A Note on Braided $T$-categories over Monoidal Hom-Hopf Algebras

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Abstract Let $\text{Aut}_{mH}(H)$ denote the set of all automorphisms of a monoidal Hom-Hopf algebra $H$ with bijective antipode in the sense of Caenepeel and Goyvaerts [2]. The main aim of this paper is to provide new examples of braided $T$-category in the sense of Turaev [14]. For this, first we construct a monoidal Hom-Hopf $T$-coalgebra $\mathcal{MHD}(H)$ and prove that the $T$-category $\text{Rep}(\mathcal{MHD}(H))$ of representation of $\mathcal{MHD}(H)$ is isomorphic to $\mathcal{MHYD}(H)$ as braided $T$-categories, if $H$ is finite-dimensional. Then we construct a new braided $T$-category $Z\mathcal{MHYD}(H)$ over $Z$, generalizing the main construction by Staic [11].

Key words: Monoidal Hom-Hopf algebra; Braided $T$-category; Diagonal crossed Hom-product, Monoidal Hom-Hopf $T$-coalgebra.

Mathematics Subject Classification: 16W30.

0. INTRODUCTION

Braided $T$-categories introduced by Turaev [14] are of interest due to their applications in homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories. As such, they are interesting to different research communities in mathematical physics (see [5, 6, 13, 15, 16]). Although Yetter-Drinfeld modules over Hopf algebras provide examples of such braided $T$-categories, these are rather trivial. The wish to obtain more interesting homotopy quantum field theories provides a strong motivation to find new examples of braided $T$-categories.

The aim of this article is to construct new examples of braided $T$-categories isomorphic to the $T$-category $\mathcal{MHYD}(H)$ in [18]. For this purpose, we prove that, if $(H, A, H)$ is a Yetter-Drinfeld Hom-datum (the second $H$ is regarded as an $H$-Hom-bimodule coalgebra) in [18], with $H$ finite dimensional, then the category $\mathcal{AMHYD}(H)$ of Yetter-Drinfeld Hom-modules is isomorphic to the category of left modules over the diagonal crossed

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Hom-product $H^* \rhd A$. Then when $H$ is finite-dimensional we construct a monoidal Hom-Hopf $T$-coalgebra $\mathcal{MHYD}(H)$, and prove that the $T$-category $\text{Rep}(\mathcal{MHYD}(H))$ of representation of $\mathcal{MHYD}(H)$ is isomorphic to $\mathcal{MHYD}(H)$ as braided $T$-categories.

The article is organized as follows.

We will present the background material in Section 1. This section contains the relevant definitions on braided $T$-categories, monoidal Hom-Hopf algebras and monoidal Hom-Hopf $T$-coalgebras necessary for the understanding of the construction. In Section 2, we define the notion of a diagonal crossed Hom-product algebra over a monoidal Hom-Hopf algebra. And then when $H$ is finite dimensional, we prove the category $A^{\mathcal{MHYD}}(H)$ is isomorphic to $H^* \rhd A^M$. Section 3, when $H$ is finite-dimensional we construct a monoidal Hom-Hopf $T$-coalgebra $\mathcal{MHYD}(H)$, and prove that the $T$-category $\text{Rep}(\mathcal{MHYD}(H))$ of representation of $\mathcal{MHYD}(H)$ is isomorphic to $\mathcal{MHYD}(H)$ as braided $T$-categories.

Section 4, we construct a new braided $T$-category $Z^{\mathcal{MHYD}}(H)$ over $Z$, generalizing the main construction by Staic [11].

1. PRELIMINARIES

Throughout, let $k$ be a fixed field. Everything is over $k$ unless otherwise specified. We refer the readers to the books of Sweedler [12] for the relevant concepts on the general theory of Hopf algebras. Let $(C, \Delta)$ be a coalgebra. We use the "sigma" notation for $\Delta$ as follows:

$$\Delta(c) = \sum c_1 \otimes c_2, \forall c \in C.$$ 

1.1. Braided $T$-categories.

A monoidal category $C = (C, \mathbb{I}, \otimes, a, l, r)$ is a category $C$ endowed with a functor $\otimes: C \times C \to C$ (the tensor product), an object $\mathbb{I} \in C$ (the tensor unit), and natural isomorphisms $a$ (the associativity constraint), where $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for all $U, V, W \in C$, and $l$ (the left unit constraint) where $l_U : \mathbb{I} \otimes U \to U$, $r$ (the right unit constraint) where $r_U : U \otimes \mathbb{I} \to U$, such that for all $U, V, W, X \in C$, the associativity pentagon $a_{U,V,W} \circ a_{U \otimes V,W,X} = (U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X} \circ (a_{U,V,W} \otimes X)$ and $(U \otimes l_V) \circ (r_U \otimes V) = a_{U,1,V}$ are satisfied. A monoidal category $C$ is strict when all the constraints are identities.

Let $G$ be a group and let $\text{Aut}(C)$ be the group of invertible strict tensor functors from
A category $C$ over $G$ is called a crossed category if it satisfies the following:

- $C$ is a monoidal category;
- $C$ is disjoint union of a family of subcategories $\{C_\alpha\}_{\alpha \in G}$, and for any $U \in C_\alpha$, $V \in C_\beta$, $U \otimes V \in C_{\alpha\beta}$. The subcategory $C_\alpha$ is called the $\alpha$-th component of $C$;
- Consider a group homomorphism $\varphi : G \to Aut(C)$, $\beta \mapsto \varphi_\beta$, and assume that $\varphi_\beta(\varphi_\alpha) = \varphi_{\alpha \beta}^{-1}$, for all $\alpha, \beta \in G$. The functors $\varphi_\beta$ are called conjugation isomorphisms.

Furthermore, $C$ is called strict when it is strict as a monoidal category.

**Left index notation:** Given $\alpha \in G$ and an object $V \in C_\alpha$, the functor $\varphi_\alpha$ will be denoted by $V(\cdot)$, as in Turaev [14] or Zunino [19], or even $\alpha(\cdot)$. We use the notation $V^\gamma(\cdot)$ for $\alpha^{-1}(\cdot)$. Then we have $V(id_U) = id_{VU}$ and $V(g \circ f) = V^g \circ V^f$. Since the conjugation $\varphi : G \to Aut(C)$ is a group homomorphism, for all $V, W \in C$, we have $V^\otimes W(\cdot) = V(W(\cdot))$ and $V^I(\cdot) = V(V(\cdot)) = id_C$. Since, for all $V \in C$, the functor $V(\cdot)$ is strict, we have $V(f \otimes g) = V(f) \otimes V(g)$, for any morphisms $f$ and $g$ in $C$, and $V^I = I$.

A braiding of a crossed category $C$ is a family of isomorphisms $(c = c_{U,V})_{U,V} \in C$, where $c_{U,V} : U \otimes V \to V \otimes U$ satisfying the following conditions:

1. For any arrow $f \in C_\alpha(U,U')$ and $g \in C(V,V')$,

$$((\alpha g) \otimes f) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g).$$

2. For all $U, V, W \in C$, we have

$$c_{U \otimes V, W} = a_{U,V,W} c_{U,V \otimes W} \circ (c_{U,V \otimes W} \otimes id_V) \circ a_{U,V,W}^{-1} \circ (i_U \otimes c_{V,W}) \circ a_{U,V,W},$$

$$c_{U,V \otimes W} = a_{U,V,W}^{-1} c_{U,V,W} \circ (i_{U,V} \otimes c_{U,V}) \circ a_{V,U,W} \circ (c_{U,V} \otimes i_W) \circ a_{U,V,W}^{-1},$$

where $a$ is the natural isomorphisms in the tensor category $C$.

3. For all $U,V \in C$ and $\beta \in G$,

$$\varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U),\varphi_\beta(V)}.$$

A crossed category endowed with a braiding is called a braided $T$-category.

### 1.2. Monoidal Hom-Hopf algebras.

Let $M_k = (M_k, \otimes, k, a, l, r)$ denote the usual monoidal category of $k$-vector spaces and linear maps between them. Recall from [2] that there is the monoidal Hom-category $\mathcal{H}(M_k) = (\mathcal{H}(M_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$, a new monoidal category, associated with $M_k$ as follows:

- The objects of the monoidal category $\mathcal{H}(M_k)$ are couples $(M, \xi_M)$, where $M \in M_k$ and $\xi_M \in Aut_k(M)$, the set of all $k$-linear automorphisms of $M$;
• The morphism $f : (M, \xi_M) \to (N, \xi_N)$ in $\mathcal{H}(M_k)$ is the $k$-linear map $f : M \to N$ in $\mathcal{M}_k$ satisfying $\xi_N \circ f = f \circ \xi_M$, for any two objects $(M, \xi_M), (N, \xi_N) \in \mathcal{H}(M_k)$;

• The tensor product is given by
  $$(M, \xi_M) \otimes (N, \xi_N) = (M \otimes N, \xi_M \otimes \xi_N)$$
  for any $(M, \xi_M), (N, \xi_N) \in \mathcal{H}(M_k)$;

• The tensor unit is given by $(k, \text{id})$;

• The associativity constraint $\tilde{a}$ is given by the formula
  $$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\xi_M \otimes \text{id}) \otimes \xi_L^{-1}) = (\xi_M \otimes (\text{id} \otimes \xi_L^{-1})) \circ a_{M,N,L},$$
  for any objects $(M, \xi_M), (N, \xi_N), (L, \xi_L) \in \tilde{H}(M_k)$;

• The left and right unit constraint $\tilde{l}$ and $\tilde{r}$ are given by
  $$\tilde{l}_M = \xi_M \circ l_M = l_M \circ (\text{id} \otimes \xi_M), \quad \tilde{r}_M = \xi_M \circ r_M = r_M \circ (\xi_M \otimes \text{id})$$
  for all $(M, \xi_M) \in \tilde{H}(M_k)$.

We now recall from [2] the following notions used later.

**Definition 1.2.1.** Let $\tilde{H}(M_k)$ be a monoidal Hom-category. A **monoidal Hom-algebra** is an object $(A, \xi_A)$ in $\tilde{H}(M_k)$ together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \to A; \quad a \otimes b \mapsto ab, \quad \xi_A \in \text{Aut}_k(A)$$

such that

$$\xi_A(ab) = \xi_A(a)\xi_A(b), \quad \alpha(1_A) = 1_A,$$  \hfill (1. 1)

$$\xi_A(a)(bc) = (ab)\xi_A(c), \quad a1_A = 1_Aa = \xi_A(a),$$  \hfill (1. 2)

for all $a, b, c \in A$.

**Definition 1.2.2.** A **monoidal Hom-coalgebra** is an object $(C, \xi_C)$ in $\tilde{H}(M_k)$ together with linear maps $\Delta : C \to C \otimes C, \Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \to k$ such that

$$\Delta(\xi_C(c)) = \xi_C(c_1) \otimes \xi_C(c_2), \quad \varepsilon(\xi_C(c)) = \varepsilon(c),$$  \hfill (1. 3)

$$\xi_C^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \xi_C^{-1}(c_2), \quad c_1\varepsilon(c_2) = \xi_C^{-1}(c) = \varepsilon(c_1)c_2,$$  \hfill (1. 4)

for all $c \in C$.

**Remark 1.2.3.** (1) Note that (1.4) is equivalent to $c_1 \otimes c_2 \otimes \xi_C(c_2) = \xi_C(c_1) \otimes c_12 \otimes c_2$. Analogue to monoidal Hom-algebras, monoidal Hom-coalgebras will be short for counital monoidal Hom-coassociative coalgebras without any confusion.
(2) Let \((C, \xi_C)\) and \((C', \xi_{C'})\) be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map \( f : (C, \xi_C) \to (C', \xi_{C'})\) is a linear map such that \( f \circ \xi_C = \xi_{C'} \circ f, \Delta \circ f = (f \otimes f) \circ \Delta \) and \( \varepsilon' \circ f = \varepsilon \).

**Definition 1.2.4.** A monoidal Hom-Hopf algebra \( H = (H, \xi_H, m_H, \Delta, \varepsilon, S) \) is a bialgebra with \( S \) in \( \tilde{H}(M_k) \). This means that \((H, \alpha, m, 1_H)\) is a monoidal Hom-algebra and \((H, \alpha, \Delta, \varepsilon)\) is a monoidal Hom-coalgebra such that \( \Delta \) and \( \varepsilon \) are morphisms of algebras, that is, for all \( h, g \in H \),

\[
\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1.
\]

\( S \) is the convolution inverse of the identity morphism \( id_H \) (i.e., \( S * id = 1_H \circ \varepsilon = id * S \)). Explicitly, for all \( h \in H \),

\[
S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).
\]

**Remark 1.2.5.** (1) Note that a monoidal Hom-Hopf algebra is by definition a Hopf algebra in \( \tilde{H}(M_k) \).

(2) Furthermore, the antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as

\[
S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon.
\]

That is, \( S \) is a monoidal Hom-anti-(co)algebra homomorphism. Since \( \xi_H \) is bijective and commutes with \( S \), we can also have that the inverse \( \xi_H^{-1} \) commutes with \( S \), that is, \( S \circ \xi_H^{-1} = \xi_H^{-1} \circ S \).

In the following, we recall the notions of actions on monoidal Hom-algebras and coactions on monoidal Hom-coalgebras.

Let \((A, \xi_A)\) be a monoidal Hom-algebra. A left \((A, \xi_A)\)-Hom-module consists of an object \((M, \xi_M)\) in \( \tilde{H}(M_k) \) together with a morphism \( \psi : A \otimes M \to M, \psi(a \otimes m) = a \cdot m \) such that

\[
\xi_A(a) \cdot (b \cdot m) = (ab) \cdot \xi_M(m), \quad \xi_M(a \cdot m) = \xi_A(a) \cdot \xi_M(m), \quad 1_A \cdot m = \xi_M(m),
\]

for all \( a, b \in A \) and \( m \in M \).

Monoidal Hom-algebra \((A, \xi_A)\) can be considered as a Hom-module on itself by the Hom-multiplication. Let \((M, \xi_M)\) and \((N, \xi_N)\) be two left \((A, \xi_A)\)-Hom-modules. A morphism \( f : M \to N \) is called left \((A, \xi_A)\)-linear if \( f(a \cdot m) = a \cdot f(m), f \circ \xi_M = \xi_N \circ f \). We denoted the category of left \((A, \xi_A)\)-Hom-modules by \( \tilde{H}(A, M_k) \).

Similarly, let \((C, \xi_C)\) be a monoidal Hom-coalgebra. A right \((C, \xi_C)\)-Hom-comodule is an object \((M, \xi_M)\) in \( \tilde{H}(M_k) \) together with a \( k \)-linear map \( \rho_M : M \to M \otimes C, \rho_M(m) = m_{(0)} \otimes m_{(1)} \) such that

\[
\xi^{-1}_M(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)}(0) \otimes m_{(0)}(1)) \otimes \xi^{-1}_C(m_{(1)}), \quad (1.5)
\]

\[
\rho_M(\xi_M(m)) = \xi_M(m_{(0)}) \otimes \xi_C(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \xi^{-1}_M(m), \quad (1.6)
\]

for all $m \in M$.

$(C, \xi_C)$ is a Hom-comodule on itself via the Hom-comultiplication. Let $(M, \xi_M)$ and $(N, \xi_N)$ be two right $(C, \xi_C)$-Hom-comodules. A morphism $g : M \to N$ is called right $(C, \xi_C)$-colinear if $g \circ \xi_M = \xi_N \circ g$ and $g(m(0)) \otimes m(1) = g(m(0)) \otimes g(m(1))$. The category of right $(C, \xi_C)$-Hom-comodules is denoted by $\widehat{\mathcal{H}}(\mathcal{M}^C)$.

Definition 1.2.6. Let $(H, m, \Delta, S, \xi_H)$ be a monoidal Hom-bialgebra and $\alpha, \beta \in \text{Aut}_{m \mathcal{H} \mathcal{D} H}^H(\alpha, \beta)$. Recall from [IS] that a \textit{left-right} $(\alpha, \beta)$-Yetter-Drinfeld Hom-module over $(H, \xi_H)$ is the object $(M, \cdot, \rho, \xi_M)$ which is both in $\mathcal{H}(H \mathcal{M})$ and $\mathcal{H}(\mathcal{M}^H)$ obeying the compatibility condition:

$$\rho(h \cdot m) = \xi_H(h_{21}) \cdot m_0 \otimes (\beta(h_{22})\xi_H^{-1}(m_1))\alpha(S^{-1}(h_1)), \quad (1.7)$$

Remark 1.2.7. (1) The category of all left-right $(\alpha, \beta)$-Yetter-Drinfeld Hom-modules is denoted by $\mathcal{H} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D} \mathcal{H}(\alpha, \beta)$ with understanding morphism.

(2) If $(H, \xi_H)$ is a monoidal Hom-Hopf algebra with a bijective antipode $S$ and $S$ commute with $\alpha, \beta$, then the above equality is equivalent to

$$h_1 \cdot m_0 \otimes \beta(h_2)m_1 = \xi_M((h_2 \cdot \xi_M^{-1}(m_0)) \otimes (h_2 \cdot \xi_M^{-1}(m_1))\alpha(h_1)), \quad (1.8)$$

for all $h \in H$ and $m \in M$.

(3) If $(M, \xi_M) \in \mathcal{H} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D} \mathcal{H}(\alpha, \beta)$ and $(N, \xi_N) \in \mathcal{H} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D} \mathcal{H}(\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta \in \text{Aut}_{m \mathcal{H} \mathcal{H}}^H(H)$, then $(M \otimes N, \xi_M \otimes \xi_N) \in \mathcal{H} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D} \mathcal{H}(\alpha \gamma^{-1}, \beta \delta^{-1})$ with structures as follows:

$$h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1}\beta \gamma(h_2) \cdot n, \quad (1.9)$$

$$m \otimes n \mapsto (m_0 \otimes n_0) \otimes n_1 m_1. \quad (1.10)$$

for all $m \in M, n \in N$ and $h \in H$.

Definition 1.2.8. Let $(H, \xi_H)$ be a monoidal Hom-algebra. A monoidal Hom-algebra $(A, \xi_A)$ is called an $(H, \xi_H)$-Hom-bicomodule algebra in [IS], with Hom-comodule maps $\rho_l$ and $\rho_r$ obeying the following axioms:

(1) $\rho_l : A \to H \otimes A$, $\rho_l(a) = a_{[-1]} \otimes a_{[0]}$, and $\rho_r : A \to A \otimes H$, $\rho_r(a) = a_{[0]} \otimes a_{[1]}$,

(2) $\rho_l$ and $\rho_r$ satisfy the following compatibility condition: for all $a \in A$,

$$a_{[0]} \otimes a_{[1]} \otimes \xi_H^{-1}(a_{[0]}) = \xi_H^{-1}(a_{[-1]}) \otimes a_{[0]} \otimes a_{[0]}. \quad (1.11)$$

Definition 1.2.9. Let $(H, \xi_H)$ be a monoidal Hom-Hopf algebra, $(A, \xi_A)$ be an $H$-Hom-bicomodule algebra. We consider the \textit{Yetter-Drinfeld Hom-datum} $(H, A, H)$ as in [IS], (the second $H$ is regarded as an $H$-Hom-bimodule coalgebra), and the \textit{Yetter-Drinfeld Hom-module category} $\mathcal{A} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D} \mathcal{H}(H)$, whose objects are $k$-modules $(M, \xi_M)$ with the following additional structure:

(1) $M$ is a left $A$-module;

(2) we have a $k$-linear map $\rho_M : M \to M \otimes H$, $\rho_M(m) = m_0 \otimes m_1,$
for all \( a \in A \) and \( m \in M \).

**Definition 1.2.10.** Let \((A, \xi_A)\) be a monoidal Hom-algebra, \((M, \xi_M)\) be a monoidal Hom-algebra. Assume that \((M, \alpha_M)\) is both a left and a right \(A\)-module algebra (with actions denoted by \(A \otimes M \to M\), \(a \otimes m \mapsto a \cdot m\) and \(M \otimes A \to M\), \(m \otimes a \mapsto m \cdot a\)). We call \((M, \xi_M)\) an \(A\)-bimodule as in [9] if the following condition is satisfied, for all \(a, a' \in A\), \(m \in M\):

\[
\xi_A(a) \cdot (m \cdot a') = (a \cdot m) \cdot \xi_A(a').
\]  

(1.14)

### 1.3. Monoidal Hom-Hopf \(T\)-coalgebras.

**Definition 1.3.1.** Let \(G\) be a group with unit 1. Then we recall from Yang Tao [17] that a monoidal Hom-\(T\)-coalgebra \((C, \xi_C)\) over \(G\) is a family of objects \(\{ (C_p, \xi_{C_p}) \}_{p \in G}\) in \(\tilde{H}(\mathcal{M}_k)\) together with linear maps \(\Delta_{p,q} : C_{pq} \to C_p \otimes C_q, c_{pq} \mapsto c_{(1,p)} \otimes c_{(2,q)}\) and \(\varepsilon : C_e \to k\) such that

\[
\xi^{-1}_{C_p}(c_{(1,p)}) \otimes \Delta_{q,r}(c_{(2,p)}) = \Delta_{p,q}(c_{(1,pq)}) \otimes \xi^{-1}_{C_p}(c_{(2,p)}), \quad \forall c \in C_{pqr},
\]

\[
c_{(1,p)} \alpha(c_{(2,p)}) = \varepsilon(c_{(1,e)}) c_{(2,p)} = \xi^{-1}_{C_p}(c_{(2,p)}), \quad \forall c \in C_p,
\]

\[
\Delta_{p,q}(\xi_{C_p}^{-1}(c_{pq})) = \xi_{C_q}^{-1}(c_{(1,p)}) \otimes \xi_{C_p}^{-1}(c_{(2,q)}), \quad \forall c \in C_{pq},
\]

\[
\varepsilon(\xi_{C_p}^{-1}(c)) = \varepsilon(c), \quad \forall c \in C_e.
\]

Let \((C, \xi_C)\) and \((C', \xi'_{C'})\) be two monoidal Hom-\(T\)-coalgebras over \(G\). A Hom-coalgebra map \(f : (C, \xi_C) \to (C', \xi'_{C'})\) is a family of linear maps \(\{ f_p \}_{p \in G}, f_p : (C_p, \xi_{C_p}) \to (C'_p, \xi'_{C_p})\) such that \(f_p \circ \xi_{C_p} = \xi'_{C_p} \circ f_p, \Delta_{p,q} \circ f_{pq} = (f_p \otimes f_q) \Delta_{p,q}\) and \(\varepsilon \circ f_e = \varepsilon\).

**Definition 1.3.2.** A monoidal Hom-Hopf \(T\)-coalgebra \((H = \bigoplus_{p \in G} H_p, \xi = \{ \xi_{H_p} \}_{p \in G})\) is a monoidal Hom-\(T\)-coalgebra where each \((H_p, \xi_{H_p})\) is a monoidal Hom-algebra with multiplication \(m_p\) and unit \(1_p\) endowed with antipode \(S = \{ S_p \}_{p \in G}, S_p : H_p \to H_{p^{-1}} \in \tilde{H}(\mathcal{M}_k)\) such that

\[
\Delta_{p,q}(h g) = \Delta_{p,q}(h) \Delta_{p,q}(g), \quad \Delta_{p,q}(1_p) = 1_p \otimes 1_q, \quad \forall h, g \in H_{pq},
\]

\[
\varepsilon(h g) = \varepsilon(h) \varepsilon(g), \quad \varepsilon(1_e) = 1_k, \quad \forall h, g \in H_e
\]

\[
S_{p^{-1}}(h_{(1,p^{-1})} h_{(2,p)}) = \varepsilon(h) 1_p = h_{(1,p)} S_{p^{-1}}(h_{(2,p^{-1})}) \quad \forall h \in H_e.
\]

Note also that the \((H_e, \xi_e, m_e, 1_e, \Delta_{e,e}, \varepsilon, S_e)\) is a monoidal Hom-Hopf algebra in the usual sense of the word. We call it the neutral component of \(H\).

**Definition 1.3.3.** A monoidal Hom-Hopf \(T\)-coalgebra \((H = \bigoplus_{p \in G} H_p, \xi = \{ \xi_{H_p} \}_{p \in G})\) is called a monoidal Hom-Hopf crossed monoidal Hom-Hopf \(T\)-coalgebra if it is endowed with a family of algebra isomorphisms \(\varphi = \{ \varphi^\alpha_\beta : H_\alpha \to H_{\beta \alpha^{-1}} \}_{\alpha, \beta \in G}\) such that
• each $\varphi_\gamma$ preserves the comultiplication and the counit i.e., for any $\alpha, \beta, \gamma \in G$, we have

$$\Delta_{\gamma \alpha \gamma^{-1}, \gamma \beta \gamma^{-1}} \circ \varphi_\gamma = (\varphi_\gamma \otimes \varphi_\gamma) \circ \Delta_{\alpha, \beta}$$

and

$$\varepsilon \circ \varphi_\gamma = \varepsilon.$$

• $\varphi$ is multiplicative, i.e., $\varphi_\beta \circ \varphi_\gamma = \varphi_{\beta \gamma}$, for any $\beta, \gamma \in G$.

It is easy to get the following identities, $\varphi_1|H_\alpha = id_\alpha$ and $\varphi^{-1}_\alpha = S_{\alpha \beta} \circ \varphi_\beta$ for all $\alpha, \beta \in G$.

2. THE DIAGONAL CROSSED HOM-PRODUCT

In this section, we define the notion of the diagonal crossed Hom-product over a monoidal Hom-Hopf algebra that are based on Hom-associative left and right coactions. If $H$ is finite dimensional, we prove the category $\mathcal{A}_{MHHD}(H)$ is isomorphic to the category of left $H^* \bowtie A$-modules, $H^* \bowtie A\mathcal{M}$, generalizing the results in [1].

In what follows, let $(H, \xi_H)$ be a monoidal Hom-Hopf algebra with the bijective antipode $S$ and let $\text{Aut}_{mHH}(H)$ denote the set of all automorphisms of a monoidal Hopf algebra $H$.

**Definition 2.1.** Let $(H, \xi_H)$ be a finite dimensional monoidal Hom-Hopf algebra, $(A, \xi_A)$ be a monoidal Hom-bicomodule algebra. Then the diagonal crossed Hom-product $H^* \bowtie A$ is defined as follows:

- as $k$-spaces, $H^* \bowtie A = H^* \otimes A$;

- multiplication is given by

\[
(f \bowtie a)(g \bowtie b) = f(a_{[1]} \rightarrow (\xi_H^2(g) \leftarrow S^{-1}(a_{[0]}<1>))) \bowtie \xi_A^2(a_{[0]}<0>)b; \quad (2.1)
\]

\[
h \rightarrow f = (f_2, \xi_H^{-1}(h))\xi_H^{*^{-2}}(f_1) \quad \text{and} \quad f \leftarrow h = (f_1, \xi_H^{-1}(h))\xi_H^{*^{-2}}(f_2); \quad (2.2)
\]

for all $a, b \in (A, \xi_A), f, g \in (H^*, \xi_H^{*^{-1}}), h \in (H, \xi_H)$.

**Proposition 2.2.** Let $(A, \xi_A)$ be an $(H, \xi_H)$-Hom-bicomodule algebra and $(H^*, \xi_H^{*^{-1}})$ be an $(H, \xi_H)$-Hom-bimodule algebra. Then the tensor space $H^* \otimes A$ is a Hom-algebra with the multiplication in the formula (2.1) and the unit $\varepsilon_H \bowtie 1_A$.

**Proof.** It is obvious that $(\varepsilon_H \bowtie 1_A)(f \bowtie a) = \xi_H^{*^{-1}}(f) \bowtie \xi_A(a)$, so $(\varepsilon_H \bowtie 1_A)$ is unit element. We have:
[(f ∇ a)(g ∇ b)]ξ_H^* ∘ A(φ ∇ c)

= [f[a_{-1}] ⇒ (∆^2_H(g) ← S^{-1}(a_{[1]}))]((ξ^2_A(a_{[0]})) b_{[-1]} ⇒ (∆^2_H(φ))
← S^{-1}((ξ^2_A(a_{[0]})) b_{[0]}))(ξ^2_A(a_{[0]})) b_{[0]}⟩ ⟨ λ_A(ξ^2_A(a_{[0]})) b_{[0]}⟩

= ξ^{-1}(f)([a_{-1}] ⇒ (∆^2_H(g) ← S^{-1}(a_{[1]})))ξ_H^*((ξ^2_H(a_{[0]})) b_{[-1]} ⇒ (∆^2_H(φ))
← S^{-1}(ξ^2_H(a_{[0]})) b_{[0]}))⟩ ⟨ λ_A(ξ^2_H(a_{[0]})) b_{[0]}⟩

= ξ^{-1}(f)([a_{-1}] ⇒ (∆^2_H(g) ← S^{-1}(a_{[1]})))ξ_H^*((ξ^2_H(a_{[0]})) b_{[-1]} ⇒ (∆^2_H(φ))
← S^{-1}(ξ^2_H(a_{[0]})) b_{[0]}))⟩ ⟨ λ_A(ξ^2_H(a_{[0]})) b_{[0]}⟩

Thus the multiplication is Hom-associative. This completes the proof.

**Example 2.3.** (1) If (A, ξ_A) = (H, ξ_H) and ρ_l = ρ_r = ∆ the formula (2.1) coincides
with the multiplication in the Drinfeld double (D(H), ξ^{-1}_H ⊗ ξ_H) = (H^* ⊗ H, ξ^{-1}_H ⊗ ξ_H),
i.e.

\[(f ∇ h)(g ∇ l) = f(h_1) ⇒ (∆^2_H(g) ← S^{-1}(h_{22}))⟩ ⟨ λ_A(ξ^2_H(a_{[0]})) b_{[0]}⟩ \]

for all f, g ∈ H^* and h, l ∈ H.

(2) Recall from Example 2.5 in [18] that α, β ∈ Aut_mH(H) and as k-vector spaces
(H(α, β), ξ_H) = (H, ξ_H), and (H(α, β), ξ_H) ∈ HμH(YD^H(α, β), with right H-Hom-
comodule structure via Hom-multiplication and left H-Hom-module structure given by:

\[h \cdot x = (∆(h_2)ξ^{-1}_H(x))α(S^{-1}(ξ_H(h_1)))\]

for all h, x ∈ H.

The diagonal crossed product (A(α, β), ξ^{-1}_H ⊗ ξ_H) = (H^* ∇ H(α, β), ξ^{-1}_H ⊗ ξ_H), whose multiplication is

\[(f ∇ h)(g ∇ l) = f(α(h_1)) ⇒ (∆^2_H(g) ← S^{-1}(β(h_{22})))⟩ ⟨ λ_A(ξ^2_H(a_{[0]})) b_{[0]}⟩ \]

for all f, g ∈ H^* and h, l ∈ H.
for all $f, g \in H^*$ and $h, l \in H$.

The Drinfeld double $D(H)$ is a Hom-Hopf algebra with coproduct $\Delta_{D(H)}$ given by

$$\Delta_{D(H)}(f \triangleright h) = (f_2 \triangleright h_1) \otimes (f_1 \triangleright h_2),$$  \hspace{0.5cm} (2.5)

for all $f \in H^*$ and $h \in H$.

**Proposition 2.4.** Let $(A, \xi_A)$ be an $(H, \xi_H)$-Hom-bicomodule algebra. Then $H^* \triangleright A$ is a $D(H)$-Hom-bicomodule algebra with two coactions $\rho_{D(H)} : H^* \triangleright A \to (H^* \triangleright A) \otimes D(H)$ and $\rho_{I(D)} : H^* \triangleright A \to D(H) \otimes (H^* \triangleright A)$ given by

$$\rho_{D(H)}(f \triangleright a) = (f_2 \triangleright a_{<0>}) \otimes (f_1 \otimes a_{<1>}),$$

$$\rho_{I(D)}(f \triangleright a) = (f_2 \otimes a_{[-1]}) \otimes (f_1 \otimes a_{[0]}),$$

where elements in $D(H)$ are written as $(f \otimes h), h \in H, f \in H^*, a \in A$.

**Proof.** In view of (2.5) the comodule axioms and the Hom-coassociative (11) are obvious. We are left to prove that $\rho_{D(H)}$ and $\rho_{I(D)}$ are Hom-algebra maps. To this end we use the following identities obviously holding for all $f \in H^*, h, l \in H$

$$\rho(h \triangleright (f \triangleright l)) = (\xi_H^{-1}(f_1) \triangleright l) \otimes (\xi_H^{-1}(h) \triangleright \xi_H^{-1}(f_2)), \hspace{0.5cm} (2.6)$$

With this we now compute

$$\rho_{D(H)}(f \triangleright a) \rho_{D(H)}(g \triangleright b)$$

$$= (f_2 \otimes a_{<0>}) \otimes (f_1 \otimes a_{<1>}) (g_2 \otimes b_{<0>}) \otimes (g_1 \otimes b_{<1>})$$

$$= (f_2(a_{<0>_{[-1]}}) \triangleright (\xi_H^{-2}(g_2) \triangleright S^{-1}(a_{<0>_{[0],1}}))) \otimes \xi_H^2(a_{<0>_{[0],0}}) b_{<0>})$$

$$\otimes (f_1(a_{<1>_{[0]}}) \triangleright (\xi_H^{-2}(g_1) \triangleright S^{-1}(a_{<1>_{[2]}}))) \otimes \xi_H^2(a_{<1>_{[2]}}) b_{<1>})$$

$$= (f_2(a_{<0>_{[0],0}}) \triangleright \xi_H^{-1}(g_2) \triangleright S^{-1}(a_{<0>_{[0],0}}) b_{<0>}) \otimes (f_1(a_{<1>_{[0],0}}) \triangleright \xi_H^{-1}(g_1) \triangleright S^{-1}(a_{<1>_{[0],0}}))$$

$$= (f_2(a_{<0>_{[0],0}}) \triangleright \xi_H^{-1}(g_2) \triangleright S^{-1}(a_{<0>_{[0],0}}) b_{<0>}) \otimes (f_1(a_{<1>_{[0],0}}) \triangleright \xi_H^{-1}(g_1) \triangleright S^{-1}(a_{<1>_{[0],0}}))$$

$$= (f_2(a_{<0>_{[0],0}}) \triangleright \xi_H^{-1}(g_2) \triangleright S^{-1}(a_{<0>_{[0],0}}) b_{<0>}) \otimes (f_1(a_{<1>_{[0],0}}) \triangleright \xi_H^{-1}(g_1) \triangleright S^{-1}(a_{<1>_{[0],0}}))$$

$$= (f_2(a_{<0>_{[0],0}}) \triangleright \xi_H^{-1}(g_2) \triangleright S^{-1}(a_{<0>_{[0],0}}) b_{<0>}) \otimes (f_1(a_{<1>_{[0],0}}) \triangleright \xi_H^{-1}(g_1) \triangleright S^{-1}(a_{<1>_{[0],0}}))$$

$$= \rho_{D(H)}(f \triangleright a) \rho_{D(H)}(g \triangleright b).$$

Hence $\rho_{D(H)}$ is a Hom-algebra map. The argument for $\rho_{I(D)}$ is analogous. \[\square\]
**Example 2.4.** Let \((H, \xi_H)\) be finite dimensional. Then \((A(\alpha, \beta), \xi_H^{\alpha-1} \otimes \xi_H)\) becomes a \(D(H)\)-bicomodule algebra, with structures

\[
\begin{align*}
H^* \triangleright H(\alpha, \beta) & \to (H^* \triangleright H(\alpha, \beta)) \otimes D(H), & f \triangleright h & \mapsto (f_2 \triangleright h_1) \otimes (f_1 \triangleright \beta(h_2)), \\
H^* \triangleright H(\alpha, \beta) & \to D(H) \otimes (H^* \triangleright H(\alpha, \beta)), & f \triangleright h & \mapsto (f_2 \triangleright \alpha(h_1)) \otimes (f_1 \triangleright h_2).
\end{align*}
\]

for all \(f \in H^*, h \in H\).

In the rest of this section we establish that if \((H, \xi_H)\) is a monoidal Hom-Hopf algebra and is finite dimensional then the category \(A\mathcal{HYD}^H(H)\) is isomorphic to the category of left \(H^* \triangleright A\)-modules, \(H^* \triangleright A\mathcal{M}\).

**Lemma 2.5.** Let \((H, \xi_H)\) be a monoidal Hom-Hopf algebra and \((H, A, H)\) a Yetter-Drinfeld Hom-datum. We have a functor \(F : A\mathcal{HYD}^H(H) \to H^* \triangleright A\mathcal{M}\), given by \(F(M) = M\) as \(k\)-module, with the \(H^* \triangleright A\)-module structure defined by

\[
(f \triangleright u) \triangleright m = \langle f, (u \cdot \xi_M^{-1}(m))_1 \rangle \xi_M^2((u \cdot \xi_M^{-1}(m))_0),
\]

for all \(f \in (H^*, \xi_H^{-1}), u \in (A, \xi_A)\) and \(m \in (M, \xi_M)\). \(F\) transforms a morphism to itself.
Proof. For all $f, g \in H^*$, $a, b \in A$ and $m \in M$, we compute:

\[
[(f \bowtie a)(g \bowtie b)] \triangleright \xi_M(m) = [(f(a_{[1]} \rightharpoonup (\xi^2_H(g) \rightharpoonup S^{-1}(a_{[0]} < [1]))) \bowtie \xi^2_A(a_{[0]} > [0]) b] \triangleright \xi_M(m)
\]

\[
= \langle g_1, S^{-1}\xi_H(a_{[0]}<[1]) \rangle \langle g_2, \xi_H^{-1}(a_{[1]}) \rangle \langle f \xi^2_H(g2_1) \bowtie \xi^2_A(a_{[0]} > [0]) b \rangle \triangleright \xi_M(m)
\]

\[
= \langle g_1, S^{-1}\xi_H(a_{[0]}<[1]) \rangle \langle g_2, \xi_H^{-1}(a_{[1]}) \rangle \langle f, (\xi^2_H(a_{[0]} > [0]) b) \cdot m \rangle_{11} \rangle \langle \xi_H^{-2}(g2_1) \rangle ((\xi^2_A(a_{[0]} > [0]) b) \cdot m)_{12} \xi^2_M((\xi^2_A(a_{[0]} > [0]) b) \cdot m)_0
\]

\[
= \langle g_1, S^{-1}\xi_H(a_{[0]}<[1]) \rangle \langle g_2, \xi_H^{-1}(a_{[1]}) \rangle \langle f, (\xi^2_H(a_{[0]} > [0]) b) \cdot m \rangle_0 \rangle \langle \xi_H^{-2}(g2_1) \rangle ((\xi^2_A(a_{[0]} > [0]) b) \cdot m)_{12} \xi^2_M((\xi^2_A(a_{[0]} > [0]) b) \cdot m)_0
\]

as needed. It is not hard to see that $(\varepsilon_H \bowtie 1_A) \triangleright m = \xi_M(m)$, for all $m \in M$, so $M$ is a left $H^* \bowtie A$-module. The fact that a morphism in $A\mathcal{M}^D(H)$ becomes a morphism in $H^* \bowtie A\mathcal{M}$ can be proved more easily, we leave the details to the reader. \hfill \blacksquare

Lemma 2.6. Let $(H, \xi_H)$ be a monoidal Hom-Hopf algebra and $(H, A, H)$ a Yetter-Drinfeld Hom-datum and assume $H$ is finite dimensional. We have a functor $G : H^* \bowtie A \mathcal{M} \rightarrow A \mathcal{M}^D(H)$, given by $G(M) = M$ as $k$-module, with the structure maps defined by

\[
u \cdot m = (\varepsilon_H \bowtie \xi_A^{-1}(u)) \triangleright m,
\]

(2.8)
\[ \rho_M : M \to M \otimes H, \quad \rho_M(m) = m_0 \otimes m_1 = \sum_{i=1}^{n} (\xi^2_H(e^i) \triangleright 1_A) \triangleright \xi^2_M(m) \otimes e_i, \quad (2.9) \]

for all \( u \in (A, \xi_A) \) and \( m \in (M, \xi_M) \). Here \( \{e_i\}_{i=1,...,n} \) is a basis of \( H \) and \( \{e^i\}_{i=1,...,n} \) is the corresponding dual basis of \( H^* \). \( G \) transforms a morphism to itself.

**Proof.** The most difficult part of the proof is to show that \( G(M) \) satisfies the relations (1, 12) or (1, 13). It is then straightforward to show that a map in \( H \otimes_A M \) is also a map in \( \mathcal{AMH} \mathcal{YD}H(H) \), and that \( G \) is a functor.

We compute:

\[
\begin{align*}
& u_{<0>} \cdot m_0 \otimes u_{<1>} m_1 \\
& = \sum_{i=1}^{n} (\varepsilon_H \triangleright \xi^{-1}_A(\xi^2_H(e^i) \triangleright 1_A) \triangleright \xi^{-2}_M(m)) \otimes u_{<1>} e_i \\
& = \sum_{i=1}^{n} (\varepsilon_H(\xi^{-2}_A(u_{<0>_{[-1]}}) \rightarrow (\xi^2_H(e^i) \leftarrow S^{-1} \xi^{-2}_H(u_{<0>_{[0]_{<1>}}})) \triangleright \xi_A(u_{<0>_{[0]_{<0>}}}) \triangleright \xi^{-1}_M(m)) \\
& \otimes u_{<1>} e_i \\
& = \sum_{i=1}^{n} (\varepsilon^i, S^{-1} \xi_H(u_{<0>_{[0]_{<1>}}}) (\varepsilon^i_22, \xi^{-1}_H(u_{<0>_{[0]_{<1>}}})) (\varepsilon^i_21, \xi^{-1}_H((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-2}_M(m))_1)) \\
& \xi^{-2}_M((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-2}_M(m))_0) \otimes u_{<1>} e_i \\
& = \sum_{i=1}^{n} (\varepsilon^i, S^{-1} \xi_H(u_{<0>_{[0]_{<1>}}}) ((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-3}_M(m))_1 \xi^{-1}_H(u_{<0>_{[-1]}})) \\
& \xi^{-2}_M((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-2}_M(m))_0) \otimes u_{<1>} e_i \\
& = \xi^{-2}_M((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-2}_M(m))_0) \otimes (\xi^{-1}_H(u_{<1>_{[-1]}})S^{-1} \xi_H(u_{<0>_{[0]_{<1>}}})) \\
& \xi^{-3}_M((\xi_A(u_{<0>_{[0]_{<0>}}}) \cdot \xi^{-2}_M(m))_0) \otimes (u_{<1>_2} S^{-1}(u_{<1>_2}) \xi^{-2}_M(m)_1 \xi_H(u_{<0>_{[-1]}})) \\
& \xi_M((\xi_A(u_{[0]_0}) \cdot \xi^{-1}_M(m))_0) \otimes (u_{[0]_0} : \xi^{-1}_M(m))_1 u_{[1]_1},
\end{align*}
\]

for all \( u \in (A, \xi_A) \) and \( m \in (M, \xi_M) \), and this finishes the proof. \( \blacksquare \)

The next result generalizes ([2], Prop. 4.3), which is recovered by taking \( H = A \).

**Theorem 2.7.** Let \((H, \xi_H)\) be a monoidal Hom-Hopf algebra and \((H, A, H)\) a Yetter-Drinfeld datum, assuming \( H \) to be finite dimensional. Then the categories \( \mathcal{AMH} \mathcal{YD}H(H) \) and \( H \otimes_A M \) are isomorphic.

**Proof.** We have to verify that the functors \( F \) and \( G \) defined in Lemmas 2.5 and 2.6 are inverse to each other. Let \( M \in \mathcal{AMH} \mathcal{YD}H(H) \). The structures on \( G(F(M)) \) are denoted by `' and \( \rho_M \). For any \( u \in (A, \xi_A) \) and \( m \in (M, \xi_M) \) we have that

\[
\begin{align*}
u \cdot m = (\varepsilon \triangleright \xi^{-1}_A(u)) \triangleright m = (\varepsilon, (\xi_A^{-1}(u) \cdot \xi^{-1}_M(m))_1) \xi^{-2}_M((\xi_A^{-1}(u) \cdot \xi^{-1}_M(m))_0) = u \cdot m.
\end{align*}
\]
We now compute for \( m \in (M, \xi_M) \) that
\[
\rho'_M(m) = \sum_{i=1}^{n} (\xi_H^{\ast 2}(e^i) \triangleright 1_A) \triangleright \xi_M^{-2}(m) \otimes e_i
\]

\[= \sum_{i=1}^{n} (\xi_H^{\ast 2}(e^i), (1_A \cdot \xi_M^{-3}(m))_1) \xi_M^{2}((1_A \cdot \xi_M^{-3}(m))_0) \otimes e_i\]
\[= \sum_{i=1}^{n} \langle e^i, m_1 \rangle m_0 \otimes e_i = \rho_M(m).\]

Conversely, take \( M \in H^{\ast \triangleright A} \mathcal{M} \). We want to show that \( F(G(M)) = M \). If we denote the left \( H^* \triangleright A \)-action on \( F(G(M)) \) by \( \mapsto \), then using Lemmas 2.5 and 2.6 we find, for all \( f \in (H^*, \xi_H^{-1}), u \in (A, \xi_A) \) and \( m \in (M, \xi_M) \):
\[
(f \triangleright u) \mapsto m = \langle f, (u \cdot \xi_M^{-1}(m))_1 \rangle \xi_M^{2}((u \cdot \xi_M^{-1}(m))_0)
\]
\[= \sum_{i=1}^{n} (f, e_i) \xi_M^{2}((\xi_H^{\ast 2}(e^i) \triangleright 1_A) \triangleright \xi_M^{-2}(u \cdot \xi_M^{-1}(m)))
\]
\[= \langle \xi_H^{\ast 2}(f), \xi_M^{-2}(u \cdot \xi_M^{-1}(m)) \rangle \xi_M^{2}(u \cdot \xi_M^{-1}(m))
\]
\[= (f \triangleright u) \triangleright m,
\]
and this finishes our proof. \(\blacksquare\)

**Proposition 2.8.** Let \((H, \xi_H)\) be finite dimensional and \(H(\alpha, \beta)\) be an \(H\)-\(\text{Hom}\)-bicomodule algebra, with an \(H\)-\(\text{Hom}\)-comodule structures showed in Example 2.9 (in \cite{IS}). Then \(H(\alpha, \beta) \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(H) \simeq_{H^{\ast \triangleright H}(\alpha, \beta)} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(H)\).

The proof is left to the reader.

Recall from Prop.2.12 in \cite{IS}, \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha, \beta) =_{H^{\ast \triangleright H}(\alpha, \beta)} H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(H)\).

**Proposition 2.9.** \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha, \beta) \simeq_{H^{\ast \triangleright H}(\alpha, \beta)} \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha, \beta)\).

We just give the correspondence as follows. If \( M \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha, \beta) \), then \( M \in H^{\ast \triangleright H}(\alpha, \beta) \mathcal{M} \) with structure
\[
(f \triangleright h) \triangleright m = f((h \cdot \xi_M^{-1}(m))_1) \xi_M^{2}((h \cdot \xi_M^{-1}(m))_0).
\]

Conversely, if \( M \in H^{\ast \triangleright H}(\alpha, \beta) \mathcal{M} \), then \( M \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha, \beta) \) with structures
\[
h \cdot m = (\varepsilon_H \triangleright \xi_H^{-1}(h)) \triangleright m,
\]
\[
\rho_M(m) = m_0 \otimes m_1 = (\sum_{i=1}^{n} \xi_H^{\ast 2}(e^i) \triangleright 1_A) \triangleright \xi_M^{-2}(m) \otimes e_i
\]
for all \( f \in H^*, h \in H, m \in M \), where \( \{e_i\}_{1, \ldots, n} \) and \( \{e^i\}_{1, \ldots, n} \) are dual bases in \( H \) and \( H^* \). The proof is left to the reader.
3. A BRAIDED T-CATEGORY $\text{Rep}(\mathcal{MH}D(H))$

Denote $G = \text{Aut}_{mHH}(H) \times \text{Aut}_{mHH}(H)$ a group with multiplication as follows: for all $\alpha, \beta, \gamma, \delta \in \text{Aut}_{mHH}(H)$,

$$(\alpha, \beta) \ast (\gamma, \delta) = (\alpha \gamma, \delta \gamma^{-1} \beta \gamma).$$

The unit of this group is $(id, id)$ and $(\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha^{-1} \beta^{-1} \alpha^{-1})$.

In this section we will construct a monoidal Hom-Hopf $T$-coalgebra over $G$, denoted by $\mathcal{MH}D(H)$, and prove that the $T$-category $\text{Rep}(\mathcal{MH}D(H))$ of representation of $\mathcal{MH}D(H)$ is isomorphic to $\mathcal{MH}YD(H)$ in [18] as braided $T$-categories.

**Proposition 3.1.** Let $(M, \xi_M) \in \mathcal{MH}YD(H)(\alpha, \beta)$ and assume that $(M, \xi_M)$ is finite dimensional. Then $(M^*, \xi_M^{-1}) = \text{Hom}(M, k)$ becomes an object in $\mathcal{MH}YD(H)(\alpha^{-1}, \alpha \beta^{-1} \alpha^{-1})$, with module structure

$$(h \cdot p)(m) = p(\beta^{-1} \alpha^{-1} S \xi_H^{-1}(h) \cdot M^{-2}(m)),$$

and comodule structure

$$\rho(p)(m) = p_0(\xi_M^{-1}(m)) \otimes H(p_1) = p(\xi_M(m_0)) \otimes S^{-1}\xi_H^2(m_1),$$

for all $h \in H, p \in M^*$ and $m \in M$. Moreover, the maps $b_M : k \to M \otimes M^*$, $b_M(1) = \sum_i c_i \otimes c^i$ (where $\{c_i\}$ and $\{c^i\}$ are dual bases in $M$ and $M^*$) and $d_M : M^* \otimes M \to k$, $d_M(p \otimes m) = p(m)$, are left $H$-module maps and right $H$-comodule maps and we have

$$(\xi_M \otimes d_M)(b_M \otimes \xi_M^{-1}) = id_M, \quad (d_M \otimes \xi_M^{-1})(\xi_M^* \otimes b_M) = id_{M^*}.$$

**Proof.** Following the idea of the proof of Panaite and Staic ([10], Prop. 3.6), we first
prove that \((M^*,\xi_M^{-1})\) is indeed an object in \(H\mathcal{M}H\mathcal{YD}^H(\alpha^{-1},\alpha \beta^{-1}\alpha^{-1})\). We compute:

\[
(\xi_H(h_{21}) \cdot p_0)(m) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22}) \xi_{H}^{-1}(p_1))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p_0(\beta^{-1} \alpha^{-1}S(h_{21}) \cdot \xi_M^2(m) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22}) \xi_{H}^{-1}(p_1))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\xi_M^2((\beta^{-1} \alpha^{-1}S(h_{21}) \cdot \xi_M^2(m))_0) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22})S^{-1}((\beta^{-1} \alpha^{-1}S(h_{21}) \cdot \xi_M^2(m))_1))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\xi_M^2(\beta^{-1} \alpha^{-1}S:\xi_H(h_{2112}) \cdot \xi_{H}^{-1}(m_0)) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22})S^{-1}((\alpha \beta^{-1} \alpha^{-1}S(h_{2111}) \xi_{H}^{-3}(m_1))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22})S^{-1}(h_{212}))
\]

\[
(\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

\[
= p(\beta^{-1} \alpha^{-1}S:\xi_H^3(h_{2112}) \cdot m_0) \otimes ((\alpha \beta^{-1} \alpha^{-1}\xi_{H}^{-1}(h_{22})\alpha \beta^{-1} \alpha^{-1}S^{-1}(h_{212}))\alpha^{-1}S^{-1}(h_1)
\]

which means that

\[
(h \cdot p)_0 \otimes (h \cdot p)_1 = (\xi_H(h_{21}) \cdot p_0) \otimes (\alpha \beta^{-1} \alpha^{-1}(h_{22})\xi_{H}^{-1}(p_1))\alpha^{-1}S^{-1}(h_1).
\]

On \(k\) we have the trivial Hom-module and Hom-comodule structure, and with these
\(k \in H \mathcal{YD}^H\). We want to prove that \(b_M\) and \(d_M\) are \(H\)-Hom-module maps. We compute:

\[
(h \cdot b_M(1))(m) = (h \cdot (\sum_i c_i \otimes c^i))(m)
\]

\[
= \sum_i \alpha^{-1}(h_1) \cdot c_i \otimes (\alpha \beta \alpha^{-1}(h_2) \cdot c^i)(m)
\]

\[
= \sum_i \alpha^{-1}(h_1) \cdot c_i \otimes c^i (\beta^{-1} \alpha^{-1}S \alpha \beta \alpha^{-1} \xi_{H}^{-1}(h_2) \cdot \xi_{M}^{-2}(m))
\]

\[
= \sum_i \alpha^{-1}(h_1) \cdot c_i \otimes c^i (S \alpha^{-1} \xi_{H}^{-1}(h_2) \cdot \xi_{M}^{-2}(m))
\]

\[
= \alpha^{-1}(h_1) \cdot (S \alpha^{-1} \xi_{H}^{-1}(h_2) \cdot \xi_{M}^{-2}(m))
\]

\[
= \alpha^{-1}(\xi^{-1}(h_1) S \xi_{H}^{-1}(h_2)) \cdot \xi_{M}^{-1}(m)
\]

\[
= (\varepsilon(h) \sum_i c_i \otimes c^i(m)
\]

\[
= (\varepsilon(h) b_M(1))(m),
\]
\[
\begin{align*}
    d_M(h \cdot (p \otimes m)) &= d_M(\alpha(h_1) \cdot p \otimes \beta^{-1}(h_2) \cdot m) \\
    &= (\alpha(h_1) \cdot p)(\beta^{-1}(h_2) \cdot m) \\
    &= p(\beta^{-1}\alpha^{-1}S\alpha \xi_H^{-1}(h_1) \cdot \xi_{M}^{-2}(\beta^{-1}(h_2) \cdot m)) \\
    &= p(\beta^{-1}(S\xi_{H}^{-2}(h_1)\xi_{H}^{-2}(h_2)) \cdot \xi_{M}^{-1}(m)) \\
    &= \varepsilon(h)d_M(p \otimes m).
\end{align*}
\]

They also are $H$-Hom-comodule maps;

\[
\begin{align*}
    ((b_M(1))_0 \otimes (b_M(1))_1)(m) &= \sum_i (c_i)_0 \otimes (c'_i)_0(m) \otimes (c'_i)_1(c_i)_1 \\
    &= \sum_i (c_i)_0 \otimes c'_i(\xi_M^2(m_0)) \otimes S^{-1}\xi_H(m_1)(c_i)_1 \\
    &= \xi_M^2(m_0) \otimes S^{-1}\xi_H(m_1)\xi_H^2(m_01) \\
    &= \xi_M(m_0) \otimes S^{-1}\xi_H^2(m_12)\xi_H^2(m_{11}) \\
    &= (b_M(1) \otimes 1)(m),
\end{align*}
\]

\[
\begin{align*}
    d_M((p \otimes m)_0) \otimes (p \otimes m)_1 &= p_0(m_0) \otimes m_1p_1 \\
    &= p(\xi_M^2(m_{00})) \otimes m_1S^{-1}\xi_H(m_01) \\
    &= p(\xi_M(m_0)) \otimes \xi_H(m_{12})S^{-1}\xi_H(m_{11}) \\
    &= d_M(p \otimes m) \otimes 1.
\end{align*}
\]

Finally, we compute:

\[
\begin{align*}
    (\xi_M \otimes d_M)(b_M \otimes \xi_M^{-1})(m) &= (\xi_M \otimes d_M)(b_M(1) \otimes \xi_M^{-1}(m)) \\
    &= (\xi_M \otimes d_M)(\sum_i (c_i \otimes c'_i) \otimes \xi_M^{-1}(m)) \\
    &= \sum_i \xi_M^2(c_i) \otimes c'_i(\xi_M^{-2}(m)) = m
\end{align*}
\]

The argument for \((d_M \otimes \xi_M^{-1})(\xi_M \otimes b_M) = id_{M^*}\) is analogous. \(\blacksquare\)

Similarly, one can obtain:

**Proposition 3.2.** Let \((M, \xi_M) \in H \mathcal{MHYD}^H(\alpha, \beta)\) and assume that \((M, \xi_M)\) is finite dimensional. Then \((M, \xi_M^{-1}) = Hom(M, k)\) becomes an object in \(H \mathcal{MHYD}^H(\alpha^{-1}, \alpha \beta^{-1} \alpha^{-1})\), with module structure

\[
(h \cdot p)(m) = p(\beta^{-1}\alpha^{-1}S\xi_H^{-1}(h) \cdot \xi_M^{-2}(m)),
\]

and comodule structure

\[
\rho(p)(m) = p_0(\xi_M^{-1}(m)) \otimes \xi_H(p_1) = p(\xi_M(m_0)) \otimes S^{-1}\xi_H^2(m_1),
\]

for all \(h \in H, \ p \in *M\) and \(m \in M\). Moreover, the maps \(b_M : k \to M \otimes *M, \ b_M(1) = \sum_i c_i \otimes c'_i\) (where \(\{c_i\}\) and \(\{c'_i\}\) are dual bases in \(M\) and \(*M\)) and \(d_M : *M \otimes M \to k, \ d_M(p \otimes m) = p(m)\), are left \(H\)-module maps and right \(H\)-comodule maps and we have

\[
(\xi_M \otimes d_M)(b_M \otimes \xi_M^{-1}) = id_M, \quad (d_M \otimes \xi_M^{-1})(\xi_M \otimes b_M) = id_{M^*}.
\]
Now, if we consider $\mathcal{MHYD}(H)_{fd}$, the subcategory of $\mathcal{MHYD}(H)$ consisting of finite dimensional objects, then by Proposition 3.1. and Proposition 3.2. we obtain:

**Corollary 3.3.** $\mathcal{MHYD}(H)_{fd}$ is a braided $T$-category with left and right dualities over $G$.

Assume now that $(H, \xi_H)$ is finite dimensional. We will construct a monoidal Hom-Hopf $T$-coalgebra over $G$, denoted by $\mathcal{MHD}(H)$, with the property that the $T$-category $\text{Rep}(\mathcal{MHD}(H))$ of representation of $\mathcal{MHD}(H)$ is isomorphic to $\mathcal{MHYD}(H)$ as braided $T$-categories.

**Theorem 3.4.** $\mathcal{MHD}(H) = \{\mathcal{MHD}(H)_{(\alpha,\beta)}\}_{(\alpha,\beta) \in G}$ is a monoidal Hom-Hopf $T$-coalgebra with the following structures:

- For any $(\alpha, \beta) \in G$, the $(\alpha, \beta)$-component $\mathcal{MHD}(H)_{(\alpha,\beta)}$ will be the diagonal crossed Hom-product algebra $H^* \bowtie H(\alpha, \beta)$ in Eq. (2.4),

- The comultiplication on $\mathcal{MHD}(H)$ is given by
  \[
  \Delta_{(\alpha,\beta),(\gamma,\delta)} : \mathcal{MHD}(H)_{(\alpha,\beta) (\gamma,\delta)} \to \mathcal{MHD}(H)_{(\alpha,\beta)} \otimes \mathcal{MHD}(H)_{(\gamma,\delta)},
  \]
  \[
  \Delta_{(\alpha,\beta),(\gamma,\delta)}(f \bowtie h) = (f_1 \bowtie \gamma(h_1)) \otimes (f_2 \bowtie \gamma^{-1}\beta\gamma(h_2)),
  \]

- The counit $\varepsilon$ is obtained by setting
  \[
  \varepsilon(f \bowtie h) = \varepsilon(h)f(1_H),
  \]

- For any $(\alpha, \beta) \in G$, the $(\alpha, \beta)^{th}$ component of the antipode of $\mathcal{MHD}(H)$ is given by
  \[
  S_{(\alpha,\beta)} : \mathcal{MHD}(H)_{(\alpha,\beta)} \to \mathcal{MHD}(H)_{(\alpha,\beta)^{-1}} = \mathcal{MHD}(H)_{(\alpha^{-1},\alpha^{-1}\beta^{-1})},
  \]
  \[
  S_{(\alpha,\beta)}(f \bowtie h) = (\varepsilon \bowtie \alpha\beta S\xi_H^{-1}(h))(S^{\alpha^{-1}\beta^{-1}\gamma^*}(f) \bowtie 1_H),
  \]

- For $(\alpha, \beta), (\gamma, \delta) \in G$, the conjugation isomorphism is given by
  \[
  \varphi_{(\gamma,\delta)}^{(\alpha,\beta)} : \mathcal{MHD}(H)_{(\gamma,\delta)} \to \mathcal{MHD}(H)_{(\alpha,\beta) (\gamma,\delta) (\alpha,\beta)^{-1}},
  \]
  \[
  \varphi_{(\gamma,\delta)}^{(\alpha,\beta)}(f \bowtie h) = (f \circ \beta \alpha^{-1} \bowtie \alpha^{-1}\beta^{-1}\gamma(h)),
  \]
  for all $f \in H^*, h \in H$. 

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Proof. We have to check the axioms of monoidal Hom-Hopf \( T \)-coalgebra. Hom-coassociativity and multiplicativity of \( \Delta \) are satisfied. We compute

\[
m_{(\alpha, \beta)}(id \otimes S(\alpha^{-1})\Delta(\alpha, \beta)) \Delta(\alpha, \beta)^{-1}(f \triangleright h) \\
= (f_1 \triangleright \alpha^{-1}(h_1))[(\varepsilon \otimes S\alpha^{-1}\xi_H^{-1}(h_2))(S^* \xi^*(f_2) \triangleright 1_H)] \\
= [(\xi_H^2(f_1) \triangleright \alpha^{-1}\xi_H^{-1}(h_1))(\varepsilon \otimes S\alpha^{-1}\xi_H^{-1}(h_2))](S^* \xi^*(f_2) \triangleright 1_H) \\
= [\xi_H^2((\alpha^{-1}\xi_H^{-1}(h_1)))(\varepsilon \otimes S\alpha^{-1}\xi_H^{-1}(h_2))](S^* \xi^*(f_2) \triangleright 1_H) \\
= (f_1 \triangleright \alpha^{-1}\xi_H^{-1}(h_1)S(h_2))(S^* \xi^*(f_2) \triangleright 1_H) \\
= \varepsilon(h)(f_1 \triangleright 1_H)(S^* \xi^*(f_2) \triangleright 1_H) \\
= \varepsilon(f \triangleright h)(\varepsilon \otimes 1_H),
\]

and similarly \( m_{(\alpha, \beta)}(S(\alpha^{-1}) \otimes id) \Delta(\alpha, \beta)^{-1}(f \triangleright h) = \varepsilon(f \triangleright h)(\varepsilon \otimes 1_H) \). This completes the proof.

Moreover, via the isomorphisms Prop.2.9. \( H\lambda\mu\nu(H, \alpha, \beta) \simeq H \otimes H(\alpha, \beta) \lambda\mu\nu \mathcal{M}, \) we obtain

\[\textbf{Theorem 3.5.} \ Rep(\mathcal{M}(H)) \simeq \mathcal{M}(\mathcal{H}(H)) \text{ as braided } T\text{-categories over } G.\]

4. A BRAIDED \textit{T-CATEGORY} \( \mathcal{Z}\mathcal{M}H\mathcal{YD}(H) \)

In this section, we will construct a new braided \( T \)-category \( \mathcal{Z}\mathcal{M}H\mathcal{YD}(H) \) over \( \mathbb{Z} \).

**Definition 4.1.** Let \((C, \xi_C)\) be a monoidal Hom-coalgebra. Then \( g \) is called a group-like element, that is

\[\xi_C(g) = g, \quad \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1,\]

for all \( g \in C \).

**Example 4.2.** Recall from Example 3.5 in [3] that \((H_4 = k\{1, g, x, y = gx\}, \xi_{H_4}, \Delta, \varepsilon, S)\) is a monoidal Hom-Hopf algebra, where the algebraic structure are given as follows:

- The multiplication “\( \circ \)” is given by

| \( \circ \) | 1_{H_4} | g | x | y |
|---|---|---|---|---|
| 1_{H_4} | g | cx | cy |
| g | g | 1_{H_4} | cy | cx |
| x | cx | -cy | 0 | 0 |
| y | cy | -cx | 0 | 0 |
• The automorphism $\xi_{H_4}$ is given by $\xi_{H_4}(1) = 1$, $\xi_{H_4}(g) = g$, $\xi_{H_4}(x) = cx$, $\xi_{H_4}(gx) = cgx$, for all $0 \neq c \in k$.

• The comultiplication $\Delta$ is defined by

$$
\Delta(1) = 1 \otimes 1, \quad \Delta(g) = g \otimes g,
\Delta(x) = c^{-1}(x \otimes 1) + c^{-1}(g \otimes x), \quad \Delta(gx) = c^{-1}(gx \otimes g) + c^{-1}(1 \otimes gx);
$$

• The counit $\varepsilon$ is defined by $\varepsilon(1) = 1$, $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, $\varepsilon(gx) = 0$.

• The antipode $S$ is given by $S(1) = 1$, $S(g) = g$, $S(x) = -gx$, $S(gx) = -x$.

Then $1_{H_4}, g$ are group-like elements of $H_4$.

In [18], the authors introduced the notion of left-right $(\alpha, \beta)$-Yetter-Drinfeld Hom-module Definition 1.2.6. We will in the section give its some special cases.

**Example 4.3.** For $(M, \xi_M) \in_H \mathcal{MHYD}^H(S^2, id)$, the left-right anti-Yetter-Drinfeld Hom-module category, i.e., the compatibility condition is

$$(h \cdot m)_0 \otimes (h \cdot m)_1 = \xi_H(h_{21}) \cdot m_0 \otimes (h_{22} \xi_H^{-1}(m_1)) S(h_1),$$

for $h \in H, m \in M$.

**Example 4.4.** In $\mathcal{MHYD}^H(S^{2n}, id)$, the object is called a left-right $n$-Yetter-Drinfeld Hom-modules, i.e., $n$-MHYD-module, for $(M, \xi_M) \in_H \mathcal{MHYD}^H(S^{2n}, id)$, the compatibility condition is

$$(h \cdot m)_0 \otimes (h \cdot m)_1 = \xi_H(h_{21}) \cdot m_0 \otimes (h_{22} \xi_H^{-1}(m_1)) S^{2n-1}(h_1),$$

for $h \in H, m \in M$.

**Example 4.5.** Similar to Panaite and Staic [10], Example 2.7, for $\alpha, \beta \in Aut_{mHH}(H)$, and assume that there is an algebra map $\theta : H \to k$ and a group-like element $\omega \in (H, \xi_H)$ such that

$$\alpha(h) = \omega^{-1}(\theta(h_{11}) \beta(h_{12}) \theta(S(h_2))) \omega, \quad \forall h \in H.$$

Then we can check that $k \in H \mathcal{MHYD}^H(\alpha, \beta)$ with structures: $h \cdot 1 = \theta(h)$ and $\rho(1) = 1 \otimes \omega$. More generally, if $V$ is any vector space, then $(V, \xi_V) \in H \mathcal{MHYD}^H(\alpha, \beta)$, with structures $h \cdot v = \theta(h) \xi_V(v)$ and $\rho(v) = v_0 \otimes v_1 = \xi_V^{-1}(v) \otimes \omega$, for all $h \in H$ and $v \in V$.

If $\alpha, \beta \in Aut_{mHH}(H)$ such that there exist $\theta, \omega$ as show in Example 4.1, we will say that $(\theta, \omega)$ is a pair in involution corresponding to $(\alpha, \beta)$ and the left-right $(\alpha, \beta)$-Yetter-Drinfeld Hom-modules $k$ and $(V, \xi_V)$ will be denoted by $g_k\omega$ and $gV\omega$, respectively.

In the following, we will show that in the presence of a pair in involution, there exists an isomorphism of categories $H \mathcal{MHYD}^H(\alpha, \beta) \simeq_H \mathcal{MHYD}^H$.

**Proposition 4.6.** Let $\alpha, \beta \in Aut_{mHH}(H)$ and assume that there exists $(\theta, \omega)$ a pair in involution corresponding to $(\alpha, \beta)$. Then the categories $H \mathcal{MHYD}^H(\alpha, \beta)$ and $H \mathcal{MHYD}^H$ are isomorphic.
**Proof.** In order to prove the isomorphism between two categories, we only need to give a pair of inverse functors. The functors pair \((F,G)\) is given as follows.

If \((M,\xi_M) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta)\), then \((F(M),\xi_M) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H\), where \(F(M) = M\) as vector space, with structures
\[
h \rightarrow m = \theta(S(h_1))\beta^{-1}(h_2) \cdot m,
\]
\[
\rho(m) =: m_{<0>} \otimes m_{<1>} = m_0 \otimes m_1\omega^{-1}.
\]

If \((N,\xi_N) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H\), then \((G(N),\xi_N) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta)\), where \(G(N) = N\) as vector space, with module and comodule structures
\[
h \rightarrow n = \theta(h_1)\beta(h_2) \cdot n,
\]
\[
\rho(n) =: n_{<0>} \otimes n_{<1>} = n_0 \otimes n_1\omega.
\]
Both \(F\) and \(G\) act as identities on morphisms.

One checks that \(F\) and \(G\) are functors and inverse to each other. ■

**Proposition 4.7.** Let \(\alpha,\beta,\gamma \in Aut_{mHH}(H)\). The categories \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H((\alpha,\beta,\gamma))\) and \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H((\alpha,\beta))\) are isomorphic.

**Proof.** A pair of inverse functors \((F,G)\) is given as follows.

If \((M,\xi_M) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\gamma)\), then \((F(M),\xi_M) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\gamma)\), where \(F(M) = M\) as vector space, with structures
\[
h \rightarrow m = \beta^{-1}(h) \cdot m,
\]
\[
\rho(m) =: m_{<0>} \otimes m_{<1>} = m_0 \otimes m_1\]

If \((N,\xi_N) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\gamma)\), then \((G(N),\xi_N) \in H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\gamma)\), where \(G(N) = N\) as vector space, with module and comodule structures
\[
h \rightarrow n = \beta(h) \cdot n,
\]
\[
\rho(n) =: n_{<0>} \otimes n_{<1>} = n_0 \otimes n_1\]
Both \(F\) and \(G\) act on morphisms as identities.

We can check that \(F\) and \(G\) are functors and inverse to each other. This completes the proof. ■

**Corollary 4.8.** For all \(\alpha,\beta \in Aut_{mHH}(H)\), we have isomorphisms of categories:
\[
H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha^{-1},\beta),
\]
\[
H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\alpha) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H,
\]
\[
H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha^{-1},\beta),
\]
\[
H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\beta,\beta) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\beta^{-1},\beta),
\]
\[
H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\beta,\alpha) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\beta^{-1},\alpha).
\]

Let again \(\alpha,\beta \in Aut_{mHH}(H)\) such that there exist \((\theta,\omega)\) a pair in involution corresponding to \((\alpha,\beta)\), and assume that \((H,\xi_H)\) is finite dimensional. Then we know that \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta) \simeq_{H^* \otimes H(\alpha,\beta)} \mathcal{M}\), \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H \simeq_{D(H)} \mathcal{M}\) (H, Proposition 4.3), and the isomorphism \(H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H(\alpha,\beta) \simeq H \mathcal{M} \mathcal{H} \mathcal{Y} \mathcal{D}^H\) constructed in the theorem is induced by a monoidal Hom-algebra isomorphism as follows.
Corollary 4.9. \((H^* \triangleright H(\alpha, \beta), \xi_H^{-1} \otimes \xi_H) \simeq (D(H), \xi_H^{-1} \otimes \xi_H)\) as monoidal Hom-algebras, given by

\[
D(H) \to H^* \triangleright H(\alpha, \beta), \quad f \otimes h \mapsto \omega^{-1} \mapsto f \triangleright \theta(\beta^{-1}(S(h_1)))\beta^{-1}(h_2), \\
H^* \triangleright H(\alpha, \beta) \to D(H), \quad f \triangleright h \mapsto f \otimes \theta(h_1)\beta(h_2).
\]

for all \(h \in H, f \in H^*\), and a group-like element \(\omega \in H\).

Finally, we consider some special cases, which are shown in Example 4.3. Similar to the cases in Staic [11], we give the following two propositions.

We define the modular pair \((\omega, \theta)\) in monoidal Hom-Hopf algebra \((H, \xi_H)\), i.e., \(\theta\) is an algebra map \(H \to k\) and \(\omega \in (H, \xi_H)\) is a group-like element satisfying \(\theta(\omega) = 1\). Defining an endomorphism \(\tilde{S}\) of \((H, \xi_H)\) by \(\tilde{S}(h) = S(h_1)\theta(h_2)\) for all \(h \in H\), then \((\omega, \theta)\) is called a modular pair in involution if \(\tilde{S}^2(h) = \omega^{-1}(h\omega)\).

Proposition 4.10. Let \((H, \xi_H)\) be a monoidal Hom-Hopf algebra, \((\omega, \theta)\) a modular pair in involution and \((M, \xi_M)\) a left-right anti-Yetter-Drinfeld Hom-module. If we define a new action of \(H\) on \(M\) as:

\[
h \to m = \theta(S(h_1))h_2 \cdot m,
\]

and a new coaction as follows:

\[
\rho(m) = m_{<0>} \otimes m_{<1>} = m_0 \otimes m_1\omega^{-1},
\]

then \((M, \to, \rho)\) is a left-right Yetter-Drinfeld Hom-module.

**Proof.** First, since \(\theta : H \to k\) is an algebra morphism and \(\omega\) is a group-like element, the module and comodule structures are given by above formulas.

We denote the involution inverse of \(\theta\) by \(\theta^{-1}\). From \(\tilde{S}^2(h) = \omega^{-1}(h\omega)\), we can get \(\theta(S(h_{11}))\tilde{S}^2(h_{12})\theta(h_{22}) = \omega^{-1}(h\omega)\) and \(\theta^{-1}(h_1)S(h_{21})\theta(h_{22}) = \omega^{-1}(\tilde{S}^{-1}(h)\omega)\):

\[
(h \to m)_{<0>} \otimes (h \to m)_{<1>}
= \theta^{-1}(h_1)(h_2 \cdot m)_0 \otimes (h_2 \cdot m)_{1}\omega^{-1}
= \theta^{-1}(h_1)\xi_H(h_{221}) \cdot m_0 \otimes ((h_{222}\xi_H^{-1}(m)_{1})S(h_{21}))\omega^{-1}
= h_{21} \cdot m_0 \otimes ((\xi_H^{-1}(h_{22})\xi_H^{-1}(m_{1}))\theta^{-1}(\xi_H(h_{11}))S(h_{12}))\omega^{-1}
= h_{21} \cdot m_0 \otimes ((\xi_H^{-1}(h_{22})\xi_H^{-1}(m_{1}))((\omega^{-1}\theta^{-1}(\xi_H(h_{111}))S(h_{121})\theta(h_{1221})\theta^{-1}(h_{1222}))\omega^{-1})
= h_{21} \cdot m_{<0>} \otimes ((h_{22}\xi_H^{-1}(m_{1}))((\omega^{-1}\theta^{-1}(\xi_H(h_{111}))S(h_{121})\theta(h_{1221})\theta^{-1}(h_{1222}))\omega^{-1}))
\theta^{-1}(\xi_H^{-2}(h_{121}))
= h_{21} \cdot m_{<0>} \otimes ((h_{22}\xi_H^{-1}(m_{1}))S^{-1}\xi_H(h_{11})\theta^{-1}(\xi_H^{-2}(h_{121}))
= \theta^{-1}(\xi_H(h_{211}))\xi_H(h_{212}) \cdot m_{<0>} \otimes ((h_{22}\xi_H^{-1}(m_{1}))S^{-1}(h_1)
= \xi_H(h_{21}) \cdot m_{<0>} \otimes ((h_{22}\xi_H^{-1}(m_{1}))S^{-1}(h_1)
\]

This means that \((M, \to, \rho)\) is a left-right Yetter-Drinfeld Hom-module. \(\blacksquare\)
By Example 4.4 and Remark 1.2.7 (3), we have the following proposition.

**Proposition 4.11.** For any integer numbers $m$ and $n$, if $(M, \xi_M)$ is a left-right $m$-Yetter-Drinfeld Hom-module and $(N, \xi_N)$ is an $n$-Yetter-Drinfeld Hom-module, then $(M \otimes N, \xi_M \otimes \xi_N)$ is a left-right $m+n$-Yetter-Drinfeld Hom-module with module structure and comodule structure as follows:

\[
(h \cdot (m \otimes n)) = S^{2n}(h_1) \cdot m \otimes h_2 \cdot n,
\]

\[
m \otimes n \mapsto (m_0 \otimes n_0) \otimes n_1 m_1.
\]

for all $m \in M$, $n \in N$ and $h \in H$.

Let $\mathcal{ZMHYD}(H)$ be the disjoint union of all categories $\mathcal{MHYD}^H(S^{2n}, id)$ of left-right $n$-Yetter-Drinfeld Hom-modules with $n \in \mathbb{Z}$, the set of integer numbers. Then by Theorem 3.7 in [15] and Proposition 4.11, the following corollary is a generalization of the main result in Staic [11].

**Corollary 4.12.** $\mathcal{ZMHYD}(H)$ is a braided T-category over $\mathbb{Z}$.

**Example 4.13.** Let $A = \langle a \rangle$ be a cyclic group of order $n$, and $Aut(A) = \{\sigma_t : \sigma_t(a) = a^t, 0 < t < n, (t, n) = 1, t \in \mathbb{Z}\}$. Then $(k[A], \xi_{k[A]})$ is a monoidal Hom-Hopf algebra with structure given by

\[
a^i \circ a^j = \xi_{k[A]}^{-1}(a^i a^j), \quad \Delta(a^i) = \xi_{k[A]}^{-1}(a^i) \otimes \xi_{k[A]}^{-1}(a^i),
\]

\[
\varepsilon(a^i) = 1_k, \quad S(a^i) = a^{-i},
\]

for all $i, j \in \mathbb{Z}$.

First, $Aut_{mHH}(k[A]) = Aut(A)$. Let $(H, \xi_H) = (k[A], \xi_{k[A]} = \sigma_2)$ is a monoidal Hom-Hopf algebra given by

\[
a^i \circ a^j = \sigma_2^{-1}(a^i a^j) = a^{i+j-2}, \quad \Delta(a^i) = \sigma_2^{-1}(a^i) \otimes \sigma_2^{-1}(a^i) = a^{i-2} \otimes a^{i-2},
\]

\[
\varepsilon(a^i) = 1_k, \quad S(a^i) = a^{-i},
\]

for all $i, j \in \mathbb{Z}$. It is easy to check that

\[
S^{2n}(a^i) = a^i,
\]

for all $n \in \mathbb{Z}$.

Let $\mathcal{ZMHYD}(k[A])$ be the disjoint union of all categories $k[A], \mathcal{MHYD}^{k[A]}(S^{2n}, id)$ of left-right $n - \mathcal{MHYD}$ with $n \in \mathbb{Z}$.

Let $(M, \xi_M)$ be an $m - \mathcal{MHYD}$-module and $(N, \xi_N)$ be an $n - \mathcal{MHYD}$-module, for all $m, n \in \mathbb{Z}$. Then $(M \otimes N, \xi_M \otimes \xi_N)$ is $m+n - \mathcal{MHYD}$-module with structures as follows:

\[
a^i \cdot (x \otimes y) = S^{2n}(a^{i-2}) \cdot x \otimes a^{i-2} \cdot y,
\]

\[
(x \otimes y) \mapsto (x_0 \otimes y_0) \otimes y_1 x_1,
\]

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for all \( x \in M, y \in N, a^i \in k[A], n \in \mathbb{Z} \).

On \((S^{2m,\text{id}})N = N\), there is an action \( \triangleright \) given by

\[
a^i \triangleright y = S^{-2m}(a^i) \cdot y,
\]

and a coaction \( \rho \) defined by

\[
y \mapsto y_0 \otimes S^{2m}(y_1),
\]

\( y \in N, a^i \in k[A], m \in \mathbb{Z} \).

Let \((M, \xi_M)\) be an \(m - \mathcal{MHYD}\)-module and \((N, \xi_N)\) be an \(n - \mathcal{MHYD}\)-module, for all \(m, n \in \mathbb{Z}\). Then the braiding

\[
c_{M,N} : M \otimes N \to (S^{2m,\text{id}}) N \otimes M
\]

is given by

\[
c_{M,N}(x \otimes y) = \xi_N(y_0) \otimes y_1 \cdot \xi_M^{-1}(x),
\]

for all \( x \in M, y \in N, m \in \mathbb{Z} \).

Then by Corollary 4.12, \( \mathcal{ZMHYD}(k[A]) \) is a new braided \( T \)-category over \( \mathbb{Z} \).

**ACKNOWLEDGEMENTS**

This work was supported by the NSF of China (No. 11371088) and the NSF of Jiangsu Province (No. BK2012736).

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