Do Vortex Filaments in a Superfluid Neutron Star Produce Gravimagnetic Forces?

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Abstract

A general analysis of the gravitational dynamics of a medium with a continuous distribution of vorticity indicates that the answer to the question raised in the title is affirmative, contrary to a recent claim.

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1) Introduction

Very soon after the formation of a neutron star, thermal energies inside it drop below the energy gap associated with neutron pairing. One therefore expects all but the outermost parts of the crust to pass rapidly into a superfluid state [1, 2, 3]. The critical angular velocity for vortex formation is negligibly small for bodies of astronomical size and therefore, as first noted by Ginzburg and Kirzhnits [4], the star’s interior will be threaded by vortex lines parallel to the axis of rotation.

A stimulating review by Kirzhnits and Yudin [5] discusses some of the interesting and surprising general-relativistic effects associated with superfluid neutron stars. Consider, for instance, an imaginary experiment in which the star is initially at rest and a spin is imparted to the crust. Lense-Thirring (frame-dragging) effects will make it appear that the superfluid interior has been set into partial co-rotation, i.e., the contravariant azimuthal velocity $v_s^\phi = d\phi/dt$ with respect to distant stationary observers will become non-zero. If the crust is separated from the fluid by a small gap, then this induced rotation can easily exceed the critical angular velocity without forming any vortices. This is because the superfluid’s orbital angular momentum, proportional to the covariant component $v_s^\phi$, remains zero. The induced rotation also makes no contribution to the value of the metric coefficient $g_{\phi t}$ (the “gravimagnetic potential”), whose non-vanishing is due entirely to the rotation of the crust.

Superfluid flow is in all circumstances irrotational. For regular axisymmetric flows this implies $v_s^\phi = 0$, i.e., the superfluid can never acquire orbital angular momentum. It is nevertheless able to entrap angular momentum within the cores of its vortex filaments, which consist of normal fluid. To simplify the description, one generally assumes that it is permissible to replace the vortex array by a fictitious “effective” orbital flow ($v_{s\phi}^{\text{eff}} \neq 0$) when averaging over macroscopic domains, i.e., domains large compared with the vortex spacing (about $10^{-2} \text{cm}$ for the Crab pulsar). The equivalence assumption is that gravitational and other physical effects of the vortex array are macroscopically indistinguishable from an effective orbital motion having the same distribution of angular momentum. This seemingly innocent assumption is called into question in a recent analysis by Kirzhnits and Yudin [6]. The authors do not succeed in finding an acceptable solution representing the contribution of a vortex line to the external gravitational field of a spherical body in the linearized Einstein theory. They conclude: “The
general formula relating the angular momentum of a body with the asymptotic behavior of its gravimagnetic field is invalid when vortex filaments are present."

This conclusion is not easy to understand intuitively, since the formation and slow migration of vortices into the superfluid from the interface with the crust is macroscopically an axisymmetric process, which conserves angular momentum locally and involves no appreciable redistribution of mass. It is therefore hard to see how it can affect the external gravitational field at all, much less suppress all gravimagnetic effects, as the angular momentum of the crust gradually becomes "vorticized". In short, there is no obvious reason why spin and orbital angular momentum should behave in a fundamentally different way as sources of gravity.

The issue is perhaps elementary but clearly important because of the potential implications for the gravitational dynamics of the binary pulsar and gravitational radiation from neutron star mergers. To remove any possible doubt it seems worth devoting some effort to an explicit verification that bulk rotation and vortex filaments are indeed equivalent in their gravitational effects, and this is the object of our paper. In section 2 we review the phenomenological form taken by the Einstein field equations for a medium which is gravitationally polarized by alignment of internal spins. The effect of this spin polarization on the gravimagnetic potential near infinity can be inferred from an integral identity due to Komar, and this is discussed in section 3. These results are exact and valid for strong fields. If the gravitational field is weak (i.e. non-relativistic) the linearized Einstein equations provide a complete solution of the problem (section 4). Finally section 5 is devoted to the simplest realisation of the superfluid neutron star within the framework of linearized gravity, namely a spherical star of constant density with purely azimuthal, locally irrotational flow. This flow corresponds to having a single vortex filament along the axis. The origin of the contribution to the far-field gravimagnetic field is traced back to the singularity of the flow along the axis of rotation.

2) **Einstein field equations for a medium with internal spin**

Analogously to the Maxwell-Lorentz equations for a dielectric medium, there is a statistically averaged form of the Einstein field equations appropriate for
the coarse-grained description of a gravitationally polarized granular medium [4]. In the case of interest to us here the polarization is a spin alignment of the granules.

At the sub-macrolevel, the material inside each granule is described by a symmetric conserved tensor $T_{\text{micro}}^{ab}$, representing the flux of 4-momentum, which couples to the Einstein tensor in the standard way. On larger scales, it becomes meaningful to split the complex motion of the material into a translational motion of the granules and a distribution of internal spins. Correspondingly the coarse-grained average $T^{ab} = \langle T_{\text{micro}}^{ab} \rangle$ splits into $T^{ab}$, the flux of translational 4-momentum and a contribution from $S_{abc}$, the flux of spin angular momentum (skew in the first two indices: $S^{abc} = S^{[abc]}$). Detailed analysis [8] shows that the coarse-grained form of the Einstein field equations is

$$G^{ab} = 8\pi T^{ab} \quad \text{where} \quad T^{ab} = T^{ab} + \nabla_c U^{abc}, \quad U^{abc} = \frac{1}{2} (S^{abc} + S^{bca} - S^{bac})$$

(1)

The translational momentum flux $T^{ab}$ is neither symmetric nor in general (in a curved spacetime) conserved. The symmetry and conservation of $T^{ab}$ are equivalent to the conditions

$$T^{[ab]} = \frac{1}{2} \nabla_c S^{abc}, \quad \nabla_b T^{ab} = -\frac{1}{2} R^{a}_{	ext{bcd}} S^{cdb}.$$  

(2)

Equations (1) are formally identical with the well-known Belinfante-Rosenfeld prescription for symmetrizing what is called (in a field-theoretic context) the “canonical” stress-energy tensor [9].

The simplest illustrative example is a gas of spinning particles. If the skew tensor $s^{ab}$ denotes the angular momentum of a particle, $p^a$ its 4-momentum and $v^a$ its 4-velocity, then

$$T^{ab}(x) = \int N(x, p, s) p^a v^b \omega_g(p, s), \quad S^{abc}(x) = \int N(x, p, s) s^{ab} v^c \omega_g(p, s),$$

where $N = N(x, p, s)$ is a distribution function and $\omega_g(p, s)$ the invariant volume form on spin-momentum space. Here the asymmetry of $T^{ab}$ arises from the non-collinearity of $p^a$ and $v^a$ for a spinning particle.
3) Tolman-Komar integral identities

To check that the spin angular momentum concealed in (1) makes its proper contribution to the gravimagnetic potential $g_{\phi t}$, it is simplest to appeal to the Tolman-Komar integral identities (see e.g. [11]). These are rigorous general-relativistic analogues of Gauss’s flux theorem, which hold for spacetimes admitting Killing-symmetries and relate volume integrals over the matter content of an isolated system to surface integrals at infinity

$$\nabla_{(a} \xi_{b)} = 0 \Rightarrow \nabla^2 \xi_b = -\xi_a R^a_b$$

$$\int_{\partial \Sigma} \nabla_a \xi_b d^2 \sigma^{ab} = -8\pi \int_{\Sigma} \xi_a (T^a_b - \frac{1}{2} \delta^a_b T) d^3 \sigma^b$$

(3)

For stationary fields ($\xi^a = \partial_t^a$) we have Tolman’s theorem

$$m = \int (T^i_i - T^t_t) \sqrt{-g} d^3 x,$$

where the integration is over a $t = \text{const}$ 3-space, $i$ runs over the three spatial coordinates, and the gravitational mass $m$ is defined by the asymptotic form

$$g_{tt} \approx (1 - \frac{2m}{r}) \quad (r \to \infty).$$

For an axisymmetric system, the gravitating angular momentum $ma$, defined by the asymptotic form

$$g_{\phi t} \approx -\frac{2ma \sin^2 \theta}{r} \quad (r \to \infty)$$

in asymptotically spherical coordinates, is similarly given by

$$ma = \int T_{\phi}^t \sqrt{-g} d^3 x,$$

(4)

For our purpose $T_{\phi}^t$ is most conveniently recast in the covariant form $\xi^a T_a^b \nabla_b t$, where $\xi^a = \partial_\phi^a$ is the axial Killing vector. Making use of (3), the identity

$$U^{abc} - U^{cba} = S^{abc},$$

(5)
and the Killing property that $\nabla_a \xi_b$ is skew, we find

$$\int_\Sigma \xi^a (T^b_a - T^b_a) d^3 \sigma_b = \int_\Sigma \nabla_c (\xi^a U^b_a) d^3 \sigma_b - \frac{1}{2} \int_\Sigma \nabla_c \xi_a S^{acb} d^3 \sigma_b$$

Since the total derivative integrates to zero, (4) becomes

$$ma = \int T^t_\phi \sqrt{-g} d^3 x - \frac{1}{2} \int S^{act} (\nabla_c \xi_a) \sqrt{-g} d^3 x,$$

in which the separate contributions of the orbital and spin angular momentum are displayed explicitly. To check that the second term has indeed the right form, we go to the non-relativistic weak-field limit, in which we can set

$$\xi_a \approx \rho^2 \nabla_a \phi \quad S^{\rho \phi t} \approx \frac{1}{\rho} \epsilon^{\rho \phi z} S_z,$$

in cylindrical coordinates ($\epsilon^{abc}$ is the three-dimensional permutation symbol). Then the second term in (6) reduces to $\int S_z d^3 x$, giving the $z$-component of the total spin angular momentum as it should.

4) **Linearized theory**

In the case of weak fields we can obtain a complete solution to the problem by integrating the linearized Einstein equations [10].

In approximately rectangular coordinates the metric perturbation $h_{ab}$ and its trace-reversed form $\bar{h}_{ab}$ are “small” quantities defined by

$$g_{ab} = \eta_{ab} + h_{ab}, \quad \bar{h}_{ab} = h_{ab} - \frac{1}{2} \eta_{ab},$$

where $\eta_{ab}$ is the Minkowski metric. The coordinate freedom allows us to impose the four (“harmonic”) gauge conditions $\partial_a \bar{h}^{ab} = 0$, with respect to which the Einstein equations and their integrability condition (the linearized conservation laws) reduce to the simple form

$$G_{ab} = -\frac{1}{2} \partial^2 \bar{h}_{ab} = 8\pi T_{ab}, \quad \partial_a T^{ab} = 0,$$

where $\partial^2$ denotes the Minkowski wave operator. We specialize to stationary fields where the solution is

$$\bar{h}_{ab}(x) = 4 \int T_{ab}(x') \frac{1}{|x - x'|} d^3 x',$$

(7)
and the conservation laws imply that (to linear order) the sources must obey the constraints
\[
\int J_i d^3x = \int T_{ij} d^3x = 0, \quad \int (x^i J^k + x^k J^i) d^3x = 0 \quad (8)
\]
where \( J^i = T^{it} \) is the momentum density and the indices \( i, k \) run from 1 to 3. (For instance the last of these follows from the integration of \( \partial_i (x^i x^k J^l) \) over a bounded source and using \( \partial \cdot J = 0 \).)

The external field of a compact source, at distances large compared to its dimensions, is thus given by
\[
\bar{h}_{ij} = 0, \quad \bar{h}_{tt} = 4m/r.
\]
To obtain the gravimagnetic vector-potential \( A_i = \bar{h}_{it} = h_{it} \) we expand \(|x - x'|^{-1}\) in (7) up to the first order in \( x' \) and note (8). This yields \( A_i \) in terms of the source’s angular momentum \( L_i \):
\[
A_i(x) = 2r^{-3}(x \times L)_i, \quad L_i = \int (x \times J(x))_i d^3x. \quad (9)
\]
For an axisymmetric source it is convenient to switch to spherical polar coordinates, for which
\[
A_i dx^i = -2r^{-3}(L^ix^k \epsilon_{ikl} dx^l) = -\frac{2ma}{r} \sin^2 \theta d\phi,
\]
where we introduced the conventional notation \( L_z = ma \). Putting all this together, we arrive at the standard form of the linearized stationary exterior metric
\[
ds^2 = (1 + \frac{2m}{r})(dx^2 + dy^2 + dz^2) - \frac{4ma}{r} \sin^2 \theta d\phi dt - (1 - \frac{2m}{r})dt^2.
\]
To exhibit directly the separate contributions of spin and orbital angular momentum to the gravimagnetic potential in (9), let us assume the orbital 4-velocity \( u^a \) non-relativistic, so that we can set \( u^t \approx 1 \) and neglect \( u^i u^t \). To this order we may set
\[
S^{abc} = S^{ab} u^c, \quad S^{ab} u_b = 0,
\]
which implies $S^{it} = S^{ik}u_k$. From (1) we then find to first order in $u^i$

$$2U^{itk} = S^{ik} = \epsilon_{ikl}S_l, \quad \partial_k U^{itk} = \frac{1}{2}(\partial \times S)^i.$$

The total momentum current $T^{it}$ in (1) therefore decomposes as

$$J^i = \rho u^i + \frac{1}{2}(\partial \times S)^i.$$

Substituting in (9) finally yields the expected result

$$L^i = \int (x \times \rho u)^i d^3x + \int S^i d^3x,$$

verifying that the angular momentum in vortices contributes in the same way as orbital angular momentum to the gravimagnetic potential.

5) A simple example: gravimagnetic field of a single vortex filament on the axis of a homogenous star

In order to give a concrete realization of the situation discussed in the previous sections let us consider the simplest possible model by taking the superfluid velocity to be the irrotational azimuthal flow $d\phi$ and assuming the mass-density to be constant within the star. The equation for the gravimagnetic potential (within the linearized theory) become

$$\ast d\ast dA = \ast d\ast F = j = \theta(a - r)d\phi$$

(10)

The fact that the “velocity” $d\phi$ becomes singular at the axis may be conveniently described by taking the distributional identity

$$d\phi = \frac{xdy - ydx}{x^2 + y^2} =: \omega_\phi \quad d\omega_\phi = 2\pi\delta^{(2)}(x)d^2x$$

(11)

into account, where $\omega_\phi$ denotes the tensor-distribution on $\mathbb{R}^3$, corresponding to $d\phi$. It is not closed and therefore certainly not exact. This fact will actually be important in the description of a vortex filament along the axis.
of rotation. Integration of the equation for the gravimagnetic potential (10) yields

\[
\begin{align*}
\text{df} = *j = \theta(a - r)dr \frac{d\theta}{\sin \theta} \\
*F = \omega_0 + \theta(a - r)(r - a)drd\phi \\
\omega_0 = 0 \Rightarrow \omega_0 = df
\end{align*}
\]

(12)

where \(\omega_0\) denotes an “integration constant” (solution of the homogeneous equation). Since the second term in (12) naively appears to be closed, one might be tempted to discard \(\omega_0\). However, in the light of (11) we find

\[
\text{df} = 0 = *df + \theta(a - r)(r - a)dr 2\pi \delta^{(2)}(x)d^2x,
\]

\[
\Delta f = *df = -2\pi \theta(a - |z|)(|z| - a)\frac{z}{|z|}\delta^{(2)}(x).
\]

(13)

This shows that we cannot take \(\omega_0\) to be zero, which would have produced a gravimagnetic field of compact support, namely within the star. It is precisely the existence of the filament, represented by concentrated terms in (13), which gives rise to contributions which are not of compact support and therefore maybe detected in the far-field. Since our model is sufficiently idealized it is possible to solve the Laplace-equation for \(f\) exactly. The calculation is considerably facilitated by taking a \(z\)-derivative of \(\Delta f\), which simplifies the inhomogeneity

\[
\Delta \partial_z f = 4\pi a\delta^{(3)}(x) - 2\pi \theta(a - |z|)\delta^{(2)}(x),
\]

and gives rise to the solution

\[
\begin{align*}
\partial_z f = -a \log \left(\frac{z}{\rho} + \sqrt{\frac{z^2}{\rho^2} + 1}\right) - \frac{1}{2} \left[(z - a) \log \left(\frac{z - a}{\rho} + \sqrt{\frac{(z - a)^2}{\rho^2} + 1}\right) \right.
\end{align*}
\]

\[
- \sqrt{(z - a)^2 + \rho^2} - (z + a) \log \left(\frac{z + a}{\rho} + \sqrt{\frac{(z + a)^2}{\rho^2} + 1}\right)
\]

\[
+ \sqrt{(z + a)^2 + \rho^2}.
\]

(14)
Inserting $\omega_0 = df$ into (12) turns the expression for $F$ into

$$F = \frac{-a}{\sqrt{\rho^2 + z^2}} \left( dz - \frac{z}{\rho^2} \tilde{x} d\tilde{x} \right) - \frac{1}{2} \left( dz \log \frac{z - a + \sqrt{(z-a)^2 + \rho^2}}{z + a + \sqrt{(z+a)^2 + \rho^2}} \right) + \left( \sqrt{(z+a)^2 + \rho^2} - \sqrt{(z-a)^2 + \rho^2} \right) \frac{\tilde{x} d\tilde{x}}{\rho^2} - \theta(a-r)(r-a) \theta \theta.$$

(15)

Integration of (15) gives rise to the potential $A$

$$A = \left\{ \frac{\rho^2}{4} \log \frac{z + a + \sqrt{(z+a)^2 + \rho^2}}{z - a + \sqrt{(z-a)^2 + \rho^2}} + \frac{z + a}{4} \sqrt{(z+a)^2 + \rho^2} - \frac{z - a}{4} \sqrt{(z-a)^2 + \rho^2} - ar - 1 \theta(a-r)(a-r) \theta \right\} d\phi$$

(16)

which is not zero in the asymptotic region.

**Conclusion**

In this paper we reviewed the influence of spin and orbital angular momentum of a material system on its gravitational field. Within the general framework of gravitationally “polarized” matter one can show that both of them contribute to the angular momentum detected by an asymptotic observer. This investigation settles the recently discussed issue about the imprint of vortex lines found in a superfluid neutron star on the asymptotic gravitational field. Specifically in the linearized approximation of general relativity, one can obtain an explicit solution which clearly exhibits the effect of the vortex line on the gravimagnetic potential.

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\footnote{In equation (17) of this reference, the factor 1/2 on the right-hand side should be 1/4. This has consequences for equations (18) and (19). We thank Yosef Verbin pointing out this error.}