Associated Probabilities in Interactive MADM under Discrimination q-Rung Picture Linguistic Environment

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Abstract: In some multi-attribute decision-making (MADM) models studying attributes’ interactive phenomena is very important for the minimizing decision risks. Usually, the Choquet integral type aggregations are considered in such problems. However, the Choquet integral aggregations do not consider all attributes’ interactions; therefore, in many cases, when these interactions are revealed in less degree, they do not perceive these interactions and their utility in MADM problems is less useful. For the decision of this problem, we create the Choquet integral-based new aggregation operators’ family which considers all pair interactions between attributes. The problem under the discrimination q-rung picture linguistic and q-rung orthopair fuzzy environments is considered. Construction of a 2-order additive fuzzy measure (TOAFM) involves pair interaction indices and importance values of attributes of a MADM model. Based on the attributes’ pair interactions for the identification of associated probabilities of a 2-order additive fuzzy measure, the Shapley entropy maximum principle is used. The associated probabilities q-rung picture linguistic weighted averaging (APs-q-RPLWA) and the associated probabilities q-rung picture linguistic weighted geometric (APs-q-RPLWG) aggregation operators are constructed with respect to TOAFM. For an uncertainty pole of experts’ evaluations on attributes regarding the possible alternatives, the associated probabilities of a fuzzy measure are used. The second pole of experts’ evaluations as arguments of the aggregation operators by discrimination q-rung picture linguistic values is presented. Discrimination q-rung picture linguistic evaluations specify the attribute’s dominant, neutral and non-dominant impacts on the selection of concrete alternative from all alternatives. Constructed operators consider the all relatedness between attributes in any consonant attribute structure. Main properties on the rightness of extensions are showed: APs-q-RPLWA and APs-q-RPLWG operators match with q-rung picture linguistic Choquet integral averaging and geometric operators for the lower and upper capacities of order two. The conjugation among the constructed operators is also considered. Connections between the new operators and the compositions of dual triangular norms \((T_p, S_q)\) and \((T_{min}, S_{max})\) are also constructed. Constructed operators are used in evaluation of a selection reliability index (SRI) of candidate service centers in the facility location selection problem, when small degree interactions are observed between attributes. In example MADM, the difference in optimal solutions is observed between the Choquet integral aggregation operators and their new extensions. The difference, however, is due to the need to use indices of all interactions between attributes.

Keywords: associated probabilities of a fuzzy measure; Choquet finite integral; fuzzy MADM; q-rung orthopair fuzzy sets; q-rung picture linguistic sets; fuzzy discriminations; selection reliability index

1. Introduction

1.1. MADM under General Orthopair Fuzzy Environments

Today we often touch with multi-attribute (criteria) decision-making (MADM/MCDM) models, methods or software to solve practical important complex problems in medicine, economics, technologies, business and others. Our work deals with MADM models under expert knowledge representations on possible alternatives. Here is some general information on MADM. MADM/MCDM is occupied by structuring and solution of the problems of...
making decisions and planning, that include several attributes/criteria. The target consists of delivering a support to DMs, which encounter such challenges. As a rule, a unique optimal solution for these problems does not exist, and often we need to use the preferences of those, who make decisions, for differentiation of the solutions. The “solution” can be understood differently. It may mean the selection of the “best” alternative from the collection of existing alternatives (where “best” can be interpreted as “most preferable alternative” for DM). Another meaning of the “solution” can be the selection of the small collection of good alternatives or grouping them in different sets of preferences. Marginal interpretation can consist in the search for all “effective” or so-called “nondominated” alternatives.

The complexity of problem is caused by the presence of several attributes/criteria. There is no more unique optimal solution of MADM problem, which can be obtained without having information about the preferences. The concept of optimal solution is frequently substituted by the collection of nondominated solutions. A solution is called nondominated, if it cannot be improved in any criterion, without certain loss in other. Therefore, for the person, who makes decision, it makes sense to select the solution from the nondominated collection. Moreover, DM can improve for some or all attributes, and not to deteriorate any others of them. However, as a rule, the collection of nondominated solutions is too big in order to be presented to DM for final selection. Consequently, the tools, which will help DMs to be concentrated during the preferential solutions (alternatives), are necessary. In this work, we construct this instrument for the orthopair fuzzy environment.

The intuitionistic fuzzy sets (IFS) theory by Atanassov [1] represents a new extension of Zadeh’s fuzzy sets (FS) theory [2]. Because for each element of IFS, an Intuitionistic fuzzy number (IFN) \( (\mu, \upsilon) \) is assigned a membership degree \( \mu \), a non-membership degree \( \upsilon \) and a hesitancy degree \( (1 - \mu - \upsilon) \), IFS is much capable of dealing with vagueness than FS. IFS theory was extensively utilized in various problems of different areas [3,4]. Definitions of main arithmetic operations on IFN are given in [1,3]. The IFS theory is found to be intensively applicable in decision-making direction of research. After consideration of the huge amount of existing materials, the authors of [5] presented a scientometric review on IFS studies. At the same time, the IFN \( (\mu, \upsilon) \) has a serious constraint—the sum of membership and non-membership degrees must be or less than 1. Nevertheless, it may happen that a DM provides such data for certain attribute that the aforementioned sum is greater than 1 (\( \mu + \upsilon > 1 \)). To cope with such a case, Yager [6,7] introduced the concept of the Pythagorean fuzzy set (PFS) as a generalization of IFS, where a Pythagorean fuzzy number (PFN) \( (\mu, \upsilon) \) has a weaker constraint—the sum of squared degrees of membership and non-membership satisfies the inequality \( \mu^2 + \upsilon^2 \leq 1 \). However, in many expert orthopair assessments, neither PFNs nor IFNs can describe fully intellectual activity, because the assessment psychology of a DM is too intricate for hard decision-making, and the attribute’s information is still problematic to express with PFNs or IFNs. This problem was solved by Yager, again [8,9]. He introduced the notion of a q-rung orthopair fuzzy set (q-ROFS), where \( q \geq 1 \), and the sum of the qth power of the degrees of membership and non-membership cannot exceed 1. For a q-rung orthopair fuzzy number (q-ROFN) we have \( (\mu^q + \upsilon^q \leq 1) \). The fundamentals of arithmetic operations on such numbers are presented in [8,10]. Obviously, the q-ROFSs are generalization of IFSs and PFSs. The IFSs and PFSs represent the particular cases of the q-ROFSs for \( q = 1 \) and \( q = 2 \). Thus, q-ROFNs appear to be more suitable and capable for expressing DM’s assessment information.

In real-world decision-making problems, when expert evaluations are given in orthopairs, we face the following two difficulties. The first difficulty: While IFSs, PFSs, and q-ROFSs can be effectively utilized in decision-making models, there still remain situations when we simply cannot use them in decision-making aggregations. For example, human voters consist of groups who: vote for, abstain, refusal of in a voting. Here we are faced with answers: yes, abstain, no and refusal. Obviously, IFSs, PFSs and q-ROFSs are not applicable in this case. Not long ago, Cuong [11] offered a concept of Picture Fuzzy Sets (PIFS), typified by a positive membership degree, a neutral membership degree and a negative membership degree. Later, PIFSs attracted a lot of attention from both researchers
and practitioners [12–19]. Thus, many authors, using the concepts of q-ROFS and PIFS, have developed a new aggregation concept in the q-rung picture fuzzy set (q-RPFS) environment. The new concept itself utilizes in aggregations the benefits of both q-ROFS and PIFS environments. The offered q-RPFS, together with expressing the degree of neutral membership, can also weaken the restriction on PIFS concerning the abovementioned sum. The restriction on q-RPFS is that the sum of qth powers of the degrees of positive, neutral and negative memberships must not exceed 1. Therefore, due to its content the q-RPFS-concept reinforces the q-ROFS-concept [9] by introducing a degree of neutral membership. For example, if a DM delivers the degrees of positive, neutral and negative memberships as 0.7, 0.2 and 0.5, then the ordered trinity (0.7, 0.2, 0.5) is not valid for q-ROFSs or PFSs, while it is in force for the offered q-RPFSs. Thus, RPFs can describe the uncertainty modeling more deeply than q-ROFSs or PFSs can. The second issue is that, for various reasons DMs prefer qualitative estimates over possible alternatives rather than quantitative ones, which naturally better corresponds to their intellectual activity. As is well-known, Zadeh’s linguistic variables [20] have these features to perform high quality modeling, although as Wang and Li [21] noted, linguistic variables can only describe the qualitative side of DMs’ intellectual activity. They cannot describe the concepts of attribution or non-attribution of an assessment element. The concepts of intuitionistic, interval-valued Pythagorean fuzzy and picture fuzzy linguistic sets are proposed in [21–23]. Afterwards, a new tool, q-rung picture linguistic set (q-RPLS) [24] is proposed for dealing with impreciseness in MADM. The new concept of q-RPLS fully describes both q-ROFS and PIFS concepts and, thus, reflects the qualitative and quantitative assessments of DMs. Subsequently, the study of aggregation operators using q-RPLS information [24] in interactive/correlated attributes decision-making models began to develop.

1.2. Literature Review of Aggregation Operators with Interaction Arguments

Below, we briefly describe the concept of q-rung orthopair fuzzy, q-rung picture fuzzy and q-rung picture linguistic sets aggregation operators with interaction phenomena among the attributes. The joint capabilities of the Choquet integral aggregations and OWA operator are given in [25] under an orthopair fuzzy environment. In this paper, the joint concept of possibility and plausibility for the general orthopair environment is also presented. The new concept has a great importance in the attributes’ interaction phenomena. Generalized Heronian mean operators under Pythagorean fuzzy environments are studied in [26]. The generalized and geometric Heronian mean operators under the q-rung orthopair fuzzy sets is studied in [27]. These aggregations provide pair interdependence between attributes’ values in MADM. Heronian mean and generalized Heronian mean operators under picture fuzzy environments are investigated in [28]. The q-RPL weighted Heronian averaging and geometric Heronian operators are investigated in [24]. In [29], the q-ROF Bonferroni averaging (BA), the q-ROF weighted BA, the q-rung orthopair fuzzy geometric BA operators [29] are studied. The Maclaurin symmetric averaging operator based on q-ROFNs is introduced in [30]. The Minkowski-type distance measures, including Hamming, Euclidean, and Chebyshev distances, for q-rung orthopair fuzzy sets are introduced in [31]. In [32], q-rung orthopair fuzzy Einstein weighted operators are presented for the description of a robust MADM technique of solving real-world problems.

1.3. Problems of Interactive Phenomena in MADM under General Orthopair Fuzzy Environments

Nowadays, modeling real complex problems with MADM (with different fuzzy environments) is connected with the development of researches which involve the interaction or dependence of attributes. Besides, expert assessments due to inevitable incomplete information should be considered as a bipolar environment based on uncertainty and imprecision. Modeling this phenomenon is one of the top priorities of qualified research. In these cases, researchers use non-additive integrals and monotone measures for constructing aggregation operators where non-additive measures describe indexes of an uncertainty, and integrable functions (compatibility functions) present an imprecision of expert’s eval-
Sugeno [33] introduced the definition of a non-additive (fuzzy) measure that instead of additivity requires only monotonicity. As it appeared, this is very efficient tool for interaction phenomena modeling [34,35] and solving decision-making problems, where a fuzzy measure used instead of probabilities. In [36], the authors reviewed a decision-maker’s behavioral patterns for such MADM problems. In such decision-making models, an identification of the monotone measure from existing expert assessments is an important problem. There are several approaches to solution of this problem, for example: linear [34] and quadratic methods [36], Sugeno’s $\lambda$ -additive measures-based method [37] and others. It is noteworthy to mention the identification of monotonous measure by optimization methods related to the Choquet integral [38]. A 2-order additive fuzzy measure identification problem is given in [39]. For a review of current works on the identification of monotonous measure, see [40]. A fuzzy non-additive integral, namely Choquet finite integral [41], is more for developing and extending instrument which reflects interaction phenomena between attributes by aggregation of fuzzy measure’s values and experts’ assessments in a scalar value.

According to our problems, the extensions of the Choquet averaging (CA) and geometric (CG) operators for different fuzzy environments become important. Therefore, we consider fuzzy-probability extensions of the Choquet integral [42] under intuitionistic [1], Pythagorean [6] and q-rung orthopair [8,9] fuzzy information. Tan and Chen [43] developed the Choquet integral based algorithms for multi-criteria decision-making models in the intuitionistic environment. Xu, in [44], used the CA operator for construction of some intuitionistic fuzzy aggregation operators. In [45], it is presented for CA operator’s extension—IF multiplicative CA operator, which describes the interdependences between the multiplicative preference information in the decision-making process. Wu et al. [46] studied the properties of the CA operator with intuitionistic fuzzy values. Peng and Yang [47] introduced the CG operator under pair-wise type Pythagorean fuzzy environment. The work [48] presents the Pythagorean fuzzy Choquet integral extensions. Based on Yager’s negation function a q-rung orthopair Choquet integral is developed in [49]. In general, the aggregation operators based on the Choquet integral consider attributes’ and their ordered position, they reflect the interactions among some consonant body of attributes [50].

1.4. Motivation of Research and Problem’s Decision Methodology

Hundreds of studies have been conducted on models of MADM that indirectly consider the interaction between attributes. They are atypical Choquet aggregations and do not use non-crisp measure. Such studies do not directly show how the interaction indices affect the selection of the optimal solution, since they are not explicitly included in the aggregations. Their use depends only on the specific experiment. They do not present fundamental research on the implantation of interaction indexes in aggregation operators, as this study presents here.

Many authors dedicated their study to the Choquet integral type extended aggregate fuzzy operators for applying them in decision-making models with interactive attributes. However, when the interactions between the attributes are exposed to a high degree there arise a significant problem with the reliability of the use of these aggregation operators. The CA operator does not reflect the overall interactions among all of attributes [51]. When the interaction between the attributes is revealed to be of high degree, then this defect becomes essential in the decision-making process. Therefore, we set a goal to solve this problem by including and considering all interactions in the aggregations. This, in turn, naturally increased the creditability of decision-making for the above circumstances. Clearly, a general solution to this problem is impossible. So, we tried to formulate this problem for specific aggregation extensions with pair interactions. Thus, we aimed at creating new operators based on the Probability Weighted Averaging (Geometric) operators [52] for the q-ROF and q-rung picture linguistic environments, when instead of probabilities we deal with associated probabilities (APs) of a fuzzy measure. Developed operators consider the overall interactions between all attributes.
The main objective of this work concerns the embedding of associated probabilities of a fuzzy measure in aggregation operators, in particular, in the Choquet type aggregations (or in aggregations where a fuzzy measure is used instead of a probabilistic measure); the connection of these embeddings to taking into account all pairwise interactions between attributes. The novelty is that the Choquet type aggregations do not consider all pair interactions; therefore in many cases, when these interactions are revealed in less degree, they do not perceive these interactions and their utility in MADM problems is less useful than the new aggregation operators’ family developed in the article. All of this is illustrated in the example of this article, which assesses the service center’s selection reliability index (SRI). These questions are discussed in Sections 3–6, where the originality of the research results and new approaches are presented.

Sometimes, in MADM problems (Section 6) the interactions appear among possible attributes. In such cases, it is expedient to use discrimination values [53] in decision-making matrix, which also reflect this interaction. Discrimination numbers (parameters) are certain transformations from experts’ evaluations. Discrimination q-rung picture linguistic values are built which indicate on the attribute’s dominant, neutral and non-dominant impact on the selection of each alternative regarding other alternatives (Section 5).

In Section 2, we shortly review some extensions of probability weighted averaging and geometric aggregation operators under the q-rung picture linguistic environment. We also present main properties of associated probabilities of a fuzzy measure and its relation to the finite Choquet integral. The variants of definition of the finite Choquet integral are presented for the q-rung picture linguistic arguments. In Section 3, we present associated probabilities q-rung picture linguistic weighted averaging (APs-q-RPLWA) and the associated probabilities q-rung picture linguistic weighted geometric (APs-q-RPLWG) aggregation operators’ minimum and maximum variants. Some important properties of the new operators are proved and propositions on the rightness of extensions are shown. The conjugate connections between built operators are also considered. In Section 4, the identification problem of APs by the interactions of attributes and attributes’ importance values is considered. In Section 5, new MADM problem with some characteristics of fuzzy discrimination measures in q-RPLNs is introduced. For illustration of the obtained results, in Section 6, a facility location selection problem under q-rung picture linguistic environment is considered. Candidate service site’s selection reliability index is specified against membership of z in C

2. Preliminaries

Definition 1 ([9]). Let Z be some ordinary set. q-rung orthopair fuzzy set C on Z is defined as a set of ordered pairs:

\[ C = \{ (\mu_C(z), \nu_C(z)) \mid z \in Z \}, \]

where the function \( \mu_C(z) \) indicates support for membership of z in C and \( \nu_C(z) \) indicates support against membership of z in C, where

\[ q \geq 1, 0 \leq \mu_C(z) \leq 1, 0 \leq \nu_C(z) \leq 1, 0 \leq (\mu_A(z))^q + (\nu_A(z))^q \leq 1. \]

\[ Hes_q(z) = (1 - ((\mu_A(z))^q + (\nu_A(z))^q))^{1/q} \]

is called a hesitancy associated with a q-rung orthopair membership grades and

\[ Str_q(z) = ((\mu_C(z))^q + (\nu_C(z))^q))^{1/q} \]

is called a strength of commitment viewed at rung q.

Yager introduced a family of logical negations—\( Neg_q(c) = (1 - c^q)^{1/q} \), \( q > 0 \) [44], which is applied for the definition of q-ROFS. It was shown [9] that Atanassov’s intuitionistic fuzzy sets [1] are \( q = 1 \)-rung orthopair and Yager’s Pythagorean fuzzy sets [6] are \( q = 2 \)-rung orthopair fuzzy
sets. For every \( z \in Z \), \( \alpha = (z, \mu_a(z), v_a(z)) \) is called a q-rung orthopair fuzzy number (q-ROFN) denoted by \( a = (\mu_a, v_a) \).

We define on the q-ROFNs the following basic operations:

**Definition 2** ([9]). Let \( a = (\mu_a, v_a) \) and \( a_1, a_2 \) be some q-ROFNs. Then:

1. \( a^c = (v_a, \mu_a) \);
2. \( a_1 \oplus_q a_2 = \left( \left( \frac{\mu_{a_1}^q + \mu_{a_2}^q - \mu_{a_1} \cdot \mu_{a_2}}{\mu_{a_1} \cdot \mu_{a_2}} \right)^{1/q}, v_{a_1} \cdot v_{a_2} \right) \);
3. \( a_1 \odot_q a_2 = \left( \frac{v_{a_1}^q + v_{a_2}^q - v_{a_1} \cdot v_{a_2}}{v_{a_1} \cdot v_{a_2}} \right)^{1/q} \);
4. \( \text{Min}(\alpha_1, \alpha_2) = (\text{Max}(\mu_{\alpha_1}, \mu_{\alpha_2}), \text{Min}(v_{\alpha_1}, v_{\alpha_2})) \);
5. \( \text{Max}(\alpha_1, \alpha_2) = (\text{Min}(\mu_{\alpha_1}, \mu_{\alpha_2}), \text{Max}(v_{\alpha_1}, v_{\alpha_2})) \);
6. \( \lambda \cdot a = \left( (1 - (1 - \mu_a^q)^{1/q}), v_a^\lambda \right), \lambda > 0 \);
7. \( a^\lambda = \left( \mu_a^{\lambda q}, (1 - (1 - v_a^q)^{1/q}) \right), \lambda > 0 \).

**Definition 3** ([9]). Let \( a = (\mu_a, v_a) \) be a q-ROFN.

(a) A score function \( Sc \) of \( a \) is defined as

\[
Sc(\alpha) = (\mu_a)^q - (v_a)^q;
\]

(b) An accuracy function \( Ac \) of \( a \) is defined as follows:

\[
Ac(\alpha) = (\mu_a)^q + (v_a)^q.
\]

Total order relation \( \leq_t \) on the q-ROFNs is:

**Definition 4** ([9]). Let \( a = (\mu_a, v_a) \) and \( \beta = (\mu_\beta, v_\beta) \) be any two q-ROFNs and \( Sc(\alpha), Sc(\beta) \) be values of the score function and \( Ac(\alpha), Ac(\beta) \) be values of the accuracy function on \( a \) and \( \beta \), respectively, then

(a) If \( Sc(\alpha) > Sc(\beta) \), then \( \beta <_t a \);
(b) If \( Sc(\alpha) = Sc(\beta) \), then

- If \( Ac(\alpha) > Ac(\beta) \), then \( \beta <_t a \);
- If \( Ac(\alpha) = Ac(\beta) \), then \( \beta =_t a \).

The picture intuitionistic fuzzy set (PIFS), constructed by a positive membership degree, a neutral membership degree as well as a negative membership degree, was originally proposed by Cuong [11].

**Definition 5** ([11]). A picture intuitionistic fuzzy set (PIFS) \( C \) defined on \( Z \) is a set of ordered triplets

\[
C = \{ <\mu_C(z), \gamma_C(z), v_C(z)> \mid z \in Z \},
\]

where \( \mu_C(z) \) is called the degree of positive membership of \( C \), \( \gamma_C(z) \) is called the degree of neutral membership of \( C \) and \( v_C(z) \) is called the degree of negative membership of \( C \), with the following condition:

\[
0 \leq \mu_C(z) \leq 1, \quad 0 \leq \gamma_C(z) \leq 1, \quad 0 \leq v_C(z) \leq 1, \quad 0 \leq \mu_C(z) + \gamma_C(z) + v_C(z) \leq 1, \quad \forall z \in Z.
\]

For \( \forall z \in Z \), \( \pi_C(z) = 1 - (\mu_C(z) + (\gamma_C(z) + v_C(z))) \) is called the degree of refusal membership of \( z \) in \( C \).

Based on the concepts of q-ROFS and PIFS, the definition of q-RPFS is given in [24].
Definition 6 ([24]). Let $X$ be an ordinary set, a $q$-rung picture fuzzy set (q-RPFS) $C$ defined on $Z$ is a set of ordered triplets

$$C = \{<\mu_C(z), \gamma_C(z), \nu_C(z)>/z \in Z\},$$

where $\mu_C(z)$ is called a degree of positive membership of $C$, $\gamma_C(z)$ is called a degree of neutral membership of $C$ and $\nu_C(z)$ is called a degree of negative membership of $C$, with the following conditions:

$$0 \leq \mu_C(z) \leq 1, 0 \leq \gamma_C(z) \leq 1, 0 \leq \nu_C(z) \leq 1,$$

$$\forall z \in Z,$$

where $\alpha = (\mu_C(z), \gamma_C(z), \nu_C(z))$.

For $\forall z \in Z$, $\pi(z) = (1 - (\mu_C(z))^q + (\gamma_C(z))^q + (\nu_C(z))^q)^{1/q}$ is called a degree of refusal membership of $z$ in $C$.

Now we present some linguistic concepts from [24]: Let $\hat{L} = \{l_i, i = 1, \ldots, t\}$ be a linguistic term set with odd cardinality and $t$ is the cardinality of $\hat{L}$. The label $l_i$ represents a possible value for a linguistic variable. Following for our future reasoning, a linguistic term set is defined as follows:

$$\hat{L} = \{l_1, l_2, \ldots, l_7\} = \{\text{very poor, poor, slightly poor, fair, slightly good, good, very good}\}.$$

Based on the concept of picture linguistic set [23], authors in [24] presented the definition of q-RPLS by combining the linguistic term set with q-RPFS.

Definition 7 ([24]). Let $Z$ be an ordinary set; $\hat{L}$ be a continuous linguistic term set of $\hat{L} = \{l_\rho, \rho = 1, \ldots, t\}$. Then a $q$-rung picture linguistic set (q-RPLS) $C$ defined on $Z$ is a set of ordered quadruples

$$C = \{<l_\rho(z), \mu_C(z), \gamma_C(z), \nu_C(z)>/z \in Z\},$$

where $l_\rho(z) \in \hat{L}$, $\mu_C(z)$ is called a degree of positive membership of $C$, $\gamma_C(z)$ is called a degree of neutral membership of $C$, and $\nu_C(z)$ is called a degree of negative membership of $C$, with the following conditions:

$$0 \leq \mu_C(z) \leq 1, 0 \leq \gamma_C(z) \leq 1, 0 \leq \nu_C(z) \leq 1,$$

$$\forall z \in Z,$$

then $<l_\rho(z), \mu_C(z), \gamma_C(z), \nu_C(z)>$ is called a q-RPLS, which can be simply denoted by $a = (l_\rho, \mu_a, \gamma_a, \nu_a)$. When $q = 1$, then $C$ is reduced to the PSLS proposed by Liu and Zhang [23].

We define on the q-RPLNs the following basic operations:

Definition 8 ([24]). Let $a = (l_\rho, \mu_a, \gamma_a, \nu_a)$, $a_1 = (l_{\rho_1}, \mu_{a_1}, \gamma_{a_1}, \nu_{a_1})$, $a_2 = (l_{\rho_2}, \mu_{a_2}, \gamma_{a_2}, \nu_{a_2})$ be some q-RPLNs. Then

1. $a^c = (l_{\rho-c}, \mu_a, \gamma_a, \nu_a);$  
2. $a_1 \oplus_q a_2 = (l_{\rho_1+\rho_2}, (\mu_{a_1} + \mu_{a_2} - \mu_{a_1} \cdot \mu_{a_2})^{1/q}, \gamma_{a_1} \cdot \gamma_{a_2}, \nu_{a_1} \cdot \nu_{a_2});$
3. $a_1 \ominus_q a_2 = (l_{\rho_1}, (\mu_{a_1}, \gamma_{a_1}, \nu_{a_1} - (\nu_{a_1} + \nu_{a_2} - \nu_{a_1} \cdot \nu_{a_2})^{1/q});$
4. $\min(a_1, a_2) = (\min(l_{\rho_1}, l_{\rho_2}), \min(\mu_{a_1}, \mu_{a_2}), \min(\gamma_{a_1}, \gamma_{a_2}), \max(\nu_{a_1}, \nu_{a_2}));$
5. $\max(a_1, a_2) = (\max(l_{\rho_1}, l_{\rho_2}), \max(\mu_{a_1}, \mu_{a_2}), \min(\gamma_{a_1}, \gamma_{a_2}), \min(\nu_{a_1}, \nu_{a_2}));$
6. $\lambda \cdot a = (l_{\lambda \cdot \rho}, (1 - (1 - \mu_{a})^\lambda)^{1/q}, \gamma_{a}^\lambda, \nu_{a}^\lambda), \lambda > 0;$
7. $a^\lambda = (l_{\rho}, (1 - (1 - \mu_{a})^\lambda)^{1/q}, (1 - (1 - \nu_{a})^\lambda)^{1/q}), \lambda > 0.$

Let $L = \{[a, b] = (a, b) \in [0, 1]^2, a \leq b\}$ be a lattice of non-empty intervals. Usually, we define the partial order relation $\leq_L$ as $[a, b] \leq_L [c, d] \iff a \leq c$ and $b \leq d$. The top and bottom
elements are \( I_L = [1; 1] \) and \( 0_L = [0; 0] \), respectively. Now we extend this definition on the lattice of all \( q \)-RPLNs—\( L_{q-RPLNs} \):

\[
(l_{p1}, \mu_1, \gamma_1, v_1) \leq l_{q-RPLNs} (l_{p2}, \mu_2, \gamma_2, v_2) \iff p_1 \leq p_2, \mu_1 \leq \mu_2, \gamma_1 \leq \gamma_2 \text{ and } v_1 \geq v_2
\]

The top and bottom elements are \( 1_{q-RPLNs} = (l_7, 1, 1, 0) \) and \( 0_{q-RPLNs} = (l_1, 0, 0, 1) \), respectively.

Score and accuracy functions of a \( q \)-RPLN are defined as:

**Definition 9 ([24]).** (a). Let \( \alpha = (l_p, \mu_a, \gamma_a, v_a) \) be a \( q \)-RPLN, then the score function of \( \alpha \) is defined as

\[
Sc(\alpha) = (\mu^d - v^d + 1) \cdot \rho,
\]

(b). Let \( \alpha = (l_p, \mu_a, \gamma_a, v_a) \) be a \( q \)-RPLN, then the accuracy function of \( \alpha \) is defined as

\[
Ac(\alpha) = (\mu^d + v^d + \gamma^d) \cdot \rho.
\]

Based on the Definition 9 a total order relation \( \preceq \) on the \( q \)-RPLNs is defined as follows.

**Definition 10 ([24]).** Let \( \alpha = (l_p, \mu_a, \gamma_a, v_a) \) and \( \chi = (l_p, \mu_\chi, \gamma_\chi, v_\chi) \) be any two \( q \)-RPLNs and \( Sc(\alpha), Sc(\chi) \) be values of the score function, and \( Ac(\alpha), Ac(\chi) \) be values of the accuracy function on \( \alpha \) and \( \chi \), respectively, then

(a) If \( Sc(\alpha) > Sc(\chi) \), then \( \chi <_t \alpha \);

(b) If \( Sc(\alpha) = Sc(\chi) \), then

If \( Ac(\alpha) > Ac(\chi) \), then \( \chi <_t \alpha \);

If \( Ac(\alpha) = Ac(\chi) \), then \( \chi =_t \alpha \).

**Definition 11.** A function \( G : L^n_{q-RPLNs} \to L_{q-RPLNs} \) is a \( q \)-RPL operator if it is monotone with respect to partial order relation \( \leq_{q-RPLNs} \) : if \( \alpha_i \leq_{q-RPLNs} \beta_i \in 1_{q-RPLNs}, i = 1, \ldots, n \), then \( G(\alpha_1, \ldots, \alpha_n) \leq_{q-RPLNs} G(\beta_1, \ldots, \beta_n) \) and \( G(0_{q-RPLNs}, \ldots, 0_{q-RPLNs}) = 0_{q-RPLNs} \).

**Definition 12.** Let \( G \) be any \( q \)-RPL operator. The Aggregation operator \( G^q_{\text{Dual}} : L^n_{q-RPLNs} \to L_{q-RPLNs} \) is called dual of \( G \) if it \( G^q_{\text{Dual}}(a_1, \ldots, a_n) = \text{Neg}_{\text{q}}(G(\text{Neg}_{\text{q}}(a_1), \ldots, \text{Neg}_{\text{q}}(a_1))) \). The probability averaging or geometric operators, or their combinations with associative weights of attributes usually are used in the stochastic decision-making environments. Merigó and others [52,54,55] have developed a P-WA and P-WG operators.

**Definition 13 ([54]).** Let \( X = \{x_1, \ldots, x_n\} \) be some finite space of elementary events and be some random variable—\( c : X \Rightarrow R, (c = (c_1, \ldots, c_n); c_i = c(x_i), i = 1, \ldots, n) \). The probability weighted averaging (P-WA) operator with respect to probability distribution \( p = \{ p_1 \}_{i=1}^{n} \), \( p_i \equiv \text{prob}(c_i), 0 < p_i < 1, \sum_{i=1}^{n} p_i = 1 \) and the aggregation weighted vector \( w = (w_1, \ldots, w_n) \) such that, \( 0 \leq w_j \leq 1, \sum_{i=1}^{n} w_j = 1 \) is called

\[
P - WA(c_1, \ldots, c_n) = \sum_{i=1}^{n} \frac{p_i c_i}{w_i}
\]

where \( \frac{p_i}{w_i} = \beta p_i + (1 - \beta) w_i \) with weighted parameter \( \beta \in [0; 1] \).

**Definition 14 ([54]).** Let \( X = \{x_1, \ldots, x_n\} \) be an elementary events’ space and \( c \) be a random variable—\( c : X \Rightarrow R, (c = (c_1, \ldots, c_n); c_i = c(x_i), i = 1, \ldots, n) \). The probability weighted
geometric (P-WG) operator with respect to probability distribution \( P = \{ p_i \}_{i=1}^n \), \( p_i \equiv \text{prob}(c_i) \), \( 0 < p_i < 1 \), \( \sum_{i=1}^n p_i = 1 \), and the aggregation weighted vector \( w = (w_1, \ldots, w_n) \) such that, \( 0 \leq w_j \leq 1 \), \( \sum_{j=1}^n w_j = 1 \), is called

\[
P - \text{WG}(c_1, \ldots, c_n) = \prod_{i=1}^n p_i^w \quad (8)
\]

where \( p_i = \beta p_i + (1 - \beta)w_i \) with weighted parameter \( \beta \in [0; 1] \).

Definitions of P-WA and P-WG operators for intuitionistic environment are given in [56], and the Pythagorean P-WA and P-WG operators are defined in [57]. The extensions of these aggregation operators for the q-ROF environment are presented in [58]. Now we begin with definition of weighted q-rung picture linguistic extensions of these operators:

**Definition 15.** Let \( \chi_1, \ldots, \chi_n, \chi^i = (\mu^i, \nu^i, \gamma^i, \mu^i \chi^i, \nu^i \chi^i) \) be values of a random variable \( \chi \) defined on \( X \).

(a) The P-q-RPLWA operator is called

\[
P - q - \text{RPLWA}(\chi_1, \ldots, \chi_n) = \bigotimes_{i=1}^n [\chi_i]^{p_i} = \left( 1 - \prod_{i=1}^n 1 - (1 - (\gamma_{\chi_i})^p) \right)^{1/q} \prod_{i=1}^n (\gamma_{\chi_i})^p \prod_{i=1}^n (\nu_{\chi_i})^p \right) . \quad (9)
\]

(b) The P-q-RPLWG operator is called

\[
P - q - \text{RPLWG}(\chi_1, \ldots, \chi_n) = \bigotimes_{i=1}^n [\chi_i]^{p_i} = \left( 1 - \prod_{i=1}^n 1 - (1 - (\gamma_{\chi_i})^p) \right)^{1/q} \prod_{i=1}^n (\gamma_{\chi_i})^p \prod_{i=1}^n (\nu_{\chi_i})^p \right) . \quad (10)
\]

where \( p_i = \beta p_i + (1 - \beta)w_i \) with weighted parameter \( \beta \in [0; 1] \);

\( P = \{ p_i \}_{i=1}^n \), \( p_i \equiv \text{prob}(\chi^i) \), is a probability distribution \( 0 < p_i < 1 \), \( \sum_{i=1}^n p_i = 1 \),

and \( w = (w_1, \ldots, w_n) \) is a weighted vector of \( \chi = (\chi_1, \ldots, \chi_n) \) \( 0 \leq w_j \leq 1 \), \( \sum_{j=1}^n w_j = 1 \).

Now we go to the study of Choquet integral [59] aggregations’ extensions under q-rung orthopair and q-rung picture linguistic information.

A fuzzy measure and its associated probabilities [60] definitions (Definitions A1 and A2), the Choquet integral aggregations’ definitions (Definitions A3 and A4), extremal capacities’ presentations by the associated probabilities (Proposition A1) are given in Appendix A. Yager’s extension of the Choquet integral aggregations under q-rung orthopair fuzzy environment (Definition A5) also is presented in Appendix A.

In contrast to the Yager’s extensions of the Choquet integral aggregations we define the Choquet integral averaging operator on q-ROFNs analogously to Definition 15.

**Definition 16 ([58]).** Let \( X = \{ x_1, x_2, \ldots, x_n \} \) be a set of attributes; \( g \) be a fuzzy measure on \( X \) and \( \chi : X \Rightarrow q - \text{RPLNs} \) be a q-rung orthopair fuzzy variable of expert evaluations, such that \( \chi(x_i) = (\mu_{x_i}, \nu_{x_i}) \in q - \text{ROFNs}, \; i = 1, 2, \ldots, n \). Then

\[
q - \text{ROFCAG}(\chi_1, \ldots, \chi_n) = \bigotimes_{i=1}^n \chi_{\chi_i}^{p_i} = \left( 1 - \prod_{i=1}^n 1 - (1 - (\chi_{x_i})^p)^1/q \right) \prod_{i=1}^n (\chi_{x_i})^p \right) \quad (11)
\]

is called a q-ROFCA operator \((q \geq 1)\) with respect to a fuzzy measure \( g \) and

\[
q - \text{ROFCG}(\chi_1, \ldots, \chi_n) = \bigotimes_{i=1}^n \chi_{\chi_i}^{p_i} = \left( \prod_{i=1}^n (\chi_{x_i})^p, \left( 1 - \prod_{i=1}^n 1 - (1 - (\chi_{x_i})^p)^1/q \right) \right) \quad (12)
\]
is called a q-ROFCG operator \((q \geq 1)\) with respect to the fuzzy measure \(g\), where

\[
p_s = g\left(\left\{ x_{i(1)}, \ldots, x_{i(s)} \right\}\right) - g\left(\left\{ x_{i(1)}, \ldots, x_{i(s-1)} \right\}\right), \quad g\left(\left\{ x_{i(0)} \right\}\right) \equiv 0,
\]

and \(x_{i(s)}\) is the \(s\)-th largest of \(x_k\), \(k = 1, 2, \ldots, n\), according to the relation \(\geq_t\).

Analogously to the previous definition we extend Choquet integral averaging operator on q-RPLNs.

**Definition 17.** Let \(X = \{x_1, x_2, \ldots, x_n\}\) be a set of attributes, \(g\) be a fuzzy measure on \(X\) and \(\chi: X \Rightarrow q - \text{RPLNs}\) be a q-rung picture linguistic variable of expert evaluations, such that \(\chi(x_i) = (l_{pi}, \mu_{pi}, \nu_{pi}, v_{pi}) \in q - \text{RPLNs}\), \(i = 1, 2, \ldots, n\). Then

\[
q - \text{RPLCA}_g(\chi_1, \ldots, \chi_n) = \bigoplus_{s=1}^{n} g(\chi_{i(s)})p_s = \\
\left( \frac{1}{n} \sum_{s=1}^{n} p_{i(s)} \right) - \left( 1 - \prod_{s=1}^{n} (1 - (\mu_{pi(s)})^{q} p)^{1/q} \right) \prod_{s=1}^{n} (\nu_{pi(s)})^{p_s} \prod_{s=1}^{n} (v_{pi(s)})^{p_s}
\]

is called a q-RPLCA operator \((q \geq 1)\) with respect the fuzzy measure \(g\) and

\[
q - \text{RPLCG}_g(\chi_1, \ldots, \chi_n) = \bigotimes_{s=1}^{n} g(\chi_{i(s)})p_s = \\
\left( \frac{1}{n} \prod_{i=1}^{n} (p_{i(s)})^{p_s} \right) - \left( 1 - \prod_{s=1}^{n} (1 - (\mu_{pi(s)})^{q} p)^{1/q} \right) \prod_{s=1}^{n} (1 - (\nu_{pi(s)})^{p_s})^{1/q} \prod_{s=1}^{n} (1 - (v_{pi(s)})^{p_s})^{1/q}
\]

is called a q-RPLCG operator \((q \geq 1)\) with respect to a fuzzy measure \(g\), where

\[
p_s = g\left(\left\{ x_{i(1)}, \ldots, x_{i(s)} \right\}\right) - g\left(\left\{ x_{i(1)}, \ldots, x_{i(s-1)} \right\}\right), \quad g\left(\left\{ x_{i(0)} \right\}\right) \equiv 0,
\]

and \(x_{i(s)}\) is the \(s\)-th largest of \(x_k\), \(k = 1, 2, \ldots, n\), according to the relation \(\geq_t\).

### 3. APs in the q-RPL Weighted Operators

The q-RPLCA and q-RPLCG operators were defined by the APC\(\{P_{\sigma}\}_{\sigma \in S_n}\) of a fuzzy measure, where the number of probability distributions on \(X\) is \(k = n\)!. It is clear that only one associated probability distribution (Definition 17) is used in definitions of q-RPLCA and q-RPLCG operators (13) and (14). Thus, in interactive MADM problems, when the attributes interact on each other in decision-making process, CA or CG operators’ q-rung orthopair fuzzy or picture linguistic (shortly OF/PL) extensions reflect the concrete consonant structure (13) and (14)

\[
\left\{ x_{i(1)} \right\}, \left\{ x_{i(1)} r x_{i(2)} \right\}, \ldots, \left\{ x_{i(1)} r x_{i(2)} r \cdots r x_{i(n)} \right\}.
\]

Our main task in this work was to construct new aggregation operators for q-ROF or q-RPL (OF/PL) environments, which reflect the overall interactions among all consonant structures of all attributes

\[
\left\{ x_{\sigma(1)} \right\}, \left\{ x_{\sigma(1)} r x_{\sigma(2)} \right\}, \ldots, \left\{ x_{\sigma(1)} r x_{\sigma(2)} r \cdots r x_{\sigma(n)} \right\}, \quad \sigma \in S_n.
\]

For this, we include an APC\(\{P_{\sigma}\}_{\sigma \in S_n}\) in the \(P - q - \text{ROFWA, P} - q - \text{ROFWG, P} - q - \text{RPLWA}\) and \(P - q - \text{RPLWG} operators. Let consider some q-rung OF/PL binary operation \(M\) based on Definitions 2 and 8 and weighted operators \(V \in \{WA, WG\}\) with its probability q-rung OF/PL extension

\[
P - q - R[U][V] \in \{P - q - \text{ROFWA, P} - q - \text{ROFWG, P} - q - \text{RPLPA, P} - q - \text{RPLWG}\},
\]

where \(U\) denotes orthopair fuzzy (OF) or picture linguistic (PL). We define a family of operators as a union of these operators:
Definition 18. For some q-rung fuzzy operation $M : (q - \text{RUN}s)^k \rightarrow (q - \text{RUN}s)$, $k = n!$, some weighted operator $V$ and fuzzy environment $U$ the $|M| \text{APs}$ fuzzy operation with respect to a fuzzy measure $g$ on $X$ is a mapping

$$[M] \text{APs} - q - R[U][V](\alpha_1, \ldots, \alpha_n) = M(\{P_{\sigma} - q - R[U][V](\alpha_1, \ldots, \alpha_n)\}) \equiv M(P_{\sigma} - q - R[U][V](\alpha_1, \ldots, \alpha_n),$$

where $P_{\sigma} - q - R[U][V](\alpha_1, \ldots, \alpha_n)$, $i = 1, \ldots, k = n!$, are values of the probability q-rung $P - q - R[U][V]$ operator calculated with respect to associated probabilities $\{P_{\sigma}\}$.

The four operators for the q-rung orthopair fuzzy values: MaxAP-q-ROFWA, MaxAP-q-ROFWG, MinAP-q-ROFWA, MinAP-q-ROFWG are investigated in [58]. In this work

Proposition 1. If $g_*$ and $g^*$ are dual fuzzy measures on $X$ then

$$[M] \text{APs} - q - RPL[V]^*|_{\alpha_1, \ldots, \alpha_n} = [M] \text{APs} - q - RPL[V]^*(\alpha_1, \ldots, \alpha_n),$$

where $APs - q - RPL[V]_*$ and $APs - q - RPL[V]^*$ are operators’ values calculated on fuzzy measures $g_*$ and $g^*$, respectively, $V = WA$ or $V = WG$, $M = \text{Max}$ or $M = \text{Min}$.

Proof. From the Definition 18 follows that operators are symmetric. Therefore, by applying the Remark A1 (Appendix A), we obtain the equalities (22). □

Proposition 2. The MaxAPs − q − RPLWA operator’s value is a q-RPLN and
The first condition is immediately derived from Definition 8. Below, we prove Equation (23) by using mathematical induction on $n$.

For $n = 2$: from the Definition 8 we have that for any permutation $\sigma \in \{(1, 2), (2, 1)\} = S_2$, since $\bar{\pi}_{\sigma(1)} \cdot \alpha_{\sigma(1)} = \left( \mathbb{I}_{\bar{\pi}_{\sigma(1)} \cdot \rho_{\sigma(1)}}, (1 - (1 - (\mu_{\sigma(1)})^q)^{1/q})^{1/q} \right)$, $(\gamma_{\alpha_{\sigma(1)}})^{\bar{\pi}_{\sigma(1)}}, (\nu_{\alpha_{\sigma(1)}})^{\bar{\pi}_{\sigma(1)}}$,
and 
\[
\left( \bar{\pi}_{\sigma(1)} \cdot \alpha_{\sigma(1)} \right) \oplus q \left( \bar{\pi}_{\sigma(2)} \cdot \alpha_{\sigma(2)} \right) = \left( \mathbb{I}_{\bar{\pi}_{\sigma(2)} \cdot \rho_{\sigma(2)}}, (1 - (1 - (\mu_{\sigma(2)})^q)^{1/q})^{1/q} \right), \]
\[
(\gamma_{\alpha_{\sigma(2)}})^{\bar{\pi}_{\sigma(2)}}, (\nu_{\alpha_{\sigma(2)}})^{\bar{\pi}_{\sigma(2)}} \right) \]
We obtain 
\[
\text{MaxAPs} - q - \text{ROFWA}(\alpha_1, \alpha_2) = \left( \mathbb{I}_{\bar{\pi}_{\sigma(2)} \cdot \rho_{\sigma(2)}}, (1 - (1 - (\mu_{\sigma(1)})^q)^{1/q})^{1/q} \right), \]
\[
\left( \gamma_{\alpha_{\sigma(2)}})^{\bar{\pi}_{\sigma(2)}}, (\nu_{\alpha_{\sigma(2)}})^{\bar{\pi}_{\sigma(2)}} \right) \]
That is, the Equation (23) holds for $n = 2$.

Suppose that the Equation (23) holds for $n = k$, i.e.,
\[
\text{MaxAPs} - q - \text{RPLWA}(\alpha_1, \ldots, \alpha_k) = \left( \mathbb{I}_{\bar{\pi}_{\sigma(k)} \cdot \rho_{\sigma(k)}}, (1 - (1 - (\mu_{\sigma(k)})^q)^{1/q})^{1/q} \right), \]
\[
\left( \gamma_{\alpha_{\sigma(k)}})^{\bar{\pi}_{\sigma(k)}}, (\nu_{\alpha_{\sigma(k)}})^{\bar{\pi}_{\sigma(k)}} \right) \]
Then, for $n = k + 1$, according to Definition 8, we have
\[
\text{MaxAPs} - q - \text{RPLWA}(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}) = \left( \mathbb{I}_{\bar{\pi}_{\sigma(k+1)} \cdot \rho_{\sigma(k+1)}}, (1 - (1 - (\mu_{\sigma(k+1)})^q)^{1/q})^{1/q} \right), \]
\[
\left( \gamma_{\alpha_{\sigma(k+1)}})^{\bar{\pi}_{\sigma(k+1)}}, (\nu_{\alpha_{\sigma(k+1)}})^{\bar{\pi}_{\sigma(k+1)}} \right) \]
\[
\begin{aligned}
&= \max_{\sigma \in S_{k+1}} \left( \sum_{i=1}^{k+1} (\mu_{a_{i+1}})_{\theta(i)} \right) \\
&= \max_{\sigma \in S_{k+1}} \left( \sum_{i=1}^{k+1} (\mu_{a_{i+1}})_{\theta(i)} \right) \\
&= \max_{\sigma \in S_{k+1}} \left( \sum_{i=1}^{k+1} (\mu_{a_{i+1}})_{\theta(i)} \right).
\end{aligned}
\]

That is, the Equation (23) is true for \( n = k + 1 \). Therefore, the Equation (23) is true for all \( n \) and the proof of the proposition is completed. \( \square \)

The proofs of the following some propositions are omitted, as they are similar to the proof of the Proposition 2.

**Proposition 3.** The following MAPs \( q - RPL[\mathcal{V}] \) operators’ values are \( q \)-RPLN and \( q \)-RPLCG operators:

\[
\begin{aligned}
&\text{MinAPs} - q - RPLWA(\alpha_1, \ldots, \alpha_n) = \\
&= \max_{\sigma \in S_{k+1}} \left( \sum_{i=1}^{k+1} (\mu_{a_{i+1}})_{\theta(i)} \right) \\
&= \max_{\sigma \in S_{k+1}} \left( \sum_{i=1}^{k+1} (\mu_{a_{i+1}})_{\theta(i)} \right).
\end{aligned}
\]

**Proposition 4.** Let \( g \) be a lower capacity of order two on \( X \) with associated probabilities \( \{p_{\theta(i)}\} \) \( \forall \alpha_1, \ldots, \alpha_n \in g - \text{RPLNs} \) be ordered by the relation \( \geq_{q \text{-RPLNs}} \). Then the MaxAPs \( q - RPLWA, \text{MinAPs} - q - RPLWG \) operator coincides with the \( q \)-RPLCA \( q \)-RPLCG operator:

\[
\begin{aligned}
&\text{MaxAPs} - q - RPLWA(\alpha_1, \ldots, \alpha_n) = q - RPLCA\mathcal{\Sigma}(\alpha_1, \ldots, \alpha_n), \\
&\text{MinAPs} - q - RPLWG(\alpha_1, \ldots, \alpha_n) = q - RPLCG\mathcal{\Sigma}(\alpha_1, \ldots, \alpha_n)
\end{aligned}
\]

where \( \mathcal{\Sigma} \) is a lower capacity of order two with associated probabilities: \( \forall \sigma \in S_n \), \( p_{\theta(s)} = \beta p_{\theta(s)} + (1 - \beta) w_s \), \( 0 < \beta \leq 1 \); \( s = 1, \ldots, n \) and \( w_s \) is a weight of \( \alpha_s \).

**Proof Case (a).** In Proposition 2 (Formula (23)) we have to consider the function \( \min \) of the following products with respect to associated probabilities

\[
\prod_{s=1}^{n} \left( 1 - (\mu_{a_{s+1}})_{\theta} \right) p_{\theta(s)}, \sigma \in S_n
\]
By the property of functions monotonicity, it is clear that the product (27) takes its minimums on the same arguments with

\[ \ln \left( \prod_{s=1}^{n} \left( 1 - (\mu_{\alpha_s(i)})^q \right)^{\mathcal{P}_{\sigma(i)}} \right) = \sum_{s=1}^{n} \mathcal{P}_{\sigma(i)} \cdot \ln \left( 1 - (\mu_{\alpha_s(i)})^q \right) \]

and

\[ \min_{\sigma \in S_n} \left[ \ln \left( \prod_{s=1}^{n} \left( 1 - (\mu_{\alpha_s(i)})^q \right)^{\mathcal{P}_{\sigma(i)}} \right) \right] = \min_{\sigma \in S_n} \left[ \sum_{s=1}^{n} \mathcal{P}_{\sigma(i)} \cdot \ln \left( 1 - (\mu_{\alpha_s(i)})^q \right) \right]. \]

By the results of Proposition A1 (Appendix A), the latest \( \min \) is a CA operator’s value on the arguments \( \ln \left( 1 - (\mu_{\alpha_s(i)})^q \right) \), \ldots, \( \ln \left( 1 - (\mu_{\alpha_s(n)})^q \right) \) under the fuzzy measure \( \mathcal{F} : \forall C \subset S, \mathcal{F}(C) = \beta \cdot g(C) + (1 - \beta) \cdot \sum_{x_i \in C} \omega_i \). It is elementary that, the fuzzy measure \( \mathcal{F} \) is also a lower capacity of order two with an APC—

\[ \{ \mathcal{F}(\pi) \} \]

\( \min \)

\[ \ln \left( 1 - (\mu_{\alpha_s(i)})^q \right) \geq \ln \left( 1 - (\mu_{\alpha_s(n)})^q \right), \quad (28) \]

then following the formulas in Prop. A1 (Appendix A), we receive:

\[ \min_{\sigma \in S_n} \left[ \ln \left( \prod_{s=1}^{n} \left( 1 - (\mu_{\alpha_s(i)})^q \right)^{\mathcal{P}_{\sigma(i)}} \right) \right] = \text{CA}\mathcal{F} \left( \ln \left( 1 - (\mu_{\alpha_s(1)})^q \right), \ldots, \ln \left( 1 - (\mu_{\alpha_s(n)})^q \right) \right). \]

From the in equalities (28) it follows that

\[ 1 - (\mu_{\alpha_s(1)})^q \geq 1 - (\mu_{\alpha_s(2)})^q \geq \cdots \geq 1 - (\mu_{\alpha_s(n)})^q, \]

and for the product function (27) we receive

\[ \min_{\sigma \in S_n} \left[ \prod_{s=1}^{n} \left( 1 - (\mu_{\alpha_s(i)})^q \right)^{\mathcal{P}_{\sigma(i)}} \right] = \prod_{s=1}^{n} \left( 1 - (\mu_{\alpha_s(i)})^q \right)^{\mathcal{P}_{\sigma(i)}}, \quad (29) \]

where \( \mathcal{P}_{\sigma(i)} = \mathcal{F}(\{ x_{\pi(1)}, \ldots, x_{\pi(s)} \}) - \mathcal{F}(\{ x_{\pi(1)}, \ldots, x_{\pi(s-1)} \}) \).

Analogously we obtain the result for the product function \( \prod_{s=1}^{n} (\nu_{\alpha_s(i)})^{\mathcal{P}_{\sigma(i)}} \):

\[ \min_{\sigma \in S_n} \left[ \prod_{s=1}^{n} (\nu_{\alpha_s(i)})^{\mathcal{P}_{\sigma(i)}} \right] = \prod_{s=1}^{n} (\nu_{\alpha_{\pi'(s)}})^{\mathcal{P}_{\pi'(i)}}, \quad (30) \]

where \( \pi' \in S_n; \mathcal{P}_{\pi'(i)} = \mathcal{F}(\{ s_{\pi'(1)}, \ldots, s_{\pi'(s)} \}) - \mathcal{F}(\{ s_{\pi'(1)}, \ldots, s_{\pi'(s-1)} \}) \) and \( \nu_{\alpha_{\pi'(1)}} \geq \nu_{\alpha_{\pi'(2)}} \geq \cdots \geq \nu_{\alpha_{\pi'(n)}} \).

Because the values \( \alpha_1, \ldots, \alpha_n \) are ordered by the relation \( \geq_{\text{RPNL}} \), we take the permutation \( \tau \in S_n \) for which

\[ \mu_{\alpha_{\tau(1)}} \leq \mu_{\alpha_{\tau(2)}} \leq \cdots \leq \mu_{\alpha_{\tau(n)}} \text{ and } \nu_{\alpha_{\tau(1)}} \geq \nu_{\alpha_{\tau(2)}} \geq \cdots \geq \nu_{\alpha_{\tau(n)}}. \]

Following the formulas (29) and (30) we can take the same permutation \( \pi = \pi' = \tau \).

Analogously to the product functions \( \prod_{s=1}^{n} (1 - (\mu_{\alpha_s(i)})^q)^{\mathcal{P}_{\sigma(i)}} \) and \( \prod_{s=1}^{n} (\nu_{\alpha_s(i)})^{\mathcal{P}_{\sigma(i)}} \) we obtain the same results for the product function \( \prod_{s=1}^{n} (\gamma_{\alpha_s(i)})^{\mathcal{P}_{\pi(i)}} \):

\[ \min_{\sigma \in S_n} \left[ \prod_{s=1}^{n} (\gamma_{\alpha_s(i)})^{\mathcal{P}_{\sigma(i)}} \right] = \prod_{s=1}^{n} (\gamma_{\alpha_{\pi(i)}})^{\mathcal{P}_{\pi(i)}}, \quad (31) \]
and for the sum function $\sum_{s=1}^{k} \mathcal{P}_{\mu(s)} \cdot \mathcal{P}_{\nu(s)}$:

$$\max_{\nu \in S_{\mu}} \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right) = \sum_{s=1}^{n} \mathcal{P}_{\pi(s)} \mathcal{P}_{\pi(s)}$$ (32)

We obtain:

$$\max_{\nu \in S_{\mu}} \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right) = \left( \prod_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \right) \left( \prod_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \right) = \left( \prod_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \right) = q - RPLCA_{\pi}(\alpha_1, \ldots, \alpha_n)$$

where $\mathcal{P}_{\mu(s)} = \mathcal{P}_{\nu(s)} \left( \left\{ s_{\pi(1)}, \ldots, s_{\pi(s)} \right\} \right) - \mathcal{P}_{\nu(s)} \left( \left\{ s_{\pi(1)}, \ldots, s_{\pi(s-1)} \right\} \right)$. The proof of the case b) is similar and it is omitted. □

Analogously to Proposition 4, we can prove the Proposition 5 and the proof also is omitted.

**Proposition 5.** Let $q$ be an upper capacity of order two on $X$ with $\mathcal{AP} = \left\{ \mathcal{P}_{\mu(s)} \right\}$. Then the $\min_{\nu \in S_{\mu}} \max_{\nu \in S_{\mu}} \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right) = q - RPLCA_{\pi}(\alpha_1, \ldots, \alpha_n)$.

**Proposition 6.** Let $g$ be an upper capacity of order two on $X$ with $\mathcal{AP} = \left\{ \mathcal{P}_{\mu(s)} \right\}$. Then the $\min_{\nu \in S_{\mu}} \max_{\nu \in S_{\mu}} \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right) = q - RPLCA_{\pi}(\alpha_1, \ldots, \alpha_n)$.

if $\alpha_1, \ldots, \alpha_n \in q - RPLNs$ are ordered by the relation $\geq_{q-RPLNs}$; $\mathcal{P}_{\mu(s)}$ is an upper capacity of order two with associated probabilities $-\forall \sigma \in S_{\mu}, \mathcal{P}_{\mu(s)} = \mathcal{P}_{\nu(s)} + (1 - \beta) w_\sigma$, $0 < \beta \leq 1$, $s = 1, \ldots, n$; $w_\sigma$ is a weight of $\alpha_\sigma$.

Now we prove the correctness of preserving of values of constructed operators under q-RPL environment.

**Proposition 7.** The aggregation operators $\min_{\nu \in S_{\mu}} \max_{\nu \in S_{\mu}} \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right)$ are averaging $q$-rung picture linguistic aggregation operators on $L_{q-RPLNs}$.

**Proof Case 1.** Let consider $M = \max_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)}$ and $\max_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \geq q - RPLWA \left( \sum_{s=1}^{n} \mathcal{P}_{\nu(s)} \mathcal{P}_{\nu(s)} \right)$ operator. From the Proposition 3 (formula 23) follows that

$$\max_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \geq q - RPLWA(\alpha_1, \ldots, \alpha_n)$$

Let us have two collections of $q$-RPLNs: $\left( \alpha_1, \alpha_2, \ldots, \alpha_n \right)$ and $\left( \beta_1, \beta_2, \ldots, \beta_n \right)$ with the partial pair ordering $\geq_{q-RPLNs}$; $\forall s \in \left\{ 1, 2, \ldots, n \right\}, \alpha_s \leq_{q-RPLNs} \beta_s \iff l_{\alpha_s} \leq l_{\beta_s}$, $\mu_{\alpha_s} \leq \mu_{\beta_s}$, $\gamma_{\alpha_s} \leq \gamma_{\beta_s}$, $\forall s \in \left\{ 1, 2, \ldots, n \right\}$, $\gamma_{\alpha_s} \leq \gamma_{\beta_s}$. We must prove that

$$\max_{\nu \in S_{\mu}} \mathcal{P}_{\nu(s)} \geq q - RPLWA(\beta_1, \ldots, \beta_n)$$

We have $\forall \sigma \in S_{\mu}$:

$$1 - \mu_{\alpha_s} \geq 1 - \mu_{\beta_s} \geq 0 \implies \prod_{s=1}^{n} \left( 1 - \left( \mu_{\alpha_s} \right)^{q} \right) \mathcal{P}_{\nu(s)} \geq \prod_{s=1}^{n} \left( 1 - \left( \mu_{\beta_s} \right)^{q} \right) \mathcal{P}_{\nu(s)}$$


and
\[
\min_{\sigma \in S_{n}} \left( \prod_{s=1}^{n} (1 - \mu_{\alpha_{s}})^{v_{\sigma}(i)} \right) \geq \min_{\sigma \in S_{n}} \left( \prod_{s=1}^{n} (1 - \mu_{\beta_{s}})^{v_{\sigma}(i)} \right) \Rightarrow \\
1 - \min_{\sigma \in S_{n}} \left( \prod_{s=1}^{n} (1 - \mu_{\alpha_{s}})^{v_{\sigma}(i)} \right) \leq 1 - \min_{\sigma \in S_{n}} \left( \prod_{s=1}^{n} (1 - \mu_{\beta_{s}})^{v_{\sigma}(i)} \right)^{1/\gamma} 
\]

(35)

This is the first inequality necessary for (34). Similarly, we receive:
\[
v_{\alpha_{s}(i)} \geq v_{\beta_{s}(i)} \geq 0 \Rightarrow (v_{\alpha_{s}})^{v_{\sigma}(i)} \geq (v_{\beta_{s}})^{v_{\sigma}(i)} \Rightarrow \prod_{s=1}^{n} (v_{\alpha_{s}})^{v_{\sigma}(i)} \geq \prod_{s=1}^{n} (v_{\beta_{s}})^{v_{\sigma}(i)}
\]

(36)

We can receive analogously iniquities for the \(l_{p} \) and \(\gamma \) parameters. The inequality (34) is derived from the inequalities (35) and (36) and analogous inequalities of the \(l_{p} \) and \(\gamma \) parameters. Therefore, the \(\text{maxAPS} - q - \text{RPLWA} \) operator is monotone with respect to \(\leq_{1_q-\text{RPLNs}}\).

It is simple to show that \(\text{MaxAPS} - q - \text{RPLWA}(0_{1_q-\text{RPLNs}}, \ldots, 0_{1_q-\text{RPLNs}}) = 0_{1_q-\text{RPLNs}} = (l_{1}, 0, 0, 1) \) and \(\text{MaxAPS} - q - \text{RPLWA}(1_{1_q-\text{RPLNs}}, \ldots, 1_{1_q-\text{RPLNs}}) = 1_{1_q-\text{RPLNs}} = (l_{7}, 1, 1, 0)\). Therefore, the function \(\text{MaxAPS} - q - \text{RPLWA} \) is a \(q\)-run linguistic aggregation operator.

The cases for the \(\text{MinAs} - P - q - \text{RPLWA} \) operator or the cases \(M = \text{Max} \) or \(M = \text{Min}\), also for the \([M]\text{APS} - q - \text{RPLWG} \) operators can be proved analogously to the case 1 and they are omitted. \(\square\)

Remark 1. By the monotonicity of the \([M]\text{APS} - q - \text{RPL}[V], M = \text{Max} / \text{Min} \) or \(V = \text{WA} / \text{WG} \) operators, we easily prove the following inequalities:

Let be given two sets of \(q\)-RPLNs: \((\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}) \) and \((\beta_{1}, \beta_{2}, \ldots, \beta_{n})\), then
\[
|M|\text{APS} - q - \text{RPL}[V]|((\text{Min}(\alpha_{1}, \beta_{1}), \ldots, \text{Min}(\alpha_{n}, \beta_{n}))) \leq 1_{q-\text{RCLNS}}
\]

\[
\text{Min}[|M|\text{APS} - q - \text{RPL}[V]|(\alpha_{1}, \ldots, \alpha_{n}), |M|\text{APS} - q - \text{RPL}[V]|(\beta_{1}, \ldots, \beta_{n})];
\]

\[
|M|\text{APS} - q - \text{RPL}[V]|((\text{Max}(\alpha_{1}, \beta_{1}), \ldots, \text{Max}(\alpha_{n}, \beta_{n}))) \geq 1_{q-\text{RCLNS}}
\]

\[
\text{Max}[|M|\text{APS} - q - \text{RPL}[V]|(\alpha_{1}, \ldots, \alpha_{n}), |M|\text{APS} - q - \text{RPL}[V]|(\beta_{1}, \ldots, \beta_{n})].
\]

Remark 2. The aggregation operators \([M]\text{APS} - q - \text{RPL}[V] , M \in \{ \text{Max, Min} \}, V \in \{ \text{WA, WG} \}\) are idempotent.

Proof. We omitted remark’s proof because this is not difficult. \(\square\)

Proposition 7. The aggregation operators \([M]\text{APS} - q - \text{RPL}[V] , M \in \{ \text{Max, Min} \}, V \in \{ \text{WA, WG} \}\) are bounded. If \(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in q - \text{RPLNs} \), then
\[
A^{-} \leq 1_{q-\text{RCLNS}} |M|\text{APS} - q - \text{RPL}[V]|(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}) \leq 1_{q-\text{RCLNS}} A^{+}
\]

(37)

where \(A^{-}\) and \(A^{+}\) are min and max elements of the set \(\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\}\) with respect to operations \(\text{Max}\) and \(\text{Min}\), respectively, and
\[
A^{-} = \min_{s=1}^{n} \left( \min_{s=1}^{n} \mu_{\alpha_{s}}, \min_{s=1}^{n} \gamma_{\alpha_{s}}, \max_{s=1}^{n} v_{\alpha_{s}} \right) = \left( l_{p^{-}}, \mu^{-}, \gamma^{-}, v^{+} \right)
\]

\[
A^{+} = \max_{s=1}^{n} \left( \max_{s=1}^{n} \mu_{\alpha_{s}}, \min_{s=1}^{n} \gamma_{\alpha_{s}}, \min_{s=1}^{n} v_{\alpha_{s}} \right) = \left( l_{p^{+}}, \mu^{+}, \gamma^{-}, v^{-} \right)
\]
Proof Case 1. Let consider $M = \text{Max}$ and the $\text{MaxAPs} - q - \text{RPLWA}$ operator. From (23) we have

$$\text{MaxAPs} - q - \text{RPLWA}(a_1, \ldots, a_n) =$$

$$\left( \frac{1}{n} \prod_{\sigma \in S_n} \left( 1 - \min_{\sigma \in S_n} \prod_{i=1}^n \left( 1 - (\mu_{a_{\sigma(i)}})^q \right)^{\frac{1}{n}} \min_{\sigma \in S_n} \prod_{i=1}^n \left( \gamma_{a_{\sigma(i)}} \right)^{\frac{1}{n}} \right) \right).$$

For $\forall \sigma \in S_n$ we have: $1 - (\mu_{a_\sigma})^q \leq 1 - (\mu^q) \Rightarrow (1 - (\mu_{a_\sigma})^q)^{\frac{1}{q}} \leq (1 - (\mu^q))^{\frac{1}{q}}$ and

$$\prod_{\sigma \in S_n} \left( 1 - (\mu_{a_\sigma})^q \right)^{\frac{1}{q}} \leq \prod_{\sigma \in S_n} \left( 1 - (\mu^q) \right)^{\frac{1}{q}} = \left( 1 - (\mu^q) \right)^{\frac{n}{q}} = (1 - (\mu^q))^{\frac{1}{q}}.$$

Therefore,

$$\left( 1 - \min_{\sigma \in S_n} \prod_{\sigma \in S_n} \left( 1 - (\mu_{a_\sigma})^q \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \geq (1 - (1 - (\mu^q)))^{\frac{1}{q}} = (\mu^q)^{\frac{1}{q}} = \mu.$$

Analogously, we obtain the inequalities:

$$\min_{\sigma \in S_n} \left( \prod_{\sigma \in S_n} \left( 1 - (\mu_{a_\sigma})^q \right)^{\frac{1}{q}} \right) \leq \nu^+, \quad \min_{\sigma \in S_n} \left( \prod_{\sigma \in S_n} \left( \gamma_{a_\sigma} \right)^{\frac{1}{n}} \right) \geq \gamma^-, \quad l_{\text{Max}} \geq l_{\text{Min}}.$$

Thus, the left inequality of (37) is proved. The right inequality in (37) can be proved analogously. The cases for the $[M] \text{APs} - q - \text{RPLW} [V]$ operators can be proved similarly. \(\square\)

Remark 3. The $[M] \text{APs} - q - \text{RPLWA} \cdot, [M] \text{APs} - q - \text{RPLW}G \cdot M = \text{Max or Min}$ operators are symmetric.

Proof. We omitted remark’s proof because this is elementary. \(\square\)

Therefore, we obtain the following result:

Remark 4. The $[M] \text{APs} - q - \text{RPLWA} \cdot, [M] \text{APs} - q - \text{RPLW}G \cdot M = \text{Max or Min}$ operators are averaging aggregation operators (monotonicity, symmetricity, boundedness and idempotency) on the $q$-RPLNs with respect to the relation $\leq_{\text{q-RPLN}}$.

Proof. Using the Proposition 7, Remarks 3 and 4, this remark can be simply proved. \(\square\)

In our following step we prove the monotonicity property of the $[M] \text{APs} - q - \text{RPLWA} \cdot, [M] \text{APs} - q - \text{RPLW}G \cdot M = \text{Max or Min}$ operators based on triangular norms. Let us consider two basic dual triangular norms [59]:

1. $t$ -norm $T_p : [0, 1] \to [0, 1]$; $T_p(s, y) = s \cdot y$ (the product) and $t$ -conorm $S_p : [0, 1] \to [0, 1]$; $S_p(s, y) = 1 - (1 - s^q)(1 - y^q)^{1/q}$, $q > 0$ (the q-rung probabilistic sum);
2. $t$ -norm $T_{\text{min}} : [0, 1] \to [0, 1]$; $T_{\text{min}}(s, y) = \min(s, y)$ (minimum) and $t$ -conorm $S_{\text{max}} : [0, 1] \to [0, 1]$; $S_{\text{max}}(s, y) = \max(s, y)$ (maximum).

These binary operations satisfy the properties of boundness, monotonicity, commutativity and associativity. The associativity of the dual triangular norms allows us to extend
these norms in a unique way to \( n \) -ary operation in the usual way by induction, defining for each \( n \)-tuple \((s_1, \ldots, s_n) \in [0; 1]^n\):

\[
T_p(s_1, \ldots, s_n) = \prod_{i=1}^n s_i, \quad S_p^q(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q};
\]

\[
T_{\min}(s_1, \ldots, s_n) = \min_{i=1, \ldots, n} s_i, \quad S_{\max}(s_1, \ldots, s_n) = \max_{i=1, \ldots, n} s_i.
\]

Assume that for any permutation \( \sigma^{(j)} \in S_n, \ j = 1, \ldots, n! \), and \( i = 1, \ldots, n \)

\[
\begin{align*}
\sigma^j(i) & \equiv \mu_{\sigma^{(j)}}(i), \quad \nu^j(i) \equiv (1 - (1 - (\mu_{\sigma^{(j)}}))q)^{1/q}; \\
\sigma''^j(i) & \equiv \nabla_{\sigma^{(j)}}(i), \quad \nu''(i) \equiv (1 - (1 - (\nabla_{\sigma^{(j)}}))q)^{1/q}; \\
\sigma'''^j(i) & \equiv \upsilon_{\sigma^{(j)}}(i), \quad \nu'''(i) \equiv (1 - (1 - (\upsilon_{\sigma^{(j)}}))q)^{1/q}; \\
\end{align*}
\]

(38)

\[z^{(j)}(i) = \sum_{i=1}^n \nu^{(j)}(i); \quad \rho^j(i)
\]

Using the results of Props. 2, 3 (formulas (23)-(26)), constructed operators can be written as some constructions of dual norms: \((T_p, S_p^q)\) and \((T_{\min}, S_{\max})\)

\[
\begin{align*}
\text{MaxAPs} - q & \leq \text{RPLWA}(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
T_{\min} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
S_{\max} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
\text{MinAPs} & \leq \text{RPLWA}(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
S_{\max} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
\text{MaxAPs} & - q \leq \text{RPLWG}(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
S_{\max} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
\text{MinAPs} & - q \leq \text{RPLWG}(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
S_{\max} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
\text{MaxAPs} & - q \leq \text{RPLWG}(s_1, \ldots, s_n) = \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
S_{\max} & \left(1 - \prod_{i=1}^n (1 - s_i^q)\right)^{1/q}; \\
\end{align*}
\]

(39)

By this we proved that new constructions of the \([\mathbb{M}]\)APs \(- q \leq \text{RPLWA}, [\mathbb{M}]\)APs \(- q \leq \text{RPLWG}, M = \text{Max or } M = \text{Min operators are defined by the compositions}

\[
(S_p^q \circ S_{\max}), \quad (S_p^q \circ T_{\min}), \quad (T_p \circ S_{\max}), \quad (T_p \circ T_{\min})
\]

(40)

but for different arguments. In the conditions of (34) and the first equality of (39), let variables \(s_1, s_2, \ldots, s_n, \nu_1, \nu_2, \ldots, \nu_n \in q \leq \text{RPLNs}\) Using the monotonicity and associativity of dual norms \((T_p, S_p^q)\) and \((T_{\min}, S_{\max})\) we obtain the similar properties for the compositions (40). One may easily prove that there exists the monotonicity between their arguments:

\[
\begin{align*}
\sigma^{(j)}(i) & \leq \sigma^{(j)}(i); \quad \nu^{(j)}(i) \geq \nu^{(j)}(i); \quad \nu^{(j)}(i) \leq \nu^{(j)}(i).
\end{align*}
\]

(41)

Using new compositional constructions of the operators, inequalities (41), properties of monotonicity of compositions (40), based on the relation \(\leq_{L_q\text{-RPLNS}}\) makes it clear that the MaxAPs \(- q \leq \text{RPLWA} \) operator is monotone. Analogous compositional connections can be received for other operators.

On the basis of the Propositions 2 and 3 and Definition 8 it is easy to prove the following three propositions:

**Proposition 8.** Let \(s_1, s_2, \ldots, s_n \in q \leq \text{RPLNs} \) be given and there exist ordering \(\leq_{L_q\text{-RPLNS}}\) on \(\{s_1, s_2, \ldots, s_n\}\). Then

\[
\begin{align*}
\text{MaxAPs} - q & \leq \text{RPL}[V] \geq L_q\text{-RPLNS}, \quad \text{MinAPs} - q \leq \text{RPL}[V],
\end{align*}
\]

(42)
Proposition 9. Let be given a set of q-RPLNs –

\[ (\text{MinAPs} - q - \text{RPLWA})^c \geq_{\text{RPL}_n} \text{MaxAPs} - q - \text{RPLWG}, \]

\[ (\text{MaxAPs} - \text{RPLWA})^c \leq_{\text{RPL}_n} \text{MaxAPs} - q - \text{RPLWG}. \]

(43)

Remark 9. If in (23)–(26) a fuzzy measure be a probability measure (\( g = P \)). It is known that [60] associated probabilities coincide and the values of each operator from (23)–(26) also coincide. Following from the idempotency of these operators we obtain: for \( M = \text{Max} \) or \( M = \text{Min} \), then

\[ [M] \text{APs} - q - \text{RPL}[V](\alpha_1, \ldots, \alpha_n) = 0 \cdot ([M] \text{APs} - q - \text{RPL}[V](\alpha_1, \ldots, \alpha_n)). \]

Proposition 10. Let be given a set of q-RPLNs \(-\alpha_1, \ldots, \alpha_n\) and a fuzzy measure \( g \) on \( X \). If \( \beta = (l_{\beta}, \mu_{\beta}, \gamma_{\beta}, v_{\beta}) \) is a q-RPLN then for the operator \( [M] \text{APs} - q - \text{RPL}[V] \) we have

\[ [M] \text{APs} - q - \text{RPL}[V](\alpha_1 \oplus_q \beta, \ldots, \alpha_n \oplus_q \beta) = [M] \text{APs} - q - \text{RPL}[V](\alpha_1, \ldots, \alpha_n) \oplus \beta. \]

Remark 5. Using Propositions 9 and 10 we receive the following formula:

\[ [M] \text{APs} - q - \text{RPL}[V](r\alpha_1 \oplus_q \beta, \ldots, r\alpha_n \oplus_q \beta) = r \cdot ([M] \text{APs} - q - \text{RPL}[V](\alpha_1, \ldots, \alpha_n) \oplus \beta). \]

Remark 6. Let in formulas (23)–(26) a fuzzy measure be a probability measure \( (g = P) \). It is known that [60] associated probabilities coincide and the values of each operator from (23)–(26) also coincide. Following from the idempotency of these operators we obtain: for \( M = \text{Max} \) or \( M = \text{Min} \) and \( V = \text{WA} \) or \( V = \text{WG} \), we have

\[ [M] \text{APs} - q - \text{RPL}[V] = q - \text{RPL}[V]. \]

Remark 7. If in formulas (23)–(26) \( \beta = 0 \), then for \( M = \text{Max} \) or \( M = \text{Min} \)

\[ [M] \text{APs} - q - \text{RPLWA} = q - \text{RPLWA}, \quad [M] \text{APs} - q - \text{RPLWG} = q - \text{RPLWG}. \]

Remark 8. If in (23)–(26) \( \beta = 1 \) and a fuzzy measure \( g \) is a probability, then for \( M = \text{Max} \) or \( M = \text{Min} \)

\[ [M] \text{APs} - q - \text{RPLWA} = P - q - \text{RPLWA}, \quad [M] \text{APs} - q - \text{RPLWG} = P - q - \text{RPLWG}. \]

Remark 9. If in (23)–(26) \( \beta = 0, \ w_s = 1, \ and \ w_j = 0, \ j \neq s \), then \( [M] \text{APs} - q - \text{RPL}[V](\alpha_1, \ldots, \alpha_n) = \alpha_s \), and these operators are step-type operators.

Now we go to the conjugate connections between new operators. In [46] Wu et al. introduced the conjugate connection between the intuitionistic fuzzy CA (IFCA) and intuitionistic fuzzy CG (IFCG) operators.

Definition 19 [46]. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of attributes; \( g \) be a fuzzy measure on \( X \) and \( \alpha : X \Rightarrow \text{IFNs} \) be an intuitionistic fuzzy variable of expert evaluations \(-\alpha(x_s) = (\mu_{x_s}, \nu_{x_s})\), \( s = 1, 2, \ldots, n \). Then a conjugate of Intuitionistic Fuzzy (IFCA) operator is called

\[ (\text{conj})\text{IFCA}_g(\alpha_1, \ldots, \alpha_n) = \text{IFCA}_g((\alpha_1)^c, \ldots, (\alpha_n)^c) \]

\[ = \left( \bigoplus_{j=1}^n \left[ p_j (\alpha_{ij})^c \right]^c \right) = \left( \prod_{j=1}^n (\mu_{x_{ij}})^{p_j}, 1 - \prod_{j=1}^n (1 - \nu_{x_{ij}})^{p_j} \right), \]

operator with respect to the fuzzy measure \( g \).
From the Definition 17 (q = 1) we have intuitionistic fuzzy environment and (conj) IFCA operator coincides with the intuitionistic fuzzy CG (IFCG) operator

\[(\text{conj}) \text{IFCA}_q(a_1, \ldots, a_n) = \text{IFCG}_q(a_1, \ldots, a_n)\]  

(45)

Now we consider the definition of conjugate operator for q-RPL environment.

**Definition 20.** Let \(G : (q - \text{RPLNs})^n \to q - \text{RPLNs}\) be a q-RPL aggregation operator. Conjugate operator of the operator \(G\) is:

\[(\text{conj}) G(\beta_1, \ldots, \beta_n) = (G((\beta_1)^c, \ldots, (\beta_n)^c))^c \quad \forall \beta_1, \ldots, \beta_n \in q - \text{RPLNs}.\]

(46)

Analogously to (45), we have conjugate connections for the newly constructed operators:

**Proposition 11.** For the operators \([M] \text{APs} - q - \text{RPL}[V], M = \text{Max} \text{ or } M = \text{Min}\) \(V = \text{WA}\) or \(V = \text{WG}\) we have

\[
\begin{align*}
(\text{conj}) \text{MaxAPs} - q - \text{RPLWA} & = \text{MinAPs} - q - \text{RPLWG}, \\
(\text{conj}) \text{MinAPs} - q - \text{RPLWA} & = \text{MaxAPs} - q - \text{RPLWG}. 
\end{align*}
\]

(47)

**Proof.** We consider a case for the \(\text{MaxAPs} - q - \text{RPLWA}\) operator. Other cases will be similar and be omitted here. Let \(\forall a_1, \ldots, a_n \in q - \text{RPLNs}, a_i = (l_{p_i}, h_{a_i}, \gamma_{a_i}, v_{a_i}), i = 1, 2, \ldots, n,\) be given. We have

\[
\begin{align*}
(\text{conj}) \text{MaxAPs} - q - \text{RPLWA} & = (\text{MaxAPs} - q - \text{RPLWA})^c, \\
\text{MaxAPs} - q - \text{RPLWA} & = (\text{MinAPs} - q - \text{RPLWA})^c, \\
\text{MinAPs} - q - \text{RPLWA} & = (\text{MaxAPs} - q - \text{RPLWA})^c. 
\end{align*}
\]

(48)

We consider the \(\text{(conj)} [M] \text{MaxAPs} - q - \text{RPLWA}\) operator with conjugate arguments. Using notations (38) in (23) we make following interchanges in the arguments:

\[
S_{\varphi(0)}(a) \leftrightarrow y_{\varphi(0)}(a), \quad y_{\varphi(0)}(a) \leftrightarrow S_{\varphi(0)}(a),
\]

(49)

Therefore, we receive the following transforms:

\[
\begin{align*}
(\text{conj}) \text{MaxAPs} - q - \text{RPLWA} & = (\text{MaxAPs} - q - \text{RPLWA})^c, \\
\text{MaxAPs} - q - \text{RPLWA} & = (\text{MinAPs} - q - \text{RPLWA})^c, \\
\text{MinAPs} - q - \text{RPLWA} & = (\text{MaxAPs} - q - \text{RPLWA})^c. 
\end{align*}
\]

(50)

The second equality in (47) may be proved analogously. \(\square\)

The constructed new four q-RPL operators create the pairs of mutually conjugate operators. Following the main sense on conjugate operators [46], we can say: In the MADM these new two pairs of conjugate operators express the DM’s preferences in a spectrum
of optimistic to pessimistic (pessimistic to optimistic) variability of decision risks under q-RPL information.

4. Associated Probabilities’ Identification by Pair Interaction Indexes of Attributes and Attributes’ Importance Values

A deep review on identification methods of a fuzzy measure using the Choquet integral aggregation is given in [39]. The overview on some recent works on reducing the complexity in the problem of identification of a fuzzy measure is given in [40]. A fuzzy measure presents an uncertainty measure in decision-making processes. It can reflect flexibly certain level of interaction among the attributes and can vary from negative interaction to positive interaction [61]. A 2-order additive fuzzy measure is widely used for applications in MADM, and its identification problem is studied in [39]. In this section, associated probabilities of the 2-order additive fuzzy measure by the pair interaction indexes of attributes and by attributes’ importance values are constructed. These connections give us an opportunity to describe new aggregation operators by the interaction indexes and importance values.

**Definition 21 [61].** Let $g$ be some set function (not only a fuzzy measure) on $X = \{x_1, \ldots, x_n\}$. The Mobius transformation of $g$ is a set function $m_g : 2^X \to \mathbb{R}$ with

$$
m_g(C) = \sum_{D \subseteq C} (-1)^{|C \setminus D|} g(D), \quad \forall C \subset X,
$$

where $|C|$ is the cardinality of the set $C$, $\mathbb{R}^1 \equiv (-\infty, +\infty)$. Mobius transformation uniquely defines a fuzzy measure (or some set function) $g$:

$$
g(C) = \sum_{D \subseteq C} m_g(D), \quad \forall C \subset X.
$$

**Definition 22 [61].** Let $g$ be a fuzzy measure on $X$. $g$ is said to be $k, k \in \{1, 2, \ldots, n\}$, $k$-order additive if its Mobius transformation satisfies $m_g(C) = 0$ for all $C \subset X$ such that $|C| > k$ and there exists at least one subset $C \subset X$ such that $m_g(C) \neq 0$.

Obviously, $k$-order additive fuzzy measure can be defined by $\sum_{l=1}^{k} \binom{C}{l}$ coefficients. In our following studies we will use the 2-order additive fuzzy measure [40,61]. For determination of 2-order additive fuzzy measure we only need $n(n+1)/2$ coefficients. Note that, fuzzy measures provide a structure for modeling the knowledge available about variables whose values are unknown, uncertain. As it appears, many different types of uncertainty can be represented in this framework.

We prefer a measure of entropy that can be used in calculation of the amount of uncertainty associated with a fuzzy measure. More precisely, in our framework we prefer Shapley entropy [62] which will be connected to the associated probabilities of a 2-order additive fuzzy measure.

**Definition 23.** Let be given a fuzzy measure $g$ on attributes set $X$. (1) [61]: The overall importance value of an attribute $x_i \in X$ is called its Shapley value (index), defined by

$$
I_i = \sum_{A \subset X \setminus \{x_i\}} \frac{|(|X| - |A| - 1)!}{(|X|)!} \cdot [g(A \cup \{x_i\}) - g(A)],
$$

(2) [61]: The interaction index of two attributes $x_i, x_j \in X$, $i \neq j$, is defined by

$$
I_{ij} = \sum_{A \subset X \setminus \{x_i, x_j\}} \frac{|(|X| - |A| - 2)!}{(|X| - 1)!} \cdot [g(A \cup \{x_i, x_j\}) - g(A \cup \{x_i\}) - g(A \cup \{x_j\}) + g(A)],
$$

(3) [62] The Shapley entropy of a fuzzy measure $g$ is defined by

$$
H_{Sh}(I_1, \ldots, I_n) = -\sum_{i=1}^{n} I_i \ln(I_i)
$$
Yager, in [62], introduced a quantification of the amount of uncertainty associated with a fuzzy measure in the spirit of the Shannon entropy. We will implement a maximal entropy principle [63] within the framework of associated probabilities of a 2-order additive fuzzy measure for the identification of importance values and interaction indexes.

By mathematical induction one may easily prove a linear relationship between values of associated probabilities of a fuzzy measure, attributes importance values and pair interaction indexes for a 2-order additive fuzzy measure.

**Proposition 12.** For a 2-order additive fuzzy measure and \( \forall \sigma = \{ \sigma(1), \ldots, \sigma(n) \} \in S_n \), \( i = 1, \ldots, n \), we have

\[
g(\{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}) = \sum_{l=1}^{i} I_{\sigma(i)} - \frac{1}{2} \sum_{l \in N_{\sigma(i)}} I_{\sigma(j)} 
\]

where \( N_{\sigma(i)} \) denotes an index subset \( N_{\sigma(i)} = \{1, \ldots, n\} \setminus \{\sigma(1), \ldots, \sigma(i)\} \).

**Proof.** Using the definition for the 2-order additive fuzzy measure \( g \) we obtain

\[
m_g(\{x_i, x_j\}) = I_{ij}, \ i \neq j.
\]

It is proved [61] that \( \sum_{i=1}^{n} I_i = 1 \). Therefore, we have

\[
g(\{x_i\}) = m_g(\{x_i\}) = I_i - \frac{1}{2} \sum_{l=1}^{n} m_g(\{x_i, x_l\}) = I_i - \frac{1}{2} \sum_{l \in N_1 \setminus \{i\}} I_{il}, \ i = 1, \ldots, n
\]

where \( N \equiv \{1, \ldots, n\} \). Now let \( A = \{x_{\sigma(1)}, x_{\sigma(2)}\} \) and we will show the formula (55) for \( s = 2 \).

\[
g(\{x_{\sigma(1)}, x_{\sigma(2)}\}) = m_g(\{x_{\sigma(1)}\}) + m_g(\{x_{\sigma(2)}\}) + m_g(\{x_{\sigma(1)}, x_{\sigma(2)}\}) = \\
I_{\sigma(1)} - \frac{1}{2} \sum_{h \in N_1 \setminus \{\sigma(1)\}} I_{\sigma(1)h} + \left(I_{\sigma(2)} - \frac{1}{2} \sum_{h \in N_1 \setminus \{\sigma(2)\}} I_{\sigma(2)h} \right) + I_{\sigma(1)\sigma(2)} = \\
I_{\sigma(1)} + I_{\sigma(2)} - \frac{1}{2} \sum_{h \in N_1 \setminus \{\sigma(1), \sigma(2)\}} (I_{\sigma(1)h} + I_{\sigma(2)h}) = \frac{1}{2} \sum_{h=1}^{2} I_{\sigma(h)} - \frac{1}{2} \sum_{j=1}^{n} \sum_{h \in N_1 \setminus \{\sigma(j)\}} I_{\sigma(j)h}.
\]

Let now the formula (55) be true for the \( s = i - 1 \), and we want to prove it for the \( s = i \). Consider

\[
g(\{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}) = \sum_{B \in \{x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}\}} m_g(B) + \sum_{|B| \leq i-1} x_{\sigma(i)} \in B \quad m_g(B) = \\
\sum_{h \in N_1 \setminus \{\sigma(i)\}} I_{\sigma(i)h} - \frac{1}{2} \sum_{h \in N_1 \setminus \{\sigma(i)\} \cup \sigma(i-1)} I_{\sigma(i)h} = \\
\sum_{h \in N_1 \setminus \{\sigma(i)\} \cup \sigma(i-1)} I_{\sigma(i)h} = \frac{1}{2} \sum_{h=1}^{i-1} I_{\sigma(h)} - \frac{1}{2} \sum_{j=1}^{i-1} \sum_{h \in N_1 \setminus \{\sigma(j)\} \cup \sigma(i-1)} I_{\sigma(j)h}.
\]

Using a definition of the associated probabilities of a fuzzy measure (Definition 17) based on the formula (55) we obtain:

\[
p_x(x_{\sigma(s)}) = g(\{x_{\sigma(1)}, \ldots, x_{\sigma(s)}\}) - g(\{x_{\sigma(1)}, \ldots, x_{\sigma(s-1)}\}) = \\
I_{\sigma(s)} + (1/2) \cdot \sum_{h=1}^{s-1} I_{\sigma(s)\sigma(h)} - (1/2) \cdot \sum_{h=s+1}^{n} I_{\sigma(s)\sigma(h)},
\]

where if \( s = 1 \), then the second addend is zero, and if \( s = n \), then the third addend is zero. Some words on the representation of the associated probability (56) by the pair interac-
tion indexes between attributes. In (56), in the positive role are taken interaction indexes of the consonant structure \( \{x_{v(1)}\}, \{x_{v(1)}, x_{v(2)}\}, \ldots, \{x_{v(s+1)}, \ldots, x_{v(n)}\} \) and in the negative role are taken interaction indexes of the consonant structure \( \{x_{v(s+1)}\}, \ldots, \{x_{v(s+1)}, \ldots, x_{v(n)}\} \). Using (56) \( n \times n! \) number of associated probabilities can be constructed by the \( n \times (n + 1)/2 \) interaction indexes \( I = \{I_{ij}\}, i \neq j, I_{ij} = I_{ji} \) and \( n \) values of overall importance values \( (I = \{I_i\}, i = 1, \ldots, n) \).

**Shapley entropy maximum principle:** Clearly, the decision makers are not actually able to determine the exact estimations of the overall importance values and the pair interaction indexes. We suppose the case when we have information about the interval estimations of these parameters: \( M_i^- \leq I_i \leq M_i^+, \quad M_{ij}^- \leq I_{ij} \leq M_{ij}^+, \ i, j, i \in \{1, \ldots, n\} \). Based on the constraints \( 0 \leq p_{x_i}(x_{v(i)}) \leq 1, \forall \sigma \in S_n, \sum_{i=1}^{n} p_{x_i}(x_{v(i)}) = 1 \) the principle of Shapley entropy maximum for numerical estimation of the mentioned values is formulated as a mathematical programming problem:

\[
\text{Max } H_{Sh}(I_1, \ldots, I_n) = -\sum_{i=1}^{n} I_i \ln(I_i),
\]

s.t.

\[
M_i^- \leq I_i \leq M_i^+, \quad i \in \{1, \ldots, n\},
\]

\[
M_{ij}^- \leq I_{ij} \leq M_{ij}^+, \quad i \neq j, \ i, j \in \{1, \ldots, n\},
\]

\[
0 \leq I_{v(s)} + (1/2) \cdot \sum_{h=1}^{s-1} I_{v(s),v(h)} - (1/2) \cdot \sum_{i=s+1}^{n} I_{v(s),v(h)} \leq 1, \forall \sigma \in S_n, s = 1, \ldots, n,
\]

\[
\sum_{s=1}^{n} \left( I_{v(s)} + (1/2) \cdot \sum_{h=1}^{s-1} I_{v(s),v(h)} - (1/2) \cdot \sum_{h=s+1}^{n} I_{v(s),v(h)} \right) = 1, \forall \sigma \in S_n.
\]

We have \( n(n + 1)(1/2 + (n - 1)! \) linear constraint conditions with the convex objective function. Not for large \( n \), which is usual in the MADM, there are schemes for the numerical solving of this problem in real time. Recent studies contrary to our approach to fuzzy measure identification in the interactive MADM environment can be found in [64–68].

In this section we used the Shapley entropy maximum principle to provide a quantification of the amount of uncertainty by the information contained in the attributes’ pair interactions for the identification of associated probabilities of a 2-order additive fuzzy measure.

5. Discrimination q-Rung Picture Linguistic Measures in MADM

Let \( SC = \{sc_1, sc_2, \ldots, sc_m\} \) denote the set of all alternatives (possible decisions) and \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of all attributes for the same MADM model. Let \( \{a_{ij}\}, i = 1, \ldots, m; j = 1, \ldots, n, \) be experts’ evaluations/ratings. Let \( a_{ij} \) be compatibility levels from [0;1] interval as an evaluation on alternative \( sc_i \) with respect to attribute \( x_j \). Based on [53] a heuristic explanation of the positive fuzzy discrimination \( (p_{ij}) \) (PFD) and negative fuzzy discrimination \( (n_{ij}) \) (NFD) measures is that \( p_{ij} \) represents the accumulated belief that attribute \( x_j \) is more indicative of alternative \( sc_i \) than any of the remaining alternatives, whilst \( n_{ij} \) represents the belief that attribute \( x_j \) is not more indicative of alternative \( sc_i \) than any of the other alternatives.

More conveniently, the PFD measure \( p_{ij} \) is an influence level of the attribute \( x_j \) on the alternative \( sc_i \) as compared to other alternatives, the NFD measure \( n_{ij} \) is a non-influence level of the attribute \( x_j \) on the alternative \( sc_i \) as compared to other alternatives. PFD and
NFD measures reflect as pairs of expert evaluations \( \alpha_{ij}^{dis} = (p_{ij}, n_{ij}) \) from the interval \([0,1]^2\). For our goal we selected the more convenient one [64]:

\[
\begin{align*}
    p_{ij} &= \left( \frac{\sum_{k \neq i} \text{LR}(\frac{a_{ij}}{a_{kj}})}{(m-1)} \right), \\
n_{ij} &= \left( \frac{\sum_{k \neq i} \text{SR}(\frac{a_{kj}}{a_{ij}})}{(m-1)} \right).
\end{align*}
\]

The authors of [58] extend this definition by the difference operation:

\[
\begin{align*}
    p_{ij} &= \left( \frac{\sum_{k \neq i} \text{LD}(a_{ij} - a_{kj})}{(m-1)} \right), \\
n_{ij} &= \left( \frac{\sum_{k \neq i} \text{SD}(a_{kj} - a_{ij})}{(m-1)} \right),
\end{align*}
\]

where LD, SD, LR and SR are fuzzy values with the following compatibility functions: \( \text{LD} \equiv'' \text{Large Difference}'' : [-1;1] \to [0;1], \text{SD} \equiv'' \text{Small Difference}'' : [-1;1] \to [0;1], \text{LR} \equiv'' \text{Large Ratio}'' : [0;+\infty] \to [0;1], \text{SR} \equiv'' \text{Small Ratio}'' : [0;+\infty] \to [0;1]. \) LD, LR are monotone increasing fuzzy sets and SD, SR are monotone decreasing fuzzy sets.

Many authors used pair discriminations in the scalar value, for example-[Large(\(p_{ij}\)) + Small(\(n_{ij}\))] / 2 [53]. However, this transformation violates the principle of duality of information related to the loss of nature of dual evaluations. In this period of time, the theory of general orthopair fuzzy sets was not developed enough and it is clear that they had to go to the Zadeh’s membership levels. It is clear that the “attribute’s influence + non-influence” is represented by some q-ROFN—\(\alpha_{ij}^{dis} = (p_{ij}, n_{ij})\) [58]. New version of the fuzzy discrimination analysis method [58] uses only expert’s data and is based on the q-ROF tabular-numerical knowledge base. If we consider interactive MADM model where the interaction phenomena between alternatives is actual, it will be more convenient and natural to use q-rung orthopair fuzzy discrimination values \(\{\alpha_{ij}^{disc} = (p_{ij}, n_{ij})\}\) instead of expert’s evaluation \(\{\alpha_{ij}\}\). The authors of [58] also extended PFD and NFD measures under q-rung orthopair fuzzy information, when experts’ evaluations \(\alpha_{ij}\) in the q-rung orthopair fuzzy numbers are presented.

\[
\begin{align*}
    p_{ij} &= \left( \frac{\sum_{k \neq i} \text{LD}(\frac{\text{Sc}(a_{ij}) - \text{Sc}(a_{kj})}{2})}{(m-1)} \right), \\
n_{ij} &= \left( \frac{\sum_{k \neq i} \text{SD}(\frac{\text{Sc}(a_{ij}) - \text{Sc}(a_{kj})}{2})}{(m-1)} \right).
\end{align*}
\]

By the principle of best approach to the experiment we used the functions [58]: \(\text{LD}(x) = \frac{1}{2} + \frac{1}{2}\arcsin(x), \text{SD}(x) = \frac{1}{2} - \frac{1}{2}\arcsin(x), x \in [-1;1]\). It is clear that if experts’ evaluations \(\alpha_{ij} = (\mu_{a_{ij}}, \upsilon_{a_{ij}})\) are usual membership levels, \(\upsilon_{a_{ij}} = 1 - \mu_{a_{ij}}\) then formulas (62) and (63) coincide.

We extend this judgment for the q-rung picture linguistic environment introducing the degree of neutral discrimination or neutral influence of the attribute \(x_j\) on the decision \(sc_i\):

\[
m_{ij} = \left( \frac{\sum_{k \neq i} \text{AZD}(\frac{\text{Sc}(a_{ij}) - \text{Sc}(a_{kj})}{2})}{(m-1)} \right),
\]
where $AZD \equiv " \text{Approximately Zero Difference}" : [-1; 1] \rightarrow [0; 1]$. By the principle of best approach to the experiment we use the function $AZD(x) = \cos(x^3 \pi / 2) \in [-1; 1]$, $\lambda$ is a positive parameter, value of which will be selected by the best dependence of the experts’ evaluations and DM’s preferences in MADM.

Therefore, in future we will consider discrimination $q$-RPL numbers $\alpha^{pl/dscr}_{ij} = (\rho_{ij}, p_{ij}, m_{ij}, n_{ij})$. However, for determining the linguistic labels $I_{\rho_{ij}}$ from the possible linguistic terms set $L = \{l_1, l_2, \ldots, l_7\} = \{\text{very poor, poor, slightly poor, fair, slightly good, good, very good}\}$ we assume a linear dependence between $\rho_{ij}$ and $Sc(a_{ij})$

$$I_{\rho_{ij}} = l_1 + 3 \cdot (1 + Sc(a_{ij})). \quad (65)$$

6. An Evaluation of Candidate Service Center’s Selection Reliability Index in the Fuzzy Facility Location Selection Problem by the $[M]APs - q - RPL[V]$ Operators

The problem of facility location in its classical version involves assessing the possibility of opening candidate centers (alternatives) for services, with respect to independently weighted factors (attributes) [69,70]. The candidate centers receiving the best evaluations from the attribute aggregations have the chance to be open. During selection of the location of the service centers, special attention is paid to the interests of different parties, such as the local population in the geographical areas, the administrative leadership of the local municipalities, etc. All this is due to the fact that beside the minimization of total costs for opening and maintenance it is also important the performance of environmentally friendly shipments and thus minimizing the negative impact on the environment and population, as well as minimizing the risks of opening service centers and etc. Many different approaches have been given in the literature for solving the facility location problems [69,70]. Multi-attribute facility location models have been studied in [71–74]. But the crucial parameters considered in these models are of a definite nature and having pre-known values, which of course does not happen in real world situations. Often, it is impossible to obtain reliable information for accurate estimates of important input parameters. This is mainly due to the insufficient statistics, and especially true when planning service centers for first aid or timely relocation of other products to areas affected by a disaster or other extreme event. Obviously, in such cases, it is necessary to use the knowledge of service network dispatchers, managers, and other experts for evaluation of the necessary input parameters in optimal service planning models. To cope such situations fuzzy theory is used [75]. Application of fuzzy theory in facility location planning in non-disaster environments has been studied in [76–81]. Fuzzy TOPSIS approaches for facility location selection problem are developed in [82–87].

Complexity and uncertainty of service centers’ location planning in decision-making problems for disaster environments inevitably leads to the fact that the information about attributes ratings with respect to candidate sites (service centers) is usually incomplete or even completely unknown. Besides, it was not a pleasant event that some interactions have already been observed between the existing attributes [87]. It is impossible to ignore this event in the problem of ranking of service centers. Therefore, in this work we focus on a multi-attribute group decision making approach for location planning for selection of service centers under uncertain and extreme environment. We develop a fuzzy interactive multi-attribute decision making approach for the service center location selection problem for which the new $[M]APs - q - RPL[V]$ aggregation operators’ approach is applied.

Attributes identification is of high significance in the facility location selection problem. We assume from the outset that the locations of all candidate service center (SC) are known beforehand. This set is denoted by $SC = \{sc_1, sc_2, \ldots, sc_m\}$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all attributes (transformed in benefit attributes) which define SCs selection. For example [58]: “access by public and special transport modes to the candidate site”, “security of the candidate site from accidents, theft and vandalism”, “connectivity of the location with other modes of transport (highways, railways, seaports, airports, etc.)”, “costs in vehicle resources, required products and etc. for the location of a candidate site”, “impact
of the candidate site location on the environment, such as important objects of Critical Infrastructure, air pollution and others”, “proximity of the candidate site location from the central locations”, “proximity of the candidate site location from customers”, “availability of raw material and labor resources in the candidate site”, “ability to conform to sustainable freight regulations imposed by managers e.g., restricted delivery hours, special delivery zones”, “ability to increase size to accommodate growing customers” and others. Let \( W = \{ w_1, w_2, \ldots, w_n \} \) be the weights of attributes. Denote by \( E = \{ e_1, e_2, \ldots, e_t \} \) the number of experts invited for evaluations. For each expert \( e_k \), \( e_k \in \) (service dispatchers and so on), let \( a_{ij}^k \) be the fuzzy rating of his/her evaluation in q-ROFNs for each candidate service center \( sc_i \), \( i = 1, \ldots, m \), with respect to each attribute \( x_j \), \( j = 1, \ldots, n \). Let experts’ ratings are presented in positive numbers—\( \omega_1, \ldots, \omega_t \). For the expert \( e_k \) we construct binary fuzzy relation \( A_k = \{ a_{ij}^k, i = 1, \ldots, m; j = 1, \ldots, n \} \) decision making matrix, elements of which are represented in q-ROFNs. If some attribute \( x_j \) is of cost type then we transform experts’ evaluations and \( a_{ij}^k \) is changed by \( (a_{ij}^k)^c \). Experts’ data must be aggregated in q-ROFNs etalon decision making matrix \( -A = \{ a_{ij}, i = 1, \ldots, m; j = 1, \ldots, n \} \). Based on the results of Section 5, (63) and (64) q-ROFNs matrix \( A \) must be transformed in discrimination q-RPLNs matrix \( A^{PL/\text{Discr}} = \{ a_{ij}^{PL/\text{Discr}}, i = 1, \ldots, m; j = 1, \ldots, n \} \). Our task is to build aggregation operators’ approach, which for each candidate service center \( sc_i \), \( i = 1, \ldots, m \), aggregates presented its objective and subjective data into scalar values—service center’s selection reliability index (SRI). This aggregation can be formally represented as an \( |M|APs - q - RPL[V] \) operator’s value defined on discrimination q-RPL numbers \( a_{ij}^{PL/\text{Discr}} = (l_{ij}, p_{ij}, m_{ij}, n_{ij}) \) induced by experts’ evaluations in q-RPLNs \( a_{ij}, j = 1, \ldots, m \).

\[
\text{SRI}(sc_i) = |M|APs - q - RPL[V](a_{ij}^{PL/\text{Discr}}, \ldots, a_{ij}^{PL/\text{Discr}}), \quad i = 1, \ldots, m
\]  

(66)

The scheme of calculation of service centers’ SRI’s includes the next steps:

**Step 1: Selection of attributes.** Refers to the location’s attributes selection for assessing possible locations for candidate service centers. The attributes are inferred from knowledge of experts and the city stakeholders’ members. Five attributes will be considered in our example \( (n = 5) \) from defined above by short names: \( x_1 = “\text{Accessibility}” \), \( x_2 = “\text{Security}” \), \( x_3 = “\text{Connectivity to multimodal transport}” \), \( x_4 = “\text{Costs}” \), \( x_5 = “\text{Proximity to customers}” \).

**Step 2: Selection of alternatives—candidate service centers (CSC).** Includes selection of possible locations for opening CSC. Proceeding from their knowledge and experience the decision makers appoint candidate locations for realization of service centers. For instance, if particular sites are prohibited by municipal administration, then they are excluded from consideration. Ideally, the candidate sites must be acceptable for all city stakeholders, city residents, logistics operators, municipal administrations, etc. As said above, in the process of selecting candidate service centers interactions between attributes are observed. Obviously, the attributes independently cannot influence on the possible opening of this or that candidate center. We choose five CSCs \( (m = 5) \).

**Step 3: Evaluation of ratings of CSCs with respect to the attributes.** Let \( A_k = \{ a_{sh}^k \in q - \text{ROFNs}, s = 1, \ldots, m; h = 1, \ldots, n \} \) be the experts’ \( e_k \) \((k = 1, 2, \ldots, t)\) evaluation ratings for each CSC—\( sc_s \) \((s = 1, 2, \ldots, m)\) with respect to attributes \( x_h \) \((h = 1, 2, \ldots, n)\). Assume that experts’ group consists of three members \((t = 3)\).

**Step 4: Computation of the q-ROF decision matrix.** Let be given experts weights as positive numbers \( \omega_k, \omega_k > 0, k = 1, \ldots, t \). Using the formulas of Definition 2 the CSC’s aggregated fuzzy ratings \( (a_{ij}) \) with respect to each attribute will be given by q-ROF weighted sum

\[
a_{ij} = \sum_{s=1}^{t} \omega_s a_{ij}^s \left( \omega_s / \sum_{i=1}^{t} \omega_i \right).
\]

(67)
Construct the q-rung fuzzy decision matrix \( \{a_{ij}\} \)

\[
\begin{bmatrix}
  x_1 & x_2 & x_n \\
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\]

and calculate functions \( \text{Sc} \) and \( \text{Ac} \) (Definition 3) on elements \( a_{ij} \).

**Step 5:** Transformation of the q-ROF decision matrix \( \{a_{ij}\} \) into discrimination q-rung picture linguistic values matrix \( \{\alpha_{ij}^{PL/discr}\} = \{l_{pij}, p_{ij}, m_{ij}, n_{ij}\} \). Using (63)–(65) make these transforms. Use some numerical method and define minimal integer \( q \) for q-RPLNs \( \alpha_{ij}^{PL/discr} \)

\[
q^{PL/discr} = \min \left( q : q \in N; \mu^{q}_{\alpha_{ij}^{PL/discr}} + \nu^{q}_{\alpha_{ij}^{PL/discr}} + \upsilon^{q}_{\alpha_{ij}^{PL/discr}} \leq 1, \ i = 1, \ldots, m; \ j = 1, \ldots, n \right). \tag{68}
\]

**Step 6:** Construction of a 2-order additive fuzzy measure and its APs. In the role of a fuzzy measure we use the 2-order additive fuzzy measure [70]. Using evaluations of attributes’ importance values \( l_s, s = 1, \ldots, n \), and the attributes’ pair interaction indexes \( l_s, s < h \), we construct APs of a 2-order additive fuzzy measure by the solution of optimization problem (57)–(61), when \( n = 5 \).

**Step 7:** Calculation of candidate service centers’ SRIs by the \( [M] \text{APs} - q - \text{RPL}[V] \) operators. Using formulas (23)–(26) of the \( [M] \text{APs} - q - \text{RPL}[V] \) operators, formulas (18)–(21) of the \( [M] \text{APs} - q - \text{ROF}[V] \) operators, formulas (13) and (14) of the q-RPLCA, q-RPLCG and formulas (3) and (4) of the q-RPLWA, q-RPLWG operators calculate CSCs’ SRIs. To be more understandable, for the calculation of candidate service centers’ SRIs by the \( [M] \text{APs} - q - [U][V] \) operators we included the following micro-algorithm:

**Step 7.1:** Select the operation \( M, M \in \{\text{Min}, \text{Max}\} \) and the ordered weighted operator \( V \in \{\text{WA}, \text{WG}\} \) in the operators’ family \( [M] \text{APs} - q - [U][V] \).

**Step 7.2:** Select the fuzzy environment \( U \in \{q - \text{ROF}, q - \text{RPL}\} \), i.e., q-rung or-thopair fuzzy or q-rung picture linguistic environment.

**Step 7.3:** Select attributes weights \( W = (w_1, \ldots, w_n) \) as an aggregation weighted vector, such that \( w_j \in [0,1], \sum_{j=1}^{n} w_j = 1 \). Select a value of weighted parameter \( \beta \).

**Step 7.4:** Select some possible alternative \( \text{sc}_{ij}, i = 1, \ldots, m \), and select the i-th row of matrix \( \{\xi_{ij} \equiv \alpha_{ij}^{PL/discr} = (l_{pij}, p_{ij}, m_{ij}, n_{ij})\} \) (Step 6, q-rung picture linguistic environment) or matrix \( \xi_{ij} \equiv \{a_{ij}\} \) (Step 4, q-rung or-thopair fuzzy environment).

**Step 7.5:** Select some permutation \( \sigma, \sigma \in S_n \) and write down associated probabilities \( \{p_{\sigma^{(i)}} = p_{\sigma^{(i)}}(\xi_{ij} = \xi_{\sigma^{(i)}})\}, i = 1, \ldots, n \) (step 6). For this permutation calculate associated probabilities \( p_{\sigma(i)} = \beta p_{\sigma(i)} + (1 - \beta) w_i, i = 1, \ldots, n \). Repeat this step for every permutation \( \sigma, \sigma \in S_n \).

**Step 7.6:** Calculate candidate \( \text{sc}_{ij}, i = 1, \ldots, m \), service centers’ SRIs by the selected \( [M] \text{APs} - q - [U][V] \) operator. I.e., calculate \( [M] \text{APs} - q - [U][V] \) operator’s value on arguments \( \{\xi_{1}, \xi_{2}, \ldots, \xi_{n}\} \) (formulas (18)–(21)).

**Numerical Example**

Assume that based on the consensus, experts evaluated the attributes’ interval weights as overall importance values:

\[
0.1 \leq l_1 \equiv w_1 \leq 0.3; \ 0.1 \leq l_2 \equiv w_2 \leq 0.3; \ 0.1 \leq l_3 \equiv w_3 \leq 0.3; \ 0.1 \leq l_4 \equiv w_4 \leq 0.3; \ 0.1 \leq l_5 \equiv w_5 \leq 0.3.
\]
Based on the consensus principle, let the experts also present interval evaluations of the pair interaction indexes (see Table 1).

Table 1. The interval matrix of attributes pair interaction indexes—\([M^--;M^+;] \) and importance values \([M^-;M^+;] \).

|     | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) | \(M^-;M^+;\) |
|-----|--------|--------|--------|--------|--------|-------------|
| \(x_1\) | -      | \([0.03;0.07]\) | \([0.08;0.12]\) | \([0.07;1.12]\) | \([0.06;0.10]\) | \([0.1;0.3]\) |
| \(x_2\) | \([0.03;0.07]\) | -      | \([0.08;0.12]\) | \([0.06;0.10]\) | \([0.10;0.15]\) | \([0.1;0.3]\) |
| \(x_3\) | \([0.08;0.12]\) | 0.10   | -      | \([0.05;0.09]\) | \([0.03;0.07]\) | \([0.1;0.3]\) |
| \(x_4\) | \([0.07;1.12]\) | 0.08   | \([0.05;0.09]\) | -      | \([0.05;0.09]\) | \([0.1;0.3]\) |
| \(x_5\) | \([0.06;0.10]\) | \([0.10;0.15]\) | \([0.03;0.07]\) | \([0.05;0.09]\) | -      | \([0.1;0.3]\) |

Based on the data from Table 1 after numerical solution of mathematical programming problem (57)–(61) (for \(n = 5\) ) we obtain the following estimations on attributes pair interaction indexes and importance values (Table 2).

Table 2. Estimations of attributes pair interaction indexes and importance values.

|     | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) | \(I_i\) |
|-----|--------|--------|--------|--------|--------|--------|
| \(x_1\) | -      | 0.049  | 0.109  | 0.088  | 0.081  | 0.244  |
| \(x_2\) | 0.049  | -      | 0.110  | 0.083  | 0.138  | 0.213  |
| \(x_3\) | 0.109  | 0.110  | -      | 0.068  | 0.061  | 0.178  |
| \(x_4\) | 0.088  | 0.083  | 0.068  | -      | 0.072  | 0.197  |
| \(x_5\) | 0.081  | 0.138  | 0.061  | 0.072  | -      | 0.168  |

Let be given appraisal matrices of experts (see Tables 3–5)

\[ A_k = \{a_{ij}^k \in q-\text{RPLNs}, \ i = 1, \ldots , 6; j = 1, \ldots , 5; k = 1, 2, 3. \]

Table 3. Appraisal matrix \(A_1\) by DM-1.

|     | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) |
|-----|--------|--------|--------|--------|--------|
| \(s_{c1}\) | \((0.7,0.5)\) | \((0.8,0.3)\) | \((0.7,0.4)\) | \((0.7,0.4)\) | \((0.8,0.4)\) |
| \(s_{c2}\) | \((0.6,0.5)\) | \((0.7,0.4)\) | \((0.6,0.4)\) | \((0.8,0.4)\) | \((0.7,0.4)\) |
| \(s_{c3}\) | \((0.8,0.5)\) | \((0.9,0.5)\) | \((0.9,0.7)\) | \((0.8,0.4)\) | \((0.8,0.5)\) |
| \(s_{c4}\) | \((0.6,0.5)\) | \((0.8,0.4)\) | \((0.8,0.5)\) | \((0.9,0.5)\) | \((0.8,0.5)\) |
| \(s_{c5}\) | \((0.8,0.6)\) | \((0.7,0.4)\) | \((0.9,0.5)\) | \((0.7,0.4)\) | \((0.8,0.6)\) |

Table 4. Appraisal matrix \(A_2\) by DM-2.

|     | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) |
|-----|--------|--------|--------|--------|--------|
| \(s_{c1}\) | \((0.7,0.5)\) | \((0.8,0.4)\) | \((0.6,0.3)\) | \((0.6,0.3)\) | \((0.7,0.4)\) |
| \(s_{c2}\) | \((0.6,0.5)\) | \((0.7,0.3)\) | \((0.7,0.4)\) | \((0.9,0.4)\) | \((0.8,0.4)\) |
| \(s_{c3}\) | \((0.8,0.5)\) | \((0.9,0.5)\) | \((0.7,0.4)\) | \((0.8,0.4)\) | \((0.7,0.2)\) |
| \(s_{c4}\) | \((0.6,0.4)\) | \((0.8,0.3)\) | \((0.9,0.6)\) | \((0.7,0.3)\) | \((0.6,0.2)\) |
| \(s_{c5}\) | \((0.9,0.7)\) | \((0.7,0.4)\) | \((0.9,0.4)\) | \((0.7,0.3)\) | \((0.9,0.6)\) |

Table 5. Appraisal matrix \(A_3\) by DM-3.

|     | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) |
|-----|--------|--------|--------|--------|--------|
| \(s_{c1}\) | \((0.7,0.4)\) | \((0.8,0.3)\) | \((0.7,0.5)\) | \((0.7,0.4)\) | \((0.9,0.5)\) |
| \(s_{c2}\) | \((0.6,0.5)\) | \((0.7,0.5)\) | \((0.7,0.3)\) | \((0.7,0.2)\) | \((0.7,0.3)\) |
| \(s_{c3}\) | \((0.6,0.2)\) | \((0.9,0.6)\) | \((0.7,0.5)\) | \((0.7,0.3)\) | \((0.6,0.3)\) |
| \(s_{c4}\) | \((0.8,0.4)\) | \((0.9,0.4)\) | \((0.8,0.5)\) | \((0.8,0.5)\) | \((0.8,0.3)\) |
| \(s_{c5}\) | \((0.9,0.7)\) | \((0.8,0.3)\) | \((0.9,0.5)\) | \((0.9,0.6)\) | \((0.7,0.4)\) |
Let the ratings of experts be $\omega_1 = 0.4$, $\omega_2 = 0.4$, $\omega_3 = 0.2$. Experts' evaluations are condensed in etalon decision making matrix $\{a_{ij}\}$ (Table 6). The problem (68) is numerically solved and we evaluated integer value $q$ of q-ROFNs $\{a_{ij}\}$ ($q_{PL/disc} = 4$).

### Table 6. Decision making matrix $\{a_{ij}\}$.

|   | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$     |
|---|-----------|-----------|-----------|-----------|-----------|
| $s_1$ | $(0.7003, 0.4782)$ | $(0.8002, 0.3366)$ | $(0.6668, 0.3728)$ | $(0.6668, 0.3565)$ | $(0.8021, 0.4183)$ |
| $s_2$ | $(0.6004, 0.5001)$ | $(0.7000, 0.3565)$ | $(0.6668, 0.3776)$ | $(0.8408, 0.3482)$ | $(0.7482, 0.3776)$ |
| $s_3$ | $(0.7756, 0.4163)$ | $(0.9001, 0.5186)$ | $(0.8175, 0.5232)$ | $(0.7846, 0.3776)$ | $(0.7365, 0.3129)$ |
| $s_4$ | $(0.6642, 0.4373)$ | $(0.8281, 0.3565)$ | $(0.8511, 0.5378)$ | $(0.8297, 0.4076)$ | $(0.7464, 0.3129)$ |
| $s_5$ | $(0.8701, 0.6581)$ | $(0.7258, 0.3776)$ | $(0.9005, 0.4573)$ | $(0.770, 0.3866)$  | $(0.8408, 0.5533)$ |

Using transformation formulas (63)–(65) we calculate decision making matrix of discrimination q-rung picture linguistic values $\{a_{ij}^{PL/disc}\} = \{p_{ij}, q_{ij}, m_{ij}, n_{ij}\}$ (in our case $\lambda = 1/15$, $q_{PL/Disc} = 3$) (Table 7).

Based on (56) and the solution of the problem (57)–(61), we calculate APs (number of associated probability distributions is $5! = 120$; calculations are omitted).

Let weighted parameter in the $|M|APs - q - \text{ROF}[V]$ ((17)–(21)) and $|M|APs - q - \text{RPL}[V]$ ((17)–(21)) operators be $\beta = 0.7$. We can already perform the last 6th step of the scheme and calculate the values of the new constructed aggregation operators as candidate service centers' SRIs.

### Table 7. Decision making matrix $\{a_{ij}^{PL/disc}\} = \{p_{ij}, q_{ij}, m_{ij}, n_{ij}\}$.

|   | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$     |
|---|-----------|-----------|-----------|-----------|-----------|
| $s_{c1}$ | $l_{1423}$ | $0.5186, 0.2834, 0.4814$ | $0.5193, 0.2721, 0.4807$ | $0.5336, 0.2297, 0.4664$ | $0.5351, 0.2163, 0.4649$ | $0.5290, 0.3356, 0.4910$ |
| $s_{c2}$ | $l_{1758}$ | $0.5338, 0.2232, 0.4662$ | $0.5299, 0.2511, 0.4711$ | $0.5431, 0.2097, 0.4569$ | $0.5250, 0.2579, 0.4750$ | $0.5065, 0.3824, 0.4993$ |
| $s_{c3}$ | $l_{2243}$ | $0.5267, 0.2635, 0.4733$ | $0.5293, 0.2418, 0.4707$ | $0.5277, 0.2527, 0.4728$ | $0.5162, 0.3933, 0.4838$ | $0.5906, 0.3888, 0.4940$ |
| $s_{c4}$ | $l_{2114}$ | $0.5196, 0.2808, 0.4804$ | $0.5213, 0.2717, 0.4767$ | $0.5259, 0.2404, 0.4705$ | $0.5169, 0.2801, 0.4809$ | $0.5053, 0.3742, 0.4947$ |
| $s_{c5}$ | $l_{2114}$ | $0.5242, 0.2706, 0.4758$ | $0.5252, 0.2610, 0.4748$ | $0.5304, 0.1875, 0.4496$ | $0.5174, 0.2856, 0.4826$ | $0.5069, 0.3532, 0.4931$ |

The presented scheme (steps 1–6) of location planning for CSCs’ with the new constructed operators gives the following results—evaluation of SRIs (Tables 8 and 9). Note that the obtained 2-order additive fuzzy measure is not a capacity of order two. Q-ROF $q - \text{ROFWA}\ (q - \text{ROFWG})$, $q - \text{ROFC}\ (q - \text{ROFG})$ and $|M|APs - q - \text{ROF}[V]$ ((66), (17)–(21)) aggregation operators’ values are presented in Table 8. Q-RPL $q - \text{RPLWA}\ (q - \text{RPLWG})$, $q - \text{RPLCA}\ (q - \text{RPLCG})$ and new $|M|APs - q - \text{RPL}[V]$ aggregation operators’ values ((17)–(21)) are presented in Table 9.

### Table 8. The evaluations of SRIs by the q-ROF operators based on q-ROFNs matrix (Table 6).

| Aggregation Operators | $s_{c1}$  | $s_{c2}$  | $s_{c3}$  | $s_{c4}$  | $s_{c5}$  |
|-----------------------|-----------|-----------|-----------|-----------|-----------|
| $q - \text{ROFWA}$   | $(0.7361, 0.3917)$ | $(0.7210, 0.4046)$ | $(0.8169, 0.4248)$ | $(0.7931, 0.4053)$ | $(0.8343, 0.4793)$ |
| $q - \text{ROFWG}$   | $(0.7235, 0.4034)$ | $(0.6943, 0.4193)$ | $(0.8029, 0.4475)$ | $(0.7752, 0.4258)$ | $(0.8174, 0.5231)$ |
| $q - \text{ROFC}$    | $(0.7082, 0.4035)$ | $(0.6779, 0.426)$ | $(0.8282, 0.4591)$ | $(0.7867, 0.4327)$ | $(0.8324, 0.5123)$ |
| $q - \text{ROFG}$    | $(0.7392, 0.4215)$ | $(0.6749, 0.4319)$ | $(0.7950, 0.4293)$ | $(0.7472, 0.4102)$ | $(0.8178, 0.5581)$ |
| Max$\text{APs} - q - \text{ROFWA}$ | $(0.7462, 0.3891)$ | $(0.7577, 0.3942)$ | $(0.8056, 0.3985)$ | $(0.8097, 0.3934)$ | $(0.8464, 0.4749)$ |
| Max$\text{APs} - q - \text{ROFWG}$ | $(0.7309, 0.3932)$ | $(0.7372, 0.3982)$ | $(0.7885, 0.4088)$ | $(0.7953, 0.4058)$ | $(0.8311, 0.4847)$ |
| Max$\text{APs} - q - \text{ROFW}$ | $(0.7315, 0.3865)$ | $(0.7347, 0.3926)$ | $(0.7931, 0.4172)$ | $(0.7996, 0.4146)$ | $(0.8334, 0.4849)$ |
| Min$\text{APs} - q - \text{ROFWA}$ | $(0.7167, 0.4005)$ | $(0.7123, 0.4090)$ | $(0.7797, 0.4385)$ | $(0.7817, 0.4349)$ | $(0.8164, 0.5149)$ |
Table 9. The evaluations of SRIs by the q-RPL operators based on discrimination q-RPLNs matrix (Table 7).

| Aggregation Operators-SRIs | sc1 | sc2 | sc3 | sc4 | sc5 |
|----------------------------|-----|-----|-----|-----|-----|
| q – RPLOWA                 | (l2.15, 0.525, 0.265, 0.477) | (l2.15, 0.525, 0.271, 0.475) | (l2.17, 0.519, 0.296, 0.482) | (l2.19, 0.518, 0.304, 0.483) | (l2.21, 0.523, 0.273, 0.477) |
| q – RPLOWG                 | (l2.12, 0.525, 0.274, 0.477) | (l2.17, 0.526, 0.289, 0.475) | (l2.21, 0.524, 0.272, 0.476) | (l2.24, 0.523, 0.267, 0.477) |
| q – RPLCA                  | (l2.13, 0.526, 0.252, 0.474) | (l2.17, 0.534, 0.231, 0.466) | (l2.21, 0.526, 0.269, 0.474) | (l2.24, 0.519, 0.294, 0.481) |
| q – RPLCG                  | (l2.15, 0.523, 0.263, 0.477) | (l2.17, 0.529, 0.249, 0.471) | (l2.21, 0.526, 0.269, 0.474) | (l2.24, 0.519, 0.294, 0.481) |
| MaxAPs – q – RPLOWA        | (l2.12, 0.522, 0.271, 0.478) | (l2.11, 0.523, 0.283, 0.477) | (l2.21, 0.526, 0.287, 0.474) | (l2.24, 0.519, 0.307, 0.481) |
| MaxAPs – q – RPLOWG        | (l2.13, 0.524, 0.274, 0.476) | (l2.12, 0.526, 0.287, 0.474) | (l2.21, 0.526, 0.287, 0.474) | (l2.24, 0.519, 0.307, 0.481) |
| MinAPs – q – RPLOWA        | (l2.11, 0.522, 0.284, 0.478) | (l2.11, 0.523, 0.302, 0.478) | (l2.21, 0.523, 0.302, 0.478) | (l2.24, 0.516, 0.324, 0.484) |
| MinAPs – q – RPLOWG        | (l2.11, 0.522, 0.284, 0.478) | (l2.11, 0.523, 0.302, 0.478) | (l2.21, 0.523, 0.302, 0.478) | (l2.24, 0.516, 0.324, 0.484) |

Total order relation $\succeq$ (Definitions 4 and 10) induces ranking order relation $\succeq \cap$ on all possible alternatives (see Tables 10 and 11):

$\text{sri}_i \succeq \text{sri}_j \iff \text{SRI(}\text{sri}_i\text{)} \geq \text{SRI(}\text{sri}_j\text{)}$

Table 10. The rankings of CSCs by the SRIs based on the relation $\succeq \cap$ (q-ROF case).

| Aggregation Operators | Ranking Order |
|-----------------------|---------------|
| q – ROFWA             | sc3 $\succ$ sc4 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| q – ROFWG             | sc3 $\succ$ sc4 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| q – ROFCA             | sc3 $\succ$ sc4 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| q – ROFCG             | sc3 $\succ$ sc4 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| MaxAPs – q – ROFWA    | sc4 $\succ$ sc3 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| MaxAPs – q – ROFWG    | sc4 $\succ$ sc3 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| MinAPs – q – ROFWA    | sc4 $\succ$ sc3 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |
| MinAPs – q – ROFWG    | sc4 $\succ$ sc3 $\succ$ sc5 $\succ$ sc1 $\succ$ sc2 |

Table 11. The rankings of CSCs by the SRIs based on the relation $\succeq \cap$ (q-RPL case).

| Aggregation Operators | Ranking Order |
|-----------------------|---------------|
| q – RPLOWA            | sc3 $\succ$ sc5 $\succ$ sc4 $\succ$ sc1 $\succ$ sc2 |
| q – RPLOWG            | sc3 $\succ$ sc5 $\succ$ sc4 $\succ$ sc1 $\succ$ sc2 |
| q – RPLCA             | sc3 $\succ$ sc5 $\succ$ sc4 $\succ$ sc1 $\succ$ sc2 |
| q – RPLCG             | sc3 $\succ$ sc5 $\succ$ sc4 $\succ$ sc1 $\succ$ sc2 |
| MaxAPs – q – RPLOWA   | sc5 $\succ$ sc4 $\succ$ sc3 $\succ$ sc1 $\succ$ sc2 |
| MaxAPs – q – RPLOWG   | sc5 $\succ$ sc4 $\succ$ sc3 $\succ$ sc1 $\succ$ sc2 |
| MinAPs – q – RPLOWA   | sc5 $\succ$ sc4 $\succ$ sc3 $\succ$ sc1 $\succ$ sc2 |
| MinAPs – q – RPLOWG   | sc5 $\succ$ sc4 $\succ$ sc3 $\succ$ sc1 $\succ$ sc2 |

Comparative analysis. Let us first consider the results of q-ROF aggregations. Table 10 shows that optimal decision (the opened service center with a higher SRI of the q-ROFWA, q-ROFWG, q-ROFCA and q-ROFCG operators is $\text{cs}_3$, which differs from the MAPs – $q$ – ROFV operators’ optimal decision $-\text{cs}_3$). This difference is due to the fact that the interactions between the attributes are observed with certain degrees (see Table 2). This phenomenon is reflected not only in additive aggregations (q-ROFWA, q-ROFWG), but also in the Choquet aggregations (ROFCA and q-ROFCG). That is, the latter operators failed to detect the small interactions that appeared in the results of the new aggregation operators.

Let us now consider the results of q-RPL aggregation. Table 11 shows that optimal decision (the opened service center with a higher SRI of the q-RPLWA, q-RPLWG, q-RPLCA and q-RPLCG operators is $\text{cs}_3$, just as it was in the results of q-ROF aggregation. At the same time this optimal solution is also different from the new MAPs – $q$ – RPLV operators’ optimal decision $-\text{cs}_5$, which reflects the interactions between the attributes. There is difference between the optimal solutions of operators MAPs – $q$ – ROFV and...
MAPs − q − RPLV. This is explained by the fact that the transformation from experts q-ROF evaluations to q-RPL evaluations resulted in the expansion of information in the linguistic variables environment, and this gave an alternative decision—the opened service center with a higher SRI $cs_5$ instead of $cs_4$. However, this change is not observed in other operators under consideration. The result is again decision $cs_3$.

Note that discrimination q-RPL numbers were taken as arguments in all q-RPL aggregation operators. The dominant impact of attributes on a given alternative with respect to other alternatives has already been taken into account here. The results of this phenomenon were still not detected in the optimal solutions of q-RPLWA, q-RPLWG, q-RPLCA and q-RPLCG operators, although there was a change in the results of the MAPs − q − RPLV operators—the decision $cs_5$. Which, of course, is due to the introduction of a discrimination model.

In conclusion, when certain degrees of interaction are observed between attributes, the newly defined aggregation operators reflect this phenomenon in the results, whereas in the Choquet type aggregations it may not be reflected. Here we say that when the degrees of interactions are very high, the Choquet aggregations really respond to this phenomenon [34,88].

7. Conclusions

The main issue of this work concerns the embedding of associated probabilities of a fuzzy measure in aggregation operators, in particular, in the Choquet type aggregations and the connection of these embeddings to taking into account all pairwise interactions between attributes. It is well known that the Choquet type aggregations do not consider all pair interactions in aggregation results. Therefore, in many cases, when these interactions are revealed in less degree, they do not perceive these interactions and their utility in MADM problems is less useful than the new aggregation operators’ family developed in the article. Which, in turn, account for all pairwise interaction indices. For the construction of new operators, the following work has been done.

In this work, we constructed and investigated new family of aggregation operators: $MAPs − q − RPLV$, $M \in \{\text{Max, Min}\}$, $V \in \{\text{WA, WG}\}$ for the q-rung picture linguistic environment. In these aggregations an uncertainty is represented by a 2-order additive fuzzy measure. New operators’ definitions are based on the APs of a 2-order additive fuzzy measure. New operators’ definitions are based on the APs of a 2-order additive fuzzy measure.

The new operators represent the extensions of the Choquet integral averaging and geometric operators in the q-rung picture linguistic environment. The propositions on the correctness of the new extensions are proved: 1. The $[M]APs − q − RPL[V]$ operators for the capacity of order two coincide with the q-RPLCA or q-RPLCG operators; 2. The $[M]APs − q − RPL[V]$ operators coincide with the q-RPLWA or q-RPLWG operators when fuzzy measure is presented as a probability one.

The constructed operators and their conjugate operators make possible to deal with interactions among all the attributes in q-RPL MADM problems. Connections between the $[M]APs − q − RPL[V]$ operators and the compositions of dual triangular norms $(T_q, S_p)$ and $(T_{\text{min}}, S_{\text{max}})$ are found.

In the examples, a 2-ordered additive fuzzy measure and its associated probabilities are identified. For this purpose, attributes’ pair interaction indexes by the Shapley maximal entropy principle are generated.

For the demonstration of applicability of new family of aggregation operators—$MAPs − q − RPLV$, $M \in \{\text{Max, Min}\}$, $V \in \{\text{WA, WG}\}$, we give an illustrative example on the fuzzy group decision-making problem on facility location selection problem. Risk management is very important in the service centers’ selection procedure. Usually risks inherent to the candidate sites are identified and assessed in the development process. New aggregations for the evaluation of service center’s selection reliability index are used. For the centers’ selection we compared the results of the weighted averaging and geometric,
also Choquet averaging and geometric operators with the new aggregation operators results (for the q-rung orthopair fuzzy and q-rung picture linguistic environments).

As the results showed, levels of attributes’ discriminations dominate on the concrete alternative—CSC as compared to other CSCs. The comparisons of discrimination models for the both environments indicate that the operators give a more opportunity to consider the DM’s viewpoint on decision risks.

New extensions for the immediate probability, probability OWA and other probabilistic operators by the APs under different fuzzy environment will be developed in our future works. Some heuristic decision-making approaches as TOPSIS, VIKOR will be developed for the q-rung picture linguistic environment. Fuzzy discrimination analysis will be developed in heuristic decision-making methodology.

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### Appendix A

Associated probabilities of a fuzzy measure and Yager’s extensions of Choquet integral aggregation operators under general orthopair fuzzy information.

**Definition A1** [89]. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set. a set function \( g : 2^X \rightarrow [0, 1] \) is said a fuzzy measure on \( X \) if it satisfies

\[
(1) \quad g(\emptyset) = 0; \quad g(X) = 1; \quad (2) \quad \forall C, D \subseteq X, \quad \text{if } C \subseteq D, \quad \text{then } g(C) \leq g(D). \quad (A1)
\]

Let us consider all possible permutations of the elements of \( X \) which form the group \( S_n \) with the cardinality \( |S_n| = n! \). A definition of APs \([50,60]\) induced by a fuzzy measure \( g \) on the group \( S_n \) is given as:

**Definition A2** [60]. The probability distribution \( P_r \) defined by

\[
P_r(x_{\sigma(1)}) = g(\{x_{\sigma(1)}\}), \ldots, P_r(x_{\sigma(n)}) = g(\{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}) - g(\{x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\}), \ldots,
\]

\[
P_r(x_{\sigma(0)}) = 1 - g(\{x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\}), \quad g(\{x_{\sigma(0)}\}) = 0
\]

for each \( \sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in S_n \), are called the associated probabilities (APs) and the Associated Probability Class (APC) \{-\{P_r\}_{r \in S_n}\} induced by the fuzzy measure \( g \).

**Remark A1** [60]. Fuzzy measures \( g_{\ast} \) and \( g^\ast \) on \( X \) are called dual if \( \forall C \subseteq X, \quad g_{\ast}(C) = 1 - g^\ast(C) \). It is not difficult to prove that dual fuzzy measures have the same APC.

**Definition A3** [44]. Let \( (g_{\ast}, g^\ast) \) be a pair of dual fuzzy measures on \( X \): \( g_{\ast} \) is a lower capacity of order two if and only if \( \forall C, D \subseteq X \quad g_{\ast}(C \cup D) + g_{\ast}(C \cap D) \geq g_{\ast}(C) + g_{\ast}(D) \); \( g^\ast \) is an upper capacity of order two if and only if \( \forall C, D \subseteq X \quad g^\ast(C \cup D) + g^\ast(C \cap D) \leq g^\ast(C) + g^\ast(D) \).

We consider important aspects of Choquet averaging (CA) and Choquet geometric (CG) operators \([38,41]\) as most typical aggregation operators with attributes interaction phenomena.
Definition A4 [41]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of attributes, $g$ be a fuzzy measure on $X$ and $c : X \Rightarrow R^n_+$ be a variable of expert evaluations, such that $c_i(x_i) \equiv c_i > 0, \ i = 1, 2, \ldots, n$. Then

$$CA_g(c_1, c_2, \ldots, c_n) = \sum_{h=1}^{n} p_h c_i(h)$$

is called the finite CA operator with respect to a fuzzy measure $g$, and

$$CG_g(c_1, c_2, \ldots, c_n) = \prod_{h=1}^{n} p^{p_i}_h$$

is called the finite CG operator with respect to a fuzzy measure $g$, where $i(\cdot)$ is an index function, such that $c_i(h)$ is the $h$-th largest of the $\{a_i\}_{i=1}^{n}$;

$$p_h = g\left(\left\{x_{i(1)}, \ldots, x_{i(h)}\right\}\right) - g\left(\left\{x_{i(1)}, \ldots, x_{i(h-1)}\right\}\right), \ g\left(\left\{x_{i(0)}\right\}\right) = 0.$$ (A5)

Proposition A1 [60]. A pair of dual fuzzy measures $(g^+, g^*)$ are lower and upper capacities of order two, respectively, if and only if for every variable $c : X \Rightarrow R^n_+$, it holds

$$CA_{g^+}(c) = \min_{\sigma \in S_n} PA_{P_{\sigma}}(c), \ CA_{g^*}(c) = \max_{\sigma \in S_n} PA_{P_{\sigma}}(c) \ or \ CG_{g^+}(c) = \min_{\sigma \in S_n} PG_{P_{\sigma}}(c), \ CG_{g^*}(c) = \max_{\sigma \in S_n} PG_{P_{\sigma}}(c).$$

In [51] an overview of CA’s extensions the for intuitionistic fuzzy environment is given. Based on the negation function in [49] R. Yager developed the CA operator under q-ROF information [8].

Definition A5 [8]. Let $a_1, \ldots, a_n$ be some set of q-ROFNs as values of a variable $a$ defined on $X$. The CA operator of q-ROFNs $\cdot a_1, \ldots, a_n$ with respect to a fuzzy measure $g$ on $X$ is an orthopair number $(a, b)$ such that

$$a = Choq(a_1, \ldots, a_n) = \sum_{i=1}^{n} p_h \mu_{\sigma_i(h)} \ and \ b = Choq^dual(a_1, \ldots, a_n) = Neg_q\left(\sum_{h=1}^{n} p_h Neg_q\left(\nu_{\eta_i(h)}\right)\right)$$

or shortly

$$q - ROFCIA_q(a_1, \ldots, a_n) = (\sum_{h=1}^{n} p_h \mu_{\eta_i(h)}) \cdot (1 - (\sum_{h=1}^{n} p_h (1 - (\nu_{\sigma_i(h)}))^q)^{\frac{1}{q}})^{\frac{1}{q}}$$

(A6)

The CG operator’s extension under q-RPL information may be defined analogously. The necessary condition for the q-ROF environment to be guaranteed is $a^q + b^q \leq 1$, i.e., aggregation result also is a q-ROFN.

References
1. Atanassov, K. Intuitionistic fuzzy sets. Fuzzy Set Syst. 1986, 20, 87–96. [CrossRef]
2. Zadeh, L.A. Fuzzy sets. Inf. Control 1965, 8, 338–353. [CrossRef]
3. Atanassov, K. Intuitionistic Fuzzy Sets: Theory and Applications; Physica-Verlag: Heidelberg, Germany, 1999.
4. Xu, Z.S. Intuitionistic Fuzzy Information Aggregation: Theory and Applications; Science Press: Beijing, China, 2008.
5. Yu, D.; Liao, H. Visualization and quantitative research on intuitionistic fuzzy studies. J. Intell. Fuzzy Syst. 2016, 30, 3653–3663. [CrossRef]
6. Yager, R.R. Pythagorean Membership Grades in Multicriteria Decision Making. IEEE Trans. Fuzzy Syst. 2014, 22, 958–965. [CrossRef]
7. Yager, R.R. Pythagorean Fuzzy Subsets. In Proceedings of the Joint IFSA Congress and NAFIPS Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 357–361.
8. Yager, R.R.; Alajlan, N.; Bazi, Y. Aspects of Generalized Orthopair Fuzzy Sets. Int. J. Intell. Syst. 2018, 33, 2154–2174. [CrossRef]
9. Yager, R.R. Generalized Orthopair Fuzzy Sets. IEEE Trans. Fuzzy Syst. 2017, 25, 1222–1230. [CrossRef]
10. Ali, M.I. Another view on q-Rung orthopair fuzzy sets. Int. J. Intell. Syst. 2018, 33, 2139–2153. [CrossRef]
11. Cuong, B.C. Picture fuzzy sets—First results. part 1. Semin. Neuro-Fuzzy Syst. Appl. 2013, 4, 201.
12. Wei, G.W. Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making. *Kybernetes* 2017, 46, 1777–1800. [CrossRef]
13. Son, L.H. Measuring analoguousness in picture fuzzy sets: From picture distance measures to picture association measures. *Fuzzy Optim. Decis. Mak.* 2017, 16, 359–378. [CrossRef]
14. Wei, G.W. Picture fuzzy aggregation operators and their application to multiple attribute decision making. *J. Intell. Fuzzy Syst.* 2017, 33, 713–724. [CrossRef]
15. Wei, G.W. Picture fuzzy cross-entropy for multiple attribute decision making problems. *J. Bus. Econ. Manag.* 2016, 17, 491–502. [CrossRef]
16. Garg, H. Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. *Arab. J. Sci. Eng.* 2017, 42, 5275–5290. [CrossRef]
17. Wei, G.W. Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Fund. Inform.* 2018, 157, 271–320. [CrossRef]
18. Li, D.X.; Dong, H.; Jin, X. Model for evaluating the enterprise marketing capability with picture fuzzy information. *J. Intell. Fuzzy Syst.* 2017, 33, 3255–3263. [CrossRef]
19. Thong, P.H. Picture fuzzy clustering: A new computational intelligence method. *Soft Comput.* 2016, 20, 3549–3562. [CrossRef]
20. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning-Part II. *Inf. Sci.* 1965, 8, 301–357. [CrossRef]
21. Wang, J.Q.; Li, J.J. The multi-criteria group decision making method based on multi-granularity intuitionistic two semantics. *Sci. Technol. Inf.* 2009, 33, 8–9.
22. Du, Y.Q.; Hou, F.J.; Zafar, W.; Yu, Q.; Zhai, Y.B. A novel method for multi-attribute decision making with interval-valued Pythagorean fuzzy linguistic information. *Int. J. Intell. Syst.* 2017, 32, 1085–1112. [CrossRef]
23. Liu, P.D.; Zhang, X.H. A novel picture fuzzy linguistic aggregation operator and its application to group decision-making. *Cogn. Comput.* 2018, 10, 242–259. [CrossRef]
24. Li, L.; Zhang, R.; Wang, J.; Shang, X.; Bai, K. A Novel Approach to Multi-Attribute Group Decision-Making with q-Rung Picture Linguistic Information. *Symmetry* 2018, 10, 172. [CrossRef]
25. Liu, P.D.; Wang, P. Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *Int. J. Intell. Syst.* 2018, 33, 259–280. [CrossRef]
26. Li, Z.; Wei, G. Pythagorean fuzzy heronian mean operators in multiple attribute decision making and their application to supplier selection. *Int. J. Knowl.-Based Intell. Eng. Syst.* 2019, 23, 77–91. [CrossRef]
27. Wei, G.; Gao, H.; Wei, Y. Some q-rung orthopair fuzzy Heronian mean operators. *Int. J. Intell. Syst.* 2018, 33, 1426–1458. [CrossRef]
28. Wei, G.; Lu, M.; Gao, H. Picture fuzzy heronian mean aggregation operators in multiple attribute decision making. *Int. J. Knowl.-Based Intell. Eng. Syst.* 2018, 22, 167–175. [CrossRef]
29. Liu, P.; Liu, J. Some q-rung orthopair fuzzy Bonferroni mean operators and their application to multi-attribute group decision making. *Int. J. Intell. Syst.* 2018, 33, 315–347. [CrossRef]
30. Liu, C.; Chen, S.M.; Wang, P. Multiple-Attribute Group Decision-Making Based on q-Rung Orthopair Fuzzy Power Maclaurin Symmetric Mean Operators. *IEEE Trans. Syst. Man Cybern Syst.* 2018, 50, 3741–3756. [CrossRef]
31. Du, W.S. Minkowski-type distance measures for generalized orthopair fuzzy sets. *Int. J. Intell. Syst.* 2018, 33, 802–817. [CrossRef]
32. Riaz, M.; Salabun, W.; Farid, H.M.A.; Ali, N. A Robust q-Rung Orthopair Fuzzy Information Aggregation Using Einstein Operations with Application to Sustainable Energy Planning Decision Management. *Energies* 2020, 13, 2155. [CrossRef]
33. Sugeno, M. Theory of Fuzzy Integral and Its Applications. Ph.D. Thesis, Tokuo Institute of Technology, Tokyo, Japan, 1974.
34. Grabisch, M. The representation of importance and interaction of features by fuzzy measures. *Pattern Recogn Lett.* 1996, 17, 567–575. [CrossRef]
35. Kojadinovic, I. Modeling interaction phenomena using fuzzy measures: On the notions of interaction and independence. *Fuzzy Sets Syst.* 2002, 135, 317–340. [CrossRef]
36. Liginial, D.; Ow, T.T. Modeling attitude to risk in human decision processes: An application of fuzzy measures. *Fuzzy Set Syst.* 2006, 157, 3040–3054. [CrossRef]
37. Marichal, J.L.; Roubens, M. Dependence between criteria and multiple criteria decision aid. In Proceedings of the 2nd International Workshop on Preferences and Decisions, Trento, Italy, 1–3 July 1998; pp. 69–75.
38. Grabisch, M.; Kojadinovic, I.; Meyer, P. A review of methods for capacity identification in Choquet integral based multi-attribute utility theory. *Eur. J. Oper. Res.* 2008, 186, 766–785. [CrossRef]
39. Wu, J.; Zhang, Q. 2-order additive fuzzy measure identification method based on diamond pairwise comparison and maximum entropy principle. *Fuzzy Optim. Decis. Mak.* 2010, 9, 433–453. [CrossRef]
40. Krishnan, A.R.; Kasim, M.M.; Bakar, E.M. A short survey on the usage of Choquet integral and its associated fuzzy measure in multiple attribute analysis. *Procedia Comput. Sci.* 2015, 59, 427–434. [CrossRef]
41. Choquet, G. Theory of capacities. *Ann. d’Institute Fourier.* 1954, 5, 131–295. [CrossRef]
42. Tan, C.Q.; Chen, X. Intuitionistic fuzzy Choquet integral operator for multi-criteria decision-making. *Expert Syst. Appl.* 2010, 37, 149–157. [CrossRef]
44. Xu, Z.S. Choquet integrals of weighted intuitionistic fuzzy information. *Inf. Sci.* **2010**, *180*, 726–736. [CrossRef]

45. Xia, M.; Xu, Z. Group decision making based on intuitionistic multiplicative aggregation operators. *Appl. Math. Model.* **2013**, *37*, 5120–5133. [CrossRef]

46. Wu, J.; Chen, F.; Nie, C.; Zhang, Q. Intuitionistic fuzzy-valued Choquet integral and its application in multicity criteria decision making. *Inf. Sci.* **2013**, *222*, 509–527. [CrossRef]

47. Peng, X.; Yang, Y. Pythagorean Fuzzy Choquet Integral Based MABAC Method for Multiple Attribute Group Decision Making. *Int. J. Intell. Syst.* **2016**, *31*, 989–1020. [CrossRef]

48. Khan, M.S.A.; Abdullah, S.; Ali, M.Y.; Hussain, I.; Farooq, M. Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment. *J. Intell. Fuzzy Syst.* **2018**, *34*, 267–282. [CrossRef]

49. Yager, R.R. On the measure of fuzziness and negation part I: Membership in the unit interval. *Int. J. Gen. Syst.* **1979**, *5*, 221–229. [CrossRef]

50. Sirbiladze, G. Extremal Fuzzy Dynamic Systems: Theory and Applications. In *IFSR International Series on Systems Science and Engineering*; Springer: New York, NY, USA; Heidelberg, Germany; Dordrecht, The Netherlands; London, UK, 2013.

51. Sirbiladze, G.; Sikharulidze, A. Extentions of Probability Intuitionistic Fuzzy Aggregation Operators in Fuzzy Environmet. *Int. J. Inf. Technol. Decis.* **2018**, *17*, 621–655. [CrossRef]

52. Mergiò, J.M. The probabilistic weighted average and its application in multiperson decision making. *Int. J. Intell. Syst.* **2012**, *27*, 457–476. [CrossRef]

53. Noriss, D.; Pilsworth, B.W.; Baldwin, J.F. Medical Diagnosis from Patient Record—A Method Using Fuzzy Discrimination and Connectivity Analysis. *Fuzzy Set Syst.* **1987**, *23*, 73–87. [CrossRef]

54. Mergiò, J.M. Fuzzy Multi-Person Decision Making with Fuzzy Probabilistic Aggregations Operators. *Int. J. Fuzzy Syst.* **2011**, *13*, 163–174.

55. Mergiò, J.M.; Casanova, M.; Yang, J.B. Group decision making with experts and uncertain generalized probabilistic weighted aggregation operators. *Eur. J. Oper. Res.* **2014**, *235*, 215–224. [CrossRef]

56. Wei, G.W.; Mergiò, J.M. Methods for strategic decision-making problems with immediate probabilities in intuitionistic fuzzy setting. *Sci. Iran.* **2012**, *19*, 1939–1946. [CrossRef]

57. Garg, H. Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment. *Int. J. Intell. Syst.* **2018**, *33*, 687–712. [CrossRef]

58. Sirbiladze, G. Associated Probabilities’ Aggregations in Interactive MADM for q-Rung Orthopair Fuzzy Discrimination Environmen. *Int. J. Intell. Syst.* **2020**, *35*, 335–372. [CrossRef]

59. Béliakov, G.; Pradera, A.; Calvo, I. *Aggregation Functions: A Guide for Practitioners*; Springer-Verlag: Berlin, Germany, 2007.

60. Campos, L.M.; Bolanos, M.N. Representation of fuzzy measures through probabilities. *Fuzzy Set Syst.* **1989**, *31*, 23–36.

61. Grabisch, M. K-order additive discrete fuzzy measures and their representations. *Fuzzy Set Syst.* **1997**, *92*, 167–189. [CrossRef]

62. Yager, R.R. On the Entropy of Fuzzy Measures. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 453–561. [CrossRef]

63. Buck, B. *Maximum Entropy in Action: A Collection of Expository Essays*; Oxford University Press: New York, NY, USA, 1991.

64. Béliakov, G.; Divakov, D. Aggregation with dependencies: Capacities and fuzzy integrals. *Fuzzy Set. Syst.* **2021**, In press (Available online 24 March 2021). [CrossRef]

65. Li, J.; Yao, X.; Sun, X.; Wu, D. Determining the fuzzy measures in multiple criteria decision aiding from the tolerance perspective. *Eur. J. Oper. Res.* **2016**, *264*, 428–439. [CrossRef]

66. Béliakov, G.; Wu, J.Z. Learning fuzzy measures from data: Simplifications and optimization strategies. *Inf. Sci.* **2019**, *494*, 100–113. [CrossRef]

67. Béliakov, G.; Cabrerizo, F.J.; Enrique Herrera-Viedma, E.; Wu, J.Z. Random generation of k-interactive capacities. *Fuzzy Set. Syst.* **2021**, In press (Available online 22 December 2020). [CrossRef]

68. Béliakov, G.; Wu, J.Z. Learning k-maxitive fuzzy measures from data by mixed integer programming. *Fuzzy Set. Syst.* **2021**, *412*, 41–52. [CrossRef]

69. Aikens, C.H. Facility location models for distribution planning. *Eur. J. Oper. Res.* **1985**, *22*, 263–279. [CrossRef]

70. Klose, A.; Drexl, A. Facility location models for distribution system design. *Eur. J. Oper. Res.* **2005**, *162*, 4–29. [CrossRef]

71. Lee, S.M.; Green, G.J.; Kim, C. A multiple criteria model for the location–allocation problem. *Comput. Oper. Res.* **1981**, *8*, 1–8. [CrossRef]

72. Puerto, J.; Fernandez, F.R. Multi-criteria minimax facility location problems. *J. Multi-Criteria Decis. Anal.* **1999**, *8*, 268–280. [CrossRef]

73. Ross, G.T.; Soland, R.M. A multicriteria approach to location of public facilties. *Eur. J. Oper. Res.* **1980**, *4*, 307–321. [CrossRef]

74. Erkut, E.; Karagiannidis, A.; Perkoulidis, G.; Tjandra, S.A. A multicriteria facility location model for municipal solid waste management in North Greece. *Eur. J. Oper. Res.* **2008**, *187*, 1402–1421. [CrossRef]

75. Anagnostopoulos, K.; Doukas, H.; Psarras, J. A linguistic multicriteria analysis system combining fuzzy sets theory, ideal and anti-ideal points for location site selection. *Expert Syst. Appl.* **2008**, *35*, 2041–2048. [CrossRef]

76. Ishii, H.; Lee, Y.L.; Yeh, Y.Y. Fuzzy facility location problem with preference of candidate sites. *Fuzzy Set Syst.* **2007**, *158*, 1922–1930. [CrossRef]

77. Yang, L.; Ji, X.; Gao, Z.; Li, K. Logistics distribution centers location problem and algorithm under fuzzy environment. *J. Comput. Appl. Math.* **2007**, *208*, 303–315. [CrossRef]
78. Kahraman, C.; Ruan, D.; Dogan, I. Fuzzy group decision-making for facility location selection. *Inf. Sci.* 2003, 157, 135–153. [CrossRef]
79. Liang, G.S.; Wang, M.J.J. A fuzzy multi-criteria decision-making method for facility site selection. *Int. J. Prod. Res.* 1991, 29, 2313–2330. [CrossRef]
80. Chen, C.T. A fuzzy approach to select the location of the distribution center. *Fuzzy Set Syst.* 2001, 118, 65–73. [CrossRef]
81. Chou, S.Y.; Chang, Y.H.; Shen, C.Y. A fuzzy simple additive weighting system under group decision making for facility location selection with objective/subjective attributes. *Eur. J. Oper. Res.* 2008, 189, 132–145. [CrossRef]
82. Chu, T.C. Facility location selection using fuzzy TOPSIS under group decisions. *Int. J. Uncertain. Fuzziness Knowl.* 2002, 10, 687–701. [CrossRef]
83. Jahanshahloo, G.R.; Hosseinzadeh, L.F.; Izadikhah, M. Extension of the TOPSIS method for decision-making problems with fuzzy data. *Appl. Math. Comput.* 2006, 181, 1544–1551. [CrossRef]
84. Saghafian, S.; Hejazi, S.R. Multi-criteria group decision making using a modified fuzzy TOPSIS procedure. In Proceedings of the International Conference on Computational Intelligence for Modeling, Control and Automation, and International Conference on Intelligent Agents, Web Technologies and Internet Commerce, Vienna, Austria, 28-30 November 2005; IEEE: Piscataway Township, NJ, USA, 2005.
85. Wang, Y.J.; Lee, H.S. Generalizing TOPSIS for fuzzy multicriteria group decision making. *Comput. Math. Appl.* 2007, 53, 1762–1772. [CrossRef]
86. Yong, D. Plant location selection based on fuzzy TOPSIS. *Int. J. Adv. Manuf. Technol.* 2006, 28, 839–844. [CrossRef]
87. Sirbiladze, G.; Ghvaberidze, B.; Matsaberidze, B.; Sikharulidze, A. Multi-Objective Emergency Service Facility Location Problem Based on Fuzzy TOPSIS. *Bull. Georgian Natl. Acad. Sci.* 2017, 11, 23–30.
88. Roubens, M. Interaction between criteria and definition of weights in MCDA problems. In Proceedings of the 44th Meeting of the European Working Group “Multicriteria Aid for Decisions”, Brussels, Belgium, 3–4 October 1996.
89. Denneberg, D. *Non-Additive Measure and Integral*; Kluwer Academic: Norwell, MA, USA, 1994.