On the Performance of Thompson Sampling on Logistic Bandits

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1 Abstract
We study the logistic bandit, in which rewards are binary with success probability \( \frac{\exp(\beta a^\top \theta)}{1 + \exp(\beta a^\top \theta)} \) and actions \( a \) and coefficients \( \theta \) are within the \( d \)-dimensional unit ball. While prior regret bounds for algorithms that address the logistic bandit exhibit exponential dependence on the slope parameter \( \beta \), we establish a regret bound for Thompson sampling that is independent of \( \beta \).

Specifically, we establish that, when the set of feasible actions is identical to the set of possible coefficient vectors, the Bayesian regret of Thompson sampling is \( \tilde{O}(d \sqrt{T}) \). We also establish a \( \tilde{O}(\sqrt{d \eta T / \lambda}) \) bound that applies more broadly, where \( \lambda \) is the worst-case optimal log-odds and \( \eta \) is the “fragility dimension,” a new statistic we define to capture the degree to which an optimal action for one model fails to satisfice for others. We demonstrate that the fragility dimension plays an essential role by showing that, for any \( \epsilon > 0 \), no algorithm can achieve \( \text{poly}(d, 1/\lambda) \cdot T^{1-\epsilon} \) regret.

Keywords: bandits, Thompson sampling, logistic regression, regret bounds.

1. Introduction
In the logistic bandit an agent observes a binary reward after each action, with outcome probabilities governed by a logistic function:

\[
P\left(\text{reward }= 1 | \text{action }= a\right) = \frac{e^{\beta a^\top \theta}}{1 + e^{\beta a^\top \theta}}.
\]

Each action \( a \) and parameter vector \( \theta \) is a vector within the \( d \)-dimensional unit ball. The agent initially knows the scale parameter \( \beta \) but is uncertain about the coefficient vector \( \theta \). The problem of learning to improve action selection over repeated interactions is sometimes referred to as the logistic bandit problem or online logistic regression.

The logistic bandit serves as a model for a wide range of applications. One example is the problem of personalized recommendation, in which a service provider successively recommends content, receiving only binary responses from users, indicating “like” or “dislike.” A growing literature treats the design and analysis of action selection algorithms for the logistic bandit. Upper-confidence-bound (UCB) algorithms have been analyzed in Filippi et al. (2010); Li et al. (2017); Russo and Van Roy (2013), while Thompson sampling (Thompson (1933)) was treated in Russo

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\footnote{In defining “log-odds,” we use base \( e^\pi \) rather than \( e \). As a result, the “log-odds” throughout this article refers to \( a^\top \theta \) instead of \( \beta a^\top \theta \).}

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| Algorithm                  | Regret Upper Bound                                      | Notes                                      |
|----------------------------|---------------------------------------------------------|--------------------------------------------|
| GLM-UCB (Filippi et al. (2010)) | $O\left(e^\beta \cdot d \cdot T^{1/2} \log^{3/2} T\right)$ | Frequentist bound.                        |
| A variation of GLM-UCB (Russo and Van Roy (2013)) | $O\left(e^\beta \log \beta \cdot d \cdot T^{1/2}\right)$ | Bayesian bound.                           |
| SupCB-GLM (Li et al. (2017)) | $O\left(e^\beta \cdot (d \log K)^{1/2} \cdot T^{1/2} \log T\right)$ | Frequentist bound, $K$ is the number of actions. |
| Thompson Sampling (Russo and Van Roy (2014b)) | $O\left(e^\beta \cdot d \cdot T^{1/2} \log^{3/2} T\right)$ | Bayesian bound.                           |
| Thompson Sampling (Abeille and Lazaric (2017)) | $O\left(e^\beta \cdot d^{3/2} \log^{1/2} \cdot d \cdot T^{1/2} \log^{3/2} T\right)$ | Frequentist bound.                        |
| Thompson Sampling (this work) | $O\left(\lambda^{-1} \cdot (d(\eta \vee d))^{1/2} \cdot T^{1/2} \log^{1/2} T\right)$ | Bayesian bound, $\lambda$ and $\eta$ are independent of $\beta$ (defined in Section 3). |

Table 1: Comparison of various results on logistic bandits. The upper bound in this work depends on $\beta$-independent parameters $\lambda$ and $\eta$, defined in Assumption 1 and Definition 2, respectively. We use the notation $a \vee b = \max\{a, b\}$.

and Van Roy (2014b) and Abeille and Lazaric (2017). Each of these algorithms has been shown to converge on the optimal action with time dependence $O(1/\sqrt{T})$, where $O$ ignores poly-logarithmic factors. However, previous analyses leave open the possibility that the convergence time increases exponentially with the parameter $\beta$, which seems counterintuitive. In particular, as $\beta$ increases, distinctions between good and bad actions become more definitive, which should make them easier to learn.

To shed light on this issue, we build on an information-theoretic line of analysis, which was first proposed in Russo and Van Roy (2016) and further developed in Bubeck and Eldan (2016) and Dong and Van Roy (2018). A critical device here is the information ratio, which quantifies the one-stage trade-off between exploration and exploitation. The information ratio has also motivated the design of efficient bandit algorithms, as in Russo and Van Roy (2014a), Russo and Van Roy (2018) and Liu et al. (2018). While prior bounds on the information ratio pertain only to independent or linear bandits, in this work we develop a new technique for bounding the information ratio of a logistic bandit. This leads to a stronger regret bound and insight into the role of $\beta$.

Our Contributions. Let $\mathcal{A}$ and $\Theta$ be the set of feasible actions and the support of $\theta$, respectively. Under an assumption that $\mathcal{A} = \Theta$, we establish a $\tilde{O}(d\sqrt{T})$ bound on Bayesian regret. This bound scales with the dimension $d$, but notably exhibits no dependence on $\beta$ or the number of feasible actions. We then generalize this bound, relaxing the assumption that $\mathcal{A} = \Theta$ while introducing dependence on two statistics of the these sets: the worst-case optimal log-odds $\lambda = \min_{\theta \in \Theta} \max_{a \in \mathcal{A}} \alpha^\top \theta$ and the fragility dimension $\eta$, which is the number of possible models such that the optimal action for each yields success probability no greater than 50% for any other. Assuming $\lambda > 0$, we establish a $\tilde{O}(\sqrt{d\eta T}/\lambda)$ bound on Bayesian regret. We also demonstrate that the fragility dimension plays an essential role, as for any function $f$, polynomial $p$, and $\epsilon > 0$, any algorithm for the logistic bandit
cannot achieve Bayesian regret uniformly bounded by \( f(\lambda)p(d)T^{1-\epsilon} \). We believe that, although \( \eta \) can grow exponentially with \( d \), in most relevant contexts \( \eta \) should scale at most linearly with \( d \).

The assumption that the worst-case optimal log-odds are positive may be restrictive. This is equivalent to assuming that for each possible model, the optimal action yields more than 50% probability of success. However, this assumption is essential, since it ensures that the fragility dimension is well-defined. When the worst-case optimal log-odds are negative, the geometry of action and parameter sets plays a less significant role than parameter \( \beta \), therefore we conjecture that the exponential dependence on \( \beta \) is inevitable. This could be an interesting direction for future research.

**Notations.** Throughout this article, for integer \( n \) we will use \([n]\) to denote the set \( \{1, \ldots, n\} \). We will also use \( B_d \) and \( S_{d-1} \) to denote the unit ball and the unit sphere in \( \mathbb{R}^d \), respectively.

### 2. Problem Settings

We consider Bayesian generalized linear bandits, defined as a tuple \( \mathcal{L} = (\mathcal{A}, \Theta, R, \phi, \rho) \), where \( \mathcal{A} \) and \( \Theta \) are the action and parameter set, respectively, \( R \) is a stochastic process representing the reward of playing each action, \( \phi \) is the link function, and \( \rho \) is the prior distribution over \( \Theta \), which represents our prior belief of the groundtruth parameter \( \theta^* \). Throughout this article, to avoid measure-theoretic subtleties, we assume that both \( \mathcal{A} \) and \( \Theta \) are finite subsets of \( B_d \). For simplicity, we assume that there exists a one-to-one mapping \(^3\) between each parameter and the corresponding optimal action. Specifically, let \( \mathcal{A} = \{a^1, \ldots, a^N\} \) and \( \Theta = \{\theta^1, \ldots, \theta^N\} \), with

\[
\arg\max_{a \in \mathcal{A}} \mathbb{E}[R(a)|\theta^* = \theta^i] = \{a_i\}, \quad \forall i = 1, \ldots, N.
\]

To specify the one-to-one mapping, for each \( \theta \in \Theta \) we define \( \alpha(\theta) \) to be the unique action that maximizes \( \mathbb{E}[R(a)|\theta^* = \theta] \). Letting \( A^* \) be the optimal action, which is a random variable under our Bayesian setting, naturally we have \( A^* = \alpha(\theta^*) \).

The reward \( R \) is related to the inner product between the action and the parameter by the link function \( \phi \), as

\[
\mathbb{E}[R(a)|\theta^* = \theta] = \phi(a^\top \theta), \quad \forall a \in \mathcal{A}, \theta \in \Theta.
\]

Specifically, in logistic bandits, the reward \( R \) is the binary process \( R_B \) and the link function is given by

\[
\phi_{\beta}(x) = \frac{\epsilon^{\beta x}}{1 + \epsilon^{\beta x}},
\]

where \( \beta > 0 \) is a parameter that characterizes the “separability” of the model. Equivalently, conditioned on \( \theta^* = \theta \), \( R_B(a) \) is a Bernoulli random variable with mean \( \phi_{\beta}(a^\top \theta) \). In the following, we will use \( \mathcal{L}_\beta \) to denote the logistic bandits problem instance with parameter \( \beta \).

At stage \( t \) the agent plays action \( A_t \) and observes reward \( R_t = R(A_t) \). Let \( \mathcal{H}_t = \sigma(A_1, R_1, \ldots, A_t, R_t) \) be the \( \sigma \)-algebra generated by the past actions and observations (rewards). A (randomized) policy \( \pi = (\pi_1, \pi_2, \ldots) \) is a sequence of functions such that for each \( t, \pi_t(\mathcal{H}_{t-1}) \)

\(^3\) Note that Thompson sampling does not consider actions that are not optimal for any parameter. If an action is optimal for multiple parameters, we can add identical copies of the action to the action set such that the mapping between each parameter and the corresponding optimal action is one-to-one.
is a probability distribution on the action set. The performance of policy \( \pi \) on problem instance \( \mathcal{L} = (\mathcal{A}, \Theta, R, \phi, \rho) \) is evaluated by the Bayesian regret, defined as

\[
\text{BayesRegret}(T; \mathcal{L}, \pi) := \mathbb{E}_{\pi, \rho} \left[ \sum_{t=1}^{T} R^* - R_t \right],
\]

where \( R^* := R(\theta^*) \), the subscripts \( \pi, \rho \) denote that \( A_t \) is drawn from \( \pi_t(\mathcal{H}_{t-1}) \) for \( t \geq 1 \) and \( A_0 \) is drawn from the prior \( \rho \). In this work, we are interested in the Thompson sampling policy \( \pi_{TS} \), characterized as

\[
\mathbb{P}(\pi_{TS}^T(\mathcal{H}_{t-1}) \in \cdot | \mathcal{H}_{t-1}) = \mathbb{P}(A^* \in \cdot | \mathcal{H}_{t-1}),
\]

i.e. the action played in each stage is drawn from the posterior of the optimal action. Since there is a one-to-one mapping between each parameter and the corresponding optimal action, the Thompson sampling policy can be equivalently carried out by sampling from the posterior of the true parameter \( \theta^* \) at each stage, and acting greedily with respect to the sampled parameter.

3. Main Results

We start off the section with a regret bound that only depends on dimension \( d \) and the number of time steps \( T \), for the particular setting where the action set \( \mathcal{A} \) is the same as the parameter set \( \Theta \).

**Theorem 1** For any \( \beta > 0 \), if \( \mathcal{L}_\beta = (\mathcal{A}, \Theta, R_B, \phi_\beta, \rho) \) is such that \( \mathcal{A}, \Theta \subset S_{d-1} \) and \( \mathcal{A} = \Theta \), then

\[
\text{BayesRegret}(T; \mathcal{L}_\beta, \pi_{TS}) \leq 40d \sqrt{T \log \left(3 + \frac{3\sqrt{2T}}{2d}\right)}.
\]

Despite nonlinearity of the link function, Theorem 1 matches the \( \tilde{O}(d \sqrt{T}) \) bound for linear bandits. It is worth noting that the this bound has no dependence on \( \beta \) or the number of arms, and also matches the \( \Omega(d \sqrt{T}) \) minimax lower bound for linear bandits in Dani et al. (2008), ignoring a \( \sqrt{\log T} \) factor. This result shows that if there exists an action that aligns perfectly with each potential parameter, the performance of Thompson sampling only depends on the problem dimension \( d \), and the dependence is at most linear.

However, as our next result shows, if the parameters do not align perfectly with their corresponding optimal actions, we have to introduce the fragility dimension to characterize the difficulty of the problem.

For our general result, we assume that the following assumption holds.

**Assumption 1** There exists constant \( \lambda \in [0, 1] \) such that for every \( \theta \in \Theta \) there is \( \alpha(\theta)^\top \theta \geq \lambda \).

For a given logistic bandit problem instance \( \mathcal{L}_\beta = (\mathcal{A}, \Theta, R, \phi_\beta, \rho) \) that satisfies Assumption 1, we show that the Bayesian regret of Thompson sampling on \( \mathcal{L}_\beta \) is closely related to its “fragility dimension,” a notion that we introduce below.

**Definition 2** For any given pair of (possibly infinite) subsets \( (\mathcal{X}, \mathcal{Y}) \) of \( B_d \), the fragility dimension, denoted by \( \eta(\mathcal{X}, \mathcal{Y}) \), is defined as the largest integer \( M \), such that there exists \( \{y_1, \ldots, y_M\} \subset \mathcal{Y} \), with

\[
f^*(y_i)^\top y_j < 0, \quad \forall i, j \in [M], i \neq j,
\]
where $f^*(y) := \arg\max_{x \in \mathcal{X}} x^\top y$. The fragility dimension of a problem instance $\mathcal{L}_0 = (\mathcal{A}_0, \Theta_0, R_0, \phi_0, \rho_0)$ is defined as the fragility dimension of $(\mathcal{A}_0, \Theta_0)$, and is denoted by $\eta(\mathcal{L}_0)$.

**Example 1** If the action set and the parameter set of $\mathcal{L}$ are identical subsets of $\mathcal{S}_{d-1}$, then for each $\theta \in \Theta$, there is $\alpha(\theta) = \theta$. We will show in Appendix D.1 that in $\mathcal{S}_{d-1}$ there exists at most $d + 1$ vectors with pairwise negative inner products. Therefore, the fragility dimension is bounded by

$$\eta(\mathcal{L}) \leq d + 1.$$  

**Remark 3** Obviously the fragility dimension cannot exceed the cardinality of the action (parameter) set. We will show in Appendix D that we can upper bound the worst-case fragility dimension by the dimensionality $d$ and the constant $\lambda$ in Assumption 1. Roughly speaking,

- If $\mathcal{L}$ is such that $\lambda = 1$, then $\eta(\mathcal{L}) \leq d + 1$ (cf. Example 1);
- For any fixed $\lambda \in (0, 1)$, if we only consider problem instances such that Assumption 1 holds with constant $\lambda$, then the worst-case fragility dimension grows exponentially with $d$.
- For any $d \geq 3$, we can find a problem instance $\mathcal{L}$ such that Assumption 1 holds with constant $\lambda = 0$, whose fragility dimension is arbitrarily large.

**Remark 4** For given finite action and parameter sets $\mathcal{A}$ and $\Theta$, we can think of each parameter as a vertex in a graph $\mathcal{G}$. Two vertices $i$ and $j$ of $\mathcal{G}$ are connected by an edge if and only if

$$\alpha(\theta_i)^\top \theta_j < 0 \text{ and } \alpha(\theta_j)^\top \theta_i < 0.$$  

Thus determining the fragility dimension of $(\mathcal{A}, \Theta)$ is equivalent to finding the maximum clique in $\mathcal{G}$. This is a widely studied NP-complete problem and there exists a number of efficient heuristics, see Tarjan and Trojanowski (1977), Tomita and Kameda (2007) and references therein.

The following general result for the performance of Thompson sampling gives a $\tilde{O}(d \eta T/\lambda)$ regret bound.

**Theorem 5** For any $\beta > 0$, if $\mathcal{L}_\beta$ is such that Assumption 1 holds with $\lambda \in (0, 1)$, then

$$\text{BayesRegret}(T; \mathcal{L}_\beta, \pi^{TS}) \leq 20 \lambda^{-1} \sqrt{2d \cdot (\eta(\mathcal{L}_\beta) \vee d) \cdot T \log \left( 3 + \frac{3\sqrt{2T}}{2d\lambda} \right)},$$

where $a \vee b = \max\{a, b\}$. It is worth noting that the fragility dimension only depends on the action and parameter sets of the problem instance, hence the right-hand side of (3) has no dependence on $\beta$.

**Remark 6** Considering Example 1, and noting that when $\mathcal{A} = \Theta$, Assumption 1 holds with $\lambda = 1$, we immediately arrive at Theorem 1.

**Remark 7** Interestingly, the fragility dimension is not monotonic with respect to the inclusion of sets, i.e. there exist sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$, such that $\mathcal{X}_1 \subset \mathcal{X}_2$ but $\eta(\mathcal{X}_1, \mathcal{Y}) > \eta(\mathcal{X}_2, \mathcal{Y})$. As we show in Appendix D.4, this fact means that by reducing the size of the action set, we could arrive at a more difficult problem. This is a somewhat surprising result that is worth noting.
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We also show that the $\eta$ term in (3) is critical, since for any fixed $\lambda < 1$, there cannot exist an $\eta$-independent upper bound that is polynomial in $d$ and sublinear in $T$.

**Theorem 8** For any fixed $\lambda \in [0,1)$, let $f(\cdot)$ be any real function, $p(\cdot)$ be any polynomial and $\epsilon > 0$ be any constant. There exists a logistic bandit problem instance $\mathcal{L}_\beta$ and integer $T_0$ such that $\mathcal{L}_\beta$ satisfies Assumption 1 with constant $\lambda$ and

$$\text{BayesRegret}(T_0; \mathcal{L}_\beta, \pi) \geq f(\lambda)p(d) \cdot T_0^{1-\epsilon}, \quad (4)$$

for any policy $\pi$.

4. Main Devices in the Proof of Theorem 5

In this section we discuss the two main devices in the proof of Theorem 5. In Section 4.1, we introduce the notion of information ratio, and present the result that relates information ratio with Bayesian regret. In Section 4.2, we highlight the role of fragility dimension. The full proof of Theorem 5 is given in Appendix B.

4.1. Information Ratio

To quantify the exploration-exploitation trade-off at stage $t$, for problem instance $\mathcal{L}$ and policy $\pi$ we define the random variable *information ratio* as the square of one-stage expected regret divided by the amount of information that the agent gains from playing an action and observing the reward, i.e.

$$\Gamma_t(\mathcal{L}, \pi) := \frac{\mathbb{E}_{t-1}[R^* - R_t]^2}{I_{t-1}(A^*; A_t, R_t)}, \quad (5)$$

where the subscript $t-1$ in the right-hand side denotes evaluation under base measure $\mathbb{P}(\cdot|\mathcal{H}_{t-1})$. If the information ratio is small at stage $t$, the agent executing the policy $\pi$ will only incur a large regret if she is about to acquire a large amount of information towards the optimal action. Past results have shown that, as long as the information ratio of Thompson sampling can be uniformly bounded, we immediately obtain a bound on the Bayesian regret of Thompson sampling.

**Proposition 9** (*Theorem 4, Dong and Van Roy (2018)*) Let $\mathcal{L}_\beta = (\mathcal{A}, \Theta, R, \phi_\beta, \rho)$ be any logistic bandit problem instance with $\inf_{\theta \in \Theta} |\alpha(\theta)\theta^\top| = \delta > 0$. Further assume that there exists constant $\bar{\Gamma}$ such that

$$\Gamma_t(\mathcal{L}_\beta, \pi^{TS}) \leq \bar{\Gamma}, \quad \text{a.s. } \forall t = 1, 2, \ldots.$$  

Then we have

$$\text{BayesRegret}(T; \mathcal{L}_\beta, \pi^{TS}) \leq \sqrt{8d\bar{\Gamma} \cdot T \log \left(3 + \frac{6\sqrt{2T}}{d} \cdot \frac{\beta e^{\beta\delta}}{(1 + e^{\beta\delta})^2}\right)}.$$  

4.2. Fragility Dimension

The one-stage expected regret can be written as

$$\mathbb{E}_{t-1}[R^* - R_t] = \mathbb{E}_{t-1}[\phi_\beta((A^*)\theta^*) - \phi_\beta(A_t^\top \theta^*)] \quad (6)$$
It is worth noting that \( A^* = \alpha(\theta^*) \) and by the definition of Thompson sampling, \( A^* \) and \( A_t \) are independent and identically distributed. Let’s first consider the simple case where \( \beta = \infty \), which motivates our analysis. When \( \beta = \infty \), we have that \( \phi_\beta(x) = 1 \) for all \( x \geq 0 \) and \( \phi_\beta(x) = 0 \) for all \( x < 0 \). By Assumption 1, we have

\[
\phi_\beta((A^*)^T \theta^*) = \phi_\beta(\alpha(\theta^*)^T \theta^*) = 1.
\]

There is also

\[
\mathbb{E}_{t-1}[\phi_\beta(A_t^T \theta^*)] = \mathbb{P}_{t-1}(A_t^T \theta^* \geq 0).
\]

Therefore, to upper bound the right-hand side of (6), we need to lower bound \( \mathbb{P}_{t-1}(A_t^T \theta^* \geq 0) \).

The proposition below shows that this term is connected critically with the fragility dimension of \((A, \Theta)\). The proof is given in Appendix A.

**Proposition 10** Let \( U, V \) be finite subsets of \( \mathcal{B}_d \). Suppose that there exists bijection \( f^* : V \mapsto U \) such that

\[
f^*(v)^T v = \max_{u \in U} u^T v, \quad \forall v \in V,
\]

and \( f^*(v)^T v > 0 \) for all \( v \in V \). Let \( V \) be any random variable supported on \( V \), \( U = f^*(V) \) and \( \hat{U} \) be an iid copy of \( U \). Then

\[
\mathbb{P}(\hat{U}^T V \geq 0) \geq \frac{1}{2\eta(U, V)}.
\]

5. **Proof Sketch of Theorem 8**

Recall that we can obtain regret bounds for linear bandits that are dependent only on the dimensionality of the problem \( d \) rather than the number of actions (such as the one in Russo and Van Roy (2016)). The reason behind such bounds is that when the link function \( \phi \) is linear, the difference between the mean rewards of two actions that are close to each other is always small. However, in logistic bandit problems, when parameter \( \beta \) is large, we could run into cases where two close actions yield diametrically different rewards, as is illustrated in Figure 1.

Specifically, suppose that our action and parameter sets are such that

\[
\alpha(\theta)^T \theta \geq 0, \quad \forall \theta \in \Theta,
\]

and

\[
a^T \theta < 0, \quad \forall a \in A, \theta \in \Theta, a \neq \alpha(\theta),
\]

that is, \( \eta(A, \Theta) = |A| = |\Theta| \). Then, when \( \beta \) is large, conditioned on each parameter being the true parameter, there is exactly one action with mean reward close to 1, while the mean rewards of all other actions are close to 0. The following proposition shows that in this problem the optimal action is inherently hard to learn, in the sense that the regret of any algorithm grows linearly in the first \( |A|/2 - 1 \) stages. The proof can be found in Appendix C.

\footnote{For the sake of simplicity, we will assume that \( \phi_\infty(0) = 1 \), while in fact \( \lim_{\beta \to \infty} \phi_\beta(0) = 1/2 \). The value of \( \phi_\infty(0) \) does not play a role in our analysis.}
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Figure 1: The difference between linear and logistic bandits. The actions $a_1$ and $a_2$ are “similar” to each other in that their embeddings in the Euclidean space are close. Under the linear link function $\phi_1$, the mean rewards of $a_1$ and $a_2$ are also similar. However, under the logistic link function $\phi_2$, the performances of the two actions are diametrical.

**Proposition 11** Let $L = (A, \Theta, R, \phi, \rho)$ be a generalized linear bandit problem such that $|A| = N < \infty$, $R$ is binary and $\rho$ is the uniform distribution over $A$. Suppose that for each $a \in A$,

$$\mathbb{E}[R(a) | A^* = a] \geq 1 - \frac{1}{N},$$

and

$$\max_{a' \neq a} \mathbb{E}[R(a') | A^* = a] \leq \frac{1}{N}.$$ 

Then for any policy $\pi$,

$$\text{BayesRegret}(t; L, \pi) \geq \frac{t}{4}, \quad \forall t \leq \frac{N}{2} - 1. \quad (12)$$

We can also show that (as in Appendix D), for any fixed $\lambda \in (0, 1)$, there exists $\gamma > 1$, such that for any $d \geq 2$ we can find a pair of action and parameter sets $(A_d, \Theta_d)$ with $A_d, \Theta_d \in \mathbb{R}^d$, $|A_d| = |\Theta_d| \geq \gamma^d$ that satisfies (10), (11) and Assumption 1 with constant $\lambda$. For any real function $f(\cdot)$, polynomial $p(\cdot)$ and constant $\epsilon \in (0, 1)$, choose $d$ large enough such that $\gamma^{ed} > 16 f(\lambda)p(d)$ and $\beta_d$ large enough such that

$$\phi_{\beta_d}(\lambda) \geq 1 - \frac{1}{|A_d|},$$

and

$$\phi_{\beta_d} \left( \max_{a \in A_d, \theta \in \Theta_d, a^\top \theta < 0} a^\top \theta \right) \leq \frac{1}{|A_d|}.$$ 

Consider the problem $L = (A_d, \Theta_d, R_B, \phi_{\beta_d}, \text{Unif}(A))$ at stage $T_0 = \gamma^d/4$, from Proposition 11 we have

$$\text{BayesRegret}(T_0; L, \pi) \geq \frac{T_0}{4} = \frac{\gamma^d}{16} = \frac{1}{4} \cdot \frac{\gamma^{ed}}{4^\epsilon} \cdot \left( \frac{\gamma^d}{4} \right)^{1-\epsilon} > f(\lambda)p(d)T_0^{1-\epsilon}, \quad (13)$$

for any policy $\pi$. 

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References

Marc Abeille and Alessandro Lazaric. Linear thompson sampling revisited. Electronic Journal of Statistics, 11(2):5165–5197, 2017.

Károly Böröczky Jr, K Böröczky, et al. Finite packing and covering, volume 154. Cambridge University Press, 2004.

Sébastien Bubeck and Ronen Eldan. Multi-scale exploration of convex functions and bandit convex optimization. In Conference on Learning Theory, pages 583–589, 2016.

Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under bandit feedback. In 21st Annual Conference on Learning Theory, pages 355–366, 2008.

Shi Dong and Benjamin Van Roy. An information-theoretic analysis for Thompson sampling with many actions. In Advances in Neural Information Processing Systems, 2018.

Sarah Filippi, Olivier Cappe, Aurélien Garivier, and Csaba Szepesvári. Parametric bandits: The generalized linear case. In Advances in Neural Information Processing Systems, pages 586–594, 2010.

Lihong Li, Yu Lu, and Dengyong Zhou. Provably optimal algorithms for generalized linear contextual bandits. In International Conference on Machine Learning, pages 2071–2080, 2017.

Fang Liu, Swapna Buccapatnam, and Ness Shroff. Information directed sampling for stochastic bandits with graph feedback. In Thirty-Second AAAI Conference on Artificial Intelligence, 2018.

Daniel Russo and Benjamin Van Roy. Eluder dimension and the sample complexity of optimistic exploration. In Advances in Neural Information Processing Systems, pages 2256–2264, 2013.

Daniel Russo and Benjamin Van Roy. Learning to optimize via information-directed sampling. In Advances in Neural Information Processing Systems, pages 1583–1591, 2014a.

Daniel Russo and Benjamin Van Roy. Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243, 2014b.

Daniel Russo and Benjamin Van Roy. An information-theoretic analysis of Thompson sampling. The Journal of Machine Learning Research, 17(1):2442–2471, 2016.

Daniel Russo and Benjamin Van Roy. Satisficing in time-sensitive bandit learning. arXiv preprint arXiv:1803.02855, 2018.

Robert Endre Tarjan and Anthony E Trojanowski. Finding a maximum independent set. SIAM Journal on Computing, 6(3):537–546, 1977.
William R Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933.

Etsuji Tomita and Toshikatsu Kameda. An efficient branch-and-bound algorithm for finding a maximum clique with computational experiments. *Journal of Global optimization*, 37(1):95–111, 2007.

Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok (in Hungarian)*, 48:436–452, 1941.

**Appendix A. Proof of Proposition 10**

We present a graph-theoretical proof of Proposition 10. For simplicity, let $\eta = \eta(U, V)$. Let $U$ and $V$ be enumerated as $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Without loss of generality, we assume that $f^*(v_i) = u_i$ for $i \in [n]$. We construct an undirected graph $G = (K, E)$, where $K = \{1, \ldots, n\}$, and for any pair $1 \leq i < j \leq n$, $i$ and $j$ are connected by an edge $(i, j) \in E$ if and only if $f^*(v_i) \top v_j < 0$ and $f^*(v_j) \top v_i < 0$.

From Definition 2, there exists no $(\eta + 1)$-clique in $G$.

Let $p$ be any probability measure on $V$. We use $p_i$ to denote the probability mass associated with $v_i$. Thus $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$. For fixed $V$, let $J(p) = \mathbb{P}_p(\hat{U} \top V < 0)$, where the subscript $p$ indicates that the distribution of $V$ is $p$. We have that

$$J(p) = \mathbb{P}_p(\hat{U} \top V < 0)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}_p(\hat{U} = u_i) \mathbb{P}_p(V = v_j) \mathbb{I}(u_i \top v_j < 0)$$

$$(a) = \sum_{i,j=1}^n p_i p_j \mathbb{I}(f(v_i) \top v_j < 0)$$

$$(b) \leq \sum_{(i,j) \in E} p_i p_j + \frac{1}{2} \sum_{(i,j) \notin E} p_i p_j$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{(i,j) \in E} p_i p_j,$$

(14)

where $(a)$ comes from that

$$\mathbb{P}_p(\hat{U} = u_i) = \mathbb{P}_p(U = u_i) = \mathbb{P}_p(V = v_i),$$

and $(b)$ is because for each $(i, j) \notin E$, at most one of $f(v_i) \top v_j$ and $f(v_j) \top v_i$ can be negative. Note that here $(i, i) \notin E$ for all $i \in [n]$.

Let $M(p) := \sum_{(i,j) \in E} p_i p_j$. We first argue that there exists probability measure $p^*$, such that

$$M(p^*) = \max_p M(p),$$

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and for any \((i, j) \notin \mathcal{E}, i \neq j\), either \(p_i^* = 0\) or \(p_j^* = 0\). In fact, let \(p\) and \((i, j) \notin \mathcal{E}\) be arbitrary. Without loss of generality, assume that
\[
\sum_{k: (i, k) \in \mathcal{E}} p_k \geq \sum_{k: (j, k) \in \mathcal{E}} p_k.
\]

We define a new measure \(p'\) as follows: \(p'_i = p_i + p_j, p'_j = 0\) and \(p'_\ell = p_\ell\) for \(\ell \neq i, j\). Then
\[
M(p') = \sum_{(\ell, k) \in \mathcal{E}} p'_\ell p'_k
= \sum_{k: (i, k) \in \mathcal{E}} p'_i p'_k + \sum_{k: (j, k) \in \mathcal{E}} p'_j p'_k + \sum_{h, \ell \neq i, j: (h, \ell) \in \mathcal{E}} p'_h p'_\ell
\geq \sum_{k: (i, k) \in \mathcal{E}} p_i p_k + \sum_{k: (j, k) \in \mathcal{E}} p_j p_k + \sum_{h, \ell \neq i, j: (h, \ell) \in \mathcal{E}} p_h p_\ell
= \sum_{(k, \ell) \in \mathcal{E}} p_\ell p_k
= M(p).
\]

Therefore, by moving all the probability mass from \(j\) to \(i\), the value \(M\) does not decrease. Thus we can always find a probability measure \(p^*\) which attains the maximum of \(M\), and at the same time satisfies \(p_i^* p_j^* = 0\) whenever \((i, j) \notin \mathcal{E}\) and \(i \neq j\).

Next we show that there can be at most \(\eta\) non-zero elements among \(\{p_1^*, \ldots, p_n^*\}\). In fact, since there exists no \((\eta + 1)\)-clique in \(G\), for any subset \(\{i_1, \ldots, i_{\eta+1}\}\) of \(V\) there must exist \((i_s, i_t) \notin \mathcal{E}\) and \(i_s \neq i_t\). This leads to \(p_{i_s}^* p_{i_t}^* = 0\). Hence \(p^*\) must be supported on at most \(\eta\) elements of \(X\).

Without loss of generality, let \(p_1^*, \ldots, p_\eta^* \geq 0\) and \(p_{\eta+1}^*, \ldots, p_n^* = 0\). Then
\[
\max_p J(p) \leq \max_p \left( \frac{1}{2} + \frac{1}{2} M(p) \right)
= \frac{1}{2} + \frac{1}{2} M(p^*)
= \frac{1}{2} + \frac{1}{2} \left( 1 - \sum_{(i, j) \notin \mathcal{E}} p_i^* p_j^* \right)
\leq 1 - \frac{1}{2} \sum_{k=1}^\eta (p_k^*)^2
= 1 - \frac{1}{2\eta},
\]

where the last inequality comes from \(\sum_{k=1}^\eta (p_k^*)^2 \geq \frac{1}{\eta} (\sum_{k=1}^\eta p_k^*)^2 = \frac{1}{\eta}\). Hence
\[
P_p(\hat{U}^\top V \geq 0) = 1 - J(p) \geq \frac{1}{2\eta}, \quad \forall p,
\]

which is the result we desire.
Remark 12 If \( U = V \) and \( f^* \) is the identity function, we can get rid of the additional \( 1/2 \) factor and show that
\[
\mathbb{P}(\hat{U}^\top V \geq 0) \geq \frac{1}{\eta}.
\]
In fact, if \( V \) is uniformly distributed on \( V \), we can recover the prestigious Turán’s theorem in graph theory:

Theorem 13 (Turán (1941)) If a graph with \( n \) vertices does not contain any \( (k + 1) \)-clique, then its number of edges cannot exceed \( (1 - \frac{1}{k}) \cdot \frac{n^2}{2} \).

By restricting the random vector \( V \) to a subset of \( \mathbb{R}^d \), we have the following corollary.

Corollary 14 Let \( \mathcal{U}, \mathcal{V} \) be finite subsets of \( B_d \). Suppose that there exists bijection \( f^* : \mathcal{V} \mapsto \mathcal{U} \) such that
\[
f^*(v)^\top v = \max_{u \in \mathcal{U}} u^\top v, \quad \forall v \in \mathcal{V},
\]
and \( f^*(v)^\top v > 0 \) for all \( v \in \mathcal{V} \). Let \( V \) be any random variable supported on \( \mathcal{V} \), \( U = f^*(V) \) and \( \hat{U} \) be an iid copy of \( U \). Then for any \( S \subseteq \mathcal{V} \),
\[
\mathbb{E}\left[ I(\hat{U}^\top V \geq 0) I(\hat{U} \in f^*(S)) I(V \in S) \right] \geq \frac{1}{2\eta(\mathcal{U}, \mathcal{V})} \mathbb{E}\left[ I(\hat{U} \in f^*(S)) I(V \in S) \right], \quad (17)
\]
where \( f^*(S) := \{ f^*(v) : v \in S \} \).
Appendix B. Proof of Theorem 5

Considering Proposition 9, and the fact that

\[ \frac{\beta e^{\beta \lambda}}{(1 + e^{\beta \lambda})^2} \leq \frac{1}{4\lambda}, \]

we only have to show

\[ \Gamma_t(\mathcal{L}_\beta, \pi^{\text{TS}}) \leq 100\lambda^{-2}(\eta(\mathcal{L}_\beta) \lor d), \quad \text{a.s., } \forall t. \tag{18} \]

We will present two separate proofs of (18) for \( \beta \leq 2 \) and \( \beta > 2 \), respectively. For \( \beta \leq 2 \), we resort to the previous Lipschitz analysis; for \( \beta > 2 \), we adopt a new line of analysis that is connected to our definition of fragility dimension.

We fix the stage index \( t \) in this section. To simplify notations, we let \( Y \) be a random variable with the same distribution as \( \theta^* \) conditioned on \( \mathcal{H}_{t-1} \). We also define \( X = \alpha(Y) \) and let \( \hat{X} \) be an iid copy of \( X \), \( \hat{Y} \) an iid copy of \( Y \). Thus \( X, Y, \hat{X} \) and \( \hat{Y} \) can be interpreted as aliases for \( A^*, \theta^*, A_t \) and \( \theta_t \), respectively. As a shorthand we use \( \eta \) in place of \( \eta(\mathcal{L}_\beta) \). We will omit the “almost surely” qualifications whenever ambiguities do not arise.

Before moving on, we introduce a result adapted from Russo and Van Roy (2016), which gives a primitive bound of information ratio.

**Proposition 15** For any generalized linear bandit problem \( \mathcal{L} = (\mathcal{A}, \Theta, R, \phi, \rho) \),

\[ \Gamma_t(\mathcal{L}, \pi^{\text{TS}}) \leq E \left[ \phi(X^\top Y) - \phi(X^\top \hat{Y}) \right]^2 \frac{2}{2 \cdot E[\text{Var}[\phi(X^\top \hat{Y}) | X]]}. \tag{19} \]

**Proof** First notice that, since \( \hat{X} \) is independent of \( Y \) and \( \hat{Y} \) is independent of \( X \), we have \( E[\phi(X^\top Y)] = E[\phi(X^\top \hat{Y})] \). Therefore (19) is equivalent to

\[ \Gamma_t(\mathcal{L}, \pi^{\text{TS}}) \leq \frac{E \left[ \phi(X^\top Y) - \phi(\hat{X}^\top Y) \right]^2}{2 \cdot E[\text{Var}[\phi(X^\top \hat{Y}) | X]]}. \tag{20} \]

Comparing (5) and (20) and , we only have to show

\[ I(X; \hat{X}, R(\hat{X})) \geq 2 \cdot E[\text{Var}[\phi(X^\top \hat{Y}) | X]]. \tag{21} \]
In fact, we have that

\[
I(X; \hat{X}, R(\hat{X})) \overset{(c)}{=} I(Y; \hat{X}, R(\hat{X})) \\
= I(Y; \hat{X}) + I(Y; R(\hat{X})|\hat{X}) \\
\overset{(d)}{=} I(Y; R(\hat{X})|\hat{X}) \\
= \sum_{x \in A} I(Y; R(x)) \mathbb{P}(\hat{X} = x) \\
\overset{(e)}{=} \sum_{y \in \Theta} I(Y; R(y)) \mathbb{P}(Y = y) \\
= \sum_{y, y' \in \Theta} D_{KL}(P(R(y')) || P(R(y')|y = y)) \cdot \mathbb{P}(Y = y) \mathbb{P}(Y = y') \\
\overset{(f)}{\geq} 2 \sum_{y, y' \in \Theta} \left( \mathbb{E}[R(y')] - \mathbb{E}[R(y')|y = y] \right)^2 \cdot \mathbb{P}(Y = y) \mathbb{P}(Y = y') \\
= 2 \sum_{y \in \Theta} \mathbb{P}(Y = y) \left\{ \sum_{y' \in \Theta} \mathbb{P}(Y = y') \left( \mathbb{E}[R(y')] - \mathbb{E}[R(y')|Y = y] \right)^2 \right\} \\
\overset{(g)}{=} 2 \cdot \mathbb{E}[\text{Var}[\phi(\hat{X}^\top Y)|\hat{X}]] \\
= 2 \cdot \mathbb{E}[\text{Var}[\phi(\hat{X}^\top Y)|X]],
\]

where we use \( R(y) \) to denote \( R(\alpha(y)) \) for \( y \in \Theta \). In (c) and (e) we use the fact that \( \alpha \) is a bijection. That (d) holds is because of the independence between \( Y \) and \( \hat{X} \). In (f) we apply the Pinsker’s inequality upon noticing that \( R \in \{0, 1\} \). The final step (g) follows from the fact that

\[
\mathbb{E}[R(y')|Y = y] = \phi(\alpha(y')^\top y),
\]

and that

\[
\mathbb{E}[R(y')] = \mathbb{E}[\phi(\alpha(y')^\top Y)].
\]

Thus we have (21).

\[ \blacksquare \]

\[ \text{B.1. Proof of (18) for Small } \beta \]

We first point out to a useful lemma.

**Lemma 16** Let \( U, V \) be random vectors in \( \mathbb{R}^d \), and let \( R, S \) be independent random variables with distributions equal to the marginals of \( U, V \), respectively. Then

\[
\mathbb{E}[|U^\top V|^2] \leq d \cdot \mathbb{E}[(R^\top S)^2].
\]
Proof Let $\Sigma = \mathbb{E}[VV^T]$, then
\[
\mathbb{E}[|U^T V|^2] \leq \mathbb{E}[\|\Sigma^{1/2}U\|_2 \cdot \|\Sigma^{-1/2}V\|_2] \\
\leq \mathbb{E}[\|\Sigma^{1/2}U\|_2^{1/2} \cdot \|\Sigma^{-1/2}V\|_2^{1/2}] \\
= \mathbb{E}[R^T \mathbb{E}[SS^T] R^1/2 \cdot \mathbb{E}[V^T \mathbb{E}[VV^T]^{-1} V]^{1/2}] \\
= \left( \mathbb{E}[\langle (R^T S)^2 \rangle \cdot d \right)^{1/2},
\]
where $(h)$ and $(i)$ result from Cauchy-Schwarz inequality and $(j)$ comes from the fact that for any random vector $W$ and non-random matrix $A$, there is $\mathbb{E}[W^T AW] = \text{Tr}(A\text{Cov}(W)) + \mathbb{E}[W^T A\mathbb{E}[W]]$. Thus we arrive at our desired result.

Proposition 17 Let $\mathcal{L} = (\mathcal{A}, \Theta, R, \phi, \rho)$ be any generalized linear bandit problem instance where $\phi$ is such that there exist constants $0 < L_1 \leq L_2$ with
\[
L_1 \leq |\phi'(x)| \leq L_2, \quad \forall x \in [-1, 1].
\]
Then we have
\[
\Gamma_t(\mathcal{L}, \pi^{TS}) \leq d \cdot \frac{L_2^2}{L_1^2}.
\]
Specifically, for the logistic bandit problem $\mathcal{L}_\beta$, there is
\[
\Gamma_t(\mathcal{L}_\beta, \pi^{TS}) \leq d \cdot \left( \frac{(1 + e^\beta)^2}{e^\beta} \right)^2.
\]

Proof From Proposition 15, we have
\[
\Gamma_t(\mathcal{L}, \pi^{TS}) \leq \frac{\mathbb{E}[\phi(X^TY) - \phi(X^T\hat{Y})]^2}{2\mathbb{E}[\text{Var}[\phi(X^TY)|X]]}.
\]
Let $\hat{Y}$ be another iid copy of $Y$, there is
\[
\mathbb{E}[\text{Var}[\phi(X^T\hat{Y})|X]] = \frac{1}{2} \cdot \mathbb{E}\left[\mathbb{E}\left[\left(\phi(X^T\hat{Y}) - \phi(X^T\hat{Y})\right)^2|X\right]\right] \\
\geq \frac{1}{2} \cdot \mathbb{E}\left[\mathbb{E}\left[(X^T\hat{Y} - X^T\hat{Y})^2|X\right]\right] \\
= \frac{L_2^2}{2} \cdot \mathbb{E}\left[(X^T\hat{Y} - X^T\hat{Y})^2\right].
\]
On the other hand, there is also
\[
\mathbb{E}[\phi(X^TY) - \phi(X^T\hat{Y})] \leq L_2 \cdot \mathbb{E}[|X^TY - X^T\hat{Y}|] \\
= L_2 \cdot \mathbb{E}[|X^T(Y - \hat{Y})|] \\
\leq L_2 \cdot \sqrt{d \cdot \mathbb{E}\left[(X^T(Y - \hat{Y}))^2\right]},
\]
(25)
where \((k)\) follows from Lemma 16. Comparing (24) and (25), we arrive at
\[
\Gamma_t(\mathcal{L}, \pi^{TS}) \leq d \cdot \frac{L_2^2}{L_1^2},
\]
which is the desired result. Plugging in \(L_\beta\) into Proposition 17 and notice that
\[
\frac{\beta e^\beta}{(1 + e^\beta)^2} \leq \phi'_\beta(x) \leq \beta, \quad \forall x \in [-1, 1],
\]
we shall arrive at
\[
\Gamma_t(\mathcal{L}_\beta, \pi^{TS}) \leq d \cdot \left( \frac{(1 + e^\beta)^2}{e^\beta} \right)^2.
\]

From Proposition 17, for \(\beta \leq 2\), there is
\[
\Gamma_t(\mathcal{L}_\beta, \pi^{TS}) \leq d \cdot \left( \frac{(1 + e^{2\beta})^2}{e^{2\beta}} \right)^2 < 100d.
\]

**B.2. Proof of (18) for Large \(\beta\)**

In this section we show (18) for \(\beta > 2\). Throughout we assume that Assumption 1 holds with constant \(\lambda \in (0, 1)\). For any \(x \in \mathcal{A}\), let \(\sigma(x) = x^T \alpha^{-1}(x)\). For \(\zeta \in \mathbb{R}\), We further define
\[
\gamma_{\beta, \lambda}(\zeta) := \phi_{\beta}(\lambda) - \phi_{\beta}(\lambda - \zeta),
\]
and let \(z_{\beta, \lambda} = \arg\max_{\zeta \in [0, 1 + \lambda]} \gamma_{\beta, \lambda}(\zeta) / \zeta\), \(w_{\beta, \lambda} = (\lambda + z_{\beta, \lambda}) / 2\) and \(\nu_{\beta}(x) = \mathbb{E}[\phi_{\beta}(X^T Y) - \phi_{\beta}(X^T \hat{Y}) | X = x]\). Under the above notations, (19) can be written as
\[
\Gamma_t(\mathcal{L}, \pi^{TS}) \leq \frac{\mathbb{E}[\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y})]^2}{2 \cdot \mathbb{E}[(\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y}) - \nu_{\beta}(X))^2]}. \tag{27}
\]

We also partition the action set \(\mathcal{A}\) into two subsets:
\[
\mathcal{D} := \{ x \in \mathcal{A} : \nu_{\beta}(x) \leq \gamma_{\beta, \lambda}(w_{\beta, \lambda}) \}
\]
and \(\bar{\mathcal{D}} = \Theta \setminus \mathcal{D}\). Suppose that we can find constants \(C_1, C_2\), such that
\[
\mathbb{E}[\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y})\mathbb{I}(X \in \mathcal{D})]^2 \leq C_1 \cdot \mathbb{E}
\left[ (\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y}) - \nu_{\beta}(X))^2 \mathbb{I}(X \in \mathcal{D}) \right]
\]
and
\[
\mathbb{E}[\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y})\mathbb{I}(X \in \bar{\mathcal{D}})]^2 \leq C_2 \cdot \mathbb{E}
\left[ (\gamma_{\beta, \sigma}(X)(\sigma(X) - X^T \hat{Y}) - \nu_{\beta}(X))^2 \mathbb{I}(X \in \bar{\mathcal{D}}) \right]
Then, from Cauchy-Schwarz inequality we have
\[
\mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})]^2 \\
= \left( \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})2I(X \in \mathcal{D})] + \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})2I(X \in \mathcal{D}^c)] \right)^2 \\
\leq 2 \left\{ \left( \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})2I(X \in \mathcal{D})] \right)^2 + \left( \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})2I(X \in \mathcal{D}^c)] \right)^2 \right\} \\
\leq 2 \max\{C_1, C_2\} \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y}) - \nu_{\beta}(X)]^2, \\
\text{(28)}
\]
Thus we can bound the right-hand side of (27) by
\[
\frac{\mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y})]^2}{2 \cdot \mathbb{E}[\gamma_{\beta,\sigma}(X)(\sigma(X) - X^\top \hat{Y}) - \nu_{\beta}(X)]^2} \leq \max\{C_1, C_2\}. \\
\text{(29)}
\]
To determine $C_1$, we first introduce a lemma.

**Lemma 18** Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be such that $f(0) = 0$ and $f(\zeta)/\zeta$ is non-decreasing over $\zeta \geq 0$ ($f(0)/0$ is interpreted as the limit of $\zeta \downarrow 0$). Then for any non-negative random variable $U$, there is
\[
\frac{\mathbb{E}[f(U)]^2}{\mathbb{E}[U]^2} \leq \frac{\mathbb{V}[f(U)]}{\mathbb{V}[U]}.
\text{(30)}
\]

**Proof** Let $g(\zeta) = f(\zeta)/\zeta$ with $g(0) = \lim_{\zeta \downarrow 0} f(\zeta)/\zeta$. By our assumption, $g(\zeta)$ is also non-negative and non-decreasing. Let $V$ be an iid copy of $U$, we have that
\[
\mathbb{E}[g(U)^2V] : \mathbb{E}[V^2] - \mathbb{E}[g(U)^2U^2] : \mathbb{E}[U] = \frac{1}{2} \left( \mathbb{E}[g(U)^2UV^2 + g(V)^2U^2V] - \mathbb{E}[g(U)^2U^2V + g(V)^2V^2U] \right) \\
= \frac{1}{2} \mathbb{E}[UV(V - U)(g(U)^2 - g(V)^2)] \\
\leq 0,
\text{(31)}
\]
where the final inequality results from the monotonicity of $g$. Therefore we have shown
\[
\frac{\mathbb{E}[g(U)^2U]}{\mathbb{E}[U]} \leq \frac{\mathbb{E}[g(U)^2U^2]}{\mathbb{E}[U]^2} = \frac{\mathbb{E}[f(U)]^2}{\mathbb{E}[U]^2}.
\text{(32)}
\]
Thus there is
\[
\frac{\mathbb{E}[f(U)]^2}{\mathbb{E}[U]^2} = \frac{\mathbb{E}[g(U)U]^2}{\mathbb{E}[U]^2} \\
\overset{(l)}{\leq} \frac{\mathbb{E}[g(U)^2U]^2}{\mathbb{E}[U]^2} \\
= \frac{\mathbb{E}[g(U)^2U]}{\mathbb{E}[U]} \\
\overset{(m)}{\leq} \frac{\mathbb{E}[f(U)]^2}{\mathbb{E}[U]^2},
\text{(33)}
\]
where \((l)\) comes from Cauchy-Schwarz inequality and \((m)\) is the consequence of \((32)\). Finally, notice that \(\text{Var}[f(U)] = \mathbb{E}[f(U)^2] - \mathbb{E}[f(U)]^2\) and \(\text{Var}[U] = \mathbb{E}[U^2] - \mathbb{E}[U]^2\), we have

\[
\frac{\mathbb{E}[f(U)]^2}{\mathbb{E}[U]^2} \leq \frac{\text{Var}[f(U)]}{\text{Var}[U]}
\]

\[
\Leftrightarrow \quad \mathbb{E}[f(U)]^2 \text{Var}[U] \leq \mathbb{E}[U]^2 \text{Var}[f(U)]
\]

\[
\Leftrightarrow \quad \mathbb{E}[f(U)]^2 (\mathbb{E}[U^2] - \mathbb{E}[U]^2) \leq \mathbb{E}[U]^2 (\mathbb{E}[f(U)^2] - \mathbb{E}[f(U)]^2)
\]

\[
\Leftrightarrow \quad \mathbb{E}[f(U)]^2 \mathbb{E}[U^2] \leq \mathbb{E}[U]^2 \mathbb{E}[f(U)^2],
\]

where the final inequality is implied by \((33)\). Hence the proof is complete.

We define function \(\bar{\gamma}_{\beta,\lambda}(\zeta)\) by

\[
\bar{\gamma}_{\beta,\lambda}(\zeta) = \begin{cases} 
\gamma_{\beta,\lambda}(\zeta) & \zeta \in [0, \zeta_{\beta,\lambda}] \\
\frac{\gamma_{\beta,\lambda}(\zeta_{\beta,\lambda})}{\bar{\gamma}_{\beta,\lambda}^{\beta,\lambda}} \cdot \zeta & \zeta \in [\zeta_{\beta,\lambda}, 1 + \lambda]
\end{cases}
\]

as is shown in Figure 2. We thus have
\[
\mathbb{E}\left[(\gamma_{\beta,\sigma(X)}(\sigma(X) - X^\top \hat{Y}) - \nu_\beta(X))^2 \mathbb{I}(X \in \mathcal{D})\right] \\
\geq \chi^2 \cdot \mathbb{E}\left[(\bar{\gamma}_{\beta,\sigma(X)}(\sigma(X) - X^\top \hat{Y}) - \nu_\beta(X))^2 \mathbb{I}(X \in \mathcal{D})\right] \\
\overset{(n)}{=} \chi^2 \cdot \mathbb{E}\left[\text{Var}(\bar{\gamma}_{\beta,\sigma(X)}(\sigma(X) - X^\top \hat{Y})|X) \mathbb{I}(X \in \mathcal{D})\right] \\
= \chi^2 \cdot \mathbb{E}\left[\text{Var}(X^\top \hat{Y})|X)Q(X)^2 \mathbb{I}(X \in \mathcal{D})\right] \\
= \chi^2 \cdot \mathbb{E}\left[(X^\top (\hat{Y} - \bar{Y}))^2 |X)Q(X)^2 \mathbb{I}(X \in \mathcal{D})\right] \\
= \chi^2 \cdot \mathbb{E}\left[\left((Q(X)\mathbb{I}(X \in \mathcal{D})X)^\top (\hat{Y} - \bar{Y})\right)^2\right] \\
\overset{(o)}{=} \frac{\chi^2}{d} \cdot \mathbb{E}\left[(Q(X)\mathbb{I}(X \in \mathcal{D})X)^\top (\hat{Y} - \bar{Y})\right]^2 \\
= \frac{\chi^2}{d} \cdot \mathbb{E}\left[(X^\top Y - X^\top \hat{Y})Q(X) \cdot \mathbb{I}(X \in \mathcal{D})\right]^2 \\
\overset{(p)}{=} \frac{\chi^2}{d} \cdot \mathbb{E}\left[\bar{\gamma}_{\beta,\sigma(X)}(X^\top Y - X^\top \hat{Y}) \cdot \mathbb{I}(X \in \mathcal{D})\right]^2 \\
\overset{(q)}{=} \frac{\chi^2}{d} \cdot \mathbb{E}\left[\gamma_{\beta,\sigma(X)}(\sigma(X) - X^\top \hat{Y}) \cdot \mathbb{I}(X \in \mathcal{D})\right]^2, \\
\overset{(r)}{=} \frac{\chi^2}{d} \cdot \mathbb{E}\left[\gamma_{\beta,\sigma(X)}(\sigma(X) - X^\top \hat{Y}) \cdot \mathbb{I}(X \in \mathcal{D})\right]^2,
\]

where

\[
\chi := \inf_{x \in \mathcal{D}, y \in \Theta} \frac{\gamma_{\beta,\sigma(x)}(\sigma(x) - x^\top y)}{\bar{\gamma}_{\beta,\sigma(x)}(\sigma(x) - x^\top y)},
\]

and

\[
Q(x)^2 := \frac{\text{Var}(\bar{\gamma}_{\beta,\sigma(x)}(\sigma(x) - x^\top \hat{Y}))}{\text{Var}(x^\top \hat{Y})}.
\]

In (n), we apply the fact that for any random variable \(W\) with \(\mathbb{E}[W^2] < \infty\) and constant \(a\), there is

\[
\mathbb{E}[(W - a)^2] \geq \text{Var}[W].
\]

In (o) we use the result in Lemma 18. In (p), we use the fact that

\[
Q(x)^2 = \frac{\text{Var}(\bar{\gamma}_{\beta,\sigma(x)}(\sigma(x) - x^\top \hat{Y}))}{\text{Var}(\sigma(x) - x^\top Y)} \\
\geq \frac{\mathbb{E}\left[\bar{\gamma}_{\beta,\sigma(x)}(\sigma(x) - x^\top \hat{Y})\right]^2}{\mathbb{E}[\sigma(x) - x^\top \hat{Y}]^2} \\
= \frac{\mathbb{E}\left[\bar{\gamma}_{\beta,\sigma(x)}(x^\top \alpha^{-1}(x) - x^\top \hat{Y})\right]^2}{\mathbb{E}[x^\top \alpha^{-1}(x) - x^\top \hat{Y}]^2}.
\]

Step (q) follows from that \(\bar{\gamma}_{\beta,\sigma(x)} \geq \gamma_{\beta,\sigma(X)}\), and the final step follows trivially from \(\sigma(X) = X^\top \alpha^{-1}(X) = X^\top Y\). Hence we can set \(C_1 = \frac{d}{\chi^2}\).
Next we turn to constant $C_2$. We have that
\[
\mathbb{E} \left[ (\gamma_{\beta, \sigma}(X)) (\sigma(X) - X^\top \hat{Y} - \nu_{\beta}(X))^2 \mathbb{I}(X \in \bar{D}) \right] \\
\geq \mathbb{E} \left[ (\gamma_{\beta, \sigma}(X)) (\sigma(X) - X^\top \hat{Y} - \nu_{\beta}(X))^2 \mathbb{I}(\sigma(X) - X^\top \hat{Y} \leq w_{\beta, \sigma}(X)) \mathbb{I}(X \in \bar{D}) \right] \\
\geq \mathbb{E} \left[ (\gamma_{\beta, \sigma}(X)) (\sigma(X) - X^\top \hat{Y} - \nu_{\beta}(X))^2 \mathbb{I}(X^\top \hat{Y} \geq 0) \mathbb{I}(\hat{Y} \in \alpha^{-1}(\bar{D})) \mathbb{I}(X \in \bar{D}) \right] \\
\geq \xi^2 \cdot \mathbb{E} \left[ \mathbb{I}(X^\top \hat{Y} \geq 0) \mathbb{I}(\hat{Y} \in \alpha^{-1}(\bar{D})) \mathbb{I}(X \in \bar{D}) \right] \\
\geq \frac{\xi^2}{2\eta} \cdot \mathbb{E} \left[ \mathbb{I}(\hat{Y} \in \alpha^{-1}(\bar{D})) \mathbb{I}(X \in \bar{D}) \right]^2 \\
= \frac{\xi^2}{2\eta} \cdot \mathbb{E} \left[ \mathbb{I}(X \in \bar{D}) \right]^2 \\
\geq \frac{\xi^2}{2\eta} \cdot \mathbb{E} \left[ \gamma_{\beta, \sigma}(X) (\sigma(X) - X^\top \hat{Y}) \cdot \mathbb{I}(X \in \bar{D}) \right]^2,
\]
with
\[
\xi^2 := \inf_{x \in \bar{D}, y \in \Theta} \left( \gamma_{\beta, \sigma}(x) (\sigma(x) - x^\top y) - \nu_{\beta}(x) \right)^2,
\]
and (s) comes from Corollary 14. Thus we can set $C_2 = \frac{2\eta}{\xi^2}$.

Finally, when $\beta \geq 2$, we have that $\chi > \xi > 0.1\lambda$. Therefore
\[
\Gamma_t(L_{\beta, \pi^{TS}}) \leq 100\lambda^{-2} \eta. \tag{38}
\]
The values of the constants are plotted in Figure 3. By combining (38) with (26), we arrive at (18).
Appendix C. Proof of Proposition 11

Suppose that for each \( a \in A \),
\[
\mathbb{E}[R(a) | A^* = a] \geq 1 - \delta,
\]
and
\[
\max_{a' \neq a} \mathbb{E}[R(a') | A^* = a] \leq \delta.
\]

Let \((\hat{a}_1, \ldots, \hat{a}_t)\) be any deterministic action sequence up to stage \( t \). Then conditioned on \( A_1 = \hat{a}_1 \ldots A_t = \hat{a}_t \), we have that \( R_1, \ldots, R_t \) are mutually independent. Hence
\[
\begin{align*}
\mathbb{P}(R_1 = \cdots = R_t = 0 | A_1 = \hat{a}_1, \ldots, A_t = \hat{a}_t) &\geq \prod_{j=1}^{t} \mathbb{P}(R_j = 0 | A_j = \hat{a}_j, A^* \neq A_j) \cdot \mathbb{P}(A^* \notin \{\hat{a}_1, \ldots, \hat{a}_t\}) \\
&= (1 - \delta)^t \cdot \left(1 - \frac{t}{N}\right),
\end{align*}
\]
where in the final step we use the fact that the prior of \( A^* \) is uniform. Let \( \mathcal{E}_t \) be the event \( \{R_1 = \cdots = R_t = 0\} \). Since (39) holds for every action sequence, we have that for any policy \( \pi \),
\[
\mathbb{P}(\mathcal{E}_t) \geq (1 - \delta)^t \cdot \left(1 - \frac{t}{N}\right).
\]

Thus
\[
\text{BayesRegret}(t; \mathcal{L}, \pi) = \sum_{j=1}^{t} \mathbb{E}[R^* - R_j]
\]
\[
= t\mathbb{E}[R^*] - \sum_{j=1}^{t} \mathbb{E}[R_j | \mathcal{E}_t] \mathbb{P}(\mathcal{E}_t) - \sum_{j=1}^{t} \mathbb{E}[R_j | \bar{\mathcal{E}}_t] \mathbb{P}(\bar{\mathcal{E}}_t)
\]
\[
\geq (1 - \delta)t - \left[1 - (1 - \delta)^t \cdot \left(1 - \frac{t}{N}\right)\right] t
\]
\[
= \left[(1 - \delta)^t \left(\frac{N - t}{N}\right) - \delta\right] t.
\]

Let \( \delta = 1/N \), we have that for \( t \leq \frac{N}{2} - 1 \),
\[
\text{BayesRegret}(t; \mathcal{L}, \pi) \geq \frac{t}{2\sqrt{e}} \geq \frac{t}{4}.
\]
Appendix D. Upper Bounds of Fragility Dimension

In this section we give worst-case bounds of fragility dimension with respect to the problem dimension \( d \). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two subsets of \( B_d \), and let \( \hat{f}^* : \mathcal{Y} \rightarrow \mathcal{X} \) be such that

\[
\hat{f}^*(y)^\top y = \max_{x \in \mathcal{X}} x^\top y, \quad \forall y \in \mathcal{Y}.
\]

Further we define \( \iota = \inf_{y \in \mathcal{Y}} \hat{f}^*(y) \). Here \( \iota \) can be interpreted as the constant \( \lambda \) in Assumption 1. We will show that the worst-case bounds vary across the three regimes \( \iota = 1 \), \( \iota \in (0, 1) \) and \( \iota = 0 \).

D.1. The Regime \( \iota = 1 \)

When \( \iota = 1 \) since we are constraining \( \mathcal{X} \) and \( \mathcal{Y} \) to be contained in the unit ball, there must be that \( \hat{f}^*(y) = y \) for each \( y \in \mathcal{Y} \). Therefore \( \eta(\mathcal{X}, \mathcal{Y}) \) is equal to the maximum integer \( M \), such that there exists \( \{y_1, \ldots, y_M\} \subseteq \mathcal{Y} \), with

\[
y_i^\top y_j < 0, \quad \forall i, j \in [M], i \neq j.
\]

The following lemma immediately implies that in this case \( \eta(\mathcal{X}, \mathcal{Y}) \leq d + 1 \).

Lemma 19 In the \( d \)-dimensional Euclidean space, there exists at most \( d + 1 \) different vectors, such that the inner-product between any pair of different vectors is negative.

Proof Suppose that there exists a set \( \mathcal{X} \) which consists of \( d + 2 \) different vectors \( x_1, \ldots x_{d+2} \), such that \( x_i^\top x_j < 0 \) for any \( 1 \leq i < j \leq d + 2 \). Let

\[
U = [x_1 \ x_2 \ \cdots \ x_{d+2}].
\]

Then the nullspace of \( U \) has dimension at least 2. Therefore we can find \( z \in \text{null}(U) \subseteq \mathbb{R}^{d+2} \), such that \( z \) has at least one positive entry and one negative entry. Without loss of generality, we have that

\[
z_1 x_1 + z_2 x_2 + \cdots + z_k x_k = -z_\ell x_\ell - z_\ell+1 x_{\ell+1} - \cdots - z_{d+2} x_{d+2},
\]

where \( 1 < k < \ell < d + 2 \) and \( z_1, \ldots z_k > 0, z_\ell \ldots z_{d+2} < 0 \). However, this gives

\[
\|z_1 x_1 + z_2 x_2 + \cdots + z_k x_k\|_2^2 = (z_1 x_1 + z_2 x_2 + \cdots + z_k x_k)^\top (z_1 x_1 + z_2 x_2 + \cdots + z_k x_k) = -(z_1 x_1 + z_2 x_2 + \cdots + z_k x_k)^\top (z_\ell x_\ell + z_\ell+1 x_{\ell+1} + \cdots + z_{d+2} x_{d+2})
\]

\[
= -\sum_{i=1}^k \sum_{j=\ell}^{d+2} z_i z_j x_i^\top x_j < 0,
\]

which is a contradiction.

\[\blacksquare\]
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D.2. The Regime \( \iota = 0 \)

We show by an example for \( d = 3 \) that when \( \iota = 0 \), the fragility dimension can be arbitrarily large. Let \( h, r \in (0, 1) \) be constants to be determined later. Consider \( X = \{x_1, \ldots, x_N\} \) and \( Y = \{y_1, \ldots, y_N\} \) where

\[
x_i = (r \cdot \cos (\frac{2\pi}{N} \cdot i), \ r \cdot \sin (\frac{2\pi}{N} \cdot i), \ \sqrt{1-r^2}), \ i = 1, \ldots, N,
\]

and

\[
y_i = (h \cdot \cos (\frac{2\pi}{N} \cdot i), \ h \cdot \sin (\frac{2\pi}{N} \cdot i), \ -\sqrt{1-h^2}), \ i = 1, \ldots, N,
\]

as is shown in Figure 4. We have that \( f^*(y_i) = x_i \) and

\[
x_k^T y_\ell = hr \cdot \cos \left(2\pi \cdot \frac{(k-\ell)}{N}\right) - \sqrt{(1-h^2)(1-r^2)}.
\]

To satisfy that \( x_k^T y_\ell < 0 \) for all \( k \neq \ell \), we only have to choose \( h \) and \( r \) such that

\[
hr \cdot \cos \left(\frac{2\pi}{N}\right) - \sqrt{(1-h^2)(1-r^2)} < 0 < hr - \sqrt{(1-h^2)(1-r^2)}.
\]

This can be done by arbitrarily choosing \( h \) and letting \( r = \sqrt{1-\gamma h^2} \) with

\[
\frac{\cos^2 \left(\frac{2\pi}{N}\right)}{1-\sin^2 \left(\frac{2\pi}{N}\right) h^2} < \gamma < 1.
\]

Notice that \( N \) can be arbitrarily large since \( \iota = 0 \). Thus \( \eta(X, Y) \) is unbounded.

D.3. The Regime \( \iota \in (0, 1) \)

In this section we show that when \( \iota \in (0, 1) \), the worst-case fragility dimension grows exponentially with \( d \). We first introduce the following result. We point readers to Böröczky Jr et al. (2004) for a detailed discussion.

**Fact 20** For any \( \epsilon \in (0, 1) \), there exists \( \gamma > 1 \), such that for all integer \( d \geq 3 \), there exist \( \gamma^d \) vectors in \( S_{d-1} \) such that the inner product of any two different vectors is at most \( \epsilon \).
For any fixed $d$, let $u, v \in (0, \frac{\pi}{2})$ and $\epsilon > 0$ be constants to be determined later. Let $z_1, \ldots, z_N \in S_{d-2}$ be such that

$$z_i^\top z_j < \epsilon, \quad \forall j, k \in [N], j \neq k.$$ 

Consider the pair of sets $\mathcal{X}, \mathcal{Y} \subset S_{d-1}$ defined by

$$\mathcal{X} := \{x_i\}_{i=1}^N, \quad x_i = (\cos u, \sin u \cdot z_i),$$

and

$$\mathcal{Y} := \{y_i\}_{i=1}^N, \quad y_i = (-\cos v, \sin v \cdot z_i).$$

Thus we have

$$x_i^\top y_i = -\cos u \cos v + \sin u \sin v = -\cos(u + v), \quad i \in [N]$$

and

$$x_j^\top y_k = -\cos u \cos v + z_j^\top z_k \sin u \sin v < -\cos(u + v) - (1 - \epsilon) \sin u \sin v, \quad j, k \in [N], j \neq k.$$ 

There is obviously $f^*(y_i) = x_i$. In order to satisfy $\inf_{y \in \mathcal{Y}} f^*(y)^\top y = \iota$, we only have to choose $u, v, \epsilon$ such that

$$\cos(u + v) \leq -\iota,$$

and

$$\cos(u + v) + (1 - \epsilon) \sin u \sin v \geq 0.$$ 

This can be done by setting

$$u = v = \frac{1}{2} \arccos(-\iota), \quad \epsilon = \frac{1 - \iota}{1 + \iota}.$$ 

Since $\iota \in (0, 1)$, we have that $\epsilon \in (0, 1)$. From Fact 20, there exists $\gamma > 1$ such that $N \geq \gamma^{d-1}$.

**D.4. Removing Actions Could Make Problem Harder**

Let $\mathcal{X}$ and $\mathcal{Y}$ be the two sets given in the example in Appendix D.2. Let the parameter set be $\Theta = \mathcal{Y}$ and consider action sets $A_1 = \mathcal{X} \cup \mathcal{Y}$ and $A_2 = \mathcal{X}$. Obviously $A_2 \subset A_1$. However, we argue that the problem $L_1$ with action and parameter sets $(A_1, \Theta)$ is easier than the problem $L_2$ with sets $(A_2, \Theta)$.

In fact, from Lemma 19, we have that $\eta(A_1, \Theta) \leq 4$. However, the argument in Appendix D.2 shows that $\eta(A_1, \Theta) = N$, where $N$ is the size of the parameter set. Therefore the regret of Thompson sampling on $L_1$ can be bounded by the result in Theorem 1, which is independent of $\beta$. However, to learn $L_2$ for a large $\beta$, we almost have to try every action to find the optimal one. Therefore, somewhat surprisingly, reducing the size of the action set can actually make the problem harder.