Q-TABLEAUX FOR IMPLICATIONAL PROPOSITIONAL CALCULUS

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Abstract. We study Q-tableaux and axiom systems that they engender, producing a new proof that the Implicational Propositional Calculus is complete.

0. INTRODUCTION

In a recent paper [4] we showed how completeness of the Implicational Propositional Calculus (IPC) may be established by means of ‘dual Q-tableaux’ and their associated axiom systems; here, we study ‘Q-tableaux’ themselves and the axiom systems to which they give rise. There are interesting differences between the two types of axiom systems: on the one hand, those arising from dual Q-tableaux are based on disjunction, which may be defined within IPC; on the other hand, those arising from Q-tableaux are based on conjunction, which only appears in IPC by (partial or residual) proxy. Among other technical differences, our proof that theorems of the Q-tableau axiom systems are provable within IPC assigns a much more pervasive rôle to the Peirce axiom scheme as a weak law of the excluded middle.

1. Q-TABLEAUX FOR IPC

The Implicational Propositional Calculus (IPC) has a single connective (⊃) and a single inference rule (modus ponens or MP) with three axiom schemes:

(IPC1) \( X \supset (Y \supset X) \)

(IPC2) \([X \supset (Y \supset Z)] \supset [(X \supset Y) \supset (X \supset Z)]\)

(Peirce) \([(X \supset Y) \supset X] \supset X\).

As usual, \( \vdash \) will signify deducibility within IPC; further, \( T \) will denote the set of theorems of IPC. In particular, the statements \( \vdash X \) and \( X \in T \) are effectively synonymous. We remark that the Deduction Theorem (DT) and Hypothetical Syllogism (HS) are valid in IPC as derived inference rules; they will be used (perhaps silently) throughout this paper. Exercises 6.3-6.5 of [2] provide a convenient do-it-yourself introduction to IPC.

Fix a (well-formed) formula \( Q \) of IPC and when \( Z \) is any IPC formula write

\( QZ := Q(Z) := Z \supset Q \)

so that \( QQZ = (Z \supset Q) \supset Q \) and so forth.

Theorem 1. Each of the following is an IPC theorem scheme:

1. \( (X \supset Y) \supset [(Y \supset Z) \supset (X \supset Z)] \)
2. \( (X \supset Y) \supset (QY \supset QX) \)
3. \( X \supset QQX \)
4. \( QQQX \supset QX \)
5. \( QQY \supset QQ(X \supset Y) \)
6. \( QQX \supset [QY \supset Q(X \supset Y)] \)
7. \( QX \supset QQ(X \supset Y) \)
8. \( (QX \supset Y) \supset [(QQX \supset Y) \supset QQY] \).

Proof. This is Exercise 6.3 in [2] so the proof is DIY. The only part requiring Peirce is (7) as noted in [2]; more than this, Peirce follows by MP from (7) with \( Q = X \) and the fact that \( X \supset X \in T \). □
Disjunction (\(\lor\)) may be defined within IPC as an abbreviation: thus,
\[
X \lor Y := (X \supset Y) \supset Y.
\]
This has the expected properties. For instance, \(X \vdash X \lor Y\) (by MP and DT: \(X, X \supset Y \vdash Y\) so \(X \vdash (X \supset Y) \supset Y\)) and \(Y \vdash X \lor Y\) (by MP and IPC\(_1\)). Moreover, the Peirce axiom scheme guarantees the following complementary property.

**Theorem 2.** If \(X \vdash Z\) and \(Y \vdash Z\) then \(X \lor Y \vdash Z\).

*Proof.* This is Theorem 3 in [3]. \(\Box\)

As an immediate consequence, \(\lor\) is ‘commutative’ in the sense that \(Y \lor X \vdash X \lor Y\). As a slightly less immediate consequence, \(\lor\) is ‘associative’ in the sense that \(X \lor (Y \lor Z) \vdash (X \lor Y) \lor Z\) and vice versa. As another consequence, we may rewrite the Peirce axiom scheme equivalently as a weak ‘law of the excluded middle’; we state this fact as a theorem, primarily for ease of reference.

**Theorem 3.** If \(Q\) and \(Z\) are IPC formulas then \(Q \lor Z\) is a theorem of IPC.

*Proof.* Rewrite! Thus:
\[
Q \lor Z = (Q \supset Z) \supset Z = [(Z \supset Q) \supset Z] \supset Z.
\]

**Remark:** We may use this to infer from \(Q \supset W \in \mathbb{T}\) and \(Z \supset W \in \mathbb{T}\) that \(W \in \mathbb{T}\). In fact, if \(Q \supset W \in \mathbb{T}\) and \(Z \supset W \in \mathbb{T}\) then \(Q \lor W \in \mathbb{T}\) by MP so that \(Q \lor Z \lor W\) by Theorem 2 and \((Q \lor Z) \supset W \in \mathbb{T}\) by DT; now MP and Theorem 3 place \(W \in \mathbb{T}\).

We shall have need of the following extension to Theorem 1.

**Theorem 4.** Each of the following is an IPC theorem scheme:

- (A\(_0\)) \(Q(X \supset Y) \supset QQX\);
- (A\(_1\)) \(Q(X \supset Y) \supset QY\);
- (B) \(QQ(X \supset Y) \supset (QQ \lor QX)\).

*Proof.* (A\(_0\)) From Theorem 1 part (7) we have \(Q(X \supset Y) \vdash QQ(X \supset Y) \in \mathbb{T}\) whence by Theorem 1 part (2) and MP we have \(QQQ(X \supset Y) \supset QQX \in \mathbb{T}\). Theorem 1 part (3) gives us \(Q(X \supset Y) \supset QQQ(X \supset Y) \in \mathbb{T}\) and an application of HS gives us \(Q(X \supset Y) \supset QQX \in \mathbb{T}\).

(A\(_1\)) Axiom scheme (IPC\(_1\)) gives us \(Y \supset (X \supset Y) \in \mathbb{T}\) and Theorem 1 part (2) gives us \(Y \supset (X \supset Y) \supset [Q(X \supset Y) \supset QY] \in \mathbb{T}\); by MP we deduce that \(Q(X \supset Y) \supset QY \in \mathbb{T}\).

(B) Note that \(QQ(X \supset Y) = [(X \supset Y) \lor Q] \supset Q = (X \supset Y) \lor Q\). We shall prove separately that \(X \lor Y \vdash QQ \lor QX\) and that \(Q \lor QQ \lor QX\); an application of Theorem 2 will then conclude the argument. Proof of \(X \lor Y \vdash QQ \lor QX\): Assume \(X \lor Y\), \(QQ \lor QX\), \(X\). Successive applications of MP yield: \(Y\) (from \(X\) and \(X \lor Y\)); \(QQY\) (from \(Y\) and \(Y \lor QQY\) in Theorem 1 part (3)); \(QX\) (from \(QQY\) and \(QQ \lor QX\)); \(Q\) (from \(X\) and \(X \lor Q = QX\)). This proves that \(X \lor Y, (QQY \lor QX), X \lor Q\) and two applications of DT yield \(X \lor Y \vdash (QQY \lor QX) \supset QX = QQ \lor QX\). Proof of \(Q \lor QQ \lor QX\): This is easy: axiom scheme (IPC\(_1\)) gives \(Q \lor X \lor Q = QX\); now \(Q, (QQ \lor QQX) \lor QX\) so that \(Q \lor (QQY \lor QX) \lor QX = QQ \lor QX\) by DT. \(\Box\)

In fact, \((QQY \lor QX) \supset QQ(X \supset Y) \in \mathbb{T}\) too: indeed, Theorem 1 part (5) tells us that \(QQY \lor QQ(X \supset Y)\) while Theorem 1 part (7) tells us that \(QX \lor QQ(X \supset Y)\); all that remains is to invoke Theorem 2 again.
Conjunction (\(\wedge\)) itself may not be definable within IPC, but a shadow of conjunction does exist and this shadow serves our purposes. Our purposes require that within IPC there be available formulas that serve as proxies for expressions of the classical form \(\sim (Z_N \wedge \cdots \wedge Z_0)\) where \(\sim\) signifies negation. It is abundantly clear from the foregoing development (in particular, Theorem 1 and Theorem 2) that the formula \(QZ = Z \supset Q\) has properties akin to those of the negation \(\sim Z\); indeed, the framework for classical Propositional Calculus presented in [1] includes a propositional constant \(\top\) (for falsity) and defines \(\sim Z\) to be \(Z \supset \top\). Taking into account this function of \(Q\) in manufacturing a partial substitute for negation, along with the classical exportation and importation rules, we accordingly make the following definition.

When \(\theta = (Z_N, \ldots, Z_0)\) is a sequence of IPC formulas, we define  
\[
C_Q(\theta) := Z_N \supset (\cdots (Z_0 \supset Q)\cdots)
\]
where \(C_Q\) suggests negated conjunction. For convenience, we may omit brackets and write simply
\[
C_Q(\theta) = Z_N \supset \cdots \supset Z_0 \supset Q
\]
with the understanding that brackets are as displayed above. Observe at once that if \(W\) is also an IPC formula and \(W, \theta\) stands for the sequence \((W, Z_N, \ldots, Z_0)\) then
\[
C_Q(W, \theta) = W \supset C_Q(\theta)
\]
which observation is of course the essence of a formal inductive definition of \(C_Q\) starting from \(C_Q(Z_0) = Z_0 \supset Q\).

The following property of this construction will be needed later.

**Theorem 5.** If \(0 \leq n \leq N \in \mathbb{N}\) then \(QZ_n \vdash C_Q(Z_N, \ldots, Z_0)\).

**Proof.** We write \(C_N = C_Q(Z_N, \ldots, Z_0)\) for convenience and break the proof into stages.

1. If \(0 \leq n \leq N \in \mathbb{N}\) then \(C_n \vdash C_N. [C_n \vdash Z_{n+1} \supset C_n = C_{n+1}\] is an instance of IPC1.]
2. If \(N \in \mathbb{N}\) then \(Q \vdash C_N. [Q \vdash Z_0 \supset Q = C_0\] is an instance of IPC1; now invoke (1).]
3. If \(0 \leq n \in \mathbb{N}\) then \(QZ_n \vdash C_n. [\) The base case \(n = 0\) is plain: \(QZ_0 = Z_0 \supset Q = C_0.\) For the inductive step, hypothesize \(QZ_n \vdash C_n. From QZ_{n+1} = Z_{n+1} \supset Q\) and \(Z_{n+1}\) we deduce \(Q\) by MP and therefore \(C_n\) by (2); thus \(QZ_{n+1} \vdash Z_{n+1} \supset C_n = C_{n+1}\) by MP and therefore \(C_n\) by (2); thus \(QZ_{n+1} \vdash Z_{n+1} \supset C_n = C_{n+1}\) by DT.]

The theorem now follows from (1) and (3). \(\square\)

Equivalently (by DT and MP) \(QZ_n \supset C_Q(Z_N, \ldots, Z_0)\) is a theorem of IPC.

Although we shall not need the following complementary pair of properties, we include them at little cost; they amount to a de Morgan law. The one property is that if \(N \in \mathbb{N}\) then
\[
QZ_0 \lor \cdots \lor QZ_N \vdash C_Q(Z_N, \ldots, Z_0)
\]
which is a fairly routine inductive consequence of Theorem 2 and Theorem 6. The other property is the following opposite deduction and perhaps calls for a more detailed argument.

**Theorem 6.** If \(N \in \mathbb{N}\) then \(C_Q(Z_N, \ldots, Z_0) \vdash QZ_0 \lor \cdots \lor QZ_N\).

**Proof.** For convenience, write \(C_N = C_Q(Z_N, \ldots, Z_0)\) and \(D_N = QZ_0 \lor \cdots \lor QZ_N\). Plainly, \(C_0 = Z_0 \supset Q = QZ_0 = D_0\). Now take \(C_N \vdash D_N\) as inductive hypothesis. The three assumptions \(C_{N+1}, D_N \vdash QZ_{N+1}, Z_{N+1}\) yield the following successive deductions: \(C_N\) (from \(C_{N+1} = Z_{N+1} \supset C_N\) and \(Z_{N+1}\) by MP); \(D_N\) (from \(C_N\) by the inductive hypothesis); \(QZ_{N+1}\) (from \(D_N \vdash QZ_{N+1}\) and \(D_N\)’; \(Q\) (from \(QZ_{N+1} = Z_{N+1} \supset Q\) and \(Z_{N+1}\)). Thus
\[
C_{N+1}, D_N \vdash QZ_{N+1}, Z_{N+1} \vdash Q
\]
and so by two applications of DT we conclude
\[
C_{N+1} \vdash (D_N \supset QZ_{N+1}) \supset QZ_{N+1} = D_{N+1}.
\]
\(\square\)
In order to introduce $Q$-tableaux, we find it convenient to assume knowledge of the theory of tableaux for signed formulas in the classical Propositional Calculus, for details of which we refer to the classic treatise [5]. IPC formulas of Type A have the form $\alpha = F(X \supset Y)$ with $\alpha_0 = TX$ and $\alpha_1 = FY$ as direct consequences, while IPC formulas of type B have the form $\beta = T(X \supset Y)$ with $\beta_0 = FX$ and $\beta_1 = TY$ as alternative consequences; symbolically,

$$\frac{\alpha}{\alpha_0} \quad \frac{\beta}{\beta_0 | \beta_1}.$$

If the IPC formula $Z$ is a tautology (true in all Boolean valuations) then $FZ$ starts a signed tableau that is closed in the sense that each of its branches contains a conjugate pair $TW$, $FW$ of signed formulas. For all of this and much more, see especially Chapter II of [5].

Now, fix a choice of IPC formula $Q$. Let $Z$ be an IPC tautology and construct a closed signed tableau $T$ starting from $FZ$; in the construction, do not abbreviate $W \supset Q$ as $QW$. Replace each node in $T$ of the form $FW$ by $QW$ and replace each node in $T$ of the form $TW$ by $QQW$. The result is a tableau $T_Q$ starting from $QZ$ with the following branching rules:

$\alpha = Q(X \supset Y)$ has direct consequences $\alpha_0 = QQX$ and $\alpha_1 = QY$;

$\beta = QQ(X \supset Y)$ has alternative consequences $\beta_0 = QX$ and $\beta_1 = QQY$.

Each branch $\theta$ of $T_Q$ is a sequence $(Z_N, \ldots, Z_0)$ with $Z_0 = QZ$ and each term $Z_n$ of the form $QW_n$ or $QQW_n$ for some IPC formula $W_n$. Each branch $\theta$ of $T_Q$ is closed in the sense that among its terms is a conjugate pair $QW$, $QQW$ for some IPC formula $W$.

**Remark:** We may instead define a $Q$-tableau for $Z$ as a tableau starting from $QZ$ with the branching rules displayed above; it was simply easier to import the standard machinery of tableaux for signed formulas.

Motivated by the construction in [6] for the classical Propositional Calculus, we associate to the IPC formula $Q$ an axiom system $U_Q$ having the following axiom schemes and inference rules, throughout which $\theta = (Z_N, \ldots, Z_0)$ stands for sequences of IPC formulas, each $Z_n$ being of the form $QW_n$ or $QQW_n$ for some IPC formula $W_n$, such a sequence $\theta$ being closed precisely when it has a conjugate pair $QW$, $QQW$ among its terms.

**Axioms:** All formulas $C_Q(\theta) = Z_N \supset \cdots \supset Z_0 \supset Q$ for which $\theta = (Z_N, \ldots, Z_0)$ is closed.

**Rule A:** If $\alpha$ is a term of $\theta$ then from $C_Q(\alpha_0, \theta)$ or $C_Q(\alpha_1, \theta)$ (separately) infer $D(\theta)$.

**Rule B:** If $\beta$ is a term of $\theta$ then from $C_Q(\beta_0, \theta)$ and $C_Q(\beta_1, \theta)$ (together) infer $D(\theta)$.

As is the case for their counterpart in the classical Propositional Calculus [6], $Q$-tableaux and their axiom systems associated to IPC formulas $Q$ facilitate a proof that the Implicational Propositional Calculus is complete, as we now proceed to show.

**Theorem 7.** Each axiom of $U_Q$ is a theorem of IPC.

**Proof.** Let the sequence $\theta = (Z_N, \ldots, Z_0)$ contain both $QW$ and $QQW$ as terms: Theorem 5 tells us that $QQW \supset C_Q(\theta) \in T$ and $QQQW \supset C_Q(\theta) \in T$; the Remark after Theorem 3 then places $C_Q(\theta)$ in $T$.

Rule A of $U_Q$ may be regarded as a derived inference rule for IPC.
Theorem 8. Let \( \theta \) have \( \alpha \) as a term. If \( C_Q(\alpha_0, \theta) \in \mathbb{T} \) or \( C_Q(\alpha_1, \theta) \in \mathbb{T} \) then \( C_Q(\theta) \in \mathbb{T} \).

Proof. Theorem 5 guarantees that \( C_Q(\alpha_0, \theta) \in \mathbb{T} \). Theorem 4 parts (A0) and (A1) guarantee that \( \alpha \supset \alpha_0 \in \mathbb{T} \) and \( \alpha \supset \alpha_1 \in \mathbb{T} \); it follows from this by HS that if \( \alpha_0 \supset C_Q(\theta) = C_Q(\alpha_0, \theta) \in \mathbb{T} \) or \( \alpha_1 \supset C_Q(\theta) = C_Q(\alpha_1, \theta) \in \mathbb{T} \) then \( \alpha \supset C_Q(\theta) \in \mathbb{T} \). Finally, the Remark after Theorem 3 places \( C_Q(\theta) \) in \( \mathbb{T} \).

Rule B of \( \mathbb{U}_Q \) may also be seen as a derived inference rule for IPC.

Theorem 9. Let \( \theta \) have \( \beta \) as a term. If \( C_Q(\beta_0, \theta) \in \mathbb{T} \) and \( C_Q(\beta_1, \theta) \in \mathbb{T} \) then \( C_Q(\theta) \in \mathbb{T} \).

Proof. Theorem 5 guarantees that \( Q\beta \supset C_Q(\theta) \in \mathbb{T} \). Theorem 4 part (B) guarantees that \( \beta \supset (\beta_0 \lor \beta_1) \in \mathbb{T} \); consequently, if \( \beta_0 \supset C_Q(\theta) = C_Q(\beta_0, \theta) \in \mathbb{T} \) and \( \beta_1 \supset C_Q(\theta) = C_Q(\beta_1, \theta) \in \mathbb{T} \) then \( (\beta_0 \lor \beta_1) \supset C_Q(\theta) \in \mathbb{T} \) by Theorem 2 so that HS yields \( \beta \supset C_Q(\theta) \in \mathbb{T} \). Finally, the Remark after Theorem 3 places \( C_Q(\theta) \) in \( \mathbb{T} \).

Taken together, Theorems 4, 8 and 9 establish that all theorems of \( \mathbb{U}_Q \) are provable within IPC.

Theorem 10. Each theorem of \( \mathbb{U}_Q \) is a theorem of IPC.

Proof. The set \( \mathbb{T} \) of all IPC theorems contains the axioms of \( \mathbb{U}_Q \) by Theorem 7; it is closed under Rules A and B according to Theorems 8 and 9.

Now, let us return to the IPC tautology \( Z \) to which we associated a closed \( Q \)-tableau \( T_Q \) starting from \( Z_0 = QZ \). As each branch \( \theta \) of \( T_Q \) is closed, each corresponding \( C_Q(\theta) \) is an axiom of \( \mathbb{U}_Q \). We prune the tableau \( T_Q \) by reversing the steps by which it was formed: pruning \( \theta \) applies an inference rule of \( \mathbb{U}_Q \) to \( C_Q(\theta) \) and so results in a theorem of \( \mathbb{U}_Q \); the final pruning lays bare the root \( Z_0 \supset Q \supset QZ \supset Q = QQZ \) which is then itself a theorem of \( \mathbb{U}_Q \). Conclusion: if \( Z \) is an IPC tautology, then \( QQZ \supset (Z \supset Q) \supset Q \) is a theorem of \( \mathbb{U}_Q \).

It is now a short step to completeness of IPC.

Theorem 11. IPC is complete.

Proof. Let \( Z \) be an IPC tautology and take \( Q := Z \). As we have just seen, \( (Z \supset Z) \supset Z \) is a theorem of \( \mathbb{U}_Z \) and therefore a theorem of IPC by Theorem 10. The proof is concluded by an application of MP to \( (Z \supset Z) \supset Z \) and the specific Peirce axiom \( [(Z \supset Z) \supset Z] \supset Z \).

In closing, we note that there are significant differences between the approach to IPC completeness via \( Q \)-tableaux offered here and the approach via dual \( Q \)-tableaux offered in [4]. One difference relates to the rôle played by the Peirce axiom scheme: Theorems 7, 8 and 9 of the present paper are all concluded by an application of the Peirce axiom scheme in its guise as a weak law of the excluded middle; by contrast, the corresponding results in [4] hinge on various parts of Exercise 6.3 in [2]. Another difference relates to conjunction (‘negated’) and disjunction: in \( Z_N \supset \cdots \supset Z_0 \supset Q \) new terms are added on the left while in \( Z_0 \lor \cdots \lor Z_N \supset Q \) they are added on the right; this difference is reflected in the definition and properties of the corresponding axiom systems. We leave the details of a full comparison to the reader.
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