LOGARITHMIC DERIVATIVES
OF LEAST DEVIATION FROM ZERO

PETR CHUNAEV

Abstract. We study least deviation of logarithmic derivatives of real-valued polynomials with a fixed root from zero on the segment $[-1; 1]$ in the uniform norm with the weight $\sqrt{1-x^2}$ and without it. Basing on results of Komarov and Novak and on a certain determinant identity due to Borchardt, we also establish a criterion for best uniform approximation of continuous real-valued functions by logarithmic derivatives in terms of a Chebyshev alternance.

1. Introduction

Logarithmic derivatives of algebraic polynomials (abbreviated to l.d.s in what follows) are rational functions of the form

$$
\rho_0(z) \equiv 0, \quad \rho_n(z) = \frac{P_n'(z)}{P_n(z)} = \sum_{k=1}^{n} \frac{1}{z-z_k}, \quad z, z_k \in \mathbb{C}, \quad n \in \mathbb{N},
$$

where $P_n(z) = \prod_{k=1}^{n}(z-z_k)$ and $n$ is the degree of the l.d. Note that l.d.s are also widely called simple partial fractions.

In rational approximation theory, attention to l.d.s was paid by works of Macintyre and Fuchs [1], Gonchar [2] and Dolzhenko [3]. Later on, approximating properties of l.d.s were studied by Korevaar [4] and Chui [5], who proposed a construction of l.d.s for approximation of analytic functions belonging to the Bergman-Bers class on simply connected domains. Besides, the poles of the l.d.s were chosen on the boundaries of the domains. Such an approximation was motivated by the fact that l.d.s specify plane electrostatic fields, consequently, it can be considered as a determination of locations $z_k$ of electrons creating a given field to a high accuracy.

Further investigations of approximating properties of l.d.s were initiated by Gorin [6], who brought up a problem of least deviation of l.d.s with a fixed pole from zero on the real line. At various times this was studied by many mathematicians (see [7] and the references given there). Then the following analogue of classical Mergelyan’s theorem on polynomial approximations was proved for l.d.s with free poles: every function continuous in the compact set $K \subset \mathbb{C}$ with connected complement and analytic in the interior of $K$ can be approximated uniformly on $K$ with l.d.s [8]. In many respects this result entailed current intensive study of approximating properties of l.d.s and several their modifications on bounded and unbounded subsets of the complex plane (see [9][23]).

Key words and phrases. Logarithmic derivatives of polynomials, best approximation, alternance, Markov-Bernstein type inequalities.
Hereinafter we consider only real-valued l.d.s $\rho_n = \rho_n(x)$, $x \in \mathbb{R}$ (sets of their poles are symmetric with respect to the real line). Moreover, from now on we set
\[
\|\rho_n\| := \max_{x \in I} |\rho_n(x)|, \quad \|\rho_n\|^* := \max_{x \in I} |\sqrt{1-x^2}\rho_n(x)|, \quad I := [-1; 1].
\]

Main results (§§2–4) of our paper are devoted to the following problem stated in [12].

Among all real-valued l.d.s of degree less than or equal to $n$ with a fixed pole find the l.d.s $\tilde{\rho}_n$ and $\bar{\rho}_n$ of least deviation from zero on $I$ in the norms $\| \cdot \|$ and $\| \cdot \|^*.$

This extremal problem is one of possible analogues of classical Chebyshev’s question about unitary polynomials of least deviation from zero on $I$ and an analogue of Gorin’s problem stated above in the case of a finite segment. We solve it in the class, being denoted by $\mathfrak{R}_n(a)$, of (real-valued) l.d.s with the fixed real pole $x = a$. Note that all results for $I$ can be extended on the case of any segment with the substitution $y = \mu x + \nu$, $\mu > 0$. At that it should be taken into account that $\rho_n(x) = \mu \rho_n(y)$, therefore $\|\rho_n\|_{I_{\mu,\nu}} = \mu^{-1}\|\rho_n\|_I$, where $I_{\mu,\nu} := [-\mu + \nu; \mu + \nu]$.

Basing on results of Komarov [20] and Novak [22] and on a certain determinant identity due to Borchardt [27], we also establish a criterion for best uniform approximation of continuous real-valued functions by l.d.s in terms of a Chebyshev alternance\footnote{Let us recall several definitions. Let $f = f(x)$ be a continuous function in $I$. Least deviation of l.d.s $\rho_n$ from $f$ on $I$ is the quantity $\inf_{\rho_n} \|f - \rho_n\|$, where the infimum is taken over all l.d.s $\rho_n = \rho_n(x)$. The l.d. $\tilde{\rho}_n$, least deviating from $f$ on $I$, is called a l.d. of best uniform approximation of $f$ on $I$. It is said that $n$ pairwise distinct points $t_k$ in $I$ form a (Chebyshev) alternance of the difference $f - \rho_n$ on $I$ if $f(t_k) - \rho_n(t_k) = \pm(-1)^k \|f - \rho_n\|$, $k = 1, \ldots, n$.} (see [25] for more details).

2. L.d.s of least deviation from zero on $I$ in the norm $\| \cdot \|^*$

In [12], there was constructed the l.d.
\[
\sin_n(A; x) = \frac{2nw}{w^2 - 1} \frac{A(2w^2 - 1)}{(Aw^2 - 1)(w^2 - A)}, \quad x = \frac{1}{2} \left( w + \frac{1}{w} \right), \quad w = e^{i\varphi},
\]
where $A \geq 1$, $x \in I$, $\varphi \in \mathbb{R}$. It was also shown there that the function $\sqrt{1-x^2}\sin_n(A; x)$ has an alternance consisting of $n$ point in $I$, and $\|\sin_n\|^* = 2nA(A^2 - 1)^{-1}$. The presence of the alternance suggests that the l.d. (2) is actually the one, which is a solution to the problem stated above with the norm $\| \cdot \|^*$. And so we prove in this section. We use the following important property of the l.d. (2) below: all its poles belong to the ellipse
\[
E_p := \{ z : z = p \cos t + \sqrt{p^2 - 1} \sin t i, \quad t \in [0; 2\pi), \quad p > 1 \},
\]
where $p = p(A) := \frac{1}{2} \left( A^{1/n} + A^{-1/n} \right)$ is a parameter. This can be easily deduced from the denominator of (2).

As usually, $T_n(x) = \cos n \arccos x$ stand for the Chebyshev polynomials of the first kind (with a leading coefficient $2^{n-1}$), and $U_n(x) = \frac{1}{n+1} T_{n+1}'(x)$ do for the Chebyshev polynomials of the second kind (with a leading coefficient $2^n$). Properties of the polynomials are well studied (see [20]), therefore we take into account below without any additional explanations for example that $|T_n(x)| \leq 1$ for $x \in I$, $T_n(1) = 1$, $T_n(-1) = (-1)^n$, and $|T_n(x)| > 1$ for $x \in \mathbb{R} \setminus I$.

We now formulate the main result of the section.
Theorem 1. For $n \geq 1$, the l.d.

$$
\hat{\rho}_n^*(a; x) = \frac{U_n-1(x)}{\int_a^x U_n-1(t)dt} = \frac{nU_n-1(x)}{T_n(x) - T_n(a)}, \quad a > \sqrt{2},
$$

is least deviating from zero on $I$ in the norm $\| \cdot \|^*$ among all those belonging to $\Re_n(a)$. Moreover,

$$
\|\hat{\rho}_n^*\|^* = \frac{n}{\sqrt{T_n^2(a) - 1}}.
$$

It is easy to check that the l.d.s $(\ref{eq:ld1})$ and $(\ref{eq:ld2})$ coincide with each other. Nevertheless, we use our representation $(\ref{eq:ld4})$ for convenience of proofs.

Let us first prove several auxiliary results.

Lemma 1. For $a > 1$ and $n \geq 1$, the function $R_n(x) := \sqrt{1 - x^2}\hat{\rho}_n^*(a; x)$ has the following properties.

(a) For $x_k = \cos \frac{2k}{n}, \ k = 0, n$, we have $R_n(x_k) = 0$.

(b) The points $a_k, \ k = 1, n, \ being \ roots \ of \ the \ equation \ (\ref{eq:lemma1})$

$$
T_n(x) = \frac{1}{T_n(a)}.
$$

form an alternance on $I$ of the function $R_n$, and

$$
|R_n(a_k)| = \frac{n}{\sqrt{T_n^2(a) - 1}}, \quad k = 1, n.
$$

(c) The poles of $R_n$ belong to the ellipse $E_a$ of the form $(\ref{eq:ellipse})$.

Proof. It is clear that the numerator of $R_n$ vanishes both in the endpoints of $I$ and in the roots of the polynomial $U_{n-1}$,

$$
U_{n-1}(x) = 0 \iff x = x_k = \cos \frac{2k}{n}, \ \ k = 1, n-1.
$$

This is (a).

To prove (b) we first find the extreme points $e_k$ of $R_n$ in the interval $I_o := (-1; 1)$.

By well-known properties

$$
(x^2 - 1)U_{n-1}(x) = nT_n(x) - xU_{n-1}(x), \quad T_n^2(x) - (x^2 - 1)U_{n-1}^2(x) = 1,
$$

after several simplifications, we obtain

$$
R_n'(x) = -n^2 \frac{1 - T_n(a)T_n(x)}{\sqrt{1 - x^2}(T_n(x) - T_n(a))^2}.
$$

The denominator of the fraction does not vanish on $I_o$ (see (c)), consequently, $e_k, \ k = 1, n, \ are \ solutions \ to \ the \ equality \ (\ref{eq:equation})$. From properties of $T_n$ on $I_o$ and the inequality $0 < 1/T_n(a) < 1$ for $a > 1$, it follows that all $e_k$ belong to $I_o$. Now let us demonstrate that $e_k$ are the points $a_k$ forming the alternance. Alternation in sign of values $R_n(e_k)$ is deduced from $(\ref{eq:ellipse})$. Indeed, the polynomial $T_n(x) - 1/T_n(a)$ attains positive and negative values alternately on $I$ due to properties of $T_n$. Furthermore,

$$
|R_n(e_k)| = \frac{n}{\sqrt{T_n^2(e_k) - T_n^2(a)}} = \frac{n}{\sqrt{T_n^2(e_k) - T_n^2(a)}} = \frac{n}{\sqrt{T_n^2(a) - 1}}.
$$

The conclusion (c) about the poles of the l.d. $(\ref{eq:ld1})$ follows from the property of the l.d. $(\ref{eq:ld2})$ indicated before Theorem 1 (see the text around the formula $(\ref{eq:ellipse})$). \qed
Lemma 2 (Komarov [19]). Let

\[ P_n(x) = \frac{p'(x)}{p(x)}, \quad Q_m(x) = \frac{q'(x)}{q(x)}, \quad m \leq n, \]

where \( p(x) = \prod_{k=1}^{n}(x - z_k) \) has only simple roots \( z_k \) and \( q(x) = \prod_{k=1}^{m}(x - \zeta_k) \).

Then there are constants \( \gamma_k \), such that

\[ P_n(x) - Q_m(x) = \frac{p(x)}{q(x)} \sum_{k=1}^{n} \frac{\gamma_k}{(x - z_k)^2}. \]

The following lemma was proved in [19] under the assumption of validity of a certain determinant identity. We show in [15] that the identity actually holds (Theorem [4]). Relying on it, we formulate the lemma in its final form.

**Lemma 3.** For \( j, k = \overline{1, n} \), let \( z_k \) be pairwise distinct real or complex conjugate numbers satisfying the condition

(8) \[ |z_k| > 1, \]

and \( c_j \) be pairwise distinct numbers belonging to \( I \). Then the determinant of the matrix \( A := ((c_j - z_k)^{-2}) \) does not vanish.

**Proof of Theorem 1.** Let us assume that there exists a l.d. \( \rho_n^*(x) := \pi'(x)/\pi(x) \) of order \( m \leq n \) belonging to \( \mathcal{H}_n(a) \), such that \( \|\rho_n^*\| < \|\tilde{\rho}_n\| \) on \( I \). We set \( r_n(x) := \sqrt{1 - x^2}\rho_n^*(x) \). According to Lemma 1 the function \( R_n(x) = \sqrt{1 - x^2}\tilde{\rho}_n^*(x) \) has an alternance on \( I \) consisting of the points \( a_k, k = \overline{1, n} \), in \( I_o \), therefore

\[ \text{sgn}(R_n(a_k) - r_n(a_k)) = \pm (-1)^k, \quad k = \overline{1, n}. \]

Consequently, the continuous function \( R_n - r_n \) (then the function \( \tilde{\rho}_n^* - \rho_n^* \)) reverses a sign at least \( n \) times on \( I_o \) and thus has at least \( n - 1 \) pairwise distinct roots \( c_j, j = \overline{1, n-1} \), in \( I_o \).

Both \( \tilde{\rho}_n^* \) and \( \rho_n^* \) belong to \( \mathcal{H}_n(a) \), thereby there exist polynomials \( p \) and \( q \) of degree \( n - 1 \) and \( m - 1 \) correspondingly, such that for the denominators of \( \tilde{\rho}_n^* \) and \( \rho_n^* \) we have \( T_n(x) - T_n(a) = p(x)(x - a) \) and \( \pi(x) = q(x)(x - a) \). From this by Lemma 2 we get

\[ \tilde{\rho}_n^*(x) - \rho_n^*(x) = \left( \frac{p'(x)}{p(x)} + \frac{1}{x - a} \right) - \left( \frac{q'(x)}{q(x)} + \frac{1}{x - a} \right) = \frac{p(x)}{q(x)} \sum_{k=1}^{n-1} \frac{\gamma_k}{(x - z_k)^2}, \]

where \( z_k \) are the (simple) roots of \( p \). Hence the points \( c_j, j = \overline{1, n-1} \), are also zeros of the sum \( \sum_{k=1}^{n-1} \gamma_k(x - z_k)^{-2} \) (the polynomial \( p \) has no real roots on \( I \) due to the conclusion (c) of Lemma 1), and the system of linear equations

(9) \[ \sum_{k=1}^{n-1} \frac{\gamma_k}{(c_j - z_k)^2} = \tilde{\rho}_n^*(c_j) - \rho_n^*(c_j), \quad j = \overline{1, n-1}, \]

with respect to the variables \( \gamma_k \) is homogeneous and the determinants \( \Delta_k, k = \overline{1, n-1} \) of the matrices with a replaced \( k \)th column are equal to zero. Let us recall that the poles of \( \tilde{\rho}_n^* \) satisfy the condition (8) here, owing to (c) of Lemma 1 and the assumption \( a > \sqrt{2} \). Hence by Lemma 3 the determinant \( \Delta \) of the matrix of (9) does not vanish, consequently, the system (9) has the trivial solution \( \gamma_k = \Delta_k/\Delta = 0 \), \( k = \overline{1, n-1} \). Thus \( \tilde{\rho}_n^* \equiv \rho_n^* \), which contradicts the assumption that there exists better approximation.

The latter conclusion of Theorem 1 follows from (b) of Lemma 1. \( \blacksquare \)
3. **L.d.s of Least Deviation from Zero on I in the Norm $\| \cdot \|$$**

Several estimates for least deviation of l.d.s from zero in the norm $\| \cdot \|$ were obtained in \[9\]. In particular, it was shown there that if $\| \rho_n \| \leq b^{-n-1}$ for some $b > 2$ then all poles of the l.d. $\rho_n$ lie outside the ellipse $E_{\rho_n}(A)$ of the form \[3\] with $A = (b/2)^n$. The following statement\[4\] is a supplement to this result.

**Theorem 2.** For $n \geq 4$ and $a > \sqrt{2} \cdot (3\sqrt{n})^{1/n}$, let the l.d. $\tilde{\rho}_n(a; x)$ be least deviating from zero on $I$ in the norm $\| \cdot \|^{*}$ among all those belonging to $\mathcal{R}_n(a)$. Then

$$\| \tilde{\rho}_n \| \leq \frac{2n}{T'_n(a) - T_{n-2}(a)}, \quad n \geq 4.$$  

For $n \to \infty$ the sign $\leq$ can be replaced by the sign $\sim$.

We first prove auxiliary results. The following lemma is an analogue of De la Vallée Poussin’s theorem (see \[24\], Ch. 1, §2) for l.d.s belonging to $\mathcal{R}_n(a)$.

**Lemma 4.** Let the l.d. $\rho_n$ belong to $\mathcal{R}_n(a)$ and have the poles $z_k$ being pairwise distinct and satisfying the condition \[3\]. Let there exist $n$ pairwise distinct points $t_k$ in $I$, such that $\rho_n(t_k) = \pm (-1)^k \lambda_k$, $\lambda_k > 0$, $k = 1, \ldots, n$. Then for the l.d. $\tilde{\rho}_n$, least deviating from zero on $I$ in the norm $\| \cdot \|$ among all those belonging to $\mathcal{R}_n(a)$,

$$\min_{k=1,\ldots,n} \lambda_k \leq \| \tilde{\rho}_n \| \leq \| \rho_n \|.$$  

**Proof.** The right hand side inequality is obvious. The non-existence of l.d.s, belonging to $\mathcal{R}_n(a)$, which norms are less than $\min_{k=1,\ldots,n} \lambda_k$ (the left hand side inequality), follows by essentially the same method as in the proof of Theorem \[1\] and we do not produce it. \[\square\]

**Lemma 5.** For $n \geq 4$, the l.d.

(10) $$\tilde{\rho}_n(a; x) = \mathcal{T}(a) = \frac{T_{n-1}(x)}{\int_a^x T_{n-1}(t)dt}, \quad a > 1 + 1/n,$$

has the following properties.

(a) For $a_k = \cos(\frac{k}{n+1} \pi)$, $k = 0, \ldots, n-1$, in $I$, we have $\tilde{\rho}_n(a_k) = \pm (-1)^k \lambda_k$, where $\lambda_k > 0$ and

(11) $$\frac{2n}{n} T_n(a) - \frac{n-2}{n} T_{n-2}(a) + \frac{n-1}{n-2} \leq \lambda_k \leq \frac{2n}{n} T_n(a) - \frac{n-2}{n} T_{n-2}(a) - \frac{n-1}{n-2}.$$  

(b) The poles of $\tilde{\rho}_n$ lie in the closure of the ellipse $E_{\rho_n}$ of the form \[3\]. Moreover, if $t_n(a) := a \cdot (3\sqrt{n})^{-1/n}$ exceeds 1 then all of them lie outside the closure of the ellipse $E_{t_n(a)}$ of the form \[3\].

**Proof.** We set

$$Q_n(a; x) := \int_a^x T_{n-1}(t)dt, \quad f(x) := \frac{1}{n} T_n(x) - \frac{1}{n-2} T_{n-2}(x).$$  

The following well-known identity $Q_n(a; x) = \frac{1}{n} (f(x) - f(a))$ is used below. Throughout the proof $a > 1 + 1/n$ and $n \geq 4$. Simple analysis of $f$ gives

(12) $$- \frac{n-1}{n(n-2)} \leq Q_n(a; x) + \frac{1}{2} f(a) \leq \frac{n-1}{n(n-2)}, \quad x \in I.$$  

\[\text{2As usually, for some positive sequences } \{a_n\} \text{ and } \{b_n\} \text{ the equivalence } a_k \sim b_k \text{ means that } a_n/b_n \to 1 \text{ for } n \to \infty, \text{ and the weak equivalence } a_n \asymp b_n \text{ does that there exist constants } \alpha \text{ and } \beta \text{ such that } 0 < \alpha \leq a_n/b_n \leq \beta < \infty \text{ for } n \geq n_0.\]
From this by the inequality \( \frac{1}{2} f(a) > \frac{n-1}{n(n-2)} \), we get \( Q_n(a; x) < 0 \) for \( x \in I \). Furthermore, in the point \( a_k, k = 0, n - 1 \), being zeros of \( \sqrt{1-x^2}U_{n-2}(x) \) (see (3)), we have \( T_{n-1}(a_k) = \cos k\pi = (-1)^k, k = 0, n - 1 \) since \( T_n(\cos \nu) = \cos n\nu \). It proves the alternation of signs of \( \tilde{P}_n(a_k) \). The identity \( |T_{n-1}(a_k)| = 1 \) and the inequality (12) yield the estimates (11).

Let us now prove (b). We first show that all roots of \( Q_n \) lie in the closure of the ellipse \( E_\alpha \) of the form (3). To do so, we make sure that the inequality \( f(a) < |f(z)| \) holds for all \( z \in I_\alpha, \) where \( \varepsilon > 0 \). Taking into account the identity

\[
T_n(z) = \frac{1}{2} \left( (z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n \right), \quad z \in \mathbb{C},
\]

where \( \sqrt{z^2 - 1} \) is the branch, for which the value equals 1 if \( z = \sqrt{2} \), by direct calculations, we obtain that \( |f(z)| \) attains a minimal value for \( z = \pm 1 \) and a maximal value for \( z = \pm i \sqrt{2} - 1 \) on the ellipse \( E_r := \{ z : |z + \sqrt{z^2 - 1}| = r \} \) with \( r > 1 \) (i.e. when \( z + \sqrt{z^2 - 1} = re^{i\phi}, \phi \in [0; 2\pi] \)). Since the ellipses \( E_{\alpha+\varepsilon} \) and \( E_r \) coincide if \( r = (a + \varepsilon) + (a + \varepsilon)^2 - 1 \), by (13) and monotone increasing of \( f(x) \) for \( x > 1 \), we thus get

\[
|f(z)|_{z \in E_r} \geq f(a + \varepsilon) > f(a), \quad \varepsilon > 0.
\]

Consequently, by Rouche’s theorem, the polynomials \( f(x) \) and \( f(x) - f(a) \) have the same number of roots inside the ellipse \( E_{\alpha+\varepsilon} \). At the same time all roots of the polynomial \( f \) as it can be easily shown, belong to \([-1 - 1/n; 1 + 1/n]\), therefore to the closure of the ellipse \( E_{\alpha+\varepsilon} \) with \( \alpha > 1 + 1/n \). By arbitrariness of \( \varepsilon \), all roots of \( f(x) - f(a) \) (then all roots of \( Q_n \)) lie in the closure of \( E_{\alpha+\varepsilon} \).

We now show that there are no roots of \( Q_n \) in the closure of \( E_{\alpha+n}(a) \) of the form (3) with \( t_n(a) = a \cdot (3\sqrt{n})^{-1/n} \). It is only sufficient to prove that \( |f(z)| < f(a) \) for \( z \in I_{\alpha+n}(a) \). To do so, we make the following observation. On the one hand, the extremal properties of \( |f(z)| \) on \( E_r, r > 1 \), and the representation (12) yield

\[
|f(z)|_{z \in E_{\alpha+n}(a)} \leq \frac{1}{n} T_n(t_n(a)) + \frac{1}{n-2} T_{n-2}(t_n(a)) < \frac{1}{n} (t_n(a) + \sqrt{t_n^2(a) - 1})^n.
\]

From the other hand, \( f(a) > (a + \sqrt{a^2 - 1})^n / (3n^{3/2}) \), which is easy to check. These imply the strengthened inequality

\[
t_n(a) + \sqrt{t_n^2(a) - 1} \leq (3\sqrt{n})^{-1/n} (a + \sqrt{a^2 - 1}),
\]

that is valid for \( t_n(a) \) mentioned, which, however, must exceed 1 (it is so for \( a > (3\sqrt{n})^{1/n} > 1 + 1/n \). Taking into account that \( f(a) \) is a constant, by Rouche’s theorem, we finally conclude that the polynomial \( f(x) - f(a) \) has no roots in the closure of \( E_{\alpha+n}(a) \).

**Proof of Theorem 2.** All the conclusions of Theorem 2 follow from the analogue of De la Vallée Poussin’s theorem for l.d.s (Lemma 1), where \( \rho_n \) is the l.d. \( \tilde{g}_n \) of the form (10), and the estimates (11). The following observation should be only taken into account. Lemma 3 requires the poles of \( \tilde{g}_n \) to satisfy the condition (8). It is easily seen from (b) of Lemma 5 that the condition is satisfied if the semi-minor axis \( \sqrt{t_n^2(a) - 1} \) of the ellipse \( E_{\alpha+n}(a) \) exceeds 1. It gives the condition \( a > \sqrt{2 \cdot (3\sqrt{n})^{1/n}}, n \geq 4 \), in Theorem 2.
4. Corollary of main theorems

In approximation theory, Markov-Bernstein type estimates connecting values of polynomials and their derivatives are well-known (see for instance a wide survey in [20] Ch. 5; Appendix A5]). Now we give a result of this type, which is a corollary of Theorems 1 and 2.

**Colloary.** Let \( n \geq 4 \) and \( P_n(x) = (x-a)p(x) \), where \( a > \sqrt{2} \cdot (3\sqrt{n})^{1/n} \) and \( p = p(x) \) is a polynomial of degree \( n-1 \) having no roots in \( I \). Then

\[
\|P_n'|^* \geq \frac{n \min_{x \in I} |P_n(x)|}{\sqrt{T_n^2(a) - 1}}, \quad \|P_n'\| \geq \frac{2n \min_{x \in I} |P_n(x)|}{T_n(a) - T_n-2(a) + 3}, \quad n \geq 4.
\]

These inequalities are asymptotically precise in the sense that there exist polynomials \( P_{n,1} \) and \( P_{n,2} \) such that for \( n \to \infty \)

\[
\frac{n \min_{x \in I} |P_{n,1}(x)|}{\sqrt{T_n^2(a) - 1}} \sim \|P_{n,1}'|^*, \quad \frac{2n \min_{x \in I} |P_{n,2}(x)|}{T_n(a) - T_n-2(a) + 3} \sim \|P_{n,2}'\|.
\]

**Proof.** From Theorem 1 under the above assumptions, it immediately follows that

\[
\frac{n}{\sqrt{T_n^2(a) - 1}} \leq \frac{\|P_n'|^*}{\|P_n'\|} \leq \frac{\|P_n'|^*}{\min_{x \in I} |P_n(x)|}.
\]

Analogously Theorem 2 (more precisely, the conclusion (a) of Lemma 5 after minor simplifications) yields the inequality for the norm \( \| \cdot \| \). Note that the former inequality in (14) holds already for \( a > \sqrt{2} \) and \( n \geq 1 \).

We now prove the asymptotic precision. For \( P_{n,1}(x) = T_n(x) - T_n(a) \) we have

\[
\|P_{n,1}'\| = n\|U_{n-1}\| = n, \quad \min_{x \in I} |P_{n,1}(x)| = T_n(a) - 1, \quad \text{thereby}
\]

\[
\frac{n \min_{x \in I} |P_{n,1}(x)|}{\sqrt{T_n^2(a) - 1}} = \frac{1 - \frac{2}{T_n(a)} + 1}{\sqrt{T_n^2(a) - 1}} = 1 - o(1), \quad o(1) > 1, \quad n \to \infty.
\]

Furthermore, for \( P_{n,2}(x) = \int_{-1}^x T_{n-1}(t) \, dt \) it is true that

\[
\|P_{n,2}'\| = \|T_{n-1}\| = 1, \quad \min_{x \in I} |P_{n,2}(x)| \geq \frac{1}{2n}(T_n(a) - \frac{n}{n-2}T_n-2(a) - 3) \quad \text{for} \ n \geq 4,
\]

and, consequently,

\[
\frac{2n \min_{x \in I} |P_{n,2}(x)|}{(T_n(a) - T_n-2(a) + 3)} \|P_{n,2}'\| \geq 1 - \frac{2(\frac{n}{n-2}T_n-2(a) - 3)}{T_n(a) - T_n-2(a) + 3} = 1 - o(1), \quad n \to \infty,
\]

where \( o(1) > 0 \).

5. Criterion for best uniform approximation by l.d.s

It is known that approximating properties of \( l.d.s \) and polynomials are similar in many respects. For instance, one has for \( l.d.s \) analogues of classical Jackson’s, Bernstein’s, Zygmund’s, Dzyadyk’s, and Walsh’s theorems for polynomials [8, 21, 22]. However, there are fundamental differences as well. So, generally speaking, there exist no direct connection between an alternance and best approximation in the case of \( l.d.s \). Moreover, a \( l.d. \) of best approximation can be non-unique. A pioneering example of this kind was given for \( n = 2 \) in [14], then it was extended in [17] to arbitrarily positive integer \( n \). Connection between best approximation by \( l.d.s \) and an alternance in the general case is still a question. Nevertheless, several analogues of Chebyshev’s alternance theorem can be obtained for \( l.d.s \) with poles under certain restrictions. In particular, the following statement is valid.
Theorem 3. Let all the poles of the l.d. \( \rho_n = \rho_n(x) \) be pairwise distinct and satisfy the condition \( \text{(8)} \). In this case \( \rho_n \) is a unique l.d. of best uniform approximation of the continuous function \( f = f(x) \) on \( I \) if and only if there exist \( n+1 \) points forming an alternance of the difference \( f - \rho_n \) on \( I \).

This theorem was formulated and proved by Komarov in \( \text{[19, 20]} \) under the essential assumption consisting in validity of the following determinant identity due to Borchardt (it was considered in \( \text{[19, 20]} \) as a conjecture).

Theorem 4 (Borchardt \( \text{[27, 28]} \)). For \( j, k = 1, n \), let \( \{z_k\} \) and \( \{c_j\} \) be arbitrary disjoint collections of complex numbers and \( A := ((c_j - z_k)^{-2}) \) and \( B := ((c_j - z_k)^{-1}) \) be matrices constructed on basis of them. Then

\[
\det A = \det B \cdot \text{per } B,
\]

where \( \text{per } B \) is a permanent\(^3\) of the matrix \( B \).

In conclusion we note that in the even earlier paper \( \text{[22]} \), Novak, considering Theorem 4 also as a conjecture, obtained an analogue of Theorem 3 for the more narrow class of l.d.s with pairwise distinct real poles out of \( I \).

References

[1] A. Macintyre, W. Fuchs, Inequalities for the logarithmic derivatives of a polynomial, J. London Math. Soc. S1-15 (3) (1940) 162-168.
[2] A.A. Goncharch, On best approximations by rational functions, Dokl. Akad. Nauk SSSR 100 (1955) 205-208 [Russian].
[3] E.P. Dolzhenko, Estimates of derivatives of rational functions, Izv. Akad. Nauk SSSR Ser. Mat. 27 (1) (1963) 9-28 [Russian].
[4] J. Korevaar, Asymptotically neutral distributions of electrons and polynomial approximation. Ann. of Math. 80 (1964) 403-410.
[5] C.K. Chui, On approximation in the Bers spaces, Proc. Amer. Math. Soc. 40 (1973) 438-442.
[6] E.A. Gorin, Partially hypoelliptic partial differential equations with constant coefficients, Siberian Math. J. 4 (1962) 506-508.
[7] V.I. Danchenko, Estimates of the distances from the poles of logarithmic derivatives of polynomials to lines and circles, Sb. Math. 82 (2) (1995) 425-440.
[8] V.I. Danchenko, D.Ya. Danchenko, Approximation by simplest fractions, Math. Notes 70 (4) (2001) 502-507.
[9] P.A. Borodin, Approximation by simple partial fractions with constraints on the poles, Sb. Math. 203 (11) (2012) 1553-1570.
[10] P.A. Borodin, O.N. Kosukhin, Approximation by the simplest fractions on the real axis, Moscow Univ. Math. Bull. 60 (1) (2005) 1-6.
[11] P.V. Chunaev, On a nontraditional method of approximation, Proc. Steklov Inst. Math. 270 (2010) 278-284.
[12] V.I. Danchenko, Estimates of derivatives of simplest fractions and other questions, Sb. Math. 197 (4) (2006) 505-524.
[13] V.I. Danchenko, E.N. Kondakova. Chebyshevs alternance in the approximation of constants by simple partial fractions, Proc. Steklov Inst. Math. 270 (2010) 80-90.
[14] V.I. Danchenko, P.V. Chunaev, Approximation by simple partial fractions and their generalizations, J. Math. Sci. 176 (6) (2011) 844-859.
[15] D.Ya. Danchenko, Some question of approximation and interpolation by rational functions. Application to equations of elliptic type, Cand. Sci. Thesis, Vladimir State Pedagogical University, Vladimir, 2001 [Russian].
[16] I.R. Kayumov, Convergence of series of simple partial fractions in \( L_p(\mathbb{R}) \), Sb. Math. 202 (10) (2011) 1493-1504.

\(^3\) Let us recall the definition of a permanent. For any square matrix \( M = (m_{i,j}) \), i,j = 1, n, per \( M = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)} \), where \( S_n = \{1, \ldots, n\} \) (cf. \( \det M = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} \)).
[17] M.A. Komarov, An example of non-uniqueness of a simple partial fraction of the best uniform approximation, Rus. Math. (Izv. VUZ. Math.) 57 (9) (2013) 22-30.
[18] M.A. Komarov, A criterion for the best approximation of constants by simple partial fractions, Math. Notes, 93 (2) (2013) 250-256.
[19] M.A. Komarov, Sufficient condition for the best uniform approximation by simple partial fractions, J. Math. Sci. 189 (3) (2013) 482-489.
[20] M.A. Komarov, Analogue of Haar’s condition for simple partial fractions, J. Math. Sci. (2014) to appear.
[21] O.N. Kosukhin, Approximation properties of the most simple fractions, Moscow Univ. Math. Bull. 56 (4) (2001) 36-40.
[22] Ya.V. Novak, Approximation and interpolation properties of simple partial fractions, Cand. Sci. Thesis, Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev, 2009 [Ukrainian].
[23] V.Yu. Protasov, Approximation by simple partial fractions and the Hilbert transform, Izv. Math. 73 (2) (2009) 333-349.
[24] V.K. Dzyadyk, Introduction into the Theory of Uniform Approximation of Functions by Polynomials, Nauka, Moscow, 1977 [Russian].
[25] J.C. Mason, D. Handscomb, Chebyshev Polynomials, Chapman & Hall / CRC Press, 2003.
[26] P. Borwein, T. Erdelyi, Polynomials and Polynomial Inequalities, Graduate Texts in Mathematics, 161, Springer-Verlag, New York, 1995.
[27] C.W. Borchardt, Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Funktion, Crelles Journal 53 (1855) 193-198 [German].
[28] H. Minc, Permanents, Reading, MA: Addison-Wesley, 1978.

CENTRE DE RECERCA MATEMÀTICA, CAMPUS DE BELLATERRA, EDIFICI C, 08193 BELLATERRA (BARCELONA) SPAIN
E-mail address: chunayev@mail.ru