BIHARMONIC SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN $S^n \times \mathbb{R}$

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Abstract. We find a Simons type formula for submanifolds with parallel mean curvature vector (pmc submanifolds) in product spaces $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a space form with constant sectional curvature $c$, and then we use it to prove a gap theorem for the mean curvature of certain complete proper-biharmonic pmc submanifolds, and classify proper-biharmonic pmc surfaces in $S^n(c) \times \mathbb{R}$.

1. Introduction

The notion of biharmonic maps was suggested in 1964 by Eells and Sampson in [14], as a natural generalization of harmonic maps. Thus, whilst a harmonic map $\psi : (M, g) \to (\bar{M}, h)$ between two Riemannian manifolds is defined as a critical point of the energy functional

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 \, v_g,$$

a biharmonic map is a critical point of the bienergy functional

$$E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 \, v_g,$$

where $\tau(\psi) = \text{trace} \nabla d\psi$ is the tension field that vanishes for harmonic maps. The Euler-Lagrange equation for the bienergy functional was derived by Jiang in 1986 (see [18]):

$$\tau_2(\psi) = \Delta \tau(\psi) - \text{trace} \bar{R}(d\psi, \tau(\psi))d\psi = 0$$

where $\tau_2(\psi)$ is the bitension field of $\psi$, $\Delta = \text{trace}(\nabla^2)^2 = \text{trace}(\nabla^2 \nabla^2 - \nabla^2_{\nabla^2})$ is the rough Laplacian defined on sections of $\psi^{-1}(T\bar{M})$ and $\bar{R}$ is the curvature tensor of $\bar{M}$, given by $\bar{R}(X,Y)Z = [\bar{\nabla}_X, \bar{\nabla}_Y]Z - \bar{\nabla}_{[X,Y]}Z$. Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called proper-biharmonic.

A biharmonic submanifold in a Riemannian manifold is a submanifold for which the inclusion map is biharmonic. In Euclidean space the biharmonic submanifolds are the same as those defined by Chen in [11], as they are characterized by the equation $\Delta H = 0$, where $H$ is the mean curvature vector field and $\Delta$ is the rough Laplacian.

Some very fertile environments for finding examples of proper-biharmonic submanifolds proved to be the unit Euclidian sphere $S^n$, and, in general, space forms.

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with positive sectional curvature. For example, whilst there are no proper-biharmonic curves and surfaces in 3-dimensional spaces with non-positive constant sectional curvature (see Chen and Ishikawa’s paper [12] and Dimitric’s paper [13] in the case of Euclidian space, and Caddeo, Montaldo and Oniciuc’s article [9] when the sectional curvature is negative) we do have examples of such submanifolds in $S^3$ in [8], where they are explicitly classified.

In the very recent paper [22], Ou and Wang studied the biharmonicity of constant mean curvature surfaces (cmc surfaces) in Thurston’s 3-dimensional geometries, amongst them being the product space $S^2 \times \mathbb{R}$.

The case of cmc surfaces in product spaces of type $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a simply connected surface with constant sectional curvature $c$, and then that of surfaces with parallel mean curvature vector field (pmc surfaces) in product spaces of type $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a space form, received a special attention (see, for example, Abresch and Rosenberg’s papers [1, 2] on cmc surfaces, and Alencar, do Carmo and Tribuzy’s article [4] on pmc surfaces). From the point of view of biharmonicity, pmc surfaces and, in general, pmc submanifolds in spheres, were studied in [9] in [7], respectively.

In his paper [23] from 1968, Simons proved a very important formula for the Laplacian of the second fundamental form of a minimal submanifold in a Riemannian manifold and then used it to characterize certain minimal submanifolds of a sphere and Euclidean space. Over the years, such formulas, called Simons type equations, also proved to be a powerful tool for studying cmc and pmc submanifolds.

In our paper, we first obtain a Simons type equation for pmc submanifolds in product spaces $M^n(c) \times \mathbb{R}$ and then we use it to prove a gap phenomenon for the mean curvature of a proper-biharmonic pmc submanifold. We also investigate the biharmonicity of pmc surfaces in product spaces and, using a reduction of codimension result of Eschenburg and Tribuzy in [16] and the above mentioned Simons type formula, we get a classification theorem. Our main results are the following two theorems.

**Theorem 4.9.** Let $\Sigma^m$ be a complete proper-biharmonic pmc submanifold in $S^n \times \mathbb{R}$, with $m \geq 2$, such that its mean curvature satisfies

$$|H|^2 > \frac{(m-1)(m^2+4) + (m-2)\sqrt{(m-1)(m-2)(m^2+m+2)}}{2m^3},$$

and the norm of its second fundamental form $\sigma$ is bounded. Then $m < n$, $|H| = 1$ and $\Sigma^m$ is a minimal submanifold of a small hypersphere $S^{n-1}(2) \subset S^n$.

**Theorem 5.6.** Let $\Sigma^2$ be a proper-biharmonic pmc surface in $S^n(c) \times \mathbb{R}$. Then either

1. $\Sigma^2$ is a minimal surface of a small hypersphere $S^{n-1}(2c) \subset S^n(c)$; or
2. $\Sigma^2$ is an (an open part of) a vertical cylinder $\pi^{-1}(\gamma)$, where $\gamma$ is a circle in $S^2(c)$ with curvature equal to $\sqrt{c}$, i.e. $\gamma$ is a biharmonic circle in $S^2(c)$.

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2. Preliminaries

Let $M^n(c)$ be a space form, i.e. a simply-connected $n$-dimensional manifold with constant sectional curvature $c$, and consider the product manifold $M = M^n(c) \times \mathbb{R}$.
The expression of the curvature tensor $\tilde{R}$ of such a manifold can be obtained from
\[
\langle \tilde{R}(X,Y)Z,W \rangle = c\{ \langle d\pi Y,d\pi Z \rangle \langle d\pi X,d\pi W \rangle - \langle d\pi X,d\pi Z \rangle \langle d\pi Y,d\pi W \rangle \},
\]
where $\pi : \tilde{M} = M^n(c) \times \mathbb{R} \to M^n(c)$ is the projection map. After a straightforward computation we get
\[
\tilde{R}(X,Y)Z = c\{ \langle Y,Z \rangle X - \langle X,Z \rangle Y - \langle Y,\xi \rangle \langle Z,\xi \rangle X + \langle X,\xi \rangle \langle Z,\xi \rangle Y + \langle X,Z \rangle \langle Y,\xi \rangle \xi - \langle Y,Z \rangle \langle X,\xi \rangle \xi \},
\]
where $\xi$ is the unit vector tangent to $\mathbb{R}$.

Let $\Sigma^m$, $m \leq n$, be an $m$-dimensional submanifold of $\tilde{M}$. From the equation of Gauss
\[
\langle R(X,Y)Z,W \rangle = \langle \tilde{R}(X,Y)Z,W \rangle + \sum_{\alpha=m+1}^{n+1} \{ \langle A_\alpha Y,Z \rangle \langle A_\alpha X,W \rangle - \langle A_\alpha X,Z \rangle \langle A_\alpha Y,W \rangle \},
\]
we obtain the expression of its curvature tensor
\[
R(X,Y)Z = c\{ \langle Y,Z \rangle X - \langle X,Z \rangle Y - \langle Y,T \rangle \langle Z,T \rangle X + \langle X,T \rangle \langle Z,T \rangle Y + \langle X,Z \rangle \langle Y,\xi \rangle \xi - \langle Y,Z \rangle \langle X,\xi \rangle \xi \},
\]
where $T$ is the component of $\xi$ tangent to $\Sigma^m$, $A$ is the shape operator defined by the equation of Weingarten
\[
\tilde{\nabla}_X V = -AVX + \nabla^\perp_X V,
\]
for any vector field $X$ tangent to $\Sigma^m$ and any normal vector field $V$. Here $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, $\nabla^\perp$ is the connection in the normal bundle, and $A_\alpha = A_{E_\alpha}$, $\{E_\alpha\}_{n+1}^{m+1}$ being a local orthonormal frame field in the normal bundle.

**Definition 2.1.** A submanifold $\Sigma^m$ of $M^n(c) \times \mathbb{R}$ is called a **vertical cylinder** over $\Sigma^{m-1}$ if $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$, where $\pi : M^n(c) \times \mathbb{R} \to M^n(c)$ is the projection map and $\Sigma^{m-1}$ is a submanifold of $M^n(c)$.

It is easy to see that vertical cylinders $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$ are characterized by the fact that $\xi$ is tangent to $\Sigma^m$.

**Definition 2.2.** If the mean curvature vector field $H$ of a submanifold $\Sigma^m$ is parallel in the normal bundle, i.e. $\nabla^\perp H = 0$, then $\Sigma^m$ is called a **pmc submanifold**.

**Remark 2.1.** It is straightforward to verify that $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$ is a pmc vertical cylinder in $M^n(c) \times \mathbb{R}$ if and only if $\Sigma^{m-1}$ is a pmc submanifold in $M^n(c)$. Moreover, the mean curvature vector field of $\Sigma^m$ is $H = \frac{n-1}{n} H_0$, where $H_0$ is the mean curvature vector field of $\Sigma^{m-1}$. It also easy to prove that the vertical cylinder $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$ is proper-biharmonic in $M^n(c) \times \mathbb{R}$ if and only if $\Sigma^{m-1}$ is proper-biharmonic in $M^n(c)$.

We end this section by recalling the following two results, which we shall use later.

**Lemma 2.3** ([3, 20]). Let $a_i$, $i = 1, \ldots, m$, be real numbers such that $\sum_{i=1}^{m} a_i = 0$ and $\sum_{i=1}^{m} a_i^2 \geq b^2$, where $b = \text{constant} \geq 0$. Then
\[
-\frac{m-2}{\sqrt{m(m-1)}} b^3 \leq \sum_{i=1}^{m} a_i^3 \leq \frac{m-2}{\sqrt{m(m-1)}} b^3.
\]
and equality holds in the right-hand (left-hand) side if and only if \((n-1)\) of the \(a_i\)'s are non-positive and equal \(((n-1)\) of the \(a_i\)'s are non-negative and equal).  

**Theorem 2.4** (Omori-Yau Maximum Principle, [24]). If \(\Sigma^m\) is a complete Riemannian manifold with Ricci curvature bounded from below, then for any smooth function \(u \in C^2(\Sigma^m)\) with \(\sup_{\Sigma^m} u < +\infty\) there exists a sequence of points \(\{p_k\}_{k \in \mathbb{N}} \subset \Sigma^m\) satisfying

\[
\lim_{k \to \infty} u(p_k) = \sup_{\Sigma^m} u, \quad |\nabla u|(p_k) < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.
\]

3. A Simons type formula for submanifolds in \(M^n(c) \times \mathbb{R}\)

Let \(\Sigma^m, m \leq n\), be an \(m\)-dimensional submanifold of \(M^n(c) \times \mathbb{R}\), with mean curvature vector field \(H\). In this section we shall compute the Laplacian of the squared norm of \(A_V\), where \(V\) is a normal vector field to the submanifold, such that \(V\) is parallel in the normal bundle, i.e. \(\nabla^\bot V = 0\), and trace \(A_V = \text{constant}\).

**Lemma 3.1.** If \(U\) and \(V\) are normal vector fields to \(\Sigma^m\) and \(V\) is parallel in the normal bundle, then \([A_V, A_U] = 0\), i.e. \(A_V\) commutes with \(A_U\).

**Proof.** The conclusion follows easily, from the Ricci equation,

\[
\langle R^\bot(X, Y)V, U \rangle = \langle [A_V, A_U]X, Y \rangle + \langle \bar{R}(X, Y)V, U \rangle,
\]

since \(R^\bot(X, Y)V = 0\) and (2.1) implies that \(\langle \bar{R}(X, Y)V, U \rangle = 0\). \(\square\)

Now, from the Codazzi equation,

\[
\langle \bar{R}(X, Y)Z, V \rangle = \langle \nabla^\bot_X \sigma(Y, Z), V \rangle - \langle \sigma(\nabla_X Y, Z), V \rangle - \langle \sigma(Y, \nabla_X Z), V \rangle - \langle \nabla^\bot_X \sigma(Z, V), \nabla_Y X \rangle + \langle \sigma(Y, \nabla_Y X), V \rangle + \langle \sigma(X, \nabla_Y Z), V \rangle,
\]

where \(\sigma\) is the second fundamental form of \(\Sigma^m\), we get

\[
\langle \bar{R}(X, Y)Z, V \rangle = X(\langle A_V Y, Z \rangle) - \langle \sigma(Y, Z), \nabla^\bot_X V \rangle - \langle A_V(\nabla_X Y), Z \rangle - \langle A_V Y, \nabla_X Z \rangle - Y(\langle A_V X, Z \rangle) + \langle \sigma(X, Z), \nabla^\bot_X V \rangle + \langle A_V(\nabla_Y X), Z \rangle + \langle A_V X, \nabla_Y Z \rangle.
\]

since \(\nabla^\bot V = 0\). Therefore, using (2.1), we obtain

\[
\langle \nabla_X A_V \rangle Y = \langle \nabla_Y A_V \rangle X + c(V, N)\langle (Y, T), X - (X, T) \rangle Y,
\]

where \(N\) is the normal part of \(\xi\).

Next, we have the following Weitzenböck formula

\[
\frac{1}{2} \Delta |A_V|^2 = |\nabla A_V|^2 + \langle \text{tr} \nabla^2 A_V, A_V \rangle,
\]

where we extended the metric \(\langle , \rangle\) to the tensor space in the standard way.

The second term in the right-hand side of (3.2) can be calculated by using a method introduced in [19], and, in the following, for the sake of completeness, we shall sketch this computation.

Let us consider

\[
C(X, Y) = \langle \nabla^2 A_V \rangle(X, Y) = \nabla_X (\nabla_Y A_V) - \nabla_{\nabla_X Y} A_V,
\]
and note that we have the following Ricci commutation formula

\begin{equation}
C(X, Y) = C(Y, X) + [R(X, Y), A_V].
\end{equation}

Next, consider an orthonormal basis \( \{e_i\}_{i=1}^m \) in \( T_p \Sigma^m \), extend \( e_i \) to vector fields \( E_i \) in a neighborhood of \( p \) such that \( \{E_i\} \) is a geodesic frame field around \( p \), and let us denote \( X = E_k \). We have

\[
(\text{trace } \nabla^2 A_V)X = \sum_{i=1}^{m} C(E_i, E_i)X.
\]

Using equation (3.1), we get, at \( p \),

\[
C(E_i, X)E_i = \nabla_{E_i}((\nabla_X A_V)E_i) = \nabla_{E_i}(\nabla_E(A_V)X) = c\nabla_{E_i}(\langle V, N \rangle(\langle E_i, T \rangle X - \langle X, T \rangle E_i))
\]

and then

\[
C(E_i, X)E_i = C(E_i, E_i)X - c\langle A_V E_i, T \rangle(\langle E_i, T \rangle X - \langle X, T \rangle E_i) + c\langle V, N \rangle(\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i),
\]

where we used \( \sigma(E_i, T) = -\nabla^i E_i N \) and \( \nabla E_i T = A_N E_i \), which follow from the fact that \( \xi \) is parallel, i.e. \( \nabla \xi = 0 \).

We also have, at \( p \),

\begin{equation}
C(X, E_i)E_i = \nabla_X(\langle \nabla_E A_V \rangle E_i),
\end{equation}

and, from (3.1), (3.4) and (3.5), we get, also at \( p \),

\[
C(E_i, E_i)X = \nabla_X(\langle \nabla_E A_V \rangle E_i) + [R(E_i, X), A_V]E_i + c\langle A_V E_i, T \rangle(\langle E_i, T \rangle X - \langle X, T \rangle E_i) - c\langle V, N \rangle(\langle A_N E_i, E_i \rangle X - \langle A_N X, E_i \rangle E_i).
\]

Since \( \nabla E_i A_V \) is symmetric, from (3.1), one obtains

\[
\langle \sum_{i=1}^{m}(\nabla_E A_V)E_i, Z \rangle = \sum_{i=1}^{m}\langle E_i, (\nabla_E A_V)Z \rangle = \sum_{i=1}^{m}\langle E_i, (\nabla Z A_V)E_i \rangle + c\langle V, N \rangle \sum_{i=1}^{m}\langle E_i, (Z, T)E_i - \langle E_i, T \rangle Z \rangle
\]

\[
= \text{trace}(\nabla Z A_V) + c(m - 1)\langle V, N \rangle \langle T, Z \rangle
\]

\[
= Z(\text{trace } A_V) + c(m - 1)\langle V, N \rangle \langle T, Z \rangle
\]

\[
= c(m - 1)\langle V, N \rangle \langle T, Z \rangle,
\]

for any vector \( Z \) tangent to \( \Sigma^m \), since trace \( A_V = \text{constant} \).

From the Gauss equation (2.2) of the surface \( \Sigma^2 \), and Lemma 3.1 we get, after a straightforward computation,

\[
\sum_{i=1}^{m} R(E_i, X)A_V E_i = c\{A_V X - (\text{trace } A_V)X + (\text{trace } A_V)(X, T)T - \langle A_V X, T \rangle T - \langle X, T \rangle A_V T + \langle A_V T, T \rangle X\} + \sum_{\alpha=m+1}^{n+1}\{A_V A^2_\alpha X - (\text{trace}(A_V A_\alpha))A_\alpha X\},
\]
and
\[
\sum_{i=1}^{m} A_i R(E_i, X)E_i = -c\{(m - 1 - |T|^2)A_V X - (m - 2)\langle X, T \rangle A_V T \}
+ \sum_{\alpha=m+1}^{n+1} \{A_V A_\alpha^2 X - (\text{trace } A_\alpha)A_V A_\alpha X \}.
\]

Therefore, we have
\[
(\text{trace } \nabla^2 A_V)X = \sum_{i=1}^{m} C(E_i, E_i)X
= \sum_{i=1}^{m} [R(E_i, X), A_v]E_i
+ c\{m(V, N)A_N X - (m - 1)\langle A_V X, T \rangle T + \langle A_V T, T \rangle X
- \langle X, T \rangle A_V T - m\langle H, N \rangle \langle V, N \rangle X \}
= c\{(m - |T|^2)A_V X + 2\langle A_V T, T \rangle X - m(A_V X, T)T
- m\langle X, T \rangle A_V T + m\langle V, N \rangle A_N X - m\langle H, N \rangle \langle V, N \rangle X
- (\text{trace } A_V)X + (\text{trace } A_V)\langle X, T \rangle T \}
+ \sum_{\alpha=m+1}^{n+1} \{(\text{trace } A_\alpha)A_V A_\alpha X - (\text{trace } (A_V A_\alpha))A_\alpha X \},
\]
and then
\[
\langle \text{trace } \nabla^2 A_V, A_V \rangle = \sum_{i=1}^{m} \langle \langle \text{trace } \nabla^2 A_V \rangle E_i, A_V E_i \rangle
= c\{(m - |T|^2)|A_V|^2 - 2m|A_V T|^2 + 3\langle \text{trace } A_V \rangle \langle A_V T, T \rangle
+ m\langle \text{trace } (A_N A_V) \rangle \langle V, N \rangle - (\text{trace } A_V)^2
- m\langle \text{trace } A_V \rangle \langle H, N \rangle \langle V, N \rangle \}
+ \sum_{\alpha=m+1}^{n+1} \{(\text{trace } A_\alpha)(\text{trace } A_\alpha^2 A_\alpha) - (\text{trace } (A_V A_\alpha))^2 \}.
\]

Thus, from (3.2), we obtain the following proposition.

**Proposition 3.2.** Let \( \Sigma^n \) be a submanifold of \( M^n(c) \times \mathbb{R} \), with mean curvature vector field \( H \) and shape operator \( A \). If \( V \) is a normal vector field, parallel in the normal bundle, with \( \text{trace } A_V = \text{constant} \), then
\[
\frac{1}{2} \Delta |A_V|^2 = |\nabla A_V|^2 + c\{(m - 2)|A_V|^2 - 2m|A_V T|^2
+ 3\langle \text{trace } A_V \rangle \langle A_V T, T \rangle
+ m\langle \text{trace } (A_N A_V) \rangle \langle V, N \rangle - (\text{trace } A_V)^2
- m\langle \text{trace } A_V \rangle \langle H, N \rangle \langle V, N \rangle \}
+ \sum_{\alpha=m+1}^{n+1} \{(\text{trace } A_\alpha)(\text{trace } A_\alpha^2 A_\alpha) - (\text{trace } (A_V A_\alpha))^2 \},
\]
where \( \{E_\alpha\}_{\alpha=m+1}^{n+1} \) is a local orthonormal frame field in the normal bundle.
4. A GAP THEOREM FOR BIHARMONIC PMC SUBMANIFOLDS IN $S^n \times \mathbb{R}$

Whereas complete biharmonic pmc submanifolds of $S^n \times \mathbb{R}$ are the subject of our first main theorem, we have the following result for compact submanifolds.

**Proposition 4.1.** If $\Sigma^m$ is a compact biharmonic submanifold in $S^n(c) \times \mathbb{R}$, then $\Sigma^m$ lies in $S^n(c)$.

*Proof.* The height function of a submanifold $\Sigma^m$ in $S^n(c) \times \mathbb{R}$ is defined by

$$ h = t \circ i : \Sigma^m \to \mathbb{R}, $$

where $t : S^n(c) \times \mathbb{R} \to \mathbb{R}$ is the projection map and $i : \Sigma^m \to S^n(c) \times \mathbb{R}$ is the inclusion map. It is easy to verify that

$$ \tau(h) = dt(\tau(i)) \quad \text{and} \quad \tau_2(h) = dt(\tau_2(i)), $$

and we see that, if $\Sigma^m$ is biharmonic, then $h$ is also a biharmonic function.

Since $\Sigma^m$ is a compact biharmonic submanifold, it follows that $h$ is a real valued biharmonic function defined on a compact manifold, which, according to a result in [18], leads to the fact that $h$ is actually a harmonic function, but then, using the maximum principle, we get that $h$ is constant, i.e. $\Sigma^m$ lies in $S^n(c)$.

We recall now the following three results which we shall use later in this paper.

**Theorem 4.2** ([21]). A proper-biharmonic cmc submanifold $\Sigma^m$ in $S^n(c)$, with mean curvature equal to $\sqrt{c}$, is minimal in a small hypersphere $S^{n-1}(2c) \subset S^n(c)$.

**Theorem 4.3** ([7]). If $\Sigma^m$ is a proper-biharmonic pmc submanifold in $S^n(c)$, with mean curvature vector field $H$ and $m > 2$, then $|H| \in (0, \frac{m-2}{m}\sqrt{c}] \cup \{\sqrt{c}\}$. Moreover, $|H| = \frac{m-2}{m}\sqrt{c}$ if and only if $\Sigma^m$ is (an open part of) a standard product

$$ \Sigma^{m-1}_1 \times S^1(2c) \subset S^n(c), $$

where $\Sigma^{m-1}_1$ is a minimal submanifold in $S^{n-2}(2c)$.

**Theorem 4.4** ([6]). A submanifold $\Sigma^m$ in a Riemannian manifold $\tilde{M}$, with second fundamental form $\sigma$, mean curvature vector field $H$, and shape operator $A$, is biharmonic if and only if

$$ \begin{cases} -\Delta H + \text{trace} \sigma(\cdot, A_H \cdot) + \text{trace}(\tilde{R}(\cdot, H) \cdot) = 0 \\ \frac{m}{2} \text{grad} |H|^2 + 2 \text{trace} A_{\nabla H} + 2 \text{trace}(\tilde{R}(\cdot, H) \cdot) = 0, \end{cases} \quad (4.1) $$

where $\Delta$ is the Laplacian in the normal bundle and $\tilde{R}$ is the curvature tensor of $\tilde{M}$.

Now, we have the following two corollaries.

**Corollary 4.5.** A pmc submanifold $\Sigma^m$ in $M^n(c) \times \mathbb{R}$, with $m \geq 2$, is biharmonic if and only if

$$ \begin{cases} H \perp \xi \\ |A_H|^2 = c(m - |T|^2)|H|^2 \\ \text{trace}(A_H A_U) = 0 \quad \text{for any normal vector} \quad U \perp H. \end{cases} \quad (4.2) $$

*Proof.* Since $\Sigma^m$ is a pmc submanifold, equations (4.1) become

$$ \begin{cases} \text{trace} \sigma(\cdot, A_H \cdot) + \text{trace}(\tilde{R}(\cdot, H) \cdot) = 0 \\ \text{trace}(\tilde{R}(\cdot, H) \cdot) = 0 \end{cases} \quad (4.1) $$
and, as from equation (2.1) we have
\[
\text{trace } \tilde{R}(\cdot, H) = c\{(m - 1)\langle H, \xi \rangle T - (m - |T|^2)H + m\langle H, \xi \rangle N\},
\]
we see that $\Sigma^m$ is biharmonic if and only if
\[
\text{trace } \sigma(\cdot, A_H \cdot) = c\{(m - |T|^2)H - m\langle H, \xi \rangle N\} \quad \text{and} \quad \langle H, \xi \rangle T = 0.
\]

Now, assume that there exists a point $p \in \Sigma^m$ such that $\langle H, \xi \rangle \neq 0$ at $p$, and then $\langle H, \xi \rangle \neq 0$ on a neighborhood of $p$. It follows that $T = 0$ on this neighborhood, i.e. $\langle X, \xi \rangle = 0$ for any tangent vector field $X$. Since $\nabla \xi = 0$, we have
\[
0 = \langle \nabla Y X, \xi \rangle = \langle \sigma(X, Y), \xi \rangle
\]
for any tangent vector fields $X$ and $Y$. Thus $\langle H, \xi \rangle = 0$ on a neighborhood of $p$, and, therefore, at $p$, which is a contradiction. Consequently, we have that $H \perp \xi$ everywhere on $\Sigma^m$. Then, one obtains
\[
\text{trace } \sigma(\cdot, A_H \cdot) = c(m - |T|^2)H,
\]
from where we get (4.2).

\textbf{Remark 4.1.} A direct consequence of Corollary 4.5 is that there are no proper-biharmonic pmc submanifolds in a product space $M^n(c) \times \mathbb{R}$ with $c \leq 0$.

\textbf{Corollary 4.6.} If $\Sigma^n$ is a proper-biharmonic cmc hypersurface in $S^n(c) \times \mathbb{R}$, then it is (an open part of) a vertical cylinder $\pi^{-1}(\Sigma^{n-1})$, where $\Sigma^{n-1}$ is a proper-biharmonic cmc hypersurface in $S^n(c)$. Moreover, if

- (1) $n = 2$, then $\Sigma^1$ is a circle in $S^2(c)$ with curvature equal to $\sqrt{c}$, and $|H| = \frac{1}{2}\sqrt{c}$;
- (2) $n = 3$, then $\Sigma^2$ is an open part of a small hypersphere $S^2(2c) \subset S^3(c)$, and $|H| = \frac{3}{4}\sqrt{c}$;
- (3) $n > 3$, then $|H| \in \{0, \frac{n-3}{n}\sqrt{c}\} \cup \{\frac{n-1}{n}\sqrt{c}\}$. Furthermore,
  - (a) $|H| = \frac{n-3}{n}\sqrt{c}$ if and only if $\Sigma^{n-1}$ is an open part of the standard product $S^{n-2}(2c) \times S^1(2c) \subset S^n(c)$;
  - (b) $|H| = \frac{n-1}{n}\sqrt{c}$ if and only if $\Sigma^{n-1}$ is an open part of a small hypersphere $S^{n-1}(2c) \subset S^n(c)$.

\textbf{Proof.} From Corollary 4.5, we get that the mean curvature vector field $H$ of our submanifold is orthogonal to $\xi$, which means that $\xi$ is tangent to $\Sigma^n$. Therefore, $\Sigma^n$ is a vertical cylinder $\Sigma^{n-1} \times \mathbb{R}$, where $\Sigma^{n-1}$ is a proper-biharmonic cmc hypersurface in $S^n(c)$, with mean curvature vector field $H_0$ satisfying $H = \frac{n-1}{n}H_0$, as we know from Remark 2.1.

Now, when $n \in \{2, 3\}$, the main result in [10] and [8, Theorem 4.8] lead to (1) and (2), respectively, and when $n > 3$, we use Theorem 4.3 to prove (3).

\textbf{Proposition 4.7.} Let $\Sigma^m$ be a proper-biharmonic pmc submanifold in $S^n(c) \times \mathbb{R}$, with $m \geq 2$. Then its second fundamental form $\sigma$ satisfies $|\sigma|^2 \geq c(m - 1)$, and the equality holds if and only if $\Sigma^m$ is a vertical cylinder $\pi^{-1}(\Sigma^{m-1})$ in $S^n(c) \times \mathbb{R}$, where $\Sigma^{m-1}$ is a proper biharmonic cmc hypersurface in $S^m(c)$.

\textbf{Proof.} From the first equation of (4.2), we have
\[
|\sigma|^2 \geq |\frac{A_H}{n}|^2 = c(m - |T|^2) \geq c(m - 1).
\]

Thus, $|\sigma|^2 = c(m - 1)$ if and only if $|T| = 1$ at every point on $\Sigma^m$, i.e. $\Sigma^m$ is a vertical cylinder $\pi^{-1}(\Sigma^{m-1})$, and $A_V = 0$ for any normal vector field $V$ orthogonal to $H$. 
Next, let us consider the subbundle \( L = \text{span}\{\text{Im}\,\sigma\} \), and we see that \( L \) is parallel in the normal bundle and \( \dim L = 1 \), since actually \( L = \text{span}\{H\} \), and that \( R(X,Y)Z \in T\Sigma^m \oplus L \), for any \( X,Y,Z \in T\Sigma^m \oplus L \). Therefore, using [16, Theorem 2], we get that the cylinder \( \Sigma^m \) lies in an \((m+1)\)-dimensional totally geodesic submanifold of \( \mathbb{S}^{n}(c) \times \mathbb{R} \), i.e. it is a vertical cylinder in \( \mathbb{S}^{n}(c) \times \mathbb{R} \). \( \square \)

**Proposition 4.8.** Let \( \Sigma^m \) be a proper-biharmonic pmc submanifold in \( \mathbb{S}^{n}(c) \times \mathbb{R} \), with \( m \geq 2 \). Then its mean curvature satisfies \( |H|^{2} \leq c \), and the equality holds if and only if \( \Sigma^m \) is minimal in a small hypersphere \( \mathbb{S}^{n-1}(2) \subset \mathbb{S}^{n} \).

**Proof.** Since \( |A_{H}|^{2} \geq m|H|^{4} \), from the first equation of (4.2), we get that
\[
c(m - |T|^{2}) \geq m|H|^{2},
\]
and then \( |H|^{2} \leq c \). The equality holds if and only if \( T = 0 \), which means that \( \Sigma^m \) lies in \( \mathbb{S}^{n} \). Thus, using Theorem 4.2, we come to the conclusion. \( \square \)

Now, for the sake of simplicity, we shall consider only the case \( c = 1 \), and we are ready to prove the first of our main results.

**Theorem 4.9.** Let \( \Sigma^m \) be a complete proper-biharmonic pmc submanifold in \( \mathbb{S}^{n} \times \mathbb{R} \), with \( m \geq 2 \), such that its mean curvature satisfies

\[
|H|^{2} > C(m) = \frac{(m - 1)(m^{2} + 4) + (m - 2)\sqrt{(m - 1)(m - 2)(m^{2} + m + 2)}}{2m^{3}},
\]

and the norm of its second fundamental form \( \sigma \) is bounded. Then \( m < n \), \( |H| = 1 \) and \( \Sigma^m \) is a minimal submanifold of a small hypersphere \( \mathbb{S}^{n-1}(2) \subset \mathbb{S}^{n} \).

**Proof.** From Corollary 4.5, we have that \( \langle H, \xi \rangle = 0 \), which implies
\[
0 = \langle \nabla_{X} H, \xi \rangle = -\langle A_{H} X, T \rangle = -\langle A_{H} T, X \rangle
\]
for any tangent vector field \( X \), and then \( A_{H} T = 0 \). Therefore, if we consider a local orthonormal frame field \( \{E_{m+1} = \frac{H}{|H|}, \ldots, E_{n+1}\} \) in the normal bundle, using Proposition 3.2 and equation (4.2), we get

\[
\frac{1}{2}\Delta |A_{H}|^{2} = |\nabla A_{H}|^{2} + m(\text{trace} A_{H}^{3}) - m^{2}|H|^{4}.
\]

Let us consider \( \phi_{H} = A_{H} - |H|^{2} I \) the traceless part of \( A_{H} \). We have
\[
\text{trace} A_{H}^{3} = \text{trace} \phi_{H}^{3} + 3|H|^{2}|\phi_{H}|^{2} + m|H|^{6},
\]
and, using the first equation of (4.2),
\[
|\phi_{H}|^{2} = |A_{H}|^{2} - m|H|^{4} = (m - |T|^{2})|H|^{2} - m|H|^{4}.
\]
Replacing in equation (4.4), one obtains
\[
\frac{1}{2}\Delta |\phi_{H}|^{2} = |\nabla \phi_{H}|^{2} + m(\text{trace} \phi_{H}^{3}) + 3m|H|^{2}|\phi_{H}|^{2} - m^{2}|H|^{4}(1 - |H|^{2}).
\]

Using Lemma 2.3, we get
\[
\text{trace} \phi_{H}^{3} \geq - \frac{m - 2}{\sqrt{m(m - 1)}} |\phi_{H}|^{3},
\]
and then, since $|T|^2|H|^4 = |\phi H T|^2 \leq |T|^2|\phi H|^2$,

$$\frac{1}{2} \Delta |\phi H|^2 \geq -\frac{m(m-2)}{\sqrt{m(m-1)^2}} |\phi H|^3 + 3m|H|^2 |\phi H|^2 - m^2 |H|^4 (1 - |H|^2)$$

$$= -\frac{m(m-2)}{\sqrt{m(m-1)^2}} |\phi H|^3 + 2m|H|^2 |\phi H|^2 - m|T|^2 |H|^4$$

(4.5)

$$\geq -\frac{m(m-2)}{\sqrt{m(m-1)^2}} |\phi H|^3 + 2m|H|^2 |\phi H|^2 - m|T|^2 |\phi H|^2$$

$$= m|\phi H|^2 \left( -\frac{m-2}{\sqrt{m(m-1)^2}} |\phi H| + 2|H|^2 - |T|^2 \right).$$

Now, we shall split our study in two cases, as $m \geq 3$ or $m = 2$.

**Case I:** $m \geq 3$. If $2|H|^2 - |T|^2 > 0$, then we can write

$$-\frac{m-2}{\sqrt{m(m-1)^2}} |\phi H|^3 + 2|H|^2 - |T|^2 = \frac{1}{m(m-1)^2} \frac{m}{\sqrt{m(m-1)^2}} |\phi H|^2 + 2|H|^2 - |T|^2,$$

where $P(t)$ is a polynomial with constant coefficients, given by

$$P(t) = m(m-1)^2 - (3m^2 - 4)|H|^2 t + m|H|^2 (m^2|H|^2 - (m-2)^2).$$

By using elementary arguments, we obtain that, if $|H|^2 > C(m)$, then $P(t) \geq P(1) > 0$ for any $t \in (-\infty, 1]$.

Since $C(m) > \frac{1}{m}$ for any $m \geq 3$, our hypothesis $|H|^2 > C(m)$ implies that $2|H|^2 - |T|^2 > 0$, and then, from (4.5), we get

$$\frac{1}{2} \Delta |\phi H|^2 \geq \frac{mP(|T|^2)}{\sqrt{m-1}((m-2)|\phi H| + \sqrt{m(m-1)^2}(2|H|^2 - |T|^2))} |\phi H|^2$$

$$\geq \frac{P(|T|^2)}{\sqrt{m-1}|H|((m-2)\sqrt{1-|H|^2} + 2\sqrt{m-1}|H|)} |\phi H|^2$$

(4.6)

$$\geq \frac{P(1)}{\sqrt{m-1}|H|((m-2)\sqrt{1-|H|^2} + 2\sqrt{m-1}|H|)} |\phi H|^2$$

$$\geq 0.$$

Next, let us consider a local orthonormal frame field $\{E_i\}_{i=1}^m$ on $\Sigma^m$, $X$ a unit tangent vector field, and $\{E_{m+1} = \frac{H}{|H|}, \ldots, E_{n+1}\}$ an orthonormal frame field in the normal bundle. Using equation (2.2), we can compute the Ricci curvature of our submanifold

$$\text{Ric } X = \sum_{i=1}^m \langle R(E_i, X)X, E_i \rangle$$

$$= \sum_{i=1}^m \{ |X|^2 - \langle X, E_i \rangle^2 - \langle X, T \rangle^2 + 2 \langle X, T \rangle \langle T, E_i \rangle \langle X, E_i \rangle \}$$

$$- \langle T, E_i \rangle |X|^2 + \sum_{\alpha=m+1}^{n+1} \{ \langle A_\alpha E_i, E_i \rangle \langle A_\alpha X, X \rangle - \langle A_\alpha X, E_i \rangle \langle A_\alpha E_i, X \rangle \}$$

$$= m - 1 - |T|^2 - (m-2)\langle X, T \rangle^2 + m \langle A_H X, X \rangle - \sum_{\alpha=m+1}^{n+1} |A_\alpha X|^2,$$

and then, it follows that

$$\text{Ric } X \geq (m - 1)(1 - |T|^2) - m|A_H X| - \sum_{\alpha=m+1}^{n+1} |A_\alpha|^2$$

$$\geq -m|A_H| - |\sigma|^2.$$
Since by hypothesis we know that $|\sigma|$ is bounded, we can see that the Ricci curvature of $\Sigma^m$ is bounded from below, and then the Omori-Yau Maximum Principle holds on our submanifold.

Therefore, we can use Theorem 2.4 with $u = |\phi_H|^2$. It follows that there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset \Sigma^m$ satisfying

$$\lim_{k \to \infty} |\phi_H|^2(p_k) = \sup_{\Sigma^m} |\phi_H|^2 \quad \text{and} \quad \Delta |\phi_H|^2(p_k) < \frac{1}{k}.$$ 

Since $P(1) > 0$, from (4.5), we get that $0 = \lim_{k \to \infty} |\phi_H|^2(p_k) = \sup_{\Sigma^m} |\phi_H|^2$, which means that $\phi_H = 0$, i.e. $\Sigma^m$ is pseudo-umbilical.

Now, since $A_H T = 0$, we have $0 = A_H T = |H|^2 T$, i.e. $T = 0$ on $\Sigma^m$, and therefore $\Sigma^m$ lies in $S^n$, which also implies that $m < n$. Since $|H|^2 > C(m) > (\frac{m-1}{m})^2 > (\frac{m-2}{m})^2$, using Theorems 4.2 and 4.3 we come to the conclusion.

Case II: $m = 2$. In this case, from equation (4.5), we have

$$\frac{1}{2} \Delta |\phi_H|^2 \geq 2 |\phi_H|^2 (2|H|^2 - |T|^2) = \frac{2|\phi_H|^2}{|H|^2} ((|\phi_H|^2 + 2|H|^2(2|H|^2 - 1)).$$

Now, since $|H|^2 > C(2) = \frac{1}{2}$, working as in the first case, we conclude. \qed

**Remark 4.2.** We note that, in the case of proper-biharmonic pmc surfaces in $S^n \times \mathbb{R}$, if we take $|H|^2 \geq C(2)$, then the conclusion of Theorem 4.9 remains unchanged.

5. **Biharmonic pmc surfaces in $S^n(c) \times \mathbb{R}$**

Before proving the second main theorem of this paper we need some preliminary results.

First, we note that the map $p \in \Sigma^2 \to (A_H - \mu I)(p)$, where $\mu$ is a constant, is analytic, and, therefore, either $\Sigma^2$ is a pseudo-umbilical surface (at every point), or $H(p)$ is not an umbilical direction for any point $p$, or $H(p)$ is an umbilical direction on a closed set without interior points. We shall denote by $W$ the set of points where $H$ is not an umbilical direction. In the second case, $W$ coincides with $\Sigma^2$, and in the third one, $W$ is an open dense set in $\Sigma^2$.

As the authors observed in [4, Lemma 1], we have that, if $\Sigma^2$ is a pmc surface in $S^n(c) \times \mathbb{R}$, with mean curvature vector field $H$, then either $\Sigma^2$ is pseudo-umbilical, i.e. $H$ is an umbilical direction everywhere, or, at any point in $W$, there exists a local orthonormal frame field that diagonalizes $A_U$ for any normal vector field $U$ defined on $W$.

If $\Sigma^2$ is a pseudo-umbilical pmc surface in $S^n(c) \times \mathbb{R}$, then it was proved in [4, Lemma 3] that it lies in $S^n(c)$, and, therefore, $\Sigma^2$ is minimal in a small hypersphere of $S^n(c)$.

**Lemma 5.1.** Let $\Sigma^2$ be a pmc surface in $S^n(c) \times \mathbb{R}$. Then $\Sigma^2$ is proper-biharmonic if and only if either

1. $\Sigma^2$ is pseudo-umbilical and, therefore, it is a minimal surface of a small hypersphere $S^{n-1}(2c) \subset S^n(c)$; or
2. the mean curvature vector field $H$ is orthogonal to $\xi$, $|A_H|^2 = c(2 - |T|^2)|H|^2$, and $A_U = 0$ for any normal vector field $U$ orthogonal to $H$.

**Proof.** As we have seen, in the first case, $\Sigma^2$ is a minimal surface in a small hypersphere of $S^n(c)$, and then the conclusion follows from [9, Theorem 3.4].

Assume now that $\Sigma^2$ is not pseudo-umbilical. In the following, we shall work on the set $W$ defined above. Let $p$ be an arbitrary point in $W$ and consider $\{e_1, e_2\}$ an orthonormal basis at $p$ that diagonalizes $A_H$ and $A_U$ for any normal vector $U$
orthogonal to $H$. Since $H \perp U$, it follows that trace $A_U = 2\langle H, U \rangle = 0$. The matrices of $A_H$ and $A_U$ with respect to $\{e_1, e_2\}$ are

$$A_H = \begin{pmatrix} a + |H|^2 & 0 \\ 0 & -a + |H|^2 \end{pmatrix} \quad \text{and} \quad A_U = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix},$$

and then, from the last biharmonic condition (11.2), we get $0 = \text{trace}(A_HA_U) = 2ab$. Since $a \neq 0$, we get $b = 0$, i.e. $A_U = 0$.

Finally, we extend the result by continuity throughout $\Sigma^2$, and we conclude. \(\Box\)

**Corollary 5.2.** If $\Sigma^2$ is a proper-biharmonic pmc surface in $S^n(c) \times \mathbb{R}$ then the tangent part $T$ of $\xi$ has constant length.

**Proof.** If the surface is pseudo-umbilical, then $T = 0$.

Now, assume that $\Sigma^2$ is non-pseudo-umbilical and we shall work on $W$. Let $p$ be an arbitrary point in $W$ and $X \in T_p \Sigma^2$. Since $\nabla \xi = 0$ and $H \perp N$, we get that $\nabla_X T = A_N X = 0$. Then, we have

$$X(|T|^2) = 2\langle \nabla_X T, T \rangle = 0.$$

By continuity, it follows that $X(|T|^2) = 0$ for any tangent vector field $X$ defined on $\Sigma^2$, and we come to the conclusion. \(\Box\)

**Remark 5.1.** We note that, if $\Sigma^2$ is a proper-biharmonic pmc surface in $S^n(c) \times \mathbb{R}$ with $T = 0$, then it lies in $S^n(c)$ and is pseudo-umbilical (see [5]).

We recall now the following two results.

**Lemma 5.3 ([H]).** Let $\Sigma^2$ be a non-pseudo-umbilical pmc surface in $M^n(c) \times \mathbb{R}$, with second fundamental form $\sigma$, and define on $W$ the subbundle $L = \text{span}\{\text{Im} \sigma \cup N\}$ of the normal bundle. Then $L$ is parallel, i.e. if $U$ is a smooth section on $L$, then $\nabla^\perp U \in L$.

**Proposition 5.4 ([17]).** If $\Sigma^2$ is a pmc surface in $M^n(c) \times \mathbb{R}$, then

$$\frac{1}{2} \Delta |T|^2 = |A_N|^2 + K|T|^2 + 2T(\langle H, N \rangle),$$

where $K$ is the Gaussian curvature of the surface.

**Corollary 5.5.** If $\Sigma^2$ is a non-pseudo-umbilical proper-biharmonic pmc surface in $S^n(c) \times \mathbb{R}$, then it is flat.

In the following, let $\Sigma^2$ be a non-pseudo-umbilical proper-biharmonic pmc surface in $S^n(c) \times \mathbb{R}$. It follows that $|T| = \text{constant} \neq 0$, i.e. $|N| = \text{constant} \in [0,1)$. Working on the set $W$, since $A_U = 0$ for any normal vector field $U$ orthogonal to $H$, we obtain $\dim \text{span}\{\text{Im} \sigma\} = 1$ and then, $\dim L = 2$. Now, we apply [16] Theorem 2] and obtain that $W$, and therefore $\Sigma^2$, lies in $S^3(c) \times \mathbb{R}$.

Further, we shall prove that $|T| = 1$ on $\Sigma^2$, i.e. $|N| = 0$.

Assume that $|N| > 0$. Then there is a global orthonormal frame field $\{E_3 = \frac{H}{|H|}, E_4 = \frac{N}{|N|}\}$, and we have $A_2 = 0$ and $|\sigma|^2 = |A_3|^2 = c(2 - |T|^2)$.

On the other hand, since the surface is flat, from (2.2), it follows that

$$0 = 2K = 2c(1 - |T|^2) + 4|H|^2 - |\sigma|^2 = -c|T|^2 + 4|H|^2$$

and then, that

$$4|H|^2 = c|T|^2.$$
From Proposition 3.2 and Corollary 3.5 in the same way as in the proof of Theorem 4.1 we have

\begin{equation}
\frac{1}{2} \Delta |A_H|^2 = |\nabla A_H|^2 + 2(\text{trace } A_H^3) - 4c|H|^4.
\end{equation}

Next, since Corollary 5.2 implies that $|A_H|^2 = c(2 - |T|^2)|H|^2$ is constant, and, from relation (5.1), it follows

\[
\text{trace } A_H^3 = \text{trace } \phi_H^3 + 3|H|^2|\phi_H|^2 + 2|H|^4 = 3c(2 - |T|^2)|H|^4 - 4|H|^6
\]

\[
= 6c|H|^4 - 16|H|^6,
\]
equation (5.2) leads to

\[
0 = |\nabla A_H|^2 + 8c|H|^4 - 32|H|^6 = |\nabla A_H|^2 + 8|H|^4(c - 4|H|^2)
\]

\[
= |\nabla A_H|^2 + 8c|H|^4|N|^2,
\]
and we get that $|\nabla A_H|^2 = 0$ and $N = 0$. Therefore, the surface is a vertical cylinder. Now, since $\Sigma^2$ is flat, we have $|H| = \frac{1}{2}\sqrt{c}$, i.e. $\Sigma^2 = \pi^{-1}(\gamma)$, where $\gamma$ is a proper-biharmonic pmc curve in $\mathbb{S}^3(c)$ with curvature $\kappa = 2|H| = \sqrt{c}$. It follows that $\gamma$ actually is a proper-biharmonic circle in $\mathbb{S}^3(c)$.

Summarizing, we have the following rigidity result.

**Theorem 5.6.** Let $\Sigma^2$ be a proper-biharmonic pmc surface in $\mathbb{S}^n(c) \times \mathbb{R}$. Then either

1. $\Sigma^2$ is a minimal surface of a small hypersphere $\mathbb{S}^{n-1}(2c) \subset \mathbb{S}^n(c)$; or
2. $\Sigma^2$ is (an open part of) a vertical cylinder $\pi^{-1}(\gamma)$, where $\gamma$ is a circle in $\mathbb{S}^2(c)$ with curvature equal to $\sqrt{c}$, i.e. $\gamma$ is a biharmonic circle in $\mathbb{S}^2(c)$.

From Theorem 5.6 we can see that equation $\nabla A_H = 0$ holds for all proper-biharmonic surfaces. From this point of view, the following result can be seen as a generalization of that theorem for higher dimensional submanifolds. Before stating the theorem, we have to mention that proper-biharmonic pmc submanifolds in $\mathbb{S}^n(c)$, with $\nabla A_H = 0$, were classified in [7].

**Theorem 5.7.** If $\Sigma^m$, with $m \geq 3$, is a proper-biharmonic pmc submanifold in $\mathbb{S}^n(c) \times \mathbb{R}$ such that $\nabla A_H = 0$, then either

1. $\Sigma^m$ is a proper-biharmonic pmc submanifold in $\mathbb{S}^n(c)$, with $\nabla A_H = 0$; or
2. $\Sigma^m$ is (an open part of) a vertical cylinder $\pi^{-1}(\Sigma^{m-1})$, where $\Sigma^{m-1}$ is a proper-biharmonic pmc submanifold in $\mathbb{S}^n(c)$ such that the shape operator corresponding to its mean curvature vector field in $\mathbb{S}^n(c)$ is parallel.

**Proof.** On the one hand, since $\nabla A_H = 0$, we have that $[R(X, Y), A_H] = 0$ for any vector fields $X$ and $Y$ tangent to $\Sigma^m$. On the other hand, since $\Sigma^m$ is a proper-biharmonic pmc submanifold, as we have seen in the proof of Theorem 4.9, we also have $A_H T = 0$, and then it follows that $A_H R(X, T) T = 0$.

Now, let us consider a local orthonormal frame field $\{E_i\}_{i=1}^{m}$ on $\Sigma^m$, and $\{E_{m+1} = \frac{H}{|H|}, \ldots, E_{n+1}\}$ an orthonormal frame field in the normal bundle. We obtain, using (2.2), Lemma 3.1 and again $A_H T = 0$,

\[
0 = \sum_{i=1}^{m} \langle A_H R(E_i, T) T, E_i \rangle
\]

\[
= c(\text{trace } A_H)|T|^2(1 - |T|^2) + \sum_{a=m+2}^{n+1} \langle \text{trace}(A_H A_a) \rangle \langle A_a T, T \rangle.
\]
From the last equation of (4.2), we know that \( \text{trace}(A_H A_\alpha) = 0 \) for \( \alpha \in \{m + 2, \ldots, n + 1\} \), and therefore we have

\[
0 = c_m |H|^2 |T|^2 (1 - |T|^2),
\]

i.e. either \( |T| = 0 \) or \( |T| = 1 \), which completes the proof. \( \square \)

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