Energetically robust large scale dynamos driven by rapidly rotating turbulent convection

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Natural dynamos generate global scale magnetic field despite the inferred presence of small scale convectively-driven turbulence. Helicity is thought to be an important ingredient for the generation of these global scale magnetic fields. Previous work finds that as the forcing is increased the relative helicity, as measured relative to the maximum possible value, decreases and the small scales of the magnetic field become energetically dominant. However, asymptotic theory predicts that energetically robust large scale dynamos, defined as large scale dynamos in which the majority of magnetic energy is contained within the largest scales of the magnetic field, are preferred when a particular set of dynamical balances are satisfied. Here we simulate rotating convection-driven dynamos in the asymptotic regime that confirms this theory, demonstrating that energetically robust large scale dynamos can be generated consistently for strongly forced turbulent convection despite small relative helicity.

Planets and stars generate magnetic field through the dynamo process that converts the kinetic energy of fluid motion into magnetic energy [e.g. 1]. These fields are often dominated by their global (e.g. dipolar) components, whereas the underlying fluid motion that powers the dynamo is turbulent and characterized by length scales much smaller than the global scale. The Coriolis force likely plays a fundamental role in the generation of such large scale structure by leading to domain-scale correlations in the flow field that generate coherent large scale magnetic field [2,4]. This correlation can be quantified by the helicity of the fluid motion [5], defined as the dot product between the velocity field and the curl of the velocity field. However, simulations tend to find that the relative contribution of the large scale component of the dynamo generated magnetic field decreases as the fluid becomes more strongly forced and turbulent [6–9]. This loss of a predominantly large scale magnetic field is associated with a decrease in the relative (as measured to a maximum value) helicity. The fundamental question pertaining to natural systems is then how a large scale dynamo is maintained in the presence of strongly forced, small scale turbulence. Asymptotic theory relevant to rapidly rotating flows suggests that the deficit in relative helicity can be overcome by enhancing the influence of magnetic diffusion on the small convective length scale [10]. In the present work we utilize direct numerical simulations (DNS) of rapidly rotating convective turbulence to confirm these asymptotic predictions, thus helping to shed light on a major problem in dynamo theory.

Buoyancy is thought to provide the main source of power for natural dynamos. Rayleigh-Bénard convection, consisting of a layer of fluid of depth $H$ heated from below and cooled from above, is a canonical system for studying buoyancy-driven turbulence and dynamos. The buoyancy force is controlled by the non-dimensional Rayleigh number,

$$Ra = \frac{g \alpha \Delta T H^4}{\nu \kappa},$$

where $g$ is the (constant) gravitational acceleration, $\alpha$ is the thermal expansion coefficient, $\Delta T$ is the temperature difference between the top and bottom boundaries, $\nu$ is the kinematic viscosity and $\kappa$ is the thermal diffusivity. For dynamos, an important parameter that characterizes the electrical properties of the fluid is the magnetic Prandtl number, $Pr_m = \nu/\eta$, where $\eta$ is the magnetic diffusivity. Dynamos require sufficiently large flow speeds to overcome, or at least balance, the resistive effects of ohmic diffusion. Therefore, the magnetic Reynolds number, $Rm = Re Pr_m$, must exceed a threshold value so that the self-induced magnetic field does not diffuse away. Planetary and stellar dynamos are in a regime of rapidly rotating magnetized convective turbulence in which $E \ll Ro \ll 1$ and $Ra \gg Ro \gg Rm \gg 1$, which is a parameter space that is challenging to study numerically given the broad range of scales characterizing the dynamics.

Asymptotic theory can provide insight into the extreme parameter space that characterizes most natural dynamos. A brief summary of this theory is provided here to aid in the interpretation of the simulation results. For details the reader is referred to Refs. [10, 11]. In the limit $E \to 0$ and $Ro \to 0$, the dynamics of convection...
depend on the reduced Rayleigh number [e.g. 12]
\[ \widetilde{Ra} \equiv RaE^{4/3}. \] (3)

At leading asymptotic order the flow is geostrophically balanced on \( O(E^{1/3}) \) horizontal convective length scales. Thus, when the governing equations are non-dimensionalized on this small convective length scale, horizontal derivatives are \( O(E^{-1/3}) \) and vertical derivatives are \( O(1) \). Variables are decomposed into horizontal averages (mean) and fluctuating quantities such that the magnetic field and velocity field become \( \mathbf{B} = \overline{\mathbf{B}}(x,y,z,t) + \mathbf{B}'(x,y,z,t) \) and \( \mathbf{u} = \overline{\mathbf{u}}(x,y,z,t) + \mathbf{u}'(x,y,z,t) \), respectively. In the plane layer geometry studied here the mean velocity field is negligibly small and so \( \mathbf{u} \approx \mathbf{u}' \). In the asymptotic regime, the small scale magnetic Reynolds number, defined as
\[ \widetilde{Rm} = RmE^{1/3}, \] (4)
plays the key role in determining the relative sizes of the mean and fluctuating magnetic field. In particular, accessing what we refer to as the energetically robust large scale dynamo regime, in which the energy contained in the large scale (mean) component of the magnetic field is asymptotically larger than the energy contained in the fluctuating magnetic field, is achieved by balancing the emf with large scale magnetic diffusion, \( \nabla \times \overline{\mathbf{u}} \times \mathbf{B}' \sim \widetilde{Rm}^{-1} \partial^2_x \mathbf{B} \) and mean stretching with small scale magnetic diffusion, \( \mathbf{B} \cdot \nabla \mathbf{u} \sim \widetilde{Rm}^{-1} \nabla^2 \mathbf{B}' \); these two balances give, respectively,
\[ \left| \mathbf{B}' \right| / \left| \mathbf{B} \right| \sim E^{1/3} / \widetilde{Rm}, \quad \left| \mathbf{B}' \right| / \left| \mathbf{B} \right| \sim \widetilde{Rm}, \] (5)
where we have used the fact that \( |\mathbf{u}'| = O(1) \) to ensure geostrophic balance at leading order [10]. For the two relations given above to be consistent we then require
\[ \widetilde{Rm} = O \left( E^{1/6} \right). \] (6)

This relationship states that the energetically robust large scale dynamo regime requires that both \( E \) and \( \widetilde{Rm} \) are small, i.e. equation (6) is the distinguished limit that allows for the large scale magnetic field to be energetically larger than the small scale magnetic field.

Connecting the asymptotic scaling of equation (6) with \( Pm \) can be made upon noting that \( \widetilde{Rm} = Pm \widetilde{Re} \), where \( \widetilde{Re} = ReE^{1/3} \) is the small scale Reynolds number. In a given simulation, \( \widetilde{Re} \) is controlled indirectly by \( \widetilde{Ra} \). Thus, reaching small values of \( \widetilde{Rm} \) requires that \( Pm \) is reduced; since \( \widetilde{Re} = O(1) \) this implies we need \( Pm = O \left( E^{1/6} \right) \). Importantly, the robust large scale dynamo regime is not limited to small values of \( \widetilde{Ra} \) (i.e. limited to near the onset of convection), and is therefore not limited to convective states in which the helicity is maximal; further discussion is provided below.

In the present study we simulate dynamo action of a Boussinesq fluid driven by RBC as both \( E \) and \( Pm \) are varied, with the goal of reducing these parameters as small as possible to demonstrate the trend toward the energetically robust large scale dynamo regime. Reduced Rayleigh numbers up to \( \widetilde{Ra} \approx 80 \) are reached for each value of \( E \); the so-called geostrophic turbulence regime occurs for \( \widetilde{Ra} \gtrsim 40 \) [12]. For Ekman numbers within the range \( 10^{-6} \leq E \leq 10^{-4} \) we use \( Pm = 1 \), and for \( E \leq 10^{-6} \) we reduce the magnetic Prandtl number down to \( Pm = 0.05 \) at the smallest Ekman number considered, \( E \approx 10^{-8} \). The simulations use a de-aliased pseudospectral method in which all flow variables are expanded as Chebyshev polynomials in the vertical dimension and Fourier series in the horizontal dimensions. A third order accurate implicit-explicit time-stepping scheme is used [e.g. 13 14]. In all cases shown the horizontal dimension of the simulation domain is scaled such that 10 critical wavelengths are present. The boundary conditions are stress-free, isothermal and the magnetic field is required to be pure vertical at the top and bottom boundaries. The thermal Prandtl number is fixed at \( Pr = \nu / \kappa = 1 \).

The helicity is denoted by \( \overline{\mathcal{H}}(z) = \overline{u'} \cdot \mathcal{C} \), where \( \mathcal{C} = \nabla \times \mathbf{u}' \) is the vorticity. Helicity can stretch and twist magnetic field lines collectively over the horizontal plane, and therefore induce large scale magnetic field. Fig. 1(a) shows the rms value of the helicity for all simulations. All combinations of \( E \) and \( Pm \) follow similar behavior with increasing \( \widetilde{Ra} \) and larger values of helicity are observed for decreasing Ekman numbers, whereas \( Pm \) tends to have only a weak influence on \( \overline{\mathcal{H}} \). In our DNS we non-dimensionalize the equations using the depth \( H \) and speed \( \nu / H \); adapting this scaling to the asymptotic theory gives \( |\mathbf{u}'| = O(E^{-1/3}) \) and \( |\mathcal{C}'| = O(E^{-2/3}) \) so that \( |\overline{\mathcal{H}}| = O(E^{-1}) \) as \( E \to 0 \). Fig. 1(b) shows the asymptotically rescaled rms helicity versus \( \widetilde{Ra} \); the collapse of the data shows that these simulations are in a quasi-geostrophic dynamical state [15].

It is further helpful to define the relative helicity \( \overline{\mathcal{H}_r} = \overline{\mathcal{H}} / (u \zeta) \), where \( u \) and \( \zeta \) are rms values, such that maximally helical flows are characterized by \( |\overline{\mathcal{H}_r}| = O(1) \). In agreement with previous studies, Fig. 2(a) shows that \( |\overline{\mathcal{H}_r}| \) becomes small as \( \widetilde{Ra} \) increases for all of the simulations, yet remains finite [e.g. 16 18]. These results suggest that natural dynamos, for which \( \widetilde{Ra} \gg 1 \), are also characterized by small values of the relative helicity. However, reducing the magnetic Prandtl number shows that despite the small relative helicity, large scale dynamo action remains possible, as predicted by theory. Figs. 2(b)-(e) compare flow visualizations for \( Pm = 0.3 \) with \( (c,e) \) of small value \( \widetilde{Ra} \). As indicated by the black arrow in Fig. 2(a), both cases are characterized by similar values of \( \overline{\mathcal{H}_r} \). The renderings of the vertical component of the vorticity shown in (b) and (c) provide visual evidence that both simulations are characterized by dynamically similar turbulent flows. Whereas the large scale Reynolds numbers are \( \widetilde{Re} \approx 4.5 \times 10^4 \) and \( \widetilde{Re} \approx 1.1 \times 10^4 \) for
FIG. 1. Helicity (rms) for all simulations: (a) $\overline{H}$ versus $Ra$; (b) asymptotically rescaled rms helicity $E\overline{H}$ versus reduced Rayleigh number, $Ra = RaE^{4/3}$. For the symbols the colors denote different values of $Pm$ and the shapes denote different values of $E$.

$E = 10^{-7}$ and $E = 10^{-8}$, respectively, the corresponding small scale Reynolds numbers for both cases are $Re \approx 21$ and $Re \approx 24$, which explains why the two flows look similar despite having large scale Reynolds numbers that are more than a factor of two different. The corresponding $x$-component of the magnetic field vector is shown for the two cases in Figs. 2(d) and (e) where we find a more obvious coherent large scale component for the $E = 10^{-8}$ ($Pm = 0.05$) case.

The relative size of the large scale magnetic field can be quantified by computing $\overline{M}/M$, where $\overline{M}$ is the magnetic energy of the mean magnetic field and $M$ is the total magnetic energy. Fig. 3(a) shows this mean energy...
fraction versus \( \tilde{R}a \). We find that smaller values of \( Pm \) typically yield larger values of \( \overline{M}/M \) for a given value of \( Ra \). For a fixed value of \( Pm \) the mean energy fraction decreases with increasing \( \tilde{R}a \). Conversely, for a fixed value of \( \tilde{R}a \), decreasing \( Pm \) typically yields larger values of \( \overline{M}/M \). Fig. 3(b) shows that a collapse of the data occurs when \( \overline{M}/M \) is plotted versus \( \tilde{R}m \). In particular, we find energetically robust large scale dynamos, as characterized by \( \overline{M}/M \to 1 \), only when \( \tilde{R}m \lesssim O(1) \).

We note that the data shown in Fig. 3(b) also demonstrates that dynamo action is achieved for smaller values of \( \tilde{R}m \) as both \( E \) and \( Pm \) are reduced. This reduction in the value of \( \tilde{R}m \) needed for dynamo action as the parameter values are made more extreme is due to the fact that these dynamos are intrinsically multiscale and anisotropic; convective motions take place over the depth \( H \) of the system, yet the small \( O(HE^{1/2}) \) horizontal length scale of the convection is crucial to these dynamics. Although \( \tilde{R}m \) is becoming smaller as both \( E \) and \( Pm \) are reduced, the large scale magnetic Reynolds number scales as \( \tilde{R}m = O(E^{-1/6}) \) and therefore becomes large even as \( \tilde{R}m = O(E^{1/6}) \) becomes small as \( E \to 0 \). Again, this property is a result of the scale separation that characterizes these rapidly rotating flows.

The relationship between the mean energy fraction and relative helicity is shown in Fig. 4. We find that robust large scale dynamo action is achieved at smaller values of the relative helicity as both \( E \) and \( Pm \) are reduced. These findings show that small relative helicity by itself does not imply that large scale dynamo action is not achievable, and that a deficit of helicity can be offset by enhancing the influence of magnetic diffusion on the small convective length scale.

In comparison to previous studies of rotating dynamos in the plane parallel geometry, the present investigation extends the parameter space to smaller Ekman numbers and smaller magnetic Prandtl numbers. Refs. 19, 20 find that so-called large scale vortices (LSVs) generated by the inverse cascade mechanism can play an important role in sustaining large scale magnetic field. However, none of the cases reported in Refs. 19, 20 show evidence of what we refer to as the energetically robust large scale dynamo regime. This finding is consistent with the trends observed in our simulations. Moreover, no general conclusion on the importance of LSVs in our simulations could be reached; whereas some cases show relatively strong LSVs, in many of the cases reported they are not energetically dominant, similar to the findings of Ref. 21. Ref. 8 found transitions in dynamo behavior for \( Pm \geq 1 \) when \( \tilde{R}m \gtrsim 13 \), though we do not observe a similar behavior that may be due to the smaller Ekman and Rossby numbers employed here.

The plane parallel geometry used in the present investigation enables a systematic exploration of parameter space with \( E \ll 1 \) and \( Pm < 1 \). This geometry contains many of the same basic physical ingredients in natural systems, but also lacks features that are known to be important in more realistic spherical geometries. Systematic variation of the relevant input parameters \( (E, Pm, Ra) \) cannot currently be done at extreme values in spherical simulations due to the computational cost required [e.g. 22]. Thus, how the present plane layer theory relates to spherical dynamos remains an open question. However, the present findings demonstrate that reducing both \( E \) and \( Pm \) to more realistic values changes the behavior of the dynamo in a fundamental manner. It will therefore be of interest to determine the influence of smaller values of \( E \) and \( Pm \) on transitions between dipo-
lar and multipolar dynamos observed in spherical simulations [e.g. 6].

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