Measurement-induced nonlocality for noninertial observers near a black hole

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We present a systematic and complementary study of quantum correlations near a black hole by considering the measurement-induced nonlocality (MIN) in the noninertial frames. The quantum measure of interest is discussed on the same footing for the fermionic, bosonic and mixed fermion-boson modes in relation to the Hawking radiation. The obtained results show that in the infinite Hawking temperature limit, the physically accessible correlations does not vanish only in the fermionic case. However, the higher frequency modes can sustain correlations for the finite Hawking temperature, with mixed system being more sensitive towards increase of the fermionic frequencies than the bosonic ones. Since the MIN for the latter modes quickly diminishes, the increased frequency may be a way to maintain nonlocal correlations for the scenarios at the finite Hawking temperature.

I. INTRODUCTION

The behavior of quantum information in the relativistic settings gained significant attention over the recent years \cite{1,2,3,4} and the advent of this interest can be associated with the pioneering work of Peres, Scudo and Terno, showing that the spin entropy is not Lorentz invariant \cite{1}. Since then, the quantum information theory faced important revisions in contact with the relativistic world. This led to the creation of the relativistic quantum information (RQI) domain and emergence of the new effects \cite{6}. For example, it was shown that the entanglement phenomenon \cite{7,8} is a observer-dependent entity \cite{9,10} and fidelity of teleportation is affected by the uniformly accelerated observers \cite{11}. Note that both of these effects are closely related to the Unruh effect, saying that the notion of vacuum (as measured by an observer) depends on the observer’s space-time path \cite{12,13}. Hence, vacuum is only invariant under the Poincaré transformations and noninertial observers measure different vacua, which is essential in the field of quantum information. As a result, the RQI appears as an interdisciplinary field that may cover various aspects of relationship between gravity and the quantum world. This includes, but is not limited to, the security in quantum cryptography against gravitational attacks \cite{14}, efficient simulations of quantum systems when relativistic effects are required \cite{15}, fast and precise quantum information processing at large distances \cite{16}, quantum communication and metrology in gravitational fields \cite{17,18} as well as fundamental considerations e.g. investigations of the gravitationally-induced entanglement \cite{19}.

In the RQI field there is also a particular interest in the quantum correlations and their quantifiers, with a special attention given to the nonlocality \cite{1,20}. Such investigations are conducted for either the conventional Bell-like non-locality measures \cite{21} or more general concepts initiated recently \cite{1}. One of such novel measures is known as the measurement-induced nonlocality (MIN) and amounts for the maximum global effect caused by locally invariant measurements \cite{22}, being somewhat dual to the quantum discord \cite{23}. Specifically, the MIN was originally introduced via the Hilbert-Smidt norm as the quantification of nonlocality from a geometric perspective, for consideration of the quantum information processing problems \cite{22}. Later on, this approach was developed further based on the relative entropy \cite{24}, the trace norm \cite{25}, the fidelity measure \cite{26} or the skew information \cite{27} in order to avoid some of the limitations of its initial formulation e.g. the noncontractivity. In a result, the MIN may currently serve as a suitable platform for the analysis of bi- and multi-partite systems, allowing comparison of various correlations as well as the discussion of related quantum effects such as the teleportation \cite{28} or the quantum steering \cite{29}.

In the relativistic frame, the MIN was so far studied for the Unruh observers and \cite{30} in the dilaton space-time \cite{31}. It was shown that at the infinite Unruh temperature limit, the correlations are non-vanishing for fermions, contrary to the bosonic modes \cite{30,31}. Although mentioned studies considered only specific setups, they pointed to the importance of addressing nonlocality and mutual information in the non-inertial systems, similarly to the investigations based on other than the MIN measures \cite{20,42,33}. However, they also shown that MIN is a general

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framework that may capture aspects of nonlocality not accessible by means of the Bell-inequality [31]. In the light of these results, an essential object to investigate MIN for the non-inertial observers appears to be a black holes with their quantum-gravitational foundations [5, 34, 35]. The pivotal role in this respect is played by the Hawking radiation, a manifestation of the quantum mechanics in the space-time of a black hole (BH) [36], which is closely related to the mentioned Unruh effect [12, 37, 38]. By employing this phenomenon it was already possible to show that quantum correlations and related effects are core for the BH thermodynamics and the information loss problem [39, 40]. Moreover, the Hawking radiation allowed studies on the behavior of quantum information in the vicinity of a BH such as the investigations of entanglement and teleportation in the background of the static [41] or the high-dimensional and rotating BHs [42]. In what follows, the BHs established themselves as a source of strong gravity field that can be employed for various investigations within the RQI field e.g. toward future advances in quantum computation and information processing [13, 44].

In this work, we attempt to present our contribution to the above domain of research by providing the general, unified and comprehensive description of the MIN in a BH space-time. Therefore, the results presented here are relevant to a large class of BHs, including the nonsingular ones [45]. For this purpose, we conventionally consider the Alice and Bob setup, where first observer is away from the BH and the latter one within its space-time. They both share maximally entangled Werner state. Such setup, allows us to extend previous discussions given in [30, 31], by describing the MIN not only for the homogenous fermion-fermion or boson-boson correlations but also in the mixed fermion-boson case, on the same footing. Moreover, the analysis is conducted here directly with respect to the Hawking radiation by considering the MIN dependence on the Hawking temperature ($T_H$) for the Werner states of choice. To this end, the MIN is additionally investigated here as a function of the fermionic/bosonic frequency, in analogy to the similar studies conducted for the entanglement [42].

The work is organized as follows: in Sec. II we describe vacuum structure of for the fermionic and bosonic fields in the BH space-time. Next, in Sec. III we derive MIN for the three correlation scenarios of interest. Finally, we conclude our discussion with some pertinent remarks and perspectives for future research.

II. VACUUM STRUCTURE OF A BLACK HOLE

The starting point for the analysis is the general, static and spherically symmetric metric of the following form [46]:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

(1)

where $f(r)$ is the radial function of choice, whose form varies depending on the considered BH type. To keep our considerations as general as possible, we do not assume any specific form of the $f(r)$ function and discuss the MIN in the model-independent manner.

Afterwards, the above metric is rewritten in terms of the Edington-Finkelstein coordinates [47]:

$$u = t - \int \frac{1}{f(r)}dr, \quad v = t + \int \frac{1}{f(r)}dr,$$

(2)

to permit propagation of signals through the space-time [34].

We also note that for any asymptotically flat BH space-time, the event horizon will be the Killing horizon, which is a null hypersurface whose vector field (the Killing vector field) is null at the surface [48]. In what follows, the normal vector ($k^\mu$) will be the Killing vector that satisfies [48]:

$$\mathcal{L}_k g_{\alpha\beta} = \nabla_\beta k_\alpha + \nabla_\alpha k_\beta,$$

(3)

where $\nabla_\beta$ denotes the usual covariant derivative, while $\mathcal{L}_k$ is the Lie derivative along the $k^\mu$. The surface gravity for the Killing vector relates the covariant directional derivative of the horizon’s vector (along itself) to $k^\mu$ i.e. $k^\mu \nabla_\mu k_\nu = \kappa k_\nu$ [49]. Note that $\kappa$ stands for the surface gravity of the horizon and in the Planck’s unit system $(c = h = G = k_B = 1)$ its unit is $1/t_P^2$, where $t_P$ is the Planck time. With this in mind, when a BH radiates, the Hawking temperature will be given by [54]:

$$T_H = \frac{\kappa}{2\pi},$$

(4)

We remind that Eq. (4) is pivotal to our analysis, since it relates general relativity to the quantum field theory and allows inclusion of the BH characteristics in our theoretical framework.
A. Fermionic modes

In order to describe vacuum state of the curved spacetime for fermions, one can start with the following Dirac equation:

\[(i\gamma^a e^a_\mu D_\mu - m)\psi = 0,\]  (5)

with \(D_\mu = \partial_\mu - \frac{i}{4} \omega^{ab}_\mu \sigma_{ab}, \sigma_{ab} = \frac{i}{2} \{\gamma_a, \gamma_b\}, e^a_\mu\) being vierbein and \(\omega^{ab}_\mu\) the spin connection. The solutions of such Dirac equation in regions I and II (i.e. outside and inside the event horizon \((r = r_H)\), respectively) are:

\[
\psi^{I+}_k = \partial e^{-i\omega_i u} , \quad \psi^{II+}_k = \partial e^{i\omega_i u},
\]  (6)

where \(\omega_i\) stands for the monochromatic frequency of the Dirac field (having the unit \(1/l_P\), where \(l_P\) is the Planck length) and \(\vartheta\) is the 4-component Dirac spinor composed of the spinorial spherical harmonics. See Fig. 1 for the graphical reference.

Since solutions of Eq. (6) are analytic inside and outside the event horizon, they completely span the orthogonal basis. Thus, the Dirac field can be written as [50]:

\[
\Psi_{\text{out}} = \sum_i \int d\mathbf{k} \left[ a^I_k \psi^{I+}_k + a^{II}_k \psi^{II+}_k + b^{I+}_k \psi^{I-}_k + b^{II-}_k \psi^{II-}_k \right],
\]  (7)

where \(a^I_k, a^{II}_k\) and \(b^{I+}_k, b^{II-}_k\) are the fermion annihilation and antifermion creation operators for the exterior (interior) region of a BH. Note that summation in Eq. (7) is over the frequencies (see Eq. (6)).

In addition to the above, it is required to describe quantum fields near the event horizon. For this purpose, after discussing the vacuum structure of a BH space-time [34, 51], we follow the formalism developed in [10, 50]. Specifically, by using the light-like Kruskal coordinates [46, 52, 53]:

\[
U = -\frac{1}{\kappa} e^{-\kappa u}, \quad V = \frac{1}{\kappa} e^{\kappa v}, \quad (r > r_H),
\]

\[
U = \frac{1}{\kappa} e^{\kappa u}, \quad V = -\frac{1}{\kappa} e^{-\kappa v}, \quad (r < r_H),
\]  (8)

we rewrite the solutions of the Dirac equation as follows:

\[
\psi^{I+}_k = \vartheta(U \kappa) e^{-i\omega_i / \kappa}, \quad \psi^{II+}_k = \vartheta(-U \kappa) e^{i\omega_i / \kappa}.
\]  (9)

Next, to construct complete basis for the positive energy modes, the analytic continuation is done for Eq. (9) by using the Damour’s technique [42, 51]. We arrive at the following solutions:

\[
\psi^{I+}_k = \vartheta(\kappa U) e^{-i\omega_i / \kappa} = \Theta(-U)(-U \kappa)^{i\omega_i / \kappa}, \quad \psi^{II+}_k = \vartheta(-U \kappa) e^{i\omega_i / \kappa} = \Theta(U)(U \kappa)^{-i\omega_i / \kappa},
\]  (10)
where $\Theta(\pm U)$ is the Heaviside step function. After that, the complete basis of interest can be expressed as the following linear combination of the modes:

$$\begin{align*}
\lambda_k^1 &= \psi_k^I + \bar{\psi}_k^I = \Theta(-U)(-U\kappa)^{i\omega_i/\kappa} + e^{\pi\omega_i/\kappa}\Theta(U)(U\kappa)^{i\omega_i/\kappa}, \\
\lambda_k^2 &= \psi_k^I - \bar{\psi}_k^I = \Theta(-U)(-U\kappa)^{-i\omega_i/\kappa} + e^{-\pi\omega_i/\kappa}\Theta(U)(U\kappa)^{-i\omega_i/\kappa},
\end{align*}$$

(11, 12)

with:

$$\bar{\psi}_k^{II} = e^{-\pi\omega_i(U\kappa)^{-i\omega_i/\kappa}}.$$  

(13)

Hence, the complete basis for the normalized modes reads:

$$\begin{align*}
\zeta_k^I &= e^{\pi\omega_i/(2\kappa)}\lambda_k^1 \psi_k^I + e^{-\pi\omega_i/(2\kappa)}\Theta(-U)(-U\kappa)^{i\omega_i/\kappa} + e^{-\pi\omega_i/(2\kappa)}\Theta(U)(U\kappa)^{i\omega_i/\kappa}, \\
\zeta_k^I &= e^{\pi\omega_i/(2\kappa)}\lambda_k^2 \psi_k^I + e^{-\pi\omega_i/(2\kappa)}\Theta(-U)(-U\kappa)^{-i\omega_i/\kappa} + e^{-\pi\omega_i/(2\kappa)}\Theta(U)(U\kappa)^{-i\omega_i/\kappa},
\end{align*}$$

(14, 15)

Finally, based on the Eqs. (14)-(15), the outgoing Dirac fields can be expanded in the Kruskal space-time:

$$\Psi_{\text{out}} = \sum_i d_k^I \frac{1}{\sqrt{2\cosh(\pi\omega_i/\kappa)}} \left[ c_k^{II} \zeta_k^I + c_k^{II} \zeta_k^{II} + d_k^{II} \zeta_k^I + d_k^{II} \zeta_k^{II} \right],$$

(16)

where $c_k$'s and $d_k$'s are creation and annihilation operators applied to the Kruskal vacuum.

To this end, via the Bogoliubov transformation, the creation and annihilation operators in the BH and the Kruskal space-times can be related to each other [34]:

$$c_k^I = \alpha a_k^I - \beta \bar{a}_k^{II},$$  

(17)

with the following Bogoliubov coefficients:

$$\alpha = \frac{1}{(e^{-\omega_i/T_H} + 1)^{1/2}}, \quad \beta = \frac{1}{(e^{\omega_i/T_H} + 1)^{1/2}}.$$  

(18)

Therefore, the vacuum and excited states of the Minkowski space-time will be related to the Kruskal’s one via the following relationships:

$$|0\rangle^+_k = \alpha |0\rangle^+_I |0_{-k}\rangle^{II} + \beta |1\rangle^+_I |_{-k}\rangle^{II}, \quad |1\rangle^+_k = |1\rangle^+_I |_{-k}\rangle^{II}. $$

(19)

B. Bosonic modes

In case of the bosonic field one is required to deal with the massless scalar field in the curved space-time, being the solution of the Klein-Gordon equation [34] [11]:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) = 0. $$

(20)

In details, by considering Eq. (20) near the event horizon, we can arrive with the incoming wave function, which is analytic for the entire space-time and given as:

$$\phi^{in} = Y_{tm} e^{i\Omega u}. $$

(21)

Similarly, the outgoing wave functions for the regions I and II of the event horizon are [11] (see Fig. 1 for the graphical reference):

$$\phi^I_{\Omega} = Y_{tm} e^{i\Omega u}, \quad \phi^{II}_{\Omega} = Y_{tm} e^{-i\Omega u}. $$

(22)
In Eqs. (21)-(22), $Y_{lm}(\theta, \varphi)$ denotes scalar spherical harmonic and $\Omega$ is the bosonic frequency of the mode with the units $1/l_P$. Analogously to the fermionic case, the $\phi^I$ and $\phi^{II}$ solutions span orthogonal basis for the Klein-Gordon field:

$$
\Phi_{\text{out}} = \sum_{lm} \int d\Omega \left[ a^l_{\Omega} \phi^l_{\Omega} + a^{II}_{\Omega} \phi^{II}_{\Omega} + b^{I*}_{\Omega} \phi^I_{\Omega} + b^{II*}_{\Omega} \phi^{II*}_{\Omega} \right],
$$

(23)

where $a^l_{\Omega}$ ($a^{II}_{\Omega}$) and $b^{I*}_{\Omega}$ ($b^{II*}_{\Omega}$) are the bosonic annihilation and creation operators for the exterior (interior) region of the considered BH. Note again, that summation in Eq. (23) is over the spherical harmonics (see Eq. (22)).

By introducing the following abbreviations:

$$
\cosh r = \frac{1}{\sqrt{1 - e^{-\Omega/T_H}}}, \quad \tanh r = e^{-\Omega/T_H},
$$

(24)

and

$$
\tanh r = t, \quad \cosh r = \frac{1}{\sqrt{1 - t^2}},
$$

(25)

one can write down the vacuum and excited bosonic modes in a more convenient form:

$$
|0\rangle_k = \sqrt{1 - t^2} \sum_{n=0}^{\infty} t^n |n\rangle_I |n\rangle_{II}, \quad |1\rangle_k = (1 - t^2) \sum_{n=0}^{\infty} \sqrt{n+1} t^n |n+1\rangle_I |n\rangle_{II}.
$$

(26)

### III. MEASUREMENT-INDUCED NONLOCALITY

The main setup considered here is conventional and based on the one originally presented in [9]. In particular, Alice is equipped with the particle detector sensitive only to mode $|n\rangle_A$, while Bob can detect only mode $|n\rangle_B$ by using his device. Alice remains stationary at the asymptotically flat region of the space-time ($r \to \infty$), while Bob will free-fall into a BH, then hovers near the event horizon. According to the Hawking effect, Bob will be affected by the thermal bath of the particles associated with the Hawking radiation near the event horizon [12, 36, 37]. To describe what Bob will detect, mode $|n\rangle_B$ must be specified in the coordinates of a BH. This situation is schematically presented in Fig. [1]

#### A. Fermionic modes

We begin with the Werner state as our initial state [31, 54]:

$$
\rho_{AB} = \rho_{AB_{II}} = \eta |\phi^+\rangle \langle \phi^+| + \frac{1 - \eta}{4} I,
$$

(27)

which is based on the following Bell state:

$$
|\phi^+\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right), \quad \frac{1}{\sqrt{2}} \left( \alpha |0_A0_I0_{II}\rangle + \beta |0_A1_I1_{II}\rangle + |1_A1_I0_{II}\rangle \right),
$$

(28)

and $I = |00\rangle \langle 00| + |11\rangle \langle 11|$. We note, that there is some freedom in choice of the initial state and herein we choose the above form to allow comparison with the results presented in [30, 31].

In order to obtain physically accessible correlations of interest, we have to trace over the region $I$ which is causally disconnected from the exterior. Such trace over the region $II$ gives:

$$
\rho_{AB_I} = \begin{pmatrix}
\frac{1}{4} \alpha^2 (1 + \eta) & 0 & 0 & \frac{1}{4} \alpha \eta \\
0 & \frac{1}{4} \beta^2 (1 + \eta) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} \beta \eta \\
\frac{1}{2} \alpha \eta & 0 & 0 & \frac{1}{4} (1 + \eta)
\end{pmatrix}.
$$

(29)

For the bipartite state $\rho$, which is shared by $A$ and $B$, the MIN is defined by the following expression:

$$
\text{MIN}(\rho) = \text{Max}_{\Pi^A} \| \rho - \Pi^A | \|^2.
$$

(30)
In Eq. (30), the maximum is taken over the local von Neumann measurements \( \prod_{l}^{A} = \{ \prod_{l}^{A} \} \) \((l = 1, 2)\) that do not disturb \( \rho^{A} \) locally. It means that \( \prod_{l}^{A} \rho \prod_{l}^{A} = \rho^{A} \) and the norm of the states is arbitrary, depending on the context. In this work, we will use the Hilbert-Schmidt norm \( \| x \| = \text{tr}X^{\dagger}X \). Note, however, that the MIN measure based on the trace norm has some potential shortcomings (see [4, 55] for more details).

\[
\text{MIN}(\rho_{AB})
\]

\[
\text{MIN}(\rho_{AB})
\]

(A) \hspace{2cm} (B)

\(T_H\) \hspace{2cm} \(\omega\)

FIG. 2: The measurement-induced nonlocality (MIN) as a function of (A) the Hawking temperature \((T_H)\) and (B) the fermionic frequency \((\omega)\) for the physically accessible correlations in the case of the fermionic modes \((\rho_{AB})\). The results are presented for the selected values of the entanglement parameter \((\eta)\), assuming (A) the fixed fermionic frequency \((\omega = 1)\) or (B) the fixed Hawking temperature \((T_H = 10)\).

To proceed, it is convenient to define arbitrary states composed of the two qubits in the Bloch decomposition:

\[
\rho = \frac{1}{4} \left( I \otimes I + \sum_i x_i \sigma_i \otimes I + \sum_i y_i \otimes I \sigma_i + \sum_{i,j} t_{ij} \sigma_i \otimes \sigma_j \right),
\]

where \(I\) is the \(2 \times 2\) identity matrix and \(\sigma_i\) are the Pauli matrices \((i, j \in \{1, 2, 3\})\) with \(x_1, x_2, x_3\) and \(y_1, y_2, y_3\) being the Bloch vectors and \(t_{ij} = \text{tr}[\sigma_i \otimes \sigma_j \rho^X]\) denoting the tensor of correlation. We notice that Eq. (29) is of the following X-shape form:

\[
\rho^X = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix},
\]

(32)

where \(\rho_{ij}\) are the real parameters for the two-qubit state. We can express the parameters of state \(\rho^X\) of the Bloch decomposition as below:

\[
\begin{align*}
x_1 &= x_2 = y_1 = t_{12} = t_{21} = t_{13} = t_{31} = t_{23} = t_{32} = 0, \\
x_3 &= \text{tr}(\sigma_z^A \rho^X) = \rho_{11} + \rho_{22} - \rho_{33} - \rho_{44}, \\
y_3 &= \text{tr}(\sigma_z^B \rho^X) = \rho_{11} - \rho_{22} + \rho_{33} - \rho_{44}, \\
t_{11} &= \text{tr}(\sigma_z^A \sigma_z^B \rho^X) = 2\rho_{14} + 2\rho_{23}, \\
t_{22} &= \text{tr}(\sigma_y^A \sigma_y^B \rho^X) = -2\rho_{14} + 2\rho_{23}, \\
t_{33} &= \text{tr}(\sigma_z^A \sigma_z^B \rho^X) = \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44}.
\end{align*}
\]

(33)

In accordance with the Theorem 3 presented in [22], the MIN of the two-qubit states \(X\) can be given by:

\[
\text{MIN}_A(\rho^X) = \begin{cases}
\frac{1}{4}(t_{11}^2 + t_{22}^2) & \text{for } x \neq 0 \\
\frac{1}{4}(t_{11}^2 + t_{22}^2 + t_{33}^2 - \delta_{\text{min}}) & \text{for } x = 0,
\end{cases}
\]

(34)
where $\delta_{\text{min}} = \min\{t_{11}^2, t_{22}^2, t_{33}^2\}$. Applying this method to Eq. (29) we arrive at the MIN for the physically accessible correlations:

$$\text{MIN}(\rho_{AB}) = \frac{\eta^2}{2(1 + e^{-\omega/T_H})}. \quad (35)$$

Note that the MIN will depend on the parameter $\eta$ as well as on the Hawking temperature, thus being sensitive to the parameters of a BH (mass, charge etc.). This fact is presented in Fig. 2 (A) where the behavior of the MIN as a function of $T_H$ for the specific values of $\eta$ is depicted. Analogously to the MIN studied for the Unruh effect in [30], one can observe there that the MIN will not be vanishing since $\lim_{T_H \to \infty} \text{MIN}(\rho_{AB}) = \frac{\eta^2}{2}$. This statement holds for the entire range of the $\eta$ parameter. It is additionally worth to remark that for $\text{MIN}(\rho_{AB}) \leq 0.25$ the Bell’s inequality is obeyed, whereas for the $\text{MIN}(\rho_{AB}) > 0.25$ it is violated, leading to the nonlocal quantum correlations [31]. This is to say, the violation of the Bell’s inequality as the Hawking temperature increases will be $\eta$-dependent. Hence, it will depend on the choice of the initial state. On the other hand, the case when the $T_H$ is fixed is presented in Fig. 2 (B) for the selected values of the parameter $\eta$. Therein, increase of the fermionic frequency clearly leads to the increase of the $\text{MIN}(\rho_{AB})$, thus high values of the MIN can be sustained. This can also be observer when noting that $\lim_{\omega \to \infty} \text{MIN}(\rho_{AB}) = \frac{\eta^2}{2}$.

### B. Bosonic modes

The starting point of the analysis of the MIN for bosons is the following Bell state:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \left( \sqrt{1 - t^2} \sum_{n=0}^{\infty} t^n |0, n, n, n\rangle + (1 - t^2) \sum_{n=0}^{\infty} \sqrt{n + 1} t^n |1, n + 1, n, n\rangle \right). \quad (36)$$

In this context, to study the MIN for the bosonic state given by the $\rho_{AB}$, one needs to use projective measurement on the Alice’s state. Such measurement can be taken over the single-qubit system. Thus, it is possible to parametrize it by the unit vector $\vec{x} = (x_1, x_2, x_3)$ with the aid of the projectors:

$$\Pi_\pm = \frac{1}{2}(I \pm \vec{x} \cdot \vec{\sigma}). \quad (37)$$

Next, these projectors can be rewritten in the following form:

$$\Pi_\pm = \frac{1}{2} \left( |1 + x_3\rangle \langle 0| + (1 + x_3) |0\rangle \langle 1| \pm (x_1 - ix_2) |0\rangle \langle 1| \pm (x_1 + ix_2) |1\rangle \langle 0| \right). \quad (38)$$

Based on the Eq. (38) one can get the post-measured final state:

$$\rho'_{AB} = \sum_{\alpha = \pm} (\Pi_\alpha \otimes I) \rho_{AB} (\Pi_\alpha \otimes I) = \sum_{\alpha = \pm} p_\alpha \Pi_\alpha \otimes \rho_{B|\alpha}, \quad (39)$$

where:

$$\rho_{B|\alpha} \equiv \text{Tr} \left((\Pi_\alpha \otimes I_B) \rho_{AB} (\Pi_\alpha \otimes I_B)\right)/p_\alpha, \quad (40)$$

is the post-measured state of the Bob’s system conditioned on the outcome $\alpha$ with the probability $p_\alpha$. Also with the aid of Eq. (38) one can obtain ($p_\pm = 1/2$) [50]:

$$\rho_{B|\pm} = \frac{1 - t^2}{2} \hat{\rho}_I, \quad (41)$$

where:

$$\hat{\rho}_I = (1 + x_3) M_{00} + (1 + x_3) M_{11} \pm (x_1 - ix_2) M_{01} \pm (x_1 + ix_2) M_{10}, \quad (42)$$

and with the following matrices for the Bob’s Hilbert space:

$$M_{00} = \frac{1}{2}(|\eta + 1\rangle t^{2n} |n\rangle \langle n|, \quad (43)$$

$$M_{10} = \eta \sqrt{n + 1} \sqrt{1 - t^2} t^{2n} |n + 1\rangle \langle n|, \quad (44)$$

$$M_{01} = M_{10}^\dagger, \quad (45)$$

$$M_{11} = \frac{1}{2}(|\eta + 1\rangle (n + 1) (1 - t^2) t^{2n} |n + 1\rangle \langle n + 1|. \quad (46)$$
After the above preparations, we can now calculate the quantity of interest:

\[
\text{Tr} \left( (\rho_{AB} - \rho'_{AB})^2 \right) = \frac{(1-t^2)^2}{4} \left[ \text{Tr}(X_{00}^2) + 2 \text{Tr}(X_{01}X_{10}) + \text{Tr}(X_{11}^2) \right],
\]

with the \(X\)'s given by:

\[
X_{00} \equiv M_{00} - \frac{1}{4}[(1 + x_3)\hat{p} + (1 - x_3)\hat{p}], \\
X_{11} \equiv M_{11} - \frac{1}{4}[(1 - x_3)\hat{p} + (1 + x_3)\hat{p}], \\
X_{01} \equiv M_{01} - \frac{1}{4}(x_1 - ix_2)(\hat{p} - \hat{p}), \\
X_{10} \equiv M_{10} - \frac{1}{4}(x_1 + ix_2)(\hat{p} - \hat{p}).
\]

We note that traces of the \(X\)'s will be given by the linear combinations of the matrices \(M\), reducing \(\text{Tr} \left( (\rho_{AB} - \rho'_{AB})^2 \right)\) to [50]:

\[
\text{Tr} \left( (\rho_{AB} - \rho'_{AB})^2 \right) = \frac{(1-t^2)^2}{8}[(1-x_3^2)\left(\text{Tr}(M_{00}^2) + \text{Tr}(M_{11}^2) - 2\text{Tr}(M_{00}M_{11})\right) + 2(1+x_3^2)\text{Tr}(M_{01}M_{10})],
\]

with their traces equal to:

\[
\text{Tr}(M_{00}^2) = \frac{1}{4}(1 + \eta^2)\sum_{n=0}^{\infty} t^{4n}, \\
\text{Tr}(M_{01}M_{10}) = \frac{\eta^2(1-t^2)}{(t^4 - 1)^2}, \\
\text{Tr}(M_{11}^2) = -\frac{(\eta + 1)^2(t^4 + 1)}{4(t^2 - 1)(t^2 + 1)^3}.
\]

For the \(x_3 = 1\), one can finally obtain the MIN for the bosonic case:

\[
\text{Tr} \left( (\rho_{AB} - \rho'_{AB})^2 \right) = \frac{\eta^2(1-t^2)^2(t^2 - 1)^3(t^4 + 1)}{8(t^4 - 1)^3}.
\]

We remark that setting \(x_3 = 0\) in Eq. [39] leads to the quantum discord [30, 50].

The behavior of the bosonic MIN is presented in Fig. [3] (A) for the Hawking temperature in the range \(T_H \in (0, 10)\) and with fixed \(\eta \in (0, 1)\). In accordance to the results presented in [30, 31], the MIN(\(\rho_{AB}\)) vanishes rapidly and \(\lim_{T_H \to \infty} \text{MIN}(\rho_{AB}) = 0\). However, by increasing the frequency of the bosonic modes it is again possible to maintain relatively high values of the MIN. This scenario is presented in Fig. [3] (B) where for \(T_H = 10\), the frequency \(\Omega\) takes values in the range \(\Omega \in (0, 100)\). Interestingly, \(\lim_{\Omega \to \infty} \text{MIN}(\rho_{AB}) = 2/3\) and coincides with the result previously obtained for the fermionic modes with frequency \(\omega\). Thus, for a high frequencies of the modes, the behavior of the bosonic and fermionic modes for the finite \(T_H\) will be similar.

### C. Mixed boson-fermion modes

In order to describe the mixed boson-fermion case, we slightly change setup used in the previous sections. In particular, Alice is still equipped with the fermionic mode detector, while Bob can detect bosons. However, Alice, just like Bob after the free fall, will now hover near the horizon of a BH. Both of them will detect Hawking temperature in their corresponding detectors. An analogous situation but with two accelerated observers and in the different context can be found in [57]. In our scenario, after performing the trace over region \(II\), the Bell state will have the following form:

\[
|\phi^+\rangle = \frac{1}{\sqrt{2}} \left( (\sqrt{1-t^2} \sum_{n=0}^{\infty} t^n |n_I, n_{II}, 1_I, 0_{II}) + (1-t^2) \sum_{n=0}^{\infty} \sqrt{n+1}t^n |n + 1_I, n_{II}) (\alpha |0_I0_{II}) + \beta |1_I1_{II}) \right),
\]
FIG. 3: The measurement-induced nonlocality (MIN) as a function of (A) the Hawking temperature \((T_H)\) and (B) the bosonic frequency \((\Omega)\) for the physically accessible correlations in the case of the bosonic modes \((\rho_{AB_I})\). The results are presented for the selected values of the entanglement parameter \((\eta)\), assuming (A) the fixed fermionic frequency \((\Omega = 1)\) or (B) the fixed Hawking temperature \((T_H = 10)\).

We would like to note that for the singlet Bell state, \(|\psi^\prime\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\) it is not possible to calculate MIN based on the trace norm \([22]\). Terms required to obtain the MIN given by Eq. \([39]\) are vanishing for the \(x_3 = 1\).

In what follows, the coefficient of the decomposition will have the following shape:

\[
\begin{align*}
M_{00} &= \sum_{n=0}^{\infty} \alpha^2 (1 + \eta) \frac{1}{2} t^{2n} |n\rangle \langle n|, \\
M_{01} &= \sum_{n=0}^{\infty} \sqrt{1 + n} \sqrt{1 - t^2 \alpha \eta t^{2n}} |n\rangle \langle n+1|, \\
M_{10} &= \sum_{n=0}^{\infty} \sqrt{1 + n} \sqrt{1 - t^2 \alpha \eta t^{2n}} |1+n\rangle \langle n|, \\
M_{11} &= \sum_{n=0}^{\infty} \frac{1}{2} (1 + \eta) (1 + n) (1 - t^2) t^{2n} |1+n\rangle \langle 1+n| + \frac{1}{2} (1 + n) t^{2n} (1 + \eta)^2 |n\rangle \langle n|, \\
\end{align*}
\]

where \(\alpha\) and \(t\) are specified by Eqs. \([18, 24, 25]\). Hence:

\[
\begin{align*}
\text{Tr}(M_{00}^2) &= \frac{\alpha^4}{4(1 - t^4)} (1 + \eta), \\
\text{Tr}(M_{01} M_{10}) &= \frac{\alpha^2 \eta^2 (1 - t^2)}{(t^4 - 1)^2}, \\
\text{Tr}(M_{00} M_{11}) &= \frac{\alpha^2 (\eta + 1)^2 \beta^2}{4(t^4 - 1)^2}, \\
\text{Tr}(M_{11}^2) &= \frac{-(\eta + 1)^2 \beta^4 (t^4 + 1)}{4(t^4 - 1)^3} - \frac{(\eta + 1)^2 (t^4 + 1)}{4(t^2 - 1)(t^2 + 1)^3}.
\end{align*}
\]

Assuming \(x_3 = 1\), one gets:

\[
\text{Tr}((\rho_{A,B_I} - \rho'_{A,B_I})^2) = \frac{1}{2} (1 - t^2)^2 \left( -\frac{\alpha^2 \eta^2 (t^2 - 1)}{(t^2 + 1)^2} \right).
\]

\(\text{Tr}((\rho_{A,I,B_I} - \rho'_{A,I,B_I})^2) = \frac{1}{2} (1 - t^2)^2 \left( -\frac{\alpha^2 \eta^2 (t^2 - 1)}{(t^2 + 1)^2} \right). \)
FIG. 4: The measurement-induced nonlocality (MIN) as a function of (A) the Hawking temperature and (B) simultaneously the fermionic ($\omega$) and bosonic ($\Omega$) frequencies for the physically accessible correlations in the case of the mixed modes ($\rho_{A,B_I}$). The results are presented for (A) the selected values of the entanglement parameter ($\eta$) and (B) for the fixed entanglement parameter ($\eta = 1$) and the fixed Hawking temperature ($T_H = 10$).

The MIN behavior for the boson-fermion case as a function of the Hawking temperature for different values of the parameter $\eta$ is presented in Fig. (4) (A). Interestingly, for the small Hawking temperature, the MIN initially behaves similar to the fermionic case. Then, the MIN will quickly decrease, leading to the $\lim_{T_H \to \infty} = 0$. As the $T_H$ increases, the bosonic contribution will dominate. An interesting situation is depicted in the Fig. (4) (B), where for the fixed $T_H = 10$ and $\eta = 1$ we plot the behavior of the MIN($\rho_{A,B_I}$) as a function of the bosonic ($\Omega$) and fermionic ($\omega$) frequencies, respectively. Just like in the fermionic case, by increasing $\omega$ it is possible to achieve rather high values of the MIN($\rho_{A,B_I}$).

D. Physically inaccessible correlations

The conducted analysis can be further supplemented by the discussion of the physically inaccessible correlations. Specifically, to describe the possibility of the destruction or the transfer of correlations, in the following part we trace over the region $B_I$. In this way, the MIN for the physically inaccessible correlations can be obtained.

For the fermion-fermion state, we obtain the following density matrix:

$$
\rho_{AB_{II}} = \begin{pmatrix}
\frac{1}{4} \alpha^2 (1 + \eta) & 0 & 0 & 0 \\
0 & \frac{1}{4} \beta^2 (1 + \eta) & \frac{1}{2} \eta \beta & 0 \\
0 & \frac{1}{2} \eta \beta & \frac{1}{4} (\eta + 1) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Next, the steps analogous to the ones conducted for the physically accessible correlations, but with the partial trace over region $B_I$ instead of region $B_{II}$, lead to the MIN($\rho_{AB_{II}}$):

$$
\text{MIN}(\rho_{AB_{II}}) = \eta^2 \frac{2}{e^{\frac{\eta^2}{T_H}} + 1}.
$$
On the other hand, for the boson-fermion situation we have the following matrices $M$:

$$
M_{00} = \sum_{n=0}^{\infty} \left[ \frac{1}{2} t^{2n} \alpha^2 |n\rangle \langle n| + \frac{1}{2} (1 + n) t^{2n} (1 - t^2)(1 + \eta) |1 + n\rangle \langle 1 + n| \right], \quad (55)
$$

$$
M_{11} = \sum_{n=0}^{\infty} \frac{1}{2} (1 + n) t^{2n} (1 + \eta) \beta |n\rangle \langle n|,
$$

$$
M_{01} = \sum_{n=0}^{\infty} (1 + n) t^{2n} \sqrt{1 - t^2} \eta \beta |n+1\rangle \langle n|, \quad M_{10} = M_{01}^*.
$$

Straightforwardly repeating the procedure presented in Eqs. (37)-(48), with the trace over $A_{II}$ and $B_I$, one can arrive at the MIN($\rho_{A_I B_{II}}$) of the mixed state:

$$
\text{MIN}(\rho_{A_I B_{II}}) = \frac{(1 - t^2)^2 (\eta^2 \beta^2 (1 - t^2) (-t^4 - 1))}{2 (t^4 - 1)^3}. \quad (56)
$$

FIG. 5: The measurement-induced nonlocality (MIN) as a function of the Hawking temperature ($T_H$) for the physically inaccessible correlations in the case of (A) the fermionic (A) and (B) the mixed modes. The results are presented for the selected values of the entanglement parameter ($\eta$) and for the fixed fermonic ($\omega = 1$) and bosonic frequencies ($\Omega = 1$).

By inspecting Fig. (A), one can observe that the physically inaccessible correlations increase under the effect of the Hawking radiation for the fermionic case, contrary to the previously analyzed region $A_B$, with the $\lim_{T_H \to \infty} \text{MIN}(\rho_{A_I B_{II}}) = \eta^2/4$. On the contrary, in the mixed fermion-boson case, the result will be finite for the infinite Hawking temperature, $\lim_{T_H \to \infty} \text{MIN}(\rho_{A_I B_{II}}) = \eta/16$, as depicted in Fig. (B). Moreover, increase of the frequencies $\omega$ for the Dirac modes leads to the destruction of the correlations for the physically inaccessible correlations. However, this is not the case for the mixed fermion-boson correlations for the region $A_I B_{II}$, where $\lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{A_I B_{II}}) = \eta^2/2(1 + e^{(1/T_H)})$.

IV. CONCLUDING REMARKS AND FUTURE PERSPECTIVES

In this work, we have presented extended and unified discussion of the nonlocality for the fermionic and bosonic fields under relativistic effects. Similarly to the discussions presented in [30, 31], the nonlocality was investigate via the quantum measure known as the measurement-induced nonlocality (MIN). Specifically, by using the generic spherical
black hole (BH) metric with the corresponding Hawking temperature, we were able to study general properties of the MIN in the model-independent way. Besides the discussion of the MIN behavior for one noninertial observer falling into a black hole, we have also analyzed properties of the MIN for the mixed fermion-boson case with the two parties influenced by Hawking radiation. It was found that, in the case of bosons (boson-fermion) their limits read
\[ \lim_{T_H \to \infty} \text{MIN}(\rho_{AB}) = 0 \] (\[ \lim_{T_H \to \infty} \text{MIN}(\rho_{A1B1}) = 0 \]), which indicates that there is no quantum correlation when \( T_H \) approaches infinity. On the other hand, for the fermionic case \( \lim_{T_H \to \infty} \text{MIN}(\rho_{A1B1}) = \eta/2 \). It means that the MIN for the Dirac fields is more robust than the bosonic or mixed one. This is to say, fermions are more resistant towards the Hawking radiation than the bosons, in analogy to the resistance against acceleration as discussed in \( \text{[37]} \). Strictly speaking, the MIN will be composed of the products of the fermionic and bosonic inverted partition functions. The bosonic and mixed correlations will diminish from the unboundness of the bosonic partition function. On the contrary, for the higher frequencies of the modes (\( \omega \) and \( \Omega \)) it is possible to sustain high values of the MIN for the finite Hawking temperature and physically accessible correlations. We shown that the infinite frequency limits for the bosonic and fermionic modes coincide, \( \lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{AB}) = \eta/2 \) being independent of the Hawking temperature. However, for the boson-fermion scenario \( \text{MIN}(\rho_{A1B1}) \) is resistant to the increase of the bosonic frequency \( \Omega \) and limit \( \lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{A1B1}) = \frac{\eta}{2} \) coincides with the result of the previous considerations of the AB1 region. For convenience, our results are summarized in the Table 1.

To this end, we note that the MIN may change rather arbitrarily through uncorrelated action of the unmeasured part. Therefore, we would like to remark that there are other forms of the MIN as well as the alternative measures of nonlocality \( \text{[24, 26, 58, 59]} \). Potential study of these various measures in the relativistic setups may shed a new light on the details and the conceptual differences between existing forms of the correlation measure. Moreover, we note that is may be also instructive to extend presented here discussion on the higher dimensional or rotating black holes, which were already extensively studied in other contexts \( \text{[9, 31, 41, 42]} \).

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### Table 1: The limits of the measurement-induced nonlocality (MIN) for the accessible and inaccessible correlations in the case of the fermionic, bosonic and mixed modes.

|                      | fermionic modes | bosonic modes | mixed modes |
|----------------------|-----------------|---------------|-------------|
| \( \lim_{T_H \to \infty} \text{MIN}(\rho_{AB}) \) | \( \eta/4 \) | 0             | –           |
| \( \lim_{T_H \to \infty} \text{MIN}(\rho_{A1B1}) \) | –               | –             | 0           |
| \( \lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{A1B1}) \) | \( \eta/2 \) | \( \eta/2 \) | –           |
| \( \lim_{T_H \to \infty} \text{MIN}(\rho_{ABII}) \) | \( \eta^2/4 \) | –             | –           |
| \( \lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{ABII}) \) | 0               | –             | –           |
| \( \lim_{T_H \to \infty} \text{MIN}(\rho_{A1B1I}) \) | –               | –             | \( \eta/16 \) |
| \( \lim_{\omega, \Omega \to \infty} \text{MIN}(\rho_{A1B1I}) \) | –               | –             | \( \eta^2/2(1 + e^{1/T_H}) \) |

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