Understanding the evolution of transverse-momentum dependent parton
densities

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Different approaches to define transverse-momentum dependent parton distribution functions are considered from the point of view of their renormalization-group properties. The associated one-loop anomalous dimensions of these quantities are presented and compared to each other. We give arguments in favor of the “pure light-like” definition, and the use of the light-cone gauge.

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Transverse-momentum dependent parton densities, or distribution functions (TMD PDFs), are used in the study of semi-inclusive hadronic reactions, e.g., semi-inclusive deep-inelastic scattering (SIDIS), or the Drell-Yan (DY) process, \((\gamma^*(q) + h_1(P) \rightarrow h_2(P') + X)\), \((h_1(P) + h_2(P') \rightarrow X)\), in order to describe the inner structure of hadrons by taking into account the longitudinal, as well as the transversal, partonic degrees of freedom \([1, 2, 3, 4]\). The explicit operator definition of TMD PDFs is quite problematic, in particular, due to a number of extra (in contrast to the ordinary integrated distributions) singularities, and is nowadays under active investigation (see, e.g., \([5, 6, 7, 8, 9, 10, 11, 12]\)). In the present work, we discuss different operator definitions of TMD PDFs from the point of view of their renormalization-group (RG) properties.

Let us start with three different definitions of a general unintegrated quark distribution:

\[
\bar{F}_{[n]}(x, k_\perp; \mu, \eta) = \frac{1}{2} \int \frac{d\xi^- d\xi_\perp}{2\pi(2\pi)^2} e^{-ik^- \xi^- + ik_\perp \cdot \xi_\perp} \langle \bar{h}(\xi^- , \xi_\perp) | [\xi^- , \xi_\perp ; \infty^- , \xi_\perp]^\dagger_n \rangle ,
\]

\[
\bar{F}_{[v]}(x, k_\perp; \mu, \zeta) = \frac{1}{2} \int \frac{d\xi^- d\xi_\perp}{2\pi(2\pi)^2} e^{-ik^- \xi^- + ik_\perp \cdot \xi_\perp} \langle h(\xi^- , \xi_\perp) | [\xi^- , \xi_\perp ; \infty^- , \xi_\perp]^\dagger_v \rangle x[\xi^- , \xi_\perp ; \infty^- , \xi_\perp]^\dagger_v (\infty^- , \infty_\perp ; 0^- , 0_\perp) \psi(0^- , 0_\perp) | h \rangle ,
\]

\[
\bar{F}_{[v_0]}(x, k_\perp; \mu, \zeta_0) = \frac{1}{2} \int \frac{d\xi^- d\xi_\perp}{2\pi(2\pi)^2} e^{-ik^- \xi^- + ik_\perp \cdot \xi_\perp} \langle h(\xi^- , \xi_\perp) | [\xi^- , \xi_\perp ; 0^- , 0_\perp]_{v_0} \psi(0^- , 0_\perp) | h \rangle ,
\]

where in all cases gauge invariance is ensured by means of the path-ordered contour-dependent Wilson-line operators (gauge links) with the generic form

\[
[y, x]_r = \mathcal{P} \exp \left[ -ig \int_{\tau_1}^{\tau_2} \, d\tau \rho^\mu A_\mu^r(\tau) \tau^a \right] , \quad \tau^\mu \tau_1 = x , \quad \tau^\mu \tau_2 = y ,
\]

and one has to distinguish between longitudinal \([,]_{[n,v,v_0]}\) and transversal \([,]_{[t]}\) gauge links \([7, 8]\). In general, the state \(|h\rangle\) is a hadron with momentum \(P\) and spin \(S\), but for the sake of simplicity we restrict ourselves in what follows to the “distribution of a quark in a quark with momentum \(p^\perp\)” sufficient for the investigation of the ultraviolet (UV) behavior (omitting color and flavor indices). The transverse gauge links, extending to light-cone infinity, are key elements in Eqs. (11–13) in order to ensure full gauge invariance, while the dependence on the hard momentum scale(s) (e.g., \(Q^2\)) is taken into account via appropriate evolution

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equation(s) (more later). Below, we present the results for the functions \(1\)–\(3\), when projected onto the \(\gamma^+\)-matrix: \(\mathcal{F} \equiv \frac{1}{2} \text{Tr} \left[\gamma^+ F\right]\). Additional variables are introduced in order to regularize the extra singularities of the TMD PDFs, viz.,

\[
\zeta = \frac{4(P \cdot v)^2}{v^2}, \quad \zeta_0 = \frac{4(P \cdot v_0)^2}{v_0^2}, \tag{5}
\]

whereas \(\eta\) is a light-cone regularization parameter to be defined later. Note that definition \(3\) involves the “direct” gauge link between two points separated by an essentially non-lightlike distance. From the point of view of kinematics and power-counting, this is forbidden and seems to have nothing to do with the (factorized) semi-inclusive processes under consideration. However, we have included it in our list because it can be used in the analysis of certain model approaches, or in lattice simulations (see the recent papers \([13, 14, 15]\)).

The definition given by Eq. \(1\) contains gauge links pointing in the light-cone directions, and depending on the hard scale \(Q^2\). This definition is consistent, at least naively, with the corresponding collinear factorization (though a formal proof is lacking). However, this definition, taken literally, produces—beyond the tree-level—UV-singularities by the standard \(\epsilon\)-expansion procedure and are controlled by renormalization-group evolution equations, e.g., in the fully integrated case it amounts to the DGLAP equation.

2. Pure rapidity divergences, which appear only in the unintegrated case. They cancel in the integrated distributions, but they are present in the TMD case giving rise to logarithmic and double-logarithmic terms of the form \(\sim \ln \zeta\), \(\ln^2 \zeta\); they have to be resumed by a consistent procedure.

3. Overlapping divergences, which contain both UV and soft singularities simultaneously. They are highly undesirable, since they break the correct UV-evolution, and furthermore depend on the parameters of the chosen gauge, thus invalidating the definition of the TMD PDFs with respect to full gauge invariance. In this case, a generic singular term has the form \(\sim \frac{1}{\zeta} \ln \zeta\), meaning that the UV pole \(\epsilon^{-1}\) mixes with a “soft” divergence, regularized by the auxiliary parameter \(\zeta\). This prevents the removal of all UV-singularities by the standard \(R\)-procedure, making it necessary to apply a special generalized renormalization procedure.

The latter two classes originate, in fact, from the uncompensated light-cone artifacts, which stem either from the lightlike gauge links (in covariant gauges), or from specific terms in the gluon propagator in the (singular) light-cone axial gauges. Let us emphasize that, while the singularities of the third class may be simply regularized by some (rapidity) cutoff, which is “separated” from other variables, the effect of the second class is more severe, because these singularities affect the UV-renormalization procedure, change the anomalous dimensions, and modify, therefore, the RG-evolution.

In order to avoid the above-mentioned problems, the following approaches have been proposed in the literature:

1. Shift in covariant gauges the gauge links off the light-cone: \(v^2 < 0\), \(v^+ \ll v^-\), or use instead the non-lightlike axial gauge \((v \cdot A) = 0\), \(v^2 < 0\). This amounts to definition \(2\).

2. Stay on the light-cone, Eq. \(1\), but subtract some specific soft factor \(R\), which is defined in such a way as to exactly cancel the extra divergences \([16, 17, 18]\). Thus, definition \(1\) is substituted by the “subtracted” function \(\mathcal{F}_n[n] \rightarrow \mathcal{F}_0[n] \cdot R^{-1}\).

3. Perform a direct regularization of the light-cone singularities in the gluon propagator \([19]\)

\[
\frac{1}{q^\pm} \rightarrow \frac{1}{|q^\pm|\eta}, \tag{6}
\]

where \(\eta\) is an additional dimensional parameter \([11]\). In this case, a generalized renormalization is in order, which is formally equivalent to multiplying the TMD PDF by a particular soft factor \([20]\): \(\mathcal{F}_n(n) \rightarrow \mathcal{F}_n(\eta) \cdot R^{-1}(\eta)\). The introduction of the small parameter \(\eta\) allows one to keep the overlapping singularities under control and treat the extra term in the UV-divergent part by means of the cusp anomalous dimension, which in turn determines the specific form of the gauge contour in the soft factor \(R\).
4. Still use the light-cone axial gauge, but supply it with the Mandelstam-Leibbrandt pole prescription \cite{21,22,23,24,25,26,27}:

\[
\frac{1}{q^+} \rightarrow \frac{1}{q^+ + i0q^−} \quad \text{or} \quad \frac{q^−}{q^+q^− + i0} .
\] (7)

Now the overlapping singularities do not appear at all, at least at the level of the one-loop order, while the contribution of the soft factor is reduced to unity, rendering the gauge-invariant definition valid \cite{28}.

Let us now list the UV-renormalization-group equations for the above definitions. The off-the-light-cone TMD PDFs \cite{2} and \cite{3} do not contain overlapping singularities. Therefore, the only source to produce their UV-divergences, when the two quark field operators are separated by a non-lightlike distance, are the divergences of these operators themselves and those entailed by the non-lightlike gauge links. Therefore, the renormalization-group equation reads \cite{29,30}:

\[
\mu \frac{d}{d\mu} F_{[v,v_0]} = \gamma_{\text{LC}} F_{[v,v_0]} , \quad \gamma_{\text{LC}} = \frac{3}{4} \frac{\alpha_s C_F}{\pi} + O(\alpha_s^2) ,
\] (8)

where \(\gamma_{\text{LC}}\) is the anomalous dimension of the two-fermion operator in the light-cone gauge. If one factorizes out the soft contribution \(R_v\), as it was proposed in Ref. \cite{29}, then the anomalous dimension changes and one has

\[
F_{[v]} \rightarrow F_{[v]} \cdot R_v^{-1} , \quad \mu \frac{d}{d\mu} \left[ F_{[v]} \cdot R_v^{-1} \right] = (\gamma_{\text{LC}} - \gamma_R) \left[ F_{[v]} \cdot R_v^{-1} \right] ,
\] (9)

where \(\gamma_R\) is the one-loop anomalous dimension of the soft factor \(R_v\).

In contrast, the anomalous dimension of the “light-cone” TMD PDF, before subtraction, deviates from \(\gamma_{\text{LC}}\) and this deviation is determined by the cusp anomalous dimension \cite{11}:

\[
\mu \frac{d}{d\mu} F_{[n]} = (\gamma_{\text{LC}} - \gamma_{\text{cusp}}) F_{[n]} .
\] (10)

Hence, the generalized renormalization procedure restores the “broken” anomalous dimensions, so that one finds

\[
F_{[n]}(\eta) \rightarrow F_{[n]}(\eta) \cdot R^{-1}(\eta) , \quad \mu \frac{d}{d\mu} \left[ F_{[n]} \cdot R^{-1} \right] = \gamma_{\text{LC}} \left[ F_{[n]} \cdot R^{-1} \right] .
\] (11)

In the light-cone gauge with the Mandelstam-Leibbrandt prescription, definition \cite{11} yields an anomalous dimension without lightlike artifacts from the very beginning, i.e.,

\[
\mu \frac{d}{d\mu} \left[ F_{[n]}^{\text{ML}} \cdot R^{-1} \right] = \mu \frac{d}{d\mu} F_{[n]}^{\text{ML}} = \gamma_{\text{LC}} \left[ F_{[n]}^{\text{ML}} \cdot R^{-1} \right] = \gamma_{\text{LC}} F_{[n]}^{\text{ML}} .
\] (12)

**Conclusions.** We presented the one-loop UV anomalous dimension of the TMD PDFs, defined in different ways expressed via Eqs. \cite{11,13}. These anomalous dimensions can be used to construct corresponding evolution equations, while the evolution in the rapidity variable—either \(\zeta\), or \(\eta\), depending on the approach applied—constitutes a separate task which will be considered elsewhere. We have also shown that using the “completely lightlike” definition \cite{11} in the light-cone gauge in conjunction with the Mandelstam-Leibbrandt pole prescription appears to be the most economic approach in the sense that it avoids—at least in the one-loop order—undesirable overlapping divergences.

**Open questions.** Let us sketch a couple of important and still unsolved problems.

(i) A major task within the TMD approach concerns the factorization of semi-inclusive processes. An all-order factorization (in a covariant gauge) was studied in Ref. \cite{29}, but in this work, the definition \cite{2} was used which contains off-the-light-cone gauge links. No explicit proof of a factorization theorem with pure lightlike TMD PDFs is known at present.

(ii) Another problem pertains to the relationship between unintegrated TMD PDFs and Feynman distribution functions, the latter appearing in factorized inclusive processes (e.g., DIS). After integrating over the transverse momentum \(k_\perp\) of a parton, one would expect that the standard integrated distribution is reproduced from the TMD PDF. It has been shown in Refs. \cite{11,28} that the “pure light-like” definition \cite{11}...
does indeed yield, after integration, an $x$-dependent distribution function that obeys the DGLAP evolution equation:

$$\int d^2 k_\perp F_{[n]}(x, k_\perp, \mu) = F_{[n]}(x, \mu), \quad \mu \frac{d}{d\mu} F_{[n]} = K_{\text{DGLAP}} \otimes F_{[n]}.$$  \hfill (13)

The reason is that the regularization via (6) and (7) does not break the light-cone properties of the function and, whence, the overlapping singularities in the real and virtual gluon contributions cancel against each other after the $k_\perp$-integration. On the other hand, performing the $k_\perp$-integral in the function (2), one cannot even expect to obtain a distribution having DGLAP evolution. In contrast, one gets a function containing off-the-light-cone gauge links along the vector $v = (v^+, v^-, 0_\perp)$, i.e.,

$$\int d^2 k_\perp F_{[v]}(x, k_\perp, \mu) = F_{[v]}(x, \mu), \quad \mu \frac{d}{d\mu} F_{[v]} = K_v \otimes F_{[v]}, \quad K_v \neq K_{\text{DGLAP}}.$$  \hfill (14)

The RG-properties of this object differ from those of the distribution with lightlike gauge links (which fulfills the DGLAP equation), the reason being that the latter produce specific UV singularities without a possibility to perform a regular transition from off-the-light-cone gauge links back to pure light-cone gauge links (see, e.g., Ref. [27]). Thus, applying definition (14), one should keep in mind that the relation to the integrated distribution is, at least, obscure.

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