Topological semimetal phase with exceptional points in 1D non-Hermitian systems

Kazuki Yokomizo\textsuperscript{1} and Shuichi Murakami\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8551, Japan
\textsuperscript{2}TIES, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8551, Japan

To describe eigenstates in non-Hermitian crystalline systems, the non-Bloch band theory has been attracting much attention in many fields of physics in the past decades. Many experimental studies have realized various physical systems with non-Hermitian effects. Among these experimental studies, appearance of exceptional points and rings where some energy eigenvalues become degenerate and the corresponding eigenstates coalesce, and intriguing phenomena have been observed. At such non-Hermitian degeneracy, since the Hamiltonian is non-diagonalizable, these degeneracies are unique to non-Hermitian systems. As is motivated by these experimental studies, existence of exceptional points, rings, and surfaces, and phenomena induced by them have been theoretically predicted in various physical systems, and under some symmetries, they are classified in terms of topology.

Recent theoretical studies have been focusing on topological systems in solid state physics. The bulk-edge correspondence has been notably under debate since it seems to be violated in contrast to Hermitian systems. The main issue of the bulk-edge correspondence in non-Hermitian systems is that there is a difference between the energy spectrum in a periodic chain and that in an open chain. This difference is caused by the non-Hermitian skin effect. In Refs. and , it was shown that while the Bloch wave number takes a real value in a periodic chain, it becomes complex in an open chain, and that the value of is confined on a loop on the complex plane so that continuum bands are reproduced in a large open chain. Then the loop of is called the generalized Brillouin zone (GBZ) , which is a generalization of the Brillouin zone in Hermitian systems. We note that is deformed as the system parameters change, and that it can have cusps. As a result, one can establish bulk-edge correspondence between the topological invariant defined in terms of and existence of the topological edge states in some cases.

In this Letter, we show that the non-Hermitian modification of the GBZ gives qualitative changes to physics of topological semimetals (TSMs). Because of this modification, the TSM phase with exceptional points is stable in one-dimensional (1D) non-Hermitian systems. The phase is topologically protected by both sublattice symmetry (SLS) and time-reversal symmetry (TRS), and therefore, it disappears if either the SLS or the TRS is broken. As system parameters change continuously, is deformed so that gapless points always lie on , and the system remains in the TSM phase. This TSM phase can be regarded as an intermediate phase between a normal insulator (NI) phase and a topological insulator (TI) phase characterized by a topological invariant. Then the creation and annihilation of the exceptional points can be related to the change of this topological invariant. We also find that the continuum bands are divided into three regions, where the energies become real, pure imaginary, and complex, depending on the symmetry of the eigenstates.

For illustration of the stabilization of the TSM phase, in Fig. we show an evolution of our model, whose details are explained later, upon a change of a system parameter . In Fig. (c)-(h), a gap in the system closes if the GBZ goes through one of gap-closing points represented by the red or blue dots and squares. In particular, one can see from Figs. (e) and (f) that as changes, is deformed so that it keeps the gap closed. We also find that the phase transitions between the NI and TI phases are qualitatively different from Hermitian systems. For example, in Fig. (b), at the phase transition, the exceptional points come to the cusps, and in total, the gap closes at three points on . These novel behaviors never occur when the Bloch wave number takes real values. Since in such a case, the Brillouin zone becomes a unit circle regardless of the values of the system parameters, the Brillouin zone cannot hold the gap-closing point (Figs. (c) and (d)), and the gap closes only at isolated points along the -axis (Figs. (b) and (e)).

In the following, we study a 1D non-Hermitian tight-binding system with both the SLS and the TRS. For a real-space Hamiltonian , these symmetries are defined as , where and are unitary matrices. Here, due to the SLS, one can write...
the Bloch Hamiltonian $\mathcal{H}(\beta)$ with $2N$ bands as

$$\mathcal{H}(\beta) = \begin{pmatrix} 0 & R_+ (\beta) \\ R_- (\beta) & 0 \end{pmatrix},$$

(1)

where $R_{\pm} (\beta)$ are $N \times N$ matrices. Then the eigenvalue equation $\det [\mathcal{H}(\beta) - E] = 0$ yields the bands symmetric with respect to $E = 0$.

Next we mention the main point of the non-Bloch band theory in Ref. [102]. In non-Hermitian systems, the Bloch wave number $k$ becomes complex so that it describes continuum bands in a long open chain. The set of $k$ forms the GBZ $C_\beta$. We note that the eigenvalue equation $\det [\mathcal{H}(\beta) - E] = 0$ is an algebraic equation for $\beta$ with an even degree $2M$ in general, and we number the $2M$ solutions of $\det [\mathcal{H}(\beta) - E] = 0$ so as to satisfy $|\beta_1| \leq \cdots \leq |\beta_{2M}|$. Then the condition for continuum bands is given by

$$|\beta_M| = |\beta_{M+1}|,$$

(2)

and the trajectories of $\beta_M$ and $\beta_{M+1}$ give $C_\beta$.

Now we describe the reason why the TSM phase is stable under the SLS and TRS. From the Bloch Hamiltonian $[\mathcal{H}(\beta)]$, a condition for a gap closing at $E = 0$ is decomposed into two equations $\det R_+ (\beta) = 0$ and $\det R_- (\beta) = 0$. Thanks to the TRS, $\det R_{\pm} (\beta)$ are polynomials of $\beta$ and $\beta^{-1}$ with real coefficients, and therefore, it follows that any complex solutions of $\det R_{\pm} (\beta) = 0$ appear in complex conjugate pairs $(\beta, \beta^*)$. Then, if we suppose $\beta_M$ and $\beta_{M+1}$ form a pair of the complex conjugate solutions of $\det R_+ (\beta) = 0$, they satisfy Eq. (2), meaning that $\beta_M$ and $\beta_{M+1}$ are on the GBZ, and the gap is zero. Even when system parameters change, the gap remains zero as long as this pair gives $M$th and $(M+1)$th largest absolute values among the $2M$ solutions. In conclusion, the gapless region in the non-Hermitian system with the SLS and the TRS is robust against the change of system parameters.

To study how the TSM phase appears, we investigate the non-Hermitian Kitaev chain, which has been studied in some previous works [104–119]. Its real-space Hamiltonian is written as

$$H = \sum_n \left[ t_6 c_{n+1}^\dagger c_n + t_f c_{n+1}^\dagger c_n - i \Delta_b c_{n+1} c_n - i \Delta_f c_{n+1} c_n - i \mu c_n^\dagger c_n \right].$$

(3)

This Hamiltonian reduces to that of the conventional Kitaev chain [120] when $t_b = t_f \equiv t \in \mathbb{R}$ and $\Delta_b = \Delta_f = \Delta \in \mathbb{C}$.

We assume that the real-space Hamiltonian $[\mathcal{H}(\beta)]$ satisfies both the SLS and the TRS, which sets all the parameters to be real. Then the Bloch Hamiltonian $\mathcal{H}(\beta)$ is expressed as the off-diagonal form $[\mathcal{H}(\beta)]$ with
In our model, the solutions of the equation $R_+ (\beta) = 0$ (or $R_- (\beta) = 0$) are gap-closing points, shown as the red and blue dots and squares in Figs. 1(c-1)-(h-1), and they become the exceptional points when the GBZ $C_\beta$ goes through them. Let $\beta = \beta^a_i$ $(i = 1, 2, \ a = +, -)$ denote the gap-closing points of $R_\alpha (\beta) = 0$, with $|\beta^a_i| < |\beta^a_2|$. In the regions A and B in Fig. 1(a), $R_-(\beta) = 0$ has two complex-conjugate gap-closing points, satisfying $\beta_1^+ = (\beta_2^-)^*$, and their common absolute value $|\beta_1^+| = |\beta_2^-|$ is between the values of $|\beta_1^-| \text{ and } |\beta_2^+|$: $|\beta_1^-| \leq |\beta_1^+| \leq |\beta_2^+|$. Thus the condition (2) is satisfied, and therefore, $\beta_1^-, \beta_2^-$, and $\bar{\beta}_1^-$ are on $C_\beta$, and are the exceptional points. The condition $\beta_2^- = (\beta_2^-)^*$ remains satisfied even when the system parameter changes because $R_- (\beta) = 0$ is an algebraic equation with real coefficients. Thus the exceptional points move along $C_\beta$ as shown in Figs. 1(c-2)-(h-2), and the system remains in the TSM phase. In the region C and D in Fig. 1(a), a similar scenario holds true, by exchanging $R_+(\beta)$ and $R_-(\beta)$.

Because of this mechanism for the stabilization of the exceptional points, annihilations (and likewise creations) of them are limited to two patterns as shown in Figs. 1(a) and (b). Figure 1(a) represents a coalescence of two exceptional points, and Fig. 1(b) represents an encounter between the gap-closing point and the cusp. In Fig. 1(a), the two exceptional points meet and become two real gap-closing points. It occurs on the real axis. This can be seen in Fig. 1(g). On the other hand, the case of Fig. 1(b) occurs when two complex-conjugate exceptional points and one gap-closing point share the same absolute value. Therefore the gap closes at three points on the GBZ, as for example, as shown in Fig. 1(d). At this point, $|\beta_1^-| = |\beta_2^-| = |\beta_2^+|$ is satisfied, and the ordering of the absolute values of these gap-closing points $\beta_1^-, \beta_2^-$ change, allowing the exceptional point to disappear and the gap to open.

We can relate the creation and annihilation of the exceptional points with the change of the value of the winding number $w$ defined in Eq. (4). For example, through the motion of the exceptional points as shown in Figs. 1(c-1)-(h-1), we find that the number of the gap-closing points inside the GBZ is changed. Then, from Eq. (4), this motion changes the values of $w_+ \text{ and } w_-$ by...
FIG. 4. Generalized Brillouin zone and continuum bands in the non-Hermitian Kitaev chain. TRS-unbroken region, STS-unbroken region, and TRS/STS-broken region are shown in green, in blue, and in orange, respectively. The values of the parameters are $t_b = 1.2$, $t_f = 0.5$, $\Delta_b = 0.3$, $\Delta_f = -0.7$, with (a) $\mu = 0.4$ and (b) $\mu = 1.1$.

1 and $-1$, respectively, resulting in the change of $w$ by 1. We show the detail of the argument in the Supplemental Material.

Finally we show that in non-Hermitian systems with the SLS and TRS, the continuum bands are divided into three regions in terms of the symmetry of the eigenstates. In the first one, the energies are real, and eigenenergies of a time-reversal pair $|\psi, T |\psi^*\rangle$ are degenerate, i.e., $E = E^*$, where $|\psi\rangle$ is an eigenstate of the Hamiltonian. In the second region, the energies are pure imaginary, and a pair of states $|\psi\rangle$ and $\Gamma T |\psi^*\rangle$ related by the sublattice-time-reversal symmetry (STS) is degenerate, i.e., $E = -E^*$. In the third region, the energies are complex, and neither the time-reversal pair nor the sublattice-time-reversal pair is degenerate. We call these three regions TRS-unbroken region, STS-unbroken region, and TRS/STS-broken region, respectively. For example, in the non-Hermitian Kitaev chain, we show the above three regions (Figs. 4(a)-2)-(b-2)).

Importantly, these regions are connected to each other at the cusps or the exceptional points. In Fig 4(a-2) and (b-2), three curves meet at one point, where the GBZ $C_3$ have cusps represented by $A_1$ and $B_1$ (or $A_2$ and $B_2$). It is consistent with the property of the cusp, where three points on $C_3$ share the same energy. Furthermore, in Fig 4(b), the green and blue lines are connected at the exceptional point with $E = 0$. Thus the exceptional point connecting the real and pure-imaginary energies becomes stable because such structure is topologically protected by the symmetries.

In summary, we show that in 1D non-Hermitian systems with both the SLS and the TRS, the TSM phase with exceptional points is stable, unlike Hermitian systems. The appearance of the TSM phase is attributed to the unique features of the GBZ. It is shown that this TSM phase can be regarded as an intermediate phase between the NI and TI phases. We also find that the continuum bands are divided into three regions in terms of the symmetry of the eigenstates, and the regions change only at the cusps and the exceptional points. Thus non-Hermiticity brings about qualitative changes to the topological phase transition.

So far, we have treated the parameters $\Delta_b$ and $\Delta_f$ as real. When $\Delta_b, \Delta_f \in \mathbb{C}$, both the SLS and the TRS are broken. In this case, this system has only pseudo particle-hole symmetry (PHS). We show that the non-Hermitian systems with the pseudo PHS is classified in terms of a $\mathbb{Z}_2$ topological invariant, and importantly, we find that the pseudo PHS cannot stabilize the TSM phase with exceptional points. We show some analyses in the Supplemental Material.

We note that whether such TSM phase appears or not depends on symmetries of systems. When the systems have either chiral symmetry, pseudo TRS, or PHS, the Bloch wave number becomes real, independent of any boundary conditions. Therefore the TSM phase cannot stably exist under either of these symmetries.

We are grateful for Ryo Okugawa and Ryo Takahashi for valuable discussions. This work was supported by JIP18H03678 and JSPS KAKENHI Grant Numbers JP16J07354; by JST, CREST (No. JP-MJCR14F1); and by the MEXT Elements Strategy Initiative to Form Core Research Center (TIES). K. Y. was also supported by JSPS KAKENHI (Grant No. 18J22113).

1. T. Eichelkraut, R. Heilmann, S. Weimann, S. Stützer, F. Dreisow, D. Christodoulides, S. Nolte, and A. Szameit, Nat. Commun. 4, 2533 (2013).
2. C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and H. Schomerus, Nat. Commun. 6, 6710 (2015).
3. J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. 115, 040402 (2015).
4. Y.-L. Xu, W. S. Pegadelli, L. Gan, M.-H. Lu, X.-P. Liu, Z.-Y. Li, A. Scherer, and Y.-F. Chen, Nat. Commun. 7, 11319 (2016).
5. S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. Makris, M. Segev, M. Rechtsman, and A. Szameit, Nat. Mater. 16, 433 (2017).
6. L. Xiao, X. Zhan, Z. Bian, K. Wang, X. Zhang, X. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, et al., Nat. Phys. 13, 1117 (2017).
7. P. St-Jean, V. Goblot, E. Galopin, A. Lemaitre, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch, and A. Amo, Nat. Photonics 11, 651 (2017).
8. B. Bahari, A. Ndao, F. Vallini, A. El Amili, Y. Fainman, and B. Kanté, Science 358, 639 (2017).
9. M. Parto, S. Wittek, H. Hohl, G. Harari, M. A. Bandres, J. Ren, M. C. Rechtsman, M. Segev,
T. Yoshida, R. Peters, and N. Kawakami, Phys. Rev. B 98, 035141 (2018).
Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018).
T. M. Philip, M. R. Hirsbrunner, and M. J. Gilbert, Phys. Rev. B 98, 155430 (2018).
Y. Chen and H. Zhai, Phys. Rev. B 98, 245130 (2018).
C.-H. Liu, H. Jiang, and S. Chen, Phys. Rev. B 99, 125103 (2019).
P. A. McClarty and J. G. Rau, Phys. Rev. B 99, 201107 (2019).
F. A. McClarty and J. G. Rau, Phys. Rev. B 100, 100405(R) (2019).
Y. Xiong, J. Phys. Commun. 2, 035043 (2018).
C. Yin, H. Jiang, L. Li, R. Lü, and S. Chen, Phys. Rev. A 97, 052115 (2018).
F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018).
S. Yao and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
S. Malzard and H. Schomerus, Phys. Rev. A 98, 033807 (2018).
S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018).
K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B 98, 165148 (2018).
K. Takata and M. Notomi, Phys. Rev. Lett. 121, 213902 (2018).
C. Yuce and Z. Oztas, Sci. Rep. 8, 17416 (2018).
K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Nat. Commun. 10, 297 (2019).
L. Jin and Z. Song, Phys. Rev. B 99, 081103(R) (2019).
K. Y. Blokh, D. Leykam, M. Lein, and F. Nori, Nat. Commun. 10, 580 (2019).
H. Wang, J. Ruan, and H. Zhang, Phys. Rev. B 99, 075130 (2019).
T. Liu, Y.-R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. 122, 076801 (2019).
E. Edvardsson, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B 99, 081302(R) (2019).
C. H. Lee and R. Thomale, Phys. Rev. B 99, 201103(R) (2019).
L. Herviou, J. H. Bardarson, and N. Regnault, Phys. Rev. A 99, 052118 (2019).
F. K. Kunst and V. Dwivedi, Phys. Rev. B 99, 245116 (2019).
T.-S. Deng and W. Yi, Phys. Rev. B 100, 035102 (2019).
M. Ezawa, Phys. Rev. B 100, 045407 (2019).
K. L. Zhang, H. C. Wu, L. Jin, and Z. Song, Phys. Rev. B 100, 045141 (2019).
L. Li, C. H. Lee, and J. Gong, Phys. Rev. B 100, 075403 (2019).
K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019).
Z.-Y. Ge, Y.-R. Zhang, T. Liu, S.-W. Li, H. Fan, and F. Nori, Phys. Rev. B 100, 054105 (2019).
N. Okuma and M. Sato, Phys. Rev. Lett. 123, 097701 (2019).
P. San-Jose, J. Cayao, E. Prada, and R. Aguado, Sci. Rep. 6, 21427 (2016).
C. Yuce, Phys. Rev. A 93, 062130 (2016).
Q.-B. Zeng, B. Zhu, S. Chen, L. You, and R. Lü, Phys. Rev. A 94, 022119 (2016).
H. Menke and M. M. Hirschmann, Phys. Rev. B 95, 174506 (2017).
M. Klett, H. Cartarius, D. Dast, J. Main, and G. Wunner, Phys. Rev. A 95, 053626 (2017).
C. Li, X. Z. Zhang, G. Zhang, and Z. Song, Phys. Rev. B 97, 115436 (2018).
K. Kawabata, Y. Ashida, H. Katsura, and M. Ueda, Phys. Rev. B 98, 085116 (2018).
M. van Caspel, S. E. T. Arze, and I. P. Castillo, SciPost Phys. 6, 26 (2019).
S. Lieu, Phys. Rev. B 100, 055110 (2019).
A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
SI. MOTION OF THE EXCEPTIONAL POINTS AND CHANGE OF THE VALUE OF THE WINDING NUMBER

In this section, we show that in one-dimensional (1D) non-Hermitian systems with both sublattice symmetry (SLS) and time-reversal symmetry (TRS), the value of the winding number changes through the creation and annihilation of the exceptional points as shown in Fig. S1. Here, for a real-space Hamiltonian $H$, the SLS and the TRS are defined as

$$\Gamma H^{-1} = -H, \ T H^* T^{-1} = H,$$

where $\Gamma$ and $T$ are unitary matrices representing the SLS and the TRS, respectively.

A. General cases

First of all, we focus on a two-band model. Due to the SLS, we can write down the Bloch Hamiltonian $\mathcal{H}(\beta)$ of this system as

$$\mathcal{H}(\beta) = \begin{pmatrix} 0 & R_+ (\beta) \\ R_- (\beta) & 0 \end{pmatrix},$$  \quad (S2)

where $\beta = e^{ik}$, $k \in \mathbb{C}$, and $R_\pm (\beta)$ are holomorphic functions for $\beta$. In the following, without loss of generality, we can write these functions as

$$R_\pm (\beta) = C_\pm \frac{2m}{\beta^m} \prod_{i=1}^{2m} (\beta - \beta_i^\pm),$$  \quad (S3)

where $C_\pm$ are real constants due to the TRS. Then the eigenvalue equation $\det [\mathcal{H}(\beta) - E] = R_+ (\beta) R_- (\beta) - E^2 = 0$ can be explicitly written as

$$C_+ C_- \frac{2m}{\beta^{2m}} \prod_{i=1}^{2m} (\beta - \beta_i^+) (\beta - \beta_i^-) = E^2,$$  \quad (S4)

which is an algebraic equation for $\beta$ with a degree $4m$. Here, by numbering the solutions of Eq. (S4) so as to satisfy $|\beta_1| \leq \cdots \leq |\beta_{2M}|$, the condition for continuum bands is given by

$$|\beta_M| = |\beta_{M+1}|,$$  \quad (S5)

and one can get the generalized Brillouin zone (GBZ) $C_\beta$ from the trajectories of $\beta_M$ and $\beta_{M+1}$. We note that $C_\beta$ always encircles the origin on the complex plane $\mathbb{C}$.

In this case, this system is classified in terms of a $Z$ topological invariant called winding number $w$, and it can be defined as

$$w = -\frac{w_+ - w_-}{2}, \quad w_\pm = \frac{1}{2\pi} \arg R_\pm(\beta)|_{C_\beta},$$

where $[\arg R_\pm(\beta)|_{C_\beta}$] means the change of the phase of the functions $R_\pm(\beta)$ as $\beta$ goes along $C_\beta$ in a counterclockwise way. We note that as long as there is a gap at $E = 0$, $R_\pm (\beta)$ never vanish along $C_\beta$, and $w$ is well defined. On the other hand, when the gap closes at $E = 0$, the gap-closing condition can be decoupled into two equations $R_+ (\beta) = 0$ and $R_- (\beta) = 0$ being satisfied somewhere on $C_\beta$. At such a point, the Bloch Hamiltonian (S2) cannot be diagonalizable, and such point is called exceptional point. In this case, $w$ is not well defined.

From Eq. (S3), we can rewrite this form as

$$w = -\frac{N_{\text{zeros}}^+ - N_{\text{zeros}}^-}{2},$$

where $N_{\text{zeros}}^\pm$ expresses the number of the solutions $\beta_i^\pm$ of $R_\pm (\beta) = 0$ inside $C_\beta$, respectively. Furthermore it is worth noting that from Ref. [2], when the system has a gap around $E = 0$, we can get

$$\frac{1}{2\pi} \int_{C_\beta} \text{d log det} \mathcal{H}(\beta) = N_{\text{zeros}} - M = 0,$$  \quad (S8)

where $N_{\text{zeros}} (= N_{\text{zero}}^+ + N_{\text{zero}}^-)$ expresses the number of solutions of the equation $\det \mathcal{H}(\beta) = 0$ inside $C_\beta$. Equation (S8) tells us that the total number of the solutions

![Fig. S1. Schematic figure of (a) coalescence of the exceptional points and (b) annihilation of the exceptional point at the cusp, respectively. The yellow (or blue) dots express the gap-closing points (or the exceptional points), and the red stars are the cusps of the generalized Brillouin zone $C_\beta$.](image_url)
of \( \det \mathcal{H}(\beta) = 0 \) inside \( C_\beta \), namely the number of the gap-closing points inside \( C_\beta \), is unchanged as long as the system has a gap.

Now we focus on two insulator phases separated by the topological semimetal (TSM) phase with exceptional points. When the system enters the TSM phase from one of the insulator phases, the exceptional points are created by the inverse process as shown in Fig. S2(a) or (b). Then, after a further change of the system parameters, the system becomes the other insulator phase from the TSM phase, and here, the exceptional points are annihilated by the process as shown in Fig. S1(a) or (b). If the creation is by the inverse process of Fig. S1(a) and the annihilation is by the process of Fig. S1(b) (or vice versa), while the total number \( N_{\text{zeros}} = N_{\text{zero}}^+ + N_{\text{zero}}^- \) of the solutions of \( \det \mathcal{H}(\beta) = 0 \) inside \( C_\beta \) is unchanged, such a motion of the exceptional points change the value of \( N_{\text{zero}}^+ \) and \( N_{\text{zero}}^- \) by 1 and -1 (or by -1 and 1), respectively, resulting in the change of \( w \) by 1 (or by -1). Therefore we conclude that these insulator phases have different values of \( w \). We note that this scenario can be extended to multi-band systems.

B. Non-Hermitian Kitaev chain

We show an example of the non-Hermitian Kitaev chain with the SLS and TRS introduced in Eq. (4) in the main text. This model has the Bloch Hamiltonian \( \mathcal{H}(\beta) \) as

\[
\mathcal{H}(\beta) = \begin{pmatrix} 0 & R_+(\beta) \\ R_-(\beta) & 0 \end{pmatrix},
\]

\( \beta \)

where

\[
R_{\pm}(\beta) = (t_0 \pm \Delta_0) - \mu + (t_f \mp \Delta_f) \beta^{-1},
\]

and all the parameters are real. We note that the gap closes at \( E = 0 \) due to the SLS when either of the two quantities \( R_{\pm}(\beta) \) is zero. We show the phase diagram of this model in Fig. S2(a) and its GBZ \( C_{\beta} \), the solutions of the equations \( R_{\pm}(\beta) = 0 \), and the motion of the exceptional points in Fig. S2(c)-(h).

In the topological phase transition along the black arrow in Fig. S2(a), we show the position of the solutions \( \beta_1^+ \) and \( \beta_2^+ \) of \( R_{+}(\beta) = 0 \) (and \( \beta_1^- \) and \( \beta_2^- \) of \( R_{-}(\beta) = 0 \)) as the red (and blue) dots and squares, respectively, as shown in Figs. S2(c-1)-(h-1). When we decrease the value of the parameter \( \mu \), at \( \mu = 2.683 \) (Fig. S2(g-1)), \( \beta_1^+ \) and \( \beta_2^+ \) change from real values to complex values via coalescence, and it corresponds to a pair creation of the exceptional points. After the coalescence, \( \beta_1^+ \) and \( \beta_2^+ \) become complex, with \( \beta_1^+ = (\beta_2^-)^* \), and their common absolute value is between the values of \( |\beta_1^-| \) and \( |\beta_2^-| \), meaning that \( \beta_1^+ \) and \( \beta_2^+ \) stay on \( C_\beta \). Then, at \( \mu = 0.5813 \) (Fig. S2(d-1)), the value of \( |\beta_2^+| \) becomes equal to that of \( |\beta_1^+| = |\beta_2^-| \), and after passing that point (i.e., \( \mu \) becomes less than 0.5813), \( |\beta_1^-| < |\beta_2^-| < |\beta_1^+| = |\beta_2^+| \), meaning that \( \beta_1^+ \) and \( \beta_2^+ \) are no longer on \( C_\beta \), and the gap opens (Fig. S2(c-1)). At the phase transition point \( \mu = 0.5813 \) (Fig. S2(d-1)), three solutions \( \beta_1^+, \beta_2^+, \beta_2^- \) share the same absolute value, and the gap closes at three points on \( C_\beta \), and at such points, \( C_\beta \) has cusps.

The change of the winding number \( w \) defined as Eq. S10 readily follows from the following argument. In the normal insulator (NI) phase with \( w = 0 \), \( C_\beta \) surrounds one solution of \( R_{+}(\beta) = 0 \) and one solution of \( R_{-}(\beta) = 0 \) (Fig. S2(h-1)). In this case, \( w = \arg R_{\pm}(\beta)_{C_{\beta}} / 2\pi = 0 \) because \( R_{\pm}(\beta) \) is proportional to \( (\beta - \beta_1^+)(\beta - \beta_2^+) / \beta \). On the other hand, in the topological insulator (TI) phase with \( w = 1 \), there exist two solutions of \( R_{-}(\beta) = 0 \) and no solutions of \( R_{+}(\beta) = 0 \) inside \( C_{\beta} \) (Fig. S2(c-1)), which leads to \( w_{-} = 1, w_{+} = -1 \), and \( w = 1 \) as expected. Therefore the creation and annihilation of the exceptional points changes the number of the solutions of \( R_{\pm}(\beta) = 0 \) inside \( C_\beta \), and the value of the topological invariant also changes.

We note that in addition to the topological phase transition between two insulator phases via the TSM phase, a direct phase transition from the NI phase to the TI phase is also possible as shown in the inset of Fig. S2(a). Here the gap closes on the real axis, where \( R_{+}(\beta) \) and \( R_{-}(\beta) \) simultaneously become zero at this value of \( \beta \).

III. 1D NON-HERMITIAN SYSTEM WITH PSEUDO PARTICLE-HOLE SYMMETRY

In this section, we investigate a 1D non-Hermitian system with pseudo particle-hole symmetry (PHS) and show that it is classified in terms of a \( Z_2 \) topological invariant. Here, for a real-space Hamiltonian \( H \), this symmetry is defined as

\[
\mathcal{C} H^\ast \mathcal{C}^{-1} = -H,
\]

where \( \mathcal{C} \) is the unitary matrix. In the following, we focus on a two-band model.

A. Two-band model

The Bloch Hamiltonian \( \mathcal{H}(\beta) \) can be written as

\[
\mathcal{H}(\beta) = \mathcal{H}_0(\beta) \sigma_0 + \sum_{i=x,y,z} \mathcal{H}_i(\beta) \sigma_i,
\]

where \( \sigma_0 \) is a \( 2 \times 2 \) identity matrix, \( \sigma_i \) (\( i = x, y, z \)) are the Pauli matrices, and the complex Bloch wave number is defined as \( \beta \equiv e^{ik}, k \in \mathbb{C} \). We assume that it satisfies

\[
\sigma_x [\mathcal{H}(\beta)]^\ast \sigma_x^{-1} = -\mathcal{H}(\beta^*) ,
\]

and then, the coefficients \( \mathcal{H}_i(\beta) \) (\( i = 0, x, y, z \)) in Eq. S12 satisfy

\[
[\mathcal{H}_i(\beta)]^\ast = -\mathcal{H}_i(\beta^*) \quad (i = 0, x, y), \quad [\mathcal{H}_z(\beta)]^\ast = \mathcal{H}_z(\beta^*). \quad (S14)
\]
FIG. S2. (a) Phase diagram in the non-Hermitian Kitaev chain with $k \in \mathbb{C}$ with the values of the parameters to be $t_k = 1.2$, $t_f = 0.5$, $\Delta_k = 0.3$. At the red star ($\mu = -1.5922$ and $\Delta_f = 0.2$), the gap closes, and a direct transition between two insulator phases with $w = 0$ (white region) and $w = 1$ (blue region) occurs. The orange regions express topological semimetal (TSM) phase. (b) Continuum band along the black arrow ($\Delta_f = -0.7$) in (a). Since it is a two-band model with the sublattice symmetry, we only show $|E|$ to see whether the gap closes. (c)-(h) Solutions of the equations $R_\pm(\beta) = 0$, generalized Brillouin zone, and motion of the exceptional points along the black arrow in (a). The red (or blue) dots and squares express the solutions of $R_+(\beta) = 0$ (or $R_-(\beta) = 0$).

It is worth noting that $\mathcal{H}_i(\beta)$ $(i = 0, x, y)$ are a pure imaginary and $\mathcal{H}_z(\beta)$ is real when $\arg \beta = 0$ and $\arg \beta = \pi$.

### B. $\mathbb{Z}_2$ topological invariant

Such a system is classified in terms of the $\mathbb{Z}_2$ topological invariant $\nu$ ($= 0, 1$), and for the Bloch Hamiltonian (S12), we can define it as

$$\nu = \frac{1}{2\pi} \int_{\beta_0}^{\beta_\pi} \frac{d\beta}{d\beta} [\arg \mathcal{R}_+(\beta) - \arg \mathcal{R}_-(\beta)] \pmod{2},$$

$$\mathcal{R}_\pm(\beta) = \mathcal{H}_z(\beta) \pm i \sqrt{\mathcal{H}_x^2(\beta) + \mathcal{H}_y^2(\beta)}, \tag{S15}$$

where $\beta_0$ (or $\beta_\pi$) is the value of $\beta$ at $\arg \beta = 0$ (or $\arg \beta = \pi$) on the GBZ $C_\beta$ (for example, see Fig. S3). In Eq. (S15), the integral contour $\beta$ goes along $C_\beta$, and we select the branch cut of the square root so that the both functions $\mathcal{R}_\pm(\beta)$ becomes continuous on $C_\beta$.

In the following, we assume that a system has a gap. Here we note that two continuum bands are separated by a line which determines a complex gap on the complex energy plane. This gap is called a line gap. In this case, one can show that $\nu$ takes only 0 or 1. To this end, we calculate the value of $\exp(2\pi i \nu)$. As mentioned in Sec. SII A, since the functions $\mathcal{R}_+(\beta_0)$ and $\mathcal{R}_+(\beta_\pi)$ take real values, we can rewrite the expression of $\exp(2\pi i \nu)$ as

$$\exp(2\pi i \nu) = \exp \{i [\arg \mathcal{R}_+(\beta_\pi) - \arg \mathcal{R}_+(\beta_0)]\} \times \exp \{i [\arg \mathcal{R}_-(\beta_\pi) - \arg \mathcal{R}_-(\beta_0)]\} \times \prod_{\beta = \beta_0, \beta_\pi, \sigma = \pm} \text{sgn} [\mathcal{R}_\sigma(\beta)].$$

Here we note that the quantities $\sum_i \mathcal{H}_i^2(\beta_0)$ and $\sum_i \mathcal{H}_i^2(\beta_\pi)$ are real. Since we assume presence of a line gap, we conclude that $\sum_i \mathcal{H}_i^2(\beta_0)$ and $\sum_i \mathcal{H}_i^2(\beta_\pi)$ have the same sign, as we prove by contradiction in the following.

Suppose $\sum_i \mathcal{H}_i^2(\beta_0)$ and $\sum_i \mathcal{H}_i^2(\beta_\pi)$ have different signs. We can set

$$\sum_{i=x,y,z} \mathcal{H}_i^2(\beta_0) > 0, \sum_{i=x,y,z} \mathcal{H}_i^2(\beta_\pi) < 0 \tag{S17}$$

without loss of generality. First of all, we assume
$H_0 (\beta) = 0$ for simplicity. At $\beta = \beta_0$, the energies are $E = \pm \varepsilon_0$, $\varepsilon_0 = \sqrt{\sum \beta_0^2}$, and we choose $E = \varepsilon_0 = \sqrt{\sum \beta_0^2}$, and we change $\beta$ along $C_0$ in a counterclockwise way from $\beta = \beta_0$ to $\beta = \beta_\pi$. Here, let $C_+$ denote this path on the complex plane. Then, at $\beta = \beta_\pi$, the energy is given by $E = \varepsilon_\pi = \sqrt{\sum \beta_\pi^2} + i \mathbb{R}$, where the branch of the square root is chosen in such a way that $E = \sqrt{\sum \beta_\pi^2}$ is continuous along $C_\pi$. Next we consider a path $C_-$ along $C_\beta$ in a clockwise way from $\beta = \beta_0$ to $\beta = \beta_\pi$. Because $[\sum \beta_\pi^2]$ is an integer. As a result, we can conclude that the value of $\beta$ at $\beta_\pi$ is $\pi$. Furthermore, we can get $\sum \beta_\pi^2 = \sum \beta_\pi^2$, and we obtain the known formula $\sum \beta_\pi^2 = 0$, but even in the case of $H_\beta (\beta) \neq 0$, because this term does not affect above argument, the above proof remains valid.

Therefore, from Eq. (S10), we can get exp $(2 \pi i \nu) = 1$ and conclude that the value of $\nu$ is an integer. As a result, $\nu$ can take only 0 or 1 (mod 2), and we conclude that $\nu$ can be interpreted as the $Z_2$ topological invariant in this system.

In particular, in Hermitian cases, we can greatly simplify the formula of $\nu$ in Eq. (S11). The Bloch wave number $k$ becomes real, and all coefficients included in Eq. (S12) become real functions. In the following, we replace $H_i (\beta)$ by $H_i (k)$ ($i = 0, x, y, z$), $k \in [-\pi, \pi]$. Here the system has the conventional PHS represented as

$$H_i (k) = -H_i (-k) \quad (i = 0, x, y) , \quad H_z (k) = H_z (-k) ,$$

and $H_i (k)$ ($i = 0, x, y$) become zero at $k = 0$ and $k = \pi$. Furthermore, we can get

$$\arg \left( H_z - i \sqrt{H_z^2 + H_0^2} \right) = -\arg \left( H_z + i \sqrt{H_z^2 + H_0^2} \right) ,$$

and then, Eq. (S15) can be rewritten as

$$\nu = \frac{1}{\pi} \int_0^{\pi} \frac{dk}{\pi} \arg \left[ H_z (k) + i \sqrt{H_z^2 (k) + H_0^2 (k)} \right]$$

$$= \frac{1}{\pi} \left[ \arg H_z (k) \right]_0^{\pi} .$$

Since Eq. (S18) tells us that both $\arg H_z (0)$ and $\arg H_z (\pi)$ take 0 or $\pi$ (mod 2$\pi$). Eq. (S20) can be further rewritten as

$$\nu = \begin{cases} 0 & \text{if sgn} \left[ H_z (0) \right] \text{sgn} \left[ H_z (\pi) \right] > 0 , \\ 1 & \text{if sgn} \left[ H_z (0) \right] \text{sgn} \left[ H_z (\pi) \right] < 0 , \end{cases}$$

and we obtain the known formula

$$(-1)^{\nu} = \text{sgn} \left[ H_z (0) \right] \text{sgn} \left[ H_z (\pi) \right] .$$

C. Generalized non-Hermitian Kitaev chain

In this subsection, we study the generalized non-Hermitian Kitaev chain. This system has the pseudo PHS, and as we discussed previously, it can be classified in terms of the $Z_2$ topological invariant $\nu$ defined in Eq. (S15). The real-space Hamiltonian of this system can be written as

$$H = \sum_n \left[ t_b c_{n+1}^\dagger c_n + t_f c_{n+1}^\dagger c_n - i \Delta_b c_{n+1}^\dagger c_n + \right.$$

$$\left. - i \Delta_f c_{n+1}^\dagger c_n + i \Delta_b c_{n+1}^\dagger c_n + i \Delta_f c_{n+1}^\dagger c_n + \right.$$

$$\left. + (i \gamma - \mu) c_n^\dagger c_n + (i \gamma + \mu) c_n c_n^\dagger \right] ,$$

where $\gamma$ represents dissipation, $t_b$ and $t_f$ are asymmetric hopping amplitudes, $\Delta_b$ and $\Delta_f$ are imbalanced pairing amplitudes, and $\mu$ is a chemical potential. $\gamma$ and $\mu$ are real and, $t_b, t_f, \Delta_b$, and $\Delta_f$ are complex. We note that this Hamiltonian reduces to that of the non-Hermitian Kitaev chain introduced in the main text when $\gamma = 0$, and $t_b, t_f, \Delta_b, \Delta_f \in \mathbb{R}$. Here, for the Bloch Hamiltonian $H (\beta)$ in the form (S12), the coefficients $H_i (\beta)$ ($i = 0, x, y, z$) are given by

$$H_0 (\beta) = i \gamma + i \Im \left( t_b \right) \beta + i \Im \left( t_f \right) \beta^{-1} ,$$

$$H_x (\beta) = - \Im \left( \Delta_b \beta - \Im (\Delta_f) \beta^{-1} ,ight) ,$$

$$H_y (\beta) = i \Im (\Delta_b) \beta - \Im (\Delta_f) \beta^{-1} ,$$

$$H_z (\beta) = \Re \left( t_b \right) \beta + \Re \left( t_f \right) \beta^{-1} - \mu .$$

Then we can explicitly write the eigenvalue equation det $[H (\beta) - E] = 0$ as

$$\left( |\Delta_b|^2 - |t_b|^2 \right) \beta^2 + |2 \mu \Re \left( t_b \right) - 2 i (E - i \gamma) \Im \left( t_b \right) \beta$$

$$- 2 \left[ \Re \left( \Delta_b \Delta_f^* \right) + \Re \left( t_b t_f^* \right) \right] + (E - i \gamma)^2 - \mu^2$$

$$+ |2 \mu \Re \left( t_f \right) - 2 i (E - i \gamma) \Im \left( t_f \right) \beta^{-1}$$

$$+ \left( |\Delta_f|^2 - |t_f|^2 \right) \beta^{-2} = 0 .$$

FIG. S3. Generalized Brillouin zone $C_3$ in the generalized non-Hermitian Kitaev chain. The values of the parameters are $\gamma = 0, t_b = 1.2, t_f = 0.5$, and $\mu = 1$; (a) $\Delta_b = 0.3$ and $\Delta_f = 0.8$, and (b) $\Delta_b = 0.3 + 0.5i$ and $\Delta_f = 0.8 + 0.8i$. The points where $C_3$ intersects the positive and negative sides of a real axis are denoted by $\beta_0$ and $\beta_\pi$, respectively.
Since Eq. (S25) is a quartic equation for \( \beta \), the condition for continuum bands can be written as \( |\beta_2| = |\beta_3| \) when the solutions of Eq. (S25) satisfy \( |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4| \). We note that the trajectories of \( \beta_2 \) and \( \beta_3 \) give the GBZ \( C_\beta \). The examples of \( C_\beta \) and the continuum bands are given in Fig. S3 and in Fig. S4 (a-3), respectively.

Let \( \ell_\pm \) denote the loops drawn by the functions \( R_\pm (\beta) \) on the complex plane when \( \beta \) goes along \( C_\beta \) in a counterclockwise way. Equation (S15) tells us how to determine the value of \( \nu \); when neither \( \ell_+ \) nor \( \ell_- \) surrounds the origin \( O \) on the complex plane, \( \nu \) is equal to 0, and when two loops simultaneously surrounds \( O \), \( \nu \) is equal to 1. For example, in the case of Fig. S4 (a), \( \nu \) becomes 1. We note that the system has the exceptional points when either \( \ell_+ \) or \( \ell_- \) passes \( O \), and \( \nu \) is not well defined in this case.

We can get the phase diagram in the generalized non-Hermitian Kitaev chain as shown in Fig. S4 (a) and can confirm that the topological edge states appear when \( \nu \) takes the nonzero value as shown in Fig. S4 (b). Therefore we can establish the bulk-edge correspondence between the \( Z_2 \) topological invariant \( \nu \) and existence of the topological edge states.

In this model, we can see that the pseudo PHS cannot protect a TSM phase with exceptional points. For simplicity, let the parameter \( \gamma \) be zero. We note that by treating the parameters \( t_b \) and \( t_f \) as complex, one can add \( H_0 (\beta) \) term to the Bloch Hamiltonian \( H (\beta) \) in the case of Fig. S4 (a). In fact, for \( \Im \left( t_b \right) = \Im \left( t_f \right) \neq 0 \) being infinitesimals as an example, we can obtain the phase diagram as shown in Fig. S4 (b). We can confirm that the TSM phase with exceptional points in Fig. S4 (a) disappears by adding \( H_0 (\beta) \). We note that the exceptional points appear on the orange lines on the phase diagram. However, this exceptional point can be removed by adding other perturbation terms. Therefore we conclude that the pseudo PHS cannot protect the TSM phase with exceptional points.

**SIII. RESTRICTION ON THE BLOCH WAVE NUMBER BY EITHER PARTICLE-HOLE SYMMETRY OR CHIRAL SYMMETRY**

In this section, we show that in 1D non-Hermitian tight-binding systems, some symmetries make the Bloch wave number real even in an open chain. The previous work showed that it takes real values even in non-Hermitian cases under pseudo time-reversal symmetry (TRS) represented as

\[
\mathcal{T} H^T \mathcal{T}^{-1} = H,
\]

where \( H \) is a real-space Hamiltonian, and \( \mathcal{T} \) is a unitary matrix. Here we show that PHS or chiral symmetry (CS) also restricts the Bloch wave number to real values. In the following, we focus on a 1D real-space tight-binding Hamiltonian

\[
H = \sum_n \sum_{i=-N}^N \sum_{\mu,\nu=1}^q t_{i,\mu\nu} c_{n+i,\mu}^\dagger c_{n,\nu}, \quad (S27)
\]

where the unit cell is composed of \( q \) degrees of freedom, and the range of hopping is \( N \). We note that this Hamiltonian can be non-Hermitian, meaning that \( t_{i,\mu\nu} \) is not necessarily equal to \( t_{i,\nu\mu}^* \). Then a Bloch Hamiltonian \( \mathcal{H} (\beta) \) is given by

\[
[\mathcal{H} (\beta)]_{\mu\nu} = \sum_{i=-N}^N t_{i,\mu\nu} \beta^i, \quad (\mu, \nu = 1, \cdots, q), \quad (S28)
\]
where the complex Bloch wave number is defined as $\beta = e^{ik}$; $k \in \mathbb{C}$. We assume that the eigenvalue equation $\det [H(\beta) - E] = 0$ is an algebraic equation for $\beta$ with an even degree $2M$, and $2M$ solutions of this equation satisfy $|\beta_1| \leq \cdots \leq |\beta_{2M}|$. Here the condition for continuum bands is written as $|\beta_{M+1}|$, of which the detail is given in the main text and Ref. 4.

A. Particle-hole symmetry

The PHS is defined as

$$CH^T C^{-1} = -H, \quad \text{(S29)}$$

where $C$ is unitary matrix. Here, for the real-space eigenvalue equation

$$H |\psi\rangle = E |\psi\rangle, \quad \text{(S30)}$$

we can get

$$H (C |\psi'\rangle) = -E (C |\psi'\rangle) \quad \text{(S31)}$$

due to the PHS. This is because the transposed matrix $H^T$ has the same eigenvalues with $H$; i.e., $H^T |\psi'\rangle = E |\psi'\rangle$. Because of Eq. (S31), the energy eigenvalues appear in pairs: $(E, -E)$. On the other hand, the hopping matrix $t_{i,\mu\nu}$ satisfies the condition

$$\sum_{\sigma,\tau=1}^q (C)_{\nu\sigma} \tau_{i,\sigma\tau} (C^{-1})_{\tau\mu} = -t_{i,\mu\nu} \quad \text{(S32)}$$

due to the PHS. Then we can obtain the constraint for the Bloch Hamiltonian $H(\beta)$ as

$$CH^T (\beta) C^{-1} = -H (\beta^{-1}), \quad \text{(S33)}$$

and by using Eq. (S33), the eigenvalue equation $\det (H(\beta) - E) = 0$ can be rewritten as

$$\det (H (\beta^{-1}) + E) = 0. \quad \text{(S34)}$$

On the other hand, $\det (H(\beta) - E) = 0$ can be also rewritten as

$$\det (H (\beta) + E) = 0. \quad \text{(S35)}$$

because the energy eigenvalues appear in pairs: $(E, -E)$. By the combination of Eqs. (S34) and (S35), it is shown that the eigenvalue equation has the solutions in pairs: $(\beta, \beta^{-1})$. Therefore the condition for continuum bands can be rewritten as

$$|\beta_{M+1}| = 1. \quad \text{(S36)}$$

This represents that the Bloch wave number $k$ takes real values because $|e^{ik}| = 1$ is satisfied.

B. Chiral symmetry

The CS is defined as

$$\Gamma H^\dagger \Gamma^{-1} = -H, \quad \text{(S37)}$$

where $\Gamma$ is unitary matrix. Here, for the real-space eigenvalue equation

$$H |\psi\rangle = E |\psi\rangle, \quad \text{(S38)}$$

we can get

$$H (\Gamma |\psi'\rangle) = -E^* (\Gamma |\psi'\rangle) \quad \text{(S39)}$$

due to the CS. This is because the complex-conjugate transposed matrix $H^\dagger$ has eigenvalues $E^*$; i.e., $H^\dagger |\psi'\rangle = E^* |\psi'\rangle$. Because of Eq. (S39), the energy eigenvalues appear in pairs: $(E, -E^*)$. On the other hand, the hopping matrix $t_{i,\mu\nu}$ satisfies the condition

$$\sum_{\sigma,\tau=1}^q (\Gamma)_{\nu\sigma} \tau_{i,\sigma\tau} (\Gamma^{-1})_{\tau\mu} = -t_{i,\mu\nu} \quad \text{(S40)}$$

due to the CS. Then we can obtain the constraint for the Bloch Hamiltonian $H(\beta)$ as

$$\Gamma H^\dagger (\beta) \Gamma^{-1} = -H (1/\beta^*), \quad \text{(S41)}$$

and by using Eq. (S41), the eigenvalue equation $\det (H(\beta) - E) = 0$ can be rewritten as

$$\det [H (1/\beta^*) + E] = 0. \quad \text{(S42)}$$

On the other hand, $\det [H(\beta) - E] = 0$ can be also rewritten as

$$\det [H (\beta) + E^*] = 0. \quad \text{(S43)}$$

because the energy eigenvalues appear in pairs: $(E, -E^*)$. By the combination of Eqs. (S42) and (S43), it is shown that the eigenvalue equation has the solutions in pairs: $(\beta, 1/\beta^*)$. Therefore the condition for continuum bands also is given by Eq. (S36) in this case.

SIV. Non-Hermitian SSH Model

Our theory can apply to any 1D systems with both the SLS and the TRS. In this section, we investigate the TSM
phase with exceptional points in the non-Hermitian Su-Schrieffer-Heeger (SSH) model as shown in Fig. S5. We studied the topological phases and the bulk-edge correspondence in Ref. 1. The real-space Hamiltonian of this system is given by

\[
H = \sum_n \left[ \left( t_1 + \frac{\gamma_1}{2} \right) c_{n,A}^\dagger c_{n,B} + \left( t_1 - \frac{\gamma_1}{2} \right) c_{n,B}^\dagger c_{n,A} \right. \
+ \left( t_2 + \frac{\gamma_2}{2} \right) c_{n+1,A}^\dagger c_{n,B} + \left( t_2 - \frac{\gamma_2}{2} \right) c_{n+1,B}^\dagger c_{n,A} \right. \
+ t_3 \left( c_{n,A}^\dagger c_{n+1,B} + c_{n+1,B}^\dagger c_{n,A} \right),
\]

(S44)

where \( t_1, t_2, t_3, \gamma_1, \) and \( \gamma_2 \) are real. We note that this system has both the SLS and the TRS represented as Eq. (S1). The Bloch Hamiltonian \( \mathcal{H}(\beta) \) can be obtained by a replacement \( e^{ik} \rightarrow \beta \), similarly to Hermitian systems, as

\[
\mathcal{H}(\beta) = \begin{pmatrix}
0 & R_+ (\beta) \\
R_- (\beta) & 0
\end{pmatrix},
\]

(S45)

where \( R_\pm (\beta) \) are given by

\[
R_+ (\beta) = \left( t_2 - \frac{\gamma_2}{2} \right) \beta^{-1} + \left( t_1 + \frac{\gamma_1}{2} \right) + t_3 \beta,
\]

\[
R_- (\beta) = t_3 \beta^{-1} + \left( t_1 - \frac{\gamma_1}{2} \right) + \left( t_2 + \frac{\gamma_2}{2} \right) \beta,
\]

(S46)

respectively, and therefore, the eigenvalue equation can be written as

\[
R_+ (\beta) R_- (\beta) = E^2.
\]

(S47)

Here, since this system has the SLS, it can be characterized by the winding number \( w \) given by Eq. (S6).

Now we expect that the TSM phase with exceptional points is stable in the non-Hermitian SSH model. In fact, we can confirm that this TSM phase appears similarly to the non-Hermitian Kitaev chain with both the SLS and the TRS. In Fig. S6, we show the motion of the solutions of the equations \( R_\pm (\beta) = 0 \) as the system parameter \( \gamma_1 \) changes. When these points are on the GBZ \( C_\beta \), the gap closes at \( E = 0 \). In this case, since either of two quantities \( R_\pm (\beta) \) is zero somewhere on \( C_\beta \), the solutions of either of \( R_\pm (\beta) = 0 \) express the exceptional points. As \( \gamma_1 \) changes between \( \gamma_1 = 1.02 \) and \( \gamma_1 = 2.706 \), \( C_\beta \) is deformed so that the exceptional points stay on \( C_\beta \). Through the creation (at \( \gamma_1 = 1.02 \)) and annihilation (at \( \gamma_1 = 2.706 \)) of the exceptional points, the number of the solutions of \( R_\pm (\beta) = 0 \) inside \( C_\beta \) changes. Therefore the value of \( w \) also changes. In fact, while \( w \) becomes zero at \( \gamma_1 = 2.8 \) (Fig. S6(g)), it takes one at \( \gamma_1 = 0.1 \) (Fig. S6(b)).

1 K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019)
2 K. Zhang, Z. Yang, and C. Fang, arXiv:1910.01131.
3 Z. Yang, K. Zhang, C. Fang, and J. Hu, arXiv:1912.05499.
4 M. Ezawa, Phys. Rev. B 100, 045407 (2019)
5 K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X 9, 041015 (2019)
6 A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001)
FIG. S6. (a) Phase diagram and continuum band in the non-Hermitian SSH model with the values of the parameters to be $t_1 = 1/3$, $t_2 = 0.8$, $t_3 = 1/5$, and $\gamma_2 = 1$ in an open chain. (b)-(g) Motions of the solutions of the eigenvalue equation (S47) for $E = 0$. The red dots express the solutions of the equation $R_+ (\beta) = 0$, and the blue dots express those of $R_- (\beta) = 0$. 