Calibrations and Intersecting Branes

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ABSTRACT

We investigate the solutions of Nambu-Goto-type actions associated with calibrations. We determine the supersymmetry preserved by these solutions using the contact set of the calibration and examine their bulk interpretation as intersecting branes. We show that the supersymmetry preserved by such solutions is closely related to the spinor singlets of the subgroup $G$ of Spin(9, 1) or Spin(10, 1) that rotates the tangent spaces of the brane. We find that the supersymmetry projections of the worldvolume solutions are precisely those of the associated bulk configurations. We also investigate the supersymmetric solutions of a Born-Infeld action. We show that in some cases this problem also reduces to counting spinor singlets of a subgroup of Spin(9, 1) acting on the associated spinor representations. We also find new worldvolume solutions which preserve 1/8 of the supersymmetry of the bulk and give their bulk interpretation.
1. Introduction

Many of the insights in the relations amongst the superstring theories and M-theory have been found by investigating the soliton-like solutions of the ten- and eleven-dimensional supergravity theories. Most attention so far has been concentrated on supersymmetric solutions, i.e. those that they preserve a proportion of the supersymmetry of the underlying theory. Some of the supersymmetric solutions also saturate a Bogomol’nyi bound and so they are BPS. It is remarkable that a large class of supersymmetric solutions of the supergravity theories can be constructed by superposing elementary soliton solutions which preserve 1/2 of the spacetime supersymmetry [1]. These elementary solutions are the various brane solutions of supergravity theories, the pp-wave and KK-monopole. After such a superposition, the resulting solutions have the interpretation of intersecting branes or branes ending on other branes [2,1,3], and whenever appropriate, in the background of a pp-wave or a KK-monopole. The small fluctuations of the (elementary) p-branes of superstrings and M-theory are described by (p+1)-dimensional world-volume actions of Dirac-Born-Infeld type. Recently, the soliton-like solutions of these worldvolume actions have been investigated [4, 5, 6]. It has been found that some of these, so called worldvolume, solutions viewed from the bulk (supergravity) perspective also have the interpretation of intersecting branes or branes ending on other branes.

One way to explain this correspondence between the supergravity solutions and those of the worldvolume actions is to consider the supergravity coupled to the worldvolume action of a D-p-brane. The latter is a (p+1)-dimensional Dirac-Born-Infeld (DBI) action. Such an action can be written schematically as

\[ S = \frac{1}{g_s^2}S_{NS\otimes NS} + S_{R \otimes R} + \frac{1}{g_s}S_{DBI} \]  \hspace{1cm} (1.1)

where \( g_s \) is the string coupling constant. The first two terms are the NS\otimes NS and R\otimes R parts of the supergravity action, respectively, while the last term is the Born-Infeld action for a D-brane propagating in a supergravity background. In the limit
that the string coupling constant becomes very large, $S_{NS\otimes NS}$ diverges faster. In order to keep the action small, we set $S_{NS\otimes NS} = 0$ which can be achieved by taking flat spacetime background and setting the field strength of the fundamental string equal to zero. This is the weak coupling limit which can be described by string perturbation theory. Note that the $S_{R\otimes R}$ remains in this limit; it can be eliminated though by choosing the D-brane field strengths to be zero. Next suppose that the string coupling becomes large. In this limit using a similar reasoning, one can set $S_{DBI} = 0$ leading to the usual supergravity (bulk) description of the various branes. Although these limits are not consistent truncations, it is expected that the various BPS configurations survive in the various limits of the string coupling constant which explains the presence of intersecting brane configurations as solutions of the Dirac-Born-Infeld action. Moreover the worldvolume solitons of other branes are related to those of D-branes by T-and S-duality transformations. Therefore, the above correspondence between bulk configurations and worldvolume ones is valid for all branes.

The worldvolume actions of branes are described by matter, vector and tensor multiplets. For convenience we shall refer to the vectors and tensors fields of the vector and tensor multiplets as Born-Infeld fields. The worldvolume actions of branes described by matter multiplets are of Nambu-Goto type. Such actions include those of the IIA string and the M-2-brane [7,8]. The worldvolume actions of branes described by vector and tensor multiplets for vanishing Born-Infeld fields can be consistently truncated to Nambu-Goto ones. Such actions include those of D-branes [9, 10, 11], M-5-brane [12, 13, 14] and NS-5-brane of IIB theory. Therefore solutions of the worldvolume theories of all branes which do not involve Born-Infeld fields are solutions of Nambu-Goto actions. The solutions of Nambu-Goto action are minimal surfaces. A large class of minimal surfaces is that constructed using calibrations [15, 16]. Such solutions saturate a bound, so they are BPS, and they preserve a proportion of the spacetime supersymmetry. The supersymmetry of calibrations in Calabi-Yau manifolds and in manifolds of exceptional holonomy has been investigated in [17, 18]. The bulk interpretation of the calibrated worldvol-
ume solutions is that of intersecting branes. Another consistent truncation of the worldvolume actions of branes is to set all the matter fields of vector and tensor multiplets to be constant. The resulting action for vector multiplets is that of non-linear electro-dynamics, *i.e.* the Born-Infeld action. The bulk interpretation of the solutions of worldvolume theories, that involve only Born-Infeld fields is that of branes within branes [19] (see [20,21] for the supergravity solutions). The associated bound states can be at or below threshold. Finally, there are worldvolume solutions that involve both Born-Infeld and matter fields. Such solutions have the bulk interpretation of branes ending at branes. The presence of Born-Infeld fields is then required by the Gauss law. We remark that all Dirac-Born-Infeld actions relevant to branes, apart from that of the M-5-brane, can be constructed by dimensionaly reducing the ten-dimensional Born-Infeld action to an appropriate dimension. Therefore all solutions of the Dirac-Born-Infeld actions can be thought as solutions of the ten-dimensional Born-Infeld action. In fact one has the striking geometric property that see e.g. [5] *any calibrated p-dimensional submanifold of* $\mathbb{E}^n$ *may be regarded as a solution of the Born-Infeld action for a* $U(1)$ *gauge field in* $\mathbb{E}^{n+p}$.

In this paper, we shall investigate the solutions of Nambu-Goto actions associated with calibrations. We shall first show that all these solutions are supersymmetric. Our proof is in three steps

(i) we shall use the Killing spinor equations [22] which are derived from the kappa-symmetry transformations of brane worldvolume actions,

(ii) we shall exploit the fact that the collection of the tangent spaces of a calibration span a subspace of the contact set, and

(iii) we shall apply the notion of branes at angles [23, 24].

The main point of the proof is that the contact set of a calibration is a subspace of a homogeneous space, $G/H$, and that $G$ leaves one-dimensional invariant subspaces (singlets) when acting on the spinor representations of $\text{Spin}(9,1)$ (or $\text{Spin}(10,1)) \supset G$. The number of such singlets determines the proportion of
the supersymmetry preserved. The groups $G$ that arise in calibrations are some of those that appear in the context of special holonomies, i.e. $G$ is $SU(n)$, $G_2$ and Spin(7). Our approach provides an alternative way to define calibrations in terms of spinors as opposed to their usual definition in terms of forms. Then we shall systematically explore the bulk interpretation of all the calibrated solutions of Nambu-Goto actions. We shall find that in most cases there is a supergravity intersecting brane configuration associated with every worldvolume solution. Moreover, we shall show that all the the supersymmetry projections associated with a bulk configuration are precisely those that arise in the corresponding worldvolume solution. Next we shall investigate a certain class of singular solutions which may arise as limits of regular ones and shall examine the supersymmetry preserved by such solutions. Then we shall investigate the supersymmetry preserved by worldvolume solutions which involve only a non-vanishing Born-Infeld field. The term in the Killing spinor equation that involves the Born-Infeld field can be interpreted as spinor rotation [22]. Using this, we shall find that the supersymmetry condition on the Born-Infeld field associated with a vector multiplet has a group theoretic interpretation. In particular the Born-Infeld field can be thought as generating an infinitesimal spinor rotation induced by a group $G$, where $G$ is again a special holonomy group. Next we shall examine solutions of the Dirac-Born-Infeld action. Such solutions involve non-vanishing vectors or tensors field as well as scalars. Although the supersymmetry condition for such solutions can also be expressed in terms of spinor rotations, there is no straightforward interpretation of these conditions in terms of calibrations. We shall summarize the solutions found which preserve $1/4$ of the supersymmetry and we shall present some new ones which preserve $1/8$ of the supersymmetry.

This paper is organized as follows: In section two, we give the definition of calibrations and summarize some of their properties. In section three, we show that all calibrations preserve a proportion of the spacetime supersymmetry. In section four, we explain the correspondence between worldvolume solutions and those of the bulk. In sections five, six and seven, we give the correspondence
between the worldvolume solutions associated with calibrations and those of the bulk. In addition, we present all the associated supersymmetry projections. In section eight, we give some of the singular worldvolume solutions. In section nine, we find the worldvolume solutions which have only Born-Infeld fields. In section ten, we relate these solutions to those associated with calibrations. In section eleven, we investigate the worldvolume solutions which include both vectors and scalars and give a new solution which preserves $1/8$ of the supersymmetry. Finally in section twelve, we present our conclusions.

2. Calibrations

2.1. The bound

To investigate the supersymmetry of certain Born-Infeld configurations and establish bounds for their energy, we shall need a few facts about calibrations [15,16]. We shall consider calibrations in $\mathbb{E}^n$ equipped with the standard Euclidean inner product. Let $G(p, \mathbb{E}^n)$ be the grassmannian, i.e. the space of (oriented) $p$-dimensional subspaces of $\mathbb{E}^n$. Given a $p$-dimensional subspace $\xi$ of $\mathbb{E}^n$, $\xi \in G(p, \mathbb{E}^n)$, we can always find an orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathbb{E}^n$ such that $\{e_1, \ldots, e_p\}$ is a basis in $\xi$. We denote the co-volume of $\xi$ by

$$\overrightarrow{\xi} = e_1 \wedge \ldots \wedge e_p.$$  \hspace{1cm} (2.1)

A $p$-form $\phi$ on a open subset $U$ of $\mathbb{E}^n$ is a calibration of degree $p$ if

(i) $d\phi = 0$

(ii) for every point $x \in U$, the form $\phi_x$ satisfies $\phi_x(\overrightarrow{\xi}) \leq 1$ for all $\xi \in G(p, \mathbb{E}^n)$ and such that the ‘contact set’

$$G(\phi) = \{\xi \in G(p, \mathbb{E}^n) : \phi(\overrightarrow{\xi}) = 1\}$$  \hspace{1cm} (2.2)

is not empty.
One of the applications of calibrations is that they provide a bound for the volume of p-dimensional submanifolds of $\mathbb{E}^n$. Let $N$ be a p-dimensional submanifold of $\mathbb{E}^n$. At every point $x \in N$, one can find an oriented orthonormal basis in $\mathbb{E}^n$ such that $\{e_1, \ldots, e_p\}$ is an oriented orthonormal basis of $T_x N$. The co-volume of $N$ at $x$ is $N = e_1 \wedge \ldots \wedge e_p$ and the volume form of $N$ at $x$ is $\mu_N = \alpha_1 \wedge \ldots \wedge \alpha_p$, where $\{\alpha_1, \ldots, \alpha_p\}$ is the dual basis to $\{e_1, \ldots, e_p\}$. The fundamental theorem of calibrations states the following:

- Let $\phi$ be a calibration of degree $p$ on $\mathbb{E}^n$. The p-dimensional submanifold $N$ for which $\phi(N) = 1$ is volume minimizing. We shall refer to such minimal submanifolds as calibrated submanifolds, or a calibrations for short, of degree $p$.

To prove the above statement, we choose an open subset $U$ of $N$ with boundary $\partial U$ and assume that there is another subspace $W$ of $\mathbb{E}^n$ and an open set $V$ of $W$ with the same boundary $\partial U = \partial V$. Using the Stokes’ theorem, we have

$$\text{vol}(U) = \int_U \phi = \int_V \phi = \int_V \phi(V) \mu_V \leq \int_V \mu_V = \text{vol}(V).$$  \hspace{1cm} (2.4)

Calibrations give a large class of bounds for the volume of subspaces in $\mathbb{E}^n$. The ‘charge’ density associated with the bound is given by the calibration form. If $X$ is the map from the ‘worldvolume’, $\mathbb{E}^p$, into $\mathbb{E}^n$. The above bound can be re-expressed as

$$\int d^p u \sqrt{\det(g_{\mu\nu})} \geq \int X^* \phi,$$  \hspace{1cm} (2.5)

where $g_{\mu\nu}$ is the induced metric on $\mathbb{E}^p$ with respect the map $X$ and $\{u^1, \ldots, u^p\}$ are local coordinates of $\mathbb{E}^p$.

The tangent spaces of a p-dimensional submanifold $N$ of $\mathbb{E}^n$, parallel transported to the origin of $\mathbb{E}^n$, span a subspace of $G(p, \mathbb{E}^n)$; this is the Gauss map. If
moreover $N$ saturates the bound associated with the calibration $\phi$, then the image of the Gauss map is in $G(\phi)$. In many cases $G(\phi) = G/H$, where $G$ is a subgroup of $SO(n)$. As we shall see, the group $G$ arises naturally in the investigation of the supersymmetry of such configurations. There are many examples of calibrations. Here we shall mention three examples which will be proved useful in the study of supersymmetry and in the construction of actual solutions of the field equations of the Nambu-Goto action.

There is a relation between calibrations and special holonomy groups. This is because special holonomy groups are characterized by the existence of certain invariant forms. These forms can be used as calibrations. Manifolds with special holonomy also admit covariantly constant spinors (see the table below). Such spinors are also invariant under the action of the holonomy groups; the invariant spinors and forms are related [25].

| Holonomy     | Forms      | Spinors | Dim  |
|--------------|------------|---------|------|
| $SU(n)$      | 2(1), n(1) | 2       | 2n   |
| $Spin(7)$    | 4$^+$ (1)  | 1       | 8    |
| $G_2$        | 3(1),4(1)  | 1       | 7    |
| $Sp(n)$      | 2(3)       | n+1     | 4n   |

Table 1: Covariantly Constant Forms and Spinors  The first column contains the special holonomy groups, the second column contains the degrees (multiplicities) of covariantly constant forms ($4^+$ denotes a self-dual four-form), the third column contains the dimension of the space of covariantly constant spinors and the last column contains the dimension of the special holonomy manifold.

The investigation of supersymmetric solutions of Nambu-Goto action provides a further connection between calibrations and special holonomy groups. This con-
nection makes use of the invariant spinors of special holonomy groups. In particular, the spinor representations\(^*\) of Cliff(9, 1) or Cliff(10, 1) have singlets when decomposed under the special holonomy subgroups of Spin(9, 1) or Spin(10, 1). As we shall show, if a calibration as solution of the Nambu-Goto action is associated with a special holonomy group, then the invariant spinors serve as Killing spinors and some of the spacetime supersymmetry is preserved. The proportion of the supersymmetry preserved is related to the number of singlets in the decomposition of the spinor representations of Spin(9, 1) under the special holonomy subgroup. We shall remark further on this relationship between special holonomy, calibrations and supersymmetry in the section eleven.

2.2. Kähler Calibrations

We begin by introducing coordinates \(\{x^i, y^i; i = 1, \ldots, n\}\) on \(\mathbb{E}^{2n}\) and the metric

\[
ds^2 = \sum_i^n (dx^i dx^i + dy^i dy^i) .
\] (2.6)

Then we choose a complex structure on \(\mathbb{E}^{2n}\) such that the associated Kähler form is

\[
\omega = \frac{i}{2} \sum_i dz^i \wedge d\bar{z}^i
\] (2.7)

where \(z^i = x^i + iy^i\). Next we set

\[
\phi = \frac{1}{p!} \omega^p .
\] (2.8)

The form \(\phi\) is a calibration of degree \(2p\) and the contact set \(G(\phi)\) is the space of complex \(p\)-dimensional planes in \(\mathbb{C}^n = \mathbb{E}^{2n}\), \(G(\phi) = G_{\mathbb{C}}(p, \mathbb{C}^n)\), where we have

\* Our metric convention is \(\eta = \text{diag}(1, \ldots, 1, -1)\)
identified $\mathbb{E}^{2n}$ with $\mathbb{C}^n$ using the above complex structure. To prove this, one uses Wirtinger’s inequality which states that

$$\phi(\xi) \leq 1$$

(2.9)

for every $\xi \in G(2p, \mathbb{E}^{2n})$ with equality if and only if $\xi \in G_{\mathbb{C}}(p, \mathbb{C}^n)$ (for the proof see [15, 16]). A consequence of this is that all complex submanifolds of $\mathbb{C}^n$ are volume minimizing. The form $\omega$, and therefore $\phi$, are invariant under $U(n)$. The contact set can be written as the coset space

$$G_{\mathbb{C}}(p, \mathbb{C}^n) = U(n)/U(p) \times U(n-p) = SU(n)/S(U(p) \times U(n-p)) .$$

(2.10)

Thus $SU(n)$ acts transitively on the space of $p$-dimensional complex planes in $\mathbb{C}^n$. As we shall see this fact will be used to find the proportion of spacetime supersymmetry preserved by the worldvolume solutions associated with Kähler calibrations.

2.3. Special Lagrangian Calibrations

To describe the special Lagrangian calibrations we begin with the metric (2.6) and the Kähler form $\omega$ (2.8) as in the previous section. In addition, we introduce the $(n,0)$-form

$$\psi = dz^1 \wedge \ldots \wedge dz^n .$$

(2.11)

The data, which include the metric, the Kähler form and the $(n,0)$-form $\psi$, are invariant under $SU(n)$. The calibration form $^*$ is in this case is

$$\phi = \text{Re}\psi .$$

(2.12)

The inequality necessary for $\phi$ to be a calibration has been demonstrated in [15, 16] and the planes that saturate the bound are called special Lagrangian. The

* We can define special Lagrangian calibrations with a more general $(n,0)$-form. However for this paper this choice will suffice.
contact set in this case is

\[ G(\phi) = SU(n)/SO(n) \quad (2.13) \]

As in the case of Kähler calibrations, \( SU(n) \) acts transitively on the space of special Lagrangian planes.

2.4. Exceptional Cases

There are three exceptional cases.

(i) The calibration form is the 3-form, \( \varphi \), in \( \mathbb{E}^7 \) invariant under the exceptional group \( G_2 \). This gives rise to a calibration of degree three in \( \mathbb{E}^7 \). The contact set is

\[ G(\varphi) = G_2/\text{SO}(4) \quad (2.14) \]

which is a subset \( G(3, \mathbb{E}^7) \).

(ii) The calibration form is the dual \( \chi \) of \( \varphi \) in \( \mathbb{E}^7 \), \( \chi = \ast \varphi \). This gives rise to a calibration of degree four in \( \mathbb{E}^7 \). The contact set is again

\[ G(\chi) = G_2/\text{SO}(4) \quad (2.15) \]

which is now thought of as the subset of \( G(4, \mathbb{E}^7) \).

(iii) The calibration form is the Spin(7)-invariant self-dual 4-form, \( \Phi \), in \( \mathbb{E}^8 \). This gives rise to a calibration of degree four in \( \mathbb{E}^8 \). The contact set is

\[ G(\Phi) = \text{Spin}(7)/H \quad (2.16) \]

which is a subset of \( G(4, \mathbb{E}^8) \), where \( H = SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2 \). This calibration is called a Cayley calibration and the 4-planes that saturate the bound are called Cayley planes.
3. Supersymmetry and calibrations

The dynamics of a large class of extended objects is described by actions of Dirac-Born-Infeld type. The fields include the embedding maps $X$ of the extended object into spacetime which we shall take it to be ten-dimensional Minkowski, $\mathbb{E}^{(9,1)}$. Apart from the embeddings maps, the action may depend on other worldvolume fields like a Born-infield 2-form field strength $F$ (as in the case of D-branes) and the various fermionic partners $\theta$ which are spacetime fermions. These actions are invariant under fermionic transformations which are commonly called kappa-symmetries. These act on $\theta$ as

$$\delta \theta = (1 + \tilde{\Gamma}) \kappa$$

where $\tilde{\Gamma} \in \text{Cliff}(9,1)$ is a traceless hermitian product structure, i.e. $\tilde{\Gamma}^2 = 1$, and $\kappa$ is the parameter which is a spacetime fermion. The product structure $\tilde{\Gamma}$ depends on the embeddings maps and the other worldvolume fields, i.e. $\tilde{\Gamma}$ depends on the worldvolume coordinates. It was shown in [22] that the Killing spinor equation in this context is the algebraic equation

$$(1 - \tilde{\Gamma}) \epsilon = 0,$$

where $\epsilon$ is the spacetime supersymmetry parameter. Since we have chosen the spacetime geometry to be Minkowski, the spacetime (supergravity) Killing spinor equations imply that $\epsilon$ is constant\(^*\). The condition for supersymmetry (3.2) is universal and applies to all types of branes, fundamental strings, solitonic 5-branes, D-branes and M-branes. Supersymmetric (worldvolume) configurations are solutions of the Born-Infeld field equations which satisfy (3.2) for some non-vanishing $\epsilon$. The proportion of the bulk supersymmetry preserved by such configuration depends on the number of linearly independent solutions of (3.2) in terms of $\epsilon$.

\(^*\) If we relax the condition that $\epsilon$ is constant, then the Killing spinor equation (3.2) has always solutions.
The product structure \( \Gamma \) for D-branes and the M-5-brane can be written \([22]\)
as
\[
\tilde{\Gamma} = e^{-\frac{a}{2}} \Gamma e^{\frac{a}{2}}
\]  
(3.3)

where \( \Gamma \) depends only on the embedding map of the worldvolume into spacetime and \( a \) contains the dependence of \( \tilde{\Gamma} \) on the Born-Infeld fields. If we take the Born-Infeld fields to vanish, then \( a = 0 \) and the product structure becomes
\[
\tilde{\Gamma} = \Gamma \ .
\]  
(3.4)

3.1. Supersymmetry without Born-Infeld fields

A consistent truncation of all Dirac-Born-Infeld type of actions is to allow the Born-Infeld fields to vanish. Let \( X \) be the embedding map of the extended object into Minkowski spacetime. The bosonic part of the truncated action is
\[
S_{NG} = \int d^{p+1}x \sqrt{|\text{det}(g_{\mu\nu})|}
\]  
(3.5)

where
\[
g_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N \eta_{MN} ,
\]  
(3.6)
is the induced metric on the worldvolume with coordinates \( \{x^\mu; \mu = 0, \ldots, p\} \), and \( \eta \) is the ten-dimensional Minkowski metric. The product structure \( \Gamma \) is then expressed in terms of the embedding maps and the spacetime Gamma matrices. The particular expression for \( \Gamma \) depends on the type of brane that we are considering. Here we shall investigate the Killing spinor equations for IIA D-branes. However our argument is universal and applies equally well to IIB D-branes, M-branes, fundamental strings and heterotic 5-branes.
To continue, we define

\[ \gamma_\mu = \partial_\mu X^M \Gamma_M \tag{3.7} \]

where \( \{ \Gamma_M; M = 0, \ldots, 9 \} \) are the spacetime gamma matrices. We remark that

\[ \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu \nu} \tag{3.8} \]

i.e. the \( \{ \gamma_\mu; \mu = 0, \ldots, p \} \) obey the Clifford algebra with respect to the induced metric. The product structure \( \Gamma \) for IIA D-branes is

\[ \Gamma = \frac{1}{(p+1)!} \sqrt{g} \epsilon^{\mu_0 \ldots \mu_p \nu_0 \ldots \nu_p} \gamma_{\mu_0} \ldots \gamma_{\mu_p} \Gamma^p_{11} . \tag{3.9} \]

For example, the product structure of a planar IIA D-p-brane which spans the first \( p \) directions of \( \mathbb{E}^{(9,1)} \) is

\[ \Gamma = \Gamma_0 \ldots \Gamma_p \Gamma^p_{11} . \tag{3.10} \]

There is another way to describe the product structure \( \Gamma \). For this we remark that there is a map \( c \) which assigns to every \( q \)-form

\[ \nu = \frac{1}{q!} \nu_{M_1 \ldots M_p} dx^{M_1} \wedge \ldots \wedge dx^{M_p} \tag{3.11} \]

in \( \mathbb{E}^{(9,1)} \) an element

\[ c(\nu) = \frac{1}{q!} \nu_{M_1 \ldots M_p} \Gamma^{M_1} \ldots \Gamma^{M_p} \tag{3.12} \]

of the Clifford algebra \( \text{Cliff}(9,1) \). Note that

\[ c(\ast \nu) = c(\nu) \Gamma_{11} \tag{3.13} \]

where \( \ast \nu \) is the Hodge dual of \( \nu \). Let \( N \) be the submanifold of \( \mathbb{E}^{(9,1)} \) that describes a solution of the Born-Infeld field equations. Then the product structure \( \Gamma \) associated
with $N$ is

$$
\Gamma = c(\mu)\Gamma^p_{11}
$$

(3.14)

where $\mu$ is the volume form of $N$. Therefore, the product structure $\Gamma$ written in terms of an oriented orthonormal frame at a point $y$ becomes the product structure of a planar p-brane tangent to $N$ at $y$. Thus to solve the Killing spinor equation of $N$ is equivalent to requiring that the Killing spinor equations of the planar $D$-p-branes tangent to $N$ at each point $y$ have a common solution.

One way to find a solution is to recall what happens when two planar branes intersect at an “angle”. Supersymmetry requires that the element of $SO(9,1)$ or $SO(10,1)$ relating the hyper-planes is in a subgroup $G \subset SO(9,1)$ or $SO(10,1)$. Such angles are called “G-angles”. For a general non-planar brane we proceed analogously. For this we choose a reference point $y_0$ and an orthonormal frame of the tangent space $T_{y_0}N$ at this point which we extend to an orthonormal frame in $\mathbb{E}^{(9,1)}$. Then we choose any other point $y$ in $N$ and introduce another orthonormal frame in the same way. The two orthonormal frames are related by a Lorentz transformation in $\mathbb{E}^{(9,1)}$ (a spatial orthogonal rotation for static configurations.). Moreover the supersymmetry condition at $y$ can be written in terms of the product structure at $y_0$ as

$$
S^{-1}\Gamma(\{y_0\})S\epsilon = \epsilon
$$

(3.15)

where $S$ is a spinor rotation induced by the Lorentz rotation that relates the above orthonormal frames at $y_0$ and $y$. For most of the solutions of the Dirac-Born-Infeld equation, the rotations required are generic elements of the ten-dimensional Lorentz group and all supersymmetry is broken. However, some cases involve rotations which lie in a subgroup of the Lorentz group. In particular, if the rotations lie in a subgroup of the Lorentz group for which the decomposition of the Majorana spinor representation of Spin$(9,1)$ has singlets, then the equation (3.15) reduces to

$$
\Gamma(\{y_0\})\epsilon = \epsilon
$$

(3.16)
provided that $\epsilon$ is a linear combination of the singlets. Therefore the Killing spinor equation of $N$ in such a case reduces to the Killing spinor equation on a single D-p-brane which can be easily solved. Examples of subgroups of the orthogonal groups for which the decomposition of the ten-dimensional spinor representation has singlets are the special holonomy groups like $SU(k)$, $Sp(k)$, $G_2$ and $Spin(7)$.

It is clear from the above arguments that preservation of a proportion of supersymmetry of the bulk by a Dirac-Born-Infeld configuration is closely related to calibrations. Let $\Phi$ be the Gauss map which takes the tangent space of $N$ at every point $y$ into the grassmannian $G(p + 1, \mathbb{E}^{(9,1)})$. If the image

$$S_\Gamma = \Phi(N)$$

of such a map in $G(p + 1, \mathbb{E}^{(9,1)})$ is a subspace of the homogeneous space $G/H$, then clearly the relevant rotations amongst the frames associated with the tangent spaces are in $G$. If $N$ is a calibration, then $S_\Gamma$ is a subspace of the contact set $G(\phi)$. In all the examples of calibrations that we have presented, the contact set is a homogeneous space with $G$ being one of the groups that appear in the context of special holonomies. From this we conclude that all these calibrations preserve a proportion of the supersymmetry of the bulk. Clearly the case of intersecting planar branes is a special case of this construction. For more about this see section eight.

### 3.2. Supersymmetry with Born-Infeld fields

The Dirac-Born-Infeld action with Born-Infeld field $F_{\mu\nu}$ is

$$I = \int d^{p+1}x \sqrt{|\det (g_{\mu\nu} + F_{\mu\nu})|}$$

where $g_{\mu\nu}$ is induced metric. This action describes the dynamics of D-p-branes as well as the dynamics of IIB NS-NS 5-branes. Here we shall investigate the Killing
spinor equations for IIA D-branes. However our argument can be easily extended to apply in the case of IIB D-branes, in the case IIB NS-NS 5-brane as well as in the case of M-5-brane. For IIA D-p-brane [22],

\[
a = Y_{\mu \nu} \gamma^\mu \gamma^\nu \Gamma_{11} ,
\]

(3.19)

where

\[
F = \text{“tan”} Y .
\]

(3.20)

The inclusion of non-vanishing Born-Infeld fields modifies the conditions for a configuration to be supersymmetric. In this case, there is no a direct relation between calibrations and supersymmetry conditions. Nevertheless the analysis for the proportion of supersymmetry preserved by a solution of the Dirac-Born-Infeld field equations proceeds as in the previous case without Born-Infeld fields. We begin with a submanifold \( N \) in \( E^{(9,1)} \) which is a solution of the Born-Infeld field equations together with a non-vanishing Born-Infeld field \( F \). Then we introduce orthonormal frames at the tangent spaces \( T_{y_0} N \) and \( T_{y_1} N \) of the points points \( y_0 \) and \( y_1 \) of \( N \) which we extend to orthonormal frames in \( E^{(9,1)} \). Again there is a Lorentz rotation in \( E^{(9,1)} \) which relates the two frames. Using a similar argument to that of the previous section, the Killing spinor equation at a point \( y \) can be written in terms of the Killing spinor equation at \( y_0 \) as follows

\[
U^{-1} \left( e^{-\frac{a(y_0)}{2}} \Gamma(y_0) e^{\frac{a(y_0)}{2}} \right) U \epsilon = \epsilon
\]

(3.21)

where

\[
U = e^{-\frac{a(y_0)}{2}} S e^{\frac{a(y)}{2}}
\]

(3.22)

and \( S \) is the induce rotation on the spinors from the Lorentz rotation that relates the two frames. These supersymmetry projections can be simplified if we assume that the Born-Infeld field vanishes at the reference point \( y_0 \). (In many
applications, the Born-Infeld field vanishes at some point usually at infinity.) The
supersymmetry projection at \( y \) then becomes

\[
e^{-\frac{a(y)}{2}} S^{-1} \Gamma(y_0) S e^{-\frac{a(y)}{2}} \epsilon = \epsilon .
\]

(3.23)

Solutions of these conditions can be found by assuming that the spinor-rotations
\( U \) leave invariant some constant Majorana spinor \( \epsilon \). In such a case, the Killing
spinor equation for \( N \) degenerates to the Killing spinor equation of the reference
point

\[
\Gamma(y_0) \epsilon = \epsilon ,
\]

(3.24)

where \( \epsilon \) is a linear combination of the singlets.

A consistent truncation of the Dirac-Born-Infeld action is to set \( \{ X^M \} = \{ x^\mu, y^i = 0 \} \). The truncated action is the Born-Infeld action

\[
I = \int d^{p+1}x \sqrt{\det(\eta_{\mu\nu} + F_{\mu\nu})} .
\]

(3.25)

For this class of configurations, \( N \) is a \((p+1)\)-dimensional Minkowski subspace of
\( \mathbb{E}^{(9,1)} \). Therefore we can choose an orthonormal frame in \( \mathbb{E}^{(9,1)} \) such that \( N \) spans
the first \( p \) spatial directions of \( \mathbb{E}^{(9,1)} \). The Killing spinor equations become

\[
e^{-\frac{a(y)}{2}} \Gamma e^{-\frac{a(y)}{2}} \epsilon = \epsilon
\]

(3.26)

where \( \Gamma \) is a constant product structure.

In the linearized limit, this condition on the Born-Infeld field \( F \) reduces to
the familiar condition for a (Yang-Mills) configuration to preserve a proportion of
supersymmetry\(^*\) together with a projection of associated with \( \Gamma \). Viewing \( F \) as
the infinitesimal transformation of \( SO(p,1) \) rotations acting on the Killing spinor

\(^*\) We have assumed that \( F \) is not a constant two-form field strength.
with the induced spinor representation, supersymmetry is preserved provided that $F$ takes values in an appropriate subalgebra of $so(p, 1)$. Such subalgebras are those of the special holonomy subgroups of $so(p, 1)$. For example $sp(1)$ is related to the self-dual connections in four dimensions, $su(k)$ is related to the Einstein-Yang-Mills connections [26], $sp(k)$ is related to the instanton solutions found in [27] (see also [31]) and similarly for $g_2$ and $spin(7)$ [28, 29, 30].

4. Static Born-Infeld Solitons, Brane Boundaries and Brane Intersections

As we have already mentioned, the worldvolume solitons of the Dirac-Born-Infeld action of a p-brane may arise as the intersection of the p-brane with other branes or the boundary of other branes ending on the p-brane. Therefore, such worldvolume solitons can be found by utilizing the bulk boundary and intersection rules for branes. These rules are known either from string theory or from supergravity. Worldvolume solitons that arise as boundaries or intersections of branes have the following qualitative properties:

- The bulk brane configuration and the associated worldvolume soliton exhibit the same manifest Poincaré invariance.

- The proportion of the bulk supersymmetry preserved by the worldvolume soliton is the same as that of the associated bulk configuration. We shall show that supersymmetric projections in both cases are identical.

- The worldvolume solitons of a p-brane that are associated with other branes ending on it have non vanishing Born-Infeld fields. This is a consequence of the conservation of the flux at the intersection.

- The number of non-vanishing scalar fields associated with a worldvolume soliton of a p-brane is equal to the number of the worldvolume directions of the other branes involved in the associated bulk configuration which are transverse to the p-brane.
For worldvolume solitons associated with the intersection of two branes, we can also impose the following boundary conditions:

- Away from the boundary or the intersection, the induced metric on the p-brane should approach that of $E^{(p,1)}$.

- Near the boundary or the intersection on a p-brane the induced metric should approach the worldvolume Minkowski metric of the ‘incoming’ brane from the bulk.

There is a very large number of brane configurations that have the interpretation of branes ending on or intersecting with other branes. We shall not attempt to present a complete list here. However, we shall mention some examples below (see [1, 32, 33]). These bulk configurations are most easily classified according to the supersymmetry that they preserve.

(i) Configurations that preserve 1/2 of the supersymmetry of the bulk are bound states of branes that lie within other branes. These bound states are below threshold. Examples of such bound states include the D-(p-2)-branes within D-p-branes and the M-2-brane within the M-5-brane.

(ii) Configurations that preserve 1/4 of the supersymmetry of the bulk include the following: In M-theory, two membranes intersecting at a 0-brane, two M-5-branes intersecting at a 3-brane, a membrane ending on a 5-brane with boundary a string. In string theory we have, the fundamental string ending on a D-p-brane with a 0-brane boundary, a D-string ending at a IIB NS-NS 5-brane with a 0-brane boundary, two D-p-branes intersecting at a D-(p-2)-brane and two NS 5-branes intersecting at a 3-brane.

(iii) Configurations that preserve 1/8 of the supersymmetry of the bulk include the following: In M-theory, three M-2-branes intersecting at a 0-brane, three M-5-branes intersecting at a string and two M-2-branes ending a M-5-brane with their string boundaries intersecting on a 0-brane.

Using the correspondence between bulk configurations and worldvolume soli-
tons, it is worth mentioning that there are two 0-brane worldvolume solitons on the D-2-brane. One is due to the IIA fundamental string ending on the D-2-brane and the other is due to the intersection of two D-2-branes at a 0-brane. The two 0-brane solitons are charged with respect to different fields. The first one is charged with respect to the Born-Infeld one-form gauge potential and the other is charged with respect to the gauge potential associated with the dual of a transverse scalar. The D-4-brane also has two different 0-brane worldvolume solitons. One is due to the fundamental string ending to the D-4-brane and it charged with respect to the Born-Infeld gauge potential. The other is due to a D-0-brane within the D-4-brane and the worldvolume solution is a Born-Infeld field instanton.

5. Static Solutions and Kähler calibrations

The worldvolume solitons associated with Kähler calibrations are complex submanifolds of $\mathbb{C}^n$. If the calibration has contact set $SU(n)/S(U(p) \times U(n-p))$, then a generic soliton will preserve $1/2^n$ of the supersymmetry of the bulk. As we shall see, the singlets of the ten-dimensional representations under $SU(n) \subset SO(9,1)$ satisfy $n-1$ relations each reducing the number of components of the Killing spinor by half. The supersymmetry projection associated with the p-brane also reduces the components of the Killing spinor by another half leading to the preservation of the $1/2^n$ of the bulk supersymmetry.

The worldvolume solitons associated with the Kähler calibrations have vanishing Born-Infeld type of fields and therefore the analysis below applies universally to all types of branes, M-branes, D-branes and NS-branes. Let us describe the solution of a k-brane soliton of a p-brane with $\ell$ transverse scalars. For this we choose the static gauge for the p-brane,

$$X^M = (x^\mu, y^i), \quad (5.1)$$

where $\{\mu = 0, \ldots, p\}$ are the worldvolume coordinates and $\{y^i; i = 1, \ldots, 9-p\}$ are the transverse coordinates of the p-brane. In this gauge, the Born-Infeld solution
that we would like to describe is a map \( y \) from \( \mathbb{E}^{p-k} \subset \mathbb{E}^{(p,1)} \) into \( \mathbb{E}^\ell \subset \mathbb{E}^{(9,1)} \). We then introduce complex structures \( I \) and \( J \) on \( \mathbb{E}^{p-k} \) and \( \mathbb{E}^\ell \), respectively, and we write the equation

\[
I^b_a \partial_b y^m = J^m_n \partial_a y^n ,
\]

(5.2)

where \( a, b = 1, \ldots, p - k \) and \( m, n = 1, \ldots, \ell \); the dimension of \( \mathbb{E}^{p-k} \) and \( \mathbb{E}^\ell \) are even. It is straightforward to show that if \( y \) satisfies (5.2) then the embedding \( \{ X^M \} = \{(x^\mu, y^m, 0)\} \) solves the Born-Infeld field equations. Choosing complex coordinates with respect to these complex structures, \( x^a = (z^\alpha, \bar{z}^{\bar{\alpha}}) \) and \( y^m = (s^A, \bar{s}^{\bar{A}}) \), the solutions of (5.2) are holomorphic functions

\[
s^A = s^A(z) .
\]

(5.3)

Therefore the above solution is a Kähler calibration in \( \mathbb{E}^{n-k+\ell} \) of degree \( p - k \). The Kähler form is given by the complex structure \( I \oplus J \) and the contact set is

\[
SU\left(\frac{p-k+\ell}{2}\right)/S(U\left(\frac{p-k}{2}\right) \times U\left(\frac{\ell}{2}\right)) .
\]

(5.4)

5.1. SU(2) Kähler calibrations

The simplest of all Kähler calibrations is the one describing an one-dimensional complex space in \( \mathbb{C}^2 \). The associated Born-Infeld brane solitons have already being found in [5]. They correspond to (p-2)-brane worldvolume solitons that appear in all p-branes \( (p \geq 2) \). From the bulk (supergravity) perspective, the existence of these solitons is due to the “(p-2)-intersection rule” which states that two p-branes intersect on a (p-2)-brane [1]. The solution can be written in an implicit form as

\[
F(s, z) = 0 .
\]

(5.5)

Different choices of \( F \) give different solutions. Some examples are the following.

(i) The 3-brane soliton of M-5-brane for \( F \) the equation of a Riemann surface of
was investigated in [34]. (ii) In [37], the 0-brane soliton of the M-2-brane for some choice of \( F \) was interpreted as the M-theory analogue of the IIB (p,q)-string triple junctions [35, 36] (for more references to earlier work see [38]). Another example is to choose

\[
F(s, z) = sz - c ,
\]

(5.6)

where \( c \) is a complex number. If \( c = 0 \), then the soliton is singular. The bulk interpretation of such a soliton is that of two planar p-branes intersecting at a (p-2)-brane with the singularity located at the intersection. Such solutions will be investigated in section 8. If \( c \neq 0 \), the singularity is blown up and the induced metric on the p-brane for this configuration has two asymptotically flat regions one at \( |z| \to \infty \) and the other at \( |z| \to 0 \). These asymptotic regions are identified with the two p-branes involved in the intersection. It turns out that the two asymptotic regions are orthogonal in the bulk metric and therefore the intersection is at right angles. Next, we take

\[
F(s, z) = (s - bz)z - c ,
\]

(5.7)

where \( b, c \) are complex numbers. The induced metric on the p-brane again has two asymptotically flat regions one at \( |z| \to \infty \) and the other at \( |z| \to 0 \) which can be identified with the p-branes involved in the intersection. However in this case the intersection is not orthogonal. This can be easily seen by observing that \( |s| \to \infty \) as \( |z| \to 0 \) and \( s \to bz \) as \( |z| \to \infty \). Therefore in the \((s, z)\) plane, one asymptotic region is along the \( s \) plane while the other is along the plane determined by the equation \( s = bz \). The two angles among these two planes are determined by the parameter \( b \); for \( b \) a real number,

\[
\tan \theta = \frac{1}{b}
\]

(5.8)

where \( \theta \) is the the angle.

As we have already mentioned all the above configurations preserve 1/4 of the bulk supersymmetry. To be more explicit let us consider the case of IIA D-branes
in more detail. We can always arrange, without loss of generality, that the product structure \( \Gamma \) at the reference point \( y_0 \) is

\[
\Gamma(y_0) = \Gamma_0 \cdots \Gamma_p (\Gamma_{11})^{p+2} .
\] (5.9)

We shall consider solutions which depend on the worldvolume coordinates \((x^{p-1}, x^p) = (X^{p-1}, X^p)\) and with transverse coordinates \((y^1, y^2) = (X^{p+1}, X^{p+2})\). We then introduce the complex coordinates \(z = x^{p-1} + ix^p\) and \(s = y^1 + iy^2\). The supersymmetry projection associated with \( y_0 \) is

\[
\Gamma_0 \cdots \Gamma_p (\Gamma_{11})^{p+2} \epsilon = \epsilon .
\] (5.10)

For such a Kähler calibration, the \( SU(2) \) rotations of the tangent planes of the solution is taking place in the space spanned by the coordinates \( \{X^{p-1}, \ldots, X^{p+2}\} \).

For the above choice of complex structure, the spinor singlets of any \( SU(2) \) rotation in these directions satisfy

\[
\Gamma_{p-1} \Gamma_p \Gamma_{p+1} \Gamma_{p+2} \epsilon = -\epsilon ;
\] (5.11)

(the sign depends on the choice of complex structure). Combining the conditions (5.10) and (5.11), we conclude that the supersymmetry preserved by the worldvolume solution is 1/4 of that of the bulk. It is worth pointing out that the two supersymmetry conditions can be rewritten as

\[
\Gamma_0 \cdots \Gamma_p (\Gamma_{11})^{p+2} \epsilon = \epsilon
\]

\[
\Gamma_0 \cdots \Gamma_{p-2} \Gamma_{p+1} \Gamma_{p+2} (\Gamma_{11})^{p+2} \epsilon = \epsilon .
\] (5.12)

These are precisely the supersymmetry conditions of the supergravity solution with the interpretation of two IIA D-p-branes intersecting on a \((p-2)\)-brane \( \star \). This turns

* The intersection can be at any \( SU(2) \) angle.
out to be a common feature of all supersymmetric worldvolume solitons. If the worldvolume soliton has a bulk interpretation, then *the supersymmetry conditions of the soliton are identical to those of the bulk configuration*. The above argument for supersymmetry with a slight modification applies to all \((p-2)\)-brane worldvolume solitons of p-branes.

5.2. SU(3) Kähler calibrations

There are two cases of Kähler calibrations associated with the group \(SU(3)\) to consider. The first case is a degree two calibration in \(\mathbb{C}^3\) and the second case is a degree four calibration in \(\mathbb{C}^3\). Both cases have the same contact set.

The worldvolume solitons of a p-brane associated with the calibration of degree two are described be the zero locus of the holomorphic functions

\[
F^1(s^1, s^2, z) = 0 \\
F^2(s^1, s^2, z) = 0 ,
\]

where \(z\) is a complex coordinate on the p-brane, \(p \geq 2\), and \(s^1, s^2\) are complex coordinates transverse to the p-brane. Next suppose that \(z = x^{p-1} + i x^p\), \(s^1 = y^1 + i y^2\) and \(s^2 = y^3 + i y^4\) (\(y^i = X^{p+i}\)). For this choice of complex structure, the spinor singlets of \(SU(3)\) acting on \(z, s^1, s^2\) satisfy the conditions

\[
\Gamma_{p-1} \Gamma_p \Gamma_{p+1} \Gamma_{p+2} \epsilon = -\epsilon \\
\Gamma_{p-1} \Gamma_p \Gamma_{p+3} \Gamma_{p+4} \epsilon = -\epsilon .
\]

We then choose the base point \(y_0\) as in the \(SU(2)\) case above so that the projector of the p-brane is given as in (5.10). Therefore a generic worldvolume soliton preserves 1/8 of the bulk supersymmetry as expected.

From the bulk perspective, the above worldvolume solitons are associated with the common intersection of three p-branes. There are several examples of such configurations. The typical example is that of three M-2-branes (or equivalently
three D-2-branes) intersecting on a 0-brane. The intersection that we are considering is not necessarily orthogonal. However to simplify notation, we shall present here the orthogonal bulk intersection which is

\[(i) \ M^{-2} : 0, 1, 2, -, -, -\]
\[(ii) \ M^{-2} : 0, -, -, 3, 4, -, -\]
\[(iii) \ M^{-2} : 0, -, -, -, 5, 6 .\]

In this notation, the numbers denote the bulk directions which are identified with the worldvolume directions of the associated brane and some of the transverse directions are denoted \(-\). From the perspective of one of the three 2-branes, say the first one, \(z\) spans the worldvolume directions 1, 2 and \(s^1, s^2\) span the worldvolume directions 3, 4, 5, 6 of the other two 2-branes. Using reduction from M-theory to IIA and T-duality, we can construct many other cases like for example that of three D-3-branes intersecting at a string.

An example of a worldvolume soliton that corresponds to the above intersection is

\[F^1 = (s^1 - b^1 z)z - c^1\]
\[F^2 = (s^2 - b^2 z)z - c^2 ,\]

where \(b^1, b^2, c^1, c^2\) are complex numbers. The induced metric on one of the M-2-branes has two asymptotically flat regions, one as \(|z| \to 0\) and the other as \(|z| \to \infty\). Identifying \(s^1, s^2\) with the coordinates of the ‘incoming’ branes, comparing the asymptotic behaviour of the solution as \(|z| \to 0\) and \(|z| \to \infty\) and using a similar argument to that of the \(SU(2)\) case, we find that the angles are determined by \(b^1\) and \(b^2\); for \(b^1, b^2\) real numbers, we have

\[\tan \theta^1 = \frac{1}{b^1}\]
\[\tan \theta^1 = \frac{1}{b^2} .\]

If the constants \(\{b^1, b^2\}\) vanish, then the branes intersect orthogonally. It is worth mentioning that the supersymmetry projections associated with this worldvolume
soliton are the same as the supersymmetry projections associated with the bulk configuration (5.15).

Next we shall consider the case of degree four calibrations in $\mathbb{C}^3$. The worldvolume solitons of a p-brane are described by the zero locus of the holomorphic function

$$F(z^1, z^2, s) = 0,$$

where $z^1, z^2$ are complex coordinates of the p-brane and $s$ is a complex coordinate transverse to the brane. The rotation group $SU(3)$ acts on the coordinates $z^1, z^2, s$. Choosing the complex structure in a way similar to that of the previous case, the spinor singlets of $SU(3)$ satisfy the same conditions as in (5.14). Using these two conditions together with the supersymmetry projection of the p-brane, one concludes that a generic worldvolume soliton will preserve $1/8$ of the bulk supersymmetry.

These worldvolume solitons correspond to intersecting brane configurations which are magnetic duals (in the sense of [1]) to the intersecting brane configurations associated with the solitons of the previous case. A typical example is the M-theory configurations of three M-5-branes pairwise intersecting on 3-branes and altogether at a string. Again here the intersection is at $SU(3)$ angles. However for simplicity we give the orthogonally intersecting configuration which is

(i) $M - 5 : 0, 1, 2, 3, 4, 5$

(ii) $M - 5 : 0, 1, 2, 3, -, -, 6, 7$

(iii) $M - 5 : 0, 1, -, -, 4, 5, 6, 7$.

This configuration is magnetic dual to the configuration of three M-2-branes intersecting on a 0-brane (5.15). The degree four calibration associated with a M-5-brane describes a string worldvolume soliton. From the perspective of one of the M-5-branes involved in the intersection (5.19), say the first one, the string is along the directions $0, 1$, the complex coordinates $z^1, z^2$ span the directions $2, 3, 4, 5$ and $s$ is along the directions $6, 7$.  

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5.3. SU(4) Kähler calibrations

There are three Kähler calibrations associated with the group $SU(4)$. These are a degree two calibrations in $\mathbb{C}^8$, degree four calibrations in $\mathbb{C}^8$ and degree six calibrations in $\mathbb{C}^8$.

The worldvolume soliton of a p-brane described by degree two Kähler calibration in $\mathbb{C}^4$ is the zero locus of the holomorphic functions

$$
F^1(s^1, s^2, s^3, z) = 0 \\
F^2(s^1, s^2, s^3, z) = 0 \\
F^3(s^1, s^2, s^3, z) = 0,
$$

(5.20)

where $z$ is a complex worldvolume coordinate and $s^1, s^2, s^3$ are complex coordinates transverse to the p-brane. Next suppose that $z = x^{p-1} + ix^p$, $s^1 = y^1 + iy^2$, $s^2 = y^3 + iy^4$ and $s^3 = y^5 + iy^6$ ($y^i = X^{p+i}$). The $SU(4)$ rotation group acts on $s^1, s^2, s^3, z$ and with this choice of complex structure the spinor singlets of $SU(4)$ satisfy

$$
\Gamma_{p-1}\Gamma_p\Gamma_{p+1}\Gamma_{p+2}\epsilon = -\epsilon \\
\Gamma_{p-1}\Gamma_p\Gamma_{p+3}\Gamma_{p+4}\epsilon = -\epsilon \\
\Gamma_{p-1}\Gamma_p\Gamma_{p+5}\Gamma_{p+6}\epsilon = -\epsilon.
$$

(5.21)

We then choose the base point $y_0$ as in the $SU(2)$ case above so that the projector of the p-brane is given as in (5.10). Using the p-brane projection operator together with (5.21), we find that the proportion of the bulk supersymmetry preserved is $1/16$ as expected. As in the $SU(2)$ case the above projections can be rewritten as the projections of four p-branes intersecting on a (p-2)-brane.

An example of such worldvolume soliton is the one that is associated with the
bulk configuration of four M-2-branes intersecting on a 0-brane, i.e.

\begin{align*}
(i) & \quad M^-2 : 0, 1, 2, - , - , - , - , - , - \\
(ii) & \quad M^-2 : 0, - , - , 3, 4, - , - , - , - \\
(iii) & \quad M^-2 : 0, - , - , - , - , 5, 6 - , - \\
(iv) & \quad M^-2 : 0, - , - , - , - , - , 7, 8.
\end{align*}

(5.22)

From the perspective of one of the four 2-branes, say the first one, \(z\) spans the worldvolume directions 1, 2 and \(s^1, s^2, s^3\) span the worldvolume directions 3, 4, 5, 6, 7, 8 of the other three M-2-branes. The explicit solutions that we have given for the case of three M-2-branes intersecting on a 0-brane can be easily generalize to this case and we shall not repeat the analysis here.

Next we shall consider the case of degree four Kähler calibrations in \(\mathbb{C}^4\). The worldvolume soliton of a p-brane associated with such calibration is described by the zero locus of two holomorphic functions

\begin{align*}
F^1(z^1, z^2, s^1, s^2) &= 0 \\
F^2(z^1, z^2, s^1, s^2) &= 0 ,
\end{align*}

(5.23)

where \(z^1, z^2\) are two complex worldvolume coordinates and \(s^1, s^2\) are two complex coordinates transverse to the p-brane. The spinor singlets of \(SU(4)\) satisfy the same conditions (5.21) of the previous case for a similar choice of complex structure. The proportion of the supersymmetry preserved is \(1/16\) of the bulk supersymmetry.

From the bulk perspective there are five intersecting brane configurations that correspond to this worldvolume solitons. These are two M-5-branes intersecting at a string, two IIA and IIB NS 5-branes intersecting at a string, two D-5-branes intersecting at a string and two D-4-branes intersecting at a 0-brane. All these configurations are related by reduction from M-theory to IIA and T-duality from IIA to IIB. So let us consider the case of two intersecting D-4-branes at a 0-brane. From the perspective of one of the D-4-branes, the 0-brane soliton is described by a degree four Kähler calibration with \(z^1, z^2\) the worldvolume coordinates of
the chosen brane and $s^1, s^2$ the worldvolume coordinates of the other. However there is a puzzle, it is well known that when the intersection of two D-4-branes is orthogonal the proportion of the supersymmetry preserved by the configuration is 1/4 of the bulk. This is unlike the cases that we have studied previously where the orthogonally intersecting configuration and the intersecting configuration at $SU(n)$ angles preserved the same proportion of supersymmetry. One resolution of the puzzle is that worldvolume soliton that is associated with the orthogonally intersecting configuration does not utilize the full $SU(4)$ group of rotations of the tangent bundle of the submanifold. We shall give such solutions in section 8. Such solutions though are singular at the intersection.

The last case is that of worldvolume solitons of p-branes associated with Kähler calibrations of degree six in $\mathbb{C}^4$. This requires that $p \geq 6$ and with at least two transverse directions. These solitons are given by the zero locus of the holomorphic function

$$F(s, z^1, z^2, z^3) = 0$$

(5.24)

where $z^1, z^2, z^3$ are complex worldvolume coordinates of the p-brane and $s$ is a complex transverse coordinate. The investigation of the properties of these solitons, like supersymmetry, is similar to that of the previous cases and we shall not pursue this further here.

6. Static Solutions and Special Lagrangian Calibrations

The special Lagrangian calibrations (SLAG) are closely related to the Kähler ones which we have investigated in the previous sections. The contact set of a SLAG calibration is $G(\phi) = SU(n)/SO(n)$. Therefore a generic worldvolume soliton associated with a SLAG calibration preserves $1/2^n$ of the supersymmetry of the bulk.

To describe the solutions of Nambu-Goto action associated with SLAG calibrations, we again choose the static gauge (5.1) of a p-brane as in the case of Kähler
calibrations. The SLAG calibration is then a map $y = \{y^i; i = 1, \ldots, n\}$ from $\mathbb{E}^n \subset \mathbb{E}^{(p,1)}$ with coordinates $\{x^i; i = 1, \ldots, n\}$ into $\mathbb{E}^n \subset \mathbb{E}^{(9,1)}$. In this gauge, the conditions on $y$ required by the SLAG calibration [15, 16] are

$$y^i = \partial_i f(x), \quad (6.1)$$

and

$$\text{Im} \left( \det (\delta_{ij} + i \partial_i \partial_j f) \right) = 0, \quad (6.2)$$

where $f$ is a real function of the coordinates $\{x^i; i = 1, \ldots, n\}$ and $\partial_i f = \frac{\partial}{\partial x^i} f$.

Some examples of SLAG calibrations have been given in [15, 16] and we shall not repeat them here.

6.1. SU(2) SLAG CALIBRATIONS

The $SU(2)$ SLAG calibrations are the same as the $SU(2)$ Kähler ones. This is because the calibration form in this case is

$$\phi = dz^1 \wedge dz^2 + dz^3 \wedge d\bar{z}^2 \quad (6.3)$$

which is the Kähler form of another complex structure $J$ on $\mathbb{E}^4$. Therefore this SLAG calibration is a Kähler calibration with respect to $J$. So the investigation of the properties of the corresponding solutions of the Born-Infeld field equations is the same as that described in section 5.1 for the corresponding Kähler calibration.

6.2. SU(3) SLAG CALIBRATIONS

The $SU(3)$ SLAG calibration is a degree three calibration in $\mathbb{E}^6$. To interpret this calibration as a soliton of a p-brane, we take three $\{x^i; i = 1, 2, 3\}$ of the coordinates of $\mathbb{E}^6$ to be worldvolume directions of a p-brane ($p > 2$) and the other three $\{y^i; i = 1, 2, 3\}$ to be transverse to it. The $SU(3)$ rotations act on $\mathbb{E}^6$ with complex coordinates $z^i = x^i + iy^i$. The supersymmetry projections of $SU(3)$
rotations of the contact set can be easily found using the above choice of complex structure of the SLAG calibration (compare with (5.14)). A straightforward computation reveals that the supersymmetry preserved by such a soliton is 1/8 of the bulk supersymmetry; 1/4 of the supersymmetry is broken by the $SU(3)$ rotations and another 1/2 is broken by the supersymmetry projection associated with the p-brane.

The simple example of a bulk configuration which is associated to the solitons of the $SU(3)$ SLAG calibration is that of three M-5-branes intersecting on a membrane, i.e.

\[(i) \text{M} - 5 : \quad 0, 1, 2, 3, 4, 5, -, -, - \]

\[(ii) \text{M} - 5 : \quad 0, 1, 2, -, -, 5, 6, 7, - \] \hfill (6.4)

\[(iii) \text{M} - 5 : \quad 0, 1, 2, -, 4, -, 6 -, 8 . \]

From the perspective of one of the three M-5-branes, say the first one, this bulk configuration is associated to a 2-brane worldvolume soliton in the directions 0, 1, 2. The transverse scalars $(y^1, y^2, y^3)$, $(y^i = X^{p+i})$, of the worldvolume soliton depend on the worldvolume coordinates $x^3, x^4, x^5$. It is straightforward to check that this configuration of M-5-branes preserves 1/8 of the bulk supersymmetry. The associated supersymmetry projections are identical with those of the 2-brane worldvolume soliton. Reducing this configuration to IIA theory along $x^1$ and T-dualizing to IIB along $x^2$, we find that the same calibration describes a string soliton on a D-4-brane and a 0-brane soliton on a D-3-brane, respectively.

The above interpretation of the worldvolume solitons of the $SU(3)$ SLAG calibrations is not unique. This calibration can also be interpreted as the soliton of two M-5-branes intersecting on a 2-brane at $SU(3)$ angles. We remark though that if the two M-5-branes are brought in an orthogonal position all supersymmetry will break.
6.3. SU(4) and SU(5) SLAG Calibrations

The investigation of the worldvolume solitons of the SU(4) and SU(5) SLAG calibrations is similar to the one described above for the worldvolume solitons of the SU(3) SLAG calibrations, so we shall not present these cases in detail. Here we shall present two examples of bulk configurations that can be associated to these worldvolume solitons the following: (i) A bulk configurations associated to SU(4) SLAG calibration and preserving 1/16 of the supersymmetry is

\[
(i) M - 5 : 0, 1, 2, 3, 4, 5, -, -, -,
(ii) M - 5 : 0, 1, -, -, 4, 5, 6, 7, -,
(iii) M - 5 : 0, 1, -, 3, -, 5, 6, 8, -
(iv) M - 5 : 0, 1, -, 3, 4, -, 6, -, 9.
\]

Another configuration can be found by placing the third M-5-brane above in the directions 0, 1, 2, 5, 7, 8. The soliton is a string on the M-5-brane and lies in the directions 0, 1. (ii) A bulk configurations associated to SU(5) SLAG calibration and preserving 1/32 of the supersymmetry is

\[
(i) M - 5 : 0, 1, 2, 3, 4, 5, -, -, -, -
(ii) M - 5 : 0, -, -, 3, 4, 5, 6, 7, -, -
(iii) M - 5 : 0, 1, -, 3, -, 5, -, 7, -
(iv) M - 5 : 0, -, 2, -, 4, 5, 6, 8, -
(v) M - 5 : 0, 1, 2, 3, -, -, -, -, 9, 10.
\]

We remark that the M-5-branes can be placed at different directions from those indicated above still leading to five M-5-branes intersecting at a 0-brane. The worldvolume soliton is a 0-brane on the M-5-brane.

Apart from the interpretation given above, SU(4) and SU(5) SLAG calibrations solitons can also be associated with two M-5-branes intersecting at SU(4) and SU(5) angles on a 1-brane and a 0-brane, respectively. The latter case may be
of interest since it is suitable for describing M-5-brane junctions but this will not be
investigated further here. The bulk configuration of two orthogonally intersecting
M-5-branes at a string preserves 1/4 of the supersymmetry and as we shall show
later that there is a singular worldvolume soliton associated with it. However the
bulk configuration of two orthogonal M-5-branes intersecting at a 0-brane breaks
all the supersymmetry and therefore for such configuration to exist the intersection
should occur at $SU(5)$ angles.

7. Exceptional calibrations

7.1. $G_2$ calibrations

The investigation of the solutions of the Born-Infeld action associated with
the group $G_2$ is similar to that of the Kähler and SLAG calibrations. However
in this case, there are not seem to be a straightforward interpretation of these
calibrations in terms of intersecting M-branes. It is tempting though to suggest
that these calibrations are associated with intersecting M-5-branes at $G_2$ angles on
a string. Supposing this, we take the M-5-brane to lie in the directions 0, 1, 2, 6, 7, 8
and the string to lie in the directions 0, 8. For the degree three calibration, we take
$E^7$ to span the directions 1, 2, 3, 4, 5, 6, 7. The spinor singlets under $G_2$ satisfy

$$G_{mn} \epsilon = 0 \quad (7.1)$$

where

$$G_{mn} = \Gamma_{mn} + \frac{1}{4} \star \varphi_{mn}^{pq} \Gamma_{pq} \quad (7.2)$$

are the generators of $G_2$, $m, n, p, q = 1, \ldots, 7$, and $\star \varphi$ is the Hodge dual of the
structure constants $\varphi$ of the octonions which we have chosen as

$$\varphi_{123} = \varphi_{246} = \varphi_{435} = \varphi_{516} = \varphi_{572} = \varphi_{471} = \varphi_{673} = 1 \quad (7.3)$$
The condition (7.1) on $\epsilon$ yields the supersymmetry projections

\[ \Gamma_{1346}\epsilon = \epsilon \]
\[ \Gamma_{2356}\epsilon = \epsilon \]
\[ \Gamma_{4567}\epsilon = \epsilon . \]  
(7.4)

There are many equivalent ways to present these projections for the above choice of $\varphi$. The consistency of these projections together with the projector associated with the M-5-brane reveals that a generic $G_2$ calibration preserves 1/16 of the bulk supersymmetry. For the degree four calibration, we again take $E^7$ to span the directions 1, 2, 3, 4, 5, 6, 7. The supersymmetry projectors associated with $G_2$ are as in (7.4). The supersymmetry preserved is also 1/16 of the bulk.

### 7.2. Spin(7) Calibrations

The Spin(7) calibrations are degree four calibrations in $E^8$ and a bulk interpretation is as two M-5-branes at Spin(7) angles intersecting on a string. As in the $G_2$ case above, we take one of the M-5-branes to lie in the directions 0, 1, 2, 6, 7, 9, the string to lie in the directions 0, 9 and the calibration to take place in the directions 1, 2, 3, 4, 5, 6, 7, 8. The spinor singlets under Spin(7) satisfy

\[ G_{IJ}\epsilon = 0 \] 
(7.5)

where

\[ G_{IJ} = \frac{3}{4}(\Gamma_{IJ} + \frac{1}{6}\Omega_{IJ}{}^{KL}\Gamma_{KL}) \] 
(7.6)

are the generators of Spin(7), $I, J, K, L = 1, \ldots, 8$, and

\[ \Omega_{mnp8} = \varphi_{mnp} \]
\[ \Omega_{mnpq} = \star\varphi_{mnpq} \] 
(7.7)

is the Spin(7)-invariant self-dual 4-form. The supersymmetry projections associ-
ated with Spin(7) acting on $\mathbb{E}^8$ spanned by the above directions are

$$\Gamma_{1346} \epsilon = \epsilon$$
$$\Gamma_{2356} \epsilon = \epsilon$$
$$\Gamma_{4567} \epsilon = \epsilon$$
$$\Gamma_{1238} \epsilon = \epsilon.$$

(7.8)

The first three projections are similar to those of $G_2$. Using the last two projections, we find that

$$\Gamma_{12345678} \epsilon = \epsilon.$$

(7.9)

This implies that if $\epsilon$ is anti-chiral in the 8-dimensional sense all supersymmetry is broken* The consistency of these projections together with the projector associated with the M-5-brane reveals that a generic Spin(7) calibration preserves $1/32$ of the bulk supersymmetry.

We summarize some of our results in the sections five-seven in the table below.

| Calibration | Contact | Supersymmetry |
|-------------|---------|---------------|
| Kähler      | $SU(n)$ | $2^{-n}$      |
| SLAG        | $S(U(p) \times U(n-p))$ | $2^{-n}$ |
| $G_2$       | $SO(n)$ | $2^{-4}$      |
| Spin(7)     | $Spin(7)$ | $2^{-5}$     |

Table 2: Calibrations and Supersymmetry  This table contains the type of calibration, the associated contact set and the proportion of the supersymmetry preserved by the calibration.

* We remark that this depends on the orientation of $\mathbb{E}^8$. If the other orientation is chosen, then chiral representation does not have singlets under Spin(7).
8. Singular Solutions and Piecewise Planar Branes

The Dirac-Born-Infeld field equations admit a large class of solutions for which the transverse scalars $y$ are piece-wise linear functions and the Born-Infeld field $F$ is piece-wise constant. In this class of solutions, we can also include double valued solutions like for example the configurations for which their graph consists of two or more geometrically intersecting planes in $\mathbb{E}^{(9,1)}$. Some of these solutions are limits of the solitons discussed in the previous sections. The bulk interpretation of such solutions is that of planar branes intersecting at angles with a non-vanishing Born-Infeld field. The singularities of these solutions are at the intersections where the transverse scalars and their first derivatives are discontinuous. We shall call such solutions of the Dirac-Born-Infeld field equations ‘singular solutions’.

The investigation of the supersymmetry preserved by a singular solution with vanishing Born-Infeld field is very similar to that of two or more planar branes placed in the ten-dimensional Minkowski background. Such analysis has already been done in [22] and we shall not repeat it here. Instead, we shall give one example to illustrate the way that such a computation can be done. For this we shall consider singular solutions of the M-2-brane action. Let $x^1, x^2$ be the two spatial worldvolume coordinates of a M-2-branes and $y^1 = X^3, y^2 = X^4$ two of its transverse scalars. A singular worldvolume solution of the M-2-brane is

$$y^i = \begin{cases} c^i, & \text{for } x^1 \cdot x^2 > 0 \\ x^i, & \text{for } x^1 \cdot x^2 < 0 , \end{cases}$$

(8.1)

where $\{c^i; i = 1,2\}$ are constants. The supersymmetry conditions associated with this solution are

$$\Gamma_0 \Gamma_1 \Gamma_2 \epsilon = \epsilon , \quad x^1 \cdot x^2 > 0$$

$$\frac{1}{2} \Gamma_0 (\Gamma_1 + \Gamma_3)(\Gamma_2 + \Gamma_4) \epsilon = \epsilon , \quad x^1 \cdot x^2 < 0 .$$

(8.2)

Using the first projector the second one can be rewritten as

$$\Gamma_0 \Gamma_3 \Gamma_4 \epsilon = \epsilon .$$

(8.3)
Therefore the supersymmetry preserve is 1/4 of the bulk. The projectors of this solution are those of two M-2-branes intersecting at a 0-brane.

We can easily include piece-wise constant Born-Infeld fields to the above singular solutions. If the Born-Infeld is everywhere constant, then it is straightforward to see that its inclusion does not break any additional supersymmetry. However if it is piece-wise constant, additional supersymmetry may be broken (see [22]).

9. Solutions with only Born-Infeld fields

A consistent truncation of the Dirac-Born-Infeld action is to fix the location of the p-brane, as in section 3.2, and reduce it to a Born-Infeld action. Choosing a Lorentz frame in the bulk adopted to the p-brane, the supersymmetry condition, given also in section 3.2, becomes

\[ e^{-\frac{\alpha}{2}} \Gamma e^{\frac{\alpha}{2}} \epsilon = \epsilon \]  \hspace{1cm} (9.1)

since \( S = 1 \); the projector \( \Gamma \) is a product of (constant) Gamma matrices. As it has already been mentioned in section 3.2 linearizing this condition in the Born-Infeld fields, it becomes the familiar supersymmetry condition of the maximally supersymmetric Maxwell multiplet in \( (p+1) \) dimensions together with the projection associated with the planar p-brane. So at least in the linear approximation, Born-Infeld fields that satisfy the self-duality condition and its generalizations, like for example the Einstein-Yang-Mills \( (SU(n)) \) condition or the \( Sp(k) \) condition, preserve a proportion of the spacetime supersymmetry. We remark that the worldvolume solitons involving only Born-Infeld fields have the bulk interpretation of branes within branes.

A general analysis for the conditions that \( Y \) (see (3.20)) should satisfy for (9.1) to admit not trivial solutions has been already given in section 3.2. Here we shall examine some special cases. First we shall consider the case where \( Y \) is self-dual...
2-form in the directions 1,2,3,4 of the p-brane \((p \geq 4)\). If this is the case, then the singlets of the rotation \(e^a\) satisfy

\[
\Gamma_{1234} \epsilon = -\epsilon .
\]  

(9.2)

We remark that the same condition on \(\epsilon\) is imposed by a self-dual Born-Infeld field in the linearized limit. If \(\epsilon\) is such a singlet, then the supersymmetry condition reduces to

\[
\Gamma \epsilon = \epsilon .
\]  

(9.3)

The compactability conditions of the two projections (9.2) and (9.3) determine the proportion of the bulk supersymmetry preserved by the configuration. It turns out that self-dual configurations on all D-p-branes \(p \geq 4\) preserve \(1/4\) of the bulk supersymmetry. It remains to express the self-duality condition on \(Y\) in terms of a condition on \(F\). It turns out that if \(Y\) is self-dual so is \(F\) as it can be easily seen using the identity

\[
Y_{ac} Y_{db} \delta^{cd} = -\frac{1}{4} Y_{cd} Y^{cd} \delta_{ab}
\]  

(9.4)

and the relation \(F = \tan Y\), where \(a,b,c,d = 1,\ldots,4\). Moreover a short calculation reveals that \(F\) satisfies the field equations of the Born-Infeld action as a consequence of the self-duality condition and the Bianchi identity.

A simple explicit example with a self-dual Maxwell field is obtained by setting

\[
F_{ab} = \partial_a A_b - \partial_b A_a
\]  

(9.5)

and

\[
A_a = I_a^b \partial_b H
\]  

(9.6)

where \(I\) is a constant complex structure of \(\mathbb{E}^4\), \(I_a^b I_b^c = -\delta_a^c\), with anti-self-dual Kähler form and \(H\) a harmonic function on \(\mathbb{E}^4\). This worldvolume solution has several bulk interpretations depending on the context that it is used. One example is that of a D-0-brane within a D-4-brane.
Other conditions on $Y$ which lead to preservation of some of the supersymmetry of the bulk have already been mentioned is section 3.2. However there does not seem to be such a simple relation between these conditions of $Y$ and those on $F$. For example, if $Y$ satisfies the Kähler-Yang-Mills condition (i.e. $Y$ is in the Lie algebra $SU(k)$), one can easily see that $F$ does not satisfy the same condition. We remark, however, that if $Y$ is tri-hermitian (i.e. $Y$ is in the Lie algebra $Sp(k)$), then $F$ is tri-hermitian as well. Such condition has been studied [24] where it was found that the supersymmetry preserved 3/16 of the bulk. Explicit solutions can be easily constructed by superposing those of (9.6) using the method developed in [31].

10. Solitons with Born-Infeld fields

Solitons of the Dirac-Born-Infeld field equations of a p-brane that involve both non-vanishing scalars and Born-Infeld fields are those that arise at the boundary of one brane ending on another. Such solitons preserve 1/4 of the bulk supersymmetry or less. We have not been able to find a calibration-like construction to deal with these solitons. So we shall use the bulk picture to determine the existence of such solutions and we shall give some new examples. It is convenient to categorize them according to the supersymmetry that they preserve.

10.1. Solitons preserving 1/4 of the supersymmetry

Most of these solitons have already being found. These include, (i) the 0-brane soliton on all D-branes due to a fundamental string ending on them [5], (ii) the 0-brane soliton on the IIB NS-5-brane due to a D-string ending on it, (iii) the self-dual string soliton [39] on the M-5-brane due to a membrane ending on it, and (iv) the 2-brane soliton [39] of the IIB NS-5-brane due to a three brane ending on it. These solitons are related using T-and S-duality transformations of the associated bulk theories [40]. Apart from these soliton solutions which are charged with respect to the standard 2-form Born-Infeld field, there are other solitons which are charged
with respect to other p-form fields. One example is the domain wall solutions on the IIB D-5- and NS-5-branes [41]. We remark that in all the above solitons, the supersymmetry projectors are those of the associated bulk configuration. For example the supersymmetry projectors of the self-dual string soliton on the M-5-brane are those on the M-5-brane and the M-2-brane. The energy bounds for some of the above solutions have been investigated in [42] and the geometry of their moduli spaces have been studied in [43].

10.2. Solitons preserving $1/8$ of the supersymmetry

There are many such solitons. This can be seen either by studying the charges that appear in the various worldvolume supersymmetry algebras or by examining the bulk intersecting brane configurations. We shall not present a complete account of all such configurations. Instead we shall investigate a class of electrically charged solutions associated with D-branes. Let $F$ be the Born-Infeld field of a D-p-brane and $\{y^i; i = 1, \ldots, 9-p\}$ be the transverse scalars. For electrically charged solutions, we use the ansatz

$$F_{0a} = -\partial_a \phi, \quad a = 1, \ldots, k$$

$$y^1 = \phi,$$  (10.1)

$$y^m = y^m(x) \quad m = 2, \ldots, 9-p$$

A standard computation using the results of [5] reveals that the field equations of the D-p-brane become

$$g^{ab} \nabla_a \partial_b \phi = 0$$

$$g^{ab} \nabla_a \partial_b y^m = 0$$  (10.2)

where

$$g_{ab} = \delta_{ab} + \partial_a y^m \partial_b y^n \delta_{mn}$$  (10.3)

is the induced metric and $\nabla$ is the Levi-Civita connection of $g$. The second equation in (10.2) can be solved by taking the maps $\{y^m\}$ to describe a degree $k$ calibration.
in $\mathbb{R}^{8+k-p}$. The first equation in (10.2) is just the harmonic function condition on the calibrated surface.

The equation (10.2) can be easily solved if the associated calibration is Kähler. In this case the induced metric $g$ on the calibrated manifold is Kähler. So the first equation of (10.2) can be solved by taking $\phi$ to be the real part of a holomorphic function $h$, i.e.

$$\phi = h(z) + \bar{h}(\bar{z}) \quad (10.4)$$

There are many bulk configuration that correspond to the above solutions by allowing for different calibrations. One example is the 0-brane worldvolume soliton on the D-3-brane corresponding to the intersection

$$(i) \quad D - 3 : \quad 0, 1, 2, 3, -, -, -$$

$$(ii) \quad D - 3 : \quad 0, 1, -, -, 4, 5, -$$

$$(iii) \quad F - 1 : \quad 0, -, -, -, -, 6 \quad (10.5)$$

The associated calibration is a degree two $SU(2)$ Kähler calibration in the directions 2, 3, 4, 5 and $y^1 = X^6$. The proportion of the supersymmetry preserved is $1/8$. We shall present a complete discussion of the worldvolume solitons that preserve $1/8$ of the supersymmetry or less elsewhere.

11. Conclusions

We have investigated the Killing spinor equations associated with the kappa-symmetry transformations of the worldvolume brane actions and shown that all the worldvolume solitons associated with calibrations are supersymmetric. The proof has been based on the properties of the contact set of a calibration. Then we have presented a bulk interpretation of all the calibrations in terms of intersecting branes. Next we have examined other supersymmetric worldvolume solutions, like singular solutions and solutions involving only Born-Infeld fields. Finally, we
found new worldvolume solutions of the Dirac-Born-Infeld action. In particular, we present a 0-brane worldvolume solution with the bulk interpretation of two intersecting D-3-brane with a fundamental string ending on them. Such solution preserves 1/8 of supersymmetry.

So far we have considered solutions of the Nambu-Goto-type of actions which are embeddings into the ten- or eleven-dimensional Minkowski spacetimes. Alternatively, we can take as spacetime any solution of ten- or eleven-dimensional supergravities. In particular, we can take $\mathcal{M}(10) = M_k \times \mathbb{E}^{(9-k,1)}$ or $\mathcal{M}(11) = M_k \times \mathbb{E}^{(10-k,1)}$ where $M_k$ is a manifold with the special holonomies of table 1.

Since for all these manifolds the Ricci tensor vanishes, the above spaces are solutions of the supergravity field equations by setting the rest of the fields either zero or constant. One can then consider solutions of the Nambu-Goto-type of actions for the spacetimes given as above (see also [17, 18]). There is a natural definition of a calibration in the manifolds with special holonomy using the covariantly constant forms (see table 1). It turns out that such solutions of the Nambu-Goto action associated with these calibrations preserve the same proportion of supersymmetry as those we have studied for Minkowski spacetimes.

We have shown that calibrations with contact set $G/H$ are associated with intersecting branes at $G$-angles. It is well known that if $G = Sp(2)$ there are supergravity solutions which have the interpretation of two M-5-branes intersecting at a string at $Sp(2)$ angles. There does not seem to be though a calibration with contact set $Sp(2)/H$, or more general $Sp(n)/H$, based on a calibrating form. However, it is straightforward to define one in terms of spinors. The relevant computation has already been done in [24]. It would be of interest to find the corresponding calibration based on forms. One option of such calibration is that for which the tangent spaces are quaternionic planes in $\mathbb{E}^{4n}$.

It would be of interest to investigate further the Dirac-Born-Infeld type of actions. In particular, there may be a generalization of calibrations which also involves non-vanishing Born-Infeld field. Unlike for calibrations, in this case it is
not clear which geometric quantity is minimized. However, the definition of such a
generalized calibration in terms of spinors is straightforward and it has been done
in section 3. Since it is expected that there should be a consistent generalization of
Dirac-Born-Infeld actions involving non-abelian Born-Infeld field (see [44]), there
may also exist a ‘non-abelian’ generalization of calibrations.

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