LARGE $Y_{k,b}$-TILINGS AND HAMILTON $\ell$-CYCLES IN $k$-UNIFORM HYPERGRAPHS

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Abstract. Let $Y_{3,2}$ be the 3-uniform hypergraph with two edges intersecting in two vertices. Our main result is that any $n$-vertex 3-uniform hypergraph with at least \( \binom{n}{3} - \binom{n-m+1}{3} + o(n^3) \) edges contains a collection of $m$ vertex-disjoint copies of $Y_{3,2}$, for $m \leq n/7$. The bound on the number of edges is asymptotically best possible. This problem generalizes the Matching Conjecture of Erdős. We then use this result combined with the absorbing method to determine the asymptotically best possible minimum $(k-3)$-degree threshold for $\ell$-Hamiltonicity in $k$-graphs, where $k \geq 7$ is odd and $\ell = (k-1)/2$. Moreover, we give related results on $Y_{k,b}$-tilings and Hamilton $\ell$-cycles with $d$-degree for some other values of $k, \ell, d$.

Keywords: Hypergraph; Hamilton cycle; Hypergraph regularity method; Absorbing method.

1. Introduction

Given $k \geq 2$, a $k$-uniform hypergraph $H$ (in short, $k$-graph) is a pair $(V, E)$, where $V$ is a vertex set and $E$ is a family of $k$-element subsets of $V$. We denote the numbers of edges in $H$ by $e(H)$. Given two $k$-graphs $F$ and $H$, an $F$-tiling in $H$ is a subgraph of $H$ consisting of vertex-disjoint copies of $F$. The number of copies of $F$ is called the size of the $F$-tiling. When $F$ is a single edge, an $F$-tiling is known as a matching. The following conjecture was proposed by Erdős in 1960s.

Conjecture 1.1. \cite{Erdos60} Let $n, s, k$ be three positive integers such that $k \geq 2$ and $n \geq k(s+1) - 1$. If $H$ is a $k$-graph on $n$ vertices which does not have a matching of size $s + 1$, then

$$e(H) \leq \max \left\{ \left( \binom{k(s+1)-1}{k} \right), \left( \binom{n}{k} - \binom{n-s}{k} \right) \right\}.$$ 

The bounds in the conjecture come from two extremal constructions: a complete $k$-graph on $k(s+1)-1$ vertices and a complete $k$-graph on $n$ vertices with a complete $k$-graph on $n-s$ vertices removed. The case $s = 1$ is the classical Erdős–Ko–Rado Theorem \cite{EKR61}. For $k = 1$ the conjecture is trivial and for $k = 2$ it was proved by Erdős and Gallai \cite{EGR60}. For $k = 3$, it was settled by Frankl, Rödl and Ruciński in the case $n \geq 4s$ in \cite{FRR95}, and by Łuczak and Mieczkowska for all $n$ and $s \geq s_0$ in \cite{LM03}, and finally, completely resolved by Frankl in \cite{Frankl05}. For general $k$, Erdős \cite{Erdos60} himself proved the conjecture for $n > n_0(k, s)$. Subsequent improvements on $n_0$ have been done by various authors \cite{FR95,FRR95,LM03,FK99} and the current state of art is $n_0 \leq \frac{2}{3}sk - \frac{2}{3}s$ for sufficiently large $s$ by Frankl and Kupavskii \cite{FK99}. Also, Frankl \cite{Frankl01} showed the conjecture for all $n \leq (s + 1)(k + \varepsilon)$, where $\varepsilon$ depends on $k$ only.

1.1. $Y_{k,b}$-tilings. We consider a generalization of this problem to $F$-tilings, and as the simplest case, we consider the $k$-graphs $F$ with two (intersecting) edges. For $k > b \geq 0$, let $Y_{k,b}$ be the $k$-graph consisting of two edges that intersect in exactly $b$ vertices. Note that a $Y_{k,0}$-tiling is just a matching, so we shall only consider $b \geq 1$. The extremal examples for Conjecture 1.1 suggest the following bounds for $k$-graphs without $Y_{k,b}$-tilings of certain size, and we formulate it as a conjecture.
Conjecture 1.2. Let $n, s, k, b$ be positive integers such that $k > b > 0$ and $n \geq (2k - b)(s + 1) - 1$. If $H$ is a $k$-graph on $n$ vertices which does not have a $Y_{k,b}$-tiling of size $s + 1$, then

$$e(H) \leq \max \left\{ \left( \binom{2k - b}{k}(s + 1) - 1 \right), \binom{n}{k} - \binom{n - s}{k} \right\} + o(n^k).$$

The case $s = 0$ is an old conjecture of Erdős [13] and was resolved by Frankl and Füredi [20] (see Theorem 5.1), which says that a $Y_{k,b}$-free $k$-graph (in which, no pair of edges intersect in exactly $b$ vertices) has $o(n^k)$ edges. For a $k$-graph $F$ and $n \in \mathbb{N}$, the Turán number of $F$ is defined as $\text{ex}(n, F) = \max\{e(H) : |V(H)| = n, F \not\subseteq H\}$. So the result of [20] says $\text{ex}(n, Y_{k,b}) = o(n^k)$.

The major terms in the conjecture above come from a complete $k$-graph on $(2k - b)(s + 1) - 1$ vertices and a complete $k$-graph on $n$ vertices with a complete $k$-graph on $n - s$ vertices removed. Moreover, in each example, one can replace the (large) independent sets by a $Y_{k,b}$-free $k$-graph. Thus, in the range where the second term above is significantly larger, it is reasonable to conjecture a tight bound $\left( \binom{n}{k} - \binom{n - s}{k} \right) + \text{ex}(n - s, Y_{k,b})$.

Overall, the study of $Y_{k,b}$-tilings is natural and important, because

- the problem generalizes the classical conjecture of Erdős on matchings;
- $Y_{k,b}$-free $k$-graphs were studied by Frankl and Füredi, which resolved a conjecture of Erdős;
- as we shall see later, $Y_{k,b}$-tilings have important applications on the Hamilton cycle problems in hypergraphs.

Now we present our results on $Y_{k,b}$-tilings. First, we use induction on $s$ and prove the following result for small $s$.

Theorem 1.3. For $k \geq 3$ and $1 \leq b \leq k - 1$ there exists $s_0 > 0$ such that the following holds whenever $s \geq s_0$. Let $H$ be an $n$-vertex $k$-graph and $n \geq (2(2k - b)^2 + 1)(k - 1)s + s$. If

$$e(H) \geq \binom{n}{k} - \binom{n - s + 1}{k} + \binom{n - 1}{k - 1} + \binom{n - 1}{k - 2}(2k - b)s,$$

then $H$ contains a $Y_{k,b}$-tiling of size $s$.

Note that the largest size $s$ of a $Y_{3,2}$-tiling when Theorem 1.3 applies is $n/67$. Our main result below verifies Conjecture 1.2 for $Y_{3,2}$-tilings of size $o(n)$ for any $0 < \alpha \leq 1/7$. Note that $Y_{3,2}$ was also denoted as $C_4^3$ (or $C_4^3$) and $C_2^3$ by other authors.

Theorem 1.4. For every $\alpha, \varepsilon \in (0, 1/7]$ there exists $n_0$ such that the following holds for integer $n \geq n_0$. Let $H$ be a $3$-graph of order $n$ such that

$$e(H) \geq \binom{n}{3} - \binom{n - \alpha n}{3} + \varepsilon n^3.$$

Then $H$ contains a $Y_{3,2}$-tiling covering $4\alpha n$ vertices.

The minimum 2-degree threshold forcing perfect $Y_{3,2}$-tilings has been studied in [34, 7] and the corresponding minimum vertex-degree threshold was determined in [27]. The study of $Y_{3,2}$-tilings is also motivated by its connection to Hamilton $\ell$-cycles in $k$-graphs (see e.g. [28]), which we describe in the next subsection. At last, we remark that optimal minimum-degree-type tiling results are rather rare, not to mention edge-density-type results (as far as we know, known results are essentially on matchings).

1Under those notation they mean 3-uniform loose cycles on four vertices, or 3-uniform loose cycles with two edges.

2Theorem 1.4 can not be implied by Conjecture 1.1 easily. Indeed, The Erdős-Matching-Conjecture with $e(H) \geq \binom{n}{3} - \binom{n - s}{3}$ gives a matching of size $s$, which after regularity, gives only a $Y_{3,2}$-tiling of size $3s/4$, or equivalently, a $Y_{3,2}$-tiling covering $3s$ vertices, rather than $4s$. 

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1.2. Hamilton cycles in \( k \)-graphs.\ As applications of Theorems 1.3 and 1.4, we prove new results on the Hamilton \( \ell \)-cycle problem. The existence of Hamilton cycles in graphs is a fundamental problem of graph theory with a rich history. The classical theorem of Dirac [11] states that every graph \( G \) on \( n \geq 3 \) vertices with minimum degree \( \delta(G) \geq n/2 \) contains a Hamilton cycle. In recent years, researchers have worked on extending this result to hypergraphs (see recent surveys, [35, 42]). Given a \( k \)-graph \( H = (V, E) \) with a vertex set \( S \in \binom{V}{\ell} \), we denote by \( N(S) \) the family of \( T \in \binom{V}{\ell} \) such that \( T \cup S \in E \) and \( \deg_H(S) := |N(S)| \). The minimum \( d \)-degree \( \delta_d(H) \) of \( H \) is the minimum of \( \deg_H(S) \) over all \( d \)-element vertex sets \( S \) in \( H \).

For \( 1 \leq \ell < k \), a \( k \)-graph is called an \( \ell \)-cycle if there exists a cyclic ordering of its vertices such that every edge is composed of \( k \) consecutive vertices, two (vertex wise) consecutive edges share exactly \( \ell \) vertices. If the ordering is not cyclic, we call it an \( \ell \)-path and we say the first and last \( \ell \) vertices are the ends of the path. A \( k \)-graph on \( n \) vertices contains a Hamilton \( \ell \)-cycle if it contains an \( \ell \)-cycle as a spanning subhypergraph, so \( (k - \ell) \mid n \). Note that an \( \ell \)-cycle on \( n \) vertices contains exactly \( n/(k - \ell) \) edges.

We define the Dirac threshold \( h^\ell_d(k, n) \) to be the smallest integer \( h \) such that every \( n \)-vertex \( k \)-graph \( H \) satisfying \( \delta_d(H) \geq h \) contains a Hamilton \( \ell \)-cycle. Let \( h^\ell_d(k) := \limsup_{n \to \infty} h^\ell_d(k, n)/\binom{n}{k-\ell} \). Confirming a conjecture of Katona and Kierstead [31], Rödl, Ruciński, and Szemerédi [44, 43] determined \( h^\ell_{k-1}(k) = 1/2 \) for any fixed \( k \), that is, \( \delta_{k-1}(H) \geq (1/2 + o(1))n \) guarantees \( (k - 1) \)-Hamiltonicity. The asymptotical Dirac threshold \( h^\ell_{k-1}(k) \) for any \( 1 \leq \ell < k \) such that \( (k - \ell) \mid k \) follows as a consequence of this result and a construction of Markström and Ruciński. When \( (k - \ell) \nmid k \), the threshold \( h^\ell_{k-1}(k) \) was determined through a series of works [34, 32, 24, 33]. We collect these results in the following theorem.

**Theorem 1.5.** [24, 32, 33, 43, 44] For any \( k > \ell \geq 1 \), we have

\[
h^\ell_{k-1}(k) = \begin{cases} 
1/2 & \text{if } (k - \ell) \mid k \\
1/(k-\ell) & \text{if } (k - \ell) \nmid k.
\end{cases}
\]

Some exact thresholds \( h^\ell_{k-1}(k, n) \) are known: for \( k = 3 \) and \( \ell = 2 \) [45] and for \( k \geq 3 \) and \( 1 \leq \ell < k/2 \) [8, 26]. For \( d = k - 2 \), the exact thresholds are known for \( 1 \leq \ell < k/2 \) [5, 28, 2, 3], but let us only state the asymptotical threshold \( h^\ell_{k-2}(k) \) here.

**Theorem 1.6.** [5, 2] For integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \), we have

\[
h^\ell_{k-2}(k) = 1 - \left( 1 - \frac{1}{2(k - \ell)} \right)^2.
\]

Recently, Polcyn, Reiner, Rödl, Ruciński, Schacht and Szemerédi [41, 39] showed the asymptotical Dirac threshold of tight Hamilton cycle in 3-graph and 4-graph, that is, \( h^3_3(3) = h^4_4(4) = 5/9 \). Lang and Sanhueza-Matamala [36] proved that \( h^\ell_{k-2}(k) = 5/9 \) for all \( k \geq 3 \) (the same result was also proved independently by Polcyn, Reigner, Rödl, and Schülke [40]) and also provided a general upper bound of \( 1 - 1/(2(k - d)) \) for \( h^\ell_{k-1}(k) \), narrowing the gap to the lower bound of \( 1 - 1/\sqrt{k - d} \) due to Han and Zhao [29]. For \( d \leq k - 3 \), Hán, Han and Zhao [23] determined the exact \( h^\ell_{d/2}(k, n) \) for any even integer \( k \geq 6 \), integer \( d \) such that \( k/2 \leq d \leq k - 1 \).

In this paper we determine more thresholds for \( d \leq k - 3 \) and \( \ell < k/2 \).

**Theorem 1.7.** Suppose that \( k \geq 3 \), \( k - \ell \leq d < 2\ell \leq k - 1 \) such that \( 2k - 2\ell \geq (2(2k - 2\ell - d)^2 + 1)(k - d - 1) + 1 \) or suppose that \( k \) is odd, \( k \geq 7 \), \( \ell = (k - 1)/2 \) and \( d = k - 3 \), then

\[
h^\ell_d(k) = 1 - \left( 1 - \frac{1}{2(k - \ell)} \right)^{k-d}.
\]
The proof of Theorem 1.7 relies on the results in Theorems 1.3 and 1.4. Indeed, under the popular framework of "absorbing method", one key step is to find a constant number of vertex-disjoint \( \ell \)-paths whose union covers almost all the vertices. To obtain a large path cover one usually uses the regularity method and reduces it to finding an almost perfect \( Y_{k,2\ell} \)-tiling\(^3\) in the reduced \( k \)-graph. In our proof, we further reduce the problem by fractional matching-covering argument to finding a large (fractional) \( Y_{k-d,b-d} \)-tiling of given size, so that we could apply Theorems 1.3 and 1.4.

The other two important components of the absorbing method are the absorbing lemma and the connecting lemma. It has been observed in [25] that the regularity approach used by Kühn, Mycroft and Osthus [33], where they derived an absorbing lemma and a connecting lemma for \( d = k - 1 \), can be adapted to prove a connecting lemma for \( d \geq \ell + 1 \). Here we further note that the approach in [33] can actually establish the absorbing lemma for \( d \geq \max\{k - \ell, \ell + 1\} \). Therefore, when \( \ell < k/2 \) and \( d \geq \max\{k - \ell, \ell + 1\} = k - \ell \), it suffices to study the path cover problem via an almost perfect \( Y_{k,2\ell} \)-tiling.

To formulate this, let us introduce the following notation. For any \( 0 < \varepsilon < \eta \leq k \geq 3, 1 \leq \ell < k/2 \) and \( 1 \leq d \leq k - 1 \), let \( t^*_\ell(k, n, \varepsilon) \) denote the minimum \( t \) such that every \( k \)-graph \( H \) of order \( n \) with \( \delta_d(H) \geq t \) contains a \( Y_{k,2\ell} \)-tiling covering all but at most \( \varepsilon n \) vertices of \( H \). Let
\[
 t(k, d, \ell) := \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} t^*_\ell(k, n, \varepsilon) \cdot \frac{\binom{n}{k-d}}{\binom{k-d}{k-d}}.
\]

We have the following result.

**Theorem 1.8.** For all \( k \geq 3, 1 \leq \ell < k/2, k - \ell \leq d \leq k - 1 \), we have \( h^*_d(k) \leq t(k, d, \ell) \).

Note that Theorem 1.8 implies that \( h^*_d(k, n) \leq (t(k, d, \ell) + \eta)\binom{n}{k-d} \). The following proposition, usually called space barriers, implies that \( h^*_d(k) \geq 1 - (1 - \frac{1}{2(k-\ell)})^{k-d} \).

**Proposition 1.9.** For all \( k \geq 3, 1 \leq \ell < k/2, 1 \leq d \leq k - 1 \) and every \( n \) with \( (k - \ell) \mid n \), there exists an \( n \)-vertex \( k \)-graph \( H_0 \) with \( \delta_d(H_0) \geq (1 - (1 - \frac{1}{2(k-\ell)})^{k-d} - o(1))\binom{n}{k-d} \) containing no Hamilton \( \ell \)-cycle. In other words, \( h^*_d(k) \geq 1 - (1 - \frac{1}{2(k-\ell)})^{k-d} \).

**Proof.** Let \( t = n/(k - \ell) \). Let \( H_0 = (V, E) \) be an \( n \)-vertex \( k \)-graph in which \( V \) is partitioned into two sets \( A \) and \( B \) such that \( |A| = \lceil t/2 \rceil - 1 \) and \( E \) consists of all \( k \)-sets that intersect \( A \). Thus
\[
 \delta_d(H_0) \geq \left( \frac{n-d}{k-d} \right) - \left( \frac{n-|A|-d}{k-d} \right) = (1 - (1 - \frac{1}{2(k-\ell)})^{k-d} - o(1))\binom{n}{k-d}.
\]

If \( H_0 \) contains a Hamilton \( \ell \)-cycle \( C \), then each vertex is contained in at most two edges of \( C \). Since \( A \) is a vertex cover of \( C \), we have \( |C| \leq 2|A| < t \), which is a contradiction. \( \square \)

In the proof above, if we decrease the size of \( A \) by \( o(n) \), then the size of a maximum \( Y_{k,2\ell} \)-tiling (which equals \( |A| \)) also decreases by the same amount, giving that \( t(k, d, \ell) \geq 1 - (1 - \frac{1}{2(k-\ell)})^{k-d} \). Known evidences help us conjecture that equalities should hold.

**Conjecture 1.10.** For all \( k \geq 3, 1 \leq \ell < k/2, k - \ell \leq d \leq k - 1 \), we have \( h^*_d(k) = t(k, d, \ell) = 1 - (1 - \frac{1}{2(k-\ell)})^{k-d} \).

By Theorem 1.8, to verify Conjecture 1.10 it suffices to prove that \( t(k, d, \ell) \leq 1 - (1 - \frac{1}{2(k-\ell)})^{k-d} \), which, by our approach (the fractional matching-covering reduction presented later), would follow from the validation of a range in Conjecture 1.2 (see Theorem 7.1).

The rest of this paper is organized as follows. We use the absorbing method to prove Theorem 1.8 in Section 2. The hypergraph regularity lemma is introduced in Section 3, which is used to prove

\( ^3 \)It is possible to use \( F \)-tilings for other appropriate \( k \)-graph \( F \) to produce the path cover.
the absorbing path lemma and the path cover lemma in Appendix A and Section 4 respectively. In Sections 5 and 6, we prove Theorems 1.3 and 1.4, and use them to derive Theorem 1.7.

2. Proof of Theorem 1.8

2.1. Auxiliary lemmas. We build the Hamilton \(\ell\)-cycle by the absorbing method, popularized by Rödl, Ruciński, and Szemerédi in [43]. More precisely, we divide the proof of Theorem 1.8 into the following lemmas: the connecting lemma (Lemma 2.1), the absorbing path lemma (Lemma 2.2), the path cover lemma (Lemma 2.3), and the reservoir lemma (Lemma 2.4).

The connecting lemma states that in any sufficiently large \(k\)-graph with large minimum \(d\)-degree, we can connect any two disjoint ordered \(\ell\)-sets of vertices by a short \(\ell\)-path. We remark that the proof of Lemma 2.1 in [25] follows closely that of a connecting lemma in [33] with minor modifications. We write \(x \ll y \ll z\) to mean that we can choose constants from right to left, that is, for any \(z > 0\), there exist functions \(f\) and \(g\) such that, whenever \(y \leq f(z)\) and \(x \leq g(y)\), the subsequent statement holds. Statements with more variables are defined similarly.

Lemma 2.1 (Connecting lemma [25], Lemma 4.1). Suppose that \(k \geq 3\) and \(1 \leq \ell < d \leq k - 1\) such that \((k - \ell) \nmid k\), and that \(1/n \ll \beta \ll \mu, 1/k\). Let \(H\) be a \(k\)-graph of order \(n\) satisfying \(\delta_d(H) \geq \mu(n^{-d})\). Then for any two disjoint ordered \(\ell\)-sets \(S\) and \(T\) of vertices of \(H\), there exists an \(\ell\)-path \(P\) in \(H\) from \(S\) to \(T\) such that \(P\) contains at most \(8k^5\) vertices.

Let \(H\) be an \(n\)-vertex \(k\)-graph, and let \(S\) be a set of \(k - \ell\) vertices of \(H\). An \(\ell\)-path \(P\) in \(H\) with ordered ends \(P^{\text{beg}}\) and \(P^{\text{end}}\) absorbs \(S\) if \(P\) does not contain any vertex of \(S\), and \(H\) contains an \(\ell\)-path \(Q\) with the same ordered ends \(P^{\text{beg}}\) and \(P^{\text{end}}\), where \(V(Q) = V(P) \cup S\). Note that the number of vertices we absorb each time must be a multiple of \(k - \ell\), because the order of an \(\ell\)-path is always in the form of \(\ell + r(k - \ell)\), where \(r \in \mathbb{N}\) is the length (i.e. the number of edges) of the path. We follow [33] and use the absorbing path of order \(b(k, \ell)\) with \(b(k, \ell) \leq k^4\), whose definition can be found in Appendix A. We say that a \((k - \ell)\)-set \(S\) of \(V(H)\) is \(c\)-good (otherwise \(c\)-bad) if \(H\) contains at least \(cnb^{b(k, \ell)}\) absorbing paths for \(S\), each on \(b(k, \ell)\) vertices.

Lemma 2.2 (Absorbing path lemma). Suppose \(k \geq 3\), \(1 \leq \ell, d \leq k - 1\) such that \(d \geq \max\{k - \ell, \ell + 1\}\) and \((k - \ell) \nmid k\), and that \(1/n \ll \alpha \ll c \ll \gamma \ll \mu' \ll \mu, 1/k\). Let \(H\) be a \(k\)-graph of order \(n\) with \(\delta_d(H) \geq \mu(n^{-d})\). Then \(H\) contains an \(\ell\)-path \(P\) on at most \(\mu'n\) vertices such that the following properties hold:

1. Every vertex of \(H - V(P)\) lies in at most \(\gamma n^{k-\ell-1}\) \(c\)-bad \((k - \ell)\)-sets,
2. \(P\) can absorb any collection of at most \(\alpha n\) disjoint \(c\)-good \((k - \ell)\)-sets of vertices of \(H - V(P)\).

The following lemma states that any sufficiently large \(k\)-graph satisfying the minimum degree condition can be almost covered by a constant number of vertex-disjoint \(\ell\)-paths.

Lemma 2.3 (Path cover lemma). Suppose that \(k \geq 3\), \(1 \leq \ell < k/2\), \(1 \leq d \leq k - 1\), and \(1/n \ll 1/D \ll \varepsilon \ll \mu, 1/k\). Let \(H\) be a \(k\)-graph of order \(n\) with \(\delta_d(H) \geq (\ell(k, d, \ell) + \mu)(n^{-d})\). Then \(H\) contains a set of at most \(D\) disjoint \(\ell\)-paths covering all but at most \(4\varepsilon n\) vertices of \(H\).

We also need the following two results in the proof of Theorem 1.8. The first result asserts that for \(1 \leq d \leq k - 1\), if \(H\) is a large \(k\)-graph with minimum \(d\)-degree and we choose \(R \subseteq V(H)\) uniformly at random, then with high probability all sets of \(d\) vertices have a large degree in \(R\), ensuring that we can use vertices of \(R\) to connect every two disjoint \(\ell\)-sets.

Lemma 2.4 (Reservoir lemma [33], Lemma 8.1). Suppose that \(k \geq 2\), \(1 \leq d \leq k - 1\), and \(1/n \ll \alpha, \mu, 1/k\). Let \(H\) be a \(k\)-graph of order \(n\) with \(\delta_d(H) \geq \mu(n^{-d})\), and let \(R\) be a subset of \(V(H)\) of size \(\alpha n\) chosen uniformly at random. Then the probability that \(|N_H(S) \cap (S^R_{k-d})| \geq \mu(n^{-d}) - n^{k-d-1/3}\) for every \(S \in \binom{V(H)}{d}\) is \(1 - o(1)\).
The second result (Daykin and Häggkvist [9]) gives a minimum vertex degree condition that guarantees the existence of a perfect matching in a uniform hypergraph.

**Lemma 2.5.** [9] Suppose that \( k \geq 2 \) and \( k \mid n \). Let \( H \) be a \( k \)-graph of order \( n \) with \( \delta_1(H) \geq \frac{k-1}{k}(\binom{n}{k-1}) \). Then \( H \) contains a perfect matching.

Assuming these lemmas, there is a routine scheme of the absorbing method to prove Theorem 1.8. In our specific case, the proof follows closely the argument in [33].

### 2.2. Proof of Theorem 1.8

Suppose we have the constants satisfying the following hierarchy

\[
\frac{1}{n} \ll \frac{1}{D} \ll \varepsilon \ll \alpha \ll c \ll \gamma \ll \gamma' \ll \eta \ll \frac{1}{k}
\]

and in particular assume that \( n \) is a multiple of \( k - \ell \). Let \( H \) be a \( k \)-graph on \( n \) vertices such that \( \delta_d(H) \geq \left( t(k,d,\ell) + \eta \right) \binom{n}{k-d} \). A Hamilton \( \ell \)-cycle of \( H \) will be constructed via the following steps.

**Build an absorbing path.** Let \( P_0 \) be an absorbing \( \ell \)-path returned by Lemma 2.2. Then in particular, \( |V(P_0)| \leq \gamma n \) and \( P_0 \) can absorb any set of at most \( \alpha n \)-good \( (k - \ell) \)-sets. Define the auxiliary \( (k - \ell) \)-graph \( G \) on \( V(H) \) such that \( E(G) \) consists of all \( \alpha n \)-good \( (k - \ell) \)-sets of \( V(H) \).

By (1) of Lemma 2.2, we infer that \( \deg_G(v) \geq \left( \frac{n-1}{k-\ell-1} \right) - \gamma n^{k-\ell-1} \geq \left( 1 - \gamma' \right) \binom{n}{k-\ell-1} \) for every \( v \in V(G) \setminus V(P_0) \).

**Choose a reservoir set.** We choose a set of \( \alpha n \) vertices uniformly at random from \( H \), denoted by \( R \). Applying Lemma 2.4 to \( H \) and \( G \), we obtain that with probability \( 1 - o(1) \),

\[
\left| N_G(v) \cap \binom{R}{k-\ell-1} \right| \geq \left( 1 - 2\gamma' \right) \binom{\alpha n}{k-\ell-1}
\]

for every \( v \in V(G) \setminus V(P_0) \) and

\[
\left| N_H(S) \cap \binom{R}{k-d} \right| \geq \left( t(k,d,\ell) + \eta/2 \right) \binom{\alpha n}{k-d}
\]

for every \( S \in \binom{V(H)}{d} \). Since \( \mathbb{E}(|R \cap V(P_0)|) = \alpha |V(P_0)| \), by Markov’s inequality, with probability at least \( 1/2 \), we have \( |R \cap V(P_0)| \leq 2\gamma' \alpha n \). Then we fix a choice of \( R \) which has all three properties above. Let \( R' = R \setminus V(P_0) \), and thus \( |R'| \geq (1 - 2\gamma') \alpha n \). Then for every \( S \in \binom{V(H)}{d} \), we have

\[
\left| N_H(S) \cap \binom{R'}{k-d} \right| \geq \left( t(k,d,\ell) + \eta/2 \right) \binom{\alpha n}{k-d} - 2\gamma' \alpha n \binom{n}{k-d-1} \geq t(k,d,\ell) \binom{|R'|}{k-d}.
\]

Similarly, \( \left| N_G(v) \cap \binom{R'}{k-\ell-1} \right| \geq \left( 1 - \eta \right) \binom{|R'|}{k-\ell-1} \) for every vertex \( v \) in \( V(G) \setminus V(P_0) \).

**Cover the majority of vertices by vertex-disjoint \( \ell \)-paths.** Let \( V' := V(H) \setminus (R \cup V(P_0)) \), and let \( H' := H[V'] \) be the restriction of \( H \) on \( V' \). As \( |R \cup V(P_0)| \leq 2\gamma' \alpha n \), we have \( \delta_d(H') \geq \left( t(k,d,\ell) + \eta/2 \right) \binom{|H'|}{k-d} \). Applying Lemma 2.3 to \( H' \), we obtain a collection of at most \( D \) vertex-disjoint \( \ell \)-paths \( P_1, \ldots, P_D \), covering all but at most \( \varepsilon n \) vertices of \( H' \). Let \( X \) denote the set of uncovered vertices. Thus \( |X| \leq \varepsilon n \).

**Connect up all \( \ell \)-paths.** Denote by \( P_i^{\text{beg}} \) and \( P_i^{\text{end}} \) the ordered ends of \( P_i \), \( 0 \leq i \leq q \). For \( 0 \leq i \leq q \), we then find disjoint \( \ell \)-paths \( P_i^{\text{end}} \) by Lemma 2.1 to connect \( P_i^{\text{end}} \) and \( P_{i+1}^{\text{beg}} \) (where subindices are taken modulo \( q + 1 \)), which will connect \( P_i \) and \( P_{i+1} \). Note that \( V(P_i') \subseteq R' \cup P_i^{\text{end}} \cup P_{i+1}^{\text{beg}} \) and \( |V(P_i')| \leq 8k^5 \). More precisely, suppose that we have chosen such \( \ell \)-paths \( P_0', \ldots, P_{i-1}' \). Let \( R_i = \left( R' \cup P_i^{\text{end}} \cup P_{i+1}^{\text{beg}} \right) \setminus \bigcup_{j=0}^{i-1} V(P_j') \). Thus

\[
\delta_d(H[R_i]) \geq \left( t(k,d,\ell) + \eta/2 \right) \binom{\alpha n}{k-d} - 8k^5(D + \gamma' n) \binom{\alpha n + 2\ell}{k-d-1} \geq \frac{t(k,d,\ell)}{2} \binom{\alpha n}{k-d}
\]

and thus we may apply Lemma 2.1 to find a desired \( \ell \)-path \( P_i' \).
Absorb the leftover. Let \( C = P_0 P_0' P_1 P_1' \cdots P_q P_q' \) be the \( \ell \)-cycle we have obtained so far. Let \( R'' := V(H) \setminus V(C) \). Then indeed \( R'' = X \cup (R' \setminus \bigcup_{q \leq q' \leq q} V(P_q')) \) and in particular, \((1 - 3\varepsilon)\alpha n \leq |R''| \leq (\alpha + \varepsilon)n\). Because \( k - \ell \) divides both \( n \) and \( |V(C)| \), we get \((k - \ell) \mid |R''|\). Moreover, for each \( v \in R'' \), we have \( |\mathcal{N}_{G[R'']}(v)| \geq (1 - 2\varepsilon)(|R''| / (k - \ell + 1)) \). Since \((k - \ell) \nmid k\), we get \((k - \ell) \geq 2\). By Lemma 2.5, the graph \( G[R''] \) contains a perfect matching which allows us to partition the set \( R'' \) into \((at most \alpha n)\) \( \ell \)-good \((k - \ell)\)-sets. Utilizing the absorbing property of \( P_0 \), there exists an \( \ell \)-path \( Q_0 \) with \( V(Q_0) = V(P_0) \cup R'' \) such that \( P_0 \) and \( Q_0 \) have the same ordered ends. Thus a Hamilton \( \ell \)-cycle \( C' = Q_0 P_0' P_1 P_1' \cdots P_q P_q' \) in \( H \) is obtained.

3. The hypergraph regularity method

We review the hypergraph regularity method, starting with some notation.

3.1. Regular complexes. A hypergraph \( H \) is a complex if whenever \( e \in E(H) \) and \( e' \in E(H) \) is a non-empty subset of \( e \) we have that \( e' \in E(H) \). All the complexes considered in this paper have the property that every vertex is contained in an edge. A complex \( H \) is a \( k \)-complex if all the edges of \( H \) consist of at most \( k \) vertices. The edges of size \( i \) are called \( i \)-edges of \( H \). For convenience, we write \(|H| := |V(H)|\) for the order of \( H \). Given a \( k \)-complex \( H \), for each \( i \in [k] \) we denote by \( H_i \) the underlying \( i \)-graph of \( H \), where the vertices of \( H_i \) are those of \( H \) and the edges of \( H_i \) are the \( i \)-edges of \( H \).

Note that a \( k \)-graph \( H \) can be naturally turned into a \( k \)-complex, denoted by \( H \leq \leq \), by replacing every edge into a complete \( i \)-graph \( K^i_k \), for each \( 1 \leq i \leq k \). Given \( k \leq s \), a \((k, s)\)-complex \( H \) is an \( s \)-partite \( k \)-complex, by which we mean that the vertex set of \( H \) can be partitioned into sets \( V_1, \ldots, V_s \) such that every edge of \( H \) meets each \( V_i \) in at most one vertex.

Now we motivate the notion of the relative density. For \( i \geq 2 \), suppose \( H_i \) is an \( i \)-partite \( i \)-graph and \( H_{i-1} \) is an \( i \)-partite \((i - 1)\)-graph, defined on the same vertex set (also under the same partition into \( i \) parts). We denote by \( \mathcal{K}_i(H_{i-1}) \) the set of \( i \)-sets of vertices which form a copy of the complete \((i - 1)\)-graph \( K^{(i-1)}_i \) on \( i \) vertices in \( H_{i-1} \). Then the density of \( H_i \) with respect to \( H_{i-1} \) is defined to be

\[
d(H_i | H_{i-1}) := \begin{cases} \frac{|\mathcal{K}_i(H_{i-1}) \cap E(H_i)|}{|\mathcal{K}_i(H_{i-1})|} & \text{if } |\mathcal{K}_i(H_{i-1})| > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

More generally, if \( Q := (Q(1), Q(2), \ldots, Q(r)) \) is a collection of \( r \) subhypergraphs of \( H_{i-1} \), then we define \( \mathcal{K}_i(Q) := \bigcup_{j=1}^r \mathcal{K}_i(Q(j)) \) and

\[
d(H_i | Q) := \begin{cases} \frac{|\mathcal{K}_i(Q) \cap E(H_i)|}{|\mathcal{K}_i(Q)|} & \text{if } |\mathcal{K}_i(Q)| > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

We say that \( H_i \) is \((d_i, \delta, r)\)-regular with respect to \( H_{i-1} \) if for all \( r \)-tuples \( Q \) with \( |\mathcal{K}_i(Q)| > \delta|\mathcal{K}_i(H_{i-1})| \) it holds that \( d(H_i | Q) = d_i \pm \delta \). Instead of \((d_i, \delta, 1)\)-regularity we simply refer to \((d_i, \delta)\)-regularity.

Given \( 3 \leq k \leq s \) and a \((k, s)\)-complex \( H \), we say that \( H \) is \((d_k, \ldots, d_2, \delta_k, \delta, r)\)-regular if the following conditions hold:

- For every \( i = 2, \ldots, k - 1 \) and every \( i \)-tuple \( K \) of vertex classes either \( H_i[K] \) is \((d_i, \delta)\)-regular with respect to \( H_{i-1}[K] \) or \( d(H_i[K] | H_{i-1}[K]) = 0 \).
- For every \( k \)-tuple \( K \) of vertex classes either \( H_k[K] \) is \((d_k, \delta_k, r)\)-regular with respect to \( H_{k-1}[K] \) or \( d(H_k[K] | H_{k-1}[K]) = 0 \).

Here we write \( H_i[K] \) for the restriction of \( H_i \) to the union of all vertex classes in \( K \). We sometimes denote \((d_k, \ldots, d_2) \) by \( d \) and simply refer to \((d, \delta_k, \delta, r)\)-regularity.
3.2. Statement of the regularity lemma. In this section we state the version of the regularity lemma for $k$-graphs due to Rödl and Schacht, which we will use to prove our connecting lemma. To prepare for this we again have to introduce some notation. Suppose that $V$ is a finite set of vertices and $\mathcal{P}^{(1)}$ is a partition of $V$ into sets $V_1, \ldots, V_{a_1}$, called clusters. Given $k \geq 3$ and any $j \in [k]$, we denote by $\text{Cross}_j = \text{Cross}_j(\mathcal{P}^{(1)})$, the set of all those $j$-subsets $J$ of $V$ such that $|J \cap V_i| \leq 1$ for all $1 \leq i \leq a_1$. For every set $A \subseteq [a_1]$ with $2 \leq |A| \leq k - 1$ we write $\text{Cross}_A$ for all those $|A|$-subsets of $V$ that meet each $V_i$ with $i \in A$. Let $\mathcal{P}_A$ be a partition of $\text{Cross}_A$. The partition classes of $\mathcal{P}_A$ is called cells. For $i = 2, \ldots, k - 1$, let $\mathcal{P}^{(i)}$ be the union of all $\mathcal{P}_A$ with $|A| = i$. Thus $\mathcal{P}^{(i)}$ is a partition of $\text{Cross}_i$.

Let $\mathcal{P}(k - 1) = \{\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k - 1)}\}$ be a family of partitions on $V$ which satisfies the following condition. Recall that $a_1$ denotes the number of clusters in $\mathcal{P}^{(1)}$. Consider any $B \subseteq A \subseteq [a_1]$ such that $2 \leq |B| < |A| \leq k - 1$ and suppose that $S, T \in \text{Cross}_A$ lie in the same cell of $\mathcal{P}_A$. Let $S_B := S \cap \bigcup_{i \in B} V_i$ and define $T_B$ similarly. Then $S_B$ and $T_B$ lie in the same cell of $\mathcal{P}_B$.

Given $1 \leq i \leq j \leq k$ with $i < k$, $J \in \text{Cross}_j$ and an $i$-set $Q \subseteq J$, we write $C_Q$ for the set of all those $i$-sets in $\text{Cross}_i$ that lie in the same cell of $\mathcal{P}^{(i)}$ as $Q$. (In particular, if $i = 1$ then $C_Q$ is the cluster containing the unique element in $Q$.) The polyad $\hat{\mathcal{P}}^{(i)}(J)$ of $J$ is defined by $\hat{\mathcal{P}}^{(i)}(J) := \bigcup_Q C_Q$, where the union is over all $i$-subsets $Q$ of $J$. That is, we can view $\hat{\mathcal{P}}^{(i)}(J)$ as a $j$-partite $i$-graph (whose vertex classes are the clusters intersecting $J$). We let $\hat{\mathcal{P}}^{(j - 1)}(J)$ be the set consisting of all the $\hat{\mathcal{P}}^{(j - 1)}(J)$ for all $J \in \text{Cross}_j$. So for each $K \in \text{Cross}_k$ we can view $\bigcup_{i=1}^{k-1} \mathcal{P}^{(i)}(K)$ as a $(k - 1, k)$-complex. We say that the family of partitions $\mathcal{P} = \mathcal{P}(k - 1)$ is $(\eta, \delta, t)$-equitable if

- there exists $d = (d_{k - 1}, \ldots, d_2)$ such that $d_i \geq 1/t$ and $1/d_i \in \mathbb{N}$ for all $i = 2, \ldots, k - 1$,
- $\mathcal{P}^{(1)}$ is a partition of $V$ into $a_1$ clusters of equal size, where $1/\eta \leq a_1 \leq t$,
- for all $i = 2, \ldots, k - 1$, $\mathcal{P}^{(i)}$ is a partition of $\text{Cross}_i$ into at most $t$ cells,
- for every $K \in \text{Cross}_k$, the $(k - 1, k)$-complex $\bigcup_{i=1}^{k-1} \mathcal{P}^{(i)}(K)$ is $(d, \delta, \delta, 1)$-regular.

Note that the last condition implies that for $i = 2, \ldots, k - 1$ the cells of $\mathcal{P}^{(i)}$ have almost equal size.

Let $\delta_k > 0$ and $r \in \mathbb{N}$. Suppose that $H$ is a $k$-graph on $V$ and $\mathcal{P} = \mathcal{P}(k - 1)$ is a family of partitions on $V$. Given a polyad $\hat{\mathcal{P}}^{(k - 1)} \in \mathcal{P}(k - 1)$, we say that $H$ is $(\delta_k, r)$-regular with respect to $\hat{\mathcal{P}}^{(k - 1)}$ if $H = (d, \delta_k, r)$-regular with respect to $\hat{\mathcal{P}}^{(k - 1)}$ for some $d$. We say that $H$ is $(\delta_k, r)$-regular with respect to $\mathcal{P}$ if

$$ \left| \bigcup_{K \in \mathcal{P}^{(k - 1)}} |K_k(\hat{\mathcal{P}}^{(k - 1)}) : H \text{ is not } (\delta_k, r)\text{-regular with respect to } \hat{\mathcal{P}}^{(k - 1)} \right| \leq \delta_k|V|^k. $$

That is, at most a $\delta_k$-fraction of the $k$-subsets of $V$ form a $K_k^{(k - 1)}$ that lies within a polyad with respect to which $H$ is not regular.

Now we are ready to state the regularity lemma.

**Theorem 3.1** (Regularity lemma [46], Theorem 17). Let $k \geq 3$ be a fixed integer. For all positive constants $\eta$ and $\delta_k$ and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to (0, 1)$, there are integers $t$ and $n_0$ such that the following holds for all $n \geq n_0$ which are divisible by $t!$. Suppose that $H$ is a $k$-graph of order $n$. Then there exists a family of partitions $\mathcal{P} = \mathcal{P}(k - 1)$ of the vertex set $V$ of $H$ such that

1. $\mathcal{P}$ is $(\eta, \delta(t), t)$-equitable and
2. $H$ is $(\delta_k, r(t))$-regular with respect to $\mathcal{P}$.

3.3. The reduced $k$-graph. The reduced $k$-graph of a regular partition is an important auxiliary $k$-graph which has been used in many applications of the regularity method\(^4\). Suppose that we have constants $1/n \ll 1/r, \delta \ll \min\{\delta_k, 1/t\} \leq \delta_k, \eta \ll d' \ll \theta \ll \mu, 1/k$ and a $k$-graph $H$ on $V$ of order $n$. We may apply the regularity lemma to $H$ to obtain a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k - 1)}\}$

\(^4\)Note also that there have been a few different notions of such auxiliary hypergraphs, depending on the contexts.
of $V$. Then the reduced $k$-graph $R = R(H, P, d')$ is defined as a $k$-graph whose vertices are the clusters of $H$, i.e. the parts of $P(1) : V_1, V_2, \ldots, V_{q}$. A $k$-tuple of clusters forms an edge of $R$ if there is some polyad $\hat{P}^{(k-1)}$ induced on these $k$ clusters such that $H$ is $(d''_i, \delta_k, r)$-regular with respect to $\hat{P}^{(k-1)}$ for some $d''_i \geq d'$. Suppose that $e$ is an edge of $R$, then there is some polyad $\hat{P}^{(k-1)}(J)$ (where $J \in \text{Cross}_k$) induced by the $k$ clusters corresponding to $e$ such that $H$ is $(d''_i, \delta_k, r)$-regular with respect to $\hat{P}^{(k-1)}(J)$ for some $d''_i \geq d'$. Let $H^*$ be the $(k, k)$-complex consisting of
\[
\bigcup_{i=1}^{k-1} \hat{P}^i(J) \cup (E(H) \cap K(\hat{P}^{(k-1)}(J))).
\]

Then $H^*$ is $(\mathbf{d}, \delta_k, \delta, r)$-regular, where $\mathbf{d} = (d''_1, d''_2, \ldots, d''_k)$, and $d''_1, \ldots, d''_k$ are implicitly given in the definition of an $(\eta, \delta, t)$-equitable family of partitions.

It is well-known that the reduced $k$-graph $R$ almost inherits the minimum degree condition and the density condition from $H$.

**Lemma 3.2.** Suppose that $k \geq 3$, $1 \leq d \leq k - 1$, and $\gamma, d' \ll \theta \ll \mu$. Let $H$ be a $k$-graph of order $n$ and $R := R(H, P, d')$ be the reduced $k$-graph of $H$, where $P$ is the family of partitions we obtained by Theorem 3.1. If $\delta_d(H) \geq (\mu + \theta)(\binom{n}{k - d})$, then all but at most $\gamma \binom{|R|}{d}$ $d$-sets $S \in \binom{|R|}{d}$ have degree at least $(\mu + \theta/2)\binom{|R|}{k - d}$. Moreover, if $e(H) \geq (\mu + \theta)(\binom{r}{k})$, then $e(R) \geq (\mu + \theta/2)(\binom{r}{k})$.

The proof of Lemma 3.2 is by now a routine calculation and thus omitted. Similar calculations can be found e.g. [33].

4. The proof of Lemma 2.3

In this section we prove Lemma 2.3. Here is a quick outline of our proof, which also follows the use of the regularity method as in previous works. We apply the regularity lemma to $H$ and obtain a regular partition together with a reduced $k$-graph $R$. Since $R$ almost inherits the minimum $d$-degree of $H$, by the definition of $t(k, d, \ell)$, we can find an almost perfect $Y_{k,2\ell}$-tiling in $R$. Thus, each copy of $Y_{k,2\ell}$ corresponds to $2k - 2\ell$ clusters in the partition and the regularity allows us to cover almost all vertices of these clusters by a bounded number of vertex-disjoint $\ell$-paths. Altogether we get the desired path cover in Lemma 2.3.

**Lemma 4.1** ([16], Lemma 3.4). Consider an $n$-vertex $k$-graph $G$, where all but $\delta(\binom{n}{d})$ of the $d$-sets have degree at least $(\mu + \eta)\binom{n - d}{k - d}$. Let $S$ be a uniformly random subset of $Q$ vertices of $G$. Then with probability at least $1 - \binom{|Q|}{d}(\delta + \exp(\Omega(n^2Q)))$, the random induced subgraph $G[S]$ has minimum $d$-degree at least $(\mu + \eta/2)\binom{Q - d}{k - d}$.

**Fact 1.** Let $P$ be an $\ell$-path on $n$ vertices with $t := \frac{n - \ell}{k - \ell}$ edges, where $t$ is odd. Then there is a partition $V_1, V_2, \ldots, V_k$ of $V(P)$ such that $|V_i| = t$, $i \in [k - 2\ell]$ and $|V_j| = (t + 1)/2$, $j \in [k] \setminus [k - 2\ell]$, and each edge of $P$ contains exactly one vertex of each $V_i, i \in [k]$.

**Proof.** Denote $P$ as $x_1, \ldots, x_n$. Color $x_i$ with $i$ for $i \in [k]$. Let
\[
C = \{1, \ldots, k - 2\ell, k - 2\ell + 1, \ldots, k - \ell, 1, \ldots, k - 2\ell, k - \ell + 1, \ldots, k\}.
\]
Then we repeatedly use the pattern $C$ ($(t - 1)/2$ times) to color $x_{k+1}, \ldots, x_n$. Note that every edge contains $k$ colors where each color of $[k - 2\ell]$ is used $t$ times and every other color is used $(t - 1)/2 + 1$ times. Let $V_i$ be the color class with color $i$, which forms a partition as desired. □

We use the following fact to build our path in a regular $k$-tuple.

**Fact 2.** Let $k \geq 3$, $1 \leq \ell < k/2$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Suppose $H$ is a $k$-partite $k$-graph with vertex set $V_1, \ldots, V_k$ each of size $m$ and $e(H) \geq \varepsilon m^k$. Then $H$ contains an $\ell$-path $P$ with $t \geq \varepsilon m/2$ edges.
whose vertex set are partitioned as in Fact 1, namely, $P$ contains $t$ vertices in each of $V_1, \ldots, V_{k-2\ell}$ and $(t-1)/2 + 1$ vertices in every other cluster.

**Proof.** We call an $\ell$-set crossing if it contains at most one vertex from each $V_i$. From $H$ we iteratively do the following: in the current $k$-graph, if there is a crossing $\ell$-set $S$ such that $\deg(S) < \varepsilon m^{k-\ell}/(2\binom{k}{\ell})$, then we remove all edges containing $S$. The resulting $k$-partite $k$-graph has at least $\varepsilon m^k - m^\ell \cdot \varepsilon m^{k-\ell}/2 > 0$ edges, and every crossing $\ell$-set has degree either zero or at least $\varepsilon m^{k-\ell}/(2\binom{k}{\ell})$.

We now find the desired path $P$ greedily, starting from an arbitrary edge. We extend the path alternately from either an $\ell$-end in $V_{k-2\ell+1} \times \cdots \times V_{k-\ell}$ or $V_{k-\ell+1} \times \cdots \times V_k$, until $P$ has $\varepsilon m/2$ edges. All extensions will succeed because the current path will block at most $(\varepsilon m/2)^{k-\ell} < \varepsilon m^{k-\ell}/(2\binom{k}{\ell})$ edges, and an $\ell$-end has degree nonzero (so at least $\varepsilon m^{k-\ell}/(2\binom{k}{\ell}))$.

**Proof of Lemma 2.3.** Choose constants such that

$$1/n \ll 1/D \ll 1/r, \delta, c \ll \min\{\delta_k, 1/t\} \leq \delta_k, \eta \ll d' \ll \gamma \ll \varepsilon \ll \mu, 1/k.$$

We may assume that $t! \mid n$. Apply Theorem 3.1 to $H$, and let $V_1, \ldots, V_{a_1}$ be the clusters of the partition obtained. Let $m = n/a_1$ be the size of each of these clusters and $R$ be the reduced $k$-graph on these clusters. The next claim says $R$ has an almost perfect $Y_{k,2\ell}$-tiling $T$.

**Claim 4.2.** The $k$-graph $R$ has a $Y_{k,2\ell}$-tiling covering all but at most $\varepsilon a_1/2$ vertices of $R$.

**Proof.** By Lemma 3.2, all but at most $\gamma^{(a_1/\mu)}$ $d$-sets $S \in \binom{V(R)}{d}$ have degree at least $(t(k,d,\ell) + \mu/2)/\binom{a_1/\mu}{k-d}$. Choose $Q$ to be large enough. In particular, by the definition of $t(k,d,\ell)$ every $k$-graph $G$ of order $Q$ with $\delta_d(G) \geq (t(k,d,\ell) + \mu/4)/\binom{Q}{k-d}$ contains a $Y_{k,2\ell}$-tiling covering all but at most $\varepsilon Q/4$ vertices of $G$. Let $\lambda = \binom{Q}{d}(\gamma + \exp^{-\Omega((\mu/2)^2Q)})$, and note that we can make $\lambda < \varepsilon/4$ by making $Q$ large.

Now, we randomly partition $V_1, \ldots, V_{a_1}$ into $\lfloor a_1/Q \rfloor$ subsets of size $Q$ (and give up at most $Q-1$ of them). By Lemma 4.1, the expected number of $Q$-sets that have minimum $d$-degree less than $(t(k,d,\ell) + \mu/4)/\binom{Q}{k-d}$ is $\lambda a_1/Q$. Thus with positive probability all but a $\lambda$-fraction of the subsets have minimum $d$-degree at least $(t(k,d,\ell) + \mu/4)/\binom{Q}{k-d}$. As discussed above, each of these $Q$-subsets $S \subseteq V(G)$ has the property that $G[S]$ has a $Y_{k,2\ell}$-tiling that leaves at most $\varepsilon Q/4$ vertices. Combining these $Y_{k,2\ell}$-tilings gives a $Y_{k,2\ell}$-tiling of $R$ covering all but at most $Q-1 + \lambda a_1 + (\varepsilon Q/4)(a_1/Q) \leq \varepsilon a_1/2$ vertices. \hfill\QED
5. Proof of Theorem 1.3

We collect known results on the Turán number of $Y_{k,b}$.

**Theorem 5.1.** [15, 20] For $k > b \geq 0$, there exists an integer $n_k$ such that for any $n \geq n_k$, $\text{ex}(n, Y_{k,b}) \leq \binom{n-1}{k-1}$ holds.

For $b = 0$, Theorem 5.1 is just the Erdős–Ko–Rado Theorem [15]; for $1 \leq b \leq k-2$, it follows from a result of Frankl and Füredi [20] and in fact, a more precise bound $O(n_{\text{max}}^{b,k-1-b})$ is known; for $b = k-1$, a simple calculation gives $\text{ex}(n, Y_{k,b}) \leq \frac{n}{k} \binom{n}{k-1}$. Next we apply induction on the size of $Y_{k,b}$-tiling and use Theorem 5.1 to show Theorem 1.3.

**Proof of Theorem 1.3.** Within the proof, we write $Y := Y_{k,b}$ for short. Let $s_0$ be sufficiently large and $s \geq s_0$ so that

$$n - 1 - (2k - b)(s - 1) \geq n_{k-1}$$

namely, we can apply Theorem 5.1. We introduce a new integer $t \in [s]$ and prove that if $n \geq (2(2k - b)^2 + 1)(k - 1)s + t$ and $e(H) \geq \binom{n}{k} - \binom{n - t + 1}{k} + \binom{n - 2}{k - 1} + \binom{n - 2}{k - 2}(2k - b)t$, then $H$ contains a $Y$-tiling of size $t$.

For the base case $t = 1$, if the theorem fails, then $H$ contains no copy of $Y$, thus $e(H) \leq \binom{n - 1}{k - 1}$ by Theorem 5.1, which is a contradiction. Suppose that the theorem holds for $t - 1$, but fails for $t \geq 2$. In particular, $H$ contains a $Y$-tiling of size $t - 1$. If there is a vertex $v$ with $\deg(v) > \binom{n - 1}{k - 1} - \binom{n - 1 - (2k - b)(t - 1)}{k - 1} + \binom{n - 2}{k - 1} + \binom{n - 2}{k - 2}(2k - b)(t - 1)$, then we have

$$e(H) - d(v) \geq e(H) - \binom{n - 1}{k - 1} > \binom{n - 1}{k} - \binom{n - t + 1}{k} + \binom{n - 2}{k - 1} + \binom{n - 2}{k - 2}(2k - b)(t - 1).$$

Thus by induction hypothesis, $H - v$ contains a $Y$-tiling of size $t - 1$. Note that the number of members of $N(v)$ intersecting $V(M)$ is at most $\binom{n - 1 - (2k - b)(t - 1)}{k - 1}$. Since $\deg(v)$ is large enough, by Theorem 5.1 and equation 5.1, $N(v)$ contains a copy of $Y_{k-1,b-1}$ which is disjoint from $V(M)$. Therefore we obtain a $Y$-tiling of size $t$, which is a contradiction.

Now we may assume a maximum degree condition of $H$

$$\Delta(H) \leq \binom{n - 1}{k - 1} - \binom{n - 1 - (2k - b)(t - 1)}{k - 1} + \binom{n - 2}{k - 1} < (2k - b)(t - 1)\binom{n - 2}{k - 2} + \binom{n - 1}{k - 2},$$

where we used $\binom{n - 1}{k - 1} - \binom{n - 1 - (2k - b)(t - 1)}{k - 1} = \binom{n - 2}{k - 2} + \binom{n - 1 - (2k - b)(t - 1)}{k - 2}$. Let $M$ be a $Y$-tiling of size $t - 1$ in $H$ and note that $V(H) \setminus V(M)$ is $Y$-free. Therefore, we have

$$e(H) \leq \binom{n - 1}{k - 1} + \Delta(H)(2k - b)(t - 1) < (2k - b)^2(t - 1)^2\binom{n - 2}{k - 2} + \binom{n - 1}{k - 1} + \binom{n - 1}{k - 2}(2k - b)(t - 1).$$

Combining with the lower bound on $e(H)$, we have

$$(t - 1)\binom{n - t + 1}{k - 2} = (t - 1)\binom{n - t + 1}{k - 1} < \binom{n - t + 1}{k} < (2k - b)^2(t - 1)^2\binom{n - 2}{k - 2},$$

where we used $\binom{n - t + 1}{k - 1} = \binom{n - t + 1}{k - 1} + \binom{n - t + 1}{k - 1} > (t - 1)\binom{n - t + 1}{k - 1}$. Rearranging we get

$$\binom{n - t - k + 3}{n - k + 1} < \binom{n - 2}{k - 2} < \frac{(2k - b)(k - 1)(t - 1)}{n - t + 1}.$$
that is, \((1 - \gamma)^{k-1} < 2(2k - b)^2(k - 1)\gamma\). Since \(n \geq (2(2k - b)^2 + 1)(k - 1)s + t\), we derive
\[
\gamma \leq \frac{t - 2}{(2(2k - b)^2 + 1)(k - 1)s + t - k + 1} \leq \frac{s - 2}{(2(2k - b)^2 + 1)(k - 1)(s - 2)}.
\]
Note that for \(0 < \gamma \leq \frac{1}{(2(2k-b)^2+1)(k-1)}\),
\[(1 - \gamma)^{k-1} > 1 - (k - 1)\gamma \geq 2(2k - b)^2(k - 1)\gamma,
\]
which is a contradiction. □

6. Large \(Y_{3,2}\)-tilings: a proof of Theorem 1.4

Let \(H = (V, E)\) be a 3-graph with \(|V| = n\). In this section, we write \(Y := Y_{3,2}\) for brevity. A \(\{Y, E\}\)-tiling is a collection of vertex-disjoint copies of \(Y\) and edges in \(H\). In the following theorem we study large \(\{Y, E\}\)-tilings, and then use it and the regularity method to show Theorem 1.4.

**Theorem 6.1.** Suppose that \(0 < \alpha \leq 1/7\). Let \(H\) be a 3-graph of order \(n\) with
\[e(H) \geq \binom{n}{3} - \binom{n - \alpha n}{3} + O(n^2).
\]
Then \(H\) contains a \(\{Y, E\}\)-tiling covering more than \(4\alpha n\) vertices.

Now we briefly discuss the approach that we use and other possible approaches. We first remark that it is not clear to us how to use the powerful shifting technique in our context, which has been a crucial tool in studying large matchings. By the standard regularity method, to build a large \(Y\)-tiling, it suffices to find a large fractional \(Y\)-tiling. However, a suboptimal (pure) fractional \(Y\)-tiling is hard to analyze so it is not easy to improve in a greedy manner or an algorithmic way. We choose to use the \(\{Y, E\}\)-tilings, a mixture of \(Y\)-tilings and matchings. A large \(\{Y, E\}\)-tiling can be converted to a large \(Y\)-tiling by the regularity method, and is much easier to manipulate.

Now we start to set up the proof. We first show that it suffices to study the case \(\alpha = 1/7\). For \(\alpha \in (0, 1/7]\), let \(H'\) be a graph obtained by adding an \(s\)-set \(S\) such that \(n - 6s = 7\alpha n\) and adding all edges which contain at least one vertex from \(S\). Note that
\[e(H') = e(H) + \binom{n + s}{3} - \binom{n}{3} \geq \binom{n + s}{3} - \binom{n - \alpha n}{3} + O(n^2) = \binom{n + s}{3} - \frac{n + s - 3\alpha n}{3} + O(n^2).
\]
Applying the theorem with \(\alpha = 1/7\) to \(H'\) gives a \(\{Y, E\}\)-tiling that covers more than \(\frac{1}{7}(n + s)\) vertices. Deleting the copies of \(Y\) and \(E\) that contain vertices from \(S\) gives a \(\{Y, E\}\)-tiling of \(H\) that covers more than \(\frac{4}{7}(n + s) - 4s = \frac{4}{7}(n - 6s) = 4\alpha n\) vertices. Therefore, it suffices to show Theorem 6.1 for \(\alpha = 1/7\).

Suppose \(T = \{Y_1, Y_2, \ldots, Y_m, E_1, E_2, \ldots, E_{m_2}\}\) is a maximum \(\{Y, E\}\)-tiling of \(H\), that is, a \(\{Y, E\}\)-tiling covering the maximum size of vertices. By adding edges to \(H\) if necessary, we may assume \(4m_1 + 3m_2 = 4\alpha n = \frac{4}{7}n\). We denote by \(U\) the set of vertices not covered by \(T\). Obviously, there are no edges in \(U\) and \(|U| = \frac{3}{7}n\). Thus we have \(|U| \geq 3m_1\) and \(|U| \geq \frac{9}{7}m_2\). Based on forbidden structures, we will bound the number of edges in \(H\) from above and derive a contradiction.

According to the size of intersections with \(U\), we classify the edges into three categories: \((1, 2)\) edges, \((2, 1)\) edges and \((3, 0)\) edges. Specifically, an edge is called an \((i, j)\) edge if it contains exactly \(i\) vertices covered by \(T\) and \(j\) vertices in \(U\) (note that there is no \((0, 3)\) edge). Denote by \(D_i\) the set of all \((i, 3 - i)\) edges for \(i \in [3]\). Note that
\[e(H) = |D_1| + |D_2| + |D_3|.
\]
To get an upper bound of \(e(H)\), we calculate \(|D_1|\) separately.
First, we consider the (1,2) edges and show
\[ |D_1| \leq m_1 \left( \frac{|U|}{2} \right) + 3m_2|U|/2. \] (6.2)

**Proof.** A (1,2) edge has exactly one vertex covered by \( T \) that is in a \( Y_i \) or an \( E_j \). Then we fix a \( Y_i \) or an \( E_j \), where \( i \in [m_1] \) and \( j \in [m_2] \), and estimate the number of such edges.

Denote the edges of \( Y_i \) by \( xyz, yzw \). Let \( Q' \) be the subgraph of \( H[V(Y_i) \cup U] \) consisting of all (1,2) edges. We claim \( \deg_{Q'}(x) = \deg_{Q'}(w) = 0 \). If not, the (1,2) edge containing the vertex \( x \), together with \( yzw \) forms a matching of size two in \( H \), contradicting the maximality of \( T \). Similarly, two disjoint edges in \( Q' \) are also impossible, from which we bound the number of edges containing \( y \) or \( z \) as follows. Indeed in \( Q' \), if one of \( y, z \) is contained in more than \( 2|U| \) edges (and less than \( \left( \frac{|U|}{2} \right) \) edges), then the degree of the other one is zero. Otherwise both \( \deg_{Q'}(y) \) and \( \deg_{Q'}(z) \) are at most \( 2|U| \), then we get
\[ \deg_{Q'}(y) + \deg_{Q'}(z) \leq 4|U| \leq \left( \frac{|U|}{2} \right). \]
as \( |U| = 3n/7 \) is large. Therefore,
\[ e(Q') = \deg_{Q'}(x) + \deg_{Q'}(y) + \deg_{Q'}(z) + \deg_{Q'}(w) \leq \left( \frac{|U|}{2} \right). \]

Denote the vertex set of \( E_j \) by \( \{u, g, h\} \). Let \( Q'' \) be the subgraph of \( H[\{u, g, h\} \cup U] \) consisting of all (1,2) edges. We observe \( \deg_{Q''}(v) \leq |U|/2, v \in \{u, g, h\} \). Suppose instead, there exist two edges \( vu_1u_2, vu_3 \), where \( u_1, u_2, u_3 \in U \), forming a copy of \( Y \), contradicting the maximality. Hence
\[ e(Q'') = \deg_{Q''}(u) + \deg_{Q''}(g) + \deg_{Q''}(h) \leq 3|U|/2. \]

Summing over all \( Y_i \) and \( E_j \) of \( T \), we have \( |D_1| \leq m_1 \left( \frac{|U|}{2} \right) + 3m_2|U|/2 \) and complete the proof. \( \square \)

The following three facts will be useful in our proof.

**Fact 3** ([38], Fact 1). For all integers \( k \geq 1, n \geq 2 \), and \( 1 \leq t \leq n - 1 \), the maximum number of edges in a \( k \)-partite \( k \)-graph with \( n \) vertices in each class and no matching of size \( t + 1 \) is \( tn^{k-1} \).

**Fact 4.** Let \( a, b \) be integers with \( b \geq a \geq 2 \). Let \( H \) be a 3-partite 3-graph on \( V_1, V_2, V_3 \) with \( |V_1| = |V_2| = a, |V_3| = b \) and no matching of size \( a \). Then \( e(H) \leq (a-1)ab \).

**Proof.** Let \( V_3 = \{v_1, \ldots, v_b\} \) and let \( V_3' = \{v_1i, v_2i, \ldots, v bi\} \subseteq V_3 \). For \( H' := H[V_1, V_2, V_3'] \), by Fact 3, we get \( e(H') = \sum_{j=1}^a \deg_{H'}(v_{ij}) \leq (a-1)a^2 \). Summing over all subsets of size \( a \) of \( V_3 \) gives
\[ e(H) \leq \frac{b}{a} \left( \frac{(a-1)a^2}{(b-1)} \right) = (a-1)ab. \] \( \square \)

**Fact 5.** Suppose \( Q = Q(V_1, V_2, V_3) \) is a 3-partite 3-graph with \( |V_1| = 2, |V_2| = t \in \{3, 4\}, |V_3| = 4 \). If \( Q \) does not have two disjoint copies of \( Y \), then \( e(Q) \leq 5t \).

**Proof.** Let \( V_1 = \{a_1, a_2\}, V_2 = \{h_1, \ldots, h_t\} \) and \( V_3 = \{g_1, g_2, g_3, g_4\} \). Let \( G := G[V_2, V_3] \) with \( E(G) = E_r(G) \cup E_b(G) \), where \( E_r(G) = \{h_igj : a_1h_igj \in E(Q)\}, E_b(G) = \{h_igj : a_2h_igj \in E(Q)\} \), and we color \( E_r(G) \) red, \( E_b(G) \) blue. Note that \( e(Q) = e(G) = e_r(G) + e_b(G) \), where \( e_r(G) := |E_r(G)| \) and \( e_b(G) := |E_b(G)| \). The assumption that \( Q \) does not have two disjoint monochromatic copies of \( P_3 \) with different colors in \( G \).

Towards a contradiction, suppose that \( e(Q) \geq 5t + 1 \geq 16 \). Without loss of generality, suppose \( e_r(G) \geq e_b(G) \), then \( 4t \geq e_r(G) \geq 8 \) and \( e_b(G) \geq t + 1 \). Denote \( \deg_r(v) (\deg_b(v)) \) by the number of blue (red) edges containing \( v \) in \( G \). We claim that for every \( h_i, (A) \deg_r(h_i) \leq 2 \) and \( (B) \deg_r(h_i) \leq 3 \).
(A) Suppose \( \deg_b(h_1) \geq 3 \), then every \( g_j \) has at most one neighbor other than \( h_1 \) in \( E_r(G) \), i.e. \( \deg_r(g_j) \leq 2 \), because of the absence of two disjoint monochromatic copies of \( P_3 \) with different colors. Since \( e_r(G) \geq 8 \), we have \( \deg_r(g_j) = 2 \) for every \( j \). Thus, \( e_r(G) = 8 \) and \( \deg_r(h_1) = 4 \). Now for every \( h_i, i > 1 \), we have \( \deg_b(h_i) \leq 1 \), as otherwise \( h_i \) gives a blue \( P_3 \) and \( h_1 \) gives a vertex-disjoint red \( P_3 \) (due to \( \deg_r(h_1) = 4 \)), a contradiction. This implies that \( e_b(G) \leq t - 1 + 4 = t + 3 \), and \( e(G) = e_b(G) + e_r(G) \leq t + 3 + 8 < 5t + 1 \), a contradiction.

(B) Suppose \( \deg_r(h_1) = 4 \), implying that \( \deg_b(h_i) \leq 1 \) for \( i \geq 2 \), due to the absence of disjoint \( P_3 \) with different colors. Thus, \( e_b(G) \leq t - 1 + 2 = t + 1 \) (using \( \deg_b(h_1) \leq 2 \) from (A) and \( \deg_b(h_i) \leq 1 \) for \( i \geq 2 \)). This implies that \( e_b(G) = t + 1 \) and \( e_r(G) = 4t \) and there are two disjoint \( P_3 \)'s with different colors, a contradiction.

However, as \( e(G) \geq 5t + 1 \), there exists \( i \) such that \( \deg_r(h_i) + \deg_b(h_i) \geq 6 \), a contradiction. \( \square \)

The analysis of \((2,1)\) and \((3,0)\) edges are much more involved. For that we introduce the following notation. Recall that \( T = \{ Y_1, Y_2, \ldots, Y_m, E_1, E_2, \ldots, E_m \} \). Let \( YYU \) represent the set of the edges \( \{ y_1, y_2, u \}, y_1 \in V(Y_i), y_2 \in V(Y_j), u \in U, i \neq j \). Let \( EYU \) represent the set of the edges \( \{ x, y, u \}, x \in V(E_i), y \in V(E_j), u \in U \) and let \( EEU \) be the set of the edges \( \{ x_1, x_2, u \}, x_1 \in V(E_i), x_2 \in V(E_j), u \in U, i \neq j \). Note that the number of \((2,1)\) edges that have two vertices covered by a single \( E_i \) or \( Y_j \) of \( T \) is \( O(n^2) \). Hence

\[
|D_2| = |YYU| + |EYU| + |EEU| + O(n^2).
\]

For \((3,0)\) edges, we denote by \( EEY \) the set of the edges \( \{ x_1, x_2, y \}, x_1 \in V(E_i), x_2 \in V(E_j), y \in V(Y_k), i \neq j \), and define \( EEE, EYU \) and \( EYY \) analogously. Other than these four types of edges, the number of remaining \((3,0)\) edges is \( O(n^2) \). Thus

\[
|D_3| = |EEE| + |EEY| + |EYY| + |YYY| + O(n^2).
\]

Similar analysis as in the proof of (6.2) would give a good estimate on \( |D_2| \), but this will leave \( |D_3| \), the number of \((3,0)\) edges, hard to analyze. To overcome this difficulty we consider \( |D_2| + |D_3| \) at the same time, by setting

\[
\begin{align*}
y_1 &:= 2|EEE| + |EEU|, \\
y_2 &:= 2|EEY| + |EEU| + |EYU|, \\
y_3 &:= 2|EYY| + |EYU|, \\
y_4 &:= 2|YYY| + 2|YYU|.
\end{align*}
\]

Note that

\[
2(|D_2| + |D_3|) = y_1 + y_2 + y_3 + y_4 + O(n^2)
\]

and we are able to bound each \( y_i \) appropriately. More precisely, we shall prove the following:

\[
\begin{align*}
y_1 &\leq 38 \binom{m_2}{3} + 3 \binom{m_2}{2} |U| + O(n^2), \\
y_2 &\leq 48 \binom{m_2}{3} m_1 + 3 \binom{m_2}{2} |U| + 6m_1 m_2 |U| + O(n^2), \\
y_3 &\leq 60 \binom{m_1}{2} m_2 + 6m_1 m_2 |U| + O(n^2), \\
y_4 &\leq 74 \binom{m_1}{3} + 14 \binom{m_1}{2} |U| + O(n^2).
\end{align*}
\]
Combining (6.1)-(6.7), we get
\[
e(H) = |D_1| + |D_2| + |D_3| \\
\le m_1 \left( \frac{|U|}{2} \right) + 7 \left( m_1 \right) |U| + 3 \left( m_2 \right) |U| + 6m_1m_2|U| \\
+ 37 \left( \frac{m_1}{3} \right) + 19 \left( \frac{m_2}{3} \right) + 30 \left( \frac{m_1}{2} \right) m_2 + 24 \left( \frac{m_2}{2} \right) m_1 + O(n^2).
\]
(6.9)

Next we write \( M_1 = 4m_1, M_2 = 3m_2 \) and \(|U| = 3n/7 = (3/4)(M_1 + M_2)\), we rewrite all terms above with \( M_1 \) and \( M_2 \). After collecting terms we obtain
\[
e(H) \le \frac{127}{64 \times 6} (M_1 + M_2)^3 + O(n^2) = \frac{127}{72} n^3 + O(n^2) = \binom{n}{3} - \binom{n - n/7}{3} + O(n^2),
\]
(6.10)
where we used \( M_1 + M_2 = 4n/7 \). (See Appendix B for a proof of this inequality).

Now it remains to prove (6.4)-(6.7).

6.1. More notation and tools. We need some more notation. Fix a triple \( T = \{P, S, R\} \), where \( P, S, R \in \mathcal{T} \). Let \( V(P) = V_1, V(S) = V_2, V(R) = V_3 \). Let \( Q_T \) be the induced tripartite subgraph of \( H \) on \( (V_1, V_2, V_3) \), and let \( G_T \) be the tripartite graph on \( (V_1, V_2, V_3) \) with \( (i, j) \in E(G_T) \) if and only if there are at least eight vertices \( u \) in \( U \), such that \( uij \in E(H) \), where \( i \in V_p, j \in V_q, p, q \in [3] \) and \( p \neq q \). Let \( EE, YY \) and \( EY \) represent the edges of \( G_T \) that are between two edges in \( T \), between two copies of \( Y \) and between an edge in \( T \) and a copy of \( Y \), respectively. Denote the number of these edges by \( |E_{EE}(G_T)|, |E_{YY}(G_T)| \) and \( |E_{EY}(G_T)| \). Thus we have \( e(G_T) = |E_{EE}(G_T)| + |E_{YY}(G_T)| + |E_{EY}(G_T)| \).

We denote by a star a graph all of whose edges share a common vertex that we call a center. We prove the following properties of \( G_T \) and \( Q_T \).

**Fact 6.** Fix a triple \( T = \{P, S, R\} \) of \( T \). Let \( G = G_T \) as defined above. If \( |V_1| = 3 \), then \( G[V_1, V_2] \) is trivial or a star, and \( e(G[V_1, V_2]) \le |V_2| \). If \( |V_1| = |V_2| = 4 \), then \( G[V_1, V_2] \) has no matching of size three and \( e(G[V_1, V_2]) \le 8 \).

**Proof.** Suppose that \( |V_1| = 3 \) and the nontrivial graph \( G[V_1, V_2] \) is not a star, then it contains two disjoint edges \( e_1, e_2 \). By the definition of \( G \), we can find vertices \( u_1, \ldots, u_4 \) such that \( e_1u_1, e_1u_2, e_2u_3, e_2u_4 \in E(H) \), which gives two disjoint copies of \( Y \) on \( V_1 \cup V_2 \cup U \). Thus, we can replace \( P \) and \( S \) by these two copies of \( Y \), and since \( |V_1| = 3 \), the resulting \( \{Y, E\} \)-tiling is larger than \( T \), a contradiction. Thus \( G[V_1, V_2] \) is a star or is trivial and thus \( e(G[V_1, V_2]) \le |V_2| \).

Suppose \( |V_1| = |V_2| = 4 \). We claim that \( G[V_1, V_2] \) has no matching of size three. Suppose instead, then similarly we can find three disjoint copies of \( Y \) on \( V_1 \cup V_2 \cup U \). Replacing \( P \) and \( S \) by them gives a larger \( \{Y, E\} \)-tiling, a contradiction. By König Theorem [10], \( G[V_1, V_2] \) has a vertex cover of size at most two, implying that \( e(G[V_1, V_2]) \le 8 \). \( \square \)

Note that \( G_T[V_1, V_2] \) is a star or the union of two stars. We call a center of a star in \( G_T[V_1, V_2] \), a center of \( G_T[V_1, V_2] \). Then \( G_T[V_1, V_2] \) has at most two centers.

**Bound y1.** Recall \( y_1 := |EEU| + 2|EEE| \). We show
\[
y_1 \le 38 \left( \frac{m_2}{3} \right) + 3 \left( \frac{m_2}{2} \right) |U| + O(n^2).
\]

**Proof.** Without loss of generality, consider a triple \( \{E_1, E_2, E_3\} \) in \( T \). Let \( V_1 = V(E_1) = \{a_1, a_2, a_3\} \), \( V_2 = V(E_2) = \{b_1, b_2, b_3\} \), \( V_3 = V(E_3) = \{c_1, c_2, c_3\} \). Let \( G = G_T \) and \( Q = Q_T \) as defined above. We consider three cases here.

**Case 1.** \( e(G) = 0 \). Obviously, \( e(Q) \le 27 \).

**Case 2.** \( e(G) \ge 1 \) and \( G \) has no matching of size two.
Note that $G$ is a star, and $e(G) \leq 6$. Let $a_1b_1 \in E(G)$ and $Q' := Q - \{a_1, b_1\}$. Since $\{a_1b_1\} \cup U$ has a copy of $Y$ and $T$ is a maximum $\{Y, E\}$-tiling, $Q'$ has no matching of size two. By Fact 4, we get $e(Q') \leq 6$.

Let $E_{a_1b_1} = \{xyz \in E(Q) : \{x, y, z\} \cap \{a_1, b_1\} \neq \emptyset\}$. Note that $|E_{a_1b_1}| \leq 15$. Thus we get $e(Q) = e(Q') + |E_{a_1b_1}| \leq 6 + 15 = 21$.

**Case 3.** $G$ has a matching of size two.

By Fact 6, we know that $G[V_i, V_j]$ is a star and $e(G[V_i, V_j]) \leq 3$, where $i, j \in [3]$ and $i \neq j$. So $e(G) \leq 9$. Without loss of generality, we may assume that $a_1b_1, b_2c_2 \in E(G)$. By the maximality of $T$, there is no edge in $Q - \{a_1, b_1, b_2, c_2\}$. Similar as Case 2 by Fact 4, we have $e(Q - \{a_1, b_1\}) \leq 6$ and $e(Q - \{b_2, c_2\}) \leq 6$. Let

$$E_{pq} := \{pqv \in E(Q) : p \in V_i, q \in V_j, v \in V_k, \{i, j, k\} = [3]\}.$$ 

Note that $|E_{a_1b_2}| \leq 3$, $|E_{a_1c_2}| \leq 3$ and $|E_{b_1c_2}| \leq 3$. Thus

$$e(Q) = |E(Q - \{a_1, b_1\}) \cup E(Q - \{b_2, c_2\})| + |E_{a_1b_2} \cup E_{a_1c_2} \cup E_{b_1c_2}|$$

$$\leq 6 \times 2 + 3 \times 3 - 2 = 19.$$ 

Note that each triple $\{E_i, E_j, E_k\}$ in $T$ satisfies one of the cases above. Suppose the number of triples in three cases above are $x_1, x_2$ and $x_3$ respectively, then $x_1 + x_2 + x_3 = \binom{m_2}{3}$. Note that we can bound the number of $EEU$ edges by $\frac{m_2}{m_2 - 2}(6x_2 + 9x_3)|U| + O(n^2)$, because every $EEU$ edge counted by $G$ is counted in exactly $m_2 - 2$ triples (so computed $m_2 - 2$ times) and the number of $EEU$ edges not reflected in $G$ is at most $7 \times 9\binom{m_2}{3}$. Thus we have

$$y_1 \leq 2(27x_1 + 21x_2 + 19x_3) + \frac{1}{m_2 - 2}(6x_2 + 9x_3)|U| + O(n^2)$$

$$\leq 38 \frac{m_2}{3} + 16x_1 + 4x_2 + 3 \frac{m_2}{2}|U| - \frac{9x_1 + 3x_2}{m_2 - 2}|U| + O(n^2).$$

We complete the proof by $\frac{|U|}{m_2 - 2} \geq \frac{|U|}{m_2} \geq \frac{3n/7}{4n/21} = 9/4$, which gives $16x_1 + 4x_2 - \frac{9x_1 + 3x_2}{m_2 - 2}|U| < 0$. □

**Bound $y_2$.** Recall $y_2 := 2|EYY| + |EEU| + |EYU|$. We show

$$y_2 \leq 48 \frac{m_2}{2}m_1 + 3 \frac{m_2}{2}|U| + 6m_1m_2|U| + O(n^2).$$

**Proof.** Without loss of generality, consider $E_1, Y_2, E_3 \in T$. Let $V_1 = V(E_1) = \{a_1, a_2, a_3\}$, $V_2 = V(Y_2) = \{h_1, h_2, h_3, h_4\}$ and $V_3 = V(E_3) = \{c_1, c_2, c_3\}$. Let $G = G_T$ and $Q = Q_T$ as defined before. By Fact 6, we get $|E_{EY}(G)| \leq 8$ and $|E_{EE}(G)| \leq 3$. In this proof we shall use the fact that there do not exist an edge $e$ of $G$, a 3-edge and a copy of $Y$ in $Q$, all of which are pairwise disjoint – since $e$ can be extended to a copy of $Y$ by two vertices in $U$, we may replace $E_1, Y_2, E_3$ by these two copies of $Y$ and one edge, contradicting the maximality of $T$. Consider the following three cases.

**Case 1.** $|E_{EY}(G)| = 0$. We use the trivial bound $e(Q) \leq 36$.

**Case 2.** $|E_{EY}(G)| \geq 1$ and $G[V_1, V_2] \cup G[V_2, V_3]$ has no matching of size two.

Note that $G[V_1, V_2] \cup G[V_2, V_3]$ is a star and thus $|E_{EY}(G)| \leq 6$. Let $a_1h_1 \in E(G)$. By the maximality of $T$, the graph $Q' := Q - \{a_1, h_1\}$ does not have an edge and a copy of $Y$ that are disjoint. Then we claim $e(Q') \leq 12$. If not, without loss of generality, we suppose $\deg_{Q'}(c_3) \geq \deg_{Q'}(c_2) \geq \deg_{Q'}(c_1)$, then $5 \leq \deg_{Q'}(c_3) \leq 6$. We may assume $a_2h_ic_3 \in E(Q')$ for every $i \in \{2, 3, 4\}$. Since

$$\deg_{Q'}(c_2) + \deg_{Q'}(c_1) \geq 7 \geq 6 \geq |\{a_2h_ic_j \in E(Q') : i \in \{2, 3, 4\}, j \in \{1, 2\}\}|,$$

we can pick an edge $e \in Q'$ such that $a_3 \in e$ and $c_j \notin e$. By our assumption $c_3$ together with the remaining vertices in $V_1$ and $V_2$ forms a copy of $Y$ in $Q'$ (disjoint with $e$), a contradiction.
Let \( E_{a_1, h_1} = \{xyz \in E(Q) : \{x, y, z\} \cap \{a_1, h_1\} \neq \emptyset\} \). Note that \(|E_{a_1, h_1}| \leq 18\). Thus we get
\[
e(Q) = e(Q') + |E_{a_1, h_1}| \leq 12 + 18 = 30.
\]

**Case 3.** \( G[V_1, V_2] \cup G[V_2, V_3] \) has a matching of size two.

Suppose \( a_1 h_1, h_2 c_2 \in E(G) \) and \( Q' := Q - \{a_1, h_1\} \). Note that \( E(Q - \{a_1, h_1, h_2, c_2\}) = \emptyset\). Then for any \( e \in E(Q') \), \( e \cap \{h_2, c_2\} \neq \emptyset\). We claim that \( e(Q') \leq 8\). Indeed, suppose that \( e(Q') \geq 9\) and \( \deg_{Q'}(h_2) \geq \deg_{Q'}(c_2) \), then \( 5 \leq \deg_{Q'}(h_2) \leq 6 \) and \( \deg_{Q - \{a_1, h_1, h_2\}}(c_2) \geq 3 \). Take a copy of \( Y \) containing \( h_2, c_1, c_3 \) in \( Q' \). Since \( \deg_{Q - \{a_1, h_1, h_2\}}(c_2) \geq 3 \), one can take an edge containing \( c_2 \) disjoint with the copy of \( Y \) above, a contradiction. Similar arguments give \( e(Q - \{h_2, c_2\}) \leq 8 \).

Similar as Case 3 in the proof of (6.4) to estimate the number of the remaining edges in \( Q \), we get \(|E_{a_1 h_2}| \leq 3\), \(|E_{a_1 c_2}| \leq 4\) and \(|E_{h_1 c_2}| \leq 3\). Then
\[
e(Q) = |E(Q - \{a_1, h_1\}) \cup E(Q - \{h_2, c_2\})| + |E_{a_1 h_2} \cup E_{a_1 c_2} \cup E_{h_1 c_2}|
\leq 8 
\leq 8 \times 2 + 3 + 4 + 3 - 2 = 24.
\]

Suppose that the number of triples of \( T \) in the three cases above is \( x_1 \), \( x_2 \) and \( x_3 \) respectively, then \( x_1 + x_2 + x_3 = m_1 + m_2 \). Note that by counting edges via triples e.g. \( E_1, Y_2, E_3 \), all \( EYU \) edges reflected by \( G \) are counted exactly \( m_2 - 1 \) times. Putting everything together we get
\[
y_2 \leq 2(36x_1 + 30x_2 + 24x_3) + 3 \left( \frac{m_2}{2} \right) |U| + \frac{1}{m_2 - 1} \left( \frac{6x_2 + 8x_3}{m_2} \right) |U| + O(n^2)
\]
\[
\leq 48 \left( \frac{m_2}{2} \right) m_1 + 24x_1 + 12x_2 + 3 \left( \frac{m_2}{2} \right) |U| + 6m_1 m_2 |U| - \frac{12x_1 + 6x_2}{m_2} |U| + O(n^2),
\]
where the error term is the number of the remaining edges in \( EEU \) and \( EYU \) not reflected in \( G \). We obtain the desired bound on \( y_2 \) because \( 24x_1 + 12x_2 - \frac{12x_1 + 6x_2}{m_2} |U| < 0 \) due to \(|U|/m_2 \geq 9/4\). \( \Box \)

**Bound \( y_3 \).** Recall \( y_3 := 2|YYE| + |EYU| \). We show
\[
y_3 \leq 60 \left( \frac{m_1}{2} \right) m_2 + 6m_1 m_2 |U| + O(n^2).
\]

**Proof.** Without loss of generality, consider \( E_1, Y_2, Y_3 \in T \). Let \( V_1 = V(E_1) = \{a_1, a_2, a_3\} \), \( V_2 = V(Y_2) = \{h_1, h_2, h_3, h_4\} \) and \( V_3 = V(Y_3) = \{g_1, g_2, g_3, g_4\} \). Let \( G = G_T \) and \( Q = Q_T \) as defined above. By Fact 6, we have \(|E_{YY}(G)| \leq 8\). In this case we use the fact that there do not exist \( i \) copies of \( Y \) in \( Q \), all of which are pairwise disjoint – since each edge of \( G \) can be extended to a copy of \( Y \) by distinct vertices in \( U \), we may replace \( E_1, Y_2, Y_3 \) by these three copies of \( Y \), contradicting the maximality of \( T \).

Consider the following three cases here.

**Case 1.** \( e(G) = 0 \). The trivial bound is \( e(Q) \leq 48 \).

**Case 2.** \( e(G) \geq 1 \) and \( |E_{YY}(G)| \leq 6 \). Note that the bound below \( e(Q) \leq 39 \) does not use the assumption \(|E_{YY}(G)| \leq 6\).

First assume \( a_1 h_1 \in E(G) \) and \( Q' = Q - \{a_1, h_1\} \). By Fact 5, we have \( e(Q') \leq 15 \). Let \( E_{a_1, h_1} := \{xyz \in E(Q) : \{x, y, z\} \cap \{a_1, h_1\} \neq \emptyset\} \). Note that \(|E_{a_1, h_1}| \leq 24\). Thus we get
\[
e(Q) = e(Q') + |E_{h_1, h_1}| \leq 15 + 24 = 39.
\]

Next we assume \( h_4 g_4 \in E(G) \). Since \( Q - \{h_4, g_4\} \) has no matching of size three, by Fact 3, we have \( e(Q - \{h_4, g_4\}) \leq 18\). The number of edges containing \( h_4 \) or \( g_4 \) is at most 21. Then
\[
e(Q) \leq 18 + 21 = 39.
\]

**Case 3.** \( e(G) \geq 1.7 \leq |E_{YY}(G)| \leq 8 \). By Fact 6, each of \( E(G[V_1, V_2]) \) and \( E(G[V_1, V_3]) \) is a star. Depending on the location of the centers and without loss of generality, we may separate the following cases and show that \( e(Q) \leq 36 \) in each of them.

**Case 3.1.** Suppose that \( E(G[V_1, V_2]) = \{a_p h_1, a_p h_2, a_p h_3, a_p h_4\} \) and \( E(G[V_1, V_3]) \supseteq \{a_q g_1, a_q g_2, a_q g_3\} \).
If \( p \neq q \), then \( Q - \{a_p, a_q\} \) has no copy of \( Y \) – otherwise we obtain two edges of \( G \) and a copy of \( Y \) in \( Q \) such that they are pairwise disjoint, a contradiction. This implies that \( Q - \{a_p, a_q\} \) has at most 4 edges. Note that the number of edges containing \( a_p \) or \( a_q \) is at most 32. Thus \( e(Q) \leq 32 + 4 = 36. \)

If \( p = q \), then let \( Q'' = Q - \{a_p\} \). We claim that \( e(Q'') \leq 20 \). Suppose instead, by Fact 5, there exist \( Y', Y'' \) that are two disjoint copies of \( Y \) in \( Q'' \). Then as \( \deg_{G}(a_p) \geq 7 \) there exists \( x \notin V(Y') \cup V(Y'') \) such that \( a_p x \in E(G) \) is disjoint with \( Y', Y'' \), a contradiction. Since the number of edges containing \( a_p \) is at most 16, we have \( e(Q) \leq 20 + 16 = 36. \)

**Case 3.2.** Suppose \( E(G[V_1, V_2]) \supseteq \{a_1h_1, a_1h_2, a_1h_3\} \) and \( E(G[V_1, V_3]) = \{g_1a_1, g_1a_2, g_1a_3\} \). Let \( Q'' = Q - \{a_1, g_1\} \). Then we have \( \deg_{G''}(a_i) \leq 3, i = 2, 3 \) – otherwise \( Q'' \) contains a copy of \( Y \), say, containing \( a_2 \) but not \( a_3 \), and at most two vertices in \( V_2 \). We can take \( g_1a_3 \) and \( a_1h_i \) for some \( h_i \) disjoint with this copy of \( Y \), which is a contradiction. Thus \( e(Q'') \leq 6. \) Since the number of edges containing \( a_1 \) or \( g_1 \) is at most 24. Thus \( e(Q) \leq 24 + 6 = 30. \)

Suppose the number of triples e.g. \( E_1, Y_2, Y_3 \) in \( T \) in three cases above are \( x_1, x_2 \) and \( x_3 \) respectively, then \( x_1 + x_2 + x_3 = m_2\binom{m_1}{2} \). Note that by counting edges via triples e.g. \( E_1, Y_2, Y_3 \), all \( EYU \) edges reflected by \( G \) are counted exactly \( m_1 - 1 \) times. Putting everything together we get

\[
y_3 \leq 2(48x_1 + 39x_2 + 36x_3) + \frac{1}{m_1 - 1}(6x_2 + 8x_3)|U| + O(n^2)
\]

\[
\leq 60\left(\frac{m_1}{2}\right)m_2 + (36x_1 + 18x_2 + 12x_3) + 6m_1m_2|U| - (12x_1 + 6x_2 + 4x_3)|U|/m_1 + O(n^2),
\]

where \( O(n^2) \) is the number of the remaining edges in \( EYU \) not reflected in \( G \). We obtain the desired bound as \( |U|/m_1 \geq (3n/7)/(n/7) = 3 \) and thus \( (36x_1 + 18x_2 + 12x_3) - (12x_1 + 6x_2 + 4x_3)|U|/m_1 \leq 0. \)

**Bound \( y_4 \).** Recall \( y_4 : = 2|YYY| + 2|YYU| \). We show

\[
y_4 \leq 74\left(\frac{m_1}{3}\right) + 14\left(\frac{m_1}{2}\right)|U| + O(n^2).
\]

**Proof.** Without loss of generality, consider \( Y_1, Y_2, Y_3 \in T \). Let \( V_1 = V(Y_1) = \{k_1, k_2, k_3, k_4\}, V_2 = V(Y_2) = \{h_1, h_2, h_3, h_4\} \) and \( V_3 = V(Y_3) = \{g_1, g_2, g_3, g_4\} \). Let \( G = G_T \) and \( Q = Q_T \) as defined before.

First we give the following claims on the structures of \( G \) and \( Q \).

**Claim 6.2.** The number of edges in \( G \) is at most 21. When \( e(G) = 21 \), there exist \( i_1, i_2, i_3 \in [4] \) such that \( k_{i_1}, h_{i_2}, g_{i_3} \) cover all edges of \( G \) with \( \deg_G(v) = 7 \) for \( v \in \{k_1, h_1, g_1\} \), see Figure 1.

**Proof.** Without loss of generality, let \( e(G[V_1, V_2]) \geq \max \{e(G[V_2, V_3]), e(G[V_1, V_3])\} \). By Fact 6, we know that \( G[V_1, V_2] \) has no matching of size three and \( e(G[V_1, V_2]) \leq 8 \). Now suppose \( e(G) \geq 21 \). First assume \( e(G[V_1, V_2]) = 8 \), and we also have \( e(G[V_1, V_3]) \geq 5 \) and \( e(G[V_2, V_3]) \geq 5 \). Note that \( G[V_1, V_2] \) has two centers, and they must be in the same part (otherwise \( e(G[V_1, V_2]) \leq 7 \).

Without loss of generality, suppose the two centers are in \( V_1 \). Take disjoint \( e_1, e_2 \in E(G[V_2, V_3]) \) and \( e_3, e_4 \in E(G[V_1, V_2 \setminus (e_1 \cup e_2)]) \) that form a matching of size four in \( G \), which can be extended to four disjoint copies of \( Y \) by adding vertices in \( U \), contradicting the maximality of \( T \). This contradiction implies that \( e(G[V_1, V_2]) = e(G[V_2, V_3]) = e(G[V_1, V_3]) = 7 \).

We next claim that \( G \) has at most three center vertices, from which the claim easily follows. Towards a contradiction, assume that \( G \) has at least four center vertices, and without loss of generality, we may assume that \( k_1, k_2 \in V_1 \) are the centers, and \( k_1 \) is a center of \( G[V_1, V_2] \), \( k_2 \) is a center of \( G[V_1, V_q] \), \( p, q \in \{2, 3\} \) and it is possible that \( p = q \). Let \( e_1, e_2 \in E(G[V_2, V_3]) \) be two
disjoint edges. Since the degree of each center is at least three (as each bipartite graph has seven edges), for \( p \neq q \), we can take two disjoint edges one containing \( k_1 \) and the other containing \( k_2 \) which together with \( e_1, e_2 \) form a matching of size four. In the case \( p = q \), as \( e(G[V_1, V_p]) = 7 \), one of \( k_1 \) and \( k_2 \) has degree three and the other has degree four in \( G[V_1, V_p] \) and thus it is also possible to pick two disjoint edges which together with \( e_1, e_2 \) form a matching of size four. Clearly the existence of the matching of size four in \( G \) contradicts the maximality of \( \mathcal{T} \) and we are done. \( \square \)

**Claim 6.3.** If \( e(G) \geq 18 \) and \( \{v_1, v_2, v_3\} \) is a vertex cover of \( G \), then \( \{v_1, v_2, v_3\} \) is also a vertex cover of \( Q \), therefore, \( e(Q) = |E_{v_1, v_2, v_3}| \), where \( E_{v_1, v_2, v_3} := \{xyz \in E(Q) : \{x, y, z\} \cap \{v_1, v_2, v_3\} \neq \emptyset \} \).

**Proof.** Fix any \( k \in V_1 \), \( h \in V_2 \), \( g \in V_3 \), \( k, h, g \notin \{v_1, v_2, v_3\} \). We will show \( khg \notin E(Q) \). Without loss of generality, let \( \deg_{G}(v_1) \geq \deg_{G}(v_2) \geq \deg_{G}(v_3) \). We separate three cases below depending on the location of \( v_1, v_2 \) and \( v_3 \).

Suppose \( v_1, v_2, v_3 \in V_1 \), then \( e(G) \leq 16 \), which is a contradiction.

Suppose \( v_1, v_2 \in V_1, v_3 \in V_3 \). Note that \( e(G[V_1, V_2] \cup G[V_1, V_3]) \leq 16 \). Then \( \deg_{G[V_2, V_3]}(v_3) \geq 2 > 1 \), there exists \( v_3w'_3 \in E(G) \) such that \( w'_3 \in V_2, w'_3 \neq h \). Since \( \deg_{G}(v_2) \geq 5 > 4, \deg_{G}(v_1) \geq 6 > 5 \).

Similar as the previous paragraph, we conclude that \( khg \notin E(Q) \).

Suppose \( v_i \in V_i \) for \( i \in [3] \). Let \( G' = G - \{v_1, v_2, v_3v_1v_3\} \). Note that \( e(G') \geq 18 - 3 = 15 \). Then \( \deg_{G'}(v_3) \geq 3 > 2 \), there exists \( v_3w'_3 \in E(G') \), where \( w'_3 \notin \{k, h\} \). Similarly, \( \deg_{G'}(v_2) \geq 5 > 3, \deg_{G'}(v_1) \geq 6 > 5 \), there exist \( w'_1, w'_2 \notin \{k, h, g, w'_3\} \), such that \( v_2w'_2 \in E(G'), v_1w'_1 \in E(G') \). Thus \( G \) has three pairwise disjoint edges which are also disjoint with \( \{k, h, g\} \). By the maximality of \( \mathcal{T} \), \( khg \notin E(Q) \).

Therefore, we conclude that \( e(Q) = |E_{v_1, v_2, v_3}| \). \( \square \)

For convenience, by a cross matching, we mean a matching \( e_1, e_2, e_3 \) in \( G_T \) such that \( e_1 \subseteq V(Y_1) \times V(Y_2), e_2 \subseteq V(Y_2) \times V(Y_3), e_3 \subseteq V(Y_3) \times V(Y_1) \). Next we show the existence of a cross matching.

**Claim 6.4.** If \( e(G) \geq 17 \), then there is a cross matching in \( G \).

**Proof.** We may assume \( e(G[V_1, V_2]) \geq e(G[V_1, V_3]) \geq e(G[V_2, V_3]), \) then \( e(G[V_1, V_2]) \geq 6 \). Suppose the two centers of \( G[V_1, V_2] \) are \( u \) and \( v \), and thus they have degree at least two in \( G[V_1, V_2] \). It suffices to find a matching \( \{e_1, e_2\} \) of size two such that \( e_1 \in E(G[V_2, V_3]), e_2 \in E(G[V_1, V_3]) \) and \(|\{e_1 \cup e_2\} \cap \{u, v\}| \leq 1 \). Indeed, as \( e_1 \cup e_2 \) contains one vertex in \( V_1 \) and in \( V_2 \), respectively, for \( u \) or \( v \) which is not contained in \( e_1 \cup e_2 \), we can pick an edge \( e_3 \in G[V_1, V_2] \) disjoint from \( e_1 \cup e_2 \). Hence we obtain a cross matching.

\[ \text{Figure 1. The structure of } G \text{ when } e(G) = 21 \]
We first assume that $u$ and $v$ are in the same part. Note that $e(G[V_1, V_3]) \geq 5$, $e(G[V_2, V_3]) \geq 1$. Then we take disjoint $e_1, e_2 \in E(G[V_1, V_3])$ and clearly $|\{u, v\} \cap (e_1 \cup e_2)| \leq 1$.

Now suppose $v \in V_1$ and $u \in V_2$, then $e(G[V_1, V_2]) \leq 7$, $e(G[V_1, V_3]) \geq 5$ and $e(G[V_2, V_3]) \geq 3$. As $e(G[V_1, V_3]) \geq 5$, there exists $wv' \in E(G)$ such that $w \in V_3$, $v' \in V_1$ and $v' \neq v$. Then we are done unless all edges in $E(G[V_2, V_3])$ contain $w$. But this implies that there exists an edge $e_1 = wv' \in E(G[V_2, V_3])$ with $v' \in V_2 \setminus \{u\}$, namely, $u \notin e_1$. As $e(G[V_1, V_3]) \geq 5$, we can pick an edge $e_2$ in $G[V_1, V_3]$ not containing $w$ and we are done. \hfill \Box

We consider five cases here.

**Case 1.** $e(G) = 0$. The trivial bound is $e(Q) \leq 64$.

**Case 2.** $1 \leq e(G) \leq 16$.

Suppose $k_3h_1 \in E(G)$, then $Q' = Q - \{k_3, h_1\}$ has no matching of size three. By Fact 4, $e(Q') \leq 24$. Let $E_{k_3, h_1} = \{xyz \in E(Q) : \{x, y, z\} \cap \{k_3, h_1\} \neq \emptyset\}$. Note that $|E_{k_3, h_1}| \leq 28$. Hence

$$e(Q) = e(Q') + |E_{k_3, h_1}| \leq 24 + 28 = 52.$$

**Case 3.** $e(G) = 17$.

By Claim 6.4, let $h_1k_1, k_2g_2, h_3g_3 \in E(G)$ and $V' = \{h_1, k_1, k_2, g_2, h_3, g_3\}$. By the definition of $G$, $V'$ together with six vertices of $U$ forms three copies of $Y$, so we have $e(Q - V') = 0$. Let $Q_1 = Q - \{k_2, g_2, h_3, g_3\}$. We may assume that $Q_1$ has no matching of size two - otherwise two disjoint edges together with $k_3g_2$ and $h_3g_3$ would contradict the maximality of $T$. Now we claim that $e(Q_1) = \deg_{Q_1 - h_1}(h_1) + \deg_{Q_1 - k_1}(k_1) + \deg_{Q_1}(h_1, k_1) \leq 6$. First, clearly $\deg_{Q_1}(h_1, k_1) \leq 2$. Note that if one of $\deg_{Q_1 - k_1}(h_1)$ and $\deg_{Q_1 - h_1}(k_1)$ is nonzero, then the other is at most two. Indeed, if $\deg_{Q_1 - k_1}(h_1)$ is nonzero, say, $h_1g_3h_3 \in E(Q_1)$, then $\deg_{Q_1 - h_1}(k_1) \leq 2$ as the only possible edges are $k_1g_3h_3$ and $k_1g_3h_4$. This implies that $\deg_{Q_1 - k_1}(h_1) + \deg_{Q_1 - h_1}(k_1) \leq 4$ and our claim is proved.

Since $Q_2 := Q - \{h_1, k_1\}$ has no matching of size three, using Fact 4, we get $e(Q_2) \leq 24$. Let $E_{k_1, h_1} = \{xyz \in E(Q) : \{x, y, z\} \cap \{k_1, h_1\} \neq \emptyset\}$. First, $|E_{k_1, h_1} \setminus E(Q_1)| \leq 4^3 - 3 \times 3 \times 4 - 10 = 18$, where we subtract from all possible edges (43) the number of edges not containing $k_1$ or $h_1$ (3×3×4 of them) and the number of edges containing $h_1$ or $k_1$ on $V(Q_1)$ (10 of them). Thus, by $e(Q_1) \leq 6$ we get $|E_{k_1, h_1}| = |E_{k_1, h_1} \setminus E(Q_1)| + e(Q_1) \leq 24$. Hence

$$e(Q) = e(Q_2) + |E_{k_1, h_1}| \leq 24 + 24 = 48.$$

**Case 4.** $18 \leq e(G) \leq 20$.

The following claim characterizes the structure of $G$. Intuitively, if $G$ has no vertex cover of size three, then it is likely to contain a matching of size four, which can be extended to four disjoint copies of $Y$, contradicting the maximality of $T$.

**Claim 6.5.** $G$ has a vertex cover of size three.

*Proof.* First suppose that there exist $v_i, v_j \in V(G)$ such that $G - \{v_i, v_j\}$ has no matching of size two. Then $G - \{v_i, v_j\}$ is a star or a triangle. If $G - \{v_i, v_j\}$ is a star, then the claim holds. So we assume $G - \{v_i, v_j\}$ is a triangle. Let $v_i, v_j \in V_1$, $v_j \in V_2$. Since $G[V_1, V_2]$ has no matching of size three, when $i_1 = i_2$, letting $i_1 = 1$, then $e(G[V_1, V_i]) \leq 3 + 2 + 1 = 6$, $i = 2, 3$, and $e(G[V_2, V_3]) = 1$ (see Figure 2). Thus $e(G) \leq 6 + 6 + 1 < 18$. When $i_1 \neq i_2$, we have $e(G[V_i, V_j]) \leq 6$, $e(G[V_i, V_k]) \leq 5$ and $e(G[V_i, V_k]) \leq 5$ for $k \neq i_1, i_2$, so $e(G) \leq 5 + 5 + 6 < 18$, a contradiction.

Thus in the rest of the proof we may assume the following.

(†) For any $v_i, v_j \in V(G)$, $G - \{v_i, v_j\}$ has a matching of size two.

Next we separate the proof into two cases in terms of the maximum degree.

**Case a.** $\Delta(G) \geq 6$.

Without loss of generality, suppose $6 \leq \deg_G(v_1) \leq 8$. Let $G^{(1)} := G - v_1$. Obviously,

$$e(G^{(1)}) = e(G) - \deg_G(v_1) \geq 18 - 8 = 10. \quad (6.11)$$


We claim that $G^{(1)}$ has no matching of size three. Indeed, if $G^{(1)}$ has a matching $M$ of size three, then since $G[V_i, V_j]$ has no matching of size three, $N(v_1) \setminus V(M) \neq \emptyset$. Thus there exists $w \in V(G)$ such that $v_1w \in E(G)$ which together with $M$ forms a matching of size four in $G$, a contradiction.

**Fact 7.** For any $v_2 \in V(G)$, $v_2 \neq v_1$ and $i, j \in [3]$, we have $\deg_{G^{(1)}}[V_i, V_j](v_2) \leq 2$ and $\deg_{G^{(1)}}(v_2) \leq 3$.

**Proof.** By (†), $G - \{v_1, v_2\}$ has a matching $M'$ of size two.

Towards a contradiction, suppose $\deg_{G^{(1)}}[V_i, V_j](v_2) \geq 3$, then there exists $w' \in V(G^{(1)})$ such that $v_2w' \in E(G^{(1)})$ together with $M'$ forming a matching of size three in $G^{(1)}$, a contradiction. We have $\deg_{G^{(1)}}[V_i, V_j](v_2) \leq 2$ and thus $\deg_{G^{(1)}}(v_2) \leq 4$.

Next suppose $\deg_{G^{(1)}}(v_2) = 4$, then for any choice of $M'$, we must have $M' \subseteq E(G[N_{G^{(1)}}(v_2)])$. If not, suppose $M'$ covers $v' \notin N_{G^{(1)}}(v_2)$. Then there exists $w'$ such that $v_2w' \in E(G^{(1)})$ together with $M'$ forming a matching of size three in $G^{(1)}$ (see Figure 3), which is a contradiction. Thus $E(G^{(1)} - \{v_2\}) \subseteq E(G[N_{G^{(1)}(v_2)})$. Note that $e(G[N_{G^{(1)}(v_2)]) \leq 4$. Hence, $e(G^{(1)}) \leq 4 + 4 < 10$, which contradicts (6.11). \qed

**Figure 2.** An illustration of the proof of Claim 6.5, where the dotted region must induce a star and thus has at most three edges.

**Figure 3.** Any edge except the four dotted ones would result a matching of size two not completely in $N_{G^{(1)}}(v_2)$.

Now we resume the proof of Case a. Suppose $\{w_1w_2, w_3w_4\}$ is a matching of size two in $G^{(1)}$. Since $G^{(1)}$ has no matching of size three, $e(G[V(G^{(1)}) \setminus \{w_1, w_2, w_3, w_4\}) = 0$. Note that $G^{(1)}$ has no vertex cover of size three, otherwise by Fact 7, $e(G^{(1)}) \leq 3 \times 3 = 9$, which contradicts (6.11). Thus every $w_i$, $i \in [4]$, is incident an edge outside $\{w_1, w_2, w_3, w_4\}$, implying that there exist $u_1, u_2, u_3, u_4$ such that $w_iu_i \in E(G^{(1)})$. By the constraint of the matching number, $u_1 = u_2$, $u_3 = u_4$ and $\deg_{G^{(1)}}(w_i) = 2$, $i \in [4]$, i.e. $G^{(1)}$ is the union of two triangles $w_1w_2u_1$ and $w_3w_4u_2$. Hence, $e(G^{(1)}) \leq 3 + 3 < 10$, contradicting (6.11).
Case b. $\Delta(G) \leq 5$.
By Claim 6.4, let $v_iw_i \in E(G)$, $i \in [3]$ be a cross matching, $V' = \{v_1, v_2, v_3, w_1, w_2, w_3\}$ and $V'' = (V_1 \cup V_2 \cup V_3) \setminus V'$. Since there is no matching of size four in $G$, $e(G[V'']) = 0$ and for $\deg_{G[V'' \cup \{v_i\}]}(v_i)$ and $\deg_{G[V'' \cup \{w_i\}]}(w_i)$, either both of them are one with the same neighborhood in $V''$, or one of them is zero.

Claim 6.6. $e(G[V'' \cup \{a_i\}] = 0, i \in [3], \text{where } a_i \in \{v_i, w_i\}$.

Proof. Note that there exists $a_i \in \{v_i, w_i\}$, $a_j \in \{v_j, w_j\}$ such that $e(G[V'' \cup \{a_i\}] = e(G[V'' \cup \{a_j\}]) = 0$. If not, suppose $G$ has two triangles, $w_1v_1a$ and $w_2v_2b$, where $a, b \in V'''$, $a \neq b$. Then $G[V, V]$ which contains $v_3w_3$ has a matching of size three, a contradiction.

Suppose $e(G[V'' \cup \{v_1\}] = e(G[V'' \cup \{w_2\}] = 0$. Then there exists $a_3 \in \{v_3, w_3\}$ such that $e(G[V'' \cup \{a_3\}] = 0$. Suppose instead, i.e. $e(G[V'' \cup \{v_3\}] = e(G[V'' \cup \{a_3\}]) = 1$. Obviously, $e(G[v_3, w_1, w_2, w_3]) \leq 5$. Since $\deg_{G}(v_i) \leq 5$ for $i \in [2]$, we get $e(G) \leq 5 + 5 \times 2 + 5 = 17 < 18$, a contradiction.

By Claim 6.6, suppose $e(G[V'' \cup \{v_i\}] = 0, i \in [3]$. Note that $e(G - \{v_1, v_2, v_3\}) = e(G\{w_1, w_2, w_3\}) \leq 3$. Since $\deg_{G}(v_i) \leq 5, i \in [3], e(G)$, $w_1, w_2, w_3$ form a triangle and $\deg_{G}(v_i) = 5$, $i \in [3]$. This further implies that $\{v_1, v_2, v_3\}$ is an independent set. Thus $\deg_{G[V'' \cup \{v_i\}]} \leq 2$ and $e(G[V'' \cup \{v_i\}] \geq 3, i \in [3]$, implying that there exists a matching $M := \{v_1w_2, v_1x, v_2y, v_3z\}$ in $G$, where $x \in V''$, $y \in V'' \setminus \{x\}$, $z \in V'' \setminus \{x, y\}$, a contradiction. Consequently, $\Delta(G) \leq 5$ is impossible.

Claim 6.5 is proved.

By Claim 6.5, $G$ has a vertex cover $\{v_1, v_2, v_3\}$. Note that $v_1, v_2, v_3$ can not be in the same $V_i$ as otherwise $e(G) \leq 8 + 8 = 16 < 18$. By Claim 6.3, $e(Q) \leq |E_{v_1, v_2, v_3}|$. If $v_i \in V_i, i \in [3]$, then we have $e(Q) \leq 64 - 27 = 37$. If $v_1, v_2 \in V_2, v_3 \in V_3$, then we have $e(Q) \leq 64 - 24 = 40$.

Case 5. $e(G) = 21$.

By Claim 6.2, $G$ has a vertex cover $\{k_{i_1}, k_{i_2}, g_{i_3}\}$. By Claim 6.3, $e(Q) = |E_{k_{i_1}, k_{i_2}, g_{i_3}}| \leq 37$.

Suppose the number of triples in each of cases above are $x_1, x_2, x_3, x_4, x_5$ respectively, then $x_1 + x_2 + x_3 + x_4 + x_5 = \binom{m_1}{3}$. Note that by counting edges via triples e.g. $Y_1, Y_2, Y_3$, all $YYU$ edges reflected by $G$ are counted exactly $m_1 - 2$ times. Putting everything together we get

\[ y_4 \leq 2(64x_1 + 52x_2 + 48x_3 + 40x_4 + 37x_5) + \frac{1}{m_1 - 2}(16x_2 + 17x_3 + 20x_4 + 21x_5)2|U| + O(n^2) \]

\[ \leq 74 \left(\binom{m_1}{3}\right) + 54x_1 + 30x_2 + 22x_3 + 6x_4 + 14 \left(\frac{m_1}{2}\right)|U| - (21x_1 + 5x_2 + 4x_3 + x_4)2\frac{|U|}{m_1} + O(n^2), \]

where the error term comes from the remaining edges in $YYU$ which are not reflected in $G$. By $|U|/m_1 \geq 3$, we have $54x_1 + 30x_2 + 22x_3 + 6x_4 - (21x_1 + 5x_2 + 4x_3 + x_4)2\frac{|U|}{m_1} \leq 0$ and we are done.

We conclude this section by deriving Theorem 1.4 from Theorem 6.1 and the regularity method.

Proof of Theorem 1.4. Define constants such that $0 < 1/n < \varepsilon' < d, \varepsilon$. By Theorem 1.3, $H$ contains a $Y$-tiling $M'$ of size $\varepsilon'n$. Apply the regularity lemma to $H' := H[V(H) \setminus V(M')]$, let $V_1, \ldots, V_K$ be the clusters of the partition obtained. Note that $e(H') \geq \binom{n - 4\varepsilon' n}{3} - \binom{n - 4\varepsilon' n(1 - \alpha)}{3} + o(n^3)$. By Lemma 3.2, the reduced hypergraph $R$ satisfies $|E(R)| \geq \binom{K}{3} - (K - \alpha K)^{3} + o(K^3)$. Applying Theorem 6.1, we get a $\{Y, E\}$-tiling $T = \{Y_1, Y_2, \ldots, Y_{m_1}, E_1, E_2, \ldots, E_{m_2}\}$ in $R$, such that $4m_1 + 3m_2 \geq 4\alpha K$.\[ \square \]
For each $Y_i \in \mathcal{T}$ with edges $i_1i_2i_3$, $i_2i_3i_4$, $E_j = \{j_1, j_2, j_3\} \in \mathcal{T}$, let $F_{Y_j} := [(1 - 2\varepsilon')m]$, $F_{E} := \frac{2}{3}[(1 - 2\varepsilon')m]$, where $m$ is the size of each cluster. Now construct a $Y$-tiling $M''$ in $H'$, by greedily adding copies of $Y$ to $M''$ using the regularity.

Recall that the Turán density of every $k$-partite $k$-graph is zero. Let $H'_1 := H'[V_{i_1}, V_{i_2}, V_{i_3}]$, $H'_2 := H'[V_{i_2}, V_{i_3}, V_{i_4}]$ and $H'_3 := H'[V_{i_1}, V_{i_2}, V_{i_3}]$. In $H'_1$ and $H'_2$, we repeatedly find disjoint copies of $Y$ each of which intersects each of $V_{i_2}$ and $V_{i_3}$ in one vertex and $V_{i_1}$ (or $V_{i_4}$) in two vertices, until $M$ contains precisely $F_Y$ disjoint copies of $Y$ in $H'[V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}]$ (see Figure 4). In $H'_3$, we find a copy of $K_{4,4,4}$ and decompose it into three copies of $Y$.

Note that at each stage of this process, the number of vertices in each $V_i \in V(R)$ that would be covered by $M''$ is at most $(1 - 2\varepsilon')m + 2 \leq (1 - \varepsilon')m$. Obtained by deleting the vertices which has been covered by $M''$, the subgraphs of $H'_i$ for $i \in [3]$ are still regular. Thus it is possible to successively add copies of $Y$ to $M''$ in order to obtain a $Y$-tiling $M''$ as desired. Note that the size of $M''$ is

$$m_1F_Y + m_2F_E \geq (1 - 2\varepsilon')m_1 + \frac{3}{4}m_2 \geq (1 - 2\varepsilon')maK$$

$$\geq (1 - 2\varepsilon')\alpha(n - 4\varepsilon'n) \geq (1 - 6\varepsilon')\alpha n \geq \alpha n - \varepsilon'n,$$

where we used $\alpha \leq 1/7$.

Consider the union of $M'$ and $M''$. Thus $H$ has a $Y$-tiling of size at least $\alpha n$, i.e. covering at least $4\alpha n$ vertices.

![Figure 4. A Y-tiling in the clusters corresponding to a copy of Y in the reduced graph](image)

### 7. The Minimum Degree Condition of $Y_{k,b}$-tiling

For $k > b \geq 0$, recall that $Y_{k,b}$ is the $k$-graph consisting of two edges that intersect in exactly $b$ vertices. For $p > 0$, fix two $k$-graphs $F$ of order $p$ and $H$, let $\mathcal{F}_{F,H} \subseteq \binom{V(H)}{p}$ be the family of $p$-sets in $V(H)$ that span a copy of $F$. A fractional $F$-tiling in $H$ is a function $\omega : \mathcal{F}_{F,H} \to [0, 1]$ such that for each $v \in V(H)$ we have $\sum_{e \in \mathcal{F}_{F,H}} \omega(e) \leq 1$. Then $\sum_{e \in \mathcal{F}_{F,H}} \omega(e)$ is the size of $\omega$. Such a fractional $F$-tiling is called perfect if it has size $n/p$.

When $F$ is a single edge, a fractional $F$-tiling is also called a fractional matching. The size of the largest fractional matching in a $k$-graph $H$ is denoted by $\nu^*(H)$. Its dual problem is to find a minimum fractional vertex cover $\tau^*(H) = \sum_{v \in V} \omega(v)$ over all functions $\omega : V \to [0, 1]$ such that for each $e \in E$ we have $\sum_{v \in e} \omega(v) \geq 1$. Then by the Duality Theorem, $\nu^*(H) = \tau^*(H)$. Let $f^*_s(F,n)$ be the minimum integer $m$ so that every $n$-vertex $k$-graph $H$ with $\delta_d(H) \geq m$ has a fractional $F$-tiling of size $s$. In particular, $\delta_0(H) = \nu(H).$
Alon et al. [1] proved the following result for fractional matchings. We extend it to fractional $Y$-tilings by similar arguments. Given a $k$-graph $H$ and a set $L$ with $|L| < k$, the link graph $H(L)$ is defined as a $(k - |L|)$-graph with vertex set $V(H) \setminus L$ and edge set $\{e \setminus L : L \subseteq e \in E(H)\}$.

**Theorem 7.1.** For all $k \geq 3$, $1 \leq d < b \leq k - 1$, and $n \geq k$,

$$f_d^{n/(2k-b)}(Y_{k,b}, n) \leq f_0^{n/(2k-b)}(Y_{k-d,b-d}, n - d).$$

**Proof.** Let $H$ be an $n$-vertex $k$-graph with $\delta_d(H) \geq f_0^{n/(2k-b)}(Y_{k-d,b-d}, n - d)$ and let $H'$ be a $(2k - b)$-graph on $V(H)$, where $E(H') = \mathcal{F}_{Y_{k,b}, H}$. By the minimum $d$-degree assumption, for every $d$-set $L$ in $H$, $e(H(L)) \geq \delta_d(H) \geq f_0^{n/(2k-b)}(Y_{k-d,b-d}, n - d)$, then there is a fractional $Y_{k-d,b-d}$-tiling $\omega_1 : \mathcal{F}_{Y_{k-d,b-d}, H(L)} \to [0, 1]$ such that for every $v \in V$ we have $\sum_{e \in \mathcal{F} \omega_1(Y) \leq 1}$ and $\sum_{e \in \mathcal{F} \omega_1(Y) \geq n/(2k-b) - e)$. Let $H'(L)$ be a $(2k - b - d)$-graph on $V(H) \setminus L$, where $E(H'(L)) = \mathcal{F}_{Y_{k-d,b-d}, H(L)}$. We claim that $\omega_1$ is also a fractional matching of $H'(L)$. Indeed, every copy of $Y_{k-d,b-d}$ in $H(L)$ together with $L$ spans a copy of $Y_{k,b}$ in $H$, that is, $\mathcal{F}_{Y_{k-d,b-d}, H(L)} \subseteq H'(L)$ and thus the claim follows. This implies that for every $d$-set $L$, $\nu^*(H'(L)) \geq n/(2k-b)$.

We will assume that there is no perfect fractional $Y_{k,b}$-tiling in $H$. Then by the definition of $H'$, there is no perfect fractional matching in $H'$. We will show that for a particular choice of $L$, $\nu^*(H'(L)) < n/(2k-b)$, contradicting the previous paragraph.

As $\tau^*(H') = \nu^*(H')$, there is a function $\omega : V \to [0, 1]$ such that $\sum_{v \in V} \omega(v) < n/(2k-b)$ and, for every $e \in E(H')$, we have $\sum_{v \in e} \omega(v) \geq 1$. Next, we define a $(2k - b)$-graph

$$H'_\omega := \left\{ e \in \left(\frac{V}{2k-b}\right) : \sum_{v \in e} \omega(v) \geq 1 \right\}.$$ 

Suppose $L$ is the set of $d$ vertices with the smallest weights in $\omega$. Without loss of generality, we may assume that the $d$ lowest values of $\omega(x)$ are all equal to each other, since otherwise we could replace them by their average. (Obviously, this would not change $\sum_{v \in V} \omega(v)$ nor the set $L$.) Note that the minimum $d$-degree $\delta_d(H'_\omega)$ is achieved by the $d$-set $L$. Let $H'_\omega(L)$ be the neighborhood of $L$ in $H'_\omega$, that is, a $(2k - b - d)$-graph on the vertex set $V \setminus L$ and with the edge set

$$\left\{ S \in \left(\frac{V}{2k-b-d}\right) : S \cup L \in E(H'_\omega) \right\}.$$ 

Let $x = \min_{v \in V} \omega(v)$ and observe that $x < 1/(2k-b)$. If $x > 0$, then apply the linear map

$$\omega' = \frac{\omega - x}{1 - (2k-b)x}.$$ 

Then, still $\sum_{v \in V} \omega'(v) < n/(2k-b)$ and $H'_\omega = H'_\omega'$. Moreover, for every $v \in L$, we have $\omega'(v) = 0$. It follows that the function $\omega'$ restricted to the set $V \setminus L$ is a fractional vertex cover of $H'(L)$, so $\nu^*(H'(L)) = \tau^*(H'(L)) < n/(2k-b)$, which completes the proof. \qed

The following lemma allows us to convert an almost perfect fractional $Y_{k,b}$-tiling into an almost perfect integer $Y_{k,b}$-tiling. Its proof is an application of the regularity method and similar to that of Lemma 2.3 and Theorem 1.4. Here we omit the proof.

**Lemma 7.2.** Suppose $k \geq 3$ and $1 \leq d < b \leq k - 1$, and $0 < 1/n \leq \varepsilon' < \varepsilon \leq \eta$. Let $H$ be a $k$-graph on $n$ vertices with

$$\delta_d(H) \geq f_d^{n/(2k-b)-\varepsilon'n}(Y_{k,b}, n) + \eta \left(\frac{n}{k-d}\right).$$

Then $H$ contains a $Y_{k,b}$-tiling covering all but at most $\varepsilon n$ vertices.

The following result is a consequence of Theorem 7.1, Lemma 7.2, Theorem 1.3 and Theorem 1.4.
Theorem 7.3. Let $0 < 1/n \ll \varepsilon \ll \eta$. Suppose $k \geq 3$, $1 \leq d < 2\ell \leq k - 1$ such that $2k - 2\ell \geq (2k - 2\ell - d)^2 + 1)(k - d - 1) + 1$ or suppose $k$ is odd, $k \geq 7, \ell = (k - 1)/2$ and $d = k - 3$. Let $H$ be a $k$-graph of order $n$ with

$$\delta_d(H) \geq \left( \frac{n}{k - d} \right) - \left( \frac{n - \frac{n}{2k - 2\ell}}{k - d} \right) + o(n^{-d}) = (1 - \frac{1}{2(k - \ell)})^{k-d} + \eta \left( \frac{n}{k - d} \right).$$

Then $H$ contains a $Y_{k,2\ell}$-tiling covering all but at most $\varepsilon n$ vertices. In particular, $t(k,d,\ell) \leq \frac{1}{1 - (\frac{1}{2(k-\ell)})^{k-d}}$.

Proof. Suppose that we have constants such that $0 < 1/n \ll \varepsilon' < \varepsilon \ll \eta$. Since

$$\delta_d(H) \geq \left( \frac{n}{k - d} \right) - \left( \frac{n - \frac{n}{2k - 2\ell}}{k - d} \right) + o(n^{-d}),$$

by Lemma 7.2, it suffices to show

$$f_{\frac{n}{k-\ell}}^{n^{\varepsilon'}}(Y_{k,2\ell}, n) \leq \left( \frac{n}{k - d} \right) - \left( \frac{n - \frac{n}{2k - 2\ell}}{k - d} \right) + o(n^{-d}). \quad (6.11)$$

If $k$ is odd, $k \geq 7, \ell = (k - 1)/2$ and $d = k - 3$, then by Theorem 7.1 and Theorem 1.4, we have

$$f_{\frac{n}{k-3}}^{n^{\varepsilon'}}(Z_{k,2\ell}, n) \leq f_{\frac{n}{n+1}}^{n^{\varepsilon'}}(Y_{3,2}, n - k + 3) \leq \left( \frac{n}{3} \right) - \left( \frac{n - \frac{n}{3}}{3} \right) + o(n^3).$$

Similarly, if $k \geq 3, 1 \leq d < 2\ell \leq k - 1$ such that $2k - 2\ell \geq (2k - 2\ell - d)^2 + 1)(k - d - 1) + 1$, then we apply Theorem 7.1 and Theorem 1.3. Thus

$$f_{d}^{\frac{n^{\varepsilon'}}{2k-2\ell}}(Y_{k,2\ell}, n) \leq f_{d}^{n/(2k-2\ell)}(Y_{k,2\ell}, n) \leq \left( \frac{n}{k - d} \right) - \left( \frac{n - \frac{n}{2k - 2\ell}}{k - d} \right) + o(n^{-d}).$$

Then we are done in both cases. Note that the “in particular” part of the theorem follows from the definition of $t(k,d,\ell)$.

Now Theorem 1.7 follows immediately from Theorem 1.8, Proposition 1.9 and Theorem 7.3.

![Diagram](image)

**Figure 5.** A diagram summarizing the flow of the proofs of our main results

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APPENDIX A. PROOF OF THE ABSORBING PATH LEMMA

We first present an extension lemma, for which we need the following notation. Suppose that $H$ is a $(k,s)$-complex with vertex classes $V_1, \ldots, V_s$ of size $m$, and also suppose that $G$ is a $(k,s)$-complex with vertex classes $X_1, \ldots, X_s$ of size at most $m$. We say that $H$ respects the partition of $G$ if whenever $G$ contains an $i$-edge with vertices in $X_{j_1}, \ldots, X_{j_i}$, then there is an $i$-edge of $H$ with vertices in $V_{j_1}, \ldots, V_{j_i}$. On the other hand, we say that a labelled copy of $G$ in $H$ is partition-respecting if for $i \in [s]$ the vertices corresponding to those in $X_i$ lie within $V_i$.

Roughly speaking, the extension lemma says that if $G'$ is an induced subcomplex of $G$, and $H$ is suitably regular, then almost all copies of $G'$ in $H$ can be extended to a large number of copies of $G$ in $H$. We write $|G'|_H$ for the number of labelled partition-respecting copies of $G'$ in $H$.

Lemma A.1 (Extension lemma, [6], Lemma 5). Let $k, s, r, b', b'', m_0$ be positive integers, where $b' < b''$, and let $c, \beta, d_2, \ldots, d_k, \delta$ be positive constants such that $1/d_i \in N$ for all $i < k$ and

$$1/m_0 \leq 1/r, \delta \leq c \leq \min\{d_2, d_4, \ldots, d_{k-1}\} \leq \delta \leq \beta, d_k, 1/s, 1/b''.$$ 

Then the following holds for all integers $m \geq m_0$. Suppose that $G'$ is a $(k,s)$-complex on $b'$ vertices with vertex classes $X_1, \ldots, X_s$ and let $G$ be an induced subcomplex of $G'$ on $b'$ vertices. Suppose also that $H$ is a $(d, \delta, \beta, \delta, r)$-regular $(k,s)$-complex with vertex classes $V_1, \ldots, V_s$, all of size $m$, which respects the partition of $G'$. Then all but at most $\beta|G'|_H$ labelled partition-respecting copies of $G$ in $H$ are extendible to at least c$m(b''-b')$ labelled partition-respecting copies of $G'$ in $H$.

Let $H$ be a $k$-graph, and let $S$ be a $(k-\ell)$-set of $V(H)$. The following absorbing path was constructed and used in [33], which we will use as well in our proof.

Proposition A.2 ([33], Proposition 6.1). Suppose that $k \geq 3$, and that $1 \leq \ell \leq k-1$ is such that $(k-\ell) \not| k$. Then there is a $k$-partite $k$-graph $\mathcal{AP}(k,\ell)$ with the following properties.

1. $|\mathcal{AP}(k,\ell)| \leq k^4$.
2. The vertex set of $\mathcal{AP}(k,\ell)$ consists of two disjoint sets $S$ and $X$ with $|S| = k-\ell$. 
\( \mathcal{AP}(k, \ell) \) contains an \( \ell \)-path \( P \) with vertex set \( X \) and ordered ends \( P^{\text{beg}} \) and \( P^{\text{end}} \).

(4) \( \mathcal{AP}(k, \ell) \) contains an \( \ell \)-path \( Q \) with vertex set \( S \cup X \) and ordered ends \( P^{\text{beg}} \) and \( P^{\text{end}} \).

(5) No edge of \( \mathcal{AP}(k, \ell) \) contains more than one vertex of \( S \).

(6) No vertex class of \( \mathcal{AP}(k, \ell) \) contains more than one vertex of \( S \).

Note that the \( k \)-graph \( \mathcal{AP}(k, \ell) \) may not be unique. For our proof we just fix an arbitrary such \( k \)-graph satisfying all properties in Proposition A.2. Let \( b(k, \ell) = |\mathcal{AP}(k, \ell)| - k + \ell \). Then \( b(k, \ell) \) is the order of the \( \ell \)-path \( P \).

Let \( c > 0 \). Recall that a \( (k - \ell) \)-set \( S \) of \( V(H) \) is \( c \)-good (otherwise \( c \)-bad) if \( H \) contains at least \( cn^{b(k, \ell)} \) absorbing paths for \( S \), each on \( b(k, \ell) \) vertices. The following lemma shows that for the values of \( k \) and \( \ell \) that we are interested in, and any small \( c \), if \( H \) is sufficiently large and has large minimum degree, then almost all \((k-\ell)\)-sets of \( V(H) \) are \( c \)-good. For convenience, let \( b := b(k, \ell) \).

**Lemma A.3.** Suppose that \( k \geq 3 \), \( 1 \leq \ell \leq k - 1 \) and that \( k - \ell \leq d \leq k - 1 \) is such that \( (k - \ell) \gamma_{m} \approx n \) and \( \frac{1}{n} \ll c \ll \gamma \ll \mu \ll k - 1 \). Let \( H \) be a \( k \)-graph on \( n \) vertices such that \( \delta_{d}(H) \geq \mu \langle \frac{n}{k - d} \rangle \). Then at most \( \gamma n^{k-\ell} \) sets \( S \) of \( k - \ell \) vertices of \( H \) are \( c \)-bad.

**Proof.** We choose more constants that satisfy the following hierarchy
\[
\frac{1}{n} \ll \frac{1}{r}, \, \delta \ll c \ll \min\{\delta_{k}, \frac{1}{\ell}\} \ll \delta_{k}, \, \eta \ll d' \ll \gamma \ll \mu' \ll \mu, \, k - 1
\]
and assume that \( t! \mid n \). Let \( H \) be a \( k \)-graph on \( n \) vertices such that \( \delta_{d}(H) \geq \mu \langle \frac{n}{k - d} \rangle \). Apply Lemma 3.1 to \( H \), and let \( V_{1}, \ldots, V_{a_{1}} \) be the clusters of the partition obtained where \( a_{i} \leq t \). Let \( m = \frac{n}{a_{1}} \) be the size of each cluster. Define \( R \) as the reduced \( k \)-graph on these clusters.

The following claim shows that for almost all \((k-\ell)\)-sets \( T \) of \( V(H) \), \( T \) is contained in clusters lying in some edge of \( R \).

**Claim A.4.** For all but at most \( \gamma n^{k-\ell}/2 \) \((k-\ell)\)-sets \( T = \{v_{1}, v_{2}, \ldots, v_{k-\ell}\} \subseteq V(H) \), there are \( V_{i_{1}}, V_{i_{2}}, \ldots, V_{i_{k}} \) which form an edge in \( R \), where \( v_{j} \in V_{i_{j}}, 1 \leq j \leq k - \ell \).

**Proof.** First, we show that \( \deg_{R}(S) \geq 1 \) holds for all but at most \( \gamma a_{1}^{k-\ell}/3 \) \((k-\ell)\)-sets \( S \subseteq V(R) \).

Indeed, by Lemma 3.2, all but at most \( \theta a_{1}^{k-\ell} \) sets \( X \subseteq V(R) \) satisfy \( \deg_{R}(X) \geq \mu \langle \frac{a_{1}}{k - d} \rangle \). Suppose that there are at least \( O(a_{1}^{k-\ell}) \) \((k-\ell)\)-sets \( S \) of \( V(R) \) which are not in any edge of \( R \), then the number of \( d \)-sets \( X \) satisfying \( \deg_{R}(X) \leq \mu \langle \frac{a_{1}}{k - d} \rangle \) is more than \( O(a_{1}^{k-\ell}) \), which is a contradiction.

By the definition of equitability, at most \( \eta n^{k-\ell} \ll \gamma n^{k-\ell}/6 \) sets \( T \) of \( k - \ell \) vertices of \( H \) do not lie in \( Cross_{k-\ell} \). For those \( T \) lying in \( Cross_{k-\ell} \), there are at most \( \gamma n^{k-\ell}/3 \) such \( T \) whose corresponding clusters do not lie in an edge of \( R \).

Now we continue the proof. Suppose that \( V_{i_{1}}, V_{i_{2}}, \ldots, V_{i_{k}} \) are clusters forming an edge of \( R \). Let \( I \) be the index set \( \{i_{1}, \ldots, i_{k}\} \). By the definition of \( R \), there exists \( J \in Cross_{I} \), such that \( H \) is \((\delta_{k}, r)\)-regular w.r.t. \( P^{k-1}(J) \). Let \( H^{*} \) be the \((k, k)\)-complex as defined in \((*)\). Thus \( H^{*} \) is \((d, \delta_{k}, \delta, r)\)-regular and satisfies the conditions of Lemma A.1. Moreover, recall that \( \mathcal{AP}(k, \ell) \subseteq \mathcal{AP}(k, \ell) \subseteq \mathcal{AP}(k, \ell) \subseteq \mathcal{AP}(k, \ell) \) by (4) and (5) of Proposition A.2. Now we can apply Lemma A.1 and conclude that all but at most \( \gamma n^{k-\ell}/2 \) ordered sets \( S' = \{v_{1}, v_{2}, \ldots, v_{k-\ell}\} \) with \( v_{j} \in V(i_{j}) \) can be extended to at least \( cnb^{b} \) labelled partition-respecting copies of \( \mathcal{AP}(k, \ell) \) in \( H^{*} \). Considering each copy \( C \) of \( \mathcal{AP}(k, \ell) \), by the structure of \( \mathcal{AP}(k, \ell) \), \( C - S' \) is an absorbing path for \( S' \). Thus, \( H^{*} \) contains at least \( cnb^{b} \) absorbing paths on \( b \) vertices for \( S' \). Thus at most \( \gamma n^{k-\ell}/2 \) such sets \( S' \) are \( c \)-bad.

Summing over all sets of \( k - \ell \) clusters in \( P \), there are at most \( \gamma n^{k-\ell}/2 \) \( c \)-bad \((k-\ell)\)-sets with no two vertices in the same cluster and the corresponding \((k-\ell)\) clusters are in some edge of \( R \). Combining with Claim A.4, we get that the number of \( c \)-bad \((k-\ell)\)-sets of \( V(H) \) is at most \( \gamma n^{k-\ell} \).\qed
Given Lemma A.3, the proof of Lemma 2.2 follows verbatim as the proof of [33, Lemma 6.3], after replacing [33, Corollary 5.4] with Lemma 2.1 and [33, Lemma 6.2] with Lemma A.3.

**Appendix B. Proof of (6.10)**

Rewriting all terms above with $M_1$ and $M_2$ in (6.9) with $M_1 = 4m_1$, $M_2 = 3m_2$ and $|U| = 3n/7 = (3/4)(M_1 + M_2)$, we get

$$e(H) \leq \frac{M_1}{4} \binom{\frac{3}{4}(M_1 + M_2)}{2} + \frac{21}{4} \binom{\frac{1}{4}M_1}{2} (M_1 + M_2) + \frac{9}{4} \binom{\frac{3}{4}M_2}{2} (M_1 + M_2) + \frac{3}{8} M_1 M_2 (M_1 + M_2)$$

$$+ 37 \binom{\frac{1}{4}M_1}{3} + 19 \binom{\frac{1}{3}M_2}{3} + 10 \binom{\frac{1}{4}M_1}{2} M_2 + 6 \binom{\frac{3}{4}M_2}{2} M_1 + O(n^2)$$

$$\leq \frac{9}{64} M_1 \binom{M_1 + M_2}{2} + \left[ \frac{21}{64} \binom{M_1}{2} (M_1 + M_2) + \frac{1}{4} \binom{M_2}{2} (M_1 + M_2) + \frac{3}{8} M_1 M_2 (M_1 + M_2) \right]$$

$$+ \left[ \frac{37}{64} \binom{M_1}{3} + \frac{19}{27} \binom{M_2}{3} + \frac{5}{8} \binom{M_1}{2} M_2 + \frac{2}{3} \binom{M_2}{2} M_1 \right] + O(n^2)$$

$$\leq \frac{27}{64} \binom{M_1 + M_2}{3} + \frac{21}{64 \times 2} (M_1 + M_2)^3 + \frac{37}{64} \binom{M_1 + M_2}{3} + O(n^2)$$

$$\leq \frac{127}{64 \times 6} (M_1 + M_2)^3 + O(n^2) = \frac{127}{73 \times 6} n^3 + O(n^2)$$

$$= \binom{n}{3} - \binom{n - n/7}{3} + O(n^2),$$

where we used $M_1 + M_2 = 4n/7$, $(M_1 + M_2)^2 = M_1^2 + M_2^2 + 2M_1 M_2$, and $\binom{M_1 + M_2}{3} = \binom{M_1}{3} + \binom{M_2}{3} + \binom{M_1}{2} M_2 + \binom{M_2}{2} M_1$.

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