TOWARDS A HEAT KERNEL EXPANSION FOR THE
ELECTROMAGNETIC FIELD INTERACTING WITH A
DIELECTRIC BODY OF ARBITRARY FORM

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The results on the heat kernel expansion for the electromagnetic field in the background of dielectric media are briefly reviewed. The common approaches to the calculation of the heat kernel coefficients are discussed from the viewpoint of their applicability to the electromagnetic field interacting with dielectric body of arbitrary form. Using the toy-model of scalar photons we develop multiple reflection expansion method which seems the most promising one when the field obeys dielectric-like matching conditions on an arbitrary interface and show that the heat kernel coefficients are expressible through geometric invariants of the latter.

1 The review of results and approaches

Studying the heat kernel $K(x, y; t; L) = \langle x | \exp(-t L) | y \rangle$ provides us with various information concerning operator $L$. Of special interest for the QFT under the influence of external conditions is the heat kernel expansion in powers of the small parameter $t$

$$K(t|L_D)|_{t \to 0} \sim (4\pi t)^{-D/2} \sum_{n=0}^{\infty} t^{n/2} B_{n/2},$$

as the coefficients $B_{n/2}$ govern the short distance behaviour of the propagator and define one-loop divergences and counterterms. Moreover the heat kernel is a powerful tool for analysing quantum anomalies and various perturbative expansions of effective action (see 1 and references therein).

To derive the heat kernel expansion for the electromagnetic field interacting with dielectric body of arbitrary form is an objective motivated primarily by the necessity of clarifying the situation with the Casimir effect in dielectrics. Many authors have devoted their efforts to the Casimir energy calculations for dielectric bodies, the results appeared to be controversial.

The electromagnetic field in unbounded media with dielectric permittivity $\varepsilon$ and magnetic permeability $\mu$ smoothly depending on coordinates may be treated as propagating in the curved space with the effective metric defined by $c(x)^2 = 1/(\varepsilon(x)\mu(x))$. Then the standard formulae 2 expressing the local coefficients $B_k(x, x)$ through the polynomials of manifold’s geometric invariants are valid 3. As the medium is unbounded the coefficients with half-integer numbers are absent in this heat kernel expansion. From the heat kernel expansion obtained in 3 it follows that there is no cancellation of ultraviolet
divergences between ghosts and "nonphysical" components of the vector potential. Moreover the coefficient $B_2$ whose vanishing in 3-dimensional space could assure the uniqueness of the renormalisation for one-loop divergences is nonzero.

If the dielectric body possesses a curved boundary, then the derivation of the heat kernel expansion is getting cumbersome. By now the expansions were obtained only for the boundaries with high symmetry. Dielectric balls and cylinders were considered in \[4\]. The non-vanishing coefficient $B_2$ is the common feature for material balls and cylinders. It means that in general case it is difficult to fix the finite part of the ground-state energy and the results of different calculations may not coincide. The account for the dispersion \[9\] also does not cure the arbitrariness which remains in the finite part of the ground state energy after the removal of the diverging contributions.

Among the balls and cylinders there are two known exceptions. Under special choice of the parameters $\varepsilon$ and $\mu$ for the body and the surrounding medium, namely when $\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^2$, all leading heat kernel coefficients except $B_{3/2}$ are equal to zero. The second exception is the vanishing of $B_2$ in the dilute approximation, i.e., to the order $(\varepsilon_1 - 1)^2$ for $\varepsilon_1 \to 1$ and $\varepsilon_2 = \mu_1 = \mu_2 = 1$.

Being concerned with the heat kernel expansion for a dielectric body of arbitrary shape one is unable to use the spherical or cylindrical symmetry of the boundary for obtaining the eigenfrequency equations in terms of the Bessel functions. Therefore the analysis of the poles for the corresponding spectral zeta function and subsequent derivation of the heat kernel coefficients \[6\], \[7\], \[8\] as the residues in these poles might hardly be accomplished.

The DeWitt iterative procedure was historically the first one used in QFT to obtain the heat kernel coefficients for curved manifolds. The generalization of this method for the manifolds with boundaries makes it too tedious and moreover requires certain assumptions regarding the general form of the heat kernel expansion. When considering the electromagnetic field with dielectric matching conditions we are lacking intuition which guided the authors of \[10\] in Dirichlet and Neumann case.

In the last decade Gilkey’s method, which involves constructing the heat kernel coefficients of all geometric invariants allowed from dimensional viewpoint and finding numerical coefficients in front of them, proved to be the most powerful for manifolds with boundaries. The advantages of this method emerge as it is applied to problems with singularities concentrated on the boundary such as $\delta_\Sigma$-shaped background potential (domain walls) or non-smooth normal derivatives of the metric (brane-world scenario).

Gilkey’s technique admits non-smooth derivatives of the metric requiring however the metric itself to be smooth. Therefore the method is inapplicable for compound dielectrics where the effective metric expressed through $\varepsilon(x)$ jumps on the boundary making the problem ill-defined. One has to replace it
by a pair of spectral problems on the sides $M^\pm$ of the boundary $\Sigma$ supplied with suitable matching conditions. After that it seems reasonable to restate obtained problems in the language of integral equations and to obtain the heat kernel as a sum of generalized multiple reflection expansion.

2 Multiple reflection expansion

The multiple reflection expansion method for the heat kernel has its origin in the potential theory for parabolic operators\[1\]

The solution of the heat equation with initial condition $K(x,y;0) = \delta(x,y)$ may be written in the form

$$K(x,y; t) = K^0(x,y; t) + \alpha_1 V(x,y; t) + \alpha_2 W(x,y; t), \quad (2)$$

$$V(x,y; t) = a^2 \int_0^t d\tau \int_\Sigma dz K^0(x,z; t-\tau) \mu(z,y; \tau),$$

$$W(x,y; t) = a^2 \int_0^t d\tau \int_\Sigma dz K^0(x,z; t-\tau) \frac{\partial}{\partial n_z} \nu(z,y; \tau)$$

where $K^0$ is the free heat kernel, $V$ and $W$ are called respectively simple and double layer heat potentials. For Dirichlet (Neumann) problem it is convenient to choose $\alpha_1 = 0$ ($\alpha_2 = 0$).

The double layer potential and normal derivative of the simple layer potential are discontinuous on the interface $\Sigma$

$$\frac{\partial}{\partial n_x} V(x,y; t) \bigg|_{\Sigma^+} = \frac{\partial}{\partial n_x} V(x,y; t) \bigg|_{\Sigma^-} \pm \frac{1}{2} \mu(x,y; t), \quad (3)$$

$$W(x,y; t) \bigg|_{\Sigma^+} = W(x,y; t) \bigg|_{\Sigma^-} \pm \frac{1}{2} \nu(x,y; t). \quad (4)$$

Substituting the solution (2) into the boundary condition under consideration with account of (3) or (4) one arrives at the integral equation defining the density $\mu$ (or $\nu$) of simple (or double) layer potential. The iterative solution of this integral equation is called multiple reflection expansion (MRE), where each $n$-th term corresponds to the $n$-th interaction with the boundary.

The MRE is convergent for most of physically reasonable boundary conditions sometimes despite of the absence of a small expansion parameter. There may exist various MREs, however as the heat kernel is unique the summation of all possible true ones should result in the same answer.

One can use the MRE as a tool for obtaining the heat kernel coefficients on condition that just a finite number of leading interactions with the boundary contributes to a coefficient with a finite index.
3 Dielectric-like toy model: scalar photons

Here we construct the MRE for the heat kernel of massless scalar field propagating in conformally flat 3D-space with conformal factor behaving like a step-function as some interface Σ is crossed (scalar photons in compound dielectric). The heat kernel \( K(x, y; t) \) may be decomposed in four parts depending on the position of the points \( x \) and \( y \)

\[
K(x, y; t) = \begin{cases} 
K_{++}(x, y; t) & x \in M_+, y \in M_+ \\
K_{+-}(x, y; t) & x \in M_+, y \in M_- \\
K_{-+}(x, y; t) & x \in M_-, y \in M_+ \\
K_{--}(x, y; t) & x \in M_-, y \in M_-
\end{cases}
\]  

which satisfy the heat equations

\[
\left[ \frac{\partial}{\partial t} - a^2 \Delta_x \right] \{ K_{++} \} = 0, \quad \left[ \frac{\partial}{\partial t} - a^2 \Delta_x \right] \{ K_{--} \} = 0
\]

and are glued together by matching conditions

\[
K_{++} \bigg|_{\Sigma^+} = K_{--} \bigg|_{\Sigma^-}, \quad \lambda_+ \frac{\partial K_{++}}{\partial n_x} \bigg|_{\Sigma^+} = \lambda_- \frac{\partial K_{--}}{\partial n_x} \bigg|_{\Sigma^-},
\]

\[
K_{+-} \bigg|_{\Sigma^+} = K_{-+} \bigg|_{\Sigma^-}, \quad \lambda_+ \frac{\partial K_{+-}}{\partial n_x} \bigg|_{\Sigma^+} = \lambda_- \frac{\partial K_{-+}}{\partial n_x} \bigg|_{\Sigma^-}.
\]

Choosing the suitable representation for the functions \( K \) in terms of simple and double layer potentials and making account for the matching conditions one obtains a system of integral equations iteratively solvable with respect to the densities of simple and double \( \mu \) and \( \nu \) layers. The substitution of the densities into the corresponding the heat kernels gives

\[
K_{++}(x, y; t) = K_{++}^0(x, y; t) + 2a^2 \sum_{n=1}^{\infty} t \int_0^t d\tau \int_{\Sigma} dz K_{++}^0(x, z; t - \tau) \mu_{n-1}(z, y; \tau),
\]

\[
K_{--}(x, y; t) = 2a^2 \sum_{n=1}^{\infty} t \int_0^t d\tau \int_{\Sigma} dz \frac{\partial}{\partial n_z} K_{--}^0(x, z; t - \tau) \nu_{n-1}(z, y; \tau)
\]

where

\[
\nu_1(x, y; t) = 2a^2 \int_0^t d\tau \int_{\Sigma} dz K_{++}^0(x, z; t - \tau) \mu_0(z, y; \tau)
\]

\[
-2a^2 \int_0^t d\tau \int_{\Sigma} dz \frac{\partial}{\partial n_z} K_{--}^0(x, z; t - \tau) \nu_1(z, y; \tau),
\]
The functions \( K \) parameterises a unit sphere, vicinity of the surface \( \Sigma \) the metric is intrinsic Ricci curvature, we are interested in the asymptotic expansion of the heat kernel trace. For our purposes it is convenient to use such coordinates that in the righthand sides of (11) the functions \( K_{++}, K_{+-}, K_{-+}, \) and \( K_{--} \) define the heat kernel in the whole space. For details see [12].

We are interested in the asymptotic expansion of the heat kernel trace \( K(t) = \int_M dx K(x, x; t) \) when \( t \to 0 \). The functions \( K_{--} \) and \( K_{++} \) do not contribute to it, thus we have to consider only \( K_{++} \) and \( K_{+-} \).

For our purposes it is convenient to use such coordinates that in the vicinity of the surface \( \Sigma \) the metric is \( g_{ij} dx^i dx^j = (dx^3)^2 + g_{ab} dx^a dx^b \) where \( x^3 \) is a coordinate on the normal to \( \Sigma \), \( x^3 = 0 \) on \( \Sigma \).

Performing the surface integrations we keep in mind that for small \( t \) the contributions of largely separated points are exponentially damped. Therefore in the vicinity of \( \Sigma \) we may replace the squared distance \( (x - z)^2 \) by several terms of its expansion in powers of the geodesic distance \( \sigma \) on the surface \( \Sigma \):

\[
(x - z)^2 = (x_3 - z_3)^2 + \sigma^2 \left\{ 1 - (x_3 + y_3)k_1 + x_3 z_3(k_1^2 + k_2^2) \right\} + \sigma^3 \left\{ -\frac{1}{3}(2z_3 + x_3)k_1' + x_3 z_3(k_1 k_1' + k_2 k_2') \right\} + \ldots \quad (12)
\]

\[
k_1 = L_{ab} \xi^a \xi^b, \quad k_2 = \frac{1}{2} (e_{a\gamma} L_{b\gamma} + e_{b\gamma} L_{a\gamma}) \xi^a \xi^b, \quad k_1' = \frac{dk_1}{d\sigma}, \quad k_2' = \frac{dk_2}{d\sigma}
\]

The surface area element is \( dz = (1 - \frac{1}{2} r_{ab} \xi^a \xi^b \sigma^2 + \ldots) \sigma d\sigma d\Omega \), \( \Omega \) parameterises a unit sphere, \( L_{ab} \) is the second fundamental form on \( \Sigma \), \( r_{ab} \) is intrinsic Ricci curvature, \( \xi \) is a unit tangent vector at \( x \) to the geodesics with the length \( \sigma \) joining \( z \) to \( x \) on \( \Sigma \).

Here we display the results for the leading terms of the MRE when \( t \to 0 \):

\[
K^{(0)}(t) = K_{++}^{(0)}(t) + K_{+-}^{(0)}(t) = \frac{t^{-3/2}}{(4\pi a_+^2)^{3/2}} M_+ + \frac{t^{-3/2}}{(4\pi a_-^2)^{3/2}} M_-,
\]

\[
K_{++}^{(1)}(t) = \frac{t^{-1} \Sigma}{8\pi a_+^2} + \frac{t^{-1/2}}{8\pi^{3/2} a_+} \int \Sigma L_{aa} + \frac{t^0}{2\pi} \int \Sigma \left[ 5 (L_0^a)^2 + L_a^b L_b^a - \frac{2}{3} r_a^a \right] + \ldots,
\]
\[ K_{++}^{(2)}(t) = -\frac{t^{-1}}{8\pi} \frac{\lambda_-}{\lambda_+} \sum \frac{1}{a_+^2} - \frac{1}{8\pi^{3/2}} \frac{\lambda_-}{\lambda_+} \frac{t^{-1/2}}{a_+} \int L_{aa} \]

\[ + \frac{t^0}{32\pi} \left\{ a_- \frac{\lambda_-}{\lambda_+} (a_+ + 2a_-) \right\} \int \left[ -(L_a^a)^2 + 4L_b^b L_a^a - r_0^a / 3 \right] \]

\[ + \left[ \frac{1}{8} \frac{\lambda_-}{\lambda_+} (a_+ + a_-)^3 \left( \frac{35}{12} a_+^3 + \frac{11}{4} a_-^2 a_+ + 2a_+^2 a_- \right) \right] \int \left[ (L_a^a)^2 + 2L_b^b L_a^a \right] + \ldots \] (13)

To obtain \( K_{++}^{(1)}(t) \) and \( K_{++}^{(2)}(t) \) one should replace \( a_+ \leftrightarrow a_- \), \( \lambda_+ \leftrightarrow \lambda_- \), \( L_a^b \rightarrow -L_a^b \). The asymptotic behaviour of the subsequent MRE terms may be found in a similar way. After that all factors appearing with the same powers of \( t \) are added up to give the heat kernel coefficients. The latter are expressed though the integrals of the surface geometric invariants.

4 Conclusion

In the present report we have shown how to formulate dielectric-like spectral problem in terms of integral equations and to find the heat kernel as their iterative solution. Our results indicate that this method being a generalization of the known multiple reflection expansion \([13][14][15]\) will finally help to derive the heat kernel coefficients for the electromagnetic field interacting with a dielectric body of arbitrary shape.

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