Perturbed Message Passing for Constraint Satisfaction Problems

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Abstract

We introduce an efficient message passing scheme for solving Constraint Satisfaction Problems (CSPs), which uses stochastic perturbation of Belief Propagation (BP) and Survey Propagation (SP) messages to bypass decimation and directly produce a single satisfying assignment. Our first CSP solver, called Perturbed Belief Propagation, smoothly interpolates two well-known inference procedures; it starts as BP and ends as a Gibbs sampler, which produces a single sample from the set of solutions. Moreover we apply a similar perturbation scheme to SP to produce another CSP solver, Perturbed Survey Propagation. Experimental results on random and real-world CSPs show that Perturbed BP is often more successful and at the same time tens to hundreds of times more efficient than standard BP guided decimation. Perturbed BP also compares favorably with state-of-the-art SP-guided decimation, which has a computational complexity that generally scales exponentially worse than our method (wrt the cardinality of variable domains and constraints). Furthermore, our experiments with random satisfiability and coloring problems demonstrate that Perturbed SP can outperform SP-guided decimation, making it the best incomplete random CSP-solver in difficult regimes.

Keywords: CSP, Message Passing, Belief Propagation, Survey Propagation, Gibbs Sampling, Decimation

1. Introduction

Probabilistic Graphical Models (PGMs) provide a common ground for recent convergence of themes in computer science (artificial neural networks), statistical physics of disordered systems (spin-glasses) and information theory (error correcting codes). In particular, message passing methods have been successfully applied to obtain state-of-the-art solvers for Constraint Satisfaction Problems [Mezard et al. 2002].

The PGM formulation of a CSP defines a uniform distribution over the set of solutions, where each unsatisfying assignment has a zero probability. In this framework, solving a CSP amounts to producing a sample from this distribution. To this end, usually an inference procedure estimates the marginal probabilities, which suggests an assignment to a subset of most biased variables. This process of sequentially fixing a subset of variables,
called *decimation*, is repeated until all variables are fixed to produce a solution. Decimation gives an incomplete solver (Kautz et al. 2009), in the sense that the procedure’s failure is not a certificate of unsatisfiability. An alternative approach is to use message passing to guide a search procedure that can back-track if a dead-end is reached (e.g., Kask et al. 2004; Parisi 2003). Here using a branch and bound technique and relying on exact solvers, one may also determine when a CSP is unsatisfiable.

The most common inference procedure for this purpose is Belief Propagation (Pearl 1988). However, due to geometric properties of the set of solutions (Krzakaa et al. 2007) as well as the complications from the decimation procedure (Coja-Oghlan 2011; Kroc et al. 2009), BP-guided decimation fails on difficult instances. Study of the change in the geometry of solutions has lead to Survey Propagation (Braunstein et al. 2002b) – a powerful message passing procedure that is slower than BP (per iteration) but typically remains convergent, even in many situations when BP fails to converge.

Using decimation, or other search schemes that are guided by message passing, usually requires estimating marginals or partition functions, a problem harder than producing a single solution (Valiant 1979). This paper introduces a message passing scheme to eliminate this requirement, therefore also avoiding the complications of applying decimation. Our alternative has advantage over both BP- and SP-guided decimation when applied to solve CSPs. Here we consider BP and Gibbs Sampling (GS) updates as operators – $\Phi$ and $\Psi$ respectively – on a set of messages. We then consider inference procedures that are convex combination – i.e., $\gamma \Psi + (1 - \gamma)\Phi$– of these two operators. Our CSP solver, Perturbed BP, starts at $\gamma = 0$ and ends at $\gamma = 1$, smoothly changing from BP to GS, and finally producing a sample from the set of solutions. This change amounts to stochastic biasing the BP messages towards the current estimate of marginals, where this random bias increases in each iteration. This procedure is often much more efficient than BP-guided decimation (BP-dec) and sometimes succeeds where BP-dec fails. Our results on random CSPs (rCSPs) show that Perturbed BP is competitive with SP-guided decimation (SP-dec) in solving difficult random instances.

Since SP can be interpreted as BP applied to an “auxiliary” PGM (Braunstein and Zecchina 2003), we can apply the same perturbation scheme to SP, which we call Perturbed SP. Note that this system, also, does not perform decimation. Here we have the choice of perturbing SP messages to directly produce a solution (without using local search). Our experiments show that Perturbed SP is often more successful than both SP-dec and Perturbed BP in finding satisfying assignments.

Stochastic variations of BP have been previously proposed to perform inference in graphical models (e.g., Ihler and Mcallester 2009; Noorshams and Wainwright 2013). However, to our knowledge, Perturbed BP is the first method to directly combine GS and BP updates. Although here we are only concerned with the application of Perturbed BP in drawing a single sample, its repeated application can be used to estimate marginals. We plan to further investigate this in the future.

In the following, Section 1.1 introduces PGM formulation of CSP using factor-graph notation. Section 1.2 reviews BP equations and decimation procedure, then Section 1.3 casts GS as a message update procedure. In Section 2 we introduce Perturbed BP as a combination of GS and BP. Section 2.1 compares BP-dec and Perturbed BP on benchmark CSP instances, showing that our method is often several folds faster and more successful in solving CSPs. Section 3 overviews the geometric properties of the set of solutions of rCSPs, then review first order Replica Symmetry Breaking Postulate and the resulting SP equations for CSP. Section 3.2 introduces Perturbed SP and Section 3.3 presents our experimental results for random satisfiability and random coloring instances close to the unsatisfiability threshold. Finally, Section 3.4 further discusses the behavior of decimation.
and perturbed BP in the light of geometric picture of the set of solutions and experimental results.

1.1 Factor Graph Representation of CSP

Let \( x = \{x_1, x_2, \ldots, x_N\} \) be a set of \( N \) discrete variables \( x_i \in \mathcal{X}_i \), where each \( \mathcal{X}_i \) is the domain of \( x_i \). Let \( I \subseteq \mathcal{N} = \{1, 2, \ldots, N\} \) denote a subset of variable indices and \( x_I = \{x_i \mid i \in I\} \) be the (sub)set of variables in \( x \) indexed by the subset \( I \). Each constraint \( C_I(x_I) : (\prod_{i \in I} \mathcal{X}_i) \to \{0, 1\} \) maps an assignment to 1 iff that assignment satisfies that constraint. Then the normalized product of all constraints defines a uniform distribution over solutions:

\[
\mu(x) \triangleq \frac{1}{Z} \prod_I C_I(x_I)
\]

(1)

where the partition function \( Z = \sum_x \prod_I C_I(x_I) \) is equal to the number of solutions. Notice that \( \mu(x) \) is non-zero iff all of the constraints are satisfied – i.e., \( x \) is a solution. With slight abuse of notation we will use probability density and probability distribution interchangeably.

Example 1 (\( q\text{-COL} \)) Here, \( x_i \in \mathcal{X}_i = \{1, \ldots, q\} \) is a \( q \)-ary variable for each \( i \in \mathcal{N} \), and we have \( M \) constraints; each constraint \( C_{i,j}(x_i, x_j) = 1 - \delta(x_i, x_j) \) depends only on two variables and is satisfied iff the two variables have different values (colors). Here \( \delta(x, x') = \) equal to one if \( x = x' \) and zero otherwise.

This model can be conveniently represented as a bipartite graph, known as factor graph [Kschischang and Frey 2001], which includes two sets of nodes: variable nodes \( x_i \), and constraint (or factor) nodes \( C_I \). A variable node \( i \) (note that we will often identify a variable \( x_i \) with its index “\( i \)”) is connected to a constraint node \( I \) if and only if \( i \in I \). We will use \( \partial \) to denote the neighbors of a variable or constraint node in the factor graph – that is \( \partial I = \{i \mid i \in I\} \) (which is the set \( I \)) and \( \partial i = \{I \mid i \in I\} \). Finally we use \( \Delta i \) to denote the Markov blanket of node \( x_i \) – i.e., \( \Delta i = \{j \in \partial I \mid I \in \partial i, j \neq i\} \).

The marginal of \( \mu(\cdot) \) for variable \( x_i \) is defined as

\[
\mu(x_i) \triangleq \sum_{x_{\mathcal{X}_i \setminus i}} \mu(x)
\]

where the summation above is over all variables but \( x_i \). Below, we use \( \hat{\mu}(x_i) \) to denote an estimate of this marginal. Finally, we use \( \mathcal{S} \) to denote the set of solutions \( \mathcal{S} = \{x \in \mathcal{X} \mid \mu(x) \neq 0\} \).

Example 2 (\( \kappa\text{-SAT} \)) All variables are binary (\( \mathcal{X}_i = \{True, False\} \)) and each clause (constraint \( C_I \)) depends on \( \kappa = |\partial I| \) variables. A clause evaluates to zero only for a single assignment out of \( 2^\kappa \) possible assignment of variables [Garey and Johnson 1979].

Consider the following 3-SAT problem over 3 variables with 5 clauses:

\[
SAT(x) = (\neg x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3)
\]

(2)

1. For eq (1) to remain valid when the CSP is unsatisfiable we define \( \delta \triangleq 0 \).
Figure 1: (a) The set of all possible assignments to 3 variables. The solutions to the 3-SAT problem of eq(2) are in white circles. (b) The factor-graph corresponding to the 3-SAT problem of eq(2). Here each factor prohibits a single assignment.

The constraint corresponding to the first clause takes the value 1, except for $x = \{\text{True, True, False}\}$, in which case it is equal to 0. The set of solutions for this problem is given by:

$S = \{\{\text{True, True, True}\}, \{\text{False, False, False}\}, \{\text{False, False, True}\}\}$. Figure 1 shows the solutions as well as the corresponding factor graph.

### 1.2 Belief Propagation-guided Decimation

Belief Propagation [Pearl 1988] is a recursive update procedure that sends a sequence of messages from variables to constraints ($\nu_{i \rightarrow I}$) and vice-versa ($\nu_{I \rightarrow i}$):

$$
\nu_{i \rightarrow I}(x_i) \propto \prod_{J \in \partial i \setminus I} \nu_{J \rightarrow i}(x_i)
$$

$$
\nu_{I \rightarrow i}(x_i) \propto \sum_{x_{i \setminus i} \in \mathcal{X}_{\partial i \setminus i}} C_I(x_I) \prod_{j \in \partial I \setminus i} \nu_{j \rightarrow I}(x_j)
$$

where $J \in \partial i \setminus I$ refers to all the factors connected to variable $x_i$, except for factor $C_I$. Similarly the summation in eq(4) is over $x_{\partial I \setminus i}$, means we are summing out all $x_j$ that are connected to $C_I$ (i.e., $x_j$ s.t. $j \in I \setminus i$) except for $x_i$.

The messages are typically initialized to a uniform or a random distribution. This recursive update of messages is usually performed until convergence – i.e., until the maximum change in the value of all messages, from one iteration to the next, is negligible (i.e., below some small $\epsilon$). At any point during the updates, the estimated marginal probabilities are given by

$$
\hat{\mu}(x_i) \propto \prod_{J \in \partial i} \nu_{J \rightarrow i}(x_i)
$$

In a factor graph without loops, each BP message summarizes the effect of the (sub-tree that resides on the) sender-side on the receiving side.

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2. In this simple case, we could combine all the constraints into a single constraint over 3 variables and simplify the factor graph. However, in general SAT, this cost-saving simplification is often not possible.
Example 3 Applying BP to the 3-SAT problem of eq(4) takes 20 iterations to converge – i.e., for the maximum change in the marginals to be below $\epsilon = 10^{-9}$. Here the message, $\nu_{C_{1}\rightarrow 1}(x_{1})$, from $C_{1}$ to $x_{1}$ is:

$$
\nu_{C_{1}\rightarrow 1}(x_{1}) \propto \sum_{x_{2},x_{3}} C_{1}(x_{1,2,3}) \nu_{2\rightarrow C_{1}}(x_{2}) \nu_{3\rightarrow C_{1}}(x_{3})
$$

Similarly, the message in the opposite direction, $\nu_{1\rightarrow C_{1}}(x_{1})$, is defined as:

$$
\nu_{1\rightarrow C_{1}}(x_{1}) \propto \nu_{C_{2}\rightarrow 1}(x_{1}) \nu_{C_{3}\rightarrow 1}(x_{1}) \nu_{C_{4}\rightarrow 1}(x_{1}) \nu_{C_{5}\rightarrow 1}(x_{1})
$$

Here BP gives us the following approximate marginals: $\hat{\mu}(x_{1} = \text{True}) = \hat{\mu}(x_{2} = \text{True}) = .319$ and $\hat{\mu}(x_{3} = \text{True}) = .522$. From the set of solutions, we know that the correct marginals are $\hat{\mu}(x_{1} = \text{True}) = \hat{\mu}(x_{2} = \text{True}) = 1/3$ and $\hat{\mu}(x_{3} = \text{True}) = 2/3$. The error of BP is caused by influential loops in the factor-graph of Figure 4(b). Here the error is rather small; it can be arbitrarily large in some instances or BP may not converge at all.

The time complexity of BP updates of eq(3) and eq(4), for each of the messages exchanged between $i$ and $I$, is $\mathcal{O}(|\partial I| |\mathcal{X}_{i}|)$ and $\mathcal{O}(|\mathcal{X}_{i}| |\partial I|)$ respectively.

Note that we can substitute eq(4) into eq(3) and eq(5) and only keep variable-to-factor messages. After this substitution, BP can be viewed as a fixed-point iteration procedure that repeatedly applies the operator $\Phi\{\nu_{i\rightarrow I}\} \triangleq \Phi_{i\rightarrow I}\{\nu_{j\rightarrow I} J_{j} (x_{j}) \}_{j \in I, J \in \partial I}\}_{i \in I}$ to the set of messages in hope of reaching a fixed point:

$$
\nu_{i\rightarrow I}(x_{i}) \propto \prod_{J \in \partial I \setminus I} \sum_{X_{i,J\setminus i}} C_{J}(x_{J}) \prod_{j \in \partial I \setminus i} \nu_{j\rightarrow I}(x_{j}) \triangleq \Phi_{i\rightarrow I}\{\nu_{j\rightarrow I} J_{j} (x_{j}) \}_{j \in I, J \in \partial I}\}_{i \in I}
$$

Also eq(5) becomes

$$
\hat{\mu}(x_{i}) \propto \prod_{I \in \partial I} \sum_{X_{i,I\setminus i}} C_{I}(x_{I}) \prod_{j \in \partial I \setminus i} \nu_{j\rightarrow I}(x_{j})
$$

where $\Phi_{i\rightarrow I}$ denotes individual message update operators. We let operator $\Phi(\cdot)$ denote the set of these $\Phi_{i\rightarrow I}$ operators.

1.2.1 Decimation

The decimation procedure can employ BP (or SP) to solve a CSP. We refer to the corresponding method as BP-dec (or SP-dec). After running the inference procedure and obtaining $\hat{\mu}(x_{i})$, $\forall i$, the decimation procedure uses a heuristic approach to select the most biased variables (or just a random subset) and fixes these variables to their most biased values (or a random $\tilde{x}_{i} \sim \hat{\mu}(x_{i})$). If it selects a fraction $\rho$ of remaining variables to fix after each convergence, this multiplies an additional $\log_{2}(N)$ to the linear (in $N$) cost$^{3}$ for each iteration of BP (or SP). The following algorithm summarizes BP-dec with a particular scheduling of updates:

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**Algorithm:** Belief Propagation-guided Decimation (BP-dec)

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3. Assuming the number of edges in the factor graph are in the order of $N$. In general, assuming a constant factor cardinality, $|\partial I|$, the cost of each iteration is $\mathcal{O}(E)$, where $E$ is the number of edges in the factor-graph.
**Input:** factor graph of a CSP.

**Output:** a satisfying assignment \(x^*\) if an assignment was found. \textit{UNSATISFIED} otherwise.

1. Initialize messages.
2. \(\tilde{N} \leftarrow N\) (set of all variable indices).
3. Repeat until \(\tilde{N}\) is empty:
   
   (a) Repeat until convergence:
   
   For each variable \(i \in \tilde{N}\):
   
   i. **Calculate** \(\{\nu_{I \rightarrow i}\}_{I \in \partial i}\) using eq(4)
   
   ii. **Return** \textit{UNSATISFIED} if \(\{\nu_{I \rightarrow i}\}_{I \in \partial i}\) are contradictory.
   
   iii. **Calculate** \(\{\nu_{i \rightarrow I}\}_{I \in \partial i}\) using eq(3)
   
   iv. **Calculate** \(\hat{\mu}(x_i)\) using eq(5).
   
   (b) **Select** \(B \subseteq \tilde{N}\) using \(\{\hat{\mu}(x_i)\}_{i \in \tilde{N}}\).
   
   (c) **Fix** \(x_j^* \leftarrow \arg_{x_j} \max \hat{\mu}(x_j) \quad \forall j \in B\).
   
   (d) **Reduce** the constraints \(\{C_I\}_{I \in \partial j}\) for every \(j \in B\).
   
   (e) \(\tilde{N} \leftarrow \tilde{N} \setminus B\).

4. **Return** \(x^* = \{x_1^*, \ldots, x_N^*\}\)

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Step 3(a)ii occurs if the product of incoming messages to node \(i\) is 0 for all \(x_i \in X_i\). This means that neighboring constraints have strict disagreement about the value of \(x_i\) and the decimation has found a contradiction. This contradiction can happen because, either (I) there is no solution for the reduced problem even if the original problem had a solution, or (II) the reduced problem has a solution but the BP messages are inaccurate.

**Example 4** To apply BP-dec to previous example, we first calculate BP marginals, reported in the example above. Here \(\hat{\mu}(x_1)\) and \(\hat{\mu}(x_2)\) have the highest bias. By fixing the value of \(x_1\) to False, the SAT problem of eq(2) collapses to:

\[
\text{SAT}(x_{(2,3)})|_{x_1=\text{False}} = (\neg x_2 \lor x_3) \land (\neg x_2 \lor \neg x_3)
\]  

(8)

BP-dec applies BP again to this reduced problem, which give \(\hat{\mu}(x_2 = \text{True}) = .14\) (note here that \(\mu(x_2 = \text{True}) = 0\)) and \(\hat{\mu}(x_3 = \text{True}) = 1/2\). By fixing \(x_2\) to False, another round of decimation yields a solution \(x^* = \{\text{False, False, True}\}\).
1.3 Gibbs Sampling as Message Update

Gibbs Sampling (GS) is a Markov Chain Monte Carlo (MCMC) inference procedure \cite{Andrieu2003} that can produce a set of samples \( \hat{x}[1], \ldots, \hat{x}[L] \) from a given PGM. We can then recover the marginal probabilities, as empirical expectations:

\[
\hat{\mu}^L(x_i) \propto \frac{1}{L} \sum_{n=1}^{L} \delta(\hat{x}[n], x_i)
\]  

(9)

Our algorithm only consider a single particle \( \hat{x} = \hat{x}[1] \). GS starts from a random initial state \( \hat{x}(t=0) \) and at each time-step \( t \), updates each \( \hat{x}_i \) by sampling from:

\[
\hat{x}^{(t)}_i \sim \mu(x_i) \propto \prod_{I \in \partial_i} C_I(x_i, \hat{x}_{\partial I \setminus i}^{(t-1)})
\]  

(10)

If the Markov chain satisfies certain basic properties \cite{Robert2005}, \( x(\infty) \) is guaranteed to be an unbiased sample from \( \mu(x_i) \) and therefore our marginal estimate, \( \hat{\mu}^L(x_i) \), becomes exact as \( L \to \infty \).

In order to interpolate between BP and GS, we establish a correspondence between a particle in GS and a set of variable-to-factor messages – i.e., \( \hat{x} \leftrightarrow \{ \nu_{i \to I}(.) \}_{I \in \partial_i} \). Here all the messages leaving variable \( x_i \) are equal to a \( \delta \)-function defined based on \( \hat{x}_i \):

\[
\nu_{i \to I}(x_i) = \delta(x_i, \hat{x}_i) \quad \forall I \in \partial_i
\]  

(11)

We define the random GS operator \( \Psi = \{ \Psi_i \} \) and rewrite the GS update of eq(10) as

\[
\nu_{i \to I}(x_i) \triangleq \Psi_i(\{ \nu_{j \to I}(x_j) \}_{j \in \Delta_i, I \in \partial_i})(x_i) = \delta(\hat{x}_i, x_i)
\]  

(12)

where \( \hat{x}_i \) is sampled from

\[
\hat{x}_i \sim \hat{\mu}(x_i) \propto \prod_{I \in \partial_i} C_I(x_i, \hat{x}_{\partial I \setminus i})
\]

\[
\propto \prod_{I \in \partial_i} \sum_{x_{\partial I \setminus i}} C_I(x_I, \hat{x}_{\partial I \setminus i}) \prod_{j \in \partial I \setminus i} \nu_{j \to I}(x_j)
\]  

(13)

Note that eq(13) is identical to BP estimate of the marginal eq(7). This equality is a consequence of the way we have defined messages in GS update and allows us to combine BP and GS updates in the following section.

2. Perturbed Belief Propagation

Here we introduce an alternative to decimation that does not require repeated application of inference. The basic idea is to use a linear combination of BP and GS operators (eq(6) and eq(12)) to update the messages:

\[
\Gamma(\{ \nu_{i \to I} \}) \triangleq \gamma \Psi(\{ \nu_{i \to I} \}) + (1 - \gamma) \Phi(\{ \nu_{i \to I} \})
\]  

(14)

The Perturbed BP operator \( \Gamma = \{ \Gamma_{i \to I} \}_{I \in \partial_i} \) updates each message by calculating the outgoing message according to BP and GS operators and linearly combines them to get the final massage. During \( T \) iterations of Perturbed BP, the parameter \( \gamma \) is gradually and linearly changed from 0 towards 1. The following algorithm summarizes this procedure.

\[\text{Algorithm: Perturbed Belief Propagation}\]
**Input:** factor graph of a CSP, number of iterations $T$.

**Output:** a satisfying assignment $x^*$ if an assignment was found. $UNSATISFIED$ otherwise.

1. Initialize messages.
2. $\gamma \leftarrow 0.$
3. Repeat $T$ times
   
   (a) For each variable $x_i$:
   
   i. Calculate $\nu_{I \rightarrow i}$ using eq(4) $\forall I \in \partial i$
   
   ii. Return $UNSATISFIED$ if $\{\nu_{I \rightarrow i}\}_{I \in \partial i}$ are contradictory.

   iii. Calculate $\hat{\mu}(x_i)$ using eq(13).

   iv. Calculate BP messages $\nu_{i \rightarrow I}$ using eq(3) $\forall I \in \partial i$.

   v. Sample $\hat{x}_i \sim \hat{\mu}(x_i)$.

   vi. Combine BP and GS messages: $\nu_{i \rightarrow I} \leftarrow \gamma \nu_{i \rightarrow I} + (1 - \gamma) \delta(x_i, \hat{x}_i)$

   (b) $\gamma \leftarrow \gamma + \frac{1}{T - 1}$

4. **Return** $x^* = \hat{x}$ as a solution.

In step 3(a)ii, if the product of incoming messages is 0 for all $x_i \in X_i$ for some $i$, different neighboring constraints have strict disagreement about $x_i$; therefore we cannot satisfy this CSP using Perturbed BP. Since the procedure is inherently stochastic, if the CSP is satisfiable, re-application of the same procedure to the problem may avoid the contradiction.

### 2.1 Experimental Results on Benchmark CSP

This section compares the performance of BP-dec and Perturbed BP on benchmark CSPs. We considered CSP instances from XCSP repository [Roussel and Lecoutre 2009, Lecoutre 2013, without global constraints or complex domains. All instances with intensive constraints (i.e., functional form) were converted into extensive format for explicit representation using dense factors. We further removed instances containing constraints with more than $10^6$ entries in their tabular form. We also discarded instances that collectively had more than $10^8$ entries in the dense tabular form of their constraints.

We used a convergence threshold of $\epsilon = .001$ for BP and terminated if the threshold was not reached after $T = 10 \times 2^{10} = 10, 240$ iterations. To perform decimation, we sort the variables according to their bias and fix $\rho$ fraction of the most biased variables in each iteration of decimation. This fraction, $\rho$, was initially set to 100%, and it was divided by 2 each time BP-dec failed on the same instance. BP-dec was repeatedly applied using the reduced $\rho$, at most 10 times, unless a solution was reached—i.e., $\rho = .1\%$ at final attempt.

For Perturbed BP, $T = 10$ at the starting attempt, which was increased by a factor of 2 in case of failure. This was repeated at most 10 times which means Perturbed BP used $T = 10, 240$ at its final attempt. Note that Perturbed BP at most uses the same number of iterations as the maximum iterations per single iteration of decimation in BP-dec.

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4. We use $T - 1$ rather than $T$ here to ensure that $\gamma$ is equal to 1 during the final iteration.
5. Since our implementation represents all factors in a dense tabular form, we had to remove many instances because of their large factor size. We anticipate that Perturbed BP and BP-dec could probably solve many of these instances using a sparse representation.
Figure 2: Comparison of time and number of iterations used by BP-dec and Perturbed BP in benchmark instances where both methods found satisfying assignments. (a,b) Maximum number of BP iterations per iteration of decimation is $T = 10,240$, equal to maximum iterations used by Perturbed BP. (c,d) Maximum number of iterations for BP in BP-dec is reduced to $T = 1000$. (e,f) Maximum number of iterations for BP in BP-dec is further reduced to $T = 100$. 

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Figure 2(a,b) compares the time and iterations of BP-dec and Perturbed BP for successful attempts where both methods satisfied an instance. The result for individual problem sets is reported in the appendix.

Overall Perturbed BP, with 284 solved instances, is more successful than BP-dec with 253 successful runs. On the other hand, the average number of iterations for successful instances of BP-dec is 41, compared to 133 iterations for Perturbed BP. This makes Perturbed BP \textit{300 times more efficient than BP-dec}.

We also ran BP-dec on all the benchmarks with maximum number of iterations set to $T = 1000$ and $T = 100$ iterations. This reduced the number of satisfied instances to 249 for $T = 1000$ and 247 for $T = 100$, but also reduced the average number of iterations to 1570 and 562 respectively, which are still several folds more expensive than Perturbed BP. Figure 2(c-f) compare the time and iterations used by BP-dec in these settings with that of Perturbed BP, when both methods found a satisfying assignment. See the appendix for a more detailed report on these results.

3. Critical Phenomena in Random CSPs

Random CSP (rCSP) instances have been extensively used in order to study the properties of combinatorial problems (Mitchell et al. 1992; Achiontas and Sorkin 2000; Krzakaa et al. 2007) as well as in analysis and design of algorithms (e.g., Selman et al. 1994; Mezard et al. 2002). Random CSPs are closely related to spin-glasses in statistical physics (Kirkpatrick and Selman 1994; Fu and Anderson 1986). This connection follows from the fact that the Hamiltonian of these spin-glass systems resemble the objective functions in many combinatorial problems, which decomposes to pairwise (or higher order) interactions, allowing for a graphical representation in the form of a PGM. Here message passing methods, such as belief propagation (BP) and survey propagation (SP), provide consistency conditions on locally tree-like neighborhoods of the graph.

The analogy between a physical system and computational problem extends to their critical behavior where computation relates to dynamics (Ricci-Tersenghi 2010). In computer science, this critical behavior is related to the time-complexity of algorithms employed to solve such problems, while in spin-glass theory this translates to dynamics of glassy state, and exponential relaxation times (Mezard et al. 1987). In fact, this connection has been used to attempt to prove the conjecture that $\mathcal{P}$ is not equal to $\mathcal{NP}$ (Deolalikar 2010).

Studies of rCSP, as a critical phenomena, focus on the geometry of the solution space as a function of the problem’s difficulty, where rigorous (e.g., Achiontas and Coja-Oghlan 2008; Cocco et al. 2002) and non-rigorous (e.g., Cavity method of Mezard and Parisi 2000; Mezard and Parisi 2003) analyses have confirmed the same geometric picture.

When working with large random instances, a scalar $\alpha$ associated with a problem instance, a.k.a. control parameter – e.g., the clause to variable ratio in SAT – can characterize that instance’s difficulty (i.e., larger control parameter corresponds to a more difficult instance) and in many situations it characterizes a sharp transition from satisfiability to unsatisfiability (Cheeseman et al. 1991).

\textbf{Example 5 (Random $\kappa$-SAT)} Random $\kappa$-SAT instance with $N$ variables and $M = \alpha N$ constraints are generated by selecting $\kappa$ variables at random for each constraint. Each constraint is set to zero (i.e., unsatisfied) for a single random assignment (out of $2^N$). Here $\alpha$ is the control parameter.

\textbf{Example 6 (Random $q$-COL)} The control parameter for a random $q$-COL instances with $N$ variables and $M$ constraints is its average degree $\alpha = \frac{2M}{N}$. We consider Erdős-Rényi random graphs and generate a random instance by sequentially selecting two distinct variables out of $N$ at random to
generate each of $M$ edges. For large $N$, this is equivalent to selecting each possible factor with a fixed probability, which means the nodes have Poisson degree distribution $P(|\partial i| = d) \propto e^{-\alpha} \alpha^d$.}

While there are tight bounds for some problems (e.g., Achlioptas et al. 2005), finding the exact location of this transition for different CSPs is still an open problem. Besides transition to unsatisfiability, these analyses has revealed several other (phase) transitions (Krzakaa et al. 2007). Figure 3(a)-(c) shows how the geometry of the set of solutions changes by increasing the control parameter.

Here we enumerate various phases of the problem for increasing values of the control parameter:

(a) In the so-called Replica Symmetric (RS) phase, the symmetries of the set of solutions (a.k.a. ground states) reflect the trivial symmetries of problem wrt variable domains. For example, for $q$-COL the set of solutions is symmetric wrt swapping all red and blue assignment. In this regime, the set of solutions form a giant cluster (i.e., a set of neighboring solutions), where two solutions are considered neighbors when their Hamming distance is one (Achlioptas and Coja-Oghlan 2008) or non-divergent with number of variables (Mezard and Parisi 2003). Local search methods (e.g., Selman et al. 1994) and BP-dec can often efficiently solve random CSPs that belong to this phase.

(b) In clustering or dynamical transition (1dRSB), the set of solutions decomposes into an exponential number of distant clusters. Here two clusters are distant if the Hamming distance between their respective members is divergent (e.g., linear) in the number of variables. (c) In the condensation phase transition (1sRSB), the set of solutions condenses into a few dominant clusters. Dominant clusters have roughly the same number of solutions and they collectively contain almost all of the solutions. While SP can be used even within the condensation phase, BP usually fails to converge in this regime. However each cluster of solutions in the clustering and condensation phase is a valid fixed-point of BP, which is called a “quasi-solution” of BP. (d) A rigidity transition (not included in Figure 3) identifies a phase in which a finite portion of variables are fixed within dominant clusters. This transition triggers an exponential decrease in the total number of solutions, which leads to (e) unsatisfiability transition. This rough picture summarizes first order Replica Symmetry Breaking’s (1RSB) basic assumptions (Mezard and Montanari 2009).

6. 1st Order dynamical RSB. The term Replica Symmetry Breaking (RSB) originates from the technique –i.e., Replica trick (Mezard et al. 1987)– that was first used to analyze this setting. According to RSB, the trivial symmetries of the problem do not characterize the clusters of solution.
7. 1st order static RSB.
8. In some problems, the rigidity transition occurs before condensation transition.
From a geometric perspective, the intuitive idea behind Perturbed BP, is to perturb the messages towards a (quasi-)solution. However, in order to achieve this, we need to initialize the messages to a proper neighborhood of a quasi-solution. Since these neighborhoods are not initially known, we resort to stochastic perturbation of messages to make local marginals more biased towards a subspace of solutions. This continuous perturbation of all messages is performed in a way that allows each BP message to re-adjust itself to the other perturbations, more and more focusing on a random subset of solutions.

3.1 1RSB Postulate and Survey Propagation

Large random graphs are locally tree-like, which means the length of short loops are typically in the order of $\log(N)$ (Janson et al. 2001). This ensures that, in the absence of long-range correlations, BP is asymptotically exact, as the set of messages incoming to each node or factor are almost independent. Although BP messages remain uncorrelated until the condensation transition (Krzakaa et al. 2007), the BP equations do not completely characterize the set of solutions after the clustering transition. This inadequacy is indicated by the existence of a set of several valid fixed points (rather than a unique fixed-point) for BP, each of which corresponds to a quasi-solution. For a better intuition, consider the cartoons of Figures 3(b) and (c). During the clustering phase (b), $x_i$ and $x_j$ (corresponding to the $x$ and $y$ axes) are not highly correlated, but they become correlated during and after condensation (c). This correlation between variables that are far apart in the PGM results in correlation between BP messages. This violates BP’s assumption that messages are uncorrelated, which results in BP’s failure in this regime.

1RSB’s approach to incorporating this clustering of solutions into the equilibrium conditions is to define a new Gibbs measure over clusters. Let $y \subset S$ denote a cluster of solutions and $Y$ be the set of all such clusters. The idea is to treat $Y$ the same as we treated $X$, by defining a distribution

$$ \mu(y) \propto |y|^m \quad \forall \ y \in Y $$

(15)

where $m \in [0, 1]$, called the Parisi parameter (Mezard et al. 1987), specifies how each cluster’s weight depends on its size. This implicitly defines a distribution over $X$

$$ \mu(x) \propto \sum_{y \ni x} \mu(y) $$

(16)

N.b., $m = 1$, corresponds to the original distribution (eq(1)).

Example 7 Going back to our simple 3-SAT example, $y^{(1)} = \{\{True, True, True\}\}$ and $y^{(2)} = \{\{False, False, False\}, \{False, False, True\}\}$ are two clusters of solutions. Using $m = 1$, we have $\mu(\{\{True, True, True\}\}) = 1/3$ and $\mu(\{\{False, False, False\}, \{False, False, True\}\}) = 2/3$. This distribution over clusters reproduces the distribution over solutions – i.e., $\mu(x) = 1/3 \forall x \in S$. On the other hand, using $m = 0$, produces a uniform distribution over clusters, but it does not give us a uniform distribution over the solutions.

This meta-construction for $\mu(y)$ can be represented using an auxiliary PGM. One may use BP to find marginals over this PGM; here BP messages are distributions over all BP messages in the original PGM– as each cluster is a fixed-point for BP. This requirement to represent a distribution over distributions makes 1RSB practically intractable. In general, each original BP message is a distribution over $X_i$ and it is difficult to define a distribution over this infinite set. However this simplifies if the original BP messages can have limited values. Fortunately if we apply max-product BP to solve a CSP, instead of sum-product BP (of eqs(3,4)), the messages can have a finite set of values.
Max-Product BP: Our previous formulation of CSP was using sum-product BP. In general, max-product BP is used to find the Maximum a Posteriori (MAP) assignment in a PGM – a single assignment with the highest probability. In our PGM, MAP assignment is a solution for CSP. The max-product update equations are

\[
\eta_{i \rightarrow t}(x_i) = \prod_{J \in \partial t \setminus I} \eta_{j \rightarrow i}(x_j), \quad \Lambda_{i \rightarrow t}(\{\eta_{j \rightarrow i}\}_{J \in \partial t \setminus I})(x_i) \tag{17}
\]

\[
\eta_{i \rightarrow t}(x_i) = \max_{x_{\partial t \setminus I}} C_t(x_i) \prod_{J \in \partial t \setminus I} \eta_{j \rightarrow i}(x_j), \quad \Lambda_{i \rightarrow t}(\{\eta_{j \rightarrow i}\}_{J \in \partial t \setminus I})(x_i) \tag{18}
\]

\[
\mu_i(x) = \prod_{J \in \partial i} \eta_{j \rightarrow i}(x), \quad \Lambda_i(\{\eta_{j \rightarrow i}\}_{J \in \partial i})(x_i) \tag{19}
\]

where \( \Lambda = \{\Lambda_{i \rightarrow t}, \Lambda_{t \rightarrow i}\}_{t \in \partial t} \) is the max-product BP operator and \( \Lambda_i \) represents the marginal estimate as a function of messages. Note that here messages and marginals are not distributions. We initialize \( \nu_{i \rightarrow t}(x_i) \in \{0, 1\} \), \( \forall i \in \partial t, x_i \in X_i \). Because of the way constraints and update equations are defined, at any point during the updates we have \( \nu_{i \rightarrow t}(x_i) \in \{0, 1\} \). This is also true for \( \hat{\mu}(x_i) \). Here any of \( \nu_{i \rightarrow t}(x_i) = 1 \), \( \nu_{\bar{i} \rightarrow t}(x_i) = 1 \) or \( \hat{\mu}(x_i) = 1 \), shows that value \( x_i \) is allowed according to a message or marginal, while \( 0 \) forbids that value. Note that \( \hat{\mu}(x_i) = 0 \forall x_i \in X_i \) if no solution was found, because the incoming messages were contradictory. The non-trivial fixed-points of max-product BP define quasi-solutions in 1RSB phase, and therefore define clusters \( \mathcal{Y} \).

Example 8 If we initialize all messages to 1 for our simple 3-SAT example, the final marginals over all the variables are equal to 1, allowing all assignments for all variables. However beside this trivial fixed-point, there are other fixed points that correspond to two clusters of solutions.

For example, considering the cluster \( \{\{\text{False}, \text{False}, \text{False}\}, \{\text{False}, \text{False}, \text{True}\}\} \), the following \( \eta_{i \rightarrow t} \) (and their corresponding \( \eta_{t \rightarrow i} \) given by eq.(18)) define a fixed-point for max-product BP:

\[
\eta_{i \rightarrow t}(\text{True}) = \hat{\mu}_i(\text{True}) = 0 \quad \eta_{i \rightarrow t}(\text{False}) = \hat{\mu}_i(\text{False}) = 1 \quad \forall i \in \partial t \tag{20}
\]

\[
\eta_{j \rightarrow t}(\text{True}) = \hat{\mu}_j(\text{True}) = 0 \quad \eta_{j \rightarrow t}(\text{False}) = \hat{\mu}_j(\text{False}) = 1 \quad \forall I \in \partial j \tag{21}
\]

\[
\eta_{k \rightarrow t}(\text{True}) = \hat{\mu}_k(\text{True}) = 1 \quad \eta_{k \rightarrow t}(\text{False}) = \hat{\mu}_k(\text{False}) = 1 \quad \forall I \in \partial k \tag{22}
\]

Here the messages indicate the allowed assignments within this particular cluster of solutions.

3.1.1 Survey Propagation

Here we define the 1RSB update equations over max-product BP messages. We skip the explicit construction of the auxiliary PGM that results in SP update equations, and confine this section to the intuition offered by SP messages. For the construction of the auxiliary-SPG see [Braunstein and Zecchina (2003)](https://doi.org/10.1109/ISIT.2003.1228364) and [Mezard and Montanari (2009)](https://doi.org/10.1109/ISIT.2009.5206370). See Maneva et al. (2004) for a different perspective on the relation of BP and SP for satisfiability problem and Kroc et al. (2012) for an experimental study of SP applied to SAT.

Let \( \mathcal{Y}_i = 2^{\mathcal{X}_i} \) be the power-set of \( \mathcal{X}_i \). Each max-product BP message can be seen as a subset of \( \mathcal{X}_i \) that contains the allowed states. Therefore \( \mathcal{Y}_i \) as its power-set contains all possible max-product BP messages. Each message \( \nu_{i \rightarrow t} : \mathcal{Y}_i \rightarrow \{0, 1\} \) in the auxiliary PGM defines a distribution over original max-product BP messages. 10

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9. The power-set of \( \mathcal{X} \) is the set of all subsets of \( \mathcal{X} \), including \( \{\} \) and \( \mathcal{X} \) itself.
10. Here our approach is different from the previous work (e.g., Braunstein et al. 2002a, Braunstein et al. 2002b) in using max-product BP rather than Warning Propagation as the basis for construction of SP. However it easy to see that two methods result in the same update equations for SP.
Example 9 (3-COL) \( \mathcal{X}_i = \{1, 2, 3\} \) is the set of colors and \( \mathcal{Y}_i = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\} \). Here \( y_i = \{\}\) corresponds to the case where none of the colors are allowed.

Applying sum-product BP to our auxiliary PGM gives entropic \( \nu(m) \) updates as:

\[
\nu_{i\rightarrow I}(y_i) \propto |y_i|^m \sum_{\{\eta_{j\rightarrow i}\}_{j \in \partial I}} \delta(y_i, \Lambda_{i\rightarrow I}(\{\eta_{j\rightarrow i}\}_{j \in \partial I})) \prod_{j \in \partial I} \nu_{j\rightarrow i}(\eta_{j\rightarrow i})
\]

(23)

\[
\nu_{I\rightarrow i}(y_i) \propto |y_i|^m \sum_{\{\eta_{j\rightarrow i}\}_{j \in \partial I}} \delta(y_i, \Lambda_{I\rightarrow i}(\{\eta_{j\rightarrow i}\}_{j \in \partial I})) \prod_{j \in \partial I} \nu_{j\rightarrow i}(\eta_{j\rightarrow i})
\]

(24)

\[
\nu_i(\{\}) := \nu_{j\rightarrow i}(\{\}) := 0 \quad \forall i, \ I \in \partial i
\]

(25)

where the summations are over all combinations of max-product BP messages. Here the \( \delta \)-function ensures that only the set of incoming messages that satisfy original BP equations make contributions. Since we only care about the valid assignments and \( y_i = \{\} \) forbids all assignments, we ignore its contribution (eq [25]).

Example 10 (3-SAT) Consider the SP message \( \nu_{1\rightarrow C_1}(y_1) \) in factor-graph of Figure 1b. Here the summation in eq (23) is over all possible combinations of incoming max-product BP messages \( \eta_{C_2\rightarrow 1}, \ldots, \eta_{C_n\rightarrow 1} \). Since each of these messages can assume one of the three valid values – e.g., \( \eta_{C_2\rightarrow 1}(x_1) \in \{\{\text{True}\}, \{\text{False}\}, \{\text{True, False}\}\} \) – for each particular assignment of \( y_1 \), a total of \( |\{\{\text{True}\}, \{\text{False}\}, \{\text{True, False}\}\}|^{\lvert \partial_1 \cap C_1 \rvert} = 3^4 \) possible combinations are enumerated in the summations of eq (23). However only the combinations that form a valid max-product message update have non-zero contribution in calculating \( \nu_{1\rightarrow C_1}(y_1) \). These are basically the messages that appear in a max-product fixed point as discussed in Example 8.

Each of original messages corresponds to a different sub-set of clusters and \( m \) (from eq [15]) controls the effect of each cluster’s size on its contribution. At any point, we can use these messages to estimate the marginals of \( \hat{\mu}(y) \) defined in eq [15]

\[
\hat{\mu}(y_i) \propto |y_i|^m \sum_{\{\eta_{j\rightarrow i}\}_{j \in \partial i}} \delta(y_i, \Lambda_{i\rightarrow i}(\{\eta_{j\rightarrow i}\}_{j \in \partial i})) \prod_{j \in \partial i} \nu_{j\rightarrow i}(\eta_{j\rightarrow i})
\]

(26)

This also implies a distribution over the original domain, which we slightly abuse notation to denote by:

\[
\hat{\mu}(x_i) \propto \sum_{y_i \supseteq x_i} \hat{\mu}(y_i)
\]

(27)

The term SP usually refers to SP(0) - i.e., \( m = 0 \) - where all clusters, regardless of their size, contribute the same amount to \( \mu(y) \). Now that we can obtain an estimate of marginals, we can employ this procedure within a decimation process to incrementally fix some variables. Here either \( \hat{\mu}(x_i) \) or \( \hat{\mu}(y_i) \) can be used by the decryption procedure to fix the most biased variables. In the former case, a variable \( y_i \) is fixed to \( y_i^* = \{x_i^*\} \) when \( x_i^* = \arg_{x_i} \max \hat{\mu}(x_i) \). In the later case, \( y_i^* = \arg_{y_i} \max \hat{\mu}(y_i) \). Here we use SP-dec(S) to refer to the former procedure (that uses \( \hat{\mu}(x_i) \) to fix variables to a single value) and use SP-dec(C) to refer to the later case (in which variables are fixed to a cluster of assignments).

The original decryption procedure for \( \kappa \)-SAT [Braunstein et al. 2002b] corresponds to SP-dec(S). SP-dec(C) for CSP with Boolean variables is only slightly different, as SP-dec(C) can choose to fix a cluster to \( y_i = \{\text{True}, \text{False}\} \) in addition to the options of \( y_i = \{\text{True}\} \) and \( y_i = \{\text{False}\} \), available to SP-dec(S). However, for larger domains (e.g., q-COL),
SP-dec(C) has a clear advantage. For example in 3-COL SP-dec(C) may choose to fix a cluster to \( y_i = \{1, 2\} \) while SP-dec(S) can only choose between \( y_i \in \{\{1\}, \{2\}, \{3\}\} \). This significant difference is also reflected in their comparative success-rate on \( q\)-COL.\(^{11}\) (See Table 1 in Section 3.3.)

During the decimation process, usually after fixing a subset of variables, SP marginals, \( \hat{\mu}(x_i) \), become uniform, indicating that clusters of solution have no preference over particular assignment of the remaining variables. The same happens when we apply SP to random instances in RS phase. At this point (a.k.a. paramagnetic phase) a local search method or BP-dec can often efficiently find an assignment to the variables that are not yet fixed by decimation. Note that both SP-dec(C) and (S) switch to local search as soon as all \( \hat{\mu}(x_i) \) become close to uniform.

The computational complexity of each SP update of eq(24) is \( \mathcal{O}(2^{\lvert X_i \rvert} - 1)^{|\partial I|} \) as for each particular value \( y_i \), SP needs to consider every combination of incoming messages, each of which can take \( 2^{\lvert X_i \rvert} \) values (minus the empty set). Similarly, using a naive approach the cost of update of eq(23) is \( \mathcal{O}(2^{\lvert X_i \rvert} - 1)^{|\partial I|} \). However by considering incoming messages one at a time, we can perform the same exact update in \( \mathcal{O}(|\partial I| \cdot 2^{\lvert X_i \rvert}) \). In comparison to the cost of BP updates, we see that SP updates are substantially more expensive for large \( |X_i| \) and \( |\partial I| \).\(^{12}\)

### 3.2 Perturbed Survey Propagation

The perturbation scheme that we use for SP is similar to what we did for BP. Let \( \Phi_{i \rightarrow I}(\{\nu_{j \rightarrow i}\}_{j \in \Delta_i, j \in \partial(y_i)}) \) denote the update operator for the message from variable \( y_i \) to factor \( C_I \). This operator is obtained by substituting eq(24) into eq(23) to get a single SP update equation. Let \( \Phi(\nu_{i \rightarrow I})_i \) denote aggregate SP operator, that applies \( \Phi_{i \rightarrow I} \) to update each individual message.

We perform GS on the auxiliary PGM, and denote the following operator by \( \Psi^C = \{\Psi_i^C\}_i \):

\[
\nu_{i \rightarrow I}(y_i) \triangleq \Psi_i^C(\{\nu_{j \rightarrow i}\}) = \delta(y_i, \hat{y}_i) \quad \text{where} \quad \hat{y}_i \sim \hat{\mu}(y_i) 
\]  

(28)

We also consider sampling from the original domain \( X \) using the implicit marginal of eq(27). We denote this random operator by \( \Psi^S = \{\Psi_i^S\}_i \):

\[
\nu_{i \rightarrow I}(y_i) \equiv \Psi_i^S(\{\nu_{j \rightarrow i}\}) \triangleq \delta(y_i, \hat{x}_i) \quad \text{where} \quad \hat{x}_i \sim \hat{\mu}(x_i) 
\]  

(29)

where the second argument of the \( \delta \)-function is a singleton set, containing a sample from the estimate of marginal.

We define the Perturbed SP operator as the convex combination of SP and either of the GS operators above:

\[
\Gamma^C(\nu_{i \rightarrow I}) \triangleq \gamma \Psi^C(\nu_{i \rightarrow I}) + (1 - \gamma) \Phi(\nu_{i \rightarrow I}) \quad \text{Perturbed SP(C)}
\]

(30)

\[
\Gamma^S(\nu_{i \rightarrow I}) \triangleq \gamma \Psi^S(\nu_{i \rightarrow I}) + (1 - \gamma) \Phi(\nu_{i \rightarrow I}) \quad \text{Perturbed SP(S)}
\]

(31)

Similar to perturbed BP, \( \gamma \) is gradually increased from 0 to 1 during iterations of Perturbed SP. The difference between two variations of Perturbed SP is that \( \Gamma^C \) (Perturbed

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\(^{11}\) Previous applications of SP-dec to \( q\)-COL by \textit{Braunstein et al.} \textit{2003} used a heuristic for decimation that is similar SP-dec (C).

\(^{12}\) Note that our representation of distributions is over-complete \( i.e. \) we are not using the fact that the distributions sum to one. However even in their more compact forms, and for general CSPs, cost of each SP update remains exponentially larger than that of BP (in \( |X_i| \), \( |\partial I| \)).
SP(C)) produces a cluster of assignments, while \( \Gamma^S \) (Perturbed SP(S)) produces a single assignment. The advantage of \( \Gamma^S \) is that it eliminates any need for further local search. In fact we may apply \( \Gamma^S \) to CSP instances in the RS phase as well, where the solutions form a single giant cluster. In contrast, applying SP-dec, to these instances simply invokes the local search method.

To demonstrate this, we applied Perturbed SP(S) to benchmark CSP instances of Table 2 in which the maximum number of elements in the factor was less than 10. Here Perturbed SP(S) solved 80 instances out of 202 cases, while Perturbed BP solved 78 instances.

### 3.3 Experiments on random CSP

We implemented all the methods above for general factored CSP using the libdai code base [Mooij (2010)]. To our knowledge this is the first general implementation of SP and SP-dec. Previous application of SP-dec to \( \kappa \)-SAT and \( q \)-COL [Braunstein et al. (2003) Mulet et al. 2002, Braunstein et al. 2002b] where specifically tailored to each of those problems.

Here we report the results on \( \kappa \)-SAT for \( \kappa \in \{3, 4\} \) and \( q \)-COL for \( q \in \{3, 4, 9\} \). We used the procedure discussed in examples in Section 3 to produce 100 random instances with \( N = 5,000 \) variables for each control parameter \( \alpha \). We report the probability of finding a satisfying assignment for different methods – i.e., the portion of 100 instances that were satisfied by each method. For coloring instances, to help decimation, we break the initial symmetry of the problem by fixing a single variable to an arbitrary value.

For BP-dec and SP-dec, we use a convergence threshold of \( \epsilon = .001 \) and fix \( \rho = 1\% \) of variables per iteration of decimation. Perturbed BP and Perturbed SP use \( T = 1000 \) iterations. Decimation-based methods use a maximum of \( T = 1000 \) iterations per iteration of decimation. If any of the methods failed to find a solution in the first attempt, \( T \) was increased by a factor of 4 at most 3 times – i.e., in the final attempt \( T = 64,000 \). To avoid blow-up in run-time, for BP-dec and SP-dec, only the maximum iteration, \( T \), during the first iteration of decimation, was increased (this is similar to the setting of Braunstein et al. (2002b) for SP-dec). For both variations of SP-dec (see Section 3.1.1) after each decimation step, if \( \max_{x_i} \mu(x_i) - \frac{1}{|X_i|} < .01 \) we consider the instance para-magnetic, and run BP-dec (with \( T = 1000 \), \( \epsilon = .001 \) and \( \rho = 1\% \)) on the simplified instance. For Perturbed SP(C) (which recall finds a cluster of assignments), we use Perturbed BP with \( T = 1000 \) as the local search method.

Figure 4 (first row) visualizes the success rate of different methods on 100 instances of 3-SAT (right) and 3-COL (left). Figure 4 (second row) reports the number of variables that are fixed by SP-dec(C) and (S) before calling BP-dec as local search. The third row shows the average amount of time that is used to find a satisfying solution. This does not include the failed attempts. For SP-dec variations, this time includes the time used by local search. The final row of Figure 4 shows the number of iterations used by each method at each level of difficulty over the successful instances. Here the area of each disk is proportional to the frequency of satisfied instances with particular number of iterations for each control parameter and inference method.

Here we make the following observations

- **Perturbed BP** is much more effective than BP-dec, while remaining ten to hundreds of time more efficient.
- As the control parameter grows larger, the chance of requiring more iterations to satisfy the instance increases for all methods.

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13. The number of iterations are rounded to the closest power of two.
Figure 4: (first row) Success-rate of different methods for 3-COL and 3-SAT for various control parameters. (second row) The average number of variables (out of $N = 5000$) that are fixed using SP-dec (C) and (S) before calling local search, averaged over 100 instances. (third row) The average amount of time (in seconds) used by the successful setting of each method to find a satisfying solution. For SP-dec(C) and (S) this includes the time used by local search. (forth row) The number of iterations used by different methods at different control parameters, when the method was successful at finding a solution. The number of iterations for each of 100 random instances is rounded to the closest power of 2. This does not include the iterations used by local search after SP-dec.
Although computationally very inefficient, BP-dec is able to find solutions for instances with larger control parameter than suggested by previous results (e.g., Mezard and Montanari (2009)).

For many instances where SP-dec(C) and (S) use few iterations the variables are fixed to a trivial cluster $y_i = x_i$, in which all assignments are allowed. This is particularly pronounced for 3-COL. For instances in which non-trivial fixes are zero, the success rate is solely due to local search (i.e., BP-dec).

While for 3-SAT, SP-dec(C) and SP-dec(S) have a similar performance, for 3-COL, SP-dec(C) significantly outperforms SP-dec(S).

Table 1 reports the success-rate as well as the average of total iterations in the successful attempts of each method. Here the number of iterations for SP-dec(C) and (S) as well as Perturbed SP(C) is the sum of iterations used by the method and the following local search. Here we observe that Perturbed BP can solve most of easier instances using only $T = 1000$ iterations (e.g., see Perturb BP’s result for 3-SAT at $\alpha = 4.1$, 3-COL at $\alpha = 4.2$ and 9-COL at $\alpha = 33.4$).

Table 1 also suggests that Perturbed SP(S) is always more successful and more efficient than Perturbed SP(C)—i.e., clamping the variables to their original domain performs better when using our perturbation scheme. This is in addition to the benefit of not requiring any further local search. For this reason we only included the results for Perturbed SP(S) in Figure 1. In contrast as we speculated in Section 3.1.1, SP-dec(C) is in general preferable to SP-dec(S), in particular when applied to the coloring problem.

The most important advantage of Perturbed BP over SP-dec and Perturbed SP is that it can be applied to instances with large factor cardinality (e.g., 10-SAT) and variable domains (e.g., 9-COL). For example for 9-COL, the cardinality of each SP message is $2^9 = 512$, which makes SP-dec and Perturbed SP impractical. Here BP-dec is not even able to solve a single instance around the dynamical transition (as low as $\alpha = 33.4$) while Perturbed BP satisfies all instances up to $\alpha = 34.1^{14}$.

### 3.4 Discussion

It is easy to check that, for $m = 1$, SP updates produce sum-product BP messages as an average case—that is, the SP updates (i.e., eqs(23,24)) reduce to that of sum-product BP updates (i.e., eqs(3,4)) where

$$\nu_{i \rightarrow l}(x_i) \propto \sum_{y_i \supsetneq x_i} \nu_{i \rightarrow l}(y_i) \quad (33)$$

This suggests that the BP equation remains correct wherever SP(1) holds, which has lead to the belief that BP-dec should perform well up to the condensation transition. However in reaching this conclusion the effect of decimation was ignored. More recent analyses (Coja-Oghlan 2011, Montanari et al. 2007, Ricci-Tersenghi and Semerjian 2009) draw a similar conclusion about the effect of decimation: At some point during the decimation process, variables form long-range correlations such that fixing one variable may imply an assignment for a portion of variables that form a loop, potentially leading to contradictions. Alternatively the same long-range correlations result in BP’s lack of convergence and error in marginals that may lead to unsatisfying assignments.

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14. Note that for 9-COL condensation transition happens after rigidity transition. So if we were able to find solutions after rigidity, it would have implied that condensation transition marks the onset of difficulty. However, this did not occur and similar to all other cases, Perturbed BP failed before rigidity transition.
Perturbed Message Passing for CSP

Table 1: Comparison of different methods on \(\{3,4\}\)-SAT and \(\{3,4,9\}\)-COL. For each method the success-rate and the average number of iterations (including local search) on successful attempts are reported. The approximate location of phase transitions are from [Montanari et al. 2008] and Zdeborova and Krzakal [2007].

| Problem | BP-dec             | SP-dec(C)          | SP-dec(S)          | Perturbed BP         | Perturbed SP(C)        | Perturbed SP(S)        |
|---------|--------------------|--------------------|--------------------|----------------------|------------------------|------------------------|
|         | avg. iters.  | success rate | avg. iters.  | success rate | avg. iters.  | success rate | avg. iters.  | success rate | avg. iters.  | success rate |
| 3-SAT   |                    |                    |                    |                      |                        |                        |
| 0.86    | dynamical and condensation transition |                    |                    |                      |                        |                        |
| 4.1     | 85.49%            | 0.000              | 102.800            | 100.0%              | 99.05%                 | 0.000                  |
| 4.15    | 401.47           | 58.5%              | 118.852            | 100.0%              | 111.854                | 36.0%                  |
| 4.2     | 93.89%            | 49.5%              | 111.828            | 100.0%              | 113.916                | 49.5%                  |
| 4.22    | 106.609%         | 12.0%              | 121.901            | 33.0%               | 114.936                | 24.0%                  |
| 4.23    | 123.118%         | 5.0%               | 109.659            | 39.0%               | 107.784                | 14.0%                  |
| 4.24    | 163.17%          | 1.0%               | 147.941            | 23.0%               | 148.824                | 19.0%                  |
| 4.25    | N/A              | 0.0%               | 127.03             | 9.0%                | 110.584                | 8.0%                   |
| 4.26    | 97.09%           | 1.0%               | 83.233             | 5.0%                | 106.362                | 3.0%                   |
| 4.28    | N/A              | 0.0%               | N/A                | N/A                 | 12.0%                  | N/A                    |
|         |                   |                    |                    |                      |                        |                        |
| 4-SAT   |                    |                    |                    |                      |                        |                        |
| 9.38    | dynamical transition |                |                    |                      |                        |                        |
| 9.16    | 94.46%           | 32.0%              | 119.483            | 15.0%               | 120.353                | 25.0%                  |
| 9.16    | 105.83%          | 1.0%               | 113.098            | 15.0%               | 86.391                 | 24.0%                  |
| 9.78    | N/A              | 0.0%               | 83.270             | 9.0%                | 139.142                | 7.0%                   |
| 9.88    | rigidity transition |                |                    |                      |                        |                        |
| 9.93    | satisfiability transition |            |                    |                      |                        |                        |
| 4-SAT   |                    |                    |                    |                      |                        |                        |
| 4.2    | 244.43%          | 95.0%              | 250.06             | 94.0%               | 246.31                 | 94.0%                  |
| 4.4    | 515.90%          | 95.0%              | 526.84             | 93.0%               | 548.57                 | 95.0%                  |
| 4.52   | 611.09%          | 20.0%              | 681.89             | 63.0%               | 547.16                 | 1.0%                   |
| 4.56   | N/A              | 0.0%               | 63.980             | 52.0%               | 133.11                 | 1.0%                   |
| 4.6    | N/A              | 0.0%               | 74.506             | 2.0%                | 16.001                 | 1.0%                   |
| 4.63   | N/A              | 0.0%               | N/A                | 0.0%                | 48.001                 | 3.0%                   |
| 4.66   | N/A              | 0.0%               | N/A                | 0.0%                | 35.001                 | 2.0%                   |
| 4.67   | satisfiability transition |            |                    |                      |                        |                        |
| 4-COL   |                    |                    |                    |                      |                        |                        |
| 8.353   |                  |                    |                    |                      |                        |                        |
| 8.4    | 84.42%           | 92.0%              | 72.59              | 58.0%               | 71.24                 | 20.0%                  |
| 8.46   |                  |                    |                    |                      |                        |                        |
| 8.7    | N/A              | 0.0%               | N/A                | 0.0%                | 62.28                 | 14.0%                  |
| 8.84   |                  |                    |                    |                      |                        |                        |
| 8.901   | satisfiability transition |            |                    |                      |                        |                        |
| 9-COL   |                    |                    |                    |                      |                        |                        |
| 33.45   |                  |                    |                    |                      |                        |                        |
| 33.4   | N/A              | 0.0%               | N/A                | 0.0%                | 106.31                | 0.0%                   |
| 33.9    | N/A              | 0.0%               | N/A                | 0.0%                | 3.001                 | 0.0%                   |
| 34.1    | N/A              | 0.0%               | N/A                | 0.0%                | 122.34                | 0.0%                   |
| 34.5    | N/A              | 0.0%               | N/A                | 0.0%                | 48.001                | 0.0%                   |
| 35.0    | N/A              | 0.0%               | N/A                | 0.0%                | N/A                   | 0.0%                   |
| 36.87   | rigidity transition |                |                    |                      |                        |                        |
| 39.08   |                  |                    |                    |                      |                        |                        |
| 39.37   | satisfiability transition |            |                    |                      |                        |                        |
Figure 5: This schematic view demonstrates the clustering during condensation phase. Here assume \( x \) and \( y \) axes correspond to \( x_1 \) and \( x_2 \). Considering the whole space of assignments, \( x_1 \) and \( x_2 \) are highly correlated. The formation of this correlation between distant variables on a PGM breaks BP. Now assume that Perturbed BP messages are focused on the largest shaded ellipse. In this case the correlation is significantly reduced.

Perturbed BP avoids the pitfalls of BP-dec in two ways: (I) Since many configurations have non-zero probability until final iteration, Perturbed BP can avoid contradictions by adapting to the most recent choices. This is in contrast to decimation in which variables are fixed once and are unable to change afterwards. A backtracking scheme suggested by [Parisi (2003)](Parisi2003) attempts to fix the same problem with SP-dec. (II) We speculate that simultaneous bias of all messages towards sub-regions over which the BP equations remain valid, prevents the formation of long-range correlations between variables that breaks BP in 1sRSB; see Figure 5.

In all experiments, we observed that Perturbed BP is competitive with SP-dec, while BP-dec often fails on much easier problems. We saw that the cost of each SP update grows exponentially faster than the cost of each BP update. Meanwhile, our perturbation scheme adds a negligible cost to that of BP – i.e., that of sampling from these local marginals and updating the outgoing messages accordingly. Considering the computational complexity of SP-dec, and also the limited setting under which it is applicable, Perturbed BP is an attractive substitute. Furthermore our experimental results also suggest that Perturbed SP(S) is a viable option for real-world CSPs with small variable domains and constraint cardinalities.

**Conclusion**

We considered the challenge of efficiently producing assignments that satisfy hard combinatorial problems, such as \( \kappa \)-SAT and \( q \)-COL. We focused on ways to use message passing methods to solve CSPs, and introduced a novel approach, Perturbed BP, that combines BP and GS in order to sample from the set of solutions. We demonstrated that Perturbed BP is significantly more efficient and successful than BP-dec. We also demonstrated that Perturbed BP can be as powerful as state-of-the-art (SP-dec), in solving rCSPs while remaining tractable for problems with large variable domains and factor cardinalities. Furthermore we provided a method to apply the similar perturbation procedure to SP, producing the Perturbed SP process that outperforms SP-dec in solving difficult rCSPs.
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Detailed Results for Benchmark CSP

In Table 2 we report the average number of iterations and average time for the attempt that successfully found a satisfying assignment – i.e., the failed instances are not included in the average. We also report the number of satisfied instances for each method as well as the number of satisfiable instance in that series of problems (if known). Further information about each data-set maybe obtained from Lecoutre (2013).
Table 2: Comparison of Perturbed BP and BP-guided decimation on benchmark CSPs.

| problem          | series | instances | # satisf. | # satisf. | avg. time (s) | avg. res | # satisf. | avg. time (s) | avg. res |
|------------------|--------|-----------|-----------|-----------|---------------|----------|-----------|---------------|----------|
| Geometric        | -      | 100       | 92        | 77        | 208.63        | 30383    | 81        | .70           | 74       |
|                  | aim-50 | 24        | 16        | 9         | 11.41         | 25344    | 14        | .07           | 181      |
|                  | aim-100| 24        | 16        | 8         | 18.2          | 16755    | 11        | .15           | 213      |
|                  | aim-200| 24        | N/A       | 7         | 401.90        | 160884   | 6         | .17           | 46       |
|                  | ssa    | 8         | N/A       | 4         | .60           | 373.25   | 4         | .50           | 86       |
|                  | janSat | 16        | 10        | 16        | 5839.86       | 141832   | 13        | 9.82          | 117      |
|                  | varDimacs | 9       | N/A       | 4         | 2.95          | 315      | 4         | 1.12          | 18       |
| Dimacs           | QCP-10 | 15        | 10        | 10        | 43.87         | 30054    | 10        | 22            | 51       |
| QCP              | QCP-15 | 15        | 10        | 3         | 5659.70       | 600741   | 4         | 9.59          | 530      |
| QCP              | QCP-25 | 15        | 10        | 0         | 0             | 0        | 0         | 0             | 0        |
|                  | ColoringExt | 17       | N/A       | 4         | .05           | 103      | 5         | .04           | 25       |
| Graph-Coloring   | school | 8         | N/A       | 0         | N/A           | N/A      | 5         | 62.86         | 153      |
|                  | mycie | 16        | N/A       | 5         | .21           | 59       | 5         | .05           | 11       |
|                  | lcs    | 13        | N/A       | 5         | 27.34         | 696      | 5         | 10.04         | 37       |
|                  | mcp    | 13        | N/A       | 5         | .068          | 313      | 4         | .004          | 11       |
|                  | register-lp sol | 25    | N/A       | 0         | N/A           | N/A      | 0         | N/A           | N/A      |
|                  | register-initkax | 25   | N/A       | 0         | N/A           | N/A      | 0         | N/A           | N/A      |
|                  | register-zeroin | 14   | N/A       | 3         | 9886.16       | 25544    | 0         | N/A           | N/A      |
|                  | register-mulsol | 49   | N/A       | 5         | 59.27         | 418      | 0         | N/A           | N/A      |
| QCP              | sgb-queen | 50     | N/A       | 7         | 33.66         | 916      | 11        | 7.56          | 81       |
| QCP              | sgb-games | 4      | N/A       | 1         | .91           | 434      | 1         | .07           | 21       |
|                | sgb-miles | 34     | N/A       | 4         | 20.86         | 371      | 2         | 4.20          | 181      |
|                | sgb-book | 26     | N/A       | 5         | 1.72          | 444      | 5         | .18           | 39       |
|                | leighton-5 | 8     | N/A       | 0         | N/A           | N/A      | 0         | N/A           | N/A      |
|                | leighton-15 | 25    | N/A       | 0         | N/A           | N/A      | 1         | 106.46        | 641      |
|                | leighton-25 | 25    | N/A       | 2         | 333.49        | 1510     | 2         | 91.11         | 244      |
| All Interval Series | series | 12        | 12        | 2         | 4.78          | 11419    | 7         | 1.85          | 520      |
| Job Shop         | cbbdr1 | 10        | 10        | 9         | 707.74        | 9195     | 5         | 37            | 257      |
| Schurr’s Lemma   | -      | 10        | N/A       | 1         | 39.89         | 120152   | 2         | .97           | 100      |
| Chessboard Coloration | -    | 14        | N/A       | 5         | 35.51         | 3111     | 5         | .66           | 27       |
| Hanoi            | -      | 3         | 3         | 3         | .48           | 12       | 3         | .52           | 14       |
| Golomb Ruler     | Arity 3 | 8         | N/A       | 2         | 1.39          | 103      | 2         | 19.78         | 660      |
| Queens           | queens | 8         | 8         | 7         | 3.30          | 159      | 8         | 2.43          | 57       |
| Multi-Knapsack   | mknaps | 2         | 2         | 2         | 2.44          | 6        | 2         | 4.41          | 10       |
| Driver           | -      | 7         | 7         | 5         | 10.14         | 1438     | 5         | 4.74          | 274      |
| Composed         | 25-10-20 | 10       | 10        | 8         | 1.62          | 695      | 5         | .17           | 38       |
| Langford         | lagford-ext | 4    | 2         | 0         | N/A           | N/A      | 1         | .002          | 10       |
|                  | lagford-2 | 22       | N/A       | 4         | .87           | 127      | 10        | 11.64         | 10       |
|                  | lagford-3 | 26       | N/A       | 0         | N/A           | N/A      | N/A       | N/A           | N/A      |