LOWER AND UPPER BOUNDS FOR NEF CONES

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ABSTRACT. The nef cone of a projective variety $Y$ is an important and often elusive invariant. In this paper we construct two polyhedral lower bounds and one polyhedral upper bound for the nef cone of $Y$ using an embedding of $Y$ into a toric variety. The lower bounds generalize the combinatorial description of the nef cone of a Mori dream space, while the upper bound generalizes the F-conjecture for the nef cone of the moduli space $M_{0,n}$ to a wide class of varieties.

1. INTRODUCTION

A central goal of birational geometry is to understand maps between projective varieties. The cone $\text{Nef}(Y)$ of divisors that nonnegatively intersect all curves on $Y$ encodes information about the possible morphisms from $Y$ to other projective varieties. The interior of this cone is the cone of ample divisors, multiples of which give rise to projective embeddings of $Y$, while divisors on the boundary of the cone determine other morphisms. This cone is hard to compute in general, and is unknown for many even elementary varieties. One contributing factor to our general ignorance is that these cones, while convex, can be very complicated, and in particular need not be polyhedral.

In this paper we construct polyhedral upper and lower bounds for nef cones of varieties. This gives (separate) necessary and sufficient conditions for a divisor to be nef: the lower bound is a polyhedral cone whose interior consists of divisors we certify to be ample, while if a divisor lives outside the polyhedral upper bound cone it is definitely not ample. Our program exploits well-chosen embeddings of the variety $Y$ into a toric variety $X_\Delta$, and unifies several different approaches found in the literature.

Equivalent recipes for the nef cone of a projective toric variety $X_\Delta$ can give rise to different cones in $\text{Pic}(X_\Delta)\otimes \mathbb{R}$ when the variety is not complete. Explicitly, on a projective toric variety, a divisor is nef if and only if it is globally generated, if and only if its pullback to every torus invariant subvariety is effective, and if and only if it nonnegatively intersects every torus invariant curve. When $X_\Delta$ is not projective, the three corresponding cones $G_\Delta$, $L_\Delta$, and $F_{\Delta,w}$ in $\text{Pic}(X_\Delta)\otimes \mathbb{R}$ satisfy $G_\Delta \subseteq L_\Delta \subseteq F_{\Delta,w}$, where each inclusion can be proper; see Proposition 2.13. The vector $w$ is a cohomological invariant that compensates for the fact that a non-projective toric variety may have no torus-invariant curves; see Definition 2.10.

Given an embedding $i: Y \to X_\Delta$, we pull these three cones in $\text{Pic}(X_\Delta)\otimes \mathbb{R}$ back to the Néron Severi space $N^1(Y)\otimes \mathbb{R}$ of $Y$. In this way, for sufficiently general embeddings, we obtain (Theorem 3.2) both lower and upper bounds $G_\Delta(Y) \subseteq L_\Delta(Y) \subseteq \text{Nef}(Y) \subseteq \text{Pic}(Y)\otimes \mathbb{R}$.
$F_\Delta(Y)$ for the nef cone of $Y$. The lower bounds hold for any toric embedding, while the upper bound requires that the induced map $i^* : \text{Pic}(X_\Delta)_\mathbb{R} \to N^1(Y)_\mathbb{R}$ is surjective, and that the fan $\Delta$ equals the tropical variety of $Y \cap T$, where $T$ is the torus of $X_\Delta$; see Section 3.

The requirement of a toric embedding for $Y$ does not impose any restrictions on $Y$; every projective variety embeds into the toric variety $\mathbb{P}^N$. The pullback of $O(1)$ on $\mathbb{P}^N$ can be considered a (not-very-informative) lower bound for $\text{Nef}(Y)$. Toric embeddings can be chosen so that the resulting lower bound is a full-dimensional subcone of the nef cone. Our bounds depend on the choice of toric embedding, and a given variety may have several useful embeddings. In addition, many interesting varieties come with natural embeddings into toric varieties satisfying all required conditions; see Sections 5 and 6.

The question of what can be deduced about an embedded variety from an ambient toric variety has been a theme in the literature, with variants of the cones appearing in special cases. This paper provides a unifying framework generalizing these constructions.

In the context of mirror symmetry, Cox and Katz conjectured a description for the toric part of the nef cone of a Calabi Yau hypersurface in a toric variety $X_\Delta$ [CK99, Conjecture 6.2.8]. We show that this is the cone $F_\Delta$; see Lemma 4.2. The subsequent counterexamples to this conjecture and its variants ([Sze02], [HLW02], [Sze03], [AB01]) give examples of the lower bound given by $G_\Delta$ not being exact.

Another interesting class is given by Mori dream spaces [HK00], important examples of which are log Fanos of general type [BCHM10]. A Mori dream space $Y$ has a natural embedding into a non-complete toric variety $X_\Delta$ for which the induced map $i^* : \text{Pic}(X_\Delta)_\mathbb{R} \to N^1(Y)_\mathbb{R}$ is an isomorphism. One may regard the ambient toric variety $X_\Delta$ as a Rosetta Stone, encoding all birational models of $Y$; see [HK00], [BH07], and [Hau08]. In this case the nef cone of $Y$ equals the pullback of $G_\Delta$ and $L_\Delta$; see Section 5. The lower bounds $G_\Delta$ and $L_\Delta$ may thus be considered as generalizations to arbitrary varieties of the construction of the nef cone of a Mori dream space.

The moduli space $\overline{M}_{0,n}$ of stable genus zero curves with $n$ marked points also has an embedding into a non-complete toric variety $X_\Delta$ with $\text{Pic}(X_\Delta)_\mathbb{R} \cong N^1(\overline{M}_{0,n})_\mathbb{R}$, where $\Delta$ is the space of phylogenetic trees from biology (see [Tev07], [GM10]). The nef cone of $\overline{M}_{0,n}$ is famously unknown, with a possible description given by the F-Conjecture. We show in Proposition 6.2 that this cone equals the pullback of $F_{\Delta,w}$. The original motivation for the F-Conjecture came from an analogy between $\overline{M}_{0,n}$ and toric varieties. It suggests that, as for complete toric varieties, the one-dimensional boundary strata of $\overline{M}_{0,n}$ should span its cone of curves. The interpretation of the conjecture as $\text{Nef}(\overline{M}_{0,n}) = F_{\Delta,w}(\overline{M}_{0,n})$ thus deepens and explains this connection.

In addition, the upper bound $F_\Delta(Y)$ can be considered to be a generalization of the F-Conjecture to varieties $Y$ that can be realized as tropical compactifications. The lower bound cone $L_\Delta(Y)$ is defined for an even wider class of varieties, and we propose that for $\overline{M}_{0,n}$ the lower bound cone $L_\Delta(\overline{M}_{0,n})$ equals the nef cone.

Explicitly, let $I = \{I \subset \{1, \ldots, n\} : 1 \in I \text{ and } |I|, |I^c| \geq 2\}$, and let $\delta_I$ denote the boundary divisor on $\overline{M}_{0,n}$ corresponding to $I \in I$. We denote by $\text{pos}(v_1, \ldots, v_r)$ the positive hull $\{\sum_{i=1}^r \lambda_i v_i : \lambda_i \geq 0\}$ of a finite set of vectors $\{v_1, \ldots, v_r\}$ in $\mathbb{R}^n$. Then
\( \mathcal{L}_\Delta(\overline{M}_{0,n}) \) equals
\[
\bigcap_{\sigma} \text{pos}(\delta_I, \pm \delta_J : I, J \in \mathcal{I} \setminus \sigma, \delta_I \cap \delta_K \neq \emptyset, \forall K \in \sigma, \text{and } \delta_J \cap \delta_L = \emptyset \text{ for some } L \in \sigma),
\]
where the intersection is over all subsets \( \sigma = \{I_1, \ldots, I_{n-3}\} \) of \( n-3 \) distinct elements of \( \mathcal{I} \) for which \( \cap_{j=1}^{n-3} \delta_{I_j} \neq \emptyset \).

For \( n \leq 6 \) we have verified that \( \mathcal{L}_\Delta(\overline{M}_{0,n}) = \text{Nef}(\overline{M}_{0,n}) = \mathcal{F}_{\Delta,w}(\overline{M}_{0,n}) \). While this may be true in general, as we are inclined to believe, we feel that showing \( \mathcal{L}_\Delta(\overline{M}_{0,n}) = \text{Nef}(\overline{M}_{0,n}) \) may be more accessible than the F-conjecture. The resulting description of \( \text{Nef}(\overline{M}_{0,n}) \) shares the main advantage of that given by the F-conjecture in that it provides a concrete polyhedral description of the nef cone, allowing detailed analysis.

We now outline the structure of the paper. The definitions of the cones \( \mathcal{G}_\Delta, \mathcal{L}_\Delta, \) and \( \mathcal{F}_{\Delta,w} \) for a toric variety \( X_\Delta \) are given in Section 2 along with several equivalent combinatorial interpretations. In Section 3 we prove the main result, Theorem 3.2. Section 4 is then devoted to applying Theorem 3.2 to many classes of examples. In Section 4.1 we consider the resulting bounds on the nef cone of a del Pezzo surface. In Section 4.2 we consider the case that \( Y \) is an ample hypersurface in a toric variety and connections with the Cox/Katz conjecture, while in Section 4.3 we consider embeddings into toric varieties of Picard rank two. The application to Mori dream spaces is described in Section 5. Finally, in Section 6 we apply the main theorem to the moduli space \( \overline{M}_{0,n} \).

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2. Cones of divisors on \( X_\Delta \)

Let \( X_\Delta \) be a normal toric variety with fan \( \Delta \). In this section we define three cones in \( \text{Pic}(X_\Delta) \):
\[ \mathcal{G}_\Delta \subseteq \mathcal{L}_\Delta \subseteq \mathcal{F}_{\Delta,w}. \]
When \( X_\Delta \) is complete all cones are equal to the nef cone of \( X_\Delta \).

We mostly follow the notation for normal toric varieties of Fulton’s book [Ful93], which we briefly recall. Fix a lattice \( N \cong \mathbb{Z}^n \), and let \( N_\mathbb{R} = N \otimes \mathbb{R} \). Let \( \mathcal{M} = \text{Hom}(N, \mathbb{Z}) \). We denote the pairing of \( u, v \in \mathcal{M} \) and \( \langle u, v \rangle \).

Throughout this paper \( \Delta \) will be a fan in \( N_\mathbb{R} \) that is not contained in any proper subspace. Often we will assume that \( \Delta \) is pure of dimension \( d \). We emphasize that almost always we will have \( d < n \), so the corresponding toric variety, which we denote by \( X_\Delta \), is not complete. We denote by \( \Delta(k) \) the set of cones of \( \Delta \) of dimension \( k \) for \( 0 \leq k \leq n \), and by \( |\Delta| \) the support \( \{v \in N_\mathbb{R} : v \in \sigma \text{ for some } \sigma \in \Delta\} \). We use the notation \( i \in \sigma \) for \( \sigma \in \Delta \) to denote that the \( i \)th ray of \( \Delta \) is a ray of the cone \( \sigma \). By a piecewise linear function \( |\Delta| \rightarrow \mathbb{R} \) we will mean one that is linear on each cone of \( \Delta \).

For \( i \in \Delta(1) \) we write \( D_i \) for the corresponding torus-invariant divisor. Any Weil divisor on \( X_\Delta \) is linearly equivalent to one of the form \( D = \sum_{i \in \Delta(1)} a_i D_i \). The divisor
$D$ is $\mathbb{Q}$-Cartier if there is a piecewise linear function $\psi: |\Delta| \to \mathbb{R}$ with $\psi(v_i) = a_i$, where $v_i$ is the first lattice point on the $i$th ray of $\Delta$. We write $\psi(v) = -\langle u(\sigma), v \rangle$ for $v \in \sigma$ and $u(\sigma) \in M_\mathbb{R}$. An element $u \in M$ defines a global linear function that corresponds to the Cartier divisor $\text{div}(\chi^u) = \sum (u, v_i) D_i$. We can thus identify $\text{Pic}(X_\Delta)_\mathbb{R}$ with the set of piecewise linear functions on $|\Delta|$ modulo global linear functions.

We denote by $\text{pos}(S)$ the cone in $\text{Pic}(X_\Delta)_\mathbb{R}$ generated by a collection of divisors $S \subseteq \text{Pic}(X_\Delta)_\mathbb{R}$, where $S$ is some set. Our first cone is the following:

**Definition 2.1.** The cone $\mathcal{G}_\Delta$ is the set

$$\mathcal{G}_\Delta = \text{pos}([D] \in \text{Pic}(X_\Delta) : D \text{ is globally generated}).$$

The cone $\mathcal{G}_\Delta$ can be computed directly from the fan $\Delta$, as Proposition 2.3 below illustrates. For the strongest result we will need the following additional hypothesis.

**Hypothesis 2.2.** There is a projective toric variety $X_\Sigma$ with $\Delta \subseteq \Sigma$.

A function $\psi: N_\mathbb{R} \to \mathbb{R}$ is convex if $\psi(\sum_{i=1}^l u_i) \leq \sum_{i=1}^l \psi(u_i)$ for all choices of $u_1, \ldots, u_l \in N_\mathbb{R}$.

**Proposition 2.3.** Fix $[D] \in \text{Pic}(X_\Delta)$ with $D = \sum_{i \in \Delta(1)} a_i D_i$. Then the following are equivalent:

1. $[D] \in \mathcal{G}_\Delta$;
2. $[D] \in \bigcap_{\sigma \in \Delta} \text{pos}([D_i] : i \not\in \sigma)$;
3. There is a piecewise linear convex function $\psi: N_\mathbb{R} \to \mathbb{R}$ that is linear on the cones of $\Delta$ with $\psi(v_i) = a_i$.
4. $[D] \in \bigcup \Sigma i_\Sigma^*(\text{Nef}(X_\Sigma))$, where the union is over all projective toric varieties $X_\Sigma$ with $\Delta \subseteq \Sigma$ and $i_\Sigma$ is the inclusion morphism of $X_\Delta$ into $X_\Sigma$. This union is equal to the union restricted to those $\Sigma$ with $\Sigma(1) = \Delta(1)$.

**Proof.**

1 $\leftrightarrow$ 2. Recall that $D$ is globally generated if and only if for each $\sigma \in \Delta$ there is a $u(\sigma) \in M$ for which $\langle u(\sigma), v_i \rangle \geq -a_i$ for all $i$, and $\langle u(\sigma), v_i \rangle = -a_i$ when $i \in \sigma$ (see [Ful93, p68]). If $[D] \in \text{pos}([D_i] : i \not\in \sigma)$ then there a representative of $[D]$ of the form $\sum_{i \not\in \sigma} a_i D_i$, where $a_i \geq 0$ for all $i$, so we can take $u(\sigma) = 0$ for this $D$.

Conversely, if $D$ is globally generated, then for each $\sigma$ we note that $D + \text{div}((\chi^u)(\sigma))$ is an effective combination of $\{D_i : i \not\in \sigma\}$, so $[D] \in \text{pos}([D_i] : i \not\in \sigma)$ for each $\sigma \in \Sigma$.

3 $\rightarrow$ 2. Let $P_D$ be the polyhedron $\{u \in M_\mathbb{R} : \langle u, v_i \rangle \geq -a_i \text{ for all } i\}$. Then $D$ is globally generated if and only if each cone $\sigma$ of $\Delta$ is contained in a cone of the inner normal fan $N$ of $P_D$. Let $\psi: N_\mathbb{R} \to \mathbb{R}$ be defined by $\psi(v) = -\min_{u \in P_D} \langle u, v \rangle$. Then $\psi$ is a convex function that is linear on the cones of $N$, and thus linear on the cones of $\Delta$, with $\psi(v_i) = a_i$, as required.

2 $\rightarrow$ 3. Suppose that there is a piecewise linear convex function $\psi$ on $N_\mathbb{R}$ that is linear on the cones of $\Delta$ with $\psi(v_i) = a_i$. Then for any fixed $\sigma \in \Delta$ there is a full-dimensional cone $\tau \subseteq N_\mathbb{R}$ containing $\sigma$ and $u \in M_\mathbb{R}$ for which $\psi(v) = -\langle u, v \rangle$ for all $v \in \tau$. Fix $v \in \text{int}(\tau)$. Since $\tau$ is full-dimensional, for any $i$ there is $0 < \lambda < 1$ with $v' = (1-\lambda)v + \lambda v_i \in \tau$. Since $\psi$ is convex, we have $\psi(v') = -\langle u, v' \rangle = -(1-\lambda)\langle u, v \rangle - \lambda \langle u, v_i \rangle \leq \psi((1-\lambda)v) + \psi(\lambda v_i) = -(1-\lambda)\langle u, v \rangle + \lambda a_i$, so $a_i + \langle u, v_i \rangle \geq 0$. 


If $v_i \in \tau$ we have $a_i + \langle u, v_i \rangle = 0$. This implies that $D' = D + \text{div}(\chi^u) \in \text{pos}([D_i] : i \not\in \sigma)$, and thus $[D] \in \sigma \in \Delta \text{ pos}([D_i] : i \not\in \sigma)$.

4 $\rightarrow$ 1  Let $i : X_\Delta \to X_\Sigma$ be an inclusion of $X_\Delta$ into a projective toric variety $X_\Sigma$. If $[D] = \iota^*([D'])$ for some nef, and thus globally generated, divisor class $[D']$ on $X_\Sigma$, then $[D]$ is globally generated, since the pullback of a globally generated divisor is globally generated. Thus $\bigcup_{\Sigma} \iota^*(\text{Nef}(X_\Sigma)) \subseteq \mathcal{G}(X_\Delta)$.

2 $\rightarrow$ 4 We first note that Hypothesis 2.2 implies that the intersection over all $\sigma \in \Delta$ of the relative interiors of $\text{pos}(D_i : i \not\in \sigma)$ is nonempty. To see this consider a projective toric variety $X_\Sigma$ with $\Delta \subset \Sigma$ whose existence is guaranteed by Hypothesis 2.2. Let $D_i'$ denote the torus invariant divisor corresponding to the $i$th ray of $\Sigma$. For each $\sigma \in \Delta$ fix a Cartier divisor $D'_\sigma = \sum_{i \in \Sigma(1)} (a_\sigma)_i D_i'$ with $(a_\sigma)_i > 0$ for $i \not\in \sigma$ and $(a_\sigma)_i = 0$ for $i \in \sigma$. Let $D'$ be an ample Cartier divisor on $X_\Sigma$. We may choose $D'$ sufficiently positive so that $D' - D'_\sigma = \sum (a_\sigma')_i D_i'$ is also ample for all $\sigma \in \Delta$. Then for any $\sigma \in \Delta$, since $\sigma \in \Sigma$ there is $u(\sigma) \in M$ with $\langle u(\sigma), v_i \rangle = -(a_\sigma)_i$ for $i \not\in \sigma$, and $\langle u(\sigma), v_i \rangle \geq -(a_\sigma)_i$ for $i \in \sigma$. Then $[D'] = [D' - D'_\sigma + \text{div}(\chi^u(\sigma))] + [D'_\sigma] = \sum_{\sigma \not\in \sigma} (a_\sigma')_i D_i'$, where $(a_\sigma')_i > 0$ for all $i \not\in \sigma$. Thus $\iota^*([D']) \in \bigcap_{\sigma \in \Delta} \text{relint}(\text{pos}(D_i : i \notin \sigma))$.

Since this intersection is nonempty, its closure is $\bigcap_{\sigma \in \Delta} \text{pos}(D_i : i \not\in \sigma)$. Thus to show that $\bigcap_{\sigma \in \Delta} \text{pos}(D_i : i \not\in \sigma) \subseteq \bigcup_{\Sigma} \iota^*(\text{Nef}(X_\Sigma))$ it suffices to show that $\bigcup_{\sigma \in \Delta} \text{relint}(\text{pos}(D_i : i \not\in \sigma)) \subseteq \bigcup_{\Sigma, \Sigma(1) = \Delta(1)} \iota^*(\text{Nef}(X_\Sigma))$, as this latter, a priori smaller, set is a finite union of closed sets. Let $D = \sum a_i D_i \in \bigcap_{\sigma \in \Delta} \text{relint}(\text{pos}(D_i : i \not\in \sigma))$. Then the regular subdivision $\Sigma'$ of $\{v_i\}$ induced by the $a_i$ contains $\sigma$ as a face for all $\sigma \in \Delta$ (see, for example, [Stu96, Chapter 8]). Let $D' = \sum a_i D'_i$ denote the corresponding divisor on $X_\Sigma$. By construction $P_{D'}$ has $\Sigma'$ as its normal fan, so $D'$ is ample; see [Ful93, pp66-70]. Thus $[D] = \iota^*([D']) \in \iota^*(\text{Nef}(X_\Sigma'))$. Note that $\Sigma'(1) = \Delta(1)$ by construction, so this shows that $\mathcal{G}_\Delta \subseteq \bigcup_{\Delta \subseteq \Sigma, \Delta(1) = \Sigma(1)} \iota^*(\text{Nef}(X_\Sigma)) \subseteq \bigcup_{\Delta \subseteq \Sigma} \iota^*(\text{Nef}(X_\Sigma))$. Since we have already shown the inclusion $\bigcup_{\Delta \subseteq \Sigma} \iota^*(\text{Nef}(X_\Sigma)) \subseteq \mathcal{G}_\Delta$, we conclude that all three sets coincide. 

Remark 2.4. To see that Hypothesis 2.2 is needed for the equivalence of the last item, consider the complete three-dimensional toric variety $X_\Delta$ whose fan intersects the sphere $S^2$ as shown in Figure 1 and for which $v_1, \ldots, v_7$ are the columns of the matrix $V$ below. In the picture, vertex 7 has been placed at infinity.

Then $\text{Pic}(X_\Delta) \cong \mathbb{Z}^4$, with an isomorphism taking $[D_i]$ to the $i$th column of the matrix $G$:

$$V = \begin{pmatrix} 3 & 0 & 0 & 2 & 1 & 1 & -1 \\ 0 & 3 & 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 3 & 1 & 1 & 2 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 & 3 & 0 \\ -1 & -1 & 2 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The cone $\mathcal{G}_\Delta$ equals $\text{pos}((1,0,0,3))$. This can be obtained from the description of Part 2 of Proposition 2.3 using software such as PORTA [TCAL]. There is no projective toric variety $X_\Sigma$ with $\Delta \subseteq \Sigma$, as we would have to have $\Sigma = \Delta$ since $\Delta$ is complete, and $X_\Delta$ is not projective.

In such cases we may recognize $\mathcal{G}_\Delta$ as the union of the pullbacks of the nef cones of all complete toric varieties $X_\Sigma$ with $\Delta \subseteq \Sigma$. Indeed, the proof of $4 \rightarrow 1$ of Proposition 2.3 goes through unchanged. The other inclusion can be obtained by modifying the second paragraph of $2 \rightarrow 4$ (ignoring the first paragraph, which no
longer applies), by taking a refined refinement of the fan $\Sigma'$ so that $\Delta$ is still a subfan. The more restrictive statement of Proposition 2.3(4), however, is more relevant in our applications in the rest of this paper.

A cone $\sigma \in \Delta$ determines a torus orbit $O(\sigma)$ which is isomorphic to $(k^*)^{n-\dim(\sigma)}$. We denote by $V(\sigma)$ the closure of $O(\sigma)$ in $X_\Delta$.

**Definition 2.5.** For $\sigma \in \Delta$ we denote by $i_\sigma : V(\sigma) \to X_\Delta$ the inclusion of the orbit closure $V(\sigma)$ into $X_\Delta$. Then

$$L_\Delta = \{ [D] \in \text{Pic}(X_\Delta) : i_\sigma^*([D]) \text{ is effective for all } \sigma \in \Delta \}.$$

**Remark 2.6.** Note that by taking $\sigma = \{0\}$ we see that $L_\Delta$ is contained in the effective cone of $X_\Delta$. Note also that when $\sigma$ is a maximal cone of $\Delta$, $V(\sigma) \cong (k^*)^{n-\dim(\sigma)}$, so the condition that $i_\sigma^*([D])$ is effective is vacuous. Let $\Delta^o$ be the subfan of $\Delta$ with all maximal cones removed. We thus have $L_\Delta = \{ [D] \in \text{Pic}(X_\Delta) : i_\sigma^*([D]) \text{ is effective for all } \sigma \in \Delta^o \}$.

Recall that the star of a cone $\sigma \in \Delta$ is the fan $\text{star}(\sigma)$ whose cones are $\{ \tau \in \Delta : \sigma \subseteq \tau \}$ together with all faces of these cones. Let $N_\sigma$ be the lattice generated by $N \cap \sigma$. For $\tau \in \text{star}(\sigma)$ we denote by $\pi$ the cone $(\tau + N_\sigma \otimes \mathbb{R})/N_\sigma \otimes \mathbb{R}$. The cones $\{ \tau : \tau \in \text{star}(\sigma) \}$ form a fan in $(N/N_\sigma) \otimes \mathbb{R}$, and the corresponding toric variety is $V(\sigma)$. The fan $\text{star}^1(\sigma)$ is the subfan of $\text{star}(\sigma)$ whose top-dimensional cones are $\{ \tau \in \text{star}(\sigma) : \dim(\tau) = \dim(\sigma) + 1 \}$. The top-dimensional cones of $\text{star}^1(\sigma)$ correspond to rays of the fan of $V(\sigma)$. We denote by $e_\tau$ the first lattice point on the ray corresponding to $\tau \in \text{star}^1(\sigma)$. For any $i \in \tau \setminus \sigma$ we have $v_i = c_i e_\tau$ for some integer $c_i > 0$, where $v_i$ denotes the image of $v_i$ in $(N/N_\sigma) \otimes \mathbb{R}$.

Fix $\sigma \in \Delta$, a Cartier divisor $D = \sum a_i D_i$, and choose $u(\sigma) \in M$ satisfying $\langle u(\sigma), v_i \rangle = -a_i$. We will use the following formula for $i_\sigma^*([D])$:

$$(1) \quad i_\sigma^*([D]) = \sum_{\tau \in \text{star}^1(\sigma)} b_\tau [D_\tau],$$

where $b_\tau = (a_i + \langle u(\sigma), v_i \rangle)/c_i$ for any $i \in \tau \setminus \sigma$. This is independent of the choice of $i$; see [Ful93, p. 97].
A function \( \psi : \Delta \to \mathbb{R} \) linear on the cones of \( \Delta \) is \textit{convex on \( \text{star}^1(\sigma) \)} if the inequality \( \psi(\sum_{i=1}^l u_i) \leq \sum_{i=1}^l \psi(u_i) \) holds for all \( u_1, \ldots, u_l \in \text{star}^1(\sigma) \) with \( \sum_{i=1}^l u_i \in \sigma \).

**Proposition 2.7.** Fix \( [D] \in \text{Pic}(\Delta \setminus \Delta) \) with \( D = \sum_{i \in \Delta(1)} a_i D_i \), and \( \sigma \in \Delta \). Let \( i_\sigma : V(\sigma) \to X_\Delta \) be the inclusion map. Then the following are equivalent:

1. \( i_\sigma^*(\psi(D)) \) is effective;
2. \( [D] = [D'] \), where \( D' = \sum a'_i D_i \) with \( a'_i = 0 \) for \( i \in \sigma \) and \( a'_i \geq 0 \) for \( i \not\in \text{star}^1(\sigma) \);
3. The piecewise linear function \( \psi_D : |\Delta| \to \mathbb{R} \) defined by setting \( \psi_D(v_i) = a_i \) and extending to be linear on each cone \( \tau \in \Delta \) is convex on \( \text{star}^1(\sigma) \);
4. \( \sum_i a_i b_i \geq 0 \) for all \( b = (b_i) \) with \( \sum_{i \in \Delta(1)} b_i v_i = 0 \) such that \( b_i = 0 \) for \( i \not\in \text{star}^1(\sigma) \), and \( b_i \geq 0 \) for \( i \in \text{star}^1(\sigma) \setminus \sigma \).

**Proof.**

1 \( \to \) 2: Suppose \( i_\sigma^*[\psi(D)] \) is effective. We may assume that the representative for \( D \) has been chosen so that \( a_i = 0 \) for \( i \in \sigma \) (by replacing \( D \) by \( D + \text{div}(\chi^{u(\sigma)}) \)), so by Equation 1, we have \( i_\sigma^*[\psi(D)] = \sum_{\tau \in \text{star}^1(\sigma)} b_\tau D_\tau \), where \( b_\tau = a_i/c_i \) for any \( i \in \tau \setminus \sigma \), and \( v_i = c_i e_\tau \in (\mathbb{N}/\mathbb{N}_\sigma) \otimes \mathbb{R} \). Since \( i_\sigma^*[\psi(D)] \) is effective there is a representative \( \psi_D(D) = \sum b'_\tau D_\tau \) with \( b'_\tau \geq 0 \) for all \( \tau \in \text{star}^1(\sigma) \). This means that there is \( u \in \text{Hom}(\mathbb{N}/\mathbb{N}_\sigma, \mathbb{R}) \) with \( b'_\tau = b_\tau + \langle u, e_\tau \rangle \). Let \( \tilde{D} \) be the lift of \( u \) to \( M_{\mathbb{R}} \) with \( \langle \tilde{u}, v_i \rangle = 0 \) for \( i \in \sigma \). Let \( D' = D + \text{div}(\chi^u) \). Then \( D' = \sum a'_i D_i \), where by construction \( a'_i = 0 \) for \( i \in \sigma \). In addition, for \( i \in \tau \setminus \sigma \) with \( \tau \in \text{star}^1(\sigma) \) we have \( a'_i = a_i + c_i \langle u, e_\tau \rangle = c_i b'_\tau \geq 0 \), so \( D' \) has the desired form.

2 \( \to \) 3: Note first that if \( D' = D + \text{div}(\chi^u) \) for some \( u \in M \) then \( \psi_D(v) = \psi_D(v + \langle u, v \rangle) \). Thus \( \psi_D \) is convex on \( \text{star}^1(\sigma) \) if and only if \( \psi_{D'} \) is. Suppose now that \( [D] = [D'] \) for \( D' = \sum a'_i D_i \) with \( a_i = 0 \) for \( i \in \sigma \) and \( a_i \geq 0 \) for \( i \not\in \text{star}^1(\sigma) \). Let \( u_1, \ldots, u_l \in \text{star}^1(\sigma) \) with \( \sum_{i=1}^l u_i \in \sigma \). Then \( \psi_{D'}(u_i) \geq 0 \) for all \( i \), and \( \psi_{D'}(\sum_{i=1}^l u_i) = 0 \), so \( \psi_{D'} \) is convex on \( \text{star}^1(\sigma) \).

3 \( \to \) 4: Suppose that \( \psi_D \) is convex on \( \text{star}^1(\sigma) \). By replacing \( D \) by \( D + \text{div}(\chi^{u(\sigma)}) \) we may assume that \( \psi_D(v) = 0 \) for all \( v \in \sigma \). Let \( b \in \mathbb{R}^{[\Delta(1)]} \) satisfy \( \sum b_i v_i = 0 \), \( b_i = 0 \) for \( i \not\in \text{star}^1(\sigma) \), and \( b_i \geq 0 \) for \( i \in \text{star}^1(\sigma) \). Let \( u_1 = b_1 v_1 \) for \( i \in \text{star}^1(\sigma) \), and \( u_0 = \sum_{j \in \sigma, b_j > 0} b_j v_j \). Then \( u_1 \in \text{star}^1(\sigma) \) for all \( i \), and \( u_0 + \sum_{i \in \text{star}^1(\sigma)} u_i = \sum_{j \in \sigma, b_j < 0} (-b_j) v_j \in \sigma \), so since \( \psi_D \) is convex on \( \text{star}^1(\sigma) \), we have \( \sum \psi_D(u_i) \geq \psi_D(\sum_{j \in \sigma, b_j < 0} (-b_j) v_j) = 0 \). Now \( \psi_D(u_i) = b_i a_i \) for \( i \in \text{star}^1(\sigma) \) \( \setminus \sigma \), and \( \psi_D(u_0) = 0 \), so \( \sum_{i \in \Delta(1)} a_i b_i = \sum_{i \in \text{star}^1(\sigma)} a_i b_i \geq 0 \) as required.

4 \( \to \) 1: Suppose that \( \sum a_i b_i \geq 0 \) for all \( b \in \mathbb{R}^{[\Delta(1)]} \) with \( \sum b_i v_i = 0 \), \( b_i \geq 0 \) for \( i \not\in \sigma \) and \( b_i = 0 \) for \( i \not\in \text{star}^1(\sigma) \). Let \( i^*_\sigma(D) = \sum d_\tau D_\tau \). To show that \( i^*_\sigma(D) \) is effective, it suffices to show that it lies on the correct side of all facet-defining hyperplanes of the effective cone, and thus that \( \sum_{\tau} b_\tau d_\tau \geq 0 \) for all choices of \( b_\tau \geq 0 \) with \( \sum_{\tau \in \text{star}^1(\sigma)} b_\tau e_\tau = 0 \). Given such a vector \( b \), we construct \( b \in \mathbb{R}^{[\Delta(1)]} \) with \( b_i v_i \) as follows. For each \( \tau \in \text{star}^1(\sigma) \) choose \( \tau \in \tau \setminus \sigma \), and set \( b_i = b_\tau/c_i \), where as above \( v_i = c_i e_\tau \). Set \( b_j = 0 \) for all other \( j \in \tau \setminus \sigma \), and for \( j \not\in \text{star}(\sigma) \). We then have \( \sum_{\tau \in \text{star}^1(\sigma)} \sum_{i \in \tau \setminus \sigma} b_i v_i \in N_\sigma \). Choose \( b_j \in \mathbb{Z} \) so that this sum is \( \sum_{i \in \tau \setminus \sigma} b_i v_i \). Then by construction \( \sum b_i v_i = 0 \), \( b_i \geq 0 \) for \( i \not\in \sigma \), and \( b_i = 0 \) for \( i \not\in \text{star}(\sigma) \). Thus \( \sum a_i b_i \geq 0 \). Now \( d_\tau = (a_i + \langle u, v_i \rangle)/c_i \) so \( \sum_{\tau} b_\tau d_\tau = \sum_{\tau} \sum_{i \in \tau \setminus \sigma} b_i c_i (a_i + \langle u, v_i \rangle) = 0 \).
as in Figure 2. Then the Picard group is generated by \( \text{torus invariant points removed} \). Name the torus invariant prime divisors \( D \) equivalent:

Definition 2.10. Fix cone \( G \) on all of \( N \).

Remark 2.9. Note that the convex function \( \psi \) is locally convex (convex on star \( |\Delta| \)) on \( \Delta \). Also, the first \( \psi \) is required to be globally convex, while the second is only locally convex (convex on \( \text{star}^1(\sigma) \)).

Corollary 2.8. Fix \([D] \in \text{Pic}(X_\Delta)\) with \( D = \sum_{i \in \Delta(1)} a_i D_i \). Then the following are equivalent:

1. \([D] \in \mathcal{L}_\Delta;\)
2. \([D] \in \bigcap_{\sigma \in \Delta^o} \text{pos}([D_i], \pm [D_j] : i \in \text{star}^1(\sigma) \setminus \sigma, j \in \Delta(1) \setminus \text{star}^1(\sigma));\)
3. The piecewise linear function \( \psi : |\Delta| \to \mathbb{R} \) defined by setting \( \psi(v_i) = a_i \) and extending to be linear on each cone \( \sigma \in \Delta \) is convex on \( \text{star}^1(\sigma) \) for all \( \sigma \in \Delta^o;\)
4. \( \sum_i a_i b_i \geq 0 \) for all \( b = (b_i) \) with \( \sum_i b_i v_i = 0 \) such that there is \( \sigma \in \Delta \) with \( b_i = 0 \) for \( i \not\in \text{star}^1(\sigma) \), and \( b_i \geq 0 \) for \( i \in \text{star}^1(\sigma) \setminus \sigma \).

Proof. This follows directly from Proposition 2.7, since the set \( \{[D] : [D] = [\sum a_i D_i] \text{ with } a_i = 0 \text{ for } i \in \sigma, a_i \geq 0 \text{ for } i \in \text{star}^1(\sigma) \} \) equals \( \text{pos}([D_i], \pm [D_j] : i \in \text{star}^1(\sigma) \setminus \sigma, j \in \Delta(1) \setminus \text{star}^1(\sigma)) \).

Remark 2.9. Note that the convex function \( \psi \) in Part 3 of Proposition 2.3 is defined on all of \( N_{\mathbb{R}} \), while the function \( \psi \) defined in Part 3 of Corollary 2.8 is only defined on \( |\Delta| \). Also, the first \( \psi \) is required to be globally convex, while the second is only locally convex (convex on \( \text{star}^1(\sigma) \)).

To see this second difference, let \( X_\Delta \) be the Hirzebruch surface \( \mathbb{F}_1 \) with the four torus invariant points removed. Name the torus invariant prime divisors \( D_1, \ldots, D_4 \) as in Figure 2. Then the Picard group is generated by \([D_1] = [D_3]\) and \([D_4]\). The cone \( \mathcal{G}_\Delta \) is \( \bigcap_{i=1}^4 \text{pos}([D_j] : j \neq i) = \text{pos}([D_1], [D_4]) \). The cone \( \mathcal{L}_\Delta \) is equal to the effective cone \( \text{pos}([D_1], [D_2]) \) of \( X_\Delta \), so \( \mathcal{G}_\Delta \subseteq \mathcal{L}_\Delta \). This is illustrated in Figure 2.

For the last cone we assume that every maximal cone in \( \Delta \) has dimension \( d \).

Definition 2.10. Fix \( w \in \text{Hom}(A_{n-d}(X_\Delta), \mathbb{R}) \) and let

\[
\mathcal{F}_{\Delta,w} = \{ D \in \text{Pic}(X_\Delta)_{\mathbb{R}} : w([D] \cdot [V(\tau)]) \geq 0 \text{ for all } \tau \in \Delta(d-1) \} = \{ D \in \text{Pic}(X_\Delta)_{\mathbb{R}} : \sum a_\sigma^* w_\sigma \geq 0 \text{ for all } \tau \in \Delta(d-1) \},
\]

where \([D] \cdot [V(\tau)] = \sum_{\sigma \in \Delta(d)} a_\sigma^*[V(\sigma)], \text{ and } w_\sigma = w(V(\sigma)).\)
Definition 2.11. Let $W = \{ w \in \text{Hom}(\mathbb{A}_{n-d}(X_\Delta), \mathbb{R}) : w(V(\sigma)) \geq 0 \text{ for all } \sigma \in \Delta(d) \}$. Let

$$F_\Delta = \bigcap_{w \in W} F_{\Delta,w}.$$  

When $A_{n-d}(X_\Delta) \cong \mathbb{Z}$, then $F_\Delta = F_{\Delta,w}$ for all $w \in W$. This is the case for many specific $\Delta$ of interest; see Propositions 4.1 and 6.4. We will use the following equivalent descriptions of $F_\Delta$.

Proposition 2.12. Fix $[D] \in \text{Pic}(X_\Delta)$ with $D = \sum_{i \in \Delta(1)} a_i D_i$. Then the following are equivalent:

1. $[D] \in F_\Delta$;
2. $i_\tau^*(D)$ is effective for the inclusions $i_\tau : V(\tau) \to X_\Delta$ with $\tau \in \Delta(d-1)$;
3. $[D] \in \bigcap_{\sigma \in \Delta(d-1)} \text{pos}([D_i], \pm [D_j] : i \in \text{star}^1(\sigma) \setminus \sigma, j \in \Delta(1) \setminus \text{star}^1(\sigma))$;
4. $\sum_i a_i b_i \geq 0$ for all $b = (b_i)$ with $\sum b_i v_i = 0$ such that there is $\sigma \in \Delta(d-1)$ with $b_i = 0$ for $i \notin \text{star}^1(\sigma)$, and $b_i \geq 0$ for $i \in \text{star}^1(\sigma) \setminus \sigma$;
5. The piecewise linear function $\psi : |\Delta| \to \mathbb{R}$ defined by setting $\psi(v_i) = a_i$ and extending to be linear on each cone $\sigma \in \Delta$ is convex on star$^1(\sigma)$ for all $\sigma \in \Delta(d-1)$.

Proof. The equivalence of Part 2 with the following ones is a direct corollary of Proposition 2.7 as in Corollary 2.8, so we thus need only show the equivalence of the first condition with the others.

Suppose first that $i_\tau^*(D)$ is effective for all $\tau \in \Delta(d-1)$. Then for all such $\tau$, the class $[D] \cdot [V(\tau)]$ lies in the cone generated by $\{ [V(\sigma)] : \sigma \in \text{star}(\tau), \dim(\sigma) = d \}$. Thus $w([D] \cdot [V(\tau)]) \geq 0$ for all $w \in W$, so $D \in F_\Delta$.

Conversely, suppose that $D \in F_\Delta$. Then for all $\tau \in \Delta(d-1)$ the class $[D] \cdot [V(\tau)]$ must have a representative $\sum_{\sigma \in \Delta(d)} a_\sigma [V(\sigma)]$, where $a_\sigma \geq 0$, as otherwise there would be $w \in W$ with $w([D] \cdot [V(\tau)]) < 0$. Since $i_\tau^*([D]) = \sum_{\sigma \in \text{star}^1(\tau)} a_\sigma D_\sigma$, this implies that $i_\tau^*([D])$ is effective.

Recall that $\Delta^o$ is the fan obtained from $\Delta$ by removing the maximal cones.

Proposition 2.13. We have

$$G_\Delta \subseteq G_{\Delta^o} \subseteq L_\Delta,$$

and if $\Delta$ is a pure fan of dimension $d$ then

$$L_\Delta \subseteq F_\Delta \subseteq F_{\Delta,w}.$$  

If $\Delta$ is the fan of a projective toric variety $X_\Delta$, then all cones coincide, and are equal to Nef$(X_\Delta)$.

Proof. By Part 2 of Proposition 2.3 $G_\Delta$ is the intersection of $G_{\Delta^o}$ with some other cones, which gives the first inclusion. Next, note that for $\sigma \in \Delta^o$, we have $\text{pos}([D_i] : i \notin \sigma) \subseteq \text{pos}([D_i], \pm [D_j] : i \in \text{star}^1(\sigma) \setminus \sigma, j \in \Delta(1) \setminus \text{star}^1(\sigma))$, so $G_{\Delta^o} \subseteq L_\Delta$ follows from Part 2 of Proposition 2.3 and Part 2 of Corollary 2.8.

Suppose now that $\Delta$ is pure of dimension $d$. The inclusion $L_\Delta \subseteq F_\Delta$ comes from Part 3 of Proposition 2.12, since the intersection for $F_\Delta$ is over only the $(d-1)$-dimensional cones of $\Delta$ rather than all cones as for $L_\Delta$. The inclusion $F_\Delta \subseteq F_{\Delta,w}$ comes from the definition of $F_\Delta$. 


If $\Delta$ is the fan of a projective toric variety then a divisor is globally generated if and only if it is nef, so $G_\Delta = \text{Nef}(X_\Delta)$. For such $\Delta$ we have $d = n$, so $A_{n-d}(X_\Delta) \cong \mathbb{Z}$, and thus $F_\Delta = F_{\Delta, w} = \{ D \in \text{Pic}(X_\Delta) : [D] : |V(\tau)| \geq 0 \text{ for all } \tau \in \Delta(n-1) \}$. The classes $\{ V(\tau) : \tau \in \Delta(n-1) \}$ generate the Mori cone of curves of $X_\Delta$, so we also have $F_\Delta = \text{Nef}(X_\Delta)$.

### 3. Bounds for nef cones

In this section we prove the main theorem of this paper, Theorem 3.2, which shows that given an appropriate embedding of a projective variety $Y$ into $X_\Delta$, the pullbacks of the cones $G_\Delta$ and $L_\Delta$ give lower bounds for $\text{Nef}(Y)$, and the pullback of the cone $F_{\Delta, w}$ for appropriate $w$ gives an upper bound.

The upper bound requires some tropical geometry, which we first review briefly.

**Definition 3.1.** Let $Y^0 \subset T$ be a subvariety of a torus $T \cong (k^*)^n$. Let $K$ be an algebraically closed field extension of $k$ with a valuation $\text{val} : K^* \to \mathbb{R}$ that is constant on $k$ and with residue field isomorphic to $k$. The tropical variety $\text{trop}(Y^0)$ is equal as a set to the closure in $\mathbb{R}^n$ of $\{ (\text{val}(y_1), \ldots, \text{val}(y_n)) \in \mathbb{R}^n : (y_1, \ldots, y_n) \in Y^0(K) \}$.

By the structure theorem for tropical varieties, the set $\text{trop}(Y^0)$ can be given the structure of a polyhedral fan of dimension $\dim(Y^0)$ ([MS, Theorem 3.3.4]). There are many possible choices of fan structure. We will fix one, which we denote by $\Sigma'$, for which the initial ideal $\text{in}_w(I(Y^0))$ is constant for all $w$ in the relative interior of a cone, where $I(Y^0)$ is the ideal in $k[T]$ defining $Y^0$. For $w \in \mathbb{R}^n$ the initial ideal $\text{in}_w(I(Y^0))$ is $\langle \text{in}_w(f) : f \in I(Y^0) \rangle$, where for a Laurent polynomial $f = \sum c_u x^u$ the initial form $\text{in}_w(f)$ is $\sum_{w \cdot u \text{ minimal}} c_u x^u$.

Given such a polyhedral fan structure $\Sigma'$ on $\text{trop}(Y^0)$, we associate to each top-dimensional cone $\sigma \in \Sigma'$ a positive integer $m_\sigma$ as follows. Choose $w$ in the relative interior of $\sigma$; then $m_\sigma = \sum_{P \in \text{Ass}(|\text{in}_w(I(Y^0))|)} \text{mult}(P, \text{in}_w(I(Y^0)))$, where $\text{Ass}(\cdot)$ denotes the set of associated primes of the ideal. Let $m = (m_\sigma)$ be the vector of multiplicities as $\sigma$ varies over top-dimensional cones. The tropical variety of $Y^0$ is the pair $(\text{trop}(Y^0), m)$. This is the “constant coefficient” case of tropical geometry. See [MS] for more information.

Given any fan $\Delta$ with $|\Delta| = \text{trop}(Y^0)$, where $\dim(Y^0) = d$, we get an element $w \in \text{Hom}(A_{n-d}(X_\Delta), \mathbb{R})$ by setting $w(V(\sigma))$ equal to the weight $m_\tau$ on any $d$-dimensional cone $\tau$ in $\Sigma'$ intersecting $\sigma$ in its relative interior. That this is well-defined follows from the balancing condition on tropical varieties; see [MS, §3.4], [FS97].

**Theorem 3.2.** Let $i : Y \to X_\Delta$ be an embedding of a $d$-dimensional projective variety $Y$ into an $n$-dimensional normal toric variety $X_\Delta$. Then we have the following inclusions of cones in $N^1(Y)_{\mathbb{R}}$:

$$i^*(G_\Delta) \subseteq i^*(L_\Delta) \subseteq \text{Nef}(Y).$$

If in addition we have $\text{trop}(Y^0) = |\Delta|$ for $Y^0 = Y \cap T$, and $i^* : \text{Pic}(X_\Delta)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$ is surjective, then

$$\text{Nef}(Y) \subseteq i^*(F_{\Delta, w}),$$

where $w \in \text{Hom}(A_{n-d}(X_\Delta), \mathbb{R})$ is as described above.
Remark 3.3. Conceptually the proof of this theorem is very straightforward. The upper bound comes from intersecting $D$ with curves obtained as the intersection of $Y$ with appropriate torus-orbit closures on $\Delta$. Care must be taken to make this rigorous however, as we do not assume that $X_\Delta$ is smooth or complete. In particular, we note that if $\dim(\Delta) < n - 1$ then $A_1(X_\Delta) = A_0(X_\Delta) = 0$, so the classes of these curves in $X_\Delta$ are zero.

Proof. The inclusion $i^*(G_\Delta) \subseteq i^*(L_\Delta)$ was shown in Proposition 2.13. To see the inclusion $i^*(L_\Delta) \subseteq \text{Nef}(Y)$, let $D' = i^*(D)$ for $D \in L_\Delta$. Choose a complete fan $\Sigma$ for which $\Delta$ is a subfan. This is possible by [Ewa96, Theorem 2.8]. Write $k: X_\Delta \rightarrow X_\Sigma$ for the inclusion morphism. After repeated stellar subdivision of cones in $\Sigma \setminus \Delta$, we can write \( \sigma \cdot [C] = \sum_{\tau \in \Delta(1)} a_{\tau} D_{\tau} \) and similarly for the other ambient spaces. Note that $j_{\tau}(\sigma) = i_{\sigma,\tau}([C]_{V_\sigma(\tau)}) = [C]_{\sigma} \in A_1(X_\Sigma)$, where $j = k \circ i$. Since $D \in L_\Delta$, by Part 2 of Proposition 2.7 we can write $[D] = \sum a_i D_i$, where $a_i = 0$ for $i \in \sigma$, and $a_i \geq 0$ for $i \in \star 1(\sigma)$. Let $\tilde{D} = \sum a_i D_i$ be the corresponding divisor on $X_\Sigma$ with $k^*(\tilde{D}) = D$.

We then have the following diagram:

$$
\begin{array}{ccc}
V_\Delta(\sigma) & \xrightarrow{k} & V_\Sigma(\sigma) \\
\downarrow{i_\sigma} & & \downarrow{i_\sigma} \\
Y & \xrightarrow{i} & X_\Delta & \xrightarrow{k} & X_\Sigma \\
& & \xrightarrow{j} & & \\
\end{array}
$$

Then $i_{\tau,\sigma}^*(\tilde{D}) = \sum_{\tau \cap \sigma \neq \emptyset} b_{\tau} D_{\tau}$, where $b_{\tau}$ is as in Equation 2. Note that for $\tau \in \Delta$, the coefficient $b_{\tau}$ is a positive multiple of $a_i$ for $i \in \tau \setminus \sigma$, and thus $b_{\tau} \geq 0$. Since $C \cap \mathcal{O}_\sigma \neq \emptyset$, $D_{\tau} \cdot [C]_{V_\Sigma(\sigma)} \geq 0$ for all torus-invariant prime divisors $D_{\tau}$ on $V_\Sigma(\sigma)$. In addition, since $C \subseteq Y \subseteq X_\Delta$, we have $D_{\tau} \cdot [C]_{V_\Sigma(\sigma)} = 0$ for $\tau \notin \Delta$. Thus $i_{\tau,\sigma}^*(\tilde{D}) \cdot [C]_{V_\Sigma(\sigma)} \geq 0$, so by the projection formula $i_{\sigma,*}(i_{\sigma}^*(\tilde{D}) \cdot [C]_{V_\Sigma(\sigma)}) = \tilde{D} \cdot [C]_{\Sigma} \geq 0$. Then $j_{\tau}(D' \cdot [C]) = \tilde{D} \cdot [C] \geq 0$, so since $j_{\tau}$ is not the zero map, $D' \cdot [C] \geq 0$ and thus $D' \in \text{Nef}(Y)$.

For the last inclusion, fix $D' \in \text{Nef}(Y)$. Since $i^*$ is assumed to be surjective we can write $D' = i^*(D)$ for some $D \in \text{Pic}(X_\Delta)$. We wish to show that $\text{w}(D \cdot V(\tau)) \geq 0$ for all $\tau \in \Delta(d-1)$.

Choose a smooth toric resolution $X_\Delta$ of $X_\Delta$ with the property that if $\tilde{Y}$ is the closure of $Y^0$ in $X_\Delta$ (so $\tilde{Y}$ is the strict transform of $Y$ with respect to the morphism $\pi: X_\Delta \rightarrow X_\Delta$), then the inclusion $\tilde{Y} \rightarrow X_\Delta$ is tropical in the sense of [Tev07]. To see that this is always possible, begin by taking the common refinement of $\Delta$ and a fixed tropical fan with support $|\Delta|$, and then taking a smooth refinement of this.
fan. The resulting morphism $\pi: \tilde{\Delta} \to \Delta$ will be proper and birational. Moreover, it defines a smooth tropical compactification as refinements of tropical fans are tropical by [Tev07, Proposition 2.5]. Let $\tilde{w} \in \text{Hom}(\text{A}_{n-d}(\tilde{X}_\Delta), \mathbb{R})$ be the induced homomorphism defined by $\tilde{w}(V(\tilde{\sigma})) = w(\pi_*(V(\tilde{\sigma})))$ for all $\tilde{\sigma} \in \tilde{\Delta}(d)$. To see that $\tilde{w}$ gives a well-defined element of $\text{Hom}(\text{A}_{n-d}(\tilde{X}_\Delta), \mathbb{R})$ consider the relations on $\text{A}_{n-d}(\tilde{X}_\Delta)$ given in [FS97, Proposition 2.1b]. Given $\tau \in \Delta(d-1)$, choose $\tilde{\tau}$ with $\pi(\tilde{\tau}) \subset \tau$, so $\pi_* (V(\tilde{\tau})) = V(\tau)$ ([Ful93, p100]). Then $w(D \cdot V(\tau)) = w(\pi_* (\pi^*(D) \cdot V(\tilde{\tau}))) = \tilde{w}(\pi^*(D) \cdot V(\tilde{\tau}))$. So it suffices to show that $\tilde{w}(\pi^*(D) \cdot V(\tilde{\tau})) \geq 0$ for all $\tilde{\tau} \in \tilde{\Delta}(d-1)$.

Choose a smooth projective toric variety $X_\Sigma$ whose fan $\tilde{\Sigma}$ contains $\tilde{\Delta}$ as a subfan so $k: X_\Delta \to X_\Sigma$ is an embedding. This is possible, assuming a suitably refined choice of $\Delta$, because the tropical fan structure on $|\Delta|$ given by the Gröbner fan has a projective compactification given by the Gröbner fan, and we can take a projective toric resolution of singularities of the resulting fan, refining $\Delta$ if necessary. As before, choose $\tilde{\Sigma}$ so that if $\sum_{i \in \tilde{\Delta}(1)} a_i D_i$ is a Cartier divisor on $X_\Delta$, then $\sum_{i \in \tilde{\Delta}(1)} a_i D_i$ is a Cartier divisor on $X_\Sigma$, and so the induced map $k^*: \text{Pic}(X_\Sigma) \to \text{Pic}(X_\Delta)$ is surjective.

Let $\tilde{j}: \tilde{Y} \to X_\Sigma$ be the composition of $\tilde{i}: \tilde{Y} \to X_\Delta$ and $k: X_\Delta \to X_\Sigma$. We thus have the following diagram.

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{j} & X_\Delta \\
\downarrow{\pi} & & \downarrow{\pi} \\
Y & \xrightarrow{i} & X_\Delta
\end{array}
$$

Fix $\tilde{\tau} \in \tilde{\Delta}(d-1)$. Since $\tilde{Y} \to X_\Delta$ is a tropical compactification, $C_{\tilde{\tau}} = \tilde{Y} \cap V(\tilde{\tau})$ is a curve on $\tilde{Y}$. Let $C_1, \ldots, C_r$ be the irreducible components of $C_{\tilde{\tau}}$. Then by [Ful93, §7.1] we have $[\tilde{Y}]_\Sigma \cdot [V(\tilde{\tau})]_\Sigma = \sum \lambda_i [C_i]_\Sigma \in A_i(X_\Sigma)$, where $\lambda_i \geq 1$.

Write $\pi^*(D) = \sum_{i \in \tilde{\Delta}(1)} a_i D_i$, and let $\bar{D} = \sum_{i \in \tilde{\Delta}(1)} a_i D_i \in \text{A}_{n-1}(X_\Sigma)$, so $k^*(\bar{D}) = \pi^*(D)$, and thus $\pi^*(D') = \tilde{j}_* (\bar{D})$. Since $D' \in \text{Nef}(Y)$, we have $\pi^*(D') \in \text{Nef}(\tilde{Y})$. Thus $\pi^*(D') \cdot [C_i]_{\tilde{Y}} \geq 0$ for $1 \leq i \leq r$. So $\tilde{D} \cdot [C_i]_{\Sigma} = j_* (\pi^*(D') \cdot [C_i]_{\tilde{Y}}) \geq 0$ by the projection formula. Thus $\tilde{D} \cdot [\tilde{Y}] \cdot [V(\tilde{\tau})] \geq 0$, where this computation takes place in $A^*(X_\Sigma)$.

Extend $\tilde{w}$ to an element of $\text{Hom}(\text{A}_{n-d}(X_\Sigma), \mathbb{R})$ by setting $\tilde{w}(V(\sigma)) = 0$ for $\sigma \in \tilde{\Sigma}(d) \setminus \tilde{\Delta}(d)$. Again, this is well-defined by [FS97, Proposition 2.1b]. Since $\tilde{Y} \to X_\Delta$ is a smooth tropical compactification, [ST08, Lemma 3.2(1)] implies that $[\tilde{Y}] \cdot ([\bar{D}] \cdot [V(\tilde{\tau})]) = \tilde{w}([\bar{D}] \cdot [V(\tilde{\tau})])$. Thus $\tilde{w}([\bar{D}] \cdot [V(\tilde{\tau})]) \geq 0$.

Note that if $\bar{D} \cdot [V(\tilde{\tau})]_{\Sigma} = \sum_{\sigma \in \tilde{\Sigma}(d)} a_{\sigma} [V(\sigma)]_{\Sigma}$ for $\tilde{\tau} \in \tilde{\Delta}(d-1)$ then $\pi^*(D) \cdot [V(\tilde{\tau})]_{\Delta} = \sum_{\sigma \in \Delta(d-1)} a_{\sigma} [V(\sigma)]_{\Delta}$, so $\tilde{w}([\bar{D}] \cdot [V(\tilde{\tau})]) = \tilde{w}([\pi^*(D)] \cdot [V(\tilde{\tau})]) \geq 0$ as required. □

**Corollary 3.4.** Let $\Delta$ be a pure $d$-dimensional polyhedral fan such that there is $w \in \text{Hom}(\text{A}_{n-d}(X_\Delta), \mathbb{R})$ with $L_\Delta = F_{\Delta,w}$. Then any subvariety $i: Y \to X_\Delta$ with $i^*: \text{Pic}(X_\Delta)_R \to N^1(Y)_R$ surjective and $\text{trop}(Y \cap T) = \Delta$ with multiplicities given by
w has \[ \text{Nef}(Y) = \text{i}^*(\mathcal{L}_\Delta) = \text{i}^*(\mathcal{F}_{\Delta,w}). \]

The nef cone of Y is thus polyhedral.

**Proof.** We have the inclusions \( \text{i}^*(\mathcal{L}_\Delta) \subseteq \text{Nef}(Y) \subseteq \text{i}^*(\mathcal{F}_{\Delta,w}) \) by Theorem 3.2, so \( \mathcal{L}_\Delta = \mathcal{F}_{\Delta,w} \) implies that both inclusions are equalities. Since pullback is a linear map, and \( \mathcal{L}_\Delta \) and \( \mathcal{F}_{\Delta,w} \) are both polyhedral, the conclusion follows. \( \square \)

**Problem 3.5.** Characterize which polyhedral fans \( \Delta \) have the property that \( \mathcal{L}_\Delta = \mathcal{F}_{\Delta,w} \) for some \( w \in \text{Hom}(A_{n-d}(X_\Delta), \mathbb{R}) \). When one such \( w \) exists, characterize the set of possible \( w \). Also give a characterization of those fans \( \Delta \) with the property that \( \mathcal{L}_\Delta = \mathcal{F}_{\Delta,w} \) for all \( w \in \text{Hom}(A_{n-d}(X_\Delta), \mathbb{R}) \).

**Remark 3.6.** While the hypothesis of Corollary 3.4 requires \( \mathcal{L}_\Delta = \mathcal{F}_{\Delta,w} \), it may be easier to check in examples the stronger condition that \( \mathcal{G}_\Delta = \mathcal{F}_{\Delta,w} \).

### 4. Examples

In this section we consider three families of examples of the cones \( \mathcal{L}_\Delta, \mathcal{G}_\Delta, \) and \( \mathcal{F}_{\Delta,w} \), highlighting where they have previously appeared in the literature in other guises.

In Section 4.1 we compute our cones for a family of del Pezzo surfaces, illustrating two phenomena: firstly, in this case, the closer the variety is to being toric, the closer the upper and lower bounds are to each other and secondly, the bounds on the nef cone given by our methods depend on the choice of toric embedding. In Section 4.2 we relate our construction to the Cox/Katz conjecture and examples of Buckley, Hassett/Lin/Wang, and Szendrői. One of their examples shows that one can have \( \mathcal{G}_\Delta(Y) \subset \text{Nef}(Y) \); see Example 4.3. Finally, in Section 4.3 we compute our cones for subvarieties of smooth Picard-rank two toric varieties. Example 4.5 shows that one does not always have \( A_{n-d}(X_\Delta) \cong \mathbb{Z} \) when \( \Delta \) is a \( d \)-dimensional fan in \( \mathbb{R}^n \), and that the cone \( \mathcal{F}_{\Delta,w} \) depends on the choice of \( w \).

The many calculations reported in this section were performed using a Macaulay 2 package available from Maclagan’s webpage [GLF], [M2].

#### 4.1. Del Pezzo Surfaces

Del Pezzo surfaces are the blow-up of \( \mathbb{P}^2 \) in at most eight general points. We consider general del Pezzo surfaces, which can be realized as tropical compactifications of complements of particular line arrangements in \( \mathbb{P}^2 \). We now review this construction and the resulting cones \( \mathcal{G}, \mathcal{L}, \) and \( \mathcal{F}_w \).

Fix \( 1 \leq r \leq 8 \) points \( p_1, \ldots, p_r \) in \( \mathbb{P}^2 \) such that no three points lie on a line, no six points lie on a conic, and no eight points lie on a cubic having a node at one of them. If \( 1 \leq r \leq 3 \), then \( Y = \text{Bl}_{p_1, \ldots, p_r}(\mathbb{P}^2) \) is a projective toric variety, and by Proposition 2.13, the three cones all coincide with \( \text{Nef}(Y) \). For \( r \geq 4 \), we let \( \mathcal{A} \) be the line arrangement consisting of all \( \binom{r}{2} \) lines through pairs of the points. We place the additional genericity condition (†) here on our choice of points that the only intersection points of three or more lines are the original \( p_i \). Such generic configurations exist for all \( r \). For \( r \leq 5 \) all configurations are generic in this sense, and for \( r > 5 \) a configuration with \( r \) points can be obtained from one with \( r - 1 \)
points by choosing the \( r \)th point not on any line joining one of the original \( r - 1 \) points to any intersection point.

The line \( \ell_{ij} \) joining \( p_i \) to \( p_j \) has the form \( \{ y = (y_0 : y_1 : y_2) \in \mathbb{P}^2 : a_{ij} \cdot y = (a_{ij})y_0 + (a_{ij})y_1 + (a_{ij})y_2 = 0 \} \) for some \( a_{ij} \in \mathbb{R} \). The complement \( Y = \mathbb{P}^2 \setminus A \) embeds as a subvariety of the torus \( T_2^{(r)} \) of \( \mathbb{P}^{(r)}_2 \) via the map \( \phi : Y \to T_2^{(r)} \) given by

\[
\phi(y) = (a_{12} \cdot y : \cdots : a_{r-1r} \cdot y) \in T_2^{(r)}.
\]

Let \( A \) be the \( 3 \times \binom{r}{2} \) matrix with columns indexed by the pairs \( \{i,j\} \) with \( 1 \leq j \leq r \) and \( ij \)th column \( a_{ij} \). Then \( \phi(Y) = \{ (z_{ij}) \in T_2^{(r)} : \sum b_{ij}z_{ij} = 0 \text{ for all } b = (b_{ij}) \in \ker(A) \} \).

We denote by \( e_{ij} \) the basis vector for \( \mathbb{R}^{(r)}_2 \) indexed by the pair \( \{i,j\} \). Let \( \Sigma_r \) be the two-dimensional fan in \( \mathbb{R}^{(r)}_2/\mathbb{R}(1, \ldots, 1) \) whose rays are generated by the images of the points \( e_{ij} \) for \( 1 \leq i < j \leq r \) and the images of the points \( f_{ij} = \sum _{j \neq i} e_{ij} \) for \( 1 \leq i \leq r \). The cones of \( \Sigma_r \) are the images of \( \text{pos}(e_{ij}, e_{kl}) + \mathbb{R}(1, \ldots, 1) \) for \( 1 \leq i \leq r \), \( j \neq i \), and \( \text{pos}(e_{ij}, e_{kl}) + \mathbb{R}(1, \ldots, 1) \) for \( \{i,j,k,l\} \subset \{1, \ldots, r\} \) with \( |\{i,j,k,l\}| = 4 \).

**Proposition 4.1.** The tropical variety \( \text{trop}(\phi(Y)) \subseteq \mathbb{R}^{(r)}_2/\mathbb{R}(1, \ldots, 1) \) is the support of \( \Sigma_r \). The closure \( \overline{Y} \subset X_{\Sigma_r} \) equals the del Pezzo surface \( \text{Bl}_{p_1, \ldots, p_r}(\mathbb{P}^2) \), and the induced map \( i^* : \text{Pic}(X_{\Sigma_r})_\mathbb{R} \to N^1(\overline{Y})_\mathbb{R} \) is an isomorphism. In addition \( A_{(r)-3}(X_{\Sigma_r}) \cong \mathbb{Z} \).

**Proof.** The description of \( \text{trop}(\phi(Y)) \) is immediate from the construction of the tropical variety of a linear space due to [AK06]; see also [MS] Chapter 4]. Note that this is the coarse fan structure on this tropical variety. The fact that the closure of \( \phi(Y) \) in \( X_{\Sigma_r} \) is \( \text{Bl}_{p_1, \ldots, p_r}(\mathbb{P}^2) \) follows from [Tev07] Example 4.1], as our genericity assumptions on the \( p_i \) ensure that we are not in the exceptional case of that example. To see that the induced map from \( \text{Pic}(X_{\Sigma_r})_\mathbb{R} \) is an isomorphism, let \( D_{fi} \) be the divisor on \( X_{\Sigma_r} \) corresponding to the ray through the image of \( f_i \), and let \( D_{ij} \) be the divisor corresponding to the ray through the image of \( e_{ij} \). We first note that \( i^*(D_{fi}) = E_i \), the exceptional divisor obtained by blowing up the point \( p_i \), and \( i^*(D_{ij}) = L_{ij} \), the strict transform of the line joining the points \( p_i \) and \( p_j \). Since the \( E_i \) and \( L_{ij} \) span \( N^1(\overline{Y}) \), \( i^* \) is surjective. Recall that \( \{E_i : 1 \leq i \leq r \} \cup \{\ell \} \) form a basis for \( N^1(\overline{Y}) \), where \( \ell \) is the pullback to \( \overline{Y} \) of a line in \( \mathbb{P}^2 \). Thus to show the isomorphism it suffices to show that \( \text{Pic}(X_{\Sigma_r}) \cong \mathbb{Z}^{r+1} \). This follow from the short exact sequence defining the class group of \( X_{\Sigma_r} \) [Ful93] Chapter 4], since \( |\Sigma_r(1)| - \binom{r}{2} = r + 1 \).

The claim that \( A_{(r)-3}(X_{\Sigma_r}) \cong \mathbb{Z} \) follows from the description of the Chow groups of a toric variety given in [FS97] Chapter 4].

We now list the relationships between \( \text{Nef}(\overline{Y}) \) and the cones associated to \( \Sigma_r \).

(1) When \( r = 4 \) we have \( i^*(\mathcal{G}_{\Sigma_4}) = i^*(\mathcal{L}_{\Sigma_4}) = \text{Nef}(\overline{Y}) = i^*(\mathcal{F}_{\Sigma_4}) = i^*(\mathcal{F}_{\Sigma_4,w}) \). In this case \( \overline{Y} \cong \text{Bl}_4(\mathbb{P}^2) \cong \mathcal{M}_{0,5} \).

(2) When \( r = 5 \) we have \( i^*(\mathcal{G}_{\Sigma_5}) = i^*(\mathcal{L}_{\Sigma_5}) = \text{Nef}(\overline{Y}) \subseteq i^*(\mathcal{F}_{\Sigma_5}) = i^*(\mathcal{F}_{\Sigma_5,w}) \). The cone \( i^*(\mathcal{L}_{\Sigma_5}) = \text{Nef}(\overline{Y}) \) equals \( i^*(\mathcal{F}_{\Sigma_5,w}) \cap \{[D] : [D] \cdot [C_5] \geq 0 \} \), where \( [C_5] = 2\ell - \sum _{i=1}^5 E_i \) is the conic through the five points \( p_1, \ldots, p_5 \).
Lemma 4.2. The cone \( i^*(G_{\Sigma_6}) = i^*(L_{\Sigma_6}) \subseteq \text{Nef}(Y) \subseteq i^*(F_{\Sigma_6}) = i^*(F_{\Sigma_6,w}) \). As in the previous case \( \text{Nef}(Y) = i^*(F_{\Sigma_6,w}) \cap \{[D] : [C] \cdot [D] \geq 0, 1 \leq i \leq 6\} \), where \( C_i \) is the conic through the five points obtained by omitting \( p_i \). The cone \( i^*(L_{\Sigma_6}) \) is the intersection of \( \text{Nef}(Y) \) (or indeed \( i^*(F_{\Sigma_6,w}) \)) with \( \{[D] : [C] \cdot [D] \geq 0\} \), where \( [C] = 2\ell - \sum_{i=1} E_i \). This class \([C]\) is not effective unless all six points lie on a conic, which will not be the case if the points are sufficiently general, so does not give a facet of \( \text{Nef}(Y) \).

(4) For \( r = 7,8 \) we also have \( i^*(G_{\Sigma_r}) = i^*(L_{\Sigma_r}) \subseteq \text{Nef}(Y) \subseteq i^*(F_{\Sigma_r}) = i^*(F_{\Sigma_r,w}) \).

4.2. Ample toric hypersurfaces. In this section we consider the case that \( Y \) is a general ample hypersurface in a simplicial projective toric variety \( X_\Sigma \), embedded by a morphism \( i \). Of particular interest is the case that \( X_\Sigma \) is Fano, and \([Y] = -K_{X_\Sigma}\), so \( Y \) is Calabi-Yau. This situation was considered in [CK99 §6.2.3], where they conjectured a description for \( \text{Nef}(Y) \cap i^*(\text{Pic}(X_\Sigma)) \). We will show that this conjectured description is \( i^*(G_\Delta) \), where \( \Delta \) is the subset of the codimension-one skeleton of \( \Sigma \) containing those cones corresponding to torus orbits intersecting \( Y \). We first recall this, and discuss the counterexamples given by Szendrői and others in our context.

A generalized flop of \( X_\Sigma \) is a simplicial projective toric variety with the same rays as \( \Sigma \) whose fan is obtained by a bistellar flip over a circuit \( \Xi = (\Xi^+,\Xi^-) \) of \( \Sigma \) (see [GKZ08 §7.2.C]), where bistellar flips are called modifications. Loosely, this replaces cones containing \( \text{pos}(v_i) : i \in \Xi^+ \) with those containing \( \text{pos}(v_i) : i \in \Xi^- \). A generalized flop is a trivial flip if \(|\Xi^+|,|\Xi^-| \geq 2\), and \( \cap_{i \in \Xi^-} D_i \cap Y = \emptyset \). The first of these conditions guarantees that \( \Sigma'(1) = \Sigma(1) \), so \( \text{Pic}(X_{\Sigma'}) \cong \text{Pic}(X_{\Sigma''}) \). We denote by \( j^* \) the induced homomorphism \( \text{Pic}(X_{\Sigma'}) \to \text{N}^1(Y)_\mathbb{R} \).

In [CK99 Conjecture 6.2.8] it was conjectured that \( \text{Nef}(Y) \cap i^*(\text{Pic}(X_\Sigma)) \) was equal to the union of \( j^*(\text{Nef}(X_{\Sigma''})) \) over all fans \( \Sigma'' \) that can be obtained from the fan \( \Sigma \) by a sequence of trivial flips.

**Lemma 4.2.** The cone \( \bigcup j^*(\text{Nef}(X_{\Sigma''})) \) equals \( i^*(G_\Delta) \), where \( \Delta \) is the subset of the codimension-one skeleton of \( \Sigma \) containing those cones corresponding to torus orbit closures intersecting \( Y \).

**Proof.** A general ample hypersurface \( Y \) in \( X_\Sigma \) will not pass through the torus-fixed points of \( X_\Sigma \), so by [Lev07 Lemma 2.2] the tropical variety \( \text{trop}(Y \cap T) \) is contained in the codimension-one skeleton of \( \Sigma \), and intersects those cones corresponding to torus-orbits closures intersecting \( Y \). Since the tropical variety has dimension \( n-1 \) and is balanced, it must actually be the union \( \Delta \) of all such cones.

To show that \( \bigcup i^*(\text{Nef}(X_{\Sigma''})) = i^*(G_\Delta) \), by Part 4 of Proposition 2.3 it suffices to show that a projective \( \Sigma' \) can be obtained from \( \Sigma \) by a sequence of trivial flips if and only if \( \Sigma' \) is the fan of a projective toric variety with \( \Sigma'(1) = \Delta(1) \) and \( \Delta \subseteq \Sigma' \). For the “only if” direction, suppose that \( \Sigma' \) is obtained by a trivial flip over a circuit \( \Xi = (\Xi^+,\Xi^-) \) from a fan \( \Sigma'' \) containing \( \Delta \). From the definition of trivial flip we have \( \Sigma'(1) = \Sigma(1) \). The condition that \( \cap_{i \in \Xi^-} D_i \cap Y = \emptyset \) implies \( \text{pos}(\Xi^-) \not\subseteq \Delta \), where \( \text{pos}(\Xi^-) \) is the cone generated by rays indexed by \( \Xi^- \). Indeed, \( \cap_{i \in \Xi^-} D_i \) equals \( V(\Xi^-) \), which is the orbit closure corresponding to \( \text{pos}(\Xi^-) \). This orbit closure intersects \( Y \) if and only if \( \Delta \) contains a cone with \( \text{pos}(\Xi^-) \) as a face, and thus if and only if
pos(Ξ) ∈ ∆. Since every cone removed from Σ" to obtain Σ' contains pos(Ξ) as a face, we conclude that no cone in ∆ is removed, so ∆ ⊆ Σ'.

For the "if" direction note that all fans Σ' with Σ'(1) = ∆(1) and ∆ ⊆ Σ' are connected by a sequence of bistellar flips through fans with the same property. This follows from the chamber description of the secondary fan, since the set of all chambers corresponding to Σ' containing ∆ form a convex cone; see [DLRS10 Section 5.3]. By the above argument, since pos(Ξ) /∈ ∆ for these bistellar flips, they are trivial flips.

In [Sze02] Szendrői gave a counterexample to [CK99 Conjecture 6.2.8], which shows that \( i^*(\mathcal{G}_\Delta) \) can be a strict lower bound for \( \text{Nef}(Y) \) in this context. This was followed by other related examples by Buckley in [AB01]. Both Hassett, Lin, and Wang [HLW02], and Szendrői [Sze03] also consider the simpler, non-Fano, example where \( X_\Sigma \) is the blow-up of \( \mathbb{P}^4 \) at two points, where the equality \( \text{Nef}(Y) \cap i^*(\text{Pic}(X_\Sigma)) = i^*(\mathcal{G}_\Delta) \) also fails.

**Example 4.3.** Let \( X_\Sigma \) be the blow-up of \( \mathbb{P}^4 \) at two torus-invariant points \( p_1 \) and \( p_2 \), whose fan \( \Sigma \) has rays spanned by the columns of the matrix \( V \) below. The corresponding torus-invariant divisors are \( \{D_0, D_1, D_2, D_3, D_4, E_1, E_2\} \), where \( E_i \) for \( i = 1, 2 \) is the exceptional divisor of the blow-up at \( p_i \). The Picard group of \( X_\Sigma \) is three-dimensional, with the classes of the torus-invariant divisors given by the columns of the matrix \( G \):

\[
V = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & -1 & 0
\end{pmatrix},
\quad
G = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}.
\]

The effective cone of \( X_\Sigma \) is the positive orthant in these coordinates, and is generated by \( D_0 = D_1 = D_2, E_1, \) and \( E_2 \). The nef cone is generated by \( D_3, D_4, \) and \( F = (1, 1, 1) = D_3 + E_1 = D_4 + E_2 \). This is the triangle labelled \( B \) in [Sze03 p3].

If \( \Delta \) is the whole 3-skeleton of \( \Sigma \), then \( \mathcal{G}_\Delta = \text{Nef}(X_\Sigma) = \text{pos}(D_3, D_4, F) \), and \( \mathcal{L}_\Delta = \text{pos}(D_3, D_4, F, D_0) \), which is the union of the triangles labelled \( B \) and \( C \) in [Sze03 p3]. In this case \( \mathcal{G}_\Delta^\vee = \mathcal{L}_\Delta \).

The group \( A_1(X_\Sigma) \cong A_1(\mathbb{P}^4) \cong \mathbb{Z}^3 \), as it is dual to the class group, which is three-dimensional. A basis is given by \( \{V(012), V(01p_1), V(01p_2)\} \), where by \( V(01p_1) \) we mean the orbit-closure corresponding to the three-dimensional cone spanned by the rays corresponding to \( D_0, D_1, \) and \( E_1 \). The set \( W = \{w \in \text{Hom}(A_1(\mathbb{P}^4), \mathbb{R}) : w(V(\sigma)) \geq 0 \text{ for all } \sigma \in \Delta(3)\} \) of Definition 2.11 is isomorphic to \( \mathbb{R}_{\geq 0}^3 \) under the map that sends \( w \) to \( (w(V(012)), w(V(01p_1)), w(V(01p_2))) \). Explicitly, for \( (\alpha, \beta, \gamma) \in \mathbb{R}_{\geq 0}^3 \) we have:

| \( w(V(\sigma)) \) | \( V(\sigma) \) |
|---------------------|----------------|
| \( \alpha \) | \( V(01p_1), V(02p_1), V(03p_1), V(12p_1), V(13p_1), V(23p_1) \) |
| \( \beta \) | \( V(01p_2), V(02p_2), V(04p_2), V(12p_2), V(14p_2), V(24p_2) \) |
| \( \gamma \) | \( V(012) \) |
| \( \alpha + \gamma \) | \( V(024), V(014), V(124) \) |
| \( \beta + \gamma \) | \( V(023), V(013), V(123) \) |
| \( \alpha + \beta + \gamma \) | \( V(034), V(134), V(234) \) |
The cone $F_{\Delta, w}$ is

$$F_{\Delta, w} = \text{pos}((1, 1, 1), (\beta, \beta, -\alpha - \gamma), (\alpha, -\beta - \gamma, \alpha)) = \text{pos}(F, \beta D_4 - (\alpha + \gamma)E_2, \alpha D_3 - (\beta + \gamma)E_1).$$

The intersection is

$$F_{\Delta} = \bigcap_{w \in W} F_{\Delta, w} = \text{pos}((1, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)) = B \cup C.$$

Note, however that for any one fixed $w$ we have $F_{\Delta, w} \not\subseteq \text{Eff}(X_\Sigma)$. This example is illustrated in cross-section in Figure 3.

4.3. Subvarieties of Picard rank two smooth toric varieties. In this section we consider subvarieties of smooth toric varieties with rank-two Picard groups. A projective such $X_\Sigma$ is determined by integers $s \geq 2$ and a sequence $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ as follows. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$, for $n = r + s - 1$. We may choose coordinates so that $\Sigma$ has rays spanned by $v_0 = -\sum_{i=1}^{s-1} e_i + \sum_{j=1}^{r} a_j e_{j+s-1}, \ v_i = e_i$ for $1 \leq i \leq s - 1, \ u_0 = -\sum_{j=1}^{r} e_{j+s-1},$ and $u_j = e_{j+s-1}$ for $1 \leq j \leq r$. The
then have the following relation between the cones $G_i$.

**Proposition 4.4.** The cone $G_{\Delta_k}$ equals $\text{pos}(1,0),(-a_i,1)$, where $i=0$ if $k \geq r$, and $r-k$ if $k < r$. The cone $L_{\Delta_k}$ equals $G_{\Delta_{k-1}}$.

**Proof.** The cones of $\Delta_k$ are indexed by a pair $\{I,J\}$ with $I \subset \{0,\ldots, s-1\}$ and $J \subset \{0,\ldots, r\}$ both nonempty and $|I|+|J| \geq r+s+1-k$. The corresponding cone is $\sigma_{IJ} = \text{pos}(v_i, u_j : i \notin I, j \notin J)$. Thus $G_{\Delta_k} = \bigcap_{\{I,J\}} \text{pos}(D_i, E_j : i \in I, j \in J)$. The cone $\text{pos}(D_i, E_j : i \in I, j \in J)$ equals $\text{pos}(D_0, E_k)$, where $k = \max\{j : j \in J\}$. Thus $G_{\Delta_k} = \text{pos}(D_0, E_l)$, where $l = \min_{\{I,J\}} \max\{j : j \in J\}$ and the minimum is taken over all pairs $\{I,J\}$ with $I,J \neq \emptyset$ and $|I|+|J| \geq r+s+1-k$. When $k \geq r$ the minimum is achieved at $I = \{0,\ldots, r+s-k-1\}$ and $J = \{0\}$, while for $k < r$ the minimum is achieved at $I = \{0,\ldots, s-1\}$ and $J = \{0,\ldots, r-k\}$, which implies the result.

The star of $\sigma_{IJ}$ contains all rays of $\Delta_k$ unless $|I| = 1$ or $|J| = 1$, in which case the ray labelled by the singleton is not contained in the star. Thus $L_{\Delta_k}$ equals the intersection of $\bigcap_{\{I,J\}} \text{pos}(D_i, E_j : i \in I, j \in J)$ with $\bigcap_{k \in \{0,\ldots, s-1\}} \bigcap_{J \subset \{0,\ldots, r\}} \text{pos}(D_i, E_j : j \in J)$ and $\bigcap_{I \subset \{0,\ldots, s-1\}} \bigcap_{J \subset \{0,\ldots, r\}} \text{pos}(D_i, \pm E_j : i \in I)$, where the first intersection is over pairs $\{I,J\}$ with $|I|,|J| \geq 2$ and $|I|+|J| \geq r+s-k$, and in the second and third intersections we have $|J| \geq r+s-1-k$ and $|I| \geq r+s-1-k$. Since $\text{pos}(\pm D_i, E_j : j \in J)$ is the upper half plane for all $j$, the second intersection is the upper half plane. For the third intersection, $\text{pos}(D_i, \pm E_j : i \in I)$ is the halfspace $\{(x,y) : x+a_jy \geq 0\}$, so the intersection over all $j$ is the cone $\text{pos}(D_0, -E_r)$. Note that this case only occurs when $s+1 \geq r+s-k$, so $k \geq r+1$. As the description of the first intersection is as in the previous paragraph, we conclude that $L_{\Delta_k} = \text{pos}(D_0, E_i)$, where $i = 0$ for $k \geq r+1$, and $i = r-k$ if $k \leq r$, so $L_{\Delta_k} = G_{\Delta_{k-1}}$. \qed

**Example 4.5.** We compute the cone $F_{\Delta_k,w}$ for $r = 1$ and $s = n$. The group $A_{n-k}(X_{\Delta_k}) \cong \mathbb{Z}^2$. One way to see this is to note that $A_{n-k}(X_{\Delta_k}) \cong A_{n-k}(X_\Sigma)$, since the generators and the relations for this group only depend on $\Delta_k$, and by [Ful93, p106] this is equal to the degree-$k$ part of the ring $\mathbb{Z}[D_0, E_0]/\langle D_0^k, E_0(D_0^k-\ldots)\rangle$. Figure 4.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Figure 4.}
\end{figure}
a_1D_0), which is rank two. We choose the basis D_0^k, DaD_0^{-1}E_0 for this group, and choose w = (w_1, w_2) \in \text{Hom}(A_{n-k}(\Delta k), \mathbb{R}) \cong \mathbb{R}^2. The classes of V(\sigma) for \sigma \in \Delta k(k) are \{D_0^k, DaD_0^{-1}E_0, DaD_0^{-1}(E_0 - a_1D_0)\} in this basis, so we must choose w_1 \geq 0, w_2 \geq a_1w_1 \geq 0. Now

\[ F_{\Delta k, w} = \{ D = aD_0 + bE_0 : w(D \cdot V(\tau)) \geq 0 \text{ for all } \tau \in \Delta(k-1) \} = \{ aD_0 + bE_0 : \text{w}(aD_0^k + bDaD_0^{-1}E_0) \geq 0, \text{w}((a + a_1b)DaD_0^{-1}E_0) \geq 0, \}
\[ = \{ aD_0 + bE_0 : aw_1 + bw_2 \geq 0, (a + a_1b)w_2 \geq 0, a(w_2 - a_1w_1) \geq 0 \}
\[ = \{ aD_0 + bE_0 : aw_1 + bw_2 \geq 0, a + a_1b \geq 0, a \geq 0 \},
\]

where the last equality comes from the fact that w_2, w_2 - a_1w_1 \geq 0. The middle inequality is redundant, so this is the cone spanned by E_0 and w_2D_0 - w_1E_0. If k \geq 2 we have G(\Delta k) = L(\Delta k) = \text{pos}(E_0, D_0) by Proposition 4.4, so the inequality L(\Delta k) \subseteq F_{\Delta k, w} is strict unless w_1 = 0.

**Remark 4.6.** Example 4.5 shows that one does not always have A_{n-d}(X_\Delta) \cong \mathbb{Z} for \Delta a d-dimensional fan in \mathbb{R}^n, and that the cone F_{\Delta, w} depends on the choice of w. The intersection F_{\Delta k} of all F_{\Delta k, w} as w varies is pos(D_0, E_0) = G_{\Delta k} = L_{\Delta k}.

5. Mori dream spaces

In this section we show that if Y is a Mori dream space then there is an embedding i: Y \rightarrow X_\Delta for which i^*(G_\Delta) = \text{Nef}(Y). Recall from [HK00] that a projective \mathbb{Q}-factorial variety Y with Pic(Y) \cong \mathbb{Z}^r is a Mori dream space if the Cox ring

\[ \text{Cox}(Y) = \bigoplus_{w \in \mathbb{Z}^r} H^0(Y, L_1^{\otimes w_1} \otimes \cdots \otimes L_r^{\otimes w_r}) \]

is finitely generated, where L_1, \ldots, L_r form a basis for Pic(Y). Important examples include log Fano varieties (see [BCHM10]).

The ring Cox(Y) has a \mathbb{Z}^r-grading given by the Picard group of Y. Choose a graded presentation for Cox(Y):

\[ \text{Cox}(Y) \cong \mathbb{k}[z_1, \ldots, z_N]/I, \]

where I is homogeneous with respect to the Pic(Y) grading. We denote by V(I) the affine subscheme of \mathbb{A}^N defined by the ideal I. The action of the torus T = Hom(Pic(Y), \mathbb{k}^*) \cong (\mathbb{k}^*)^r on \mathbb{A}^N descends to an action on V(I). Linearizations of this action correspond to characters of T, and thus to line bundles on Y. If L is an ample line bundle, then

\[ Y = \text{Proj}(\bigoplus_{k \geq 0} \text{Cox}(Y)_{kL}) = V(I) \parallel_L T. \]

This gives an embedding i: Y = V(I) \parallel_L T \rightarrow \mathbb{A}^N \parallel_L T. This latter space is a normal toric variety which we denote by X_\Sigma. Let \Delta be the subfan of \Sigma containing those cones for which the corresponding T-orbit closure intersects Y. The embedding i restricts to an embedding i: Y \rightarrow X_\Delta.
Proposition 5.1. Let $Y$ be a Mori Dream Space, and let $i: Y \to X_\Delta$ be the toric embedding described above. Then

$$\text{Nef}(Y) = i^*(G_\Delta) = i^*(\mathcal{L}_\Delta).$$

Proof. It suffices to show that $\text{Nef}(Y) = i^*(G_\Delta)$, as the equality with $i^*(\mathcal{L}_\Delta)$ then follows from Theorem 3.2 in [HK00]. Proposition 2.9 of [Hu and Keel] shows that the description of Equation 2 satisfies the condition of [HK00, Theorem 2.3], and so the Mori chambers of $Y$ are equal to the GIT chambers. Since $\text{Nef}(Y)$ is a Mori chamber, the proof reduces to showing that $i^*(G_\Delta)$ is a GIT chamber of the GIT description for $Y$ given in Equation 2.

For $L \in \text{Pic}(Y)$ we denote by $V(I)_L^{ss}$ the semistable locus for $V(I)$ with respect to the linearization of $T$ labelled by $L$. The GIT chamber of $Y$ corresponding to $L$ is $\text{pos}(V(I)_L^{ss}) \subseteq \text{Pic}(Y)$. Now $V(I)_L^{ss}$ is the intersection of $V(I)$ with $(\mathbb{A}^N)^{ss}_L$. The GIT chamber of $\mathbb{A}^N \parallel L$ with respect to the linearization by $L$ equals the nef cone of the toric variety $\mathbb{A}^N \parallel L$, whose fan we denote by $\Sigma_L$. Thus the GIT chamber of $V(I) \parallel L$ containing a fixed ample $L$ is the union of the nef cones of those toric varieties $X_{\Sigma_{L'}} = \mathbb{A}^N \parallel L / L'$ for which $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) = (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$. To show that this equals $i^*(G_\Delta)$ it suffices by Part 4 of Proposition 2.3 to show that this condition is equivalent to $\Delta \subseteq \Sigma_{L'}$.

Let $v_i$ be the first lattice point on the $i$th ray of $\Sigma_L$. If $\sigma \in \Delta$ is a maximal cone, then $Y \cap O(\sigma) \neq \emptyset$, so then there is $x \in (\mathbb{A}^N)^{ss} \cap V(I)$ with $\sigma = \text{pos}(v_i : x_i = 0)$. Now note that for any $L'$ with $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) = (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$ this $x$ lies in $(\mathbb{A}^N)^{ss}_{L'}$, so $\text{pos}(v_i : x_i = 0) \in \Sigma_{L'}$. Thus $\Delta \subseteq \Sigma_{L'}$.

Conversely, suppose that $\Delta \subseteq \Sigma_{L'}$ for some $L'$. If $x \in (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$, then $\text{pos}(v_i : x_i = 0) \in \Delta$, so $\text{pos}(v_i : x_i = 0) \in \Sigma_{L'}$. This means that $x \in (\mathbb{A}^N)^{ss}_{L'}$, and so $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) \subseteq (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$.

Now $Y = V(I) \parallel L, T$ is the closure of $Y \cap T$ in $X_{\Sigma_L}$, and similarly $Y' = V(I) \parallel L', T$ is the closure of $Y \cap T$ in $X_{\Sigma_{L'}}$. Since the closure of $Y \cap T$ in $X_{\Sigma_{L'}}$ is contained in $X_\Delta$, the same must be true for the closure of $Y \cap T$ in $X_{\Sigma_{L'}}$. If, however, this inclusion $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) \subseteq (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$ were proper, then there would be a point $x \in (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$ with $\sigma = \text{pos}(v_i : x_i = 0)$ satisfying $\sigma \in \Sigma_{L'}$, $\sigma \notin \Delta$, and $(V(I) \parallel L', T) \cap O(\sigma) \neq \emptyset$, where the intersection takes place in $X_{\Sigma_{L'}}$. From this contradiction we conclude that $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) = (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$. Thus $(\mathbb{A}^N)^{ss}_{L'} \cap V(I) \subseteq (\mathbb{A}^N)^{ss}_{L'} \cap V(I)$ if and only if $\Delta = \Sigma_{L'}$ as required.

Remark 5.2. We note that this combinatorial description of the nef cone of a Mori dream space was already known, using other language, by Berchtold and Hausen; see [BH07].

Example 5.3. Let $Y = \text{Bl}_6(\mathbb{P}^2)$ be the blow-up of $\mathbb{P}^2$ at six general points $p_1, \ldots, p_6$. This is a del Pezzo surface of degree 3. The Cox ring of $Y$ has a generator of degree $E_i$ for each of the six exceptional divisors $E_i$, one of degree $E - E_i - E_j$ (the strict transform of the line joining $p_i$ and $p_j$) for each of the fifteen such lines, and a generator of degree $2E - \sum_{k \neq i} E_k$ for each of the six conics through five of the points. The ideal $I$ of relations is generated in degree 2 (see for example [STV07]). We thus get an embedding $i: Y \to X_\Delta \subset X_\Sigma = \mathbb{A}^{27} \parallel (k^*)^7$ of the surface $Y$ into a 20-dimensional toric variety $X_\Delta$ with 27 rays and $\text{Nef}(Y) = i^*(G_\Delta) = i^*(\mathcal{L}_\Delta)$. 

This contrasts with the embedding of Section 4.1 where \( Y \) is embedded into a 14-dimensional toric variety with 20 rays and \( i^*(G_\Delta) = i^*(L_\Delta) \subseteq \text{Nef}(Y) \). The missing generators in this case correspond to the cones through sets of five points. This example illustrates that the bound on \( \text{Nef}(Y) \) obtained depends on the choice of toric embedding.

6. BOUNDS FOR THE NEF CONE OF \( \overline{M}_{0,n} \)

In this section we apply the main theorem to obtain bounds for the nef cone of the moduli space \( \overline{M}_{0,n} \) of stable \( n \)-pointed curves of genus zero. Kapranov’s construction of \( \overline{M}_{0,n} \) as a Chow or Hilbert quotient of the Grassmannian \( G(2, n) \) by an algebraic torus \([\text{Kap93a}], [\text{Kap93b}]\) gives rise to a natural embedding of the moduli space into a toric variety \([\text{Tev07}], [\text{GM10}]\). In Proposition 6.2 we give simple and explicit descriptions of the three corresponding cones of divisors \( G_\Delta(\overline{M}_{0,n}) \), \( L_\Delta(\overline{M}_{0,n}) \), and \( F_\Delta(\overline{M}_{0,n}) \) that give lower and upper bounds for \( \text{Nef}(\overline{M}_{0,n}) \). We also show that \( F_\Delta(\overline{M}_{0,n}) \) is the cone of F-divisors, which the F-Conjecture asserts is equal to \( \text{Nef}(\overline{M}_{0,n}) \). Finally, we propose that the cone \( L_\Delta(\overline{M}_{0,n}) \) is an equally likely, and useful, polyhedral description of \( \text{Nef}(\overline{M}_{0,n}) \).

We first recall the F-Conjecture. See, for example, \([\text{KV07}]\) for further background on \( \overline{M}_{0,n} \). An F-curve on \( \overline{M}_{0,n} \) is any curve that is numerically equivalent to a component of the locus of points in \( \overline{M}_{0,n} \) corresponding to curves having at least \( n - 4 \) nodes. An F-divisor on \( \overline{M}_{0,n} \) is any divisor that nonnegatively intersects every F-curve. The F-Conjecture on \( \overline{M}_{0,n} \) says that a divisor is nef if and only if it is an F-divisor. The F-conjecture can be stated for \( \overline{M}_{g,n} \) for all \( g \), and the case \( g > 0 \) was shown in \([\text{GKM02}]\) to be implied by the F-conjecture for \( S_g \)-symmetric divisors on \( \overline{M}_{0,g+n} \).

In order to state Proposition 6.2 we use the following simplicial complex \( \tilde{\Delta} \).

**Definition 6.1.** Let \( \mathcal{I} = \{ I \subseteq \{1, \ldots, n\} : 1 \in I \text{ and } |I|, |J'| \geq 2 \} \). Let \( \tilde{\Delta} \) be the simplicial complex on the vertex set \( \mathcal{I} \) for which \( \sigma \in \tilde{\Delta} \) if for all \( I, J \in \sigma \) we have \( I \subseteq J, J \subseteq I, \) or \( I \cup J = \{1, \ldots, n\} \).

The maximal cones of \( \Delta \) have dimension \( n - 3 \), and the simplices of \( \tilde{\Delta} \) are in bijection with boundary strata of \( \overline{M}_{0,n} \). See, for example, \([\text{AC08}], [\text{KV07}]\) for a description of the boundary divisors \( \delta_I \) on \( \overline{M}_{0,n} \).

**Proposition 6.2.** There are three cones \( G_\Delta(\overline{M}_{0,n}), L_\Delta(\overline{M}_{0,n}), \text{ and } F_\Delta(\overline{M}_{0,n}) \) in \( N^1(\overline{M}_{0,n})_\mathbb{R} \) that bound \( \text{Nef}(\overline{M}_{0,n}) \):

\[
G_\Delta(\overline{M}_{0,n}) \subseteq L_\Delta(\overline{M}_{0,n}) \subseteq \text{Nef}(\overline{M}_{0,n}) \subseteq F_\Delta(\overline{M}_{0,n}).
\]

These are described as follows.

1. \( G_\Delta(\overline{M}_{0,n}) = \bigcap_{\sigma \in \tilde{\Delta}} \text{pos}(\delta_I : I \in \mathcal{I} \setminus \sigma) \).
2. \( L_\Delta(\overline{M}_{0,n}) = \bigcap_{\sigma \in \tilde{\Delta}} \text{pos}(\delta_I, \pm \delta_J : I, J \in \mathcal{I} \setminus \sigma, \delta_I \cap \delta_J \neq \emptyset, \forall K \in \sigma, \text{ and } \delta_I \cap \delta_L = \emptyset \text{ for some } L \in \sigma) \).
3. \( F_\Delta(\overline{M}_{0,n}) = \bigcap_{\sigma \in \tilde{\Delta} : |\sigma| = n-4} \text{pos}(\delta_I, \pm \delta_J : I, J \in \mathcal{I} \setminus \sigma, \delta_I \cap \delta_K \neq \emptyset, \forall K \in \sigma, \text{ and } \delta_I \cap \delta_L = \emptyset \text{ for some } L \in \sigma) \).
Moreover, \( F_\Delta(M_{0,n}) \) is equal to the cone of \( F \)-divisors on \( M_{0,n} \).

The key to proving Proposition 6.2 is to recognize \( \Delta \) as the simplicial complex corresponding to a fan \( \Delta \subset \mathbb{R}^{(2)}_{-n} \). This fan, known as the space of phylogenetic trees, arises in the consideration of \( M_{0,n} \) as a Chow or Hilbert quotient, and there is an embedding of \( M_{0,n} \) into the associated toric variety \( X_\Delta \). We summarize the necessary information in the following proposition; see [Tev07] or [GM10, §5] for more information.

**Proposition 6.3.** There is a collection \( \{ r_I : I \in \mathcal{I} \} \) of lattice points in \( \mathbb{R}^{(2)}_{-n} \) for which the collection of cones \( \{ \text{pos}(r_I : I \in \sigma) : \sigma \in \Delta \} \) is an \((n-3)\)-dimensional polyhedral fan \( \Delta \). The associated toric variety \( X_\Delta \) is smooth. In addition, there is an embedding of \( M_{0,n} \) into \( X_\Delta \) with \( M_{0,n} \cap T = M_{0,n} \), where \( T \) is the torus of \( X_\Delta \), and the support of \( \Delta \) is the tropical variety of \( M_{0,n} \subset T \).

An important ingredient in the proof of Proposition 6.2 is the following isomorphism of Chow rings. Note that while \( X_\Delta \) is not complete, it is smooth, so there is a ring structure on \( A^*(X_\Delta) = \bigoplus k A^k(X_\Delta) \), where \( A^k(X_\Delta) = A_{d-k}(X_\Delta) \) for \( d = \binom{n}{2} - n \).

**Proposition 6.4.** Let \( i : M_{0,n} \to X_\Delta \) be the embedding of \( M_{0,n} \) into the toric variety \( X_\Delta \) given in Proposition 6.3. The pullback
\[
i^* : A^*(X_\Delta) \to A^*(M_{0,n})
\]
is an isomorphism, and in particular, \( A_{d-(n-3)}(X_\Delta) \cong \mathbb{Z} \), where \( d = \binom{n}{2} - n \).

The second assertion of Proposition 6.4 follows from the first, but it can also be proved with an explicit toric computation. The only difficulty of this strategy is the combinatorial bookkeeping. Instead, we opt for the following conceptual proof that relies on the realization of \( M_{0,n} \) as a De Concini/Procesi wonderful compactification of a particular hyperplane arrangement complement.

**Proof of Proposition 6.4.** In [FY04, pp533-555] Feichtner and Yuzvinsky construct a smooth toric variety \( X_{\Sigma(L,G)} \) and a ring isomorphism \( \phi : A^*(X_{\Sigma(L,G)}) \to A^*(M_{0,n}) \). This construction is a special case of the identification of the Chow ring of any wonderful compactification of a hyperplane arrangement complement with the Chow ring of a toric variety.

The fan \( \Sigma(L,G) \) lies in \( \mathbb{R}^{(2)}_{-n+1} \), and has a ray for each ray of \( \Delta \), plus one additional ray spanned by \((1, \ldots, 1)\). If \( D^I \) is the torus-invariant divisor corresponding to the ray of \( \Sigma(L,G) \) indexed by \( I \), then \( \phi(D^I) = \delta_I \). A collection of rays span a cone in \( \Delta \) if and only if the corresponding collection of rays, plus the ray through \((1, \ldots, 1)\), span a cone in \( \Sigma(L,G) \), and the identification of \( \mathbb{R}^{(2)}_{-n} \) with \( \mathbb{R}^{(2)}_{-n+1} / \mathbb{R}(1, \ldots, 1) \) induces a map of fans \( \pi : \Sigma(L,G) \to \Delta \).

In [Oda93] and [Par93] it is shown that the Stanley-Reisner presentation of [Ful93, p106] describes the Stanley-Reisner ring of any smooth toric variety, even if it is not complete. The fact that \( \Delta \) is the projection under \( \pi \) of \( \Sigma(L,G) \) implies that the Stanley-Reisner ring of \( \Sigma(L,G) \) has one more generator than that of \( X_\Delta \), corresponding to the ray through \((1, \ldots, 1)\). It has the same monomial generators, and one more linear relation, which involves the generator corresponding to the ray through \((1, \ldots, 1)\).
The induced map $\pi^* : A^*(X_\Delta) \to A^*(X_{(L,G)})$ is thus an isomorphism, so $i^* = \phi \circ \pi^*$ is the desired isomorphism.

\[ \square \]

We are now able to prove Proposition 6.2.

\textbf{Proof of Proposition 6.2.} We use the fact that $A^*(X_\Delta)$ is the cone of $F$-divisors. Since $A_d^{(n-3)}(X_\Delta)$ is one-dimensional, we can write $A_d^{(n-3)}(X_\Delta) = \{ D : D \cdot V(\tau) \geq 0 \text{ for all } \tau \in \Delta(n-4) \}$. By Proposition 6.4, there is an isomorphism $i^* : A^*(X_\Delta) \to A^*(M_{0,n})$. Now $D \cdot [V(\tau)] \geq 0$ if only if $i^*(D) \cdot i^*(V(\tau)) \geq 0$. Since $X_\Delta$ is smooth, $[V(\tau)] = D_{I_1} \cdots D_{I_{n-4}}$, where $\tau$ is the cone generated by rays labelled by $I_1, \ldots, I_{n-4}$. Thus $i^*(D_{I_1}) \cdots i^*(D_{I_{n-4}}) = \delta_{I_1} \cdots \delta_{I_{n-4}}$. The intersection of these boundary divisors is the class of the F-curve $C_\tau$ whose dual graph is the tree corresponding to $\tau$. The set of all classes of F-curves is the set of $[C_\tau]$ for $\tau \in \Delta(n-4)$, so

$$i^*(F_\Delta) = \{ i^*(D) : D \cdot V(\tau) \geq 0 \text{ for all } \tau \in \Delta(n-4) \} = \{ i^*(D) : i^*(D) \cdot i^*(V(\tau)) \geq 0 \text{ for all } \tau \in \Delta(n-4) \} = \{ i^*(D) : i^*(D) \cdot [C_\tau] \geq 0 \text{ for all } \text{F-curves } C_\tau \},$$

which is the cone of F-divisors.

\[ \square \]

\textbf{Example 6.5.} One can check easily by hand for $n = 5$, and by computer for $n = 6$, that

$$G_\Delta(M_{0,n}) = L_\Delta(M_{0,n}) = \text{Nef}(M_{0,n}) = F_\Delta(M_{0,n}).$$

One original motivation for the F-Conjecture was the expectation that cycles on $M_{0,n}$ should behave like those on a toric variety, and thus the cone of effective $k$-cycles should be generated by classes of the $k$-dimensional strata. This was shown to be false for divisors ($k = n-4$) independently by Keel and Vermeire [GKM02, p4], [Ver02]. The F-conjecture is the case $k = 1$. Proposition 6.2 enriches the connection of $M_{0,n}$ with toric varieties, by showing that the boundary strata are pullbacks of torus-invariant loci of the noncomplete toric variety $X_\Delta$.

In light of Proposition 6.2, one way to prove the F-conjecture would be to show that $L_\Delta = F_\Delta$. This has been computationally verified for $n \leq 6$. This suggests the following question.

\textbf{Question 6.6.} Is $L_\Delta(M_{0,n}) = \text{Nef}(M_{0,n})$?

Even if $L_\Delta = F_\Delta$, the description of this polyhedral cone given by $L_\Delta$ may be more accessible than the facet-description given by $F_\Delta$. 

\[ \square \]
Another way to find the nef cone of $\overline{M}_{0,n}$ would be to show that it is a Mori dream space, and give generators for its Cox ring. This has been done for $n = 6$ by Castravet \cite{Cas09}. The resulting toric embedding $\overline{M}_{0,n} \to X_\Sigma$ is different from the embedding into $X_\Delta$, as the effective cone of $X_\Sigma$ equals that of $\overline{M}_{0,n}$, which is strictly larger than that of $X_\Delta$. However for $n = 6$ we have $L_\Delta = L_\Sigma = \text{Nef}(\overline{M}_{0,n})$. This suggests that the phenomenon illustrated earlier in the case of the del Pezzo surface $\text{Bl}_5(\mathbb{P}^2) \subseteq X_\Sigma$ that different toric embeddings may give the same cone $L_\Delta$ may hold for $\overline{M}_{0,n}$.

The main goal of the F-conjecture or of Question 6.6 is to have a concrete description of the nef cone of $\overline{M}_{0,n}$ in order to study its birational geometry. The cone $L_\Delta$ would give such a description.

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