Bifurcation Analysis of a Bistable Nonlinear Vibration Energy Harvester with Elastic Boundary

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Abstract. This paper presents a novel bistable vibration energy harvester with an elastic boundary (BVEH_EB). The bistable nonlinearity of the BVEH_EB is realized by an inclined spring, which could induce a large-amplitude inter-well response to pursue high harvesting efficiency. The elastic boundary brings additional dynamic coupling with the inclined spring to reduce the depth of the potential energy well, which could enhance the inter-well response. The bifurcation responses in terms of different parameters such as the magnet mass, the inclined spring stiffness, the excitation frequency, and the excitation amplitude are numerically investigated. Abundant nonlinear phenomena, such as intra-well oscillation, inter-well oscillation, chaos, etc., are observed. The design guidelines of the BVEH_EB are developed, which could provide a novel harvesting method.

1. Introduction

Harvesting ambient energy has received growing interest over the last two decades for the convenience of powering electronics such as distributed sensors and portable devices [1]. Since vibration energy is one of the most widespread forms of wasted ambient energy, the design and exploration of the energy harvester (EH) based on ambient vibration energy is worthwhile. The transduction mechanisms, such as electromagnetic, magnetostrictive, electrostatic, and piezoelectric were introduced into different types of EHs to convert the wasted vibration energy into electrical energy. The ubiquitous used vibration energy harvesters (VEHs) consist of a linear oscillator based on the above transduction mechanisms, which could only obtain a considerable output if the ambient vibration frequency is in the vicinity of the EH’s natural frequency. Output performance reduces sharply for mistuned (away from the natural frequency) linear EH. Afterward, the frequency tuning techniques were introduced in EH to adjust the natural frequency to obtain the desired output. And the tuning of EH’s natural frequency could be achieved by geometrical tuning, extensional mode resonator, variable stiffness, etc. However, to tune the EH’s frequency, a complex machinery design is needed and an external intervention that could be automated through actuators or human, which was found to be inefficient in some situations [2].

Another method to expand the bandwidth of a VEH is employing multi-frequency structures. This kind of VEH has two or more natural frequencies, and once the ambient vibration frequency is near either of the natural frequencies, a considerable output power will be obtained. Toyabur et al. [3] experimentally validated a hybrid multimodal piezoelectric-electromagnetic VEH, and the proposed VEH could achieve multiple close resonant modes in a certain frequency range which significantly
improved the output power. Wu et al. [4] proposed a 2-DOF VEH consisting of two cantilever beams. The first two natural frequencies could be tuned and significant magnitudes were obtained. They thought the proposed structure is more adaptive and functional in a random vibration environment. However, it is seen that a larger area is needed for a multi-frequency structure, which hinders its applications in small-scale devices.

In addition, harvesting the ambient vibration energy from more than one direction is an effective solution to improve the harvesting efficiency as well, since most of the harvesters were designed to harvest from a single direction. Fan et al. [5] presented a bi-directional hybrid VEH to scavenge wasted vibration energy from ultra-low frequency mechanical vibration. The handshaking test showed that the bi-directional VEH has a better charging performance than 1-DOF VEH. Liu et al. [6] presented an electromagnetic VEH consisting of a permanent magnet and a circular suspension structure on an EH chip. It could harvest the vibration energy from all 6-DOF.

Furthermore, introducing nonlinearity into a VEH was also validated to be an effective solution to enhance the output power. Monostable [7], bistable [8], tristable [9], quadstable [10], etc. are different characteristics of nonlinear VEH, which could make considerable improvements to the efficiency of harvesting, and bistable nonlinearity is the most ubiquitous one. There were several ways to create bistable nonlinearity in a VEH, such as magnetic interaction, nonlinear spring, buckled beam, etc. Yang et al. [11] studied the harvesting performance of a double-beam piezo-magneto-elastic wind energy harvester under wind excitation. The harvester consisted of two piezoelectric beams, and both beams were supported by a prism-like bluff embedded with a permanent magnet. And the bistability was induced by the magnetic repulsion force. Harne et al. [12] investigated a nonlinear-spring-based bistable energy harvester attached to an additional linear oscillator for harvesting enhancement. They found the additional DOF of the linear oscillator is an effective way to enhance the EH performance and robustness. Panyam et al. [13] investigated the potential of a bistable-beam-based harvesting vibration energy. The bistable beam was a clamped-clamped beam exposed to an axial preload, and two stable equilibria symmetrical about the axis were obtained, i.e., bistability. A broadband frequency response could be achieved compared to traditional linear resonance.

Bifurcation happens when a small change is made to the vibration system parameter, which causes a sudden topological or 'qualitative' change in its dynamic behavior. It is necessary to investigate the bifurcation phenomena of a VEH based on nonlinearity, which could develop design guidelines for VEH. Cao et al. [14] studied the bifurcation diagrams in terms of fractional order of damping, excitation frequency, and excitation amplitude in a piezoelectric EH. Abundant nonlinear behaviors, such as chaos, periodic motion, etc., were observed.

Enlightened by the above literatures, this paper proposes a bistable vibration energy harvester based on nonlinear spring with elastic boundary (BVEH_EB). The BVEH_EB employs the electromagnetic mechanism to transform ambient vibration energy into electric energy, and it introduces the bistable nonlinearity through an inclined spring. The governing equations of the BVEH_EB are formulated. The bifurcation analyses in terms of the magnet mass, the inclined spring stiffness, the excitation frequency, and the excitation amplitude are presented, which show aplenty nonlinear characteristics. The phase portraits and the Poincare maps are introduced as well. The design guidelines of the BVEH_EB will be developed, which could provide a novel harvesting method.

2. Dynamic Modeling

Figure 1 presents the schematic diagram of the proposed BVEH_EB. The BVEH_EB consists of an inclined spring bistable structure, an elastic boundary, a linear track which winded by coils, and the coils are connected with an electrical resistor \( R \). The bistable part comprises an inclined spring with linear stiffness \( k_1 \) having undeformed length \( L_1 \), and a magnet \( m_1 \) embedded with the linear coil track. \( m_1 \) slides on the linear track. One end of the inclined spring is attached to the magnet, while the opposite end is connected to a mass block (\( m_2 \)) in the elastic boundary. The elastic boundary consists of a spring with linear stiffness \( k_2 \) and undeformed length \( L_2 \), a damper with linear damping constant \( c_2 \), and \( m_2 \). The vertical distance from the bottom of \( m_2 \) (when \( k_2 \) is undeformed) to the top of \( m_1 \) is
and satisfies \( d < L_1 \). Hence, the inclined spring is compressed, which could induce two stable equilibria, i.e., bistable nonlinearity. Besides, this study supposes that \( c_1 \) is an equivalent linear damping constant coming from the linear track and all the springs are massless.

**Figure 1.** Schematic diagram of the BVEH_EB.

When the BVEH_EB is under external vibration excitation \( \ddot{Z} = -pcos(2\pi\Omega t) \), the kinetic energy is

\[
T_1 = \frac{1}{2}m_1(\dot{x} + \dot{Z})^2 + \frac{1}{2}m_2\dot{y}^2
\]  

where \( x \) is the relative displacement of \( m_1 \) to the center line, \( y \) is the variation of \( L_2 \) (relative displacement of \( m_2 \) to the position where \( k_2 \) is undeformed), \( p \) and \( \Omega \) are the excitation amplitude and the frequency, and the operator \( (\dot{\dot{\cdot}}) \) is the derivative with respect to time \( t \).

The potential energy of the BVEH_EB is

\[
V_1 = \frac{1}{2}k_1(\sqrt{(x^2 + (d - y)^2)} - L_1)^2 + \frac{1}{2}k_2y^2 - mg\gamma
\]  

where \( g \) is the gravitational acceleration. By combining Eqs. (1)(2) and applying the Lagrange equations and the electromechanical coupling equation, the governing equations of BVEH_EB could be expressed as

\[
m_1(\ddot{x} + \ddot{Z}) + c_1\dot{x} + k_1x \left[ 1 - \frac{L_1}{\sqrt{(x^2 + (d - y)^2)}} \right] + YI = 0
\]

\[
m_2\ddot{y} + c_2\dot{y} + k_2y + k_1(y - d) \left[ 1 - \frac{L_1}{\sqrt{(x^2 + (d - y)^2)}} \right] - m_2g = 0
\]

\[
R_0\dot{I} + RI - Y\dot{x} = 0
\]

where \( I \) is the current flow through resistor \( R \), \( L_0 \) is the electromagnetic harvester inductance, and \( Y \) is the electromagnetic coupling constant.

3. Bifurcation Analyses

To develop insights into the nonlinear bistable dynamic behaviors of BVEH-EH, the bifurcation responses of BVEH_EB in terms of \( m_1, k_1, \Omega, \) and \( p \) are numerically investigated in this section. Eqs. (3)-(5) are calculated by the 4th Runge-Kutta algorithm. Through bifurcation analyses, various nonlinear dynamics phenomena are observed, such as periodic motions and chaos. Since the initial conditions are extremely important in a nonlinear system for the possible responses, the initial position of both \( m_1 \) and \( m_2 \) are set as random. The phase portraits and the Poincare maps are given to clearly illustrate the intriguing nonlinear behaviors which is different from that depicted in the bifurcation diagram. The red points in the phase portraits represent the Poincare maps under the corresponding
situation. Besides, the stroboscopic time (1/Ω) is employed in both the bifurcation diagrams and Poincare maps. m_2 = 0.01 kg and k_2 = 2000 N/m are employed in all simulations.

3.1. Bifurcation Diagram for (k_1, x)
Figure 2(a) depicts the bifurcation diagram of the BVEH_EB about k_1 and x when m_1 = 0.1 kg, p = 0.5 m/s^2, and Ω = 3 Hz. k_1 is in the range of [200, 20000] N/m with a step of 20 N/m. It could be observed that k_1 has significant impacts on the m_1 displacement amplitude. Besides, the characteristics uncovered by the phase portraits and the Poincare maps are extremely different from the bifurcation diagram, which indicates the initial conditions matter. Figure 2(b) shows a period-3 inter-well oscillation of the m_1 displacement response, and three independent points are there on the Poincare map. Figures 2(c)-(e) illustrate three fundamental periodic inter-well responses, i.e., periodic-1 inter-well responses. It is seen that when BVEH_EB exhibits fundamental periodic inter-well response, a large-amplitude response could be obtained. Consequently, it is the most desirable dynamic behavior for a VEH to harvesting the ambient vibration energy in a high efficiency. Besides, it could be found that, with the increase of k_1, the amplitude of the displacement response decreases, which implies there may be optimal k_1 in corresponding situations.

![Figure 2](image)

Figure 2. Bifurcation diagram of displacement amplitude versus k_1. (a) bifurcation diagram; (b)-(e) the phase portraits and the Poincare maps of BVEH_EB under different k_1.

3.2. Bifurcation Diagram for (m_1, x)
Figure 3(a) shows the bifurcation diagram of the BVEH_EB about m_1 and x when p = 0.5 m/s^2, Ω = 3 Hz, and k_1 = 2000 N/m. m_1 is ranged from 0.01 to 1 kg with a step of 0.01 kg. It is seen that several quasi-periodic responses are obtained in the bifurcation diagram. When m_1 is small, it is more likely for the BVEH_EB to realize intra-well response, and figure 3(b) is an example of m_1 = 0.01 kg. With the increase of m_1, the inter-well responses could be obtained. The m_1 displacement response could be large-amplitude fundamental periodic inter-well oscillation or other types of quasi multi-periodic inter-well oscillation under different initial conditions, as shown in figures 3(c)-(e).

3.3. Bifurcation Diagram for (Ω, x)
Figure 4(a) shows the bifurcation diagram of m_1 displacement amplitude (x) versus excitation frequency (Ω). m_1 = 0.1 kg, p = 0.5 m/s^2, and k_1 = 2000 N/m are set. Ω is ranged from 1 to 10 Hz with a step of 0.01 Hz. The nonlinear response that BVEH_EB undergoes in the bifurcation diagram is abundant and intriguing. Take the phase portraits and the Poincare maps as examples. When Ω = 1 Hz, BVEH_EB exhibits chaos, and plenty of irregular dispersed points are shown in figure 4(b). Two strange attractors representing chaos are shown in the Poincare map, which means the BVEH_EB is exhibiting an orderly disorder motion, and the magnet (m_1) vibrates around the two strange attractors.
(i.e., two equilibria) in a disordered way but with a clear boundary. In figure 4(c), the BVEH_EB realizes a large period-1 inter-well oscillation. Besides, a periodic-3 inter-well oscillation is depicted in figure 4(d) when $\Omega = 4 \text{ Hz}$. The amplitude is slightly lower than the periodic-1 inter-well oscillation. Keeping increase the excitation frequency, it may be found that the BVEH_EB could only exhibit low-efficiency intra-well oscillation. Figure 4(e) is an example of phase portrait when $\Omega = 6 \text{ Hz}$, the magnet vibrates around one of the two equilibria.

![Figure 3. Bifurcation diagram of displacement amplitude versus $m_1$. (a) bifurcation diagram; (b)-(e) the phase portraits and the Poincare maps of BVEH_EB under different $m_1$.](image)

![Figure 4. Bifurcation diagram of displacement amplitude versus $\Omega$. (a) bifurcation diagram; (b)-(e) the phase portraits and the Poincare maps of BVEH_EB under different $\Omega$.](image)

3.4. Bifurcation Diagram for $(p, x)$  

Figure 5(a) shows the bifurcation diagram of the BVEH_EB about $p$ and $x$ when $m_1 = 0.1 \text{ kg}$, $k_1 = 2000 \text{ N/m}$, and $\Omega = 3 \text{ Hz}$. It is seen in figure 5(a) that the BVEH_EB seems to undergo abundant nonlinear dynamic responses with the increase of excitation amplitude. However, the bifurcation diagram misleads us. The phase portraits in figures 5(b)-(e) are distinct examples, which shows the periodic-1 response happens with a large occurring probability. With the increase of excitation amplitude, the response bifurcates from intra-well oscillation to inter-well oscillation. The greater the excitation amplitude is, the larger the displacement amplitude will be obtained. Besides, it is worth noting that the amplitude of the response increases does not match the amplitude of the excitation increases, which intrigues us to pursue the optimal parameter combinations.
Figure 5. Bifurcation diagram of displacement amplitude versus $p$. (a) bifurcation diagram; (b)-(e) the phase portraits and the Poincare maps of BVEH_EB under different $p$.

4. Conclusion
This paper investigates the bifurcation features of a nonlinear bistable energy harvester coupled with an elastic boundary (BVEH_EB). The BVEH_EB employs an inclined spring to induce bistable nonlinearity, which could obtain large-amplitude inter-well oscillation. The coupled elastic boundary and the inclined spring to further enhance the inter-well oscillation by lowering the potential well depth. The bifurcation responses of BVEH_EB in terms of the magnet mass ($m_1$), the inclined spring stiffness ($k_1$), the excitation frequency ($\Omega$), and the excitation amplitude ($p$) are numerically investigated. The results indicate that the BVEH_EB has abundant bistable dynamic characteristics, including periodic inter-well oscillation, periodic intra-well oscillation, and chaotic responses. Besides, the initial condition determines the final nonlinear dynamic responses of the BVEH_EB. Furthermore, it is found that $k_1$ has a significant impact on the harvesting efficiency, large $m_1$ seems to hinder the harvesting process, greater $p$ contributes to large amplitude responses, and high $\Omega$ leads to inefficiency. The design guidelines of the BVEH_EB has been formed through detailed bifurcation analyses, which could provide a novel harvesting mean.

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