The lambda-q calculus can efficiently simulate quantum computers

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Abstract

We show that the lambda-q calculus can efficiently simulate quantum Turing machines by showing how the lambda-q calculus can efficiently simulate a class of quantum cellular automaton that are equivalent to quantum Turing machines. We conclude by noting that the lambda-q calculus may be strictly stronger than quantum computers because NP-complete problems such as satisfiability are efficiently solvable in the lambda-q calculus but there is a widespread doubt that they are efficiently solvable by quantum computers.

1. Introduction

We show that the $\lambda^q$-calculus defined in [1] can efficiently simulate the one-dimensional partitioned quantum cellular automata (1d-PQCA) defined in [2]. By the equivalence of 1d-PQCA and quantum Turing machines (QTM) proved in [2], the $\lambda^q$-calculus can efficiently simulate QTM.

We assume familiarity with both the $\lambda^q$-calculus and 1d-PQCA as defined in the above papers.

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2. Simulation

To show that 1d-PQCA can be efficiently simulated by the $\lambda^q$-calculus, we need to exhibit a $\lambda^q$-term $M$ for a given 1d-PQCA $A$ such that $A$ after $k$ steps is in the same superposition as $M$ after $P(k)$ steps, with $P$ a polynomial.

We assume for now that the 1d-PQCA has transition amplitudes not over the complex numbers, but over the positive and negative rationals. It has been shown that this is equivalent to the general model in QTM.

To express $A$ in $M$, we need to do the following things.

1. Translate states of $A$ into $\lambda^q$-terms that can be compared (e.g. into Church numerals).
2. Translate the acceptance states and the integer denoting the acceptance cell into $\lambda^q$-terms.
3. Create a $\lambda^q$-term $P$ to mimic the operation of the permutation $\sigma$.
4. Translate the local transition function into a transition term. For 1d-PQCA this means translating the matrix $\Lambda$ into a term $L$ comparing the initial state with each of the possible states and returning the appropriate superposition.
5. Determine an injective mapping of configurations of $A$ and configurations of $M$.

Although we will not write down $M$ in full, we note that within $M$ are the mechanisms described above that take a single configuration, apply $P$, and return the superposition as described by $L$.

We recall that the contextual closure of the $\beta^q$-relation is such that $M, N \rightarrow^\beta M', N'$ where $M \rightarrow^\beta M'$ and $N \rightarrow^\beta N'$. Thus there is parallel reduction within superpositions. By inspection of the mechanisms above it follows that $k$ steps of $A$ is equivalent to a polynomial of $k$ steps of $M$.

Steps 1, 2, and 3 are easy. We will use the following abbreviatory notation for $\lambda^q$-superpositions. We let $[(M_i : n_i)]$ be a rewriting of the term $\left[\sum_{i \in I} M_i^n\right]$ such each of the $M_i$ are distinct and the integer $n_i$ represents the count of each $M_i$. We can also write this as $[(M_i : a_i, b_i, n_i)]$ such that $M_i \neq M_j$ and $M_i \neq M_j$ for $i \neq j$, all of the $M_i$ are of positive sign, the integer $a_i$ denotes the count of $M_i$, the integer $b_i$ denotes the count of $M_i$, and $n_i = a_i - b_i$. 

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Then for step 5, the $\lambda^q$-superposition $[(M_i : a_i, b_i, n_i)]$ (let $n = \sum n_i$) will be equivalent to the 1d-PQCA-superposition $\sum \frac{n_i}{n} |c(M_i)\rangle$, where $c$ takes $\lambda^q$-terms and translates them into 1d-PQCA configurations. Essentially this means stripping off everything other than the data, that is to say, the structure containing the contents. Note that $c$ is not itself a $\lambda^q$-term. It merely performs a fixed syntactic operation, removing extraneous information such as $P$ and $L$, and translating the Church numerals that represent states into the 1d-PQCA states. This is injective because the mapping from states of A into numerals is injective. Thus, step 5 is complete.

Step 4 requires translating the $\Lambda$ matrix into a matrix of whole numbers, and translating an arbitrary 1d–PQCA superposition into a $\lambda^q$-superposition. The latter is done merely by multiplying each of the amplitudes by the product of all of the denominators of all of the amplitudes, to get integers. We call the product of the denominators here $d$. We perform a similar act on the $\Lambda$ matrix, multiplying each element by the product of all of the denominators of $\Lambda$. We call this constant $b$. Then we have that $T = b\Lambda$ is a matrix over integers. This matrix can be considered notation for the $\lambda^q$-term that checks if a given state is a particular state and returns the appropriate superposition. For instance, if

$$\Lambda = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$$

then

$$T = b\Lambda = 9\Lambda = \begin{pmatrix} 6 & 3 \\ 0 & 9 \end{pmatrix}$$

which we can consider as alternate notation for

$$Q \equiv \lambda s. \text{IF (EQUAL } s1) \ (1,1,1,1,1,1,1,2,2,2,2)$$

$$\ (\text{IF (EQUAL } s2) \ (2,2,2,2,2,2,2,2,2,2))$$

Then it follows that if $c$ is a superposition of configuration of $A$, applying $\Lambda k$ times results in the same superposition as applying $T k$ times to the representation of $c$ in the $\lambda^q$-calculus.

3. Conclusion

The $\lambda^q$-calculus can efficiently simulate 1d-PQCA, which can efficiently simulate QTM. Therefore the $\lambda^q$-calculus can efficiently simulate QTM. However, the $\lambda^q$-calculus can efficiently solve NP-complete problems such as satisfiability [1], while
there is widespread belief (e.g. [3]) that QTM cannot efficiently solve satisfiability. Thus, the greater the doubt that QTM cannot solve NP-complete problems, the greater the justification in believing that the λ̂-calculus is strictly stronger than QTM.

References

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