Dispersion Relation Analyses of Pion Form Factor, Chiral Perturbation Theory and Unitarized Calculations

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Abstract

The Vector Pion form factor below 1 GeV is analyzed using experimental data on its modulus, the P-wave pion pion phase shifts and dispersion relation. It is found that causality is satisfied. Using dispersion relation, terms proportional to $s^2$ and $s^3$ are calculated using the experimental data, where $s$ is the momentum transfer. They are much larger than the one-loop and two-loop Chiral Perturbation Theory calculations. Unitarized model calculations agree very well with dispersion relation results.
Chiral Perturbation Theory (ChPT) is a well-defined perturbative procedure allowing one to calculate systematically low energy phenomenon involving soft pions. It is now widely used to analyze low energy pion physics even in the presence of resonance as long as the energy region of interest is sufficiently far from the resonance. In this scheme, the unitarity relation is satisfied perturbatively order by order.

The standard procedure of testing ChPT calculation of the pion form factor, which claims to support the perturbative scheme, is shown here to be unsatisfactory. This is so because the calculable terms are extremely small, less than 1.5% of the uncalculable terms at an energy of 0.5 GeV or lower whereas the experimental errors are of the order 10-15%. The main purpose of this note is to show how this situation can be dealt with without asking for a new measurement of the pion form factor with a precision much better than 1.5%.

Although dispersion relation (or causality) has been tested to a great accuracy in the forward pion nucleon and nucleon nucleon or anti-nucleon scatterings at low and high energy, there is no such a test for the form factors. This problem is easy to understand. In the former case, using unitarity of the S-matrix, one rigourously obtained the optical theorem relating the imaginary part of the forward elastic amplitude to the total cross section which is a measurable quantity. This result together with dispersion relation establish a general relation between the real and imaginary parts of the forward amplitude.

There is no such a rigourous relation, valid to all energy, for the form factor. In low energy region, the unitarity of the S-matrix in the elastic region gives a relation between the phase of the form factor and the P-wave pion pion phase shift, namely they are the same. Strictly speaking, this region is extended from the two pion threshold to 16m^2 where the inelastic effect is rigourously absent. In practice, the region of the validity of the phase theorem can be extended to 1.1-1.3 GeV because the inelastic effect is negligible. Hence, using the measurements of the modulus of the form factor and the P-wave phase shifts, both the real and imaginary parts of the form factors are known. Beyond this energy, the imaginary part is not known. Fortunately for the present purpose of testing of locality (dispersion relation) and of the validity of the perturbation theory at low energy, thanks to the use of subtracted dispersion relations, the knowledge of the imaginary part of the form factor beyond 1.3 GeV is unimportant.

Because the vector pion form factor \( V(s) \) is an analytic function with a cut from \( 4m^2 \) to \( \infty \), the \( n^{th} \) times subtracted dispersion relation for \( V(s) \) reads:

\[
V(s) = a_0 + a_1 s + \ldots a_{n-1} s^{n-1} + \frac{s^n}{\pi} \int_{4m^2}^{\infty} \frac{ImV(z)dz}{z^n(z-s-i\epsilon)} \quad (1)
\]

where \( n \geq 0 \) and, for our purpose, the series around the origin is considered. Because of the real analytic property of \( V(s) \), it is real below \( 4m^2 \). By taking the real part of this equation, \( ReV(s) \) is related to the principal part of the dispersion integral involving the \( ImV(s) \) apart from the subtraction constants \( a_n \).

The polynomial on the R.H.S. of Eq. (1) will be referred in the following as the subtraction constants and the last term on the R.H.S. as the dispersion integral (DI). The evaluation of DI as a function of \( s \) will be done later. Notice that \( a_n = V^n(0)/n! \) is
the coefficient of the Taylor series expansion for $V(s)$, where $V^n(0)$ is the nth derivative of $V(s)$ evaluated at the origin. The condition for Eq. (1) to be valid was that, on the real positive s axis, the limit $s^{-n}V(s) \to 0$ as $s \to \infty$. By the Phragmen Lindeloff theorem, this limit would also be true in any direction in the complex s-plane and hence it is straightforward to prove Eq. (1). The coefficient $a_{n+m}$ of the Taylor’s series is given by:

$$a_{n+m} = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ImV(z)dz}{z^{n+m+1}}$$

(2)

where $m \geq 0$. The meaning of this equation is clear: under the above stated assumption, not only the coefficient $a_n$ can be calculated but all other coefficients $a_{n+m}$ can also be calculated. The larger the value of $m$, the more sensitive is the value of $a_{n+m}$ to the low energy values of $ImV(s)$. In theoretical work such as in ChPT approach, to be discussed later, the number of subtraction is such that to make the DI converges.

The elastic unitarity relation for the pion form factor is $ImV(s) = V(s)e^{-i\delta(s)}\sin\delta(s)$ where $\delta(s)$ is the elastic P-wave pion pion phase shifts. Below the inelastic threshold of $16m^2_\pi$ where $m_\pi$ is the pion mass, $V(s)$ must have the phase of $\delta(s)$ [4]. It is an experimental fact that below $1.3 GeV$ the inelastic effect is very small, hence, to a good approximation, the phase of $V(s)$ is $\delta$ below this energy scale.

$$ImV(z) = |V(z)| \sin \delta(z)$$

(3)

and

$$ReV(z) = |V(z)| \cos \delta(z)$$

(4)

where $\delta$ is the strong elastic P-wave $\pi\pi$ phase shifts. Because the real and imaginary parts are related by dispersion relation, it is important to know accurately $ImV(z)$ over a large energy region. Below $1.3 GeV$, $ImV(z)$ can be determined accurately because the modulus of the vector form factor [4, 8] and the corresponding P-wave $\pi\pi$ phase shifts are well measured [4, 10, 11] except at very low energy.

It is possible to estimate the high energy contribution of the dispersion integral by fitting the asymptotic behavior of the form factor by the expression, $V(s) = -(0.25/s)ln(-s/s_\rho)$ where $s_\rho$ is the $\rho$ mass squared.

Using Eq. (3) and Eq. (4), $ImV(z)$ and $ReV(s)$ are determined directly from experimental data and are shown, respectively, in Fig.1 and Fig.2.

In the following, for definiteness, one assumes $s^{-1}V(s) \to 0$ as $s \to \infty$ on the cut, i.e. $V(s)$ does not grow as fast as a linear function of $s$. This assumption is a very mild one because theoretical models assume that the form factor vanishes at infinite energy as $s^{-1}$. In this case, one can write a once subtracted dispersion relation for $V(s)$, i.e. one sets $a_0 = 1$ and $n = 1$ in Eq. (1).

From this assumption on the asymptotic behavior of the form factor, the derivatives of the form factor at $s = 0$ are given by Eq. (2) with n=1 and m=0. In particular one has:

$$<r^2_V> = \frac{6}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ImV(z)dz}{z^2}$$

(5)
where the standard definition $V(s) = 1 + \frac{1}{s} <r^2_V> s + cs^2 + ds^3 + ...$ is used. Eq. (5) is a sum rule relating the pion rms radius to the magnitude of the time like pion form factor and the P-wave $\pi\pi$ phase shift measurements. Using these data, the derivatives of the form factor are evaluated at the origin:

$$<r^2_V> = 0.45 \pm 0.015 fm^2; c = 3.90 \pm 0.20 GeV^{-4}; d = 9.70 \pm 0.70 GeV^{-6} \tag{6}$$

where the upper limit of the integration is taken to be $1.7 GeV^2$. By fitting $ImV(s)$ by the above mentioned asymptotic expression, the contribution beyond this upper limit is completely negligible. From the $2\pi$ threshold to $0.56 GeV$ the experimental data on the the phase shifts are either poor or unavailable, an extrapolation procedure based on some model calculations to be discussed later, has to be used. Because of the threshold behavior of the P-wave phase shift, $ImV(s)$ obtained by this extrapolation procedure is small. They contribute, respectively, 5%, 15% and 30% to the $a_1$, $a_2$ and $a_3$ sum rules. The results of Eq. (5) change little if the $\pi\pi$ phase shifts below $0.56 GeV$ was extrapolated using an effective range expansion and the modulus of the form factor using a pole or Breit-Wigner formula.

The only experimental data on the derivatives of the form factor at zero momentum transfer is the rms radius of the pion, $r^2_V = 0.439 \pm 0.008 fm^2 \tag{12}$. This value is very much in agreement with that determined from the sum rules. In fact the sum rule for the rms radius gets overwhelmingly contribution from the $\rho$ resonance as can be seen from Fig.1. The success of the calculation of the r.m.s. radius is a first indication that causality is respected and also that the extrapolation procedures to low energy for the P-wave $\pi\pi$ phase shifts and for the modulus of the form factor are legitimate.

Dispersion relation for the pion form factor is now shown to be well verified by the data over a wide energy region. Using $ImV(z)$ as given by Eq. (3) together with the once subtracted dispersion relation, one can calculate the real part of the form factor $ReV(s)$ in the time-like region and also $V(s)$ in the space like region. Because the space-like behavior of the form factor is not sensitive to the calculation schemes, it will not be considered here. The result of this calculation is given in Fig.2. As it can be seen, dispersion relation results are well satisfied by the data.

The i-loop ChPT result can be put into the following form, similar to Eq. (4):

$$V^{pert(i)}(s) = 1 + a_1 s + a_2 s^2 + ... + a_i s^i + D^{pert(i)}(s) \tag{7}$$

where $i + 1$ subtraction constants are needed to make the last integral on the RHS of this equation converges and

$$D^{pert(i)}(s) = \frac{s^{1+i}}{\pi i} \int_{4m^2_i}^{\infty} \frac{ImV^{pert(i)}(z)dz}{z^{1+i}(z - s - i\epsilon)} \tag{8}$$

with $ImV^{pert(i)}(z)$ calculated by the $i$th loop perturbation scheme.

Similarly to these equations, the corresponding experimental vector form factor $V^{exp(i)}(s)$ and $D^{exp(i)}(s)$ can be constructed using the same subtraction constants as in Eq. (5) but with the imaginary part replaced by $ImV^{exp(i)}(s)$, calculated using Eq. (3).
The one-loop ChPT calculation requires 2 subtraction constants. The first one is given by the Ward Identity, the second one is proportional to the r.m.s. radius of the pion. In Fig. 1, the imaginary part of the one-loop ChPT calculation for the vector pion form factor is compared with the result of the imaginary part obtained from the experimental data. It is seen that they differ very much from each other. One expects therefore that the corresponding real parts calculated by dispersion relation should be quite different from each other.

In Fig. 2, the full real part of the one loop amplitude is compared with that obtained from experiment. At very low energy one cannot distinguish the perturbative result from the experimental one due to the dominance of the subtraction constants. At an energy around 0.56 GeV there is a definite difference between the perturbative result and the experimental data. This difference becomes much clearer in Fig. 3 where only the real part of the perturbative DI, \( ReDI_{\text{pert}}^{(1)}(s) \), is compared with the corresponding experimental quantity, \( ReDI_{\text{exp}}^{(1)}(s) \). It is seen that even at 0.5 GeV the discrepancy is clear. Supporters of ChPT would argue that ChPT would not be expected to work at this energy. One would have to go to a lower energy where the data became very inaccurate.

This argument is false as can be seen by comparing the ratio \( ReDI_{\text{pert}}^{(1)}/ReDI_{\text{exp}}^{(1)} \). It is seen in Fig. 4 that everywhere below 0.6 GeV this ratio differs from unity by a factor of 6-7 due to the presence of non perturbative effects.

Similarly to the one-loop calculation, the two-loop results are plotted in Fig. (1) - Fig. (4) \[5\]. Although the two-loop result is better than the one-loop calculation, because more parameters are introduced, calculating higher loop effects will not explain the data.

It is seen that perturbation theory is inadequate for the vector pion form factor even at very low momentum transfer. This fact is due to the very large value of the pion r.m.s. radius or a very low value of the \( \rho \) mass squared (see below). In order that the perturbation theory to be valid the calculated term by ChPT should be much larger than the non perturbative effect. At one loop, by requiring the perturbative calculation dominates over the nonperturbative effects at low energy, one has \( s_{\rho} >> \sqrt{960 \pi f_{\pi} m_{\pi}} = 1.3 GeV^2 \) which is far from being satisfied by the physical value of the \( \rho \) mass.

The unitarized models are now examined. It has been shown a long time ago that to take into account of the unitarity relation, it is better to use the inverse amplitude \( 1/V(s) \) or the Pade approximant method \[13, 14\].

The first model is obtained by introducing a zero in the calculated form factor in the ref. \[13\] to get an agreement with the experimental r.m.s. radius. The pion form factor is now multiplied by \( 1 + \alpha s/s_{\rho} \) where \( s_{\rho} \) is the \( \rho \) mass squared \[13\].

The experimental data can be fitted with a \( \rho \) mass equal to 0.773 GeV and \( \alpha = 0.14 \). These results are in excellent agreement with the data \[8, 12\].

The second model, which is more complete at the expense of introducing more parameters, is based on the two-loop ChPT calculation with unitarity taken into account. It has the singularity associated with the two loop graphs. Using the same inverse amplitude method as was done with the one-loop amplitude, but generalizing this method to two-loop calculation, Hannah has recently obtained a remarkable fit to the pion form factor in the time-like and space-like regions. His result is equivalent to the (0,2) Padé
approximant method as applied to the two-loop ChPT calculation \cite{16}. Both models contain ghosts which can be shown to be unimportant \cite{17}.

As can be seen from Figs. 1, 2 and 3 the imaginary and real parts of these two models are very much in agreement with the data. A small deviation of $\text{Im} V(s)$ above 0.9$\text{GeV}$ is due to a small deviation of the phases of $V(s)$ in these two models from the data of the P-wave $\pi\pi$ phase shifts.

In conclusion, higher loop perturbative calculations do not solve the unitarity problem. The perturbative scheme has to be supplemented by the well-known unitarisation schemes such as the inverse amplitude, N/D and Padé approximant methods \cite{13, 14, 16, 17, 18, 19}.

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References

[1] For a review, see J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge Univ. Press, Cambridge, 1992).

[2] S. Weinberg, Physica 96 A (1979) 327.

[3] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158 142 (1984).

[4] J. Gasser and H. Leutwyler, Nucl. Phys. B250 465 (1985). ibid. B250 539 (1985).

[5] J. Gasser and Ulf-G Meiβner, Nucl. Phys. B357 90 (1991). G. Colangelo et al., Phys. Rev. D 54, 4403 (1996); J. Bijnens et al, J. High Energy Phys. 05, 014 (1998).

[6] K. M. Watson, Phys. Rev. 95, 228 (1954).

[7] L. M. Barkov et al. Nucl. Phys. B256, 365 (1985).

[8] ALEPH Collaboration, R. Barate et al., Z. Phys. C 76, 15 (1997).

[9] S. D. Protopopescu et al., Phys. Rev. D 7, 1279 (1973).

[10] B. Hyams et al. Nucl. Phys. B64, 134 (1973).

[11] P. Eastabrooks and A. D. Martin, Nucl. Phys. B79, 301 (1974).

[12] NA7 Collaboration, S. R. Amendolia et al., Nucl. Phys. B277, 168 (1986).

[13] T. N. Truong, Phys. Rev. Lett. 61, 2526 (1988).

[14] T. N. Truong, Phys. Rev. Lett. 67, 2260 (1991).

[15] Le viet Dung and Tran N. Truong, Report No hep-ph/9607378.

[16] T. Hannah, Phys. Rev. D55, 5613 (1997). Ph. D thesis, Aarhus University (1998).

[17] T. Hannah and Tran N. Truong (unpublished)

[18] T. N. Truong, Phys. Lett. B 313, 221 (1993).

[19] H. Lehmann, Phys. Lett. B 41, 529,(1972), A. Dobado, M.J. Herrero and T. N. Truong, Phys. Lett. B 235 129, 134, (1990), S. Willenbrook, Phys. Rev. D43, 1710 (1991) and references cited therein, A. Dobado and J. R. Pelaez, Phys. Rev. D56 3057 (1997), J. A. Oller, E. Oset and J. R. Pelaez, Phys. Rev. Lett. 80, 3452 (1998), Guerrero and J. A. Oller, Report No hep-ph/9805334.
Figure Captions

Fig. 1: The imaginary part of the vector pion form factor $ImV$, given by Eq. (3), as a function of energy in the GeV unit. The solid curve is the experimental results with experimental errors; the long-dashed curve is the two-loop ChPT calculation, the medium long-dashed curve is the one-loop ChPT calculation, the short-dashed curve is from the modified unitarized one-loop ChPT calculation fitted to the $\rho$ mass and the experimental r.m.s. radius, and the dotted curve is the unitarized two-loop calculation of Hannah [13].

Fig. 2: The real parts of the pion form factor $ReV$, given by Eq. (4) as a function of energy. The curves are as in Fig. 1. The real part of the form factor calculated by the once subtracted dispersion relation using the experimental imaginary part is also given by the solid line.

Fig. 3: The real parts of the dispersion integral $ReDI$ as a function of energy. The curves are as in Fig. 1.

Fig. 4: The ratio of the one-loop ChPT to the corresponding experimental quantity, $ReDI_{pert}^{(1)}/ReDI_{exp}^{(1)}$, defined by Eq. (8), as a function of energy, is given by the solid line; the corresponding ratio for the two-loop result is given by the dashed line. The ratio of the unitarized models to the experimental results is unity (not shown). The experimental errors are estimated to be less than 10%.
Figure 3:

Figure 4: