100% of the non-trivial zeros of $\zeta(s)$ are on the critical line

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Abstract

In this manuscript we denote by $N(T)$ the number of zeros $\rho$ of $\zeta(s)$ such that $0 < \Im(\rho) < T$. Denote by $N_0(T)$ the number of zeros $\rho$ of $\zeta(s)$ such that $\Re(\rho) = \frac{1}{2}$ and $0 < \Im(\rho) < T$. Denote by $N_\epsilon(T)$ the number of zeros $\rho$ of $\zeta(s)$ such that $\frac{1}{2} - \epsilon < \Re(\rho) < \frac{1}{2} + \epsilon$ and $0 < \Im(\rho) < T + \epsilon$ where $\epsilon > 0$ is chosen later. We consider a rectangle $R_\epsilon = \{\sigma + it \mid \frac{1}{2} - \epsilon \leq \sigma \leq \frac{1}{2} + \epsilon, 0 \leq t \leq T + \epsilon\}$ where $\epsilon > 0$ is chosen so that $\xi(s)$ has no zeros on the boundary. Since $\xi(s)$ is non-zero on the real axis, so choose $\epsilon > 0$ such that $\xi(s)$ has no zeros on the boundary of rectangle $R_\epsilon$. So we use the Cauchy’s argument principle on $\xi(s)$ on the rectangle $R_\epsilon$ along with properties of Riemann zeta function and Riemann xi function and prove that as $T \to \infty$

$$N_0(T) = \frac{T}{2\pi} \log \left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} - \frac{1}{8} + O(\log T)$$

Also if we have

$$\kappa = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}$$

then we prove as a consequence that $\kappa = 1$.

Keywords: Riemann zeta function, Riemann xi function, Functional equation, Riemann Hypothesis, Cauchy’s argument principle, Schwarz Reflection Principle, Jensen’s formula.

Mathematics Subject Classification: 11M26, 11M06, 11M32

1 Introduction

The Riemann zeta function, $\zeta(s)$ is defined as the analytic continuation of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges in the half plane $\Re(s) > 1$. The Riemann zeta function is a meromorphic function on the whole complex $s$-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. The Riemann Hypothesis states that all the non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$. The Riemann xi function is defined as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$\xi(s)$ is an entire function whose zeros are the non trivial zeros of $\zeta(s)$ (see [7]). Further $\xi(s)$ satisfies the functional equation (see [7])

$$\xi(s) = \xi(1-s)$$

Denote by $N(T)$ the number of zeros $\rho$ of $\zeta(s)$ such that $0 < \Im(\rho) < T$. Denote by $N_0(T)$ the number of zeros $\rho$ of $\zeta(s)$ such that $\Re(\rho) = \frac{1}{2}$ and $0 < \Im(\rho) < T$. Then Riemann (see [4]) in his 1859 paper
conjectured that \( N_0(T) = N(T) \). We have the following asymptotic for \( N(T) \) (see [1, p.134])

\[
N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + O(\log T) \quad \text{as } T \to \infty
\]

Selberg (see [5]) in 1942 proved that

\[
k = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)} > 0
\]

where \( k \) is the proportion of the zeros of \( \zeta(s) \) on the critical line. Conrey (see [3]) in 1989 proved that more than two fifths of the zeros of the Riemann zeta function are on the critical line. Pratt et al. (see [6]) in 2019 proved that more than five twelfths of zeros of \( \zeta(s) \) are on the critical line. Suman and Das (see [10]) proved a series equivalent of the Riemann Hypothesis.

## 2 Main Result

We denote by \( N(T) \) the number of zeros \( \rho \) of \( \zeta(s) \) such that \( 0 < \Im(\rho) < T \). Denote by \( N_0(T) \) the number of zeros \( \rho \) of \( \zeta(s) \) such that \( \Re(\rho) = \frac{1}{2} \) and \( 0 < \Im(\rho) < T \). Consider the rectangle \( R_\epsilon = \{ s = \sigma + it \in \mathbb{C} \mid \frac{1}{2} - \epsilon \leq \sigma \leq \frac{1}{2} + \epsilon, \ 0 \leq t \leq T + \epsilon \} \) where \( \epsilon > 0 \) is arbitrarily small. Also we denote by \( N_\epsilon(T) \) the number of zeros \( \rho \) of \( \zeta(s) \) such that \( \frac{1}{2} - \epsilon < \Re(\rho) < \frac{1}{2} + \epsilon \) and \( 0 < \Im(\rho) < T + \epsilon \) where \( \epsilon > 0 \) is chosen so that \( \zeta(s) \) has no zeros on the boundary of rectangle \( R_\epsilon \) (this is proved in the following Lemma). The goal of this note is to prove the following result:

**Theorem:** As \( T \to \infty \) we have the following

\[
N_0(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{1}{8} + O(\log T)
\]

Also if

\[
k = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}
\]

then we have

\[
k = 1
\]

**Proof:** First we prove a Lemma:

**Lemma:** We can choose \( \epsilon > 0 \) such that \( \zeta(s) \) has no zeros on the boundary of rectangle \( R_\epsilon \).

**Proof:** Let \( \partial R_\epsilon \) be the boundary of the rectangle \( R_\epsilon \). Let \( C_\epsilon \) be \( \partial R_\epsilon \setminus \mathbb{R} \). Suppose on the contrary such an \( \epsilon \) does not exist. Then for all \( \epsilon > 0 \) there is a \( z \) on \( \partial R_\epsilon \). But \( \zeta(s) \) is non zero on \( \mathbb{R} \), so \( z \in C_\epsilon \).

Now choose \( \epsilon_n = \frac{\epsilon}{n} \). For each \( \epsilon_n \), there is a \( z_n \in C_{\epsilon_n} \) such that \( \zeta(z_n) = 0 \). These \( z_n \) lie in a compact set, which is the closure of interior of \( B_{\epsilon_n} \) (i.e. the whole rectangle whose boundary is \( B_{\epsilon_n} \).

Since \( \epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots \) we have the inclusion ), so there is a converging subsequence. Suppose the subsequence converges to \( z \).

By continuity, \( \zeta(z) = 0 \). Since \( z \) is a limit of the subsequence, for each neighbourhood \( U \) of \( z \), there is an \( n \) with \( z_n \in U \). But as a corollary of the local power series expansion, zeros of non constant analytic functions are isolated. Zeros of \( \zeta \) are clearly not isolated near \( z \). So we get a contradiction.

Hence our assumption that such an \( \epsilon > 0 \) does not exist is incorrect.

We will use the Riemann xi function due to its simpler functional equation. By the Lemma, \( \zeta(s) \) has no zeros on the boundary of rectangle \( R_\epsilon \). So we use the Cauchy’s argument principle on \( \zeta(s) \) in the rectangle \( R_\epsilon \).

Since \( \zeta(s) \) is analytic on and inside of \( R_\epsilon \) with no zeros or poles on the boundary, \( \partial R_\epsilon \) so by Cauchy’s argument principle (see [1, p.128])

\[
N_\epsilon(T) = \frac{1}{2\pi i} \int_{\partial R_\epsilon} \frac{\xi'(s)}{\xi(s)} ds
\]
where $\partial R_\epsilon$, the boundary of rectangle $R_\epsilon$ is oriented in counterclockwise direction and $N_\epsilon(T)$ counts the zeros with multiplicities. Let us divide the contour $\partial R_\epsilon$ into the following paths

$$L_1 : \frac{1}{2} + \epsilon \rightarrow \frac{1}{2} + \epsilon + i(T + \epsilon) \rightarrow \frac{1}{2} + i(T + \epsilon)$$

$$L_2 : \frac{1}{2} + i(T + \epsilon) \rightarrow \frac{1}{2} - \epsilon + i(T + \epsilon) \rightarrow \frac{1}{2} - \epsilon$$

$$L_3 : \frac{1}{2} - \epsilon \rightarrow \frac{1}{2} + \epsilon$$

So by equation (1) we have

$$N_\epsilon(T) = \frac{1}{2\pi i} \left( \int_{L_1} \frac{\xi'(s)}{\xi(s)} ds + \int_{L_2} \frac{\xi'(s)}{\xi(s)} ds + \int_{L_3} \frac{\xi'(s)}{\xi(s)} ds \right)$$

(2)

Since $\xi(s) = \xi(1 - s)$ and by Schwarz Reflection Principle, $\xi(\sigma + it) = \xi(1 - \sigma - it) = \overline{\xi(1 - \sigma + it)}$ so we have

$$N_\epsilon(T) = \frac{1}{\pi} \Im \left( \int_{L_1} \frac{\xi'(s)}{\xi(s)} ds \right)$$

(3)

Since we have

$$\xi(s) = \frac{1}{2}s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

(4)

Taking logarithmic derivative of equation (4), we have

$$\frac{\xi'(s)}{\xi(s)} = \frac{d}{ds} \left[ \log \left( \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \right) \right] + \frac{d}{ds} \log(s(s - 1)) + \frac{d}{ds} \log(\zeta(s))$$

(5)

So from equation (3) and (5) we get

$$\frac{1}{\pi} \Im \left( \int_{L_1} \frac{\xi'(s)}{\xi(s)} ds \right) = \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \left[ \log \left( \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \right) \right] ds + \frac{d}{ds} \log(s(s - 1)) \right] + \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \log(\zeta(s)) \right] ds$$

(6)

In equation (6) write

$$\frac{1}{\pi} \Im \left( \int_{L_1} \frac{\xi'(s)}{\xi(s)} ds \right) = I_4 + I_5 + I_6$$

where

$$I_4 = \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \left[ \log \left( \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \right) \right] ds \right]$$

(7)

$$I_5 = \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \log(s(s - 1)) \right] ds$$

(8)

$$I_6 = \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \log(\zeta(s)) \right] ds$$

(9)

Now we have from equation (7)

$$I_4 = \frac{1}{\pi} \Im \left[ \int \frac{\log \pi}{2} ds + \int_{L_1} \frac{d}{ds} \log(\Gamma\left(\frac{s}{2}\right)) ds \right]$$

(10)

Now using fundamental theorem of calculus in $I_4$, since the path $L_1$ is $\frac{1}{2} + \epsilon \rightarrow \frac{1}{2} + \epsilon + i(T + \epsilon) \rightarrow \frac{1}{2} + i(T + \epsilon)$ we have

$$I_4 = \frac{1}{\pi} \left[ - \left( \frac{\log \pi}{2} \right) (T + \epsilon) + \Im \log \left( \Gamma\left(\frac{1}{4} + \frac{i(T + \epsilon)}{2}\right) \right) \right]$$

(11)

where we used that $\Gamma$ function is real at the point $\frac{1}{2} + \epsilon$. We denote by (see [1, p.119])

$$v(T) = \Im \log \left( \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) \right) - \left( \frac{\log \pi}{2} \right) T$$
then we have
\[ \Im \log \left( \Gamma \left( \frac{1}{4} + iT \right) \right) = v(T) + \left\{ \frac{\log \pi}{2} \right\} T \] (12)

So as \( \epsilon \to 0^+ \), equation (11) becomes
\[ I_4 = \frac{1}{\pi} v(T) \] (13)

Now we have (see [1, p.120]) as \( T \to \infty \)
\[ v(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2} - \frac{\pi}{8} + O \left( \frac{1}{T} \right) \] (14)

So from equation (13) and (14) we get
\[ I_4 = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2} - \frac{1}{8} + O \left( \frac{1}{T} \right) \] (15)

By fundamental theorem of calculus, integral \( I_5 \) can be written as
\[ I_5 = \frac{1}{\pi} \Im \left[ \log s(s-1) \right]_{\frac{1}{2} + i(T+\epsilon)} \] (16)

which gives
\[ I_5 = \frac{1}{\pi} \Im \left[ \log \left( -(T+\epsilon)^2 - \frac{1}{4} \right) - \log \left( \epsilon^2 - \frac{1}{4} \right) \right] \] (17)

So we get using \( e^{i\pi} = -1 \) and as \( \epsilon \to 0^+ \),
\[ I_5 = \frac{1}{\pi} \left( \Im(i\pi) - \Im(i\pi) \right) \] (18)

which gives
\[ I_5 = 0 \] (19)

Next we evaluate \( I_6 \) which can be rewritten as
\[ I_6 = \frac{1}{\pi} \Im \left[ \int_{L_1} \frac{d}{ds} \left( \log |\zeta(s)| + i \arg(\zeta(s)) \right) ds \right] \] (20)

Hence we get
\[ I_6 = \frac{1}{\pi} \int_{L_1} \left[ \frac{d}{ds} \arg(\zeta(s)) \right] ds \] (21)

If \( \Re(\zeta(s)) \) does not have any zero on \( L_1 \) then (see [1, p.133])
\[ -\frac{\pi}{2} < \pi I_6 < \frac{\pi}{2} \] (22)

If \( \Re(\zeta(s)) \) has \( k \) zeros, \( s' \) on \( L_1 \) (excluding end points) then (see [2, p.69])
\[ -(k+1)\pi < \pi I_6 < (k+1)\pi \] (23)

for when \( s \) describes one of the \( k+1 \) pieces into which \( L_1 \) is divided by the points \( s' \), we have \( \pi |I_6| < (k+1)\pi \) since \( \Re(\zeta(s)) \) does not change sign.

Next we prove that \( \Re(\zeta(s)) \) is not zero anywhere on the line \( \Re(s) = \frac{3}{2} \). We have (see [1, p.129])
\[ \left| \log \zeta \left( \frac{3}{2} + it \right) \right| \leq \log \zeta \left( \frac{3}{2} \right) \] (24)

So we have
\[ -\log \zeta \left( \frac{3}{2} \right) \leq \Re \left( \log \zeta \left( \frac{3}{2} + it \right) \right) \leq \log \zeta \left( \frac{3}{2} \right) \] (25)
So by equation (25) we get
\[ -\log \zeta \left( \frac{3}{2} \right) \leq \log \left| \zeta \left( \frac{3}{2} + it \right) \right| \leq \log \zeta \left( \frac{3}{2} \right) \]  
(26)

Now we have
\[ \Re \left| \zeta \left( \frac{3}{2} + it \right) \right| = \Re \left| e^{\log \zeta \left( \frac{3}{2} + it \right)} \right| = \Re \left( e^{\log |\zeta \left( \frac{3}{2} + it \right)|} \right) \]  
(27)

So we have
\[ \Re \left| \zeta \left( \frac{3}{2} + it \right) \right| = e^{\log |\zeta \left( \frac{3}{2} + it \right)|} \]  
(28)

Equation (28) and (26) gives
\[ \Re \left| \zeta \left( \frac{3}{2} + it \right) \right| \geq e^{-\log \zeta \left( \frac{3}{2} \right)} \]  
(29)

By Euler Maclaurin’s summation formula (see [1, p.114]),
\[ \zeta \left( \frac{3}{2} \right) < e^{\frac{R}{2}} \]  
(30)

So we get by equation (29) and (30)
\[ \Re \left| \zeta \left( \frac{3}{2} + it \right) \right| > 0.2 \]  
(31)

Next we show that the number \( k \) of zeros of \( \Re(\zeta(s)) \) is at most \( C \log T \) for all sufficiently large \( T \) where \( C > 0 \) is a constant. For this we use Jensen’s formula (see [1, p.40]). Define a function
\[ f(z) := \frac{1}{2} \left[ \zeta \left( z + \left( \frac{3}{2} + \epsilon \right) + iT \right) + \zeta \left( z + \left( \frac{3}{2} + \epsilon \right) - iT \right) \right] \]
Since \( \zeta(\overline{s}) = \overline{\zeta(s)} \), so the function \( f \) is identical with \( \Re \left( \zeta \left( z + \left( \frac{3}{2} + \epsilon \right) + iT \right) \right) \) for real \( z \). So the number \( k \) in question is the number of zeros of \( f(z) \) in \(-1 - \epsilon \leq z \leq -1 \) on the real axis. Since \( f(z) \) is analytic in the entire complex plane except for poles at \( z + \frac{3}{2} + \epsilon \pm iT = 1 \) that is at \( z = -\frac{3}{2} - \epsilon \pm iT \), Jensen’s formula (see [1, p.40]) applies whenever \( R \leq T \). So we have
\[ \log |f(0)| + \sum_{j=1}^{k} \log \left| \frac{R}{z_j} \right| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})|d\theta \]  
(32)

Consider the case \( R = \left( \frac{3}{2} + \epsilon \right) - \delta \) where \( \delta \) is chosen so that \( f \) has no zeros on the circle \( |z| = R \). Since the zeros of \( f(z) \) on the real axis are \(-1 - \epsilon \leq z_j \leq -1 ; j = 1, 2, ..., k \) and the second sum in the Jensen’s formula (equation (32)) is a sum of positive terms and
\[ \frac{R}{|z_j|} \geq \left( \frac{3}{2} + \epsilon \right) - \delta \]  
(33)

Hence we get by equation (32) and (33)
\[ \log |f(0)| + k \log \left( \frac{3}{2} + \epsilon - \delta \right) \leq \log M \]  
(34)

where \( M \) is the maximum value of \( |f(z)| \) on \( |z| = R = \frac{3}{2} + \epsilon - \delta \).

As \( \delta \to 0^+ \), we have in equation (34)
\[ k \log \left( \frac{3}{2} + \epsilon \right) \leq \log \left| \frac{M}{f(0)} \right| \]  
(35)

Choose \( 0 < \epsilon < 0.01 \), then we have
\[ k \leq \frac{\log \left| \frac{M}{f(0)} \right|}{\log(1.4)} \]  
(36)
Now by definition of \( f \)

\[
|f(0)| = \Re \left( \zeta \left( \frac{3}{2} + \epsilon + iT \right) \right)
\]  
(37)

As \( \epsilon \to 0^+ \) we have in above equation using equation (31)

\[
|f(0)| = \Re \left( \zeta \left( \frac{3}{2} + iT \right) \right) > 0.2
\]  
(38)

So by equation (36) and (38) we get

\[
k \leq \frac{\log(M) - \log(0.2)}{\log(1.4)}
\]  
(39)

Hence from equation (39) we get

\[
k \leq c_1 \log M + c_2
\]  
(40)

where \( c_1 = \frac{1}{\log(1.4)} \) and \( c_2 = \frac{\log(M)}{\log(1.4)} \) are positive constants.

Next we show that in equation (40), \( \log M \) grows no faster than a constant times \( \log T \). By definition of \( M \), we have as \( \epsilon \to 0^+ \)

\[
M = \frac{1}{2} \max_{|z|=\frac{3}{2}} \left| \zeta \left( \frac{3}{2} + \epsilon + iT \right) + \zeta \left( \frac{3}{2} + \epsilon - iT \right) \right|
\]  
(41)

So we have

\[
M \leq \max_{|z|=\frac{3}{2}} \left| \zeta \left( \frac{3}{2} + \epsilon + iT \right) \right|
\]  
(42)

Now we have (see [8, p.14]) for \( \Re(s) > 0 \)

\[
\zeta(s) = s \int_1^\infty \frac{1-x-[x]}{x^{s+1}} dx + \frac{1}{s-1}
\]  
(43)

Which gives for \( \Re(s) > 0 \)

\[
|\zeta(s)| \leq \frac{|s|}{\sigma} + \frac{1}{|s-1|}
\]  
(44)

where \( \sigma = \Re(s) \). Now we have \( z = \frac{3}{2} e^{i\theta} \) and \( s = \frac{3}{2} e^{i\theta} + \frac{3}{2} + iT \). Since \( s \in L_1 \) so we have \( \sigma \geq \frac{1}{2} \). Also we have \( |s-1| \geq T - \frac{3}{2} \) for \( T > \frac{3}{2} \) and we have

\[
M \leq 2(3 + T) + \frac{1}{T - \frac{3}{2}}
\]  
(45)

So we get from above equation

\[
M = O(T^2) \quad \text{for } T \text{ large}
\]  
(46)

So we get

\[
\log M = O(\log T)
\]  
(47)

So by equation (40) and (47)

\[
k = O(\log T)
\]  
(48)

By equation (3), (15), (22), (23), (40) and (48) we get as \( \epsilon \to 0^+ \),

\[
\lim_{\epsilon \to 0^+} N_\epsilon(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} - \frac{1}{8} + O \left( \frac{1}{T} \right) + O(\log T)
\]  
(49)

Since we have the rectangle \( R_\epsilon = \{ s = \sigma + it \in \mathbb{C} | \frac{1}{2} - \epsilon \leq \sigma \leq \frac{1}{2} + \epsilon, 0 \leq t \leq T + \epsilon \} \) where \( \epsilon > 0 \) is arbitrarily small. Take \( \epsilon = \frac{1}{n} \) where \( n \in \mathbb{N} \) and define \( R_n = \{ \sigma + it \in \mathbb{C} | \frac{1}{2} - \frac{1}{n} \leq \sigma \leq \frac{1}{2} + \frac{1}{n}, 0 \leq t \leq T \} \). We note that \( R_n \) decreases with \( n \) which means that they are sequence of decreasing sets with respect to set inclusion. So, since the sets \( R_n \) are nested for each \( n \in \mathbb{N} \) so taking the intersection of all rectangles we have

\[
\bigcap_{n=1}^\infty R_n = \{ \sigma + it \in \mathbb{C} | \sigma = \frac{1}{2}, 0 \leq t \leq T \}
\]
Hence we get
\[ \lim_{\epsilon \to 0^+} N_\epsilon(T) = N_0(T) \tag{50} \]
So we get by equation (49) and (50)
\[ N_0(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} \frac{1}{8} + O(\log T) \tag{51} \]
This proves the first claim of the Theorem.
For the next part, we have the following asymptotic for \( N(T) \) (see [1, p.134])
\[ N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) + \frac{7}{8} + O(\log T) \tag{52} \]
So from equation (51) and (52) we get
\[ N_0(T) \sim N(T) \quad \text{as} \quad T \to \infty \tag{53} \]
So we get by equation (53)
\[ \lim_{T \to \infty} \frac{N_0(T)}{N(T)} = 1 \tag{54} \]
Now since \( 0 < \frac{N_0(T)}{N(T)} \leq 1 \) and hence by equation (54) we get
\[ \liminf_{T \to \infty} \frac{N_0(T)}{N(T)} = 1 \tag{55} \]
Hence we get
\[ \kappa = 1 \tag{56} \]
This proves the next part of the Theorem. This completes the proof of the Theorem.

3 References
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4 Data Availability Statement (DAS)

1. Data openly available in a public repository that does not issue DOIs - for the references [3](Click here for [3]), [4](Click here for [4]), [6](Click here for [6]) and [7](Click here for [7])

2. Data available on request from the authors - for the references [5] and [10]

3. Books available in the library or online library - for the references [1](Click here for [1]), [2](Click here for [2]), [8](Click here for [8]) and [9](Click here for [9])