A Tensor Product of Kantorovich-Stancu Type Operators with Shifted Knots and their $k$th Order Generalization

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Abstract. In this paper, we introduce a tensor product of the Stancu-Kantorovich type operators defined by Içöz [11]. The rate of convergence of these operators is obtained in terms of the modulus of continuity and the Peetre’s K-functional. Further, we consider a generalization of the above operators via Taylor’s polynomials and examine their approximation behavior. Some applications of these two dimensional generalized Stancu-Kantorovich type polynomials are also discussed. Finally, we present some numerical examples and illustrations to show the convergence behavior of the operators under study using MATLAB algorithms.

1. Introduction

For $f \in C(I)$, the space of all continuous functions on $I = [0, 1]$ with sup-norm, Stancu [15] proposed a sequence of polynomials

$$S_m^{(\alpha, \beta)}(f; x) = \sum_{j=0}^{m} f\left(\frac{j + \alpha}{m + \beta}\right)b_{m,j}(x),$$

where the Bernstein basis functions $b_{m,j}(x)$ are defined by

$$b_{m,j}(x) = \binom{m}{j} x^j (1-x)^{m-j}, \quad x \in I,$$

and showed that these polynomials converge to the function $f(x)$ uniformly in $x \in I$. It is obvious that whenever $\alpha = \beta = 0$, the operators defined by equation (1) reduce to the classical Bernstein operators defined by Bernstein [6]. Gadjiev and Ghorbanalizadeh [10] constructed Bernstein-Stancu type polynomials with shifted knots involving some non-negative real numbers $\theta$ and $\theta_i$, $i = 1, 2, 3$, as

$$G_{m,\theta}^{(\alpha, \beta)}(f; x) = \left(\frac{m + \theta}{m}\right)^m \sum_{j=0}^{m} \Omega_{m,j}^{(\alpha, \beta)}(x)f\left(\frac{j + \theta_1}{m + \theta_2}\right).$$

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where the basis functions \( \Omega^{(0,\phi)}_{m,j}(x) \) are defined by
\[
\Omega^{(0,\phi)}_{m,j}(x) = \frac{m!}{j!(m+j)!} \frac{m+\phi_2}{m+\phi_1} (x - \phi_2) (m+\phi_2)^{m-j},
\]
(4)
\[ x \in \left[ \frac{\phi_2}{m+\phi_1}, \frac{m+\phi_2}{m+\phi_1} \right] \] and \( 0 \leq \phi_3 \leq \phi_2 \leq \phi_1 \leq 1 \). It is obvious that whenever \( \phi = \phi_i = 0 ; i = 1, 2, 3 \), the operators defined by equation (1.3) include the classical Bernstein operators. Wang et al. [17] obtained some direct results and a converse result in approximation by the polynomials defined in (3). To make it possible to approximate the Lebesgue integrable functions on \( I \), Kantorovich [12] proposed a modification of the Bernstein polynomials as
\[
K_m(f; x) = (m + 1) \sum_{j=0}^{m} b_{m,j}(x) \int_{t^1}^{t^2} f(t)dt.
\]
Inspired by the above idea, Icöz [11] introduced a Kantorovich variant of the Bernstein-Stancu type polynomials with shifted knots given by (3) as follows:
\[
k^{(0)}_{m,j}(f; x) = \frac{(m+\phi_1)^m}{m} (m+\phi_2)^{m+\phi_1+1} \sum_{j=0}^{m} \Omega^{(0,\phi)}_{m,j}(x) \int_{-\frac{m+\phi_1}{m+\phi_2}}^{1} \int_{-\frac{m+\phi_2}{m+\phi_1}}^{1} f(t_1, t_2)dt_1dt_2,
\]
(5)
where the basis functions \( \Omega^{(0,\phi)}_{m,j}(x) \) are defined in (4) and \( x \in \left[ \frac{\phi_2}{m+\phi_1}, \frac{m+\phi_2}{m+\phi_1} \right] \). Evidently, in the particular case, \( \phi_i = \phi_j = 0 ; i = 1, 2, 3 \), the operators \( k^{(0)}_{m,j} \) reduce to the operators \( K_m \). The author [11] established some approximation results for the operators (5) in the continuous functions space with the aid of the usual modulus of continuity and the Peetre’s K-functional and also investigated the approximation properties of a \( k^{th} \) order generalization of these operators. For other contributions, in the direction of the above study, we refer the reader to (cf. [1], [3]-[5], [14] etc.).

In this article, we introduce the following tensor product of Kantorovich-Stancu type polynomials on the rectangle \( \square = \left[ \frac{\phi_2}{m+\phi_1}, \frac{m+\phi_2}{m+\phi_1} \right] \times \left[ \frac{\phi_3}{m+\phi_2}, \frac{m+\phi_3}{m+\phi_2} \right] \) as:
\[
\Omega^{(0,\phi)}_{n,m,j}(f; t_1, t_2; x, y) = \left( \frac{n+\phi_3}{m+\phi_2} \right)^m (n+\phi_3)^m (m+\phi_3+1)(m+\phi_3+1) \left( \frac{m+\phi_3}{m+\phi_2} \right)^n (m+\phi_3)^n \sum_{j=0}^{m} \Omega^{(0,\phi)}_{m,j}(x) \Omega^{(0,\phi)}_{n,j}(y) \int_{-\frac{m+\phi_1}{m+\phi_2}}^{1} \int_{-\frac{m+\phi_2}{m+\phi_1}}^{1} f(t_1, t_2)dt_1dt_2,
\]
(6)
where the basis functions \( \Omega^{(0,\phi)}_{n,m,j}(x) \) and \( \Omega^{(0,\phi)}_{n,m,j}(y) \) are as defined in (4).

We investigate the uniform convergence of these operators in the space \( C(I^2) \) where \( I^2 = I \times I \) and then determine the degree of convergence by these operators using the modulus of continuity and the Peetre’s K-functional. We also define a \( k^{th} \) order generalization of these operators to study the approximation of continuous functions having \( k^{th} \) order continuous partial derivatives on \( I^2 \) and present some applications of this study to bi-variate Bernstein type operators on a simplex. Finally, we validate the results of this paper by some graphs and error estimation tables using MATLAB.

2. Auxiliary results

In our future consideration, \( ||\cdot||_{C(I^2)} \) denotes the sup-norm on \( I^2 \).

**Lemma 2.1.** Let \( e_i(t_1, t_2) = l_i^1 t_1^2 \), where \( i, j \in \mathbb{N} \cup \{0\} \). For \( x, y \in \Omega \), the Kantorovich type generalized Bernstein-Stancu operators \( \Omega^{(0,\phi)}_{n,m,j}(f; x, y) \), defined by (6), possess the following properties:

(i) \( \Omega^{(0,\phi)}_{n,m,j}(e_{i0}; x, y) = 1; \)
Following ([11], Thm.1), the proof of this lemma easily follows. Consequently, in view of Theorems 2 and 4 of [11], we are led to:

**Lemma 2.2.** For the operator \( R_{\theta, \phi}^{t_1, t_2}(f; x, y) \), following hold good:

(i) \( R_{m, n, \theta, \phi}^{t_1, t_2}(t_1 - x; x, y) = \frac{m+\phi}{m+\phi+1} x + \frac{\theta + \phi}{m+\phi+1} + \frac{1}{2(n+\theta+1)} \).

(ii) \( R_{m, n, \theta, \phi}^{t_1, t_2}(t_2 - y; x, y) = \frac{m+\phi}{m+\phi+1} y + \frac{\phi - \phi_2}{m+\phi+1} + \frac{1}{2(n+\theta+1)} \).

(iii) \( R_{m, n, \theta, \phi}^{t_1, t_2}((t_1 - x)^2; x, y) = \frac{1}{n+\theta+1}\left\{ x(\theta - \theta_1 - 1) - (\theta_2 - \theta_3 - 1) \right\} + \frac{(n+\theta)^2}{n} \left\{ x - \frac{\theta_1}{n+\theta} \right\} \left\{ x - \frac{\theta_2}{n+\theta} \right\} + \frac{1}{n+\theta+1} \).

Also, if \( \theta_2 - \theta_3 \geq 1 \) and \( \theta - \theta_1 \geq \theta_2 - \theta_3 \), we get

\[
\| R_{m, n, \theta, \phi}^{t_1, t_2}((t_1 - x)^2; x, y) \|_{C(\bar{\Omega})} \leq \frac{(\theta - \theta_1)^2 + \frac{n}{4} + 2}{(n + \theta + 1)^2};
\]

In the case \( \theta - \theta_1 < \theta_2 - \theta_3 \) such that \( \theta_2 - \theta_3 \geq 1 \), we obtain

\[
\| R_{m, n, \theta, \phi}^{t_1, t_2}((t_1 - x)^2; x, y) \|_{C(\bar{\Omega})} \leq \frac{(\theta_2 - \theta_3)^2 + \frac{n}{4} + 2}{(n + \theta + 1)^2}.
\]

(iv) \( R_{m, n, \theta, \phi}^{t_1, t_2}((t_2 - y)^2; x, y) = \frac{1}{m+\phi+1}\left\{ y(\phi - \phi_1 - 1) - (\phi_2 - \phi_3 - 1) \right\} + \frac{(m+\phi)^2}{m} \left\{ y - \frac{\phi_1}{m+\phi} \right\} \left\{ y - \frac{\phi_2}{m+\phi} \right\} - y(\phi - \phi_1 - 1) - (\phi_3 - \phi_2 + \frac{3}{2}).
\]

Also, if \( \phi_2 - \phi_3 \geq 1 \) and \( \phi - \phi_1 \geq \phi_2 - \phi_3 \), we get

\[
\| R_{m, n, \theta, \phi}^{t_1, t_2}((t_2 - y)^2; x, y) \|_{C(\bar{\Omega})} \leq \frac{(\phi - \phi_1)^2 + \frac{n}{4} + 2}{(m + \phi + 1)^2};
\]

In the case \( \phi - \phi_1 < \phi_2 - \phi_3 \) such that \( \phi_2 - \phi_3 \geq 1 \), we obtain

\[
\| R_{m, n, \theta, \phi}^{t_1, t_2}((t_2 - y)^2; x, y) \|_{C(\bar{\Omega})} \leq \frac{(\phi_2 - \phi_3)^2 + \frac{n}{4} + 2}{(m + \phi + 1)^2}.
\]
3. Rate of convergence by \( R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) \)

In this section, we first give the following Korovkin type theorem on the convergence of \( R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) \) to \( f(x, y) \).

**Theorem 3.1.** Let \( f \in C(I^2) \). Then

\[
\lim_{n,m \to \infty} \max_{(x,y) \in \Box} |R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) - f(x, y)| = 0.
\]

**Proof.** Taking into consideration the equalities in Lemma 2.1, we obtain

\[
\lim_{n,m \to \infty} \max_{(x,y) \in \Box} |R_{n,m,d,\phi}^{\theta,\phi} (e_i; x, y) - e_i| = 0,
\]

for \((i, j) \in \{(0, 0), (1, 0), (0, 1)\}\). Further

\[
\lim_{n,m \to \infty} \max_{(x,y) \in \Box} |R_{n,m,d,\phi}^{\theta,\phi} (e_{20} + e_{02}; x, y) - x^2 - y^2| = 0.
\]

Let us define

\[
R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) = \begin{cases} R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) & \text{if } (x, y) \in \Box, \\ f(x, y) & \text{if } (x, y) \in I^2 \setminus \Box. \end{cases}
\]

Considering the above definition of the operators, we easily get

\[
\|R_{n,m,d,\phi}^{\theta,\phi} (f) - f\|_{C(I^2)} = \max_{(x,y) \in \Box} |R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) - f(x, y)|.
\]

Now, using (7)-(8), we immediately get

\[
\lim_{n,m \to \infty} \|R_{n,m,d,\phi}^{\theta,\phi} (e_i) - e_i\|_{C(I^2)} = 0,
\]

for \((i, j) \in \{(0, 0), (1, 0), (0, 1)\}\) and

\[
\lim_{n,m \to \infty} \|R_{n,m,d,\phi}^{\theta,\phi} (e_{20} + e_{02}) - x^2 - y^2\|_{C(I^2)} = 0.
\]

Applying the two dimensional Korovkin’s type theorem (see [16]) to the sequence of operators \( R_{n,m,d,\phi}^{\theta,\phi} \), we obtain

\[
\lim_{n,m \to \infty} \|R_{n,m,d,\phi}^{\theta,\phi} (f) - f\|_{C(I^2)} = 0,
\]

for every continuous function \( f \in C(I^2) \). Therefore (9) gives

\[
\lim_{n,m \to \infty} \max_{(x,y) \in \Box} |R_{n,m,d,\phi}^{\theta,\phi} (f; x, y) - f(x, y)| = 0.
\]

This completes the proof. \( \Box \)

In order to discuss the next results, we recall some definitions of the modulus of continuity.
Definition 3.2. For \( f \in C(\Phi) \) and \( \delta > 0 \), the complete modulus of continuity is defined as

\[
\omega^{(c)}(f; \delta) = \sup \frac{\|f(t_1, t_2) - f(x, y)\|}{\sqrt{\|t_1 - x\|^2 + \|t_2 - y\|^2}} : (t_1, t_2), (x, y) \in \Phi.
\]

The partial moduli of continuity of \( f \) with respect to \( x \) and \( y \) is given by

\[
\omega^{(1)}(f; \delta) = \sup_{|x_1 - x_2| \leq \delta} \sup_{y \in \Phi} \|f(x_1, y) - f(x_2, y)\|
\]

and

\[
\omega^{(2)}(f; \delta) = \sup_{x \in \Phi} \sup_{|y_1 - y_2| \leq \delta} \|f(x, y_1) - f(x, y_2)\|
\]

respectively. We shall use the following property of the complete modulus of continuity:

\[
|f(t_1, t_2) - f(x, y)| \leq \omega^{(c)}(f; \delta) \left(1 + \frac{\sqrt{(t_1 - x)^2 + (t_2 - y)^2}}{\delta}\right).
\]

It is known that these definitions satisfy the properties analogous to the usual modulus of continuity. For more details, we refer to [2].

In the next result, we obtain an estimate of the rate of convergence in terms of the complete modulus of continuity for the operators defined by (6).

Theorem 3.3. Let \( f \in C(\Phi) \). If \( \theta_2 - \theta_3 \geq 1 \) and \( \phi_2 - \phi_3 \geq 1 \), then the following inequalities hold:

\[
\|\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f) - f\| \leq \left\{\begin{array}{ll}
\frac{3}{2} \omega^{(c)}(f; \sqrt{\frac{4(\theta - \theta_1)^2 + n + 8}{(n + \theta_1 + 1)^2} + \frac{4(\phi - \phi_1)^2 + m + 8}{(m + \phi_1 + 1)^2}})}; & \text{if } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\
\frac{3}{2} \omega^{(c)}(f; \sqrt{\frac{4(\theta_2 - \theta_3)^2 + n + 8}{(n + \theta_3 + 1)^2} + \frac{4(\phi_2 - \phi_3)^2 + m + 8}{(m + \phi_3 + 1)^2}})}; & \text{if } \theta - \theta_1 < \theta_2 - \theta_3 \text{ and } \phi - \phi_1 < \phi_2 - \phi_3.
\end{array}\right.
\]

Proof. From the linearity and positivity of the operators (6), Cauchy-Schwarz inequality and Lemma 1, the property (12) of the complete modulus of continuity gives

\[
\|\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f; x, y) - f(x, y)\| \leq \omega^{(c)}(f; \delta) \left(1 + \frac{\sqrt{\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f; x, y)}}{\delta}\right).
\]

where \( \delta > 0 \). Therefore considering Lemma 2.2, for \( \theta - \theta_1 \geq \theta_2 - \theta_3 \geq 1 \) and \( \phi - \phi_1 \geq \phi_2 - \phi_3 \geq 1 \), we have

\[
\|\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f) - f\| \leq \omega^{(c)}(f; \delta) \left(1 + \frac{\sqrt{\frac{4(\theta - \theta_1)^2 + n + 8}{(n + \theta_1 + 1)^2} + \frac{4(\phi - \phi_1)^2 + m + 8}{(m + \phi_1 + 1)^2}}}{\delta}\right).
\]

Now choosing \( \delta = \sqrt{\frac{4(\theta - \theta_1)^2 + n + 8}{(n + \theta_1 + 1)^2} + \frac{4(\phi - \phi_1)^2 + m + 8}{(m + \phi_1 + 1)^2}} \), we obtain

\[
\|\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f) - f\| \leq \frac{3}{2} \omega^{(c)}(f; \sqrt{\frac{4(\theta - \theta_1)^2 + n + 8}{(n + \theta_1 + 1)^2} + \frac{4(\phi - \phi_1)^2 + m + 8}{(m + \phi_1 + 1)^2}}).
\]

Analogously, taking into account Lemma 2.2, for \( \theta - \theta_1 < \theta_2 - \theta_3 \) and \( \phi - \phi_1 < \phi_2 - \phi_3 \) such that \( \theta_2 - \theta_3, \phi_2 - \phi_3 \geq 1 \) (with \( \delta = \sqrt{\frac{4(\theta_2 - \theta_3)^2 + n + 8}{(n + \theta_3 + 1)^2} + \frac{4(\phi_2 - \phi_3)^2 + m + 8}{(m + \phi_3 + 1)^2}} \)), we are led to

\[
\|\mathcal{R}_{n,m,\theta,\phi}^{\theta,\phi}(f) - f\| \leq \frac{3}{2} \omega^{(c)}(f; \sqrt{\frac{4(\theta_2 - \theta_3)^2 + n + 8}{(n + \theta_3 + 1)^2} + \frac{4(\phi_2 - \phi_3)^2 + m + 8}{(m + \phi_3 + 1)^2}}).
\]
Theorem 3.4. Let $f \in C(I^2)$. If $\theta_2 - \theta_3 \geq 1$ and $\phi_2 - \phi_3 \geq 1$, then the following inequalities hold:

$$
\|R_{n,m,\theta,\phi}(f) - f\| \leq \left\{ \begin{array}{ll}
2(\omega^{(1)}(f; \sqrt{\frac{4(0-\theta_2)^2+n+8}{2(n+\theta_1+1)}}) + \omega^{(2)}(f; \sqrt{\frac{4(0-\theta_2)^2+n+8}{2(n+\theta_1+1)}})) & \text{if } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\
2(\omega^{(1)}(f; \sqrt{\frac{4(0-\theta_2)^2+n+8}{(n+\theta_1+1)^2}}) + \omega^{(2)}(f; \sqrt{\frac{4(0-\theta_2)^2+n+8}{(m+\phi_1+1)^2}})) & \text{if } \theta - \theta_1 < \theta_2 - \theta_3 \text{ and } \phi - \phi_1 < \phi_2 - \phi_3.
\end{array} \right.
$$

Proof. From linearity and monotonicity of the operators $R_{n,m,\theta,\phi}$ and the definitions of the partial moduli of continuity with respect to $x$ and $y$ as defined in (11), we have

$$
\left| R_{n,m,\theta,\phi}^{(i)}(f(t_1,t_2);x,y) - f(x,y) \right| \leq R_{n,m,\theta,\phi}^{(i)}(f(t_1-x);x,x) + R_{n,m,\theta,\phi}^{(i)}(f(t_2-y);y,y).
$$

Now using the property of modulus of continuity similar to (12) and the Cauchy-Schwarz inequality, for $\delta_1, \delta_2 > 0$, we get

$$
\left| R_{n,m,\theta,\phi}^{(i)}(f(t_1,t_2);x,y) - f(x,y) \right| \leq \left\{ \begin{array}{ll}
1 + \frac{1}{\delta_1} \left( R_{n,m,\theta,\phi}^{(i)}((t_1-x)^2;2x,2x) + \omega^{(1)}(f;\delta_1) \right) + \left( 1 + \frac{1}{\delta_2} \left( R_{n,m,\theta,\phi}^{(i)}((t_2-y)^2;2y,2y) + \omega^{(2)}(f;\delta_2) \right) \right) & \text{for all } (x,y) \in I^2.
\end{array} \right.
$$

This proves the first assertion of our result. Similarly, for $\theta - \theta_1 < \theta_2 - \theta_3$ and $\phi - \phi_1 < \phi_2 - \phi_3$ such that $\theta_2 - \theta_3, \phi_2 - \phi_3 \geq 1$, using Lemma 2 with $\delta_1 = \frac{4(0-\theta_3)^2+n+8}{2(n+\theta_1+1)}$ and $\delta_2 = \frac{4(0-\theta_3)^2+n+8}{(m+\phi_1+1)}$, we immediately find the second assertion.

We study the rate of convergence of the bi-variate Bernstein-Stancu-Kantorovich type operators $R_{n,m,\theta,\phi}$ for elements of the Lipschitz class $Lip_\gamma(I^2)$, for $0 < \gamma \leq 1$. We recall the following definition:

Definition 3.5. A function $f \in C(I^2)$ is said to be in $Lip_\gamma(I^2)$ if

$$
|f(t_1,t_2) - f(x,y)| \leq M((t_1-x)^2 + (t_2-y)^2)^{\frac{\gamma}{2}},
$$

holds for all $(t_1, t_2), (x, y) \in I^2$.

Theorem 3.6. If $\theta_2 - \theta_3 \geq 1$ and $\phi_2 - \phi_3 \geq 1$, then for all $f \in Lip_\gamma(I^2)$, the following inequalities hold:

$$
\|R_{n,m,\theta,\phi}(f) - f\| \leq M \left\{ \begin{array}{ll}
\left( \frac{(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2} + \frac{(\phi-\phi_1)^2+n+8}{(m+\phi_1+1)^2} \right)^{\frac{\gamma}{2}} & \text{for } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\
\left( \frac{(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2} + \frac{(\phi-\phi_1)^2+n+8}{(m+\phi_1+1)^2} \right)^{\frac{\gamma}{2}} & \text{for } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3.
\end{array} \right.
$$

where $0 < \gamma \leq 1$ and $M$ is a positive constant.
Theorem 3.8. For all \( f \in \text{Lip}_M(\gamma) \), we have

\[
\left| \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi}(f(t_1,t_2);x,y) - f(x,y) \right| \leq M \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi}((t_1-x)^2 + (t_2-y)^2)^{\frac{\gamma}{2}}; x,y.
\]

Now, applying the Hölder’s inequality with \( p = \frac{2}{\gamma}, q = \frac{2}{2-\gamma} \) and Lemma 2.1, we obtain

\[
\left| \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi}(f(t_1,t_2);x,y) - f(x,y) \right| \leq M \left( \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi}((t_1-x)^2 + (t_2-y)^2); x,y \right)^{\frac{\gamma}{2}}.
\]

Finally using Lemma 2.2 and considering sup-norm, we reach to the desired result. \( \Box \)

Let \( C^2(\mathbb{R}^2) \) be the space of all continuous function \( f \) having continuous partial derivatives upto the second order. We consider the following norm on \( C^2(\mathbb{R}^2) \):

\[
\| f \|_{C^2(\mathbb{R}^2)} = \| f \|_{C(\mathbb{R}^2)} + \sum_{j=1}^{2} \left( \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{C(\mathbb{R}^2)} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{C(\mathbb{R}^2)} + \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_{C(\mathbb{R}^2)} \right).
\]

We use the following definition in our upcoming result.

**Definition 3.7.** Let \( f \in C^2(\mathbb{R}^2) \) and \( \delta > 0 \). The Peetre’s K-functional and second-order modulus of smoothness of \( f \) are given by

\[
K(f; \delta) = \inf_{g \in C(\mathbb{R}^2)} \left\{ \| f - g \|_{C(\mathbb{R}^2)} + \delta \| g \|_{C(\mathbb{R}^2)} \right\},
\]

and

\[
\omega_2(f; \delta) = \sup_{\| \nabla^2 f \| \leq \delta} \left\| \Delta^2 f(x,y) \right\|,
\]

where \( \Delta^2 f(x,y) = \sum_{j=0}^{2} (-1)^{2-j} \partial^j f(x+jt, y+js), \) respectively.

In the next result, we establish an order of approximation for the bi-variate operator \( \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi} \) in terms of the Peetre’s K-functional and the complete modulus of continuity.

**Theorem 3.8.** For all \( f \in C(\mathbb{R}^2) \) and \( \theta_2 - \theta_3, \phi_2 - \phi_3 \geq 1 \), the following inequalities hold

\[
\| \mathcal{R}_{n,m,	heta,\phi}^{\delta,\phi}(f) - f \| \leq \begin{cases} 4K(f, \delta_1) + \omega_2(f; \theta_2), & \text{if } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3, \\ 4K(f, \delta_2) + \omega_2(f; \theta_2), & \text{if } \theta - \theta_1 \leq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \leq \phi_2 - \phi_3,
\end{cases}
\]

where

\[
\delta_1 = \frac{1}{8} \left[ \left( \frac{\theta - \theta_1 + \frac{n + 2}{4} + 2}{n + \theta_1 + 1^2} \right)^2 + \left( \frac{\phi - \phi_1 + \frac{m + 2}{4} + 2}{m + \phi_1 + 1^2} \right)^2 \right] + \left( \frac{\theta - \theta_1 + \theta_2 - \theta_3 - \frac{1}{2}}{n + \theta_1 + 1} + \frac{\phi - \phi_1 + \phi_2 - \phi_3 - \frac{1}{2}}{n + \phi_1 + 1} \right)^2,
\]

\[
\delta_2 = \frac{1}{8} \left[ \left( \frac{\theta_2 - \theta_3 + \frac{n + 2}{4} + 2}{n + \theta_1 + 1^2} \right)^2 + \left( \frac{\phi_2 - \phi_3 + \frac{m + 2}{4} + 2}{m + \phi_1 + 1^2} \right)^2 \right] + \left( \frac{\theta - \theta_1 + \theta_2 - \theta_3 - \frac{1}{2}}{n + \theta_1 + 1} + \frac{\phi - \phi_1 + \phi_2 - \phi_3 - \frac{1}{2}}{n + \phi_1 + 1} \right)^2,
\]

and

\[
\Delta^2 = \left( \frac{\theta - \theta_1 + \theta_2 - \theta_3 - \frac{1}{2}}{n + \theta_1 + 1} + \frac{\phi - \phi_1 + \phi_2 - \phi_3 - \frac{1}{2}}{n + \phi_1 + 1} \right)^2.
\]
Proof. We consider the following auxiliary operators:
\[ \hat{S}_{n,m,0,0}^{\theta,\phi}(f; x, y) = R_{n,m,0,0}^{\theta,\phi}(f(x, y) + f(x, y) - f(S_{n,m,0,0}^{\theta,\phi}(t_1; x, y), S_{n,m,0,0}^{\theta,\phi}(t_2; x, y)). \]

From Taylor expansion, for any \( h \in C^2(I^2) \), we have
\[
h(t_1, t_2) - h(x, y) = \frac{\partial h(x, y)}{\partial x} (t_1 - x) + \iint_x (t_1 - \eta) \frac{\partial^2 h(\eta, \xi)}{\partial \eta^2} d\eta + \frac{\partial h(x, y)}{\partial y} (t_2 - y) + \iint_x \iint_y \frac{\partial^2 h(u, v)}{\partial u \partial v} dudv, \tag{14}
\]
and, let \( \psi_1^{i,j}(t_1, t_2) = \left( \int_x (t_1 - \eta) \frac{\partial^2 h(\eta, \xi)}{\partial \eta^2} d\eta \right) \left( \int_y (t_2 - \xi) \frac{\partial^2 h(\eta, \xi)}{\partial \xi^2} d\xi \right) \).

Applying the auxiliary operator \( \hat{S}_{n;m,0,0}^{\theta,\phi} \) on the equation (14) and taking \( \hat{S}_{n;m,0,0}^{\theta,\phi}(1; x, y) = 1; S_{n,m,0,0}^{\theta,\phi}(t_1 - x; x, y) = 0 = \hat{S}_{n,m,0,0}^{\theta,\phi}(t_2 - x; y, y), \) we have
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(h; x, y) - h(x, y)| \leq |\hat{S}_{n,m,0,0}^{\theta,\phi}(\psi_1^{3,1}(t_1, t_2); x, y)| + |\hat{S}_{n,m,0,0}^{\theta,\phi}(\psi_1^{1,1}(t_1, t_2); x, y)| + |\hat{S}_{n,m,0,0}^{\theta,\phi}(\int_x \iint_y \frac{\partial^2 h(u, v)}{\partial u \partial v} dudv; x, y)|. \tag{15}
\]

Further, applying the auxiliary operator \( \hat{S}_{n,m,0,0}^{\theta,\phi} \) on \( \psi_1^{1,0} \) gives us
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(\psi_1^{1,0}(t_1, t_2); x, y)| \leq \frac{|\hat{h}|_{C^2(I^2)}}{2} \left( |S_{n,m,0,0}^{\theta,\phi}(t_1 - x)^2; x, y) + |S_{n,m,0,0}^{\theta,\phi}(t_1 - x, y)|^2 \right) = \frac{|\hat{h}|_{C^2(I^2)}}{2} [\mu_{2,x} + \mu_{1,y}],
\]
where \( \mu_{2,x} \) and \( \mu_{1,y} \) are the second and first order central moments, respectively. Similarly,
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(\psi_1^{0,1}(t_1, t_2); x, y)| \leq \frac{|\hat{h}|_{C^2(I^2)}}{2} [\nu_{2,y} + \nu_{1,y}],
\]
where \( \nu_{2,y} \) and \( \nu_{1,y} \) are the second and first order central moments, respectively. Also,
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(\int_x \iint_y \frac{\partial^2 h(u, v)}{\partial u \partial v} dudv; x, y)| \leq |\hat{h}|_{C^2(I^2)} \left( |S_{n,m,0,0}^{\theta,\phi}(t_1 - x, t_2 - y; x, y) + |S_{n,m,0,0}^{\theta,\phi}(e_1, x; x, y) - x| \right) \left| S_{n,m,0,0}^{\theta,\phi}(e_1, x; x, y) - y \right|,
\]
hence using the Cauchy-Schwarz inequality
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(\int_x \iint_y \frac{\partial^2 h(u, v)}{\partial u \partial v} dudv; x, y)| \leq \frac{|\hat{h}|_{C^2(I^2)}}{2} \left( |S_{n,m,0,0}^{\theta,\phi}(t_1 - x)^2; x, y) \right)^{\frac{1}{2}} \times \left( |S_{n,m,0,0}^{\theta,\phi}(t_2 - y)^2; x, y) \right)^{\frac{1}{2}} + \frac{|\hat{h}|_{C^2(I^2)}}{2} |S_{n,m,0,0}^{\theta,\phi}(t_1 - x, y) + |S_{n,m,0,0}^{\theta,\phi}(t_2 - y, y)| \right) \left( |\hat{h}|_{C^2(I^2)}[\mu_{2,y}^{1/2} + \nu_{1,y}^{1/2}]^2 + |\mu_{1,y}| \right). \tag{16}
\]
Consequently, from the equation (14)
\[
|\hat{S}_{n,m,0,0}^{\theta,\phi}(h; x, y) - h(x, y)| \leq \frac{|\hat{h}|_{C^2(I^2)}}{2} \left( (\mu_{2,x}^{1/2} + \nu_{1,y}^{1/2})^2 + (|\mu_{1,y}| + |\mu_{1,y}|)^2 \right). \tag{17}
\]
Now, from the definition of auxiliary operator and equation (17), we may write
\[
|\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(f; x, y) - f(x, y)| \leq |\hat{\mathcal{R}}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(f - h; x, y)| + |\hat{\mathcal{R}}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(h; x, y) - h(x, y)| + |(f - h)(x, y)| + 4\|f - h\|_{C^2(P)} + \|h\|_{C^2(P)}
\]
\[
\leq 4\|f - h\|_{C^2(P)} + \|h\|_{C^2(P)} + \left\{ (|\mu_1| + |\nu_1|)^2 \right\}
\]
\[
+ \omega^2(f; \sqrt{|\mu_1|^2 + |\nu_1|^2}).
\]

Now, for \(\theta - \theta_1 \geq \theta_2 - \theta_3 \geq 1\) and \(\phi - \phi_1 \geq \phi_2 - \phi_3 \geq 1\), using Lemma 2.2 and taking infimum over all \(h \in C^2(P)\), we get
\[
|\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(f) - f| \leq 4K(f, \delta_1) + \omega^2(f; \Delta).
\]

By a similar reasoning, for the other case \(\theta - \theta_1 \leq \theta_2 - \theta_3\) and \(\phi - \phi_1 \leq \phi_2 - \phi_3\) such that \(\theta_2 - \theta_3, \phi_2 - \phi_3 > 1\), we have
\[
|\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(f) - f| \leq 4K(f, \delta_2) + \omega^2(f; \Delta).
\]

This proves the required result. \(\square\)

**Corollary 3.9.** Considering the well-known relation [8] that
\[
K(f; \delta) \leq C \omega^2(f; \sqrt{\delta}), \quad \text{for any } \delta > 0,
\]
where \(C\) is some positive constant, the result of the Theorem 3.8 takes the following form:
\[
|\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(f) - f| \leq \left\{ \begin{array}{ll}
\frac{5}{2} \omega^2(f; \sqrt{\delta}) + \omega^2(f; \Delta), & \text{if } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\
\frac{5}{2} \omega^2(f; \sqrt{\delta}) + \omega^2(f; \Delta), & \text{if } \theta - \theta_1 \leq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \leq \phi_2 - \phi_3,
\end{array} \right.
\]

4. A \(k^{th}\) order generalization of the operators \(\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}\)

In this section, we use the method of Kirov and Popova [13] to introduce and investigate approximation properties of a \(k^{th}\) order generalization of our bi-variate Bernstein-Stancu-Kantorovich type operator \(\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(; x, y)\) defined in (6). Let \(C^k(P), k \in \mathbb{N} \cup \{0\}\), denote the set of all functions \(f : P \rightarrow \mathbb{R}\) having continuous partial derivatives upto the \(k^{th}\) order on the box \(P\). We now define, for any function \(f \in C^k(P)\), the \(k^{th}\) order generalization of Bernstein-Stancu-Kantorovich type polynomials \(\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi)}(; x, y)\) as

\[
\mathcal{R}_{n,m,0,\phi}^{\Omega,(\theta,\varphi),k}(f(u, v); x, y) = \left( \frac{n + \theta}{n} \right)^m \left( \frac{m + \phi}{m} \right)^m (n + \theta_1 + 1)(m + \phi_1 + 1)
\]
\[
\times \sum_{s=0}^{n} \sum_{r=0}^{m} \Omega_{n,m,(\theta,\varphi),s} \Omega_{m,(\phi,\varphi),r}(y) \int_{\frac{n-s}{n+1}}^{\frac{n-r}{m+1}} \int_{\frac{n-s}{n+1}}^{\frac{n-r}{m+1}} \sum_{l=0}^{k} \frac{d^l f(u, v)}{dudv} \bigg|_{(y, x)} dudv,
\]

where \(d^l f(u, v) = \sum_{l=0}^{k} \left( \int_{\mathbb{R}^2} \frac{\partial^l f(u, v)}{\partial x^l \partial y^l} (x - u)^l(y - v)^l \right)\).

Now, there is a unit vector \((\mu, \eta)\) for which \((x - u, y - v) = w(\mu, \eta)\) where \(w > 0\). Let
\[
P(w) = f(u + w\mu, v + w\eta) = f(u + (x - u), v + (y - v)) = f(x, y).
\]

Following remarks can be made from the equations (18) and (19).
Remark 4.1. Note that, when $k = 0$ in the equation (18), we immediately get the operator defined in (6), i.e.

$$S_{n,m,0,0}^{0,\phi_{0}}(f; x, y) = S_{n,m,0,0}^{0,\phi_{0}}(f; x, y)$$

Remark 4.2. The $k^{th}$ order derivative of the function $P(w)$ has the following form (See chapter 3 in [7])

$$P^{k}(w) = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^{k} f(u + w \mu, v + w \eta)}{\partial x^{k-i} y^{i}} \mu^{k-i} \eta^{i}, \quad (k \in \mathbb{N}).$$

(20)

Also, using the equation (20), we can easily deduce that the Taylor’s formula for $P(w)$ at $w = 0$ is the same as that of $f(x, y)$ at $(u, v)$.

The following intermediate result is useful in the proof of some important corollaries which provide us a deeper insight into the approximation behavior of the operators defined by (18):

**Theorem 4.3.** For any $m, n, k \in \mathbb{N},$ and for all $f \in C^{k}(I^{2})$ such that $P^{k}(w) \in \text{Lip}_{M}(\gamma)$, we have

$$\|f - S_{n,m,0,0}^{0,\phi_{0}}(f)\|_{C(I^{2})} \leq \frac{M}{(k-1)!} \frac{(\gamma + k)}{\gamma} B(\gamma, k) \times \|R_{n,m,0,0}^{0,\phi_{0}}(n - u, y - v)\|^{k+\gamma}_{C(I^{2})},$$

where $0 < \gamma \leq 1, M > 0$ and $B(\gamma, k)$ denotes the usual Beta function.

Proof. Let $f \in C^{k}(I^{2})$ and $(x, y) \in I^{2}$. By the definition of the operators $S_{n,m,0,0}^{0,\phi_{0}}(f; x, y)$ in (18), we see that for any $m, n, k \in \mathbb{N},$

$$f(x, y) - S_{n,m,0,0}^{0,\phi_{0}}(f; u, v), x, y) = \left(\frac{n + \theta_{1}}{n}\right)^{n} \left(\frac{m + \phi_{1}}{m}\right)^{m} (n + \theta_{1} + 1)(m + \phi_{1} + 1)$$

$$\times \sum_{i=0}^{n} \sum_{j=0}^{m} \Omega_{n,m}^{(0,\phi_{0})}(x) \Omega_{n,m}^{(0,\phi_{0})}(j) \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} f(x, y) - \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^{k} f(u + z(x - u), v + z(y - v))}{\partial x^{k-i} y^{i}} (x - u)^{k-i} (y - v)^{i} dz.$$

(21)

It is known from Taylor’s integral remainder formula for $f(x, y)$ at $(u, v)$ (see [7]) that

$$f(x, y) - \sum_{i=0}^{k} \frac{d^{i} f(u, v)}{i!} = \frac{1}{(k-1)!} \int_{0}^{1} (1 - z)^{k-1} \times \left(\sum_{i=0}^{k} \binom{k}{i} \frac{\partial^{k} f(u + z(x - u), v + z(y - v))}{\partial x^{k-i} y^{i}} (x - u)^{k-i} (y - v)^{i} \right) dz.$$

(22)

Using Remark 4.2, the equation (22) takes the form

$$P(u) - \sum_{i=0}^{k} P^{i}(0) w^{i} = \frac{w^{k}}{(k-1)!} \int_{0}^{1} (1 - z)^{k-1} [P^{k}(wz) - P^{k}(0)] dz.$$

Since $P^{k}(w) \in \text{Lip}_{M}(\gamma)$, it follows that

$$\left|f(x, y) - \sum_{i=0}^{k} \frac{d^{i} f(u, v)}{i!}\right| = \left|P(u) - \sum_{i=0}^{k} P^{i}(0) w^{i}\right| \leq \frac{Mw^{k+\gamma}}{(k-1)!} \int_{0}^{1} z^{\gamma} (1 - z)^{k-1} dz.$$

(23)

From the definition of Beta function, we have

$$\int_{0}^{1} z^{\gamma} (1 - z)^{k-1} dz = B(1 + \gamma, k) = \frac{\gamma B(\gamma, k)}{\gamma + k},$$

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Hence, the equation (23) takes the following from
\[ \left| f(x, y) - \sum_{j=0}^{k-1} \frac{d^j f(u, v)}{j!} \right| \leq \frac{M}{(k-1)!} \gamma B(y, k) \left| (x - u, y - v) \right|^k \]  
(24)

Finally, using (24) in (21) and taking supremum over all \((x, y) \in I^2\), we obtain the desired result. \(\Box\)

Let \(g \in C(I^2)\) be a function defined by
\[ g(u, v) = \left| (u, v) - (x, y) \right|^k \]  
(25)

Since \(g \in C(I^2)\) and \(g(x, y) = 0\), Theorem 3.1 yields
\[ \|R_n^{0,0;k}(g; x, y)\|_{C(I^2)} \to 0 \text{ as } m, n \to \infty. \]

Thus, Theorem 4.3 yields that for all \(f \in C^k(I^2)\) such that \(P^k(w) \in \text{Lip}_M(y)\),
\[ \|R_n^{0,0;k}(f; x, y) - f(x, y)\|_{C(I^2)} \to 0 \text{ as } m, n \to \infty. \]

Taking into consideration Theorem 2, one can deduce the following result from Theorem 4.3 immediately:

**Corollary 4.4.** If \(\theta_2 - \theta_3 \geq 1\) and \(\phi_2 - \phi_3 \geq 1\), then for each \(m, n \in \mathbb{N}\), and for all \(f \in C^k(I^2)\) such that \(P^k(w) \in \text{Lip}_M(y)\) we have
\[ \|f - R_n^{0,0;k}(f)\|_{C(I^2)} \leq \frac{2M}{(k-1)!} \gamma B(y, k) \times \begin{cases} \omega_d^{(0)}(g; \gamma) \left( \frac{4(0-0)^2+i+8}{(m+1)\gamma} + \frac{4(0-0)^2+i+8}{(m+1)\gamma} \right); & \text{for } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\ \omega_d^{(0)}(g; \gamma) \left( \frac{4(0-0)^2+i+8}{(m+1)\gamma} + \frac{4(0-0)^2+i+8}{(m+1)\gamma} \right); & \text{for } \theta - \theta_1 < \theta_2 - \theta_3 \text{ and } \phi - \phi_1 < \phi_2 - \phi_3. \end{cases} \]

where \(g\) is given by (25).

Applying Theorem 3.6, the following result is immediate from Theorem 4.3:

**Corollary 4.5.** For each \(m, n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}\) and \(f \in C^k(I^2)\) such that \(f^{(k)} \in \text{Lip}_M(y)\), and assuming that \(g \in \text{Lip}_M^{2+\gamma}(y)\) in Theorem 3.6, we have
\[ \|f - R_n^{0,0;k}(f)\|_{C(I^2)} \leq \frac{2M}{(k-1)!} \gamma B(y, k) \times \begin{cases} \left( \frac{(\theta-0)^2+i}{(m+1)\gamma} + \frac{(\theta-0)^2+i}{(m+1)\gamma} \right); & \text{for } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\ \left( \frac{(\theta-0)^2+i}{(m+1)\gamma} + \frac{(\theta-0)^2+i}{(m+1)\gamma} \right); & \text{for } \theta - \theta_1 < \theta_2 - \theta_3 \text{ and } \phi - \phi_1 < \phi_2 - \phi_3. \end{cases} \]

Lastly, taking into account Theorem 3.8, we can easily deduce the following from Theorem 4.3:

**Corollary 4.6.** For all \(f \in C^k(I^2)\) such that \(f^{(k)} \in \text{Lip}_M(y)\), if \(\theta_2 - \theta_3, \phi_2 - \phi_3 \geq 1\), then we obtain
\[ \|f - R_n^{0,0;k}(f)\| \leq \frac{M}{(k-1)!} \gamma B(y, k) \times \begin{cases} 4K(g, \delta_1) + \omega_d^{(0)}(f; \Delta); & \text{if } \theta - \theta_1 \geq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \geq \phi_2 - \phi_3 \\ 4K(g, \delta_2) + \omega_d^{(0)}(f; \Delta); & \text{if } \theta - \theta_1 \leq \theta_2 - \theta_3 \text{ and } \phi - \phi_1 \leq \phi_2 - \phi_3. \end{cases} \]

where \(\delta_1, \delta_2, \Delta\) are given in Theorem 3.8 and \(g\) is defined by (25).
Example 1. Let \( f(x, y) = (x + 2)^3 y^4 \) and \( \theta_1 = 1, \theta_2 = 2, \theta_1 = 3, \theta = 4 \) and \( \phi_3 = 1, \phi_2 = 2, \phi_1 = 3, \phi = 4 \). The convergence of the operators \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) and \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) to the function \( f \) for \( n = m = 5 \) and \( k = 2 \) and \( k = 5 \) is illustrated in Figure 1 and Figure 2 respectively. It is seen that if \( f \) is differentiable \( k \) times then \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) yields a better convergence in comparison to the classical Bernstein-Stancu-Kantorovich operator \( R_{n,m,\theta,\phi}^{0,\phi}(f) \). In Table 1, we obtain estimates of the maximum absolute errors in the approximation of the function \( f(x, y) = (x + 2)^3 y^4 \) by using the operators \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) as defined in (6) and \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) as given in (18), namely \( E_{n,m,\theta,\phi}^{0,\phi,k} = ||R_{n,m,\theta,\phi}^{0,\phi,k}f - f||_{C[\Omega]}, \) and \( E_{n,m,\theta,\phi}^{0,\psi,k} = ||R_{n,m,\theta,\phi}^{0,\psi,k}f - f||_{C[\Omega]} \), respectively.

![Figure 1: Illustration of error bound](image)

**Table 1: Comparison of** \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) **and** \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) **for** \( n = m = 5 \) **and some values of** \( k \)

| \( m, n \) | Error bound for \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) | Derivative order \( k \) | Error bound \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) |
|-----------|---------------------------------|-----------------|----------------|
| 5,5       | 2.2878                          | 2               | 0.1323         |
| 5,5       | 2.2878                          | 3               | 0.0263         |
| 5,5       | 2.2878                          | 4               | 0.0015         |
| 5,5       | 2.2878                          | 5               | 0.0002         |

Example 2. For \( m = n = 5 \) and \( \theta_1 = 1, \theta_2 = 2, \theta_1 = 3, \theta = 4 \) and \( \phi_3 = 1, \phi_2 = 2, \phi_1 = 3, \phi = 4 \), the estimates of the maximum absolute errors in the approximation of the function \( f(x, y) = (x + 3)^2 e^{-y} \) by using operators \( R_{n,m,\theta,\phi}^{0,\phi}(f) \) and \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) are listed in Table 2. The convergence of the operators \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) and \( R_{n,m,\theta,\phi}^{0,\psi,k}(f) \) to the function \( f \) for \( k = 2 \) and \( k = 5 \) is illustrated in Figure 2. Further from the figure 2 and Table 2 it follows that, depending on the order of the derivative \( k \), \( R_{n,m,\theta,\phi}^{0,\phi,k}(f) \) gives better approximation to the function \( f \) in comparison to the Bernstein-Stancu-Kantorovich operators \( R_{n,m,\theta,\phi}^{0,\phi}(f) \).
5. Applications

We shall now consider some further generalized Bernstein type polynomials. To obtain an approximation process for \(k^{th}\) order generalization of the operator of Bernstein-type, we introduce some examples;

5.1. Bivariate Bernstein operators in rectangle

In [10], Gadjiev and Ghorbanalizadeh also introduced two dimensional Bernstein polynomials on the rectangle \(\Box = \{(x, y) | \frac{m}{m+\theta_1}, \frac{n}{n+\phi_1}, \frac{m+\theta_2}{m+\theta_1}, \frac{n+\phi_2}{n+\phi_1} \} \) and the polynomials \(B_{m,n}^{(\theta,\phi)}\) defined as follows:

\[
B_{m,n}^{(\theta,\phi)}(f; x, y) = \left(\frac{m + \theta}{m}\right)^m \left(\frac{n + \phi}{n}\right)^n \sum_{j=0}^{m} \sum_{r=0}^{n} \Omega_{m,s}^{(\theta,\phi)}(x) \Omega_{n,r}^{(\phi,\phi)}(y) f\left(\frac{s + m\theta_1 + \phi_1}{m + \theta_1}, \frac{r + n\phi_2}{n + \phi_2}\right),
\]

where the basis functions \(\Omega_{m,s}^{(\theta,\phi)}(x)\), \(\Omega_{n,r}^{(\phi,\phi)}(y)\); \((x, y) \in \Box\) are as defined in (4) and \(\theta, \phi, \theta_1, \phi_1, i = 1, 2, 3\) are non-negative real numbers satisfying \(0 \leq \theta_2 \leq \theta_1 \leq \theta\) and \(0 \leq \phi_3 \leq \phi_2 \leq \phi_1 \leq \phi\).

We consider the following generalization \(B_{m,n}^{(\theta,\phi,k)}(f; x, y)\) of the above linear positive operators:

\[
B_{m,n}^{(\theta,\phi,k)}(f; x, y) = \left(\frac{m + \theta}{m}\right)^m \left(\frac{n + \phi}{n}\right)^n \sum_{j=0}^{m} \sum_{r=0}^{n} \Omega_{m,s}^{(\theta,\phi)}(x) \Omega_{n,r}^{(\phi,\phi)}(y) \sum_{i=0}^{k} \frac{d^i f\left(\frac{s + m\theta_1 + \phi_1}{m + \theta_1}, \frac{r + n\phi_2}{n + \phi_2}\right)}{i!},
\]  

(26)

where

\[
d^i f\left(\frac{s + \theta_3}{m + \theta_1}, \frac{r + \phi_3}{n + \phi_1}\right) = \sum_{i=0}^{l} \binom{l}{i} \frac{\partial^i f\left(\frac{s + m\theta_1 + \phi_1}{m + \theta_1}, \frac{r + n\phi_2}{n + \phi_2}\right)}{\partial x^{l-i} \partial y^i} \left(x - \frac{s + \theta_3}{m + \theta_1}\right)^{l-i} \left(y - \frac{r + \phi_3}{n + \phi_1}\right)^i.
\]  

(27)
Example 3. For $\theta_3 = 1$, $\theta_2 = 2$, $\theta_1 = 3$, $\theta = 4$, $f(x,y) = (x + 3)^2 e^{-y}$ and $\varphi_3 = 1$, $\varphi_2 = 2$, $\varphi_1 = 3$, $\varphi = 4$, the convergence of the operators $B^{(0,\varphi,\phi)}_{m,n}(f)$ towards the function $f(x,y)$ for $k = 0, 2, 5$ is illustrated in Fig.3. From Fig 3 it is clear that the operators $B^{(0,\varphi,\phi)}_{m,n}(f)$ provides better approximation than the operator $B^{(0,\varphi,\phi)}_{m,n,0}(f)$ for both $k = 2, 5$. In Table 3, we observe that as the value of the order $k$ of the derivative increases, the error in the approximation of function $f$ by the operator $B^{(0,\varphi,\phi)}_{m,n,0}(f)$ becomes smaller.

| $m, n$ | Error bound for $B^{(0,\varphi,\phi)}_{m,n}$ | Derivative order $k$ | Error bound $B^{(0,\varphi,\phi)}_{m,n,0}$ |
|--------|---------------------------------|------------------|----------------------------------|
| 5,5    | 0.4079                          | 2                | 0.0103                           |
| 5,5    | 0.4079                          | 5                | 0.000004456                      |

Figure 3: $B^{(0,\varphi,\phi)}_{m,n,0,0}(f)$ approximates $f(x,y)$ much better than $B^{(0,\varphi,\phi)}_{m,n,0}(f)$

5.2. Bivariate-Stancu type operators in a triangle

Gadjiev and Ghorbanalizadeh [10] defined another bivariate Bernstein-Stancu type operators on the triangle $\Delta$ for the functions $f : \Delta = \{(x,y) : x + y \leq \frac{m+\theta}{m+\theta} ; x, y \geq \frac{\theta}{m+\theta}\} \rightarrow \mathbb{R}$. More precisely, in [10], they considered $R^{(0,\varphi,\phi)}_{m,n,0,0}$ with:

$$
R^{(0,\varphi,\phi)}_{m,n,0,0}(f;x,y) = \left(\frac{m+\theta}{m}\right)^n \sum_{s=0}^{m} \sum_{r=0}^{m-s} \Omega^{(0,\varphi,\phi)}_{m,n,s,r}(x,y) f\left(\frac{s + \theta_3}{m + \theta_1}, \frac{r + \phi_3}{m + \phi_1}\right),
$$

where the basis functions $\Omega^{(0,\varphi,\phi)}_{m,n,s,r}(x)$ are defined by

$$
\Omega^{(0,\varphi,\phi)}_{m,n,s,r}(x) = \left(\frac{m}{s}\right)\left(\frac{m-s}{r}\right)^{m-s} \left(x - \frac{\theta_2}{m + \theta}\right)^r \left(y - \frac{\theta_2}{m + \theta}\right)^s \left(m + 2\theta_2 - x - y\right)^{m-s-r},
$$

and $\theta, \varphi, \theta_i, \varphi_i, i = 1, 2$ are the positive numbers satisfying $0 < \theta_2 \leq \theta_3 \leq \theta_1 \leq \theta$ and $0 < \varphi_2 \leq \varphi_3 \leq \varphi_1 \leq \varphi$.

The authors [10] derived the rate of convergence in terms of the complete and partial moduli of continuity for operators $R^{(0,\varphi,\phi)}_{m,n,0,0}$. 
We now introduce the $k$th order generalization of the operators $B_{m,\theta,\phi}^{0,\phi,k}$:

$$B_{m,\theta,\phi}^{0,\phi,k}(f; x, y) = \left(\frac{m + \theta}{m}\right)^m \sum_{s=0}^{m-m} \sum_{r=0}^{m-s} \Omega_{m,s,r}(x, y) \sum_{l=0}^{k} \frac{d^l f}{l!} \left(\frac{s+\theta_{1}}{m+\theta_{1}}, \frac{r+\phi_{1}}{m+\phi_{1}}\right),$$

where $d^l f$ is given by (27).

**Example 4.** Let $\theta_{3} = 1, \theta_{2} = 2, \theta_{1} = 3, \theta = 4, f(x, y) = y^3 e^{-2x}$ and $\phi_{3} = 1, \phi_{2} = 2, \phi_{1} = 3, \phi = 4$, and $m = 5$. In Fig. 4, the comparison of convergence of the operators $B_{m,\theta,\phi}^{0,\phi}$ and $B_{m,\theta,\phi}^{0,\phi,k}$, $k = 2, 5$ towards the function $f(x, y)$ is illustrated. From Table 4, it is clear that the Bernstein-Stancu-Taylor operators $B_{m,\theta,\phi}^{0,\phi,k}$ give us a better approximation to $f(x, y)$ compared to Bernstein-Stancu operators $B_{m,\theta,\phi}^{0,\phi}$. Further, it may be remarked that the parameters $\theta_{3}, \theta_{2}, \theta_{1}, \theta$ and $\phi_{3}, \phi_{2}, \phi_{1}, \phi$, play an important role to achieve a better approximation.

![Figure 4: $B_{m,\theta,\phi}^{0,\phi,k}(f)$ approximates $f(x, y)$ much better than $B_{m,\theta,\phi}^{0,\phi}(f)$](image)

**Table 4: Comparison of $B_{m,\theta,\phi}^{0,\phi}(f)$ and $B_{m,\theta,\phi}^{0,\phi,k}(f)$ for $m = 5$ and some values of $k$**

| $m$ | Error bound for $B_{m,\theta,\phi}^{0,\phi}$ | Derivative order $k$ | Error bound $B_{m,\theta,\phi}^{0,\phi,k}$ |
|-----|---------------------------------------|---------------------|--------------------------------------|
| 5   | 0.1675                                 | 2                   | 0.0340                               |
| 5   | 0.1675                                 | 5                   | 0.0004425                            |

### 5.3. Bivariate Stancu-Kantorovich operators in a triangle

Inspired by the work [10], we present the following bivariate extension of the operators (28) on the triangle $\Delta = \{(x, y) : x + y \leq \frac{m+2\theta_{1}}{m+\theta_{1}}, x, y \geq \frac{\theta_{2}}{m+\theta_{2}}\}$:

$$B_{m,\theta,\phi}^{0,\phi}(f; x, y) = (m + \phi_{1} + 1)(m + \theta_{1} + 1) \left(\frac{m + \theta}{m}\right)^m \sum_{j=0}^{m} \sum_{l=0}^{m-j} \Omega_{m,j,l}(x, y) \int_{\frac{j+\theta_{1}}{m+\theta_{1}}}^{\frac{j+\theta}{m+\theta}} \int_{\frac{l+\phi_{1}}{m+\phi_{1}}}^{\frac{l+\phi}{m+\phi}} f(u, v) du dv,$$

where the basis functions $\Omega_{m,j,l}(x, y)$ are as defined by (28). At last, we define the Bernstein-Stancu-Kantorovich-Taylor extension of these operators as follows:
For $f \in C^k(l^2)$, $k \in \mathbb{N} \cup \{0\}$, we propose

$$
\Psi^{m,\theta,k}_{m,\theta,\phi}(f(u,v);x,y) = (m + \phi_1 + 1)(m + \theta_1 + 1)\left(\frac{m + \theta}{m}\right)^m \sum_{j=0}^{m} \sum_{l=0}^{m-j} \Omega_{m,j,l}^{(l),\phi}(x,y) \times \int_{r_{m+2k+1}}^{\infty} \int_{r_{m+2k+1}}^{\infty} \sum_{r=0}^{k} \frac{d^r f(u,v)}{r!} du dv,
$$

(31)

where $d^r f(u,v) = \sum_{j=0}^{r} \left(\int_{i}^{j} \frac{\partial^i f(u,v)}{\partial x^i \partial y^j}(x-u)^{-i}(y-v)^{j}\right)$.

**Remark 5.1.** It is remarked that the results analogous to Theorem 4.3 and the resulting corollaries can be easily deduced for the above $k^{th}$ order generalizations (26), (29) and (31).

**Example 5.** Since $f(x, y) = e^{-2x}y^3$ is infinitely continuously differentiable on $\mathbb{R}^2$, we can use Bernstein-Stancu-Kantorovich-Taylor operators to study the approximation of $f$ on $l^2$. It is observed that, we achieve a better approximation by these operators in comparison to Bernstein-Stancu-Kantorovich operators, if we make a suitable choice of the parameters. For $m = 5$, $k = 2, 5$ and $\theta_2 = 3$, $\theta_3 = 4$ and $\phi_1 = 1$, $\phi_2 = 2$, the illustrative graphics of $\Psi^{0,\phi}_{m,\theta,\phi}$, $\Psi^{1,\phi}_{m,\theta,\phi}$, $\Psi^{2,\phi}_{m,\theta,\phi}$ and the function $f(x, y) = e^{-2x}y^3$ are shown together in Fig. 5. From the estimates of the absolute maximum errors in the approximation of $f(x, y)$ by the operators $\Psi^{0,\phi}_{m,\theta,\phi}$ in (30) and $\Psi^{1,\phi}_{m,\theta,\phi}$ in (31) for $m = 5$ and $k = 2, 5$ presented in Table 5, it turns out that as the value of $k$ increases, the error becomes smaller.

![Figure 5](image.png)

**Table 5:** Comparison of $\Psi^{0,\phi}_{m,\theta,\phi}(f)$ and $\Psi^{1,\phi}_{m,\theta,\phi}(f)$ for $m = 5$ and some values of $k$

| $m$ | Error bound for $\Psi^{0,\phi}_{m,\theta,\phi}$ | Derivative order $k$ | Error bound $\Psi^{1,\phi}_{m,\theta,\phi}$ |
|-----|-----------------------------------------------|----------------------|-------------------------------|
| 5   | 0.1030                                        | 2                    | 0.0228                         |
| 5   | 0.1030                                        | 5                    | 0.0002907                      |
6. Conclusion

The Stancu-Kantorovich operators and the $k$th order generalization of Bernstein-Stancu-Kantorovich type operators for functions of two variables are constructed with the help of modified Bernstein basis functions with shifted knots for $x, y \in \left[ \frac{\theta}{m} \cdot \frac{n+\theta}{m} \right] \times \left[ \frac{\phi}{m} \cdot \frac{m+\phi}{m} \right]$. By introducing the parameters $\theta, \phi, \theta_i, \phi_i, i = 1, 2, 3$ we enable the shift of Bernstein basis functions over the subintervals of $I$. A simulation was performed through MATLAB and it was shown that depending on the order of the derivative $k$, the $k$th order generalization of Bernstein-Stancu-Kantorovich type polynomials $K_{\theta, \phi}^{h, k}(x, y)$ shows much better approximation results to a function compared to Bernstein-Stancu-Kantorovich operators which are presented in Examples 1 and 2. Finally, the $k$th order generalizations of the generalized bivariate Bernstein type polynomials are studied and elaborated by means of some examples.

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