Scaling Study of the Leptonic Decay Constants of Heavy-Light Mesons: A Consumers Report on Improvement Factors

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A high statistics calculation, performed at $\beta = 5.74$, 6.00 and 6.26, enables us to study the variation of the leptonic decay constants $f_P$ of heavy pseudoscalar mesons with the lattice spacing $a$. We observe only a weak $a$ dependence when the standard $\sqrt{2}\kappa$ normalization is used for the quark fields, whereas application of the Kronfeld-Mackenzie normalization induces a stronger variation with $a$. Increasing the meson mass from 1.1GeV to 2.3GeV this situation becomes even more pronounced.

1. INTRODUCTION

The prediction of the leptonic decay constants of the $D$ and the $B$ meson within the framework of lattice QCD is a challenging but also very delicate problem, since in the region of heavy mesons, the inverse of the respective masses comes close to currently reachable lattice resolutions. Therefore large discretization effects may contaminate the results.

The question how to suppress these unphysical contributions has been tackled from various sides\cite{1,2,3}. Using meanfield arguments, Kronfeld and Mackenzie suggested that the replacement of the standard $\sqrt{2}\kappa$ normalization of Wilson quark fields by $\sqrt{1 - 3\kappa/4\kappa_c}$ should significantly reduce the effects of finite lattice spacing.

The present study takes a rather empirical approach to the issue of finite $a$ effects in the region of heavy mesons. We vary the lattice spacing in the currently accessible range and analyse the corresponding variation of $f_P$. Clearly a strong dependence of $f_P$ on $a$ would indicate the presence of large discretisation errors, whereas a weak dependence would be associated with smaller contaminations. In this sense our approach is perfectly suited to judge on the efficiency of different quark field normalizations within a given $a$ region.

We will finally perform an extrapolation of $f_P$ to the continuum, assuming that its functional dependence on $a$ is linear in the leading part, as suggested by the data.

2. PREPARATION

In order to visualize unambiguously the $a$ dependence of $f_P$, we have to take care that the finite $a$ effects are not hidden in the statistical noise or distorted by incomplete groundstate projection of the meson propagator and effects due to the finite size of the lattice. Therefore we have done our calculation with high statistics, keeping the errors of the raw data below 5%. We have varied the lattice size from about 0.7 fm to 2 fm and have smeared the quark fields with the well established Gauss like Wuppertal wavefunction ($n = 100, \alpha = 4$). In table I we display the lattice parameters together with the lattice spacing, taken from the stringtension $\sigma$. The influence of the finite lattice extension on $f_P$ was checked by comparing the results at different lattice sizes and fixed lattice constant. We find that finite size effects are small once the lattice extension becomes as large as 1.4fm.
Table 1
Lattice parameters

| $\beta$ | $a^{-1}_\sigma$ | $\sigma = 1.118(9)$ | $\beta$ | $a^{-1}_\sigma$ | $\sigma = 1.876(19)$ | $\beta$ | $a^{-1}_\sigma$ | $\sigma = 2.775(18)$ |
|---------|-----------------|----------------------|---------|-----------------|----------------------|---------|-----------------|----------------------|
| $4.24$  | $1404$          | $6.36$               | $227$   | $12.48$         | $76$                 | $10.24$ | $213$           | $18.36$              |
| $6.24$  | $131$           |                      | $18.48$ | $76$            |                      | $10.24$ | $213$           | $18.36$              |
| $8.24$  | $175$           | $12.36$              | $204$   | $18.48$         | $76$                 | $10.24$ | $213$           | $18.36$              |
| $10.24$ | $213$           | $18.36$              | $9$     |                 |                      | $10.24$ | $213$           | $18.36$              |

3. RESULTS

3.1. Finite $a$ effects

In fig. 1 we show the leptonic decay constant $f_P$ as a function of $a$ both in the $\sqrt{2}\kappa$ normalization (open symbols) and in the Kronfeld-Mackenzie normalization (closed symbols). The light quark mass has been extrapolated to the chiral limit and we have interpolated between the results at adjacent heavy quark masses (c.f. table 2) in order to keep the meson mass (in GeV) fixed when the $a$ dependence of $f_P(M_P, a)$ is investigated. Due to their small statistical errors, we have used stringtension measurements \[\text{[5]}\] to relate our data to a physical scale. The renormalization factor $Z_A$ was taken from perturbation theory \[\text{[6]}\] with an effective coupling $\bar{g}^2 = 3g_0^2/ \langle tr P_{\mu\nu} \rangle$, recommended in ref. \[\text{[3]}\] ($P_{\mu\nu} \equiv 1 \times 1$ Wilson loop).

It goes without saying that the $a$ dependence of $f_P$ must be different in the two normalizations. Very surprisingly, however, fig. 1 shows clearly that - in contrast to the $\sqrt{2}\kappa$ normalized results - the variation with $a$ becomes stronger and stronger with increasing meson mass when the KroMac normalization is used. This means that – at least in the displayed $a$ and $M_P$ range – the KroMac normalization does a bad job: Instead of suppressing finite $a$ effects it enhances them.

In order to connect our results to the (physical) continuum, we followed the behavior suggested by the data in both normalizations and extrapolated linearly\[\text{[6]}\] to $a = 0$. As can be seen from fig. 1 we obtain nice agreement of the results, although the KroMac normalization has induced considerably larger errors.

3.2. Heavy mass extrapolation

The most 'natural' scale for $f_P$ is $f_\pi$, since the uncertainty originating from the renormalization constant $Z_A$ cancels out in this case. Lattice measurements of $f_\pi$ are generally affected with large statistical errors and therefore we have decided to convert our results to this scale only after having performed the $a \rightarrow 0$ extrapolation of $f_P$. To achieve this we have decoupled the extrapolations according to $f_\pi / f_P(a \rightarrow 0) = f_\pi / f_P(a \rightarrow 0) / f_\pi / f_P(a \rightarrow 0)$.

\[1\] $Z_A = 1 - 0.133g^2$ for standard normalization and $Z_A = 1 - 0.0248g^2$ in the case of KroMac normalization.

\[2\] Since the $a$ dependence cannot be exactly linear for both normalizations at the same time, we have excluded those points from the fit where $\sqrt{1 - 3\sigma_{a=0}/\sqrt{2\kappa}} > 1.6$. 

Figure 1. $f_P$ as a function of $a$. Points connected only by dashed lines were not used in the extrapolation.
Table 2
Decay constant and meson mass in lattice units. The light quark has been extrapolated to \( \kappa_c \).

| \( \beta = 5.74 \) | \( \beta = 6.00 \) | \( \beta = 6.26 \) |
|----------------|----------------|----------------|
| \( \kappa_h \) | \( f_P/Z_A \) | \( M_P \) | \( \kappa_h \) | \( f_P/Z_A \) | \( M_P \) | \( \kappa_h \) | \( f_P/Z_A \) | \( M_P \) |
| 0.06 | 0.1197(102) | 2.502(13) | 0.10 | 0.0873(17) | 1.498(10) | 0.09 | 0.0437(32) | 1.579(19) |
| 0.09 | 0.1629(52) | 1.871(13) | 0.115 | 0.0983(18) | 1.197(7) | 0.10 | 0.0486(31) | 1.375(14) |
| 0.125 | 0.1890(33) | 1.205(7) | 0.125 | 0.1038(18) | 0.995(5) | 0.120 | 0.0609(33) | 0.965(10) |
| 0.140 | 0.1907(38) | 0.904(6) | 0.135 | 0.1085(19) | 0.780(7) | 0.135 | 0.0689(29) | 0.636(6) |
| 0.150 | 0.1829(53) | 0.684(5) | 0.145 | 0.1032(31) | 0.551(2) | 0.145 | 0.0711(24) | 0.382(4) |

Although the \( O(\bar{g}^4) \) uncertainty in \( Z_A \) does not cancel out exactly if one first extrapolates and then takes the ratio, its effect should be roughly the same in numerator and denominator. To obtain the denominator of this ratio we used both our own data and the results quoted in refs. [7,8,2,9]. Since the \( a \) dependence of \( f_\pi/\sqrt{\sigma} \) is weak, a linear extrapolation to \( a = 0 \) is well justified and leads to \( \hat{f}_\pi/\sqrt{\sigma}(a = 0) = 0.269(12) \).

In figure 2 we display our final results at \( a = 0 \) in the form \( \hat{f}_P(1/M_P) \), together with our static value from ref. [3]. The new data appears to depend only weakly on \( M_P \). Because of the various extra- and interpolations however, the data points carry error bars of order 25% and therefore do not exclude a stronger variation in \( M_P \). Given this situation we draw an error band that links the conventional results with the static point. The \( M_P \) dependence of the error band was chosen according to the ansatz \( \hat{f}_P = c_0 + c_1 M_P + c_2 M_P^2 \).

At the location of the B and D meson the error band corresponds to the bounds \( 155 \text{MeV} \leq f_B \leq 242 \text{MeV} \), \( 150 \text{MeV} \leq f_D \leq 200 \text{MeV} \). It is evident from figure 2 that these bounds are strongly affected by the size and uncertainty of \( f_{\text{stat}} \). More work is necessary to obtain an accurate prediction for \( f_B \).

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