Analytic bootstrap at large spin

Apratim Kaviraj∗ Kallol Sen† and Aninda Sinha‡

Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India.

Abstract

We use analytic conformal bootstrap methods to determine the anomalous dimensions and OPE coefficients for large spin operators in general conformal field theories in four dimensions containing a scalar operator of conformal dimension \(\Delta_\phi\). It is known that such theories will contain an infinite sequence of large spin operators with twists approaching \(2\Delta_\phi+2n\) for each integer \(n\). By considering the case where such operators are separated by a twist gap from other operators at large spin, we analytically determine the \(n, \Delta_\phi\) dependence of the anomalous dimensions. We find that for all \(n\), the anomalous dimensions are negative for \(\Delta_\phi\) satisfying the unitarity bound, thus extending the Nachtmann theorem to non-zero \(n\). In the limit when \(n\) is large, we find agreement with the AdS/CFT prediction corresponding to the Eikonal limit of a 2-2 scattering with dominant graviton exchange.

Contents

1 Introduction 2

2 Review of the analytical approach 5

3 The \(\ell \gg n\) case 8
   3.1 Case I: \(\tau_m = 2, \ell_m = 0\) ................................. 10
   3.2 Case II: \(\tau_m = 2, \ell_m = 2\) ............................... 11
   3.3 Comment on the \(\mathcal{N} = 4\) result ................... 12

4 The \(n \gg \ell \gg 1\) case 12
   4.1 Large \(h, \bar{h}\) limit of the bootstrap equation .............. 13
      4.1.1 Case I: \(\ell \gg n \gg 1\) or \(h \gg \bar{h} \gg 1\) ............ 14
      4.1.2 Case II: \(n \gg \ell \gg 1\) or \((h - \bar{h})/2h \ll 1\) .......... 16

5 Comparison with AdS/CFT 17

6 Correction to OPE coefficients for \(\ell \gg n \gg 1\) 18

∗apratin@cts.iisc.ernet.in
†kallol@cts.iisc.ernet.in
‡asinha@cts.iisc.ernet.in
1 Introduction

Over the last few years there has been a resurgent interest in conformal bootstrap methods \[1,2,3\] using the seminal work on conformal blocks by Dolan and Osborn \[4\]. Using numerical methods, interesting constraints have been placed on conformal field theories in diverse dimensions \[1\]. Applications have been found in diverse field theories ranging from supersymmetric conformal field theories \[5\] to the 3d-Ising model at criticality \[6\]. The lessons learnt using these methods transcend any underlying Lagrangian formulation and are hoped to be very general. Our aim in this paper is to present new analytic results for conformal field theories in four dimensions.

Analytic bootstrap methods have been used in \[7, 8\] to study the four point function of four identical scalar operators. It has been shown that there must exist towers of operators at large spins with twists $2\Delta_\phi + 2n$ with $\Delta_\phi$ being the conformal dimension of the scalar and $n \geq 0$ is an integer. For the case where a single tower of operator exists with twists $2\Delta_\phi + 2n$ and there is a twist gap between these operators and any other operator, one can calculate the anomalous dimensions of such operators. In four dimensions, the anomalous dimensions in the large spin ($\ell \gg 1$) limit for these operators for $n = 0$ are given by \[7, 8, 9\],

$$\gamma(0, \ell) = -\frac{c_0}{\ell^2},$$

(1.1)

where $c_0 > 0$. This conclusion is consistent with the Nachtmann theorem \[10\] which predicts that the leading operators at a given $\ell$ should have twists increasing with $\ell$. However it is not known if this behaviour persists for arbitrary $n$ introduced above (for a recent study\[ see \[11\]).

Recently it has been pointed out that in the context of the AdS/CFT correspondence, there is a connection between the CFT anomalous dimensions and the bulk Shapiro time delay \[12, 13, 14, 15\]. In \[15\] it was argued that to preserve causality, the Shapiro time delay should be positive and hence the anomalous dimensions of double trace operators negative. Thus it is of interest to see what happens to $\gamma(n, \ell)$ for $n > 0$. In the literature, it has been shown using input from AdS/CFT that using the results for the four point functions of dimension-2 and dimension-3 half-BPS multiplets in $\mathcal{N} = 4$ supersymmetric SU($N$) Yang-Mills theories, to leading order in $1/N^2$, $\gamma(n, \ell) \leq 0$ for all

---

\footnote{In \[11\] the dependence of $n$ in the limit $\ell \gg n \gg 1$ is extracted numerically from a recursion relation but from that approach it is not possible to make general conclusions.}
n—see [16] for a recent calculation for the dimension-2 case and [17] for earlier work related to the dimension-3 case. Furthermore, in [12, 13, 14], using Eikonal approximation methods pertaining to 2-2 scattering with spin-\(\ell_m\) exchange in the gravity dual, the anomalous dimensions of large-\(\ell\) and large-\(n\) operators have been calculated.

In this paper we examine \(\gamma(n, \ell)\) and OPE coefficients for general CFTs following [7, 8]. Our findings are consistent with AdS/CFT predictions [12, 13, 14] where it was found that for \(\ell \gg n \gg 1\), \(\gamma(n, \ell) \propto -n^4/\ell^2\) while for \(n \gg \ell \gg 1\), \(\gamma(n, \ell) \propto -n^3/\ell\) for graviton exchange dominance in the five dimensional bulk.

**Summary of the results:**

As we will summarize below, we can calculate the anomalous dimensions and OPE coefficients for the single tower of twist \(2\Delta\phi + 2n\) operators with large spin-\(\ell\) which contribute to one side of the bootstrap equation in an appropriate limit with the other side being dominated by certain minimal twist operators. In this paper we will focus on the case where the minimal twist \(\tau_m = 2\). One can consider various spins \(\ell_m\) for these operators. We will present our findings for various spins separately; the case where different spins \(\ell_m\) contribute together can be computed by adding up our results. We begin by summarizing the \(\ell \gg n \gg 1\) case first. We note that, as was pointed out in [7], in this limit we do not need to have an explicit \(1/N^2\) expansion parameter to make these claims. The \(1/\ell^2\) suppression in both the anomalous dimensions and OPE coefficients does the job of a small expansion parameter\(^3\).

For the dominant \(\tau_m = 2, \ell_m = 0\) contribution, the anomalous dimension becomes independent of \(n\) and is given by,

\[
\gamma(n, \ell) = -\frac{P_m(\Delta\phi - 1)^2}{2\ell^2}.
\]

while the correction to the OPE coefficient can be shown to approximate to,

\[
C_n = \frac{1}{\hat{q}_{\Delta\phi,n}} \partial_n(\hat{q}_{\Delta\phi,n}\gamma_n),
\]

in the large \(n\) limit similar to the observation made in [2]. The coefficient \(\hat{q}_{\Delta\phi,n}\) is related to the MFT coefficients as shown in (2.12) later. Here \(P_m\) is related to the OPE coefficient corresponding to the \(\tau_m = 2, \ell_m = 0\) operator. For the dominant \(\tau_m = 2 = \ell_m\) contribution, the anomalous dimension is given by,

\[
\gamma(n, \ell) = \frac{\gamma_n}{\ell^2},
\]

where,

\[
\gamma_n = -\frac{15P_m}{\Delta\phi^2}[6n^4 + \Delta\phi^2(\Delta\phi - 1)^2 + 12n^3(2\Delta\phi - 3) + 6n^2(11 - 14\Delta\phi + 5\Delta\phi^2) + 6n(2\Delta\phi - 3)(\Delta\phi^2 - 2\Delta\phi + 2)].
\]

\(^3\)Strictly speaking we will need \(\ell^2 \gg n^4\) for this to hold. Otherwise we will assume that there is a small expansion parameter.
Using the standard AdS/CFT normalization (see [8]), $P_m = 2\Delta^2 \phi / (45N^2)$ and hence $P_m / \Delta^2 \phi$ becomes independent of $\Delta \phi$. Thus for $n \gg 1$, $\gamma(n, \ell)N^2 \approx -4n^4/\ell^2$, independent of $\Delta \phi$. The coefficients $\gamma_n$ are negative for arbitrary $n$ and $\Delta \phi \geq 1$. Interestingly some $\gamma_n$’s can become positive if $0 < \Delta \phi < 1$, i.e., for $\Delta \phi$ violating the unitarity bound. For general $\ell_m$ we find that the anomalous dimension behaves like

$$\gamma(n, \ell) \propto -\frac{n^{2\ell_m}}{\ell^2},$$

for large $n$. The proportionality constant is related to the corresponding OPE coefficient. Even for this case, the anomalous dimensions are all negative for $\Delta \phi$ respecting the unitarity bound and can be positive otherwise. Thus there appears to be an interesting correlation between CFT unitarity and bulk causality (in the sense that the sign of the anomalous dimension is correlated with the bulk Shapiro time delay [15]).

Let us make some observations. If we assume that $\ell_m \leq 2$ as in [7], our results suggest that since the $\Delta \phi$ dependence drops out in $\gamma_n$ for $n \gg 1$, the findings are universal for any 4d CFT with a scalar of conformal dimension $\Delta \phi$ and where in the $\ell \gg 1$ limit the spectrum is populated with a single tower of operators with twists $2\Delta \phi + 2n$ separated by a twist gap from other operators. The explicit results given in [16, 17] are indeed consistent with the universal form of our result at large $n$. Furthermore our result is consistent with the AdS/CFT calculations in the Eikonal approximation. This gives credence to our finding that in the limit $\ell \gg n \gg 1$ the anomalous dimensions and the OPE coefficients for the $\ell_m = 2$ exchange indeed take on a universal form provided we choose the AdS/CFT normalization for $P_m$.

In the other interesting limit $n \gg \ell \gg 1$ which falls in the purview of the AdS/CFT calculations in [12, 13, 14], we will give an argument based on saddle point approximations that the bootstrap results are indeed consistent with the AdS/CFT calculation. In this case, however we will need to assume a small expansion parameter $1/N^2$ in the large-$N$ limit since $n^4/\ell \gg 1$. This limit corresponds to the scattering problem with a small impact parameter $\rho$ and hence we expect to see stringy effects from the dual gravity perspective. Nevertheless for large $\Delta \phi$ and for $\rho \sim \ell/n > \ell_0/n_{\text{max}}$ ($\ell_0$ is a lower cutoff on $\ell$ while $n_{\text{max}}$ is an upper cutoff on $n$; both are needed for our analysis to be valid) we will give a saddle point argument that the $n$ dependence from the bootstrap equation is identical to what arises in the calculations of [12, 13, 14].

Our paper is organized as follows: we start with the review of the analytical bootstrap methods used in [7, 8] in section (2). In section (3) we apply these methods in the limit when the spin is much larger than the twist, to cases where the $\text{lhs}$ of the bootstrap equation is dominated by either the twist-2, spin-2 operator exchange or a twist-2 scalar operator exchange. In section (4) we address the other limit where the twist is much larger than the spin. This section aims to provide an unified approach to handle both the limits ($\ell \gg n$ and $n \gg \ell$) using a saddle point analysis. In section (5) we compare our results with the ones from AdS/CFT. Specifically we find that our results are in agreement with the results in [12, 13, 14] in both the limits. In section (6) we discuss

---

3To be precise, in our derivation we will need $\Delta \phi > 1$ for certain approximations to hold so in our case $\gamma_n$ is always negative. However we can ask what happens to $\gamma_n$ if we consider $0 < \Delta \phi < 1$. Another point to note is that if we consider $\tau_m = 1, \ell_m = 2$ which would violate the unitarity bound, then one can explicitly check that some $\gamma_n$’s change sign.
the behaviour of the corrections to the OPE coefficients \( C_n \) for \( \ell \gg n \) limit where we show that asymptotically the coefficients \( C_n \) approach the relation (1.3) while at low \( n \) there are deviations.

We end the paper with a brief discussion of open questions in (7). Certain useful relations and formulae used for (2) are discussed in appendices (A) and (B). In the last appendix (C) we give a brief detail of the \( n \) dependence of the coefficients \( \gamma_n \) for \( \ell_m > 2 \) cases.

## 2 Review of the analytical approach

We begin by reviewing the key results of [7] (see also [8]) which will help us set the notation as well.

Consider the scalar 4-point correlation function \( \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \). In an arbitrary conformal field theory, we have a \( 12 \to 34 \) OPE decomposition (s-channel) given by,

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_\mathcal{O} P_\mathcal{O} g_{\tau_\mathcal{O},\ell_\mathcal{O}}(u,v). \tag{2.1}
\]

Here we have used the notation \( x_{ij} = x_i - x_j \). The variables \( u \) and \( v \) are the conformal cross ratios defined by,

\[
u = \frac{x_{12}^2 x_{34}^2}{x_{24}^2 x_{13}^2}, \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{24}^2 x_{13}^2}. \tag{2.2}
\]

The functions \( g_{\tau_\mathcal{O},\ell_\mathcal{O}}(u,v) \) are called conformal blocks or conformal partial waves [4], and they depend on the spin \( \ell_\mathcal{O} \) and twist \( \tau_\mathcal{O} \) of the operators \( \mathcal{O} \) appearing in the OPE spectrum. The twist is given by \( \tau_\mathcal{O} = \Delta_\mathcal{O} - \ell_\mathcal{O} \), where \( \Delta_\mathcal{O} \) is the conformal dimension of \( \mathcal{O} \). \( P_\mathcal{O} \) is a positive quantity related to the OPE coefficient. The sum goes over all the twists \( \tau \) and spins \( \ell \) that characterize the operators.

The 4-point function will also have a decomposition in the \( 14 \to 23 \) channel (t-channel), and equating the two channels we will have the bootstrap equation,

\[
1 + \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(u,v) = \left( \frac{u}{v} \right)^{\Delta_\phi} \left( 1 + \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u) \right). \tag{2.3}
\]

We will work in the limit \( u \ll v < 1 \). In this limit the leading term on the \( lhs \) is the 1. However on the \( rhs \) \( g_{\tau,\ell} \) has no negative power of \( u \) in the small \( u \) limit and all terms are vanishingly small. So we cannot reproduce the leading 1 from the \( rhs \) from a finite number of terms. In mean field theory it was shown [7] that the large \( \ell \) operators produce the leading term. For a general CFT, the authors of [7] argued that in order to satisfy the leading behavior,

\[
1 \approx \left( \frac{u}{v} \right)^{\Delta_\phi} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u), \tag{2.4}
\]

the twists \( \tau \) must have the same pattern as in MFT. To show this we have to look at the large \( \ell \)
and small $u$ limit of the conformal blocks,

$$g_{\tau,\ell}(v, u) = k_{2\ell}(1 - z)v^{\tau/2}F^{(d)}(\tau, v), \quad \text{(when } |u| \ll 1 \text{ and } \ell \gg 1\text{)}$$

$$k_\beta(x) = x^{\beta/2}F_1(\beta/2, \beta/2, \beta, x). \quad (2.5)$$

Here $z$ is defined by $u = z\bar{z}, v = (1 - z)(1 - \bar{z})$; and $F^{(d)}(\tau, v)$ is a positive and analytic function near $v = 0$ whose exact expression is not necessary for the discussion. We derive the above result later in this section. For now, we just use this to rewrite (2.4),

$$1 \approx \sum_{\tau} \left( \lim_{z \to 0} z^{\Delta_\phi} \sum_{\ell} P_{\tau,\ell}k_{2\ell}(1 - z) \right) v^{\tau/2 - \Delta_\phi}(1 - v)^{\Delta_\phi} F^{(d)}(\tau, v). \quad (2.6)$$

The term in brackets are independent of $z$ and $\ell$ after taking the limit and doing the sum (over $\ell$). Then what is left is just a function of $\tau$ with a sum over $\tau$. The function $F^{(d)}(\tau, v)$ around small $v$ begins with a constant. Thus we must have $\tau/2 = \Delta_\phi$ in the spectrum. Next since $F^{(d)}(\tau, v)$ has terms with higher powers in $v$, we must have $\tau = 2\Delta_\phi + 2n$ for every integer $n$, to cancel these terms. This shows that there are operators with twists $\tau = 2\Delta_\phi + 2n$. Since these are operators in MFT, $P_{\tau,\ell} = P_{\tau,\ell}^{\text{MFT}}$ at leading order. We will now focus our attention on the subleading terms of the bootstrap equation.

The subleading corrections to the bootstrap equation are characterized by the anomalous dimension $\gamma(n, \ell)$ and corrected OPE coefficients $C_n$. We will assume that for each $\ell$ there is a single operator having twist $\tau \approx 2\Delta_\phi + 2n$. The bootstrap equation takes the form $\mathbb{E}^{\text{c}}$.

$$1 + \sum_{\ell_m} \frac{P_{\ell_m}}{4} u^{\tau_m/2} f_{\tau_m,\ell_m}(0, v) \approx \sum_{\tau,\ell} P_{\tau,\ell} v^{\tau/2 - \Delta_\phi} u^{\Delta_\phi} f_{\tau,\ell}(v, u), \quad (2.7)$$

which is valid up to subleading corrections in $u$ as $u \to 0$. Note that the $\text{lhs}$ demands the existence of an operator of minimal twist $\tau_m = \Delta_m - \ell_m$ which is non-zero. We set $u = z(1 - v) + O(z^2)$ and consider $u \to 0$ to be $z \to 0$. The explicit form of the function $f_{\tau_m,\ell_m}(v)$ is given by,

$$f_{\tau_m,\ell_m}(v) = \frac{\Gamma(\tau_m + 2\ell_m)}{\Gamma(\tau_m + \ell_m)} (1 - v)^{\ell_m} \sum_{n=0}^{\infty} \left( \frac{\tau_m + 2\ell_m}{n!} \right)^n v^n \left[ 2[n + 1] - \psi(\tau_m + \ell_m + n) \right] \log v. \quad (2.8)$$

Later we will set $\tau_m = 2$ because we are particularly interested in the twist 2 primary operator or the stress tensor in the theory.

Let us now focus on the $\text{rhs}$ where we have an infinite sum over all twists and spins. In the limit $\ell \gg n \gg 1$ we can simplify the $\text{rhs}$ considerably. Note that we will be working in $d = 4$ since in the $d = 2$ case there is no minimal twist operator with a twist gap from the identity operator ($\tau_{\text{min}}^{d=2} = 0$). To proceed we first need to find the behaviour of the conformal blocks in the above limit (in other words $\tau_m = 2$) and when $|u| \ll |v| < 1$. With $u = z(1 - v) + O(z^2)$ since

---

4Our conventions for $P_m$ differ from [7] by a factor of 1/4.
\[ \tilde{z} = (1 - v) + O(z), \] we can form a small \( z \) expansion around \( z = 0 \) and then a small \( v \) expansion.

To find the anomalous dimension \( \gamma(n, \ell) \) for each \( \ell \) we need to match the coefficients of the terms \( v^n \log v \) on both sides of (2.7). Considering \( \tau(n, \ell) = 2\Delta_\phi + 2n + \gamma(n, \ell) \), we can see that the \( \log v \) arises from the next to the leading term in the perturbative expansion around small \( v \) given by,

\[ v^{\tau(n, \ell)/2 - \Delta_\phi} \to \frac{\gamma(n, \ell)}{2} v^n \log v. \quad (2.9) \]

The MFT coefficients take the following form in the \( \ell \gg n \) limit,

\[ P_{2\Delta_\phi + 2n, \ell} \approx q_{\Delta_\phi, n} \frac{\sqrt{\pi}}{2^{2\Delta_\phi + 2n + 2\ell}} \ell^{2\Delta_\phi - 3/2}, \quad (2.10) \]

where the coefficient \( q_{\Delta_\phi, n} \) is given by,

\[ q_{\Delta_\phi, n} = \frac{8}{\Gamma(\Delta_\phi)^2} \frac{(1 - d/2 + \Delta_\phi)^2}{n!(1 - d + n + 2\Delta_\phi)}. \quad (2.11) \]

Here \((a)_b = \Gamma(a + b)/\Gamma(a)\) is the Pochhammer symbol. We will also use another notation for convenience in the later part of the work,

\[ \tilde{q}_{\Delta_\phi, n} = 2^{-2\Delta_\phi - 2n} q_{\Delta_\phi, n}. \quad (2.12) \]

The \( d = 4 \) crossed conformal blocks are given by

\[ g_{\tau, \ell}(v, u) = \frac{(1 - z)(1 - \tilde{z})}{\tilde{z} - z} [k_{2\ell+\tau}(1 - z)k_{\tau-2}(1 - \tilde{z}) - k_{2\ell+\tau}(1 - \tilde{z})k_{\tau-2}(1 - z)], \quad (2.13) \]

where we have already defined \( k_\beta(x) \) in (2.5). As already mentioned, in the large \( \ell \) limit, the conformal blocks simplify to give (2.5). For \( \ell \gg n \) we can decompose \( k_{2\ell+\tau}(1 - z) \) even further to get,

\[ k_{2\ell+\tau}(1 - z) \to \frac{2^{\tau+2\ell-1} \ell^{1/2}}{\sqrt{\pi}} K_0(2\ell \sqrt{z}). \]

We will also need the expression for \( F^{(d)}(\tau, v) \). In \( d = 4 \) we have,

\[ F^{(4)} = \frac{2^\tau}{1 - v} 2F_1 \left[ \frac{\tau}{2} - 1, \frac{\tau}{2} - 1, \tau - 2, v \right]. \quad (2.15) \]

With this, the entire (\( \log v \) dependent part of) rhs of (2.7) in the limit \( \ell \gg n \) can be organized into the following form,

\[ \sum_{\tau, \ell} P_{\tau, \ell} v^{\tau/2 - \Delta_\phi} u^{\Delta_\phi} f_{\tau, \ell}(v, u) = \sum_{n=0, \ell=0}^{\infty} \frac{q_{\Delta_\phi, n} \ell^{2\Delta_\phi - \frac{3}{2}}}{2} \left[ \frac{\gamma(n, \ell)}{2} \right] v^n \log v \ \ell^{1/2} K_0(2\ell \sqrt{z}) z^{\Delta_\phi} \]

\[ (1 - v)^{\Delta_\phi - 1} 2F_1(\Delta_\phi + n - 1, \Delta_\phi + n - 1, 2\Delta_\phi + 2n - 2; v). \quad (2.16) \]

Now the overall factor of \( u^{\Delta_\phi} \) sitting on the rhs of (2.7) is translated into an overall factor of \( z^{\Delta_\phi}(1 - v)^{\Delta_\phi} \). We assume that the anomalous dimension has the form \( \gamma(n, \ell) = \gamma_n/\ell^a \). Now in
the large $\ell$ limit we can convert the sum over $\ell$ in (2.16) into an integral given by,

$$
\int_{\ell_0}^{\infty} d\ell \ell^{-1-\alpha+2\Delta_\phi} z^{\Delta_\phi} K_0(2\ell \sqrt{z}) \approx \frac{z^{\alpha/2}}{4} \Gamma^2 \left( \Delta_\phi - \frac{\alpha}{2} \right) + O(z^{\Delta_\phi \log z}).
$$

(2.17)

In order to do this integral, it is convenient to use an upper cutoff $L$. The integral works out to be in terms of regularized Hypergeometric functions. By expanding the result assuming $L \sqrt{z} \gg 1$ and $\ell_0 \sqrt{z} \ll 1$ we get the leading and subleading terms in the above equation. For $\Delta_\phi > 1$, the $O(z^{\Delta_\phi} \log z)$ terms can be ignored. This reproduces the factor of $z^{\tau m}$ exactly if $\alpha = \tau m$. If we take the minimal nonzero twist to be $\tau_m = 2$, the anomalous dimension behaves as,

$$
\gamma(n, \ell) = \frac{\gamma_n}{\ell^2}.
$$

(2.18)

Once again the interested reader should refer to [7, 8] for the mathematical details of the above algebra and approximations. In the next section, we demonstrate how the expression for $\gamma_n$ can be given in terms of an exact sum for all $n$. This sum enables us to extract the exact behaviour of the anomalous dimensions for all $n$ when $\ell \gg n$. Later, we will explore how $\gamma(n, \ell)$ behaves in the other limit $n \gg \ell$.

3 The $\ell \gg n$ case

We begin by determining $\gamma_n$ appearing in (2.18) in the limit $\ell \gg n$. To get $\gamma_n$, we have to match the power of $v^n \log v$ on both sides of (2.7). To do that we take the $(1 - v)^{\tau m + \tau m/2 - \Delta_\phi + 1}$ of (2.16) to the $\text{lhs}$ of (2.7) and expand $(1 - v)^{\tau m + \tau m/2 - \Delta_\phi + 1}$ in powers of $v$. Thus the $\text{lhs}$ of (2.7) becomes,

$$
-(1 - v)^{\tau m + \tau m/2 - \Delta_\phi + 1} P_m \frac{\Gamma(2\ell_m + \tau m)}{4 \Gamma(\ell_m + \tau m/2)} \sum_{n=0}^{\infty} \left( \frac{\Gamma(\tau m/2 + \ell_m n)}{n!} \right)^2 v^n \log v.
$$

(3.1)

Expanding the term $(1 - v)^{\tau m + \tau m/2 - \Delta_\phi + 1}$, we get,

$$
(1 - v)^{\tau m + \tau m/2 - \Delta_\phi + 1} = \sum_{\alpha=0}^{\infty} (-1)^k \frac{b!}{\alpha!(b - \alpha)!} v^\alpha \quad \text{where} \quad b = \ell_m + \frac{\tau m}{2} + 1 - \Delta_\phi.
$$

(3.2)

Now set $n + \alpha = k$ whereby the $\text{lhs}$ can be arranged as $\sum_{n=0}^{\infty} L_n v^n \log v$ where to find $L_k$ we need to perform the $\alpha$ sum explicitly.

This gives, the coefficient of $v^n \log v$ to be,

$$
L_n = -4 P_m \frac{\Gamma(\tau m + 2\ell_m)}{\Gamma \left( \frac{\tau m}{2} + \ell_m \right)} \sum_{\alpha=0}^{\infty} (-1)^\alpha \left( \frac{(\tau m/2 + \ell_m)_{(n-\alpha)}}{(n-\alpha)!} \right)^2 \frac{b!}{(b - \alpha)!\alpha!},
$$

(3.3)

where we have multiplied the $\text{lhs}$ of (2.7) with an overall numerical factor of 16 coming from the

\footnote{Note that for $\Delta_\phi = 1$ and $\tau_m = 2$, this does not work as the Gamma function blows up. This is presumably indicative of a log $\ell$ scaling for the operators [13] in this case.}

8
This finite sum is given by,

\[
L_n = -\frac{4P_m \Gamma(2\ell_m + \tau_m) \Gamma \left( n + \ell_m + \frac{\tau_m}{2} \right) \, _3F_2 \left( \begin{array}{c} -n, -n, -1 - \ell_m + \Delta_\phi - \frac{\tau_m}{2} \\ 1 - n - \ell_m - \frac{\tau_m}{2}, 1 - n - \ell_m - \frac{\tau_m}{2} \end{array} \right)}{\Gamma(1 + n)^2 \Gamma \left( \ell_m + \frac{\tau_m}{2} \right)^4}.
\]

To get the same coefficient of \( v^n \log v \) on the rhs of (2.7), we expand the hypergeometric function in powers of \( v \) given by

\[
_2F_1 \left( \frac{\tau}{2} - 1, \frac{\tau}{2} - 1, -2, v \right) = \sum_{\alpha=0}^{\infty} \frac{\left( \frac{\tau}{2} - 1 \right)^{\alpha}}{(\tau - 2)_{\alpha} \, \alpha!} v^\alpha,
\]

where \((a)_b\) is the Pochhammer symbol given by \((a)_b = \Gamma(a + b)/\Gamma(a)\). On the rhs we have two infinite sums \( \sum_{k=0}^{\infty} \sum_{\alpha=0}^{\infty} f_{\alpha,k} v^{k+\alpha} \). To put the rhs in the form \( \sum_{n=0}^{\infty} R_n v^n \) we will regroup the terms in the double sum in increasing powers of \( v^n \). This is achieved by setting \( k + \alpha = n \) where \( \alpha \) runs from 0 to \( n \) giving,

\[
\text{rhs} = \sum_{n=0}^{\infty} R_n \, v^n \log v,
\]

where, the coefficients \( R_k \) can be written as

\[
R_n = \Gamma(\Delta_\phi - \frac{\tau_m}{2})^2 \sum_{\alpha=0}^{n} q_{\Delta_\phi,n-\alpha} \gamma_{n-\alpha} \left( \frac{\left( \frac{\tau}{2} - 1 \right)^{n-\alpha}}{(n-\alpha)!(\tau - 2)_{n-\alpha}} \right),
\]

where the extra factor of \( \frac{1}{2} \) comes from the normalization \( 2^{2\ell + \tau - 1} \) when we consider the large \( \ell \) approximation of the conformal blocks. Equating the coefficients \( R_n = L_n \) we can find the corresponding coefficients \( \gamma_n \). Thus, in principle, we would know \( \gamma_n \) if we know \( \gamma_k \) for all \( k \leq n - 1 \). In figure (1) we have plotted the log \( \gamma_n \) vs. log \( n \) for a twist-2 scalar and a twist-2 and spin-2 field.

We find that the slope of the curve for the twist-2, spin-2 exchange is \( \approx 4 \) while that for the twist-2 scalar is a constant. So \( \gamma_n \sim n^4 \) for large values of \( n \) for spin-2 field. To show this behavior explicitly, we notice that \( \gamma_n \) can be written as an exact sum over the coefficients \( R_m \) appearing on the lhs. This formula can be guessed by looking at the first few \( \gamma_n \)'s. We give the form of the first few \( \gamma_n \)'s. These take the form\(^6\)

\[
\begin{align*}
\gamma_0 &= \frac{(\Delta_\phi - 1)^2}{8} L_0, \\
\gamma_1 &= -\frac{(\Delta_\phi - 1)^2}{8} L_0 + \frac{\Delta_\phi - 1}{4} L_1, \\
\gamma_2 &= \frac{(\Delta_\phi - 1)^2}{8} L_0 - \frac{2\Delta_\phi - 1}{4} L_1 + \frac{2\Delta_\phi - 1}{2\Delta_\phi} L_2 \text{ etc.}
\end{align*}
\]

\(^6\)We will assume \( \Delta_\phi > 1 \). See footnote 4.
Figure 1: log $|\gamma_n|$ vs. log $n$ plot showing the dependence of $\gamma_n$ on $n$ for $n \gg 1$. $\gamma_T$ is the anomalous dimension for the spin-2 operator exchange and $\gamma_S$ for the scalar operator exchange. The slope of the blue straight line for spin-2 exchange is 3.998 while the red line denotes the scalar exchange for which $\gamma_n$ is constant for all $n$. We have used $\Delta_\phi = 2$ in the above plots.

We observe that the above terms follow a definite pattern which can be written as,

$$\gamma_n = - \sum_{m=0}^n a_{n,m} \quad \text{with} \quad a_{n,m} = c_{n,m} L_m.$$  \hspace{1cm} (3.9)

where for general $\tau_m$ and $\ell_m$ the coefficients $c_{n,m}$ are given by,

$$c_{n,m} = \frac{1}{8} \left( \frac{\Gamma(\Delta_\phi)}{\Gamma(\Delta_\phi + m - 1)} \right)^2 \frac{(2\Delta_\phi + n - 3)_m (-1)^{n+m} n!}{(n-m)!} \left( \frac{\Gamma(\Delta_\phi - 1)}{\Gamma(\Delta_\phi - \tau_m/2)} \right)^2.$$  \hspace{1cm} (3.10)

We have checked the analytic expression for the coefficients $\gamma_n$ agrees with the solutions of $\gamma_n$ found from solving the equations $R_k = L_k$ order by order for arbitrary values of $n$.

3.1 Case I: $\tau_m = 2$, $\ell_m = 0$

We now consider the case where the lhs of (2.7) is dominated by the exchange of a twist-2 scalar operator. For this case

$$3F_2 \left[ \begin{array}{c} -m, -m, -2 + \Delta_\phi \\ -m, -m \end{array}, 1 \right] = \sum_{k=0}^m \frac{\Gamma(k + \Delta_\phi - 2)}{\Gamma(\Delta_\phi - 2)k!}.$$  \hspace{1cm} (3.11)

The coefficients $a_{n,m}$ can thus be written as,

$$a_{n,m} = \frac{P_m (-1)^{m+n} (\Delta_\phi - 1) \Gamma(n+1) \Gamma(\Delta_\phi) \Gamma(2\Delta_\phi + m + n - 3)}{2 \Gamma(m+1) \Gamma(n + 1 - m) \Gamma(\Delta_\phi + m - 1) \Gamma(2\Delta_\phi + n - 3)}.$$  \hspace{1cm} (3.12)
We sum over the coefficients $a_{n,m}$ to get,

$$
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{P_m}{2}(\Delta_\phi - 1)^2.
$$

(3.13)

Note that the coefficients $\gamma_n$ appearing in the expression for the anomalous dimension become independent of $n$ in this case. The details can be found in appendices (A) and (B).

3.2 Case II: $\tau_m = 2$, $\ell_m = 2$

Here we consider the case where the lhs of (2.7) is dominated by the exchange of a twist-2 and spin-2 operator exchange. In the language of AdS/CFT, the particle is a graviton that dominates the scattering amplitude in the Eikonal limit $[12, 13, 14]$. As in the previous case the anomalous dimension goes as $\sim 1/\ell^2$ for large spin in the rhs of (2.7). Performing the $\ell$ integration we are left with a single sum on the rhs from which we can determine the coefficients $\gamma_n$ as a function of $n$. Using relation (3.9) we can evaluate the coefficients $L_m$ for the case when $\tau_m = 2$ and $\ell_m = 2$ respectively which we proceed to show below. We defer the details of the calculation to the appendix and present here with only the final results. First we write

$$
3F_2\left[\begin{array}{c}
-m, -m, -4 + \Delta_\phi \\
-2 - m, -2 - m
\end{array}\right, 1] = \sum_{k=0}^{m} \frac{(m+1-k)^2(m+2-k)^2\Gamma(\Delta_\phi - 4 + k)}{(m+1)^2(m+2)^2\Gamma(k+1)\Gamma(\Delta_\phi - 4)}
$$

(3.14)

The combined coefficients $a_{n,m}$, after putting in the proper normalizations, can be written as,

$$
a_{n,m} = -(-1)^{m+n}15\frac{P_m}{\Delta_\phi}(6m^2 + 6m(\Delta_\phi - 1) + \Delta_\phi(\Delta_\phi - 1))
\times \frac{\Gamma(n+1)\Gamma(\Delta_\phi)\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n+1-m)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)}.
$$

(3.15)

We can now perform the summation, over the coefficients $a_{n,m}$ to get,

$$
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{15P_m}{\Delta_\phi^2}(6n^4 + \Delta_\phi^2(\Delta_\phi - 1)^2 + 12n^3(2\Delta_\phi - 3) + 6n^2(11 - 14\Delta_\phi + 5\Delta_\phi^2)
+ 6n(2\Delta_\phi - 3)(\Delta_\phi^2 - 2\Delta_\phi + 2)\].
$$

(3.16)

The above formula negative and monotonic for all values of $n$ and $\Delta_\phi > 1$ (see appendices (A) and (B) for details). Until this point we did not need the explicit form of the coefficient $P_m$ but we can choose the conventions $[8]$. $P_m$ for any general $d$ is given by

$$
P_m = \frac{d^2}{(d-1)^2 C_T}\Delta_\phi^2.
$$

(3.17)
This result follows from the conformal Ward identity\(^7\), as a consequence the \(\Delta_\phi\) independence of the \(n^4\) term in the anomalous dimension is a general result. For our case we put \(d = 4\) and \(C_T = 40N^2\), which correspond to the AdS\(_5\)/CFT\(_4\) normalization and where \(C_T\) is the central charge. Putting all these together, we get, \(P_m = \frac{2}{45N^2}\Delta_\phi^2\). Note that the \(n^4\) term in \(\gamma_n\) becomes independent of \(\Delta_\phi\) using this convention. Thus when \(n\) is large, the result is independent of \(\Delta_\phi\) and hence universal.

### 3.3 Comment on the \(\mathcal{N} = 4\) result

In [16], the authors showed that for dimension-2 half-BPS multiplet the anomalous dimension in \(\mathcal{N} = 4\) SYM, for \(\Delta_\phi = 2\), has the form,

\[
\gamma(n, \ell)N^2 = -\frac{4(n+1)(n+2)(n+3)(n+4)}{(\ell+1)(\ell+6+2n)}. \tag{3.18}
\]

To compare this with our result (3.16) we put \(P_m = 2/(45N^2)\Delta_\phi^2\) (See for eg. [8]), and set \(\Delta_\phi = 4\). This gives,

\[
\gamma(n, \ell)N^2 \approx -\frac{4(n+1)(n+2)(n+3)(n+4)}{\ell^2}. \tag{3.19}
\]

for large values of \(\ell\). Quite curiously this form matches with the supergravity result, for large spin and finite \(n\). The reason for this agreement is not clear to us although [16] made a similar observation that the extra solutions to the bootstrap equation they find (for \(\Delta_\phi = 2\)) match exactly with the solutions in [2] for \(\Delta_\phi = 4\).

### 4 The \(n \gg \ell \gg 1\) case

In this section we will deal with the other limit where \(n \gg \ell \gg 1\) implying we are still in the large spin limit but the twists are even larger. We will assume a general form of the anomalous dimension in \(n\) and \(\ell\) given by,

\[
\gamma(n, \ell) \approx n^b\ell^a, \tag{4.1}
\]

where for the two cases (\(\ell \gg n \gg 1\) and \(n \gg \ell \gg 1\)), we want to show that \(b = 4\), \(a = -2\) and \(b = 3\), \(a = -1\) respectively. We start by a transformation of variables,

\[
h = \Delta_\phi + n + \ell, \quad \tilde{h} = \Delta_\phi + n. \tag{4.2}
\]

These are the variables originally used in the papers [12, 13, 14]. In the limit where \(n, \ell \gg \Delta_\phi\) we assume that, \(h \approx n + \ell\), and \(\tilde{h} \approx n\). In terms of the variables \(h, \tilde{h}\) we can write,

\[
\gamma_{h, \tilde{h}} \approx \tilde{h}^b(h - \tilde{h})^a. \tag{4.3}
\]

\(^7\)We thank Joao Penedones for reminding us of this fact.
In the next subsection we will elaborate on how the transformation of the original variables $n$ and $\ell$ to $h$ and $\bar{h}$ can be used to our advantage to demonstrate the behaviour of the anomalous dimension in both the limits $\ell \gg n \gg 1$ and $n \gg \ell \gg 1$.

### 4.1 Large $h, \bar{h}$ limit of the bootstrap equation

The coefficient $P_{MFT}^{\text{MFT}}$ in terms of the conformal dimensions $h$ and $\bar{h}$ is given by,

$$
P_{MFT}^{\text{MFT}} = \frac{2^{\Delta - 2(h + \bar{h})} (h + \bar{h} - 2)(h - \bar{h} + 1)\pi}{\Gamma(\Delta_{\phi})^2 \Gamma(\Delta_{\phi} - 1)^2} \frac{\Gamma(h) \Gamma(\bar{h} - 1) \Gamma(h + \Delta_{\phi} - 2) \Gamma(\bar{h} + \Delta_{\phi} - 3)}{\Gamma(h - 1/2) \Gamma(h - 3/2) \Gamma(h - \Delta_{\phi} + 2) \Gamma(h - \Delta_{\phi} + 1)}.
$$

The Stirling approximation of the $\Gamma$-function is given by,

$$
\Gamma(a + b) \approx \sqrt{2\pi} a^{a-b} \left( \frac{a}{e} \right)^b.
$$

Using this we can show that the MFT coefficients in the large $h, \bar{h}$ limit behave as,

$$
P_{MFT}^{\text{MFT}} \approx \frac{2^{\Delta - 2(h + \bar{h})} \pi (h - \bar{h} + 1)(h + \bar{h} - 2)}{\Gamma(\Delta_{\phi})^2 \Gamma(\Delta_{\phi} - 1)^2} (h\bar{h})^{2\Delta_{\phi} - \frac{7}{2}}.
$$

Also in this limit and taking $z \rightarrow 0$, the conformal blocks (crossed channel) takes the form,

$$
g_{h,\bar{h}}(v, u) = 2^{h - 1} h^{1/2} K_0(2h\sqrt{z}) v^\bar{h} \frac{1}{1 - v^2} {}_2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v).
$$

The rhs of the (2.7) can be written as,

$$
\frac{1}{\Gamma(\Delta_{\phi})^2 \Gamma(\Delta_{\phi} - 1)^2} \int_0^\infty dh \, h^{2\Delta_{\phi} - 3}(h - \bar{h})^{a+1} (h + \bar{h}) K_0(2h\sqrt{z})
\times \int_0^\infty dh' \, h'^{2\Delta_{\phi} - 3}(h' - \bar{h}) \frac{2 - 2h + \pi}{4} h'^{2\Delta_{\phi} - 7/2} v^{\bar{h}} \frac{1}{1 - v^2} {}_2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v).
$$

The essential idea is that the leading order $z$-dependence should come from performing only the $h$ integral while the other integral over $\bar{h}$ can be converted into a sum to determine the coefficients $\gamma_n$ as a function of $n$. We now proceed to compute the $h$ integral by the saddle point approximation method as follows. The $h$ integral becomes,

$$
\int_0^\infty dh \, h^{2\Delta_{\phi} - 3}(h - \bar{h})^{a+1} (h + \bar{h}) K_0(2h\sqrt{z}).
$$

We can further approximate the modified Bessel function for $h\sqrt{z} \gg 1$ as

$$
K_0(2h\sqrt{z}) \approx \frac{\sqrt{\pi}}{2h^{1/2}} e^{-2h\sqrt{z}}.
$$
Combining all these together, the $h$ integral can be written as,

$$2\sqrt{\frac{\pi}{2}} \bar{h}z^{-1/4} \int_{\bar{h}}^{\infty} dh \ h^{2\Delta_\phi - 7/2} (h - \bar{h})^{a+1} \left(1 + \frac{h - \bar{h}}{2h}\right)e^{-2h\sqrt{z}}. \quad (4.11)$$

Note in advance that there are two different limits that we will be considering. One for $\ell \gg n \gg 1$ case where $h \gg \bar{h}$ so that,

$$1 + \frac{h - \bar{h}}{2h} \approx \frac{h}{2\bar{h}}, \quad (4.12)$$

and the other limit $n \gg \ell \gg 1$ where $(h - \bar{h})/2\bar{h} \ll 1$ will be neglected. For these two limits the integrands will change accordingly as we list below for clarity.

1. Case I: $\ell \gg n \gg 1$ or $h \gg \bar{h} \gg 1$ where we approximate $h + \bar{h} \approx h$.

$$\sqrt{\frac{\pi}{2}} z^{-1/4} \int_{\bar{h}}^{\infty} dh \ h^{2\Delta_\phi - 5/2} (h - \bar{h})^{a+1} e^{-2h\sqrt{z}}. \quad (4.13)$$

2. Case II: $n \gg \ell \gg 1$. In this case $(h - \bar{h})/2\bar{h} \ll 1$ and can be neglected. Thus the integral becomes,

$$\sqrt{\pi \bar{h}z^{-1/4}} \int_{\bar{h}}^{\infty} dh \ h^{2\Delta_\phi - 7/2} (h - \bar{h})^{a+1} e^{-2h\sqrt{z}}. \quad (4.14)$$

In both the above cases we can approximate the leading order behaviour of the term $(h - \bar{h})^{a+1} \approx h^{a+1} + O(\bar{h}/h)$.

**4.1.1 Case I: $\ell \gg n \gg 1$ or $h \gg \bar{h} \gg 1$**

To perform the saddle point in this case, we put the function as, $e^{g(h)}$, where,

$$g(h) = -2h\sqrt{z} + (2\Delta_\phi - 7/2) \log h + (a + 1) \log(h - \bar{h}) + \log \left[1 + \frac{h - \bar{h}}{2h}\right]. \quad (4.15)$$

The first order derivative gives,

$$g'(h) = -2\sqrt{z} + \frac{2\Delta_\phi - 7/2}{h} + \frac{a + 1}{h - \bar{h}} + \frac{1}{h + \bar{h}}. \quad (4.16)$$

Equating this to 0 we get the approximate saddle-point (assuming that it will be such that $h \gg \bar{h} \gg 1$) at,

$$h_0 = \frac{2\Delta_\phi - 3/2 + a}{2\sqrt{z}}. \quad (4.17)$$

Note that the crucial fact is that the saddle-point goes as $\sim 1/\sqrt{z}$. One important point is that we approximated $K_0(2h\sqrt{z}) \propto e^{-2h\sqrt{z}}$ which needs,

$$2h_0\sqrt{z} = 2\Delta_\phi - 3/2 + a \gg 1, \quad \Rightarrow \Delta_\phi \gg \frac{3}{4} - \frac{a}{2}. \quad (4.18)$$
However we have shown in section 2, that \( \Delta_\phi > 1 \) is sufficient for our results for \( \ell \gg n \) to hold. Thus although for the rest of the section we will use \( \Delta_\phi \gg 7/4 \) (which follows from above with \( a = -2 \)) this is strictly not needed unlike in the other limit \( n \gg \ell \gg 1 \). To do the saddle-point approximation we need \( g''(h_0) \) given by \( g''(h_0) = -4z/(2\Delta_\phi - 3/2 + a) \). Putting in all these together we can evaluate the \( h \) integral in the limit \( \ell \gg n \gg 1 \) as,

\[
\bar{z}^{-1/4} \frac{\sqrt{\pi}}{2} \int_0^\infty dh \ h^{2\Delta_\phi - 5/2}(h - \bar{h})^{a+1} e^{-2h\sqrt{z}} \delta \phi = \frac{\pi}{4} z^{-\Delta_\phi-a/2}\sqrt{2(2\Delta_\phi - 3/2 + a)}^{1/2} \left( \frac{2\Delta_\phi - 3/2 + a}{e} \right)^{2\Delta_\phi-3/2+a} \ e^{-2(2\Delta_\phi-3/2+a)}. \tag{4.19}
\]

Comparing the leading order \( z \)-dependence on the lhs of (2.7), we get \( a = -2 \). We also must have \( \Delta_\phi \gg 7/4 \). This will fix the leading \( \ell \) dependence of the anomalous dimensions in the limit of large spins. After doing the \( h \) integral, the remaining integral on \( \bar{h} \) can be converted into the sum as follows. Let us write down the full \( \bar{h} \)-integral after multiplying with \( (u/v)^{\Delta_\phi} = z^{\Delta_\phi}(1-v)^{\Delta_\phi}v^{-\Delta_\phi} \), one more time for the convenience of the reader.

\[
\frac{1}{4} z c_{\Delta_\phi} v^{-\Delta_\phi} \int_0^\infty dh \ h^{b+2\Delta_\phi-7/2} 2^{7/2} \sqrt{2}\bar{h}^{5/2-2\Delta_\phi}(1-v)^{\Delta_\phi-1} F_1(\bar{h} - 1, \bar{h} - 1; 2\bar{h} - 2; v), \tag{4.20}
\]

where,

\[
c_{\Delta_\phi} = \frac{\sqrt{\pi}}{2} \frac{2^{5/2-2\Delta_\phi}\Gamma(2\Delta_\phi - 5/2)}{\Gamma(\Delta_\phi)^2\Gamma(\Delta_\phi - 1)^2}. \tag{4.21}
\]

The coefficient \( c_{\Delta_\phi} \) is the same as the overall coefficient which appeared when we did the calculation for \( \ell \gg n \) limit if we have approximated \( K_0(2h\sqrt{z}) \sim e^{-2h\sqrt{z}} \). To convert the above integral back into its summation form, first note that the factor \( \bar{h}^{2\Delta_\phi-7/2}2^{-7/2} \) is the asymptotic form of the function,

\[
q_{\Delta_\phi, n} = \frac{8\Gamma(\Delta_\phi + n - 1)^2\Gamma(n + 2\Delta_\phi - 3)}{\Gamma(n + 1)^2\Gamma(2\Delta_\phi + 2n - 3)} \approx 1 \ \frac{n^{2\Delta_\phi-7/2}2^{-7/2}}{\Gamma(\Delta_\phi)^2\Gamma(\Delta_\phi - 1)^2}. \tag{4.22}
\]

where for \( n \gg \Delta_\phi \) we can take \( \bar{h} = \Delta_\phi + n \approx n \). We can further replace the part \( \bar{h}^b \) by \( \gamma_n \) since this part comes from assuming a form for the coefficients \( \gamma_n \) of the anomalous dimensions \( \gamma(n,l) \). Apart from this, the other factors in the integrand of \( \bar{h} \) integral are exactly the same as for the \( n \) summation we have encountered earlier. Performing a change of variables, \( \bar{h} = \Delta_\phi + n \) we can see that the integrand (without the factor of \( z \)) can be put into the summation form,

\[
\Gamma(\Delta_\phi)^2\Gamma(\Delta_\phi - 1)^2 \frac{c_{\Delta_\phi}}{4} \sum_{n=1}^\infty \gamma_n q_{\Delta_\phi, n} v^n (1-v)^{\Delta_\phi-1} F(\Delta_\phi + n, v) = lhs, \tag{4.23}
\]

where \( F(\Delta_\phi + n, v) = F_1(\Delta_\phi + n - 1, \Delta_\phi + n - 1, 2\Delta_\phi + 2n - 2, v) \). We already know the result of the above summation from the previous section. We thus find that the summation that leads to the exact expression for \( \gamma_n \) given in (3.16) is the same summation that comes when we replace the integral over \( \bar{h} \) with the sum over \( n \).
As the main aim of the section is to draw a unified conclusion about both the limits \( \ell \gg n \gg 1 \) and \( n \gg \ell \gg 1 \) from the saddle-point approach, we have also to calculate the integrand in the other limit which we proceed to do in the next section. Since the details are exactly the same as for this section we will omit some of the intermediate mathematical steps and quote the results for convenience.

4.1.2 Case II: \( n \gg \ell \gg 1 \) or \( (h - \bar{h})/2\bar{h} \ll 1 \)

Similarly for the other limit \( n \gg \ell \gg 1 \) we can take the integral in (4.14) for which the saddle point is around,

\[
h_0 = \frac{2\Delta_\phi - 5/2 + a}{2\sqrt{z}}.
\]

(4.24)

Here also note that the saddle point goes as \( \sim 1/\sqrt{z} \) and to keep the saddle within the domain of integration we are to choose,

\[
2h_0\sqrt{z} = 2\Delta_\phi - \frac{5}{2} + a \gg 0, \Rightarrow \Delta_\phi \gg \frac{5}{4} - \frac{a}{2}.
\]

(4.25)

To match the powers of \( z \) on both sides we will need \( g''(h_0) \) which is given by,

\[
g''(h_0) = -\frac{2\Delta_\phi - 5/2 + a}{h_0^2} = -\frac{4z}{2\Delta_\phi - 5/2 + a}.
\]

(4.26)

Putting all these together in (4.14) we get,

\[
2z^{-1/4}\sqrt{\frac{\pi}{2}} \int_{\bar{h}}^\infty dh \frac{h^{2\Delta_\phi - 7/2}}{(h - \bar{h})^{a+1}}e^{-2h\sqrt{z}}
\]

\[
= (2\bar{h})^{\frac{\pi}{4}} z^{1/2-\Delta_\phi - a/2} \sqrt{2}(2\Delta_\phi - 5/2 + a)^{1/2}\left(\frac{2\Delta_\phi - 5/2 + a}{e}\right)^{2\Delta_\phi - 5/2 + a} 2^{-(2\Delta_\phi - 5/2 + a)},
\]

(4.27)

which will match with the leading power of \( z \) for \( a = -1 \). So setting \( a = -1 \) and \( \Delta_\phi \gg 7/4 \) fixes the \( \ell \) dependence of the anomalous dimension in the limit \( n \gg \ell \gg 1 \) case. Now following the same method as in the previous subsection we have to find the (approximate) \( n \) dependence.

The remaining integral over \( \bar{h} \) can be written after multiplying the factor of \( (u/v)^{\Delta_\phi} \) as,

\[
\frac{z}{2}c_{\Delta_\phi}v^{-\Delta_\phi} \int_0^\infty d\bar{h} \bar{h}^{b+2\Delta_\phi - 5/2}2^{7-2\bar{h}}v^{\bar{h}(1-v)}\Delta_\phi^{-1/2}F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v),
\]

(4.28)

where note that we have absorbed the factor of \( 2\bar{h} \) in the integral and \( c_{\Delta_\phi} \) is the same coefficient as given in (4.21). The rest of the steps are exactly the same as followed in the previous subsection but we demonstrate here once again for completeness. Once again, note that the factor \( \bar{h}^{2\Delta_\phi - 7/2}2^{7-2\bar{h}} \) is the asymptotic form of \( q_{\Delta_\phi,n} \) as given by (4.22). Considering \( \bar{\gamma}_n = n^b = \bar{h}^b \) we can immediately see that the integral over \( \bar{h} \) is nothing but the familiar sum in the bootstrap equation, for the \( \ell \gg n \gg 1 \) case. To see that clearly we change the variables from \( \bar{h} = \Delta_\phi + n \) and in the domain where \( n \gg \Delta_\phi \) we obtain the summation form of the above integral (modulo the overall factor of
z) as,
\[
\Gamma(\Delta_\phi)^2/\Gamma(\Delta_\phi - 1)^2 \frac{c_{\Delta_\phi}}{4} \sum_{n} \infty (2n\tilde{\gamma}_n)q_{\Delta_\phi,n}v^n(1 - v)^{\Delta_\phi - 1} F(\Delta_\phi + n, v) = \text{lhs},
\]
(4.29)
where as before \( F(\Delta_\phi + n, v) = 2F_1(\Delta_\phi + n - 1, \Delta_\phi + n - 1; 2\Delta_\phi + 2n - 2; v) \). Note that this is the same sum as \([4.23]\) if we consider \( \gamma_n = 2n\tilde{\gamma}_n \) and with the exact same coefficients on the \( \text{lhs} \). Thus without calculating the sum over again, we find that the combined coefficient \( 2n\tilde{\gamma}_n \) will also follow the same polynomial behaviour as the \( \gamma_n \) for the \( \ell \gg n \gg 1 \) limit. However the result is not valid if \( n \) is not large. The coefficients in the numerator of the anomalous dimensions in the two different limits (in both limits \( n \gg 1 \)) are related in a simple way:
\[
\tilde{\gamma}_n \approx \frac{\gamma_n}{2n}.
\]
(4.30)
Thus explicit forms of the anomalous dimensions in both the limits are given by,
\[
\gamma(n, \ell)N^2 \approx -4\frac{n^4}{\ell^2}, \forall \ell \gg n \gg 1, \quad \text{and} \quad \gamma(n, \ell)N^2 \approx -2\frac{n^3}{\ell}, \forall n \gg \ell \gg 1.
\]
(4.31)
Since the \( n \gg \ell \gg 1 \) limit probes short distance physics in the dual gravity description, it may appear confusing why this limit is captured by a minimal twist exchange on the \( \text{lhs} \) of the bootstrap equation. We should point out that the saddle point analysis in this limit required (a) \( \ell > \ell_0 \gg 1 \) since these operators are supposed to play a role only for large \( \ell \), (b) \( n < n_{\text{max}} \) for \( \ell_m = 2 \) since the anomalous dimension contributes \(- (2n/\ell)(n^2/N^2) \) to the total dimension \( \Delta = 2\Delta_\phi + 2\ell + \gamma \) and the perturbative assumption will break down unless \( n < \ell^{1/2}N \)–note that if \( \ell_m = 0 \) then there is no \( n_{\text{max}} \) and (c) \( \Delta_\phi \gg 7/4 \) which was needed for the saddle to be within the range of integration\(^9\)

Thus the impact parameter from the dual gravity perspective satisfies \( \rho > \rho_c \sim \ell_0/n_{\text{max}} \). In [15], it was conjectured that for \( \rho > \rho_c \sim 1/\Delta_{\text{gap}} \), with \( \Delta_{\text{gap}} \) being some scale at which stringy effects become important, the anomalous dimension for double trace operators in the \( n \gg \ell \gg 1 \) would behave precisely like what we find. If we say that \( \Delta_{\text{gap}} \sim 1/\sqrt{\alpha'} \sim \lambda^{1/4} \), with \( \lambda \) being the ’t Hooft coupling, then to keep \( \ell_0/n_{\text{max}} \sim 1/\Delta_{\text{gap}} \) small we will need \( \lambda > 1 \)–which is consistent with the finding that the minimal twist operator dominates. We can estimate \( \ell_0 \sim N^2\lambda^{-1/4} \) which follows from \( \ell_0/n_{\text{max}} \sim \ell_0/N^2 \sim \lambda^{-1/4} \) with \( n_{\text{max}} = \ell^{1/2}N \sim n_{\text{max}}^{1/2}N \) so that \( n_{\text{max}} \sim N^2 \). It will be interesting to see stringy effects from the CFT (in other words seeing the necessity of going beyond the minimal twist operator on the \( \text{lhs} \) of the bootstrap equation), by considering \( \ell/n < \ell_0/n_{\text{max}} \) or \( \Delta_\phi \sim O(1) \). We will leave a more careful investigation of this interesting question for future work.

### 5 Comparison with AdS/CFT

AdS/CFT provides us with a formula for the anomalous dimensions in terms of the variables \( \tilde{h} = \Delta_\phi + n, h = \tilde{h} + \ell \). In the limit \( h, \tilde{h} \to \infty \), the form of the anomalous dimension is given by

\(^8\)We thank Juan Maldacena and Joao Penedones for discussions on this issue.

\(^9\)Furthermore, we can show that the correction to the anomalous dimensions from the saddle point approximation will behave as \( 1/\Delta_\phi^{1/2} \) which will be small only if \( \Delta_\phi \gg 1 \).
where $\ell_m$ is the spin of the minimal twist operator, $\Pi(h, \bar{h})$ is a particular function of $h, \bar{h}$ and $G = \frac{\pi}{2N^2}$ (where the radius of AdS is unity). In $d = 4$ the function $\Pi(h, \bar{h})$ is given by

$$
\Pi(h, \bar{h}) = \frac{2^{2\ell_m-2} \bar{h}^{2\ell_m}}{N^2 (h^2 - \bar{h}^2)},
$$

Neglecting the factor of $\Delta_\phi$ when both $n, \ell \gg 1$ we can write the above formula in terms $n, \ell$ giving,

$$
\gamma(n, \ell) = -\frac{2^{2\ell_m-2} n^{2\ell_m}}{N^2 \ell (2n + \ell)}.
$$

In the limit $\ell \gg n \gg 1$ we can see that the above formula reduces to $\gamma(n, \ell) = -(2^{2\ell_m-3}/N^2)(n^{2\ell_m-1}/\ell)$ while in the opposite limit it gives, $\gamma(n, \ell) = -(2^{2\ell_m-3}/N^2)(n^{2\ell_m-1}/\ell)$, where $\ell_m$ is the spin of the minimal twist operator. We can see that for $\ell_m = 2$ our results for the two limits match exactly with the above prediction from AdS/CFT. Also for $\ell_m > 2$ the $n$ and $\ell$ dependence of the above expression is the same as given by our analysis (see appendix C).

6 Correction to OPE coefficients for $\ell \gg n \gg 1$

We now turn to the question about what happens to the leading corrections to the OPE coefficients for the $\ell \gg n \gg 1$ case. The starting point of the calculation is,

$$
\sum_{n,\ell} P^{MFT}_{2\Delta_\phi+2n,\ell} \left( \delta P_{2\Delta_\phi+2n,\ell} + \frac{1}{2} \gamma(n, \ell) \frac{\partial}{\partial n} \right) v^n 4^{\ell/2} K_0(2\ell \sqrt{z}) F(4)[2\Delta_\phi+2n, v] = \sum_{\alpha} A_{\alpha} v^\alpha,
$$

where we are now only considering the terms without the log $v$ term in (2.8). As before we can perform the integration over the spins to eliminate one of the sums. To get the same leading order in $z$ as explained in [7], the coefficients $\delta P_{2\Delta_\phi+2n,\ell}$ should go like,

$$
\delta P_{2\Delta_\phi+2n,\ell} = \frac{c_n}{\ell_m}.
$$

Thus the above equation becomes, after performing the $\ell$ integration,

$$
\frac{1}{8} \Gamma \left( \Delta_\phi - \frac{\tau_m}{2} \right)^2 \sum_n q_{\Delta_\phi,n} \left[ C_n + \frac{1}{2} \gamma_n \frac{\partial}{\partial n} \right] v^n F(4)[2\Delta_\phi+2n, v] = \sum_{\alpha} A_{\alpha} v^\alpha.
$$
Acting the derivatives of $n$ on $v^n$ obtains a $v^n \log v$ term and the terms containing only $v^n$ come from considering,

$$
\frac{1}{8} \Gamma \left( \Delta_\phi - \frac{\tau_m}{2} \right)^2 \sum_n q_{\Delta_\phi,n} \left( C_n F^{(4)}[2\Delta_\phi + 2n, v] + \frac{1}{2} \gamma_n \partial_n F^{(4)}[2\Delta_\phi + 2n, v] \right) v^n = \sum_{\alpha} A_\alpha v^\alpha. \hspace{1cm} (6.4)
$$

At this point note that the function $F^{(4)}[2\Delta_\phi + 2n, v] = 2^\tau_2 F_1(\Delta_\phi + n - 1, \Delta_\phi + n - 1, 2\Delta_\phi + 2n - 2; v)$ has a separate $n$ dependent part coming from the $2^\tau$. So the $n$-derivative should act on this part as well. Thus equation (6.4) becomes,

$$
\frac{1}{8} \Gamma \left( \Delta_\phi - \frac{\tau_m}{2} \right)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q_{\Delta_\phi,n} d_{n,k} (C_n + \gamma_n (\log 2 + g_{n,k})) v^{n+k} = \sum_{\alpha=0}^{\infty} A_\alpha v^\alpha. \hspace{1cm} (6.5)
$$

where the function $g_{n,k}, d_{n,k}$ are defined as,

$$
g_{n,k} = \psi(2\Delta_\phi + 2n - 2) + \psi(n + \Delta_\phi + k - 1) - \psi(\Delta_\phi + n - 1) - \psi(2\Delta_\phi + 2n + k - 2) \hspace{1cm} (6.6)
$$

$$
d_{n,k} = \frac{(\Delta_\phi + n - 1)^2}{(2\Delta_\phi + 2n - 2)k!}, \hspace{1cm} (6.7)
$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. To regroup the terms in (6.5) increasing powers of $v^\alpha$, we set $n + k = \alpha$ and the lhs of the above equation becomes $\sum_{\alpha=0}^{\infty} f_{\alpha,\Delta_\phi} v^\alpha$ where,

$$
f_{\alpha,\Delta_\phi} = \sum_{k=0}^{\alpha} q_{\alpha-k,\Delta_\phi} d_{\alpha-k,k} C_{\alpha-k} + b_{\alpha}, \hspace{1cm} \text{where} \hspace{1cm} b_{\alpha} = \sum_{k=0}^{\alpha} q_{\alpha-k,\Delta_\phi} d_{\alpha-k,k} \gamma_{\alpha-k} (\log 2 + g_{\alpha-k,k}). \hspace{1cm} (6.8)
$$

By equating the two sides of the above equation via $f_{\alpha,\Delta_\phi} = A_\alpha$, we can get the coefficients $C_n$ once we know the anomalous dimensions $\gamma_n$. On the lhs of (6.4), the coefficients $A_\alpha$ are determined as follows. We have absorbed the term $(1 - v)\Delta_\phi^{-1}$ in to the lhs of (2.7) to obtain,

$$
(1 - v)^{\tau_m/2 + \ell_m + 1 - \Delta_\phi} P_m \Gamma(\ell_m + 2\tau_m) \sum_{n=0}^{\infty} \left( \frac{(\ell_m + \tau_m/2)n}{n!} \right)^2 (2(\psi(n + 1) - \psi(\tau_m/2 + \ell_m + n))) v^n = \sum_{\alpha=0}^{\infty} A_\alpha v^\alpha. \hspace{1cm} (6.9)
$$

The coefficients $A_\alpha$ can be written (after transposing the overall factor of 1/8 to the rhs of (6.5) for the two cases of scalar and spin-2 operators as,

$$
A_\alpha = \begin{cases} 
0 & \ell_m = 0 \\
-2P_m \frac{3\Gamma(\tau_m + 2\ell_m)}{\Gamma(\tau_m/2 + \ell_m)^2 \Gamma(\Delta_\phi - 1)^2} \frac{(\Delta_\phi + 2\alpha - 1)\Gamma(\Delta_\phi + \alpha - 1)}{\Gamma(\alpha + 1)\Gamma(\Delta_\phi)} & \ell_m = 2
\end{cases}
$$
Figure 2: Plot for $C_n$ for three cases. The blue curve is for $\mathcal{N} = 4$, the red curve for the twist-2, spin-2 operator exchange and the yellow for the twist-2 scalar. We have scaled down the OPE coefficients by a factor $10^8$ in this figure.

We can thus write (6.8) as,

$$\alpha \sum_{k=0}^{\alpha} q_{\alpha-k,\Delta_\phi} d_{\alpha-k,k} C_{\alpha-k} = A_\alpha - b_\alpha \equiv B_\alpha,$$  

with $b_\alpha$ given in (6.8). This relation can be inverted in the same spirit as we did for the anomalous dimensions. After inversion the corrections to the OPE coefficients can be written as,

$$C_n = \Gamma(\Delta_\phi - 1)^2 \sum_{m=0}^{n} c_{n,m} B_m,$$  

where we have defined the coefficients $B_\alpha$ above and $c_{n,m}$ is the same coefficient as given in (3.10).

Unfortunately to extract a closed form for the coefficients $C_n$ from the above sum appears difficult. Nevertheless the behaviour of the OPE corrections can be inferred from (6.11). In figure (2) below we have done a comparative study of the OPE corrections for $\mathcal{N} = 4$ SYM [10], when the lhs of (2.7) is dominated by a twist-2, spin-2 operator and for twist-2 scalar operators. From the figure we see that at large $n$, $C_n$ tend to follow the relation,

$$C_n = \frac{1}{2\bar{q}_{\Delta_\phi,n}} \partial_n (\bar{q}_{\Delta_\phi,n} \gamma_n).$$  

whereas for small $n$ there are deviations from the $\mathcal{N} = 4$ case. From the inset in figure (2) we see
that for low lying values of \( n \), \( C_n \) for the twist-2, spin-2 operator exchange becomes negative while those for the \( \mathcal{N} = 4 \) case are positive. \( C_n \) for the scalar exchange case is a constant positive value.

We were unable to extend our calculations to the \( n \gg \ell \gg 1 \) case. The reason is that in order to compute the coefficient \( C_n \) using the methods in this section we would need to know all the coefficients \( C_0 \cdots C_{n-1} \). This is not possible since we only know the leading order form of \( \gamma_n \) in this limit.

7 Discussion

We conclude by listing some open problems.

- It will be nice to extend our results to other dimensions, especially odd dimensions where the conformal blocks are not known in closed form.
- It will be interesting to understand the restriction \( \ell_m \leq 2 \) better. Since we found that the anomalous dimensions for large-\( n \) will be proportional to \( n^{2\ell_m} \), it could be that this behaviour will be incompatible with unitarity for \( \ell_m > 2 \). For instance, one can try to see if an introduction of a gap as advocated in [15] and examined further in [16] can make this case consistent as well.
- It will be interesting to capture stringy effects (or see why they are necessary) from CFT in the limit \( n \gg \ell \gg 1 \). Namely, it will be nice to have a rigorous argument when we need a double expansion in terms of \( 1/N^2 \) and \( 1/\Delta_{\text{gap}} \) as advocated by [15] [16] and to identify what \( \Delta_{\text{gap}} \) is. This will be the analog of \( g_s, \alpha' \) expansion in string theory. For this it appears that we have to examine the bootstrap equation in the limit \( n \gg \ell \gg 1 \) in the regime \( \ell/n < \ell_0/n_{\text{max}} \) or \( \Delta_\phi \sim O(1) \) which are regimes that our saddle point analysis cannot capture.
- Our result used the scalar four point function as the starting point. Whether a similar conclusion can be reached by bootstrapping other four point functions of operators with spin \( \ell \neq 0 \) is an interesting open problem.
- Our results agreed exactly with the large-\( n \) behaviour found using the Eikonal approximation in AdS/CFT. On the dual gravity side, one can try to get the subleading terms in \( n \) for the case \( \ell \gg n \).
- It will be interesting to see if Nachtmann’s original proof [10] can be extended to the \( n \neq 0 \) case.

8 Acknowledgements

We thank Agnese Bissi, Justin David, Zohar Komargodski, Tomasz Lukowski, Juan Maldacena, João Penedones and Sheer-El-Showk for discussions and useful comments. We thank Agnese Bissi and Tomasz Lukowski for generously sharing their mathematica notebook which helped us understand the results of [16]. We also thank Fernando Rejon-Barrera for pointing out a few
typos in our earlier version and confirming several of our results. AS acknowledges support from
a Ramanujan fellowship, Govt. of India.

A Calculation details

To clearly see the expressions for the anomalous dimensions discussed in the main text we now
take a mathematical detour a little to explain some of the steps and the useful formulae that goes
into the derivation of the above expressions. Note that in the following calculations we will not
put the overall factor of \( 4P_m \) for convenience. Each of the above expressions use the summation
of the generic type

\[
a(x, m, \epsilon) = \sum_{k=0}^{m} \frac{\Gamma(x + k)}{k! \Gamma(x)} \epsilon^k.
\]

(A.1)

Using the integral representation of the \( \Gamma \)-function, the summation on the rhs can be converted
into,

\[
a(x, m, \epsilon) = \frac{1}{\Gamma(x)} \int_0^\infty dt \ e^{-t} \sum_{k=0}^{m} \frac{t^{x+k-1}}{k!} \epsilon^k.
\]

(A.2)

The summation inside the integral can be written as,

\[
\sum_{k=0}^{m} \frac{t^{x+k-1}}{k!} \epsilon^k = e^{\epsilon t^x - 1} \int_0^\infty \frac{\Gamma(m + 1, \epsilon t^x)}{\Gamma(m + 1)} dz \int_0^\infty z^m e^{-z} dz,
\]

(A.3)

where \( \Gamma(a, x) \) is the incomplete Gamma function given by \( \Gamma(a, x) = \int_x^\infty z^{a-1}e^{-z}dz \). Thus the
function \( a(x, m) \) becomes after the above substitution as,

\[
a(x, m, \epsilon) = \frac{1}{\Gamma(x) \Gamma(m + 1)} \int_0^\infty dt \ e^{(t-1)\epsilon t^x - 1} \int_0^\infty dz \ z^m e^{-z}.
\]

(A.4)

At this point we do a change of variable from \( z \) to \( z = y + \epsilon t \) whereby we notice that the limits of
the integral on \( z \) changes to \( y = 0 \) and \( y = \infty \) respectively. Thus we get,

\[
a(x, m, \epsilon) = \frac{1}{\Gamma(x) \Gamma(m + 1)} \int_0^{\infty} dt \int_0^{\infty} dy \ (y + \epsilon t)^m e^{-(t+y)\epsilon t^x - 1}.
\]

(A.5)

Whatever summation formulae we have derived in the text are linear combinations of the above
function and its derivatives. For example,

\[
a(x, m, \epsilon = 1) = \frac{\Gamma(x) \Gamma(m + x + 1)}{\Gamma(x + 1) \Gamma(m + 1)}.
\]

(A.6)
Again a polynomial arranged like,

\[ \sum_{k=0}^{m} [c_0 + c_1 k + c_2 k(k-1) + c_3 k(k-1)(k-2) + c_4 k(k-1)(k-2)(k-3) + \cdots] \frac{\Gamma(k+x)}{k!\Gamma(x)} \]

\[ = c_0 a(x, m, \epsilon)|_{\epsilon=1} + c_1 \partial_{\epsilon} a(x, m, \epsilon)|_{\epsilon=1} + c_2 \partial^2_{\epsilon} a(x, m, \epsilon)|_{\epsilon=1} + c_3 \partial^3_{\epsilon} a(x, m, \epsilon)|_{\epsilon=1} + c_4 \partial^4_{\epsilon} a(x, m, \epsilon)|_{\epsilon=1} + \cdots, \quad (A.7) \]

where,

\[ \partial^i_{\epsilon} a(x, m, \epsilon)|_{\epsilon=1} = \sum_{k=0}^{m} k(k-1) \cdots (k-i+1) \frac{\Gamma(x+k)}{k!\Gamma(x)} \frac{\Gamma(-2+k+\Delta)}{k!(x+i)\Gamma(m-i+1)\Gamma(x)}. \quad (A.8) \]

### B Verification of some useful formulae

With the definitions of the formula in the previous section we can now apply them to our cases specific to the exchange of the twist-2 scalar and a spin-2, twist-2 field.

#### B.1 $\ell_m = 0$ and $\tau_m = 2$

We will first deal with the case of a twist-2 scalar exchange. The formulae are much simpler for this case.

1. \[
\frac{(-m)^2 (-1 - \ell_m + \Delta - \frac{\tau_m}{2})}{(1 - \ell_m - m - \frac{\tau_m}{2})^2 k!} = \frac{\Gamma(-2 + k + \Delta)}{k!\Gamma(-2 + \Delta)}. \quad (B.1)
\]

   This formula needs no verification. We can simply put $\ell_m = 0$ and $\tau_m = 2$ to see that the rhs is produced.

2. \[
\sum_{k=0}^{m} \frac{\Gamma(x+k)}{k!\Gamma(x)} = \frac{\Gamma(1+m+x)}{\Gamma(1+m)\Gamma(1+x)}. \quad (B.2)
\]

   To see this we recall from the previous section that

   \[
\sum_{k=0}^{m} \frac{\Gamma(x+k)}{k!\Gamma(x)} = a(x, m, \epsilon = 1). \quad (B.3)
\]

   Performing the integrals at $\epsilon = 1$, fixes the form on the rhs of the above formula.

3. \[
\gamma_n = \sum_{m=0}^{n} a_{n,m}, \quad (B.4)
\]

   In this case the coefficients $a_{n,m}$ are given by,

   \[
a_{n,m} = -\frac{(-1)^{m+n}}{8} \frac{(\Delta_\phi - 1)\Gamma(n+1)\Gamma(\Delta_\phi)\Gamma(2\Delta_\phi + n + m - 3)}{m!(n-m)!\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)}. \quad (B.5)
\]

23
We will now use the reflection formula for the Γ-functions to obtain,

\[ \Gamma(m + \Delta - 1) = (-1)^{-(m+1)} \frac{\pi}{\sin(\pi\Delta) \Gamma(2 - \Delta - m)} . \]  

(B.6)

Separating out the \( m \) independent parts and using the integral representation of the product of the Γ-functions given by,

\[ \Gamma(n + m + 2\Delta - 3)\Gamma(2 - \Delta - m) = \int_0^\infty \int_0^\infty dx dy e^{-(x+y)}x^{m+n+2\Delta-4}y^{-m-1-\Delta} , \]  

(B.7)

we can perform the sum over \( m \) to get,

\[ \sum_{m=0}^n \frac{(x/y)^m}{m!(n-m)!} n! = \frac{1}{n!} \left( \frac{x+y}{y} \right)^n \equiv b(n, x, y) . \]  

(B.8)

Hence the coefficient \( \gamma_n \) associated with the anomalous dimensions become,

\[ \gamma_n = \frac{(-1)^{n+1} \sin(\pi\Delta) (\Delta - 1)\Gamma(n + 1)\Gamma(\Delta - 1)}{8\Gamma(n + 2\Delta - 3)} \int_0^\infty \int_0^\infty dx dy b(n, x, y)e^{-(x+y)}x^{n+2\Delta-4}y^{1-\Delta} . \]  

(B.9)

Using the transformation of variables for \( x = r^2 \cos^2 \theta \) and \( y = r^2 \sin^2 \theta \) and performing the integral over only the first quadrant, the integration limits change from \( r = 0 \) to \( r = \infty \) and \( \theta = 0 \) to \( \theta = \pi/2 \). The integral thus becomes,

\[ \int_0^\infty \int_0^\infty dx dy b(n, x, y)e^{-(x+y)}x^{n+2\Delta-4}y^{1-\Delta} = \frac{(-1)^{n-1} \pi \csc(\pi\Delta)\Gamma(n + 2\Delta - 3)}{\Gamma(n + 1)\Gamma(\Delta - 1)} . \]  

(B.10)

Putting this with the overall factors we get,

\[ \gamma_n = -\frac{1}{8}(\Delta - 1)^2 . \]  

(B.11)

which is independent of \( n \). Here we have not taken into account the overall factor of \( 4P_m \) that we should multiply with the expression for \( \gamma_n \) to match the result with the main text.

\section*{B.2 \( \ell_m = 2 \) and \( \tau_m = 2 \)}

We list below the derivation of important formulae required pertaining to this case.

1. \[ \frac{(-m)^2(-1 - \ell_m + \Delta - \frac{\tau_m}{2})}{(1 - \ell_m - m - \frac{\tau_m}{2})k!} = \frac{(1 - k + m)^2(2 - k + m)^2\Gamma(-4 + k + \Delta)}{(1 + m)^2(2 + m)^2\Gamma(1 + k)\Gamma(-4 + \Delta)} . \]  

(B.12)

As in the scalar case we put \( \tau_m = 2 \) and \( \ell_m = 2 \) for this case to retrieve the \( \text{rhs} \) of the above formula. 

24
2.
\[
\sum_{k=0}^{m} \frac{(1-k+m)^2(2-k+m)^2}{k!(1+m)^2(2+m)^2} \frac{\Gamma(x+k)}{\Gamma(x)} = \frac{4(6m^2 + 6m(3 + x) + (3 + x)(4 + x))\Gamma(3 + m + x)}{(1 + m)(2 + m)\Gamma(3 + m)\Gamma(5 + x)}.
\]
(B.13)

To get to this, we will appeal to (A.7), by noticing that the factor \((1-k+m)^2(2-k+m)^2\)
can be arranged as,
\[
(1-k+m)^2(2-k+m)^2 = Ak(k-1)(k-2) + Bk(k-1) + Ck(k-2) + Dk + E,
\]
(B.14)
where \(A = 1\), \(B = -4m\), \(C = 6m^2 + 6m + 2\), \(D = -4(m+1)^3\) and \(E = (2 + 3m + m^2)^2\). Thus the sum becomes,
\[
\sum_{k=0}^{\infty} \frac{(1-k+m)^2(2-k+m)^2}{(m+1)^2(m+2)^2} \frac{\Gamma(x+k)}{k!\Gamma(x)} = A\partial_x^4 a(x, m, \epsilon)|_{\epsilon=1} + B\partial_x^3 a(x, m, \epsilon)|_{\epsilon=1}
\]
\[
+ C\partial_x^2 a(x, m, \epsilon)|_{\epsilon=1} + D\partial_x a(x, m, \epsilon)|_{\epsilon=1}
\]
\[
+ Ea(x, m, \epsilon)|_{\epsilon=1}.
\]
(B.15)

We know how the each of the terms go by looking at (A.8). By combining the coefficients we find that the rhs is produced.

3.
\[
\gamma_n = \sum_{m=0}^{n} a_{n,m}.
\]
(B.16)

We will now prove the final piece of the analytic puzzle as follows. First note that \(a_{n,m}\) for \(\ell_m = 2\) and \(\tau_m = 2\) is given in a closed form expression as
\[
a_{n,m} = (-1)^{n+m} \frac{15(6m^2 + 6m(\Delta_{\phi} - 1) + \Delta_{\phi}(\Delta_{\phi} - 1))}{4\Delta_{\phi}} \times \frac{\Gamma(n+1)\Gamma(\Delta_{\phi})\Gamma(n+m+2\Delta_{\phi}-3)}{m!(n-m)!\Gamma(n+2\Delta_{\phi} - 3)}.
\]
(B.17)

We will now use the reflection formula for the \(\Gamma\)-functions to obtain,
\[
\Gamma(n + \Delta_{\phi} - 1) = (-1)^{(m+1)} \frac{\pi}{\sin(\pi \Delta_{\phi})\Gamma(2 - \Delta_{\phi} - m)}.
\]
(B.18)

Separating out the \(m\)-independent parts we have
\[
\gamma_n = \frac{(-1)^{n+1} \sin(\pi \Delta_{\phi})}{\pi} \frac{15\Gamma(n+1)\Gamma(\Delta_{\phi})}{\Gamma(n+2\Delta_{\phi} - 3)4\Delta_{\phi}} \sum_{m=0}^{n} \frac{1}{m!(n-m)!}[6m^2 + 6m(\Delta_{\phi} - 1) + \Delta_{\phi}(\Delta_{\phi} - 1)]\Gamma(n + m + 2\Delta_{\phi} - 3)\Gamma(2 - \Delta_{\phi} - m).
\]
(B.19)
The integral representation of the product of the two \( \Gamma \)-functions is given by
\[
\Gamma(n + m + 2\Delta_\phi - 3)\Gamma(2 - \Delta_\phi - m) = \int_0^\infty \int_0^\infty dxdye^{-(x+y)x^{m+n+2\Delta_\phi-4}y^{m+1-\Delta_\phi}}. \tag{B.20}
\]
Performing the sum over \( m \) inside the integral for a polynomial multiplying the \( \Gamma \)-functions of the form \( f(m) = c_0 + c_1 m + c_2 m^2 \) we get,
\[
\sum_{m=0}^n \left(\frac{x}{y}\right)^m \frac{f(m)}{m!(n-m)!} = \left(\frac{x+y}{y}\right)^n \frac{c_0(x+y)^2 + c_1 nx(x+y) + c_2 nx(nx+y)}{(x+y)^2n!} \equiv b(n, x, y). \tag{B.21}
\]
Thus the expression for \( \gamma_n \) becomes,
\[
\gamma_n = \frac{(-1)^{n+1} \sin(\pi \Delta_\phi)}{\pi} \frac{15\Gamma(n+1)\Gamma(\Delta_\phi)}{\Gamma(n+2\Delta_\phi-3)4\Delta_\phi} \int_0^\infty dxdy b(n, x, y)e^{-(x+y)x^{n+2\Delta_\phi-4}y^{1-\Delta_\phi}}. \tag{B.22}
\]
Using the transformation of variables for \( x = r^2 \cos^2 \theta \) and \( y = r^2 \sin^2 \theta \) and performing the integral over only the first quadrant, the integration limits change from \( r = 0 \) to \( r = \infty \) and \( \theta = 0 \) to \( \theta = \pi/2 \). Thus, putting the values of \( c_0 = \Delta_\phi(\Delta_\phi - 1) \), \( c_1 = 6(\Delta_\phi - 1) \) and \( c_2 = 6 \), we have
\[
\int_0^\infty dxdy b(n, x, y)e^{-(x+y)x^{n+2\Delta_\phi-4}y^{1-\Delta_\phi}} = -\frac{(-1)^{n-1}\pi \csc(\pi \Delta_\phi)}{\Gamma(n+1)\Gamma(\Delta_\phi+1)} \frac{15\Gamma(n+2\Delta_\phi-3)}{(2\Delta_\phi+n(n+2\Delta_\phi-3))} \left[6n(n+2\Delta_\phi-3)(2-\Delta_\phi+n(n+2\Delta_\phi-3))
\right.
\left. + \Delta_\phi(\Delta_\phi-1)(\Delta_\phi(\Delta_\phi-1) + 6n(n+2\Delta_\phi-3))\right]. \tag{B.23}
\]
Multiplying this by the overall \( n \)-dependent factors outside we have,
\[
\gamma_n = \frac{-15}{4\Delta_\phi^2} \left[6n^4 + \Delta_\phi^2(\Delta_\phi - 1)^2 + 12n^3(2\Delta_\phi - 3) + 6n^2(11 - 14\Delta_\phi + 5\Delta_\phi^2)
\right.
\left. + 6n(2\Delta_\phi - 3)(\Delta_\phi^2 - 2\Delta_\phi + 2)\right], \tag{B.24}
\]
which is the precise formula for \( \gamma_n \) in \( d = 4 \) dimensions. Note that the final expression for \( \gamma_n \) derived above needs to be multiplied by an overall factor of \( 4P_m \) to match with that in the main text.

C \( n \) dependence of \( \gamma_n \) for \( \ell_m > 2 \)

In this section we will give an overview on the leading \( n \)-dependence of the coefficients of the anomalous dimensions \( \gamma_n \). We will consider two cases with twist-2 and spins \( \ell_m = 4, 6 \). For
\( \ell_m = 4 \), the coefficients \( a_{n,m} \) are given by,

\[
a_{n,m} = -\frac{315P_m(-1)^{m+n}\Gamma(n+1)\Gamma(\Delta_\phi)^2\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(\Delta_\phi + 3)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)} \\
\times \left[ 70m^4 + 140m^3(\Delta_\phi - 1) + 10m^2(9\Delta_\phi^2 - 15\Delta_\phi + 11) + 10m(2\Delta_\phi^3 - 3\Delta_\phi^2 + 5\Delta_\phi - 4) \right] \\
\Delta_\phi(\Delta_\phi^2 - 1)(\Delta_\phi + 2)].
\] (C.1)

To calculate the leading \( n \) dependence in the coefficient \( \gamma_n \), we take the leading term proportional to \( m^4 \) in \( a_{n,m} \) and do the sum over \( m \) to get,

\[
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{2205P_m n^8}{\Delta_\phi^2(\Delta_\phi + 1)^2(\Delta_\phi + 2)^2} - \cdots .
\] (C.2)

Thus the leading \( n \) dependence of the coefficients \( \gamma_n \) for \( \ell_m = 4 \) is \( \sim -n^8 \). Similarly for \( \ell_m = 6 \), the coefficients \( a_{n,m} \) are given by,

\[
a_{n,m} = -\frac{6006P_m(-1)^{m+n}\Gamma(n+1)\Gamma(\Delta_\phi)^2\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(\Delta_\phi + 5)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)} \\
\times \left[ 924m^6 + 2772m^5(\Delta_\phi - 1) + 210m^4(15\Delta_\phi^2 - 27\Delta_\phi + 26) + 420m^3(\Delta_\phi - 1) \\
4\Delta_\phi^2 - 5\Delta_\phi + 15) + 42m^2(10\Delta_\phi^4 - 20\Delta_\phi^3 + 95\Delta_\phi^2 - 145\Delta_\phi + 88) + 42m(\Delta_\phi^5 \\
15\Delta_\phi^3 - 30\Delta_\phi^2 + 38\Delta_\phi - 24) + (\Delta_\phi + 4)(\Delta_\phi + 3)(\Delta_\phi + 2)(\Delta_\phi + 1) \Delta_\phi(\Delta_\phi - 1) \right].
\] (C.3)

Again, we take the leading term in \( m \) in \( a_{n,m} \) and sum over \( m \) to get,

\[
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{5549544P_m n^{12}}{\Delta_\phi^2(\Delta_\phi + 1)^2(\Delta_\phi + 2)^2(\Delta_\phi + 3)^2(\Delta_\phi + 4)^2} - \cdots .
\] (C.4)

All the above expressions for \( \gamma_n \) are up to overall normalization constants. Thus for a generic \( \ell_m \) we find that the coefficient \( \gamma_n \) has an \( n \) dependence given by,

\[
\gamma_n \sim -n^{2\ell_m} .
\] (C.5)

References

[1] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, JHEP 0812, 031 (2008) [arXiv: 0807.0004].
R. Rattazzi, S. Rychkov, and A. Vichi, Phys. Rev. D 83 046011 (2011) [arXiv: 1009.2725].
R. Rattazzi, S. Rychkov, and A. Vichi, J. Phys. A 44 035402 (2011) [arXiv: 1009.5985].
D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, Phys. Rev. D 86 105043 (2012) [arXiv: 1208.6449].
D. Poland, D. Simmons-Duffin and A. Vichi, JHEP 1205, 110 (2012) [arXiv:1109.5176 [hep-th]].

27
Y. Nakayama and T. Ohtsuki, Phys. Lett. B 734, 193 (2014) [arXiv:1404.5201 [hep-th]].

[2] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, JHEP 0910 (2010) 079 [arXiv: 0907.0151[hep-th]].

[3] M. Hogervorst, H. Osborn and S. Rychkov, JHEP 1308 (2013) 014 [arXiv: 1305.1321[hep-th]].
M. Hogervorst and S. Rychkov, Phys. Rev. D D87 (2013) 106004 [arXiv: 1303.1111[hep-th]].

[4] F. A. Dolan and H. Osborn, Nucl. Phys. B 629 (2002) 3-73 [hep-th/0011040].
F. A. Dolan and H. Osborn, Annals Phys. 321 (2006) 581-626 [hep-th/0309180].
F. A. Dolan and H. Osborn, [arXiv: 1108.6194[hep-th]].

[5] C. Beem, L. Rastelli and F. Passerini, Phys. Rev. Lett. 111 (2013) 071601 [arXiv: 1304.1803[hep-th]].
C. Beem, M. Lemos, P. Liendo, L. Rastelli and B. C. van Rees, [arXiv: 1412.7541[hep-th]].
F. A. Dolan, M. Nirschl and H. Osborn, Nucl. Phys. B 749 (2006) 109-152 [hep-th/0601148].
F. A. Dolan and H. Osborn, Nucl. Phys. B 629 (2002) 3-73 [hep-th/0112251].
F. A. Dolan and H. Osborn, Nucl. Phys. B 593 (2001) 599-633 [hep-th/0006098].

[6] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffins and A. Vichi, Phys. Rev. D 86 (2012) 025022 [arXiv: 1203.6064[hep-th]].
S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffins and A. Vichi, J. Stat. Phys. 157 (2014) 869 [arXiv: 1403.4545[hep-th]].
F. Kos, D. Poland, D. Simmons-Duffin, JHEP 1411 (2014) 109 [arXiv: 1411.7932[hep-th]].
F. Gliozzi and A. Rago JHEP 1410 (2014) 42 [arXiv: 1403.6003[hep-th]].

[7] A. L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, JHEP 1312, 004 (2013) [arXiv:1212.3616 [hep-th]].
[8] Z. Komargodski and A. Zhiboedov, JHEP 1311, 140 (2013) [arXiv:1212.4103 [hep-th]].
[9] L. F. Alday and J. M. Maldacena, JHEP 0711, 019 (2007) [arXiv:0708.0672 [hep-th]].
[10] O. Nachtmann, Nucl. Phys. B 63, 237 (1973).
[11] G. Vos, arXiv:1411.7941 [hep-th].
[12] L. Cornalba, M. S. Costa and J. Penedones, JHEP 0709, 037 (2007) [arXiv:0707.0120 [hep-th]].
[13] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, Nucl. Phys. B 767, 327 (2007) [hep-th/0611123].
[14] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, JHEP 0708, 019 (2007) [hep-th/0611122].
[15] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, arXiv:1407.5597 [hep-th].
[16] L. F. Alday, A. Bissi and T. Lukowski, arXiv:1410.4717 [hep-th].
[17] G. Arutyunov, F. A. Dolan, H. Osborn and E. Sokatchev, Nucl. Phys. B 665, 273 (2003) [hep-th/0212116].
[18] L. F. Alday and A. Bissi, JHEP 1310 (2013) 202 [arXiv: 1305.4604[hep-th]].