ASYMPTOTIC BEHAVIOR OF A SECOND-ORDER SWARM SPHERE MODEL AND ITS KINETIC LIMIT

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Abstract. We study the asymptotic behavior of a second-order swarm model on the unit sphere in both particle and kinetic regimes for the identical cases. For the emergent behaviors of the particle model, we show that a solution to the particle system with identical oscillators always converge to the equilibrium by employing the gradient-like flow approach. Moreover, we establish the uniform-in-time ℓ_2-stability using the complete aggregation estimate. By applying such uniform stability result, we can perform a rigorous mean-field limit, which is valid for all time, to derive the Vlasov-type kinetic equation on the phase space. For the proposed kinetic equation, we present the global existence of measure-valued solutions and emergent behaviors.

1. Introduction. Emergence of collective behavior has been widely studied not only in applied mathematics, but also in other scientific disciplines, for instance, control theory in engineering community [24, 33, 36], active matter in statistical physics [3, 28, 31] and swarming behavior in quantitative biology [14, 17, 34]. In spite of its crucial role in biological processes, it has been only fifty years since the mathematical study of such collective motion started after seminal work of Winfree [38] and Kuramoto [25]. Among well-known models describing collective oscillatory behavior, to name a few, the Cucker-Smale model [13], the Kuramoto model [25] and the Vicsek model [37], our main interest lies in the emergent dynamics on the d-dimensional unit sphere S^{d-1} embedded in the d-dimensional Euclidean space R^d. For the simplest case, the first-order dynamics has been used to describe aggregation of particles whose norms are preserved for all time:

\[
\begin{aligned}
\dot{x}_i &= \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i), \quad t > 0, \\
x_i(0) &= x_i^0, \quad |x_i^0| = 1, \quad i = 1, \cdots, N.
\end{aligned}
\]

(1)

Here, \langle \cdot, \cdot \rangle and | \cdot | denote the standard inner product in R^d and its induced norm in R^d, respectively, \Omega_i \in \text{Skew}_d(R) is a d \times d skew-symmetric matrix so that each norm of x_i is conserved along the dynamics, and \kappa stands for a positive coupling strength. In addition, the communication term in the right-hand side can be interpreted in

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the following way [32]: each agent moves towards the projected average position of its neighbors on a sphere. See the last paragraph in Section 2.1 for literature review for (1).

Recently, Ha and Kim proposed a second-order extension to the first-order model (1) in [18] by incorporating an inertial force and centripetal force:

\[
\begin{align*}
\dot{m}\ddot{x}_i &= -m\langle \dot{x}_i, \dot{x}_i \rangle x_i - \gamma \dot{x}_i + \Omega_i |x_i|^2 + \frac{\kappa}{N} \sum_{k=1}^{N} \left( x_k - \langle x_i, x_k \rangle x_i \right), \quad t > 0, \\
(x_i, \dot{x}_i)(0) &= (x_i^0, \dot{x}_i^0), \quad i = 1, \ldots, N,
\end{align*}
\]

where \( m \) and \( \gamma \) are nonnegative mass and friction coefficient, respectively. For the newly proposed model (2), it can be heuristically derived from Hamiltonian dynamics theory together with a frictional force, an one-body force field and an interaction force field. For more detailed derivation, we refer the reader to Appendix A in [35] or Section 2.1 in [18]. Same as the first-order model (1), we see that the unit sphere is positively invariant under the flow (2) (see Lemma 2.2). Then, (2) can be reduced to the model on \( S^{d-1} \times \mathbb{R}^d \):

\[
\begin{align*}
\dot{m}\ddot{x}_i &= -m|\dot{x}_i|^2 x_i - \gamma \dot{x}_i + \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i), \quad t > 0, \\
(x_i, \dot{x}_i)(0) &= (x_i^0, \dot{x}_i^0), \quad |x_i^0| = 1, \quad (x_i^0, \dot{x}_i^0) = 0, \quad i = 1, \ldots, N.
\end{align*}
\]

In [18], they provided some restricted classes of initial data leading to the complete aggregated state and incoherent state for attractive coupling and repulsive coupling regimes, respectively.

The main result of this paper consists of two parts: the particle model (3) and its kinetic model (4). Our first main result concerns with asymptotic behavior for the particle model. More precisely, we utilize the gradient-like flow approach to show that a solution to (3) always converges to the equilibrium (see Theorem 3.1). However, this does not imply that the complete aggregation occurs for all initial data (see Definition 2.1 for the definition of the complete aggregation). Together with the linear stability analysis performed in Section 4.2 of [18], we would say that the complete aggregation emerges for generic initial data. Note that the gradient-like flow approach to the second-order system is first studied in [23] and after then, this work has been applied in the context of collective dynamics. See for instance [11, 12] in which the gradient-like flow approach is used for the inertial Kuramoto model.

On the other hand, uniform-in-time \( \ell_2 \)-stability with respect to the initial data is established in which we a priori assume that the complete aggregation occurs. This kind of stability theorem states that difference of any two solutions can be uniformly controlled by difference of initial data. To be more specific, for any two solutions \( Z \) and \( \tilde{Z} \) to (3), there exists a uniform constant \( C \) which does not depend on \( N \) and \( t \) such that

\[
\sup_{0 \leq t < \infty} |Z(t) - \tilde{Z}(t)| \leq C |Z(0) - \tilde{Z}(0)|.
\]

It follows from the literature that finite-in-time estimate always holds when the system admits a sufficiently regular solution, in other words, for any \( T > 0 \), the uniform stability estimate holds until such a time, that is, it holds on \([0, T)\). However, \( T \) does not allow infinity in general. To investigate asymptotic behavior of a solution towards \( t \to \infty \), we need \( T = \infty \). In fact, in Theorem 3.3, we show that the uniform \( \ell_2 \)-stability is valid for all time, hence \( T \) can achieve infinity.
Our second main result deals with rigorous derivation of a kinetic description for (3) using uniform-in-time $\ell_2$-stability. As a result, we obtain the kinetic equation for a density $f = f(t, \omega, v, \Omega)$ corresponding to (3):

\[
\begin{align*}
\partial_t f + v \cdot \nabla_\omega f + \nabla_v \cdot (L[f]f) &= 0, \\
f(0, \omega, v, \Omega) &= f_0(\omega, v, \Omega),
\end{align*}
\]

where the interaction term is defined as

\[
L[f](t, \omega, v, \Omega) = \frac{1}{m} \left( -\gamma v - m|v|^2\omega + \Omega \omega \right) + \kappa \int_D (\mathcal{P}(\omega)\omega_\ast) f(t, \omega_\ast, v_\ast, \Omega_\ast) dS_{\omega_\ast} dv_\ast d\Omega_\ast.
\]

Indeed, by following the same procedure of the first-order model, we first identify the kinetic equation for (2) defined on $\mathbb{R}^d \times \mathbb{R}^d \times \text{Skew}_d(\mathbb{R})$ and then it follows from the invariance of manifold $D$ that our target equation (4) is derived from the method of characteristic (see Section 2.3). Once we attain the kinetic equation from the particle system using mean-field limit, then the global existence of measure-valued solution for the kinetic equation directly follows by measure-theoretical formulation (see Theorem 3.4). For detailed description, we refer the reader to Section 5 of [22] for the measure-valued solutions for the Cucker-Smale dynamics. Moreover, asymptotic behavior of the measure-valued solutions can be obtained by using the method of characteristic and by handling the particle and kinetic models in a common framework. Thus, it suffices to lift the results for the particle model to the kinetic model (see [9] for a parallel story of the inertial Kuramoto model). For this, we derive the energy estimate to show that the kinetic energy converges zero with an integrable decay rate. In addition, we see that the measure-valued solution tends to a bi-polar state (see Theorem 3.5 and Corollary 1).

The rest of this paper is organized as follows. In Section 2, we briefly introduce recent progress on the first-order swarm sphere model and present previous results on (3). Moreover, we discuss the basic properties of our main model (3) and its kinetic model (4). Section 3 contains the summary of the main results whose proofs will be completed in the following two sections. In Section 4, we first study the emergent behavior of the particle model in which the gradient-like flow approach and uniform-in-time $\ell_2$-stability are used. Section 5 deals with the kinetic model derived from the particle model and the global existence of a measure-valued solution. Moreover, we present the asymptotic behavior of the measure-valued solution using energy estimate. Finally, Section 6 is devoted to a brief summary of our main results and remaining issues for future work.

Notation. For vectors $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$ in $\mathbb{R}^d$, we set the $\ell_p$-norm and inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d$:

\[
\|x\|_p := \left( \sum_{i=1}^d |x^i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty) \quad \text{and} \quad \langle x, y \rangle := \sum_{i=1}^d x^i y^i.
\]

In particular, we denote $|x| := \|x\|_2 = \sqrt{\langle x, x \rangle}$ for the simplicity.

2. Preliminaries. In this section, we begin with a brief story of the first-order swarm sphere model which gives an outline of the paper, and discuss the basic
properties of the particle model (3) and the kinetic model (4). Moreover, we present the previous results of (3) and introduce some preparatory lemmas for later use, and provide the relation between (3) and the other models for collective dynamics such as the inertial Kuramoto model and the first-order swarm sphere model.

2.1. A swarm sphere model. Here, we report recent progress on the first-order swarm sphere model which generalizes the well-known Kuramoto model into high-dimensional space: for $x_i \in \mathbb{R}^d$, 

$$
\begin{align*}
\dot{x}_i &= \Omega_i \frac{x_i}{|x_i|^2} + \frac{\kappa}{N} \sum_{k=1}^{N} \left( x_k - \frac{x_i \langle x_i, x_k \rangle}{\langle x_i, x_i \rangle} \right), \quad t > 0, \\
x_i(0) &= x_i^0, \quad i = 1, \ldots, N.
\end{align*}
$$

(5)

Here, $\Omega_i \in \text{Skew}_d(\mathbb{R})$ is a $d \times d$ skew-symmetric matrix so that each norm of $x_i$ is conserved along time, and $\kappa$ denotes a positive coupling strength. Hence, if we choose the initial data satisfying $x_i^0 \in S^{d-1}$, that is, $|x_i^0| = 1$ for all $i = 1, \ldots, N$, then system (1) reduces to the model on $S^{d-1}$: 

$$
\begin{align*}
\dot{x}_i &= \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i), \quad t > 0, \\
x_i(0) &= x_i^0 \in S^{d-1}, \quad i = 1, \ldots, N.
\end{align*}
$$

(6)

In fact, the communication term in the right-hand side is represented as the orthogonal projection onto the tangent plane [21, 30]. Hence, one can rewrite (2) as the gradient flow (see Section 4.1). We here recall the definition of complete aggregation.

**Definition 2.1.** [7] Let $X = (x_1, \ldots, x_N)$ be a solution to (2), and define the following quantities which measure the degree of aggregation: for $t \geq 0$, 

$$
D(X(t)) := \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)|, \quad x_c(t) := \frac{1}{N} \sum_{i=1}^{N} x_i(t) \quad \text{and} \quad \rho(t) := |x_c(t)|.
$$

Then, we say that system (3) or (2) exhibits the complete aggregation, if the following relation holds: 

$$
\lim_{t \to \infty} D(X(t)) = 0, \quad \text{or, equivalently,} \quad \lim_{t \to \infty} \rho(t) = 1.
$$

On the other hand, in [19], the continuum equation for the one-particle distribution function $F = F(t, x, \Omega)$ corresponding to (1) is formally derived using standard BBGKY hierarchy:

$$
\begin{align*}
\partial_t F + \nabla_x \cdot (\mathcal{L}[F] F) = 0, \quad (x, \Omega) \in \mathbb{E} := \mathbb{R}^d \times \text{Skew}_d(\mathbb{R}), \\
\mathcal{L}[F](t, x, \Omega) := \Omega x \frac{x}{|x|^2} + \kappa \int_{\mathbb{E}} \left( x_s - \frac{x \langle x, x_s \rangle}{\langle x, x \rangle} \right) F(t, x_s, \Omega_s) dx_s d\Omega_s.
\end{align*}
$$

(7)

As in the particle model, we can also check that the domain $E := S^{d-1} \times \text{Skew}_d(\mathbb{R})$ is invariant so that continuum equation (3) can be reduced to the following equation...
defined on $E$:
\[
\begin{aligned}
\partial_t f + \nabla_\omega \cdot (L[f] f) &= 0, \quad (\omega, \Omega) \in E = S^{d-1} \times \text{Skew}_d(\mathbb{R}), \\
L[f](t, \omega, \Omega) &= \Omega \omega + \kappa \int_E (\omega_s - \langle \omega, \omega_s \rangle \omega) f(t, \omega_s, \Omega_s) d\omega_s d\Omega_s.
\end{aligned}
\]  
\tag{8}

Moreover, when the identical oscillators are considered, that is, $\Omega_i \equiv \Omega$ for $i = 1, \cdots, N$, rigorous uniform-in-time derivation of (4) from (2) is obtained in [19]. To the best of our knowledge, uniform-in-time mean-field limit is first presented in [20] where uniform $\ell_{p,q}$-stability estimate for the Cucker-Smale flocking model is established.

The first-order model (2) and (4) have been extensively studied by several groups, to name a few, Frouvelle and Liu [16], Ha [7, 19, 21], Lohe [26, 27], Markdahl [29, 30], Olfati-Saber [32], Ott [5, 6], Piccoli [4] and Zhu [39, 40]. Below, we present brief literature review. For the particle model (2) with identical oscillators, emergence of the complete aggregation has been considered in different contexts. In [32], they first classified the equilibrium of (2) with $\Omega_i \equiv O$ into three types: completely aggregated state, bi-polar state and dispersed (or incoherent) state. Then, they performed a (formal) linear stability analysis of the equilibrium for the system to see that the last two equilibria are unstable. Note that (2) has been also studied in the opinion dynamics [1, 4]. In [4], they employed a specific network structure and a control term into the model to study how they affect the asymptotic behavior of (2). See also [1] for the opinion dynamics on the compact Riemannian manifold. In [39], they constructed a potential function associated to (2) and applied LaSalle’s invariance principle to verify that all vectors $\{x_i\}_{i=1}^N$ converges to the same point. In [29], they formulated system (2) with $\Omega_i \equiv O$ as the gradient flow and use Lojasiewicz inequality to show that the complete aggregation occurs for generic initial data. In [21], they obtained the similar results as shown in [29] simultaneously, and provided a dichotomy for all initial data. More precisely, if $\rho(0) = 0$, then $\rho(t) \equiv 0$ for all time. Otherwise, if $\rho(0) > 0$, only two cases are possible: completely aggregated state and bi-polar state with only one antipodal position. For the latter case, they showed that there exists one and only one single particle that converges to the point, say $x_*$, and the remained $N-1$ particles converge to the antipodal points $-x_*$. On the other hand for the continuum model (4), in [19], its rigorous and uniform-in-time derivation of (4) is obtained using uniform stability estimate. In [16], they studied the asymptotic behavior of a measure-valued solution to (4) and showed that the measure-valued solution tends to a bi-polar state. In [5], they observed that discontinuous phase transition occurs nonhysteretically in odd dimension $d \geq 3$, whereas only continuous phase transition can arise in the Kuramoto model.

2.2. Basic properties. In this subsection, we study the basic properties of (3), such as the invariance of underlying manifolds, orthogonal invariance and solution splitting property.

Lemma 2.2. Let $X = (x_1, \cdots, x_N)$ be a solution to (2) satisfying the initial data:
\[
\langle x_i^0, x_i^0 \rangle = 0, \quad i = 1, \cdots, N.
\]  
\tag{9}

Then, the modulus of $x_i$ is a constant of motion.
\[
\frac{d}{dt} |x_i|^2 = 0, \quad t > 0.
\]
Thus, one has

\[ |x_0^i| = 1 \implies |x_i(t)| = 1, \quad t \geq 0, \quad i = 1, \ldots, N. \]

**Proof.** Although its proof can be found in Lemma 2.1 of [18], we provide its proof for the consistency of the paper. We take an inner product (2) with \( x_i \) to obtain

\[ m\langle \dot{x}_i, x_i \rangle = -m|\dot{x}_i|^2 - \gamma \langle \dot{x}_i, x_i \rangle. \]  

(10)

For the handy notation, we set

\[ u_i := \frac{1}{2} |x_i|^2. \]

By differentiating \( u_i \), one has

\[ \ddot{u}_i = \langle \ddot{x}_i, x_i \rangle, \quad \dot{u}_i = \langle \dot{x}_i, x_i \rangle + |\dot{x}_i|^2. \]

Then, the initial condition (5) is rewritten as

\[ \dot{u}_i(0) = 0. \]  

(11)

We also rewrite (6) in terms of \( u_i \):

\[ m\ddot{u}_i = -\gamma \dot{u}_i, \quad \text{or, equivalently,} \quad \dot{u}_i(t) = \dot{u}_i(0)e^{-\gamma t}, \quad t > 0. \]  

(12)

Hence, in (8), we use the initial condition (7) to obtain the desired result:

\[ \dot{u}_i(t) \equiv 0 \quad \text{or} \quad \frac{1}{2} \frac{d}{dt} |x_i(t)|^2 = 0, \quad t > 0. \]

Due to Lemma 2.2, system (2) can be written as the simplified system (3) defined on \( S^{d-1} \times \mathbb{R}^d \):

\[
\begin{aligned}
& \begin{cases}
  m\ddot{x}_i = -m|\dot{x}_i|^2 x_i - \gamma \dot{x}_i + \Omega_i x_i + \frac{N}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i), \quad t > 0,
  \\
  (x_i, \dot{x}_i)(0) = (x_0^i, \dot{x}_0^i), \quad |x_0| = 1, \quad \langle x_0^i, \dot{x}_0^i \rangle = 0, \quad i = 1, \ldots, N.
\end{cases}
\end{aligned}
\]

Next, we show that system (3) with \( \Omega_i \equiv 0 \) has the rotational symmetry. For a \( d \times d \) orthogonal matrix \( U \), we defined transformed variables:

\[ \tilde{x}_i(t) := Ux_i(t), \quad t > 0, \quad i = 1, \ldots, N. \]  

(13)

**Lemma 2.3.** Let \( X = (x_1, \ldots, x_N) \) be a solution to (3) and \( U \) be a \( d \times d \) orthogonal matrix. Then, the transformed variable \( \tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_N) \) is also a solution to (3) with \( \Omega_i \equiv 0 \).

**Proof.** Since \( U \) is a \( d \times d \) orthogonal matrix, we find for \( v, w \in \mathbb{R}^d \),

\[ \langle Uv, Uw \rangle = \langle v, w \rangle, \quad |Uv|^2 = |v|^2. \]  

(14)

Thus, for \( \Omega_i \equiv 0 \), we multiply (3) with \( U \) to obtain

\[
\begin{aligned}
& mU\ddot{x}_i = -m|\dot{x}_i|^2 Ux_i - \gamma U\dot{x}_i + \frac{N}{N} \sum_{k=1}^{N} (Ux_k - \langle x_i, x_k \rangle Ux_i)

= -|U\dot{x}_i|^2 Ux_i - \gamma U\dot{x}_i + \frac{N}{N} \sum_{k=1}^{N} (Ux_k - \langle x_i, Ux_k \rangle Ux_i).
\end{aligned}
\]

Finally, we use the notation (9) to show the desired assertion. \( \square \)
Remark 1. (i) In fact, the orthogonal invariance property does not hold in general for system (3) with distinct \( \Omega_t \).
(ii) It is worthwhile to mention that system (3) itself does not satisfy the solution splitting property as it is. In order for the property to be valid, we have to modify (3) as follows:

\[
m\ddot{x}_i = -m|\dot{x}_i|^2x_i - \gamma \dot{x}_i + \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i) + \Omega_t x_i + \frac{2m}{\gamma} (\Omega_t \dot{x}_i - \langle x_i, \Omega_t \dot{x}_i \rangle x_i) - \frac{m}{\gamma^2} (\Omega_t^2 x_i - \langle x_i, \Omega_t^2 x_i \rangle x_i).
\]

(15)

Note that the last two terms are added to preserve the solution splitting property. More precisely, we set \( y_i := e^{-\frac{\Omega_t}{\gamma}}x_i \). Then, its derivatives become

\[
\begin{align*}
\dot{y}_i &= e^{-\frac{\Omega_t}{\gamma}} \dot{x}_i - \frac{\Omega}{\gamma} e^{-\frac{\Omega_t}{\gamma}} x_i - \frac{\Omega}{\gamma} y_i, \\
\ddot{y}_i &= e^{-\frac{\Omega_t}{\gamma}} \ddot{x}_i - 2\frac{\Omega}{\gamma} e^{-\frac{\Omega_t}{\gamma}} \dot{x}_i + \frac{\Omega^2}{\gamma^2} e^{-\frac{\Omega_t}{\gamma}} x_i = e^{-\frac{\Omega_t}{\gamma}} \dot{\gamma} x_i - \left( \frac{2\Omega}{\gamma} y_i + \frac{\Omega^2}{\gamma^2} y_i \right) + \frac{\Omega^2}{\gamma^2} y_i.
\end{align*}
\]

(16)

Since \( e^{-\frac{\Omega_t}{\gamma}} \) is an orthogonal matrix, we can use the property (10). For \( \Omega_t \equiv \Omega \), we multiply (11) with \( e^{-\Omega t} \) to find

\[
m e^{-\frac{\Omega t}{\gamma}} \dot{x}_i + m|\dot{x}_i|^2 e^{-\frac{\Omega t}{\gamma}} x_i
\]

\[
= -\gamma e^{-\frac{\Omega t}{\gamma}} \dot{x}_i + \sum_{k=1}^{N} (e^{-\frac{\Omega t}{\gamma}} x_k - \langle e^{-\frac{\Omega t}{\gamma}} x_i, e^{-\frac{\Omega t}{\gamma}} x_k \rangle e^{-\frac{\Omega t}{\gamma}} x_i)
\]

\[
+ \Omega e^{-\frac{\Omega t}{\gamma}} \dot{x}_i + \frac{2m}{\gamma} (\Omega e^{-\frac{\Omega t}{\gamma}} x_i - \langle e^{-\frac{\Omega t}{\gamma}} x_i, e^{-\frac{\Omega t}{\gamma}} \Omega^2 x_i \rangle e^{-\frac{\Omega t}{\gamma}} x_i)
\]

\[
- \frac{m}{\gamma^2} (e^{-\frac{\Omega t}{\gamma}} \Omega^2 x_i - \langle e^{-\frac{\Omega t}{\gamma}} x_i, e^{-\frac{\Omega t}{\gamma}} \Omega^2 x_i \rangle e^{-\frac{\Omega t}{\gamma}} x_i).
\]

(17)

We use (12) to represent (13) in terms of \( y_i \):

\[
m \ddot{y}_i + \frac{2m\Omega}{\gamma} \dot{y}_i + \frac{m\Omega^2}{\gamma^2} y_i = -m \left( \ddot{y}_i + \frac{\Omega}{\gamma} y_i \right)^2 y_i - \gamma \dot{y}_i + \Omega \dot{y}_i + \sum_{k=1}^{N} (y_k - \langle y_i, y_k \rangle y_i)
\]

\[
+ \frac{2m}{\gamma} (\Omega \dot{y}_i + \frac{\Omega^2}{\gamma} y_i - \langle y_i, \Omega \dot{y}_i + \frac{\Omega^2}{\gamma} y_i \rangle y_i)
\]

\[
- \frac{m}{\gamma^2} (\Omega^2 y_i - \langle y_i, \Omega^2 y_i \rangle y_i),
\]

which can be rearranged as

\[
m \ddot{y}_i + m|\dot{y}_i|^2 y_i + \gamma y_i - \sum_{k=1}^{N} (y_k - \langle y_i, y_k \rangle y_i)
\]

\[
= -\frac{2m\Omega}{\gamma} \dot{y}_i - \frac{m\Omega^2}{\gamma^2} y_i - \frac{m}{\gamma^2} \langle y_i, \Omega^2 y_i \rangle y_i - \frac{2m}{\gamma} (\dot{y}_i, \Omega y_i) y_i.
\]
Lemma 2.4. Let $X = (x_1, \cdots, x_N)$ be a solution to (11) and suppose that
\[
\Omega_i \equiv \Omega, \quad i = 1, \cdots, N.
\]
Then, $y_i := e^{-\frac{\Omega t}{m}} x_i$ is a solution to the following reduced equation:
\[
m_i \ddot{y}_i = -m|\dot{y}_i|^2 y_i - \gamma y_i + \frac{K}{N} \sum_{k=1}^N (y_k - \langle y_i, y_k \rangle y_i).
\]

2.3. From particle swarm to kinetic swarm. We investigate a kinetic model which can be naturally derived from (2) using standard BBGKY hierarchy. After performing formal mean-field limit, we consequently identify the desired equation for kinetic density $F = F(t, x, v, \Omega)$ on the (extended) phase space $\Xi := \mathbb{R}^d \times \mathbb{R}^d \times \text{Skew}_d(\mathbb{R})$:
\[
\begin{align*}
&\partial_t F + v \cdot \nabla_x F + \nabla_v \cdot (\mathcal{L}[F]F) = 0, \quad (x, v, \Omega) \in \Xi, \quad t > 0, \\
&\mathcal{L}[F](t, x, v, \Omega) = \frac{1}{m} \left( -\gamma v - \frac{|v|^2}{|x|^2} x + \Omega \frac{x}{|x|^2} ight) \\
&+ \kappa \int_{\Xi} \left( x_s - \frac{(x, x_s)}{(x, x)} x \right) F(t, x_s, v_s, \Omega_s) dx_s dv_s d\Omega_s,
\end{align*}
\]
which can be naturally derived from (2) using standard BBGKY hierarchy. After performing formal mean-field limit, we consequently identify the desired equation for kinetic density $F = F(t, x, v, \Omega)$ on the (extended) phase space $\Xi := \mathbb{R}^d \times \mathbb{R}^d \times \text{Skew}_d(\mathbb{R})$:
\[
\begin{align*}
&\partial_t F + v \cdot \nabla_x F + \nabla_v \cdot (\mathcal{L}[F]F) = 0, \quad (x, v, \Omega) \in \Xi, \quad t > 0, \\
&\mathcal{L}[F](t, x, v, \Omega) = \frac{1}{m} \left( -\gamma v - \frac{|v|^2}{|x|^2} x + \Omega \frac{x}{|x|^2} ight) \\
&+ \kappa \int_{\Xi} \left( x_s - \frac{(x, x_s)}{(x, x)} x \right) F(t, x_s, v_s, \Omega_s) dx_s dv_s d\Omega_s,
\end{align*}
\]
(18)

Lemma 2.2 yields that the $d$-dimensional unit sphere $\mathbb{S}^{d-1}$ is positively invariant under the flow (2). By a similar fashion, we show that our domain $\mathbb{D} = \mathbb{S}^{d-1} \times \mathbb{R}^d \times \text{Skew}_d(\mathbb{R})$ is a positively invariant manifold for the kinetic model (14). In order to distinguish the notation for the densities, we set $f := F|_{\mathbb{D}}$ to derive our kinetic model on the unit sphere.

Lemma 2.5. Let $F = F(t, x, v, \Omega)$ be a continuously differentiable solution to (14) so that the calculation below can be performed. Then, the following two assertions hold.

(i) If $\Omega$-support of $F_{0}$ is supported in the radius $r^\infty$-ball $B_{r^\infty}(0)$, then $\Omega$-support of $F(t, \cdot, \cdot, \cdot)$ is also supported on the set $B_{r^\infty}(0)$ for $t > 0$.

(ii) If $x$-support of $F_{0}$ is supported on the unit sphere $\mathbb{S}^{d-1}$, then $x$-support of $F(t, \cdot, \cdot, \cdot)$ is also supported on the unit sphere $\mathbb{S}^{d-1}$ for $t > 0$.

Proof. First, we rewrite (14) into a quasilinear form:
\[
\partial_t F + v \cdot \nabla_x F + \mathcal{L}[F] \cdot \nabla_v F = - (\nabla_v \cdot \mathcal{L}[F]) F.
\]
For a given $x \in \mathbb{R}^d$ and $\Omega \in B_{r^\infty}(0)$, we set a forward characteristic associated to (2)
\[
(X(s), V(s), \Omega(s)) := (X(s; x, v, \Omega), V(s; x, v, \Omega), \Omega(s; x, v, \Omega))
\]
Proposition 1. Suppose that the system parameters and the initial data satisfy

\[ m > 0, \quad \gamma > 0, \quad \kappa > 0, \quad \mathcal{L}_+^0 < \infty, \]

and let \( X \) be a solution to (1). Then, there exists a function \( M_V \in (L^1 \cap L^\infty)(0, \infty) \) such that

\[ \max_{1 \leq i \leq N} |\dot{x}_i(t)| < M_V(t) \quad \text{and} \quad \int_0^\infty M_V(t) \, dt < \infty. \]
Hence, we have the zero convergences of the velocity variables with an integrable decay rate:

\[
\lim_{t \to \infty} \max_{1 \leq i \leq N} |\dot{x}_i(t)| = 0 \quad \text{and} \quad \sup_{t \geq 0} \max_{1 \leq i \leq N} |\dot{x}_i(t)| \leq \|M_V\|_{\infty}.
\]

Next, we consider the asymptotic behavior for the position variable \(x_i\). We observe the identity:

\[
|x_i - x_j|^2 = 2(1 - \langle x_i, x_j \rangle).
\]

Hence, it suffices to focus on the dynamics of relative angles \(h_{ij} := \langle x_i, x_j \rangle\). For the evolution of them, we introduce two frameworks (large \(m\kappa\) and small \(m\kappa\) regimes) leading the complete aggregation. Let \(\delta \in (0, 1)\) be a fixed (small) positive number and \(G = G(t)\) be the maximal diameter for \(1 - h_{ij}\):

\[
G(t) := \max_{1 \leq i,j \leq N} |1 - \langle x_i, x_j \rangle(t)|, \quad t \geq 0.
\]

• Framework \((F_A)\) (large \(m\kappa\) regime):
  - \((F_A1)\): Parameters \(m, \gamma, \kappa\) and \(\delta\) satisfy
    \[
    \gamma^2 - 8m\kappa(1 - \delta) < 0.
    \]
  - \((F_A2)\): There exists small numbers \(\varepsilon^k_A\), \(k = 1, 2, 3\) depending only on the system parameters such that
    \[
    G(0) < \varepsilon^1_A < \delta, \quad \dot{G}(0) + \varepsilon^2_A G(0) < \varepsilon^3_A.
    \]

• Framework \((F_B)\) (small \(m\kappa\) regime):
  - \((F_B1)\): Parameters \(m, \gamma, \kappa\) and \(\delta\) satisfy
    \[
    \gamma^2 - 8m\kappa(1 - \delta) > 0.
    \]
  - \((F_B2)\): There exists small numbers \(\varepsilon^k_B\), \(k = 1, 2, 3\) depending only on the system parameters such that
    \[
    G(0) < \varepsilon^1_B < \delta, \quad \dot{G}(0) + \varepsilon^2_B G(0) > \varepsilon^3_B.
    \]

For exact values of the constants \(\varepsilon^k_A\) and \(\varepsilon^k_B\), we refer the reader to Section 4.3 in [18]. Under exactly one of these frameworks \((F_A)\) and \((F_B)\), we show that the complete aggregation occurs. We refer the reader to Theorem 4.10 in [18] for the proof.

**Theorem 2.6.** [18] Suppose that exactly one of frameworks \((F_A)\) and \((F_B)\) holds, and let \(X\) be a solution to (1). Then, there exists a function \(M_X \in (L^1 \cap L^\infty)(0, \infty)\) such that

\[
G(t) < M_X(t) \quad \text{and} \quad \int_0^\infty M_X(t)dt < \infty.
\]

Hence, the system exhibits the complete aggregation with an integrable decay rate, in other words,

\[
\lim_{t \to \infty} G(t) = 0.
\]
2.5. Preparatory lemmas. For later use, we recall Barbalat’s lemma and Gronwall-type lemmas.

**Lemma 2.7.** [2] Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a real-valued function. Then, the following assertions hold.

(i) If \( f \) is uniformly continuous and
\[ \lim_{t \to \infty} \int_0^t f(s) \, ds \]
exists, then, \( f \) tends to zero as \( t \to \infty \):
\[ \lim_{t \to \infty} f(t) = 0. \]

(ii) If \( f \) is continuously differentiable, \( \lim_{t \to \infty} f(t) \in \mathbb{R} \), and \( f' \) is uniformly continuous, then \( f' \) tends to zero as \( t \to \infty \):
\[ \lim_{t \to \infty} f'(t) = 0. \]

**Lemma 2.8.** Suppose that \( f \in (L^1 \cap L^\infty)(0, \infty) \) is an integrable and bounded function, and \( a, b \) and \( c \) are positive constants. Let \( y = y(t) \) be a nonnegative \( C^2 \)-function satisfying exactly one of the following differential inequalities:

(i) \( \dot{y} + ay \leq f, \quad t > 0, \quad y(0) = y_0. \)

(ii) \( a\ddot{y} + b\dot{y} + cy \leq f, \quad t > 0, \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0. \)

Then, in any cases, we have \( y \in (L^1 \cap L^\infty)(0, \infty) \). In other words,
\[ \lim_{t \to \infty} y(t) = 0 \] with an integrable decay rate.

**Proof.** Convergence to zero of \( y \) in both cases can be found in Lemma A.1 for [8] and Lemma 4.9 for [18], respectively. We here discuss the integrability of \( y \). For the first assertion, it follows from [8] that
\[ y(t) \leq \frac{1}{a} \max_{s \in [t/2, t]} |f(s)| + y_0 e^{-at} + \frac{\|f\|_{L^\infty}}{a} e^{-\frac{at}{2}}, \quad t \geq 0. \] (22)

Since the exponential function and \( f \) are integrable, we see that \( y \) is also integrable.

For the second assertion, we consider the two cases: \( b^2 - 4ac < 0 \) and \( b^2 - 4ac > 0 \). If \( b^2 - 4ac < 0 \), then there exist positive constants \( C^k_n, k = 1, \cdots, 4 \) such that
\[ y(t) \leq C^1_n e^{-\frac{bt}{2}} + C^2_n t e^{-\frac{bt}{2}} + C^3_n e^{-\frac{bt}{2}} + C^4_n \max_{s \in [t/4, t]} |f(s)|, \quad t \geq 0. \]

Thus, integrability of \( y \) follows from the integrability of \( e^{-t}, te^{-t} \) and \( f \). On the other hand, if \( b^2 - 4ac > 0 \), then there exist positive constants \( C^k_p, k = 1, \cdots, 4 \) such that
\[ \dot{y} + C^2_p y \leq C^3_p e^{-\frac{ct_1}{2}} + C^4_p e^{-\frac{ct_1}{2}} + \frac{1}{a} \max_{s \in [t/2, t]} |f(s)|, \quad t \geq 0. \] (23)

Then, since the right-hand side of (19) is integrable, one applies (18) to obtain the desired result. \( \square \)

Before we end this section, we briefly discuss the relation between (3) with other two collective models, namely the inertial Kuramoto model [10, 15] and the first-order swarm sphere model (1). First, we present the connection with the inertial
Kuramoto model. Let \( x_i \in \mathbb{R}^2 \) be a solution to (3) with \( |x_i| = 1 \). Then, we can represent \( x_i \) in terms of a polar coordinate:

\[
x_i = e^{i\theta_i}, \quad \Omega_i = iv_i, \quad (\theta_i, v_i) \in \mathbb{R}^2.
\]  

(24)

Then, we observe

\[
\dot{x}_i = i\dot{\theta}_i e^{i\theta_i}, \quad \ddot{x}_i = i\ddot{\theta}_i e^{i\theta_i} - |\dot{\theta}_i|^2 e^{i\theta_i}.
\]  

(25)

By substituting the ansatz (20) and (21) into (3), one has

\[
m\ddot{\theta}_i e^{i\theta_i} = -\gamma \dot{\theta}_i e^{i\theta_i} + iv_ie^{i\theta_i} + \frac{\kappa}{N} \sum_{k=1}^{N} \left( e^{i\theta_k} - e^{i(\theta_i - \theta_k)} e^{i\theta_i} \right).
\]

Now, we multiply the above relation by \( e^{-i\theta_i} \) to see

\[
m\ddot{\theta}_i = -i\gamma \dot{\theta_i} + iv_i + \frac{\kappa}{N} \sum_{k=1}^{N} \left( e^{i(\theta_k - \theta_i)} - e^{i(\theta_i - \theta_k)} \right)
\]

\[
= -i\gamma \dot{\theta}_i + iv_i + \frac{2i\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i).
\]  

(26)

We finally compare the imaginary parts of the both sides of (22) to obtain the inertial Kuramoto model:

\[
m\ddot{\theta}_i + \gamma \dot{\theta}_i = iv_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i), \quad \kappa = 2\kappa.
\]

On the other hand, if we formally set \( m = 0 \), then (3) reduces to the first-order model:

\[
\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_1, x_k \rangle x_i), \quad \Omega_i := \frac{\Omega_i}{\gamma}, \quad \kappa := \frac{\kappa}{\gamma}.
\]  

(27)

Thus, we would say that our model (3) is a second-order extension of (23).

3. Description of main results. In this section, we discuss our main results without detailed proofs, which will be presented in the following two sections, and the main results can be divided into two parts: the particle model in Section 3.1 and the kinetic one in Section 3.2. In what follows, unless otherwise stated, we mainly deal with the identical case, and due to the solution splitting property, we may assume that \( \Omega_i \equiv O \) so that the identical particle model becomes

\[
\begin{cases}
    m\ddot{x}_i = -m|x_i|^2 x_i - \gamma \dot{x}_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_1, x_k \rangle x_i), & t > 0, \\
    (x_i, \dot{x}_i)(0) = (x_i^0, \dot{x}_i^0), & |x_0| = 1, \quad \langle x_i^0, x_k^0 \rangle = 0, \quad i = 1, \cdots, N.
\end{cases}
\]  

(28)

and hence its corresponding kinetic model reads as follows: for \( f = f(t, \omega, v) \),

\[
\begin{cases}
    \partial_t f + v \cdot \nabla_\omega f + \nabla_v \cdot (L[f] f) = 0, & (\omega, v) \in D := S^d-1 \times \mathbb{R}^d, \quad t > 0, \\
    L[f](t, \omega, v) = \frac{1}{m} \left( -\gamma v - m|v|^2 \omega + \kappa \int_D \langle \mathbb{P}(\omega) \omega, f(t, \omega, v) \rangle dS_{\omega}, dv \right), \\
    f(0, \omega, v) = f_0(\omega, v), & \mathbb{P}(\omega) \omega := \omega - \langle \omega, \omega \rangle \omega.
\end{cases}
\]  

(29)

However, if we divide (1) by \( m \), then \( \gamma/m \) and \( \kappa/m \) play the same roles as \( \gamma \) and \( \kappa \), respectively. Thus, without loss of generality, we henceforth set \( m = 1 \) in both (1) and (2) to avoid possible cluttered estimates.
3.1. **Particle model.** First, we provide gradient-like flow formulation of (3). Since our system contains the centripetal force term $|\dot{x}_i|^2 x_i$, the classical theorem (Theorem 4.1) in [22] could not be applied as it is. Hence, we follow the proof of the theorem and slightly modify it to show our first main result.

**Theorem 3.1.** Let $y_i \in W^{1, \infty}(\mathbb{R}_+; S^{(d-1)})$ be a solution to
\begin{equation}
\ddot{y}_i + g(\dot{y}_i) + |\dot{y}_i|^2 y_i = \nabla F(y_i), \quad t \geq 0, \quad i = 1, \ldots, N.
\end{equation}
Suppose that $g$ and $F$ satisfy the structural condition: there exist positive constants $0 < C_1 \leq C_2 < \infty$ such that
\begin{enumerate}[(i)]
\item $F$ is analytic.
\item $|g(v)| \leq C_1 |v|^2$ and $|g(v)| \leq C_2 |v|$, $\forall v \in \mathbb{R}^N$.
\end{enumerate}
Then, there exists $y_\infty^i \in S := \{ x \in \mathbb{R}^N : \nabla F(x) = 0 \}$ such that
\begin{equation}
\lim_{t \to \infty} \left( |\dot{y}_i(t)| + |y_i(t) - y_\infty^i| \right) = 0, \quad i = 1, \ldots, N.
\end{equation}

**Proof.** The detailed justification will be given in Section 4.1.

Before we state our second main result, we introduce the definition of the uniform $\ell_2$-stability with respect to the initial data. Recall $| \cdot |$ denotes the usual $\ell_2$-norm in the Euclidean space.

**Definition 3.2.** We say that system (1) is uniformly $\ell_2$-stable with respect to the initial data if for any two solutions $Z$ and $\tilde{Z}$ with initial data $Z^0$ and $\tilde{Z}^0$, respectively, there exists a uniform positive constant $C$ which does not depend on $N$ and $t$ such that
\begin{equation}
\sup_{0 \leq t < \infty} |Z(t) - \tilde{Z}(t)| \leq C |Z^0 - \tilde{Z}^0|.
\end{equation}

**Remark 2.** If the maximal lifespan of time satisfying the estimate (5) is finite, then we say that the estimate is local-in-time or system (1) is local-in-time stable.

Then, we find the desired uniform constant $G$ when we a priori assume that the complete aggregation occurs.

**Theorem 3.3.** Suppose that the damping coefficient is larger than the inertia assumed to be 1, that is,
\begin{equation}
\gamma > 1,
\end{equation}
and that the framework $(\mathcal{F}_A)$ or $(\mathcal{F}_B)$ holds so that the emergence of the complete aggregation is guaranteed. Let $Z$ and $\tilde{Z}$ be two solutions to (1) with initial data $Z^0$ and $\tilde{Z}^0$, respectively. Then, there exists a uniform positive constant $G$ satisfying (5). In other words, system (1) is uniformly $\ell_2$-stable in the sense of Definition 3.2.

**Proof.** The detailed proof can be found in Section 4.2.

3.2. **Kinetic model.** It follows from the uniform-in-time $\ell_2$-stability in Theorem 3.3 that global existence and uniform-in-time stability of a measure-valued solution to (2) directly follow. For a metric space $\mathcal{X}$, we denote $\mathcal{P}_2(\mathcal{X})$ as the set of Borel probability measures on $\mathcal{X}$ with finite moments of order two.

**Theorem 3.4.** Suppose that the framework $(\mathcal{F}_A)$ or $(\mathcal{F}_B)$ holds. Then, the following assertions hold.
\begin{enumerate}[(i)]
\item (Global existence): There exists a unique measure-valued solution $d\mu_t = f(t) d\omega dv \in L^\infty([0, \infty); \mathcal{P}_2(D))$ to (2) with the initial data $d\mu_0 = f_0 d\omega dv$. Moreover, $d\mu$ can
be approximated by the empirical measure $\delta(f^N)$ in the Wasserstein-2 distance uniformly-in-time:

$$\limsup_{N \to \infty} \sup_{t \in [0, \infty)} W_2(f^N(t), f(t)) = 0.$$ 

(ii) (Uniform stability): If $f$ and $\tilde{f}$ are two solutions to (2) with the initial data $f_0$ and $\tilde{f}_0$, respectively, then there exists a uniform constant $G$ independent of $t$ such that

$$W_2(f(t), \tilde{f}(t)) \leq GW_2(f_0, \tilde{f}_0), \quad t \geq 0.$$

**Proof.** We postpone the rigorous justification in Section 5.1.

We finally state our last main result which concerns with the asymptotic behavior of a measure-valued solution whose global existence is guaranteed from Theorem 3.4. For this, we consider the characteristic system associated to (2) with identical oscillators:

$$\begin{align*}
\frac{\partial}{\partial t} q(t; 0, \omega, v) &= p(t; 0, \omega, v), \quad (\omega, v) \in D = S^{d-1} \times \mathbb{R}^d, \\
\frac{\partial}{\partial t} p(t; 0, \omega, v) &= L[f](t, q(t; 0, \omega, v), p(t; 0, \omega, v)), \\
(q(0; 0, \omega, v), p(0; 0, \omega, v)) &= (\omega, v),
\end{align*}$$

and define the order parameter

$$R_f(t) := \int_D q_t(\omega, v)f_0(\omega, v)dS_\omega dv, \quad t > 0, \quad R_f(0) = R^0_f.$$

**Theorem 3.5.** Suppose that the initial data satisfy

$$\mathcal{E}_K(0) < \kappa |R^0_f|^2,$$

and let $f \in C_w(\mathbb{R}_+; \mathcal{P}_2(D))$ be a measure-valued solution to (2). Then, the following assertions hold:

(i) If $R^0_f = 0$, then we have

$$R_f(t) \equiv 0, \quad t \geq 0.$$ 

(ii) If $R^0_f > 0$, then the measure-valued solution $d\mu_t = f(t)dS_\omega dv$ tends to the bipolar state. More precisely, there exists a constant $m_0 \in (0, 1)$ and a vector $u \in S^{d-1}$ such that $f$ tends to

$$((1 - m_0)\delta_u(\omega) + m_0\delta_{-u}(\omega)) \otimes \delta_0(v),$$

where $\delta$ denotes the Dirac measure.

**Proof.** We provide the proof in Section 5.2. 

4. Asymptotic behavior of the particle swarm sphere model. In this section, we study the asymptotic behavior of the identical second-order particle swarm sphere model (1), and present the proofs of Theorems 3.1 and 3.3.
4.1. Proof of Theorem 3.1. In this subsection, we provide the proof of Theorem 3.1 which deals with the gradient-like flow approach. In [21, 30], a first-order swarm sphere model (1) can be represented as a gradient flow only if the natural frequencies are identical to the zero matrix, i.e., $\Omega_i \equiv O$. More precisely, we define an analytic potential function $V$ associated to a solution for (1):

$$V(X) := \frac{1}{N} \sum_{i,j=1}^{N} \langle x_i, x_j \rangle.$$  

Then, one can rewrite (1) with $\Omega_i \equiv O$ as

$$\dot{x}_i = \nabla x_i V \bigg|_{T_{x_i} S^{d-1}}, \quad t > 0, \quad i = 1, \cdots, N.$$  

(35)

Note that the right-hand side denotes the orthogonal projection of the gradient vector $\nabla x_i V \in \mathbb{R}^d$ onto the tangent plane $T_{x_i} S^{d-1}$ at $x_i$, which becomes the induced gradient vector on the manifold $S^{d-1}$. It is worthwhile to mention that the orthogonal projection of the usual gradient onto the tangent plane defines the Riemannian gradient. To be more specific, for a Riemannian submanifold $M$ of $\mathbb{R}^d$, let $\bar{f}$ be a function defined on $\mathbb{R}^d$ and $f$ be the restriction of $\bar{f}$ to $M$, that is, $f = \bar{f} |_M$. Then, the Riemannian gradient can be calculated as

$$\text{grad} f(x) = \text{Proj}_x \nabla \bar{f}(x), \quad x \in M,$$

where grad denotes the Riemannian gradient and $\text{Proj}_x : \mathbb{R}^d \rightarrow T_x M$ is the orthogonal projection onto the tangent space at $x \in M$. In a similar fashion to (1), we rewrite (1) a gradient-like system:

$$\ddot{x}_i = -|\dot{x}_i|^2 x_i - \gamma \dot{x}_i + \nabla x_i V, \quad i = 1, \cdots, N, \quad t > 0.$$  

(36)

Before we provide the proof of Theorem 3.1, we recall the classical theorem in [23] which concerns with the convergence of a solution to the gradient-like system.

**Theorem 4.1.** [23] Let $Y = (y_1, \cdots, y_N) \in W^{1, \infty}(\mathbb{R}_+, \mathbb{R}^N)$ be a solution to

$$\dot{Y} + g(\dot{Y}) = \nabla F(Y), \quad t \geq 0.$$  

(37)

Suppose that $g$ and $F$ satisfy the conditions (4). Denote the set of critical points of the gradient vector field $\nabla F$:

$$S := \{x \in \mathbb{R}^N : \nabla F(x) = 0\}.$$  

(38)

Then, there exists $a \in S$ such that

$$\lim_{t \to \infty} \left( |\dot{Y}(t)| + |Y(t) - a| \right) = 0.$$

In fact, we notice that Theorem 4.1 is formulated on the Euclidean space $\mathbb{R}^d$, whereas our system (1) is defined on the unit sphere. Moreover, system (1) cannot be written as the form of (3). In other words, since (2) contains the centripetal force term $|\dot{x}_i|^2 x_i$ so that it cannot be a function of only $\dot{x}_i$, we cannot associate such a function $g$ depending only on $\dot{x}_i$ as in Theorem 4.1. Thus, we cannot apply Theorem 4.1 as it is. To overcome the technical problems, we follow the proof of Theorem 4.1 step by step to recover the same result. For this, we recall the Lojasiewicz inequality in a Riemannian manifold whose proof can be found in Theorem 5.1 of [21].
Theorem 4.2. [21] Let \((\mathcal{M}, g)\) be a Riemannian \(C^\infty\)-manifold with a Riemannian metric \(g\). Then, for an open subset \(U \subset \mathcal{M}\), a point \(p \subset U\), and a real analytic function \(f : U \to \mathbb{R}\), there exists constants \(\theta \in (0, 1/2)\), \(C_L\) and an open neighborhood \(V\) satisfying \(p \in V \subset U\) such that
\[
|f(z) - f(p)|^{1-\theta} \leq C_L|\nabla f(z)|, \quad \forall z \in V.
\]

Together with Theorems 4.1 and 4.2, we now present the proof of Theorem 3.1.

(Proof of Theorem 3.1) Our proof consists of two steps: convergences of \(\dot{y}_i\) and \(y_i\).

• Step A (convergence of \(\dot{y}_i\)): we take an inner product (3) with \(\dot{y}_i\) and integrate the resulting relation with respect to time to find
\[
\frac{1}{2}|\dot{y}_i(t)|^2 - \frac{1}{2}|\dot{y}_i^0|^2 + \int_0^t \langle g(\dot{y}_i), \dot{y}_i \rangle(s) ds = F(y_i(t)) - F(y_i^0), \quad t \geq 0,
\]
where we used
\[
\langle y_i, \dot{y}_i \rangle(t) = 0, \quad t \geq 0.
\]

Then, we recall (4)(i) to see
\[
F(\dot{y}_i^0) + C_1 \int_0^t |\dot{y}_i(s)|^2 ds \leq F(y_i(t)) - F(y_i^0) + \frac{1}{2}|\dot{y}_i^0|^2, \quad t \geq 0.
\]

Since \(F\) is analytic, we have \(|\dot{y}_i|^2 \in L^1(0, \infty)\). On the other hand, since \(y_i\) and \(\dot{y}_i\) are uniformly bounded, it follows from (3) that \(\dot{y}_i\) is also bounded. This implies that \(|\dot{y}_i|^2\) is uniformly bounded and hence uniformly continuous. Thus, Barbalat’s lemma yields the first desired convergence:
\[
\lim_{t \to \infty} |\dot{y}_i(t)| = 0, \quad i = 1, \cdots, N. \tag{39}
\]

• Step B (convergence of \(y_i\)): For a given initial data \((y_i^0, \dot{y}_i^0)\), we define an \(\omega\)-limit set:
\[
\omega(y_i^0, \dot{y}_i^0) := \{ u \in \mathbb{S}^{d-1} : \exists t_n \to \infty \text{ such that } y_i(t_n) \to u \}.
\]

Then, since \(\mathbb{S}^{d-1}\) is compact, the \(\omega\)-limit set \(\omega(y_i^0, \dot{y}_i^0)\) is also a nonempty compact set and hence a connected set. In addition, if we use the definition of the \(\omega\)-limit set, the governing equation (3) and the convergence (5), then one can check that
\[
\omega(y_i^0, \dot{y}_i^0) \subseteq S.
\]

This shows
\[
\lim_{t \to \infty} \text{dist}(y_i(t), S) = 0.
\]

To verify the desired convergence, we use the analytic condition on \(F\). First, without loss of generality, we can set
\[
F(y_i^\infty) = 0 \quad \text{and} \quad \nabla F(y_i^\infty) = 0,
\]
by setting \(\tilde{F}(x) := F(x) - F(y_i^\infty)\) which also implies \(\nabla \tilde{F}(y_i^\infty) = \nabla F(y_i^\infty)\). For a small positive number \(\varepsilon \in (0, 1)\) which will be determined later in (6) and (8), we associate the energy functional to the system
\[
E_i(t) := \frac{1}{2}|\dot{y}_i|^2 - F(y_i) - \varepsilon \langle \nabla F(y_i), \dot{y}_i \rangle, \quad i = 1, \cdots, N.
\]
We differentiate the functional $E_i$ to obtain
\[
\dot{E}_i(t) = \langle \dot{y}_i, \dot{y}_i \rangle - \langle \nabla F(y_i), \dot{y}_i \rangle - \epsilon \langle \nabla^2 F(y_i) \dot{y}_i, \dot{y}_i \rangle - \epsilon \langle \nabla F(y_i), \dot{y}_i \rangle
\]
\[
= -(g(y_i), \dot{y}_i) - \epsilon \langle \nabla^2 F(y_i) \dot{y}_i, \dot{y}_i \rangle - \epsilon \langle \nabla F(y_i), -g(y_i) - |\dot{y}_i|^2 y_i + \nabla F(y_i) \rangle,
\]
where $\nabla^2 F(y_i)$ denotes the Hessian matrix of $F$. We choose $\epsilon$ to satisfy
\[
\epsilon \|\nabla^2 F(y_i)\|_{\infty} \leq \frac{C_1}{2},
\]
where $\| \cdot \|_{\infty}$ denotes the maximum norm of a given matrix. Then, we observe
\[
\dot{E}_i(t) \leq -C_1 |\dot{y}_i|^2 + \frac{C_1}{2} |\dot{y}_i|^2 - \epsilon |\nabla F(y_i)|^2 + \epsilon |\nabla F(y_i)||g(\dot{y}_i)| + \epsilon |\dot{y}_i|^2 |\nabla F(y_i)|
\]
\[
\leq -C_1 |\dot{y}_i|^2 - \epsilon |\nabla F(y_i)|^2 + \epsilon (C_2 + M) |\nabla F(y_i)||\dot{y}_i|
\]
\[
\leq -\frac{C_1}{2} |\dot{y}_i|^2 - \epsilon |\nabla F(y_i)|^2 + 2\delta \epsilon (C_2 + M) |\nabla F(y_i)|^2 + \frac{\epsilon (C_2 + M)}{2\delta} |\dot{y}_i|^2
\]
\[
= \left( \frac{\epsilon (C_2 + M)}{2\delta} - \frac{C_1}{2} \right) |\dot{y}_i|^2 + \epsilon (2\delta (C_2 + M) - 1) |\nabla F(y_i)|^2
\]
\[
< -\nu (|\dot{y}_i| + |\nabla F(y_i)|)^2,
\]
where we used the uniform bound, say $M$, of $\max_i \dot{y}_i$ in the second inequality and the young’s inequality is used in the third inequality. Moreover, the positive constant $\nu$ is defined as
\[
\nu := \frac{1}{2} \min \left\{ \frac{C_1}{2} - \frac{\epsilon (C_2 + M)}{2\delta}, \frac{\epsilon (1 - 2\delta (C_2 + M))}{C_1 \delta} \right\}.
\]
For the negative sign of the right-hand side for (7) or the positivity for $\nu$, we choose $\delta$ and $\epsilon$ to satisfy
\[
\delta < \frac{1}{2(C_2 + mM)}, \quad \epsilon < \min \left\{ \frac{C_2 + mM}{C_1 \delta}, 1 \right\}.
\]
Thus, we find
\[
\dot{E}_i(t) \leq 0, \quad t \geq 0.
\]
Since $\omega(y_i^0, \dot{y}_i^0)$ is a nonempty compact set, we choose $y_i^\infty \in \omega(y_i^0, \dot{y}_i^0)$. Hence, we attain
\[
\lim_{t \to \infty} E_i(t) = -F(y_i^\infty) = 0.
\]
On the other hand, for $\theta \in (0, 1/2]$ in Theorem 4.2, we have
\[
- \frac{d}{dt} E^\theta = -\theta \dot{E} E^{\theta - 1},
\]
and it follows from the Cauchy-Schwarz inequality that
\[
E^{1-\theta} = \left( \frac{1}{2} |\dot{y}_i|^2 - F(y_i) - \epsilon (\nabla F(y_i), \dot{y}_i) \right)^{1-\theta}
\]
\[
\leq |\dot{y}_i|^{2(1-\theta)} + |F(y_i)|^{1-\theta} + |\nabla F(y_i)|^{1-\theta} |\dot{y}_i|^{1-\theta}
\]
\[
\leq |\dot{y}_i|^{2(1-\theta)} + |F(y_i)|^{1-\theta} + |\nabla F(y_i)| |\dot{y}_i|^{1-\theta}.
\]
Since $y_i^\infty \in \omega(y_i^0, \dot{y}_i^0)$, there exists a sequence $(t_n)_{n \geq 1}$ such that
\[
\lim_{n \to \infty} y_i(t_n) = y_i^\infty.
\]
Thus, for arbitrary small $\sigma > 0$ and $C_0 := \max\{2, 1 + C_L\}$, there exists a large number $N_0$ such that for $n \geq N_0$,

$$\begin{align}
(i) \ |y_n(t) - y_n^\infty| < \frac{\sigma}{2}, \quad (ii) \ \frac{C_0}{\theta \nu} E(t_{N_0})^\theta < \frac{\sigma}{2}, \quad (iii) \ |\dot{y}_n(t)| < 1, \quad t \geq t_{N_0}.
\end{align}$$

(46)

Define a time $T_*$ as

$$T_* := \sup\{t \geq t_{N_0} : |y_n(t) - y_n^\infty| < \sigma \text{ on } [t_{N_0}, t]\}.$$ 

From the definition of $T_*$, one has

$$\lim_{t \to T_*} |y_n(t) - y_n^\infty| = \sigma.$$ 

(47)

In (11), we use Theorem 4.2 and (12)(iii) to obtain

$$E(t)^{1-\theta} \leq C_0(|\nabla F(y) + |\dot{y}||), \quad t \in (t_{N_0}, T_*).$$

(48)

Now, we consider two cases: first, if there exists $t_0 \in \mathbb{R}_+$ such that $E(t_0) = 0$, $t \geq t_0$. Then, the relation (9) yields

$$E(t) = 0 \text{ for } t \geq t_0 \text{ and this shows that our solution converges to a stationary state.}$$

Otherwise, in (10), we combine (7) and (14) to obtain

$$-\frac{d}{dt} E^\theta \geq \frac{\theta \nu}{C_0} \int_{t_{N_0}}^{T_*} |\dot{y}(s)| + |\nabla F(y(s))| ds,$$

(49)

We integrate (15) over the interval $(t_{N_0}, T_*)$ to find

$$-E(T_*)^\theta + E(t_{N_0})^\theta \geq \frac{\theta \nu}{C_0} \int_{t_{N_0}}^{T_*} |\dot{y}(s)| + |\nabla F(y(s))| ds,$$

which yields

$$\int_{t_{N_0}}^{T_*} |\dot{y}(s)| ds \leq \frac{C_0}{\theta \nu} E(t_{N_0})^\theta < \frac{\sigma}{2}.$$ 

(50)

Now, suppose to the contrary that $T_* < \infty$. Then, we use (12) and (16) to see

$$|y_n(T_*) - y_n^\infty| = \int_{t_{N_0}}^{T_*} |\dot{y}(s)| ds - y_n^\infty + y_n(t_{N_0}) \leq \int_{t_{N_0}}^{T_*} |\dot{y}(s)| ds + |y_n(t_{N_0}) - y_n^\infty| \leq \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma,$$

which contradicts (13). Hence, $T_* = \infty$. Finally, if we invoke (16), then we can show that the limit of $y_i$ exists:

$$\lim_{t \to \infty} y_i(t) = \int_{t_{N_0}}^{\infty} \dot{y}_i(s) ds \leq \frac{C_0}{\theta \nu} E(t_{N_0})^\theta < \infty.$$ 

This completes the proof.
4.2. Proof of Theorem 3.3. In this subsection, we establish the uniform-in-time $\ell_2$-stability with respect to the initial data of the second-order particle swarm model (1) in which the proof of Theorem 3.3 is provided. For this, we rewrite the system into the first-order dynamics introducing the velocity variable $v_i := \dot{x}_i$. Then, our system reads as

\[
\begin{aligned}
\dot{x}_i &= v_i, \quad t > 0, \\
\dot{v}_i &= -|v_i|^2 x_i - \gamma v_i + \frac{\kappa}{N} \sum_{k=1}^{N} \left( x_k - \langle x_i, x_k \rangle x_i \right), \\
|v_i^0| &= 1, \quad \langle x_i^0, v_i^0 \rangle = 0, \quad i = 1, \cdots, N.
\end{aligned}
\]

(51)

Let $(X, V)$ and $(\hat{X}, \hat{V})$ be two solutions to (17). For notational simplicity, we set

\[ h_{ik} := \langle x_i, x_k \rangle, \quad \hat{h}_{ik} := \langle \hat{x}_i, \hat{x}_k \rangle, \]

\[ U := \|X - \hat{X}\|^2 = \sum_{i=1}^{N} |x_i - \hat{x}_i|^2, \quad W := \|V - \hat{V}\|^2 = \sum_{i=1}^{N} |v_i - \hat{v}_i|^2. \]

For the moment, we here use the notation $\| \cdot \|$ for vectors in $\mathbb{R}^{dN}$ to distinguish the notation $| \cdot |$ for those in $\mathbb{R}^d$. Then, we observe

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} |v_i - \hat{v}_i|^2
&= -\gamma |v_i - \hat{v}_i|^2 + \left( |v_i|^2 \langle x_i, \hat{v}_i \rangle + |\hat{v}_i|^2 \langle \hat{x}_i, v_i \rangle \right) \\
&\quad + \frac{\kappa}{N} \sum_{k=1}^{N} \left( (x_k - \hat{x}_k, v_i - \hat{v}_i) - h_{ik} \langle x_i - \hat{x}_i, v_i - \hat{v}_i \rangle + (\hat{h}_{ik} - h_{ik}) \langle \hat{x}_i, v_i - \hat{v}_i \rangle \right) \\
&=: -\gamma |v_i - \hat{v}_i|^2 + I_{21} + \frac{\kappa}{N} \sum_{k=1}^{N} I_{22}.
\end{aligned}
\]

(52)

Below, we present estimates of $I_{21}$ and $I_{22}$, respectively.

\diamond (Estimate of $I_{21}$) We recall the relation in Lemma 2.2:

\[ \langle x_i, v_i \rangle = \langle \hat{x}_i, \hat{v}_i \rangle = 0. \]

For handy notation, we set

\[ F_V(t) := \max_{1 \leq i \leq N} \{|v_i(t)|, |\hat{v}_i(t)|\} \leq M_V(t), \quad t \geq 0, \]

and it follows from Proposition 1 that

\[ \lim_{t \to \infty} M_V(t) = 0 \quad \text{and} \quad \int_0^{\infty} M_V(t) dt < \infty. \]
Then, we estimate $\mathcal{I}_{21}$ as follows:

$$\mathcal{I}_{21} = \left( |v_i|^2 \langle x_i, \tilde{v}_i \rangle + |\tilde{v}_i|^2 \langle \tilde{x}_i, v_i \rangle \right) = \langle |v_i|^2 x_i - |\tilde{v}_i|^2 \tilde{x}_i, v_i - \tilde{v}_i \rangle$$

$$= \langle |v_i|^2 - |\tilde{v}_i|^2 \rangle x_i, v_i - \tilde{v}_i \rangle + \langle |\tilde{v}_i|^2 (x_i - \tilde{x}_i), v_i - \tilde{v}_i \rangle$$

$$\leq |v_i|^2 - |\tilde{v}_i|^2 ||v_i - \tilde{v}_i|| + |\tilde{v}_i|^2 ||x_i - \tilde{x}_i|| ||v_i - \tilde{v}_i||$$

$$\leq 2||v_i - \tilde{v}_i||^2 M_V(t) + M_V(t)^2 ||x_i - \tilde{x}_i|| ||v_i - \tilde{v}_i||.$$  \hspace{1cm} (53)

\(\diamond \) (Estimate of $\mathcal{I}_{22}$) Note that the following relations hold:

$$\langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle = \frac{1}{2} \frac{d}{dt} |x_i - \tilde{x}_i|^2,$$

$$\langle \tilde{x}_i, v_i - \tilde{v}_i \rangle = \langle x_i, \tilde{v}_i - v_i \rangle = -\langle x_i, v_i - \tilde{v}_i \rangle = \frac{1}{2} \langle \tilde{x}_i - x_i, v_i - \tilde{v}_i \rangle.$$  

Hence, the term $\mathcal{I}_{22}$ can be calculated in the following way:

$$\mathcal{I}_{22} = \left( \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - h_{ik} \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle + (\tilde{h}_{ik} - h_{ik}) \langle \tilde{x}_i, v_i - \tilde{v}_i \rangle \right)$$

$$= \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - h_{ik} \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle + \frac{1}{2} (\tilde{h}_{ik} - h_{ik}) \langle \tilde{x}_i - x_i, v_i - \tilde{v}_i \rangle$$

$$= \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - \frac{h_{ik} + \tilde{h}_{ik}}{2} \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle$$

$$= \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - \frac{h_{ik} + \tilde{h}_{ik}}{4} \frac{d}{dt} |x_i - \tilde{x}_i|^2.$$  \hspace{1cm} (54)

We recall the complete aggregation estimate in Theorem 2.6:

$$\max \left\{ \max_{1 \leq i, k \leq N} |1 - h_{ik}(t)|, \max_{1 \leq i, k \leq N} |1 - \tilde{h}_{ik}(t)| \right\} \leq M_X(t), \quad t \geq 0.$$  \hspace{1cm} (55)

Then, we can further estimate $\mathcal{I}_{22}$ in (20):

$$\frac{\kappa}{N} \sum_{k=1}^{N} \mathcal{I}_{22} = \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - \frac{h_{ik} + \tilde{h}_{ik}}{4} \frac{d}{dt} |x_i - \tilde{x}_i|^2$$

$$\leq \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - \frac{\kappa}{2} \frac{d}{dt} |x_i - \tilde{x}_i|^2 + \kappa M_X(t) |x_i - \tilde{x}_i||v_i - \tilde{v}_i|.$$  \hspace{1cm} (56)

In (18), we combine (19) and (22) to obtain

$$\frac{1}{2} \frac{d}{dt} |v_i - \tilde{v}_i|^2 \leq -\gamma |v_i - \tilde{v}_i|^2 + 2M_V(t) |v_i - \tilde{v}_i|^2 + M_V(t)^2 |x_i - \tilde{x}_i||v_i - \tilde{v}_i|$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle - \frac{\kappa}{2} \frac{d}{dt} |x_i - \tilde{x}_i|^2 + \kappa M_X(t) |x_i - \tilde{x}_i||v_i - \tilde{v}_i|.$$  \hspace{1cm} (57)
We observe
\[
\frac{\kappa}{N} \sum_{i,k=1}^{N} \langle x_k - \tilde{x}_k, v_i - \tilde{v}_i \rangle = \frac{\kappa}{N} \left( \sum_{i=1}^{N} x_i - \tilde{x}_i, \sum_{i=1}^{N} v_i - \tilde{v}_i \right) \]
\[
\leq \frac{\kappa}{2N} \frac{d}{dt} \left( \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right)^2 ,
\]
and it follows from the Cauchy-Schwarz inequality that
\[
\sum_{i=1}^{N} |x_i - \tilde{x}_i| |v_i - \tilde{v}_i| \leq \left( \sum_{i=1}^{N} |x_i - \tilde{x}_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} |v_i - \tilde{v}_i|^2 \right)^{\frac{1}{2}} = \|X - \tilde{X}\| \|V - \tilde{V}\| .
\]
Hence, we sum (23) over all the index \(i = 1, \ldots, N\) and use (24)–(25) to find
\[
\frac{1}{2} \frac{d}{dt} \|V - \tilde{V}\|^2 + \frac{\kappa}{2} \frac{d}{dt} \|X - \tilde{X}\|^2 \leq -\gamma \|V - \tilde{V}\|^2 + 2M_V(t)\|V - \tilde{V}\|^2 + M_V(t)^2 \|X - \tilde{X}\| \|V - \tilde{V}\| + \frac{\kappa}{2N} \frac{d}{dt} \left( \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right)^2 + \kappa M_X(t) \|X - \tilde{X}\| \|V - \tilde{V}\| .
\]
We summarize the previous estimates in the following lemma.

**Lemma 4.3.** Let \((X, V)\) and \((\tilde{X}, \tilde{V})\) be two solutions to (17). Then, we have
\[
\frac{1}{2} \frac{d}{dt} \|V - \tilde{V}\|^2 + \frac{\kappa}{2} \frac{d}{dt} \|X - \tilde{X}\|^2 - \frac{\kappa}{2N} \frac{d}{dt} \left( \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right)^2 \leq -\gamma \|V - \tilde{V}\|^2 + 2M_V(t)\|V - \tilde{V}\|^2 + M_V(t)^2 \|X - \tilde{X}\| \|V - \tilde{V}\| + \kappa M_X(t) \|X - \tilde{X}\| \|V - \tilde{V}\| .
\]
We now consider the time-evolution of \(\langle X - \tilde{X}, V - \tilde{V} \rangle\). For this, we first see
\[
\frac{d}{dt} \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle
\]
\[
= |v_i - \tilde{v}_i|^2 + \langle x_i - \tilde{x}_i, -|v_i|^2 x_i + |\tilde{v}_i|^2 \tilde{x}_i \rangle - \gamma \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle
\]
\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, (x_k - \tilde{x}_k - h_{ik} x_i + \tilde{h}_{ik} \tilde{x}_i) \rangle
\]
\[
= |v_i - \tilde{v}_i|^2 - \gamma \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle + \mathcal{I}_{23} + \mathcal{I}_{24} .
\]
Below, we present estimates of \(\mathcal{I}_{23}\) and \(\mathcal{I}_{24}\), separately.

- **(Estimate of \(\mathcal{I}_{23}\)):** We use the following identity:
\[
\langle x_i, x_i \rangle = \langle \tilde{x}_i, \tilde{x}_i \rangle = 1 \quad \text{and} \quad \langle x_i, \tilde{x}_i \rangle = 1 - \frac{|x_i - \tilde{x}_i|^2}{2} .
\]
to calculate $\mathcal{I}_{23}$:

$$
\mathcal{I}_{23} = \langle x_i - \tilde{x}_i, -|v_i|^2 x_i + |\tilde{v}_i|^2 \tilde{x}_i \rangle = -|v_i|^2 - |\tilde{v}_i|^2 + |v_i|^2 \langle x_i, \tilde{x}_i \rangle + |\tilde{v}_i|^2 \langle x_i, \tilde{x}_i \rangle
$$

$$
= -\frac{1}{2} (|v_i|^2 + |\tilde{v}_i|^2) |x_i - \tilde{x}_i|^2.
$$

(62)

• (Estimate of $\mathcal{I}_{24}$): We observe

$$
\mathcal{I}_{24} = \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, x_i - \tilde{x}_i \rangle - \frac{\kappa}{N} \sum_{k=1}^{N} h_{ik} |x_i - \tilde{x}_i|^2
$$

$$
+ \frac{\kappa}{N} \sum_{k=1}^{N} (\tilde{h}_{ik} - h_{ik}) \langle \tilde{x}_i, x_i - \tilde{x}_i \rangle
$$

$$
= \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, x_i - \tilde{x}_i \rangle - \frac{\kappa}{N} \sum_{k=1}^{N} h_{ik} |x_i - \tilde{x}_i|^2
$$

$$
+ \frac{\kappa}{2N} \sum_{k=1}^{N} (\tilde{h}_{ik} - h_{ik}) |x_i - \tilde{x}_i|^2
$$

$$
= \frac{\kappa}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, x_i - \tilde{x}_i \rangle - \frac{\kappa}{2N} \sum_{k=1}^{N} (h_{ik} + \tilde{h}_{ik}) |x_i - \tilde{x}_i|^2.
$$

In (27), we collect (28)–(29) and multiply the resulting relation by $\varepsilon_1 \in (0,1)$ to find

$$
\varepsilon_1 \frac{d}{dt} \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle
$$

$$
= \varepsilon_1 |v_i - \tilde{v}_i|^2 - \gamma \varepsilon_1 \langle x_i - \tilde{x}_i, v_i - \tilde{v}_i \rangle - \frac{\varepsilon_1}{2} (|v_i|^2 + |\tilde{v}_i|^2) |x_i - \tilde{x}_i|^2
$$

$$
+ \frac{\kappa \varepsilon_1}{N} \sum_{k=1}^{N} \langle x_k - \tilde{x}_k, x_i - \tilde{x}_i \rangle - \frac{\kappa \varepsilon_1}{2N} \sum_{k=1}^{N} (h_{ik} + \tilde{h}_{ik}) |x_i - \tilde{x}_i|^2
$$

$$
\leq \varepsilon_1 |v_i - \tilde{v}_i|^2 - \frac{\gamma \varepsilon_1}{2} \frac{d}{dt} |x_i - \tilde{x}_i|^2 - \frac{\varepsilon_1}{2} (|v_i|^2 + |\tilde{v}_i|^2) |x_i - \tilde{x}_i|^2
$$

$$
+ \frac{\kappa \varepsilon_1}{N} \sum_{k=1}^{N} |x_k - \tilde{x}_k|^2 |x_i - \tilde{x}_i|^2 - \frac{\kappa \varepsilon_1}{2N} \sum_{k=1}^{N} (h_{ik} + \tilde{h}_{ik}) |x_i - \tilde{x}_i|^2.
$$

(64)

Then, by summing (30) with respect to $i = 1, \cdots, N$ and using the relations (21) and (25) to obtain

$$
\varepsilon_1 \frac{d}{dt} \langle X - \tilde{X}, V - \tilde{V} \rangle \leq \varepsilon_1 \|V - \tilde{V}\|^2 - \frac{\gamma \varepsilon_1}{2} \frac{d}{dt} \|X - \tilde{X}\|^2 + \kappa \varepsilon_1 M_X(t) \|X - \tilde{X}\|^2.
$$

Our estimate for $\langle X - \tilde{X}, V - \tilde{V} \rangle$ can be stated in the following lemma.

**Lemma 4.4.** Let $(X, V)$ and $(\tilde{X}, \tilde{V})$ be two solutions to (17). Then, we have

$$
\varepsilon_1 \frac{d}{dt} \langle X - \tilde{X}, V - \tilde{V} \rangle + \frac{\gamma \varepsilon_1}{2} \frac{d}{dt} \|X - \tilde{X}\|^2 \leq \varepsilon_1 \|V - \tilde{V}\|^2 + \kappa \varepsilon_1 M_X(t) \|X - \tilde{X}\|^2.
$$

(65)
Now, we add (26) and (31) to yield

\[
\frac{d}{dt} \left( \frac{1}{2} \| V - \tilde{V} \|^2 + \frac{\kappa + \gamma \varepsilon_1}{2} \| X - \tilde{X} \|^2 + \varepsilon_1 \langle X - \tilde{X}, V - \tilde{V} \rangle \right) - \frac{\kappa}{2N} \left| \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right|^2 \leq \left( -\gamma + 2M_V(t) + \varepsilon_1 \right) \| V - \tilde{V} \|^2 + \kappa \| X - \tilde{X} \|^2 + \varepsilon_1 \langle X - \tilde{X}, V - \tilde{V} \rangle
\]

\[
\leq \left( -\gamma + 2M_V(t) + \varepsilon_1 \right) \| V - \tilde{V} \|^2 + \kappa \| X - \tilde{X} \|^2 + \varepsilon_1 \langle X - \tilde{X}, V - \tilde{V} \rangle.
\]

(66)

We finally prove Theorem 3.3 applying Lemmas 4.3 and 4.4.

**Proof of Theorem 3.3** To avoid the cluttered mathematical expressions, we rewrite (32):

\[
\frac{d}{dt} I_{25} \leq I_{26},
\]

and then provide estimates of \( I_{25} \) and \( I_{26} \), respectively.

- **(Estimate of \( I_{25} \)):** It follows from the Cauchy-Schwarz inequality that

\[
\varepsilon_1 \langle X - \tilde{X}, V - \tilde{V} \rangle \geq -\varepsilon_1 \| X - \tilde{X} \| \| V - \tilde{V} \| \geq -\frac{\varepsilon_1}{2} (\| X - \tilde{X} \|^2 + \| V - \tilde{V} \|^2),
\]

\[
\frac{\kappa}{2N} \left| \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right|^2 \leq \frac{\kappa}{2} \sum_{i=1}^{N} |x_i - \tilde{x}_i|^2 = \frac{\kappa}{2} \| X - \tilde{X} \|^2.
\]

Then, \( I_{25} \) can be estimated as follows:

\[
I_{25} = \frac{1}{2} \| V - \tilde{V} \|^2 + \frac{\kappa + \gamma \varepsilon_1}{2} \| X - \tilde{X} \|^2 + \varepsilon_1 \langle X - \tilde{X}, V - \tilde{V} \rangle - \frac{\kappa}{2N} \left| \sum_{i=1}^{N} (x_i - \tilde{x}_i) \right|^2
\]

\[
\geq \frac{1}{2} \| V - \tilde{V} \|^2 + \frac{\kappa + \gamma \varepsilon_1}{2} \| X - \tilde{X} \|^2 - \frac{\varepsilon_1}{2} (\| X - \tilde{X} \|^2 + \| V - \tilde{V} \|^2) - \frac{\kappa}{2} \| X - \tilde{X} \|^2
\]

\[
= \frac{1}{2} (1 - \varepsilon_1) \| V - \tilde{V} \|^2 + \frac{\varepsilon_1}{2} (\gamma - 1) \| X - \tilde{X} \|^2
\]

\[
\geq \beta (\| X - \tilde{X} \|^2 + \| V - \tilde{V} \|^2),
\]

where the constant \( \beta > 0 \) is defined by

\[
\beta := \min \left\{ \frac{1}{2} (1 - \varepsilon_1), \frac{\varepsilon_1}{2} (\gamma - 1) \right\}.
\]

Note that since we assume and \( \varepsilon_1 \in (0, 1) \) the condition (6), \( \beta \) is positive.
Then, it follows from Proposition 1 and Theorem 2.6 that
\[
J \leq 2M_V(t)\|V - \tilde{V}\|^2 + \kappa \varepsilon_1 M_X(t)\|X - \tilde{X}\|^2 \\
+ (M_V(t)^2 + \varepsilon_1 \kappa M_X(t)) \|X - \tilde{X}\|\|V - \tilde{V}\| \\
\leq \left( 2F(t) + \frac{1}{2} M_V(t)^2 + \frac{\varepsilon_1 \kappa M_X(t)}{2} \right) \|V - \tilde{V}\|^2 \\
+ \left( \kappa \varepsilon_1 M_X(t) + \frac{1}{2} M_V(t)^2 + \frac{\varepsilon_1 \kappa M_X(t)}{2} \right) \|X - \tilde{X}\|^2
\]
(69)
where the function \(J = J(t)\) is defined through the relation:
\[
J(t) := \max \left\{ 2M_V(t) + \frac{1}{2} M_V(t)^2 + \frac{\varepsilon_1 \kappa M_X(t)}{2}, \right. \\
\left. \kappa \varepsilon_1 M_X(t) + \frac{1}{2} M_V(t)^2 + \frac{\varepsilon_1 \kappa M_X(t)}{2} \right\}.
\]

Then, it follows from Proposition 1 and Theorem 2.6 that \(J\) is integrable:
\[
\int_0^\infty J(t)dt =: J_\infty < \infty.
\]

We integrate (33) to see
\[
I_{25}(t) - I_{25}(0) \leq \int_0^t I_{26}(s)ds,
\]
and then use (34)–(35) to obtain
\[
\beta(\|X - \tilde{X}\|^2 + \|V - \tilde{V}\|^2) \leq I_{25}(0) + \int_0^t J(s)(\|X - \tilde{X}\|^2 + \|V - \tilde{V}\|^2)ds.
\]

We recall the definition of \(I_{25}\) to find
\[
I_{25}(0) \leq \frac{1}{2} \|V^0 - \tilde{V}^0\|^2 + \frac{\kappa + \gamma \varepsilon_1}{2} \|X^0 - \tilde{X}^0\|^2 + \varepsilon_1 \langle X^0 - \tilde{X}^0, V^0 - \tilde{V}^0 \rangle \\
\leq \left( \frac{\kappa + \gamma \varepsilon_1 + \varepsilon_1}{2} \right) \|X^0 - \tilde{X}^0\|^2 + \frac{(1 + \varepsilon_1)}{2} \|V^0 - \tilde{V}^0\|^2.
\]

Finally, we apply the Grönewall’s lemma to find our desired uniform constant:
\[
\|X(t) - \tilde{X}(t)\|^2 + \|V(t) - \tilde{V}(t)\|^2 \\
\leq \exp \left( \frac{1}{\beta} \int_0^\infty J(t)dt \right) \left[ \left( \frac{\kappa + \gamma \varepsilon_1 + \varepsilon_1}{2\beta} \right) \|X^0 - \tilde{X}^0\|^2 + \frac{(1 + \varepsilon_1)}{2\beta} \|V^0 - \tilde{V}^0\|^2 \right] \\
\leq e^{\frac{\varepsilon_0}{2\beta}} \left( (\kappa + \gamma \varepsilon_1 + \varepsilon_1) \|X^0 - \tilde{X}^0\|^2 + (1 + \varepsilon_1) \|V^0 - \tilde{V}^0\|^2 \right) \\
\leq C \left( \|X^0 - \tilde{X}^0\|^2 + \|V^0 - \tilde{V}^0\|^2 \right), \quad t \geq 0,
\]
where the positive constant \(C\) is defined as
\[
C := e^{\frac{\varepsilon_0}{2\beta}} \max \left\{ \kappa + \gamma \varepsilon_1 + \varepsilon_1, \ (1 + \varepsilon_1) \right\}.
\]

This establishes the proof.
5. Asymptotic behavior of the kinetic swarm sphere model. In this section, we study the asymptotic behavior of a measure-valued solution to the kinetic swarm sphere model (2), and present the proof of Theorems 3.4 and 3.5.

5.1. Proof of Theorem 3.4. In this subsection, we present the proof of Theorem 3.4. For this, we provide a measure-theoretic preliminaries for our discussion. First, we recall the definition of the measure-valued solution to (4).

Definition 5.1. For $T \in [0, \infty)$, $\mu \in C^\omega([0, T); \mathcal{P}_2(\mathbb{D}))$ is a measure-valued solution with initial measure $\mu_0 \in \mathcal{P}_2(\mathbb{D})$, if the following three assertions hold.

(i) Total mass is normalized: $\langle \mu_t, 1 \rangle = 1$.

(ii) $\mu$ is weakly continuous in $t$: $t \mapsto \langle \mu_t, \phi \rangle$ is continuous in $t$ for all $\phi = \phi(t, \omega, v, \Omega) \in C^1_0(\mathbb{D} \times [0, T])$.

(iii) $\mu$ satisfies the kinetic equation (4) in the following weak sense: for all test functions $\varphi \in C^1_0(\mathbb{D} \times [0, T])$,

$$\langle \mu_t, \varphi(t, \cdot, \cdot, \cdot) \rangle - \langle \mu_0, \varphi(0, \cdot, \cdot, \cdot) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + v \cdot \nabla \omega \varphi + L[\mu_t] \cdot \nabla_v \varphi \rangle ds,$$

where $L[\mu_t]$ is defined as

$$L[\mu_t](t, \omega, v, \Omega) = -\gamma v - |v|^2 \omega + \kappa \int_{\mathbb{D}} (F(\omega)w_*) \mu_t(d\omega, dv, d\Omega_*).$$

Here, we adopt a standard duality relation: for $f \in C_0(\mathbb{D})$ and $\mu \in \mathcal{P}_2(\mathbb{D})$,

$$\langle \mu, f \rangle := \int_\mathbb{D} f(\omega, v, \Omega) d\mu(\omega, v, \Omega).$$

Below, as mentioned in Section 1, we see that (3) and (4) can be treated in a common setting. For any solution $(\omega_i, v_i)$ to (17), we associate the empirical measure

$$f^N_N := \frac{1}{N} \sum_{i=1}^N \delta(\omega_i, v_i, \Omega_i).$$

Then, if the empirical measure $f^N$ is acted on (4), then one can recover (3). This allows us to deal with (3) and (4) in the same framework. We now determine how to define the distance between two probability measures. Among many candidates, we adopt Wasserstein-2 distance, denoted as $W_2$, in the space $\mathcal{P}_2(\mathbb{D})$, the set of Borel probability measures on $\mathbb{D}$ with finite moments of order two.

Definition 5.2. (i) For two measures $\mu, \nu \in \mathcal{P}_2(\mathbb{D})$, we define the Wasserstein-2 distance between $\mu$ and $\nu$ as

$$W_2(\mu, \nu) := \left( \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{D} \times \mathbb{D}} |x - y|^2 d\gamma(x, y) \right)^{\frac{1}{2}},$$

where $\Gamma(\mu, \nu)$ represents the collection of all measures in $\mathbb{D} \times \mathbb{D}$ with marginals $\mu$ and $\nu$.

(ii) The kinetic equation (4) is derivable in $[0, T)$ from the particle level (3) if the following properties hold.
(A): For a given initial measure \( \mu_0 \in \mathcal{P}_2(\mathbb{D}) \) to (4), \( \mu_0 \) can be approximated by the initial empirical measure \( \mu_0^N \) corresponding to (3):

\[
\lim_{N \to \infty} W_2(\mu_0, \mu_0^N) = 0.
\]

(B): For \( t \in [0, T) \), there exists a unique measure-valued solution \( \mu_t \) of (4) with initial data \( \mu_0 \), and such solution \( \mu_t \) can be approximated by the empirical measure \( \nu_t^N \) of (3) uniformly in time interval \([0, T]\), where \( \mu_t^N \) is the measure-valued solution of the particle system (3):

\[
\limsup_{t \to 0} \lim_{N \to \infty} W_p(\mu_t, \mu_t^N) = 0.
\]

Then, together with the uniform stability result in Theorem 3.3 and Definition 5.2, we directly obtain the global-in-time existence of a measure-valued solution to (2). We now present the proof of Theorem 3.4.

(Proof of Theorem 3.4) (i) For the first part, we refer the reader to Corollary 1.1 in [20] or Theorem 3 in [19] for the detailed steps. Instead of providing all the details, we just mention how the uniform stability estimate at the particle level in Theorem 3.3 directly yields global existence of a measure-valued solution. First, for a given initial datum \( \mu_0 \), there exists a sequence of empirical measures \( \{\mu_0^N\}_{N=1}^\infty \) which approximates \( \mu_0 \):

\[
\lim_{N \to \infty} W_2(\mu_0, \mu_0^N) = 0.
\]

Hence, for any small \( \varepsilon > 0 \), there exists a large number \( N(\varepsilon) \) such that

\[
W_2(\mu_0^n, \mu_0^\ell) < \varepsilon, \quad n, \ell > N(\varepsilon), \quad (70)
\]

where \( \mu_0^n \) and \( \mu_0^\ell \) are empirical measures concentrated on the initial configuration \((x_{i0}, v_{i0})\) and \((\tilde{x}_{i0}, \tilde{v}_{i0})\), respectively. Then, after a rough argument, we have

\[
\left| W_2^2(\mu_0^n, \mu_0^\ell) - \frac{1}{nLD} \sum_{k=1}^{nLD} \left( |x_k^0 - \tilde{x}_k^0|^2 + |v_k^0 - \tilde{v}_k^0|^2 \right) \right| < \varepsilon,
\]

where \( D \) is a constant. Consequently, one has

\[
W_2^2(\mu_0^n, \mu_0^\ell) \leq \frac{1}{nLD} \sum_{k=1}^{nLD} \left( |x_k(t) - \tilde{x}_k(t)|^2 + |v_k(t) - \tilde{v}_k(t)|^2 \right) \quad \leq \frac{G^2}{nLD} \sum_{k=1}^{nLD} \left( |x_k^0 - \tilde{x}_k^0|^2 + |v_k^0 - \tilde{v}_k^0|^2 \right) \leq \mathcal{O}(1)\varepsilon. \quad (71)
\]

Here, the uniform stability result is used in the second inequality. Thus, we show that \( \{\mu_t^N\}_{N=1}^\infty \) is a Cauchy sequence in \( \mathcal{P}_2(\mathbb{D}) \) and there exists the limit, say \( \mu_t := \lim_{N \to \infty} \mu_t^N \). By taking limit \( m \to \infty \) in (1), we find

\[
W_2^2(\mu_t, \mu_t^N) < \varepsilon, \quad t \geq 0. \quad (72)
\]

We can finally check \( \mu_t \) is indeed a unique measure-valued solution to (2) with the initial datum \( \mu_0 \).

(ii) For the second part, let \( \mu_t \) and \( \nu_t \) be measure-valued solutions which belong to \( C_w(0, \infty); \mathcal{P}_2(\mathbb{D}) \) with initial data \( \mu_0 \) and \( \nu_0 \), respectively. Then, we use (3) to see

\[
W_2(\mu_t, \mu_0^n) < \varepsilon, \quad W_2(\nu_t, \nu_0^n) < \varepsilon, \quad n \gg 1.
\]
Hence, it follows from the triangle inequality and (2) that
\[ W_2(\mu_t, \nu_t)^2 \leq (W_2(\mu_t, \mu_n^t) + W_2(\mu_n^t, \nu_n^t) + W_2(\nu_t, \nu_n^t))^2 \leq O(1) \varepsilon. \]

**Remark 3.** Note that the previous uniform stability result in Theorem 3.3 crucially depend on the condition:
\[ \Omega_i = \Omega, \quad i = 1, \ldots, N, \quad (73) \]
which guarantees the emergence of the complete aggregation. However, if we do not impose the condition (4), then we would not achieve uniform-in-time results. For instance, consider the general non-identical case, that is, \( \Omega_i \not= \Omega_j \) for some \( i \not= j \). Then, the estimate (5) in Theorem 3.3 can be stated in the following way: for \( T \in (0, \infty) \), there exists a positive constant \( G = G(T) \) such that
\[
\sup_{0 \leq t < T} \left( \|X(t) - \tilde{X}(t)\|^2 + \|V(t) - \tilde{V}(t)\|^2 \right) \leq G(T) \left( \|X_0 - \tilde{X}_0\|^2 + \|V_0 - \tilde{V}_0\|^2 \right).
\]
Thus, the measure-valued solution to kinetic equation (2) exists globally but not globally-in-time. In other words, global-in-time existence of a measure-valued solution is possible for (2), however, it would not be possible for (4).

### 5.2. Proof of Theorem 3.5

In this subsection, we provide the proof of Theorem 3.5 which concerns with the asymptotic behavior of the kinetic equation (2). For our estimate, we a priori assume that our measure-valued solution \( f \in C_w(\mathbb{R}_+; P_2(D)) \).

Consider the characteristic system associated to (2) with identical oscillators:
\[
\begin{cases}
\frac{\partial}{\partial t} q(t; 0, \omega, v) = p(t; 0, \omega, v), \quad (\omega, v) \in D = S^{d-1} \times \mathbb{R}^d, \\
\frac{\partial}{\partial t} p(t; 0, \omega, v) = L[f](t, q(t; 0, \omega, v), p(t; 0, \omega, v)), \\
(q(0; 0, \omega, v), p(0; 0, \omega, v)) = (\omega, v).
\end{cases}
\]

Since the force filed \( L[f] \) in (5) is globally Lipschitz, the characteristic flow (5) is well-defined for all time. We observe
\[
(q(t; 0, \cdot, \cdot), p(t; 0, \cdot, \cdot)) \# f_0 = f(t), \quad t \geq 0.
\]
For the notational simplicity, we set
\[
q_t(\omega, v) := q(t; 0, \omega, v), \quad p_t(\omega, v) := p(t; 0, \omega, v), \quad t \geq 0.
\]
Then, we can write
\[
L[f](t, q_t(\omega, v), p_t(\omega, v)) \\
= -\gamma p_t(\omega, v) - |p_t(\omega, v)|^2 q_t(\omega, v) + \kappa \int_D \mathbb{P}(q_t(\omega, v))(w_*) f dS_w, dv_* \\
= -\gamma p_t(\omega, v) - |p_t(\omega, v)|^2 q_t(\omega, v) + \kappa \int_D \mathbb{P}(q_t(\omega, v))(q_t(\omega_*, v_*)) f_0(\omega_*, v_*) dS_{\omega_*} dv_*.
\]
As in the particle level, we define the energy functional to (2) which is reminiscent of (17):
\[
E(t) := \int_D |p_t(\omega, v)|^2 f_0(\omega, v) dS_\omega dv + \frac{\kappa}{2} \int_D |q_t(\omega_*, v_*) - q_t(\omega, v)|^2 f_0(\omega, v) dS_\omega dv_\omega dv_\omega dv
=: E_K(t) + E_P(t),
\]
where \( E_K \) and \( E_P \) denote kinetic and potential energies. Below, we present the energy estimate of the kinetic equation (5).

**Lemma 5.3.** Let \( d\mu_t = f(t) dS_\omega dv \in C_w(\mathbb{R}_+; P_2(D)) \) be a measure-valued solution to (5). Then, the following estimate holds: for \( t > 0 \),

(i) (Positive invariance of the domain): \( |q_t(\omega, v)|^2 = 1 \) and \( q_t(\omega, v) \cdot p_t(\omega, v) = 0 \).

(ii) (Energy estimate): \( \frac{d}{dt} E = -2\gamma E_K \).

**Proof.** (i) We observe
\[
\frac{1}{2} \frac{d^2}{dt^2} |q_t(\omega, v)|^2 = \frac{d}{dt} (q_t(\omega, v) \cdot p_t(\omega, v)) = \partial_t q_t(\omega, v) \cdot p_t(\omega, v) + q_t(\omega, v) \cdot \partial_t p_t(\omega, v)
= |p_t(\omega, v)|^2 - \gamma p_t(\omega, v) \cdot q_t(\omega, v) - |p_t(\omega, v)|^2 |q_t(\omega, v)|^2
+ \kappa (1 - |q_t(\omega, v)|^2) q_t(\omega, v) \cdot \int_D q_t(\omega_*, v_*) f_0(\omega_*, v_*) dS_\omega_\omega dv_\omega dv.
\]

(75)

For notational simplicity, we set
\[
u(t; 0, \omega, v) := u_t(\omega, v) := 1 - |q_t(\omega, v)|^2, \quad t > 0.
\]

Then, it follows from (6) that \( u_t \) satisfies
\[
\frac{1}{2} \frac{d^2}{dt^2} u_t(\omega, v) + \gamma \frac{d}{dt} u_t(\omega, v)
+ \left( |p_t(\omega, v)|^2 + q_t(\omega, v) \cdot \int_D q_t(\omega_*, v_*) f_0(\omega_*, v_*) dS_\omega_\omega dv_\omega \right) u_t(\omega, v) = 0.
\]

(76)

Since the coefficients in (7) are all globally Lipschitz, (7) admits the global unique solution. Note that the initial data satisfy
\[
u(0; 0, \omega, v) = 1 - |\omega|^2 = 0, \quad \dot{u}(0; 0, \omega, v) = \langle \omega, v \rangle = 0.
\]

(77)

On the other hand, a trivial solution \( u_t(\omega, v) \equiv 0 \) satisfies the autonomous equation (7) and initial condition (8). Hence, the uniqueness yields the desired identity:
\[
u_t(\omega, v) \equiv 0, \quad \text{or, equivalently,} \quad |q_t(\omega, v)|^2 = 1, \quad t > 0.
\]
(ii) By differentiating the kinetic energy, one has
\begin{align*}
\frac{m}{2} \frac{d}{dt} \int_D |p_t(\omega, v)|^2 f_0(\omega, v) dS_\omega d\omega \\
= m \int_D p_t(\omega, v) \cdot \partial_t p_t(\omega, v) f_0(\omega, v) dS_\omega d\omega \\
= -\gamma \int_D |p_t(\omega, v)|^2 f_0(\omega, v) dS_\omega d\omega \\
&+ \kappa \int_{D \times D} p_t(\omega, v) \cdot q_t(\omega, v^*) f_0(\omega, v) dS_\omega d\omega + \frac{\kappa}{2} \frac{d}{dt} \left| \int_D q_t(\omega, v) f_0(\omega, v) dS_\omega d\omega \right|^2.
\end{align*}

On the other hand, we find
\begin{align*}
\int_D |q_t(\omega, v^*) - q_t(\omega, v)|^2 f_0(\omega, v^*) f_0(\omega, v) dS_\omega d\omega d\omega \\
= \int_D \left( |q_t(\omega, v)|^2 + |q_t(\omega, v^*)|^2 - 2q_t(\omega, v^*) \cdot q_t(\omega, v) \right) \\
\times f_0(\omega, v^*) f_0(\omega, v) dS_\omega d\omega d\omega \\
= 2 - 2 \left| \int_D q_t(\omega, v) f_0(\omega, v) dS_\omega d\omega \right|^2.
\end{align*}

Finally, we substitute (10) into (9) to find the desired energy relation:
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_D |p_t(\omega, v)|^2 f_0(\omega, v) dS_\omega d\omega \\
= -\gamma \int_D |p_t(\omega, v)|^2 f_0(\omega, v) dS_\omega d\omega \\
&- \kappa \int_D |q_t(\omega, v^*) - q_t(\omega, v)|^2 f_0(\omega, v^*) f_0(\omega, v) dS_\omega d\omega d\omega.
\end{align*}

As a direct consequence of Lemma 5.3(ii), we see that the kinetic energy tends to zero.

**Corollary 1.** Let \(d\mu = f(t)dS_\omega d\omega \in C_w(\mathbb{R}_+; \mathcal{P}_2(D))\) be a measure-valued solution to (5). Then, the kinetic energy converges to zero with unknown but integrable decay rate:
\[ \lim_{t \to \infty} \mathcal{E}_K(t) = 0. \]

**Proof.** By integrating the relation in Lemma 5.3(ii), we find
\[ 2\gamma \int_0^t \mathcal{E}_K(s) ds \leq \mathcal{E}(t) + 2\gamma \int_0^t \mathcal{E}_K(s) ds = \mathcal{E}(0), \quad t \geq 0. \]

This yields
\[ \int_0^\infty \mathcal{E}_K(t) dt \leq \frac{1}{2\gamma} \mathcal{E}(0). \]
In order to apply Barbalat’s lemma, we need to check that $\mathcal{E}_K$ is uniformly continuous. For this, we use Hölder’s inequality to see

$$
\int_{D \times D} p_t(\omega, v) \cdot q_t(\omega, v) f_0(\omega, v) f_0(\omega, v) dS_\omega dS_v dS_\omega dS_v 
\leq \left( \int_D |p_t(\omega, v)| f_0(\omega, v) dS_v \right)^{1/2} \left( \int_D f_0(\omega, v) dS_v \right)^{1/2} 
= \mathcal{E}_K^2.
$$

(81)

In (9), we use (12) together with the notation $y := \mathcal{E}_K^2$ to get

$$
\frac{1}{2} \frac{d}{dt} y^2 \leq -\gamma y^2 + \kappa y, \quad \text{or, equivalently,} \quad \frac{d}{dt} y \leq -\gamma y + \kappa, \quad t > 0.
$$

Hence, we find the uniform upper bound for $\mathcal{E}_K$:

$$
\sup_{0 \leq t < \infty} \mathcal{E}_K(t) \leq \max \left\{ \frac{\kappa^2}{\gamma^2}, \mathcal{E}_K(0)^2 \right\} =: \mathcal{E}_K^\infty.
$$

In addition, $\dot{\mathcal{E}}_K$ is uniformly bounded:

$$
\sup_{0 \leq t < \infty} |\dot{\mathcal{E}}_K(t)| \leq 2 (\gamma \mathcal{E}_K^\infty + \kappa \sqrt{\mathcal{E}_K^\infty}).
$$

Thus, we verify that $\mathcal{E}_K$ is uniformly continuous so that the relation (11) yields the desired convergence:

$$
\lim_{t \to \infty} \mathcal{E}_K(t) = 0.
$$

Lemma 5.3 and Corollary 1 concern with the evolution of the velocity variable.

In what follows, we consider the evolution of the order parameter which gives the information of the position variable:

$$
R_f(t) := \int_{D} q_t(\omega, v) f_e(\omega, v) dS_\omega dS_v, \quad t \geq 0.
$$

Before we prove our main theorem, we first classify all stationary solutions to the kinetic equation (2).

**Proposition 2.** The function $f_e(\omega, v)$ is a stationary solution to (2) if and only if one of the following relations hold:

(i) $R_f := \int_{D} q_0(\omega, v) f_0(\omega, v) dS_\omega dS_v \equiv 0$.

(ii) There exists $m_0 \in (0, 1)$ and $u \in S^{d-1}$ such that

$$
f_e(\omega, v) = ((1 - m_0) \delta_u(\omega) + m_0 \delta_{-u}(\omega)) \otimes \delta_0(v).
$$

**Proof.** We recall Proposition 4.2 in [18] which states that the equilibria of the second-order dynamics (3) with $\Omega_i \equiv 0$ coincides with those of the first-order dynamics with $\Omega_i \equiv 0$. The results of the particle level can be lifted to the for kinetic level. More precisely, for the first-order kinetic level, it follows from Theorem 1 and Remark 2 in [16] that the stationary solution has the form

$$
f_e(\omega) = (1 - m) \delta_u(\omega) + m \delta_{-u}(\omega), \quad m \in (0, 1), \quad u \in S^{d-1}.
$$
Hence, we just associate the Dirac measure at zero for the velocity variable to obtain the desired classification of a stationary solution.

Now, we prove our last main theorem which shows that a solution to (2) converges to the bi-polar state, except the trivial case.

(Proof of Theorem 3.5) (i) It follows from the initial condition (7) that
\[ \mathcal{E}_K(0) = 0, \]
which directly yields
\[ \int_D |v|^2 f_0(\omega, v) dS_\omega dv = 0. \]
This implies that the supp_{v \in \mathbb{R}^d} f_0 has measure zero. Since \( p_t \) is absolutely continuous, we have
\[ \text{supp}_{v \in \mathbb{R}^d} f(t) = 0, \quad t \geq 0. \]
Hence, we conclude
\[ R_f(t) \equiv 0, \quad t \geq 0. \]

(ii) We integrate the energy relation in Lemma 5.3(ii) to get
\[ \mathcal{E}_K(t) + \kappa (1 - |R_f(t)|^2) + 2\gamma \int_0^t \mathcal{E}_K(s) ds = \mathcal{E}_K(0) + \kappa (1 - |R_f|^2), \quad (82) \]
where we represent the potential energy \( \mathcal{E}_P \) in terms of the order parameter as in (10). Our claim is to show that there exists a positive number \( R_* > 0 \) such that
\[ R_f(t) \geq R_*, \quad t \geq 0. \quad (83) \]
Suppose to the contrary that there exists a positive time \( t_* \in (0, \infty] \) such that
\[ \lim_{t \to t_*^-} R_f(t) = 0. \]
By letting \( t \to t_*^- \) in (13), we have
\[ \mathcal{E}_K(t_*) + \kappa + 2\gamma \int_0^{t_*} \mathcal{E}_K(s) ds = \mathcal{E}_K(0) + \kappa (1 - |R_f|^2). \quad (84) \]
Again the condition (7) yields that the relation (15) implies
\[ \mathcal{E}_K(t_*) + 2\gamma \int_0^{t_*} \mathcal{E}_K(s) ds \leq 0. \]
Since \( \mathcal{E}_K \) is non-negative and continuously differentiable, one has
\[ \mathcal{E}_K(t) \equiv 0, \quad t \in [0, t_*]. \]
Then, the energy relation (13) gives
\[ 0 = R_f(t_*) = R_f(0), \quad t \in [0, t_*]. \]
However, this contradicts the initial assumption (7). Hence, our claim (14) holds.

On the other hand, we use the definition of the order parameter to see
\[ \frac{d}{dt} |R_f| \leq \int_D |p_t(\omega, v)| f_0(\omega, v) dS_\omega dv \leq \left( \int_D |p_t(\omega, v)|^2 f_0 dS_\omega dv \right)^{\frac{1}{2}}. \quad (85) \]
Then, it follows from Corollary 1 that
\[ \int_0^\infty \left( \int_D |p_t(\omega, v)|^2 f_0 dS_\omega dv \right)^{\frac{1}{2}} < \infty. \quad (86) \]
We combine (16) and (17) to find
\[
\left| \int_0^\infty \dot{R}_f \, dt \right| \leq \int_0^\infty \left| \frac{d}{dt} R_f \right| \, dt \leq \int_0^\infty \frac{d}{dt} |R_f| \, dt < \infty,
\]
and (18) yields the existence of the limit for \(R_f\):
\[
\lim_{t \to \infty} R_f(t) = R_0 + \int_0^\infty \dot{R}_f \, dt < \infty.
\]
Thanks to Barbalat’s lemma, we also see that \(\dot{R}_f\) converges to zero. Finally, our desired assertion follows from the classification of a stationary solution in Proposition 2. This completes the proof.

6. Conclusion. In this paper, we have studied the emergent behavior of the identical second-order swarm sphere model at both particle and kinetic levels. More precisely, we adopt the gradient-like flow approach to show that a solution to identical particle model always converges to the equilibrium. Here, we cannot use the classical theorem as it is due to the presence of the nonlinear centripetal force term and this technical difficulty can be overcome once we follow the proof of the classical theorem step by step. In addition, we establish uniform-in-time \(\ell_2\)-stability with respect to the initial data using the complete aggregation estimate. As discussed before, such uniform-in-time results are rarely found in the literature of the collective dynamics. For the kinetic model, we rigorously derive the mean-field kinetic equation as a by-product of the uniform stability estimate, and the global-in-time existence of a measure-valued solution also directly follows from the measure-theoretic setting. In addition, we investigate the emergent behavior of the kinetic equation by lifting the corresponding results for the particle model. To be more specific, we show that under some initial framework, a solution to the kinetic equation converges to the bi-polar state. In fact, there are still many interesting problems and in particular, emergent dynamics for non-identical problems is largely open. Hence, extension of the presented results to the non-identical case will be pursued in future work.

REFERENCES

[1] A. Aydoğ, S. T. McQuade and N. P. Duteil, Opinion dynamics on a general compact Riemannian manifold, *Netw. Heterog. Media*, 12 (2017), 489–523.
[2] I. Barbalat, Systèmes d’équations différentielles d’oscillations non linéaires, *Rev. Math. Pures Appl.*, 4 (1959), 267–270.
[3] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot and D. Bartolo, Emergence of macroscopic directed motion in populations of motile colloids, *Nature*, 503 (2013), 95–98.
[4] M. Caponigro, A. C. Lai and B. Piccoli, A nonlinear model of opinion formation on the sphere, *Discrete Contin. Dyn. Syst.*, 35 (2015), 4241–4268.
[5] S. Chandra, M. Girvan and E. Ott, Continuous versus discontinuous transitions in the D-dimensional generalized Kuramoto model: Odd D is different, *Phys. Rev. X*, 9 (2019), 011002.
[6] S. Chandra and E. Ott, Observing microscopic transitions from macroscopic bursts: Instability-mediated resetting in the incoherent regime of the D-dimensional generalized Kuramoto model, *Chaos*, 29 (2019), 033124, 13pp.
[7] D. Chi, S.-H. Choi and S.-Y. Ha, Emergent behaviors of a holonomic particle system on a sphere, *J. Math. Phys.*, 55 (2014), 052703, 18pp.
[8] J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko, Emergence of bi-cluster flocking for the Cucker-Smale model, *Math. Models Methods Appl. Sci.*, 26 (2016), 1191–1218.
[9] Y.-P. Choi, S.-Y. Ha and J. Morales, Emergent dynamics of the Kuramoto model under the effect of inertia, *Discrete Contin. Dyn. Syst.*, 38 (2018), 4875–4913.
[10] Y.-P. Choi, S.-Y. Ha and S.-B. Yun, Complete synchronization of Kuramoto oscillators with finite inertia, *Phys. D*, **240** (2011), 32–44.

[11] Y.-P. Choi and Z. Li, Synchronization of nonuniform Kuramoto oscillators for power grids with general connectivity and dampings, *Nonlinearity*, **32** (2019), 559–583.

[12] Y.-P. Choi, Z. Li, S.-Y. Ha, X. Xue and S.-B. Yun, Complete entrainment of Kuramoto oscillators with inertia on networks via gradient-like flow, *J. Differential Equations*, **257** (2014), 2591–2621.

[13] F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control*, **52** (2007), 852–862.

[14] T. Danino, O. Mondragon-Palomino, L. Tsimring and J. Hasty, A synchronized quorum of genetic clocks, *Nature*, **463** (2010), 326–330.

[15] G. B. Ermentrout, An adaptive model for synchrony in the firefly *Pteroptyx malaccae*, *J. Math. Biol.*, **29** (1991), 571–585.

[16] A. Frouvelle and J.-G. Liu, Long-time dynamics for a simple aggregation equation on the sphere, in *Stochastic Dynamics Out of Equilibrium. IHPStochDyn 2017*, (eds. G. Giacomin, S. Olla, E. Saada, H. Spohn, G. Stoltz), Springer Proceedings in Mathematica & Statistic, Springer, Cham, **282** (2019), 457–479.

[17] T. Gregor, K. Fujimoto, N. Masaki and S. Sawai, The onset of collective behavior in social amoebae, *Science*, **328** (2010), 1021–1025.

[18] S.-Y. Ha and D. Kim, A second-order particle swarm model on a sphere and emergent dynamics, *SIAM J. Appl. Dyn. Syst.*, **18** (2019), 80–116.

[19] S.-Y. Ha, D. Kim, J. Lee and S. E. No, Particle and kinetic models for swarming particles on a sphere and stability properties, *J. Stat. Phys.*, **174** (2019), 622–655.

[20] S.-Y. Ha, J. Kim and X. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, *Kinet. Relat. Models*, **11** (2018), 1157–1181.

[21] S.-Y. Ha, D. Ko and S. Ryoo, On the relaxation dynamics of Lohe oscillators on the Riemannian manifold, *J. Stat. Phys.*, **172** (2018), 1427–1478.

[22] S.-Y. Ha and J.-G. Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, *Comm. Math. Sci.*, **7** (2007), 297–325.

[23] A. Haraux and M. A. Jendoubi, Convergence of solutions of second-order gradient-like systems with analytic nonlinearities, *J. Differential Equations*, **144** (1998), 313–320.

[24] M. Hung and S. N. Givigi, A q-learning approach to flocking with UAVs in a stochastic environment, *IEEE Trans. Cybern.*, **47** (2017), 186–197.

[25] Y. Kuramoto, Self-entrainment of a population of coupled non-linear oscillators, in *International Symposium on Mathematical Problems in Mathematical Physics*, Lecture Notes in Theoretical Physics, **39** (1975), 420–422.

[26] M. A. Lohe, Non-Abelian Kuramoto model and synchronization, *J. Phys. A*, **42** (2009), 395101, 25pp.

[27] M. A. Lohe, High-dimensional generalizations of the Watanabe-Strogatz transform for vector models of synchronization, *J. Phys. A*, **51** (2018), 225101, 24pp.

[28] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao and R. Aditi Simha, Hydrodynamics of soft active matter, *Rev. Mod. Phys.*, **85** (2013), 1143–1189.

[29] J. Markdahl and J. Gonçalves, Global convergence properties of a consensus protocol on the n-sphere, *Proc. of the 55th IEEE conference on Decision and Control*, (2016), 2487–2492.

[30] J. Markdahl, J. Thunberg and J. Gonçalves, Almost global consensus on the n-sphere, *IEEE Trans. Automat. Contr.*, **63** (2018), 1664–1675.

[31] F. J. Nédélec, T. Surry, A. C. Maggas and S. Leibler, Self-organization of microtubules and motors, *Nature*, **389** (1997), 305–308.

[32] R. Olufi-Saber, Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks, *Proc. of the 45th IEEE Conference on Decision and Control*, (2006), 5060–5066.

[33] L. Perea, P. Elsoegui and G. Gomez, Extension of the Cucker-Smale control law to space flight formations, *J. Guid. Control*, **32** (2009), 526–537.

[34] M. Rubenstein, A. Cornejo and R. Nagapal, Programmable self-assembly in a thousand-robot swarm, *Science*, **345** (2014), 795–799.

[35] R. Sknepnek and S. Henkes, Active swarms on a sphere, *Phys. Rev. E*, **91** (2015), 022306.

[36] Y. Sun, W. Li and D. Zhao, Realization of consensus of multi-agent systems with stochastically mixed interactions, *Chaos*, **26** (2016), 073112, 8pp.
[37] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett., 75 (1995), 1226–1229.

[38] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, J. Theor. Biol., 16 (1967), 15–42.

[39] J. Zhu, Synchronization of Kuramoto model in a high-dimensional linear space, Phys. Lett. A, 377 (2013), 2939–2943.

[40] J. Zhu, J. Zhu and C. Qian, On equilibria and consensus of the Lohe model with identical oscillators, SIAM J. Appl. Dyns. Syst., 17 (2018), 1716–1741.

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