On the classification of quantum W-algebras

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\textbf{ABSTRACT}

In this paper we consider the structure of general quantum W-algebras. We introduce the notions of deformability, positive-definiteness, and reductivity of a W-algebra. We show that one can associate a reductive finite Lie algebra to each reductive W-algebra. The finite Lie algebra is also endowed with a preferred \textit{sl}(2) subalgebra, which gives the conformal weights of the W-algebra. We extend this to cover W-algebras containing both bosonic and fermionic fields, and illustrate our ideas with the Poisson bracket algebras of generalised Drinfeld-Sokolov Hamiltonian systems. We then discuss the possibilities of classifying deformable W-algebras which fall outside this class in the context of automorphisms of Lie algebras. In conclusion we list the cases in which the W-algebra has no weight one fields, and further, those in which it has only one weight two field.
1 Introduction

In the last few years remarkable progress has been made in the understanding of two-dimensional field theories that are conformally invariant. A key to completing this program is the classification of extended conformal algebras, or W-algebras. The first examples of W-algebras were the conformal algebra, various superconformal algebras and Kac-Moody algebras [1]. Later it was realised that a wider variety of algebras could be constructed from GKO coset theories [2–5]. Interest then moved to Toda theories, which provided many examples of W-algebras [6]; later these were re-incorporated into the Drinfeld-Sokolov scheme of Hamiltonian reductions [7–10]. Most recently it has been shown that the generalised Drinfeld-Sokolov reduction of WZW models can be used to produce a new class of algebras. In this construction one gauges a WZW model associated with a Lie algebra \( g \) by the currents associated with some nilpotent subalgebra. For bosonic W-algebras with fields of integer conformal weight, the nilpotent subalgebra can be labelled by an (integral) \( su(1,1) \) embedding in \( g \) [11]. This latter method has the advantage that the properties of the algebra obtained are easily related to the finite Lie algebraic ingredients of the construction, while the corresponding relationship in the GKO case is much more mysterious for the present.

However illuminating the examples cited above may be, we cannot hope to obtain a classification scheme for W-algebras if we tie ourselves to any one construction. It is this that motivates us to adopt a more general standpoint in this paper. The study of W-algebras is hampered by their infinite-dimensional nature. Worse still, their commutation relations are generally non-linear in the generating fields. Some progress has been made by looking for examples using algebraic computing techniques [12, 13], but the calculations involved are complex, and so the searches are restricted to examples containing two or three fields of low conformal weight.

In this paper we shall restrict our attention to ‘deformable’ algebras. By deformable we mean that the algebra satisfies the Jacobi identity for a continuous range of values of the central charge \( c \). In general the structure constants of the algebra are allowed to be functions of \( c \). The algebras excluded by this restriction are a very complicated set of objects, which in principle include, for example, a large number of W-algebras which can be constructed from lattices using vertex operators [14, 15]. There is some hope that these algebras are extensions of deformable algebras, occurring when a generically non-integer weighted primary field becomes integer weighted for particular values of \( c \).

The main result of our paper is contained in section 2, where we demonstrate the existence of a finite subalgebra associated with each classical W-algebra. Perhaps of more importance is that we can extend this result to the quantum case if we demand that the quantum algebra have a ‘good classical limit’. After discussing what precisely this means, we demonstrate the existence of a similar finite subalgebra in the limit that the central charge \( c \to \infty \). This provides us with an easily computable characteristic for W-algebras. A special role is played by an \( su(1,1) \) subalgebra of this finite algebra, which corresponds to the modes \( L_1, L_0, L_{-1} \) of the Virasoro algebra which generate Möbius transformations.

The characteristic we have derived is a finite dimensional Lie algebra, and an \( su(1,1) \) embedding. This is precisely the data used in the generalised Drinfeld-Sokolov reduction method. In sections 3 and 4 we clarify this connection. After a review of the generalised
Drinfeld-Sokolov construction of $W$-algebras, we calculate the structure constants of the $W$-algebra obtained by this method up to linear order in the fields, and use this to demonstrate that in this case the finite subalgebra is simply the Lie algebra $g$ associated with the WZW model that we are reducing and that, further, the $su(1,1)$ embedding is the same as that used in the construction. This provides us with a proof of the existence of the classical $W$-algebras associated with each finite Lie algebra $g$, and $su(1,1)$ embedding.

Armed with these results we discuss certain features of the $W$-algebras that are constructed in this way in section 5. In particular we give a complete list of such algebras which have no Kac-Moody components, and those with only one spin-2 field. We also comment on the possible use of automorphisms of the finite Lie algebra to generate homomorphisms of the $W$-algebra. The resulting algebras will be deformable, but will not have a good classical limit.

Finally, we conclude with some comments on the relevance of our approach to the classification of deformable $W$-algebras.

2 Finite algebras from $W$-algebras

If we want to classify extended conformal symmetries, or $W$-algebras, we should like to attribute to them some easily computable characteristics which specify the algebra completely. In this section we shall construct a finite Lie algebra associated with classical $W$-algebras and their quantum counterparts. Although we do not prove that this specifies the $W$-algebra, it does reveal something of its structure, and may ultimately form part of some classification scheme. To start with we shall consider the relationship between a general quantum $W$-algebra and its classical counterpart. After discussing the ‘vacuum preserving algebra’ (vpa) for both types of algebra, we show that in the classical case this contains a finite subalgebra if we define a ‘linearised’ Poisson bracket. We then extend this result to the quantum case by showing that the corresponding finite algebra decouples in the limit that the central charge goes to infinity.

Let us begin by discussing the relationship between quantum $W$-algebras and their classical counterparts. A quantum $W$-algebra comprises a set of modes $W^a(z)$, a notion of normal ordering, and a Lie Bracket. $W$-algebras are usually presented in the form of an operator product expansion, which we may represent schematically as

\begin{align}
W^a(z)W^b(z') &= g^{ab}(z - z')^{-\Delta_a - \Delta_b} \\
&+ \sum_c f_{ab}^c(z - z')^{\Delta_c - \Delta_b}[W^c(z') + g_{bc}(z - z')\partial W^c(z') + \ldots] \\
&+ \sum_{c,d} f_{ab}^{cd}(z - z')^{\Delta_c + \Delta_d - \Delta_b} [\circ W^c(z')W^d(z')_0 + \ldots + \ldots |z| > |z'|]
\end{align}

Here we have arranged the right hand side according to degree in $W$. Since we are interested in conformal field theories, we assume that the algebra contains the Virasoro algebra

\begin{equation}
L(z)L(z') = \frac{c}{2}(z - z')^{-4} + 2L(z')(z - z')^{-2} + \partial L(z')(z - z')^{-1} + O(1),
\end{equation}
as a subalgebra. We also assume that the algebra is generated by a finite number of primary fields $W^a(z)$ which obey

$$L(z)W^a(z') = \Delta^a W^a(z')(z - z')^{-2} + \partial W^a(z')(z - z')^{-1} + O(1) ,$$

(2.3)

where $\Delta^a$ is the weight of $W^a$. The commutation relations of the modes $W^a_m$ which are given by $W^a(z) = \sum \ W^a_m z^{-\Delta_a - m}$ can be deduced in the standard manner from the operator product expansion by a double contour integral. These take the form

$$[W^a_m, W^b_n] = g^{ab} P(\Delta_a, \Delta_b, 0, m, n) + f^{ab}_{(1)c} P(\Delta_a, \Delta_b, \Delta_c, m, n) W^c_{m+n} + \ldots ,$$

(2.4)

where $P$ is some known polynomial. In terms of modes (2.3) becomes

$$[L_m, W^a_n] = [(\Delta_a - 1)m - n] W^a_{m+n} .$$

(2.5)

Any field that obeys (2.5) for all $m$ is called primary, and any field that obeys it for $m = -1, 0, 1$ is called quasi-primary. The modes of a quasi-primary field $O^a_m$ form an indecomposable representation of the $su(1, 1)$ algebra generated by $L_{-1}, L_0, L_1$. One can use the representation theory of $su(1, 1)$ to show that the quasi-primary fields and their derivatives span the space of fields in the algebra, and that the polynomials $P$ are related to Clebsch-Gordan coefficients.

We define the hermitian conjugate by

$$(W^a_m)^\dagger = W^a_{-m} ,$$

(2.6)

and this induces a natural inner product on the states of the quantum theory. The requirement that this inner product be positive definite and the representation theory of the Virasoro algebra requires that the central charge $c > 0$ and that the fields all have positive definite weight. This then implies that the metric $g^{ab}$ is only non-vanishing of fields of equal weight, that it is positive definite, and that a basis of fields satisfying (2.6) can be chosen for which the metric is diagonal. We call such a W-algebra positive-definite. With this choice of basis, the algebra (2.4) takes the form

$$[W^a_m, W^b_n] = [(c/(2\Delta_a - 1)!)\delta^a_{\Delta_a} m(m^2 - 1) \ldots (m^2 - (\Delta_a - 1)^2) \delta_{m+n,0} + f^{ab}_{(1)c} P(\Delta_a, \Delta_b, \Delta_c + \Delta_d, m, n) W^c_{m+n} + \ldots ,$$

(2.7)

where $f$ are constants, and we have used that $P(\Delta_a, \Delta_a, 0, m, n) = m(m^2 - 1) \ldots (m^2 - (\Delta_a - 1)^2) \delta_{m+n,0}$ . We do not as yet require that the algebra be defined for more than one value of $c$. Examples of algebras which are not positive-definite include Kac-Moody algebras based on non-compact groups, and algebras including ‘ghost’ fields with strange statistics, such as the bosonic $N = 2$ superalgebra recently considered in [16].

We should remark on the definition of normal ordering $\phi_\circ \circ$ which we use here. In meromorphic conformal field theory, we assign a field uniquely to each state by

$$\phi(z) \leftrightarrow \phi(0)|0\rangle = |\phi\rangle = \phi_{-\Delta_\circ} |0\rangle .$$

(2.8)
We can define the normal ordered field $\bar{x}\phi_\phi'\bar{x}$ by

$$\bar{x}\phi_\phi'\bar{x} \leftrightarrow \phi_{-\Delta}\phi_{-\Delta}'|0\rangle .$$

(2.9)

However this is not the only possible normal ordering. Following Nahm, [17, 13], we have introduced the normal ordering $\circ\phi_\phi'$, by

$$\circ\phi_\phi' = \mathcal{P}\times\phi_\phi' ,$$

(2.10)

where $\mathcal{P}$ is the projector onto $su(1, 1)$ highest weight fields so that the resulting composite fields are quasi-primary. Further ambiguities arise when we try to normal order more than two fields, since this product is not associative. One can, for example, decide to order the fields by conformal weight and index $a$, and then always nest the normal orderings from the left. However, there are many choices of basis. We shall call a particular choice of basis a presentation of the W-algebra as a commutator algebra. The underlying structure, which is that of a meromorphic conformal field theory, is the same for each presentation, but the structure constants will be different. The point that we should like to stress is that the classical limit of different presentations are identical. This is a consequence of the observation that the difference in two orderings can be written as a commutator, and thus must be an $O(\hbar)$ term which vanishes in the classical limit. (In fact, the projection operator $\mathcal{P}$ does not simply amount to a reordering, but the difference between the two normal orderings can be seen to be the Virasoro descendents of commutators, so that the result is true in this case too.)

Let us now consider classical $W$-algebras. This is a Poisson bracket algebra of fields $W^a(x)$ of one variable which closes on (differential) polynomials and central terms. We can represent the Poisson bracket schematically as

$$\{W^a(x), W^b(y)\} = g_{ab}^c \partial^{\Delta_a + \Delta_b - 1} \delta(x - y)$$

$$+ \sum_c f^a_{(1)c} \partial^{\Delta_b - \Delta_c}\delta(x - y)W^c(y) + g_{c}^{ab} \partial^{\Delta_b + \Delta_c - 2}\delta(x - y)\partial W^c(y) + \ldots$$

$$+ \sum_{c,d} f_{(2)c}^{ab} \partial^{\Delta_b - \Delta_d - 1}\delta(x - y)[W^c(y)W^d(y) + \ldots] + \ldots ,$$

(2.11)

where the right hand side has been ordered according to degree in $W$, and $f^{ab}_{(i)c.d}$ may be functions of the central charge $c$. Alternatively, if we take the space on which the fields are defined to be the unit complex circle, we can expand the fields in modes exactly as in the quantum case. Identical mode algebras are generated if, in the equations (2.1), (2.11) we use the correspondence

$$(z - z')^{-N} \rightarrow (-1)^{N-1}2\pi i\delta^{N-1}(z - z')/(N - 1)! ,$$

(2.12)

although if we use identical structure constants we do not expect that both quantum and classical commutator algebras satisfy the Jacobi identity if they are non-linear. We shall assume that the classical algebras have a number of properties that they would inherit automatically as the classical limit of the quantum algebras. They contain a classical version of the Virasoro algebra

$$\frac{1}{2\pi}\{L(x), L(y)\} = -\frac{c}{12}\delta''(x - y) - 2L(y)\delta'(x - y) + \partial L(y)\delta(x - y) ,$$

(2.13)
and the generating fields obey the classical version of (2.5). Further, the only terms in the Poisson bracket algebra which are independent of $W^a$ are taken to be of the form $m(m^2 - 1) \cdots (m^2 - (\Delta_a - 1)^2) \delta_{m+n,0} \delta^{ab}$. By analogy, we refer to such algebras as positive-definite classical algebras.

We now discuss the relationship between a classical $W$-algebra and its quantised version. We shall see that an extremely important criterion for a quantum algebra to have a classical limit is that it is well-defined for all values of the central charge $c$, with the exception perhaps of a few isolated values or closed intervals. By this we mean that there is an operator product algebra for a set of fields $W^a(z)$ of fixed conformal weights $\Delta_a$, with structure constants $f^{(i)}$ which are continuous functions of $c$, which is associative for a continuous range of $c$ values. We call such algebras deformable. We shall see that deformability is however not sufficient for a quantum algebra to have a classical limit.

As an example let us first consider the classical Virasoro algebra, (2.13). Quantising this algebra yields

$$[L'_m, L'_n] = \hbar \frac{c'}{12} m(m^2 - 1) \delta_{m+n,0} + \hbar (m - n)L'_{m+n},$$

where the prime $'$ indicates that we have the normalisation inherited from the classical Poisson bracket structure. To recover the standard normalisation we must substitute

$$L' = \hbar L, \quad c' = \hbar c.$$  

Similarly, for a general quantum $W$-algebra, we can re-introduce $\hbar$ by the substitutions

$$L \to L'/\hbar, c \to c'/\hbar, W^a \to W'^a/\hbar^a,$$

where the constants $\alpha_a$ are to be determined. The classical limit is given by the usual correspondence

$$\{W^a, W^b\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [W'^a, W'^b].$$

For this limit to make sense we require that the quantum operator product algebra remain associative as $\hbar \to 0$, or equivalently, as $c \to \infty$. This is why the $W$-algebra must be deformable. Substituting (2.16) into (2.7) we obtain schematically

$$[W'_a, W'_b] = c' \hbar^{2\alpha_a - 1} \delta + f^{(1)} W'_c \hbar^{\alpha_a + \alpha_b - \alpha_c} + f^{(2)} \circ W'_c W'^f \circ \hbar^{\alpha_a + \alpha_b - \alpha_f - \alpha_e} + O(W'^3).$$

For this to be the quantisation of a classical $W$-algebra, we require that the right-hand side be $O(\hbar)$. The $\hbar$ dependence comes both explicitly from $W \to W'\hbar^a$, but also implicitly, from $f^{(i)}(c) \to f^{(i)}(c'/\hbar)$. If we have

$$f^{ab}_{(i)(c_i)} = O(\hbar^{\gamma(a,b,c)}(c_i) )$$

as $\hbar \to 0$,

and all the fields $\circ(W'^i)^p \circ$ are $O(1)$, then we must impose that for each term in the singular part of the operator product algebra expansion of $W^a$ and $W^b$

$$\min_{(a,b,c_i)} (\gamma^i + \alpha_a + \alpha_b - \sum_{j=1}^i \alpha_{c_i}) \geq 1.$$
If it is not possible to find such constants $\alpha_a$, then we say that the W-algebra has no classical limit. The restriction (2.20) only restricts the couplings to fields which appear in the commutation relation, or equivalently to fields in the singular part of the operator product of $W^a$ and $W^b$. It is obvious that we have no restriction on the regular terms since

$$W^a(z)W^b(z') = \ldots + \circ W^a(z') W^b(z') \circ + \ldots , \quad (2.21)$$

where the coupling to $\circ W^a W^b \circ$ is $O(1)$.

If the classical limit of a W-algebra is positive-definite, we call the quantum algebra reductive. To examine what restrictions this implies, we must consider the central terms in (2.18). Fixing the behaviour of the central term we require

$$\alpha_a = 1 \text{ for all } a . \quad (2.22)$$

For this choice of $\alpha_a$ the requirement (2.20) becomes, for each term which appears in the singular part of the operator product expansion,

$$f_{(i)} = O(c^{1-i}) \text{ as } c \to \infty . \quad (2.23)$$

If it is not possible to impose (2.20), then the only possibility of recovering a classical W-algebra is that the normal ordered products are no longer $O(1)$. Generically for a W-algebra which comes from the quantisation of a classical algebra we have

$$\circ W^a(x) W^b(y) \circ = W^a(x) W^b(y) + O(\hbar) . \quad (2.24)$$

It is possible for the first term to vanish if the bosonic fields $W^a, W^b$ can be written as composite fermionic fields,

$$W^a = d(x) f(x), \quad W^b = d(x) e(x) , \quad (2.25)$$

where classically $d(x)d(x) = 0$, and quantum mechanically $\circ d(x)d(y) \circ = O(\hbar)$. A simple example of this possibility may be seen by considering the first two W(4,6) algebras of ref [12]. The coupling constants of these algebras do not meet the requirements (2.23) or even (2.20). In fact one of these two algebras can be constructed as the bosonic ‘reduction’ of a fermionic W-algebra [18], the $N = 1$ superconformal algebra. This yields a W-algebra with fields of spins 4 and 6, with zero central charge classically. We shall discuss such reductions further in section 5.

Let us now turn to the question of constructing the advertised finite Lie algebra from classical and quantum W-algebras. Although the full set of modes of a quasi-primary operator $O^i$ only form an indecomposable representation of $su(1,1)$, the subset of modes

$$\left\{ O^i_m : |m| < \Delta(O^i) \right\} \quad (2.26)$$

form an irreducible representation of $su(1,1)$. The set of all such modes for a W-algebra forms a closed subalgebra. We call this the vacuum-preserving algebra (vpa), since in the quantum case these are precisely the modes which annihilate both the right and left $su(1,1)$ invariant vacua. Although this algebra involves only a finite number of modes of each field, for a non-linear algebra it will only close on the modes associated with an infinite
number of such quasi-primary fields, so that it is not a finite Lie algebra. For the linear Virasoro algebra the vpa is the set \{L_1, L_0, L_{-1}\} which form the algebra \(su(1, 1) = sl(2, \mathbb{R})\); for the superconformal algebra, the vpa is the algebra \(osp(1, 2)\). These subalgebras give useful information about the structure constants of fields in conformally invariant and superconformally invariant theories respectively and we would like to define a similar finite Lie algebra associated to a general W-algebra.

Let us consider first a classical W-algebra. The assumption of positive-definiteness says that the algebra takes the form

\[
\{W^a_m, W^b_n\} = \frac{c}{((2\Delta_a - 1)!\Delta_a)m(m^2 - 1)\ldots(m^2 - (\Delta - 1)^2)\delta^{ab}\delta_{m+n}} + f_{(1)c}^{ab} P(\Delta_a, \Delta_b, \Delta_c, m, n)W^c_{m+n} + \ldots
\]

(2.27)

If we restrict attention to the vpa, we see immediately that the central terms are absent. We can easily check that the Jacobi identity is satisfied by this new bracket when we restrict to the vpa. This is because for \(|m| < \Delta_a P(\Delta_a, \Delta_b, 0, m, n) = 0\) and consequently

\[
\{W^a_m, \sum_p W^b_{n+p} W^c_{-p}\} = \sum_p f_{(1)d}^{ab} P(\Delta_a, \Delta_b, \Delta_d, m, p+n)W^d_{m+n+p} W^c_{-p} + f_{(1)d}^{ac} P(\Delta_a, \Delta_c, \Delta_d, m, -p)W^d_{m-p} W^b_{n+p} + O(W^3),
\]

(2.28)

so that in the classical case the contributions to the Jacobi identity from the quadratic and higher order terms which we have neglected do not contribute, and so the restricted bracket

\[
\{W^a_m, W^b_n\}_D = f_{(1)c}^{ab} P(\Delta_a, \Delta_b, \Delta_c, m, n)W^c_{m+n}.
\]

(2.29)

is a closed Lie algebra, \(g\). Since we have, by assumption, included the modes \(L_{\pm 1}, L_0\) in the vpa, we see that we automatically have an \(su(1, 1)\) embedding \(su(1, 1) \subset g\) given by a classical W-algebra. In the case of the Zamolodchikov algebra \(WA_2\), we have the modes \(\{Q_{\pm 2}, Q_{\pm 1}, Q_0, L_{\pm 1}, L_0\}\) forming the algebra \(sl(3)\) with \(su(1, 1)\) in the maximal regular embedding.

We should like to attempt the extension of this argument to the quantum case. In the classical case, the contribution from composite terms to the linear terms in the double commutator vanished for the vpa because the only possible ‘contraction’ from three fields to one arose from the central term which decoupled precisely for these modes. However, in the quantum case there are other contributions to this term which arise from the need to normal order composite fields. As an illustrative example we return to the Zamolodchikov algebra \(WA_2\), this time in its quantised version. The quantum commutation relations are

\[
[L_m, Q_n] = (2m - n)Q_{m+n}
\]

(2.30)

\[
[Q_m, Q_n] = \frac{c m(m^2 - 1)(m^2 - 4)}{3} \delta_{m+n} + \frac{(m-n)}{30} (2m^2 - mn + 2n^2 - 8)L_{m+n} + \beta(m-n)\Delta_{m+n},
\]

(2.31)
where
\[ \beta = \frac{16}{22+5c}, \quad \Lambda(z) = \frac{z}{6} T(z) T(z)^{-1}. \] (2.32)

The only non-trivial double commutators are \([L_p, Q_m, Q_n]\) and \([Q_p, Q_m, Q_n]\) and the only composite field appearing in the intermediate channel is \(\Lambda\), so we need only consider the linear terms in \([L_p, \Lambda_{m+n}], [Q_p, \Lambda_{m+n}]\). These are
\[ [L_p, \beta \Lambda_{m+n}] = \frac{16}{5} \frac{p(p^2 - 1)}{3!} L_{m+n+p} + \cdots \] (2.33)
\[ [Q_p, \beta \Lambda_{m+n}] = \beta \frac{4(5p^3 - 5p^2(m+n) + 3p(m+n)^2 - (m+n)^3 - 17p + 9(m+n))}{35} Q_{p+m+n} + \cdots \] (2.34)

The first of these commutators vanishes when we take \(p = -1, 0, 1\). In the second commutator there is a contribution from the \(\Lambda\) term in \([Q, Q]\) which does not vanish even when we restrict to the vpa, and a more careful consideration of this term shows that it arises from the need to normal order composite fields. Instead we can ensure that this term does not violate the consistency of the ‘linearised’ Jacobi identity by taking the limit \(c \to \infty\). In this case \(\beta \to 0\) and the vpa linearises, again to give \(sl(3, \mathbb{R})\). Note that we need to combine the limit \(c \to \infty\) and the restriction to the vpa to ensure that both commutators vanish.

We are now in a position to prove this feature, namely that the vpa linearises to give a finite Lie algebra as \(c \to \infty\), assuming that the W-algebra is reductive. Let us denote a generic composite field composed of \(i\) basic fields as \(\delta(W)^{i_0}_{i_1}\). The contribution to the Jacobi identity from the coupling through such terms in \([W^a, [W^b, W^c]]\) is
\[ [W^a_m, [W^b_n, W^c_p]] = \sum_i [W^a_m, f^{bc}_{(i)} P(\Delta_b, \Delta_c, \Delta(i), n, p) \delta(W)^{i_0}_{i_1} W^c_{m+n+p}] . \] (2.35)

Let us also write the linear part of the contribution from the commutator of \(W^a\) with \(W^{(i)}\),
\[ [W^a_m, \delta(W)^{i_0}_{i_1} W^c_{o_n}] = g^{ae(i)}_{(i)} P(\Delta_a, \Delta(i), \Delta(e), m, n) W^e_{m+n} . \] (2.36)

At this point we must split our argument into two cases, depending on whether \(\delta(W)^{i_0}_{i_1}\) appears in the singular or regular part of the operator product expansion of \(W^a\) with \(W^e\). If \(\delta(W)^{i_0}_{i_1}\) appears in the singular part of the operator product expansion then we can calculate the order of \(g^{ae(i)}\) by taking the three point function
\[ C_{abi} = \langle W^a W^e \delta(W)^{i_0}_{i_1} \rangle . \] (2.37)

This can be written in two ways. We have that
\[ C_{abi} = f^{ae(i)}_{(i)} \langle \delta(W)^{i_0}_{i_1} \rangle = O(c) , \] (2.38)
using (2.23) and evaluating the leading contribution in \(c\) to \(\langle 0 | (W)^i_{\Delta(i)} (\Delta(i)^{-1} | 0) \rangle\). We also obtain
\[ C_{abi} = g^{ae(i)}(W^a W^e) \sim g^{ae(i)} c , \] (2.39)
using (2.7) and (2.36), and suppressing non-zero constants. Thus we see that \(g^{ae(i)} = O(1)\) and so the contribution to the Jacobi identity of three basic fields \([W^a, [W^b, W^c]]\) to the field
$W^e$ from the term in $[W^b, W^c]$ of form $\circ(W)^i \circ$ is $O(g^{ae}_{(i)} f^{be}_{(j)}) = O(c^{i-e})$, if the field $\circ(W)^i \circ$ appears in the singular part of the operator product expansion of $W^a$ with $W^e$. However, if the field $\circ(W)^i \circ$ does not, then we cannot apply (2.23) to deduce the order of the coupling $g^{ae}_{(i)}$. In this case we have that $\circ(W)^i \circ$ has conformal weight $\Delta_{(i)} \geq \Delta_a + \Delta_e$. However, the polynomials $P(\Delta_a, \Delta_{(i)}, \Delta_e, m, n)$ vanish identically if $|m| < \Delta^a, \Delta_{(i)} \geq \Delta^a + \Delta^e$ (see ref. [19]), and we can use this fact to bypass our ignorance of the coupling $f^{ae}_{(i)}$.

This shows that if we consider Jacobi’s identity for the vpa modes of the generating fields in the limit $c \to \infty$, then all contributions to linear terms from composite fields in the intermediate channel drop out. Since the commutator algebra is a Lie algebra for all $W$ fields in the limit $c \to \infty$, then all contributions to linear terms from composite fields in the intermediate channel drop out. Since the commutator algebra is a Lie algebra for all $c$ values by the assumption of deformability, the only obstruction to the vpa algebra restricted to the generating fields satisfying the Jacobi identity was from such contributions. Thus, this algebra in the $c \to \infty$ limit of a reductive W-algebra is a finite Lie algebra.

We have now shown how to recover finite Lie algebras from positive-definite classical W-algebras and reductive quantum W-algebras. We shall call this the linearised vpa algebra. By the Levi-Malcev theorem, the most general form for a finite Lie algebra would be the semidirect product of a semi-simple Lie algebra with its radical, which is its maximal solvable ideal. However, we can use the positive-definiteness of the classical algebra to show that the maximal solvable ideal of the finite Lie algebra we have constructed is in fact its centre, or, in other words, that the linearised vpa is the direct sum of a semisimple Lie algebra with an abelian Lie algebra (For results on the structure of Lie algebras used here, see e.g. [20]).

To do this, let us consider the maximal solvable ideal $a$ of the linearised vpa of a reductive W-algebra. Let us suppose that a particular mode $W^a_m \in a$. The modes $L_{\pm 1}, L_0$ are always in the linearised vpa, and we know the commutation relations of $W^a_m$ with $L_m$ to be of the form

$$[L_m, W^a_n] = ((\Delta_a - 1)m - n)W^a_{m+n}.$$  
(2.40)

Since $a$ is an ideal, eqn. (2.40) implies that $W^a_m \in a \Rightarrow W^a_n \in a$ for all $|n| < \Delta_a$. With the standard normalisation for a positive-definite quantum W-algebra we have

$$W^a(z)W^a(z') = (c/\Delta)(z - z')^{-2\Delta_a} + 2L(z')(z - z')^{-2\Delta_a+2} + \ldots.$$  
(2.41)

Using the Virasoro Ward identities, (see e.g. appendix B of ref. [21]), we can deduce the coefficients of all the terms $\partial^j L(z')$ in this operator product and we can deduce that

$$[W^a_{\Delta-1}, W^a_{1-\Delta}] = 12(\Delta - 1)/(\Delta(2\Delta - 1))L_0 + \ldots$$  
(2.42)

For $\Delta_a > 1$ this is non-zero, and so, for $\Delta_a > 1$ we have $L_0 \in a$. If $L_0 \in a$, then we immediately we get that $L_0, L_{\pm 1} \in a$. which is a contradiction since a solvable ideal cannot contain a semi-simple algebra.

So, if $W^a_m \in a$, where $a$ is a solvable ideal, then $\Delta_a \leq 1$. If $\Delta_a < 1$ then it contributes no modes to the vpa; if $\Delta_a = 1$ then $m = 0$, and we can thus denote the elements of $a$ as $U^0_\beta$, the zero modes of a set of weight one fields $U^i(z)$. These zero modes $U^0_\beta$ form a solvable Lie algebra. However, it has been known for a long time (see e.g. [22]) that the requirement that the inner product on the primary fields of weight one is positive definite forces them to have a Kac-Moody algebra based on a compact semi-simple Lie algebra plus some $u(1)^a$ current algebra. Thus the zero modes of weight one fields in cft form a finite
dimensional Lie algebra which is the direct product of a semisimple Lie algebra with an abelian algebra, and we see that the ideal \( a \) is abelian.

The only possibility left open to us now is that the linearised vpa has the structure of a semi-simple Lie algebra semidirect product with an abelian algebra. Suppose that \( U_0 \in a \). Then \( [U_0, W_0^a] \in a \) for all \( W_0^a \) in the vpa. If \( [U_0, W_0^a] = X_m \), then the operator product expansion of \( U(z) \) with \( W^a(z) \) must be of the form

\[
U(z)W^a(z') = \ldots + X(z')(z - z')^{-1} + O(1),
\]

since \( U_0 = \int \frac{dz}{(2\pi i)} U(z) \). However, from (2.43) we see that the field \( X \) must have conformal weight equal to that of \( W^a \). Since \( X_m \in a \), we see that \( X \) must have weight one, and so \( W^a \) must have weight one. We already know that \( a \) commutes with the zero modes of the spin one fields, so in fact \( a \) is the centre of the vpa. This completes the proof that the linearised vpa of the classical limit of a reductive W-algebra is the direct sum of a semisimple Lie algebra with an abelian algebra.

The above discussion has been for a purely bosonic W-algebra. If we wish to include fermions then we must also consider Lie superalgebras, since the vpa of fermionic fields will contain anti-commutators of the modes of fermionic fields. We shall use the notation of [23] for Lie superalgebras, with the algebra decomposition

\[
g = g_0 \oplus g_1
\]

where the bosonic generators are in \( g_0 \) and the fermionic in \( g_1 \). We define the grade of a generator \( X \) to be \( g(X) = j \) if \( X \in g_j \). The (super)Lie bracket then takes the form

\[
[X, Y] = XY - (-1)^{g(X)g(Y)} YX
\]

The bosonic fields have modes in \( g_0 \) and the fermionic fields have modes in \( g_1 \). It will also be the case that the bosonic fields will have integral conformal weight and the fermionic fields half-integral conformal weight, for unitarity. A fermionic field of weight \( \Delta \) will have mode decomposition

\[
\psi(z) = \sum_{n \in \mathbb{Z} + 1/2} \psi_n z^{-n-\Delta}
\]

As for bosonic fields, the vpa contains the modes of the fermionic fields

\[
\psi_m : |m| < \Delta
\]

We see that for a free fermionic field of conformal weight \( 1/2 \), there are no modes in the vpa. Thus analysis of the vpa will yield no information on the free fermion content of a theory. However, this is not obstacle since free fermions have already been shown to factorise from the Hilbert space by Goddard and Schwimmer [24].

Since the fermionic fields will have half integer modes, the \( su(1,1) \) decomposition must be compatible in the sense that the decomposition takes the form

\[
g_0 = \oplus_{j \in \mathbb{Z}} D_j, \quad g_1 = \oplus_{j \in \mathbb{Z} + 1/2} D_j
\]

where \( D_j \) is the representation of dimension \( 2j + 1 \). We can also prove that the superalgebra consists of the direct sum of simple (super)-algebras and an abelian Lie algebra in an analogous manner to that above for purely bosonic W-algebra vpa’s.
This means that the field content of any positive-definite \( W \)-algebra which is defined for all \( c \) values must comprise a set of free fermion fields of weight one half (which do not contribute to the vpa; such fields have already been shown to factorise [24]), a set of bosonic free fields of weight one \( (u(1)^n \text{ current algebra}) \) and a set of fields whose weights are given by an \( sl(2) \) embedding in a semisimple Lie superalgebra which is compatible with the grading of the superalgebra as in (2.48).

For a purely bosonic \( W \)-algebra, the field content will comprise a set of weight one fields and a set of bosonic fields whose conformal spins are given by an integral \( su(2) \) embedding in a semi-simple Lie algebra.

We shall now go on to show that this is indeed the case for the generalised Drinfeld Sokolov constructions mentioned earlier, and then to consider various cases of particular interest. The rest of this paper will be concerned only with the case of bosonic algebras for simplicity.

### 3 Hamiltonian systems and co-adjoint orbits

The analysis of the previous section showed that to each reductive \( W \)-algebra one could associate a finite Lie algebra with some \( su(1, 1) \) embedding specified. This is reminiscent of the data that is required for a generalisation of the Drinfeld-Sokolov construction of Hamiltonian structures that have been studied recently [11, 25, 26]. In the next section we shall show that this data is recovered as the finite Lie algebra we constructed. This provides us with an existence proof for the classical \( W \)-algebras associated with each Lie algebra and \( su(1, 1) \) embedding. In this section we give a brief review of this construction.

The classical Hamiltonian systems of Drinfeld and Sokolov [8] are based on a Poisson bracket structure on \( g^* \), the dual to the Lie algebra \( g \), and the extension of this to \( \hat{g} \), the centrally-extended Kac-Moody algebra related to \( g \). An element of \( \hat{g} \) consists of a pair,

\[
(j(z), c) ,
\]

where \( c \) is a number and \( j \) is a field on \( S^1 \) valued in \( g \). The coordinate on \( S^1 \), we denote by \( z \), with \( 0 \leq z < 2\pi \). With this definition, the Lie bracket of two elements of \( \hat{g} \) is given by

\[
[(j^1(z), c^1), (j^2(z), c^2)] = ([j^1(z), j^2(z)], k \int \text{Tr}\{\partial j^1(z) j^2(z)\} dz) ,
\]

where the second term corresponds to the cocycle of \( \hat{g} \). An element of \( \hat{g}^* \) is given by a pair \( (q, \lambda) \), where \( q \) is a \( g \)-valued field on \( S^1 \) and \( \lambda \) is a number. The action of this element on \( (j, c) \) in \( \hat{g} \) is given by

\[
\langle (q, \lambda), (j, c) \rangle = \int \text{Tr}\{qj\} dz + c\lambda .
\]

With this, we may identify \( \hat{g} \) and \( \hat{g}^* \), and we obtain a canonical action of \( \hat{g} \) on \( \hat{g}^* \), the coadjoint action \( ad^* \). If \( (q, \lambda) \in \hat{g}^* \), \( (j^i, c^i) \in \hat{g} \), then we have

\[
ad^*_{(j^i, c^i)}(q, \lambda)[(j^2, c^2)] = (q, \lambda)[ad_{(j^1, c^1)}(j^2, c^2)]
\]

\[
= \langle (q, \lambda), ([j^1, j^2], k \int \text{Tr}\{(j^1)' j^2\}) \rangle
\]
\[ \int \text{tr} \{ [q, j^1] j^2 + k \lambda \partial j^1 j^2 \} . \]

Thus we obtain
\[ \text{ad}^*_U(q, \lambda) = ([q + k \lambda \partial, j^1], 0) . \]

This is simply an infinitesimal gauge transformation of \( q \). This phase space also has a canonical action of \( \hat{G} \), the coadjoint action \( \text{Ad}^* \), given by
\[ \text{Ad}^*_U \cdot q(h) = q(\text{Ad}^*_U h) , \quad (3.4) \]
\[ \text{Ad}^*_U \cdot (q, \lambda) = (U^{-1} q U + k \lambda U^{-1} \partial U, \lambda) . \quad (3.5) \]

There is a canonical Poisson bracket structure on \( \hat{g}^* \), the Berezin-Kirillov-Kostant-Lie-Poisson bracket. If \( U, V \) are two functionals on \( \hat{g}^* \), then their Poisson bracket is also a functional on \( \hat{g}^* \). When evaluated on \( q \in \hat{g}^* \) it is explicitly given by
\[ \{ U, V \}_q = \langle q, [d_q U, d_q V]_{KM} \rangle , \quad (3.6) \]
\[ \text{where} \quad d_q \equiv \frac{d}{dq} . \]

We shall usually suppress the \( j \) suffix if it is clear from context.

We may accordingly evaluate the Poisson brackets of the components of \( \hat{g}^* \). If \( \{ T^i \} \) form a basis of the generators of \( \hat{g} \) with \( \text{Tr} \{ T^i T^j \} = g^{ij} \), then we can define the functionals \( \hat{T}^i(x), \hat{\epsilon} \) on \((q, \lambda) \in \hat{g}^* \) by
\[ \hat{T}^i(x)[(q, \lambda)] = \text{Tr} \{ T^i(x) \} , \quad \hat{\epsilon}[(q, \lambda)] = \lambda . \quad (3.8) \]

We have
\[ d\hat{T}^i(x) = (T^i \delta(x - y), 0) , \quad \hat{\epsilon} = (0, 1) . \quad (3.9) \]

Thus we can evaluate the Poisson brackets of these functionals
\[ \{ \hat{T}^i(x), \hat{T}^j \}_{(q, \lambda)} = \int dz \, \text{Tr} \{ q [T^i \delta(x - z), T^j \delta(y - z)] + k \lambda \partial_x (T^i \delta(x - z)) T^j \delta(y - z) \} . \]

Thus we see that
\[ \{ \hat{T}^i(x), \hat{T}^j \}_{(q, \lambda)} = f^{ij} k \hat{T}^k(y) \delta(x - y) - k \hat{\epsilon} g^{ij} \delta'(x - y) . \quad (3.10) \]

This is the Kac-Moody algebra \( \hat{g} \). In particular we shall often denote the zero grade subalgebra functional \( \hat{J}^i \) by \( j^i \) and \( \hat{\epsilon} \) by \( 1 \), with the Poisson brackets
\[ \{ j^i(x), j^j(y) \} = f^{ij} k \hat{T}^k(y) \delta(x - y) - \frac{k}{2} \delta^{ij} \delta'(x - y) . \quad (3.11) \]

The method of hamiltonian reduction involves constraining currents associated with nilpotent elements of the algebra. In the traditional reduction associated with Toda theory or the standard KdV hierarchy one gauged the maximal nilpotent algebra associated
with, say, all the positive roots of $g$. It was then realised that one could generalise this construction by gauging some smaller set of currents, and moreover, that this set could be succinctly labelled by some $su(1, 1)$ embedding. Since we are interested in bosonic positive-definite $W$-algebras, we may assume that the $su(1, 1)$ embedding is integral. Non-integral embeddings result in bosonic fields of half-integral weight and the resulting $W$-algebras are not positive-definite.

Let us consider some modified Cartan-Weyl basis for $g$,\begin{equation}
g = g^- \oplus h \oplus g^+. \tag{3.12}\end{equation}
Here\begin{equation}g^\pm = \oplus C E^{\pm \alpha}, \quad h = \oplus C H^i, \tag{3.13}\end{equation}
with the commutation relations\begin{equation}[E^{\alpha}, E^{-\alpha}] = (2/\alpha^2) \alpha^i H^i, \quad [H^i, E^\beta] = \beta^i E^\beta. \tag{3.14}\end{equation}
One can always conjugate any $su(1, 1)$ subalgebra of $g$ so that $I_+ \in g^+, I_0 \in h, I_- \in g^-$ where $I_+, I_-, I_0$ are the usual raising, lowering and diagonal basis of $su(1, 1)$. We may write $I_0 = \rho^\vee \cdot H$. If we use the standard normalisation for the $su(1, 1)$ algebra,\begin{equation}[I_0, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = \sqrt{2} I_0, \tag{3.15}\end{equation}
then we may define the characteristic of the $su(1, 1)$ embedding to be $(\rho^\vee \cdot e_1, ..., \rho^\vee \cdot e_i)$, where $e_j$ are the simple roots of $g$. It is a fact that the entries of the characteristic are $0, 1/2, 1$. For integral embeddings they must either be 0 or 1. The standard reduction is associated with the principal embedding whose characteristic contains all ones.

We may grade $g$ with respect to the $\rho^\vee \cdot H$ eigenvalue as\begin{equation}g = \oplus_m g_m. \tag{3.16}\end{equation}
The elements of $g$ which are highest weight states for this $su(1, 1)$ action form a commuting subalgebra of $g$. We denote these highest weights by $E^{(e_i)}$, and the corresponding lowest weights by $E^{(-e_i)}$. The highest weights are annihilated by $I_+$ and the lowest weights by $I_-$. We denote the subalgebra $\oplus_{n \geq 0} g_n$ by $p^+$ and the subalgebra $\oplus_{n > 0} g_n$ by $n^+ \subset g^+$. Similarly for $p^-, n^-$. For the standard reduction associated with the principal reduction, $n^\pm = g^\pm$.

Since $\hat{G}$ acts on $\hat{g}^*$, we may perform a classical Hamiltonian reduction [27] with respect to the subgroup $\hat{N}^-$, where $N^-$ is the subgroup of $G$ which has the nilpotent subalgebra $n^-$ as its Lie algebra. In this procedure one chooses an image of the momentum map $\pi$ and the phase space consists of equivalence classes under the residual symmetry of the inverse image of $\pi$. Here $\pi$ is essentially the projection map $g \mapsto n_+$. We can choose the image of $\pi$ in such a way that the inverse image consists of elements of $\hat{g}^*$ of the form\begin{equation}(b(z) + I_+, 1), \tag{3.17}\end{equation}
where $b(z) \in p^-$. We call this space $M$.

The action of $\hat{N}^-$ is now an equivalence relation on $M$. From the form of $M$, we may choose coordinates on this space to be gauge invariant differential polynomials of the entries
in the matrices $b(z)$. In particular, there is a unique gauge transformation $Ad_N^* = \exp(ad_n^*)$ with $n \in \hat{n}^-$ which gives
\[ N^{-1}(b + I_+)N + kN^{-1}\partial N = I_+ + \sum_n W_n E^{(n)}, \tag{3.18} \]
where $E^{(n)}$ span the kernel of $I_-$. To show this gauge transformation is unique, take components in $g_m$. As a result, the entries of $n$ are uniquely determined polynomials in the entries of $b(z)$.

The Poisson bracket structure on gauge invariant functionals $\phi, \psi$ on classes of $q \in \hat{M}$ is given by
\[ \{\phi, \psi\} = \langle q, [\nabla_q \phi, \nabla_q \psi]_{KM} \rangle, \tag{3.19} \]
where $j = \nabla_q \phi$ is any element of $\hat{g}$ such that
\[ \phi(q + \delta q) = \phi(q) + \langle \delta q, \nabla_q \phi \rangle + O(\delta q^2), \tag{3.20} \]
for all $q \in M$. Thus $\nabla_q \phi$ is determined up to an element of $\hat{n}^-$; one such choice is $\nabla_q \phi = d_q \phi$. It is easy to show that this bracket is well defined and satisfies the Jacobi identity [8].

Further, we can imbed this structure in the Lie-Poisson bracket structure by considering the map $\mu$ from $\hat{G}_0$, corresponding to zero-graded Lie algebra $g_0$, to gauge invariant functionals on $M$ given by
\[ \mu : (g_0 + I_+, 1) \mapsto \{W_i(g_0)\}, \tag{3.21} \]
where
\[ N^{-1}(g_0 + I_+)N = I_+ + \sum_n W_n E^{(n)}. \tag{3.22} \]
Since this gauge transformation is unique, the polynomials $W_n(g_0)$ are a choice of coordinates on the manifold $\hat{M} = M/\hat{N}^-$. It can also be shown [8] that the Poisson bracket structure on $\hat{g}$ and $\hat{M}$ are compatible in the sense that
\[ \{\mu^*\phi, \mu^*\psi\} = \mu^*\{\phi, \psi\}, \tag{3.23} \]
where $\mu^*\phi$ is a functional on $\hat{g}^*$. $\mu^*$ is called the generalised Miura transformation. For the standard reduction corresponding to the principal embedding, $g_0 = h$, and so the Miura transformation provides a free field representation of the Poisson bracket algebra (3.19).

For more general reductions we obtain a construction in terms of the currents of the zero-graded (non-abelian) algebra. Since the polynomials $W_n(g_0)$ are a choice of coordinates on $\hat{M}$, this algebra is closed, although not necessarily on linear combinations of the original coordinates.

4 Linearised Poisson brackets for classical $W$-algebras

We are now in a position to calculate the Poisson brackets of the $W$-algebra given by the Drinfeld-Sokolov reduction associated with some Lie algebra $g$ and a particular integral $su(1, 1)$ embedding. The purpose of this section is to show that the finite subalgebra and
su(1, 1) embedding of section 2 associated with this W-algebra coincide with those chosen to specify the reduction. This will demonstrate the existence of the classical W-algebra associated to each such pair. We can check this in this case by using the expressions we deduced in the previous section for the W-algebra Poisson brackets. The finite subalgebra is simply the ‘linearised’ vpa for the generating fields, so we will only need to calculate the Poisson brackets to linear order in the fields. This is the feature which makes the calculation tractable.

We need to calculate the Poisson brackets of the functionals \( \mathcal{H} \),

\[
\mathcal{H} = \int \frac{dz}{2\pi i} f^a(z) W^a(z),
\]

where \( W^a \) is a W-algebra field and now \( z \) is a complex coordinate. If we denote the W-algebra fields of weight \( \Delta_a = a + 1 \) by \( W^a_m \), then the modes \( W^a_m \) are given by \( \mathcal{H} \) for \( f(z) = z^{m+a} \). We shall not differentiate between fields of equal conformal weight, to avoid proliferation of indices. From (3.18) we know that the gauge invariant fields correspond to the lowest weights of the \( sl(2) \) embedding. If we wish to identify a particular field of a given weight, then we shall use the notation \( W[X] \), where \( X \) is a generator of \( g \) which is a lowest weight of the \( sl(2) \).

The results of the last section tell us that the Poisson bracket algebra is given by

\[
\{W^a_m, W^b_n\} = \int \frac{dz}{2\pi i} \text{Tr}(j(z)[dW^a_m, dW^b_n]) + k \int \frac{dz}{2\pi i} \text{Tr}(dW^a_m(dW^b_n)) .
\]

If we are interested in the term in this Poisson bracket which is linear in the fields \( W \), then we clearly only need to calculate \( dW^a_m \) to linear order in the fields \( W \).

Consider the arbitrary element of the space \( M \) to be of the form

\[
j = b(z) + I_+ .
\]

We may decompose \( g \) with respect to the \( sl(2) \) subalgebra to find the highest weights of this \( sl(2) \) which we shall denote by \( E^a \). Then bases for \( g, g^- \) and \( n^- \) are given by

\[
\begin{align*}
g & = \bigoplus_a \bigoplus_{i=0}^{2a+1} \mathbb{C}E^{i,a}, \\
g^- & = \bigoplus_a \bigoplus_{i=0}^{a+1} \mathbb{C}E^{i,a}, \\
n^- & = \bigoplus_a \bigoplus_{i=0}^{a} \mathbb{C}E^{i,a}.
\end{align*}
\]

where

\[
E^{i,a} = ad^i(I_+) \circ E^a .
\]

Then we may write

\[
j = I_+ + j_0 + I_+ + \sum_a \sum_{i=0} a j_i^a(z) E^{i,a} .
\]

We shall now consider a gauge transformation of the form

\[
j^l = \exp(ad(l)) \circ (j + k\partial),
\]

15
where \( l \) is some current in \( n^- \). If \( l \) is the transformation which puts \( j \) into the highest weight gauge, then \( l \) is defined implicitly as a polynomial function in the components of \( j_0 \) and their derivatives, by

\[ j^i = j^W = I_+ + \sum_a W^a E^a. \quad (4.10) \]

An important step in the argument is to decompose \( l \) into components which are homogeneous in the components of \( j_0 \), and further into components of \( E^i,a \):

\[ l = \sum_j l^{(j)} \quad (4.11) = \sum_j \sum_a \sum_{i=0}^{a-1} l^{j|ia} E^{i,a}, \quad (4.12) \]

where \( l^{j|ia} \) is homogeneous in \( j_0 \) of degree \( p \). If we decompose the gauge invariant functionals \( W^a \) into homogeneous pieces \( W^{p|a} \) of degree \( p \), then upon substituting (4.12) into eqn. (4.10), we obtain

\[ W^{1|a} = j^{0,a} - k(l^{0|0a})' \quad (4.13) \]
\[ W^{2|a} = \text{Tr} \left( E^{2a,a}[l^{(1)}, (j_0 + \frac{1}{2}[l^{(1)}, I_+])] \right) - k(l^{2|0a})', \quad (4.14) \]

where

\[ l^{1|ia} = \sum_{p=i+1}^a (-k\partial)^{p-i-1} j^{p,a} \quad (4.15) \]
\[ l^{2|i-1,a} = \sum_{p=i}^{a-1} (k\partial)^{p-i} (-1)^p \left[ \text{Tr} \left( E^{2a-p,a}[l^{(1)}, (j_0 + \frac{1}{2}[l^{(1)}, I_+]]) \right) \right]. \quad (4.16) \]

We have chosen a normalisation for the \( sl(2) \) highest weight vectors

\[ \text{Tr}(E^{2a,a}E^{0,a}) = 1, \quad (4.17) \]

For simplicity we have used notation which assumes that there is a unique field of each weight, but the generalisation is straightforward. The normalisation (4.17) means that the generators \( I_\pm, [I_+, I_-] \) obey \( su(1,1) \) commutation relations with nonstandard normalisation factors.

We can now deduce \( d\mathcal{H} \) for \( \mathcal{H} = \int W^a(z) f(z) \frac{dz}{2\pi i} \). This will in general be a function of the entries \( j^{ia} \), and since we are interested in the Poisson brackets of gauge-invariant quantities, we may simply substitute \( j \) by \( j^W \), after we have evaluated \( dW \). This makes the evaluation of \( dW \) very easy since \( l^{[1]} = 0, j^{ia} = \delta^i_0 W^a \). We can thus identify the terms which will contribute to \( d\mathcal{H} \) where \( \mathcal{H} = \int f W^a \). Using the notation

\[ \left\{ \begin{array}{ccc} a & b & c \\ j & k & l \end{array} \right\} = \text{Tr}\{E^{j,a}[E^{k,b}, E^{l,c}]\}, \quad (4.18) \]

we have

\[ d\mathcal{H} = \int \sum_{p=0}^a (k\partial)^p f dy^{ia} \]
\[ + \int \sum_{p=0}^{a-1} \sum_{q=0}^{b-1} \sum_{r=q+1}^b (k\partial)^{r-q-1} [W^c(-k\partial)^p f] \left\{ \begin{array}{ccc} a & b & c \\ 2a-p & q & 0 \end{array} \right\} dj^{r,b}. \quad (4.19) \]
Using \( \text{Tr}(jd^{i,a}) = j^{i,a} \), we can easily see that \( d^{i,a} = (-i)^{j}E^{2a-i,a} \). We are really only interested in the case \( f = z^{a+m} \) where \( \mathcal{H} = W_{m}^{a} \), and so finally we obtain

\[
dW_{m}^{a} = \sum_{i=0}^{a} (a + m)_{i}(-k)^{i}z^{m+a-i}E^{2a-i,a} + \sum_{b,c} \sum_{i=0}^{a-1} \sum_{q=0}^{c-q} (-1)^{q+j} \partial^{j-1}(W^{b}\partial^{i}z^{m+a}) \left\{ \begin{array}{ccc} b & a & c \\ 0 & 2a - i & m \end{array} \right\} E^{2a-j,q,c}k^{i+j-1} + O(W^{2}) ,
\]

where we denote

\[
(b)(b - 1) \ldots (b - c + 1) \text{ by } (b)_{c} .
\]

We are now in a position to evaluate the Poisson brackets of the modes \( W_{m}^{a} \) to linear order in the fields. We shall decompose \( dW \) into its homogeneous pieces, as

\[
dW_{m}^{a} = dW_{m}^{a(0)} + dW_{m}^{a(1)} + O(W^{2}) .
\]

Using (4.2) we see that the terms which contribute to the linear piece of the Poisson bracket are

\[
\{ W_{m}^{a}, W_{n}^{b} \} = k \int \frac{dz}{2\pi i} \text{Tr}(dW_{m}^{a(0)}(dW_{n}^{b(0)})') + k \int \frac{dz}{2\pi i} \text{Tr}(j^{W(z)}[dW_{m}^{a(0)},dW_{n}^{b(0)}]) + k \int \frac{dz}{2\pi i} \text{Tr}(dW_{m}^{a(1)}(dW_{n}^{b(0)})') + O(W^{2}) .
\]

O’Raifeartaigh et al. in [25, 11] have shown that \( L = \alpha W[I_{-}] + \sum_{a} \beta^{a}W[U^{a}]^{2} \) is a Virasoro algebra for the system we have described, where \( W[I_{-}] \) is the field corresponding to the representation of the embedded \( sl(2) \) itself, and \( W[U^{a}] \) are fields corresponding to the singlets in the decomposition of \( q \) with respect to the embedded \( sl(2) \). The fields \( W[E^{0,a}] \) transform as primary fields of weight \( a + 1 \) w.r.t. this Virasoro algebra. The central term in the Virasoro algebra is generated purely by the field \( W[I_{-}] \), since we have already shown that there are no central terms in the Poisson brackets of composite fields. In the case \( W = W[I_{-}] \), eqn. (4.20) becomes exact,

\[
dW[I_{-}]_{m} = z^{m+1}[I_{+},[I_{+},I_{-}]] - k(m + 1)z^{m}[I_{+},I_{-}] .
\]

We can now evaluate the Poisson bracket

\[
\{ W[I_{-}]_{m}, W[I_{-}]_{n} \} = k\lambda(m - n)W[I_{-}]_{m+n} + k^{3}m(m^{2} - 1)\delta_{m+n} ,
\]

which corresponds to a rescaled Virasoro algebra. In (4.25) \( \lambda \) is defined by

\[
[I_{+},[I_{+},I_{-}]] = \lambda I_{+} .
\]

Thus, if we re-scale \( W[I_{-}] \) to return (4.25) to the standard normalisation, we recover \( c = 12k/\lambda^{2} = 6k(\rho^{\vee})^{2} \) where \( \rho^{\vee} \) defines the \( su(1,1) \) embedding.
Thus the Poisson brackets (4.24) represent a W-algebra. To establish that this is a positive-definite W-algebra, note that the first term in (4.24) corresponds to the central term and is easy to compute;

$$\int \frac{dz}{2\pi i} \sum_{i=0}^{a} \sum_{j=0}^{b} (a + m)(b + n)(j + 1)(-k)^{i+j+1} z^{m+n+a+b-i-j-1} \text{Tr}(E^{2a-i,a} E^{2b-j,b}) .$$

The trace in this term gives $\delta^{ab} \delta_{ij}^2 (-1)^a$ and so we get now

$$= \int \frac{dz}{2\pi i} \delta^{ab}(a + m)(a + n)_{a+1} z^{m+n-1} k^{2a+1} (-1)^a$$

$$= -k^{2a+1} m(m^2 - 1) \ldots (m^2 - a^2) \delta_{m+n,0} \quad (4.27)$$

Thus the algebra (4.24) is a positive-definite W-algebra since the central term is non-degenerate. Following the theoretical framework we laid out before, we can now restrict our attention to the algebra of the modes of the vpa. For the vpa the central term vanishes and the second term in (4.24) is now easy to compute.

$$\int \frac{dz}{2\pi i} \text{Tr}(j^W(z) [dW^a_m(z), dW^b_n(z)])$$

$$= \int \frac{dz}{2\pi i} \text{Tr} \left( \sum_c W^c(z) E^c \right)$$

$$\times \left[ \sum_{i=0}^{a} (a + m)_i (-k)^i z^{m+a-i} E^{2a-i,a} + \sum_{j=0}^{b} (b + n)_j (-k)^j z^{n+b-j} E^{2b-j,b} \right] k^{i+j}$$

$$= \sum_c W^c_{m+n} \sum_{i=0}^{a} \sum_{j=0}^{b} (a + m)_i (b + n)_j (-k)^{i+j} \text{Tr}(E^c [E^{2a-i,a}, E^{2b-j,b}]) k^{i+j} ,$$

$$= \sum_c W^c_{m+n} \min(a+b,b+c) \sum_{\lambda=\max(b,c)} (a + m)(a+b-\lambda)(b + n)(\lambda-c) \left\{ \begin{array}{cccc} c & b & a \\ c-b \lambda \ldots (c-b+c) a \\ c-b \lambda \ldots (c-b+c) b \\ \end{array} \right\} k^{a+b-c} ,$$

remembering that the trace gives a delta function $\delta(c + a + b, 2a + 2b - j - i)$.

The third term is more complicated,

$$T_3 \equiv k \int \frac{dz}{2\pi i} \text{Tr}(dW^a_m(z) dW^b_n(z))$$

$$= \int \frac{dz}{2\pi i} \sum_{i=0}^{a} z^{a+m-l}(a + m)_i (-1)^l$$

$$\times \sum_{d,c} \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{p=0}^{c} (-1)^{p+j} \partial^j (W^d(z) \partial^j z^p+b) \left\{ \begin{array}{cccc} d & b & a \\ 0 & 2b-c-p \lambda \ldots (c-b+c) a \\ \end{array} \right\} \text{Tr}(E^{2a-l,a} E^{2c-j-p,c}) k^{l+j+l}$$

The last trace gives us $\delta(l, a), \delta(a, c), \delta(j, c - p)$ and so we get now

$$T_3 = \int \frac{dz}{2\pi i} \sum_{d} \sum_{i=0}^{a} \sum_{p=0}^{a} (-1)^a z^m \partial^{a-p} (W^d(z) \partial^j z^p+b) \left\{ \begin{array}{cccc} d & b & a \\ 0 & 2b-c-p \lambda \ldots (c-b+c) a \\ \end{array} \right\} k^{l+2a-p}(a + m)_a$$

$$= \int \frac{dz}{2\pi i} \sum_{d} \sum_{i=0}^{a} \sum_{p=0}^{a} (-1)^a \partial^{a-p} (W^d(z) \partial^j z^p+b) \left\{ \begin{array}{cccc} d & b & a \\ 0 & 2b-c-p \lambda \ldots (c-b+c) a \\ \end{array} \right\} k^{l+2a-p}(a + m)_a$$

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This last term in curly braces represents a trace, which gives us \( \delta(i,b+p-d-c) \), and so we get, with \( \lambda = 2b+d-p \),

\[
T_3 = \sum_d W^d_{m+n} \sum_{\lambda=b+d+1}^{2b+d} (a+m)_{(a+b-\lambda)(b+n)(\lambda-d)} \left\{ \begin{array}{ccc} d & a & b \\ 0 & a-b+\lambda & 2b+d-\lambda \end{array} \right\} k^{a+b-d} \]

where we used the fact that for \( \lambda > b+d+n \) the second of the two Pochhammer symbols vanishes.

By similar reasoning we can evaluate the fourth term of (4.24) and when we put them all together we obtain for the vpa terms \( \{W^a_m, |m| \leq a\} \),

\[
\{W^a_m, W^b_n\} = \sum_c W^c_{m+n} \sum_{\lambda=\max(c,b-m)}^{\min(b+c+n,a+b)} (a+m)_{(a+b-\lambda)(b+n)(\lambda-c)} \left\{ \begin{array}{ccc} c & a & b \\ 0 & a-b+\lambda & 2b+c-\lambda \end{array} \right\} k^{a+b-c} + O(W^2) \quad (4.29)
\]

This establishes the linearised vpa commutation relations. We can now compare them with the commutation relations of \( g \) in a particular basis. We know that \( W[I_{\pm1,0}] \) form the \( sl(2) \) embedded inside \( g \), and so it is convenient for us to take the modes of the \( sl(2) \) primary fields \( W^a_m \) to correspond to the representation of \( sl(2) \) with highest weight \( E^a \). If we take the correlation to be

\[
W^a_m \cong E^{a-m,a} f(a,m) \quad (4.30)
\]

where \( f(a,m) \) are some constants, then the fact that \( W[I_{\pm1,0}] \) are the embedded \( sl(2) \) tells us that \( f(a,m) \) is given by

\[
f(a,m) = \mu(a)(-k\lambda)^{a+m}(a+m)!/(2a)! , \quad (4.31)
\]

where \( \lambda \) is defined in equation (4.26). We may now evaluate the commutator of two of these elements of \( g \), and we find that we recover the commutation relations (4.29) exactly for \( \mu(a) = (2a)! \), and so

\[
W^a_m \cong E^{a-m,a}(-k\lambda)^{a+m}(a+m)! \quad (4.32)
\]

completes the identification of the vacuum preserving modes of the \( W \)-algebra with the algebra \( g \) itself, up to quadratic terms in the fields \( W \). Moreover the modes \( L_1, L_0, L_- \) are clearly associated to \( I_+, I_0, I_- \). If one examines the commutation relations more carefully and normalises the algebra correctly according to (2.7), one sees that indeed the couplings \( f^{ab}_c \) are \( O(1) \) in the central charge, thus bearing out our expectations for a classical \( W \)-algebra.

This clarifies the relationship between the linearised vpa and the generalised Drinfeld-Sokolov reduction. It follows that there exists at least one classical \( W \)-algebra for every
and every integral $su(1,1)$ embedding. If this W-algebra is unique, then the Drinfeld-
Sokolov reductions completely saturate the possibilities for bosonic integrally weighted
W-algebras. We return to this point in the conclusion.

5 Lie algebras and $sl(2)$ embeddings

In this section, we shall use the theory of finite Lie algebras and their three-dimensional
subalgebras together with what we have learnt in the preceding sections to look at various
aspects of reductive W-algebras. We briefly comment on a possible relation between au-
tomorphisms of the linearised vpa and homomorphisms of the W-algebra. Then we give
a complete list of positive-definite W-algebras without Kac-Moody components by classi-
ifying all $su(2)$ embeddings of semi-simple Lie algebras with trivial center. We enumerate
the algebras which, in addition, contain no generating spin-2 fields besides the Virasoro
algebra.

First, let us consider the case where the $su(1,1)$ is not a maximal subalgebra of $g$;
that is there exists some algebra $h$ such that $su(2) \subset h \subset g$. In the limit that $c \to \infty$
we expect that some subalgebra of generating fields which includes the Virasoro algebra
closes. This, however, does not imply that these generating fields generate some $W$-algebra
associated with $su(2) \subset h$, which is a subalgebra of the $W$-algebra associated with $su(2) \subset
g$. As an example we can consider the algebra $WA_3$ associated with the principal $su(2)$
embedding in $A_3$ which is generated by fields of weight 2, 3, and 4. However we can write
$su(2) \subset B_2 \subset A_3$ in this case, where the spin 2, 4 fields are associated with the first
embedding. An inspection of the operator product expansion of the spin 4 field with itself
reveals that it contains a term which is the composite field associated with the square of
the spin 3 field. The coupling to this term vanishes in the limit that $c \to \infty$, but the spin
2, 4 algebra does not close on itself for finite central charge, and is distinct from $WB_2$
which we would associate with the first embedding. However, we suspect that a weaker
statement is true. If $\tau$ is an automorphism of $g$ for which $h$ is the stable subalgebra, this
gives an automorphism of the vacuum preserving modes in the limit that $c \to \infty$. Since
$\tau(L) = L$ it defines some homomorphism which maps a generating primary field to some
linear combination of primary fields of the same weight. We shall assume that this can
be used to define a homomorphism on the Verma module of the W-algebra associated
with $g$. The subalgebra which is stable under this homomorphism will contain only those
generating fields associated with $h$, but will in general require additional generators which
will be composites in all the generating fields of the algebra. In the example above, if we
choose the simple roots of $A_3$ to be $e_1 - e_2, e_2 - e_3, e_3 - e_4$ then the automorphism which
preserves its $B_2$ subalgebra is given by

$$
e_1 \to -e_4$$
$$e_2 \to -e_3$$

and the induced homorphism on the $WA_3$ algebra is simply spin-3 $\to$ $-\text{spin-3}$. The subal-
gebra which is stable under this homomorphism is the smallest closed algebra containing
the spin 2, 4 fields. If it is the case that the resulting W-algebra does not have a good
classical limit, which we know to be true in the case that we reduce a super W-algebra
in this way, then this sort of consideration may provide a powerful tool for constructing deformable algebras which do not have a good classical limit.

As a second topic of interest, we shall now give a complete list of the possibilities for bosonic W-algebras with good classical limit which have no Kac-Moody components. If we want there to be no spin 1 fields in the W-algebra we require that there are no singlets in the decomposition of the adjoint representation of $g$ under $su(1,1)$. Alternatively, if we decompose $g$ w.r.t. the $I_0$ member of the preferred $sl(2)$, we may express this by saying that $\dim g_1 = \dim g_0$, or by saying that the centraliser of $su(1,1)$ in $g$ is zero. We classify all the examples of $su(1,1)$ embeddings where this is so.

First we need to borrow some notation from Dynkin [28]. For ease of exposition we shall revert to the real compact form of the Lie algebras, since the form of the algebra is not important for the argument. A regular subalgebra of $g$ is a subalgebra whose root system is simply a subset of the root system of $g$. A subalgebra of $g$ is called an $R$-subalgebra if it is contained in some proper regular subalgebra of $g$. Otherwise it is called an $S$-subalgebra. $S$-subalgebras have the properties that

(i) they are integral
(ii) $\dim g_1 = \dim g_0$.

Thus the condition that an $su(2)$ subalgebra by an $S$-subalgebra is sufficient for producing a W-algebra without Kac-Moody components, but as it turns out it is not necessary. In the other cases there must exist some proper regular subalgebra of $g$ which contains $su(2)$ and moreover its rank must be equal to that of $g$, since otherwise it is easy to prove that some member of the Cartan subalgebra will commute with it. The classification of all $su(2)$ subalgebras of the exceptional Lie algebras whose centraliser vanishes has been given in [28]. We reproduce these results in Table 1. The algebra $g$ is given in the first column. The second column gives the characteristic specifying the $su(2)$ embedding while the third summarises the weights (with degeneracies in parentheses) of the generating fields of the corresponding W-algebra. The next column gives the minimal subalgebra(s) of $g$ which contain the $su(2)$. A $P$ in the final column indicates that the embedding is principal. We now deal with the remaining classical examples.

(i) $su(n)$

The only $su(2)$ $S$-subalgebra of $A_n$ is given by the principal embedding. Any other candidate is a subalgebra of one of the maximal regular subalgebras of $su(n)$ and hence we can write $su(2) \subset su(p) \oplus su(q) \oplus u(1) \subset su(n)$. Furthermore it is clear that $su(2) \subset su(p) \oplus su(q)$ so that the $u(1)$ factor commutes with it. Thus no other algebra has zero centraliser.

(ii) $sp(n)$

Again the only $su(2)$ $S$-subalgebra of $sp(n)$ is the principal subalgebra. For the other subalgebras we may write

$$su(2) \subset c(p_1) \oplus c(p_2) \subset c(n) \quad p_1 + p_2 = n$$

(5.1)

If the copy of the $su(2)$ is not principal in one of the $c(p_i)$ we can decompose this further
till we have
\[ su(2) \subset c(p_1) \oplus ... \oplus c(p_i) \subset c(n) \quad \sum_j p_j = n \quad (5.2) \]

where the copies of \( su(2) \) in each \( c(p_i) \) are principal. We can decompose the adjoint representation of \( c(n) \) with respect to this direct sum of \( c(p_i) \) within which the \( su(2) \) are principal. We find that

\[
\text{adjc}(n) = (\text{adjc}(p_1) \otimes 1 \otimes ... \otimes 1) \oplus ... \oplus (1 \otimes ... \otimes 1 \otimes \text{adjc}(p_i)) \\
\oplus_{j,k} 1 \otimes ... \otimes 2p_j \otimes 1 \otimes 1 \otimes 2p_k \otimes 1 \otimes ... \otimes 1,
\]

where we have denoted the \( 2p \) dimensional representation of \( c(p_i) \) by \( 2p_i \). The adjoint representation of \( c(p_i) \) contains no singlets when decomposed with respect to a principal \( su(2) \). The \( 2p_i \) representations decomposes into a single irreducible representation of this \( su(2) \) and so the tensor product \( 2p_i \otimes ... 2p_i \) decomposes with respect to the diagonal \( su(2) \) to give \( |2p_i - 2p_j| \oplus ... \oplus 2p_i + 2p_j \). From this we can see that \( \text{adjc}(n) \) contains a trivial representation of the diagonal \( su(2) \) subalgebra if and only if \( p_j = p_k \) for some \( j \neq k \).

(iii) \( so(n) \)

The argument in this case is a little more involved. \( b_n \) again possesses no other \( su(2) \) \( S \)-subalgebras other than its principal. \( d_n \) possesses \( \text{int}[(n-2)/2] \) \( S \)-subalgebras, but in fact none of these are maximal (not even the principal) and they correspond to the embeddings
\[ su(2) \subset so(p) \oplus so(q) \subset so(2n) \]
where \( p + q = 2n \) and \( p, q \) are both odd. For our purposes it will be convenient to think of \( so(4) \) as simple. Its principal subalgebra is maximal.

Now starting with any \( su(2) \) embedding in \( so(n) \) it is straightforward to show that
\[ su(2) \subset so(p_1) \oplus so(p_2) \oplus ... \oplus so(p_i) \subset so(n) \quad (5.3) \]

where \( p_i = 3, 4, 5, 7, 9, ... \) and the copy of \( su(2) \) in each simple ideal is principal. For \( \bigoplus_j so(p_j) \) to be maximal in \( so(n) \) and hence to have trivial centraliser we need that \( n - \sum_j p_j > 2 \). Again we can decompose the adjoint representation of \( so(n) \) with respect to this direct product of principal \( su(2) \) subalgebras and we find that

\[
\text{adjso}(n) = \bigoplus_j 1 \otimes 1 \otimes ... \otimes \text{adjso}(p_i) \otimes 1 \otimes ... \otimes 1 \\
\oplus_{j,k} 1 \otimes ... \otimes p_j \otimes 1 \otimes 1 \otimes p_k \otimes 1 \otimes ... \otimes 1,
\]

where we have denoted the \( p_j \) dimensional representation of \( so(p_j) \) by \( p_j \). Again this will have a trivial representation with respect to the diagonal \( su(2) \) if and only if \( p_j = p_k \) for some \( j \neq k \). Notice that if one of the \( p_j = 4 \), the associated embedding is non-integral. This is the only example of a \( W \)-algebra with no Kac-Moody components which is not positive-definite and can be obtained in this way. The results for \( su(2) \) embeddings in classical algebras are summarised in Table 2.

It is also straightforward to extend the above analysis to search for \( W \)-algebras with no Kac-Moody components and no other spin-2 fields than that associated with the Virasoro algebra. If there exists some algebra \( h \) such that \( su(2) \subset h \subset g \), \( h \) is not simple, and
$su(2)$ is embedded diagonally in more than one of the simple ideals of $h$ then there exists more than one spin 2 field. This is because the decomposition of the direct product of $N$ copies of $su(2)$ with respect to its diagonal subalgebra contains $N$ spin-1 representations. The cases where no such $h$ exist are easy to read off from the tables, and are marked with a check-mark in Table 1. For the classical algebras, it is clear that only the principal embeddings result in only one spin-2 field.
| $G$ | Index | Characteristic | Spins | $H$ | spin-2 |
|-----|-------|----------------|-------|-----|--------|
| $E_8$ | 40 | 0 0 0 1 0 0 0 0 | 2(10),3(10),4(10),5(6),6(4) | $A_4 \oplus A_4$ | |
| $E_8$ | 88 | 1 0 0 0 1 0 0 0 | 2(4),3(4),4(5),5(3),6(6) 7(2),8(3),9 | $E_6 \oplus A_2$ | |
| $E_8$ | 120 | 0 1 0 0 1 0 0 0 | 2(3),3,4(5),5(3),6(3) 7(3),8(3),9,10(2) | $B_5 \oplus B_2$ | |
| $E_8$ | 160 | 1 1 0 0 1 0 0 0 | 2(4),3,4(2),5(3),6(3) 8(3),9,10(2) | $E_7 \oplus A_1$ | |
| $E_8$ | 184 | 0 1 0 0 1 0 1 0 | 2(3),3,4,5,6(4),7(2) 8(3),9,10,11,12(2) | $B_6 \oplus A_1$ | |
| $E_8$ | 232 | 1 1 0 0 1 0 1 0 | 2(2),3,4,5,6(2),6(3) 7(2) 8(2),9(2),10,11,12(2),14 | $E_7 \oplus A_1$ | |
| $E_8$ | 280 | 1 0 1 0 1 0 1 0 | 2(3),4,5,6(2),8(3) 9,10(2),12(2),14,15 | $B_7$ | ✓ |
| $E_8$ | 400 | 1 1 1 0 1 0 1 0 | 2(2),5,6(2),8,9,10(2) 12,14(2),15,18 | $E_7 \oplus A_1$ | |
| $E_8$ | 520 | 1 1 0 1 0 1 1 1 | 2(2),4,6,8,9,10,12(2) 14,15,18,20 | $E_8$ | ✓ |
| $E_8$ | 760 | 1 1 1 1 0 1 1 1 | 2(2),6,8,10,12,14,15,18 20,24 | $E_8$ | ✓ |
| $E_8$ | 1240 | 1 1 1 1 1 1 1 1 | 2(2),6,8,10,12,14,17,20,24,30 | $E_8$ | ✓ P |
| $E_7$ | 39 | 1 0 0 1 0 0 0 0 | 2(6),3(4),4(5),5(3),6(3) | $A_5 \oplus A_2$ | |
| $E_7$ | 63 | 1 0 0 1 0 1 0 0 | 2(4),3(2),4(3),5(2),6(4) 7,8 | $D_6 \oplus A_1$ | |
| $E_7$ | 111 | 1 1 0 1 0 1 0 0 | 2(2),3,4(2),5,6(3) 8(2),9,10 | $D_6 \oplus A_1$ | |
| $E_7$ | 159 | 1 0 1 0 1 1 1 | 2(2),4,5,6(2),8(2) 9,10,12 | $F_4 \oplus A_1$ | |

Table 1
| G     | Index   | Characteristic | Spins                                    | H  | spin-2 |
|-------|---------|----------------|------------------------------------------|----|--------|
| \(E_7\) | 231     | 1 1 1 0 1 1 1 | 1, 2, 4, 6(2), 8, 9, 10, 12, 14          | \(E_7\) | ✓      |
| \(E_7\) | 399     | 1 1 1 1 1 1 1 | 1, 2, 6, 8, 10, 12, 14, 18              | \(E_7\) | ✓ P    |
| \(E_6\) | 36      | 1 0 1 0 1 1 0 1 | 2, 3, 4(3), 5(2), 6(2)                 | \(A_5 \oplus A_1\) |       |
| \(E_6\) | 84      | 1 1 0 1 1 1 1 | 2, 3, 4, 5, 6(2), 8, 9                 | \(C_4, G_2\) | ✓      |
| \(E_6\) | 156     | 1 1 1 1 1 1 1 1 | 2, 5, 6, 8, 9, 12                      | \(F_4\) | ✓ P    |
| \(F_4\) | 12      | 0 1 \(\equiv\) 0 0 | 2, 6, 3, 4(4), 4(2)                  | \(A_3 \oplus A_1\) |       |
| \(F_4\) | 36      | 0 1 \(\equiv\) 0 1 | 2, 3, 4, 5, 6(2)                     | \(C_3 \oplus A_1\) |       |
| \(F_4\) | 60      | 1 1 \(\equiv\) 0 1 | 2, 3, 4, 6(2), 8                   | \(B_4\) | ✓      |
| \(F_4\) | 156     | 1 1 \(\equiv\) 1 1 | 2, 6, 8, 12                          | \(F_4\) | ✓ P    |
| \(G_2\) | 4       | 1 \(\equiv\) 0 | 2, 3, 3                                | \(A_1 \oplus A_1\) |       |
| \(G_2\) | 28      | 1 \(\equiv\) 1 | 2, 6                                    | \(G_2\) | ✓ P    |

Table 1 (cont)

| G     | Index | Spins                                    | H         |
|-------|-------|------------------------------------------|-----------|
| \(A_n\) | \(\frac{n(n+1)(n+2)}{6}\) | 2, 3, 4, ..., n+1 | \(A_n\) |
| \(B_n\) | \(\frac{n(n+1)(2n+1)}{3}\) | 2, 4, 6, ..., 2n | \(B_n\) |
| \(C_n\) | \(\frac{n(2n+1)(2n-1)}{3}\) | 2, 4, 6, ..., 2n | \(C_n\) |
| \(D_n\) | \(\frac{n(n-1)(2n-1)}{3}\) | 2, 4, 6, ..., 2n-2, n | \(B_n\) |
| \(C_n\) | \(\sum_i \frac{n_i(2n_i+1)(2n_i-1)}{3}\) | \(\sum_i 2, ..., 2n_i + \sum_{i>j} |n_i - n_j| + 1, ..., n_i + n_j + 1\) | \(\bigoplus_i C_{n_i}\) |
| \(so(N)\) | \(\sum_i \frac{n_i(n_i-1)(n_i+1)}{12} + \sum_i n_i = 4 2\) | \(\sum_i 2, ..., \frac{n_i-1}{2} + 2 \sum_{i:n_i=4} 2 + \sum_{i>j} |n_i - n_j| + 1, ..., n_i + n_j + 1\) | \(\bigoplus_i so(n_i)\) |

Table 2
6 Conclusions

In this paper we have found a connection between a general class of W-algebras and finite Lie algebras. A crucial role in our arguments was played by the vacuum-preserving-algebra (vpa) which is the closed subalgebra of modes which annihilate both right and left vacuua. For ‘linear’ W-algebras one finds that the vpa contains a finite subalgebra which provides a useful tool for studying the properties of theories invariant under these W-algebras. To extend this idea to more general non-linear quantum W-algebras, it became necessary to consider deformable W-algebras which are defined for a range of $c$ values in the classical limit $c \to \infty$, and the subclass of algebras which behave ‘well’ under this limit. A natural criterion which arises is that of positive-definiteness of the W-algebra, which essentially ensures that all the fields are important to the structure of the algebra. Reductive algebras are algebras which have positive-definite classical limits. As a result we were able to assign to each reductive W-algebra a finite Lie (super-)algebra and an embedding of $su(1, 1)$. The field content of the W-algebra is encoded in this embedding, with each representation of $su(1, 1)$ in the decomposition of the Lie algebra being associated to one of the Virasoro primary fields; the weight of that field being equal to $(1 + \text{the dimension})/2$. By considering the structure of the commutation relations of the W-algebra, combined with the Virasoro Ward identities, we were also able to show that this finite algebra was restricted to be of the form of a direct sum of a semi-simple algebra and an abelian algebra, namely a reductive Lie algebra. This condition places considerable restrictions on the possible field contents of W-algebras and on their commutation relations. As an example of the ideas presented, we considered the classical Poisson bracket algebras of generalised Drinfeld-Sokolov type. The analysis here held out our theoretical predictions – to each such W-algebra we were able to assign a finite Lie algebra and an $su(1, 1)$ embedding in that algebra. Conversely we used the construction to demonstrate the existence of a classical W-algebra associated with each such pair.

Our work suggests that W-algebras can be divided broadly into three categories: the reductive algebras considered in this paper, other deformable algebras, and ‘non-deformable’ algebras which are only associative for specific values of $c$. There are several questions which present themselves concerning each category.

Firstly, although we have demonstrated the existence of a classical W-algebra associated with each finite algebra and $su(1, 1)$ embedding, it is not clear as yet whether one can actually find a quantum W-algebra for each such embedding. The quantisation of these models has a lengthy history and is by no means over yet [29–32], although there seem to be good arguments in favour of their existence. If one can find quantum W-algebras which satisfy the conditions of section 2, namely having a good classical limit which is positive-definite, then the question obviously arises, are they unique? That is, to each such embedding can one uniquely ascribe a quantum W-algebra? We know of no counterexamples.

Our conditions, although they catch many of the W-algebras which have been studied to date which have proven useful in conformal field theory, still exempt many W-algebras. Indeed we present such an exception with fields of spins 2, 4, and 6. In section 5 we have presented what we hope will be a useful approach to the study of these algebras, namely automorphisms of Lie algebras which preserve a subalgebra. For the case we presented this
was a $\mathbb{Z}_2$ automorphism which preserved the $B_2$ subalgebra of $A_3$. We hope that we can extend this to other cases. Certainly the idea of dividing out by a finite group action is not new, but rather the idea that we may be able to ascribe a W-algebra uniquely to each such action, and even reconstruct the larger algebra from the smaller, is. There are good reasons to believe that the absence of a good classical limit implies strong constraints on the unitary representations of W-algebras, and we hope to return to this and other topics in the future.

As even more distant projects we can mention the idea that one may be able to show that each W-algebra which occurs for a specific set of $c$-values is simply the extension of a \textit{deformable} W-algebra by primary fields of integer spin. Thus, it may be instructive to look for ‘maximal’ deformable subalgebras of such W-algebras.

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