Abstract

Learning in a lifelong setting, where the dynamics continuously evolve, is a hard challenge for current reinforcement learning algorithms. Yet this would be a much needed feature for practical applications. In this paper, we propose an approach which learns a hyper-policy, whose input is time, that outputs the parameters of the policy to be queried at that time. This hyper-policy is trained to maximize the estimated future performance, efficiently reusing past data by means of importance sampling, at the cost of introducing a controlled bias. We combine the future performance estimate with the past performance to mitigate catastrophic forgetting. To avoid overfitting the collected data, we derive a differentiable variance bound that we embed as a penalization term. Finally, we empirically validate our approach, in comparison with state-of-the-art algorithms, on realistic environments, including water resource management and trading.

1 Introduction

In the most common setting, Reinforcement Learning (RL, Sutton and Barto 2018) considers the interaction between an agent and an environment in a sequence of episodes. The agent progressively adapts its policy, but the dynamics of the environment, typically, remain unchanged. Most importantly, the agent can experience multiple times the same portion of the environment. However, this usual setting is sometimes not met in real applications. Hence several modifications have been proposed to model different, more realistic scenarios. One of them is non-stationary RL (Bowerman 1974), which considers that the episodes can follow different distributions, or even that the distribution changes within each episode. The change can either be abrupt, when a clear separation between tasks evolving through time is present, or smooth, when the environment evolution displays some regularity w.r.t. time. Non-stationarity can arise from diverse causes and can be interpreted as a form of partial knowledge on environment (Khetarpal et al. 2020). Learning in non-stationary environments has been diffusely addressed in the literature (Garcia and Smith 2000; Ghate and Smith 2013; Lesner and Scherrer 2015). Nevertheless, in these works, the agent-environment interaction based on episodes is preserved, so that the same region of non-stationary behavior can be experienced multiple times by the agent.

Still moving towards a more realistic setting, another modification is the lifelong interaction with the environment (Silver, Yang, and Li 2013; Brunskill and Li 2014). Here, the separation in episodes vanishes and, therefore, there is no clear distinction between learning and testing. Moreover, given the never-ending nature of this interaction, the agent is not allowed to reset the environment and, consequently, it might not be possible to visit twice some portions of the environment. Thus, the agent aims at exploiting the experience collected in the past to optimize its future performance. In this sense, Lifelong Learning (LL) can be considered closer to the intuitive idea of learning for human agents. More technically, LL requires the agent to readily adapt its behavior to the environment evolution, as well as keeping memory of past behaviors in order to leverage this knowledge on future similar phases (Khetarpal et al. 2020). This represents, indeed, a critical trade-off, peculiar of the lifelong setting. Indeed, if the agent displays a highly non-stationary behavior, the samples collected in the past would be poorly informative and, consequently, hardly usable to estimate the future performance. Instead, preferring a more stationary behavior would favor sample reuse, at the cost of sacrificing the optimality of the learned behavior.

In this paper, we consider the RL problem with a lifelong interaction between an agent and the environment, where the environment’s dynamics smoothly evolve over time. We address this problem by designing a hyper-policy, responsible for selecting the best policy to be played at time $t$. This way, we decouple the problem of learning in a non-stationary setting, by assigning to the hyper-policy level the management of the dependence on time and to the policy level the action to be played given a state (Section 3). This hyper-policy is trained with an objective composed of the future performance, the ultimate quantity to be maximized, and the past performance. Although the past performance is not the direct interest of our agent, it is included to constrain the hyper-policy to perform well on past samples, thus mitigating catastrophic forgetting. Future performance is estimated through multiple importance sampling. To avoid overfitting, we additionally penalize the hyper-policy for the variance of the estimations. Rather than estimating this quantity, that would inject further uncertainty, we derive a differentiable
upper-bound allowing a gradient based optimization. This
penalization, involving a divergence between past and future
hyper-policies, has the indirect effect of quantifying and con-
trolling the “amount” of non-stationarity selected by the
agent (Section 4). We propose a practical policy-grad-
ient optimization of the objective, which we name POLIS, for Pol-
icy Optimization in Lifelong learning Through Importance
Sampling. After having revised the literature (Section 5), we
provide an experimental evaluation on realistic domains, in-
cluding a trading environment and a water resource manage-
ment problem, in comparison with state-of-the-art baselines
(Section 6). The proofs of the results presented in the main
paper are reported in Appendix A.

2 Preliminaries

In this section, we report the necessary background that will
be employed in the following sections.

Lifelong RL A Non-Stationary Markov Decision Process
(MDP, Puterman 2014) is defined as \( M = (X, \mathcal{A}, P, R, \gamma, D_0) \), where \( X \) and \( \mathcal{A} \) are the state and action
spaces respectively, \( P = (P_t)_{t \in \mathbb{N}} \) is the transition model that
for every decision epoch \( t \in \mathbb{N} \) and \( (x, a) \in X \times A \) assigns
a probability distribution over the next state \( x' \sim P_t(\cdot|x, a) \),
\( R = (R_t)_{t \in \mathbb{N}} \) is the reward distribution assigning for every
\( t \in \mathbb{N} \) and \( (x, a) \in X \times A \) the reward \( r \sim R_t(\cdot|x, a) \) such
that \( ||r||_{\mathbb{R}} \leq R_{\max} < \infty \), \( \gamma \in [0, 1] \) is the discount fac-
tor, and \( D_0 \) is the initial state distribution. A non-stationary
policy \( \pi = (\pi_t)_{t \in \mathbb{N}} \) assigns for every decision epoch \( t \in \mathbb{N} \)
and state \( x \in X \) a probability distribution over the actions
\( a_t \sim \pi_t(\cdot|x) \). Let \( T \in \mathbb{N} \) be the current decision epoch, let
\( \beta \in \mathbb{N}_{\geq 1} \), we define the \( \beta \)-step ahead expected return as:

\[
J_{T, \beta}(\pi) = \sum_{t=T+1}^{T+\beta} \gamma^t \mathbb{E}^\pi_t [r_t],
\]

where \( \gamma^t = \gamma^{t-T-1} \) and we denote with \( \mathbb{E}^\pi_t \) the expec-
tation under the visitation distribution induced by policy \( \pi \)
in MDP \( M \) after \( t \) decision epochs. A policy \( \pi_{T, \beta}^* \) is \( \beta \)-step
ahead optimal if \( \pi_{T, \beta}^* \in \arg \max_{\pi \in \Pi} J_{T, \beta}(\pi) \), where \( \Pi \)
is the set of non-stationary policies operating over \( \beta \) decision
epochs. In classical RL, the agent’s goal consists in maxi-
mizing \( J_{0, H} \), where \( H \) is the (possibly inﬁnite) horizon of the
task, having the possibility to collect multiple episodes (not
necessarily of length \( H \)). Instead, from the lifelong RL
perspective, the agent is interested in maximizing the \( \infty \)-step
ahead expected return \( J_{T, \infty}(\pi) \), having observed in the past
only one episode of length \( T \), i.e., optimizing for the future.

Multiple Importance Sampling Importance Sampling (IS, Owen
2013) allows estimating the expectation
\( \mu = \mathbb{E}_{x \sim P} [f(x)] \) of a function \( f \) under a target
distribution \( P \) having samples collected with a sequence of behav-
ioral distributions \( (Q_j)_{j \in \{1, J\}} \) such that \( P \ll Q_j \), i.e., \( P \) is
absolutely continuous w.r.t. \( Q_j \), for all \( j \in \{1, J\} \). Let \( p \) and
\( (q_j)_{j \in \{1, J\}} \) be the density functions corresponding to \( P \) and
\( (Q_j)_{j \in \{1, J\}} \), then, the resulting unbiased estimator is:

\[
\hat{\mu} = \sum_{j=1}^J \frac{1}{N_j} \sum_{i=1}^{N_j} \beta_j(x_{ij}) p(x_{ij}) q_j(x_{ij}) f(x_{ij}),
\]

where \( \{x_{ij}\}_{i=1}^{N_j} \sim Q_j \) and \( (\beta_j(x))_{j \in \{1, J\}} \) are a partition
of the unit for every \( x \in X \). A common choice for the latter is
the balance heuristic (BH, Veerach and Guibas 1995),
yielding \( \beta_j(x) = \frac{N_j q_j(x)}{\sum_{k=1}^{J} N_k q_k(x)} \). Using BH, samples can be
regarded as obtained from the mixture of the \( (Q_j)_{j \in \{1, J\}} \)
distributions as \( \Phi = \sum_{k=1}^J \frac{N_k}{N} Q_k \), with \( N = \sum_{j=1}^J N_j \).

Rényi divergence Let \( \alpha \in [0, \infty) \), the \( \alpha \)-Rényi diver-
gence between two probability distributions \( P \) and \( Q \) such
that \( P \ll Q \) is deﬁned as:

\[
D_\alpha(P||Q) = \frac{1}{\alpha-1} \log \int_X p(x)\alpha q(x)^{1-\alpha} dx.
\]

We denote with \( d_\alpha(P||Q) = \exp(D_\alpha(P||Q)) \) the expo-
nential \( \alpha \)-Rényi divergence, linked to the \( \alpha \)-moment of the
importance weight, i.e., \( \mathbb{E}_{x \sim \Phi} \left[ \left( \frac{p(x)}{\Phi(x)} \right)^\alpha \right] = d_\alpha(P||Q)^{\alpha-1} \).

3 Lifelong Parameter-Based Policy
Optimization

In this paper, we consider the Policy Optimization (PO, Deisenroth,
Neumann, and Peters 2013) setting in which the policy belongs to a set of parametric polici-
yes \( \Pi_\theta = \{ \pi_\theta : \theta \in \Theta \subseteq \mathbb{R}^d \} \). In particular, we focus on the parameter-
based PO\(^1\) in which the policy parameter \( \theta \) is sampled from a
hyper-policy \( \nu_\theta \) belonging, in turn, to a parametric set
\( \mathcal{N}_P = \{ \nu_\theta : \theta \in \nu \subset \mathbb{R}^d \} \) (Sehnke et al. 2008). As op-
posed to action-based PO in which policies \( \pi_\theta \) needs to be
stochastic for exploration reasons, in parameter-based PO
we move the stochasticity to the hyper-policy \( \nu_\theta \) level and
\( \pi_\theta \) can be deterministic.

Optimizing the \( \beta \)-step ahead expected return in Equation
(1) requires, in general, considering non-stationary polici-
es. From the PO perspective, this requirement can be ful-
filled in two ways. The traditional way consists in augment-
ing the state \( x \) with time \( t \) and, consequently, considering a
policy of the form \( \pi_\theta(\cdot|x, t) \). This approach highlights the
direct dependence of the action \( a_t \sim \pi_\theta(\cdot|x_t, t) \) on the
time \( t \). However, in several cases, it is convenient to track
the evolution of the policy parameters \( \theta \) as a function of the
time \( t \), whose dependence might be simpler compared to
that of the action. In this latter approach, the one we adopt
in this work, the policy parameter is sampled from a time-
dependent hyper-policy \( \pi_\theta(\cdot|t) \), \( \nu_\theta(\cdot|t) \) and the policy depends
on the state only \( \pi_\theta(\cdot|x_t, t) \). We will refer to this setting as
lifelong parameter-based PO. Refer to Figure 1 for a com-
parison of the graphical models of the two approaches.

In this setting, we aim at learning a hyper-policy parameter
\( \rho_{T, \beta}^* \) maximizing the \( \beta \)-step ahead expected return:

\[
\rho_{T, \beta}^* \in \arg \max_{\rho \in \mathcal{P}} J_{T, \beta}(\rho) = \sum_{t=T+1}^{T+\beta} \gamma^t \mathbb{E}_\rho^\pi_t [r_t],
\]

where \( \mathbb{E}_\rho^\pi_t [\cdot] \) is a shorthand for \( \mathbb{E}_{x \sim \rho(\cdot|t)} [\mathbb{E}_\pi^\rho_t [\cdot]] \).

\(^1\)We follow the taxonomy of (Metelli et al. 2018).
4 Lifelong Parameter-Based PO via Multiple Importance Sampling

In this section, we propose an estimator for $J_{T,\beta}(\rho)$ (Section 4.1), we analyze its bias (Section 4.2) and variance (Section 4.3), and we propose a novel surrogate objective accounting for the estimation uncertainty (Section 4.4).

4.1 $\beta$-Step Ahead Expected Return Estimation

The main challenge we face in estimating $J_{T,\beta}(\rho)$ is that it requires evaluating hyper-policy $\nu_\rho$ in the future, while having samples from the past only. Since the environment evolves smoothly, it is reasonable to use the past data to approximate the future dynamics and IS to correct the hyper-policy behavior mismatch from past to future. More specifically, in this section, we study how to leverage the history of the past $\alpha$ samples $H_{T,\alpha} = (\theta_1, \tau_1)_{t \in [T-\alpha+1, T]}$ in order to estimate the $\beta$-step ahead expected return $J_{T,\beta}(\rho)$.

As a preliminary step, we illustrate the estimation of the $s$-step ahead expected reward $E_\rho^S[r_s]$. For every $s \in [T + 1, T + \beta]$, we employ the following MIS estimator that makes use of the history $H_{T,\alpha}$:

$$\hat{r}_s = \frac{\sum_{t=T}^{T+1} \omega^{T-t} \nu_\rho(\theta_t | s)}{\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta_k | k)}, \quad (3)$$

where $\omega \in [0, 1]$ is an exponential weighting parameter. The importance sampling correction $\frac{\omega^{T-t} \nu_\rho(\theta_t | s)}{\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta_k | k)}$ addresses the mismatch between the hyper-policies in the future $\nu_\rho(\cdot | s)$ and those in the past $\nu_\rho(\cdot | k)$. The reader may have noticed that these importance weights are not using the exact BH weights. Indeed, we have adapted the heuristic to include our knowledge that the environment is smoothly changing. With BH, each past sample would have been weighted equally whereas our sampling mixture probability, proportional to $\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta_k | k)$, gives more weight to recent samples thanks to the parameter $\omega$ which exponentially discounts samples as they are collected far from current time $T$ (Jagerman, Markov, and de Rijke 2019).

Using $\hat{r}_s$ as building block, we now propose the estimator for the $\beta$-step ahead expected return $\hat{J}_{T,\alpha,\beta}(\rho)$ that is obtained as the discounted sum of the $s$-step ahead expected reward estimators of Equation 3:

$$\hat{J}_{T,\alpha,\beta}(\rho) = \frac{\sum_{s=T}^{T+\beta} \hat{r}_s}{\sum_{s=T}^{T+\beta} \hat{r}_s}$$

This estimator could be, in principle, employed in an optimization algorithm but, as common in IS-based estimators, we would incur in the following undesired effect. In order to increase $\hat{J}_{T,\alpha,\beta}(\rho)$, the agent can either increase the probability of good actions for the future policies $\nu_\rho(\theta_t | s)$ (the numerator of the importance weight) or decrease the probability of the same good actions for the policies from the past $\nu_\rho(\theta_k | k)$ (the denominator of the importance weight). The latter phenomenon, akin to catastrophic forgetting, is clearly undesired, but can be easily spotted by looking at the past reward. Specifically, we propose to adjust the objective function with the return of the last $\alpha$ steps, called $\alpha$-step behind expected return:

$$\hat{J}_{T,\alpha}(\rho) = \frac{1}{C_\omega} \sum_{t=T-\alpha+1}^{T} \omega^{T-t} \hat{r}_t,$$

where $\gamma_t = \frac{\omega}{1 - \omega}$ if $\omega < 1$ otherwise $C_\omega = 1$. Putting all together, we obtain the objective:

$$\hat{J}_{T,\alpha,\beta}(\rho) = \hat{J}_{T,\alpha}(\rho) + \hat{J}_{T,\beta}(\rho).$$

4.2 Bias Analysis

In this section, we analyze the bias of estimator $\hat{J}_{T,\alpha,\beta}(\rho)$, under suitable regularity conditions on the environment and on the hyper-policy model. In particular, we will require that the environment and the hyper-policy are smoothly changing. We formalize the intuition in the following assumptions.

Assumption 4.1 (Smoothly Changing Environment). For every $t, t' \in N$, and for every policy $\pi$ it holds for some Lipschitz constant $0 < L_M < \infty$:

$$|E_\pi^S[r_t] - E_\pi^S[r_{t'}]| \leq L_M |t - t'|.$$

Assumption 4.2 (Smoothly Changing Hyper-policy). For every $t, t' \in N$, and for every time-dependent hyper-policy $\rho \in P$ it holds for some Lipschitz constant $0 < L_\rho < \infty$:

$$|\nu_\rho(\cdot | t) - \nu_\rho(\cdot | t')| \leq L_\rho |t - t'|.$$

Thus, Assumption 4.1 prescribes that executing the same policy $\pi$ in different time instants $t$ and $t'$ results in an expected reward that can be bounded proportionally to the time distance. A similar requirement is requested by Assumption 4.2, involving the total variation distance between time-dependent hyper-policies. Under these assumptions, we provide the following bias bound. We denote with $E_\rho^T[r]$ the expectation under the probability distribution induced by the joint hyper-policy $\prod_{t=T-\alpha+1}^{T} \nu_\rho(\cdot | t)$ in the MDP.

Lemma 4.1. Under Assumptions 4.1 and 4.2, the bias of the estimator $\hat{J}_{T,\alpha,\beta}(\rho)$, for $\omega < 1$, can be bounded as:

$$|J_{T,\beta}(\rho) - E_\rho^T[\hat{J}_{T,\alpha,\beta}]| \leq (L_M + 2R_{\text{max}} L_\rho) C_\gamma \left( \frac{\omega}{1 - \omega} + \frac{1}{1 - \gamma} \right),$$

where $C_\gamma(\xi) = \frac{1 - \gamma^\xi}{1 - \gamma}$ if $\gamma < 1$ otherwise $C_\gamma(\xi) = \xi$ for $\xi \geq 1$. 

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Figure 1: Graphical models of the two approaches to model non-stationarity of the environment: the non-stationarity is handled at the parameter selection level (left), the non-stationarity is handled at the action selection level (right).
A tighter, but more intricate bias bound, and a derivation for the case $\omega = 1$ can be found in Appendix A. Some observations are in order. First, we note the role of $\omega$ in controlling the bias: the smaller $\omega$, the smaller the bias. Second, the bound is proportional to the Lipschitz constants governing the non-stationarity of the environment and of the hyper-policy. It is worth noting that in a fully stationary setting the non-stationarity of the environment and of the hyper-bound is proportional to the Lipschitz constants governing $\omega$.

4.3 Variance Analysis

Before showing the construction of the surrogate objective, we derive in this section a bound on the variance of $J_{T,\alpha,\beta}(\rho)$ that involves the Rényi divergence. To this purpose, we denote with $\text{Var}_{T,\alpha}^p$ the variance under the probability distribution induced by the joint hyper-policy $\prod_{t=T-\alpha+1}^{T} \nu_\rho(\cdot|t)$ in the MDP.

**Lemma 4.2.** The variance of the objective $J_{T,\alpha,\beta}$ can be bounded as:

$$\text{Var}_{T,\alpha}^p[J_{T,\alpha,\beta}(\rho)] \leq 2R_{\max}^{2}(C_\alpha(\alpha)^2 + C_\gamma(\beta)^2) \times d_2\left(\frac{1}{C_\gamma(\beta)} \sum_{s=T+1}^{T+\beta} \tilde{\gamma}_s \nu_\rho(\cdot|s) \left| \frac{1}{C_\omega} \sum_{t=T-\alpha+1}^{T} \omega^{T-t} \nu_\rho(\cdot|t) \right|^2\right).$$

The variance bound resembles the ones usually provided in the context of off-policy estimation and learning (e.g., Metelli et al. 2018; Papini et al. 2019; Metelli et al. 2020). The first addendum accounts for the variance of the estimator component $J_{T,\alpha}(\rho)$ that does not involve importance sampling, whereas the second refers to $\hat{J}_{T,\alpha,\beta}(\rho)$, based on importance sampling. Indeed, this latter term comprises the exponentiated 2-Rényi divergence between two mixture hyper-policies. Unfortunately, even in presence of convenient distributions, like Gaussians, the Rényi divergence between mixtures does not admit a closed form (Papini et al. 2019). In Appendix B, we discuss several approaches, based on variational upper-bounds, to provide a usable version of such a divergence. In the following, we report the upper-bound that we will use in practice.

**Lemma 4.3.** The divergence between mixtures of Lemma 4.2 can be bounded as:

$$d_2\left(\frac{1}{C_\gamma(\beta)} \sum_{s=T+1}^{T+\beta} \tilde{\gamma}_s \nu_\rho(\cdot|s) \left| \frac{1}{C_\omega} \sum_{t=T-\alpha+1}^{T} \omega^{T-t} \nu_\rho(\cdot|t) \right|^2\right) \leq \frac{C_\omega}{C_\gamma(\beta)^2} \left(\sum_{s=T+1}^{T+\beta} \tilde{\gamma}_s \left(\sum_{t=T-\alpha+1}^{T} \omega^{T-t} \nu_\rho(\cdot|t) \right)^2 \right).

4.4 Surrogate Objective

The direct optimization of the objective $J_{T,\alpha,\beta}(\rho)$ makes the hyper-policy overfit the non-stationary process on the last $\alpha$ steps. To allow for a better generalization on future unseen variations, following the idea of Metelli et al. (2018), we regularize the objective with the bound on the variance of Lemma 4.3. The following concentration bound, based on Cantelli’s inequality, is the theoretical grounding of our surrogate objective.

**Theorem 4.1.** For every $\delta \in (0, 1)$, with probability at least $1 - \delta$, it holds that:

$$\mathbb{E}_{T,\alpha}^p[J_{T,\alpha,\beta}(\rho)] \geq J_{T,\alpha,\beta}(\rho) - \sqrt{\frac{1 - \delta}{\delta} 2R_{\max}^{2}(C_\alpha(\alpha)^2 + C_\omega B_{T,\alpha,\beta}(\rho))}.$$ 

We are now ready to construct the surrogate objective function and show how to optimize it. Following the path of Metelli et al. (2018), we take an uncertainty-average approach, by maximizing a probabilistic lower bound of the quantity of interest, i.e., the one presented in Theorem 4.1. Renaming $\lambda = \sqrt{\frac{1 - \delta}{\delta} 2R_{\max}^{2}}$, and treating it as a hyperparameter, we get the following surrogate objective:

$$L_\lambda(\rho) = J_{T,\alpha,\beta}(\rho) - \lambda \sqrt{C_\alpha(\alpha)^2 + C_\omega B_{T,\alpha,\beta}(\rho)}. \quad (4)$$

In order to optimize this objective, we use a policy-gradient approach that is discussed in Appendix A.1. We call this algorithm POLIS (Policy Optimization in Lifelong learning through Importance Sampling) and provide its pseudocode in Algorithm 1.

5 Related Works

The problem of lifelong RL is not new to the community. Nevertheless, the term encapsulates problems which can have slightly differing definitions, thus hindering the comparison between existing approaches. In this section, we will briefly discuss the most relevant ones. A more complete overview can be found in (Padakandla 2021). Lifelong RL approaches handle finite-horizon settings (Hallak, Di Castro, and Mannor 2015; Ortner, Gajane, and Auer 2020) as well as infinite-horizon settings. In this second case, many works focus on the detection of abrupt changes in the dynamics (da Silva et al. 2006; Hadoux, Beynier, and Weng 2014) or on scenarios where the non-stationarity arises from switching between stationary dynamics, where the number of such dynamics is known (Choi, Yeung, and Zhang 2000).

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**Algorithm 1: Lifelong learning with POLIS**

**Input:** steps behind $\alpha$, steps ahead $\beta$, regularization $\lambda$, discount factor $\omega$, training period $h$, training epochs $N$.

1: Initialize $\nu_\rho$, $t \leftarrow 0$
2: while True do
3: Sample $\theta_i \sim \nu_\rho(t)$
4: Collect new state $s_t$ and reward $r_t$ using $\pi(\theta_i)$
5: if $t \mod h = 0$ and $t > 1$ then
6: for $i \in \{1, \ldots, N\}$ do
7: Compute $\hat{J}_{T,\alpha}(\rho)$, $J_{T,\alpha,\beta}(\rho)$ and $B_{T,\alpha,\beta}(\rho)$
8: $\rho \leftarrow \arg \max_{\rho} L_\lambda(\rho)$
9: $t \leftarrow t + 1$
Non-stationarity in the MDP dynamics is not only bound to LL, since it is also a core element of *Continual Learning*. To adapt to the evolving environment, continually learning agents need to find structure in the world to tackle new tasks by decomposing them in smaller sub-problems through function composition (Griffiths et al. 2019) or by extracting meaningful information in the form of abstract concepts (Zhang, Satija, and Pineau 2018; Francois-Lavet et al. 2019). Other approaches focus on capturing task-agnostic underlying dynamics of the world, by building auxiliary tasks like reward prediction (Jaderberg et al. 2017) or using inverse dynamics prediction (Shelhamer et al. 2017) to provide dense training signal. A general overview of continual RL approaches can be found in (Khetarpal et al. 2020).

Another relevant approach to Lifelong RL is Meta RL, which leverages past experience to learn new skills more efficiently, i.e. using a small amount of new data. Usual Meta RL algorithms can be adapted to non-stationarity by modelling the consecutive tasks as a Markov chain model (Al-Shedivat et al. 2018) or using experience replay (Riemer et al. 2019).

Lastly, more similar to our approach, is the one of Chandak et al. (2020) where the policy is trained to optimize the future predicted performance. To this end, the past performance is first of all estimated through importance sampling and then used to forecast future performance. All these steps are differentiable, which allows optimizing the policy through gradient ascent. The authors propose two algorithms, Pro-OLS, forecasting the performance using an ordinary least-squares regression and Pro-WLS, where the regression takes into account the importance weights inside a weighted least-squares. The latter reduces the variance of the estimates at the expense of adding some bias. One major difference with our approach is that their method is designed for episodic RL, where the non-stationarity arises from one episode to the next. We instead consider a truly lifelong framework, where there are no episodes and non-stationarity arises at the single step level. Our approach is different for three other reasons. First, while our estimate of the ω-step ahead performance has in common with the aforementioned paper the use of importance sampling, our objective is however greatly different as we add an extra discounting parameter to control the bias due to non-stationarity and two terms, the ω-step behind performance and a variance regularization. Second, our surrogate objective can be optimized at any point in time, meaning that if a significant shift in dynamics is detected, one has the opportunity to retrain the algorithm suddenly. Third, we consider a parameter-based approach in which the hyper-policy depends only upon time, the policy may thus change at every step. Chandak et al. (2020) consider an action-based approach where policy’s parameters are fixed during an episode.

6 Experiments

In this section, we report the experimental evaluation of our algorithm in comparison with state-of-the-art baselines.

6.1 Lifelong Learning Framework

The schedule for a lifelong interaction with the environment is divided in two periods. In the first, which we refer to as behavioural period, a behavioural hyper-policy is queried to sample data in order to gather enough samples to compute the first α-step behind expected return. In the second period, referred to as target period, the agent continues interacting with same environment, but is now training its hyper-policy every few steps (50 in all experiments) for a given number of gradient steps (100 in all experiments). For all tasks, we set γ = ω = 1.2

We consider a particular subclass of non-stationary environments, frequently encountered in practice. The state \( x = (x^c, x^u) \) is decomposed into a controllable \( x^c \) and a non-controllable \( x^u \) part. The controllable part evolves according to stationary dynamics and depends on the action \( P^c((x')^c|x^c, a) \). The non-controllable part instead is not affected by the action and follows non-stationary dynamics \( P^u((x')^u|x^u) \).

**Assumption 6.1.** The transition model \( P = (P_t)_{t \in \mathbb{N}} \) factorizes as follows for every \( x = (x^c, x^u) \in \mathcal{X}, a \in \mathcal{A}, \) and \( t \in \mathbb{N} \):

\[
P_t(x'|x, a) = P^c((x')^c|x^c, a^u, a)P^u((x')^u|x^u). \tag{5}
\]

Under the assumption, we can sample new trajectories from the last α steps, where the non-stationary part of the state \( x^u \) is kept fixed but the sampled policy at each time and therefore the stationary controllable part of the state \( x^c \) changes. Therefore, we have access to the value of the gradient of the α-step behind expected return by direct estimation, without requiring importance sampling.

6.2 Trading Environment

The first task is the daily trading of the EUR-USD (€/S) currency pair from the Foreign Exchange (Forex). Following (Bisi et al. 2020), we allow the agent to trade up to a fixed quantity of 100k€ USD with a per-transaction fee \( f \) of 1$. The agent has a continuous actions space in the size, while \( 0 \leq x \leq 1 \). The reward is defined as \( r_t = \alpha_t(x_{t+1} - x_t) - f|a_t - x_t| \).

We consider three datasets of historical data, 2009-2012, 2013-2016, and 2017-2020, each period having a little more than 1000 data points. \( \alpha \) is set to 500 and we consider a target period of 500 steps.

Finding a satisfactory set of hyperparameters (in the sense of parameters of the algorithm itself, not parameters of the hyper-policy) can be problematic in our lifelong scenario. Indeed, here, there is no distinction between training and training.
testing since the parameters are continually updated. Selecting hyperparameters for future interactions with the environment by evaluating past performances is thus prone to overfitting on the past performance. To account for this problem, we compare two hyperparameter selection approaches. In the first, we select the best performing hyperparameters from the dataset 2009-2012 and evaluate the selection on the other two datasets. In the second approach, we both select the hyperparameters and evaluate on the last two datasets.

The trading of the EUR-USD currency pair is a highly complex task. To give more chance to the algorithm to exploit potential patterns of the series, we provide another trading task on a simulated series. The framework is the same, only changes the underlying rate process which will now be a Vasicek process. In this scenario, the rate \((p_t)_{t \geq 1}\) satisfies \(p_{t+1} = 0.9 p_t + u_t\), where \(u_t \sim N(0, 1)\). On this task we will test the set of hyperparameters selected on the EUR-USD.

### 6.3 Dam Environment

The second environment is a water resource management problem. A dam is used to save water from rains and possibly release it to meet a certain demand for water (e.g., the needs of a town). We model the environment following (Castelletti et al. 2010; Tirinzoni et al. 2018). The inflow (e.g., rain) is the non-stationary process and the agent has obviously no impact on it, thus satisfying Assumption 6.1. The mean inflow follows one of either 3 profiles given in Appendix C.2. The state observed by the agent is the day’s lake level. The agent does not observe the day of the year, contrarily to (Tirinzoni et al. 2018), in order to ensure non-stationarity. The agent’s action is continuous and corresponds to selecting the daily amount of water to release in order to avoid flooding and meet the demand. Considering the flooding level \(F = 300\) and the daily demand for water \(D = 10\), the penalty that the agent gets for each is respectively \(c_F = (\max(x-F, 0))^2\) and \(c_D = (\max(a-D, 0))^2\), where \(a\) is the action of the agent and \(x\) the current lake level. The final cost is a convex combination of those costs, whose weights depend on the inflow profile (see Appendix C.2). In this environment, \(a\) is set to 1000 in order to include enough years of past data in the estimator. We provide results for a target period of 500 steps. Because the results for this environment are less sensitive to the choice of hyperparameters, we only select them according to the performance given the first inflow profile.

### 6.4 The hyper-policy and policy

We now describe the hyper-policy used in all experiments. It is composed of two modules. The first is positional encoding introduced in (Vaswani et al. 2017). It embeds its input, time, as a vector of Fourier basis. Therefore, it does not add learnable parameters to the hyper-policy. This module has two main advantages. First, its output dimension is free to be chosen, allowing to control the input size of the next module. Second, while time eventually becomes large, the output of positional encoding is bounded, which is a valuable property when then fed to a neural network. The second module are convolutions scanning through time. We chose convolutions as they generally excel in finding patterns in time series. Moreover, they allow processing inputs of variable length and are easily parallelizable. We use a particular type of convolutions, temporal convolutions (Oord et al. 2016) which preserve time causality. Obviously, the convolutions require several time-steps in order to scan through with their kernel. However our hyper-policy takes only the current time as input, \(\nu_p(\cdot|t)\). Nevertheless, we can freely decide to consider the positional encoding of \(t\) and a few previous times to reach the length of the receptive field. Its length is \(b = 2^{l-1}(k-1)\), where \(l\) is the number of layers and \(k\) the kernel size of the temporal convolution. Another advantage of using temporal convolutions is that the computation of the policy parameters \(\theta\) can be made in parallel. This is an interesting property in practice as between two updates of the hyper-policy, we can already sample in parallel all the policy parameters to be used. The output of the temporal convolutions is the mean \(\mu_t\) of the normally distributed policy parameter \(\theta_t\). The standard deviation of each entry of \(\theta_t\) is not time dependent and can be either learned or fixed during training. A schematic representation of the hyper-policy is given in Figure 2.

At the policy level, in all the experiments, we use an affine policy with bounded outputs.

### 6.5 Baselines

The first baseline is the stationary hyper-policy which can be seen as a special case of POLIS when \(\nu_p(\cdot|t) = \nu_p(\cdot)\). We consider this hyper-policy along with the same affine policy. Note that, although stationary between trainings, the hyper-policy’s parameters are retrained every 50 steps.

We then consider baselines from the literature, including Pro-OLS and Pro-WLS (Chandak et al. 2020). In their experiments, Chandak et al. (2020) use a baseline which they refer to as ONPG, replicating the idea of Al-Shedivat et al. (2018). We also include this baseline and thank Chandak et al. (2020) for providing their code.

### 6.6 Results

**Trading environment** The cumulative returns obtained for the hyperparameters selected on 2009-2012 are given in Figure 4. Interestingly, on the period 2013-2016, POLIS has a
performance similar to the stationary policy, which is comparable or superior to baselines. On the period 2017-2020, POLIS under-performs the baselines, but the stationary one. When selecting the set of hyperparameters from the testing dataset, we obtain the results shown in Figure 7. This time, POLIS obtains more similar performance to the baselines, closing the gap on the period 2017-2020.

The cumulative returns obtained for the trading on the Vasicek process are reported in Figure 5. On this task, specifically designed to highlight the smooth non-stationarity, POLIS is clearly superior to baselines, particularly the stationary one. Note also its smaller variance.

**Dam environment** We report the results of the experiment in Table 1. Surprisingly, the stationary policy is able to significantly outperform the other baselines for each inflow profile, with the exception of ONPG on the second profile. However, we note the higher standard deviation of ONPG in this case. Remarkably, POLIS is able to match the stationary policy’s performance on each inflow profile. This indicates that our approach is able to avoid extra non-stationarity in tasks where it is not needed.

### 7 Conclusion

In this paper, we proposed to address the *lifelong* RL problem by using a hyper-policy mapping time to policy parameters. To grasp the objective of LL, i.e., the future performance, we designed an estimator of such quantity, making use of the past collected experience via importance sampling. The estimator has a controllable bias which vanishes as the environment and the hyper-policy become stationary. Besides, we add two terms to the objective: an estimation of the past performance preventing catastrophic forgetting and a penalization based on an upper-bound on the variance, which prevents overfitting the past and favors generalization to future non-stationarity. We proposed an implementation of such hyper-policy which we tested in several scenarios, demonstrating that our approach can exploit predictable non-stationarity, control for its variance and avoid excessive non-stationarity when non necessary. Our approach tackled exploration via the stochasticity of the hyper-policy. Future work include a more principled and explicit treatment of the exploration problem in the lifelong RL setting.

| Inflow 1 | Inflow 2 | Inflow 3 |
|----------|----------|----------|
| POLIS    | $-2.2 \pm 0.2$ | $-1.5 \pm 0.0$ | $-3.2 \pm 0.2$ |
| Stationary | $-2.2 \pm 0.1$ | $-1.5 \pm 0.0$ | $-3.2 \pm 0.2$ |
| ONPG     | $-5.1 \pm 3.2$ | $-1.4 \pm 0.2$ | $-4.1 \pm 0.5$ |
| Pro-OLS  | $-2.6 \pm 0.4$ | $-5.2 \pm 5.1$ | $-3.8 \pm 0.7$ |
| Pro-WLS  | $-5.5 \pm 4.3$ | $-8.5 \pm 9.7$ | $-8.4 \pm 4.2$ |

Table 1: Lifelong learning on the Dam environment for each of 3 inflow profile. Mean return on the target period and standard deviation over 3 seeds. Reported results are divided by an order of $1e3$ for aesthetic.
References

Al-Shedivat, M.; Bansal, T.; Burda, Y.; Sutskever, I.; Mordatch, I.; and Abbeel, P. 2018. Continuous Adaptation via Meta-Learning in Nonstationary and Competitive Environments. In ICLR 2018.

Bisi, L.; Liotet, P.; Sabbioni, L.; Reho, G.; Montali, N.; Corno, C.; and Restelli, M. 2020. Foreign Exchange Trading: A Risk-Averse Batch Reinforcement Learning Approach. In ICAIF ‘20.

Bowerman, B. L. 1974. Nonstationary Markov decision processes and related topics in nonstationary Markov chains.

Brunskill, E.; and Li, L. 2014. PAC-inspired Option Discovery in Lifelong Reinforcement Learning. In ICML 2014.

Castelliti, A.; Galelli, S.; Restelli, M.; and Soncini-Sessa, R. 2010. Tree-based reinforcement learning for optimal water reservoir operation. Water Resources Research, 46(9).

Chandak, Y.; Theocharous, G.; Shankar, S.; Mahadevan, S.; White, M.; and Thomas, P. S. 2020. Optimizing for the Future in Non-Stationary MDPs. ICML 2020.

Choi, S. P.; Yeung, D.-Y.; and Zhang, N. L. 2000. Hidden-mode markov decision processes for nonstationary sequential decision making. In Sequence Learning, 264–287. Springer.

da Silva, B. C.; Basso, E. W.; Bazzan, A. L. C.; and Engel, P. M. 2006. Dealing with Non-Stationary Environments Using Context Detection. In ICML 2006.

Deisenroth, M. P.; Neumann, G.; and Peters, J. 2013. A Survey on Policy Search for Robotics. Foundations and Trends in Robotics, 2(1-2).

Francois-Lavet, V.; Bengio, Y.; Precup, D.; and Pineau, J. 2019. Combined Reinforcement Learning via Abstract Representations. In AAAI 2019.

Garcia, A.; and Smith, R. L. 2000. Solving nonstationary infinite horizon stochastic production planning problems. Oper. Res. Lett., 27(3).

Ghate, A.; and Smith, R. L. 2013. A Linear Programming Approach to Nonstationary Infinite-Horizon Markov Decision Processes. Oper. Res., 61(2): 413–425.

Griffiths, T. L.; Callaway, F.; Chang, M. B.; Grant, E.; Krueger, P. M.; and Lieder, F. 2019. Doing more with less: meta-reasoning and meta-learning in humans and machines. Current Opinion in Behavioral Sciences, 29: 24–30. Artificial Intelligence.

Hadoux, E.; Beynier, A.; and Weng, P. 2014. Sequential decision-making under non-stationary environments via sequential change-point detection. In Learning over multiple contexts (LMCE).

Hallak, A.; Di Castro, D.; and Mannor, S. 2015. Contextual markov decision processes. arXiv preprint arXiv:1502.02259.

Jaderberg, M.; Mnih, V.; Czarnecki, W. M.; Schaul, T.; Leibo, J. Z.; Silver, D.; and Kavukcuoglu, K. 2017. Reinforcement Learning with Unsupervised Auxiliary Tasks. In ICLR 2017.

Jagerman, R.; Markov, I.; and de Rijke, M. 2019. When people change their mind: Off-policy evaluation in non-stationary recommendation environments. In Proceedings of the Twelfth ACM International Conference on Web Search and Data Mining, 447–455.

Khetarpal, K.; Riemer, M.; and Li, L. 2019. Loss is its own Reward: Self-Supervision for Reinforcement Learning. In ICLR 2017.

Ortner, R.; Gajane, P.; and Auer, P. 2020. Variational Regret Bounds for Reinforcement Learning. In UAI 2020.

Owen, A. B. 2013. Monte Carlo theory, methods and examples.

Padakandla, S. 2021. A survey of reinforcement learning algorithms for dynamically varying environments. ACM Computing Surveys (CSUR), 54(6): 1–25.

Metelli, A. M.; Papini, M.; Montali, N.; and Restelli, M. 2020. Importance Sampling Techniques for Policy Optimization. Journal of Machine Learning Research, 21(141): 1–75.

Oord, A. v. d.; Dieleman, S.; Zen, H.; Simonyan, K.; Vinyals, O.; Graves, A.; Kalchbrenner, N.; Senior, A.; and Kavukcuoglu, K. 2016. Wavenet: A generative model for raw audio. arXiv preprint arXiv:1609.03499.

Riemer, M.; Cases, I.; Ajemian, R.; Liu, M.; Rish, I.; Tu, Y.; and Tesarou, G. 2019. Learning to Learn without Forgetting by Maximizing Transfer and Minimizing Interference. In In International Conference on Learning Representations (ICLR).

Shenhke, F.; Osendorfer, C.; Rückstieß, T.; Graves, A.; Peters, J.; and Schmidhuber, J. 2008. Policy Gradients with Parameter-Based Exploration for Control. In Kurková, V.; Neruda, R.; and Koutník, J., eds., ICANN 2008.

Silver, D. L.; Yang, Q.; and Li, L. 2013. Lifelong Machine Learning Systems: Beyond Learning Algorithms. In Lifelong Machine Learning. Papers from the 2013 AAAI Spring Symposium, Palo Alto, California, USA, March 25-27, 2013.

Tesauro, G. 2019. Learning to Learn without Forgetting by Maximizing Transfer and Minimizing Interference. In In International Conference on Learning Representations (ICLR).

Tieleman, T.; and Hinton, G. 2012. Lecture 6.5—RmsProp: Divide the gradient by a running average of its recent magnitude. COURSERA: Neural Networks for Machine Learning.

Tirinzoni, A.; Sessa, A.; Pirotta, M.; and Restelli, M. 2018. Importance Weighted Transfer of Samples in Reinforcement Learning. In ICML 2018.

Williams, R. J. 1992. Simple statistical gradient-following algorithms for connectionist reinforcement learning. Mach Learn, 8(3): 229–256.

Zhang, A.; Satija, H.; and Pineau, J. 2018. Decoupling dynamics and reward for transfer learning. In ICLR (workshop) 2018.
A Proofs and Derivations

In this appendix, we report the proofs of the results presented in the main paper.

A.1 Gradient of the \( \beta \)-Step Ahead Expected Return

Using similar derivations as in PGT (Williams 1992), we derive the gradient of the \( \beta \)-step ahead expected return.

\[
\nabla_\rho \hat{J}_{T,\alpha,\beta}(\nu_\rho) = \nabla_\rho \mathbb{E}_{T,\alpha}^p \left[ \sum_{t=T-\alpha+1}^{T} r_t \omega^{T-t} \frac{\sum_{s=T+1}^{T+\beta} \nu_\rho(\theta|s) \omega^{T-k} \nu_\rho(\theta|k) \nu_\rho(\theta|t) \nu_\rho(\theta|t) \nu_\rho(\theta|t)}{\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta|k) \nu_\rho(\theta|t)} \right]
\]

\[
= \sum_{t=T-\alpha+1}^{T} \nabla_\rho \mathbb{E}_t^\theta \left[ r_t \omega^{T-t} \frac{\sum_{s=T+1}^{T+\beta} \nu_\rho(\theta|s) \nu_\rho(\theta|t) \nu_\rho(\theta|t) \nu_\rho(\theta|t)}{\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta|k) \nu_\rho(\theta|t)} \right]
\]

\[
= \mathbb{E}_{T,\alpha}^p \left[ \sum_{t=T-\alpha+1}^{T} r_t \omega^{T-t} \frac{\sum_{s=T+1}^{T+\beta} \nu_\rho(\theta|s) \nu_\rho(\theta|t) \nu_\rho(\theta|t) \nu_\rho(\theta|t)}{\sum_{k=T-\alpha+1}^{T} \omega^{T-k} \nu_\rho(\theta|k) \nu_\rho(\theta|t)} \right]
\]

A.2 Proof of Lemma 4.2

Lemma 4.2. The variance of the objective \( \hat{J}_{T,\alpha,\beta} \) can be bounded as:

\[
\text{Var}_{T,\alpha} \left[ \hat{J}_{T,\alpha,\beta}(\rho) \right] \leq 2R_{\max}^2 \left( C_\gamma(\alpha)^2 + C_\gamma(\beta)^2 \right)
\]
Proof.
For every Theorem 4.1.

**Lemma 4.3.** The divergence between mixtures of Lemma 4.2 can be bounded as:

$$D_0(s_0) \sum_{t=1}^{T} P_t(s_{t+1} \mid s_t, \pi_{\theta_t}(s_t))v_{\rho} (\theta_t \mid \theta)\right) \left( 2 \right)^2,$$

where in line (6) we applied Cauchy-Schwarz inequality to the summation. Let us now consider the expectation $E_{T,\alpha}^\rho$ that is computed under the distribution:

$$D_0(s_0) \sum_{t=1}^{T} P_t(s_{t+1} \mid s_t, \pi_{\theta_t}(s_t))v_{\rho} (\theta_t \mid \theta).$$

Now, consider an individual term at time $t$ of the summation:

$$E_{T,\alpha}^\rho \left[ \ell(\theta_t)^2 \right] = \int_{\theta_0}^{} \int_{\theta_0}^{} \cdots \int_{\theta_0}^{} \int_{\theta_0}^{} D_0(s_0) \sum_{t=1}^{T} P_t(s_{t+1} \mid s_t, \pi_{\theta_t}(s_t))v_{\rho} (\theta_t \mid \theta)\right) \ell(\theta_t)^2 \, d\theta_0 \cdots d\theta_T,$$

where the last line is obtained by observing that all integrals cancel out apart from the one regarding $\theta_t$. Thus, we have:

$$E_{T,\alpha}^\rho \left[ \frac{1}{C_{\omega}} \sum_{t=T-\alpha+1}^{T} \omega_{T-t} \ell(\theta_t)^2 \right] = \frac{1}{C_{\omega}} \sum_{t=T-\alpha+1}^{T} \omega_{T-t} \int_{\theta_t}^{} v_{\rho} (\theta_t \mid \theta)\right) \ell(\theta_t)^2 \, d\theta_t,$$

$$= \int_{\theta_t}^{} \frac{1}{C_{\omega}} \sum_{t=T-\alpha+1}^{T} \omega_{T-t} v_{\rho} (\theta_t \mid \theta)\right) \ell(\theta_t)^2 \, d\theta_t,$$

where in line (7) we exploited the fact that $\theta_t$ is a dummy variable. The result is obtained by observing that the last expression corresponds to the exponentiated 2-Rényi divergence.

**A.3 Proof of Lemma 4.3**

**Lemma 4.3.** The divergence between mixtures of Lemma 4.2 can be bounded as:

$$d_2 \left( \frac{1}{C_{\gamma}(\beta)} \sum_{s=T+1}^{T+\beta} \gamma^s v_{\rho} (\cdot \mid s) \right) \left( \frac{1}{C_{\omega}} \sum_{t=T-\alpha+1}^{T} \omega_{T-t} v_{\rho} (\cdot \mid t) \right)^2 \leq \frac{C_{\gamma}}{C_{\gamma}(\beta)^2} \left( \sum_{s=T+1}^{T+\beta} \frac{\gamma^s}{\sum_{t=T-\alpha+1}^{T} \omega_{T-t}} \right)^{1/2},$$

where $B_{T,\alpha,\beta}(\rho)$

**Proof.** We apply Proposition B.2 to our bound with $\zeta_i = \frac{\gamma^s}{C_{\gamma}(\beta)}$, $\mu_j = \frac{\omega_{T-j}}{C_{\omega}}$, $\alpha = 2$ and change the summation from $i \in \{1, \ldots, L\}$ to $i \in \{T + 1, \ldots, T + \beta\}$ and $j \in \{1, \ldots, K\}$ to $j \in \{T - \alpha + 1, \ldots, T\}$.}

**A.4 Proof of Theorem 4.1**

**Theorem 4.1.** For every $\delta \in (0, 1)$, with probability at least $1 - \delta$, it holds that:

$$E_{T,\alpha}^\rho \left[ J_{T,\alpha,\beta}(\rho) \right] \geq J_{T,\alpha,\beta}(\rho) - \sqrt{\frac{1 - \delta}{\delta} 2R_{\alpha}^2 (C_{\gamma}(\alpha)^2 + C_{\omega} B_{T,\alpha,\beta}(\rho))}.$$

**Proof.** As in (Metelli et al. 2018), we use a Cantelli’s inequality on the random variable $J_{T,\alpha,\beta}(\rho)$,

$$P \left( J_{T,\alpha,\beta}(\rho) - E[J_{T,\alpha,\beta}(\rho)] \geq \lambda \right) \leq \frac{1}{1 + \frac{\lambda^2}{\text{Var}[J_{T,\alpha,\beta}(\rho)]}}.$$
we define $\delta = \frac{1}{1 + \var_{\omega} [T_{\alpha, \beta}(\rho)]}$ and consider the complementary event, which yields that with probability at least $1 - \delta$,

$$E[J_{T, \alpha, \beta}(\rho)] \geq J_{T, \alpha, \beta}(\rho) - \frac{1 - \delta}{\delta} \var_{\omega} [J_{T, \alpha, \beta}(\rho)].$$

We now replace with the bound of the variance from Lemma 4.2 and use the variational bound for the mixture of Rényi-\alpha-divergences from Lemma 4.3 to obtain

$$E[J_{T, \alpha, \beta}(\rho)] \geq J_{T, \alpha, \beta}(\rho) - \frac{1 - \delta}{\delta} 2||R||_2^2 \left( C_\gamma(\alpha)^2 + C_\omega(\beta)^2 \delta_2 \left( \frac{1}{C_\omega} \sum_{s = T + 1}^{T + \beta} \tilde{\gamma}^s \nu_\rho(\cdot|s) \right) \frac{1}{C_\omega} \sum_{t = T - \alpha + 1}^{T} \omega^{T - t} \nu_\rho(\cdot|t) \right)^2.$$ 

The result is obtained by recalling the definition of $B_{T, \alpha, \beta}(\rho)$.

### A.5 Bias Analysis and Proof of Lemma 4.1

We first derive a first result involving a tighter bound than the one provided in Lemma 4.1, but with a more intricate expression. The bound also holds for $\omega = 1$.

**Lemma A.1.** Under Assumptions 4.1 and 4.2, the bias of the estimator $\hat{J}_{T, \alpha, \beta}(\rho)$, for $0 < \omega \leq 1$, can be bounded as:

$$|J_{T, \beta}(\rho) - E_{T, \alpha}^\rho [\hat{J}_{T, \alpha, \beta}]| \leq (L_M + 2 R_{\max} L_\nu) C_\gamma(\beta) \left( \frac{1 - \alpha \omega^{\alpha - 1} + (\alpha - 1) \omega^{\alpha}}{(1 - \omega)(1 - \omega^{\alpha})} + \frac{1}{1 - \gamma} \right),$$

where $C_\gamma(\xi) = \frac{1 - \gamma x}{1 - \gamma}$ if $\gamma < 1$ otherwise $C_\gamma(\xi) = \xi$ for $\xi \geq 1$. In particular, when $\omega = 1$, the bound is the limit at $\omega \to 1$ of the previous expression and reads

$$|J_{T, \beta}(\rho) - E_{T, \alpha}^\rho [\hat{J}_{T, \alpha, \beta}]| \leq (L_M + 2 R_{\max} L_\nu) C_\gamma(\beta) \left( \frac{\alpha - 1}{2} + \frac{1}{1 - \gamma} \right).$$

**Proof.** Let us express explicitly the estimator:

$$E_{T, \alpha}^\rho [\hat{J}_{T, \alpha, \beta}(\rho)] = \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \frac{\sum_{t = T - \alpha + 1}^{T} \omega^{T - t} \nu_\rho(\theta|t) \mathbb{E}_t^\theta [r]}{\sum_{k = T - \alpha + 1}^{T} \omega^{T - k} \nu_\rho(\theta|k)} d\theta.$$

Let us now observe that $J_{T, \beta}(\rho) = \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \mathbb{E}_t^\theta [r] d\theta$. Thus, we have:

$$|E_{T, \alpha}^\rho [\hat{J}_{T, \alpha, \beta}(\rho)] - J_{T, \beta}(\rho)| = \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \sum_{t = T - \alpha + 1}^{T} \omega^{T - t} \nu_\rho(\theta|t) \mathbb{E}_t^\theta [r] - \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \mathbb{E}_t^\theta [r] d\theta.$$

Now we proceed as follows, by renaming $\nu_\rho(\theta|s) = \sum_{k = T - \alpha + 1}^{T} \omega^{T - k} \nu_\rho(\theta|k)$:

$$\left| \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \sum_{t = T - \alpha + 1}^{T} \omega^{T - t} \nu_\rho(\theta|t) \mathbb{E}_t^\theta [r] - \sum_{k = T - \alpha + 1}^{T} \omega^{T - k} \nu_\rho(\theta|k) \right| 
\leq \sum_{s = T + 1}^{T + \beta} \gamma^s \int_{\Theta} \nu_\rho(\theta|s) \left( \sum_{t = T - \alpha + 1}^{T} \omega^{T - t} \nu_\rho(\theta|t) \mathbb{E}_t^\theta [r] - \sum_{k = T - \alpha + 1}^{T} \omega^{T - k} \nu_\rho(\theta|k) \right) d\theta.$$
We consider the two terms separately. Let us start from (a):

\[
\begin{align*}
(a) = & \frac{1}{C_\omega} \left| \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \int_{\Theta} \left[ \omega^{T-t} \nu_\rho(\theta) \left( E_{\rho}^\omega[r] - E_{\rho}^\omega[r] \right) \right] d\theta \right| \\
& \leq \frac{1}{C_\omega} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \int_{\Theta} \omega^{T-t} \nu_\rho(\theta) \left| E_{\rho}^\omega[r] - E_{\rho}^\omega[r] \right| d\theta \\
& \leq \frac{L_M}{C_\omega} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \omega^{T-1} \int_{\Theta} \nu_\rho(\theta) |t-s| d\theta \\
& \leq \frac{L_M}{C_\omega} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \omega^{T-1} |t-s|,
\end{align*}
\]

where we employed \( \int_{\Theta} \nu_\rho(\theta) d\theta = 1 \) in the last passage and Assumption 4.1 in the last passage but one. Let us now move to (b):

\[
\begin{align*}
(b) & \leq 2 R_{\text{max}} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \int_{\Theta} \left| \nu_\rho(\theta) \right| \left( E_{\rho}^\omega[r] - E_{\rho}^\omega[r] \right) \omega^{T-t} \nu_\rho(\theta) \frac{C_\omega}{C_\omega} d\theta \\
& = 2 R_{\text{max}} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \int_{\Theta} \left| \nu_\rho(\theta) \right| \omega^{T-1} d\theta \\
& \leq 2 R_{\text{max}} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \frac{1}{C_\omega} \int_{\Theta} \omega^{T-1} \nu_\rho(\theta) |t-s| d\theta \\
& \leq 2 R_{\text{max}} L_{\nu} \frac{C_\omega}{C_\omega} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \omega^{T-1} |t-s|,
\end{align*}
\]

where we used Assumption 4.2 in the last passage.

We now use a similar derivation as in Lemma 3.4 of (Jagerman, Markov, and de Rijke 2019). Observe that, setting \( m = s - T \) and \( n = T - t \),

\[
\frac{1}{C_\omega} \sum_{t=T+\alpha+1}^{T} \omega^{T-1}(s-t) = \frac{1}{C_\omega} \sum_{n=0}^{\alpha-1} \omega^{n}(m + n) = m + \frac{1}{C_\omega} \sum_{n=1}^{\alpha-1} \omega^n n.
\]

If \( \omega < 1 \), we have:

\[
\frac{1}{C_\omega} \sum_{t=T+\alpha+1}^{T} \omega^{T-1}(s-t) = m + \frac{1}{C_\omega} \omega \frac{d}{d\omega} \sum_{n=1}^{\alpha-1} \omega^n \\
= m + \frac{1}{C_\omega} \omega \frac{1 - \omega \alpha^{-1} + (\alpha - 1) \omega^n}{(1 - \omega)^2} \\
= m + \omega \frac{1 - \omega \alpha^{-1} + (\alpha - 1) \omega^n}{(1 - \omega)(1 - \omega^n)},
\]

which yields

\[
\begin{align*}
\left| J_{T,\beta}(\rho) - \mathbb{E}_{T,\alpha}^{\rho} \left[ J_{T,\alpha,\beta} \right] \right| & \leq \frac{L_M + 2 R_{\text{max}} L_{\nu}}{C_\omega} \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \omega^{T-1}(s-t) \\
& = (L_M + 2 R_{\text{max}} L_{\nu}) \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \left( m + \omega \frac{1 - \omega \alpha^{-1} + (\alpha - 1) \omega^n}{(1 - \omega)(1 - \omega^n)} \right) \\
& = (L_M + 2 R_{\text{max}} L_{\nu}) \sum_{s=T+1}^{T+\beta} \hat{\gamma}^s \left( (s-T) + \omega \frac{1 - \omega \alpha^{-1} + (\alpha - 1) \omega^n}{(1 - \omega)(1 - \omega^n)} \right) \\
& = (L_M + 2 R_{\text{max}} L_{\nu}) \left( 1 - \gamma^\beta \omega \frac{1 - \omega \alpha^{-1} + (\alpha - 1) \omega^n}{(1 - \omega)(1 - \omega^n)} + \sum_{k=0}^{\beta-1} \gamma^k (k+1) \right)
\end{align*}
\]
unsuccessful idea is to take uniform values. We list six alternative approaches to finding a better bound for 

\[ \phi \]

B.1 Direct Convex Optimization

is repeated each time the variance bound is needed in the main algorithm. We refer to this approach as direct optimization with

\[ \text{reset} \]

With Reset

The first possibility is to start from a uniform distribution over the variational parameters then follow the gradient for a given number of steps and return the values of the variational parameters that will then be used in the bound. The process is repeated each time the variance bound is needed in the main algorithm. We refer to this approach as direct optimization with reset.

Lemma B.1

In this appendix, we discuss different approaches to obtain a useful bound on the Rényi divergence between mixture distributions. Let \( \Psi = \sum_{i=1}^{L} \zeta_i P_i \) and \( \Phi = \sum_{j=1}^{K} \mu_j Q_j \) with \( \forall i \in \{1, L\}, \zeta_i \in [0, 1], \forall j \in \{1, K\}, \mu_j \in [0, 1] \), \( \sum_{i=1}^{L} \zeta_i = 1 \) and \( \sum_{j=1}^{K} \mu_j = 1 \) be two mixtures of probabilities. We are interested in finding an upper-bound of \( d_\alpha(\Psi \| \Phi) \) for \( \alpha \geq 1 \). We first recall a cornerstone results from (Papini et al. 2019).

Lemma B.1 (Lemma 4, (Papini et al. 2019)). Let \( \{\psi_{ij}\}_{i=1}^{L \in \{1, L\}} \) and \( \{\phi_{ij}\}_{j=1}^{K \in \{1, K\}} \) be two sets of variational parameters s.t.

\[ \psi_{ij} \geq 0, \phi_{ij} \geq 0, \sum_{i=1}^{L} \psi_{ij} = \mu_j \text{ and } \sum_{j=1}^{K} \phi_{ij} = \zeta_i. \]

Then for any \( \alpha \geq 1 \), and for the previously defined mixture of probabilities it holds that:

\[
d_\alpha(\Psi \| \Phi) \leq \sum_{i=1}^{L} \sum_{j=1}^{K} \phi_{ij}^\alpha \psi_{ij}^{1-\alpha} d_\alpha(P_i \| Q_j)^{\alpha-1}. 
\]

We wish to find the values of \( \{\psi_{ij}\} \) and \( \{\phi_{ij}\} \) which minimize the upper-bound as to make it tighter. A straightforward yet unsuccessful idea is to take uniform values. We list six alternative approaches to finding a better bound for \( d_\alpha(\Psi \| \Phi) \). We then compare them on a toy problem and select the most promising one.

B.1 Direct Convex Optimization

Optimizing for the best \( \phi_{ij} \geq 0 \) and \( \psi_{ij} \geq 0 \) under their constraints for a given pair of mixtures is a convex problem.

In the case \( \omega = 1 \), the Equation (9) becomes:

\[
\frac{1}{C_\omega} \sum_{t=T-\alpha+1}^{T} \omega^{T-t} (s - t) = m + \frac{\alpha - 1}{2}.
\]

Thus, the bound becomes:

\[
\left| J_{T, \alpha, \beta}(p) - \mathbb{E}_{T, \alpha}^{p} \left[ \hat{J}_{T, \alpha, \beta} \right] \right| \leq \left( L_M + 2R_{\max}L_v \right) \sum_{s=T+1}^{T+\beta} \gamma^s \left( m + \frac{\alpha - 1}{2} \right)
\]

\[
= \left( L_M + 2R_{\max}L_v \right) C_\gamma(\beta) \left( \frac{\alpha - 1}{2} + \sum_{k=0}^{\beta - 1} \gamma^k (k + 1) \right)
\]

\[
= \left( L_M + 2R_{\max}L_v \right) C_\gamma(\beta) \left( \frac{\alpha - 1}{2} + \frac{1}{1 - \gamma} \right).
\]

Lemma 4.1. Under Assumptions 4.1 and 4.2, the bias of the estimator \( \hat{J}_{T, \alpha, \beta}(p) \), for \( \omega < 1 \), can be bounded as:

\[
\left| J_{T, \alpha, \beta}(p) - \mathbb{E}_{T, \alpha}^{p} \left[ \hat{J}_{T, \alpha, \beta} \right] \right|
\]

\[
\leq \left( L_M + 2R_{\max}L_v \right) C_\gamma(\beta) \left( \frac{\omega}{1 - \omega} + \frac{1}{1 - \gamma} \right),
\]

where \( C_\gamma(\xi) = \frac{1 - \gamma^k}{1 - \gamma} \) if \( \gamma < 1 \) otherwise \( C_\gamma(\xi) = \xi \) for \( \xi \geq 1 \).

Proof. We start from the bound is Lemma A.1 and observe that \( \frac{1 - \alpha \omega^{\alpha-1} + (\alpha - 1) \omega^{\alpha}}{(1 - \omega)(1 - \omega^{\alpha})} \leq \frac{1}{1 - \omega} = C_\omega \) which yields the result.

B On the Variational Bounds of Rényi Divergence between Mixture Distributions

In this appendix, we discuss different approaches to obtaining a useful bound on the Rényi divergence between mixture distributions. Let \( \Psi = \sum_{i=1}^{L} \zeta_i P_i \) and \( \Phi = \sum_{j=1}^{K} \mu_j Q_j \) with \( \forall i \in \{1, L\}, \zeta_i \in [0, 1] \), \( \forall j \in \{1, K\}, \mu_j \in [0, 1] \), \( \sum_{i=1}^{L} \zeta_i = 1 \) and \( \sum_{j=1}^{K} \mu_j = 1 \) be two mixtures of probabilities. We are interested in finding an upper-bound of \( d_\alpha(\Psi \| \Phi) \) for \( \alpha \geq 1 \). We first recall a cornerstone results from (Papini et al. 2019).

Let \( \psi_{ij} \in \{1, L\} \) and \( \phi_{ij} \in \{1, K\} \) be two sets of variational parameters s.t.

\[
\phi_{ij} \geq 0, \phi_{ij} \geq 0, \sum_{i=1}^{L} \psi_{ij} = \mu_j \text{ and } \sum_{j=1}^{K} \phi_{ij} = \zeta_i. \]

Then for any \( \alpha \geq 1 \), and for the previously defined mixture of probabilities it holds that:

\[
d_\alpha(\Psi \| \Phi) \leq \sum_{i=1}^{L} \sum_{j=1}^{K} \phi_{ij}^\alpha \psi_{ij}^{1-\alpha} d_\alpha(P_i \| Q_j)^{\alpha-1}. 
\]
**Without Reset**  The idea is the same but the value of the variational parameters is not reset in between optimization, instead, the value of the variational parameters is kept in memory and reused as initialization each time the variance bound is needed. We refer to this approach as *direct optimization without reset*.

### B.2 Two Steps Minimization

Since solving an optimization problem whenever computing the bound might be inefficient, we propose in the following an alternative approach based on a bound. We again state a useful results from (Papini et al. 2019).

**Theorem B.1** (Theorem 5, Papini et al. (2019)). Let $P$ be a probability measure and consider the previous mixture $\Phi$, then for any $\alpha \geq 1$, one has:

$$d_\alpha(P \parallel \Phi) \leq \frac{1}{\sum_{j=1}^{K} p_j d_\alpha(P_j \parallel Q_j)}.$$  

Notice that the bound is the harmonic mean of the Rényi -divergences between $P$ and the component of the mixture. We now derive a result for the case $d_\alpha(\Psi \parallel Q)$, where $\Psi$ is a mixture distribution.

**Proposition B.1.** Let $Q$ be a probability measure and consider the previous mixture $\Psi$, then for any $\alpha \geq 1$, one has:

$$d_\alpha(\Psi \parallel Q) \leq \left( \frac{L}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}} \right)^{\frac{\alpha}{\alpha-1}}.$$  

**Proof.** Since the variational bound in Equation (10) is convex in $\{\psi_i\}$, we can find the optimal value of $\{\psi_i\}$ via Lagrange multipliers following a similar reasoning as in (Papini et al. 2019):

$$\psi_i = \frac{\zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}}.$$  

Then, we replace in the original problem:

$$d_\alpha(\Psi \parallel Q)^{\alpha-1} \leq \frac{L}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}} \left( \frac{\zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}} \right)^{1-\alpha} d_\alpha(P_i \parallel Q)^{\alpha-1}$$

$$= \sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{1-\alpha} \frac{\alpha-1}{\alpha} \left( \frac{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}} \right)^{1-\alpha}$$

$$= \frac{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}} \left( \sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}} \right)^{1-\alpha}$$

$$= \sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel Q)^{\frac{\alpha-1}{\alpha}}.$$  

$$\square$$

Note that this is now the weighted power mean of exponent $\frac{\alpha-1}{\alpha}$, We can combine Theorem B.1 and Proposition B.1 to find a bound for $d_\alpha(\Psi \parallel \Phi)$. There are 2 ways of doing so, presented in the following paragraphs.

### B.3 Two Steps $\psi$ First

We first use Proposition B.1 then Theorem B.1. We call this approach *two steps $\psi$ first*. Doing so, we have:

**Proposition B.2.** Under the same assumptions of Lemma B.1, one has:

$$d_\alpha(\Psi \parallel \Phi) \leq \left( \sum_{i=1}^{L} \zeta_i d_\alpha(P_i \parallel \Psi)^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}}$$

$$\leq \left( \frac{L}{\sum_{i=1}^{L} \zeta_i} \frac{1}{\sum_{j=1}^{K} p_j d_\alpha(P_j \parallel Q_j)^{\frac{\alpha-1}{\alpha}}} \right)^{\frac{\alpha}{\alpha-1}}.$$  

We can now apply this results to the bound in Lemma 4.2 with $\alpha = 2$ and it yields the result from Theorem 4.1.
B.4 Two Steps $\phi$ First
Alternatively, we can use Proposition B.1 first and then Theorem B.1. We call this approach two steps $\phi$ first. This yields:

**Proposition B.3.** Under the same assumptions of Lemma B.1,

$$d_{\alpha}(\Psi \mid \Phi) \leq \frac{1}{\sum_{j=1}^{K} d_{\alpha}(\Psi(Q_j))} \leq \frac{1}{\sum_{j=1}^{K} \mu_j \left(\sum_{i=1}^{L} \zeta_i d_{\alpha}(P_i \| Q_j) \right)}.$$

We now apply this results to the bound in Lemma 4.2 to yield the following result.

**Proposition B.4** (Lower bound with two steps $\phi$ first). For $\delta > 0$, with probability at least $1 - \delta$, it holds that

$$\mathbb{E}\left[J_{T,\alpha,\beta} \right] \geq J_{T,\alpha,\beta} - \left[\frac{1 - \delta}{\delta} 2\|R\|_{\rho}^2 \left(C_\gamma(\alpha)^2 + C_\omega \sum_{k=T-\alpha+1}^{T} \omega^{T-k} \left(\sum_{i=T+\beta}^{T} \zeta_i \rho_{\phi}(\cdot|s) \|\rho_{\phi}(\cdot|k)\|^2 \right)^{-\frac{1}{2}} \right)^2 \right].$$

B.5 One Step then Uniform
Another possible approach is to find the optimal value of the variational parameters $\psi_{ij}$ as a function of the parameters $\phi_{ij}$ and then replace it in Equation (10). It is also possible to do the other way around. The result is given in the next proposition.

**Proposition B.5.** Under the same conditions as in Lemma B.1. The optimal values of $\psi_{ij}$ are

$$\psi_{ij} = \mu_j \frac{\phi_{ij} d_{\alpha}(P_i \| Q_j)\alpha^{-1}}{\sum_{i=1}^{K} \phi_{ij} d_{\alpha}(P_i \| Q_j)\alpha^{-1}}.$$

and the optimal values of $\phi_{ij}$ are

$$\phi_{ij} = \frac{\zeta_i \psi_{ij} d_{\alpha}(P_i \| Q_j)}{\sum_{k=1}^{K} \psi_{ik} d_{\alpha}(P_i \| Q_k)^{-1}}.$$

**Proof.** By Equation (10), one knows that

$$d_{\alpha}(\Psi \mid \Phi)\alpha^{-1} \leq \sum_{i=1}^{L} \sum_{j=1}^{K} \phi_{ij} \psi_{ij} \alpha^{-1} d_{\alpha}(P_i \| Q_j)^{\alpha^{-1}}.$$

We write the Lagrangian of the optimisation problem:

$$\mathcal{L}(\phi_{ij}, \psi_{ij}, \lambda_{ij}^\alpha, \lambda_{ij}^\beta) = \sum_{i=1}^{L} \sum_{j=1}^{K} \phi_{ij} \psi_{ij} \alpha^{-1} d_{\alpha}(P_i \| Q_j)^{\alpha-1} - \sum_{i=1}^{L} \lambda_{ij}^\alpha \left(\sum_{j=1}^{K} \phi_{ij} - \zeta_i\right) - \sum_{j=1}^{K} \lambda_{ij}^\beta \left(\sum_{i=1}^{L} \psi_{ij} - \mu_j\right).$$

We now look at the zero of the derivative with respect to the variational variables:

$$\frac{\partial \mathcal{L}}{\partial \phi_{ij}} = \alpha \phi_{ij}^{-1} \psi_{ij} \alpha^{-1} d_{\alpha}(P_i \| Q_j)^{\alpha^{-1}} - \lambda_{ij}^\alpha = 0.$$

It implies that:

$$\phi_{ij} = \frac{\lambda_{ij}^\alpha}{\alpha^{-1} d_{\alpha}(P_i \| Q_j)^{\alpha^{-1}}}.$$

Recall that $\sum_{j} \phi_{ij} = \zeta_i$ so:

$$\left(\frac{\lambda_{ij}^\alpha}{\alpha^{-1}} \sum_{k=1}^{K} \frac{\psi_{ik} d_{\alpha}(P_i \| Q_k)}{\alpha^{-1}} = \zeta_i.$$

This gives the value of $\lambda_{ij}^\beta$:

$$\lambda_{ij}^\beta = \frac{\zeta_i \psi_{ij} d_{\alpha}(P_i \| Q_j)\alpha^{-1}}{\sum_{k=1}^{K} \psi_{ik} d_{\alpha}(P_i \| Q_k)^{-1}}.$$
Finally, by replacing,
\[
\phi_{ij} = \frac{\zeta_i \psi_{ij}}{d_\alpha(P_i|Q_j)} \left( \sum_{k=1}^K \frac{\psi_{ik}}{d_\alpha(P_k|Q_j)} \right)^{-1}.
\]

We do the same for \(\psi_{ij}\):
\[
\phi_{ij} = \frac{\zeta_i \psi_{ij}}{d_\alpha(P_i|Q_j)} \left( \sum_{k=1}^K \frac{\psi_{ik}}{d_\alpha(P_k|Q_j)} \right)^{-1} - \lambda_j^\alpha = 0.
\]

It implies that
\[
\psi_{ij} = \left( \frac{1 - \alpha}{\lambda_j^\beta} \right) \frac{\phi_{ij} d_\alpha(P_i|Q_j)}{\alpha - 1}.
\]

Recall that \(\sum_i \psi_{ij} = \beta_j\) so:
\[
\left( \frac{1 - \alpha}{\lambda_j^\beta} \right) \frac{\phi_{ij} d_\alpha(P_i|Q_j)}{\alpha - 1} = \mu_j.
\]

This gives the value of \(\lambda_j^\beta\):
\[
\lambda_j^\beta = \frac{1 - \alpha}{\mu_j^\alpha} \left( \sum_{i=1}^L \phi_{ij} d_\alpha(P_i|Q_j) \right)^{\frac{\alpha - 1}{\alpha}}.
\]

Finally, by replacing,
\[
\psi_{ij} = \frac{\phi_{ij} d_\alpha(P_i|Q_j)}{\sum_{i=1}^L \phi_{ij} d_\alpha(P_i|Q_j)}^{\frac{\alpha - 1}{\alpha - 1}}.
\]

\[\square\]

**Uniform \(\psi\)** We use the optimal value of \(\phi_{ij}\) from Equation (11) inside Equation (10) and then use a uniform value of \(\psi_{ij}\). We call this approach *uniform \(\psi\)*. Following this approach we have the following proposition.

**Proposition B.6.** Under similar assumption as in Lemma B.1, one has
\[
d_\alpha(\Psi|\Phi) \leq \sum_{i=1}^L \left( \sum_{j=1}^K \frac{\psi_{ij}}{d_\alpha(P_i|Q_j)} \right)^{1 - \alpha}.
\]

**Proof.** We inject the value of \(\phi_{ij}\) from Equation (11) inside Equation (10) to get
\[
d_\alpha(\Psi|\Phi) \leq \sum_{i=1}^L \sum_{j=1}^K \frac{\psi_{ij}}{d_\alpha(P_i|Q_j)} \left( \sum_{k=1}^K \frac{\psi_{ik}}{d_\alpha(P_k|Q_j)} \right)^{-\alpha}.
\]

Then we wish now to find values of \(\psi_{ij}\) to minimize this bound. Recall that the constraints are \(\psi_{ij} \geq 0\) and \(\sum_{i=1}^L \psi_{ij} = \mu_j\). We choose a uniform value, that is \(\psi_{ij} = \frac{\mu_j}{L}\). This gives the result. \[\square\]

We now apply this results to the bound in Lemma 4.2 to yield the following result.

**Proposition B.7 (Lower bound with uniform \(\psi\)).** For \(\delta > 0\), with probability at least \(1 - \delta\), it holds that
\[
\mathbb{E} \left[ J_{T,\alpha,\beta} \right] \geq J_{T,\alpha,\beta} - \frac{1 - \delta}{\delta} \cdot R(\infty) \left( C_\gamma(\alpha)^2 + \beta C_\omega \sum_{s=T+1}^{T+\beta} \phi_{ij} \left( \sum_{k=1}^{T-\alpha+1} \frac{\omega^{T-k}}{d_2(\nu_\rho(|s|),\nu_\rho(|k|))} \right)^{-1} \right).
\]
Uniform $\phi$  We repeat the previous approach but switching the role of $\phi_{ij}$ and $\psi_{ij}$. We derive the following proposition.

Proposition B.8. Under similar assumption as in Lemma B.1, one has

$$d_{\alpha}(|\Psi|) \leq \sum_{j=1}^{K} \frac{\mu_j}{K} \left( \sum_{i=1}^{L} \frac{\zeta_i d_{\alpha}(P_i \| Q_j)}{\xi} \right)^{\frac{\alpha-1}{\alpha}}.$$

Proof. We inject the value of $\psi_{ij}$ from Equation (11) inside Equation (10) to get

$$d_{\alpha}(|\Psi|) \leq \sum_{i=1}^{L} \sum_{j=1}^{K} \zeta_i \phi_{ij} d_{\alpha}(P_i \| Q_j)^{\frac{\alpha-1}{\alpha}}.$$

Then we wish now to find values of $\phi_{ij}$ to minimize this bound. Recall that the constraints are $\phi_{ij} \geq 0$ and $\sum_{j=1}^{K} \phi_{ij} = \zeta_i$. We choose a uniform value, that is $\phi_{ij} = \frac{\zeta_i}{K}$. This gives the result. \qed

We now apply this results to the bound in Lemma 4.2 to yield the following result.

Proposition B.9 (Lower bound with uniform $\psi$). For $\delta > 0$, with probability at least $1 - \delta$, it holds that

$$E[J_{T,\alpha,\beta}] \geq J_{T,\alpha,\beta} - \sqrt{\frac{1-\delta}{\delta}2C2\alpha \| R \|_2^2 \left( \frac{C_\gamma \alpha}{\alpha^2} + \sum_{k=1}^{T-\alpha+1} \frac{1}{\omega_{T-k}} \sum_{s=T+1}^{T+\beta} \tilde{d}_2(\nu_{\rho}(.|s), \nu_{\rho}(.|k))^{2} \right)}.$$

B.6 Comparison of the Bounds

In this section, we discuss, from the six lower-bounds of the variance of the total return $J_{T,\alpha,\beta}(\rho)$, which is tighter and, when used in the optimization, which constrains the hyper-policy toward being stationary.

To do so, we design the following test. Our hyper-policy will be sinusoidal, that is, it will output the following mean for the policy parameter $\theta$:

$$\theta_t = A \sin(\phi t + \psi) + B.$$

We choose such a hyper-policy since the scale parameter $A$ is the parameter which controls non-stationarity. Ideally, optimizing our lower-bound, we should see $A$ converging to 0. The environment is not relevant for this study since we optimize the hyper-policy solely on the variance term, discarding $J_{T,\alpha,\beta}(\rho)$. However, for completeness, we report that the environment is a contextual bandit, where the context follows a sinusoidal function.

From Figure 8 we see that the most efficient methods when it comes to making the hyper-policy stationary, and thus push $A$ toward 0, are uniform $\Psi$, two steps $\Psi$ first, two steps $\Phi$ first and the direct optimization with reset.

From Figure 9 we see however that the upper-bound is less smooth for the convex optimization based approaches. Moreover, the upper-bound is tighter for uniform $\Psi$ and two steps $\Psi$ first. The final choice of our bound in practice will thus be made between those two. The bounds have similarities but we believe that the two steps $\Psi$ first may better adapt to other scenarios, since setting one set of the variational parameters to a uniform distribution as in uniform $\Psi$ doesn’t seem to be a robust choice.

C Experiments Details

All experiments are conducted on Python using Pytorch (Paszke et al. 2019) as deep learning library. Below, we give more details the parameters of each algorithm used in our experiments.

C.1 Hyper-parameters

POLIS The hyper-policy has 1046 parameters for the Trading environment and 1040 for the Dam. This is due to the different size of the environment state in the Dam and Trading environments. Indeed, the state has two entries for the Trading and one for the Dam. This makes 3 parameters for the affine policy for Trading and 2 for the Dam.

Recall that we have set $\alpha = 500$, $\gamma = \omega = 1$. For the gradient optimization, we use RMSprop (Tieleman and Hinton 2012) with a learning rate of 1e$-3$, smoothing constant $\alpha_{\text{RMSprop}} = 0.9$ and parameter $\epsilon_{\text{RMSprop}} = 1e - 10$ for numerical stability. Before each gradient step, we sample 100 replayed trajectory on the last $\alpha$ steps in order to take into account the hyper-policy’s stochasticity. We randomly initialize the hyper-policy’s convolutional parameters using Pytorch’s default initialization. We set the size of the channels inside the temporal convolution of the hyper-policy to $[8, 8, 4]$, their kernel size to 3 and the positional encoding’s dimension to 8. We use the same hyper-policy in the behavioural and target period at the exception of the
Figure 8: Evolution of the scale parameter for several approaches on the upper-bounds of the variance. Note that the value of $A$ for the two steps $\Psi$ first and uniform $\Psi$ are confounded.

Figure 9: Evolution of the variational upper-bound on the variance for several approaches. Here again, the log upper-bound for the two steps $\Psi$ first and uniform $\Psi$ are confounded.

Table 2: Optimal parameters for the POLIS algorithm on each dataset. The Trading (1) has hyper-parameters selected on the return of 2009-2012 dataset while Trading (2)’s hyper-parameters are selected on the mean return of 2013-2016 and 2017-2020.

|        | Dam | Trading (1) | Trading (2) |
|--------|-----|-------------|-------------|
| $\lambda$ | 100 | 10          | 10          |
| $\beta$  | 50  | 500         | 100         |
| Fix $\sigma$ | False | True       | True       |

Table 3: Optimal parameters for the Stationary hyper-policy on each dataset. The Trading (1) has hyper-parameters selected on the return of 2009-2012 dataset while Trading (2)’s hyper-parameters are selected on the mean return of 2013-2016 and 2017-2020.

|        | Dam | Trading (1) | Trading (2) |
|--------|-----|-------------|-------------|
| Fix $\sigma$ | False | True       | True       |

vector of policy parameters’ standard deviation $\sigma$. For the behavioural hyper-policy it is set for all entries to $e^{\frac{1}{2}}$. For the target hyper-policy, whether it will be fixed during training or not, its initial value is set to a vector of $e^{-1}$.

The remaining parameters are the one which will vary during hyper-parameter search. They are the variance regularization level $\lambda$, the steps for estimation of future performance $\beta$ and whether or not to fix $\sigma$ during training. We tested values from $\{10, 100, 1000\}$ for $\lambda$, $\{10, 100, 50\}$ for $\beta$ and whether or not to fix $\sigma$ on the Trading environment. For the Dam, we also tested whether or not to fix $\sigma$. We explored values of $\lambda$ in $\{0, 10, 100\}$ and $\beta$ in $\{10, 50, 100\}$.

We report the best performing set of hyper-parameters in Table 2.

**Stationary Hyper-policy**  The stationary hyper-policy shares most of its parameters with POLIS. Of course, it doesn’t use the temporal convolution and the positional encoding and its optimization involves only the $\alpha$-step behind expected return, so $\beta$ and $\lambda$ are not used. The standard deviation $\sigma$ can still be learned or fixed, we report the best performing set of hyper-parameters in Table 3.

|        | Dam | Trading (1) | Trading (2) |
|--------|-----|-------------|-------------|
| Fix $\sigma$ | False | True       | True       |

Table 3: Optimal parameters for the Stationary hyper-policy on each dataset. The Trading (1) has hyper-parameters selected on the return of 2009-2012 dataset while Trading (2)’s hyper-parameters are selected on the mean return of 2013-2016 and 2017-2020.

**Baselines**  Although the three baselines Pro-OLS, Pro-WLS and ONPG implement different ideas, their parameters are similar. They all proceed to a number of inner optimizations steps each time the policy is updated, which we set to 10. We also set the standard deviation of the normal distribution under which the action is selected to 0.5. The policy learns features from the state with a neural network which we refer to as state representation module. We refer to the policy module as the neural network which takes the output features of the state representation and outputs the actions. We set the learning rate of both the state representation module and the policy module to $1e^{-2}$. The policy module uses a hidden layer of 16 neurons. The number of neurons of the state representation module is discussed below. We use a buffer size of 1000 and a maximum horizon inside the buffer of 150.

We then proceed to a grid search over following the recommendations for the hyper-parameters values of (Chandak et al. 2020). For the entropy regularization parameter $\lambda_{\text{entropy}}$, we consider the set $\{0, 1e^{-3}, 1e^{-2}\}$. For the importance clip-
Figure 10: The three inflow profiles for the Dam experiment.

Figure 12: Value of the rate of the EUR-USD on the period 2009-2020, divided in the three datasets used in experiments.

The state representation module’s neurons per layer $n_{\text{neurons}}$ is explored in $\{32, 64, [32, 32]\}$, where $[32, 32]$ corresponds to 2 layers of 32 neurons. We consider the set $\{5, 7\}$ for the size of the extrapolator Fourier basis $k_{\text{Fourier}}$ which is used in the performance prediction. We consider the set $\{1, 3, 5\}$ for the number of step ahead to predict the performance $\delta$.

|               | Dam          | Trading (1)  | Trading (2)  |
|---------------|--------------|--------------|--------------|
| $\lambda_{\text{entropy}}$ | $0 \ 1e-2 \ 1e-2$ | $1e-3 \ 1e-2 \ 0$ | $0 \ 1e-2 \ 1e-3$ |
| $t_{IS}$      | $10 \ 15 \ 15$ | $10 \ 15 \ 10$ | $15 \ 15 \ 15$ |
| $n_{\text{neurons}}$ | $[32,32]$ $[32,32]$ $[32,32]$ | $64 \ 32 \ 32$ | $[32,32] \ [32,32] \ 64$ |
| $k_{\text{Fourier}}$ | $7 \ 7 \ 7$ | $7 \ 7 \ 5$ | $7 \ 7 \ 5$ |
| $\delta$     | $1 \ 5 \ 5$ | $5 \ 1 \ 1$ | $1 \ 5 \ 5$ |

Table 4: Optimal parameters for the Pro-OLS, Pro-WLS and ONPG baselines. The Trading (1) has hyper-parameters selected on the return of 2009-2012 dataset while Trading (2)’s hyper-parameters are selected on the mean return of 2013-2016 and 2017-2020.

C.2 Description of Dataset

**Dam** Recall that the cost that the agent gets is a convex combination of the cost related to flooding and the one for not meeting the daily demand. The parameters of this convex combination are, respectively, 0.3 and 0.7 for the first inflow profile, 0.8 and 0.2 for the second and 0.35 and 0.65 for the last one. We give the mean inflow throughout the year for each profile in Figure 10.

The three inflows used for the Dam environment are given in

**Trading** The Trading dataset is composed of the day price of the EUR-USD rate as given in Figure 12.
C.3 Further Experiments

We report in this section extra experiments for the POLIS algorithm on the Vasicek process. We study values of $\lambda$ in the range $[1, 100]$ and values of $\beta$ in the range $(1, 100]$ ($\beta = 1$ is not considered as it does not involve a mixture of distribution in the variance bound). We are interested in the trade-off between the return and the standard deviation of the rewards. The results are reported in Figure 14 and Figure 16. The result suggest that a smaller $\beta$ allows for a general better return a the expense of a higher standard deviation. The dependence on $\lambda$ is less clear. However it can be seen that generally, smaller values of $\lambda$ have the same effect as the smaller values of $\beta$. For a clearer view of this experiment, we report the same plots but for only one value of the other parameter in Figure 18.
Figure 16: Return (left) and standard deviation of the rewards (right) of POLIS on the Vasicek process for different values of $\lambda$ and $\beta$.

Figure 18: Plot of the standard deviation of the rewards and return of POLIS on the Vasicek process for different values of $\lambda$ and $\beta$. The value of one parameter is shown on a color scale while the other parameter is labelled on each point. Left figure uses a color scale for $\lambda$ and $\beta$ is fixed to 50 while right figure uses a color scale for $\beta$ and $\lambda$ is fixed to 50.