A Study on Nonnegative Tubal Matrices

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Abstract

Tubal scalars are usual vectors, and tubal matrices are matrices with every element being a tubal scalar. Such a matrix is often recognized as a third-order tensor. The product between tubal scalars, tubal vectors, and tubal matrices can be done by the powerful t-product. In this paper, we define nonnegative/positive/strongly positive tubal scalars/vectors/matrices, and establish several properties that are analogous to their matrix counterparts. In particular, we introduce the irreducible tubal matrix, and provide two equivalent characterizations. Then, the celebrated Perron-Frobenius theorem is established on the nonnegative irreducible tubal matrices. We show that some conclusions of the PF theorem for nonnegative irreducible matrices can be generalized to the tubal matrix setting, while some are not. One reason is the defined positivity here has a different meaning to its usual sense. For those conclusions that can not be extended, weaker conclusions are proved. We also show that, if the nonnegative irreducible tubal matrix contains a strongly positive tubal scalar, then most conclusions of the matrix PF theorem hold.

Keywords: tensor; tubal matrix; eigenvalue; t-product; Perron-Frobenius theorem

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1 Introduction

The celebrated Perron-Frobenius theorem is a fundamental result in the nonnegative matrix theory. Due to the study of higher-order tensors, the PF theorem was extended to the largest H-eigenvalue of a nonnegative irreducible tensor first by Lim [9], and later the result was strengthened by Chang, Pearson, and Zhang [2], which is quite close to its matrix counterpart. Friedland, Gaubert, and Han extended the PF theorem to nonnegative multi-linear forms [5]. Chang, Pearson, and Zhang generalized the PF theorem to the Z-eigenvalue setting [4]. Besides, a variety of research was focused on studying properties of nonnegative tensors, see e.g., [3, 6, 15].

A tubal scalar is a usual vector, a tubal vector is a vector whose every element is a tubal scalar, and a tubal matrix is a matrix whose every element is also a tubal scalar. Such

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a matrix is often recognized as a third-order tensor. In [1,7,8], Kilmer, Martin, Braman, and their coauthors introduced the t-product between third-order tensors (tubal matrices), enabling one to study the properties of third-order tensors, such as transpose, inverse, orthogonality, range and kernel spaces, and SVD, as their matrix counterpart. Further properties are studied then. For example, Miao, Qi, and Wei introduced the t-eigenvalue, t-Jordan form, and t-Drazin inverse [13]. The tensor t-functions were studied in [7,12]. The t-positive (semi)definiteness tensors and related properties were investigated by Zheng, Huang, and Wang [16]. The t-eigenvalue and t-eigenvector were studied by Liu and Jin [11]. The SVD was deeply investigated for tubal matrices by Qi and Luo [14], where the definition of tubal matrices there is more general than the current setting.

In this paper, we still use the notion of a tubal matrix to represent a third-order tensor for convenience. We first define nonnegative/positive/strongly positive tubal scalars/vectors/matrices. The meaning of some of these notions is different from the usual sense. For instance, a tubal matrix with one frontal slice being a positive matrix while the other frontal slices being zero matrices is called positive in the current definition, while such a tubal matrix is only recognized as a nonnegative third-order tensor before. Using the t-product, we derive several properties of nonnegative tubal matrices. In particular, the concept of reducibility/irreducibility is extended to the tubal matrix case. In our definition, even a tubal matrix is irreducible, its block curculant representation can be reducible. Such a definition is also different from that in [2] for tensors. As will be shown in Proposition 3.8 and Remark 3.5, in the third-order tensor case, our definition of irreducibility covers a wider range of tensors than that of [2]. Similar to the matrix case, equivalent characterizations of the reducibility/irreducibility based on permutations or tensor powers are provided. We then study the PF theorem for nonnegative irreducible tubal matrices. Some conclusions of the matrix PF theorem can be generalized, while some are not. For those that can not be extended, we prove weaker results. We also prove that, if the nonnegative irreducible tubal matrix contains a strongly positive tubal scalar, then most conclusions of the matrix PF theorem hold.

The rest is organized as follows. Sect. 2 introduce preliminaries concerning the t-product. Nonnegative tubal scalars/vectors/matrices and irreducible tubal matrices are defined in Sect. 3, with properties given. Sect. 4 studies the PF theorem for nonnegative irreducible tubal matrices.

2 Preliminaries

The notations and definitions throughout this paper mainly follow those of [1,7,8].

The tensors considered are in the field $\mathbb{R}^{n\times n\times p}$. We call such a type of tensors square, namely, its first and second modes have equal dimension. The $i$-th frontal slice of $\mathbf{A} \in \mathbb{R}^{n\times n\times p}$ is denoted as $\mathbf{A}^{(i)} \in \mathbb{R}^{n\times n\times 1}$, the $j$-th horizontal slice is denoted as $\mathbf{A}_{j,} \in \mathbb{R}^{1\times n\times p}$, and the $k$-th lateral slice is denoted as $\mathbf{A}_{,k} \in \mathbb{R}^{n\times 1\times p}$. $\mathbf{a}_{j,k} \in \mathbb{R}^{1\times 1\times p}$ represents the $j,k$-th tube of $\mathbf{A}$ and $\mathbf{a}^{(i)}_{j,k}$ is the $i$-th entry (frontal slice) of $\mathbf{a}_{j,k}$. In Matlab, the above notations are respectively given as $\mathbf{A}^{(i)} = \mathbf{A}(;,:,i), \mathbf{A}_{j,} = \mathbf{A}(j,:,;), \mathbf{A}_{,k} = \mathbf{A}(;,:,;), \text{ and } \mathbf{a}_{j,k} = \mathbf{A}(j,k,:)$.

An element $\mathbf{a} \in \mathbb{R}^{1\times 1\times p}$ is recognized as a tubal scalar of length $p$. The space of all such tubal scalars is denoted as $\mathbb{K}_p$. Similarly, an element $\mathbf{A} = (\mathbf{a}_j) \in \mathbb{R}^{n\times 1\times p}$ is called a tubal
vector consisting of \( n \) tubal scalars. The space of all such tubal vectors is denoted as \( K^p_n \).

Likewise, an element \( A = (a_{j,k}) \in \mathbb{R}^{n \times n \times p} \) can be seen as a tubal matrix consisting of \( n \) tubal vectors. We denote by \( K^{n \times n \times p} = \mathbb{R}^{n \times n \times p} \). For tubal scalars, vectors, and matrices, the superscript \( \cdot^{(i)} \) means the \( i \)-th frontal slice of this scalar, vector, or matrix, respectively.

Let \( a \in K^p_n \). Define

\[
\text{circ} (a) := \begin{bmatrix}
a^{(1)} & a^{(p)} & a^{(p-1)} & \cdots & a^{(2)} \\
a^{(2)} & a^{(1)} & a^{(p)} & \cdots & a^{(3)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a^{(p)} & a^{(p-1)} & \cdots & a^{(2)} & a^{(1)} \\
\end{bmatrix}.
\] (2.1)

Given \( a, b \in K^p_n \), the multiplication \( \ast \) between \( a \) and \( b \) is defined as [1]

\[
a \ast b := \text{circ} (a) \, b = \begin{bmatrix}
a^{(1)} & a^{(p)} & a^{(p-1)} & \cdots & a^{(2)} \\
a^{(2)} & a^{(1)} & a^{(p)} & \cdots & a^{(3)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a^{(p)} & a^{(p-1)} & \cdots & a^{(2)} & a^{(1)} \\
\end{bmatrix} \begin{bmatrix} b^{(1)} \\
b^{(2)} \\
\vdots \\
b^{(p)} \end{bmatrix}.
\]

It follows from [1] that \( (K^p_n, +, \ast) \) is a commutative ring with unity, where + is the usual addition. The unity is \( e \), the tubal scalar whose first entry is 1 while the other ones are zero. For any \( a \in K^p_n \), \( e \ast a = a \). Let \( 0 \) denote the zero tubal scalar, i.e., every entry of \( 0 \) is zero. Then for any \( a \in K^p_n \), \( 0 \ast a = 0 \).

Let \( A \in K^{n \times n}_p \). Define the following notations:

\[
\text{bcirc} (A) := \begin{bmatrix}
A^{(1)} & A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} \\
A^{(2)} & A^{(1)} & A^{(p)} & \cdots & A^{(3)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} & A^{(1)} \\
\end{bmatrix}, \text{unfold} (A) = \begin{bmatrix} A^{(1)} \\
A^{(2)} \\
\vdots \\
A^{(p)} \end{bmatrix}.
\] (2.2)

The fold operator satisfies

\[
\text{fold} (\text{unfold} (A)) = A.
\]

Given \( A, B \in K^{n \times n}_p \), the t-product \( \ast \) between \( A \) and \( B \) is defined as [8]

\[
A \ast B := \text{fold} (\text{bcirc} (A) \, \text{unfold} (B)).
\]

The t-product can be implemented fast by using the fast Fourier transform [8].

Similar to the usual product between matrices, the \( j \)-th horizontal slice of \( A \ast B \) is equal to the t-product between the \( j \)-th horizontal slice of \( A \) and \( B \):

\[
(A \ast B)_{j,} = A_{j,} \ast B.
\]

Correspondingly, for \( A \in \mathbb{R}^{1 \times n \times p} \) and \( B \in \mathbb{R}^{n \times 1 \times p} \), which are respectively represented as \( A = [a_1, \ldots, a_n] \) and

\[
B = \begin{bmatrix} b_1 \\
\vdots \\
b_n \end{bmatrix},
\]

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where \( a_i, b_i \in \mathbb{K}_p \), there holds

\[
A \ast B = \sum_{i=1}^{n} a_i \ast b_i.
\]

For \( A_1, A_2 \in \mathbb{K}_p^{n \times n} \), if they are partitioned as

\[
A_1 = \begin{bmatrix}
B & C \\
D & E
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
F & G \\
H & I
\end{bmatrix},
\]

where \( B, F \in \mathbb{K}_p^{n_1 \times n_1}, E, I \in \mathbb{K}_p^{n_2 \times n_2}, C, G \in \mathbb{K}_p^{n_1 \times n_2}, \) and \( D, H \in \mathbb{K}_p^{n_2 \times n_1} \), with \( n_1 + n_2 = n \), then [12]

\[
A_1 \ast A_2 = \begin{bmatrix}
B \ast F + C \ast H & B \ast G + C \ast I \\
D \ast F + E \ast H & D \ast G + E \ast I
\end{bmatrix}.
\]

In particular, for \( A = (a_{i,j}) \in \mathbb{K}_p^{n \times n} \) and \( X = (x_i) \in \mathbb{K}_p^n \),

\[
(A \ast X)_i = \sum_{j=1}^n a_{i,j} \ast x_j.
\]

The transpose of \( a \in \mathbb{K}_p \) (\( A \in \mathbb{K}_p^{n \times n} \), or \( A \in \mathbb{K}_p^{n \times n} \)) is given by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through \( p \), namely,

\[
a^\top := \text{fold} \left( \begin{bmatrix}
a^{(1)} \\
a^{(p)} \\
\vdots \\
a^{(2)}
\end{bmatrix} \right), A^\top := \text{fold} \left( \begin{bmatrix}
(A^{(1)})^\top \\
(A^{(p)})^\top \\
\vdots \\
(A^{(2)})^\top
\end{bmatrix} \right), \quad A^\top := \text{fold} \left( \begin{bmatrix}
(A^{(1)})^\top \\
(A^{(p)})^\top \\
\vdots \\
(A^{(2)})^\top
\end{bmatrix} \right).
\]

They are respectively of sizer \( 1 \times 1 \times p, 1 \times n \times p, \) and \( n \times n \times p \). There holds the relation:

\[
\text{bcirc} (A^\top) = \text{bcirc} (A)^\top.
\] (2.3)

If \( A, B \in \mathbb{K}_p^{n \times n} \), then \( (A \ast B)^\top = B^\top \ast A^\top \).

The identity tensor (tubal matrix) in \( \mathbb{K}_p^{n \times n} \) is the tensor whose first frontal slice is the identity matrix, and whose other frontal slices are all zeros. For any \( A \in \mathbb{K}_p^{n \times n}, I_{nnp} \ast A = A \ast I_{nnp} = A \).

The permutation tensor (tubal matrix) in our context is slightly different from [8]. We call \( P \in \mathbb{K}_p^{n \times n} \) a permutation tensor, if its first frontal slice is a permutation matrix, and other frontal matrices are all zeros. For any \( A \in \mathbb{K}_p^{n \times n}, I_{nnp} \ast A = A \ast I_{nnp} = A \).

The permutation tensor (tubal matrix) in our context is slightly different from [8]. We call \( P \in \mathbb{K}_p^{n \times n} \) a permutation tensor, if its first frontal slice is a permutation matrix, and other frontal matrices are all zeros. For any \( A \in \mathbb{K}_p^{n \times n} \), it holds that

\[
P \ast A \ast P^\top := \text{fold} \left( \begin{bmatrix}
P^{(1)} A^{(1)} (P^{(1)})^\top \\
P^{(1)} A^{(2)} (P^{(1)})^\top \\
\vdots \\
P^{(1)} A^{(p)} (P^{(1)})^\top
\end{bmatrix} \right).
\] (2.4)

**Proposition 2.1.** Let \( P \in \mathbb{K}_p^{n \times n} \) be a permutation tubal matrix. Then \( P^\top \ast P = P \ast P^\top = I_{nnp} \).
The following proposition shows that the permutation tubal matrix acting on a tubal vector performs similar to that in the matrix case.

**Proposition 2.2.** Let $X = (x_i) \in \mathbb{K}^{n \times p}$ with some $x_i = 0$. Then there exists a permutation tubal matrix $P$, such that $P \cdot X$ takes the form \[
\begin{bmatrix}
X_1 \\
0 
\end{bmatrix},
\] where 0 in this tubal vector denotes a zero tubal vector.

**Proof.** The assumption shows that every frontal slice $X^{(i)} \in \mathbb{R}^n$ of $X$ contains zero entries, some of which appear at the same positions. Then there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$, such that $PX^{(i)} = \begin{bmatrix} X^{(i)}_1 \\
0 
\end{bmatrix} \in \mathbb{R}^n$, $i = 1, \ldots, p$, where 0 here means a zero vector.

Now, define $P$ such that its first frontal slice is $P$, and other frontal slices are zero matrices. The required result follows immediately. 

3 Nonnegative Tubal Scalars, Vectors, and Matrices

The modulus of $a \in \mathbb{C}^{1 \times 1 \times p}$ is defined as $\|a\| = \sqrt{\sum_{i=1}^{p} |a^{(i)}|^2}$. We call $a = 0$ if $\|a\| = 0$.

In the sequel, the meaning of 0 depends on the context: it can be a tubal scalar, vector, or matrix, every entry of which is zero. Its size also depends on the context.

**Definition 3.1** (Nonnegative tubal scalars). We call a tubal scalar $a \in \mathbb{K}_p$ nonnegative, if every entry $a^{(i)}$ are nonnegative; we call it positive, if its modulus is not zero; we call it strongly positive, if every entry $a^{(k)} > 0$.

The set of nonnegative/positive/strongly positive tubal scalars are respectively denoted as $\mathbb{K}_p^+, \mathbb{K}_p^{++}$, and $\mathbb{K}_p^{+++}$.

**Remark 3.1.** Note that $\mathbb{K}_p^+ = \mathbb{K}_p^{++} \cup \{0\}$. There holds the relation

$\mathbb{K}_p^{+++} \subset \mathbb{K}_p^{++} \subset \mathbb{K}_p^+ \subset \mathbb{K}_p$.

**Proposition 3.1.** Let $a, b \in \mathbb{K}_p$. If $a, b \in \mathbb{K}_p^+$, then $a \ast b \in \mathbb{K}_p^+$; if $a, b \in \mathbb{K}_p^{++}$, then $a \ast b \in \mathbb{K}_p^{++}$; if one of $a$ or $b$ is strongly positive, while the other one is positive, then $a \ast b \in \mathbb{K}_p^{+++}$.

**Proof.** The results follow immediately from the definition of $a \ast b$ and Definition 3.1. 

Proposition 3.1 and Remark 3.1 immediately give that:

**Proposition 3.2.** Let $a \in \mathbb{K}_p^n$ and $b \in \mathbb{K}_p^{n \times p}$. Then $a \ast b = 0$ if and only if $a = 0$.

**Definition 3.2** (Nonnegative tubal vectors). We call a tubal vector $A \in \mathbb{K}_p^n$ nonnegative/positive/strongly positive, if every element of $A$ is nonnegative/positive/strongly positive.

The set of nonnegative/positive/strongly positive tubal vectors are respectively denoted as $\mathbb{K}_p^n, \mathbb{K}_p^{n \times p}$, and $\mathbb{K}_p^{n \times p}$.

From Proposition 3.1 we have the following observation:
Remark 3.3. It follows from the definition of the transpose that
\[ (X^\top)^\top = Y \quad \text{for every } X, Y. \]

Proposition 3.6. Let \( A \in K_p^n \). Then \( A^\top \) is nonnegative/positive/strongly positive, if every element of \( A \) is nonnegative/positive/strongly positive.

Definition 3.3 (Nonnegative tubal matrices). We call a tubal matrix \( A \in K_p^n \) nonnegative/positive/strongly positive, if every element of \( A \) is nonnegative/positive/strongly positive.

The set of nonnegative/positive/strongly positive tubal matrices are respectively denoted as \( K_p^n \), \( K_p^+ \), and \( K_p^{++} \).

Remark 3.2. Under Definition 3.3, even if \( A \) is a positive tubal matrix, it can be highly sparse; for example, consider \( A = (a_{i,j}) \) where every \( a_{i,j} \) containign only one positive \( a_{i,j}^{(k)} \).

It is easy to see that:

Proposition 3.4. Let \( A \in K_p^np \) and \( X \in K_p^n \). Then \( A \ast X \in K_p^n \).

Proposition 3.5. Let \( A \in K_p^np \) and \( X \in K_p^np \). Then \( A \ast X = 0 \) if and only if \( A = 0 \).

Proof. This follows from Proposition 3.2.

The following result is similar to its matrix counterpart.

Proposition 3.6. Let \( A \in K_p^np \). Then \( A = (a_{i,j}) \in K_p^np \) if and only if for every \( X = (x_i) \in K_p^n \), \( X \neq 0 \), \( A \ast X \in K_p^np \).

Proof. If \( A \in K_p^np \) and \( X \in K_p^n \), \( X \neq 0 \), then \( a_{i,j} \in K_p^+ \) for every \( i,j \), and there exists at least an \( x_j \in K_p^+ \). So, \( (A \ast X)_i = A_{i,j} \ast x_j = \sum_{j=1}^n a_{i,j} \ast x_j \in K_p^+ \), where the last relation follows from Proposition 3.1.

If for every \( X = (x_j) \in K_p^n \), \( X \neq 0 \), \( A \ast X \in K_p^np \), while \( A \notin K_p^np \), assume without loss of generality that \( a_{1,1} = 0 \). Take \( X = (x_j) \) with \( x_1 \in K_p^+ \) and \( x_j = 0 \), \( j = 2, \ldots, n \). Then \( A_1 \ast X = \sum_{j=1}^n a_{i,j} \ast x_j = 0 \), deducing a contradiction.

Remark 3.3. It follows from the definition of the transpose that \( a \in K_p^+(K_p^+, K_p^{++}, \text{or } K_p^{+++}) \), then so is \( a^\top \). The same observations hold for the tubal vector \( A^\top \) and the tubal matrix \( A^\top \).

We then define irreducible tubal matrices. Let \([n] := \{1, \ldots, n\}\).

Definition 3.4 (Reducibility and irreducibility). We call a tubal matrix \( A = (a_{i,j}) \in K_p^{n \times n} \) reducible, if there is a nonempty proper index subset \( I \subset [n] \) such that
\[ a_{i,j} = 0, \quad \forall i \in I, \forall j \notin I. \]

If \( A \) is not reducible, then we call \( A \) irreducible.

Remark 3.4. When \( p = 1 \), the above definition boils down exactly to the reducibility/irreducibility of a matrix.
We provide some examples.

**Example 3.1.** Let $A \in \mathbb{K}^{n \times n}$ with $A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^{(2)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. It is clear that $a_{1,2}, a_{2,1} \in \mathbb{K}_{p+}$, and so $A$ is irreducible.

**Example 3.2.** Let $A \in \mathbb{K}^{n \times n}$ with $A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Let $I = \{2\}$. Then $a_{i,j} = 0$, $\forall i \in I$, $\forall j \not\in I$, and so $A$ is reducible.

Restricting to third-order tensors, the previous reducibility/irreducibility was defined as follows:

**Definition 3.5.** [2] Let $A \in \mathbb{R}^{n \times n \times n}$. $A$ is called reducible if there is a nonempty proper index subset $I \subset [n]$ such that

$$A_{ijk} = 0, \forall i \in I, \forall j, k \not\in I.$$

If $A$ is not reducible, then $A$ is called irreducible.

It follows from the definition of $A^\top$ that:

**Proposition 3.7.** If $A \in \mathbb{K}^{n \times n}$ is reducible/irreducible, then so is $A^\top$.

We have the following relation between the two definitions of irreducibility.

**Proposition 3.8.** Let $A \in \mathbb{K}^{n \times n}$. If it is irreducible in the sense of Definition 3.5, then it is also irreducible in the sense of our Definition 3.4.

**Proof.** It suffices to show that if $A$ is reducible in the sense of our Definition 3.4, then it is reducible in the sense of Definition 3.5.

Write $A = (A_{ijk}) = (a_{i,j})$, where $A_{ijk} \in \mathbb{R}$ and $a_{i,j} \in \mathbb{K}_p$. Assume that there is $I \subset [n]$ such that $a_{i,j} = 0$, $\forall i \in I, \forall j \not\in I$. This means that

$$A_{ijk} = a^{(k)}_{i,j} = 0, \forall i \in I, \forall j \not\in I, \forall k \in [n],$$

which is clearly reducible in the sense of Definition 3.5.

**Remark 3.5.** Example 3.1 provides an example which is irreducible in the sense of Definition 3.4 while is reducible in the sense of Definition 3.5. In fact, we see that $A_{122} = a_{1,2}^{(2)} = 0$ in this example, showing its reducibility in the sense of Definition 3.5.

Therefore, our definition of irreducibility covers a wider range of tubal matrices than that of [2].

Similar to the matrix setting, we have the following equivalent characterizations of non-negative irreducibility.

**Theorem 3.1.** Let $A \in \mathbb{K}^{n \times n}$. Then $A$ is reducible if and only if
• Every frontal slice $A^{(i)}$ is a reducible matrix; more specifically, there exists a permutation matrix $P$, such that

$$PA^{(i)}P^T = \begin{bmatrix} B^{(i)} & C^{(i)} \\ 0 & D^{(i)} \end{bmatrix}, \ i = 1, \ldots, p,$$

where the bottom left is a zero matrix, and $B^{(i)}$ and $D^{(i)}$ are square matrices.

• There exists a permutation tensor $P$, such that

$$P^*A^*P^\top = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where the right-hand side means that $P^*A^*P^\top$ takes a block structure, whose bottom left is a zero tensor; $B$ and $D$ are square tensors.

Proof. The first claim easily follows from Definition 3.4, while the second one holds by taking $P$ with $P^{(1)} = P$ and the other frontal slices being zero and noticing (2.4).

Theorem 3.2. Let $A \in \mathbb{K}^{n \times n}_p$. Then $A$ is nonnegative irreducible if and only if

$$(A + I_{nnp})^{n-1} = \underbrace{(A + I_{nnp}) \ast \cdots \ast (A + I_{nnp})}_{n-1 \text{ times}} \in \mathbb{K}^{n \times n}_{p^{++}}.$$

Proof. Necessity: According to Proposition 3.6, it suffices to show that for any $X \in \mathbb{K}^{n \times n}_{p^{++}}$, $X \neq 0$, it holds that

$$(A + I_{nnp})^{n-1} \ast X \in \mathbb{K}^{n \times n}_{p^{++}}.$$

For any $X \in \mathbb{K}^{n \times n}_{p^{++}}$, $X \neq 0$, denote $X_0 := X$. Since the associative law holds for the t-product ( [8, Lemma 3.3]), It then follows that we can recursively define

$$X_{k+1} := (A + I_{nnp}) \ast X_k = (A + I_{nnp})^{k+1} X_0, \ k = 0, 1, \ldots.$$

Proposition 3.4 shows that all the $X_k$ are in $\mathbb{K}^{n \times n}_{p^{++}}$. Represent $X_k$ as $X_k = (x_{k,i})$, $i = 1, \ldots, n$. Denote $M_k$ as the number of zero tubal scalars of $X_k$, i.e., $M_k := \text{card} \{i \mid x_{k,i} = 0\}$. Since $X_{k+1} = A \ast X_k + I_{nnp} \ast X_k = A \ast X_k + X_k$, by Proposition 3.4, there holds

$$M_{k+1} \leq M_k \leq \cdots \leq M_0 \leq n - 1,$$

where the last relation follows from $X_0 \neq 0$. We then wish to show that if $M_k > 0$, then $M_{k+1} < M_k$.

Otherwise, suppose that there exists a $k$ such that $M_{k+1} = M_k$; from the construction of $X_{k+1}$ and by Proposition 2.2, there exists a permutation tubal matrix $P \in \mathbb{K}_{p^{++}}^{n \times n}$, such that

$$P \ast X_{k+1} = \begin{bmatrix} A \\ 0 \end{bmatrix}, \ P \ast X_k = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

where $A, B \in \mathbb{K}_{p^{++}}^{n-M_k}$, and $0 \in \mathbb{K}_{p^{++}}^{M_k}$. Since

$$P \ast X_{k+1} = P \ast A \ast X_k + P \ast X_k = P \ast A \ast P^\top \ast P \ast X_k + P \ast X_k,$$
where the last relation uses Proposition 2.1, it follows that
\[
\begin{bmatrix}
A
\end{bmatrix} = \begin{bmatrix}
B & C
\end{bmatrix} \ast \begin{bmatrix}
B
\end{bmatrix} + \begin{bmatrix}
B
\end{bmatrix} \ast \begin{bmatrix}
0
\end{bmatrix},
\]
where we denote
\[
P \ast A \ast P^\top = \begin{bmatrix}
B & C
\end{bmatrix},
\]
where the block tensors \(B, C, D, E\) are partitioned according to the size of \(B\) and \(0\). In particular, \(B, C, D, E\) are square tubal matrix. From the orthogonality of \(P\), we then have
\[
P \ast (A + I_{nnp}) \ast P^\top = I_{nnp} + \begin{bmatrix}
B & C
\end{bmatrix} = \begin{bmatrix}
B + I_1 & C
\end{bmatrix} = \begin{bmatrix}
0 & 0
\end{bmatrix}
\]
where \(I_1\) and \(I_2\) are identity tubal matrices of proper size. Then for any \(k\),
\[
P \ast (A + I_{nnp})^k \ast P^\top = (P \ast (A + I_{nnp}) \ast P^\top)^k = \begin{bmatrix}
B + I_1 & C
\end{bmatrix}^k = \begin{bmatrix}
D + I_2 & C
\end{bmatrix}^k.
\]
Note that for any \(k\), the right-hand side is always an upper triangular tubal matrix, which means that \((A + I_{nnp})^k\) always contains zero tubal scalar(s), which contradicts the condition that \((A + I_{nnp})^{n-1} \in \mathbb{K}_{pp}^{n \times n}\).

As a consequence, \(M_{k+1} < M_k\), for any \(k\) with \(M_k > 0\). Therefore, by the definition of \(M_k\), \(M_{n-1} = 0\), and so \(X_{n-1} = (A + I_{nnp})^{n-1} \ast X \in \mathbb{K}_{pp}^{n \times n}\).

Sufficiency: Suppose on the contrary that \(A\) is reducible. By Theorem 3.1, there exists a permutation tubal matrix \(P \in \mathbb{K}_{pp}^{n \times n}\), such that
\[
P \ast A \ast P^\top = \begin{bmatrix}
B & C
\end{bmatrix},
\]
where \(B\) and \(D\) are square tubal matrix. From the orthogonality of \(P\), we then have
\[
P \ast (A + I_{nnp}) \ast P^\top = I_{nnp} + \begin{bmatrix}
B & C
\end{bmatrix} = \begin{bmatrix}
B + I_1 & C
\end{bmatrix} = \begin{bmatrix}
0 & 0
\end{bmatrix}
\]
where \(I_1\) and \(I_2\) are identity tubal matrices of proper size. Then for any \(k\),
\[
P \ast (A + I_{nnp})^k \ast P^\top = (P \ast (A + I_{nnp}) \ast P^\top)^k = \begin{bmatrix}
B + I_1 & C
\end{bmatrix}^k = \begin{bmatrix}
D + I_2 & C
\end{bmatrix}^k.
\]
Note that for any \(k\), the right-hand side is always an upper triangular tubal matrix, which means that \((A + I_{nnp})^k\) always contains zero tubal scalar(s), which contradicts the condition that \((A + I_{nnp})^{n-1} \in \mathbb{K}_{pp}^{n \times n}\).

We present two examples to illustrate the above results.

**Example 3.3.** Consider \(A = (a_{ij}) \in \mathbb{K}_{3 \times 3}^{3 \times 3}\) with
\[
A^{(1)} = \begin{bmatrix}
0 & 1 & 0
0 & 0 & 0
0 & 0 & 0
\end{bmatrix},\ A^{(2)} = \begin{bmatrix}
0 & 0 & 0
0 & 1 & 0
0 & 0 & 0
\end{bmatrix},\ A^{(3)} = \begin{bmatrix}
0 & 0 & 0
0 & 0 & 0
1 & 0 & 0
\end{bmatrix};
\]
that is, \(a_{1,2} = \text{fold} \left(\begin{bmatrix}0 & 1
\end{bmatrix}\right), a_{2,3} = \text{fold} \left(\begin{bmatrix}0 & 1
\end{bmatrix}\right), a_{3,1} = \text{fold} \left(\begin{bmatrix}0 & 0
\end{bmatrix}\right)\), and \(a_{i,j} = 0\) for other pairs of \((i,j)\). Let \(B = A + I_{33}\). Then \(b_{1,1} = b_{2,2} = b_{3,3} = \text{fold} \left(\begin{bmatrix}1 & 0
\end{bmatrix}\right)\) while \(b_{i,j} = a_{i,j}\) for other pairs of \((i,j)\). Let \(C = B^2\). After computation we get
\[
C^{(1)} = \begin{bmatrix}
1 & 2 & 0
1 & 0 & 0
0 & 1 & 1
\end{bmatrix},\ C^{(2)} = \begin{bmatrix}
0 & 0 & 1
0 & 0 & 2
0 & 0 & 0
\end{bmatrix},\ C^{(3)} = \begin{bmatrix}
0 & 0 & 0
0 & 0 & 0
2 & 0 & 0
\end{bmatrix},
\]
namely, \(C \in \mathbb{K}_{pp}^{n \times n}\). Thus \(A\) is irreducible.
Example 3.4. Consider $\mathcal{A} = (a_{i,j}) \in \mathbb{K}_{333}^{3\times3}$ with
\[
\begin{align*}
\mathcal{A}^{(1)} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{A}^{(2)} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{A}^{(3)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};
\end{align*}
\]
that is, $a_{1,2} = \text{fold}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$, $a_{2,3} = \text{fold}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$, and $a_{i,j} = 0$ for other pairs of $(i,j)$. Let $\mathcal{B} = \mathcal{A} + \mathbb{I}_{333}$. Then $b_{1,1} = b_{2,2} = b_{3,3} = \text{fold}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$ while $b_{i,j} = a_{i,j}$ for other pairs of $(i,j)$. Let $\mathcal{C} = \mathcal{B}^2$. Then
\[
\begin{align*}
\mathcal{C}^{(1)} &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\mathcal{C}^{(2)} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{C}^{(3)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};
\end{align*}
\]
from which we see that $c_{2,1} = c_{3,1} = c_{3,2} = 0$, namely, $\mathcal{C} \in \mathbb{K}_{p^+ \times p^+}^{n \times n} \setminus \mathbb{K}_{p \times p}^{n \times n}$. Thus $\mathcal{A}$ is reducible.

Remark 3.6. Note that the irreducibility of $\mathcal{A}$ does not mean that $\text{bcirc}(\mathcal{A})$ is an irreducible matrix; an example is given in Example 4.1. This leads to that some nice properties of nonnegative irreducible matrices do not hold for nonnegative irreducible tubal matrices. This will be studied in the next section.

4 Perron-Frobenius Theorem for Nonnegative Tubal Matrices

The t-eigenvalues and t-eigenvectors were defined as follows.

Definition 4.1. (c.f. [11, 13]) Let $\mathcal{A} \in \mathbb{K}_{p \times p}^{n \times n}$. If there exists a scalar $\lambda \in \mathbb{C}$ and a tubal vector $X \in \mathbb{C}^{n \times 1 \times p}$ such that
\[
\mathcal{A} * X = \lambda X,
\]
then $\lambda$ is called a t-eigenvalue of $\mathcal{A}$, with its associated t-eigenvector. The set of t-eigenvalues of $\mathcal{A}$ is denoted as $\text{spec}(\mathcal{A})$, and the spectral radius is denoted as $\rho(\mathcal{A}) := \max\{|\lambda| : \lambda \in \text{spec}(\mathcal{A})\}$.

Remark 4.1. From the definition of t-product, $\mathcal{A} * X = \lambda X$ if and only if $\text{bcirc}(\mathcal{A}) \text{unfold}(X) = \lambda \text{unfold}(X)$. Thus $\text{spec}(\mathcal{A}) = \text{spec}(\text{bcirc}(\mathcal{A}))$ and $\rho(\mathcal{A}) = \rho(\text{bcirc}(\mathcal{A}))$.

Similar to the matrix case, in this paper we call $X$ in Definition 4.1 the right t-eigenvector of $\mathcal{A}$, and we can define left t-eigenvectors.

Definition 4.2. If there is a scalar $\lambda \in \mathbb{C}$ and a tubal vector $Y \in \mathbb{C}^{n \times 1 \times p}$ such that
\[
\mathcal{A}^\top * Y = \lambda Y,
\]
then $Y$ is called a right t-eigenvector of $\mathcal{A}$.

Remark 4.2. Using (2.3), $\text{unfold}(Y)$ is a left eigenvector of $\text{bcirc}(\mathcal{A})^\top$. Thus $\text{spec}(\mathcal{A}^\top) = \text{spec}(\mathcal{A})$, and $\rho(\mathcal{A}^\top) = \rho(\mathcal{A})$. 

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When context permits, we use the t-eigenvector to mean a right t-eigenvector.

**Theorem 4.1** (Perron-Frobenius theorem for nonnegative tubal matrices). Let \( A \in \mathbb{K}^n_{p+} \), \( A \neq 0 \) be a nonnegative tubal matrix. Then \( \rho(A) \) is a t-eigenvalue of \( A \), with a nonnegative t-eigenvector \( X \in \mathbb{K}^n_{p+}, X \neq 0 \) corresponding to it.

**Proof.** Since \( A \in \mathbb{K}^{n \times n}_{p+} \), \( \text{bcirc}(A) \in \mathbb{R}^{n^2 \times np} \) is also a nonnegative matrix, By the Perron-Frobenius theorem for nonnegative matrices, \( \rho(\text{bcirc}(A)) \) is an eigenvalue of \( \text{bcirc}(A) \), with a nonnegative eigenvector \( x \in \mathbb{R}^{np}, x \neq 0 \) corresponding to it. Noticing Remark 4.1 and folding \( x \) to a tubal vector \( X \in \mathbb{K}^n_{p+} \) give the desired result. \( \square \)

**Remark 4.3.** Theorem 4.1 also holds for the right t-eigenvectors of \( A \).

Next we consider PF theorem for nonnegative irreducible tubal matrices. Some preparations are needed.

**Lemma 4.1.** Let \( A \in \mathbb{K}^{n \times n}_{p+}, A \neq 0 \) admit a positive left t-eigenvector \( Y \in \mathbb{K}^n_{p++} \) corresponding to \( \rho(A) \). If \( X \in \mathbb{K}^n_{p+}, X \neq 0 \) satisfies \( A^*X - \rho(A)X \in \mathbb{K}^n_{p+} \), then \( A^*X = \rho(A)X \), i.e., \( X \) is a t-eigenvector corresponding to \( \rho(A) \).

**Proof.** Since \( A^*X - \rho(A)X \in \mathbb{K}^n_{p+} \), if it is identically zero, then we are done. Otherwise, since \( Y \in \mathbb{K}^n_{p++} \), by Proposition 3.3 we have \( Y^\top * (A^*X - \rho(A)X) \in \mathbb{K}^n_{p++} \). However, as \( Y^\top * A = \rho(A)Y^\top \), we have \( Y^\top * (A^*X - \rho(A)X) = 0 \), deducing a contradiction. Thus the assertion holds. \( \square \)

For \( A \in \mathbb{C}^{n \times n \times p} \), denote \( |A| \in \mathbb{K}^{n \times n \times p}_{p+} \) as the tubal matrix whose every element is the magnitude of the corresponding element of \( A \). The same notation applies also to tubal vectors and scalars, and the usual matrices, vectors, and scalars. The following lemma is useful.

**Lemma 4.2.** If \( A \in \mathbb{C}^{n \times n \times p} \) and \( X \in \mathbb{C}^{n \times 1 \times p} \), then \( |A| * |X| - |A * X| \in \mathbb{K}^{n \times p}_{p+} \).

**Proof.** By noting the relation between \( A * X \) and \( \text{bcirc}(A) \text{unfold}(X) \), and realizing that \( |\text{bcirc}(A) \text{unfold}(X)| \leq |\text{bcirc}(A)| |\text{unfold}(X)| \), the result follows. \( \square \)

**Theorem 4.2** (Perron-Frobenius theorem for nonnegative irreducible tubal matrices). Let \( A \in \mathbb{K}^{n \times n}_{p+} \) be a nonnegative irreducible tubal matrix. Then

1. \( \rho(A) \) is a t-eigenvalue of \( A \);
2. \( \rho(A) > 0 \);
3. Every nonnegative t-eigenvector \( X \in \mathbb{K}^n_{p+}, X \neq 0 \), must be a positive t-eigenvector, i.e., \( X \in \mathbb{K}^n_{p++} \);
4. There exists a positive t-eigenvector \( X \in \mathbb{K}^n_{p++} \) corresponding to \( \rho(A) \);
5. All the t-eigenvectors corresponding to \( \lambda \) with \( |\lambda| = \rho(A) \) do not contain zero tubal scalars;
6. All the real t-eigenvectors corresponding to $\rho(A)$ cannot take the following form: Some of its tubal elements are positive, while the other ones are not;

7. if $\lambda$ is a t-eigenvalue with a strongly positive t-eigenvector $Y \in K_{p++}^n$, then $\lambda = \rho(A)$.

All the above results hold for the left t-eigenvectors.

Proof. Item 1 follows from Theorem 4.1.

Let $A*X = \rho(A)X$ with $X \in K_{p+}^n$, $X \neq 0$ the corresponding nonnegative t-eigenvector. To show $\rho(A) > 0$, it suffices to show that $A*X \neq 0$. Represent $A = (a_{i,j})$, $X = (x_i)$, $a_{i,j}, x_i \in K_{p+}^n$. Let $I := \{i \mid x_i = 0\}$, and $\bar{I} = [n] \setminus I$. If $A*X = 0$, then

$$\sum_{j=1}^n a_{i,j} \cdot x_j = 0, \forall i \in [n],$$

which together with the nonnegativity of $a_{i,j}$ and $x_j$ gives that

$$a_{i,j} \cdot x_j = 0, \forall i \in [n], \forall j \in \bar{I},$$

In particular, when $j \in \bar{I}$, $x_j \in K_{p++}^n$, which by Proposition 3.2 implies that $a_{i,j} = 0, \forall i \in [n], \forall j \in \bar{I}$, which contradicts the irreducibility of $A$. Thus $A*X \neq 0$, and so $\rho(A) > 0$. This proves Item 2.

Let $X \in K_{p+}^n, X \neq 0$ be a nonnegative t-eigenvector corresponding to a t-eigenvalue $\lambda$. Similar to the proof of Item 2 we have $\lambda > 0$. If $X \not\in K_{p++}^n$, then let $I$ and $\bar{I}$ be defined as those in Item 2. Then

$$0 = \lambda x_i = \sum_{j=1}^n a_{i,j} \cdot x_j, \forall i \in I \iff 0 = a_{i,j} \cdot x_j, \forall i \in I.$$

Since when $j \in \bar{I}$, $x_j \in K_{p++}^n$, it follows again from Proposition 3.2 that

$$a_{i,j} = 0, \forall i \in I, \forall j \in \bar{I},$$

deducing a contradiction. Thus $I$ is empty and so $X \in K_{p++}^n$. This proves Item 3.

Item 4 follows from Theorem 4.1 and Item 3.

To prove Item 5, let $X \in C^{n \times 1 \times p}$ be a t-eigenvector with t-eigenvalue $\lambda \in C$ and $|\lambda| = \rho(A)$, such that $A*X = \lambda X$. Then

$$|A \cdot X| - \rho(A) |X| = A \cdot |X| - |\lambda X| = A \cdot |X| - |A \cdot X| \in K_{p+}^n, \quad (4.6)$$

It follows from Proposition 3.7 that $A^\top$ is also nonnegative irreducible; then Item 4 tells us that there is a positive left t-eigenvector $Y \in K_{p++}^n$ corresponding to $\rho(A)$ of $A^\top$. Applying Lemma 4.1 to (4.6) shows that

$$A \cdot |X| = \rho(A) |X|,$$
i.e., $|X|$ is a nonnegative $t$-eigenvector of $\mathcal{A}$. Then Item 3 shows that $|X| \in \mathbb{K}^{n}_{++}$, namely, $|X|$ does not contain zero tubal scalars. The same situation also holds for $X$.

We then prove Item 6. Let $X \in \mathbb{K}^{n}_{p}$ with $\mathcal{A} \ast X = \rho(\mathcal{A})X$. Write $X = (x_i)$. Item 5 tells us that $x_i \neq 0$, $i = 1, \ldots, n$. If some of $x_i \in \mathbb{K}^{p}_{++}$ while the other ones are not, without loss of generality, we assume that

$$
\begin{align*}
x_i \in \mathbb{K}^{p}_{++}, & \quad i \in I \subset [n], I \neq \emptyset, \\
\notin \mathbb{K}^{p}_{+}, & \quad i \in \bar{I}.
\end{align*}
$$

On the other hand, the proof of Item 5 shows that $\mathcal{A} \ast |X| = \rho(\mathcal{A}) |X| = |\rho(\mathcal{A})X| = |\mathcal{A} \ast X|$.

Then, if $i \in \mathbb{K}^{p}_{++}$, we have

$$
\begin{align*}
(A \ast X)_i = \rho(\mathcal{A})x_i = \rho(\mathcal{A})|x_i| = |(A \ast X)_i| = (A \ast |X|)_i,
\end{align*}
$$

which means that

$$
\sum_{j=1}^{n} a_{i,j} \ast (|x_j| - x_j) = 0, \quad \forall i \in I.
$$

It follows from $|x_j| - x_j \in \mathbb{K}^{p}_{++}$ that

$$
a_{i,j} \ast (|x_j| - x_j) = 0, \quad \forall i \in I.
$$

Note that $x_j \notin \mathbb{K}^{p}_{+}$ when $j \in \bar{I}$; i.e., there exist some $x_{j}^{(k)} < 0$, which implies that $|x_j| - x_j \in \mathbb{K}^{p}_{++}$. Proposition 3.2 then tells us that

$$
a_{i,j} = 0, \quad \forall i \in I, \forall j \in \bar{I},
$$

contradicting the irreducibility of $\mathcal{A}$. This completes the proof of Item 6.

To show Item 7, let $\mathcal{A} \ast Y = \lambda Y$ with $Y \in \mathbb{K}^{n}_{p++}$. Similar to the proof of Item 2, we have $\lambda > 0$. Define

$$
\delta_{X,Y} := \sup_{s \geq 0} \{ s \mid Y - sX \in \mathbb{K}^{n}_{p+} \}.
$$

Then $\delta_{X,Y} > 0$ since $Y \in \mathbb{K}^{n}_{p++}$. Thus

$$
\lambda Y - \rho(\mathcal{A})\delta_{X,Y} X = \mathcal{A} \ast (Y - \delta_{X,Y}X) \in \mathbb{K}^{n}_{p+},
$$

where the last relation follows from Proposition 3.4. Since $\lambda > 0$, this means that $Y - \frac{\rho(\mathcal{A})}{\lambda} \delta_{X,Y} X \in \mathbb{K}^{n}_{p+}$, which together with the definition of $\delta_{X,Y}$ implies that $\rho(\mathcal{A})/\lambda \leq 1$. Thus we must have $\rho(\mathcal{A}) = \lambda$.\hfill \Box

We present an example to illustrate Theorem 4.2.

Example 4.1. Consider $\mathcal{A} \in \mathbb{K}^{2 \times 2}_{2}$, with

$$
\mathcal{A}^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$
Then \( \text{bcirc}(A) \) is

\[
\text{bcirc}(A) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

Note that \( A \) is an irreducible tubal matrix, but \( \text{bcirc}(A) \) is a reducible matrix. Since \( A^\top \) is symmetric, its left and right t-eigenvectors are the same.

\( A \) then has four t-eigenvalues: 1 (with multiplicity 2) and \(-1\) (with multiplicity 2). Thus \( \rho(A) = 1 > 0 \) is a t-eigenvalue. The t-eigenvectors are

- \( A \) with \( a_1 = [0, 1], a_2 = [-1, 0] \);
- \( B \) with \( b_1 = [1, 0], b_2 = [0, -1] \);
- \( C \) with \( c_1 = [1, 0], c_2 = [0, 1] \);
- \( D \) with \( d_1 = [0, 1], d_2 = [1, 0] \).

Here \( A \) and \( B \) correspond to \(-1\), while \( C \) and \( D \) correspond to \( \rho(A) = 1 \). Every linear combination of \( A \) and \( B \) is also a t-eigenvector of \(-1\), and every linear combination of \( C \) and \( D \) is also a t-eigenvector of 1.

From the structure of \( C \) and \( D \), we can see that \( C \) and \( D \) are positive; moreover, all the nonnegative t-eigenvectors must be positive. These illustrate Items 3 and 4 of Theorem 4.2. These also tell us that the positive t-eigenvectors corresponding to \( \rho(A) \) might not be unique.

Besides \( C \) and \( D \), we can also see that \( A \) and \( B \) do not contain zero tubal scalars, and any linear combination of \( A \) and \( B \) does not contain zero tubal scalars either. This confirms Item 5 of Theorem 4.2.

From again the structure of \( C \) and \( D \), we can check that any linear combination of \( C \) and \( D \) is either positive, or does not belong to \( \mathbb{R}_2^+ \), as proved in Item 6 of Theorem 4.2.

To show Item 7 of Theorem 4.2, let \( C = \alpha A + \beta B \) where \( \alpha, \beta \in \mathbb{R} \). It is clearly seen that if \( c_1 \) is strongly positive, then \( c_2 \) cannot be; if \( c_2 \) is strongly positive, then \( c_1 \) cannot be. While any positive linear combination of \( C \) and \( D \) is a strongly positive t-eigenvector. Thus any strongly positive t-eigenvector can only correspond to \( \rho(A) \).

We then discuss the differences between Theorem 4.2 and PF theorem for nonnegative irreducible matrices. Recall the PF theorem:

**Theorem 4.3.** Let \( A \in \mathbb{R}^{n \times n} \) be nonnegative irreducible. Then

1. \( \rho(A) > 0 \) is an eigenvalue;
2. There is a positive eigenvector corresponding to \( \rho(A) \);
3. If \( \lambda \) is an eigenvalue with a nonnegative eigenvector, then \( \lambda = \rho(A) \);
4. \( \rho(A) \) is a simple root.

Comparing Theorem 4.2 with Theorem 4.3, we see that Items 1 and 2 of Theorem 4.3 are inherited by nonnegative irreducible tubal matrices. Item 3 of Theorem 4.3 cannot be completely generalized to the tubal matrix case, where Item 7 of Theorem 4.2 gives a
weaker version. Item 4 of Theorem 4.3 does not hold for tubal matrices, as illustrated in Example 4.1; what is worse is that \( \rho(A) \) may have t-eigenvectors that are neither positive nor negative. The only thing currently we can make sure is that the real t-eigenvectors corresponding to \( \rho(A) \) is either positive, or cannot contain nonnegative tubal scalars, as shown in Item 6 of Theorem 4.2.

Theorem 4.2 can be enhanced if there is an additionally relatively mild assumption is satisfied.

**Theorem 4.4 (Enhanced PF theorem).** Let \( A \in K^{n \times n}_+ \) be a nonnegative irreducible tubal matrix. If there exists a strongly positive tubal scalar \( a_{i,j} \) in \( A \), then

1. \( \rho(A) > 0 \) is a t-eigenvalue;
2. Every nonzero nonnegative t-eigenvector must be strongly positive;
3. There is a strongly positive t-eigenvector corresponding to \( \rho(A) \); such a strongly positive t-eigenvector is unique up to a multiplicative constant;
4. If \( \lambda \) is a t-eigenvalue with a nonzero nonnegative t-eigenvector, then \( \lambda = \rho(A) \).

**Proof.** Item 1 has been proved in Theorem 4.2.

Let \( X = (x_i) \in K^n_{p+}, X \neq 0 \) with \( A \ast X = \lambda X \). The proof of Item 2 of Theorem 4.2 shows that \( \lambda > 0 \). Item 3 of Theorem 4.2 shows that \( X \in K^n_{p++} \). We then show that \( X \in K^n_{p++} \) in the current setting. Since \( X \in K^n_{p++} \) and there exists an \( a_{i,j} \in K_{p++} \), Proposition 3.1 shows that there exists an \( x_i \in K_{p++} \). Therefore, let \( I := \{ i \mid x_i \in K_{p++} \} \) and \( \bar{I} = [n] \setminus I \). Then \( \bar{I} \neq \emptyset \). We will prove that \( I = \emptyset \). If not, consider \( i \in I \),

\[
\sum_{j=1}^{n} a_{i,j} \ast x_j = \lambda x_i, \forall i \in I.
\]

Since when \( j \in \bar{I}, x_j \in K_{p++} \); if \( a_{i,j} \neq 0 \), then Proposition 3.1 shows that \( a_{i,j} \ast x_j \in K_{p++} \), and so the LHS above is also in \( K_{p++} \), namely, \( x_i \in K_{p++} \), which contradicts the definition of \( I \). Thus the only case can happen is that

\[
a_{i,j} = 0, \forall i \in I, \forall j \in \bar{I}.
\]

However, this also leads to a contradiction. Thus, \( I = \emptyset \), and so \( X \in K^n_{p++} \).

To show Item 3, note that the existence of a strongly positive t-eigenvector follows from Item 4 of Theorem 4.2 and Item 2 above. Assume that \( A \ast X = \rho(A)X \) and \( A \ast Y = \rho(A)Y \) with \( X, Y \in K^n_{p++} \), and \( X \neq \alpha Y, \alpha > 0 \). Denote \( \delta_{XY} \) the same as that in the proof of Item 7 of Theorem 4.2. Then \( \delta_{XY} > 0 \). Denote \( Z := Y - \delta_{XY}X \). It is clear that \( Z \in K^n_{p++} \setminus K^n_{p++} \), namely, there is at least a tubal scalars \( z_i \notin K_{p++} \). However, \( Z \) is a nonnegative t-eigenvector corresponding to \( \rho(A) \). Thus Item 2 above tells us that the only case now can happen is \( Z = 0 \), i.e., \( X = \alpha Y \) for some \( \alpha > 0 \). This proves the uniqueness of the strongly positive t-eigenvector.

Item 4 follows from Item 7 of Theorem 4.2 and Item 2 above.

Comparing with the PF theorem for nonnegative irreducible matrices, all the conclusions except the simplicity of \( \rho(A) \) hold now.
5 Conclusions

This paper treats third-order tensors as tubal matrices, and defines nonnegative/positive/strongly positive tubal scalars/vectors/matrices. The meaning of positivity here is different from its usual sense. For example, a positive tubal matrix might be very sparse. Based on the nice properties of the t-product, we derive several properties for nonnegative tubal scalars/vectors/matrices. The irreducibility is defined with equivalent characterizations given. Some conclusions of the Perron-Frobenius theorem for nonnegative irreducible matrices have been generalized to nonnegative irreducible tubal matrices setting, while some fail. With an additionally relatively mild assumption, the obtained conclusions are enhanced, which are more close to the original PF theorem.

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