Collisionless dynamics of dilute Bose gases: 
Role of quantum and thermal fluctuations

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We study the low-energy collective oscillations of a dilute Bose gas at finite temperature in the collisionless regime. By using a time-dependent mean-field scheme we derive for the dynamics of the condensate and noncondensate components a set of coupled equations, which we solve perturbatively to second order in the interaction coupling constant. This approach is equivalent to the finite-temperature extension of the Beliaev approximation and includes corrections to the Gross-Pitaevskii theory due both to quantum and thermal fluctuations. For a homogeneous system we explicitly calculate the temperature dependence of the velocity of propagation and damping rate of zero sound.

In the case of harmonically trapped systems in the thermodynamic limit, we calculate, as a function of temperature, the frequency shift of the low-energy compressional and surface modes.

I. INTRODUCTION

The study of collective excitations is one of the main areas of interest for the experimental and theoretical research activity in trapped Bose-condensed gases (for a review of experimental and theoretical investigations see respectively \[1\] and \[2\]). At low temperatures, the frequencies of the low-energy collective oscillations of the condensate have been measured with great accuracy \[3\], and found in very good agreement with the predictions of the mean-field Gross-Pitaevskii theory \[4\], \[5\]. In a series of experiments carried out at JILA \[7\] and MIT \[8\] the excitations of a trapped Bose gas have also been explored as a function of temperature. The main features are: on the one hand oscillations of both the condensate and the thermal cloud are visible and, on the other hand, the oscillations are increasingly damped as temperature is raised and temperature dependent frequency shifts are also observed. A theoretical description which correctly accounts for these phenomena has not yet been fully developed.

At finite temperature the dynamics of Bose-condensed systems is complicated. Depending on the temperature, density and frequency of the excitations one is probing different regimes (for an exhaustive discussion see the books \[9\] and \[10\]). If the frequency \(\omega\) is much smaller than the inverse of the typical collision time \(\tau_c\): \(\omega \tau_c \ll 1\), the excitations are collective collisional modes, which are described by the theory of two-fluid hydrodynamics. In terms of length scales this regime is equivalently defined by the condition: \(\lambda_{ex} \gg \ell_{mfp}\), where \(\lambda_{ex}\) is the wavelength of the excitation and \(\ell_{mfp}\) is its mean free path. At low temperatures and low values of the density the mean free path becomes comparable with the size of the system. In this case, which corresponds to the condition \(\omega \tau_c \gg 1\), one is probing the collisionless regime, which is properly described by mean-field theories. Collisionless modes can be further distinguished into collective and single-particle excitations, depending on whether the excitation energy lies respectively well below or above the chemical potential \(\mu\). Single-particle excitations have wavelength much smaller than the healing length of the condensate, which is defined as \(\xi = 1/\sqrt{\pi \xi_0}\), where \(\xi\) is the s-wave scattering length and \(n_0\) is the condensate density. On the contrary, collective modes satisfy the condition: \(\lambda_{ex} \gg \xi\). Finally, in harmonically trapped systems, collective modes can behave semiclassically if their energy is much larger than the typical trapping energy: \(\hbar \omega_{ho} \ll \hbar \omega \ll \mu\), where \(\omega_{ho}\) is the harmonic oscillator frequency. If instead \(\hbar \omega \sim \hbar \omega_{ho}\), the discretization of levels becomes important and one is not allowed to treat the excitation as quasiclassical.

Collective modes in the collision-dominated regime have been investigated in harmonically trapped systems by several authors \[11\]. The present work is focused on the study of collective excitations in the collisionless regime. In the last years a large number of theoretical papers have appeared in the literature addressing this problem. Mean-field approaches, which extend to finite temperature the Gross-Pitaevskii equation for the order parameter, have been put forward \[12\] and applied to the calculation of the low-energy modes in traps \[13\], \[14\]. However, in these approaches the noncondensate component is treated as a static thermal bath and its dynamic coupling to the oscillations of the condensate is neglected. The results obtained for the collective modes do not adequately reproduce the features observed in experiments, in particular these approaches do not account for the damping of the excitations.

More accurate time-dependent mean-field schemes have been proposed \[15\], \[16\], which describe the coupled dynamics of the condensate and noncondensate components. These methods have been applied to the study of damping in
trapped systems and agree with results obtained from perturbation theory. Explicit calculation of the damping rate of the low-energy modes in harmonic traps has been carried out in and found in good agreement with experiments. Similar methods have also been applied to the calculation of the temperature dependence of the frequency shifts in the collisionless regime. Bijlsma and Stoof have used a variational ansatz to describe the time evolution of the condensate and the thermal cloud and have calculated the frequencies of the coupled modes in which the two components move either in phase or out of phase. These authors also suggest that the avoided crossing between the in and out of phase modes might be the reason of the features observed for the $m = 0$ mode at JILA. Olshanii has explicitly analyzed the JILA $m = 2$ mode, suggesting that the observed temperature dependence of the excitation frequency might be due to a strong resonance between the oscillation frequency of the condensate and one of the eigenfrequencies of the thermal cloud. Fedichev and Shlyapnikov have developed a Green’s function perturbation scheme for inhomogeneous Bose-condensed gases at finite temperature and have calculated energy shifts and damping rates of quasiclassical collective modes, which satisfy the condition $\hbar \omega_{ho} \ll \epsilon \ll \mu$, being $\epsilon$ the energy of the excitation. Very recently, Reidl et al. have calculated by the dielectric formalism the frequency shift of the $m = 0$ and $m = 2$ modes, and compared the results with the JILA experiments.

In the present work we derive, within a time-dependent mean-field scheme, coupled equations for the dynamics of the condensate and noncondensate components. These equations are solved perturbatively to second order in the interaction coupling constant. For homogeneous systems this approach is equivalent to the finite-temperature extension of the Beliaev approximation discussed in. In the homogeneous case we give explicit results for the temperature dependence of the velocity of zero sound, which include effects beyond the Bogoliubov theory. We also apply our analysis to harmonically trapped systems in the thermodynamic limit. In this regime, which is reached for systems with a very large number of trapped particles, one can use the Thomas-Fermi approximation for the condensate and neglect finite-size effects. Under these conditions, which are not difficult to realize in experiments (see e.g. ), the frequencies of the collective modes are found to change with temperature due to static and dynamic correlations beyond the Gross-Pitaevskii theory. We calculate, as a function of temperature, the frequency shifts of the lowest compressional and surface modes. We find that at the intermediate temperatures $T \sim 0.6-0.7 T_c$, where $T_c$ is the transition temperature, the fractional shift due to beyond-mean-field effects is of the order of 5%. This result should be compared with the corresponding correction predicted at very low temperatures and arising from quantum fluctuations, which turns out to be typically of the order of 0.5%.

The structure of the paper is as follows. In Sec. II we develop the time-dependent mean-field scheme and derive coupled equations for the small-amplitude oscillations of the condensate and noncondensate components. Sec. III is devoted to spatially homogeneous systems. First we develop the perturbation scheme and hence we calculate to second order in the interaction coupling constant the equation of state of the system and the speed and damping rate of zero sound. In Sec. IV we apply the same perturbation scheme to harmonically trapped systems in the thermodynamic limit. We calculate the temperature dependence of the frequency shift of the low-energy collective modes and discuss the comparison with experiments. Finally, we show that at zero temperature our approach reproduces the hydrodynamic equations of superfluids.

II. TIME-DEPENDENT MEAN-FIELD SCHEME

Our starting point is the grand-canonical Hamiltonian of the system in the presence of an inhomogeneous external potential $V_{ext}(r)$. In terms of the creation and annihilation particle field operators $\psi^\dagger(r, t)$ and $\psi(r, t)$, the Hamiltonian takes the form

$$H' \equiv H - \mu N = \int \! dr \, \psi^\dagger(r, t) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(r) - \mu \right) \psi(r, t)$$

$$+ \frac{\hbar}{2} \int \! dr \, \psi^\dagger(r, t) \psi^\dagger(r, t) \psi(r, t) \psi(r, t) \ .$$

(1)

In the above equation we have assumed a point-like interaction between particles $V(r - r') = g \delta(r - r')$, with the coupling constant $g$ given by the expression $g = 4\pi \hbar^2 a/m$, to lowest order in the $s$-wave scattering length $a$. The equation of motion for the particle field operator follows directly from the Heisenberg equation and reads

$$i\hbar \frac{\partial}{\partial t} \psi(r, t) = [\psi(r, t), H']$$

$$= \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(r) - \mu \right) \psi(r, t) + g \psi^\dagger(r, t) \psi(r, t) \psi(r, t) \ .$$

(2)
The dynamic equations derived in this section correspond to the linearized time-dependent Hartree-Fock-Bogoliubov (TDHFB) approximation. This self-consistent mean-field scheme is based on the following prescriptions (we use the notations of Ref. [12]):

\[ \psi(r, t) = \Phi(r, t) + \tilde{\psi}(r, t) \]
\[ \Phi(r, t) = \langle \psi(r, t) \rangle \]
\[ \langle \tilde{\psi}(r, t) \rangle = 0 \]

\[ \langle \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) \rangle = \tilde{n}(r, t) \]
\[ \langle \tilde{\psi}(r, t) \tilde{\psi}(r, t) \rangle = \tilde{\tilde{n}}(r, t) \]

\[ \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) \tilde{\psi}(r, t) = 4 \tilde{n}(r, t) \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) + \tilde{\tilde{n}}(r, t) \tilde{\psi}^\dagger(r, t) + \tilde{\tilde{n}}^*(r, t) \tilde{\psi}(r, t) \tilde{\psi}(r, t) \]

\[ \langle \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) \psi(r, t) \rangle = 0 \]
\[ \langle \tilde{\psi}(r, t) \tilde{\psi}(r, t) \psi(r, t) \rangle = 0 \]

The averages \( \langle \ldots \rangle \) in a), b) and d) are nonequilibrium averages, while time-independent equilibrium averages are indicated in this paper with the symbol \( \langle \ldots \rangle_0 \). The prescription a) is the usual decomposition of the field operator into a condensate and a noncondensate component and defines the condensate wave function \( \Phi(r, t) \). Prescription b) defines the normal, \( \tilde{n}(r, t) \), and anomalous, \( \tilde{\tilde{n}}(r, t) \), noncondensate particle densities. In terms of these quantities the interaction term in the Hamiltonian (1) quartic in the noncondensate components of \( \psi(r, t) \) can be approximated using the factorization given by prescription c). Finally, in prescription d) all averages of cubic products of noncondensate operators are set to zero. This is expected to be a good approximation for dilute systems. The inclusion of the triplet correlations \( \langle \tilde{\psi} \tilde{\psi}^\dagger \rangle \) and \( \langle \tilde{\psi} \tilde{\psi} \psi \rangle \) in a time-dependent self-consistent mean-field scheme is discussed in [16,19]. By using the above prescriptions one gets the following equation of motion for the condensate wave function

\[
\frac{i\hbar}{\partial t} \Phi(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(r) - \mu \right) \Phi(r, t) + g \Phi^2(r, t) \Phi(r, t) + 2g \Phi(r, t) \tilde{n}(r, t) + g \Phi^*(r, t) \tilde{\tilde{n}}(r, t) .
\]

(3)

This equation includes the dynamic coupling between the condensate and the noncondensate particles. If we neglect these effects, \( \tilde{n} = \tilde{\tilde{n}} = 0 \), equation (3) reduces to the usual Gross-Pitaevskii (GP) equation.

We are interested in the small-amplitude oscillations of the condensate, which is only slightly displaced from its stationary value \( \Phi_0(r) = \langle \psi(r) \rangle_0 \)

\[
\Phi(r, t) = \Phi_0(r) + \delta \Phi(r, t) ,
\]

(4)

where \( \delta \Phi(r, t) \) is a small fluctuation. In the same way, we consider small fluctuations of the normal and anomalous particle densities

\[
\tilde{n}(r, t) = \tilde{n}_0^0(r) + \delta \tilde{n}(r, t)
\]
\[
\tilde{\tilde{n}}(r, t) = \tilde{n}_0^0(r) + \delta \tilde{\tilde{n}}(r, t)
\]

(5)

around their equilibrium values \( \tilde{n}_0^0(r) = \langle \tilde{\psi}^\dagger(r) \tilde{\psi}(r) \rangle_0 \) and \( \tilde{n}_0^0(r) = \langle \tilde{\psi}(r) \tilde{\psi}(r) \rangle_0 \).

The real wave function \( \Phi_0(r) \) satisfies the stationary equation [31]

\[
\left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(r) - \mu + gn_0(r) + 2g\tilde{n}_0^0(r) + gm\tilde{n}_0^0(r) \right) \Phi_0(r) = 0 ,
\]

(6)

where \( n_0(r) = |\Phi_0(r)|^2 \) is the condensate density. The time-dependent equation for \( \delta \Phi(r, t) \) is obtained by linearizing the equation of motion [32]

\[
\frac{i\hbar}{\partial t} \delta \Phi(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(r) - \mu + 2gn_0(r) \right) \delta \Phi(r, t)
\]
\[
+ \left( gn_0(r) + g\tilde{n}_0^0(r) \right) \delta \Phi^*(r, t) + 2g\Phi_0(r) \delta \tilde{n}(r, t) + g\Phi_0(r) \delta \tilde{\tilde{n}}(r, t) ,
\]

(7)
where we have introduced the total equilibrium density \( n(r) = n_0(r) + \tilde{n}^0(r) \).

Both the stationary wave function \( \Phi_0 \) and the fluctuations \( \delta \Phi \) depend through Eqs. (1) and (2) on the normal and anomalous noncondensate particle densities, for which we need independent equations for their equilibrium values and time evolution. To this purpose it is convenient to express the noncondensate operators \( \tilde{\psi}_r, \tilde{\psi}^\dagger_r \) in terms of quasiparticle operators \( \alpha_i, \alpha_i^\dagger \) by means of the generalization to inhomogeneous systems of the Bogoliubov canonical transformations [32]

\[
\begin{align*}
\tilde{\psi}(r, t) &= \sum_i \left( u_i(r)\alpha_i(t) + v_i^*(r)\alpha_i^\dagger(t) \right) , \\
\tilde{\psi}^\dagger(r, t) &= \sum_i \left( u_i^*(r)\alpha_i^\dagger(t) + v_i(r)\alpha_i(t) \right) .
\end{align*}
\]

(8)

The normalization condition for the functions \( u_i(r), v_i(r) \), which ensures that the quasiparticle operators \( \alpha_i, \alpha_i^\dagger \) satisfy Bose commutation relations, reads

\[
\int \! dr \left[ u_i^*(r)u_j(r) - v_i^*(r)v_j(r) \right] = \delta_{ij} .
\]

(9)

The time evolution of \( \tilde{n}(r, t) \) and \( \tilde{m}(r, t) \) can be obtained from the Heisenberg equations for the products of quasiparticle operators \( \alpha_i^\dagger(t)\alpha_j(t) \) and \( \alpha_i(t)\alpha_j(t) \)

\[
\begin{align*}
\frac{i\hbar}{\partial t} \langle \alpha_i^\dagger(t)\alpha_j(t) \rangle &= \left[ \langle \alpha_i^\dagger(t)\alpha_j(t), H \rangle \right] , \\
\frac{i\hbar}{\partial t} \langle \alpha_i(t)\alpha_j(t) \rangle &= \left[ \langle \alpha_i(t)\alpha_j(t), H \rangle \right] .
\end{align*}
\]

(10)

In the above equations the commutators are calculated using the mean-field prescriptions a)-d) and the canonical transformation \( \Phi \). The calculation can be easily done by noticing that in the Hamiltonian (1) only the terms quadratic and quartic in the noncondensate operators \( \tilde{\psi}, \tilde{\psi}^\dagger \) give non vanishing contributions, as we set to zero averages of single and cubic products of noncondensate operators.

At equilibrium, we take the occupation of quasiparticle levels to be diagonal, \( \langle \alpha_i^\dagger\alpha_j \rangle = \delta_{ij}f_i^0 \), while anomalous averages of quasiparticles are zero, \( \langle \alpha_i\alpha_j \rangle = 0 \). With these conditions, the stationary equations \( \left[ \langle \alpha_i^\dagger(t)\alpha_j(t), H \rangle \right]_0 = \left[ \langle \alpha_i(t)\alpha_j(t), H \rangle \right]_0 = 0 \) yield the following coupled equations for the quasiparticle amplitudes \( u_i(r), v_i(r) \)

\[
\begin{align*}
\mathcal{L}u_i(r) + \left[ gn_0(r) + g\tilde{m}^0(r)\right]v_i(r) &= \epsilon_iu_i(r) , \\
\mathcal{L}v_i(r) + \left[ gn_0(r) + g\tilde{m}^0(r)\right]u_i(r) &= -\epsilon_iv_i(r) ,
\end{align*}
\]

(11)

where we have introduced the hermitian operator

\[
\mathcal{L} = -\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r) - \mu + 2gn(r) .
\]

(12)

The coupled Eqs. (11) correspond to the static Hartree-Fock-Bogoliubov (HFB) equations as described in Ref. [12], and the \( \epsilon_i \) are the quasiparticle energies which fix the quasiparticle occupation numbers at equilibrium \( f_i^0 = [e^{\epsilon_i/k_BT} - 1]^{-1} \). The equilibrium values of the normal and anomalous particle densities are written as

\[
\begin{align*}
\tilde{n}^0(r) &= \sum_i \left\{ \left[ |u_i(r)|^2 + |v_i(r)|^2 \right]f_i^0 + |v_i(r)|^2 \right\} , \\
\tilde{m}^0(r) &= \sum_i \left\{ 2u_i(r)v_i^*(r)f_i^0 + u_i^*(r)v_i(r) \right\} .
\end{align*}
\]

(13)

Out of equilibrium we define the following normal and anomalous quasiparticle distribution function

\[
\begin{align*}
f_{ij}(t) &= \langle \alpha_i^\dagger(t)\alpha_j(t) \rangle - \delta_{ij}f_i^0 , \\
g_{ij}(t) &= \langle \alpha_i(t)\alpha_j(t) \rangle ,
\end{align*}
\]

(14)

in terms of which the fluctuations of \( \tilde{n} \) and \( \tilde{m} \) take the form
\[\delta \hat{n}(r, t) = \sum_{ij} \left\{ u_i^*(r)u_j(r) + v_i^*(r)v_j(r) \right\} f_{ij}(t) + u_i(r)v_j(r)g_{ij}(t) + u_i^*(r)v_j^*(r)g_{ij}^*(t) \] ,
\[\delta \hat{m}(r, t) = \sum_{ij} \left\{ 2v_i^*(r)u_j(r)f_{ij}(t) + u_i(r)u_j(r)g_{ij}(t) + v_i^*(r)v_j^*(r)g_{ij}^*(t) \right\} .\] (15)

Up to linear terms in the fluctuations \(\delta \Phi\), \(\delta \hat{n}\) and \(\delta \hat{m}\), the equation of motion for \(f_{ij}\) gives the result
\[i\hbar \frac{\partial}{\partial t} f_{ij}(t) = (\epsilon_j - \epsilon_i) f_{ij}(t) + 2g(f_i^0 - f_j^0) \times \int dr \Phi_0 \left[ \delta \Phi(t) \left( u_i u_j^* + v_i v_j^* + v_i u_j^* \right) + \delta \Phi^*(t) \left( u_i u_j^* + v_i v_j^* + u_i v_j^* \right) \right] \]
\[+ g(f_i^0 - f_j^0) \int dr \left[ 2\delta \hat{n}(t) \left( u_i u_j^* + v_i v_j^* \right) + \delta \hat{m}(t)u_i u_j^* + \delta \hat{m}^*(t)u_i v_j^* \right] .\] (16)

Analogously, for the time evolution of the anomalous quasiparticle distribution function \(g_{ij}\), one obtains in the linear regime
\[i\hbar \frac{\partial}{\partial t} g_{ij}(t) = (\epsilon_j - \epsilon_i) g_{ij}(t) + 2g(1 + f_i^0 + f_j^0) \times \int dr \Phi_0 \left[ \delta \Phi(t) \left( u_i u_j^* + v_i v_j^* + u_i u_j^* \right) + \delta \Phi^*(t) \left( u_i u_j^* + v_i v_j^* + v_i u_j^* \right) \right] \]
\[+ g(1 + f_i^0 + f_j^0) \int dr \left[ 2\delta \hat{n}(t) \left( u_i u_j^* + v_i v_j^* \right) + \delta \hat{m}(t)u_i u_j^* + \delta \hat{m}^*(t)u_i v_j^* \right] .\] (17)

The first term on the r.h.s. of Eqs. (16), (17) describes the free evolution of the quasiparticle states, corresponding to quasiparticle operators which evolve in time according to \(\alpha_i(t) = e^{-i\epsilon_i t/\hbar}\alpha_i\). The second and the third term describe, respectively, the coupling to the oscillations of the condensate and to the fluctuations \(\delta \hat{n}\) and \(\delta \hat{m}\) of the normal and anomalous particle density. Above the Bose-Einstein transition temperature, where the system becomes homogeneous, the HFB equations reduce to the so-called HFB-Popov equation. In the semiclassical limit TDHF is equivalent to the collisionless Boltzmann equation for the particle distribution function.

In the framework of mean-field theories, coupled time-dependent equations for the condensate and noncondensate components of a Bose gas have been discussed by many authors. The TDHF scheme is discussed in [17]. Moreover, coupled equations of motion for the condensate and for correlation functions of pairs and triplets of noncondensate particles have been derived in [14], and studied in the linear response regime in [14]. Similar kinetic equations were derived by Kane and Kadanoff [34] using the formalism of non-equilibrium Green’s functions developed in [36]. Recently, the approach of Kane and Kadanoff has been extended to deal with a trapped Bose-condensed gas in [37].

The stationary Eq. (3) for \(\Phi_0\), with the normal and anomalous particle densities at equilibrium given by (13), and Eqs. (14) for the quasiparticles correspond to the static self-consistent HFB approximation as reviewed by Griffin in [12].

If one neglects the anomalous particle density, \(\tilde{n}^0 = 0\), the HFB equations reduce to the so-called HFB-Popov approximation [14],[22], which is gapless and in the high temperature regime coincides with the Hartree-Fock scheme [10]. The HFB-Popov approximation has been applied by several authors to interacting bosons in harmonic traps, both to calculate the frequencies of the collective modes [13] and to study the thermodynamic properties of the system [11],[14]. Gapless static mean-field approximations, alternative to HFB-Popov, have been put forward and discussed in [14],[15].

Finally, by neglecting both the normal and the anomalous particle density, \(\tilde{n}^0 = \tilde{m}^0 = 0\), the HFB equations reduce to the Gross-Pitaevskii theory. From Eq. (4) one recovers the stationary GP equation, while Eqs. (11) follow from the linearization of the time-dependent GP equation. At very low temperatures, where effects arising from the depletion of the condensate are negligible, the Gross-Pitaevskii theory is well suited to describe dilute gases in traps. For these systems the linearized GP equation has been solved by many authors [15],[16] and successfully compared to experiments.

The linearized TDHFB mean-field approximation is a closed set of self-consistent equations which describe the small oscillations of the system around the static HFB solution. The dynamic Eq. (7) describes the fluctuation \(\delta \Phi\) of the condensate around the stationary solution \(\Phi_0\), while Eqs. (13) and (17) describe the small oscillations \(\delta \hat{n}\), \(\delta \hat{m}\) of
the normal and anomalous particle density around their equilibrium values \( \tilde{n}_0, \tilde{m}_0 \). Since the equations for the time evolution of \( \delta \Phi, \delta \tilde{n} \) and \( \delta \tilde{m} \) are derived from the corresponding exact Heisenberg equations, it can be easily checked that the linearized TDHFB approach preserves important conservation laws, such as number of particles and energy conservation. This is a general feature of time-dependent mean-field approximations [13,16]. Another important feature of linearized TDHFB is that, even though the quasiparticle energies obtained from Eqs. (11) exhibit a gap at long wavelengths: \( \epsilon_p \rightarrow \Delta \) as \( p \rightarrow 0 \), the self-consistent solution of Eq. (11) is gapless. In fact, one can show that this self-consistent solution satisfies the Hugenholtz-Pines theorem [17].

There are some questions one should address before embarking on the difficult task of a self-consistent solution of the linearized TDHFB equations. A first problem concerns the equilibrium anomalous density \( \tilde{n}_0 \) [see Eq. (13)], which is ultraviolet divergent. To second order in the interaction and for homogeneous systems the ultraviolet divergence is canceled by the renormalization of the coupling constant (see e.g. [18]): \( g \rightarrow g \left(1 + g \frac{1}{m} \sum p/m^2 \right) \). How to include the renormalization of \( g \) in a self-consistent calculation and how to extend this renormalization procedure to inhomogeneous systems is still an open problem. Recently, Burnett and coworkers [14,15] have shown that there is no need of renormalization of \( g \) if one uses, instead of a contact potential, an effective interaction, the many-body T-matrix, which includes self-consistently the effects arising from the anomalous density. Another problem concerns the gap exhibited by the quasiparticle energies in Eqs. (11). The self-consistent solution of these equations defines the equilibrium state of the system: it fixes the noncondensate densities \( \tilde{n}_0 \) and \( \tilde{m}_0 \) through Eqs. (13), and the chemical potential \( \mu \) and the condensate wavefunction \( \Phi_0 \) through Eq. (11). Even though the small oscillations around the static HFB solutions give rise to a gapless spectrum for the fluctuations of the condensate, properties such as the phonon velocity and their damping rate will be affected by an incorrect description of the system at equilibrium, originating from the unphysical gap \( \Delta \) in the quasiparticle energies. In particular, if \( k_B T \sim \Delta \), one expects a strong influence of the gap on the temperature dependence of these properties. In the present work we will not solve the linearized TDHFB equations self-consistently, instead, we solve them perturbatively to order \( g^2 \). By this method we avoid the problems mentioned above, in particular, the quasiparticle states in (11) are properly described by Bogoliubov theory, which is gapless.

Another point which deserves some comments is the Kohn mode (dipole mode). As it is well known, in the presence of harmonic confinement the center of mass degrees of freedom separate from all other degrees of freedom, giving rise to a collective mode of the system, the Kohn mode, in which the equilibrium density profile oscillates rigidly at the trap frequency. The linearized TDHFB equations obtained in this section do not describe this mode, as they do not account for the motion of the center of mass. In fact, these equations correctly describe excitations for which the center of mass is at rest, and we will restrict our analysis to this class of excitations. The Kohn mode is associated with broken translational symmetry and is often referred to as a “spurious” mode. For a discussion of spurious states and their appearance in linearized time-dependent mean-field theories see Ref. [16].

### III. Spatially Homogeneous System

In this section we will develop, starting from the dynamic equations for the condensate and noncondensate components derived in the previous section, a perturbation scheme for the elementary excitations in a homogeneous system. Explicit results for the temperature dependence of the chemical potential, damping rate and speed of zero sound are given in the limit \( a^3 n_0 \ll 1 \), where \( n_0 = n_0(T) \) is the equilibrium value of the condensate density at temperature \( T \). At zero temperature, the calculation presented here is equivalent to the Beliaev second-order approximation of the single-particle Green’s function [19]. In the high temperature regime, \( k_B T \gg \mu \), our approach corresponds to the finite-temperature extension of the Beliaev approximation recently employed in Refs. [20,21] (the former reference gives also a systematic account of the various perturbation schemes for a uniform Bose gas). The perturbation expansion carried out in the present work holds to order \( (a^3 n_0)^{1/2} \) for any temperature in the condensed phase, except clearly very close to the transition, where the time-dependent mean-field equations we used as starting point break down.

In a spatially homogeneous system the condensate wave function at equilibrium is constant, \( \Phi_0(\mathbf{r}) = \sqrt{n_0} \). The stationary equation [11] reads then

\[
\mu = gn_0 + 2g \tilde{n}^0 + g \tilde{m}^0 ,
\]

and represents the equation of state of the system, which fixes the chemical potential as a function of the condensate density \( n_0 \) and the temperature \( T \). By using the above result for the chemical potential, equation (11) for the fluctuations of the condensate becomes
\[
\hat{\delta} \hat{\Phi}(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + gn_0 - g\tilde{m}^0 \right) \delta \Phi(r, t) + (gn_0 + g\tilde{m}^0) \delta \Phi^*(r, t) \\
+ 2g\sqrt{n_0}\dot{\delta}n(r, t) + g\sqrt{n_0}\delta\dot{n}(r, t) \ .
\] (19)

In the above equation the terms involving the equilibrium anomalous density \(\tilde{m}^0\) account for the coupling between the fluctuations of the condensate and the static distribution of noncondensate atoms, while the terms containing \(\delta n\) and \(\delta\dot{n}\) describe the dynamic coupling between the condensate and the fluctuations of the noncondensate component.

### A. Perturbation scheme

The perturbation scheme applied to Eqs. (18) and (19) consists in treating the terms which give the static and dynamic coupling to the noncondensate component to second order in \(g\). It means that the static and fluctuating parts of the normal and anomalous density need to be calculated only to first order in \(g\). To accomplish this task one must retain in the equations for the quasiparticles (11), (16) and (17) only the terms which describe the coupling to the condensate and neglect all terms arising from the coupling to the noncondensate component.

Let us suppose that the condensate oscillates with frequency \(\omega\) and wave vector \(q/\hbar\)

\[
\delta \Phi(r, t) = \frac{\delta \Phi_1(q)}{\sqrt{V}} e^{iq \cdot r/\hbar} e^{-i\omega t} \ , \quad \delta \Phi^*(r, t) = \frac{\delta \Phi_2(q)}{\sqrt{V}} e^{iq \cdot r/\hbar} e^{-i\omega t} .
\] (20)

Furthermore, the quasiparticle amplitudes be described by plane-wave functions

\[
u_p(r) = \frac{n_p}{\sqrt{V}} e^{ip \cdot r/\hbar} \ , \quad v_p(r) = \frac{\nu_p}{\sqrt{V}} e^{ip \cdot r/\hbar} .
\] (21)

To first order in \(g\) the quasiparticle equations (11) become

\[
(p^2/2m + gn_0)\nu_p + gn_0 v_p = \epsilon_p \nu_p \ , \\
(p^2/2m + gn_0)\nu_p + gn_0 u_p = -\epsilon_p v_p .
\] (22)

These coupled equations coincide with the well-known Bogoliubov equations for the real quasiparticle amplitudes \(\nu_p\), \(v_p\), which satisfy the following relations

\[
u_p^2 = 1 + \epsilon_p^2 = \frac{(\epsilon_p^2 + g^2 n_0^2)^{1/2} + \epsilon_p}{2\epsilon_p} \ , \\
u_p v_p = \frac{gn_0}{2\epsilon_p} .
\] (23)

with the quasiparticle energy \(\epsilon_p\) given by the Bogoliubov spectrum

\[
\epsilon_p = \left( (\epsilon_p^0 + gn_0)^2 - g^2 n_0^2 \right)^{1/2} ,
\] (24)

being \(\epsilon_p^0 = p^2/2m\) the free-particle energy. Notice that, by employing the equation of state (18), the full HFB equations (11) would coincide with the matrix equation (22) apart from a term \(g\dot{n}_0\). This term would appear with the minus sign in the diagonal term and with the plus sign in the off-diagonal term, and is responsible for the gap in \(\epsilon_p\) as \(p \to 0\). Since we use the Bogoliubov result (24), we avoid the problem of the gap in the quasiparticle spectrum.

In the same approximation one must neglect in Eqs. (16) and (17) the terms which describe the coupling to the fluctuations \(\delta n\) and \(\delta\dot{n}\) of the noncondensate component. Due to the coupling to the condensate, which acts as a time-dependent external drive, the only elements of the matrices \(f_{p,p'}\), \(g_{p,p'}\) and \(g_{p,p'}^{*}\) which oscillate at the frequency \(\omega\) are given by

\[
f_{p,q+p}(\omega) = g\sqrt{n_0} \frac{f_{q+p}^0 - f_{q+p}^0}{\sqrt{V}} \frac{f_{q+p}^0}{\hbar \omega + (\epsilon_p - \epsilon_{q+p}) + i0} \left[ (\delta \Phi_1 - \delta \Phi_2) \left( v_p \nu_{q+p} - \nu_p v_{q+p} \right) + (\delta \Phi_1 + \delta \Phi_2) \left( 2v_p \nu_{q+p} + 2v_p v_{q+p} + \nu_p v_{q+p} + \nu_p v_{q+p} \right) \right] ,
\]
\[ g_{p,q-p}(\omega) = g \frac{\sqrt{n_0}}{\sqrt{\hbar \omega}} \left( 1 + f^0_p + f^0_{q-p} \right) \left( \delta \Phi_1 - \delta \Phi_2 \right) \left( u_{p,q-p} - v_{p,q-p} \right) + \left( \delta \Phi_1 + \delta \Phi_2 \right) \left( 2u_{p,q-p} + 2v_{p,q-p} + u_{p,q-p} + v_{p,q-p} \right) \]  
\[ \text{for the self-energy } \Sigma \]

\[ \text{In the above equations the frequency } \omega \text{ has been chosen with an infinitesimally small component on the positive imaginary axis.} \]

As it is well known (see e.g. Ref. [48]), to treat consistently to order \( g \) the properties of a Bose-condensed gas one must include the proper renormalization of the coupling constant \( g \rightarrow g \left( 1 + g \frac{1}{2} \sum_n m/p^2 \right) \). This renormalization of \( g \) is crucial because it cancels exactly the large-\( p \) ultraviolet divergence exhibited by the equilibrium anomalous density \( \hat{m}^0 \). In fact, by using the renormalized value of \( g \), the term \( g n_0 + g \hat{m}^0 \) present in Eqs. (18), (19) becomes

\[ gn_0 + g \hat{m}^0 \rightarrow gn_0 + g^2 n_0 \frac{1}{V} \sum_p \left( \frac{m}{p^2} - \frac{1 + 2f^0_p}{2\epsilon_p} \right) \]

and is well behaved at large \( p \).

To order \( g^2 \), Eq. (24) for the fluctuations of the condensate can be finally written in the form

\[ \hbar \omega \left( \delta \Phi_1 + \delta \Phi_2 \right) = \frac{g^2}{2m} \left( \delta \Phi_1 - \delta \Phi_2 \right) + \frac{\Sigma_{11}(q,\omega) - \Sigma_{11}(q,-\omega)}{2} \left( \delta \Phi_1 + \delta \Phi_2 \right) + \left[ \Sigma_{11}(q,\omega) + \Sigma_{11}(q,-\omega) - \Sigma_{12}(q,\omega) \right] \left( \delta \Phi_1 - \delta \Phi_2 \right) , \]

\[ \hbar \omega \left( \delta \Phi_1 - \delta \Phi_2 \right) = \left( \frac{g^2}{2m} + 2gn_0 \right) \left( \delta \Phi_1 + \delta \Phi_2 \right) + \frac{\Sigma_{11}(q,\omega) - \Sigma_{11}(q,-\omega)}{2} \left( \delta \Phi_1 - \delta \Phi_2 \right) \]

In the above equations the self-energies \( \Sigma_{11}(q,\omega) \) and \( \Sigma_{12}(q,\omega) \) are proportional to \( g^2 \). They are obtained from Eq. (24) through the expressions (25), which give \( \delta \hat{m} \) and \( \delta \hat{m} \) in terms of the matrices \( f_{p,p'} \) and \( g_{p,p'} \), and through Eqs. (26) and (27). After some straightforward algebra one finds for the self-energy \( \Sigma_{11} \)

\[ \Sigma_{11}(q,\omega) = g^2 n_0 \frac{1}{V} \sum_p \left( \frac{m}{p^2} - \frac{1 + 2f^0_p}{2\epsilon_p} \right) + g^2 n_0 \frac{1}{V} \sum_p \left( f^0_p - f^0_k \right) \]

\[ \times \left( \frac{2B_p A_k + 8C_p A_k + 4B_p B_k + 4C_p C_k}{\hbar \omega + (\epsilon_p - \epsilon_k + i0)} - \frac{2A_p B_k + 8C_p B_k + 4A_p A_k + 4C_p C_k}{\hbar \omega + (\epsilon_p - \epsilon_k + i0)} \right) \]

\[ + g^2 n_0 \frac{1}{V} \sum_p \frac{1 + 2f^0_p}{\epsilon_p} + g^2 n_0 \frac{1}{V} \sum_p \left( 1 + f^0_p + f^0_k \right) \]

\[ \times \left( \frac{2A_p A_k + 8C_p A_k + 4A_p B_k + 4C_p C_k}{\hbar \omega + (\epsilon_p + \epsilon_k + i0)} - \frac{2B_p B_k + 8C_p B_k + 4B_p A_k + 4C_p C_k}{\hbar \omega + (\epsilon_p + \epsilon_k + i0)} \right) , \]

where we have introduced the notations: \( k = q + p, A_p = u_{p,v}^2, B_p = v_{p,v}^2 \) and \( C_p = u_{p,v}v_{p,v} \). Analogously for \( \Sigma_{12} \) one has

\[ \Sigma_{12}(q,\omega) = g^2 n_0 \frac{1}{V} \sum_p \left( \frac{m}{p^2} - \frac{1 + 2f^0_p}{2\epsilon_p} \right) + g^2 n_0 \frac{1}{V} \sum_p \left( f^0_p - f^0_k \right) \]

\[ \times \left( \frac{4C_p B_k + 4C_p A_k + 4B_p B_k + 6C_p C_k}{\hbar \omega + (\epsilon_p - \epsilon_k + i0)} - \frac{4C_p A_k + 4C_p B_k + 4A_p A_k + 6C_p C_k}{\hbar \omega + (\epsilon_p - \epsilon_k + i0)} \right) \]
\[ g^2 n_0 \frac{1}{V} \sum_p (1 + f_0^p + f_0^k) \times \left( \frac{4C_p B_k + 4C_p A_k + 4A_p B_k + 6C_p C_k}{\hbar \omega - (\epsilon_p + \epsilon_k) + i0} - \frac{4C_p B_k + 4C_p A_k + 4B_p A_k + 6C_p C_k}{\hbar \omega + (\epsilon_p + \epsilon_k) + i0} \right). \] (29)

The above expressions for \( \Sigma_{11} \) and \( \Sigma_{12} \) coincide with the second-order self-energies explicitly calculated at finite temperature by Shi and Griffin using diagrammatic methods [28]. At zero temperature they correspond to Beliaev’s results [49], while in the high-temperature regime, \( k_BT \gg gn_0 \), they have been recently discussed by Fedichev and Shlyapnikov [27].

By neglecting in (27) the terms proportional to the self-energies, one is left with the equations for the fluctuations of the condensate to first order in \( g \). These equations coincide with the quasiparticle equations (22). The solution is then given by Beliaev’s rate \( \gamma \). Notice that, concerning the damping rate, result (32) coincides with the calculation carried out within the Popov approximation [Eq. (39) of [18]], where the condition \( \tilde{n}_0 = 0 \) was assumed. However, as discussed in [18], the frequency shift \( \delta E \) is not given correctly to order \( g^2 \) by the Popov approximation. In fact, the static anomalous density \( \tilde{n}_0 \) and the renormalized coupling constant contribute to the real part of Eq. (22).

**B. Equation of state**

Let us first discuss the equation of state (18). By calculating \( \tilde{n}_0 \) and \( \tilde{m}_0 \) using the equilibrium expressions (13) with the first-order quasiparticle amplitudes and energies given by (23) and (24), one finds to order \( g^2 \)

\[ \mu = gn_0 + 2gn_0^0 + gn_0(\alpha^2 n_0)^{1/2} H(\tau) \] . (34)
In the above equation \( n_0^0 = \zeta(3/2)\lambda_T^{-3} \approx 2.612 \lambda_T^{-3} \) is the noncondensate density of an ideal gas, which is fixed by the thermal wavelength \( \lambda_T = \hbar / (2\pi mk_B T) \). Moreover, \( H(\tau) \) is a dimensionless function of the reduced temperature \( \tau = k_B T / gn_0 \) given by

\[
H(\tau) = \frac{40}{3\sqrt{\pi}} + \frac{\sqrt{32}}{\sqrt{\pi}} \int_0^\infty dx \frac{1}{e^x - 1} \left( \sqrt{u - 1} \frac{2u - 1}{u} - 2\sqrt{\tau}x \right),
\]

where we have introduced the quantity \( u = \sqrt{1 + \tau^2x^2} \). Result (34) gives the chemical potential as a function of the equilibrium condensate density \( n_0 \) and the temperature \( T \) to second order in \( g \). It coincides with the result of Shi and Griffin [Eq. (7.9) of 28]. Notice that the sum of the first two terms on the r.h.s. of (34) corresponds to the chemical potential calculated to first order in \( g \). In Fig. 1 the dimensionless function \( H(\tau) \) is plotted as a function of the reduced temperature \( \tau \).

At low temperatures, \( \tau \ll 1 \), the function \( H(\tau) \) can be expanded as

\[
H(\tau) \approx \frac{40}{3\sqrt{\pi}} - \sqrt{32}\zeta(3/2)\tau^{3/2} + \frac{2\pi^{3/2}}{3} \tau^2 + \frac{\pi^{7/2}}{10} \tau^4.
\]

In the same regime of temperatures, the condensate density is given in terms of the total density \( n = n_0 + \tilde{n}^0 \) by the following relation

\[
n_0 \approx n \left[ 1 - (a^3n_0)^{1/2} \left( \frac{8}{3\sqrt{\pi}} + \frac{2\pi^{3/2}}{3} \tau^2 - \frac{\pi^{7/2}}{30} \tau^4 \right) \right],
\]

valid to order \( g^2 \). In the above expression the first term in brackets corresponds to the quantum depletion of the condensate, while the other two terms account for the thermal depletion caused by phonon-type excitations. By using Eqs. (34) and (37), one gets the following result for the low-temperature behavior of the chemical potential in terms of the density \( n \)

\[
\mu \approx gn \left[ 1 + (a^3n_0)^{1/2} \left( \frac{32}{3\sqrt{\pi}} + \frac{2\pi^{7/2}}{15} \tau^4 \right) \right]
\]

\[
\approx \mu(T = 0) + \frac{\pi^2}{60} \left[ (k_B T)^4 \right],
\]

where \( \mu(T = 0) = gn[1 + 32(a^3n_0)^{1/2}/3\sqrt{\pi}] \) is the value of the chemical potential at zero temperature [4], and \( c_B = \sqrt{gn_0/m} \) is the Bogoliubov velocity of sound. The \( T^4 \) term in (38) coincides with the result obtained from the thermodynamic relation \( \mu = (\partial F / \partial N)_{V,T} \), where \( F \) is the low-temperature expansion of the free energy of a Bose gas [54, 15].

At high temperatures, \( \tau \gg 1 \), the function \( H(\tau) \) yields the asymptotic result \( H(\tau) \to -12\sqrt{\tau} \), and for the chemical potential in this regime of temperatures one gets

\[
\mu \approx gn_0 + 2gn_0^0 - 12\sqrt{\pi}(a^3n_0)^{1/2}k_B T,
\]

which coincides with the result obtained by Popov [8, 28, 27].

C. Zero sound: damping rate and velocity of propagation

Zero sound, or Bogoliubov sound, is a collective oscillation of the system in the collisionless regime, for which the restoring force acting on a given particle comes from the mean-field created by the other particles. At zero temperature, zero sound coincides with ordinary sound and the velocity of propagation \( c \) is fixed by the compressibility of the system \( mc^2 = (\partial P / \partial n)_{T=0} \). At finite temperature one can have both zero sound and hydrodynamic modes, depending on the temperature and the wavelength of the excitation. In this case the velocity of zero sound can not be fixed only by thermodynamic relations. For an exhaustive discussion of collisionless and hydrodynamic collective modes we refer to the books [9, 10].

To first order in \( g \), the excitation energy of the zero-sound mode is given by the long-wavelength limit of the Bogoliubov dispersion relation (24), which corresponds to phonons \( \epsilon_q = c_B q \) propagating with the velocity \( c_B = \sqrt{gn_0/m} \) fixed by the condensate density. Starting from Eq. (32), we will explicitly calculate the damping rate and
the temperature dependence of the speed of zero sound to order $g^2$. The damping of phonons in homogeneous systems has been recently investigated by Liu [51], using functional-integration methods, and by Pitaevskii and Stringari [20] by means of perturbation theory. The temperature dependence of the speed of zero sound has been investigated by Payne and Griffin [52] within the framework of the dielectric formalism, and by Shi and Griffin [28] and Fedichev and Shlyapnikov [27] using diagrammatic methods.

1. Damping of zero sound

The calculation of the damping of phonons from Eq. (32) has been already carried out in [18]. Here we will only review the main results.

In the quantum regime, $\epsilon_q \gg k_B T$, the damping is governed by the Beliaev mechanism, in which a phonon decays into a pair of excitations. This mechanism is described in Eq. (32) by the imaginary part of the term containing the matrices $B$ and $\tilde{B}$. The damping rate is given by

$$\frac{\gamma}{\epsilon_B q} = \frac{3q^4}{640\hbar^3 m_0 c_B}.$$  \hspace{1cm} (40)

This result has been first obtained by Beliaev using diagrammatic techniques [49].

In the thermal regime, $\epsilon_q \ll k_B T$, the phonon decay is dominated by a different damping process, in which a phonon with energy $\epsilon_q$ is absorbed by a thermal excitation with energy $\epsilon_p$ and is turned into another thermal excitation with energy $\epsilon_{q+p}$. This mechanism is known as Landau damping and is described in Eq. (32) by the imaginary part of the term involving the matrix $A$. The result is given by (see Refs. [20,18])

$$\frac{\gamma}{\epsilon_B q} = \left(\frac{a^3 n_0}{2}\right)^{1/2} F(\tau),$$  \hspace{1cm} (41)

In the above equation $\tau = k_B T/gn_0$ is the reduced temperature and $F(\tau)$ is the dimensionless function

$$F(\tau) = 4\sqrt{\pi} \int_0^\infty dx \left(e^{x/2} - e^{-x/2}\right)^{-2} \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2,$$  \hspace{1cm} (42)

where $u$ is defined as in (35).

For temperatures $\tau \ll 1$ the function $F$ takes the asymptotic limit $F \to 3\pi^{9/2}/5$ and one recovers the known result for the phonon damping [53–55,20]

$$\frac{\gamma}{\epsilon_B q} = \frac{3\pi^{3/2} (k_B T)^4}{40 m_0 \hbar^3 c_B^3}.$$  \hspace{1cm} (43)

In the opposite regime of temperatures, $\tau \gg 1$, one finds the asymptotic limit $F \to 3\pi^{3/2}/4$, and the damping coefficient is given by

$$\frac{\gamma}{\epsilon_B q} = \frac{3\pi k_B T a}{8 \hbar c_B}.$$  \hspace{1cm} (44)

The damping of phonons in this regime of temperatures has been first investigated by Szépfalusy and Kondor [56].

In Fig. 2 the dimensionless function $F(\tau)$ is plotted as a function of $\tau$ together with its asymptotic behaviour both at small and large $\tau$’s. One can see that $F$ departs rather soon from the low-temperature $\tau^4$ behaviour, while it approaches the high-temperature linear law very slowly.

2. Velocity of zero sound

Differently from the calculation of the damping rates, all terms on the r.h.s. of (32) contribute to the frequency shift $\delta E$. The first two terms, which involve $n_0$ and the renormalization of the coupling constant, are referred to as static terms. Concerning the other two terms, for excitations with $\epsilon_q \ll gn_0$, the relevant region in the calculation of the real part of the term which contains the matrices $B$ and $\tilde{B}$ corresponds to energies $\epsilon_p \gg \epsilon_q$. This term is referred to as non-resonant term. On the contrary, the resonance region gives an important contribution to the real part of the term involving the matrix $A$. This term is referred to as resonant term.

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Let us start by analyzing the non-resonant term. The contribution of this term to the energy shift $\delta E$ can be written as

$$
\delta E_{NR} = -\frac{\epsilon_q g^2 n_0}{\epsilon_q^2} \frac{1}{V} \sum_p \left[ f^0_p \frac{\epsilon_p - \epsilon_k}{2\epsilon_p\epsilon_k} + \frac{1 + 2f^0_p}{2\epsilon_p} - (1 + 2f^0_p) \frac{(\epsilon_k^0 - \epsilon_p^0)^2}{4\epsilon_p\epsilon_k(\epsilon_p + \epsilon_k)} \right] 
$$

$$
+ \epsilon_q g^2 n_0 \frac{1}{V} \sum_p \left[ (1 + 2f^0_p) \frac{\epsilon_p + \epsilon_k}{\epsilon_q} - (\epsilon_p + \epsilon_k)^2 \right] \left[ \epsilon^2_q 2(2u_p v_k + 2v_p u_k + u_p u_k + v_p v_k)^2 \right] 

+ 2\frac{u_p u_k - u_p v_k}{\epsilon_p + \epsilon_k} \left(2u_p v_k + 2v_p u_k + u_p u_k + v_p v_k\right) + \frac{\epsilon^2_q}{\epsilon_q^2} \left(\frac{u_p u_k - u_p v_k}{\epsilon_p + \epsilon_k}\right)^2 \right), \tag{45}
$$

where $\mathbf{k} = \mathbf{q} + \mathbf{p}$. The above result is valid for any excitation energy $\epsilon_q$, and is not limited to the long-wavelength regime $\epsilon_q \ll g n_0$. In the phonon regime, result (45) can be simplified and one gets

$$
\frac{\delta E_{NR}}{e B q} = -\frac{1}{V} \sum_p \left[ (1 + 2f^0_p) \left[ \frac{(\epsilon_k^0)^2}{4\epsilon_p^2} - \frac{g n_0 \epsilon^0_p}{6\epsilon_p^2} \right] - \frac{1}{\epsilon_q^2} g^2 n_0 \frac{1}{V} \sum_p \frac{1 + 2f^0_p}{2\epsilon_p} \right]. \tag{46}
$$

The contribution to $\delta E$ from the resonant term can be calculated in the same way and one gets the general result

$$
\delta E_R = \frac{\epsilon_q g^2 n_0}{\epsilon_q^2} \frac{1}{V} \sum_p \left[ f^0_p \frac{\epsilon_p - \epsilon_k}{2\epsilon_p\epsilon_k} - \frac{f^0_p - f^0_k}{\epsilon_p - \epsilon_k} \frac{(\epsilon_k - \epsilon_p)^2}{4\epsilon_p\epsilon_k} \right] + \epsilon_q g^2 n_0 \frac{1}{V} \sum_p \frac{f^0_p - f^0_k}{\epsilon_q + \epsilon_p - \epsilon_k} \left[ \frac{\epsilon^0_q}{\epsilon^2_q} \left(2u_p u_k + 2v_p v_k + u_p u_k + v_p v_k\right) \right] 

+ \frac{\epsilon^2_q}{\epsilon_q^2} \left(\frac{u_p u_k - u_p v_k}{\epsilon_p - \epsilon_k}\right)^2 \right), \tag{47}
$$

In the limit $\epsilon_q \ll g n_0$ the above expression reduces to

$$
\frac{\delta E_R}{e B q} = \frac{g}{2V} \sum_p \frac{\partial f^0_p}{\partial \epsilon_p} \left[ \left(\frac{\epsilon^0_p}{\epsilon_p} + \frac{\partial \epsilon^0_p}{\partial \epsilon_p}\right)^2 \left(1 - \frac{c_B}{c_B - \cos \theta \partial \epsilon_p / \partial p}\right) \right] \left[ 1 - \frac{c_B}{c_B - \cos \theta \partial \epsilon_p / \partial p}\right] \right] \left[ \frac{2 g n_0 \epsilon^0_p}{3 \epsilon^2_p} \right] 

+ \frac{1}{\epsilon_q^2} g^2 n_0 \frac{1}{V} \sum_p \frac{f^0_p \epsilon_p - \epsilon_k}{2\epsilon_p\epsilon_k} \right], \tag{48}
$$

where $\theta$ is the angle the momentum $\mathbf{p}$ forms with the direction of $\mathbf{q}$.

Notice that the last terms on the r.h.s. of (44) and (47) are equal and opposite in sign, thus they cancel in the sum $\delta E_R + \delta E_{NR}$. Moreover, the second term on the r.h.s. of (44), which is ultraviolet divergent, is canceled by a corresponding term arising from the contribution to $\delta E$ of the static terms. In conclusion, the final result for the energy shift $\delta E$ in the phonon regime is well behaved and proportional to $\epsilon_q$. One finds

$$
\frac{\delta E}{e B q} = \frac{g}{2V} \sum_p \frac{\partial f^0_p}{\partial \epsilon_p} \left[ \left(\frac{\epsilon^0_p}{\epsilon_p} + \frac{\partial \epsilon^0_p}{\partial \epsilon_p}\right)^2 \left(1 - \frac{c_B}{c_B - \cos \theta \partial \epsilon_p / \partial p}\right) \right] \left[ 1 - \frac{c_B}{c_B - \cos \theta \partial \epsilon_p / \partial p}\right] \right] \left[ \frac{2 g n_0 \epsilon^0_p}{3 \epsilon^2_p} \right] 

+ \frac{g}{2V} \sum_p \left[ \left(\frac{m}{p^2} - \frac{1}{2\epsilon_p} + \frac{4 g n_0 \epsilon^0_p}{3 \epsilon^3_p} \right) - \frac{f^0_p}{\epsilon_p} \left(1 - \frac{8 g n_0 \epsilon^0_p}{3 \epsilon^2_p}\right) \right], \tag{49}
$$

The r.h.s. of the above equation gives the correction to the Bogoliubov velocity of zero sound. By rearranging the integrals over momenta, one gets the relevant result

$$
c = c_B \left[ 1 + (a^3 n_0)^{1/2} G(\tau) \right]. \tag{50}
$$
Here $G(\tau)$ is the following dimensionless function of the reduced temperature $\tau = k_B T / g n_0$

$$G(\tau) = \frac{28}{3\sqrt{\pi}} + \frac{\sqrt{32}}{\sqrt{\pi}} \int_0^\infty dx \frac{1}{e^{x^2} - 1} \frac{\sqrt{u - 1}}{u} \frac{5 - 3u}{6(u + 1)} + \frac{\sqrt{32}}{\sqrt{\pi}} \int_0^\infty dx \frac{1}{(e^{x^2/2} - e^{-x^2/2})^2}$$

\begin{align}
\times \left[ 1 + \frac{3}{2} \left( \frac{u + 1}{u} \right) \left( 1 + \frac{u}{2\sqrt{2u}} \log \frac{\sqrt{2u - \sqrt{u + 1}}}{\sqrt{2u + \sqrt{u + 1}}} \right) \right],
\end{align}

(51)

where $u$ is defined as in (52).

It is interesting to study Eq. (52) in particular regimes of temperature. At zero temperature, the function $G$ takes the value: $G(\tau = 0) = 28/(3\sqrt{\pi})$, and $n_0$ is related to the total density $n$ by the expression: $n_0 = n[1 - 8/(3\sqrt{\pi})(a^3 n_0)^{1/2}]$, which accounts for the quantum depletion. The result for the sound velocity is:

$$c(T = 0) = \sqrt{\frac{gn_0}{m}} \left[ 1 + \frac{8}{\sqrt{\pi}} (a^3 n_0)^{1/2} \right].$$

(52)

The above result, which was first found by Beliaev [49], coincides with the one obtained from the thermodynamic relation $c(T = 0) = |n(\partial \mu(T = 0)/\partial n)/m|^{1/2}$, where $\mu(T = 0)$ is given in (53).

At low temperatures, $\tau \ll 1$, one finds the following expansion of the $G$ function

$$G(\tau) \simeq \frac{28}{3\sqrt{\pi}} + \frac{\pi^{3/2}}{3} \tau^2 + \frac{3\pi^{7/2}}{5} \tau^4 \log(1/\tau^2).$$

(53)

In this regime of temperatures the condensate density is given by the expression (54) in terms of the density $n$, and the velocity of zero sound turns out to be

$$c = c(T = 0) + \frac{3\pi^2}{40} \frac{(k_B T)^4}{mn_0 h^3 c_B^4} \log[m^2 c_B^4/(k_B T)^2].$$

(54)

This result was first obtained by Andreev and Khalatnikov [5] using kinetic equations, and later by Ma et al. [57] within the framework of the dielectric formalism.

Finally, in the high temperature regime $\tau \gg 1$, the function $G$ is linear in $\tau$: $G(\tau) \to G(\infty)\tau$, with the numerical coefficient $G(\infty)$ given by the following expression

$$G(\infty) = \frac{\sqrt{\pi}}{3} (9\sqrt{2} - 28) + \frac{1}{\sqrt{\pi}} \int_0^1 dx \frac{x^3 + 3x^2 - 4}{\sqrt{1 - x^2}(1 + x)} \log \left( \frac{\sqrt{2} - \sqrt{x(1 + x)}}{\sqrt{2} + \sqrt{x(1 + x)}} \right) \simeq -7.4.$$

(55)

For the speed of zero sound in this regime of temperatures one gets the result [28, 57]

$$c = c_B + \frac{G(\infty) k_B T a}{\sqrt{\pi} \hbar},$$

(56)

where the numerical coefficient $G(\infty)/(2\sqrt{\pi}) \simeq -2.1$ agrees with the finding of [27], while is about a factor 6 larger than the one calculated in [28].

The proper description of the cross-over between the low and high-temperature regime is provided by Eq. (56). The dimensionless function $G(\tau)$ is plotted in Fig. 3. In the experiments on trapped gases the gas parameter in the center of the trap is typically $a^3 n_0 \sim 10^{-5} - 10^{-4}$. For temperatures of the order of the chemical potential, which means $\tau \sim 1$, the correction to the Bogoliubov speed of sound amounts to about 2-5%.

IV. SPATIALLY INHOMOGENEOUS SYSTEM

In this section we generalize the perturbation scheme developed for a homogeneous Bose-condensed gas to the case of inhomogenous systems trapped by a harmonic confining potential

$$V_{\text{ext}}(r) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right).$$

(57)

The relevant length scale associated to the external potential (57) is the harmonic oscillator length defined as
\[ a_{ho} = \left( \frac{\hbar}{m \omega_{ho}} \right)^{1/2}, \]  

where \( \omega_{ho} = (\omega_x \omega_y \omega_z)^{1/3} \) is the geometric average of the oscillator frequencies. The length scale \( a_{ho} \) gives the average width of the Gaussian which describes the ground state of non-interacting particles in the harmonic potential \((55)\). The shape of the potential \( V_{ext} \) fixes the symmetry of the problem. So far all experiments on trapped Bose gases have been realized using axially symmetric traps. In this case there are only two distinct oscillator frequencies: \( \omega_\perp = \omega_x = \omega_y \) and \( \omega_z \). The ratio between the axial and radial frequencies, \( \lambda = \omega_z/\omega_\perp \), fixes the asymmetry of the trap. For \( \lambda < 1 \) the trap is cigar shaped, whereas for \( \lambda > 1 \) is disk shaped and \( \lambda = 1 \) refers to spherically symmetric traps.

In our analysis we only consider systems with repulsive interactions \( (a > 0) \) in the thermodynamic limit. As extensively discussed in \([2]\), for harmonically trapped Bose systems the thermodynamic limit is obtained by letting the total number of trapped particles \( N \to \infty \) and \( \omega_{ho} \to 0 \), while keeping the product \( N \omega_{ho}^3 \) constant. With this definition the Bose-Einstein transition temperature \( k_B T_c^0 = \hbar \omega_{ho}(N/\zeta(3))^{1/3} \) is well defined in the thermodynamic limit.

In the thermodynamic limit, the condition \( N_0(T)a/\omega_{ho} \gg 1 \), which ensures the validity of the Thomas-Fermi (TF) approximation for the condensate with occupation number \( N_0 \), is always guaranteed below the transition temperature. In the TF approximation one neglects the quantum-pressure term proportional to \( V^2 \Phi_0(r) \) in the stationary equation \((6)\), and the equilibrium profile of the condensate density is fixed by the following equation

\[ g n_0(r) = \mu - V_{ext}(r) - 2g \tilde{n}(r) - g \tilde{m}(r), \]  

in the central region of the trap where the r.h.s. of the above equation is positive, whereas outside this region one has \( n_0(r) = 0 \). The chemical potential in Eq. \((59)\) is fixed by the normalization condition \( \int dr n_0(r) = N_0(T) \), with \( N_0(T) \) the equilibrium condensate occupation number at temperature \( T \). To lowest order in the interaction, the profile of the condensate density has the form of the inverted parabola

\[ n_{TF}(r) = g^{-1} [\mu_{TF}(N_0) - V_{ext}(r)], \]  

where

\[ \mu_{TF}(N_0) = \frac{\hbar \omega_{ho}}{2} (\frac{15 N_0 a}{a_{ho}})^{2/5} \]  

is the temperature dependent TF chemical potential. Moreover, in the thermodynamic limit, one can show that the equilibrium properties of the system can be expressed in terms of two parameters: the reduced temperature \( t = T/T_c^0 \) and the interaction parameter \( \eta \) defined as the ratio

\[ \eta = \frac{\mu_{TF}(N)}{k_B T_c^0}, \]  

between the TF chemical potential at \( T = 0 \) and the transition temperature.

The time-dependent equation \((5)\) for the fluctuations of the condensate only needs to be solved in the region where \( n_0(r) \neq 0 \), according to Eq. \((18)\). One finds

\[ i \hbar \frac{\partial}{\partial t} \delta \Phi(r,t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + g n_0(r) - g \tilde{n}(r) \right) \delta \Phi(r,t) + \left( g n_0(r) + g \tilde{m}(r) \right) \Phi^*(r,t) \]

\[ + 2g \Phi_0(r) \delta \tilde{n}(r,t) + g \Phi_0(r) \delta \tilde{m}(r,t). \]  

For a trapped system in the thermodynamic limit the above equations \((59)\) and \((63)\) replace respectively Eqs. \((18)\) and \((19)\), holding for a homogeneous system.

We are interested in the lowest-lying collective modes of the system with excitation energy \( \hbar \omega \ll \mu \). To lowest order in the interaction these modes are the solution of the following coupled equations

\[ i \hbar \frac{\partial}{\partial t} [\delta \Phi(r,t) + \delta \Phi^*(r,t)] = -\frac{\hbar^2 \nabla^2}{2m} [\delta \Phi(r,t) - \delta \Phi^*(r,t)], \]

\[ i \hbar \frac{\partial}{\partial t} [\delta \Phi(r,t) - \delta \Phi^*(r,t)] = 2 g n_{TF}(r) [\delta \Phi(r,t) + \delta \Phi^*(r,t)]. \]  

\[ \text{Eq. (64)} \]
These equations are obtained from (53) by neglecting all coupling terms to noncondensate particles and neglecting also the term proportional to $\nabla^2(\delta \Phi + \delta \Phi^*)$, which is of higher order for the low-lying modes we are considering [58].

The oscillating solution defined by $\delta \Phi(r, t) = \delta \Phi_0(r) e^{-i\omega t}$, $\delta \Phi^*(r, t) = \delta \Phi_0^*(r) e^{-i\omega t}$, with the Fourier components fixed by the relations

$$\langle \delta \Phi_0^0(r) + \delta \Phi_0^0(r) \rangle = \sqrt{\frac{\hbar \omega}{2gn_{TF}(r)}} \chi_0(r),$$

$$\langle \delta \Phi_0^0(r) - \delta \Phi_0^0(r) \rangle = \sqrt{\frac{2gn_{TF}(r)}{\hbar \omega}} \chi_0(r),$$

reduces the coupled equations (64) to the following equation for the function $\chi_0(r)$ [58]

$$m\omega^2 \chi_0(r) + \nabla \left[ g n_{TF}(r) \nabla \chi_0(r) \right] = 0. \quad (66)$$

The normalization condition $\int dr \left( |\delta \Phi_0^0(r)|^2 - |\delta \Phi_0^0(r)|^2 \right) = 1$ satisfied by the Fourier components $\delta \Phi_0^0$ and $\delta \Phi_0^0$, implies the normalization condition $\int dr \chi_0^* \chi_0 = 1$ on the function $\chi_0(r)$.

Equation (66) was first derived at $T = 0$ by Stringari [3] using the hydrodynamic theory of superfluids, and it has then been studied by many authors [68,69]. For spherically symmetric traps the excitation energies $\hbar \omega \equiv \epsilon_{TF}$ obey the dispersion law [4]

$$\epsilon_{TF}(n_r, l) = \hbar \omega_{\lambda} \left( 2n_r^2 + 2n_r l + 3n_r + l \right)^{1/2}, \quad (67)$$

where $n_r$ and $l$ are respectively the radial and the angular momentum quantum numbers. In the case of axially symmetric traps, analytic results for the excitation energies are obtained for the $m = 0$ low and high mode [4]

$$\epsilon_{TF}(m = 0)_{L,H} = \hbar \omega_{\lambda} \left( 2 + \frac{3}{2} \lambda^2 \pm \frac{1}{2} \sqrt{9 \lambda^4 - 16 \lambda^2 + 16} \right)^{1/2}, \quad (68)$$

and for the surface modes of the form $\chi_m \propto r^{|m|} e^{im\phi}$, for which one has

$$\epsilon_{TF}(m) = \hbar \omega_{\lambda} \sqrt{|m|}. \quad (69)$$

A general feature of Eq. (64), which is explicitly reflected in the above results for $\epsilon_{TF}$, is that the excitation energies do not depend on interaction and are proportional to the oscillator frequencies of the harmonic potential. At finite temperature, where $g n_{TF} = \mu_{TF} [N_0(T)] - V_{ext}$, this fact implies that $\epsilon_{TF}$ does not depend on temperature either. This is an important difference with respect to the homogeneous case where the corresponding excitations have the dispersion law $\epsilon_q = 4g n_{0}/m q$, and depend on temperature through the condensate density. The behavior exhibited in the harmonic trap is well understood if one notes that the values of $q$ are fixed by the boundary and vary as $1/R$, where $R$ is the size of the condensate. In the Thomas-Fermi limit, $R \sim \sqrt{\mu_{TF}/m \omega_{\lambda}^2}$ and the radius $R$ explicitly depends on the chemical potential. On the other hand, the sound velocity is also fixed by the value of the chemical potential: $c_B \sim \sqrt{\mu_{TF}/m}$. One finally finds that in the product $c_B q$ the chemical potential cancels out, so that the collective frequency is proportional to the oscillator frequency $\omega_{\lambda}$.

Provided that finite size effects can be neglected, the explicit dependence of the collective frequency on the interaction parameter $\eta$ as well as on the reduced temperature $t = T/T_c$ arises due to quantum and thermal fluctuations which are of order $g^2$. These fluctuations have the same physical origin as the corrections to the Bogoliubov speed of sound given in the homogeneous case by result [54]. The difference, however, is that in the case of harmonic traps, beyond mean-field effects give corrections to a collective frequency which is fixed only by the oscillator frequency: a much better situation from the experimental point of view. In the following part of this section we will explicitly calculate the effects of quantum and thermal fluctuations on the frequencies of the lowest compressional and surface modes.

A. Perturbation scheme

The perturbation scheme we employ for trapped systems follows the same lines as the one developed in the homogeneous case. However, there are two important differences: first of all the quasiparticle states are not exact plane waves, secondly the condensate density is not fixed by a single parameter but depends on position.
Concerning the quasiparticle states, we make use of the local density (semiclassical) approximation which amounts to setting \([11,12,61]\)

\[
\begin{align*}
  u_i(r) &= \frac{\tilde{u}_i(r)}{\sqrt{V}} e^{i\varphi_i(r)} , \\
  v_i(r) &= \frac{\tilde{v}_i(r)}{\sqrt{V}} e^{i\varphi_i(r)} ,
\end{align*}
\]  

(70)

where \(V\) is a large volume containing the system and the real functions \(\tilde{u}_i\), \(\tilde{v}_i\) satisfy the normalization condition \(\tilde{u}_i^2(r) - \tilde{v}_i^2(r) = 1\). The factor \(e^{i\varphi_i(r)}\) represents the rapidly varying part of the functions \(u_i\) and \(v_i\), while the functions \(\tilde{u}_i\), \(\tilde{v}_i\) are assumed to be smooth functions of the position. The phase \(\varphi_i(r)\), which is also assumed to be a smooth function of \(r\), characterizes the local impulse \(p = \hbar\nabla \varphi_i\) of the quasiparticle. When summations over quasiparticle states are involved, these are replaced in the semiclassical approximation by sums over momenta, \(\sum_i \rightarrow \sum_p \ldots\)

and \(\tilde{u}_i(r) \rightarrow u_p(r), \tilde{v}_i(r) \rightarrow v_p(r)\), where the functions \(u_p(r)\) and \(v_p(r)\) are given, in the region of the condensate, by the following expressions

\[
u_p^2(r) = 1 + v_p^2(r) = \left(\epsilon_p^2(r) + g^2 n_{TF}^2(r)\right)^{1/2} + \epsilon_p(r) ,
\]

\[
u_p(r)v_p(r) = -\frac{g n_{TF}(r)}{2\epsilon_p(r)} ,
\]

(71)

where the position-dependent quasiparticle energies \(\epsilon_p(r)\) are given by

\[
\epsilon_p(r) = \left(\left[\epsilon_p^0 + g n_{TF}(r)\right]^2 - \left[g n_{TF}(r)\right]^2\right)^{1/2} .
\]

(72)

For each position \(r\) the above equations coincide with the Bogoliubov expressions \([23,24]\) with a local condensate density given by the TF density profile \(n_{TF}(r)\) defined in \([51]\). The semiclassical approximation for the excited states of a trapped Bose gas has been extensively used in the theoretical study of the thermodynamic properties of the system \([11,32-43]\). It gives a very good description of the system for temperatures \(k_B T \gg \hbar \omega_{ho}\), but is also valid at \(T = 0\) if the relevant energies in the summation over excited states are much larger than the oscillator energy \(\hbar \omega_{ho}\) \([43]\). For large systems the oscillator energy is the smallest energy scale and vanishes in the thermodynamic limit, as a consequence, in this limit, the semiclassical approximation becomes a rigorous treatment.

The equilibrium noncondensate densities \(\tilde{n}_i^0(r)\) and \(\tilde{m}_i^0(r)\) are readily calculated employing the semiclassical approximation. One obtains

\[
\tilde{n}_i^0(r) = \frac{1}{V} \sum_i \left\{ [\tilde{u}_i^2(r) + \tilde{v}_i^2(r)] f_i^0 + \tilde{v}_i^2(r) \right\} = \frac{1}{V} \sum_p \left\{ [u_p^2(r) + v_p^2(r)] f_p^0(r) + v_p^2(r) \right\} ,
\]

\[
\tilde{m}_i^0(r) = \frac{1}{V} \sum_i \tilde{u}_i(r) \tilde{v}_i(r) \left(1 + 2 f_i^0\right) = \frac{1}{V} \sum_p u_p(r) v_p(r) \left[1 + 2 f_p^0(r)\right] ,
\]

(73)

where \(f_p^0(r) = (e^{\epsilon_p(r)/\hbar k_B T} - 1)^{-1}\) is the local equilibrium quasiparticle distribution function.

By inserting in Eq. (59) the above expressions for the noncondensate densities and using the renormalization \([20]\) of the coupling constant, one obtains the following result for the profile of the condensate density valid to order \(g^2\)

\[
g n_0(r) = g n_{TF}(r) + \delta \mu - g n_{TF}(r) [a^3 n_{TF}(r)]^{1/2} H(\tau(r)) ,
\]

(74)

where \(H(\tau)\) is the dimensionless function \([33]\) of the local reduced temperature \(\tau(r) = k_B T / g n_{TF}(r)\). In the above equation \(\delta \mu = \mu - \mu_{TF}(N_0) - 2 g n_{TF}^0\), with \(n_i^0 = \zeta(3/2) \lambda_T^{-3}\) as in \([34]\), is the shift in the chemical potential corresponding to the change in the condensate density profile.

The application of the semiclassical approximation to the last two terms on the r.h.s. of Eq. (38), which describe the dynamic coupling to the noncondensate particles, needs a careful treatment. Thus, for the moment, we calculate them in terms of the \(u_i\) and \(v_i\) functions. Similarly to the homogeneous case [see Eqs. (23)], one must neglect in Eqs. (16), (17) the terms proportional to \(\delta \tilde{n}\) and \(\delta \tilde{m}\). In Fourier space, one finds for the components of the matrices \(f_{ij}\), \(g_{ij}\) and \(g^{ij}_{ij}\) oscillating at the frequency \(\omega\)

\[
f_{ij}(\omega) = g \frac{f_i^0 - f_j^0}{\hbar \omega + (\epsilon_i - \epsilon_j) + i \hbar} \int dr \Phi_0 \left[ (\delta \Phi_1 - \delta \Phi_2)(v_i u_j^* - u_i v_j^*) + (\delta \Phi_1 + \delta \Phi_2)(2 u_i v_j^* + 2 v_i u_j^* + v_i u_j^* + u_i v_j^*) \right] ,
\]

(75)
From Eqs. (76), by treating the corrections to Eqs. (64) as small perturbations, one gets the result
\[ f \]
the noncondensate component, which contain

In the above equations one can recognize the terms arising from the dynamic coupling between the condensate and homogeneous case. In Eqs. (76) we have neglected, as in Eqs. (64), the term proportional to

We have taken

Following the analysis carried out in the homogeneous case, we write the excitation energy as

To order \( g \) the equation for the low-lying oscillations of the condensate can be written in the form

In the above equations one can recognize the terms arising from the dynamic coupling between the condensate and the noncondensate component, which contain \( f_{ij}(\omega) \), \( g_{ij}(\omega) \) and \( g_{ij}^*(\omega) \), the terms arising from the coupling to the static anomalous density \( \tilde{n}^0 \) and from the renormalization of \( g \), and, finally, the terms proportional to \( \delta \mu \) and \( H(\tau) \) which come from the change in the density profile of the condensate. The last terms have no counterpart in the homogeneous case. In Eqs. (74) we have neglected, as in Eqs. (74), the term proportional to \( \nabla^2(\delta \Phi_1 + \delta \Phi_2) \).

Following the analysis carried out in the homogeneous case, we write the excitation energy as

From Eqs. (74), by treating the corrections to Eqs. (64) as small perturbations, one gets the result

holding for the low-lying modes with \( \epsilon_{TF} \ll \mu \). Notice that the shift \( \delta \mu \) of the chemical potential does not enter result (74). In fact, in the Thomas-Fermi limit, the excitation frequencies obtained from Eq. (66) do not depend on the value of \( \mu \). In Eq. (77), the matrix elements \( A_{ij} \), \( B_{ij} \) and \( \tilde{B}_{ij} \) are defined, in analogy to the homogeneous case, as

\[ A_{ij} = \frac{1}{2} \int \frac{d\mathbf{r}}{\sqrt{\nu_{TF}}} \left[ (\delta \Phi_1^0 + \delta \Phi_2^0)(2u^*_i u^*_j + 2v^*_i v^*_j + u^*_i u^*_j + v^*_i v^*_j) + (\delta \Phi_1^0 - \delta \Phi_2^0)(v_i u^*_j - u_i v^*_j) \right], \]

\[ B_{ij} = \frac{1}{2} \int \frac{d\mathbf{r}}{\sqrt{\nu_{TF}}} \left[ (\delta \Phi_1^0 + \delta \Phi_2^0)(2u^*_i v^*_j + 2v^*_i u^*_j + u^*_i v^*_j + v^*_i u^*_j) + (\delta \Phi_1^0 - \delta \Phi_2^0)(u^*_i u^*_j - v^*_i v^*_j) \right], \]

(78)
\[ \tilde{B}_{ij} = \frac{1}{2} \int d\mathbf{r} \sqrt{\nu_{TF}} \left[ (\delta \Phi_1^0 + \delta \Phi_2^0)(2u_i v_j + 2v_i u_j + u_i u_j + v_i v_j) - (\delta \Phi_1^0 - \delta \Phi_2^0)(u_i u_j - v_i v_j) \right]. \]

Starting from Eq. (71), one can study both the damping and the frequency shift of the low-lying modes. The calculation of the damping rates has been carried out by several authors [23, 24]. In Refs. [23, 24] the damping of the \( m = 0 \) and \( m = 2 \) mode has been calculated as a function of temperature and found in good agreement with experiments [38]. Concerning the frequency shifts, a calculation based on a method similar to ours has been carried out in [27], but only for quasiclassical modes which satisfy the condition \( \hbar \omega_0 \ll \epsilon_{TF} \ll \mu \). In the present work we study the frequency shift of the lowest-lying collective modes with \( \epsilon_{TF} \sim \hbar \omega_0 \). These modes have also been studied within the dielectric formalism in [24].

### B. Frequency shift of the collective modes

#### 1. Non-resonant contribution

Similarly to the homogeneous case, the non-resonant contribution to the frequency shift \( \delta E \) is defined as

\[ \delta E_{NR} = 2g^2 \sum_{ij} (1 + f_i^0 f_j^0) \left( \frac{|B_{ij}|^2}{\epsilon_{TF} - \epsilon_i - \epsilon_j} - \frac{|\tilde{B}_{ij}|^2}{\epsilon_{TF} + \epsilon_i + \epsilon_j} \right). \]  

(79)

The matrix elements of \( B \) and \( \tilde{B} \) are given in (78). By using the semiclassical approximation (76) for the quasiparticle functions \( u_i \) and \( v_i \), one can write \( \delta E_{NR} \) in the following form

\[ \frac{\delta E_{NR}}{\epsilon_{TF}} = -\frac{g}{2V^2} \sum_{ij} \int d\mathbf{r} d\mathbf{s} e^{i\mathbf{r} \cdot \mathbf{\nabla} [\varphi_i(r) + \varphi_j(r)]} \chi_0(r) \left[ K_{1ij}^{NR}(r, r + s) + \tilde{K}_{1ij}^{NR}(r, r + s) \right] \chi_0^*(r + s), \]

(80)

where the functions \( \chi_0(r) \) are the solutions of (66), and in \( e^{i\mathbf{r} \cdot \mathbf{\nabla} [\varphi_i(r) + \varphi_j(r)]} \) we have neglected second derivatives of the phase \( \varphi \). The smoothly varying kernels \( K_{1ij}^{NR} \) and \( \tilde{K}_{1ij}^{NR} \) are defined as

\[ K_{1ij}^{NR}(r, r') = \frac{1 + f_i^0 + f_j^0}{\epsilon_i + \epsilon_j} \left( \frac{2g_{n_{TF}}(r)}{\epsilon_i + \epsilon_j} b_{ij}(r) \right) \left( a_{ij}(r') + \frac{2g_{n_{TF}}(r')}{\epsilon_i + \epsilon_j} b_{ij}(r') \right), \]

\[ \tilde{K}_{1ij}^{NR}(r, r') = \frac{1 + f_i^0 + f_j^0}{\epsilon_i + \epsilon_j} \frac{2g_{n_{TF}}(r)}{\epsilon_{TF}} b_{ij}(r) \frac{2g_{n_{TF}}(r')}{\epsilon_{TF}} b_{ij}(r'), \]

(81)

where we have introduced the matrices

\[ a_{ij}(r) = 2\tilde{u}_i(r)\tilde{u}_j(r) + 2\tilde{v}_i(r)\tilde{u}_j(r) + \tilde{u}_i(r)\tilde{u}_j(r) + \tilde{v}_i(r)\tilde{v}_j(r), \]

\[ b_{ij}(r) = \tilde{u}_i(r)\tilde{v}_j(r) - \tilde{v}_i(r)\tilde{u}_j(r). \]

(82)

In the limit \( \epsilon_{TF} \ll \mu \) we can use the following gradient expansion

\[ \chi_0(r)K_{1ij}^{NR}(r, r + s)\chi_0^*(r + s) \simeq K_{1ij}^{NR}(r, r) |\chi_0(r)|^2, \]

(83)

and

\[ \chi_0(r)\tilde{K}_{1ij}^{NR}(r, r + s)\chi_0^*(r + s) \simeq \tilde{K}_{1ij}^{NR}(r, r) \left[ |\chi_0(r)|^2 + \frac{1}{2} \chi_0(r) (s \cdot \nabla) |\chi_0(r)|^2 \right] \]

\[ + \frac{1}{2} \chi_0(r) (s \cdot \nabla) \tilde{K}_{1ij}^{NR}(r, r) (s \cdot \nabla) |\chi_0(r)|. \]

(84)

Since the kernel \( K_{1ij}^{NR} \) is already zeroth order in \( \epsilon_{TF}/\mu \), we can neglect higher order terms in the expansion (83). On the contrary, \( \tilde{K}_{1ij}^{NR} \) is of order \( (\mu/\epsilon_{TF})^2 \) and we need the expansion (84) to second order in the displacement \( s \). Notice also that terms in the expansion (84) containing odd powers of \( s \) vanish in (84) due to geometry. Moreover, we have neglected in (84) second derivatives of the slowly varying functions \( \tilde{u}_i, \tilde{v}_i \). By the replacement \( \sum_i \to \sum_p \), and after integration by parts, one gets the result
where \( k = q + p \). In the first term on the r.h.s. of (85) the integration over \( s \) gives \( \delta_{qp} \), while in the second term one writes \( s^2 e^{-i\mathbf{q} \cdot \mathbf{s}/\hbar} = -\hbar^2 \nabla_q^2 e^{-i\mathbf{q} \cdot \mathbf{s}/\hbar} \) and integrates by parts over \( q \). After some algebra one gets the result

\[
\frac{\delta E_{NR}}{\epsilon_{TF}} = -\int d\mathbf{r} \frac{g}{\mathcal{V}} \sum_{p} \left( 1 + 2f_0^p \right) \left[ \left| \chi_0 \right|^2 \frac{\left( f_0^p \right)^2}{4\epsilon_p} - \left| \nabla \chi_0 \right|^2 \frac{\hbar^2 g^2 \epsilon_{TF}^2}{m \epsilon_{TF}} \frac{f_0^p}{6\epsilon_p} \right] - \int d\mathbf{r} \left| \chi_0 \right|^2 \frac{\hbar^2 g^2 \epsilon_{TF}^2}{2\epsilon_{TF}^2} \frac{1}{\mathcal{V}} \sum_{p} \frac{1 + 2f_0^p}{\epsilon_p} \right] - \frac{1}{6} \int d\mathbf{r} \left| \nabla \chi_0 \right|^2 \frac{\hbar^2 g^2 \epsilon_{TF}^2}{2\epsilon_{TF}^2} \frac{1}{\mathcal{V}} \sum_{q} \int d\mathbf{s} e^{-i\mathbf{q} \cdot \mathbf{s}/\hbar} \nabla_q^2 \frac{1}{4\epsilon_{q+p}} \sum_{p} f_0^p \epsilon_{q+p} . \tag{86}
\]

In the above equation \( \epsilon_p = \epsilon_p(\mathbf{r}) \) and \( f_0^p = f_0^p(\mathbf{r}) \), according to (72). We notice that in the homogeneous limit, where \( \chi_0(\mathbf{r}) = e^{i\mathbf{q} \cdot \mathbf{r}/\hbar}/\sqrt{V} \) and \( \epsilon_{TF} = \epsilon_B q \), the first two terms on the r.h.s. of (85) coincide with the corresponding terms in (40).

2. Resonant contribution

The resonant contribution to \( \delta E \) is defined as

\[
\frac{\delta E_R}{\epsilon_{TF}} = 4g^2 \sum_{ij} \left( f_0^i - f_0^j \right) \frac{|A_{ij}|^2}{\epsilon_{TF} + \epsilon_i - \epsilon_j} , \tag{87}
\]

where the matrix elements \( A_{ij} \) are given in (78). Following the method used in the analysis of the non-resonant terms one has

\[
\frac{\delta E_R}{\epsilon_{TF}} = \frac{g}{2\mathcal{V}^2} \sum_{ij} \int d\mathbf{r} d\mathbf{s} e^{-i\mathbf{q} \cdot \nabla(\phi_i - \phi_j)} \chi_0(\mathbf{r}) \left[ \tilde{K}_{1ij}(\mathbf{r}, \mathbf{r} + \mathbf{s}) + \tilde{K}_{2ij}(\mathbf{r}, \mathbf{r} + \mathbf{s}) \right] \chi_0^*(\mathbf{r} + \mathbf{s}) \tag{88}
\]

The kernels in the above equation are defined by

\[
\tilde{K}_{1ij}(\mathbf{r}, \mathbf{r}') = \frac{f_0^i - f_0^j}{\epsilon_{TF} + \epsilon_i - \epsilon_j} \left( c_{ij}(\mathbf{r}) - \frac{2g\epsilon_{TF}(\mathbf{r})}{\epsilon_i - \epsilon_j} d_{ij}(\mathbf{r}) \right) \left( c_{ij}(\mathbf{r}') - \frac{2g\epsilon_{TF}(\mathbf{r}')}{\epsilon_i - \epsilon_j} d_{ij}(\mathbf{r}') \right) , \tag{89}
\]

\[
\tilde{K}_{2ij}(\mathbf{r}, \mathbf{r}') = \frac{f_0^i - f_0^j}{\epsilon_i - \epsilon_j} \frac{2g\epsilon_{TF}(\mathbf{r})}{\epsilon_{TF}^2} d_{ij}(\mathbf{r}) \frac{2g\epsilon_{TF}(\mathbf{r}')}{\epsilon_{TF}^2} d_{ij}(\mathbf{r}') ,
\]

where we have introduced the matrices

\[
c_{ij}(\mathbf{r}) = 2\tilde{u}_i(\mathbf{r})\tilde{u}_j(\mathbf{r}) + 2\tilde{v}_i(\mathbf{r})\tilde{v}_j(\mathbf{r}) + \tilde{v}_i(\mathbf{r})\tilde{u}_j(\mathbf{r}) + \tilde{u}_i(\mathbf{r})\tilde{v}_j(\mathbf{r}) ,
\]

\[
d_{ij}(\mathbf{r}) = \tilde{v}_i(\mathbf{r})\tilde{u}_j(\mathbf{r}) - \tilde{u}_i(\mathbf{r})\tilde{v}_j(\mathbf{r}) . \tag{90}
\]

In the limit \( \epsilon_{TF} \ll \mu \) the term in Eq. (88) containing the kernel \( \tilde{K}_{2ij} \) can be treated using the gradient expansion (54). One gets thus

\[
\frac{g}{2\mathcal{V}^2} \sum_{ij} \int d\mathbf{r} d\mathbf{s} e^{-i\mathbf{q} \cdot \nabla(\phi_i - \phi_j)} \chi_0(\mathbf{r}) \tilde{K}_{2ij}(\mathbf{r}, \mathbf{r} + \mathbf{s}) \chi_0^*(\mathbf{r} + \mathbf{s}) =
\]

\[
- \int d\mathbf{r} \left| \nabla \chi_0 \right|^2 \frac{\hbar^2 g^2 \epsilon_{TF}^2}{m \epsilon_{TF}} \frac{1}{\mathcal{V}} \sum_{p} \frac{\partial f_0^p}{\partial \epsilon_p} \frac{\epsilon_0^p}{3\epsilon_0^p} + \frac{1}{6} \int d\mathbf{r} \left| \nabla \chi_0 \right|^2 \frac{\hbar^2 g^2 \epsilon_{TF}^2}{\epsilon_{TF}^2} \frac{1}{\mathcal{V}} \sum_{q} \int d\mathbf{s} e^{-i\mathbf{q} \cdot \mathbf{s}/\hbar} \nabla_q^2 \frac{1}{4\epsilon_q} \sum_{p} \frac{f_0^p}{\epsilon_{q+p}} . \tag{91}
\]
In the homogeneous limit, the first term on the r.h.s. of \( \text{(11)} \) coincides with the second term in the square bracket on the r.h.s. of \( \text{(8)} \). Moreover, the last term on the r.h.s. of \( \text{(8)} \) and \( \text{(11)} \) are equal and opposite in sign, and cancel out in the sum \( \Delta E_{NR} + \delta E_R \). A similar cancellation is also present in the homogeneous case [see Eqs. \( \text{(6)} \) and \( \text{(8)} \)].

The contribution to \( \Delta E_R \) arising from \( \bar{K}_{ij}^R \) is more delicate. In fact, all terms in the gradient expansion give contributions which are of the same order in the limit \( \epsilon_{TF} \ll \mu \). However, if we restrict ourselves to modes for which \( \nabla^2 \chi_0 = \text{const} \), the expansion \( \text{(8)} \) is still appropriate. In fact, higher order terms in \( \text{(8)} \) contain derivatives of \( \nabla^2 \chi_0 \) and vanish for modes with constant Laplacian. To this class of modes belong, for example, all surface modes, for which \( \nabla^2 \chi_0 = 0 \), and the lowest breathing modes. For the above mentioned term one finds

\[
\frac{g}{2V^2} \sum_{ij} \int dr ds e^{-i\mathbf{s} \cdot \nabla \varphi_i(r) - \varphi_j(r)} \chi_0(r) K_{1ij}^R(r, r + s) \chi_0^*(r + s) = \frac{-1}{12} \sum_{pq} e^{ij} \sum p q e^{-i\mathbf{q} \cdot \mathbf{s}/\hbar^2} \nabla_q^2 \left[ \chi_0(r) \nabla^2 \chi_0^*(r) \bar{K}_{1pk}(r, r) + \chi_0(r) \nabla \chi_0^*(r) \cdot \nabla \bar{K}_{1pk}(r, r) \right].
\]

(92)

The Laplacian in momentum space can be easily calculated once the low-q behavior of \( \bar{K}_{1pk}(r, r) \) has been obtained. A straightforward calculation yields:

\[
\frac{g}{V} \sum_p \bar{K}_{1pk}^R \rightarrow \frac{2g^2}{3m \epsilon_{TF}} \frac{1}{V} \sum_p \left( -\frac{\partial f_p}{\partial \epsilon_p} \right) \frac{e^0}{1 + \frac{e^0}{\epsilon_p} \partial \epsilon_p}^2.
\]

(93)

After some algebra, one obtains the following result for the contribution to \( \Delta E_R \)

\[
\frac{g}{2V^2} \sum_{ij} \int dr ds e^{-i\mathbf{s} \cdot \nabla \varphi_i(r) - \varphi_j(r)} \chi_0(r) K_{1ij}^R(r, r + s) \chi_0^*(r + s) = \frac{\sqrt{2} \hbar^2 g}{3\sqrt{\pi} \epsilon_{TF}} \times \int dr |\nabla \chi_0|^2 (a_{TF}^3)^{3/2} \int dx \left[ \frac{u - 1}{u + 1} \left( \frac{2u + 1}{u + 1} \right)^2 - 4 \tau x \right],
\]

(94)

where \( u = \sqrt{1 + \tau^2(r)} \) and \( \tau(r) = \frac{k_B T}{g n_{TF}(r)} \). We stress that the above contribution to the frequency shift is peculiar of collective modes with constant Laplacian of harmonically trapped systems in the thermodynamic limit. In the homogeneous case, where \( \chi_0 \propto e^{i\mathbf{q} \cdot \mathbf{r}} \) and \( \nabla^2 \chi_0 \neq \text{const} \), the contribution of this term is different and is given by the first term in the square bracket on the r.h.s. of \( \text{(8)} \).

3. Results

We are now in a position to calculate the shift \( \delta E \) to order \( g^2 \), by summing the various contributions to the real part of Eq. \( \text{(77)} \). One gets the relevant result

\[
\frac{\delta E}{\epsilon_{TF}} = -\frac{4}{3\sqrt{\pi}} \frac{g}{m \omega_{TF}^2} \int \frac{dr}{a_{TF}^3} \chi_0 \nabla^2 \chi_0^* + \chi_0^* \nabla^2 \chi_0
\]

\[
+ \int dr (a_{TF}^3)^{3/2} |\chi_0|^2 G_1(\tau(r)) - \frac{g}{m \omega_{TF}^2} \int dr (a_{TF}^3)^{3/2} |\nabla \chi_0|^2 G_2(\tau(r)),
\]

(95)

where \( \omega_{TF} = \epsilon_{TF}/\hbar \), and the functions \( G_1(\tau) \), \( G_2(\tau) \) of the local reduced temperature \( \tau(r) = \frac{k_B T}{g n_{TF}(r)} \) are defined as follows

\[
G_1(\tau) = \frac{\sqrt{32}}{3\sqrt{\pi}} \tau \int_0^\infty dx \frac{1}{e^x - 1} \left( \frac{\sqrt{\tau x} - \sqrt{u - 1}}{u} \frac{u^2 + u - 1}{u + 1} \right),
\]

(96)

and

\[
G_2(\tau) = \frac{\sqrt{32}}{3\sqrt{\pi}} \left\{ \int_0^\infty dx \frac{x}{(e^x/2 - e^{-x/2})^2} \left[ 4 \tau x - \frac{\sqrt{u - 1}}{u(u + 1)} + \frac{(u - 1)^{3/2}}{u} \left( \frac{2u + 1}{u + 1} \right)^2 \right] - \int_0^\infty dx \frac{1}{e^x - 1} \frac{\sqrt{u - 1}}{u(u + 1)} \right\},
\]

(97)
where, as usual, \( u = \sqrt{1 + \tau^2 x^2} \). The functions \( G_1(\tau) \) and \( G_2(\tau) \) are plotted in Fig. 4. Both are positive for any value of \( \tau \). As a consequence, \( G_1(\tau) \) gives an upward shift of the excitation frequency, while \( G_2(\tau) \) gives a downward shift. The above result holds for collective modes which do not excite the center of mass degrees of freedom. In fact, as discussed in Sec. II, the theoretical approach developed in the present work does not describe the center of mass motion and, in particular, the dipole mode.

At \( T = 0 \), both \( G_1 \) and \( G_2 \) are zero and one is left with the result

\[
\frac{\delta E}{\epsilon_T} = -\frac{4}{3\sqrt{\pi}} \frac{g}{m\omega_T^2} \int dx (an_T)^{3/2} \left( \chi_0^* \nabla^2 \chi_0^* + \chi_0^* \nabla^2 \chi_0 \right) ,
\]

which coincides with the findings of Ref. [23], obtained from the hydrodynamic theory of superfluids, and of Ref. [30]. Notice that only non-resonant terms give contribution at \( T = 0 \), as a consequence result (108) holds in general for collective modes with \( \epsilon_T \ll \mu \), and is not restricted to modes which have constant Laplacian. For the monopole (breathing) mode in a spherically symmetric trap (\( \lambda = 1 \)), characterized by the frequency \( \omega_M = \sqrt{5} \omega_{ho} \), one has from Eq. (98) the fractional shift [24,30]

\[
\frac{\delta \omega_M}{\omega_M} = \frac{21\sqrt{2}}{320\sqrt{3}} \eta^3 ,
\]

expressed in terms of the parameter \( \eta \). For the \( m = 0 \) modes in an axially symmetric trap, which have excitation frequency given by (108), one finds the result [24,30]

\[
\frac{\delta \omega_{m=0}}{\omega_{m=0}} = \frac{21\sqrt{2}}{320\sqrt{3}} \eta^3 f_\pm(\lambda) ,
\]

where

\[
f_\pm(\lambda) = \frac{1}{2} \pm \frac{8 + \lambda^2}{6\sqrt{9\lambda^4 - 16\lambda^2 + 16}} ,
\]

and the index \( \pm \) refers to the high (+) and low (–) \( m = 0 \) mode. As discussed in Ref. [23], these frequency shifts are very small. For \( \eta = 0.4 \), which is a typical value for the interaction parameter in experiments, one gets a fractional shift of the order of 0.5%.

At finite temperature the terms involving the \( G_1 \) and \( G_2 \) functions contribute to the frequency shift, and, differently from the \( T = 0 \) case, also surface excitations with \( \nabla^2 \chi_0 = 0 \) are affected by the correction (102). We consider first spherically symmetric traps. The monopole oscillation has the form \( \chi_M \propto (r^2 - 3R^2/5) \), where \( R = \sqrt{2\mu_T(N_0)/m\omega_{ho}^2} \) is the condensate radius. The temperature dependence of the fractional shift \( \delta \omega_M/\omega_M \) is given by the equation

\[
\frac{\delta \omega_M}{\omega_M} = \frac{21\sqrt{2}}{320\sqrt{3}} \eta^3 \left( \frac{N_0}{N} \right)^{1/5} \left[ 1 + \frac{16}{9\sqrt{\pi}} \int_0^1 dx x^{1/2} \sqrt{1-x} (2 - 5x)^2 G_1[\tau(x)] \right. \]
\[
- \frac{160}{9\sqrt{\pi}} \int_0^1 dx x^{3/2} (1-x)^{3/2} G_2[\tau(x)] \right] ,
\]

where \( N_0/N \) is the equilibrium value of the condensate fraction, which is fixed by the parameter \( \eta \) and the reduced temperature \( t = T/T_c \). The argument \( \tau(x) \) of the \( G_1 \) and \( G_2 \) functions is given by the expression \( \tau(x) = [(N_0/N)^{-2/5} t/\eta]^{1/x} \). In Fig. 6 the monopole frequency shift (102) is shown as a function of the reduced temperature \( t \) for the value \( \eta = 0.4 \) of the interaction parameter. Already at relatively low temperatures, \( t \simeq 0.3 \), the monopole frequency is found to be about 1% smaller than \( \omega_M = \sqrt{5} \omega_{ho} \). In fact, even for such low temperatures, the local reduced temperature is large at the boundary of the condensate, as \( \tau(x) > 1 \) if \( x \to 0 \), and the contribution of this region dominates the shift (102). If \( \eta \ll t \), one can approximate the functions \( G_1 \) and \( G_2 \) in (102) with their asymptotic behavior for \( \tau \gg 1 \). One has

\[
G_1(\tau) \to G_1(\infty) \tau ,
\]

\[
G_2(\tau) \to G_2(\infty) \tau ,
\]

where \( G_1(\infty) = 5\sqrt{\pi} \) and

\[
G_2(\infty) = \frac{4\sqrt{2}}{3\sqrt{\pi}} \int_0^1 dx \left[ \frac{8 + 5x - x^2}{\sqrt{x} \sqrt{1-x(1+x)^3}} + \frac{4}{x^{3/2}(1-x)^{3/4}} \left( 1 - \frac{(1-x)^{1/4}}{(1+x)^{1/4}} \right) \right] \approx 22 .
\]
In this case the monopole shift is given by
\[
\frac{\delta \omega_M}{\omega_M} = \frac{7\sqrt{2} \pi}{960 \zeta(3)} \frac{n^2 t}{(1 - t^3)^{1/5}} \left[ 17G_1(\infty) - 10G_2(\infty) \right] \simeq -1.1 \frac{n^2 t}{(1 - t^3)^{1/5}},
\] (105)
where for the condensate fraction we have used the ideal gas law \( N_0/N = 1 - t^3 \). Result (103) gives a reasonably good approximation to the frequency shift \( \delta \omega_M \) also when \( n \sim t \), for example, for \( n = 0.4 \) and \( t = 0.8 \) Eq. (105) gives \( \delta \omega_M/\omega_M \simeq -0.16 \) and the calculation based on Eq. (102) gives \( -0.11 \).

In a surface mode the oscillation of the condensate has the form \( \chi_{lm} \propto r^l Y_{lm}(\theta, \phi) \), and the excitation frequency is given by \( \omega_l = \sqrt{\omega_{ha}} \). For these modes one finds the following fractional shift
\[
\frac{\delta \omega_l}{\omega_l} = \frac{\sqrt{2}(2l + 3)}{30 \sqrt{15} \zeta(3)} \frac{n^3 \left( \frac{N_0}{N} \right)^{1/5}}{(1 - t^3)^{1/5}} \left[ \int_0^1 dx x^{1/2} \left( 1 - x \right)^{l+1/2} G_1[\tau(x)] \\ - \left( l + 1/2 \right) \int_0^1 dx x^{3/2} \left( 1 - x \right)^{l-1/2} G_2[\tau(x)] \right].
\] (106)

In the limit \( n \ll t \) one gets, by using (103), the following result
\[
\frac{\delta \omega_{m=0}}{\omega_{m=0}} = -21^{1/3} 2 \frac{n^3 \left( \frac{N_0}{N} \right)^{1/5}}{320 \sqrt{15} \zeta(3)} \left[ f_\pm(\lambda) - \frac{160}{9 \sqrt{\pi}} \int_0^1 dx x^{1/2} (1 - x)^{3/2} G_2[\tau(x)] \right],
\] (107)
\[
\frac{\delta \omega_{m=0}}{\omega_{m=0}} = -21^{1/3} \frac{n^3 \left( \frac{N_0}{N} \right)^{1/5}}{320 \sqrt{15} \zeta(3)} \left[ f_\pm(\lambda) - \frac{160}{9 \sqrt{\pi}} \int_0^1 dx x^{1/2} (1 - x)^{3/2} G_2[\tau(x)] \right],
\] (108)
where \( f_\pm(\lambda) \) is defined in (101). In the limit \( n \ll t \) the above result reduces to
\[
\frac{\delta \omega_{m=0}}{\omega_{m=0}} = \frac{7\sqrt{2} \pi}{960 \zeta(3)} \frac{n^2 t}{(1 - t^3)^{1/5}} \left[ (20 - 3f_\pm(\lambda))G_1(\infty) - 10G_2(\infty) \right].
\] (109)

On the contrary, surface excitations of the form \( \chi_m \propto r^l Y_{lm}(\theta, \phi) \) and with excitation energy \( \omega_m = \sqrt{|m|} |\omega_\perp| \), exhibit the fractional shift (104) with \( t \) replaced by \( |m| \).

In Fig. 5 (Fig. 6) we show the fractional shift of the mode \( m = 0 \) low (high) as a function of the reduced temperature \( T = T/T_c \) and for the value \( n = 0.4 \) of the interaction parameter. We notice that in the case of the \( m = 0 \) high mode (Fig. 6), the size of the fractional shift is maximum for spherically symmetric traps (\( \lambda = 1 \)) and is minimum for disk-shaped traps (\( \lambda \gg 1 \)). On the contrary, for the \( m = 0 \) low mode (Fig. 5), \( |\delta \omega/\omega| \) is maximum for \( \lambda \gg 1 \), while it is minimum in the \( \lambda = 1 \) case. However, by changing the geometry of the trap, the curve of the fractional shift remains qualitatively the same, and at intermediate temperatures, \( T \sim 0.5T_c \), one finds downward shifts ranging from 1 to 4% for both the mode \( m = 0 \) low and high. In Fig. 7 we show the results for the surface modes \( m = 2 \) and \( m = 4 \) for the same value, \( n = 0.4 \), of the interaction parameter. In the case of surface modes the fractional shift is independent of the deformation parameter \( \lambda \) and we find that the size of the shift increases by increasing \( m \). An explanation of this behavior can be found in the fact that modes with higher \( m \) are more localized at the surface of the condensate where \( k_B T > g n_0(\mathbf{r}) \), being \( g n_0(\mathbf{r}) \) the local chemical potential. Thus, thermal effects are more pronounced for such modes.

Experiments on the temperature dependence of the collective modes have been carried out both at JILA [7] and MIT [8]. The JILA group has measured, as a function of temperature, the frequency of the \( m = 2 \) and \( m = 0 \) low modes [7]. However, in these experiments, the number of trapped particles is about \( 10^4 \) and beyond Thomas-Fermi effects are expected to play a significant role. Nevertheless, our results for the fractional shift of the \( m = 2 \) mode in disk-shaped geometries, shown in Fig. 7, both qualitatively and quantitatively agree with the observed behavior. In the case of the \( m = 0 \) low mode other effects, not included in the present analysis, might be responsible for the features observed in the experiment. The frequency of collective excitations in the Thomas-Fermi regime has been measured by the MIT group for the \( m = 0 \) low mode in a cigar-shaped trap [8]. In Fig. 8 we show the comparison
between the experimental results and our theoretical prediction. The calculation has been carried out with the value \( \eta = 0.4 \) of the interaction parameter, which is close to the experimental conditions of [8]. In Fig. 8, the experimental data have been plotted as a function of the reduced temperature \( T/T_c \) [62]. This is possible only for temperatures above 0.5 \( \mu \) K, as lower temperatures were not measurable in [63].

### C. Hydrodynamic equations at \( T = 0 \)

At zero temperature, superfluid systems are described by the equations of hydrodynamics (for a general discussion see the book [63]). These equations involve the total density \( n \) of the system and the superfluid velocity \( \mathbf{v}_s \), which is related to the gradient of the phase of the order parameter. The hydrodynamic picture has been successfully employed in [62] to obtain the frequencies of the collective modes in the Thomas-Fermi regime and later in [64] to calculate the corrections to these frequencies due to beyond mean-field effects. We have already verified [see Eq. (98)] that our perturbation scheme reproduces at \( T = 0 \) the results obtained from hydrodynamic theory. However, since we start from dynamic equations written in terms of the condensate wavefunction and the noncondensate density, it is important to understand whether these equations reduce at zero temperature to the hydrodynamics of superfluids.

At zero temperature, superfluid systems are described by the equations of hydrodynamics (for a general discussion see the book [63]). By writing the condensate wavefunction in terms of a modulus and a phase \( \Phi(r,t) = \sqrt{n_0(r,t)} e^{i\varphi(r,t)} \), one has the following identifications

\[
\begin{align*}
\delta n_0(r,t) &= \Phi_0(r) [\delta \Phi(r,t) + \delta \Phi^*(r,t)] , \\
i \delta \varphi(r,t) &= \frac{1}{2\Phi_0(r)} [\delta \Phi(r,t) - \delta \Phi^*(r,t)] ,
\end{align*}
\]

(110)

between the fluctuations of \( n_0 \) and \( \varphi \) and the fluctuations of the order parameter. The coupled equations (64), holding in the Thomas-Fermi regime, are then equivalent to

\[
\begin{align*}
\frac{\partial \delta n_0}{\partial t} + \nabla \cdot (n_{TF} \mathbf{v}_s) &= 0 , \\
m \frac{\partial \mathbf{v}_s}{\partial t} + g \nabla \delta n_0 &= 0 ,
\end{align*}
\]

(111)

where \( \mathbf{v}_s = \hbar \nabla \varphi/m \) is the superfluid velocity. At \( T = 0 \), if one neglects quantum depletion, the condensate density coincides with the total density and Eqs. (111) coincide with the linearized hydrodynamic equations. The former of Eqs. (111) corresponds to the equation of continuity and the latter to Euler equation with the pressure \( P \) fixed by \( \partial P/\partial n_0 = g n_{TF} \).

Beyond Gross-Pitaevskii theory one must replace Eqs. (64) by (74), which include the corrections to order \( g^2 \). At \( T = 0 \) these equations reduce to

\[
\begin{align*}
\hbar \omega (\delta \Phi_1 + \delta \Phi_2) &= -\frac{\hbar^2}{2m} (\delta \Phi_1 - \delta \Phi_2) - 2 \hbar \bar{m} (\delta \Phi_1 - \delta \Phi_2) \\
&\quad + g \sqrt{n_{TF}} \sum_{ij} \left[ \left( u_i u_j - v_i v_j \right) g_{ij}(\omega) - \left( u_i^* u_j^* - v_i^* v_j^* \right) g_{ij}^*(\omega) \right] , \\
\hbar \omega (\delta \Phi_1 - \delta \Phi_2) &= 2 (g n_{TF} + \delta \mu) (\delta \Phi_1 + \delta \Phi_2) \\
&\quad + 2 g n_{TF} \sum_{ij} \left[ \frac{1}{V} \sum_p \frac{m}{p^2} - \frac{40}{3 \sqrt{\pi}} (a^3 n_{TF})^{1/2} \right] (\delta \Phi_1 + \delta \Phi_2) \\
&\quad + g \sqrt{n_{TF}} \sum_{ij} \left( 2 u_i u_j + v_i v_j + u_i^* u_j^* + v_i^* v_j^* \right) g_{ij}(\omega) \\
&\quad + \left( 2 u_i^* v_j^* + 2 v_i^* u_j^* + u_i^* u_j^* + v_i^* v_j^* \right) g_{ij}^*(\omega) ,
\end{align*}
\]

(112)

where \( \delta \mu = \mu - \mu_{TF}(N_0) \) is the change in the chemical potential [see Eq. (74)] and the matrices \( g_{ij}(\omega) \) and \( g_{ij}^*(\omega) \) are given in (74) with \( f_0^0 = f_{00}^0 = 0 \). By using the semiclassical approximation (70) for the quasiparticle states, the above equations are written in terms of the variables \( \varphi \) and \( n_0 \) as

\[
- i \hbar \omega \delta n_0(\mathbf{r}) = -\hbar \nabla \cdot [n_0(\mathbf{r}) \mathbf{v}_s(\mathbf{r})] - 4 g n_{TF}(\mathbf{r}) \bar{m}(\mathbf{r}) \delta \varphi(\mathbf{r})
\]

23
\[ + \frac{2g^2}{\sqrt{2}} \sum_{ij} \frac{n_{TF}(r)\delta n_0(r)}{\epsilon_i + \epsilon_j} \int ds \, e^{is \cdot \nabla [\varphi_i(r) + \varphi_j(r) - \frac{i\hbar}{\epsilon_i + \epsilon_j} b_{ij}(r + s)]} \]

\[ \times \left[ \frac{i\hbar \omega \delta n_0(r + s)}{\epsilon_i + \epsilon_j} \left( a_{ij}(r + s) + \frac{2gn_{TF}(r + s)}{\epsilon_i + \epsilon_j} b_{ij}(r + s) \right) - 2n_{TF}(r + s) b_{ij}(r + s) \delta \varphi(r + s) \right], \]

and

\[ i\hbar \omega \delta \varphi(r) = g \left( 1 + \frac{1}{2} \sum_P \frac{m}{p^2} \right) \delta n_0(r) \]  \hspace{1cm} (114)

\[ - \frac{g^2}{\sqrt{2}} \sum_{ij} \frac{a_{ij}(r)}{\epsilon_i + \epsilon_j} \int ds \, e^{is \cdot \nabla [\varphi_i(r) + \varphi_j(r)]} \delta n_0(r + s) \left( a_{ij}(r + s) + \frac{2gn_{TF}(r + s)}{\epsilon_i + \epsilon_j} b_{ij}(r + s) \right), \]

where the matrices \( a_{ij} \) and \( b_{ij} \) have been defined in [24]. By using the gradient expansion employed in the previous section for the calculation of the non-resonant contributions to \( \delta E \), we get the result

\[ i\omega \left( 1 + \frac{4}{\sqrt{\pi}} (a^3 n_{TF}(r))^{1/2} \right) \delta n_0(r) = \nabla \cdot \left[ n_0(r) \left( 1 + \frac{8}{3\sqrt{\pi}} (a^3 n_{TF}(r))^{1/2} \right) \mathbf{v}_s(r) \right] \]  \hspace{1cm} (115)

and

\[ i\hbar \omega \delta \varphi = g \left( 1 + \frac{20}{\sqrt{\pi}} (a^2 n_{TF})^{1/2} \right) \delta n_0(r) \]  \hspace{1cm} (116)

If one takes into account the effect of quantum depletion, the local relation between condensate density and total density is given by \( n = n_0[1 + 8(a^3 n_{TF})^{1/2}/(3\sqrt{\pi})] \), and for the fluctuations of the two densities \( \delta n = \delta n_0[1 + 4(a^3 n_{TF})^{1/2}/\sqrt{\pi}] \). Finally, the change in the local chemical potential \( \mu_l = gn[1 + 32(a^3 n_{TF})^{1/2}/(3\sqrt{\pi})] \) induced by a density fluctuation is given by the following expression \( \delta n \partial \mu_l/\partial n = \delta n_0[1 + 20(a^3 n_{TF})^{1/2}/\sqrt{\pi}] \). It is now straightforward to recognize Eqs. (115, 116) as the linearized hydrodynamic equations

\[ \frac{\partial \delta n}{\partial t} + \nabla \cdot (n \mathbf{v}_s) = 0, \]

\[ m \frac{\partial \mathbf{v}_s}{\partial t} + \nabla \cdot \left( \frac{\partial \mu_l}{\partial n} \delta n \right) = 0, \]  \hspace{1cm} (117)

which involve the total density \( n \) and the superfluid velocity \( \mathbf{v}_s \).

\section*{V. CONCLUDING REMARKS}

In this paper we have studied the collisionless collective modes of a dilute Bose gas beyond the Gross-Pitaevskii theory. In particular, for harmonically trapped systems in the thermodynamic limit, we have calculated the corrections to the excitation frequencies of the low-lying collective modes. We find that, not far below the Bose-Einstein transition temperature, the fractional frequency shift is of the order of few percent for typical experimental conditions and can be measured. A direct comparison with experimental data obtained by the MIT group with large condensates looks very good. Similarly to what happened to Gross-Pitaevskii theory, the study of collective excitations can become a useful bench-mark also for theories beyond mean-field approximation.

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**FIG. 1.** Dimensionless function $H$ as a function of the reduced temperature $\tau = k_B T/g n_0$.

**FIG. 2.** Dimensionless function $F$ as a function of the reduced temperature $\tau = k_B T/g n_0$ (solid line). The asymptotic behaviors for $\tau \ll 1$ (dashed line) and for $\tau \gg 1$ (long-dashed line) are also reported.

**FIG. 3.** Dimensionless function $G$ as a function of the reduced temperature $\tau = k_B T/g n_0$.

**FIG. 4.** Dimensionless functions $G_1$ and $G_2$ as a function of the reduced temperature $\tau = k_B T/g n_0$.

**FIG. 5.** Fractional shift of the $m = 0$ low mode as a function of the reduced temperature $T/T_0^c$. The value of the interaction parameter is $\eta = 0.4$. For a spherical trap ($\lambda = 1$) this mode corresponds to the $l = 2$ quadrupole mode.

**FIG. 6.** Fractional shift of the $m = 0$ high mode as a function of the reduced temperature $T/T_0^c$. The value of the interaction parameter is $\eta = 0.4$. For a spherical trap ($\lambda = 1$) this mode corresponds to the monopole (breathing) mode.

**FIG. 7.** Fractional shift of the $m = 2$ and $m = 4$ surface modes as a function of the reduced temperature $T/T_0^c$. The value of the interaction parameter is $\eta = 0.4$.

**FIG. 8.** Temperature dependence of the frequency of the $m = 0$ low mode. The experimental points are taken from [8]. The theoretical calculation (solid line) corresponds to cigar-shaped traps and to the value $\eta = 0.4$ of the interaction parameter. The dashed line corresponds to the hydrodynamic prediction $\nu = \sqrt{5/2} \nu_s$ from Eq. (58).
Fig. 1

$H(\tau)$

$\tau$
Fig. 3

\[ G(\tau) \]
Fig. 4

The graph shows two linear functions:

- $G_2(\tau)$
- $G_1(\tau)$

The graph is plotted against the variable $\tau$. The y-axis represents the value of the functions, and the x-axis represents the variable $\tau$. The functions are linear and increase as $\tau$ increases.
Fig. 5

\[ \frac{\delta \omega}{\omega} \]

\[ \frac{T}{T_C^0} \]

- \( \lambda = 1 \) (quadrupole)
- \( \lambda \gg 1 \)
- \( \lambda \ll 1 \)
Fig. 6

\[ \frac{\delta \omega}{\omega} = \begin{cases} 0 & \text{for } \lambda >> 1 \\ \lambda = 1 \text{ (monopole)} & \text{for } \lambda << 1 \end{cases} \]

\[ T/T_c^0 \]
Fig. 7

\[ \delta \omega / \omega \]

\[ T / T_c^0 \]
Fig. 8