Critical phenomena of collapsing massless scalar wave packets

Anzhong Wang *
Departamento de Astrofísica, Observatório Nacional – CNPq,
Rua General José Cristino 77, São Cristóvão, 20921-400 Rio de Janeiro – RJ, Brazil

Henrique P. de Oliveira
Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro,
Rua São Francisco Xavier 524, Maracanã, 20550-013 Rio de Janeiro – RJ, Brazil

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Abstract

An analytical model that represents the collapse of a massless scalar wave packet with continuous self-similarity is constructed, and critical phenomena are found. In the supercritical case, the mass of black holes is finite and has the form $M \propto (p - p^*)^\gamma$, with $\gamma = 1/2$.

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*e-mail address: wang@on.br; a.wang@vmesa.uerj.br
I. Introduction

Recently, Choptuik [1] studied the collapse of a massless scalar field with spherical symmetry, and found the following intriguing features: Let the initial distribution of the massless scalar field be parametrized smoothly by a parameter $p$ that characterizes the strength of the initial conditions, such that the collapse of the scalar field with the initial data $p > p^*$ forms a black hole, while the one with $p < p^*$ does not. Then, it was found that: (i) the critical solution with $p = p^*$ is universal in the sense that in all the one-parameter families of the solutions considered it approaches an identical spacetime; (ii) the critical solution has a discrete self-similarity (DSS); (iii) near the critical solution (but with $p > p^*$), the black hole mass is given by

$$M_{BH} = K(p - p^*)^\gamma,$$

where $K$ is a family-dependent constant, but $\gamma$ is an apparently universal scaling exponent, which has been numerically determined as $\gamma \approx 0.37$. These phenomena were soon found also in the collapse of axisymmetric gravitational waves [2] as well as in the one of radiation fluid [3]. Therefore, it seems that the phenomena are not due to the particular choice of the matter fields but rather are generic features of General Relativity. Further numerical evidences to support this conclusion are given in [4].

Parallelly to the above numerical investigations, there have been analyt-
ical efforts to understand the physics behind these phenomena [5, 6, 7, 8, 9, 10]. While the universality of the critical solution and its self-similarity (echoing) have been found in most cases considered, there exists dispute over the universal scaling exponent $\gamma$. Maison [8] showed that $\gamma$ is matter-dependent. For the collapse of the perfect fluid with the equation of state $P = k\rho$, it strongly depends on $k$, where $P$ and $\rho$ are respectively the pressure and energy density of the fluid, and $k$ is a constant. This dependence is further shown in [10]. The same conclusion was also obtained numerically by Eardley and co-workers [11]. Thus, one might expect that $\gamma$ is universal only within a particular family of matter fields. However, even in this sense the analytical results are not consistent with the numerical ones. In particular, it was shown [5, 6] that for the massless scalar field the exponent $\gamma$ is 0.5, instead of its numerical value $\gamma \approx 0.37$. Moreover, Maison [8] showed that the value $k = 0.88$ is the maximal one for which a regular self-similar solution could be found for the perfect fluid. As we know, a massless scalar field is energetically equivalent to a perfect fluid with $k = 1$ [12], thus, Maison’s results seem also in conflict with the ones obtained in [5, 6].

It should be noted that the model considered in [5, 6], which will be referred to as the BONT model, is not asymptotically flat, and the exponent $\gamma$ is obtained in a rather unusual way. Moreover, except for the one of Gundlach [9], all the analytic models, including the BONT model, have continuous
self-similarity (CSS), in contrast to the original model of Choptuik that has DSS \[\text{(I)}\]. Thus, it is not clear whether the difference in \(\gamma\) obtained above is due to the different self-similarities or due to the non-asymptotic flatness of the spacetime. If it is because of the former, we will have a new classification to the critical behavior, and many new solutions that represent critical phenomena with different similarities need to be found. To resolve the above problem, in this paper we shall make a “surgery” to the BONT model \[\text{(I)}\]. That is, we shall cut the spacetime along a null hypersurface and then join it to an out-going Vaidya solution, so the resulted spacetime is asymptotically flat and has a finite mass. Clearly, such obtained solution will represent the collapse of a massless scalar wave packet, which is radiating as it collapses. This model is more realistic and more comparable with the one of Choptuik but with different self-similarities. Specifically, the paper is organized as follows: In Sec. II, we briefly review the main properties of the BONT model, while in Sec. III we cut the BONT spacetime along a null hypersurface and then join it with an out-going Vaidya solution. In Sec. IV we consider a particular case of the solutions obtained in Sec. III, which represents the collapse of a scalar wave packet. Critical phenomena are found with the exponent \(\gamma = 1/2\). Finally, the paper is closed by Sec. V, where our main conclusions are presented.
II. The BONT model

The BONT model is described by the solutions

\[ ds^2 = -G(u,v)dudv + r^2(u,v)d^2\Omega, \]

where \( d^2\Omega \equiv d\theta^2 + \sin^2 \theta d\varphi^2 \), and the metric coefficients are given by

\[ r(u,v) = \frac{1}{2} \left[ u^2 - 2uv + 4b_2v^2 \right]^{1/2} H(v) + \frac{1}{2} \left[ 2(a(v) - a(0)) - u \right] H(-v), \]

\[ G(u,v) = H(v) + 2a'(v)H(-v), \]

where \( b_2 \) is an arbitrary constant, and \( H(x) \) denotes the Heaviside function, which is one for \( x \geq 0 \) and zero for \( x < 0 \). A prime denotes the ordinary differentiation, and \( a(v) \) is an arbitrary function subject to \( a'(v) > 0 \) and \( a'(0) = 1/2 \). One can show that the hypersurface \( v = 0 \) is free of any matter and represents a boundary surface \[ \text{[14]} \]. The corresponding massless scalar field is given by

\[ \phi = \pm \frac{1}{\sqrt{2}} \ln \left| \frac{(u - v) - \sqrt{1 - 4b_2v}}{(u - v) + \sqrt{1 - 4b_2v}} \right| H(v). \]

Note the slight difference in the notations used here and the ones used in \[ \text{[5, 6]} \].

From the above expressions one can show that the spacetime is Minkowski in the region \( u < 0, v < 0 \), while in the region \( u < 0, v > 0 \) it represents a critical phenomena. This class of solutions was first found in \[ \text{[13]} \], but their physical interpretation was first given in \[ \text{[5, 6]} \]. The Roberts solutions are continuous self-similar, since they possess the Killing vector, \( X = u\partial_u + v\partial_v \) with the property \( L_Xg_{\mu\nu} = 2g_{\mu\nu} \), where \( L_X \) denotes the Lie derivative with respect to \( X \).
collapsing massless scalar wave. When \( b_2 < 0 \), the scalar wave collapses into a spacetime singularity on the hypersurface \( u = -[\sqrt{1 - 4b_2} - 1]v \), which is preceded by an apparent horizon at \( u = 4b_2v \). Thus, the corresponding solutions represent the formation of black holes and are supercritical. When \( b_2 = 0 \), the singularity coincides with the apparent horizon on \( u = 0 \) and becomes null. This solution is critical, which separates the supercritical solutions from the subcritical ones. The subcritical solutions are those with \( 0 < b_2 < 1/4 \). It can be shown that in the latter case the scalar wave first collapses and then disperses into infinity, without forming black holes but leaving behind a Minkowski region \( u,v > 0 \), in which the metric takes the form of Eq.(4) with

\[
G(u,v) = 4\sqrt{b_2} b'(u), \quad r = \sqrt{b_2} v - [b(u) - b(0)],
\]

\[
\phi(u,v) = \pm \frac{1}{\sqrt{2}} \ln \left[ \frac{1 + \sqrt{1 - 4b_2}}{1 - \sqrt{1 - 4b_2}} \right], \quad (u,v > 0)
\]

where \( b(u) \) is an arbitrary function, subject to \( b'(u) > 0, b'(0) = 1/(4\sqrt{b_2}) \).

One can show that the hypersurface \( u = 0, v > 0 \) is also a boundary surface.

In the region \( u < 0, v > 0 \), the local mass of the scalar field is given by

\[
m(u,v) = \frac{r}{2} \left( 1 - r_{\alpha\beta} g^{\alpha\beta} \right) = -\frac{(1 - 4b_2)uv}{8r}, \quad (5)
\]

where \( r \) is given by Eq.(2). Clearly, on the apparent horizon \( u = 4b_2v \) the mass becomes unbounded as \( v \to +\infty \) for \( b_2 < 0 \). That is, the spacetime fails to be asymptotically flat for the supercritical case. As a result, the total
mass of the black hole can not be written in a power-law form in terms of initial data, whereby the exponent $\gamma$ can be read out. For more details, we refer the readers to [3, 4].

III. Matching the Roberts solutions to the outgoing Vaidya dust solution

To circle the above problem, we shall cut the spacetime along the hypersurface, say, $v = v_0 > 0$, and keep the region $v \leq v_0$, while the region $v \geq v_0$ will be replaced by that of out-going dust Vaidya solution [13]. To do the matching, following Barrabés and Israel [16], we first write the Roberts solutions in the form

$$ds^2_{-} = -e^{\psi_-} dv(f_- e^{\psi_-} dv - 2dr) + r^2 d^2 \Omega,$$  \hspace{1cm} (6)

where $v$ is the Eddington advanced time, and for Roberts’ solutions (2) in the region $u < 0, v > 0$, the metric coefficients are given by

$$f_- = \frac{[(1 - 4b_2)v^2 + 4r^2]^{1/2}}{4r^2} \left\{ \left[(1 - 4b_2)v^2 + 4r^2\right]^{1/2} - (1 - 4b_2)v \right\},$$  \hspace{1cm} (7)

$$e^{\psi_-} = \frac{2r}{[(1 - 4b_2)v^2 + 4r^2]^{1/2}}.$$

Now we restrict Eq.(7) valid only in the region $u < 0, 0 \leq v \leq v_0$. Then, the normal to the hypersurface $\Phi^- = v - v_0 = 0$ is $n_{\mu}^- = \alpha^{-1} \partial_{\mu} \Phi^- = \alpha^{-1} \delta_{\mu}^v$, where $\alpha$ is a negative function. From $n_{\mu}^-$ we can introduce a “transverse” null vector $N_{\mu}^-$ by requiring $N_{\lambda}^- N^{-\lambda} = 0$, and $N_{\lambda}^- n^{-\lambda} = -1$. Without loss
of generality, we assume that $N_\mu^-$ takes the form $N_\mu^- = N_\nu^- \delta_\mu^\nu + N_r^- \delta_\mu^r$, and choose the arbitrary function $\alpha$ as $\alpha = -\exp\{-\psi_\cdot\}$. Then, it is easy to show that $N_\mu^-$ is given by

$$N_\mu^- = -\frac{f}{2} e^{\psi - \delta_\mu^v + \delta_\mu^r}. \quad (8)$$

Choosing the coordinates $r, \theta, \varphi$ as the three intrinsic coordinates $\xi^a \equiv (r, \theta, \varphi), \ (a = 1, 2, 3)$ on the hypersurface $v = v_0$, we find

$$e_{(1)}^- = \delta_r^\mu, \quad e_{(2)}^- = \delta_\theta^\mu, \quad e_{(3)}^- = \delta_\varphi^\mu, \quad (9)$$

where $e_{(a)}^- = \partial x_-^\mu / \partial \xi^a$. Then, it can be shown that the “transverse” extrinsic curvature, defined by [16]

$$\mathcal{R}_{ab} = -N_{\mu} e_{(b)}^\nu \left( \nabla_{\nu} e_{(a)}^\mu \right), \quad (10)$$

takes the form

$$\mathcal{R}_{ab}^- = \text{diag.}\left\{ -\frac{\partial \psi_\cdot}{\partial r}, \ \frac{rf_-}{2}, \ \frac{rf_-}{2} \sin^2 \theta \right\}. \quad (11)$$

Note that in calculating the above equation, we have not used the particular expressions (7) for the functions $\psi_\cdot$ and $f_-$. Thus, it is valid for the general case.

On the other hand, the out-going Vaidya solution [15] can be written in the form

$$ds_+^2 = -e^{\psi_+} dU (f_+ e^{\psi_+} dU + 2 dr) + r^2 d\Omega, \quad (12)$$

8
where

\[ f_+ = 1 - \frac{2m(U)}{r}, \quad \psi_+ = 0, \quad (13) \]

and \( U \) is the Eddington retarded time, which is in general the function of \( u \) appearing in Eq.(1), and \( m(U) \) is the local mass of the out-going Vaidya dust. The corresponding energy-momentum tensor is given by

\[ T_{\mu\nu}^+ = -\frac{2}{r^2} \frac{dm(U)}{dU} \delta^U \delta^U. \quad (14) \]

In the following, we shall take metric (12) as valid only in the region \( u < 0, v \geq v_0 \). To have our results more applicable, for the moment we shall not restrict ourselves to the particular solution (13). The hypersurface \( v = v_0 \) in the \((U, r)\)-coordinates can be written as \( \Phi^+ = U - U_0(r) = 0 \), where \( U_0(r) \) is a solution of the equation

\[ \frac{dU_0}{dr} = -\frac{2}{f_+} e^{-\psi_+}, \quad (v = v_0). \quad (15) \]

Then, the normal to the surface is given by

\[ n_\mu^+ = \beta^{-1} \partial_\mu \Phi^+ = \beta^{-1} \left( \delta_\mu^U + \frac{2}{f_+} e^{-\psi_+} \delta_\mu^r \right), \]

where \( \beta \) is a negative otherwise arbitrary function. From \( n_\mu^+ \) we can also introduce the “transverse” null vector \( N_\mu^+ \), by requiring \( N_\lambda^+ N^{+\lambda} = 0 \), and \( N_\lambda^+ n^{+\lambda} = -1 \). It can be shown that it takes the form

\[ N_\mu^+ = \frac{\beta f_+}{2} e^{2\psi_+} \delta_\mu^U. \quad (16) \]
On the other hand, we also have

\[ e^+_{(1)} = -\frac{2}{f_+}e^{-\psi_+} \delta U^0 + \delta r^0, \quad e^+_{(2)} = \delta \theta^0, \quad e^+_{(3)} = \delta \phi^0, \quad (17) \]

where \( e^+_{(a)} \equiv \partial x^a_+ / \partial \xi^a \). To be sure that the two “transverse” vectors \( N^\pm_\mu \)
defined in the two faces of the hypersurface \( v = v_0 \) represent the same vector,
we need to impose the condition

\[ N^+_\lambda e^+_{(a)} \big|_{v = v_0} = N^-_\lambda e^+_{(a)} \big|_{v = v_0}, \]

which requires that the function \( \beta \) has to be \( \beta = -e^{\psi_+} \). Once \( N^+_\lambda \)
and \( e^+_{(a)} \) are given, using Eq. (10) we can calculate the corresponding “transverse”
etrinsic curvature, which in the present case takes the form

\[ \mathcal{R}^+_ab = \text{diag.} \left\{ -\frac{2e^{-\psi_+}}{f_+^2} \frac{\partial f_+}{\partial U}, \quad \frac{rf_+}{2}, \quad \frac{rf_+}{2} \sin^2 \theta \right\}. \quad (18) \]

Then, from Eqs. (11) and (18), we find that

\[ \gamma_{ab} = 2 \left( \mathcal{R}^+_ab - \mathcal{R}^-_{ab} \right) = \frac{2e^{-\psi_+}}{f_+^2} \frac{\partial f_+}{\partial U} \delta^r_a \delta^r_b + r(f_+ - f_-) \left( \delta^\theta_a \delta^\theta_b + \sin^2 \theta \delta^\phi_a \delta^\phi_b \right). \quad (19) \]

Once \( \gamma_{ab} \) is given, using the formula \[ (16) \]

\[ \tau^{ab} = -S^{ab} = \frac{1}{16\pi} \left( g_s^{ac} l^b l^d + g_s^{bd} l^a l^c - g_s^{ab} l^c l^d - g_s^{cd} l^a l^b \right) \gamma_{cd}, \quad (20) \]

we can calculate the surface energy-momentum tensor \( \tau^{ab} \) on the null hyper-
surface \( v = v_0 \), which now can be written as

\[ \tau^{ab} = \sigma l^a l^b + P g_s^{ab}, \quad (21) \]
\[ \sigma = \frac{f_+ - f_-}{8\pi r}, \]
\[ P = \frac{1}{8\pi} \left( \frac{2e^{-\psi^+}}{f_+^2} \frac{\partial f_+}{\partial U} - \frac{\partial \psi_+}{\partial r} \right), \]  
(22)

and

\[ g^{ab}_* = r^{-2} \left( \delta^a_\theta \delta^b_\theta + \sin^{-2} \theta \delta^a_\varphi \delta^b_\varphi \right), \]
\[ l^a = \delta_r^a, \quad l^a l_b = 0. \]  
(23)

The function \( \sigma \) represents the surface energy density of the null shell, and
\( P \) the pressures in the \( \theta \)- and \( \varphi \)-directions. Note that in [16] the case of a
null shell was also considered. But, there they used the same null coordinate
in both sides of the shell, while here we use the retarded null coordinate in
one side of the shell and the advanced null coordinate in the other side [cf.
Eqs. (13) and (12)].

For the particular solutions given by Eqs. (7) and (13), Eq. (22) yields

\[ \sigma = \frac{1}{4\pi r^2} \left\{ M(r) - \frac{(1 - 4b_2)v_0}{8r} \left[ \sqrt{(1 - 4b_2)v_0^2 + 4r^2} - v_0 \right] \right\}, \]
\[ P = \frac{1}{4\pi (r - 2M)} \left\{ \frac{dM(r)}{dr} - \frac{(1 - 4b_2)v_0^2(r - 2M)}{2r[(1 - 4b_2)v_0^2 + 4r^2]} \right\}, \]  
(24)

where

\[ M(r) \equiv m(U)|_{U=U_0(r)}, \]  
(25)

and \( U_0(r) \) is a solution of Eq. (13).
IV. Critical phenomena for the case $P = 0$

To study the general shell given by Eq. (24), it is found very complicate. In this section, we shall consider the case where $P = 0$, i.e.,

$$
\frac{dM(r)}{dr} = \frac{(1 - 4b_2)v_0^2(r - 2M)}{2r[(1 - 4b_2)v_0^2 + 4r^2]}.
$$

Integrating the above equation, we find that

$$
M(r) = \frac{1}{r} \left\{ p\sqrt{4p^*r^2 + r^2 - 2p^*} \right\},
$$

where $p$ is the integration constant, and

$$
p^* \equiv \frac{\sqrt{1 - 4b_2}}{4}v_0.
$$

At the past null infinity ($r \to +\infty$), Eq. (27) shows that

$$
M(r \to +\infty) = p.
$$

That is, the parameter $p$ represents the total initial mass of the massless scalar wave packet and the null shell, with which they collapse. As $r \to 0^+$, $M(r)$ has the following asymptotic behavior

$$
M(r) \to \begin{cases} 
+\infty, & p > p^*, \\
0, & p = p^*, \\
-\infty, & p < p^*.
\end{cases}
$$

On the other hand, it is well-known that the apparent horizon at $r = 2M(r)$ of the out-going Vaidya solution always coincides with its future event horizon. Thus, by comparing the mass $M(r)$ with $r/2$ we can tell whether the
collapse forms a black hole or not,

\[ M(r) - \frac{r}{2} = \left( \frac{4p^*^2 + r^2}{4r^2[p + (4p^*^2 + r^2)]} \right)^{1/2} \left[ 4(p^2 - p^*^2) - r^2 \right]. \quad (31) \]

Clearly, only when \( p > p^* \), the scalar field and the null shell will collapse inside the event horizon at

\[ r_{AH} = 2\sqrt{p^2 - p^*^2}. \quad (32) \]

When \( p = p^* \), \( M(r) = r/2 \) is possible only at the origin, \( r = 0 \), where a zero-mass singularity is formed. Thus, the solution with \( p = p^* \) represents the critical solution that separates the supercritical solutions (\( p > p^* \)) from the subcritical ones (\( p < p^* \)). In the subcritical case, \( M(r) \) is always greater than \( r/2 \), and the collapse never forms a black hole [cf. Fig.1]. In the latter case, the region \( u, v > 0 \) should be replaced by the Minkowski solution (4). As shown in the last section, the matching across the hypersurface \( u = 0, \ 0 \leq v \leq v_0 \) is smooth, i.e., no matter appears on it. To show that it is also the case on the hypersurface \( u = 0, \ v \geq v_0 \), which separates the Vaidya solution (13) from the Minkowski (4), we first make the coordinate transformation \( U = U(u) \), and then write the metric (12) in terms of \( u \). Using the results obtained in (16), one can show that to have a smooth matching we have to impose the condition

\[ U'(0) = \frac{1}{\sqrt{4b_2}}, \quad M(r) \big|_{u=0} = 0. \]
Clearly, by properly choosing the function-dependence of $U$ on $u$, the first condition can be always satisfied. On the other hand, from Eqs. (2) and (27) one can show that the last condition is also satisfied identically. Therefore, the matching of the Vaidya solution to the Minkowski one across the hypersurface $u = 0, v \geq v_0$ is always possible for $p < p^*$. The corresponding Penrose diagram for each of the three cases are shown in Fig. 2.

On the apparent horizon $r = r_{AH}$, the total mass of the scalar wave packet and the shell is

$$M_{BH} = \frac{r_{AH}}{2} = K(p - p^*)^{1/2},$$

where $K \equiv (p + p^*)^{1/2}$. The above expression shows that the black hole mass takes a power-law form with its exponent $\gamma$ being equal to 0.5. This is exactly the same value as that obtained in [5, 6]. This result seems a little bit surprising, since our model is different from the one of [5, 6]. In particular, in our case the collapse consists of two parts, one is the massless scalar wave packet and the other is the null shell, while in [5, 6] only a collapsing scalar wave exists. However, a more detailed investigation shows that the existence of the shell does not affect our main conclusions. In fact, by properly choosing the parameters in our model the shell can be made completely disappear, without affecting the main properties of the collapse.

To show this, let us write the total mass $M(r)$ as

$$M(r) = M_\phi + M_{shell},$$

14
where \( M_\phi \) denotes the total mass of the scalar wave packet, and \( M_{shell} \) the total mass of the shell, given respectively by

\[
M_\phi = \frac{2p^*}{v_0^2} \left\{ 2\sqrt{4p^*r^2 + r^2} - v_0 \right\},
\]

\[
M_{shell} = \frac{v_0p - 4p^*}{v_0^2} \left\{ \frac{4p^* + r^2}{r^2} \right\}^{1/2}.
\]  

(35)

Clearly, if we choose the parameters such that

\[
v_0 = \frac{4p^*}{p},
\]

(36)

the null shell disappears. Moreover, using Eqs. (28) and (36), we find

\[
p - p^* = \left[ \frac{4p^*}{p + p^*} \right] (-b_2).
\]  

(37)

Therefore, when the null shell is absent, the massless scalar wave packet will collapse to form a black hole for \( b_2 < 0 \), and first collapses and then disperses to infinity, without forming a black hole for \( 1/4 > b_2 > 0 \)

V. Concluding remarks

In this paper, by “cut-paste” method, we have constructed a physically more realistic model for the collapse of a massless scalar wave packet, which is usually accompanied by a collapsing null dust shell. Critical phenomena are also found in these solutions, and the mass of the black holes is finite and takes the form \( M \propto (p - p^*)^{1.5} \). In our model, the exponent \( \gamma \) is 0.5,
which is the same as that found in [5, 6], but different from the numerical one of Choptuik, which is 0.37. However, the results do not contradict and rather show the fact that $\gamma$ is not only matter-dependent but also symmetry-dependent. In Choptuik’s model, the critical solution has DSS, while in [5, 6] and ours it has CSS.

On the other hand, quite recently Choptuik, Chmaj, and Bizoń [17] studied the collapse of a Yang-Mills field, and found two distinct critical solutions at the threshold of black hole formation. In one case, the critical solution has DSS, and the mass of black holes for the supercritical solutions takes a power-law form with the exponent $\gamma \approx 0.20$. This class of solutions was referred to as Type II solutions. In the other case, the critical solution is the static $n = 1$ Bartnik-McKinnon sphaleron [18], and the formation of black holes always turns on at finite mass. This class of solutions was referred to as Type I solutions.

Summarizing all the above results, the following seems to emerge: The solutions of gravitational collapse can be at least divided into three different types, a) Type II.A solutions. In these solutions, the mass of black holes always takes the form $M \propto (p-p^*)^{\gamma_A}$. b) Type II.B solutions. In this class of solutions, the black hole mass also takes the power-law form, $M \propto (p-p^*)^{\gamma_B}$, but the only difference to Type II.A is that the critical solution of Type II.A has DSS, while the one of Type II.B has CSS. Because of this difference, the
exponent $\gamma$ is in general also different. c) Type I solutions. These solutions have no self-similarities, neither DSS nor CSS, and the formation of black holes always turns on at finite mass. If the above classification is universal, a natural question is that: Do these three types of solutions saturate all the solutions that represent the formation of black holes in gravitational collapse?

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Figure Captions

Fig. 1 The qualitative behavior of the total mass $M(r)$ defined by Eq.(27) in the text for the three different cases: (a) The supercritical case ($p > p^*$); (b) the critical case ($p = p^*$); and (c) the subcritical case ($p < p^*$).

Fig. 2 The Penrose diagram of the solutions with $P = 0$ for the three different cases: (a) The supercritical case; (b) the critical case; and (c) the subcritical case. Double lines represent spacetime singularities.
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