instead, we allow critical orbits that behave arbitrarily for several levels before settling in the desired $Q$-recurrent pattern. This critical behavior is referred to as *generalized $Q$-recurrency*. The assertion of Theorem 6.3 follows; it can be interpreted as saying that the geometry of most Julia sets is replicated near arbitrary locations of the boundary of $M$.

**Theorem 6.3.** Let $c_1, c_2 \in \partial M$ be two parameters such that $f_{c_2}$ has no indifferent periodic orbits that are rational or linearizable. Then there exists a sequence of parapieces $\{\Upsilon_1, \Upsilon_2, \ldots\}$ (most likely not nested) converging to $c_1$ as compact sets, but such that $\Upsilon_n \longrightarrow K_{c_2}$ in shape.

**Proof.** It is not difficult to obtain the result of Theorem 5.2 in more generality. In fact, inside any ball $B_\varepsilon(c)$ with $c \in \partial M$, we can find a system of parameters for which the first return maps converge (after scaling) to a given superattracting map $Q$.

To see this, simply consider a tuned copy of $M$ contained in $B_\varepsilon$. All parameters in this copy $M'$, are renormalizable by the same combinatorics. In particular, there will be parameters whose renormalization is hybrid equivalent to a $Q$-recurrent map. For these parameters a high level of the frame will contain a substructure whose graph is isomorphic to $\Gamma_0(Q)$ and we can start the same construction as in the proof of Theorem 5.2 to produce frame-like structures whose graphs are isomorphic to $\Gamma_n(Q)$. Since the combinatorics is prescribed by a polynomial, there can be no obstructions just as in the original case. Then, the rescaled first return maps will converge to $Q$ as before. Moreover, we can translate the shape property to the parameter plane.

Note that this argument is equivalent to prescribing the itineraries of nest pieces arbitrarily on the initial levels and then proving that they can be admissibly extended on subsequent levels to match the pattern given in Formula 10.

Now consider the filled Julia set of $f_{c_2}$. We know from [11] that there is a sequence $\{Q_1, Q_2, \ldots\}$ of superattracting polynomials in a prime hyperbolic component of $M$ such that $K_{Q_n}$ can be arbitrarily approximated by filled Julia sets: $K_{Q_n} \longrightarrow K_{c_2}$. To fix ideas, let us choose subindices so that the Hausdorff distance is $\dist_H(K_{Q_n}, K_{c_2}) < \frac{1}{2^m}$.
For any $n$, consider the ball $B_n(c_1)$ and locate a generalized $Q_n$-recurrent parameter $s_n$. By going to a deep enough level, we can find some parapiece $\gamma_n \subset B_{1/n}$ around $s_n$ whose shape is $(\frac{1}{2n})$-close to $K_{Q_n}$; that is, so that there is a rescaling $\tilde{\gamma}_n$ of $\gamma_n$ for which $\text{dist}_H(\tilde{\gamma}_n, K_{Q_n}) < \frac{1}{2n}$.

Since $\gamma_n \subset B_{1/n}$, the sequence $\{\gamma_n\}$ consists of parapieces that get arbitrarily small and converge to $c_1$, while at the same time $\text{dist}_H(\tilde{\gamma}_n, K_{Q_n}) < \frac{1}{2n}$, so $\gamma_n \rightarrow K_{c_2}$ in shape.

\section*{Appendix}

The theorems of previous sections rely on several notions and results of complex analysis. In this Appendix we will describe the necessary ideas on which the text relies. For references and proofs of these results, the reader can consult [A], [DH3] and [LV].

\section{Carathéodory topology}

A sequence of pointed disks $\{(U_n, x_n)\}$ is said to converge to the pointed disk $(U, x)$ in the Carathéodory topology if:

(a) $x_n \rightarrow x$.

(b) For every compact $K \subset U$ there is an $N$ such that $K \subset U_n$ for $n > N$.

(c) If $V \ni x$ is an open connected region and $V \subset U_n$ for infinitely many $n$, then $V \subset U$.

The interpretation of convergence in this topology is as follows. Consider the complements $\hat{C} \setminus U_n$ which are compact sets converging to $X$ in the Hausdorff topology. To satisfy the above conditions, $x \notin X$ and $U$ is the component of $\hat{C} \setminus X$ that contains $x$.

We can describe a similar convergence for a sequence of annuli $\{A_n\}$. In this case, we require that $\hat{C} \setminus A_n \rightarrow Y$ in the Hausdorff sense and that the core geodesics $\gamma_n$ of the annuli $A_n$ converge to a non-degenerate loop $\gamma \subset Y$. The Carathéodory limit will be the doubly connected component of $\hat{C} \setminus Y$ containing $\gamma$.

\section{Modulus and capacity as conformal invariants}

Let $R \subset \mathbb{C}$ be a doubly connected region in the plane. Consider the family $\Gamma$ of curves $\gamma \subset R$ whose endpoints are on the boundary of $R$. Given a conformal metric $\rho$ on $R$, we can define the length of $\Gamma$ as $L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_{\gamma} \rho |dz|$ and the area of $R$ as $A_\rho(R) = \int \int_A \rho^2 dx dy$.

\textbf{Definition 6.3.} The modulus of $R$ is

$$\text{mod}(R) = \sup_\rho \frac{(L_\rho(\Gamma))^2}{A_\rho(R)}$$

where the supremum is taken over all conformal metrics $\rho$ with non-degenerate area: $A_\rho(R) \neq 0, \infty$.

It can be shown that the modulus is a conformal invariant. As a consequence, we can give an alternative definition as follows. Consider the Riemann map $\varphi : R \rightarrow S$ where $S$ is the round annulus $S = \{1 < z < r\}$. Then $\text{mod}(R) = 2\pi \ln r$.

Another conformal invariant, this time of topological disks, is the \textit{capacity} or \textit{conformal radius}.\textbf{Definition 6.4.} Let $U \subset \mathbb{C}$ be a simply connected domain, $z \in U$ and $\varphi : (\mathbb{D}, 0) \rightarrow (U, z)$ the Riemann map from the unit disk to $U$ that satisfies $\varphi'(0) > 0$. The \textbf{capacity} of $U$ with respect to $z$ is

$$\text{cap}_z(U) = \ln \varphi'(0).$$

Of interest to us, will be the following property of capacities and moduli.

\textbf{Theorem B.1.} Both capacity and modulus are quantities that vary continuously with respect to the Carathéodory topology.
C. **Grötzsch inequality.** The following result (and its quantitative version) is essential to estimate the modulus of an annulus that is split in subannuli.

**Theorem C.1. Extended Grötzsch Inequality:** Let $K \subset \mathbb{C}$ be a simply connected compact set and denote $\text{int}_0 K$ the component of its interior that contains 0.

(a) Consider two topological disks $0 \in U \subset \text{int}_0 K \subset V$. Then

$$\text{mod}(V \setminus U) \geq \text{mod}(K \setminus U) + \text{mod}(V \setminus K).$$

(b) Let $\{U_n\}$ and $\{V_n\}$ be two sequences of nested topological disks satisfying

- $0 \in U_n \subset \text{int}_0 K$ and $\text{diam} U_n \searrow 0$
- $K \subset V_n$ and $\text{dist}(K, \partial V_n) \nearrow \infty$.

Then the deficit in the Grötzsch inequality tends to

$$\lim_{n \to \infty} \left( \text{mod}(V_n \setminus U_n) - \text{mod}(K \setminus U_n) - \text{mod}(V_n \setminus K) \right) = |\text{cap}_0(\text{int}_0 K)| + |\text{cap}_\infty(K)|.$$

An important observation is the fact that equality in (a) is achieved if and only if $\partial K \subset V \setminus U$ maps to a centered circle under the Riemann map.

D. **Koebe distortion Theorem.**

**Definition 6.5.** Given an analytic univalent map $\varphi$ between regions $U$ and $V$, the **distortion** of $\varphi_0$ is defined as:

$$\text{Dist}(\varphi_0) = \sup_{x,y \in U} \frac{\varphi_0'(x)}{\varphi_0'(y)}.$$  

Koebe’s Theorem provides great control of the distortion when there is enough space between $U$ and $V$.

**Theorem D.1.** Let $U$ and $V$ be two topological disks with $U \setminus V$. Then there is a constant $C$ such that for any univalent map $\varphi(U) = V$,

$$\text{Dist}(\varphi) < C.$$  

Moreover, $C = 1 + O(e^{-\text{mod}(V \setminus U)})$ as the modulus goes to $\infty$.

E. **Teichmüller space.** The Teichmüller space of a Riemann surface carries a great deal of structural information. Here we focus in the case that the surface $S$ is the complex plane punctured at a finite set $\mathcal{O}$. Then, the Teichmüller space $T_S$ can be described as a quotient of the space of quasiconformal deformations of $S$ (i.e. the family of maps $\{h : S \to \mathbb{C} | h$ is a qc homeomorphism$\}$), where two deformations $h_1$ and $h_2$ are identified if and only if there is a conformal change of coordinates $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ h_1$ is isotopic to $h_2$ relative to the puncture set $h_2(\mathcal{O})$.

**Note.** The coordinate changes $\varphi$ are affine maps, so the deformation of $\mathcal{O}$ within a class is determined up to translation and complex scaling. Therefore, we can normalize a deformation $h$ by requiring that $h$ fixes two distinguished points in $\mathcal{O}$. These could be, for instance, the critical point and critical value of $Q$ in the case that $\mathcal{O}$ is the postcritical set of a hyperbolic map $Q$.

It is fundamental to consider an alternate description of $T_S$ in terms of Beltrami differentials. Fix two almost complex structures on $S$ determined by their Beltrami coefficients $\mu_0$ and $\nu_0$. Assume that they are related by $\nu = \overline{h} \cdot \mu$, where $h : S \to S$ is a quasiconformal self homeomorphism of $S$ which is homotopic to id relative to $\mathcal{O}$. Then, the straightening maps $h_\mu$ and $h_\nu$ are two quasiconformal deformations of $S$ in the same equivalence class in $T_S$. 29
Conversely, we can associate to a deformation $h$ the almost complex structure $h^*\sigma = \frac{\partial h}{\partial z} \bar{\frac{\partial h}{\partial \bar{z}}}$ where $\sigma$ is the standard structure. It is easy to verify that this correspondence lifts to the equivalence classes where it induces a bijection.

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