Entanglement detection in the vicinity of arbitrary Dicke states

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Dicke states represent a class of multipartite entangled states that can be generated experimentally with many applications in quantum information. We propose a method to experimentally detect genuine multipartite entanglement in the vicinity of arbitrary Dicke states. The detection scheme can be used to experimentally quantity the entanglement depth of many-body systems and is easy to implement as it requires to measure only three collective spin operators. The detection criterion is strong as it heralds multipartite entanglement even in cases where the state fidelity goes down exponentially with the number of qubits.

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Quantum entanglement provides the most useful resource for implementation of many quantum information protocols. To test fundamentals of quantum mechanics and to realize quantum information processing, a big experimental drive is to get more and more particles prepared into massively entangled states.\textsuperscript{[1–4]} There are different types of entangled states for many qubits.\textsuperscript{[3–5]} Experiments so far typically center around two kinds of entangled states: Dicke states and graph states.\textsuperscript{[3–4]} Dicke states represent a class of multipartite entangled states that can be generated experimentally with many applications in quantum information.\textsuperscript{[3, 4, 10, 11]} The proposed scheme has the following advantages: firstly, only three collective spin operators are required to detect entanglement, which is pretty robust to experimental noise, and can show significant entanglement depth of the system even in cases where the state fidelity has been exponentially small with the number of qubits.

Dicke states are co-eigenstates of the collective spin operators. Each qubit is described by a Pauli matrix \(\sigma\). For \(N\) qubits, we define the collective spin operator \(J\) as \(J = \sum_{i=1}^{N} \mathbf{\sigma} / 2\). The Dicke state \(|\frac{N}{2}, \frac{n}{2}\rangle\) is defined as a coeigenstate of the operators \(J^2 = J_x^2 + J_y^2 + J_z^2\) and \(J_z\), with the eigenvalues \(N(N+1)/4\) and \(n/2\) (\(n = -N/2, -N/2 + 1, \ldots, N/2\)), respectively. The Dicke states can be conveniently generated in experiments without the need of separate addressing.\textsuperscript{[2–4]} For the trivial case of \(n = \pm N\), the Dicke states is a multipartite entangled state with interesting applications in both precision measurements and quantum information.\textsuperscript{[2–4, 14, 15]}

To construct an entanglement detection criterion in the vicinity of Dicke states, we note that the variances of the collective spin operators \(J_x, J_y, J_z\) have very special properties for these states. The variance of \(J_z\) is minimized (ideally it should be zero) while the variances of \(J_x, J_y\) are maximized under the constraint of \(\langle J_z \rangle\). So, to detect entanglement, we should construct an inequality to bound the variances of \(J_x, J_y\) with the variance of \(J_z\) for any separable states or insufficiently entangled states, and at the same time this inequality should be violated by the states sufficiently close to a Dicke state.

For a composite system of \(N\) qubits, we note that its density operator \(\rho\) can always be written into the following form if \(\rho\) does not contain genuine \(N\)-qubit entanglement: \textsuperscript{[16]}

\[\rho = \sum_{\mu} p_\mu \rho_\mu,\]

with \(p_\mu \geq 0, \sum_{\mu} p_\mu = 1\), and

\[\rho_\mu = \rho_{1\mu} \otimes \rho_{2\mu} \otimes \cdots \otimes \rho_{k\mu},\]
where $\rho_{i\mu}$ ($i = 1, 2, \cdots , k$) represents a component state of $m_{i\mu}$ ($m_{i\mu} \geq 1$) qubits with $\sum_{i=1}^{k} m_{i\mu} = N$. In other words, for each component $\mu$, the $N$ qubits are divided into $k$ groups with $m_{i\mu}$ qubits for the $i$th group, and the component state $\rho_{\mu}$ is a tensor product of the states for each group. For a fixed component $\mu$, each qubit uniquely belongs to one group, however, for different $\mu$, the group division of the qubits can be different. If all $m_{i\mu} = 1$ (and corresponding $k = N$), $\rho$ reduces to a separable state. If the maximum of $m_{i\mu}$ is $m_0$, we conclude that the state $\rho$ has no genuine $(m_0 + 1)$-qubit entanglement [16]. With a smaller $m_0$, the state $\rho$ gets less entangled.

We now show that for any states in the form Eqs. (1-2), the variance of the collective spin operators are severely bounded, while this bound is violated by the Dicke states. For each group division $\mu$ of the $N$ qubits, the total collective spin operators $\mathbf{J}$ can be written as $\mathbf{J} = \sum_{i=1}^{k} \mathbf{J}_i$, where $\mathbf{J}_i = \sum_{j=1}^{m_{i\mu}} \sigma_j/2$ is the collective spin operator for $m_{i\mu}$ qubits in the $i$th group. Through addition of the angular momenta, we know the maximum spin of $\mathbf{J}_i$ is $m_{i\mu}/2$, so the moments of $J_{\alpha i}$ ($\alpha = x, y, z$) are bounded by

$$
\langle J_{\alpha i}^2 \rangle \leq m_{i\mu}^2/4, \mathrm{and} \langle \mathbf{J}_i^2 \rangle \leq m_{i\mu}(m_{i\mu} + 2)/4. \tag{3}
$$

Under state $\rho$, we have $\langle J_{x i}^2 \rangle = \sum_\mu p_\mu \langle J_{x i}^2 \rangle_\mu$ and

$$
\langle J_{x i}^2 \rangle_\mu = \sum_{i_1, i_2} \langle J_{x i_1} \rangle_\mu \langle J_{x i_2} \rangle_\mu + \sum_i \langle (\Delta J_{xi})^2 \rangle_\mu. \tag{4}
$$

Using the uncertainty relation $\langle (\Delta J_{yi})^2 \rangle_\mu \leq \langle (\Delta J_{zi})^2 \rangle_\mu \leq \langle J_{zi}^2 \rangle_\mu/4$, we can bound the term $\sum_{i_1, i_2} \langle J_{x i_1} \rangle_\mu \langle J_{x i_2} \rangle_\mu$ as

\[
\sum_{i_1, i_2} \langle J_{x i_1} \rangle_\mu \langle J_{x i_2} \rangle_\mu \leq \sum_{i_1, i_2} \left[ \langle (\Delta J_{y i_1})^2 \rangle_\mu + \langle (\Delta J_{y i_2})^2 \rangle_\mu \right] \leq \sum_{i_1, i_2} \left[ \langle (\Delta J_{y i_1})^2 \rangle_\mu + \langle (\Delta J_{y i_2})^2 \rangle_\mu \right] = 4 \langle (\Delta J_{y})^2 \rangle_\mu \sum_i \langle (\Delta J_{zi})^2 \rangle_\mu, \tag{5}
\]

where we have used the relation $\langle (\Delta J_{zi})^2 \rangle_\mu = \sum_i \langle (\Delta J_{zi})^2 \rangle_\mu$ for the state in the form of Eqs. (1-2).

Combining Eqs. (4) and (5), we get

$$
\langle J_{x i}^2 \rangle \leq \sum_\mu p_\mu \left[ \langle (\Delta J_{zi})^2 \rangle_\mu + \langle (\Delta J_{y i})^2 \rangle_\mu \right]. \tag{6}
$$

Using the relation $\langle (\Delta J_{zi})^2 \rangle_\mu \leq \langle J_{zi}^2 \rangle_\mu \leq m_{i\mu}^2/4$ (see Eq. (3)) and $\langle (\Delta J_{zi})^2 \rangle_\mu \geq \sum_\mu p_\mu \langle (\Delta J_{zi})^2 \rangle_\mu$, we can bound $\langle J_{x i}^2 \rangle$ by

$$
\langle J_{x i}^2 \rangle \leq \left[ 1 + 4 \langle (\Delta J_{zi})^2 \rangle \right] \max_{\{m_{i\mu}\}} \left( \sum_{i=1}^{k} m_{i\mu}^2/4 \right), \tag{7}
$$

where the maximum is taken over all the group division $\{m_{i\mu}\}$ ($m_{i\mu}$ are positive integers) of the $N$ qubits with the constraint $\sum_{i=1}^{k} m_{i\mu} = N$ and $m_{i\mu} \leq m_0$. The maximum value is obtained by choosing $k = \lfloor N/m_0 \rfloor$ ($\lfloor N/m_0 \rfloor$ denotes the smallest integer no less than $N/m_0$), $m_{1\mu} = N - m_0(k - 1)$, and all the other $m_{i\mu} = m_0$ ($i = 2, \cdots , k$). Correspondingly, Eq. (10) reduces to

$$
\langle J_{x i}^2 \rangle \leq \left[ 1 + 4 \langle (\Delta J_{zi})^2 \rangle \right] m_0 N/4, \tag{8}
$$

where we have used the relation $m_{1\mu}^2 + m_{0}^2(k - 1) \leq m_0[m_{1\mu} + m_0(k - 1)] = m_0 N$. So, for any states without genuine $(m_0 + 1)$-qubit entanglement, the moment $\langle J_{x i}^2 \rangle$ (and similarly also $\langle J_{y i}^2 \rangle$) will be bounded by the inequality (8). When $m_0 \geq 2$, we can derive a stronger bound. Note that $\langle J_{y i}^2 \rangle$ satisfies an inequality similar to Eq. (6), but with the indices $x$ and $y$ exchanged. If we add up the inequalities for $\langle J_{x i}^2 \rangle$ and $\langle J_{y i}^2 \rangle$, and use the relation $\langle (\Delta J_{zi})^2 \rangle_\mu + \langle (\Delta J_{y i})^2 \rangle_\mu \leq \langle J_{y i}^2 \rangle_\mu \leq m_{i\mu}(m_{i\mu} + 2)/4$ (see Eq. (3)), we obtain

$$
\langle J_{x i}^2 \rangle + \langle J_{y i}^2 \rangle \leq \left[ 1 + 4 \langle (\Delta J_{zi})^2 \rangle \right] N (m_0 + 2)/4. \tag{9}
$$

We can use violation of the inequality (8) with $m_0 = 1$ to experimentally prove entanglement of the system and then use the following criterion to quantify its entanglement depth:

**Criterion 1:** We can experimentally measure the following quantity $\xi$ through detection of the collective spin operator $\mathbf{J}$:

$$
\xi = \frac{\langle J_{x i}^2 \rangle + \langle J_{y i}^2 \rangle}{N (1/4 + \langle (\Delta J_{zi})^2 \rangle)} - 1. \tag{10}
$$

If $\xi > m$, it is confirmed that the system has genuine $m$-qubit entanglement.
For the Dicke state $|N/2,0\rangle$, we have $\langle J_x^2 \rangle = \langle J_y^2 \rangle = N(N+2)/8$ and $\langle (\Delta J_x)^2 \rangle = 0$, so in the ideal case, $\xi = N + 1 > N$, and from measurement of $\xi$, we can confirm that all the qubits are in a genuine $N$-qubit entangled state. The noise in experiments will degrade the entanglement depth of the system. First, we consider dephasing noise which is a major source of noise in many experiments. The detection criterion in Eq. (10) is very robust to dephasing noise. To see this, we note the state $|N/2,0\rangle$ is a big superposition state with $\binom{N}{N/2} = (N/2)! \frac{(N/2)!(N-N/2)!}{N!}$ terms in the computational basis. All the superpositions terms have $J_z = 0$, so the dephasing error only degrades the moments $\langle J_x^2 \rangle + \langle J_y^2 \rangle$, but does not increase $\langle (\Delta J_x)^2 \rangle$. For each superposition term of the state $|N/2,0\rangle$, we know $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \sum_{i=1}^{N} (1 - (1 - p)^2/4) = N/4$. So, if coherence is completely gone, $\xi$ reduces to 1, and the state has no entanglement as expected. However, under incomplete dephasing, we can experimentally prove a significant entanglement depth of the system by measuring $\xi$ even if the state fidelity becomes exponentially small. For instance, with a dephasing error rate $p$ for each qubit, the state fidelity goes down exponentially roughly by $p^N$ for $N$ qubits with $N \gg 1$. To estimate the value of $\xi$, we note that with a probability $\left(\frac{1}{2}\right)^{N/2} (1 - p)^{N^2/4}$ (according to the binormal distribution), $i$ qubits are decohered among the $N$ qubits, which contribute a value of $i/2$ to $\langle J_x^2 \rangle + \langle J_y^2 \rangle$. The remaining $N - i$ qubits still have coherence, which contribute a value of $(N - i)(N - i + 2)/4 - \langle J_x^2 \rangle_{N-i}$ to $\langle J_x^2 \rangle + \langle J_y^2 \rangle$. Since initially the $N$ qubits are in the $J_z = 0$ eigenstate, the mean value of $\langle J_x^2 \rangle_N-i$ for the $N - i$ qubits is equal to $\langle J_x^2 \rangle_i$ for the decohered $i$ qubits. For the decohered $i$ qubits, $\langle J_x^2 \rangle_i = \sum_{k=1}^{i} \langle (\sigma_{kz})^2 \rangle = i/4$. So the value of $\xi$ is estimated by $\xi \approx 4/N \sum_{i=0}^{N} \left(\frac{1}{2}\right)^{N/2} (1 - p)^{N^2/4} (i/2 + [(N - i)(N - i + 2)/4 - i/4]) - 1 = (1 - p) N + 1 - p^2$, we can thus experimentally prove a significant entanglement depth of $(1 - p)$ $N$ qubits by measuring $\xi$.

The detection criterion in Eq. (10) is more sensitive to the bit-flip error as this type of error significantly increases $\langle (\Delta J_x)^2 \rangle$. With a bit flip error rate $p_b$ for each qubit, the variance of $J_z$ is estimated by $\langle (\Delta J_z)^2 \rangle \sim Np(1 - p)$. We need $Np(1 - p) < 1/4$ to minimize change to $\xi$. For tens of qubits, we can tolerate bit-flip error rate at a percent level to keep the qubits in a genuine multipartite entangled state. Alternatively, in the limit of large $N$ with $Np(1 - p) \gg 1/4$, the value of $\xi$ is estimated by $\xi \approx 1/4[2N(1 - p)] - 1$. With a percent of bit flip error rate for each qubit, we can experimentally prove an entanglement depth of more than 20 qubits by measuring $\xi$.

The criterion 1 is most appropriate for detection of the entanglement depth in the vicinity of the Dicke state $|N/2,0\rangle$. It becomes weaker for other Dicke states $|N/2,n\rangle$ with increasing $|n|$. For the state $|N/2,n/2\rangle$, the moments of $J_x$ and $J_y$ are bounded by $\langle J_x^2 \rangle + \langle J_y^2 \rangle = \langle J^2 \rangle - \langle J_x^2 \rangle = N(N+2)/4 - n^2/4$. The criterion 1 does not take into account this bound due to a finite $\langle J_z \rangle$. To derive a stronger detection criterion for the Dicke states $|N/2,n\rangle$, we start from Eq. (6) and a similar bound for $\langle J_y^2 \rangle$. When we add up the inequalities for $\langle J^2 \rangle$ and $\langle J_y^2 \rangle$ both in the form of Eq. (6), we want to find a better bound for $\langle (\Delta J_x)^2 \rangle + \langle (\Delta J_y)^2 \rangle$ under a finite $\langle J_z \rangle$.

Using the relation $\langle (\Delta J_x)^2 \rangle + \langle (\Delta J_y)^2 \rangle \leq \langle J^2 \rangle - \langle J_z \rangle$ and $\langle J^2 \rangle = \langle \sum_{i=1}^{N} J_{z_{i}}^2 \rangle \leq k \langle J_{z_{i}}^2 \rangle$, we obtain

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \sum_{\mu} p_{\mu} \left[ 1 + 4 \langle (\Delta J^2)_{\mu} \rangle + \sum_{i} m_{\mu i}(m_{\mu i} + 2)/4 - \langle J_{z_{i}}^2 \rangle_{\mu} \right].$$

To bound the right side of Eq. (11), we consider the two-fold average $\sum_{\mu} p_{\mu} \langle (\Delta J^2)_{\mu} \rangle \langle J^2 \rangle_{\mu}$, where $\langle \cdots \rangle$ denotes the average over $\mu$ with the weight function $p_{\mu}$. For any two variables $A$ and $B$, we know their average satisfies the following property:

$$\langle AB \rangle = \langle A \rangle \langle B \rangle + \langle \Delta A \Delta B \rangle \geq \langle A \rangle \langle B \rangle - \sqrt{\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle}.$$
In the second line of Eq. (15), we use again the property in Eq. (12). Substituting Eq. (13) into Eq. (11), we finally obtain the following bound for any state in the form of Eqs. (1-2)

\[
\langle J_z^2 \rangle + \langle J_y^2 \rangle \leq \left[ 1 + 4 \langle (\Delta J_z)^2 \rangle \right] \times \max_{\{m, \chi\}} \left[ \sum_{i} m_{i\mu} (m_{i\mu} + 2) / 4 - \chi / k \right]
\]

where \( \chi \) is defined by

\[
\chi = \langle J_z^2 \rangle - \left[ 1 / 4 + \langle (\Delta J_z)^2 \rangle \right]^{-1} \langle (\Delta J_z^2)^2 \rangle (1 + 2\alpha).
\]

The parameter \( \chi \) is determined experimentally by measuring the operator \( J_z \), and its value is basically given by the first term \( \langle J_z^2 \rangle \), with small correction from the fluctuation of \( J_z^2 \) when the real state deviates from the Dicke state (the latter has \( \langle (\Delta J_z^2)^2 \rangle = 0 \)). Summarizing the result, we arrive at the following criterion

**Criterion 2.** We can experimentally measure the values of \( \xi \) and \( \chi \) (defined by Eqs. (10,17)) through detection of the collective spin operator \( J \). The system has genuine \( m \)-qubit entanglement if

\[
\xi > f (m, \chi) = \frac{4}{N} \max_{\{m, \chi\}} \left( \sum_{i=1}^{k} m_{i\mu} (m_{i\mu} + 2) / 4 - \chi / k \right)^{-1},
\]

where the maximum is taken under the constraint of \( m_{i\mu} \leq m - 1 \) and \( \sum_{i} m_{i\mu} = N \).

With a known \( \chi \), it is typically easy to calculate the function of \( f (m, \chi) \). For instance, for the state \( |N/2, n/2 \rangle \), \( \chi \approx n^2/4 \), and \( f (m, \chi) \approx m - (m - 1) n^2 / N^2 \) for the simple case when \( m - N \) divides \( N \) and \( (m - 1) n^2 < 2N^2 \). Similar to the discussion made for the state \( |N/2, 0 \rangle \), the entanglement detection criterion 2 is pretty robust to noise, in particular the dephasing noise, and appropriate for entanglement detection in the vicinity of the Dicke states \( |N/2, n/2 \rangle \) with nonzero \( n \).

In summary, we have proposed powerful detection criteria to experimentally prove entanglement and quantify the entanglement depth for many-body systems in the vicinity of arbitrary Dicke states. The criteria are based on simple measurements of the collective spin operators and ready to be implemented in future experiments.

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