Phase Space of Compact Bianchi Models with Fluid

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The structure of phase space is determined for spatially compact and locally homogeneous universe models with fluid. Analysis covers models with all possible space topologies except for those covered by $S^3$, $H^3$ or $S^2 \times \mathbb{R}$ which have no moduli freedom. We show that space topology significantly affects the number of dynamical degrees of freedom of the system. In particular, we give a detailed proof of the result that for the systems modeled on the Thurston types $H^2 \times \mathbb{R}$ and $\tilde{SL}_2 \mathbb{R}$, which have locally the Bianchi type III or VIII symmetry, the number of dynamical degrees of freedom increases without bound when the space topology becomes more and more complicated, which was first pointed out by Koike, Tanimoto and Hosoya in an incomplete form.

§1. Introduction

Bianchi models provide the basis of modern cosmological models as the zeroth approximation. They have been also frequently used as the simplest models called minisuperspace models in the study of quantum gravity and quantum cosmology. In many of the investigations in these fields, however, topological features of models were neglected, and models with the simplest space topology were treated. Such treatments are allowed if one is interested only in local structures and dynamics of models. However, if one wants to discuss global structures and dynamics of models, one must take into account the effects of space topology. In particular, when one constructs a minisuperspace model in quantum gravity, one has to use a Bianchi model with compact space in order to make the canonical variables well-defined.

When one considers Bianchi models with non-trivial topologies, various new features arise. First, space topology is strongly restricted by the Bianchi type. In particular, there exists no spatially compact model which has the local symmetry of the Bianchi type IV or VI$_h$($h \neq 0, -1$)\footnote{9}. Second, there appear new dynamical degrees of freedom called the moduli, which describe the degrees of freedom in deforming the global structure of space preserving the local structure and the topology\footnote{1, 11}. This is a higher-dimensional analogue of the well-known moduli freedom for the flat 2-dimensional torus. This appearance of the moduli freedom is very important in the arguments of minisuperspace models based on Bianchi models, because it significantly alters the canonical structure and dynamics of the systems. From this standpoint, we developed a formalism to construct the diffeomorphism-invariant phase space and determine its canonical structure for locally homogeneous systems on compact closed 3-manifolds on the basis of the Thurston classification of the maximal geometries with compact quotients, and applied it to vacuum systems of the Thurston types $E^3$, $Nil$ and $Sol$, corresponding to the Bianchi types I, VII$_0$, II and VI$_0$, in our previous paper (Paper I)\footnote{10}.

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One of the main purpose of the present paper is to extend the analysis in Paper I to systems on spaces of the Thurston types $H^2 \times \mathbb{R}$ and $\widetilde{SL}_2 \mathbb{R}$ corresponding to the Bianchi types III and VIII. By this the analysis of the moduli freedom for all compact Bianchi models is completed, because the compact Bianchi models of the types V, VII, and IX have no moduli freedom $^9, ^{11}, ^3, ^4$. Another purpose of the present paper is to extend the analysis of the vacuum systems in Paper I to systems with perfect fluid. Its main motivation is the generality problem of Bianchi models. As is well known, among the Bianchi models with simply connected space, the types VI$_h (h \neq 0, -1)$, VII$_h (h \neq 0)$, VIII and IX are the most generic models both for the vacuum and the fluid cases in the sense that the parameter count, i.e., the dimension of the solution space, for them are the largest: 4 for the vacuum case and 8 for the fluid case $^8, ^{15}$. Further, the Bianchi type VII$_0$ is more general than the type I in the same sense. This generality argument based on the parameter count is the starting point in the isotropization problem in the spatially homogeneous cosmology $^6, ^7, ^{12}, ^2, ^5, ^{19}, ^{13}$, and also plays the basic role in the mathematical cosmology $^{18}$. However, the situation drastically changes when the space is compactified. In particular, the type I becomes the most generic among the types I, II, V, VI$_0$, VII$_0$, VII, and IX for the vacuum case as was shown in Paper I. Hence it is interesting to see how the introduction of fluid alters the parameter count and to extend the analysis to the Bianchi type III and VIII. Partial answers to these problems were published in the joint work by the author and John D. Barrow $^3, ^4$. In the present paper we give complete answers with detailed accounts on their derivations.

The paper is organized as follows. In the next section, we give the outline of the formalism developed in Paper I to determine the diffeomorphism-invariant phase space of a locally homogeneous system on a compact and closed 3-manifold. Then we apply it to the fluid model on orientable compact 3-manifolds of all possible topologies with non-trivial moduli freedom in the order of the Thurston types $E^3 (§3)$, Nil (§4), Sol (§5), $H^2 \times \mathbb{R}$ (§6) and $\widetilde{SL}_2 \mathbb{R}$ (§7). Since the fluid system cannot be put in the canonical form, we do not discuss the canonical structure of the phase space. Section 8 is devoted to summary and discussion.

**Notations:** Throughout the paper, $R(\theta)$ and $R_i(\theta)(i = 1, 2, 3)$ denote the rotation matrix of angle $\theta$ in the 2-plane and that around the $i$-th coordinate axis in the 3-dimensional Euclidean space, respectively. $R_i(\pi)$ is often simply written as $I_i$. The block diagonal matrix with block matrices $A, B, \cdots$ is denoted as $D(A, B, \cdots)$. For a three dimensional object $X$ such as a vector and a square matrix, its two dimensional part is often written as $\hat{X}$. In particular, for a 3-vector $\mathbf{v} = (v^1, v^2, v^3)$, $\hat{\mathbf{v}} = (v^1, v^2)$. This symbol is also used to denote the 3-vector $(v^1, v^2, 0)$ if there is no possibility of confusion. Further, the row vector expression and the column vector expression of a vector with the same components are often denoted by the same symbol. Finally, the linear transformation $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{a}$ is often denoted as $(\mathbf{a}, A)$. 
§2. General Framework

As in Paper I, we define the spatially locally homogeneous system as a globally hyperbolic spacetime whose universal covering is spatially homogeneous, and its invariance group $G$ as the invariance group of its universal covering. We often call such a system with compact space simply a compact Bianchi model. Under this definition, a compact Bianchi model is specified by topology of the compact space, an invariance group of transformations on the universal covering spacetime, and a set of fields on it. In the present paper, we mainly consider systems with perfect fluid, for which the fields consist of the spacetime metric and the 4-velocity and density of each component of fluid. We further restrict considerations to spacetimes with orientable spaces as in Paper I.

2.1. Compact Bianchi models

| Space   | $G_{\text{max}}$ | $G_{\text{max}}^+$ | $G_{\text{min}}$ | Bianchi type |
|---------|------------------|-------------------|-----------------|-------------|
| $E^3$   | IO(3)            | ISO(3)            | $\mathbb{R}^3(=I)$ | I           |
|         |                  |                   | $\text{VII}_0(\pm)$ |             |
| $S^3$   | O(4)             | SO(4)             | SU(2)(= IX)      | IX          |
| $H^3$   | $O_+(3,1)$       | PSL$_2^\mathbb{C}$ | $G_{\text{max}}^+$ | V, VII$_h(h \neq 0)$ |
| $Nil$   | $Nil \times O(2)$ | $G_{\text{max}}$ | $Nil(=\text{II})$ | II          |
| $Sol$   | $Sol \times D_4$ | $Sol \times D_2$ | $Sol(=\text{VI}_0)$ | VI$_0$     |
| $\tilde{SL}_2\mathbb{R}$ | $\tilde{SL}_2\mathbb{R} \times O(2)$ | $G_{\text{max}}$ | $\tilde{SL}_2\mathbb{R}(=\text{VIII})$ | VIII       |
| $H^2 \times E^1$ | $O_+(2,1) \times IO(1)$ | $(PSL_2\mathbb{R} \times \mathbb{R}) \times \mathbb{Z}_2$ | PSL$_2\mathbb{R} \times \mathbb{R}$ | III        |
| $S^2 \times E^1$ | $O(3) \times IO(1)$ | $(SO(3) \times \mathbb{R}) \times \mathbb{Z}_2$ | SO(3) $\times \mathbb{R}$ | Kantowski-Sachs models |

At present, topologies of generic compact and closed 3-manifolds are not classified yet, but those which admit locally homogeneous metrics are well classified. According to Thurston\cite{thurston1980geometry,thurston1982three,thurston1984geometry}, such a 3-manifold $\Sigma$ is specified by a maximal geometry $(\tilde{\Sigma}, G_{\text{max}})$ on its universal covering $\tilde{\Sigma}$ and a discrete subgroup $K$ of $G_{\text{max}}$ isomorphic to the fundamental group $\pi_1(\Sigma)$. Here, $G_{\text{max}}$ is a maximal transformation group on $\tilde{\Sigma}$ among all possible isometry groups of metrics on $\tilde{\Sigma}$. There exist only 8 maximal geometries for which the quotient $\tilde{\Sigma}/K$ is a smooth compact closed 3-manifold for some $K \subset G_{\text{max}}$: $E^3$, $S^3$, $H^3$, $Nil$, $Sol$, $H^2 \times \mathbb{R}$, $\tilde{SL}_2\mathbb{R}$ and $S^2 \times \mathbb{R}$ (see Table I). Possible choices of the discrete subgroup $K$ are completely determined for all Thurston types except for $H^3$\cite{thurston1984geometry}. Their explicit expressions for the Thurston
types $E^3$, $Nil$ and $Sol$ were given in Paper I and those for $H^2 \times \mathbb{R}$ and $\widetilde{SL}_2\mathbb{R}$ will be given in §6.2.1 and §7.2.1.

The invariance group $G$ of a locally homogeneous system contains some Bianchi group by definition, and at the same time it is a subgroup of one of the 8 Thurston type maximal symmetry groups. The latter maximal geometry is uniquely determined by the topology of the space. However, the invariance group does not uniquely determine the Thurston type. Hence the correspondence between the Thurston types and the Bianchi types is not one-to-one as shown in Table I. That is, the Thurston type $E^3$ contains the Bianchi types I and VII$_0$ as subgeometries and the Thurston type $H^3$ contains the Bianchi types V and VII$_h (h \neq 0)$. Further, the Bianchi type III belongs to the two Thurston types $H^2 \times \mathbb{R}$ and $\widetilde{SL}_2\mathbb{R}$. The former degeneracy in the correspondence implies that locally homogeneous systems with different Bianchi symmetries can be implemented on spacetimes with the same space topology. The same feature appears in non-compact Bianchi models, in which the eight Bianchi types, I $\sim$ VIII, are realized on the same space $\mathbb{R}^3$. Compared with this non-compact case, the degeneracy in the correspondence between the topology and the Bianchi type for the compact case is rather small. In other words, the space topology strongly restricts the local symmetry of a system in compact Bianchi models. On the other hand, the degeneracy in the correspondence concerning the Bianchi type III is a new feature for compact Bianchi models, and indicates the possibility that two spacetimes with different space topologies may have the same local symmetry. However, since such systems always have symmetries larger than the Bianchi III symmetry as will be shown in §6.1.4, such possibility is not realized.

Another important feature in the correspondence between the Thurston types and the Bianchi types is the fact that the Bianchi groups IV and VI$_h$ do not belong to any Thurston type as shown in Table I. This implies that the Bianchi types IV and VI$_h$ cannot be compactified. From a geometrical viewpoint, this follows from the fact that each of IV and VI$_h$ is really the maximal connected subgroup of a maximal symmetry, but any transformation preserving this symmetry group leaves invariant a vector field with non-vanishing divergence (see the discussion in §6.1.4 on this special vector). For a similar reason, compact Bianchi models with the type V or VII$_h (h \neq 0)$ symmetry are always isotropic, hence their spaces are quotients of the constant curvature space $H^3$. Further, such space has no moduli freedom due to Mostow’s rigidity theorem$^{9, 3, 4}$. Hence the phase space of a system on such a space is trivial. Spaces obtained as quotients of $S^3$ do not have the moduli freedom either, although they allow local anisotropy. The phase space of a system on such a space coincides with that for its universal covering.$^{*}$

From these observations we see that the phase space of a compact Bianchi model has a non-trivial structure different from that of its universal covering only for the

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$^{*}$ A compact quotient of $S^2 \times \mathbb{R}$ does not have the moduli freedom in our definition given in §2.2. This definition is different from that adopted in Ref. 11). In their definition, compact quotients of $S^2 \times \mathbb{R}$ have non-trivial moduli degree of freedom. Although it is stated in Ref. 11) that $S^2 \times S^1$ has two moduli degrees of freedom in their definition, it is not not correct. The correct number is one.
Bianchi types I, II, III, VI₀, VII₀ and VIII, or equivalently for the Thurston types $E^3$, Nil, Sol, $H^2 \times \mathbb{R}$ and $\tilde{SL}_2 \mathbb{R}$.

2.2. Invariant phase space

Let $(M, g, \cdots)$ be a compact Bianchi model with matter satisfying the Einstein equations. Then its universal covering $(\tilde{M}, \tilde{g}, \cdots)$ is invariant under some transformation group $G$ whose orbits are spatial hypersurfaces $\tilde{\Sigma}(t)$ in $\tilde{M}$ where $t$ is a label representing the time coordinate. In this paper we only consider the case in which $G$ contains some subgroup $G_s$ which acts simply transitively on each $\tilde{\Sigma}(t)$. As is well-known, we can then always find the space coordinates $\mathbf{x}$ of $\tilde{M}$ such that the action of $G$ is represented by transformations of the $\mathbf{x}$ coordinates independent of $t$. In other words, $\tilde{M}$ is written as $\tilde{\Sigma} \times \mathbb{R}$, where $\tilde{\Sigma} \approx \tilde{\Sigma}(t)$, and the action of $G$ on $\tilde{M}$ is induced from that on $\tilde{\Sigma}$. Here, the original spacetime $M$ is represented as the quotient $\tilde{M}/K$ of $\tilde{M}$ by a discrete subgroup $K$ of $G$. Hence $M$ is written as $\Sigma \times \mathbb{R}$ with $\Sigma = \tilde{\Sigma}/K$.

Since we are considering a space-orientable spacetime, $\Sigma$ is assumed to be an orientable 3-manifold, and $G$ is contained in the orientation-preserving diffeomorphism group of $\Sigma$, Diff$^+(\Sigma)$.

In terms of an appropriate set of variables on $\tilde{\Sigma}$, $X(t)$, constructed from the values of the metric, the matter variables and their time derivatives on $\Sigma(t)$, the Einstein equations are represented as a set of evolution equations, first-order in time, supplemented with the Hamiltonian and the momentum constraints. Since $X(t)$ is invariant under $G_s$, its expansion coefficients in terms of the invariant basis with respect to $G_s$ are functions only of $t$, and the Einstein equations reduce to a set of first-order ordinary differential equations for them. Hence, the whole solution on $\tilde{M}$ is uniquely determined by the initial data for $X$ at some time $t_0$. Therefore, when the pair $(X, K)$ of a $G$-invariant initial data set $X$ on $\tilde{\Sigma}$ satisfying the constraints and a discrete subgroup $K$ of $G$ such that $\tilde{\Sigma}/K \approx \Sigma$ is given, a locally $G$-homogeneous solution to the Einstein equations on $M$ is uniquely specified. This specification, however, is not complete because two solutions connected by a diffeomorphism should be identified physically in general relativity.

As discussed in Paper I, when the topology of $\Sigma$ is given, a maximal geometry $(\tilde{\Sigma}, G_{\text{max}})$ giving $\Sigma$ as a compact quotient is uniquely determined up to diffeomorphism, and when a standard realization of the maximal geometry is fixed, the invariance group $G$ can be always chosen as a subgroup of the fixed maximal symmetry group $G_{\text{max}}$ by choosing an appropriate projection from $\tilde{\Sigma}$ to $\Sigma$. Further, if $G'$ is a subgroup of $G_{\text{max}}$ which is conjugate to $G$ in Diff$^+(\tilde{\Sigma})$, i.e., if there is a diffeomorphism $f \in $Diff$^+(\tilde{\Sigma})$ such that $G' = f G f^{-1}$, a $G'$-invariant system and a $G$-invariant system are equivalent. Hence, we can fix $G$ to one representative group in its conjugate class.

Under this setting, we only have to consider diffeomorphisms preserving the invariance group $G$ and the decomposition of the spacetime $\tilde{M}$ into space and time $\tilde{\Sigma} \times \mathbb{R}$ in the identification of solutions. They are further classified into two types. First one is the one-parameter family of diffeomorphisms corresponding to time translations. Since the evolution equations for $X$ are first-order in time and autonomous, time translations are represented as a one-dimensional transformation group in the
initial data space. Second one is the diffeomorphisms which preserve each \( G \)-orbit, i.e., \( \tilde{\Sigma}(t) \). Such diffeomorphisms are induced from \( f \)'s on \( \tilde{\Sigma} \) such that \( fGf^{-1} = G \).
We call such a transformation a homogeneity-group preserving diffeomorphism or simply a HPD of \( G \). All HPDs of \( G \) form a group \( \mathcal{N}(G) \) called the normalizer of \( G \), whose subgroup consisting of orientation-preserving HPDs is denoted by \( \mathcal{N}^+(G) \).
In the initial data space, each HPD \( f \) acts on \( (X,K) \) as \( (f_*X,fKf^{-1}) \). We call the space obtained from the space of the initial data \( (X,K) \) by the identification in terms of HPDs the diffeomorphism-invariant phase space for the system \((\Sigma,G)\), and denote it by \( \Gamma_{\text{inv}}(\Sigma,G) \).
Here, we only impose the diffeomorphism constraints on \( X \) to construct \( \Gamma_{\text{inv}}(\Sigma,G) \) as in Paper I, because the canonical structure always becomes degenerate in the subspace determined by the Hamiltonian constraint if the canonical description is possible as in the vacuum case. This degeneracy is removed only by imposing a gauge-condition on the time coordinate. Hence, \( \Gamma_{\text{inv}}(\Sigma,G) \) is expressed as
\[
\Gamma_{\text{inv}}(\Sigma,G) = \left( \Gamma_D(\tilde{\Sigma},G) \times \mathcal{M}(\Sigma,G) \right) / \mathcal{N}^+(G),
\]
where \( \Gamma_D(\tilde{\Sigma},G) \) is the set of \( G \)-invariant covering data \( X \) on \( \tilde{\Sigma} \) satisfying the diffeomorphism constraints, and \( \mathcal{M}(\Sigma,G) \) is the set of subgroups \( K \) of \( \tilde{\Sigma} \) such that \( \tilde{\Sigma}/K \approx \Sigma^* \). In order to determine the dimension of the space of physically distinct solutions, \( N_s \), we must subtract two from the dimension of \( \Gamma(\tilde{\Sigma},G) \), taking into account the identification by time translations and the Hamiltonian constraint:
\[
N_s(\Sigma,G) = \dim \Gamma_{\text{inv}}(\Sigma,G) - 2.
\]

As mentioned above, the homogeneous data set \( X \) on \( \tilde{\Sigma} \) is represented by a set of components in an invariant basis \( \chi^I (I = 1, 2, 3) \) with respect to a Bianchi group \( G_s \). For fluid systems considered in the present paper, they are given by the matrix representing the space metric \( Q = (Q_{IJ}; 1 \leq I, J \leq 3) \), its conjugate momentum \( P = (P^I_J; 1 \leq I, J \leq 3) \), and the spatial velocity \( u_I (1 \leq I \leq 3) \) and the energy density \( \rho \) of the fluid. For the invariant basis satisfying the Mauer-Cartan equation
\[
d\chi^I = -\frac{1}{2} C^I_{JK} \chi^J \wedge \chi^K,
\]
the momentum constraints are expressed in terms of these variables as
\[
H_I \equiv 2C_{IJ}P^J_I + 2C^K_{IJ}P^K_J + cu_I = 0,
\]
where \( C_I = C^J_{IJ}, P^I_J = P^K_{JK}Q_{KJ} \), and \( c \) is expressed as \( c_0(\rho + p)u_0 \) in terms of a constant \( c_0 \) and the total energy density \( \rho \), the total pressure \( p \) and the time component of the center of mass 4-velocity \( u_0 \) of the fluid.

\(^*\) Some of these notations are different from those adopted in Paper I. There, \( \Gamma_{\text{inv}}(\Sigma,G), \Gamma_D(\tilde{\Sigma},G) \) and \( \mathcal{N}^+(G) \) were denoted by \( \Gamma_{\text{inv}}^{(\text{Li})}(\Sigma,G), \Gamma_D^{(\text{Li})}(\tilde{\Sigma},G) \) and \( \text{HPDG}^{(\text{Li})}(\tilde{\Sigma},G) \), respectively. Further, the symmetry group \( G \) of a system was regarded as an abstract group, and \( G \) and its realization \( \tilde{G} \) as a transformation group on \( \tilde{\Sigma} \) were distinguished in Paper I. We do not do such distinction in the present paper, and regard \( \tilde{G} \) as the invariance group of the locally homogeneous system, denoting it simply as \( G \).
When we construct the moduli space $\mathcal{M}(\Sigma,G)$ in the following sections, we start from the set of monomorphisms $\psi : \pi_1(\Sigma) \rightarrow G$ whose image $K$ is a discrete subgroup consisting of transformations without a fixed point, $\text{Mono}_*(\pi_1(\Sigma),G)$. Then, since $\psi_1, \psi_2 \in \text{Mono}_*(\pi_1(\Sigma),G)$ give the same $K$ if and only if they are related by an automorphism of $\pi_1(M)$, i.e., a modular transformation, $\mathcal{M}(\Sigma,G)$ is given by

$$\mathcal{M}(\Sigma,G) = \text{Mono}_*(\pi_1(\Sigma),G)/\text{Modular transformations}. \quad (2.5)$$

Further, in order to parametrize the invariant phase space, we select a subspace of $\text{Mono}_*(\pi_1(\Sigma),G)$, called the reduced moduli space or the moduli sector of the invariant phase space, which intersects with each $\mathcal{N}^+(G)$-orbit at a single point, and denote it by $\mathcal{M}_0(\Sigma,G)$. Let $\mathcal{N}_{\Sigma 0}(G)$ be the isotropy group of the action of $\mathcal{N}(G)$ on $\mathcal{M}_0(\Sigma,G)$, which consists of transformations in $\mathcal{N}^+(G)$ induced from those in the maximal connected subgroup of $\text{Diff}^+(\Sigma)$. Then, the invariant phase space is expressed as

$$\Gamma_{\text{inv}}(\Sigma,G) = (\Gamma_{\text{dyn}}(\Sigma,G) \times \mathcal{M}_0(\Sigma,G)) / H_{\text{mod}}, \quad (2.6)$$

where

$$\Gamma_{\text{dyn}}(\Sigma,G) = \Gamma_D(\tilde{\Sigma},G)/\mathcal{N}_{\Sigma 0}(G), \quad (2.7)$$

and $H_{\text{mod}}$ is the discrete group consisting of the combinations of HPDs and modular transformations which map $\mathcal{M}_0(\Sigma,G)$ onto itself. We call $\Gamma_{\text{dyn}}(\Sigma,G)$ the dynamical sector of the invariant phase space because the variables describing the moduli sector $\mathcal{M}_0(\Sigma,G)$ become constants of motion (see Theorem 2.3 and the argument following it in Paper I). In the present paper, when $(\Gamma_{\text{dyn}} \times \mathcal{M}_0)/H_{\text{mod}}$ is isomorphic to $\Gamma_{\text{dyn}} \times (\mathcal{M}_0/H_{\text{mod}})$, we write $\mathcal{M}_0(\Sigma,G)/H_{\text{mod}}$ simply as $\mathcal{M}_0(\Sigma,G)$. As we will see in the following sections, $\mathcal{N}_{\Sigma 0}(G)$ becomes trivial in many cases for an appropriate choice of $\mathcal{M}_0(\Sigma,G)$.

To summarize, the invariant phase spaces for compact Bianchi models whose space $\Sigma$ is modelled on a maximal geometry $(\tilde{\Sigma},G_{\text{max}})$ is determined by the following steps:

1. Determine $\text{Mono}_*(\pi_1(\Sigma),G_{\text{max}})$.
2. Make the list of all conjugate classes of possible invariance groups which contain a simply transitive subgroup and are contained in $G_{\text{max}}$, and select a representative element $G$ from each conjugate class.
3. Determine the normalizer $\mathcal{N}^+(G)$ of each representative invariance group $G$.
4. Pick up an appropriate reduced moduli space $\mathcal{M}_0(\Sigma,G)$ and determine the isotropy group $\mathcal{N}_{\Sigma 0}(G)$ and the discrete transformation group $H_{\text{mod}}$ consisting of modular+HPD transformations preserving $\mathcal{M}_0(\Sigma,G)$.
5. Construct the dynamical sector $\Gamma_{\text{dyn}}(\Sigma,G)$ taking account of the momentum constraints.

Among these steps, the most cumbersome ones are the second and the third. In Appendix A, we have given some propositions which are useful in executing these tasks.

Finally we give some comments on notations. In all of the cases considered in the present paper, the simply connected space $\tilde{\Sigma}$ of each maximal geometry can be
identified with some group $G_s$ which acts simply transitively on it. Under this identification, the group structure of $G_s$ naturally defines two types of transformations on $\tilde{\Sigma}$. First one is induced from the left multiplication of a fixed element $a$ in $G_s$:

$$L_a : G_s \ni g \mapsto ag \in G_s,$$

which we call a left transformation. Second one is a right transformation induced from the right multiplication

$$R_a : G_s \ni g \mapsto ga \in G_s.$$

$L_a$ and $R_b$ commute with each other for any $a, b \in G_s$. We denote the group of left transformations and that of right transformations by attaching the subscripts $L$ and $R$ to the group symbol as $G_L$ and $G_R$, respectively. These two transformation groups are isomorphic to the original group $G_s$. Throughout the paper, we regard $G_s$ itself as the left transformation group, and we omit the subscript $L$ when the distinction is not important.

§3. $E^3$

3.1. Maximal geometry

The maximal geometry $E^3$ is given by the isometry group of the standard 3-dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2,$$

that is $IO(3)$ on $\mathbb{R}^3$. Its orientation preserving part is given by $ISO(3)$, and its Lie algebra is generated by

$$T_i = \partial_i, \quad J_i = \epsilon_{ijk}x^j\partial_k$$

with the commutation relations

$$[\mathbf{a} \cdot T, \mathbf{b} \cdot T] = 0,$$

$$[\mathbf{a} \cdot J, \mathbf{b} \cdot T] = -(\mathbf{a} \times \mathbf{b}) \cdot T,$$

$$[\mathbf{a} \cdot J, \mathbf{b} \cdot J] = -(\mathbf{a} \times \mathbf{b}) \cdot J.$$

3.2. Invariance groups

As is shown in Table I, $IO(3)$ contains the two simply transitive groups of the Bianchi types I and VII$_0$, up to conjugations with respect to $\mathcal{N}(IO(3))$.

The Bianchi type I subgroup $\mathbb{R}^3$ is generated by $\xi_i = T_i(i = 1, 2, 3)$ and the invariant basis is given by

$$\chi^1 = dx, \quad \chi^2 = dy, \quad \chi^3 = dz;$$

$$d\chi^1 = 0, \quad d\chi^2 = 0, \quad d\chi^3 = 0.$$

For the translation $L_a : x \mapsto x + a(a \in \mathbb{R}^3)$, the condition $fLaf^{-1} \in \mathbb{R}^3$ is equivalent to the condition $f(x + a) = f(x) + b$ for any $x$ and $a$, where $b$ depends only
on $a$. From this it follows that $f \in \mathcal{N}(\mathbb{R}^3)$ is given by the linear transformation $f(x) = Ax + a \in IGL(3)$, under which the invariant basis transforms as

$$f^*\chi^i = A^i_j\chi^j. \quad (3.6)$$

As was shown in Paper I, any invariance group $G$ whose maximal connected subgroup $G_0$ is given by $\mathbb{R}^3$ is conjugate to a semi-direct product of $\mathbb{R}^3$ and a subgroup of $D_4 = \{\pm 1, \pm I_1, \pm I_2, \pm I_3\}$. Further, $I_1$ and $I_2$ are conjugate to $I_3$ by rotations. Hence, conjugate classes of $G$ other than $\mathbb{R}^3$ are represented by the semi-direct product of $\mathbb{R}^3$ with $\{1, -1\}$, $\{1, I_3\}$, $\{\pm 1, \pm I_3\}$, $D_2$ and $D_4$, where $D_2$ is the dihedral group consisting of the orientation preserving elements of $D_4$. The normalizers of these groups are listed in Table II. In this table, o/u in the second column indicates whether $G$ consists of only orientation-preserving transformations (o) or contains orientation-reversing ones (u). Note that all invariance groups in this table are invariant under the space reflection, hence the conjugate class of each group in $\text{Diff}^+(\Sigma)$ and that in $\text{Diff}(\Sigma)$ coincide.

In contrast, subgroups of the type $\text{VII}_0$ are not invariant under the space reflection, and if we restrict the arguments to orientation preserving transformations, the conjugate class of $\text{VII}_0$ is divided into two subclasses $\text{VII}_0^\epsilon (\epsilon = \pm)$, whose canonical generators are given by

$$\xi_1 = T_1, \quad \xi_2 = T_2, \quad \xi_3 = -\epsilon T_3 - J_3, \quad (3.7)$$

\[ [\xi_1, \xi_2] = 0, \quad [\xi_3, \xi_1] = \xi_1, \quad [\xi_3, \xi_2] = -\xi_2. \quad (3.8) \]
where $\hat{N}$preserving transformations:

$$
\hat{\text{IO}}_L
$$

where

$$
\begin{aligned}
\text{Under this transformation, the invariant basis (3.10) transforms as}

\hat{b} = (\hat{0}, \hat{0})

\text{is expressed as}

\begin{align*}
\text{normalizers are the subgroups of } G \text{ generating by the same} \\
\text{connected subgroups, which is isomorphic to } \text{ISO} \text{ denotes as subgroups. Since } \text{ISO} \text{ contains both } \text{ISO} \text{ and VII}_0 \text{ as subgroups. Since } \text{ISO} \text{ is the only 3-dimensional Abelian subgroup of } \text{ISO}(2), \text{N(ISO}(2)) \text{ preserves } \mathbb{R}^3. \text{ Hence it consists of } f \in \mathcal{N}(\mathbb{R}^3) \text{ such}
\end{align*}

\begin{align*}
\begin{array}{cccc}
G & \mathrm{o/u} & f \in \mathcal{N}(G) & f \hat{g} f^{-1} \\
\text{VII}_0^*(0) & 0 & \{1, I_1\} f_{a,b,k,\theta} & (3.15) \\
\text{VII}_0^*(0) \times \{1, I_3\} & \hat{b} = 0 & f I_3 f^{-1} = I_3 L_c; c = -2(a^1, \pm a^2, 0) \\
\text{VII}_0^*(0) \times \{1, I_1\} & R(\theta) = \pm 1, \hat{b} = (b, 0) & f I_1 f^{-1} = I_1 L_c; c = (I_1 - R_3(-2a^3)) a \\
\text{VII}_0^*(0) \times D_2 & \theta = \frac{a^3}{2}, \hat{b} = 0 & R_3(\pi/2) I_1 R_3(\pi/2) = I_2 \\
\end{array}
\end{align*}

A natural invariant basis is given by

$$
\left( \begin{array}{c} \chi^1 \\ \chi^2 \end{array} \right) = R(\epsilon \epsilon) \left( \begin{array}{c} dx \\ dy \end{array} \right), \ \chi^3 = dz; \quad (3.9)
$$

$$
d\chi^1 = -\chi^2 \wedge \chi^3, \ d\chi^2 = \chi^2 \wedge \chi^3, \ d\chi^3 = 0. \quad (3.10)
$$

The normalizer of VII_0^* does not depend on $\epsilon$ and consists only of orientation preserving transformations: $\mathcal{N}^+(\text{VII}_0^*) = \mathcal{N}(\text{VII}_0^*) = \mathcal{N}(\text{VII}_0)$. Its generic element is expressed as

$$
f = \{1, I_1\} f_{a,b,k,\theta}; \ f_{a,b,k,\theta} := L_a R_b D(k, k, 1) R_3(\theta), \quad (3.11)
$$

where $L_a$ and $R_a$ are the transformations corresponding to the left and the right multiplications of the element $a$ in VII_0 respectively,

$$
L_a : x \mapsto R_3(\epsilon a^3) x + a, \quad (3.12)
$$

$$
R_a : x \mapsto x + R_3(\epsilon z) a, \quad (3.13)
$$

$\hat{b} = (\hat{b}, 0)$ and $D(k, k, 1)(k > 0)$ is the linear transformation given by the same diagonal matrix. Under this transformation, the invariant basis (3.10) transforms as

$$
f'^* \chi^i = F'^j \chi^j : F = \{1, I_1\} \times \left( \begin{array}{c} kR(\theta) \\ \epsilon \hat{b}^* \\ 1 \end{array} \right), \quad (3.14)
$$

where $\hat{b}^* = \{(-b^2, b^3)$. The conjugate transformation of $L_c$ is given by

$$
f L_c f^{-1} = L_c'; \\
\overrightarrow{c'} = \{1, I_1\} \left[ \left( \begin{array}{cc} kR(\theta + \epsilon a^3) & 0 \\ 0 & 1 \end{array} \right) c + (1 - R_3(\epsilon c^3)) a \right]. \quad (3.15)
$$

An invariance group $G$ generated by $G_0 = \text{VII}_0^*$ and discrete transformations in $\text{IO}(3)$ is conjugate to the semi-direct product of VII_0^* with one of $\{1, I_3\}$, $\{1, I_1\}$ and $D_2$ as discussed in Paper I. All of these groups are orientation preserving and their normalizers are the subgroups of $\mathcal{N}(\text{VII}_0^*)$ listed in Table III.

In addition to these subgroups, $\text{IO}(3)$ has one conjugate class of 4-dimensional connected subgroups, which is isomorphic to $\text{ISO}(2)$ generated by $\mathbb{R}^3$ and $R_3(\theta)$. It contains both $\mathbb{R}^3$ and VII_0^* as subgroups. Since $\mathbb{R}^3$ is the only 3-dimensional Abelian subgroup of $\text{ISO}(2)$, $\mathcal{N}(\text{ISO}(2))$ preserves $\mathbb{R}^3$. Hence it consists of $f \in \mathcal{N}(\mathbb{R}^3)$ such
The normalizers of these groups are the same and are generated by manifolds modeled on transformations.

The normalizers of all these groups coincide with $N$ is conjugate to one of the 4 groups listed in the same table. The normalizers of all these groups coincide with $N(ISO(2))$. The explicit form of $f$ is given in Table II. An invariance group $G$ with $G_0 = ISO(2)$ is generated by $ISO(2)$ and elements in $N(ISO(2))$, and is conjugate to one of the 4 groups listed in the same table. The normalizers of all these groups coincide with $N(ISO(2))$.

Finally, an invariance group $G$ with $G_0 = ISO(3)$ is given by $ISO(3)$ or $IO(3)$. The normalizers of these groups are the same and are generated by $IO(3)$ and dilation transformations.

3.3. Phase space

There exist six diffeomorphism classes for the orientable compact closed 3-manifolds modeled on $E^3$ as listed in Table IV, where the fundamental group and its embedding into $Isom^+(E^3)$ of each class are given. Although it is more natural

| Space | Fundamental group and representation |
|-------|-------------------------------------|
| $T^3$ | $< \alpha, \beta, \gamma | [\alpha, \beta] = 1, [\beta, \gamma] = 1, [\gamma, \alpha] = 1 >$ |
|       | $\alpha = (a, 1), \beta = (b, 1), \gamma = (c, 1);$ |
|       | $(a, b, c) \in GL(3, \mathbb{R})$. |
| $T^3/\mathbb{Z}_2$ | $< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} \alpha = 1, \gamma \beta \gamma^{-1} \beta = 1 >$ |
|       | $\alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma);$ |
|       | $R_\alpha, R_\beta$ and $R_\gamma$ are rotations of the angle $\pi$ about mutually orthogonal axes, such that $R_\beta a + R_\alpha b + R_\gamma c = 0,$ and |
|       | $(a + R_\alpha a, b + R_\beta b, c + R_\gamma c) \in GL(3, \mathbb{R})$ |
| $T^3/\mathbb{Z}_3$ | $< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \gamma^{-1} \beta^{-1} >$ |
|       | $\alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma);$ |
|       | $R_\gamma = R_{a \times b}(\frac{2\pi}{3})$ |
|       | $b = R_\gamma a, (a, b, c) \in GL(3, \mathbb{R})$. |
| $T^3/\mathbb{Z}_4$ | $< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta^{-1}, \gamma \beta \gamma^{-1} = \alpha >$ |
|       | $\alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma);$ |
|       | $R_\gamma = R_{a \times b}(\frac{\pi}{3})$ |
|       | $b = R_\gamma a, (a, b, c) \in GL(3, \mathbb{R})$. |
| $T^3/\mathbb{Z}_6$ | $< \alpha, \beta, \gamma | [\alpha, \beta] = 1, \gamma \alpha \gamma^{-1} = \beta, \gamma \beta \gamma^{-1} = \alpha^{-1} \beta >$ |
|       | $\alpha = (a, 1), \beta = (b, 1), \gamma = (c, R_\gamma);$ |
|       | $R_\gamma = R_{a \times b}(\frac{\pi}{3})$ |
|       | $b = R_\gamma a, (a, b, c) \in GL(3, \mathbb{R})$. |
from the geometrical point of view to discuss possible symmetries of the system and the structure of phase space for each space topology, we instead examine possible topologies and the structure of phase space for each invariance group as in Paper I, because it makes arguments simpler. In this section, we call the matrix \( K = (a \ b \ c) \) consisting of the translational vectors associated with the three generators of the fundamental group in Table IV the moduli matrix, because the moduli sector of the invariant phase space is uniquely parametrized by that matrix after putting it into some canonical form by HPDs.

### 3.3.1. \( G \supset \mathbb{R}^3 \)

For an invariance group \( G \) containing \( \mathbb{R}^3 \), (3.4) is the most natural invariant basis to represent homogeneous covering data on \( \mathbb{R}^3 \). Since all the structure constants vanish for this basis, the momentum constraints for the single-component perfect fluid system are expressed as

\[
H_I \equiv cu_I = 0. \tag{3.16}
\]

Hence, the fluid velocity is always orthogonal to the constant-time slices, and the phase space for the system is simply given by adding the energy density \( \rho \) to that for the vacuum system discussed in Paper I. In particular, \( G \) always contains \( \mathbb{R}^3 \rtimes D_2 \) as a subgroup.

In contrast, for a multi-component system, the momentum constraints only restrict the center-of-mass velocity, and the 4-velocity of each component can be tilted. In this case, the system can have the lower symmetries, \( \mathbb{R}^3 \) and \( \mathbb{R}^3 \rtimes \{1, I_3\} \). However, the number of dynamical degrees of freedom in the gravitational sector for such cases is the same as that for \( G = \mathbb{R}^3 \rtimes D_2 \).

In fact, for \( G = \mathbb{R}^3 \), only \( T^3 \) is allowed as the space topology. In this case, since \( N(\mathbb{R}^3) = IGL(3) \), the moduli matrix \( K = (a \ b \ c) \) can be put to the unit matrix by a HPD and a modular transformation, and the isotropy group \( N_{\Sigma 0}(\mathbb{R}^3) \) of the action of \( N^+(\mathbb{R}^3) \) at \( K = 1 \) in the moduli space is trivial. Hence \( N_Q = N_P = 6 \) and \( N_m = 0 \), and the total parameter count \( N_s \) is given by \( 10 + 3(n_f - 1) + n_f = 7 + 4n_f \), where \( n_f \) is the number of fluid components. On the other hand, for \( G = \mathbb{R}^3 \rtimes \{1, I_3\}, T^3 \) and \( T^3/\mathbb{Z}_2 \) are allowed. Now, from Table II, \( K \) for \( T^3 \) can be put to the form

\[
K = \begin{pmatrix}
1 & 0 & c^1 \\
0 & 1 & c^2 \\
a^3 & b^3 & 1
\end{pmatrix}
\tag{3.17}
\]

by a HPD and a modular transformation, for which \( N_{\Sigma 0}(\mathbb{R}^3) \) is trivial. On the other hand, by the symmetry, \( Q_{13} = Q_{23} = P_{13} = P_{23} = 0 \) and the second and the third components of all fluid velocities vanish. Hence \( N_Q = N_P = N_m = 4 \) and \( N_s = 10 + (n_f - 1) + n_f = 9 + 2n_f \). The argument for \( T^3/\mathbb{Z}_2 \) is the same except that \( K \) can be put to the unit matrix, hence \( N_Q = N_P = N_m = 0 \) and \( N_s = 5 + 2n_f \).

Finally, note that for higher symmetries for which \( G \supset \mathbb{R}^3 \rtimes D_2 \), the 4-velocity of every component must be orthogonal to the constant time slices and the dynamical degrees of freedom in the fluid sector are simply given by the energy density of each component. Hence \( N_s \) is obtained by adding the number of components to the corresponding vacuum value.
Table V. The parameter counts for type $E^3$.

| Space | Symmetry | $N_Q$ | $N_P$ | $N_m$ | $N_f$ | $N$ | $N_s$ | $N_s$(vacuum) |
|-------|-----------|-------|-------|-------|-------|-----|-------|----------------|
| $\mathbb{R}^3$ | $\mathbb{R}^3 \rtimes D_2$ | 0 | 3 | 0 | 1 | 4 | 2 | 1 |
| | VII$_0$ | 2 | 3 | 0 | 4 | 9 | 7 | - |
| | VII$_0 \rtimes Z_2$ | 2 | 3 | 0 | 2 | 7 | 5 | - |
| | VII$_0 \rtimes D_2$ | 2 | 3 | 0 | 1 | 6 | 4 | 3 |
| | ISO(2) $\times \{1, I_1\}$ | 0 | 2 | 0 | 1 | 3 | 1 | 0 |
| | ISO(3) | 0 | 1 | 0 | 1 | 2 | 0 | 0 |
| $T^3$ | $\mathbb{R}^3 \rtimes D_2$ | 3 | 3 | 6 | 1 | 13 | 11 | 10 |
| | VII$_0$ | 3 | 3 | 4 | 4 | 14 | 12 | - |
| | VII$_0 \rtimes Z_2$ | 3 | 3 | 4 | 2 | 12 | 10 | - |
| | VII$_0 \rtimes D_2$ | 3 | 3 | 4 | 1 | 11 | 9 | 8 |
| | ISO(2) $\times \{1, I_1\}$ | 2 | 2 | 4 | 1 | 9 | 7 | 6 |
| | ISO(3) | 1 | 1 | 5 | 1 | 8 | 6 | 5 |
| $T^3/\mathbb{Z}_2$ | $\mathbb{R}^3 \rtimes D_2$ | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | VII$_0$ | 3 | 3 | 2 | 4 | 12 | 10 | - |
| | VII$_0 \rtimes Z_2$ | 3 | 3 | 2 | 2 | 10 | 8 | - |
| | VII$_0 \rtimes D_2$ | 3 | 3 | 2 | 1 | 9 | 7 | 6 |
| | ISO(2) $\times \{1, I_1\}$ | 2 | 2 | 3 | 1 | 8 | 6 | 5 |
| | ISO(3) | 1 | 1 | 3 | 1 | 6 | 4 | 3 |
| $T^3/\mathbb{Z}_2 \times Z_2$ | $\mathbb{R}^3 \rtimes D_2$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| | VII$_0 \times \{1, I_1\}$ | 3 | 3 | 1 | 2 | 9 | 7 | - |
| | VII$_0 \rtimes D_2$ | 3 | 3 | 1 | 1 | 8 | 6 | 5 |
| | ISO(2) $\times \{1, I_1\}$ | 2 | 2 | 1 | 1 | 6 | 4 | 3 |
| | ISO(3) | 1 | 1 | 2 | 1 | 5 | 3 | 2 |
| $T^3/\mathbb{Z}_k(k = 3, 4, 6)$ | VII$_0$ | 3 | 3 | 0 | 4 | 10 | 8 | - |
| | VII$_0 \rtimes Z_2$ | 3 | 3 | 0 | 2 | 8 | 6 | - |
| | VII$_0 \rtimes D_2$ | 3 | 3 | 0 | 1 | 7 | 5 | 4 |
| | ISO(2) $\times \{1, I_1\}$ | 2 | 2 | 0 | 1 | 5 | 3 | 2 |
| | ISO(3) | 1 | 1 | 1 | 1 | 4 | 2 | 1 |

3.3.2. $G_0 = \text{VII}_0$

For $G_0 = \text{VII}_0$, the momentum constraint with respect to the invariant basis (3.10) is given by

\[
H_1 \equiv 2P_2^0 + cu_1 = 0, \ H_2 \equiv -2P_1^3 + cu_2 = 0, \ H_3 \equiv 2(P_1^2 - P_2^1) + cu_3 = 0. \ (3.18)
\]

For the vacuum system, it follows from this constraint that the system always has the higher symmetry VII$_0 \rtimes D_2$, but for the fluid system the lower symmetries listed in Table III are all allowed.

A. $G = \text{VII}_0$

Since the left transformation $L_d$ is a linear transformation $(d, R_3(ed^3)) \in IGL(3)$, $\pi_1(\Sigma)$ can be embedded into VII$_0$ only for $\Sigma = T^3$ or $T^3/\mathbb{Z}_k(k = 2, 3, 4, 6)$ from Table
IV.  

A-i) $T^3$: $L_d$ becomes a pure translation only when $d^3/2\pi$ is an integer. Hence the moduli matrix $K = (a\ b\ c)$ has the form

$$K = \begin{pmatrix} \hat{a} & \hat{b} & \hat{c} \\ 2\pi l & 2\pi m & 2\pi n \end{pmatrix}$$  \hspace{1cm} (3.19)

where $\hat{a}, \hat{b}$ and $\hat{c}$ are two-component vectors, and $l, m$ and $n$ are integers. By a modular transformation $K \mapsto KZ$ with $Z \in GL(3, \mathbb{Z})$, we can always put $l = m = 0$ and $n > 0$. Further, from (3.15), $f = f_{a,b,k,\theta} \in \mathcal{N}(\text{VII}_0)$ transforms $L_d$ with $d = (\hat{d}, 2\pi p)(p \in \mathbb{Z})$ to $L_{d'}$ with $d' = kR(\theta + ea^3)\hat{d}$. Hence $K$ can be put by a HPD to

$$K = \begin{pmatrix} 1 & X & V \\ 0 & Y & W \\ 0 & 0 & 2\pi n \end{pmatrix},$$  \hspace{1cm} (3.20)

where $Y > 0$. Let us denote the set of $K$ with this form by $\mathcal{M}_1$. Then $\mathcal{M}_1$ intersects with each HPD orbit in the total space of the moduli matrix at a single point, and the isotropy group $\mathcal{N}_{T^30}(\text{VII}_0)$ at $K$ consists of $f_{a,b,k,\theta}$ with $k = 1$ and $R(\theta + ea^3) = 1$.

In the present case, a modular transformation represented by the matrix

$$Z = \begin{pmatrix} \hat{Z} & p \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (3.21)

maps $\mathcal{M}_1$ onto $\mathcal{M}_1$ when it is combined with an appropriate HPD, where $\hat{Z} \in SL(2, \mathbb{Z})$ and $p$ is a 2-vector with integer components. Hence $H_{\text{mod}}$ in (2.6) is isomorphic to $SL(2, \mathbb{Z}) \times \mathbb{Z}^2$. Although the HPDs associated with each transformation in $H_{\text{mod}}$ have rather complicated structures, their effective action on the invariant basis is simply given by the product of a $D(k, k, 1)$ transformation with $k > 0$ and a transformation induced from $f$ in $\mathcal{N}_{T^30}(\text{VII}_0)$. Further, although the action of some transformations in $H_{\text{mod}}$ on $\mathcal{M}_1$ have fixed points, the isotropy group at each fixed point induces transformations of the invariant basis with $k = 1$. Hence, taking into account that a bundle with the structure group $\mathbb{R}_+$ (the multiplicative group of positive numbers) is always trivial, the invariant phase space is simply given by the product $\Gamma_{\text{dyn}} \times \mathcal{M}_0$ with $\mathcal{M}_0 = \mathcal{M}_1/H_{\text{mod}}$.

As mentioned above, the action of $H_{\text{mod}}$ on $\mathcal{M}_1$ has non-trivial isotropy group $H_K$ at discrete points. Although there are infinite number of such points, they project to 6 points on $\mathcal{M}_0$ by the identification by the action of $H_{\text{mod}}$:

i) $(X, Y, V, W) = (0,1,1/2,0)$: $H_K = \mathbb{Z}_2; \hat{Z} = -1, p = (-1,0)$.

ii) $(X, Y, V, W) = (0,1,0,0)$: $H_K = \mathbb{Z}_4; \hat{Z} = R(\pi/2), p = (0,0)$.

iii) $(X, Y, V, W) = (0,1,1/2,1/2)$: $H_K = \mathbb{Z}_4; \hat{Z} = R(\pi/2), p = (0,1)$.

iv) $(X, Y, V, W) = (1/2, \sqrt{3}/2, 1/2, 0)$: $H_K = \mathbb{Z}_2; \hat{Z} = -1, p = (-1,0)$.

v) $(X, Y, V, W) = (1/2, \sqrt{3}/2, 1/2, 1/2\sqrt{3})$: $H_K = \mathbb{Z}_3; \hat{Z} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, p = (-1,0)$.

vi) $(X, Y, V, W) = (1/2, \sqrt{3}/2, 0, 0)$: $H_K = \mathbb{Z}_6; \hat{Z} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, p = (0,0)$.  

Here \((\hat{Z}, p)\) represents the parameter of \(Z\) generating \(H_K\). The reduced moduli space has the orbifold singularities of the type specified by the isotropy groups at these fixed points. Except for these points, it is smooth and given by

\[
\mathcal{M}_0(T^3, \Pi_0) \approx \mathbb{N} \times \mathcal{M}(T^2) \times T^2, \tag{3.22}
\]

where \(\mathbb{N}\) is the set of positive integers, and \(\mathcal{M}(T^2)\) is the standard moduli space of the flat 2-torus.

As mentioned above, the moduli sector is invariant under the HPD \(f = f_{a, b, k, \theta} \in \mathcal{N}(\Pi_0)\) with \(k = 1\) and \(a^2 = -e^2\). From (3.14), it is easy to see that in terms of this residual HPD we can transform the metric component matrix \(Q\) into the diagonal form \(Q = D(Q_1, Q_2, Q_3)\) with \(Q_1 \geq Q_2\). If \(Q_1 > Q_2\) after this diagonalization, the residual HPDs induce only the \(I_3 = R_3(\pi)\) transformation of the invariant basis, and the off-diagonal components of the \(P\) matrix are determined by the fluid velocity \(u_I\) through the momentum constraints (3.18). In particular, if two components of \(u_I\) vanish, the system acquires a higher symmetry. On the other hand, if \(Q_1 = Q_2\), \(u_3\) must vanish and there remains the \(R_3(\theta)\) HPD symmetry, in terms of which we can put \(P^{12} = 0\). If \(u_1 = 0\) or \(u_2 = 0\), \(P^{23} = 0\) or \(P^{13} = 0\) and the system has a higher symmetry again. Therefore, the dynamical sector of the invariant phase space is given by

\[
\Gamma_{\text{dyn}}(T^3, \Pi_0) = \{(Q_1, Q_2, Q_3; P^{11}, P^{22}, P^{33}; u_1, u_2, u_3, \rho) | Q_1 \geq Q_2 > 0,
\]

\[
Q_3 > 0, u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_2^2 \neq 0\}/\{1, I_3\}, \tag{3.23}
\]

from which we obtain the parameter count in Table V.

The extension of the argument to a multi-component fluid system is quite simple. The parameter count for \(N_s\) is simply given by adding 4 for each extra component to the value for the single component system.

**A-ii) \(T^3/\mathbb{Z}_k\):** First we consider the case \(k = 2\). In this case two generators of the fundamental group, \(\alpha\) and \(\beta\), are represented by translations \(a\) and \(b\), and the third generator \(\gamma\) is the combination of a translation \(c\) and the rotation of the angle \(\pi\) around the axis \(a \times b\). Since \(\gamma\) belongs to \(\Pi_0\), this implies that \(a\) and \(b\) are orthogonal to the \(z\)-axis and \(c^2/2\pi\) is an odd integer. Then, since \(R_3(c^2) = R_3(\pi)\), we can put \(c^1 = c^2 = 0\) by the conjugate transformation (3.15). Hence, by the same argument as in the previous case, the moduli matrix \(K\) can be put into the canonical form

\[
K = \begin{pmatrix}
1 & X & 0 \\
0 & Y & 0 \\
0 & 0 & (2n - 1)\pi
\end{pmatrix}, \tag{3.24}
\]

where \(n\) is a positive integer, and \(Y > 0\). The discrete group of transformations consisting of HPDs and modular transformations which preserve this form of \(K\) is identical to the modular transformation group of the flat torus. Further, the HPD associated with a modular transformation fixing \(K\) is given by \(f_{a, b, k, \theta}\) with \(k = 1\). Hence, by the same argument as in the previous case, the invariant phase space is written as a product of the dynamical sector and the moduli sector, and the latter
has the topological structure
\[ \mathcal{M}_0(T^3/\mathbb{Z}_2, \text{VII}_0) \approx \mathbb{N} \times \mathcal{M}(T^2). \]  
(3.25)

The argument for \( k = 3, 4, 6 \) is the same except for two points. First, \( c^3 \) takes values of the form \( 2\pi(n + \epsilon/k) \) with integer \( n \). Second, since \( b \) and \( a \) are related by \( b = R_3(2\pi/k)a \), the value of \((X,Y)\) is fixed to \((\cos(2\pi/k), \sin(2\pi/k))\). Hence, the moduli sector becomes a set of discrete points and there exists no continuous moduli freedom:
\[ \mathcal{M}_0(T^3/\mathbb{Z}_k, \text{VII}_0) = \mathbb{Z} \quad (k = 3, 4, 6). \]  
(3.26)

Since the isotropy group of the action of \( \mathcal{N}(\text{VII}_0) \) in the moduli space does not depend on the space topology, the dynamical sector of the invariant phase space is the same as that for \( \Sigma = T^3 \). Hence the parameter count \( N_\epsilon \) is simply given by replacing \( N_m \) for \( T^3 \) by \( N_m = 2 \) for \( k = 2 \) and \( N_m = 0 \) for \( k = 3, 4, 6 \). The extension of the parameter count to multi-component systems is also the same.

B. \( G = \text{VII}_0 \times \{1, I_3\} \)

Since a transformation in \( G \) is a translation or a linear transformation with a rotation around the \( z \)-axis, \( \pi_1(\Sigma) \) can be embedded into \( G \) only for \( \Sigma = T^3 \) or \( T^3/\mathbb{Z}_k \) as in the case \( G = \text{VII}_0 \).

B-i) \( T^3 \): In the present case, in addition to \( I_4 \) with \( d^3 \equiv 0(\text{mod}2\pi) \), \( I_3 L_4 \) also becomes a pure translation when \( d^3/2\pi \) is an odd integer. Hence the generic moduli matrix is given by the expression (3.19) with \( 2l, 2m \) and \( 2n \) replaced by integers \( l, m \) and \( n \), respectively. Apart from this difference, the same argument on the reduction of the moduli matrix \( K \) into the canonical form as that for \((T^3, \text{VII}_0)\) applies to the present case, because transformations in \( \mathcal{N}(\text{VII}_0 \times \{1, I_3\}) \) are simple linear transformations, and their effective action on the moduli parameter \( K \) is the same. Hence the canonical form for \( K \) is given by replacing \( 2n \) by a positive integer \( n \) in (3.20) and the moduli sector of the invariant phase space has the same topological structure as that for \((T^3, \text{VII}_0)\):
\[ \mathcal{M}_0(T^3, \text{VII}_0 \times \{1, I_3\}) \approx \mathbb{N} \times \mathcal{M}(T^2) \times T^2. \]  
(3.27)

In contrast, the dynamical sector of the phase space has a different structure due to the higher symmetry. First, \( I_3 \) invariance requires that \( Q_{13} = Q_{23} = P_{13} = P^{23} = 0 \) and \( u_1 = u_2 = 0 \). The transformations of the invariant basis induced from the residual HPDs after fixing the moduli parameter are now given by rotations \( R_3(\theta) \), by which we can diagonalize \( Q \) to \( D(Q_1, Q_2, Q_3) \) with \( Q_1 \geq Q_2 \). After this diagonalization, the momentum constraints \( H_1 = 0 \) and \( H_2 = 0 \) become trivial. If \( Q_1 = Q_2 \), we can further diagonalize \( P \). Since \( u_3 = 0 \) from the momentum constraint, this implies that the system has a higher symmetry in this case. Hence \( Q_1 > Q_2 \) and \( P^{12} \) is determined by \( u_3 \) via the momentum constraint \( H_3 = 0 \). Thus the dynamical sector of the invariant phase space is given by
\[ \Gamma_{\text{dyn}}(T^3, \text{VII}_0 \times \{1, I_3\}) = \{(Q_1, Q_2, Q_3; P_{11}, P_{22}, P_{33}; u_3, \rho)| \quad Q_1 > Q_2 > 0, Q_3 > 0, u_3 \neq 0 \}. \]  
(3.28)
B-ii) $T^3/Z_k$: As in the previous case, the argument on the moduli sector is the same as that for $(T^3/Z_k, \Sigma^0)$ except for the range of the discrete parameter $c^3$: $c^3 = n\pi (n > 0)$ for $k = 2$ and $c^3 = \pi (n = 2k)$ for $k = 3, 4, 6$ with an integer $n$. Therefore the moduli sector of the invariant phase space is the same:

$$\mathcal{M}_0(T^3/Z_k, \Sigma^0 \times \{1, I_3\}) \cong \mathcal{M}_0(T^3/Z_k, \Sigma^0).$$

(3.29)

Further, since the residual HPDs after fixing the moduli parameter does not depend on the space topology in the present case, the dynamical sector of the phase space is the same as that for $T^3$.

Since the symmetry requires $u_1 = u_2 = 0$ for any component, the parameter count for the multi-component system is simply obtained by adding 2 for each extra component of the fluid, irrespective of the topology.

C. $G = \Sigma^1 \times \{1, I_1\}$

For $G = \Sigma^1 \times \{1, I_1\}$, the fundamental group of any compact quotient of the Thurston type $E^3$ can be embedded in $G$.

C-i) $T^3$ and $T^3/Z_k$: The image of $\pi_1(\Sigma)$ is contained in $\Sigma^0$ for these spaces except for $T^3/Z_2$. Further, the group of conjugate transformations of the moduli parameter induced from $\mathcal{N}(\Sigma^1 \times \{1, I_1\})$ is the same as that for $G = \Sigma^0$. Hence for $k \neq 2$, the moduli sector of the invariant phase space is also the same as that for $G = \Sigma^0$:

$$\mathcal{M}_0(T^3/Z_k, \Sigma^0 \times \{1, I_1\}) = \mathcal{M}_0(T^3/Z_k, \Sigma^0) \quad (k = 1, 3, 4, 6).$$

(3.30)

On the other hand, for $\Sigma = T^3/Z_2$, embeddings for which the rotation axis of the generator $\gamma$ is orthogonal to the $z$-axis are allowed, because $R_1(\pi)R_3(c^3) = R_3(-c^3/2)R_1(\pi)R_3(c^3/2)$. For such an embedding, the rotation axis of $\gamma$ can be rotated to any direction orthogonal to $z$-axis by the conjugate transformation with respect to $f = L_0 \in \mathcal{N}(\Sigma^1 \times \{1, I_1\})$, because from Table III the rotation matrix associated with $fI_1f^{-1}$ is given by $I_1R_3(-2d^3)$. Hence, by an appropriate HPD, $R_\gamma$ can be transformed to $R_1(\pi)$, for which $\gamma$ is represented by $I_1L_c$ and the moduli matrix $K$ takes the form

$$K = \begin{pmatrix} 0 & 0 & c^1 \\ a^2 & b^2 & c^2 \\ 2l\pi & 2m\pi \end{pmatrix},$$

(3.31)

where $l, m$ and $n$ are integers. By a modular transformation among the generators $\alpha$ and $\beta$, $l$ can be put to zero. Further, $m$ and $c^2$ can be put to zero by the HPD $f = L_d$ with $d^3 = m\pi$. Finally, by the HPD $f = D(\pm k, \pm k, 1)$ and the modular transformations $\alpha \to \alpha^{-1}, \beta \to \beta^{-1}$ and $\gamma \to \gamma^{-1}$, we can put $K$ to

$$K = \begin{pmatrix} 0 & 0 & 1 \\ X & Y & 0 \\ 0 & 2n\pi & 0 \end{pmatrix},$$

(3.32)

where $n$ is a positive integer and $X > 0$. The combinations of modular transformations and HPDs which preserve this form of $K$ are just the modular transformations.
$Y \to Y + pX$ with $p \in \Z$. Hence, the moduli sector of the invariant phase space is given by

$$\mathcal{M}_0(T^3/\mathbb{Z}_2, \mathbb{V}_0^I \times \{1, I_1\}) \approx \mathbb{N} \times (\mathcal{M}(T^3) \cup \mathbb{R}_+ \times S^1).$$  \hspace{1cm} (3.33)

The symmetry requires that $Q_{12} = Q_{13} = P_{12} = P_{13} = 0$ and $u_2 = u_3 = 0$. The transformations of the invariant basis induced by the residual HPDs are given by $F$ in (3.14) with $kR(\theta) = 1$ and $\hat{b} = (b, 0)$. We can put $Q_{23}$ to zero and diagonalize $Q$ by these transformations. After this diagonalization, the momentum constraints $H_2 = 0$ and $H_3 = 0$ become trivial, and $P^{23}$ is determined by $u_1$ via $H_1 = 0$. Hence, the dynamical sector of the invariant phase space is given by

$$\Gamma_{\text{dyn}}(\mathbb{R}^3, \mathbb{V}_0^I \times \{1, I_1\}) = \{(Q_1, Q_2, Q_3; P^{11}, P^{22}, P^{33}, u_1, \rho)| Q_1, Q_2, Q_3 > 0, u_1 \neq 0\}. \hspace{1cm} (3.34)$$

C-ii) $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$: In this case, the generators of the fundamental group are represented by three glind rotations, $\alpha = (a, R_\alpha), \beta = (b, R_\beta)$ and $\gamma = (c, R_\gamma)$, where $R_\alpha, R_\beta$ and $R_\gamma$ are rotations of angle $\pi$ around mutually orthogonal axes, one of which has to be the $z$-axis. As explained in the argument for $T^3/\mathbb{Z}_2$, we can always set $R_\alpha = I_1, R_\beta = I_2$ and $R_\gamma = I_3$ by a HPD, for which $a^3 = 2l\pi, b^3 = (2m-1)\pi, c^3 = (2n-1)\pi$, and the relation $R_\beta a + R_\gamma b + R_\alpha c = 0$ gives the constraints

$$a^1 + b^2 = c^1, \quad b^2 + c^2 = a^1, \quad l + n = m, \hspace{1cm} (3.35)$$

and $a^1 b^2(2n-1) \neq 0$. From the formula in Table III, the conjugate transformation by $L_d \in \mathcal{N}(G)$ preserves the above form of the generators when $d^3 = p\pi$ with an integer $p$. Under this condition, $a$ and $c$ transform as

$$a \mapsto ((-1)^pa^1, (-1)^pa^2 + 2d^2, 2(l+p)\pi), \hspace{1cm} (3.36)$$

$$c \mapsto ((-1)^pc^1 + 2d^1, (-1)^pc^2 + 2d^2, (2n-1)\pi). \hspace{1cm} (3.37)$$

Hence $a^2, l$ and $c^1$ can be put to zero. Further $a^1 b^2$ can be made positive by the transformation $f = I_1$ if necessary, and then $b^1$ can be put to unity by $f = D(\pm k, \pm k, 1)$. Hence, by taking account of the above constraints, the moduli matrix $K$ can be put into the canonical form

$$K = \begin{pmatrix} X & -X & 0 \\ 0 & 1 & -1 \\ 0 & -(2n-1)\pi & (2n-1)\pi \end{pmatrix}, \hspace{1cm} (3.38)$$

where $X > 0$. Since there remains no modular transformation freedom, the moduli sector of the invariant phase space is given by

$$\mathcal{M}_0(T^3/\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{V}_0^I \times \{1, I_1\}) \approx \mathbb{Z} \times \mathbb{R}_+. \hspace{1cm} (3.39)$$

Since the isotropy group of $\mathcal{N}(G)$ at $K$ is the same as that in the previous case, the dynamical sector of the invariant phase space is given by (3.34).

As in the case of $G = \mathbb{V}_0^I \times \{1, I_3\}$, the parameter count for a multi-component system is given by adding two for each extra component to the value for the single component system, irrespective of the topology.
D. $G = \text{VII}_0 \ltimes D_2$

For this invariance group, the symmetry requires that $u_I = 0$. Hence, as in the case $G \supset \mathbb{R}^3$, the moduli sector has the same structure as that for the vacuum system, and the invariant phase space is obtained simply by adding the fluid energy density to that for the vacuum system, both for the single and multiple component systems.

§4. Nil

4.1. Maximal geometry and invariance groups

Nil is the maximal geometry $(\mathbb{R}^3, \text{Isom}(\text{Nil}))$ whose representative metric is given by

$$ds^2 = dx^2 + dy^2 + Q_3 \left[ dz + \frac{1}{2}(ydx - xdy) \right]^2. \quad (4.1)$$

It contains the Bianchi type II group as the subgeometry. To see this, recall that the type II group is the Heisenberg group with the multiplication structure given by

$$(a, b, c)(x, y, z) = (a + x, b + y, c + z + \frac{ay - bx}{2}). \quad (4.2)$$

The left transformation $L_a$ and the right transformation $R_a$ defined from this multiplication are represented by linear transformations in $\mathbb{R}^3$:

$$L_a = (a, tN(\hat{a}^*/2)) \in IGL(3), \quad (4.3)$$
$$R_a = (a, tN(-\hat{a}^*/2)) \in IGL(3), \quad (4.4)$$

where $\hat{a}^* = (a^2, a^1)$, and $N(b)$ is the matrix

$$N(b) = \begin{pmatrix} 1 & 0 & b^1 \\ 0 & 1 & b^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.5)$$

which forms an Abelian group isomorphic to $\mathbb{R}^2$ with respect to the multiplication. In particular, the infinitesimal left transformations are generated by

$$\xi_1 = \partial_x + \frac{1}{2}y\partial_z, \quad \xi_2 = \partial_y - \frac{1}{2}x\partial_z, \quad \xi_3 = \partial_z, \quad (4.6)$$

with the commutation relations for the Bianchi type II group

$$[\xi_1, \xi_2] = -\xi_3, \quad [\xi_1, \xi_3] = 0, \quad [\xi_2, \xi_3] = 0. \quad (4.7)$$

A natural invariant basis is given by

$$\chi^1 = dx, \quad \chi^2 = dy, \quad \chi^3 = dz + \frac{1}{2}(ydx - xdy); \quad (4.8)$$
$$d\chi^1 = 0, \quad d\chi^2 = 0, \quad d\chi^3 = -\chi^1 \wedge \chi^2. \quad (4.9)$$
invariant basis. First, since \( \xi_2 \) is invariant under the rotation around the \((N(I))\), it follows that \( fL_\epsilon f^{-1} = L_{\epsilon'} \) and \( \epsilon' = (A\epsilon, \Delta\epsilon^3 + \hat{a}^* \cdot \hat{A}\epsilon) \).

Further, it is easy to see that it is also invariant under the rotation around the \(z\)-axis. Hence the metric has 4-dimensional isometries generated by \( L_a \) and \( R_3(\theta) \). Since the 3-space whose isometry group has a dimension greater than 4 is always a constant curvature space, this 4-dimensional group gives \( \text{Isom}_0(\text{Nil}) \):

\[
\xi_4 = -y\partial_x + x\partial_y, \quad (4.10)
\]

\[
[\xi_4, \xi_1] = -\xi_2, \quad [\xi_4, \xi_2] = \xi_1, \quad [\xi_4, \xi_3] = 0. \quad (4.11)
\]

In order to determine extra discrete isometries, we need the information on the normalizer group \( \mathcal{N}(\text{Isom}_0(\text{Nil})) \). Since \( \mathcal{L}[\text{Isom}(\text{Nil})], \mathcal{L}[\text{Isom}(\text{Nil})] = \mathcal{L}(\Pi_L) \), \( \mathcal{N}(\text{Isom}_0(\text{Nil})) \) is a subgroup of \( \mathcal{N}(\Pi_L) \) from Prop.A.3. Hence, let us first determine \( \mathcal{N}(\Pi_L) \), which is the set of transformations inducing linear transformations of the invariant basis. First, since \( \xi_3 \) is a generator of the center of \( \Pi_L \), \( f_\epsilon \xi_3 = k \xi_3 \), i.e., \( \partial_x f^1 = \partial_x f^2 = 0 \) and \( \partial_x f^3 = k \) with a constant \( k \), for \( f = (f^1, f^2, f^3) \in \mathcal{N}(\Pi_L) \). Hence, \( (f^1, f^2) \) are required to be a linear transformation \( (\mathbf{b}, \hat{A}) \) in the \((x, y)\) plane. Further, from the condition that \( f^\ast \chi^3 \) is a linear combination of \( \chi^I \), it follows that \( k = \det \hat{A} \). Since the linear transformation \( D(\hat{A}, \det \hat{A}) \) belongs to \( \mathcal{N}(\Pi_L) \), we can make \( (f^1, f^2) \) a pure translation and put \( \partial_z f^3 \) to unity by combining \( f \) with \( D(\hat{A}, \det \hat{A})^{-1} \). Then \( f^3 - z \) becomes linear in \( x \) and \( y \). This final form of \( f \) is realized as an appropriate combination of the left and the right transformations. To summarize, a generic element \( f \) of \( \mathcal{N}(\Pi_L) \) is expressed as

\[
f = L_a R_b D(\hat{A}, \Delta); \Delta = \det \hat{A}. \quad (4.12)
\]

The invariant basis \( (4.8) \) transforms by this HPD as

\[
f^\ast \chi^i = F^i, j \chi^j : F = \iota \mathcal{N}(\mathbf{b}^\ast) D(\hat{A}, \Delta). \quad (4.13)
\]

It is directly checked that the conjugate transformation by \( f \) maps a rotation around the \(z\)-axis into \( \text{Isom}_0(\text{Nil}) \) if and only if \( \mathbf{b} = 0 \) and \( \hat{A} \) is written as \( \{1, I_1\} kR_3(\theta) \).

### Table VI. Normalizers for the invariance groups \( G \) containing \( \Pi_L \).

| \( G \) | o/u | \( f \in \mathcal{N}(G) \) | \( fgf^{-1} \) |
|---|---|---|---|
| \( \Pi_L \) | o | \( L_a R_b D(\hat{A}, \Delta) \) | \( fL_\epsilon f^{-1} = L_{\epsilon'} \) |
| \( \Pi_L \times \{1, I_1\} \) | o | \( \mathbf{b} = 0 \) | \( fI_3 f^{-1} = I_3 L_{-2\mathbf{b}} \) |
| \( \Pi_L \times \{1, I_1\} \) | o | \( \hat{A} = D(p,q), \mathbf{b} = (b,0) \) | \( fI_1 f^{-1} = I_1 L_b \) |
| \( \Pi_L \times D_2 \) | o | \( \{1, J\} L_a D(p,q,pq); J = R_3(\pi/2)I_1 \) | \( L_a I_1 L_{a}^{-1} = I_1 L_{a}; \) |
| | | | \( c_1 = (0,-2a^2,-2a^3-a^3a^2), \) |
| | | | \( c_2 = (-2a^1,0,-2a^3+a^3a^2), \) |
| | | | \( c_3 = (-2a^1,-2a^2,0), \) |
| | | | \( J I_1 J = I_2, J I_3 J = I_3 \) |
| \( \text{Isom}_0(\text{Nil}) \) | o | \( \{1, I_1\} L_a R_3(\theta) D(k,k,k^2) \) | \( I_1 R_3(\phi) I_1 = R_3(-\phi), \) |
| | | | \( L_a R_3(\phi) L_a^{-1} = R_3(\phi) L_3(R_3(-\phi)-1) \) |
| \( \text{Isom}(\text{Nil}) \) | o | ibid | ibid |
Hence \( f \in N(\text{Isom}_0(\text{Nil})) \) is written as \( f = \{1, I_1\} L_a D(k, k, k^2) R_3(\theta) \). From this we find that

\[
\text{Isom}(\text{Nil}) = \text{Isom}_0(\text{Nil}) \times \{1, I_1\} \cong \Pi_L \times O(2),
\]

Further from the formulas on the conjugate transformations of \( I_1 \) given in Table VI we can confirm that \( N(\text{Isom}(\text{Nil})) = N(\text{Isom}_0(\text{Nil})). \)

Next we determine the conjugate classes of invariance group \( G \) such that \( G_0 = \Pi_L \) and their normalizers. Since \( \text{Isom}(\text{Nil}) \subset N(\Pi_L) \), possible discrete transformations to be added to \( \Pi_L \) are \( I_1, R_3(\theta) \) and their combinations. As in the type \( E^3 \) case, the invariance group contains \( \text{Isom}_0(\text{Nil}) \) if it contains a transformation \( R_3(\theta) \) with \( \theta \neq 0, \pi (\text{mod} 2\pi) \). Further \( I_1 R_3(\theta) \) is conjugate to \( I_1 \) for any \( \theta \). Hence the conjugate classes of \( G \) is given by \( \Pi_L \) or the semi-direct product of \( \Pi_L \) with one of the discrete groups \( \{1, I_1\}, \{1, I_3\} \) and \( D_2 = \{1, I_1, I_2, I_3\} \). By examining the conjugate transformation of \( I_1 \) by \( f \in N(\Pi_L) \), one finds that the normalizer of these groups are given by those listed in Table VI.

4.2. Phase space

Diffeomorphism classes of the orientable compact closed 3-manifold modeled on \( \text{Nil} \) are grouped into 7 subclasses, each of which is further classified by a positive integer, as listed in Table VII. Although the representations of the fundamental groups have rather complicated structures for many of them, we have to treat only topologies with rather simple structures in the present paper because the phase space for the other complicated topologies is easily determined by that for the corresponding vacuum system discussed in Paper I.

To see this, first consider the case in which the invariance group \( G \) contains \( \Pi_L \times D_2 \), i.e., \( G = \Pi_L \times D_2 \) or \( G = \text{Isom}(\text{Nil}) \). In this case, all the spatial components of the fluid velocity must vanish. Hence, as in the corresponding case for the type \( E^3 \), the invariant phase space for the fluid system is simply given by adding the energy density of each component of the fluid to that for the vacuum system. Next, since the momentum constraint is expressed in terms of the invariant basis (4.8) as

\[
H_1 \equiv 2P_3^2 + cu_1 = 0, \quad H_2 \equiv -2P_3^1 + cu_2 = 0, \quad H_3 \equiv cu_3 = 0,
\]

\( u_1 = u_2 = u_3 = 0 \) if the system has \( I_3 \) symmetry. Hence, from the argument on the vacuum system in Papar I, the system is invariant under \( \Pi_L \times D_2 \) if the fluid has a single component.

A similar situation arises for \( G = \text{Isom}_0(\text{Nil}) \), which can be realized only for \( \Sigma = T^3(u)/Z_k \) (\( k = 1, 2, 3, 4, 6 \)). Since \( N(\text{Isom}_0(\text{Nil})) = N(\text{Isom}(\text{Nil})) \), the argument for the moduli sector for \( G = \text{Isom}_0(\text{Nil}) \) is the same as that for \( G = \text{Isom}(\text{Nil}) \), and the matrices \( Q \) and \( P \) become diagonal due to symmetry for both cases. Hence the difference in the structure of the phase space comes from the fluid sector, for which \( u_1 = u_2 = 0 \) by symmetry. In the single component case, \( u_3 = 0 \) is further required by \( H_3 = 0 \), hence the system is Isom(\( \text{Nil} \)) invariant if it is invariant under Isom(\( \text{Nil} \)). This does not hold for a multi-component system because \( u_3 \) of each fluid component may not be zero, but it does not affect the argument on the geometrical sector because HPDs fixing the moduli parameter does not affect \( u_3 \). Hence, the phase space for
Table VII. Fundamental groups and their representation in $\text{Isom}^+(\text{Nil})$ of compact closed orientable 3-manifolds of type $\text{Nil}$. In this table $\Delta(a,b) = a^3b^2 - a^2b^3$.

| Space | Fundamental group and representation |
|-------|--------------------------------------|
| $T^3(n)$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |
| $K^3(n)$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |
| $T^3(n)/\mathbb{Z}_2$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |
| $T^3(n)/\mathbb{Z}_3$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |
| $T^3(n)/\mathbb{Z}_4$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |
| $T^3(n)/\mathbb{Z}_6$ | $\alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma^n > (n \in \mathbb{N})$ |

a multi-component system with $G = \text{Isom}_0(\text{Nil})$ is obtained by adding the variables $(u_3, \rho)$ for each extra-component of fluid to that for the single component system with $G = \text{Isom}(\text{Nil})$.

Thus, we only have to consider in detail the cases $G = \Pi_L$ or $G = \Pi_L \times \{1, I_1\}$ and a multi-component system with $G = \Pi_L \times \{1, I_3\}$. From Table VII, we immediately see that $\pi_1(\Sigma)$ can be embedded into $G$ only when $\Sigma = T^3(n)$ for $G = \Pi_L$, $\Sigma = T^3(n)$, $K^3(n)$ for $G = \Pi_L \times \{1, I_1\}$, and $\Sigma = T^3(n), T^3(n)/\mathbb{Z}_2$ for $G = \Pi_L \times \{1, I_3\}$. Since the moduli freedom is uniquely specified by the matrix $K = (a\ b\ c)$ in these cases, we call it the moduli matrix as before.
4.2.1. $G = \Pi_L$

In this case $\Sigma = T^3(n)$, and from the formula on $fLdf^{-1}$ in Table VII it is easy to show that the moduli matrix $K$ can be put into the canonical form

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{pmatrix}.$$ 

(4.16)

Hence the moduli sector is trivial. This matrix is invariant under $f \in \mathcal{N}(\Pi_L)$ only when $f = R_b$. From (4.13), we can put $Q_{13} = Q_{23} = 0$ by this residual HPD. On the other hand, $u_3 = 0$, and $P^{13}$ and $P^{23}$ are determined by $u_1, u_2$ and $Q_{33}$. Since $Q$ can be always diagonalized by HPDs if we neglect the moduli sector, the system has a higher symmetry when $u_1$ or $u_2$ vanishes. Hence, the dynamical sector of the invariant phase space for the single component fluid system is given by

$$\Gamma_{\text{dyn}}(T^3(n), \Pi_L) = \{(\hat{Q}, Q_{33}; \hat{P}, P^{33}; u_1, u_2, \rho \mid \det \hat{Q} > 0, Q_{33} > 0, u_1u_2 \neq 0)\}.$$ 

(4.17)

For a multi-component system, each extra component adds 4 to $N_s$.

4.2.2. $G = \Pi_L \rtimes \{1, I_1\}$

A. $T^3(n)$: In this case, $\pi_1(\Sigma)$ is embedded in $\Pi_L$, and we can show that the moduli matrix $K$ can be always put by a HPD into the form

$$K = \begin{pmatrix} 1 & X & 0 \\ 1 & Y & 0 \\ 0 & 0 & Y-X/n \end{pmatrix},$$

(4.18)

where $X \neq Y$. The modular transformation group of $\pi_1(T^3(n))$ consists of the transformations $(\alpha, \beta) \mapsto (\alpha^p\beta^q\gamma^u, \alpha^r\beta^s\gamma^v)$, where $Z = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ is a matrix in $GL(2, \mathbb{Z})$, and $u$ and $v$ are integers. By appropriate combinations with HPDs, they induce a discrete transformation group isomorphic to $GL(2, \mathbb{Z})$ which preserves the above form of $K$.

In the dynamical sector, the symmetry requires that $Q_{12} = Q_{13} = P^{12} = P^{13} = 0$ and $u_2 = u_3 = 0$. We can put $Q_{23} = 0$ and diagonalize $Q$ with the help of the residual HPD $R_b$ with $\hat{b} = (b, 0)$. Then the momentum constraints $H_2 = 0$ and $H_3 = 0$ become trivial, and $H_1 = 0$ can be used to express $P^{23}$ by $u_1$. Hence, taking account of the residual discrete transformations, the invariant phase space is given by

$$\Gamma_{\text{inv}}(T^3(n), \Pi_L \rtimes \{1, I_1\}) = \{(Q_1, Q_2, Q_3; P^{11}, P^{22}, P^{33}; u_1, \rho; X, Y) \mid Q_1, Q_2, Q_3 > 0, u_1 \neq 0, X \neq Y \}/GL(2, \mathbb{Z}).$$ 

(4.19)

Here it can be checked that the action of $GL(2, \mathbb{Z})$ is properly discontinuous and the invariant phase space has a smooth manifold structure.

B. $K^3(n)$: Since no transformation in $G$ contains the $R_3(\theta)$ factor, $R(\theta)$ in the representation of $\pi_1(K^3(n))$ in Table VII must be unity, and hence $a^1 = 0$. Further,
Table VIII. The parameter counts for type Nil.

| Space   | Symmetry         | $N_Q$ | $N_P$ | $N_m$ | $N_f$ | $N$ | $N_s$ | $N_s$(vacuum) |
|---------|------------------|-------|-------|-------|-------|-----|-------|---------------|
| $\mathbb{R}^3$ | $\Pi_L$          | 1     | 3     | 0     | 3     | 7   | 5     | -             |
|         | $\Pi_L \rtimes \{1, I_1\}$ | 1     | 3     | 0     | 2     | 6   | 4     | -             |
|         | $\Pi_L \rtimes D_2$        | 1     | 3     | 0     | 1     | 5   | 3     | 2             |
|         | Isom(Nil)            | 1     | 2     | 0     | 1     | 4   | 2     | 1             |
| $T^3(n)$ | $\Pi_L$          | 4     | 4     | 0     | 3     | 11  | 9     | -             |
|         | $\Pi_L \rtimes \{1, I_1\}$ | 3     | 3     | 2     | 2     | 10  | 8     | -             |
|         | $\Pi_L \rtimes D_2$        | 3     | 3     | 2     | 1     | 9   | 7     | 6             |
|         | Isom(Nil)            | 2     | 2     | 2     | 1     | 7   | 5     | 4             |
| $K^3(n)$ | $\Pi_L \rtimes \{1, I_1\}$ | 3     | 3     | 0     | 2     | 8   | 6     | -             |
|         | $\Pi_L \rtimes D_2$        | 3     | 3     | 0     | 1     | 7   | 5     | 4             |
|         | Isom(Nil)            | 2     | 2     | 1     | 1     | 6   | 4     | 3             |
| $T^3(n)/\mathbb{Z}_2$ | $\Pi_L \rtimes D_2$ | 3     | 3     | 2     | 1     | 9   | 7     | 6             |
|         | Isom(Nil)            | 2     | 2     | 2     | 1     | 7   | 5     | 4             |
| $T^3(n)/\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ | $\Pi_L \rtimes D_2$ | 3     | 3     | 0     | 1     | 7   | 5     | 4             |
|         | Isom(Nil)            | 2     | 2     | 1     | 1     | 6   | 4     | 3             |
| $T^3(n)/\mathbb{Z}_k(k = 3, 4, 6)$ | Isom(Nil)            | 2     | 2     | 0     | 1     | 5   | 3     | 2             |

with the help of the formulas for the conjugate transformations in Table VI, we can show that by a HPD $K$ can be put to the canonical form

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1/n \end{pmatrix}. \quad (4.20)$$

The argument on the dynamical sector is the same as that for $T^3(n)$. Hence the invariant phase space is given by

$$\Gamma_{\text{inv}}(K^3(n), \Pi_L \rtimes \{1, I_1\}) = \{(Q_1, Q_2, Q_3; P_{11}, P_{22}, P_{33}; u_1, \rho) | Q_1, Q_2, Q_3 > 0, u_1 \neq 0\}. \quad (4.21)$$

4.2.3. $G = \Pi_L \rtimes \{1, I_3\}$

As discussed above, this case occurs only for a multi-component fluid system.

**A. $T^3(n)$:** In this case $\pi_1(\Sigma)$ is contained in $\Pi_L$ and the effective action of $\mathcal{N}(\Pi_L \rtimes \{1, I_3\})$ is the same as that of $\mathcal{N}(\Pi_L)$. Hence the moduli sector is trivial. In the dynamical sector, the invariance and the momentum constraints require that $Q_{13} = Q_{23} = P_{13} = P_{23} = 0$ and $u_I = 0$. Therefore, $N_Q = 4, N_P = 4, N_m = 0$ and $N_f = 2(n_f - 1) + 1$, where $n_f$ is the number of the fluid components. Note that for $n_f = 1$, the total degrees of freedom $N$ coincides with that for $(T^3(n), \Pi_L \rtimes D_2)$ as expected, but the assignments of the degrees of freedom among $N_Q, N_P$ and $N_m$ are different. What is happening is that for $n_f = 1$, the system really has the higher
By a HPD in \(N(\Pi_L \times \{1, I_3\})\), these two constants of motion can be transferred to the moduli freedom.

**B. \(T^3(n)/\mathbb{Z}_2\):** In this case the representation of the generator \(\gamma\) contains the \(I_3\) factor. From the formulas in Table VI, we can show that \(c^1\) and \(c^2\) can be put to zero by \(f = L_{\hat{a}}\). Then \(a^3\) and \(b^3\) also vanish. Further, we can put \((\hat{a}, \hat{b})\) to the unit matrix by \(f = D(\hat{A}, \Delta)\). Hence the moduli matrix \(K\) can be transformed to the constant matrix (4-16). The argument on the dynamical sector is the same as that for the previous case \(T^3(n)\). Hence the degrees of freedom are again given by \(N_Q = N_P = 4, N_m = 0\) and \(N_f = 2n_f - 1\).

§5. **Sol**

5.1. **Maximal geometry and invariance groups**

**Sol** is the maximal geometry \((\mathbb{R}^3, \text{Isom}(\text{Sol}))\) obtained by extension of the Bianchi type VI\(_0\) symmetry. In order to determine \(\text{Isom}(\text{Sol})\), let us first summarize basic properties of the Bianchi type VI\(_0\) group.

The VI\(_0\) group has the multiplication structure

\[
(a, b, c)(x, y, z) = (a + e^{-c}x, b + e^c y, c + z).
\] (5.1)

The left and the right transformations on VI\(_0\) defined by this multiplication are expressed as

\[
L_a = (a, B(a^3)) \in IGL(3),
\] (5.2)
\[
R_b(x) = x + B(z)b,
\] (5.3)

where \(B(c) = D(e^{-c}, e^c, 1)\). The Lie algebra of the left transformation group is generated by

\[
\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z - x\partial_x + y\partial_y,
\] (5.4)

with the commutation relations

\[
[\xi_1, \xi_2] = 0, \quad [\xi_3, \xi_1] = \xi_1, \quad [\xi_3, \xi_2] = -\xi_2.
\] (5.5)

We adopt the following invariant basis in this paper:

\[
\chi^1 = e^zdx + e^{-z}dy, \quad \chi^2 = e^zdx - e^{-z}dy, \quad \chi^3 = dz;
\] (5.6)
\[
d\chi^1 = \chi^3 \wedge \chi^2, \quad d\chi^2 = \chi^3 \wedge \chi^1, \quad d\chi^3 = 0.
\] (5.7)

The normalizer of \((\text{VI}_0)_L\) is determined in the following way. First, since \(\xi_1\) and \(\xi_2\) are the generators of linearly independent one-dimensional invariant subspaces of the adjoint representation of VI\(_0\), \(f_*(\xi_1, \xi_2)\) is written as either \((k_1\xi_1, k_2\xi_2)\) or \((k_1\xi_2, k_2\xi_1)\) for \(f \in N((\text{VI}_0)_L)\). The commutation relations require that \(f_*(\xi_3)\) must be written as \(\xi_3 + a^1\xi_1 + a^2\xi_2\) in the former case and as \(-\xi_3 + a^1\xi_2 + a^2\xi_1\) in the latter case. Hence the automorphism induced by \(f\) has the form

\[
f_*(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3) \{1, J\} N(\hat{a}) D(k_1, k_2, 1),
\] (5.8)
First, we transform the invariant metric $Q$ group. From (5.11) we can make $D_N$ in $f$ further put $Q$ of a 2-dimensional Lorentz transformation and a dilation, in terms of which we can of $N$ 

It is easy to see that $f = L_\alpha$ induces $N(\alpha)$, and the linear transformations represented by the matrices $J$ and $D(k_1, k_2, 1)$ induce the automorphisms represented by the same matrices, respectively. Hence the generic element $f$ of $N((V_{I_b})_L)$ is written as $f = \{1, J\} L_\alpha R_b D(k_1, k_2, 1)$, which can be rewritten as

$$ f = \{1, J\} L_\alpha R_b D(k_1, k_2, 1) $$

(5.10)

by appropriate redefinitions of the parameters. By this HPD the invariant basis (5.6) transforms as

$$ f^* \chi^i = F^i j \chi^j; \quad F = \{1, I_1\} $$

(5.11)

where

$$ k_\pm = \frac{1}{2} (k_1 \pm k_2), \quad b_\pm = -b_1 \pm b_2. $$

(5.12)

The corresponding formulas for the conjugate transformations are given in Appendix B.

We can show that there exists no higher dimensional isometry group containing $V_{I_b}$ by examining the structure of possible 4-dimensional Lie algebra containing $V_{I_b}$. Hence $\text{Isom}(Sol)$ is the semi-direct product of $(V_{I_b})_L$ with a discrete subgroup of $N((V_{I_b})_L)$. Such additional discrete symmetry is found in the following way. First, we transform the invariant metric $Q_{IJ} \chi^I \chi^J$ into some special form by HPDs in $N((V_{I_b})_L)$ because we are only interested in the conjugate class of the invariance group. From (5.11) we can make $Q_{13}$ and $Q_{23}$ vanish by $f = R_b$ to obtain $Q = D(Q, Q_3)$. Then the transformation matrix $F$ with $\dot{b} = 0$ acts on $\dot{Q}$ as the combination of a 2-dimensional Lorentz transformation and a dilation, in terms of which we can further put $Q = D(Q_1, 1/Q_1, Q_3)$ (see the argument on the diagonalization of $Q$ by

---

Table IX. Normalizers for the invariance groups with $G_0 = Sol$. The formulas for conjugate transformations are given in Appendix B.

| $G$              | $u/o$ | $f \in N(G)$                      |
|------------------|-------|-----------------------------------|
| $(V_{I_0})_L$    | o     | $\{1, J\} R_b L_a D(k_1, k_2, 1)$|
| $(V_{I_0})_L \times \{1, I_3\}$ | o     | $\dot{b} = 0$                     |
| $(V_{I_0})_L \times \{1, J\}$   | o     | $\dot{b}^1 = b^2, k_1 = k_2$     |
| $(V_{I_0})_L \times \{1, I_3J\}$ | o     | $\dot{b}^1 = -b^2, k_1 = k_2$   |
| Isom$^+$$(Sol)$  | o     | $\dot{b} = 0, k^1 = k^2$         |
| $(V_{I_0})_L \times \{1, I_3, -I_1, -I_2\}$ | u     | $\dot{b} = 0$                     |
| $(V_{I_0})_L \times \{1, I_3, -I_1, J, -I_2J\}$ | u     | $\dot{b} = 0, k_1 = k_2$         |
| Isom$(Sol)$      | u     | ibid                              |
| $(V_{I_0})_L \times \{1, -I_1\}$   | u     | $R_{(0,\dot{b})} L_a D(k_1, k_2, 1)$ |
the Lorentz transformation in §7.3). There remains no continuous HPD freedom preserving this metric form. Hence a canonical form of the Sol metric is given by

$$ds^2 = Q_1(x^1)^2 + Q_1^{-1}(x^2)^2 + Q_3(x^3)^2.$$  \hfill (5.13)

Clearly the transformation $F$ of the invariant basis preserving this form of the metric is either $I_1, I_3, -J$ or their multiplication and is induced from the HPD $f = J, I_3, -I_1$ or their multiplication, respectively. These HPDs altogether form the discrete group

$$\mathcal{D}_4 = \{1, I_3, J, I_3J, -I_1, -I_2, -I_1J, -I_2J\}.$$  \hfill (5.14)

The above metric is invariant by this group, if and only if $Q_1 = 1$. Hence the maximal geometry including $\text{VI}_0$ is given by

$$\text{Isom}(\text{Sol}) = (\text{VI}_0)_L \rtimes \mathcal{D}_4,$$  \hfill (5.15)

and the corresponding canonical metric is written as

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + Q_3 dz^2.$$  \hfill (5.16)

Here note that $\text{Isom}(\text{Sol})$ contains orientation reversing transformations and that $Q_1 = 1$ is required by the invariance under such transformations. Hence, when we consider only the orientation-preserving transformations, the isometry group is reduced to

$$\text{Isom}^+(\text{Sol}) = (\text{VI}_0)_L \rtimes \mathcal{D}_2,$$  \hfill (5.17)

where

$$\mathcal{D}_2 = \{1, I_3, J, I_3J\},$$  \hfill (5.18)

and the most generic invariant metric is given by (5.13).

Since the invariance group $G$ is always a semi-direct product of $\text{VI}_0$ and a subgroup of $\mathcal{D}_4$, its conjugate class is easily classified. The result is given in Table IX with the corresponding normalizer groups.

### 5.2. Phase space

A compact quotient of $\text{Sol}$ is a torus bundle over a circle which is obtained by gluing the two boundaries of $T^2 \times [0, 1]$ by a hyperbolic large diffeomorphism of $T^2$ \hfill (3.11). Hence its diffeomorphism class is classified by the $\text{SL}(2, \mathbb{Z})$-conjugate class of the $\text{SL}(2, \mathbb{Z})$ matrix $Z = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $|p + s| > 2$ specifying the automorphism of $\pi_1(T^2)$ for the gluing. The fundamental groups with different values of $n = p + s$ are not isomorphic, and each class is uniquely specified by $n = p + s$ and the roots $\omega_1$ and $\omega_2$ of the characteristic equation $rx^2 + (s - p)x - q = 0$, but there exists finitely many different classes with the same $n$ in general (see the explanation in Paper I for further details).

As was discussed in the previous subsection, $Q$ can be diagonalized by a HPD in $\mathcal{N}(\text{VI}_0)$. Further, with respect to the invariant basis (5-6), the momentum constraints are written as

$$H_1 \equiv -2P_2^3 + cu_1 = 0, \; H_2 \equiv -2P_1^3 + cu_2 = 0, \; H_3 \equiv 2(P_2^1 + P_2^2) + cu_3 = 0.$$  \hfill (5.19)
Table X. Fundamental groups and their representation in Isom⁺(Sol) of compact, closed orientable 3-manifolds of type Sol.

| Space | Fundamental group and representation |
|-------|---------------------------------------|
| Sol(n;ω₁,ω₂) | [α, β] = 1, γαγ⁻¹ = α⁺β⁺, γβγ⁻¹ = α⁺β⁺; \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) ∈ SL(2, Z) |
| n = p + s, ω₁ = \( \frac{p-s+\sqrt{n^2-4}}{2r} \), ω₂ = \( \frac{p-s-\sqrt{n^2-4}}{2r} \) |
| n > 2 : | α = Lₐ; \( a = (b⁰₁, b⁰₂, 0) \) |
| | β = Lₜ; \( b = (b¹₁, b¹₂, 0) \) (\( b¹b² ≠ 0 \)) |
| | γ = Lₜ; \( e^c = \frac{n+\sqrt{n^2-4}}{2} \) |
| n < −2 : | α = Lₐ; \( a = (b⁰₁, b⁰₂, 0) \) |
| | β = Lₜ; \( b = (b¹₁, b¹₂, 0) \) (\( b¹b² ≠ 0 \)) |
| | γ = R₀(τ)Lₜ; \( e^c = \frac{|n|+\sqrt{n^2-4}}{2} \) |

Hence in the vacuum case for which \( c = 0 \), \( P \) also becomes diagonal, and the system always has the maximal symmetry Isom⁺(Sol). However, for the fluid system with \( (u_I) ≠ 0 \), this symmetry is broken, and the system can have lower symmetries listed in Table IX. Conversely, if the system is invariant under the maximal group, all the velocities vanish and the structure of the phase space for a fluid system is easily determined by that for the vacuum system as in the other Thurston types discussed so far.

5.2.1. \( G = (\text{VI₀})_L \)

From the structure of the fundamental group in Table X, only the space \( \text{Sol}(n) \) with \( n > 2 \) can have this invariance group. From the formula for \( L_dL_cL_d⁻¹ \) in the appendix, we see that \( c^1 \) and \( c^2 \) of \( γ = L_c \) can be put to zero by the transformation \( f = L_d \) because \( c^3 > 0 \). Further \( b₁ \) and \( b₂ \) can be put to \( ±1 \) by the transformation \( D(k₁,k₂)(k₁k₂ > 0) \) in \( N⁺((\text{VI₀})_L) \). Hence the moduli matrix \( K \) can be put to the canonical form

\[
K = \begin{pmatrix} \omega₁ & 1 & 0 \\ ±\omega₂ & ±1 & 0 \\ 0 & 6 & c³ \end{pmatrix},
\]

which implies that the moduli sector of the invariant phase space consists of two points representing two different orientations of the same space.

Because \( L_d \) with \( d = (0, 0, d) \) induces the conjugate transformation \( B(d) = D(e⁻^d, e^d, 1) \), the isotropy group of the action of \( N((\text{VI₀})_L) \) at the above moduli matrix in the moduli space is generated by \( Rₜ \) and \( B(−d) \). In terms of the former transformation we can put \( Q \) to the form \( D(\hat{Q}, Q₃) \). Further, since the latter transformation induces a Lorentz boost, we can diagonalize \( \hat{Q} \) by it. Then, by the momentum constraints, the non-diagonal entries of \( P \) are expressed in terms of \( u_I \). Therefore, taking into account that the system has a higher symmetry if two com-
ponents of \((u_I)\) vanishes, we find that the invariant phase space is given by

\[
\Gamma_{\text{inv}}(\text{Sol}(n)(n > 2), \text{VI}_0) = \{(Q_1, Q_2, Q_3; P^{11}, P^{22}, P^{33}; u_1, u_2, u_3, \rho)|
Q_1, Q_2, Q_3 > 0, u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_2^2 \neq 0\} \times \mathbb{Z}_2. \tag{5.21}
\]

For a multi-component system, \(N_f\) simply increases by 4 for each extra component.

5.2.2. \(G = (\text{VI}_0)_L \rtimes \{1, I_3\}\)

First, let us consider the space \(\text{Sol}(n)\) with \(n > 2\). For this space, the fundamental group is still embedded into \((\text{VI}_0)_L\), and the effective action of HPDs on the moduli is the same as for \(G = (\text{VI}_0)_L\). Hence the moduli sector is again given by two points. The argument on the dynamical sector is also the same except that the symmetry requires that \(Q = D(Q, Q_3), P = (P, P^3)\) and \(u_1 = u_2 = 0\), for which the only non-trivial momentum constraint is \(H_3 = 0\).

Next for \(\text{Sol}(n)\) with \(n < -2\), the generator \(\gamma\) contains the \(I_3\) transformation. However, this does not introduce any problem and essentially the same argument as that for \(n > 2\) applies. Therefore, irrespective of the sign of \(n\), the invariant phase space is given by

\[
\Gamma_{\text{inv}}(\text{Sol}(n), \text{VI}_0 \rtimes \{1, I_3\}) = \{(Q_1, Q_2, Q_3, P^{11}, P^{22}, P^3; u_3, \rho)|
Q_1, Q_2, Q_3 > 0, u_3 \neq 0\} \times \mathbb{Z}_2. \tag{5.22}
\]

For a multi-component system, \(N_f\) simply increases by 2 for each extra component.

5.2.3. \(G = (\text{VI}_0)_L \rtimes \{1, J\}\)

This symmetry is realized only for \(\text{Sol}(n)\) with \(n > 2\). Now, \(D(k_1, k_2, 1)\) belongs to \(N(G)\) only when \(k_1 = k_2\). However, since \(L_{(0,0,d)}\) induces the \(D(e^{-d}, e^d, 1)\) transformation of the moduli vector, the moduli sector is reduced to two points as in the case \(G = (\text{VI}_0)_L\).

In the dynamical sector, since \(f = J\) induces the \(I_1\) transformation of the invariant basis, the symmetry requires that \(Q = (Q_1, \hat{Q}), P = (P^1, \hat{P})\) and \(u_2 = u_3 = 0\). \(Q\) can be completely diagonalized in terms of the HPD \(R_{(b,b)}\). Further, the non-trivial momentum constraint \(H_1 = 0\) can be used to express \(P^{23}\) by \(u_1\). Hence the structure of the invariant phase space is the same as that for \(G = (\text{VI}_0)_L \rtimes \{1, I_3\}\) irrespective of the number of the fluid components:

\[
\Gamma_{\text{inv}}(\text{Sol}(n)(n > 2), \text{VI}_0 \rtimes \{1, J\}) = \{(Q_1, Q_2, Q_3, P^1, P^{22}, P^{33}; u_1, \rho)|
Q_1, Q_2, Q_3 > 0, u_1 \neq 0\} \times \mathbb{Z}_2. \tag{5.23}
\]

5.2.4. \(G = (\text{VI}_0)_L \rtimes \{1, JI_3\}\)

This symmetry is realized only for \(\text{Sol}(n)\) with \(n > 2\) again. The argument in the moduli sector is essentially the same as that for the previous case because the effective action of HPDs in the moduli sector is the same. Further, since \(JI_3\) induces the \(I_2\) transformation of the invariant basis and \(R_{(b,-b)}\) shifts \(Q_{13}\), the argument on the dynamical sector is just the repetition of the previous one with replacing the role
Table XI. The parameter count for type Sol.

| Space            | Symmetry | \( N_Q \) | \( N_P \) | \( N_m \) | \( N_f \) | \( N_s \) | \( N_z \) | \( N_s(\text{vacuum}) \) |
|------------------|----------|----------|----------|----------|----------|----------|----------|--------------------------|
| \( \mathbb{R}^3 \) | \( Vl_0 \) | 2        | 3        | 0        | 4        | 9        | 7        | –                        |
|                  | \( Vl_0 \times \mathbb{Z}_2 \) | 2        | 3        | 0        | 2        | 7        | 5        | –                        |
|                  | \( \text{Isom}^{+}(\text{Sol}) \) | 2        | 3        | 0        | 1        | 6        | 4        | 3                        |
| \( \text{Sol}(n) \) (\( n > 2 \)) | \( Vl_0 \) | 3        | 3        | 0        | 4        | 10       | 8        | –                        |
|                  | \( Vl_0 \times \mathbb{Z}_2 \) | 3        | 3        | 0        | 2        | 8        | 6        | –                        |
|                  | \( \text{Isom}^{+}(\text{Sol}) \) | 3        | 3        | 0        | 1        | 7        | 5        | 4                        |
| \( \text{Sol}(n) \) (\( n < -2 \)) | \( Vl_0 \times \mathbb{Z}_2 \) | 3        | 3        | 0        | 2        | 8        | 6        | –                        |
|                  | \( \text{Isom}^{+}(\text{Sol}) \) | 3        | 3        | 0        | 1        | 7        | 5        | 4                        |

of \( \chi^1 \) and \( \chi^2 \). Hence,

\[
\Gamma_{\text{inv}}(\text{Sol}(n)(n > 2), VI_0 \times \{1, JI_3\}) = \left\{ (Q_1, Q_2, Q_3, P^{11}, P^2, P^{33}, u_2, \rho) \mid Q_1, Q_2, Q_3 > 0, u_2 \neq 0 \right\} \times \mathbb{Z}_2 \times \mathbb{R}^6.
\]

§6. \( H^2 \times \mathbb{R} \)

6.1. Invariance subgroups and normalizers

6.1.1. Bianchi type III group

Space \( H^2 \times \mathbb{R} \) has the group structure of the Bianchi type III. To see this, let us embed the hyperbolic surface in the 3-dimensional Minkowski space \( E^{2,1} \),

\[
H^2 : T = (1 + X^2 + Y^2)^{1/2},
\]

\[
ds^2 = -dT^2 + dX^2 + dY^2,
\]

into the space of symmetric \( SL_2 \mathbb{R} \) matrices with positive diagonal elements by

\[
S = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix}.
\]

This matrix can be always uniquely written as \( S = Z^t Z \)

in terms of a matrix \( Z \) in \( TSL_2^+ \), where

\[
TSL_2^+ = \left\{ Z = \begin{pmatrix} p & 0 \\ q & 1/p \end{pmatrix} \mid p > 0, q \in \mathbb{R} \right\}.
\]

Conversely, for any \( Z \in TSL_2^+ \), \( S = Z^t Z \) is always written in the form (6.3). Further, \( TSL_2^+ \) forms a 2-dimensional closed subgroup of \( SL_2 \mathbb{R} \). Hence the embedding gives a one-to-one correspondence between \( H^2 \) and the group \( TSL_2^+ \) and defines a group structure in the hyperbolic space \( H^2 \).
In this and the next sections, we use the complex variable \( \zeta \) defined by
\[
\zeta = \frac{q}{p} + \frac{i}{p^2},
\] (6.6)
to parametrize the \( TSL_2^+ \) matrix \( Z \) as \( Z(\zeta) \). This parametrization gives a one-to-one mapping from the hyperbolic surface to the upper half part of the complex plane \( \{ \zeta = x + iy \mid x \in \mathbb{R}, y > 0 \} \). Through this mapping, we identify the hyperbolic surface \( H^2 \) with the upper half complex plain. Since the original Minkowski coordinates are expressed in terms of \( \zeta = x + iy \) as
\[
T = \frac{1 + |\zeta|^2}{23 \zeta}, \quad X = \frac{1 - |\zeta|^2}{23 \zeta}, \quad Y = \frac{\Re \zeta}{23 \zeta} ,
\] (6.7)
the hyperbolic metric on \( H^2 \) is written in terms of \( \zeta = x + iy \) as
\[
ds^2 = \frac{dx^2 + dy^2}{4y^2}.
\] (6.8)

The group structure of \( H^2 \) defined in this way together with the natural Abelian group structure of \( \mathbb{R} \) makes \( H^2 \times \mathbb{R} \) a 3-dimensional group. In terms of the coordinates \( (x, y, z) \in H^2 \times \mathbb{R} \), the corresponding multiplication is written as
\[
(a, b, c) \cdot (x, y, z) = (a + bx, by, z + c).
\] (6.9)

Infinitesimal left transformations defined by this multiplication are generated by
\[
\xi_1 = \partial_x, \quad \xi_2 = x \partial_x + y \partial_y, \quad \xi_3 = \partial_z
\] (6.10)
which form the Bianchi type III Lie algebra,
\[
[\xi_1, \xi_2] = \xi_1, \quad [\xi_3, \xi_1] = 0, \quad [\xi_3, \xi_2] = 0.
\] (6.11)
We denote this left transformation group of \( H^2 \times \mathbb{R} \) by \( \text{III}_L \). The bases of \( \text{III}_L \)-invariant 1-forms are given by
\[
\chi_1 = \frac{dx}{y}, \quad \chi_2 = \frac{dy}{y}, \quad \chi_3 = dz;
\] (6.12)
\[
d\chi_1 = \chi_1 \wedge \chi_2, \quad d\chi_2 = 0, \quad d\chi_3 = 0.
\] (6.13)

Let us determine the normalizer of \( \text{III}_L \) for later uses. Since the center of \( \text{III}_L \) is generated by \( \xi_3 \), and \( [\text{III}_L, \text{III}_L] \) is generated by \( \xi_1 \), from Prop.A.3, \( \phi \in \text{Aut}(\mathcal{L}(\text{III}_L)) \) has the form
\[
\phi(\xi_1) = a \xi_1, \quad \phi(\xi_2) = b \xi_1, \quad \phi(\xi_3) = c \xi_3.
\] (6.14)
From \( [\phi(\xi_2), \phi(\xi_2)] = ab^2 \xi_1 = \phi(\xi_1) \), the condition for \( \phi \) to be an automorphism is given by \( b^2 = 1 \). HPDs generating these automorphisms are determined as follows. First, the discrete transformation \((-I_1) : (x, y, z) \mapsto (-x, y, z) \) induces
(-I1) : (ξ1, ξ2, ξ3) → (-ξ1, ξ2, ξ3). Second, ZL(α) ∈ H2L with α = α1 + iα2 as a transformation in IIIL induces the automorphism

\[(ZL)_*(ξ1, ξ2) = (ξ1, ξ2) \left( \begin{array}{cc} 1/α2 & α1/α2 \\ 0 & 1 \end{array} \right), (ZL)_*(ξ3) = ξ3. \tag{6.15} \]

Third, the linear transformation

\[D(1, 1, c) : (x, y, z) \mapsto (x, y, cz) \tag{6.16} \]

gives the automorphism (ξ1, ξ2, ξ3) → (ξ1, ξ2, cξ3). Finally, the transformation

\[N'(b) : (x, y, z) \mapsto (x, y, z + b \ln y) \tag{6.17} \]

induces the automorphism (ξ1, ξ2, ξ3) → (ξ1, ξ2, bξ3, ξ3). Combining these transformations, we can generate all automorphisms of the form (6.14). Hence, from Prop.A.4, \(f \in N(III_L)\) is written as

\[f = \{1, -I1, I2, -I3\} LαN'(b)D(1, 1, k)R^b, \tag{6.18} \]

where \(R^b = Z_R(b_1 + ib_2) \in III_R\), and we have used the fact that the group consisting of translations in \(R\) of \(H^2 \times R\) coincides with the center of III, III \(_L\) ∩ III \(_R\).

6.1.2. Symmetry of \(H^2\)

Since the hyperbolic surface is a maximally symmetric surface with the isometry group given by the Lorentz transformation group \(O^+(2, 1)\), the metric (6.8) should have the same invariance. In order to express this isometry group Isom(\(H^2\)) in terms of the coordinates \(ζ = x + iy\), we utilize the standard representation of \(SL_2R\) onto \(SO^+(2)\) given by

\[SL_2 \ni V : S \mapsto VS \overline{V}. \tag{6.19} \]

For \(S = Z(ζ)^tZ(ζ)\), after the decomposition

\[VZ(ζ) = Z(ζ')R; \ ζ' \in H^2, R \in SO(2), \tag{6.20} \]

\(VS \overline{V}\) is expressed as \(Z(ζ')^tZ(ζ')\). Hence each \(SO^+(2, 1)\) transformation is represented by the \(PSL_2R\) transformation \(Z(ζ) \rightarrow Z(ζ')\) determined by this decomposition. To be explicit, it is expressed as

\[ζ' = V * ζ := \frac{dζ + c}{bζ + a}; \ V = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2. \tag{6.21} \]

Since this transformation is holomorphic in \(ζ\), it preserve the orientation of \(H^2\).

By definition, this transformation coincides with a left transformation in \(H^2\) for \(V \in TSL^+(2)\). Hence, Isom\(_0\)(\(H^2\)) ≅ \(PSL_2R\) is generated by this left transformation and the transformation \(R_H(θ)\) corresponding to \(V = R(θ) \in SO(2)\):

\[H^2_L : \ ξ1, ξ2, \tag{6.22} \]

\[SO(2) : ξ4 = \frac{1}{2}(1 + x^2 - y^2)∂x + xy∂y. \tag{6.23} \]
The full isometry group $\text{Isom}(H^2)$ is generated by $\text{Isom}_0(H^2)$ and the space reflection $(-I_1)$.

Next we determine the normalizer of $\text{Isom}(H^2)$ in $\text{Diff}(H^2)$. In terms of the basis $\eta_1 = \xi_1 - \xi_4, \eta_2 = \xi_2, \eta_3 = \xi_4$, the commutation relations of $L(\text{Isom}(H^2)) \cong L(SO_+(2,1))$ are expressed in the standard form,

$$[\eta_1, \eta_2] = \eta_3, [\eta_2, \eta_3] = -\eta_1, [\eta_3, \eta_1] = -\eta_2.$$  \hspace{1cm} (6.24)

Hence the adjoint representation of the algebra is given by

$$\text{Ad}(X \cdot \eta) = \begin{pmatrix} 0 & X^3 & -X^2 \\ -X^3 & 0 & X^1 \\ -X^2 & X^1 & 0 \end{pmatrix}. \hspace{1cm} (6.25)$$

From this it follows that the Killing form is proportional to the metric of $E^{2,1}$,

$$\gamma(X, X) = 2[(X^1)^2 + (X^2)^2 - (X^3)^2]. \hspace{1cm} (6.26)$$

From Prop A.2, this implies that an automorphism $\phi$ of this algebra is represented by a matrix $A$ in $O(2,1)$ as $\phi(a \cdot \eta) = (Aa) \cdot \eta$. This linear transformation belongs to $L(SO(2,1))$ when $\det A = 1$ from

$$[\phi(a \cdot \eta), \phi(b \cdot \eta)] = (\det A)\phi([a \cdot \eta, b \cdot \eta]). \hspace{1cm} (6.27)$$

Hence

$$\text{Aut}(L(\text{Isom}(H^2))) = SO(2,1). \hspace{1cm} (6.28)$$

Clearly the maximal connected subgroup of this group, $SO_+(2,1)$, is generated by the inner automorphisms, i.e., by the transformations in $\text{Isom}_0(H^2)$. On the other hand, the space reflection $(-I_1) : (x,y) \mapsto (-x, y)$ induces the automorphism $I_2 : (\eta_1, \eta_2, \eta_3) \mapsto (-\eta_1, \eta_2, -\eta_3)$, which together with $SO_+(2,1)$ generates $SO(2,1)$. Therefore, from Prop A.4, $N(\text{Isom}_0(H^2))$ is contained in $\{1, (-I_1)\} \rtimes (\text{Isom}_0(H^2) \times H^2_\mathbb{R})$. Here, from Eq.(6.9), for $f = Z_R(\alpha) \in H^2_\mathbb{R}$ and the isometry $R_H(\theta) \in \text{Isom}_0(H^2)$ corresponding to $V = R(\theta)$, $f R_H(\theta) f^{-1}$ is given by the transformation

$$x + iy \mapsto a + bR(\theta) \ast (x + ya'), \hspace{1cm} (6.29)$$

where $\alpha = a + ib$ and $\alpha' = (i - a)/b$. If this transformation belongs to $\text{Isom}_0(H^2)$, it should be written as $\zeta = x + iy \mapsto V \ast \zeta$ with some $V \in \text{SL}_2\mathbb{R}$, which is holomorphic in $\zeta$. This implies that $\alpha' = i$, i.e., $f = Z_R(i) = \text{id}$. Hence

$$N(\text{Isom}_0(H^2)) = \text{Isom}(H^2) = \{1, (-I_1)\} \rtimes \text{PSL}_2. \hspace{1cm} (6.30)$$

Since $N(\text{Isom}(H^2)) \subset N(\text{Isom}_0(H^2))$, it follows from this that

$$N(\text{Isom}(H^2)) = \text{Isom}(H^2). \hspace{1cm} (6.31)$$
6.1.3. Maximal geometry

The hyperbolic metric (6.8) on $H^2$ and the natural metric on $\mathbb{R}$ define the invariant metric on $H^2 \times \mathbb{R}$,

$$ds^2 = Q(x_1^2 + x_2^2) + x_3^2 = Q \frac{dx^2 + dy^2}{y^2} + dz^2,$$

(6.32)

where $Q$ is an arbitrary positive constant. This is clearly invariant under the transformation group $\text{Isom}(H^2) \times IO(1)$, where $IO(1)$ is the isometry group of the line $\mathbb{R}$. Actually, it precisely gives the maximal symmetry of $H^2 \times \mathbb{R}$.

$$\text{Isom}(H^2 \times \mathbb{R}) = \text{Isom}(H^2) \times \text{Isom}(\mathbb{R}).$$

(6.33)

This follows from the following consideration.

First, note that $\text{Isom}_0(H^2 \times \mathbb{R})$ is given by

$$\text{Isom}_0(H^2 \times \mathbb{R}) = \text{Isom}_0(H^2) \times \mathbb{R},$$

(6.34)

because $H^2 \times \mathbb{R}$ is not a maximally symmetric space and therefore the dimension of its isometry group is not greater than 4. Hence, $\text{Isom}(H^2 \times \mathbb{R})$ is generated by $\text{Isom}_0(H^2 \times \mathbb{R})$ and discrete transformations in $\mathcal{N}(\text{Isom}_0(H^2 \times \mathbb{R}))$.

The Lie algebra of $\text{Isom}_0(H^2 \times \mathbb{R})$ is generated by $\xi_1, \xi_2, \xi_3$ in Eq.(6.10) and the infinitesimal rotation $\xi_4$ of $H^2$. Their commutators are given by

$$[\xi_4, \xi_1] = -\xi_2, \ [\xi_4, \xi_2] = \xi_1 - \xi_4, \ [\xi_4, \xi_3] = 0.$$  

(6.35)

From these commutation relations, it follows that the center of this algebra is generated by $\xi_3$ and $[\mathcal{L}(\text{Isom}(H^2 \times \mathbb{R})), \mathcal{L}(\text{Isom}(H^2 \times \mathbb{R}))$ coincides with the subalgebra $\mathcal{L}(\text{PSL}_2\mathbb{R})$ generated by $\xi_1, \xi_2$ and $\xi_4$. Hence, from Prop.A.3 and the argument on $\text{Aut}(\mathcal{L}(\text{Isom}(H^2)))$, the automorphism $\phi \in \text{Aut}(\mathcal{L}(\text{Isom}(H^2 \times \mathbb{R})))$ has the structure

$$\phi(\xi_3) = k\xi_3,$$

$$\phi(\xi_1 - \xi_4, \xi_2, \xi_4) = (\xi_1 - \xi_4, \xi_2, \xi_4)A; \ A \in SO(2,1).$$

(6.36)  

(6.37)

Since this automorphism is induced from a transformation in the group $\mathcal{N}(\text{Isom}(H^2)) \times \mathcal{N}(\text{Isom}(\mathbb{R}))$, it follows from Prop.A.4 that for any $f \in \mathcal{N}(\text{Isom}_0(H^2 \times \mathbb{R}))$, there is $f_0 \in \mathcal{N}(\text{Isom}(H^2)) \times \mathcal{N}(\text{Isom}(\mathbb{R}))$ such that $g = f_0^{-1}f$ is some right transformation in $H^2 \times \mathbb{R}$. Here, since $\mathbb{R}$ is an Abelian group, for which the right and the left transformations coincide, we can assume that $g$ belongs to $H^2_R$. Then the condition $g \in \mathcal{N}(\text{Isom}_0(H^2 \times \mathbb{R}))$ is equivalent to the condition $g \in \mathcal{N}(\text{Isom}_0(H^2))$. However, as shown before, $H^2_R \cap \mathcal{N}(\text{Isom}_0(H^2))$ is trivial. This implies that $g$ is an identity transformation, and we find

$$\mathcal{N}(\text{Isom}(H^2 \times \mathbb{R})) = \mathcal{N}(\text{Isom}_0(H^2 \times \mathbb{R})) = \text{Isom}(H^2) \times \mathcal{N}(\text{Isom}(...))$$

(6.38)

It is clear that a transformation $\mathcal{N}(\text{Isom}(\mathbb{R}))$ induces an isometry of $H^2 \times \mathbb{R}$ only when it belongs to $\text{Isom}(\mathbb{R})$. This proves the equality (6.33).
Finally we give one formula. A generic transformation in \( \mathcal{N}(\text{Isom}(H^2 \times \mathbb{R})) \) is written as
\[
f = \{1, -I_1, I_2, -I_3\} L(a,b,c) R_H(\theta) D(1,1,k).
\]
(6.39)

From (6.21), we find that this induces the transformation of the invariant basis (6.12) given by
\[
f^* \chi^j = F^j F^I \chi^I : F = \{1, -I_1, I_2, -I_3\} R_3(H(\theta, \zeta)) D(1,1,k)
\]
(6.40)
at the point \((x, y, z)\), where \(\zeta = x + iy\) and \(H(\theta, \zeta)\) is the angle defined mod. \(2\pi\) by
\[
e^{-iH(\theta, \zeta)} = \frac{\cos \theta - \zeta \sin \theta}{|\cos \theta - \zeta \sin \theta|}.
\]
(6.41)

In the next section we will give a definition of \(H\) without the ambiguity of mod. \(2\pi\), but in the present section this ambiguity has no significance.

6.1.4. Invariance group

We first determine the possible invariance groups \(G\) with \(G_0 = \text{Isom}_0(H^2 \times \mathbb{R})\). Since the maximal isometry group of \(H^2 \times \mathbb{R}\) is written as
\[
\text{Isom}(H^2 \times \mathbb{R}) = \{1, -I_1, I_2, -I_3\} \rtimes \text{Isom}_0(H^2 \times \mathbb{R}),
\]
(6.42)
where \(-I_1, I_2, -I_3\) are the discrete linear transformations of \((x, y, z)\) represented by the matrices \(-I_1, I_2, -I_3\), respectively, \(G\) is given by \(G_0\) or the semi-direct product of \(G_0\) and one of the discrete groups \(\{1, I_2\}, \{1, -I_1\}, \{1, -I_3\}\), and \(\{1, -I_1, I_2, -I_3\}\). Among these, \(\text{Isom}_0(H^2 \times \mathbb{R})\) and \(\{1, I_2\} \rtimes \text{Isom}_0(H^2 \times \mathbb{R})\) are orientation preserving. It is easy to see that the normalizer of all of these groups coincide and is given by Eq.(6.38).

Next we consider the group \(G\) with \(G_0 = \text{III}_L\). Since an invariance group with \(G_0 = \text{III}_L\) is a subgroup of \(\mathcal{N}(\text{III}_L) \cap \text{Isom}(H^2 \times \mathbb{R}) = \{1, -I_1, I_2, -I_3\} \text{III}_L\), it is

---

Table XII. Normalizers for the invariance groups with \(G_0 = \text{III}_L, \text{Isom}_0(H^2 \times \mathbb{R})\). The formulas for conjugate transformations are given in Appendix B

| \(G\) | \(o/u\) | \(f \in \mathcal{N}(G)\) |
|---|---|---|
| \(\text{III}_L\) | 0 | \(\{1, -I_1, I_2, -I_3\} L(a,b,c) R_H(\theta) D(1,1,k)\) |
| \(\text{III}_L \rtimes \{1, -I_3\}\) | 0 | \(d = 0\) |
| \(\text{III}_L \rtimes \{1, -I_1\}\) | 0 | \(d = 0\) |
| \(\text{III}_L \rtimes \{1, I_2\}\) | 0 | \(d = 0\) |
| \(\text{III}_L \rtimes \{1, -I_1, I_2, -I_3\}\) | 0 | \(\text{ibid}\) |
| \(\text{Isom}_0(H^2 \times \mathbb{R})\) | 0 | \(\{1, -I_1, I_2, -I_3\} L(a,b,c) R_H(\theta) D(1,1,k)\) |
| \(\text{Isom}_0(H^2 \times \mathbb{R}) \rtimes \{1, I_2\}\) | 0 | \(\text{ibid}\) |
| \(\text{Isom}_0(H^2 \times \mathbb{R}) \rtimes \{1, -I_3\}\) | 0 | \(\text{ibid}\) |
| \(\text{Isom}(H^2 \times \mathbb{R})\) | 0 | \(\text{ibid}\) |
given by $G_0$ itself or the semi-direct product of $G_0$ with one of \{1, $-I_3$\}, \{1, $I_2$\}, \{1, $-I_1$\} and \{1, $-I_1$, $I_2$, $-I_3$\}. Among these $\text{III}_L$ and $\text{III}_L \rtimes \{1, I_2\}$ are orientation preserving.

Although we listed the normalizers of these invariance groups with $G_0 = \text{III}_L$ in the table XII, we do not need those because these groups do not appear as an invariance group of a compact Bianchi model for the following reason. The group $\text{III}$ is classified as class B and $c_I = c^L_{IJ}X_J$ does not vanish. From this constant, we can construct a vector field $V = Q^{IJ}c_I X_J$, which is invariant under $\text{III}_L$. Since this vector field has a constant divergence $D_I V^I = Q^{IJ}c_I c_J$, if the space $\Sigma$ is given by $\mathbb{R}^3/K$ with a discrete group $K$ of isometries, there must be some transformation $f \in K$ such that $f_* V \neq V$, because otherwise $V$ is a pull back of some vector field $V'$ on $\Sigma$ with the same constant divergence, which leads to the contradiction

$$0 = \int_{\Sigma} d^3v D_I V'^I = D_I V'^I \int_{\Sigma} d^3v. \quad (6.43)$$

In the present case, $(c_I) = (0, -1, 0)$ and $V = -Q^{2I}X_I$. If we restrict to orientation preserving transformations, the transformation which may move this vector is only $I_2$. However, if the metric is $I_2$-invariant, $Q^{21}$ and $Q^{23}$ vanish, and $V$ is parallel to $X_2$, which is invariant under $I_2$. Therefore we cannot compactify $H^2 \times \mathbb{R}$ by a discrete group $K$ contained in a group $G$ with $G_0 = \text{III}_L$. On the other hand, since $R_H(\theta)$ moves the vector $X_2$, we can compactify a system whose invariance group contains $\text{Isom}_0(H^2 \times \mathbb{R})$, as we see soon.

6.2. Compact topologies

6.2.1. Fundamental group

As we saw in §6.1.3, any isometry of $H^2 \times \mathbb{R}$ is written as a product of isometries of $H^2$ and $\mathbb{R}$. This implies that, if $H^2 \times \mathbb{R}$ is compactified to $\Sigma$ by some discrete group $K \subset \text{Isom}(H^2 \times \mathbb{R})$, each transformation in $K$ preserves the bundle structure $H^2 \times \mathbb{R} \rightarrow H^2$. Hence $\Sigma$ has also a bundle structure $\pi: \Sigma \rightarrow X = H^2/K'$ with fibre $S^1$, where $K'$ is a discrete transformation group of $H^2$ induced by $K$. To be precise, $\Sigma$ is not a bundle in the narrow sense in general, because a transformation in $K'$ may have a fixed point. If the fixed point is isolated, the isotropy group becomes an Abelian group of a finite order isomorphic to $\mathbb{Z}_p$, and the base space $X$ has a conical singularity of the form $\mathbb{R}^2/\mathbb{Z}_p$ there. On the other hand, if the fixed point is not isolated, it belongs to a curve of fixed points. Such a situation arises when $K'$ contains a reflection with respect to a geodesic in $H^2$. In this case the reflection curve becomes a boundary of the base space. Thus the base space is represented by a surface with conical singularities and reflection curves, which is called an orbifold. The integer $p$ characterizing the structure of a conical singularity is called the orbifold index of that point.

Let $D'$ be a small disk around a point $x$ in $X$ (or a half disk in the case $x$ is on a reflector curve) and $V$ be a connected component of the preimage in $H^2 \times \mathbb{R}$ of
\[ U = \pi^{-1}(D') \subset \Sigma. \] Then from the diagram,

\[
\begin{array}{ccc}
H^2 \times \mathbb{R} & \xrightarrow{\pi} & \Sigma \\
\downarrow & & \downarrow \pi \\
H^2 & \xrightarrow{\gamma} & X
\end{array}
\]

(6.44)

\( V \) is expressed as \( V = D \times \mathbb{R} \), where \( D \) is a disk in \( H^2 \) projected onto \( D' \) in \( X \). Further, let \( L \) be the kernel of the homomorphism \( j : K \to K', \) \( K_y \) be the isotropy group of \( K' \) at the point \( y \) in \( D \) projected to \( x \), and \( K_y \) be the preimage of \( K'_y \) in \( K \) by \( j \). Then, \( L \) is isomorphic to \( \mathbb{Z} \) generated by a translation \( \ell \) along \( \mathbb{R} \) in \( H^2 \times \mathbb{R} \), and the following exact sequence holds:

\[
0 \to L \xrightarrow{i} K_y \xrightarrow{j} K'_y \to 0 \text{ (exact).} \quad (6.45)
\]

Since \( K_y \) coincides the set of transformations in \( K \) which maps \( V \) to \( V \), \( K_y \) is a subgroup of the isometry group of \( V \cong D \times \mathbb{R} \) which is generated by rotations of \( D \), translations along \( \mathbb{R} \) and reflections in \( D \) and \( \mathbb{R} \), and \( U \) is written as \( V/K_y \). Since a combination of a translation and a reflection at a point in \( \mathbb{R} \) is always a reflection at some point, such a transformation always has fixed points in \( V \) and cannot be contained in \( K_y \). On the other hand, a combination of a translation in \( \mathbb{R} \) and reflection in \( D \) does not have a fixed point. However, such an orientation-reversing transformation makes \( \Sigma \) unorientable. Since we are only interested in orientable 3-spaces, we will not consider such cases in the present paper. Under this restriction, the base orbifold has no reflector curve. Further, since a properly discontinuous subgroup generated by translations and rotations in \( D \times \mathbb{R} \) is always an infinite cyclic group, \( K_y \) is isomorphic to \( \mathbb{Z} \). This implies that \( U \) is homeomorphic to a solid torus.

Here note that the conjugate transformation \( a\ell a^{-1} \) of \( L \) by an element \( a \) of \( K_y \) depends only on \( j(a) \) because \( L \) is Abelian. Further, since the automorphism of a free cyclic group is given by either the identity transformation or the inversion transformation, \( a\ell a^{-1} \) coincides with \( \ell \) or \( \ell^{-1} \). Since we are only considering the case \( \Sigma \) is orientable, the former happens when \( a \) projects to a orientation preserving transformation of \( H^2 \) and the latter when \( a \) projects to a orientation reversing one.

If \( x \) is a regular point, \( K'_y \) is trivial and \( K_y \) coincides with \( L \). Hence, \( U \) is written as \( V/L \cong V/\mathbb{Z} \) and has a regular bundle structure isomorphic to the solid torus \( D' \times S^1 \). On the other hand, if \( x \) is a conical singularity with index \( p \), \( K'_y \) is isomorphic to \( \mathbb{Z}_p \) and \( K_y \) does not coincide with \( L \). Let \( \gamma \) be a transformation in \( K \) which projects to the generator \( \tilde{\gamma} \) of \( K'_y \) corresponding to the rotation of angle \( 2\pi/p \) in \( H^2 \). Then, since \( j(\gamma^p) = \tilde{\gamma}^p = 1 \), there exists an integer \( r \) such that \( \gamma^p \ell^r = 1 \) and \( p \) and \( r \) are coprime. We can make \( r \) to be \( 0 < r < p \) by replacing \( \gamma \) by \( \gamma \ell^c \) with some integer \( c \). Under this normalization, \( \gamma \) represents the translation along \( \mathbb{R} \) corresponding to \( -r/p \) times \( \ell \) and rotation of angle \( 2\pi/p \) around the central line in \( V = D \times \mathbb{R} \). Here let \( (q, s) \) be the pair of coprime integers uniquely determined by the condition that \( 0 < q < p \) and \( ps + qr = 1 \). Then it is easy to show that for \( \lambda = \gamma^{-q}\ell^s \), \( \ell = \lambda^p \) and \( \gamma = \lambda^{-r} \). Hence \( K_y \) is a free cyclic group generated by \( \lambda \).
This implies that $U$ is homeomorphic to the solid torus as before, but it is not a regular $S^1$ bundle over $D$ because $\lambda$ corresponds to a combination of the translation of $1/p$ times $\ell$ and the rotation $-2\pi q/p$ around the central line in $D \times \mathbb{R}$. By this transformation $\lambda$, the central line is identified to $S^1$, but the other lines are identified to $S^1$ only by $\lambda^p$. The $S^1$ corresponding to this central line is called a critical fibre, and the others regular fibres. In the torus $U$, if one goes around the critical fiber once, regular fibres rotate by the angle $2\pi q/p$ around it. The latter close only by $p$ turns.

Thus we have shown that a compact quotient of $H^2 \times \mathbb{R}$ is fibred by $S^1$, and a small neighborhood of each fibre is isomorphic to a solid torus with a twisted fibering. The latter case occurs when the fibre is critical and projects to a conic singularity in the base orbifold. A manifold of this type of $S^1$-fibering structure is called a Seyfert fibre space. The pair of coprime integers $(p, q)$ or $(p, r)$ characterizing the twist around the critical fibre is called the Seifert index of that fibre. To be exact, a Seyfert fibre space is in general allowed to contain a critical fibre whose neighborhood is isomorphic to a solid Klein bottle. In such a case, the base orbifold contains a reflector curve and the total space becomes unorientable.

In general, a Seifert fibred space may allow a different fibring structure which is not isomorphic to the original one. Fortunately, however, it can be shown that a compact quotient of $H^2 \times \mathbb{R}$ or $SL_2 \mathbb{R}$ admit a unique Seifert fibering\(^{14}\). Hence we can determined the structure of the fundamental group $\pi_1(\Sigma) \cong K$ by the Seifert bundle structure of $\Sigma$ over the base orbifold $X$ described above in the following way.

First, we consider the case in which the base orbifold is an orientable closed surface of genus $g$ with $k$ conical singular points $x_i$ with indices $(p_i, r_i)$. Let $D_i$ be a small disk around $x_i$ and $N$ be the closure of $X - \bigcup_i D_i$. Then, $\Sigma' = \pi^{-1}(N)$ contains only regular fibres and has a normal $S^1$-bundle structure over $N$. Then the covering transformation in $H^2 \times \mathbb{R}$ corresponding to the $S^1$ fibre is given by the generator $\ell$ of $L$, and from the exact sequence for homotopies groups, we have

$$0 \to L \xrightarrow{\iota} \pi_1(\Sigma') \xrightarrow{j} \pi_1(N) \to 1. \tag{6.46}$$

Here, $\pi_1(N)$ is expressed in terms of $g$-pairs of isometries $(\bar{\alpha}_a, \bar{\beta}_a)$ corresponding to generators of $\pi_1(X)$ and $\ell$ isometries $\bar{\gamma}_i$ corresponding to the boundary of $D_i$ as

$$\pi_1(N) = \langle \bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_g, \bar{\beta}_g, \bar{\gamma}_1, \ldots, \bar{\gamma}_k \ | \ [\bar{\alpha}_1, \bar{\beta}_1] \cdots [\bar{\alpha}_g, \bar{\beta}_g] \bar{\gamma}_1 \cdots \bar{\gamma}_k = 1 \rangle. \tag{6.47}$$

Let $\alpha_a, \beta_a$ and $\gamma_i$ be generators of $\pi_1(\Sigma')$ which project to $\bar{\alpha}_a, \bar{\beta}_a$ and $\bar{\gamma}_i$, respectively. Then the relation among the generators in $\pi_1(N)$ is lifted to $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = \ell^b$ with some integer $b$. We must add relations between $\ell$ and the other generators to specify $\pi_1(\Sigma')$. In the case $k > 0$ it is simple. Since $N$ is homotopic to a join of circles in this case, and since an orientable $S^1$-bundle over each circle is always trivial, all elements in $\pi_1(\Sigma')$ commute with $\ell$. Therefore, $\pi_1(\Sigma')$ is given by

$$\pi_1(\Sigma') = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k, \ell \ | \ [\alpha_a, \ell] = [\beta_a, \ell] = [\gamma_i, \ell] = 1, [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = \ell^b \rangle. \tag{6.48}$$

From the van Kampfen theorem, the fundamental group of the total space $\Sigma$ is obtained from this and the fundamental groups of solid tori around the critical fibres.
by adding relations of their generators in \( \pi_1(\Sigma) \) to the free product of them. In the present case, since each critical fibre is homotopic to some product of \( \ell \)'s and \( \gamma_i \)'s as shown above, \( \pi_1(\Sigma) \) is obtained from \( \pi_1(\Sigma') \) by adding relations among their generators in \( \pi_1(\Sigma) \). Since \( \bar{\alpha}_a \) and \( \bar{\beta}_a \) are independent in \( \pi_1(X) \), such relations only contains \( \ell \) and \( \gamma_i \). From the explanation above on the homotopy relations among \( \ell, \gamma \) and \( \lambda \) in a twisted torus, we can adopt as these relations the normalization conditions \( \gamma_i^\ell \ell^r = 1 \) for the \( \ell \)-multiplication freedom in the choice of \( \gamma_i \). Note that under these normalization conditions, \( b \) becomes a topological invariant which is independent of rescaling of the other generators by \( \ell \). Thus we obtain

\[
\pi_1(\Sigma) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k, \ell \mid [\alpha_a, \ell] = [\beta_a, \ell] = [\gamma_i, \ell] = 1, \\
\gamma_i^\ell \ell^r = 1, [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = \ell^b \rangle. \quad (6.49)
\]

This expression also holds in the case \( k = 0 \), i.e., when the orbifold \( X \) is a regular surface and \( \Sigma \) is a \( S^1 \) bundle over it. To see this, we take an arbitrary small disk in \( X \) and define \( N \) as the closure of \( X - D \). Then \( \pi_1(\Sigma') \) is given by the expression (6.48) with a single \( \gamma \) generator. Now, the original bundle \( \Sigma \) is obtained by gluing the solid torus \( D \times S^1 \), instead of a twisted torus, to \( \Sigma' \) along their boundaries. After gluing, \( \gamma \) becomes homotopic to \( \ell' \) with some integer \( c \), because \( \ell \) is the generator of the gluing torus in the present case. Hence, by eliminating \( \gamma \) in terms of this relation and redefining \( b \), we obtain the expression (6.49) with no \( \gamma \) generator. In this case, \( -b \) coincides with the Euler number of the \( S^1 \) bundle.

Next we consider the case in which the base orbifold is unorientable. The derivation of the fundamental group in this case is similar to the above case except for the following two points. First, the fundamental group of an unorientable closed surface with genus \( g \) is generated by \( g \) loops \( \bar{\alpha}_a \) in stead of \( 2g \) loops in the orientable case, and the relation for the generators of \( \pi_1(N) \) are replaced by \( \bar{\alpha}_1^2 \cdots \bar{\alpha}_g^2 \bar{\gamma}_1 \cdots \bar{\gamma}_k = 1 \). Second, although the \( S^1 \) bundle over \( N \) is still determined by its restriction on each loop \( \bar{\alpha}_a \), this time we must choose the Klein bottle as the bundle over each \( \bar{\alpha}_a \) to make the total space orientable, because a normal neighborhood of \( \bar{\alpha}_a \) in \( N \) is a Möbius band. This implies that the relation between \( \ell \) and \( \alpha_a \) is given by \( \alpha_a \ell \alpha_a^{-1} = \ell^{-1} \).

Taking these points, we find that the fundamental group of the Seifert bundle over an unorientable orbifold is given by

\[
\pi_1(\Sigma) = \langle \alpha_1, \ldots, \alpha_g, \gamma_1, \ldots, \gamma_k, \ell \mid \alpha_a \ell \alpha_a^{-1} = \ell^{-1}, [\gamma_i, \ell] = 1, \gamma_i^\ell \ell^r = 1, \alpha_1^2 \cdots \alpha_g^2 \gamma_1 \cdots \gamma_k = \ell^b \rangle. \quad (6.50)
\]

The argument on the fundamental group so far is quite general and applies to any orientable Seifert bundle over an orbifold with no reflector curve. This implies that some additional constraints must be imposed on the indices appearing in the expressions for \( \pi_1(\Sigma) \) above to obtain the fundamental groups for compact quotients of each Thurston type, because the fundamental groups for compact quotients of different Thurston types are not isomorphic due to Thurston’s theorem \(^{14} \). For the Thurston type \( H^2 \times \mathbb{R} \), they are given by the conditions \( \chi < 0 \) and \( e = 0 \), where \( \chi \) is the Euler characteristic of the orbifold defined in terms of the standard Euler
characteristic $\chi_0(X)$ of the orbifold as a topological space and the orbifold index $p_i$ by

$$\chi(X) = \chi_0(X) - \sum_i \left(1 - \frac{1}{p_i}\right), \quad (6.51)$$

and $e$ is the Euler number of the Seifert bundle defined in terms of the Seifert index $(p_i, r_i)$ and $b$ as

$$e = -b - \sum_i \frac{r_i}{p_i}. \quad (6.52)$$

From now on we generally denote the Seifert fibred spaces with the fundamental groups $(6.49)$ and $(6.50)$ by $S^+(g, e; \{(p_1, r_1), \ldots, (p_k, r_k)\})$ and $S^-(g, e; \{(p_1, r_1), \ldots, (p_k, r_k)\})$, respectively. They will be also written simply as $S^+(g, e; k)$, when we do not write the Seifert indices explicitly.

6.2.2. Moduli freedom

In order to determine the moduli freedom of each compact quotient of $H^2 \times \mathbb{R}$, we must classify the conjugate class of embeddings of $\pi_1(\Sigma)$ into each invariance group $G$ with respect to $\mathcal{N}^+(G)$. As explained in §6.1.4, a possible invariance group $G$ is $\text{Isom}_0(H^2 \times \mathbb{R}) \cong PSL_2 \times \mathbb{R}$ or $\text{Isom}^+(H^2 \times \mathbb{R}) \cong \text{Isom}_0(H^2 \times \mathbb{R}) \times \{1, I_2\}$.

For $\Sigma = S^+(g, 0; k)$, in which the base orbifold is orientable, the image of $\pi_1(\Sigma)$ is always contained in $\text{Isom}_0(H^2 \times \mathbb{R})$. Further, $\mathcal{N}^+(G)$ is the same for both invariance groups. Hence we only have to consider the case $G = \text{Isom}_0(H^2 \times \mathbb{R})$.

First, from the argument above, the generator $\ell$ must be represented by a translation $T_1 : z \mapsto z + l$ along $\mathbb{R}$. Then, from the relation $\gamma_i^{p_i} \ell^{r_i} = 1$, $\gamma_i$ is uniquely determined if we specify the center position of the rotation in $H^2$ for $\gamma_i$. Further, when $2g$ elements $\alpha_a$ and $\beta_a$ of $\text{Isom}^+(H^2)$ satisfying the relation

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = 1 \quad (6.53)$$

are given, $\alpha_a$ and $\beta_a$ are expressed as $T_{x_a, \ell} \alpha_a, T_{y_a, \ell} \beta_a$, where $x_a$ and $y_a$ are arbitrary real numbers. Here, the number of degrees of freedom in the choice of the rotation center of each $\gamma_i$ is two, that of $\alpha_a$ or $\beta_a$ as an element of $PSL_2 \mathbb{R}$ is three, and the relation (6.53) gives three independent scalar equations. Hence, the number of the total degrees of freedom in the embedding $K$ is given by

$$1 + 2k + (3 + 1) \times 2g - 3 = 8g + 2k - 2. \quad (6.54)$$

On the other hand, $\mathcal{N}^+(\text{Isom}(H^2 \times \mathbb{R}))$ is given by $\{1, I_2\} \text{Isom}^+(H^2) \times \left\{D(1, 1, k)\right\} \times \{T_c\}$. Among the transformations in this group, $I_2$ changes $T_1$ to $T_{-1}$. Hence, this freedom is eliminated by requiring $l > 0$. Further, $D(1, 1, k)(k > 0)$ transforms $T_1$ to $T_{kl}$, but does not affect the transformations in $\text{Isom}(H^2)$. Finally, $\text{Isom}^+(H^2)$ acts on the space of the set $(\alpha_a, \beta_a, \gamma_i)$ freely at generic points, while $T_c$ does not affect the moduli. Hence, the dimension of each orbits of the action $\mathcal{N}^+(\text{Isom}(H^2 \times \mathbb{R}))$ on the space of $K$ is $1 + 3 = 4$, and the number of the moduli degrees of freedom is given by

$$\mathcal{N}_{m+} = \dim \left(\text{Mono}(\pi_1(S^+(g, 0; k)), \text{Isom}_0(H^2 \times \mathbb{R}))/\text{ad}(\mathcal{N}^+(\text{Isom}_0(H^2 \times \mathbb{R})))\right) = 8g + 2k - 6. \quad (6.55)$$
Table XIII. The parameter count for the type $H^2 \times \mathbb{R}$.

| Space           | Symmetry       | $Q$ | $P$ | $N_m$ | $N_f$ | $N$ | $N_s$ | $N_s$ (vacuum) |
|-----------------|----------------|-----|-----|-------|-------|-----|-------|----------------|
| $\mathbb{R}^3$ | Isom($H^2 \times \mathbb{R}$) | 1   | 2   | 0     | 1     | 4   | 2     | 1              |
| $M^\pm(g, k; 0)$ | Isom($H^2 \times \mathbb{R}$) | 2   | 2   | $N_m^\pm$ | 1     | $5 + N_m^\pm$ | 3 + $N_m^\pm$ | 2 + $N_m^\pm$ |

The isotropy group $N_{\Sigma 0}(G)$ of the action of $N^+(\text{Isom}_0(H^2 \times \mathbb{R}))$ on the moduli space is generated by $T_c$ in $\Pi L$.

Here note that, since $\chi_0(X)$ for an orientable closed orbifold with genus $g$ is given by $2 - 2g$, the condition $\chi < 0$ requires that for $g = 0, 1$ orbifolds always have conic singularities:

$$k \geq \begin{cases} 3 & : g = 0 \\ 1 & : g = 1 \end{cases}.$$  \hfill (6.56)

Further, the condition $e = 0$ yields the strong restriction on the possible values of the Seifert indices that $\sum_i r_i / p_i$ is an integer. It should be noted that this condition is essential to guarantee that $b$ becomes always an integer regardless of the choice of the parameter $l$.

Next let us consider the case $\Sigma = S^-(g, 0; k)$ for which the base orbifold is unorientable. Since we need transformations which reverse the orientation of $H^2$ in this case, $\pi(\Sigma)$ can be embedded only in $\text{Isom}^+(H^2 \times \mathbb{R})$. $\ell$ is still represented as $T_\ell$, and $\gamma_i$ is uniquely specified by a rotation $\bar{\gamma}_i$ in $H^2$, but from the relation $\alpha_\ell \alpha_{\ell^{-1}} = \ell^{-1}$, $\alpha_\ell$ is now represented as $I_2 T_{x_a} \tilde{\alpha}_a$ with $x_a \in \mathbb{R}$ and $\tilde{\alpha}_a \in \text{Isom}^+(H^2)$ satisfying the condition

$$\tilde{\alpha}_1^2 \cdots \tilde{\alpha}_g^2 \bar{\gamma}_1 \cdots \bar{\gamma}_k = 1.$$  \hfill (6.57)

Hence, the number of the degrees of freedom in $K$ is now given by $4g + 2k - 2$. Further, since $T_\ell$ in $N^+(\text{Isom}^+(H^2 \times \mathbb{R}))$ transforms $I_2$ as $T_\ell I_2 T_{-\ell} = I_2 T_{-2\ell}$, the dimension of HPD orbits becomes 5. Hence the number of the moduli degrees of freedom for $S^-(g, 0; k)$ is given by

$$N'_{m-} = \dim \left( \text{Mono}(\pi_1(S^-(g, 0; k)), \text{Isom}^+(H^2 \times \mathbb{R}))/\text{ad}(N^+(\text{Isom}^+(H^2 \times \mathbb{R}))) \right)$$  
$$= 4g + 2k - 7,$$  \hfill (6.58)

and $N_{\Sigma 0}(G)$ becomes trivial.

We also obtain a constraint on the number of conic singularities like (6.56). Since $\chi_0(X)$ for the unorientable closed surface with genus $g$ is given by $2 - g(g \geq 1)$, the constraint is expressed as

$$k \geq \begin{cases} 2 & : g = 1 \\ 1 & : g = 2 \end{cases}.$$  \hfill (6.59)

6.3. Phase Space

As we have shown in the previous subsection, the isotropy group $N_{\Sigma 0}(G)$ of the action of $N^+(G)$ on the moduli space is given by a subgroup of $\Pi L$. However,
since the action of $\mathbb{III}_L$ on the space of homogeneous data on the covering space $\tilde{\Sigma} = H^2 \times \mathbb{R}$ is trivial, the degrees of freedom for the homogeneous data is simply determined by the invariance of the data by $G$.

First, note that if the system is invariant under $\text{Isom}_0(H^2 \times \mathbb{R})$, the components of any quantity with respect to the invariant basis are invariant under $R_3(\theta)$ for any $\theta$ from Eq.(6.40). Hence $Q_{IJ}, P_{IJ}$ must have the diagonal forms $Q = D(Q_1, Q_1, Q_3)$ and $P = (P_1, P_1, P_3)$, respectively, and $u_1 = u_2 = 0$. Further from the diffeomorphism constraint

$$H_3 = cu_3 = 0,$$

$u_3$ also vanishes. Hence, the system with a single-component fluid always has the maximal symmetry $G = \text{Isom}_0^+(H^2 \times \mathbb{R}) = \text{Isom}_0(H^2 \times \mathbb{R}) \rtimes \{1, I_2\}$, and the phase space for the homogeneous data is given by

$$\Gamma_D(H^2 \times \mathbb{R}, G) = \{(Q_1, Q_3; P_1, P_3; \rho) \mid Q_1, Q_3 > 0\}. \quad (6.61)$$

Thus, putting together this result with the argument on the moduli freedom in the previous subsection, we find that the number of the total dynamical degrees of freedom is given by

$$N = 5 + N'_{m\pm}. \quad (6.62)$$

For a multi-component system, $\text{Isom}_0(H^2 \times \mathbb{R})$ is also allowed as the invariance group, for which $N$ is increased by 2 for each extra component of fluid. For $G = \text{Isom}_0^+(H^2 \times \mathbb{R})$, the contribution of each extra component to $N$ is just one.

§7. Type $\widehat{SL_2\mathbb{R}}$

7.1. Invariance subgroups and normalizers

7.1.1. Bianchi type VIII group

Any matrix $V \in \widehat{SL_2\mathbb{R}}$ is uniquely decomposed into the product of a lower triangle matrix $Z(\zeta) \in TSL_2^+$ and a $\text{SO}(2)$ matrix $R(z)$ as

$$V = Z(\zeta)R(z). \quad (7.1)$$

From this Iwazawa decomposition, we see that $\widehat{SL_2\mathbb{R}}$ is homeomorphic to $H^2 \times S^1$, and that an element of the universal covering group $\widehat{SL_2\mathbb{R}}$ of this group is parametrized by $\zeta \in H^2$ and $z \in \mathbb{R}$ as $\tilde{V}(\zeta, z)$. This element is mapped to $Z(\zeta)R(z)$ by the natural projection $p : \widehat{SL_2\mathbb{R}} \to SL_2\mathbb{R}$. Let us denote $\tilde{V}(\zeta, 0)$ by $\tilde{V}(\zeta)$ and $\tilde{V}(i, z)$ by $\tilde{R}(z)$. Then a general element of $\widehat{SL_2\mathbb{R}}$ is expressed as

$$\tilde{V} = \tilde{Z}(\zeta)\tilde{R}(z). \quad (7.2)$$

Here note that $\tilde{Z}(\zeta)$ and $Z(\zeta)$ are in one-to-one correspondence, while $\tilde{R}(z + 2\pi) \neq \tilde{R}(z)$ and $\tilde{R}(z)$ is in one-to-one correspondence to $z \in \mathbb{R}$. Thus $\tilde{Z}(\zeta)$’s form a group isomorphic to $H^2 \cong TSL_2^+$, and $\tilde{R}(z)$ form a group isomorphic to $\mathbb{R}$. However, $\widehat{SL_2\mathbb{R}}$ is not isomorphic to $H^2 \times \mathbb{R}$ because $\tilde{Z}$ and $\tilde{R}$ do not commute. Using the fact
that $SL_2\mathbb{R}$ and $\hat{SL}_2\mathbb{R}$ have the same group structure near the unit element, we can show that the commutation relation is given by

$$\tilde{R}(z)\tilde{Z}(\zeta) = \tilde{Z}(\zeta')\tilde{R}(z'); \quad (7.3)$$

$$\zeta' = R(z)\ast \zeta, \quad (7.4)$$

$$z' = H(z, \zeta), \quad (7.5)$$

where the function $H(z, \zeta)$ is defined by

$$H(z, \zeta) = \Im \int_{0}^{z} \frac{d\phi}{|\cos \phi - \zeta \sin \phi|^2}. \quad (7.6)$$

This function satisfies the relation (6.41) and

$$H(n\pi, \zeta) = n\pi, \quad \forall n \in \mathbb{Z}, \forall \zeta \in H^2, \quad (7.7)$$

$$H(z, i) = z, \quad \forall z \in \mathbb{R}. \quad (7.8)$$

From this it follows that the group multiplication law is given by

$$(a + ib, c) \cdot (x + iy, z) = (a + bR(c) \ast (x + iy), z + H(c, x + iy)) \quad (7.9)$$

In particular, the left transformation $L_{(a + ib, c)}$ corresponding to the left multiplication of $\tilde{V}(a + ib, c)$ is expressed as

$$L_{(a + ib, c)} : (\zeta, z) \mapsto (a + bR(c) \ast \zeta, z + H(c, \zeta)), \quad (7.10)$$

and the right transformation $R_{(a + ib, c)}$ as

$$R_{(a, c)} : (x + iy, z) \mapsto (x + yR(z) \ast \alpha, c + H(z, \alpha)). \quad (7.11)$$

From these expressions, we see that $L_{(\alpha, c)} = R_{(\alpha', c')}$ if and only if $\alpha = \alpha' = i$ and $c = c' = n\pi$ for some $n \in \mathbb{Z}$. This implies that the center of $\hat{SL}_2\mathbb{R}$ is given by

$$C(\hat{SL}_2\mathbb{R}) \equiv \mathcal{VIII}_L \cap \mathcal{VIII}_R = \{ \tilde{R}(n\pi) \mid n \in \mathbb{Z} \}. \quad (7.12)$$

Here note that $R_{(i, c)}$ gives just the translation of $z$,

$$T_c = R_{(i, c)} : (x + iy, z) \mapsto (x + iy, z + c), \quad (7.13)$$

because $R(z) \ast i = i$. The center is contained in the intersection of this subgroup of right transformations with the corresponding subgroup of left transformations $\{ L_{(i, c)} \}$.

From (7.10), the infinitesimal left transformations of $\hat{SL}_2\mathbb{R}$ are generated by

$$\xi_1 = \frac{1}{2}(1 - x^2 + y^2)\partial_x - xy\partial_y - \frac{1}{2}y\partial_z, \quad (7.14)$$

$$\xi_2 = x\partial_x + y\partial_y, \quad (7.15)$$

$$\xi_3 = \frac{1}{2}(1 + x^2 - y^2)\partial_x + xy\partial_y + \frac{1}{2}y\partial_z, \quad (7.16)$$
which form the Bianchi type VIII Lie algebra
\[ [\xi_1, \xi_2] = \xi_3, \ [\xi_3, \xi_1] = -\xi_2, \ [\xi_3, \xi_2] = \xi_1. \]  
(7.17)

From (7.11), the invariant vector fields, i.e., the infinitesimal right transformations of \( SL_2\mathbb{R} \), are generated by
\[ X_1 = y(\cos 2z \partial_x + \sin 2z \partial_y) - \frac{1}{2} \cos 2z \partial_z, \]  
(7.18)
\[ X_2 = y(-\sin 2z \partial_x + \cos 2z \partial_y) + \frac{1}{2} \sin 2z \partial_z, \]  
(7.19)
\[ X_3 = \frac{1}{2} \partial_z, \]  
(7.20)
and their dual basis is given by
\[ \chi^1 = \frac{1}{y}(\cos 2zd\bar{x} + \sin 2zd\bar{y}), \]  
(7.21)
\[ \chi^2 = \frac{1}{y}(-\sin 2zd\bar{x} + \cos 2zd\bar{y}), \]  
(7.22)
\[ \chi^3 = \frac{d\bar{x}}{y} + 2d\bar{z}; \]  
(7.23)
\[ d\chi^1 = -\chi^2 \wedge \chi^3, \ d\chi^2 = \chi^1 \wedge \chi^3, \ d\chi^3 = \chi^1 \wedge \chi^2. \]  
(7.24)

Finally, we determine the normalizer of \( VIII_L \). Since \( \mathcal{L}(VIII_L) \) is isomorphic to \( \mathcal{L}(SL_2\mathbb{R}) \cong \mathcal{L}(\text{Isom}(H^2)) \), its automorphism group is given by \( SO(2,1) \), whose connected component \( SO_+(2,1) \) is generated by inner-automorphisms of \( SL_2\mathbb{R} \), as was shown in the previous section. The remaining discrete transformation is induced from \( I_2 : (x,y,z) \mapsto (-x,y,-z) \). Hence, from Prop.A.4, the normalizer of \( VIII_L \) is given by
\[ N(VIII_L) = VIII_L \cdot VIII_R \rtimes \{1, I_2\}. \]  
(7.25)
Here we have used the symbol \( \cdot \) instead of \( \times \) to emphasize that \( VIII_L \) and \( VIII_R \) share the non-trivial center. These HPDs induce the following transformations of the invariant basis. First, for \( f = R(\alpha,c) \in \text{VIII}_R \), since the action of \( f \) on \( \xi^I \) and that of \( f^* \) on \( \chi^I \) is expressed by the same matrix, we obtain
\[ R^*(\alpha,c)\chi^I = \Lambda(\tilde{V}(\alpha,c))^I J \chi^J, \]  
(7.26)
where \( \Lambda(\tilde{V}) \) is the representation of \( SL_2\mathbb{R} \) to \( \text{SO}_+(2,1) \) induced from (6.19). Second, \( I_2 \) transforms the basis as
\[ (I_2)^* : (\chi^1, \chi^2, \chi^3) \mapsto (-\chi^1, \chi^2, -\chi^3). \]  
(7.27)

7.1.2. Maximal geometry

The most symmetric metric with \( VIII_L \) invariance is given by
\[ ds^2 = Q_1 [(\chi^1)^2 + (\chi^2)^2] + Q_3 (\chi^3)^2 = Q_1 \frac{dx^2 + dy^2}{y^2} + Q_3 \left(2dz + \frac{dx}{y}\right)^2. \]  
(7.28)
Table XIV. Normalizers for the invariance groups with $G_0 = \text{VIII}_L$, $\text{Isom}_0(\widetilde{SL}_2\mathbb{R})$.

| $G$                  | $o/u$ | $f \in \mathcal{N}(G)$                      |
|----------------------|-------|---------------------------------------------|
| $\text{VIII}_L$      | o     | $\{1, I_2\} L_{(\alpha, c)} R_{(\beta, d)}$ |
| $\text{VIII}_L \times \{1, I_2\}$ | o     | $\{1, I_2\} \{1, T_{\pi/2}\} L_{(\alpha, c)} \tilde{Z}_R(ie^\gamma)$ |
| $\text{VIII}_L \cdot \{1, T_{\pi/2}\}$ | o     | $\{1, I_2\} L_{(\alpha, c)} T_\theta$      |
| $\text{VIII}_L \cdot D_2$ | o     | $\{1, I_2\} \{1, T_{\pi/2}\} L_{(\alpha, c)}$ |
| $\text{Isom}(\widetilde{SL}_2\mathbb{R})$ | o     | $\{1, I_2\} L_{(\alpha, c)} T_\theta$      |
| $\text{Isom}(SL_2\mathbb{R})$ | o     | ibid                                        |

It is easy to see that this metric is also invariant under a translation in $z$, i.e., $T_c$. Since $\widetilde{SL}_2\mathbb{R}$ is not a constant curvature space, this implies that

$$\text{Isom}_0(\widetilde{SL}_2\mathbb{R}) = \text{VIII}_L \cdot \mathbb{R} \ni L_{(\alpha, c)} T_\theta,$$

which is generated by $\xi_1, \xi_2, \xi_3$ and

$$\xi_4 = \partial_z$$

with

$$[\xi_4, \xi_1] = 0, \quad [\xi_4, \xi_2] = 0, \quad [\xi_4, \xi_3] = 0.$$  

(7.31)

Since $[\mathcal{L}(\text{Isom}(\widetilde{SL}_2\mathbb{R})), \mathcal{L}(\text{Isom}(\widetilde{SL}_2\mathbb{R}))] = \mathcal{L}(\widetilde{SL}_2\mathbb{R})$, $f \in \mathcal{N}(\text{Isom}(\widetilde{SL}_2\mathbb{R}))$ preserves $SL_2\mathbb{R}$, i.e., $\mathcal{N}(\text{Isom}(\widetilde{SL}_2\mathbb{R})) \subset \mathcal{N}(\text{VIII}_L)$. Among transformations in $\mathcal{N}(\text{VIII}_L)$, $I_2$ transforms $T_\theta$ as

$$I_2 T_\theta I_2 = T_{-\theta}.$$  

(7.32)

Hence $I_2 \in \mathcal{N}(\text{Isom}(\widetilde{SL}_2\mathbb{R}))$. On the other hand, if $R_{(\alpha, c)} T_\theta R_{(\alpha, c)}^{-1} = T_{\theta'}$ holds for any $\theta$, its projection to $SL_2\mathbb{R}$ gives

$$R(-c)Z(\alpha)^{-1} R(\theta') Z(\alpha) R(c) = R(\theta') \iff Z(\alpha)^{-1} R(\theta') Z(\alpha) = R(\theta').$$  

(7.33)

This holds only when $Z(\alpha) \in SO(2)$, i.e., $\alpha = i$. Therefore the normalizer of $\text{Isom}_0(\widetilde{SL}_2\mathbb{R})$ is given by

$$\mathcal{N}(\text{Isom}(\widetilde{SL}_2\mathbb{R})) = \{1, I_2\} \rtimes \text{VIII}_L \cdot \mathbb{R},$$  

(7.34)

where $\mathbb{R}$ is the group $\{T_c \mid c \in \mathbb{R}\}$. Since the metric (7.28) is clearly invariant under this group, we obtain

$$\text{Isom}(\widetilde{SL}_2\mathbb{R}) = \mathcal{N}(\text{Isom}(\widetilde{SL}_2\mathbb{R})) = \mathcal{N}(\text{Isom}_0(\widetilde{SL}_2\mathbb{R})).$$  

(7.35)

7.1.3. Invariance groups

From the argument in the previous subsection, invariance groups with $G_0 = \text{Isom}_0(\widetilde{SL}_2\mathbb{R})$ are given by $\text{Isom}_0(\widetilde{SL}_2\mathbb{R})$, and $\text{Isom}(\widetilde{SL}_2\mathbb{R})$. 
The invariance subgroups $G$ with $G_0 = \text{VIII}_L$ are determined as follows. Since $\text{Isom}(\widetilde{SL}_2\mathbb{R}) \subset \mathcal{N}(\text{VIII}_L)$, candidates of discrete isometries to be added are $I_2$ and $T_\theta$. Clearly $\text{VIII}_L \times \{1, I_2\}$ is a subgroup of $\text{Isom}(\widetilde{SL}_2\mathbb{R})$. On the other hand, since $T_\theta$ transforms the invariant basis as $T_\theta^* \chi = R(2\theta)^{IJ} \chi^J$, (7.36) any $\text{VIII}_L$-invariant tensor quantity becomes $\text{Isom}_0(\widetilde{SL}_2\mathbb{R})$ invariant if it is invariant under $T_\theta$ unless $2\theta = n\pi (n \in \mathbb{Z})$. Further, $T_{n\pi} = L_{(i,n\pi)}$. Hence, the only possible element of the form $T_\theta$ in $G$ which does not belong to $\text{VIII}_L$ is $T_{\pi/2}$. It commutes with the left transformations and with $I_2$ mod. $\text{VIII}_L$, $I_2T_{\pi/2} = T_{\pi/2}I_2T_\pi$. Hence the possible invariance groups with $G_0 = \text{VIII}_L$ are given by $\text{VIII}_L$, $\text{VIII}_L \times \{1, I_2\}$, $\text{VIII}_L \cdot \{1, T_{\pi/2}\}$, and $\text{VIII}_L \cdot \widetilde{D}_2$, where $\widetilde{D}_2 = \{I, I_2, T_{\pi/2}, I_2T_{\pi/2}\}$. (7.37)

The normalizers of these groups are given in Table XIV.

Finally note that $\text{Isom}_0(\widetilde{SL}_2\mathbb{R})$ also contains $\text{III}_L$ as a transitive subgroup, say $\{L_\alpha T_c \hat{a} | \hat{a} \in \mathbb{R}^2, c \in \mathbb{R}\}$. However, we do not consider this subgroup separately, because a system with the local $\text{III}_L$ invariance on a compact quotient always has an invariance group of higher dimension as shown in §6.1.4. Thus, in the present case, it is contained in the cases with $G \supset \text{Isom}_0(\widetilde{SL}_2\mathbb{R})$.

7.2. Compact topologies

7.2.1. Fundamental group

Through the natural homomorphisms

$$\widetilde{SL}_2\mathbb{R} \to SL_2\mathbb{R} \to \text{Isom}_0(H^2) \cong PSL_2\mathbb{R},$$

(7.38)

$\widetilde{SL}_2\mathbb{R}$ acts transitively on $H^2$. Hence, $\widetilde{SL}_2\mathbb{R}$ can be regarded as a fibre bundle over $H^2$ with a fibre $\mathbb{R}$, which is isomorphic to the isotropy group $\{R(z) | z \in \mathbb{R}\}$. In terms of the parametrization $\tilde{V}(\zeta, z) \in \widetilde{SL}_2\mathbb{R}$, the projection of this bundle structure is expressed as

$$\tilde{\pi} : \tilde{V}(\zeta, z) \mapsto \zeta \in H^2.$$

(7.39)

Since the action of $L_{(a,c)}T_\theta \in \text{Isom}(\widetilde{SL}_2\mathbb{R})$ on $\widetilde{SL}_2\mathbb{R}$ is written as

$$L_{(a,c)}T_\theta : (\zeta, z) \mapsto (a + bR(c) \star \zeta, c + \theta + H(z, a)),$$

(7.40)

$\text{Isom}(\widetilde{SL}_2\mathbb{R})$ preserves this bundle structure. Hence, by the same arguments in §6.2.1, we can show that any compact quotient $\Sigma$ of $\widetilde{SL}_2\mathbb{R}$ has a Seifert bundle structure over an orbifold $X$ covered by $H^2$. Further, it can be shown that $\Sigma$ has a unique Seifert bundle structure up to isomorphism. Therefore, the fundamental group of $\Sigma$ is given by Eq.(6.49) or Eq.(6.50), depending on whether the base orbifold is orientable or not. The only difference from the $H^2 \times \mathbb{R}$ type is that now the condition on $e$ is given by $e < 0$. 


7.2.2. Moduli freedom

In the compactification of $H^2 \times \mathbb{R}$, discrete transformations in $\text{Isom}(H^2)$ to compactify $H^2$ did not belong to the group $TSL_2^+ \cong H^2$ and were associated with rotations in $H^2$. This feature required the invariance group $G$ to include $\text{Isom}_0(H^2 \times \mathbb{R})$. In contrast, in the present case, $\text{Isom}^+(H^2)$ is covered by the left translation group $\text{VIII}_L$ itself and no rotation in $\widetilde{SL_2} \mathbb{R}$ is required to compactify the base space $H^2$. Further, although the generator $\ell$ must be represented by a translation along the fibre, $T_\ell \in \text{VIII}_R$, in the present case as well from the argument in §6.2.1, we can now represent $\ell$ as a transformation in $\text{VIII}_L$, because $\text{VIII}_L$ and $\text{VIII}_R$ share the non-trivial center. Hence, in the present case, the fundamental group can be embedded even into the smallest group $\text{VIII}_L$.¹)

First we consider the embedding of $\pi_1(S^+(g,e;k))$ in $G = \text{VIII}_L$ or $G = \text{VIII}_L \times \{1, I_2\}$. The latter case is the same as for the former, because the transformation group must preserve the orientation of the orbifold, hence the fundamental group is contained in $\text{VIII}_L$. In the embedding of $\pi_1(\Sigma)$ given by Eq.(6.49) into $\text{VIII}_L$, the freedom in the choice of the generators $\bar{\alpha}_a, \bar{\beta}_a, \bar{\gamma}_i$ is the same as in the case of $H^2 \times \mathbb{R}$. However, the freedom in taking their lifts to $G$ and in the choice of $\ell$ is different. In fact, from the exact sequence

$$0 \to \{T_{n\pi}\} \xrightarrow{i} \text{VIII}_L \xrightarrow{j} \text{Isom}_0(H^2) \to 1,$$

(7.41)

$l$ for $\ell = T_i$ is restricted to $l = n\pi$ with non-zero integer $n$. Similarly, the freedom in the lifts of the generators $\bar{\alpha}_a, \bar{\beta}_a \in \text{Isom}_0(H^2)$ are represented by the multiplication of translations of the form $T_{m\pi}$ with $m \in \mathbb{Z}$. This restriction to discrete values of the freedom in the lifts decreases the number of the moduli degrees of freedom by $1 + 2g$, compared with the $H^2 \times \mathbb{R}$ case. On the other hand, $\mathcal{N}(G)$ is now contained in $\{1, I_2\} \text{VIII}_L \cdot \text{VIII}_R$, in which the action of the subgroup $\text{VIII}_R$ on the moduli is trivial, while the action of $\text{VIII}_L$ on the moduli has no continuous isotropy group. Hence the effective HPD freedom in the moduli is smaller by 1 than the $H^2 \times \mathbb{R}$ case. Thus, the number of the moduli freedom is given by

$$N_{m+} = N_{m+}^l - 2g = 2k + 6g - 6.$$  

(7.42)

Since $I_2$ reverses the direction of $\ell$ and $R_a$ commutes with the action of $\text{VIII}_L$, the isotropy group $\mathcal{N}_{\Sigma 0}(G)$ of the action of $\mathcal{N}(G)$ in the moduli space is given by $\mathcal{N}(G) \cap \text{VIII}_R$.

So far we have not touched upon the lift of $\bar{\gamma}_i$. Because the normalization condition $\bar{\gamma}_i^P \ell^P = 1$ always fixes the lift uniquely, it does not affect the argument on the degrees of freedom. However, it is not guaranteed that there exits a lift satisfying this condition in general. In fact, in the present case, this condition yields a restriction on topologies whose fundamental group can be embedded in $\text{VIII}_L$. To

¹) In the appendix of Ref. 11) it was claimed that $\widetilde{SL_2} \mathbb{R}$ cannot be compactified by a discrete group $K$ contained in $\text{VIII}_L$. But the proof there is not correct because they assumed that the transformation group of the base space $H^2$ induced from the action of $K$ through the bundle structure $\widetilde{SL_2} \mathbb{R} \to H^2$ was contained in $H^2_L$. In reality, $\text{VIII}_L$ induces $\text{Isom}_0(H^2)$, as shown above.
see this, note that $\gamma_i$ is represented by the left multiplication of an element in $SL_2\mathbb{R}$ of the form $Z(\zeta_i)\hat{R}(\pi/p_i)Z(\zeta_i)^{-1}$ by the $I_2$ transformation if necessary, whose lift is given by $\hat{Z}(\zeta_i)\hat{R}(\pi/p_i + m_i\pi)\hat{Z}(\zeta_i)^{-1}$ with some $m_i \in \mathbb{Z}$. Thus the normalization condition is expressed as $1 + m_ip_i = -nr_i$. This equation has an integer solution for $m_i$ only when the condition
\[ (1 + nr_i)/p_i \in \mathbb{Z} \]
(7.43) is satisfied for all $i = 1, \cdots, k$. In terms of the pair of coprime integers $(q_i, s_i)$ such that $p_is_i + q_ir_i = 1$, this condition is expressed as
\[ (n + q_i)/p_i \in \mathbb{Z}. \]
(7.44) It can be shown by induction that there exists an integer $n$ satisfying this condition if and only if the greatest common divisor of $p_i$ and $p_j$ divides $q_i - q_j$ for any pair $i \neq j$:
\[ (p_i, p_j)|(q_i - q_j). \]
(7.45) Further if $n = n_0$ is the smallest positive integer satisfying the condition, $n = n_0 + mL$ satisfies the condition for any integer $m$, where $L$ is the least common multiple of $p_i$. For example, $n = 1$ satisfies the condition if and only if $r_i = p_i - 1$ for all $i = 1, \cdots, k$.

Further, the condition that $b$ is integer also restricts possible topologies. From the structure of the invariance group, $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_k$ is always written as $T_{h\pi}$ with some integer $h$. Let $h_0$ be the value of $h$ for $\gamma_i = \hat{Z}(\zeta_i)\hat{R}(\pi/p_i)\hat{Z}(\zeta_i)^{-1}$, and let $n = n_0 + mL$ be integers satisfying the above condition. Then $h$ for $n = n_0 + mL$ is written as $h = h_0 - \sum_i (1 + nr_i)/p_i$, and $b$ is given by $h = bn$. Hence, when an integer $b$ with $e < 0$ is given, we can find an embedding of the corresponding fundamental group only when the following condition is satisfied:
\[ \frac{h_0 - \sum_i 1/p_i + n_0e}{eL} \in \mathbb{Z}, \]
(7.46) where $e = -(b + \sum_i r_i/p_i)$ is the Euler number. Since $n_0$ and $h_0$ are determined by the genus of the orbifold and the Seifert indices, the allowed values of $b$ (or $e$) are also determined by them. It is very difficult to find a general expression for $h_0$. Hence, we cannot give an explicit criterion on the indices for the fundamental group to be embedded in $\text{VIII}_L$, but it is certain that for any genus and any set of orbifold indices $p_i$ there exists at least one choice of $r_i$ such that the embedding is possible. It is because $n_0 = 1$ for $r_i = p_i - 1$ and $b = h_0 - k$ satisfies the above condition.

Next we consider the case $G = \text{VIII}_L \cdot \{1, T_{\pi/2}\}$ or $G = \text{VIII}_L \cdot \tilde{D}_2$, the latter of which is reduced to the former case. The only difference in the case $G = \text{VIII}_L \cdot \{I, T_{\pi/2}\}$ from the previous case is that from the exact sequence
\[ 0 \rightarrow \{T_{n\pi/2}\} \xrightarrow{i} G \xrightarrow{j} \text{Isom}_0(H^2) \rightarrow 1, \]
(7.47) translations in the $z$ direction must be integer multiples of $\pi/2$ instead of $\pi$. So the generator $\ell$ is represented as $\ell = T_{l}$ with $l = \pi n/2 (n \in \mathbb{Z})$, and the normalization condition for $\gamma_i$ gives the restriction
\[ (2q_i + n)/p_i \in \mathbb{Z}, \]
(7.48)
which has a solution for \( n \) if and only if the greatest common divisor of \( p_i \) and \( p_j \) divides \( 2(q_i - q_j) \) for any pair \( i \neq j \),

\[
(p_i, p_j) | 2(q_i - q_j), \tag{7.49}
\]

and \( b \) becomes an integer if and only if

\[
2h_0 - 2 \sum_{e \in L} 1/p_i + n_\theta e \in \mathbb{Z}. \tag{7.50}
\]

Both of these conditions are weaker than the previous ones, hence a slightly wider class of topologies are allowed. Apart from this difference in the restriction on the topology, the count of the moduli degrees of freedom is the same as in the previous case and given by \( \mathcal{N}(G) \cap \text{VIII}_R \).

The argument on the embedding of \( \pi_1(S^-(g, e; k)) \) into \( G \) with \( G_0 = \text{VIII}_L \) is quite similar. Now the invariance group \( G \) should contain a transformation which reverses the orientation of \( H^2 \), hence \( G = \text{VIII}_L \times \{1, I_2\} \) or \( G = \text{VIII}_L \cdot D_2 \). In the former case, \( \ell \) is represented as \( T_{\pi} \), with an non-vanishing integer \( n \), and the lift of \( \bar{\alpha}_a \) is represented by \( I_2 \tilde{Z}(\zeta_a) \tilde{R}(\theta_a) T_{\pi} \), with some integer \( x_a \). Now, there is no IPD which changes \( x_a \) continuously unlike in the case of \( H^2 \times \mathbb{R} \). Hence, the number of the moduli degrees of freedom is smaller by \( g - 1 \) than that for the \( S^-(g, 0; k) \) case of \( H^2 \times \mathbb{R} \):

\[
N_{m-} = N_{m-}' = g + 1 = 2k + 3g - 6. \tag{7.51}
\]

The restriction on the indices characterizing the fundamental group is the same as that for the embedding of \( S^+(g, e; k) \) in \( G = \text{VIII}_L \) and is given by (7.45) and (7.46). For \( G = \text{VIII}_L \cdot D_2 \), the number of the moduli degrees of freedom is given by the same expression (7.51), but the possible topologies are restricted by (7.49) and (7.50). Since \( T_{\pi/2} \) in \( \mathcal{N}(G) \) transforms \( I_2 \) non-trivially as \( T_{\pi/2} I_2 T_{-\pi/2} = I_2 T_{-\pi} \), \( \mathcal{N}_{\pi_0}(G) \) is given by \( \{ \tilde{Z}_R(ie^\gamma) \} \) for \( G = \text{VIII}_L \times \{1, I_2\} \), while it becomes trivial for \( G = \text{VIII}_L \cdot D_2 \).

Finally we consider the case \( G = \text{Isom}_0(\text{SL}_2 \mathbb{R}) \) or \( G = \text{Isom}(\text{SL}_2 \mathbb{R}) \). First, the image of \( \pi_1(S^+(g, e; k)) \) is always contained in \( \text{Isom}_0(\text{SL}_2 \mathbb{R}) \). Since \( \mathcal{N}(G) \) is the same for \( G = \text{Isom}_0(\text{SL}_2 \mathbb{R}) \) and \( G = \text{Isom}(\text{SL}_2 \mathbb{R}) \), the argument for the latter is reduced to that for the former as for \( G_0 = \text{VIII}_L \). Now \( G \) contains translations in \( z \)-direction with distances represented by arbitrary real numbers. Hence, \( \ell \) for \( \ell = T_l \) can take arbitrary non-zero real number, the degrees of freedom in the lifts of \( \alpha_a \) and \( \beta_a \) increase by \( 2g \) compared to the case \( G = \text{VIII}_L \), and no restriction like (7.45) and (7.46) arises. In fact, if we start from the choice \( T_{\pi} \) for \( \ell \) and \( (\gamma_i)_0 = \tilde{Z}(\zeta_i) \tilde{R}(\pi/p_i) \tilde{Z}(\zeta_i)^{-1} \), \( \gamma_i = (\gamma_i)_0 T_{\pi} \) with \( x_i = -(1 + xr_i)/p_i \) satisfies the normalization condition for \( \gamma_i \). Further, after this rescaling, \( [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k \) is given by \( T_{(h_0 + \sum x_i)} \). Hence, for a given \( b \), we obtain the equation \( x = \sum \frac{1}{p_i} - h_0 \).

\[
xe = \sum \frac{1}{p_i} - h_0. \tag{7.52}
\]
Since \( e < 0 \), this equation always has a unique solution. Thus the fundamental group of any \( S^+(g,e;k) \) can be embedded into \( G \). Since \( T_\theta \) commutes with the action of \( \text{Isom}_0(SL_2\mathbb{R}) \), the dimension of \( N(G) \)-orbits in the moduli space is 3. Hence, the number of the moduli degrees of freedom is given by

\[
N''_{m+} = (2k + 6g - 3) + (2g + 1) - 3 = 2k + 8g - 5, \quad (7.53)
\]

and the isotropy group \( N^{0}(G) \) is again given by \( N(G) \cap \text{VIII}_L \).

The argument for \( \Sigma = S^{-}(g,e;k) \) is almost the same. Now the fundamental group can be embedded only into \( G = \text{Isom}(SL_2\mathbb{R}) \) and the \( \alpha_a \) is represented by a transformation with \( I_2 \). Due to this, the action of \( T_\theta \) on the moduli becomes effective, and the dimension of \( N(G) \)-orbits in the moduli space is given by 4. Apart from this difference, the argument is just a combination of those given so far. Hence, the number of the moduli degrees of freedom is given by

\[
N''_{m-} = (2k + 3g - 3) + (g + 1) - 4 = 2k + 4g - 6, \quad (7.54)
\]

and the isotropy group \( N^{0}(G) \) becomes trivial. No restriction on topology exists apart from the conditions \( \chi < 0 \) and \( e < 0 \).

7.3. Phase Space

From the general argument in \( \S \) 2.2, the invariant phase space is expressed as (2.6). The dimension of the reduced moduli space \( M_0(\Sigma,G) \) was determined in the previous subsection, but its topological structure was not discussed there. It is because that problem is too difficult to address. For the same reason, we do not discuss the structure and the action of the possible residual discrete transformation group \( H_{\text{mod}} \) in the expression (2.6), and only determine the structure of the dynamical sector \( \Gamma_{\text{dyn}}(\Sigma,G) \).

First we consider the case \( G = \text{VIII}_L \), which can be realized only for \( \Sigma = S^+(g,e,k) \). In this case \( N_{\Sigma 0}(G) = \text{VIII}_R \cap N(G) \) coincides with \( \text{VIII}_R \), which transforms the invariant basis for \( \text{VIII}_L \) as (7.26) and (7.27). Let \( \eta \) be the diagonal matrix \( D(-1,1,1) \). Then, for the matrix \( Q \) representing the components of the metric, \( \eta Q \) always has an eigenvector \( v_1 \), \( \eta Q v_1 = \lambda_1 v_1 \), in the complex 3-vector space. Since \( \left| \begin{array}{c} Q v_1 \\
_1 v_1 \end{array} \right| > 0 \), which implies that \( \lambda_1 \) is a non-vanishing real number. Hence we can take \( v_1 \) as a real 3-vector with unit length with respect to \( \eta \). By repeating this argument in the subspace orthogonal to \( v_1 \) with respect to the metric \( \eta \) in \( \mathbb{R}^3 \), we finally obtain the set of three eigenvectors and three eigenvalues such that \( \eta Q (v_1 v_2 v_3) = (v_1 v_2 v_3)^{\nu} D(\lambda_1, \lambda_2, \lambda_3) \). From this we find that for \( A = (v_1, v_2, v_3) \in SO_+(2,1), \ A Q^\dagger A = D(Q_1, Q_2, Q_3), \) where \( Q_i = \pm \lambda_i > 0 \). Hence, we can diagonalize \( Q \) by \( \text{VIII}_R \). We can further put \( Q_1 \geq Q_2 \) by \( F = R_3(\pi/2) \) corresponding to the \( T_{\pi/4} \) transformation if necessary. After this diagonalization, the momentum constraints are expressed as

\[
H_1 = 2P^{23}(Q_2 + Q_3) + cu_1 = 0, \quad (7.55)
\]

\[
H_2 = -2P^{13}(Q_1 + Q_3) + cu_2 = 0, \quad (7.56)
\]

\[
H_3 = 2P^{12}(Q_1 - Q_2) + cu_3 = 0. \quad (7.57)
\]
For the vacuum system with $c = 0$, we obtain the constraint $P_{13}^3 = P_{23}^3 = 0$. This implies that the invariance group contains $T_{\pi/2}$ inducing $F = I_3$, hence is larger than $\text{VIII}_L$. Further, when $Q_1 > Q_2$, $P_{12}^2$ also vanishes and the invariance group $G$ of the system includes $\text{VIII}_L \cdot \tilde{D}_2$. On the other hand, when $Q_1 = Q_2$, $Q$ is invariant under $F = \Lambda = R_3(\theta)$, in terms of which $P$ can be diagonalized. Hence, $G \supset \text{VIII}_L \cdot \tilde{D}_2$ again.

In contrast, for the fluid system, the constraints simply determine $P_{13}^3$ and $P_{23}^3$ by $u_1$ and $u_2$, and further $P_{12}^2$ by $u_3$ if $Q_1 > Q_2$. In the special case $Q_1 = Q_2$, for which we obtain the constraint $u_3 = 0$, we can put $P_{12}^2 = 0$ by the residual HPDs. Including this special case, $G = \text{VIII}_L$ if and only if two of $u_i$ do not vanish. Hence, the dynamical sector of the invariance phase space for the single-component fluid system is given by

$$
\Gamma_{\text{dyn}}(S^+, \text{VIII}) = \{(Q_1, Q_2, Q_3; P_{11}^1, P_{22}^2, P_{33}^3; u_1, u_2, u_3, \rho) | Q_1 \geq Q_2 > 0, Q_3 > 0, u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2 \neq 0\} / \mathbb{Z}_2(7.58)
$$

where $\mathbb{Z}_2$ represents the group $\{1, I_3\}$ with $I_3$ induced from $T_{\pi/2}$.

Next for $G = \text{VIII}_L \times \{1, I_2\}$, the symmetry requires that $Q_{12} = Q_{23} = P_{12}^2 = P_{32}^2 = u_1 = u_3 = 0$, and a transformation in $\mathcal{N}_{\Sigma_0}(G)$ is written as $\{1, T_{\pi/2}\} \tilde{Z}_R(ie^\gamma)$ for $\Sigma = S^+(g, e; k)$ and by $\tilde{Z}_R(ie^\gamma)$ for $\Sigma = S^-(g, e; k)$. From the definition of the representation (6.19), we find that $\Lambda(\tilde{Z}(ie^\gamma))$ is given by the Lorentz boost in the $T - X$ plane or the $\chi^1 - \chi^3$ plane,

$$
\begin{pmatrix}
\chi^1
\
\chi^3
\end{pmatrix} = \begin{pmatrix}
\cosh \gamma & -\sinh \gamma \\
-\sinh \gamma & \cosh \gamma
\end{pmatrix} \begin{pmatrix}
\chi^1
\
\chi^3
\end{pmatrix}.
$$

(7.59)

We can diagonalize $Q$ by this transformation to $Q = D(Q_1, Q_2, Q_3)$. Then, the momentum constraint $H_2 = 0$ determines $P_{13}^3$ by $u_2$. If $u_2 = 0$, the system has a higher symmetry, so $u_2 \neq 0$. For $\Sigma = S^+(g, e; k)$, we can put $u_2 > 0$ by the $I_3$ transformation induced from $T_{\pi/2}$. Hence

$$
\Gamma_{\text{dyn}}(S^+, \text{VIII} \times \{1, I_2\}) = \{(Q_1, Q_2, Q_3; P_{11}^1, P_{22}^2, P_{33}^3; u_2, \rho) | Q_1, Q_2, Q_3 > 0, u_2 > 0\}.
$$

(7.60)

On the other hand, for $\Sigma = S^-(g, e; k)$, $T_{\pi/2}$ is not contained in $\mathcal{N}_{\Sigma_0}(G)$. Hence,

$$
\Gamma_{\text{dyn}}(S^-, \text{VIII} \times \{1, I_2\}) = \{(Q_1, Q_2, Q_3; P_{11}^1, P_{22}^2, P_{33}^3; u_2, \rho) | Q_1, Q_2, Q_3 > 0, u_2 \neq 0\}.
$$

(7.61)

The argument for $G = \text{VIII}_L \cdot \{1, T_{\pi/2}\}$ with $\Sigma = S^+(g, e; k)$ is quite similar to the previous case. Since $T_{\pi/2}$ induces the $I_3$ transformation of the invariant basis, the symmetry requires $Q_{13} = Q_{23} = P_{13}^3 = P_{23}^3 = u_1 = u_2 = 0$. By HPD $T_{\theta}$ in $\mathcal{N}_{\Sigma_0}(G) = \mathcal{N}(G) \cap \text{VIII}_R$, we can diagonalize $Q$ as $Q = D(Q_1, Q_2, Q_3)$ with $Q_1 \geq Q_2$. If $Q_1 = Q_2$, we can further put $P_{12}^2 = 0$, which together with $H_3 = 0$ requires the fluid velocity to vanish. This implies that the system has an invariance
group larger than $\text{VIII}_L \cdot \tilde{D}_2$. Hence $Q_1 > Q_2$, and $P^{12}$ is determined by $u_3$ through $H_3 = 0$. The dynamical sector of the invariant phase space is given by

$$
\Gamma_{\text{dyn}}(S^+, \text{VIII} \cdot \{1, T_{\pi/2}\}) = \{(Q_1, Q_2, Q_3; P^{11}, P^{22}, P^{33}, u_3, \rho) \mid Q_1 > Q_2 > 0, Q_3 > 0, u_3 \neq 0\}. \tag{7.62}
$$

The $G = \text{VIII}_L \cdot \tilde{D}_2$ case is very simple. Since $\tilde{D}_2$ induces the transformation group $D_2 = \{1, I_1, I_2, I_3\}$ of the invariant basis, the invariance under $\tilde{D}_2$ requires that both $Q$ and $P$ are diagonal and the fluid velocity vanishes. Hence, the existence of fluid does not play an essential role in the analysis and simply adds the matter density freedom to the invariant phase space of the vacuum system. The system does not have a higher symmetry if and only if $Q_1 \neq Q_2$ or $P^1 \neq P^2$. Thus, for the single-component fluid system,

$$
\Gamma_{\text{dyn}}(S^\pm, \text{VIII} \cdot \tilde{D}_2) = \{(Q_1, Q_2, Q_3; P^1, P^2, P^3; \rho) \mid Q_1, Q_2, Q_3 > 0, Q_1 \neq Q_2 \text{ or } P^1 \neq P^2\}. \tag{7.63}
$$

Finally for $G \supset \text{Isom}(\widetilde{SL_2\mathbb{R}})$, the argument is the same as that for the previous case apart from the symmetry conditions $Q_1 = Q_2$ and $P^1 = P^2$, and the system with a single-component fluid has always the maximal symmetry $\text{Isom}(\widetilde{SL_2\mathbb{R}})$. Hence, the dynamical sector of its invariant phase space is given by

$$
\Gamma_{\text{dyn}}(S^\pm, \text{Isom}(\widetilde{SL_2\mathbb{R}})) = \{(Q_1, Q_3; P^1, P^3; \rho) \mid Q_1, Q_3 > 0\}. \tag{7.64}
$$

So far we have considered only the single-component fluid system, but the extension to a multi-component system is quite simple. First, for $G = \text{VIII}_L$, each extra component of fluid adds the four variables $(u_1, \rho)$ to $\Gamma_{\text{dyn}}$. Second, for $G = \text{VIII}_L \times \mathbb{Z}_2$, two variables $(u_2, \rho)$ or $(u_3, \rho)$ are added for each extra component. Third, for

| Space          | Symmetry | $Q$ | $P$ | $N_m$ | $N_I$ | $N$ | $N_s$ | $N_i$ (vacuum) |
|----------------|----------|-----|-----|-------|-------|-----|-------|----------------|
| $\mathbb{R}^3$| VIII     | 3   | 3   | 0     | 4     | 10  | 8     | –              |
| $\text{VIII} \times \mathbb{Z}_2$| 3     | 3   | 0   | 2     | 8     | 6   | –     |               |
| $\text{VIII} \cdot \tilde{D}_2$| 3     | 3   | 0   | 1     | 7     | 5   | 4     |               |
| $\text{Isom}(\widetilde{SL_2\mathbb{R}})$| 2     | 2   | 0   | 1     | 5     | 3   | 2     |               |
| $S^+(g,e;k)$   | VIII     | 3   | 3   | $N_{m+}$ | 4   | $10 + N_{m+}$ | $8 + N_{m+}$ | –             |
| $\text{VIII} \times \mathbb{Z}_2$| 3     | 3   | $N_{m+}$ | 2   | $8 + N_{m+}$ | $6 + N_{m+}$ | –             |
| $\text{VIII} \cdot \tilde{D}_2$| 3     | 3   | $N_{m+}$ | 1   | $7 + N_{m+}$ | $5 + N_{m+}$ | $4 + N_{m+}$ |
| $\text{Isom}(\widetilde{SL_2\mathbb{R}})$| 2     | 2   | $N'_{m+}$ | 1   | $5 + N''_{m+}$ | $3 + N''_{m+}$ | $2 + N''_{m+}$ |
| $S^-(g,e;k)$   | VIII     | 3   | 3   | $N_{m-}$ | 2   | $8 + N_{m-}$ | $6 + N_{m-}$ | –             |
| $\text{VIII} \times \mathbb{Z}_2$| 3     | 3   | $N_{m-}$ | 1   | $7 + N_{m-}$ | $5 + N_{m-}$ | $4 + N_{m-}$ |
| $\text{VIII} \cdot \tilde{D}_2$| 3     | 3   | $N_{m-}$ | 1   | $5 + N_{m-}$ | $3 + N_{m-}$ | $2 + N_{m-}$ |
$G \supset \text{VIII}_L \cdot \tilde{D}_2$, each extra component only add its energy density. Finally for $G = \text{Isom}_0(\text{SL}_2\mathbb{R})$, $\Gamma_{\text{dyn}}(S^+, G)$ is obtained by adding $(u_3, \rho)$ for each extra component to $\Gamma_{\text{dyn}}(S^+, \text{Isom}(\text{SL}_2\mathbb{R}))$ of the single-component system.

The parameter counts for the vacuum and the single-component fluid system of the type $\text{SL}_2\mathbb{R}$ obtained in this section is summarized in Table XV.

§8. Summary and Discussion

In this paper, we have systematically determined the structure of the diffeomorphism invariant phase space, i.e., the initial data space, of compact Bianchi models with fluid for all possible space topologies and local invariance groups except for those modeled on the Thurston types $S^3$, $H^3$ and $S^2 \times \mathbb{R}$ which have no moduli freedom. The main results are summarized in Tables V, VIII, XI, XIII and XV. Although these tables list only the parameter counts for the vacuum and the single-component fluid system, they can be easily extended to multi-component fluid systems as was described in detail in the arguments of individual systems. It is also easy to extend the analysis to systems with scalar fields. Since scalar fields have to be spatially constant if they are spatially homogeneous locally, their contribution to the phase space does not depend on the space topology or details of the invariance group and simply increases the parameter count by two times the number of fields.

We can read various things from these tables. First, although it is quite natural that the parameter count increases when the space is compactified due to the appearance of the moduli freedom, the degree of increase is quite sensitive to whether fluid exists or not as well as the local symmetry and the space topology of the system. In particular, although the parameter count $N_s$ for the locally $\mathbb{R}^3$-symmetric system is larger than that for the locally VII$_0$-symmetric system in the vacuum case, the latter becomes larger than the former for the system with fluid. Hence, the Bianchi type VII$_0$ symmetry becomes more generic than the Bianchi type I symmetry for compact Bianchi models with fluid as for non-compact models. This inversion occurs because the locally $\mathbb{R}^3$-symmetric system always has the higher local symmetry $\mathbb{R}^3 \rtimes D_2$, while the minimum locally symmetry VII$_0 \rtimes D_2$ of the compact vacuum Bianchi VII$_0$ model is broken down to VII$_0$ when fluid is introduced.

Second, the local spatial homogeneity often requires a local isotropy when the space has a complicated topology, even for the class A Bianchi models. For example, Bianchi-type I models with space $T^3/\mathbb{Z}_k(k = 3, 4, 6)$ and Bianchi-type II models with space $T^3(n)/\mathbb{Z}_k(k = 3, 4, 6)$ always have local rotational symmetry. There are also models for which additional discrete symmetries are required. They are the Bianchi-type VII$_0$ model with fluid on $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, Bianchi-type II models on $K^3(n)$, $T^3(n)/\mathbb{Z}_k(k = 2, 3, 4, 6)$ and $T^3(n)/\mathbb{Z}_2 \times \mathbb{Z}_2$, Bianchi-type VI$_0$ models on $\text{Sol}(n)$ with $n < -2$ and Bianchi-type VIII models on $S^-(g, e; k)$. For these models, although additional discrete symmetries do not affect the geometrical degrees of freedom, they require the fluid velocity to align toward a special direction.

This feature in general makes models with complicated topologies less probable if we assume that a model with a larger parameter count has a higher probability to
be realized. However, there are very important exceptions to this general tendency. They are models with spaces covered by $H^2 \times \mathbb{R}$ or $SL_2 \mathbb{R}$. For these models, the number of the moduli degrees of freedom increases without bound as the genus $g$ or the number $k$ of conical singularities in the base orbifold increases, as is clear from (6.55), (6.58), (7.42), (7.42), (7.53) and (7.54). It should be noted here that although fully anisotropic Bianchi-type VIII models have larger degrees of freedom in the fluid sector than locally rotationally symmetric models, the total parameter count $N_s$ for the latter becomes larger than that for the former when $g \geq 2$ since $N_s(S^+, SL_2 \mathbb{R}) - N_s(S^+, \text{VIII}) = 2g - 2$ and $N_s(S^-, \widetilde{SL_2 \mathbb{R}}) - N_s(S^-, \text{VIII} \rtimes \mathbb{Z}_2) = g - 1$. Hence, the LRS Bianchi-type III and VIII models are the most generic among all compact Bianchi models.

This result has an important cosmological consequence. In most of the studies on cosmological models with compact space, in particular, in the investigations of the effect of space topology on the CMB anisotropy, space topologies of the Thurston types $E^3$ and $H^3$ have been considered. This is because many people believe that the Bianchi symmetries admitting the spatially homogeneous and isotropic models should be imposed to accord with the observed isotropy of the universe. From this viewpoint, compact Bianchi models of the type III or VII should be rejected, because they only admit LRS models at best. However, if a sufficient inflation occurs in the early stage of the universe, a universe with any spatial symmetry can be made sufficiently flat and isotropic on the horizon scale at present. Hence, there is no reason to consider only models with spatial isotropy. From this viewpoint, the above result rather suggests that the space topology of our universe may be of the Thurston type $H^2 \times \mathbb{R}$ or $\widetilde{SL_2 \mathbb{R}}$ if our universe has compact space.

Of course, it is dangerous to say something definite about the real universe with infinite degrees of freedom simply based on the analysis of locally homogeneous systems with finite degrees of freedom. However, it is also hard to think that the strong influence of the space topology on the isotropy and the dynamical degrees of freedom found in the present paper is just an accidental feature of the locally homogeneous systems. We hope that this point will be clarified in future studies.

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**Appendix A**

**Propositions on the normalizer**

In this appendix we give some useful general propositions, which are used to determine the normalizer groups in the present paper.

In general a transformation group $G$ is not connected. The normalizer of such a group can be easily determined if one knows the normalizer of the maximal connected subgroup $G_0$, from the following proposition.
Proposition A.1 The normalizer of $G$ is contained in that of $G_0$. Hence, if $G$ is generated by $G_0$ and a set of discrete transformations $S$, $N(G)$ consists of $f \in N(G_0)$ such that $fgf^{-1} \in G$ for any $g \in S$.

Proof. For any $f \in N(G)$, $fG_0f^{-1}$ is a connected subgroup of $G$ and $fG_0f^{-1} \cap G_0 \neq \emptyset$. Hence, $fG_0f^{-1} \subset G_0$. This implies that $f \in N(G_0)$.

Each HPD $f \in N(G)$ gives an isomorphism of $G$, which induces an isomorphism $f_s$ of $L(G)$. That is, $f_s$ is contained in $\text{Aut}(L(G))$ which is the set of linear transformations $\phi$ satisfying the condition

$$[\phi(X), \phi(Y)] = \phi([X, Y]) \quad \forall X, Y \in L(G). \quad (A.1)$$

In general, it is easier to determine $\text{Aut}(L(G))$ than $N(G)$, because the former is an algebraic problem. In particular, if the Killing form is non-degenerate, it can be used to restrict possible candidates for the linear transformations in $\text{Aut}(L(G))$. Let $\gamma(X, Y)(X, Y \in L(G))$ be the Killing form of $L(G)$ defined by

$$\gamma(X, Y) = \text{Tr}(\text{Ad}(X)\text{Ad}(Y)), \quad (A.2)$$

where $\text{Ad}(X)$ is the linear transformation defined by $\text{Ad}(X)Y = [X, Y]$. Then, since $\phi \in \text{Aut}(L(G))$ preserves the matrix representation of $\text{Ad}(X)$, the following proposition holds.

Proposition A.2 Each linear transformation of $\text{Aut}(L(G))$ preserves the Killing form. That is, $\text{Aut}(L(G))$ is a subgroup of the orthogonal transformation group with respect to the Killing form.

If there is a special subspace of $L(G)$, it can be also used to restrict the form of $\phi$. For example, the center of $L(G)$, $L(L(G))$, consisting of elements which commute with all elements of $L(G)$, is preserved by $\phi$ because $L(L(G))$ is the unique subspace with that property. Similarly, because $\phi(L(G)) = L(G)$, $\phi([L(G), L(G)]) = [\phi(L(G)), \phi(L(G))] = [L(G), L(G)]$. Thus

Proposition A.3 An automorphism of $L(G)$ preserves its center and $[L(G), L(G)]$.

Let $\xi_j = \xi_j^i(x)\partial_i$ be a basis of $L(G)$. Then the condition that a transformation $x'^i = f^i(x)$ induces the automorphism

$$\phi_{\xi_j} = \xi_j^i \phi^j I \quad (A.3)$$

is expressed as

$$\xi_j^i(x)\partial_i f^i(x) = \xi_j^i(f(x)) \phi^j I. \quad (A.4)$$

In general, it is not an easy task to solve this set of differential equations directly. However, special solutions are often found easily by guess or other methods. In such cases, the following proposition is very useful in determining the general solution to this equation.

Proposition A.4 Let $G$ be a transformation group containing a simply transitive group $G_s$ on a connected manifold $M$. If $f \in N(G)$ induces the identity transformation of $L(G_s)$, $f$ coincides with some right transformation $R_0$ with respect to $G_s$. 

Proof. Let $g_t$ be a one-parameter subgroup of $G_s$. Then the infinitesimal transformation for $f g_t f^{-1}$ coincides with that for $g_t$ from the assumption. Hence $f g_t f^{-1} = g_t$. Since any element of the connected group $G_s$ is contained in such a subgroup, this implies that $f$ commute with any left transformation $L_a$. Let $o$ be a fixed point in $M$, and identify $G_s$ with $M$ by the diffeomorphism $F_0 : G_s \to M$ defined by $F_0(a) = L_o a$. Then, for any point $x$ in $M$, there exists a unique element $g_x$ in $G_s$ such that $F_0(g_x) = L_o x$. Now, let $b$ be an element in $G_s$ defined by $b = F_0^{-1}(f(o))$. Then, since the right transformation is defined by $R_b F_0(a) = F_0(ab)$,

$$R_b(o) = F_0(b) = f(o).$$

Hence, $f(x) = f L_{g_x}(o) = L_{g_x} f(o) = L_{g_x} R_b(o) = R_b L_{g_x}(o) = R_b(x)$. \[\square\]

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### Appendix B

**Conjugate transformations**

B.1. Sol

\begin{align*}
  f L_c f^{-1} &= L_{f(c)} \text{ for } f = \{1, J\} D(k_1, k_2, 1), \quad (B.1) \\
  L_a L_c L_a^{-1} &= L_c' = B(a'^3)c + a - B(c'^3)a, \quad (B.2) \\
  J(-I_1)J &= (-I_2), \quad JI_3 = I_3 J, \quad (B.3) \\
  L_a(-I_1)L_a^{-1} &= (-I_1)L_c; \quad c = (-2a^1, 0, 0), \quad (B.4) \\
  L_a(-I_2)L_a^{-1} &= (-I_2)L_c; \quad c = (0, -2a^2, 0), \quad (B.5) \\
  L_a I_3 L_a^{-1} &= I_3 L_{-a}, \quad (B.6) \\
  L_a J L_a^{-1} &= J L_c; \quad c = (J - B(-2a^3))a, \quad (B.7) \\
  R_b(-I_1)R_b^{-1} &= (-I_1)R_{b'}; \quad b' = (-2 b^1 e^{b^3}, 0, 0), \quad (B.8) \\
  R_b(-I_2)R_b^{-1} &= (-I_2)R_{b'}; \quad b' = (0, -2 b^2 e^{-b^3}, 0), \quad (B.9) \\
  R_b I_3 R_b^{-1} &= I_3 R_{b'}; \quad b' = (-2 b^1 e^{b^3}, -2 b^2 e^{-b^3}, 0), \quad (B.10) \\
  R_b J R_b^{-1} &= J R_{b'}; \quad b' = (-b^1 + b^2 e^{b^3}, b^1 - b^2 e^{-b^3}, -2 b^3). \quad (B.11)
\end{align*}

B.2. $H^2 \times \mathbb{R}$

\begin{align*}
  \mathcal{N}(III_L): \\
  L(a,b,c)L(x,y,z)L_{(a,b,c)}^{-1} &= L(a+bx-ay, y, z), \quad (B.12) \\
  (-I_1)L(x,y,z)(-I_1) &= L(-x, y, -z), \quad (B.13) \\
  I_2 L(x,y,z)I_2 &= L(-x, -y, -z), \quad (B.14) \\
  (-I_3)L(x,y,z)(-I_3) &= L(x, y, -z), \quad (B.15) \\
  \mathcal{N}'(d)L(x,y,z)\mathcal{N}'(d)^{-1} &= L(x, y, z+d \ln y), \quad (B.16) \\
  D(1, 1, k)L(x,y,z)D(1, 1, k)^{-1} &= L(x, y, kz), \quad (B.17) \\
  D(1, 1, k)L_a &= L_a D(1, 1, k), \quad (B.18) \\
  L(a,b,c)(-I_1)L_{(a,b,c)}^{-1} &= (-I_1)L(-2a, 0, 0), \quad (B.19)
\end{align*}
\[ L_{(a,b,c)} I_2 L^{-1}_{(a,b,c)} = I_2 L(-2a,0,-2c), \]  
\[ L_{(a,b,c)} (-I_3) L^{-1}_{(a,b,c)} = (-I_3)L(0,0,-2c), \]  
\[ (-I_1)N'(d) = N'(d)(-I_1). \]  

\( \mathcal{N}(\text{Isom}_0(H^2 \times \mathbb{R})):\)

\[ L_{(a,b,c)} R_H(\theta) L^{-1}_{(a,b,c)} = Z_L (R(\theta) \ast \gamma) R_H(H(\theta,-\gamma)); \]
\[ \gamma = \frac{-a + i}{b}, \]  
\[ R_H(\theta) L_{(a,b,c)} R_H(\theta)^{-1} = Z_L(R(\theta) \ast (a + ib)) R_H(H(\theta,\alpha) - \theta), \]  
\[ R_H(\theta)(-I_1) R_H(\theta)^{-1} = (-I_1) R_H(-2\theta), \]  
\[ R_H(\theta) I_2 R_H(\theta)^{-1} = I_2 R_H(-2\theta), \]  
\[ R_H(\theta)(-I_3) = (-I_3) R_H(\theta), \]  
\[ R_H(\theta) D(1,1,k) = D(1,1,k) R_H(\theta). \]  

B.3. \( \widetilde{SL_2\mathbb{R}} \)

\( \mathcal{N}(\text{VIII}_L):\)

\[ L_{(\alpha,c)} L_{(\xi,z)} L^{-1}_{(\alpha,c)} = L_{(\xi',z')}; \]  
\[ \zeta' = Z(\alpha) \ast Z(R(c) \ast \zeta) \ast R(H(c,\zeta) + z - c) \ast \frac{i - a}{b}, \]  
\[ z' = H \left( H(c,\zeta) + z - c, \frac{i - a}{b} \right), \]  
\[ I_2 L_{(\xi,z)} I_2 = L_{(-\xi,-z)}, \]  
\[ I_2 R_{(\xi,z)} I_2 = R_{(-\xi,-z)}, \]  
\[ L_{(\alpha,c)} I_2 L^{-1}_{(\alpha,c)} = I_2 L_{(\alpha',c')}, \]  
\[ R_{(\alpha,c)} I_2 R^{-1}_{(\alpha,c)} = I_2 R_{(\alpha',c')}; \]  
\[ \alpha' = -a + b R(-2c) \ast \frac{-a + i}{b}, \]  
\[ c' = H(-c,\gamma) - H(c,\gamma), \]  
\[ \gamma = R(-c) \ast \frac{-a + i}{b}. \]

\( \mathcal{N}(\text{Isom}_0(\widetilde{SL_2\mathbb{R}})):\)

\[ R_{(i,\theta)} I_2 R^{-1}_{(i,\theta)} = I_2 R_{(i,-2\theta)}. \]  

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