On pure Yang-Mills theory in $3+1$ dimensions: Hamiltonian, vacuum and gauge invariant variables

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Abstract

In this work we discuss an analytic approach towards the solution of pure Yang-Mills theory in $3+1$ dimensional spacetime which strongly suggests that the recent strategy already applied to pure Yang-Mills theory in $2+1$ can be extended to $3+1$ dimensions. We show that the local gauge invariant variables introduced by Bars gives a natural generalisation to any dimension of the formalism of Karabali and Nair which recently led to a new understanding of the physics of QCD in dimension $2+1$. After discussing the kinematics of these variables, we compute the jacobian between the Yang-Mills and Bars variables and propose a regularization procedure which preserves a generalisation of holomorphic invariance. We discuss the construction of the QCD hamiltonian properly regularized and compute the behavior of the vacuum wave functional both at weak and strong coupling. We argue that this formalism allows the developpement of a strong coupling expansion in the continuum by computing the first local eigenstate of the kinetic part of Yang-Mills hamiltonian.

1 Introduction

In a remarkable series of papers Karabali and Nair \cite{1} and Karabali, Kim and Nair \cite{2} (see \cite{3} for an introductory review) have developed a novel and powerful method in order to adress the confinement problem of pure Yang-Mills in dimension $2+1$ (I will refer to this work as KKN for short). Their approach,

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inspired by the work of Feynman on the Schrödinger representation of QCD, has led to an analytic determination of the string tension and the possibility to devise a strong coupling expansion in the continuum and evidence for a mass gap. The main idea behind this work is to formulate the Yang-Mills hamiltonian in terms of local gauge invariant variables. This parametrization heavily uses the power of the complex structure which is available in 2D.

This work has been followed by a deeper investigation of the vacuum wave functional in the Large N limit by Leigh, Minic and Yelnikov. Basing their analysis on a quadratic ansatz for the wave functional, these authors have investigated the structure of the Schrödinger equation and the action of the kinetic term on a restricted class of operators. From this study they have proposed an interpolating kernel capturing the behavior of the vacuum wave function both in the infrared and the ultraviolet. This kernel, expressed simply as a ratio of Bessel functions, exhibits an infinite set of resonances arguing in favor of a constituent picture for glueballs which becomes apparent in the KKN formalism. This have led them to a theoretical prediction of the glueball spectra simply related to the zeros of Bessel function which is in striking agreement with the large N lattice data.

This series of work open many new questions and avenues in the study of QCD, first in 2+1 dimensions where among others things one can wonder about the possibility of exploring along these lines the mesons spectra, the inclusion of a non zero temperature and where a deeper understanding of the renormalization group in this context is needed. But one of the most pressing issue is to know wether such a methodology can be extended and applied to the case of pure Yang-Mills in dimension 3+1. The main message of the present work is to show that this is indeed the case.

The first obstacle to overcome lies in the fact that the KKN formalism uses the full strength of complex analysis which is available in 2 space dimensions but not in 3D. The key and simple insight obtained in collaboration with R. Leigh and D. Minic is the fact that the KKN gauge invariant variables are related to lines integral from infinity to a point $x$ and admit a natural generalization in any dimension which was first proposed by I. Bars some time ago and baptized ‘corner variables’.

In this work we explore the consequences of this insight and develop a new set of techniques allowing us to work in terms of the Bars variables and extract a first set of physical information from this formalism. In section 2
we explore the kinematics of Bars variables and reestablish some important
formulae relating them to the usual Yang-Mills variables, we also identify the
key notion of an ‘holomorphic symmetry’. In section 3 we present one of our
first main result which is the computation of the jacobian of the transformation
between Yang-Mills and Bars variables. This computation is possible once we
devise a natural regularization scheme which preserves holomorphic symmetry.
The jacobian is then shown to be trivial in the case of real coordinates and
given in terms of a generalization of the WZW action in the case of semi-
complex coordinates. This result implies that the trivial wave functional is
normalizable and hence the theory is amenable to a strong coupling expansion.
This strongly resonates with the results obtained in lattice QCD but in the
context of a continuum formulation.

In section 4 we present the construction of the regularized hamiltonian in
Bars variables and as a check give the construction of the wave functional in
the ultraviolet regime.

In section 5 we show that the regularized kinetic term acts diagonally on
the potential term which is our second main result. In contrast with 2+1 D
the eigenvalue, having a dimension of a mass, is cutoff dependent. This allows
us to construct the vacuum wave function at first order in a strong coupling expansion and using an argument of dimensional reduction we show how this
determines the value of the string tension in this regime. We conclude on the
open issues, one of the main one being the problem of the continuum limit.

In a joint companion paper [8] we explore the physical consequences of the
results obtained here and strongly argue in favor of a deep similarity between
the 2+1 and 3+1 dimensional case.

2 The kinematics of Bars Corner variables

We denote by $A_i, i = 1, 2, \cdots, D$ the spatial components of the Yang-Mills
connection. These are taken to be anti-hermitian and our main interest is $D =
3$, but we keep it free in the first part of the paper. One of the key ingredient of
Karabali Kim and Nair (KKN for short) formalism is a parametrization of the
configuration space of 2d Yang-Mills in terms of local gauge invariant variables
which uses heavily at first sight the available complex structure of 2 dimension.
In fact [8] the KKN gauge invariant variables can be understood as Wilson
lines integral from infinity to a point \( x \) and admit a natural generalization in any dimension which was proposed by I. Bars \([9]\). We denote them \( M_i(x) \), they satisfy the defining equation

\[
A_i = -\partial_i M_i M_i^{-1}.
\]  

(1)

No summation over repeated indices is assumed here and after. A solution is given by

\[
M_j(x) = \hat{\exp}\left[- \int_{-\infty}^{x} A \right]
\]

(2)

where the integral is a straight spatial contour for fixed \( x^i \) for \( i \neq j \) and \( \hat{\exp} \) is the path ordered exponential, explicitly

\[
M_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \int_{x_1 > t_1 > \cdots > t_n} A_1(t_1, x_2, x_3) \cdots A_1(t_n, x_2, x_3) dt_1 \cdots dt_n.
\]

(3)

Gauge transformations

\[
A_i \rightarrow A_i^g = g A_i g^{-1} + g \partial_i g^{-1}
\]

(4)

act on \( M \)'s as

\[
M_i \rightarrow g M_i.
\]

(5)

One can then define gauge invariant local variables

\[
H_{ij} = M_i^{-1} M_j.
\]

(6)

Note that \( H_{jj} = 1 \) and \( H_{ji} = H_{ij}^{-1} \) –this just means traversing the corner in the opposite direction is precisely the inverse element. The \( H_{ij} \) are unitary in a real coordinate basis. There is also a constraint (here written for \( D = 3 \))

\[
H_{ij} H_{ji} = 1, \quad H_{ij} H_{jk} H_{ki} = 1.
\]

(7)

The translation to 2+1 KN variables is obtained if one uses a complex basis

\[
M_2 = M, \quad M_\bar{z} = (M^\dagger)^{-1}, \quad H \equiv H_\bar{z} = M^\dagger M
\]

(8)

Note that if we had used a real coordinate basis, then we would have had a pair of (unrelated) unitary matrices \( M_1, M_2 \). In 3+1 there is no complex structure and we will work in a real basis where \( M_i \) are unitary matrices. One
could also choose to work in a “semi-complex” coordinate basis \{u, z, \bar{z}\} for 
\(D = 3\), then we could parameterize this as

\[
H_{uz} = H \quad (9) \\
H_{\bar{z}u} = H^\dagger \quad (10) \\
H_{\bar{z}\bar{z}} = H^\dagger H \quad (11)
\]

For example, one could use the notation 
\(M_z = M, \ M_{\bar{z}} = (M^\dagger)^{-1}, \ M_u^\dagger M_u = 1, \)
with \(H = M_u^\dagger M\). The constraint takes the form \(H_{zu} H_{uz} H_{\bar{z}\bar{z}} = 1\). In other
words, there is in \(D = 3\) in the semi-complex coordinate basis a complex \(H\)-field (compared to a Hermitian field in \(D = 2\)); thus there are twice as many
degrees of freedom.

A very important feature of this parametrization is a generalization to any
dimension of holomorphic invariance, and even if there is no complex structure
in dimension 3 we will refer to this symmetry as holomorphic symmetry. This
extra symmetry acts as

\[
M_i \mapsto M_i h_i^{-1}(x^j), \quad j \neq i \quad (12)
\]

The condition \(j \neq i\) on the function \(h_i\) is the analogue of holomorphy. This
leaves the gauge fields invariant, and one finds

\[
H_{ij} \mapsto h_i H_{ij} h_j^{-1} \quad (13)
\]

In the complex basis, we would have \(M_u \to M_u h_u^{-1}(z, \bar{z}), \ M^\dagger \to h(u,z)M^\dagger, \)
\(M \to M h^\dagger(u, \bar{z})\) and so \(H \to h_u(z, \bar{z}) H h^\dagger(u, \bar{z}), \ H^\dagger \to h(u,z) H^\dagger h_u^{-1}(z, \bar{z})\).
Note that one can introduce currents

\[
J_{ij} = (\partial_j H_{ij}) H_{ij}^{-1}, \quad (14)
\]

(in \(2+1\), \(J \sim J_{\bar{z}z}\) and \(J^\dagger \sim -J_{z\bar{z}}\)). The \(J_{ij}\) transform as (‘holomorphic’) connections

\[
J_{ij} \mapsto h_i J_{ij} h_i^{-1} + \partial_j h_i \ h_i^{-1} \quad (15)
\]

(This is not a typo – it only depends on \(h_i\).)

In the real coordinate basis, it appears that there are six currents that are
apparently distinct. However, there is a ‘reality’ condition on their derivatives
of the form

\[
\partial_i J_{ij} = -H_{ij}(\partial_j J_{ji}) H_{ij}^{-1} \quad (16)
\]
(in $D = 2$, this reads $\bar{\partial}J = H(\partial J^\dagger)H^{-1}$.) By defining $\bar{J}_{ij} = -H_{ij}J_{ji}H^{-1}$, we may rewrite this as

$$\partial_iJ_{ij} = \partial_j\bar{J}_{ij} - [J_{ij}, \bar{J}_{ij}]$$

(17)

and so there are covariant derivatives $D_{ij} = \partial_j - J_{ij}$. These currents are related to the magnetic field $F_{ij} = \partial_iA_j - \partial_jA_i + [A_i, A_j]$ by

$$\partial_iJ_{ij} = -M^{-1}_iF_{ij}M_i.$$ 

(18)

Note that the covariant derivatives can be written in terms of the usual derivative and $M_i$ as

$$\nabla_i = \partial_i + A_i = M_i\partial_iM^{-1}_i.$$ 

(19)

Let us finally note that Wilson loops observables can be expressed entirely in terms of the currents: Let us consider $C$ a closed curve in the 12 plane. The Wilson loop observable in the representation $R$ can be expressed as

$$W_R(C) = \text{Tr}_R e^{\frac{-i}{\hbar}\int_C A_{idx_i}} = \text{Tr}_R e^{\frac{-i}{\hbar}\int_C J_{12(x(t))dx_2}}.$$ 

(20)

This is easily obtained once we write the connection as a gauge transformation of the current

$$A_{1dx_1} + A_{2dx_2} = -dx_1\partial_1M_1M_1^{-1} - dx_2\partial_2M_2M_2^{-1}$$

(21)

$$= -(dx_1\partial_1M_1M_1^{-1} + dx_2(M_1J_{12}M_1^{-1} - \partial_2M_1M_1^{-1}))$$

(22)

If one works out what this correspondence imply for a rectangular plaquette $C$ in the plane $y_3 = \text{cste}$ with corners $(0, 0)(y_1, 0)(0, y_2)(y_1, y_2)$ oriented clockwise one simply get

$$W_R(y) = \text{Tr}_R \left( e^{-\frac{i}{\hbar}\int_0^{y_2} J_{12(y_1,x_2)dx_2}} e^{\frac{i}{\hbar}\int_0^{y_2} J_{12(0,x_2)dx_2}} \right)$$

$$= \text{Tr}_R \left( H_{12}(y_1, 0)H_{12}^{-1}(y_1, y_2)H_{12}(0, y_2)H_{12}^{-1}(0, 0) \right)$$

(23)

In order to continue and work out some formulae necessary for the computation of the hamiltonian in these variables lets introduce some notations and conventions. The connection is expanded in terms of anti-hermitian generators $T_a$ (in order to avoid unnecessary factors of $i$), $A_i = A^a_iT_a$ satisfying the algebra $[T_a, T_b] = f_{ab}^cT_c$, we also denote by $\text{Tr}$ the trace in the vectorial representation, so that $\text{Tr}(1) = N$. by $-2\text{Tr}(T_aT_b) = \delta_{ab}$ (the minus sign is because of the antihermiticity and the 2 is the standard convention). The
Yang-Mills action is taken to be

\[ S_{YM} = \frac{1}{2g^2} \int \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \]

and the Hamiltonian in the Yang-Mills variables is

\[ H = \sum_{i,a} \int -\frac{g^2}{2} \left( \frac{\delta}{\delta A_i^a} \right)^2 + \frac{1}{2g^2} \left( F_i^a \right)^2 \]

with \( F_i^a = \frac{1}{2} \epsilon_{ijk} F_j^a k \). This should be supplemented by the Gauss law constraint

\[ \nabla_i^a \frac{\delta}{\delta A_i^a} = 0. \]

In the adjoint representation the generators are given by \((T^a)_{bc} = -f_{abc}\), where index are raised or lowered with the metric \( \delta_{ab} \) and the trace in the adjoint is denoted by \( \text{Tr}_{ad} \). A group element \( M \) is represented in the adjoint by

\[ M_{ab} = -2\text{Tr}(T^a M_i M^{-1}_i), \]

clearly we have \((M^{-1})_{ab} = M_{ba}\) and also \( M T_b M^{-1} = T_a M^a b \).

We denote \( G_i(x, y) \) the inverse of \( \partial_i \), \( \partial_i G_i(x, y) = \delta(x, y) \) (no summation on repeated indices). Since \( \partial_i \) admits zero modes, its inverse is not uniquely defined and we will work with the explicit choice \( G_1(x) = \theta(x_1) \delta(x_2) \delta(x_3) \) where \( \theta \) is the heaviside function and \( G_i(x, y) \equiv G_i(x - y) \) (which is not antisymmetric). This choice is not arbitrary, it is the unique choice consistent with the definition of the variables \( M_i \) as an ordered exponential. Indeed we can rewrite (3) in terms of the propagator as

\[ M_i(x) = \sum_n (-1)^n \int dy (G_i A_i)^n(x, y), \]

where \((G_i A_i)^2(x, y) = \int dz G_i(x, z) A_i(z) G_i(z, y) A_i(y) \) etc...

One can first compute the derivative of \( M_j \) with respect to \( A_i \). Starting from

\[ A_i^a = 2\text{Tr}(T^a \partial_i M_i M^{-1}_i), \]

one obtains the relation

\[ \delta A_i^a = -(M_i)^{ab} \partial_i (M^{-1}_i \delta M_i)^b \]

with the obvious notation \( M_i^{-1} \delta M_i = (M_i^{-1} \delta M_i)^b T_b \). Inverting this relation gives

\[ \frac{(M_i^{-1} \delta M_i)^b}{\delta A_i^a(x)} = -\delta_j^i G_i(y, x) (M_i^{-1}(x))^b_a = \delta_i^j (M_i G_i^j)(x, y)^b_a. \]
where we introduced the notation

\[ G^t(x, y) \equiv -G(y, x). \tag{27} \]

\( t \) stands for transpose and the minus sign insures that \( \partial^x_i G^t(x, y) = \delta(x, y) \). In other words we can express \( A_i \) derivatives in terms of right derivative on the group

\[ [P^i_a(x), M_j(y)] \equiv \delta^i_j M_i(y) T_a \delta(x, y) \]

as

\[ \frac{\delta}{\delta A^a_i(x)} = (M_i(x))_a^b \int \! dy \, G^t_i(x, y) P^i_b(y) \equiv (M_i G^t_i P^i)_a(x). \tag{28} \]

The next step is to express \( A \) derivatives in terms of the currents, from their definitions one has

\[ J_{ij} = \partial_j H_{ij} H^{-1}_{ij} = -M^{-1}_i (A_j + \partial_j M_i M^{-1}_i) M_i = -M^{-1}_i A_j M_i - M^{-1}_i \partial_j M_i, \]

taking its variation one gets

\[ \delta J_{ij} = -M^{-1}_i \delta A_j M_i - [D_{ij}, M^{-1}_i \delta M_i] \tag{29} \]

with \( D_{ij} = \partial_j - J_{ij} \). If one takes (26) into account this reads

\[ \delta J_{ij} = -M^{-1}_i \delta A_j M_i + [D_{ij}, G_i M^{-1}_i \delta A_i M_i], \tag{30} \]

from which we get

\[ \frac{\delta J^b_{ij}(y)}{\delta A^a_i(x)} = - (M_j(x))_a^b \delta(x, y), \tag{31} \]

\[ \frac{\delta J^b_{ij}(y)}{\delta A^a_i(x)} = (D^b_{ij})_c G_i(y, x) (M^{-1}_i(x))_c^a \tag{32} \]

\[ \frac{\delta J^b_{jk}(y)}{\delta A^a_i(x)} = 0, \quad i \neq j, k \tag{33} \]

where

\[ D^{ab}_{ij} = \partial_j \delta^{ab} + J^c_{ij} f^{ab}_c \]

(note that \( (T_c)_{ab} = -f_{cab} \)).

So far everything is similar to 2D. Before going on, it is useful to dwell further on the kinematical structure and write down more precisely the structure of phase space in this new variables. In order to do so we need some
notations: one introduces the momenta variables which replace the electric field generators in our variables and the corresponding operator smeared with a Lie algebra valued 1-form field \( \Phi_i(x) = \Phi^a_i(x) T_a \)

\[ \Pi^i_a(x) \equiv (G^j_a P^j)_{a}(x), \quad \Pi(\Phi) \equiv \int dx \Phi^a_i(x) \Pi^i_a(x). \tag{34} \]

Obviously,

\[ \Pi(\partial \Phi) = -P(\Phi), \quad \Pi(\Phi) = -P(G\Phi), \tag{35} \]

with \((\partial \phi)_i \equiv \partial_i \phi_i\), \((G\Phi)_i \equiv G_i \Phi_i\) and \(P(\Phi)\) the smeared version of \(P^i_a\). The algebra in terms of the \(P, M\) variables is simply

\[ [P(\Phi), M_i(x)]_q = M_i \Phi_i(x), \quad [P(\Phi), P(\Psi)]_q = P([\Phi, \Psi]). \tag{36} \]

Where the bracket index \(q\) denotes quantum commutators to be distinguished from the Lie algebra commutators. It will be useful to know also the commutator with the currents

\[ [P(\Phi), J_{ij}]_q = H_{ij} \partial_j \phi_j H_{ij}^{-1} - D_{ij} \phi_i. \tag{37} \]

The generators of gauge symmetries can be naturally expressed in terms of these, if \(X(x) = X^a(x) T_a\) labels the parameter of the infinitesimal gauge transformation one can write

\[ G_X = -\int \text{Tr} \left( X \nabla_i \frac{\delta}{\delta A_i} \right) = P(M_i X M_i^{-1}). \tag{38} \]

The generator of the holomorphic symmetry can be written in a similar form in terms of Lie algebra elements \(H_i = H^a_i T_a\) satisfying \(\partial H_i = 0\),

\[ G_H = P(H_i). \tag{39} \]

One sees that the conditions of holomorphic invariant \(G_H \sim 0\) is necessary in order for the momentum \(\Pi\) to be well defined since if \(H_i\) is an holomorphic transformation \(\Pi(\partial H) = 0 = P(H)\). Moreover the observables \(\Pi, J\) are gauge invariant

\[ [G_X, P(\Phi)]_q = [G_X, J_{ij}]_q = 0. \tag{40} \]

When written in terms these observables the algebra reads

\[ [\Pi(\Phi), \Pi(\Psi)]_q = -\Pi(\partial G(\Phi, G\Psi)), \tag{41} \]

\[ [\Pi(\Phi), J_{ij}(x)]_q = -(H_{ij} \Phi_j H_{ij}^{-1}(x) + (D_{ij} G_i \Phi_i)(x), \tag{42} \]

\[ [J_{ij}, J_{kl}]_q = 0. \tag{43} \]
where \([\Phi,\Psi], \equiv [\Phi_i,\Psi_i]\) and obviously \(\partial[G\Phi,G\Psi] = [\Phi,G\Psi] + [G\Phi,\Psi]\). It might be interesting to note that this algebra possesses strong similarity with a centrally extended algebra, this is clear if one takes the derivative of \(\Pi(\Phi)\) leading to

\[
[\Pi(\Phi),\partial_i J_{ij}] = [(G\Phi)_i,\partial_i J_{ij}] + D_{ij}\Phi_i - \partial_i(H_{ij}\Phi_j H_{ij}^{-1}).
\]  

(44)

3 Determinant and regularization

The wave functions in the Schrödinger representation of pure Yang-Mills are gauge invariant functionals of \(A_i\) and the scalar product is given by

\[
||\Psi||^2 = \int_{A/G} D\mu(A)\bar{\Psi}(A)\Psi(A),
\]  

(45)

where the integral is over the space of gauge connections modulo gauge transformations and \(D\mu(A) = \frac{DA}{Vol(G)} = \frac{\prod_i DA_i}{Vol(G)}\). From the previous section we know that we can equivalently describe gauge invariant wave functionals as holomorphic invariant wave functionals of \(H_{ij}\) or \(J_{ij}\). In order to express the physical scalar product in this new variables we need to compute the jacobian. Since \(A_i = -\partial_i M_i M_i^{-1}\), \(\delta A_i = -(\nabla_i \delta M_i) M_i^{-1}\), the change of variables involve a determinant

\[
e^\Gamma \equiv \det\left(\frac{\delta A_i}{\delta M_i M_i^{-1}}\right) = \det(\nabla_1 \nabla_2 \nabla_3).
\]  

(46)

The variational derivative of the action is given by

\[
\frac{\delta \Gamma}{\delta A_i^a(x)} = \text{Tr}_{ad} \left[ (\nabla_i)^{-1}(x,x)T_a \right]
\]  

(47)

the trace being in the adjoint representation. Since \(\nabla_i = M_i \partial_i M_i^{-1}\) the covariant propagator can be expressed in terms of the standard propagator as a coincident limit

\[
\frac{\delta \Gamma}{\delta A_i^a(x)} = \lim_{x\rightarrow y} \left( M_i(x)G_i(x,y)M_i^{-1}(y) \right)^{bc} f_{abc}.
\]  

(48)

A similar coefficient arises when one expresses the kinetic term in terms of the group variables using \(\delta\)

\[
\frac{\delta}{\delta A_i^a(x)}\frac{\delta}{\delta A_i^a(x)} = \lim_{x\rightarrow y} \left( M_i(x)^{-1}G_i(x,y)M_i(y)T_b \right)^{bc} (G_i P_c^i)(y) + (G_i P_a^i)(x)(G_i P_a^i)(x).
\]  

(49)
Integration over $x$ and summations over indices should be understood. The
relation between these coefficients is not a coincidence, indeed the presence of
$e^{\Gamma}$ in the integration measure insures that (49) is a self adjoint operator.

Of course, in order to make sense of these statements, one needs a regu-
larization of the propagator which preserves all the symmetries, namely the
holomorphic symmetry $M_i \mapsto M_i h_i^{-1}$. Since $\nabla_i = M_i \partial_i M_i^{-1}$ is invariant un-
der holomorphic transformations this means that the regularized propagator
should transform as $G^\mu_i(x,y) \mapsto h_i(x)G^\mu_i(x,y)h_i^{-1}(y)$, where $\mu$ denotes the
momentum cut-off scale (this is obviously true for the unregulated propagator
which satisfies $h_i(x)G_i(x,y)h_i^{-1}(y) = G_i(x,y)$).

The key ingredient in the construction of the regularized propagator are
the group valued functionals $\Lambda_i(x,y)$ defined to be

$$\Lambda_1(x,y) \equiv H_{12}(y_1, x_2, x_3)H_{23}(y_1, y_2, x_3)H_{31}(y_1, y_2, y_3), \quad (50)$$

and cyclic permutation for $\Lambda_2, \Lambda_3$. These group elements transform under
holomorphic transformation $H_{ij} \mapsto h_i H_{ij} h_j^{-1}$ as

$$\Lambda_1(x,y) \mapsto h_1(x_2, x_3)\Lambda_1(x,y)h_1^{-1}(y_2, y_3) \quad (51)$$

and similarly for $\Lambda_2, \Lambda_3$. They also satisfy the key properties

$$\partial^\mu_i \Lambda_i(x,y) = 0, \quad \text{and} \quad \Lambda_i(x,x) = 1. \quad (52)$$

We can now propose a regularization scheme which preserves holomorphic
invariance :

$$G^\mu_i(x,y) \equiv \int dz \ G_i(x,z)\Lambda_i(z,y)\delta_\mu(z,y) \quad (53)$$

where the regularised delta function is given by (say)

$$\delta_\mu(x,y) = \prod_{i=1}^{3} \delta_\mu(x_i - y_i), \quad \delta_\mu(x_i) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2 x_i^2}. \quad (54)$$

This regularised propagator satisfies

$$\partial^\mu_i G^\mu_i(x,y) = \Lambda_i(x,y)\delta_\mu(x,y). \quad (55)$$

Moreover the integral (53) can be explicitly performed and one gets

$$G^\mu_i(x,y) = \Lambda_1(x,y)\delta_\mu(x_1 - y_1)\delta_\mu(x_2 - y_2)\delta_\mu(x_3 - y_3) \equiv \Lambda_1(x,y)G^\mu_1(x,y), \quad (56)$$
where \( \theta_\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mu x} dt e^{-t^2} \) is one half plus half the error function.

Now that we have a regularization which preserves gauge invariance and holomorphic symmetry we can evaluate the propagator at coincident points and compute \( G^\mu_1(x, x) = \frac{\mu^2}{2\pi} \). This propagator is proportional to the identity hence \( \frac{\partial}{\partial A_\mu} = 0 \) and

\[
|\det(\nabla_1 \nabla_2 \nabla_3)| = |\det(\partial_1 \partial_2 \partial_3)|
\] (57)

the determinant is independent of the connection. This is independent on the form of the regulated delta function. We have proven this results here in dimension 3 but the formalism is valid in any dimension therefore in any dimension \( D \)

\[
|\det(\nabla_1 \cdots \nabla_D)| = |\det(\partial_1 \cdots \partial_D)|.
\] (58)

Since this is for us an important conclusion, let us give an other totally independent proof of the same result using lattice gauge theory. Putting gauge theory on a lattice provides a gauge invariant regularization of the theory. We start with a gauge theory on a square periodic \( D \)-dimensional lattice, we choose one origin and an orientation and denote the displacement vector of one lattice unit in the direction \( i \) by \( e_i \), the sites of the lattice are labeled by \( x = \sum_i x_i e_i \) with \( x + Ne_i \) being identified with \( x \). The gauge connection is encoded in terms of group elements \( g_i(x) \) associated with the link \( (x, x + e_i) \). The analogous of the Bars variables are given by holonomies starting at one corner of the lattice. Let \( x = \sum_i x_i e_i, \) \( 0 \leq x_i < N \), then

\[
M_i(x) = g_i(x)^{-1} g_i(0)^{-1} g_i^{-1}(x - e_i) \cdots g_i^{-1}(x - x_i e_i),
\] (59)

so that the starting element is always on the plane \( x_i = 0 \). We introduce the discrete derivative \( \partial_i f(x) \equiv f(x) - f(x - e_i) \), the \( M_i \) satisfy the difference equation

\[
\partial_i M_i(x) M_i^{-1}(x) = (1 - g_i(x)), \quad M_i(x) = 1, \text{ for } x_i = 0.
\] (60)

which is clearly the discrete analog of (11). It can be equivalently written as \( \nabla_i M_i = 0 \), where the discrete covariant derivative is \( \nabla_i M(x) \equiv g_i(x) M(x) - M(x - e_i) \). Now, we want to compute the Jacobian of the transformation \( g_i \rightarrow M_i \), since \( g_i(x) = M_i(x - e_i) M_i^{-1}(x) \) we first remark that \( M_i \) depends only on \( g_i \) so that the transformation matrices is block diagonal and the problem is essentially one dimensional. A direct computation gives the variation

\[
g_i(x)^{-1} \delta g_i(x) = (\nabla_i \delta M_i(x)) M_i^{-1}(x) = M_i(x) (\partial_i M_i^{-1} \delta M_i(x)) M_i(x).
\] (61)
Therefore

\[
|\det(\nabla_1 \cdots \nabla_D)| = \prod_{i=1}^{D} |\det(\nabla_i)| = \prod_{i=1}^{D} |\det(M_i \partial_i M_i^{-1})|
\]

\[
= \prod_{i=1}^{D} |\det(\partial_i)| = |\det(\partial_1 \cdots \partial_D)|
\]

(62)

Where we have used the block diagonal form of the transformation matrix, and also the factorization and invariance under conjugation of the determinants which are all valid operations since we are in a finite dimensional context.

There is an important subtlety here which is hidden in the domain of definition of the operator \(\partial_i\). This comes from the fact that \(M_i(x)\) is not a periodic function on the lattice, since on one side of the lattice \(M_i(0) = 1\) and on the other side \(M_i(N e_i)\) is arbitrary. This means that the determinant of \(\partial_i\) which arises from the change of variables should be computed in the space of Lie algebra valued functions satisfying \(\phi(0) = 0\) and \(\phi(N e_i)\) arbitrary and not on the space of periodic Lie algebra valued functions. One can easily see that then the residual derivative determinant can be exactly evaluated and is in fact equal to unity. This simple answer can be traced back to the lattice regularization which starts from group variables and doesn’t affect our main statement which is that the full determinant is independent of the connection.

### 3.1 Gauge invariant measure

This results allows us, up to a gauge field independent determinant, to make the change of variables from \(A_i\) to \(M_i\). The induced measure \(\mathcal{D}M_i\) on the fields \(M_i\) should be thought as being the product over spacetime points of the Haar measure \(\mathcal{D}M = \prod_x dM(x)\).

More precisely, the proper way to define the measure is to give its moments evaluated on a proper class of functionals. The functionals which have a well defined integration under this measure are called cylindrical functions \(F(M(x)) = F_n(M(x_1), \ldots, M(x_n))\) which depends on the value of the field only in a finite number of points and the measure \(\mathcal{D}M\) is defined by its value on cylindrical functions to be \(\int \mathcal{D}M F(M(x)) = \int dM_1 \ldots dM_n F_n(M_1, \ldots, M_n)\) the integral being over a product over Haar measure. The Haar measure is right-left invariant and one can write (since the measure is ultralocal, in the
sense just defined, we can look at only a fix $x$)

$$\int dM_1dM_2dM_3f(M_1, M_2, M_3) = \int dM_1dH_{12}dH_{13}f(M_1, M_1H_{12}, M_1H_{13})$$

(the order of integration being important). If $f$ is gauge invariant the integral over $M_1$ factorises, supposing the measure on the unitary group to be normalized we get $\int dH_{12}dH_{13}f(H_{12}, H_{13})$. This shows that the measure on $A/G$ is

$$\int DH_{12}DH_{13} = \int DH_{12}DH_{13}DH_{23}\delta_1(H_{12}H_{13}H_{23})$$

where in the last equality we insert a delta function on the group to emphasize the symmetric form of the measure under permutation of indices. The interpretation of this delta function constraint should be clear: it is the integrated version of the Bianchi identity $\epsilon^{ijk}\nabla_i F_{jk} = 0$ expressed in terms of gauge invariant observables.

Now, the identity functional $\psi(H) = 1$ can be viewed as a limit of identity cylindrical functional the integral of which being always one if one choose the normalized Haar measure. So we can conclude from this analysis that the trivial wave functional is integrable.

This result is purely kinematical but it captures an essential feature of the formalism, namely the fact that even if we are working in the continuum, the choice of Bars variables allows us to recover results which are easily obtained in the lattice formulation, and as we will see, allows us to devise a strong coupling expansion. If one computes the expectation value of Wilson lines observables in this trivial kinematical vacuum one sees, for rectangular Wilson loops and from the definition of the measure, that

$$\langle W_R(y) \rangle = \int DH_{12}DH_{13}W_R(y) = \delta_{R,0}. \quad (63)$$

Wilson loop expectation values are non zero only for trivial representation, this confirms the interpretation that this trivial vacuum corresponds to an infinite string tension, which is what one expects in a crude $g \to \infty$ limit. Now, one could say that such a result seems at odds with the well known fact that in 2D there is a non trivial determinant in the KKN formalism, namely the hermitian WZW action $[1, 2]$. This apparent contradiction can be easily resolved since, thanks to the work of Gawedzki and Kupianen $[10]$, we know that there is a deep relationship between hermitian and unitary model.
More precisely if we specify their results to the genus zero case, the relation between correlation functions of WZW theory level $k$ with $SU(N)$ group and field $g(x)$ and the hyperbolic model with field $h(x) = m(x)^{\dagger}m(x)$ is given by

$$
\int N \prod_{i=1}^{N} (g_{R_i})_{\alpha_i}^{\beta_i}(x_i) e^{-kS(g)} Dg \int \prod_{n=1}^{N} (h_{R_i})_{\alpha_i}^{\beta_i}(x_i) e^{(k+2N)S(h)} \delta_1(h(x_0)) dh = \mathcal{N}_V
$$

where $\mathcal{N}_V$ is the number of conformal blocks (independent of $x_i$ and the sphere metric). When extrapolated to $k = 0$ clearly this identity shows the normalizability of the trivial wave functional.

Now it is also well known that the correlation functions of the unitary WZW model are zero unless the highest weight $\Lambda_R$ is integrable, that is $(\Lambda_R, \theta) \leq k$ where $\theta$ is the highest weight of the adjoint representation $(\theta, \theta) = 2$. If we extrapolate these results to $k = 0$ this means that only primary fields associated to the trivial representation and its descendant are non zero in agreement with (63). Moreover as noticed by KKN [2] the correlators of the currents of the unitary WZW model can be obtained by analytic continuation $(k + N) \rightarrow -(k + N)$ from the hermitian ones. The reason being that both satisfy the KZ equations with opposite $(k + N)$ and both are invariant combinations under monodromies, which fix them uniquely up to normalization.

### 3.2 Semi-complex coordinates

The determinant is trivial in the real basis where $H$ are unitary matrices. However, if one choose to work in the semi complex basis we find a non trivial determinant which is a generalization of WZW model in higher dimension and is related to the magnetic mass term introduced in [11]. For completeness we present this computation here.

We have seen in the first section that in the semi-complex coordinates $X = (x, \bar{x}, u_x)$ the gauge invariant data is entirely encoded into a complex $SL(N, \mathbb{C})$ field $H \equiv H_{u_x}$. The unregulated propagators are given by

$$
G_u(X, Y) = \theta(u_x - u_y)\delta^2(x - y), \quad (64)
$$

$$
G_{\bar{z}}(X, Y) = \frac{1}{\pi(\bar{x} - \bar{z})}\delta(u_x - u_y), \quad (65)
$$

$$
G_{\bar{x}}(X, Y) = \frac{1}{\pi(x - \bar{z})}\delta(u_x - u_y). \quad (66)
$$
The integrals (53) defining the regulated propagators can be explicitly performed leading to

$$G_\mu(X,Y) = \theta_\mu(u_x - u_y)\delta^2(x - y)\Lambda_u(X,Y),$$  \hfill (67)

$$G_z(X,Y) = \frac{\delta_\mu(u_x - u_y)}{\pi(x - y)}(1 - e^{-\mu^2|x - y|^2}\Lambda_z(X,Y)),$$  \hfill (68)

$$G_{\bar{z}}(X,Y) = \frac{\delta_\mu(u_x - u_y)}{\pi(x - y)}(1 - e^{-\mu^2|x - y|^2}\Lambda_{\bar{z}}(X,Y)).$$  \hfill (69)

with $|x|^2 = x\bar{x}$ and

$$\Lambda_u(X,Y) = H(x, \bar{x}, u_y)(H^\dagger H)^{-1}(y, \bar{y}, u_y)H(y, \bar{y}, u_y),$$  \hfill (70)

$$\Lambda_z(X,Y) = (H^\dagger H)^{-1}(y, \bar{y}, u_x)H(y, \bar{y}, u_x)H(y, \bar{y}, u_y),$$  \hfill (71)

$$\Lambda_{\bar{z}}(X,Y) = H^\dagger(x, \bar{y}, u_x)H(x, \bar{y}, u_y)(H^\dagger H)^{-1}(y, \bar{y}, u_y).$$  \hfill (72)

The value of the propagator at coincident point is given by

$$G_\mu(X,X) = \frac{\mu^2}{2\pi^2},$$

$$G_z(X,X) = -\frac{\mu}{\pi^\frac{3}{2}}(H^\dagger H)^{-1}\partial(H^\dagger H), \quad G_{\bar{z}}(X,X) = \frac{\mu}{\pi^\frac{3}{2}}\partial(H^\dagger H)(H^\dagger H)^{-1}.\hfill (72)$$

The variational derivative of the determinant (48) can be explicitly computed and after some algebra one gets

$$\delta\Gamma = -\frac{2N\mu}{\pi^\frac{3}{2}}\int \text{Tr} \left\{ (H^\dagger H)^{-1}\partial(H^\dagger H)\partial \left( (H^\dagger H)^{-1}\partial H^\dagger H \right) \right\}$$  \hfill (73)

This equation can be easily integrated out in terms of a three dimensional generalization of a Wess-Zumino-Witten action

$$-\frac{\pi^\frac{3}{2}}{2N\mu} \Gamma = \frac{1}{2} \int dud^2z \text{Tr} \left( \partial(H^\dagger H)\partial(H^\dagger H)^{-1} \right) + i \frac{1}{12} \int du \int_{B_u} \text{Tr} \left( [(H^\dagger H)^{-1}d(H^\dagger H)]^3 \right).$$

The last integral being over a three ball $B_u$ bounding the plane $u = \text{cste}$.  

4 Hamiltonian

In this section we write down the Hamiltonian in terms of gauge invariant variables. From now on and in the following one shall stick to three (space) dimensions, a real basis and we introduce some notations well adapted to this
case. The vacuum wave functional we look for, is expressed as a functional of the currents $J_{12}, J_{23}, J_{31}$. These should be thought as a vector (or three dimensional holomorphic connection) and we denote

$$H_i \equiv H_{i-1i}, \quad J_i \equiv J_{i-1i} = \partial_i H_i H_i^{-1}, \quad D_i \equiv D_{i-1i} = \partial_i - J_i$$  \hspace{1cm} (74)

and we introduce the ‘magnetic field’\footnote{It is related to the true magnetic field $F_i$ by $B_{i-1} = -M_i^{-1} F_{i-1} M_i$}

$$B_{i-1} = \partial_i J_{i+1}.$$  \hspace{1cm} (75)

(all indices are modulo three indeed) The potential term is readily expressed in these variables

$$\mathcal{V} = -\sum_i \int \text{Tr} \left( \left( \partial_i J_{i+1} \right)^2 (x) \right) \text{d}x = -\sum_i \int \text{Tr}(B_i(x))^2 \text{d}x$$ \hspace{1cm} (76)

and with the help of the holomorphic regularization one can compute the kinetic term \footnote{If one look at the detailed structure of the kinetic term one encounters a disturbing infrared divergence hidden in it, namely if one look at (78) the structure of each term is given by}

$$\mathcal{F} = -\frac{1}{2} \sum_i \int P_i^a(y) \tilde{\Theta}_i^{ab}(y,z) P_i^b(z) \text{d}y \text{d}z$$  \hspace{1cm} (78)

Here for later convenience we have introduced a (unregularized at this stage) kernel

$$\tilde{\Theta}_i^{ab}(y,z) = \delta^{ab} (G_i G_i^i)(y,z) = \delta^{ab} \int G_i(y,x) G_i(z,x) \text{d}x.$$  \hspace{1cm} (79)

and we have used that $(G_i^i P_a^i)^i(x) = (P_a^i G_i)(x)$.\footnote{It is related to the true magnetic field $F_i$ by $B_{i-1} = -M_i^{-1} F_{i-1} M_i$}
using the identity \( 1 = \theta(x, y) + \theta(y, x) \) we can express this term as

\[
\frac{1}{2} \int d^2x d^2y \left( \int_{z_1}^{+\infty} P_1^a dy_1 \right) \left( \int_{-\infty}^{+\infty} P_1^a dz_1 \right) (x_2, x_3) \tag{81}
\]

\[
+ \frac{1}{2} \int d^3x dy_1 dz_1 \theta(y_1, x_1) \theta(x_1, z_1) P_1^a(y_1, x_2, x_3) P_1^a(z_1, x_2, x_3). \tag{82}
\]

The first term in this expansion contains

\[
\int_{-\infty}^{+\infty} P_1^a(y_1, x_2, x_3) dy_1 = \int \text{Tr}(P_1(y) \phi_a^x(y)) dy
\]

which is a generator of an infinitesimal holomorphic symmetry with generator \( \phi_a^x(y) = -2 T^a \delta(x_2 - y_2) \delta(x_3 - y_3), \partial_1 \phi_a^x(y) = 0 \) and therefore acts trivially on holomorphic invariant states. We can therefore drop this term and consider the kinetic term to be given by the second term \( \tag{82} \). This amounts to replace in all the previous expressions \( \tilde{\Theta}_i = GG^d \) by the convolution product \( \Theta_i = GG \). This second term is now free of any infrared problem, the integral over \( x_1 \) can be performed and we are left with

\[
\frac{1}{2} \int dy_1 dz_1 d^2x dy_1 |y_1 - z_1| \text{Tr}(P_1(y_1, x_2, x_3)P_1(z_1, x_2, x_3)) \tag{83}
\]

which exhibits a stringy nature of the kinetic term with some linearly rising potential. There is even a deeper justification arguing also in favor of this second form of the kinetic term which comes from the derivation, in the connection variables, of its matrix elements given by

\[
\langle \Psi | \mathcal{T} | \Psi \rangle = \int_{\mathcal{A}/G} DA \left( \frac{\delta \Psi(A)}{\delta A^a_i} \right) \frac{\delta \Psi(A)}{\delta A^a_i}.
\]

We also want our theory to be CPT invariant which means that instead of conjugating with simple complex conjugation \( \Psi \rightarrow \overline{\Psi} \) we can equivalently replace this conjugation by a CPT transformation \( \Psi \rightarrow CPT(\Psi) \). The CPT transformation of the gauge field is given \( CPT(A_i(x)) = -A_i^1(-x) = A_i(-x) \). When we change variables from \( A_i \) to \( M_i \) we need to integrate over one dimension from \( -\infty \) to \( x \), this breaks parity and the CPT conjugate of \( M_i \) is not just the hermitian conjugate: it is given by a path ordered integral which starts from \( +\infty \) that is \( CPT(M_i(x)) = (M_i(-x) h_i)^\dagger \). Where \( h_i(x_2, x_3) = \frac{e^{\gamma}}{\epsilon} \int_{-\infty}^{+\infty} A_1(-x_1, x_2, x_3) dx_1 \), is the holonomy across space at fixed \( x_2, x_3 \), and \( \partial h_i = \)
0. Of course if the wave function is holomorphic invariant it doesn’t depend on $h_i$ and CPT invariance is satisfied. Now if one starts from a form of the scalar product with is manifestly CPT invariant (that is $CPT(\Psi)$ is used instead of $\bar{\Psi}$) and compute from there the form of the kinetic term one can see that the $A$ derivative acting on the right can be expressed in terms of the $M$ and the propagator $G$ but the derivatives acting on the left should be expressed in terms of the propagator $G^t$ and in this case the form of the kinetic term is expressed in terms of $\Theta = GG$. This form of the kinetic term being the one in which CPT invariance is manifest. That is the kinetic term we will use is
\[
T = \frac{1}{2} \sum_{i,a,b} \int P^i_a(y) \Theta^{ab}_i(y, z) P^i_b(z) dy dz
\] (85)

Here the unregularised kernel is
\[
\Theta^{ab}_i(y, z) = \delta^{ab}(G_i G_i)(y, z) = \delta^{ab} \int G_i(y, x) G_i(x, z) dx.
\] (86)

This kernel satisfies
\[
\partial_y^i \Theta^{ab}_i(y, z) = \delta^{ab} G_i(y, z), \quad \partial_z^i \Theta^{ab}_i(y, z) = -\delta^{ab} G_i(y, z), \quad (87)
\]
\[
\partial_y^i \partial_z^i \Theta^{ab}_i(y, z) = -\delta^{ab} \delta(y, z). \quad (88)
\]

We can now express the kinetic term in the current variables, first recalling for comfort and according to our new notations the commutator
\[
[P^k_a(x), J^b_i(y)] = \delta^k_i (H_k(y))^b \partial_y^i \delta(x, y) - \delta^{k+1}_i (D_{k+1}^y)^b \delta(x, y).
\] (89)

Therefore
\[
T = \frac{1}{2} \sum_{i,j,b,c} \int dy dz \Omega^{bc}_{ij}(y, z) \frac{\delta}{\delta J^b_i(y)} \frac{\delta}{\delta J^c_j(z)}
\] (90)

with
\[
\Omega^{bc}_{ii}(y, z) = \left[ (D^y_i)^{ba} (D^z_i)^{ca} \Theta_{i-1}(y, z) - \delta^{bc} \delta(y, z) \right], \quad (91)
\]
\[
\Omega^{bc}_{i+1,i}(y, z) = \frac{1}{2} \left[ (D^y_{i+1} H_{i+1}^{-1}(z))^{bc} (G_i(y, z) - G_i(z, y)) \right]. \quad (92)
\]

The full QCD Hamiltonian is
\[
\mathcal{H} = g^2 T + \frac{1}{g^2} V.
\] (93)
4.1 Regularised Hamiltonian

In order to define mathematically the theory we need to include a regulator. Preserving holomorphic invariance is a strong constraint on the form of the regulator and we have seen that there is a natural holomorphic invariant regularization with scale parameter $\mu$ which is available in our context. This regularization amounts to replace $\delta^{ab} G_i(x,y)$ by its holomorphic invariant regulated version (53) $G^{\mu}G_i^{\mu}(x,y)$, the holomorphic regularization of the kernel $\Theta$ is denoted $\Theta^{\mu}$ and is given by

$$\Theta^{\mu}_{i}(y,z) \equiv (G_i G_i^{\mu})_{bc}(y,z) = \int dx \left( \Lambda_i(y,x)\Lambda_i(x,z) \right)^{bc} G^{\mu}_{i}(y,x)G^{\mu}_{i}(x,z)$$

(94)

with

$$G^{\mu}_{i}(y,x) = \theta_{\mu}(y_1 - x_1)\delta_{\mu}(y_2 - x_2)\delta_{\mu}(y_3 - x_3),$$

and (see section 3),

$$\Lambda_i(y,x) = H_2(x_1, y_2, y_3)H_3(x_1, x_2, y_3)H_1(x_1,x_2,x_3)$$

(95)

plus cyclic permutations. Under holomorphic transformations we have

$$M_i \rightarrow M_i h_i^{-1}, \quad H_i \rightarrow h_i^{-1}H_i h_i^{-1}, \quad P^{i}_{a} \rightarrow (h_i)_{b}^{\ d} P^{i}_{b}, \quad B_{i-1} \rightarrow h_i B_{i-1} h_i^{-1}$$

$$\Lambda_i(y,x) \rightarrow h_i(y)\Lambda_i(y,x) h_i(x)^{-1}$$

(96)

and $\Theta^{\mu}_{i}(y,z) \rightarrow h_i(y)\Theta^{\mu}_{i}(y,z) h_i(z)^{-1}$.

The regularised kinetic term written in a manifestly hermitian form is given by

$$\mathcal{T} = \frac{1}{4} \sum_{i,a} \int (G^{\mu}_{i} P^{i})_{a}(x)(G^{\mu}_{i} P^{i})_{a}(x) + (G^{\mu}_{i} P^{i})_{a}(x)(G^{\mu}_{i} P^{i})_{a}(x)dx,$$

(97)

$$= \frac{1}{4} \sum_{i,b,c} \int P^{i}_{b}(y)(\Theta^{\mu bc}(y,z) + \Theta^{\mu cb}(z,y))P^{i}_{c}(z)dydz,$$

(98)

$$= \frac{1}{4} \sum_{i} \int [P^{i}(\Theta^{\mu} + \Theta^{\mu})]^{i} (y,z)dydz,$$

(99)

where we used the transposed kernels

$$(G^{\mu}_{i})^{ab}(x,y) \equiv -(G^{\mu}_{i})^{ba}(y,x), \quad (\Theta^{\mu})^{ab}(x,y) \equiv (\Theta^{\mu})^{ba}(y,x).$$
It is convenient for our subsequent computations to move all the derivatives to the right; in doing so we encounter a commutator

\[ [P_i^b(y), \Theta^{\mu bc}(y,z)] \equiv \lim_{\mu \to \infty} [P_{\mu i}^b(y), \Theta^{\mu bc}(y,z)], \quad (100) \]

where \( P_i^b \) is a left invariant derivative regulated according to our prescription: it satisfies \( (G_i^\mu P_i^a) = (G_i P_i^a) \) and is explicitly given by

\[ P_{\mu i}^a(y) \equiv \int dz \Lambda_i^{ab}(y,z)\delta_{\mu i}(y,z)P_i^b(z). \quad (101) \]

The calculation of this commutator gives

\[ \int [P_i^b(y), \Theta^{\mu bc}(y,z)]dy = -\Theta_i^{bc}(z,z), \quad (102) \]

where \( \Theta_i^{bc} \equiv \Theta_i^{ab}f_{ab}^{\, bc} \). This term is of order \( O\left(\frac{1}{\mu^2}\right) \). For the other commutator we find

\[ \int [P_i^b(y), \Theta_i^{\mu i b}(z,y)]dy = \rho_i^c(z), \quad (103) \]

with

\[ \rho_i^c(z) = \int dx(G_i^\mu(z,x)G_i^\mu(x,z)) \delta_{\mu i}(x,z). \quad (104) \]

where \( z_{x_i} = (x_1, z_2, z_3) \), \( z_{x_2} = (z_1, x_2, z_3) \) etc... This term is also of order \( O\left(\frac{1}{\mu^2}\right) \) and independent of \( z_i \), it corresponds to an holomorphic transformation. We can now write the kinetic term in terms of the currents: in order to do so one needs to evaluate the double commutator

\[ \int \Theta_i^{\mu bc}(y,z)[P_i^b(y), [P_i^a(z), J_k^a(x)]]dydz \quad (105) \]

which is also of order \( O\left(\frac{1}{\mu^2}\right) \). The conclusion is that up to terms \( \int dz \Omega_i^c(z)\delta_{\mu i}(x,z) \) with \( \Omega_i^c(z) = O\left(\frac{1}{\mu^2}\right) \) the regularised kinetic term takes the form \( (90) \) with

\[ \Omega_i^{bc}(y,z) = [D_i^b D_i^c \Theta_i^{1-1}(y,z) + H_i^c(y)(\partial_i^b \partial_i^c \Theta_i(y,z))(H_i(z))^{-1}]^{bc}, \quad (106) \]

\[ \Omega_i^{bc}(y,z) = -\frac{1}{2} [D_i^b H_i^{-1}(z)\partial_i^c (\Theta_i(y,z) + \Theta_i^b(y,z))]^{bc}, \quad (107) \]

where \( (\Theta^i)^{ab}(x,y) = \Theta^{ba}(z,x) \) and \( (DD\Theta)^{bc} \equiv D^b_a D^c_d \Theta^{ad}. \)
5 Vacuum wave functional

In this section we show how one can, in our formalism, determine the behavior of the vacuum wave function both in the weak and strong coupling regime.

5.1 Weak coupling regime

Let us first look at the weak coupling limit of the theory: as is usual one first rescale the currents $J \rightarrow J/g$ and then expand the Hamiltonian in power series in $g$, $\mathcal{H} = \sum_i g^i \mathcal{H}_i$. After field rescaling, the potential term $\mathcal{V}$ is of order zero, therefore this expansion can be realized entirely as an expansion in the kinetic term and more precisely in the coefficients $\Omega$. This expansion is an expansion in the number of currents $J$. Since $J$ has a dimension of mass it always enters the expansion in the combination $J/p$, if one think in terms of Fourier modes. This means that we should interpret this expansion as being valid in the ultraviolet sector $p \rightarrow \infty$ of the theory and $\mathcal{H}_0$ is the Hamiltonian governing the gluon dynamic in the deep ultraviolet, this Hamiltonian being free.

The diagonal terms $\Omega_{ii}$ contains only a finite number of terms up to $g^4$ but the off diagonal component due to the presence of the group element $H$ contains an infinite number of terms in this expansion. We are now interested only in the structure of the first non trivial term $\mathcal{H}_0$ and the coefficients of the kinetic term at zero order are given by

$$\Omega^{(0)bc}_{ii} (y,z) = \left[ (\partial^y_i) (\partial^z_i) \Theta_{i-1}(y,z) - \delta(y,z) \right] \delta^{bc}, \quad \Omega^{(0)bc}_{i+1i} (y,z) = \frac{1}{2} \left[ \partial^y_i (G_i(y,z) - G_i(z,y)) \right] \delta^{bc}. \quad (108)$$

This expression is not particularly illuminating at first sight, however if one writes down this Hamiltonian in Fourier space $B(p) = \int dx e^{-ipx} B(x)$ and takes as fundamental variables the magnetic fields $B_i = \partial_i+1 J_i-1$, after simple algebra the Hamiltonian becomes

$$\mathcal{H}_0 = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ - (2\pi)^6 (p^2 \delta_{ij} - p_i p_j) \frac{\delta}{\delta B_{i+1}(p)} \frac{\delta}{\delta B_{j-1}(\mathbf{-p})} + B_{a+1}(p) B_{a-1}(\mathbf{-p}) \right].$$

The Poincaré invariance of the theory becomes manifest. The matrix contracting the derivatives has eigenvalues $p^2, p^2, 0$ the eigenstates associated to
$p^2$ correspond to the two gluons polarization perpendicular to $p$. The eigenstate corresponding to the eigenvalue 0 is puzzling at first sight since we are working in a gauge invariant formalism. In order to properly understand it one needs to remember that the gauge invariant measure constructed previously contains only two scalar independent variables per Lie algebra generator, this is encoded in the constraint $H_1 H_2 H_3 = 1$ or in differential form in the Bianchi identity $\nabla_i F^i = 0$. This means that $J_1, J_2, J_3$ are not independent variables, and that any operator proportional to this constraint is strongly 0. When expressed in terms of our local variables and in the limit $g \to 0$ this constraint imposed by the measure reads $\sum_i p_i B_i(p) = 0$.

We can now easily diagonalize the free hamiltonian $H_0$ subject to this constraint and give explicitly the ground state in this regime

$$\Psi(B) = \exp - \frac{1}{2g^2} \int \frac{d^3p}{(2\pi)^3} \frac{B_i^a(p)B_i^a(-p)}{|p|}$$

which is indeed the one expected and where we have reinstalled the dependence in $g$ by rescaling to the original variables.

### 5.2 Evidence for confinement

The previous result is not very impressive by itself since it just shows that our formalism can handle appropriately the perturbative regime which is well understood by other means. However, if one look back at the formulation in terms of Bars variables this formalism, even if it is local, is very similar to the lattice formulation of QCD which is easily amenable to strong coupling expansion. It shows that the route we have chosen allows us to deal also with the strong coupling or infrared regime of the theory.

In the strong coupling regime $g \to \infty$ the hamiltonian $H$ is dominated by its kinetic term, as emphasized for instance in the hamiltonian formulation of lattice QCD [12]. At first order in the strong coupling expansion the vacuum state is just the constant wave functional which is normalizable as we have seen even if we work in the continuum. In order to go beyond this naive result and construct the vacuum state in the infrared regime, we establish as a first key result that the potential term $\mathcal{V}$ viewed as a wave functional is an eigenstate of the kinetic term. Since the eigenvalue has a dimension of a mass we see the
emergence of a dynamically generated mass scale, namely

\[ g^2 \mathcal{T} : \mathcal{V} : = M : \mathcal{V} : \quad \text{with} \quad M = \frac{g^2 N \mu}{2(2\pi)^2}, \quad (111) \]

where \( N \) is the dual coxeter number \( N \delta^a_b = f^{apq} f_{bpq} \) (with our convention on Lie algebra generators) the \( \cdot \cdot \) denotes normal ordering of the operator and \( \cdot \) denotes the action of the operator \( \mathcal{T} \) and \( \mu \) is our ultraviolet cutoff regulator. It is important to keep in mind that this result is valid only in the strong coupling regime, the true QCD mass scale should appear in the continuum limit as the regulator is removed \( \mu \to \infty \).

The first step of the computation follows directly from the definition of the regulated Hamiltonian (90, 106) which is defined in terms of the holomorphically invariant regulated \( \Theta^\mu_i(y,z)^{ab} \) and its derivative. From this definition, one obtained after integration by part and noticing that the off diagonal terms do not play any role

\[ g^2 \mathcal{T} \cdot \int -\text{Tr}(\partial_{i-1} J_i)^2(x) \, dx \]

\[ = \frac{g^2}{2} \int \left[ \partial^a_{i-1} \partial^z_{i-1} \left( D^a_i D^z_i \Theta^\mu_i(y,z) + H_i(y)(\partial^a_i \partial^z_i \Theta^\mu_i(y,z))(H_i(z))^{-1} \right) \right]_{y=z} \, dy. \]

The RHS of (112) is the sum of two terms which can be computed separately, this relatively lengthy calculation is presented in detail in the appendix. The computation of the first term of (112) gives

\[ - \left[ \partial^y_{i-1} \partial^z_{i-1} \text{Tr}_{ad} \left( D^y_i D^z_i \Theta^\mu_i(y,z) \right) \right]_{y=z} = \]

\[ \frac{\mu}{4(2\pi)^{3/2}} \left[ -2N \text{Tr}(B^2_{i+1}) - 4\mu^2(N^2 - 1) \right] + O \left( \frac{1}{\mu} \right). \]

The computation of the second term of (112) gives the same result !

\[ - \left[ \partial^y_{i-1} \partial^z_{i-1} \text{Tr}_{ad} \left( H_i(y)(\partial^y_i \partial^z_i \Theta^\mu_i(y,z))(H_i(z))^{-1} \right) \right]_{y=z} = \]

\[ \frac{\mu}{4(2\pi)^{3/2}} \left[ -2N \text{Tr}(B^2_{i+1}) - 4\mu^2(N^2 - 1) \right] + O \left( \frac{1}{\mu} \right). \]

The factor \( N \) in front of the trace is the dual coxeter number coming from the relation between trace in the adjoint and in the vectorial \( \text{Tr}_{ad}(T^a T^b) = 2N \text{Tr}(T^a T^b) \). Summing over \( i \) shows that (112) is

\[ \frac{g^2 N \mu}{2(2\pi)^{3/2}} \left( - \sum_i \text{Tr}(B^2_i) - 6\mu^2 N^2 - \frac{1}{N} \right) + O \left( \frac{1}{\mu} \right). \]

(115)
Therefore if one defines the normal ordered operator

\[ : \mathcal{V} : \equiv \mathcal{V} - 6\mu^2 \frac{N^2 - 1}{N}, \]  

(116)

then one gets the announced result (111). It is interesting to note that \( T \) acts diagonally not only on the Lorentz invariant combination \( \mathcal{V} \) but also on each term in the sum that is

\[ T \cdot : \text{Tr}(B_i^2) : = M : \text{Tr}(B_i^2) :, \quad : \text{Tr}(B_i^2) : \equiv \text{Tr}(B_i^2) + 2\mu^2 \frac{N^2 - 1}{N}. \]  

(117)

This result is extremely simple and strongly resonates with the 2 dimensional results. It suggests that the action of the kinetic term on local gauge invariant operators is analogous to a kind of dilation operator which counts the number of \( J \) constituents of the operators it acts on. In order to validate such a picture higher order computation should be performed.

5.3 strong coupling expansion

We can now exploit the full strength of our formalism and show that the strong coupling expansion can be performed following a strategy devised in 2+1 by KN and strongly reminiscent of the cluster expansion of Lattice QCD [13]. The QCD hamiltonian reads (lets take a finite dimensional analog for the presentation of the argument)

\[ H = g^2 T + \frac{1}{g^2} \mathcal{V}, \text{ with } T = \Omega^{ij} \partial_i \partial_j \]  

(118)

Then we see that in the strong coupling regime \( g \to \infty \) the hamiltonian reduces to its kinetic term and the trivial wave function \( \Psi = 1 \) is a solution in this regime to the equation \( H \psi = 0 \) up to order \( 1/g^2 \), this solution is normalisable as we have seen in the previous section. In order to go further lets suppose that

\[ \Omega^{ij} \delta_i \delta_j V = MV, \]  

(119)

which is what we have proven in the last section in the context of QCD, and lets consider the wave function \( \Psi = e^{-\frac{1}{g^2 M} \mathcal{V}} \Phi \), using the previous equality we have

\[ H e^{-\frac{1}{g^2 M} \mathcal{V}} \Phi = e^{-\frac{1}{g^2 M} \mathcal{V}} \tilde{H} \Phi, \]  

(120)
where
\[ \tilde{H} = g^2 T - \frac{2}{M} \Omega^{ij} \partial_i V \partial_j + \frac{1}{g^4 M^2} \Omega^{ij} \partial_i V \partial_j V \quad (121) \]
and we see that the wave function \( \Phi = 1 \) or \( \Psi = e^{-\frac{1}{g^2 M} V} \) is a vacuum solution at order \( 1/g^4 \) and therefore describes the vacuum in the strong coupling regime.

The higher order terms in this expansion can be recursively constructed in principle. For instance at the next order the vacuum wave functional reads
\[ \Psi = e^{-\frac{1}{g^2 M} (V + \mu M^2)} \]
where \( \mu \) satisfies the equation
\[ \frac{g^2}{M} [[T, V], V] \equiv \frac{g^2}{M} T \cdot P = \frac{1}{2} \int \text{Tr}(F_i \Delta F_i) - \epsilon^{ijk} \text{Tr}(F_i [F_j, F_k]) \quad (122) \]
with \( \Delta \) the covariant laplacian. Now one can see from the definition of the regularised kinetic term that \( \frac{g^2}{M} T \) when acting on gauge invariant operators of order 6 reproduces a linear combination of such operators plus a sum over lower dimensional terms \( \mu^2 V \) and \( \mu^6 \) which contributes to a renormalization of the vacuum energy and coefficients of \( V \). Thus up to renormalization, which can be reabsorbed at this order in a redefinition of \( M \rightarrow M(1 + \frac{\alpha}{g^4 N^2}) \), with \( \alpha \) a numerical constant, we have
\[ P = \int a \text{Tr}(F_i \Delta F_i) - b \epsilon^{ijk} \text{Tr}(F_i [F_j, F_k]). \quad (123) \]
where \( a, b \) are numerical coefficients (independent of \( N, \mu, g \)). These coefficients are uniquely computable from the regularization prescription we have given. This calculus is quite lengthy but otherwise straightforward and will be presented elsewhere. This procedure can be implemented at higher order in a strong coupling expansion.

The conclusion is then that in the infrared regime (more precisely for slowly varying and low amplitude field) and in a strong coupling expansion the vacuum wave functional is given by
\[ \Psi = e^{-\frac{1}{g^4 m} \int \text{Tr}(B^2)}. \quad (124) \]
where the mass scale \( m \) computed here only to first order in \( 1/g^4 N^2 \) can be recursively computed in a strong coupling expansion. That is \( m = M(1 + \frac{\alpha}{g^4 N^2} + \cdots) \) where the dots refer to terms of higher order in \( 1/(g^2 N)^2 \). We can now give easily the physical interpretation of the dynamical scale \( m \) entering
the definition of the infrared vacua wave functional \([14]\). Let us compute the expectation value of a large Wilson loop \(W_C\) with area \(A_C\)

\[
\langle W_C \rangle = \langle \Psi_{\text{vac}} | W_C | \Psi_{\text{vac}} \rangle.
\]

This average should be dominated in the limit \(A_C \to \infty\) by the expectation value with respect to the infrared vacua \([124]\) and is therefore given by

\[
\langle W_C \rangle \sim \int_{A/G} DA e^{-\frac{2}{g^2 m} \int \text{Tr}(B^2)} W_C(A). \tag{125}
\]

We recognize the partition function of 2+1 Euclidean Yang-Mills theory with coupling \(g_{2+1}^2 = g^2 m\). Now this expectation value reduces in the limit of large area to the expectation value of the Wilson line in the 2+1 Yang-Mills vacua. This vacua has been recently exactly constructed at least in the large \(N\) limit where no screening due to \(n\)-ality is expected \([15]\) and since 2+1 Yang-mills shows the property of confinement one gets

\[
\langle W_C \rangle \sim e^{-\sigma A_C} \tag{126}
\]

where the string tension is given by

\[
\sigma = g_{2+1}^4 \frac{N^2 - 1}{8\pi}. \tag{127}
\]

Translated back into 3+1 Yang-mills coupling one gets

\[
\sigma = m^2 g^4 \frac{N^2 - 1}{32\pi}. \tag{128}
\]

One should insist here that this results has been only establish firmly only in a strong coupling regime at first order, that is \(m = M\), and is not yet shown to survive the continuous limit.

### 6 Conclusion

We have seen in this work that it is possible to give a formulation of pure QCD in terms of local gauge invariant observables and computed explicitly the jacobian for this transformation which shows that the trivial wave function is normalisable. We have also constructed the regularised hamiltonian
and expressed the vacuum wave function at first order in a strong coupling expansion. This expansion can be extended to higher order as an expansion in terms of higher dimensional local operators.

Several problems in this framework have not been explored yet. The most formidable is the problem of the continuum limit. As we have seen a mass scale makes its appearance in the construction of the vacuum wave functional but this scale is cutoff dependent. In order to establish on a robust basis the existence of a dynamical mass scale in the infrared one would need to show that it is possible to remove the cut-off without destroying the presence of a non zero string tension. This would mean first showing that no phase transition makes its appearance while going from strong to weak coupling (that is \( M > 0 \) from strong to weak coupling) and moreover that the scaling of the dynamical mass is such that its continuum value is controlled by the usual renormalization group scaling \[16\]

\[
\frac{(4\pi)^2}{g^2 N} = \frac{(4\pi)^2}{g^2(\Lambda)N} + \frac{11}{3} \ln \left( \frac{\mu^2}{\Lambda^2} \right) + \frac{34}{11} \ln \left( \ln \left( \frac{\mu^2}{\Lambda^2} \right) \right). \tag{129}
\]

Notice that the detail study of the continuum limit in 2+1 pure QCD from the KKN approach needs to be completed too.

A related project which is less ambitious but phenomenologically more interesting concerns the possibility to construct in 3+1 pure Yang-Mills the kernel interpolating between the infrared and ultraviolet behavior of the vacuum wave functional and extract from this an estimate for the mass gap in terms of the string tension \[8\].

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## 7 appendix

In this appendix we want to establish the main technical result of this paper, that is the proof of the fact that the kinetic term acts diagonally on the potential term which is the lowest order local gauge invariant operator \([11]\).
This amounts, has shown in the main body of the paper, to the computation of
\[
\partial_{i-1}^\mu \partial_{i-1}^\nu \text{Tr}_\text{Ad} (\Omega_i^\mu(y, z)) \big|_{z=y}.
\] (130)

In order to perform this calculation it is convenient to introduce some conventions and establish some basic results. First one recall the definition of the current \( J_i \)
\[
\partial_i H_i = J_i H_i,
\] (131)

One also introduce an auxiliary current \( \tilde{J}_i \)
\[
\partial_i H_{i+1} \equiv H_{i+1} \tilde{J}_i.
\] (132)

The integrated Bianchi identity \( H_1 H_2 H_3 = 1 \) allows us to express the third type of \( H \) derivatives in terms of these currents
\[
\partial_i H_{i-1} = -H_{i-1} J_i - \tilde{J}_i H_{i-1}.
\] (133)

Finally, derivatives of the currents \( \tilde{J} \) can be purely expressed in terms of \( J \) derivatives:
\[
\partial_{i+1} \tilde{J}_i = H_{i+1}^{-1} \partial_i J_{i+1} H_{i+1} = H_{i+1}^{-1} B_{i-1} H_{i+1},
\] (134)

and
\[
D_{i-1} \tilde{J}_i = H_{i-1} \partial_{i-1} J_i H_{i-1}^{-1} - \partial_i J_{i-1}
\] (135)

The computation of (130) splits in two parts, first one needs to evaluate
\[
I_i(y) \equiv \left[ \partial_{i-1}^\mu \partial_{i-1}^\nu \text{Tr}_\text{ad} (H_i(y)(\partial_i^\mu \partial_i^\nu \Theta(y, z))(H_i(y))^{-1}) \right]_{y=z}
\] (136)
\[
= \int \text{d}x \text{Tr}_\text{ad} \left( \left[ \partial_{i-1}^\mu (H_i(y)(\partial_i^\mu G_i^\mu(y, x))) \right] \left[ \partial_{i-1}^\mu (\partial_i^\mu G_i^\mu(x, y)) H_i^{-1}(z) \right] \right)
\]

One starts by the evaluation of
\[
\partial_{i-1}^\mu (H_i(y)(\partial_i^\mu G_i^\mu(y, x)))
\] (137)
\[
= (\partial_{i-1}^\mu H_i(y)\Lambda_i(y, x))\delta_\mu(y, x) + H_i(y)\Lambda_i(y, x)\partial_{i-1}^\mu \delta_\mu(y, x)
\]
where we have used the property (52) \( \partial_i^\mu \Lambda_i(y, x) = 0 \). Moreover, choosing \( i = 1 \) without loss of generality and starting from the definitions of \( \Lambda_1(y, x) \) and the currents one can compute
\[
\partial_3^\mu H_1(y)\Lambda_1(y, x) = H_1(y) \left[ \tilde{J}_3(y_1, y_2, y_3) - \tilde{J}_3(x_1, y_2, y_3) + H_2(x_1, y_2, y_3) (J_3(x_1, x_2, y_3) - J_3(x_1, y_2, y_3)) H_2^{-1}(x_1, y_2, y_3) \right] \Lambda_1(y, x).
\] (138)
We can Taylor expand (137) and get

\[ \partial_\nu (H_1(y)(\partial_\mu G_i^\mu(y, x))) \] (139)

\[ = \left[-X_1B_2(y) + X_2(H_3^{-1}B_1H_3)(y) + 2\mu^2X_3\right] \delta_\mu(X)H_1(y) + O(X^2) \]

with \(X_i \equiv x_i - y_i\) and where we have made use of the relations (132), \(\Lambda(x, x) = 1\) and the explicit form of the regulated delta function derivative: \(\partial_\nu \delta_\mu(x, y) = 2\mu^2(x_3 - y_3)\delta_\mu(x, y)\).

One continues with the evaluation of

\[ \partial_{\nu-1} \left( (\partial_\mu G_i^\mu(x, y))H_1^{-1}(y) \right) \] (140)

\[ = -\Lambda_1(x, y)H_1^{-1}(y)\partial_{\nu-1} \delta_\mu(x, y) + \partial_{\nu-1} \left( \partial_\mu \Lambda_1(x, y) \right)H_1^{-1}(y)G_i^\mu(x, y) \]

with

\[ \partial_\mu \Lambda_1(x, y) = \Lambda_1(x, y)H_1^{-1}(y) \left[J_1(y_1, y_2, y_3) - J_1(y_1, y_2, x_3)\right]H_1(y) \]

\[ +H_2(y_1, x_2, x_3) \left[J_1(y_1, x_2, x_3) - \tilde{J}_1(y_1, y_2, x_3)\right]H_2^{-1}(y_1, x_2, x_3)\Lambda_1(x, y), \]

and

\[ \partial_\nu \left( \partial_\mu \Lambda_1(x, y) \right)H_1^{-1}(y) = \Lambda_1(x, y)H_1^{-1}(y)\partial_\nu J_1(y). \] (141)

We have also used the property

\[ \partial_\nu \Lambda_1(x, y)H_1^{-1}(y) = 0. \] (142)

The Taylor expansion of (140) is therefore given by

\[ \partial_{\nu-1} \left( H_i(y)(\partial_\mu G_i^\mu(x, y)) \right) \] (143)

\[ = H_1^{-1}(y) \left[-2\mu^2X_3\delta_\mu(X) + B_2(y)G_i^\mu(X) \right] \]

\[ +[X_3B_2 - X_2(H_1B_3H_1^{-1})](y)\partial_\nu G_i^\mu(X) \] \[ + O(X^2) \]

we have denoted \(G_i^\mu(X) \equiv G_i^\mu(x, y)\) and similarly for \(\delta_\mu\).

Putting these results together one obtains

\[ I_1 = \int \mathrm{d}X \mathrm{Tr}_{ad} \left( \left[-X_1B_2(y) + X_2(H_3^{-1}B_1H_3)(y) + 2\mu^2X_3\right] \delta_\mu(X) \right) \]

\[ \left[-2\mu^2X_3\delta_\mu(X) + B_2(y)(1 - 2\mu^2X_3^2)G_i^\mu(X) + 2\mu^2X_2X_3(H_1B_3H_1^{-1})(y)G_i^\mu(X) \right) \]

where we have neglected inside the parenthesis in the integrand all the term of cubic and higher orders in \(X_i\) and express \(\partial_3 G_1 = -2\mu^2X_3G_1\). Due to parity
symmetry only the terms containing $X_i^2\delta_\mu^\alpha(X)$, $X_iG_i\delta_\mu$ or $X_j^2X_iG_i\delta_\mu$ are non zero, the corresponding integrals can be explicitly evaluated

$$\int dXX_i^2\delta_\mu^\alpha(X) = \frac{\mu}{4(2\pi)^{\frac{3}{2}}}, \quad (144)$$

$$\int dXX_iG_i(\delta_\mu^\alpha(X)) = 2\frac{\mu}{4(2\pi)^{\frac{3}{2}}}, \quad (145)$$

$$2\mu^2\int dXX_j^2X_iG_i(\delta_\mu^\alpha(X)) = \frac{\mu}{4(2\pi)^{\frac{3}{2}}}, \quad (146)$$

$$2\mu^2\int dXX_j^2G_i(\delta_\mu^\alpha(-X)) = \frac{\mu}{4(2\pi)^{\frac{3}{2}}}. \quad (147)$$

(the last integral will be used later.) Moreover for the same reason the cubic terms which have been neglected do not contribute to the integral, the first non trivial contribution starts at quartic order and the corresponding integral is of order $1/\mu$, which justifies, a posteriori, our approximation. Eventually, using the relation $\text{Tr}_{ad}(B^2) = 2N\text{Tr}(B^2)$, $\text{Tr}_{ad}(1) = N^2 - 1$ one gets

$$I_1 = \frac{\mu}{4(2\pi)^{\frac{3}{2}}}(-2N\text{Tr}(B_2^2) - 4\mu^4(N^2 - 1)) + O\left(\frac{1}{\mu}\right) \quad (148)$$

which proves (114).

For the second part of the computation one needs to evaluate

$$II_i(y) \equiv \left[\partial_{i-1}^\mu D_i^\mu \Theta_{i-1}^\alpha(y, z)\right]_{y=z} \quad (149)$$

$$\int dx\text{Tr}_{ad}\left([\partial_{i-1}^\mu D_i^\mu G_{i-1}^\alpha(x, y)][\partial_{i-1}^\mu \bar{D}_i^\mu G_{i-1}^\alpha(x, y)]\right)$$

One first focus on the derivative (take $i = 2$)

$$[\partial_i^\mu D_i^\mu G_1^\mu(y, x)]$$

$$= (\partial_i^\mu D_i^\mu \Lambda_1(y, x))G_1^\mu(y, x) + (D_i^\mu \Lambda_1(y, x))\delta_\mu(y, x) + \Lambda_1(y, x)\partial_i^\mu \delta_\mu(y, x), \quad (150)$$

with

$$D_i^\mu \Lambda_1(y, x) = H_2(y)(\partial_i^\mu H_2^{-1}(y)\Lambda_1(y, x)) \quad (151)$$

$$= (J_2(x_1, y_2, y_3) - J_2(y_1, y_2, y_3))\Lambda_1(y, x) \quad (152)$$

and

$$\partial_i^\mu D_i^\mu \Lambda_1(y, x) = -\partial_1 J_2(y)\Lambda_1(y, x) = -B_3(y)\Lambda_1(y, x). \quad (153)$$
The equality (151) follows from the definition of $D$, and the next one from a direct computation. The Taylor expansion of the first term is therefore given by

$$[\partial_\mu^D \bar{B}^\mu \mathbf{G}^\mu_1 (y, x)]$$

(154)

$$= (B_3(y)(X_1\delta_\mu(X) - G_1(-X)) + 2\mu^2 X_2\delta_\mu(X)) + O(X^2)$$

(155)

with $X_i = x_i - y_i$.

The second term of (149) is given by

$$[\partial_\mu^D \bar{B}^\mu \mathbf{G}^\mu_1 (x, y)] = (\partial_\mu^D \bar{B}^\mu \Lambda_1(x, y))G^\mu_1(x, y) - (\bar{D}^\mu \Lambda_1(x, y))\delta_\mu(x, y) + (\partial_\mu^D \Lambda_1(x, y))\partial_\mu^G G^\mu_1(x, y) - \Lambda_1(x, y)\partial_\mu^\delta \delta_\mu(x, y),$$

with

$$\bar{D}_2 \Lambda_1(x, y) = \partial_\mu^D (\Lambda_1 H_2(x, y))H_2^{-1}(y)$$

(156)

$$= H_2(y_1, x_2, x_3)H_3(y_1, y_2, y_3)[\tilde{J}_2(y_1, y_2, x_3) - \tilde{J}_2(y_1)]H_1(y)$$

$$= X_3(H_2 B_1 H_2^{-1})(y) + O(X^2)$$

(157)

and

$$\partial_\mu^D D_2 \Lambda_1(x, y) = X_3[H_2 \bar{D}_1 B_1 H_2^{-1}](y) + O(X^2),$$

$$\partial_1 \Lambda_1(x, y) = X_2 B_3(y) - X_3(H_1^{-1} B_2 H_1)(y) + O(X^2),$$

(158)

(159)

with $\bar{D}_i = \partial_i + \tilde{J}_i$. Thus, up to terms of order $1/\mu$ we have

$$II_2(y) = \int dX \text{Tr}_{ad} \left( [B_3(y)(X_1\delta_\mu(X) - G_1(-X)) + 2\mu^2 X_2\delta_\mu(X)] \right)$$

(160)

$$\times \left[ X_3\{H_2 \bar{D}_1 B_1 H_2^{-1}G^\mu_1(X) - H_2 B_1 H_2^{-1}\delta_\mu(X) - H_1 B_2 H_1 2\mu^2 G^\mu_1(X) \} + B_3 2\mu^2 X_2^2 G^\mu_1(X) - 2\mu^2 X_2\delta_\mu(X) \right]$$

Due to parity symmetry the term proportional to $X_3$ do not contribute, we are left with

$$II_2(y) = \text{Tr}_{ad}(B_3^2(y)) \int dXX_2^2G_1(X)(X_1\delta_\mu(X) - G_1(-X))$$

$$- 4\mu^4 \text{Tr}_{ad}(1) \int dXX_2^2\delta_\mu(X)^2$$

$$= \frac{\mu}{4(2\pi)^2} \left( -2N \text{Tr}(B_3^2(y)) - 4\mu^4(N^2 - 1) \right).$$

This proves (111).
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