Non-singular Morse–Smale flows on $n$-manifolds with attractor–repeller dynamics

O V Pochinka and D D Shubin

National Research University Higher School of Economics, Russia
E-mail: schub.danil@yandex.ru

Received 6 April 2021, revised 29 December 2021
Accepted for publication 17 January 2022
Published 2 March 2022

Abstract
In the present paper the exhaustive classification up to topological equivalence of non-singular Morse–Smale flows on $n$-manifolds $M^n$ with exactly two periodic orbits is presented. Denote by $G_2(M^n)$ the set of such flows. Let a flow $f^t : M^n \to M^n$ belong to the set $G_2(M^n)$. Hyperbolicity of periodic orbits of $f^t$ implies that among them one is an attracting and the other is a repelling orbit. Due to the Poincaré–Hopf theorem, the Euler characteristic of the ambient manifold $M^n$ is zero. Only the torus and the Klein bottle can be ambient manifolds for $f^t$ in case of $n = 2$. The authors established that there are exactly two classes of topological equivalence of flows in $G_2(M^2)$ if $M^2$ is the torus and three classes if $M^2$ is the Klein bottle. For all odd-dimensional manifolds the Euler characteristic is zero. However, it is known that an orientable three-manifold admits a flow from $G_2(M^3)$ if and only if $M^3$ is a lens space $L_{p,q}$. In this paper it is proved that every set $G_2(L_{p,q})$ contains exactly two classes of topological equivalence of flows, except the case when $L_{p,q}$ is homeomorphic to the three-sphere $S^3$ or the projective space $\mathbb{R}P^3$, where such a class is unique. Also, it is shown that the only non-orientable $n$-manifold (for $n > 2$), which admits flows from $G_2(M^n)$ is the twisted I-bundle over the $(n - 1)$-sphere $S^{n-1} \times S^1$. Among orientable $n$-manifolds only the product of the $(n - 1)$-sphere and the circle $S^{n-1} \times S^1$ can be ambient manifolds for flows from $G_2(M^n)$ and $G_2(S^{n-1} \times S^1)$ splits into two topological equivalence classes.

Keywords: Morse–Smale flows, nonsingular flows, topological classification
Mathematics Subject Classification numbers: 37D15.

(Some figures may appear in colour only in the online journal)
1. Introduction and statement of results

This article will focus on non-singular Morse–Smale flows (abbreviated as NMS-flows), which are Morse–Smale flows without fixed points, given on closed \( n \)-manifolds \( M^n \), \( n > 1 \). The authors obtained the exhaustive topological classification of NMS-flows \( f^t : M^n \to M^n \) with exactly two periodic orbits. A general theory of hyperbolic dynamical systems (see e.g. [13]) implies that the ambient manifold \( M^n \) for such a flow \( f^t \) is the union of the stable and the unstable manifolds of these orbits. It immediately implies that one of these orbits is attracting (denote it \( A \)) and the other is a repelling (denote it \( R \)).

Let \( G_2(M^n) \) be the class on NMS-flows with exactly two periodic orbits. In cases where the results are fundamentally different for orientable and non-orientable manifolds we will use the notation \( M^n_+, M^n_- \) for orientable and non-orientable manifolds respectively.

Recall that a periodic orbit is called twisted if at least one of its invariant manifolds is non-orientable. Otherwise, we call the orbit untwisted. The Poincaré–Hopf theorem implies that the Euler characteristic of an NMS-flow ambient manifold equals zero. Considering two-dimensional surfaces we immediately get that this constraint leaves us with only the torus and the Klein bottle (actually, both admit NMS-flows). The classification of such flows follows from the classification of Morse–Smale flows on surfaces (see e.g. [8, 11, 12]). We provide an independent classification in the class \( G_2(M^2) \) in section 4.

**Theorem 1.**

(a) The set \( G_2(M^2_+) \) splits into two topological equivalence classes of flows (see figure 1), both with untwisted orbits.

(b) The set \( G_2(M^2_-) \) splits into three topological equivalence classes of flows (see figure 2), two with twisted orbits and one with untwisted orbit.

The Euler characteristic of any odd-dimensional manifold is zero then \textit{a priori}, such a manifold \( M^n \) admits a flow from \( G_2(M^n) \). For \( n = 3 \) necessary and sufficient conditions for the topological equivalence of three-dimensional NMS-flows follows from [14], where a larger class of Morse–Smale flows were considered. However, this classification does not allow us to say anything about the admissible topology of the ambient manifolds. In the case of a small number of periodic orbits the topology of the ambient manifold and exhaustive topological classification can be established.

Recall, that a lens space is defined as the topological space obtained by gluing two solid tori by a homeomorphism of their boundaries and is denoted as \( L_{p,q} \), \( p, q \in \mathbb{Z} \), where \( \langle p, q \rangle \) is the homotopy type of the meridian image under the gluing homeomorphism. Some well known three-manifolds are lens spaces, for example, three-sphere \( S^3 = L_{1,0} \), the manifold \( S^2 \times S^1 = L_{0,1} \), the projective space \( \mathbb{R}P^3 = L_{1,2} \).

It follows from the proposition below that only lens spaces can be ambient manifolds for NMS-flows with a small number of the periodic orbits.

**Proposition 1 ([2]).** Let \( M \) be an orientable, simple\(^1\), closed three-manifold without boundary. If \( M \) admits an NMS-flow with 0 or 1 saddle periodic orbit, then \( M \) is a lens space.

Since it is obligatory that every NMS-flow has at least one attracting and at least one repelling orbit then the complete number of orbits for a flow, satisfying proposition 1, is at least \( \sum_i n_i \).
Figure 1. Phase portraits of non topologically equivalent flows on the torus.

Figure 2. Phase portraits of pairwise non topologically equivalent flows on the Klein bottle: 1 with untwisted orbits; 2–3 with twisted orbits.

Figure 3. Phase portraits of equivalent flows on three-sphere.

two (exactly two when there are no saddle orbits). Existence and uniqueness up to topological equivalence of a flow in the set $G_2(S^3)$ follows from the proposition below.

**Proposition 2 ([15]).** Up to topological equivalence, there exists exactly one NMS-flow $f^+ : S^3 \to S^3$ whose periodic orbits are composed of an attractor $A$ and a repeller $R$. Moreover, the periodic orbits $A \sqcup R$ form the Hopf link in $S^3$ (see figure 3).

In the present paper the exhaustive topological classification of the class $G_2(M^3)$ is done.

**Theorem 2.**

(a) A manifold $M^3_+$ admits a flow from the set $G_2(M^3_+)$ if and only if $M^3_+$ is a lens space. The set $G_2(M^3_+)$ splits into two topological equivalence classes of flows (see figure 4), except
the case when \( M^3 \) is the three-sphere \( S^3 \) or the projective space \( \mathbb{R}P^3 \), where such a class is unique. The periodic orbits are untwisted in any case.

(b) The only non-orientable manifold which admits flow from the set \( G_2(M^3) \) is the twisted I-bundle over the two-sphere \( S^2 \times S^1 \). The set \( G_2(S^2 \times S^1) \) splits into two topological equivalence classes of flows, both periodic orbits of such flows are twisted.

The exhaustive classification of \( G_2(M^n) \), \( n > 3 \) follows from the theorem below.

**Theorem 3.**

(a) A manifold \( M^+_n \) admits a flow from the set \( G_2(M^+_n) \) if and only if \( M^+_n \) is homeomorphic to \( S^{n-1} \times S^1 \). The set \( G_2(S^{n-1} \times S^1) \) splits into two topological equivalence classes of flows, both periodic orbits of such flows are untwisted.

(b) A manifold \( M^-_n \) admits a flow from the set \( G_2(M^-_n) \) if and only if \( M^-_n \) is homeomorphic to \( S^{n-1} \times S^1 \). The set \( G_2(S^{n-1} \times S^1) \) splits into two topological equivalence classes of flows, both periodic orbits of such flows are twisted.

2. General properties of NMS-flows

In this section we provide properties of the NMS-flows which are necessary for the subsequent proofs.

Flows \( f^t \) and \( f'^t \) on a manifold \( M^n \) are said to be *topologically equivalent* if there is a homeomorphism \( h : M^n \to M^n \) which sends orbits of \( f^t \) into orbits of \( f'^t \) and preserves the orientation on the orbits.

To describe the behaviour of a flow \( f^t : M^n \to M^n \) in a neighbourhood of an attracting or repelling hyperbolic orbit we use the following notion of a *suspension*.

Define a diffeomorphism \( a_\pm : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by

\[
a_\pm(x_1, x_2, \ldots, x_{n-1}) = (\pm 2x_1, 2x_2, \ldots, 2x_{n-1}).
\]

Let \( g_\pm : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism defined by

\[
g_\pm(x, r) = (a_\pm(x), r - 1).
\]
Let \( \Pi_\pm = \mathbb{R}^n / (g_n) \) and denote the natural projection by \( v_\pm : \mathbb{R}^n \to \Pi_\pm \). Define a flow \( \bar{b}' \) on \( \mathbb{R}^n \) by the system of the following differential equations:

\[
\begin{align*}
\dot{x}_1 &= 0, \\
\vdots \\
\dot{x}_{n-1} &= 0, \\
\dot{x}_n &= 1.
\end{align*}
\]

The natural projection \( v_\pm \) induces a flow \( b'_\pm = v_\pm \bar{b}' v_\pm^{-1} : \Pi_\pm \to \Pi_\pm \) which is called suspension.

**Proposition 3 ([5]).** Every hyperbolic repelling orbit \( R \) of a flow \( f^t : M^n \to M^n \) possesses the unstable manifold \( W^u_R = \{ x \in S | f^t(x) \to R \quad \text{as} \quad t \to -\infty \} \) with the following properties:

(a) There is a value \( \delta_R \in \{-, +\} \) and a homeomorphism \( h_R : \Pi_{\delta_R} \to W^u_R \) which provides the topological equivalence of the flows \( b'_\delta \) and \( f^t|_{W^u_R} \). The orbit \( R \) is twisted, if \( f^t|_{W^u_R} \) is equivalent to \( b'_\delta \) and is untwisted otherwise.

(b) \( W^u_R \) is diffeomorphic to \( \mathbb{R}^{n-1} \times S^1 \) if \( R \) is untwisted and is diffeomorphic to \( \mathbb{R}^{n-1} \times \mathbb{S}^1 \) if \( R \) is twisted.

A similar statement holds for the stable manifold \( W^s_A = \{ x \in S | f^t(x) \to A \quad \text{as} \quad t \to +\infty \} \) of the hyperbolic attracting orbit \( A \) which states that a flow \( b''_{\delta_A} \), \( \delta_A \in \{-, +\} \) is topologically equivalent to the flow \( f^t|_{W^s_A} \) by means of a homeomorphism \( h : W^s_A \to \Pi_{\delta_A} \).

For \( r > 0 \) let (see figure 5)

\[
\bar{V}_r^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n-1}^2 \leq r 2^{-\delta_A}\}, \quad \forall \bar{V}_r^n = v_\pm(\bar{V}_r^n).
\]
By the construction, the quotient space $V_n^+ + r$ is homeomorphic to the **generalized solid torus** $D^{n-1} \times S^1$ and the quotient $V_n^- r$ is homeomorphic to the **generalized solid Klein bottle** $D^{n-1} \tilde{\times} S^1$.

Let $V^n = V^n_1$, $V^n_\pm = u_\pm(V^n)$, $\mathbb{I}^n_\pm = u_\pm(Ox_0)$, where $Ox_i$ is the coordinate axis. Consider a homeomorphism $j : \partial V^n_\pm \to \partial V^n_\pm$, two copies $V^n_\pm \times \mathbb{I}^n_\pm$ of the manifold $V^n_\pm$ and a homeomorphism $J : \partial V^n_\pm \times \{0\} \to \partial V^n_\pm \times \{1\}$, defined by $J(s, 0) = (j(s), 1)$. Let

$$M^n_j = V^n_\pm \times \{0\} \cup_j V^n_\pm \times \{1\}.$$  

Denote the natural projection by $p_j : V^n_\pm \times \mathbb{I}^n_\pm \to M^n_j$ and let $f^n_j : M^n_j \to M^n_j$ be a topological flow defined by

$$f^n_j(x) = \begin{cases} p_jb^n_\pm(p_j|_{V^n_\pm \times \{0\}})^{-1}(x), & x \in p_j|_{V^n_\pm \times \{0\}}, \ t \leq 0 \\ p_jb^n_\mp(p_j|_{V^n_\pm \times \{1\}})^{-1}(x), & x \in p_j|_{V^n_\pm \times \{1\}}, \ t \geq 0 \end{cases}.$$  

We call $f^n_j : M^n_j \to M^n_j$ *n*-dimensional model flows. For any model flow let (see figure 6).

$$R_j = p_j(\mathbb{I}^n_\pm \times \{0\}), \quad A_j = p_j(\mathbb{I}^n_\pm \times \{1\}),$$  

$$V_{R_j} = p_j(V^n_\pm \times \{0\}), \quad V_{A_j} = p_j(V^n_\pm \times \{1\}),$$  

$$\Sigma_j = p_j(\partial V^n_\pm \times \{0\}) = p_j(\partial V^n_\pm \times \{1\}).$$  

**Lemma 1.** Every flow $f^i \in G_2(M^n)$ is topologically equivalent to some model flow $f^n_j : M^n_j \to M^n_j$.

**Proof.** Let $f^i \in G_2(M^n)$ and $A, R$ be the attracting and the repelling hyperbolic orbits respectively. Due to proposition 3 there is a homeomorphism $h_R : \Pi_\pm \to W^n_\pm$ which provides the topological equivalence of the flows $b^\pm$ and $f^i|_{W^n_\pm}$. Also, there is a homeomorphism $h_A : \Pi_\pm \to W^n_\pm$ which provides the topological equivalence of the flows $b^\pm$ and $f^i|_{W^n_\pm}$. Let $V_A = h_A(V^n_\pm)$ and $H_A = h_A|_{V^n_\pm}$. We can choose $r > 0$ such that a neighbourhood $V^n_R = h_R(V^n_\pm)$ of $R$ is disjoint with $V_A$. Since the non-wandering set of $f^i$ consists of $A$ and $R$ then (see e.g. [13])

$$M^n = W^n_R \cup A = W^n_A \cup R$$

and consequently the set $M^n\cap({\text{int}}(V_A \cup V^n_R))$ consists of segments of wandering trajectories of the flow $f^i$, which have their boundary points on different connected components of the boundary.
Figure 7. The neighbourhoods \( V_A \) and \( V'_R \) on \( M^n \).

\[ \partial M^n \setminus (V_A \cup \mathbb{V}'_R) \] (see figure 7). Let \( V_R = M^n \setminus \text{int} V_A \). Then the homeomorphism \( h_R |_{\mathbb{V}'_R} \) can be extended to the homeomorphism \( H_R : \mathbb{V}'_R \to V_R \) which provides the topological equivalence of the flows \( h'_R \) and \( f' \). Define a homeomorphism \( j : \partial \mathbb{V}'_R \to \partial \mathbb{V}'_R \) by

\[ j = H_R^{-1} \partial \mathbb{V}'_R. \]

Define a homeomorphism \( H : M^n_j \to M^n \) by

\[
H(x) = \begin{cases} 
H_R p_j^{-1}(x), & x \in V_A_j, \\
H_R p_j^{-1}(x), & x \in V_B_j.
\end{cases}
\]

It is directly verified that \( H \) provides the topological equivalence of the flows \( f'_j \) and \( f' \). □

Thus, the classification of the flows with the attractor–repeller dynamics can be reduced to the classification of the model flows. Using methods of the previous proof it is easy to show that it is sufficient to consider some special class of homeomorphisms providing the topological equivalence of the model flows.

**Lemma 2.** If the model flows \( f'_j : M^n_j \to M^n_j \) and \( f'_j : M^n_j' \to M^n_j' \) are topologically equivalent, then there is a topological equivalence of these flows given by a homeomorphism \( H : M^n_j \to M^n_j' \) such that \( H(\Sigma_j) = \Sigma_j' \).

3. A criterion for topological equivalence of model flows

In this section we give a criterion for topological equivalence of model flows, from which the complete description of equivalence classes in \( \mathcal{G}_2(M^n) \) follows.

Denote by \( \hat{\alpha} \) the connected component of the set \( \partial \mathbb{V}_R \cap O_{X_2} x_n \) containing the point \( (0,1,\ldots,0) \). We consider \( \hat{\alpha} \) as the curve oriented in the direction of the increasing \( x_n \) coordinate. Let \( \hat{\beta} = \partial \mathbb{V}_R \cap O_{X_1 \ldots x_{n-1}} \) (see figure 8) and

\[ \alpha_\pm = v_\pm(\hat{\alpha}), \quad \beta_\pm = v_\pm(\hat{\beta}). \]

Denote by \( i : \partial \mathbb{V}'_R \to \mathbb{V}'_R \) the inclusion map and by \( i_* : \pi_1(\partial \mathbb{V}'_R) \to \pi_1(\mathbb{V}'_R) \) the induced isomorphism. Since the group \( \langle g_\pm \rangle \) is isomorphic to \( \mathbb{Z} \) and acts freely and discontinuously on the
simply connected space $\tilde{V}^n$, the fundamental group $\pi_1(V^n_\pm)$ is also isomorphic to the group $\mathbb{Z}$
(see e.g. [7]) and $\alpha_\pm$ is its generator$^2$.

**Theorem 4 (Criteria for topological equivalence).** Two model flows $f^i_j : M^n_i \to M^n_j$, $f^i_j : M^n_j \to M^n_i$, are topologically equivalent if and only if there is a homeomorphism $h_0 : \partial V^n_\pm \to \partial V^n_\pm$ such that

$$i_*h_{0*} = i_* \quad (1)$$

and the homeomorphism $h_1 = f^i_0 h_0^{-1}$ possesses the property

$$i_*h_1* = i_* \quad (2)$$

**Proof.** Necessity. Let the flows $f^i_j$ and $f^j_i$ be topologically equivalent by means of a homeomorphism $H : M^n_i \to M^n_j$. By lemma 2 without loss of generality we assume that $H(\Sigma_\pm) = \Sigma_\pm$.

Define a homeomorphism $H_k : \mathbb{V}^n_\pm \to \mathbb{V}^n_\pm$, $k \in \mathbb{Z}$, by $(H_k(s),k) = p^i_j H p^j_i|\mathbb{V}^n_\pm \times \{k\} \to \mathbb{V}^n_\pm \times \{k\}$.

Let $h_k = H_k(\partial \mathbb{V}^n_\pm)$.

Notice that the curves $\mathbb{L}_\pm$ and $\alpha_{\pm}$ are homotopic in $\mathbb{V}^n_\pm$ as they bound a two-dimensional annulus $\mathbb{V}^n_\pm \cap O_2(x_n)$ in $\mathbb{V}^n_\pm$. As $H$ provides the topological equivalence of the flows $f^i_j$ and $f^j_i$, then $H_0(R_i) = R_j$, that implies $H_0([R_i]) = [R_j]$. Considering the fact that $\pi_1(\mathbb{V}^n_\pm) \cong \langle \alpha_\pm \rangle$ we can deduce that $H_0* = id$ which implies equality (1) $i_*h_{0*} = i_*$.

It follows from the definition of the model flow that $h_1 = f^i_0 h_0^{-1}$. The equality (2) $i_*h_1* = i_*$ for the map $h_1$ can be proved as above.

Sufficiency. Assume that there is a homeomorphism $h_0 : \partial \mathbb{V}^n_\pm \to \partial \mathbb{V}^n_\pm$ such that $i_*h_{0*} = i_*$ and the homeomorphism $h_1 = f^i_0 h_0^{-1}$ satisfies $i_*h_1* = i_*$. Since $i_*h_{0*} = i_*$ and due to [3] [proposition 10.2.26], homeomorphism $h_0$ admits a lift $\tilde{h}_0 : \partial V^n \to \partial V^n$, which commutes with $g_\pm$. Let $\tilde{\beta} = h_0(\beta)$. Choose a positive integer $n_0$ such that $\tilde{\beta} \subset \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : 0 < x_n < n_0\}$ (see figure 9). Let us extend the homeomorphism $h_0$ to a homeomorphism $H_0 : V^n \to V^n$ commuting with $g_\pm : \mathbb{R}^n \to \mathbb{R}^n$ which provides the topological equivalence of the flow $\tilde{\beta}'$ with itself.

Recall that $g_{\pm}(x,r) = (a_{\pm}(x), r - 1)$. Let $y = (x_1, \ldots, x_{n-1})$ and define a map $p_0 : \mathbb{R}^n \to Oy$ by

$$p_0(y, x_n) = y.$$  

$^2$The space $\partial \mathbb{V}^n_\pm$ consists of two connected components. In this case $i_*$ is a map composed of the induced isomorphisms for each connected component.
By the construction $p_0|_{\partial V^n} : \partial V^n \rightarrow Oy\setminus O$ (hyperplane $x_n = 0$ without the point $O$) is a diffeomorphism for any $r > 0$. Define the homeomorphism $\tilde{w} : Oy\setminus O \rightarrow Oy\setminus O$ by

$$\tilde{w} = p_0 \tilde{h}_0(p_0|_{\partial V^n})^{-1}.$$

Since $i, h_{0, i} = i_*$, the homeomorphism $\tilde{w}$ can be continuously extended to the point $O$ by $\tilde{w}(O) = O$. Assume that $y' = \tilde{w}(y), y \in Oy$, $\beta_0^+ = \beta_0(\tilde{\beta}^+)$ and $\beta_0 = p_0(\bar{g}_0^n(\tilde{\beta})$. Let $B, B_0, B^*_0$ denote the disks in $Oy$ which are bounded by the spheres $\beta, \beta_0, \beta_0^+$ respectively. Thus, $\tilde{w}(B) = B^*_0$.

By virtue of the annulus conjecture and the fact that spheres $\beta_0$ and $\beta_0^+$ are disjoint and are cylindrically embedded they bound an annulus $K_0 \subset Oy$ (see figure 9). Let $\tau : S^{n-2} \times [0, 1] \rightarrow K_0$ be a homeomorphism such that $\tau(S^{n-2} \times \{0\}) = \beta_0$ and $\tau(S^{n-2} \times \{1\}) = \beta_0^+$. For $t \in [0, 1]$ put $c_t = \tau(S^{n-2} \times \{t\})$, $r_t = t + 2^{-m}(1 - t)$ and $C_t = (p_0|_{\partial V^n})^{-1}(c_t)$. Let us define a disk $B'$ by

$$B' = \bigcup_{t \in [0, 1]} C_t \cup B_0.$$

Since $i, h_{0, i} = i_*$, the homeomorphism $h_{0, i}$ sends the part $\Gamma$ of the cylinder $\partial V^n$ lying between spheres $\beta, \bar{g}_0^n(\tilde{\beta})$ to the part $\Gamma^*$ of cylinder $\partial \bar{V}^n$ lying between the spheres $\beta^+, \bar{g}_0^n(\tilde{\beta})$. Denote by $W(W')$ a compact subset of $\bar{V}^n$ bounded by $B, g_0^{-1}(B)$ and $\Gamma^*$ (by $B', g_0^{-1}(B')$ and $\Gamma^*$ respectively).

For every $y_0 \in Oy$ let $L_{x_0} = \{y, x_n \in \mathbb{R}^2 : y = y_0\}$. For $(y_0, 0) \in B$ let $I_{y_0} = L_{x_0} \cap W$ and $I'_{y_0} = L_{x_0} \cap W'$. Denote boundary points of segments $I_{y_0}$ and $I'_{y_0}$ by $A_{y_0}, B_{y_0}$ and $A'_{y_0}, B'_{y_0}$, where $A_{y_0} = (y_0, 0), B_{y_0} = (y_0, b_{y_0})$ and $A'_{y_0} = (y_0, a'_{y_0}), B'_{y_0} = (y_0, b'_{y_0}), a'_{y_0} \leq b'_{y_0}$. Define a homeomorphism $\tilde{h}_{y_0} : I_{y_0} \rightarrow I'_{y_0}$ by

$$\tilde{h}_{y_0}(y_0, x_n) = \left( y_0, x_n, \frac{b'_{y_0} - a'_{y_0}}{b_{y_0}} + a'_{y_0} \right).$$

By virtue of the fact that $W = \bigcup_{y_0 \in B} I_{y_0}$ we get a homeomorphism $h_W : W \rightarrow W'$, composed of $\tilde{h}_{y_0}$, which provides the topological equivalence of the flow $b'_{y_0}|_W$ with $b'_{y_0}|_{W'}$. Extend $h_{W}$ to $V^n$ by

$$\tilde{h}_0(x_1, \ldots, x_{n-1}, x_n) = g_{\lfloor x_n \rfloor}(g_{\lfloor x_n \rfloor}(x_1, \ldots, x_{n-1}, x_n)),$$

where $[x_n]$ denotes the integer part of the number $x$. By the construction $H_0 g_{\pm} = g_{\pm} H_0$. By virtue of [3] this fact implies, that $H_0 = v_{\pm}^{-1} H_0 v_{\pm}$ is a homeomorphism and the following equality holds $H_0 b' = b'H_0$.

In the same way the homeomorphism $h_1$ can be extended to a homeomorphism $H_1 : V^m_{\pm} \rightarrow V_{\pm}^m$ commuting with $g_{\pm}$ and providing the topological equivalence of the flow $b'_{y_0}$ with itself. Thus, the required homeomorphism $H : M^m_j \rightarrow M^m_{j'}$ is defined by

$$H(x) = \begin{cases} p_j^* H_0 p_j^{-1}(x), & x \in p_{j'}(V^m_{\pm} \times \{0\}) \\ p_j^* H_1 p_j^{-1}(x), & x \in p_{j'}(V^m_{\pm} \times \{1\}) \end{cases}.$$
4. Classification of surface model flows

In this section we prove theorem 1.

Let \( f_j : M^2_j \to M^2_j \) be a two-dimensional model flow. Then the ambient surface \( M^2_j \) has the form \( M^2_j = \mathbb{V}^2_+ \times \{0\} \cup \mathbb{V}^2_- \times \{1\} \), where \( J : \partial \mathbb{V}^2_+ \times \{0\} \to \partial \mathbb{V}^2_- \times \{1\} \) is a homeomorphism defined as \( J(s, 0) = (j(s), 1) \) for some homeomorphism \( j : \partial \mathbb{V}^2_+ \to \partial \mathbb{V}^2_- \).

If the periodic orbit is untwisted then its tubular neighbourhood is an annulus and its boundary has two connected components each of which is homeomorphic to the circle. If the periodic orbit is twisted then its tubular neighbourhood is a Möbius band and its boundary is homeomorphic to the circle. Let \( S^1 = \{ e^{i\varphi}, \varphi \in \mathbb{R} \}, S^0 = \{-1, +1\} \). Define the following diffeomorphisms on the manifolds \( \partial \mathbb{V}^2_+ \cong S^1 \times S^0 \):

(a) \( j_1(\varphi, \pm 1) = (\varphi, \pm 1) \);
(b) \( j_2(\varphi, \pm 1) = (-\varphi, \pm 1) \);
(c) \( j_3(\varphi, -1) = (-\varphi, -1), j_3(\varphi, +1) = (\varphi, +1) \).

Define the following diffeomorphisms on the manifold \( \partial \mathbb{V}^2_- \cong S^1 \):

(d) \( j_4(\varphi) = \varphi \);
(e) \( j_5(\varphi) = -\varphi \).

Figures 10 and 11 provide the phase portraits of the model flows corresponding to the given maps. The sign ‘+’ means gluing with the map \( \varphi \) and the sign ‘−’ with the map \(-\varphi\).
So, theorem 1 is easily derived from the following lemma.

**Lemma 3.**

(a) Every model flow \( f^j_1 : M^2_+ \to M^2_+ \) is topologically equivalent either to \( f^j_1 \) or to \( f^j_2 \), herein \( f^j_1 \) and \( f^j_2 \) are non topologically equivalent to each other.

(b) Every model flow \( f^j_1 : M^2_+ \to M^2_- \) is topologically equivalent either to \( f^j_1 \) or to \( f^j_2 \), or to \( f^j_3 \), herein \( f^j_1, f^j_2, f^j_3 \) are pairwise non topologically equivalent.

**Proof.** Since the fundamental group of the circle is isomorphic to the group \( \mathbb{Z} \) then every orientation-preserving homeomorphism induces identical action in the fundamental group and the orientation-reversing homeomorphism induces action which changes the homotopy type of the curve to the opposite (which corresponds to multiplication of the integer by \(-1\)). Then, by virtue of theorem 4, flows \( f^j_1 : M^2_+ \to M^2_+ \), \( f^j_1 : M^2_+ \to M^2_- \), are topologically equivalent if and only if the orientation-preserving homeomorphism \( h_0 : \partial V^3_+ \times \{0\} \to \partial V^3_- \times \{1\} \) such that the homeomorphism \( h_1 = f^j_1 h_0 j^{-1} \) preserves orientation. That is equivalent to the fact that \( f^j_1 j^{-1} \) preserves orientation.

The exhaustive search of all possible combinations of the orientation type of the homeomorphism \( j \) on the connected components gives that the homeomorphism \( j^j_1 j^{-1} \) preserves the orientation exactly for unique value \( i \in \{1, \ldots, 5\} \). Moreover, if \( i = 1, 2 \) the ambient manifold is the torus, but for \( i = 3, 4, 5 \) it is the Klein bottle. Finally, if \( i = 4, 5 \) then both orbits are twisted but untwisted when \( i = 1, 2, 3 \).

\[ \square \]

5. **Classification of three-dimensional model flows**

Let \( f^j_1 : M^3_+ \to M^3_+ \) be a three-dimensional model flow. Then the ambient manifold \( M^3_+ \) has a form \( M^3_+ = V^3_+ \times \{0\} \cup J V^3_+ \times \{1\} \) where \( J : \partial V^3_+ \times \{0\} \to \partial V^3_- \times \{1\} \) is a homeomorphism defined by \( J(s, 0) = (j(s), 1) \) for \( j : \partial V^3_+ \to \partial V^3_- \). It is easy to see that the manifold \( V^3_+ \) is the solid torus whereas \( V^3_- \) is the solid Klein bottle. In the first case, the ambient manifold \( M^3_+ \) is a lens space, which is orientable and, by proposition 4 below, there are countably many pairwise non-homeomorphic lens spaces. All the manifolds obtained by gluing the solid Klein bottles, on the contrary, are homeomorphic to the \( S^2 \times S^1 \) (see e.g. [4, section 3.1(c)]).

Let us prove theorem 2 separately for orientable and non-orientable manifolds.
5.1. Orientable case

On the torus $\partial V^3_+$ the curves $\alpha_+$ and $\beta_+$ are generators in the torus fundamental group. The oriented curves $\alpha_+$, $\beta_+$ on the torus $\partial V^3_+$ are said to be the longitude and meridian respectively (figure 12). The action of the homeomorphism $j$ in the fundamental group of the torus is uniquely defined by an unimodular integer matrix

$$j_\ast = \begin{pmatrix} r & p \\ s & q \end{pmatrix}. $$

Recall that the presentation of the lens space in the form $M^3_j = V^3_+ \times \{0\} \cup V^3_+ \times \{1\}$ is called a Heegaard decomposition and $\Sigma_j$ is called a Heegaard torus. Using the uniqueness up to isotopy of the Heegaard torus in every lens space (see, for example, [4]), we will suppose below that every homeomorphism $h: M^3_j \to M^3_j'$ possesses a property

$$\eta_1(h(p_j(V^3_+ \times \{0\}))) = p'_j(V^3_+ \times \{0\}),$$

here $\eta_t: M^3_j \to M^3_j'$, $t \in [0, 1]$ is an isotopy which moves $h(\Sigma_j)$ to $\Sigma_j'$. The classification of the lens spaces up to such homeomorphisms has the following description.

**Proposition 4 (Lens space classification, [1]).** Two lens spaces $M^3_j$ and $M^3_j'$, are homeomorphic if and only if the induced isomorphisms $j_\ast = \begin{pmatrix} r & p \\ s & q \end{pmatrix}$, $j'_\ast = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix}$ satisfy the conditions $|p'| = |p|$ and $q' \equiv \pm q \pmod{|p|}$.

The following lemma is a refinement of proposition 4 which we need to prove theorem 4.

**Lemma 4.** Two homeomorphisms $h_0, h_1: \partial V^3_+ \to \partial V^3_+$ such that $i \circ h_k = i_k$ for $k = 0, 1$ and $h_1 = f h_0 j^{-1}$ exist if and only if the induced isomorphisms $j_\ast = \begin{pmatrix} r & p \\ s & q \end{pmatrix}$, $j'_\ast = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix}$ satisfy the conditions $|p'| = |p|$, $q' \equiv \pm r \pmod{|p|}$.

**Proof.** Necessity. Assume there is a homeomorphism $h_0: \partial V^3_+ \to \partial V^3_+$ such that for $h_0$ and $h_1 = f h_0 j^{-1}$ the condition $i \circ h_k = i_k$, $k = 0, 1$ holds. Then the homeomorphism $h_k$ acts in the

Considering the fact that matrices $j_\ast$, $j'_\ast$ are unimodular it is easy to establish one more criteria for two lens spaces to be homeomorphic. Namely, two lens spaces $M^3_j$ and $M^3_j'$ are homeomorphic if and only if the induced isomorphisms $j_\ast = \begin{pmatrix} r & p \\ s & q \end{pmatrix}$, $j'_\ast = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix}$ satisfy the relations $|p'| = |p|$ and $r' \equiv \pm r \pmod{|p|}$. 

![Figure 12. Longitude and meridian pre-images on $V^3$](image-url)
fundamental group of the torus with the matrix

\[
h_k = \begin{pmatrix} 1 & 0 \\ m_k & \delta_k \end{pmatrix}
\]

where \( m_k \) is integer, \( \delta_k \in \{0, 1\} \) and

\[
h_1, j_1 = f_1 h_{0*}.
\]

The last relation can be written in the following matrix form:

\[
\begin{pmatrix} 1 & 0 \\ m_1 & \delta_1 \end{pmatrix} \begin{pmatrix} r & p \\ s & q \end{pmatrix} = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_0 & \delta_0 \end{pmatrix}.
\]

(3)

So, we get the equalities \( p = \delta_0 p' \), \( r = r' + m_0 p' \), which are equivalent to \( |p'| = |p| \) and \( r' \equiv r \pmod{|p|} \).

Sufficiency. Let the elements of the matrices \( j_* = \begin{pmatrix} r & p \\ s & q \end{pmatrix} \), \( j'_* = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix} \) satisfy the relations \( |p'| = |p| \) and \( r' \equiv r \pmod{|p|} \). Thus

\[
p = \delta_0 p', \quad r = r' + m_0 p'
\]

for some \( \delta_0 \in \{-1, 1\} \), \( m_0 \in \mathbb{Z} \). Let \( h_0 : \partial\mathbb{V}^3_+ \rightarrow \partial\mathbb{V}^3_+ \) be the algebraic torus automorphism defined by the matrix \( h_{0*} = \begin{pmatrix} 1 & 0 \\ m_0 & \delta_0 \end{pmatrix} \). Then \( i_* h_{0*} = i_* \). Formula (3) and the fact that all the matrices are unimodular give us that the homeomorphism \( h_1 = j'_1 h_0 j_1^{-1} : \partial\mathbb{V}^3_+ \rightarrow \partial\mathbb{V}^3_+ \) induces the isomorphism, defined by the matrix \( h_{1*} = \begin{pmatrix} 1 & 0 \\ m_1 & \delta_1 \end{pmatrix} \) for some \( \delta_1 \in \{-1, 1\} \), \( m_1 \in \mathbb{Z} \). Thus \( i_* h_{1*} = i_* \). □

Lemma 5. Up to topological equivalence there is only one flow on both \( S^3 \) and \( \mathbb{R}P^3 \) and two flows on the remaining lens spaces.

Proof. By virtue of proposition 4 and lemma 4, on the same lens space there are either one or two topological equivalence classes of flows from \( G_2(M^3_3) \) and the cases are distinguished by the following condition: whether for two coprime numbers \( p \geq 0, r \in \mathbb{Z}_p \) there are two numbers \( n_1, n_2 \) such that

\[
r + n_1 p = -r + n_2 p
\]

(5)

Check whether condition (5) holds for all the possible values of \( p \).

(a) If \( p = 0 \) than condition (5) does not hold, since the equality is true only if \( r = 0 \) but in this case \( r, p \) are not coprime;
(b) If \( p = 1 \) the condition holds for \( r = 0 \), which means that up to topological equivalence there is a unique flow in \( G_2(S^3) \);
(c) If \( p = 2k \) then \( r = k(n_2 - n_1) \). Considering the fact that \( (r, p) = 1 \) one deduce that \( k = 1, r = 1 \), which means that up to topological equivalence there is a unique flow in \( G_2(\mathbb{R}P^3) \);
(d) For \( p = 2k + 1, k > 0 \) condition (5) never holds since \( n_2 - n_1 \) is even and \( (r, p) \neq 1 \). □
5.2. Non-orientable case

By [9] every homeomorphism \( j : \partial V^3_+ \rightarrow \partial V^3_+ \) satisfies either \( i_*(j_*(\{c\})) = i_*(\{c\}) \) or \( i_*(j_*(\{c\})) = -i_*(\{c\}) \). Then theorem 4 implies that there are two topological equivalence classes of the flows in \( G_2(S^2 \times S^1) \).

6. Classification of \( n \)-dimensional model flows for \( n > 3 \)

According to [10], every homeomorphism \( j : \partial V^n_+ \rightarrow \partial V^n_+ \) can be extended to a homeomorphism \( \phi : V^n_+ \rightarrow V^n_+ \). Thus the only manifold obtained by gluing two copies of \( V^n_+ \) along the boundaries is \( S^{n-1} \times S^1 \). Similarly, the results of the article [6] imply that the only manifold obtained by gluing two copies of \( V^n_- \) along the boundaries is \( S^{n-1} \tilde{\times} S^1 \).

Since the fundamental group \( \pi_1(\partial V^n_{\pm}) \) is isomorphic to the group \( \mathbb{Z} \) then any homeomorphism \( j : \partial V^n_{\pm} \rightarrow \partial V^n_{\pm} \) satisfies either \( i_*(j_*(\{c\})) = i_*(\{c\}) \) or \( i_*(j_*(\{c\})) = -i_*(\{c\}) \). Thus, theorem 4 implies that each of the manifolds \( S^{n-1} \times S^1 \) and \( S^{n-1} \tilde{\times} S^1 \) admits two topological equivalence classes of the flows from \( G_2(M^n), n > 3 \).

Acknowledgments

This work was supported by the Russian Science Foundation (Project 21-11-00010) except for the section 3 which is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, by Ministry of Science and Higher Education of the Russian Federation (Ag. 075-15-2019-1931) and section 4 which was prepared within the framework of the Academic Fund Program at the HSE University in 2021–2022 (Grant No. 21-04-004).

ORCID iDs

O V Pochinka  https://orcid.org/0000-0002-6587-5305
D D Shubin  https://orcid.org/0000-0002-8495-4826

References

[1] Bonahon F and Otal J-P 1982 Scindements de Heegaard des espaces lenticulaires. (French. English summary) [Heegaard splittings of lens spaces] C. R. Acad. Sci. Paris Sér. I Math. 294 585–7
[2] Campos B, Cordero A, Martínez Alfaro J and Vindel P 2004 NMS flows on three-dimensional manifolds with one saddle periodic orbit Acta Math. Sin. 20 47–56
[3] Grines V, Medvedev T and Pochinka O 2016 Dynamical Systems on two- and three-manifolds (Berlin: Springer)
[4] Hatcher A 2007 Notes on basic three-manifold topology (http://math.cornell.edu/hatcher/3M/3M.pdf)
[5] Irwin M 1970 A classification of elementary cycles Topology 9 35–48
[6] Jahren B and Kwasik S 2011 Free involutions on \( S^n \times S^1 \) Math. Ann. 351 281–303
[7] Kosniowski C 1980 A First Course in Algebraic Topology (Cambridge: Cambridge University Press)
[8] Kruglov V, Malyshev D and Pochinka O 2018 Topological classification of \( \Omega \)-stable flows on surfaces by means of effectively distinguishable multigraphs Discrete Contin. Dyn. Sys. A 38 4305–27
[9] Lickorish W B R 1963 Homeomorphisms of non-orientable two-manifolds Proc. Camb. Phil. Soc. 59 307–17
[10] Max N 1967 Homeomorphisms of \( S^n \times S^1 \) Bull. Amer. Math. Soc. 73 939–42
[11] Oshemkov A A and Sharko V V 1998 On the classification of Morse–Smale flows on two-dimensional manifolds Sb. Math. 189 1205–50
[12] Peixoto M M 1973 On a classification of flows on two-manifolds Proc. Symp. Dyn. Syst. Salvador 389–492
[13] Smale S 1967 Differentiable dynamical systems Bull. Am. Math. Soc. 73 747–817
[14] Umanskii Y L 1990 Necessary and sufficient conditions for topological equivalence of three-dimensional Morse–Smale dynamical systems with a finite number of singular trajectories Sb. Math. 181 212–39
[15] Yu B 2016 Behavior of nonsingular Morse–Smale flows on $S^3$ Discrete Contin. Dyn. Syst. 36 509–40