A GEOMETRIC VIEW ON IWASAWA THEORY

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Abstract. These notes expand on the presentation given by the second author at the Iwasawa 2019 conference in Bordeaux of our joint work on the geometry of the \( p \)-adic eigencurve at a weight one CM form \( f \) irregular at \( p \), namely its implications in Iwasawa and in Hida theories. Novel features include the determination of

- the Fourier coefficients of the infinitesimal deformations of \( f \) along each Hida family containing it in terms of \( p \)-adic logarithms of algebraic numbers.
- the “mysterious” cross-ratios of the ordinary filtrations of the Hida families containing \( f \).

Introduction

0.1. Historical background. In the 1960s Kubota and Leopoldt used Kummer’s congruences involving Bernoulli numbers to define the \( p \)-adic zeta function \( \zeta_p \in \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \) and Iwasawa formulated his Main Conjecture postulating that \( \zeta_p \) generates the characteristic ideal of a certain \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \)-module controlling the class group growth of the number fields contained in the cyclotomic \( \mathbb{Z}_p \)-extension \( \mathbb{Q}_\infty \) of \( \mathbb{Q} \). Around the same time Serre [41] observed that ordinary Eisenstein series vary \( p \)-adic analytically in the weight, hence can be interpolated over the \( p \)-adic analytic weight space \( \mathcal{W}(\mathbb{C}_p) = \text{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \), thus providing a modular interpretation of the Kubota–Leopoldt \( p \)-adic zeta as a constant term of a \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \)-adic Eisenstein series.

In the 1980s H. Hida gave a new impulse to the subject by \( p \)-adically interpolating in [34] ordinary cuspforms of weight at least 2. He introduced a space of \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \)-adic cuspform which is in a perfect duality of finite free \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \)-modules with the corresponding Hecke algebra and proved a Control Theorem (see [35] when \( p \) is odd and [28] when \( p = 2 \)). Furthermore, Hida showed that every eigenform as above is the specialization of a unique \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \)-adic eigenform, also called a Hida family. The Galois theoretic properties of these families, namely the restriction at the decomposition group at \( p \), were studied by B. Mazur and A. Wiles who also made a crucial, for us, extension of Hida Theory to allow classical weight one forms.

Coming back to Iwasawa theory, Eisenstein Hida families play a prominent role in the proof of the Main Conjecture over \( \mathbb{Q} \) by Mazur and Wiles. They proved one divisibility using Ribet’s Eisenstein ideal method and then deduced the equality via the class number formula. In their own words, while being ‘explicit’ from a certain modular perspective, the approach does not allow to determine whether or not \( \zeta_p \) admits multiple zeros (one merely knows that if that were the case then the characteristic series of the Iwasawa module would have a zero of
the same multiplicity). Instead, they relied on the Ferrero–Greenberg Theorem showing that the ‘trivial’ zeros of \( \zeta_p \) are simple.

Another remarkable class of Hida families have complex multiplication (CM) by an imaginary quadratic field \( K \), and are obtained by \( p \)-adic interpolation of classical theta series. Their adjoint \( p \)-adic \( L \)-functions is essentially equal to Katz’ anti-cyclotomic \( p \)-adic \( L \)-function, and the corresponding anti-cyclotomic Main Conjecture over \( K \) has been proven by K. Rubin and independently by H. Hida and J. Tilouine. Similarly to the Eisenstein case, both proofs proceed by proving one divisibility and by invoking the class number formula, but without determining the possible zeros and their multiplicities. The question, analogous to the Ferrero–Greenberg Theorem, of whether the trivial zeros of Katz’ anti-cyclotomic \( p \)-adic \( L \)-functions are simple, has remained open while being reformulated in terms of certain Iwasawa modules, via the mysterious bridge envisioned by Iwasawa. In [7] we use Hida Theory together with Mazur’s Galois Deformations Theory, to show that these trivial zeros are indeed simple, provided that a certain anti-cyclotomic \( L \)-invariant does not vanish, as predicted by the Four Exponents Conjecture in Transcendence Theory.

Explicit results in Hida theory have been notoriously difficult to obtain since the Hecke-Hida algebras are finite and flat over \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \), a complete local algebra of Krull dimension 2 isomorphic to \( \mathbb{Z}_p[X] \) by a theorem of J.P. Serre. What has made our approach successful is the passage to the discrete valuation ring \( \mathbb{Q}_p[X] \) and even to its strict henselization \( \Lambda = \overline{\mathbb{Q}_p[X]} \), allowing the use of the rigid geometry tools brought to the modular world by Coleman and Mazur in the 1990s. A central object in this theory is the Coleman–Mazur eigencurve \( \mathcal{C} \to \mathcal{W} \) first introduced in [15] (under some technical assumptions which were later removed by Buzzard [11]) as the rigid analytic curve parametrizing systems of Hecke eigenvalues of overconvergent \( p \)-adic modular forms of finite slope (see [1] for a detailed presentation).

0.2. **Intersection numbers and \( p \)-adic \( L \)-functions.** The local geometry at classical points has significant impact on Iwasawa theory since \( p \)-adic \( L \)-functions have acquired a new variable ‘the weight’ in addition to the cyclotomic variable considered in classical Iwasawa theory. R. Greenberg and G. Stevens [30] have brilliantly illustrated how the weight variable can be used to shed light on problems involving \textit{a priori} only the cyclotomic variable, such as the Mazur–Tate–Teitelbaum Conjecture [39] on the central trivial zeros of the \( p \)-adic \( L \)-function of modular forms. More recently J. Bellaïche [2] pushed the idea of using the weight variable even further to palliate the shortage of critical values needed for the construction of \( p \)-adic \( L \)-functions for modular forms of critical slope, provided that \( \mathcal{C} \) is smooth at such points.

At points where the weight map \( w : \mathcal{C} \to \mathcal{W} \) is etale, one can use the weight space \( \mathcal{W} \) to parametrize a Coleman family, whereas at singular points it is a challenge in itself to attach a \( p \)-adic \( L \)-function to a family. Let us recall that Hida’s Control Theorem [35] for ordinary forms, extended by Coleman [16] to all forms of non-critical slope, implies that \( w : \mathcal{C} \to \mathcal{W} \) is etale at classical non-critical \( p \)-regular points of weight \( \geq 2 \). J. Bellaïche and one of us showed in [3] that the eigencurve is smooth at classical points of weight 1 which are \( p \)-regular,
using Galois deformations and Transcendence Theory, namely the Baker–Brumer Theorem on the linear independence of logarithms of algebraic numbers, to elucidate the geometry of the eigencurve. The geometry of \( \mathcal{C} \) at points which are either \( p \)-irregular or have critical slope is expected to be more complicated, and \( [2] \) presents the state of the art on such matters.

Another link between the geometry of \( \mathcal{C} \) and Iwasawa theory, is the expectation that local intersection numbers of \( \mathcal{C} \) should be directly related to the adjoint \( p \)-adic \( L \)-functions, as shown in Hida’s trilogy \([31, 32, 33]\) and vast subsequent research by numerous authors. In particular, a vanishing of the adjoint \( p \)-adic \( L \)-functions, including for trivial reasons, should detect interesting geometric phenomena. This has been the leitmotiv in \([8]\), resp. \([7]\), where the geometry of \( \mathcal{C} \) at certain \( p \)-irregular Eisenstein, resp. CM, weight 1 points is related to trivial zeros of the Kubota–Leopoldt, resp. Katz anti-cyclotomic, \( p \)-adic \( L \)-function.

In the CM case summarized in \( \mathcal{S}2.7 \) we determine the completed local ring of \( \mathcal{C} \) at the weight 1 point and use the congruence ideal between CM and non-CM families passing through that point, to provide an upper bound for the order of the trivial zero of the corresponding branch \( \zeta^{-} \in \overline{\mathbb{Z}}_{p}[X] \) of the anti-cyclotomic Katz \( p \)-adic \( L \)-function. The exactness of this upper bound is predicted by widely accepted conjectures in Transcendence Theory. Furthermore, thanks to the Six Exponentials Theorem we know that at least one amongst \( \zeta^{-} \) or \( \overline{\zeta}^{-} \) has a simple trivial zero.

In the Eisenstein case summarized in \( \mathcal{S}2.5 \) one can determine the local geometry of \( \mathcal{C} \) at a weight 1 Eisenstein cuspidal overconvergent point unconditionally, and deduce from there Gross’ formula for the derivative of the Kubota–Leopoldt \( p \)-adic zeta at a trivial zero. The non-trivial zeros all occur at cuspidal overconvergent Eisenstein points having non-classical weight. By assuming Greenberg’s pseudo-null conjecture, C. Wang-Erickson and P. Wake showed in \([32]\) that the cuspidal eigencurve is smooth at these points if, and only if, the zeros are simple. The question about the etaleness of the weight map at these points is still open.

Regarding generalizations to groups of higher rank, in a collaboration with S.-C. Shih \([9]\), we recently extended the main results of \([8]\) to irregular weight one Eisenstein points on Hilbert eigenvarieties. The investigation of Eisenstein Hida families in this setting goes at least back to Wiles’ work \([33]\) on the Iwasawa Main Conjecture over totally real number fields, and also plays a prominent role in the proof \([20, 21]\) of the Gross–Stark Conjecture on the derivative of the Deligne–Ribet \( p \)-adic \( L \)-function at a trivial zero.
0.3. Overconvergent generalized eigenforms. Fourier coefficients of classical eigenforms are related to arithmetic functions such as the partition function or the Dedekind eta function, and are motivic in nature as the corresponding two-dimensional Galois representations occur in the étale cohomology of proper smooth varieties. Amongst non-classical overconvergent forms, the closest in nature to a classical eigenform \( f \), are those belonging to its generalized eigenspace \( S^I_{\psi(f)}[f] \). The very exclusive club of genuine overconvergent generalized eigenforms \( S^I_{\psi(f)}[f]_0 \) is a natural supplement of the classical subspace in \( S^I_{\psi(f)}[f] \). In his quest \([2,1]\) to attach \( p \)-adic \( L \)-functions to classical eigenforms of critical slope, J. Bellaïche classified the possible such examples in weight \( k \geq 2 \), and concluded that conjecturally the only genuine overconvergent generalized eigenforms are critical CM forms, whose Fourier coefficients were recently computed by Hsu \([36]\). The first to take up the task in weight 1 were H. Darmon, A. Lauder and V. Rotger \([18]\) who expressed the Fourier coefficients of a certain \( p \)-adic overconvergent weight 1 generalized eigenform in terms of \( p \)-adic logarithms of algebraic numbers in ring class fields of real quadratic fields. The underlying classical weight 1 form has real multiplication (RM) and according to \([4]\) is the only case in which a 1-adic overconvergent weight \( \varphi \) should observe that \((1)\) is true for \( \varphi \) not. Under these assumptions, the newform on \( GL_2 / \mathbb{Q} \) obtained by automorphic induction from \( K \) to \( \mathbb{Q} \) of \( \psi \) is a weight 1 theta series \( \theta_\psi \) of level \( D \cdot N_{K / \mathbb{Q}}(\text{cond}(\psi)) \) and central character \( \varepsilon = \varepsilon_K \cdot \chi_\psi \circ \text{Ver} \). Here \( \varepsilon_K \) denotes the quadratic Dirichlet character attached to \( K / \mathbb{Q} \), and \( \text{Ver} \) denotes the transfer homomorphism. Let \( f = \sum_{n \geq 1} a_n q^n \) denote the unique \( p \)-stabilization of \( \theta_\psi \) (see \( \text{(7)} \)).

In order to state our main result concerning the irregular CM case, we need to fix some notations which will be used throughout the paper. We denote by \( G_L = \text{Gal}(\bar{L}/L) \) the absolute Galois group of a field \( L \). The choice of an embedding \( \iota_p : \mathbb{Q} \to \bar{\mathbb{Q}}_p \) allows one to see \( G_{\mathbb{Q}_p} \) as a decomposition subgroup of \( G_{\mathbb{Q}} \). Let \( K \) be an imaginary quadratic field having fundamental discriminant \(-D\) in which \( p \) splits. Given a finite order character \( \psi : G_K \to \bar{\mathbb{Q}}_p^* \to \bar{\mathbb{Q}}_p^* \), we consider the anti-cyclotomic character \( \varphi = \psi \cdot \bar{\psi}^{-1} \), where \( \bar{\psi} \) denotes the internal Galois conjugate of \( \psi \) by the complex conjugation \( \tau \in G_{\mathbb{Q}} \backslash G_K \). We assume that \( \varphi|_{G_{\mathbb{Q}_p}} \) is trivial, but \( \varphi \) is not. Under these assumptions, the newform on \( GL_2 / \mathbb{Q} \) obtained by automorphic induction from \( K \) to \( \mathbb{Q} \) of \( \psi \) is a weight 1 theta series \( \theta_\psi \) of level \( D \cdot N_{K / \mathbb{Q}}(\text{cond}(\psi)) \) and central character \( \varepsilon = \varepsilon_K \cdot \chi_\psi \circ \text{Ver} \). Here \( \varepsilon_K \) denotes the quadratic Dirichlet character attached to \( K / \mathbb{Q} \), and \( \text{Ver} \) denotes the transfer homomorphism. Let \( f = \sum_{n \geq 1} a_n q^n \) denote the unique \( p \)-stabilization of \( \theta_\psi \) (see \( \text{(7)} \)).

Let \( \mathcal{L}_1 \) and \( \mathcal{L}_{\varphi,\lambda} \) be the \( p \)-adic logarithms of some explicit \( \ell \)-units defined in \((10)\) and \((10)\), respectively. Throughout the paper, we make the following assumption on the anti-cyclotomic \( \mathcal{L} \)-invariants from \([7] \) \( \S 1 \)

\begin{equation}
\mathcal{L}(\varphi) \cdot \mathcal{L}(\bar{\varphi}) \cdot (\mathcal{L}(\varphi) + \mathcal{L}(\bar{\varphi})) \neq 0,
\end{equation}

and we choose a square root \( \xi \) of \( \mathcal{L}(\bar{\varphi}) \mathcal{L}(\varphi)^{-1} \mathcal{S}_\varphi^{-1} \) where the slope \( \mathcal{S}_\varphi \) is as in \((8)\). One should observe that \((1)\) is true for \( \varphi \) quadratic, and that in all other cases the Schanuel Conjecture predicts that \( \mathcal{L}(\varphi) \) and \( \mathcal{L}(\bar{\varphi}) \) are algebraically independent.
Theorem A. There exists a basis \{f^\ell_\varnothing, f^{\ell,\dagger}_\varnothing\} of \(S^I_{w(f)}[f]_0\) whose \(q\)-expansion is as follows:

(i) For any prime \(\ell \neq p\) splitting in \(K\) as \(\mathfrak{p} \cdot \mathfrak{q}\), one has \(a_\ell(f^\ell_\varnothing) = 0\) and

\[
a_\ell(f^{\ell,\dagger}_\varnothing) = (\mathcal{L}_\ell - \mathcal{L}_\varnothing)(\psi(0) - \psi(0)).
\]

(ii) For any prime \(\ell | D\), letting \(\nu = 2\) if \(p = 2\), and \(\nu = 1\) otherwise, one has \(a_\ell(f^\ell_\varnothing) = 0\) and

\[
a_\ell(f^{\ell,\dagger}_\varnothing) = \frac{2\psi(0) \cdot \xi \cdot \mathcal{L}(\varnothing) \cdot \mathcal{L}_{\psi,\ell}}{\log_p(1 + p^\nu) \cdot (\mathcal{L}(\varnothing) + \mathcal{L}(\varnothing))}.
\]

(iii) For any prime \(\ell\) inert in \(K\), if \(\ell | N\) then \(a_\ell(f^\ell_\varnothing) = a_\ell(f^{\ell,\dagger}_\varnothing) = 0\), whereas if \(\ell \nmid N\) then

\[
a_\ell(f^\ell_\varnothing) = 0, \text{ and } a_\ell(f^{\ell,\dagger}_\varnothing) = \frac{2 \cdot \xi \cdot \mathcal{L}(\varnothing) \cdot \mathcal{L}_{\psi,\ell}}{\log_p(1 + p^\nu) \cdot (\mathcal{L}(\varnothing) + \mathcal{L}(\varnothing))}.
\]

(iv) Any form \(\sum_{n \geq 1} a_n^\ell \cdot q^n \in S^I_{w(f)}[f]_0\) satisfies \(a_1^\ell = a_p^\ell = 0\) and the recursive relations:

\[
a_{mn}^\ell = a_m a_n^\ell + a_n a_m^\ell, \text{ for all } (n, m) = 1, \text{ and }\]

\[
a_r^\ell = \begin{cases} a_r^{\ell-1} + a_{r-1}^{\ell-1} - \epsilon(\ell) a_{r-2}^\ell, & \text{for all primes } \ell \nmid Np \text{ and all } r \geq 2, \\ ra_r^{\ell-1} a_\ell^\ell, & \text{otherwise}. \end{cases}
\]

Most of the paper is organized around the proof of the above Theorem. In §1 we summarize the basic properties of \(\mathcal{C}\) while making a detour to define a geometric \(q\)-expansion of a Coleman family at a cusp of the ordinary locus using the overconvergent modular sheaf constructed by V. Pilloni [40]. Using the resulting \(q\)-expansion Principle we prove a perfect Hida duality between the space of Coleman families and the corresponding Hecke algebra. Exploiting this duality and the results of [7] on the local geometry of \(\mathcal{C}\) at \(f\), allows us in §3 to compute infinitesimally the Fourier coefficients of all families passing through \(f\). The resulting formulas involve \(p\)-adic logarithms of algebraic numbers in the field cut out by the projective Galois representation attached to \(f\).

0.4. A mysterious cross-ratio. Let us first formulate the precise problem. We let \(f\) be a \(p\)-ordinary stabilization of a newform of weight \(k \geq 1\) and level \(\Gamma_1(N)\), and denote by \(\alpha \neq 0\) its \(U_p\)-eigenvalue. If \(f\) is \(p\)-regular then one knows that (see §2.2 for more details)

- \(f\) belongs to a unique, up to Galois conjugacy, Hida family \(\mathcal{F}\) and the \(\Lambda = \mathcal{O}_{W,w}^\wedge(f) \simeq \hat{Q}_p[X]\)-algebra \(\mathcal{O}_{\mathcal{F},\ell}^w\) is isomorphic to \(\hat{Q}_p[Y]\), where \(Y^e = X\) for some \(e \geq 1\),
- the corresponding Galois representation \(\rho_{\mathcal{F}} : G_Q \to GL_2(\hat{Q}_p[Y])\) is \(p\)-ordinary and its \(G_{Q_{p'}}\)-stable line reduces modulo \(Y\) to the unique \(G_{Q_{p'}}\)-stable line in \(\rho_f\) such that the arithmetic Frobenius \(\text{Frob}_p\) acts by \(\alpha\) on the (unramified) quotient.

If \(f\) irregular at \(p\) and \(k = 1\), then the restriction of \(\rho_f\) to \(G_{Q_p}\) is scalar, given by an unramified character sending \(\text{Frob}_p\) to \(\alpha\). Galois conjugacy classes of Hida families \(\mathcal{F}\) containing \(f\) are in bijection with the irreducible components of \(\mathcal{C}\) containing \(f\), and

- either \(\mathcal{O}_{\mathcal{F},\ell}^w\) is not a regular ring (e.g. it is not even normal) and \(\rho_{\mathcal{F}}\) might not even admit an ordinary filtration,
or $O_{\mathcal{F}, f} \approx \bar{\mathbb{Q}}_p[Y]$ is a discrete valuation ring and thus $\rho_{\mathcal{F}}$ does admit a (unique) ordinary filtration yielding, when reduced modulo $(Y)$, a well defined line in $\rho_f$, i.e. an element of $\mathbb{P}(\rho_f)$.

The choice of basis for $\rho_f$ allows one to identify $\mathbb{P}(\rho_f)$ with $\mathbb{P}^1(\bar{\mathbb{Q}}_p)$ and each of the finitely many “regular” Hida families containing $f$ picks a well-defined element in it. The aim of this section is to illustrate this phenomenon with some non-trivial examples.

When $f$ is a weight 1 Eisenstein series which is irregular at $p$, it is natural to choose a basis in which $\rho_f$ is reducible and semi-simple. There are two Eisenstein Hida families containing $f$, one having residual slope 0 and the other one $\infty$. The main result in [8] shows that there is a unique cuspidal Hida family $\mathcal{F}$ containing $f$, whose residual slope belongs to $\bar{\mathbb{Q}}_p^\times$. Since one can rescale the vectors of the reducible basis, all values in $\bar{\mathbb{Q}}_p^\times = \mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus \{0, \infty\}$ are allowed, the forbidden values 0 and $\infty$ corresponding to the two $G_{\mathbb{Q}}$-stable lines. One can then recover $\rho_{\mathcal{F}}$ as the universal ordinary deformation of $\rho_f$ endowed with any non-$G_{\mathbb{Q}}$-stable line. A tame analogue of such deformation problems was used by F. Calegari and M. Emerton in [12] to establish an $R = T$ theorem for the weight 2 Hecke algebra at Eisenstein primes.

The situation is more rigid when $\rho_f$ is irreducible. Since $\rho_f$ is odd, a canonical pair of elements of $\mathbb{P}(\rho_f)$ is given by the eigenspaces for the complex conjugation $\tau \in G_{\mathbb{Q}}$.

We now assume that $f$ is a weight 1 cuspform having CM which is $p$-irregular, and use the notations introduced immediately before Theorem A, in particular the number $\xi \in \bar{\mathbb{Q}}_p^\times$.

**Theorem B.** Under the assumption [11], the ordinary lines of the four families containing $f$ are pairwise distinct and their cross-ratio belongs to $\{-1, 1, \frac{1}{2}\}$. Moreover the cross-ratio of the line fixed by the complex conjugation and the three lines in $\rho_f$ obtained by reducing the ordinary lines of $\mathcal{F}$ and the two CM families containing $f$, belongs to the set $\left\{\xi, \frac{1}{\xi}, 1 - \frac{1}{\xi}, 1 - \xi, \frac{1}{\xi - 1}, \xi - 1, \frac{\xi - 1}{\xi}\right\}$.

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References
1. Background on the p-adic eigencurve \( \mathcal{C} \)

1.1. Overconvergent modular forms. Let \( p \) be any prime number. For an integer \( N \geq 4 \) relatively prime to \( p \), we let \( \mathcal{X} \) be the proper smooth modular curve of level \( \Gamma_1(N) \) over \( \mathbb{Z}_p \) and \( \mathcal{E} \to \mathcal{X} \) be the universal generalized elliptic curve. The fiber of \( \mathcal{E} \) above any cusp is given by a certain Néron polygon endowed with \( \Gamma_1(N) \)-level structure.

The invertible sheaf \( \omega \) on \( \mathcal{X} \) is defined as the pull-back of the relative differentials \( \Omega_{\mathcal{E}/\mathcal{X}} \) along the zero section of \( \mathcal{E} \to \mathcal{X} \). The space of classical modular forms of weight \( k \in \mathbb{Z}_{\geq 1} \), level \( \Gamma_1(N) \) and coefficients in a \( \mathbb{Z}_p \)-algebra \( A \) is defined as \( M_k(N; A) = H^0(\mathcal{X}_A, \omega_A^k) \). The \( A \)-module \( M_k(N; A) \) is functorial in \( A \) and commutes with flat base change.

Let \( X^{an} \) be the rigid analytification of the generic fibre \( X = \mathcal{X}_{\mathbb{Q}_p} \) of \( \mathcal{X} \). Note that the properness of \( \mathcal{X} \) over \( \mathbb{Z}_p \) implies that \( X^{an} \) is also the rigid space in the sense of Raynaud. The analytification of the line bundle \( \omega \) is a line bundle on \( X^{an} \) and will still be denoted by \( \omega \).

The ordinary locus \( X(0) \) is the complement of the supersingular residue classes in \( X^{an} \) and can be characterized as the locus where the truncated valuation of the Hasse invariant is 0. More generally, for \( v \in \mathbb{Q}_{>0} \), let \( X(v) \) denote the strict overconvergent neighborhood in \( X^{an} \) of the ordinary locus \( X(0) \) where the (truncated) valuation of the Hasse invariant is \( \leq v \). The space Katz \( p \)-adic modular forms of weight \( k \in \mathbb{Z} \) is the infinite dimensional \( \mathbb{C}_p \)-vector space \( H^0(X(0)_{\mathbb{C}_p}, \omega_{{\mathbb{C}_p}}^k) \), whereas the space of \( p \)-adic overconvergent modular forms was defined by Coleman as the Fréchet \( \mathbb{C}_p \)-vector space:

\[
M^1_k = \lim_{v \to 0} H^0(X(v)_{\mathbb{C}_p}, \omega_{{\mathbb{C}_p}}^k).
\]

The weight space \( \mathcal{W}_p \) is the rigid space over \( \mathbb{Q}_p \) representing homomorphisms \( \mathbb{Z}_p^* \to \mathcal{G}_m \). We consider \( \mathbb{Z} \) as a subset of \( \mathcal{W}_p \) by sending \( k \in \mathbb{Z} \) to the algebraic character \( x \mapsto x^k \). Then \( \mathbb{Z}_{\geq 1} \) is very Zariski dense in the sense that for any affinoid \( \mathcal{U} \) of \( \mathcal{W}_p \), the set \( \mathbb{Z}_{\geq 1} \cap \mathcal{U} \) is either empty or Zariski dense in \( \mathcal{U} \). The same is also true for locally algebraic weights. The space \( \mathcal{W}_p \) is a disjoint union, indexed by the characters of \( (\mathbb{Z}/2p\mathbb{Z})^* \), of copies of the rigid open unit disk \( \{ |z - 1|_p < 1 \} \) representing homomorphisms \( 1 + 2p\mathbb{Z}_p \to \mathcal{G}_m \) (via the image of the topological generator \( 1 + 2p \)). The latter is admissibly covered by the closed disks \( B_m = \{ |z - 1|_p < p^{-1/m} \} \), where the increasing sequence of positive integers \( (n_m)_{m \geq 1} \) is chosen so that the following holds. For any \( m \in \mathbb{Z}_{\geq 1} \) there exists a (unique) character extending the universal character \( \kappa_m : \mathbb{Z}_p^* \to (1 + p^m\mathcal{O}_{\mathbb{C}_p})^* \) and whose restriction to \( (1 + p^m\mathcal{O}_{\mathbb{C}_p})^* \) is analytic.

The invertible sheaf \( \omega \) is representable by \( \text{Hom}_\mathcal{X}(\mathcal{O}_\mathcal{X}, \omega) \) and we denote by \( \pi : \mathcal{I} = \text{Isom}_\mathcal{X}(\mathcal{O}_\mathcal{X}, \omega) \to \mathcal{X} \) the corresponding \( \mathcal{G}_m \)-torsor. The fibers of the rigid analytification \( \pi^{an} : \mathcal{I}^{an} \to X^{an} \) are naturally isomorphic to \( \mathbb{C}_p^* \). For \( k \in \mathbb{Z} \) one can recover \( \omega^k \) as \( \pi^{an}_*(\mathcal{O}_{\mathcal{I}^{an}})[k] \), where \([k] \) means the \( k \)-equivariant sections for the action of \( \mathcal{G}_m \).
In [40 §3], Pilloni showed that there exists \( v_m > 0 \) and an invertible sheaf \( \omega_m \) on \( X(v_m) \times \mathcal{B}_m \) specializing for any \( k \in \mathbb{Z}_{>1} \cap \mathcal{B}_m \) to the automorphic line bundle \( \omega^{\otimes k} \) on \( X(v_m) \). Namely, he constructed an open \( \mathcal{J}_m \) of \( \mathcal{T}^{an}_{X(v_m)} \) endowed with \( \mathbb{Z}_p^\times \)-action such that

\[
\omega_m = (\pi_{\mathcal{J}_m}^* \mathcal{O}_{\mathcal{T}_m} \tilde{\otimes} \mathcal{O}_{\mathcal{B}_m}) [\kappa_m],
\]

where \([\kappa_m] \) means the \( \kappa_m \)-equivariant sections for the action of \( \mathbb{Z}_p^\times \). More precisely, by loc. cit. \( \mathcal{J}_m \) is locally isomorphic for the etale topology on \( X(0) \) to the union of disks

\[
\mathbb{Z}_p^\times \cdot (1 + p^m \mathcal{O}_C) = \bigcup_{y \in (\mathbb{Z}/p^n \mathbb{Z})^\times} B(\tilde{y}, p^{-m}) \subset \mathbb{C}_p^\times
\]

where \( \tilde{y} \in \mathbb{Z}_p^\times \) denotes a lift of \( y \) and \( B(\tilde{y}, p^{-m}) = \{ z \in \mathbb{C}_p | z - \tilde{y} |_p \leq p^{-m} \} \). By definition \( \omega_m \) is locally isomorphic for the etale topology on \( X(0) \) to the eigenspace of \( \mathcal{O}(\mathcal{B}_m) \)-valued locally analytic functions on \( \mathbb{Z}_p^\times \cdot (1 + p^m \mathcal{O}_C) \) which are \( \kappa_m \)-equivariant for the action of \( \mathbb{Z}_p^\times \). This space is clearly generated by the section corresponding to \( \tilde{\kappa}_m \).

It follows that given any open affinoid \( \mathcal{U} \) of \( \mathcal{W}_p \) which is admissible (so it is contained in some \( \mathcal{B}_m \)) there exists an invertible sheaf \( \omega_\mathcal{U} \) on \( X(v) \times \mathcal{U} \) for all \( v > 0 \) sufficiently small.

The correspondence \( U_p \) is defined on the locus \( X \left( \frac{1}{p+1} \right) \) where the canonical subgroup of \( \mathcal{E} \) exists, and it sends \( X(v) \) to \( X \left( \frac{1}{p} \right) \). One can show that it induces a compact operator \( U_p \) on the \( \mathcal{O}(\mathcal{U}) \)-Banach module \( H^0(X(v) \times \mathcal{U}, \omega_\mathcal{U}) \) and that the latter admits a slope decomposition. The \( \mathcal{O}(\mathcal{U}) \)-module of Coleman families of slope at most \( s \in \mathbb{Q}_{>0} \) is defined as

\[
M^{1, \leq s}_\mathcal{U} = \lim_{v \to 0} H^0(X(v) \times \mathcal{U}, \omega_\mathcal{U})^{\leq s}.
\]

By the slope decomposition theorem \( M^{1, \leq s}_\mathcal{U} \) is locally free of finite type as \( \mathcal{O}(\mathcal{U}) \)-module.

**Remark 1.1.** One can extend the definition to the tame levels \( N \leq 3 \), by considering the analogous objects on the modular curve of level \( \Gamma_1(N') \) for \( N' \) a multiple of \( N \) larger than 4 and relatively prime to \( p \), and then taking \( \Gamma_1(N) \)-invariants.

### 1.2. \( q \)-expansions of Coleman families.

In this subsection we describe how one can attach to a Coleman family a geometric \( q \)-expansion, show that it interpolates the \( q \)-expansions of its classical specializations and satisfy the \( q \)-expansion Principle.

The generalized elliptic curve Tate(\( q \)) over \( \mathbb{Z}_p[[q]] \), with 1-gon special fiber and natural \( \Gamma_1(N) \)-level structure defines a morphism \( \text{Spec}(\mathbb{Z}_p[[q]]) \to \mathcal{X} \) (the section \( q = 0 \) corresponds to \( \infty \)). Let \( d^*t \) be the canonical differential of Tate(\( q \)), i.e., a canonical \( \mathcal{O}_{\mathcal{X}, \infty} \)-basis of the completed stalk \( \omega_{\infty}^\wedge \) of \( \omega \) along the section \( \infty : \text{Spec}(\mathbb{Z}_p) \to \mathcal{X} \). In this way, for \( k \in \mathbb{Z} \), one can identify \( (\omega_{\infty}^\wedge)^{\otimes k} \) with \( \mathcal{O}_{\mathcal{X}, \infty} = \mathbb{Z}_p[[q]] \) using the canonical basis \( (d^*t)^{\otimes k} \), allowing one to geometrically define the \( q \)-expansion of an overconvergent modular form.

**Proposition 1.2.** There exists a neighborhood \( \mathcal{V} \) for the etale topology of the cusp \( \infty \in X(0) \) and a section of \( \omega_\mathcal{U}(\mathcal{V} \times \mathcal{U}) \) specializing at any \( k \in \mathbb{Z}_{>1} \cap \mathcal{U} \) to the canonical differential \( (d^*t)^{\otimes k} \). In particular, one can attach to any Coleman family \( \mathcal{F} \in M^{1, \leq s}_\mathcal{U} \) a \( q \)-expansion
\[ \sum_{n \geq 0} a_n(F)q^n \in \mathcal{O}(U)[q] \] interpolating the q-expansions of its classical specializations. Moreover, the q-expansion Principle holds, i.e., the q-expansion map is injective

\[ M^{1, s}_{\mathcal{U}} \to \mathcal{O}(U)[q]. \]

Proof. As already observed, \( \omega_{\mathcal{U}} \) is locally isomorphic for the etale topology on \( X(0) \) to the eigenspace of \( \mathcal{O}(U) \)-valued locally analytic functions on \( \mathbb{Z}_p^* \cdot (1 + p^n \mathbb{O}_{\mathbb{C}_p}) \) which are \( \mathbb{Z}_p^* \)-equivariant with respect to the action of the universal character \( \kappa_{\mathcal{U}} : \mathbb{Z}_p^* \to \mathcal{O}(U)^\times \), a basis being given by the locally analytic character \( \bar{\kappa}_{\mathcal{U}} \). Here \( m \) is chosen so that \( U \subset B_m \).

Since \( \text{Tate}(q) \) is ordinary at \( p \), one can choose a neighborhood \( V \) for the etale topology of the cusp \( \infty \) with a local trivialization of \( \omega \) given by \( d^s t \) and such that the section \( \bar{\kappa}_{\mathcal{U}} \) generates \( \omega_{\mathcal{U}}(V \times \mathcal{U}) \). The specialization of \( \bar{\kappa}_{\mathcal{U}} \) at any \( k \in \mathbb{Z} \cap \mathcal{U} \) corresponds under this construction to the canonical differential \( (d^s t)^{\otimes k} \) generating \( \omega^{\otimes k}(V) \), thus providing the sought-for \( p \)-adic analytic interpolation. This yields the desired trivialization \( \omega_{\mathcal{U}}(V \times \mathcal{U}) = \mathcal{O}(V)\overline{\mathcal{O}(U)} \) together with the natural injection \( \mathcal{O}(V)\overline{\mathcal{O}(U)} \to \mathcal{O}(U)[q] \) given by the localization \( \mathcal{O}(V) \to \mathcal{O}_{X(0), \infty} = \overline{\mathbb{Q}_p}[q] \) at \( \infty \). In this manner we have associated to any Coleman family \( F \in M^{1, s}_{\mathcal{U}} \) a q-expansion \( \sum_{n \geq 0} a_n(F)q^n \in \mathcal{O}(U)[q] \) interpolating the q-expansions of its classical specializations, and thus satisfying a q-expansion Principle. \( \square \)

Remark 1.3. Using similar techniques one can prove a statement analogous to Proposition 1.2 at an arbitrary cusp of \( X(0) \).

1.3. Basic global properties of \( \mathcal{C} \). The eigencurve \( \mathcal{C} \) of tame level \( N \) is admissibly covered by the affinoids attached to the \( \mathcal{O}(U) \)-algebras \( T_{\mathcal{U}} \subset \text{End}_{\mathcal{O}(U)}(M^{1, s}_{\mathcal{U}}) \) generated by the Hecke operators \( T_\ell, (\ell), \ell \mid Np \) and \( U_p \), where \( s \in \mathbb{Q}_{\geq 0} \) and the admissible open affinoid \( \mathcal{U} \subset \mathcal{W}_p \) are both allowed to vary. Henceforth we will use the weight space \( \mathcal{W} \) representing the continuous homomorphisms:

\[ \mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^s \to \mathbb{G}_m, \]

which is endowed with shifted forgetful map to \( \mathcal{W}_p \) and is locally generated over the latter by the diamond operators \( (a), a \in (\mathbb{Z}/N\mathbb{Z})^s \). The shift is made so that \( k \in \mathbb{Z} \) henceforth corresponds to the character \( x \mapsto x^{k-1} \) of \( \mathcal{W} \), and on the level of Iwasawa algebras is given by the automorphism of \( \mathbb{Q}_p[1 + 2p\mathbb{Z}_p] \) sending \( [1 + 2p] \) to \( (1 + 2p)[1 + 2p] \).

The eigencurve \( \mathcal{C} \) is reduced, and it follows from its construction that there exists a flat and locally finite morphism \( w : \mathcal{C} \to \mathcal{W} \), called the weight map. Thanks to the above shift, the classical weight 1 forms, which are the focal point of our study, map under \( w \) to finite order characters equal to the determinant of the corresponding Galois representation (pre-composed with the Artin reciprocity map).

The classical points \( \mathcal{C}^{cl} \) are very Zariski dense in \( \mathcal{C} \) and it has been shown in \( \mathbb{[23]} \) that \( \mathcal{C} \) is proper over the weight space.
By construction of $\mathcal{C}$, there exist bounded global sections $\{T_\ell, U_p\}_{\ell \nmid Np} \subset \mathcal{O}_\ell^\times(\mathcal{C})$ such that the usual application “system of eigenvalues”

$$x \in \mathcal{C}(\mathbb{Q}_p) \mapsto \{T_\ell(x), U_p(x)\}_{\ell \nmid Np}$$

is injective, and produces all systems of eigenvalues for $\{T_\ell, U_p\}_{\ell \nmid Np}$ acting on the space of overconvergent forms with coefficients in $\mathbb{Q}_p$, of tame level $N$, having weight in $\mathcal{W}(\mathbb{Q}_p)$ and a non-zero $U_p$-eigenvalue.

A fundamental arithmetic tool in the study of the geometry of $\mathcal{C}$ is the universal 2-dimensional pseudo-character

$$(\mathcal{C})$$

which is unramified at all $\ell \nmid Np$ and such that $\tau_\mathcal{C}$ maps an arithmetic Frobenius $\text{Frob}_\ell$ to $T_\ell$. This pseudo-character interpolates $p$-adically the traces of semi-simple $p$-adic Galois representations attached to the classical points of $\mathcal{C}$. While these Galois representations are De Rham at $p$, the the semi-simple $p$-adic Galois representation attached to an arbitrary specialization of $\tau_\mathcal{C}$ is only trianguline at $p$.

1.4. **Hecke operators at primes dividing the level.** For each $\ell \mid N$ the module $M_{\mathcal{U}}^{\mathcal{C},\text{cs}}$ is endowed with a $\mathcal{O}(\mathcal{U})$-linear operator $U_\ell$, commuting with $\mathcal{T}_{\mathcal{U}}$. It can be either defined geometrically as a correspondence between modular curves of levels prime to $p$, or by the usual formulas on $q$-expansions, i.e., using $p$-adic interpolation of classical forms. By adding those operators to $\mathcal{T}_{\mathcal{U}}$ one can define the full eigencurve, whose ordinary part is directly related to the Hida Hecke algebras in their most classical definition. The main advantage of working with the full Hecke algebra would become transparent in the next section. The disadvantage is that this bigger algebra is not necessarily reduced. Luckily for us, when working in a neighborhood of a classical cuspidal point we don’t have to choose, as the next proposition shows that the two are locally isomorphic.

**Proposition 1.4.** Any classical cuspidal point $f$ of $\mathcal{C}$ has a neighborhood $\mathcal{U}$ such that for any $\ell \mid N$, one has $U_\ell, \mathcal{T}_{\mathcal{U}}$. 

**Proof.** One follows closely the proof from [4, Prop.7.1] (see also [24]), the main point being the existence (by Nyssen–Rouquier as $\rho_f$ is irreducible) of a two dimensional $\mathcal{O}(\mathcal{U})$-valued Galois representation $\rho_{\mathcal{U}}$ whose trace equals the push-forward $\tau_{\mathcal{U}}$ of the universal pseudo-character $\tau_\mathcal{C}$. The proof of loc.cit. then goes through *mutatis mutandis* (including in possible irregular cases), except for the case dealing with the special (or Steinberg) representations. More precisely, given $\ell \nmid p$ not dividing the level of $f$, one has to exclude the possibility of $f$ being a $p$-adic limit of representation special at $\ell$. If that was the case, then $\rho_{f|\mathbb{Q}_p}$ would be extension of two characters whose quotient is unramified and sends $\text{Frob}_\ell$ to $\ell$. This would contradict, if $f$ is cuspidal, the Ramanujan conjecture, proved by P. Deligne, for $\rho_f$. \[\square\]
1.5. Cuspidal Hida duality. The cuspidal overconvergent submodule of $M^{1,\ell}_{\mathcal{U}}$ is defined as

$$S^{1,\ell}_{\mathcal{U}} = \lim_{\nu \to 0} H^0(X(\nu) \times \mathcal{U}, \omega_{\mathcal{U}}(-D))^\ell_s,$$

where $D$ is the cuspidal divisor of the ordinary locus $X(0)$. Note that the $\mathcal{O}(\mathcal{U})$-submodule $S^{1,\ell}_{\mathcal{U}}$ is $\mathcal{T}_{\mathcal{U}}$-stable and one defines the cuspidal Hecke $\mathcal{O}(\mathcal{U})$-algebra $\mathcal{T}_{\mathcal{U}}^{\text{cusp}}$ as the quotient of $\mathcal{T}_{\mathcal{U}}$ acting faithfully on it. Using $\mathcal{T}_{\mathcal{U}}^{\text{cusp}}$ instead of $\mathcal{T}_{\mathcal{U}}$ one defines the cuspidal eigencurve $\mathcal{C}^{\text{cusp}}$ which is endowed with a closed immersion $\mathcal{C}^{\text{cusp}} \to \mathcal{C}$ of reduced flat rigid curves over $\mathcal{W}$.

Let $f$ be a classical cuspidal point in $\mathcal{C}$. As well known $\mathfrak{w}(f)$ has a neighborhood $\mathcal{U}$ such that $S^{1,\ell}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{U}}^{\ell_s}$ are both free of finite rank as $\mathcal{O}(\mathcal{U})$-modules. For $T \in \mathcal{T}_{\mathcal{U}}^{\ell_s}$ and $\mathscr{G} \in S^{1,\ell}_{\mathcal{U}}$ we consider Hida’s pairing defined by $$(T, \mathscr{G}) = a_1 \left( T, \mathcal{I} \right).$$

**Proposition 1.6.** The pairing $(\cdot, \cdot) : \mathcal{T}_{\mathcal{U}}^{\ell_s} \times S^{1,\ell}_{\mathcal{U}} \to \mathcal{O}(\mathcal{U})$ is $\mathcal{O}(\mathcal{U})$-linear and perfect.

**Proof.** By Proposition 1.4 and the abstract recurrence relations between Hecke operators, one knows that for all $n \in \mathbb{Z}_{\geq 1}$ one has $T_n \in \mathcal{T}_{\mathcal{U}}^{\ell_s}$. Since

$$(T, T_n(\mathscr{G})) = (T_n T, \mathscr{G}) = a_1 \left( T_n T, \mathcal{I} \right) = a_n (T, \mathcal{I})$$

for any $n \geq 1$, the $q$-expansion Principle from Proposition 1.2 shows that the natural $\mathcal{O}(\mathcal{U})$-linear maps

$$S^{1,\ell}_{\mathcal{U}} \to \text{Hom}_{\mathcal{O}(\mathcal{U})\text{-mod}} \left( \mathcal{T}_{\mathcal{U}}^{\ell_s}, \mathcal{O}(\mathcal{U}) \right), \mathscr{G} \mapsto (T \mapsto (T, \mathcal{I}))$$

are both injective and that $S^{1,\ell}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{U}}^{\ell_s}$ have locally the same rank. To prove that $(\mathcal{C})$ is an isomorphism for some (sufficiently small) neighborhood $\mathcal{U}$ of $\mathfrak{w}(f)$, it suffices to proceed by localization at $\mathfrak{w}(f)$. Furthermore, by Nakayama’s lemma one has to prove that the cokernels of $(\mathcal{C})$ vanish residually. Since $S^{1,\ell}_{\mathcal{U}, \mathfrak{w}(f)}$ and $\mathcal{T}_{\mathcal{U}, \mathfrak{w}(f)}^{\ell_s}$ are both free with the same finite rank as $\mathcal{O}(\mathcal{U})_{\mathfrak{w}(f)}$-modules, it suffices to prove that $(\mathcal{C})$ is an isomorphism when $\mathcal{U} = \{ \mathfrak{w}(f) \}$. We then conclude by observing that both $S^{1,\ell}_{\mathfrak{w}(f)}$ and $\mathcal{T}_{\mathfrak{w}(f)}^{\ell_s}$ are finite dimensional $\mathbb{C}_p$-vector spaces having the same dimension, and therefore the surjectivity follows from the injectivity. □

**Remark 1.7.** Similar techniques were used in [3] to establish a perfect duality between $M^{1,\ell}_{\mathcal{U}}^{\ell_0}$ and $\mathcal{T}_{\mathcal{U}}^{\ell_0}$ for some neighborhood $\mathcal{U}$ of a weight one irregular Eisenstein series, the issue being the control of the constant terms. To that effect, we introduced the notion of evaluation of Hida families at the cusps of $X(0)$, and used it to write a “fundamental exact sequence”.
2. Generalized eigenforms at classical points of the eigencurve

2.1. Classical points of \( \mathcal{C} \). A point of \( \mathcal{W} \) is said to be classical if its restriction to some open subgroup of \( \mathbb{Z}_p^k \) is given by the homomorphism \( (x \mapsto x^{-1}) \), for some \( k \in \mathbb{Z}_{>1} \) (such characters are locally algebraic). We let \( \mathcal{W}^{cl} \subset \mathcal{W}(\bar{\mathbb{Q}}_p) \) denote the subset of classical weights. A classical point of \( \mathcal{C} \) always maps to a point of \( \mathcal{W}^{cl} \), but the converse is not necessarily true.

In order to describe the classical points of \( \mathcal{C} \) let us first recall the notion of a \( p \)-stabilization. Given a newform \( f \) of weight \( k \in \mathbb{Z}_{>1} \), and given a non-zero root \( \alpha \) of its Hecke polynomial at \( p \), the corresponding \( p \)-stabilization \( f_\alpha \) is the unique normalized eigenform sharing the same eigenvalues as \( f \) for Hecke operators relatively prime to \( p \) and such that \( U_p \cdot f_\alpha = \alpha f_\alpha \). It can be explicitly constructed as follows. Suppose that the newform \( f(z) = \sum_{n \geq 0} a_n e^{2\pi i nz} \) has level \( \Gamma_1(Mp^l) \), with \( t \in \mathbb{Z}_{>0} \) and \( M \) dividing \( N \), central character \( \varepsilon \), and let \( f^{(p)}(z) = f(pz) \). Then

(i) if \( t = 0 \), then the Hecke polynomial \( X^2 - a_pX + \varepsilon(p)p^{k-1} \) of \( f \) at \( p \) has two (necessarily non-zero, but not necessarily distinct) roots, denoted \( \alpha \) and \( \beta \). The corresponding \( p \)-stabilizations \( f_\alpha = f - \beta f^{(p)} \) and \( f_\beta = f - \alpha f^{(p)} \) both have level \( \Gamma_1(M) \cap \Gamma_0(p) \) and define points \( f_\alpha \) and \( f_\beta \) in \( \mathcal{C}^{cl} \). If those points are distinct, we call them regular, if not, irregular. By an abuse of language we sometime say that \( f \) is regular/irregular at \( p \).

(ii) if \( t > 0 \), then \( f \) is already a \( U_p \)-eigenvector with eigenvalue \( \alpha \), and if it has finite slope, i.e., \( \alpha \neq 0 \), then it defines a point \( f_\alpha \in \mathcal{C}^{cl} \). In this case, the Hecke polynomial being \( X(X - \alpha) \), the point is regular.

The set of classical points \( \mathcal{C}^{cl} \subset \mathcal{C}(\bar{\mathbb{Q}}_p) \) consists of all points \( f_\alpha \) as above (considered with coefficients in \( \bar{\mathbb{Q}}_p \) via \( \iota_p \)) where \( k \in \mathbb{Z}_{>1} \), \( t \in \mathbb{Z}_{>0} \) and \( M \) divides \( N \).

The \( p \)-adic valuation of the \( U_p \)-eigenvalue of a point in \( \mathcal{C}(\bar{\mathbb{Q}}_p) \) is called its slope, and by definition belongs to \( \mathbb{Q}_{>0} \). The slope of a classical weight \( k \) point cannot exceed \( k - 1 \) and is called critical when the equality is reached. The points having slope 0 are called ordinary. The locus of \( \mathcal{C} \) where \( |U_p| = 1 \) is open and closed in \( \mathcal{C} \), is called the ordinary locus of \( \mathcal{C} \) and denoted by \( \mathcal{C}^{\text{ord}} \). The ordinary locus \( \mathcal{C}^{\text{ord}} \) has a formal model given by the universal \( p \)-ordinary reduced Hida Hecke algebra of tame level \( N \) generated by the Hecke operators \( T_\ell \) for all primes \( \ell \nmid Np \) and \( U_p \). Its irreducible components can be described in terms of Hida families (see [21]).

2.2. Local geometry of \( \mathcal{C} \) at classical points of weight at least 2. Recast in rigid geometry, Hida’s famous Control Theorem [35] states that any point of \( \mathcal{C}^{\text{ord}} \) having classical weight \( k \geq 2 \) is classical. Note that ordinary forms of weight with \( k \geq 2 \) are necessarily regular and, when a second \( p \)-stabilization exists, then it is necessarily critical. Coleman’s Control Theorem generalizing Hida’s result, states that any non-critical point of \( \mathcal{C} \) having weight in \( \mathcal{W}^{cl} \) is classical (see [16]). Those results imply that \( \mathcal{C} \) is etale over \( \mathcal{W} \) at any non-critical, regular classical point of weight \( k \geq 2 \), hence it is also smooth at these points. There are no known examples of classical forms of weight \( k \geq 2 \) which are irregular, i.e., for which the Hecke polynomial at \( p \) has a double root. There are only three cases in which \( \mathcal{W} \) could fail to be etale.

at a classical regular point $f$ of weight $k \geq 2$, potentially providing a cuspidal-overconvergent
generalized eigenform.

(i) **Critical slope (a.k.a. evil) Eisenstein points.** Bellaïche and Chenevier proved in $[3]$ that $C$ is smooth and is conjecturally étale over $\mathcal{W}$ at such points.

(ii) **Critical slope CM points.** Bellaïche showed in $[1]$ that the eigencurve is smooth, although ramified over the weight space, at such points. He further showed that Jannsen’s conjecture in Galois cohomology implies that the ramification degree is 2.

(iii) **Critical slope non-CM points.** Breuil and Emerton proved in $[10]$ (see also $[27]$ for a partial result and $[5]$ for a different proof) that $w$ ramifies at a classical weight $k \geq 2$ point of critical slope if, and only if, there exists an ordinary companion form and thus the restriction to $G_{Q_p}$ of the attached $p$-adic Galois representation splits. It is a folklore conjecture, often attributed to R. Coleman and to R. Greenberg, that such non-CM points should not exist (see for example $[13]$).

Note that if an irregular classical weight $k \geq 2$ points were to exist, they would be of non-critical slope and Coleman’s Classicality Theorem $[16]$ would imply that the corresponding generalized eigenspace consists only of classical forms. Thus the second case above is (conjecturally) the only one yielding cuspidal-overconvergent generalized eigenforms, and their $q$-expansions have recently been computed by Hsu $[36]$ (note that the technical condition preceding Theorem 1.1 in loc. cit. appears to be superfluous).

2.3. **Geometry of $C$ at classical points of weight 1 and Hida theory.** Classical weight 1 points in $C$ all belong to $C^{\leq 0}$, while having a critical slope, hence lie are outside the reach of Hida’s Control Theorem. Thus, specializations in weight 1 of a Hida family need not be classical, and according to a result of E. Ghate and V. Vatsal $[29]$ only CM Hida families admit infinitely many classical weight 1 specializations. An explicit bound for the number of classical weight 1 specializations of a non-CM family is given in $[28]$, where the authors also study the converse question, namely as to how many Hida families contain a given classical weight 1 eigenform.

One of the main interest in studying the geometry of the eigencurve at weight one points arises from the question of determining whether there is a unique, up to Galois conjugacy, $p$-adic ordinary Hida family specializing to a given $p$-stabilization of a weight one eigenform. It admits the following geometric reformulation: the uniqueness of such a $p$-adic ordinary Hida family is a consequence of the smoothness of $C$ at that classical weight one point. It is observed in $[25]$ that if a classical weight 1 form is of Klein type which is irregular at $p$, then it is contained in two Hida families having CM by different imaginary quadratic fields, and therefore non-Galois conjugate. It turns out that this is part of a more general phenomenon. Indeed, it is observed in $[27]$ that any $p$-irregular weight 1 form having CM by $K$, belongs to two (non-Galois conjugate) Hida families having CM by $K$, and to at least one another family having no CM by $K$. Thus $p$-irregular weight 1 forms systematically provide counter-examples to the uniqueness of the Hida family.
By a theorem of Deligne and Serre [22, Prop.4.1], to any classical weight one point $f$ one can attach a representation with finite image $\rho_f : \mathbb{Q}_\ell \to \text{GL}_2(\mathbb{Q}_p)$, which is irreducible if, and only if, $f$ is cuspidal. While conjecturally classical points of weight $k \geq 2$ are all expected to be regular (see [17]), the Chebotarev Density theorem applied to $\rho_f$, shows that for each classical weight 1 point $f$ there are infinitely many irregular primes $p$, providing many classical points at which $w$ is not etale (and not even smooth if $f$ has CM).

The main result of [4] asserts that all weight 1 regular points of $\mathcal{C}$ are smooth. Furthermore, if the weight map $w$ is not etale at a regular point $f$ if, and only if, $f$ has multiplication by a real quadratic field. The next subsection is devoted to this case.

2.4. Regular RM case. Let $f$ be the $p$-stabilization of a $p$-regular weight 1 theta series having multiplication by a real quadratic field in which $p$ splits, and let $m$ denote the corresponding maximal ideal of the Hecke algebra. In this case the classical subspace of the generalized eigenspace $S^1_{w(f)}[m] \subset S^1_{w(f)}$ is given by the line $\mathbb{Q}_p \cdot f = S_1(Np)[m]$ which has a natural supplement $S^1_{w(f)}[m^2]_0$ in $S^1_{w(f)}[m^2]$, consisting of cuspforms whose first Fourier coefficient vanishes ($a_1 = 0$). Since $S^1_{w(f)}[m^2]_0$ is naturally isomorphic to the relative tangent space of $\mathcal{C}$ over $W$ at $f$ (i.e., the tangent space of the fiber of $w(f^{-1}(w(f)))$ at $f$), the results of [4] show that $S^1_{w(f)}[m^2]_0$ is a line, having a canonical basis $f^\dagger$. Based on the cohomological computations of [4], the main result of Darmon-Lauder-Rotger [18] computes the $q$-expansion of this genuine generalized weight 1 eigenform $f^\dagger$ and draws some parallels with the famous and still open Hilbert’s twelfth problem over real quadratic fields.

We use the present opportunity to observe that, contrarily to what was claimed in [4] based on a misinterpretation of a result by Cho and Vatsal [14], the expected equality $S^1_{w(f)}[m] = S^1_{w(f)}[m^2]$ remains an open question, equivalent to showing that the ramification index of $w$ at $f$ equals 2 (see [3] for a thorough study of this case).

The geometry of the eigencurve is expected to be more intricate at a weight 1 irregular points and to lead to fascinating applications in Iwasawa theory.

2.5. Eisenstein case. We refer to [3] for a detailed study of this case. It is observed there that regular weight 1 Eisenstein points belong to a unique irreducible component of $\mathcal{C}$ which is Eisenstein and etale over the weight space at such points. In contrast to that, an irregular weight 1 Eisenstein point $f$, attached to an odd Dirichlet character $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}_p^\times$ such that $\phi(p) = 1$, belongs to exactly two Eisenstein components and is in addition cuspidal-overconvergent, i.e. vanishes at all cusps of the multiplicative ordinary locus of the modular curve $X(\Gamma_0(p) \cap \Gamma_1(N))$ corresponding to the $\Gamma_0(p)$-orbit of $\infty$. Thus $f$ belongs to the cuspidal eigencurve $\mathcal{C}^{\text{cusp}}$ which is shown in loc. cit. to be etale at $f$ over the weight space, and hence there exists a unique, up to a Galois conjugacy, cuspidal Hida family $\mathcal{F}$ specializing to $f$, whose Fourier coefficients were determined infinitesimally. Furthermore it is shown that $\mathcal{C}^{\text{cusp}}$ intersects transversally each of the two Eisenstein components containing $f$ (note that the evil weight Eisenstein series of weight $\geq 2$ do not belong to Eisenstein components). Finally,
one computes in loc. cit. the $q$-expansions of a basis $\{f^{\dagger}_{1,\phi}, f^{\dagger}_{\phi,1}\}$ of the space of generalized weight 1 overconvergent eigenforms in terms of $p$-adic logarithms of algebraic numbers, and one sees that these forms are not cuspidal-overconvergent. Indeed, one has $a_0(f^{\dagger}_{1,\phi}) = (\mathcal{L}(\phi) + \mathcal{L}(\phi^{-1}))(L(\phi,0) L(\phi,0))$, where $\mathcal{L}(\phi)$ denotes the cyclotomic $\mathcal{L}$-invariant appearing in the derivative at a trivial zero of the Kubota–Leopoldt $p$-adic $L$-function, thus providing a geometric flavored proof of Gross’ formula $L'_p(\phi_\omega, 0) = -\mathcal{L}(\phi) \cdot L(\phi, 0)$ (see [8] §5).

2.6. Irregular weight one case. Let us first say that this case is still the subject of active research. A weight one cuspidal newform which is irregular at $p$ yields a unique point $f$ on $\mathcal{C}$.

Let $T$ be the completed local ring of $\mathcal{C}$ at $f$ and $S^\dagger_{\mathcal{C}}$ be the $T$-module obtained by localizing and completing $S^\dagger_{\mathcal{C}}$ at the maximal ideal $m$ of $T^0$ corresponding to the system of Hecke eigenvalues of $f$, where $U$ is an admissible affinoid of $\mathcal{W}$ containing $w(f)$. The localization at $m$ of the pairing defined in Proposition [10] gives rise to a perfect pairing

$$T \times S^\dagger_m \rightarrow \Lambda,$$

where $\Lambda$ is the completed local ring of $\mathcal{W}$ at $w(f)$. Specializing in weight $w(f)$, corresponding to the maximal ideal $(X)$ of $\Lambda$, yields a natural isomorphism

$$S^\dagger_{w(f)}[f] := S^\dagger_m/m^i S^\dagger_m \sim \text{Hom}_{\bar{Q}_p} \left( T/(X \cdot T), \bar{Q}_p \right).$$

Since $T/X \cdot T$ is an Artinian $\bar{Q}_p$-algebra, the space $S^\dagger_{w(f)}[f]$ of overconvergent weight $w(f)$ generalized eigenforms is by definition the union over all $i \geq 1$ of its subspaces $S^\dagger_{w(f)}[m^i]$ annihilated by the ideal $m^i$.

Let us observe that the classical subspace of the generalized eigenspace $S^\dagger_{w(f)}[f]$ is given by the plane $\bar{Q}_p \cdot f \oplus \bar{Q}_p \cdot f^{(p)} = S_1(\Gamma_0(p) \cap \Gamma_1(N))[m^2]$ which has a natural supplement $S^\dagger_{w(f)}[m^2]$ of overconvergent weight $w(f)$, consisting of cuspforms whose first and $p$-th Fourier coefficients both vanish ($a_1 = a_p = 0$). As in [24] one can reasonably conjecture that $S^\dagger_{w(f)}[m] = S^\dagger_{w(f)}[m^2]$.

In [19] Darmon, Lauder and Rotger constructed a map

$$S^\dagger_{w(f)}[m^2] \rightarrow H^1(Q, ad^0 \rho)$$

and conjectured that it is an isomorphism. A first evidence was found by Hao Lee [37] in the Klein case, i.e. when $f$ has multiplication by three quadratic fields (two imaginary and one real). The only other case where a full study of the local geometry has been successfully completed is the case of an irregular weight 1 CM form, presented in the following subsection.

2.7. Irregular CM case. We will use without recalling the notations and assumptions introduced in the paragraph preceding Theorem [13].

Since $\psi \neq \bar{\psi}$, the reducibility $\rho_{f|G_K} = \psi \oplus \bar{\psi}$ allows to choose a basis of eigenvectors $(e_1, e_2)$ which is uniquely defined up to individual scaling. Using the complex conjugation $\tau$ to further
impose that \( e_2 = \rho(\tau)e_1 \) determines projectively uniquely this basis and one has:

\[
(6) \quad \rho_{f|G_K} = \begin{pmatrix} \psi & 0 \\ 0 & \overline{\psi} \end{pmatrix}, \quad \rho_{f|G_{\mathfrak{q}}\backslash G_K} = \begin{pmatrix} 0 & \psi(\tau) \\ \psi(-\tau) & 0 \end{pmatrix}.
\]

The local-global compatibility for \( \rho_f = \mathrm{Ind}_K^Q \psi \) yields that

\[
(7) \quad a_\ell = \begin{cases} \psi(l) + \psi(l) &, \text{if } (\ell) = \mathfrak{O} \neq (p) \text{ splits in } K, \\ 0 &, \text{if } \ell \text{ is inert in } K, \\ \psi(l) &, \text{if } | \ell | pD, \end{cases}
\]

which together with the usual recurrence relations \( a_{\ell+1} = a_\ell \cdot a_{\ell} - \varepsilon(\ell)a_{\ell-1} \), for \( r \in \mathbb{Z}_{\geq 1} \), and \( a_{mn} = a_m a_n \), for \( m, n \) relatively prime, uniquely determine \( f \). The above formulas are understood with the convention that \( \psi(l) = 0 \), if \( l \) divides the conductor of \( \psi \), and similarly \( \varepsilon(\ell) = 0 \), if \( \ell \) divides the conductor of \( \rho_f \).

It has been shown in [7] that in addition to belonging to two components of \( \mathcal{C} \) having CM by \( K \), \( f \) also belongs to one or two other components, i.e., \( \mathcal{O}_{\mathcal{C}, f} \), as well as the Hida Hecke algebra localized at \( f \), have exactly three or four minimal primes. According to Hida’s work in Iwasawa theory, crossing points between CM and non-CM families are expected to correspond to zeros of anti-cyclotomic Katz \( p \)-adic \( L \)-functions. In our weight 1 situation, we are in the presence of a so-called “trivial” zero, and one ignores whether this zero is simple or not, i.e. there is no an “anti-cyclotomic” analogue of the famous Ferrero–Greenberg Theorem on the Kubota–Leopoldt \( p \)-adic \( L \)-functions of a Dirichlet character encountered in the Eisenstein case described in §2.5. By using \( p \)-adic geometry, commutative algebra and Galois theoretic tools together, it is shown in [7] that these “anti-cyclotomic” trivial zeros are simple whenever a certain \( \mathcal{L} \)-invariant does not vanish, as predicted by the Strong Four Exponentials Conjecture.

A corollary of the main results of [7] is a proof of the Darmon-Lauder-Rotger Conjecture on \( \mathcal{O} \) being an isomorphism, for all irregular weight 1 CM forms. In the next section we will compute the \( q \)-expansions of a basis \( \{ f^{1}_x, f^{1}_q \} \) of the genuine generalized eigenspace \( S^{1}[m^2]_0 \).

### 3. Overconvergent \( q \)-expansions at irregular CM forms

The purpose of this section is to prove Theorem [A] which is a natural expansion of the results of [7]. We keep the setting and the notations from §2.7 (in particular we recall that \( \nu = 2 \) if \( p = 2 \), and \( \nu = 1 \) otherwise), and we let

\[
\mathcal{L} = \frac{\mathcal{L}(\varphi)}{\log_p (1 + p^\nu) \cdot (\mathcal{L}\nabla(\varphi) + \mathcal{L}(\varphi))}, \quad \text{and} \quad \tilde{\mathcal{L}} = \frac{\mathcal{L}(\varphi)}{\log_p (1 + p^\nu) \cdot (\mathcal{L}(\varphi) + \mathcal{L}(\varphi))}.
\]

#### 3.1. Infinitesimal non-CM Hida families. Under the assumption [H], the results of [7] recalled in §2.7 imply that \( f \) belongs to exactly four irreducible components of \( \mathcal{C} \), all etale over \( \Lambda \), two having CM by \( K \), while the other two corresponding to a Hida family \( \mathcal{F} = \sum_{n \geq 1} a_n(\mathcal{F}) q^n \in \Lambda[q] \) without CM by \( K \), and to its quadratic twist \( \mathcal{F} \otimes \varepsilon_K \).
Consider the unique co-chains $\eta_\varphi : G_K \to \bar{Q}_p$ representing the co-cycles $[\eta_\varphi] \in H^1(K, \varphi)$, normalized so that $\text{res}_p([\eta_\varphi]) = \log_p \in H^1(\bar{Q}_p, \bar{Q}_p)$ and $\eta_\varphi(\gamma_0) = 0$ for a fixed element $\gamma_0 \in G_K$ such that $\varphi(\gamma_0) \neq 0$ (see [2] §1). The slope $\mathcal{J}_\varphi$ is a non-zero $p$-adic number defined by $\text{res}_p([\eta_\varphi]) = \mathcal{J}_\varphi \cdot [\log_p]$. As explained in loc. cit., one has the following formula

$$\mathcal{J}_\varphi = -\frac{\log_p(u_\varphi)}{\log_p(\tau(u_\varphi))},$$

where $\bar{Q} \cdot u_\varphi = (\bar{Q} \otimes \mathcal{O}_H^*)[\varphi]$, and moreover $\mathcal{J}_\varphi = \mathcal{J}_\varphi^{-1}$.

**Proposition 3.1** ([2] §2). Any $G_K/Q$-stable ordinary infinitesimal deformation of $\rho = \begin{pmatrix} \psi & \psi \eta_p \\ 0 & \bar{\psi} \end{pmatrix}$ is reducible, i.e., of the form $\begin{pmatrix} 1+dX & bX \\ 0 & 1+dX \end{pmatrix} \cdot \rho \mod X^2$ and one has a natural isomorphism:

$$t_{\text{ord}} \cong H^1(K, \bar{Q}_p), \quad [(\begin{smallmatrix} a \\ b \\ d \end{smallmatrix})] \mapsto d.$$

Recall the basis $\{\eta_p, \eta_\bar{p}\}$ of $H^1(K, \bar{Q}_p)$ where $\eta_p$, resp. $\eta_\bar{p}$, is unramified outside $p$, resp. $\bar{p}$, and $\text{res}_p(\eta_p) = \text{res}_p(\eta_\bar{p}) = \log_p$. One has $\eta_p + \eta_\bar{p} = \eta_p|_{G_K}$, where $\eta_p, \eta_\bar{p} \in H^1(Q, \bar{Q}_p)$ is such that $\text{res}_p(\eta_p) = \log_p$. Writing $d = x \cdot \eta_p + y \cdot \eta_\bar{p}$ one can represent the relevant subspaces of $t_{\text{ord}}$.

We note that $t_{\mathcal{F}}$ corresponds to the tangent space of a $\Lambda$-adic deformations of $\rho$ having no CM by $K$, whereas the line $x + y = \log_p^{-1}(1 + p')$ corresponds to the deformations whose determinant is infinitesimally equal to $\det(\rho) \cdot \left(1 + \frac{\eta_p}{\log_p(1 + p')}\right)X$. See the closing Remark for a definition of the genuine overconvergent generalized eigenform $f_\psi^1$.

**Proposition 3.2.** There exists a basis where $\rho_{\mathcal{F}}(\tau) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod X^2$, and

$$\rho_{\mathcal{F}|G_K} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathcal{L}\eta_p + \mathcal{L}\eta_\bar{p} \\ \xi \mathcal{L}\eta_\varphi \mathcal{L}\eta_p + \mathcal{L}\eta_\bar{p} \end{pmatrix} \cdot X \mod X^2,$$

where $\xi \in \bar{Q}_p$ is such that $\xi^2 = \mathcal{J}_\varphi \mathcal{F}_{\mathcal{F}}$. Moreover, the $G_{K_p}$-stable line of $\rho_{\mathcal{F}}$ is generated residually by $\xi \cdot e_1 + e_2$ and the corresponding unramified quotient is given by the character

$$\chi_{\mathcal{F}} \equiv \psi \cdot (1 + \mathcal{L}(\eta_p - \eta_\varphi) + \mathcal{L}\eta_\bar{p}) \cdot X \mod X^2.$$

**Remark 3.3.** The choice of $\xi$ is equivalent to a choice between $\mathcal{F}$ and its twist $\mathcal{F} \otimes \xi_K$.
Proof. The claim about $\rho_{\mathcal{E}|G_K}$ follows from the above diagram, the equality $(\xi \mathcal{L})^2 = \mathcal{I}_p \mathcal{L} \mathcal{L}$ and the fact, proved in [7, §3], that its reducibility ideal equals $(X^2)$. One should be careful to observe that $[\eta_\varphi] = \mathcal{I}_p[\eta_\varphi]$, whereas $\eta_\varphi = \mathcal{I}_p\eta_{\varphi}$, and therefore the replacement of $\eta_\varphi$ from loc. cit. with $\mathcal{I}_p\eta_{\varphi}$ requires an infinitesimal change of basis. However $\eta_\varphi = \mathcal{I}_p\eta_{\varphi}$ on $G_K$, which is used to compute the ordinary filtration.

It follows from loc. cit. that $\rho_{\mathcal{E}}(\tau) \equiv \left(0 \cdot \mu^{-1} \mid 0\right)$ mod $X$, for some $\mu \in \bar{\mathbb{Q}}_p^\times$. By rescaling the basis by an element of $1 + \bar{\mathbb{Q}}_p X$ (which would not alter the expression of $\rho_{\mathcal{E}|G_K}$ mod $X^2$), one can find $\mu' \in \bar{\mathbb{Q}}_p$ such that

$$
\rho_{\mathcal{E}}(\tau) \equiv \left(\mu'X \cdot \mu^{-1} \mid -\mu'X\right) \mod X^2.
$$

Computing $\rho_{\mathcal{E}|G_K}(\tau \cdot \tau)$ one finds that:

$$
(\xi \mathcal{L}_{\eta_\varphi + \xi \mathcal{L}_{\eta_\varphi}})(1 \varphi) = \left(0 \cdot \mu^{-1} \mid 0\right) \left(\xi \mathcal{L}_{\eta_\varphi + \xi \mathcal{L}_{\eta_\varphi}}(\varphi)\right) - \left(\xi \mathcal{L}_{\eta_\varphi + \xi \mathcal{L}_{\eta_\varphi}}(\varphi)\right)\left(0 \cdot \mu^{-1} \mid 0\right) + \left(\mu \mu'(\varphi^{-1}) \cdot \mu^{-1} \mu'(\varphi^{-1}) \mid 0\right).
$$

Since $\varphi$ is non-trivial, one finds the above equality implies that $\nu_\ell(0) = 0$ and $\mu^2 = 1$, and if $\mu = 1$, then one can change the signs of $\mu$ and $\xi$ simultaneously. 

\[\square\]

3.2. Some $\ell$-units. Recall the $\mathcal{L}$-invariant $\mathcal{L}_\ell = \frac{-\log_p(u_\ell)}{\text{ord}_1(u_\ell)}$ defined in [7], where $u_\ell \in \mathcal{O}_K[\frac{1}{p}]^\times$ is any element having non-zero $p$-adic valuation. Analogously given a prime $I$ of $K$ we let

$$
\mathcal{L}_I = \frac{-\log_p(u_I)}{\text{ord}_I(u_I)},
$$

where $u_I \in \mathcal{O}_K[\frac{1}{p}]^\times$ is any element whose $I$-adic valuation $\text{ord}_I(u_I)$ is non-zero. Clearly, $\mathcal{L}_I$ only depends on $I$ (and not on the particular choice of $u_I$), and it equals $-\log_p(\ell)$ (resp. $\frac{1}{2} \log_p(\ell)$) when $\ell$ is inert (resp. ramified) in $K$.

We let $H$ denote the splitting field of the anti-cyclotomic character $\varphi$.

Assume now that $\ell$ is inert or ramified in $K$, and that $\varphi$ is unramified at the unique prime $I$ of $K$ above $\ell$. One then has

$$
\varphi|_{G_{K_I}} \text{ is trivial},
$$

i.e. $I$ splits completely in $H$. Choose a prime $\lambda$ of $H$ above $I$. Using (11) one can prove that $(\bar{\mathbb{Q}} \otimes \mathcal{O}_H[\frac{1}{\lambda}]^\times)[\varphi]$ is a $\bar{\mathbb{Q}}$-plane with a basis consisting of $u_{\varphi}$ and

$$
u_{\varphi, \lambda} = \sum_{h \in \text{Gal}(H/K)} \varphi^{-1}(h) \otimes h(u_\lambda) \in (\bar{\mathbb{Q}} \otimes \mathcal{O}_H[\frac{1}{\lambda}]^\times)[\varphi],
$$

where $u_\lambda \in \mathcal{O}_H[\frac{1}{\lambda}]^\times$ is any element whose $\lambda$-adic valuation $\text{ord}_\lambda(u_\lambda)$ is non-zero. We let:

$$
\mathcal{L}_{\varphi, \lambda} = \frac{-\log_p(u_{\varphi, \lambda}) + \mathcal{I}_p \log_p(\tau(u_{\varphi, \lambda}))}{\text{ord}_\lambda(u_\lambda)}
$$

which by (8) does not depend on the particular choice of $u_\lambda$ (see [8]).

Let us now investigate how $\mathcal{L}_{\varphi, \lambda}$ depends on $\lambda$, under the additional assumption that $\ell D \nmid N$, i.e. $\psi$ is unramified at $I$. Since $\text{Gal}(H/K)$ acts transitively on the primes $\lambda$ of $H$ above $\ell$, any such prime is of the form $h(\lambda)$ for some $h \in G_K$. One clearly has $h(u_\lambda) = u_{h(\lambda)}$,
hence \( u_{\varphi,h}(\lambda) = \varphi(h) \cdot u_{\varphi,\lambda} \) and \( \mathcal{L}_{\varphi,h}(\lambda) = \varphi(h) \cdot \mathcal{L}_{\varphi,\lambda} \). With a little more work, one sees that for all \( h \in G_K \) and for any choice of a Frobenius element \( \text{Frob}_\ell \in G_\mathbb{Q} \setminus G_K \) one has
\[
(14) \quad \psi(\tau h \text{Frob}_\ell h^{-1}) = \psi(\tau h \tau \cdot \tau \text{Frob}_\ell h^{-1}) = \varphi(h)^{-1} \psi(\tau \text{Frob}_\ell).
\]
and therefore \( \psi(\tau h \text{Frob}_\ell h^{-1}) \cdot \mathcal{L}_{\varphi,h}(\lambda) = \psi(\tau \text{Frob}_\ell) \cdot \mathcal{L}_{\varphi,\lambda} \). Finally, since \( \tau \text{Frob}_\ell \in G_K \),
\[
(15) \quad \psi(\text{Frob}_\ell \tau) \cdot \mathcal{L}_{\varphi,\tau(\lambda)} = \psi(\tau \text{Frob}_\ell \varphi(\tau \text{Frob}_\ell)^{-1} \cdot \mathcal{L}_{\varphi,\tau(\lambda)} = \psi(\tau \text{Frob}_\ell) \cdot \mathcal{L}_{\varphi,\tau(\text{Frob}_\ell(\lambda))}.
\]
We deduce the following Definition-Lemma.

**Lemma 3.4.** Let \( \ell \) be a prime which is inert of ramified in \( K \) and such that \( \ell D \nmid N \). Choose a prime \( \lambda \) of \( H \) above \( \ell \) and let \( \text{Frob}_\ell \in G_\mathbb{Q} \) be any Frobenius whose image in \( \text{Gal}(H/\mathbb{Q}) \) equals the non-trivial element of its (order 2) decomposition group at \( \lambda \). Then the number
\[
(16) \quad \mathcal{L}_{\psi,\ell} = \psi(\tau \text{Frob}_\ell) \cdot \mathcal{L}_{\varphi,\lambda} \in \bar{\mathbb{Q}}_p
\]
is canonical in that it only depends on \( \ell \) and on \( \psi \), and not on the particular choices of \( \lambda \) and of \( \text{Frob}_\ell \) (as long as the latter two choices are compatible as described above).

### 3.3. Computation at split primes
Assume that the prime \( \ell \) splits in \( K \), and write \( (\ell) = \mathfrak{f} \) with \( \mathfrak{f} \neq \mathfrak{f} \mathfrak{l} \). We recall that
\[
a_{\ell} = \psi(\mathfrak{f}) + \psi(\mathfrak{l}), \quad \text{for } \ell \neq p \quad \text{and} \quad a_p = \psi(p) = \psi(\mathfrak{p}).
\]
We will now determine \( a_{\ell}(\mathcal{F}) \) infinitesimally.

**Proposition 3.5.** For any \( \ell \) splitting in \( K \) as \( \mathfrak{f} \mathfrak{l} \), one has \( \eta_p(\text{Frob}_\ell) = \mathcal{L}_{\mathfrak{f}} \) and
\[
\frac{d}{dX} \bigg|_{X=0} a_{\ell}(\mathcal{F}) = \begin{cases} 
\psi(\mathfrak{f}) \cdot (\mathcal{L}_{\mathfrak{f}} + \mathcal{L}_{\mathfrak{l}}) + \psi(\mathfrak{l}) \cdot (\mathcal{L}_{\mathfrak{f}} + \mathcal{L}_{\mathfrak{l}}) & , \text{if } \ell \neq p, \\
\frac{-\psi(p)}{\log_\mathfrak{f}(1+p^r)} \left( \mathcal{L}_{\mathfrak{f}}(\varphi) \cdot \mathcal{L}_{\mathfrak{f}}(\varphi) + \mathcal{L}_{\mathfrak{l}} \right) & , \text{if } \ell = p.
\end{cases}
\]

**Proof.** By Class Field Theory one has an exact sequence
\[
0 \to \text{Hom}(G_K, \bar{\mathbb{Q}}_p) \to \text{Hom}(\mathcal{O}_K^{\mathfrak{f}p} \times \mathcal{O}_K^{\mathfrak{l}p} \times K_\mathfrak{f}^*, \bar{\mathbb{Q}}_p) \to \text{Hom}(\mathcal{O}_K[\mathfrak{f}]^*, \bar{\mathbb{Q}}_p)
\]
whose first map sends \( \eta_p \) to \( (\log_\mathfrak{f}(0), \eta_p(\text{Frob}_\mathfrak{f}) \cdot \text{ord}_\mathfrak{f}) \). Hence
\[
\log_\mathfrak{f}(u_\mathfrak{f}) + \eta_p(\text{Frob}_\mathfrak{f}) \text{ord}_\mathfrak{f}(u_\mathfrak{f}) = 0
\]
as claimed (see \([\text{10}]\)). Similarly \( \eta_p(\text{Frob}_\mathfrak{l}) = \mathcal{L}_{\mathfrak{l}} \).

One can attach to \( \mathcal{F} \) a \( p \)-ordinary \( \Lambda \)-adic representation \( \rho_\mathcal{F} : G_\mathbb{Q} \to \text{GL}_2(\Lambda) \) whose trace is given by the pushforward of \( \tau_\mathcal{F} \) introduced in \([\text{2}]\). It follows that for any \( \ell \neq p \)
\[
a_{\ell}(\mathcal{F}) = \text{tr} \rho_\mathcal{F}^L(\text{Frob}_\ell), \quad \text{and} \quad a_p(\mathcal{F}) = \chi_\mathcal{F}(\text{Frob}_p),
\]
where \( \chi_\mathcal{F} \) is the unramified character acting on the unramified \( G_{K_\mathfrak{f}} \)-quotient of \( \rho_\mathcal{F} \). The computation of \( \frac{d}{dX} \bigg|_{X=0} a_{\ell}(\mathcal{F}) \) then follows directly from the infinitesimal expression for \( \rho_\mathcal{F} |_{G_K} \) given in Proposition \([\text{3.2}]\). The value \( \frac{d}{dX} \bigg|_{X=0} a_p(\mathcal{F}) \) is computed similarly using the formulas
\[
(\eta_\mathcal{F} - \eta_p)(\text{Frob}_p) = \mathcal{L}_{\mathfrak{f}}(\varphi) + \mathcal{L}_{\mathfrak{l}} \quad \text{and} \quad \eta_p(\text{Frob}_p) = -\mathcal{L}_{\mathfrak{l}} \text{ established in } [\text{7}]. \quad \square
3.4. Computation at inert primes. We now turn to the case of a prime $\ell$ which is inert in $K$. If $\ell \nmid N$, then we let $\text{Frob}_\ell \in G_Q \backslash G_K$ denote a Frobenius element and let $\lambda$ denote the resulting place of $H$ above $\ell$, i.e. $\text{Frob}_\ell^J \in G_H$ defines a Frobenius element at $\lambda$, denoted $\text{Frob}_\lambda$. We have seen that $a_\ell = 0$ and the aim of this section is to express infinitesimally $a_\ell(\mathcal{F})$ in terms of logarithms of $\ell$-units.

**Proposition 3.6.** Assume that $\ell$ is inert in $K$. If $\ell \mid N$, then $a_\ell(\mathcal{F}) = 0$. If $\ell \nmid N$ then

(i) $\eta_\ell(\text{Frob}_\lambda) = \mathcal{L}_{\psi,\lambda}$, and

(ii) $\left. \frac{d}{dx} \right|_{x=0} a_\ell(\mathcal{F}) = \xi \mathcal{L} \cdot \mathcal{L}_{\psi,\ell}$.

**Proof.** If $\ell$ divides $N$ then it also divides the conductor of $\psi$, hence $\rho_f^J = \{0\}$ and $a_\ell = 0$. Then $a_\ell(\mathcal{F}) = 0$ as well, since by [24, §6] all classical specializations of a Hida family $\mathcal{F}$ share the same local type at $\ell$.

(i) Letting $\pi \mid p$ be the prime of $H$ given by $\iota_p$, Class Field Theory yields an exact sequence

$$0 \to \text{Hom}(G_H, \bar{Q}_p) \to \text{Hom} \left( \prod_{\text{hGal}(H/K)} (O_H^\times, \bar{Q}_p), \text{Hom}(\mathcal{O}_H[\frac{1}{X}], \bar{Q}_p) \right) \to \text{Hom}(\mathcal{O}_H[\frac{1}{X}], \bar{Q}_p),$$

sending $\eta_{\psi G_H} \in \text{Hom}(G_H, \bar{Q}_p)[\varphi^{-1}]$ to $((\varphi^{-1}(h)(1, \mathcal{F}_\psi) \log_p)_{\text{hGal}(H/K)}, \eta_{\psi}(\text{Frob}_\lambda) \cdot \text{ord}_\lambda)$. The triviality on $u_\lambda \in \mathcal{O}_H[\frac{1}{X}]^\times$ yields by [12] and [13] the desired equality

$$\log_p(u_{\psi,\lambda}) + \mathcal{F}_\psi \log_p(\tau(u_{\psi,\lambda})) + \eta_{\psi}(\text{Frob}_\lambda) \text{ord}_\lambda(u_\lambda) = 0.$$

(ii) By Proposition 3.2 for any $g \in G_Q \backslash G_K$ one has $\rho_g(g) = \left( \begin{array}{cc} 0 & \psi(g \tau) \\ \psi(g)(\tau) & 0 \end{array} \right)$ and therefore

$$\rho_{\mathcal{F}}(g^2) = \rho_{\mathcal{F}}(g) \cdot \rho_{\mathcal{F}}(g) \equiv \left( \begin{array}{cc} * & \psi(g \tau) \cdot \text{tr}(\rho_{\mathcal{F}}(g)) \\ \psi(\tau g) \cdot \text{tr}(\rho_{\mathcal{F}}(g)) & * \end{array} \right) \quad (\text{mod } X^2).$$

Comparing the expression of the upper right coefficient of $\rho_{\mathcal{F}}(g^2)$ with Proposition 3.2 one finds:

$$\text{tr} \rho_{\mathcal{F}}(g) = \xi \mathcal{L} \psi(\tau g) \eta_{\psi}(g^2) X \quad (\text{mod } X^2),$$

for all $g \in G_Q \backslash G_K$.

Applying this to $g = \text{Frob}_\ell$, so that $g^2 = \text{Frob}_\lambda$, one deduces that

$$\left. \frac{d}{dx} \right|_{x=0} a_\ell(\mathcal{F}) = \xi \mathcal{L} \psi(\tau \text{Frob}_\ell) \eta_{\psi}(\text{Frob}_\lambda).$$

The claim then follows from (i), in view of [16].

3.5. Computation at ramified primes. Finally, we turn to the case of a prime $\ell$ which is ramified in $K$. We let $\text{Frob}_\lambda \in G_H$ denote a Frobenius element at a place $\lambda$ of $H$ above $\ell$. We have seen that $a_\ell = \psi(1)$ and the aim of this section is to express infinitesimally $a_\ell(\mathcal{F})$ in terms of logarithms of $\ell$-units.

**Proposition 3.7.** Assume that $\ell$ is ramified in $K$.

(i) $\eta_\ell(\text{Frob}_\lambda) = \eta_\ell(\text{Frob}_\ell) = \mathcal{L}_\ell$ and $\eta_\ell(\text{Frob}_\lambda) = \mathcal{L}_{\psi,\lambda}$,

(ii) $\left. \frac{d}{dx} \right|_{x=0} a_\ell(\mathcal{F}) = \psi(1) \cdot \left( \frac{-\log_\ell(\ell)}{2 \log_\ell(1 + \psi)} + \xi \mathcal{L} \cdot \mathcal{L}_{\psi,\ell} \right)$. 

\[\square\]
\[(iii) \quad \frac{d}{dx}\bigg|_{x=0} a_e(\mathcal{F} \otimes \varepsilon_K) = \psi(1) \cdot \left(\frac{-\log_p(2\ell)}{2\log_p(1+p^\nu)} - \xi \mathcal{L} \cdot \mathcal{L}_\psi, \ell\right)\text{.}
\]

**Proof.** If \(\ell D \mid N\), then our convention as that \(\psi(1) = 0\) and applying the same argument as in Proposition 3.5 yields \(a_e(\mathcal{F}) = 0\). Henceforth we assume that \(\ell D \nmid N\).

(i) The restriction of \(\eta_p, \eta_\wp\) and \(\eta_\psi\) (see (11)) to the inertia at \(\wp\) belongs to \(\text{Hom}(\mathcal{O}_K^\times, \bar{\mathbb{Q}}_p) = \{0\}\).

Therefore their values at \(\text{Frob}_\wp\) are well defined (depending on the choice of \(\Lambda | I\) for \(\eta_\psi\)) and can be computed using Class Field Theory exactly as in Propositions 3.3 and 3.6.

(ii) It follows from (i) and Proposition 3.2 that \(\rho_\wp \equiv \rho_\mathcal{F} \pmod{\mathbb{X}^2}\) has an unramified line generated by the vector \(e_1 + \psi(\text{Frob}_\wp)\), where the restriction of \(\text{Frob}_\wp \in \mathbb{G}_\mathbb{Q}\) to \(H\) is the non-trivial element of \(\text{Gal}(H\lambda/\mathbb{Q}_\ell) = I(H\lambda/\mathbb{Q}_\ell)\). Using again (i) and Proposition 3.2 together with (10), yields

\[
a_e(\mathcal{F}) = (1, 0) \rho_\mathcal{F}(\text{Frob}_\lambda) \left(\psi(\tau \text{Frob}_\wp)\right) \equiv \psi(1)(1, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \xi \mathcal{L}_\psi \eta_\wp(\text{Frob}_\lambda) & (\xi \mathcal{L} \eta_\wp(\text{Frob}_\lambda)) \mathcal{L} \eta_\wp + (\xi \mathcal{L}_\psi \eta_\wp(\text{Frob}_\lambda)) \mathcal{L} \eta_\wp = 1 \end{pmatrix} \psi(\tau \text{Frob}_\wp) \equiv \psi(1) \cdot (\mathcal{L} + \mathcal{L}_\psi) \cdot \mathcal{L} \cdot \mathcal{L}_\psi, \lambda \cdot X \quad (\text{mod } \mathbb{X}^2)\text{.}
\]

(iii) One proceeds as in (ii) noting that the unramified line of \(\rho_\wp \otimes \varepsilon_K \equiv \rho_\mathcal{F} \otimes \varepsilon_K\) (mod \(\mathbb{X}\)) is generated by \(e_1 - \psi(\text{Frob}_\wp)\). Since the Hida family \(\mathcal{F}\) has local type \(\mathbb{1} \oplus \varepsilon_K\) at \(\ell\), one deduces that \(\frac{a_e(\mathcal{F})a_e(\mathcal{F} \otimes \varepsilon_K)}{\psi(1)} = \left(\frac{\det \rho_\mathcal{F}}{\det \rho_\wp}\right)(\text{Frob}_\lambda)\) which concords with the above congruence. \(\square\)

3.6. **On the \(q\)-expansion of CM families specializing to \(f\).** Let \(\mathcal{E}_K^{(p)}(p^\infty)\) denotes the \(p\)-primary part of the ray class group of \(K\) of conductor \(p^\infty\). The torsion-free quotient \(\mathcal{E}_K^{(p)}(p^\infty)_{/\text{tor}}\) is the Galois group of the unique \(p\)-ramified \(\mathbb{Z}_p\)-extension of \(K\). We introduced in \(\mathcal{H} 3\) a \(p\)-adic avatar \(G_K \to \mathcal{E}_K^{(p)}(p^\infty)_{/\text{tor}} \to \mathbb{Z}_p^\times\) of a Hecke character of infinity-type \((1, 0)\), further used to define of a universal character \(G_K \to \mathcal{E}_K^{(p)}(p^\infty)_{/\text{tor}} \to \mathbb{Z}_p^\times[\mathcal{E}_K^{(p)}(p^\infty)_{/\text{tor}}]^\times\) interpolating \(p\)-adically Hecke characters of \(K\) of infinite-type \(\{(k, 0); k \in \mathbb{Z}\}\). Its localization yields a character \(\chi_p : G_K \to \mathbb{A}^\times\) such that

\[
\chi_p \equiv 1 + \frac{\eta_p}{\log_p(1 + p^\nu)} X \quad (\text{mod } \mathbb{X}^2)\text{.}
\]

It is easy to check that \(\chi_p \bar{\chi}_p\) extends to \(G_\mathbb{Q}\) and equals the universal cyclotomic character \(\chi_p\), up to an automorphism of \(\Lambda\).

There exist two CM families \(\Theta_\psi\) and \(\Theta_\psi\) specializing to \(f\) in weight one and such that their attached \(\Lambda\)-adic representation is given by \(\text{Ind}_K^\mathbb{Q} \psi \chi_p\) and \(\text{Ind}_K^\mathbb{Q} \bar{\psi} \chi_p\) respectively, hence in particular their Fourier coefficients vanish at all primes \(\ell\) inert in \(K\). Since the CM-line given by \(y = 0\) in (19) corresponds to the tangent space of \(\Theta_\psi\), if one lets \(\mathcal{L} = 0\) in Proposition 3.5 then one obtains that for any prime \(\ell \neq p\) splitting in \(K\)

\[
\frac{d}{dx}\bigg|_{x=0} a_e(\Theta_f) = \frac{\psi(1) \mathcal{L} + \psi(1) \mathcal{L}_\psi}{\log_p(1 + p^\nu)}, \quad \text{and} \quad \frac{d}{dx}\bigg|_{x=0} a_p(\Theta_\psi) = -\frac{\psi(p) \mathcal{L}_\psi}{\log_p(1 + p^\nu)}\text{.}
\]
Finally, for any $\ell \mid D$, one has (by letting $\mathcal{L} = 0$ in Proposition 3.7):

$$
\left. \frac{d}{dx} \right|_{x=0} a_\ell(\Theta_\psi) = \frac{\psi(i) \mathcal{L}_i}{\log p (1+p^\nu)}.
$$

### 3.7. Generalized eigenforms at weight 1 CM points of the eigencurve.

Recall from §2.6 that one can attach to $f$ a generalized eigenspace $S_{w(f)}^1[f] = S_{w(f)}^1[m]$.

One clearly has $S_{w(f)}^1[m] = \mathbb{O}_p \cdot f$ and we have already observed that classical subspace of $S_{w(f)}^1[f]$ has a basis $\{f, \theta_\psi\}$ whose elements belong to $S_{w(f)}^1[m^2]$.

Under the running assumption (1), it is shown in [7] that $\dim S_{w(f)}^1[f] = \dim S_{w(f)}^1[m^2] = 4$ and hence the space $S_{w(f)}^1[f,0] = S_{w(f)}^1[m^2,0]$ of genuine overconvergent generalized eigenforms defined in §2.6 is two-dimensional. We consider the following forms in $S_{w(f)}^1[f]$:

$$
\begin{align*}
 f^\dagger_{\mathcal{F}} &= \frac{d}{dx} \bigg|_{x=0} \left( \mathcal{F} - \mathcal{F} \otimes \varepsilon_K \right), \quad \text{and} \quad f^\dagger_\Theta = \log_p (1+p^\nu) \cdot \frac{d}{dx} \bigg|_{x=0} (\Theta_\psi - \Theta_{\bar{\psi}}).
\end{align*}
$$

A direct computation, based on the $q$-expansions Principle (see Proposition 1.2) implies the following linear relation in $S_{w(f)}^1[f]$:

$$
\begin{align*}
 \frac{d}{dx} \bigg|_{x=0} \left( \log_p (1+p^\nu) \right) (\mathcal{F} + \mathcal{F} \otimes \varepsilon_K) - \mathcal{L}_\Theta - \mathcal{L}_{\bar{\Theta}} &= \frac{\mathcal{L}(\phi) \cdot \mathcal{L}(\bar{\phi})}{\mathcal{L}(\phi) + \mathcal{L}(\bar{\phi})} (f - \theta_\psi)
\end{align*}
$$

**Proof of Theorem A**  Parts (i)-(iii) are direct consequence of Propositions 3.5, 3.6 and 3.7 in view of the $q$-expansion Principle (see Lemma 3.3 for the definition of $\mathcal{L}_{\psi,\ell}$). Part (iv) results from the well-known relations between the abstract Hecke operators.

**Proof of Theorem B**  In the basis $(e_1, e_2)$ from §2.7, $\rho_f(\tau)$ fixes the vector $e_1 + e_2$, while the ordinary line of $\Theta_\psi$, resp. $\Theta_{\bar{\psi}}$ is spanned by $e_1$, resp. $e_2$. Finally by Proposition 3.2 the ordinary line of $\mathcal{F}$ is spanned residually by $\xi \cdot e_1 + e_2$, allowing us to compute the desired cross-ratio as follows:

$$
[e_1 + e_2, \xi \cdot e_1 + e_2; e_1, e_2] = \left[ \begin{array}{c} 1 \ 1 \\ 1 \ 0 \end{array} \right] \cdot \left[ \begin{array}{c} 1 \ 1 \\ \xi \ 0 \end{array} \right] = \xi.
$$

**Remark 3.8**  (The point $f^\dagger_\psi$ on (23)). The Galois representation $\rho$ from Proposition 3.1 whose semi-simplification is given by $\rho_f|_{G_K}$ has a unique ordinary line and the tangent space of the corresponding deformation problem is represented in (23). It is clear that it has a unique CM deformation given by $\Theta_\psi$ and that both $\mathcal{F}$ and its twist $\mathcal{F} \otimes \varepsilon_K$ yield the same non-CM deformation. Then $f^\dagger_\psi = \frac{d}{dx} \bigg|_{x=0} \left( \mathcal{F} - \Theta_\psi \right)$ is a genuine overconvergent generalized eigenform, and its Fourier coefficient at a prime $\ell$ is given by

$$
\begin{align*}
 a_\ell(f^\dagger_\psi) = \mathcal{L}_\ell \cdot \\
 \left\{ \begin{array}{ll}
 (\psi(i) - \psi(i)) \cdot (\mathcal{L}_i - \mathcal{L}_i^\dagger), & \text{if } \ell \neq p \text{ splits as } 1 \cdot \overline{1}, \\
 \xi \cdot \mathcal{L}_{\psi,\ell}, & \text{if } \ell \nmid N \text{ is inert} \\
 -\psi(p) \cdot \mathcal{L}(\phi), & \text{if } \ell = p, \\
 \psi(i) \cdot \xi \cdot \mathcal{L}_{\psi,\ell}, & \text{if } \ell \mid D.
\end{array} \right.
\end{align*}
$$
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