Geometric classification of real ternary octahedral quartics
Noémie Combe

To cite this version:
Noémie Combe. Geometric classification of real ternary octahedral quartics. Jeunes chercheurs en singularités, Feb 2014, Marseille, France. hal-01128553

HAL Id: hal-01128553
https://hal.science/hal-01128553v1
Submitted on 9 Mar 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
3. Example of a geometric classification for \( n = 3 \)

**Definitions**

- Let \( G \) be a finite group and \( \rho : G \to GL(n, \mathbb{R}) \) a representation of \( G \).
- Then \( G \) acts on the vector space \( V = \mathbb{R}^n \) through linear transformations, and this may be extended to \( \mathbb{F}[V] \) by the formula \( (g\rho)(v) := f(\rho(g^{-1})v) \forall v \in V \). One of the basic objects of study in invariant theory is the set of \( G \)-invariant polynomials \( \mathbb{F}[V]^G := \{ f \in \mathbb{F}[V] | g\rho f = f \forall g \in G \} \).
- \( \mathbb{F} \) is the field of real numbers.
- \( G = B_3 \), the Coxeter group.

**Example of a geometric classification for \( n = 3 \)**

Let \( n = 3 \). The following equation defines an octahedral quartic.

\[
f_{-\alpha} = \beta (x^2y^2 + y^2z^2 + z^2x^2) - \left( x^2 + y^2 + z^2 + \frac{1}{2} \right)^2 + \frac{1 - \beta}{4} = 0. \tag{1}
\]

1. \( 0 < \beta < 3 \)
   - \( \beta < \frac{3}{4} \): \( Q^{--}(k) = C \), cuboid vanishing in \( \{0\} \).
2. \( \beta = 3 \)
   - \( \beta < 0 \) the surface is an eight branched star
3. \( 3 < \beta < 4 \)
   - \( \beta < 1 \): \( Q^{--}(k) = \# \check{H}_4 \# C \), \# the surface is the connected sum of a cube with hyperbolic sheets having the diagonal of the cube as axis.
   - \( \beta = 3 \): \( Q^{--}(\frac{3}{4}) \), singular quartic, analogus to the Cayley’s cubic: cube with cones at each vertex. this surface has eight conical singular points at the vertices of a cube.
   - \( \beta < 0 \): \( Q^{--}(k) = \check{H}_4 \cup \check{H}_4 \cup C \).
4. \( \beta = 4 \)
   - \( \beta < 0 \): \( Q^{--}(k) = \check{S}_4 \cup \check{S}_4 \) six disjoint hyperbolic sheets with section of square’s type \( \check{S}_4 \), and axes being the coordinate axes \( | \).
   - \( \beta = 0 \): \( Q^{--}(0) = \check{S}_4 \cup \check{S}_4 \cup \{0\} \).
   - \( 0 < \beta < 1 \): \( Q^{--}(k) = \check{S}_4 \cup \check{S}_4 \cup C \) is the disjoint union of six hyperbolic sheets \( \check{S}_4 \), and an octahedron centered at the origin \( \{0\} \).

**References**

1. W. Barth *Two projective surfaces with many nodes, admitting the symmetries of the icosahedron* Journal of Algebraic Geometry 5 (1996), 173-186
2. R. S. Burington. *A classification of quadrics in affine n-space by mean of arithmetic invariant* Amer. Math. Monthly. 39, (1932), 527-532.
3. J.S.Cassels An Introduction to the geometry theory of numbers. Classics in Mathematics Springer (1971)

**Figures in \( \mathbb{R}^3 \)**

1. Figure 1: Figure of two nested compact connected components
2. Figure 2: Compact singular surface, 12 conic singularities

**Conclusions and results**

- Let \( f \in \mathbb{R}[x_1, x_2, \ldots, x_k]^{B_n}, \deg(f) = 4 \). The octahedral surface defined by \( f \) has at most \( \frac{k^2}{2} \) connected components for a given family of parameters.
- The \( B_4 \) quartic hyper-surface has at most \( \frac{k^2}{4} \) compact connected components.
- If \( B_4 \) quartic hyper-surface has at most two connected components, then they are necessarily necessary.
- The maximal bound on the number of connected components of an octahedral quartic surface is \( 8 \) in \( \mathbb{R}^2 \). These compact connected components in \( \mathbb{R}^2 \) are homeomorphic to 2-spheres.
- The maximal number of isolated singularities of an octahedral quartic is 12.