SUPER-MOONSHINE FOR CONWAY’S LARGEST SPORADIC GROUP

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ABSTRACT. We study a self-dual \( N = 1 \) super vertex operator algebra and prove that the full symmetry group is Conway’s largest sporadic simple group. We verify a uniqueness result which is analogous to that conjectured to characterize the Moonshine vertex operator algebra. The action of the automorphism group is sufficiently transparent that one can derive explicit expressions for all the McKay–Thompson series. A corollary of the construction is that the perfect double cover of the Conway group may be characterized as a point-stabilizer in a spin module for the Spin group associated to a 24 dimensional Euclidean space.

1. Introduction

The preeminent example of the structure of vertex operator algebra (VOA) is the Moonshine VOA denoted \( V^\sharp \) which was first constructed in [FLM88] and whose full automorphism group is the Monster sporadic group.

Following [Höh96] we say that a VOA is nice when it is \( C_2 \)-cofinite and satisfies a certain natural grading condition (see §2.1), and we make a similar definition for super vertex operator algebras (SVOAs). We say that a VOA is rational when all of its modules are completely reducible (see §2.2). Then conjecturally \( V^\sharp \) may be characterized among nice rational VOAs by the following properties:

- self-dual
- rank 24
- no small elements

where a self-dual VOA is one that has no non-trivial irreducible modules other than itself, and we write “no small elements” to mean no non-trivial vectors in the degree one subspace, since in a nice VOA this is the \( L(0) \)-homogeneous subspace with smallest degree that can be trivial. In this note we study what may be viewed as a super analogue of \( V^\sharp \). More specifically, we study an object \( A^{1/2} \) characterized among nice rational \( N = 1 \) SVOAs by the following properties:

- self-dual
- rank 12
- no small elements

An \( N = 1 \) SVOA is an SVOA which admits a representation of the Neveu–Schwarz superalgebra, and now “no small elements” means no non-trivial vectors with degree 1/2. (We define an SVOA to be rational when its even sub-VOA is rational, and a self-dual SVOA is an SVOA with no irreducible modules other than itself.)

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We find that the full automorphism group of $A^{f^2}$ is Conway’s largest sporadic group $Co_1$. Thus by considering the graded traces of elements of $\operatorname{Aut}(A^{f^2})$ acting on $A^{f^2}$, that is, the McKay–Thompson series, one can associate modular functions to the conjugacy classes of $Co_1$, and we obtain moonshine for $Co_1$. This is directly analogous to the moonshine which exists for the Monster simple group, was first observed in [CN79] and is to some extent explained by the existence of $V^3$.

The main results of this paper are the following three theorems.

**Theorem 4.10.** The space $A^{f^2}$ admits a structure of self-dual rational $N = 1$ SVOA.

**Theorem 4.11.** The automorphism group of the $N = 1$ SVOA structure on $A^{f^2}$ is isomorphic to Conway’s largest sporadic group, $Co_1$.

**Theorem 5.28.** The $N = 1$ SVOA $A^{f^2}$ is characterized among nice rational $N = 1$ SVOAs by the properties: self-dual, rank 12, trivial degree 1/2 subspace.

The earliest evidence in the mathematical literature that there might be an object such as $A^{f^2}$ was given in [FLMS85] where it was suggested to study a $\mathbb{Z}/2$-orbifold of a certain SVOA $V_L^f = A_L \otimes V_L$ associated to the $E_8$ lattice. Here $L$ is a lattice of $E_8$ type, $V_L$ denotes the VOA associated to $L$, and $A_L$ denotes the Clifford module SVOA associated to the space $\mathbb{C} \otimes_2 L$. Since the lattice $L = E_8$ is self-dual, $V_L^f$ is a self-dual SVOA, and one finds that the graded character of $V_L^f$ satisfies

\[
\operatorname{tr}|_{V_L^f} q^{L(0) - c/24} = \frac{\theta_{E_8}(\tau)\eta(\tau)^8}{\eta(\tau/2)^8 \eta(2\tau)^8} = q^{-1/2}(1 + 8q^{1/2} + 276q + 2048q^{3/2} + \ldots)
\]

and is a Hauptmodul for a certain genus zero subgroup of the Modular group $\tilde{\Gamma} = \operatorname{PSL}(2, \mathbb{Z})$. One can check that, but for the constant term, these coefficients exhibit moonshine phenomena for $Co_1$. For example, we have that 276 is the dimension of an irreducible module for $Co_1$, and 2048 = 1 + 276 + 1771 is a possible decomposition of the degree 3/2 subspace into irreducibles for $Co_1$. The only problem being that there is no irreducible representation of $Co_1$ of dimension 8, and the space corresponding to the constant term in the character of $V_L^f$ would have to be a sum of trivial modules were it a $Co_1$-module at all. As observed in [FLMS85], orbifolding $V_L^f$ by a suitable lift of $-1$ on $L$, one obtains a space $V^{f^2}$ with $\operatorname{tr}|_{V^{f^2}} q^{L(0)} = \operatorname{tr}|_{V_L^f} q^{L(0) - c/24} - 8$; that is, a space with the correct character for $Co_1$.

\[
\operatorname{tr}|_{V^{f^2}} q^{L(0) - c/24} = \frac{\theta_{E_8}(\tau)\eta(\tau)^8}{\eta(\tau/2)^8 \eta(2\tau)^8} - 8 = q^{-1/2}(1 + 276q + 2048q^{3/2} + \ldots)
\]

Also one finds that $V^{f^2}$ admits a reasonably transparent action by a group of the shape $2^{1+8}(\mathbb{W}_E/\{\pm 1\})$ which is the same as that of a certain involution centralizer and maximal subgroup in $Co_1$. (Here $\mathbb{W}_E$ denotes the derived subgroup of the Weyl group of type $E_8$.)

We note here that the existence of $V^{f^2}$ has certainly been known to Richard E. Borcherds for some time. Also, an action of $Co_1$ on the SVOA underlying what we will call $A^{f^2}$ was considered in [BR96].

The space $V^{f_2}$ may be described as

\[
V^{f_2} = (V_L^f)^0 \oplus (V_L^f)_0
\]
where \((V_L^f)_\theta\) denotes a \(\theta\)-twisted \(V_L^f\)-module and \(\theta = \theta_f \otimes \theta_b\) is an involution on \(V_L^f\) obtained by letting \(\theta_f\) be the parity involution on \(A_L\), and letting \(\theta_b\) be a lift to \(\text{Aut}(V_L)\) of the \(-1\) symmetry on \(L\). The \(\theta\)-twisted module \((V_L^f)_\theta\) may be described as a tensor product of twisted modules \((V_L^f)_\theta = (A_L)_\theta \otimes (V_L)_\theta\), and \((V_L^f)_\theta^0\) and \((V_L^f)_\theta^\theta\) denote \(\theta\)-fixed points. Now one is in a situation very similar to that which gives rise to the Moonshine VOA \(V^\natural\) via the Leech lattice VOA as is carried out in [FLMSS], and one can hope that a similar approach as that used in [FLMSS] would yield the desired result: an SVOA structure on \(V^\natural\) with an action by \(Co_1\).

We find it convenient to pursue a different approach. We construct an SVOA \(A^{f_3}\) whose graded character coincides with that of \(V^\natural\), and we do so using a purely fermionic construction; that is, using Clifford module SVOAs. It turns out that this fermionic construction enables one to analyze the symmetries in quite a transparent way. We find that there is a specific vector in the degree \(3/2\) subspace of \(A^{f_3}\) such that the corresponding point stabilizer in the full group of SVOA automorphisms of \(A^{f_3}\) is precisely the group \(Co_1\). Furthermore, this vector naturally gives rise to a representation of the Neveu–Schwarz superalgebra on \(A^{f_3}\), and thus \(Co_1\) is realized as the full group of automorphisms of a particular \(N = 1\) SVOA structure on \(A^{f_3}\).

These results are presented in [4].

In [5] we consider an analogue of the uniqueness conjecture for \(V^\natural\). It turns out that the \(N = 1\) SVOA structure on \(A^{f_3}\) is unique in the sense that the vectors in \((A^{f_3})_{3/2}\) that give rise to a representation of the Neveu–Schwarz superalgebra on \(A^{f_3}\) form a single orbit under the action of \(\text{Spin}_{24}(\mathbb{R})\) on \(A^{f_3}\). Making use of modular invariance for \(n\)-point functions on a self-dual SVOA, due to [Zhu96] and [H090918], and utilizing also some ideas of [DM04] and [DM04b], one can show that the SVOA structure underlying \(A^{f_3}\) is characterized up to some technical conditions by the by now familiar properties:

- self-dual
- rank 12
- no small elements

Combining these results, we find that \(A^{f_3}\) is characterized among \(N = 1\) SVOAs by the above three properties. This uniqueness result accentuates the analogy between \(V^\natural\) and certain other celebrated algebraic structures: the Golay code, the Leech lattice, and (conjecturally) the Moonshine VOA \(V^\natural\).

The homogeneous subspace of \(A^{f_3}\) of degree \(3/2\) may be identified with a half-spin module over \(\text{Spin}_{24}(\mathbb{R})\). The main idea behind the construction of \(A^{f_3}\) is to realize this spin module in such a way that the vector giving rise to the \(N = 1\) structure on \(A^{f_3}\) is as obvious as possible. Essentially, we achieve this by using the Golay code to construct a particular idempotent in the Clifford algebra of a 24 dimensional vector space. The automorphism group of \(A^{f_3}\) turns out to be the quotient by center of the subgroup of \(\text{Spin}_{24}(\mathbb{R})\) fixing this idempotent. Thus a curious corollary of the construction is that the group \(Co_0\) (the perfect double cover of \(Co_1\)) may be characterized as a point stabilizer in a spin module over \(\text{Spin}_{24}(\mathbb{R})\).

The Golay code is characterized among length 24 doubly-even linear binary codes (see [14]) by the conditions of self-duality and having no “small elements” (that is, no weight 4 codewords). The Golay code is an important ingredient in the construction of \(A^{f_3}\), and these defining properties yield direct influence upon the structure of \(A^{f_3}\). For example, the condition “no small elements” allows one to
conclude that $\text{Aut}(A^{f_2})$ is finite, and the uniqueness of the Golay code ultimately entails the uniqueness of the $N = 1$ structure on $A^{f_2}$. The uniqueness of the $N = 1$ structure in turn allows one to formulate the following characterization of $Co_0$.

**Theorem 5.29.** Let $M$ be a spin module for $\text{Spin}_{24}(\mathbb{R})$ and let $t \in M$ such that $\langle xt, t \rangle = 0$ whenever $x \in \text{Spin}_{24}(\mathbb{R})$ is an involution with $\text{tr} \cdot 24x = 16$. Then the subgroup of $\text{Spin}_{24}(\mathbb{R})$ fixing $t$ is isomorphic to $Co_0$.

It is perhaps interesting to note that the Leech lattice, which has automorphism group $Co_0$ and which furnishes a popular definition of this group [Con69], does not figure directly in our construction of $A^{f_2}$. Although the Golay code does play a prominent role, the uniqueness of $A^{f_2}$, or alternatively the above Theorem 5.29, provide definitions for the group $Co_0$ relying neither on the Leech lattice nor the Golay code.

The SVOA $V^{f_2}$ constructed from the $E_8$ lattice admits a natural structure of $N = 1$ SVOA in analogy with the way in which a usual lattice VOA is naturally equipped with a Virasoro element. Thus a corollary of the uniqueness result for $A^{f_2}$ is that $A^{f_2}$ is isomorphic to the $N = 1$ SVOA $V^{f_2}$ discussed above. In the penultimate section §6 we consider the construction of $V^{f_2}$ in more detail, and we indicate how to construct an explicit isomorphism with $A^{f_2}$.

In the final section §7 we consider the McKay–Thompson series associated to elements of $Co_0$ acting on $A^{f_2}$. One can derive explicit expressions for each of the McKay–Thompson series associated to elements of $Co_0$ in terms of the frame shapes of the corresponding preimages in $Co_0$. These expressions are recorded in Theorem 7.1.

### 1.1. Notation.

If $M$ is a vector space over $\mathbb{F}$ and $\mathbb{E}$ is a field containing $\mathbb{F}$, we write $\mathbb{E}M = \mathbb{E} \otimes_\mathbb{F} M$ for the vector space over $\mathbb{E}$ obtained by extension of scalars. For the remainder we shall use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. We choose a square root of $-1$ in $\mathbb{C}$ and denote it by $i$. For $q$ a prime power $\mathbb{F}_q$ shall denote a field with $q$ elements. For $G$ a finite group we write $\mathbb{F}G$ for the group algebra of $G$ over $\mathbb{F}$.

For $\Sigma$ a finite set, we denote the power set of $\Sigma$ by $P(\Sigma)$. The set operation of symmetric difference (which we denote by $+$) equips $P(\Sigma)$ with a structure of $\mathbb{F}_2$-vector space, and with this structure in mind, we sometimes write $\mathbb{F}P_2$ in place of $P(\Sigma)$. Suppose that $\Sigma$ has $N$ elements. The space $\mathbb{F}P_2$ comes equipped with a function $w : \mathbb{F}P_2 \to \{0, 1, \ldots, N\}$ called *weight*, which assigns to an element $\gamma \in \mathbb{F}P_2$ the cardinality of the corresponding element of $P(\Sigma)$. An $\mathbb{F}_2$-subspace of $\mathbb{F}P_2$ is called a *linear binary code of length* $N$. A linear binary code $C$ is called *even* if $2|w(C)$ for all $C \subseteq \Sigma$, and is called *doubly-even* if $4|w(C)$ for all $C \subseteq \Sigma$. For $C \subseteq \mathbb{F}P_2$ a linear binary code, the *co-code* $C^*$ is the space $\mathbb{F}P_2/C$. We write $X \mapsto \bar{X}$ for the canonical map $\mathbb{F}P_2 \to C^*$. The weight function on $\mathbb{F}P_2$ induces a function $w^*$ on $C^*$ called *co-weight*, which assigns to $\bar{X} \in C^*$ the minimum weight amongst all preimages of $\bar{X}$ in $\mathbb{F}P_2$. Once a choice of code $C$ has been made, it will be convenient to regard $w^*$ as a function on $\mathbb{F}P_2$ by setting $w^*(X) := w^*(\bar{X})$ for $X \in \mathbb{F}P_2$.

The Virasoro algebra is the universal central extension of the Lie algebra of polynomial vector fields on the circle (see [Kac22] for an algebraic formulation). In the case that a vector space $M$ admits an action by the Virasoro algebra, and the action of $L(0)$ is diagonalizable, we write $M = \bigsqcup_n M_n$ where $M_n = \{v \in M \mid L(0)v = nv\}$.
We call $M_u$ the homogeneous subspace of degree $n$, and we write $\deg(u) = n$ for $u \in M_u$.

When $M$ is a super vector space, we write $M = M_0 \oplus M_1$ for the superspace decomposition. For $u \in M$ we set $|u| = k$ when $u$ is $\mathbb{Z}/2$-homogeneous and $u \in M_k$ for $k \in \{0, 1\}$. The dual space $M^*$ has a natural superspace structure such that $M_k^* = (M_r)^*$ for $\gamma \in \{0, 1\}$.

There are various vector spaces throughout the paper that admit an action by a linear automorphism of order two denoted $\theta$. Suppose that $M$ is such a space. Then we write $M^k$ for the $\theta$-eigenspace with eigenvalue $(-1)^k$. Note that $M$ may be a super vector space, and $M = M^0 \oplus M^1$ may or may not coincide with the superspace grading on $M$.

We denote by $D_z$ the operator on formal Laurent series which is formal differentiation in the variable $z$, so that if $f(z) = \sum f_r z^{-r-1} \in V\{z\}$ is a formal Laurent series with coefficients in some space $V$, we have $D_z f(z) = \sum (-r) f_{r-1} z^{-r-1}$. For $m$ a positive integer, we set $D_m = \frac{1}{m!} D_z^m$.

As is customary, we use $\eta(\tau)$ to denote the Dedekind eta function.

$$(1.1.1) \quad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

Here $q = e^{2\pi i \tau}$ and $\tau$ is a variable in the upper half plane $h = \{ \sigma + it \mid t > 0 \}$.

We write $Co_1$ for an abstract group isomorphic to Conway’s largest sporadic group. We write $Co_0$ for an abstract group isomorphic to the perfect double cover of $Co_1$. It is well known that $Co_0$ is isomorphic to the automorphism group of the Leech lattice $\text{[Con6]}$, and in particular, admits an orthogonal representation of degree 24 writable over $\mathbb{Z}$.

The most specialized notations arise in $\text{[8]}$. We include here a list of them, with the relevant subsections indicated in brackets.

- $u$ A real or complex vector space of even dimension with non-degenerate bilinear form, assumed to be positive definite in the real case ($\text{3.3}$).
- $\{e_i\}_{i \in \Sigma}$: A basis for $u$, orthonormal in the sense that $(e_i, e_j) = \delta_{ij}$ for $i, j \in \Sigma$ ($\text{3.1}$).
- $e_I$ We write $e_I$ for $e_{i_1} \cdots e_{i_k} \in \text{Cliff}(u)$ when $I = \{i_1, \ldots, i_k\}$ is a subset of $\Sigma$ and $i_1 < \cdots < i_k$ ($\text{3.1}$).
- $g(\cdot)$ We write $g \mapsto g(\cdot)$ for the natural homomorphism $\text{Spin}(u) \to \text{SO}(u)$. Regarding $g \in \text{Spin}(u)$ as an element of $\text{Cliff}(u)$ we have $g(u) = gug^{-1}$ in $\text{Cliff}(u)$ for $u \in u$. More generally, we write $g(x)$ for $gxg^{-1}$ when $x$ is any element of $\text{Cliff}(u)$ ($\text{3.4}$).
- $e_I(\cdot)$ When $I$ is even, $e_I \in \text{Cliff}(u)$ lies also in $\text{Spin}(u)$, and $e_I(u)$ denotes $e_I(ue_I^{-1})$ when $u \in \text{Cliff}(u)$.
- $\alpha$ The main anti-automorphism of a Clifford algebra ($\text{3.1}$).
- $\beta$ We denote $e_\Sigma \in \text{Spin}(u)$ also by $\beta$ ($\text{3.2}$).
- $\delta$ The map which is $-\text{Id}$ on $u$, or the parity involution on $\text{Cliff}(u)$ ($\text{3.1}$), or the parity involution on $A(u)_{\text{even}}$ ($\text{3.3}$).
- $1_E$ A vector in $\text{CM}(u)_E$ such that $x 1_E = 1_E x$ for $x \in E$ ($\text{3.5}$).
- $1_{E'}$ The vector corresponding to $1_E$ under the identification between $\text{CM}(u)_E$ and $\left( A(u)_{\theta, E}/\mathbb{Z}/8 \right)$ when $u$ has dimension $2N$ ($\text{3.6}$).
- $\Sigma$ A finite ordered set indexing an orthonormal basis for $u$ ($\text{3.1}$).
- $\xi$ A label for the basis $\{e_i\}_{i \in \Sigma}$ ($\text{3.1}$).
- $E$ A subgroup of $\text{Cliff}(u)$ homogeneous with respect to the $F_2$ grading on $\text{Cliff}(u)$ ($\text{3.5}$).
- $A(u)$ The Clifford module SVOA associated to the vector space $u$ ($\text{3.4}$).
- $A(u)_{\theta}$ The canonically $\theta$-twisted module over $A(u)$ ($\text{3.4}$).
- $A(u)_{\theta, E}$ The $\theta$-twisted module $A(u)$ realized in such a way that the subspace of minimal degree is identified with $\text{CM}(u)_E$ ($\text{3.4}$).
- $A(u)_{\theta,0}$ The direct sum of $A(u)$-modules $A(u) \oplus A(u)_{\theta}$ ($\text{3.4}$).
\( \mathcal{C}(E) \) The linear binary code on \( \Sigma \) consisting of elements \( I \) in \( \mathbb{F}_2^E \) for which \( E \) has non-trivial intersection with \( \mathcal{F}_E \subset \text{Cliff}(u) \) \((\S3.6)\).

\( \text{CM}(u)_E \) The module over \( \text{Cliff}(u) \) induced from a trivial module over \( E \) \((\S3.8)\).

\( \text{Cliff}(u) \) The Clifford algebra associated to the vector space \( u \) \((\S3.3)\).

\( \langle \cdot, \cdot \rangle \) A non-degenerate symmetric bilinear form on \( u \) or on \( \text{Cliff}(u) \) \((\S3.1)\), or on the \( \text{Cliff}(u) \)-module \( \text{CM}(u)_E \) \((\S3.8)\). In the case that \( u \) is real all of these forms will be positive definite.

\( \langle \cdot | \cdot \rangle \) A non-degenerate symmetric bilinear form on \( A(u)_\alpha \) \((\S3.4)\).

From \( \S 4 \) onwards we restrict to the case that

\( u = 1 \) is a 24 dimensional real vector space with positive definite symmetric bilinear form \( \langle \cdot, \cdot \rangle \);

\( \Sigma = \Omega \) is an ordered set with 24 elements indexing an orthonormal basis \( \mathcal{E} = \{ e_i \}_{i \in \Omega} \) for \( \Sigma \);

\( E = G \) is a subgroup of \( \text{Spin}(l) \) such that \( \mathcal{C}(G) \) is a copy of the Golay code \( \mathcal{G} \);

since this is the situation that will be relevant for the construction of \( A_f^3 \).

1. SVOA STRUCTURE

\[ 2.1. \text{ SVOAs.} \] Suppose that \( U = U_0 \oplus U_1 \) is a super vector space over \( \mathbb{F} \). For an SVOA structure on \( U \) we require the following data.

- **Vertex operators**: a map \( U \otimes U \to U((z)) \) denoted \( u \otimes v \mapsto Y(u, z)v \) such that, when we write \( Y(u, z)v = \sum u_n v z^{-n-1} \), we have \( u_n v \in U_{\gamma+n} \) when \( u \in U \) and \( v \in U_0 \), and such that \( Y(u, z)v = 0 \) for all \( v \in U \) implies \( u = 0 \).

- **Vacuum**: a distinguished vector \( 1 \in U_0 \) such that \( Y(1, z)u = u \) for \( u \in U \), and \( Y(u, z)1|_{z=0} = u \).

- **Conformal element**: a distinguished vector \( \omega \in U_0 \) such that the operators \( L(n) = \omega_{n+1} \) furnish a representation of the Virasoro algebra on \( U \)(c.f. \( \S 3.3 \)).

This data furnishes an SVOA structure on \( U \) just when the following axioms are satisfied.

1. **Translation**: for \( u \in U \) we have \( [L(-1), Y(u, z)] = D_z Y(u, z) \).
2. **Jacobi Identity**: the following Jacobi identity is satisfied for \( \mathbb{Z}/2 \) homogeneous \( u, v \in U \).
3. **L(0)-grading**: the action of \( L(0) \) on \( U \) is diagonalizable with rational eigenvalues bounded below, by \( -N \) say, and thus defines a \( \mathbb{Q}_{> -N} \)-grading \( U = \prod U_n \) on \( U \). This grading is such that the \( L(0) \)-homogeneous subspaces \( U_n = \{ u \in U \mid L(0)u = nu \} \) are finite dimensional.

In the case that \( U = U_0 \) we are speaking of ordinary VOAs.

An SVOA \( U \) is said to be \( C_2 \)-cofinite in the case that \( U_{-2} = \{ u_{-2}v \mid u, v \in U \} \) has finite codimension in \( U \). Following \( \text{Hoh96} \) we say that an SVOA \( U \) is nice when it is \( C_2 \)-cofinite, the eigenvalues of \( L(0) \) are non-negative and contained in
rational and the degree zero subspace $U_0$ is one dimensional and spanned by the vacuum vector. All the SVOAs we consider in this paper will be nice SVOAs.

By definition the coefficients of $Y(\omega, z)$ define a representation of the Virasoro algebra on $U$ (c.f. \cite{22}). Let $c \in \mathbb{C}$ be such that the central element of the Virasoro algebra acts as $\text{cld}$ on $U$. Then $c$ is called the rank of $U$, and we denote it by $\text{rank}(U)$.

2.2. SVOA Modules. For an SVOA module over $U$ we require a $\mathbb{Z}/2$-graded vector space $M = M_0 \oplus M_1$ and a map $U \otimes M \to M((z))$ denoted $u \otimes v \mapsto Y^M(u, z)v$ such that, when we write $Y^M(u, z)v = \sum_n u^M_nvz^{-n-1}$, we have $u^M_n v \in M_{\gamma+\delta}$ when $u \in U_\gamma$ and $v \in M_\delta$. The pair $(M, Y^M)$ is called an admissible $U$-module just when the following axioms are satisfied.

1. Vacuum: the operator $Y^M(1, z)$ is the identity on $M$.
2. Jacobi Identity: for $\mathbb{Z}/2$-homogeneous $u, v \in U$ we have
   \[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^M(u, z_1)Y^M(v, z_2) \]
   \[-(-1)^{|u||v|}z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y^M(v, z_2)Y^M(u, z_1) \]
   \[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y^M(Y(u, z_0)v, z_2) \]
   where $u \in U_\gamma$, $v \in U_\delta$.
3. Grading: The space $M$ carries a $\mathbb{Z}/2$-grading $M = \coprod_r M(r)$ bounded from below such that $u_nM(r) \subset M(m + r - n - 1)$ for $u \in U_m$.

We say that an admissible $U$-module $(M, Y^M)$ is an ordinary $U$-module if there is some $h \in \mathbb{C}$ such that the $\mathbb{Z}/2$-grading $M = \coprod_r M(r)$ satisfies $M(r) = \{m \in M \mid \text{L}(0)m = (r + h)m\}$, and the spaces $M(r)$ are finite dimensional for each $r \in \mathbb{Z}/2$. In this case we set $M_{h+r} = M(r)$ and write $M = \coprod_r M_{h+r}$.

All the SVOA modules that we consider will be ordinary modules. We understand that unless otherwise qualified, the term module shall mean ordinary module.

Remark 2.1. The significance of the notion of admissible module is the result of \cite{DLM98} that if every admissible module over a VOA is completely reducible, then there are finitely many irreducible admissible modules up to equivalence, and every finitely generated admissible module is an ordinary module.

A VOA is called rational if each of its admissible modules are completely reducible. We define an SVOA to be rational if its even sub-VOA is rational. An SVOA $U$ is called simple if it is irreducible as a module over itself. We say that a rational SVOA is self-dual if it is simple, and has no irreducible modules other than itself.

2.3. Intertwiners. Let $V$ be a VOA, and let $(M^i, Y^i)$, $(M^j, Y^j)$ and $(M^k, Y^k)$ be three $V$-modules. Suppose given a map $M^i \otimes M^j \to M^k\{z\}$, and employ the notation $u \otimes v \mapsto Y_{ij}^k(u, z)v = \sum_n u_nvz^{-n-1}$. We assume that for any $u \in M^i$ and $v \in M^j$ we have $u_nv = 0$ for $n$ sufficiently large, and we assume that $Y_{ij}^k(u, z) = 0$ only for $u = 0$. Then the map $Y_{ij}^k$ is called an intertwining operator of type $(i^j_k)$ just when the following axioms are satisfied.

1. Translation: for $u \in M^i$ we have $Y_{ij}^k(L^i(-1)u, z) = D_zY_{ij}^k(u, z)$. 

We say that an admissible canonically twisted $U$-module $M$ admits an action by $U$ which is the identity on twisted module. This shall mean “ordinary canonically twisted module”. In the sense of [DZ05], and an SVOA that is rational in our sense is both rational and $\sigma$-rational in the sense of [DZ05].

2.4. Twisted SVOA modules. Any SVOA $U$ say, admits an order two involution which is the identity on $U_0$ and acts as $-1$ on $U_1$. We refer to this involution as the canonical involution on $U$, and denote it by $\sigma$.

For a structure of canonically twisted or $\sigma$-twisted $U$-module on a vector space $M$ we require a map $Y^M : U \otimes M \rightarrow M((z^{1/2}))$ such that when we write $Y^M(u, z) = \sum_n u_n^M z^{-n-1}$ for $\mathbb{Z}/2$-homogeneous $u \in U$, then $u_n^M = 0$ unless $u \in U_k$ and $n \in \mathbb{Z} + \frac{k}{2}$. The pair $(M, Y^M)$ is called an admissible canonically twisted $U$-module when it satisfies just the same axioms as for untwisted admissible $U$-modules except that we modify the Jacobi identity axiom and the grading condition as follows.

* Twisted Jacobi identity: For $\mathbb{Z}/2$-homogeneous $u, v \in U$ we require that

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^M(u, z_1) Y^M(v, z_2)$$

$$(2.4.1) \quad - (-1)^{|u||v|} z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y^M(v, z_2) Y^M(u, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-k/2} Y^M(Y(u, z_0)v, z_2)$$

when $u \in U_k$.

* Twisted grading: The space $M$ carries a $\mathbb{Z}$-grading $M = \bigsqcup_r M(r)$ bounded from below such that $u_n M(r) \subset M(m + r - n - 1)$ for $u \in U_m$.

We say that an admissible canonically twisted $U$-module $(M, Y^M)$ is an ordinary canonically twisted $U$-module if there is some $h \in \mathbb{C}$ such that the $\mathbb{Z}$-grading $M = \bigsqcup_r M(r)$ satisfies $M(r) = \{ m \in M \mid L(0)m = (r + h)m \}$, and the spaces $M(r)$ are finite dimensional for each $r \in \mathbb{Z}$. In this case we set $M_{h+r} = M(r)$ and write $M = \bigsqcup_r M_{h+r}$.

As in [22] we convene that unless otherwise qualified, the term “canonically twisted module” shall mean “ordinary canonically twisted module”.

A canonically twisted module $(M, Y^M)$ over an SVOA $V$ is called $\sigma$-stable if it admits an action by $\sigma$ compatible with that on $V$, so that we have $\sigma Y^M(u, z)v = Y^M(\sigma u, z)\sigma v$ for $u \in U, v \in M$.

An important result we will make use of is the following.

**Theorem 2.2 ([DZ05]).** If $V$ is a self-dual rational $C_2$-cofinite SVOA then $V$ has a unique irreducible $\sigma$-stable $\sigma$-twisted module.

**Remark 2.3.** Recall from [22] that we define an SVOA to be rational in case it’s even sub-VOA is rational. This is a stronger condition than the notion of rationality in [DZ05], and an SVOA that is rational in our sense is both rational and $\sigma$-rational in the sense of [DZ05].
For $V$ satisfying the hypotheses of Theorem 2.4 we will write $(V_\tau, Y_\tau)$ for the unique irreducible canonically twisted $V$-module this theorem guarantees.

2.5. $N=1$ SVOAs. The Neveu–Schwarz superalgebra is the Lie superalgebra spanned by the symbols $L_m, G_{m+1/2}$, and $c$, for $m \in \mathbb{Z}$, and subject to the following relations [KW94].

\begin{align}
(2.5.1) \quad [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}, \\
(2.5.2) \quad [G_{m+1/2}, L_n] &= \left( m + \frac{1}{2} - \frac{n}{2} \right) G_{m+n+1/2}, \\
(2.5.3) \quad \{G_{m+1/2}, G_{n-1/2}\} &= 2L_{m+n} + \frac{m^2+m}{3}\delta_{m+n,0}c, \\
(2.5.4) \quad [L_m, c] &= [G_{m+1/2}, c] = 0.
\end{align}

Note that this algebra is generated by the $G_{m+1/2}$ for $m \in \mathbb{Z}$. The subalgebra generated by the $L_n$ is the Virasoro algebra.

Suppose that $U$ is an SVOA. We say that $U$ is an $N=1$ SVOA if there is an element $\tau \in U_{3/2}$ such that the operators $G(n+\frac{1}{2}) = \tau_{n+1}$ generate a representation of the Neveu–Schwarz superalgebra on $U$. We refer to such an element $\tau$ as a superconformal element for $U$. In general, a superconformal element for an SVOA $U$ is not unique and may not even exist, but for any superconformal $\tau$ we have that $\frac{1}{2}\tau_0\tau = \frac{1}{2}G(-\frac{1}{2})\tau$ is a conformal element. That is to say, the coefficients of $Y\left(\frac{1}{2}G\left(-\frac{1}{2}\right)\tau, z\right)$ generate a representation of the Virasoro algebra on $U$. Given an SVOA $U$, we will always assume that any superconformal element for $U$ is chosen so that $\frac{1}{2}G\left(-\frac{1}{2}\right)\tau = \omega$.

Note that the commutation relations for the Neveu–Schwarz superalgebra of central charge $c$ are equivalent to the following operator product expansions [DGHSS].

\begin{align}
(2.5.5) \quad Y(\omega, z_1)Y(\omega, z_2) &\sim \frac{c/2}{(z_1-z_2)^4} + \frac{2Y(\omega, z_2)}{(z_1-z_2)^2} + \frac{D_{z_2}Y(\omega, z_2)}{z_1-z_2}, \\
(2.5.6) \quad Y(\omega, z_1)Y(\tau, z_2) &\sim \frac{3}{2} \frac{Y(\tau, z_2)}{(z_1-z_2)^2} + \frac{D_{z_2}Y(\tau, z_2)}{z_1-z_2}, \\
(2.5.7) \quad Y(\tau, z_1)Y(\tau, z_2) &\sim \frac{2c/3}{(z_1-z_2)^3} + \frac{2Y(\omega, z_2)}{(z_1-z_2)}.
\end{align}

Consequently, we have the following

**Proposition 2.4.** Suppose $U$ is an SVOA with conformal element $\omega$ and central charge $c$. Then $\tau \in (U)_{3/2}$ is a superconformal element for $U$ so long as $\tau_2\tau = \frac{3}{4}c\mathbf{1}$, $\tau_1\tau = 0$ and $\tau_0\tau = 2\omega$.

Given an $N=1$ SVOA $U$ with superconformal element $\tau$ and conformal element $\omega = \frac{1}{2}\tau_0\tau$, we write $\text{Aut}_{SVOA}(U)$ for the group of automorphisms of the SVOA structure on $U$, and we write $\text{Aut}(U)$ for group of automorphisms of the $N=1$ SVOA structure on $U$. That is, for $U$ an $N=1$ SVOA, $\text{Aut}(U)$ denotes the subgroup of $\text{Aut}_{SVOA}(U)$ comprised of automorphisms that fix $\tau$.

2.6. **Adjoint operators.** Suppose that $U$ is a nice SVOA. For $M$ a module over $U$, let $M'$ denote the restricted dual of $M$, and let $\langle \cdot, \cdot \rangle_M$ be the natural pairing
\(M' \times M \to \mathbb{C}\). We define the adjoint vertex operators \(Y' : U \otimes M' \to M'\{z\}\) by requiring that
\[
(Y'(u, z)w', w)_{M} = (-1)^{n} \left< w', Y(\epsilon zL(1)z^{-2L(0)}u, z^{-1})w \right>_{M}
\]
for \(u \in U_{n-1/2} \oplus U_{n}\) with \(n \in \mathbb{Z}\), where \(w' \in M'\) and \(w \in M\). As in [FHL98] we have

**Proposition 2.5.** The map \(Y'\) equips \(M'\) with a structure of \(U\)-module.

Let \(\langle \cdot | \cdot \rangle : U \otimes U \to \mathbb{C}\) be a bilinear form on \(U\) such that \(\langle U_{m} | U_{n} \rangle \subset \{0\}\) unless \(m = n\). Then there is a unique grading preserving linear map \(\varphi : U \to U'\) determined by the formula \(\langle \varphi(u), v \rangle_{U} = \langle u \mid v \rangle\), and \(\varphi\) is a \(U\)-module equivalence if and only if
\[
\langle Y(u, z)w_{1} \mid w_{2} \rangle = \left< w_{1} \mid Y(\epsilon zL(1)z^{-2L(0)}u, z^{-1})w_{2} \right>
\]
for all \(u, w_{1}\) and \(w_{2}\) in \(U\). When this identity is satisfied we say that the bilinear form \(\langle \cdot | \cdot \rangle\) is an invariant form for \(U\).

**Proposition 2.6.** Suppose that \(\langle \cdot | \cdot \rangle\) is an invariant form for \(U\). Then it is symmetric.

Just as in the untwisted case, we can define the adjoint canonically twisted vertex operators. For \((M, Y)\) a canonically twisted \(U\) module, we define operators \(Y' : U \otimes M' \to M'\{z\}\) by requiring that
\[
(Y'(u, z)w', w)_{M} = (-1)^{n} \left< w', Y(\epsilon zL(1)z^{-2L(0)}u, z^{-1})w \right>_{M}
\]
for \(u \in U_{n-1/2} \oplus U_{n}\) with \(n \in \mathbb{Z}\), where \(w' \in M'\), \(w \in M\), and \(\langle \cdot , \cdot \rangle_{M}\) is the natural pairing \(M' \times M \to \mathbb{C}\). As in the untwisted case, we have

**Proposition 2.7.** The map \(Y'\) equips \(M'\) with a structure of canonically twisted \(U\)-module.

### 2.7. Lattice SVOAs

Let \(L\) be a positive definite integral lattice, and recall that \(\mathbb{F}\) denotes \(\mathbb{R}\) or \(\mathbb{C}\). Then the following standard procedure associates to \(L\), an SVOA defined over \(\mathbb{F}\) and we shall denote it by \(\hat{\mathcal{V}}_{L}\). We refer the reader to [FLM88] for more details.

Let \(\hat{\mathfrak{h}} = \mathbb{F} \otimes \mathbb{Z} L\) and let \(\hat{\mathfrak{h}}\) denote the Heisenberg Lie algebra described by
\[
\hat{\mathfrak{h}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{F} t^{n} \oplus \mathbb{F}c, \quad [h(m), h'(n)] = m\langle h, h' \rangle \delta_{m+n,0}, \quad [h(m), c] = 0.
\]
We denote by \(\hat{\mathfrak{h}}\) and \(\hat{\mathfrak{b}}\), the (commutative) subalgebras of \(\hat{\mathfrak{h}}\) given by
\[
\hat{\mathfrak{b}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{F} t^{n} \oplus \mathbb{F}c, \quad \hat{\mathfrak{b}}' = \bigoplus_{n \in \mathbb{Z}_{< 0}} \mathbb{F} t^{n}.
\]
Let \(\hat{L}\) be the unique up to equivalence extension of \(L\) by a group \(\langle \kappa \rangle\) with generator \(\kappa\) of order two, such that the commutators in \(\hat{L}\) satisfy
\[
aba^{-1}b^{-1} = \kappa(\bar{a}, \bar{b}) + \langle \bar{a}, \bar{a} \rangle(\bar{a}, \bar{b})
\]
where \(a \to \bar{a}\) denotes the natural homomorphism \(\hat{L} \to L\). We have the following short exact sequence.
\[
1 \to \langle \kappa \mid \kappa^{2} = 1 \rangle \to \hat{L} \to L \to 1
\]
Let $\mathbb{F}\{L\}$ denote the $\hat{L}$-module obtained by factoring $\mathbb{F}\hat{L}$ by the subalgebra generated by $\kappa + 1$. We write $\iota(a)$ for the image of $a \in \hat{L}$ in $\mathbb{F}\{L\}$ under the composition of maps $\hat{L} \hookrightarrow \mathbb{F}\hat{L} \rightarrow \mathbb{F}\{L\}$. The space $\mathbb{F}\{L\}$ is again an algebra, and is linearly isomorphic to $\mathbb{F}L$. The algebra $\mathbb{F}\{L\}$ may be equipped with a structure of $\mathfrak{g}\mathfrak{b}$-module as follows.

\begin{equation}
(2.7.5) \quad h(m) \cdot \iota(a) = 0 \text{ for } m > 0, \quad h(0) \cdot \iota(a) = \langle h, a \rangle \iota(a), \quad c \cdot \iota(a) = \iota(a).
\end{equation}

As an $\mathfrak{g}\mathfrak{b}$-module, $\mathfrak{g}V_L$ is defined to be that induced from the $\mathfrak{g}\mathfrak{b}$-module structure on $\mathbb{F}\{L\}$.

\begin{equation}
(2.7.6) \quad \mathfrak{g}V_L = U(\mathfrak{g}\hat{h}) \otimes_{U(\mathfrak{g}\hat{b})} \mathbb{F}\{L\}
\end{equation}

We identify $\mathbb{F}\{L\}$ with the subspace $1 \otimes \mathbb{F}\{L\}$ of $\mathfrak{g}V_L$, and we set $1 = 1 \otimes \iota(1)$. There is a natural isomorphism of $\mathfrak{g}\mathfrak{b}'$-modules, $\mathfrak{g}V_L \simeq S(\mathfrak{g}\mathfrak{b}') \otimes \mathbb{F}\{L\}$.

Now we define the vertex operators on $\mathfrak{g}V_L$. Let $h \in \mathfrak{g}h$. We define generating functions $h(z)$ and $l(h, z)$ of operators on $\mathfrak{g}V_L$ by

\begin{equation}
(2.7.7) \quad h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}, \quad l(h, z) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{h(n)}{-n} z^{-n}
\end{equation}

Then for $h \in \mathfrak{g}h$ and $a \in \hat{L}$, the vertex operators associated to $\iota(a) = 1 \otimes \iota(a)$ and $h(-n - 1) = h(-n - 1) \otimes \iota(1)$ are given by

\begin{equation}
(2.7.8) \quad Y(h(-n - 1), z) = D^{(m)}_z h(z), \quad Y(\iota(a), z) = \exp \left( (l(a, z)) a z^{\omega(0)} \right)
\end{equation}

respectively, where the colons denote the Bosonic normal ordering: that all operators $h(m)$ with $m \geq 0$ be commuted to the right of all other operators. The remaining vertex operators are determined by the requirement that $Y$ be a linear map, and that

\begin{equation}
(2.7.9) \quad Y(u_{-1}v, z) = Y(u, z)Y(v, z) : \quad \text{for } u, v \in \mathfrak{g}V_L.
\end{equation}

If $\{h_i\}$ is a basis for $\mathfrak{g}h$ and $\{h'_i\}$ is the dual basis, let $\omega = \frac{1}{2} \sum h_i(1) h'_i(-1)$. Then $\omega$ is independent of the choice of basis, and we have the following

**Theorem 2.8 [FLMSS].** The quadruple $(\mathfrak{g}V_L, Y, 1, \omega)$ is an SVOA and the rank of $\mathfrak{g}V_L$ coincides with the rank of $L$.

We refer to $\mathfrak{g}V_L$ as the SVOA over $\mathbb{F}$ associated to $L$ via the standard construction. The superspace decomposition of $\mathfrak{g}V_L$ is given by $\mathfrak{g}V_L = \mathfrak{g}V_{L_0} \oplus \mathfrak{g}V_{L_1}$ where $L_0$ is the sublattice of $L$ consisting of elements with even norm squared, and $L_1$ is the (unique) coset of $L_0$ in $L$, consisting of elements with odd norm squared.

2.7.1. The involution $\theta$ for $\mathfrak{g}V_L$. The lattice $L$ admits a non-trivial involution that acts by $\alpha \mapsto -\alpha$ for $\alpha \in L$. This involution lifts naturally to automorphisms of $\mathfrak{g}h$ and $\mathfrak{g}b$ and hence to $U(\mathfrak{g}h)$. We denote it by $\theta$. We may define an automorphism of $\mathbb{F}\{L\}$ by

\begin{equation}
(2.7.10) \quad \iota(a) \mapsto (-1)^{(\bar{a}, a)/2+(\bar{a}, a)/2} \iota(a^{-1})
\end{equation}

for $a \in \hat{L}$. We denote it also by $\theta$. Recalling that $\mathfrak{g}V_L$ was constructed as

\begin{equation}
(2.7.11) \quad \mathfrak{g}V_L = U(\mathfrak{g}\hat{h}) \otimes_{\mathfrak{g}\hat{b}} \mathbb{F}\{L\}
\end{equation}

we may define an automorphism of $\mathfrak{g}V_L$ by $\theta \otimes \theta$ where the $\theta$ on the left is that defined for the left tensor factor of $\mathfrak{g}V_L$, and the one on the right is that just defined
on the right tensor factor. Since all these automorphisms may be regarded as lifts of \(-1\), we denote \(\theta \otimes \theta\) also by \(\theta\). Then \(\theta\) is an automorphism of the VOA structure on \(\mathcal{F} V_L\), and we may refer to it as a lift to \(\text{Aut}(\mathcal{F} V_L)\) of the \(-1\) symmetry on \(L\).

**Remark 2.9.** It is a result of DGH\textsuperscript{+} that all lifts of \(-1\) are conjugate under the action of \(\text{Aut}(\mathcal{F} V_L)\).

The space \(\mathcal{F} V_L\) decomposes into eigenspaces for the action of \(\theta\), and we express this decomposition as \(\mathcal{F} V_L = \mathcal{F} V_L^0 \oplus \mathcal{F} V_L^1\) where \(\mathcal{F} V_L^k\) is the eigenspace with eigenvalue \((-1)^k\) for the action of \(\theta\).

### 2.7.2. Real form for \(\mathcal{F} V_L\)

The adjoint operators on \(\mathcal{F} V_L\) determine an invariant bilinear form, which is given by

\[
(u \mid v) = \text{Res}_{z=0} z^{-1}(-1)^n Y(e^{zL(1)} z^{-2L(0)} u, z^{-1} v)
\]

when \(u\) is in \((\mathcal{F} V_L)_n\) or \((\mathcal{F} V_L)_{n-1/2}\) for some \(n \in \mathbb{Z}\). Consider the case that \(\mathbb{F} = \mathbb{R}\). Then the form \(\langle \cdot \mid \cdot \rangle\) is not positive definite on \(\mathbb{R} V_L\). In fact, the form is positive definite on the subspace \(\mathbb{R} V_L^0\), and is negative definite on \(\mathbb{R} V_L^1\). Suppose now that we view \(\mathbb{R} V_L\) as embedded in \(\mathbb{C} V_L\) curtesy of the the natural inclusion \(\mathbb{R} \hookrightarrow \mathbb{C}\). Let us set \(V_L\) to be the \(\mathbb{R}\) subspace of \(\mathbb{C} V_L\) described by

\[
V_L = \mathbb{R} V_L^0 \oplus i \mathbb{R} V_L^1 = \{u + iv \mid u \in \mathbb{R} V_L^0, v \in \mathbb{R} V_L^1\}
\]

Then \(V_L\) closes under the vertex operators \(Y\) associated to \(\mathbb{C} V_L\), and the form \(\langle \cdot \mid \cdot \rangle\) restricts to be positive definite on \(V_L\). From now on we write \(V_L\) for the real VOA with positive definite bilinear form obtained in this way, by restricting the form and vertex operator algebra structure from \(\mathbb{C} V_L\).

### 3. Clifford algebras

The construction of SVOAs that we will use is based on the structure of Clifford algebra modules. In this section we recall some basic properties of Clifford algebras and we exhibit a construction of modules over finite dimensional Clifford algebras using doubly-even linear binary codes (see §1.4). In §3.4 we recall the construction of SVOA module structure on modules over certain infinite dimensional Clifford algebras.

#### 3.1. Clifford algebra structure.

In this section \(\mathbb{F}\) denotes either \(\mathbb{R}\) or \(\mathbb{C}\). For \(u\) an \(\mathbb{F}\)-vector space with non-degenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\) we write \(\text{Cliff}(u)\) for the Clifford algebra over \(\mathbb{F}\) generated by \(u\). More precisely, we set \(\text{Cliff}(u) = T(u)/I(u)\) where \(T(u)\) is the tensor algebra of \(u\) over \(\mathbb{F}\) with unit denoted \(1\), and \(I(u)\) is the ideal of \(T(u)\) generated by all expressions of the form \(u \otimes u + \langle u, u \rangle 1\) for \(u \in u\). The natural algebra structure on \(T(u)\) induces an associative algebra structure on \(\text{Cliff}(u)\). The vector space \(u\) embeds in \(\text{Cliff}(u)\), and when it is convenient we identify \(u\) with its image in \(\text{Cliff}(u)\). We also write \(\alpha\) in place of \(\alpha 1 + I(u) \in \text{Cliff}(u)\) for \(\alpha \in \mathbb{F}\) when no confusion will arise. For \(u \in u\) we have the relation \(u^2 = -|u|^2\) in \(\text{Cliff}(u)\). Polarization of this identity yields \(uv + vu = -2\langle u, v \rangle\) for \(u, v \in u\).

The linear transformation on \(u\) which is \(-1\) times the identity map lifts naturally to \(T(u)\) and preserves \(I(u)\), and hence induces an involution on \(\text{Cliff}(u)\) which we denote by \(\theta\). The map \(\theta\) is often referred to as the *parity involution*. We have \(\theta(u_1 \cdots u_k) = (-1)^k u_k \cdots u_1\) for \(u_1 \cdots u_k \in \text{Cliff}(u)\) with \(u_i \in u\), and we write \(\text{Cliff}(u) = \text{Cliff}(u)^0 \oplus \text{Cliff}(u)^1\) for the decomposition into eigenspaces for \(\theta\). Define a bilinear form on \(\text{Cliff}(u)\), denoted \(\langle \cdot, \cdot \rangle\), by setting \(\langle 1, 1 \rangle = 1\), and requiring that
for $u \in \mathfrak{u}$, the adjoint of left multiplication by $u$ is left multiplication by $-u$. Then the restriction of $\langle \cdot, \cdot \rangle$ to $u$ agrees with the original form on $u$.

\begin{equation}
\langle u, u \rangle = -(1, u^2) = -(1, |u|^2 \mathbf{1}) = |u|^2
\end{equation}

More generally, the adjoint of $u = u_1 \cdots u_k$ for $u_i \in \mathfrak{u}$ is $(-1)^k u_k \cdots u_1$. The main anti-automorphism of $\text{Cliff}(u)$ is the map we denote $\alpha$, which acts by sending $u_1 \cdots u_k$ to $u_k \cdots u_1$ for $u_i \in \mathfrak{u}$.

### 3.2. Spin groups.

Let us write $\text{Cliff}(u)^\times$ for the group of invertible elements in $\text{Cliff}(u)$. For $x \in \text{Cliff}(u)^\times$ and $a \in \text{Cliff}(u)$, we set $x(a) = xax^{-1}$. We will define the Pinor and Spinor groups associated to $u$ slightly differently according as $u$ is real or complex: in the case that $u$ is real, we define the Pinor group $\text{Pin}(u)$ to be the subgroup of $\text{Cliff}(u)^\times$ comprised of elements $x$ such that $x(u) \subset u$ and $\alpha(x)x = \pm 1$; in the case that $u$ is complex we define $\text{Pin}(u)$ to be the set of $x \in \text{Cliff}(u)^\times$ such that $x(u) \subset u$ and $\alpha(x)x = 1$. In both cases we define the Spinor group by setting $\text{Spin}(u) = \text{Pin}(u) \cap \text{Cliff}(u)^0$.

Let $x \in \text{Pin}(u)$. Then we have $\langle x(u), x(v) \rangle = \langle u, v \rangle$ for $u, v \in \mathfrak{u}$, and thus the map $x \mapsto x(\cdot)$, which has kernel $\pm \mathbf{1}$, realizes the Pinor group as a double cover of $O(u)$. (If $u \in \mathfrak{u}$ and $\langle u, u \rangle = 1$, then $x(\cdot)$ is the orthogonal transformation of $u$ which is minus the reflection in the hyperplane orthogonal to $u$.) The image of $\text{Spin}(u)$ under the map $x \mapsto x(\cdot)$ is just the group $SO(u)$.

In the case that $u$ is real with definite bilinear form, we have $\alpha(x)x = 1$ for all $x \in \text{Spin}(u)$, and the group $\text{Spin}(u)$ is generated by the (well-defined) expressions $\exp(\lambda e_i e_j) \in \text{Cliff}(u)^\times$ for $\lambda \in \mathbb{R}$ and $\{e_i\}$ an orthonormal basis of $u$. The Spinor group of the complexified space $\mathfrak{c} u$ is then generated by the $\exp(\lambda e_i e_j)$ for $\lambda \in \mathbb{C}$.

### 3.3. Clifford algebra modules.

For the remainder of this section we suppose that $u$ is a finite dimensional real vector space with positive definite symmetric bilinear form, and also that $\Sigma = \{e_i\}_{i \in \Sigma}$ is an orthonormal basis for $u$, indexed by a finite set $\Sigma$. For $S = (s_1, \ldots, s_k) \in \Sigma^k$, a $k$-tuple of elements from $\Sigma$ for any $k$, we write $e_S$ for the element $e_{s_1} e_{s_2} \cdots e_{s_k}$ in $\text{Cliff}(u)$. We suppose that $\Sigma$ is equipped with some ordering. If $S = \{s_1, \ldots, s_k\}$ is a subset of $\Sigma$ (that is, an unordered subset), we denote by $\hat{S}$ the $k$-tuple given by $\hat{S} = (s_1, \ldots, s_k)$ just when $s_1 < \cdots < s_k$, so that $e_{\hat{S}} = e_{s_1} \cdots e_{s_k}$. We then abuse notation to write $e_S$ for $e_{\hat{S}}$.

In this way we obtain an element $e_S$ in $\text{Cliff}(u)$ for any $S \subset \Sigma$. (We set $e_{\emptyset} = \mathbf{1}$.)

This correspondence depends on the choice of ordering, but our discussion will be invariant with respect to this choice. Note that $e_{S+R} = \pm e_{S+R}$ for any $S, R \subset \Sigma$, and the set $\{e_S \mid S \subset \Sigma\}$ furnishes an orthonormal basis for $\text{Cliff}(u)$.

We now obtain an $\mathbb{F}_2^\Sigma$-grading on $\text{Cliff}(u)$ by decreeing that for $S \subset \Sigma$, the homogeneous subspace of $\text{Cliff}(u)$ with degree $S$ is just the $\mathbb{F}$-span of the vector $e_S$.

\begin{equation}
\text{Cliff}(u) = \bigoplus_{S \subset \Sigma} \text{Cliff}(u)^S, \quad \text{Cliff}(u)^S = \mathbb{F} e_S, \quad \text{Cliff}(u)^S \text{Cliff}(u)^R \subset \text{Cliff}(u)^{S+R}.
\end{equation}

Since this grading depends on the choice of orthonormal basis $\mathcal{E}$, we will refer to it as the $\mathbb{F}_2^\Sigma$-grading, and we refer to the homogeneous elements $e_S$ as $\mathbb{F}_2$-homogeneous elements. A given subset of $\text{Cliff}(u)$ is called $\mathbb{F}_2$-homogeneous if all of its elements are $\mathbb{F}_2$-homogeneous.

Suppose that $E$ is an $\mathbb{F}_2$-homogeneous subgroup of $\text{Spin}(u)$. Then $E$ is a union of elements of the form $\pm e_C$ for $C \subset \Sigma$, and hence is finite, and has exponent four.
Furthermore, the set of $C$ for which $+e_C$ is in $E$ determines a linear binary code on $\Sigma$. For $E$ an $F_2^n$-homogeneous subgroup of $\text{Spin}(u)$, we write $\mathcal{C}(E)$ for the associated linear binary code on $\Sigma$. The following result is straightforward.

**Proposition 3.1.** Suppose that $-1 \notin E$ and $\mathcal{C}(E)$ is a doubly-even code. Then the map $E \to \mathcal{C}(E)$ such that $+e_S \mapsto S$ is an isomorphism of abelian groups, and the sub-algebra of $\text{Cliff}(u)$ generated by $E$ is naturally isomorphic to the group algebra $\mathbb{R}E$.

Suppose now that $E$ is an $F_2^n$-homogeneous subgroup of $\text{Spin}(u)$ such that $-1 \notin E$ and $\mathcal{C}(E)$ is a self-dual doubly-even code. (Note that this forces dim$(u)$ to be a multiple of eight.) Then we write $\text{CM}(E)$ for the Clifford module defined by $\text{CM}(E)_E = \text{Cliff}(u) \otimes_{\mathbb{R}E} \mathbb{R}_1$ where $\mathbb{R}_1$ denotes a trivial $E$-module. Let us set $1_E = 1 \otimes 1 \in \text{CM}(E)_E$. Then $\text{CM}(E)_E$ admits a bilinear form defined so that $\langle 1_E, 1_E \rangle = 1$, and the adjoint to left multiplication by $u \in u \mapsto \text{Cliff}(u)$ is left multiplication by $-u$.

**Proposition 3.2.** The Clifford module $\text{CM}(u)_E$ is irreducible, and a vector-space basis for $\text{CM}(u)_E$ is naturally indexed by the elements of the co-code $\mathcal{C}(E)^\ast$. The bilinear form on $\text{CM}(u)_E$ is non-degenerate.

**Proof.** We have $e_{S+C}1 = \pm e_S1$ for any $S \subseteq \Sigma$ when $C \in \mathcal{C}(E)$. This shows that a basis for $\text{CM}(u)_E$ is indexed by the elements of the co-code $\mathcal{C}(E)^\ast = P(\Sigma)/\mathcal{C}(E)$, and it follows that the irreducible submodules of $\text{CM}(u)_E$ are indexed by the cosets of $\mathcal{C}(E)$ in its dual code $\mathcal{C}(E)^\perp = \{ S \in P(\Sigma) \mid |S \cap C| \equiv 0 \pmod{2}, \forall C \in \mathcal{C}(E) \}$.

Since $\mathcal{C}(E)$ is self-dual, $\text{CM}(u)_E$ is irreducible. \hfill \Box

Note that the vector $1_E \in \text{CM}(u)_E$ is such that $g1_E = 1_E$ for all $g \in E$, and $\text{CM}(u)_E$ is spanned by the $a1_E$ for $a \in \text{Cliff}(u)$. We have the following

**Proposition 3.3.** Suppose that $\text{CM}(u)_0$ is a non-trivial irreducible Clifford module with a vector $1_0$ such that $g1_0 = 1_0$ for all $g \in E$. Then $\text{CM}(u)_0$ is equivalent to $\text{CM}(u)_E$, and a module equivalence is furnished by the map $\phi : \text{CM}(u)_E \to \text{CM}(u)_0$ defined so that $\phi(a1_E) = a1_0$ for $a \in \text{Cliff}(u)$.

**Proof.** Recall our assumption that $\mathcal{C}(E)$ is self-dual so that $2d = \text{dim}(u)$ is divisible by eight, and $|\mathcal{C}(E)| = 2^d$. In this case we have that $\text{Cliff}(u)$ is isomorphic to $M_{2d}(\mathbb{R})$ so that there is a unique non-trivial irreducible Clifford module up to equivalence and it has dimension $2^d$. It suffices then to show that $\phi$ is well defined, and this follows from the universal mapping property of the induced module $\text{CM}(u)_E$. \hfill \Box

We will sometimes wish to replace $u$ with its complexification $\mathbb{C}u$ in the above, in which case we shall understand $\text{CM}(\mathbb{C}u)_E$ to be the complexification $\mathbb{C}\text{CM}(u)_E$ of $\text{CM}(u)_E$. Then $\text{CM}(\mathbb{C}u)_E$ is an irreducible module over $\text{Cliff}(\mathbb{C}u)$.

### 3.4. Clifford module construction of SVOAs

Let $u$ be a finite dimensional vector space over $\mathbb{R}$ with positive definite symmetric bilinear form. In this section we review the construction of SVOA structure on modules over certain infinite dimensional Clifford algebras associated to $u$. The construction is quite standard and one may refer to [FPR91] for example, for full details. Our setup is somewhat different from that in [FPR91] in that we prefer to be able to work over $\mathbb{R}$, and we must therefore use an alternative construction of the canonically twisted SVOA module, since a polarization of $u$ does not exist in this case. On the other hand, all
one requires is an irreducible module over the (finite dimensional) Clifford algebra \( \text{Cliff}(u) \), and the arguments of \[ \text{FPR91} \] then go through with only cosmetic changes.

So that the reader can translate between this section and the exposition in \[ \text{FPR91} \], we recall that a say, is a complex vector space with non-degenerate bilinear form \( \langle \cdot, \cdot \rangle_0 \), and a polarization \( a = a^+ + a^- \) such that \( a^\pm \) is spanned by vectors \( a_i^\pm \), which satisfy \( \langle a_i^+, a_j^- \rangle_0 = \delta_{ij} \).

They consider the Clifford algebra \( \text{Cliff}_0(a) \) defined by \( \text{Cliff}_0(a) = T(a)/I_0(a) \) where \( I_0(a) \) is the ideal spanned by elements of the form \( uv + vu \) for \( u, v \in \text{Cliff}_0 \). In this section we take \( u \) to be a real vector space of even dimension with positive definite bilinear form and orthonormal basis \( \{ e_i \} \). Let \( d = \dim(u)/2 \), and in the complexification \( \mathbb{C}u \) of \( u \) set \( a^\pm_j = \frac{1}{2}(ie_j \mp e_{j+d}) \). Then we have an identification of vector spaces \( a = \mathbb{C}u \).

We also have \( \langle a_i^+, a_j^- \rangle_0 = -\frac{1}{2} \delta_{ij} \) so that \( \{ a_i^+, a_j^- \} = -2 \langle a_i^+, a_j^- \rangle_0 = \delta_{ij} \) in \( \text{Cliff}(\mathbb{C}u) \), and setting \( \langle \cdot, \cdot \rangle_0 = -2\langle \cdot, \cdot \rangle_0 \) we have an identification of algebras \( \text{Cliff}_0(a) = \text{Cliff}(\mathbb{C}u) \).

We now proceed with the construction. We assume that \( u \) admits an orthonormal basis \( \{ e_i \}_{i \in \Sigma} \) indexed by a finite set \( \Sigma \). For simplicity we suppose that the dimension of \( u \) is divisible by eight so that maximal self-orthogonal doubly-even codes on \( \Sigma \) are self-dual. Let \( u_\theta \) denote the infinite dimensional inner product spaces described as follows.

\[
\text{(3.4.1)}
\quad u = \bigoplus_{m \in \mathbb{Z}} u \otimes t^{m+1/2}, \quad u_\theta = \bigoplus_{m \in \mathbb{Z}} u \otimes t^m,
\]

\[
\text{(3.4.2)}
\quad \langle u \otimes t^r, v \otimes t^s \rangle = \langle u, v \rangle \delta_{r+s,0}, \quad \text{for } u, v \in u \text{ and } r, s \in \frac{1}{2} \mathbb{Z}.
\]

We write \( u(r) \) for \( u \otimes t^r \) when \( u \in u \) and \( r \in \frac{1}{2} \mathbb{Z} \). We consider the Clifford algebras \( \text{Cliff}(u) \) and \( \text{Cliff}(u_\theta) \). The inclusion of \( u \) in \( u_\theta \) given by \( u \mapsto u(0) \) induces an embedding of algebras \( \text{Cliff}(u) \hookrightarrow \text{Cliff}(u_\theta) \). For \( S = (i_1, \ldots, i_k) \) an ordered subset of \( \Sigma \) we write \( e_S(r) \) for the element \( e_{i_1}(r) \cdots e_{i_k}(r) \), which lies in \( \text{Cliff}(u) \) or \( \text{Cliff}(u_\theta) \) according as \( r \) is in \( \mathbb{Z} + \frac{1}{2} \mathbb{Z} \) or \( \mathbb{Z} \). With this notation \( e_S(0) \) coincides with the image of \( e_S \) under the embedding \( \text{Cliff}(u) \hookrightarrow \text{Cliff}(u_\theta) \). Let \( E \) be an \( \mathbb{F}_2 \)-homogeneous subgroup of \( \text{Cliff}(u)^{\times} \) such that \( C(E) \) is a self-dual doubly-even code on \( \Sigma \).

We write \( B(u) \) for the subalgebra of \( \text{Cliff}(u) \) generated by the \( u(m + \frac{1}{2}) \) for \( u \in u \) and \( m \in \mathbb{Z}_{>0} \). We write \( B(u_\theta)_E \) for the subalgebra of \( \text{Cliff}(u_\theta) \) generated by \( E \subset \text{Cliff}(u) \), and the \( u(m) \) for \( u \in u \) and \( m \in \mathbb{Z}_{>0} \). Let \( \mathbb{R}_1 \) denote a one-dimensional module for either \( B(u) \) or \( B(u_\theta)_E \), spanned by a vector \( 1_E \), such that \( u(r)1_E = 0 \) whenever \( r \in \frac{1}{2} \mathbb{Z}_{>0} \), and such that \( g(0)1_E = 1_E \) for \( g \in E \). We write \( A(u) \) (respectively \( A(u)_E \)) for the \( \text{Cliff}(u) \)-module (respectively \( \text{Cliff}(u_\theta)_E \)-module) induced from the \( B(u) \)-module structure (respectively \( B(u_\theta)_E \)-module structure) on \( \mathbb{R}_1 \).

\[
\text{(3.4.3)}
\quad A(u) = \text{Cliff}(u) \otimes_{B(u)} \mathbb{R}_1, \quad A(u)_E = \text{Cliff}(u_\theta) \otimes_{B(u_\theta)_E} \mathbb{R}_1.
\]

We write \( 1 \) for 1 \( \otimes 1_E \in A(u) \), and we write \( 1_\theta \) for 1 \( \otimes 1_E \in A(u)_E \). When no confusion will arise, we simply write \( A(u)_\theta \) in place of \( A(u)_E \).

The space \( A(u) \) supports a structure of SVOA. In order to define the vertex operators we require the notion of fermionic normal ordering for elements in \( \text{Cliff}(u) \) and \( \text{Cliff}(u_\theta) \). The fermionic normal ordering on \( \text{Cliff}(u) \) is the multi-linear operator defined so that for \( u_i \in u \) and \( r_i \in \mathbb{Z} + \frac{1}{2} \) we have

\[
\text{(3.4.4)}
\quad u_1(r_1) \cdots u_k(r_k) := -\text{sgn}(\sigma)u_{\sigma 1}(r_{\sigma 1}) \cdots u_{\sigma_k}(r_{\sigma_k})
\]
Finally, for arbitrary elements in \( \text{Cliff}(\hat{\lambda}) \) the fermionic normal ordering is defined in steps by first setting

\[
(3.4.5) \quad u_i(0) \cdot \cdots \cdot u_k(0) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) u_{\sigma 1}(0) \cdots u_{\sigma k}(0)
\]

for \( u_i \in u \). Then in the situation that \( n_i \in \mathbb{Z} \) are such that \( n_i \leq n_{i+1} \) for all \( i \), and there are some \( s \) and \( t \) (with \( 1 \leq s \leq t \leq k \)) such that \( n_j = 0 \) for \( s \leq j \leq t \), we set

\[
(3.4.6) \quad u_1(n_1) \cdots u_k(n_k) := u_1(n_1) \cdots u_{s-1}(n_{s-1}) : u_s(0) \cdots u_t(0) : u_{t+1}(n_{t+1}) \cdots u_k(n_k)
\]

Finally, for arbitrary \( n_i \in \mathbb{Z} \) we set

\[
(3.4.7) \quad u_1(n_1) \cdots u_k(n_k) := \text{sgn}(\sigma) : u_{\sigma 1}(n_1) \cdots u_{\sigma k}(n_{\sigma k}) :
\]

where \( \sigma \) is again any permutation of the index set \( \{1, \ldots, k\} \) such that \( n_{\sigma 1} \leq \cdots \leq n_{\sigma k} \), and we extend the definition multi-linearly to \( \text{Cliff}(\hat{\lambda}) \).

Now for \( u \in u \) we define the generating function, denoted \( u(z) \), of operators on \( A(u) = A(u) \oplus A(u)_{\omega} \) by \( u(z) = \sum_{r \in \mathbb{Z}} u(r)z^{-r-1/2} \). Note that \( u(r) \) acts as 0 on \( A(u) \) if \( r \in \mathbb{Z} \), and acts as 0 on \( A(u)_{\omega} \) if \( r \in \mathbb{Z} + \frac{1}{2} \). To an element \( a \in A(u) \) of the form \( a = u_1(-m_1 - \frac{1}{2}) \cdots u_k(-m_k - \frac{1}{2}) \mathbb{1} \) for \( u_i \in u \) and \( m_i \in \mathbb{Z}_{\geq 0} \), we associate the operator valued power series \( \mathcal{Y}(a, z) \), given by

\[
(3.4.8) \quad \mathcal{Y}(a, z) =: D_z^{(m_1)}u_i(z) \cdots D_z^{(m_k)}u_k(z):
\]

We define the vertex operator correspondence

\[
(3.4.9) \quad Y(\cdot, z) : A(u) \otimes A(u)_{\omega} \to A(u)_{\omega}(\langle z^{1/2} \rangle)
\]

by setting \( Y(a, z)b = \mathcal{Y}(a, z)b \) when \( b \in A(u) \), and by setting \( Y(a, z)b = \mathcal{Y}(e^{\Delta_z}a, z)b \) when \( b \in A(u)_{\omega} \), where \( \Delta_z \) is the expression defined by

\[
(3.4.10) \quad \Delta_z = \frac{1}{4} \sum_{m, n \in \mathbb{Z}_{\geq 0}} C_m e_i(m + \frac{1}{2})e_i(n + \frac{1}{2})z^{-m-n-1}
\]

\[
(3.4.11) \quad C_{mn} = \frac{1}{2} \frac{(m-n)}{m+n+1} \left( \frac{\frac{1}{2}}{m} \right) \left( \frac{-1}{n} \right)
\]

Set \( \omega = -\frac{1}{4} \sum_{i} e_i(-\frac{1}{2})e_i(-\frac{1}{2}) \mathbb{1} \in A(u)_{\omega} \). Then one has the following

**Theorem 3.4 (FFR91).** The map \( Y \) defines a structure of self-dual rational SVOA on \( A(u) \) when restricted to \( A(u) \oplus A(u) \). The Virasoro element is given by \( \omega \), and the rank is \( \frac{1}{2} \dim(u) \). The map \( Y \) defines a structure of \( \theta \)-twisted \( A(u) \)-module on \( A(u)_{\omega} \) when restricted to \( A(u) \oplus A(u)_{\omega} \).

The definition of \( Y(a, z)b \) for \( b \in A(u)_{\omega} \) appears quite complicated, but all we need to know about these operators is contained in the following

**Proposition 3.5.** Let \( b \in A(u)_{\omega} \).

1. If \( a = \mathbb{1} \in A(u)_{\omega} \) then \( Y(a, z)b = b \).
2. If \( a \in A(u)_{\omega} \) then \( \Delta_z a = 0 \) so that \( Y(a, z)b = \mathcal{Y}(a, z)b \).
3. If \( a \in A(u)_{\omega} \) then \( \Delta_z a = 0 \) and \( Y(a, z)b = \mathcal{Y}(a, z)b \) unless \( a \) has non-trivial projection onto the span of the vectors \( e_i(-\frac{1}{2})e_i(-\frac{1}{2}) \mathbb{1} \) for \( i \in \Omega \).
4. For \( a = e_i(-\frac{1}{2})e_i(-\frac{1}{2}) \mathbb{1} \) we have \( \Delta_z a = -\frac{1}{4}z^{-2} \Delta_z^2 a = 0 \) so that \( Y(a, z)b = \mathcal{Y}(a, z)b - \frac{1}{4}bz^{-2} \) in this case.
As a corollary of Proposition 3.4.12 we have that \( Y(\omega, z)1_\theta = \frac{1}{16}\dim(u)1_\theta z^{-2} \), and consequently the \( L(0) \) grading on \( A(u)_\Theta \) is given by
\[
(3.4.12) \quad A(u) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A(u)_n, \quad A(u)_\theta = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (A(u)_\theta)_{h+n},
\]
where \( h = \frac{1}{16}\dim(u) \). Given a specific choice of \( E \), the embedding of \( \text{Cliff}(u) \) in \( \text{Cliff}(u_\theta) \) gives rise to an isomorphism of \( \text{CM}(u)_E \) with \( (A(u)_\theta)_E)_{h_n} \), and it will be convenient to consider these spaces as identified. We may have occasion to replace \( u \) by its complexification \( cu \) in the above; in such a situation we shall understand \( A(cu)_\Theta \) to be the complexified space \( cA(u)_\Theta \).

4. Structure of \( A^{f_2} \)

4.1. Construction. Suppose that \( \Omega \) is some finite set with cardinality 24 and an arbitrary ordering. Let \( I \) be a 24 dimensional vector space over \( \mathbb{R} \) with positive definite bilinear form, and let \( E = \{ e_i \}_{i \in \Omega} \) be an orthonormal basis for \( I \). The goal of this section is to show that the space \( A^{f_2} \) given by
\[
(4.1.1) \quad A^{f_2} = A(0)^0 \oplus A(0)_\theta^0
\]
adopts a structure of self-dual rational \( N = 1 \) SVOA.

Let \( G \subset P(\Omega) \) be a copy of the Golay code. Let \( G \) be an \( \mathbb{F}_2^I \)-homogeneous subgroup of \( \text{Spin}(I) \) such that \( G \) does not contain \(-1 \in \text{Spin}(I) \) and the associated code \( C(G) \) is \( G \). Curtesy of \( \mathbb{F} \) we have the \( \text{Cliff}(I)_G = \text{Cliff}(I) \otimes_{\mathbb{R}} \mathbb{F} \), and this space can be used to give an explicit realization of the \( A(I)_G \)-module \( A(I)_{\theta} \). From now on we set \( A(I)_\theta = A(I)^0_G \). Notice that the odd parity subspace of \( A^{f_2} \) is precisely \( A(I)^0_\theta \). Since \( I \) has dimension 24, the \( L(0) \)-homogeneous subspace of \( A(0)^0 \) with minimal degree is \( (A(I)^0_\theta)_3/2 \), and this space is identified with the \( \text{Cliff}(I)_G \)-module \( C(G) \). Note that \( 1_\theta \leftrightarrow 1_G \) under this identification. Also, the bilinear form on \( A^{f_2} \) coincides with that on \( \text{Cliff}(I)_G \) when restricted to \( (A^{f_2})^3/2 \), and in particular, is normalized so that \( (1_\theta|1_\theta) = 1 \).

We require a vertex operator correspondence \( Y: A^{f_2} \otimes A^{f_2} \rightarrow A^{f_2}((z)) \) and as yet this map is defined only on \( (A^{f_2})^0 \otimes (A^{f_2})^0 \) and on \( (A^{f_2})^0 \otimes (A^{f_2})_1 \). Such a map \( Y \) must satisfy skew-symmetry if it exists, so for \( u \otimes v \in (A^{f_2})^0_1 \otimes (A^{f_2})_0 \) we define \( Y(u, z)v \) by
\[
(4.1.2) \quad Y(u, z)v = e^{zL(-1)}Y(v, -z)u
\]
(since \( |u|v| = 0 \) in this case). Suppose now that \( u \otimes v \in (A^{f_2})^1_1 \otimes (A^{f_2})_1 \). Motivated by \( \[2\] \) we define \( Y(u, z)v \) by requiring that for any \( w \in (A^{f_2})_0 \) we should have
\[
(4.1.3) \quad \langle Y(u, z)v \mid w \rangle = (-1)^n \langle v \mid Y(e^{z(L-1)}(z^{-2L+0})u, z^{-1})w \rangle
\]
whenever \( u \in (A^{f_2})_{n-1/2} \) for \( n \in \mathbb{Z} \). Now the operator on the right of \( \[4\] \) is defined by \( \[5\] \). We can use this later expression to rewrite \( \[5\] \) in terms of the operator \( Y \) defined on \( (A^{f_2})^0_1 \otimes A^{f_2} \) in \( \[4\] \) and doing so we obtain the following convenient working definition for the operator \( Y \) on \( (A^{f_2})^0_1 \otimes (A^{f_2})^1_1 \). For \( u \in (A^{f_2})_{n-1/2} \) with \( n \in \mathbb{Z} \), for \( v \in (A^{f_2})^1_1 \) and \( w \in (A^{f_2})^0_1 \) we have
\[
(4.1.4) \quad \langle Y(u, z)v \mid w \rangle = (-1)^n \langle e^{-zL(1)}v \mid Y(w, -z^{-1})e^{-z(L)(z^{-2L+0})u} \rangle
\]

**Proposition 4.1.** The map \( Y: A^{f_2} \otimes A^{f_2} \rightarrow A^{f_2}((z)) \) defines a structure of rank 12 self-dual rational SVOA on \( A^{f_2} \).
Proof. Let \( \mathbb{C}A^{f_2} \) denote the complexification of \( A^{f_2} \). Then \( (\mathbb{C}A^{f_2})_\bar{0} \) is a simple VOA of rank 12, and \( (\mathbb{C}A^{f_2})_1 \) is an irreducible module over \( (\mathbb{C}A^{f_2})_\bar{0} \). Let us write \( Y_{k,l} \) for the restriction of \( Y \) to \( (\mathbb{C}A^{f_2})_k \otimes (\mathbb{C}A^{f_2})_l \) for \( k,l \in \{0,1\} \).

By the Boson-Fermion correspondence [Pre81] (see also [DM94]) we have that \( (\mathbb{C}A^{f_2})_\bar{0} \) is isomorphic to a lattice VOA \( \mathbb{C}V_{M_0} \) where \( M_0 \) is an even lattice of type \( D_{12} \). The irreducible modules over a lattice VOA \( \mathbb{C}V_L \) for \( L \) an even lattice are known to be indexed by the cosets of \( L \) in its dual \( L^* = \{ u \in \mathbb{R} \otimes \mathbb{Z} \mid \langle u, L \rangle \subset \mathbb{Z} \} \) [Don93]. In particular, a lattice VOA is rational. Further, it is known that the fusion algebra associated to the modules over \( \mathbb{C}V_L \) coincides with the group algebra of \( L^*/L \) in the natural way. (One may refer to [DL93] for a thorough treatment.)

In the case that \( L = M_0 \), there are exactly three non-trivial cosets, and for any one of these \( M_0 + \mu \) say, the set \( M_0 \cup (M_0 + \mu) \) forms an integral lattice in \( \mathbb{R}L = \mathbb{R} \otimes \mathbb{Z} \), and in particular, \( M_0^*/M_0 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) has exponent two.

From this we conclude that \( \mathbb{C}A^{f_2} \) is isomorphic to \( \mathbb{C}V_M \) for \( M = M_0 \cup (M_0 + \mu) \) for some \( \mu \in M_0^* \setminus M_0 \), that there exist unique up to scale intertwiners of types \( (\frac{1}{10}) = (M_0 + \mu, M_0) \) and \( (\frac{0}{3}) = (M_0 + \mu, M_0 + \mu) \) for \( \mathbb{C}V_{M_0} \), and that these intertwiners are just those obtained by restricting the SVOA structure on \( \mathbb{C}V_M \). In particular, there is a unique structure of rank 12 SVOA on \( \mathbb{C}A^{f_2} \). On the other hand it is known from [DM98] for example, that the maps \( Y_{10} \) and \( Y_{11} \) defined by equations \( 4.1.2 \) and \( 4.1.4 \), respectively, yield intertwiners of types \( (\frac{1}{10}) = (M_0 + \mu, M_0) \) and \( (\frac{0}{3}) = (M_0 + \mu, M_0 + \mu) \), respectively, for \( (\mathbb{C}A^{f_2})_\bar{0} \). By uniqueness, they must coincide with those inherited from the SVOA structure on \( \mathbb{C}V_M \) up to some scalar factors, and in any case the map \( Y \) defined above furnishes \( \mathbb{C}A^{f_2} \) with a structure of rational rank 12 SVOA. We have chosen scalars in such a way that \( A^{f_2} \) is a real form for \( \mathbb{C}A^{f_2} \). One can check directly that \( M \) is a self-dual lattice (given that \( (A^{f_2})_1/2 \) is trivial, \( M \) must be a copy of the \( D_{12}^+ \) lattice — the unique self-dual lattice of rank 12 with no vectors of unit norm [CS93 Ch.19]), and it follows from the above that \( \mathbb{C}V_M \) and \( \mathbb{C}A^{f_2} \) are then self-dual SVOAs. This completes the proof of the proposition. \[ \square \]

Remark 4.2. The method used here to extend the vertex operator map from a VOA to the sum of the VOA and a module over it was given earlier in [Hua96].

The following proposition gives a convenient criterion for when a vector in \( (A^{f_2})_{3/2} \) is superconformal. In section 4 it will be shown that all such vectors are equivalent up to the action of \( \text{Spin}(l) \).

Proposition 4.3. Suppose that \( t \in (A^{f_2})_{3/2} \) is such that \( \langle t|t \rangle = 8 \) and \( \langle e_0(0)|t|t \rangle = 0 \) whenever \( C \subset \Omega \) has cardinality two or four. Then \( t \) is a superconformal vector for \( A^{f_2} \).

Proof. We should compute \( t_n t \) for \( n = 0, 1 \) and \( n = 2 \), and then compare with the results of Proposition 4.4. Using \( 4.1.4 \) and recalling \( t \in (A^{f_2})_{2-1/2} \) we find that for arbitrary \( u \in A^{f_2} \) we have

\[
\langle u|Y(t,z)t \rangle = \langle Y(u,-z^{-1})e^{zL(1)}z^{-2L(0)}t|e^{z^{-1}L(1)}t \rangle = \langle Y(u,-z^{-1})t|t \rangle z^{-3}
\]

Then for \( t_n t \) we obtain

\[
\langle u|t_n t \rangle = \text{Res}_{z=0} \langle Y(u,-z^{-1})t|t \rangle z^{n-3} = \langle u_{-n+1}t|t \rangle (-1)^{n-2}
\]
For \( L(0) \)-homogeneous \( u \in A^{f_2} \) the expression \( \mathbf{e}^{11,0} \) is zero unless \( u \in (A^{f_2})_{-n+2} \). In order to determine \( t_n t \), we should compute \( u_m t \) for various \( u \in (A^{f_2})_0 \), and apply the equation \( (1.1.0) \). To compute \( u_m t \) we will use the results of Proposition 3.5.

For \( n = 2 \) the expression \( \mathbf{e}^{11,0} \) is zero unless \( u \in (A^{f_2})_0 = \mathbb{R}1 \), in which case we obtain \( \langle 1|2t \rangle = \langle 1|t \rangle \). Since \( Y(1, z) = 1 \) and \( \langle 1|1 \rangle = 1 \), we find that \( t = (t|t)1 \).

For the case that \( n = 1 \) we should consider the \( u \) in \( (A^{f_2})_1 \). Suppose \( u = e_i(-\frac{1}{2})e_j(-\frac{1}{2})1 \) for \( i \neq j \in \Omega \). Then

\[
(4.1.7) \quad Y(u, z)t = \overline{Y}(u, z)t = e_{ij}(0)tz^{-1} + \ldots
\]

By hypothesis we have that \( e_{ij}(0)t \) is orthogonal to \( t \), so \( t_1t = 0 \).

Finally, when \( n = 0 \) we are concerned with \( u_1t \) for \( u \in (A^{f_2})_2 \). The space \( (A^{f_2})_2 \) is spanned by the vectors of the form \( e_i(-\frac{3}{2})e_j(-\frac{1}{2})1 \) for \( i, j \in \Omega \), and also by the \( e_C(-\frac{1}{2})1 \) for \( C = \{i_1, i_2, i_3, i_4\} \subset \Omega \). For \( u \) one of these vectors we have \( Y(u, z)t = \overline{Y}(u, z)t = e_C(0)tz^{-2} + \ldots \) for \( C \subset \Omega \) of size two or four unless \( u = e_i(-\frac{3}{2})e_i(-\frac{1}{2})1 \). In the former cases, \( u_1t \) is orthogonal to \( t \) by hypothesis, and in the later case we have

\[
(4.1.8) \quad Y(u, z)t = \overline{Y}(u, z)t - \frac{1}{4}tz^{-2} = -\frac{1}{t}tz^{-2} + \ldots
\]

The expression \( \mathbf{e}^{11,0} \) now reduces to \( \langle u|t_0t \rangle = -\frac{1}{4}(t|t) = -2 \), and we conclude that \( t_0t = -\frac{1}{4} \sum_\Omega e_i(-\frac{3}{2})e_i(-\frac{1}{2})1 \) since we have \( \langle u|u \rangle = 4 \) for \( u = e_i(-\frac{3}{2})e_i(-\frac{1}{2})1 \) for any \( i \in \Omega \).

We have verified that \( t_0t = 8 \), \( t_2t = 0 \) and \( t_0t = 2\omega \). Since the rank of \( A^{f_2} \) is 12 and \( 8 = \frac{2}{3}12 \), an application of Proposition 2.4 confirms that \( t \) is superconformal for \( A^{f_2} \).

Set \( \tau_A = \sqrt{3}1_{\theta} \in (A^{f_2})_{3/2} \). Then \( \langle \tau_A|\tau_A \rangle = 8 \), and we have

**Corollary 4.4.** The vector \( \tau_A \) is a superconformal vector for \( A^{f_2} \).

**Proof.** That \( \tau_A \) satisfies the hypotheses of Proposition 4.3 is a consequence of the fact that the Golay code \( G = C(G) \) has minimum weight 8. \( \Box \)

We record the results of Proposition 1.1 and Corollary 4.4 in the following

**Theorem 4.5.** The quadruple \((A^{f_2}, Y, 1, \tau_A)\) is a self-dual rational \( N = 1 \) SVOA.

### 4.2. Symmetries.

In this section we show that the automorphism group of the \( N = 1 \) SVOA structure on \( A^{f_2} \) is isomorphic to Conway’s largest sporadic group, \( Co_1 \).

The operators \( x_0 \) for \( x \in A(1)_1 \) of the Lie algebra of type \( D_{12} \) in \( \text{End}(A(1)_1) \), and the exponentials \( \exp(x_0) \) for \( x \in A(1)_1 \) generate a group \( S \) say, which acts as \( A(1)_0 \)-module automorphisms of \( A(1)_9 = A(1) \oplus A(1)_9 \). This group \( S \) is isomorphic to the group \( \text{Spin}(l) \), and we may choose the isomorphism so that \( \exp(x_0) \) in \( S \) corresponds to \( \exp(\frac{a}{b}(ab - ba)) \) in \( \text{Spin}(l) \subset \text{Cliff}(l)^\times \) when \( x = a(-\frac{1}{2})b(-\frac{1}{2})1 \in A(1)_1 \) for some \( a, b \in l \). The action of \( S \) on \( A(1)_9 \) commutes with the action of \( \theta \), and so preserves the subspace \( A^{f_2} = A(1)_9 \oplus A(1)_9 \). The kernel of this action is the group of order 2 generated by \( \theta \). Taking the complexification \( c \) in place of \( l \) in the above we obtain an action of the complex Lie group \( \text{Spin}(c) \) on the complexified SVOA \( cA^{f_2} = A(c)_0 \oplus A(c)_0 \). Let us write \( cS \) for this copy of \( \text{Spin}(c) \) generated by exponentials \( \exp(x_0) \) with \( x \in A(c)_1 \).
Proposition 4.6. The group $cS$ maps surjectively onto the group of SVOA automorphisms of $cA^{f_2}$.

Proof. From the proof of Proposition 4.4, we may regard $cA^{f_2}$ as the (complex) lattice SVOA $cV_M$ where $M$ is a lattice of type $D_{12}^2$. Let us write $M^0$ for the even sublattice of $M$ (of type $D_{12}$), and $M^1$ for the unique coset of $M^0$ in $M$; we may write $cV_{M^0} \oplus cV_{M^1}$ for the superspace decomposition of $cV_M$. Let us write $G$ (not to be confused with the $G$ of Proposition 4.4) for the group of SVOA automorphisms of $cV_M$, and $G^0$ for the group of VOA automorphisms of $cV_{M^0}$. Let $S$ denote the image of $cS$ in $G = \text{Aut}_{\text{SVOA}}(cA^{f_2})$. We wish to show that $S = G$.

Any element of $G$ preserves the superspace structure on $cV_M$, so we have a natural map $\phi : G \to G^0$. By a similar token, any element of $G^0$ preserves the Lie algebra structure on the degree 1 subspace of $cV_{M^0}$ (we denote this Lie algebra by $g$) so we have also a natural map $G^0 \to \text{Aut}(g)$. In fact, this map is faithful and onto since $cV_{M^0}$ is generated by its subspace of degree 1 elements, in the sense that we have

\begin{equation}
\tag{4.2.1}
cV_{M^0} = \text{Span}_C \{ x_{-n_1}^1, x_{-n_2}^2, \ldots, x_{-n_r}^r, 1 \mid \text{deg}(x_i) = 1, n_i \in \mathbb{Z}_{>0} \}
\end{equation}

so that any element $g \in \text{Aut}(g)$ extends to an element of $\text{Aut}(cV_{M^0}) = G^0$ once we decree

\begin{equation}
\tag{4.2.2}
g : x_{-n_1}^1, x_{-n_2}^2, \ldots, x_{-n_r}^r, 1 \mapsto (gx_1^1)_{-n_1}(gx_2^2)_{-n_2} \cdots (gx_r^r)_{-n_r}, 1
\end{equation}

and any element of $G^0$ that fixes $g$ fixes all of $cV_{M^0}$. Thus we may identify $\text{Aut}(g)$ with $G^0$.

We claim that $\phi(S) = \phi(G)$. In [DN99] it is proved that the automorphism group of a lattice VOA (for an even positive definite lattice, such as $M^0$) is generated by exponentials of zero-modes of degree 1 elements and by lifts of automorphisms of the lattice. In our situation this means $G^0 = \langle \phi(S), \text{O}(M^0) \rangle$ where $\text{O}(M^0)$ denotes the group of automorphisms of $cV_{M^0}$ generated by lifts of elements of $\text{Aut}(M^0)$. On the other hand, we know that the group $\text{Inn}(g)$ of inner automorphisms of $g$ (this is just our group $\phi(S)$) has index 2 in $\text{Aut}(g)$, since $g$ is a simple complex Lie algebra of type $D_{12}$. So there is some $x \in \text{O}(M^0)$ of order 2, such that $G^0 = \phi(S) \cup x\phi(S)$, and either $\phi(G) = \phi(S)$, or $\phi(G) = G^0$. Let $\bar{x}$ denote the canonical image of $x$ in $\text{Aut}(M^0)$. The coset $x\phi(S)$ corresponds to a so-called diagram automorphism of $D_{12}$, and $\bar{x}$ acts non-trivially on the coset space $(M^0)^*/M^0$ interchanging the two cosets with minimal norm 3 (one of which is $M^1$). In particular, $\bar{x}$ does not preserve the lattice $M = M^0 + M^1$, and thus $x$ cannot be extended to an automorphism of $cV_M$ (c.f. [DN99] Lemma 2.3). We conclude that $\phi(S) = \phi(G)$.

Next we claim that $\ker(\phi)$ is contained in $S$. For suppose $g \in \ker(\phi)$. Then $g$ fixes all elements of $g$, and therefore commutes with the action of $S$ on $(cV_M)_{3/2} = \text{CM}(cV_M)_G$. The space $\text{CM}(cV_M)_G$ is irreducible for the action of $S$, so $g$ acts as scalar multiplication by $\zeta \in C$ say, on $(cV_M)_{3/2}$, and indeed, on all of $cV_{M^1}$ (since $cV_{M^1}$ is generated by the action of $C_{M^0}$ on $(cV_M)_{3/2}$). Then for $x, y \in cV_{M^1}$, we have $g(x(n)y) = (gx)(n)(gy) = \zeta^2 x(n)y$, and also $g(x(n)y) = x(n)y$ since $x(n)y$ lies in $cV_{M^0}$. It follows that $\zeta = \pm 1$ and $\ker(\phi)$ has order 2. In the non-trivial case that $g|_{cV_{M^0}} = \text{Id}$ and $g|_{cV_{M^1}} = -\text{Id}$, we have that $g$ is realized by the image of the element $-1 \in S$ in $S$. This proves the claim.

We have shown that $\phi(S) = \phi(G)$ and $\ker(\phi) < S$, and it follows that $S = G$, which is what we required. \hfill \square
From now on it will be convenient to regard Spin(ℓ) and Spin(ℓ1) as groups of SVOA automorphisms of A^f2 and cA^f2, respectively.

Recall that the ordering on Ω is chosen so that e₀ ∈ Spin(ℓ) lies in G. We denote e₀ also by 3. Then the action of e₀ on cA^f2 is trivial, the kernel of the map Spin(ℓ1) → Aut_{SVOA}(cA^f2) is {1, 3}, and the full group of SVOA automorphisms of cA^f2 is Spin(ℓ1)/{3}. Any automorphism of A^f2 extends to an automorphism of the complexification cA^f2, so Spin(ℓ1)/{3} contains the group of SVOA automorphisms of the real form A^f2. On the other hand, this latter group contains Spin(ℓ1)/{3} which is maximal compact in Spin(ℓ1)/{3}. We conclude that Spin(ℓ1)/{3} is the full group of SVOA automorphisms of the real SVOA A^f2.

Let F denote the subgroup of Spin(ℓ1) that fixes 1_G ∈ CM(ℓ1)_G^0 ⊆ (cA^f2)_{3/2}. Then the full group of automorphisms of the N = 1 SVOA structure on cA^f2 is F/{3}. Let us write C' for the span of the vectors u|_G ∈ CM(ℓ1)_G for u ∈ ℓ, and cℓ' for the complexification of ℓ', regarded as a subspace of CM(ℓ1)_G. Then cℓ' has dimension 24, and F embeds naturally in SO(cℓ') since xu|_G = xux^{-1}|_G = x(u)|_G for u ∈ ℓ and x ∈ F, and x(u) ⊂ u for x ∈ Spin(ℓ1). We will now show

**Proposition 4.7.** The group F contains a group isomorphic to C₀.

*Proof.* Recall that the natural map Spin(ℓ1) → SO(ℓ1) is denoted x → x(.). Recall that G is an F^2 homogeneous lift of the Golay code G to Spin(ℓ) (see [8R]), so that for each C ∈ G there is a unique g_C ∈ G such that g_C(·) is −1 on e_i for i ∈ C, and +1 on e_i otherwise. Let C₀ be a subgroup of SO(ℓ) isomorphic to C₀ such that C₀ contains g(·) for each g ∈ G < Spin(ℓ). The Golay code construction of the Leech lattice given in [8U], for example, shows that this is possible.

Let C be the preimagine of C₀ in Spin(ℓ1). The group C₀ has trivial Schur multiplier [8U,85], so there exists a group C' in Spin(ℓ1), a subgroup of index 2 in C, such that the map x → x(·) restricts to an isomorphism of C' with C₀ < SO(ℓ). Set G' = {g_C, g_C · 3 | C ∈ G}, and set G' = G ∩ C'. Then we have G' = {γ_Cg_C | C ∈ G} where C ↦→ γ_C is a map G → {±1} such that γ_Cγ_D = γ_{C+D}. In particular, C → γ_C is a homomorphism, and there must be some S ⊂ Ω such that we have γ_C = (−1)^(|S|). Then C is isomorphic to C₀, and contains G. In particular, C contains the central element 3, and C/{3} must be isomorphic to C₀. The space CM(ℓ1)^0 is then a C/{3}-module of dimension 2048 and since the only C₀ irreducibles with dimension less than 2048 have dimension 1, 276, 299 and 1771 [8U,85], the space CM(ℓ1)^0 must have a fixed point t say, for the action of C. We may assume that t has unit norm. Since G is contained in C, the vector t is also invariant for G, and this forces t = 1_G by Proposition 3.3. We conclude that C is a subgroup of F isomorphic to C₀, and this completes the proof. □

The proof of Proposition 4.7 shows that a copy of C₀ may be found even inside the intersection F ∩ Spin(ℓ).

**Proposition 4.8.** The group F is finite.

*Proof.* F is a subgroup of the algebraic group Spin(ℓ1) = Spin_24(ℂ). The condition that a subspace be stabilized by a linear transformation is polynomial, so F too is algebraic, since it is by definition the stabilizer of a subspace in a representation of Spin(ℓ1). At the same time, F is a subgroup of SO(ℓ1) = SO_{24}(ℂ) containing the algebraic group C₀ by Proposition 4.7. Since the latter group acts irreducibly
on \( C' \), we conclude that \( F \) is reductive. We now check if there is any non-trivial semisimple complex algebraic group or algebraic torus that can occur as a factor of the connected component of the identity in \( F \). Any such group would have a non-trivial Lie algebra \( \mathfrak{t} \), say, with an embedding in the Lie algebra \( \mathfrak{g} \) of \( \text{Spin}(C) \), which we may identify with the degree one subspace of \( \text{CM}(C) \) (equipped with the bracket \( [x, y] = x_0 y \)). Now for all \( x \in \mathfrak{t} \) we have \( \exp(x_0) 1_G = 1_G \), and this implies \( x_0 1_G = 0 \) for some non-trivial \( x \in \mathfrak{t} \). We claim that if \( x \in \mathfrak{g} \) satisfies \( x_0 1_G = 0 \) then \( x = 0 \). For consider the map \( \mathfrak{g} \to \text{CM}(C) \) given by \( x \mapsto x_0 1_G \), and write \( \mathfrak{g}' \) for the image of \( \mathfrak{g} \) under this map. Then the dimension of \( \mathfrak{g}' \) is at most 276. On the other hand \( \mathfrak{g}' \) contains the span of the vectors \( \{ e_{ij} 1_G \} \) for \( i < j \), and since the Golay code has minimum weight 8 these vectors are linearly independent, and we see that \( \mathfrak{g}' \) has dimension not less than 276. It follows that the map \( x \mapsto x_0 1_G \) is a linear isomorphism from \( \mathfrak{g} \) to \( \mathfrak{g}' \), and in particular, the kernel is trivial. This verifies the claim. We conclude that \( \dim(F) = 0 \), whence \( F \) is finite. \( \square \)

We have shown that \( F \) is a finite subgroup of \( \text{Spin}(C) \) such that \( F \cap \text{Spin}(I) \) contains a copy of \( C_{01} \). Our last main task for this section is to show that \( F \) itself is isomorphic to \( C_{01} \). With this in mind, we offer the following proposition, the proof of which owes much to the methods used in Theorems 5.6 and 6.5 of \[\text{NRS01}\].

In particular, we utilize the notion of primitive matrix group: a group \( G \leq \text{GL}(V) \), for \( V \) a vector space, is said to be \textit{primitive} if there is no non-trivial decomposition \( V = V_1 \oplus \cdots \oplus V_k \) into subspaces permuted by the action of \( G \). Note that if \( N \) is normal in \( G \), then \( G \) permutes the isotypic components of the restricted module \( V|_N \), so that \( V|_N \) must be multiple copies of a single irreducible representation for \( N \) in the case that \( G \) is primitive.

**Proposition 4.9.** The group \( C_{01} \) is a maximal subgroup of \( \text{SO}_{24}(\mathbb{C}) \) subject to being finite.

**Proof.** Any compact subgroup of \( \text{SO}_n(\mathbb{C}) \) is realizable over \( \mathbb{R} \) (c.f. [Ser77, §13.2]), so it suffices to show that \( C_{01} \) is a maximal finite subgroup of \( \text{SO}_{24}(\mathbb{R}) \). Let \( V \) denote a real vector space of dimension 24, equipped with a non-degenerate symmetric bilinear form. Suppose that \( F \) is a finite subgroup of \( \text{SO}(V) \) properly containing a copy \( C \) of the group \( C_{01} \). Then \( C \) is not normal in \( F \), since \( C \) acts absolutely irreducibly on \( V \), and if \( C \) were normal in \( F \) then \( F/C \) would embed in the outer automorphism group of \( C \), which is trivial [CCN75]. Let us write \( Z \) for the center \( \{ \pm 1 \} \) of \( C \) (and \( F \)).

We claim that \( F \) has no non-trivial normal \( p \)-subgroups for \( p \) odd, and the only non-trivial normal 2-subgroup is \( Z \). For suppose \( N \) is a normal \( p \)-subgroup of \( F \) for some prime \( p \). Then \( C_F(N) \cap C \) (we write \( C_F(N) \) for the centralizer in \( F \) of \( N \)) is normal in \( C \) and contains \( Z \), so that \( C_F(N) \cap C \) is either \( Z \) or \( C \). In the former case we have that \( C/Z \cong C_{01} \) is a subgroup of \( \text{Aut}(N) \). In the latter case, \( N \) is centralized by (the absolutely irreducible action of) \( C \), and hence must consist of scalar matrices. It follows that \( N \) is trivial unless \( p = 2 \), in which case \( N \) is either trivial or \( N = Z \). We suppose then that \( N \) is a normal \( p \)-subgroup of \( F \) such that \( \text{Aut}(N) \) contains a copy of \( C_{01} \). The group \( C \) acts primitively on \( V \), and even on the complexification \( CV = C \otimes_{\mathbb{R}} V \), and hence so does \( F \). It follows that \( CV|_N \) is an isotypic module for \( N \); i.e. several copies of a single irreducible module \( M \) say, for \( N \). Since \( N \) is by definition a subgroup of \( \text{SO}_{24}(\mathbb{R}) \), it follows that \( M \) is the complexification of an irreducible \( N \)-module \( \mathbb{R}M \) say, defined over...
\( \mathbb{R} \), and that the action of \( N \) on \( \mathbb{R} M \) is faithful. Any \( p \)-group has non-trivial center, and a central subgroup of a group acts by scalar multiplications on any irreducible module for that group. We conclude that \( p \) is not odd, since a \( p \)-group for odd \( p \) has central elements which must act as multiplication by primitive (and non-real) \( p \)-th roots of unity. We see also that any abelian normal subgroup of \( F \) is cyclic. The irreducible representations of a \( p \)-group are of \( p \)-power order, so \( \deg(\mathbb{F} M) \) is a power of 2 dividing 24. Without loss of generality, we suppose \( \deg(\mathbb{F} M) = 8 \). Note that \( N \) can contain no noncyclic characteristic abelian subgroups, since such a subgroup would be a noncyclic normal abelian \( p \)-subgroup of \( F \). A \( p \)-group with this property is said to be of symplectic type, and such groups are classified by a Theorem of P. Hall (c.f. \cite{Tie97}, (23.9)). In particular, there are no 2-groups of symplectic type that both embed in \( \text{SO}_8(\mathbb{R}) \) and admit a non-trivial action by \( \text{Co}_1 \) as automorphisms. The claim follows.

Now we seek a contradiction. Any finite group is realizable over a cyclotomic number field (c.f. \cite{Ser77} Thm 24)). In fact we may assume that \( F \) is a subgroup of \( \text{SO}_{24}(K) \) where \( K \) is a totally real abelian number field (e.g. \( K = \mathbb{Q}(\zeta_m + \zeta_m^{-1}) \) for \( m \) such that \( x^m = 1 \) for all \( x \in F \) and \( \zeta_m = \exp(2\pi i/m) \) — c.f. \cite{Dre75} Prop 5.6)). Let \( K \) be a minimal such field, and let \( R \) be the ring of integers in \( K \). Then \( F \) preserves an \( RC \)-lattice, and such a lattice is of the form \( I \otimes \Lambda \) for \( \Lambda \) a copy of the Leech lattice and \( I \) a fractional ideal of \( R \), since any \( C \) invariant lattice in \( \mathbb{Q}^{24} \) is isometric to \( \Lambda \) (c.f. \cite{Asc01}). It follows that \( F \) preserves the lattice \( R \otimes \Lambda \). (To see this, note that if \( F \) preserves a lattice \( L \), then it also preserves \( aL \) for any \( a \in K \).

Also, if \( F \) preserves lattices \( L_1 \) and \( L_2 \), then it preserves the sum \( L_1 + L_2 \). Now take \( L = I \otimes \Lambda \) for \( I \) a fractional ideal of \( R \), and let \( \{ y_i \} \) be a set of generators for the inverse fractional ideal. Then \( R = \sum y_i I \), and \( \sum y_i L = R \otimes \Lambda \) is also invariant for \( F \).) We may now regard \( F \) as a group of matrices with entries in the ring \( R \). If \( K = \mathbb{Q} \) then \( F = C \) and we are done. If not, then there is some rational prime \( p \) that ramifies in \( K \). Let \( p \) be a prime ideal of \( R \) lying above \( p \), and let \( \Gamma_p \) denote the subgroup of \( \text{Gal}(K/\mathbb{Q}) \) consisting of automorphisms \( \sigma \) such that \( \sigma(a) \equiv a \pmod{p} \) for all \( a \in R \). (This is the first inertia group. It stabilizes \( p \), and has order equal to the ramification index of \( p \) over \( p \) — c.f. \cite{FT33} III:4) The Galois group of \( K \) over \( \mathbb{Q} \) acts on \( F \) by acting component-wise on matrices. Let \( F_p = \{ g \in F \mid g \equiv \text{Id} \pmod{p} \} \), let \( \sigma \) be a non-trivial element of \( \Gamma_p \), and let \( \phi : F \to F_p \) be the map defined by \( \phi(g) = g^{-1}\sigma(g) \) for \( g \in F \). The group \( F_p \) is a normal \( p \)-subgroup of \( F \), and we have shown that such a group is trivial except possibly in the case that \( p = 2 \). If \( F_p \) is trivial then \( \sigma \) fixes \( F \), and this contradicts the minimality of \( K \).

So suppose \( p = 2 \) and \( F_p \) is the group \( Z = \{ \pm \text{Id} \} \). Then \( \phi \) is in fact a group homomorphism \( F \to Z \) (since \( g^{-1}\sigma(g) \) is now central for all \( g \in F \)). Since the image of \( \phi \) is abelian, the derived subgroup \( F^{(1)} \) of \( F \) lies in the kernel of \( \phi \), and is thus fixed by \( \sigma \). If \( F = F^{(1)} \) then \( F \) is realizable over the subfield of \( K \) fixed by \( \sigma \), contradicting the minimality of \( K \). So \( F \) properly contains \( F^{(1)} \), and the argument thus far shows that any finite subgroup of \( \text{SO}_{24}(\mathbb{R}) \) properly containing \( C \), properly contains its own derived subgroup. Consider now the descending chain \( F \supseteq F^{(1)} \supseteq F^{(2)} \supseteq \cdots \) where \( F^{(k+1)} \) is the derived subgroup of \( F^{(k)} \). Each term contains \( C \) since \( F > C \) and \( C = C^{(1)} \), and thus each containment \( F^{(k)} \supseteq F^{(k+1)} \) is proper unless \( F^{(k)} = C \). Since \( F \) is finite, not all containments are proper, and thus we have \( F^{(k)} = C \) for some \( k \). Then \( C \) is a characteristic subgroup of \( F \), and in particular, is normal in \( F \), and this is again a contradiction.
We conclude that $Co_0$ is a maximal subgroup of $SO_{24}(\mathbb{C})$ subject to being finite. □

We have established the following

**Theorem 4.10.** The subgroup of $\text{Spin}(\mathbb{C})$ fixing $1_G \in \text{CM}((\mathbb{C})G$ is isomorphic to $Co_0$.

Recall that $\text{Aut}(\mathbb{C})^F = F/\langle z \rangle$. The group $\langle z \rangle$ is the center of $F$, and thus $F/\langle z \rangle$ is isomorphic to $Co_1$. By construction this copy of $Co_1$ is contained in $\text{Aut}(A^f)$. We have therefore established

**Theorem 4.11.** There are isomorphisms of groups $\text{Aut}(\mathbb{C})^F \cong \text{Aut}(A^f) \cong Co_1$.

**Remark 4.12.** It was noted in the Introduction that an action of $Co_1$ on the SVOA underlying $A^f(\mathbb{C})$ was considered earlier in [BR96]. In fact, an action of $Co_0$ (the perfect double cover of $Co_1$) on the SVOA underlying $A^f(\mathbb{C})$ was also considered in [BR96], and in our setting, this action arises naturally by considering the action of the group $F$ on the object $A$ given by

\begin{equation}
A^f = A(0)^0 \oplus A(0)^1
\end{equation}

where we realize $A(0)^0$ as $A(0)_{G,\theta}$ with $G$ as in [4.2]. We have seen that the group $F$ is isomorphic to the quasi-simple group $Co_0$, and in contrast to the situation with $A^f$, the central element of $F$ acts non-trivially on $A^f$. The same method used in [4.1] shows that $A^f$ has a unique structure of SVOA, and also that $A^f$ and $A^f$ are isomorphic, as SVOAs. There is however no $Co_0$ invariant vector in the degree 3/2 subspace of $A^f$, and hence no $Co_0$ invariant $N = 1$ structure on $A^f$.

5. **Uniqueness**

In this section we prove a uniqueness result for $A^f$. In the first subsection we verify that any nice rational $N = 1$ SVOA satisfying

- self-dual
- rank 12
- no small elements

is isomorphic to $\mathbb{C}A^f$ as an SVOA. To do this we first recall the modularity results for trace functions on VOAs due to Zhu (see [Zhu90], [Zhu96]), and their extension to the SVOA case given in [Hohn96]. Then we make use of some techniques from [DN06] and [DN08], replacing VOA concepts with their SVOA analogues as necessary. The guiding principle that we adopt from these two papers is that one may use modular invariance results for a VOA $V$ to deduce properties about the Lie algebra structure on $V_1$, the degree one subspace of $V$.

In the second subsection we show that the $N = 1$ structure on $A^f$ is unique in the sense that if $\tau \in (A^f)_{3/2}$ is a superconformal vector then there is some SVOA automorphism of $A^f$ mapping $\tau$ to $\tau_A$.

5.1. **SVOA structure.** Recall from [2.1] and [2.2] the definitions of niceness and rationality for an SVOA. Recall also from [2.2] that a rational SVOA has finitely many irreducible modules up to equivalence.
5.1.1. **Theta group.** Let $\Gamma = \text{SL}(2, \mathbb{Z})$ and recall that the modular group $\bar{\Gamma} = \Gamma/\{\pm 1\} \cong \text{PSL}(2, \mathbb{Z})$ acts faithfully on the upper half plane $h = \{\sigma + it \mid t > 0\} \subset \mathbb{C}$, with the action generated by modular transformations $S$ and $T$ where $S: \tau \mapsto -1/\tau$ and $T: \tau \mapsto \tau + 1$. We identify $\bar{\Gamma}$ with its image in the isometry group of $h$ and set $\bar{\Gamma}_\theta = \langle S, T^2 \rangle$. The compactification of the quotient space $\bar{\Gamma}\backslash h$ is topologically a sphere, and the same is true for $\bar{\Gamma}_\theta\backslash h$. The space $\bar{\Gamma}_\theta\backslash h$ has two cusps, with representatives $1$ and $\infty$. There is a unique holomorphic function on $h$ that is invariant under $\bar{\Gamma}_\theta$, has a $q$ expansion of the form $q^{-1/2} + a + bq^{1/2} + cq + \ldots$, and vanishes at $1$. We denote this function by $J_\theta(\tau)$ since it is an analogue of the $J$ function, which generates the field of functions on the compactified curve $\bar{\Gamma}\backslash h$. The function $J_\theta$ furnishes a bijective map from the compactification of $\bar{\Gamma}_\theta\backslash h$ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. One has the following expression for $J_\theta(\tau)$.

\[
J_\theta(\tau) = \frac{\eta(\tau)^{24}}{\eta(\tau/2)^{24} \eta(2\tau)^{24}} = q^{-1/2} + 24 + 276q^{1/2} + 2048q + 11202q^{3/2} + 49152q^2 + \ldots
\]

To see the behavior of $J_\theta$ at $1$, note that $T S \tau \to 1$ as $\tau \to \infty$. For $J_\theta|_{TS}$ we have

\[
J_\theta|_{TS} = J_\theta(-1/\tau + 1) = -2^{12} \frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} = -(4096q + 98304q^2 + 1228800q^3 + \ldots)
\]

confirming that $J_\theta$ vanishes as $\tau \to 1$. We write $\Gamma_\theta$ for the preimage of $\bar{\Gamma}_\theta$ in $\Gamma$.

5.1.2. **Modular Invariance.** Suppose now that $(V, Y, 1, \omega)$ is a nice rational VOA. Following [Zhu96], we may define a new VOA structure on the space $V$ as follows. We define the genus one VOA associated to $V$ to be the four-tuple $(V, Y[^1], 1, \bar{\omega})$ where $\bar{\omega} = (2\pi i)^2(\omega - c/24)$ and the linear map $Y[^1]: V \otimes V \to V((z))$ is defined so that

\[
Y[u, z] = \sum_{n \in \mathbb{Z}} u[n]z^{-n-1} = Y(u, e^{2\pi i z} - 1)e^{\text{deg}(u)2\pi iz}
\]

for $u$ an $L(0)$-homogeneous element in $V$. The object thus defined is again a VOA and is isomorphic to $(V, Y, 1, \omega)$ [Zhu96]. In particular the coefficients of $Y[\bar{\omega}, z]$ define a representation of the Virasoro algebra with central charge $c$. We write

\[
L[z] = Y[\bar{\omega}, z] = \sum_{n} L[n]z^{-n-2}
\]

and for $n \in \mathbb{Z}$, we set $V[n] = \{u \in V \mid L[0]u = nu\}$. Note that for $u$ an $L(0)$-homogeneous element in $V$ we have

\[
Y[u, z] = \sum_{n} u[n](e^{2\pi iz} - 1)^{-n-1}e^{\text{deg}(u)2\pi iz}
\]

\[
= \sum_{n} u[n](2\pi iz + \frac{1}{2}(2\pi iz)^2 + \ldots)^{-n-1}e^{\text{deg}(u)2\pi iz}
\]

\[
= \sum_{n} (2\pi i)^{-n-1}u[n]z^{-n-1}(1 + \frac{1}{2}2\pi iz + \ldots)^{-n-1}e^{\text{deg}(u)2\pi iz}
\]

and in particular, $u[n] = (2\pi i)^{-n-1}u[n] + \sum_{k>0} c_ku[n+k]$ for some constants $c_k \in \mathbb{C}$. If $u, v \in V_1$ then $u_1v = (u|v)\mathbf{1}$ and $u_nv = 0$ for $n > 1$ so we have the following
Lemma 5.1. Let $V$ be a nice rational VOA and let $u, v \in V_1$. Then $u_{[1]} v = - (4 \pi^2)^{-1} \langle u | v \rangle 1$.

Recall the Eisenstein series $G_2(\tau)$ given by

\begin{equation}
G_2(\tau) = \frac{\pi^2}{3} + \sum_{m \neq 0} \sum_{n} \frac{1}{(m \tau + n)^2}
\end{equation}

The function $G_2(\tau)$ has a $q$ expansion which may be expressed in the form

\begin{equation}
G_2(\tau) = \frac{\pi^2}{3} - 8 \pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n
\end{equation}

where $\sigma_1(n)$ is the sum of the divisors of $n$. We denote the corresponding formal power series (that is, element of $\mathbb{C}[[q]]$) by $\tilde{G}_2(q)$.

We define a linear function $o(\cdot) : V \to \text{End}(V)$ by setting $o(u) = u_{\deg(u)} - 1$ for $L(0)$-homogeneous $u \in V$. The following result is a special case of Proposition 4.3.5 in [Zhu96].

Proposition 5.2. Let $V$ be a nice rational SVOA and $M$ a finitely generated $V$-module. Then for $u, v \in V_1$ we have

\begin{equation}
\text{tr}_{M} o(u) o(v) q^{L(0)} = \text{tr}_{|M} o(u_{-1}) v q^{L(0)} - \tilde{G}_2(q) \text{tr}_{M} o(u_{[1]} v) q^{L(0)}
\end{equation}

Let $V$ be a nice rational SVOA and let $M$ be a finitely generated $V$-module. For an $n$-tuple $(u_1, \ldots, u_n)$ of $L(0)$-homogeneous elements in $V$ we define the following formal series.

\begin{equation}
\tilde{F}_M((u_1, x_1), \ldots, (u_n, x_n); q) = x_1^{\deg(u_1)} \cdots x_n^{\deg(u_n)} \text{tr}_{|M} Y(u_1, x_1) \cdots Y(u_n, x_n) q^{L(0)}
\end{equation}

We extend the definition of $\tilde{F}_M$ to arbitrary $n$-tuples of elements from $V$ by linearity. As in [Zhu96, Th 4.2.1] one can show that this series $\tilde{F}_M$ converges to a holomorphic function in the domain

\begin{equation}
\{(x_1, \ldots, x_n, q) \mid 1 > |x_1| > \ldots > |x_n| > |q|\}
\end{equation}

and can be continuously extended to be meromorphic in the domain

\begin{equation}
\{(x_1, \ldots, x_n, q) \mid x_i \neq 0, |q| < 1\}
\end{equation}

We denote the meromorphic function so obtained by $F_M$. We substitute variables $x_i$ with $e^{2\pi i x_i}$, and $q$ with $e^{2\pi i \tau}$, and we set

\begin{equation}
T_M((u_1, z_1), \ldots, (u_n, z_n); \tau) = q^{-c/24} F_M((u_1, x_1), \ldots, (u_n, x_n); q)
\end{equation}

Following [Zhu96] and [Höhn96] we call $T_M((u_1, z_1), \ldots, (u_n, z_n); \tau)$ the $n$-point correlation function on the torus with parameter $\tau$ for the operators $Y(u_i, z_i)$ and the module $M$.

Proposition 5.3. The function $T_M((u_1, z_1), \ldots, (u_n, z_n); \tau)$ is doubly periodic in each variable $z_i$ with periods 1 and $2 \tau$, and possible singularities only at the points $z_i = z_j + k + l \tau$ for $i \neq j$, $k, l \in \mathbb{Z}$. For $j \in \{1, \ldots, n\}$ and $u_j$ an $L(0)$-homogeneous element of $V$ we have

\begin{equation}
T_M((u_1, \ldots, u_j + \tau), \ldots; \tau) = (-1)^{p(u_j)} T_M((u_1, \ldots, u_j, z_j), \ldots; \tau)
\end{equation}
For a permutation $\sigma \in S_n$ we have

\begin{equation}
T_M((u_1, z_1), \ldots, (u_n, z_n); \tau) = (-1)^w T_M((u_{\sigma(1)}, z_{\sigma(1)}), \ldots, (u_{\sigma(n)}, z_{\sigma(n)}); \tau)
\end{equation}

where $w$ is the number of permutations of the elements $u_i$ that lie in $V_1$.

We denote by $T_M$ the mapping defined on the set $\bigcup_{n=1}^{\infty} ((V \times \mathbb{C})^n \times h)$ that sends $((u_1, z_1), \ldots, (u_n, z_n); \tau)$ to $T_M((u_1, z_1), \ldots, (u_n, z_n); \tau)$.

Suppose now that $\{M^1, \ldots, M^r\}$ is a complete list of irreducible $V$-modules. The superconformal block on the torus associated to the SVOA $V$ is the $\mathbb{C}$-vector space spanned by the $r$ mappings $T_M$. We denote it by $SB_V$.

The following result is an analogue for SVOAs of a celebrated theorem due to Zhu concerning the modularity properties of $n$-point correlation functions on the torus associated to the vertex operators on VOAs. As is indicated in [Höh96], this analogue may be proven in a manner directly analogous to that of the VOA version given in [Zhu96], and one should use the SVOA analogues of Zhu algebras defined in [KW94].

**Theorem 5.4** ([Zhu96], [KW94], [Höh96]). Let $V$ be a nice rational SVOA and suppose that $\{M^1, \ldots, M^r\}$ is a complete list of irreducible $V$-modules. Then the superconformal block on the torus associated to $V$ is $r$-dimensional and the functions $T_M$ form a basis. Moreover, there exists a representation $\rho$ of $\Gamma_0$ on $SB_V$ such that for Virasoro highest weight vectors $\{u_1, \ldots, u_n\} \in V$ the $n$-point correlation functions on the torus for the operators $Y(u_i, z_i)$ and the modules $M^j$ satisfy the following transformation property

\begin{equation}
T_{M^j}(\left(\frac{u_1, z_1}{c\tau + d}, \ldots, \frac{u_n, z_n}{c\tau + d}\right); a\tau + b) = (c\tau + d)^{\deg(u)} \sum_{j} \rho(A)_{ij} T_{M^j}(\left(\frac{u_1, z_1}{c\tau + d}, \ldots, \frac{u_n, z_n}{c\tau + d}\right))
\end{equation}

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma_0$ and $\rho(A)_{ij}$ is the matrix representing $\rho(A) \in \text{End}(SB_V)$ with respect to the basis $\{T_{M^j}\}$.

In the case that $n = 1$ the function $T_M((u, z); \tau)$ is elliptic in the variable $z$ and without poles, and is therefore constant with respect to $z$. We may therefore set $T_M(u; \tau) = T_M((u, z); \tau)$. Note that $T_M(u; \tau) = \text{tr}|_M o(u) q^{L(0)-c/24}$, and in particular $T_M(1; \tau) = \text{tr}|_M q^{L(0)-c/24}$.

**Corollary 5.5.** Let $V$ be a self-dual nice rational SVOA. Then $SB_V$ is one dimensional, and for $u \in V_0$ a Virasoro highest weight vector with $\deg(u) = k$, the function $\text{tr}|_M o(u) q^{L(0)-c/24}$ is a weight $k$ modular form on $\Gamma_0$, possibly with character.

**Remark 5.6.** Corollary 5.5 now appears as a special case of the more general Theorem 3 of [DZ05] which incorporates also $g$-twisted SVOA modules for $g$ in a finite group of automorphisms of a suitable SVOA $V$.

We apply Corollary 5.5 immediately with $u = 1$ in order to determine the character of an SVOA satisfying our hypotheses.
Proposition 5.7. Suppose that $V$ is a self-dual nice rational SVOA of rank 12. Then we have

$$
\text{tr}|_{V}q^{L(0)-c/24} = J_0(\tau) + \dim(V_{1/2}) - 24
$$

for the character of $V$.

Proof. Let us set $f(\tau) = \text{tr}|_{V}q^{L(0)-c/24}$. By hypothesis, $f(\tau)$ admits a Fourier expansion of the form $q^{-1/2} + \sum_{n \geq 0} f_n q^{n/2}$ with all the $f_n$ non-negative integers. By Corollary 2.4, we know that $f(\tau)$ is holomorphic on $\mathbb{H}$ and is invariant for the action of $\Gamma_0$. It follows that $f(\tau) = P(J_0)/Q(J_0)$ for some polynomials $P(X),Q(X) \in \mathbb{C}[X]$, with $\deg(P) = \deg(Q) + 1$, and we may assume that $P$ and $Q$ are both monic and have no common factors. The function $J_0(\tau)$ is a surjective map from $\mathbb{H}$ to $\mathbb{C} \setminus \{0\}$, so that for $f(\tau)$ to be holomorphic we must have $Q(X) = X^m$ for some $m$. Then we have

$$
(5.1.17) \quad f(\tau) = J_0 + a_m + a_{m-1}J_0^{-1} + \cdots + a_0J_0^{-m}
$$

for $P(X) = X^{m+1} + a_mX^m + \cdots + a_0$ with $a_0 \neq 0$ unless possibly if $m = 0$. We claim that $m = 0$ and $a_m = a_0 = \dim(V_{1/2}) - 24$. Certainly, $a_m = \dim(V_{1/2}) - 24$, since the first two terms of (5.1.17) determine the first two Fourier coefficients of $f(\tau)$. Let us write $J_0(\tau)^{-d} = \sum_{n} r_{-d}(n)q^{n/2}$. Then the sequence $\{r_{-d}(n)\}_n$ alternates in sign when $d$ is positive, as can be seen from the following identity.

$$
(5.1.18) \quad J_0(\tau)^{-d} = \frac{n(\tau/2)^{24d}n(2\tau)^{24d}}{q(\tau)^{12d}} = q^{d/2} \prod_{1 \leq n \leq \sqrt{d}} \frac{1}{(1 + q^{n+1/2})^{24d}}
$$

We see also from this that for $d > 0$, the value of $|r_{-d}(n)|$ is the number of partitions of $n - d$ into odd parts with $24d$ colors. The asymptotic behavior of such functions is described by Theorem 1 of [Mei54], and we will quote this result presently. The value of $r_1(n)$ is the number of partitions of $n + 1$ into odd parts of 24 colors without replacement, and for the asymptotics of this function we refer to Proposition 1 of [Hwa01]. The result is that we have

$$
(5.1.19) \quad r_1(n) \sim C_1 e^{2\pi \sqrt{n}} \quad \text{and} \quad |r_{-d}(n)| \sim C_{-d} e^{2\pi \sqrt{2d\sqrt{n}}} \quad \text{for} \ d > 0,
$$

for some constants $C_k$. Evidently, the growth of the $r_{-m}(n)$ outstrips that of the $r_{-d}(n)$ for $1 \geq -d \geq -m + 1$ if $m > 0$. In particular, the $f_n$ can be all non-negative integers only if $m = 0$. \hfill \Box

As demonstrated in [DZ05], one may recover a modular invariance under the full modular group for the trace functions associated to an SVOA by considering canonically twisted modules together with untwisted modules. The following result is a special case of Theorem 1 of [DZ05] where we take $V$ to be self-dual, and $G$ to be the group of SVOA automorphisms generated by the canonical automorphism of $V$. Recall from [DZ05] that if $V$ is a self-dual rational $C_2$-cofinite SVOA, then $V_\gamma$ denotes the unique $\sigma$-stable $\sigma$-twisted $V$-module, and $\sigma$-stable here means that $V_\sigma$ admits a compatible action by $\sigma$.

Proposition 5.8 ([DZ05]). Let $V$ be a self-dual rational $C_2$-cofinite SVOA. Let $w \in V$ such that $w \in V_{(k)}$ for some $k$. Then for $\gamma \in \Gamma$, we have

$$
(5.1.20) \quad \text{tr}|_{V}q^{L(0)-c/24}\gamma = (c\tau + d)^k \rho(\gamma)\text{tr}|_{V}q^{L(0)-c/24}
$$

for the character of $V$. \hfill \Box
for some $\rho(\gamma) \in \mathbb{C}$ independent of $w$, where $W = V_\sigma$ if $\sigma^{1+a+c} = \sigma$, and $W = V$ otherwise, and $\gamma$ is the matrix $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$.

In Proposition 5.7 we determined that the character of a self-dual nice rational SVOA of rank 12 is $J_0(7) + \dim(V_{1/2}) - 24$. Applying Proposition 5.8 with $\gamma = TS$ (so that $(a, b, c, d) = (1, -1, 1, 0)$) we find that the character $\text{tr}|_{V_\sigma} q^{L(0) - c/24}$ of the canonically twisted module $V_\sigma$ over such an SVOA is just $\alpha(J_0|TS + \dim(V_{1/2}) - 24)$ for some $\alpha \in \mathbb{C}$. Recalling from Proposition 5.11 that the $q$ expansion of $J_0|TS$ involves only positive integer powers of $q$, we have

**Proposition 5.9.** Let $V$ be a self-dual nice rational SVOA of rank 12 with $V_{1/2} = 0$. Then $(V_\sigma)_n$ vanishes unless $n > 0$ and $n \in \mathbb{Z} + \frac{1}{2}$.

5.1.3. Structure of $V_1$. Our plan now is to study the structure of the Lie algebra on $V_1$ for $V$ satisfying suitable hypotheses. By the end of this section, knowledge of $V_1$ will determine $V$ uniquely under the conditions we consider. Much of our method follows that employed in certain sections of [DM04b] and [DM04a], and is a manifestation of the principle established there that modular invariance for an SVOA $V$ can be used to make strong conclusions about the structure of $V_1$.

**Proposition 5.10.** Suppose that $V$ is a nice $N = 1$ SVOA with $V_{1/2} = 0$. Then $V$ has a unique non-degenerate invariant bilinear form.

**Proof.** By the results of [Schn98], the space of invariant bilinear forms on $V$ is in natural correspondence with the space $V_0/L(1)V_1$. We have that $V_0$ is one dimensional by hypothesis, so we require to show that $L(1)V_1 = 0$. From the commutation relations of the Neveu–Schwarz superalgebra we have $G(\frac{1}{2})^2 = L(1)$ so that $L(1)V_1 \subset G(\frac{1}{2})V_{1/2}$. Since $V_{1/2} = 0$ the result follows.

From now we assume $V$ to be a nice rational $N = 1$ SVOA with $V_{1/2} = 0$. Then the import of Proposition 5.10 is that $V_0$ is a strongly rational VOA in the sense of [DM04b]. In particular Theorem 1 of [DM04b] yields the following

**Theorem 5.11** (DM04b). The Lie algebra $V_1$ is reductive.

Since $V_1$ is reductive, the Lie rank of $V_1$ is well defined. Suppose in addition now that $V$ is self-dual. We will follow the technique used to prove Theorem 2 in [DM04b] to establish the following

**Theorem 5.12.** The Lie rank of $V_1$ is bounded above by $\text{rank}(V)$.

**Proof.** We set $c = \text{rank}(V)$. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $V_1$ consisting of semisimple elements. The Lie rank of $V_1$ is the dimension of $\mathfrak{h}$ and we denote this value by $l$. The bilinear form $\langle \cdot, \cdot \rangle$ restricts to be non-degenerate on $\mathfrak{h}$, and thus the vertex operators $Y(h, z)$ for $h \in \mathfrak{h}$ generate an affine Lie algebra $\mathfrak{h}$, and we can decompose $V$ as

\[(5.1.21) \quad V = M(1) \otimes \Omega_V\]

where $M(1) \simeq S(h_{-m} \mid h \in \mathfrak{h}, m > 0)$ is the Heisenberg VOA of rank $l$ associated to the space $\mathfrak{h}$, and $\Omega_V$ is the vacuum space consisting of vectors $u \in V$ such that $h_m u = 0$ for all $h \in \mathfrak{h}$ and $m \geq 0$. Both factors on the right hand side of (5.1.21) are invariant under the action of $L(0)$, so that (5.1.21) holds even as a decomposition of $L(0)$-graded spaces. Taking the trace of $q^{L(0) - c/24}$ on each side of
multiplying both sides by $\eta(q)^c$ and noting that $\text{tr}|_{M(1)}q^{L(0)} = q^{l/24}\eta(q)^{-l}$ we have

$$
\eta(q)^c\text{tr}|_Vq^{L(0) - \epsilon/24} = q^{(l-c)/24}\eta(q)^c\text{tr}|_Vq^{L(0)}
$$

The expression on the left hand side of (5.1.22) is a holomorphic modular form on $\Gamma_0$ (of weight $c/2$). It is a classical result that the Fourier coefficients $r(n)$ say, of such a function satisfy $r(n) = O(n^3)$. On the other hand, the Fourier coefficients of $\eta(q)^{-s}$ grow like $n^{-s/4-3/4}\exp(\pi\sqrt{2s/3}\sqrt{n})$ whenever $s > 0$ (c.f. [Mei54, Thm 1]), so we must have $c - l \geq 0$. This is what we required to show.

Remark 5.13. Certainly one expects Theorem 5.12 to hold not just for the case that $V$ is self-dual, but the present result is sufficiently strong for our interests.

The following proposition is an analogue for self-dual rational SVOAs of rank 12 of Corollary 2.3 in [DM04a], and we repeat here the method of proof used there.

**Proposition 5.14.** Suppose that $V$ is a self-dual nice rational $N = 1$ SVOA of rank 12 with $V_{1/2} = 0$. Then the Killing form on $V_1$ satisfies $\kappa(\cdot, \cdot) = 44\langle \cdot | \cdot \rangle$.

**Proof.** By Proposition 5.7 we have $\text{tr}|_Vq^{L(0) - \epsilon/24} = J_0(\tau) - 24$, and in particular, $\dim(V_1) = 276$. Now we apply Proposition 5.2 to $V$ with $M = V$ and use Lemma 5.1 to rewrite the conclusion as

$$
\text{tr}|_V o(u)v q^{L(0) - 1/2} = \text{tr}|_V o(u_{1/2}) q^{L(0) - 1/2} + \frac{\langle u|v \rangle}{4\pi^2} \hat{G}_2(q) \text{tr}|_Vq^{L(0) - 1/2}
$$

The leading term of the second summand on the right hand side of (5.1.23) is therefore $\frac{1}{12}(u|v)q^{-1/2}$, but the leading term on the left hand side of (5.1.23) is $\kappa(u, v)q^{1/2}$ where $\kappa(\cdot, \cdot)$ is the Killing form on $V_1$. We conclude that the leading term of the first summand on the right hand side of (5.1.23) is $-\frac{1}{12}(u|v)q^{-1/2}$. For $u, v \in V_1$ set $X_{u, v}(\tau) = \text{tr}|_V o(u_{1/2}v)q^{L(0) - \epsilon/24}$. We claim that $X_{u, v}(\tau) = \frac{1}{6}\langle u|v \rangle qD_q \text{tr}|_Vq^{L(0) - \epsilon/24}$. This is certainly true if $\langle u|v \rangle = 0$. Suppose not then, and note firstly that $X_{u, v}(\tau)$ is a weight 2 modular form for $\bar{\Gamma}_0$ by Corollary 5.14, and secondly that $X_{u, v}(\tau) = -\frac{1}{12}(u|v)q^{-1/2} + 0 + aq^{1/2} + \ldots$ for some $a \in \mathbb{C}$ by the above. Applying Proposition 5.8 with $w = u_{1/2}v$ and $\gamma = TS$, we find

$$
X_{u, v}(-1/\tau + 1)\tau^{-2} = \alpha \text{tr}|_V o(u_{1/2}v)q^{L(0) - \epsilon/24}
$$

for some $\alpha \in \mathbb{C}$, and this $q$ series belongs to $\mathbb{C}[\langle q \rangle]$ by Proposition 5.9. From this we note thirdly, that $X_{u, v}(\tau)$ is holomorphic at the cusp represented by 1. We claim that these three properties determine $X_{u, v}(\tau)$ uniquely, for if $X'(\tau)$ is another such function, then $Z(\tau) = X_{u, v}(\tau) - X'(\tau)$ is a weight two modular form for $\bar{\Gamma}_0$ that is holomorphic at both cusps, and vanishes at $\infty$. The space of weight 2 modular forms that are holomorphic at both cusps is spanned by the theta function of the lattice $\mathbb{Z}^2$ (c.f. [Ran77]), but this function does not vanish at $\infty$, and the claim follows. It is easy to check that $\frac{1}{6}\langle u|v \rangle qD_q \text{tr}|_Vq^{L(0) - \epsilon/24}$ satisfies the three
properties of $X_{u,v}(\tau)$ so we may rewrite (5.1.25) as follows.

\begin{equation}
(5.1.25) \quad \text{tr}|_V o(u)o(v)q^{L(0)-1/2} = \frac{\langle u|v \rangle}{6} qD_4 \text{tr}|_V q^{L(0)-1/2} + \frac{\langle u|v \rangle}{4\pi^2} \tilde{G}_2(q) \text{tr}|_V q^{L(0)-1/2}
\end{equation}

Equating the coefficients of $q^{1/2}$ on each side we have $\kappa(\cdot, \cdot) = 44\langle \cdot | \cdot \rangle$. \hfill \Box

**Theorem 5.15.** Let $V$ be a self-dual nice rational $N = 1$ SVOA with rank 12 and $V_{1/2} = 0$. Then $V$ is isomorphic to $\mathcal{C}A^{f_2}$ as an SVOA.

**Proof.** By Proposition 5.10 the bilinear form defined by the adjoint operators is non-degenerate, and Theorems 5.11 and 5.12 show that $V_1$ is a reductive Lie algebra with Lie rank bounded above by 12. From Proposition 5.14 we find that $V_1$ is of dimension 276 and the Killing form $\kappa(\cdot, \cdot)$ on $V_1$ satisfies $\kappa(\cdot, \cdot) = 44\langle \cdot | \cdot \rangle$. In particular, the Killing form is non-degenerate, and $V_1$ is a semi-simple Lie algebra.

Suppose then that $g$ is a simple component of $V_1$ with level $k$ and dual Coxeter number $h$. By the main theorem of [DM06] we have that $k$ is an integer. Suppose that $\langle \cdot, \cdot \rangle$ is the bilinear form on $g$ normalized so that $(\alpha, \alpha) = 2$ for a long root $\alpha$. Then we have $\langle u|v \rangle = k(u,v)$ for $u, v \in g$, and thus also $\kappa(u,v) = 44k(u,v)$ for $u, v \in g$. Taking $u = v = \alpha$ we obtain $h/k = 22$ since $\kappa(\alpha, \alpha) = 4h$. This argument is independent of the choice of simple component and so the ratio $h/k$ must hold for each simple component. By inspection the only possibility then is that $g$ is of type $D_{12}$ with level $k = 1$.

Thus we find that $V_1$ is a semisimple Lie algebra of type $D_{12}$, and the VOA $V_0$ is isomorphic to the lattice VOA $V_{M_0}$ for $M_0$ a copy of the $D_{12}$ lattice. It follows then that the SVOA $V$ is isomorphic to a lattice VOA $V_M$ for some positive definite integral lattice $M = M_0 \cup M_1$. Since $V$ is self-dual of rank 12, $M$ is self-dual of rank 12, and the fact that $V_{1/2} = 0$ implies that $M$ has no vectors of unit length. There is one such lattice up to isomorphism; namely, the lattice $D_{12}^-$. We conclude that any self-dual nice rational SVOA with rank 12 and trivial degree 1/2 subspace is isomorphic to $V_M$, where $M$ is a copy of the lattice $D_{12}^-$. From the proof of Proposition 5.13 we see that the SVOA $\mathcal{C}A^{f_2}$ is also such an object, and this completes the proof of the theorem. \hfill \Box

**Remark 5.16.** We sketch here an alternative approach to Theorem 5.15 that was described to us by Gerald Höhn. Suppose that $V$ is as in the statement of Theorem 5.15 and let us write $U_0$ for a copy of the lattice VOA associated to the lattice of type $D_4$. This VOA has three irreducible modules beyond itself; we pick one of them and denote it $U_1$. We then set $W = U_0 \otimes V_0 \oplus U_1 \otimes V_1$, so that $W$ is a module for the VOA $U_0 \otimes V_0$ with only integral weights. Using knowledge of the fusion of modules for $V_0$ and $U_0$ it can be shown that the VOA structure extends in a unique way from $U_0 \otimes V_0$ to the whole space. Then $W$ is a self-dual VOA of rank 16, and one may invoke Theorem 2 of [DM04a] to conclude that $W$ is a lattice VOA $W_L$ for $L$ one of the two self-dual lattices of rank 16; namely, $E_8 \oplus E_8$ or $D_{16}^+$. One then shows that the $D_4$ lattice VOA $U_0$ can only be embedded in $W_L$ in such a way that $V_0$ must also be a lattice VOA, and the lattice must be of type $D_{12}$. 


5.2. \textbf{N = 1 structure.} We now wish to demonstrate that the \( N = 1 \) structure on \( A^2 \) is unique. More precisely, suppose that \( t \in \text{CM}(l)_G^0 \) satisfies \( \langle e gt, t \rangle = 0 \) for any \( S \subset \Omega \) with \( 0 < w(S) \leq 4 \). Citing Proposition 5.18 as justification, we call such a vector superconformal. We wish to show that if \( t \) is superconformal with \( |t| = 1 \) then \( t = x1_G \) for some \( x \in \text{Spin}(l) \). This will be achieved in Theorem 5.28 after we establish a few preliminary lemmas. For the benefit of the reader we now include a few words about the idea behind the proof of this theorem.

5.2.1. \textbf{Strategy.} Our strategy is the following. Suppose that \( t \in \text{CM}(l)_G^0 \) is a superconformal vector of unit norm, and define a function \( f_t : \text{Spin}(l) \rightarrow [-1, 1] \) by \( f_t(x) = \langle x1_G, t \rangle \). Since the bilinear form on \( \text{CM}(l)_G^0 \) is non-degenerate we are done as soon as we find an \( x \in \text{Spin}(l) \) such that \( f_t(x) = 1 \). We will show that for any \( x \in \text{Spin}(l) \) with \( f_t(x) < 1 \) there exists some \( x' \in \text{Spin}(l) \) such that \( f_t(x') > f_t(x) \). The function \( f \) is certainly continuous and \( \text{Spin}(l) \) is compact, so showing that \( f_t \) can always be made closer to 1 suffices to show that \( f_t \) attains the value 1. In fact, since \( at \) is superconformal for \( x \in \text{Spin}(l) \) whenever \( t \) is, it suffices to show only that for any superconformal \( t \) there is some \( x \in \text{Spin}(l) \) such that \( f_t(x) > f_t(1) \) where 1 denotes the identity in \( \text{Spin}(l) \). Thus the following results up to and including Theorem 5.28 are dedicated to showing that for any superconformal \( t \) in \( \text{CM}(l)_G^0 \) there is some \( x \in \text{Spin}(l) \) such that \( \langle x1_G, t \rangle > \langle 1_G, t \rangle \).

We next recall some facts about the Golay co-code, and then introduce some useful notation and terminology before presenting Propositions 5.18 and 5.20 which will be the main tools we use to implement the stated strategy. A superconformal vector \( t \) may be regarded as an element of the unit ball in 2048 dimensional space, and Proposition 5.18 provides a way of regarding \( t \) as an element of the unit ball in \( 2048/|\Gamma| \) dimensional space via a kind of linearization over the cosets of certain subgroups \( \Gamma < G^* \). The Proposition 5.20 is a generalization of this result which arises essentially because \( G^* \) has many distinct lifts to \( \mathbb{F}_2^{|\Omega|} \). We sometimes refer to Propositions 5.18 and 5.20 as the coset contraction results.

5.2.2. \textbf{Golay co-code.} Recall that the Golay co-code is the space \( G^* = \mathbb{F}_2^{|\Omega|}/G \), and recall the co-weight function \( w^* \) on \( \mathbb{F}_2^{|\Omega|} \) from §1.4. We write \( X \mapsto \bar{X} \) for the canonical map \( \mathbb{F}_2^{|\Omega|} \rightarrow G^* \). The Golay code corrects three errors and the range of the co-weight function is the set \{0, 1, 2, 3, 4\}. Let \( X \in \mathbb{F}_2^{|\Omega|} \). We say that \( X \) and \( \bar{X} \) are co-even if \( 2|w^*(X)| \), and we say that \( X \) and \( \bar{X} \) are doubly co-even if \( 4|w^*(X)| \).

If \( w^*(X) = 2 \) then there is a unique pair of points \( i, j \in \Omega \) such that \( X + G = \{ij\} + G \). If \( w(X) = w^*(X) = 4 \) then there are exactly five weight 8 words (octads) in \( G \) containing \( X \), and we have \( w(Y) = w^*(Y) = 4 \) and \( \bar{X} = \bar{Y} \) just when \( X + Y \) is one of these. Thus for \( X \in G^* \) with \( w^*(X) = 4 \), the six sets of cardinality four in \( \mathbb{F}_2^{|\Omega|} \) that lift \( \bar{X} \) constitute a partition of \( \Omega \) into six disjoint four sets. Such a partition is called a sextet, and the four sets in a given sextet are called tetrads.

Let \( \bar{T}, \bar{T}' \in G^* \) with co-weight 4, and let \( S = \{T_i\} \) and \( S' = \{T'_j\} \) be the sextets determined by \( \bar{T} \) and \( \bar{T}' \) respectively. Then there are essentially four different ways that the sextets \( S \) and \( S' \) can overlap.

(1) \( |T_i \cap T'_j| \in \{0, 4\} \) for all \( i, j \).
(2) \( |T_i \cap T'_j| \in \{0, 1, 3\} \) for all \( i, j \).
(3) \( |T_i \cap T'_j| \in \{0, 1, 2\} \) for all \( i, j \).
(4) \( |T_i \cap T'_j| \in \{0, 2\} \) for all \( i, j \).
The first case is just the case that $S$ and $S'$ are the same sextet. The second case is the case that $\bar{T} + \bar{T}'$ has co-weight 2. In the third and fourth cases $\bar{T} + \bar{T}'$ has co-weight 4, and the last case is distinguished by the property that any lift of the set $\{\bar{T}, \bar{T}'\}$ is isotropic. We refer to the sextets $S$ and $S'$ as commuting sextets when $|T_i \cap T_j'|$ is even for all $i$ and $j$, and we refer to them as non-commuting otherwise.

In this setting, we refer to tetrads determined by any pair of tetrads in $\bar{\Delta}$ are commuting in the sense of $\bar{\Delta}$ is doubly co-even and balanced. We then say that $\bar{\Delta}$ is commuting if the sextets $\langle T_i \rangle$ and $\langle T_i' \rangle$ are commuting, and we refer to them as non-commuting otherwise.

5.2.3. Co-code lifts. Let $\Delta$ be a subset of $F^0_2$ such that the map $F^0_2 \to G^*$ induces a bijection $\Delta \leftrightarrow \Delta = \{X \mid X \in \Delta\}$. We then call $\Delta$ a lift of $\Delta$ to $F^0_2$. If further we have that $w(X) = \omega(X)$ for all $X \in \Delta$, we say that $\Delta$ is a balanced lift of $\bar{\Delta}$. Recall that the space $\bar{\Delta}$ admits a bilinear form $F^0_2 \times F^0_2 \to F_2$ defined so that $\langle X, Y \rangle = |X \cap Y|$ (mod 2). In some cases a subset $\Delta < G^*$ has a lift $\Delta \subset F^0_2$ such that $\langle X, Y \rangle \equiv 0$ for any $X, Y \in \Delta$, and we call such a lift isotropic. Suppose that $\Delta$ is doubly co-even and balanced. We then say that $\Delta$ is commuting if the sextets determined by any pair of tetrads in $\Delta$ are commuting in the sense of $\bar{\Delta}$.

Suppose now that $\Sigma$ is a balanced lift of $(G^*)^0$, the even part of $G^*$. Then the set $\Sigma$ is in natural bijective correspondence with $(G^*)^0$, and the group structure on the later may be lifted via this correspondence so as to define a group structure on the former. We denote this group operation with $\tilde{+}$, so that $X + Y = Z$ just when $X + Y + Z \in G$. Note that the bilinear form on $\bar{\Sigma}$ is not bilinear with respect to $\tilde{+}$, so that in general $\langle A + B, X \rangle \neq \langle A, X \rangle + \langle B, X \rangle$ for example. A multiplicative 2-cocycle $\sigma$ with values in $\{\pm 1\}$ is defined on the group $\Sigma = (\Sigma, \tilde{+} )$ by requiring that $e_X e_Y 1_G = \sigma(X, Y) e_{X + Y} 1_G$ for $X, Y \in \Sigma$.

**Proposition 5.17.** We have

\[(5.2.1) \quad \sigma(X, X) = (-1)^{\omega(X)/2}\]
\[(5.2.2) \quad \sigma(X, Y) = (-1)^{\langle X, Y \rangle} \sigma(Y, X)\]
\[(5.2.3) \quad \sigma(X, Y) = \sigma(X, X) \sigma(X, X + Y)\]

for all $X, Y \in \Sigma$. In particular $\sigma(X, X + Y) = \sigma(X, X + Y)$ for all $Y \in \Sigma$ just when $X$ is doubly co-even.

**Proof.** For any $X, Y \in \Sigma$ we have $e_X e_Y 1_G = (-1)^{\langle X, Y \rangle} e_{X-Y} 1_G$, and this implies $\sigma(X, Y) = (-1)^{\langle X, Y \rangle} \sigma(Y, X)$. Left multiplying both sides of $e_X e_Y 1_G = \sigma(X, Y) e_{X+Y} 1_G$ by $e_X$ yields $\sigma(X, Y) e_{X+Y} 1_G = e_X^2 e_X 1_G$. On the other hand $e_X e_{X+Y} 1_G = \sigma(X, X + Y) e_{X+Y} 1_G$. Since $e_X^2 = \sigma(X, X)$, this verifies the claim. $\square$

When $\Sigma$ is a lift of $(G^*)^0$, the set $\{e_X 1_G \mid X \in \Sigma\}$ constitutes an orthonormal basis for $CM(l)^0_G$. We then have $t = \sum_{X} t_X e_X 1_G$ for unique $t_X \in \mathbb{R}$ such that $\sum t_X^2 = 1$ when $t$ is a unit vector in $CM(l)^0_G$. Note that $f_L(1) = t \emptyset$. We write $\text{supp}(t)$ for the set of $X \in G^*$ such that $t_X \neq 0$.

One way to obtain a balanced lift of $(G^*)^0$ is the following. Choose an element in $\Omega$ and denote it by $\infty$. Let $\Delta = \Delta_0 \cup \Delta_2 \cup \Delta_4$ where $\Delta_0$ contains just the emptyset, $\Delta_2$ is the set of pairs of elements from $\Omega$, and $\Delta_4$ is the set of subsets of $\Omega$ of size four containing $\infty$. Then $\Delta$ is a balanced lift of $(G^*)^0$. A doubly co-even subgroup $\Gamma < \Delta$ is then isotropic just when it is commuting.
5.2.4. Coset contraction. Let $\Sigma$ be a balanced lift of $(G^*)^0$, suppose that $\bar{\Gamma}$ is a subgroup of $G^*$, and suppose that $W \in \Sigma$ is chosen so that the coset $W + \bar{\Gamma}$ of $\bar{\Gamma}$ in $G^*$ is doubly co-even. Suppose also that the corresponding lift $W + \Gamma \subset \Sigma$ obtained by restriction from $\Sigma$ is isotropic. Then the 2-cocycle $\sigma$ is symmetric on $W + \Gamma$. Let $\lambda : \Sigma \rightarrow \pm 1$ be a function such that

\begin{equation}
\lambda(A + W)\sigma(A + W, Z) = \chi(Z)\lambda(A + W + Z), \quad \forall A \in \Gamma, \ Z \in \Sigma.
\end{equation}

Then by Proposition 5.18 we have $\sigma(A + W, Z) = \sigma(A + W, A + W + Z)$ so that the invariance under swapping $Z$ with $A + W + Z$ is evident for both sides of the expression (5.2.4). The assumption that $W + \Gamma$ be isotropic ensures that (5.2.4) can be satisfied for $Z$ in $W + \Gamma$.

In the case that $W + \bar{\Gamma} = \Gamma$ the condition (5.2.4) implies that the restriction $\lambda|_{\bar{\Gamma}}$ of $\lambda$ to $\bar{\Gamma}$ is a 1-cocycle with coboundary $\sigma|_{\bar{\Gamma} \times \bar{\Gamma}}$ since we have $\sigma(A + B, A) = \sigma(A, B)$ for $A, B \in \Gamma$ when $\Gamma$ is isotropic. The values of $\lambda$ on a coset $Z + \Gamma$ of $\Gamma$ in $\Sigma$ are determined by those on $\Gamma$ together with $\chi(Z)$ since $\lambda(Z + A) = \chi(A)\sigma(A, Z)/\chi(Z)$. Any two such functions $\lambda : \Sigma \rightarrow \pm 1$ therefore differ by a single element of $\bar{\Gamma}$ in $\Sigma$. Recall that the maps $X \mapsto (-1)^{(D, X)}$ for $D \in G$ exhaust the homomorphisms $G^* \rightarrow \pm 1$. Indeed, $(-1)^{(D, X)}$ is independent of the choice of lift $X$ for $\bar{X}$, and thus any homomorphism $(\Sigma, +) \rightarrow \pm 1$ is of the form $X \mapsto (-1)^{(D, X)}$ for some $D \in G$.

To a function $\lambda$ satisfying (5.2.4) and to any given $Z \in \Sigma$ we associate the element $u_{\lambda, Z} = \sum_{A \in \Gamma} \chi(Z + A)e_{Z + A}$ in Cliff$(\Gamma)$, and for ease of notation we set $u_\lambda = u_{\lambda, \emptyset}$. The following proposition is our main tool for classifying superconformal vectors in $\text{CM}(\Gamma)$.  

**Proposition 5.18.** Let $t = \sum_{X} t_X e_X 1_G$ be superconformal with $|t| = 1$. Let $\bar{\Gamma}$ be a subgroup of $G^*$ and suppose that $W \in G^*$ is chosen so that $W + \bar{\Gamma}$ is doubly co-even. Suppose also that $W + \Gamma \subset \Sigma$ is an isotropic lift of $W + \bar{\Gamma}$. Then for $\lambda : \Sigma \rightarrow \pm 1$ satisfying (5.2.4) and for $\Gamma$ any transversal of $\Gamma$ in $\Sigma$ we have

\begin{equation}
\sum_{Z \in \Gamma} \langle u_{\lambda, Z} 1_G, t \rangle \langle u_{\lambda, W + Z} 1_G, t \rangle = \langle u_\lambda 1_G, t \rangle \begin{cases} 1 & \text{if } W \in \Gamma, \\ 0 & \text{if } W \notin \Gamma. \end{cases}
\end{equation}

In particular, for the case that supp$(t)$ is a doubly co-even group with an isotropic lift $\Gamma$ we have $\sum_{A \in \Gamma} \chi(A)t_A = \pm 1$ for any 1-cocycle $\lambda : \Gamma \rightarrow \pm 1$ with coboundary $\sigma|_{\Gamma \times \Gamma}$.

**Proof.** Let us consider the expression $\langle u_\lambda 1_G, t \rangle$. Since $t$ is superconformal, we have $\langle e_X t, t \rangle = 0$ whenever $w(X) < 8$, so that $\langle u_\lambda 1_G, t \rangle = \chi(\emptyset)t(t, t) = 1$. On the other hand we have

\begin{equation}
\begin{aligned}
\langle u_\lambda 1_G, t \rangle &= \sum_{A \in \Gamma, Z \in \Sigma} \chi(A)t_Z e_{A e_Z 1_G} \\
&= \sum_{A \in \Gamma, Z \in \Sigma} \chi(A)\sigma(A, Z)t_Z e_{Z + A 1_G} \\
&= \sum_{A \in \Gamma, Z \in \Sigma} \chi(Z)\lambda(Z + A)t_Z e_{Z + A 1_G}
\end{aligned}
\end{equation}

and then $1 = \langle u_\lambda 1_G, t \rangle = \sum_{A \in \Gamma} \chi(Z)\lambda(Z + A)t_Z e_{Z + A}$. From the fact that $\langle u_{\lambda, Z} 1_G, t \rangle = \sum_{A \in \Gamma} \chi(Z + A)t_Z e_{Z + A}$ we see that the left hand side of (5.2.5) coincides with $\langle u_\lambda 1_G, t \rangle$, and the equation (5.2.4) follows. This handles the case that $W \in \Gamma$, and the
case that \( W \notin \Gamma \) is similar. The case that all \( t_X \) vanish for \( X \notin \Gamma \) then yields \((\sum_X \chi(X)t_X)^2 = 1\), and the last part follows from this. \(\square\)

Suppose that \( \Sigma \) and \( \Sigma' \) are balanced lifts of \( \mathcal{G}^* \) to \( \mathbb{F}^*_2 \). We denote the group operations on \( \Sigma \) and \( \Sigma' \) both by \( + \). There is a correspondence \( \Sigma \leftrightarrow \Sigma' \) such that we have \( X \leftrightarrow X' \) if and only if \( \bar{X} = \bar{X}' \in \mathcal{G}^* \). We then have \( X + Y = Z \) in \( \Sigma \) just when \( X' + Y' = Z' \) in \( \Sigma' \). For \( X \in \Sigma \) we have \( \langle X, X' \rangle = 0 \) and \( X + X' \in \mathcal{G} \). Define \( g : \mathcal{G}^* \rightarrow \pm 1 \) so that \( g_x e_X e_Y 1_G = 1_G \). We abuse notation to write \( g_X \) for \( g_x \) whenever \( X \in \Sigma \). Let \( \sigma' \) denote the multiplicative 2-cocycle on \( (\Sigma', +) \) such that \( e_X e_Y 1_G = \sigma'(X', Y')e_Z 1_G \) for \( Z' = X' + Y' \). The following lemma gives the relationship between \( \sigma \) and \( \sigma' \).

**Lemma 5.19.** For \( X, Y \in \Sigma \) and \( Z = X + Y \) we have

\[
(5.2.7) \quad \sigma'(X', Y') = (-1)^{\langle X+X', Y \rangle} g_X g_{Y} g_{X+Y} \sigma(X, Y)
\]

As before we assume that \( W + \Gamma \) is doubly co-even and isotropic, and we assume now the same for \( W' + \Gamma' \). We also assume that \( \langle X', Y \rangle = \langle X, Y' \rangle \) for all \( X, Y \in W + \Gamma \). Given \( \chi : \Sigma \rightarrow \pm 1 \) satisfying \((5.2.4)\), we define \( \chi' : \Sigma' \rightarrow \pm 1 \) by setting \( \chi'(X') = \psi(X)\chi(X) \) for \( X' \in \Sigma' \) where \( \psi : \Sigma \rightarrow \pm 1 \) is chosen to satisfy

\[
(5.2.8) \quad \psi(X)(-1)^{\langle X + X', Z \rangle} g_X g_{X+Z} \psi(Z) = \psi(X+Z), \quad \forall X \in W + \Gamma, Z \in \Sigma.
\]

Then from Lemma 5.19 we obtain that \( \chi'(X')\sigma'(X', Z') = \chi'(Z')\chi'(X'+Z') \) whenever \( X' \in W' + \Gamma' \) and \( Z' \in \Sigma' \). Now we may apply Proposition 5.18 with \( \Sigma' \) in place of \( \Sigma \), and \( \chi' \) in place of \( \chi \), and we obtain the following generalization of that proposition.

**Proposition 5.20.** Let \( \Sigma, \Sigma' \) be balanced lifts of \( \mathcal{G}^* \), and suppose that \( \Gamma < \Sigma \) and \( W \in \Sigma \) are chosen so that both \( W + \Gamma \) and \( W' + \Gamma' \) are doubly co-even and isotropic. Suppose also that \( \langle X', Y \rangle = \langle X, Y' \rangle \) for all \( X, Y \in W + \Gamma \). Then for \( \chi : \Sigma \rightarrow \pm 1 \) satisfying \((5.2.4)\), for \( \psi : \Sigma \rightarrow \pm 1 \) satisfying \((5.2.8)\), and for \( \Gamma \) a transversal of \( \Gamma \) in \( \Sigma \), we have

\[
(5.2.9) \quad \sum_{Z \in \Gamma} (u_{\chi''} Z 1_G, t) (u_{\chi''} W + Z 1_G, t) = \begin{cases} 1 & \text{if } W \in \Gamma, \\ 0 & \text{if } W \notin \Gamma. \end{cases}
\]

where \( \chi'' : \Sigma \rightarrow \pm 1 \) is given by \( \chi''(X) = g_X \psi(X)\chi(X) \), and \( u_{\chi''} Z \) is defined by \( u_{\chi''} Z = \sum_{A \in \Gamma} \chi''(A + Z) e_{A+Z} \) for \( Z \in \Sigma \).

### 5.2.5. Superconformal vectors

We now embark upon the task of realizing the strategy summarized in \((5.2.4)\) that is, the task of showing that for any superconformal vector \( t \in \text{CM}(\mathbb{F}^*_2) \), there is some \( x \) in \( \text{Spin}(\mathbb{F}) \) such that \( f_t(x) > t_0 \). In practice, we treat all possible unit vectors \( t \) on a case by case basis using the coset contraction results to narrow down the possibilities for the coefficients \( t_X \) of \( t = \sum_X t_X e_X 1_G \) (given a balanced lift \( \Sigma \) of \( (\mathcal{G}^*)^0 \) to \( \mathbb{F}^*_2 \)) that can make \( t \) superconformal. In the course of doing so we find superconformal vectors in the \( \text{Spin}(\mathbb{F}) \) orbit of \( 1_G \) other than those of the form \( e_X 1_G \) for \( X \in \mathbb{F}^*_2 \), or \( \exp(re_X) 1_G \) for \( r \in \mathbb{R} \) and \( w(X) = 2 \), and ultimately we find that \( t \) either has projection on one of these vectors exceeding \( t_0 \) or cannot be superconformal.

Suppose that \( |t_X| > t_0 \) for some \( X \in \Sigma \). Then we have \( \langle X 1_G, t \rangle = \sigma(X, X)t_X \) for \( x = e_X \). Multiplying by \(-1\) if necessary, we have \( f_t(x) > t_0 \). Thus from now on we may suppose that \( t_0 \geq |t_X| \) for all \( X \in \Sigma \).
Suppose that \( t_X \neq 0 \) for some \( X \) with \( w^*(X) = 2 \). Setting \( \exp(re_X) = \cos(r) + \sin(r) e_X \) we have \( \exp(re_X) = (1 - \frac{1}{2}r^2) t_0 - rt_X + o(r^2) \) so that \( f_t(x) > t_0 \) for \( x = \exp(re_X) \) and suitably chosen \( r \). From now on we assume that \( t_X = 0 \) whenever \( w^*(X) = 2 \). That is we may assume that \( \text{supp}(t) \) is a doubly co-even subset of \( G^* \).

Suppose that \( \text{supp}(t) \) is contained in a doubly co-even subgroup \( \Gamma \) of \( G^* \) and suppose that \( \Gamma \) has a balanced isotropic lift \( \Gamma \). We may assume that \( \Sigma \) is a balanced lift of \( (G^*)^0 \) containing \( \Gamma \). Since it is useful, we now state the following result, which is obtained by direct application of Proposition 5.20 to our present situation.

**Proposition 5.21.** Let \( \Gamma \) be a doubly co-even subgroup of \( G^* \) and suppose that \( \Gamma \) and \( \Gamma^\prime \) are balanced isotropic lifts of \( \Gamma \) such that \( \langle A', B \rangle = \langle A, B' \rangle \) for all \( A, B \in \Gamma \).

Then for \( \chi : \Gamma \to \pm 1 \) a 1-cocycle with coboundary \( \sigma|_{\Gamma \times \Gamma} \), and for \( \psi : \Gamma \to \pm 1 \) satisfying

\[
\psi(A)(-1)^{\langle A + A', B \rangle} \theta_A \theta_B \theta_{A + B} = \psi(B) \psi(A + B)
\]

for all \( A, B \in \Gamma \) we have \( \langle u_{\chi''}|_{G}, t \rangle^2 = 1 \) where \( \chi'' : \Gamma \to \pm 1 \) is given by \( \chi''(A) = \theta_A \psi(A) \chi(A) \), and \( u_{\chi''} = \sum_{A \in \Gamma} \chi''(A)e_A \).

The requirement that \( \Gamma \) be doubly co-even is quite strong. Any maximal doubly co-even subgroup of \( G^* \) has order 16 or 32, and every doubly co-even subgroup of order 16 or less has an isotropic lift. A convenient way to generate doubly co-even partitions the 12 non-zero coordinates into six pairs \( \{A_i\} \). Then the 15 elements \( A_i + A_j \in \Gamma \) are the non-trivial elements in a doubly co-even subgroup \( \Gamma \) say, of \( G^* \) of order 16. Furthermore, the set \( \{0, A_i + A_j\} \) furnishes a balanced isotropic lift of \( \Gamma \). For some partitions there is a sextet \( S = \{T_i\} \) such that each pair \( A_i \) is contained in a tetrad of \( S \). In this case, the addition of one of the \( T_i \) extends \( \Gamma \) to be a balanced isotropic lift of a doubly co-even 32 group in \( G^* \).

Suppose then that \( \text{supp}(t) \) is contained in a two group \( \Gamma \) with balanced lift \( \Gamma = \{0, A\} \). Then since \( t \) is a unit vector we have \( t_0^2 + t_A^2 = 1 \). On the other hand from Proposition 5.21 we have \( (t_0 \pm t_A)^2 = 1 \) and thus \( t_0 + t_A = \pm 1 \) and \( t_0 - t_A = \pm 1 \). The only solutions are \( t_0 = \pm 1 \) and \( t_A = \pm 1 \). Since we have assumed \( t_0 \geq |t_X| \) for all \( X \in \Sigma \), we have \( t_0 = 1 \) and \( t = t_G \).

Suppose now that \( \text{supp}(t) \) is contained in a four group \( \Gamma \) with balanced isotropic lift \( \Gamma = \{0, A, B, C\} \). Similar to before we have \( t_0^2 + t_A^2 + t_B^2 + t_C^2 = 1 \). A function \( \chi_0 : \Gamma \to \pm 1 \) suitable for an application of Proposition 5.21 may be given arbitrary values on generators of \( \Gamma \), and then the remaining values are determined by \( \sigma \). For example, we may set \( \chi_0(\emptyset) = \chi_0(A) = \chi_0(B) = 1 \) and \( \chi_0(C) = \sigma(A, B) \chi_0(A) \chi_0(B) = \sigma(A, B) \). Any other suitable 1-cocycle \( \chi \) differs from \( \chi_0 \) by some element of \( \Gamma^* \), and Proposition 5.21 (with \( \Sigma' = \Sigma \)) now yields

\[
t_0 + (\mu(A)t_A + \mu(B)t_B + \sigma(A, B) \mu(C)t_C) = \pm 1
\]

where \( \mu \) is any homomorphism \( \mu : \Gamma \to \pm 1 \). Summing over 5.21.11 for various choices of \( \mu \) we find that \( t_X \pm t_Y \in \{0, \pm 1\} \) for each \( X, Y \in \Gamma \), and then \( t_X \in \{0, \pm 1\} \) for each \( X \in \Gamma \). Thus if one of the \( t_X \) vanishes then the remaining \( t_Y \) must lie in \( \{0, \pm 1\} \), and then \( t = \pm t_Y \) for some \( Y \in \Gamma \). If one of the \( t_X \) is \( \pm 1 \) then the remaining \( t_Y \) must lie in \( \pm \frac{1}{2} \) and there is some restriction on the signs: any two solutions differ by an element of \( \Gamma^* \), and one solution is given by \( s = \frac{1}{2}(1 + e_A + e_B - \sigma(A, B)e_C)1_G \). Taking \( \Sigma' \) different from \( \Sigma \) we can say more:
if there is some lift \( \Gamma' \) of \( \bar{\Gamma} \) such that \( \langle X', Y \rangle \neq 0 \) for some \( X, Y \in \Gamma \), then we obtain another equation like \( \sigma_{X'} \), with signs not differing by an element of \( \Gamma' \), and this extra restriction is enough to rule out the possibility that any \( t_\chi \) has norm \( \frac{1}{2} \). Such a lift \( \Gamma' \) exists just when two of the sextets corresponding to \( \{ \bar{A}, \bar{B}, \bar{C} \} \) are non-commuting (see (5.2.13)). We are left with the question of whether of not a vector \( s \in \text{CM}(1)'_G \) of the form \( s = \frac{1}{2}(1 + \mu(A)e_A + \mu(B)e_B - \sigma(A,B) \mu(C)e_C)1_G \) with \( \mu \in \Gamma^* \) is in the \( \text{Spin}(l) \) orbit containing \( 1_G \) given that the sextets in \( \Gamma \) are commuting. The answer is affirmative, as the following lemma demonstrates.

**Lemma 5.22.** Suppose \( \Gamma = \{ \emptyset, A, B, C \} \) is a balanced lift of a commuting four group in \( \mathcal{G}^* \), and

\[
(5.2.12) \quad s = \frac{1}{2} (1 + \mu(A)e_A + \mu(B)e_B - \sigma(A,B) \mu(C)e_C)1_G
\]

for some \( \mu \in \Gamma^* \). Then \( s \) is in the \( \text{Spin}(l) \) orbit containing \( 1_G \).

**Proof.** For \( S \in \mathcal{G} \) let \( g_S = \pm e_S \) with the sign chosen so that \( g_S \in G \). For any given \( S \in \mathcal{G} \) either \( \langle S, X \rangle = 0 \) for all \( X \in \Gamma \), or there is a unique non-trivial \( X \in \Gamma \) such that \( \langle S, X \rangle = 0 \). We define a group \( G' = \{ g'_S \mid S \in \mathcal{G} \} < \text{Spin}(l) \) by setting \( g'_S = Xe_Xg_S \) when \( X \) is the unique non-trivial element of \( \Gamma \) such that \( \langle S, X \rangle = 0 \), and setting \( g'_S = g_S \) when \( \langle S, X \rangle = 0 \) for all \( X \in \Gamma \). Then a simple computation shows that \( g'_Sg_S = s \) for all \( S \in \mathcal{G} \). The group \( G' \) is a \( F^2_4 \)-homogeneous subgroup of \( \text{Spin}(l) \) whose associated code is doubly-even self-dual and has no short roots. In other words, \( G' \) is a lift of a Golay code on \( \Omega \). Noting that both \( G \) and \( G' \) contain the volume element \( e_\Omega \). It follows from the uniqueness of the Golay code that there is some coordinate permutation in \( \text{Spin}(l) \) that sends \( s \) to \( 1_G \).

The method illustrated above for the case that \( \bar{\Gamma} \) has order four is a model for the cases of higher order. For this reason we will summarize only the results for the higher order cases that we need, and refrain from burdening the reader with all the details.

**Lemma 5.23.** In the case that \( \text{supp}(t) \) is contained in an eight group \( \bar{\Gamma} \) with balanced isotropic lift \( \Gamma = \langle A, B, C \rangle \), either \( \text{supp}(t) \) is contained in a commutative four group, or \( \bar{\Gamma} \) is totally commutative and \( t \) is of the form

\[
(5.2.13) \quad t = \frac{1}{4} (3 + (A)e_A + (B)e_B + (C)e_C + (A + B)\sigma(A,B)e_Ae_B + \ldots)1_G
\]

for some \( \mu \in \Gamma^* \), and \( t \) belongs to the \( \text{Spin}(l) \) orbit containing \( 1_G \).

**Lemma 5.24.** In the case that \( \text{supp}(t) \) is contained in a 16 group \( \bar{\Gamma} \) with balanced isotropic lift \( \Gamma \), either \( \text{supp}(t) \) is contained in some commutative eight group, or there is a dodecad in \( \mathcal{G} \) and a partition \( P \) of its non-trivial coordinates into six pairs \( P = \{ A_0, \ldots, A_5 \} \) such that \( \Gamma = \{ \emptyset, A_{ij} \} \) where we write \( A_{ij} \) for the tetrad \( A_i + A_j \in F^2_4 \). We may assume that \( e_{A_0}e_{A_1} \cdots e_{A_5} \in G \). Then \( \sigma(A_{ij}, A_{ik}) = -1 \) and \( \sigma(A_{ij}, A_{kl}) = 1 \) for distinct \( i, j, k, l \). The vector \( t \) is then of the form

\[
(5.2.14) \quad t = \frac{1}{4} (1 + (A_0)e_{A_0} + \ldots + (A_4)e_{A_0} - (A_{12})e_{A_{12}} - \cdots - (A_{45})e_{A_{45}})1_G
\]

for some \( \mu \in \Gamma^* \), and \( t \) belongs to the \( \text{Spin}(l) \) orbit containing \( 1_G \).
Lemma 5.25. In the case that supp(t) is contained in a 32 group with balanced isotropic lift, either supp(t) is contained in some 16 group with isotropic lift, or \( f_1(x) > t_0 \) for \( x \mathcal{1}_G = s \) a superconformal vector with supp(s) contained in a doubly co-even 16 group with isotropic lift.

We must now treat the case that supp(t) is not contained in any doubly co-even group with isotropic lift. We remind here that we assume \( t_0 \geq |t_X| \) for all \( X \in \Sigma \), and \( t_X = 0 \) whenever \( w^*(X) = 2 \). We claim that for such \( t \), either \( f_1(x) > t_0 \) for some \( x \mathcal{1}_G = s \) with \( s \) supported on a doubly co-even group with isotropic lift as given in Lemmata \( \ref{5.22} \) through \( \ref{5.24} \) or \( t \) is not superconformal.

So let us assume that \( f_1(x) \leq t_0 \) for any \( x \in \text{Spin}(l) \) such that \( x \mathcal{1}_G = s \) for some non-vanishing co-weight 4 components appearing in Lemmata \( \ref{5.22} \) through \( \ref{5.24} \). This condition amounts to putting upper bounds on the moduli of the coefficients \( t_X \) that are non-zero. For example, take \( s \) and \( \Gamma \) as in Lemma \( \ref{5.22} \). Then \( \langle s, t \rangle \leq t_0 \) for all \( \mu \in \Gamma^* \) is equivalent to the inequalities

\[
0 \leq \frac{1}{2} (t_0 + \mu(A)t_A + \mu(B)t_B + \mu(C)s(A,B)t_C), \quad \forall \mu \in \Gamma^*,
\]

which in turn imply that the smallest of \( |t_A|, |t_B| \) and \( |t_C| \) is not greater than \( \frac{1}{2} \), given that all are non-zero. Also, one can construct elements \( x \in \text{Spin}(l) \) of the form \( x = \exp(\theta_1 x_1) \cdots \exp(\theta_6 x_6) \) for \( w(x_i) = 2 \) such that \( f_1(x) > t_0 \) so long as not all the non-vanishing \( t_X \) in \( t \) are too small. On the other hand, Proposition \( \ref{5.18} \) applied in the case that \( W \notin \Gamma \) can be used to show that the non-vanishing of some co-weight 4 coefficients \( t_X \) implies the non-vanishing of others. The simplest result of this kind is the following.

Lemma 5.26. If \( t_A \neq 0 \) then there is some \( B \in \Sigma \) such that \( t_B t_{A + B} \neq 0 \).

Proof. We take \( \Gamma = \{ \emptyset \} \) and \( W = X \) in Proposition \( \ref{5.18} \). We then obtain

\[
0 = \langle e_A t, t \rangle = \sum_{X \in \Sigma} \langle tx e_A x \mathcal{1}_G, t \rangle = 2t_0 t_A + \sum_{X \in \Sigma \backslash \emptyset, A} \sigma(A, X) t_X t_{A + X}
\]

and this implies the claim. \( \square \)

We require to show then the coefficients \( t_X \) for \( w^*(X) = 4 \) cannot all be too small. Since some non-vanishing co-weight 4 components implies more non-vanishing co-weight 4 components, let us consider the extreme case that \( t_X \neq 0 \) for all \( X \in \Sigma \) with \( w^*(X) = 4 \). Suppose that \( t_Z = \epsilon \) is the greatest among these (we may assume \( t_Z > 0 \)) so that \( \epsilon \geq |t_X| \) for all \( X \in \Sigma \) with \( w^*(X) = 4 \). Then since \( \sum t_X^2 = 1 \) we have \( t_0 > t_0^2 > 1 - N \epsilon^2 \) where \( N = 1771 \) is the number of co-weight 4 elements in \( \mathcal{G}^* \). On the other hand Proposition \( \ref{5.18} \) with \( \Gamma = \emptyset \) and \( W = Z \) yields \( 0 = \langle e_Z t, t \rangle \), and we then have

\[
0 = \langle e_Z t, t \rangle > 2(1 - N \epsilon^2) \epsilon - M \epsilon^2
\]

where \( M \) is the number of \( X \in \Sigma \) with \( w^*(X) = 4 \) such that \( w^*(Z + X) = 4 \). We have \( 2(1 - N \epsilon^2) \epsilon - M \epsilon^2 > 0 \) (that is, a contradiction) just when \( 2 > M \epsilon + 2N \epsilon^2 \), so that \( \epsilon \) cannot be smaller than \( 1/\sqrt{2N} \) for example. In this way we find that any \( t \) which does not satisfy \( f_1(x) > t_0 \) for some superconformal \( x \mathcal{1}_G = s \) already constructed is not superconformal. That is, we have the following.

Theorem 5.27. The superconformal vectors in \( \text{CM}(l)_G \) form a single orbit under the action of \( \text{Spin}(l) \). This orbit contains \( 1_G \).
From Theorems 5.15 and 5.27, we deduce the following characterization of $A^{f_2}$.

**Theorem 5.28.** Let $V$ be a self-dual nice rational $N = 1$ SVOA with rank 12 and $V_{1/2} = 0$. Then $V$ is isomorphic to $cA^{f_2}$ as an $N = 1$ SVOA.

For $x \in \text{Spin}_{24}(\mathbb{R})$ let us write $\text{tr}|_{24}x$ for the trace of $x$ in the representation of $\text{Spin}_{24}(\mathbb{R})$ on $\mathbb{R}^{24}$. We have $\text{tr}|_{24}(e_X) = 24 - 2n$ when $w(X) = n$. Combining Theorems 4.10 and 5.27, we obtain the following characterization of the group $\text{Co}_0$.

**Theorem 5.29.** Let $M$ be a spin module for $\text{Spin}_{24}(\mathbb{R})$ and let $t \in M$ such that $\langle xt, t \rangle = 0$ whenever $x \in \text{Spin}_{24}(\mathbb{R})$ is an involution with $\text{tr}|_{24}x = 16$. Then the subgroup of $\text{Spin}_{24}(\mathbb{R})$ fixing $t$ is isomorphic to $\text{Co}_0$.

### 6. Structure of $V^{f_2}$

In this section we summarize the construction of $V^{f_2}$, mentioned in the introduction, and we indicate how to construct an explicit isomorphism with $A^{f_2}$.

**6.1. Lattice $N = 1$ SVOAs.** There is a standard construction which assigns an $N = 1$ SVOA to a positive definite integral lattice, and we summarize that construction now. Suppose that $L$ is a positive definite integral lattice. Let $F$ be $\mathbb{R}$ or $\mathbb{C}$, and recall from 2.7 the SVOA $\mathbb{C}V_L$ associated to $L$ via the standard construction. Let $\mathfrak{a} = F \otimes_{\mathbb{Z}} L$, and let us denote $A(\mathfrak{a})$ by $\mathfrak{a}L$. Define $\mathbb{C}V^f_L$ to be the tensor product of SVOAs

$$(6.1.1) \quad \mathbb{C}V^f_L = \mathfrak{a}L \otimes \mathbb{C}V_L.$$ 

For $F = \mathbb{C}$ the SVOA $\mathbb{C}V^f_L$ admits a natural structure of $N = 1$ SVOA. To define the superconformal element, let $h_i$ be an orthonormal basis of $\mathfrak{a} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and let $e_i$ be the corresponding elements of $\mathfrak{a}L$ under the identification $\mathfrak{a}L = \mathfrak{a} = \mathbb{C} \otimes_{\mathbb{Z}} L$. Then we set $\tau$ to be the element in $(\mathbb{C}V^f_L)^{3/2}$ given by

$$(6.1.2) \quad \tau = \frac{i}{\sqrt{8}} \sum_i e_i(-\frac{1}{2})h_i(-1)\mathbf{1}$$

where we suppress the tensor product from our notation. From [KO03], [Sch98] for example, we have the following

**Theorem 6.1.** The element $\tau$ is a superconformal vector for $\mathbb{C}V^f_L$. In particular, $\mathbb{C}V^f_L$ admits a natural structure of $N = 1$ SVOA.

Just as in [DM94], we can obtain a real form $V^f_L$ for $\mathbb{C}V^f_L$ by setting $V^f_L = \mathfrak{a}L \otimes \mathbb{C}V_L$. Noting that $ih_i(-1) \in \mathfrak{a}V^f_L \subset V^f_L$ we see that $\tau \in V^f_L$, and the $N = 1$ structure on $\mathbb{C}V^f_L$ restricts so as to furnish an $N = 1$ structure on $V^f_L$.

For simplicity, let us suppose that the rank of $L$ is even. By the Boson-Fermion correspondence [Fre81], [DM94], we have an isomorphism of SVOAs $\mathbb{C}A_L \cong \mathbb{C}V_{Z_{n}}$ where $n = \text{rank}(L)/2$, so that $\mathbb{C}A_L$ is self-dual as an SVOA. The tensor product $\mathbb{C}V^f_L$ is therefore isomorphic to a lattice SVOA $\mathbb{C}V_{Z_{n^2} \oplus L}$. The lattice $Z_{n^2} \oplus L$ is self-dual just when $L$ is, so we conclude that the $N = 1$ SVOA associated to any self-dual lattice is self-dual as an SVOA. More generally, the irreducible $\mathbb{C}V^f_L$ modules are indexed by the cosets of $L$ in its dual.
6.2. The case that \( L = E_8 \). From now on we take \( L \) to be a lattice of \( E_8 \) type so that \( cV_L^f \) is a realization of the \( N = 1 \) SVOA associated to the \( E_8 \) lattice. Since \( L \) is a self-dual lattice, \( cV_L^f \) is a self-dual \( N = 1 \) SVOA. The idea is that the \( N = 1 \) SVOA \( cV_{f^2} \) should be a \( \mathbb{Z}/2 \)-orbifold of \( cV_L^f \). More particularly, we wish to define the space underlying \( cV_{f^2} \) to be \((cV_L^f)^0 \oplus (cV_L^f)^0_\theta \) where \( \theta \) is a suitably chosen involution on \( cV_L^f \), the space \((cV_L^f)^\theta \) is an \( \theta \)-twisted \( cV_L^f \)-module, and the superscripts outside the brackets indicate \( \theta \)-fixed points. In order to construct \( cV_{f^2} \) we must therefore specify the involution \( \theta \), and construct a \( \theta \)-twisted module. This will be the objective of the next two subsections.

6.3. Twisting. The SVOA \( cV_L^f \) admits an automorphism \( \theta \) given by \( \theta = \theta_f \otimes \theta_b \) where \( \theta_f \) denotes the parity involution on \( \mathcal{C}A_L \), and \( \theta_b \) denotes a lift of \(-1\) on \( L \) to \( \text{Aut}(cV_L^f) \). Observe that both \( \theta_f \) and \( \theta_b \) may be regarded as a lift of \(-1\) on \( L \). We have \( \theta(\tau_V) = \tau_V \), so \( \theta \) is an automorphism of the \( N = 1 \)-structure on \( cV_L^f \). Also, the real form of \( cV_L^f \) is just

\[
V_L^f = \left\{ u + i v \mid u \in (\mathbb{R}V_L^f)^0, \ v \in (\mathbb{R}V_L^f)^1 \right\} \subset cV_L^f
\]

where \( \mathbb{R}V_L^f = (\mathbb{R}V_L^f)^0 \oplus (\mathbb{R}V_L^f)^1 \) indicates the decomposition into \( \theta \)-eigenspaces.

An \( \theta \)-twisted \( cV_L^f \)-module \( (cV_L^f)^\theta \) is of the form

\[
(cV_L^f)^\theta = (\mathcal{C}A_L)_{\theta_f} \otimes (cV_L)^{\theta_b}
\]

where \( (\mathcal{C}A_L)_{\theta_f} \) is a canonically twisted \( \mathcal{C}A_L \)-module and may be constructed as in [3.4] and \( (cV_L)^{\theta_b} \) is a \( \theta_b \)-twisted module over \( cV_L \). There is a well known method for constructing \( \theta_b \)-twisted \( cV_L \) modules for \( \theta_b \) a lift of \(-1\) on \( L \), and one may refer to [FLMSS] for a thorough treatment. It turns out that for the case we are interested in there is a simpler approach using only modules over lattice VOAs, and this in turn can be viewed from the point of view of Clifford module SVOAs. Such an approach is convenient for our purpose.

Recall that the lattice \( L \) contains a sublattice of the form \( K \oplus K \) where \( K \) is a lattice of \( D_4 \) type. Let \( K^* \) denote the dual lattice to \( K \), and let \( K_\gamma \) for \( \gamma \in \Gamma = \{0, 1, \omega, \bar{\omega}\} \) be an enumeration of the cosets of \( K \) in \( K^* \). We decree that \( K_0 = K \). The remaining cosets \( K_\gamma \) for \( \gamma \neq 0 \) are permuted by automorphisms of \( K^* \) preserving \( K \), and for this reason it is natural to regard \( \Gamma \) as a copy of the field of order 4. We may assume that the lattice \( L \) decomposes as \( L = \bigcup_{\Gamma} K_\gamma \oplus K_\gamma^* \) into cosets of \( K \oplus K \). Then the VOA \( cV_L \) has a decomposition

\[
cV_L = \bigoplus_{\gamma \in \Gamma} cV_{K_\gamma} \oplus cV_{K_\gamma^*}
\]

The VOA \( cV_K \), being a VOA of \( D_4 \) type, may be realized using Clifford module SVOAs, and similarly for its modules \( cV_{K_\gamma} \). In fact we may take \( cV_K \) to be a copy of \( A(\mathcal{C}a)^0 \), and then \( \bigoplus_{\Gamma} cV_{K_\gamma} \) is isomorphic as an \( A(\mathcal{C}a)^0 \)-module to the space \( A(\mathcal{C}a) \oplus A(\mathcal{C}a)_{\theta_f} \).

The corresponding construction of \( cV_L \), using \( A(\mathcal{C}a)^0 \)-modules in place of \( cV_{K_\gamma} \)-modules, was achieved in [PPR91]. Indeed, they provided more than this, describing certain twisted modules over \( cV_L \) and proving that the direct sum of these twisted and untwisted structures may be equipped with a certain generalization of VOA structure; namely para-VOA structure. It turns out that the twisting involutions considered in [PPR91] are conjugate to the involution \( \theta_b \) under the action
of \(\text{Aut}(cV_L) \cong E_8(\mathbb{C})\), and in particular, we may use one of these in place of \(\theta_b\).

Before describing precisely the involution we will use, we set up some new notation, and recall the relevant results of [FFR91].

6.4. Clifford construction of \(cV_L\). Recall that \(F = F \otimes_{\mathbb{Z}} L\) for \(F = \mathbb{R}\) or \(\mathbb{C}\).

The extended Hamming code is the unique up to equivalence doubly-even self-dual code of length 8. Let \(\Pi\) be some set of cardinality 8, and let \(H\) be a copy of the extended Hamming code, which we regard at once as a subset of \(\mathcal{P}(\Pi)\), and as a subspace of \(\mathbb{F}_2^\Pi\). Let \(E = \{e_i\}_{i \in \Pi}\) be an orthonormal basis for \(\mathbb{R}a \subset \mathcal{A}\), and let \(H\) be an \(\mathbb{F}_2^2\)-homogeneous lift of \(H\) to \(\text{Spin}(\mathbb{R}a)\). We may then realize an \(\theta_f\)-twisted \(A(\mathfrak{a})\)-module explicitly by setting \(A(\mathfrak{a})_{\theta_f} = A(\mathfrak{a})_{\theta_f,H}\).

We define \(\gamma_0\) to be the VOA \(A(\mathfrak{a})^0\) and we enumerate the \(\gamma_0\)-modules \(\gamma_\gamma\) for \(\gamma \in \Gamma\) by setting

\[(6.4.1)\] \(\gamma_0 = A(\mathfrak{a})^0, \quad \gamma_1 = A(\mathfrak{a})^1, \quad \gamma_\omega = A(\mathfrak{a})^\omega, \quad \gamma_\omega = A(\mathfrak{a})^\omega_\gamma\).

Then \(cU_0\) is isomorphic to the \(D_4\) lattice VOA, and the \(cU_{\gamma}\) are its irreducible modules. We set \(U = \bigoplus_{\gamma \in \Gamma} \gamma_\gamma\). From [FFR91] we have \(cU_0\)-module intertwining operators \(I_{\gamma,\delta} : \gamma_\delta \otimes \gamma_\delta \rightarrow \gamma_{\delta + \gamma}(z^{1/2})\) such that the map \(I = (I_{\gamma,\delta}) : \gamma \otimes \gamma \rightarrow \gamma_{\gamma + \gamma}(z^{1/2})\) furnishes \(\gamma\) with a structure of para-VOA. We refer the reader to [FFR91] for detailed information about para-VOAs, and we note here that the restriction of \(I\) to \(cU_0 \oplus cU_{\gamma}\) equips that space with a structure of SVOA for any \(\gamma \neq 0\).

For \(k \in \{1, 2, 3\}\) let \(\mathfrak{a}^k\) be a copy of the space \(\mathfrak{a}\) with orthonormal basis \(E^k = \{e_i^k\}_{i \in \Pi}\), and let \(\gamma^k\) be a copy of the space \(\gamma\). Suppose we define spaces \(\gamma^k W_L\) and \(\gamma^k W_L^f\) by setting

\[(6.4.2)\] \(\gamma^k W_L = \bigoplus_{\gamma \in \Gamma} \gamma U_\gamma^2 \otimes \gamma U_\gamma^3, \quad \gamma^k W_L^f = \bigoplus_{\gamma \in \Gamma} \gamma U_\gamma^2 \otimes \gamma U_{\gamma + \omega}^3\).

The main result from [FFR91] that we will use is that the para-VOA structure on \(cU\) induces a VOA structure on \(cW_L\) isomorphic to \(cV_L\), and induces a structure of \(\theta_b^\gamma\)-twisted \(cW_L\)-module on \(cW_L^f\), where \(\theta_b^\gamma = 1 \otimes \theta_f\). Furthermore, we may assume that the isomorphism is chosen so that the action of \(\theta_b^\gamma\) on \(cW_L\) corresponds to that of \(\theta_b\) on \(cV_L\). Indeed, a Cartan subalgebra of \(c(W_L)_{11}\) is spanned by the \(i e_i^k (z^{1/2}) e_i^k (z^{1/2})\) for \(i \in \Pi\), and \(\theta_b^\gamma\) acts as \(-1\) on this space.

From now on we will regard the VOAs \(cW_L\) and \(cV_L\) as identified via some VOA isomorphism such that \(\theta_b\) corresponds to \(\theta_b^\gamma\), and we will write \(cV_L\) in place of \(cW_L\) and \(\theta_b\) in place of \(\theta_b^\gamma\). Then for a \(\theta_b\)-twisted \(cV_L\)-module we may take \((cV_L)_{\theta_b} = cW_L^f\). Note that \(\theta_b\) acts naturally on the \(\theta_b\)-twisted module \((cV_L)_{\theta_b}\). A real form \(V_L\) for \(cV_L\) may be described by \(V_L = \mathbb{R}W_L = \bigoplus_{\gamma \in \Gamma} \mathbb{R}U_\gamma^2 \otimes \mathbb{R}U_{\gamma + \omega}^3\).

We may now express the spaces \(cV_L^f\) and \((cV_L^f)_{\theta_b}\) in the following way as sums of tensor products of the \(cU_\gamma^k\):

\[(6.4.3)\] \(cV_L^f = (cU_0^1 \oplus cU_1^1) \otimes \left(\bigoplus_{\gamma \in \Gamma} cU_\gamma^2 \otimes cU_\gamma^3\right)\)

\[(6.4.4)\] \((cV_L^f)_{\theta_b} = (cU_0^1 \oplus cU_1^1) \otimes \left(\bigoplus_{\gamma \in \Gamma} cU_\gamma^2 \otimes cU_{\gamma + \omega}^3\right)\)

We obtain real forms \(V_L^f\) and \((V_L^f)_{\theta_b}\) by replacing \(\mathbb{C}\) with \(\mathbb{R}\) in the subscripts of all the \(cU_\gamma^k\). Note that the super-conformal element \(\tau_V \in cV_L^f\) may now be written in
the following form.

\[(6.4.5) \quad \tau_V = \frac{1}{\sqrt{8}} \sum_{i} e_i^1(-\frac{1}{2})e_i^2(-\frac{1}{2})e_i^3(-\frac{1}{2})1\]

**Remark 6.2.** One can see that the Clifford module SVOA $A(u)$ has an $N = 1$ structure whenever $\dim(u)$ is divisible by 3.

We now define the space $cV^f_k$ and its real form $V^f_k$ as follows.

\[(6.4.6) \quad cV^f_k = (cV^f_L)^0 + (cV^f_R)^0, \quad V^f_k = (V^f_L)^0 + (V^f_R)^0.\]

Then in terms of the $\mathcal{U}_\gamma^{(k)}$ we have

\[(6.4.7) \quad cV^f_k = \bigoplus_{\gamma_k \in \{0, \omega\}, \sum \gamma_k = 0} cU_{\gamma_1 \gamma_2 \gamma_3} + \bigoplus_{\gamma_k \in \{1, \omega\}, \sum \gamma_k = 1} cU_{\gamma_1 \gamma_2 \gamma_3}\]

where we use an abbreviated notation to write $cU_{\gamma_1 \gamma_2 \gamma_3}$ for $cU^1_{\gamma_1} \otimes cU^2_{\gamma_2} \otimes cU^3_{\gamma_3}$, and there is a similar expression for the real form $V^f_k$ obtained by replacing $\mathbb{C}$ with $\mathbb{R}$ in the subscripts of the $cU^i_k$. By a similar argument to that used in \[FFR91\] to equip $cW_L$ with VOA structure via the para-VOA structure on $cU$, one may also equip the spaces $cV^f_k$ and $V^f_k$ with $N = 1$ SVOA structure. Since our main focus is to study the $N = 1$ SVOA $V^f_k$ via its realization $A^f_k$, we omit a verification of this claim and proceed directly to the task of indicating how one may arrive at an $N = 1$ SVOA isomorphism between $V^f_k$ and $A^f_k$.

### 6.5. **Isomorphism.**

We will concentrate on finding a correspondence between the real $N = 1$ SVOAs $V^f_k$ and $A^f_k$. Recall that $V^f_k$ may be described as follows.

\[(6.5.1) \quad V^f_k = \bigoplus_{\gamma_k \in \{0, \omega\}, \sum \gamma_k = 0} \mathbb{R}U_{\gamma_1 \gamma_2 \gamma_3} + \bigoplus_{\gamma_k \in \{1, \omega\}, \sum \gamma_k = 1} \mathbb{R}U_{\gamma_1 \gamma_2 \gamma_3}\]

On the other hand, the space underlying $A^f_k$ is described as $A(1)^0 \oplus A(0)^0$ where $I$ is real vector space of dimension 24, and in particular, for a suitable identification of $I$ with $\bigoplus_{\mathbb{R}a^k}$, we may identify $\mathbb{R}U_{000} = A(\mathbb{R}a^1)^0 \oplus A(\mathbb{R}a^2)^0 \oplus A(\mathbb{R}a^3)^0$ with a subspace of $A(\bigoplus_{\mathbb{R}a^k})^0 = A(0)^0$. As a sum of modules over this sub-VOA $A^f_k$ admits the following description.

\[(6.5.2) \quad \bigoplus_{\gamma_k \in \{0, 1\}, \sum \gamma_k = 0} \mathbb{R}U_{\gamma_1 \gamma_2 \gamma_3} + \bigoplus_{\gamma_k \in \{\omega, \bar{\omega}\}, \sum \gamma_k = \omega} \mathbb{R}U_{\gamma_1 \gamma_2 \gamma_3}\]

Thus it is evident that our method of constructing $V^f_k$ has almost delivered us an isomorphism with $A^f_k$ already. We require to find some way of interchanging 1 with $\omega$ in the subscripts on the right hand side of (6.5.1), and to do so we will invoke the results of \[FFR91\] once more. It is well known that the type $D_4$ Lie algebra admits an $S_3$ group of outer automorphisms that has the effect of permuting transitively the three inequivalent irreducible non-adjoint $D_4$ modules. As shown in \[FFR91\] this action extends to the corresponding VOA modules, and applying the outer automorphism that preserves the spaces $U_0$ and $U_{\bar{\omega}}$, and interchanges $U_1$ with $U_{\omega}$ simultaneously to each tensor factor on the right hand side of (6.5.1), we obtain an
isomorphism of $V^{fs}$ with an $N = 1$ SVOA $V^{fs'}$ whose underlying $\mathbb{R}U_{000}$-module structure is as in (6.3.2).

\[(6.5.3)\quad V^{fs'} = \bigoplus_{\gamma_k \in \{0, 1\}} \mathbb{R}U_{\gamma_1\gamma_2\gamma_3} \oplus \bigoplus_{\gamma_k \in \{\omega, \bar{\omega}\}} \mathbb{R}U_{\gamma_1\gamma_2\gamma_3}\]

We have seen that the spaces $(V^{fs'})_0$ and $(A^{fs})_0$ are isomorphic VOAs due to the fact that $A(l) = A(\bigoplus_k \mathbb{R}a^k)$ and $\bigotimes_k A(\mathbb{R}a^k)$ are naturally isomorphic. Similarly, $(V^{fs'})_1$ is naturally isomorphic to a canonically-twisted $A(l)$-module, and the same is true for $(A^{fs})_1$ by construction. The difference between $(V^{fs'})_1$ and $(A^{fs})_1$ is that the former may be naturally identified with the $A(l)_0$-module $A(l)^0_{\theta, H}$ where $\tilde{H}$ is an $\mathbb{F}_2^3$-homogeneous lift of a direct sum of three copies of the Hamming code $H^{[3,3]}$, and the later is realized as a the $A(l)_0$-module $A(l)_0^0_{\theta, G}$ for $G$ a lift of the Golay code $G$. Canonically twisted modules over $A(l)$ are unique up to isomorphism, so we can be assured that $(V^{fs'})_1$ and $(A^{fs})_1$ are isomorphic as $A(l)_0$-modules, and the proof of Theorem 6.3.1 shows that the SVOA structures on $V^{fs'}$ and $A^{fs}$ are essentially unique.

What is perhaps not so clear is whether or not the $N = 1$ structures on $V^{fs'}$ and $A^{fs}$ coincide. Recall the following description of the superconformal element in $V^{fs}$.

\[(6.5.4)\quad \tau_V = -\frac{1}{\sqrt{8}} \sum_{\Pi} c_1^1(-\frac{1}{2})c_2^2(-\frac{1}{2})c_3^3(-\frac{1}{2})1 \in \mathbb{R}U_{111}\]

Since $H$ may be defined as a quadratic residue code, it is convenient to take $\Pi$ to be the points of the projective line over $\mathbb{F}_7$ so that $\Pi = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$ say. Then we may choose the isomorphism $V^{fs} \to V^{fs'}$ in such a way that the image $\tau_V'$ of $\tau_V$ in $V^{fs'}$ is the following.

\[(6.5.5)\quad \tau_V' = -\frac{1}{\sqrt{8}} \sum_{\Pi} c_{\infty}^1 c_{\infty}^2 c_{\infty}^3 1_{\tilde{H}} \in (A(l)_0^0_{\theta, H})_{3/2}\]

On the other hand, the superconformal element in $A^{fs}$ is given by $\tau_A = 1_G \in (A(l)_0^0_{\theta, G})_{3/2}$. The spaces $(A(l)_0^0_{\theta, H})_{3/2}$ and $(A(l)_0^0_{\theta, G})_{3/2}$ are different realizations of the spin module over Clifford, and we may assume that $\tilde{H}$ and $G$ are $\mathbb{F}_2^3$-homogeneous subgroups of Spin(l). In the notation of 6.3.3 the spaces $(A(l)_0^0_{\theta, H})_{3/2}$ and $(A(l)_0^0_{\theta, G})_{3/2}$ are identified with $\text{CM}(l)_H$ and $\text{CM}(l)_G$, respectively. These two modules are equivalent and irreducible as Clifford-modules, and in particular, the action of Clifford(l) on any non-zero vector generates the entire module in each case. The vector $\tau_A = 1_G$ in $\text{CM}(l)_G$ is determined by the property that $g1_G = 1_G$ for any $g \in G$. It is a remarkable fact that the correspondence between $l$ and $\bigoplus_k \mathbb{R}a^k$ may be chosen in such a way that $\tau_V$ also satisfies the property $g\tau_V = \tau_V$ for any $g \in G$. Consequently we obtain explicit Clifford(l)-module and Spin(l)-module equivalences between $\text{CM}(l)_H$ and $\text{CM}(l)_G$ such that $\tau_V'$ corresponds to $\tau_A$. Using this isomorphism we can construct an explicit $A(l)$-module equivalence between $(A(l)_0^0_{\theta, H})$ and $(A(l)_0^0_{\theta, G})$. This is the last piece of information needed to construct an isomorphism of $N = 1$ SVOAs $V^{fs'} \to A^{fs}$, and consequently we obtain

**Theorem 6.3.** There is an isomorphism of $N = 1$ SVOAs $V^{fs} \cong A^{fs}$. 
7. McKay–Thompson series

In this section we consider the McKay–Thompson series associated to elements of $\mathcal{C}_0$ acting on $A^f_2$.

7.1. Series via $A^f_2$. Let $g \in \mathcal{C}_0$, and suppose that $g^m = 1$. Then there are unique integers $p_k$ for $k | m$ such that for $\det(g - x)$, the characteristic polynomial of $g$, we have $\det(g - x) = \prod_{k | m} (1 - x^k)^{p_k}$. This data can be expressed using a kind of formal permutation notation as $\prod_{k | m} k^{p_k}$, and this expression is called the frame shape for $g$. Recall $\eta(\tau)$, the Dedekind eta function \ref{eq:dedekind_eta}, and let $\phi(\tau)$ be the function on the upper half plane given by

\begin{equation}
\phi(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)} = q^{-1/48} \prod_{n=0}^\infty (1 - q^{n+1/2})
\end{equation}

For $g \in \mathcal{C}_0$ with frame shape $\prod k^{p_k}$, we set

\begin{equation}
\phi_g(\tau) = \prod_{k | m} \phi(\tau)^{p_k}, \quad \eta_g(\tau) = \prod_{k | m} \eta(\tau)^{p_k}.
\end{equation}

The group \( \mathcal{C}_0 \) has unique up to equivalence irreducible representations of dimensions 1, 24, 276, 2024 and 1771 \cite{CCN+85}. With \( N \) any one of these numbers, we write $\chi_N$ for the trace function $\chi_N : \mathcal{C}_0 \to \mathbb{C}$ on an irreducible \( \mathcal{C}_0\)-module of dimension \( N \). Let us also write $\chi_G$ for the trace function on the $\mathcal{C}_0$-module CM($l$).

\begin{theorem}
For $\bar{g} \in \text{Aut}(A^f_2)$, let $\pm g$ be the preimages of $\bar{g}$ in $\text{SO}_{24}(\mathbb{R})$. Then we have

\begin{equation}
\text{tr}_{|A^f_2} g q^{L(0) - C/24} = \frac{1}{2} (\phi_g(\tau) + \phi_{-g}(\tau)) + \frac{1}{2} (\chi_G(g) \eta_{-g}(\tau) + \chi_G(-g) \eta_g(\tau))
\end{equation}

\end{theorem}

\begin{proof}
Suppose that $g \in \mathcal{C}_0$ is of order $m$ and has frame shape $\prod_{k | m} k^{p_k}$. Then $g^{-1}$ has the same frame shape. Let $\{f_i\}_{i=1}^{24}$ be a basis for $\mathcal{C} = \mathbb{C} \otimes \mathbb{R} f$ consisting of eigenvectors of $g$ with eigenvalues $\{\xi_i\}_{i=1}^{24}$. Then we have

\begin{equation}
\det(g - x) = \prod_{i} (\xi_i - x) = \prod_{k | m} (1 - x^k)^{p_k}
\end{equation}

and we note also that $\sum_{k | m} k p_k = 24$.

Recall that $A^f_2$ may be described as $A^f_2 = A(0)^0 \oplus A(0)^\theta$ where $A(0)$ is the Clifford module SVOA associated to a 24-dimensional inner product space $\mathbb{I}$, and $A(0)^\theta$ is a canonically twisted $A(0)$-module. It is not hard to derive the following expressions for the trace of $g$ on the complexified spaces $\mathbb{C} A(0)$ and $\mathbb{C} A(0)^\theta$.

\begin{equation}
\text{tr}_{|A(0)} (-g) q^{L(0) - C/24} = q^{-1/2} \prod_{n \geq 0} \prod_{i} (1 - \xi_i q^{n+1/2})
\end{equation}

\begin{equation}
\text{tr}_{|A(0)^\theta} (-g) q^{L(0) - C/24} = \chi_G(-g) q \prod_{n \geq 0} \prod_{i} (1 - \xi_i q^{n+1})
\end{equation}

Substituting $q^r$ for $x$ in \ref{eq:det_expression} and using the fact that $\prod_i \xi_i = 1$ we obtain

\begin{equation}
\prod_{i} (1 - \xi_i q^r) = \prod_{k | m} (1 - (q^k)^r)^{p_k}.
\end{equation}
Then for $\text{tr}|_{C(l)}(-g)q^{L(0)}c^{-24/2}$ for example, we have

$$\text{tr}|_{C(l)}(-g)q^{L(0)}c^{-24/2} = q^{-1/2} \prod_{n \geq 0} \prod_{i}(1 - \xi_i q^{n+1/2})$$

(7.1.8)

$$= \prod_{k|m} \left( q^{-kp_k/48} \prod_{n \geq 0} (1 - (q^{k})^{n+1/2})p_k \right) = \phi_g(\tau)$$

and similarly, we obtain $\text{tr}|_{C(l)\theta}(-g)q^{L(0)}c^{-24/2} = \chi_G(-g)\eta_\theta(\tau)$. To compute the traces of $\bar{g} \in Co_1 = Co_0/\{\pm 1\}$ on $A(l)0$ we should average over the traces of its preimages $g$ and $-g$ on $A(l)$, and similarly for the trace of $\bar{g}$ on $A(l)\theta$. This completes the verification of (7.1.3). \qed

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