ALMOST EVERYWHERE STRONG SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES

GYÖRGY GÁT AND USHANGI GOGINA V

Abstract. In this paper we study the a.e. strong convergence of the quadratical partial sums of the two-dimensional Walsh-Fourier series. Namely, we prove the a.e. relation
\[
\left( \frac{1}{n} \sum_{m=0}^{n-1} |S_{nm}f - f|^p \right)^{1/p} \to 0
\]
for every two-dimensional functions belonging to $L\log L$ and $0 < p \leq 2$. From the theorem of Getsadze it follows that the space $L\log L$ can not be enlarged with preserving this strong summability property.

1. Introduction

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}\!:=\!\mathbb{P}\!\cup\!\{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let $G$ be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of $G$ are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ ($k \in \mathbb{N}$). The group operation on $G$ is the coordinate-wise addition, the measure (denote by $\mu$) and the topology are the product measure and topology. The compact Abelian group $G$ is called the Walsh group. A base for the neighborhoods of $G$ can be given in the following way:

\[
I_0(x) := G, \quad I_n(x) := I_n(x_0, ..., x_{n-1}) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...), x \in G, n \in \mathbb{N}\}.
\]

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of $G$, $I_n := I_n(0)$ ($n \in \mathbb{N}$). Set $e_n := (0, ..., 0, 1, 0, ...)$ $\in G$ the $n$th coordinate of which is 1 and the rest are zeros ($n \in \mathbb{N}$).

For $k \in \mathbb{N}$ and $x \in G$ denote by
\[
r_k(x) := (-1)^{rx_k} \quad (x \in G, k \in \mathbb{N})
\]

02010 Mathematics Subject Classification: 42C10

Key words and phrases: two-dimensional Walsh system, strong Marcinkiewicz means, a.e. convergence.

Research was supported by project TÁMOP-4.2.2.A-11/1/KONV-2012-0051 and by Shota Rustaveli National Science Foundation grant no.31/48 (Operators in some function spaces and their applications in Fourier analysis).
the $k$-th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ (i.e. $n$ is expressed in the number system of base 2. For $n > 0$ denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{n-1} n_k x_k} \quad (x \in G, n \in \mathbb{P}),$$

and $w_0 := 1$. The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [13] and [32])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in T_n. \end{cases}$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on the $G \times G$. The notation $a \lessapprox b$ in the whole paper stands for $a \leq c \cdot b$, where $c$ is an absolute constant.

The rectangular partial sums of the 2-dimensional Walsh-Fourier series are defined as follows:

$$S_{M,N}(x,y,f) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) w_i(x) w_j(y),$$

where the number

$$\hat{f}(i,j) = \int_{G \times G} f(x,y) w_i(x) w_j(y) d\mu(x,y)$$

is said to be the $(i,j)$th Walsh-Fourier coefficient of the function $f$.

Denote

$$S_{n}^{(1)}(x,y,f) := \sum_{l=0}^{n-1} \hat{f}(l,y) w_l(x),$$

$$S_{n}^{(2)}(x,y,f) := \sum_{r=0}^{m-1} \hat{f}(x,r) w_r(y),$$

where

$$\hat{f}(l,y) = \int_{G} f(x,y) w_l(x) d\mu(x)$$

and

$$\hat{f}(x,r) = \int_{G} f(x,y) w_r(y) d\mu(y).$$
The norm (or quasinorm) of the space $L_p(G \times G)$ is defined by

$$\|f\|_p := \left( \int_{G \times G} |f(x,y)|^p \, d\mu(x,y) \right)^{1/p} \quad (0 < p < +\infty).$$

We denote by $L \log L(G \times G)$ the class of measurable functions $f$, with

$$\int_{G \times G} |f| \log^+ |f| < \infty,$$

where $\log^+ u := \log(1, \infty) \log u$, where $\mathbb{I}_E$ is character function of the set $E$.

Denote by $S^T_n(x,f)$ the partial sums of the trigonometric Fourier series of $f$ and let

$$\sigma^T_n(x,f) = \frac{1}{n+1} \sum_{k=0}^{n} S^T_k(x,f)$$

be the $(C,1)$ means. Fejér [11] proved that $\sigma^T_n(f)$ converges to $f$ uniformly for any $2\pi$-periodic continuous function. Lebesgue in [17] established almost everywhere convergence of $(C,1)$ means if $f \in L_1(T)$, $T := [-\pi, \pi)$. The strong summability problem, i.e. the convergence of the strong means

$$(2) \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |S^T_k(x,f) - f(x)|^p, \quad x \in T, \quad p > 0,$$

was first considered by Hardy and Littlewood in [14]. They showed that for any $f \in L_r(T)$ ($1 < r < \infty$) the strong means tend to 0 a.e., if $n \to \infty$. The Fourier series of $f \in L_1(T)$ is said to be $(H,p)$-summable at $x \in T$, if the values [2] converge to 0 as $n \to \infty$. The $(H,p)$-summability problem in $L_1(T)$ has been investigated by Marcinkiewicz [22] for $p = 2$, and later by Zygmund [41] for the general case $1 \leq p < \infty$. Oskolkov in [24] proved the following: Let $f \in L_1(T)$ and let $\Phi$ be a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and

$$(3) \quad \ln \Phi(t) = O(t/ \ln t) \quad (t \to \infty).$$

Then for almost all $x$

$$(4) \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(|S^T_k(x,f) - f(x)|\right) = 0.$$
for a. e. $x \in \mathbb{T}$.

Karagulyan [15] proved that the following is true.

**Theorem B.** Suppose that a continuous increasing function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$, satisfies the condition

$$
\limsup_{t \to +\infty} \frac{\log \Phi(t)}{t} = \infty.
$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$
\limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi \left( |S_{kk}^T(x, f)| \right) = \infty
$$

holds everywhere on $\mathbb{T}$.

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [23] has proved, that if $f \in L \log L (\mathbb{T}^2), \mathbb{T} := [-\pi, \pi)^2$, then

$$
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( S_{kk}^T(x, y, f) - f(x, y) \right) = 0
$$

for a. e. $(x, y) \in \mathbb{T}^2$. Zhizhiashvili [39] improved this result showing that class $L \log L (\mathbb{T}^2)$ can be replaced by $L_1(\mathbb{T}^2)$.

From a result of Konyagin [16] it follows that for every $\varepsilon > 0$ there exists a function $f \in L \log^{1-\varepsilon} (\mathbb{T}^2)$ such that

$$
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left| S_{kk}^T(x, y, f) - f(x, y) \right| \neq 0 \text{ for a. e. } (x, y) \in \mathbb{T}^2.
$$

These results show that in the case of one dimensional functions the $(C,1)$ summability and $(C,1)$ strong summability we have the same maximal convergence spaces. That is, in both cases we have $L_1$. But, the situation changes as we step further to the case of two dimensional functions. In other words, the spaces of functions with almost everywhere summable Marcinkiewicz and strong Marcinkiewicz means are different.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see Schipp [28, 29, 30], Fridli [2, 3], Leindler [17, 18, 19, 20, 21], Totik [33, 34, 35], Fridli and Schipp [3], Rodin [26], Weisz [37, 38], Gabisonia [4].

The problems of summability of cubical partial sums of multiple Fourier series have been investigated by Gogoladze [10, 11, 12], Wang [36], Zhag [40], Gukhov [7], Goginava [8], Gát, Goginava, Tkebuchava [7], Goginava, Gogoladze [9].

For Walsh system Schipp [27] proved that the following is true.
Theorem C. Let \( f \in L_1(G) \). Then for any \( A > 0 \)
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \exp \left( A \left| S_k(x,f) - f(x) \right| \right) - 1 \right) = 0
\]
for a.e. \( x \in G \).

Schipp in [27] introduce the following operator
\[
V_n f(x) := \left( \frac{1}{2^n} \int_G \left( \sum_{j=0}^{n-1} 2^{j-1} \mathbb{1}_{I_j}(t) S_{2^n} f(x + t + e_j) \right)^2 d\mu(t) \right)^{1/2}.
\]

Let \( Vf := \sup_n V_n f \).

The following theorem is proved by Schipp.

Theorem D ([27]). Let \( f \in L_1(G) \). Then
\[
\mu \{ |Vf| > \lambda \} \lesssim \frac{\|f\|_1}{\lambda}.
\]

In [9] it is studied the exponential uniform strong approximation of the Marcinkiewicz means of the two-dimensional Walsh-Fourier series. We say that the function \( \psi \) belongs to the class \( \Psi \) if it increase on \([0, +\infty)\) and
\[
\lim_{u \to 0} \psi(u) = \psi(0) = 0.
\]

Theorem E ([9]). a) Let \( \varphi \in \Psi \) and let the inequality
\[
\lim_{u \to \infty} \frac{\varphi(u)}{\sqrt{u}} < \infty
\]
hold. Then for any function \( f \in C(G \times G) \) the equality
\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \left( e^{\varphi(|S_i(f)-f|)} - 1 \right) \right\|_C = 0
\]
is satisfied.

b) For any function \( \varphi \in \Psi \) satisfying the condition
\[
\lim_{u \to \infty} \frac{\varphi(u)}{\sqrt{u}} = \infty
\]
there exists a function \( F \in C(G \times G) \) such that
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \left( e^{\varphi(|S_i(0,0,F)-F(0,0)|)} - 1 \right) = +\infty.
\]
For the two-dimensional Walsh-Fourier series Weisz [38] proved that if \( f \in L^1(G \times G) \) then

\[
\frac{1}{n} \sum_{j=0}^{n-1} (S_{jj}(x, y; f) - f(x, y)) \to 0
\]

for a. e. \((x, y) \in G \times G\).

In the paper we consider the strong means

\[
H^p_n f := \left( \frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_{mm} f|^p \right)^{1/p}
\]

and the maximal strong operator

\[
H^*_n f := \sup_{n \in \mathbb{N}} H^p_n f.
\]

We study the a. e. convergence of strong Marcinkiewicz means of the two-dimensional Walsh-Fourier series. In particular, the following is true.

**Theorem 1.** Let \( f \in L \log L(G \times G) \) and \( 0 < p \leq 2 \). Then

\[
\mu \{H^p_n f > \lambda\} \lesssim \frac{1}{\lambda} \left( 1 + \iint_{G \times G} |f| \log^+ |f| \right).
\]

The weak type \( (L \log^+ L, 1) \) inequality and the usual density argument of Marcinkiewicz and Zygmund imply

**Theorem 2.** Let \( f \in L \log L(G \times G) \) and \( 0 < p \leq 2 \). Then

\[
\left( \frac{1}{n} \sum_{m=0}^{n-1} |S_{mm}(x, y, f) - f(x, y)|^p \right)^{1/p} \to 0 \text{ for a.e. } (x, y) \in G \times G \text{ as } n \to \infty.
\]

We note that from the theorem of Getsadze [6] it follows that the class \( L \log L \) in the last theorem is necessary in the context of strong summability in question. That is, it is not possible to give a larger convergence space (of the form \( L \log L \phi(L) \) with \( \phi(\infty) = 0 \)) than \( L \log L \). This means a sharp contrast between the one and two dimensional strong summability.

We also note that in the case of trigonometric system Sjölin proved [31] that for every \( p > 1 \) and two variable function \( f \in L_p(\mathbb{T}^2) \) the almost everywhere convergence \( S_{nn} f \to f \) \((n \to \infty)\) holds. Since this issue with respect to the Walsh system is still open, then in this point of view Theorem 2 may seem more interesting.

### 2. Proof of Theorems

Let \( f \in L_1(G \times G) \). Then the dyadic maximal function is given by

\[
Mf(x, y) := \sup_{n \in \mathbb{N}} 2^{2n} \left| \int_{I_n \times I_n} f(x + s, y + t) \, d\mu(s, t) \right|.
\]
For a two-dimensional integrable function $f$ we need to introduce the following hybrid maximal functions

$$M_1 f (x, y) := \sup_{n \in \mathbb{N}} 2^n \int_{I_n} |f (x + s, y)| \, d\mu (s),$$

$$M_2 f (x, y) := \sup_{n \in \mathbb{N}} 2^n \int_{I_n} |f (x, y + t)| \, d\mu (t),$$

(7) \quad V_1 (x, y, f)

$$: = \sup_{n \in \mathbb{N}} \left( \frac{1}{2^n} \int_{G} \left( \sum_{j=0}^{n-1} 2^{j-1} \mathbb{1}_{I_j} (t) S^{(1)}_{2^n} f (x + t + e_j, y) \right)^2 \, d\mu (t) \right)^{1/2},$$

(8) \quad V_2 (x, y, f)

$$: = \sup_{n \in \mathbb{N}} \left( \frac{1}{2^n} \int_{G} \left( \sum_{j=0}^{n-1} 2^{j-1} \mathbb{1}_{I_j} (t) S^{(2)}_{2^n} f (x, y + t + e_j) \right)^2 \, d\mu (t) \right)^{1/2}.$$
Analogously, we can prove that
\[
\mu \{ (x, y) \in G \times G : V_2 f (x, y) > \lambda \} \lesssim \frac{\|f\|_1}{\lambda}.
\]

For Dirichlet kernel Schipp proved the following representation [27, page 622]
\[
D_m (x) = \sum_{k=0}^{n-1} \mathbb{I}_{k \setminus k+1} (x) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m (x + e_j) - \frac{1}{2} w_m (x) + (m + 1/2) \mathbb{I}_n (x),
\]
where \( m < 2^n \) and
\[
\varepsilon_{kj} = \begin{cases} 
-1, & \text{if } j = 0, 1, ..., k - 1, \\
+1, & \text{if } j = k.
\end{cases}
\]

**Proof of Theorem** [7] First, we prove that the following estimation holds
\[
\left( \frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_{mm} (x, y, f)|^2 \right)^{1/2} \lesssim V_2 (x, y, M_1 f) + V_1 (x, y, M_2 f) + M f (x, y)
+ V_2 (x, y, A) + V_1 (x, y, A) + \|f\|_1,
\]
where \( A \) is an integrable on \( G \times G \) function of two variable which will be defined below.

It is easy to show that
\[
\left( \sum_{m=0}^{2^n-1} |S_{mm} (x, y, f)|^2 \right)^{1/2} = \left( \sum_{m=0}^{2^n-1} |S_{mm} (x, y, S_{2^n, 2^n} f)|^2 \right)^{1/2}
= \left( \sum_{m=0}^{2^n-1} \left| \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) D_m (s) D_m (t) d\mu (s, t) \right|^2 \right)^{1/2}
\leq \sup_{\{\alpha_{mn} (x, y)\}} \left| \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) D_m (s) D_m (t) d\mu (s, t) \right|
\]
by taking the supremum over all \( \{\alpha_{mn} (x, y)\} \) for which
\[
\left( \sum_{m=0}^{2^n-1} |\alpha_{mn} (x, y)|^2 \right)^{1/2} \leq 1.
\]
From (13) we can write

\[(15) \quad \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) D_m (s) D_m (t) d\mu (s, t)\]

\[= \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} k_1 \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} I_{k_1 \setminus I_{k_1+1}} (s)\]

\[\times \int \int_{k_2 \setminus I_{k_2+1}} (t) \varepsilon_{k_1, j_1} \varepsilon_{k_2, j_2} 2^{j_1+j_2-2}\]

\[-\frac{1}{2} \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{k_1=0}^{n-1} k_1 \sum_{j_1=0}^{k_1} I_{k_1 \setminus I_{k_1+1}} (s)\]

\[\times \varepsilon_{k_1, j_1} 2^{j_1-1} \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_1} + e_{j_2}) d\mu (s, t)\]

\[+ \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{k_1=0}^{n-1} k_1 \sum_{j_1=0}^{k_1} I_{k_1 \setminus I_{k_1+1}} (s)\]

\[\times \varepsilon_{k_1, j_1} 2^{j_1-1} I_n (t) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + e_{j_1}) (m + 1/2) d\mu (s, t)\]

\[-\frac{1}{2} \int \int_{G \times G} S_{2^n, 2^n} f (x + s, y + t) \sum_{k_2=0}^{n-1} k_2 \sum_{j_2=0}^{k_2} I_{k_2 \setminus I_{k_2+1}} (t)\]

\[\times \varepsilon_{k_2, j_2} 2^{j_2-1} \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_2}) d\mu (s, t)\]

\[+ \frac{1}{4} \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t) d\mu (s, t)\]

\[-\frac{1}{2} \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s)\]

\[\times \left( m + \frac{1}{2} \right) I_n (t) d\mu (s, t)\]

\[+ \int \int_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{k_2=0}^{n-1} k_2 \sum_{j_2=0}^{k_2} I_{k_2 \setminus I_{k_2+1}} (t)\]
\[ x_{k,j} 2^{j-1} \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + \epsilon_j) \]

\[ \times \left( m + \frac{1}{2} \right) \mathbb{I}_{I_n} (s) d\mu (s, t) \]

\[ - \frac{1}{2} \iint_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t) \]

\[ \times \left( m + \frac{1}{2} \right) \mathbb{I}_{I_n} (s) d\mu (s, t) \]

\[ + \iint_{G \times G} S_{2^n, 2^n} (x + s, y + t, f) \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) \]

\[ \times \left( m + \frac{1}{2} \right)^2 \mathbb{I}_{I_n} (s) \mathbb{I}_{I_n} (t) d\mu (s, t) \]

\[ := \sum_{k=1}^{9} J_k. \]

It is easy to show that

\[ |J_5| \lesssim \left( \sum_{m=0}^{2^n-1} |\alpha_{mn} (x, y)|^2 \right)^{1/2} \]

\[ \times 2^{(5/2)n} \iint_{I_n \times I_n} |f (x + s, y + t)| d\mu (s, t) \]

\[ \lesssim 2^{n/2} M f (x, y), \]

\[ (16) \]

\[ |J_5| \lesssim 2^{n/2} \left( \sum_{m=0}^{2^n-1} |\alpha_{mn} (x, y)|^2 \right)^{1/2} \| f \|_1 \lesssim 2^{n/2} \| f \|_1. \]

\[ (17) \]

\[ |J_8| \lesssim \iint_{I_n \times G} S_{2^n, 2^n} (x + s, y + t, f) \]

\[ \times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t) (m + 1/2) d\mu (s, t) \]

\[ = \iint_{I_n \times G} \left( 2^n \iint_{I_n} |f (x + s, y + t + v)| d\mu (v) \right) \]

\[ \times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t) (m + 1/2) d\mu (s, t) \]

\[ \](18)
= \int_G \left( \int_{J_n} \left( 2^n \int_{I_n} |f(x+s, y+t+v)| \, d\mu(s) \right) \, d\mu(v) \right) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) \, w_m(t) \, (m + 1/2) \, d\mu(t) \\
\lesssim \int_G \left( \int_{J_n} M_1 f(x, y+t+v) \, d\mu(v) \right) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) \, w_m(t) \, (m + 1/2) \, d\mu(t) \\
\lesssim 2^{-n} \int_G S_{2^n}^{(2)}(x, y, t, M_1 f) \\
\times \left( \int \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) \, w_m(t) \, (m + 1/2) \right|^2 \, d\mu(t) \right)^{1/2} \\
\lesssim 2^{n/2} \left( \sum_{m=0}^{2^n-1} |\alpha_{mn}(x, y)|^2 \right)^{1/2} \, V_2(x, y, M_1 f) \\
\lesssim 2^{n/2} V_2(x, y, M_1 f).

Analogously, we can prove that

(19) \quad |J_6| \lesssim 2^{n/2} V_1(x, y, M_2 f).

Now, we estimate $J_7$. Since

$$
\int_{I_n} S_{2^n, 2^n} (x+s, y+t, |f|) \, d\mu(s) = 2^{-n} S_{2^n, 2^n} (x, y+t, |f|)
$$

we can write

(20) \quad |J_7| \lesssim \sum_{j_2=0}^{n-1} \sum_{k_2=j_2}^{n-1} 2^{j_2-1} \int_{I_n \times (I_{k_2} \backslash I_{k_2+1})} S_{2^n, 2^n} (x+s, y+t, |f|)$$
\[
\begin{align*}
&\times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_2})(m + 1/2) \left| d\mu(s, t) \right| \\
&\lesssim \sum_{j_2=0}^{n-1} 2^{j_2-1} \iint_{I_n \times I_{j_2}} S_{2^n, 2^n}(x + s, y + t, |f|) \\
&\times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_2})(m + 1/2) \left| d\mu(t) \right| \\
&\lesssim 2^{-n} \sum_{j_2=0}^{n-1} 2^{j_2-1} \int_{I_{j_2}} S_{2^n, 2^n}(x, y + t, |f|) \\
&\times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_2})(m + 1/2) \left| d\mu(t) \right| \\
&\lesssim 2^{-n} \sum_{j_2=0}^{n-1} 2^{j_2-1} \int_{I_{j_2}} S_{2^n, 2^n}(x, y + t + e_{j_2}, |f|) \\
&\times \left( \int_{G} \left( \sum_{j_2=0}^{n-1} 2^{j_2-1} \|I_{j_2} \| S_{2^n, 2^n}(x, y + t + e_{j_2}, |f|) \right)^2 d\mu(t) \right)^{1/2}.
\end{align*}
\]

Since
\[
S_{2^n, 2^n}(x, y + t + e_{j_2}, |f|) = 2^n \int_{I_n} \left( 2^n \int_{I_n} |f(x + u, y + t + e_{j_2} + v)| d\mu(u) \right) d\mu(v)
\]
\[
\lesssim 2^n \int_{I_n} M_1f (x, y + t + e_{j_2} + v) \, d\mu(v) \\
= S_{2^n}^{(2)} (x, y + t + e_{j_2}, M_1f)
\]

From (20) we can write

\[
|J_7| \lesssim \left( \int_G \left( \sum_{j_2=0}^{n-1} 2^{j_2-1} \mathbb{1}_{I_{j_2}} (t) S_{2^n}^{(2)} (x, y + t + e_{j_2}, M_1f) \right)^2 \, d\mu(t) \right)^{1/2} \\
\lesssim 2^{n/2} V_2 (x, y, M_1f).
\]

Analogously, we can prove that

\[
|J_3| \lesssim 2^{n/2} V_1 (x, y, M_2f).
\]

For \(J_1\) we can write

\[
J_1 \lesssim \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} 2^{j_1+j_2-2} \\
\times \int \int_{(I_{k_1} \setminus I_{k_1+1}) \times (I_{k_2} \setminus I_{k_2+1})} S_{2^n, 2^n} (x + s, y + t, |f|) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_1} + e_{j_2}) \, d\mu(s, t) \\
\lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} 2^{j_1+j_2-2} \int \int_{I_{j_1} \times I_{j_2}} S_{2^n, 2^n} (x + s, y + t, |f|) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_1} + e_{j_2}) \, d\mu(s, t) \\
= \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} 2^{j_1+j_2-2} \int \int_{I_{j_1} \times I_{j_2}} S_{2^n, 2^n} (x + s, y + t, |f|) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_1} + e_{j_2}) \, d\mu(s, t) \\
+ \sum_{j_1=0}^{n-1} \sum_{j_2=j_1+1}^{n-1} 2^{j_1+j_2-2} \int \int_{I_{j_1} \times I_{j_2}} S_{2^n, 2^n} (x + s, y + t, |f|) \\
\times \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (s + t + e_{j_1} + e_{j_2}) \, d\mu(s, t)
\]
\[ = J_{11} + J_{12}. \]

It is easy to show that \( s + t + e_{j_2} \in I_{j_2} \) for \( s \in I_{j_1}, t \in I_{j_2} \) and \( j_2 \leq j_1 \). Hence, we can write

\[
(24) \quad J_{11} \lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_1+j_2-2} \int_I S_{2^n,n} (x + s, y + t + s + e_{j_2}, |f|) \]

\[
\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + e_{j_1}) \right| d\mu (s, t) \]

\[
\lesssim 2^{2n} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_1+j_2-2} \int_I \int_{I_n} \int_{I_n} \int_{I_n} \int_I |f (x + s + u, y + t + s + e_{j_2} + v)| d\mu (u, v) \]

\[
\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + e_{j_1}) \right| d\mu (s, t) \]

\[
\lesssim 2^{2n} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2} \]

\[
\times \int_I \int_{I_n} \left( 2^{j_1} \int_{I_{j_1}} |f (x + s + u, y + t + s + e_{j_2} + v)| d\mu (s) \right) d(u, v) \]

\[
\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + e_{j_1}) \right| d\mu (t) \]

\[
\lesssim 2^{2n} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2} \]

\[
\times \int_I \int_{I_n} \left( 2^{j_1} \int_{I_{j_1}} |f (x + s, y + t + s + e_{j_2} + u + v)| d\mu (s) \right) d(u, v) \]

\[
\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + e_{j_1}) \right| d\mu (t) \]

\[
\lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2} \]

\[
\times \int_I \left( 2^n \int_{I_n} \left( 2^{j_1} \int_{I_{j_1}} |f (x + s, y + t + s + e_{j_2} + v)| d\mu (s) \right) d(v) \right) \]
\[
\times \left\lfloor \sum_{m=0}^{2^n-1} \alpha_{mn} (x, y) w_m (t + e_{j_1}) \right\rfloor d\mu (t) .
\]

Set
\[
A_{j_1} (x, y) := 2^{j_1} \int_{I_{j_1}} |f (x + s, y + s)| d\mu (s).
\]

Then it is evident that
\[
A_{j_1} (x, y + x) = 2^{j_1} \int_{I_{j_1}} |f (x + s, y + x + s)| d\mu (s)
\]
\[= 2^{j_1} \int_{I_{j_1}} |F_2 (x + s, y)| d\mu (s),
\]
where
\[
F_2 (x, y) := f (x, y + x).
\]

From the condition of the theorem it is evident that \(F_2 \in L \log L (G \times G)\).

On the other hand,
\[
\sup A_j (x, x + y) \lesssim M_1 F_2 (x, y).
\]

Let
\[
A (x, y) := \sup_j A_j (x, y).
\]

It is evident that
\[
\int\int_{G \times G} A (x, y) d\mu (x, y) = \int\int_{G \times G} A (x, y + x) d\mu (x, y)
\]
\[\lesssim \int\int_{G \times G} M_1 F_2 (x, y) d\mu (x, y)
\]
\[\lesssim 1 + \int\int_{G \times G} |F_2 (x, y)| \log^+ |F_2 (x, y)| d\mu (x, y)
\]
\[\lesssim 1 + \int\int_{G \times G} |f (x, y)| \log^+ |f (x, y)| d\mu (x, y).
\]

Then from (24) we have
\[
|J_{11}| \lesssim \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2}
\]
\[\times \int_{I_{j_2}} \left( 2^n \int_{I_n} A (x, y + t + v + e_{j_2}) \right) d(v)
\]
\[
\sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_1}) d\mu(t) \times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_1}) d\mu(t)
\leq \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_2-2} \int_{I_{j_2}} S_{2^n}^{(2)}(x, y + t + e_{j_2}, A) \times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_1}) d\mu(t)
\leq \sum_{j_1=0}^{n-1} \int_{G} \sum_{j_2=0}^{j_1} 2^{j_2-2} \mathbb{1}_{I_{j_2}}(t) S_{2^n}^{(2)}(x, y + t + e_{j_2}, A) \times \sum_{m=0}^{2^n-1} \alpha_{mn}(x, y) w_m(t + e_{j_1}) d\mu(t)
\leq \sum_{j_1=0}^{n-1} \left( \int_{G} \sum_{j_2=0}^{j_1} 2^{j_2-2} \mathbb{1}_{I_{j_2}}(t) S_{2^n}^{(2)}(x, y + t + e_{j_2}, A) \right)^2 d\mu(t)^{1/2}
\leq \sum_{j_1=0}^{n-1} 2^{j_1/2} V_2(x, y, A) \lesssim 2^{n/2} V_2(x, y, A),
\]

where

\[A \in L_1(G \times G).\]

Analogously, we can prove that

\[(27) \quad J_{12} \lesssim 2^{n/2} V_1(x, y, A).\]

Combining (23), (26) and (27) we conclude that

\[(28) \quad |J_1| \lesssim 2^{n/2} V_1(x, y, A) + 2^{n/2} V_2(x, y, A).\]

Analogously, we can prove that

\[(29) \quad |J_2| + |J_4| \lesssim 2^{n/2} V_1(x, y, A) + 2^{n/2} V_2(x, y, A).\]

Combining (15), (16)-(22), (28), (29) we obtain of estimation (14).

Since

\[H_p f \leq H_2^2 f \quad (0 < p \leq 2),\]

and

\[\mu \{Mf > \lambda\} \lesssim \frac{\|f\|_{L_1}}{\lambda}\]
Almost everywhere strong summability

from (9), (10), (11), (12), (13), (25) and Theorem D we conclude that

\[ \mu \{ H_p^* f > \lambda \} \]

\[ \lesssim \frac{1}{\lambda} \left( \| M_1 f \|_1 + \| M_2 f \|_1 + \| A \|_1 + \| f \|_1 \right) \]

\[ \lesssim \frac{1}{\lambda} \left( 1 + \int \int_{G \times G} |f| \log^+ |f| \right) . \]

Theorem [1] is proved. \qed

References

[1] Fejér L., Untersuchungen über Fouriersche Reihen, Math. Annalen, 58 (1904), 501–569.
[2] Fridli S., Schipp F., Strong summability and Sidon type inequalities, Acta Sci. Math. (Szeged) 60 (1995), no. 1-2, 277–289.
[3] Fridli S., Schipp F., Strong approximation via Sidon type inequalities, J. Approx. Theory 94 (1998), 263–284.
[4] Gabisonia O. D., On strong summability points for Fourier series, Mat. Zametki. 5, 14 (1973), 615–626.
[5] Gát G., Goginava U., Tkebuchava G., Convergence in measure of logarithmic means of quadratical partial sums of double Walsh-Fourier series, J. Math. Anal. Appl. 323 (2006), no. 1, 535–549.
[6] Getsadze R., On the boundedness in measure of sequences of superlinear operators in classes \( L_\Phi (L) \), Acta Sci. Math. (Szeged) 71 (2005), no. 1-2, 195–226.
[7] Glukhov V. A., Summation of multiple Fourier series in multiplicative systems (Russian), Mat. Zametki 39 (1986), no. 5, 665–673.
[8] Goginava U., The weak type inequality for the maximal operator of the Marcinkiewicz-Fejér means of the two-dimensional Walsh-Fourier series, J. Approx. Theory 154 (2008), no. 2, 161–180.
[9] Goginava U., Gogoladze L., Strong approximation by Marcinkiewicz means of two-dimensional Walsh-Fourier series, Constr. Approx. 35 (2012), no. 1, 1–19.
[10] Golubov B. I., Efimov A.V., Skvortsov V.A., Series and transformations of Walsh, Moscow, 1987 (Russian); English translation, Kluwer Academic, Dordrecht, 1991.
[11] Hardy G. H., Littlewood J. E., Sur la série de Fourier d’une fonction a carré sommable, Comptes Rendus (Paris) 156 (1913), 1307–1309.
[12] Karagulyan G. A., Everywhere divergent \( \Phi \)-means of Fourier series (Russian), Mat. Zametki 80 (2006), no. 1, 50–59; translation in Math. Notes 80 (2006), no. 1-2, 47–56.
[13] Lebesgue H., Recherches sur la sommabilité forte des séries de Fourier, Math. Annalen 61 (1905), 251–280.
[14] Leindler L., Über die Approximation im starken Sinne, Acta Math. Acad. Hungar, 16 (1965), 255–282.
[19] Leindler L., On the strong approximation of Fourier series, Acta Sci. Math. (Szeged) 38 (1976), 317–324.
[20] Leindler L., Strong approximation and classes of functions, Mitteilungen Math. Seminar Giessen, 132 (1978), 29–38.
[21] Leindler L., Strong approximation by Fourier series, Akadémiai Kiadó, Budapest, 1985.
[22] Marcinkiewicz J., Sur la sommabilité forte de séries de Fourier (French), J. London Math. Soc. 14, (1939), 162–168.
[23] Marcinkiewicz J., Sur une méthode remarquable de sommation des series doublefes de Fourier, Ann. Scuola Norm. Sup. Pisa, 8 (1939), 149–160.
[24] Oskolkov K. I., Strong summability of Fourier series. (Russian) Studies in the theory of functions of several real variables and the approximation of functions, Trudy Mat. Inst. Steklov. 172 (1985), 280–290, 355.
[25] Rodin, V. A., The space BMO and strong means of Fourier series, Anal. Math. 16 (1990), no. 4, 291–302.
[26] Rodin V. A., BMO-strong means of Fourier series, Funct. anal. Appl. 23 (1989), 73–74, (Russian)
[27] Schipp F., On the strong summability of Walsh series, Publ. Math. Debrecen 52 (1998), no. 3-4, 611–633.
[28] Schipp F., Über die starke Summation von Walsh-Fourier Reihen, Acta Sci. Math. (Szeged), 30 (1969), 77–87.
[29] Schipp F., On strong approximation of Walsh-Fourier series, MTA III. Oszt. Kozl. 19(1969), 101–111 (Hungarian).
[30] Schipp F., Ky N. X., On strong summability of polynomial expansions, Anal. Math. 12 (1986), 115–128.
[31] Sjölin, P., Convergence almost everywhere of certain singular integrals and multiple Fourier series, Ark. Mat. 9 (1971), 65–90.
[32] Schipp F., Wade W., Simon P., Pal P., Walsh Series, an Introduction to Dyadic Harmonic Analysis, Adam Hilger, Bristol, New York, 1990.
[33] Totik V., on the strong approximation of Fourier series, Acta Math. Sci. Hungar 35 (1980), 151–172.
[34] Totik V., On the generalization of Fejér’s summation theorem, Functions, Series, Operators; Coll. Math. Soc. J. Bolyai (Budapest) Hungary, 35, North Holland, Amsterdam-Oxford-New-Yourk, 1980, 1195–1199.
[35] Totik V., Notes on Fourier series: Strong approximation, J. Approx. Theory, 43 (1985), 105–111.
[36] Wang, Kun Yang. Some estimates for the strong approximation of continuous periodic functions of the two variables by their sums of Marcinkiewicz type (Chinese), Beijing Shifan Daxue Xuebao 1981, no. 1, 7–22.
[37] Weisz F., Strong Marcinkiewicz summability of multi-dimensional Fourier series, Ann. Univ. Sci. Budapest. Sect. Comput. 29 (2008), 297–317.
[38] Weisz F., Convergence of double Walsh–Fourier series and Hardy spaces, Approx. Theory Appl. (N.S.) 17:2 (2001), 32–44.
[39] Zhizhiashvili L. V., Generalization of a theorem of Marcinkiewicz, Izvest.AN USSR, ser. matem. 32(1968), 1112–1122 (Russian).
[40] Zhang Y., He X., On the uniform strong approximation of Marcinkiewicz type for multivariable continuous functions, Anal. Theory Appl. 21 (2005), 377–384.
[41] Zygmund A., Trigonometric series. Cambridge University Press, Cambridge, 1959.
U. Goginava, Department of Mathematics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia

E-mail address: zazagoginava@gmail.com