Free algebras in Boolean algebras with operators

Tarek Sayed Ahmed
Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

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Abstract. We study atomicity of free algebras in varieties of Boolean algebras with operations, and we give some applications to cylindric-like algebras, mostly simplifying existing proofs in the literature due to Németi, Tarski and Givant. We obtain a new result concerning Pinters algebra, namely that the free n dimensional representable algebra with m free generators, can be generated but not freely with a (redundant) set of m elements.

Cylindric and relation algebras were introduced by Tarski to algebraize first order logic. The structures of free cylindric and relation algebras are quite rich since they are able to capture the whole of first order logic, in a sense. One of the first things to investigate about these free algebras is whether they are atomic or not, i.e. whether their boolean reduct is atomic or not. By an atomic boolean algebra we mean an algebra for which below every non-zero element there is an atom, i.e. a minimal non-zero element. Throughout n will denote a countable cardinal (i.e. $n \leq \omega$). More often than not, $n$ will be finite. $\mathsf{CA}_n$ stands for the class of cylindric algebras of dimension $n$. For a class $K$ of algebras, and a cardinal $\beta > 0$, $\mathfrak{Fr}_\beta K$ stands for the $\beta$-generated free $K$ algebra. In particular, $\mathfrak{Fr}_\beta \mathsf{CA}_n$ denotes the $\beta$-generated free cylindric algebra of dimension $n$. The following is known: If $\beta \geq \omega$, then $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is atomless (has no atoms) [Pigozzi [3] 2.5.13]. Assume that $0 < \beta < \omega$. If $n < 2$ then $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is finite, hence atomic, [3] 2.5.3(i). $\mathfrak{Fr}_\beta \mathsf{CA}_2$ is infinite but still atomic [Henkin, [3] 2.5.3(ii), 2.5.7(ii.).] If $3 \leq n < \omega$, then $\mathfrak{Fr}_\beta \mathsf{CA}_n$ has infinitely many atoms [Tarski, [3] 2.5.9], and it was posed as an open question, cf. [3] problem 4.14, whether it is atomic or not. Here we prove, as a partial solution of problem 4.14 in [3], and among other things, that $\mathfrak{Fr}_\beta \mathsf{CA}_n$ is not atomic for $\omega > \beta > 0$ and $\omega > n \geq 4$. Here we investigate atomicity or non atomicity of free algebras in (often discriminator varieties) of Boolean algebras with operators.
1 Free algebras in a broad context

In cylindric algebra theory, whether the free algebras are atomic or not is an important topic. In fact, Németi proves that for \( n \geq 3 \) the free algebras of dimension \( n \) on a finite set of generators are not atomic, and this is closely related to Godel’s incompleteness theorems for the finite \( n \)-variable fragments of first order logic. We first start by proving slightly new results concerning free algebras of classes of BAO’s.

**Definition 1.1.** Let \( K \) be variety of BAO’s. Let \( \mathcal{L} \) be the corresponding multimodal logic. We say that \( \mathcal{L} \) has the Godel’s incompleteness property if there exists a formula \( \phi \) that cannot be extended to a recursive complete theory. Such formula is called incompletable.

Let \( \mathcal{L} \) be a general modal logic, and let \( \mathcal{F}_m \equiv \) be the Tarski-Lindenbaum formula algebra on finitely many generators.

**Theorem 1.2.** (Essentially Nemeti’s) If \( \mathcal{L} \) has G.I, then the algebra \( \mathcal{F}_m \equiv \) is not atomic.

**Proof.** Assume that \( \mathcal{L} \) has G.I. Let \( \phi \) be an incompletable formula. We show that there is no atom in the Boolean algebra \( \mathcal{F}_m \equiv \) below \( \phi/ \equiv \). Note that because \( \phi \) is consistent, it follows that \( \phi/ \equiv \) is non-zero. Now, assume to the contrary that there is such an atom \( \tau/ \equiv \) for some formula \( \tau \). This means that \( (\tau \land \phi)/ \equiv \tau/ \equiv \). Then it follows that \( \vdash (\tau \land \phi) \implies \phi \), i.e. \( \vdash \tau \implies \phi \). Let \( T = \{\tau, \phi\} \) and let \( \text{Conseq}(T) = \{\psi \in Fm : T \vdash \psi\} \). \( \text{Conseq}(T) \) is short for the consequences of \( T \). We show that \( T \) is complete and that \( \text{Conseq}(T) \) is decidable. Let \( \psi \) be an arbitrary formula in \( \mathcal{F}_m \). Then either \( \tau/ \equiv \psi/ \equiv \) or \( \tau/ \equiv \neg\psi/ \equiv \) because \( \tau/ \equiv \) is an atom. Thus \( T \vdash \psi \) or \( T \vdash \neg \psi \). Here it is the exclusive or i.e. the two cases cannot occur together. Clearly \( \text{Conseq}(T) \) is recursively enumerable. By completeness of \( T \) we have \( \mathcal{F}_m \setminus \text{Conseq}(T) = \{\neg \psi : \psi \in \text{Conseq}(T)\} \), hence the complement of \( \text{Conseq}(T) \) is recursively enumerable as well, hence \( T \) is decidable. Here we are using the trivial fact that \( \mathcal{F}_m \) is decidable. This contradiction proves that \( \mathcal{F}_m \equiv \) is not atomic.

In the following theorem, we give a unified perspective on several classes of algebras, studied in algebraic logic. Such algebras are cousins of cylindric algebras; though the differences, in many cases, can be subtle and big.

(1) holds for diagonal free cylindric algebras, cylindric algebras, Pinter’s substitution algebras (which are replacement algebras endowed with cylindric-fiers) and quasipolyadic algebras with and without equality when the dimension is \( \leq 2 \). (2) holds for Boolean algebras; we do not know whether it extends any further. (3) holds for such algebras for all finite dimensions.
In fact, (1) holds for any discriminator variety $V$ of $BAO$'s, with finitely many operators, when $V$ is generated by the discriminator class $SirK$, of subdirectly indecomposable algebras having a discriminator term. To prove the latter, we start by a (well-known) lemma:

**Lemma 1.3.** Let $L \supseteq L_{BA}$ be a functional signature, and $V$ a variety of $L - BAO$'s. Let $d(x)$ be a unary $L$ term. Then the following are equivalent:

1. $d$ is a discriminator term of $SirV$, so that $V$ is a discriminator variety.

2. all equations of the following for are valid in $V$:
   1. $x \leq d(x)$
   2. $d(d(x)) \leq d(x)$
   3. $f(x) \leq d(x)$ for all $f \in L \sim L_{BA}$

**Theorem 1.4.** Let $K$ be a variety of Boolean algebras with finitely many operators.

1. Assume that $K = V(\text{Fin}(K))$, and for any $\mathfrak{B} \in K$ and $b' \in \mathfrak{B}$, there exists a regular $b \in \mathfrak{B}$ such that $\lg \mathfrak{B} \{b'\} = \lg \mathfrak{B} \{b\}$. If $\mathfrak{A}$ is finitely generated, then $\mathfrak{A}$ is atomic, hence the finitely generated free algebras are atomic. In particular, if $K$ is a discriminator variety, with discriminator term $d$, then finitely generated algebras are atomic. (One takes $b' = d(b)$).

2. Assume that $V$ is a $BAO$ and that the condition above on principal ideals, together with the condition that that if $b_1'$ and $b_2'$ are the generators of two given ideals happen to be a partition (of the unit), then $b_0, b_1$ can be chosen to be also a partition. Then $\mathfrak{F}_{\beta}K_{\alpha} \times \mathfrak{F}_{\beta}K_{\alpha} \cong \mathfrak{F}_{\beta+1}K$. In particular if $\beta$ is infinite, and $\mathfrak{A} = \mathfrak{F}_{\beta}K$, then $\mathfrak{A} \times \mathfrak{A} \cong \mathfrak{A}$.

3. Assume that $\beta < \omega$, and assume the above condition on principal ideals. Suppose further that for every $k \in \omega$, there exists an algebra $\mathfrak{A} \in K$, with at least $k$ atoms, that is generated by a single element. Then $\mathfrak{F}_{\beta}K$ has infinitely many atoms.

4. Assume that $K = V(\text{Fin}(K))$. Suppose $\mathfrak{A}$ is $K$ freely generated by a finite set $X$ and $\mathfrak{A} = \mathfrak{S}gY$ with $|Y| = |X|$. Then $\mathfrak{A}$ is $K$ freely generated by $Y$.

**Proof.** (1) Assume that $a \in A$ is non-zero. Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of $\mathfrak{A}$ into a finite algebra $\mathfrak{B}$ such that $h(a) \neq 0$. Let $I = \ker h$. We claim that $I$ is a finitely generated ideal. Let $R_I$ be the congruence relation corresponding to $I$, that is $R_I = \{(a, b) \in A \times A : h(a) = h(b)\}$. 

3
Let $X$ be a finite set such that $X$ generates $\mathfrak{A}$ and $h(X) = \mathfrak{B}$. Such a set obviously exists. Let $X' = X \cup \{x + y : x, y \in X\} \cup \{-x : x \in X\} \cup \bigcup_{f \in \{f(x) : x \in X\}}. \ Let \ R = \mathfrak{S}g^{\mathfrak{A}}(R_I \cap X \times X')$. Clearly $R$ is a finitely generated congruence and $R_I \subseteq R$. We show that the converse inclusion also holds.

For this purpose we first show that $R(X) = \{a \in A : \exists x \in X, a \in R\} = \mathfrak{A}$. Assume that $xRa$ and $yRb$, $x, y \in X$ then $x + yRa + b$, but there exists $z \in X$ such that $h(z) = h(x + y)$ and $zR(x + y)$, hence $zR(a + b)$, so that $a + b \in R(X)$. Similarly for all other operations. Thus $R(X) = A$. Now assume that $a, b \in A$ such that $h(a) = h(b)$. Then there exist $x, y \in X$ such that $xRa$ and $xRb$. Since $R \subseteq \ker h$, we have $h(x) = h(a) = h(b) = h(y)$ and so $xRy$, hence $aRb$ and $R_I \subseteq R$.

So $I = \lg\{b\}$ for some element $b$. Then there exists $b \in \mathfrak{A}$ such that $\lg^{\mathfrak{B}}\{b\} = \lg\{b\}$. Since $h(b) = 0$ and $h(a) \neq 0$, we have $a. -b \neq 0$. If $a. -b = 0$, then $h(a). -h(b) = 0$.

Now $h(\mathfrak{A}) \cong \mathfrak{A}/\lg^{\mathfrak{B}}\{b\}$ as $K$ algebras. Let $\mathfrak{N}_{-b}\mathfrak{A} = \{x : x \leq -b\}$. Let $f : \mathfrak{A}/\lg^{\mathfrak{B}}\{b\} \to \mathfrak{N}_{-b}\mathfrak{A}$ be defined by $\bar{x} \mapsto x. -b$. Then $f$ is an isomorphism of Boolean algebras (recall that the operations of $\mathfrak{N}_{-b}\mathfrak{B}$ are defined by relativizing the Boolean operations to $-b$.) Indeed, the map is well defined, by noting that if $x\delta y \in \lg^{\mathfrak{B}}\{b\}$, where $\delta$ denotes symmetric difference, then $x. -b = y. -b$ because $x, y \leq b$.

Since $\mathfrak{N}_{-b}\mathfrak{A}$ is finite, and $a. -b \in \mathfrak{N}_{-b}\mathfrak{A}$ is non-zero, then there exists an atom $x \in \mathfrak{N}_{-b}\mathfrak{A}$ below $a$, but clearly $\text{At}(\mathfrak{N}_{-b}\mathfrak{A}) \subseteq \text{At}\mathfrak{A}$ and we are done.

(2) Let $(g_i : i \in \beta + 1)$ be the free generators of $\mathfrak{A} = \mathfrak{Fr}_{\beta + 1}K$. We first show that $\mathfrak{N}_{g_\beta}\mathfrak{A}$ is freely generated by $\{g_\beta : i < \beta\}$. Let $\mathfrak{B}$ be in $K$ and $y \in \beta^\mathfrak{B}$. Then there exists a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $f(g_i) = y_i$ for all $i < \beta$ and $f(g_\beta) = 1$. Then $f \upharpoonright \mathfrak{N}_{g_\beta}\mathfrak{A}$ is a homomorphism such that $f(g_i, g_\beta) = y_i$. Similarly $\mathfrak{N}_{-g_\beta}\mathfrak{A}$ is freely generated by $\{g_i, -g_\beta : i < \beta\}$. Let $\mathfrak{B}_0 = \mathfrak{N}_{g_\beta}\mathfrak{A}$ and $\mathfrak{B}_1 = \mathfrak{N}_{-g_\beta}\mathfrak{A}$. Let $t_0 = g_\beta$ and $t_1 = -g_\beta$. Let $x_i$ be such that $J_i = \lg\{t_i\} = \lg^{\mathfrak{B}_i}\{x_i\}$, and $x_0.x_1 = 0$. Exist by assumption. Assume that $z \in J_0 \cap J_1$. Then $z \leq x_i$, for $i = 0, 1$, and so $z = 0$. Thus $J_0 \cap J_1 = \{0\}$. Let $y \in A \times A$, and let $z = (y_0.x_0 + y_1.x_1)$, then $y_i.x_i = z.x_i$ for each $i = \{0, 1\}$ and so $z \in \bigcap y_0/J_0 \cap y_1/J_1$. Thus $A/J_i \cong \mathfrak{B}_i$, and so $\mathfrak{A} \cong \mathfrak{B}_0 \times \mathfrak{B}_1$.

(3) Let $\mathfrak{A} = \mathfrak{Fr}_\beta K$. Let $\mathfrak{B}$ have $k$ atoms and generated by a single element. Then there exists a surjective homomorphism $h : \mathfrak{A} \to \mathfrak{B}$. Then, as in the first item, $\mathfrak{A}/\lg^{\mathfrak{B}}\{b\} \cong \mathfrak{B}$, and so $\mathfrak{N}_b\mathfrak{B}$ has $k$ atoms. Hence $\mathfrak{A}$ has $k$ atoms for any $k$ and we are done.
If we look at order. Here $\psi$ is one such that variables in its atomic subformulas occur only in their natural order. We shall construct three restricted formulas $\phi$. Then $f$ can extended to a homomorphism $f' : A \to B$. Let $\bar{f} = f' | Y$. If $f, g \in X B$ and $\bar{f} = \bar{g}$, then $f'$ and $g'$ agree on a generating set $Y$, so $f' = g'$, hence $f = g$. Therefore we obtain a one to one mapping from $X B$ to $Y B$, but $|X| = |Y|$, hence this map is surjective. In other words for each $h \in Y B$, there exists a unique $f \in X B$ such that $f = h$, then $f'$ with domain $A$ extends $h$. Since $\mathfrak{fr}_X K = \mathfrak{fr}_X (Fin(K))$ we are done. 

\[\Box\]

## 2 Two new results on Pinter’s algebras

**Example 2.1.** Let $U, n$ be finite, such that each has at least two elements, and $n > 2$. Let $\mathfrak{B} = \wp(nU) \in SC_n, X = \{s \in nU : s_0 < s_1\}$, and $\mathfrak{A} = \wp g^{\mathfrak{A}} \{X\}$. Define by recursion, $Y_0 = n U, Y_1 = c_0 X$ and $Y_{m+1} = C_0 (C_1 (Y_m \sim X \cap X)$. Then it is clear that $Y_m = s : \lambda \leq s_1, |R y Y| = |U| + 1$. The $\mathfrak{A}$ is finite and is simple and generated by a single element. From the above we get that the free algebras have infinitely many atoms.

**Theorem 2.2.** For every finite $n > 2$, and $\beta > 0$, there is an irredundant $\beta$ element generator set of $\mathfrak{fr}_n RCA_{\beta}$ which does not generate it freely. The same holds for Pinters substitution algebras.

**Proof.** For the first part, we take $n = 3$, which is the most difficult case, because the corresponding logic has the least number of variables. This part is due to Nemeti, though to the best of our knowledge it was not published in this form, which is also due to Nemeti in a preprint of his. Let $\mathcal{L}$ be a language with 3 variables, and one ternary relation. The formulas that we will construct will be restricted meaning that variables occur only in their natural order. We shall construct three restricted formulas $\phi, \psi$ and $\eta$ such that $\models R(x, y, z) \iff \psi(R/\phi), \models \eta(R/\phi)$ but not $\models \eta$. A restricted formula is one such that variables in its atomic subformulas occur only in their natural order. Here $\psi(R/\phi)$ is the formula obtained from $\psi$ by replacing all occurrences of $R$ with $\phi$ and $x, y, z$ are the variables $v_0, v_1, v_2$ of $\mathcal{A}$ respectively. In the following we write $R$ instead of $R(x, y, z)$. We may write $R y x$ for $R(x, y)$. Let

$$su(x, y) = \forall z ([R y z \land z \neq y] \iff [R x z \lor z = x])$$

If we look at $R$ as a binary relation symbol interpreted as an order, then $su(x, y)$ says that $y$ is the element after $x$. $A$ is the following set of formulas

$$[R \iff \exists z R, R y x \land R y z \Rightarrow x = y, x \neq y \Rightarrow (R(x) \lor R(yz)), (\forall x)(\exists y) su(x, y), \exists y (R y y \land \forall x [R x x \rightarrow R y y])]$$


Now $Ax$ says that $R$ is binary, and is a discrete ordering without endpoints and has a greatest fixed point. Call such a relation good. Let

$$\phi = R \lor (Ax \land x = y \land \exists z[\text{succ}(x, z) \land Rzz \land (\forall x)(Rxz \to Rxz)].$$

Now $\phi$ says that if $R$ is good then $\phi$ represents $\bar{R}$ where $\bar{R}$ is $R \cup \{\text{the successor of the greatest fixed point of } R\}$ as a new fixed point, otherwise $\phi$ is $R$.

$$\psi = (\neg Ax \land R) \lor (Ax \land R \land [x = y \to \exists y(x \neq y \land Rxy \land Ryy)]),$$

$\psi$ is $R$ without the greatest fixed point if $R$ is good, otherwise it is $R$.

$$\eta = Ax \to \exists xy(x \neq y \land Rxx \land Ryy).$$

$\eta$ says that if $R$ is good then it has at least two fixed points. Then $\psi(R/\phi)$ is equivalent to $R$ since $R$ can be recovered from $\bar{R}$ by omitting its greatest fixed point. $\eta(R/\phi)$ is true since if $R$ is good then $\bar{R}$ has two fixed points. Clearly for every infinite set $M$ there is a model $\mathcal{M}$ with universe $M$ such that not $\mathcal{M} \models \eta$. Then $\phi$, $\psi$ and $\eta$ are as required. By using the correspondence between terms and restricted formulas we obtain three terms $\tau(x)$, $\sigma(x)$ and $\delta(x)$ such that $\text{RCA}_3 \models \sigma(\tau(x)) = x$ and $\text{RCA}_3 \models \delta(\tau(x)) = 1$ but not $\text{CS}_3 \models \delta(x) = 1$. Then for every $n \geq 3$ we have (a) $\text{RCA}_n \models \sigma(\tau(x)) = x$ and (b) $\text{RCA}_n \models \delta(\tau(x)) = 1$ but not $\text{CS}_n \models \delta(x) = 1$. Let $0 < \beta$, and $n \geq 3$ and let $\{g_i : i < \beta\}$ be an arbitrary generator set of $\bar{\text{RCA}}_n$. Then $\{\tau(g_0)\} \cup \{g_i : 0 < i < \beta\}$ generates $\bar{\text{RCA}}_n$ by (a) but not freely by (b).

For the second part, we add a binary relation to our language and we pretend that it the membership relation in set theory; in fact will be the real membership relation when semantically interpreted, which will be the case.

The idea is to translate any formula with equality to one having an extra binary relation, that acts as equality such that the two are equivalent modulo a certain strong congruence, and the second is equality free using the existential axiom of set theory.

The proof is purely semantical, which makes life easier. However, there is a syntactical proof too, using the pairing technique of Tarski substantially modified by Németi, giving the same result for $SC_3$, but we omit this much more involved proof. This pairing technique, implemented via a recursive translation function for $L_{w, \omega}$ to $L_3$ preserves meaning, hence providing a completeness theorem for $\text{CA}_3$. (Larger $n$ is much easier, below we will deal with paring function in dimension 4 a technique invented by Tarski.) Such a procedure enables one to transfer results proved for the representable algebras to the
abstract ones, though the distance between them is infinite, in some precise
sense (a result of Monk).

Let $Ax_{eq}$ and $Ax_{cong}$ be as in [1] and $tr$ be the function that takes every
formula to an equality free formula. The latter is an adjoint function, and it
clearly preserves meaning.

For $\{x, y, z\} = \{v_0, v_1, v_2\}$, these are defined as follows:

$$Ax_{eq} = \{\forall x \forall y(x = y \leftrightarrow (\forall z(z \in x \leftrightarrow z \in y))\}$$

$$Ax_{cong} = \{\forall xy(\forall z(z \in x \leftrightarrow z \in y)) \rightarrow (\forall z(x \in z \leftrightarrow y \in z))\}$$

For a formula $\phi$ with equality, $tr(\phi)$ is obtained from $\phi$ by replacing all of the
occurrences of $x = y$ by $\forall z(z \in x \leftrightarrow z \in y)$. Notice that such formulas can be
defined by algebraic terms, the former in cylindric algebras and the second in
Pinters algebras. Then, we have $Ax_{eq} \vdash \phi \leftrightarrow tr(\phi)$.

For a model $M$ a model for a language without equality, define the Leibniz congruence
$\sim$ by $a \sim b \iff \forall z(z \in a \leftrightarrow z \in b)$.

It is not hard to check that $\sim$ is a strong congruence; it preserves $\in$ in both
directions.

Then for any formula with equality using 3 variables, and $M$ a model
without equality of $Ax_{cong}$, we have $M \models tr(\phi)$ iff $M/\sim \models \phi$. Notice that $\phi$
has equality, we have $a/\sim = b/\sim$ iff $a \sim b$, so that is is meaningful to talk
about equality here.

Let $\psi$ be any formula with equality and let $tr(\phi)$ be the equality free corre-
sponding formula, using the membership binary relation. A piece of notation:
If $M$ is a model for the language with equality, let $\mathfrak{A}_M \in \mathfrak{C}_{n}$ be the corre-
sponding set algebra, and same for models without equality; in this last case,
we denote the corresponding Pinter’s set algebra corresponding to $N$ by $\mathfrak{B}_N$.

Let $M$ be a model for the language with equality such that $M \models Ax_{eq}$, then there exists $N$
a model for the language without equality such that $M \cong N/\sim$. Then there is an an induced base isomorphism between $\mathfrak{B}_N \to \mathfrak{A}_M$.

Now we use the correspondence between formulas and terms, we lift the
translation function to the level of terms. If $\tau$ corresponds to $\psi$ then let $tr(\tau)$
be that corresponding to $tr(\phi)$. Frm the above we have and $\text{RCA}_0 \models \tau = 1$
iff $\text{RSC}_\alpha \models tr(\tau) = 1$. and we done, from the first part of the proof.

**Theorem 2.3.** There is a formula $\psi \in L_4$ such that no consistent recursive
extension $T$ of $\psi$ is complete, and moreover, $\psi$ is hereditory inseparable mean-
ing that no recursive extension of $\psi$ separates the $\vdash$ consequences of $\psi$ from
the $\psi$ refutable sentences.

**Proof.** We assume that we have only one binary relation and we denote our
language by $L_4(E, 2)$. This is implicit in the Tarski Givant approach, when
they interpreted ZF in RA. L₄ is very close to RA but not quite RA, it is a little bit stronger (for example there are four variable terms that cannot be expressed in RA terms). The technique is called the pairing technique, which uses quasi projections to code extra variable, establishing the completeness theorem above for ⊢₄.

We have one binary relation E in our language; for convenience, we write x ∈ y instead of E(x, y), to remind ourselves that we are actually working in the language of set theory. We define certain formulas culminating in formulating the axioms of a finite undecidable theory, better known as Robinson’s arithmetic in our language. These formulas are taken from Németi. We formulate the desired hereditary inseparable ψ in L₄(E, 2).

For 4 variables, we need the following ’translation’ result of Tarski which states a basic property of Tarski’s pairing functions, namely we can code up, or represent, any sequence of variables in terms of a single variable, thus effectively reducing the number of variables to one. In more detail, we have:

**Fact.** Let p₀(x, y) and p₁(x, y) be in L₃(E, 2) and let

\[
\pi = (\forall x)(\forall y)(\forall z)[(p₀(x, y) \land p₀(x, z) \implies y = z) \land
\]

\[
p₁(x, y) \land p₁(x, z) \implies y = z) \land
\]

\[
\exists z(p₀(z, x) \land p₁(z, y)].
\]

be the formula stipulating that they are quasiprojections. Then there is a recursive function tr : Lω(E, 2) → L₃ such that (i) – (iii) below hold for every φ ∈ L₃(Em2)

(i) π ⊨ φ ←→ trφ

(ii) tr(¬φ) = ¬tr(φ),

First we interpret usual Robinson arithmetic in the usual language with ω many variables, using the standard interpretation of Peano arithmetic into set theory relativized to finite hereditary sets (that is Peano arithmetic with axiom of infinity). This part is semantical, in nature, so it is not too difficult to implement:

\[
x = \{y\} =: y ∈ x \land (∀z)(z ∈ x \implies z = y)
\]

\[
\{x\} ∈ y =: ∃z(z = \{x\} \land z ∈ y)
\]

\[
x = \{\{y\}\} =: ∃z(z = \{y\} \land x = \{z\})
\]

\[
x ∈ ∪y := ∃z(x ∈ z \land z ∈ y)
\]

\[
pair(x) =: ∃y[\{y\} ∈ x \land (∀z)(\{z\} ∈ x → z = y)] \land ∀yz[(z ∈ ∪x \land \{z\} \notin x ∧
\]

pair(z) ∈ ∪x \land z = y)
\]
\[ y \in \cup x \land \{ y \} \notin x \rightarrow z = y \] \land \forall z \in x \exists y (y \in z).

Now we define the pairing functions:

\[ p_0(x, y) =: \text{pair}(x) \land \{ y \} \in x \]
\[ p_1(x, y) =: \text{pair}(x) \land [x = \{ \{ y \} \} \lor (\{ y \} \notin x \land y \in \cup x)]. \]

\[ p_0(x, y) \] and \[ p_1(x, y) \] are defined.

\( x \in \text{Ord} =: \) “\( x \) is an ordinal, i.e. \( x \) is transitive and \( \in \) is a total ordering on \( x \),

\( x \in \text{Ford} =: x \in \text{Ord} \land \) “every element of \( x \) is a successor ordinal ”

i.e. \( x \) is a finite ordinal .

\( x = 0 =: \) “\( x \) has no element ”

\[ sx = z =: z = x \cup \{ x \}, \]
\[ x \leq y =: x \subseteq y, \]
\[ x < y =: x \leq y \land x \neq y, \]
\[ x + y = z =: \exists v(z = x \cup v \land x \cap v = 0 \land \) “there exists a bijection between \( v \) and \( y \)”\]

\[ x \cdot y = z =: \) “there is a bijection between \( z \) and \( x \times y \)”

\( x_{\exp y} = z : \) there is a bijection between \( z \) and the set of all functions from \( y \) to \( x \)”

Now \( \lambda ' \) is the formula saying that: 0, s, +, \( \cdot \), exp are functions of arities 0, 1, 2, 2, 2 on \( \text{Ford} \) and

\[ (\forall xy \in \text{Ford})(sx \neq 0 \land sx = sy \rightarrow x = y) \land (x < sy \leftrightarrow x \leq y) \land \]
\[ \neg(x < 0) \land (x < y \lor x = y \lor y < x) \land (x + 0 = x) \land (x + sy = s(x + y)) \land (x.0 = 0) \]
\[ \land (x \cdot sy = x \cdot y + x) \land (x_{\exp 0} = s0) \land (x_{\exp y} = x_{\exp y} \cdot x)]. \]

Now the existence of the desired incompletable \( \lambda \) readily follows: \( \lambda \in Fm_\omega^0 \).

Let \( \text{RT} \) be the absolutely free relation algebra on one generator. Let \( p = r(p_0(x, y)) \) and \( q = r(p_1(x, y)) \), where \( r \) is the recursive function mapping \( L_3(E, 2) \) into \( \text{RT} \) that also preserves meaning. Here \( \text{RT} \) is the set terms in the language of relation algebras with only one generator.

\[ \pi_{\text{RA}} = (\hat{p}; p \rightarrow \text{Id}) \cdot (\hat{q}; q \rightarrow \text{Id}) \cdot (\hat{p}; q). \]

This is just stipulation that \( p \) and \( r \) are quasi projections, in the language of relation algebras. Then, we have \( \pi_{\text{RA}} \in \text{RT} \) since \( p_i(x, y) \in L_3(E, 2) \)
Let $\lambda \in L_\omega$ be the inseparable sentence, that is the conjunction of finite axioms of Robinson arithmetic) constructed above and let $\psi = (r(\text{tr}(\lambda))) \cdot \pi_{\text{RA}}$. From the definition of $r$ and $\text{tr}$ we have $\eta \in \text{RT}$. Let $F_4$ be the algebra built on $L_4(E, 2)$. Let $G$ be the absolutely free RA algebra on one generator $g$. Let $h : G \to \text{Ra}_F^4$ be the homomorphism that takes the free generator of $G$ to $x \in y$. Let $\psi = h(\eta)$. Then $\psi \in F_4$ and furthermore, it can be checked that $\psi$ is the desired formula. (Here we use that the RA reduct of a CA is a relation algebra.)

The same idea can be implemented by avoiding the path from CA to RA and then back to CA, using Simon’s result, and a very deep result of Németi’s.

Németi defines a set of axioms $\text{Ax}$ that is semantically equivalent to $\pi$ but stronger (proof theoretically). The idea to translate all 3 variable usual first order formula into the QRA fragment of $L_3(E, 2)$. We have the quasiprojections $p_0, p_1$ and the set of axioms $\text{Ax}$; which say that $p_0$ and $p_1$ are quasiprojections, and it can prove a strong form of associativity of relations.

We also know that in very QRA, for each $n \in \omega$, there sits in a CA$_n$, and there are cylindric algebras of various increasing finite dimensions synchronized by the neat reduct functor, so that the CA$_4$ sitting there, has the cylindric neat embedding property (it neatly embeds, and indeed faithfully so in cylindric algebras of arbitrary larger finite dimensions theorem). Yet again, by Henkin’s neat embedding theorem, this algebra call it $\mathfrak{C}$ is representable.

Define $f : \mathfrak{F}_3 \to \mathfrak{C}$ be the homomorphism defined the usual way. Then define the translation map as follows: $\text{tr}(\phi) = \text{Ax} \to f(\phi)$. This functions covers the infinite gap between $\vdash_3$ and $\models$. The above proof for 3 dimensions can be done by $\pi$ instead of $\text{Ax}$.

For cylindric algebras, diagonal free cylindric algebras Pinter’s algebras and quasipolyadic equality, though free algebras of $> 2$ dimensions contain infinitely many atoms, they are not atomic. (The diagonal free case of cylindric algebras is a very recent result, due to Andréka and Németi, that has profound repercussions on the foundation of mathematics.) We, next, state two theorems that hold for such algebras, in the general context of BAO’s. But first a definition.

**Definition 2.4.** Let $K$ be a class of BAO with operators $(f_i : i \in I)$. Let $\mathfrak{A} \in K$. An element $b \in A$ is called hereditary closed if for all $x \leq b$, $f_i(x) = x$.

In the presence of diagonal elements $d_{ij}$ and cylindrifications $c_i$ for indices $< 2$, $-c_0 - d_{01}$, is hereditary closed.

**Theorem 2.5.** (1) Let $\mathfrak{A} = \mathfrak{F}_gX$ and $|X| < \omega$. Let $b \in \mathfrak{A}$ be hereditary closed. Then $\text{At}\mathfrak{A} \cap \text{Ra}_b \mathfrak{A} \leq 2^n$. If $\mathfrak{A}$ is freely generated by $X$, then $\text{At}\mathfrak{A} \cap \text{Ra}_b \mathfrak{A} = 2^n$. 

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(2) If every atom of \( \mathfrak{A} \) is below \( b \), then \( \mathfrak{A} \cong \mathcal{R}t_{b} \mathfrak{A} \times \mathcal{R}l_{b} \mathfrak{A} \), and \( |\mathcal{R}l_{b} \mathfrak{A}| = 2^{2^{n}} \). If in addition \( \mathfrak{A} \) is infinite, then \( \mathcal{R}l_{b} \mathfrak{A} \) is atomless.

**Proof.** Assume that \( |X| = m \). We have \( |At \mathfrak{A} \cap \mathcal{R}l_{b} \mathfrak{A}| = |\{\prod Y \sim \sum (X \sim Y).b\}| \leq m^{2} \). Let \( \mathfrak{B} = \mathcal{R}l_{b} \mathfrak{A} \). Then \( \mathfrak{B} = \mathcal{S}g^{\mathfrak{B}}\{x_{i}.b : i < m\} = \mathcal{S}g^{B_{2}}\{x_{i}.b : i < \beta\} \) since \( b \) is hereditary fixed. For \( \Gamma \subseteq m \), let

\[
x_{\Gamma} = \prod_{i \in \Gamma} (x_{i}.b). \prod_{i \in m-\Gamma} (x_{i}.-b).
\]

Let \( \mathfrak{C} \) be the two element algebra. Then for each \( \Gamma \subseteq m \), there is a homomorphism \( f : \mathfrak{A} \rightarrow \mathfrak{C} \) such that \( fx_{i} = 1 \) if \( i \in \Gamma \). This shows that \( x_{\Gamma} \neq 0 \) for every \( \Gamma \subseteq m \), while it is easily seen that \( x_{\Gamma} \) and \( x_{\Delta} \) are distinct for distinct \( \Gamma, \Delta \subseteq m \). We show that \( \mathfrak{A} \cong \mathcal{R}l_{b} \mathfrak{A} \times \mathcal{R}l_{b} \mathfrak{A} \).

Let \( \mathfrak{B}_{0} = \mathcal{R}l_{b} \mathfrak{A} \) and \( \mathfrak{B}_{1} = \mathcal{R}l_{b} \mathfrak{A} \). Let \( t_{0} = b \) and \( t_{1} = -b \). Let \( J_{i} = \mathcal{I}g\{t_{i}\} \) Assume that \( z \in J_{0} \cap J_{1} \). Then \( z \leq t_{i} \), for \( i = 0, 1 \), and so \( z = 0 \). Thus \( J_{0} \cap J_{1} = \{0\} \). Let \( y \in \mathfrak{A} \times \mathfrak{A} \), and let \( z = (y_{0}.t_{0} + y_{1}.t_{1}) \), then \( y_{i}.x_{i} = z.x_{i} \) for each \( i = \{0, 1\} \) and so \( z \in \bigcap y_{0}/J_{0} \cap y_{1}/J_{1} \). Thus \( \mathfrak{A}/J_{i} \cong \mathfrak{B}_{i} \), and so \( \mathfrak{A} \cong \mathfrak{B}_{0} \times \mathfrak{B}_{1} \).

The above theorem holds for free cylindric and quasi-polyadic equality algebras. The second part (all atoms are zero-dimensional) is proved by Madárasz and Németi.

The following theorem holds for any class of \( BAO \)’s.

**Theorem 2.6.** The free algebra on an infinite generating set is atomless.

**Proof.** Let \( X \) be the infinite freely generating set. Let \( a \in A \) be non-zero. Then there is a finite set \( Y \subseteq X \) such that \( a \in \mathcal{S}g^{Y} \mathfrak{Y} \). Let \( y \in X \sim Y \). Then by freeness, there exist homomorphisms \( f : \mathfrak{A} \rightarrow \mathfrak{B} \) and \( h : \mathfrak{A} \rightarrow \mathfrak{B} \) such that \( f(\mu) = h(\mu) \) for all \( \mu \in Y \) while \( f(y) = 1 \) and \( h(y) = 0 \). Then \( f(a) = h(a) = a \). Hence \( f(a.y) = h(a. - y) = a \neq 0 \) and so \( a.y \neq 0 \) and \( a. - y \neq 0 \). Thus \( a \) cannot be an atom.

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