ON THE T-EQUIVARIANT COHOMOLOGY OF HESSENBURG VARIETIES

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ABSTRACT. For an endomorphism \( s : V \to V \) of a finite dimensional complex vector space and an action of a torus \( T \) on the full flag variety \( \text{GL}_n(\mathbb{C})/B \), we give a description of its fixed point set when \( s \) is semisimple or regular nilpotent. We also compute the one dimensional orbits of this action on the Hessenberg subvariety \( \text{Hess}(s, h) \subseteq \text{GL}_n(\mathbb{C})/B \) for any Hessenberg function \( h \). For the action of the one dimensional torus \( S \) and a regular nilpotent endomorphism \( N : V \to V \), we give a new computation of the equivariant cohomology of the Hessenberg variety \( \text{Hess}(N, h) \) for any Hessenberg function using determinantal conditions.

1. INTRODUCTION

Let \( V \) be a \( \mathbb{C} \)-vector space of finite dimension \( \dim V = n \), and let \( \text{GL}_n(\mathbb{C})/B \) denote the full flag variety, where \( B \) is the subgroup of upper-triangular matrices. Let \( g, b \) denote the Lie algebras corresponding to \( \text{GL}_n(\mathbb{C}) \) and \( B \) respectively. For \( X \in g \) and \( H \subset g \) a \( b \)-submodule such that \( b \subset H \), consider the Hessenberg variety

\[
\text{Hess}(X,H) = \{ gB \in \text{GL}_n(\mathbb{C})/B : \text{Ad}(g^{-1}) \cdot X \in H \},
\]

where \( \text{Ad} : \text{GL}_n(\mathbb{C}) \to \text{GL}_n(\mathbb{C}) \) is the adjoint representation. There is another description \( \text{Hess}(X,h) \) of a Hessenberg variety using a Hessenberg function \( h : \{1, \ldots, n\} \to \{1, \ldots, n\} \) (see Section 2) with \( \text{Hess}(X,h) = \text{Hess}(X,H) \) for \( H \) a subspace that depends on \( h \). We consider the \( n \)-dimensional torus \( T \subseteq B \) given by the diagonal matrices and its action on \( \text{GL}_n(\mathbb{C})/B \) by left multiplication. Let \( t \) denote its Lie algebra. By Tymoczko [6, Proposition 2.8] there is a description of the fixed point set \( (\text{GL}_n(\mathbb{C})/B)^T \) and the one dimensional orbits associated to the action of \( T \). Goresky-MacPherson in [1] give a description \( H^*_S(\mathbb{Y}; \mathbb{C}) \) for \( \mathbb{Y} \) a GM-space and a one-dimensional torus \( S \simeq \mathbb{C}^* \). Since \( \text{GL}_n(\mathbb{C})/B \) is a GM-space, in [2] Insko computes \( H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C}) \simeq \mathbb{C}[x_1, \ldots, x_n, t]/I_1 \) where \( I_1 \) is an ideal that gives a combinatorial characterization of the set \( (\text{GL}_n(\mathbb{C})/B)^T \). Assuming that \( N \) is nilpotent and that \( H^*_S(\text{Hess}(N,h); \mathbb{C}) \) is generated by its cohomology classes of degree two, then \( \text{Hess}(N,h) \) is a GM-subspace of \( \text{GL}_n(\mathbb{C})/B \) and therefore

\[
H^*_S(\text{Hess}(N,h); \mathbb{C}) \simeq H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C})/I
\]

for some ideal \( I \). In [2, Theorem 5.11] Insko defines an ideal \( I_1 \in \mathbb{C}[x_1, \ldots, x_n, t] \) using Hessenberg diagrams to describe the ideal \( I_1 = I_{1,1/I_1} \) and computes \( H^*_S(\text{Hess}(N,h); \mathbb{C}) \).

Since \( \text{Hess}(X,h) \) is a closed subvariety of \( \text{GL}_n(\mathbb{C})/B \), our contribution starts by considering which of the above properties for the full flag variety are inherited to \( \text{Hess}(X,h) \). If \( X \) is a semisimple operator \( X = \text{diag}(\lambda_1, \ldots, \lambda_n) \) the action of \( T \) restricts to an action on \( \text{Hess}(X,h) \) by Theorem 3.2 and \( (\text{Hess}(X,h))^T \subseteq (\text{GL}_n(\mathbb{C})/B)^T \). Theorem 3.4 identifies \( (\text{Hess}(X,h))^T = (\text{GL}_n(\mathbb{C})/B)^T \) and the one dimensional orbits in \( \text{Hess}(X,h) \). On the other hand, if \( N \) is a regular nilpotent operator, there are some similar results as in the

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semisimple case although the action of $\mathbb{T}$ does not restrict to an action on $\text{Hess}(N, h)$. In Theorem 4.1 we compute which fixed points of $\text{GL}_n(\mathbb{C})/B$ are contained in $\text{Hess}(N, h)$. Now, if instead of an $n$-torus we take a one dimensional torus $S = \{\text{diag}(t, t^2, \ldots, t^n) : t \in \mathbb{C}^*\}$, then $\text{Hess}(N, h)$ is $S$-invariant under the $S$ action on $\text{GL}_n(\mathbb{C})/B$ and we identify $(\text{Hess}(N, h))^S$ in Corollary 5.1. Also, Theorem 4.1 allow us to give another description of the fixed points of $\text{Hess}(N, h)$ using an ideal $\tilde{I}_n \subset \mathbb{C}[x_1, \ldots, x_n]$. We define $\tilde{I}_h$ by monomials whose zeros contain the values of a permutation $w$ in $S_n$. Indeed, if $S_n = \{w = (w(1), \ldots, w(n)) : w \in S_n\}$, then $\overline{w} \in V(\tilde{I}_n) \cap S_n$ if and only if $wB \in \text{Hess}(N, h)^S$, where $V(\tilde{I}_n)$ is the (projective) zero-set of the ideal $\tilde{I}_n$. This description of $\text{Hess}(N, h)^S$ via $\tilde{I}_h$ allow us to identify $V(\tilde{I}_h) \cap S_n$ with $V(I_1 + I_n)$ under a suitable evaluation and therefore to compute its $S$-equivariant cohomology $H^*_S(\text{Hess}(N, h); \mathbb{C})$ using $\tilde{I}_h$.

The paper is organized as follows: Section 2 recalls two basic definitions of the Hessenberg variety and contains a description of the defining ideal of $\text{Hess}(X, H)$ due to Insko [3, Theorem 10]. We give a simpler proof of this theorem using determinantal conditions. In addition, in this section we reinterpret some results of GKM-theory applied to an $n$-torus $\mathbb{T} \subset B$ acting on the full flag variety and we compute the resulting moment graph.

For the natural action of a full dimensional algebraic torus $\mathbb{T} \subset B$ on the flag variety $\text{GL}_n(\mathbb{C})/B$, in Section 3 we start by recalling how this action restricts to an action of $\mathbb{T}$ on the Hessenberg closed subvariety of $\text{GL}_n(\mathbb{C})/B$ in Proposition 3.1. Theorems 3.2 and 3.4 describe the one-dimensional orbits of the action of $\mathbb{T}$ on $\text{Hess}(X, H)$.

Section 4 considers a regular nilpotent operator $N$, with corresponding Hessenberg variety $\text{Hess}(N, h)$ for $h$ any Hessenberg function. In this section we continue considering the action of $\mathbb{T}$ on $\text{GL}_n(\mathbb{C})/B$. We analyze when a fixed point of $\text{GL}_n(\mathbb{C})/B$ under $\mathbb{T}$ belongs to $\text{Hess}(N, h)$ although the action of $\mathbb{T}$ does not restrict to $\text{Hess}(N, h)$. Theorem 4.1 characterizes the flags $[w] = wB \in (\text{GL}_n(\mathbb{C})/B)^{\mathbb{T}}$ such that $[w] \in \text{Hess}(N, h)$ for $w \in S_n$.

Finally, in Section 5 we consider the action of $S = \{\text{diag}(t, t^2, \ldots, t^n) : t \in \mathbb{C}^*\}$ on $\text{GL}_n(\mathbb{C})/B$. In this case, $\text{Hess}(N, h)$ is an $S$-invariant subvariety and Corollary 5.1 describes the fixed point set $\text{Hess}(N, h)$. This section includes a Corollary 5.3, a consequence of [2, Theorem 4.14] that computes the $S$-equivariant class of $\text{Hess}(N, h)$. Also, once the ideal $\tilde{I}_h$ is defined using Corollary 5.1 we give a new proof of [2, Theorem 5.11].

## 2. Preliminaries

### Definition 2.1

Let $V$ be a complex vector space of finite dimension $\dim V = n$. A **full flag** $V_\bullet := V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_{n-1} \subsetneq V$ is a sequence of nested subspaces such that $\dim V_i = i$. The collection of all full flags in $V$ is a projective variety, indeed a determinantal variety. Choosing a basis of $V \simeq \mathbb{C}^n$, there is an isomorphism between the full flag variety and the homogeneous space $\text{GL}_n(\mathbb{C})/B$, where $B$ is the subgroup of upper-triangular matrices.

For $V = \mathbb{C}^n$ and $g \in \text{GL}_n(\mathbb{C})$, let $V_\bullet(g)$ be the full flag where each subspace is the span $V_j(g)$ of the first $j$ column vectors of $g$. Let $S_n$ be the group of permutations of $n$ letters and fix the standard basis $\{e_1, \cdots, e_n\}$ of $V$. For $w$ a permutation in $S_n$ viewed as a permutation matrix $w \in \text{GL}_n(\mathbb{C})$ we let $[w] \in \text{GL}_n(\mathbb{C})/B$ denote the flag given by $we_i = e_{w(i)}$. For the transposition $s_{j,k} \in S_n$ that interchanges $j$ and $k$ and for $0 \neq c \in \mathbb{C}$,
let $G_{i,k}(c) \in \text{GL}_n(\mathbb{C})$ be the matrix given by

$$
(G_{i,k}(c))_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
c & \text{if } i = j \text{ and } k = 1, \\
0 & \text{otherwise}.
\end{cases}
$$

For an $n$-torus $T \subset B$ acting on $\text{GL}_n(\mathbb{C})/B$ by left multiplication we are interested on the $T$-equivariant cohomology of the flag variety $\text{GL}_n(\mathbb{C})/B$. GKM theory gives techniques for computing the $T$-equivariant cohomology, and one useful result is given by Tymoczko [6]: Let $T \subset B \subset \text{GL}_n(\mathbb{C})$ an $n$-torus, $T = \{ \text{diag}(T_1, \cdots , T_n) : T_i \in \mathbb{C}^* \}$ and $g, b$ and $t$ be Lie algebras corresponding to $\text{GL}_n(\mathbb{C})$, $B$, $T$ respectively, where $t = \{ (t_1, \cdots , t_n) : t_i \in \mathbb{C} \}$. In [6, Proposition 2.8], Tymoczko proved the next proposition:

**Proposition 2.2** (Tymoczko [6, Proposition 2.8]). (1) In $\text{GL}_n(\mathbb{C})/B$, the vertices of the moment graph are the points $[w] \in \text{GL}_n(\mathbb{C})/B$ for $w \in S_n$.

(2) If $j < k$, there is an edge of the moment graph from $[w]$ to $[s_{j,k}w]$ if and only if $w^{-1}(j) > w^{-1}(k)$. The points of this edge correspond to the flags $[G_{j,k}(c)w] \in \text{GL}_n(\mathbb{C})/B$ with $c \neq 0$. This edge is labeled $t_j - t_k$.

**Definition 2.3.** A **Hessenberg function** $h$ is a function $h : \{1, 2, \cdots , n\} \rightarrow \{1, 2, \cdots , n\}$ such that $j \leq h(j)$ for all $1 \leq j \leq n$ and $h(j) \leq h(j) + 1$ for all $1 \leq j \leq n - 1$. For a fixed element $X$ in the Lie algebra $\mathfrak{gl}_n$ of $\text{GL}_n(\mathbb{C})$, the corresponding **Hessenberg variety** is the flag variety

$$
\text{Hess}(X, h) = \{ gB \in G/B : XV_h(g) \subseteq V_{h(j)}(g) \}.
$$

C. Procesi [4] has a more general definition of a Hessenberg variety: Let $G$ be a semisimple reductive algebraic group over $\mathbb{C}$, $B \subseteq G$ a Borel subgroup, $\mathfrak{g}$ the Lie algebra of $G$ and $b \subseteq \mathfrak{g}$ the Lie subalgebra corresponding to $B$. A **Hessenberg space** is a $b$-submodule $H \subseteq \mathfrak{g}$ such that $b \subseteq H$. Given a Hessenberg space $H$ and a fixed element $X \in \mathfrak{g}$, the corresponding **Hessenberg variety** is the subvariety of $G/B$ defined by

$$
\text{Hess}(X, H) = \{ gB \in G/B : \text{Ad}(g^{-1}) \cdot X \in H \},
$$

where $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ is the adjoint representation of $G$. When $G = \text{GL}_n(\mathbb{C})$ definition (2.2) is equivalent to (2.1) noticing that for $g = \mathfrak{gl}_n$ and $b = \mathfrak{b}_n$ the Borel subalgebra of upper triangular matrices, a Hessenberg function $h$ defines the Hessenberg space

$$
H_h = b \oplus \bigoplus_{j \leq i \leq h(j)} \mathfrak{g}_{e_i - e_j} = \{ (x_{ij}) \in \mathfrak{gl}_n : x_{ij} = 0 \text{ for all } i > h(j) \},
$$

and for an element $X \in \mathfrak{gl}_n$, using that for $G = \text{GL}_n(\mathbb{C})$ the adjoint representation is given by conjugation there is an equality $\text{Hess}(X, h) = \text{Hess}(X, H_h)$ as sets and as varieties as shown by E. Insko, J. Tymoczko and A. Woo [3, Proposition 8]. In [3, Theorem 10] there is a description of the defining ideal of the Hessenberg variety as

$$
I_{X, H_h} = \langle d_{(1, \cdots , n)}(u_1, \cdots , u_{i-1}, xu_j, u_{i+1}, \cdots , u_n) : i > h(j) \rangle
$$

where $u_1, \cdots , u_n$ are the column vectors of a generic $n \times n$ matrix $Z = (z_{ij})$ and $d_{(1, \cdots , n)}(u_1, \cdots , u_{i-1}, Xu_j, u_{i+1}, \cdots , u_n)$ is the determinant of the matrix with columns $u_1, \cdots , u_{i-1}$, $Xu_j$, $u_{i+1}, \cdots , u_n$, namely $d \in R = \mathbb{C}[z_{ij}]$. Since we are considering flags, then $d \neq 0$ and $I_{X, H_h} \subset \mathbb{C}[z_{ij}, d^{-1}]$. We start by giving a different description of $I_{X, H_h}$ and to do
this we start by giving a description of \( \text{Hess}(X, h) \). Given \( gB \in G/B \), let \( v_1, \ldots, v_n \) be the columns of \( g \) and \( V_\bullet(g) \) the flag defined by \( gB \); observe that
\[
V_\bullet(g) \in \text{Hess}(X, h) \iff X \cdot V_j(g) \subseteq V_{h(j)}(g) \text{ for all } 1 \leq j < n
\]

(2.4)
\[
\iff \langle v_1, \ldots, v_{h(j)}, Xv_1, \ldots, Xv_j \rangle = V_{h(j)} \text{ for all } 1 \leq j < n.
\]

This is equivalent to say that every size \( h(j) + 1 \) minor of the matrix whose columns are the vectors \( \{v_1, \ldots, v_{h(j)}, Xv_1, \ldots, Xv_j\} \) vanish. Now, for \( l < j \) then \( h(l) \leq h(j) \), so \( V_{h(l)}(g) \subseteq V_{h(j)}(g) \). Then, for \( V_\bullet(g) \in \text{Hess}(X, h) \) and for each \( l \) and \( l + 1 \) we have that \( XV_l(g) \subseteq V_{h(l)}(g) \subseteq V_{h(l+1)}(g) \). Therefore, the inclusion \( XV_{l+1}(g) \subseteq V_{h(l+1)}(g) \) is determined by the condition \( XV_{l+1} \in V_{h(l+1)}(g) \). Hence,
\[
X \cdot V_j(g) \subseteq V_{h(j)}(g) \iff \langle v_1, \ldots, v_{h(j)}, Xv_1, \ldots, Xv_j \rangle = V_{h(j)}(g)
\]
\[
\iff Xv_j \in V_{h(j)}(g) \text{ for all } 1 \leq j < n.
\]

The last condition implies that every size \( h(j) + 1 \) minor of the matrix formed by the columns \( \{v_1, \ldots, v_{h(j)}, Xv_j\} \) vanish and hence the sets
\[
\mathcal{B}_{j,k} = \{v_1, \ldots, v_{h(j)}, Xv_j, v_{h(j)+1}, \ldots, \hat{v}_k, \ldots, v_n\}
\]
are linearly dependent, where \( v_1, \ldots, v_n \) are the column vectors of the matrix \( g \) and \( \hat{v}_k \) means removing \( v_k \). Reciprocally, if for all \( 1 \leq j \leq n - 1 \) and for all \( k \) such that \( h(j) + 1 \leq k \leq n \) all sets \((2.6)\) are linearly dependent then \((2.5)\) is consequence of the following elementary lemma:

**Lemma 2.4.** Let \( V \) be a complex vector space of finite dimension \( n \), \( B = \{v_1, \ldots, v_n\} \) a basis of \( V \) and \( v \in V \) non zero such that \( v \neq v_j \) for all \( 1 \leq j \leq n \). Then, for a fixed \( j, v \in V_j = \{v_1, \ldots, v_j\} \) if and only if for all \( k \) with \( j + 1 \leq k \leq n \) the sets \( \mathcal{B}_{j,v,k} = (B - \{v_k\}) \cup \{v\} \) are linearly dependent. \( \square \)

Now, we note that the conditions \((2.4), (2.5)\) and \((2.6)\) have associated equations that arise from certain determinants: Indeed, let \( u_1, \ldots, u_n \) denote the column vectors of a generic square \( n \times n \) matrix \( Z = (z_{ij}) \) and for a linear operator \( X : \mathbb{C}^n \rightarrow \mathbb{C}^n \) consider the column vectors \( Xu_i \), for \( 1 \leq i \leq n \). For a positive integer \( m \leq n \) let \( R = (R_1, \ldots, R_m) \) with \( R_i \in \{1, \ldots, n\} \) and \( R_1 < \cdots < R_m \) and let \( C = \{C_1, \ldots, C_m\} \) an ordered set of column vectors in \( \mathbb{C}^n \). Let \( d_{R,C} \in \mathbb{C}[z_{ij}] \) denote the \( m \times m \) minor obtained by choosing the rows of the columns of \( C \) according to \( R \), that is, the determinant of the \( m \times m \) matrix whose \((i,j)\)-th entry is the \( R_i\)-th entry of \( C_j \). Now, for a Hessenberg variety \( \text{Hess}(X, h) \) associated to a Hessenberg function \( h \) and an endomorphism \( X \in \mathfrak{gl}_n \) consider the following ideals
\[
J_{X,h(j)} = \langle d_{R,C} : R \subseteq \{1, 2, \ldots, n\} \text{ and } C \subseteq \{u_1, \ldots, u_{h(j)}\}, Xu_1, \ldots, Xu_j \rangle
\]
\[
\text{such that } |R| = |C| = h(j) + 1,
\]
\[
J_{X,h(j)+1} = \langle d_{R,C} : R = \{1, \ldots, h(j) + 1\}, C = \{u_1, \ldots, u_{h(j)}\}, Xu_j \rangle,
\]
and note that:
- Condition \((2.4)\) has associated the following ideal
\[
J_{X,h} = \sum_{j=1}^{n} J_{X,h(j)}.
\]
We observe since

\[ J'_{X,h} = \sum_{j=1}^{n} J'_{X,j,h(j)+1} \]

• and condition (2.6) has associated the following ideal

\[ I_{X,H,h} = \langle d_{[1,\ldots,n]}, x_{t_1,\ldots,x_{t_{k-1}}, x_{t_{k+1}}, \ldots, x_{t_n}} : 1 \leq j \leq n, y > h(j) \rangle. \]

Finally, by the equivalence of Conditions (2.4) and (2.5), the ideals \( J'_{X,h} = J'_{X,h} \), and since Hess\([X,h]\) is a closed subvariety of the complete flag variety, by Lemma 2.4 \( J'_{X,h} \) and \( I_{X,H,h} \) are ideals of \( \mathbb{C}[z_{ij}, d^{-1}] \), where \( d \) is the determinant function. Since \( J'_{X,h} \) is the ideal defining Hess\([X,h]\) and \( I_{X,H,h} \) is the defining ideal of Hess\([X,H,h]\), as ideals in \( \mathbb{C}[z_{ij}, d^{-1}] \), the equivalences (2.4) \( \iff \) (2.5) \( \iff \) (2.6) and Lemma 2.4 gives another proof of:

**Theorem 2.5 ([3, Theorem 10]).** For any Hessenberg function \( h \)

\[ I_{X,H,h} = J_{X,h} \]

as ideals in \( \mathbb{C}[z_{ij}, d^{-1}] \). \( \square \)

### 3. One-dimensional orbits in a Hessenberg variety. The semisimple case

In this section we will give the one-dimensional orbits from a \( T \)-action on Hess\([X,h]\) in the case when \( X \) is semisimple, \( (X \in \mathfrak{t}) \) say \( X = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and start with the example \( h(j) = j + 1 \). Let \( B \subseteq \text{GL}_n \) the subgroup of upper triangular matrices (a Borel subgroup), \( T \subseteq B \) an \( n \)-torus (diagonal matrices) and \( g, b \) and \( t \) the Lie algebras corresponding. Thus, \( T = \{ s = \text{diag}(T_1, \ldots, T_n) : T_i \in \mathbb{C}^* \} \) and \( t = \{ \sigma = \text{diag}(t_1, \ldots, t_n) : t_i \in \mathbb{C} \} \). Consider the \( T \)-action on \( \text{GL}_n(\mathbb{C})/B \) defined by left multiplication \( s \cdot [g] = [sg] \). By Proposition 2.2 the \( T \)-fixed points under the action are \( [w] \in \text{GL}_n(\mathbb{C})/B \) such that \( w \in S_n \). We will show that Hess\([X,h]\) is invariant under the \( T \)-action and once this is proven it determines which \( T \)-fixed points of \( \text{GL}_n(\mathbb{C}) \) belong to Hess\([X,h]\).

**Proposition 3.1.** Let \( T \) be an \( n \)-torus acting on \( \text{GL}_n(\mathbb{C})/B \) by left multiplication. Let Hess\([X,h]\) be a Hessenberg variety with \( X \in \mathfrak{g} \) semisimple and \( h(j) = j + 1 \). Then Hess\([X,h]\) is invariant under the \( T \)-action and \( (\text{GL}_n(\mathbb{C})/B)^T = (\text{Hess}(X,h))^T \).

**Proof.** We observe since \( X \) is diagonal and \( T \) is given by invertible diagonal matrices, \( X \) commutes with all \( s \in T \) and hence the \( T \)-action on \( G/B \) induces an action on Hess\([X,h]\). Additionally, the \( T \)-fixed points are the same, Hess\([X,h]^T = (G/B)^T \). Indeed, by Proposition 2.2 the \( T \)-fixed points of \( G/B \) are the \( [w] \in G/B \) such that \( w \in S_n \). On the other hand \( Xv_j = \lambda_{w(t)}v_j \) with \( w(l) = j \), and thus

\[ \{ v_1, \ldots, v_j, Xv_j, v_{j+1}, \ldots, v_k, \ldots, v_n \} \]

is linearly dependent for all \( k \) with \( j + 1 < k < n \). Therefore, by Theorem 2.5, \( [w] \in \text{Hess}(X,h) \). \( \square \)

The next theorem gives a description of the one-dimensional orbits of Hess\([X,h]\) under the action of \( T \) given by Proposition 3.1. This description is analogous to the description of the one-dimensional orbits on the full flag variety cited in Proposition 2.2. Let \( O_{w,s_{j,k}w} \) denote the one-dimensional orbit of the full flag variety whose closure contains the \( T \)-fixed points \( [w] \) and \( [s_{j,k}w] \). This orbit consists of the flags

\[ (3.1) \quad O_{w,s_{j,k}w} = \{ [Gj_{k}(c)w] \in \text{GL}_n(\mathbb{C})/B : c \in \mathbb{C}^* \} \]
where \([G_{jk}(c)w]\) and \(s_{j,k}\) were defined in Section 2.

**Theorem 3.2.** Let \(O_{w,s_{j,k}} \in \{[G_{jk}(c)w] : c \in \mathbb{C}^*\}\) be a one-orbit under the action of \(T\) on \(G/B\) as before. If \(w(w^{-1}(k) + 1) = j\), then \(O_{w,s_{j,k}} \cap \text{Hess}(X,h) \neq \emptyset\). Moreover, for the \(T\)-action on \(\text{Hess}(X,h)\) given by the restriction of the action of \(T\) on \(G/B\) as in Proposition 3.1, then \(O_{w,s_{j,k}} \subset \text{Hess}(X,h)\).

**Proof.** For each flag \([G_{jk}(c)w]\) with representative in \(G/B\) given by the matrix

\[
(G_{jk}(c)w)_{\ell} = \begin{cases}
1 & \text{if } w(\ell) = i, \\
c & \text{if } i = j \text{ and } w^{-1}(k) = \ell, \\
0 & \text{otherwise},
\end{cases}
\]

for its column \(v_{\ell}\), with \(\ell \neq w^{-1}(k)\), we have \(Xv_{\ell} = \lambda_{w(\ell)}v_{\ell}\). It follows that

\[
\{v_1, \ldots, v_{\ell-1}, Xv_{\ell}, \ldots, \hat{v}_{\ell}, \ldots, v_n\}
\]

is a linearly dependent set. In particular, for each \(r > h(\ell)\), it follows that

\[
d_{1, \ldots, n} = d_v = 0.
\]

We only have to analyze the column vector \(v_{w^{-1}(k)}\) where \(a_{jw^{-1}(k)} = c\) and \(a_{kw^{-1}(k)} = 1\). In these cases we have that

\[
Xv_{w^{-1}(k)} = (0, \ldots, \lambda_{w^{-1}(k)}c, 0, \ldots, \lambda_k, 0, \ldots, 0).
\]

By hypothesis we have \(w(w^{-1}(k) + 1) = j\), and this implies that \(v_{w^{-1}(k)+1}\) has the entry \(a_{jw^{-1}(k)+1} = 1\). Since \(h(\ell) = \ell + 1\), we have for all \(r\) such that \(r > h(w^{-1}(k)) = w^{-1}(k) + 1\), the determinant

\[
\hat{d} = d_{1, \ldots, n} = 0.
\]

Now, since the column vector \(v_{\ell}\) has entry \(a_{\ell r} = 1\) with \(\ell = w(\tau)\) and \(\ell \neq j\) so by substituting \(v_{\tau}\) with \(Xv_{w^{-1}(k)}\), the \(\ell\) row of \(d\) consists of zeros. We have shown that the column vectors of \([G_{jk}(c)w]\) satisfy the condition of the defining ideal of \(\text{Hess}(X,h)\), and thus \([G_{jk}(c)w] \in \text{Hess}(X,h)\) for all \(c \in \mathbb{C}^*\). It follows that \(O_{w,s_{j,k}} \subset \text{Hess}(X,h)\).

**Remark 3.3.** Let \(v_{\tau}\) be a column vector such that \(a_{\tau r} = 1\) \((w(\tau) = j)\). If \(r > w^{-1}(k) + 1\), then there is \(c \in \mathbb{C}^*\) such that

\[
\hat{d} = d_{1, \ldots, n} = 0.
\]

this is because

\[
\hat{d} = (-1)^{r+j}c \text{ sgn}(\eta) + (-1)^{r+k} \text{ sgn}(\tau \eta)
\]

where \(\tau, \eta\) are in \(S_{n-1}\) and \(\tau\) is a transposition. Thus \([G_{jk}(c)w] \notin \text{Hess}(X,h)\).

The next theorem is a generalization of Theorem 3.2 for an arbitrary Hessenberg function. Theorem 3.4 allow us to determine which one-dimensional orbits of \(\text{GL}_n(\mathbb{C})/B\) are contained in the corresponding Hessenberg variety:

**Theorem 3.4.** Let \(T\) be an \(n\)-torus acting on \(\text{GL}_n(\mathbb{C})/B\) by left multiplication. Let \(\text{Hess}(X,h)\) be a Hessenberg variety with \(X \in \mathfrak{g}\) semisimple and \(h\) any Hessenberg function. Then

1. \(\text{Hess}(X,h)\) is invariant under the \(T\)-action.
2. \((\text{GL}_n(\mathbb{C})/B)^T = (\text{Hess}(X,h))^T\).
(3) If \( \ell = j \), then \( \ell \leq h(w^{-1}(k)) \) if and only if for all \( c \in \mathbb{C}^* \), \( [G_{jk}(c)w] \in \text{Hess}(X, h) \), that is
\[
O_{w, s_j k w} \subset \text{Hess}(X, h).
\]

**Proof.** Parts (1) and (2) are proved as in Proposition 3.1. For Part (3), if \( \ell \neq w^{-1}(k) \), \( X_{\ell t} = \lambda_{w(\ell)} \), by a similar argument as in the proof of Theorem 3.2, we have the linear dependence condition of (3.3), thus, we have
\[
\hat{d} = \hat{d}(\ell, \cdots, n, \lambda_1, \ldots, \lambda_k) = 0.
\]
Again, we have to analyze \( w^{-1}(k) \), where \( X_{w^{-1}(k)} = (0, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, 0) \). Then, for all \( r > h(w^{-1}(k)) \) we have
\[
\hat{d} = \hat{d}(\ell, \cdots, n, \lambda_1, \ldots, \lambda_k) = 0.
\]

Assume now that \( O_{w, s_j k w} \subset \text{Hess}(X, h) \), that is, \( [G_{jk}(c)w] \in \text{Hess}(X, h) \) for all \( c \in \mathbb{C}^* \). Let \( \{v_1, \ldots, v_n\} \) be the column vectors of the matrix \( G_{jk}(c)w \) which represents the flag \( [G_{jk}(c)w] \in \mathbb{C}^n \). Then, the ideal \( I_{X \otimes H_n} \) of the Theorem 2.5 vanishes on the columns \( v_t \). Now, each \( v_t \) with \( \ell \neq w^{-1}(k) \) and \( X_{\ell t} = \lambda_{w(\ell)}v_t \) the following determinants vanish
\[
\hat{d} = \hat{d}(\ell, \cdots, n, \lambda_1, \ldots, \lambda_k) = 0.
\]

and for \( \ell = w^{-1}(k) \), we have \( X_{w^{-1}(k)} = \lambda_1cv_{w^{-1}(j)} + \lambda_2v_{w^{-1}(k)} \). Then, by hypothesis and by Theorem 2.5, for all \( r > h(w^{-1}(k)) \) we have the first equality in:
\[
0 = \hat{d} = \hat{d}(\ell, \cdots, n, \lambda_1, \ldots, \lambda_k) = 0.
\]

The last equality implies that
\[
B_{w^{-1}(j)} = \{v_1, \ldots, v_{w^{-1}(k)}, \ldots, v_{h(w^{-1}(k))}, \lambda_1cv_{w^{-1}(j)}, \ldots, v_r, \ldots, v_n\}
\]
is a linearly dependent set for all \( r > h(w^{-1}(k)) \). This linear dependence condition occurs if \( v_{w^{-1}(j)} \) and \( \lambda_1cv_{w^{-1}(j)} \) belong to \( B_{w^{-1}(j)} \) for all \( r > h(w^{-1}(k)) \). This happens whenever \( w^{-1}(j) \leq h(w^{-1}(k)) \) by Lemma 2.4. Thus, if \( O_{w, s_j k w} \subset \text{Hess}(X, h) \) then \( \ell \leq h(w^{-1}(k)) \), this proves Part (3). \( \square \)

**Example 3.5.** We consider the full flag variety \( GL_n(\mathbb{C})/B \) and \( T \) an \( n \)-torus acting on \( GL_n(\mathbb{C}) \) as Proposition 2.2. Let \( X \in \mathfrak{g} \) be semisimple and \( h \) Hessenberg function defined by \( h(i) = n \) for all \( i \leq i \leq n \). For all one-dimensional orbit \( O_{w, s_j k w} \) of \( GL_n(\mathbb{C})/B \), we have \( O_{w, s_j k w} \subset \text{Hess}(X, h) \). Indeed, for all pair \( j, k \) with \( j < k \) and \( w^{-1}(k) < w^{-1}(j) \), the condition (3) of 3.4 holds since \( w^{-1}(j) \leq n = h(w^{-1}(k)) \). Particularly, when \( X = \text{Id} \), we have Proposition 2.2.
Remark 3.6. If \( h \) is the Hessenberg function \( h(i) = i \) for all \( i \) and \( X \in \mathfrak{g} \) is diagonal, for the corresponding Hessenberg variety \( \text{Hess}(X, h) \subset \text{GL}_n(\mathbb{C})/B \), consider an \( n \)-torus \( T \) acting on \( \text{GL}_n(\mathbb{C})/B \) by left multiplication. By Proposition 3.1 \( \text{Hess}(X, h) \) is invariant under this \( T \)-action. However, for every one-dimensional orbit \( O_{w,s_j,k,w} \) of \( \text{GL}_n(\mathbb{C})/B \) with \( w(\ell) = j \), by construction \( \ell > w^{-1}(k) \), and since \( h(i) = i \) then \( \ell > h(w^{-1}(k)) \). By a similar argument as in Remark 3.3 there is \( c \) such that

\[
\tilde{d} = d_{(1,...,n)},(v_1,...,v_{w^{-1}(k)},...,v_{h(w^{-1}(k))},x_{v_{w^{-1}(k)}},...v_1,...v_n) \neq 0,
\]

hence \([G_{jk}(c)w] \notin \text{Hess}(X, h)\) and thus the \( T \)-action on \( \text{Hess}(X, h) \) does not have one-dimensional orbits.

4. ONE DIMENSIONAL ORBITS IN A HESSENBERG VARIETY. THE NILPOTENT CASE

In this section we obtain the one-dimensional orbits from a \( T \)-action on the Hessenberg variety \( \text{Hess}(N, h) \) when \( N \) is a regular nilpotent operator. Thus, \( N \) is a matrix whose Jordan form has one block with corresponding eigenvalue equal to zero. We obtain an analogue description as in the diagonal case in Theorem 3.4. Here, \( T \) is an \( n \)-torus acting on flag variety \( \text{GL}_n(\mathbb{C})/B \) by left multiplication. First, we determine which fixed points in \((\text{GL}_n(\mathbb{C})/B)^T \) belong to \( \text{Hess}(N, h) \). Next, we do same for the one dimensional orbits of \( \text{GL}_n(\mathbb{C})/B \).

By Proposition 2.2, the \( T \)-fixed points \([w] \in \text{GL}_n(\mathbb{C})/B \) correspond to \( w \in S_n \). For \([w] \in (\text{GL}_n(\mathbb{C})/B)^T \) with column vectors \( v_1, ..., v_n \), \( Nv_{w^{-1}(i+1)} = e_i \) for all \( 1 \leq i < n \), where \( e_i \) are the canonical base, and for \( i = 1 \), \( Nv_{w^{-1}(1)} = N \), \( \tilde{0} \), the null vector. By Theorem 2.5, a \( T \)-fixed point \([w] \) belongs to \( \text{Hess}(N, h) \) if and only if its column vectors vanish in the ideal \( I_{N,H_h} \). This is equivalent to \( w^{-1}(i) \leq h(w^{-1}(i+1)) \) for all \( 1 \leq i < n \). This last claim follows from the fact that \( Nv_{w^{-1}(i+1)} \) and \( v_{w^{-1}(i+1)} \) are linearly independent since

\[
\tilde{d} = d_{(1,...,n)},(v_1,...,v_{w^{-1}(i+1)},...,v_{h(w^{-1}(i+1))},x_{v_{w^{-1}(i+1)}},...v_1,...v_n) = 0
\]

if and only if \( e_i = v_{w^{-1}(i)} \) belongs to

\[
B_{w^{-1}(i+1)},r = \{v_1, ..., v_{h(w^{-1}(i+1))}, Nv_{w^{-1}(i+1)}, ..., \tilde{v}_r, ..., v_n\}
\]

for all \( r > h(w^{-1}(i+1)) \), and (4.1) is equivalent to \( w^{-1}(i) \leq h(w^{-1}(i+1)) \).

Now, we compute the one dimensional orbits of \( \text{Hess}(N, h) \). From Proposition 3.1 and Equation (3.2) the one dimensional orbits of \( \text{GL}_n(\mathbb{C})/B \) are \( O_{w,s_j,k,w} \) with \( w \in S_n \) and the transposition \( s_j,k \). For \( w \in S_n \) with \([w] \in \text{GL}_n(\mathbb{C})/B \) the corresponding flag, in (3.1) we defined

\[
O_{w,s_j,k,w} = \{[G_{jk}(c)w] \in \text{GL}_n(\mathbb{C})/B : c \in \mathbb{C}^*\}
\]

with \( j < k \) such that \( w^{-1}(j) > w^{-1}(k) \). To determine which of the orbits (4.2) belong to \( \text{Hess}(N, h) \), first note that \([w] \) must belong to \( \text{Hess}(N, h) \) and if that is the case, \( O_{w,s_j,k,w} \subset \text{Hess}(N, h) \) if every flag \([G_{jk}(c)w] \) of \( O_{w,s_j,k,w} \) vanish in the ideal \( I_{N,H_h} \) and the proof of this last claim starts by noting that

\[
Nv_{w^{-1}(k)} = \begin{cases} e_{k-1} + ce_{j-1} & \text{if } j, k \neq 1, \\ e_{k-1} & \text{if } j = 1, \text{ note that } k > j = 1 \end{cases}
\]
and for $j, k \neq 1$ in (4.3), $[G_{1,k}(\cdot)^w]$ vanish in $I_{N,H_h}$ if and only if for all $r > h(w^{-1}(k))$ we have the first equality in (4.4), where the next equalities follow from a direct computation:

$$0 = \tilde{d} = d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n)$$

$$= d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n)$$

(4.4)

We observe now the right-hand side of the last equality in (4.4) is zero if and only if for all $r > h(w^{-1}(k))$ both determinants are simultaneously zero, that is

$$0 = d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n)$$

(4.5)

Indeed, by Lemma 2.4 there is $r' > h(w^{-1}(k))$ such that the two determinants in the last equality in (4.4) are nonzero and cancel each other if and only if $w^{-1}(k-1)$ and $w^{-1}(j-1)$ are greater than $h(w^{-1}(k))$. Now, for $r = w^{-1}(k-1)$ we have

$$0 \neq d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n)$$

and hence

$$0 = d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n).$$

(4.6)

Similarly, for $r = w^{-1}(j-1)$, Equation 4.7 is nonzero and Equation 4.6 is zero. Thus, both determinants cannot be nonzero simultaneously proving (4.5). Lastly, the vanishing of both summands in (4.5) holds if and only if

$$w^{-1}(k-1) \leq h(w^{-1}(k)) \quad \text{and} \quad w^{-1}(j-1) \leq h(w^{-1}(k)).$$

Next, for the case $j = 1$ in (4.3) equality (4.8) holds

$$0 = d_{(1,...,n)}(\nu_1,...,\nu_{w^{-1}(k)}), v_{h(w^{-1}(k))}, v_{w^{-1}(k)}, N v_{w^{-1}(k)}, \nu_r,...,\nu_n)$$

(4.8)

if and only if $w^{-1}(k-1) \leq h(w^{-1}(k))$. These arguments characterize when an open orbit $O_{w,s_{j,k}^w}$ is contained in $\text{Hess}(N, h)$. To determine when the closure of a one dimensional orbit is also contained in $\text{Hess}(N, h)$ we need to characterize when $[s_{j,k} w] \in \text{Hess}(N, h)$. For this, observe that $[s_{j,k} w]$ is the permutation matrix that interchanges the rows $j$ and $k$ of $[w]$. In other words, $[w^{-1}s_{j,k}]$ interchanges the columns $w^{-1}(j)$ and $w^{-1}(k)$ of $[w^{-1}]$ leaving the remaining columns remain fixed. Hence, it suffices to verify that the columns $w^{-1}(j)$ and $w^{-1}(k)$ vanish in $I_{N,H_h}$. For the column $w^{-1}(k)$ this is equivalent to show that $(s_{j,k} w)^{-1}(j) \leq h(w^{-1}(j+1))$. To check this, observe that for $k \neq j + 1$,

$$(s_{j,k} w)^{-1}(j) = (w^{-1}s_{j,k}(j)) = w^{-1}(k) \leq h((s_{j,k} w)^{-1}(j+1)) = h(w^{-1}(j+1))$$

and for $k = j + 1$, we have $w^{-1}(j+1) \leq h(w^{-1}(j))$. Similarly, for the column $w^{-1}(j)$,

$$(s_{j,k} w)^{-1}(k) = (w^{-1}s_{j,k}(k)) = w^{-1}(j) \leq h((s_{j,k} w)^{-1}(k+1)) = h(w^{-1}(k+1)).$$

(4.9)

Hence, $O_{w,s_{j,k}^w} \subset \text{Hess}(N, h)$ if and only if

1. $[w] \in \text{Hess}(N, h)^T$
2. $[s_{j,k} w] \in \text{Hess}(N, h)^T$
3. $w^{-1}(k-1) \leq h(w^{-1}(k))$ and $w^{-1}(j-1) \leq h(w^{-1}(k))$
and we have proved the following:

**Theorem 4.1.** Let $T$ be an $n$-torus acting on $GL_n(\mathbb{C})/B$ by left multiplication. Let $Hess(N, h)$ be a Hessenberg variety with $N \in \mathfrak{g}$ regular nilpotent operator and $h$ any Hessenberg function. Then $\mathcal{O}_{w, s_{1,2}} \subset Hess(N, h)$ if and only if

1. $w^{-1}(i) \leq h(w^{-1}(i+1))$ for all $i$ such that $1 \leq i < n$,
2. $w^{-1}(k) \leq h(w^{-1}(j+1))$ for $k \neq j + 1$ and for $k = j + 1$ we have $w^{-1}(j+1) \leq h(w^{-1}(j))$. It must also be satisfied $w^{-1}(j) \leq h(w^{-1}(k+1))$ if $k < n$.
3. $w^{-1}(k-1) \leq h(w^{-1}(k))$ and $w^{-1}(j-1) \leq h(w^{-1}(j))$.

**Example 4.2.** For $GL_3(\mathbb{C})/B$ and $h(i) = i + 1$, consider the nilpotent operator

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $T$ be the 3-torus in $GL_3(\mathbb{C})$ and consider its action on $GL_3(\mathbb{C})/B$ by left multiplication. The $T$-fixed points of $GL_3(\mathbb{C})/B$ are

$$[e] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ [w_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ [w_2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$[w_3] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ [w_4] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ [w_5] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

By Theorem 4.1 we find the $T$-fixed points and one dimensional orbits of $Hess(N, h)$. The identity $e$ belong to $Hess(N, h)$ and

1. for $[w_1]$ we have
   - $w^{-1}_1(1) = 1 \leq h(w^{-1}_1(2)) = h(3) = 3$, then $[w_1] \in Hess(N, h)$
2. for $[w_2]$ we have
   - $w^{-1}_2(1) = 2 = h(w^{-1}_2(2)) = h(1) = 2$, then $[w_2] \in Hess(N, h)$
3. for $[w_3]$ we have
   - $w^{-1}_3(1) = 3 > h(w^{-1}_3(2)) = h(1) = 2$, then $[w_3] \notin Hess(N, h)$
4. for $[w_4]$ we have
   - $w^{-1}_4(2) = 3 > h(w^{-1}_4(3)) = h(1) = 2$, then $[w_4] \notin Hess(N, h)$
5. for $[w_5]$ we have
   - $w^{-1}_5(1) = 3 \leq h(w^{-1}_5(2)) = h(2) = 3$, then $[w_5] \in Hess(N, h)$.

Thus, the $T$-fixed points of $Hess(N, h)$ are $\{e, [w_1], [w_2], [w_3]\}$.

By Theorem 4.1, $\mathcal{O}_{w, s_{1,2}}$ is contained in $Hess(N, h)$ for $[w]$, $[s_{1,2} w]$ fixed points. Then, we compute these orbits for $\{e, [w_1], [w_2], [w_3]\}$. We observe for $e$ there are not pairs $(j, k)$. Now:

1. For $[w_1]$ it has the pair $(2, 3)$ and $[s_{2,3} w_1]$. Then
Remark. For Hess we analyze the case Hess. Let \( N \) be a regular nilpotent operator (say \( N = 3 \)). Then \( O_{\omega_1, s_2, 3, w_1} = \text{Hess}(N, h) \). Thus \( O_{\omega_1, s_2, 3, w_1} \subset \text{Hess}(N, h) \).

(2) For \( [w_2] \) it has the pair \((1, 2)\) and \([s_1, 2, w_2] \). Then
\[
\omega^{-1}(1) = h(w^{-1}(1)) = h(2) = 3
\]
hence \([s_1, 2, w_2] \subset \text{Hess}(N, h) \). Now we verify condition (3) of Theorem 4.1 for \( O_{\omega_1, s_2, 3, w_1} \)
\[
\omega^{-1}(1) = 2 = h(w^{-1}(2)) = h(1) = 2
\]
then \( O_{\omega_2, s_1, 2, w_2} \subset \text{Hess}(N, h) \). Thus \( O_{\omega_2, s_1, 2, w_2} \subset \text{Hess}(N, h) \).

(3) For \([w_5] \) we have the pairs \((1, 2), (1, 3)\) and \((2, 3)\). Then
\[
\text{for } [s_1, 2, w_5], \omega^{-1}(1) = 3 > h(w^{-1}(3)) = h(1) = 2 \text{ thus } [s_1, 2, w_5] \notin \text{Hess}(N, h).
\]
\[
\text{for } [s_2, 3, w_5], \omega^{-1}(1) = 3 > h(w^{-1}(3)) = h(1) = 2 \text{ thus } [s_2, 3, w_5] \notin \text{Hess}(N, h).
\]
\[
\text{for } [s_1, 3, w_5], \omega^{-1}(2) = 2 = h(w^{-1}(3)) = h(1) = 2. \text{ Thus } [s_1, 3, w_5] \in \text{Hess}(N, h).
\]

hence \([s_1, 3, w_5] \in \text{Hess}(N, h) \). Finally we show condition (3) of Theorem 4.1 for \( O_{\omega_1, s_1, 3, w_1} \)
\[
\omega^{-1}(2) = 2 = h(w^{-1}(3)) = h(2) = 3
\]
then \( O_{\omega_2, s_1, 3, w_1} \subset \text{Hess}(N, h) \). Thus \( O_{\omega_2, s_1, 3, w_1} \subset \text{Hess}(N, h) \).

The corresponding moment graph is represented by Figure 4.1

![Figure 4.1](image)

**Figure 4.1.** Moment graph of Hess\((N, h)\).

**Example 4.3.** We consider \( GL_n(\mathbb{C})/B \) and let \( T \) be the \( n \)-torus acting on \( GL_n(\mathbb{C})/B \) by left multiplication. Let \( N \) be a regular nilpotent operator (say \( N \) as in Example 4.2 in the case \( n = 3 \)). We define \( h(i) = i \) for all \( 1 \leq i \leq n \) and we consider Hess\((N, h)\). We observe that Hess\((N, h)\) has only one fixed point, \([e]\). Indeed, for all \( i \) we have
\[
w^{-1}(i) < w^{-1}(i + 1) = h(w^{-1}(i + 1))
\]
and this is true only for \([e]\).

**Remark 4.4.** We consider \( h(i) = i \) for all \( 1 \leq i \leq n \). In the section 3, the Remark 3.6 we analyze the case Hess\((X, h)\) with X diagonal. Its fixed points are the same \( GL_n(\mathbb{C})/B \). For Hess\((N, h)\) with N nilpotent, has only one fixed point, \([e]\), as proved in Example 4.3.
5. The equivariant cohomology of a Hessenberg variety in the nilpotent case with an action of a 1-dimensional torus

In this section, we consider the action by left multiplication of the one dimensional torus \( S = \{ \text{diag}(t, t^2, \ldots, t^n) : t \in \mathbb{C}^* \} \) on \( \text{Hess}(N, h) \). In the whole section, we will assume that the cohomology \( H^*_S(\text{Hess}(N, h); \mathbb{C}) \) is generated by its two degree cohomology classes (this last assumption is a conjecture).

First, consider the action of the torus on \( \text{GL}_n(\mathbb{C})/B \) by left multiplication. For a regular nilpotent operator \( N \) and \( h \) any Hessenberg function the variety \( \text{Hess}(N, h) \) is \( S \)-invariant and the \( S \)-fixed points of \( \text{GL}_n(\mathbb{C})/B \) are given by the permutation matrices, \([w] \in \text{GL}_n(\mathbb{C})/B\) such that \( w \in S_n \). The proof of part (1) of Theorem 4.1 also gives which fixed points of the action of \( S \) on \( \text{GL}_n(\mathbb{C})/B \) are fixed points in the corresponding Hessenberg subvariety:

**Corollary 5.1.** Let \( \text{Hess}[N, h] \) be the Hessenberg variety for \( N \in \mathfrak{g} \) a regular nilpotent operator and \( h \) any Hessenberg function and with the action of the one dimensional torus \( S \). Then, for all \([w] \in (\text{GL}_n(\mathbb{C})/B)^S\), \([w] \in \text{Hess}(N, h)^S\) if and only if \( w^{-1}(i) \leq h(w^{-1}(i+1)) \) for all \( i \) such that \( 1 \leq i < n \).

E. Insko [2, Theorem 4.14] gives a description of the \( S \)-equivariant cohomology class of any \( \text{Hess}(N, h) \) in the \( S \)-equivariant cohomology ring of \( \text{GL}_n(\mathbb{C})/B \):

**Theorem 5.2** (Insko, Theorem 4.14). For \([wB] = [w]\) an \( S \)-fixed points of \( \text{GL}_n(\mathbb{C})/B \), if \([wB] \) is a point in \( \text{Hess}(N, h) \), the localization in the \( S \)-equivariant cohomology ring of \( \text{GL}_n(\mathbb{C})/B \) of the \( S \)-equivariant cohomology class of \([\text{Hess}(N, h)]S\) at \([wB] \) is

\[
\prod_{\{1 \leq i \leq k, 1 \leq j \leq n : h(j) < w^{-1}(i)\}} ((1 - w(j) + 1)t).
\]

With this result Insko computes the \( S \)-equivariant class of \( \text{Hess}(N, h) \) in

\[
H^*_S(\text{GL}_n(\mathbb{C})/B)^S \simeq \bigoplus_{w \in W} (\mathbb{C}[t])
\]

and proves that \([\text{Hess}(N, h)]S\) is the tuple in \( \bigoplus_{w \in W} (\mathbb{C}[t]) \) consisting of the localizations at \([wB] = [w]\) for \([w] \in \text{Hess}(N, h) \) and zero elsewhere. We can improve this result using our description of the \( S \)-fixed point of \( \text{Hess}(N, h) \):

**Corollary 5.3.** Let \( \text{Hess}[N, h] \) be a Hessenberg variety with the action the one-dimensional torus \( S \) as before. Then, the localization of the non zero equivariant cohomology classes \([wB] \in (\text{GL}_n(\mathbb{C})/B)^S\) are given by

\[
\prod_{\{1 \leq i \leq k, 1 \leq j \leq n : h(j) < w^{-1}(i)\}} ((1 - w(j) + 1)t).
\]

The difference between Theorem 5.2 and Corollary 5.3 is that the former needs to know which fixed points belong to \( \text{Hess}(N, h) \) and the corollary does not since by Corollary 5.1 the condition \( w^{-1}(i) \leq h(w^{-1}(i+1)) \) means that \([w] \in \text{Hess}(N, h) \).

Now, we give another description of the fixed points of \( \text{Hess}(N, h) \). Condition 1 of Theorem 4.1 is equivalent to:

\[
(5.1) \quad [wB] \in \text{Hess}(N, h)^S \iff w(i) - 1 = w(k) \text{ with } k \leq h(i), \text{ if } w(i) \neq 1.
\]

Equivalently, there is \( k \) with \( 1 \leq k \leq h(i) \) such that \( Nv_i = v_k \), that is, \( w(i) - 1 = w(k) \). Hence, we must find an ideal \( I_h \subseteq \mathbb{C}[x_1, \ldots, x_n] \) such that its zero set \( V(I_h) \) contains some permutations \( w \in S_n \). The indeterminate \( x_r \) will be interpreted as the value of the
permutation \( w \) in \( \tau \). This represents the entry \( w(\tau) \) of the column vector \( v_\tau \). Using the equivalence (5.1) we will prove that \( \tilde{I}_v \) detects the fixed points of \( \text{Hess}(N, h) \) for the \( \text{S} \)-action. Let \( wB \in \text{Hess}(N, h) \) be a fixed point with column vectors \( \{v_1, \ldots, v_n\} \). Since \( wB \) satisfies the Insko’s ideal of the Theorem 2.5, we can assume without loss of generality that for all \( i \)

\[
0 = d_{(1, \ldots, n), (v_1, v_2, \ldots, v_{h(i)}, Nv_1, \ldots, Nv_n)}
\]

with \( r = h(i) + 1 \). Starting with \( v_1 \), if \( v_1 \neq e_1 \) and \( h \neq h_0 \), where \( h_0(i) = i \) for all \( 1 \leq i \leq n \), we have

\[
Nv_1 = v_{k_1} \quad \text{with} \quad w(1) - 1 = w(k_1) \quad \text{and} \quad 2 \leq k_1 \leq h(1).
\]

By (5.2), Condition (5.3) is equivalence to considering the following determinant

\[
0 = d_{(1, \ldots, n), (v_1, v_2, \ldots, v_{h(1)}, Nv_1, \ldots, v_n)}.
\]

Since \( 2 \leq k_1 \leq h(1) \), for \( v \neq e_1 \) all solutions to conditions (5.3) and (5.4) are detected by the polynomial

\[
\tilde{g}_i = (x_1 - x_2 - 1) \cdots (x_1 - x_{h(1)} - 1).
\]

Now, for \( Nv_1 = 0 \) or \( h_{0j} \), both cases imply \( v_1 = e_1 \). This condition is detected by the monomial \((x_1 - 1)\). Hence we define

\[
\tilde{g}_i = (x_1 - 1)(x_1 - x_2 - 1) \cdots (x_1 - x_{h(1)} - 1).
\]

For the next column vectors \( v_\ell \) we do a similar analysis but now we need to consider the nesting condition also. This means, in addition to \( Nv_\ell = v_{k_\ell} \), \( 1 \leq k_\ell \leq h(i) \) we must also have \( Nv_j = v_\ell \) with \( 1 \leq \ell \leq h(i), j \neq i \), and \( j_\ell \leq j \leq i - 1 \) where \( j_\ell \) is a largest integer such that \( j_\ell < \ell \), \( h(j_\ell - 1) < \ell \) and \(\ell \leq h(j_\ell)\). Now, observe that if \( wB \) is such that \( Nv_\ell = v_{k_\ell} \) with \( j_\ell \leq k_\ell < i \), then the fixed point \( \tau_{i,k_\ell}wB \) satisfies Condition (5.1) for the column vector \( v_{k_\ell} \). In this case, it is enough to detect \( wB \) or \( \tau_{i,k_\ell}wB \) since it is equivalent to consider the image \( Nv_\ell \) or the preimage \( Nv_\ell \) = \( v_\ell \). Also, we observe that \( \tau_{i,k_\ell}wB \) satisfies that \( Nv_\ell = v_\ell \) with \( \ell \leq i \leq h(\ell) \). Hence it is enough to consider the case \( Nv_\ell = v_{k_\ell} \) with \( i \leq k_\ell \leq h(i) \). Finally, \( wB \) does not belong to \( \text{Hess}(N, h) \) if and only if there is \( i \) such that \( Nv_\ell = k_\ell \) with \( k_\ell > h(i) \). Then \( \tau_{i,k_\ell}wB \) belongs to \( \text{Hess}(N, h) \). Thus, all these fixed points are already included. We summarize this information as follows:

1. If \( h = h_0 \) and \( Nv_\ell = v_{\ell - 1} \), then \( v_\ell = e_\ell \),
2. If \( h \neq h_0 \), we have \( Nv_\ell = v_{k_\ell} \) with \( i + 1 \leq k_\ell \leq h(i) \) and
3. For all \( v_\ell \) such that \( i \leq \ell \leq h(i) \), we have \( Nv_\ell = v_\ell \) with \( j_\ell \leq j \leq i - 1 \).

The first condition is equivalent to \((x_1 - i) = 0\). The remaining conditions are the next two determinants

\[
0 = d_{(1, \ldots, n), (v_1, v_2, \ldots, v_{h(i)}, Nv_1, \ldots, v_n)}
\]

and

\[
0 = \prod_{\substack{Nv_\ell = v_\ell \\
i \leq \ell \leq h(i - 1) \leq h(i - 1)}} \left( \prod_{j = \ell}^{i - 1} d_{(1, \ldots, n), (v_1, v_2, \ldots, v_{h(i)}, Nv_1, \ldots, v_n)} \right).
\]

Hence, in general we have the determinant

\[
\tilde{g}_i = (x_1 - i) \prod_{k = i + 1}^{h(i)} (x_1 - x_k - 1) \prod_{\ell = 1}^{i - 1} \left( \prod_{j = \ell}^{i - 1} (x_1 - x_j - 1) \right) .
\]
The polynomial \( \tilde{g}_i \) allows us to identify all possible images of \( Nv_i \) and the preimages of \( v_\ell \), \( i \leq \ell \leq h(i) \).

Defining

\[ (5.10) \quad \tilde{I}_h = \langle \tilde{g}_1, \ldots, \tilde{g}_n \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n] \]

it follows that \( V(\tilde{I}_h) \cap S_n \), where \( S_n = \{ w = (w(1), \ldots, w(n)) : w \in S_n \} \) is the \( S \)-fixed point set of \( \text{Hess}(N, h) \). By Theorem 4.1, we have

\[ (5.11) \quad wB \in \text{Hess}(N, h)^S \iff w \in V(\tilde{I}_h) \cap S_n. \]

This means, \( \tilde{I}_h \) identifies the fixed points of \( \text{Hess}(N, h) \). On the other hand, for the polynomials

\[ (5.12) \quad g_i = (x_i - it) \left( \prod_{k<i,j<i} (x_k - x_j - t) \right) \left( \prod_{j>i} (x_i - x_j - t) \right). \]

of \( \mathbb{C}[x, t] = \mathbb{C}[x_1, \ldots, x_n, t] \), where \( \mathbb{C}[t] \) is the Lie algebra of the torus \( S \), the ideal of [2, Algorithm 5.10]

\[ I_h = \langle g_1, \ldots, g_n \rangle \subseteq \mathbb{C}[x, t] \]

satisfies that, if \( x_i = w(i)t \) for \( w \in S_n \), we have

\[ (5.13) \quad g_i = t \tilde{g}_i. \]

Our goal is to give a different proof of [2, Theorem 5.11] using the ideal \( \tilde{I}_h \). By [2, Chapter 5]

\[ (5.14) \quad H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C}) \simeq \mathbb{C}[x, t]/E_n(x, t), \]

where \( E_n(x, t) = (e_1(x_1, \ldots, x_n) - e_i(t)) : 1 \leq i \leq n \) is the ideal generated by the elementary symmetric functions \( e_i(x) \) in the variables \( x_1, \ldots, x_n \), and the elementary symmetric functions \( e_i(t) = e_i(t^1, \ldots, t^n) \) in powers of \( t \). The ideal \( E_n(x, t) \) detects the \( S \)-fixed points on \( \text{GL}_n(\mathbb{C})/B \) by [1, Theorem 3.1]. Moreover,

\[ (5.15) \quad \text{Spec}(H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C})) \simeq \bigcup_{wB \in \text{GL}_n(\mathbb{C})/B} t_w \subset H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C}) \cong \mathbb{C}^n \]

with \( t_w \) the line corresponding to the fixed point \( wB \in \text{GL}_n(\mathbb{C})/B \). Since all zeros of \( E_n(x, t) \) are of the form \( wB \) with \( w \in S_n \), all zeros of \( E_n(x, t) + I_h \) are of the form \( wB \) with \( w \in S_n \). Then by (5.13), \( w \in V(\tilde{I}_h) \cap S_n \) if and only if \( wB \in V(E_n(x, t) + I_h) \). Hence, by (5.11) \( E_n(x, t) + I_h \) identifies \( \text{Hess}(N, h)^S \). Since \( \text{Hess}(N, h) \) is a GM-subspace of \( \text{GL}_n(\mathbb{C})/B \) it follows that \( \text{Spec}(H^*_S(\text{Hess}(N, h); \mathbb{C})) \) is a subarrangement of \( \text{Spec}(H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C})) \). Thus,

\[ (5.16) \quad \text{Spec}(H^*_S(\text{Hess}(N, h); \mathbb{C})) \cong \bigcup_{wB \in \text{Hess}(N, h)} t_w \subset \text{Spec}(H^*_S(\text{GL}_n(\mathbb{C})/B; \mathbb{C})/I) \cong \bigcup_{wB \in \text{Hess}(N, h)} t_w \]

where \( I = E_n(x, t) + I_h \) identifies the fixed points in \( \text{Hess}(N, h) \) since

\[ (5.17) \quad wB \in \text{Hess}(N, h)^S \iff w \in V(\tilde{I}_h) \cap S_n \iff wB \in V(E_n(x, t) + I_h). \]

Thus, by (5.16), it follows

\[ E_n(x, t) + I_h = \bigcap_{wB \in \text{Hess}(N, h)} t_w. \]

We have proved:
Theorem 5.4 ([2, Theorem 5.11]). Let \( I(t_w) \) denote the ideal of the line \( t_w \) of (5.15) for each \( w \in (\text{GL}_n(\mathbb{C})/B)^S \). Then,

\[
E_n(x, t) + I_h = \bigcap_{w \in \text{Hess}(N, h)} I(t_w)
\]

In the other words, the equivariant cohomology of the regular nilpotent Hessenberg variety \( \text{Hess}(N, h) \) is

\[
(5.18) \quad H^*_\mathbb{C}(\text{Hess}(N, h); \mathbb{C}) \simeq \frac{\mathbb{C}[x, t]}{\bigcap_{w \in \text{Hess}(N, h)} I(t_w)} \simeq \mathbb{C}[x, t]/(E_n(x, t) + I_h).
\]

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