Derivative Formula and Harnack Inequality for Degenerate Functional SDEs

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September 20, 2011

Abstract

By constructing successful couplings, the derivative formula, gradient estimates and Harnack inequalities are established for the semigroup associated with a class of degenerate functional stochastic differential equations.

AMS subject Classification: 60H10, 47G20.

Keywords: Coupling, derivative formula, gradient estimate, Harnack inequality, functional stochastic differential equation.

1 Introduction

In recent years, the coupling argument developed in \cite{1} for establishing dimension-free Harnack inequality in the sense of \cite{13} has been intensively applied to the study of Markov semigroups associated with a number of stochastic (partial) differential equations, see e.g. \cite{3, 4, 6, 7, 8, 9, 14, 16, 18, 19, 20, 22} and references within. In particular, the Harnack inequalities have been established in \cite{4, 19} for a class of non-degenerate functional stochastic differential equations (SDEs), while the (Bismut-Elworthy-Li type) derivative formula and applications have been investigated in \cite{5} for a class of degenerate SDEs (see also \cite{21, 23} for the study by using Malliavin calculus). The aim of this paper is to establish the derivative formula and (log-)Harnack inequalities for degenerate functional SDEs. The derivative formula implies explicit gradient estimates of the associated semigroup, while a number of

\textsuperscript{*}Supported in part by SRFDP and the Fundamental Research Funds for the Central Universities.
applications of the (log-)Harnack inequalities have been summarized in [17, §4.2] on heat kernel estimates, entropy-cost inequalities, characterizations of invariant measures and contractivity properties of the semigroup.

Let \( m \in \mathbb{Z}_+ \) and \( d \in \mathbb{N} \). Denote \( \mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d \), where \( \mathbb{R}^m = \{0\} \) when \( m = 0 \). For \( r_0 > 0 \), let \( \mathcal{C} := C([-r_0,0];\mathbb{R}^{m+d}) \) be the space of continuous functions from \([-r_0,0]\) into \( \mathbb{R}^{m+d} \), which is a Banach space with the uniform norm \( \| \cdot \|_\infty \). Consider the following functional SDE on \( \mathbb{R}^{m+d} \):

\[
\begin{align*}
\text{(E1)} \quad & \begin{cases} 
\text{d}X(t) = \{AX(t) + MY(t)\} \text{d}t, \\
\text{d}Y(t) = \{Z(X(t),Y(t)) + b(X_t,Y_t)\} \text{d}t + \sigma \text{d}B(t), 
\end{cases}
\end{align*}
\]

where \( B(t) \) is a \( d \)-dimensional Brownian motion, \( \sigma \) is an invertible \( d \times d \)-matrix, \( A \) is an \( m \times m \)-matrix, \( M \) is an \( m \times d \)-matrix, \( Z : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d \) and \( b : \mathcal{C} \to \mathbb{R}^d \) are locally Lipschitz continuous (i.e. Lipschitzian on compact sets), \( (X_t,Y_t)_{t \geq 0} \) is a process on \( \mathcal{C} \) with \( (X_t,Y_t)(\theta) := (X(t + \theta),Y(t + \theta)), \theta \in [-r_0,0] \). We assume that there exists an integer number \( 0 \leq k \leq m - 1 \) such that

\[
\begin{align*}
\text{(R1)} \quad & \text{Rank}[M,AM,\cdots,A^kM] = m.
\end{align*}
\]

When \( m = 0 \) this condition automatically holds by convention. Note that when \( m \geq 1 \), this rank condition holds for some \( k > m - 1 \) if and only if it holds for \( k = m - 1 \).

Let \( \nabla, \nabla^{(1)} \) and \( \nabla^{(2)} \) denote the gradient operators on \( \mathbb{R}^{m+d}, \mathbb{R}^m \) and \( \mathbb{R}^d \) respectively, and let

\[
Lf(x,y) := \langle Ax + My, \nabla^{(1)} f(x,y) \rangle + \langle Z(x,y), \nabla^{(2)} f(x,y) \rangle + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial y_i \partial y_j} f(x,y), \quad (x,y) \in \mathbb{R}^{m+d}, f \in C^2(\mathbb{R}^{m+d}).
\]

Since both \( Z \) and \( b \) are locally Lipschitz continuous, due to [12] the equation (E1) has a unique local solution for any initial data \( (X_0,Y_0) \in \mathcal{C} \). To ensure the non-explosion and further regular properties of the solution, we make use of the following assumptions:

\[(A) \quad \text{There exist constants } \lambda, l > 0 \text{ and } W \in C^2(\mathbb{R}^{m+d}) \text{ of compact level sets with } W \geq 1 \text{ such that}\]

\[
\begin{align*}
\text{(A1)} & \quad LW \leq \lambda W, \quad |\nabla^{(2)} W| \leq \lambda W; \\
\text{(A2)} & \quad \langle b(\xi), \nabla^{(2)} W(\xi(0)) \rangle \leq \lambda \|W(\xi)\|_\infty, \quad \xi \in \mathcal{C}; \\
\text{(A3)} & \quad |Z(z) - Z(z')| \leq \lambda |z - z'| W(z'), \quad z, z' \in \mathbb{R}^{m+d}, |z - z'| \leq 1; \\
\text{(A4)} & \quad |b(\xi) - b(\xi')| \leq \lambda \|\xi - \xi'\|_\infty \|W(\xi')\|_\infty, \quad \xi, \xi' \in \mathcal{C}, \quad \|\xi - \xi'\|_\infty \leq 1.
\end{align*}
\]
Comparing with the framework investigated in [5, 23], where $b = 0$, $A = 0$ and $\text{Rank}[M] = m$ are assumed, the present model is more general and the segment process we are going to investigate is an infinite-dimensional Markov process. On the other hand, unlike in [5] where the condition $|\nabla_z W| \leq \lambda W$ is not used, in the present setting this condition seems essential in order to derive moment estimates of the segment process (see the proof of Lemma 2.1 below). Moreover, if $|\nabla W| \leq cW$ holds for some constant $c > 0$, then (A3) and (A4) hold for some $\lambda > 0$ if and only if there exists a constant $\lambda' > 0$ such that $|\nabla Z| \leq \lambda'W'$ and $|\nabla b| \leq \lambda'\|W\|_\infty$ holds on $\mathbb{R}^{m+d}$ and $\mathcal{C}$ respectively.

It is easy to see that (A) holds for $W(z) = 1 + |z|^2$, $l = 1$ and some constant $\lambda > 0$ provided that $Z$ and $b$ are globally Lipschitz continuous on $\mathbb{R}^{m+d}$ and $\mathcal{C}$ respectively. It is clear that (A1) and (A2) imply the non-explosion of the solution (see Lemma 2.1 below). In this paper we aim to investigate regularity properties of the Markov semigroup associated with the segment process:

$$P_t f(\xi) = \mathbb{E}^\xi f(X_t, Y_t), \quad f \in \mathcal{B}_b(\mathcal{C}), \xi \in \mathcal{C},$$

where $\mathcal{B}_b(\mathcal{C})$ is the class of all bounded measurable functions on $\mathcal{C}$ and $\mathbb{E}^\xi$ stands for the expectation for the solution starting at the point $\xi \in \mathcal{C}$. When $m = 0$ we have $X_t \equiv 0$ and $\mathcal{C} = \{0\} \times \mathcal{C}_2 \equiv \mathcal{C}_2 := C([-r_0, 0]; \mathbb{R}^d)$, so that $P_t f$ can be simply formulated as $P_t f(\xi) = \mathbb{E}^\xi f(Y_t)$ for $f \in \mathcal{B}_b(\mathcal{C}_2), \xi \in \mathcal{C}_2$. Thus, (1.3) also includes non-degenerate functional SDEs. For any $h = (h_1, h_2) \in \mathcal{C}$ and $z \in \mathbb{R}^{m+d}$, let $\nabla_h$ and $\nabla_z$ be the directional derivatives along $h$ and $z$ respectively. The following result provides an explicit derivative formula for $P_t, T > r_0$.

**Theorem 1.1.** Assume (A) and let $T > r_0$. Let $v : [0, T] \to \mathbb{R}$ and $\alpha : [0, T] \to \mathbb{R}^m$ be Lipschitz continuous such that $v(0) = 1, \alpha(0) = 0, v(s) = 0, \alpha(s) = 0$ for $s \geq T - r_0$, and

$$h_1(0) + \int_0^t e^{-sA}M\phi(s)ds = 0, \quad t \geq T - r_0,$$

where $\phi(s) := v(s)h_2(0) + \alpha(s)$. Then for any $h = (h_1, h_2) \in \mathcal{C}$ and $f \in \mathcal{B}_b(\mathcal{C})$,

$$\nabla_h P_T f(\xi) = \mathbb{E}^\xi \left\{ f(X_T, Y_T) \int_0^T \langle N(s), (\sigma^*)^{-1}dB(s) \rangle \right\}, \quad \xi \in \mathcal{C}$$

holds for

$$N(s) := (\nabla\Theta(s)Z)(X(s), Y(s)) + (\nabla\Theta_b)(X_s, Y_s) - v'(s)h_2(0) - \alpha'(s), \quad s \in [0, T],$$

where

$$\Theta(s) = \Theta^{(1)}(s), \quad \Theta^{(2)}(s) := \begin{cases} h(s), & \text{if } s \leq 0, \\ (e^{As}h_1(0) + \int_0^se^{(s-r)A}M\phi(r)dr, \phi(s)), & \text{if } s > 0. \end{cases}$$
A simple choice of $v$ is
\[ v(s) = \frac{(T - r_0 - s)^+}{T - r_0}, \quad s \geq 0. \]

To present a specific choice of $\alpha$, let
\[ Q_t := \int_0^t \frac{s(T - r_0 - s)^+}{(T - r_0)^2} e^{-sA^*} M M^* e^{-sA^*} ds, \quad t > 0. \]

According to [11] (see also [21, Proof of Theorem 4.2(1)]), when $m \geq 1$ the matrix $Q_t$ is invertible with
\[ \|Q_t^{-1}\| \leq c(T - r_0)(t \wedge 1)^{-2(k+1)}, \quad t > 0 \]
for some constant $c > 0$.

**Corollary 1.2.** Assume (A) and let $T > r_0$. Then (1.4) holds for $v(s) = \frac{(T - r_0 - s)^+}{T - r_0}$ and
\[ \alpha(s) = -\frac{s(T - r_0 - s)^+}{(T - r_0)^2} M^* e^{-sA^*} Q_t^{-1} \left( h_1(0) + \int_0^{T-r_0} \frac{(T - r_0 - r)^+}{T - r_0} e^{-rA} M h_2(0) dr \right), \]
where by convention $M = 0$ (hence, $\alpha = 0$) if $m = 0$.

The following gradient estimates are direct consequences of Theorem 1.1.

**Corollary 1.3.** Assume (A). Then:

1. There exists a constant $C \in (0, \infty)$ such that
   \[ |\nabla h P_T f(\xi)| \leq C \sqrt{P_T f^2(\xi)} \left\{ |h(0)| \left( 1 + \frac{\|M\|}{(T - r_0)^{2k+1} \wedge 1} \right) \right. 
   + \left. \|W(\xi)\|_\infty \sqrt{T \wedge (1 + r_0)} \left( \|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T - r_0)^{2k+1} \wedge 1} \right) \right\} \]
   holds for all $T > r_0, \xi, h \in C$ and $f \in \mathcal{B}_b(C)$;

2. Let $|\nabla^{(2)} W|^2 \leq \delta W$ hold for some constant $\delta > 0$. If $l \in [0, 1/2)$ then there exists a constant $C \in (0, \infty)$ such that
   \[ |\nabla h P_T f(\xi)| \leq r \left\{ P_T f \log f - (P_T f) \log P_T f \right\}(\xi) 
   + \frac{C P_T f(\xi)}{r} \left\{ |h(0)|^2 \left( \frac{1}{(T - r_0) \wedge 1} + \frac{\|M\|^2}{\{(T - r_0) \wedge 1\}^{4k+3}} \right) 
   + \|h\|_\infty^2 \|W(\xi)\|_\infty + \left( \|h\|_\infty^2 + \frac{|h(0)|^2 \|M\|^2}{\{(T - r_0) \wedge 1\}^{4k+2}} \right)^{\frac{1}{2k+2}} \left( \frac{r^2}{\|h\|_\infty^2 \wedge 1} \right) \right\} \]
   holds for all $r > 0, T > r_0, \xi, h \in C$ and positive $f \in \mathcal{B}_b(C)$;
(3) Let $|\nabla^{(2)}W|^2 \leq \delta W$ hold for some constant $\delta > 0$. If $l = \frac{1}{2}$ then there exist constants $C, C' \in (0, \infty)$ such that

$$
|\nabla_h P_T f(\xi)| \leq r \left\{ P_T f \log f - (P_T f) \log P_T f \right\}(\xi) + \frac{C P_T f(\xi)}{r} \left\{ |h(0)|^2 \left( \frac{1}{(T - r_0) \wedge 1} + \frac{\|M\|^2}{((T - r_0) \wedge 1)^{4k+3}} \right) + \|W(\xi)\|_\infty \left( \|h\|^2_\infty + \frac{\|M\|^2 |h(0)|^2}{((T - r_0) \wedge 1)^{4k+2}} \right) \right\}
$$

holds for

$$
r \geq C' \left( \|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{((T - r_0) \wedge 1)^{2k+1}} \right),
$$

all $T > r_0, \xi, h \in \mathcal{C}$ and positive $f \in \mathcal{B}_b(\mathcal{C})$.

When $m = 0$ the above assertions hold with $\|M\| = 0$.

According to [2], the entropy gradient estimate implies the Harnack inequality with power, we have the following result which follows immediately from Corollary [13] (2) and [5] Proposition 4.1. Similarly, Corollary [13] (3) implies the same type of Harnack inequality for smaller $\|h\|_\infty$ comparing to $T - r_0$.

**C1.4** Corollary 1.4. Assume (A) and let $|\nabla^{(2)}W|^2 \leq \delta W$ hold for some constant $\delta > 0$. If $l \in [0, \frac{1}{2})$ then there exists a constant $C' \in (0, \infty)$ such that

$$
(P_T f)^p(\xi + h) \leq P_T f^p(\xi) \exp \left[ \frac{C p}{p - 1} \left\{ \|h\|^2_\infty \int_0^1 \|W(\xi + sh)\|_\infty \, ds + \left( \|h\|^2_\infty + \frac{\|M\|^2 |h(0)|^2}{((T - r_0) \wedge 1)^{4k+2}} \right)^{\frac{1}{p-1}} \left( \frac{(p - 1)^2}{\|h\|^2_\infty} \vee 1 \right)^{\frac{2}{p-1}} \right\} \right]
$$

holds for all $T > r_0, p > 1, \xi, h \in \mathcal{C}$ and positive $f \in \mathcal{B}_b(\mathcal{C})$. If $m = 0$ then the assertion holds for $\|M\| = 0$.

Finally, we consider the log-Harnack inequality introduced in [10] [15]. To this end, as in [5], we slightly strengthen (A3) and (A4) as follows: there exists an increasing function $U$ on $[0, \infty)$ such that

(A3') $|Z(z) - Z(z')| \leq \lambda |z - z'| \{ W(z')^l + U(|z - z'|) \}, \quad z, z' \in \mathbb{R}^{m+d};$

(A4') $|b(\xi) - b(\xi')| \leq \lambda \|\xi - \xi'\|_\infty \{ \|W(\xi')\|_\infty + U(\|\xi - \xi'\|_\infty ) \}, \quad \xi, \xi' \in \mathcal{C}.$

Obviously, if

$$
W(z)^l \leq c \{ W(z')^l + U(|z - z'|) \}, \quad z, z' \in \mathbb{R}^{m+d}
$$

holds for some constant $c > 0$, then (A3) and (A4) imply (A3') and (A4') respectively with possibly different $\lambda$. 

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Theorem 1.5. Assume \((A1), (A2), (A3')\) and \((A4')\). Then there exists a constant \(C \in (0, \infty)\) such that for any positive \(f \in B_b(\mathcal{C}), T > r_0\) and \(\xi, h \in \mathcal{C}\),

\[
P_T \log f(\xi + h) - \log P_T f(\xi) \leq C \left\{ \|W(\xi + h)\|_\infty^2 + U^2 \left( C\|M\| \cdot |h(0)| \right) \right\}^2 \|h\|_\infty^2 + \frac{|h(0)|^2}{(T - r_0) \wedge 1} + \frac{\|M\|^2 |h(0)|^2}{\{(T - r_0) \wedge 1\}^{4k+3}}.
\]

If \(m = 0\) then the assertion holds for \(\|M\| = 0\).

For applications of the Harnack and log-Harnack inequalities we are referred to [17 §4.2].

The remainder of the paper is organized as follows: Theorem 1.1 and Corollary 1.2 are proved in Section 2, while Corollary 1.3 and Theorem 1.5 are proved in Section 3; in Section 4 the assumption \((A)\) is weakened for the discrete time delay case, and two examples are presented to illustrate our results.

2 Proofs of Theorem 1.1 and Corollary 1.2

Lemma 2.1. Assume \((A1)\) and \((A2)\). Then for any \(k > 0\) there exists a constant \(C > 0\) such that

\[
\mathbb{E}^\xi \sup_{-r_0 \leq s \leq t} W(X(s), Y(s))^k \leq 3\|W(\xi)\|_\infty^k e^{Ct}, \quad t \geq 0, \; \xi \in \mathcal{C}
\]

holds. Consequently, the solution is non-explosive.

Proof. For any \(n \geq 1\), let

\[
\tau_n := \inf\{t \in [0, T] : |X(t)| + |Y(t)| \geq n\}.
\]

Moreover, let

\[
\ell(s) := W(X, Y)(s), \quad s \geq -r_0.
\]

By the Itô formula and using the first inequality in \((A1)\) and \((A2)\) we may find a constant \(C_1 > 0\) such that

\[
\ell(t \wedge \tau_n)^k = \ell(0)^k + k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \langle \nabla^2 W(X, Y)(s), \sigma dB(s) \rangle
\]

\[
+ k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \left\{ LW(X, Y)(s) + \langle b(X_s, Y_s), \nabla^2 W(X, Y)(s) \rangle
\right. \left. + \frac{1}{2} (k-1) \ell(s)^{-1} |\sigma^* \nabla^2 W(X, Y)(s)|^2 \right\} ds
\]

\[
\leq \ell(0)^k + k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \langle \nabla^2 W(X, Y)(s), \sigma dB(s) \rangle + C_1 \int_0^{t \wedge \tau_n} \sup_{r \in [-r_0, s]} \ell(r)^k ds.
\]

Noting that by the second inequality in \((A1)\) and the Burkholder-Davis-Gundy inequality we obtain
Proposition 2.2. Let $\phi(s) := v(s)h_2(0) + \alpha(s)$, $s \in [0, T]$, and the conditions of Theorem 1.1 hold. Then

\begin{align*}
(X^\varepsilon(t), Y^\varepsilon(t)) &= (X(t), Y(t)) + \varepsilon \Theta(t), \quad \varepsilon, t \geq 0
\end{align*}

holds for

\begin{align*}
\Theta(t) := (\Theta^{(1)}(t), \Theta^{(2)}(t)) := \begin{cases}
    h(t), & \text{if } t \leq 0, \\
    (e^{Mt}h_1(0) + \int_0^t e^{(t-r)A} M\phi(r)dr, \phi(t)), & \text{if } t > 0.
\end{cases}
\end{align*}

In particular, $(X_{T}^\varepsilon, Y_T^\varepsilon) = (X_T, Y_T)$. 

\[ k\mathbb{E}^\xi \sup_{s \in [0, t]} \left| \int_0^{t \wedge \tau_n} \ell(r)^{k-1} (\nabla W(X, Y)(s), \sigma dB(r)) dr \right| \leq C_2 \mathbb{E}^\xi \left( \int_0^t \ell(s \wedge \tau_n)^{2k} ds \right)^{1/2} \]

\[ \leq C_2 \mathbb{E}^\xi \left\{ \sup_{s \in [0, t]} \ell(s \wedge \tau_n)^k \left( \int_0^t \ell(s \wedge \tau_n)^k ds \right)^{1/2} \right\} \]

\[ \leq \frac{1}{2} \mathbb{E}^\xi \sup_{s \in [0, t]} \ell(s \wedge \tau_n)^k + \frac{C_2}{2} \int_0^t \sup_{r \in [0, s]} \ell(r \wedge \tau_n)^k ds \]

for some constant $C_2 > 0$. Combining this with (2.1) and noting that $(X_0, Y_0) = \xi$, we conclude that there exists a constant $C > 0$ such that

\[ \mathbb{E}^\xi \sup_{-r_0 \leq s \leq t} \ell(s \wedge \tau_n)^k \leq 3\|W(\xi)\|^k_\infty + C \mathbb{E}^\xi \int_0^t \sup_{s \in [-r_0, t]} \ell(s)^k ds, \quad t \geq 0. \]

Due to the Gronwall lemma this implies that

\[ \mathbb{E}^\xi \sup_{-r_0 \leq s \leq t} \ell(s \wedge \tau_n)^k \leq 3\|W(\xi)\|^k_\infty e^{Ct}, \quad t \geq 0, n \geq 1. \]

Consequently, we have $\tau_n \uparrow \infty$ as $n \uparrow \infty$, and thus the desired inequality follows by letting $n \to \infty$. 

To establish the derivative formula, we first construct couplings for solutions starting from $\xi$ and $\xi + \varepsilon h$ for $\varepsilon \in (0, 1]$, then let $\varepsilon \to 0$. For fixed $\xi = (\xi_1, \xi_2)$, $h = (h_1, h_2) \in \mathcal{C}$, let $(X(t), Y(t))$ solve (1.1) with $(X_0, Y_0) = \xi$; and for any $\varepsilon \in (0, 1]$, let $(X^\varepsilon(t), Y^\varepsilon(t))$ solve the equation

\begin{align*}
\textbf{E2} \quad \begin{cases}
    dX^\varepsilon(t) &= \{AX^\varepsilon(t) + MY^\varepsilon(t)\} dt, \\
    dY^\varepsilon(t) &= \{Z(X(t), Y(t)) + b(X_t, Y_t)\} dt + \sigma dB(t) + \varepsilon \{v'(t)h_2(0) + \alpha'(t)\} dt
\end{cases}
\end{align*}

with $(X_0^\varepsilon, Y_0^\varepsilon) = \xi + \varepsilon h$. By Lemma 2.1 and (2.3) below, the solution to (2.2) is non-explosive as well.

Proposition 2.2. Let $\phi(s) := v(s)h_2(0) + \alpha(s)$, $s \in [0, T]$, and the conditions of Theorem 1.1 hold. Then

\[ (X^\varepsilon(t), Y^\varepsilon(t)) = (X(t), Y(t)) + \varepsilon \Theta(t), \quad \varepsilon, t \geq 0 \]

holds for

\[ \Theta(t) := (\Theta^{(1)}(t), \Theta^{(2)}(t)) := \begin{cases}
    h(t), & \text{if } t \leq 0, \\
    (e^{Mt}h_1(0) + \int_0^t e^{(t-r)A} M\phi(r)dr, \phi(t)), & \text{if } t > 0.
\end{cases} \]
Lemma 2.3. By (2.2) and noting that \( v(0) = 1 \) and \( v(s) = 0 \) for \( s \geq T - r_0 \), we have \( Y^\varepsilon(t) = Y(t) + \varepsilon \phi(t) \) and

\[
X^\varepsilon(t) = X(t) + \varepsilon e^{\varepsilon A} h_1(0) + \varepsilon \int_0^t e^{(t-s)A} M \phi(s) ds, \quad t \geq 0.
\]

Thus, (2.3) holds. Moreover, since \( \alpha(s) = v(s) = 0 \) for \( s \geq T - r_0 \), we have \( \Theta^{(2)}(s) = \phi(s) = 0 \) for \( s \geq T - r_0 \). Moreover, by (1.3) we have \( \Theta^{(1)}(s) = 0 \) for \( s \geq T - r_0 \). Therefore, the proof is finished.

Since according to Proposition 2.2 we have \((X^\varepsilon_T, Y^\varepsilon_T) = (X_T, Y_T)\). Noting that \((X^\varepsilon_0, Y^\varepsilon_0) = (\xi + \varepsilon h, \xi + \varepsilon h)\), if (2.2) can be formulated as (1.1) using a different Brownian motion, then we are able to link \( P_T f(\xi) \) to \( P_T f(\xi + \varepsilon h) \) and furthermore derive the derivative formula by taking derivative w.r.t. \( \varepsilon \) at \( \varepsilon = 0 \). To this end, let

\[
\Phi^\varepsilon(s) = Z(X(s), Y(s)) - Z(X^\varepsilon(s), Y^\varepsilon(s)) + b(X_s, Y_s) - b(X^\varepsilon_s, Y^\varepsilon_s) + \varepsilon \{ v'(s) h_2(0) + \alpha'(s) \}.
\]

Set

\[
R^\varepsilon(s) = \exp \left[ - \int_0^s <\sigma^{-1} \Phi^\varepsilon(r), dB(r)> - \frac{1}{2} \int_0^s |\sigma^{-1} \Phi^\varepsilon(r)|^2 dr \right],
\]

and

\[
B^\varepsilon(s) = B(s) + \int_0^s \sigma^{-1} \Phi^\varepsilon(r) dr.
\]

Then (2.2) reduces to

\[
\begin{aligned}
\text{(2.4)}
\end{aligned}
\]

According to the Girsanov theorem, to ensure that \( B^\varepsilon(t) \) is a Brownian motion under \( \mathbb{Q}^\varepsilon := \mathbb{P}^{R^\varepsilon(T)} \), we first prove that \( R^\varepsilon(t) \) is an exponential martingale. Moreover, to obtain the derivative formula using the dominated convergence theorem, we also need \{ \frac{R^\varepsilon(T) - 1}{\varepsilon} \}_{\varepsilon \in (0,1)} \) to be uniformly integrable. Therefore, we will need the following two lemmas.

**Lemma 2.3.** Let (A) hold. Then there exists \( \varepsilon_0 > 0 \) such that

\[
\sup_{s \in [0,T], \varepsilon \in (0,\varepsilon_0)} \mathbb{E}[R^\varepsilon(s) \log R^\varepsilon(s)] < \infty,
\]

so that for each \( \varepsilon \in (0,1) \), \( (R^\varepsilon(s))_{s \in [0,T]} \) is a uniformly integrable martingale.

**Proof.** By (2.3), there exists \( \varepsilon_0 > 0 \) such that

\[
(2.5) \quad \varepsilon_0 |\Theta(t)| \leq 1, \quad t \in [-r_0, T].
\]

For any \( \varepsilon \in [0, \varepsilon_0] \), define

\[
\tau_n := \inf\{ t \geq 0 : |X(t)| + |Y(t)| + |X^\varepsilon(t)| + |Y^\varepsilon(t)| \geq n \}, \quad n \geq 1.
\]
We have \( \tau_n \uparrow \infty \) as \( n \uparrow \infty \) due to the non-explosion. By the Girsanov theorem, the process \( \{R^\varepsilon(s \land \tau_n)\}_{s \in [0,T]} \) is a martingale and \( \{B^\varepsilon(s)\}_{s \in [0,T \land \tau_n]} \) is a Brownian motion under the probability measure \( Q_{\varepsilon,n} := R^\varepsilon(T \land \tau_n) \mathbb{P} \). By the definition of \( R^\varepsilon(s) \) we have

\[
\mathbb{E}[R^\varepsilon(s \land \tau_n) \log R^\varepsilon(s \land \tau_n)] = \mathbb{E}_{Q_{\varepsilon,n}}[\log R^\varepsilon(s \land \tau_n)] \leq \frac{1}{2} \mathbb{E}_{Q_{\varepsilon,n}} \int_0^{T \land \tau_n} |\sigma^{-1} \Phi^\varepsilon(r)|^2 dr.
\]

By (2.5), (A3) and (A4),

\[
|\sigma^{-1} \Phi^\varepsilon(s)|^2 \leq c\varepsilon^2\|W(X^\varepsilon, Y^\varepsilon)\|_{2}^{2},
\]

holds for some constant \( c \) independent of \( \varepsilon \). By the weak uniqueness of the solution to (1.1) and (2.4), the distribution of \( (X^\varepsilon(s), Y^\varepsilon(s))_{s \in [0,T \land \tau_n]} \) under \( Q_{\varepsilon,n} \) coincides with that of the solution to (1.1) with \( (X_0, Y_0) = \xi + \varepsilon h \) up to time \( T \land \tau_n \), we therefore obtain from Lemma 2.1 that

\[
\mathbb{E}[R^\varepsilon(s \land \tau_n) \log R^\varepsilon(s \land \tau_n)] \leq c\|W(\xi + \varepsilon h)\|_{2}^{2} \int_0^{T} e^{Ct} dt < \infty, \quad n \geq 1, \varepsilon \in (0, \varepsilon_0).
\]

Then the required assertion follows by letting \( n \to \infty \).

\[\square\]

**Lemma 2.4.** If (A) holds, then there exists \( \varepsilon_0 > 0 \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \mathbb{E}\left( \frac{R^\varepsilon(T) - 1}{\varepsilon} \log \frac{R^\varepsilon(T) - 1}{\varepsilon} \right) < \infty.
\]

Moreover,

\[
\lim_{\varepsilon \to 0} \frac{R^\varepsilon(T) - 1}{\varepsilon} = \int_0^T \left\langle (\nabla \Theta(s) Z)(X(s), Y(s)) + (\nabla \Theta, b)(X_s, Y_s) - \nu'(s)h_2(0) - \alpha'(s), (\sigma^*)^{-1} dB(s) \right\rangle.
\]

**Proof.** Let \( \varepsilon_0 \) be such that (2.5) holds. Since (2.8) is a direct consequence of (2.3) and the definition of \( R^\varepsilon(T) \), we only prove the first assertion. By [5] we know that

\[
\frac{R^\varepsilon(T) - 1}{\varepsilon} \log \frac{R^\varepsilon(T) - 1}{\varepsilon} \leq 2R^\varepsilon(T) \left( \frac{\log R^\varepsilon(T)}{\varepsilon} \right)^2.
\]

Since due to Lemma 2.3 \( \{B^\varepsilon(t)\}_{t \in [0,T]} \) is a Brownian motion under the probability measure \( Q_{\varepsilon} := R^\varepsilon(T) \mathbb{P} \), and since

\[
\log R^\varepsilon(T) = - \int_0^T \langle \sigma^{-1} \Phi^\varepsilon(r), dB(r) \rangle - \frac{1}{2} \int_0^T |\sigma^{-1} \Phi^\varepsilon(r)|^2 dr
\]

\[
= - \int_0^T \langle \sigma^{-1} \Phi^\varepsilon(r), dB^\varepsilon(r) \rangle + \frac{1}{2} \int_0^T |\sigma^{-1} \Phi^\varepsilon(r)|^2 dr,
\]

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it follows from (2.7) that

\[
\mathbb{E}\left(\frac{R^e(T) - 1}{\varepsilon} \log \frac{R^e(T) - 1}{\varepsilon}\right) \leq \mathbb{E}\left(2R^e(T)\left(\log \frac{R^e(T)}{\varepsilon}\right)^2\right) = 2\mathbb{E}_{Q_\varepsilon}\left(\left(\log \frac{R^e(T)}{\varepsilon}\right)^2\right)
\]

\[
\leq \frac{4}{\varepsilon^2}\mathbb{E}_{Q_\varepsilon}\left(\int_0^T \langle \sigma^{-1}\Phi^e(r), dB^e(r) \rangle\right)^2 + \frac{1}{\varepsilon^2}\mathbb{E}_{Q_\varepsilon}\left(\int_0^T |\sigma^{-1}\Phi^e(r)|^2\, dr\right)^2
\]

\[
\leq \frac{4}{\varepsilon^2}\int_0^T \mathbb{E}_{Q_\varepsilon}|\sigma^{-1}\Phi^e(r)|^2\, dr + \frac{T}{\varepsilon^2}\int_0^T \mathbb{E}_{Q_\varepsilon}|\sigma^{-1}\Phi^e(r)|^4\, dr
\]

\[
\leq c \int_0^T \mathbb{E}_{Q_\varepsilon}\|W(X^e, Y^e)\|_\infty\, dr
\]

holds for some constant \(c > 0\). As explained in the proof of Lemma 2.3, the distribution of \((X^e, Y^e)_{s\in[0,T]}\) under \(Q_\varepsilon\) coincides with that of the segment process of the solution to (1.1) with \((X_0, Y_0) = \xi + \varepsilon h\), the first assertion follows by Lemma 2.4.

**Proof of Theorem 1.1.** Since Lemma 2.3 together with the Girsanov theorem, implies that \{\(B^e(s)\)\}_{s\in[0,T]}\ is a Brownian motion with respect to \(Q_\varepsilon := R^e(T)\mathbb{P}\), by [2.4] and \((X_T, Y_T) = (X^e_T, Y^e_T)\) we obtain

\[
P_T f(\xi + \varepsilon h) = \mathbb{E}_{Q_\varepsilon} f(X^e_T, Y^e_T) = \mathbb{E}\{R^e(T) f(X_T, Y_T)\}. \tag{2.9}
\]

Thus,

\[
P_T f(\xi + \varepsilon h) - P_T f(\xi) = \mathbb{E}R^e(T) f(X_T, Y_T) - \mathbb{E} f(X_T, Y_T) = \mathbb{E}[(R^e(T) - 1) f(X_T, Y_T)].
\]

Combining this with Lemma 2.4 and using the dominated convergence theorem, we arrive at

\[
\nabla_h P_T f(\xi, \eta) = \lim_{\varepsilon \to 0} \frac{P_T f(\xi + \varepsilon h) - P_T f(\xi)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathbb{E}[(R^e(T) - 1) f(X_T, Y_T)]}{\varepsilon}
\]

\[
= \mathbb{E} \left\{ f(X_T, Y_T) \int_0^T \langle N(s), (\sigma^*)^{-1} dB(s) \rangle \right\}.
\]

**Proof of Corollary 1.2.** It suffices to verify (1.3) for the specific \(v\) and \(\alpha\). Since when \(m = 0\) we have \(h_1 = M = 0\) so that (1.3) trivially holds, we only consider \(m \geq 1\). In this case, (1.3) is satisfied since according to the definition of \(\phi(s)\) and \(\alpha(s)\) we have for \(t \geq T - r_0\),

\[
\int_0^t e^{-sA}M\phi(s)\, ds = \int_0^{T-r_0} e^{-sA}M\phi(s)\, ds
\]

\[
= \int_0^{T-r_0} v(s)e^{-sA}Mh_2(0)\, ds - Q_{T-r_0}Q_{T-r_0}^{-1} \left(h_1(0) + \int_0^{T-r_0} v(s)e^{-sA}Mh_2(0)\, ds\right)
\]

\[
= -h_1(0).
\]


3 Proofs of Corollary 1.3 and Theorem 1.5

To prove the entropy-gradient estimates in Corollary (2) and (3), we need the following simple lemma which seems new and might be interesting by itself.

L3.1 Lemma 3.1. Let $\ell(t)$ be a non-negative continuous semi-martingale and let $\mathcal{M}(t)$ be a continuous martingale with $\mathcal{M}(0) = 0$ such that

$$d\ell(t) \leq d\mathcal{M}(t) + c\bar{\ell}_t dt,$$

where $c \geq 0$ is a constant and $\bar{\ell}_t := \sup_{s \in [0,t]} \ell(s)$. Then

$$E\exp \left[ \frac{\varepsilon}{Te^{1+cT}} \int_0^T \bar{\ell}_t dt \right] \leq e^{\varepsilon \ell(0) + 1 \left( EE^{2\varepsilon^2 \mathcal{M}(T)} \right)^{1/2}}, \ T, \varepsilon \geq 0.$$

Proof. Let $\bar{\mathcal{M}}_t := \sup_{s \in [0,t]} \mathcal{M}(t)$. We have

$$\bar{\mathcal{M}}_t + c \int_0^t \bar{\ell}_s ds \geq \bar{\ell}_t - \ell(0).$$

Thus,

$$\frac{\ell_T}{e^{1+cT}} - \ell(0) \leq \frac{\mathcal{M}_T + c \int_0^T \bar{\ell}_t dt}{e^{1+cT}} - (1 - e^{-(1+cT)}) \ell(0)$$

$$= \int_0^T d \left( e^{-(c+T^{-1})t} \left( \bar{\mathcal{M}}_t + c \int_0^t \bar{\ell}_s ds \right) \right) - (1 - e^{-(1+cT)}) \ell(0)$$

$$= \int_0^T e^{-(T^{-1}+c)t} d\bar{\mathcal{M}}_t + \int_0^T e^{-(c+T^{-1})t} \left( c\bar{\ell}_t - (T^{-1} + c)(\bar{\ell}_t - \ell(0)) \right) dt - (1 - e^{-(1+cT)}) \ell(0)$$

$$\leq \mathcal{M}_T + \int_0^T e^{-(c+T^{-1})t} \left( c\bar{\ell}_t - (T^{-1} + c)(\bar{\ell}_t - \ell(0)) \right) dt - (1 - e^{-(1+cT)}) \ell(0)$$

$$\leq \mathcal{M}_T - \frac{1}{T e^{1+cT}} \int_0^T \bar{\ell}_t dt.$$

Combining this with

$$Ee^{\varepsilon \bar{\mathcal{M}}_t} \leq Ee^{1+\varepsilon \mathcal{M}(T)} \leq e\left( EE^{2\varepsilon^2 \mathcal{M}(T)} \right)^{1/2},$$

we complete the proof. $\square$

C3.1 Corollary 3.2. Assume (A) and let $|\nabla^{(2)}W|^2 \leq \delta W$ hold for some constant $\delta > 0$. Then there exists a constant $c > 0$ such that

$$E^\xi \exp \left[ \frac{1}{2\|\sigma\|^2 \delta T^2 e^{2+2cT} + 2} \int_0^T \|W(X_t, Y_t)\|_\infty dt \right]$$

$$\leq \exp \left[ 2 + \frac{W(\xi(0))}{\|\sigma\|^2 \delta T e^{1+cT}} + \frac{r_0\|W(\xi)\|_\infty}{2\|\sigma\|^2 \delta T^2 e^{2+2cT}} \right], \ T > r_0.$$
Proof. By (A) and the Itô formula, there exists a constant $c > 0$ such that

$$dW(X,Y)(s) \leq \langle \nabla^{(2)}W(X,Y)(s), \sigma dB(s) \rangle + c\|W(X_s,Y_s)\|_{\infty}ds.$$ 

Let

$$\mathcal{M}(t) := \int_0^t \langle \nabla^{(2)}W(X,Y)(s), \sigma dB(s) \rangle, \quad l(t) := W(X,Y)(t),$$

and let $\varepsilon = (2\|\sigma\|^2 \delta T e^{1+cT})^{-1}$ such that

$$\frac{\varepsilon}{Te^{1+cT}} = 2\|\sigma\|^2 \varepsilon^2.$$ 

Then by Lemma 3.1 and $|\nabla^{(2)}W| \leq \delta W$, we have

$$\mathbb{E}^\xi \exp \left[ \frac{\varepsilon}{Te^{1+cT}} \int_0^T \bar{l}_t dt \right] \leq e^{\varepsilon l(0) + \left( \mathbb{E}^\xi e^{2\varepsilon\delta T} (\mathcal{M}(T)) \right)^{1/2}} \leq e^{1+\varepsilon l(0)} \left( \mathbb{E}^\xi e^{2\varepsilon\delta T} \int_0^T \bar{l}_t dt \right)^{1/2}.$$ 

By using stopping times as in the proof of Lemma 2.1 we may assume that

$$\mathbb{E}^\xi \exp \left[ \frac{\varepsilon}{Te^{1+cT}} \int_0^T \bar{l}_t dt \right] < \infty$$

so that

$$\mathbb{E}^\xi \exp \left[ \frac{\varepsilon}{Te^{1+cT}} \int_0^T \bar{l}_t dt \right] \leq e^{2+2\varepsilon l(0)}.$$ 

This completes the proof by noting that

$$\frac{1}{2\|\sigma\|^2 \delta T e^{2+2cT}} \int_0^T \|W(X_t,Y_t)\|_{\infty} dt \leq \frac{r_0\|W(\xi)\|_{\infty}}{2\|\sigma\|^2 \delta T e^{2+2cT}} + \frac{\varepsilon}{Te^{1+cT}} \int_0^T \bar{l}_t dt.$$ 

Proof of Corollary 1.3. Let $v$ and $\alpha$ be given in Corollary 1.2. By the semigroup property and the Jensen inequality, we will only consider $T - r_0 \in (0,1]$.

(1) By (1.5) and the definitions of $\alpha$ and $v$, there exists a constant $C > 0$ such that

$$|v'(s)h_2(0) + \alpha'(s)| \leq C1_{[0,T-r_0]}(s)|h(0)| \left( \frac{1}{T-r_0} + \frac{\|M\|}{(T-r_0)^{2k+1}} \right), \quad s \in [0,T],$$

(3.1) $|\Theta(s)| \leq C|h(0)| \left( 1 + \frac{\|M\|}{(T-r_0)^{2k+1}} \right), \quad s \in [0,T],$

$$\|\Theta_s\|_{\infty} \leq C \left( \|h\|_{\infty} + \frac{\|M\| \cdot |h(0)|}{(T-r_0)^{2k+1}} \right), \quad s \in [0,T].$$
Therefore, it follows from (A3) and (A4) that

\[ |N(s)| \leq C1_{[0,T-r_0]}(s)|h(0)|\left(\frac{1}{T-r_0} + \frac{\|M\|}{(T-r_0)^{2(k+1)}}\right) \]

\[ + C\left(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r_0)^{2k+1}}\right)\|W(X_s, Y_s)\|_\infty^l \]

holds for some constant \(C > 0\). Combining this with Theorem [3], we obtain

\[ |\nabla_P h P_T f(\xi)| \leq C \sqrt{P_T f^2(\xi)} \left(\mathbb{E}^\xi \int_0^T |N(s)|^2 ds \right)^{1/2} \]

\[ \leq C \sqrt{P_T f^2(\xi)} \left\{ |h(0)| \left(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\right) \right. \]

\[ \left. + \left(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r_0)^{2k+1}}\right) \left(\int_0^T \mathbb{E}^\xi \|W(X_s, Y_s)\|_\infty^2 ds \right)^{1/2} \right\}, \]

This completes the proof of (1) since due to Lemma [1], one has

\[ \mathbb{E}^\xi \|W(X_s, Y_s)\|_\infty^2 \leq 3\|W(\xi)\|_\infty^2 e^{Cs}, \quad s \in [0, T] \]

for some constant \(C > 0\).

(2) By Theorem [4] and the Young inequality (cf. [2], Lemma 2.4), we have

\[ |\nabla_P h P_T f(\xi)| \leq r \left\{ P_T f \log f - (P_T f) \log P_T f \right\}(\xi) \]

\[ + r P_T f(\xi) \log \mathbb{E}^{1/2}\left(\mathbb{E}^{x/2} f_h(\mu(\xi), (\sigma^*)^{-1} dB(s))\right), \quad r > 0. \]

Next, it follows from (3.2) that

\[ \left(\mathbb{E}^\xi \exp \left[\frac{1}{r^2} \int_0^T \langle N(s), (\sigma^*)^{-1} dB(s) \rangle \right]\right)^2 \leq \mathbb{E}^\xi \exp \left[\frac{2\|\sigma^{-1}\|_2^2}{r^2} \int_0^T |N(s)|^2 ds \right] \]

\[ \leq \exp \left[\frac{C_1 |h(0)|^2}{r^2} \left(\frac{1}{T-r_0} + \frac{\|M\|^2}{(T-r_0)^{4k+3}}\right)\right] \]

\[ \times \mathbb{E}^\xi \exp \left[\frac{C_2 |h|^2 + \|M\|^2 |h(0)|^2}{(T-r_0)^{4k+2}} \right] \left(\int_0^T \|W(X_s, Y_s)\|_\infty^2 ds \right), \quad T \in (r_0, 1 + r_0] \]

holds for some constant \(C_1 \in (0, \infty)\). Since \(2l \in [0, 1)\) and \(T \leq 1 + r_0\), there exists a constant \(C_2 \in (0, \infty)\) such that

\[ \beta \|W(X_s, Y_s)\|_\infty^{2l} \leq \left(\frac{\|h\|_\infty^2}{r^2} \wedge 1\right)\|W(X_s, Y_s)\|_\infty^{2l} + C_2 \beta^{\frac{2l}{r^2}} \left(\frac{\|h\|_\infty^2}{r^2} \wedge 1\right)^{-\frac{2}{r^2}}, \quad \beta > 0. \]
Taking
\[ \beta = C_1 \left( \|h\|^2 + \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}} \right), \]
and applying Corollary 3.2 we arrive at
\[
\mathbb{E}^\epsilon \exp \left[ \beta \int_0^T \|W(X_s, Y_s)\|_\infty^2 ds \right] \leq \exp \left[ C_2 \beta^{\frac{1}{2r^2}} \left( \frac{\|h\|^2}{r^2} \wedge 1 \right)^{-\frac{2r^2}{2r^2}} \right] \times \left( \mathbb{E}^\epsilon \exp \left[ \frac{1}{2\|\sigma\|^2 \delta T^2 \epsilon^{2+2} r^2} \int_0^T \|W(X_s, Y_s)\|_\infty^2 ds \right] \right)^{\|h\|^2 \wedge 1} \\
\leq \exp \left[ C_3 \left\{ \|h\|^2_\infty \|W(\xi)\|_\infty + \left( \|h\|^2_\infty + \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}} \right)^{\frac{1}{2r^2}} \right\} \right] \]
for some constant \( C_3 \in (0, \infty) \) and all \( T \in (r_0, 1+r_0] \). Therefore, the desired entropy-gradient estimate follows by combining this with (3.3) and (3.4).

(3) Let \( C' > 0 \) be such that \( r \geq C' \left( \|h\|_\infty + \frac{\|M\|^2|h(0)|}{(T - r_0)^{4k+2}} \right) \) implies
\[
\frac{C_1}{r^2} \left( \|h\|^2_\infty + \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}} \right) \leq \frac{1}{2\|\sigma\|^2 \delta T^2 \epsilon^{2+2} r^2},
\]
so that by Corollary 3.2
\[
\mathbb{E}^\epsilon \exp \left[ \frac{C_1}{r^2} \left( \|h\|^2_\infty + \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}} \right) \int_0^T \|W(X_s, Y_s)\|_\infty^2 ds \right] \\
\leq \left( \mathbb{E}^\epsilon \exp \left[ \frac{1}{2\|\sigma\|^2 \delta T^2 \epsilon^{2+2} r^2} \int_0^T \|W(X_s, Y_s)\|_\infty^2 ds \right] \right)^{\frac{2C_1}{\|h\|^2_\infty} \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}}} \\
\leq \exp \left[ \frac{C'}{r^2} \left( \|h\|^2_\infty + \frac{\|M\|^2|h(0)|^2}{(T - r_0)^{4k+2}} \right) \right]
\]
holds for some constant \( C > 0 \). Then proof is finished by combining this with (3.3) and (3.4).

Proof of Theorem 1.4. Again, we only prove for \( T \in (r_0, 1+r_0) \). Applying (2.9) to \( \varepsilon = 1 \) and using \( \log f \) to replace \( f \), we obtain

(3.5) \[ P_T \log f(\xi + h) = \mathbb{E}\{ R_T(T) \log f(X_T, Y_T) \} \leq \log P_T f(\xi) + \mathbb{E}(R_T \log R_T)(T). \]

Next, taking \( \varepsilon = 1 \) in (2.6) and letting \( n \uparrow \infty \), we arrive at

(3.6) \[ \mathbb{E}(R_T \log R_T)(T) \leq \frac{1}{2} \mathbb{E}_{\xi_1} \int_0^T |\sigma^{-1}\Phi^1(r)|^2 dr. \]
By (A3'), (A4'), (5.1) and the definition of \(\Phi^1\), we have

\[
|\sigma^{-1}\Phi^1(s)|^2 \leq C_1 \left\{ \|W(X^1, Y_s^1)\|^2_\infty + U^2 \left( C_1 \|h\|_\infty + \frac{C_1 \|M\| \cdot |h(0)|}{(T - r_0)^{2k+1}} \right) \right\} \|h\|^2_\infty + C_1 |h(0)|^2 \left( \frac{1}{(T - r_0)^2} + \frac{\|M\|^2}{(T - r_0)^{4(k+1)}} \right) 1_{[0,T-r_0]}(s)
\]

for some constant \(C_1 > 0\). Then the proof is completed by combining this with (3.5), (3.6) and Lemma 2.1 (note that \((X^1(s), Y^1(s))\) under \(Q_1\) solves the same equation as \((X_s, Y_s)\) under \(\mathbb{P}\)).

## 4 Discrete Time Delay Case and Examples

In this section we first present a simple example to illustrate our main results presented in Section 1, then relax assumption (A) for the discrete time delay case in order to cover some highly non-linear examples.

### Example 4.1.

For \(\alpha \in C([-r_0, 0]; \mathbb{R})\), consider functional SDE on \(\mathbb{R}^2\)

\[
\begin{align*}
    dX(t) &= -\{X(t) + Y(t)\}dt \\
    dY(t) &= dB(t) + \left\{ -\varepsilon Y^2(t) + Y(t - r_0) + \int_{-r_0}^0 \alpha(\theta)X(t + \theta)d\theta \right\}dt
\end{align*}
\]

with initial data \(\xi = (\xi_1, \xi_2) \in C([-r_0, 0]; \mathbb{R}^2)\), where \(\varepsilon \geq 0\) and \(n \in \mathbb{N}\) are constants. For \(z = (x, y) \in \mathbb{R}^2\), let \(W(x, y) = 1 + |x|^2 + |y|^2\) and set \(Z(z) = -y^3\) and \(b(\xi) = \int_{-r_0}^0 \alpha(\theta)\xi_1(\theta)d\theta + \xi_2(-r_0)\). By a straightforward computation one has for \(x, y \in \mathbb{R}\)

\[
LW(x, y) = 1 - 2x(x + y) - 2\varepsilon y^{2n} \leq 3W(x, y)
\]

and for \(\xi \in C([-r_0, 0]; \mathbb{R}^2)\)

\[
\langle b(\xi), \nabla^{(2)}W(\xi(0)) \rangle \leq 2 \left| \int_{-r_0}^0 \alpha(\theta)\xi_1(\theta)d\theta + \xi_2(-r_0) \right| |\xi_2(0)|
\]

\[
\leq 2 \left( 1 + \int_{-r_0}^0 \alpha(\theta)d\theta \right) \|\xi\|^2_\infty.
\]

Then conditions (A1) and (A2) hold. Next, there exists a constant \(c > 0\) such that for any \(z = (x, y)\) and \(z' = (x', y') \in \mathbb{R}^2\),

\[
|Z(z) - Z(z')| = \varepsilon |y^3 - y'^3| \leq c|y - y'|(|y'|^2 + |y - y'|^2).
\]

Finally, for \(\xi = (\xi_1, \xi_2), \xi' = (\xi_1', \xi_2') \in C([-r_0, 0]; \mathbb{R}^2)\),

\[
|b(\xi) - b(\xi')| \leq \sqrt{2} \left( \int_{-r_0}^0 |\alpha(\theta)|d\theta + 1 \right) \|\xi - \xi'\|_\infty.
\]

So, (A3) holds for \(l = 1\) whenever \(|y - y'| \leq 1\) and (A4) holds for any \(l \geq 0\). Moreover, (A3') and (A4') hold for \(U(|z|) = |z|^2\), \(z \in \mathbb{R}^2\). Therefore, Theorem 1.1, Theorem 1.5 and Corollary 1.3 hold.
To derive the entropy-gradient estimate and the Harnack inequality as in Corollary 1.4 we need to weaken the assumption (A). To this end, we consider a simpler setting where the delay is time discrete. Consider

\begin{equation}
(4.2) \begin{cases}
  \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\
  \mathrm{d}Y(t) = Z(X(t), Y(t)) + \tilde{b}(X(t - r_0), Y(t - r_0))\mathrm{d}t + \sigma\mathrm{d}B(t),
\end{cases}
\end{equation}

with initial data \( \xi \in \mathcal{C} \), where \( Z, \tilde{b} : \mathbb{R}^{m+d} \to \mathbb{R}^d \). If we define \( b(\xi) = \tilde{b}(\xi(-r_0)) \) for \( \xi = (\xi_1, \xi_2) \in \mathcal{C} \), then equation (4.2) can be written as equation (1.1). For \( (x, y), (x', y') \in \mathbb{R}^{m+d} \), define the diffusion operator associated with (4.2) by

\[ \mathcal{L}W(x, y; x', y') = LW(x, y) + \tilde{b}(x', y'), \nabla^{(2)}W(x, y). \]

**Theorem 4.2.** Assume that there exist constants \( \alpha, \beta, \gamma > 0 \) with \( \beta \geq \gamma \), functions \( W \in C^2(\mathbb{R}^{m+d}) \) with \( W \geq 1 \) and \( U \in C(\mathbb{R}^{m+d}; \mathbb{R}_+) \) such that for \( (x, y), (x', y') \in \mathbb{R}^{m+d} \)

\begin{equation}
(4.3) \quad \mathcal{L}W(x, y; x', y') \leq \alpha\{W(x, y) + W(x', y')\} - \beta U(x, y) + \gamma U(x', y').
\end{equation}

Assume further that there exists \( \nu > 0 \) such that for \( z = (x, y) \), \( z' = (x', y') \in \mathbb{R}^{m+d} \) with \( |z - z'| \leq 1 \)

\begin{equation}
|Z(z) - Z(z')|^2 \geq |\tilde{b}(z) - \tilde{b}(z')|^2 \leq \nu|z - z'|^2 W(z').
\end{equation}

Then for \( \delta := (\alpha r_0 + 1)\|W(\xi)\|_{\infty} + \gamma r_0\|U(\xi)\|_{\infty} \) and \( t \geq 0 \)

\begin{equation}
\mathbb{E}^xW(X(t), Y(t)) \leq \delta e^{\gamma t},
\end{equation}

and

\begin{equation}
|\nabla_h P_T f(\xi)| \leq C \sqrt{\frac{PTf^2(\xi)}{\mathbb{E}^xP_T f^2(\xi)}} \left\{ |h(0)| \left( 1 + \frac{\|M\|}{(T-r_0)^{2k+1}} \right) + \frac{r_0^2 \|W(\xi)\|_{\infty}}{\|h\|_{\infty}} \right\}
\end{equation}

for all \( T > r_0, \xi, h \in \mathcal{C} \) and \( f \in \mathcal{B}_b(\mathcal{C}) \), where \( C > 0 \) is some constant. If moreover there exist constants \( K, \lambda_i \geq 0, i = 1, 2, 3, 4 \), with \( \lambda_1 \geq \lambda_2 \) and \( \lambda_3 \geq \lambda_4 \), functions \( W \in C^2(\mathbb{R}^{m+d}) \) with \( \tilde{W} \geq 1 \) and \( \tilde{U} \in C(\mathbb{R}^{m+d}; \mathbb{R}_+) \) such that for \( (x, y), (x', y') \in \mathbb{R}^{m+d} \)

\begin{equation}
(4.7) \quad \frac{\mathcal{L}\tilde{W}(x, y; x', y')}{\tilde{W}(x, y)} \leq K - \lambda_1 W(x, y) + \lambda_2 W(x', y') - \lambda_3 \tilde{U}(x, y) + \lambda_4 \tilde{U}(x', y'),
\end{equation}

then there exist constants \( \delta_0, C > 0 \) such that for \( r \geq \delta_0/(T-r_0)^{2k+1} \), \( \xi, h \in \mathcal{C} \) and positive \( f \in \mathcal{B}_b(\mathcal{C}) \)

\begin{equation}
|\nabla_h P_T f(\xi)| \leq r \left\{ P_T f \log f - (P_T f) \log P_T f \right\}(\xi) + \frac{C 2P_T f}{2r} \left\{ \left| h(0) \right|^2 \left( \frac{1}{(T-r_0)^{2k+1}} \right) + \frac{\|M\|^2}{\{(T-r_0)^{2k+1}\}^{4k+3}} \right\} + \frac{\{1 + \|M\|^2\}|h(0)|^2}{\{(T-r_0)^{2k+1}\}^{4k+2}} \left( \lambda_2 r_0 W(\xi)_{\infty} + \lambda_4 r_0 \tilde{U}(\xi)_{\infty} + KT + \log \tilde{W}(\xi(0)) \right).\end{equation}
Proof. By the Itô formula one has for any \( t \geq 0 \)

\[
\mathbb{E}^x W(X(t), Y(t)) \leq W(\xi(0)) + \alpha \mathbb{E}^x \int_0^t \{ W(X(s), Y(s)) + W(X(s - r_0), Y(s - r_0)) \} ds
\]

\[
- \beta \mathbb{E}^x \int_0^t U(X(s), Y(s)) ds + \gamma \mathbb{E}^x \int_0^t U(X(s - r_0), Y(s - r_0)) ds
\]

\[
\leq W(\xi(0)) + \alpha \int_{-r_0}^0 W(X(s), Y(s)) ds + \gamma \int_{-r_0}^0 U(X(s), Y(s)) ds
\]

\[
+ 2\alpha \mathbb{E}^x \int_0^t W(X(s), Y(s)) ds
\]

\[
\leq \delta + 2\alpha \mathbb{E}^x \int_0^t W(X(s), Y(s)) ds.
\]

Then (1.3) follows from the Gronwall inequality.

By Theorem 1.1 for \( T - r_0 \in (0, 1] \) and some \( C > 0 \) we can deduce that

\[
|\nabla_h P_T f(\xi)| \leq C \sqrt{P_T f^2(\xi)} \left( \mathbb{E}^x \int_0^T |N(s)|^2 ds \right)^{1/2},
\]

where for \( s \in [0, T] \)

\[
N(s) := (\nabla_{\Theta(s)} Z)(X(s), Y(s)) + (\nabla_{\Theta(s - r_0)} \tilde{b})(X(s - r_0), Y(s - r_0)) - v'(s) h_2(s) - \alpha'(s).
\]

Recalling the first two inequalities in (3.1) and combining (4.4) yields that for some \( C > 0 \)

\[
|\nabla_h P_T f(\xi)| \leq C \sqrt{P_T f^2(\xi)} \left\{ \left( \int_0^T |v'(s) h_2(s) + \alpha(s)|^2 ds \right)^{1/2}
\]

\[
+ \left( \mathbb{E}^x \int_0^T |\Theta(s)|^2 W(X(s), Y(s)) ds \right)^{1/2}
\]

\[
+ \left( \mathbb{E}^x \int_0^T |\Theta(s - r_0)|^2 W(X(s - r_0), Y(s - r_0)) ds \right)^{1/2} \right\}
\]

\[
\leq C \sqrt{P_T f^2(\xi)} \left\{ |h(0)| \left( 1 + \frac{\|M\|}{(T - r_0)^{2k+1}} \right) + r_0 \frac{\delta}{2} \|\xi\|_{2k} \|h\|_{\infty}
\]

\[
+ |h(0)| \left( 1 + \frac{\|M\|}{(T - r_0)^{2k+1}} \right) \left( \mathbb{E}^x W(X(s), Y(s)) ds \right)^{1/2} \right\}.
\]

This, together with (4.5), leads to (4.6).

Due to (4.6) and (4.1) we can deduce that there exists \( C > 0 \) such that for arbitrary \( r > 0 \) and \( T - r_0 \in (0, 1] \)

\[
|\nabla_h P_T f(\xi)| \leq r \left\{ P_T f \log f - (P_T f) \log P_T f \right\}(\xi)
\]

\[
+ \frac{r P_T f(\xi)}{2} \left\{ C |h(0)|^2 \left( 1 + \frac{\|M\|^2}{(T - r_0)^{4k+3}} \right) + \frac{C \|h\|^2_{\infty} \|W(\xi)\|_{\infty} r_0}{r^2}
\]

\[
+ \log \mathbb{E}^x \exp \left\{ \frac{C (1 + \|M\|^2 |h(0)|^2}{r^2 (T - r_0)^{4k+2}} \int_0^T W(X(s), Y(s)) ds \right\} \right\}.\]
Moreover, since for $s \in [0, T]$

$$
\hat{W}(X(s), Y(s)) \exp \left( - \int_0^s \frac{\mathcal{L}\hat{W}(X(r), Y(r), X(r-r_0), Y(r-r_0))}{\hat{W}(X(r), Y(r))} \,dr \right)
$$

is a local martingale by the Itô formula, in addition to $\hat{W} \geq 1$, we obtain from (4.7) that

$$
\mathbb{E}^\xi \exp \left[ (\lambda_1 - \lambda_2) \int_0^T W(X(s), Y(s))ds - \lambda_2 r_0\|W(\xi)\|_\infty \right]
\leq \mathbb{E}^\xi \exp \left[ \int_0^T \left( \lambda_1 W(X(s), Y(s)) - \lambda_2 W(X(s-r_0), Y(s-r_0)) \right)ds \right]
\leq \mathbb{E}^\xi \exp \left[ \text{KT} - \int_0^T \frac{\mathcal{L}\hat{W}(X(s), Y(s); X(s-r_0), Y(s-r_0))}{\hat{W}(X(s), Y(s))} \,ds \right]
\leq \exp(\lambda_1 r_0\|\hat{U}(\xi)\|_\infty + \text{KT})
\times \mathbb{E}^\xi \left[ \hat{W}(X(T), Y(T)) \exp \left( - \int_0^T \frac{\mathcal{L}\hat{W}(X(s), Y(s); X(s-r_0), Y(s-r_0))}{\hat{W}(X(s), Y(s))} \,ds \right) \right]
\leq \exp(\lambda_1 r_0\|\hat{U}(\xi)\|_\infty + \text{KT})\hat{W}(\xi(0)).
$$

Combining (4.9) and (4.10), together with the Hölder inequality, yields (4.8).

The next example shows that Theorem 4.2 applies to the equation (4.2) with a highly non-linear drift.

**Example 4.3.** Consider delay SDE on $\mathbb{R}^2$

$$
\begin{cases}
    dX(t) = -\{X(t) + Y(t)\}dt \\
    dY(t) = dB(t) + \left\{ - Y^3(t) + \frac{1}{4} Y^3(t-r_0) + \frac{1}{2} X(t) - Y(t) \right\}dt
\end{cases}
$$

(4.11)

with initial data $\xi \in C([-r_0, 0]; \mathbb{R}^2)$. In this example for $z = (x, y), z' = (x', y') \in \mathbb{R}^2$ let $Z(z) = \frac{1}{2}x - y - y^3$ and $b(z') = \frac{1}{4}y^3$. For $W(x, y) = 1 + x^2 + y^4$ it is easy to see that

$$
\mathcal{L} W(x, y; x', y') = -2x(x + y) + 4y^3 \left( \frac{1}{2}x - y - y^3 + \frac{1}{4}y^3 \right)
\leq -2x^2 + y^2 - 4y^4 - 4y^6 + y^3y^3 + 2y^3x
\leq y^2 - 4y^4 - \frac{5}{2}y^6 + \frac{1}{2}y^6.
$$

Then (4.3) holds for $\beta = \frac{5}{2}, \gamma = \frac{1}{2}$ and $U(x, y) = y^6$. Moreover for $z = (x, y), z' = (x', y') \in \mathbb{R}^2$ there exists $c > 0$ such that

$$
|Z(z) - Z(z')|^2 \vee |b(z) - b(z')|^2 \leq c|z - z'|^2(|y - y'|^4 + |y'|^4).
$$
Thus condition (4.4) holds, Therefore, by Theorem 4.2 we obtain (4.6).

To derive (4.8), we take $w(x, y) = \frac{1}{4}(x^2 + y^4) + \frac{1}{10}xy$ and set $\tilde{W}(x, y) = \exp(w(x, y) - \inf w)$. Compute for $(x, y, x', y') \in \mathbb{R}^4$

$$\frac{\mathcal{L} \tilde{W}}{W}(x, y, x', y') = \mathcal{L} \log \tilde{W}(x, y) + \frac{1}{2} |\partial_y \log \tilde{W}|^2(x, y)$$

$$\leq - \left(\frac{1}{2}x + \frac{1}{10}y\right)(x + y) + \left(y^3 + \frac{1}{10}x\right)\left(\frac{1}{2}x - y - y^3 + \frac{1}{4}y^3\right) + \frac{3}{2}y^2 + \frac{1}{2}(y^3 + \frac{1}{10}x)^2$$

$$\leq 0.5((0.35)^2/\epsilon + 1.4)^2 - (0.2325 - \epsilon)x^2 - 0.5y^4 - 0.175y^6 + 0.1375y^6,$$

where $\epsilon > 0$ is some constant such that $0.2325 - \epsilon > 0$. Then condition (4.7) holds. Therefore, by Theorem 4.2 we obtain (4.8), which implies the Harnack inequality as in Corollary 1.4 according to [5, Proposition 4.1].

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