SYMMETRIES OF LINE BUNDLES AND NOETHER THEOREM FOR TIME-DEPENDENT NONHOLONOMIC SYSTEMS

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ABSTRACT. We consider Noether symmetries of the equations defined by the sections of characteristic line bundles of nondegenerate 1-forms and of the associated perturbed systems. It appears that this framework can be used for time-dependent systems with constraints and nonconservative forces, allowing a quite simple and transparent formulation of the momentum equation and the Noether theorem in their general forms.

1. Introduction. The Noether theorem on integrals related to invariant variational problems is one of the basic theorems in mechanics, both for finite and infinite-dimensional systems [24, 29]. There has been a lot of efforts on its generalization and our reference list [1]–[30] covers just a part of contributions for finite-dimensional systems.

Recently, we presented the problem for non-constrained systems through the perspective of contact geometry [22]. We considered Noether symmetries of the equation

\[
\dot{x} = Z, 
\]

where \(Z\) is a section of the characteristic line bundle of a nondegenerate 1-form \(L = \ker d\alpha\), as symmetries that preserve the action functional \(A_\alpha[\gamma] = \int_\gamma \alpha\). In the case of time-dependent Hamiltonian systems, Noether symmetries are transformations that preserve Poincaré–Cartan (modulo addition of a closed 1-form) and, via Legendre transformation, this is similar to the notion of symmetry of Lagrangian systems given by Crampin in [10].

It appears that the above framework can be used for time-dependent systems with constraints and nonconservative forces, allowing a quite simple and transparent formulation of the momentum equation and of the Noether theorem in their general forms.

We briefly recall on the notion of Noether symmetries for systems defined by the sections of non-degenerate 1-forms studied in [22] and formulate the statements for perturbed systems

\[
\dot{x} = Z + P, 
\]
where $P$ is not a section of $\mathcal{L} = \ker da$ (Theorems 2.2, 2.3, Section 2). In Section 3 we apply Theorem 2.2 and obtain the main results: a general momentum equation and Noether theorem for time-dependent nonholonomic systems subjected to nonconservative forces (Theorems 3.2, 3.3). In particular, when we deal with symmetries that are prolongation of time-dependent vector fields on the configuration space, we get a time-dependent variant of the so called gauge symmetries studied in [2, 3, 14, 16] (see Corollary 2) and the moving energy integral given in [16, 17] (see Corollary 3).

2. Noether symmetries of characteristic line bundles.

2.1. Dynamical systems defined by characteristic line bundles. Let $(M, \alpha)$ be a $(2n+1)$-dimensional manifold endowed with a 1-form $\alpha$, such that $da$ has the maximal rank $2n$. The kernel of $da$ defines a one dimensional distribution

$$\mathcal{L} = \bigcup_x \mathcal{L}_x, \quad \mathcal{L}_x = \ker da|_x$$

of the tangent bundle $TM$ called characteristic line bundle.

Also, at every point $x \in M$ we have the horizontal space $\mathcal{H}_x = \ker \alpha|_x$. In the case when $\alpha$ differs from zero on $M$, then the collection of horizontal subspaces

$$\mathcal{H} = \bigcup_x \mathcal{H}_x = \bigcup_x \ker \alpha|_x$$

is a nonintegrable $2n$-dimensional distribution of $TM$, called horizontal distribution. If, in addition, $\alpha \wedge (da)\wedge \neq 0$, then $\alpha$ is a contact form, $(M, \alpha)$ is a strictly contact manifold, and $\mathcal{H}$ is a contact distribution [26].

The integral curves $\gamma : [a, b] \to M$ of the characteristic line bundle $\mathcal{L}$ are extremals of the action functional

$$A_\alpha[\gamma] = \int_\gamma \alpha = \int_a^b \alpha(\dot{\gamma}) dt \quad (3)$$

in a class of variations $\gamma_s$ with fixed endpoints. Recall that a variation of a curve $\gamma : [a, b] \to M$ is a family of curves $\gamma_s(t) = \Gamma(t, s)$, where $\Gamma : [a, b] \times [0, \epsilon] \to M$ is a mapping, such that $\gamma(t) = \Gamma(t, 0), \ t \in [a, b]$. The endpoints are fixed if $\gamma_s(a) \equiv \gamma(a), \ \gamma_s(b) \equiv \gamma(b)$.

Consider the equation (1), where $Z$ is a section of $\mathcal{L}$. We say that a vector field $\zeta$ is a Noether symmetry of equation (1) if the induced one-parameter group of diffeomorphisms $g^\zeta_\epsilon$ preserves the 1-form $\alpha$. Then, by analogy with the classical formulation [8, 29], $g^\zeta_\epsilon$ preserves the action functional (3).

Note that $\mathcal{L}$ is determined by the cohomology class $[\alpha]$ ($\mathcal{L} = \ker da'$, where $\alpha' = \alpha + \beta$, $\beta$ is a closed 1-form on $M$), while $\mathcal{H}$ depends on $\alpha$. Thus, the integral curves of $\mathcal{L}$ are also extremals the action (3) with $\alpha$ replaced by $\alpha' \in [\alpha]$. We say that $\zeta$ is a weak Noether symmetry of equation (1) if we have the invariance of the perturbation $\alpha' = \alpha + \beta, \ d\beta = 0$, modulo the differential of a function $f$:

$$L_\zeta(\alpha') = L_\zeta(\alpha + \beta) = df. \quad (4)$$

That is, $g^\zeta_\epsilon$ preserves the action $A_{\alpha'}[\gamma] = \int_\gamma \alpha'$ modulo $f$:

$$\frac{d}{ds} A_{\alpha'}[\gamma_s] \bigg|_{s=0} = \frac{d}{ds} \left( \int_{\gamma_s} \alpha + \beta \right) \bigg|_{s=0} \quad (5)$$

Here the usual assumption that the endpoints are fixed can be relaxed: we can consider also the variations $\gamma_s(t)$, such that $\delta \gamma(a)$ and $\delta \gamma(b)$ are horizontal vectors, where $\delta \gamma(t)$ denotes the vector field $\frac{d}{ds} \bigg|_{s=0} \in T_{\gamma(t)} M$, e.g., see [22].
\[
\int_{\gamma_s} df\bigg|_{s=0} = \int_a^b df(\gamma(t)) = f(\gamma(b)) - f(\gamma(a)),
\]
where \(\gamma : [a, b] \to M\) is an arbitrary smooth curve and the variation \(\gamma_s\) is determined by the one-parameter group of diffeomeomorphisms \(g_s^\zeta\), \(\gamma_s = g_s^\zeta(\gamma)\).

Now, let \(\gamma : [a, b] \to M\) be a trajectory of (1) and \(\gamma_s = g_s^\zeta(\gamma)\). The relation \(\dot{\gamma}(t) \in \mathcal{L}_{\gamma(t)} = \ker d\alpha|_{\gamma(t)}\) and Cartan’s formula,

\[
L_{\zeta} = i_{\zeta} \circ d + d \circ i_{\zeta},
\]

imply

\[
\frac{d}{ds} A_{\alpha'}[\gamma_s]\bigg|_{s=0} = \int_a^b d(\alpha + \beta)(\zeta|_{\gamma(t)}, \dot{\gamma}(t)) dt + \int_a^b d((\alpha + \beta)(\zeta|_{\gamma(t)}))
\]

\[
= (\alpha + \beta)(\zeta|_{\gamma(b)}) - (\alpha + \beta)(\zeta|_{\gamma(a)}),
\]

and by comparing (5) and (7), we obtain the identity

\[(\alpha + \beta)(\zeta|_{\gamma(a)}) - f(\gamma(a)) = (\alpha + \beta)(\zeta|_{\gamma(b)}) - f(\gamma(b)).\]

Thus, the weak Noether symmetries induce conservation quantities described in the following statement (see [22]; for the Lagrangian setting and \(M = \mathbb{R} \times TQ\), see [8, 10]).

**Theorem 2.1.** Let \(\zeta\) be a weak Noether symmetry of equation (1) that satisfies (4). Then:

(i) The function

\[J = i_\zeta(\alpha + \beta) - f\]

is a first integral of (1).

(ii) \(J\) is preserved under the flow of \(g_s^\zeta\) as well: \(L_{\zeta}(J) = 0\).

(iii) The commutator of vector fields \([Z, \zeta]\) is a section of \(\mathcal{L}\), i.e., \(g_s^\zeta\) permutes the trajectories of (1) modulo reparametrization.

It is natural to refer to (8) as a Noether function associated to the week Noether symmetry \(\zeta\).

2.2. Noether symmetries of time-dependent Hamiltonian equations. The basic example is the extended phase space endowed with the Poincaré–Cartan 1-form

\[
(M, \alpha) = (\mathbb{R} \times T^*Q, pdq - Hdt).
\]

Namely, sections of \(\mathcal{L} = \ker d(pdq - Hdt)\) are of the form \(Z_\mu = \mu(t, q, p)Z\), where \(Z\) is the vector field defining the Hamiltonian flow of \(H\) in the extended phase space (e.g., see [26])

\[
Z = \frac{\partial}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p^i}\right).
\]

The action functional (3) for \(\alpha = pdq - Hdt\) implies Poincaré’s variant of the Hamiltonian principle of least action [1], while Theorem 2.1 is a natural generalization of the classical Noether theorem (see subsection 3.3). Recently, a similar approach to the higher order Lagrangian problems is given in [18].
2.3. Noether integrals for perturbed systems. Consider a perturbation (2) of equation (1) by a vector field $P$, which is not a section of $L = \ker d\alpha$. Also, let $M \subset \mathcal{M}$ be an invariant submanifold of the system (2), so we can consider the system restricted to $M$. The following observation, although quite elementary, is fundamental in our considerations.

**Theorem 2.2.** Assume that $\zeta$ satisfies (4) restricted to $M$. Then the derivative of the Noether function (8) along the flow of (2) equals

$$\frac{d}{dt}J|_{M} = d\alpha(P, \zeta)|_{M}.$$  

**Proof.** From the definition (4) and Cartan’s formula (6) we have

$$i_{\zeta}d\alpha = -d(i_{\zeta}(\alpha + \beta)) + df = -dJ,$$

implying the statement:

$$L_{Z+P}J = i_{Z+P}(-i_{\zeta}d\alpha) = d\alpha(Z, \zeta) + d\alpha(P, \zeta) = d\alpha(P, \zeta).$$  

**Corollary 1.** Assume that $P$ is $d\alpha$–orthogonal to the weak Noether symmetry field $\zeta$ restricted to $M$. Then $J|_{M}$ is a first integral of system (2).

Next, from the proof of Theorem 2.2, we see that we can relax the assumption that the vector field $\zeta$ is a weak Noether symmetry.

**Theorem 2.3.** Assume that $\zeta$ is a vector field and $\gamma$ is a 1-form satisfying

$$L_{\zeta}(\alpha + \beta)|_{M} = df + \gamma|_{M},$$

$$d\alpha(P, \zeta) + i_{Z+P}\gamma = 0|_{M}.$$  

Then the Noether function (8) is preserved along the flow of (2).

**Proof.** Now, from the definition (14) and Cartan’s formula we get

$$i_{\zeta}d\alpha = -d(i_{\zeta}(\alpha + \beta)) + df + \gamma = -dJ + \gamma.$$

Therefore

$$L_{Z+P}J = i_{Z+P}(-i_{\zeta}d\alpha) + i_{Z+P}\gamma = d\alpha(P, \zeta) + i_{Z+P}\gamma,$$

which proves the statement. 

Note that for $P = 0$, $M = \mathcal{M}$, Theorem 2.3 implies the following variant of Theorem 2.1: if $\zeta$ satisfies (14), where a 1-form $\gamma$ annihilates the line bundle $L$, then the Noether function (8) is a first integral of (1).

3. Time-dependent systems with constraints and nonconservative forces.

3.1. Equations. Consider a Lagrangian system $(Q, L, F)$, where $Q$ is a configuration space, $L(t, q, \dot{q})$ is a time-dependent Lagrangian, $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$, and $F$ is a non-conservative force. Assume that the motion of the system is subjected to $s$, in general time-dependent, independent ideal holonomic constraints

$$f^{l}(t, q) = 0, \quad l = 1, \ldots, s,$$

defining a $(n - s)$-dimensional time-dependent constraint submanifold $\Sigma_{t} \subset Q$. Therefore, the velocities of the system satisfy the constraints

$$a^{0}_{0}(t, q) + a^{1}_{1}(t, q)q_{1} + \cdots + a^{n}_{n}(t, q)q_{n} = 0, \quad l = 1, \ldots, s.$$  


where
\[ a_0^l(t,q) = \frac{\partial f^l}{\partial t}(t,q), \quad a_i^l(t,q) = \frac{\partial f^l}{\partial \dot{q}_i}(t,q), \quad i = 1, \ldots, n, \]
and \( q = (q_1, \ldots, q_n) \) are local coordinates on \( Q \).

In addition, suppose \( k - s \) additional independent ideal (nonholonomic) constraints are given
\[ a_0^l(t,q) + a_1^l(t,q)\dot{q}_1 + \cdots + a_n^l(t,q)\dot{q}_n = 0, \quad l = s + 1, \ldots, k. \tag{18} \]
As a result, the velocities of the system belong to a \((n - k)\)-dimensional time-dependent affine distribution \( D_t \subset T\Sigma_t Q \).

Together with \( D_t \), we consider the associated \((n - k)\)-dimensional distribution of virtual displacements \( D^0_t \subset T\Sigma_t \subset T\Sigma_t Q \) defined by the homogeneous equations
\[ \sum_{i=1}^n a_i^l(t,q)\xi_i = 0, \quad l = 1, \ldots, k. \tag{19} \]
The motion of the system on the constrained space \( D = \{(t, D_t) \mid t \in \mathbb{R}\} \subset \mathbb{R} \times TQ \)
is described by the Euler–Lagrange–d’Alembert equations
\[ \sum_{i=1}^n \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - F_i \right)\xi_i = 0, \tag{21} \]
for all time-dependent vector fields \( \xi = \sum_i \xi_i \partial / \partial q_i \), which satisfy the homogeneous constraints (19) (so called virtual displacements).

Equivalently, equations (21) can be rewritten in the form
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i + R_i, \quad i = 1, \ldots, n. \tag{22} \]
Here, \( F_i \) and
\[ R_i = \sum_{l=1}^k \lambda_l(t,q,\dot{q})a_i^l(t,q), \quad i = 1, \ldots, n, \tag{23} \]
are the components of the nonconservative and the reaction force, respectively.
The Lagrange multipliers \( \lambda_i(t,q,\dot{q}) \) are determined from the condition that a motion \( q(t) \) satisfy the constraints (16), (17), (18). We skip a discussion on the existence and the uniqueness of the Lagrange multipliers.

Let \( \text{FL}_t : TQ \to T^*Q \) be the Legendre transformation
\[ \text{FL}_t(t,q,\xi) \cdot \eta = \frac{d}{ds}{\bigg|}_{s=0} L(t,q,\xi + s\eta) \iff p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \ldots, n, \tag{24} \]
where \( \xi, \eta \in T_q Q \) and \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) are canonical coordinates of the cotangent bundle \( T^*Q \). In order to have a Hamiltonian description of the dynamics we suppose that the Legendre transformation (24) is a diffeomorphism.

Let
\[ \mathcal{M} = \{(t, M_t) \mid t \in \mathbb{R}\} \subset \mathbb{R} \times T^*Q, \quad M_t|_q = \text{FL}_t(D_t|_q) \subset T^*_q Q. \]

\(^2\)As an example, we can take \( Q \) to be the configuration space \( \mathbb{R}^{3N} \) of \( N \) free material points, see [1, 25].
be the constrained manifold (20) within $\mathbb{R} \times T^*Q$. It is defined by the equations (16) and
\[
a_0^l(t, q) + a_1^l(t, q) \frac{\partial H}{\partial p_1} + \cdots + a_n^l(t, q) \frac{\partial H}{\partial p_n} = 0, \quad l = 1, \ldots, k. \tag{25}
\]

In the canonical coordinates $(q, p)$ of the cotangent bundle $T^*Q$, the equations (22) read:
\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i(t, q, p) + R_i(t, q, p), \quad i = 1, \ldots, n. \tag{26}
\]

Here the Hamiltonian function $H(t, q, p)$ is the Legendre transformation of $L$
\[
H(t, q, p) = \mathcal{F}L(t, q, \dot{q}) \cdot \dot{q} - L(t, q, \dot{q})|_{\dot{q} = \mathcal{F}L^{-1}(t, q, p)}, \tag{27}
\]
and $F_i(t, q, p) = F_i(t, q, \dot{q})|_{\dot{q} = \mathcal{F}L^{-1}(t, q, p)}$, $R_i(t, q, p) = R_i(t, q, \dot{q})|_{\dot{q} = \mathcal{F}L^{-1}(t, q, p)}$.

In other words, on the constrained manifold $\mathcal{M}$ we have a system of the form (2), where $Z$ is a section of the characteristic line bundle $\ker d(pdq - Hdt)$ given by (10) and the perturbation vector field is
\[
P = \sum_{i=1}^{n} (F_i(t, q, p) + R_i(t, q, p)) \frac{\partial}{\partial p_i}.
\]

3.2. The reaction-annihilator distribution. Let $(t, q, p) \in \mathcal{M}$. Define distributions $\mathcal{V}$ and $\mathcal{R}$ of $T(\mathbb{R} \times T^*Q)$ at the points of $\mathcal{M}$ by
\[
\mathcal{V}_{(t, q, p)} = \{\zeta = \tau \frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial q_i} + \eta_i \frac{\partial}{\partial p_i} | a_0^l(t, q) \tau + \sum_{i=1}^{n} a_1^l(t, q) \xi_i = 0, \ l = 1, \ldots, k\},
\]
\[
\mathcal{R}_{(t, q, p)} = \{\zeta = \tau \frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial q_i} + \eta_i \frac{\partial}{\partial p_i} | \sum_{i=1}^{n} R_i(t, q, p)(\xi_i - \frac{\partial H}{\partial p_i} \tau) = 0\}.
\]

We refer to $\mathcal{V}$ as an admissible distribution over $\mathcal{M}$, since the velocity of a curve $(t, q(t), p(t))$ belongs to $\mathcal{V}$ if and only if $q(t)$ satisfies the constraints (16), (18). The rank of $\mathcal{V}$ is $2n + 1 - k$. On the other hand, the distribution $\mathcal{R}$ need not be of a constant rank: $\mathcal{R}_{(t, q, p)}$ is either a hyperplane or a whole tangent space $T_{(t, q, p)}(\mathbb{R} \times T^*Q)$, if reaction forces vanish at $(t, q, p)$.

Following [14], we call $\mathcal{R}$ a reaction-annihilator distribution over $\mathcal{M}$ (see Section 4 below).

Lemma 3.1. We have the inclusion $\mathcal{V} \subset \mathcal{R}$.

Proof. Let $\zeta \in \mathcal{V}_{(t, q, p)}$. Then, from (23) we have
\[
\sum_{i=1}^{n} R_i \xi_i = \sum_{i=1}^{n} \sum_{l=1}^{k} \lambda_0^l a_0^l \xi_i = -\tau \sum_{l=1}^{k} \lambda_0^l a_0^l.
\]

On the other hand, from (25) we get
\[
\sum_{i=1}^{n} R_i \frac{\partial H}{\partial p_i} = \sum_{i=1}^{n} \sum_{l=1}^{k} \lambda_0^l a_0^l \frac{\partial H}{\partial p_i} = -\sum_{l=1}^{k} \lambda_0^l a_0^l, \tag{28}
\]
which implies $\zeta \in \mathcal{R}_{(t, q, p)}$. \qed
Let
\[ \zeta = \tau (t, q, p) \frac{\partial}{\partial t} + \sum_i \xi_i (t, q, p) \frac{\partial}{\partial q_i} + \eta_i (t, q, p) \frac{\partial}{\partial p_i}, \]
be a weak Noether symmetry of the Hamiltonian system with the Hamiltonian (27) restricted to \( M \):
\[ L_\zeta (p dq - H dt + \beta) = df|_\mathcal{M}, \]
with respect to a closed 1-form \( \beta \) and a smooth function \( f \) in the extended phase space \( \mathbb{R} \times T^* Q \).

From Theorem 2.2 we get.

**Theorem 3.2.** (i) The derivative of
\[ J = i_\zeta \left( p dq - H dt + \beta \right) - f = \sum_i p_i \xi_i - H \tau + \beta (\zeta) - f \]
along the flow of (26) equals
\[ \frac{dJ}{dt}|_\mathcal{M} = \sum_{i=1}^n (F_i + R_i)(\xi_i - \dot{q}_i \tau)|_\mathcal{M}. \]

(ii) If \( \zeta|_\mathcal{M} \) is a section of the admissible distribution \( \mathcal{V} \), or more generally, a section of \( \mathcal{R} \), the derivative of the Noether function is given by
\[ \frac{dJ}{dt}|_\mathcal{M} = \sum_{i=1}^n F_i (\xi_i - \dot{q}_i \tau)|_\mathcal{M}. \]

In particular, if \( F \equiv 0 \), \( J \) is preserved along the flow of (26) if and only if \( \zeta \) is a section of \( \mathcal{R} \).

**Proof.** (i) We have
\[ d\alpha (P, \zeta) = (dp \wedge dq - dH \wedge dt)(P, \zeta) = \sum_{i=1}^n dp_i (P)dq_i (\zeta) - dH (P) dt (\zeta) \]
\[ = \sum_{i=1}^n (F_i + R_i)(\xi_i - \tau \frac{\partial H}{\partial p_i}) = \sum_{i=1}^n (F_i + R_i)(\xi_i - \dot{q}_i \tau), \]
where we used \( dq_i (P) = dt (P) = 0 \).

(ii) The proof follows directly from item (i) and Lemma 3.1.

**3.3. Classical Noether theorem and prolongations of time-dependent vector fields.** In the classical Noether theorem, without constraints and nonconservative forces, one considers the invariance of the action functional
\[ A = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) dt \]
under the transformations induced by the \( \mathbb{R} \times TQ \)-prolongation (e.g., see [8])
\[ \hat{\xi}_{\mathbb{R} \times TQ} = \tau \frac{\partial}{\partial t} + \sum_i \xi_i \frac{\partial}{\partial q_i} + \nu_i \frac{\partial}{\partial \dot{q}_i}, \]
\[ \nu_i = \frac{\partial \xi_i}{\partial t} - \dot{q}_i \frac{\partial \tau}{\partial t} + \sum_j \left( \frac{\partial \xi_i \dot{q}_j - \dot{q}_i \frac{\partial \tau \dot{q}_j}}{\partial q_j} \right), \]
(31)
of a time-dependent vector field
\[
\dot{\xi} = \tau \frac{\partial}{\partial t} + \xi = \tau(t, q) \frac{\partial}{\partial t} + \sum_i \xi_i(t, q) \frac{\partial}{\partial q_i}.
\] (32)

The action (30) is preserved if the Lagrangian satisfies the invariance condition:
\[
\sum_i \left( \frac{\partial L}{\partial q_i} \xi_i + \frac{\partial L}{\partial \dot{q}_i} \nu_i \right) + \tau \frac{\partial}{\partial t} + \sum_j \frac{\partial \tau}{\partial \dot{q}_j} \dot{q}_j = 0,
\] (33)
and then
\[
J(t, q, \dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}_i} (\xi_i - \tau \dot{q}_i) + L \tau
\] (34)
is a first integral of the Euler-Lagrange equations (e.g., see [8]).

On the Hamiltonian side, firstly we need to take some natural prolongation of
(32) to \(\mathbb{R} \times T^*Q\). We define the prolongation
\[
\tilde{\zeta} = \tau \frac{\partial}{\partial t} + \sum_i \xi_i \frac{\partial}{\partial q_i} + \sum_i \left( H \frac{\partial \tau}{\partial q_i} - \sum_j \frac{\partial \xi_j}{\partial q_i} p_j \right) \frac{\partial}{\partial p_i}
\] (35)
from the condition that the Lie derivative \(L_{\tilde{\zeta}}(pdq - Hdt)\) is proportional to the one-form \(dt\) (see Proposition 2.1 [22]). Then \(\tilde{\zeta}\) is a Noether symmetry of the Poincaré–Cartan 1-form
\[
L_{\tilde{\zeta}}(pdq - Hdt) = 0
\] if and only if the following invariance condition is satisfied (see the proof of Proposition 2.2, [22]):
\[
L_{\tilde{\zeta}} H = p \frac{\partial \xi}{\partial t} - H \frac{\partial \tau}{\partial t} = \sum_{i=1}^n p_i \frac{\partial \xi_i}{\partial \dot{q}_i} - H \frac{\partial \tau}{\partial t}.
\] (36)

Moreover, under the Legendre transformation, the invariance conditions (33) and (36) are equivalent, and the Noether integral
\[
J(t, q, p) = i_{\tilde{\zeta}}(pdq - Hdt) = \sum_{j=1}^n \xi_j(q, t) p_j - \tau(t, q) H(t, q, p)
\] (37)
takes the usual form (34) (Proposition 2.2 [22]).

Obviously, the first integrals of many classical problems are not of the form (34). In order to extend the class of examples of Noether integrals, the gauge terms and some modifications of the action (30) are considered (see the references in [8]). Note that a function \(f(t, q, p)\) in our definition of a week Noether symmetry plays the role of a gauge term in the classical formulation.

The next natural step was the extending the class of symmetries. The framework where the components of \(\xi\) additionally depend of velocities was introduced in [12, 13], and then a geometrical setting for the equivalence of the first integrals and Noether symmetries of the Lagrangian given by vector fields on \(\mathbb{R} \times TQ\) was formulated in [8, 10].

In our notation, if \(\alpha\) is a contact form and \(F\) is the integral of (1), we have the inverse Noether theorem directly, without using the gauge terms: the contact Hamiltonian vector field of the function \(F\) defines the Noether symmetry \(\zeta\) of the equation (1) with the Noether integral \(F = i_{\zeta} \alpha\).
For mechanical problems, the Poincaré–Cartan 1-form is contact in a domain $U_H \subset \mathbb{R} \times T^*Q$ defined by the condition

$$\rho = i_Z(pdq - Hdt) = p\frac{\partial H}{\partial p} - H \neq 0,$$

where $Z$ is given by (10). Thus, on $U_H$ we have a simple explicit expression for the Noether symmetry (see Theorem 4.1 and examples given in [22]). In this sense, the form $\beta$ should be taken such that $pdq - Hdt + \beta$ is a contact form on $\mathbb{R} \times T^*Q$, i.e., $\rho + i_Z\beta \neq 0$. We left the terms $f$ and $\beta$ in the formulation of Theorem 3.2, although we do not use them in the examples given below.

3.4. Noether theorem for prolonged vector fields. Now we return to the constrained system (26). Let

$$\Sigma = \{(t, \Sigma_t)| t \in \mathbb{R}\} \subset \mathbb{R} \times Q$$

be a $(n - s + 1)$-dimensional submanifold of $\mathbb{R} \times Q$ defined by the holonomic constraints (16). Consider a time-dependent vector field (32) and its prolongation (35). Since $\hat{\xi}$ does not depend on the momenta, the conditions on $\zeta|_M$ to be a section of the admissible distribution $\mathcal{V}$, or a section of the reaction-annihilator distribution $\mathcal{R}$, reduce to the conditions on the vector field $\hat{\xi}|_\Sigma$.

It is clear that $\zeta|_M$ belongs to $\mathcal{V}$ if and only if $\hat{\xi}|_\Sigma$ belongs to a $(n + 1 - k)$-dimensional distribution $\hat{\mathcal{V}}$ over $\Sigma$:

$$\hat{\mathcal{V}} = \{\hat{\xi} \in T_{(t,q)}\Sigma | a^i_0\tau + \sum_{i=1}^{n} a^i_i\xi_i = 0, \quad i = 1, \ldots, k, \quad (t,q) \in \Sigma\}.$$ 

Since the first $s$ equations in the definition of $\hat{\mathcal{V}}$ determine the tangent bundle $T\Sigma$, we have

$$\hat{\mathcal{V}} \subset T\Sigma \subset T\Sigma(\mathbb{R} \times Q).$$

On the other hand, by using the expressions for $R_i$ given by (23) and relation (28), $\zeta|_M$ is a section of $\mathcal{R}$, if and only if

$$\sum_{i=1}^{n} R_i(\xi_i - \tau \frac{\partial H}{\partial p_i}) = \sum_{i=1}^{k} \lambda_i(t,q,p)(a^i_0\tau + \sum_{i=1}^{n} a^i_i\xi_i) = 0|_M. \quad (38)$$

That is, $\xi|_{(t,q)}$, $(t,q) \in \Sigma$, belongs to the subspace $\hat{\mathcal{R}}_{t,q}$ of $T_{(t,q)}(\mathbb{R} \times Q)$ determined by equations (38) for all momenta $p$ in the space

$$\mathcal{M}_t|_q = F L_t(D_t|_q). \quad (39)$$

Let $\hat{\mathcal{R}}$ be the collection of spaces $\mathcal{R}_{t,q}$, $(t,q) \in \Sigma$. We refer to $\hat{\mathcal{V}}$ and $\hat{\mathcal{R}}$ as an admissible and a reaction-annihilator distribution over $\Sigma$, respectively. Obviously,

$$\hat{\mathcal{V}} \subset \hat{\mathcal{R}} \subset T\Sigma(\mathbb{R} \times Q).$$

From Theorem 3.2, we have:

**Theorem 3.3.** Let (32) be a symmetry of the Hamiltonian: $H$ satisfies (36) on $\mathcal{M}$, where $\xi$ is the prolongation of (32) given by (35). Then the derivative of the

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3 According to Proposition 2.2 [22], this is equivalent to the assumption that the Lagrangian $L$ satisfies (33) at the constrained manifold $D \subset \mathbb{R} \times TQ$. Then we have the following analogous statement in the Lagrangian setting: if $F \equiv 0$, the Noether function (34) is conserved along the flow of the system (22) if and only if $\xi|_\Sigma$ is a section of $\hat{\mathcal{R}}$. 

Noether function equals
\[ \frac{d}{dt}J|_M = \frac{d}{dt} \left( \sum_{j=1}^{n} \xi_j(q,t)p_j - \tau(t,q)H(t,q,p) \right)|_M = \sum_{i=1}^{n} F_i(\xi_i - \dot{q}_i\tau)|_M \] (40)

if and only if \( \hat{\xi}|_\Sigma \) is a section of the reaction-annihilator distribution \( \hat{R} \). Thus, for \( F \equiv 0 \), the Noether function (34) is preserved if and only if \( \hat{\xi}|_\Sigma \) is a section of \( \hat{R} \). In particular, \( J \) is an integral if \( \hat{\xi}|_\Sigma \) is a section of the admissible distribution \( \hat{V} \).

Note that the left hand side of (40) simplifies if \( F \) is a gyroscopic force
\[ \sum_{i=1}^{n} F_i\dot{q}_i = \sum_{i=1}^{n} F_i \frac{\partial H}{\partial p_i} = 0. \] (41)

In the next section we shall consider two important special cases of Theorem 3.3, which are variants of some well known statements in nonholonomic mechanics.

4. Examples.

4.1. Nonholonomic Noether theorem. If \( \tau \equiv 0 \) in the symmetry field (32), we have a time-dependent vector field \( \xi = \sum_j \xi_j(t,q)\partial/\partial q_j \) on \( Q \) and the Noether function linear in momenta
\[ J = \sum_{j=1}^{n} \xi_j(t,q)p_j. \] (42)

It is well known that the one-parameter group of diffeomorphisms \( g^\xi \) of the vector field \( \xi \) (for a fixed \( t \)) have natural lifts to \( TQ \) (used in considering the Noether symmetries in [14]) and \( T^*Q \). A well known expression for the cotangent lift of \( \xi \) is
\[ \zeta = \sum_i \xi_i \frac{\partial}{\partial q_i} - \sum_{i,j} \frac{\partial \xi_i}{\partial q_j} p_j \frac{\partial}{\partial p_i}, \] (43)

which is the Hamiltonian vector field of the Noether function (42) with respect to the canonical symplectic form \( dp \wedge dq \) on \( T^*Q \) (e.g., see Proposition 1.9, Ch IV, [26]).\(^4\) Thus, the invariance condition (36) becomes
\[ L_\zeta H = p_\xi \frac{\partial H}{\partial t} = \sum_{i=1}^{n} p_i \frac{\partial \xi_i}{\partial t}. \] (44)

In particular, when \( \xi \) does not depend on time, (44) reduces to the invariance of the Hamiltonian function with respect to the flow of (43) on the cotangent bundle \( T^*Q \), or equivalently, the invariance of the Lagrangian \( L \) under the action of the flow of \( \xi \) that is extended to the tangent bundle \( TQ \). In that case, (42) is the standard momentum map of the action of the symmetry field \( \xi \) (see [1, 26]).

For \( \tau \equiv 0 \), the equation (38) takes the form
\[ \sum_{i=1}^{n} R_i\xi_i = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_i(t,q,p)a'_i(t,q)\xi_j = 0|_M. \] (45)

\( ^4\)Another way to see the prolongations of \( \xi \) to \( TQ \) and \( T^*Q \) is simply the substitution of \( \tau \equiv 0 \) in (31) and (35), respectively.
As above, since $\xi$ does not depend on the momenta, $\zeta|_{\mathcal{M}}$ belongs to $\mathcal{R}$ if and only if $\xi|_q$, $q \in \Sigma_t$, belongs to the subspace $\mathcal{R}_{t,q}^0$ of $T_q\mathcal{Q}$ defined by equations (45) for all momenta $p$ in (39). Let

$$\mathcal{R}_{t}^0 = \cup_{q \in \Sigma_t} \mathcal{R}_{t,q}^0 \subset T_{\Sigma_t} \mathcal{Q}.$$  

In the case of time-independent constrained Lagrangian systems, the distribution $\mathcal{R}_{t}^0$ is defined by Fassò, Ramos, and Sansonetto in [14, 16], where it is referred as a reaction-annihilator distribution (the sections of $\mathcal{R}_{t}^0$ annihilate the 1-form of reaction forces $\sum_i R_i dq_i$). We keep the same notation.

The distribution of virtual displacements $\mathcal{D}_{t}^0$ defined by (19) is a subset of $\mathcal{R}_{t}^0$, and it takes the role of the admissible distribution. Thus, we get

**Corollary 2.** Let a time-dependent vector field $\xi$ on $\mathcal{Q}$ be a symmetry of the Hamiltonian $H$: the invariance equation (44) is satisfied on $\mathcal{M}$, where $\zeta$ is the prolongation (43) of $\xi$. The momentum equation

$$\frac{d}{dt} J|_{\mathcal{M}} = \frac{d}{dt} (\sum_{i=1}^{n} \xi_i(q,t)p_i)|_{\mathcal{M}} = \sum_{i=1}^{n} F_i \xi_i|_{\mathcal{M}}$$  

(46)

holds if and only if $\xi|_{\Sigma_t}$ is a section of the reaction-annihilator distribution $\mathcal{R}_{t}^0$ for all $t$. Thus, for $F \equiv 0$, the Noether function (42) is a first integral of the constrained system (26) if and only if $\xi|_{\Sigma_t}$ is a section of $\mathcal{R}_{t}^0$, $t \in \mathbb{R}$. In particular, it is an integral if $\xi|_{\Sigma_t}$ is a section of the distribution of virtual displacements $\mathcal{D}_{t}^0$, $t \in \mathbb{R}$.

This statement is referred as a nonholonomic Noether theorem, usually formulated for the time-independent case and under the condition that $\xi$ is a symmetry of the nonconstrained system as well, induced by the action of an one-parameter subgroup of the Lie group $G$ of symmetries acting on the configuration space $\mathcal{Q}$ (see [1, 4]).

One of the first variants of the nonholonomic Noether theorem is given by Kozlov and Kolesnikov [25]. They considered a natural mechanical system of $N$ material points, $\mathcal{Q} = \mathbb{R}^{3N}$, with symmetries that are infinitesimal isometries of $\mathbb{R}^{3N}$ with respect to the metric induced by the kinetic energy. For the rigid body examples, see, e.g., [6]. The Chaplygin rolling ball problem in $\mathbb{R}^3$ is an illustrative example of the reduction of symmetries of the preserved nonholonomic momentum map, see [20, 21].

The situation where the function (42) is not an integral of the nonconstrained system, while it is an integral of the constrained one appears in the case of gauge symmetries of nonholonomic systems studied in [2, 3, 14, 15, 16]. Corollary 2 for the Lagrangian systems $(\mathcal{Q}, L)$ with $F \equiv 0$ and time-independent nonholonomic constraints (18) can be found in [14, 15] (the homogeneous case) and [16] (the affine case).

Thus, we can consider Theorems 3.2, 3.3 as a generalization of gauge symmetries to the time-dependent symmetries, as well as to the symmetries which are related to integrals that are not linear in momenta.

**Example 4.1.** As an illustration, consider the motion of a material point with the position vector $\mathbf{r} = (x, y, z)$, mass $m$ and electric charge $q$ in $\mathbb{R}^3$ under the influence of a homogeneous gravitational field $\mathbf{F} = mgk$ in the direction of a unit vector $\mathbf{k} = (k_x, k_y, k_z)$ and a magnetic field $\mathbf{B} = Be_z = (0, 0, B)$. The Newtonian equations of motion are

$$\dot{\mathbf{p}} = mgk + e\mathbf{r} \times e_z.$$
where $\epsilon = qB$, $\mathbf{p} = m\dot{\mathbf{r}} = (p_x, p_y, p_z)$. For the Hamiltonian we take the kinetic energy $H_0 = \frac{1}{2m}(\mathbf{p}, \mathbf{p})$ and treat the gravitational field as a nonconservative force.

Now, assume the motion is subjected to a time-dependent nonholonomic constraint:

$$a(t)y\dot{x} - \dot{z} + b(t) = 0.$$  

The symmetry vector field $\xi = e_y$ belongs to the distribution of virtual displacements. From the corresponding momentum equation

$$\dot{p}_y = mgk_y - \epsilon\dot{x},$$

we get that $p_y - mgk_yt + \epsilon x$ is an integral of the system.

Consider the vector field $\xi = fe_x + a(t)yfe_z$, which also belongs to the distribution of virtual displacements. Let $\zeta$ be a prolongation of $\xi$:

$$\zeta = \dot{f}e_x + a(t)yfe_z - \left( \frac{\partial f}{\partial y}p_x + a(t)fp_x + a(t)y\frac{\partial f}{\partial y}p_z \right)e_{pz},$$

for $f = f(t, y)$. The invariance condition (44) takes the form

$$\frac{1}{m}\left( \frac{\partial f}{\partial y}p_x p_y + afp_xp_z + ay\frac{\partial f}{\partial y}p_y p_z \right) = p_x \frac{\partial f}{\partial t} + p_z y\left( \frac{\partial a}{\partial t}f + a\frac{\partial f}{\partial t} \right),$$

and for $b \equiv 0$, under the condition

$$p_x = a(t)yp_x,$$

we have a solution

$$f(t, y) = 1/\sqrt{1 + a(t)^2y^2}$$

($\xi$ is a time-dependent gauge symmetry).

Thus, the momentum equation reads

$$\frac{dp_x}{dt}\left( \frac{p_x}{\sqrt{1 + a(t)^2y^2}} + \frac{a(t)yp_x}{\sqrt{1 + a(t)^2y^2}} \right)|_{p_z = a(t)yp_z} = mgk_x + a(t)yk_z + \epsilon\dot{y} \left( \frac{p_x}{\sqrt{1 + a(t)^2y^2}} + \frac{a(t)yp_x}{\sqrt{1 + a(t)^2y^2}} \right)$$

and for $k_x = k_z = \epsilon = 0$, we have the additional gauge integral

$$\left( \frac{p_x}{\sqrt{1 + a(t)^2y^2}} + \frac{a(t)yp_x}{\sqrt{1 + a(t)^2y^2}} \right)|_{p_z = a(t)yp_z}.$$

### 4.2. Conservation of energy.

As the next important case, we suppose $\tau \equiv 1$ and that there is no influence of nonconservative forces ($F \equiv 0$). Consider a time-dependent vector field

$$\dot{\xi} = \frac{\partial}{\partial t} + \xi = \frac{\partial}{\partial t} + \sum_i \xi_i(t, q)\frac{\partial}{\partial q_i},$$

and its prolongation (see (35)):

$$\zeta = \frac{\partial}{\partial t} + \sum_i \xi_i \frac{\partial}{\partial q_i} - \sum_{i,j} \frac{\partial \xi_j}{\partial q_i} p_j \frac{\partial}{\partial p_i}.$$  

As in the previous subsection, it is a Noether symmetry if the invariance condition (44) holds (but now with $\zeta$ given by (48) instead by (43)) and the associated Noether function takes the form

$$J(t, q, p) = i_\zeta(pdq - Hdt) = \sum_{j=1}^{n} \xi_j(q, t)p_j - H(t, q, p).$$
It is well known that if the Hamiltonian $H$ does not depend on time, then $\zeta = \partial/\partial t$ ($\zeta \equiv 0$) is a Noether symmetry and the Hamiltonian multiplied by $-1$ ($J = -H$) is the corresponding Noether integral of the non-constrained system.

Note that the affine distribution $D_t \subset T_{\Sigma}Q$ can be seen as a sum $D_t^0 + \xi^0$, where $\xi^0 = \sum_j \xi_j^0(t,q)\partial/\partial q_j$ is a section of the affine distribution $D_t$ over $\Sigma_t$, $t \in \mathbb{R}$.

**Corollary 3.** (i) Let $(47)$ be a symmetry vector field of the Hamiltonian $H$: the equation $(44)$ is satisfied on $\mathcal{M}$, where $\zeta$ is the prolongation $(48)$ of $\xi$. Then the constrained system has a Noether integral $(49)$ if and only if $\xi|_{\Sigma}$ is a section of the affine distribution $\xi^0 + \mathcal{R}_t^0$ for all $t$. In particular, if $\xi|_{\Sigma}$ is a section of the affine bundle $D_t$, $t \in \mathbb{R}$, then $J$ is an integral of the system.

(ii) Assume that the Hamiltonian $H$ does not depend of time. It is preserved if and only if the vector field $\xi^0$ is a section of the reaction-annihilator distribution $\mathcal{R}_t^0$, $t \in \mathbb{R}$. In particular, if the holonomic constraints $(16)$ do not depend on time and the nonholonomic constraints $(18)$ are homogeneous (and possibly time-dependent), the Hamiltonian is a first integral of the system.

**Proof.** (i) According to Theorem 3.3, the constrained system has a Noether integral $(49)$ if and only if $\hat{\xi}|_{\Sigma}$ is a section of $\mathcal{R}$, i.e., $\hat{\xi}|_{\Sigma}$ is a solution of equation $(38)$. Since $\xi^0$ satisfies the constraints $(17), (18)$,

$$a_0^0(t,q) + a_1^0(t,q)\xi^0_1 + \cdots + a_n^0(t,q)\xi^0_n = 0,$$

it follows that the vector field $\xi|_{\Sigma} - \xi^0$ is a solution of $(45)$ (i.e., $\xi|_{\Sigma} - \xi^0$ is a section of $\mathcal{R}_t^0$, $t \in \mathbb{R}$) if and only if $\hat{\xi}|_{\Sigma}$ is a solution of equation $(38)$.

Further, for the last statement, we note that $\hat{\xi}|_{\Sigma}$ is a section of the admissible distribution $\mathcal{V}$ if and only if $\xi|_{\Sigma}$ is a section of $D_t$, $t \in \mathbb{R}$.

(ii) The statement follows from item (i) by taking $\xi \equiv 0$ in $(47)$: the zero section is a section of $\xi^0 + \mathcal{R}_t^0$ if and only if $\xi^0$ is a section of $\mathcal{R}_t^0$. In particular, if the holonomic constraints $(16)$ do not depend on time and the nonholonomic constraints $(18)$ are homogeneous, then $D_t \equiv D_t^0$ and we can take $\xi^0 \equiv 0$.

A variant of Corollary 3 is given in [16, 17], where the Noether function $J = \sum_j \xi_j \rho_j - H$ is referred as a moving energy integral. For the conservation of energy in systems with affine constraints see also [7].

**Remark 1.** The Hamiltonian is conserved also if in item (i) of Corollary 3 we assume the action of a gyroscopic force $F$ (see $(41)$).

**Example 4.2.** Let

$$H = H_0 + V = \frac{1}{2m}(p,p) - mg(k,r)$$

be the total energy of a material point considered in Example 4.1. According to item (i) of Corollary 3 and the above remark, it is conserved for the nonholonomic problem with a time-dependent homogeneous constraint

$$a(t)y\dot{x} - \dot{z} = 0.$$

**Remark 2.** It would be interesting to have a geometrical setting for the Noether theorem in quasi-coordinates given in [12, 28, 30]. The case, where the symmetries are induced by the symmetry vector field $\xi$ on the configuration space $Q$ is treated, e.g., in [5]. Our goal is also the analysis of a symmetry and integrability of the multidimensional rolling spheres problems introduced in [23].
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