RENORMALISATION AND LOCALITY: BRANCHED ZETA VALUES

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Abstract. Multivariate renormalisation techniques are implemented in order to build, study and then renormalise at the poles, branched zeta functions associated with trees. For this purpose, we first prove algebraic results and develop analytic tools, which we then combine to study branched zeta functions. The algebraic aspects concern universal properties for locality algebraic structures, some of which had been discussed in previous work; we "branch/lift" to trees operators acting on the decoration set of trees, and factorise branched maps through words by means of universal properties for words which we prove in the locality setup. The analytic tools are multivariate meromorphic germs of pseudodifferential symbols with linear poles which generalise the meromorphic germs of functions with linear poles studied in previous work. Multivariate meromorphic germs of pseudodifferential symbols form a locality algebra on which we build various locality maps in the framework of locality structures. We first show that the finite part at infinity defines a locality character from the latter symbol valued meromorphic germs to the scalar valued ones. We further equip the locality algebra of germs of pseudodifferential symbols with locality Rota-Baxter operators given by regularised sums and integrals. By means of the universal properties in the framework of locality structures we can lift Rota-Baxter operators to trees, and use the lifted discrete sums in order to build and study renormalised branched zeta values associated with trees. By construction these renormalised branched zeta values factorise on mutually independent (for the locality relation) trees.

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Introduction

Trees offer a useful tool to understand the hierarchical structure underlying renormalisation in quantum field theory. They provide a toy model to analyse subdivergences arising in Feynman diagrams [K] for integrals associated with trees reflect the structure of nested divergences. Our objects of study in this paper are their discrete counterpart, nested sums associated with trees, which alongside their nested structure yield interesting generalisations of multizeta functions, which we call branched zeta functions. The latter generalise the arborified zeta values studied in [M] using J. Ecalle’s “arborification” procedure, viewed as a surjective Hopf algebra morphism from the Hopf algebra of decorated rooted forests onto a Hopf algebra of shuffles or quasi-shuffle, which amounts to what we call the “flatenening procedure”.

Multiple zeta functions $\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \cdots > n_k > 0} s_1^{-n_1} \cdots s_k^{-n_k}$ can be interpreted as sums associated either with (Chen) cones or with rooted (ladder) trees, involving the pseudodifferential symbols $\sigma_{s_i}, i = 1, \ldots$. In [GPZ1] and [GPZ2] we generalised multiple zeta functions to sums on general convex cones leading to conical zeta functions. Here we study their generalisation to non planar rooted trees leading to branched zeta-functions. In contrast to cones, which offer a relative flexibility we dealt with in [MP] pseudodifferential symbols $\sigma$, subdivisions, rooted trees present a certain rigidity and enjoy a universal property which we implement at different stages of the construction.

Our starting point is the Riemann zeta function $\zeta(s)$ defined as the meromorphic extension $s \mapsto \sum_{n=1}^{\infty} n^{-s}$ of the holomorphic map $s \mapsto \sum_{n=1}^{\infty} n^{-s}$ on the half-plane $\Re(s) > 1$, where $\sum_{n=1}^{\infty} n^{-s} := \lim_{N \to \infty} \sum_{n=1}^{N} n^{-s}$ is the cut-off regularised sum as defined in [P1], see also [MP]. Following the same line of thought as [MP], starting from the (polyhomogeneous) pseudodifferential symbol $\sigma_s(x) = x^{-s}\chi(x)$ on $\mathbb{R}_{>0}$, where $\chi$ is a smooth excision function at zero, we consider holomorphic families $z \mapsto \sigma_{s+z}(x)$ of (polyhomogeneous) symbols on $\mathbb{R}_{>0}$. The Riemann zeta function $\zeta(s)$ at a pole $s$, corresponds to the evaluation at $z = 0$ of a regularised cut-off sum

$$\zeta(s) := \text{ev}_{z=0} \left( \lim_{N \to \infty} \sum_{n=1}^{N} \sigma_{s+z}(n) \right), \quad \text{resp. } \zeta^*(s) := \text{ev}_{z=0} \left( \lim_{N \to \infty} \sum_{n=1}^{N-1} \sigma_{s+z}(n) \right).$$

We reinterpret this expression by means of summation Rota-Baxter operators $\mathcal{G}_\lambda$ with $\lambda \in \{-1, 0, 1\}$ (resp. $\mathcal{I}$), that to a symbol $\sigma$ on $\mathbb{R}_{>0}$ assign another symbol $\mathcal{G}_\lambda(\sigma)$ (resp. $\mathcal{I}(\sigma)$). For $\lambda = \pm 1$, the operator $\mathcal{G}_\lambda$ coincides on any positive integer $n$ with the discrete summation map $n \mapsto S(\sigma)(n) := \sum_{k=1}^{n-1} \sigma(k)$ or $n \mapsto S(\sigma)(n) := \sum_{k=1}^{n} \sigma(k)$ according to whether $\lambda = -1$ or $\lambda = 1$. For $\lambda = 0$, $\mathcal{G}_0 = I(\sigma)$ is the integral map $x \mapsto I(\sigma)(x) := \int_{0}^{x} \sigma(y) \, dy$ defined for $x \geq 0$. For fixed $z \in \mathbb{C}$, we take the finite part at infinity of the map $N \mapsto \mathcal{G}_\lambda(\sigma_{s+z})(N)$ to build the regularised cut-off sum (compare with (1))

$$\zeta(s) = \text{ev}_{z=0} \left( \lim_{N \to \infty} \mathcal{G}_{-1}(\sigma_{s+z})(N) \right), \quad \text{resp. } \zeta^*(s) = \text{ev}_{z=0} \left( \lim_{N \to \infty} \mathcal{G}_1(\sigma_{s+z})(N) \right).$$

This serves as a starting point to build higher zeta functions associated with trees by means of an algebra $\Omega$ (Definition 4.13) of (multivariate meromorphic germs of) symbols.
on \( \mathbb{R}_{\geq 0} \) used to decorate the trees and thereby regularise the multiple integrals and multiple sums involved in the construction. We adopt a multivariate renormalisation approach already used in the toy model of [CGPZ2] and work in the locality setup discussed in [CGPZ1]. In view of multivariate regularisation, we consider the algebra \( \mathcal{F}_\Omega \) of rooted trees decorated with multivariate meromorphic germs of pseudodifferential symbols in \( \Omega \) and view it as an \( \Omega \)-operated algebra. The algebra \( \mathcal{F}_\Omega \) of rooted trees is equipped with a locality algebra structure involving a partial product on independent germs, and an independence relation \( \top_{\mathcal{F}_\Omega} \) inherited from the independence relation \( \top_\Omega \) on the decorating locality set \( \Omega \).

In the algebraic Part 1 of the present paper, we use a locality version of universal properties proved in [CGPZ2], of the algebra \( \mathcal{F}_\Omega \) of rooted trees \([F]\) decorated by a locality set \( \Omega \). From [CGPZ2] we borrow Corollary 1.24 which yields a lift of any locality map \( \phi : \Omega \rightarrow \Omega \) on the decoration locality algebra \( \Omega \) to a locality morphism \( \widehat{\phi} : \mathcal{F}_\Omega \rightarrow \Omega \) of \( \Omega \)-operated locality algebras. The first part is mainly dedicated to relating the constructions on trees of [CGPZ2] to new constructions on locality algebras of words. It is written in the locality set category; all the results nevertheless hold in the ordinary set category, which can be recovered in viewing a set \( X \) as a locality set \((X, \top)\) with the trivial locality structure \( \top = X \times X \). In particular, we

- establish a universal property for a locality Rota-Baxter operator on (proper) words (Theorem 2.7).
- establish a correspondence (Theorem 2.9) between the \( \lambda \)-Rota-Baxter property of maps \( \mathcal{G}_\lambda \) on the decoration algebra and the multiplicative property of theirs lifts to (decorated) words for a \( \lambda \)-shuffle product;
- introduce a “flatening” map in Definition 2.10, which “flats” decorated trees to words corresponding to an “arborification” à la Écalle discussed in [M];
- use the “flatening” map to relate (Theorem 2.13) the lift to trees of Rota-Baxter operators with their lift to words.

Part 2 is dedicated to the study of the decoration algebra \( \Omega \) (Definition 4.13) of multivariate meromorphic germs of (polyhomogeneous) pseudodifferential symbols with linear poles, which contains the algebra \( \mathcal{M} \) of multivariate meromorphic germs of functions with linear poles introduced in [GPZ3]. We equip \( \Omega \) with an independence relation inherited from an ambient inner product, similar to the one defined on \( \mathcal{M} \) in [CGPZ1] and show (Proposition 4.15) that it is a locality algebra for the pointwise product on symbols. We further prove (Proposition 4.17) that the renormalised evaluation at infinity \( \text{fp} : \Omega \rightarrow \mathcal{M} \) is a locality morphism for this independence relation. Finally, we build (Theorem 4.20) for \( \lambda \in \{ \pm 1, 0 \} \) locality \( \lambda \)-Rota-Baxter operators on \( \Omega \) which generalise to the multivariate setup (keeping the same notations), the summation maps \( \mathcal{S}_\lambda, \lambda \in \{ -1, 1 \} \) as well as the integration map \( \mathcal{I} \) on univariate meromorphic germs of symbols mentioned above.

In Part 3, we combine the results of Parts 1 and 2 to build and study branched zeta-functions as multivariate meromorphic functions with linear poles and their renormalised counterparts. To carry out this programme, we implement Corollary 1.24 to build the corresponding branched maps which yield locality morphisms \( \mathcal{S}_\lambda : \mathcal{F}_\Omega \rightarrow \Omega \). Implementing the finite part \( \text{fp} \) on \( \Omega \) then gives rise to locality morphisms \( \text{fp} \circ \mathcal{S}_\lambda : \mathcal{F}_\Omega \rightarrow \mathcal{M} \) on the algebra of trees properly decorated by \( \Omega \) to multivariate meromorphic germs of functions.
This gives rise to discrete summation locality morphisms $\zeta^{\text{reg}, \pm 1} : \mathcal{F}_\Omega \to \mathcal{M}$ called regularised branched functions (Proposition 5.5). The locality morphism property ensures the multiplicativity of the regularised branched zeta functions on mutually independent (for the locality relation) pairs of decorated trees (Theorem 5.8). In order to study the pole structure of the regularised branched functions and investigate the rationality of the renormalised branched zeta values, we use the “flattening” to express branched zeta-functions as rational linear combinations of multiple zeta functions. As a consequence of the linear pole structure of multiple zeta functions, the poles of any branched zeta function are also linear. Assuming rationality of the inner product underlying the multivariate renormalisation procedure, we show that the renormalised values at their poles are rational (Theorem 5.11).

To conclude, we were able to study the poles of branched zeta functions thanks to universal properties of localised Rota-Baxter algebras and then renormalise the resulting multivariate meromorphic germs by means of a multivariate minimal subtraction scheme we had already implemented in [CGPZ1] for exponential sums on convex cones leading to conical zeta functions. To implement the multivariate subtraction scheme, we introduced a locality algebra of multivariate meromorphic germs of polyhomogeneous symbols. The relative rigidity of tree structures when compared to the relative flexibility of cone structures, enabled us to “lift” the ordinary discrete summation operator $\sigma \mapsto \sum_{k=1}^{n} \sigma(k)$ on polyhomogeneous symbols to a branched discrete summation operator on this algebra or meromorphic symbols. This branched summation operator is shown to be a locality morphism, which ensures multiplicativity of the resulting renormalised branched zeta values on disjoint trees.

So it is the very special tree structure reflected in the pole structure of the discrete branched sums of multivariate meromorphic symbols cut-off at infinite (via the finite part map) that enabled us a good control of the poles and hence to renormalise appropriately. The next stage we hope to carry out in forthcoming work is to provide a precise description of the tree structure of the poles and to identify a larger class of “branched multivariate meromorphic germs” that hosts such cut-off discrete branched sums of multivariate meromorphic symbols to which we can extend similar multivariate minimal subtraction schemes.
Part 1. Algebraic aspects

An algebraic formulation of the locality principle was provided in [CGPZ1] in the context of the algebraic approach to perturbative quantum field theory initiated by Connes and Kreimer [CK]. It was shown in [CGPZ2] that the space spanned by decorated rooted forests equipped with an appropriate independence relation inherited from the one on the decorating set, is the initial object in the category of locality operated algebras. We establish for words similar universal properties, which we then use to lift to words $\lambda$-Rota-Baxter maps on the decoration algebra. We further show that a “flatening” map—corresponding to an “arborification” procedure à la Écalle described in [M]—which “flattens” decorated trees to words, defines a locality map, which we use to relate their branched lifts to trees with their lift to words.

1. Locality operated sets and algebras

We recall the concepts of locality structures from [CGPZ1] and locality operated structures from [CGPZ2], locality operated semigroups and monoids, and locality operated algebras, successively.

1.1. Locality sets, magmas and algebras. We first recall the concept of a locality set introduced in [CGPZ1]. A locality set is a couple $(X, \top)$ where $X$ is a set and $\top \subseteq X \times X$ is a symmetric binary relation on $X$. For $x_1, x_2 \in X$, denote $x_1 \top x_2$ if $(x_1, x_2) \in \top$. We also use the alternative notations $X \times \top X$ and $X^\top$ for $\top$. In general, for any subset $U \subset X$, let $U^\top := \{ x \in X \mid (x, U) \subseteq \top \}$ denote the polar subset of $U$. For integers $k \geq 2$, we set $X^\top_k := X \times \top \cdots \top X := \{(x_1, \cdots, x_k) \in X^k \mid x_i \top x_j \text{ for all } 1 \leq i \neq j \leq k\}$.

We call two subsets $A$ and $B$ of a locality subset $(X, \top)$ independent, if $A \times B \subset \top$. This induces an independence relation on the power set $\mathcal{P}(X)$, which we denote by the same symbol $\top$. Then $(\mathcal{P}(X), \top)$ is a locality set with $\mathcal{P}(X)^\top = \mathcal{P}(X^\top)$.

We also recall the concepts of locality monoids and locality algebras. The following definition is a special instance of a “partial magma”, which is to a magma what a partial algebra is to an algebra [Gr], namely a set equipped with a partial product defined only for certain pairs with arguments in the set. See e.g. [EnM].

Remark 1.1. The condition for a locality magma is more restrictive than that of a partial magma in that the former requires that the pairs for which the partial product is defined stem from a symmetric relation.

Definition 1.2. (i) A partial magma is a locality set $(G, \top)$ together with a product law defined on $\top$:

$$m_G : G \times \top G \to G$$

For notational convenience, we usually abbreviate $m_G(x, y)$ by $x \cdot y$ or simply $xy$. 
Example 1.3. For a given subset $A \subset G$ in an arbitrary magma $(G, \star)$, the relation

$$\alpha \top_A \beta \iff \alpha \star \beta \notin A$$

defines a partial magma.

We recall the notion of locality semi-group introduced in [CGPZ1].

Definition 1.4. (i) A locality magma is a partial magma $(G, \top, m_G)$ whose product law is compatible with the locality relation on $G$ in the following sense:

$$(\star) \quad \text{for all } U \subseteq G, \quad m_G((U^\top \times U^\top) \cap \top) \subseteq U^\top.$$

(ii) A locality semigroup is a locality magma whose product law is associative in the following sense:

$$(\star) \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \text{ for all } (x, y, z) \in G \times_\top G \times_\top G.$$

Note that, because of the condition (4), both sides of Eq. (5) are well-defined for any triple in the given subset.

(iii) A locality semigroup is commutative if $m_G(x, y) = m_G(y, x)$ for $(x, y) \in \top$.

(iv) A locality monoid is a locality semigroup $(G, \top, m_G)$ together with a unit element $1_G \in G$ given by the defining property

$$\{1_G\}^\top = G \quad \text{and} \quad m_G(x, 1_G) = m_G(1_G, x) = x \quad \text{for all } x \in G.$$

We denote the locality monoid by $(G, \top, m_G, 1_G)$.

Counterexample 1.5. The set $\mathbb{Q}$ equipped with the relation

$$x \top y \iff x + y \notin \mathbb{Z}$$

is a partial semigroup for the addition $+: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, but it is neither a locality semigroup nor a locality magma. Indeed, the locality condition for semi-groups does not hold: indeed, for $U = \{1/3\}$ we have $(1/3, 1/3) \in ((U^\top \times U^\top) \cap \top)$ but $1/3 + 1/3 = 2/3 \notin U^\top$.

Here is a related example which will be useful for later purposes.

Counterexample 1.6. Let us consider a subset $A$ of $\mathbb{C}$ such that $A + Z \subset A$. We equip the power set $\mathcal{P}(\mathbb{C})$ of $\mathbb{C}$ with the following relation: for $U, V \in \mathcal{P}(\mathbb{C})$, $U \top_A V \iff U + V \subset \mathbb{C} \setminus A$. In particular $U \top_A V \iff U + V - \mathbb{Z}_{\geq 0} \subset \mathbb{C} \setminus A$. The locality set $(\mathcal{P}(\mathbb{C}), \top_A)$ equipped with the map

$$\top_A : \mathcal{P}(\mathbb{C}) \times \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$$

$$(U, V) \mapsto U + V - \mathbb{Z}_{\geq 0}$$
Definition 1.7. (i) An **locality vector space** is a vector space $V$ equipped with a locality relation $\top$ which is compatible with the linear structure on $V$ in the sense that, for any subset $X$ of $V$, $X^\top$ is a linear subspace of $V$.

(ii) Let $V$ and $W$ be vector spaces and let $\top := V \times_\top W \subseteq V \times W$. A map $f : V \times_\top W \to U$ to a vector space $U$ is called a **locality bilinear** map if

$$f(v_1 + v_2, w_1) = f(v_1, w_1) + f(v_2, w_1), \quad f(v_1, w_1 + w_2) = f(v_1, w_1) + f(v_1, w_2),$$

$$f(kv_1, w_1) = kf(v_1, w_1), \quad f(v_1, kw_1) = kf(v_1, w_1)$$

for all $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $k \in K$ such that all the pairs arising in the above expressions are in $V \times_\top W$.

(iii) A (not necessarily unitary nor associative) **locality algebra** over $K$ is a locality vector space $(A, \top)$ over $K$ together with a locality bilinear map

$$m_A : A \times_\top A \to A$$

such that $(A, \top, m_A)$ is a locality magma.

(iv) A (not necessarily unitary) associative algebra over $K$ is a locality vector space $(A, \top)$ over $K$ together with a locality bilinear map

$$m_A : A \times_\top A \to A$$

such that $(A, \top, m_A)$ is a locality semi-group.

(v) A **locality (unitary and associative) algebra** is a locality algebra $(A, \top, m_A)$ together with a unit $1_A : K \to A$ in the sense that $(A, \top, m_A, 1_A)$ is a locality monoid. We shall omit explicitly mentioning the unit $1_A$ and the product $m_A$ unless this generates an ambiguity.

Combining the locality vector space and locality magma structure, we build locality algebras and related structures.

Proposition 1.8. Let $(A, \top, m_A)$ (resp. $(A, \top, m_A), (A, \top, m_A, 1_A)$) be a locality magma (resp. semigroup, algebra). The independence relation $\top$, resp. the product $m_A$ on $A$ extends by linearity to an independence relation $\tilde{\top}$ on $KA$, resp. a bilinear form $\tilde{m}_A : \tilde{\top} \to KA$. Then $(KA, \tilde{\top}, \tilde{m}_A)$ (resp. $(KA, \tilde{\top}, \tilde{m}_A), (KA, \tilde{\top}, \tilde{m}_A, 1_A)$) is a nonunitary nonassociative locality algebra (resp. a nonunitary locality algebra, a locality algebra).

Remark 1.9. We shall often drop the symbol $\top$ over $\top$ denoting the bilinearly extended locality relation by the same symbol $\top$.

We next construct free objects in related categories. Let $W(\Omega)$ denote the set of words, called $\Omega$-**words**, including the empty word $1$, from the alphabet set $\Omega$. So

$$W(\Omega) := \{ \omega_1 \cdots \omega_k \mid \omega_i \in \Omega, 1 \leq i \leq k, k \geq 1 \} \cup \{ 1 \}.$$  

Also let $W(\Omega)^*$ denote the set of non-empty $\Omega$-words. So $W(\Omega) = W(\Omega)^* \cup \{ 1 \}$.

The following results are well-known.

Proposition 1.10. Let $\Omega$ be a set. The set $W(\Omega)^*$ (resp. $W(\Omega)$) is the free semigroup (resp. monoid) on $\Omega$.

We next extend the construction to the locality context.
Definition 1.11. Let \((\Omega, \tau)\) be a locality set. An \(\Omega\)-word \(w\) is called an \((\Omega, \tau)\)-proper word if any pair of letters in \(w\) are independent for \(\tau\). Let \(W_{\Omega, \tau}^\top\) denote the set of \((\Omega, \tau)\)-proper words. We denote the linear span of \(W_{\Omega, \tau}^\top\) by \(KW_{\Omega, \tau}^\top\). Similar notions can be defined for the set \(W(\Omega)^\top\) of non-empty \(\Omega\)-words.

The set \(W_\Omega\) is equipped with the independence relation \(\tau_{W_\Omega}\) defined for any pair of words \(w = \omega_1 \cdots \omega_n, w' = \omega'_1 \cdots \omega'_m \in W_\Omega\) by

\[
w \tau_{W_\Omega} w' \iff \{\omega_1, \ldots, \omega_n\} \tau_{\Omega} \{\omega'_1, \ldots, \omega'_m\}.
\]

This independence relation restricts to an independence relation \(\tau_{KW_\Omega}\) on \(W_{\Omega, \tau}^\top\), and extends to an independence relation \(\tau_{KW_\Omega}\) on the linear span \(KW_{\Omega, \tau}^\top\) of \(W_{\Omega, \tau}^\top\), the restriction of \(\tau_{KW_\Omega}\) to \(KW_{\Omega, \tau}^\top\) will be denoted by \(\tau_{KW_\Omega}\). Again similar notions can be defined for the set \(W(\Omega)^\top\) of non-empty \(\Omega\)-words.

Then Proposition 1.10 has the following locality variate which can be proved by the same argument.

Proposition 1.12. Let \((\Omega, \tau)\) be a locality set and let \(W_{\Omega, \tau}^\top\) be the locality set of \((\Omega, \tau)\)-proper words. With the concatenation product on \(W_{\Omega} \times W_{\Omega}^\top\) restricted to \(W_{\Omega} \times_{\tau} W_{\Omega}, W_{\Omega, \tau}^\top\) is a locality monoid. Further it is the free locality monoid on \(\Omega, \tau\), characterised by the universal property: for any locality monoid \((U, \tau_U)\) and locality map \(f : (\Omega, \tau) \to (U, \tau_U)\), there is unique morphism \(\bar{f} : (W_{\Omega, \tau}^\top, \tau_{W_\Omega}) \to (U, \tau_U)\) such that \(\bar{f}i = f \) where \(i : (\Omega, \tau) \to (W_{\Omega, \tau}^\top, \tau_{W_\Omega})\) is the natural inclusion.

Similarly, with the restriction of the concatenation product, \(W_{\Omega, \tau}^\top\) is the free locality semigroup on \((\Omega, \tau)\). Furthermore, the linear spans \(KW_{\Omega}\) and \(KW_{\Omega, \tau}^\top\) are the free locality unitary and nonunitary \(K\)-algebras on \((\Omega, \tau)\).

Corollary 1.13. Let \((\Omega_i, \tau_i), i = 1, 2\) be two locality sets. A locality map \(\phi : \Omega_1 \to \Omega_2\) uniquely lifts to a locality monoid morphism

\[
\phi_{\Omega_2}^\circ : W_{\Omega_1} \to W_{\Omega_2}
\]

and we have

\[
\phi_{\Omega_2}^\circ(\omega_1 w_1) = \phi(\omega_1) \phi_{\Omega_2}^\circ(w_1) \quad \text{for all } \omega_1 \in \Omega_1, w_1 \in W_{\Omega_1}.
\]

Proof. The corollary is a consequence of Proposition 1.16 applied to \(\Omega = \Omega_1, U = W_{\Omega_2}^\top\) and \(f = \phi_i_{\Omega_2} : \Omega_1 \to W_{\Omega_2}^\top\) where \(i_{\Omega_2} : \Omega_2 \to W_{\Omega_2}^\top\) is the natural inclusion. It follows from the universal property of the \(W_{\Omega_1}\) that for \(\omega_1 \cdots \omega_k \in W_{\Omega_1}\) with \(\omega_1, \cdots, \omega_k \in \Omega_1\), we have

\[
\phi_{\Omega_2}^\circ(\omega_1 \cdots \omega_k) = \phi(\omega_1) \cdots \phi(\omega_k).
\]

This gives the equation in the corollary.

\(\square\)

1.2. Locality operated structures.

Definition 1.14. Let \((\Omega, \tau)\) be a locality set. An \((\Omega, \tau)\)-operated locality set or simply a locality operated set is a locality set \((X, \tau_X)\) together with a partial action on a subset \(\tau_{\Omega, X} := \Omega \times_{\tau} X \subseteq \Omega \times X\)

\[
\beta : \Omega \times_{\tau} X \to X, (\omega, x) \mapsto \beta^\omega(x)
\]

satisfying the following conditions
(i) $\beta \times \text{Id}_X$ yields a map
\[ \Omega \times_\tau X \times_\tau X \xrightarrow{\beta \times \text{Id}_X} X \times_\tau X, \]
where
\[ \Omega \times_\tau X \times_\tau X := \{(\omega, u, u') \in \Omega \times X \times X \mid (u, u') \in \tau_X, (\omega, u), (\omega, u') \in \Omega \times_\tau X\}. \]

In other words,
\[ (\omega, u, u') \in \Omega \times_\tau X \times_\tau X \implies \beta^\omega(u) \tau_X u'. \]

(ii) $\text{Id}_\Omega \times \beta$ yields a map
\[ \Omega \times_\tau \Omega \times_\tau X \xrightarrow{\text{Id}_\Omega \times \beta} \Omega \times_\tau X. \]
that is, if $(\omega', \omega) \in \tau_\Omega, (\omega, u), (\omega', u) \in \Omega \times_\tau X$, then $(\omega', \beta^\omega(u)) \in \Omega \times_\tau X$.

The sets $\mathcal{W}(\Omega)^*$ and $\mathcal{W}(\Omega)$ also have the following universal properties.

**Proposition 1.15.** Let $X$ be a set. The Cartesian product $\mathcal{W}(\Omega) \times X$ is an $\Omega$-operated set with the operations given by
\[ \beta_{\mathcal{W}(\Omega) \times X} : \Omega \times \mathcal{W}(\Omega) \times X \to \mathcal{W}(\Omega) \times X, \quad (\omega, w, x) \mapsto (\omega, w, x). \]
Further, together with the map
\[ i : X \to \mathcal{W}(\Omega) \times X, \ x \mapsto (1, x), \]
$\mathcal{W}(\Omega) \times X$ is the free $\Omega$-operated set on $X$. More precisely, for any $\Omega$-operated set $(U, \beta_U)$ and set map $f : X \to U$, there is unique homomorphism $\tilde{f} : \mathcal{W}(\Omega) \times X \to U$ of $\Omega$-operated sets such that $\tilde{f} \circ i = f$.

Extending Proposition 1.15 to the locality context needs restriction on the set $X$ because of the compatibility of the locality conditions on the set and on the operator set. But this restriction is broad enough for later applications.

**Proposition 1.16.** Let $(\Omega, \tau)$ be a locality set. Denote
\[ \Omega \times_\tau \mathcal{W}_{\Omega, \tau}^* := \{ (\omega, w) \in \Omega \times \mathcal{W} \mid (\omega, \omega_1, \cdots, \omega_k) \in \Omega_\tau^{(k+1)} \text{ where } w = \omega_1 \cdots \omega_k \}. \]
The set $\mathcal{W}_{\Omega, \tau}^*$ is an $(\Omega, \tau)$-operated set with the operations given by
\[ \beta_{\mathcal{W}_{\Omega, \tau}^*} : \Omega \times_\tau \mathcal{W}_{\Omega, \tau}^* \to \mathcal{W}_{\Omega, \tau}^*, \quad (\omega, w) \mapsto \omega w. \]
Further, together with the natural inclusion
\[ i : \Omega \to \mathcal{W}_{\Omega, \tau}^*, \omega \mapsto \omega, \]
$\mathcal{W}_{\Omega, \tau}^*$ is the free $(\Omega, \tau)$-operated set on $\Omega$. More precisely, for any $(\Omega, \tau)$-operated set $(U, \beta_U)$ and any locality map $f : \Omega \to U$, there is unique homomorphism $\tilde{f} : \mathcal{W}_{\Omega, \tau}^* \to U$ of $\Omega$-operated sets such that $\tilde{f} \circ i = f$.

**Proof.** Taking $X = \Omega$ in Proposition 1.15, we first note the natural identification
\[ \mathcal{W}(\Omega) \times \Omega \cong \mathcal{W}(\Omega)^*, (w, \omega) \mapsto w\omega, (1, \omega) \mapsto \omega \text{ for all } w \in \mathcal{W}(\Omega), \omega \in \Omega, \]
which allows the identification of $\beta_{\mathcal{W}(\Omega) \times \Omega}$ in Proposition 1.15 with
\[ \beta_{\mathcal{W}(\Omega)^*} : \Omega \times \mathcal{W}(\Omega)^* \cong \Omega \times \mathcal{W}(\Omega) \times \Omega \to \mathcal{W}(\Omega) \times \Omega \cong \mathcal{W}(\Omega)^*, \]
of which the \((\Omega, \top)\)-action \(\beta^{\omega}_{\Omega, \top}\) in Proposition 1.16 is simply the restriction to \(\Omega \times \top \W_{\Omega, \top}\). Further, the map \(i\) in Eq. (8) is identified with the map \(i\) in Eq. (9). With these identifications in mind, let an \((\Omega, \top)\)-operated set \((U, \beta_U)\) and a locality map \(f : (\Omega, \top) \to (U, \top)\) be given. Then by Proposition 1.15, there is unique morphism \(\bar{f} : \W(\Omega)^* \to U\) such that \(\bar{f}i = f\). Then restricting \(\bar{f}\) to \(\W_{\Omega, \top}\) gives the existence of the desired morphism \(\bar{f}\). Its uniqueness follows from the same inductive proof on the length of the proper \(\Omega\)-words for Proposition 1.15.

**Definition 1.17.** Given a locality set \((\Omega, \top)\), we call

(i) locality \((\Omega, \top)\)-operated semigroup a quadruple \((U, \top U, \beta, m_U)\) where \((U, \top U, m_U)\)

is a locality semigroup and \((U, \top U, \beta)\) is a \((\Omega, \top)\)-operated locality set such that

\[(\omega, u, u') \in \Omega \times \top U \times \top U \implies (\omega, uu') \in \Omega \times \top U;\]

(ii) locality \((\Omega, \top)\)-operated monoid a quintuple \((U, \top U, \beta, m_U, 1_U)\) where \((U, \top U, m_U, 1_U)\)

is an locality monoid and \((U, \top U, \beta, m_U)\) is a \((\Omega, \top)\)-operated locality semigroup, and \(\Omega \times 1_U \subset \Omega \times \top U\).

(iii) \((\Omega, \top)\)-operated locality nonunitary algebra (resp. \((\Omega, \top)\)-operated locality unitary algebra) a quadruple \((U, \top U, \beta, m_U)\) (resp. quintuple \((U, \top U, \beta, m_U, 1_U)\)) which is at the same time a locality algebra (resp. unitary algebra) and a locality \((\Omega, \top)\)-operated semigroup (resp. monoid), satisfying the additional condition that for any \(\omega \in \Omega\), the set \(\omega^{\top : u, v} := \{ u \in U | \omega \top u \in v \}\) is a subspace of \(U\) on which the action of \(\omega\) is linear (resp. \(\omega \times 1_U \subset \Omega \times \top U\)). Explicitly, the linearity condition reads

\[\text{for any } u_1, u_2 \in \omega^{\top : u, v}, k_1, k_2 \in K, \text{ we have } k_1 u_1 + k_2 u_2 \in \omega^{\top : u, v} \text{ and } \beta^\omega (k_1 u_1 + k_2 u_2) = k_1 \beta^\omega (u_1) + k_2 \beta^\omega (u_2).\]

**Definition 1.18.** A morphism of locality operated locality structures (sets, semigroups, monoids, nonunitary algebras, algebras) \((\Omega_i, \top_i)\)-operated structures \((U_i, \top U_i, \beta_i), i = 1, 2\) is a couple \((\phi, f)\) with \(\phi : \Omega_1 \to \Omega_2\) and \(f : U_1 \to U_2\) such that

- \(\phi\) is a locality map and \(f\) is a morphism of locality structures;
- \(f \circ \beta^\omega_1 = \beta^\omega_2 \circ f\).

For a given locality operated set \((\Omega, \top)\), the collection of \((\Omega, \top)\)-operated locality semigroups, with the morphisms in Definition 1.18 taking \((\Omega_i, \top_i) = (\Omega, \top)\), form a category.

The following lemma will be useful in the sequel.

**Lemma 1.19.** Let \((\Omega, \top, m_\Omega, 1_\Omega)\) be a locality monoid. A locality map \(\phi : \Omega \to \Omega\) independent of \(I \Omega\) induces an \((\Omega, \top \Omega)\)-operated structure on \((\Omega, \top, \beta_\phi, m_\Omega, 1_\Omega)\), with

\[\beta^\omega_\phi : \top \Omega \to \Omega, (\omega, \omega') \mapsto \beta^\omega_\phi (\omega') := \phi (m_\Omega (\omega, \omega')).\]

Thus as a direct consequence of Proposition 1.16, we obtain

**Corollary 1.20.** Let \((\Omega, \top, \bullet)\) be a locality semigroup. For any locality operator \(P : (\Omega, \top) \to (\Omega, \top)\), there is unique locality map

\[\hat{P}^\omega : \W_{\Omega, \top} \to \Omega\]

such that \(\hat{P}^\omega (\omega) := P(\omega)\) for any \(\omega \in \Omega\), and \(\hat{P}^\omega (\omega w) := P(\omega \bullet \hat{P}^\omega (w))\) for any \(\omega w \in \W_{\Omega, \top}\). This gives rise to a map

\[\Phi^\omega : \L_\top(\Omega, \Omega) \to \L_\Omega, \Omega (\W_{\Omega, \top}, \Omega), \quad P \mapsto \hat{P}^\omega,\]
where $L_{\Omega, \Omega}(W_{\Omega, \Omega}^\ast, \Omega)$ is the set of morphisms from $W_{\Omega, \Omega}^\ast$ to $\Omega$ of locality sets.

1.3. Properly decorated forests and their universal properties. The locality $\Omega$-operated monoid and locality algebra of properly decorated rooted forests are built in [CGPZ2]. We recall its definition and its universal property in the category of locality operated monoids and algebras.

**Definition 1.21.** Let $(\Omega, \top)$ be a locality set. A $\Omega$-properly decorated (planar) rooted forest is a decorated (planar) rooted forest $(F, d_F)$, such that the vertices are decorated by mutually independent elements.

Let $\mathcal{F}_{\Omega, \top}$ (resp. $\mathcal{P}_{\Omega, \top}$) denote the set of $\Omega$-properly decorated rooted forests (resp. planar rooted forests), and $K \mathcal{F}_{\Omega, \top}$ (resp. $K \mathcal{P}_{\Omega, \top}$) be its linear span. $\mathcal{F}_{\Omega, \top}$ (resp. $\mathcal{P}_{\Omega, \top}$) inherits the independence relation $\top_{\mathcal{F}_{\Omega, \top}}$ of $\mathcal{F}_{\Omega}$ (resp. $\top_{\mathcal{P}_{\Omega, \top}}$) of $\mathcal{P}_{\Omega}$, which we denote by $\top_{\mathcal{F}_{\Omega, \top}}$ (resp. $\top_{\mathcal{P}_{\Omega, \top}}$), and $K \mathcal{F}_{\Omega, \top}$ (resp. $K \mathcal{P}_{\Omega, \top}$) inherits the independence relation $\top_{K \mathcal{F}_{\Omega, \top}}$ of $K \mathcal{F}_{\Omega}$ (resp. $\top_{K \mathcal{P}_{\Omega, \top}}$) of $K \mathcal{P}_{\Omega}$ which we also denote by $\top_{K \mathcal{F}_{\Omega, \top}}$ (resp. $\top_{K \mathcal{P}_{\Omega, \top}}$).

It is easy to see that taking disjoint union of forests in $\mathcal{F}_{\Omega}$ (resp. $\mathcal{P}_{\Omega}$) defines a locality monoid structure on $\mathcal{F}_{\Omega, \top}$ (resp. $\mathcal{P}_{\Omega, \top}$), and hence a locality algebra structure in $K \mathcal{F}_{\Omega, \top}$ (resp. $K \mathcal{P}_{\Omega, \top}$). This leads to the following straightforward yet fundamental result which we quote from [CGPZ2].

**Proposition 1.22.** [CGPZ2] Let $(\Omega, \top_{\Omega})$ be a locality set. Then

(i) $(\mathcal{F}_{\Omega, \top_{\Omega}}, \top_{\mathcal{F}_{\Omega, \top_{\Omega}}}, B_{++}, 1)$ is a locality $(\Omega, \top_{\Omega})$-operated commutative monoid;

(ii) $(K \mathcal{F}_{\Omega, \top_{\Omega}}, \top_{K \mathcal{F}_{\Omega, \top_{\Omega}}}, B_{++}, 1)$ is a locality $(\Omega, \top_{\Omega})$-operated commutative algebra;

(iii) $(\mathcal{P}_{\Omega, \top_{\Omega}}, \top_{\mathcal{P}_{\Omega, \top_{\Omega}}}, B_{++}, 1)$ is a locality $(\Omega, \top_{\Omega})$-operated monoid;

(iv) $K \mathcal{P}_{\Omega, \top_{\Omega}}, \top_{K \mathcal{P}_{\Omega, \top_{\Omega}}}, B_{++}, 1)$ is a locality $(\Omega, \top_{\Omega})$-operated algebra.

Given two locality sets $(\Omega_i, \top_{\Omega_i})$, $i = 1, 2$, let

- $L_{\top}(\Omega_1, \Omega_2)$ denote the set of locality maps $\phi : \Omega_1 \to \Omega_2$;
- $L_{\Omega_1, \Omega_2}(U_1, U_2)$ denote the set of morphisms between $(\Omega_1, \top_{\Omega_1})$-operated locality structures $(U_1, \top_{U_1}, \beta)$ of the same type.

All these sets are equipped with the independence relation of maps: $\phi, \psi : (A, \top_A) \to (B, \top_B)$

$\phi \top \psi \iff (a_1 \top_A a_2 \implies \phi(a_1) \top_B \psi(a_2))$.

**Theorem 1.23.** [CGPZ2] Let $(\Omega_1, \top_{\Omega_1})$, $(\Omega_2, \top_{\Omega_2})$ be two locality sets and let $\phi : (\Omega_1, \top_{\Omega_1}) \to (\Omega_2, \top_{\Omega_2})$ be a locality map. For any commutative locality algebra $(U, \top_U, \beta_U, m_U, 1_U)$ over $(\Omega_2, \top_{\Omega_2})$, $\phi$ uniquely lifts to a morphism of operated commutative locality algebras $\phi^\ast : \mathcal{F}_{\Omega_1, \top_{\Omega_1}} \to U$, which gives rise to a map

$\phi^\ast : (L_{\top}(\Omega_1, \Omega_2), \top) \to (L_{\Omega_1, \Omega_2}(K \mathcal{F}_{\Omega_1, \top_{\Omega_1}}, U), \top)$

$\phi \mapsto \phi^\ast$.

We call $\phi^\ast$ the lifted $\phi$-map, which by construction is characterised by the following properties

$\phi^\ast(\emptyset) = 1_U$,

$\phi^\ast(F_1, d_1) \cdots (F_n, d_n)) = \phi^\ast(F_1, d_1) \cdots \phi^\ast(F_n, d_n)$,

$\phi^\ast(B^\ast_F(F, d)) = \beta_U^{\phi^\ast}(\phi^\ast(F, d))$, $\phi^\ast$.
for any mutually independent properly \( \Omega_1 \)-decorated rooted (planar) forests \((F_1, d_1), \cdots, (F_n, d_n)\) and any \(\omega \in \Omega_1\) independent of \((F, d)\).

**Corollary 1.24.** Let \((\Omega, \top_\Omega)\) be a commutative locality monoid (resp. a unital commutative locality algebra, a locality monoid, a unital locality algebra). A map \(\phi : (\Omega, \top_\Omega) \rightarrow (\Omega, \top_\Omega)\) such that

\[
\phi \top_\Omega \text{Id}_\Omega,
\]

induces a unique morphism of locality commutative monoids (resp. locality unital commutative algebras, locality monoids, locality unital algebras)

\[
\hat{\phi} : F_{\Omega, \top_\Omega} \rightarrow (\Omega, \top_\Omega),
\]

(resp.

\[
\hat{\phi} : K F_{\Omega, \top_\Omega} \rightarrow (\Omega, \top_\Omega),
\]

\[
\hat{\phi} : P_{\Omega, \top_\Omega} \rightarrow (\Omega, \top_\Omega),
\]

\[
\hat{\phi} : K P_{\Omega, \top_\Omega} \rightarrow (\Omega, \top_\Omega),
\]

\(\hat{\phi}\) is called the **branched \(\phi\)-map**. By construction it is characterised by the following properties:

\[
\hat{\phi}(\emptyset) = 1_{\Omega}
\]

\[
\hat{\phi}((F_1, d_1) \cdots (F_n, d_n)) = \hat{\phi}(F_1, d_1) \cdots \hat{\phi}(F_n, d_n)
\]

\[
\hat{\phi}(B_{\omega}(F, d)) = \phi\left(\omega\left(\hat{\phi}(F, d)\right)\right),
\]

for any mutually independent properly decorated forests where \((F_1, d_1), \cdots, (F_n, d_n) \in F_{\Omega, \top_\Omega}\), and any \(\omega \in \Omega_1\) which is independent of \((F, d)\).

### 2. FROM ROOTED TREES TO WORDS

#### 2.1. Locality quasi-shuffle algebra.

**Proposition-Definition 2.1.** Let \((\Omega, \top_\Omega, \cdot)\) be a locality semigroup. For \(\lambda \in K\) we define the **\(\lambda\)-locality quasi-shuffle product** (for short **locality quasi-shuffle product** when \(\lambda\) is fixed) on \(K_{\Omega, \top_\Omega}\)

\[
\star_{\lambda} : K_{\Omega, \top_\Omega} \times_{\top_\Omega} K_{\Omega, \top_\Omega} \rightarrow K_{\Omega, \top_\Omega}
\]

as the linear map whose action on the basis elements is inductively defined on words by

\[
1 \star_{\lambda} w = w \star_{\lambda} 1 = w,
\]

and for \((\omega, \omega', w, w') \in W_{\Omega, \top_\Omega}^4\) with \(\omega, \omega' \in \Omega\)

\[
(\omega w) \star_{\lambda} (\omega' w') = \omega(w \star_{\lambda} (\omega' w')) + \omega'(((\omega w) \star_{\lambda} w') + \lambda(\omega \cdot \omega')(w \star_{\lambda} w'))
\]

and extended by bilinearity to \(K_{\Omega, \top_\Omega}\), this product is well-defined and associative. Thus \((K_{\Omega, \top_\Omega}, \top_{KW_{\Omega, \top_\Omega}}, \star_{\lambda}, 1)\) is an locality algebra.

**Proof.** Simply put, \(\star_{\lambda}\) is the restriction of the usual quasi-shuffle product on \(KW_{\Omega}\) to independent pairs of words. We give some details to ensure that the restriction is well-defined.

**Claim 2.2.** Let \(w\) and \(w'\) be independent words. Then \(w \star_{\lambda} w'\) in Eq. (16) is well defined. For any subset \(U \subseteq KW_{\Omega}\), if \(w\) and \(w'\) are in \(U^\top\), then \(w \star_{\lambda} w'\) is in \(U^\top\).
We prove the claim by induction on the sum $n$ of the lengths of $w$ and $w'$. If $n = 0$ we have $w = w' = 1$ and $w \ast \lambda w' = 1$ is well defined and is independent of any subset $U$.

For $n = 1$, we have $w = \omega \in \Omega$ and $w' = 1$ or $w' = \omega' \in \Omega$ and $w = 1$. In each case the product is well defined and independent of any subset that is independent of $w$ and $w'$.

Assume that we have shown that the claim has been verified for independent words whose sum of lengths is equal or less than $n \geq 1$. Let $w$, $w'$ be two independent words of sum of lengths $n + 1$. If $w = 1$ or $w' = 1$, then the claim is true by definition. Otherwise we can write $w = \omega u$ and $w' = \omega'u$. By the induction hypothesis, the term $u \ast \lambda w'$ (resp. $w \ast \lambda w'$, resp. $u \ast \lambda w'$) is well-defined and is independent of any subset $U$ which are independent of $u$ and $w'$ (resp. $w$ and $u'$, resp. $u$ and $u'$). Take $V$ to be any subset of $KW_{\Omega}$ independent of $w$ and $w'$. Then $V \cup \{\omega\}$ is independent of $u$ and $w'$. By the induction hypothesis, $V \cup \{\omega\}$ is independent of $(u \ast \lambda w')$. Then $\omega(u \ast \lambda w')$ is well defined and is independent of $V$. Likewise, the other two terms on the right hand side of Eq. (16) are well-defined and are independent of $V$. This completes the induction. \hfill \Box

Remark 2.3. Taking $\lambda = 1$ in the above definition gives a locality version of the usual stuffle product. Taking $\lambda = 0$ gives the shuffle product.

2.2. Free locality commutative Rota-Baxter algebras. We quote further concepts and results from [CGPZ2]

Definition 2.4. A linear operator $P : A \to A$ on a commutative locality algebra $(A, T)$ over a field $K$ is a locality Rota-Baxter operator of weight $\lambda \in K$ if it is a locality morphism, independent of $1_{A}$, and satisfies the following locality Rota-Baxter relation:

\[ P(a)P(b) = P(P(a)b + P(a)P(b)) + \lambda P(ab) \quad \forall (a, b) \in T. \]

We call the triple $(A, T, P)$ a locality Rota-Baxter algebra.

Let $(A, T_{A}, P_{A})$ and $(B, T_{B}, P_{B})$ be two locality Rota-Baxter algebras of weight $\lambda$. A map $f : A \to B$ is a Rota-Baxter morphism if it is a local algebra morphism such that $f \circ P_{A} = P_{B} \circ f$.

We now take $\Omega$ to be a locality monoid with unit $1_{\Omega}$. Note that $1_{\Omega}$ is not the empty word 1 of the free monoid $W_{\Omega}$ or the free locality monoid $W_{\Omega, T_{\Omega}} = (W_{\Omega}, T_{W_{\Omega}})$. To avoid confusion, we sometimes use $\sqcup$ to denote the concatenation product in $W_{\Omega}$ in contrast to the product $\cdot$ in $\Omega$. Thus for example, for $w \in W_{\Omega}$, we have $1 \sqcup w = w$ but $1 \sqcup w \neq w$. Also for $\omega \in \Omega$, we have $1_{\Omega} \cdot \omega = \omega$, but $1 \cdot \omega$ is not defined.

Let $A := A_{\Omega} := K \Omega$ be the semigroup algebra. Then $1_{A} = 1_{\Omega}$. Denote $W_{\Omega}^{*} = W_{\Omega} \setminus \{1\}$.

Let us recall two definitions (adapted here to the locality setting) of [G, page 93].

Definition 2.5. Let $(\Omega, T_{\Omega}, 1_{\Omega})$ be a locality commutative monoid and let $A := K \Omega$. Let $(A, T, m_{A}, 1_{A})$ be the resulting locality commutative algebra over $K$. called the locality monoid algebra on $(\Omega, T)$. Denote

$\mathfrak{m}(A) := K W_{\Omega}^{*}$ and $\mathfrak{m}_{T}(A) := K W_{\Omega, T}^{*}$.

Define the linear map

\[ P_{A} : \mathfrak{m}_{T}(A) \to \mathfrak{m}_{T}(A), \quad P_{A}(w) := 1_{\Omega}w = 1_{\Omega} \sqcup w, w \in W_{\Omega, T}^{*}. \]

Let $\diamond_{\lambda} : W_{\Omega, T}^{*} \times_{T_{W_{\Omega}}} W_{\Omega, T}^{*} \to \mathfrak{m}_{T}(A)$ be the map defined by

\[ (a \sqcup w) \diamond_{\lambda} (b \sqcup w') := (a \cdot b) \sqcup (w \ast \lambda w'). \]
for any proper words \( a \sqcup w, b \sqcup w' \) in \( W_{Ω,Τ} \). By the definition of \( ⊗_{Τw_n} \) in [CGPZ1], \( ⊙_λ \) extends by linearity to a locality linear map
\[
⊙_λ : III_λ(A) ⊗_{Τw_n} III_λ(A) \to III_λ(A).
\]

**Remark 2.6.** Since \( 1_A ∈ A^Τ \) we have \( 1_A ∈ (W^{*_A}_A)^{Τw_λ} \), therefore \( P_A(W^{*_A}_A,Τ) ⊆ W^{*_A}_A,Τ \) as claimed on the definition.

**Theorem 2.7.** Let \((Ω, Τ, m_Ω, 1_Ω)\) be a locality commutative monoid and let \((A = K, Ω, Τ, m_A, 1_A)\) be the resulting locality commutative monoid algebra over \( K \). The quadruple \((III_λ(A), ⊙_{λ}, P_A)\), together with the natural embedding \( i_A : A \to III_λ(A) \) is a free locality commutative Rota-Baxter algebra of weight \( λ \) over \( A \). More precisely, for any locality commutative Rota-Baxter algebra \((R, Τ_R, m_R, P)\) of weight \( λ \) over \( K \) and any locality algebra homomorphism \( f : A \to R \) there is a unique homomorphism \( \tilde{f} : (III_λ(A), ⊙_{λ}, P_A) \to (R, Τ_R, m_R, P) \) of locality Rota-Baxter algebra of weight \( λ \) such that \( f = \tilde{f} \circ i_A \).

**Remark 2.8.** This result is the locality version, and hence generalisation, of [G, Theorem 3.2.1]. The proof in the locality setup is close to the one of [G], only making sure at every step that the products are well-defined in view of the partial algebra structures.

**Proof.** In this proof, we shall write \( ⊙ \) instead of \( ⊙_λ \) to improve readability.

Throughout this proof, we will repeatedly use that for any pair of nonempty proper words \( w, w' \) such that \( w ⊔ w' \), each letter in \( w \) is independent (in \( Ω \)) of each of the letters in \( w' \). In particular, the first letter of \( w \) is independent of the first letter of \( w' \), and thus \( a · b \) is well-defined. Furthermore, by the semigroup property, \( a · b \) is independent of all the words appearing in \( w \star_λ w' \), showing that \((a · b) ⊔ (w \star_λ w')\) is a proper word. We carry out the rest of the proof in several steps.

(I) **Well-definedness of \( ⊙ \):** First, we show that \((III_λ(A), ⊙_{λ}, P_A)\) is a locality commutative Rota-Baxter algebra of weight \( λ \) over \( K \).

- Since \( 1_A ∈ A^Τ \) we have \( 1_A ∈ (W^{*_A}_A)^{Τw_λ,Τ} \). Therefore \( P_A ⊙_{λ} P_A Id_{W^{*_A}_A,Τ} \).
- \( P_A \) is a Rota-Baxter operator of weight \( λ \) by direct computation as in the proof of [G, Theorem 3.2.1]: let \( w_a, w_b ∈ W^{*_A}_A \) such that \( w_a ⊔ w_b \). We write \( w_a = a ⊔ w'_a \) and \( w_b = b ⊔ w'_b \) for some \( a, b \) in \( Ω \) and \( w'_a, w'_b \) in \( W^{*_A}_A,Τ \). Then we have
  \[
  w_a ⊙ P_A(w_b) = [a ⊔ w'_a] ⊙ [(1_A ⊔ w_a) ⊔ w_b] = a ⊔ (w'_a ⊙ λ w_b)
  \]
  \[
P_A(w_a) ⊙ w_b = [1_A ⊔ w_a] ⊙ [b ⊔ w'_b] = b ⊔ (w_a ⊙ w'_b),
  \]
  Then we have
  \[
P_A(w_a) ⊙ P_A(w_b) = (1_A ⊔ w_a) ⊙ (1_A ⊔ w_b) \quad \text{(by definition of } P_A) \]
  \[
  = 1_A ⊔ (w_a ⊙ λ w_b) \quad \text{(by definition of } ⊙_λ) \]
  \[
  = 1_A ⊔ (a ⊔ (w'_a ⊙ λ w_b) + b ⊔ (w_a ⊙ λ w'_b)) \quad \text{(by definition of } ⊙_λ) \]
  \[
  + λ(a · b) ⊔ (w'_a ⊙ λ w'_b) \quad \text{(by definition of } ◦_λ) \]
  \[
  = P_A(w_a) + P_A(w_b) + λP_A(w_a) + λP_A(w_b).
  \]

(II) **Construction of \( \tilde{f} \):** To prove the universal property, let us start by explicitly constructing the map \( \tilde{f} \). We let \((B, Τ_B, m_B, P)\) be a unital commutative locality Rota-Baxter algebra of weight \( λ \) and \( f : A → B \) a locality algebra homomorphism. We
inductively define $\bar{f} : W_{\Omega,T}^* \to B$ from:

$$\bar{f}(a) := f(a)$$

for any proper words $a$ of length 1 (thus for any $a$ in $\Omega$). Then $\bar{f} = f \circ i_A$ by definition.

Now, assume $\bar{f}$ has been defined on words of length between 1 and $n \geq 1$. The map $\bar{f}$ restricts to a locality map on nonempty proper words of length up to $n$ so that for any nonempty proper words $w, w'$ of length no larger than $n$, we have $w \top_{W_{\Omega}} w'$ implying $\bar{f}(w) \top_{W_{\Omega}} \bar{f}(w')$. For any word $a \sqcup w$ of length $n + 1$ we define

$$\bar{f}(a \sqcup w) := \bar{f}(a) P(\bar{f}(w)) = f(a) P(\bar{f}(w)).$$

This product in $B$ is well-defined combining the induction hypothesis with the fact that $f$ and $P$ and the composition of local maps are all local maps. It follows from the induction assumption combined with the semigroup property, that for any pair of nonempty proper words $w, w'$ of length no larger than $n + 1$, $w \top_{W_{\Omega}} w'$ implies $\bar{f}(w) \top_{W_{\Omega}} \bar{f}(w')$ since both $\bar{f}$ and $P$ are local.

Thus we have defined a locality map $\bar{f} : W_{\Omega,T}^* \to B$ and hence $\bar{f} : \mathfrak{M}_T(A) \to B$ by linearity extension.

(III) Compatibility of $\bar{f}$ with Rota-Baxter operators: For any nonempty proper word $w$ a direct computation gives

$$\bar{f}(P_A(w)) = \bar{f}(1_A \sqcup w) = f(1_A) P(\bar{f}(w)) = 1_B P(\bar{f}(w)) = P(\bar{f}(w)).$$

Thus $\bar{f} \circ P_A = P \circ \bar{f}$.

(IV) Multiplicativity of $\bar{f}$: We now prove that $\bar{f}$ is a locality algebra homomorphism, namely that $\bar{f}(w \odot w') = f(w) \odot \bar{f}(w')$ by induction on $n + m \geq 2$, the sum of the lengths of the nonempty words $w$ and $w'$.

- If $n + m = 2$, then $n = m = 1$ and we write $w = a, w' = b$. Then

  $$\bar{f}(a \odot b) = \bar{f}(a \cdot b) = f(a \cdot b) = f(a) f(b) = \bar{f}(a) \bar{f}(b),$$

  where we have used that $f$ is a locality algebra homomorphism.

- Assume now that for any pair of independent nonempty proper words $w, w'$ whose sum of lengths is no larger than $k$, we have $\bar{f}(w \odot w') = f(w) \bar{f}(w')$. Let $w_a$ and $w_b$ be any two independent nonempty proper words whose sum of lengths is equal to $k + 1$. Since $P_A$ also acts on empty words, we can write $w_a = (a) \sqcup w$ and $w_b = (b) \sqcup w'$. We use the fact that

  $$w_a \odot w_b = (a \cdot b) \sqcup [w \star_\lambda w'] = (a \cdot b) \sqcup [1_A \sqcup [w \star_\lambda w']] = (a \cdot b) \odot P_A(w) \odot P_A(w')$$

  where we have used the associativity of $\odot$. Then

  $$\bar{f}(w_a \odot w_b) = \bar{f}((a \cdot b) \odot P_A(w) \odot P_A(w'))$$

  $$= \bar{f}((a \cdot b) \odot P_A(w) \odot P_A(w') + P_A(w) \odot w' + \lambda w \odot w')).$$

We now use the fact $P_A$ is a locality Rota-Baxter operator of weight $\lambda$. Noticing that the image under $P_A$ of a word of length $p$ is a word of length $p + 1$, and that if $w$ is of length $p$ and $w'$ is of length $q$, then $w \odot q$ is a word of length $p + q - 1$. Thus we can use the induction hypothesis to write

$$\bar{f}(w_a \odot w_b) = f(a \cdot b) (\bar{f} \circ P_A)(w \odot P_A(w') + P_A(w) \odot w' + \lambda w \odot w')$$

$$= f(a \cdot b) (P \circ \bar{f})(w \odot (1_A \sqcup w') + ((1_A) \sqcup w) \odot w' + \lambda w \odot w').$$
Using once again the induction hypothesis we deduce that

\[ \bar{f}(w_a \circ w_b) = f(a)f(b)P(\bar{f}(w)\bar{f}((1_A) \cup w') + \bar{f}((1_A) \cup w)\bar{f}(w') + \lambda \bar{f}(w)\bar{f}(w')) \]

\[ = f(a)f(b)P(\bar{f}(w)f(1_A)P(\bar{f}(w')) + f(1_A)P(\bar{f}(w))\bar{f}(w') + \lambda \bar{f}(w)\bar{f}(w')) \]

\[ = f(a)f(b)P(\bar{f}(w)P(\bar{f}(w')) + P(\bar{f}(w))\bar{f}(w') + \lambda \bar{f}(w)\bar{f}(w')) \]

\[ = f(a)P(\bar{f}(w))f(b)P(\bar{f}(w')) \]

\[ = \bar{f}(w_a)\bar{f}(w_b). \]

This ends our induction.

(V) Uniqueness of \( \bar{f} \): Finally we prove the uniqueness of \( \bar{f} \) by induction. Assume we have two different such maps \( \bar{f}_1 \) and \( \bar{f}_2 \). Then for any \( a \in A \), \( \bar{f}_1(a) = f(a) = \bar{f}_2(a) \). Thus \( \bar{f}_1 \) and \( \bar{f}_2 \) coincide on proper words of length 1. Assume that \( \bar{f}_1 \) and \( \bar{f}_2 \) coincide on proper words of length \( n \geq 1 \). Let \( a \cup w \) be a nonempty proper word of length \( n + 1 \). Since \( a \cup w = a \circ P_A(w) \) we get

\[ \bar{f}_1(a \cup w) = \bar{f}_1(a \circ P_A(w)) \]

\[ = \bar{f}_1(a)\bar{f}_1(P_A(w)) \]

\[ = \bar{f}_2(a)\bar{f}_2(P_A(w)) \]

\[ = \bar{f}_2(a \cup w). \]

Thus \( \bar{f}_1 \) and \( \bar{f}_2 \) coincide on \( W^*_{\Omega, \tau} \). \( \square \)

2.3. The universal property and quasi-shuffle algebras. Now let us assume that \( (\Omega, \tau_\Omega, \bullet) \) is a commutative (non necessarily unital) locality algebra. Then the map \( \hat{P}_w \) of Corollary 1.20 can be extended linearly as a linear map

\[ \hat{P}_w : K W^*_{\Omega, \tau_\Omega} \rightarrow \Omega. \]

**Theorem 2.9.** Let \( (G, \tau_G, \bullet) \) be a commutative locality monoid. Let \( A := KG \) and let \( (A, \tau_A, 1_A) \) be the corresponding commutative unital locality algebra. Let \( P : A \rightarrow A \) be a locality linear map. The following statements are equivalent:

(i) \( P \) is a locality Rota-Baxter operator on \( A \) of weight \( \lambda \).

(ii) \( \hat{P}_w : (KW^*_A, \tau_A, \bullet, 1_A) \rightarrow (A, \tau_A, \bullet) \) is a morphism of unital algebras. Namely, for any mutually independent words \( w \) and \( w' \) we have

\[ \hat{P}_w(w \ast_\lambda w') = \hat{P}_w(w) \bullet \hat{P}_w(w'). \]

(19)
Theorem 2.9 can be expressed as the commutative diagram

\[
\begin{array}{ccc}
\text{A} & \xrightarrow{\Pi} & \text{A} \\
\text{Id} \downarrow & & \downarrow \text{Id} \\
\text{A} & \xrightarrow{\Pi} & \text{A} \\
\end{array}
\]

\[
(\mathcal{K}W^*_A, \circ_\lambda) \xrightarrow{P_A} (\mathcal{K}W^*_A, \star_\lambda)
\]

**Proof.** We carry out the proof in the usual setup dropping the locality, since the proof in the general case is similar, the various locality assumptions ensuring that at every step the products are well-defined.

\((\iff):\) Assume that Eq. (19) holds. Applying it to two words of length 1 gives the Rota-Baxter relation.

\((\implies):\) Now if \(P_\lambda : A \to A\) is a locality Rota-Baxter operator of weight \(\lambda\), then \((A, T, P_\lambda)\) is a locality Rota-Baxter algebra of weight \(\lambda\). By Theorem 2.7 there is a unique homomorphism

\[
\overline{\text{Id}} : (\mathcal{K}W^*_A, \circ_\lambda, P_A) \longrightarrow (A, \bullet, P_\lambda),
\]

of Rota-Baxter algebras with \(P_A\) as in Eq. (18), such that \(\overline{\text{Id}} i = \text{Id}\) for the inclusion \(i : A \to \mathcal{K}W^*_A, T\). and the algebra morphism \(\text{Id} : A \to A\). Thus the composition \(\overline{\text{Id}} P_A : \mathcal{K}W^*_A, T \to A\) is a locality algebra homomorphism.

Let us check that \(\overline{\text{Id}} P_A = \overline{\mathcal{P}}_\lambda\). First for \(\omega \in A\), we have

\[
\overline{\text{Id}} P_A(\omega) = P(\overline{\text{Id}}(i(\omega))) = P(\omega).
\]

Next for \(\omega w\) with \(\omega, w \in W^*_A, T\), we have \(\omega w = \omega \circ_\lambda P_A(w)\). Since \(\overline{\text{Id}}\) defined in Eq. (20) is a homomorphism of locality Rota-Baxter algebras, we have

\[
(\overline{\text{Id}} P_A)(\omega w) = P(\overline{\text{Id}}(\omega \circ_\lambda P_A(w))) = P(\overline{\text{Id}}(\omega) \bullet (\overline{\text{Id}} P_A(w))) = P(\omega \bullet (\overline{\text{Id}} P_A)(w)).
\]

Thus \(\overline{\text{Id}} P_A\) satisfies the same defining properties as \(\overline{\mathcal{P}}_\lambda\), yielding \(\overline{\text{Id}} P_A(\omega) = \overline{\mathcal{P}}_\lambda\).

Since

\[
P_A(\omega \star_\lambda w') = 1_A (1_A \star_\lambda w') = (1_A \star_\lambda 1_A) = P_A(\omega) \circ_\lambda P_A(w'),
\]

Eq. (19) can be verified as follows:

\[
\overline{\mathcal{P}}_\lambda(\omega \star_\lambda w') = (\overline{\text{Id}} P_A(\omega) \bullet (\overline{\text{Id}} P_A(w'))) = \overline{\mathcal{P}}_\lambda\overline{\mathcal{P}}_\lambda.
\]

Adapting the proof to the locality setup, shows that Eq. (19) holds for any independent \(w, w' \in W^*_A, T\).

**2.4. Factorisation through words.** Let \((\mathcal{K}W_{\Omega, T, \Omega}, \star_\Omega, C_+, \star_\lambda, 1)\) be the locality algebra with quasi-shuffle product of weight \(\lambda\), introduced in Proposition-Definition 2.1. For \(\omega \in \Omega\), the map

\[
\beta^\omega : \mathcal{K}W_{\Omega, T, \Omega} \to \mathcal{K}W_{\Omega, T, \Omega}
\]

defined by \(w \mapsto \omega w = \omega \cup w\) for all \(w \in W_{\Omega, T, \Omega}\), and linearly extended to \(\mathcal{K}W_{\Omega, T, \Omega}\) defines an \((\Omega, \star_\Omega, C_+, \star_\lambda, 1)\)-action on \(\mathcal{K}W_{\Omega, T, \Omega}\). Thus applying Theorem 1.123, we define
**Definition 2.10.** The \( \lambda \)-flatening operator

\[ f_\lambda = \text{Id}^\sharp : (\mathcal{F}_\Omega, \top, \beta_B, 1) \rightarrow (KW_\Omega, \top, \beta_C, 1) \]

is the unique morphism of \((\Omega, \top)\)-operated commutative locality algebras defined as in Theorem 1.23. In other words, it is characterised by the following properties

\[ f_\lambda(1) = 1, \]
\[ f_\lambda(B_\omega (F, d)) = \omega \sqcup f_\lambda (F, d), \]
\[ f_\lambda((F_1, d_1) \cdot (F_2, d_2)) = f_\lambda(F_1, d_1) \ast_\lambda f_\lambda(F_2, d_2). \]

We state a simple, yet important, result concerning the flatening maps.

**Lemma 2.11.** Let \((\Omega, \top, .)\) be a locality semigroup. Then \(f_\lambda\) is a locality map and maps properly decorated forests to linear combinations of properly decorated words.

**Proof.** The proof is an easy induction on the number of vertices of the forests. The statement clearly holds for the empty forest. Assuming it holds for properly decorated forests with \(n\) vertices, let \((F, d)\) be a properly decorated forest with \(n + 1\) vertices.

If \((F, d) = (F_1, d_1)(F_2, d_2)\) with \(F\) nonempty, we have that \(f_\lambda(F_1, d_1) \top W f_\lambda(F_2, d_2)\) by the induction hypothesis and the result follows since \((W_\Omega, \top, \cdot\) is a locality semigroup.

If \((F, d) = B_\omega (F_1, d_1)\) then the result follows from the induction hypothesis by Eq. (22) and the definition of \(\top W\).

\( \square \)

**Remark 2.12.** In what follows, \((G, \top, .)\) is a commutative locality monoid and we set as before \(A := KG\), which becomes a unital commutative locality algebra \((A, \top, \cdot, 1_A)\). We extend \(\widehat{P}^w\) to \(W_{A, \top}\) by setting \(\widehat{P}^w(1) := 1_A\).

The subsequent theorem states that an operated algebra homomorphism from the free (commutative) operated algebra to a Rota-Baxter algebra factors through the free (commutative) Rota-Baxter algebra.

**Theorem 2.13.** Given \(\lambda \in K\) and a commutative locality algebra \((A, \top, \cdot)\), let \(P : A \rightarrow A\) be a linear locality map, \(f_\lambda : \mathcal{F}_A, \top \rightarrow W_{A, \top}\) be the flatening locality morphism of \((A, \top)\)-operated commutative locality algebras, \(i_w : (A, \top) \rightarrow W_{A, \top}\) be the natural morphism of locality sets and let \(\widehat{P} : \mathcal{F}_A \rightarrow A\) be the locality morphism of \((A, \top)\)-operated locality algebras built from \(P\). The following statements are equivalent

\[ (i) \text{ The map } \widehat{P}^w : (W_{A, \top}, \top, \ast_\lambda) \rightarrow (A, \top, \cdot) \text{ is a morphism of commutative } (A, \top)\)-operated locality algebras; \]
\[ (ii) \text{ } \widehat{P} \text{ is a locality } \lambda\text{-Rota-Baxter operator,} \]
\[ (iii) \text{ } \widehat{P} \text{ factorises through words, that is } \widehat{P} = \widehat{P}^w \circ f_\lambda. \]

In this case, the following diagram of locality maps between locality sets, whose maps in the r.h.s. triangle are locality morphisms of \((A, \top)\)-operated locality algebras, commutes.
Proof. Before proving the equivalence of the assertions, let us briefly comment on the commutativity of the diagramme. All subdiagrammes outside the r.h.s one commute by construction of the various maps. The commutativity of the r.h.s follows from (iii).

Assertions (i) and (ii) are equivalent by Theorem 2.9. We prove the equivalence of (ii) and (iii).

\((iii) \implies (ii)\): Let us assume that \(\hat{P}\) factorises through words. Then, for any pair \((\omega, \omega') \in \mathbb{T}_A\) we have on the one hand

\[
\hat{P}(\bullet \omega \bullet \omega') = P(\omega)P(\omega').
\]

On the other hand we have

\[
\hat{P}(\bullet \omega \bullet \omega') = \hat{P}^w(\omega \star \lambda \omega') \quad \text{by definition of } f_\lambda
\]

\[
= P(\omega P(\omega')) + P(\omega' P(\omega)) + \lambda P(\omega \omega')
\]

by definition of \(\star \lambda\) and \(\hat{P}^w\), and by linearity of \(P\).

\((ii) \implies (iii)\): We prove this implication by induction on the number \(n\) of vertices of forests. If \(n = 1\) we directly have \(\hat{P}(\bullet \omega) = P(\omega) = \hat{P}^w(\omega)\). Assuming that the result holds for all properly decorated forests of at most \(n\) vertices, let \((F,d)\) be a properly decorated forest of \(n + 1\) vertices. If \((F,d) = B^*_\omega((F_1, d_1))\) we have

\[
\hat{P}((F,d)) = P(\omega \cdot \hat{P}((F_1, d_1)))
\]

\[
= P(\omega \cdot \hat{P}^w(f_\lambda((F_1, d_1)))) \quad \text{(by the induction hypothesis)}
\]

\[
= \hat{P}^w(C^w_\omega(f_\lambda((F_1, d_1)))) \quad \text{(by definition of } \hat{P}^w\)
\]

\[
= \hat{P}^w(f_\lambda((F,d)))
\]

by definition of \(f_\lambda\). If \((F,d) = (F_1, d_1)(F_2, d_2)\) with \((F_1, d_1)\) and \((F_2, d_2)\) non-empty we have

\[
\hat{P}((F,d)) = \hat{P}((F_1, d_1)) \cdot \hat{P}((F_2, d_2))
\]

\[
= \hat{P}^w(f_\lambda((F_1, d_1))) \cdot \hat{P}^w(f_\lambda((F_2, d_2))) \quad \text{(by the induction hypothesis)}
\]

\[
= \hat{P}^w(f_\lambda((F_1, d_1)) \star \lambda f_\lambda((F_2, d_2))) \quad \text{(by Theorem 2.9)}
\]

by definition of \(f_\lambda\). Notice that in both cases, every product is well-defined as we are dealing with locality maps and by Lemma 2.11.

\(\square\)
Let as before and with the above notations $i_W$ be the canonical locality embedding $\mathcal{W}_{A,\top} \hookrightarrow \mathcal{F}_{A,\top}$ of words as ladder trees. The following identity follows from the above theorem and the fact that $f_\lambda \circ i_W = \text{Id}_W$:

$$\hat{P} \circ i_W = \hat{P}^W.$$

Let us recall a result from [CGPZ1].

**Proposition 2.14.** [CGPZ1, Proposition 3.22] Let $(A, \top, m_A)$ be a locality algebra. Let $P : A \rightarrow A$ be a locality linear idempotent operator in which case there is a linear decomposition $A = A_1 \oplus A_2$ with $A_1 = \text{Ker}(\text{Id} - P)$ and $A_2 = \text{Ker}(P)$ where $P$ is the projection onto $A_1$ along $A_2$. The following statements are equivalent:

(i) $P$ is a locality Rota-Baxter operator;

(ii) $A_1$ and $A_2$ are locality subalgebras of $A$ and $A_1 \top A_2$.

Furthermore, $P$ is a locality multiplicative map if and only if, in addition to Items (i) and (ii), $A_2$ is a locality ideal of $A$.

**Corollary 2.15.** Let $(A, \top_A)$ be a locality algebra and $P : A \rightarrow A$ be a locality linear idempotent linear map. The following statements are equivalent.

(i) $A_1 := \text{Ker}(\text{Id} - P)$ and $A_2 := \text{Ker}(P)$ are locality subalgebras of $A$ and $A_1 \top A_2$.

(ii) The branched operator $\hat{P} : \mathcal{F}_{A,\top_A} \rightarrow A$ factorises through words.

**Proof.** By Proposition 2.14, the first item is equivalent to $P$ being a Rota-Baxter operator which in turn is equivalent to the second item by Theorem 2.13. □

**Example 2.16.** Recall [CGPZ1] that the $\mathcal{M}(\mathbb{C}^\infty)$ of meromorphic germs at zero with linear poles equipped with the relation $\perp^Q$ induced by an inner product $Q$ is a locality monoid. In [CGPZ1], we showed that the inner product $Q$ gives rise to a locality projection map $\pi^Q : \mathcal{M}(\mathbb{C}^\infty) \rightarrow \mathcal{M}^Q(\mathbb{C}^\infty)$ along the space $\mathcal{M}_+$ of holomorphic germs at zero onto the space $\mathcal{M}^Q(\mathbb{C}^\infty)$ of polar germs at zero, which defines a locality Rota-Baxter operator. It follows from the above corollary, that the branched projection map $\hat{\pi}^{Q,\top} : \mathcal{F}_{\mathcal{M}(\mathbb{C}^\infty)} \rightarrow \mathcal{M}^Q(\mathbb{C}^\infty)$ defined on forests decorated by meromorphic germs, factors through a locality morphism $\hat{\pi}^{Q,W} : \mathcal{W}_{\mathcal{M}(\mathbb{C}^\infty)} \rightarrow \mathcal{M}^Q(\mathbb{C}^\infty)$ on words decorated by meromorphic germs.
Part 2. Analytic aspects

In [GPZ3] we studied $k$-variate meromorphic germs $\sigma(z_1, \cdots, z_k)$ of functions at zero which form an algebra generated by compositions $f \circ \ell$ of singlevariate meromorphic germs $f(z)$ at zero composed with non-zero multivariate linear forms $\ell(z_1, \cdots, z_k)$ on $\mathbb{C}^k$. In the present paper, we consider the algebra of symbol-valued $k$-variate meromorphic germs at zero, generated by compositions $\sigma \circ \ell$ of symbol-valued singlevariate meromorphic germs $\sigma(z)$ at zero (the symbols are polyhomogeneous on some cone $\Lambda$, here $\mathbb{R}_{>0}$) composed with non-zero multivariate linear forms $\ell(z_1, \cdots, z_k)$ on $\mathbb{C}^k$. The latter relate to the former by evaluating the symbol at some point; evaluating a singlevariate meromorphic germ of symbols $\sigma(z)$ at some point $x$ gives rise to a singlevariate meromorphic germ of functions $z \mapsto \delta_x \circ \sigma(z)$ and a $k$-variate meromorphic germ of symbols $\sigma(z_1, \cdots, z_k)$ gives rise to a $k$-variate meromorphic germ of functions $(z_1, \cdots, z_k) \mapsto \delta_x \circ \sigma(z_1, \cdots, z_k)$. Letting $x$ tend to $+\infty$, we can build the finite part at infinity, which takes $\sigma(z)$ to a singlevariate meromorphic germ of functions $z \mapsto +\infty \circ \sigma(z)$, a $k$-variate meromorphic germ of symbols $\sigma(z_1, \cdots, z_k)$ to a $k$-variate meromorphic germ of functions $(z_1, \cdots, z_k) \mapsto +\infty \circ \sigma(z_1, \cdots, z_k)$. In the context of renormalisation, one can view the composition with non-zero multivariate linear forms as a blow-up to resolve singularities.

The locality structure $\perp^Q$ on the class of multivariate meromorphic germs of functions studied in [GPZ3] induces one on the class of multivariate meromorphic germs of (polyhomogeneous) symbols defined on $\Lambda$ via the evaluation map:

$$\sigma \perp^Q \sigma' \iff (\delta_x \circ \sigma) \perp^Q (\delta_x \circ \sigma') \quad \forall x \in \Lambda.$$ 

We show that the evaluation map $fp$ at infinity is a locality character and that the integration map and an interpolated summation yield locality Rota-Baxter operators on this locality algebra.

3. Locality Rota-Baxter operators on symbols with constrained order

3.1. A locality structure on a class of symbols with constrained order.

**Definition 3.1.** A smooth function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is called a symbol (with constant coefficients) on $\mathbb{R}_{\geq 0}$ if there exists a real number $r$ such that the condition $(C_r)$ below is satisfied.

$$(C_r) \quad \forall k \in \mathbb{Z}_{\geq 0}, \ \exists D_k \in \mathbb{R}_{>0} : \forall x \in \mathbb{R}_{\geq 0}, \ |\partial_x^k \sigma(x)| \leq D_k(x)^r-k$$

with $\langle x \rangle := \sqrt{x^2 + 1}$. The set of symbols on $\mathbb{R}_{\geq 0}$ satisfying the condition $C_r$ is denoted by $S^r(\mathbb{R}_{\geq 0})$, which is a real vector space.

Notice that $(r \leq r') \implies (C_r \implies C_{r'})$. So $(r \leq r') \implies (S^r(\mathbb{R}_{\geq 0}) \subset S^{r'}(\mathbb{R}_{\geq 0}))$.

An element of

$$(24) \quad S^{-\infty}(\mathbb{R}_{\geq 0}) := \bigcap_{r \in \mathbb{R}} S^r(\mathbb{R}_{\geq 0})$$

is called smoothing.

**Remark 3.2.** Note that $S^{-\infty}(\mathbb{R}_{\geq 0})$ corresponds to the algebra of Schwartz functions on $\mathbb{R}_{\geq 0}$. Thus a symbol is smoothing if and only if it is a Schwartz function.
By an **excision function** around zero, we mean a smooth function \( \chi : \mathbb{R}_{\geq 0} \to \mathbb{R} \) such that \( \chi \) is identically zero in a neighborhood of zero and identically equal to one outside some interval containing \([0,1]\).\(^1\) The excision function is there to avoid divergences at zero, the so-called infrared divergences in physics.

**Example 3.3.** For any complex number \( \alpha \), \( (x)^{\alpha} \) is a symbol in \( S^{R(\alpha)}(\mathbb{R}_{\geq 0}) \). For any excision function \( \chi \) around zero, and any complex number \( \alpha \), \( \chi(x)x^{\alpha} \) is a symbol in \( S^{R(\alpha)}(\mathbb{R}_{\geq 0}) \). The function \( x \mapsto \log(x) \) is an element of \( S^{r}(\mathbb{R}_{\geq 0}) \) for any \( r > 0 \), but it is not an element of \( S^{0}(\mathbb{R}_{\geq 0}) \).

**Proposition 3.4.** Let \( \sigma : \mathbb{R}_{\geq 0} \to \mathbb{C} \) be a symbol. There is at most one pair \( (\alpha, \{a_j\}) \) with \( \alpha \in \mathbb{C} \) and \( a_j \in \mathbb{C}, j \in \mathbb{Z}_{\geq 0}, a_0 \neq 0 \), such that

(i) \( \sigma \in S^{R(\alpha)}(\mathbb{R}_{\geq 0}) \), and

(ii) there is an excision function \( \chi \) around zero, such that for any \( N \in \mathbb{Z}_{\geq 1} \), the map

\[
(25) \quad x \mapsto \sigma^x_{(N)}(x) := \sigma(x) - \sum_{j=0}^{N-1} \chi(x) a_j x^{\alpha-j}
\]

lies in \( S^{R(\alpha)-N}(\mathbb{R}_{\geq 0}) \).

**Proof.** If there are pairs \( (\alpha, \{a_j\}) \) and \( (\beta, \{b_k\}) \) with the given conditions, then \( \sigma(x) \langle x \rangle^{-\alpha} \) converges to the nonzero constant \( a_0 \) and \( \sigma(x) \langle x \rangle^{-\beta} \) converges to the nonzero constant \( b_0 \). This forces \( \alpha = \beta \) and \( a_0 = b_0 \). Then \( a_j = b_j, j \geq 1 \), follows inductively on \( j \geq 1 \) from the fact that

\[
\sigma^x_{(N)}(x) \langle x \rangle^{N-\alpha} \xrightarrow{x \to \infty} a_N
\]

for any \( N \in \mathbb{Z}_{\geq 0} \).

We further notice that the coefficients \( a_j, j \in \mathbb{Z}_{\geq 0} \) are independent of the particular choice of the excision function \( \chi \). Indeed, given another excision function \( \chi' \), the difference \( \sigma^x_{(N)} - \sigma^x_{(N)} \) is a Schwartz function. \( \square \)

**Definition 3.5.** For a symbol \( \sigma : \mathbb{R}_{\geq 0} \to \mathbb{C} \), if the pair in Proposition 3.4 exists, hence is unique, the symbol is called a **polyhomogeneous** (also classical) symbol of **order \( \alpha \)** with **asymptotic expansion** \( \sum_{j=0}^{\infty} a_j x^{\alpha-j} \). We write

\[
(26) \quad \sigma(x) \sim \sum_{j=0}^{\infty} a_j x^{\alpha-j}.
\]

The set of polyhomogeneous symbols on \( \mathbb{R}_{\geq 0} \) of order \( \alpha \) will be denoted by \( S^\alpha_{\text{ph}}(\mathbb{R}_{\geq 0}) \) and its linear span by \( E^\alpha_{\text{ph}}(\mathbb{R}_{\geq 0}) \).

By definition, \( S^\alpha_{\text{ph}}(\mathbb{R}_{\geq 0}) \) and hence \( E^\alpha_{\text{ph}}(\mathbb{R}_{\geq 0}) \) are contained in \( S^{R(\alpha)}(\mathbb{R}_{\geq 0}) \) and for any \( \alpha \in \mathbb{C} \):

\[
k \in \mathbb{Z}_{\geq 0} \implies S^{\alpha-k}_{\text{ph}}(\mathbb{R}_{\geq 0}) \subset S^\alpha_{\text{ph}}(\mathbb{R}_{\geq 0}).
\]

**Example 3.6.** For any nonnegative integer \( k \) the set \( \mathcal{P}^k(\mathbb{R}_{\geq 0}) \) of real polynomial functions of degree \( k \) restricted to \( \mathbb{R}_{\geq 0} \) is a subset of \( S^k_{\text{ph}}(\mathbb{R}_{\geq 0}) \).

\(^1\) Without loss of generality, we can take an excision function to be identically one outside the unit interval, which we shall do unless otherwise specified.
Since $S^\alpha_{ph}(\mathbb{R}_{\geq 0}) \cdot S^\beta_{ph}(\mathbb{R}_{\geq 0}) \subset S^{\alpha+\beta}_{ph}(\mathbb{R}_{\geq 0})$, the union $\cup_{\alpha \in \mathbb{C}} S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ forms a monoid. Let $S_{ph}(\mathbb{R}_{\geq 0}) := \sum_{\alpha \in \mathbb{C}} S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ be the linear span of classical symbols of all orders, then $S_{ph}(\mathbb{R}_{\geq 0})$ is an algebra.

**Remark 3.7.**

(i) The subspace $S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ generated by polyhomogeneous symbols of **integer order** is a subalgebra of $S_{ph}(\mathbb{R}_{\geq 0})$.

(ii) The algebra $P(\mathbb{R}_{\geq 0}) := \cup_{k=0}^{\infty} P^k(\mathbb{R}_{\geq 0})$ of real polynomial functions restricted to $\mathbb{R}_{\geq 0}$ is a subalgebra of $S_{ph}(\mathbb{R}_{\geq 0})$.

(iii) The space $S^{-\infty}(\mathbb{R}_{\geq 0})$ forms a subalgebra of $S_{ph}(\mathbb{R}_{\geq 0})$ since it lies in $S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ for any $\alpha \in \mathbb{C}$. We have $S^{-\infty}(\mathbb{R}_{\geq 0}) = \cap_{\alpha \in \mathbb{C}} S^\alpha_{ph}(\mathbb{R}_{\geq 0}) = \cap_{k \in \mathbb{Z}} S^k_{ph}(\mathbb{R}_{\geq 0})$.

We now define classes of polyhomogeneous symbols on $\mathbb{R}_{\geq 0}$ with **constrained order**. Given a subset $A \subset \mathbb{C}$, we consider the linear span $S^A_{ph}(\mathbb{R}_{\geq 0}) := \sum_{\alpha \in A} S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ of polyhomogeneous symbols of order in $A$, and we denote by $S^{\infty A}_{ph}(\mathbb{R}_{\geq 0}) := \sum_{\alpha \in \mathbb{C} \setminus A} S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ the linear span of polyhomogeneous symbols of order not in $A$.

For a subset $A \subset \mathbb{C}$ with $A + \mathbb{Z} = A$, we have a direct sum decomposition

$$S^{\infty A}_{ph}(\mathbb{R}_{\geq 0}) \oplus S^A_{ph}(\mathbb{R}_{\geq 0}) = S_{ph}(\mathbb{R}_{\geq 0}).$$

When specializing to $A = \mathbb{Z}$, let

$$\Sigma(\mathbb{R}_{\geq 0}) := S^\infty_{ph}(\mathbb{R}_{\geq 0}) + P(\mathbb{R}_{\geq 0}); \quad \Sigma(\mathbb{Z}_{\geq 0}) := \{\sigma|_{\mathbb{Z}_{\geq 0}} \mid \sigma \in \Sigma(\mathbb{R}_{\geq 0})\}.$$

**Definition 3.8.** For any $A \subset \mathbb{C}$, consider a relation $\top_A$ on $S_{ph}(\mathbb{R}_{\geq 0})$ by

$$\sigma \top_A \tau \iff \sigma \cdot \tau \in S^{\infty A}_{ph}(\mathbb{R}_{\geq 0}).$$

More precisely, let $\sigma = \sum \sigma_i$, $\tau = \sum \tau_j$ with $\sigma_i \in S^{\alpha_i}_{ph}(\mathbb{R}_{\geq 0})$, $\tau_j \in S^{\beta_j}_{ph}(\mathbb{R}_{\geq 0})$. Then $\sigma \top_A \tau$ means $\alpha_i + \beta_j - \mathbb{Z}_{\geq 0} \cap A = \emptyset$.

3.2. **The finite part at infinity.** A symbol in $S_{ph}(\mathbb{R}_{\geq 0})$ lies in $L^1(\mathbb{R}_{\geq 0})$ if it is a linear combination of polyhomogeneous symbols of orders with negative real parts, in which case we have $\lim_{x \to +\infty} \sigma = 0$ as a consequence of $(C_\sigma)$. So, for a symbol $\sigma$ in $S_{ph}(\mathbb{R}_{\geq 0})$ with polyhomogeneous asymptotic expansion given by Eq. (26), with the notations of Eq. (25) we have

$$N > \Re(\alpha) \implies \lim_{x \to +\infty} \sigma^N_{\alpha}(\cdot) = 0.$$

The following definition taken from [P1] was also used in [MP].

**Definition 3.9.** For a symbol $\sigma$ in $S^\alpha_{ph}(\mathbb{R}_{\geq 0})$ with polyhomogeneous asymptotic expansion given by Eq. (26) we set

$$\text{fp} \sigma := \sum_{j=0}^{\infty} a_j \delta_{\alpha-j,0},$$

(with $\delta_{\alpha,0}$ the Kronecker symbol) called the **finite part at infinity** of $\sigma$. Then fp can be viewed as a map from $S^\alpha(\mathbb{R}_{\geq 0})$ to $\mathbb{C}$, we extend it by linearity to $S_{ph}(\mathbb{R}_{\geq 0})$, and we call it the finite part (of infinity) map. We write it $\text{fp}_{x \to +\infty}$ whenever we want to stress the dependence in $x$. 
Remark 3.10. The sum on the r.h.s. is clearly finite since it consists of at most one term, which we refer to as the constant term.

Lemma 3.11. The kernel of the finite part map contains \( S_{ph}^{\mathbb{Z}^{\geq 0}}(\mathbb{R}_{\geq 0}) \).

Proof. The fact that the finite part at infinity vanishes on \( S_{ph}^{\mathbb{Z}}(\mathbb{R}_{\geq 0}) \) follows from Eq. (30) combined with the following trivial observation \( \alpha \notin \mathbb{Z}_{\geq 0} \implies \alpha - j \notin \mathbb{Z}_{\geq 0} \ \forall j \in \mathbb{Z}_{\geq 0} \). □

Example 3.12. (i) For \( \sigma \in S_{ph}(\mathbb{R}_{\geq 0}) \cap L^1(\mathbb{R}_{\geq 0}) = \sum_{\Re(\alpha) < -1} S_{ph}^{\alpha}(\mathbb{R}_{\geq 0}) \), we have \( \text{fp} \sigma = 0 \).

(ii) For a polynomial \( P \) in \( \mathcal{P}(\mathbb{R}_{\geq 0}) \), we have \( \text{fp} \ P = P(0) \).

We investigate the behaviour of the finite part at infinity under pull-back by translations. For \( a \in \mathbb{R}_{+} \), let \( t_a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) denote the translation \( x \mapsto x + a \). The pull-back by the translation map \( t_a^* : \sigma \mapsto \sigma \circ t_a \) stabilises \( \mathcal{P}(\mathbb{R}_{\geq 0}) \) yet it does not preserve the finite part at \( +\infty \) on \( \mathcal{P}(\mathbb{R}_{\geq 0}) \) since for any polynomial \( P \), the finite part \( \text{fp} t_a^* P(x) = P(a) + \infty \) depends on \( a \). We nevertheless have the following statement.

Proposition 3.13. For any \( a \in \mathbb{R}_{+} \) and any \( \alpha \in \mathbb{C} \), the pull-back \( t_a^* \) by the translation map stabilises \( S_{ph}^{\alpha}(\mathbb{R}_{\geq 0}) \). In general, the finite part \( +\infty \text{fp} \) is not invariant under the pull back \( t_a^* \), yet its restriction to \( S_{ph}^{\mathbb{Z}^{\geq 0}}(\mathbb{R}_{\geq 0}) \) is, since \( S_{ph}^{\mathbb{Z}^{\geq 0}}(\mathbb{R}_{\geq 0}) \) lies in the kernel of \( +\infty \text{fp} t_a^* \).

Proof. The stability of \( S_{ph}^{\alpha}(\mathbb{R}_{\geq 0}) \) under the pull-back by the translation map was shown in [P2, Proposition 1] (see also [P1, Proposition 2.52]) using a Taylor expansion at zero in \( \frac{2}{x} \). The fact that \( S_{ph}^{\mathbb{Z}^{\geq 0}}(\mathbb{R}_{\geq 0}) \) lies in the kernel of \( +\infty \text{fp} t_a^* \) then follows from the previous lemma. □

The finite part at infinity map defines a character on \( \mathcal{P}(\mathbb{R}_{\geq 0}) \) since \( +\infty \text{fp} (P \ Q) = (P \ Q)(0) = P(0) \ Q(0) = +\infty \text{fp} (P) +\infty \text{fp} (Q) \). Yet on \( \Sigma(\mathbb{R}_{\geq 0}) \), it only defines a partial character.

Proposition 3.14. For any two symbols \( \sigma, \tau \in \Sigma(\mathbb{R}_{\geq 0}) \), if \( \sigma \top_{\mathbb{Z}} \tau \), then \( +\infty \text{fp} (\sigma \ \tau) = +\infty \text{fp} (\sigma \ \text{fp} (\tau)) = 0 \).

Proof. By linearity, we only need to prove this for \( \sigma, \tau \in S_{ph}^{\alpha}(\mathbb{R}_{\geq 0}) \) or \( \mathcal{P}^{k}(\mathbb{R}_{\geq 0}) \). Let \( \alpha \) be the order of \( \sigma \), \( \beta \) of \( \tau \) so that their product \( \sigma \ \tau \) is of order \( \alpha + \beta \). By definition, \( \sigma \top_{\mathbb{Z}} \tau \) if and only if \( \alpha + \beta \notin \mathbb{Z} \), a condition which is fulfilled whenever

(i) \((\alpha, \beta) \in (\mathbb{Z} \times (\mathbb{C} \setminus \mathbb{Z})) \cup ((\mathbb{C} \setminus \mathbb{Z}) \times \mathbb{Z}) \), or

(ii) \( \alpha \notin \mathbb{Z} \land \beta \notin \mathbb{Z} \land \alpha + \beta \notin \mathbb{Z} \) holds.

Both cases are verified on the grounds of Lemma 3.11. □

3.3. Differentiation and integration maps on symbols. We single out classes of symbols stable under differentiation and integration, quoting results from [MP] and [P1]. Clearly, \( \mathcal{P}(\mathbb{R}_{\geq 0}) \) is stable under differentiation \( \sigma \mapsto \partial_{x} \sigma \) and integration \( \sigma \mapsto \int_{0}^{x} \sigma \).

Proposition 3.15. (i) Differentiation \( \mathcal{D} : \sigma \mapsto \partial_{x} \sigma \) maps \( S^{r}(\mathbb{R}_{\geq 0}) \) to \( S^{r-1}(\mathbb{R}_{\geq 0}) \) for any real number \( r \). It furthermore maps \( S_{ph}^{\alpha}(\mathbb{R}_{\geq 0}) \) to \( S_{ph}^{\alpha-1}(\mathbb{R}_{\geq 0}) \) for any \( \alpha \in \mathbb{C} \) and therefore stabilises \( S_{ph}(\mathbb{R}_{\geq 0}) \).
(ii) Integration $\mathcal{J}$: $\sigma \mapsto \int_0^x \sigma$ maps $S^r(\mathbb{R}_0)$ to $S^{r+1}(\mathbb{R}_0) + \mathbb{C}$ for any real number $r \neq -1$.

(b) $S^0_{ph}(\mathbb{R}_0)$ to $S^0_{ph}(\mathbb{R}_0) + \mathbb{C}$ for any $\sigma \in \mathbb{C} \setminus \mathbb{Z}_{\geq -1}$, so that the integration map

$\mathcal{J}: \sigma(x) \mapsto \int_0^x \sigma(t) \ dt$ stabilises $\Sigma(\mathbb{R}_0)$.

Proof. (i) It is easy to check on the grounds of condition $(C_r)$ that differentiation $\partial_x$ maps $S^r(\mathbb{R}_0)$ to $S^{r-1}(\mathbb{R}_0)$ for any real number $r$.

Consequently, for any $\sigma \in S^0_{ph}(\mathbb{R}_0)$, the remainder term $\sigma_x^\chi_N$, which lies in $S^{\Re(\alpha) - N}(\mathbb{R}_0)$ is mapped to $\partial_x \sigma_x^\chi_N \in S^{\Re(\alpha) - N - 1}(\mathbb{R}_0)$. Combining this with the fact that the homogeneous components $x^{\alpha-j}$ are mapped to $\partial_x x^{\alpha-j} = (\alpha-j) x^{\alpha-j-1}$ of homogeneity degree $\alpha-j$, and the excision function $\chi$ is mapped to a smooth function $\partial_x \chi$ with compact support, we conclude that

$$
\partial_x \sigma = \sum_{j=0}^{N-1} (\alpha-j) a_j \chi(x) x^{\alpha-j-1} + \sum_{j=0}^{N-1} \partial_x \chi(x) a_j x^{\alpha-j} + \partial_x \sigma_x^\chi_N \sim \sum_{j=0}^{\infty} (\alpha-j) a_j x^{\alpha-j-1}
$$

lies in $S^{\alpha-1}_{ph}(\mathbb{R}_0)$.

(ii) (cfr. [P1, Exercise 3.1], see also [MP, Proposition 2])

(a) For any real number $r < -1$, by condition $(C_r)$, we know

$$
|\sigma(x)| \leq D_0 \langle x \rangle^r,
$$

so $\int_0^\infty \sigma(y) dy$ converges and

$$
\left| \int_0^x \sigma(y) dy - \int_0^\infty \sigma(y) dy \right| = \left| \int_x^\infty \sigma(y) dy \right| \leq \int_x^\infty |\sigma(y)| dy \leq D_0 \int_x^\infty \langle y \rangle^r dy.
$$

It is easy to check that $\int_x^\infty \langle y \rangle^r dy \in S^{r+1}(\mathbb{R}_0)$. Therefore there is a constant $D'_0$ such that

$$
\left| \int_0^x \sigma(y) dy - \int_0^\infty \sigma(y) dy \right| \leq D'_0 \langle x \rangle^{r+1}.
$$

Together with the fact that $\partial_x^k \circ \int_0^x = \partial_x^{k-1}$ for $k \in \mathbb{Z}_{\geq 1}$, we know, $x \mapsto \int_0^x \sigma(y) dy - \int_0^\infty \sigma(y) dy$ is in $S^{r+1}(\mathbb{R}_0)$, that is $\int_0^x \sigma(y) dy$ is in $S^{r+1}(\mathbb{R}_0) + \mathbb{C}$. If $r > -1$, then

$$
|\sigma(x)| \leq D_0 \langle x \rangle^r.
$$

So

$$
\left| \int_0^x \sigma(y) dy \right| \leq \int_0^x |\sigma(y)| dy \leq D_0 \int_0^x \langle y \rangle^r dy.
$$

Now it is easy to check that $\int_0^x \langle y \rangle^r dy \in S^{r+1}(\mathbb{R}_0)$, thus $\int_0^x \sigma(y) dy$ is in $S^{\Re(\alpha) - N}(\mathbb{R}_0) = S^{\Re(\alpha) - N + 1}(\mathbb{R}_0) + \mathbb{C}$.

(b) Consequently, for any $\sigma \in S^0_{ph}(\mathbb{R}_0)$ of order $\alpha \notin \mathbb{Z}_{\geq -1}$, and with the notations of Eq. (25), the remainder term $\sigma_x^\chi_N$ (here $\chi$ is an excision function identically one outside the open unit interval), which lies in $S^{\Re(\alpha) - N}(\mathbb{R}_0)$, is mapped to $\int_0^x \sigma_x^\chi_N(y) dy \in S^{\Re(\alpha) - N + 1}(\mathbb{R}_0) + \mathbb{C}$ when $N \neq \Re(\alpha) + 1$, which is the case when $N > \Re(\alpha) + 1$. 
Now for
\[ \sigma(x) = \sum_{j=0}^{N-1} \chi(x) a_j x^{\alpha-j} + \sigma_{(N)}^{\chi}(x), \]
If \( \chi(x) \) is identically 1 when \( x \geq x_0 \), then for \( x \geq x_0 \),
\[ \int_0^x \chi(y) y^{\alpha-j} dy = \int_0^{x_0} \chi(y) y^{\alpha-j} dy + \int_{x_0}^x \chi(y) y^{\alpha-j} dy = \frac{x^{\alpha-j+1} - x_0^{\alpha-j+1}}{\alpha-j+1} + \int_0^{x_0} \chi(y) y^{\alpha-j} dy. \]

Thus, for any excision function \( \tilde{\chi} \), which vanishes on \([0,r]\) and is identically one on some interval \([r+\delta, +\infty)\) with \( \delta > 0 \), since by assumption \( \alpha-j \neq -1 \) for any \( j \in \mathbb{Z}_{\geq 0} \), we have
\[ \int_0^x \chi(y) y^{\alpha-j} dy = \tilde{\chi}(x) \left( \frac{x^{\alpha-j+1} - x_0^{\alpha-j+1}}{\alpha-j+1} + \int_0^{x_0} \chi(y) y^{\alpha-j} dy \right) + (1-\tilde{\chi}(x)) \int_0^{x_0} \chi(y) y^{\alpha-j} dy. \]

Thus we have shown that the map \( x \mapsto \int_0^x \chi(y) y^{\alpha-j} dy \) lies in \( S_{\text{ph}}^{\alpha-j+1}(\mathbb{R}_{\geq 0}) + \mathbb{C} \) for any \( j \in \mathbb{Z}_{\geq 0} \).

Using Eq. (25) we now write \( x \mapsto \sigma(x) = \sum_{j=0}^{N-1} \chi a_j x^{\alpha-j} + \sigma_{(N)}^{\chi}(x) \).

The above argument in Part b) tells us that \( \sum_{j=0}^{N-1} \int_0^x \chi(y) y^{\alpha-j} dy \) lies in \( S_{\text{ph}}^{\alpha+1}(\mathbb{R}_{\geq 0}) + \mathbb{C} \). Part a) tells us that for large enough \( N \), the symbol \( x \mapsto \int_0^x \sigma_{(N)}^{\chi} \) lies in \( S_{\mathbb{R}}^{\alpha-N}(\mathbb{R}_{\geq 0}) \).

Summing the two we conclude that
\[ \int_0^x \sigma \sim \text{constant} + \sum_{j=0}^{\infty} \frac{a_j}{\alpha-j+1} x^{\alpha-j-1} \]
lies in \( S_{\text{ph}}^{\alpha+1}(\mathbb{R}_{\geq 0}) + \mathbb{C} \). Since the integration map clearly stabilises \( \mathcal{P}(\mathbb{R}_{\geq 0}) \), it follows that it stabilises \( \Sigma(\mathbb{R}_{\geq 0}) \).

Since the integration stabilises \( \mathcal{P}(\mathbb{R}_{\geq 0}) \), it stabilises \( \Sigma(\mathbb{R}_{\geq 0}) \).

\[ \square \]

**Remark 3.16.**
- We want to single out a class of symbols stable under integration: if one insists on avoiding the occurrence of logarithms while integrating more general symbols, one needs to avoid integrating powers \( x^{-1} \), hence the natural class to consider is \( \Sigma(\mathbb{R}_{\geq 0}) \).
- Let us nevertheless point out that an alternative point of view adopted in [MP] would be to extend the algebra of polyhomogeneous symbols to log-polyhomogeneous ones; we chose to avoid this extension which would involve more technicalities.

We saw in Proposition 3.15 that the algebra \( S_{\text{ph}}(\mathbb{R}_{\geq 0}) \) is stable under differentiation and the class \( \Sigma(\mathbb{R}_{\geq 0}) \) is stable under integration. So we can define the finite part at infinity of an integrated polyhomogeneous symbol and set the following definition.

**Definition 3.17.** For any \( \sigma \in \Sigma(\mathbb{R}_{\geq 0}) \),
\[ (31) \quad \int_0^\infty \sigma := \int_0^\infty \sigma(x) dx := \text{fp}_{x \to +\infty} \int_0^x \sigma(y) dy \]
is called the cut-off integral of $\sigma$.

**Example 3.18.** We have $\int_0^\infty Q = 0$ for any polynomial $\sigma = Q$, since $\text{fp } P = P(0)$ vanishes if $P(x) = \int_0^x Q$.

**Example 3.19.** By the proof of Proposition 3.15, we know for a classical symbol $\sigma$ of order $< -1$, $\int_0^\infty \sigma = \int_0^\infty \sigma$.

An explicit computation derived from splitting the integral $\int_x^0 \sigma = \int_x^1 \sigma + \int_1^0 \sigma$ for large $x$ and the fact that $\text{fp } x^{\alpha-j+1} = 0$ for $\alpha \neq j - 1$ yields the following expression for any $\sigma \in \Sigma(\mathbb{R}_{\geq 0})$ of order $\alpha$ (we use the notations of Eq. (25)

(32)

$$\mathcal{I}(\sigma)(x) = \int_0^x \sigma(y) dy = \sum_{j=0}^{N-1} a_j \left( \int_0^1 \left( y^{\alpha-j}(y) dy + \frac{x^{\alpha-j+1}}{\alpha-j+1} - \frac{1}{\alpha-j+1} \right) + \int_0^x \sigma^\chi(N)(y) dy, \right.$$ which after taking the finite part at $+\infty$ yields (for $N$ sufficiently large):

(33)

$$\int_0^\infty \sigma(y) dy = \sum_{j=0}^{N-1} a_j \left( \int_0^1 \left( \chi(x) x^{\alpha-j}(x) dx - \frac{1}{\alpha-j+1} \right) + \int_0^\infty \sigma^\chi(N)(x) dx. \right.$$ This quantity is clearly independent of the choice of the excision function $\chi$ and the choice of the integer $N$ as long as it is chosen sufficiently large.

### 3.4. Summation of symbols

To a symbol $\sigma$ in $S_{ph}(\mathbb{R}_{\geq 0})$, we assign to any positive integer $N$ the sum

$$S(\sigma)(N) := \sum_{n=0}^{N} \sigma(n),$$

and for $\lambda \in \{\pm 1\}$ the sum

$$S_\lambda(\sigma)(N) = S(\sigma) \left( N + \frac{\lambda - 1}{2} \right), \text{ so that } S_{-1}(\sigma)(N) = S(\sigma)(N - 1), \quad S_1 = S.$$

The Euler-MacLaurin formula (see [H, Formula (13.1.1)]) relates the sum over $[0, N] \cap \mathbb{Z}$ and the corresponding integral over $[0, N]$. Let $\overline{B_k}(x) = B_k(x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ stands for the integer part of the real number $x$, and $B_k(x)$ is the $k$-th Bernoulli polynomial. Then

$$S(\sigma)(N) = \int_0^N \sigma(x) dx + \frac{1}{2} (\sigma(N) + \sigma(0))$$

$$+ \sum_{k=2}^{K} \frac{B_k}{k!} \left( \sigma^{(k-1)}(N) - \sigma^{(k-1)}(0) \right) + \frac{(-1)^{K+1}}{K!} \int_0^N \overline{B_K}(x) \sigma^{(K)}(x) dx.$$ Note that this expression is independent of the choice of the integer $K \geq 2$. 

Following [MP] we interpolate the discrete sum $S(\sigma)$ by a smooth function $\mathcal{G}(\sigma) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as

$$\mathcal{G}(\sigma) = \mathcal{I}(\sigma) + \mu(\sigma),$$

(35) $$\mu(\sigma)(x) := \frac{1}{2} (\sigma(x) + \sigma(0)) + \sum_{k=2}^{K} \frac{B_k}{k!} \left( \sigma^{(k-1)}(x) - \sigma^{(k-1)}(0) \right) + \frac{(-1)^{K+1}}{K!} \int_{0}^{x} B_K(t) \sigma^{(K)}(t) \, dt.$$

It follows from Eq. (35) that $\mathcal{G}(\sigma)(N) = S(\sigma)(N)$ for any positive integer $N$.

**Definition 3.20.** For convenience we define for $\lambda \in \{ \pm 1 \}$

(36) $$\mathcal{G}_\lambda : \sigma \longmapsto \left( x \mapsto \mathcal{G}(\sigma)(x + \frac{\lambda - 1}{2}) \right)$$

and set $\mathcal{G}_0 = \mathcal{I}$.

We have the following generalisation of [MP, Proposition 8 and Formula(36)].

**Proposition 3.21.** For $\lambda \in \{ 0, \pm 1 \}$, the map $\mathcal{G}_\lambda$ stabilises $\Sigma(\mathbb{Z}_{\geq 0})$.

**Proof.** This conclusion follows from the fact that $\Sigma(\mathbb{R}_{\geq 0})$ is stable under pull-back and $\mathcal{G}$ stabilises $\Sigma(\mathbb{R}_{\geq 0})$. We now prove that $\mathcal{G}$ stabilises $\Sigma(\mathbb{R}_{\geq 0})$:

- we know from Proposition 3.15 that the integration map $\mathcal{G}_0 = \mathcal{I}$ enjoys this property;
- the term $\sum_{k=2}^{K} \frac{B_k}{k!} \sigma^{(k-1)}$ in the Euler-Maclaurin expansion interpreted as a linear combination of differentiation maps $\partial_2^j$ applied to $\sigma$ also lies in $\Sigma(\mathbb{R}_{\geq 0})$ if $\sigma$ does; indeed, it follows from Proposition 3.15 2.b. that $\partial_2^j$ maps a classical symbol (resp. polynomial) of order $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ (resp. degree $m$) to a classical symbol (resp. polynomial) of order $\alpha - j \in \mathbb{C} \setminus \mathbb{Z}$ (resp. degree $m - j$);
- Let $\tau_K(x) := \int_{0}^{x} B_K(t) \sigma^{(K)}(t) \, dt$. If $a$ denotes the order of $\sigma$, for any $J \in \mathbb{Z}_{\geq 0}$ we have $|\sigma^{(J)}(t)| \leq C_J \langle t \rangle^{\Re(a) - J}$ for some constant $C_J$. For $K > \Re(a) + 1$, the map $t \mapsto \langle t \rangle^{\Re(a) - K - i}$ is $L^1$ for any $i \in \mathbb{Z}_{\geq 0}$. Since the map $B_K$ is 1-periodic and smooth on any segment $[i, i + 1]$, it follows that the map $\tau_K : x \mapsto \int_{0}^{x} B_K(t) \sigma^{(K)}(t) \, dt$ is smooth with derivative $\tau'_K = B_K \sigma^{(K)}$ which is a classical symbol of order $a - K$ with real part $\Re(a) - K < -1$. Thus $\tau_K$ differs by a constant $c$ from a classical symbol of order $a - K + 1$. This proves that $\sigma \in \Sigma(\mathbb{R}_{\geq 0}) \implies \tau_K \in \Sigma(\mathbb{R}_{\geq 0})$ for any $K > \Re(a) + 1$.

□

4. **Locality Rota-Baxter operators on symbol-valued multivariate meromorphic germs**

4.1. **The locality algebra of symbol-valued meromorphic germs.** We generalise the space of meromorphic multivariate germs of functions with linear poles at zero considered in [GPZ3] to the space of multivariate meromorphic germs of polyhomogeneous symbols (on $\mathbb{R}_{\geq 0}$) with linear poles at zero.

We adopt notations close to those of [GPZ3]. For the filtered Euclidean space $$(V, Q) = \lim_{k} (\mathbb{R}^k, Q_k),$$
let \( L_k := (\mathbb{R}^k)^* \) and \( \mathcal{L} = \lim_{k \to \infty} \mathcal{L}_k \) be the direct limit of spaces of linear forms. The space of holomorphic germs at zero with linear poles and real coefficient is denoted by \( \mathcal{M}_+(\mathbb{C}^\infty) \), and the space of meromorphic germs at zero with linear poles and real coefficient is denoted by \( \mathcal{M}(\mathbb{C}^\infty) \).

Definition 4.1. Let \( U \) be a domain in \( \mathbb{C}^n \).

- We call a family \( (\sigma(z))_{z \in U} \) of classical symbols **holomorphic (of affine order \( \alpha(z) \))** if
  
  \((i)\) \( \forall z \in U, \; \sigma(z) \in S_{ph}^\alpha(\mathbb{R}_{\geq 0}) \);
  
  \((ii)\) \( \alpha(z) = L(z) + c \) with \( c \in \mathbb{R} \) and \( L \in \mathcal{L}_k \);
  
  \((iii)\) for any \( N \in \mathbb{Z}_{\geq 1} \) and any excision function \( \chi \), the remainder (see Eq. (26))
  
  \[
  z \mapsto \sigma^{\chi}_{(N)}(z) := \sigma(z) - \sum_{j=0}^{N-1} \chi(x) a_j(z) x^ {\alpha(z) - j},
  \]
  
  satisfies the following uniform estimation: for any \( k \in \mathbb{Z}_{\geq 0} \), and for any \( x \in \mathbb{R}_{\geq 0} \), the derivatives \( \partial_x^k \sigma^{\chi}_{(N)} \) are holomorphic functions on \( U \), and for any compact subset \( K \) of \( U \), and any \( n \in \mathbb{Z}_{\geq 0} \) there is a positive constant \( C_{k,n,N}(K) \) such that
  
  \[
  \left| \partial_x^n \left( \partial_x^k \sigma^{\chi}_{(N)}(z) \right) \right| \leq C_{k,n,N}(K) \langle x \rangle^{\Re(\alpha(z)) - n - k + \epsilon} \quad \forall z \in K \subset U \quad \forall \epsilon > 0.
  \]

  Such a family is called a **simple holomorphic family of symbols (of affine order \( \alpha \))**.

- Let \( \sigma_j(z), j = 1, \ldots, J \) be simple holomorphic families of symbols. Then \( \sigma(z) = \sum_{j=1}^J \sigma_j(z) \) is called a **holomorphic family of symbols**.

The subsequent straightforward property is nevertheless of importance for the following.

Lemma 4.2. The product \( \sigma(z) := \sigma_1(z) \sigma_2(z) \) of two simple holomorphic families of symbols \( \sigma_i(z) \) of affine orders \( \alpha_i(z) \) is a simple holomorphic family of symbols of affine order \( \alpha_1(z) + \alpha_2(z) \).

Definition 4.3. A simple **symbol-valued holomorphic germ** or **holomorphic germ of symbols** at zero (with affine order \( \alpha(z) \)) is an equivalence class of simple holomorphic families around zero of symbols of affine order \( \alpha(z) \) under the equivalence relation:

\[
(\sigma(z))_{z \in U} \sim (\tau(z))_{z \in V} \iff \exists W, \; 0 \in W \subset U \cap V, \; \sigma(z) = \tau(z), \quad \forall z \in W.
\]

For any positive integer \( k \), any \( \alpha(z) = L(z) + c \) with \( c \in \mathbb{R}, L \in \mathcal{L}_k \), let \( \mathcal{M}_+ S^\alpha(\mathbb{C}^k) \) denote the linear space generated by simple holomorphic germs of symbols of order \( \alpha \), and \( \mathcal{M}_+ S(\mathbb{C}^k) \) denote the linear space generated by simple holomorphic germs of symbols.

Remark 4.4. Clearly, two equivalent families of symbols have the same (affine) order so it makes sense to define the (affine) order of a holomorphic germ of such symbols.

Definition 4.5. Let \( U \) be a domain of \( \mathbb{C}^k \) containing the origin. A simple **meromorphic family** on \( U \) of polyhomogeneous symbols with linear poles (with real coefficients) and affine order \( \alpha(z) \) is a holomorphic family \( (\sigma(z))_{z \in U \setminus X} \) with affine order \( \alpha(z) \) of symbols on \( U \setminus X \), for which

- \( X = \cup_{i=1}^k \{ L_i = 0 \} \) with \( L_1, \ldots, L_n \in \mathcal{L}_k \),
there exists a simple holomorphic family \((\tau(z))_{z \in U}\) with affine order \(\alpha(z)\) and nonnegative integers \(s_1, \cdots, s_n\), such that
\[
L_{s_1}^{e_1} \cdots L_{s_n}^{e_n} \sigma(z) = \tau(z)
\]
on \(U \setminus X\).

A simple symbol-valued \textbf{meromorphic germ} or meromorphic germ of symbols at zero on \(\mathbb{C}^k\) with linear poles and affine order \(\alpha(z)\) is an equivalence class of meromorphic families around zero with linear poles of symbols of affine order \(\alpha(z)\) under the equivalence relation:
\[
(\sigma(z)_{z \in U \setminus X}) \sim (\tau(z)_{z \in V \setminus Y}) \iff \exists W, \ 0 \in W \subset U \cap V, \ \sigma(z) = \tau(z), \ \forall z \in W \setminus (X \cup Y),
\]
where \(U\) and \(V\) are domains of \(\mathbb{C}^k\) containing the origin. Let \(\mathcal{M}^\alpha_s(\mathbb{C}^k)\) denote the linear space generated by simple symbol-valued meromorphic germ at zero on \(\mathbb{C}^k\) with linear poles and affine order \(\alpha(z)\), and \(\mathcal{M}(\mathbb{C}^k)\) denote the linear space generated by simple symbol-valued meromorphic germ at zero.

Composing with the projection \((\mathbb{C}^{k+1})^* \to (\mathbb{C}^k)^*\) dual to the inclusion \(\iota_k : \mathbb{C}^k \to \mathbb{C}^{k+1}\), and the isomorphism induced by the inner product \(\langle \cdot, \cdot \rangle : (\mathbb{C}^k)^* \cong \mathbb{C}^k\), yields the embeddings \(\mathcal{M}_+(\mathbb{C}^k) \hookrightarrow \mathcal{M}_+ S(\mathbb{C}^{k+1})\) (resp. \(\mathcal{M}(\mathbb{C}^k) \hookrightarrow \mathcal{M}(\mathbb{C}^{k+1})\)), thus giving rise to the direct limits:
\[
\begin{align*}
\mathcal{M}^\alpha_+(\mathbb{C}^\infty) &:= \lim_{\rightarrow k=1} \mathcal{M}^\alpha_+(\mathbb{C}^k) = \bigcup_{k=1}^{\infty} \mathcal{M}^\alpha_+(\mathbb{C}^k), \\
\mathcal{M}_+(\mathbb{C}^\infty) &:= \lim_{\rightarrow k=1} \mathcal{M}_+(\mathbb{C}^k) = \bigcup_{k=1}^{\infty} \mathcal{M}_+(\mathbb{C}^k), \\
\text{(resp. } \mathcal{M}(\mathbb{C}^\infty) \text{):} &\lim_{\rightarrow k=1} \mathcal{M}(\mathbb{C}^k) = \bigcup_{k=1}^{\infty} \mathcal{M}(\mathbb{C}^k),
\end{align*}
\]
where \(\alpha(z) = L(z) + c\) with \(c \in \mathbb{R}\) and \(L \in \mathcal{L}_k\).

\textbf{Example 4.6.} For any \(\ell \in \mathcal{L}_k\),
\[
z \mapsto \langle x \mapsto \langle x \rangle^{\ell(z)} \rangle
\]
defines a simple holomorphic germ of symbols of order \(\ell\).

\textbf{Proposition 4.7.} Under pointwise function multiplication, \(\mathcal{M}(\mathbb{C}^\infty)\) is a complex algebra and we have the following inclusions of subalgebras
\[
\mathcal{M}_+(\mathbb{C}^\infty) \subset \mathcal{M}(\mathbb{C}^\infty); \quad \mathcal{M}(\mathbb{C}^\infty) \cdot \mathcal{P}(\mathbb{R}_{\geq 0}) \subset \mathcal{M}(\mathbb{C}^\infty).
\]

\textbf{4.2. Dependence space of a meromorphic germ of symbols.} The notion of dependence space defined in [CGPZ1, Definition 2.13] for meromorphic germs extends to meromorphic germs of symbols since the arguments used there to justify the definition apply in the same way.

\textbf{Definition 4.8.} Let \(\sigma(z)\) be a meromorphic family of symbols with affine order on an open neighborhood \(U\) of 0 in \(\mathbb{C}^n\). If there are linear forms \(L_1, \cdots, L_k\) on \(\mathbb{C}^n\) and a meromorphic family of symbols \(\tau(w)\) on an open neighborhood \(W\) of 0 in \(\mathbb{C}^k\), such that \(\sigma(z) = \tau(\phi(z))\) on \(U \cap \phi^{-1}(W)\), where \(\phi = (L_1, \cdots, L_k) : \mathbb{C}^n \to \mathbb{C}^k\), then we say that \(\sigma\) \textbf{depends} on the (linear) subspace of \((\mathbb{C}^n)^*\) spanned by \(L_1, \cdots, L_k\). We say that a meromorphic germ of
symbols at \( 0 \in \mathbb{C}^n \) depends on a subspace \( W \subset (\mathbb{C}^n)^* \) if one of its representatives in the equivalence class given by the germ does.

The dependence subspace \( \text{Dep}(\sigma) \) of a meromorphic family of symbols \( \sigma \) on an open neighborhood \( U \) of \( 0 \) in \( \mathbb{C}^n \), is the smallest subspace of \( (\mathbb{C}^n)^* \) on which it depends. For a meromorphic germ, the dependence subspace is the dependence subspace of any of its representing element.

**Lemma 4.9.** For any \( \sigma \in \Omega \) we have

\[
\text{Dep}(\partial^2 \sigma) \subset \text{Dep}(\sigma) \quad \text{and} \quad \text{Dep}(I(\sigma)) \subset \text{Dep}(\sigma),
\]

where \( I \) is the integration map defined in Eq. (32).

**Proof.** The first inclusion follows from the fact that differentiation commutes with multiplication by meromorphic germs of functions. The second inclusion follows from inspection of the explicit formula (32) for \( I \). \( \square \)

**Proposition 4.10.** If a meromorphic germ of symbols \( \sigma(z) \) depends on a space \( V \), and \( \sigma(z) \sim \sum a_n(z) x^{\alpha(z)-n} \)

then so do \( \alpha(z), a_n(z), n \in \mathbb{Z}_{\geq 0} \) depend on \( V \).

**Proof.** Assuming that the polyhomogeneous symbol \( \sigma(z) \sim \sum a_n(z) x^{\alpha(z)-n} \) reads \( \sigma(z) = \tau(\phi(z)) \) for some symbol \( \tau(z) \sim \sum b_n(z) x^{\beta(z)-n} \), it follows from the uniqueness of the coefficients in the asymptotic expansion of a polyhomogeneous symbol that \( a_n = b_n \circ \phi \) and \( \alpha = \beta \circ \phi \) for any non negative integer \( n \). The statement then follows. \( \square \)

We are now ready to extend the independence relation on meromorphic germs of functions introduced in [CGPZ1, Definition 2.14] to meromorphic germs of symbols.

**Definition 4.11.** Two meromorphic germs of symbols \( \sigma_1 \) and \( \sigma_2 \) in \( \Omega \) are said to be independent whenever

\[
\sigma_1 \perp^Q \sigma_2 :\iff \text{Dep}(\sigma_1) \perp^Q \text{Dep}(\sigma_2).
\]

**Example 4.12.**

(i) Given two functions \( f, g \in M(\mathbb{C}^\infty) \) and two polynomials \( P, Q \) on \( \mathbb{R}_{\geq 0} \), we have

\[
(f \cdot P) \perp^Q (g \cdot Q) \iff f \perp^Q g.
\]

(ii) Given \( \ell_i, L_i \in \mathcal{L} \) with \( i = 1, 2 \), we have

\[
\frac{\langle x \rangle_{\ell_1}}{L_1} \perp^Q \frac{\langle x \rangle_{\ell_2}}{L_2} \iff \{\ell_1, L_1\} \perp^Q \{\ell_2, L_2\}.
\]

**Definition 4.13.** We call the complex linear space generated by the set \( \cup_{\alpha(0) \neq 0} \mathcal{MS}^\alpha(\mathbb{C}^k) \) and the linear space \( M(\mathbb{C}^\infty) \cdot \mathcal{P}(\mathbb{R}_{\geq 0}) \) the space of admissible meromorphic germ of symbols, and denote it by \( \Omega \).

**Remark 4.14.**

(i) \( \Omega \) is not an algebra for \( c + c' \notin \mathbb{Z}_{\geq 0} \Longrightarrow \chi(x) x^{qz+c} x^{-qz+c'} \notin \Omega \) for any excision function \( \chi \) around zero. Yet it can be equipped with a locality algebra structure as we shall see from the subsequent proposition.
(ii) The fact that we exclude meromorphic germs of symbols with constant order which are not polynomial, is motivated by the fact that allowing for negative integer powers can give rise to logarithmic symbols after integration. Here like in quantum field theory, we want to avoid such logarithmic symbols.

**Proposition 4.15.** The triple \((\Omega, \perp^Q, m_\Omega)\) is a commutative and unital locality algebra, with unit given by the constant function \(1\) and \(m_\Omega\) is the restriction of the pointwise function multiplication to the graph \(\perp^Q \subset \Omega \times \Omega\).

**Proof.** Assume that \(\sigma_1, \sigma_2 \in \Omega, \sigma_1 \perp^Q \sigma_2\). By checking the order, we know \(\sigma_1 \sigma_2 \in \Omega\).

If \(\text{Dep}(\sigma_1) = \langle L_1, \ldots, L_{k_1} \rangle\) and \(\text{Dep}(\sigma_2) = \langle L_{k_1+1}, \ldots, L_{k_1+k_2} \rangle\), so that

\[
\sigma_1(z) = \tau_1(L_1(z), \ldots, L_{k_1}(z)), \\
\sigma_2(z) = \tau_2(L_{k_1+1}(z), \ldots, L_{k_1+k_2}(z))
\]

for some meromorphic germs of symbols \(\tau_i(w_i)\), then the product reads

\[
\sigma(z) = \tau_1(L_1(z), \ldots, L_{k_1}(z)) \tau_2(L_{k_1+1}(z), \ldots, L_{k_1+k_2}(z)) = \tau_1(\phi_1(z)) \tau_2(\phi_2(z)),
\]

where we have set \(\phi_1(z) := (L_1(z), \ldots, L_{k_1}(z))\) and \(\phi_2(z) := (L_{k_1+1}(z), \ldots, L_{k_1+k_2}(z))\).

This shows that \(\text{Dep}(\sigma_1 \sigma_2) \subset \text{Dep}(\sigma_1) \oplus \text{Dep}(\sigma_2)\).

So

\[
(\text{Dep}(\sigma_1) \perp^Q \text{Dep}(\sigma_3) \wedge \text{Dep}(\sigma_2) \perp^Q \text{Dep}(\sigma_3)) \implies (\text{Dep}(\sigma_1 \sigma_2) \perp^Q \text{Dep}(\sigma_3)),
\]

from which it follows that \((\Omega, \perp^Q, m_\Omega)\) is a locality algebra. Moreover, \(\Omega\) is clearly commutative and has unit given by the constant function. \(\square\)

### 4.3. Finite part at infinity on \(\Omega\). We can now define a finite part at infinity map on \(\mathcal{MS}(\mathbb{C}^\infty)\) by using the finite part at infinity map on \(\mathcal{S}_{\text{ph}}(\mathbb{R}_{\geq 0})\). For a generator of the form

\[
\sigma(z) \sim \sum_{j=0}^{\infty} a_j(z) x^{\alpha(z)-j},
\]

on \(U, \alpha(z) = L(z) + \alpha_0\), if \(L \in \mathcal{L}\), we set it to be

\[
\sigma(z) \mapsto \text{fp}_+^\infty (\sigma(z))
\]

on \(U \setminus \{\alpha(z) + j = 0, j \in \mathbb{Z}_{\geq 0}\}\), if \(L \neq 0\);

and

\[
\sigma(z) \mapsto \text{fp}_+^\infty (\sigma(z))
\]

on \(U\), if \(L = 0\).

**Lemma 4.16.** This defines a linear map

\[
\text{fp}_+^\infty : \mathcal{MS}(\mathbb{C}^\infty) \to \mathcal{M}(\mathbb{C}^\infty),
\]

on \(\Omega\), it sends simple meromorphic germs to zero except on polynomials \(P(z) = \sum_{j=0}^{k} a_j(z) x^j\) for which \(\text{fp}_+^\infty (P(z)) = a_0(z)\).
Proof. By linearity, we only need to check simple meromorphic germs of symbols. For a simple meromorphic germ of symbols with order \( \alpha(z) = L(z) + \alpha_0, L \neq 0 \), represented by

\[
\sigma(z) \sim \sum_{j=0}^{\infty} a_j(z) x^{\alpha(z) - j},
\]
on \( U \setminus X \) with \( U \) contained in some open ball centered at zero, and \( X = \{ L_1 = \cdots = L_n = 0 \} \), \( a_j(z), j \in \mathbb{Z}_{\geq 0} \) holomorphic on \( U \setminus X \). By definition, on \( U \setminus \{ L_1 = \cdots = L_n = \alpha - j = 0, j \in \mathbb{Z}_{\geq 0} \} \),

\[
fp_{+\infty} \sigma = 0.
\]

By our choice of \( U, U \) and \( \{ \alpha - j = 0 \} \) have no intersection when \( j \) is big enough, so \( U \setminus \{ L_1 = \cdots = L_n = \alpha - j = 0, j \in \mathbb{Z}_{\geq 0} \} \) is \( U \setminus \) finite hyperplanes. So in this case \( fp_{+\infty} \sigma \) extends to the zero function on \( U \), thus the 0 germ.

For a simple meromorphic germ of symbols with order \( \alpha(z) = \alpha_0 \), represented by

\[
\sigma(z) \sim \sum_{j=0}^{\infty} a_j(z) x^{\alpha_0 - j},
\]
on \( U \setminus X \) with \( U \) contained in some open ball centered at zero, and \( X = \{ L_1 = \cdots = L_n = 0 \} \), \( a_j(z), j \in \mathbb{Z}_{\geq 0} \) holomorphic on \( U \setminus X \), and

\[
L_1^{s_1} \cdots L_n^{s_n} \sigma = \tau(z) \sim \sum_{j=0}^{\infty} b_j(z) x^{\alpha(z) - j}
\]
on \( U \setminus X \) with \( X = \{ L_1 = \cdots = L_n = 0 \} \), and \( b_j(z), j \in \mathbb{Z}_{\geq 0} \) holomorphic on \( U \). By the uniqueness of the asymptotic expansion, \( L_1^{s_1} \cdots L_n^{s_n} a_j = b_j \), so \( a_j \in \mathcal{M}(\mathbb{C}^\infty) \).

By definition, on \( U \setminus \{ L_1 = \cdots = L_n = 0 \} \),

\[
fp_{+\infty} \sigma = \sum_{+\infty} a_j \delta_{\alpha_0 - j, 0}.
\]

Since this is only one term at most, \( fp_{+\infty} \sigma \in \mathcal{M}(\mathbb{C}^\infty) \). We have the conclusion. \( \square \)

We know that \( fp_{+\infty} \) is a partial character on \( \Sigma(\mathbb{R}_{\geq 0}) \) (see Proposition 3.14). By Proposition 4.10, we have

**Proposition 4.17.** The finite part at infinity

\[
fp_{+\infty} : \Omega_{+\infty} \rightarrow \mathcal{M}(\mathbb{C}^\infty),
\]

\[
\sigma(z) \mapsto fp_{+\infty} (\sigma)(z)
\]
is a morphism of locality algebras i.e., for any two independent germs of symbols \( \sigma_1 \) and \( \sigma_2 \) in \( \Omega_{+\infty} \), then

\[
fp_{+\infty} (\sigma_1 \sigma_2) = fp_{+\infty} (\sigma_1) \cdot fp_{+\infty} (\sigma_2).
\]

**Proof.** In view of the linearity of the map \( fp_{+\infty} \) and the fact that it only affects the \( x \)-variable, it is sufficient to consider products of simple holomorphic germs.
Let $\sigma_1$ and $\sigma_2$ be two independent simple holomorphic germs of symbols in $\Omega$ with respective orders $\alpha_1$ and $\alpha_2$ respectively. If for $i = 1, 2$,
\[
\sigma_i(z) \sim \sum_{n_i=0}^{\infty} a_{i n_i} (z) x^{\alpha_i(z) - n_i},
\]
it follows from Proposition 4.15 that $\sigma_1 \perp^Q \sigma_2$ and $a_{1 n_1} \perp^Q a_{2 n_2}$ for any $(n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$.

- Assume that $\alpha_1$ and $\alpha_2$ are affine nonconstant respectively. Consequently, the holomorphic map $\alpha : (z_1, z_2) \mapsto \alpha_1(z_1) + \alpha_2(z_2)$ corresponding to the order of $\sigma_1 \sigma_2$ is also affine nonconstant. By Lemma 4.16, we know that $z_1 \mapsto \text{fp} (\sigma_1)(z_1)$, $z_2 \mapsto \text{fp} (\sigma_2)(z_2)$ and $(z_1, z_2) \mapsto \text{fp} (\sigma_1 \sigma_2)(z_1, z_2)$ all vanish as meromorphic functions respectively. It follows that Eq. (41) holds in that case.

- We now consider the case when one of the two symbols is polynomial so, let us assume that $\alpha_1$ is affine nonconstant in $z_1$ and $\sigma_2$ is a polynomial of degree $d_2$. In that case $\sigma_1(z_1) \sigma_2(z_2)$ is a simple holomorphic germ of order $\alpha_1(z_1) + d_2$ which is also affine nonconstant. The rest of the above reasoning goes through in a similar manner.

- If the two symbols are polynomials say $\sigma_1(z) = \sum_{j_i=0}^{d_i} a_{i j_i} (z) x^{d_i}$, so is their product $\sigma_1(z_1) \sigma_2(z_2) = \sum_{k=1}^{d_1+d_2} c_k(z_1, z_2) x^{d_1+d_2}$ with $c_k(z_1, z_2) = \sum_{j_1+j_2=k} a_{1 j_1}(z_1) a_{2 j_2}(z_2)$ and we have
\[
\text{fp} (\sigma_1(z_1) \sigma_2(z_2)) = c_0(z_1, z_2) = a_{0 1}^0(z_1) a_{0 2}^0(z_2) = \text{fp} (\sigma_1) \text{fp} (\sigma_2).
\]
\[
\square
\]

**Remark 4.18.** Notice that Eq. (41) does not hold for any germ of symbols. For non-independent germs of symbols, the argument of Remark 4.14, (i) provides a counterexample.

An easy observation:

**Proposition 4.19.** The finite part at infinity map equips $\mathcal{M}(\mathbb{C}^\infty)$ a $(\Omega, \perp^Q)$-operated locality algebra structure:

$\Omega \times \mathcal{M}(\mathbb{C}^\infty) \to \mathcal{M}(\mathbb{C}^\infty)$

$(\sigma, f) \mapsto f \text{fp} (\sigma)$.

**4.4. Locality Rota-Baxter operators on $\Omega$.**

**Theorem 4.20.** (i) The integration map $\mathcal{I}$ extends to a map on $\Omega$ and the triple $(\Omega, \perp^Q, m_\Omega, \mathcal{I})$ is a locality Rota-Baxter commutative algebra of weight zero, and we have $\text{Id} \perp^Q \mathcal{I}$.

(ii) The summation maps $\mathcal{S}_{\mp 1}$ extend to a map on $\Omega$, the triple $(\Omega, \perp^Q, m_\Omega, \mathcal{S}_{\mp 1})$ is a locality Rota-Baxter commutative algebra of weight $\mp 1$, and we have $\text{Id} \perp^Q \mathcal{S}_{\mp 1}$.

**Proof.** Since $\Omega$ consists of sums of simple meromorphic germs of symbols with nonconstant affine order and polynomials with meromorphic coefficients and since the integration map clearly stabilises the algebra of such polynomials, it suffices to consider the integration map on simple meromorphic germs of symbols with nonconstant affine order and it is sufficient to consider a holomorphic germ of symbols with nonconstant affine order.
Let $\rho(z)$ be such a holomorphic family of order $\alpha(z)$,
$$
\rho(z) \sim \sum_{j=0}^{\infty} a_j(z) x^{\alpha(z)-j}.
$$

- We want to show that $\mathcal{I}(\rho(z)) \in \Omega$. For $z \notin \alpha^{-1}(\mathbb{Z})$, the symbol $\rho(z)$ whose order $\alpha(z)$ lies in $\mathbb{C} \setminus \mathbb{Z}$, is an element of $\Sigma(\mathbb{R}_{\geq 0})$ so that we can implement the explicit expression (32) of the integration map which yields

$$
\mathcal{I}(\rho(z))(x) = \sum_{j=0}^{N-1} a_j(z) \left( \int_0^1 \chi(y) y^{\alpha(z)-j}(y) \, dy + \frac{x^{\alpha(z)-j+1}}{\alpha(z)-j+1} \right)
$$

(42) $+ \int_0^x \tau_{\chi(z)}(y) \, dy.$

The r.h.s defines a meromorphic germ of polyhomogeneous symbols of order $\alpha(z)+1$ whose poles lie in $\alpha^{-1}([-1, +\infty) \cap \mathbb{Z})$ and thus defines an element $\mathcal{I}(\rho(z))$ in $\Omega$.

A similar procedure using the explicit expression (35) shows the corresponding assertion for $S_{\pm 1}(\rho(z))$.

- The fact that $\mathcal{I}$ is a locality map, namely

$$
\sigma_1 \perp Q \sigma_2 \Rightarrow \mathcal{I}(\sigma_1) \perp Q \mathcal{I}(\sigma_2),
$$

follows from combining Eq. (42) with the first item of Proposition 4.15. To deduce the locality of $S_{\pm 1}$ from that of $\mathcal{I}$, it is useful to observe that the derivatives of any two independent holomorphic germs of symbols are also independent

$$
\rho_1 \perp Q \rho_2 \Rightarrow \partial_x^{(k)} \rho_1 \perp Q \partial_x^{(k)} \rho_2, \quad \forall k \in \mathbb{N}.
$$

It is then easy to deduce from the locality of the map $\mathcal{I}$ and the Euler-Maclaurin formula, that $\mathcal{S}_{\pm 1}$ are locality maps.

- The locality $\lambda$-Rota-Baxter property of the maps $\mathcal{I}$ and $\mathcal{S}_{\pm 1}$ for $\lambda = 0$ and $\lambda = \pm 1$ respectively, is a consequence of the corresponding known usual Rota-Baxter properties.

Since the operators $\mathcal{S}_0 = \mathcal{I}$ and $\mathcal{S}_{\pm 1}$ stabilize $\Omega$, we can compose them on the left with the finite part at infinity investigated in Proposition 4.17.

**Corollary 4.21.** (i) The cut-off integral $\int_0^\infty := \text{fp} \circ \mathcal{I}$ defines a linear map

$$
\int_0^\infty : \Omega \rightarrow \mathcal{M}(\mathbb{C}^\infty)
$$

compatible with the filtration on the source and target space.

(ii) The cut-off sum $\sum_0^\infty := \text{fp} \circ \mathcal{S}_{\pm 1}$ defines a linear map

$$
\sum_0^\infty : \Omega \rightarrow \mathcal{M}(\mathbb{C}^\infty)
$$

compatible with the filtration on the source and target space.
Part 3. Branched zeta values

We bring together the algebraic and analytic aspects of this paper to define and renormalise branched zeta values. These are higher zeta functions which generalise the usual multiple zeta values; a detailed study of convergent branched zeta values was carried out in [M].

5. Branched zeta functions

For the rest of the paper, we will take $K$ to be $\mathbb{C}$.

5.1. Branching the Rota-Baxter operators $\mathfrak{S}_\lambda$. Let $\Omega$ be the algebra of admissible meromorphic germs of symbols defined in Subsection 4.3 equipped with the independence relation $\perp^Q$ for the canonical inner product $Q$, which we will often write it as $\top_\Omega$. With the notations of Definition 1.21, we consider the locality algebra $\mathcal{F}_{\Omega, \top_\Omega}$ of properly $\Omega$-decorated forests, which we refer to as the $(\Omega, \top_\Omega)$—locality algebra of forests properly decorated by meromorphic multivariate germs of symbols. A properly $\Omega$-decorated forest will be denoted by $F := (F, \sigma_F)$.

Let $\lambda \in \{\pm 1\}$. In view of the properties proved in Theorem 4.20 of the maps $\mathfrak{S}_\lambda$ introduced in Definition 3.20 we can apply to them Corollary 1.24 and build the corresponding branched map

\[(43) \quad \widehat{\mathfrak{S}}_\lambda : \mathbb{C}\mathcal{F}_{\Omega, \top_\Omega} \rightarrow \Omega,\]

which is a morphism of locality unital algebras.

**Proposition 5.1.** For $\lambda \in \{\pm 1\}$, the map

\[Z^\lambda : \mathbb{C}\mathcal{F}_{\Omega, \top_\Omega} \rightarrow \mathcal{M}(\mathbb{C}\infty), \quad F := (F, \sigma_F) \mapsto Z^\lambda_F := \text{fp}^+ _{+\infty} (\widehat{\mathfrak{S}}_\lambda(F)),\]

is a morphism of locality algebras.

**Proof.** Combining the facts that $\widehat{\mathfrak{S}}_\lambda$ and the finite part map $\text{fp}^+ _{+\infty}$ are local morphisms of locality algebras (by Corollary 1.24 and Proposition 4.17 respectively) yields that the composition $F \mapsto Z^\lambda_F$ gives a morphism of locality algebras. \qed

The subsequent properties are a straightforward consequence of the fact that $Z^\lambda$ is a morphism of locality algebras and $\widehat{\mathfrak{S}}_\lambda$ a morphism of locality operated algebras.

**Corollary 5.2.** The following identities of meromorphic functions hold:

(i) For mutually independent properly decorated $\Omega$-forests $F_1 \top_\Omega F_2$, we have

\[Z^\lambda_{F_1, F_2} = Z^\lambda_{F_1} \cdot Z^\lambda_{F_2},\]

i.e. $Z^\lambda$ is a locality character.

(ii) For a properly decorated $\Omega$-forest $F = (F, \sigma_F)$ and $\sigma \perp_\Omega \sigma_F(v)$ for all $v \in V(F)$, thus making $B^\sigma_+(F)$ a properly decorated tree, we have

\[Z^\lambda_{B^\sigma_+(F)} = \text{fp}^+ _{+\infty} (\mathfrak{S}_\lambda \left( \sigma \widehat{\mathfrak{S}}_\lambda(F) \right)).\]
Proof. The first property is a direct consequence of the locality morphism property and of Proposition 3.14. The last property follows from the fact that $\tilde{\mathcal{S}}_\lambda$ is a morphism of operated locality algebras for the operation defined in Lemma 1.19:

$$\tilde{\mathcal{S}}_\lambda(B^e_+(F)) = \mathcal{S}_\lambda\left(\sigma \tilde{\mathcal{S}}_\lambda(F)\right).$$

\[ \square \]

5.2. Branched zeta functions. We next set $E = \mathbb{Z}_{\geq 1} \times \mathbb{C}$ endowed with the independence relation $(k, s) \mathrel{\top} (\ell, t) :\iff k \neq \ell$.

Remark 5.3. As it will become clear from the following Lemma, $\mathbb{Z}_{\geq 1}$ serves as a label set on vertices to attach a specific element of $\Omega$ to each vertex, while $\mathbb{C}$ hosts the “weights” of the branched zeta functions yet to be defined.

Let $\chi(x)$ be an excision function which is identically 1 on $[1, \infty)$. 

Definition 5.4. Let $\mathcal{R}_\chi : \mathbb{Z}_{\geq 1} \times \mathbb{C} \to \Omega$ be the map defined by

$$\mathcal{R}_\chi(\ell, s) := (x \mapsto \chi(x)x^{s-z_{\ell}})$$

with $(z_1, z_2, \cdots)$ the canonical coordinates of $\mathbb{C}^\infty$.

On the grounds of Theorem 1.23, the locality map $\mathcal{R}_\chi : (E = \mathbb{Z}_{\geq 1} \times \mathbb{C}, \mathrel{\top} E) \to (\Omega, \mathrel{\top} \Omega)$ lifts to a morphism

$$\mathcal{R}_\chi^2 : \mathcal{C}\mathcal{F}_{E, \mathrel{\top} E} \to \mathcal{C}\mathcal{F}_{\Omega, \mathrel{\top} \Omega}$$

of locality algebras.

Given $\lambda \in \{\pm 1\}$, composing the two morphisms $\mathcal{R}_\chi^2$ and $Z_\lambda$ of locality algebras, we obtain

Proposition-Definition 5.5. The map

$$\zeta_{\chi, \lambda}^{\text{reg}} : \mathcal{F}_{E, \mathrel{\top} E} \to \mathcal{M}(\mathbb{C}^\infty), \quad F \mapsto Z_\lambda^{\mathcal{R}_\chi^2}(F) = (Z_\lambda \circ \mathcal{R}_\chi^2)(F),$$

defines a morphism of locality algebras, which to a properly $\mathbb{Z}_{\geq 1} \times \mathbb{C}$-decorated tree $F = (F, d_F)$, assigns a multivariate meromorphic germ at zero, denoted by $\zeta_{\chi, \lambda}^{\text{reg}}(F)$ and which we call regularised branched zeta functions. Extending the conventions commonly used for multiple zeta functions, when $\lambda = -1$, we drop the upper index writing $\zeta_{\chi, \text{reg}}$ and when $\lambda = 1$ we write $\zeta_{\chi, \text{reg,}*}$.

The subsequent statement follows on inspection of the concrete formula.

Proposition 5.6. The map $\zeta_{\chi, \lambda}^{\text{reg}}$ does not depends on the choice of excision function $\chi$ as long as it is identically 1 on $[1, \infty)$. Therefore we will drop the subindex $\chi$ from now on.

5.3. Renormalised branched zeta values. We choose the inner product on $\mathbb{C}^\infty$ to be the canonical inner product.

Definition 5.7. Let $\lambda \in \{\pm 1\}$ and let $F = (F, \hat{k})$ be a $\mathbb{Z}_{\geq 1} \times \mathbb{C}$-decorated forest. The renormalised branched zeta value (or renormalised BZV) associated to the decorated tree $F$ is defined as

$$\zeta_{\lambda}^{\text{ren}}(F) := \pi_+ \circ \zeta_{\lambda}^{\text{reg}}(F)|_{\hat{z}=0} = \text{ev}_0 \circ \pi_+ \circ \zeta_{\lambda}^{\text{reg}}(F),$$
where \( \text{ev}_0 \) is the evaluation at zero and \( \pi_+ \) is the projection operator defined in [GPZ3] associated to the canonical inner product \( Q \), making the subsequent diagram commutative.

\[
\begin{array}{c}
\mathbb{C} F_{Z \geq 1, \mathbb{T}} \xrightarrow{\zeta^{\text{ren}, \lambda}} \mathbb{C} \\
\downarrow \pi^+ \quad \mathbb{C} F_{\Omega, +} \xrightarrow{\zeta^{\text{reg}, \lambda}} M_+ (\mathbb{C}^\infty) \\
\uparrow \tilde{p}_\lambda \quad \mathbb{C} F_{\Omega, +} \xrightarrow{\tilde{p}} M(\mathbb{C}^\infty)
\end{array}
\]

The following theorem ensures the multiplicativity of the regularised branched zeta functions on mutually independent elements.

**Theorem 5.8.** The map \( \zeta^{\text{ren}, \lambda} : \mathbb{C} F_{Z \geq 1, \mathbb{T}} \rightarrow \mathbb{C} \) is a locality algebra homomorphism.

**Proof.** The map \( \zeta^{\text{ren}, \lambda} : F \mapsto \text{ev}_0 \circ \pi^+ \circ \zeta^{\text{reg}, \lambda}(F) \) is a morphism on the locality algebra \( \mathbb{C} F_{\Omega, +} \) as the composition of locality morphisms of locality algebras, namely \( \text{ev}_0 : M_+ (\mathbb{C}^\infty) \rightarrow \mathbb{C} \) is clearly a locality character, \( F \mapsto \zeta^{\text{reg}, \lambda}(F) : \mathbb{C} F_{\Omega, +} \rightarrow M(\mathbb{C}^\infty) \) is a locality morphism by Proposition 5.5 and \( \pi^+ \) is a locality morphism by [CGPZ1, Example 3.9]. \( \square \)

By a similar argument to the one used in [CGPZ2], we have the following statement.

**Proposition 5.9.** For properly \( \mathbb{Z} \geq 1 \times \mathbb{C} \)-decorated forest \( F \), \( \zeta^{\text{ren}, \lambda}(F) \) does not depend on the \( \mathbb{Z} \geq 1 \) decorations.

### 5.4. Branched zeta-functions in terms of multiple zeta functions.

We now relate our constructions to that of multiple zeta functions carried out in [MP], identifying a proper word with a properly decorated ladder tree.

**Proposition 5.10.** Let \( F = (F, \tilde{k}) \) be a ladder tree \( F \) with \( k \) vertices decorated from bottom to top by \( (\ell, s_\ell) \in \mathbb{Z} \geq 1 \times \mathbb{C} \).

(i) If \( \Re(s_1) > 1 \) and \( \Re(s_i) \geq 1, i = 2, \ldots, k \), then

\[
\zeta^{\text{reg}}(F) = \zeta(s_1 - z_1, \ldots, s_k - z_k) := \sum_{1 \leq n_k < \cdots < n_1} n_k^{-s_k + z_k} \cdots n_1^{-s_1 + z_1},
\]

for the multiple zeta function. Similarly,

\[
\zeta^{\text{reg}, \ast}(F) = \zeta^\ast(s_1 - z_1, \ldots, s_k - z_k) := \sum_{1 \leq n_k \leq \cdots \leq n_1} n_k^{-s_k + z_k} \cdots n_1^{-s_1 + z_1},
\]

for the multiple star zeta function.
In general, \( \zeta^{\text{reg}}(F) \) and \( \zeta^{\text{reg}, \ast}(F) \) coincide with the meromorphic germs
\[
\sum_{1 \leq n_1 < n_{k-1} < \cdots < n_1} n_1^{s_1 - z_1} \cdots n_k^{s_k - z_k}, \quad \text{resp.} \quad \sum_{1 \leq n_k \leq n_{k-1} \leq \cdots \leq n_1} n_1^{s_1 - z_1} \cdots n_k^{s_k - z_k}
\]
considered in [MP, Theorem 9].

Proof. (i) This follows from the fact that \( \hat{P} \) restricted to words coincides with \( \hat{P}^{\text{W}} \) (see (Eq. (23)) applied to the Rota-Baxter operator \( \hat{S}_\lambda \).

(ii) Since the cut-off Chen sums considered in [MP, Theorem 9] were built by means of the composition \( f_\lambda \circ \hat{S}_\lambda \big|_{\mathcal{W}_{\Omega, \tau_\Omega}} \), the statement follows from the fact that
\[
\hat{S}_\lambda \big|_{\mathcal{W}_{\Omega, \tau_\Omega}} = \hat{S}_\lambda \big|_{\mathcal{W}_{\Omega, \tau_\Omega}}.
\]

□

Theorem 5.11. Let \( F = (F, \vec{k}) \in \mathcal{F}_{\mathbb{Z}, \mathbb{T}} \) be any properly \( \mathbb{Z} \geq 1 \times \mathbb{C} \)-decorated forest.

(i) \( \zeta^{\text{ren}, \lambda}(F) \) is a \( \mathbb{Q} \)-linear combination of renormalised (ordinary) multiple zeta functions \( \zeta \) (resp. \( \zeta^{\ast} \)) if \( \lambda = 1 \) (resp. if \( \lambda = -1 \)).

(ii) Provided the inner product \( Q \) is rational, the renormalised branched zeta values \( \zeta^{\text{ren}, \lambda}(F) \) associated to any decorated tree \( F = (F, \vec{k}) \) are rational whenever \( s_v \in \mathbb{Z} \leq -1, \forall v \in V(T) \).

Proof. (i) The first statement follows from combining \( \hat{S}_\lambda = \hat{S}_\lambda^{\text{W}} \circ f_\lambda \) with the fact that ladder trees give rise to multiple zeta functions.

(ii) The second statement follows from the first one combined with the fact that renormalized multiple zeta values are rational ([MP, Theorem 9]). Alternatively, one shows by an induction on the number of vertices of \( F \) that \( \hat{S}_\lambda \) has rational coefficients in the sense of [GPZ3]. In view of the rationality of the inner product \( Q \) the projection map \( \pi^Q_\lambda \) preserves rationality so that \( \zeta^{\text{reg}, \lambda}(F) \) has rational coefficients and thus \( \zeta^{\text{ren}, \lambda}(F) \) is rational.

□
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REFERENCES

[CGPZ1] P. Clavier, L. Guo, S. Paycha, B. Zhang, An algebraic formulation of the locality principle in renormalisation, European Journal of Math. (2018). https://doi.org/10.1007/s40870-018-0255-8, arXiv:1711.00884. 3, 4, 5, 6, 7, 15, 21, 31, 32, 39

[CGPZ2] P. Clavier, L. Guo, S. Paycha, B. Zhang, Locality and renormalisation: universal properties and integrals on trees, in preparation. 4, 6, 12, 14, 39

[CK] A. Connes, D. Kreimer, Hopf algebras, Renormalisation and Noncommutative Geometry, Comm. Math. Phys. 199 (1988) 203-242. 6

[EnM] https://www.encyclopediaofmath.org/index.php/Magma 6

[F] L. Foissy, Les algèbres de Hopf des arbres enracinées décorés I,II, Bull. Sci. Math. 126 (2002) 193-239 and 249-288. 4

[G] L. Guo, An introduction to Rota-Baxter algebra, SMM4, Surveys of Modern Mathematics, 2012. 14, 15

[GPZ1] L. Guo, S. Paycha and B. Zhang, Renormalization and the Euler-Maclaurin formula on cones, Duke Math. J., 166, N.3 (2017) 537–571. 3

[GPZ2] L. Guo, S. Paycha, B. Zhang, Renormalised conical zeta values, to appear in a publication (Springer Verlag) of the Scuola Normale Superiore in Pisa. 3

[GPZ3] L. Guo, S. Paycha, B. Zhang, A conical approach to Laurent expansions for multivariate meromorphic germs with linear poles, arXiv:1501.00426v2 (2017). 4, 22, 29, 39, 40

[Gr] G. Grätzer, Universal Algebra, 2nd ed. Springer, 2008. 6

[H] G. Hardy, Divergent Series, Oxford University Press (1967). 28

[K] D. Kreimer, Chen’s iterated integral represents the operator product expansion, Adv. Theor. Math. Phys. 3 (1999) 627-670. 3

[M] D. Manchon, Arborified multiple zeta values, Proceedings of ”New approaches to Multiple Zeta Values”, ICMAT, Madrid, 2013. ArXiv: 1603.01498 [math.CO]. 3, 4, 6, 37

[MP] D. Manchon, S. Paycha, Nested Sums of Symbols and Renormalized Multiple Zeta Values, Int. Math. Res. Notices (2010) 4628-4697. ArXiv: 0702135v3 [math.NT]. 3, 24, 25, 26, 27, 29, 39, 40

[P1] S. Paycha, Regularised integrals, sums and traces; analytic aspects, American Mathematical Society University Lecture Notes, Vol. 59 (2012). 3, 24, 25, 26

[P2] S. Paycha, Affine transformations on symbols, In Analysis, Geometry and Quantum Field Theory (Ed. C. L. Aldana, M. Braverman, B. Iochum, C. Neira-Jimenez), Contemporary Mathematics 584, Amer. Math. Soc. p. 199–222 (2012). 25

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