Abstract

Physical processes ranging from the Lamb shift to the energy loss $dE/dx$ of a charged particle traversing a plasma entail processes that occur over a wide range of energy or length scales. Different physical mechanisms dominate at one or the other end of this range. For example, in the energy loss problem, soft collisions that are screened by collective effects are important at large distances, while at short distances hard collisions are important where the exact details of the single-particle interactions must be taken into account. We introduce a novel application of dimensional continuation. The soft processes dominate at all scales when the spatial dimension $\nu$ is less than 3, and we use them to compute the result to leading order for $\nu < 3$. On the other hand, the hard processes dominate at all scales for $\nu > 3$, and we use them to compute the result to leading order for these spatial dimensions. We then explain why the sum of the analytic continuation of these disparate mechanisms yields the correct leading-order result for the physical limit at $\nu = 3$ dimensions. After applying this new method to the energy loss problem in some detail, we then show how it also provides a very short and easy way to compute the Lamb shift.
I. INTRODUCTION

The purpose of this paper is to introduce a new application of dimensional continuation to physical problems that involve simultaneously both short-distance, ultraviolet processes and long-distance, infrared processes. Such problems typically involve a logarithm of a large ratio of two different scales. Although the overall coefficient of the logarithm is usually easy to compute, the constant under the logarithm is often difficult to find. Our new method makes the computation of this constant under the logarithm easy. We shall illustrate the simplicity of the method and its wide range of applicability by using it to solve two very diverse physical problems, the $dE/dx$ energy loss of a fast particle traversing a fully ionized, but non-relativistic dilute plasma and the Lamb shift of hydrogen-like atoms. We shall first work out the energy loss problem in some detail because it involves rather elementary physics, and a self-contained exposition can be presented within a short space. The plasma example also does share some common features with relativistic plasmas such as those that appear in QCD and weak interaction physics. Indeed, this simple problem serves as a useful test of the validity of methods used in those more complex problems. In this regard, it should be noted that although our non-relativistic, Abelian example is conceptionally simpler than those in the relativistic, non-Abelian gauge theory, the non-relativistic plasma involves a Debye length and a plasma frequency that are quite distinct parameters while, in the extreme relativistic theory, these parameters become essentially the same (in lowest-order perturbation theory.) Thus a verification of the treatment of different physical processes may be more difficult in the relativistic theory, and the non-relativistic case may well serve as a quite useful testing bed for methods used in the relativistic case. Here our aim is to explain our method, and although we shall re-derive results that have been obtained before, the derivations will clearly describe and illustrate the power of the method. It has been applied recently to plasma energy-loss problems in which the constants under the logarithms were not known [1], [2].

Dimensional regularization is widely employed in relativistic quantum field theory to make otherwise divergent expressions finite and well behaved, and then to implement the renormalization procedure. It should be emphasized that here we are making use of a different and novel application of continuation to spatial dimensions $\nu \neq 3$ to compute results that are always well-defined and finite at the physical $\nu = 3$ dimension. We are not using dimensional continuation to render infinities finite so as to perform renormalizations as one does in quantum field theory. Moreover, our purpose here is to introduce and describe this new application of dimensional continuation by two very different but well-known physical problems in order to illustrate its range of application. What is new is the method.

It is worth first illustrating our method with a trivial mathematical example, the behavior of the modified Hankel function $K_{\nu}(z)$ in the small argument $z$ limit with the index $\nu$ also small. The argument $z$ will play the role of the small parameter in our work below; the index $\nu$ will play the role of the dimensionality except that in this simple Bessel function example we shall examine the region where $\nu$ is near zero, not three. In general, the Hankel function

\[ K_{\nu}(z) \]



1In the Lamb shift example at the end of this paper, we work with renormalized quantities that are well-defined and finite.
function has the integral representation

\[ K_\nu(z) = \frac{1}{2} \int_0^\infty \frac{dk}{k} k^\nu \exp \left\{ -\frac{z}{2} \left( k + \frac{1}{k} \right) \right\}. \]  

(1)

Although \( k \) is simply a dummy integration variable, it is convenient to think of it as a wave number or momentum variable. When \( z \) is small, \( \exp \left\{ -\frac{z}{2} \left( k + \frac{1}{k} \right) \right\} \) may be replaced by 1 except when one or the other of the factors \( \exp \left\{ -\frac{zk}{2} \right\} \) or \( \exp \left\{ -\frac{z}{2k} \right\} \) is needed to make the \( k \) integration converge in the neighborhood of one of its end points. When \( \nu \) is slightly less than zero, the integral (1) is dominated by the small \( k \), “infrared or long-distance”, region. In this case, only the \( \exp \left\{ -\frac{zk}{2} \right\} \) factor is needed to provide convergence, and we have

\[ \nu < 0 : \]

\[ K_\nu(z) \simeq \frac{1}{2} \int_0^\infty \frac{dk}{k} k^\nu \exp \left\{ -\frac{z}{2k} \right\}. \]

(2)

The variable change \( k = z/(2t) \) places this integral in the form of the standard representation of the gamma function, and we thus find that the leading term for small \( z \) in the region \( \nu < 0 \) is given by

\[ \nu < 0 : \]

\[ K_\nu(z) \simeq \frac{1}{2} \left( \frac{z}{2} \right)^\nu \Gamma(-\nu) \]

\[ \simeq -\frac{1}{2\nu} \left( \frac{z}{2} \right)^\nu (1 + \nu\gamma), \]

(3)

where \( \gamma = 0.5772 \cdots \) is Euler’s constant. Note that the second line describes the behavior for \( \nu < 0 \) near \( \nu = 0 \) including the correct finite constant as well as the singular pole term.

When \( \nu \) is slightly greater than zero, the integral (1) is dominated by the large \( k \), “ultraviolet or short-distance” regions. In this case, only the \( \exp \{ -zk/2 \} \) factor is needed to provide convergence, and we have

\[ \nu > 0 : \]

\[ K_\nu(z) \simeq \frac{1}{2} \int_0^\infty \frac{dk}{k} k^\nu \exp \left\{ -\frac{zk}{2} \right\}. \]

(4)

The integral again defines a gamma function, and so

\[ \nu > 0 : \]

\[ K_\nu(z) \simeq \frac{1}{2^\nu} \left( \frac{z}{2} \right)^{-\nu} (1 - \nu\gamma), \]

(5)

with again the result containing the correct finite constant as well as the singular pole term.

The result (3) for \( \nu < 0 \) can be analytically continued into the region \( \nu > 0 \). In this region it involves a higher power of \( z \) than that which appears in the other evaluation (5), and hence this analytic continuation of the leading result for \( \nu < 0 \) into the region \( \nu > 0 \) becomes sub-leading here. Similarly, the result (3) for \( \nu > 0 \) may be analytically continued into the region \( \nu < 0 \) where it now becomes sub-leading. An examination of the defining
integral representation (1) shows that these sub-leading analytic continuation terms are, in fact, the dominant, first-subleading terms. For \( \nu > 0 \) one term is leading and the other subleading, while for \( \nu < 0 \) their roles are interchanged. Thus their sum

\[
K_\nu(z) \simeq \frac{1}{2\nu} \left\{ \left( \frac{z}{2} \right)^{-\nu} [1 - \nu \gamma] - \left( \frac{z}{2} \right)^{\nu} [1 + \nu \gamma] \right\}
\]  

contains both the leading and the first subleading terms for both \( \nu > 0 \) and \( \nu < 0 \). In the limit \( \nu \to 0 \) the (“infrared” and “ultraviolet”) pole terms in this sum cancel, with the variation of the residues of the poles producing a logarithm, yielding the familiar small \( z \) result

\[
K_0(z) = -\ln(z/2) - \gamma.
\]  

It must be emphasized that the correct constant terms \([\ln 2 - \gamma]\) are obtained by this method in addition to the logarithm \(-\ln z\) which is large for small \( z \). The result (6) was derived from the analytic continuation of results that were easy to compute in one or another region where either “infrared” or “ultraviolet” terms dominated. This is the essence of our method. Of course, the general result (6) could be obtained by a more careful computation of both the leading and first-subleading terms in either of the separate \( \nu > 0 \) or \( \nu < 0 \) regions as was done in the previous footnote. In some of the physical examples that we shall work out, such an extraction of the subdominant terms would be very difficult indeed, although possible in principle. Thus our method acquires real power for the physical problems.

In the physical examples described below, one could object that we do not explicitly prove that larger subleading terms are not present. In the limit of the typical small parameter \( z \to 0 \) that concerns us (with now \( \nu \to 3 \)), for our physical examples we shall find (after extracting some overall factor of \( z \)) leading terms of order \( z^{(3-\nu)} \) for \( \nu > 3 \) and \( z^{(\nu-3)} \) for \( \nu < 3 \). One may then ask if an additional term that has a power dependence between \( z^{(\nu-3)} \) and \( z^{(3-\nu)} \) can appear. However, simple dimensional analysis shows that such terms of intermediate order cannot appear. The point is that these physical examples involve only two different physical mechanisms that dominate at large and small scales, and that the two different mechanisms involve different combinations of the basic physical parameters and hence give quite different dependencies on the small parameter when the dimension \( \nu \)

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2For example, subtracting the leading term (3) for \( \nu < 0 \) from the integral representation (1) gives

\[
K_\nu(z) - \frac{1}{2} \left( \frac{z}{2} \right)^{\nu} \Gamma(-\nu) = \frac{1}{2} \int_0^\infty \frac{dk}{k} k^{\nu} \left[ e^{-z k/2} - 1 \right] e^{-z/(2k)}.
\]

For \( 0 > \nu > -1 \), the integral on the right-hand-side of the equation converges when the the final exponential factor in the integrand is replaced by unity, the \( z \to 0 \) limit of this factor. Hence this final factor may be omitted in the evaluation of the first sub-leading term. A partial integration presents the result as

\[
\frac{z}{4\nu} \int_0^\infty dk k^{\nu} e^{-zk/2},
\]

whose evaluation gives precisely the analytic continuation of the leading term (3) for \( \nu > 0 \).
departs from $\nu = 3$. Incidentally, it should go without saying that the physical examples that we shall study have a basic theoretical description that is valid for a range of spatial dimensions $\nu$ about $\nu = 3$.

II. ENERGY LOSS IN A PLASMA

The usual method for obtaining the energy loss for a charged particle moving through matter is to divide the calculation into two parts: The long-distance, soft collisions and the short-distance, hard collisions. Collective effects are important in the long-distance part, and it is evaluated from the $\mathbf{j} \cdot \mathbf{E}$ power loss of a particle moving in a dielectric medium. The hard collisions are described by Coulomb scattering. The rub is to join the disparate pieces together. For the case of classical scattering, this is often done by computing the energy loss in Coulomb scattering out to some impact parameter, and then adding the $\mathbf{j} \cdot \mathbf{E}$ energy loss for all larger impact parameters. Although such methods do yield the correct large logarithm without much difficulty, the logarithm of the ratio of the two scales which is large, the purely numerical constants (which one expects to be of order one) that accompany the logarithm are harder to compute. Here we describe an easily applied method that yields a unique result – the result including the constants in addition to the large logarithm. The new idea is to compute the energy loss from Coulomb scattering over all angles, but for dimensions $\nu > 3$ where there are no infrared divergences. A separate calculation of the energy loss using the $\mathbf{j} \cdot \mathbf{E}$ heating is done for $\nu < 3$, where the volume integration may be extended down to the particle’s position without encountering an ultraviolet divergence. Both of these results have a simple pole at $\nu = 3$, but they both may be analytically continued beyond their initial range of validity. In their original domain of dimension $\nu$, both calculations are performed to the leading order in the plasma density. As will be seen, although the Coulomb scattering result is the leading order contribution for $\nu > 3$, it is of subleading order when $\nu < 3$. Conversely, the $\mathbf{j} \cdot \mathbf{E}$ heating is subleading for $\nu > 3$ but leading for $\nu < 3$. Hence, the sum of the two (analytically continued) processes gives the leading and (first) subleading terms in the plasma density for all dimensions $\nu$, and thus, in the limit of this sum at $\nu = 3$, the pole terms must cancel with the remainder yielding the correct physical limit to leading order in the plasma density.

The fully ionized, classical plasma with which we are concerned is described exactly by a coupled set of kinetic equations, the well-known BBGKY hierarchy as described, for example, in Section 3.5 of ref. [3]. We are interested, however, in the computation to leading order in the plasma density of the energy loss of a fast particle traversing the plasma. The correct equations that govern the leading order low-density behavior change as the spatial dimensionality $\nu$ changes. For $\nu < 3$, the long-distance, collective effects dominate, and the formula derived by Lenard and Balescu applies [4], [5]. This formula describes the interaction of the various species that the plasma may contain. In the limit in which one species is very dilute, as is our case in which we examine the motion of a single, fast “test particle” moving through the plasma, the energy lost in the particle motion is described by its $\mathbf{j} \cdot \mathbf{E}$ Joule heating with the background plasma response given by the permittivity of a collisionless plasma. On the other hand, when the spatial dimension $\nu$ is greater than 3, the short-distance, hard Coulomb collisions dominate. For these dimensions, the leading low density limit of the BBGKY hierarchy is described by the familiar Boltzmann equation.
The Boltzmann equation is derived, for example, also in Section 3.5 of ref. [3]. We use the Boltzmann equation to obtain the leading order energy loss rate when \( \nu > 3 \). Again, since we are concerned with the motion of a single, fast “test particle”, the Boltzmann equation reduces to the product of the energy loss weighted cross section times the plasma density. The derivations that we have just described, which start from first principles, justify the methods outlined in the previous paragraph, the methods that we shall use.

Since we are only interested in describing the new method, we simplify the discussion by treating only the electrons in a classical plasma (electron recoil gives the dominant energy loss since they are light), and by taking the moving projectile velocity \( v_p \) to be much larger than the electron velocities in the plasma so that the latter may be neglected relative to \( v_p \). We shall assume, however, that the projectile velocity is small in comparison with the velocity of light so that this particle produces a simple Coulomb field (as modified by the plasma) and that non-relativistic mechanics applies.

\[ A. \ \nu < 3 \]

We first compute the heating with \( \nu < 3 \). The electric field \( \mathbf{E} \) is produced by the point projectile of charge \( e_p \) moving with velocity \( \mathbf{v}_p \), which gives the charge density

\[
\rho(r, t) = e_p \delta(\nu)(r - \mathbf{v}_p t)
\]

\[
= e_p \int \frac{(d^\nu k)}{(2\pi)^\nu} \exp \{i \mathbf{k} \cdot (r - \mathbf{v}_p t)\}.
\]

In our non-relativistic limit, the electric field is curl free while \( \nabla \cdot \mathbf{D} = \rho \), and so

\[
\mathbf{E}(r, t) = e_p \int \frac{(d^\nu k)}{(2\pi)^\nu} \frac{-i \mathbf{k}}{k^2} \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_p) \exp \{i (\mathbf{k} \cdot \mathbf{r} - \mathbf{k} \cdot \mathbf{v}_p t)\},
\]

where \( \epsilon(\mathbf{k}, \omega) \) is the wavenumber and frequency dependent electric permittivity of the plasma. Note that we use rationalized Gaussian units so that electrostatic potential of a point charge in three dimensions has the form \( \phi = e/(4\pi r) \). Since the current of the projectile is given by \( j(\mathbf{r}, t) = \mathbf{v}_p \rho(\mathbf{r}, t) \), this energy loss mechanism gives \( dE/dt = -e_p \mathbf{v}_p \cdot \mathbf{E}(\mathbf{v}_p t, t) \), or

\[
\frac{dE_c}{dt} = e_p^2 \int \frac{(d^\nu k)}{(2\pi)^\nu} \frac{i}{k^2} \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_p).
\]

The electric permittivity is the boundary value of an analytic function, \( \epsilon(\mathbf{k}, \omega) = \epsilon(\mathbf{k}, \omega + i\eta) \), \( \eta \to 0^+ \), with \( \epsilon(\mathbf{k}, z) = \epsilon(\mathbf{k}, -z) \) an even function of \( z \) which also depends only on the modulus \( |k| \) of \( \mathbf{k} \) (by rotational invariance). [This is demonstrated in the explicit form (12) below.] Thus, in view of the factor \( \mathbf{k} \cdot \mathbf{v}_p \) which extracts the odd part of the rest of the integrand in Eq. (10), we may write this energy loss as

\[
\frac{dE_c}{dt} = e_p^2 \int \frac{(d^\nu k)}{(2\pi)^\nu} \frac{1}{k^2} \Im \left\{-\mathbf{k} \cdot \mathbf{v}_p / \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_p)\right\},
\]

in which \( \Im \) denotes the imaginary part.
For our calculation to leading order in the plasma density, the permittivity function may be taken in the first (one-loop) approximation \[6\] (the ring graph of quantum statistical mechanics)

\[
\epsilon(k, \omega) = 1 - \frac{e^2 k^2}{2\pi \hbar^2} \int \frac{(d^\nu p)}{(2\pi \hbar)^\nu} n_e(p) \frac{2 \left[ (p + \hbar k)^2/(2m_e) - p^2/(2m_e) \right]}{\hbar^2 (\omega + i\eta)^2 - [(p + \hbar k)^2/(2m_e) - p^2/(2m_e)]^2},
\]

where \(\eta \to 0^+\) in the denominator corresponds to a retarded response, and where

\[
\int \frac{(d^\nu p)}{(2\pi \hbar)^\nu} n_e(p) = n_e
\]

is the electron number density, and \(m_e\) is the mass of the electron. Since we are examining the limit in which the projectile is moving very rapidly with respect to the average thermal velocity of the electrons, the electron motion in the plasma may neglected. This corresponds to setting \(p = 0\) in the remainder of the integrand in Eq. (12), which gives

\[
\epsilon(k, \omega) = 1 - \frac{\omega^2_e}{(\omega + i\eta)^2 - (\hbar k^2/2m_e)^2},
\]

where \(\omega_e\) is the plasma frequency for the electrons defined by

\[
\omega_e^2 = \frac{e^2 n_e}{m_e}.
\]

With \(\omega = k \cdot v_p\), two length scales appear, \(v_p/\omega_e\) and \(\hbar/(m_e v_p)\). It is easy to check that the scale for the wave number integration in the energy loss \[11\] (with, of course, \(\nu < 3\)) is set by the former, classical length. Hence the latter quantum length appears as a correction involving the dimensionless parameter \(\hbar^2 \omega_e^2/(m_e v_p^2)^2\). Since \(\omega_e^2\) is proportional to the electron density, and we are working to leading order in this density, we must omit this small parameter and use the purely classical limit \[16\]

\[
\epsilon(0, \omega) = 1 - \frac{\omega^2_e}{(\omega + i\eta)^2},
\]

This is the limit to be used in the energy loss \[11\]. In this limit,

\[
\text{Im} \left\{ \frac{-\omega}{\epsilon(0, \omega)} \right\} = \pi \omega_e^3 \delta \left( \omega^2 - \omega_e^2 \right).
\]

Hence, performing the integration over the component of \(k\) parallel to \(v_p\), and writing \(dx = v_p dt\) gives

\[
3^3\text{Note that our dimensional regularization method yields the correct leading-order result, unac-}
\]

accompanied by any higher-order terms that only give a part of the higher-order corrections and thus represent spurious corrections.
\[
\frac{dE_\prec}{dx} = \frac{e_p^2}{2} \int \frac{(d^{\nu-1}k)}{(2\pi)^{\nu-1}} \frac{\omega_e^2}{\omega_e^2 + v_p^2 k^2}.
\]

Exponentiating the denominator via
\[
D^{-1} = \int_0^\infty dse^{-sD},
\]
interchanging integrals, performing the resulting \(\nu - 1\) Gaussian \(k\) integrals, and recognizing the final \(s\) integral as a standard representation of the \(\Gamma\) function gives
\[
\frac{dE_\prec}{dx} = \frac{e_p^2 \omega_e^2}{4\pi v_p^2} \left( \frac{\omega_e^2}{4\pi v_p^2} \right)^{\frac{\nu-1}{2}} \Gamma \left( \frac{3 - \nu}{2} \right),
\]
or, with the neglect of terms which vanish when \(\nu \to 3\),
\[
\frac{dE_\prec}{dx} = \frac{e_p^2 \omega_e^2}{4\pi v_p^2} \left( \frac{\omega_e^2}{4\pi v_p^2} \right)^{\frac{\nu-1}{2}} \left\{ \frac{1}{3 - \nu} - \frac{\gamma}{2} \right\}.
\]
The pole in this expression, which becomes negative when \(\nu > 3\), corresponds to the ultraviolet divergence which appears when \(\nu \to 3\) in the wavenumber integral (18).

**B. \(\nu > 3\)**

We turn now to the \(\nu > 3\) case where the energy loss is computed by single-particle scattering. By the conservation of energy, the energy loss in the scattering of the projectile velocity \(v_p \to v_p'\) on electrons whose initial velocity may be neglected is
\[
\Delta E = -\frac{m_p}{2} \left[ v_p'^2 - v_p^2 \right] = \frac{m_e}{2} v_e'^2,
\]
where \(v_e'\) is the speed of the scattered electron. Since the initial electron has negligible momentum, this can be written in the invariant form \(\Delta E = q^2/(2m_e)\), where \(q\) is the electron momentum transfer in the scattering process. With the initial electron at rest, the differential rate of scattering is \(v_p n_e d\sigma\), where \(n_e\) is the electron density in the plasma and \(d\sigma\) is the cross section element. Since \(dx = v_p dt\), the energy loss for \(\nu > 3\) is given by
\[
\frac{dE_\succ}{dx} = \frac{n_e}{2m_e} \int d\sigma q^2.
\]
We first evaluate this scattering contribution when the interaction is weak, when \(\eta = e_p e/\hbar v_p \ll 1\). In this case, the quantum-mechanical Born approximation result is appropriate with, in \(\nu > 3\) dimensions,
\[
\int d\sigma_B q^2 = \int \frac{(d^\nu P')}{(2\pi \hbar)^\nu} 2\pi \hbar \delta \left( \frac{p'^2}{2m} - \frac{p^2}{2m} \right) \left( \frac{\hbar e_p e}{q^2} \right)^2 \frac{1}{v} q^2.
\]
Here \((1/m) = (1/m_e) + (1/m_p)\) defines the reduced mass \(m\) and \(v\) is the relative velocity between the electron and the projectile. Writing \(q^2 = 4m^2v^2\sin^2\theta/2\), and
\[
(d'p') = m p^{(\nu-2)} d(p^2/2m) \Omega_{\nu-2} \sin^{\nu-2} \theta \, d\theta,
\]
with \(\sin^{\nu-2} \theta = [2 \cos \theta/2 \sin \theta/2]^{\nu-2}\), and noting that the solid angle \(\Omega_{\nu-2}\) is given by
\[
\frac{\Omega_{\nu-2}}{2\pi} = \frac{\pi^{(\nu-3)/2}}{\Gamma\left(\frac{\nu-1}{2}\right)},
\]
we get, on setting \(\chi = \theta/2\),
\[
\int d\sigma_B q^2 = \left(\frac{e_p e}{2\pi v^2}\right)^2 \left(\frac{m^2 v^2}{\pi \hbar^2}\right)^{(\nu-3)/2} \frac{1}{\Gamma\left(\frac{\nu-1}{2}\right)} \int_0^{\pi/2} d\chi \cos^{\nu-2} \chi \sin^{\nu-4} \chi.
\]
The integral which appears here has the value \((\nu - 3)^{-1} + O(\nu - 3)\) as one can show by dividing it into two parts with a suitable partial integration or by expressing it in terms of the standard integral representation of the Beta function. Since we neglect the motion of the initial electron, the relative velocity \(v\) may be replaced by the projectile velocity \(v_p\), and so using the result in Eq. (23) gives
\[
\frac{dE_{Qm}}{dx} = \frac{e_p^2 \omega_e^2}{4\pi v_p^2} \left(\frac{m^2 v_p^2}{\pi \hbar^2}\right)^{\frac{\nu-3}{2}} \left\{ \frac{1}{\nu - 3} + \frac{\gamma}{2} \right\}.
\]
The pole in this expression, which becomes negative when \(\nu < 3\), corresponds to the infrared divergence of the momentum integral (24) in the \(\nu \rightarrow 3\) limit.

C. \(\nu = 3\)

When the result (28) is added to that in Eq. (21) the divergent pole terms cancel, and the physical limit \(\nu \rightarrow 3\) is
\[
\frac{dE_{Qm}}{dx} = \frac{e_p^2 \omega_e^2}{4\pi v_p^2} \ln \left(\frac{2m v_p^2}{\hbar\omega_e}\right).
\]
using this delta function to eliminate the polar angle $\theta$ of the solid angle integration gives a remaining integral over the magnitude $k$ of $|k|$, $\int dk/k$. The leading terms for small $\omega_c$ of the upper and lower limits of this logarithmic integral give the result (29), except that the correct reduced mass $m$ in Eq. (29) is replaced by the electron mass $m_e$ since the current $j$ describes the motion of a very heavy projectile particle. This sort of calculation was done some time ago by Lindhard [7]. Although the reduced mass correction is negligible when the projectile is a heavy ion, it does represent a conceptual shortcoming of the quantum-corrected, joule heating treatment. Moreover, this treatment completely breaks down when the projectile is itself an electron. This sort of dielectric treatment is also restricted to the case of a cold plasma whose electron velocities are much less than that of the projectile. On the other hand, our method is easily extended [1] to treat the case of a hot plasma where this restriction is not imposed, and again a complete calculation can be performed which includes the constants in addition to the logarithm.

Although, as we have just seen, the $j \cdot E$ calculation can be improved to obtain the correct energy loss (except for the replacement of the reduced by the electron mass), with the computation always done in three dimensions, we do not know of a similar improvement of the Boltzmann equation in three dimensions which yields the correct result. One might be tempted to replace the Coulomb potential by the screened Debye potential. This alteration changes the $1/(q^2)^2$ factor in the cross section formula (24) by

$$\left( \frac{1}{q^2} \right)^2 \rightarrow \left( \frac{1}{q^2 + \hbar^2 \kappa^2} \right)^2,$$

in which $\kappa^2 = e^2 n/T$ is the squared Debye wave number for the plasma. This alteration removes the long-distance infra-red divergence, and the cross section formula (24) now converges in three dimensions. Exponentiating this denominator using the integral representation (19) with an additional factor of $s$ in the integrand to produce the square makes the remainder of the calculation easy, and one finds that

$$\frac{dE_D}{dx} = \frac{e^2 \omega}{4\pi v_p^2} \ln \left[ \left( \frac{2m v_p}{\hbar \kappa} \right) - \frac{1}{2} \right].$$

Although the constant out in front of the logarithm is again the correct over-all constant, the argument of the logarithm is quite different from the correct form given in Eq. (29). This should have been expected at the outset because Debye screening describes the static screening of a particle at rest in the plasma, not a dynamical screening of a fast moving particle which is the case that we are examining. As far as I know, such a dynamical screening within a Boltzmann equation context cannot be done.

Our method can be used to extend the result (29) to arbitrary values of $\eta = e e_p / (4\pi \hbar v_p)$, always retaining the correct additional constants. To do this, we use some clever mathematics of Lindhard and Sorensen [8], but in a manner which justifies that these constants have been kept. Namely, we compute

$$\Delta \frac{dE_D}{dx} = \frac{n_e}{2m_e} \int (d\sigma - d\sigma_B) q^2.$$

This difference is well behaved in the limit $\nu \rightarrow 3$ since the pole at $\nu = 3$ produced by the cross section integral comes from soft, infrared physics which is completely contained in the
Born approximation $d\sigma_B$. Hence the three-dimensional partial wave decomposition of the scattering amplitude may be used, and then standard manipulations yield

$$
\int (d\sigma - d\sigma_B) q^2 = 2\pi \hbar^2 \sum_{l=0}^{\infty} (l + 1) \left\{ 2 - e^{2i(\delta_l - \delta_{l+1})} - e^{-2i(\delta_l - \delta_{l+1})} \right\}
- \left\{ 2 - e^{2i(\delta_l - \delta_{l+1})} - e^{-2i(\delta_l - \delta_{l+1})} \right\} B.
$$

(33)

For the Coulomb potential

$$
e^{2i\delta_l} = \frac{\Gamma(l + 1 + i\eta)}{\Gamma(l + 1 - i\eta)} e^{i\phi},
$$

(34)

where the phase $\phi$ is independent of $l$. Using $\Gamma(z + 1) = z\Gamma(z)$ and a little algebra, we find that

$$
\int (d\sigma - d\sigma_B) q^2 = 4\pi^2 \hbar^2 \sum_{l=0}^{\infty} \left[ \frac{1}{l + 1 + i\eta} + \frac{1}{l + 1 - i\eta} - \frac{2}{l + 1} \right]
= -\frac{e^2e_p^2}{4\pi v_p^2} 2 \left[ \text{Re} \psi(1 + i\eta) + \gamma \right],
$$

(35)

where $\psi(z)$ is the logarithmic derivative of the gamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$, and $\text{Re}$ denotes the real part. Recalling the definition (15) of the plasma frequency, we now have

$$
\Delta \frac{dE_{\gamma}}{dx} = -\frac{e^2e_p^2}{4\pi v_p^2} 2 \left[ \text{Re} \psi(1 + i\eta) + \gamma \right],
$$

(36)

with the energy loss for all $\eta$ values given by

$$
d\frac{dx}{dx} = d\frac{E_{Qm}}{dx} + \Delta \frac{dE_{\gamma}}{dx}
= \frac{e^2e_p^2}{4\pi v_p^2} \left\{ \ln \left( \frac{2mv_p^2}{\hbar\omega_e} \right) - \left[ \text{Re} \psi(1 + i\eta) + \gamma \right] \right\}.
$$

(37)

In the classical case, $\eta = ee_p/(4\pi h\nu_p)$ becomes large. Using the limit

$$
|z| \to \infty : \quad \psi(1 + z) = \ln z + O(z^{-1}),
$$

(38)

Eq. (37) yields the classical form

$$
\frac{dE_{Cl}}{dx} = \frac{e^2e_p^2}{4\pi v_p^2} \ln \left( 2e^{-\gamma} \frac{4\pi mv_p^3}{e_p e\omega_e} \right).
$$

(39)

This result, including the proper constant $2e^{-\gamma}$ that appears within the logarithm, was obtained long ago by Kramers [10]. It may also be obtained directly with our dimensional continuation methods by using the classical Coulomb scattering cross section for dimension $\nu > 3$ in the scattering energy loss expression (23).
III. LAMB SHIFT

Essentially the same method applied here has been used before in my Quantum Field Theory book [12] to calculate the Lamb shift for hydrogen-like atoms, with the small parameter role of the plasma density replaced by the nuclear charge $Z e$. That exposition, however, was presented in a somewhat mystical manner, and it unfortunately did not bring out the essence of the method. This will be rectified now and the process will provide another example of how the method works. Section 8.7 of ref. [12] explains in detail how a radiative energy correction may be expressed as a matrix element of the electron self-energy operator $\Sigma(E)$ in Coulomb, bound-state Dirac wave functions of energy $E$. The Lamb shift is an energy difference that has both infrared and ultraviolet contributions just as in the more elementary plasma energy loss example explained above. In $\nu$ spatial dimensions, these become two distinct physical processes that scale in different ways with a characteristic atomic energy $E$: Removing a common overall factor, the ultraviolet contribution behaves as $E^\nu$, while the infrared contribution goes as $E^{\nu-2}$. Since the energy $E$ vanishes when $Ze^2$ vanishes, we may take $E$ (implicitly divided by some fixed energy scale to yield a dimensionless number) as our small parameter. Thus, just as in the previous plasma case, the infrared part dominates when $\nu < 3$, the ultraviolet part dominates when $\nu > 3$, the sum of the two contributions analytically extended in the vicinity of $\nu = 3$ always contains both the dominant and leading sub-dominant terms, and so the $\nu \to 3$ limit of this sum yields the correct leading-order Lamb shift. To tame the infrared divergences which are prevalent when $\nu < 3$, the binding of the electron must be accounted for. To tame the ultraviolet divergences that may appear when $\nu > 3$, a relativistic treatment must be made. We turn now to sketch this calculation. The detailed expressions that we shall need are derived and presented in ref. [12].

A. $\nu < 3$

With soft (virtual) photon exchange, the leading terms are given by the non-relativistic limit of the electron motion. In this non-relativistic limit, the calculation is most easily performed in the radiation gauge. The Coulomb self-energy contribution is removed by a mass renormalization. The photon exchange contribution to the electron self-energy is properly renormalized by a subtraction so that it vanishes for a free particle. An elementary computation yields

\[4In this formulation, the vacuum polarization contribution to the Lamb shift appears as a separate modification of the Coulomb potential. This is a simple correction which does not involve an interplay between long and short distances that concerns us in this paper, so we omit the effect of vacuum polarization here.

5Simple dimensional analysis shows that characteristic atomic energy in $\nu$ spatial dimensions is given by $E = (\hbar^2/m)(Z^2e^4 m^2/h^4)^{(4-\nu)^{-1}}$, which reduces to the familiar scale $E = Z^2e^4 m/\hbar^2$ in three dimensions. Since the scaling behavior of $\Sigma(E)$ is more simply expressed in terms of $E$ rather than a dimensionless parameter formed from $Ze^2$, we use $E$ as our small parameter.
\[ \Sigma_<(E) = e^2 \int \frac{(d^\nu k)}{(2\pi)^\nu} \left( \delta_{lm} - \hat{k}_l \hat{k}_m \right) \frac{1}{2k^2} \frac{p}{mc} \cdot \frac{H - E}{H - k \hbar c - i\epsilon} \cdot \frac{p}{mc}. \]  

(40)

Here \( H \) is the non-relativistic Hamiltonian for the hydrogen-like atom with nuclear charge \( Ze \). This result, which may be obtained from old-fashioned second-order time-dependent perturbation theory, involves the atomic Coulomb exchange to all orders as shown in Fig. 1. It is just Eq. (5) of the original Lamb shift paper of Bethe [13] except that it is written in \( \nu \) rather than 3 dimensions. It is also essentially Eq. (8.7.43) of ref. [12]. Performing the integrations (as are explicitly done in the ref. [12]) gives, with the neglect of terms that vanish at \( \nu = 3 \),

\[ \Sigma_<(E) = \frac{2}{3\pi} \frac{e^2}{4\pi \hbar c} \left[ \frac{1}{3 - \nu} + \frac{5}{6} - \frac{\gamma}{2} \right] \frac{p}{mc} \cdot (H - E) \left[ \frac{H - E - i\epsilon}{\sqrt{\pi} \hbar c} \right]^{\nu - 3} \frac{p}{mc}. \]  

(41)

In describing the scaling of the results with respect to the small parameter \( \mathcal{E} \), we implicitly consider matrix elements of the self-energy operator in a bound-state energy eigenfunction and omit the scale associated with the two \( p \) operators that always appear in the expressions. Here, the two \( p \) flank the operator \( (H - E)^{\nu - 2} \) which has the characteristic atomic size \( \mathcal{E}^{\nu - 2} \). The operator \( (H - E)^{\nu - 2} \) has this typical scale for any intermediate state when a complete set of intermediate states are inserted within the matrix element. Since the whole expression converges, it has the size \( \mathcal{E}^{\nu - 2} \). Thus we confirm the that the leading term for \( \nu < 3 \) in the Lamb shift is of order \( \mathcal{E}^{\nu - 2} \) as stated before. The divergence that appears when \( \nu \) approaches 3 is, in view of the structure of the integral (41), an ultraviolet divergence.

**B. \( \nu > 3 \)**

In leading order, for spatial dimensionality greater than three, the leading correction to the electron self energy involves a single Coulomb interaction between the electron and the nucleus as shown in Fig. 2. The correction entails the radiative modification of this interaction which are described by the order \( q^2 \) term in the electric form factor \( F_1(q^2) \) of the electron, \( F_1'(0) \) (the charge radius), and a relativistic anomalous magnetic moment effect
FIG. 2. Feynman diagram for the electron self-energy operator corrected by a single Coulomb exchange with the nucleus. The waving line stands for the transverse virtual photon exchange of the radiation gauge. The dashed line ending in a cross stands for the single Coulomb exchange with the nucleus.

given by the magnetic form factor at zero momentum transfer, $F_2(0)$. The $q^2$ wave number factor gives the Laplacian of the Coulomb potential, $\nabla^2 V$. Using the Dirac equation to reduce the $F_2(0)$ contribution yields a term involving $\nabla^2 V$ and a spin-orbit interaction. The Schrödinger energy eigenstate matrix elements of the operator $2\mathbf{p} \cdot (\mathbf{H} - E) \mathbf{p}$. The results are derived in ref. [12] and presented there in Eq's. (8.7.31) and (8.7.36), which we now write together as

$$
\Sigma_>(E) = \frac{2}{3\pi} \frac{e^2}{4\pi \hbar c} \left[ \frac{1}{\nu - 3} + \frac{\gamma}{2} \right] \left[ \frac{mc}{2\sqrt{\pi} \hbar} \right]^{\nu-3} \mathbf{p} \cdot \frac{\mathbf{P}}{mc} \cdot (\mathbf{H} - E) \mathbf{P} \frac{\mathbf{P}}{mc} + \frac{e^2}{4\pi \hbar c} \frac{1}{4\pi mc^2} \left[ \frac{mc}{2\sqrt{\pi} \hbar} \right]^{\nu-3} \sigma \cdot i[\mathbf{p}, V] \times \mathbf{p} , \tag{42}
$$

where again terms that vanish at $\nu = 3$ are omitted. Note that, with our conventions, this result scales as $\mathcal{E}$, again confirming an assertion made before. The divergence in the first term on the right-hand side of Eq. (12) when $\nu$ approaches 3 comes from the contribution of $F_1'(0)$ which contains an infrared divergence in three spatial dimensions [12].

C. $\nu = 3$

Since the two effects that we have listed contain the leading and first sub-leading terms for $\nu$ near 3, their sum

$$
\Sigma(E) = \Sigma_<(E) + \Sigma_>(E) \tag{43}
$$

evaluated at $\nu = 3$ must produce the Lamb shift to leading order. Indeed, the pole terms cancel as they must with the variation of the residues producing a logarithmic contribution, and one finds that
\[
\Sigma(E) = \frac{2\alpha}{3\pi} \frac{p}{mc} \cdot (H - E) \left\{ \ln \left[ \frac{m c^2}{2 (H - E - i\epsilon)} \right] + \frac{5}{6} \right\} \frac{p}{mc} + \frac{\alpha}{4\pi m^2 c^2} \sigma \cdot \bar{h} \nabla V \times p,
\]

where now \( \alpha = e^2 / (4\pi \hbar c) \simeq 1/137 \) may now be identified with the fine structure constant. This is the familiar form\(^6\) of the Lamb shift operator. This form appears in Eq. (8.7.63) in Ref. [12], and its consequences are explained there. The correct factor of \( 5/6 \) has an interesting history in the computation of the Lamb shift, as related in footnote 13 of ref. [14].

It is worth noting that this calculation of the Lamb shift using our dimensional continuation method is simpler than that done using the methods of effective quantum field theory \[15\] which would entail an additional matching calculation. It is much simpler than the conventional, old-fashioned method which utilizes a fictitious photon mass and a cumbersome joining process with an intermediate, non-covariant photon momentum cutoff.\(^7\)

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\(^6\)The imaginary part gives the width or lifetime of the level.

\(^7\)This hoary procedure is still presented in detail in modern texts on quantum field theory. See, for example, Section 7-3-2 of Itzykson and Zuber \[16\], or Section 14.3 of Weinberg \[17\].
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