A Q-ANALOGUE OF KEMPF’S VANISHING THEOREM

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Abstract. We use deep properties of Kashiwara’s crystal basis to show that the induction functor $H^0(\cdot)$ introduced by Andersen, Polo and Wen satisfies an analogon of Kempf’s vanishing Theorem.

1. Introduction

1.1. Let $G$ be a reductive connected algebraic group and let $B$ be a Borel subgroup. One of central themes of the representation theory of $G$ is the study of the induction functor $H^0$ from $B$ representations to $G$ representations. Many of the features of $H^0$ in the characteristic zero case also hold in the modular case, e.g. the properties that $H^0(\lambda) \neq 0$ if and only if $\lambda \in P^+$, and that the weights of $H^0(\lambda)$ are all less than or equal to $\lambda$. On the other hand the Borel-Weil-Bott theorem fails in general in the modular case, and hence the simplicity of $H^0(\lambda)$ also breaks down in general. Still, we consider the $H^0(\lambda)$’s to be the fundamental objects of study, the reason being that their characters, like in the characteristic zero case, are given by the Weyl character formula. This fact in turn relies on the Kempf vanishing theorem, i.e. that

$$H^i(\lambda) = 0 \text{ for } i > 0 \text{ and } \lambda \in P^+$$

1.2. In 1979 Andersen and Haboush independently found a short proof of this vanishing, see [A] and [H]. Their idea was to show the following isomorphism

$$H^i((p^r - 1)\rho + p^r \lambda) \cong St_r \otimes H^r(\lambda)^{(r)}$$

where $St_r$ is a Steinberg module and the superscript denotes the $r$-order Frobenius twist. Because of ampleness properties the left hand

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side is $O$ for $r$ sufficiently big; hence $H^i(\lambda)^{(r)}$ must be zero, and thus also $H^i(\lambda)$.

1.3. In [APW 1,2] and [AW] an induction functor $H^0$ for quantum groups is constructed and studied in great detail. Many of the results in these papers rely on specialization to the modular case. In the mixed case however, i.e. the case where the ground field is of positive characteristic prime to $l$, these methods fail to give a generalization of the Kempf vanishing theorem when $l < h$, the Coxeter number. And as higher ordered Frobenius twists do not exist for quantum groups, also the classical method sketched above fails.

1.4. Our approach to the quantum Kempf vanishing theorem is based on some properties of the crystal basis proved by Kashiwara in order to obtain the refined Demazure character formula in [K2]. In section 2 and 3 we discuss these results and in section 4 and 5 we show how they can be applied to deduce the Kempf vanishing theorem for the quantum $H^0$ in all cases.

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1.6. Since its appearance and distribution as a preprint in may 1994, there has been a couple of developments closely related to and in part dependent of our work, we here mention [Ka1], [Ka2] and [W]. As pointed out by M. Kaneda the paper [AP], is incorrect in the generality stated. It is on the other hand a key reference for certain ampleness properties that are needed in section 4 of our work. This gap was filled in [Ka1, Ka2]. In [Ka2] the universal coefficient theorems are set up that make it possible to obtain the vanishing theorem over the ground ring $\mathbb{Z}[q, q^{-1}]$. Finally, Woodcock [W] shows that the idea of using the above mentioned properties of the crystal basis to obtain the quantized Kempf’s vanishing theorem also works from a Schur algebra point of view.

2. Notation and some fundamental results

2.1. Let $\mathfrak{g}$ be a semisimple complex finite dimensional Lie-algebra. In [K1] $\mathfrak{g}$ is allowed to be a general Kac-Moody algebra, but otherwise we shall more or less follow the terminology of that paper. In particular $\{\alpha_i\}_{i}$ is the set of simple roots of $\mathfrak{g}$, $\{h_i\}_{i}$ is the set of simple coroots,
$P$ is the weight lattice, $U_q(\mathfrak{g})$ is the quantized $\mathbb{Q}(q)$-algebra generated by $e_i, f_i$ where $i \in I$ and $q^h$ where $h \in P^*$, $A$ is the subring of $\mathbb{Q}(q)$ consisting of rational function regular at $q = 0$, $V(\lambda)$ is the Weyl module for $U_q(\mathfrak{g})$ with $v_\lambda$ as a highest weight vector.

2.2. Assume $\mathfrak{g}$ has rank one. Then the weight lattice $P$ is equal to $\mathbb{Z}$. For $\lambda \geq -1$ the dimension of $V(\lambda)$ is $\lambda + 1$: a basis is $\{f^{(k)}_i v_\lambda \mid 0 \leq k \leq \lambda\}$. The action of $U_q(\mathfrak{g})$ on this is given by the following formulas:

\[
f f^{(k)}_i v_\lambda = [k + 1] f^{(k+1)}_i v_\lambda \tag{2.2.1}
\]

\[
e f^{(k)}_i v_\lambda = [\lambda - k + 1] f^{(k-1)}_i v_\lambda \tag{2.2.2}
\]

\[
q^h f^{(k)}_i v_\lambda = q^{\lambda - 2k} f^{(k)}_i v_\lambda \tag{2.2.3}
\]

where by convention $f^{(-1)}_i v_\lambda = f^{(\lambda+1)}_i v_\lambda = 0$.

2.3. By $U_q(\mathfrak{sl}_2)$-theory any $v \in V(\lambda)$ can be written uniquely in the form

\[v = \sum_n f^{(n)}_i u_n\]

where $u_n \in V(\lambda)$ is a weight vector satisfying $e_i u_n = 0$. Then the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $V(\lambda)$ are defined in the following way

\[\tilde{e}_i v := \sum_n f^{(n-1)}_i u_n, \quad \tilde{f}_i v := \sum_n f^{(n+1)}_i u_n\]

The $A$-lattice $L(\lambda) \subset V(\lambda)$ is then defined by

\[L(\lambda) := A\langle \tilde{f}_{i_1} \tilde{f}_{i_2} \ldots \tilde{f}_{i_k} v_\lambda \mid i_j \in I, k \geq 0 \rangle\]

And $B(\lambda)$ is defined as follows

\[B(\lambda) := \pi(A\langle \tilde{f}_{i_1} \tilde{f}_{i_2} \ldots \tilde{f}_{i_k} v_\lambda \mid i_j \in I, k \geq 0 \rangle \subset L(\lambda)/qL(\lambda)\]

where $\pi$ is the canonical map $\pi : L(\lambda) \to L(\lambda)/qL(\lambda)$. One of the main results of [K1] is then that $(L(\lambda), B(\lambda))$ forms a lower crystal basis of $V(\lambda)$; this means among other things that $\tilde{e}_i$ and $\tilde{f}_i$ induce operators on $B(\lambda) \cup \{0\}$, see theorem 2 of [K1].
2.4. The functions $\varepsilon_i, \varphi_i$ and $w_i : B(\lambda) \to \mathbb{Z}$ are defined in the following way:

$$\varepsilon_i(b) := \max\{n|\tilde{e}_ib \neq 0\}, \quad \varphi_i(b) := \max\{n|\tilde{f}_ib \neq 0\}, \quad w_i := \langle \text{weight}(b), h_i \rangle$$

We have the following relations between them

$$w_i(b) = \varphi_i(b) - \varepsilon_i(b) \quad (2.4.1)$$
$$\varepsilon_i(\tilde{f}_ib) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_ib) = \varphi_i(b) - 1 \quad (2.4.2)$$

and $(L(\lambda), B(\lambda))$ is a normal crystal with respect to $\varepsilon_i, \varphi_i$ and $w_i$.

2.5. The $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $f_i(n), e_i(n)$ and $q^h, \left\{ \frac{q^h}{n} \right\}$ for $h \in \mathbb{P}^*$ is denoted $U^\mathbb{Z}(\mathfrak{g})$ and the $U^\mathbb{Z}(\mathfrak{g})$-submodule $V_\mathbb{Z}(\lambda)$ of $V(\lambda)$ is by definition $U^\mathbb{Z}(\mathfrak{g})v_\lambda$. Furthermore $-\sigma$ is the $\mathbb{Q}$-automorphism of $U^\mathbb{Z}(\mathfrak{g})$ given by the formulas

$$\overline{e}_i = e_i, \quad \overline{f}_i = f_i, \quad \overline{q^h} = q^{-h}, \quad \overline{q} = q^{-1}$$

Then Kashiwara has shown, (G2) in section 7.2. of [K1], that

$$V_\mathbb{Z}(\lambda) \cap L(\lambda) \cap L(\lambda) \overset{\pi}{\cong} V_\mathbb{Z}(\lambda) \cap L(\lambda)/V_\mathbb{Z}(\lambda) \cap qL(\lambda) \quad (2.5.1)$$

where $\pi$ is the canonical map. The inverse of $\pi$ is denoted $G_\lambda$.

2.6. It is known that the crystal $B(\lambda)$ is contained in and gives a $\mathbb{Z}$-basis of $V_\mathbb{Z}(\lambda) \cap L(\lambda)/V_\mathbb{Z}(\lambda) \cap qL(\lambda)$. From 2.5.1 we now get the following results by applying Lemma 7.1.2 of [K1]:

$$V_\mathbb{Z}(\lambda) \cap L(\lambda) \cap L(\lambda) \cong \bigoplus_{b \in B(\lambda)} \mathbb{Z}G_\lambda(b) \quad (2.6.1)$$

$$V_\mathbb{Z}(\lambda) \cong \bigoplus_{b \in B(\lambda)} \mathbb{Z}[q, q^{-1}]G_\lambda(b) \quad (2.6.2)$$

$$V(\lambda) \cong \bigoplus_{b \in B(\lambda)} \mathbb{Q}(q)G_\lambda(b) \quad (2.6.3)$$

$$L(\lambda) \cong \bigoplus_{b \in B(\lambda)} AG_\lambda(b) \quad \overline{L(\lambda)} \cong \bigoplus_{b \in B(\lambda)} \overline{AG_\lambda(b)} \quad (2.6.4)$$

Due to these properties $\{G_\lambda(b)|b \in B(\lambda)\}$ is said to be a global basis.

2.7. For $\Gamma$ a $\mathbb{Z}[q, q^{-1}]$-algebra the quantum group $U^\mathbb{Z}(\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \Gamma$ is denoted $U_{\Gamma}(\mathfrak{g})$. We use the notation $U_{\Gamma}(b)$ and $U_{\Gamma}(b^-)$ for the corresponding Borel subalgebras.
2.8. Let $C$ be the [APW] category of integrable $U_q^\mathbb{Z}(\mathfrak{g})$-modules $M$. Its objects are the $U_q^\mathbb{Z}(\mathfrak{g})$-modules $M$ having a weight space decomposition

$$M = \bigoplus_{\lambda} M_\lambda$$

where for $\lambda \in P$ the $\lambda$’th weight space $M_\lambda$ of $M$ is

$$M_\lambda = \left\{ m \in M \mid q^h m = q^{<h,\lambda>} m, \left\{ q^h \right\} m = \left\{ q^{<h,\lambda>} \right\} m \text{ for all } h \in P^* \right\}$$

and such that the following local nilpotency condition holds:

$$\forall m \in M \text{ and } i \in I : \quad e_i^{(n)} m = f_i^{(n)} m = 0 \text{ for } n >> 0$$

Similarly, we consider the category $C^\geq$ of $U(b)$-modules that admit a weight space decomposition and satisfy:

$$\forall m \in M \text{ and } i \in I : \quad e_i^{(n)} m = 0 \text{ for } n >> 0$$

The category $C^\leq$ is defined likewise.

The analogous categories of modules for the specialized quantum groups $U_\Gamma(g), U_\Gamma(b)$ and $U_\Gamma(b^-)$ are denoted $C_\Gamma, C^\geq_\Gamma$ and $C^\leq_\Gamma$.

For $M$ a $U_q^\mathbb{Z}(\mathfrak{g})$-module, we define $F(M)$ as the largest submodule of $M$ which belongs to $C$.

If $M \in C$ we let $P(M)$ denote the set of weights of $M$.

2.9. Let $N \in C^\geq$. Consider the $\mathbb{Z}[q, q^{-1}]$-module $\text{Hom}_{U_q^\mathbb{Z}(b)}(U_q^\mathbb{Z}(g), N) := \{ f \in \text{Hom}_{\mathbb{Z}[q, q^{-1}]}(U_q^\mathbb{Z}(g), N) \mid f(ub) = S^{-1}(b)f(u) \quad \forall u \in U_q^\mathbb{Z}(g), \forall b \in U_q^\mathbb{Z}(b) \}$ where $S$ is the antipode map of $U_q^\mathbb{Z}(b)$. It has the structure of a $U_q^\mathbb{Z}(g)$-module through $(uf)(m) := f(S(u)m)$. Then the [APW] induction $H^0(N)$ is defined as

$$H^0(N) := F(\text{Hom}_{U_q^\mathbb{Z}(b)}(U_q^\mathbb{Z}(g), N))$$

(Contrary to what APW do ( and what is the tradition for algebraic groups ) we are here inducing from positive Borel groups, this is the reason for the difference between our definition and the one in [APW] ). The map $Ev : H^0(N) \rightarrow N ; f \mapsto f(1)$ is a $U_q^\mathbb{Z}(b)$-linear map; it induces the Frobenius reciprocity isomorphism:

$$\text{Hom}_{U_q^\mathbb{Z}(b)}(E, N) = \text{Hom}_{U_q^\mathbb{Z}(g)}(E, H^0(N)) \quad \forall N \in C^\geq, E \in C$$

This is the universal property of $H^0$. There is also a tensor product theorem for $H^0$. 


For any \( \mathbb{Z}[q, q^{-1}] \)-algebra \( \Gamma \) we have likewise an induction functor \( H^0_\Gamma : \mathcal{C}^\geq_{\Gamma} \to \mathcal{C}_{\Gamma} \).

3. \( W \)-filtrations and crystal bases.

3.1. In this section we shall improve on some of the results of [K2]. In that paper all theorems deal with the rings \( \mathbb{Q}(q) \) and \( \mathbb{Q}[q, q^{-1}] \); however we need the results to hold also for \( \mathbb{Z}[q, q^{-1}] \).

3.2. Throughout the rest of this section we fix an \( i \in I \) and consider the corresponding \( \mathfrak{sl}_2 \)-component \( \mathfrak{g}_i \) of \( \mathfrak{g} \). Let \( W^l(\lambda) \) be the sum of all \( U_q(\mathfrak{g}_i) \)-submodules of \( V(\lambda) \) of dimension greater than or equal to \( l \). Furthermore \( W^l(B(\lambda)) \) and \( I^l(B(\lambda)) \) are defined as the following subsets of \( B(\lambda) \):

\[
W^l(B(\lambda)) := \{ b \in B(\lambda) \mid \varepsilon_i(b) + \varphi_i(b) \geq l \}
\]

\[
I^l(B(\lambda)) := \{ b \in B(\lambda) \mid \varepsilon_i(b) + \varphi_i(b) = l \}
\]

where \( \varepsilon_i \) and \( \varphi_i \) are the functions mentioned in 2.3. Then (3.3.1) of [K2] says that

\[
W^l(\lambda) = \bigoplus_{b \in W^l(B(\lambda))} \mathbb{Q}(q)G_\lambda(b)
\]

We can improve this to the following Lemma:

**Lemma 3.1.** \( W^l(\lambda) \cap \mathbb{Z}(\lambda) = \bigoplus_{b \in W^l(B(\lambda))} \mathbb{Z}[q, q^{-1}]G_\lambda(b) \)

**Proof.** The inclusion \( \supset \) follows from 3.2.1 together with 2.6.2. For the other inclusion assume \( w \in W^l(\lambda) \cap \mathbb{Z}(\lambda) \). Then using 3.2.1 and 2.6.2 once more \( w \) can be written as

\[
w = \sum_{b \in W^l(B(\lambda))} f_b G_\lambda(b) = \sum_{b \in B(\lambda)} g_b G_\lambda(b), \quad f_b \in \mathbb{Q}(q), \quad g_b \in \mathbb{Z}[q, q^{-1}]
\]

But the \( G_\lambda(b) \)’s are independent so we can conclude that \( f_b = g_b \). The Lemma is proved. \( \square \)

3.3. For \( b \in I^l(B(\lambda)) \) we have the following formulas, 3.1.2 of [K2]:

\[
f^{(k)}_i G_\lambda(b) \equiv \left[ \begin{array}{c} \varepsilon_i(b) + k \\ k \end{array} \right] G_\lambda(\bar{f}^{k}i b) \mod W^{l+1}(\lambda) \quad (3.3.1)
\]

\[
e^{(k)}_i G_\lambda(b) \equiv \left[ \begin{array}{c} \varepsilon_i(b) + k \\ k \end{array} \right] G_\lambda(\bar{e}^{k}i b) \mod W^{l+1}(\lambda) \quad (3.3.2)
\]

**Lemma 3.2.** The above formulas also hold mod \( W^{l+1}(\lambda) \cap \mathbb{Z}(\lambda) \).
Proof. We have that \( G_\lambda(b) \in V_\mathbb{Z}(\lambda) \ \forall b \in B(\lambda) \) and hence \( f_i^{(k)}G_\lambda(b) - \left[ \varepsilon_i(b) + \frac{k}{k} \right] G_\lambda(\tilde{f}_i^k b) \in V_\mathbb{Z}(\lambda) \). This proves the Lemma. \( \square \)

3.4. Let us now consider a \( U^\mathbb{Z}_q(b_i) \)-module \( N \subset V(\lambda) \) and a \( U^\mathbb{Z}_q(b_i) \)-submodule \( N_\mathbb{Z} \). Assume furthermore the existence of a \( B_N \subset B(\lambda) \) such that

\[
N_\mathbb{Z} \cong \bigoplus_{b \in B_N} \mathbb{Z}[q, q^{-1}]G_\lambda(b) \quad (3.4.1)
\]

\[
N = \bigoplus_{b \in B_N} \mathbb{Q}(q)G_\lambda(b) \quad (3.4.2)
\]

According to Lemma 3.1.2 of \([K2]\) \( B_N \) must then satisfy that

\[
\tilde{e}_i B_N \subset B_N \cup \{0\} \quad (3.4.3)
\]

We now make the following definitions:

\[
\tilde{N}_\mathbb{Z} = \sum_n f_i^{(n)}N_\mathbb{Z}, \quad \tilde{N} = \sum_n f_i^{(n)}N, \quad \tilde{B}_N = \bigcup_n \tilde{f}_i^n B_N
\]

Then Kashiwara has shown, Theorem 3.1.1 of \([K2]\), that

\[
\tilde{N} = \bigoplus_{b \in B_N} \mathbb{Q}(q)G_\lambda(b) \quad (3.4.4)
\]

We wish to improve this to a statement about \( \tilde{N}_\mathbb{Z} \).

Lemma 3.3. \( \tilde{N}_\mathbb{Z} = \bigoplus_{b \in \tilde{B}_N} \mathbb{Z}[q, q^{-1}]G_\lambda(b) \)

Proof. The intersection of the right hand side of 3.4.4 with \( V(\lambda)_\mathbb{Z} = \bigoplus_{b \in B(\lambda)} \mathbb{Z}[q, q^{-1}]G_\lambda(b) \) equals the right hand side of the Lemma. Hence we must prove that

\[
\tilde{N} \cap V_\mathbb{Z}(\lambda) = \tilde{N}_\mathbb{Z} \quad (3.4.5)
\]

The inclusion \( \supset \) is clear. For the other inclusion choose an \( n \in \tilde{N} \cap V_\mathbb{Z}(\lambda) \). Then we can write \( n \) in the following two ways

\[
(*) \quad n = \sum_{k \geq 0, b \in B_N} c_{k,b} f_i^{(k)}G_\lambda(b) = \sum_{\beta \in B(\lambda)} d_\beta G_\lambda(\beta)
\]

where \( c_{k,b} \in \mathbb{Q}(q) \) and \( d_\beta \in \mathbb{Z}[q, q^{-1}] \).

We wish to modify the first sum so that the occurring \( b \)'s all satisfy \( \tilde{e}_i b = 0 \). Assume that \( b \) occurs in the sum and that \( \tilde{e}_i b \neq 0 \). Choose \( l \) minimal such that \( b \in W^l(B(\lambda)) \). By 3.2.1 we then have \( G_\lambda(b) \in W^l(\lambda) \)
and thus $G_\lambda(b) \in W^l(\lambda) \cap \tilde{N} = W^l(\tilde{N})$; so it follows from $\mathfrak{sl}_2$-theory that

$$G_\lambda(b) \equiv cf_i e_i G_\lambda(b) \mod W^{l+1}(\tilde{N}) \quad (3.4.6)$$

where $c \in \mathbb{Q}(q)$. On the other hand 3.2.1 and 3.3.2 give that

$$e_i G_\lambda(b) = c_1 G_\lambda(\tilde{e}_i b) + \sum_{b \in W^{l+1}(B(\lambda))} c_b G_\lambda(b) \quad (3.4.7)$$

with $c_1, c_b \in \mathbb{Q}(q)$. As $e_i G_\lambda(b) \in N$, it can also be written as a $\mathbb{Q}(q)$-combination of the $G_\lambda(b)$'s with $b \in B_N$; hence we get from the independence of the $G_\lambda(b)$'s that $\tilde{e}_i b$ along with the $b$'s in the sum belong to $B_N$. (This is actually the proof of 3.4.3). Thus, writing 3.4.7 in the form

$$e_i G_\lambda(b) \equiv c_1 G_\lambda(\tilde{e}_i b) \mod W^{l+1}(\tilde{N})$$

we deduce that

$$cf_i e_i G_\lambda(b) \equiv cc_1 f_i G_\lambda(\tilde{e}_i b) \mod W^{l+1}(\tilde{N})$$

Combining this with 3.4.6 we obtain

$$G_\lambda(b) \equiv cc_1 f_i G_\lambda(\tilde{e}_i b) \mod W^{l+1}(\tilde{N}) \quad (3.4.8)$$

Using this and descending induction on $l$ we can write $n$ as promised

$$n = \sum_{b \in B_N, \tilde{e}_i b = 0} c_{k,b} f_i^{(k)} G_\lambda(b)$$

Using $\mathfrak{sl}_2$-theory once more and an induction like the previous one we can furthermore ensure that the occurring $f_i^{(k)} G_\lambda(b)$ satisfy $k \leq w_i(b)$. Now from 3.3.1 and 3.4.4 we have that

$$f_i^{(k)} G_\lambda(b) \equiv G_\lambda(\tilde{f}_i^{(k)} b) \mod W^{l+1}(\tilde{N})$$

Thus the set of vectors in $\tilde{N}$

$$\{ f_i^{(k)} G_\lambda(b) \mid k \leq w_i(b), b \in B_N, \tilde{e}_i b = 0 \}$$

is a $\mathbb{Q}(q)$-basis of $\tilde{N}$ and the base change matrix from $\{ G_\lambda(b) \mid b \in \tilde{B}_N \}$ is triangular with ones on the diagonal with respect to a proper indexing of the basis. But then in (*) we must have that $c_{k,b} \in \mathbb{Z}[q, q^{-1}]$ and thus $n \in \tilde{N}_\mathbb{Z}$. This proves the Lemma.
3.5. For $w \in W$ the $U_q^Z(b)$-submodule $V_w^Z(\lambda)$ of $V^Z(\lambda)$ is defined in the following recursive way

$$V_1^Z(\lambda) := v_\lambda, \quad V_w^Z(\lambda) := U_q^Z(g_w)V_{sw}^Z(\lambda) \quad sw < w$$

It is shown in Lemma 3.3.1 of [K2] that $V_w^Z(\lambda)$ also has the following description

$$V_w^Z(\lambda) = U_q^Z(g) v_{w\lambda}$$

where $v_{w\lambda} \in V_w^Z(\lambda)$ is defined in the following recursive way

$$v_{1\lambda} := v_\lambda, \quad v_{w\lambda} := f_{s}^{(m)} v_{sw\lambda} \quad sw < w$$

where $m := \langle \alpha_i, sw\lambda \rangle$. By the quantum Verma relations this is independent of the choice of reduced expression of $w$ and hence also $V_w^Z(\lambda)$ is independent of the reduced expression of $w$.

Applying now the above Lemma to $N_Z = V_{sw}^Z(\lambda)$ we obtain the existence of a $B_w(\lambda) \subset B(\lambda)$ such that

$$V_w^Z(\lambda) = \bigoplus_{b \in B_w(\lambda)} \mathbb{Z}[q, q^{-1}] G_\lambda(b) \quad (3.5.1)$$

Then $B_w(\lambda)$ has the following properties

$$\tilde{e}_i B_w(\lambda) \subset B_w(\lambda) \cup \{0\} \quad (3.5.2)$$

$$B_w(\lambda) = \bigcup_k f^k B_{sw}(\lambda) \quad (3.5.3)$$

The first property is a consequence of 3.4.3 and the second one follows from Lemma 3.3. Let $S$ be an $i$-string, i.e. a subset of $B(\lambda)$ of the form

$$S = \{ f^k b \mid k \geq 0, b \in B(\lambda), \tilde{e}_i b = 0 \}$$

where $b$ is called the highest weight vector. Then $B_w(\lambda)$ has the following property

$$B_w(\lambda) \cap S \text{ is either } S \text{ or } b \text{ or the empty set} \quad (3.5.4)$$

This is rather deep; it is the content of Theorem 3.3.3 of [K2].

3.6. We shall investigate the consequences of these properties for the $V_w^Z(\lambda)$:

**Lemma 3.4.** There is a $U_q^Z(b_i)$-filtration of $V_w^Z(\lambda)$

$$0 = W^l(\lambda) \cap V_w^Z(\lambda) \subset W^{l-1}(\lambda) \cap V_w^Z(\lambda) \subset \ldots \subset W^0(\lambda) \cap V_w^Z(\lambda) = V_w^Z(\lambda)$$

such that the quotients are direct sums of Weyl $U_q^Z(g_i)$-modules restricted to $U_q^Z(b_i)$ and of rank one $U_q^Z(b_i)$-modules having dominant weights.
Proof. We can construct a $U_q^Z(b_i)$-filtration of $V_w^Z(\lambda)$ in the following way
\[ 0 = W^l(\lambda) \cap V_w^Z(\lambda) \subset W^{l-1}(\lambda) \cap V_w^Z(\lambda) \subset \ldots \subset W^0(\lambda) \cap V_w^Z(\lambda) = V_w^Z(\lambda) \]
where $l$ is chosen big enough for the first equality to hold. By Lemma 3.1, 3.5.1 and the definition of $I_l$ the quotients are
\[ W^k(\lambda) \cap V_w^Z(\lambda)/W^{k+1}(\lambda) \cap V_w^Z(\lambda) \cong \bigoplus_{b \in I_l(B_w(\lambda))} \mathbb{Z}[q, q^{-1}] G_\lambda(b) \]
If $S$ is an $i$-string then $\varepsilon_i(b) + \varphi_i(b)$ is constant on $S$; this follows from 2.4.2. But then $I^k(S)$ is either $S$ or the empty set and we conclude that $I^k(B_w(\lambda))$ inherits the string property 3.5.4. If the intersection is $S$, the formula 3.3.2 together with Lemma 3.2 and the description of the Weyl modules for $U_q^Z(g_i)$ in Section 2.2 show that \{ $G_\lambda(b), b \in S$ \} gives rise to an $U_q^Z(g_i)$ Weyl module restricted to $U_q^Z(b_i)$. If the intersection is a highest weight vector \{ $b$ \} then it has the weight $l > 0$: $\varepsilon_i(b) = 0$ whence $\varphi_i(b) = \varphi_i(b) + \varepsilon_i(b) = l$ and then 2.4.1 gives $w_i(b) = \varphi_i(b) - \varepsilon_i(b) = l$. The Lemma is proved. \hfill \Box

Remark 3.5. For $N$ a $U_q^Z(b_i)$-module the filtration of it by the $N \cap W^l(\lambda)$ is denoted the $W$-filtration of $N$.

3.7. Let $k$ be a field of characteristic $p > 0$ which is made into a $\mathbb{Z}[q, q^{-1}]$-algebra by sending $q$ to an $l$'th root of unity. ( Later on we shall appeal to results of [AW], hence we should really impose the restrictions on $l$ that occur in that paper. However, in [AP] it is shown that the Frobenius map of [L] can be employed to get rid of these restrictions ). Then, as the filtration quotients in Lemma 3.4 are free $\mathbb{Z}[q, q^{-1}]$-modules we get by tensoring a filtration of $V^k_w(\lambda) := V^Z_w(\lambda) \otimes_{\mathbb{Z}[q, q^{-1}]} k$ having the same properties as the one of $V^Z_w(\lambda)$.

4. JOSEPH’S INDUCTION FUNCTOR

4.1. In this section we shall compare the induction functor $D$ of Joseph, see [J], with the [APW] functor $H^0$. Let $k$ be as in Section 3.7. Then $D$ is defined in the following way

Definition 4.1. Let $N$ be a finite dimensional $U_k(b)$-module and let $U_k \supset U_k(b)$ be a parabolic ( in the sense of [APW] ) quantum group. Then $DN$ is
\[ DN := D(U_k \otimes_{U_k(b)} N) \]
where the tensor product has structure as a $U_k$-module through left multiplication and $D$ is the functor from $U_k$-modules to finite dimensional $U_k$-modules that takes an $M$ to the largest finite dimensional quotient of $M$ by a $U_k$-submodule.

Remark 4.2. Recall that if $M$ is an integrable $U_k$-module then (the relevant) Weyl group acts on the weights of $M$, see [AW] Proposition 1.7. Hence, arguing as in [APW], 1.14, we find that there is a unique submodule of $U_k \otimes_{U_k(b)} \mathcal{N}$ such that the quotient is of maximal dimension, i.e. $D$ is well defined.

The universal property of $D$ is given by the following Frobenius property

$$\text{Hom}_{U_k(b)}(N, E) = \text{Hom}_{U_k}(DN, E)$$

where $E, N$ are finite dimensional $U_k, U_k(b)$-modules. The isomorphism is induced by the natural $U_k(b)$-map $\sigma : N \mapsto DN$. Furthermore, $D$ satisfies a tensor product theorem.

4.2. For $M$ a $U_k$-module we set $M^{\text{dual}} := \text{Hom}_k(M, k)$. Then $M^{\text{dual}}$ has two structures as a $U_k$-module, namely $M^*$ and $M^t$ defined as

$$M^* : (uf)(m) := f(S(u)m)$$

$$M^t : (uf)(m) := f(S^{-1}(u)m)$$

where $S$ is the antipodal map of the Hopf algebra $U_k$. When $M$ is of finite dimension we have the isomorphisms $M^{*t} \cong M^{ts} \cong M$.

4.3. Using this we can deduce the following Lemma

Lemma 4.3. Let $N$ be a finite dimensional $U_k(b)$-module. Then

$$(DN)^* \cong H^0_k(N^*)$$

Proof. Let $\Phi \in \text{Hom}_{U_k}( (DN)^*, H^0_k(N^*) )$ be the map corresponding to $\sigma \in \text{Hom}_{U_k}( N, DN )$ under the isomorphisms

$$\text{Hom}_{U_k(b)}( N, DN ) \cong \text{Hom}_{U_k(b)}( (DN)^*, N^* ) \cong \text{Hom}_{U_k(b)}( (DN)^*, H^0_k(N^*) )$$

The second isomorphism was Frobenius reciprocity for $H^0_k$.

Let $\Psi \in \text{Hom}_{U_k(b)}( (H^0_k(N^*), (DN)^* )$ be the map corresponding to $Ev \in \text{Hom}_{U_k(b)}( H^0_k(N^*), N^* )$ under the isomorphisms

$$\text{Hom}_{U_k}( H^0_k(N^*), (DN)^* ) \cong \text{Hom}_{U_k}( DN, H^0_k(N^*)^t ) \cong$$

$$\text{Hom}_{U_k(b)}( N, H^0_k(N^*)^t ) \cong \text{Hom}_{U_k(b)}( H^0_k(N^*), N^* )$$
Here the second isomorphism was Frobenius reciprocity for $D$; it can be applied since $H^0_k(N^*)$ according to [AW] is finite dimensional. We can describe $\Phi$ and $\Psi$ in the following way

$\Phi : f \mapsto [u \mapsto (n \mapsto f(u(\sigma n)))]$

$\Psi : f \mapsto [u\sigma(n) \mapsto f(u)(n)]$

Using these descriptions one checks that $\Phi \circ \Psi = Id$ and $\Psi \circ \Phi = Id$. □

4.4. The functor $D$ is coinvariant and right exact so we would like to introduce its derived functors. However, as the category of finite dimensional $U_k$-modules does not have enough projectives one cannot proceed as normal when defining these. To overcome this obstacle, Joseph proposes to use objects of the form $E \otimes \lambda$, where $E$ is finite dimensional and $\lambda$ is dominant, as substitutes for projectives [J]. We shall show that this definition also makes sense in our context. Our methods were here inspired by [P].

**Lemma 4.4.** Let $M$ be a finite dimensional $U_k(b)$-module. Then there exists a $\lambda \in P^+$ and a finite dimensional $U_k$-module $E$ such that $M$ is a quotient of $E \otimes \lambda$.

**Proof.** In [AW] the following ampleness property of $H^0$ is shown

$$H^i(\lambda) = 0 \text{ for } i > 0 \text{ and } \lambda << 0 \quad (4.4.1)$$

(We are here inducing from positive Borel groups, hence dominant is replaced by antidominant). As $M$ is finite dimensional, we can use the above to find a $\lambda$ such that $P(M \otimes -\lambda) \subset P^-$ and such that

$$H^1(\lambda) = 0 \quad \forall \mu \in P(M \otimes -\lambda) \quad (4.4.2)$$

We now proceed by induction on the cardinality of $P(M \otimes -\lambda)$. Choose $\nu$ maximal in $P(M \otimes -\lambda)$; then $\nu \subset M \otimes -\lambda$ as $U_k(b)$-modules. From this we obtain a commutative diagram as follows

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0_k(\nu) & \rightarrow & H^0_k(M \otimes -\lambda) & \rightarrow & H^0_k((M \otimes -\lambda)/\nu) & \rightarrow & 0 \\
\downarrow_{Ev} & & \downarrow_{Ev} & & \downarrow_{Ev} & & \downarrow_{Ev} & & \\
0 & \rightarrow & \nu & \rightarrow & M \otimes -\lambda & \rightarrow & (M \otimes -\lambda)/\nu & \rightarrow & 0
\end{array}
$$

The rows are both exact, the first one by 4.4.2, the second by construction. The first vertical map is surjective by definition and the third is surjective by induction hypothesis. But then also the second vertical map must be surjective and we are done. □
4.5. We can now in the usual way construct resolutions of all finite dimensional $U_k$-modules $N$; these will be on the form $(E \otimes \lambda)^* \rightarrow N$ and in general infinite. We can furthermore assume that in $(E \otimes \lambda)^* \rightarrow N$ all $-\lambda$ satisfy $H^0_k(\lambda) = 0$ for $i > 0$, this is possible by 4.4.1.

Lemma 4.5. Let $(E \otimes \lambda)^* \rightarrow N$ be a resolution like the above. Then the cohomology $D^\bullet$ of the complex $D((E \otimes \lambda)^* \rightarrow 0$ is independent of the choice of resolution. Furthermore there is an isomorphism of $U_k$-modules

$$D^i(N)^* \cong H^i_k(N^*)$$

Proof. Dualizing the resolution $(E \otimes \lambda)^* \rightarrow N$ we get the resolution $N^* \hookrightarrow (E^* \otimes -\lambda)^*$. It is acyclic for $H^0_k$ because the tensor identity gives that

$$H^i_k(E^* \otimes \lambda) \cong E^* \otimes H^i_k(\lambda) \cong 0$$

But then $H^i_k(N^*)$ is the $i$'th cohomology of $0 \hookrightarrow H^0_k(E^* \otimes \lambda)^*$, which by Lemma 4.3 is the $i$'th cohomology of $0 \hookrightarrow (D(E \otimes \lambda))^\bullet$. The Lemma is proved. □

5. The vanishing theorem

5.1. Assume that we are in the rank one case. i.e. $g = sl_2$. We then have the following well known results

Lemma 5.1. (1) Let $\lambda \geq -1$. Then $D^j\lambda = 0$ for $j > 0$, while $D\lambda$ has dimension $\lambda + 1$, a basis being $f^{(k)} \otimes v_\lambda$. The action of $U_q(g)$ is as in 2.2

(2) Let $\lambda$ be as above and let $Q$ be the $U_k(b)$-module $D\lambda/\lambda$. Then $DQ = 0$ and $D^jQ = 0$ for $j > 0$.

Proof. (1) follows from the corresponding Proposition 4.2 in [APW] and the preceding Lemma. As for (2) consider the long exact sequence of $U_k(g)$-modules

$$\cdots \rightarrow D^1 D\lambda \rightarrow D^1 Q \rightarrow D\lambda \rightarrow g \rightarrow D\lambda \rightarrow DQ \rightarrow 0$$

which arises from the application of $D$ to the sequence defining $Q$. The definition of derived functors in the last section gives that $D^jD\lambda$ for $j > 0$; thus the $j > 1$ case of (2) follows by combining with (1). Now, $g$ must be a nonzero scalar times the identity map on $D\lambda$ because it is the map corresponding to $\sigma \neq 0$ under Frobenius reciprocity and

$$\text{Hom}_{U_k(g)}(D\lambda, D\lambda) \cong \text{Hom}_{U_k(b)}(\lambda, D\lambda) \cong k$$

Hence $D^1Q$ as well as $DQ$ must be zero. The Lemma is proved. □
5.2. We still consider the rank one case. For a copy of \( \text{sl}_2 \) in \( U_k \) corresponding to the simple reflection \( s \) we denote the corresponding Joseph induction \( D_s \).

**Theorem 5.2.**  
(1) Let \( V^k_w(\lambda) \) be as in 3.5. Then \( D_s^j V^k_w(\lambda) = 0 \) for \( j > 0 \).  
(2) Let \( sw < w \). Then \( D_s V^k_{sw}(\lambda) = V^k_w(\lambda) \)

**Proof.** \( \text{ad } (1). \) Let \( i \in I \) be the index corresponding to \( s \) and consider the \( W \)-filtration of \( V^k_w(\lambda) \) from Lemma 3.4 with respect to this \( i \). The quotients \( Q \) of this all satisfy \( D_s^j Q = 0 \) for \( j > 0 \); if \( Q \) is a dominant line this is because of Lemma 5.1 and if \( Q \) is the restriction of an \( \text{sl}_2 \)-module it follows from the definition of \( D^j \) in the previous section. Using induction on the filtration length we conclude that \( D_s^j V^k_w(\lambda) = 0 \) for \( j > 0 \).

\( \text{ad } (2). \) Let the \( U_k(\mathfrak{b}_i) \)-module \( R \) be defined as such that the following is exact

\[
0 \longrightarrow V^k_{su}(\lambda) \longrightarrow V^k_w(\lambda) \longrightarrow R \longrightarrow 0 \tag{5.2.1}
\]

We know from 3.5 that \( V^k_w(\lambda) \) has a \( k \)-basis on the form \( \{G_\lambda(b)|b \in B_w(\lambda)\} \); hence \( R \) has a basis on the form \( \{G_\lambda(b)|b \in B_w(\lambda) \setminus B_{sw}(\lambda)\} \). In the \( W \)-filtration of \( R \), the quotient between the \( l \)'th and \((l+1)\)'th term hence has basis \( \{G_\lambda(b)|b \in I^l(B_w(\lambda)) \setminus I^{l+1}(B_{sw}(\lambda))\} \).

We saw in the proof of Lemma 3.4 that the \( I^l(B_g(\lambda)) \) satisfy string properties like 3.5.4. If \( S \) is an \( i \)-string and \( S \cap I^l(B_{sw}(\lambda)) \neq \emptyset \), then by 3.5.4 we have \( S \subset I^l(B_{sw}(\lambda)) \). Now \( V^k_w(\lambda) \) is a \( U_k(\mathfrak{g}_i) \)-module, so \( I^l(B_w(\lambda)) \) is the disjoint union of \( i \)-strings; and if \( S \subset I^l(B_w(\lambda)) \) is such a string then from 3.5.4 we get \( S \cap I^l(B_{sw}(\lambda)) \neq \emptyset \). Putting these things together we see that \( I^l(B_w(\lambda)) \setminus I^{l+1}(B_{sw}(\lambda)) \) is the union of \( i \)-strings with the highest weight vector omitted; this must furthermore be a of \( \text{sl}_2 \)-weight \( l \). From the formula 3.3.2 we then see that the span of the global basis elements of \( I^l(B_w(\lambda)) \setminus B_{sw}(\lambda) \) form a \( U_k(\mathfrak{g}_i) \)-module of type \( \mathcal{D} \lambda/\lambda \). And then an induction on the filtration length, the induction start being provided by (2) of Lemma 5.1 proves that \( D_s^1 R = D_s R = 0 \).

Now, \( V^k_w(\lambda) \) is a \( U_k(\mathfrak{g}_i) \)-module so we have \( D^1 R = D_s R = 0 \) we may finish the proof of (2) by applying \( D \) to 5.2 (1). \( \square \)

In Theorem 5.2 the induction \( D_s \) was induction from a rank one Borel subgroup. However, \( V^k_w(\lambda) \) has a module structure for the full Borel group \( U_k(\mathfrak{b}) \). Denote by \( U_k(\mathfrak{g}_i) \) the minimal parabolic quantum group generated by this and the \( f_i^{(n)} \)'s and by \( D_s' \) the induction from \( U_k(\mathfrak{b}) \) to
$U'_k(g_i)$. As $k$-vector spaces and $U_k(g_i)$-modules $D'_s V^k_w(\lambda)$ and $D_s V^k_w(\lambda)$ are isomorphic, namely both equal to the largest $f_i$-finite quotient of $U_k(g_i) \otimes_{U_k(n)} V^k_w(\lambda)$

We have the following Lemma

**Lemma 5.3.** There is an isomorphism of $U'_k(g_i)$-modules

$$D'_s V^k_w(\lambda) \cong V^k_w(\lambda)$$

**Proof.** There is a commutative diagram

\[
\begin{array}{ccc}
V^k_w(\lambda) & \xrightarrow{id} & V^k_w(\lambda) \\
\sigma \downarrow & & \downarrow i \\
D'_s V^k_w(\lambda) & \xrightarrow{\varphi} & V^k_w(\lambda)
\end{array}
\]

where $i$ is the inclusion map, $\sigma$ is the canonical map – these are $U'_k(g_i)$-linear – and $\varphi$ is the $U_k(g_i)$-linear map obtained from Theorem 5.2 (2) together with the remarks in the beginning of this section.

Any element of $D'_s V^k_w(\lambda)$ can be written as a linear combination of elements on the form $u\sigma(v)$ where $u \in U'_k(g_i), v \in V^k_w(\lambda)$. We must show that $\varphi$ commutes with $e_j$ for $j \neq i$; it suffices to do so for $u\sigma(v)$. On the one hand we have

$$\varphi(e_j(u\sigma(v))) = e_j(e_j(u\sigma(v))) = e_ju\sigma(e_jv) = e_jui(e_jv) = e_jui(\sigma(v))$$

where we used the commutativity of the diagram and of $e_j$ and $f_i$. On the other hand

$$e_j\varphi(u\sigma(v)) = e_j \varphi(v) = e_jv$$

We see that the two sides are equal. \qed

**5.3.** We omit from now on the primes on the $D$, inductions will be from the full Borel subalgebra. For $w_0 = s_{i_1} s_{i_2} \ldots s_{i_n}$ a reduced expression of the longest element of $W$ we define $D_{w_0}$ as $D_{i_1} D_{i_2} \ldots D_{i_n}$, (apriori this may depend on the chosen expression). The Weyl module $V_k(\lambda)$ is by definition $V_Z(\lambda) \otimes k$. We now obtain the following Theorem

**Theorem 5.4.** For $\lambda \in P^+$ there are isomorphisms of $U_k(b)$-modules

$$D_{w_0} \lambda \cong D\lambda \cong V_k(\lambda)$$

**Proof.** As $V_k(\lambda)$ has finite dimension there is by definition of $D$ a surjection $\varphi : D\lambda \twoheadrightarrow V_k(\lambda)$. Now, successive applications of Theorem 5.2 and Lemma 5.3 give the isomorphism $D_{w_0} \lambda \cong V_k(\lambda)$. And applying Frobenius reciprocity successively, the canonical map $\sigma : \lambda \rightarrow D\lambda$ induces a $U_k(b)$-linear map $\psi : D_{w_0} \lambda \cong V_k(\lambda) \rightarrow D\lambda$. But any $U_k(b)$-linear map must also be a $U_k(g)$-linear map as one sees from Frobenius reciprocity:
$V_k(\lambda)$ and $D\lambda$ are both $U_k(\mathfrak{g})$-modules. Thus the composition $\varphi \circ \psi$ is a $U_k(\mathfrak{g})$-linear endomorphism of $V_k(\lambda)$ and one checks that it is nonzero on the $\lambda$'th weight space. But $V_k(\lambda)$ is a highest weight module, hence $\varphi \circ \psi$ must be a nonzero scalar times the identity. This shows that $D\lambda \cong V_k(\lambda) \oplus M$ for $M$ some $U_k(\mathfrak{g})$-module. Now, $D\lambda$ is indecomposable being a highest weight module as well, and we get a contradiction unless $M = 0$. The Theorem is proved.  

5.4. We can now prove our main Theorem.

**Theorem 5.5.** *(Kempf vanishing).* Let $\lambda \in P^-$. Then $H_k^i(\lambda) = 0$ for $i > 0$.

**Proof.** By Lemma 4.5 the Theorem is equivalent to $D^i(-\lambda) = 0$. Let $(E \otimes \nu)^* \to -\lambda \to 0$ be a resolution of $-\lambda$ as in Section 4.5. Then the complex $(D_{sn}(E \otimes \nu))^* \to D_{sn}(-\lambda) \to 0$ is exact by Lemma 5.1 (1). It is also acyclic for $D_{sn-1}$ because by the tensor identity and Theorem 5.2 we have

$$D_{sn-1}^i(D_{sn}(E \otimes \nu)) \cong E \otimes D_{sn-1}^i(D_{sn}(\nu)) \cong E \otimes D_{sn-1}^i(D_{sn}(V_k^s(\nu))) \cong 0$$

Thus the application of $D_{sn-1}$ to $(D_{sn}(E \otimes \nu))^* \to D_{sn}(-\lambda) \to 0$ gives a complex that evaluates $D_{sn-1}^i(D_{sn}(-\lambda))$. But we know from Theorem 5.2 that these cohomology groups are zero so the complex $D_{sn-1}(D_{sn}(E \otimes \nu))^* \to D_{sn-1}(D_{sn-1}(\nu)) \to 0$ is exact. Furthermore, the argument from before shows that it is acyclic for $D_{sn-2}$. Continuing we eventually reach the sequence $(D_{w_0}(E \otimes \nu))^* \to D_{w_0}(-\lambda) \to 0$ which thus is exact. By Theorem 5.4 it is isomorphic to $(D(E \otimes \nu))^* \to D(-\lambda) \to 0$ and we are done.  

5.5. We have a couple of corollaries to this Theorem.

**Corollary 5.6.** *(Demazure vanishing).* For $\lambda \in P^-$ we have

$$D_{sn-1}^i(D_{sn-1}D_{sn-2} \cdots D_1(\lambda)) \cong 0$$

**Proof.** This is contained in the last proof.  

**Corollary 5.7.** The modular Kempf and Demazure vanishing theorems.

**Proof.** The result follows from the Theorem and the Corollary by specializing $q$ to 1; there are base change theorems controlling this.  

**Corollary 5.8.** Demazure’s character formula in terms of the $H^k$.

**Proof.** The classical proof carries over, see [A1]  

□
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