Exactly Soluble Model of a 3D Symmetry Protected Topological Phase of Bosons with Surface Topological Order

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We construct an exactly soluble Hamiltonian on the D=3 cubic lattice, whose ground state is a topological phase of bosons protected by time reversal symmetry, i.e. a symmetry protected topological (SPT) phase. In this model anyonic excitations are shown to exist at the surface but not in the bulk. The statistics of these surface anyons is explicitly computed and shown to be identical to the 3-fermion $\mathbb{Z}_2$ model, a variant of $\mathbb{Z}_2$ topological order which cannot be realized in a purely D=2 system with time reversal symmetry. Thus the model realizes a novel surface termination for SPT phases, which only becomes available in D=3: that of a fully symmetric gapped surface with topological order. The 3D phase found here is also outside the group cohomology classification that appears to capture all SPT phases in lower dimensions. It is identified with a phase previously predicted from a field theoretic analysis, whose surface is roughly ‘half’ of a Kitaev $E_8$ phase. Our construction utilizes the Walker-Wang prescription to create a 3D confined phase with surface anyons, which can be extended to other symmetry classes and topological orders.

\textbf{Introduction} Recently, there has been much interest in studying gapped phases of interacting bosons that, despite having only conventional gapped excitations in the bulk, are topologically distinct from the trivial phase. Such short range entangled (SRE) phases are significantly simpler than topologically ordered phases, such as fractional quantum Hall states and gapped spin liquids, whose bulk anyonic excitations reflect their long range entangled nature\cite{1,2}. Nevertheless, SRE topological phases display a host of interesting phenomena: for example, like the noninteracting fermionic topological insulators, they possess exotic surface states\cite{3–5}. However, in contrast to the fermionic case, topological phases of bosons cannot be realized without interactions. Often, SRE topological phases are well defined only in the presence of a symmetry. In those cases, they are termed Symmetry Protected Topological (SPT) phases. The best known example is the Haldane antiferromagnetic chain of S=1 moments in 1+1D, which is in a SPT phase protected by spin rotation symmetry\cite{6,7}. Bosonic topological phases in 1+1 dimension occur only in the presence of a symmetry group $G$, and are rigorously argued to be classified by the second cohomology group $H^2(G,U(1))$\cite{8,12}.

Recently it has been realized that analogous states exist in higher dimensions\cite{13,17}, and could potentially be realized as ground states of frustrated magnetic insulators or ultra cold systems of lattice bosons\cite{18–21}. In 2+1D the situation is especially rich, since SRE topological phases that exist independent of symmetry are possible. Bosonic phases with $c_- = 8n$ chiral bosonic edge modes (with $n$ any integer) were shown by Kitaev to be compatible with SRE. This leads to a series of topological phases with quantized thermal Hall conductivity $\kappa_{xy}/T = 8n\$\cite{22,23}. Thus different integers $n$ lead to physically distinct transport properties, giving a sharp criterion that can be used to distinguish these phases. We will refer to the $n = 1$ member of this sequence as the Kitaev $E_8$ state. In the presence of symmetry, more topological phases have been predicted, and it was proposed that SPT phases are classified by the higher dimensional cohomology groups $H^{d+1}(G,U(1))$ in $d + 1$ dimensions\cite{13,14}. The predicted set of phases from this group cohomology approach was reproduced using other techniques in a number of specific cases: in 2D via dualities\cite{15} and the Chern Simons K-matrix theory\cite{16,24}, and by various field theoretical approaches in 3D\cite{17,29}.

Intriguingly, however, the field theoretical approach\cite{17} predicted new SPT phases in 3+1 D protected by time reversal symmetry $\mathcal{T}$. Thus, in contrast to the single nontrivial phase predicted by group cohomology, two distinct types of nontrivial phases were obtained. One of these nontrivial phases has a close connection to the 2D Kitaev $E_8$ state: a domain wall between opposite $\mathcal{T}$ breaking regions on the surface carries a chiral central charge $c_- = 8$, i.e. 8 chiral boson modes, and corresponds to the edge of the Kitaev $E_8$ state. Thus each domain may be associated with half this number of chiral edge states, a feature that is forbidden in a purely 2D system in the absence of topological order. This bosonic state is analogous to the 3+1D free fermion topological superconductor (class DIII), where a surface domain wall between regions of opposite $\mathcal{T}$ breaking is associated with a single chiral Majorana mode. Hence we will refer to the bosonic phase as the 3D Bosonic Topological Superconductor (BTSc). This state was proposed to be outside the group cohomology classification, and although it was discussed as a physical possibility in Ref. 17, more
direct evidence for its existence would be welcome. This would shed light on the exact relation between group cohomology and SPT phases. Here we irrefutably demonstrate its existence by providing an exactly soluble model that realizes this phase.

Surface states of 2+1D SPT phases must either be gapless or spontaneously break symmetry. This is analogous to the surface states of free fermion topological insulators which are required either to be metallic or break one of the protecting symmetries. On the 2D surface of a 3D SPT phase, a novel possibility may arise - a fully gapped and symmetric state is allowed if the surface develops topological order [17]. However, the resulting topologically ordered surface is anomalous - i.e. it implements the global symmetries in a way that cannot be realized in a strictly 2+1D phase [17]. In the context of the 3D SPT phase, the topologically ordered surface is predicted to realize the “3-fermion $Z_2$ state”, with $\mathcal{T}$. Recall that the regular $Z_2$ topological order, as embodied by the toric code or a $Z_2$ gauge theory, has three nontrivial anyons: the electric ($e$) and magnetic ($m$) charges and their bound state ($\epsilon$). $e$ and $m$ are bosons, and $\epsilon$ is a fermion; all excitations have mutual $\pi$ statistics. In the 3-fermion $Z_2$ state, all three nontrivial anyons are fermions, again with mutual $\pi$ statistics. Ref. [17] proposed that this is a possible surface termination of the 3D BTSc with time reversal symmetry. A strictly 2D realization of the 3-fermion $Z_2$ state always breaks time reversal symmetry since it is associated with $c_- = 4$ chiral edge boson modes [22]. Since this surface state in the presence of $\mathcal{T}$ can never be realized as a 2D system, it qualifies as the protected surface state of a 3+1D SPT phase. Our exact solution produces precisely the 3-fermion $Z_2$ state in the presence of a boundary, while preserving $\mathcal{T}$. Thus not only does our solvable model exhibit an interesting bulk phase; it also turns out to have the most exotic possible surface state. The key observation is that although a 2D realization of the 3-fermion $Z_2$ phase always breaks $\mathcal{T}$, the self and mutual statistics of the anyons only involve real $\pm 1$ phases, allowing us to construct a $\mathcal{T}$ symmetric 3D model.

We use the Walker-Wang [27, 28] prescription for constructing an exactly soluble model of a 3+1D phase given a 2+1 dimensional topological quantum field theory (TQFT) [29]. Let us assume that the anyons of the topological theory are such that the only particle that has trivial mutual statistics with all the others is the identity particle (a Modular Tensor Category). In this case the Walker-Wang models realize a confined 3+1D phase [27, 28]. However, the surfaces are naturally endowed with the topological order of the 2+1D TQFT. Since the bulk is not topologically ordered, and, in the absence of symmetry, there is no obstruction to destroying the topological order at the surface, these models lead to trivial insulators [30]. This is an ideal starting point for constructing SPT phases, which, in the absence of symmetry, are connected to the trivial insulator. As we will see, in the presence of a global symmetry, the Walker-Wang models can have topologically protected surface states.

In the remainder of this paper, we first discuss some background about the 3-fermion $Z_2$ model and the Walker-Wang construction. We explain why in these models the bulk is confined, but deconfined excitations of a specific kind exist at the boundary. Since these arguments are rather general and intuitive we review them before presenting a microscopic model. We then discuss a 3D model on the cubic lattice based on the 3-fermion $Z_2$ state, which is composed of a sum of commuting projectors and hence exactly soluble. The model is shown to be $\mathcal{T}$ symmetric, confined in the bulk but with deconfined excitations at the surface, which are explicitly shown to have the same statistics as the 3-fermion $Z_2$ state.

**Background and Walker-Wang Construction**

We begin by describing the 3 fermion state in 2D, introduced by Ref. [22]. We may think of the three types of fermions as fermionic $Z_2$ charges ($e$), fermionic $Z_2$ fluxes ($m$), and a bound state of a charge and a flux ($\epsilon$). Because the charges acquire a $\pi$ Berry phase upon encircling the $Z_2$ fluxes, it can be checked that $\epsilon$ is also a fermion. Moreover, the three species of anyonic excitations all have mutual semionic statistics, i.e. braiding one around another induces a phase factor of $-1$. Due to the symmetry in these statistics, we call this topological state the ‘three fermion $Z_2$ model’; in practice it is irrelevant which one of the labels $\{e, m, \epsilon\}$ we assign to the flux and which to the ‘original’ fermionic charge. We will use the label 1 to designate the vacuum.

One important property of this three fermion model in 2D is that it breaks time reversal symmetry. This becomes evident when the system is put on a disc whose boundary will have chiral edge modes. In particular, using the relation between the topological properties of the bulk anyons and the edge chiral central charge

$$\sum_\alpha d^2_\alpha \theta_\alpha / \sqrt{\sum_\alpha d^2_\alpha} = e^{2\pi \mathcal{c}_- / 8} \quad (1)$$

it is easy to check that $c_- = 4 \, (\text{mod } 8)$, with quantum dimensions $d_\alpha = 1$, and topological spin $\theta_\alpha = -1$ for $e, m$, and $\epsilon$, corresponding to fermions. Therefore, the bulk topological property dictates that this state can not be realized in a 2D time reversal invariant system.

An explicit field theory of this 2D state can be written using an Abelian Chern Simons theory with four $U(1)$
gauge fields:
\[
4\pi S_{\text{TQFT}} = \int d^3x \sum_{I,J=1}^{4} K_{IJ}^{SO(8)} \epsilon_{\mu\nu\lambda} a^I_{\mu} \partial_\nu a^J_\lambda
\]  
(2)

\[
K^{SO(8)} = \begin{pmatrix}
2 & -1 & -1 & -1 \\
-1 & 2 & 0 & 0 \\
-1 & 0 & 2 & 0 \\
-1 & 0 & 0 & 2 \\
\end{pmatrix}
\]  
(3)

This is the Cartan matrix of \( SO(8) \). Note that the inverse matrix is:
\[
\left[K^{SO(8)}\right]^{-1} = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 1 & 1/2 & 1/2 \\
1 & 1/2 & 1 & 1/2 \\
1 & 1/2 & 1/2 & 1 \\
\end{pmatrix}
\]  
(4)

which clearly demonstrates the mutual statistics of the 3 fermion model, obtained by inner products \( \theta_{ij} = 2\pi l_i^T \cdot K^{-1} l_j \), while self statistics is given by: \( \theta_i = \pi l_i^T \cdot K^{-1} l_i \) where \( l_i \) are integer vectors representing the quasiparticles. The eigenvalues of \( K^{SO(8)} \) are all positive, indicating that all four edge modes are chiral (propagate in the same direction). Therefore the 3 fermion state explicitly breaks time reversal symmetry, when realized in 2D.

To realize such a surface state, we will use a Hamiltonian based on the Walker-Wang\cite{27} construction. This leads to a ground state wave function which is a superposition of loops (more precisely “string nets”, in the sense of \cite{29}) labelled by the anyon types. The amplitude for a given configuration of these loops \( C \) in the 3+1D wave function \( \Psi_{3D}(C) \) is determined by the expectation value of the corresponding Wilson loop operators in the 2+1D TQFT; i.e. we set:

\[
\Psi_{3D}(C) = \langle W(C) \rangle_{2+1\text{TQFT}}
\]

This is similar in spirit to eg. Quantum Hall wavefunctions, which are related to the space-time correlations of their edge states. Here, since we demand a topologically ordered boundary state, the expectation values are taken in the boundary TQFT. Below we will argue more physically why the bulk wave function encodes the statistical interactions of the surface anyons, but why the bulk is itself not topologically ordered.

The wave function hence contains three different colors of loops corresponding to the three species of fermions. Any two colors can merge into the third, in accordance with the fusion rules of the TQFT. Each loop configuration comes with a specific phase due to the twisting and intertwining of the fermion world-lines. Since this twisting may depend on the angle of view, in order to calculate the phase, we need to pick a particular projection of the 3D world-lines onto 2D ones. The projection we will use is shown in Fig.4. Having fixed a projection, the phase factor can be obtained using the braiding rules (as shown in Fig.1) given by the \( R \) matrix:

\[
\begin{align*}
R_{e,\mu} &= -1, & \mu &= e, m, \varepsilon \\
R_{e,m} &= R_{m,e} = R_{e,\varepsilon} = -1 \\
R_{m,\varepsilon} &= R_{e,m} = R_{e,\varepsilon} = 1
\end{align*}
\]  
(5)

Isolated loops can shrink to the vacuum without an ex-

\[
\begin{array}{c}
\text{e}(m,\varepsilon) \quad \text{e}(m,\varepsilon) \\
\text{e}(m,\varepsilon) \quad \text{e}(m,\varepsilon)
\end{array}
\]

FIG. 1. (color online) Braiding rules for strings in the ground state wave function. (a) applies to strings of the same color while (b) applies to strings of different colors.

tra phase factor (the quantum dimension of our fermions is 1). The ground state wave function is a superposition of all allowed loop configurations weighted by the corresponding phase factor. When the system has a surface, the same graphical rules can be used to determine the wave function for the loop gas after fixing the projection.

An important feature of this state is that, because the braiding rules involve no complex numbers, the ground state wave function is real, hence symmetric under the time reversal operator \( T \) that acts by complex conjugation. This is even true for the wave function on a 3D manifold with boundary. Therefore, no time reversal symmetry breaking occurs either in the bulk or on the surface.

On the other hand, the graphical rules allow us to see that the surface of the 3D model has the anyon content of a state which would violate time reversal in 2D. In particular, anyonic excitations can be created by adding open strings to the surface. The wave function becomes a superposition of all string configurations in which strings end at the positions of the excitations. Then we can check the statistics of the excitations by tracking these open strings. Suppose we exchange two string ends of the same color (say red, as show in Fig.2(a)) by crossing two red string segments to the surface. (Fig.2(a) shows one possible configuration.) This twist in the string configuration (relative to the string configuration before exchange) can be removed to bring the strings back to their original form, but this results in a \((-)\) sign. Therefore, exchanging end of strings of the same color adds a \((-)\) sign to the total wave function, which is equivalent to saying that the ends of the strings are fermions. Similarly one can check that string ends of different colors have mutual semionic statistics by braiding them with linked loops on the surface as shown in Fig.2.

Open strings in the bulk also create excitations in the ground state. However, the excitation energy grows linearly with the string length, leading to confinement of the
particles at the ends of the strings. To see the confinement, consider an open string (for example blue) in the bulk which is circled by a small ring of a different color (for example red), as shown in Fig. 3 Using the braiding rule between the loops we find that the linking between the ring and the open string can be removed together with a (−) sign. That is, the open string changes the quantum fluctuation phase factors of small loops along its length, which costs finite energy. Therefore, the end of strings are confined in the bulk.

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FIG. 2. (color online) The anyonic excitations on the surface are created by open strings. At the ends of the strings are three species of fermions (corresponding to three colors of open strings) which have mutual semionic statistics. This can be seen from the braiding statistics of the strings generating (a) the exchange and (b) the braiding of the end of strings.

FIG. 3. (color online) Open strings in the bulk change the quantum fluctuation phase factors of loops along its length, which costs finite energy. Therefore, the end of strings are confined in the bulk.

removed without breaking time reversal symmetry or going through a bulk phase transition. To see this, we can imagine destroying the topological order by adding a 2D surface layer with the same types of excitations and binding the corresponding pairs of excitations together; however, as this 2D system has chiral fermionic edge modes, this operation must break time reversal symmetry. In fact, if we perform the above surface confinement operation using opposite $T$ breaking states in different halves of the sample, this creates a domain wall. Since each 2D state has edge chiral central charge $c_- = 4$, the combination leads to a net chiral central charge $c_− = 8$ along the domain wall. This is expected for the non fractionalized surface of the 3D Bosonic Topological Superconductor discussed in [17]. It also indicates that when we destroy the topological order of the surface, this phase breaks $T$ symmetry and displays a thermal analog of the quantized magneto-electric effect [3–5].

One may directly argue that realizing a topologically ordered phase which transforms under symmetry in a way this is forbidden in 2D immediately leads to a protected surface state. Assume for the moment that the opposite is true - i.e. there is some way to confine the 3-fermion $Z_2$ state without breaking $T$ symmetry. Then, one can make a slab of the 3D phase with well separated top and bottom surfaces, and eliminate the surface state on the bottom side. Now, consider shrinking the slab till the 2D limit is reached. Since the bottom side and bulk are gapped, it should be possible to retain the original surface state on the top surface, without changing the symmetry. Now we have achieved a contradiction, by having produced a 2D realization of an ‘impossible’ 2D state. Therefore our assumption that it is possible to eliminate the surface state without breaking the symmetry must have been incorrect.

Exactly Soluble Model of a 3D SPT Phase with Surface Topological Order

We now describe an exactly solvable 3D model which realizes these properties. The Hamiltonian we consider is defined on the cubic lattice. The Hilbert space consists of 4 possible states on each edge, which we call $|1\rangle, |e\rangle, |m\rangle$, and $|\varepsilon\rangle$. The Hamiltonian is a sum of commuting vertex ($V$) and plaquette ($P$) terms:

$$H = - \sum_V A_V - \sum_P B_P$$

(6)

Here the first sum is over all vertices $V$ of the cubic lattice, and $A_V$ gives an energy penalty to certain combinations of the possible states of the 6 edges adjoining $V$. The combinations that are energetically cheap are dictated by the “fusion rules” for the 3-fermion model: in the ground state, combining the anyons on all edges adjoining $V$ must give the identity. Thus closed loops of a single anyon type are allowed, as are vertices where the three nontrivial anyons fuse together. It is useful to think of the states $|e\rangle$ and $|m\rangle$ as carrying the non-trivial
charge of two $\mathbb{Z}_2$ symmetries ($\mathbb{Z}_2^{(e)}$ and $\mathbb{Z}_2^{(m)}$), and the $|\varepsilon\rangle$ state as charged under both $\mathbb{Z}_2^{(e)}$ and $\mathbb{Z}_2^{(m)}$, while $|1\rangle$ carries no charge. The vertex term $A_V$ then simply enforces the conservation of $\mathbb{Z}_2^{(e)}$ and $\mathbb{Z}_2^{(m)}$ charges at $V$. Some of the configurations that satisfy this condition are shown in Fig. [4].

The plaquette term is a little more involved. First, define operators $\hat{S}^\mu$ by

$$\hat{S}^\mu|1\rangle = |\mu\rangle, \quad \hat{S}^\mu|\nu\rangle = \delta_{\mu\nu}|1\rangle + |\epsilon\rangle$$

where $(\mu, \nu, \lambda \in \{e, m, \varepsilon\})$, with $|\epsilon\rangle = 1$ when $\mu, \nu,$ and $\lambda$ are some permutation of $e, m,$ and $\varepsilon,$ and 0 otherwise. We can think of $\hat{S}^\mu$ as raising the appropriate $\mathbb{Z}_2$ charge carried by the edge upon which it acts. For example, $\hat{S}^e$ adds a non-trivial $\mathbb{Z}_2^{(e)}$ charge to the link, changing an $|e\rangle$ edge into an edge that carries trivial charge under both symmetries (i.e. the $|1\rangle$ edge), the $|m\rangle$ edge into an edge that is charged under both symmetries (the $|\varepsilon\rangle$ edge), and so on.

The plaquette operator $B_P$ acts on $\partial P$, the set of edges bordering $P$, and for all, and let $O$ and $U$ be the two edges which adjoin a vertex of the plaquette $P$ and project inside it under the 2d projection (Fig. [5]). $O$ is the edge which sits under it. Let $|\alpha\rangle$ and $|\beta\rangle$ be the states sitting on the $O$ and $U$ edges respectively. Then

$$B_P = \sum_{\mu = e, m, \varepsilon} \Theta^O_{\mu, \alpha} \Theta^U_{\mu, \beta} \prod_{i \in \partial P} \hat{S}^\mu_i$$

Here $\Theta^O$ and $\Theta^U$ are 4 by 4 matrices of signs, indexed by the labels $1, e, m, \varepsilon$, and $\alpha, \beta$ are indices corresponding to the states on the $O$ and $U$ edges respectively. These are related to the $R$ matrices introduced in Eq. (5) by

$$\Theta^U_{\mu, \alpha} = R_{\mu, \alpha}, \quad \Theta^O_{\mu, \alpha} = R_{\alpha, \mu}$$

Explicitly, the $e, m, \varepsilon$ parts are given by

$$\Theta^U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

and $\Theta^O = (\Theta^U)^T$. Since the matrix elements of $H$ are real, the Hamiltonian is invariant under time reversal $T$, where $T$ is defined to be complex conjugation of the many body wavefunction in our 1, $e, m, \varepsilon$ basis. Note that this time reversal operator satisfies $T^2 = 1$.

With the definition (8), all of the terms in the Hamiltonian commute. Since $B_P$ raises the $\mathbb{Z}_2$ charge on a pair of edges at each vertex, it conserves both $\mathbb{Z}_2$ charges, and thus commutes with $A_V$, at every vertex $V$. To see that $[B_P, B_P] = 0$, we must check that the signs of all matrix elements are independent of the order in which the plaquette terms act. If $P_1$ contains no $O$ or $U$ edges of $P_2$ and vice versa, this follows immediately. Further, if $B_{P_1}$ raises the $U$ edge of $P_2$, then $B_{P_2}$ also raises the $O$ edge of $P_1$ (and vice versa). In this case the phase terms which could potentially depend on the operator ordering are:

$$\Theta^O_{\mu_1, \mu_2 \times i_O} \Theta^U_{\mu_2, i_U} \Theta^U_{\mu_2, i_U} \Theta^O_{\mu_1, i_O}$$

where $\mu_2 \times i$ indicates the state on edge $i$ after acting with $\hat{S}^\mu_i$. One can check using Eq. (10) that these are equal for every choice of $\mu_1, \mu_2, i_O$ and $i_U$, so that $[B_{P_1}, B_{P_2}] = 0$.

Because all operators in the Hamiltonian commute, we can solve for the spectrum exactly. Before turning to this, let us develop further the algebraic formulation suggested by the mapping to Ising charges $\mathbb{Z}_2^{(e,m)}$. Let the Pauli matrix $\sigma_i^x$ ($\tau^x$) measure the Ising charge $\mathbb{Z}_2^{(e)}$ ($\mathbb{Z}_2^{(m)}$) flowing through the link $i$. Then, the vertex term can be written as:

$$A_V = \left( \prod_i \sigma_i^x + \prod_i \tau_i^x \right)$$

FIG. 4. (color online) The Low energy Hilbert space of the lattice model consists of loops of three colors that satisfy fusion rules at the vertices - i.e. they are either closed loops of a single color, or segments of three colors can meet at a vertex.
where the product is taken over the six links emanating from a vertex. The plaquette operators in Eq. (8)

\[ B_\mu = \sum_{\mu=e, m, e} \hat{\Theta}_\mu \hat{\Theta}_\mu \prod_{i \in \partial P} S_1^\mu \]  

(13)

are a product of three operators \( \hat{S}^\mu, \hat{\Theta}_\mu, \) and \( \hat{\Theta}_\mu \), where

\begin{align*}
\hat{S}^\mu &= \begin{pmatrix} \sigma_x & \sigma_x \\ \sigma_z & \sigma_z \end{pmatrix}, \\
\hat{\Theta}^O &= \begin{pmatrix} \sigma_x & \tau_x \\ \tau_x & \sigma_x \end{pmatrix}, \\
\hat{\Theta}^U &= \begin{pmatrix} \sigma_x \tau_x & \tau_x \\ \tau_x & \sigma_x \end{pmatrix}
\end{align*}

(14)

It is readily seen that the different plaquette operators \( B_\mu \) commute with one another: for any two different links 1, 2, \( \hat{S}_1^\mu \hat{\Theta}_2^O \) commutes with \( \hat{S}_2^\mu \hat{\Theta}_1^O \) for all pairs \( \mu, \nu \).

Excitations in the Bulk: We now show that there are no low energy point-like excitations in the bulk, by establishing all bulk excitations are confined. This points to a short range entangled bulk state without topological entanglement entropy or 3 dimensional topological order. Essentially, the string operator that creates point excitations also creates a string of excited plaquettes, leading to linear confinement of the ends of the string[28]. We will also show that deconfined excitations exist on the surface.

To create a pair of point particles we consider an operator that violates the vertex term \( A_V \) at vertices \( V_1 \) and \( V_2 \). The simplest possibility is to act with \( \hat{S}_1^\mu \) on all edges in some path \( C_{12} \) that connects \( V_1 \) to \( V_2 \) (see Fig. 6). For example, we can attempt to create a pair of \( e \) particles by acting with \( \prod_{i \in C_{12}} \sigma_i^z \). However, in doing so we necessarily change the state on the \( O \) and \( U \) edges of certain plaquettes along this path, creating a pair of strings of defective plaquettes running parallel to \( C_{12} \). For a path in the \( y \) direction, affected plaquettes are those in the \( xz \) plane that lie above and to the right (U edges) or below and to the left (O edges) of the path. Therefore the energy cost to separate the ends of the string operator grows linearly with distance. We can reduce the number of affected plaquettes per unit separation from 2 to 1 by using the modified string operator:

\[ \hat{S}_{C_{12}}^\mu = \prod_{i \in C_{12}} \hat{S}_i^\mu \prod_{j \in C_{12}^{\text{over}}} \hat{\Theta}_{\mu,j} \prod_{k \in C_{12}^{\text{under}}} \hat{\Theta}_{\mu,k} \]  

(15)

where \( * \) is the set of edges that are crossed by a curve that runs parallel to \( C_{12} \), but is displaced slightly in the \(-\hat{x} + \hat{y} + \hat{z}\) direction, and \( * \) are the subsets of edges that cross over (under) the path \( C_{12} \) in our projection (colored purple and green respectively in Figure 6).

The \( \Theta \) terms in \( \hat{S}_1^\mu \) exactly cancel the phases that lead to non-commutativity with the plaquettes whose \( O \) and \( U \) edges are raised; however, the new operator fails to commute with the string of plaquettes (shaded blue plaquettes in Fig. 6) threaded by the curve used to determine \( * \) (dashed blue line in Fig. 6). For example, to create a pair of \( e \) particles at \( V_{1,2} \) we act with \( \hat{S}_1^\mu \) along the bonds connecting \( V_{1,2} \), but also with the operators \( \Theta^O = \sigma_x \) and \( \Theta^U = \sigma_x \tau_x \) on the purple and green bonds respectively in Figure 6. It can be explicitly checked that this string operator does not commute with the shaded plaquettes but does commute with all other plaquettes. It is not possible to further reduce the number of violated plaquettes for a given \( C_{12} \) [27, 28]. Observe that if we terminate the system just above the plane of the curve \( C_{12} \), the defective plaquettes would not be included in the 3D lattice, leading to deconfined surface excitations. This is indeed the case as we will see below. In fact, the surface has three species of deconfined fermion excitations with mutually semionic statistics, corresponding exactly to the excitations in the 3 fermion model.

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(15)

where \( * \) is the set of edges that are crossed by a curve that runs parallel to \( C_{12} \), but is displaced slightly in the \(-\hat{x} + \hat{y} + \hat{z}\) direction, and \( * \) are the subsets of edges that cross over (under) the path \( C_{12} \) in our projection (colored purple and green respectively in Figure 6).

The \( \Theta \) terms in \( \hat{S}^\mu \) exactly cancel the phases that lead to non-commutativity with the plaquettes whose \( O \) and \( U \) edges are raised; however, the new operator fails to commute with the string of plaquettes (shaded blue plaquettes in Fig. 6) threaded by the curve used to determine \( * \) (dashed blue line in Fig. 6). For example, to create a pair of \( e \) particles at \( V_{1,2} \) we act with \( \hat{S}^\mu \) along the bonds connecting \( V_{1,2} \), but also with the operators \( \Theta^O = \sigma_x \) and \( \Theta^U = \sigma_x \tau_x \) on the purple and green bonds respectively in Figure 6. It can be explicitly checked that this string operator does not commute with the shaded plaquettes but does commute with all other plaquettes. It is not possible to further reduce the number of violated plaquettes for a given \( C_{12} \) [27, 28]. Observe that if we terminate the system just above the plane of the curve \( C_{12} \), the defective plaquettes would not be included in the 3D lattice, leading to deconfined surface excitations. This is indeed the case as we will see below. In fact, the surface has three species of deconfined fermion excitations with mutually semionic statistics, corresponding exactly to the excitations in the 3 fermion model.

FIG. 6. Excitations in the bulk are confined. The path \( C_{12} \) is shown in red; the displaced path used to determine \( * \) (dashed blue line) is indicated with a dashed blue line. Edges that cross under (over) this path are colored green (purple). The violated plaquettes (shaded blue) are those that are threaded by the dashed blue line.

Deconfined Surface Excitations: Consider terminating the 3D system with horizontal plaquettes in the \( x-y \) plane as shown in Figure 7. The Hamiltonian for the surface vertices and plaquettes is derived from the bulk Hamiltonian by simply retaining terms for the existing bonds on the surface. Thus both the horizontal plaquette and vertex terms at the surface each involve a product of five bond operators. With these boundary conditions the Hamiltonian is again a sum of commuting terms, and we can determine its spectrum exactly.

Deconfinement occurs at the surface because, if \( C_{12} \) lies entirely on the boundary, acting with an operator of the form (15) creates a pair of vertex defects but commutes with all plaquette projectors. (Note we may omit the \( \Theta^O \) operators, since those bonds are absent). Thus, creating a pair of \( e \) excitations corresponds to multiply-
ing by $\sigma^z$ on the red and $\sigma^x \tau^x$ on the green bonds. These combinations can be readily shown to commute with the Hamiltonian along the length of the string, the only non-commutativity being associated with the ends. Hence the energy cost to create the point defects at the surface is finite even for large separation.

![FIG. 7. Deconfined fermionic excitations on the boundary. The path $C_{12}$ shown in red connects a pair of vertices $V_1$ and $V_2$, each of which is associated with a fermionic excitation. The displaced path used to determine $C_{12}$ is indicated with a dashed blue line; edges that cross under this path are shown in green. Unlike in the bulk (Fig. 6), on the surface the dashed blue line does not thread any plaquettes, and there is no string of defective plaquettes connecting the vertices.](image)

**Statistics of Deconfined Surface Excitations:**

![FIG. 8. Mutual Statistics: The operation of braiding a pair of anyons (say the $e$ and $m$ particles) is captured by first creating a pair of $e$ particles (red) followed by a pair of $m$ particles (blue) in the manner shown. We now annihilate first the $e$ and then the $m$ particles to return to the vacuum, and examine the resulting phase.](image)

We now show that on the surface, the excitations created at the end-points of the string operators $\hat{S}_{C_{ij}}^e$, $\hat{S}_{C_{ij}}^m$, and $\hat{S}_{C_{ij}}$ in Eq. (15) (where $C_{ij}$ is an open curve in the boundary) have exactly the mutual statistics of the fermions $e$, $m$, and $\varepsilon$ respectively in our three fermion model. The crucial point is that if $C_{12}$ and $C_{34}$ are paths that cross once, the operators $\hat{S}_{C_{12}}^e$ and $\hat{S}_{C_{12}}$ ($\mu \neq \nu$) do not commute. Consider, for example, the mutual statistics of the $e$ and $m$ particle. This can be obtained by first creating a pair of $e$ particles, and then a pair of $m$ particles as shown in Figure 8 and next destroying first the $e$ particles and then the $m$ particles. The net phase incurred during this process encodes the mutual statistics of this pair of particles. The $e$ particles in Fig. 8 are created by $\hat{S}_{13}^e = \sigma_1^e \sigma_3^e \tau_1^e$ while the $m$ particles are created by $\hat{S}_{24}^m = \tau_2^m \tau_4^m$. It is readily seen that $\hat{S}_{24}^e \hat{S}_{13}^e \hat{S}_{24}^m \hat{S}_{13}^m = -I$, demonstrating that $e$ and $m$ have mutual semionic statistics.

![FIG. 9. Fermionic statistics: The sequence of operations described in the text to exchange a pair of fermions. The positions of the fermions are indicated by the red dots. (The open red circles in (b) - (d) indicate the original positions of the fermions). Solid red arrows indicate the edge acted on by a string operator to move the fermion at this step; the solid red lines show where the string operators have acted at previous steps. The edges crossed by the dashed blue line are in $C$: there is a phase of $-\frac{1}{12}$ every time a dashed blue line crosses a solid red line.](image)

Now consider the self statistics. The exchange of two anyons of the same type can be realized as shown in Fig. 8. We begin with two anyons ($a$ and $b$) of the same type at vertices $i$ and $i+x$ respectively. The first step in the exchange is to move the anyon $a$ from $i$ to $i-y$: then move anyon $b$ from $i+x$ to $i-x$; next move anyon $a$ from $i-y$ to $i+x$; and finally move the anyon $b$ from $i-x$ to $i$. This process exchanges the two anyons. The whole procedure is realized by string operator $\hat{S}_{C_{i-j}^x}^\mu \hat{S}_{C_{i-j}^y}^\mu \hat{S}_{C_{i-j}^z}^\mu \hat{S}_{C_{i-j}^\mu}^\nu$. We can explicitly check that this operator is equal to $-I$ for $\mu = e, m, \varepsilon$. For example, when $\mu = e$, $\hat{S}_{C_{i-j}^e} = \sigma_1^e \sigma_3^e \tau_1^e$, $\hat{S}_{C_{i-j}^e}^x = \sigma_3^e \sigma_3^e \sigma_1^e \tau_1^e \tau_2^e$, $\hat{S}_{C_{i-j}^e}^y = \sigma_3^e \sigma_3^e \sigma_1^e \tau_1^e \tau_2^e$, $\hat{S}_{C_{i-j}^e}^z = \sigma_3^e \tau_1^e \tau_1^e$. The total exchange string operator is then equal to $-I$. Similar checks can be performed for $\mu = m$ and $\varepsilon$.

**Conclusion** In this paper, we presented an exactly solvable model for the 3D bosonic SPT phase with time reversal symmetry ($T^2 = 1$). Our construction is based on the Walker-Wang prescription, which results in a topologically trivial 3D bulk state but a topologically nontrivial 2D surface state that has three species of fermionic excitations with mutual semionic statistics.
Such a state cannot be realized in 2D without breaking time reversal symmetry, while our 3D model is explicitly time reversal symmetric. Therefore, the surface realizes a state that cannot be time-reversal invariant in a purely 2D system, clearly indicating that the model has nontrivial SPT order in the bulk. The underlying principle of realizing a topologically ordered surface and a SRE bulk state with 2D fermion model is the $\Theta$ matrix which is equal to the $R$ matrix for the braiding of the electric quasiparticle and magnetic quasiparticle, respectively, with time reversal symmetry\cite{17}.

The Hilbert space on each edge is again four dimensional with basis states labeled $|\varepsilon\rangle$, $|e\rangle$, $|m\rangle$, and $|\bar{e}\rangle$. If Pauli operators $\sigma^x_i$ ($\tau^+_i$) measure the Ising charges $Z_2^{(\varepsilon)}$ ($Z_2^{(m)}$) flowing through the link $i$, then the exactly solvable bulk Hamiltonian can be written as

$$H = -\sum_V A_V - \sum_P B_P$$

where

$$A_V = - (\prod_i \sigma^z_i + \prod_i \tau^+_i)$$

$$B_P = \sum_{\mu=\varepsilon, m, \bar{e}} \Theta^\mu V^\mu \prod_{i \in \partial P} \hat{S}_i^\mu$$

with

$$\hat{S} = \left( \begin{array}{ccc} \sigma^z & \tau^z \\ \tau^z & \sigma^z \end{array} \right); \quad \Theta^O = \left( \begin{array}{cc} \tau^z & 1 \\ 1 & \tau^z \end{array} \right); \quad \hat{U} = \left( \begin{array}{cc} 1 & \sigma^x \\ \sigma^x & 1 \end{array} \right)$$

Open string operators are then given by

$$\hat{S}^\mu_{C_{ij}} = \prod_{i \in C_{12}} \hat{S}^\mu_i \prod_{j \in C_{12}}^{\Sigma_{\text{term}}^S} \tilde{\Theta}^O_{\mu,j} \prod_{k \in C_{12}} \tilde{\Theta}^U_{\mu,k}$$

It can then be explicitly checked that the end of strings are confined in the bulk but deconfined on the surface and the surface anyons have the same statistics as the $Z_2$ gauge theory in 2D. Of course, this theory can also appear in 2D with time reversal symmetry as defined. However, if anyons transform in more nontrivial ways this can lead to additional SPT phases \cite{17}. This is left for future work.

**APPENDIX 1: SURFACE $Z_2$ GAUGE THEORY**

Similar construction of 3D loop gas state can give rise to a SRE bulk state with 2D $Z_2$ gauge theory on the surface. The only difference from the definition of the 3 fermion model is the $\Theta$ matrix which is equal to the $R$ matrix for the braiding of the electric quasiparticle $e$, the magnetic quasiparticle $m$ and their bound state $\varepsilon$

$$\Theta^U_{Z_2} = \left( \begin{array}{ccc} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \right)$$

and $\Theta^O_{Z_2} = (\Theta^U_{Z_2})^T$.

The Hilbert space on each edge is again four dimensional with basis states labeled $|\varepsilon\rangle$, $|e\rangle$, $|m\rangle$, and $|\bar{e}\rangle$. If Pauli operators $\sigma^x_i$ ($\tau^+_i$) measure the Ising charges $Z_2^{(\varepsilon)}$ ($Z_2^{(m)}$) flowing through the link $i$, then the exactly solvable bulk Hamiltonian can be written as

$$H = -\sum_V A_V - \sum_P B_P$$

where

$$A_V = - (\prod_i \sigma^z_i + \prod_i \tau^+_i)$$

$$B_P = \sum_{\mu=\varepsilon, m, \bar{e}} \Theta^\mu V^\mu \prod_{i \in \partial P} \hat{S}_i^\mu$$

with

$$\hat{S} = \left( \begin{array}{ccc} \sigma^z & \tau^z \\ \tau^z & \sigma^z \end{array} \right); \quad \Theta^O = \left( \begin{array}{cc} \tau^z & 1 \\ 1 & \tau^z \end{array} \right); \quad \hat{U} = \left( \begin{array}{cc} 1 & \sigma^x \\ \sigma^x & 1 \end{array} \right)$$

Open string operators are then given by

$$\hat{S}^\mu_{C_{ij}} = \prod_{i \in C_{12}} \hat{S}^\mu_i \prod_{j \in C_{12}}^{\Sigma_{\text{term}}^S} \tilde{\Theta}^O_{\mu,j} \prod_{k \in C_{12}} \tilde{\Theta}^U_{\mu,k}$$

It can then be explicitly checked that the end of strings are confined in the bulk but deconfined on the surface and the surface anyons have the same statistics as the $Z_2$ gauge theory in 2D. Of course, this theory can also appear in 2D with time reversal symmetry as defined. However, if anyons transform in more nontrivial ways this can lead to additional SPT phases \cite{17}. This is left for future work.

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