Notes on Sliding Mode Control of Two-Level Quantum Systems

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Abstract: In [Dong and Petersen, Sliding Mode Control of Two-Level Quantum Systems, arXiv:1009.0558, quant-ph, 2010], a sliding mode control approach has been proposed for two-level quantum systems to deal with bounded uncertainties in the system Hamiltonian. This paper further extends these results in two directions. One extension is to consider the effect of uncertainties during the process of driving the system’s state back to the sliding mode domain from outside and we propose two approaches to accomplish this control task. The other extension generalizes the previous design approach to consider the uncertainties described as perturbations in the free Hamiltonian.

1. INTRODUCTION

Developing control theory for quantum systems is becoming an active research area (Dong and Petersen [2010a], Rabitz et al. [2000], D’Alessandro [2007]). In particular, there exist many types of uncertainties (including noise, disturbance, decoherence, etc.) for most practical quantum systems and the robust control problem of quantum systems has been recognized as a key issue in developing practical quantum technology (Zhang and Rabitz [1994], Doherty et al. [2000], Pravia et al. [2003], Yamamoto and Bouren [2009], Dong et al. [2010]). Several methods have been proposed for the robust control of quantum systems. For example, James et al. [2008] have formulated and solved a quantum robust control problem using the H∞ method for linear quantum stochastic systems. In (Dong and Petersen [2009], Dong and Petersen [2010b]), we develop a sliding mode control approach to enhance the robustness of quantum systems. In particular, two approaches based on sliding mode design have been proposed for the control of quantum systems in (Dong and Petersen [2009]) and potential applications of sliding mode control to quantum information processing have been presented. Ref. (Dong and Petersen [2010c]) provides a detailed sliding mode control method for two-level quantum systems to deal with bounded uncertainties in the system Hamiltonian. This paper focuses on the sliding mode control design of two-level quantum systems and extend the results in (Dong and Petersen [2010c]) to deal with additional types of uncertainties.

Sliding mode control generally includes two main steps: selecting a sliding surface (sliding mode) and controlling the system’s state to and maintaining it in this sliding surface. We select an eigenstate |0⟩ of the free Hamiltonian of a two-level quantum system as a sliding mode. Being in the sliding mode guarantees that the quantum system has the desired dynamics. Furthermore, we also define a sliding mode domain D in which the system’s state has a small probability to collapse out of D when making a measurement. Then, such a sliding mode control problem includes two important subtasks: (I) design a control law to maintain the system’s state in D; (II) design a control law to drive the system’s state back to D if a measurement operation takes it away from D. In (Dong and Petersen [2010c]), we assumed that there exist no uncertainties during the control phase (II) and proposed a Lyapunov-based approach to accomplish such a subtask. This paper first proposes two controller design approaches (dependent on different situations) to accomplish the subtask (II) where the uncertainties are not ignored. For the subtask (I), assuming that there are no uncertainties which are described as perturbations in the free Hamiltonian, Dong and Petersen [2010c] presented a periodic measurement based method to guarantee the desired robustness and also gave an approach to design the measurement period. In this paper, we give modified measurement periods which need to be used when we consider uncertainties described as perturbations in the free Hamiltonian.

This paper is organized as follows. Section 2 introduces the sliding mode control problem for two-level quantum systems. In Section 3, we present a controller design method to accomplish subtask (II) taking into account uncertainties. Section 4 presents modified measurement periods when we consider uncertainties described as perturbations in the free Hamiltonian. Concluding remarks are given in Section 5.

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2. SLIDING MODE CONTROL OF TWO-LEVEL QUANTUM SYSTEMS

The quantum control model under consideration can be described as (we have assumed $\hbar = 1$ by using atomic units)

$$i\dot{\psi}(t) = (H_0 + H_\Delta + H_u)|\psi(t)\rangle,$$

where the quantum state $|\psi(t)\rangle$ corresponds to a two-dimensional unit complex vector in a Hilbert space, the free Hamiltonian $H_0 = \frac{1}{2}\sigma_z$, the uncertainties $H_\Delta = \delta(t)I_x + \epsilon_\Delta(t)I_y + \epsilon_\Delta(t)I_y$ ($\delta(t), \epsilon_\Delta(t), \epsilon_\Delta(t) \in \mathbb{R}$), the control Hamiltonian $H_u = \sum_{k=x,y,z} u_k(t)I_k$, $(u_k(t) \in \mathbb{R}, I_k = \frac{1}{2}\sigma_k)$ and the Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ take the following form:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Furthermore, we assume that the uncertainties are bounded:

$$|\delta(t)| \leq \delta, \quad (\delta \geq 0), \quad \sqrt{\epsilon_t^2(t) + \epsilon_y^2(t)} \leq \epsilon, \quad (\epsilon > 0).$$

In practical applications, we often use the density operator $\rho$ to describe the quantum state of a quantum system. For a two-level quantum system, the state $\rho$ can be represented in terms of the Bloch vector $r = (x, y, z) = (\text{tr}(|\rho\sigma_z|), \text{tr}(\rho\sigma_y), \text{tr}(\rho\sigma_z))$:

$$\rho = \frac{1}{2} (I + r \cdot \sigma). \quad (3)$$

The dynamical equation of $\rho$ can be written as

$$\dot{\rho} = -i[H, \rho],$$

where $[A, B] = AB - BA$. After we represent the state $\rho$ with the Bloch vector, the pure states (satisfying $\rho = |\psi\rangle\langle\psi|$) of a two-level quantum system correspond to the surface of the Bloch sphere, where $(x, y, z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \theta \in [0, \pi], \varphi \in [0, 2\pi]$. An arbitrary pure state $|\psi\rangle$ for a two-level quantum system can be represented as

$$|\psi\rangle = \cos \theta |0\rangle + e^{i\varphi} \sin \theta \frac{1}{\sqrt{2}} |1\rangle, \quad (5)$$

where $|0\rangle$ and $|1\rangle$ are eigenstates of $H_0$.

To deal with the uncertainties $H_\Delta$, Dong and Petersen [2010c] have proposed a sliding mode control approach where the eigenstate $|0\rangle$ is identified as the sliding mode. We further define a sliding mode domain $D = \{|\psi\rangle : |0\rangle\langle 0|\psi\rangle^2 \geq 1 - p_0, 0 < p_0 < 1\}$, where $p_0$ is a given constant. The definition of the sliding mode domain implies that the system’s state has a probability of at most $p_0$ (which we call the probability of failure) to collapse out of $D$ when making a measurement with the measurement operator $\sigma_z$. We expect to drive and then maintain a two-level quantum system’s state in a sliding mode domain $D$. However, the uncertainties $H_\Delta$ may take the system’s state away from $D$. Since a measurement operation unavoidably makes the measured system’s state collapse, we expect that our control laws will guarantee that the system’s state remains in $D$ except that a measurement operation may take it away from $D$ with a small probability (not greater than $p_0$).

In [Dong and Petersen [2010c]], we divide the control task under consideration into three main subtasks. Since we can make a measurement on any initial state to drive it into $|0\rangle$ or $|1\rangle$, we only consider the following two important subtasks: (I) design a control law to maintain the system’s state in $D$; (II) design a control law to drive the system’s state back to $D$ if a measurement operation takes it away from $D$. Ref. [Dong and Petersen [2010c]] ignores the uncertainties $H_\Delta$ during the control process (II) and the uncertainties $\delta(t)I_x$ (i.e., $\delta(t) \equiv 0$) for the subtask (I). This paper will relax these two assumptions. First, we propose two controller design approaches for the subtask (II) with uncertainties. Then we present modified measurement periods to guarantee the desired robustness for the subtask (I) when $\delta(t)I_x$ ($|\delta(t)| \leq \delta$) exists.

3. CONTROL DESIGN FOR THE SUBTASK (II) WITH UNCERTAINTIES

In [Dong and Petersen [2010c]], we have proposed a Lyapunov-based design approach to drive the quantum system’s state back to the sliding mode domain when we ignore the uncertainties during the control process. This section will consider the case where the uncertainties are not ignored and propose a simple method to accomplish subtask (II). For an arbitrary initial state, we first make a projective measurement to drive it to $|0\rangle$ or $|1\rangle$. The general control algorithm used in this section can be described as follows: (i) Select an eigenstate $|0\rangle$ of $H_0$ as a sliding mode $S(|\psi\rangle, H) = 0$, and define the sliding mode domain as $D = \{|\psi\rangle : |0\rangle\langle 0|\psi\rangle^2 \geq 1 - p_0\}$. (ii) For the initial state $|1\rangle$, design a control law to drive the system’s state into $D$ using information on $\epsilon$. (iii) For given $p_0$ and $\epsilon$, design the period $T$ for periodic projective measurements. (iv) For an initial state, make a projective measurement, then repeat the following operations: If the result is $|0\rangle$, make periodic projective measurements with a period $T$ to maintain the system’s state in $D$; If the measurement result corresponds to $|1\rangle$, use the corresponding control law to drive the state back into $D$.

In the control algorithm, the design of a control law in (ii) and the measurement period $T$ in (iii) are two important tasks. We use the results of [Dong and Petersen [2010c]] to design the measurement period $T$. Hence, this section focuses on the design of a control law in (ii). In the following, we consider two situations $H_\Delta = \epsilon(t)I_x$ ($\xi = x$ or $y$) and $H_\Delta = \epsilon(t)(\sin \varphi I_x - \cos \varphi I_y)$ ($\varphi \in [0, 2\pi]$ is a constant).

3.1 Control design for $H_\Delta = \epsilon(t)I_x$ ($\xi = x$ or $y$)

For the uncertainties $H_\Delta = \epsilon(t)I_x$ ($\xi = x$ or $y, |\epsilon(t)| \leq \epsilon$), we first use a simple constant Hamiltonian $H_u = uI_x$ (where we assume that the constant $u$ satisfies $u > \epsilon$) to drive the system’s state from $|1\rangle$ into $D$. Then, we make a projective measurement to guarantee that the probability of failure is not greater than $p_0$. In particular, we have the following theorem:

**Theorem 1.** For a two-level quantum system with the initial state $|\psi(0)\rangle = |1\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = I_x + [u + \epsilon(t)]I_x$ (where $\xi = x$ or $y, |\epsilon(t)| \leq \epsilon$ and $u > \epsilon$). If $t_1 \leq t_2$, where

$$t_1 = \frac{1}{\sqrt{1 + (u - \epsilon)^2}} \arccos \frac{2p_0 - 1}{1 + (u - \epsilon)^2} - 1,$$

$$t_2 = \frac{1}{\sqrt{1 + (u - \epsilon)^2}} \arccos \frac{2p_0 - 1}{1 + (u - \epsilon)^2} + 1,$$

and the system's state is in $D$, then the probability of failure is not greater than $p_0$. If $t_1 > t_2$, then the system's state is not in $D$.
when one makes a projective measurement with the measurement operator $\sigma_z$ at the time $t \in [t_1, t_2]$, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than $p_0$.

Using Pontryagin’s minimum principle, this Theorem can be proven by considering two cases of maximizing $z_f$ (equivalently minimizing $-z_f$) and minimizing $z_f$. Here we omit the details of proof.

**Remark 2.** For a two-level quantum system with uncertainties $H_\Delta = \epsilon(t)I_\xi$ ($|\epsilon(t)| \leq \epsilon$ and $\xi = x$ or $y$), we first make a projective measurement. If the result is $|0\rangle$, we remain the system’s state in the sliding mode domain $D$ by implementing periodic projective measurement with the measurement period $T(1) = \frac{\arccos(1-2p_0)}{\epsilon}$ (or $T(2) = \frac{\arccos(1-2p_0)}{\epsilon}$ if $p_0 \in (0, \frac{\epsilon^2}{1+\epsilon^2}]$) (Dong and Petersen 2010c). When the result is $|1\rangle$, we find a constant control $u$ (i.e., control Hamiltonian $H_u = u\phi_I$) to satisfy the relationship $t_1 \leq t_2$ using (6) and (7). Then we can apply the control to the system and make a projective measurement when $t \in [t_1, t_2]$, which can guarantee that the system’s state collapses into the sliding mode with a small probability of failure (not greater than $p_0$). Fig. 1 gives the relationships between $\Delta t = t_2 - t_1$, $\epsilon$ and $u$ for $p_0 = 0.01$. This shows that a larger $u$ is required for a larger $\epsilon$ to guarantee $t_2 \geq t_1$. The relationships between $\Delta t$, $p_0$ and $u$ for $\epsilon = 0.2$ are also shown in Fig. 2. It is clear that a larger $u$ is required for smaller $p_0$ to ensure $t_2 \geq t_1$.

**Example 3.** For a two-level quantum system with uncertainties $H_\Delta = \epsilon(t)I_\xi$ ($|\epsilon(t)| \leq \epsilon$ and $\xi = x$ or $y$), we consider the same setting as the example in (Dong and Petersen 2010c): $p_0 = 0.01$, $\epsilon = 0.02$ or $\epsilon = 0.2$. For $\epsilon = 0.02$, the measurement period is $T = 10.017$. For $\epsilon = 0.2$, $T = 1.002$. Now we consider the control process of driving the system’s state $|1\rangle$ back into $D$. From Fig. 3, it is clear that $u \geq 11$ can guarantee $t_2 > t_1$ for $\epsilon = 0.02$ and $u \geq 13$ can ensure $t_2 > t_1$ for $\epsilon = 0.2$. For $\epsilon = 0.02$, we select $u = 14$, $t_1 = 0.2141$, $t_2 = 0.2236$. We may first use $H_u = 14\phi_I$ on the system and then make a projective measurement at $t \in [0.2141, 0.2236]$. For $\epsilon = 0.2$, we select $u = 20$, $t_1 = 0.1497$, $t_2 = 0.1553$. We first apply $H_u = 20\phi_I$ to the system and then make a projective measurement at $t \in [0.1497, 0.1553]$. For the uncertainties $H_\Delta = \epsilon(t)(\sin \varphi I_x - \cos \varphi I_y)$, we first use the control Hamiltonian $H_u = -I_\xi + u(\sin \varphi I_x - \cos \varphi I_y)$ ($u > \epsilon$ is a constant) to drive the system’s state from $|1\rangle$ into $D$. Then, we make a projective measurement to guarantee the desired robustness. In particular, we have the following theorem:

**Theorem 4.** For a two-level quantum system with the initial state $|\psi(0)\rangle = |1\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = [u + \epsilon(t)](\sin \varphi I_x - \cos \varphi I_y)$ (where $|\epsilon(t)| \leq \epsilon$, $\epsilon > 0$, $\varphi$ is a constant and $u > \epsilon$). If $t \in [t_1, t_2]$, where $t_1 = \frac{1}{u - \epsilon} \arccos(2p_0 - 1)$, $t_2 = \frac{\pi}{u + \epsilon}$, when one makes a projective measurement with the measurement operator $\sigma_z$ at the time $t \in [t_1, t_2]$, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than $p_0$. 

**Fig. 1.** The relationships between $\Delta t = t_2 - t_1$, $\epsilon$ and $u$ in Theorem 1 for $p_0 = 0.01$.

$$t_2 = \frac{\pi}{\sqrt{1 + (u + \epsilon)^2}},$$

(7)
Proof. For $H^A = [u + \epsilon(t)][\sin \varphi I_x - \cos \varphi I_y]$, using \( \dot{\rho} = -i[H^A, \rho] \) and (3), we have
\[
\dot{r}_T^T = \begin{pmatrix} 0 & 0 & -(u + \epsilon(t)) \cos \varphi & 0 \\ (u + \epsilon(t)) \cos \varphi & 0 & -(u + \epsilon(t)) \sin \varphi \end{pmatrix} r_T^T,
\]
where \( r_T = (x_T, y_T, z_T) \) and \( r_0 = (x_0, y_0, z_0) = (0, 0, -1) \).

We consider two cases of maximizing \( z_T \) (equivalently minimizing \(-z_T\)) and minimizing \( z_T \). First, we take \( \epsilon(t) \) as a control input and select the performance measure as
\[
J(\epsilon) = -z_T.
\]
(11)

Also, we introduce the Lagrange multiplier vector \( \lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T \) and obtain the corresponding Hamiltonian function as follows:
\[
H(r_t, \epsilon(t), \lambda(t), t) = \lambda^T(t) \dot{r}_T
\]
(12)
\[
= \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \cos \varphi (u + \epsilon(t)) \sin \varphi
\]
where
\[
F(t) = (\lambda_3(t) x_t - \lambda_1(t) z_t) \cos \varphi + (\lambda_2(t) y_t - \lambda_2(t) z_t) \sin \varphi.
\]

According to Pontryagin’s minimum principle (Kirk [1970]), a necessary condition for \( \epsilon^* \) to minimize \( J(\epsilon) \) is
\[
H(r_t^*, \epsilon^*(t), \lambda^*(t), t) = \frac{\partial}{\partial t} (H(r_t, \epsilon(t), \lambda(t), t), t)
\]
(15)
Using a similar argument as in (Dong and Petersen [2010c]), we can exclude the possibility of existing singular cases (i.e., \( F(t) \equiv 0 \)). Hence, the optimal control \( \epsilon^* \) should be chosen as follows:
\[
\epsilon^*(t) = -\text{sgn}[F(t)].
\]
(16)

That is, the optimal control strategy for \( \epsilon(t) \) takes the form of \( \epsilon(t) = +\epsilon \) or \(-\epsilon \). Now we consider \( H_B = (u + \epsilon)(\sin \varphi I_x - \cos \varphi I_y) \) and have the state equation
\[
(x_{\tilde{t}}, y_{\tilde{t}}, z_{\tilde{t}})^T = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 0 & 0 & -(u + \epsilon) \cos \varphi \\ (u + \epsilon) \cos \varphi & 0 & -(u + \epsilon) \sin \varphi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix},
\]
(17)
where \( (x_0, y_0, z_0) = (0, 0, -1) \). The corresponding solution is
\[
\begin{pmatrix} x_{\tilde{t}} \\ y_{\tilde{t}} \\ z_{\tilde{t}} \end{pmatrix} = \begin{pmatrix} \cos \varphi \sin(u + \epsilon) t \\ \sin \varphi \sin(u + \epsilon) t \\ -\cos(u + \epsilon) t \end{pmatrix}.
\]
(18)

Now, consider the optimal control problem with a fixed final time \( t_f \) and a free final state \( r_f = (x_f, y_f, z_f) \). According to Pontryagin’s minimum principle (Kirk [1970]), \( \lambda^*(t) = \frac{\partial}{\partial t} r_T \). From this, it is straightforward to verify that \( \lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f) = (0, 0, -1) \). Now let us consider another necessary condition \( \lambda(t) = \frac{\partial}{\partial t} (r_T, \epsilon(t), \lambda(t), t) \) which leads to the following relationships:
\[
\lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix}
\]
(19)
\[
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cos \varphi (u + \epsilon) \sin \varphi
\]
where \( (\lambda_1(t_f), \lambda_2(t_f), \lambda_3(t_f)) = (0, 0, -1) \). The corresponding solution is
\[
\begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix} = \begin{pmatrix} -\cos \varphi \sin(u + \epsilon)(t_f - t) \\ -\sin \varphi \sin(u + \epsilon)(t_f - t) \\ -\cos(u + \epsilon)(t_f - t) \end{pmatrix}.
\]
(20)

From this, we obtain
\[
F(t) = -\sin(u + \epsilon) t_f.
\]
(21)

It is easy to show \( F(t) \leq 0 \) when \( t_f \in [0, \frac{\pi}{u + \epsilon}] \) and \( t \in [0, t_f] \).

From the above analysis, \( \epsilon(t) = \epsilon \) is the optimal control when \( t \in [0, \frac{\pi}{u + \epsilon}] \). Hence \( z_t^1 = z_t(\epsilon(t)) \leq z_t(\epsilon) = z_t^2 \). Assume that the state at time \( t_f \) is \( \rho_f \). When we make measurements on this system, the probability \( p \) that it will collapse into \( |1 \rangle \) (the probability of failure) is
\[
p = (1 - |\rho_f|) = 1 - \frac{z_t^2}{2}.
\]
(22)

From (22), it is clear that the probabilities of failure satisfy \( p_t^A = \frac{1 - z_t^A}{2} \geq p_t^B = \frac{1 - z_t^B}{2} \). That is, the probability of failure \( p_t^B \) is not greater than \( p_t^A \) for \( t \in [0, \frac{\pi}{u + \epsilon}] \). When \( t \in [0, \frac{\pi}{u + \epsilon}] \), \( z_t^B \) is monotonically increasing in \( t \) from (18) and \( p_t^B \) is monotonically decreasing with \( t \). When \( t = t_2 = \frac{\pi}{u + \epsilon} \), \( p_t^B \) reaches its minimum 0.

For \( J(\epsilon) = z_f \), using a similar analysis to that for \( J(\epsilon) = -z_f \), we have
\[
\begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix} = \begin{pmatrix} \cos \varphi \sin(u + \epsilon)(t_f - t) \\ \sin \varphi \sin(u + \epsilon)(t_f - t) \\ \cos(u + \epsilon)(t_f - t) \end{pmatrix}.
\]
(23)

(24)

The optimal control is \( \epsilon(t) = -\epsilon \). Hence, we can estimate the upper bound on \( z_t \) using \( H = (u + \epsilon)(\sin \varphi I_x - \cos \varphi I_y) \) and estimate the lower bound on \( z_t \) using \( H = (u - \epsilon)(\sin \varphi I_x - \cos \varphi I_y) \). For \( H = (u - \epsilon)(\sin \varphi I_x - \cos \varphi I_y) \), let \( p_0 = \frac{1 - z_t^A}{2} \), and it is easy to obtain
\[
t_1 = \frac{1}{u - \epsilon} \arccos(2p_0 - 1).
\]
(25)

Based on the above analysis, we can obtain the conclusion of Theorem 4.

Remark 5. For a two-level quantum system with uncertainties \( \Delta x = \epsilon(t)(\sin \varphi I_x - \cos \varphi I_y) \), we first make a projective measurement. If the result is \( |0 \rangle \), we retain its state in the sliding mode domain \( D \) by implementing periodic projective measurements with the measurement period \( T = \frac{\arccos(1-2p_0)}{2} \). When the result is \( |1 \rangle \), we find the control Hamiltonian \( H_\theta = -I_x + u I_x \) to satisfy the relationship \( t_2 \geq t_1 \) in Theorem 4. Then, we can apply the control to the system and make a projective measurement at \( t \in [t_1, t_2] \), which guarantees that the system’s state collapses into the sliding mode with a small probability of failure (not greater than \( p_0 \)). Since \( u > \epsilon > 0, \Delta t = t_2 - t_1 \) has the same sign as \( u \Delta t \). Fig. 4 shows the relationships between \( u \Delta t, p_0 \) and \( \frac{\pi}{u + \epsilon} \) for \( \epsilon = 0.2 \). When \( \epsilon = 0.2 \) and \( p_0 = 0.01 \), we may design the measurement period \( T = 1.002 \). From Fig. 5(b), we can select \( u = 10 \), \( t_1 = 0.30001 \), \( t_2 = 0.30800 \). For \( \epsilon = 0.02 \) and \( p_0 = 0.01 \), we can design the measurement period \( T = 10.017 \) (Dong and Petersen [2010c]) and select \( u = 1.5 \), \( t_1 = 1.9873 \), \( t_2 = 2.0668 \) from Fig. 5(a).
Fig. 4. The relationships between $u\Delta t$ ($\Delta t = t_2 - t_1$), $\epsilon/u$ and $p_0$ for $\epsilon = 0.2$ in Theorem 4.

Fig. 5. The relationships between $\Delta t = t_2 - t_1$ and $u$ in Theorem 4. (a) $p_0 = 0.01$, $\epsilon = 0.02$. (b) $p_0 = 0.01$, $\epsilon = 0.2$.

Remark 6. For general uncertainties $H_\Delta = \epsilon z(t)I_x + \epsilon_l(t)I_y$ ($\sqrt{\epsilon_z^2(t) + \epsilon_l^2(t)} \leq \epsilon$), it is difficult to establish a sufficient condition for finding a control Hamiltonian like $H_u = -I_z + uI_x$. However, it is easy to obtain some necessary conditions on the relationship between $u$, $p_0$ and $\epsilon$, which may be useful for finding a control law to ensure the required robustness. For example, we consider the control Hamiltonian $H_u = -I_z + uI_x$, the uncertainty is

$$H_\Delta = \begin{cases} \epsilon I_y & \text{when } z_l \leq 0; \\ 0 & \text{when } z_l > 0. \end{cases}$$

The system will evolve from $z_0 = 1$ to $z_1 = 0$ under $H(t) = uI_x + \epsilon I_y$ and then the Hamiltonian switches to $H(t) = uI_x$ at $z_1 = 0$. We can obtain the following necessary condition for the required robustness

$$u \geq \frac{(1 - 2p_0)\epsilon}{2\sqrt{p_0(1 - p_0)}}.$$

4. Modified Measurement Period for the Subtask (I)

4.1 Modification of $T^{(1)}$

This section will extend the results in (Dong and Petersen [2010c]) along another direction. In (Dong and Petersen [2010c]), we have assumed that the uncertainties are of the form $H_\Delta = \epsilon z(t)I_x + \epsilon_l(t)I_y$ ($\sqrt{\epsilon_z^2(t) + \epsilon_l^2(t)} \leq \epsilon$). In practical applications, uncertainties $H_\Delta = \delta(t)I_x$ ($|\delta(t)| \leq \delta$) are unavoidable for such quantum systems as spin systems in nuclear magnetic resonance (Li and Khaneja [2009]). Hence, here we consider $H_\Delta = \delta(t)I_x + \epsilon z(t)I_x + \epsilon_l(t)I_y$. We have the following theorem:

**Theorem 7.** For a two-level quantum system with the initial state $|\psi(0)\rangle = |0\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = \delta(t)I_x + \epsilon z(t)I_x + \epsilon_l(t)I_y$ (where $|\delta(t)| \leq \delta$, $\sqrt{\epsilon_z^2(t) + \epsilon_l^2(t)} \leq \epsilon$ and $\epsilon > 0$). If $t \in [0, T^{(1)}]$, where

$$T^{(1)} = \arccos(1 - 2p_0),$$

the system’s state will remain in $\mathcal{D} = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1 - p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator $\sigma_z$ at the time $t$, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than $p_0$.

**Proof.** Since the proof is very similar to that of Theorem 1 in (Dong and Petersen [2010c]), we omit the proof details. We only present two lemmas (Lemmas 8 and 10) and one corollary (Corollary 9), which are used in the proof of Theorem 7.

**Lemma 8.** For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to $(x_A^1, y_A^1, z_A^1)$ and $(x_B^1, y_B^1, z_B^1)$ under the action of $H_A = \delta_0I_x + \epsilon_0\cos\gamma_0I_x + \epsilon_0\sin\gamma_0I_y$ ($\delta_0$ is a constant and $\epsilon_0$ is a nonzero constant) and $H_B = \epsilon_0\cos\gamma_0I_x + \epsilon_0\sin\gamma_0I_y$, respectively. For arbitrary $t \in [0, \frac{T^{(1)}}{2}]$, $z_A^1 \geq z_B^1$.

Let this $\gamma_0 = 0$, and we have the following corollary.

**Corollary 9.** For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to $(x_A^1, y_A^1, z_A^1)$ and $(x_B^1, y_B^1, z_B^1)$ under the action of $H_A = \delta_0I_x + \epsilon_0I_x$ (where $\delta_0$ is a constant and $\epsilon_0$ is a nonzero constant) and $H_B = \epsilon_0I_x$, respectively. For arbitrary $t \in [0, \frac{T^{(1)}}{2}]$, $z_A^1 \geq z_B^1$.

**Lemma 10.** For a two-level quantum system with the initial state $(x_0, y_0, z_0) = (0, 0, 1)$ (i.e., $|0\rangle$), the system evolves to $(x_A^1, y_A^1, z_A^1)$ and $(x_B^1, y_B^1, z_B^1)$ under the action of $H_A = \delta(t)I_x + \epsilon z(t)I_x$ (where $|\delta(t)| \leq \delta$ and $|\epsilon t| \leq \epsilon$) and $H_B = \epsilon I_x$, respectively. For arbitrary $t \in [0, \frac{T^{(1)}}{2}]$, $z_A^1 \geq z_B^1$.

From Theorem 7, we know that the period is the same as that in (Dong and Petersen [2010c]). This fact also justifies the assumption that $\delta(t) \equiv 0$ when designing $T^{(1)}$ in (Dong and Petersen [2010c]). Using Theorem 7, we may try to maintain the system’s state in $\mathcal{D}$ (i.e., the subtask (I)) by implementing periodic projective measurements with the measurement period $T = \tilde{T}^{(1)}$. 

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Fig. 6. A demonstration of the relationships between $T(1)$, $T(2)$ and $\tilde{T}(2)$ for $\delta = 0.1$ and $p_0 \in [0, \frac{\epsilon^2}{1+\epsilon^2}]$, where $T^*(1)$, $T^*(2)$ and $\tilde{t}ldT_2$, respectively, correspond to $T_1$, $T_2$ and $\tilde{T}(2)$. (a) $\epsilon = 0.2$. (b) $\epsilon = 0.02$.

4.2 Modification of $T(2)$

Now we assume that the uncertainty is $H_\Delta = \delta(t)I_x + \epsilon(t)I_\xi$ ($\xi = x$ or $y$, $|\delta(t)| \leq \delta$ and $\delta \in [0,1]$) and $p_0 \in (0, \frac{\epsilon^2}{1+\epsilon^2}]$. We have the following theorem.

**Theorem 11.** For a two-level quantum system with the initial state $|\psi(0)\rangle = |0\rangle$ at the time $t = 0$, the system evolves to $|\psi(t)\rangle$ under the action of $H(t) = \delta(t)I_x + \epsilon(t)I_\xi$ (where $\xi = x$ or $y$, $|\epsilon(t)| \leq \epsilon$, $\epsilon > 0$, $|\delta(t)| \leq \delta$ and $\delta \in [0,1]$). If $p_0 \in (0, \frac{\epsilon^2}{1+\epsilon^2}]$ and $t \in [0, \tilde{T}(2)]$, where

$$\tilde{T}(2) = \frac{\arccos[1 - 2(1 + \frac{1-\delta^2}{\epsilon^2})p_0]}{\sqrt{(1 - \delta)^2 + \epsilon^2}}$$  \hspace{1cm} (28)

the system’s state will remain in $D = \{|\psi\rangle : |\langle 0|\psi\rangle|^2 \geq 1 - p_0\}$ (where $0 < p_0 < 1$). When one makes a projective measurement with the measurement operator $\sigma_\xi$ at time $t$, the probability of failure $p = |\langle 1|\psi(t)\rangle|^2$ is not greater than $p_0$.

This theorem can be proven using a similar argument as that in (Dong and Petersen [2010c]). We can also prove that $\tilde{T}(2) \geq \tilde{T}(1)$ for $p_0 \in (0, \frac{\epsilon^2}{1+\epsilon^2}]$ and $T(2) \geq \tilde{T}(2)$ for $0 < p_0 \leq \frac{\epsilon^2}{1+\epsilon^2}$.

**Remark 12.** Theorem 11 means that we are required to modify the measurement period to $T = \tilde{T}(2)$ in (28) from $T = T(2)$ in (Dong and Petersen [2010c]) when we do not ignore the uncertainties $\delta(t)I_x$. The selection rule for $T$ is summarized as follows: For $H_\Delta = \delta(t)I_x + \epsilon(t)I_\xi$ ($|\delta(t)| \leq \delta$, $\sqrt{\epsilon^2(t) + \delta^2(t)} \leq \epsilon$) and $0 < p_0 < 1$, $T = \tilde{T}(1)$; For $H_\Delta = \delta(t)I_x + \epsilon(t)I_\xi$, ($|\delta(t)| \leq \delta$, $\delta \in [0,1]$, $|\epsilon(t)| \leq \epsilon$, and $\xi = x$ or $y$), if $0 < p_0 \leq \frac{\epsilon^2}{1+\epsilon^2}$, $T = \tilde{T}(2)$, otherwise $T = \tilde{T}(1)$. Fig. 6 shows the relationships between $T(1)$, $T(2)$ and $\tilde{T}(2)$ for $\epsilon = 0.2$ and $\epsilon = 0.02$ ($\delta = 0.1$). For example, when $\epsilon = 0.2$, $\delta = 0.1$ and $p_0 = 0.01$, $T(1) = 1.0017$, $T(2) = 1.0494$, $\tilde{T}(2) = 1.0393$.

5. CONCLUDING REMARKS

This paper extends the results in (Dong and Petersen [2010c]) on sliding mode control design of two-level quantum systems. We consider more possible uncertainties in the control design. The results make our sliding mode control approach more practical for the control of quantum systems with uncertainties.

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