The connections between the properties of associative rings that are Lie-solvable (Engel, n-Engel, locally finite, respectively) and the properties of their adjoint subgroups are investigated.

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1 Introduction

Let \( R = (R, +, \cdot) \) be an associative ring (not necessary with unity). The set of all elements of \( R \) forms a semigroup with respect to the circle operation “\( \circ \)” defined by the rule \( a \circ b = a + b + a \cdot b \) for each \( a, b \in R \). The set

\[
R^\circ = \{ a \in R \mid a \circ b = 0 = b \circ a \text{ for some } b \in R \}
\]

is a group (so-called the adjoint group of \( R \)). If \( R \) has unity and \( U(R) \) is the unit group of \( R \), then

\[
R^\circ \ni a \mapsto 1 + a \in U(R)
\]

is a group isomorphism. If \( R = R^\circ \), then \( R \) is called radical.

We study properties of associative rings and their adjoint groups which are connected with solvability, Engel conditions and periodicity.

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We always assume that $p$ is a prime number, $\mathbb{N}$ is the set of positive integers, $\mathbb{F}$ is a field, $\mathbb{Z}_n$ is the ring of integers modulo $n$. Let $n, k \in \mathbb{N}$, $m \in \mathbb{Z}$ and let $x, g \in R$. We introduce the following notation:

\[
\mu_n(x) = \sum_{k=1}^{n} \binom{n}{k} x^k,
\]

\[
[x, g] = [x, g] = x \cdot g - g \cdot x, \quad [x, n+1 g] = [x, n g], g,
\]

$C_R(g) = \{x \in R \mid xg = gx\}$ is the centralizer of $g$ in $R$,

$J(R)$ is the Jacobson radical of $R$,

$N(R)$ is the set of all nilpotent elements of $R$,

$g^{(m)}$ is the $m$-th power of $g$ in $R$,

$F(R) = \{x \in R \mid x$ is of finite order in $R^+\}$ is the torsion part of the additive group $R^+$,

$\text{char } R$ is the characteristic of $R$,

$Z(R)$ is the center of $R$,

$\mathbb{P}(R)$ is the prime radical of $R$ (i.e., the intersection of all prime ideals of $R$),

$N^*_R(R)$ is the nil radical of $R$ (i.e., the sum of all nil ideals),

$N_f(R)$ is the sum of all nil right ideals of $R$ (moreover, $N_f(R)$ is the sum of all nil left ideals of $R$ and, therefore, $N_f(R)$ is a two-sided ideal of $R$),

$[A, B]$ is the additive subgroup of $R^+$ generated by all $[a, b]$, where $a \in A, b \in B$ and $A, B \subseteq R$,

$C(R)$ is the commutator ideal of $R$ (i.e., an ideal of $R$ generated by all $[g, x]$),

$D = \text{Der } R$ is the set of all derivations of $R$,

$\Delta(R)$ is the ideal of $R$ generated by all $\delta(R)$, where $\emptyset \neq \Delta \subseteq \text{Der } R$ and $\delta \in \Delta$,

$\gamma_1 R = [R, R]$ and $\gamma_{n+1} R = [\gamma_n R, R]$,

$\delta_0 R = R$ and $\delta_{n+1} R = [\delta_n R, \delta_n R]$,

$\langle X \rangle_{rg}$ is a subring of $R$ generated by $X \subseteq R$ (if $X = \emptyset$, then $\langle X \rangle_{rg} = 0$).

If $\Delta = (\Delta, +[-, -])$ is a Lie ring, then

\[
\gamma_1 \Delta := \Delta, \ldots, \gamma_{k+1} \Delta := [\gamma_k \Delta, \Delta], \ldots \quad (k \in \mathbb{N}).
\]

and $\Delta^{(1)} := \Delta, \ldots, \Delta^{(n+1)} := [\Delta^{(n)}, \Delta^{(n)}]$.

Let $G$ be a group and let $\tau(G)$ be the set of all torsion elements of $G$.

Recall that a ring $R$ is called:

- **nil** if each $x \in R$ is nilpotent, i.e., there exists $n = n(x) \in \mathbb{N}$ such that $x^n = 0$; if there exists $n \in \mathbb{N}$ such that $x^n = 0$ for any $x \in N(R)$, then $R$ is of bounded index of nilpotency (of bounded index for short),

- **local** if $R \ni 1$ and $R/J(R)$ is a simple ring,

- **right Artinian** in case for each ascending chain

  \[
  I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots
  \]

  of right ideals $I_j$ of $R$ ($j = 1, 2, \ldots$), there exists $n \in \mathbb{N}$ such that $I_{n+1} = I_n$,

- **right Noetherian** in case for each descending chain

  \[
  I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots
  \]

  of right ideals $I_j$ of $R$ ($j \in \mathbb{N}$), there exists $n \in \mathbb{N}$ such that $I_{n+1} = I_n$,

- **semilocal** if $R/J(R)$ is a left Artinian ring,

- **right Goldie** if it has no infinite direct sum of left ideals and has the ascending chain condition on right annihilators,

- **Lie nilpotent** of class $n$ if $n$ is a minimal positive integer such that $\gamma_{n+1} R = 0$,

- **locally nilpotent** if each its finitely generated subring is nilpotent.
locally Lie nilpotent if each finitely generated subring of $R$ is Lie nilpotent,

- Lie soluble of length at most $n$ if $\delta_n R = 0$,
- Lie metabelian if $\delta_2 R = 0$,
- Lie centrally metabelian if $\delta_2 R \subseteq Z(R)$,
- Engel (or equivalently $R$ satisfies the Engel condition) if, for each $x$, $y \in R$, there exists $n = n(x, y) \in \mathbb{N}$ such that $[x, n y] = 0$,
- $n$-Engel (or equivalently $R$ satisfies the $n$-Engel condition or $R$ is bounded Engel) if $[x, n y] = 0$ for any $x$, $y \in R$,
- semiprime if it has no nonzero nilpotent ideals,
- prime if the product of each two nonzero ideals is nonzero,
- simple if $R^2 \neq 0$ and 0, $R$ are the only ideals of $R$,
- reduced if $N(R) = 0$,
- 2-primal if $\mathbb{P}(R) = N(R)$,
- right quasi-duo if each maximal right ideal is two-sided,
- of stable range 1 if $R \ni 1$ and, for each $a$, $b$, $x$, $y \in R$ with $ax + by = 1$, there exists $h \in R$ such that $a + bh \in U(R)$,
- abelian if its all idempotents are central,
- $\pi$-regular if, for each $a \in R$, there exists $b \in R$ such that $a^n = a^n ba^n$ for some $n \in \mathbb{N}$,
- right weakly $\pi$-regular if, for each $a \in R$, there exists $n = n(a) \in \mathbb{N}$ such that $a^n R = (a^n R)^2$.

We shall use freely the following well known facts: Right Artinian rings are $\pi$-regular and $\pi$-regular rings are weakly $\pi$-regular. An associative ring $R$ is Lie nilpotent (Engel, respectively) if and only if $R^L$ is nilpotent (Engel, respectively). Each (Lie or associative) locally nilpotent ring is Engel. Each Lie nilpotent ring of nilpotency class $n$ is $n$-Engel. A radical ring $R$ is Lie nilpotent if and only if its adjoint group $R^\circ$ is nilpotent [38].

In certain papers (see, for example, [5, 13, 58, 72] and others) by many authors was investigated properties of Lie soluble rings and its relations with groups. Each radical ring $R$ with soluble adjoint group $R^\circ$ is Lie soluble [4, Theorem A]. As a consequence, if $R$ is a semilocal ring with the soluble unit group $U(R)$, then $R$ is Lie soluble.

Each 2-torsion-free Lie soluble ring $R$ has a nilpotent ideal $I$ such that $R/I$ is Lie centre-by-metabelian (and so $R^\circ$ is soluble) (see [58, 72]). There exists a Lie center-metabelian (and so it is Lie soluble) total $(2 \times 2)$-matrices ring $M_2(R)$ over an infinite commutative domain $R$ of characteristic 2, but its adjoint group $M_2(R)^\circ$ is non-solvable. The group of units $U(R)$ of a Lie metabelian unitary ring $R$ is metabelian (see [59] and [43, Theorem 1]).

Our result is the following.

**Theorem 1** Let $R$ be a ring. The following statements hold:

(i) if $R$ is Lie nilpotent, then the adjoint group $R^\circ$ is nilpotent-by-abelian;

(ii) if $R^\circ$ is soluble-by-finite, then $[J(R), R] \subseteq \mathbb{P}(R)$ and $J(R)C(R) \subseteq \mathbb{P}(R)$;

(iii) if $R$ is 2-torsion-free Lie soluble, then:

(a) $C(R) \subseteq \mathbb{P}(R) = N(R)$ (i.e., $R$ is 2-primal and right quasi-duo), $N(R)$ is the locally nilpotent ideal of $R$ and $R^\circ$ is (locally nilpotent)-by-abelian. Moreover, if $R$ has unity, then it is abelian;

(b) if $R^\circ$ is torsion, then it is locally nilpotent;

(c) if $N(R)^+$ is torsion-free divisible, then $R^\circ$ is nilpotent-by-abelian.

In the local case we obtain the following.
Theorem 2 Let $R$ be a local ring. The following statements hold:

(i) if the unit group $U(R)$ is solvable, then:
   
   (a) $C(R) \subseteq L(R) = N(R)$ is a locally nilpotent ideal of $R$, where $L(R)$ is the Levitzki radical of $R$;
   
   (b) $R$ is Lie solvable;
   
   (ii) if $R$ is a Lie solvable $\mathbb{Q}$-algebra, then $R^\circ$ is nilpotent-by-abelian.

Recall that the Levitzki radical $L(R)$ of $R$ is its unique maximal locally nilpotent ideal.

An additive map $\delta : R \rightarrow R$ is called a derivation of $R$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. The set $D$ of all derivations of $R$ is a Lie ring. Properties of a ring $R$ which induced by the Engel condition of the derivation ring $D$ gives the following.

Theorem 3 Let $R$ be a ring. If $\emptyset \neq \Delta \subseteq D$, then

(i) if $\Delta$ is a nilpotent Lie ring and $R\Delta \subseteq \Delta$ modulo $C(R)$, then $\Delta(R)^m \subseteq C(R)$ (in particular, if $D$ is Lie nilpotent, then $D(R)^m \subseteq C(R)$) for some $m \in \mathbb{N}$;

(ii) if $\Delta$ is a nilpotent Lie ring of the nilpotent length $n$ and $R\Delta \subseteq \Delta$ modulo $C(R)$, then $d^n(R)d \subseteq C(R)$ for some $d \in \Delta^{(n-1)}$;

(iii) if $\Delta$ is an Engel Lie ring and $R\Delta \subseteq \Delta$ modulo $C(R)$, then, for each $a \in R$ and $\delta \in \Delta$, there exists $n = n(\delta, a) \in \mathbb{N}$ such that $\delta^n(a) \delta(R) \subseteq C(R)$.

If the unit group $U(R)$ of a semi-local ring $R$ is $m$-Engel, then $U(R)$ is locally nilpotent and, furthermore, $R$ is $n$-Engel provided that $R$ is generated by $U(R)$ (bibliography in this way see in [6, 50]). Moreover, a local ring $R$ is Lie nilpotent if and only if $U(R)$ is nilpotent [64] and that is this case the classes of nilpotency of both structures coincide. We prove the next.

Theorem 4 Let $R$ be a $n$-Engel local ring. If $F(R) = 0$, then $R$ is Lie nilpotent.

A ring $R$ is called locally finite if each finite subset of $R$ generates a finite semigroup multiplicatively. The class of locally finite rings is closed under formation of subrings, homomorphic images and direct sums (see [37, Proposition 2.1]). A finite subset of a locally finite ring generates a finite subring (not necessary with unity) [37, Theorem 2.2] and a locally finite ring is strongly $\pi$-regular [37, Lemma 2.4(ii)]. Recall that a ring $R$ is called strongly $\pi$-regular if, for each $a \in R$, there exist $n = n(a) \in \mathbb{N}$ and $b \in R$ such that $a^n = a^{n+1}b$. A ring $R$ is strongly $\pi$-regular [9] if and only if it satisfies the descending chain condition on principal right ideals of the form

$$aR \supseteq a^2R \supseteq \cdots \supseteq a^nR \supseteq \cdots \quad (\forall a \in R).$$

Local rings with the nil Jacobson radical and semilocal rings with the nil Jacobson radical of bounded index are strongly $\pi$-regular [37, Lemma 3.1 and Corollary 3.3]. The Jacobson radical $J(R)$ of a locally finite ring $R$ is a locally nilpotent ring (in view of [2, Corollaries 1 and 4] and Lemma 16(ii)). We precise [37, Propositions 2.5, 2.10 and 2.11] in the following

Proposition 5 Let $R$ be a 2-torsion-free locally finite ring with unity. The following statements hold:

(i) each prime ideal of $R$ is maximal as a right ideal (and so $R$ is $\pi$-regular);

(ii) $R$ is an abelian exchange ring of stable range 1, $C(R) \subseteq P(R) = N(R) = J(R)$ and $R/J(R)$ is a subdirect product of locally finite fields (so each element of $R$ is a sum of a unit and a central element).
Each absolute field (i.e., a field in which each nonzero element is a root of 1) is a locally finite ring. In a locally finite field \( \mathbb{F} \) every finite subset \( X \subseteq \mathbb{F} \) generates a finite subfield. Since the unit group \( U(\langle X \rangle_{rg}) \) is cyclic, we deduce that a locally finite field is absolute. A locally finite right Noetherian ring is right Artinian (see Proposition 22).

Rings with torsion adjoint groups were intensively studied in [2, 30–32, 44, 45, 60, 65] and others. It is well known [31, Theorem 8] that a division ring \( D \) with the torsion multiplicative group \( D^* \) is commutative. Moreover, a torsion normal subgroup of the multiplicative group \( U(D) \) of a skew field \( D \) is central [71, Lemma 10]. Each torsion subgroup of a linear group over a field is locally finite by classical results of W. Burnside and I. Schur. A torsion subgroup of the unit group \( U(R) \) of a unitary \( PI \)-ring is locally finite by results of C. Procesi and A.I. Shirshov. Each locally finite subgroup of the adjoint group \( R^\circ \) of a radical ring \( R \) is locally nilpotent [2, Corollary 1].

We have the following.

**Proposition 6** Let \( R \) be a ring such that \( R^\circ \) is torsion and \( F(R) = 0 \). The following statements hold:

(i) \( R \) is commutative or without zero-divisors,

(ii) if \( R \) is prime with unity, then \( R \) is a domain such that \( J(R) = 0 \) and the unit group \( U(R) \) is finite of one of the following types:

(a) \( U(R) \) is a cyclic group of order 2 such that \( \langle U(R) \rangle_{rg} \cong \mathbb{Z} \);

(b) \( U(R) \) is a cyclic group of order 4 such that \( \langle U(R) \rangle_{rg} \cong \mathbb{Z}[i] \) is the ring of Gaussian integers;

(c) \( U(R) \) is a cyclic group of order 6 such that \( \langle U(R) \rangle_{rg} \cong \mathbb{Z}[\zeta_3] \) is the subring of integer elements of the Eisenstein field \( \mathbb{Q}[\sqrt{3}] \);

(d) \( U(R) \) is the quaternion group of order 8 such that \( \langle U(R) \rangle_{rg} \cong \mathbb{Z}[i, j] \) is the ring of quaternions with integer coefficients;

(e) \( U(R) = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle \) is the dicyclic group of order 12 such that \( \langle U(R) \rangle_{rg} \cong \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \alpha + \mathbb{Z} \cdot \beta + \mathbb{Z} \cdot \gamma \) is the ring with the following Cayley table of multiplication:

\[
\begin{array}{cccc}
\alpha & \beta & \gamma \\
\alpha & -1 & \gamma & -\beta \\
\beta & -\alpha - \gamma & -1 - \beta & \alpha \\
\gamma & 1 + \beta & -\alpha - \gamma & -1 \\
\end{array}
\]

(f) \( U(R) = \langle a, b \mid a^3 = b^3 = (ab)^2 = 1 \rangle \) is the binary tetrahedral group of order 24 such that \( \langle U(R) \rangle_{rg} \) is a subring of the skew field \( \mathbb{Q}(i, j) \) of quaternions which is generated by \( i, j \) and \( \frac{1+i+j+k}{2} \).

Any unexplained terminology is standard as in [25, 55].

## 2 Semiprime rings

The unit group \( U(R) \) of a unitary \( n \)-Engel ring \( R \) is \( m \)-Engel for some \( m = m(n) \in \mathbb{N} \) depending on \( n \) (see [54, Corollary 1] and [57, Corollary]). The adjoint group \( R^\circ \) of a radical ring \( R \) is \( n \)-Engel if and only if \( R \) is an \( m \)-Engel ring for some \( m = m(n) \in \mathbb{N} \) [4, Main Theorem]. Each \( n \)-Engel Lie algebra is locally nilpotent and each \( n \)-Engel Lie algebra over a field of characteristic zero is nilpotent [73–75].
If a group $G$ contains a non-trivial $p$-element and the unit group $U(\mathbb{F}[G])$ of the group algebra $\mathbb{F}[G]$ is bounded Engel, then $\mathbb{F}[G]$ is bounded Engel [14] (see also [15–17]). An unitary associative bounded Engel algebra $A$ over a field of prime characteristic has the bounded Engel group $U(A)$ [57], which is locally nilpotent [57, Remark]. In the case of zero characteristic, $U(A)$ is nilpotent and $A$ is Lie nilpotent (see [40, 54]). Each bounded Engel subgroup of the adjoint group $R^\circ$ of a radical ring $R$ is locally nilpotent [2, Corollary 1].

It is known that $N_r(R) \subseteq R^\circ$, $N^a(R) \subseteq J(R) \subseteq R^\circ$,
$$\mathbb{P}(R) \subseteq L(R) \subseteq N^a(R) \subseteq N_r(R) \subseteq N(R).$$
If $R$ is an $n$-Engel ring, then its commutator ideal $C(R)$ is nil (for example, see [28, Application 2]) and, additionally, $C(R) \subseteq L(R)$ [4, Lemma 3.1].

We use the following.

**Lemma 7** [47, Theorem 1] Let $L \neq 0$ be a left ideal of a prime ring $R$. Let $D$ be the Lie ring of all derivations of $R$. Let $k, n \in \mathbb{N}$ and let $0 \neq \delta \in D$. If
$$[\delta(x^k), n x^k] = 0, \quad (\forall x \in L)$$
then $R$ is commutative.

As a consequence, we have the next.

**Corollary 8** Let $R$ be an $n$-Engel ring. The following statements hold:

(i) $C(R) \subseteq \mathbb{P}(R) = N(R) \subseteq J(R)$ (i.e., $R$ is 2-primal and right quasi-duo);

(ii) if $R \ni 1$, then $R$ is abelian;

(iii) the adjoint group $R^\circ$ is (locally nilpotent)-by-abelian and $N(R)$ is a locally nilpotent ideal of $R$;

(iv) if $F(N(R)) = 0$, then $R^\circ$ is nilpotent-by-abelian.

**Proof** (i) The quotient ring $R/P$ is $n$-Engel for each prime ideal $P$ of $R$ and each inner derivation $\delta$ of $R/P$ satisfies (2.1), so we conclude that $R/P$ is commutative by Lemma 7. That yields $C(R) \subseteq \mathbb{P}(R) = N(R)$ and $N(R)$ is locally nilpotent as a $PI$-ring. Thus, each maximal right ideal of $R$ is two-sided.

(ii) If $R$ has unity, then it is abelian by [69, 3.20].

(iii) The set $N(R)$ is an ideal in view of the part (i), $N(R)^\circ$ is $m$-Engel for some $m \in \mathbb{N}$ [4, Main Theorem] and $N(R)^\circ$ is locally nilpotent [4, Lemma 2.2]. Then $N(R)$ is locally nilpotent by [2, Lemma 3].

(iv) If $F(N(R)) = 0$, then $N(R)^\circ$ is torsion-free. Since $N(R)^\circ$ is $m$-Engel for some $m \in \mathbb{N}$ [4, Main Theorem], it is nilpotent [74].

**Lemma 9** (See [22, Theorem 3]) Let $R$ be a ring with $N_r(R) = 0$. If, for given $x, y \in R$ there exist positive integers $m = m(x, y), n = n(x, y)$ and $k = k(x, y)$ such that
$$[x^m, k y^n] = 0,$$
then $R$ is commutative.

An unitary associative $PI$-algebra $R$ with the Engel condition over a field of any characteristic has the nil commutator ideal by [53, Proposition 2.3]. Furthermore, each nil ring is Engel by [66, Proposition 4.2], but there exist nil rings $R$ (which also are algebras over arbitrary fields) such that their adjoint groups $R^\circ$ are not Engel [61, Theorem 1.5]. Inasmuch as an Engel ring $R$ satisfies (2.2), we obtain the following.
Corollary 10 Let R be an Engel ring. The following statements hold:

(i) \(C(R) \subseteq N_r(R) = N(R) \subseteq J(R)\), \(N(R)\) is an ideal of \(R\) (i.e., \(R\) is right (left) quasi-duo) and \(R^o\) is Engel-by-abelian;

(ii) if \(R\) is locally Lie nilpotent, then the adjoint group \(R^o\) is (locally nilpotent)-by-abelian;

(iii) if \(R\) has unity, then it is abelian.

Proof The part (i) follows from Lemma 9. Assume that \(R\) is locally Lie nilpotent. If \(x, y \in R\), then the subring \(\langle x, y \rangle_{rg}\) is Lie nilpotent and so there exist \(n = n(x, y) \in \mathbb{N}\) such that \([x, n y] = 0\). This means that \(R\) is Engel, the adjoint group \(N(R)^o\) is locally nilpotent in view of [3, Main Theorem] and so \(R^o\) is (locally nilpotent)-by-abelian. All maximal right ideals in unitary Engel ring \(R\) are two-sided so \(R\) is abelian by [69, 3.20].

Every domain of characteristic 0 that is Engel (as a Lie ring) is commutative [11, Theorem 4]. If the unit group \(U(D)\) is \(m\)-Engel, then a division ring \(D\) is commutative by [6, Lemma 4.1]. We obtain an affirmative answer on [24, Question 1.2].

Proposition 11 Each Engel division ring is commutative.

Proof The assertion holds from Corollary 10.

Our next result confirm a conjecture of [24, Hypothesis 1.1].

Proposition 12 An Engel adjoint group \(R^o\) of a right Artinian ring \(R\) is nilpotent.

Proof The Jacobson radical \(J(R)\) is a nilpotent ideal of \(R\), so \(J(R)^o\) is a nilpotent group. We can assume that \(J(R)^2 = 0\). Since \(R\) is a right Noetherian [25, Theorem 18.3, in Russian translation], \(J(R)\) is a finite direct sum of minimal ideals of \(R\) and \(J(R) \subseteq Z(R)\) by [24, Lemma 2.1]. Thus \(R^o\) is a nilpotent group by a well-known Ph. Hall’s Theorem.

3 Torsion subgroups

It is well known the following.

Lemma 13 Let \(R\) be a ring with 1. The following statements hold:

(i) \(U(R)\) is a torsion group if and only if the Jacobson radical \(J(R)\) is nil with the torsion additive group \(J(R)^+\) and \(U(R/J(R))\) is torsion [44, Lemma 1.1];

(ii) if \(F(R) = 0\) and \(U(R)\) is torsion, then \(J(R) = 0\) and \(R\) is reduced [44, Corollary 1.2];

(iii) if \(F(R) = 0\), then \(U(R)\) is torsion if and only if \(U(R)\) is locally finite ([44, Theorem 3.3] and [60, Proposition 2]).

We precise [70, Corollary 2.10] as the following.

Proposition 14 Let \(R\) be a ring with the additive \(p\)-group \(R^+\). The set \(N(R)\) is a subring of \(R\) if and only if \(N(R)^o\) is a normal Sylow \(p\)-subgroup of \(R^o\).

Proof Clearly, \(pR\) is an ideal of \(R\) such that \(pR \subseteq N(R)\). The groups \((R/pR)^o\) and \(R^o/(pR)^o\) are isomorphic, so we can assume that \(pR = 0\).

\((\Rightarrow)\) Suppose that \(N(R)\) is a subring. Then \(N(R)^o\) is a \(p\)-subgroup of \(R^o\) in view of [1, Lemma 2.4] and \(N(R)^o\) is contained in some maximal (Sylow) \(p\)-subgroup \(S\) of \(R^o\). If \(g \in S \setminus N(R)^o\), then

\[0 = g(p^n) = \mu_{p^n}(g) = g^{p^n}\]
for some $n \in \mathbb{N}$, so $g \in N(R)$, a contradiction. Hence $N(R)^\circ$ is a Sylow $p$-subgroup of $R^\circ$. If $S_1$ is a maximal (Sylow) $p$-subgroup of $R^\circ$ and $h \in S_1$ has order $p^m$, then $0 = h^{p^m} = h^m$, so $h \in N(R)$ and $S_1 = S$. Consequently, $S$ is normal in $R^\circ$.

(\iff) Since $N(R)$ is closed under the circle operation “\circ”, $N(R)$ is a subring of $R$ by [70, Theorem 2.1].

Let $Dz(R)$ be the set of all left and right zero divisors and $0 \in Dz(R)$.

**Proof of Proposition 6**

(i) Let $R$ be not commutative and let $g \in N(R)$. If $g^2 = 0$, then

\[ 0 = g^{(n)} = \mu_n(g) = ng \]

for some $n \in \mathbb{N}$ and so $g = 0$. This means that $N(R) = 0$. It follows that if $ab = 0$ for some nonzero $a, b \in R$, then $(ba)^2 = 0$ what implies that $ba = 0$. Therefore $Dz(R)$ is commutative and, consequently, $Dz(R)$ is an ideal of $R$ by [42, Theorem 5.5]. Moreover, $Dz(R)C(R) = 0$ [42, Lemma 5.4], so $C(R)^3 = 0$. Thus $C(R) = 0$, which is a contradiction.

(ii) Indeed, $J(R) = N(R) = 0$ by Lemma 13. If $ac = 0$ for some $a, c \in R$, then $(ca)^2 = c(ac)a = 0$ implies that $ca = 0$. As a consequence, $0 = a(cR) = cRa$ and $c = 0$ or $a = 0$ by the primeness of $R$. Hence $R$ is a domain. The rest follows from [60, Proposition 4].

**Proposition 15** Let $R$ be an unitary domain of characteristic $p > 0$. If $U(R)$ is torsion, then it is a $p'$-group and the following statements hold:

(i) $R^\circ\setminus\{0\} \subseteq U(R)$ and $I \cap R^\circ = 0$ for any proper right (left) ideal $I$ of $R$; in particular, $J(R) = 0$;

(ii) if $S$ is a subring of $R$ and $S \cap R^\circ \neq 0$, then $1 \in S$.

**Proof** Obviously, $N(R) = 0$. Since $g^{(p)} = \mu_p(g) = g^p$ for $g \in R^\circ$, we deduce that $g^{(p)} \neq 0$ and $R^\circ$ is a $p'$-group.

(i) Let $I$ be a proper right (left) ideal of $R$. If $0 \neq b \in I \cap R^\circ$, then

\[ 0 = b^{(n)} = \mu_n(b) = b \left(n + \sum_{k=2}^{n} \binom{n}{k} b^{k-1}\right) \]

for some $n \in \mathbb{N}$. If the great common divisor $\text{GCD}(n, p) = 1$, then $n \in I$ and there exist $u, v \in \mathbb{Z}$ such that $r = (nr)u + (pr)v \in I$ for any $r \in R$. Consequently, $I = R$, a contradiction. This implies that $I \cap R^\circ = 0$.

Inasmuch as $x \in (Rx)^\circ \cap R^\circ$ ($x \in (xR)^\circ \cap R^\circ$, respectively) for each $x \in R^\circ$, we conclude that $xR = R = Rx$, so each nonzero quasi-invertible element is invertible in $R$.

(ii) If $S$ is a nonzero subring of $R$ and $0 \neq b \in S \cap R^\circ$, then, as above, $n \in S$ and consequently $1 \in S$.

According to Proposition 15, we can ask the following questions:

Q1. Does there exist a unitary infinite non-commutative simple ring $R$ of characteristic $p > 0$ with the torsion unit group $U(R)$?

Q2. Does there exist a unitary (infinite) non-commutative ring $R$ which is not a skew field, such that $R^\circ\setminus\{0\} \subseteq U(R)$?
4 Locally finite rings

We start with some properties of locally finite rings.

**Lemma 16** If $R$ is a locally finite ring with unity, then the following statements hold:

(i) $R^+$ is a torsion $\pi$-group for some set $\pi$ of primes;
(ii) $U(R)$ is locally finite;
(iii) $1 + J(R)$ is a locally nilpotent $\pi$-group.

**Proof** (i) Obviously.
(ii) If $X$ is a finite subset of $U(R)$, then $\langle X \rangle \subseteq \langle X \rangle^R$ and so the subgroup $\langle X \rangle$ is finite.
(iii) The unipotent subgroup $1 + J(R)$ of $U(R)$ is locally nilpotent [2, Corollary 1]. Since $J(R)$ is nil, $1 + J(R)$ is a $\pi$-group in view of [1, Lemma 2.4].

\[\square\]

**Lemma 17** If $P$ is a minimal prime ideal of a 2-torsion-free ring $R$, then $\text{char } R/P \neq 2$.

**Proof** Let $X = \{2^n a \mid a \in R \setminus P \text{ and } n \in \mathbb{N} \cup \{0\}\}$. Clearly, $X$ is non-empty, $0 \notin X$ and $X$ is an $m$-system (in the sense of [49]). Therefore, there exists a two-sided ideal $M$ of $R$ which is maximal to being disjoint from $X$ (then $M$ is prime by [49, Lemma 4] and $M \subseteq R \setminus X$). Since $R \setminus P \subseteq X$, we conclude that $M \subseteq P$ and consequently $M = X$. Hence $\text{char } R/P \neq 2$.

If for each $x \in R$ there exists $n \in \mathbb{N}$ with $x^n = x$, then $R$ is commutative by a well-known theorem of N. Jacobson. A ring $R$ is called periodic if, for each $x \in R$, there exist different positive integers $m$ and $n$, such that $x^m = x^n$.

**Lemma 18** Let $R$ be a locally finite ring with unity. The following statements hold:

(i) $R$ is periodic;
(ii) if $R$ is 2-torsion-free semiprime, then it is commutative;
(iii) if $R$ is prime of $\text{char } R \neq 2$, then it is a field;
(iv) if $R$ is 2-torsion-free, then $C(R) \subseteq \mathbb{P}(R) = N(R) = J(R)$ (i.e., $R$ is 2-primal and right quasi-duo).

**Proof** (i) For the proof, see [36, Corollary 2].
Let $R$ be a 2-torsion-free ring.
(ii) It holds in view [12, Thereom 4.5] and the part (i).
(iii) It follows from the part (i) and the fact that any periodic domain is a field.
(iv) It is a consequence of parts (ii)–(iii) and Lemma 17.

\[\square\]

**Proof of Proposition 5.** (i) The quotient ring $R/P$ is a field for each prime ideal $P$ of $R$ by [37, Corollary 2.6] and Lemma 18(ii). Thus the part (i) holds.
(ii) Since $R$ is strongly $\pi$-regular and $J(R) \subseteq N(R)$ by [35, Theorem 1], we conclude that $J(R) = N(R)$ in view of Lemma 18(ii). Hence $R$ is exchange of stable range 1 by [69, Theorem 5.23 and Proposition 5.6]. Moreover, $R$ is abelian by [69, 3.20(3)], $R/J(R)$ is a subdirect product of fields and so each element of $R$ is a sum of an invertible and a central elements by [69, Thereom 6.29].

**Corollary 19** A locally finite 2-torsion-free ring $R$ is right (left) Ore, i.e., there exists the classical right (left) quotient ring $Q(R)$. 
Proof The assertion holds in view of Lemma 18(iii), Proposition 5(ii) and [41, Theorem 2.1 and Proposition 1.9(5)]. □

Since each 2-torsion-free locally finite ring is abelian π-regular, we provide the following.

Corollary 20 An abelian π-regular ring \( R \) satisfies the Köthe’s conjecture, i.e., the sum of two nil left ideals is always nil.

Proof The set \( N(R) \) of nilpotent elements is an ideal of \( R \) by [10, Theorem 2]. The rest follows from [41, Lemma 1.4(2) and Theorem 2.1(2)]. □

Lemma 21 [25, Lemma 18.34B] Let \( R \) be a right Noetherian ring. If \( R/P \) is an Artinian ring for each prime ideal \( P \) of \( R \), then \( R \) is a prime ring or \( R \) is a right Artinian ring.

Proposition 22 A ring \( R \) is locally finite right Noetherian if and only if it is a locally finite right Artinian.

Proof (⇐) Each right Artinian ring is right Noetherian by [25, Theorem 18.13, in Russian translation].

(⇒) Since \( R/P \) is a field for each prime ideal \( P \) of \( R \) (see Lemma 18(iii)), we deduce that \( R \) is right Artinian in view of Lemmas 21 and 18(iii). □

Proposition 23 Let \( R \) be a semilocal ring. The following conditions are equivalent:

(i) \( R \) is a locally finite ring;
(ii) the unit group \( U(R) \) is locally finite;
(iii) \( R^+ \) is a torsion group, \( J(R) \) is a locally nilpotent ideal and \( R/J(R) = \bigoplus_{i=1}^{n} M_{m_i}(D_i) \) is a finite direct sum of rings of \( m_i \times m_i \) matrices over locally finite fields \( D_i \) with \( i = 1, \ldots, n \).

Proof (i) ⇒ (ii) It follows from Lemma 16(ii).

(ii) ⇒ (i) It is clear that \( R/J(R) \) is a finite direct ring sum and each direct summand is a locally finite field or a finite total matrix ring. The unipotent group \( 1 + J(R) \) is locally nilpotent group. That yields the subring \( (J(R)^0)_{rg} = J(R) \) is locally nilpotent by [2, Lemma 3].

Let \( X \) be a finite subset of \( R \). There exists an additive group isomorphism

\[
(\langle X \rangle_{rg} + J(R))/J(R) \cong \langle X \rangle_{rg}/(\langle X \rangle_{rg} \cap J(R))
\]

and

\[
(\langle X \rangle_{rg} + J(R))/J(R) = \langle X + J(R) \rangle_{rg}
\]

is a finite subring of \( R/J(R) \). The subring \( B := \langle X \rangle_{rg} \cap J(R) \) is finitely generated by [48, Theorem 2]. Clearly, it is nilpotent and so \( B/B^2 \) is a finitely generated \( \mathbb{Z}_n \)-module for some \( n \in \mathbb{N} \). It implies that the subring \( B^2 \) is finitely generated by [48, Theorem 2]. Using induction on the nilpotency index of \( B \), we obtain that \( B \) (and consequently \( \langle X \rangle_{rg} \)) is finite.

(ii) ⇒ (iii) For each \( D_i \) \((i = 1, \ldots, n)\) there exists a chain \( F_1 \subseteq F_2 \subseteq \cdots \) such that \( D_i = \bigcup_j F_j \) and each \( F_j \) is a finite subfield of the field \( F_{j+1} \). Thus \( (R/J(R))^0 \) is locally finite. Moreover, the adjoint group \( J(R)^0 \) is locally finite, \( (R/J(R))^0 \cong R^0/J(R)^0 \) and so \( R^0 \) is locally finite.

(iii) ⇒ (ii) It is obvious. □

Corollary 24 A locally finite semilocal ring is semiperfect.
5 Properties induced by derivations

Proposition 25. Let $R$ be a commutative ring with unity. If $R$ has a derivation $\delta$ with the finite kernel $\text{Ker} \, \delta$, then $R$ is a locally finite ring. The prime radical $\mathcal{P}(R)$ has finite index in $R$ and $\delta(R) \subseteq \mathcal{P}(R)$.

Proof. Assume that $R$ is infinite. Obviously, $\delta(1) = 0$ and $\text{Ker} \, \delta \neq 0$. This implies that $nR = 0$ for some $n \in \mathbb{N}$ and $R$ is a finite direct ring sum of $p$-components $F_p$, where the prime $p$ divides $n$ and

$$F_p = \{ r \in R \mid p^kr = 0 \text{ for some } k = k(r) \in \mathbb{N} \cup \{0\} \}.$$ 

Consequently, without loss of generality, we can assume that $n = p^s$ for some $s \in \mathbb{N}$. Since

$$\varphi : R/pR \ni a + pR \mapsto p^k a + p^{k+1}R \in p^k R/p^{k+1}R$$

is an additive group isomorphism, $p^k R/p^{k+1}R$ is infinite for any $k = 0, \ldots, s - 1$. If $\delta(R) \subseteq pR$, then $\delta(p^{s-1}R) = 0$ and so $p^{s-1}R \subseteq \text{Ker} \, \delta$, a contradiction. Hence $\delta(R) \not\subseteq pR$.

Inasmuch as $\delta(pR) \subseteq pR$, the rule

$$\Delta : R/pR \ni a + pR \mapsto \delta(a) + pR \in R/pR$$

determines a nonzero derivation $\Delta$ of $R/pR$ and $\text{Ker} \, \Delta$ is finite. Then $\Delta(\beta p) = 0$ for any $\beta \in R/pR$ and so the set $\{\alpha_p \mid \alpha \in R/pR\}$ is finite. If $\alpha, \beta \in R/pR$ are distinct elements and $\alpha p - \beta p = 0$, then $\alpha - \beta \in \mathcal{P}(R/pR)$. This implies that the index $|R/pR : \mathcal{P}(R/pR)| < \infty$.

However $pR \subseteq \mathcal{P}(R)$ and so $|R : \mathcal{P}(R)| < \infty$.

Since $\mathcal{P}(R)$ is nil with the torsion additive group $\mathcal{P}(R)^+$, we conclude that the adjoint group $\mathcal{P}(R)^+$ is locally finite. Thus $R$ is a semiperfect ring with the torsion unit group $U(R)$, so $R$ is locally finite by Proposition 23.

Proof of Theorem 3. (i) Assume that $R$ is commutative, $d \in \Delta$, $\delta \in Z(\Delta)$ and $a \in R$.

Then

$$0 = [\delta, ad] = \delta(a)d + a[\delta, d] = \delta(a)d. \quad (5.1)$$

The ideal of $R$ generated by the set $\{\mu(R) \mid \mu \in Z(\Delta)\}$ we denote by $\Delta_1(R)$. Then $\Delta_1(R)^2 = 0$ in view of (5.1). Since $d(\delta(R)) = \delta(d(R)) \subseteq \delta(R)$, we conclude that $\Delta_1(R)$ is a $\Delta$-ideal of $R$ and so

$$\overline{\Delta} : R/\Delta_1(R) \ni a + \Delta_1(R) \mapsto d(a) + \Delta_1(R) \in R/\Delta_1(R) \quad (5.2)$$

is a derivation of $B := R/\Delta_1(R)$. Then $\overline{\Delta} = \{\overline{\delta} \mid d \in \Delta\}$ is a subring of the Lie ring Der $B$ and a left $B$-module. As before, $\Delta_1(B)$ (and its inverse image in $R$) is nilpotent, where $\Delta_1 := \{\overline{\delta} \mid d \in \Delta_2\}$ and $\Delta_2$ is an inverse image of $Z(\Delta/Z(\Delta))$ in $\Delta$. Thus $\Delta(R)$ is nilpotent according to the induction on nilpotent length of $\Delta$.

Now, assume that $R$ is not necessary commutative. If $\delta \in \Delta$, then $\delta(C(R)) \subseteq C(R)$ and the rule

$$\overline{\delta} : R/C(R) \ni x + C(R) \mapsto \delta(x) + C(R) \in R/C(R)$$

determines a derivation of $R/C(R)$. Since $\overline{\Delta} = \{\overline{\delta} \mid \delta \in \Delta\}$ is a left $(R/C(R))$-module, $\overline{\delta}(R/C(R))^m = 0$ for some $m \in \mathbb{N}$ and consequently $\Delta(R)^m \subseteq C(R)$. 

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(ii) Suppose that \( \gamma_{n+1}(\Delta) = 0 \) and \( R \) is commutative. If \( d \in \Delta^{(n-1)}, a \in R \), then
\[
\begin{align*}
    d(a)d &= [d, ad] \in \gamma_2(\Delta), \\
    d^2(a)d &= [d, d(a)d] \in \gamma_3(\Delta), \\
    &\vdots \\
    d^{n-1}(a)d &= [d, d^{n-2}(a)d] \in \gamma_n(\Delta), \\
    d^n(a)d &= [d, d^{n-1}(a)d] \in \gamma_{n+1}(\Delta) = 0.
\end{align*}
\]
This implies that \( d^n(R)d = 0 \). Since \( d(C(R)) \subseteq C(R) \), the result can be obtained similarly to that of the part (i).

(iii) Let \( \delta \in \Delta \). If \( R \) is commutative, then \( a\delta \in \Delta \) for any \( a \in R \) and so
\[
(-1)^n\delta^n(a)\delta = (-1)^n[a\delta, \underbrace{\delta, \ldots, \delta}_{n \text{ times}}] = 0
\]
for some \( n \in \mathbb{N} \). Hence \( \delta^n(a)\delta = 0 \).

Now, assume that \( R \) is not necessarily commutative. Then \( \delta(C(R)) \subseteq C(R) \) and, by the same argument as in the part (i), there exist \( n = n(\delta, a) \in \mathbb{N} \) such that \( \delta^n(a)\delta(R) \subseteq C(R) \) and the assertion holds.

\[\square\]

If \( x \in R \), then the rule \( \partial_x : R \ni a \mapsto (ax - xa) \in R \) determines a derivation \( \partial_x \) of \( R \); this derivation is called an inner derivation of \( R \) (induced by \( x \)). The set \( \text{IDer} R \) of all inner derivations of \( R \) is an ideal of the Lie ring \( D \).

**Proposition 26** Let \( R \) be a ring. The following statements hold:

(i) \( R \) is Lie solvable (Lie nilpotent, \( n \)-Engel, Engel, locally Lie nilpotent, locally Lie solvable, respectively) if and only if the Lie ring \( \text{IDer} R \) is solvable (nilpotent, \( n \)-Engel, Engel, locally nilpotent, locally solvable, respectively);

(ii) if \( D \) is solvable (nilpotent, \( n \)-Engel, Engel, locally nilpotent, locally solvable, respectively), then \( R \) is Lie solvable (Lie nilpotent, \( n \)-Engel, Engel, locally Lie nilpotent, locally Lie solvable, respectively);

(iii) if \( R \) is 2-torsion-free semiprime and \( D \) (\( \text{IDer} R \), respectively) is solvable, then \( D = 0 \) (\( R \) is commutative, respectively);

(iv) if \( R \) is with unity of characteristic 0 (\( \text{char}(R/\mathbb{P}(R)) = 0 \), respectively) and \( D \) is solvable, then \( C(R) \subseteq D(R) \subseteq \mathbb{P}(R) \);

(v) if \( R \) consists from countable many elements and \( D \) (\( \text{IDer} R \), respectively) is Lie solvable, then \( D = 0 \) (\( R \) is commutative, respectively);

(vi) if \( R \) is commutative and \( D \) is locally nilpotent, then, for each \( a \in R \) and \( \delta \in D \) there exists \( n = n(a, \delta) \in \mathbb{N} \) such that \( \delta^n(a)\delta = 0 \). Moreover, if \( R \) is reduced, then \( \delta^n(a) = 0 \);

(vii) if \( R \) is a semiprime ring with the \( n \)-Engel derivation ring \( D \) (\( \text{IDer} R \), respectively), then \( R \) is commutative and, for each \( \delta \in D \) (\( \delta \in \text{IDer} R \), respectively), there exists \( n = n(\delta) \in \mathbb{N} \) such that \( \delta^n(R) = 0 \). Moreover, if \( F(R) = 0 \), then \( D = 0 \).

**Proof** (i) Since \( [\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}] = \partial_{[x_1, x_2, \ldots, x_n]} \) for any \( x_1, x_2, \ldots, x_n \in R \) and
\[
\text{IDer} \ni \partial_x \mapsto x + Z(R) \in R^L/Z(R)
\]
is a Lie ring isomorphism, the result is obvious.

(ii) It is immediately.
(iii) If $A \subseteq R$, then by $\Delta_A$ we denote the set $\{\partial_a \mid a \in A\}$. If $A$ is a Lie ideal of $R$, then $\Delta_A$ is an ideal of $\text{IDer} R$. Assume that $D$ is solvable of length $n > 1$. If $D^{(n-1)} \cap \text{IDer} R \neq 0$, then

$$I_{D^{(n-1)}} := \{a \in R \mid \partial_a \in D^{(n-1)} \cap \text{IDer} R\}$$

is a Lie ideal of $R$ and

$$0 = [D^{(n-1)}, D^{(n-1)}] \supseteq [\Delta_{I_{D^{(n-1)}}}, \Delta_{I_{D^{(n-1)}}}] = \Delta_{[I_{D^{(n-1)}}, I_{D^{(n-1)}}]}$$

what gives that $[I_{D^{(n-1)}}, I_{D^{(n-1)}}] \subseteq Z(R)$. Thus $I_{D^{(n-1)}} \subseteq Z(R)$ by [30, Lemma 1] and consequently $\Delta_{I_{D^{(n-1)}}} = 0$, a contradiction. Hence $D^{(n-1)} \cap \text{IDer} R = 0$. This implies that $R$ is commutative, $D(R)^m = 0$ for some $m \in \mathbb{N}$ by Theorem 3(i) and $D(R) = 0$ by the semiprimeness of $R$. We conclude the result.

(iv) Each minimal prime ideal of $R$ is closed with respect to each $\delta \in D$ by [26, Proposition 1.3]. The map

$$\overline{\delta} : R/P \ni a + P \mapsto \delta(a) + P \in R/P \quad (5.3)$$

is a derivation of a prime ring $R/P$ of characteristic $\neq 2$ by Lemma 17 and $\text{Der}(R/P) = 0$ by the part (iii). Thus $C(R) \subseteq D(R) \subseteq \mathbb{P}(R)$ ($C(R) \subseteq \mathbb{P}(R)$, respectively).

(v) There exists a collection of prime ideals $P_{\beta}$ ($\beta \in \Gamma$) by [21, Theorem 2.1] such that

$$\bigcap_{\beta \in \Gamma} P_{\beta} = 0 \quad \text{and} \quad \delta(P_{\beta}) \subseteq P_{\beta} \quad (\forall \delta \in D \text{ and/or } \forall \delta \in \text{IDer} R, \text{ respectively}).$$

Consequently, $\overline{\delta}$ defined by the rule (5.3) is a derivation of $R/P$ for any $P = P_{\beta}$ and $\text{Der}(R/P) = 0$ by the part (iii). Hence

$$C(R) \subseteq D(R) \subseteq \bigcap_{\beta \in \Gamma} P_{\beta} = 0$$

and the assertion holds.

(vi) The subring of $D$ generated by derivations $d$ and $ad$ ($a \in R$ and $d \in D$) is nilpotent and the result holds by the same argument as in the proof of Theorem 3(ii).

(vii) Since $R$ is $n$-Engel it is commutative by Corollary 8. If $\delta \in D$ ($\delta \in \text{IDer} R$, respectively), then by the same argument, as in the proof of Theorem 3(ii), we obtain that $\delta^n(R) = 0$ for some $n \in \mathbb{N}$.

Let $F(R) = 0$. We prove that $\delta = 0$ using induction by $n$. If $\delta^2(R) = 0$, then $\delta = 0$ by [19, Corollary 1]. Let $n > 2$ and suppose that $\delta^{n-1}(R) = 0$ implies that $\delta = 0$. Assuming $\delta^n(R) = 0$ we see that

$$0 = \delta^n(a\delta^{n-2}(b)) = (n-1)\delta^{n-1}(a)\delta^{n-1}(b) \quad (\forall a, b \in R)$$

what implies that $(\delta^{n-1}(R))^2 = 0$ and hence $\delta^{n-1}(R) = 0$ by the semiprimeness of $R$. Thus $\delta = 0$. \hfill $\square$

Recall that the commutator ideal $C(R)$ of a 2-torsion-free Lie solvable ring $R$ is nil (see [58, Theorem 2.1] and [72, Theorem]). Proposition 26(v) precise this result in the countable case.

**Corollary 27** Let $R$ be an algebra over a field $\mathbb{F}$ of characteristic 0. If the Lie $\mathbb{F}$-algebra $D$ ($\text{IDer} R$, respectively) is Engel, then $R$ is Lie nilpotent.
Proof Since $\text{IDer } R$ is an Engel Lie algebra over $Z(R)$ and $\text{IDer } R$ and $R/Z(R)$ are isomorphic as Lie algebras, $R$ is Lie Engel and so it is nilpotent by [52, Theorem B]. □

Note that examples of the rings derivations $\text{Der } R$ of certain rings $R$ can be found in [8].

6 Solvability

Lemma 28 Each Lie solvable nil ring $R$ is locally nilpotent. Moreover, if $R$ is a $\mathbb{Q}$-algebra, then it is Lie nilpotent.

Proof The ring $R$ contains a nilpotent ideal $I$ such that $R/I$ satisfies the identity

$$[x_1, [x_2, x_3], [x_4, x_5]] = 0$$

(see [58, Theorem 2.1] and [72, Theorem]). This implies that every finitely generated subring of $R/I$ is nilpotent (see e.g. [56, Theorems 6.3.3 and 6.3.39]), so $R$ is a locally nilpotent ring. If $R$ is a $\mathbb{Q}$-algebra, then $R$ is locally nilpotent as an algebra and, by [52, Theorem B], it is Lie nilpotent. □

Proof of Theorem 1. (i) The quotient ring $R/\mathbb{P}(R)$ is semiprime and so its adjoint group $(R/\mathbb{P}(R))^\circ$ is abelian by Corollary 8. Moreover, $(R/\mathbb{P}(R))^0 \cong R^0/\mathbb{P}(R)^0$ and $\mathbb{P}(R)^0$ is nilpotent by [38].

(ii) Let $P$ be a prime ideal of $R$. If $J(R/P)$ is nonzero, then $[J(R), R] \subseteq P$ in view of [20, Theorem A]. This implies that $[J(R), R] \subseteq \mathbb{P}(R)$. Since $R/\mathbb{P}(R)$ is semiprime, then $J(R) \subseteq \mathbb{P}(R)$ or $J(R/\mathbb{P}(R))$ is commutative in view of [20, Theorem B]. From this it follows that $J(R/\mathbb{P}(R))^2 \cdot C(R/\mathbb{P}(R)) = 0$ and consequently $J(R/\mathbb{P}(R)) \cdot C(R/\mathbb{P}(R)) = 0$ what gives that

$$J(R) \cdot C(R) \subseteq \mathbb{P}(R).$$

(iii) Let $R$ be a 2-torsion-free Lie solvable ring.

(a) First, assume that $R$ is prime of solvable length $n > 1$. Since $[R^{(n-1)}, R^{(n-1)}] = 0$, we conclude that $R^{(n-1)} \subseteq Z(R)$ by [30, Lemma 1]. But then $[R^{(n-2)}, R^{(n-2)}] \subseteq R^{(n-1)}$ and we obtain a contradiction in view of [30, Lemma 1]. Hence $R$ is commutative. This implies that $C(R) \subseteq \mathbb{P}(R) = N(R)$ in general case. If $R$ has unity, then $R$ is abelian in view of [69, 3.20] and $N(R)$ is a locally nilpotent ring by Lemma 28.

(b) If $R^0$ is torsion, then it is locally finite (and so it locally nilpotent by [2, Corollary 2]).

(c) If $N(R)^+$ is torsion-free divisible, then $N(R)$ is a locally nilpotent $\mathbb{Q}$-algebra and so it satisfies the Engel condition. The algebra $N(R)$ is Lie nilpotent by [52, Theorem B] and $N(R)^0$ is nilpotent by [38], as required. □

Corollary 29 Let $R$ be a right Goldie ring (or $R$ satisfies the ascending chain condition on both left and right annihilators) with unity. If $R$ satisfies one of the conditions:

(i) $R$ is Engel as a Lie ring;

(ii) $R$ is 2-torsion-free locally finite,

then $R$ is Lie solvable and the unit group $U(R)$ is nilpotent-by-abelian.

Proof We have that $C(R) \subseteq N(R)$ by Corollary 10 (by Lemma 18, respectively), and so $N(R)$ is an ideal of $R$. Since $N(R)$ is nilpotent by [46, Theorem 1] (by [33, Theorem 1], respectively) and $R/N(R)$ is commutative, $R$ is Lie solvable and the unit group $U(R)$ is nilpotent-by-abelian. □
Remark 30 If $R$ is a right Goldie $n$-Engel ring of prime characteristic $p > 0$ and $n < p$, then it is Lie nilpotent in view of [34].

7 Local rings

Proof of Theorem 2. (i) The group unit $U(R/J(R))$ is solvable what gives that $[R, R] \subseteq J(R)$ by [71, Theorem 2]. Since $J(R)^{o}$ is solvable, we deduce that $J(R)$ (and consequently $R$) is Lie solvable by [5, Theorem A]. Moreover, the Levitzki radical $L(R)$ of $R$ is a $PI$-ring by [5, Theorem B(2)] and so it is locally nilpotent. If $B = R/L(R)$, then $J(B)$ is commutative in view of [5, Theorem B] what implies that $C(B)^{3} = 0$ and consequently $B$ is commutative. Hence $C(R) \subseteq L = N(R)$.

(ii) Since $N(R)$ is a locally nilpotent ideal of $R$, the $\mathbb{Q}$-algebra $N(R)$ is Lie nilpotent by [52, Theorem B] and so the adjoint group $N(R)^{o}$ is nilpotent by [38].

Proposition 31 Let $R$ be a local ring. The following statements are equivalent:

(i) the unit group $U(R)$ is torsion;
(ii) $J(R)$ is nil and $R/J(R)$ is an absolute field of characteristic $p > 0$;
(iii) $U(R) \cong (1 + J(R)) \rtimes U(R/J(R))$, where $1 + J(R)$ is a $p$-group and $U(R/J(R))$ is a $p^{k}$-group.

Therefore, a local ring $R$ with the torsion unit group $U(R)$ is a locally finite ring.

Proof (i) $\Rightarrow$ (ii) Since $R/J(R)$ is a skew field and the unit group $U(R/J(R))$ is torsion, we deduce that $J(R)$ is commutative and $p(R/J(R)) = 0$ for some prime $p$. Hence $pR \subseteq J(R)$ and $p^{k}R = 0$ for some $k \in \mathbb{N}$ in view of Lemma 13(i). Since $U(R/J(R))$ is torsion, $R/J(R)$ is an absolute field.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obviously. □

Lemma 32 Let $R$ be a local ring which is Engel as a Lie ring. If $F(R) = 0$, then either $R$ is a Lie nilpotent $\mathbb{Q}$-algebra or $pR \subseteq J(R)$ for some prime $p$, $C(R) \subseteq N_{r}(R) = N(R)$ and $N(R)^{o}$ is a torsion-free group. Moreover, if in the last case, $R$ is $n$-Engel, then it is Lie solvable and $N(R)^{o}$ is nilpotent (the $R^{o}$ is nilpotent-by-abelian and $R$ is Lie solvable).

Proof If $R^{+}$ is a divisible group, then $R$ is a $\mathbb{Q}$-algebra and so it is Lie nilpotent by [52, Theorem B]. Therefore, we assume in the next that $pR$ is proper in $R$ for some prime $p$. Then $pR \subseteq J(R)$ and $C(R) \subseteq N_{r}(R) = N(R)$ by Corollary 10. Consequently, $N(R)$ is an ideal of $R$ and the adjoint group $N(R)^{o}$ is torsion-free by [1, Lemma 2.4].

Assume that $R$ is $n$-Engel. Since $N(R)^{o}$ is $m$-Engel for some $m \in \mathbb{N}$ by [4, Main Theorem], it is nilpotent by the theorem of Zelmanov [74]. Then $N(R)$ is Lie nilpotent by [38] and consequently $R$ is Lie solvable. □

Proof of Theorem 4. In view of Lemma 32, we assume that $pR \subseteq J(R)$ for some prime $p$. Since $C(R) \subseteq N(R) \subseteq J(R)$ and $(J(R)/C(R))^{o} \cong J(R)^{o}/C(R)^{o}$ is abelian, we deduce that $N(R)$ is an ideal of $R$ and $J(R)^{o}$ is a solvable group. Moreover, $J(R)^{o}$ is $m$-Engel group for some $m \in \mathbb{N}$ depending on $n$ by [4, Main Theorem]. Then the adjoint group $J(R)^{o}$ is locally nilpotent by [27, Theorem 1].

If $0 \neq a \in \tau(J(R)^{o})$, then

$$0 = a^{(n)} = a \left( n + \sum_{k=2}^{n} \binom{n}{k} a^{k-1} \right) \quad \text{(for some } n \in \mathbb{N} \text{)}.$$
Hence $n \in J(R)$. Obviously, the order of each element of $U(R/J(R))$ is relatively prime with $p$ and, thus, $1 \in J(R)$, a contradiction. Hence $\tau(J(R)^0) = 0$ and, by theorem of Zelmanov [74], $J(R)^0$ is nilpotent as a locally nilpotent $m$-Engel torsion-free group. Since $pR \subseteq J(R)$, $\gamma_{n+1}(pR) = 0$ for some integer $n \geq 0$ and

$$\gamma_1(pR) = pR, \quad \gamma_2(pR) = [pR, pR] = p^2[R, R] = p^2\gamma_2(R),$$

$$\vdots$$

$$\gamma_{n+1}(pR) = [\gamma_n(pR), \gamma_n(pR)] = p^{n+1}[\gamma_n(R), \gamma_n(R)] = p^{n+1}\gamma_{n+1}(R).$$

Thus we conclude that $\gamma_{n+1}(R) = 0$, i.e., $R$ is Lie nilpotent.

**Lemma 33** Let $R$ be a local ring with the nil Jacobson radical $J(R)$. If $R$ is Engel and $\text{char} R/J(R) = 0$, then $R$ is Lie nilpotent.

**Proof** If $0 \neq a \in R$ and $pa = 0$ for some prime $p$, then $a \cdot pR = 0$ and therefore $pR \subseteq J(R)$, a contradiction. Hence $F(R) = 0$. If $qR \neq R$ for some prime $q$, then $qR \subseteq J(R)$ and, for any $x \in R$, there exists $k = k(x) \in \mathbb{N}$ such that $q^kx^k = 0$, a contradiction. Hence $R^+$ is a divisible group. As a consequence, $R$ is an algebra over the rational numbers field $\mathbb{Q}$ and $R$ is Lie nilpotent by [52, Theorem B]. \qed

### 8 Corollaries

There are large number of articles which extend the Cohen’s Theorem [23] and Kaplansky Theorem [39, Theorem 12.3] (see e.g. [51, 68] and others). We also present the following generalizations of these theorems.

**Corollary 34** Let $R$ be a ring with unity. If the commutator ideal $C(R)$ is nil (in particular, $R$ is Engel), then the following statements hold:

(i) if prime ideals of $R$ are finitely generated as right ideals, then the quotient ring $R/C(R)$ is a commutative Noetherian ring;

(ii) if $R$ is a right Noetherian ring and each its maximal right ideal is principal, then $R/C(R)$ is a commutative principal ideal ring;

(iii) if each prime ideal of $R$ is principal as a right ideal, then $R/C(R)$ is a commutative principal ideal ring;

(iv) if $R$ is right Noetherian and $R/P(R)$ is finite, then $R/C(R)$ is finite.

**Proof** (i)–(iii) Since $1 + C(R)$ is an ideal of $R$ for its any right ideal $I$, the result follows from Cohen’s Theorem [23] and Kaplansky Theorem [39, Theorem 12.3].

(iv) Without loss of generality, assume that $P(R) \neq 0$. Obviously, $C(R)$ is nilpotent by [46, Theorem 1] and Corollary 8 and $C(R) \subseteq P(R) \subseteq P$ for any nonzero prime ideal $P$ of $R$. Finally, $R$ is Artinian by Lemma 21 and so $R/C(R)$ is finite by [7, Lemma 22]. \qed

**Remark 35** If $R$ is a Lie nilpotent algebra over a field of characteristic 0 and $I = I^2$ is a f.g. ideal (as a one-sided ideal) of $R$, then $I = eR$ for some central idempotent $e \in R$.

In fact, this follows from [67, Theorem 1] and Corollary 8(ii).

**Proposition 36** Let $R$ be a nil ring. The following statements hold:
(i) if \( R \) is \( n \)-Engel as a Lie ring and \( F(R) = 0 \), then \( R^o \) is nilpotent (and so \( R \) is Lie nilpotent);
(ii) if \( R \) is \( n \)-Engel of bounded index, then \( R/F(R) \) is Lie nilpotent (and so the adjoint group \( R^o \) is locally nilpotent and torsion-by-(torsion-free nilpotent)).

**Proof**  
(i) The adjoint group \( R^o \) is \( m \)-Engel for some \( m \in \mathbb{N} \) by [4, Main Theorem] and, therefore, it is locally nilpotent by [4, Lemma 2.2]. As a consequence, \( R^o \) is nilpotent by [74] (see e.g. [2, Corollary 1]) and \( R \) is Lie nilpotent by [38].

(ii) In view of the result of Levitzky [29, Lemma 1.1], the ring \( R \) is locally nilpotent and so \( R^o \) is a locally nilpotent group. If \( F(R) = 0 \), then \( R^o \) is nilpotent by [74] and hence \( R \) is Lie nilpotent by [38].

\[ \blacksquare \]

**Proposition 37** Let \( R \) be a ring such that \( Dz(R) \) is commutative. The following holds:

(i) either \( Dz(R)^2 \neq 0 \) and \( R \) is Lie metabelian (then the adjoint group \( R^o \) is metabelian) or \( Dz(R)^2 = 0 \) (i.e., \( R \) is 2-primal); in the last case \( R \) is commutative or \( Dz(R) \) is completely prime ideal of \( R \);
(ii) if \( R \) is Engel (2-torsion-free Lie solvable, 2-torsion-free locally finite, respectively) with unity, then it is Lie metabelian and \( C(R)^3 = 0 \).

**Proof**  
Assume that \( R \) is non-commutative.

(i) The set \( N(R) \) is an ideal of \( R \) by [42, Theorem 5.7]. If \( Dz(R)^2 \neq 0 \), then \( R \) is Lie metabelian by [42, Theorem 5.7].

Now, assume that \( Dz(R)^2 = 0 \). Thus \( Dz(R) = N(R) \) and \( ab \in Dz(R) \) implies that \( a \in Dz(R) \) or \( b \in Dz(R) \) for each \( a, b \in R \) what means that \( Dz(R) \) is completely prime.

(ii) If \( Dz(R)^2 = 0 \), then \( N(R)^2 = 0 \) and therefore \( R \) is Lie metabelian in view of Corollary 10 (Theorem 1(iii), Lemma 18(iv), respectively).

From [42, Theorem 5.7] it follows that \( R \) is Lie metabelian also in the case \( Dz(R)^2 \neq 0 \).

If \( D(R)^2 \neq 0 \), then \( D(R)^2 C(R) = 0 \) by [42, Lemma 5.4] and so \( C(R)^3 = 0 \).

Finally, \( Dz(R)^2 C(R) = 0 \) by [42, Lemma 5.4] and so \( C(R)^3 = 0 \).  

\[ \blacksquare \]

**Remark 38** Let \( R \) be a ring such that the set \( N(R) \) is commutative. If \( R \) is an Engel (2-torsion-free Lie solvable, 2-torsion-free locally finite, respectively), then \( N(R) \) is an ideal of \( R \), \( C(R) \subseteq N(R) \) and \( N(R)^2 \cdot C(R) = 0 \).

In fact, \( C(R) \subseteq N(R) \) by Corollary 10 (by Theorem 1, Lemma 18(iv), respectively) what gives that \( N(R) \) is an ideal of \( R \) and the result follows.

**Corollary 39** Let \( R \) be a ring with unity. The following statements hold:

(i) if \( R \) is right Noetherian \( \pi \)-regular and satisfies the \( n \)-Engel condition (is 2-torsion-free Lie solvable, respectively), then it is a field or right Artinian;
(ii) if \( R \) is an Engel ring of bounded index and \( J(R) \) is nil, then
\[
C(R) \subseteq \mathbb{P}(R) = N(R) = J(R),
\]

\( N(R) \) is a locally nilpotent ideal of \( R \) and \( U(R) \) is (locally nilpotent)-by-abelian.

**Proof**  
(i) The ring \( R \) is abelian by Corollary 8 (by Theorem 1, respectively) and, therefore, \( R/P \) is a field for any prime ideal \( P \) by [10, Theorem 5]. Hence each prime ideal is a maximal right ideal. In view of Lemma 21 and Corollary 8(i), \( R \) is a field or right Artinian.
(ii) We obtain that $C(R) \subseteq N(R) = J(R)$ by Corollary 10. Then $R/P$ is commutative for each prime ideal $P$ of $R$ and so $C(R) \subseteq P(R)$. The ideal $N(R)$ is locally nilpotent in view of [29, Lemma 1.1] and so the unit group $U(R)$ is (locally nilpotent)-by-abelian.

Finitely generated non-commutative radical rings need not be nil [62]. Each nil ring is Engel [66, Proposition 4.2] and the adjoint group $R^o$ of a nil algebra $R$ is locally graded (i.e., each finitely generated infinite subgroup of $R^o$ contains a proper subgroup of finite index) [63, Theorem 1]. We extend this result in the following way.

**Corollary 40** For each Engel (2-torsion-free Lie solvable, 2-torsion-free locally finite, respectively) algebra $R$, its adjoint group $R^o$ is locally graded.

**Proof** The set $N(R)$ is an ideal of $R$ by Corollary 10 (by Theorem 1(iii) and Proposition 5(ii), respectively) and $N(R)^o$ is locally graded by [63, Theorem 1]. Inasmuch as $R^o/N(R)^o$ is abelian, we deduce that $R^o$ is locally graded.

**Proposition 41** If $R$ is a ring of bounded index, then the following statements hold:

(i) $P(R) = N(R)$ is a locally nilpotent ring (and so $P(R)^o$ is a locally nilpotent group);

(ii) if $R$ is Engel (2-torsion-free locally solvable, 2-torsion-free locally finite, respectively), then $R^o$ is a (locally nilpotent)-by-abelian group;

(iii) if $R$ is $n$-Engel and $F(R) = 0$, then $N(R)$ is nilpotent and $R^o$ is (torsion-free nilpotent)-by-abelian;

(iv) if $R$ is $n$-Engel, then $N(R)^o$ is (torsion locally nilpotent)-by-(torsion-free nilpotent).

**Proof** (i) Obviously, $P(R) \subseteq N(R)$ and so $N(R/P(R)) = 0$ in view of [29, Lemma 1.1]. Thus $P(R) = N(R)$. Since each nonzero homomorphic image of $P(R)$ contains a nonzero nilpotent ideal, $P(R)$ is locally nilpotent. If $S = \langle g_1, \ldots, g_n \rangle$ for $g_1, \ldots, g_n \in N(R)$, then $S \subseteq (\langle g_1, \ldots, g_n \rangle_{rg})^o$. Hence $S$ is nilpotent and $P(R)^o$ is locally nilpotent.

(ii) Inasmuch as $C(R) \subseteq N(R) = P(R)$ by Corollary 10 (Theorem 1(iii), Lemma 18(iv), respectively), the ideal $N(R)$ is locally nilpotent in view of [29, Lemma 1.1] and the result follows.

(iii) We get that $C(R) \subseteq P(R) = N(R)$ by Corollary 8 and $N(R)^o$ is a torsion-free locally nilpotent $m$-Engel group for some $m \in \mathbb{N}$ in view of the part (i) and [4, Main Theorem]. Consequently, $N(R)^o$ is nilpotent in view of [74] and $(R/N(R))^o$ is abelian.

(iv) The quotient group $N(R)^o/F(N(R))^o$ is torsion-free locally nilpotent $m$-Engel for some $m \in \mathbb{N}$ and so it is nilpotent. Furthermore, $F(N(R))^o$ is locally nilpotent by [2, Corollary 2].

Finally, we have also the following.

**Proposition 42** If the set $N(R)$ of nilpotent elements of a ring $R$ is finite and $C(R) \subseteq N(R)$, then the following statements hold:

(i) $R$ is an FC-ring (i.e., the centralizer $C_R(a) = \{ r \in R \mid ra = ar \}$ is of finite index in the additive group $R^+$ for any $a \in R$ [7, 18]);

(ii) $R^o$ is a finite-by-abelian group with the finite commutator subgroup (i.e., $R^o$ is a BF$C$-group (see e.g. [55])).

**Proof** (i) The quotient group $R^+ / \ker \partial_x$ is isomorphic to the image $\text{Im} \partial_x$ for any $x \in R$. Inasmuch as $\text{Im} \partial_x \subseteq C(R)$, we conclude that $R$ is an FC-ring.
(ii) Since $N(R)^o$ is a finite normal subgroup of $R^o$ and $R^o/N(R)^o \cong (R/N(R))^o$, the assertion is true.

\begin{flushright}
\Box
\end{flushright}

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