SIMPLE GRAPHS OF ORDER 12 AND MINIMUM DEGREE 6 CONTAIN $K_6$ MINORS

RYAN ODENEAL AND ANDREI PAVELESCU

Abstract. We prove that every simple graph of order 12 which has minimum degree 6 contains a $K_6$ minor.

1. Introduction

In [2], Jørgensen proved that every simple graph of order at most 11 with minimal degree $\delta(G)$ at least 6 contains a $K_6$ minor. Following this theorem, he conjectured that every 6-connected graph which contains no $K_6$ minor is 1-apex. In [3], Kawarabayashi, Norine, Thomas, and Wollan proved this conjecture for sufficiently large graphs of bounded tree-width. Little is known about small order graphs. In this paper, we extend Jørgensen’s result by proving the following:

Theorem 1. Let $G$ be a simple graph of order 12 and assume that $\delta(G) \geq 6$. Then $G$ contains a $K_6$ minor.

Note that the theorem implies Jørgensen’s conjecture is vacuously true for graphs of order 12.

2. Notation and Definitions

All the graphs considered in this article are simple (non-oriented, no loops, no multiple edges). For a graph $G$, $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. For $n \geq 1$, $K_n$ denotes the complete graph of order $n$ and $K_n^-$ denotes the complete graph of order $n$ with one edge removed. If $v_1, v_2, \ldots, v_k$ are vertices of $G$, then $\langle v_1, v_2, \ldots, v_k \rangle_G$ denotes the subgraph of $G$ induced on these vertices. If $v$ is a vertex of $G$, $N(v)$ is the subgraph of $G$ induced by $v$ and the vertices adjacent to $v$ in $G$. If $v$ is a vertex of $G$, $G - v$ is the subgraph of $G$ obtained by deleting vertex $v$ and all edges of $G$ that $v$ is incident to. The minimum of the degrees of all vertices of $G$ is denoted by $\delta = \delta(G)$. A graph is called $k$-regular if every vertex has degree $k$.

For two graphs $G_1$ and $G_2$, $G_1 \ast G_2$ denotes the graph with vertex set $V(G_1) \sqcup V(G_2)$ and edge set $E(G_1) \sqcup E(G_2) \sqcup E'$, where $E'$ is the set of edges with one
endpoint in $V(G_1)$ and the other endpoint in $V(G_2)$. In $G_1 * v$, we call $v$ a cone over $G_1$.

A graph $G$ is the clique sum of $G_1$ and $G_2$ over $K_p$ if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and the subgraphs induced by $V(G_1) \cap V(G_2)$ in both $G_1$ and $G_2$ are complete of order $p$.

A graph $G$ is called planar if it can be embedded into the plane. A graph $G$ is called maximal planar if adding an edge between any two non-adjacent vertices causes $G$ to be not planar. A maximal planar graph of order $n$ has exactly $3n - 6$ edges. A graph $G$ is called $k$-apex if there is a choice of $k$ vertices of $G$ which, if removed, produce a planar subgraph of $G$. A 1-apex graph is called apex.

The minimal number of vertices of a graph $G$, whose removal disconnects $G$, or creates a graph with a single vertex, is called the vertex connectivity of $G$, and it is denoted by $\kappa(G)$. For all $k \leq \kappa(G)$, we say that $G$ is $k$-connected. The set of vertices removed in order to disconnect the graph or reduce it to a single point is called a vertex cut. Any minimal vertex cut has size $\kappa(G)$. Since removing all the neighbors of a vertex produces either a disconnected graph or a graph of order 1, for any simple graph $G$, $\kappa(G) \leq \delta(G)$.

For a graph $G$, a minor of $G$ is any graph that can be obtained from $G$ by a sequence of vertex deletions, edge deletions, and simple edge contractions. A simple edge contraction means identifying its endpoints, deleting that edge, and deleting any double edges thus created.

The reverse operation to producing minors is called inflation. We used Diestel's [2] notation and definition for this concept. For a simple graph $H$, replace each vertex $x$ in $V(H)$ with disjoint connected finite simple graphs $G_x$ and the edges $xy$ of $E(H)$ with non-empty sets of $G_x - G_y$. We call this new graph an inflated $H$ or $IH$. If a graph $G$ contains an $IH$ as a subgraph, then $H$ is a minor of $G$. We call the inflated $H$ a model of $H$ in $G$. This is equivalent to saying that $H$ is a minor of $G$ if and only if there exists a map $\chi$ from a subset of $V(G)$ onto $V(H)$ such that for every vertex $h \in H$, its inverse image $\chi^{-1}(h)$ is connected in $Y$ and for every edge $hh' \in H$ there is an edge in $G$ between the one of the vertices in the pre-images of $\chi^{-1}(h)$ and $\chi^{-1}(h')$. Consider a partition of the set $V = \{1, 2, ..., n\}$ into blocks $B_1, B_2, ..., B_j$, where the blocks are ordered in the increasing order of their smallest element. One can encode this partition into a string of $n$ numbers from the set $\{1, 2, ..., j\}$. In their paper [6], Dinneen and Xiong define a restricted growth string (or RG string) as a string $a[1...n]$ where $a[i]$ is the block in which element $i$ occurs.

3. Main Theorem

In [4], Mader proved a minor theorem for graphs of minimum degree at least 5.

**Theorem 2** (W. Mader, 1968). Let $G$ be a simple graph and assume $\delta(G) \geq 5$. Then $G$ contains as a minor either $K_6^-$, or the icosahedral graph.
In the same paper, Mader also showed:

**Theorem 3** (W. Mader, 1968). For every integer $2 \leq t \leq 7$ and every simple graph $G$ of order $n \geq t - 1$ which has no minor isomorphic to $K_t$, $G$ has at most \((t - 2)n - \binom{t-1}{2}\) edges.

This result has two immediate implications.

**Lemma 4.** Let $G$ be a simple graph of order $n$ and size $4n - 10$. If $G \setminus v$ is planar, then $v$ cones over $G \setminus v$.

**Proof.** Since $G \setminus v$ is planar of order $n - 1$, it has at most $3(n - 1) - 6 = 3n - 9$ edges. This implies that $v$ has at least $4n - 10 - (3n - 9) = n - 1$ neighbors, and the conclusion follows. \(\square\)

For order 12, Theorem 3 implies that any graph of order 12 and size at least 39 contains a $K_6$ minor. On the other hand, for $\delta(G) \geq 6$, the Handshaking Lemma implies the size of the graph must be at least 36.

In [1], Jørgensen classified the graphs of order $n$ and size $4n - 10$.

**Theorem 5.** Let $p$ be a natural number, $5 \leq p \leq 7$. Let $G$ be a graph with $n$ vertices and $(p - 2)n - \binom{p}{2}$ edges that is not contractible to $K_p$. Then either $G$ is an $MP_{p-5}$-cockade or $p = 7$ and $G$ is the complete 4-partite graph $K_{2,2,2,3}$.

In our case, we only need to use $MP_1$-cockades.

**Definition 6.** $MP_1$-cockades are defined recursively as follows:

1. $K_5$ is an $MP_1$-cockade and if $H$ is a 4-connected maximal planar graph then $H \ast K_1$ is an $MP_1$-cockade.
2. Let $G_1$ and $G_2$ be disjoint $MP_1$-cockades, and let $x_1, x_2, x_3$, and $x_4$ be the vertices of a $K_4$ subgraph of $G_1$ and let $y_1, y_2, y_3$, and $y_4$ be the vertices of a $K_4$ subgraph of $G_2$. Then the graph obtained from $G_1 \cup G_2$ by identifying $x_j$ and $y_j$, for $j = 1, 2, 3, 4$, is an $MP_1$-cockade.

We shall now prove Theorem 1. The proof is organized based on the size of the graph.

**Proof.** Case 1. Assume $|E(G)| = 38$. By Theorem 5 either $G$ contains a $K_6$ minor, or $G$ is apex, or $G$ is the clique sum over $K_4$ of two $MP_1$ cockades.

If $G$ is isomorphic to $H \ast K_1$, where $H$ is a maximal planar graph on 11 vertices, then $\delta(H) \geq 5$ and, by Theorem 2 it follows that $H$ has a $K_6$ minor and thus $G$ has a $K_6$ minor.

Assume that $G$ is the clique sum over $S \simeq K_4$ of two $MP_1$-cockades. If $G \setminus S$ has more than two connected components, $Q_1, Q_2, \ldots$, then at least one of them, say $Q_1$, has at most two vertices. But this contradicts the fact that $\delta(G) \geq 6$. So $G \setminus S = Q_1 \sqcup Q_2$. Furthermore, unless $|Q_1| = |Q_2| = 4$, the graph either contains
a $K_7$ subgraph (if $|Q_1| = 3$), or $\delta(G) < 6$ (if $1 \leq |Q_1| \leq 2$). Since $\delta(G) \geq 6$, it follows that each vertex of $Q_i$ connects to at least two other vertices of $Q_i$, for $i = 1, 2$ respectively.

Without loss of generality, let $Q_1 = <v_1, v_2, v_3, v_4>_G$, $Q_2 = <v_5, v_6, v_7, v_8>_G \simeq K_4$, and $Q_2 = <v_9, v_{10}, v_{11}, v_{12}>_G$. If $Q_i$ is not isomorphic to $K_4$ for any of $i = 1, 2$, say $v_1v_2 \notin E(Q_1)$, since $v_1$ and $v_2$ must both connect to both $v_3$ and $v_4$, then contracting the edges $v_1v_3$ and $v_2v_4$ produces a minor of $G$ which contains a $K_6$ subgraph induced by $v_1, v_2$, and the four vertices of $S$. If, on the other hand, $Q_1 \simeq K_4$, as $\delta(G) \geq 6$, it follows that there are at least 12 edges between each of the $Q_s$ and $S$. That would imply that $E(|G|) \geq 6 + 12 + 6 + 12 + 6 = 42$, a contradiction. It follows that for $|E(G)| = 38$, $G$ has a $K_6$ minor.

**Case 2.** Assume that $|E(G)| = 37$. Since $\delta(G) \geq 6$, it follows that the possible degree sequences for the vertices of $G$ are either $(6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8)$ or $(6, 6, 6, 6, 6, 6, 6, 6, 7, 7)$. In either of the situations, we shall need the following lemma:

**Lemma 7.** Let $M$ denote a graph of order 11 and size 34, such that $\delta(M) \geq 5$. Assume that $M$ is not 1-apex and has at most two vertices of degree 5. Then $M$ contains a $K_6$ minor.

**Proof.** By Theorem 3, either $M$ contains a $K_6$ minor or is a $MP_1$-cockade. Since $M$ is not 1-apex, it follows that $M$ is the clique sum over $S \simeq K_4$ of two $MP_1$-cockades. If $M \setminus S$ has more than two connected components, $Q_1, Q_2, \ldots$, then at least one of them, say $Q_1$, has at most two vertices. As $|Q_1| = 1$ would violate the condition $\delta(M) \geq 5$, it follows that $|Q_1| = 2$ and the subgraph of $M$ induced by $Q_1$ and $S$ forms a $K_6$. So $M \setminus S = Q_1 \sqcup Q_2$ and, without loss of generality, $Q_1 = <v_1, v_2, v_3>_M$, $S = <v_4, v_5, v_6, v_7>_M \simeq K_4$, and $Q_2 = <v_8, v_9, v_{10}, v_{11}>_M$. Unless $Q_1 \simeq K_3$, since $\delta(M) \geq 5$, it follows that at least two vertices of $Q_1$ connect to all the vertices of $S$ and thus, via an edge contraction, they induce a $K_6$ minor of $M$.

If $Q_1 \simeq K_3$, there have to be exactly 9 edges connecting the vertices of $Q_1$ to those of $S$. If there are more than nine, the subgraph induced by the vertices of $Q_1$ and $S$ has 7 vertices and more than $3 + 9 + 6 = 18$ edges, thus it contains a $K_6$ minor by Theorem 3. If there are less than 9, then at least one of the vertices of $Q_1$ has degree less than 5. But this shows that $M$ has more than two vertices of degree 5, a contradiction. \qed

Assume the vertex degree sequence for $G$ is $(6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8)$. Furthermore, without loss of generality, we may assume that $\operatorname{deg}_G(v_1) = 6$, $\operatorname{deg}_G(v_8) = 8$, and that $N_G(v_1) = \{v_1, \ldots, v_7\}$. Let $N = <v_2, v_3, \ldots, v_7>_G$, $H = <v_8, \ldots, v_{12}>_G$, and let $L$ denote the set of edges of $G$ with one endpoint in $N$ and the other in $H$. The Handshaking Lemma provides the following relations between the sizes
of $E(N)$, $L$, and $E(H)$:

$$2|E(N)| + |L| = 30, \quad 2|E(H)| + |L| = 32.$$ 

If $|E(H)| = 10$, that is $H \simeq K_5$, as every vertex of $H$ must have at least two neighbors in $N$ ($\delta(G) \geq 6$), contracting the edges $v_iv_i$, for $2 \leq i \leq 7$, produces a $K_6$ minor of $G$.

If $|E(H)| \leq 9$, then $|L| \geq 14$ and thus $|E(N)| \leq 8$. It follows that there is a vertex of $N$, say $v_2$, such that $\deg_N(v_2) \leq 2$. If $\deg_N(v_2) < 2$, contracting the edge $v_1v_2$ would produce a minor of $G$ of order 11 and size at least 35, which would contain a $K_6$ minor by Theorem 3.

If $\deg_N(v_2) = 2$, then contracting the edge $v_1v_2$ would produce a minor $M$ of $G$ of order 11 and size precisely 34. Furthermore, since $v_2$ neighbors exactly 3 vertices of $H$, the maximum degree of $M$ is 8, so it cannot be 1-apex, according to Lemma 4. Lastly, $M$ has at most two vertices of degree 5, since $\deg_N(v_2) = 2$. By Lemma 7, $M$ has a $K_6$ minor, and therefore so does $G$.

Assume the vertex degree sequence for $G$ is $(6,6,6,6,6,6,6,6,6,6,6,7,7)$. If the degree 7 vertices are connected in $G$, deleting the edge connecting them would produce a 6 regular subgraph of order 36, to be dealt with in the last case of the proof. So, without loss of generality, assume that $\deg_G(v_1) = 6$, $\deg_G(v_8) = 7$, and $N_G(v_1) = \{v_1, \ldots, v_7\}$. Let $N = v_2, v_3, \ldots, v_7 \supset G$, $H = v_8, \ldots, v_{12} \supset G$, and let $L$ denote the set of edges of $G$ with one endpoint in $N$ and the other in $H$. If $\deg(v_i) = 7$, for some $9 \leq i \leq 12$, the same argument as before shows that $|E(N)| \leq 8$ and thus $G$ contains a $K_6$ minor. So we may assume $\deg_G(v_7) = 7$ and $v_7v_8 \notin E(G)$. Using the Handshaking Lemma, we get:

$$2|E(N)| + |L| = 31, \quad 2|E(H)| + |L| = 31.$$ 

If $|E(H)| = 10$, contracting the edges $v_iv_i$, for $2 \leq i \leq 7$, produces a $K_6$ minor of $G$.

If $|E(H)| \leq 9$, then $|L| \geq 13$ and thus $|E(N)| \leq 9$. If $|E(N)| \leq 8$, it follows that there is a vertex of $N$, $v_i$, such that $\deg_N(v_i) \leq 2$. If $\deg_N(v_i) < 2$, contracting the edge $v_1v_i$ would produce a minor of $G$ of order 11 and size at least 35, which would contain a $K_6$ minor by Theorem 3.

Assume $\deg_N(v_1) = 2$. Contracting the edge $v_1v_1$ produces a minor $M$ of $G$ of order 11 and size 34. Moreover, since for $2 \leq j \leq 7$, $v_j$ neighbors at most for of the vertices of $H$, $M$ cannot be 1-apex. Lastly, $M$ has at most two vertices of degree 5, since $\deg_N(v_1) = 2$. By Lemma 7, $M$ has a $K_6$ minor, and therefore so does $G$.

It follows that $N$ is 3-regular, $L = 13$ and that $|H| = 9$; that is, $H \simeq K_5$. If the missing edge of $H$ has $v_8$ as its endpoint, and since $\deg_G(v_8) = 7$, it follows that $v_8$ neighbors at least 4 vertices of $N$. As the other endpoint, say $v_9$, neighbors at least 3 vertices of $N$, it follows that there exists $2 \leq i \leq 6$ such that $v_i$ is a common neighbor of $v_8$ and $v_9$. Contracting the edges $v_iV_8$ and $v_1v_j$, for
2 \leq j \leq 7, j \neq i$, one obtains a $K_6$ minor of $G$, as every vertex of $H$ neighbors at least one of the $v_j$s.

If the missing edge of $H$ is $v_9v_{10}$, as $N$ is 3-regular, $deg_G(v_7) = 7$ and $v_7$ does not neighbor $v_8$, it follows that $v_7$ neighbors $v_j$, for all $9 \leq j \leq 12$. Since every vertex of $H$ neighbors at least two vertices of $N$, it follows that contracting the edges $v_7v_9$ and $v_1v_i$, for $2 \leq i \leq 6$, produces a $K_6$ minor of $G$.

**Case 3.** Assume $|G| = 36$, that is $G$ is a 6-regular graph. There are 7849 non isomorphic 6-regular graphs on 12 vertices as discovered by M. Meringer in [7]. For the purpose of this article, we generated these graphs and stored them using his GENREG program.

We implement an algorithm described by Xiong Liu and Michael J. Dinneen in [6] to check all 7849 cases. The goal is to check if there is a model of $K_6$ in $G$. Let $V(K_6) = \{k_1, k_2, k_3, k_4, k_5, k_6\}$ and $V(G) = \{g_1, g_2, ..., g_{12}\}$. Since all 6-regular graphs on 12 vertices are connected, it is sufficient to consider all surjective maps $\chi : V(G) \rightarrow V(K_6)$ and try to verify if for each edge $k_ik_j \in E(K_6)$, there exists at least one corresponding edge between a vertex in $\chi^{-1}(k_i)$ and $\chi^{-1}(k_j)$.

It is known that there is a bijective correspondence between the set of surjective functions from one set with cardinality $n$ to another with cardinality $k$ and the set of permutations of the partitions of the set with cardinality $n$ into $k$ blocks which is $S(n, k)!$ where $S(n, k)$ is the Stirling number of second kind. So we generate the set of permutations of all restricted growth strings where for each permuted RG string $a[1..n]$, if $a[i] = j$ where $1 \leq i \leq n$ and $1 \leq j \leq 6$, then $\chi(g_i) = k_j$. We then check if the inverse image has the properties that we want. This process ends if we find a model of $k_6$ in $G$ or if we exhaust our list of permuted RG strings.

In our case, by [6], this algorithm has worst case running time of

$$12^2 |E(K_6)| \sum_{j=1}^{6} (-1)^{6-j} \binom{6}{j} j^{12}.$$ 

A python implementation of this algorithm running on a Linux environment with 6GB of ram and an Intel(R) Core(TM) i5-4300 CPU @ 1.90GHz took approximately 7.92 hours to confirm that all 6-regular graphs on 12 vertices have a $K_6$ minor. \[\square\]

4. **Future Explorations**

(1) It is conceivable that one could prove that the statement of the main result is true for 6-regular graphs without computer assistance, by trying to carefully choose two edges which, when contracted, would produce a graph of order 10 and at least 30 edges. An adjusted version of Lemma 7 would finish the argument.
In his paper [2], Jørgensen proved that in a 6-regular graph, if the open neighborhood of any vertex is 3-regular, then any connected component of the graph is isomorphic to either $K_{3,3,3}$ or the complement of the Petersen graph. Since both contain $K_6$ minors, this argument solves one of the cases for 6-regular graphs of order 12. It is also conceivable that the sheer number of cases to consider would vouch for a computer search.

(2) Is it true that any simple graph of order 13 and minimum degree at least 6, which is not apex, contains a $K_6$-minor? Paired with the result of this paper, answering this question would provide a better understanding for the techniques required to tackle Jørgensen’s conjecture for small order graphs.

(3) The result of this paper shows that, for graphs of order 12, weaker assumptions are needed for the conclusion of Jørgensen’s conjecture to be true. What is the minimum $n > 12$ for which the minimum degree 6 condition is no longer sufficient and the 6-connected condition is needed? Such $n$ would have to be at most 26, as the following example demonstrates.

Let $G_1 \simeq G_2 \simeq K_1 \star Ic$, where $Ic$ denotes the icosahedral graph (5-regular, maximal planar, order 12). Let $G$ be the graph obtained by joining $G_1$ and $G_2$ with a single edge. Then $\delta(G) = 6$, $G$ is not apex and it has no $K_6$ minor.

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Department of Mathematics, University of South Alabama, Mobile, AL 36688, USA.