ACM sheaves on a nonsingular quadric hypersurface in \( \mathbb{P}^4_k \)

Elena Drozd

Abstract

We prove that on a nonsingular quadric hypersurface \( Q \) in \( \mathbb{P}^4_k \) even CI liaison classes of ACM curves are in bijective correspondence with the stable equivalence classes (up to shift in degree) of maximal Cohen-Macaulay graded modules over the coordinate ring \( R \) of \( Q \), which in turn, are in bijective correspondence with stable equivalence classes (up to shift in degree) of ACM sheaves on \( Q \). In the situation of a nonsingular quadric hypersurface \( Q \in \mathbb{P}^4_k \) work of Knörrer \[7\] shows that there is a unique nonfree indecomposable MCM module over \( R \). We also describe the ACM sheaf corresponding to this MCM module and give its cohomology table and its Hilbert polynomial.

Keywords: CI liaison, maximal Cohen-Macaulay modules, quadric hypersurface, ACM sheaves.

In classifying algebraic space curves, the special class of Arithmetically Cohen-Macaulay (ACM) curves plays a significant role. C.Peskine and L.Szpiro \([9]\) showed that any ACM scheme of codimension two in \( \mathbb{P}^4 \) is in the CI-liaison class of a complete intersection. In higher codimension this result is no longer true for CI-liaison.

In this work our attention is restricted to the ACM curves that lie on a nonsingular quadric hypersurface \( Q \in \mathbb{P}^4_k \).

We prove that on a nonsingular quadric hypersurface \( Q \) in \( \mathbb{P}^4_k \) even CI liaison classes of ACM curves are in bijective correspondence with the stable equivalence classes (up to shift in degree) of Maximal Cohen-Macaulay graded modules over the coordinate ring \( R \) of \( Q \), which in turn, are in bijective correspondence with stable equivalence classes (up to shift in degree) of ACM sheaves on \( Q \) \([2]\). In the situation of a nonsingular quadric hypersurface \( Q \in \mathbb{P}^4_k \) work of Knörrer \[7\] shows that there is a unique nonfree indecomposable MCM module over \( R \). In this work we describe the ACM sheaf corresponding to this MCM module. We give its cohomology table and its Hilbert polynomial. This information is very helpful in determining which ACM sheaf corresponds to a given curve. Some examples are discussed in \([4]\).
Let $X$ be a complete (connected) arithmetically Gorenstein subscheme of $\mathbb{P}^n_k$ of dimension $r \geq 2$ with a very ample line bundle $\mathcal{L}$. Suppose that $H^1(X, \mathcal{L}^n) = 0$ for all $n$. (For example $X$ is an arithmetically Gorenstein subcanonical: $u_X \cong O_X(c)$ for some integer $c$ since $X$ is arithmetically Gorenstein.) We start with describing relations between ACM curves on $Q$ and MCM modules on $R$.

**Definition 1.** Let $Z$ be a subscheme of $X$. Then a resolution

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow \mathcal{I}_Z \rightarrow 0$$

where $\mathcal{L} = \bigoplus O_X(-a_i)$ for some $a_i \in \mathbb{Z}$ and $\mathcal{E}$ is a locally free sheaf on $X$ with $H^1(\mathcal{E}) = 0$ is called an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$. For any sheaf $\mathcal{F}$ we denote $H^1(X, \mathcal{F}(t))$ for all $t \in \mathbb{Z}$ by $H^1_* (\mathcal{F})$.

**Lemma 1.** Let $Z \subset X$ be locally Cohen-Macaulay equidimensional and $\operatorname{codim}_X Z = 2$. Then there exists an $\mathcal{E}$-type resolution $\mathcal{I}_Z$.

**Proof.** Let $S(X)$ be the homogeneous coordinate ring of $X$ and $S(Z)$ be the homogeneous coordinate ring of $Z$, let $I_Z$ be the ideal of $Z$ in $S(X)$, and let

$$(1) \quad 0 \rightarrow E \rightarrow L \xrightarrow{\alpha} I_Z \rightarrow 0$$

be a presentation where $L$ is a free graded $S(X)$-module (i.e. $\bigoplus S(-a_i)$), whose generators are sent by $\alpha$ onto the generators of the homogeneous ideal of $Z$ in $S(X)$. Let $E = \ker \alpha$. By sheafifying one obtains a resolution

$$(2) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow \mathcal{I}_Z \rightarrow 0,$$

where $\mathcal{L}$ is a direct sum of line bundles on $X$. To show that $\mathcal{E}$ is locally free sheaf on $X$, consider any closed point $y \in Z$. We know that $\operatorname{projdim} O_{Z,y} = \operatorname{dim} X - \operatorname{depth} O_{Z,y} = 2$, since $O_{X,y}$ is regular and $Z$ is locally Cohen-Macaulay. So, $\operatorname{projdim} \mathcal{I}_Z = 1$. This implies that $\mathcal{E}$ is locally free. To show that (2) is an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$ we need to prove that $H^1_* (\mathcal{E}) = 0$. From (2) we get the following exact sequence:

$$(3) \quad 0 \rightarrow H^0_* (\mathcal{E}) \rightarrow H^0_* (\mathcal{L}) \rightarrow H^0_* (\mathcal{I}_Z) \rightarrow H^1_* (\mathcal{E}) \rightarrow H^1_* (\mathcal{L}) \rightarrow \ldots.$$ 

$H^1_* (\mathcal{L}) = 0$ since $\mathcal{L} = \bigoplus_{i=1}^m O(-a_i)$. $H^0_* (\mathcal{L}) = L$; $H^0_* (\mathcal{I}_Z) = I_Z$. Therefore sequence (3) is sequence (1). Thus $H^1_* (\mathcal{E}) = 0$ and sequence (3) is an $\mathcal{E}$-type resolution of $\mathcal{I}_Z$. This concludes the proof of the lemma.

**Definition 2.** Two vector bundles $\mathcal{F}$ and $\mathcal{G}$ on $X$ are called stably equivalent if

$$\mathcal{F} \oplus \bigoplus_{i=1}^t O_X(a_i) \cong \mathcal{G} \oplus \bigoplus_{i=1}^s O_X(b_i)$$

for some $c, t, d, a_i, b_i \in \mathbb{Z}$.
Theorem 1 ([10] 1.9). Let $X$ be a nonsingular quadric hypersurface in $\mathbb{P}^4_k$. Then the even liaison classes of curves $Z$ on $X$ are in bijective correspondence with the stable equivalence classes modulo twists of vector bundles $F$ on $X$ with the property that $H^1_*(F) = 0$. The correspondence is given by $Z \mapsto F$, where $F$ is the kernel of an $E$-type resolution of $I_Z$.

We note here that theorem 1 is the statement cited since the nonsingular quadric hypersurface in $\mathbb{P}^4_k$ is a complete connected Gorenstein scheme of dimension at least two. $O_X(1)$ is a very ample locally free sheaf with $H^i_*(O_X) = 0$ for $i = 1, 2$. Finally, $L$-stable equivalence of [10] 1.9 is stable equivalence up to shift in degree defined in this work.

Lemma 2. Let $C$ be a curve on a nonsingular hypersurface $Q$ in $\mathbb{P}^4_k$. Let

$$0 \to F \to L \to I_C \to 0$$

be an $E$-type resolution of $I_C$. Then the curve $C$ is ACM if and only if $H^i_*(F) = 0$ for $i = 1, 2$.

Proof. From theorem 1 we know that $H^1_*(F) = 0$. Recall that $C$ is ACM is equivalent to $H^1_*(I_C) = 0$, where $I_C$ is the ideal sheaf of $C$. Taking cohomology in (4) we get an exact sequence:

$$H^1_*(F) \to H^1_*(L) \to H^1_*(I_C) \to H^2_*(F) \to H^2_*(L) \to \ldots,$$

in which $H^1_*(I_C) = 0$ since $C$ is ACM curve, and $H^2_*(L) = 0$ since $L$ is a direct sum of locally free sheaves. Therefore, if $C$ is ACM, $H^i_*(F) = 0$ for $i = 1, 2$. Reversing this argument we get the converse of the statement. \[\square\]

Definition 3 ([5]). Let $B$ be a local ring and $M$ be a finitely generated $B$-module. Then we say $M$ is a Maximal Cohen-Macaulay module if $\text{depth} M = \dim B$.

Definition 4. On a nonsingular 3-fold $Q$ a locally free sheaf $F$ with the property that $H^i_*(F) = 0$ for $i = 1, 2$ is called an ACM sheaf.

Theorem 2. Let $Q$ be a nonsingular hypersurface in $\mathbb{P}^4_k$. Then the following classes are in bijective correspondence:

1. even CI-liaison classes of ACM curves
2. stable equivalence classes up to shift in degree of ACM sheaves on $Q$
3. stable equivalence classes up to shift in degree of Maximal Cohen-Macaulay graded modules over the ring $R = k[x_0, \ldots, x_4]/I_Q$, where $I_Q$ is the ideal of $Q$.  

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Proof. (1) $\leftrightarrow$ (2) is Theorem [1] plus Lemma [2].

(1) to (3): For an ACM curve $C$ there exists the following resolution:

$$0 \rightarrow E \rightarrow L \overset{\alpha}{\rightarrow} I_C \rightarrow 0$$

where $I_C$ is the ideal of $C$, $L$ is a graded free $R$-module whose generators are sent by $\alpha$ onto the generators of $I_C$ and $E = \text{Ker } \alpha$. Let $m$ be the irrelevant maximal ideal $R_+$ of $R$. We associate to the curve $C$, the module $E$.

Then, in order to prove that $E$ is MCM we need to show that depth $E = \dim R$. Now, dim $Q = 3$, therefore dim $R = 4$; let $A_C$ be the coordinate ring of $C$. Then dim $A_C = 2$.

Consider the following short exact sequence:

$$0 \rightarrow I_C \rightarrow R \rightarrow A_C \rightarrow 0$$

From it, taking local cohomology, we will calculate depth $I_C$:

- depth $A_C = 2$ since $C$ is ACM, therefore $H^i_m(A_C) = 0$, $i = 0, 1$, $H^2_m(A_C) \neq 0$
- depth $R = 4$, therefore $H^i_m(R) = 0$, $i \leq 3$

Consider

$$0 \rightarrow H^0_m(I_C) \rightarrow H^0_m(R) \rightarrow H^0_m(A_C) \rightarrow H^1_m(I_C) \rightarrow$$

$$\rightarrow H^1_m(R) \rightarrow H^1_m(A_C) \rightarrow H^2_m(I_C) \rightarrow H^2_m(R) \rightarrow H^2_m(A_C) \rightarrow$$

$$\rightarrow H^3_m(I_C) \rightarrow H^3_m(R) \rightarrow H^3_m(A_C) \rightarrow \ldots .$$

Therefore $H^i_m(I_C) = 0$ $i = 0, 1, 2$, $H^3_m(I_C) \neq 0$, therefore depth $I_C = 3$ by the cohomolgical interpretation of depth.

Similarly consider

$$0 \rightarrow E \rightarrow L \rightarrow I_C \rightarrow 0$$

Taking local cohomology:

$$0 \rightarrow H^0_m(E) \rightarrow H^0_m(L) \rightarrow H^0_m(I_C) \rightarrow H^1_m(E) \rightarrow$$

$$\rightarrow H^1_m(L) \rightarrow H^1_m(I_C) \rightarrow H^2_m(E) \rightarrow H^2_m(L) \rightarrow H^2_m(I_C) \rightarrow$$

$$\rightarrow H^3_m(E) \rightarrow H^3_m(L) \rightarrow H^3_m(I_C) \rightarrow H^4_m(E) \rightarrow H^4_m(L) \rightarrow \ldots ,$$

- depth $L = 4$, therefore $H^i_m(L) = 0$, $i \leq 3$, $H^4_m(E) \neq 0$
- depth $I_C = 3$, therefore $H^i_m(I_C) = 0$, $i \leq 2$, $H^3_m(I_C) \neq 0$

Therefore, $H^i_m(E) = 0$, $i = 0, 1, 2, 3$, $H^4_m(E) \neq 0$, therefore depth $E = 4$, therefore $E$ is MCM as was to be shown.

(3) to (2): Let $E$ be a graded MCM $R$-module. Let $F = \widetilde{E}$ be the associated sheaf over $Q$. To show that $H^i(F) = 0$ for $i = 1, 2$ we need the following result:
Theorem 3 ([5], p.693). Let $R$ be a graded ring. If $M$ is a graded $R$-module, then there is a natural exact sequence (where $m$ is irrelevant maximal ideal $R_+$)

$$0 \rightarrow H^0_m(M) \rightarrow M \rightarrow H^0_*(M) \rightarrow H^1_m(M) \rightarrow 0$$

and $H^i_*(M) \cong H^{i+1}_m(M)$ for $i > 0$, where $E$ is $\tilde{M}$.

Applying this theorem for $F = \tilde{E}$ we obtain:

$$H^i_*F \cong H^{i+1}_m(E) \quad i > 0, \quad H^0_*(F) = E,$$

which means that $H^1_*(F) = H^2_m(E); H^2_*(F) = H^3_m(E)$. Now $E$ is MCM, therefore depth $E = \dim R = 4$, therefore $H^i_m(E) = 0, i < 4$, therefore $H^2_m(E) = H^3_m(E) = 0$, therefore $H^i_*(F) = 0, i = 1, 2$ as was to be shown.

This concludes the proof of the theorem.

Corollary 1. Let $Q$ be a nonsingular hypersurface in $\mathbb{P}^4_k$. Let $C$ be an ACM curve on $Q$. If $0 \rightarrow E \rightarrow \mathcal{L} \rightarrow \mathcal{I}_C \rightarrow 0$ is an $E$-type resolution of $\mathcal{I}_C$, then there exists a curve $C'$ with an $E$-type resolution of $\mathcal{I}_{C'}$ of the form

$$0 \rightarrow E \oplus \bigoplus_{i=1}^k \mathcal{O}(a_i) \rightarrow \mathcal{L}' \rightarrow \mathcal{I}_{C'} \rightarrow 0,$$

if and only if $C'$ is CI-linked to $C$.

If $0 \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow \mathcal{I}_C \rightarrow 0$ is an $N$-type resolution of $\mathcal{I}_C$, then there exists a curve $C'$ with an $N$-type resolution of $\mathcal{I}_{C'}$ of the form

$$0 \rightarrow \mathcal{L}' \rightarrow \mathcal{N} \oplus \bigoplus_{i=1}^k \mathcal{O}(a_i) \rightarrow \mathcal{I}_{C'} \rightarrow 0,$$

if and only if $C'$ is CI-linked to $C$.

Now as equivalence of CI-biliaison classes of ACM curves and stable equivalence classes up to shift in degree of ACM sheaves is established, we describe ACM sheaves on a nonsingular quadric hypersurface $Q$ in $\mathbb{P}^4_k$.

Theorem 4 ([1]). Let $k = \overline{k}, \quad \text{char } k \neq 2$. Let $Q$ be a quadratic form on a vector space $V$ over $k$ considered as an element of $S_2(V^*)$, which is regular in the sense that $R = k[V^*]/Q$ has only an isolated singularity. Then the Maximal Cohen-Macaualy modules over $R$ are such that there are always one or two nonfree indecomposable Maximal Cohen-Macaulay $R$-modules depending on whether $\dim V$ is odd or even; that they both have the same rank and are syzygies of one another when there are two; and that writing $m = \frac{\dim V}{2} - 1$, the rank of their direct sum is $2^m$. 

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**Corollary 2.** In the case of a nonsingular quadric hypersurface in $\mathbb{P}^4_k$ there is only one indecomposable Maximal Cohen-Macaulay module over $R$ and its rank is 2.

Now we describe this unique MCM module over $R$.

**Proposition 1.** Let $L$ be a line in $Q \subseteq \mathbb{P}^4_k$. Then it has an $\mathcal{E}$-type resolution of the form:

\[(5) 0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Q^3(-1) \rightarrow \mathcal{I}_L \rightarrow 0\]

*Proof.* Let $I_L$ be an ideal of a line $L$ in $R$ where $R$ is the homogeneous coordinate ring of $Q$. Then there is an exact sequence:

\[0 \rightarrow E \rightarrow \mathcal{R}^3(-1) \xrightarrow{\beta} I_L \rightarrow 0\]

where $E = \ker \beta$. By sheafifying we obtain the following exact sequence:

\[(6) 0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Q^3(-1) \rightarrow \mathcal{I}_L \rightarrow 0\]

Following the proof of 1 we get that (6) is an $\mathcal{E}$-type resolution of $\mathcal{I}_L$. By 2 this implies that $\mathcal{E}$ is an ACM sheaf on $Q$. Also note that rank $\mathcal{E} = 2$. \(\square\)

**Definition 5.** We denote by $\mathcal{E}_0$ the ACM rank 2 sheaf appearing in the $\mathcal{E}$-type resolution of $\mathcal{I}_L$.

**Theorem 5.** The maximal Cohen-Macaulay module $\mathcal{E}$ corresponding to $\mathcal{E}_0$ is the unique nonfree indecomposable MCM module on $R$, and $\mathcal{E}_0$ is the unique (up to shift) indecomposable ACM sheaf on $Q$ that is not isomorphic to a direct sum of line bundles.

*Proof.* Let us show that $\mathcal{E}_0$ is not isomorphic to a direct sum of two locally free sheaves.

Suppose $\mathcal{E}_0 = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_i$ are locally free sheaves. By 2 this would imply that a line on $Q$ is in the biliaison class of a complete intersection, which is not the case since deg $L$ is odd while degree of a complete intersection on $Q$ is even and CI-liaison preserves parity. Therefore $\mathcal{E}_0$ is not a direct sum of locally free sheaves. Therefore $\mathcal{E}_0$ is an indecomposable ACM sheaf corresponding to an indecomposable MCM module.

By the theorem 4 there is only one isomorphism class of nonfree indecomposable graded MCM modules over $R$. Therefore $H^0_*(\mathcal{E}_0)$ is the unique MCM given by that theorem. \(\square\)

Theorem 5 and corollary 2 imply the following:

**Corollary 3.** Let $Q$ be a nonsingular quadric hypersurface in $\mathbb{P}^4_k$. Let $\mathcal{E}$ be an ACM sheaf on $Q$. Then

\[\mathcal{E} = \bigoplus_{i=1}^{k_1} \mathcal{E}_0(a_i) \oplus \bigoplus_{j=1}^{k_2} \mathcal{O}_X(b_j)\]

for some $a_i, b_j \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}$. 

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We now calculate dimensions of cohomology groups of $E_0$ and its Hilbert polynomial.

Proposition 2.

1. The dimensions of cohomology groups of $E_0$ are:

| $n$ | $h^0(Q, E_0(n))$ | $h^1(E_0)$ | $h^2(E_0)$ | $h^0(\mathcal{O}_Q(n))$ |
|-----|------------------|-------------|-------------|------------------------|
| $< 0$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 5 | 35 |
| 1 | 1 | 15 | 70 | 126 |
| 2 | 5 | 35 | 70 | 210 |
| 3 | 126 | 330 | 210 | 495 |
| 4 | 210 | 495 | 210 | 630 |
| 5 | 330 | 555 | 210 | 1008 |
| 6 | 555 | 885 | 210 | 1524 |
| 7 | 885 | 1335 | 210 | 2190 |
| 8 | 1335 | 2190 | 210 | 2994 |

$h^1(E_0) = h^2(E_0) = 0$ and $h^0(\mathcal{O}_Q(n)) = 0$ for $n < 0$;

2. $E_0(n)$ is generated by global section for all $n \geq 2$;

3. $E_0^\vee = E_0(3)$.

Proof. 1) Since $E_0$ is an ACM sheaf, we have $h^1(E_0) = h^2(E_0) = 0$. Taking cohomology in the short exact sequence

$$0 \rightarrow I_Q(n) \rightarrow \mathcal{O}_{\mathbb{P}^4}(n) \rightarrow \mathcal{O}_Q(n) \rightarrow 0$$

we arrive at:

$$0 \rightarrow H^0(I_Q(n)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^0(\mathcal{O}_Q(n)) \rightarrow H^1(I_Q(n)) = 0,$$

where $H^1(I_Q(n)) = 0$ since $Q$ is ACM. Therefore dimensions are as follows:

| $n$ | $h^0(I_Q(n))$ | $h^0(\mathcal{O}_{\mathbb{P}^4}(n))$ | $h^0(\mathcal{O}_Q(n))$ |
|-----|---------------|-------------------------------|------------------------|
| $< 0$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 2 | 5 | 5 | 5 |
| 3 | 35 | 35 | 35 |
| 4 | 70 | 70 | 70 |
| 5 | 126 | 126 | 126 |
| 6 | 210 | 210 | 210 |
| 7 | 330 | 330 | 330 |
| 8 | 495 | 495 | 495 |

Similarly, from the short exact sequence

$$0 \rightarrow I_L \rightarrow \mathcal{O}_Q \rightarrow \mathcal{O}_L \rightarrow 0$$
twisting and taking cohomology we get

\[0 \to H^0(I_L(n)) \to H^0(O_Q(n)) \to H^0(O_L(n)) \to H^1(I_L(n)) = 0,\]

where \(H^1(I_L(n)) = 0\) since a line \(L\) is an ACM curve. Then dimensions of cohomology groups are:

\[
\begin{array}{cccc}
 n & h^0(I_L(n)) & h^0(O_Q(n)) & h^0(O_L(n)) \\
< 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 3 & 5 & 2 \\
2 & 11 & 14 & 3 \\
3 & 26 & 30 & 4 \\
4 & 50 & 55 & 5 \\
5 & 85 & 91 & 6 \\
6 & 133 & 140 & 7 \\
7 & 196 & 204 & 8 \\
8 & 296 & 285 & 9 \\
\end{array}
\]

And similarly from the short exact sequence

(7) \[0 \to \mathcal{E}_0 \to O_Q^3(-1) \to I_L \to 0\]
defining \(\mathcal{E}_0\), twisting and taking chomology we get

\[0 \to H^0(\mathcal{E}_0(n)) \to H^0(O_Q^3(n - 1)) \to H^0(I_L(n)) \to 0\]
since \(H^1(\mathcal{E}_0(n)) = 0\) for the Maximal Cohen-Macaulay module \(\mathcal{E}_0\). Then dimensions of cohomology groups are:

\[
\begin{array}{cccc}
 n & h^0(\mathcal{E}_0(n)) & h^0(O_Q^3(n - 1)) & h^0(I_L(n)) \\
< 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 3 & 3 \\
2 & 4 & 15 & 11 \\
3 & 16 & 42 & 26 \\
4 & 40 & 90 & 50 \\
5 & 80 & 165 & 85 \\
6 & 140 & 273 & 133 \\
7 & 224 & 420 & 196 \\
8 & 336 & 612 & 276 \\
\end{array}
\]
2) Note that $\mathcal{E}_0$ is 2-regular [8, 1.1.4] and thus, by Castelnuovo-Mumford regularity [8, 1.1.5 (3)] $\mathcal{E}_0(n)$ is generated by its global section for all $n \geq 2$.

3) Since $\mathcal{E}_0$ is a locally free sheaf of rank 2, $\mathcal{E}_0^\vee = \mathcal{E}_0(-c_1)$, where $c_1$ is the first Chern class of $\mathcal{E}_0$. We calculate $c_1$ from the exact sequence (7):

$$c_1(\mathcal{E}_0) + c_1(I_L) = c_1(\mathcal{O}_Q^3(-1)) = -3.$$ 

$L$ is of codimension 2 on $Q$, thus $c_1(I_L) = 0$, wherefrom $c_1(\mathcal{E}_0) = -3$. 

\[ \square \]

**Proposition 3.** The Hilbert polynomial of $\mathcal{E}_0$ is

$$P_{\mathcal{E}_0}(n) = 4\left(\frac{n + 1}{3}\right).$$

**Proof.** Consider

$$0 \rightarrow I_Q(n) \rightarrow \mathcal{O}_{\mathbb{P}_k^4}(n) \rightarrow \mathcal{O}_Q(n) \rightarrow 0.$$ 

The Hilbert polynomial of $\mathcal{O}_Q(n)$ is

$$P_Q(n) = \left(\frac{n + 4}{4}\right) - \left(\frac{n + 2}{4}\right).$$

From the short exact sequence

$$0 \rightarrow I_L \rightarrow \mathcal{O}_Q \rightarrow \mathcal{O}_L \rightarrow 0$$

we get that the Hilbert polynomial of $I_L$ is

$$P_L(n) = \left(\frac{n + 4}{4}\right) - \left(\frac{n + 2}{4}\right) - n - 1.$$ 

Thus from the sequence

$$0 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{O}_Q^3(-1) \rightarrow I_L \rightarrow 0.$$ 

we get the Hilbert polynomial of $\mathcal{E}_0$:

$$P_{\mathcal{E}_0}(n) = 3\left[\left(\frac{n + 3}{4}\right) - \left(\frac{n + 1}{4}\right)\right] - \left(\frac{n + 4}{4}\right) + \left(\frac{n + 2}{4}\right) + n + 1 = 4\left(\frac{n + 1}{3}\right).$$ 

\[ \square \]

**Lemma 3.** There is an $\mathcal{E}$-type resolution of $\mathcal{E}_0$ of the form

$$0 \rightarrow \mathcal{E}_0(-1) \rightarrow \mathcal{O}_4(-2) \rightarrow \mathcal{E}_0 \rightarrow 0$$
Proof. The dimension of \( H^0(\mathcal{E}(2)) \) is 4 and \( \mathcal{E}(2) \) is generated by global sections by proposition 2. Thus, there is a surjective map
\[
\mathcal{O}^4 \longrightarrow \mathcal{E}(2) \longrightarrow 0.
\]
Its kernel \( \mathcal{E} \) is a rank two ACM sheaf. Thus, by proposition 3 \( \mathcal{E} \) is equal to either \( \mathcal{E}(a) \) or \( \mathcal{O}(a) \oplus \mathcal{O}(b) \) for some \( a, b \in \mathbb{Z} \). By 2 we must have \( \mathcal{E} = \mathcal{E}(1) \). Thus, the sequence
\[
0 \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{O}^4 \longrightarrow \mathcal{E}(2) \longrightarrow 0
\]
is exact and it gives an \( \mathcal{E} \)-type resolution of \( \mathcal{E}(2) \). 

References

[1] R.O. Buchweitz, D. Eisenbud, J. Herzog: Cohen-Macaulay modules on quadrics Lecture Notes 1273 Springer (1987) 58-116.

[2] M. Casanellas, E. Drozd, R. Hartshorne: Gorenstein Liaison and ACM Sheaves accepted by Crelles Journal, (2004).

[3] E. Drozd: Curves on a nonsingular quadric hypersurface in \( \mathbb{P}_k^4 \): existence and liaison theory, Ph.D thesis, UC Berkeley, (2003).

[4] E. Drozd: On irreducibility of the family of ACM curves of degree 8 and genus 4 in \( \mathbb{P}_k^4 \) to be published.

[5] D. Eisenbud: Commutative Algebra with a View Toward Algebraic Geometry, Springer (1999).

[6] R. Hartshorne: Algebraic Geometry, Springer (1977).

[7] H. Knorrer: Cohen-Macaulay modules on hypersurface singularities I, Invent. Math. 88 (1987) 153-164.

[8] J.C. Migliore: Gorenstein liaison theory and deficiency modules, Progress in Mathematics 165 Birkhäuser(1998).

[9] C. Peskine, L.Szpiro: Liaison des variétés algébriques, I, Invent. Math. 26 (1974) 271-302.

[10] P. Rao: Liaison equivalence classes, Math Ann. 258 (1981) 169-173.