On Müntz-type formulas related to the Riemann zeta function

Hélder Lima

ABSTRACT. The Mellin transform and several Dirichlet series related with the Riemann zeta function are used to deduce some identities similar to the classical Müntz formula [1]. These formulas are derived in the critical strip and in the half-plane $\text{Re}(s) < 0$. As particular cases, integral representations for products of the gamma and zeta functions are exhibited.

Keywords: Arithmetic functions, Dirichlet series, Mellin transform, Riemann zeta function, Euler gamma function, Müntz formula, Müntz-type formulas.

AMS Subject Classifications: 11M06, 11M26, 33B15, 42A38, 42B10, 44A05.
1 Introduction

The Mellin transform $\mathcal{M}$ of a function $f$ is defined by

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx$$

(1.1)

and its inverse transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)x^{-s}ds.$$  

(1.2)

The following proposition establishes reciprocity between (1.1) and (1.2) under favourable conditions (see section 1.29 of [3]).

**Proposition 1.1.** Suppose that $a, b \in \mathbb{R}$, $a < b$, $f^*(s)$ is an analytic function in the strip $a < \text{Re}(s) < b$ such that, for each $a < \sigma < b$, $f^*(s) \in L_1(\sigma \pm i\infty) \equiv L_1(\sigma - i\infty, \sigma + i\infty)$ and $f(x)$ is defined by (1.2). Then $f(x)x^{\sigma-1} \in L_1(0, \infty)$, for all $a < \sigma < b$, and the Mellin transform of $f$ is equal to $f^*(s)$ in the strip $a < \text{Re}(s) < b$.

As it is known, the Riemann zeta function $(\zeta(s))$ is analytic in the entire complex plane except the point $s = 1$, where it has a simple pole such that $\text{res}_{s=1} \zeta(s) = \lim_{s \to 1}(s-1)\zeta(s) = 1$.

Moreover, we have the following representations of expressions involving the Riemann zeta function in form of Dirichlet series absolutely convergent in the half-plane $\text{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s};$$  

(1.3)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s};$$  

(1.4)

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, k \in \mathbb{N};$$  

(1.5)

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s};$$  

(1.6)
\[
\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}; (1.7)
\]

\[
\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}; (1.8)
\]

\[
\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d(n))^2}{n^s}. (1.9)
\]

These Dirichlet series involve several arithmetic functions. The Möbius function \(\mu(n)\) is defined by \(\mu(1) = 1; \mu(n) = (-1)^k\), if \(n\) is the product of \(k\) distinct primes; and \(\mu(n) = 0\), if there exists any prime \(p\) such that \(p^2 | n\). The function \(\omega(n)\) represents the number of distinct prime factors of \(n\). The Dirichlet divisor function \((d(n))\) expresses the number of divisors of \(n\). For any fixed \(k \in \mathbb{N}\), \(d_k(n)\) denotes the number of different ways of writing \(n\) as the product of \(k\) natural factors where expressions with the same factors in different orders are counted as distinct. Observe that \(d_2(n) = d(n)\).

Finally (see [2]), fixed \(t_0 > 1\), there exists \(M \in \mathbb{R}^+\) such that, for all \(t \geq t_0\),

\[
|\zeta(\sigma \pm it)| \leq \begin{cases} 
M, & \text{if } \sigma \geq 2; \\
M \ln(t), & \text{if } 1 \leq \sigma \leq 2; \\
Mt^{\frac{1}{2}\sigma} \ln(t), & \text{if } 0 \leq \sigma \leq 1; \\
Mt^{\frac{1}{2}-\sigma} \ln(t), & \text{if } \sigma \leq 0. 
\end{cases} \quad (1.10)
\]
2 A new family of classes of functions

The M"untz-type class of functions $\mathcal{M}_\alpha$, where $\alpha > 1$, is introduced in [6]. Here we generalise this class, defining the following family of classes of functions.

**Definition 2.1.** A function $f(x)$, defined for $x \in \mathbb{R}^+_0$, belongs to the generalized M"untz-type class of functions $\mathcal{M}_{\alpha,k}$, where $\alpha > 1$ and $k \in \mathbb{N}_0$, if $f \in C^{(k)}(\mathbb{R}^+_0)$ and $f^{(j)}(x) = \mathcal{O}(x^{-\alpha-j})$, $x \to \infty$, for all $j = 0, 1, \ldots, k$.

This definition is a generalisation of the class $\mathcal{M}$, where $\mathcal{M}_{\alpha,2} = \mathcal{M}_{\alpha}$, for any $\alpha > 1$. Note that, if $k \geq l$ and $\beta \geq \alpha$, then $\mathcal{M}_{\beta,k} \subseteq \mathcal{M}_{\alpha,l}$.

The next theorem shows that the Mellin transform of a function in a $\mathcal{M}_{\alpha,k}$ class is analytic in the strip $-k < \text{Re} \ s < \alpha$ except some finite (at most $k$) singularity points.

**Theorem 2.2.** Let $f \in \mathcal{M}_{\alpha,k}$. Then, for any $n = 0, 1, \ldots, k$, $f^{(n)}(x)x^{\sigma+n-1} \in L_1(0, \infty)$, for all $-n < \sigma < \alpha$, and the Mellin transform of $f$, $f^*(s)$, is an analytic function in the strip $0 < \text{Re} \ s < \alpha$, which can be analytically continued to the strip $-n < \text{Re} \ s < \alpha$ by

$$f^*(s) = \frac{(-1)^n}{(s)_n} \int_0^\infty f^{(n)}(x)x^{s+n-1}dx,$$

(2.1)

where $(s)_n$ is the Pochhammer symbol, defined by $(s)_0 = 1$ and $(s)_n = s(s+1)\cdots(s+(n-1))$.

Moreover, $f^*(s)$ is analytic in the strip $-k < \text{Re} \ s < \alpha$, except at the points $s = -n$, with $n = 0, 1, \ldots, k-1$, where $f^*(s)$ either has a simple pole with residue $\frac{f^{(n)}(0)}{n!}$, if $f^{(n)}(0) \neq 0$, or has a removable singularity, if $f^{(n)}(0) = 0$.

**Proof.** Fix $n = 0, 1, \ldots, k$. Then, if $-n < \sigma < \alpha$, $f^{(n)}(x)x^{\sigma+n-1} = \mathcal{O}(x^{\sigma+n-1})$, $x \to 0$ and $f^{(n)}(x)x^{\sigma+n-1} = \mathcal{O}(x^{\sigma-\alpha-1})$, $x \to \infty$, so $f^{(n)}(x)x^{\sigma+n-1} \in L_1(0, \infty)$. As a consequence, the integral $\int_0^\infty f^{(n)}(x)x^{s+n-1}dx$ defines an analytic function in the strip $-n < \text{Re} \ s < \alpha$.

In particular, if $n = 0$, it can be deduced that $f^*(s)$ is analytic in the strip $0 < \text{Re} \ s < \alpha$ and its derivatives are obtained differentiating inside the integral (1.1).

Now we derive (2.1) in the strip $0 < \text{Re} \ s < \alpha$, for all $n = 0, 1, \ldots, k$. If $n = 0$, (2.1) coincides with the definition of $f^*(s)$. Otherwise, if $n = 1, 2, \ldots, k$, (2.1) can be deduced from the case $n-1$, using integration by parts and eliminating the integrated terms due to the asymptotic behaviour of $f^{(n-1)}(x)$ at the infinity.
Moreover, \( (2.1) \) gives the analytic continuation of \( f^*(s) \) to the strip \(-k < \text{Re}(s) < \alpha\), except at the zeros of \((s)_k = s(s+1) \cdots (s+k-1)\): the points \( s = -n, \ n = 0,1, \cdots,k-1. \)

Furthermore, \( \lim_{s \to -n} (s+n)f^*(s) = \lim_{s \to -n} \frac{(-1)^{n+1}(s+n)}{(s)_{n+1}} \int_0^\infty f^{(n+1)}(x)x^{s+n}dx. \) Besides that, \( \frac{(s+n)}{(s)_{n+1}} = \frac{1}{(s)_n} \) and \( (-n)_n = (-1)^n n! \) so \( \lim_{s \to -n} \frac{(-1)^{n+1}(s+n)}{(s)_{n+1}} = \frac{(-1)^{n+1}}{(-n)_n} = -\frac{1}{n!} \) and the integral \( \int_0^\infty f^{(n+1)}(x)x^{s+n}dx \) defines an analytic function in the strip \(-n+1 < \text{Re}(s) < \alpha\)

Therefore \( \lim_{s \to -n} \int_0^\infty f^{(n+1)}(x)x^{s+n}dx = \int_0^\infty f^{(n+1)}(x)dx = -f^{(n)}(0). \)

The following proposition establishes an upper bound for the Mellin transform of a function in a \( \mathcal{M}_{\alpha,k} \) class and, as a result, it gives us sufficient conditions for the absolute convergence of its integral over vertical lines of the complex plane.

**Proposition 2.3.** Let \( f \in \mathcal{M}_{\alpha,k}. \) Then, for any \(-k < \sigma < \alpha\), there exists \( C(\sigma) \in \mathbb{R} \) such that \( |f^*(\sigma + it)| \leq \frac{C(\sigma)}{|t|^k} \), for all \( t \in \mathbb{R} \setminus \{0\} \). Moreover, if \( k \geq 2, f^*(s) \in L_1(\sigma \pm i\infty), \) for any \(-k < \sigma < \alpha\) such that \( \sigma \neq 0, -1, \cdots, -(k-1), \) and \( f(x) \) can be represented by \( (1.2) \), for any \( 0 < \sigma < \alpha. \)

**Proof.** For any \( s \in \mathbb{C}, |(s)_k| \geq |\text{Im}(s)|^k \) so, if \(-k < \text{Re}(s) < \alpha\) and \( \text{Im}(s) \neq 0\), we replace \( n = k \) in \( (2.1) \) to deduce that

\[
|f^*(s)| \leq \frac{1}{|(s)_k|} \int_0^\infty |f^{(k)}(x)x^{s+k-1}|dx \leq \frac{1}{|\text{Im}(s)|^k} \int_0^\infty |f^{(k)}(x)||x^{\text{Re}(s)+k-1}|dx. \quad (2.2)
\]

Fix \(-k < \sigma < \alpha. \) Then, by theorem 2.2, \( f^{(k)}(x)x^{\sigma+k-1} \in L_1(0, \infty) \), so we can define \( C(\sigma) = \int_0^\infty |f^{(k)}(x)||x^{\sigma+k-1}|dx \) and we derive that \( |f^*(\sigma + it)| \leq \frac{C(\sigma)}{|t|^k} \), for all \( t \in \mathbb{R} \setminus \{0\}. \)

Furthermore, if \( \sigma \neq 0, -1, \cdots, -(k-1), f^*(s) \) is continuous on the line \( \text{Re}(s) = \sigma. \) As a result, if \( k \geq 2, f^*(s) \in L_1(\sigma \pm i\infty) \) and, because \( f^*(s) \) is analytic in the strip \( 0 < \text{Re}(s) < \alpha, \) \( f(x) \) is the Mellin inverse transform of \( f^*(s) \) in that strip.

\[\square\]
3 Müntz-type formulas in the critical strip

For functions \( f \) with suitable properties (see section 2.11 of [4]), the Müntz formula
\[
\zeta(s) \int_0^\infty f(y)y^{s-1}dy = \int_0^\infty \left( \sum_{n=1}^\infty f(nx) - \frac{1}{x} \int_0^\infty f(t)dt \right) x^{s-1}dx
\]
is valid in the critical strip \( 0 < \text{Re}(s) < 1 \). Here we derive several identities similar to (3.1) in the \( M_{\alpha,k} \) classes.

The following theorem generates, for each Dirichlet series exhibited in the end of our introduction, an equality between an integral over a vertical line in the half-plane \( \text{Re}(s) > 1 \) and a series where appears an arithmetic function (see [5]).

**Theorem 3.1.** Suppose that \( f \in M_{\alpha,k}, k \geq 2, \phi(n) \) is an arithmetic function and \( \Phi(s) \) is defined in the half-plane \( \text{Re}(s) > 1 \) by the absolutely convergent Dirichlet series \( \sum_{n=1}^\infty \frac{\phi(n)}{n^s} \).

Then, for any \( 1 < \sigma < \alpha \), \( \Phi(s)f^*(s) \in L_1(\sigma \pm i\infty) \),
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)f^*(s)x^{-s}ds = \sum_{n=1}^\infty \phi(n)f(nx)
\]
and the Mellin transform of \( \left( \sum_{n=1}^\infty \phi(n)f(nx) \right) \) in the strip \( 1 < \text{Re}(s) < \alpha \) is \( \Phi(s)f^*(s) \).

**Proof.** Fix \( 1 < \sigma < \alpha \). By definition of \( \Phi(s) \),
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)f^*(s)x^{-s}ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n=1}^\infty \frac{\phi(n)}{n^s} f^*(s)x^{-s}ds.
\]

Moreover, the Dirichlet series that defines \( \Phi(s) \) is absolutely convergent in the half-plane \( \text{Re}(s) > 1 \) and, by proposition 2.3 \( f^*(s) \in L_1(\sigma \pm i\infty) \), so \( \Phi(s)f^*(s) \in L_1(\sigma \pm i\infty) \), because
\[
\int_{\sigma-i\infty}^{\sigma+i\infty} |\Phi(s)f^*(s)ds| \leq \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n=1}^\infty \frac{\phi(n)}{n^s} f^*(s)ds \leq \sum_{n=1}^\infty \frac{|\phi(n)|}{n^\sigma} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)ds < \infty.
\]

Next we change the order of summation and integration to obtain
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)f^*(s)x^{-s}ds = \sum_{n=1}^\infty \frac{\phi(n)}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)(xn)^{-s}ds.
\]
Furthermore, again by proposition 2.3, \( f(x) \) is equal to the inverse Mellin transform of \( f^*(s) \) over the vertical line \( \text{Re}(s) = \sigma \), so we can deduce (3.2) from the previous formula.

Finally, using proposition 1.1, we deduce that the Mellin transform of \( \left( \sum_{n=1}^{\infty} \phi(n)f(nx) \right) \) exists and is equal to \( \Phi(s)f^*(s) \) in the strip \( 1 < \text{Re}(s) < \alpha \).

\[ \square \]

To derive our M"untz-type formulas it will be necessary to move integrals of the type on (3.2), first to the critical strip and later to half-plane \( \text{Re}(s) < 0 \). To this purpose, we show the following result similar to the residue theorem for integrals over vertical lines of the complex plane.

**Theorem 3.2.** Suppose that \( a, b, c, d \in \mathbb{R} \), with \( c < a < b < d \), \( F(s) \) is an analytic function in the strip \( c < \text{Re}(s) < d \) except at a point \( s = x_0 \), with \( a < x_0 < b \), and there exists \( t_0 \in \mathbb{R}^+ \) and a continuous function \( g(t) \) integrable at the infinity such that \( |F(s)| \leq g(\text{Im}(s)) \), for all \( s \in \mathbb{C} \) such that \( a \leq \text{Re}(s) \leq b \) and \( |\text{Im}(s)| \geq t_0 \).

Then \( F(s) \in L_1(a \pm i\infty) \), \( F(s) \in L_1(b \pm i\infty) \) and, for any \( x \in \mathbb{R}^+ \),

\[
\text{res}_{s=x_0} (F(s)x^{-s}) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s)x^{-s} ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)x^{-s} ds.
\]  \hspace{1cm} (3.3)

**Proof.** The functions \( F(a + it) \) and \( F(b + it) \) are continuous and upper bounded at infinity by the integrable function \( g(t) \), thus \( F(s) \in L_1(a \pm i\infty) \) and \( F(s) \in L_1(b \pm i\infty) \).

Fix \( x \in \mathbb{R}^+ \). For any \( T \in \mathbb{R}^+ \), we define \( \Omega_T \) as the rectangle (positively oriented) whose sides are segments of the vertical lines \( \text{Re}(s) = a \) and \( \text{Re}(s) = b \) and the horizontal lines \( \text{Im}(s) = T \) and \( \text{Im}(s) = -T \). Then, using the Cauchy residue theorem, we obtain

\[
\text{res}_{s=x_0} (F(s)x^{-s}) = \frac{1}{2\pi i} \int_{\Omega_T} F(s)x^{-s} ds = \frac{1}{2\pi i} \left( \int_{b-iT}^{b+iT} - \int_{a+iT}^{b+iT} - \int_{a-iT}^{a+iT} + \int_{a-iT}^{b-iT} \right) F(s)x^{-s} ds.
\]

Besides that, as \( g(t) \) is continuous and integrable at the infinity, \( \lim_{t \to \infty} g(t) = 0 \) so

\[
\left| \int_{a \pm iT}^{b \pm iT} F(s)x^{-s} ds \right| \leq \int_{a}^{b} |F(u \pm iT)|x^{-u} du \leq \begin{cases} (b - a)x^{-a} g(T), & \text{if } x \geq 1 \\ (b - a)x^{-b} g(T), & \text{if } x \leq 1 \end{cases} \to 0 \quad \text{as} \ T \to \infty.
\]

Finally we pass to the limit \( T \to \infty \) on the formula for \( \text{res}_{s=x_0} (F(s)x^{-s}) \) to deduce (5.3).

\[ \square \]
3.1 Müntz-type formulas in the critical strip involving $\zeta^k(s)$

Here we derive a family of Müntz-type formulas where appears $\zeta^k(s)$, $k \in \mathbb{N}$, in the critical strip $0 < \Re(s) < 1$. We observe that the case $k = 1$ of these identities is the classical Müntz formula and we determine explicitly the case $k = 2$, deducing a Müntz-type formula involving $\zeta^2(s)$.

Fix $k \in \mathbb{N}$ and suppose that $f \in \mathcal{M}_{\alpha,m}$, $\alpha > 1$ and $m \in \mathbb{N}_0$. Then $\zeta^k(s)f^*(s)$ is analytic in the strip $0 < \Re(s) < \alpha$, except at the point $s = 1$ where it has a pole of order at most $k$ (or a removable singularity). Moreover, if $m \geq 2$, using (1.5) and theorem 3.1, we claim that, for all $1 < \sigma < \alpha$, $\zeta^k(s)f^*(s) \in L_1(\sigma \pm i\infty)$ and, if $x \in \mathbb{R}^+$,

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \zeta^k(s)f^*(s)x^{-s}ds = \sum_{n=1}^{\infty} d_k(n)f(nx).$$

(3.4)

The following proposition establishes sufficient conditions to move the integral in (3.4) to the left, using theorem 3.2.

**Proposition 3.3.** Let $k \in \mathbb{N}$ and $f \in \mathcal{M}_{\alpha,m}$, $m \geq 1 + \frac{k}{2}$. Then, if we fix $t_0, u_1, u_2 \in \mathbb{R}$ such that $t_0 > 1$ and $\frac{1}{2} - \frac{m-1}{k} < u_1 < u_2 < \alpha$, there exists a continuous function $g_k(t)$ integrable at the infinity such that $|\zeta^k(s)f^*(s)| \leq g_k(\text{Im}(s))$, if $u_1 \leq \Re(s) \leq u_2$ and $|\text{Im}(s)| \geq t_0$.

**Proof.** Fix $t_0, u_1, u_2 \in \mathbb{R}$ such that $t_0 > 1$ and $\frac{1}{2} - \frac{m-1}{k} < u_1 < u_2 < \alpha$.

For any $u_1 \leq u \leq u_2$,

$$\int_0^{\infty} |f^{(m)}(x)| x^{u+m-1}dx \leq \int_0^{1} |f^{(m)}(x)| x^{u_1+m-1}dx + \int_1^{\infty} |f^{(m)}(x)| x^{u_2+m-1}dx =: C,$$

so, remembering (2.2), we obtain, for all $t > 0$,

$$|f^*(u \pm it)| \leq \frac{1}{t^m} \int_0^{\infty} |f^{(m)}(x)| x^{u+m-1}dx \leq \frac{C}{t^m}.$$  

By (1.10), there exists $M \in \mathbb{R}^+$ such that, for all $t \geq t_0$: $|\zeta(u \pm it)| \leq M \ln(t)$, if $u \geq 1$; $|\zeta(u \pm it)| \leq Mt^\frac{1+u}{2} \ln(t)$, if $0 \leq u \leq 1$; and $|\zeta(u \pm it)| \leq Mt^\frac{1-u}{2} \ln(t)$, if $u \leq 0$.

Therefore, for any $t \geq t_0$ and $u_1 \leq u \leq u_2$, $|\zeta^k(u \pm it)f^*(u \pm it)| \leq g_k(t) = M^k C(\ln(t))^{k} p(u_1)$, where $p(u_1) = -m$, if $1 \leq u_1 < \alpha$; $p(u_1) = -m + \frac{k}{2}(1-u_1)$, if $0 \leq u_1 \leq 1$; and $p(u_1) = -m + \frac{1}{2} - \frac{m-1}{k}$, if $\frac{1}{2} - \frac{m-1}{k} < u_1 \leq 0$. Finally, $p(u_1) < -1$, for any of the possible values for $u_1$, so $g_k(t)$ is a (continuous) function integrable at the infinity.
Suppose that $f \in \mathcal{M}_{\alpha,m}$, $m \geq 1 + \frac{k}{2}$. Fix $0 < c_0 < 1$ and $1 < \sigma < \alpha$. Then theorem 3.2 and proposition 3.3 can be used to deduce that $\zeta^k(s)f^*(s) \in L_1(c_0 \pm i\infty)$ and

$$
\frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \zeta^k(s)f^*(s)x^{-s}ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \zeta^k(s)f^*(s)x^{-s}ds - \text{res}_{s=1} \left( \zeta^k(s)f^*(s)x^{-s} \right).
$$

Besides that,

$$
\text{res}_{s=1} \left( \zeta^k(s)f^*(s)x^{-s} \right) = \int_0^\infty f(xy)P_{k-1}(\ln(y))dy,
$$

where $P_{k-1}(x)$ is a certain monic polynomial of degree $k-1$ (see [6]).

Therefore, remembering (3.4), we obtain

$$
\frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \zeta^k(s)f^*(s)x^{-s}ds = \sum_{n=1}^\infty d_k(n)f(nx) - \int_0^\infty f(xy)P_{k-1}(\ln(y))dy.
$$

Finally, applying proposition 1.1 to (3.5), we derive the following theorem.

**Theorem 3.4.** Let $k \in \mathbb{N}$ and $f \in \mathcal{M}_{\alpha,m}$, $m \geq 1 + \frac{k}{2}$. Then the Müntz-type formula

$$
\zeta^k(s)f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty d_k(n)f(nx) - \frac{1}{x} \int_0^\infty f(t)dt \right) x^{s-1}dx
$$

is valid in the critical strip $0 < \text{Re}(s) < 1$.

If $k = 1$, $P_0(x) = 1$ so

$$
\text{res}_{s=1} \left( \zeta(s)f^*(s)x^{-s} \right) = \int_0^\infty f(xy)dy = \frac{1}{x} \int_0^\infty f(t)dt = \frac{f^*(1)}{x}.
$$

Therefore, because $d_1(n) = 1$, for all $n \in \mathbb{N}$, we deduce from the previous theorem that the Müntz formula (3.1) is valid in the critical strip, for any function $f \in \mathcal{M}_{\alpha,m}$, $m \geq 2$.

If $k = 2$, $P_1(x) = x + 2\gamma$ (where $\gamma$ is the Euler-Mascheroni constant [1]) so

$$
\text{res}_{s=1} \left( \zeta^2(s)f^*(s)x^{-s} \right) = \int_0^\infty f(xy)P_1(\ln(y))dy = 2\gamma \int_0^\infty f(xy)dy + \int_0^\infty f(xy)\ln(y)dy.
$$

Besides that, making a change of variable $t = xy$,

$$
\int_0^\infty f(xy)\ln(y)dy = \frac{1}{x} \int_0^\infty f(t)\ln(t)dt - \frac{\ln(x)}{x} \int_0^\infty f(t)dt = \frac{(f^*)'(1)}{x} - \frac{f^*(1)\ln(x)}{x}.
$$
As a result,
\[
\text{res}_{s=1} \left( \zeta^2(s) f^*(s) x^{-s} \right) = \int_0^\infty f(xy) P_1(\ln(y)) dy = \frac{1}{x} \left( \left( (f^*)(1) + 2 \gamma f^*(1) \right) - f^*(1) \ln(x) \right)
\] (3.8)

and, as \( d_2(n) = d(n) \), for all \( n \in \mathbb{N} \), we derive the M"untz-type formula
\[
\zeta^2(s) f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty d(n) f(nx) + \frac{1}{x} (f^*(1) \ln(x) - ((f^*)'(1) + 2 \gamma f^*(1))) \right) x^{s-1} dx \] (3.9)
valid in the critical strip \( 0 < \Re(s) < 1 \), for any function \( f \in \mathcal{M}_{\alpha,m}, m \geq 2 \).

### 3.2 M"untz-type formulas in the critical strip involving \( \frac{\zeta^k(s)}{\zeta(2s)} \)

In this section we derive M"untz-type formulas where appears \( \frac{\zeta^k(s)}{\zeta(2s)} \), \( k = 1, 2, 3, 4 \).

Let \( k \in \mathbb{N} \) and \( f \in \mathcal{M}_{\alpha,m} \). Then, because \( \frac{1}{\zeta(2s)} \) is analytic in the half-plane \( \Re(s) > \frac{1}{2} \), \( \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \) is an analytic function in the strip \( \frac{1}{2} < \Re(s) < \alpha \) except at the point \( s = 1 \), where it has a pole of order at most \( k \) (or a removable singularity).

Moreover, if \( m \geq 2 \), remembering formulas (1.6)-(1.9) and theorem 3.1 we deduce that, for all \( 1 < \sigma < \alpha \), \( \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \in L_1(\sigma \pm i\infty) \) and, for any \( x \in \mathbb{R}^+ \),
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) x^{-s} ds = \sum_{n=1}^\infty |\mu(n)| f(nx); \] (3.10)
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta^2(s)}{\zeta(2s)} f^*(s) x^{-s} ds = \sum_{n=1}^\infty 2^{\omega(n)} f(nx); \] (3.11)
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta^3(s)}{\zeta(2s)} f^*(s) x^{-s} ds = \sum_{n=1}^\infty d(n^2) f(nx); \] (3.12)
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta^4(s)}{\zeta(2s)} f^*(s) x^{-s} ds = \sum_{n=1}^\infty (d(n))^2 f(nx). \] (3.13)
Let \( u_1 \in \mathbb{R} \) and \( s \in \mathbb{C} \) such that \( u := \Re(s) \geq u_1 > \frac{1}{2} \). Then, using (1.3) and (1.4),

\[
\frac{1}{\zeta(2s)} \leq \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2u}} = \zeta(2u) \leq \zeta(2u_1).
\]

Therefore \( \left| \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \right| \leq \zeta(2u_1) \left| \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \right| \) and we may derive the following result from proposition 3.3, defining \( g_k(t) := \zeta(2u_1)g(t) \), where \( g_k(t) \) is the function obtained in proposition 3.3.

**Proposition 3.5.** Let \( k \in \mathbb{N} \) and \( f \in M_{\alpha,m}, m \geq 1 + \frac{k}{2} \). Then, if we fix \( t_0, u_1, u_2 \in \mathbb{R} \) such that \( t_0 > 1 \) and \( \frac{1}{2} < u_1 < u_2 < \alpha \), there exists a continuous function \( \tilde{g}_k(t) \) integrable at the infinity such that \( \left| \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \right| \leq \tilde{g}_k(\Im(s)) \), if \( u_1 \leq \Re(s) \leq u_2 \) and \( |\Im(s)| \geq t_0 \).

Suppose that \( f \in M_{\alpha,m}, m \geq 1 + \frac{k}{2} \). Fix \( \frac{1}{2} < \epsilon_0 < 1 \) and \( 1 < \sigma < \alpha \). Then theorem 3.2 and proposition 3.5 can be used to deduce that \( \frac{\zeta^k(s)}{\zeta(2s)} f^*(s) \in L_1(\epsilon_0 \pm i\infty) \) and

\[
\frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{\zeta^k(s)}{\zeta(2s)} f^*(s)x^{-s} ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta^k(s)}{\zeta(2s)} f^*(s)x^{-s} ds = \lim_{\epsilon \to 0} \left( \frac{\zeta^k(s)/\zeta(2s)}{f^*(s)x^{-s}} \right) ds = \mathcal{R}_{s=1} \left( \frac{\zeta^k(s)}{\zeta(2s)} f^*(s)x^{-s} \right).
\]

(3.14)

Next we calculate this residue for \( k = 1 \) and \( k = 2 \).

If \( k = 1 \), \( \frac{\zeta(s)}{\zeta(2s)} f^*(s)x^{-s} \) has a simple pole (or a removable singularity) at the point \( s = 1 \) and then, remembering (3.7) and \( \zeta(2) = \frac{\pi^2}{6} \),

\[
\mathcal{R}_{s=1} \left( \frac{\zeta(s)}{\zeta(2s)} f^*(s)x^{-s} \right) = \frac{1}{\zeta(2)} \mathcal{R}_{s=1} \left( \zeta(s)f^*(s)x^{-s} \right) = \frac{6f^*(1)}{\pi^2 x}.
\]

(3.15)

If \( k = 2 \), \( \frac{\zeta^2(s)}{\zeta(2s)} f^*(s)x^{-s} \) has (at most) a double pole at the point \( s = 1 \), so

\[
\mathcal{R}_{s=1} \left( \frac{\zeta^2(s)}{\zeta(2s)} f^*(s)x^{-s} \right) = \lim_{s \to 1} \frac{d}{ds} \left( (s-1)^2 \frac{\zeta^2(s)}{\zeta(2s)} f^*(s)x^{-s} \right)
\]

\[
= \left| \frac{1}{\zeta(2)} \mathcal{R}_{s=1} \left( \zeta^2(s)f^*(s)x^{-s} \right) + \frac{1}{x} f^*(1) \left( \lim_{\epsilon \to 0} \left( \delta \zeta(s) \right) \right)^2 \frac{d}{ds} \left( \frac{1}{\zeta(2s)} \right) \right|.
\]
Besides that \( \zeta'(2) = \frac{\pi^2}{6} \left( \gamma + \ln \left( \frac{2\pi}{A^{12}} \right) \right) \), where \( A \) is the Glaisher-Kinkelin constant, so
\[
\frac{d}{ds} \left( \frac{1}{\zeta(2s)} \right) \bigg|_{s=1} = -2 \frac{\zeta'(2)}{\zeta^2(2)} = \frac{12}{\pi^2} \left( \ln \left( \frac{A^{12}}{2\pi} \right) - \gamma \right). 
\]

Therefore, remembering (3.8), we obtain
\[
\text{res}_{s=1} \left( \frac{\zeta^2(s)}{\zeta(2s)} f^*(s) x^{-s} \right) = \frac{6}{\pi^2 x} \left( (f^*)'(1) + f^*(1) \ln \left( \frac{A^{24}}{4\pi^2 x} \right) \right). 
\]

Finally, the following theorem can be derived from (3.14), (3.10)-(3.11), (3.15)-(3.16) and proposition 1.1.

**Theorem 3.6.** Let \( f \in \mathcal{M}_{\alpha,m}, m \geq 2 \). Then the M"untz-type formulas
\[
\frac{\zeta(s)}{\zeta(2s)} f^*(s) = \int_0^\infty \left( \sum_{n=1}^{\infty} \mu(n) f(nx) - \frac{6 f^*(1)}{\pi^2 x} \right) x^{s-1} dx, 
\]
and
\[
\frac{\zeta^2(s)}{\zeta(2s)} f^*(s) = \int_0^\infty \left( \sum_{n=1}^{\infty} 2^{\omega(n)} f(nx) + \frac{6}{\pi^2 x} \left( f^*(1) \ln \left( \frac{4\pi^2 x}{A^{24}} \right) - (f^*)'(1) \right) \right) x^{s-1} dx 
\]
are valid in the strip \( \frac{1}{2} < \text{Re}(s) < 1 \).

Analogously, using (3.12), (3.13) and (3.14), one may deduce the following theorem (but we will not calculate the residues appearing here).

**Theorem 3.7.** Let \( f \in \mathcal{M}_{\alpha,m}, m \geq 3 \). Then the M"untz-type formulas
\[
\frac{\zeta^3(s)}{\zeta(2s)} f^*(s) = \int_0^\infty \left( \sum_{n=1}^{\infty} d(n^2) f(nx) - \text{res}_{s=1} \left( \frac{\zeta^3(s)}{\zeta(2s)} f^*(s) x^{-s} \right) \right) x^{s-1} dx 
\]
and
\[
\frac{\zeta^4(s)}{\zeta(2s)} f^*(s) = \int_0^\infty \left( \sum_{n=1}^{\infty} d^2(n) f(nx) - \text{res}_{s=1} \left( \frac{\zeta^4(s)}{\zeta(2s)} f^*(s) x^{-s} \right) \right) x^{s-1} dx 
\]
are valid in the strip \( \frac{1}{2} < \text{Re}(s) < 1 \).

This section ends with a remark about how the Riemann hypothesis may affect the strip of validity of the formulas exhibited above.

**Remark 3.8.** If the Riemann hypothesis holds true, the formulas given by theorems 3.6 and 3.7 are not only valid in the strip \( \frac{1}{2} < \text{Re}(s) < 1 \) but in the entire strip \( \frac{1}{4} < \text{Re}(s) < 1 \).
4 Müntz-type formulas in the half-plane \( \text{Re}(s) < 0 \)

In the previous section we moved some integrals of the type on (3.2) to the critical strip \( 0 < \text{Re}(s) < 1 \). Now we move the integrals of \( \zeta^k(s)f^*(s)x^{-s} \) \( (k \in \mathbb{N}) \) to the half-plane \( \text{Re}(s) < 0 \) and we derive some Müntz-type formulas in that half-plane.

Suppose that \( f \in \mathcal{M}_{\alpha,m}, \alpha > 1 \) and \( m \in \mathbb{N} \). By theorem 2.2 \( f^*(s) \) is analytic in the strip \(-m < \text{Re}(s) < \alpha \) except at the points \( s = -j, j = 0, 1, \ldots, m - 1 \), where \( f^*(s) \) either has a simple pole, if \( f^{(j)}(0) \neq 0 \), or a removable singularity, if \( f^{(j)}(0) = 0 \). Besides that, \( \zeta(s) \) is analytic in the entire complex plane except at the point \( s = 1 \) and \( \zeta(-2n) = 0 \), for all \( j \in \mathbb{N} \), so these zeros cancel the eventual simple poles of \( f^*(s) \) at the points \( s = -2n, 1 < 2n < m \). Therefore, for any fixed \( k \in \mathbb{N} \), \( \zeta^k(s)f^*(s)x^{-s} \) is an analytic function in the strip \(-m < \text{Re}(s) < 1 \), except at the points \( s = -j \), with \( j = 0 \) or \( j = 2n - 1 < m \) \( (n \in \mathbb{N}) \), where \( \zeta^k(s)f^*(s)x^{-s} \) either has a simple pole, if \( f^{(j)}(0) \neq 0 \), or a removable singularity, if \( f^{(j)}(0) = 0 \). Moreover, we can use theorem 2.2 to calculate the residues of \( \zeta^k(s)f^*(s)x^{-s} \) at these possible singularities, because

\[
\text{res}_{s=-j} \left( \zeta^k(s)f^*(s)x^{-s} \right) = \zeta^k(-j) x^j \text{res}_{s=-j} f^*(s) = \frac{\zeta^k(-j)}{j!} f^{(j)}(0) x^j, \tag{4.1}
\]

for all \( j \in \mathbb{N}_0 \) such that \( j < m \). In particular, if \( j = 0 \), then, because \( \zeta(0) = -\frac{1}{2} \),

\[
\text{res}_{s=0} \left( \zeta^k(s)f^*(s)x^{-s} \right) = \frac{(-1)^k}{2^k} f(0). \tag{4.2}
\]

Now we move the integral of \( \zeta^k(s)f^*(s)x^{-s} \) to the strip \(-1 < \text{Re}(s) < 0 \) to obtain a Müntz-type formula in that strip.

Let \( k \in \mathbb{N} \) and \( f \in \mathcal{M}_{\alpha,m}, m \geq 1 + \frac{3k}{2} \). Fix \(-1 < \sigma_0 < 0 \) and \( 0 < c_0 < 1 \). Using proposition 3.3 and theorem 3.2 we claim that \( \zeta^k(s)f^*(s) \in L_1(\sigma_0 \pm i\infty) \) and

\[
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta^k(s)f^*(s)x^{-s} ds = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \zeta^k(s)f^*(s)x^{-s} ds - \text{res}_{s=0} \left( \zeta^k(s)f^*(s)x^{-s} \right). \tag{4.3}
\]

Then, remembering (3.5) and (4.2), we deduce that

\[
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta^k(s)f^*(s)x^{-s} ds = \sum_{n=1}^{\infty} d_k(n)f(nx) - \int_{0}^{\infty} f(xy)P_{k-1}(\ln(y))dy + \frac{(-1)^{k+1}}{2^k} f(0). \tag{4.4}
\]
Therefore, the following theorem can be obtained applying proposition 1.1 to (4.4).

**Theorem 4.1.** Let \( k \in \mathbb{N} \) and \( f \in \mathcal{M}_{\alpha,m}, \ m \geq 1 + \frac{3k}{2} \). Then the M"{u}ntz-type formula

\[
\zeta^k(s)f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty d_k(n)f(nx) - \int_0^\infty f(xy)P_{k-1}(\ln(y))dy + \frac{(-1)^{k+1}}{2^k} f(0) \right) x^{s-1}dx
\]

(4.5)
is valid in the strip \(-1 < \text{Re}(s) < 0\).

Replacing \( k = 1 \) and \( k = 2 \) on the previous theorem and remembering (3.7) and (3.8), we derive the M"{u}ntz-type formulas

\[
\zeta(s)f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty f(nx) - \frac{f^*(1)}{x} + \frac{f(0)}{2} \right) x^{s-1}dx
\]

(4.6)
and

\[
\zeta^2(s)f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty d(n)f(nx) + \frac{1}{x} \left( f^*(1) \ln(x) - ((f^*)'(1) + 2\gamma f^*(1)) \right) - \frac{f(0)}{4} \right) x^{s-1}dx
\]

(4.7)
valid in the strip \(-1 < \text{Re}(s) < 0\), for any function \( f \in \mathcal{M}_{\alpha,m} \), where \( m \geq 3 \) and \( m \geq 4 \), respectively.

Next we move the integral of \( \zeta^k(s)f^*(s)x^{-s} \) to the half-plane \( \text{Re}(s) < -1 \) to deduce M"{u}ntz-type formulas in strips of that half-plane.

Let \( k, m \in \mathbb{N} \) and \( f \in \mathcal{M}_{\alpha,l}, \ l \geq 1 + k \left( 2m + \frac{3}{2} \right) \). Fix \( \sigma_n, \ n = 0, 1, \cdots, m \), such that \(-1 < \sigma_0 < 0 \) and \(-2n - 1 < \sigma_n < -2n + 1 \), for all \( n = 1, 2, \cdots, m \). Using proposition 3.3 and theorem 3.2, we claim that, for any \( n = 1, 2, \cdots, m \), \( \zeta^k(s)f^*(s) \in L_1(\sigma_n \pm i\infty) \) and

\[
\frac{1}{2\pi i} \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \zeta^k(s)f^*(s)x^{-s}ds = \frac{1}{2\pi i} \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \zeta^k(s)f^*(s)x^{-s}ds - \text{res}_{s=1-2n} \left( \zeta^k(s)f^*(s)x^{-s} \right).
\]

Then, remembering formulas (4.1) and (4.4), we may deduce that

\[
\frac{1}{2\pi i} \int_{\sigma_m - i\infty}^{\sigma_m + i\infty} \zeta^k(s)f^*(s)x^{-s}ds = (P_{k,m}f)(x),
\]

(4.8)
where \((P_{k,m}f)(x) \ (x \in \mathbb{R}^+)\) is defined as

\[
\sum_{n=1}^\infty d_k(n)f(nx) - \int_0^\infty f(xy)P_{k-1}(\ln(y))dy + \frac{(-1)^{k+1}f(0)}{2^k} - \sum_{n=1}^m \zeta^k(1-2n) \left( \frac{1}{2^{n-1}} \right) f(2n-1)(0)x^{2n-1}.
\]
Finally, the following theorem can be obtained applying proposition 1.1 to (4.8).

**Theorem 4.2.** Let \( k, m \in \mathbb{N} \) and \( f \in M_{\alpha,l}, \ l \geq 1 + k \left( 2m + \frac{3}{2} \right) \). Then, if we define \((P_{k,m}f)(x)\) as above, the Müntz-type formula

\[
\zeta^k(s)f^*(s) = \int_0^\infty (P_{k,m}f)(x)x^{s-1}dx \tag{4.9}
\]

is valid in the strip \(-2m - 1 < \text{Re}(s) < -2m + 1\).

Fixing \( m \in \mathbb{N} \), replacing \( k = 1 \) and \( k = 2 \) on the previous theorem and remembering (3.7) and (3.8), we derive the Müntz-type formulas

\[
\zeta(s)f^*(s) = \int_0^\infty \left( \sum_{n=1}^\infty f(nx) - \frac{f^*(1)}{x} + \frac{f(0)}{2} - \sum_{n=1}^m \frac{\zeta(1-2n)}{(2n-1)!} f^{(2n-1)}(0) x^{2n-1} \right) x^{s-1}dx \tag{4.10}
\]

and

\[
\zeta^2(s)f^*(s) = \int_0^\infty (P_{2,m}f)(x)x^{s-1}dx, \tag{4.11}
\]

where \((P_{2,m}f)(x)\ (x \in \mathbb{R}^+)\) is defined as equal to

\[
\sum_{n=1}^\infty d(n)f(nx) + \frac{1}{x} \left( f^*(1) \ln(x) - \left( (f^*)'(1) + 2\gamma f^*(1) \right) \right) - \frac{f(0)}{4} - \sum_{n=1}^m \frac{\zeta^2(1-2n)}{(2n-1)!} f^{(2n-1)}(0) x^{2n-1},
\]

valid in the strip \(-2m - 1 < \text{Re}(s) < -2m + 1\), for any function \( f \in M_{\alpha,l} \), where \( l \geq 2m + 3 \) and \( l \geq 4m + 4 \), respectively.

This section ends with a remark about the relation between the Müntz-type formulas in the strip \(-1 < \text{Re}(s) < 0\) and some classical summation formulas.

**Remark 4.3.** The classical Poisson and Voronoi summation formulas (see [3] and [2], respectively) can be derived in the \( M_{\alpha,m} \) classes, replacing \( k = 1 \) and \( k = 2 \), respectively, on (4.4) and using the functional equation of the Riemann zeta function [4] to calculate

\[
\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \zeta^k(s)f^*(s)x^{-s}ds, \ -1 < \sigma_0 < 0.
\]
5 Identities involving the gamma and zeta functions

Perhaps the most important example of a function belonging to the generalized Müntz-type classes of functions is $f(x) = e^{-x} \in M_{\alpha,k}$, for all $\alpha > 1$ and $k \in \mathbb{N}_0$. Its Mellin transform is the gamma function $\Gamma(s)$. In this section we replace $f(x) = e^{-x}$ and $f^*(s) = \Gamma(s)$ in the previously derived formulas to obtain integral representations in vertical strips of the complex plane for products of the gamma and zeta functions.

Observe that, if $f(x) = e^{-x}$ and $f^*(s) = \Gamma(s)$, then $f^*(1) = \Gamma(1) = 1$ and

$$\sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x-1}.$$ 

Therefore we can derive from the Müntz formula the following integral representation for $\zeta(s)\Gamma(s)$ in the critical strip $0 < \text{Re}(s) < 1$, which may be found in section 2.7 of [4],

$$\zeta(s)\Gamma(s) = \int_0^\infty \left( \frac{1}{e^x-1} - \frac{1}{x} \right) x^{s-1} dx. \quad (5.1)$$

Similarly, we may deduce more formulas relating the gamma and zeta functions. For instance, making $f(x) = e^{-x}$ and $f^*(s) = \Gamma(s)$, we have $f^*(1) = 1$ and $(f^*)'(1) = \Gamma'(1) = -\gamma$, so $f^*(1) \frac{\ln(x)}{x} - (2\gamma f^*(1) + (f^*)'(1)) \frac{1}{x} = \frac{\ln(x)}{x} - \frac{\gamma}{x}$. As a result, we obtain from (3.9) the integral representation

$$\zeta^2(s)\Gamma(s) = \int_0^\infty x^{s-1} \left( \sum_{n=1}^{\infty} d(n)e^{-nx} + \frac{\ln(x)}{x} - \frac{\gamma}{x} \right) dx \quad (5.2)$$

valid in the critical strip $0 < \text{Re}(s) < 1$.

Analogously, we derive from theorem 3.6 the integral representations

$$\frac{\zeta(s)}{\zeta(2s)} \Gamma(s) = \int_0^\infty x^{s-1} \left( \sum_{n=1}^{\infty} |\mu(n)|e^{-nx} - \frac{6}{\pi^2x} \right) dx \quad (5.3)$$

and

$$\frac{\zeta^2(s)}{\zeta(2s)} \Gamma(s) = \int_0^\infty x^{s-1} \left( \sum_{n=1}^{\infty} 2\omega(n)e^{-nx} + \frac{6}{\pi^2x} \left( \ln \left( \frac{4\pi^2x}{A^{24}} \right) + \gamma \right) \right) dx \quad (5.4)$$

valid in the strip $\frac{1}{2} < \text{Re}(s) < 1$ (or in the strip $\frac{1}{4} < \text{Re}(s) < 1$, if the Riemann hypothesis holds true).
Moreover, if \( f(x) = e^{-x} \) then \( f(0) = 1 \), so we deduce from (4.6) and (4.7) the integral representations

\[
\zeta(s)\Gamma(s) = \int_{0}^{\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1} dx
\]  

(5.5)

and

\[
\zeta^2(s)\Gamma(s) = \int_{0}^{\infty} \left( \sum_{n=1}^{\infty} d(n)e^{-nx} + \frac{\ln(x)}{x} - \frac{\gamma}{x} - \frac{1}{4} \right) x^{s-1} dx
\]  

(5.6)

valid in the strip \(-1 < \text{Re}(s) < 0\).

Finally, if \( f(x) = e^{-x} \) then, for any \( j \in \mathbb{N}_0 \), \( f^{(j)}(x) = (-1)^j e^{-x} \) so \( f^{(j)}(0) = (-1)^j \). In particular, \( f^{(2n-1)}(0) = -1 \), for all \( n \in \mathbb{N} \). Therefore, for any fixed \( m \in \mathbb{N} \), we obtain from (4.10) and (4.11) the integral representations

\[
\zeta(s)\Gamma(s) = \int_{0}^{\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} + \sum_{n=1}^{m} \frac{\zeta(1-2n)}{(2n-1)!} x^{2n-1} \right) x^{s-1} dx
\]  

(5.7)

and

\[
\zeta^2(s)\Gamma(s) = \int_{0}^{\infty} \left( \sum_{n=1}^{\infty} d(n)e^{-nx} + \frac{\ln(x)}{x} - \frac{\gamma}{x} - \frac{1}{4} + \sum_{n=1}^{m} \frac{\zeta^2(1-2n)}{(2n-1)!} x^{2n-1} \right) x^{s-1} dx
\]  

(5.8)

valid in the strip \(-2m - 1 < \text{Re}(s) < -2m + 1\).

**Acknowledgements**

The author is deeply grateful to Semyon Yakubovich for fruitful discussions of these topics and his useful suggestions, which rather improved the presentation of this paper.
References

[1] J. Havil. *Gamma, Exploring Euler’s constant*. Princeton University Press, Princeton, NJ, 2003.

[2] A. Ivic. *The Riemann Zeta-function*. John Wiley & Sons, Inc., New York, 1985.

[3] E.C. Titchmarsh. *Introduction to the Theory of Fourier Integrals*. Chelsea Publishing Co., New York, Second edition, 1948.

[4] E.C. Titchmarsh. *The Theory of the Riemann Zeta-function*. The Clarendon Press, Oxford University Press, New York, Second edition, 1986.

[5] S.B. Yakubovich. Integral and series transformations via Ramanujan’s identities and Salem’s type equivalences to the Riemann hypothesis. *Integral Transforms and Special Functions*, 25(4):255–271, 2014.

[6] S.B. Yakubovich. New summation and transformation formulas of the Poisson, Müntz, Möbius and Voronoi type. *Integral Transforms and Special Functions*, 26(10):768–795, 2015.