HARMONIC MORPHISMS FROM 
FOUR-DIMENSIONAL LIE GROUPS

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ABSTRACT. We consider 4-dimensional Lie groups with left-invariant Riemannian metrics. For such groups we classify left-invariant conformal foliations with minimal leaves of codimension 2. These foliations produce local complex-valued harmonic morphisms.

1. Introduction

This article develops an interplay between homogeneous spaces and extremal mappings rather along the lines of Kaluza-Klein theory, whereby the gravitational field (the metric) and electromagnetism are unified in terms of a projection from a higher dimensional space to a lower dimensional one. Although here, the dimensions are not as in classical KK theory, see [1].

We are interested in the existence of complex-valued harmonic morphisms from 4-dimensional Riemannian homogeneous spaces. S. Ishihara has in [12] shown that any such space is either symmetric or a Lie group equipped with a left-invariant metric, see also [5]. It is well-known that any 4-dimensional symmetric space carries local harmonic morphisms and even global solutions exist if the space is of non-compact type, see [9]. This means that we can focus our attention on the Lie group case.

The current authors have in [10] classified the 3-dimensional Riemannian Lie groups $G$ carrying complex-valued harmonic morphisms. The classification is based on the fact that any such local solution induces a global left-invariant conformal foliation on $G$ with minimal leaves of codimension 2.

In [8], the authors introduce a general method for producing left-invariant conformal foliations, on higher dimensional Riemannian Lie groups, with minimal leaves. This is then used to construct many new examples, in particular, in the case when the leaves are of codimension 2. This is important since such foliations produce local complex-valued harmonic morphisms.

J. Nordström considers in [11] a family of homogeneous Hadamard manifolds of any dimension greater than 2. They are all Riemannian Lie groups with a rather simple bracket structure. He proves that none of these groups carry a left-invariant conformal foliation with minimal fibres of codimension...
2. This makes it interesting to understand what algebraic and geometric conditions are necessary or sufficient for existence.

In the first part of this paper, we classify the left-invariant conformal foliations on 4-dimensional Riemannian Lie groups, with minimal leaves of codimension 2. We prove the following result.

**Theorem 1.1.** Let $G$ be a 4-dimensional Lie group equipped with a left-invariant Riemannian metric. Let $F$ be a left-invariant conformal foliation on $G$ with minimal leaves of codimension 2. Then the Lie algebra $\mathfrak{g}$ of $G$ belongs to one of the families $\mathfrak{g}_1, \ldots, \mathfrak{g}_{20}$ given below and $F$ is the corresponding foliation generated by $\mathfrak{g}$.

We show that most of the complex-valued harmonic morphisms constructed in this paper are not holomorphic with respect to any (integrable) Hermitian structure on their domains.

In the second part, we consider Riemannian Lie groups of higher dimensions. On those we construct new conformal foliations with minimal leaves of codimension 2. Our examples show, for the first time, that Theorem 1.2 does not hold if the dimension of $M$ is greater than 3.

**Theorem 1.2.** [3] Let $(M, g)$ be a 3-dimensional Riemannian manifold with a conformal foliation $F$ with minimal leaves of codimension 2. Let $\{X, Y\}$ be a local orthonormal frame for the horizontal distribution $\mathcal{H}$. Then the Ricci curvature satisfies

$$\text{Ric}(X, X) = \text{Ric}(Y, Y) \quad \text{and} \quad \text{Ric}(X, Y) = 0.$$  

This once more makes pertinent the connection with Kaluza-Klein theories, where the Ricci tensor plays a crucial role.

For the general theory of harmonic morphisms between Riemannian manifolds we refer to the excellent book [4] and the regularly updated on-line bibliography [7].

## 2. Harmonic Morphisms and Minimal Conformal Foliations

Let $M$ and $N$ be two manifolds of dimensions $m$ and $n$, respectively. A Riemannian metric $g$ on $M$ gives rise to the notion of a Laplacian on $(M, g)$ and real-valued harmonic functions $f : (M, g) \to \mathbb{R}$. This can be generalized to the concept of harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds, which are solutions to a semi-linear system of partial differential equations, see [4].

**Definition 2.1.** A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and T. Ishihara. For the definition of horizontal (weak) conformality we refer to [4].
Theorem 2.2. [6] [13] A map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

Let \((M, g)\) be a Riemannian manifold, \(\mathcal{V}\) be an involutive distribution on \(M\) and denote by \(\mathcal{H}\) its orthogonal complement distribution on \(M\). As customary, we also use \(\mathcal{V}\) and \(\mathcal{H}\) to denote the orthogonal projections onto the corresponding subbundles of \(TM\) and denote by \(\mathcal{F}\) the foliation tangent to \(\mathcal{V}\). The second fundamental form for \(\mathcal{V}\) is given by
\[
B^\mathcal{V}(U, V) = \frac{1}{2} \mathcal{H}(\nabla_U V + \nabla_V U) \quad (U, V \in \mathcal{V}),
\]
while the second fundamental form for \(\mathcal{H}\) is given by
\[
B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).
\]
The foliation \(\mathcal{F}\) tangent to \(\mathcal{V}\) is said to be conformal if there is a vector field \(V \in \mathcal{V}\) such that
\[
B^\mathcal{H} = g \otimes V,
\]
and \(\mathcal{F}\) is said to be Riemannian if \(V = 0\). Furthermore, \(\mathcal{F}\) is said to be minimal if trace \(B^\mathcal{V} = 0\) and totally geodesic if \(B^\mathcal{V} = 0\). This is equivalent to the leaves of \(\mathcal{F}\) being minimal and totally geodesic submanifolds of \(M\), respectively.

It is easy to see that the fibres of a horizontally conformal map (resp. Riemannian submersion) give rise to a conformal foliation (resp. Riemannian foliation). Conversely, the leaves of any conformal foliation (resp. Riemannian foliation) are locally the fibres of a horizontally conformal map (resp. Riemannian submersion), see [4].

The next result of Baird and Eells gives the theory of harmonic morphisms, with values in a surface, a strong geometric flavour.

Theorem 2.3. [2] Let \( \phi : (M^m, g) \to (N^2, h) \) be a horizontally conformal submersion from a Riemannian manifold to a surface. Then \( \phi \) is harmonic if and only if \( \phi \) has minimal fibres.

3. 4-DIMENSIONAL LIE GROUPS

Let \( G \) be a 4-dimensional Lie group equipped with a left-invariant Riemannian metric. Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \{X, Y, Z, W\} \) be an orthonormal basis for \( \mathfrak{g} \). Let \( Z, W \in \mathfrak{g} \) generate a 2-dimensional left-invariant and integrable distribution \( \mathcal{V} \) on \( G \) which is conformal and with minimal leaves. We denote by \( \mathcal{H} \) the horizontal distribution, orthogonal to \( \mathcal{V} \), generated by \( X, Y \in \mathfrak{g} \). Then it is easily seen that the Lie bracket relations for \( \mathfrak{g} \) are of the form
\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + z_2 Z + w_2 W,
\end{align*}
\]
\[
[W, X] = aX + bY + z_3 Z - z_1 W,
\]
\[
[W, Y] = -bX + aY + z_4 Z - z_2 W,
\]
\[
[Y, X] = rX + \theta_1 Z + \theta_2 W
\]

with real structure constants. For later reference we state the following easy result describing the geometry of the situation.

**Proposition 3.1.** Let \( G \) be a 4-dimensional Lie group and \( \{X, Y, Z, W\} \) be an orthonormal basis for its Lie algebra as above. Then

(i) \( \mathcal{F} \) is totally geodesic if and only if \( z_1 = z_2 = z_3 + w_1 = z_4 + w_2 = 0 \),

(ii) \( \mathcal{F} \) is Riemannian if and only if \( \alpha = a = 0 \), and

(iii) \( \mathcal{H} \) is integrable if and only if \( \theta_1 = \theta_2 = 0 \).

On the Riemannian Lie group \((G, g)\) there exist, up to sign, exactly two invariant almost Hermitian structure \( J_1 \) and \( J_2 \) which are adapted to the orthogonal decomposition \( g = V \oplus H \) of the Lie algebra \( g \). They are determined by

\[
J_1 X = Y, \quad J_1 Y = -X, \quad J_1 Z = W, \quad J_1 W = -Z,
\]
\[
J_2 X = Y, \quad J_2 Y = -X, \quad J_2 W = Z, \quad J_2 Z = -W.
\]

An elementary calculation involving the Nijenhuis tensor shows that \( J_1 \) is integrable if and only if

\[
2z_1 - z_4 - w_2 = 2z_2 + z_3 + w_1 = 0
\]

and the same applies to \( J_2 \) if and only if

\[
2z_1 + z_4 + w_2 = 2z_2 - z_3 - w_1 = 0.
\]

This means that most of the complex-valued harmonic morphisms constructed in this paper are *not* holomorphic with respect to any Hermitian structure on the corresponding Lie groups.

**Remark 3.2.** This is interesting in the light of a result of J. C. Wood, see [14]. He shows that a submersive harmonic morphism from an orientable 4-dimensional Einstein manifold \( M^4 \) to a Riemann surface, or a conformal foliation of \( M^4 \) by minimal surfaces, determines an (integrable) Hermitian structure with respect to which it is holomorphic.

For any \( V \in \mathcal{V} \), the adjoint action of \( V \) on \( \mathcal{H} \) is conformal i.e.

\[
\langle \text{ad}_V X, Y \rangle + \langle X, \text{ad}_V Y \rangle = \rho \cdot \langle X, Y \rangle \quad (X, Y \in \mathcal{H})
\]

or put differently

\[
\mathcal{H} \text{ ad}_V \bigg|_{\mathcal{H}} \in \mathbb{R} \cdot \text{Id}_{\mathcal{H}} + \mathfrak{so}(\mathcal{H}) = \mathfrak{co}(\mathcal{H}).
\]

Note that this is indeed a Lie algebra representation of \( \mathcal{V} \), so that

\[
\mathcal{H} \text{ ad}_{[Z,W]} \bigg|_{\mathcal{H}} = [\mathcal{H} \text{ ad}_Z \bigg|_{\mathcal{H}}, \mathcal{H} \text{ ad}_W \bigg|_{\mathcal{H}}] \quad (Z, W \in \mathcal{V}),
\]

where the bracket on the right-hand side is just the usual bracket on the space of endomorphisms on \( \mathcal{H} \). This follows from the Jacobi identity, and from the fact that \( \mathcal{V} \) is integrable. Now, since \( \mathfrak{co}(\mathcal{H}) \) is abelian, we see
from this formula, that the adjoint action of \([V, V]\) has no \(\mathcal{H}\)-component. This means that our analysis branches into two different cases depending on whether \(V\) is abelian or not.

4. The Case of Non-Abelian Vertical Distribution \((\lambda \neq 0)\)

We now assume that the vertical distribution \(V\) generated by the left-invariant vector fields \(Z, W \in \mathfrak{g}\) is not abelian. Then there exists an orthonormal basis \(\{Z, W\}\) for \(V\) such that

\[
[W, Z] = \lambda W
\]

for some real constant \(\lambda \neq 0\) and \(\text{ad}_W\) has no \(\mathcal{H}\)-component. These conditions give the following bracket relations

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + z_2 Z + w_2 W, \\
[W, X] &= z_3 Z - z_1 W, \\
[W, Y] &= z_4 Z - z_2 W, \\
[Y, X] &= r X + \theta_1 Z + \theta_2 W.
\end{align*}
\]

An elementary calculation shows that the two Jacobi equations

\[
\begin{align*}
([W, Z], X) + ([X, W], Z) + ([Z, X], W) &= 0, \\
([W, Z], Y) + ([Y, W], Z) + ([Z, Y], W) &= 0
\end{align*}
\]

are equivalent to the following relations for the real structure constants

\[
\begin{pmatrix}
\beta & \lambda - \alpha \\
\lambda - \alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

4.1. Case (A) - \((\lambda \neq 0 \text{ and } (\lambda - \alpha)^2 + \beta^2 \neq 0)\). Applying equation (4.3)

we see that \(z = 0\) so the Lie brackets satisfy

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + w_2 W, \\
[Y, X] &= r X + \theta_1 Z + \theta_2 W.
\end{align*}
\]

In this situation it is easily seen that the Jacobi equations

\[
\begin{align*}
([X, Y], Z) + ([Z, X], Y) + ([Y, Z], X) &= 0, \\
([X, Y], W) + ([W, X], Y) + ([Y, W], X) &= 0
\end{align*}
\]

are equivalent to

\[
\theta_1 = r \alpha = r \beta = 0 \quad \text{and} \quad \theta_2 (\lambda + 2 \alpha) = rw_1.
\]
Example 4.1 \((g_1(\lambda, r, w_1, w_2))\). If \(r \neq 0\) then clearly \(\alpha = \beta = 0\) and \(rw_1 = \lambda \theta_2\). This gives a 4-dimensional family of solutions satisfying the following Lie bracket relations

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= w_1 W, \\
[Z, Y] &= w_2 W, \\
\lambda[Y, X] &= \lambda r X + rw_1 W.
\end{align*}
\]

On the other hand, if \(r = 0\) then clearly \(\theta_1 = \theta_2(\lambda + 2\alpha) = 0\) providing us with the following two examples.

Example 4.2 \((g_2(\lambda, \alpha, \beta, w_1, w_2))\). For \(r = \theta_1 = \theta_2 = 0\) we have the family \(g = g_2(\lambda, \alpha, \beta, w_1, w_2)\) given by

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + w_2 W.
\end{align*}
\]

Example 4.3 \((g_3(\alpha, \beta, w_1, w_2, \theta_2))\). If \(r = \theta_1 = 0\) and \(\theta_2 \neq 0\) then \(\lambda = -2\alpha\) provides us with the family \(g = g_3(\alpha, \beta, w_1, w_2, \theta_2)\) of solutions satisfying

\[
\begin{align*}
[W, Z] &= -2\alpha W, \\
[Z, X] &= \alpha X + \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + w_2 W, \\
[Y, X] &= \theta_2 W.
\end{align*}
\]

4.2. Case (B) - \((\lambda \neq 0 \text{ and } (\lambda - \alpha)^2 + \beta^2 = 0)\). Under the assumptions that \(\alpha = \lambda\) and \(\beta = 0\) we have the following bracket relations

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \lambda X + z_1 Z + w_1 W, \\
[Z, Y] &= \lambda Y + z_2 Z + w_2 W, \\
[W, X] &= z_3 Z - z_1 W, \\
[W, Y] &= z_4 Z - z_2 W, \\
[Y, X] &= r X + \theta_1 Z + \theta_2 W.
\end{align*}
\]

The Jacobi equations (4.4) and (4.5) are easily seen to be equivalent to

\[
z_1 = z_3 = z_4 = \theta_1 = 0, \quad z_2 = -r \quad \text{and} \quad \lambda \theta_2 = -z_2 w_1.
\]

Example 4.4 \((g_4(\lambda, z_2, w_1, w_2))\). In this case we have the family of solutions given by

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \lambda X + w_1 W, \\
[Z, Y] &= \lambda Y + z_2 Z + w_2 W, \\
[W, Y] &= -z_2 W,
\end{align*}
\]
\[ \lambda[Y, X] = -z_2\lambda X - z_2w_1W. \]

5. **The case of abelian vertical distribution (\( \lambda = 0 \))**

We now assume that the vertical distribution \( V \) generated by the left-invariant vector fields \( Z, W \in g \) is abelian. Under this natural algebraic condition, we have the general Lie bracket relations:

\[
\begin{align*}
[Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + z_2 Z + w_2 W, \\
[W, X] &= \alpha X + b Y + z_3 Z - z_1 W, \\
[W, Y] &= -b X + a Y + z_4 Z - z_2 W, \\
[Y, X] &= r X + \theta_1 Z + \theta_2 W.
\end{align*}
\]

An elementary calculation shows that the Jacobi equations (4.1), (4.2), (4.4) and (4.5) are equivalent to the following homogeneous systems of quadratic relations for the real structure constants:

\[
\begin{align*}
-2\begin{pmatrix}
  z_1 & w_1 \\
  z_2 & w_2 \\
  z_3 & -z_1 \\
  z_4 & -z_2
\end{pmatrix}
\begin{pmatrix}
  \alpha & \beta \\
  a & b
\end{pmatrix}
= r
\begin{pmatrix}
  \beta & \alpha \\
  \alpha & -\beta \\
  b & a \\
  a & -b
\end{pmatrix}
\end{align*}
\]

and

\[
\begin{pmatrix}
  \alpha \theta_1 \\
  -2 a \theta_2 \\
  z_3 w_2 - z_4 w_1 \\
  2 z_3 z_2 - 2 z_1 z_4 \\
  2 z_1 w_2 - 2 z_2 w_1
\end{pmatrix}
= \begin{pmatrix}
  2 a \theta_2 + rz_1 \\
  2 a \theta_1 - rz_3 \\
  2 a \theta_2 - rw_1
\end{pmatrix}
\]

5.1. **Case (C) - (\( \lambda = 0, r \neq 0 \) and \( (a\beta - ab) \neq 0 \)).** According to equation (5.1) we can now solve \((z, w)\) in terms of \((\alpha, \beta, a, b)\) and get

\[
\begin{pmatrix}
  z_1 & w_1 \\
  z_2 & w_2 \\
  z_3 & -z_1 \\
  z_4 & -z_2
\end{pmatrix}
= \frac{r}{2(a\beta - ab)}
\begin{pmatrix}
  \beta - \alpha a \\
  \alpha a + \beta b \\
  b^2 - a^2 \\
  2ab
\end{pmatrix}
\]

By substituting these expressions for \((z, w)\) into equation (5.2) we then obtain the useful relations:

\[
2(a\beta - ab)
\begin{pmatrix}
  a \theta_2 \\
  a \theta_1 \\
  \alpha \theta_2
\end{pmatrix}
= r^2
\begin{pmatrix}
  a \alpha \\
  -a^2 \\
  \alpha^2
\end{pmatrix}
\]

and \( \alpha \theta_1 = -a \theta_2 \).

By solving those we get

\[
\theta_1 = \frac{-ar^2}{2(a\beta - ab)} \quad \text{and} \quad \theta_2 = \frac{ar^2}{2(a\beta - ab)}.
\]
Example 5.1 \((g_5(\alpha, a, \beta, b, r))\). In this case the Lie bracket relations take the following form

\[
\begin{align*}
[Z, X] &= \alpha X + \beta Y + \frac{r(\beta b - \alpha a)}{2(a\beta - ab)} Z + \frac{r(a^2 - \beta^2)}{2(a\beta - ab)} W, \\
[Z, Y] &= -\beta X + \alpha Y + \frac{r(\alpha b + \beta a)}{2(a\beta - ab)} Z - \frac{r\alpha \beta}{(a\beta - ab)} W, \\
[W, X] &= aX + bY + \frac{r(b^2 - a^2)}{2(a\beta - ab)} Z + \frac{r(\alpha a - \beta b)}{2(a\beta - ab)} W, \\
[W, Y] &= -bX + aY + \frac{rab}{(a\beta - ab)} Z - \frac{r(\alpha b + \beta a)}{2(a\beta - ab)} W, \\
[Y, X] &= rX - \frac{ar^2}{2(a\beta - ab)} Z + \frac{\alpha r^2}{2(a\beta - ab)} W.
\end{align*}
\]

5.2. Case (D) - \((\lambda = 0, r \neq 0\) and \((a\beta - ab) = 0\)). Picking the appropriate determinants from equation (5.1) we see that

\[
\begin{align*}
r^2(a^2 + \beta^2) &= 4(z_2 w_1 - z_1 w_2)(ab - a\beta) = 0, \\
r^2(a^2 + b^2) &= 4(z_2 z_3 - z_1 z_4)(ab - a\beta) = 0.
\end{align*}
\]

This means that in this case we have \(\alpha = a = \beta = b = 0\) so the corresponding foliations are all Riemannian. The equation (5.1) is automatically satisfied and the system (5.2) takes the form

\[
\begin{pmatrix}
z_3 w_2 - z_4 w_1 \\
2z_2 z_3 - 2z_1 z_4 \\
2z_2 w_1 - 2z_2 w_1
\end{pmatrix} = r
\begin{pmatrix}
z_1 \\
z_3 \\
-w_1
\end{pmatrix}.
\]

Applying basic algebraic manipulations on (5.5) we obtain

\[
\begin{align*}
z_1^2 + w_1 z_3 &= 0 \\
2z_1 z_2 + z_4 w_1 + z_3 w_2 &= 0.
\end{align*}
\]

Example 5.2 \((g_6(z_1, z_2, z_3, r, \theta_1, \theta_2))\). When \(z_1^2 = -w_1 z_3 \neq 0\) it immediately follows that

\[
z_4 = \frac{z_3 (r + 2z_2)}{2z_1}, \quad w_1 = -\frac{z_1^2}{z_3}, \quad w_2 = \frac{z_1 (r - 2z_2)}{2z_3}.
\]

This provides the following family of solutions

\[
\begin{align*}
[Z, X] &= z_1 Z - \frac{z_1^2}{z_3} W, \\
[Z, Y] &= z_2 Z + \frac{z_1 (r - 2z_2)}{2z_3} W, \\
[W, X] &= z_3 Z - z_1 W, \\
[W, Y] &= \frac{z_3 (r + 2z_2)}{2z_1} Z - z_2 W, \\
[Y, X] &= rX + \theta_1 Z + \theta_2 W.
\end{align*}
\]
When $z_1 = 0$ the system (5.5) is equivalent to
\begin{equation}
(5.7) \quad z_3w_2 = z_4w_1, \quad z_3(2z_2 + r) = 0, \quad w_1(2z_2 - r) = 0.
\end{equation}

**Example 5.3** ($g_7(z_2, w_1, w_2, \theta_1, \theta_2)$). When $z_1 = 0$ and $w_1 \neq 0$ it follows from (5.7) that $z_3 = z_4 = 0$ and $r = 2z_2$ so we have the following solutions
\begin{align*}
[Z, X] &= w_1W, \\
[Z, Y] &= z_2Z + w_2W, \\
[W, Y] &= -z_2W, \\
[Y, X] &= 2z_2X + \theta_1Z + \theta_2W.
\end{align*}

When $z_1 = w_1 = 0$ it follows that $z_3w_2 = 0$ and $z_3(2z_2 + r) = 0$. The two possible cases are given in Examples 5.4 and 5.5.

**Example 5.4** ($g_8(z_2, z_4, w_2, r, \theta_1, \theta_2)$). When $z_1 = z_3 = w_1 = 0$ we have the solutions
\begin{align*}
[Z, Y] &= z_2Z + w_2W, \\
[W, Y] &= z_4Z - z_2W, \\
[Y, X] &= rX + \theta_1Z + \theta_2W.
\end{align*}

**Example 5.5** ($g_9(z_2, z_3, z_4, \theta_1, \theta_2)$). The conditions $z_1 = w_1 = w_2 = 0$ and $r = -2z_2$ produce the following solutions
\begin{align*}
[Z, Y] &= z_2Z, \\
[W, X] &= z_3Z, \\
[W, Y] &= z_4Z - z_2W, \\
[Y, X] &= -2z_2X + \theta_1Z + \theta_2W.
\end{align*}

5.3. **Case (E)** - ($\lambda = 0$, $r = 0$ and $ab - a\beta \neq 0$). In this situation the equation (5.1) takes the following simple form
\begin{equation}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
-\alpha \beta \\
\alpha \beta
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}.
\end{equation}

As an immediately consequence of $\alpha \beta - a\beta \neq 0$ we see that $z = w = 0$. This means that the system (5.2) is equivalent to
\[
\alpha \theta_1 = a\theta_2 = a\theta_1 = a\theta_2 = 0.
\]
Applying $ab \neq a\beta$ we see that $\theta_1 = \theta_2 = 0$ so we have the following.

**Example 5.6** ($g_{10}(\alpha, a, \beta, b)$). If $ab - a\beta \neq 0$ then the solutions are given by
\begin{align*}
[Z, X] &= \alpha X + \beta Y, \\
[Z, Y] &= -\beta X + aY, \\
[W, X] &= aX + bY, \\
[W, Y] &= -bX + aY.
\end{align*}
5.4. Case (F) - \((\lambda = 0, \ r = 0 \text{ and } ab - a\beta = 0)\). In this section we assume that both \(\lambda\) and \(r\) are zero. This means that the special choice of bases \(\{X, Y\}\) and \(\{Z, Y\}\) for \(H\) and \(V\), we made above, is irrelevant and that we get the following symmetric system of bracket relations

\[
\begin{align*}
[Z, X] &= \alpha X + \beta Y + z_1Z + w_1W, \\
[Z, Y] &= -\beta X + \alpha Y + z_2Z + w_2W, \\
[W, X] &= aX + bY + z_3Z - z_1W, \\
[W, Y] &= bX + aY + z_4Z - z_2W, \\
[Y, X] &= \theta_1Z + \theta_2W.
\end{align*}
\]

The Jacobi identities now take the following form

\[
(5.13) \quad \begin{pmatrix}
\alpha z_1 & \beta z_1 \\
\alpha z_2 & \beta z_2 \\
\alpha z_3 & \beta z_3 \\
\alpha z_4 & \beta z_4
\end{pmatrix}
= \begin{pmatrix}
-aw_1 & -bw_1 \\
-aw_2 & -bw_2 \\
a z_1 & b z_1 \\
a z_2 & b z_2
\end{pmatrix}

\begin{pmatrix}
-z_3w_2 - z_4w_1 \\
z_2z_3 - z_1z_4 \\
z_1w_2 - z_2w_1
\end{pmatrix}
= \begin{pmatrix}
\alpha \theta_1 \\
2a \theta_2 \\
a \theta_1 \\
\alpha \theta_2
\end{pmatrix}.
\]

Our analysis divides into disjoint cases parametrized by \(\Lambda = (\alpha, a, \beta, b)\). The variables are assumed to be zero if and only if they are marked by 0. For example, if \(\Lambda = (0, a, \beta, 0)\) then the two variable \(\alpha\) and \(b\) are assumed to be zero and \(a\) and \(\beta\) to be non-zero.

Let us first consider the case when \(\Lambda = (0, 0, 0, 0)\). Then the Jacobi equations are equivalent to

\[
\begin{pmatrix}
z_3w_2 - z_4w_1 \\
z_2z_3 - z_1z_4 \\
z_1w_2 - z_2w_1
\end{pmatrix}
= \begin{pmatrix}0 \\
0 \\
0
\end{pmatrix}
\]

and all the possible cases are covered by Examples 5.7-5.10.

**Example 5.7** \((g_{11}(z_1, z_2, z_3, w_1, \theta_1, \theta_2))\). If \(\Lambda = (0, 0, 0, 0)\) and \(z_1 \neq 0\) then we have the following solutions

\[
\begin{align*}
[Z, X] &= z_1Z + w_1W, \\
[Z, Y] &= z_2Z + \frac{z_2w_1}{z_1}W, \\
[W, X] &= z_3Z - z_1W, \\
[W, Y] &= \frac{z_2z_3}{z_1}Z - z_2W, \\
[Y, X] &= \theta_1Z + \theta_2W.
\end{align*}
\]

If \(z_1 = 0\) then

\[
z_3w_2 = z_4w_1, \quad z_2z_3 = 0, \quad z_2w_1 = 0.
\]

**Example 5.8** \((g_{12}(z_3, w_1, w_2, \theta_1, \theta_2))\). If \(\Lambda = (0, 0, 0, 0)\), \(z_1 = 0\) and \(w_1 \neq 0\) then

\[
z_2 = 0 \quad \text{and} \quad z_4 = \frac{z_3w_2}{w_1}
\]
so possible solutions are given by

\[
\begin{align*}
[Z, X] &= w_1 W, \\
[Z, Y] &= w_2 W, \\
[W, X] &= z_3 Z, \\
[W, Y] &= \frac{z_3 w_2}{w_1} Z, \\
[Y, X] &= \theta_1 Z + \theta_2 W.
\end{align*}
\]

**Example 5.9** \((g_{13}(z_3, z_4, \theta_1, \theta_2))\). If \(\Lambda = (0, 0, 0, 0)\), \(z_1 = w_1 = 0\) and \(z_3 \neq 0\) then \(z_2 = w_2 = 0\) producing the solutions

\[
\begin{align*}
[W, X] &= z_3 Z, \\
[W, Y] &= z_4 Z, \\
[Y, X] &= \theta_1 Z + \theta_2 W.
\end{align*}
\]

**Example 5.10** \((g_{14}(z_2, z_4, w_2, \theta_1, \theta_2))\). The conditions \(\Lambda = (0, 0, 0, 0)\) and \(z_1 = z_3 = w_1 = 0\) provide the solutions

\[
\begin{align*}
[Z, Y] &= z_2 Z + w_2 W, \\
[W, Y] &= z_4 Z - z_2 W, \\
[Y, X] &= \theta_1 Z + \theta_2 W.
\end{align*}
\]

The rest of this section is devoted to the situation \(\Lambda \neq (0, 0, 0, 0)\). Up to obvious similarities, all the possible cases are listed in Examples 5.11-5.16.

**Example 5.11** \((g_{15}(\alpha, w_1, w_2))\). For \(\Lambda = (\alpha, 0, 0, 0)\) the Jacobi identities \((5.13)\) give

\[
z_1 = z_2 = z_3 = z_4 = \theta_1 = \theta_2 = 0.
\]

This yields the 3-dimensional family of solutions

\[
\begin{align*}
[Z, X] &= \alpha X + w_1 W, \\
[Z, Y] &= \alpha Y + w_2 W.
\end{align*}
\]

It follows from the symmetry of \((5.13)\) that \(\Lambda = (0, a, 0, 0)\) gives similar solutions.

**Example 5.12** \((g_{16}(\beta, w_1, w_2, \theta_1, \theta_2))\). If \(\Lambda = (0, 0, \beta, 0)\) then the Jacobi identities are equivalent to

\[
z_1 = z_2 = z_3 = z_4 = 0.
\]

This provides us with the 5-dimensional family of solutions satisfying

\[
\begin{align*}
[Z, X] &= \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + w_2 W, \\
[Y, X] &= \theta_1 Z + \theta_2 W.
\end{align*}
\]

The case \(\Lambda = (0, 0, 0, b)\) is similar.
Example 5.13 \((g_{17}(\alpha, a, w_1, w_2))\). In the case when \(\Lambda = (\alpha, a, 0, 0)\) the Jacobi equations give

\[
\begin{align*}
    z_1 &= -\frac{aw_1}{\alpha}, \quad z_2 = -\frac{aw_2}{\alpha}, \quad z_3 = -\frac{a^2w_1}{\alpha^2}, \quad z_4 = -\frac{a^2w_2}{\alpha^2}, \quad \theta_1 = 0, \quad \theta_2 = 0
\end{align*}
\]

so we obtain the 4-dimensional family of solutions

\[
\begin{align*}
    [Z, X] &= \alpha X - \frac{aw_1}{\alpha} Z + w_1 W, \\
    [Z, Y] &= \alpha Y - \frac{aw_2}{\alpha} Z + w_2 W, \\
    [W, X] &= aX - \frac{a^2w_1}{\alpha^2} Z + \frac{aw_1}{\alpha} W, \\
    [W, Y] &= aY - \frac{a^2w_2}{\alpha^2} Z + \frac{aw_2}{\alpha} W.
\end{align*}
\]

Example 5.14 \((g_{18}(\beta, b, z_3, z_4, \theta_1, \theta_2))\). In the case when \(\Lambda = (0, 0, \beta, b)\) the Jacobi equations give

\[
\begin{align*}
    z_1 &= \beta z_3 b, \quad z_2 = \beta z_4 b, \quad w_1 = -\frac{\beta^2z_3}{b^2}, \quad w_2 = -\frac{\beta^2z_4}{b^2}
\end{align*}
\]

so we have the following family of solutions

\[
\begin{align*}
    [Z, X] &= \beta Y + \frac{\beta z_3}{b} Z - \frac{\beta^2z_3}{b^2} W, \\
    [Z, Y] &= -\beta X + \frac{\beta z_4}{b} Z - \frac{\beta^2z_4}{b^2} W, \\
    [W, X] &= bY + z_3 Z - \frac{\beta z_3}{b} W, \\
    [W, Y] &= -bX + z_4 Z - \frac{\beta z_4}{b} W, \\
    [Y, X] &= \theta_1 Z + \theta_2 W.
\end{align*}
\]

Example 5.15 \((g_{19}(\alpha, \beta, w_1, w_2))\). When \(\Lambda = (\alpha, 0, \beta, 0)\) we get from the Jacobi equations that

\[
\begin{align*}
    z_1 = z_2 = z_3 = z_4 = \theta_1 = \theta_2 = 0.
\end{align*}
\]

This produces the solutions

\[
\begin{align*}
    [Z, X] &= \alpha X + \beta Y + w_1 W, \\
    [Z, Y] &= -\beta X + \alpha Y + w_2 W.
\end{align*}
\]

The case \(\Lambda = (0, a, 0, 0)\) is similar.

Example 5.16 \((g_{20}(\alpha, \alpha, a, \beta, \alpha, \beta, w_1, w_2))\). For \(\Lambda = (\alpha, a, \beta, b)\) the Jacobi relation give

\[
\begin{align*}
    z_1 &= -\frac{aw_1}{\alpha}, \quad z_2 = -\frac{aw_2}{\alpha}, \quad z_3 = -\frac{a^2w_1}{\alpha^2}, \quad z_4 = -\frac{a^2w_2}{\alpha^2}, \quad b = \frac{\beta a}{\alpha}, \quad \theta_1 = 0, \quad \theta_2 = 0.
\end{align*}
\]
In this case we have solutions of the form
\[
[Z, X] = \alpha X + \beta Y - \frac{aw}{w_1} Z + w_1 W,
\]
\[
[Z, Y] = -\beta X + \alpha Y - \frac{aw_2}{w_1} Z + aw_1 W,
\]
\[
[W, X] = \alpha X + \beta a Y - \frac{aw_1}{w^2} Z + \frac{a}{w_1} W,
\]
\[
[W, Y] = -\beta a X + aY - \frac{aw_2}{w^2} Z + \frac{a}{w_1} W.
\]

6. The Ricci Operator

In this section we produce Riemannian Lie groups carrying conformal foliations with minimal leaves. We start with an infinite series of Riemannian Lie groups which are all nilpotent.

**Example 6.1.** For a positive integer \(n \in \mathbb{Z}^+\) let \(\text{Nil}^{n+2}\) be the simply connected nilpotent Lie group with Lie algebra \(\mathfrak{n}_{n+2} = \mathbb{R} \ltimes \mathbb{R}^{n+1}\) generated by the left-invariant vector fields \(W, X_1, \ldots, X_{n+1}\) satisfying
\[
[W, X_k] = X_{k+1}, \quad k = 1, 2, \ldots, n.
\]

Equip \(\text{Nil}^{n+2}\) with the left-invariant Riemannian metric such that \(\{W, X_1, \ldots, X_{n+1}\}\) is an orthonormal basis for the Lie algebra \(\mathfrak{n}_{n+2}\). It is easily shown that for each \(k = 1, 2, \ldots, n\) the orthogonal decomposition
\[
\mathcal{H}_k \oplus \mathcal{V}_k = <W, X_1, \ldots, X_k> \oplus <X_{k+1}, \ldots, X_{n+1}>
\]
gives an example of a minimal Riemannian foliation, with non-integrable horizontal distribution and leaves which are not totally geodesic.

A standard calculation of the Ricci operator \(\text{Ric} : \mathfrak{n}_{n+2} \to \mathfrak{n}_{n+2}\) of \(\text{Nil}^{n+2}\) shows that
\[
\text{Ric}(X_2) = \cdots = \text{Ric}(X_n) = 0,
\]
\[
\text{Ric}(X_1) = -\frac{1}{2} X_1, \quad \text{Ric}(X_{n+1}) = \frac{1}{2} X_{n+1}, \quad \text{Ric}(W) = -\frac{n}{2} W.
\]

This implies that for the orthogonal decomposition
\[
\mathcal{H}_1 \oplus \mathcal{V}_1 = <W, X_1> \oplus <X_2, \ldots, X_{n+1}>
\]
the Ricci curvature tensor satisfies
\[
\text{Ric}(X_1, X_1) - \text{Ric}(W, W) = \frac{n-1}{2}.
\]

We complete this section with an infinite series of solvable Riemannian Lie groups with conformal minimal foliations.
Example 6.2. Let $Sol^{n+1}$ be the simply connected Lie group with Lie algebra

$$\mathfrak{s}_{n+1} = \mathbb{R} \ltimes \mathbb{R}^n$$

generated by the left-invariant vector fields $W, X_1, \ldots, X_n$ satisfying

$$[W, X_k] = \alpha_k X_k, \quad k = 1, 2, \ldots, n, \quad \alpha_k \in \mathbb{R}.$$ 

Then for each $k = 1, 2, \ldots, n - 2$ the orthogonal decomposition

$$\mathcal{H}_k \oplus \mathcal{V}_k = < W, X_1, \ldots, X_k > \oplus < X_{k+1}, \ldots, X_n >$$

gives a solution if and only if

$$\alpha_{k+1} + \cdots + \alpha_n = 0.$$ 

A standard calculation of the Ricci operator $\text{Ric} : \mathfrak{s}_{n+1} \to \mathfrak{s}_{n+1}$ of $Sol^{n+1}$ shows that

$$\text{Ric}(X_k) = -\alpha_k (\alpha_1 + \cdots + \alpha_n) X_k, \quad \text{Ric}(W) = -|\alpha|^2 W.$$ 

If $k = 1$ and $n \geq 3$ we can easily find $\alpha_1, \ldots, \alpha_n$ such that $\alpha_2 + \cdots + \alpha_n = 0$ but $\alpha_2^2 + \cdots + \alpha_n^2 \neq 0$. Again this shows that Theorem 1.2 does not hold in any dimension greater than 3.

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