Spectral Triples and Generalized Crossed Products

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Abstract

We give a construction for lifting spectral triples to crossed products by Hilbert bimodules. The spectral triple one obtains is a concrete unbounded representative of the Kasparov product of the spectral triple and the Pimsner-Toeplitz extension associated to the crossed product by the Hilbert module. To prove that the lifted spectral triple is the above-mentioned Kasparov product, we rely on operator-∗-algebras and connexions.

Keywords: Spectral triple, Pimsner algebra, Kasparov product, connexion, generalized crossed product, K-homology, KK-theory, boundary map.

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1 Introduction

The starting point of this article is double: on the one hand, a standard way to construct new $C^*$-algebras is given by the crossed product construction; on the other hand, a “smooth” structure on a $C^*$-algebra is – according to Connes’ philosophy – given by a so-called spectral triple. When the $C^*$-algebra is commutative, this corresponds – under appropriate hypotheses – to the choice of a spin structure and associated Dirac operator, as is shown by the reconstruction theorem [Con13]. It is therefore natural to ask in which way the two notions can be combined, in other words, to study the permanence properties of spectral triples. Examples for spectral triples on crossed products by ordinary automorphisms have been around for a long time, recently they have been studied more intensively for example in [BMR10] and [Pat12].

The present paper aims to give a conceptual way for constructing spectral triples on a certain class of crossed product-like algebras, which contains the crossed products by $\mathbb{Z}$ and the (commutative) $S^1$-principal bundles – see Proposition 5.8. This class fits within the general realm of Pimsner algebras [Pim97], but our approach is more simply phrased within the setting of Abadie, Eilers and Exel in [AEE98]. In fact, a construction of Pimsner in [Pim97] (which is based on a construction from Jones’ paper [Jon83]) shows that one can always write a Pimsner algebra as a generalized crossed product – at the cost of changing the algebras and modules involved.

To sketch the basic idea, let us however start with the simplest case – that of crossed products. Suppose for the moment that $\alpha$ is an isomorphism of a unital $C^*$-algebra $B$. Recall first of all that Pimsner and Voiculescu proved in [PV80] the existence of the so-called Pimsner-Voiculescu six-term exact sequence, which we state in its $K$-homological version:

\[
\begin{array}{ccccc}
K^0(B) & \to & K^0(B) & \to & K^0(B \rtimes_\alpha \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
K^1(B \rtimes_\alpha \mathbb{Z}) & \to & K^1(B) & \to & K^1(B)
\end{array}
\]

Here the vertical boundary maps are generalizations of the Fredholm index and are, in fact, given by the Kasparov product with the $KK^1$-class corresponding to the so-called Toeplitz-extension of $B$ by $B \rtimes_\alpha \mathbb{Z}$ obtained by replacing the unitary in the crossed product by an isometry and identifying $KK^1$ with Ext.

Recalling that in the unbounded (or Baaj-Julg picture) of $KK$-theory [BJ83] the $KK$-classes are actually given by spectral triples, it is natural to view this boundary map as a lifting procedure along the Toeplitz-extension for spectral triples on the base algebra $B$. And in fact, representing the crossed product canonically on $l^2(\mathbb{Z}) \otimes H$, where $H$ is some faithful nondegenerate representation of $B$, one finds the completely natural operator $N \otimes \gamma + 1 \otimes D$ associated to any unbounded representative $D$ with grading $\gamma$ of a $K^0$-class (here $N$ denotes the unbounded operator in $l^2(\mathbb{Z})$ defined by the identity function on $\mathbb{Z}$). Under some natural hypotheses, this operator is almost immediately seen to represent the Kasparov product of the class of the Toeplitz extension with the class $[D] \in KK_*(A, \mathbb{C})$ determined by $D$. 

References
Suppose now that $E$ is a $B$-$B$-$C^*$-correspondence, i.e. $E$ is a Hilbert module over $B$ and we are given a homomorphism $\phi : B \to \mathbb{B}_B(E)$. Suppose further that $E$ is full over $B$ and $\phi$ is injective. Then one may construct (see [Pim97] for details) an extension $\gamma_E$

$$0 \to I \to \mathcal{T}_E \to \mathcal{O}_E \to 0.$$ 

Here the ideal $I$ is closely related to the base algebra $B$, and in good cases Morita equivalent to it. Furthermore $A := \mathcal{O}_E$ corresponds to the crossed product (if $E = B$ and $B$ acts on the left via an automorphism $\alpha$, then $\mathcal{O}_E \approx B \rtimes_\alpha \mathbb{Z}$).

Using the same idea as above, we get a long exact sequence in $K$-homology and a boundary map $\partial_E : K^i(I) \to K^{i+1}(B)$ (which is given by the Kasparov product with the class of the extension). Thus we may try to lift spectral triples from $I$ to $A$ by just choosing an explicit representative of the Kasparov product of some fixed spectral triple on $I$ and the class $\partial_E$.

Now as mentioned above, already in the setting of crossed products there is a regularity condition a spectral triple on the base algebra $B$ has to satisfy in order to be “liftable” in a simple manner. In the more general setting of Pimsner algebras, this condition is more difficult to track down. For example, it is easily seen that for a covering with more than one leaf, the commutator with the Dirac on the base is not sufficiently well behaved to allow the simple lifting $N \otimes \gamma + 1 \otimes D$ indicated above. Moreover, the Cuntz algebra $\mathcal{O}_n$ is a Pimsner algebra but as it is purely infinite, it is traceless; this is another example which illustrates the difficulties involved in the construction of such liftings.

The regularity condition we use in this paper in order to construct liftings is the existence of certain connexions for the operator $D$ on the base algebra. This connexion allows us to lift the spectral triple on the base algebra to an operator on the Pimsner algebra. This is certainly not the only possible approach, but the availability of techniques from [KL13], which originate from [Mes12], have lead us to this approach. Ultimately, these results are based on [Kuc97]. However, we need to adopt the results from [KL13] to the case of Kasparov products of odd and even unbounded modules which causes some technicalities deferred to the appendix.

Recall that by Kasparov’s stabilization theorem [Kas80] any countably generated Hilbert module over $B$ is a direct summand of the standard module $H_B$. Our main result concerning the Pimsner algebras $\mathcal{O}_E$ can then be formulated as follows (see Theorem 7.4 for the precise definition of a Rieffel-spectral triple we refer the reader to section 3):

**Theorem 1.1.** Suppose that $B$ is unital and equipped with a Rieffel spectral triple $(H, D_h)$ with grading $\gamma_h$ and that $E$ is a finitely generated projective Hilbert bimodule; assume $E$ is equipped with a two-sided Hermitian $D_h$-connexion $\nabla$, then the lifting along the Pimsner-Toeplitz extension of of $(H, D_h)$ can be represented by the spectral triple

$$(X \otimes_B H, D_h \otimes \gamma + 1 \otimes \nabla D_h)$$

where $X$ is the Hilbertmodule-completion of $B \rtimes_E \mathbb{Z}$ for the scalar product associated to the conditional expectation $E(a) := \int_S \lambda.a \, d\lambda$ and $\nabla$ is a connexion explicitly constructed from $\nabla$ (compare 6.9).

Here $\lambda.a$ refers to the natural circle-action on $B \rtimes_E \mathbb{Z}$.

The standard example (see Example 6.3) of such a situation is given by a Hermitian line bundle $V$ over a locally compact Hausdorff base space $X$ by taking
$E = \Gamma(V)$ and $B = C(X)$. The action by multiplication of $B$ on the left on $E$ may then be twisted by an isomorphism $\sigma$ of $B$ by setting $b\xi := \sigma^{-1}(b)\xi$ and $B\langle \xi|\eta \rangle := \sigma(\langle \xi|\eta \rangle_B)$. For example, the quantum Heisenberg manifolds introduced by Rieffel \cite{Rie89} fall into this category. This standard example motivates the notation $D_h$ (horizontal class) for the spectral triple on the base and $D_v$ (vertical class) for the Kasparov module determined by the Toeplitz extension.

The article is organized as follows:

- **Section 2** contains preliminary results regarding operator $\ast$-algebras and operator $\ast$-modules, as well as some technical results concerning stability under holomorphic functional calculus. We put some emphasis on closability of derivations and connexions as this issue is sometimes overlooked. In fact, only under this closability condition the canonical operator-$\ast$-algebra closure \cite[(Proposition 2.6)]{KL13} of the domain of a $\ast$-derivation on a $C^\ast$-algebra is a subalgebra of the $C^\ast$-algebra. Along the same lines, we construct operator-$\ast$-module closures for every connexion associated to a derivation (see Corollary 2.16).

- **Section 3** recalls some basic facts regarding spectral triples and associated derivations. It then introduces a class of spectral triples associated to Lie group actions. These spectral triples were studied in \cite{GG}, where it is shown that they can be associated to ergodic Lie group actions and include many natural examples. However, the quantum Heisenberg manifolds are not included in this framework as shown in loc. cit., and are one of the motivating examples for this article.

- In **Section 5**, we give a brief account of a particularly well-behaved class of Pimsner algebras, the so-called generalized crossed products from \cite{AEE98}. These algebras carry a natural $S^1$-action, which however admits a non-empty fixed point subalgebra $B$; we thus do not obtain a spectral triple from this Lie group action, but rather a spectral triple “with coefficients”, i.e. a $KK$-class. This class is the vertical class (introduced properly in Definition 7.1) which will be used later on in order to lift spectral triples from the base $B$ to the generalized crossed product. Along the way we give a complete characterization of commutative generalized crossed products.

- **The Section 6** contains the main technical tools of this article. We introduce two-sided connexions and prove how they can be extended to generalized crossed products. We show that under the condition of existence of a generalized frame and under some natural conditions on the connexion itself the extended connexion is closable – a property which is essential as noted above. Using essentially the same idea, we also obtain a derivation on the generalized crossed product and thus a sub-$\ast$-algebra.

- In **Section 7** we finally tackle the main theorem already stated above. We thus define the vertical class and calculate its products with a given Rieffel spectral triple on the base algebra. In order to do so, we show that the connexion constructed from two-sided connexion in Section 5 provides us with a correspondence from the vertical class to the class of the spectral triple. We may then apply the (modified) criterion of Kaad and Lesch, which is proved in the appendix, to conclude.

- **The last Section 8** analyses the construction for the case of quantum Heisenberg manifolds. We show that the spectral triple obtained in this case coincides
with a spectral triple constructed from scratch in [CS03]. The result could be
formulated in a somewhat more general framework, but we refrain from doing
so here in order to keep to a moderate size.

• We have relegated to the appendix the somewhat technical task of adapting
the results of Kaad and Lesch to the case of odd-even Kasparov products which
we need in order to apply our results to quantum Heisenberg manifolds. The
modifications are mostly purely algebraic, depending basically on some yoga
of graded tensor products.

2 Connexions and operator modules

In this section, we study the notion of operator $\ast$-module introduced in [KL13]. We
give explicit examples of such modules, based on derivations of the base algebra.

We refer to [Lan95, Bla05] for missing details about Hilbert modules. We remind
the reader of the following:

Definition 2.1. A finite family $(\xi_j)_{j=1}^m$ in a (right) Hilbert $A$-module $E$ is called a
(right) frame if and only if

$$\sum_{j=1}^m \xi_j \langle \cdot, \cdot \rangle = \text{id}_E.$$ 

Remark 2.2. A right Hilbert module $E$ admits a frame if and only if it is finitely
generated (f.g.) and projective (see Proposition 3.9 p.89 of [GVF01]).

We also need the notion of $C^\ast$-correspondence. Since several definitions of “cor-
respondences” exist in the literature, we make explicit the definition we use. It is
called a “Hilbert bimodule” in [KL13] – but we keep this term for a more symmetric
structure (see Definition 5.1 below).

Definition 2.3. Given $C^\ast$-algebras $A$ and $B$, an $A$-$B$ $C^\ast$-correspondence $E$ is

• a right Hilbert module over $B$ whose scalar product we denote $\langle \cdot, \cdot \rangle_B$ or $\langle \cdot, \cdot \rangle$
  when the context is clear;

• a map $\pi: A \to B(\mathcal{B}_B(E))$, where $\mathcal{B}_B(E)$ is the set of maps $T: E \to E$ such that
  there is a $T^*: E \to E$ with $\forall \xi, \eta \in E,$

  $$\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle.$$ 

In particular, $T$ is right $B$-linear and bounded.

We write $a\xi$ instead of $\pi(a)\xi$ when this notation is unambiguous.

There is a straightforward notion of tensor product of $C^\ast$-correspondences (see
[Bla05], II.7.4 p.147 or [Lan95]):

Definition 2.4. Given a (right-) $A$-Hilbert module $E$ and an $A$-$B$ $C^\ast$-correspondence
$F$, the inner tensor product over $A$, denoted $E \otimes_A F$, is the $B$-Hilbert module ob-
tained from the quotient of the algebraic tensor product $E \otimes \mathcal{B}_B(F)$ by the subspace
generated by

$$\{\xi a \otimes \eta - \xi \otimes \pi(a)\eta | \xi, \eta \in E, a \in A\}$$

by completing it for (the norm induced by) the scalar product:

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle_B := \langle \eta, \langle \xi, \xi' \rangle_B \cdot \eta' \rangle_B.$$
If $E$ is actually a $C^*\text{-}A C^*\text{-}AC^*$-correspondence, then the resulting tensor product $E \otimes_A F$ is a $C^*\text{-}B C^*$-correspondence.

We can iterate this construction to obtain $A C^*$-correspondences

$$E \otimes_A E \otimes_A \cdots \otimes_A E$$

for any positive integer $k$ (the tensor product contains $k$ factors) which we denote by $E^{\otimes k}$.

**Definition 2.5.** Let $A$ be a $C^*$-algebra. A Banach $A$-bimodule $X$ is a Banach space equipped with continuous left- and right actions from $A$.

In this paper, we focus more precisely on two types of such Banach $A$-bimodules:

(i) the case of a $A\text{-}AC^*$-correspondence $X$;

(ii) given two $A\text{-}B C^*$-correspondences $F_1$ and $F_2$, set $X := B_B(F_1 \rightarrow F_2)$. This is a Banach $A$-bimodule for the left- and right-actions are provided post- and pre-composition with $\pi_j(a) \in B_B(F_j)$. It is readily checked that these actions are continuous.

Here (i) is actually just a special case of (ii), as is shown by the following construction:

**Definition 2.6.** If $F_1$ and $F_2$ are two $A\text{-}B C^*$-correspondences, $F_1 \oplus F_2$ their $A\text{-}B C^*$-correspondence direct sum, i.e. the inner sum $B$-Hilbert module equipped with the diagonal $A$-action. The algebra $B_B(F_1 \oplus F_2)$ decomposes naturally into a direct sum which we can write in matrix notation:

$$B_B(F_1 \oplus F_2) = \begin{pmatrix} B_B(F_1) & B_B(F_1, F_2) \\ B_B(F_1, F_2) & B_B(F_2) \end{pmatrix}.$$

In particular, if $A$ is unital and $\mathcal{X}$ is an $A\text{-}AC^*$-correspondence, then setting $F_1 := A$ and $F_2 := \mathcal{X}$, we recover $B_A(A \rightarrow \mathcal{X}) = \mathcal{X} \subseteq B_A(A \oplus \mathcal{X})$.

Indeed, $A$ is unital and any $\varphi \in B_A(A \rightarrow \mathcal{X})$ is fully determined by $\varphi(1) = \xi \in \mathcal{X}$. This bijection preserves the norm and therefore gives an isomorphism.

In the sequel, we will only write $B(F)$ instead of $B_A(F)$ when the context is clear.

**Remark 2.7.** This is just a variation on the theme of linking algebras as introduced by Rieffel in [Rie74] (see for instance [Bla05], II.7.6.9 p.152 for an overview).

**Definition 2.8.** A derivation $\delta$ on a $C^*$-algebra $A$ with values in a Banach $A$-bimodule $\mathcal{X}$ is a linear map defined on a dense $*$-subalgebra $\mathcal{A} \subseteq A$ with values in $\mathcal{X}$ which satisfies for all $a, b \in \mathcal{A}$,

$$\delta(ab) = \delta(a)b + a\delta(b).$$

Such a derivation is closed if the closure of its graph in $A \times \mathcal{X}$ is the graph of a function. In other words, any sequence $a_n \in \mathcal{A}$ such that

$$a_n \rightarrow 0 \quad \text{and} \quad \delta(a_n) \rightarrow y$$

(convergence in $A$ and $\mathcal{X}$, resp.) satisfies $y = 0$. 
If $X$ admits an involution which is compatible with the action of $A$, $\delta$ is a $*$-derivation if $\delta(\alpha^*) = \delta(\alpha^*)$ whenever the involved terms are well-defined.

Given a derivation $\delta$ in the sense of Definition 2.8 defined on a subalgebra $\mathcal{A}$, we want to construct the associated “$C^*$-functions” (more precisely almost everywhere Lipschitz functions, see Lemma 1 in [Con94]). More formally, we construct an operator $*$-algebra $A_1$ (see Definition 2.3 p.8 of [KL13]), notion which in turn depends on that of operator space. This later concept is well-understood (see e.g. [Rua88, Ble95]), suffice it to say here that these are Banach spaces which admit a suitable extension of their norms to finite matrices and that they are precisely (norm-)closed subspaces of $C^*$-algebras. Naturally associated to these structures are completely bounded morphisms, i.e. morphisms that extend to matrices of arbitrary size while keeping a bounded norm. An operator $*$-algebra, as introduced first in Definition 3.2.3 of [Mes12] and Definition 3.3 of [Iva11]:

**Definition 2.9.** An operator $*$-algebra $A_1$ is an operator space s.t.

- its multiplication $m$: $A_1 \times A_1 \to A_1$ is completely bounded;
- there is a completely bounded involution $\dagger$: $A_1 \to A_1$ – the extension to matrices being provided by transposing matrices (and applying $\dagger$ entrywise);

An important example of such operator $*$-algebra is provided by the following adaptation of Proposition 2.6 in loc. cit.

**Proposition 2.10.** Let $F_1$ and $F_2$ be two $B$-$C$ $C^*$-correspondences with faithful left-actions $\pi_1, \pi_2$ of $B$ and a derivation $\delta$ from $B$ to $\mathbb{B}_C(F_1 \to F_2)$ which is defined on the dense subalgebra $\mathcal{B} \subseteq B$. Define an algebra morphism from $\mathcal{B}$ to $\mathbb{B}(F_1 \oplus F_2)$ by

$$\rho(b) = \begin{pmatrix} \pi_1(b) & 0 \\ \delta(b) & \pi_2(b) \end{pmatrix}.$$  

The completion $B_1$ of $\mathcal{B}$ for $\|b\|_1 := \|\rho(b)\|$ is a dense subalgebra of $B$ if and only if $\delta$ is closable. In this case, $B_1$ has the following properties:

1. Both the inclusion $B_1 \hookrightarrow B$ and $b \mapsto \delta(b)$ are completely bounded.
2. $B_1$ is stable under holomorphic calculus.
3. If $\mathcal{B}$ is a $*$-algebra, $F_1 = F_2 =: F$ and $\delta(b^*) = U\delta(b)^*U$ for some unitary $U \in \mathbb{B}_C(F)$ which commutes with $\pi(b)$ for $b \in \mathcal{B}$, then $B_1$ is an operator $*$-algebra.

**Proof.** We start by the proof of the inclusion $B_1 \hookrightarrow B$. Considering $\rho$ as above, by definition $B_1$ is included in $\mathbb{B}(F_1 \oplus F_2)$. We have two continuous linear maps from $B_1$ to $\mathbb{B}(F_1)$ and $\mathbb{B}(F_1 \to F_2)$ given on elements of $\mathcal{B}$ by $\phi(b) = \pi_1(b)$ and $\psi(b) = \delta(b)$. By definition of $B_1$, these maps are continuous for $\| \cdot \|_1$ and therefore extend to $B_1$.

However, if $\delta$ is not closable, $\phi$ is not injective. Indeed, in this case there is a sequence $b_n \in \mathcal{B}$ s. t. $b_n \to 0$ and $\partial(b_n) \to y \neq 0$. Denote $b$ the limit of this Cauchy sequence in $B_1$, then $\phi(b) = 0$ but $\psi(b) = y \neq 0$.

Conversely, if $\delta$ is closable, since $\pi_1$ is faithful, the image of $\mathcal{B}$ by the map $b \mapsto (\phi(b), \psi(b))$ is (isomorphic to) the graph of $\delta$. $B_1$ is then (isomorphic to) the closure of this graph in $B \times \mathbb{B}(F_1 \to F_2)$. If this closure is still the graph of a
function, then \( \phi \) is injective and \( B \to B \). Moreover, \( \mathcal{B} \) is dense and \( \mathcal{B} \subseteq B_1 \) – which proves the density of \( B_1 \) in \( B \).

The stability under holomorphic functional calculus is then a straightforward consequence of the “standard result” Lemma 2 p.247 of [Con94], 6.0.

The inclusion \( B_1 \to B \) is completely bounded: the tensor product of \( B_1 \) by finite matrices comes with the restriction of the unique \( C^\ast \)-norm on \( \mathbb{B}(F_1 \otimes F_2) \otimes M_N(\mathbb{C}) \).

The map from \( B_1 \otimes M_N(\mathbb{C}) \) to \( B \otimes M_N(\mathbb{C}) \) is thus contractive.

For the last point, we refer to Proposition 2.6 in [KL13]. \( \square \)

We now introduce connexions associated to derivations, i.e. our definition is slightly different from [Con94] III.1. Definition 5 p.227.

**Definition 2.11.** Given a (right) \( A \)-Hilbert module \( E \) and a derivation \( \delta : \mathcal{A} \to \mathcal{X} \) on \( A \) with values in an \( A \)-\( A \) \( C^\ast \)-correspondence \( \mathcal{X} \), an associated (right) connexion on \( E \) is a linear map \( \nabla : \mathcal{E} \to E \otimes_B \mathcal{X} \) defined on a dense subset \( \mathcal{E} \subseteq E \) such that

- \( \mathcal{E} \) is a right \( \mathcal{A} \)-module and
- for all \( a \in \mathcal{A}, \xi \in \mathcal{E}, \)

\[
(1) \quad \nabla(\xi a) = \nabla(\xi) a + \xi \otimes \delta(a).
\]

The connexion \( \nabla \) is called closable if \( \delta \) is closable and for any sequence \( \xi_n \in \mathcal{E} \) which satisfies \( \xi_n \to 0 \) and \( \nabla(\xi_n) \to \eta \), we have \( \eta = 0 \).

If \( \nabla \) is closable, we denote the domain of its closure by \( \overline{\text{Dom}(\nabla)} \).

**Remark 2.12.** It is readily checked that \( \overline{\text{Dom}(\nabla)} \) is a \( B_1 \)-module. We use the symbol \( \nabla \) to denote both the connexion and its closure.

**Proposition 2.13.** Any connexion \( \nabla \) on a Hilbert \( B \)-module \( E \), associated to a derivation \( \delta \), induces a derivation \( \partial \) on \( \mathbb{B}_B(E) \) with values in \( \mathbb{B}_B(E \to E \otimes \mathcal{X}) \) defined by

\[
(2) \quad (\partial(T))(\xi) = \nabla(T(\xi)) - (T \otimes \text{id}_X)(\nabla(\xi)).
\]

Moreover, if \( \nabla \) is closed, defined on a dense subset \( E_1 = \overline{\text{Dom}(\nabla)} \subseteq E \), \( E \) is finitely generated projective and \( (E_1, E_1) \subseteq B_1 \) then \( \partial \) is densely defined and is closable.

The domain of its closure is:

\[
(3) \quad \overline{\text{Dom}(\partial)} := \{ T \in \mathbb{B}_B(E) | T(\overline{\text{Dom}(\nabla)}) \subseteq \overline{\text{Dom}(\nabla)} \},
\]

which is therefore stable under holomorphic functional calculus.

**Proof.** If \( \overline{\text{Dom}(\nabla)} \) is the domain of the connexion \( \nabla \), then we set:

\[
\overline{\text{Dom}(\partial)} := \{ T \in \mathbb{B}_B(E) | T(\overline{\text{Dom}(\nabla)}) \subseteq \overline{\text{Dom}(\nabla)} \}.
\]

and define \( \partial T \) by (2).

It remains to prove that \( \partial \) is a derivation:

\[
\partial(TT')(\xi) = \nabla(TT'(\xi)) - (TT' \otimes 1)(\nabla\xi)
\]
\[
= \nabla(TT'(\xi)) - (T \otimes 1)v(T'(\xi)) + (T \otimes 1)v(T'(\xi)) - (TT' \otimes 1)v\xi
\]
\[
= (\partial(T))v(T'(\xi)) + (T \otimes 1)[v(T'(\xi)).
\]
Whenever $\partial T$ is defined, it is $\mathcal{B}$-linear:

$$
(\partial T)(\xi b) = \nabla(T(\xi b)) - (T \otimes 1)(\nabla(\xi b)) = \nabla(T(\xi))b + T(\xi)\delta(b) - (T \otimes 1)(\nabla(\xi))b - T(\xi)\delta(b) = \nabla(T(\xi))b - (T \otimes 1)(\nabla(\xi))b = \partial(T)(\xi)b.
$$

Under the additional assumptions, we prove the density of $\text{Dom}(\partial)$: since $E$ is f.g. projective, any element $T \in \mathbb{B}_B(E)$ can be written

$$
T = \sum_{j=1}^{m} \eta_j \langle \xi_j, \cdot \rangle,
$$

where $\eta_j$ and $\xi_j$ are elements of $E$. Since $E$ is dense in $E$, we can find “perturbations” $\eta'_j$ and $\xi'_j$ in $E$ of $\eta_j$ and $\xi_j$ respectively. It is then easy to prove using the Hilbert module norm that if $\eta'_j$ and $\xi'_j$ are chosen “close” to $\eta_j$ and $\xi_j$, then

$$
T' := \sum_{j=1}^{m} \eta'_j \langle \xi'_j, \cdot \rangle
$$

is “close” to $T$. Furthermore, $(\mathcal{E}, \mathcal{E}) \subseteq B_1$ ensures that $T' \in \text{Dom}(\partial)$. This concludes the proof that $\partial$ is densely defined.

If we apply the previous argument to $T = \text{id}_E$, we get an element $T' \in \text{Dom}(\partial)$ which is invertible in $\mathbb{B}_B(E)$. This implies in particular that $(\eta'_j)$ generates $E$ over $B$. This property ensures that we can extend $\partial T'$ — which is \textit{a priori} only defined on $\text{Dom}(\nabla)$ — by $B$-linearity to an element of $\mathbb{B}(E \to E \otimes \mathcal{X})$. Indeed, if $V$ is the inverse (in $\mathbb{B}(E)$) of $T$, then

$$
\xi = T'V(\xi) = \sum_{j} \eta'_j \langle \xi'_j, V(\xi) \rangle.
$$

This provides a \textit{unique} decomposition of $\xi$ in terms of $\eta'_j$. We then extend $\partial T$ by setting:

$$
(\partial T)(\xi) := \sum_{j} (\partial T)(\eta'_j) \langle \xi'_j, V(\xi) \rangle.
$$

Since for any fixed $j$, the map $\xi \mapsto \langle \xi'_j, V(\xi) \rangle$ is continuous, the map $\partial T$ thus extended is continuous and coincides with the “natural definition” of $\partial T$ on the dense subset $\text{Dom}(\nabla)$. This continuous extension is therefore unique and $\partial T \in \mathbb{B}(E \to E \otimes \mathcal{X})$.

We now prove that if $\nabla$ is closable, then $\partial$, defined naturally on

$$
\text{Dom}(\partial) := \left\{ T \in \mathbb{B}_C(E) \big| T(\text{Dom}(\nabla)) \subseteq \text{Dom}(\nabla) \right\},
$$

is closable: if we have a sequence $T_n$ in $\text{Dom}(\partial)$ such that both $T_n \to 0$ and $\nabla T_n \to V$ (convergence in $\mathbb{B}_B(E)$ and $\mathbb{B}_B(E \to E \otimes \mathcal{X})$, respectively), then $V = 0$.

The proof is very similar to that of Lemma 3.2: we consider $\xi \in \text{Dom}(\nabla)$ and denote $\eta := \nabla(\xi) \in E \otimes \mathcal{X}$. Since $T_n \to 0$, we have $T_n \nabla(\xi) \to 0$ and consequently

$$
(\partial T_n)(\xi) = \nabla(T_n(\xi)) - (T_n \otimes 1)\nabla(\xi) \to V(\xi).
$$

The sequence $T_n(\xi) \in \text{Dom}(\nabla)$ tends to $0$ and $\nabla(T_n(\xi))$ admits the limit $V(\xi)$ for $n \to 0$. As $\nabla$ is closed, we must have $V(\xi) = 0$. This is true on the dense subspace $\text{Dom}(\nabla)$ of $E$, therefore $V = 0$ and $\partial$ is closable.
Finally, the inclusion of sets given by \( \{ \mathcal{T} \} \) is actually an equality: consider
\[
\text{graph}(\partial) := \{(T, \partial T), T(\text{Dom}(\nabla)) \subseteq \text{Dom}(\nabla)\} \subseteq \mathcal{B}_B(E) \times \mathcal{B}_B(E \to E \otimes \mathcal{X}).
\]
A Cauchy sequence \((T_n, \partial T_n)\) in graph(\(\partial\)) is a sequence such that both \(T_n\) and \(\partial T_n\) converge in \(\mathcal{B}_B(E)\) and \(\mathcal{B}_B(E \to E \otimes \mathcal{X})\), respectively. Denote \(T_\infty\) the limit of \(T_n\) in \(\mathcal{B}_B(E)\). For any \(\xi \in E_1\), we have \(T_k \xi \to T_\infty \xi\). Moreover, the definition (2) can be read:
\[
\partial(T_k(\xi)) = (\partial T_k)(\xi) + (T_k \otimes 1)(\partial \xi).
\]
By hypothesis, \(\partial T_k\) is norm convergent and \((T_k \otimes 1)(\partial \xi) \to (T_\infty \otimes 1)(\partial \xi)\), hence \(\partial(T_k(\xi))\) converges in \(E \otimes \mathcal{X}\). This implies that \(T_k(\xi)\) actually converges in \(E_1\) and thus \(T_\infty(\xi) \in E_1\). This in turn implies that \(T_\infty(E_1) \subseteq E_1\), i.e. \(T_\infty \in \text{Dom}(\partial)\).

The last property of the Lemma is an immediate consequence of our previous results and of the “standard result” Lemma 2 p.247 of [Con94], 6.a.

For our purposes, the notion of operator \(*\)-module is the essential companion of operator \(*\)-algebra as introduced in Definition 2.9 above. We call standard module \(A_1^\infty\) over an operator \(*\)-algebra \(A_1\) the closure of the finite column-vectors with entries in \(A_1\). We are now ready to state (compare [KL13, Definition 3.4]):

**Definition 2.14.** An operator \(*\)-module \(E_1\) over an operator \(*\)-algebra \(A_1\) is an operator space together with

- a completely bounded (right) action of \(A_1\) on \(E_1\);
- a completely bounded pairing \(\langle \cdot, \cdot \rangle\) with values in \(A_1\), which satisfies the same algebraic conditions as for a Hilbert module;
- a completely bounded self-adjoint idempotent \(P: A_1^\infty \to A_1^\infty\) s.t. \(PA_1^\infty\) is isomorphic to \(E_1\) (as operator space).

**Remark 2.15.** The abstract notion underlying our constructions is more clearly expressed in terms of the following alternative definitions: call derivation on an algebra \(A\) a pair of linear maps \((\pi, \delta)\) where \(\pi: A \to B\) is a homomorphism to an algebra \(B\) and \(\delta: \mathcal{A} \to B\) is a linear map defined on a subalgebra \(\mathcal{A} \subseteq A\) such that for all \(a, a' \in \mathcal{A}\):
\[
\delta(aa') = \delta(a)\pi(a') + \pi(a)\delta(a')
\]
One obtains an associated notion of connexion as follows: a \(\delta\)-connexion on \(E\) is given by an \(\mathcal{A}\)-submodule \(\mathcal{E} \subseteq E\) and linear maps \(\pi_E: E \to D, \nabla: \mathcal{E} \to D\) such that for all \(a \in \mathcal{A}\) and \(\xi \in \mathcal{E}\):
\[
\nabla(\xi a) = \nabla(\xi)\pi_A(a) + \pi_E(\xi)\delta(a).
\]
\(\mathcal{E}\) will be referred to as the domain of \(\nabla\). This yields a submodule and subalgebra of \(M_2(D)\) by identifying
\[
a \approx \begin{pmatrix}
\pi_A(a) & 0 \\
\delta(a) & \pi_A(a)
\end{pmatrix}, \quad \xi \approx \begin{pmatrix}
\pi_E(\xi) & 0 \\
\nabla(\xi) & \pi_E(\xi)
\end{pmatrix}
\]
which thus carry natural structures of an operator algebra and operator module. However, such definition would require a careful study of the dependence on \(D, \delta\) and \(\nabla\) which we want to avoid for the moment.
The previous Proposition 2.13 has the following consequences:

**Corollary 2.16.** If

- \( \delta \) and \( \nabla \) are respectively a densely defined and closable derivation on \( B \) and a \( \delta \)-connexion on a finitely generated projective \( C^* \)-correspondence \( E \);
- we have for \( E_1 := \overline{\text{Dom}(\nabla)} \), the domain of the closure of \( \nabla \), that \( \langle E_1, E_1 \rangle \subseteq B_1 \),

then

- (i) there is a frame of \( E \) consisting of elements of \( E_1 \),
- (ii) both the inclusion \( E_1 \hookrightarrow E \) and the map \( E_1 \to E, \xi \mapsto \nabla(\xi) \) are completely bounded.

**Remark 2.17.** This implies in particular that \( E_1 \) is an operator module in the sense of [KL13], Definition 3.1. Moreover, the frame of elements of \( E_1 \) induces a projection with entries in \( B_1 \) – as required by Definition 3.4 in [KL13].

**Proof.** Regarding the point (i), we use Remark 2.2 to obtain a (finite) frame \( \xi_i \) in \( E \), i.e.

\[
\sum_{i} \xi_i \cdot \langle \xi_i, \cdot \rangle = \text{id}_E.
\]

Just as in the proof of Proposition 2.13 we consider a perturbation \( \zeta_j \) of \( \xi_i \) which is “close enough” to \( \xi_i \). We thus obtain an operator \( \sum_{j=1}^n \zeta_j \cdot \langle \zeta_j, \cdot \rangle := T \in B(B(E)) \) which is “close enough” to \( 1_E \) – and in particular invertible in \( B(B(E)) \). It is also clear that \( T \) sends \( \text{Dom}(\nabla) \) to itself.

We can then consider \( V := T^{-1/2} \), which is defined by holomorphic functional calculus and is therefore in \( \overline{\text{Dom}(\nabla)} \) – by Proposition 2.13. By construction, \( V \) belongs to the algebra generated by \( T \) and since \( T = T^* \), \( V \) commutes with \( T \) and \( V = V^* \). We can evaluate:

\[
VTV = \text{id}_E = \sum_{j} (V\zeta_j) \cdot \langle V\zeta_j, \cdot \rangle
\]

where we used that \( V^* = V \). This equation precisely means that \( (V\zeta_j) \) is a frame for \( E \). The equality \( 2 \) proves that \( V\zeta_j \in E_1 \) and therefore concludes the proof of point (i).

Point (ii) requires to introduce \( F := B \oplus E \). It is easy to see that \( \nabla' := \delta \oplus \nabla \) is a connexion on \( F \). Since \( \delta \) and \( \nabla \) are closable, so is \( \nabla' \). In the same way, since \( E \) is f.g. projective, so is \( F \). The compatibility of \( F_1 \) and \( B_1 \) with the scalar product follows from that of \( E_1 \) and \( B_1 \). We then rely on the linking algebra as in Definition 2.6 applying Proposition 2.13 and Proposition 2.10 to \( \nabla' \), we get a completely bounded inclusion \( \mathcal{B}(F)_1 \hookrightarrow \mathcal{B}(F) \) and a complete bounded map extending the derivation \( \partial' \) on \( \mathcal{B}(F) \).

We can restrict and corestrict the injection to \( E \subseteq \mathcal{B}(F) \) as in Definition 2.6 thereby obtaining a completely bounded inclusion \( E_1 \hookrightarrow E \). Consider now the restriction of \( \partial' \) to \( E \subseteq \mathcal{B}(F) \): if \( \xi \in E \simeq \mathcal{B}(B \to E) \), we can evaluate \( \partial'(\xi) \) on \( 1 \in B_1 \):

\[
\partial'(\xi)(1) = \nabla(\xi)(1) - \xi\delta(1) = \nabla(\xi).
\]

By restricting and corestricting \( \partial' \), we hence prove that \( \xi \mapsto \nabla(\xi) \) from \( E_1 \) to \( E \otimes X \) is completely bounded. \( \Box \)
3 Dirac-Rieffel operators

3.1 Spectral triples

In this article, our main object of interest will be spectral triples in the following sense:

Definition 3.1. Let $A$ be a unital $C^\ast$-algebra. An odd spectral triple, also called odd unbounded Fredholm module, is a triple $(\pi, H, D)$ where:

- $H$ is a Hilbert space and $\pi: A \to \mathcal{B}(H)$ a faithful $\ast$-representation of $A$ by bounded operators on $H$;
- a selfadjoint unbounded operator $D$ – which we will call the Dirac operator – defined on the domain $\text{Dom}(D)$;

such that

(i) $(1 + D^2)^{-1/2}$ is compact,

(ii) the subalgebra $\mathcal{A}$ of all $a \in A$ such that $[\pi(a), D]$ is bounded is dense in $A$.

An even spectral triple is given by the same data, but we further require that a grading $\gamma$ be given on $H$ such that (i) $A$ acts by even operators, (ii) $D$ is odd.

We recall for the readers’ convenience that a commutator of bounded operator $T$ with an unbounded operator $D$ is said to be bounded (written $[D, T] \in \mathcal{B}(H)$ if $T \text{Dom}(D) \subseteq \text{Dom}(D)$ and $[D, a]$ is a bounded operator on its domain. As is common practice, we denote the closure of $[D, a]$ again by $[D, a]$. We caution the reader that this condition is often obscured (see e.g. [Bla93]), and that this definition is indeed the only reasonable one for the definition of $KK$-theory (compare [Hil10], Section 4).

Spectral triples provide the standard examples of closed derivations on $\mathcal{A}$ with values in $\mathcal{B}(H) = \mathbb{B}_c(H)$:

Lemma 3.2. Let $(\pi, H, D)$ be a spectral triple on $A$, then $D$ determines a closable derivation on $\mathcal{A}$ with values in $\mathcal{B}(H)$ by $\delta_D(a) := [D, a]$. Its closure is the derivation $\bar{\delta}_D$ with domain $\text{Dom}(\bar{\delta}_D) = \{a \in A \mid [D, a] \in \mathbb{B}(H)\}$ defined by $\bar{\delta}_D(a) = [D, a]$.

Remark 3.3. It follows that $\delta'(a) := i[D, a]$ is a $\ast$-derivation.

Proof. The proof is similar to that of Proposition 2.13. The operator $D$ is selfadjoint and thus closed. Let $a \in \text{Dom}(\delta_D)$, i.e. there exists a sequence $a_n \in \mathcal{A}$ with $a_n \to a$ in $A$ and $\delta_D(a_n)$ convergent in $\mathbb{B}(H)$. By hypothesis, $a_n \text{Dom}(D) \subseteq \text{Dom}(D)$. Given any $\xi \in \text{Dom}(D), [D, a_n]\xi$ converges and $a_n D\xi \to a D\xi$, thus $D(a_n\xi)$ is convergent. As $D$ is closed and $a_n\xi \to a\xi$, $a\xi \in \text{Dom}(D)$ and $D a\xi = \lim D a_n \xi$. It follows that for all $\xi \in \text{Dom}(D)$:

$$\bar{\delta}_D(a)\xi = \lim \delta_D(a_n)\xi = \lim D a_n \xi - a_n D\xi = [D, a]\xi$$

thus $[D, a] \in \mathbb{B}(H)$ and $\bar{\delta}_D(a) = [D, a]$. \hfill $\Box$

It is natural to define $\Omega_D^1$, the 1-forms associated to $D$ (see [Con94], VI.1. Proposition 4 p.549), as the closed linear span in $\mathbb{B}(H)$:

$$\Omega_D^1 := \text{Span}\{a_j^0[D, a_j^1]a_j^0, a_j^1 \in \mathcal{A}\}. $$(5)
The space of 1-forms $\Omega^1_D$ is clearly an $\mathcal{A}$-bimodule via the representation $\pi: \mathcal{A} \to B(H)$. It is a Banach bimodule of the type mentioned in the point (ii) of Definition 2.5. In this context, we obtain a “concrete” (or “represented”) version of Definition 2.11:

**Definition 3.4.** (compare Chapter 6.1, Definition 8 of [Con94]) Let $(\pi, H, D)$ be a spectral triple on a unital $C^*$-algebra $A$, $E$ a Hilbert $A$-module. Denote by $\delta_D$ the derivation on $A$ with values in $\Omega^1_D$ defined by $\delta_D(a) = [D, a]$. We call $D$-connexion on $E$ a connexion on $E$ with respect to $\delta_D$.

### 3.2 Rieffel Spectral Triples

We recall briefly the definition of the complexified Clifford algebra of dimension $n$ (see [LM89] for further details):

**Definition 3.5.** For $n \in \mathbb{N}$, we denote $\mathbb{C}l(n)$ the universal unital $C^*$-algebra generated by $n$ selfadjoint elements $e_j$ which satisfy the relations:

$$e_j e_k + e_k e_j = 2\delta_{jk}. \tag{6}$$

A $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{C}l(n)$ is induced by the automorphism $h$ defined by $h(e_i) = -e_i$.

In the rest of this article, we will consider a Clifford module $S$, i.e. a module $S$ equipped with a representation $\pi_{\mathbb{C}l(n)}$ of $\mathbb{C}l(n)$ and denote by $\gamma_i$ the operators $i\pi_{\mathbb{C}l(n)}(e_j) \in B(S)$. To unclutter notation, we will most of the time suppress the representation $\pi_{\mathbb{C}l(n)}$. For simplicity, we take $S$ to be finite dimensional.

**Remark 3.6.** The operators $\gamma_j$ satisfy:

$$\gamma_j^* = -\gamma_j \quad \gamma_j \gamma_k + \gamma_k \gamma_j = -2\delta_{jk}. \tag{7}$$

The present work focuses on a particular form of Dirac operators:

**Definition 3.7.** We say that $(\pi, H, D)$ is a Rieffel spectral triple if

- $H = H_0 \otimes S$ for some Hilbert space $H_0$ and Clifford module $S$;
- the operator $D$ is a Dirac-Rieffel operator, i.e. with respect to the previous decomposition, it can be written:

$$D = \sum_{j=1}^n \partial_j \otimes \gamma_j, \tag{8}$$

with $\gamma_j$ as in Definition 3.5 and unbounded operators $\partial_j$ on $H$ such that $\partial_j^* = -\partial_j$;

- $\partial_j^{\mathcal{A}}(a) := [\partial_j, a]$ for all $a \in \mathcal{A}$ defines $*$-derivations on $A$ with values in $A$.

**Remark 3.8.** A direct consequence of the above definition is: for all $a \in \mathcal{A}$,

$$[D, a] = \sum_{j} \partial_j^{\mathcal{A}}(a) \otimes \gamma_j. \tag{9}$$

In our previous article [GG], we proved that such Dirac-Rieffel operators can be constructed from ergodic actions of compact Lie groups – see Theorem 5.4 therein. In the present article, we want to investigate the “permanence properties” of this class of spectral triples.
4 Operator ∗-modules

We study connexions associated to Rieffel spectral triples as introduced in Definition 3.7. For the rest of this section, let \((\pi, H, D)\) be a Rieffel spectral triple and \(\delta_D := [D, \cdot]\) the associated derivation (Lemma 3.2). As we will see, in this setting, there is a particularly simple notion of Hermitian connexion. At the root of this lie the following results:

**Lemma 4.1.** The space of 1-forms \(\Omega^1_D\) satisfies:

\[
\Omega^1_D \subseteq \pi(B) \otimes \mathbb{C} \langle \gamma_1, \ldots, \gamma_n \rangle \subseteq B(H)
\]

and has a natural \(B\)-Hilbert module structure.

If \(\nabla\) is a connexion defined on \(\mathcal{E}\) and associated to \(\delta_D\), then \(\nabla(\xi) = \sum_j \nabla_j(\xi) \otimes \gamma_j\) for certain "components" \(\nabla_j\) with the following properties:

- \(\nabla_j : \mathcal{E} \to \mathcal{E}\);
- for all \(j\), the following equality holds for \(\xi \in \mathcal{E}\) and \(b \in \mathcal{B}\):

\[
\nabla_j(\xi b) = \nabla_j(\xi) b + \xi \partial_j(b).
\]

Conversely, given maps \(\nabla_j\) defined on \(\mathcal{E}\) which satisfy Equation (11)

\[
\nabla(\xi) := \sum \nabla_j(\xi) \otimes \gamma_j.
\]

defines a concrete connexion associated to \(D\).

Finally, \(\nabla\) is closable if and only if all its components are closable.

**Proof.** The proof of this lemma is essentially algebraic. The inclusion (11) is a direct consequence of Equation (9). We can then define a \(B\)-valued scalar product on \(\Omega^1_D\) by setting:

\[
(b \otimes \gamma_j, b' \otimes \gamma_k) = \delta_{j,k} b^* b' = (\text{id}_B \otimes \text{Tr})((b \otimes \gamma_j)^* (b' \otimes \gamma_k))
\]

where \(\text{Tr}\) is the unique normalised trace on the finite dimensional matrix algebra \(B(S)\). Hence in this particular case, the Banach bimodule \(\mathcal{X}\) actually fits within case (i) of Definition 2.5.

An immediate consequence of (11) is the inclusion:

\[
E \otimes_B \Omega^1_D = \text{Span} \{\xi \otimes [D, b] | \xi \in \mathcal{E}, b \in B_1\} \subseteq E \otimes \mathcal{E} B(S).
\]

For any given \(\xi\), there are \(n\) unique elements \(\xi_j \in \mathcal{E}\) such that

\[
\nabla(\xi) = \sum \xi_j \otimes \gamma_j.
\]

Moreover, since the family \(\gamma_j\) is free in \(B(S)\), these \(\xi_j\) are uniquely defined. It then suffices to define the maps \(\nabla_j\) by \(\xi \mapsto \xi_j\) and Equation (11) reads:

\[
\sum \nabla_j(\xi b) \otimes \gamma_j = \sum \nabla_j(\xi) b \otimes \gamma_j + \xi \sum \partial_j(b) \otimes \gamma_j.
\]

Since \(\gamma_j\) is free in \(B(S)\), it is equivalent to say that for all \(j\),

\[
\nabla_j(\xi b) = \nabla_j(\xi) b + \xi \partial_j(b)
\]

where the equalities take place in \(\mathcal{E}\).
Now, assume that the family $\nabla_j$ satisfies Equation (11), let us show that (12) defines a connexion associated to $D$:
\[
\nabla(\xi b) = \sum \nabla_j(\xi b) \otimes \gamma_j = \sum (\nabla_j(\xi) b + \xi \partial_j(b)) \otimes \gamma_j = \nabla(\xi) b + \xi [D, b].
\]
Finally, the result on closability is a straightforward consequence of the decomposition of $\nabla$ into components. \hfill \square

Remark 4.2. Using Lemma 4.1 we can give a meaning to the scalar products
\[
\langle \xi, \nabla \eta \rangle := \sum \langle \xi, \nabla_j(\eta) \rangle \otimes \gamma_j, \quad \langle \nabla \xi, \eta \rangle := - \sum \langle \nabla_j(\xi), \eta \rangle \otimes \gamma_j.
\]
It is readily checked that $\langle \nabla \xi, \eta \rangle^* = \langle \eta, \nabla \xi \rangle$ – and this accounts for the minus sign in $\langle \nabla \xi, \eta \rangle$.
Comparing the above definition with the pairing appearing in [KL13] (4.2), we see that both expressions are compatible.

We proceed with Hermitian connexions, following [KL13], Definition 4.3 and [Con94] VI.1 p.553:

Definition 4.3. Consider a connexion $\nabla$ on $E$, associated to $\delta_D := [D, \cdot]$ as in Lemma 4.1 which is defined on $E$ such that the $B$-valued scalar product on $E$ restricts to $E$ in a compatible way, i.e. $\langle E, E \rangle \subseteq B_1$. Then $\nabla$ is Hermitian if for all $\xi, \eta \in E$
\[
[D, \langle \xi, \eta \rangle] = \langle \xi, \nabla(\eta) \rangle - \langle \nabla(\xi), \eta \rangle.
\]

Remark 4.4. It is easily seen, by decomposing with respect to the free family $(\gamma_j)$ of $B(S)$ that (13) is equivalent to for all $\xi, \eta \in E$
\[
[D, \langle \xi, \eta \rangle] = \langle \xi, \nabla(\eta) \rangle - \langle \nabla(\xi), \eta \rangle.
\]

Moreover, if $\nabla$ is Hermitian on $E$, then $\langle E_1, E_1 \rangle \subseteq B_1$ and (13) is satisfied on $E_1$.

Proposition 4.5. If
\begin{itemize}
  \item $\nabla$ is a closable connexion on a f.g. projective $C^*$-correspondence $E$,
  \item $\text{Dom}(\nabla)$ is dense in $E$ and for $E_1 := \text{Dom}(\nabla)$, $\langle E_1, E_1 \rangle \subseteq B_1$,
  \item $\nabla$ is associated to the derivation $\delta_D$ induced by a Rieffel spectral triple,
  \item $\nabla$ is Hermitian,
\end{itemize}
then $E_1$ is an operator $*$-module (for the standard involution $\dagger = \ast$ – see Definition 3.2 p.12 of [KL13]).

Proof. All hypotheses are satisfied to apply Cor. 2.16 to $\nabla$ and $E$. This existence of a frame suffices to obtain the isomorphism with $PA^\infty$ as required in [KL13] p.13. Indeed, consider the finite matrix $P := \left( \langle \zeta_j, \zeta_k \rangle \right)_{j,k}$. This is a selfadjoint operator:
\[
\langle Pa_i, b_i \rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} (P_j a_j)^* b_i = \sum_{i,j=1}^{m} \langle \zeta_i, \zeta_j a_j \rangle^* b_i
\]
\[
= \sum_{i,j=1}^{m} \langle \zeta_j a_j, \zeta_i \rangle b_i = \sum_{i,j=1}^{m} a_j^* \langle \zeta_j, \zeta_i \rangle b_i = \langle a_j, P b_j \rangle.
\]
The two representations extend naturally to $\Omega^1_{id}$ and $B\sum_\pi$ representation $B$ have a meaning in $P$ is also an idempotent:

$$
\sum_{k=1}^{m}\langle \xi_k, \zeta_k \rangle = \sum_{k=1}^{m}\langle \xi_k, \xi_k \zeta_k \rangle = \langle \xi_k, \zeta_k \rangle
$$

since $\langle \zeta_j \rangle$ is a frame. $E_1$ is isomorphic to $PA^n_1$, in the sense that there are maps $\Phi : E_1 \rightarrow PA^n_1$ and $\Psi : PA^n_1 \rightarrow E_1$ defined by:

$$
\Phi(\xi) = \langle (\zeta_j, \xi) \rangle_j \quad \quad \quad \Psi(a_j) = \sum \zeta_j a_j
$$

such that $\Phi \circ \Psi = id_{PA^n}$ and $\Psi \circ \Phi = id_{E_1}$.

Finally, the scalar product induced on $PA^n_1$ coincides with the one induced by the isomorphism with $E_1$:

$$
\langle \Psi(a_j), \Psi(b_i) \rangle = \left\langle \sum \zeta_j a_j, \sum \zeta_k b_k \right\rangle = \sum a_j^* \zeta_j \sum \zeta_k b_k = \sum a_j^* b_j
$$

since $\sum \zeta_k b_k = b_j$.

To prove the complete boundedness property of the pairing, we appeal to the Hilbert space $H$. We set $E' := B \oplus E$ and equip it with the closed connexion $\delta_{B'} \nabla$. We can apply Proposition 2.13 to $\mathbb{B}(E') := B'$, thereby getting a derivation $\delta'$ on $B'$ with values in $\mathbb{B}(E' \rightarrow E' \Omega_{B'})$.

Introduce $H' := E' \Omega_{B'}^1$ $H = H \oplus E \Omega_B$, whose decomposed tensors we write $x \oplus \eta \oplus y$. We can realise $\mathbb{B}(E' \rightarrow E' \Omega_{B'})$ inside $B(H')$. Indeed, given $T \in \mathbb{B}(E' \rightarrow E' \Omega_{B'})$ we let it act on $H'$ by:

$$
T(x \oplus \eta \oplus y) = T(\eta)y
$$

where $T(\eta) \in E' \Omega_{B'}^1$ acts on $y \in H$ via the inclusion $\Omega_{B'}^1 \subseteq \mathbb{B}(H)$. This enables us to apply Proposition 2.10 with $\mathbb{B}(H' \oplus H')$ as underlying algebra.

To summarise, $B$ and $E$ sit inside $B'$ and are represented on $B(H')$ via the representation $\pi$ of the spectral triple, which we extend by $\pi(\xi)(x \oplus \eta \oplus y) = \xi \otimes x$ and $\pi(\eta)(x \oplus \eta \oplus y) = \pi_0(b)(x \oplus \eta \oplus y) = bx \oplus (b\eta) \otimes y$.

The two representations extend naturally to $\Omega_{B'}$ and $E \otimes B \Omega_B$: it suffices to let $\text{id}_{H_0} \otimes \gamma_j$ act on $H = H_0 \otimes S$. This gives a meaning to the action of $\nabla(\xi) = \sum \nabla_j(\xi) \otimes \gamma_j$ on $H'$. Since we are using faithful representations of $C^*$-algebras, the norms on $E_1$ and $B_1$ coincide with those obtained from Corollary 2.10.

The expressions $\pi(\xi)$, $\delta'(\pi(\xi)) = \nabla(\pi(\xi))$ (abbreviated to $\nabla(\xi)$) and $\nabla(\xi)^*$ all have a meaning in $B(H')$. It is readily checked that $\pi(\xi)^* \pi(\eta) = \pi_0(\langle \xi, \eta \rangle)$ and we can then write:

$$
\pi(\xi)^* \nabla \eta - (\nabla \xi)^* \pi(\eta) = \pi(\xi)^* \left( \sum \pi(\partial_j(\eta)) \otimes \gamma_j \right) - \sum \pi(\partial_j(\xi) \otimes \gamma_j)^* \pi(\eta)
$$

$$
= \sum \pi_0(\langle \xi, \partial_j(\eta) \otimes \gamma_j \rangle) + \sum \pi_0(\langle \partial_j(\xi), \eta \otimes \gamma_j \rangle) = \pi_0([D, \langle \xi, \eta \rangle])
$$
This leads to
\[
\begin{pmatrix}
   \pi_0(\langle \xi, \eta \rangle) \\
   \pi_0(\langle \xi, \eta \rangle) \\
   \pi_0(\langle \xi, \eta \rangle)
\end{pmatrix}
= \begin{pmatrix}
   \pi(\xi)^* \pi(\eta) & 0 \\
   \pi(\xi)^* \nabla \eta - (\nabla \xi)^* \pi(\eta) & \pi(\xi)^* \pi(\eta)
\end{pmatrix}
\begin{pmatrix}
   \pi(\eta) & 0 \\
   \nabla(\eta) & \pi(\eta)
\end{pmatrix}
\]

The leftmost matrix is (isometric to) the one we used to define $B_1$ in Proposition 2.10. To make contact with the norm on $E_1$ we remark that:
\[
\begin{pmatrix}
   \xi^* & 0 \\
   -\nabla(\xi)^* & \xi^*
\end{pmatrix}
= \begin{pmatrix}
   \xi & -\nabla(\xi) \\
   0 & \xi
\end{pmatrix}^* = \begin{pmatrix}
   0 & -1 \\
   1 & 0
\end{pmatrix} \begin{pmatrix}
   \xi & 0 \\
   \nabla(\xi) & \xi
\end{pmatrix} \begin{pmatrix}
   0 & 1 \\
   -1 & 0
\end{pmatrix}
\]
which proves that $\|\langle \xi, \eta \rangle\|_1 \leq \|\xi\|_1 \|\eta\|_1$.

We conclude this section with the following proposition, which by exception doesn’t require finiteness of the original $C^*$-correspondence $E$. It is a sort of converse to Corollary 2.16. It requires the following

**Definition 4.6.** Let $E$ be a $C^*$-correspondence which is projective. A (possibly infinite) sequence $(\xi_j)$ is a “frame” for $E$ if

\[
T_N := \sum_{k=1}^N \xi_j \langle \xi_j, \cdot \rangle
\]
satisfies $T_N \xi \to \xi$ for any $\xi \in E$ (convergence in $E$).

Examples of such frames will appear naturally in connexion with generalized crossed products by finitely generated projective modules below (see Proposition 6.9).

**Proposition 4.7.** Let

- $\partial$ be a derivation on $B$ associated to a Rieffel spectral triple,
- and $\nabla$ be an associated Hermitian connexion defined on the dense and $B_1$-stable subset $\mathcal{E} \subseteq E$.

If $(\xi_j) \in \mathcal{E}$ is a “frame” for $E$ then the connexion $\nabla$ is closable.

**Remark 4.8.** This is in particular true for the Grassmannian connexion (see [KL13], Section 4.1).

**Proof.** We start from a sequence $\zeta_n$ of elements in Dom($\nabla$) such that $\zeta_n \to 0$ and $\nabla(\zeta_n) \to \eta$. We want to prove that $\eta = 0$.

From Definition 4.6 we see that it suffices that $\langle \xi_j, \eta \rangle = 0$ for all $j$ to obtain $\eta = 0$. We estimate $\langle \xi_j, \eta \rangle$:
\[
\langle \xi_j, \nabla(\zeta_n) \rangle = \partial(\langle \xi_j, \zeta_n \rangle) + \langle \nabla(\xi_j), \zeta_n \rangle
\]
Since $\nabla(\xi_j)$ is a fixed quantity and $\zeta_n \to 0$, we have $\langle \nabla(\xi_j), \zeta_n \rangle \to 0$. Moreover, since both $\xi_j$ and $\zeta_n$ are elements of Dom($\nabla$), $b_n = \langle \xi_j, \zeta_n \rangle \in \text{Dom}(\partial)$ and this sequence tends to 0 in norm. Our previous computation ensures that $\partial b_n \to \langle \xi_j, \eta \rangle$. Since $\partial$ is closed, we get $\eta = 0$. 

\[\square\]
5 Generalised crossed products: a review

Our aim in this paper is to construct a spectral triple for Generalised Crossed Products (GCP), we therefore introduce this notion.

**Definition 5.1.** A Hilbert bimodule $E$ over a $C^*$-algebra $A$ is a bimodule $E$ over $A$ such that

- $E_A$ (right $A$-module) is a right-Hilbert module over $A$ for the scalar product $\langle \cdot, \cdot \rangle_A$;
- $AE$ is a left-Hilbert module over $A$ for $\langle \cdot, \cdot \rangle$;
- for all $\xi, \eta, \zeta \in E$,
  $$A(\xi, \eta)\zeta = \xi(\eta, \zeta)_A.$$ 

**Remark 5.2.** Several notions of “Hilbert bimodule” can be found in the literature. Our convention is different from the one used in [Pim97], for instance.

To illustrate this notion, consider the following simple example: let $X$ be a compact space, and denote by $E := C(X)$ the $C^*$-algebra of complex continuous functions on $X$. We further assume that we have a (locally trivial) line bundle $\mathcal{L} \to X$. According to Serre–Swan’s Theorem (see for instance [GVF01] Theorem 2.10 p.59), the $B$-module $E$ of sections of $\mathcal{L}$ is a f.g. projective bundle over $B$.

If moreover, $E$ is equipped with a fibrewise scalar product $(\cdot | \cdot) : E_x \times E_x \to \mathbb{C}$ (Hermitian line bundle), then we can define a Hilbert module structure on $E$ (see Definition II.7.1.1 p.137 of [Blad05]). Adding an automorphism of $X$, we can define a Hilbert bimodule structure on $E$:

**Example 5.3.** Consider $B = C(X)$, a commutative unital $C^*$-algebra and $\sigma$, an automorphism of $B$. We denote by $\tilde{\sigma} : X \to X$, the homeomorphism of $X$ such that $\sigma(b)(x) = b(\tilde{\sigma}(x))$. A Hilbert module $E$ over $B$ yields a Hilbert bimodule structure by setting, for all $b \in B, \xi \in E$:

$$b \cdot \xi := \xi \sigma^{-1}(b) \quad \quad B\xi, \eta := \sigma(\eta, \xi)_B.$$ 

It is clear that Hilbert bimodules are in particular $C^*$-correspondences. Moreover, it is easily checked that given two Hilbert bimodules, the tensor product $E \otimes_B F$ (see Definition 2.4 above) is also a Hilbert bimodule. The left-scalar product is defined by:

$$B(\xi \otimes \eta, \xi' \otimes \eta') := B(\xi, \eta) \cdot B(\eta', \xi').$$ 

Iterating the construction, we obtain $B$-Hilbert bimodules $E^n := E \otimes_B E \otimes_B \cdots \otimes_B E$ for $n \in \mathbb{N}$, where the tensor product contains $n$-factors.

In the particular case described in Example 5.3, the left- and right-products are linked by the relation: for $\xi_1, \ldots, \xi_n \in E$ and $b \in B$,

$$b \cdot (\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\xi_1 \cdot \sigma^{-1}(b)) \otimes (\xi_2 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes (\xi_2 \cdot \sigma^{-2}(b)) \otimes \cdots \otimes (\xi_n \cdot \sigma^{-n}(b)).$$

Let $A$ be a $C^*$-algebra equipped with a pointwise continuous $S^1$-action $\gamma$. We call such $\gamma$ a gauge action and it yields spectral subspaces $A^{(k)}$, $k \in \mathbb{Z}$, defined by:

$$A^{(k)} := \{ T \in A | \forall z \in S^1 \subseteq \mathbb{C}, \gamma_z(T) = z^k T \}.$$
$T \in A^{(k)}$ is called a gauge-homogeneous element of gauge $k$. It is readily checked that $A^{(0)}$ is a $C^*$-algebra and that given any $k \in \mathbb{Z}$, $A^{(k)}$ has a natural structure of Hilbert bimodule over $A^{(0)}$.

We now define GCP, which are our main topic of interest. The following appeared as Definition 2.1 in [AEE98]:

**Definition 5.4.** Given a Hilbert bimodule $E$ over a $C^*$-algebra $B$, a representation of $E$ on a $C^*$-algebra $C$ is a $C^*$-algebra morphism $\pi$ together with a linear map $S : E \to C$ such that for all $\xi, \zeta \in E$,

1. $S(\xi)^*S(\zeta) = \pi(\langle \xi, \zeta \rangle_B)$
2. $S(\xi)\pi(b) = S(b\xi)$
3. $\pi(b)S(\xi) = S(b\xi)$
4. $S(\xi)S(\zeta)^* = \pi(B\langle \xi, \zeta \rangle_B)$

**Definition 5.5.** Let $B \rtimes _E E$ be a $B$-Hilbert bimodule. The Generalized Crossed Product (GCP) $B \rtimes _E E$ of $B$ by $E$ is the universal $C^*$-algebra generated by the representations of $E$, in the sense that (see [Bla05], II.8.3):

- there is a representation $(\pi_E, S_E)$ of $E$ on $B \rtimes _E E$;
- $B \rtimes _E E$ is generated by $\pi(E) \cup S(E)$;
- for any representation $(\pi, S)$ of $E$ on $C$, there is a $C^*$-morphism $\rho(\pi, S) : B \rtimes _E E \to C$ such that the following diagram commutes:

We say that $F \in B \rtimes _E E$ is algebraic if it is in the involutive algebra generated by $\pi(E)$ and $S(E)$ – without taking the closure. By definition, algebraic elements are dense in the GCP.

**Example 5.6.** Given a compact space $X$, a Hermitian line bundle $L$ over $X$ and an automorphism $\sigma$ of $B = C(X)$, we call GCP associated to $(X, L, \sigma)$ the GCP generated by $B$ and the Hilbert bimodule $E$ constructed from the sections of $L$, as described in Example 5.3.

The notion of GCP is closely related to that of (Cuntz-)Pimsner algebra [Pim97]. An important characterization of GCP is the following Theorem 3.1 of [AEE98]:

**Theorem 5.7.** A $C^*$-algebra $A$ equipped with a gauge action $\gamma$ is the GCP of $A^{(0)}$ by $A^{(1)}$ if and only if it is generated as a $C^*$-algebra by $A^{(0)}$ and $A^{(1)}$.

Using this characterisation, we can prove that commutative GCP correspond essentially to principal $S^1$-bundles:

**Proposition 5.8.** If $A$ is a unital commutative $C^*$-algebra carrying an $S^1$-action such that

- $E := A^{(1)}$ generates $A$ as $C^*$-algebra,
• $E$ is a f.g. projective module over $B := A^{(0)}$, then

(i) $B = C(X)$ for some compact space $X$,
(ii) $E$ is the module of sections of some line bundle $\mathcal{L} \to X$,
(iii) $A$ is isomorphic to $C(P)$ where $P \to X$ is the principal $S^1$-bundle over $X$
associated to $\mathcal{L}$ and the gauge action comes from the principal $S^1$-action.

Remark 5.9. The previous hypotheses ensure that $A$ and $B$ with the gauge action
form a Hopf-Galois extension, as first introduced in [KT81] and later related to
“noncommutative principal bundle” – see e.g. [BM93, BM95, Haj96, Ell00], among
numerous others. We could deduce the previous result from these general directions
of research, however we include the short proof below for self-containment and clarity.

Proof. Since $A$ is unital, there is a compact space $P$ such that $A = C(P)$; $X$ is
of course the spectrum of $A$. It is also clear that $1_A \in B$. Consider the characters
$\varphi : A \to \mathbb{C}$ of $A$: they identify with points of $P$. Any such $\varphi \in P$ induces by
restriction a character on $B$, i.e. $\varphi(b) = b(x_0)$ for all $b \in B$ and some $x_0 \in X$. This
defines the map $P \to X$.

We can now apply the Serre-Swan Theorem to $E := A^{(1)}$ and get a vector bundle
$\mathcal{L} \to X$ whose module of sections is $E$. Since $\mathcal{L}$ is a bundle over $X$, for any $x_0 \in X$
we can define a map $ev_{x_0} : \mathcal{L} \to \mathcal{L}_{x_0}$ where $\mathcal{L}_{x_0}$ is the fiber of $\mathcal{L}$ over $x_0$ – in fact,
this map extends to a $C^*$-algebra morphism $ev_{x_0} : B \otimes_{E} \mathbb{Z} \to \mathbb{C} \otimes \mathcal{L}_{x_0} \mathbb{Z}$,
thereby defining a notion of “local” equality. $A$ is commutative and contains $E \otimes_{B} E$, hence
we must have $\xi \otimes \eta = \eta \otimes \xi$ for any $\xi, \eta \in \mathcal{L}_{x_0}$. This is only possible if the rank
of $\mathcal{L}$ is 1 or 0 – the latter is impossible since $E$ generates $A$, unless $A = \{0\}$.

Thus, for any $x_0 \in X$, we can find a section $\xi_1$ of $\mathcal{L}$ such that $\langle \xi_1, \xi_1 \rangle = 1$ locally
around $x_0$. Consequently $\varphi(\xi_1)^* \varphi(\xi_1) = \varphi(\langle \xi_1, \xi_1 \rangle) = 1$, which shows that $\varphi(\xi_1)$ is a
complex number of module 1. Moreover, any algebraic $F \in A$ can be written locally
around $x_0$ as a sum

$$F = \sum b_n(\xi_1)^n$$

Thus $\varphi$ is totally determined by $x_0$ and $\lambda := \varphi(\xi_1) \in U(1) \subseteq \mathbb{C}$. Moreover, any such
choice $(x, \lambda)$ for $x$ around $x_0$ defines a character of $A$.

Finally, the gauge action $\gamma$ is given by

$$\varphi(\gamma_z(b)) = b \quad \varphi(\gamma_z(\xi_1)) = z \xi_1$$

for any $b \in B$ and complex number $z$ with $|z| = 1$, therefore completing the proof.

\[\square\]

6 Derivations and extensions of connexions

The next step of our construction is to link the spectral triple on the base algebra
with the Hilbert bimodule, via a connexion satisfying certain conditions (see
Definition 6.1 below). For the rest of this section, we fix a Rieffel spectral triple
$(\pi, D, H)$ on a unital $C^*$-algebra $B$ with components $\partial_j$ and associated subspace $\mathcal{B}$
(see Definition 3.1 and 3.7). We denote again by $B_1 \subseteq B$ the operator-$*$-algebra
closure of $\mathcal{B}$ associated to the derivation $\delta_D = [D, \cdot]$, see Proposition 2.10.
Definition 6.1. A two-sided Hermitian $D$-connexion on a Hilbert bimodule $E$ over $B$ is a map $\nabla: \mathcal{E} \to E \otimes_B \Omega_D^1$ defined on a dense subspace $\mathcal{E} \subseteq E$ which is stable under both left and right multiplication by $\mathcal{B}$ such that

- $\nabla$ is a Hermitian $D$-connexion in the sense of Definition 4.3,
- $B(\mathcal{E}, \mathcal{E}) \subseteq B_1$ and for all $j$ and $\xi, \eta \in \mathcal{E}$

\begin{equation}
\nabla_j(b\xi) = b\nabla_j(\xi) + \partial_j(b)\xi,
\end{equation}

as well as for all $j$, $\xi \in \mathcal{E}$ and $b \in \mathcal{B}$

\begin{equation}
\nabla(\xi,\eta) = \partial(\xi,\eta),
\end{equation}

where $\nabla_j : \mathcal{E} \to E$ denote the components of $\nabla$ in the sense of Lemma 4.1.

Remark 6.2. The main difference between (14) and (17) is that the second uses the left scalar product whereas the first involves the right scalar product.

Remark 6.3. We can define in $\Omega_D^1$ the following scalar products:

\begin{equation}
B(\nabla\xi,\eta) := \sum B(\nabla_j(\xi),\eta) \otimes \gamma_j, \quad B(\xi,\nabla\eta) := -\sum B(\xi,\nabla_j(\eta)) \otimes \gamma_j.
\end{equation}

It is easy to check that $(B(\nabla\xi,\eta))^* = B(\eta,\nabla\xi)$. Just like in Remark 4.4, the condition (17) is equivalent to:

\begin{equation}
B(\nabla\xi,\eta) - B(\nabla\eta,\xi) = [D, B(\xi,\eta)].
\end{equation}

This definition of the scalar products on $\Omega_D^1$ is coherent with the pairing defined in (4.2) p. 17 of [KL13].

Example 6.4. If $G$ is a given Lie group, $B$ is equipped with a $G$-action $\alpha$ and $E$ is a Hilbert module over $B$ endowed with a compatible Hilbert bimodule action $\beta$ of $G$, i.e. $\beta_g(\xi) = \beta_g(\xi)\alpha_g(b)$, $\beta_g(b\xi) = \alpha_g(b)\beta_g(\xi)$ and

\begin{equation}
B(\beta_g(\xi),\beta_g(\eta)) = \alpha_g(B(\xi,\eta)), \quad B(\beta_g(\xi),\beta_g(\eta)) = \alpha_g(B(\xi,\eta)),
\end{equation}

then the $G$-smooth elements $\mathcal{E}$ of $E$ and the infinitesimal generators $\nabla_j$ of the action $\beta$ define a two-sided Hermitian connexion (this an easy generalisation of Proposition 4.9 in [Gab13]). We can make it concrete using the Dirac-Rieffel operator as in Definition 4.3.

Remark 6.5. In the situation described in [GG], i.e. if the Dirac operator arises from an ergodic action $\forall g \in G, \alpha_g(b) = b \implies b \in \mathcal{C}^1$ of a compact Lie group $G$, the results of [Gos09] show that all $G$-$B$-Hilbert modules are embeddable, i.e. $E$ is $G$-equivariantly embedded into $B \otimes H_0$ for some Hilbert space $H_0$ endowed with a $G$-action.

Consider now a fixed Hilbert bimodule $E$ over $B$ and denote by $A := B \rtimes E \mathbb{Z}$ the associated GCP. Suppose $\nabla$ is a two-sided Hermitian $D$-connexion on $E$. We denote by $E_1 \subseteq E$ the operator space obtained from $E$ and $\nabla$ by Proposition 4.5.

Lemma 6.6. Denote by $\mathfrak{A}$ the sub-$*$-algebra of $A$ generated $*$-algebraically from $E_1$ and $B_1$. The components $\partial_j$ and $\nabla_j$ of $D$ and $\nabla$, respectively, extend uniquely into $*$-derivations $\sum_j$ defined on $\mathfrak{A}$ with values in $A$. These extensions $\sum_j$ preserve the gauge-action on $\mathfrak{A}$.
Proof. If such derivations exist, then given any ∗-algebraic combination Ξ of elements ξ ∈ E₁ and b ∈ B₁ (e.g. Ξ = ξ₁b₁ξ₂b₂), the ∗-derivation property gives a unique expression for \( \nabla_j(Ξ) \). On our example, we get:

\[
\nabla_j(Ξ) = \nabla_j(ξ₁)b₁ξ₂b₂ + ξ₁\partial_j(b₁)ξ₂b₂ + ξ₁b₁\nabla_j(ξ₂)^*b₂ + ξ₁b₁ξ₂\partial_j(b₂).
\]

This proves unicity. We now need to show that these expressions indeed define a derivation on \( A \). To this end, it suffices to check that the analogs of relations (i) to (iv) of Definition 5.4 behave suitably under the derivation. This is in turn ensured by the assumptions and Definition 6.1. For instance, regarding (i):

\[
\nabla_j(ξ^*ζ) = \partial_j(⟨ξ,ζ⟩_B) = ⟨\nabla_j(ξ),ζ⟩_B + ⟨ξ,\nabla_j(ζ)⟩_B = Σ_j(ξ)^*ζ + ξ^*Σ_j(ζ),
\]

where we used (14). Regarding (ii), we have:

\[
\nabla_j(ξb) = \nabla_j(ξ)b + ξ\partial_j(b)
\]

by Equation (17).

For a formal proof, decompose \( \mathcal{A} \) into an algebraic sum of spectral subspaces (see equation (16)) and use induction on the degree to prove first that \( \nabla_j \) is well-defined, second that it is compatible with products.

Definition 6.7. Let \( B \) be a \( C^* \)-algebra, \( E \) a Hilbert bimodule over \( B \) and \( A := B ⋊_E \mathbb{Z} \). We define a \( B \)-Hilbert module \( X \) as the \( C^* \)-module-completion of \( A \) for the \( B \)-valued scalar product:

\[
⟨Ξ,Ξ'⟩_B := E(Ξ^*Ξ'),
\]

where the conditional expectation \( E : A → B \) is induced by the gauge action.

Remark 6.8. That is, \( E \) is the application defined by

\[
E(a) := \int_{S¹} λ.a dλ.
\]

Equivalently, \( X \) is the Hilbert sum obtained as direct sum of the spectral subspaces \( A(V)^k, k \in \mathbb{Z} \) (as defined equation in (16)). Since the conditional expectation \( E \) is faithful on \( A \), we have an injection \( A↪ X \). Moreover, the conditional expectation is norm-contracting, therefore this injection is continuous.

We suppose that the Rieffel spectral triple \( (H,D) \) on \( B \) is associated to a Clifford module \( S \) over \( \mathbb{C}l(n) \) and again denote the images of the generators of \( \mathbb{C}l(n) \) in \( \mathbb{B}(S) \) by \( γ_i \). Let \( H = H₀ ⊗ S \) be the underlying Hilbert space. Suppose furthermore that \( \nabla \) is a two-sided Hermitian \( D \)-connexion on \( X \).

Proposition 6.9. Under the above hypothesis, the derivations \( \nabla_j : \mathcal{A} ↪ A \) from Lemma 6.6 combine to form a Hermitian \( D \)-connexion \( \nabla : \mathcal{A} ↪ X ⊗_B \mathbb{B}(H) \) on setting

\[
\nabla(Ξ) := \sum_{j=1}^{n} \nabla_j(Ξ) ⊗ id_{H₀} ⊗ γ_j.
\]

If \( E \) is finitely generated and projective both as a left- and right-\( B \)-module, then

(i) \( X \) admits a “frame” in the sense of Definition 4.6.
(ii) \( \Sigma \) is closable and the \( B_1 \)-module \( X_1 \) obtained from \( A \) by Proposition 7.9 is an operator \( * \)-module over \( B_1 \). It is dense in \( X \) and the inclusion \( X_1 \hookrightarrow X \) is completely bounded.

**Proof.** We know that the \( \Sigma_j \) are well-defined maps from \( A \) to \( A \subseteq X \otimes B(S) \) – see Lemma 6.6.

We prove first that \( \Sigma \) defines a \( D \)-connexion (Definition 4.3). To this end, we calculate for all \( \Xi \in A \) and \( b \in B \)

\[
\Sigma(\Xi b) = \sum \Sigma_j(\Xi) \otimes \text{id}_{H_0} \otimes \gamma_j = \sum (\Sigma_j(\Xi)b + \Xi \Sigma_j(b)) \otimes \text{id}_{H_0} \otimes \gamma_j
\]

\[
= \Sigma(\Xi)b + \Xi \otimes \left( \sum \partial_j(b) \otimes \gamma_j \right) = \Sigma(\Xi)b + \Xi \otimes [D, b].
\]

We now evaluate \( [D, \langle \Xi_1, \Xi_2 \rangle] \) for \( \Xi_1, \Xi_2 \in A \) in \( A \) in order to prove that \( \Sigma \) is Hermitian (Definition 4.3). Without loss of generality, we can assume that \( \Xi_1 \) and \( \Xi_2 \) are gauge homogeneous and of same degree – otherwise, the scalar product vanishes and since \( \Sigma \) preserves the gauge action, the equality holds. We then have:

\[
[D, \langle \Xi_1, \Xi_2 \rangle] = \sum \partial_j ((\Xi_1, \Xi_2)) \otimes \gamma_j = \sum \partial_j (\Xi_1 \Xi_2) \otimes \gamma_j
\]

\[
= \sum (\Xi_j(\Xi_1)^* \Xi_2 + \Xi_1 \Sigma_j(\Xi_2)) \otimes \gamma_j
\]

\[
= \sum \langle \Xi_1, \Sigma_j(\Xi_2) \rangle \cdot \text{id}_{H_0} \otimes \gamma_j - \left( \sum \langle \Xi_2, \Sigma_j(\Xi_1) \rangle \cdot \text{id}_{H_0} \otimes \gamma_j \right)^*
\]

\[
= \langle \Xi_1, \Sigma(\Xi_2) \rangle - \langle \Sigma(\Xi_1), \Xi_2 \rangle,
\]

where we used the derivations \( \Sigma_j \) on \( A \), together with the definition of the scalar product between elements of \( X \) and \( X \otimes B \mathcal{B}(H) \) given in Remark 6.3.

To prove (i), remark that given a right \( B \)-module \( F \) and a \( B \)-bimodule \( F' \), it is easy to show that if \( (\xi_j)_{j=1}^n \) and \( (\zeta_k)_{k=1}^m \) are right frames for \( F \) and \( F' \), then \( (\xi_j \otimes \zeta_k) \) is a right frame for \( F \otimes_B F' \). Consequently, if \( E \) is finitely generated and projective as a right-module, all \( A^{(k)} \) for \( k \in \mathbb{N} \) admit a frame whose elements are in \( A \). The same argument applies to \( A^{(-k)} \) for \( k \in \mathbb{N} \), provided that \( E \) be f.g. projective as left \( B \)-module. We denote \( (\xi_j)^{\otimes \infty}_{j=-\infty} \) the elements thus obtained, organised in blocks of elements of \( A^{(k)} \) for increasing \( k \). We now prove that

\[
T_N := \sum_{k=-N}^N \epsilon_k \langle \xi_k, \cdot \rangle
\]

satisfies \( T_N \Xi \to \Xi \) for all \( \Xi \in X \).

It is clear from the definition that any \( \Xi \in X \) is \( \varepsilon \)-close to a \( \Xi_0 \in A \) (where \( \Xi_0 \) contains only elements of a finite number of \( A^{(k)} \) involved, i.e. an element of the algebraic direct sum). For a fixed \( \Xi_0 \), we clearly have \( T_N \Xi_0 = \Xi_0 \) for \( N \) large enough. If we can prove \( \|T_N\| \leq 1 \) for all \( N \), we are done. From the definition of \( X \) and because \( T_N \) preserves the degree, \( \|T_N\| = \sup \{ \|T_N \uparrow A^{(k)}\| \mid k \in \mathbb{Z} \} \). But for any
fixed $k$, $T_N \upharpoonright A^{(k)}$ is an increasing finite sum of positive operators (in $B_B(A^{(k)})$).
For $N$ large enough this sum is the identity. Thus for $N' \leq N$ we have
$$0 \leq (T_{N'} \upharpoonright A^{(k)}) \leq (T_N \upharpoonright A^{(k)}) = 1,$$
which implies that $\|T_{N'}\| \leq 1$ and proves that $T_N$ is a “frame” in the sense of Definition 4.6.

We can then apply Proposition 4.7 and prove that $\nabla$ is closable. The hypotheses
of Proposition 4.5 are thus satisfied for each of the modules $A^{(k)}$, hence all $A^{(k)}$ for
$k \in \mathbb{Z}$ are operator $\ast$-modules. Denote $P_k$ the associated matrix in $M(B)$ – following
the notations of [KL13]. The operator $\ast$-module $X_1$ is then obtained by Corollary 2.16. The associated projector is the direct sum $\bigoplus_{k \in \mathbb{Z}} P_k$.

We note the following:

**Corollary 6.10.** Combining the derivations $\nabla_j$ of Lemma 6.6, we obtain on $\mathcal{A}$ a closable derivation defined by
$$\delta(a) = \sum_{j=1}^{n} \nabla_j(a) \otimes \gamma_j$$
with values in $X \otimes_B \mathbb{B}(H)$.

**Proof.** We consider a sequence $a_n \in \mathcal{A}$ such that $a_n \rightarrow a$ and $\delta(a_n) \rightarrow y$. By Remark 6.8 this implies that $a_n$ and $\delta(a_n)$ seen as sequences in $X$ and $X \otimes_B \mathbb{B}(H)$ are convergent. But by Proposition 6.9 $\nabla$ is closable and therefore $y = 0$, which completes the proof.

**Definition 6.11.** Let $A_1 \subseteq A$ be the operator $\ast$-algebra closure of $\mathcal{A}$ with respect to $\delta$ (Proposition 2.10). It is therefore a subalgebra closed under holomorphic functional calculus.

**7 Conditional expectation, vertical class and main results**

Let $B$ be a $C^*$-algebra and $E$ a Hilbert bimodule over $B$. Denote $A := B \rtimes_E \mathbb{Z}$ the (generalized) crossed product of $B$ by $E$.

**Definition 7.1.** We will call vertical class the odd Kasparov $(A,B)$-module defined by the canonical representation of $A$ on the $A$-$B$ $C^*$-correspondence $X = \bigoplus_{k \in \mathbb{Z}} A^{(k)}$ of Definition 6.7 by left multiplication, and the operator $D_v$ determined by
$$(D_v \xi)_n = n \xi_n.$$
introduced by Pimsner on page 195 of the fundamental article [Pim97]. Pimsner starts with a general A-A-C∗-correspondence E which necessitates the construction of a Hilbert bimodule over the larger algebra $F_E$—our situation is simpler in as far as we start out with a Hilbert bimodule.

Let now $B$ be equipped with a Rieffel-spectral triple $(H, D_h)$ and operator+algebra $B_1$. Further let $\nabla : E \to E \otimes_B \mathbb{B}(H)$ be a two-sided $D_h$-connexion, $\nabla$ the extension to a connexion on $X$, and $X_1$ the associated operator+module closure constructed in Proposition 6.9.

We are now ready to prove the following:

**Proposition 7.3.** The pair $(X_1, \nabla)$ defines a correspondence from $(X, D_c)$ to $(H, D_h)$ (compare 6.3 of [KL13]) if we use the natural algebra $B$ for $X$. Hence, it only remains to identify: $\nabla$ in Definition 6.3 and they therefore have the same domain. Even better, they share $\nabla$. We assumed that $B$ is unital, thus it is clear that the unbounded Kasparov module $(H, D_h)$ is essential. Checking two out of four points of Definition of correspondence is straightforward:

1. Proposition 6.9 (ii) ensures that $X_1$ is an operator module and Corollary 2.16 ensures that the map $\nabla$ is a completely bounded Hermitian $D_h$-connexion from $X_1$ to $X \otimes_B \mathbb{B}(H)$.

2. By definition, any $b \in B_1$ lies in the domain of the closure of $\delta_{D_h}$. Thus by Lemma 3.2 we have $b(\text{Dom}(D_h)) \subseteq \text{Dom}(D_h)$ and $[D_h, b] : B_1 \to \mathbb{B}(H)$ is completely bounded.

Before treating the last two points, we establish that $\mathcal{C} := \mathcal{A} \otimes_{B_1} \text{Dom}(D_h)$ (algebraic tensor product) is a core for $1 \otimes_B D_h$. Recall that the algebra $\mathcal{A}$ used in the definition of $\mathcal{C}$ was introduced in Lemma 6.6.

Consider the selfadjoint operator $\text{Diag}(D_h) = D_h \otimes \text{id}$ acting on the Hilbert space $H \otimes \ell^2(\mathbb{Z})$. The algebraic tensor product of $\text{dim}(D_h)$ with vectors of compact support in $\ell^2(\mathbb{Z})$ is a core $\mathcal{C}'$ of $\text{Diag}(D_h)$ by definition. Upon identifying $X$ with $PB_1^\infty$, the operator $1 \otimes_{\nabla^{\mathcal{A}}} D_h$ coincides with the compression $P \text{Diag}(D_h)|P$ whose domain is $P \text{Dom}(\text{Diag}(D_h))$. It is readily checked that $P\mathcal{C}'$ is then a core for this compression.

The operator $1 \otimes_{\nabla} D_h$ is just a bounded perturbation of $1 \otimes_{\nabla^{\mathcal{A}}} D_h$ (see Theorem 5.4 in [KL13]) and they therefore have the same domain. Even better, they share the same cores. Hence, it only remains to identify:

$$\mathcal{A} \otimes_{B_1} \text{Dom}(D) \simeq PB_1^{\text{alg, }\infty} \otimes_{B_1} \text{Dom}(D) \simeq P \text{Dom}(D)^{\text{alg, }\infty} \simeq P\mathcal{C}'$$

which holds because $1 \in B_1$ and $B_1 \otimes_{B_1} \text{Dom}(D) = \text{Dom}(D)$. Here we denote by $B_1^{\text{alg, }\infty}$ the sequences of finite support in $B^{\infty}$ which were denoted $c_0(B)$ in [KL13].

Definition 3.3. Hence $\mathcal{C} = \mathcal{A} \otimes_{B_1} \text{Dom}(D_h)$ is indeed a core for $1 \otimes_{\nabla} D_h$. We can now proceed with the remaining two points:

3. As mentioned in [KL13] below Definition 6.3, given the core $\mathcal{C}$ for $1 \otimes_{\nabla} D$, it suffices to prove that

- any $a$ in the subalgebra $\mathcal{A}$ sends $\mathcal{C}$ to itself,
- $[1 \otimes_{\nabla} D, a]$ defined from $\mathcal{C}$ to $X \otimes_B H$ extends to a bounded operator on $X \otimes_B H$. 

The definition of $C$ ensures that $\forall a \in \mathcal{A}, a^* C \subseteq C$. Moreover on simple tensors in $C$, the operator $1 \otimes \nabla D$ acts by (compare Lemma 5.1 p.21 of [KL13]):

$$(1 \otimes \nabla D)(x \otimes \xi) = x \otimes D(\xi) + \nabla(x)\xi,$$

where $\nabla(x) \in X \otimes_B \mathcal{B}(H)$ acts naturally on $\xi \in H$. We can then calculate:

$$[1 \otimes \nabla D, a](x \otimes \xi) = (1 \otimes \nabla D)(ax \otimes \xi) - a(1 \otimes \nabla D)(x \otimes \xi) = \nabla(ax)\xi - a\nabla(x)\xi,$$

since the last part of the computation takes place in $\mathcal{A}$, where we know that $\nabla$ is a derivation. For $a \in \mathcal{A}$, the operator $\nabla(a)$ is clearly the restriction of a bounded operator on $X \otimes_B H$.

(4) This last condition is an abbreviation for points (1) and (2) of Theorem 1.3 in [KL13]. On the “algebraic elements” of $C$, the required stability properties are clear. The extension of $[\partial_t, 1 \otimes \nabla D](\partial_t - i\mu)^{-1} : \mathcal{C} \to X \otimes_B H$ to a bounded operator is clear, since $[\partial_t, 1 \otimes \nabla D](x \otimes \xi) = 0$ for any simple tensor of $\mathcal{C}$.

We can now state our main result:

**Theorem 7.4.** Let $E$ be a Hilbert bimodule over a $C^*$-algebra $B$, which is finitely generated as both left- and right-module. Suppose that $B$ is unital and equipped with an odd Rieffel spectral triple $(H,D_h)$.

If $\nabla : E \to E \otimes_B \Omega_B^1 D_v$ is a two-sided Hermitian $D_h$-connexion on $E$ and $\nabla$ the associated $D_h$-connexion on $X$ according to Proposition 6.9, then the operator

$$
\begin{pmatrix}
0 & D_v \otimes 1 - i 1 \otimes \nabla D_h \\
D_v \otimes 1 + i 1 \otimes \nabla D_h & 0
\end{pmatrix}
$$

on $X \otimes_B H \oplus X \otimes_B H$ equipped with the standard grading defines a spectral triple on $A = B \rtimes^g \mathbb{Z}$ with respect to the natural action on the left by multiplication.

If $(\tau, H, D_h)$ is an even Rieffel spectral triple with grading operator $\gamma$, then

$$(X \otimes_B H, D_v \otimes \gamma + 1 \otimes \nabla D_h)$$

is a spectral triple on $A = B \rtimes^g \mathbb{Z}$.

In both cases, this spectral triple is a representative of the Kasparov product of the vertical and horizontal class.

**Proof.** This is now a consequence of Theorem 9.3 from the appendix and Proposition 7.3.

8 Examples

Our first examples are Quantum Heisenberg Manifolds (QHM), which were introduced by Rieffel in [Ric89] as one of the original examples of Rieffel deformations. They were subsequently studied among others by Abadie (see e.g. [Aba95, AEE98],
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Chakraborty (see [CS03, Cha05]) and ourselves [Gab13, GG13] – for a more complete survey of literature about QHM, see the introduction of [Gab13]. Quantum Heisenberg manifolds form a family of $C^*$-algebras $D^c_{\mu,\nu}$ indexed by $c \in \mathbb{Z}$ and $\mu, \nu \in \mathbb{R}$.

Here, we follow Definition 1 of [CS03] to introduce the QHM. Given $c \in \mathbb{Z}$, $\mu, \nu \in \mathbb{R}$, $D^c_{\mu,\nu}$ is the enveloping $C^*$-algebra of the $*$-algebra generated by

$$\left\{ F \in C_c(\mathbb{Z} \to C_b(\mathbb{R} \times S^1)) \mid F(x + 1, y, p) = e(cy)F(x, y, p) \right\}$$

equipped with multiplication:

$$(20) \quad (F_1 \cdot F_2)(x, y, p) = \sum_{q \in \mathbb{Z}} F_1(x - (q - p)\mu, y - (q - p)\nu, q)F_2(x - q\mu, y - q\nu, p - q)$$

and involution:

$$F^*(x, y, p) = \overline{F(x, y, -p)}.$$

This $C^*$-algebra carries a pointwise continuous action $\alpha$ of the Heisenberg group $H_3$, i.e. the subgroup of $GL_3(\mathbb{R})$ of the matrices

$$(21) \quad \begin{pmatrix} 1 & s & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, r, s, t \in \mathbb{R}.$$ 

The explicit expression of the action $\alpha$ is (compare [CS03], equation (2.4)):

$$\alpha_{(r,s,t)}(F)(x, y, p) = e(p(t + cs(x - r)))F(x - r, y - s, p)$$

where $x, y \in \mathbb{R}$ and $p \in \mathbb{Z}$. Considering only $\alpha_{(0,0,t)}$ actually yields an action of $S^1 = \mathbb{R}/(2\pi)\mathbb{Z}$, which we consider as the gauge action for our situation. This definition leads to the spectral subspaces:

$$D^c_{\mu,\nu}(k) = \left\{ F \in D^c_{\mu,\nu} \mid F(x, y, p) = \delta_{k,p}F(x, y, k) \right\},$$

where $k \in \mathbb{Z}$ and $\delta_{k,p}$ is 1 if $k = p$ and 0 else. In particular, the gauge-invariant elements of $D^c_{\mu,\nu}$ are just the algebra $B = C(T^2)$. It is easily seen that $D^c_{\mu,\nu}(1)$ is a Hilbert bimodule over $B$: the left and right actions are restrictions of the product on $D^c_{\mu,\nu}$, while the left and right $B$-valued scalar products are $\langle \xi, \eta \rangle_B := \xi^*\eta$ and $B\langle \xi, \eta \rangle = \xi\eta^*$.

We denote by $E: D^c_{\mu,\nu} \to B$ the conditional expectation induced by the gauge action, i.e.

$$E(F) := \int_{S^1} \alpha_{(0,0,t)}(F)dt.$$ 

There is a unique $T^2$-invariant and normalised trace on $B$, which is

$$\tau_0(b) = \int_{T^2} b(x, y) d\lambda(x, y)$$

where $\lambda$ is the Lebesgue measure. Composing these two maps, we obtain a $H_3$-invariant trace $\tau$ on $D^c_{\mu,\nu}$ (compare [CS03], p.427):

$$\tau(F) := \tau_0(E(F)) = \int_{T^2} F(x, y, 0)dxdy.$$ 

Let us now prove that $D^c_{\mu,\nu}$ is a GCP using Theorem 5.7 and the following Proposition 2.6 of [Gab13]:
Lemma 8.1. There are elements $\xi_1, \xi_2 \in D_{\mu, \nu}^c$ with $\xi_1^* \xi_1 + \xi_2^* \xi_2 = 1$.

Proof. For self-containment, we sketch the proof. We can find two real-valued, 1-periodic functions $\chi_1, \chi_2$ on $\mathbb{R}$ such that $\chi_1^2(x) + \chi_2^2(x) = 1$ while $\chi_1(x) = 1$ for $x \in [-1/6, 1/6]$ and $\chi_2(x) = 1$ for $x \in [1/3, 2/3]$. We can then define $\xi_1$ in $D_{\mu, \nu}$ by $\xi_1(x) = e^{i\alpha_x}$ for $x \in [-1/2, 1/2]$, which we extend into elements of $D_{\mu, \nu}$. Applying the same construction to $\chi_2$ on $[0, 1]$, we obtain $\xi_2$. It is easy to check using suitable covers of $T^2$ that $\xi_1^* \xi_1 + \xi_2^* \xi_2 = 1$.

On the gauge-invariant subalgebra $B \simeq C(T^2) \subseteq D_{\mu, \nu}$, the action of $H_3$ factors through an action of $G = T^2$ — which is just the canonical action of $T^2$ on $B$. The usual differential structure on $T^2$ corresponds to $\mathcal{B} := C^\infty(T^2)$. It can equivalently be obtained from the $T^2$-smooth elements of $B$. The natural spectral triple structure on $B$ can be obtained for instance by the method of [GG]. We thus recover an unbounded operator

$$D := \partial_1 \otimes \gamma_1 + \partial_2 \otimes \gamma_2,$$

which is acting on $L^2(T^2) \otimes \mathbb{C}^2$. The left action of $\mathcal{B}$ on $L^2(T^2)$ is provided by pointwise multiplication. The matrices $\gamma_1, \gamma_2$ satisfy $\gamma_i \gamma_k + \gamma_k \gamma_j = 2\delta_{jk}$ and $\gamma_j^* = -\gamma_j$. The spectral triple thus obtained is even with grading operator $\gamma_3 = i\gamma_1 \gamma_2$ which commutes with both $\gamma_1$ and $\gamma_2$.

Let $\partial_1, \partial_2, \partial_3 \in \mathfrak{h}_3$ in the Lie algebra of $H_3$ be the infinitesimal generators associated to the parametrisation [21]. It is easy to see that they satisfy the commutation relations:

$$[\partial_j, \partial_k] = 0 \quad \text{for} \quad j = 1, 2, 3 \quad \text{and} \quad [\partial_1, \partial_2] = \partial_3$$

for $j = 1, 2$. We introduce a smooth version of $M_{\mu, \nu}^c$ by setting:

$$\mathcal{M}_{\mu, \nu}^c := \{ \xi \in M_{\mu, \nu}^c : (r, s, t) \mapsto \alpha_{r, s, t}(\xi) \text{ is in } C^\infty(H_3, D_{\mu, \nu}^c) \}.$$  

It follows immediately from the definition that $\mathfrak{h}_3$ acts by derivations on $\mathcal{M}_{\mu, \nu}^c$. We will write $\partial_\xi$ for the action of the infinitesimal generator $\partial_\xi$ on this “smooth module”.

Proposition 8.2. A connexion $\nabla$ on $\mathcal{M}_{\mu, \nu}^c$ associated to the canonical spectral triple on $B$ is defined by

$$\nabla(\xi) = \partial_1(\xi) \otimes \gamma_1 + \partial_2(\xi) \otimes \gamma_2.$$  

In other words, the “components” of $\nabla$ defined in Lemma 8.1 are $\partial_j$. Moreover, $\nabla$ is a two-sided Hermitian connexion in the sense of Definition 2.11 and closable.

Proof. This connexion is associated to the spectral triple induced by the unbounded operator

$$D = \partial_1 \otimes \gamma_1 + \partial_2 \otimes \gamma_2.$$

The commutators of $D$ with elements $b$ of $B$ is $[D, b] = \partial_1 \otimes \gamma_1 \gamma_2 + \partial_2 \otimes \gamma_2 \gamma_1$. We first prove that $\nabla$ is indeed a connexion in the sense of Definition 2.11. Since $\alpha$ is pointwise continuous, $\mathcal{M}_{\mu, \nu}^c$ (also denoted $\mathcal{M}$ for brevity) is dense in $M_{\mu, \nu}^c$. Moreover, it is clear from (21) that $\mathcal{M}$ is stable under left and right multiplication by elements of $\mathcal{B}$. Let $\xi \in \mathcal{M}$ and $b \in \mathcal{B}$, then

$$\nabla_\xi(\xi b) = \nabla_\xi(\xi) b + \xi \partial_\xi(b).$$
since the action of $H_3$ restricts to that of $T^2$ on $B$ i.e. $\partial_j^{\mu,\nu}(b) = \partial_j^\mu(b)$ for any $b \in B$ and $j = 1, 2$. This proves that the equation (11) is satisfied and thus that $\nabla$ is a connexion (per Lemma 4.1). The same argument applies to $\partial_j^\mu(b\xi)$, thereby proving that $\nabla$ is also a left-connexion. To prove Hermicity, we rely on the fact that $\partial_j$ are derivations on $D^c_{\mu,\nu}$. First of all, the scalar products $\langle \mathcal{M}, \mathcal{M} \rangle_B$ and $B(\mathcal{M}, \mathcal{M})$ are both contained in $\mathcal{B}$. We then have:

$$\partial_j^\mu(\xi^*\eta) = \xi^*\partial_j(\eta) + \partial_j(\xi)^*\eta \quad \partial_j^\mu(\xi\eta^*) = \xi\partial_j(\eta^*) + \partial_j^\mu(\xi)^*\eta.$$  

This proves that $\nabla$ is indeed a two-sided Hermitian connexion in the sense of Definition 6.1. It remains to show that $\nabla$ is closable, but this is an immediate consequence of Proposition 4.7 and Lemma 8.1.

**Remark 8.3.** The previous argument applies more generally to generalized crossed products equipped with a Lie group action such that the restriction of this action to the basis algebra $B$ actually induces the spectral triple structure on $B$ e.g. ergodic actions as in [CC] Theorem 5.4.

**Proposition 8.4.** The spectral triple we obtain by applying our Theorem 7.4 to the connexion defined in Proposition 8.2 coincide with the one constructed in [CS03] Theorem 10.

**Proof.** The Hilbert space which we obtain from the Kasparov product is $X \otimes_B L^2(T^2)$, where $X$ is the $B$-Hilbert module induced from $D^c_{\mu,\nu}$ by using the conditional expectation $\mathbb{E} : D^c_{\mu,\nu} \rightarrow B$. The Hilbert space $L^2(T^2)$ arises from the GNS construction on $B$ with the trace $\tau_0(f) = \int_{T^2} f(x)d\lambda(x)$.

Therefore, $X \otimes_B L^2(T^2)$ is the completion of $F \otimes b$ for $F \in D^c_{\mu,\nu}$ and $b \in B$ for the scalar product:

$$\langle F_1 \otimes b_1, F_2 \otimes b_2 \rangle = \langle b_1, \mathbb{E}(F_1^*F_2)b_2 \rangle = \tau_0(b_1^*\mathbb{E}(F_1^*F_2)b_2) = \tau((F_1b_1)^*F_2b_2).$$

In other words, $X \otimes_B L^2(T^2) \approx \text{GNS}(D^c_{\mu,\nu}, \tau)$ where $\tau$ is the previously introduced normalised $B_3$-invariant trace on $D^c_{\mu,\nu}$. Thus, the Hilbert spaces and the associated representations of $D^c_{\mu,\nu}$ appearing in [CS03] Theorem 10 and by application of our Theorem 7.4 are the same.

## 9 Appendix

### 9.1 Preliminaries

We briefly recall some basics on $KK$-theory (see the original paper [BJ83] for more details). We restrict to the case of ungraded algebras.

**Definition 9.1.** An even unbounded $(A,B)$-Kasparov module is a pair $(E, D)$ such that $E$ is a graded $B$-Hilbert module carrying an action by bounded even operators of $A$ on the right and $D$ an unbounded regular odd operator on $E$ such that:

- $a(1 + D^2)^{-1}$ extends to a compact operator on $E$ for all $a \in A$
- the set of $a \in A$ that preserve the domain of $D$ and such that $[a, D]$ is bounded is dense in $A$.

An odd unbounded $(A,B)$-Kasparov module is given by an even Kasparov $(A, B \otimes \mathbb{C}[t])$-module.
It was shown by Baaj and Julg in [BJ83] that there is a natural map

\[ D \mapsto q(D) := \frac{D}{\sqrt{1 + D^2}} \]

associating to each such unbounded Kasparov module a bounded Kasparov module (which becomes a bijection on homotopy classes).

As for bounded Kasparov modules, the odd unbounded Kasparov module have a particularly simple description, referred to as the **Fredholm picture** in the bounded setting in [Bla98], Section 17.5. An **unbounded odd Fredholm module** is, by definition, given by a pair \((E, D)\), where \(E\) is a (trivially graded) Hilbert \(B\)-module carrying an action by bounded operators of \(A\) on the left and \(D\) an unbounded operator on \(E\) such that

- \(a(1 + D^2)^{-1}\) is compact for all \(a \in A\);
- the set of all \(a \in A\) that preserve the domain of \(D\) and such that \([a, D]\) is bounded is dense in \(A\).

Using a stabilization, it is easy to see that every unbounded odd Kasparov module \(x\) corresponds yields canonically an unbounded Fredholm module \(\hat{x}\) (compare Section 17.5 in Blackadar).

On the other hand, an unbounded Fredholm module \((E, D)\) yields an unbounded \((A \otimes \mathbb{C}l_1, B)\)-Kasparov module \(\hat{x}\) by setting

\[ \hat{x} := (E \otimes \mathbb{C}l_1, D \otimes c) \]

where \(E \otimes \mathbb{C}l_1\) is viewed as a \(B\)-module and \(c\) is the canonical self-adjoint generator of the Clifford algebra (as an algebra) acting by multiplication. We will refer to \(\hat{x}\) as the **left hand Fredholm picture**, in order to distinguish it from \((E \otimes \mathbb{C}l_1, D \otimes c)\) as an \((A, B \otimes \mathbb{C}l_1)\)-Kasparov module (the **right hand Fredholm picture**). It is easy to see that the class of \(\hat{x}\) coincides with the class of \((E \otimes \mathbb{C}l_1, D \otimes c)\) viewed as an \((A, B \otimes \mathbb{C}l_1)\)-Kasparov module (using Morita invariance of KK-theory and formal Bott-periodicity).

### 9.2 The Unbounded Kasparov Product

In this subsection, we will use the notational conventions from [KL13].

Throughout, we fix two unbounded Kasparov modules: \((X, D_1)\) over \((A, B)\) and \((Y, D_2)\) over \((B, C)\) as in [KL13]. We denote by \(\gamma_1\) and \(\gamma_2\), respectively, the (possibly trivial) grading operators on \(X\) and \(Y\), i.e. in case that the Kasparov module is odd, we assume that it is given in the Fredholm picture. We assume throughout that \((Y, D_2)\) is essential, i.e. \(B\) is dense in \(Y\) and \(B\Omega_{D_2}^3\) is dense in \(\Omega_{D_2}^3\) (see Definition 6.2 in loc. cit.). We further assume that the Conventions 4.1 and 4.2 of loc. cit. are satisfied.

We are interested in the case where \((X, D_1)\) is odd and \((Y, D_2)\) even or odd.

Convention 4.2 from loc. cit. is in force verbatim – thus as we started out with a (possibly even) Kasparov module \((Y, D_2)\), the representation of \(B\) on \(Y\) is by even operators and \(D_2\) is an odd unbounded operator.

Furthermore, all tensor products are supposed to be equipped with the grading corresponding to the grading operator \(\gamma_1 \otimes \gamma_2\). The contraction map

\[ c: (X \hat{\otimes}_B L(Y)) \otimes Y \to X \hat{\otimes}_B Y, \; x \otimes T \otimes y \mapsto x \otimes Ty \]
is then even.

Let now $\nabla$ be a $D_2$-connexion on $X$. Just as in Section 5.1. of loc. cit. we may now define on $X \hat{\otimes} A Y$ the unbounded operator $1 \otimes \nabla D_2$ with domain $\text{Dom}(\text{Diag}(D_2)) \cap X \hat{\otimes} Y$ by

$$(1 \otimes \nabla D_2)(x \otimes y) = x \otimes D_2(y) + c(\nabla)(x \otimes y).$$

Note that if $(Y, D_2)$ is even then this defines an odd operator only if the connexion is an odd map! We will only consider such connections. In any case, it follows from Theorem 5.4 in loc. cit. that this is a regular selfadjoint operator if the connexion is Hermitian.

**Definition 9.2.** Let $\nabla : X \rightarrow X \hat{\otimes} B Y$ be a completely bounded $D_2$-connexion. The definition of the operator $D_1 \times \nabla D_2$ depends on the parities of $(X, D_1)$ and $(Y, D_2)$ and denotes the following operator:

1. If $(X, D_1)$ and $(Y, D_2)$ are odd,

$$\begin{pmatrix}
0 & D_1 \otimes 1 - i 1 \otimes \nabla D_2 \\
D_1 \otimes 1 + i 1 \otimes \nabla D_2 & 0
\end{pmatrix}$$

on $(X \hat{\otimes} B Y) \oplus (X \hat{\otimes} B Y)$ equipped with the standard grading and domain 

$$((\text{Dom}(D_1) \otimes B Y) \cap \text{Dom}(\text{Diag}(D_2))) \otimes \mathbb{C}^2.$$

2. If $(X, D_1)$ is odd and $(Y, D_2)$ is even,

$$D_1 \otimes \gamma_2 + 1 \otimes \nabla D_2$$

on $X \hat{\otimes} B Y$ equipped with the trivial grading and domain $(\text{Dom}(D_1) \otimes B Y) \cap \text{Dom}(\text{Diag}(D_2))$.

It follows as in Theorem 5.5 in loc. cit. that this defines selfadjoint and regular operators if the connexion is Hermitian.

For the definition of correspondence, we refer the reader to Definition 6.3 of loc. cit. However when $(X, D_1)$ is even and $(Y, D_2)$ is odd we replace in condition (4) the operator $D_1 \otimes 1$ by the operator $D_1 \otimes \gamma$.

Using essentially the same techniques as in loc. cit., one obtains (compare Theorem 6.7 and Theorem 7.5 therein):

**Theorem 9.3.** Suppose that $x = (X, D_1)$, $y = (Y, D_2)$ admits a correspondence $(X_1, \nabla^0)$, then for any (completely bounded) Hermitian $D_2$-connexion $\nabla$, the Fredholm module with operator $D_1 \times \nabla D_2$ defines a class $\gamma$ in $KK$ which is an unbounded representative of the product of $x$ and $y$.

**Remark 9.4.** That is, the bounded transform $q(\gamma)$ is a Kasparov product of $q(\bar{x})$ and $q(\bar{y})$.

**Proof.** It remains to prove the case of $x$ is odd and $y$ is even, the case where both modules are odd being shown in loc. cit. We first need to check that $(X \hat{\otimes} B Y, D_1 \times \nabla D_2)$ defines an unbounded Fredholm module. First of all, the operator $1 \otimes \nabla D_2$ (compare Theorem 5.4. of loc. cit. and [Con94] Chapter 6, Section 3, Lemma 1) defined by

$$(1 \otimes \nabla D_2)(\xi \otimes \eta) = \xi \otimes D_2 \eta + \nabla \xi(\eta)$$

is then even.
is selfadjoint and regular by Theorem 5.4 in [KL13]. Now as we have replaced condition (4) in the definition of correspondence, we may apply Theorem 1.3 of loc. cit. to conclude that $D_1 \otimes \gamma + 1 \otimes \nabla_s D_2$ is selfadjoint and regular. So this follows also for the operator $D_1 \times \nabla D_2$, as it is a bounded perturbation of the former operator. It follows as in the proof of Theorem 6.7 in loc. cit. that we actually obtain a Kasparov module.

Let us now show that this Fredholm module actually represents the Kasparov product of $(X, D_1)$ and $(Y, D_2)$. Since the analytic details are almost verbatim the same as in loc. cit., we will only treat the different algebraic aspects. As in the proof of Theorem 7.5 in loc. cit., we may assume that $\nabla = \nabla_0$ by a perturbation argument. Denote by $\gamma$ the grading operator on $\mathcal{C}l_1$, by $Z := (\mathcal{C}l_1 \otimes X) \otimes_B Y$ the tensor product of $\mathcal{C}l_1 \otimes X$ with $Y$ over $B$ with grading operator $\gamma \otimes 1 \otimes 1$, and by $Z'$ the same tensor product with grading operator $\gamma \otimes 1 \otimes \gamma_2$ (note that we view $Z$ and $Z'$ as a $(\mathcal{C}l_1 \otimes A, C)$-correspondence). Let $c = (1, -1) \in \mathcal{C}l_1$, $d := -c$ and define a graded unitary isomorphism $\theta : Z' \to Z$ by setting

$$\theta(x \otimes \xi \otimes \eta) := \begin{cases} x \otimes \xi \otimes \eta & \text{if } \partial(x) = 0 = \partial(\eta) \\ cx \otimes \xi \otimes \eta & \text{if } \partial(x) = 1 = \partial(\eta) \\ x \otimes \xi \otimes \eta & \text{if } \partial(x) = 1 = \partial(\eta) = 0 \\ dx \otimes \xi \otimes \eta & \text{if } \partial(x) = 0, \partial(\eta) = 1 \end{cases}$$

Let $D := c \otimes D_1 \times \nabla \otimes D_2$, $D' := \theta^{-1} D\theta$ and $x := (Z, D)$. We will show that $\theta^* \partial_{\theta^{-1}}(x) = (Z', D')$ is a representative of the Kasparov product $x \otimes_B y$. Note that $D'$ defines an odd operator on $Z'$, $\theta$ being a graded isomorphism and $c \otimes D_1 \times \nabla \otimes D_2$ an odd operator on $Z$. As in loc. cit., we have to check the conditions of [Kuc97], Theorem 13. The connexion condition (i) follows by a case-by-case calculation. For example, assume that $v \otimes \xi \in \mathcal{C}l_1 \otimes X$ with $\partial v = 1$, $\xi \in \operatorname{Dom}(D_1) \cap X_1$ and $\partial \eta = 0$, then if we have for all $\xi \in \operatorname{Dom}(D_2)$:

$$T_v \otimes_\xi D_2(\eta) - (-1)^{pr} D'(v \otimes \xi \otimes \eta) = T_v \otimes_\xi D_2(\eta) + \theta^{-1}(c \otimes D_1 \times \nabla \otimes D_2) \theta(v \otimes \xi \otimes \eta)$$

$$= v \otimes \xi \otimes D_2 + \gamma - \theta^{-1}(c \otimes D_1 \otimes \gamma + c \otimes 1 \otimes \nabla D_2)(v \otimes \xi \otimes \eta)$$

$$= v \otimes \xi \otimes D_2 + \theta^{-1}(cv \otimes \xi \otimes D_2 \eta + cv \otimes \nabla \eta \xi + cv \otimes D_2 \xi \otimes \eta)$$

$$= \theta^{-1}(cv \otimes \nabla \eta \xi + cv \otimes D_2 \xi \otimes \eta).$$

The other cases are similar. We leave the remaining (analytic) details to the reader – these follow as in Section 7 of loc. cit. \hfill \Box

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