On decay properties and asymptotic behavior of solutions
to a non-local perturbed \textit{KdV} equation

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April 2, 2019

Abstract

We consider the \textit{KdV} equation with an additional non-local perturbation term defined through the Hilbert transform, also known as the OST-equation. We prove that the solutions \(u(t, x)\) has a pointwise decay in spatial variable: \(|u(t, x)| \lesssim \frac{1}{1 + |x|^2}\), provided that the initial data has the same decaying and moreover we find the asymptotic profile of \(u(t, x)\) when \(|x| \to +\infty\).

Next, we show that decay rate given above is optimal when the initial data is not a zero-mean function and in this case we derive an estimate from below \(\frac{1}{|x|^2} \lesssim |u(t, x)|\) for \(|x|\) large enough. In the case when the initial datum is a zero-mean function, we prove that the decay rate above is improved to \(\frac{1}{1 + |x|^{2+\varepsilon}}\) for \(0 < \varepsilon \leq 1\). Finally, we study the local-well posedness of the OST-equation in the framework of Lebesgue spaces.

Keywords: \textit{KdV} equation; OST-equation; Hilbert transform; Decay properties; Persistence problem.

1 Introduction

In this article we consider the following Cauchy’s problem for a non-locally perturbed \textit{KdV} equation

\[
\begin{aligned}
\partial_t u + u\partial_x u + \partial_x^2 u + \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^2 u) &= 0, & \eta > 0, &\text{on } [0, +\infty) \times \mathbb{R}, \\
u(0, \cdot) &= u_0.
\end{aligned}
\]

(1)

where the function \(u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}\) is the solution, \(u_0 : \mathbb{R} \to \mathbb{R}\) is the initial datum and \(\mathcal{H}\) is the Hilbert transform defined as follows: for \(\varphi \in \mathcal{S}(\mathbb{R})\),

\[
\mathcal{H}(\varphi)(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(x - y)}{y} dy.
\]

(2)

Equation (1), also called the Ostrovsky, Stepanyams and Tsimring equation (OST-equation), was derived by Ostrovsky \textit{et al.} in [18] [19] to describe the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer. It deserves remark that when \(\eta = 0\) we...
obtain the well-know KdV equation. The parameter $\eta > 0$ represents the importance of amplification and damping relative to dispersion. Indeed, the fourth term in equation \( \textbf{1} \) represents amplification, which is responsible for the radiational instability of the negative energy wave, while the fifth term in equation \( \textbf{1} \) denotes damping (see [17] for more details). Both of these two terms are described by the non-local integrals represented by the Hilbert transform \( \mathcal{H} \).

One rewrites Equation \( \textbf{1} \) in the equivalent Duhamel formulation (see [1]):

\[
    u(t, x) = K_{\eta}(t, \cdot) * u_0(x) - \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) * \partial_x (u^2)(\tau, \cdot)(x) d\tau,
\]

where the kernel $K_{\eta}(t, x)$ is given by

\[
    K_{\eta}(t, x) = \mathcal{F}^{-1}\left(e^{(i\xi^3 t - \eta |\xi|^3 - |\eta|)}\right)(x),
\]

and where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Well-posedness results for the Cauchy problem \( \textbf{3} \) was extensively studied in the framework of Sobolev spaces. The first work on this problem was carried out by B. Alvarez Samaniego in his PhD thesis \( \textbf{1} \) (see also the article \( \textbf{2} \) by the same author). Alvarez Samaniego proved the local well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$, using properties of the semi-group associated with the linear problem. He also obtained a global solution in $H^s$ for $s \geq 1$, making use of the standard energy estimates. This result was improved by several authors: X. Carvajal & M. Scialon proved in their article \( \textbf{7} \), through Strichartz-type estimates and smoothing effects, the local well-posedness (LWP) of the Cauchy’s problem \( \textbf{3} \) in $H^s(\mathbb{R})$ for $s \geq 0$ and global well-posedness (GWP) in $L^2(\mathbb{R})$. After, X. Zhao & S. Cui proved in \( \textbf{20} \) and \( \textbf{21} \) the LWP of problem \( \textbf{1} \) in $H^s(\mathbb{R})$ for $s > -\frac{3}{2}$ and GWP for $s \geq 0$. Finally, in recent works A. Esfahani and H. Wang \( \textbf{10, 11} \) used purely dissipative approaches based on the method of bilinear estimates in the Bourgain-type spaces (see also \( \textbf{15} \) for more references on these spaces) to show that the Cauchy problem \( \textbf{3} \) is LWP in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{2}$ and moreover, it is shown that $H^{-\frac{3}{2}}$ is the critical Sobolev space for the LWP.

On the other hand, since equation \( \textbf{1} \) is a nonlinear dissipative equation, it is natural to ask for the existence of solitary waves. Numerical studies done in \( \textbf{8} \) by B.F. Feng & T. Kawahara shows that for every $\eta > 0$ there exists a family of solitary waves which experimentally decay as $\frac{1}{1 + |x|^2}$ when $|x| \to +\infty$. This numerical decay of solitary waves suggests the theoretical study of the decay in spacial variable of solutions $u(t, x)$ of equation \( \textbf{1} \) and, in this setting, B. Alvarez Samaniego showed in the last part of his PhD thesis (see Theorem 5.2 of \( \textbf{1} \)) that if the initial datum $u_0$ verifies $u_0 \in H^2(\mathbb{R}) \cap L^2(1 + |\cdot|^2, dx)$ then there exists $u \in C([0, \infty], H^2(\mathbb{R}) \cap L^2(1 + |\cdot|^2, dx))$ a unique solution of equation \( \textbf{1} \). This result is intrinsically related to the nature of the functional spaces above in which the Fourier Transform plays a very important role: kernel $K_{\eta}(t, x)$ given in \( \textbf{1} \) associated with the equation is explicitly defined in frequency variable. Furthermore, remark that this spatially-decaying of solution is studied in the setting of the weighted-$L^2$ space and therefore it’s a weighted average decay.

The general aim of this paper is to study spatial decay estimates of the solution $u(t, x)$. Our methods are inspired by L. Brandolese \textit{et. al.} \( \textbf{3, 4} \) which are essentially based on well-know properties of the kernel associated to the linear equation, however, our approach to find these estimates is a little different. Indeed, using the explicit definition in the frequency variable of $K_{\eta}(t, x)$ and the inverse Fourier transform, we deduce some sharp spatially-decaying properties for this kernel and for its derivatives. It is worth remarking that these methods are technically different with respect to previous works on equation \( \textbf{1} \) since in those works the kernel is studied in the frequency variable and not in the spatial variable.
On the other hand, this approach permits to study the equation \( \text{1} \) in other functional spaces which, to the best of our knowledge, have not been considered before. More precisely, we prove that the properties in the spatial variable of kernel \( K_\eta(t, x) \) allow us to prove that the integral equation \( \text{3} \) is LWP for small initial datum in the framework of Lebesgue spaces.

**Organization of the paper.** In Section 2 we state all the results obtained. In Section 3 we study the optimal decay in spatial variable of the kernel \( K_\eta(t, x) \). Section 4 is devoted to the study of pointwise decaying and asymptotic behavior in the spatial variable of solutions of equation \( \text{1} \). The last section 5 is devoted to the studies of the LWP of equation \( \text{1} \) in the framework of Lebesgue spaces.

**Acknowledgments.** We thank the referees for the useful remarks and comments which allow us to improve our work.

## 2 Statement of the results

### 2.1 Pointwise decay and asymptotic behavior in spatial variable

The first purpose of this paper is to obtain a pointwise decay in the spacial variable of solution \( u(t, x) \). More precisely, we prove that if the initial datum \( u_0 \in H^s(\mathbb{R}) \) (with \( \frac{3}{2} < s \leq 2 \)) verifies \( |u_0(x)| \leq c_1 + |x|^2 \), then there exist a unique global in time solution \( u(t, x) \) of the integral equation \( \text{3} \) which fulfills the same decay of the initial datum \( u_0 \). Moreover, we show that the solution \( u(t, x) \) of the integral equation \( \text{3} \) is smooth enough and then this solution verifies the differential equation \( \text{1} \) in the classical sense.

**Theorem 1** Let \( \frac{3}{2} < s \leq 2 \) and let \( u_0 \in H^s(\mathbb{R}) \) be an initial datum, such that \( |u_0(x)| \leq c \frac{1}{1 + |x|^2} \). Then, the equation \( \text{1} \) possesses a unique solution \( u \in C([0, +\infty[; C^\infty(\mathbb{R})) \) arising from \( u_0 \), such that for all time \( t > 0 \) there exists a constant \( C(t, \eta, u_0, u) > 0 \) such that for all \( x \in \mathbb{R} \) the solution \( u(t, x) \) verifies:

\[
|u(t, x)| \leq \frac{C(t, \eta, u_0, \|u\|_{H^s})}{1 + |x|^2}.
\] (5)

**Remark 1** Estimate (5) is valid only in the setting of the perturbed KdV equation \( \text{7} \) when the parameter \( \eta \) is strictly positive.

Indeed, with respect to the parameter \( \eta \) the constant \( C(t, \eta, u_0, u) > 0 \) behaves like the following expression (see formula (65) for all the details): \( \frac{1}{\eta^3} \left( 1 + \left( \frac{1}{\eta} + 2 \right)^2 \right) + 1 \), and this expression is not controlled when \( \eta \to 0^+ \).

Recall that in the case \( \eta = 0 \) the equation \( \text{1} \) becomes the KdV equation. In this framework T. Kato [14] showed the following persistence problem: if \( u_0 \in H^{2m}(\mathbb{R}) \cap L^2(|x|^{2m}, dx) \), where \( m \in \mathbb{N} \) is strictly positive, then the Cauchy problem for the KdV equation is globally well-posed in the space \( C([0, +\infty[; H^{2m}(\mathbb{R}) \cap L^2(|x|^{2m}, dx)) \) and then the solution of the KdV equations decays at infinity as fast as the initial datum. For related results see also [12] and [16].

Getting back to the perturbed KdV equation \( \text{1} \) a natural question arises: is the spatial decay given in the formula (65) optimal? and concerning this question B Alvarez Samaniego has shown in [3] that the solution cannot have a weight average decay faster than \( \frac{1}{1 + |x|^4} \); and in this case we have a loss of persistence in the spatial decay. This results suggests that the optimal decay rate in spatial variable of solution \( u(t, x) \) must be of the order \( \frac{1}{1 + |x|^s} \) with \( 2 \leq s < 4 \).
The second purpose of this paper is to study how sharp is the decay rate of solution given in Theorem 1. For this purpose, in the following theorem we start by studying the asymptotic profile of solution \( u(t,x) \) and we prove that if the initial datum \( u_0 \) decays a little faster than \( \frac{1}{1 + |x|^2} \), then the solution \( u(t,x) \) associated to \( u_0 \) has the following asymptotic behavior in the spatial variable.

**Theorem 2** Let \( \frac{3}{2} < s \leq 2 \) and let \( u_0 \in H^s(\mathbb{R}) \) be an initial datum such that for \( \varepsilon > 0 \) we have

\[ |u_0(x)| \leq \frac{c}{1 + |x|^{2+s}} \quad \text{and} \quad \frac{d}{dx} u_0(x) \leq \frac{c}{1 + |x|^2}. \]

Then, the solution \( u(t,x) \) of the equation (1) given by Theorem 1 has the following asymptotic development when \( |x| \) is large enough:

\[ u(t,x) = K_0(t,x) \left( \int_{\mathbb{R}} u_0(y)dy \right) + \int_0^t K_0(t-\tau,x) \left( \int_{\mathbb{R}} u(\tau,y)\partial_y u(\tau,y)dy \right) d\tau + o(t) \left( \frac{1}{|x|^2} \right), \quad (6) \]

where the kernel \( K_0(t,x) \) is given in (4), and where the quantity \( o(t) \left( \frac{1}{|x|^2} \right) \) is such that for all \( t > 0 \)

\[ \lim_{|x| \rightarrow +\infty} \frac{o(t) \left( \frac{1}{|x|^2} \right)}{\frac{1}{|x|^2}} = 0. \quad (7) \]

This asymptotic development of solution \( u(t,x) \) provides us interesting information on the behavior of this solution in spatial variable. Remark first that all the information respect to the spatial variable relies on the information (in the spatial variable) of the kernel \( K_0(t,x) \). More precisely, concentrating our attention in the first term on the right-hand side of this identity we may observe that this term is not zero when the initial datum \( u_0 \) is not a zero-mean function (\( \int_{\mathbb{R}} u_0(y)dy \neq 0 \)). Moreover, in Proposition 3.1 below, we show that this kernel has an optimal decay rate of the order \( \frac{1}{1 + |x|^2} \) and this fact suggests that the decay of solution \( u(t,x) \) given in Theorem 1 must be sharp when the initial datum verifies a non zero-mean condition.

On the other hand, in the case of a zero-mean initial datum (\( \int_{\mathbb{R}} u_0(y)dy = 0 \)) and for \( |x| \) large enough, observe that the solution behaves essentially as the second term on the right-hand side of identity (6) which comes from the nonlinear term in equation (1). In this case we shall prove that the decay rate of solution given in the formula (6) actually is not sharp and it can be improved.

Our next result summarizes these statements.

**Theorem 3** Under the same hypothesis of Theorem 1

1) Assume that \( \int_{\mathbb{R}} u_0(y)dy \neq 0 \). Then there exists \( M > 0 \) and there exists a constant \( 0 < c_{\eta,t} < c_\eta e^{4\eta t} \) such that for \( |x| > M \) we have the estimate from below:

\[ \frac{c_{\eta,t}}{2|x|^2} \int_{\mathbb{R}} u_0(y)dy \leq |u(t,x)|. \quad (8) \]

2) Assume that \( \int_{\mathbb{R}} u_0(y)dy = 0 \). Then the solution \( u(t,x) \) of the equation (1) given by Theorem 1 has the following decay: for \( 0 < \varepsilon \leq 1 \)

\[ |u(t,x)| \leq \frac{C'(\eta,\varepsilon,t,u_0,u)}{1 + |x|^{2+\varepsilon}}, \quad (9) \]

where the constant \( C'(\eta,\varepsilon,t,u_0,\|u\|_{H^s}) > 0 \) does not depend on the variable \( x \).

**Remark 2** It should be emphasized that while the non zero-mean condition \( \int_{\mathbb{R}} u_0(y)dy \neq 0 \) is verified, even if the initial datum is a smooth, compact-support function the arising solution \( u(t,x) \) cannot decay at infinity faster than \( \frac{1}{|x|^2} \) and in this case the decay rate of solution given in Theorem 2 is optimal.
Remark 3 When \( \int_{\mathbb{R}} u_0(y)dy = 0 \), the estimate from below \((8)\) is not more valid and moreover the decay rate of solution \( u(t, x) \) given in Theorem \((1)\) is improved in estimate \((9)\). Thus, the persistence problem is valid for \( 0 < \epsilon \leq 1 \).

However, with respect to optimality of this estimate, our approaches do not seem to be sufficient to derive an estimate from below of the type \( \frac{1}{|x|^2} \lesssim |u(t, x)| \). This fact remains an interesting open question.

Remark 4 For \( \epsilon > 1 \), the persistence problem studied in point 2) of Theorem \((3)\) does not seem to be valid. Indeed, roughly speaking, inequality \((9)\) relies on sharp estimates for the linear and the nonlinear term in the integral formulation of the solution given in \((3)\). The estimate done on the linear term actually can be improved as

\[
|K_\eta(t, x) * u_0(x)| \lesssim \frac{1}{1 + |x|^{2+\epsilon}},
\]

with \( \epsilon > 1 \), provided that the initial datum is a zero-mean function which decays fast enough, but, the nonlinear term is estimated as

\[
\left| \int_0^t \partial_x K_\eta(t - \tau, \cdot) * u^2(\tau, \cdot)(x)d\tau \right| \lesssim \frac{1}{1 + |x|^3},
\]

see estimate \((10)\) for the details. As the expression \( \partial_x K_\eta(t, x) \) has a sharp decay of the order \( \frac{1}{|x|^3} \) this proposes that this term cannot decay faster than \( \frac{1}{|x|^3} \) and, to the best of our knowledge, we do not know a better estimate.

2.2 The local well-posedness in Lebesgue spaces

The third purpose of this paper is to study the existence and uniqueness of mild solutions for the Cauchy problem \((1)\) in the framework of Lebesgue spaces when the initial datum \( u_0 \) is small enough. We start by recalling that we refer to a mild solution \( u(t, x) \) when this solution is written as the integral formulation \((3)\) and it is obtained by a fixed-point argument.

It is worth remarking here that the following theorem is just a first study in the setting of Lebesgue spaces and we think that this result could be improved in further investigations.

**Theorem 4** Let \( 1 \leq p \leq +\infty \) and let \( u_0 \in L^p(\mathbb{R}) \) be an initial datum. Let \( T > 0 \). Then, there exists \( \delta = \delta(T) > 0 \) such that if \( \|u_0\|_{L^p} < \delta \) then the integral equation \((3)\) possesses a unique solution local in time \( u \in L^\infty([0, T], L^p(\mathbb{R})) \) which verifies

\[
\sup_{0 \leq t \leq T} t^{\frac{1}{p}} \|u(t, \cdot)\|_{L^p} < +\infty.
\]

**Remark 5** The value of the parameter \( p = 2 \) is of particular interest since in this case the result above gives a new proof for the LWP obtained by X. Carvajal & M. Scialons in \((7)\), which relies essentially on smoothing effects and Strichartz-type estimates.

We finish the statement of our results with the following interesting remark.

**Remark 6** All our results given for the equation \((1)\) are still valid (under some technical modifications in the proofs) for the non-local perturbation of the Benjamin-Ono (npBO) equation:

\[
\begin{cases}
\partial_t u + u \partial_x u + \mathcal{H} \partial_x^2 u + \eta(\mathcal{H} \partial_x u + \mathcal{H} \partial_x^3 u) = 0, & \eta > 0, \text{ on } ]0, +\infty[ \times \mathbb{R}, \\
 u(0, \cdot) = u_0.
\end{cases}
\]

(10)
On the other hand, it is easy to see that the kernel \( F_\eta(t, x) \) associated to equation (10) in explicitly defined in the frequency variable as
\[
\hat{F}_\eta(t, \xi) = e^{i\xi|\xi|t + \eta t(|\xi| - |\xi|^3)},
\]
and thus, our methods can be adapted without any problem. Indeed, our results are purely based on estimates on the non-complex exponential part \( e^{\eta t(|\xi| - |\xi|^3)} \) which is exactly the same for the kernel \( \hat{K}_\eta(t, \xi) \).

### 3 kernel estimates

In this section we study the properties decay in spatial variable of the kernel \( K_\eta(t, x) \) which will be useful in the next sections.

**Proposition 3.1** Let \( K_\eta(t, x) \) be the kernel defined in the expression (4).

1) There exists a constant \( c_\eta > 0 \), given in the formula (22) and which only depends on \( \eta > 0 \), such that for all time \( t > 0 \) we have
\[
|K_\eta(t, x)| \leq c_\eta \frac{1}{t^\frac{1}{3} + |x|^2}.
\]

2) Moreover, the kernel \( K_\eta(t, x) \) cannot decay at infinity faster than
\[
\frac{1}{1 + |x|^2}.
\]

**Proof.**

1) First, we will estimate the quantity \( |K_\eta(t, x)| \) and then we will estimate the quantity \( |x|^2|K_\eta(t, x)| \).

We write
\[
|K_\eta(t, x)| \leq \|K_\eta(t, \cdot)\|_{L^\infty} \leq \|\hat{K}_\eta(\cdot, \cdot)\|_{L^1},
\]
and then we must study the term \( \|\hat{K}_\eta(\cdot, \cdot)\|_{L^1} \). By the expression (4), we have
\[
\hat{K}_\eta(t, \xi) = e^{i\xi^3 t - \eta t(|\xi|^3 - |\xi|)}
\]
and we can write
\[
\|\hat{K}_\eta(t, \cdot)\|_{L^1} = \int_{\mathbb{R}} |e^{i\xi^3 t}| |e^{-\eta t(|\xi|^3 - |\xi|)}| d\xi = \int_{\mathbb{R}} e^{-\eta t(|\xi|^3 - |\xi|)} d\xi = \int_{|\xi| \leq \sqrt{\eta t}} e^{-\eta t(|\xi|^3 - |\xi|)} d\xi + \int_{|\xi| > \sqrt{\eta t}} e^{-\eta t(|\xi|^3 - |\xi|)} d\xi
\]
\[
= I_1 + I_2.
\]

In order to estimate the integral \( I_1 \), remark that if \( |\xi| \leq \sqrt{\eta t} \) then we have \(-(|\xi|^3 - |\xi|) \leq |\xi| \) and thus we can write
\[
I_1 \leq \int_{|\xi| \leq \sqrt{\eta t}} e^{-\eta t|\xi|} d\xi \leq c e^{\sqrt{\eta t}} \leq c e^{2\eta t}.
\]

Now, in order to estimate the integral \( I_2 \), remark that if \( |\xi| > \sqrt{\eta t} \) then we have \(-(|\xi|^3 - |\xi|) < -\frac{|\xi|^3}{2} \) and thus, we write
\[
I_2 \leq \int_{|\xi| > \sqrt{\eta t}} e^{-\eta t|\xi|^3} d\xi \leq \int_0^{+\infty} e^{-\eta t\xi^3} d\xi \leq \frac{c}{(\eta t)^{\frac{3}{2}}}.
\]

With these estimates, we get back to the identity (12) and we write
\[
\|\hat{K}_\eta(\cdot, \cdot)\|_{L^1} \leq c e^{2\eta t} + \frac{c}{(\eta t)^{\frac{3}{2}}} \leq c e^{2\eta t} (\eta t)^{\frac{1}{2}} + 1 \leq c e^{3\eta t} + 1 \leq C e^{3\eta t},
\]

(13)
Thus, following the same computation done in identity (15) and since
\[
\lim_{\xi \to \infty} e^{it\xi^3 - \eta(\xi^3 - \xi)} = 0 \quad \text{and} \quad \lim_{\xi \to -\infty} e^{it\xi^3 - \eta(\xi^3 - \xi)} = 0
\]
then, we have
\[
\frac{1}{2\pi i x} \int_{\xi < 0} \partial_\xi (e^{2\pi i \xi} e^{i t \xi^3 - \eta(\xi^3 + \xi)}) d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} \partial_\xi (e^{2\pi i \xi} e^{i t \xi^3 - \eta(\xi^3 - \xi)}) d\xi = \frac{1}{2\pi i x} \int_{\xi < 0} e^{2\pi i \xi} \partial_\xi (e^{i t \xi^3 - \eta(\xi^3 + \xi)}) d\xi - \frac{1}{2\pi i x} \int_{\xi > 0} e^{2\pi i \xi} \partial_\xi (e^{i t \xi^3 - \eta(\xi^3 - \xi)}) d\xi = (a).
\]
Thus, following the same computation done in identity (15) and since \( \partial_\xi (e^{2\pi i \xi}) = 2\pi i x e^{2\pi i \xi} \), then we write
\[
(a) = -\frac{1}{(2\pi i x)^2} \int_{\xi < 0} e^{2\pi i \xi} \partial_\xi (e^{i t \xi^3 - \eta(\xi^3 + \xi)}) d\xi + \frac{1}{(2\pi i x)^2} \int_{\xi > 0} \partial_\xi (e^{2\pi i \xi}) \partial_\xi (e^{i t \xi^3 - \eta(\xi^3 - \xi)}) d\xi
\]
where we will estimate both expressions \( I_1 \) and \( I_2 \). For expression \( I_1 \), remark that we have
\[
\lim_{\xi \to \infty} (e^{i t \xi^3 - \eta(-\xi^3 + \xi)}(3it\xi^2 - \eta(-3\xi^2 + 1))) = 0,
\]
and integrating by parts, we can write
\[
I_1 = -\frac{1}{(2\pi i x)^2} \left( -\eta t - \int_{\xi < 0} e^{2\pi i \xi} \partial_\xi \left( (e^{i t \xi^3 - \eta(-\xi^3 + \xi)}(3it\xi^2 - \eta(-3\xi^2 + 1))) d\xi \right) \right)
\]
\[
= \eta t \left( \frac{1}{(2\pi i x)^2} + \frac{1}{(2\pi i x)^2} \int_{\xi < 0} e^{2\pi i \xi} \partial_\xi \left( (e^{i t \xi^3 - \eta(-\xi^3 + \xi)}(3it\xi^2 - \eta(-3\xi^2 + 1)))) d\xi \right) \right) = I_a.
\]
Thus, with identities (17) and (18) at hand, we get back to the identity (16) and we write

\[ I_2 = -\frac{1}{(2\pi i)^2} \left( -\eta - \int_{\xi>0} e^{2\pi in\zeta} \partial_\zeta \left( (e^{it\xi^2-\eta(\xi^2+\zeta)})(3it\xi^2 - \eta(3\xi^2 - 1)) \right) d\zeta \right) \]

\[ = \frac{\eta t}{(2\pi i)^2} + \frac{1}{(2\pi i)^2} \int_{\xi>0} e^{2\pi in\zeta} \partial_\zeta \left( (e^{it\xi^2-\eta(\xi^2+\zeta)})(3it\xi^2 - \eta(3\xi^2 - 1)) \right) d\zeta. \]

(18)

Thus, with identities (17) and (18) at hand, we get back to the identity (16) and we write

\[ I_1 + I_2 = \frac{2\eta t}{(2\pi i)^2} + \frac{1}{(2\pi i)^2} (I_a + I_b), \]

(19)

and then, getting back to the identity (15) we have

\[ |K_\eta(t, x)| = \left| \frac{2\eta t}{(2\pi i)^2} + \frac{1}{(2\pi i)^2} (I_a + I_b) \right| \leq c \frac{\eta t}{x^2} + C \frac{1}{x^2} |I_a + I_b|. \]

(20)

We still need to estimate the term \(|I_a + I_b| \) above and for this we have the following technical lemma, which we will prove later in the appendix.

**Lemma 3.1** There exists a numerical constant \( c > 0 \), which does not depend on \( \eta > 0 \), such that for all \( t > 0 \), we have \(|I_a + I_b| \leq c \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t} \).

With this estimate, we get back to the equation (20) and we get

\[ |K_\eta(t, x)| \leq c \frac{\eta t}{x^2} + c \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t} \leq c \left( \frac{1}{\eta} + 2 \right)^2 \frac{\eta t}{x^2} + c \left( \frac{1}{\eta} + 2 \right)^2 \frac{e^{4\eta t}}{x^2} \]

\[ \leq c \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t} + c \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t} \leq C \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t}. \]

Hence, we can write

\[ |x|^2 |K_\eta(t, x)| \leq C \left( \frac{1}{\eta} + 2 \right)^2 e^{4\eta t}. \]

(21)

Thus, with estimates (14) and (21), we can write

\[ |K_\eta(t, x)| + |x|^2 |K_\eta(t, x)| \leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{3}{2}}} + C \left( \frac{1}{\eta} + 2 \right)^2 \frac{e^{4\eta t}}{x^2} \leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{3}{2}}} + C \left( \frac{1}{\eta} + 2 \right)^2 \left( \frac{\eta t}{x^2} \right)^{\frac{1}{2}} \frac{e^{4\eta t}}{x^2} \]

\[ \leq C \frac{e^{5\eta t}}{(\eta t)^{\frac{3}{2}}} + C \left( \frac{1}{\eta} + 2 \right)^2 \frac{e^{5\eta t}}{(\eta t)^{\frac{3}{2}}} \leq C \left( 1 + \left( \frac{1}{\eta} + 2 \right)^2 \right) \frac{e^{5\eta t}}{t^{\frac{3}{2}}}. \]

Finally, from now on we set the constant

\[ c_\eta = \frac{C}{\eta^2} \left( 1 + \left( \frac{1}{\eta} + 2 \right)^2 \right) > 0, \]

(22)

and we get the desired estimate.
2) We will suppose that there exists $\varepsilon > 0$ and $M > 0$ such that for all $|x| > M$, we have $|K_\eta(t, x)| \lesssim \frac{1}{|x|^{2+\varepsilon}}$ and then we will arrive to a contradiction. Indeed, if we suppose this estimate then we can prove that the function $xK_\eta(t, x)$ belongs to the space $L^1(\mathbb{R})$: we write

$$\int_\mathbb{R} |xK_\eta(t, x)| \, dx = \int_{|x| \leq M} |xK_\eta(t, x)| \, dx + \int_{|x| > M} |xK_\eta(t, x)| \, dx = I_1 + I_2.$$ 

In order to estimate the term $I_1$, recall that from point 1) of Proposition $3.1$ we have: for all $t > 0$, $K_\eta(t, \cdot) \in L^1(\mathbb{R})$. Thus, we have

$$I_1 \leq M \int_{|x| \leq M} |K_\eta(t, x)| \, dx \leq M \|K_\eta(t, \cdot)\|_{L^1} < +\infty.$$ 

Now, we estimate the term $I_2$ and since we have $|K_\eta(t, x)| \lesssim \frac{1}{|x|^{2+\varepsilon}}$, for all $|x| > M$, then we can write

$$I_2 \lesssim \int_{|x| > M} |x| \frac{1}{|x|^{2+\varepsilon}} \, dx \lesssim \int_{|x| > M} \frac{dx}{|x|^{1+\varepsilon}} < +\infty.$$ 

Thus, the function $xK_\eta(t, x)$ belongs to the space $L^1(\mathbb{R})$ and then by the properties of the Fourier transform we get that $\partial_x \tilde{K}_\eta(t, \xi)$ is a continuous function. Moreover, recall that we have $K_\eta(t, \cdot) \in L^1(\mathbb{R})$ and then $\tilde{K}_\eta(t, \xi)$ is also a continuous function and thus, for all time $t > 0$, we have $\tilde{K}_\eta(t, \cdot) \in C^4(\mathbb{R})$, but this fact is not possible. Indeed, by identity $\mathbb{H}$, we have $\tilde{K}_\eta(t, \xi) = e^{i\xi t}e^{-\eta|\xi|}e^{\eta|\xi|}$, but observe that the term $e^{\eta|\xi|}$ is not differentiable at the origin and then $\tilde{K}_\eta(t, \cdot)$ cannot belong to the space $C^4(\mathbb{R})$.

\section{Pointwise decaying and asymptotic behavior in spacial variable}

\subsection{Proof of Theorem 1}

Let $\frac{2}{3} < s \leq 2$ fix and let $u_0 \in H^s(\mathbb{R})$ be the initial datum and suppose that this function verifies

$$|u_0(x)| \leq \frac{c}{1 + |x|^2}. \quad (23)$$

We start by studying the existence of a local in time solution $u$ of integral equation $\mathbb{M}$. 

\subsubsection{Local in time existence}

Let $T > 0$ and consider the functional space $Y_T = \left\{ u \in \mathcal{S}'([0, T] \times \mathbb{R}) : \sup_{0 \leq t \leq T} t^\frac{s}{2} \|(1 + |\cdot|^2)u(t, \cdot)\|_{L^\infty} < +\infty \right\}$ and then define the Banach space

$$F_T = Y_T \cap C([0, T], H^s(\mathbb{R})), \quad (24)$$
doted with the norm

$$\| \cdot \|_{F_T} = \sup_{t \in [0, T]} t^\frac{s}{2} \|(1 + |\cdot|^2)(\cdot)\|_{L^\infty(\mathbb{R})} + \sup_{t \in [0, T]} \| \cdot \|_{H^s(\mathbb{R})}. \quad (25)$$

Remark that this norm is composed of two terms: the first term in the right side in $\mathbb{M}$ will allow us to study the decay in spatial variable of the solution $u$. In this term we can observe a weight in time variable $t^\frac{s}{2}$ which the reason to add this weight is purely technical and it allows us to carry out the estimates which
we shall need later.

On the other hand, the second term on the right side in (25) will allow us to study the regularity of solution u and this will be done later in Section 4.1.3.

**Theorem 4.1** There exists a time \( T_0 > 0 \) and a function \( u \in F_{T_0} \) which is the unique solution of the integral equation (5).

**Proof.** We write
\[
\left\| u \right\|_{F_T} = \left\| K_\eta(t, \cdot) * u_0 - \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) * \partial_x(u^2)(\tau, \cdot) d\tau \right\|_{F_T}
\]
\[
\leq \left\| K_\eta(t, \cdot) * u_0 \right\|_{F_T} + \frac{1}{2} \int_0^t \left\| K_\eta(t - \tau, \cdot) * \partial_x(u^2)(\tau, \cdot) d\tau \right\|_{F_T},
\]
and we will estimate each term in the right side.

**Proposition 4.1** There exist a constant \( C_{1,\eta} > 0 \) given in the formula (32), which only depends on \( \eta > 0 \), such that we have:
\[
\left\| K_\eta(t, \cdot) * u_0 \right\|_{F_T} \leq C_{1,\eta} e^{5\eta T} \left( \left\| (1 + \cdot)^2 \right\|_{L^\infty} + \left\| u_0 \right\|_{H^s} \right).
\]

**Proof.** By the definition of the quantity \( \| \cdot \|_{F_T} \) given in the equation (25) we write
\[
\left\| K_\eta(t, \cdot) * u_0 \right\|_{F_T} = \sup_{t \in [0, T]} t^\frac{1}{2} \left\| (1 + \cdot)^2 \right\|_{L^\infty} K_\eta(t, \cdot) * u_0 \right\|_{L^\infty} + \sup_{t \in [0, T]} \left\| K_\eta(t, \cdot) * u_0 \right\|_{H^s},
\]
and we start by estimating the first term on the right side. For all \( x \in \mathbb{R} \) we write
\[
\left| K_\eta(t, \cdot) * u_0(x) \right| \leq \int_{\mathbb{R}} \left| K_\eta(t, x - y) \right| \left| u_0(y) \right| dy \leq \int_{\mathbb{R}} \left| K_\eta(t, x - y) \right| \frac{1 + |y|^2}{1 + |y|^2} |u_0(y)| dy
\]
\[
\leq \left\| (1 + \cdot)^2 \right\|_{L^\infty} \int_{\mathbb{R}} \frac{\left| K_\eta(t, x - y) \right|}{1 + |y|^2} dy.
\]
We need to study the term \( \int_{\mathbb{R}} \frac{\left| K_\eta(t, x - y) \right|}{1 + |y|^2} dy \). Remark that from point 1) of Proposition 3.1 we have the estimate \( \left| K_\eta(t, x - y) \right| \leq \frac{c_\eta e^{5\eta T}}{t^\frac{1}{2}} \frac{1}{1 + |x - y|^2} \), and then we can write
\[
\int_{\mathbb{R}} \frac{\left| K_\eta(t, x - y) \right|}{1 + |y|^2} dy \leq \frac{c_\eta e^{5\eta T}}{t^\frac{1}{2}} \int_{\mathbb{R}} \frac{dy}{(1 + |x - y|^2)(1 + |y|^2)}.
\]
where the last term on the right side verifies
\[
\int_{\mathbb{R}} \frac{dy}{(1 + |x - y|^2)(1 + |y|^2)} \leq c \frac{1}{1 + |x|^2}.
\]
Now, we get back to (29) and we have \( \left| K_\eta(t, \cdot) * u_0(x) \right| \leq \left\| (1 + \cdot)^2 \right\|_{L^\infty} \frac{c_\eta e^{5\eta T}}{t^\frac{1}{2}} \frac{1}{1 + |x|^2} \).
Thus, the first term on the right side in (28) is estimated as follows:
\[
\sup_{t \in [0, T]} t^\frac{1}{2} \left\| (1 + \cdot)^2 \right\|_{L^\infty} \leq c_\eta e^{5\eta T} \left\| (1 + \cdot)^2 \right\|_{L^\infty}.
\]
Now, we go to study the second term on the right side in (25) and we will prove the following estimate
\[
\sup_{t \in [0,T]} \| K_{\eta}(t, \cdot) \ast u_0 \|_{H^s} \leq c e^{5\eta T} \| u_0 \|_{H^s},
\] (33)
where \( c > 0 \) is a numerical constant which does not depend on \( \eta > 0 \). This estimate relies on the following technical estimate given in Lemma 2.2, (page 10) of [1]: let \( s_1 \in \mathbb{R}, \phi \in H^{s_1}(\mathbb{R}) \) and let \( s_2 \geq 0 \). Then, for all \( t > 0 \), we have
\[
\| K_{\eta}(t, \cdot) \ast \phi \|_{H^{s_1} + s_2} \leq c e^{5\eta t} \| \phi \|_{H^{s_1}}.
\] (34)
In this estimate we set \( \phi = u_0 \in H^s(\mathbb{R}), s_1 = s \) and \( s_2 = 0 \); and then, for all \( 0 \leq t \leq T \), we get
\[
\| K_{\eta}(t, \cdot) \ast u_0 \|_{H^s} \leq c e^{5\eta t} \| u_0 \|_{H^s} \leq c e^{5\eta T} \| u_0 \|_{H^s},
\]
and hence, we have the estimate (33). Now, by estimates (32) and (33) we set the constant \( C_{1, \eta} > 0 \) as
\[
C_{1, \eta} = c_\eta + c,
\] (35)
where \( c_\eta > 0 \) is the constant given in the formula (23), and then we have the estimate given in (27). Proposition 1.1 is proven.

Now, we estimate the second term on the right side in the equation (26).

**Proposition 4.2** There exists a constant \( C_{2, \eta} > 0 \) given in the formula (46), which depends only on \( \eta > 0 \), such for all \( u \in F_T \) we have
\[
\left\| \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) d\tau \right\|_{F_T} \leq C_{2, \eta} e^{5\eta T} \max(T^{\frac{2}{3}}, T^\frac{5}{6}) \| u \|_{F_T} \| u \|_{F_T}.
\] (36)

**Proof.** By definition of the norm \( \| \cdot \|_{F_T} \) given in (25), we write
\[
\left\| \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) d\tau \right\|_{F_T} = \sup_{t \in [0,T]} \int_0^t \left\| (1 + |\cdot|^2)^{\frac{1}{2}} \left( \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty}
\]
\[
+ \sup_{t \in [0,T]} \left\| \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) d\tau \right\|_{H^s},
\] (37)
and we will estimate each term in the right side.

For the first term in (37), for all \( t \in [0, T] \) we have
\[
\int_0^t \left\| (1 + |\cdot|^2)^{\frac{1}{2}} \left( \frac{1}{2} \int_0^t K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty} \leq t^\frac{1}{2} \int_0^t \left\| (1 + |\cdot|^2)^{\frac{1}{2}} \frac{1}{2} K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) \right\|_{L^\infty} d\tau,
\]
and now we need to prove the following estimate:
\[
\left\| (1 + |\cdot|^2)^{\frac{1}{2}} \frac{1}{2} K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot) \right\|_{L^\infty} \leq c_\eta \frac{e^{5\eta (t - \tau)}}{(t - \tau)^\frac{5}{6}} \| u \|_{F_T} \| u \|_{F_T}.
\] (38)
Indeed, we will study first the quantity \( \frac{1}{2} K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) \). Remark that we have \( \frac{1}{2} \partial_x (u^2) = u \partial_x u \) and then for all \( x \in \mathbb{R} \) we write
\[
\frac{1}{2} K_{\eta}(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) \leq | K_{\eta}(t - \tau, \cdot) \ast (u(\tau, \cdot) \partial_x u(\tau, \cdot)) (x) |
\]
\[
\leq \int_{\mathbb{R}} | K_{\eta}(t - \tau, x - y) | u(\tau, y) \partial_y u(\tau, y) dy.
\] (39)
Now, recall that by point 1) of Proposition 3.1 we have \( |K_\eta(t - \tau, x - y)| \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x - y|^2} \), and then in the last term above, we can write

\[
\int_\mathbb{R} |K_\eta(t - \tau, x - y)||u(\tau, y)||\partial_y u(\tau, y)|dy \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x - y|^2} \int_\mathbb{R} |u(\tau, y)||\partial_y u(\tau, y)|dy \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x - y|^2} \int_\mathbb{R} |u(\tau, y)| \left( \frac{1}{1 + |y|^2} \right)^\frac{3}{2} \left( \frac{1}{1 + |x - y|^2} \right) dy \quad (40)
\]

where we have to study the terms (a) and (b). For term (a) we have

\[
(a) \leq \frac{c}{\tau^\frac{3}{2}} \|u\|_{F_T} \|u\|_{F_T}. \quad (42)
\]

Indeed, recall first that we have the inclusion \( H^{s-1}(\mathbb{R}) \subset L^\infty(\mathbb{R}) \) (since \( s - 1 > \frac{1}{2} \)). Hence, we can write

\[
\|\partial_y u(\tau, \cdot)\|_{L^\infty} \leq c\|\partial_x u(\tau, \cdot)\|_{H^{s-1}} \leq c\|u(\tau, \cdot)\|_{H^s}. \quad (43)
\]

Thus, we have

\[
(a) \leq \|(1 + |\cdot|^2)u(\tau, \cdot)\|_{L^\infty} \|u(\tau, \cdot)\|_{H^s} \leq \frac{c}{\tau^\frac{3}{2}} \left( \frac{1}{\tau^\frac{3}{2}} \|1 + |\cdot|^2\|_{L^\infty} \right) (\|u(\tau, \cdot)\|_{H^s}),
\]

and by definition of the norm \( \| \cdot \|_{F_T} \) given in (25) we can write the estimate given in (42).

For term (b) in (41), recall that this was already estimated at (31).

Then, in the estimate (41), by estimates (42) and (31) we have

\[
\int_\mathbb{R} |K_\eta(t - \tau, x - y)||u(\tau, y)||\partial_y u(\tau, y)|dy \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x - y|^2} \|1\} \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x - y|^2} \|u\|_{F_T} \|u\|_{F_T},
\]

and now, we get back to estimate (39) and we write

\[
\left| \frac{1}{2} K_\eta(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) \right| \leq c_\eta \frac{e^{5\eta(t - \tau)}}{(t - \tau)^\frac{3}{2} + |x|^2} \|u\|_{F_T} \|u\|_{F_T} \quad (43)
\]

Thus, we get the estimate (39).

Once we dispose of this estimate, for all \( t \in [0, T] \), we can write

\[
t^\frac{3}{2} \int_0^t \left( \frac{1}{t - \tau} \right)^\frac{3}{2} \|u\|_{F_T} \left| \frac{d}{d\tau} \right| \quad (44)
\]

\[
\leq c_\eta T^\frac{3}{2} e^{5\eta T} \left( T^\frac{3}{2} \right) \|u\|_{F_T} \|u\|_{F_T} \quad (45)
\]

\[
\leq c_\eta e^{5\eta T} T^\frac{3}{2} \|u\|_{F_T} \|u\|_{F_T},
\]

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and then we have
\[
\sup_{t \in [0,T]} \left\| \left(1 + | \cdot |^2 \right) \left( \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x(u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty_x} \leq c_\eta e^{5\eta T} T^\frac{\eta}{2} \| u \|_{F_T} \| u \|_{F_T}.
\] (44)

Now, we estimate the second term in identity (37). For all \( t \in [0, T] \), we write
\[
\left\| \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x(u^2)(\tau, \cdot) d\tau \right\|_{H^s} \leq \int_0^t \| K_\eta(t - \tau, \cdot) \ast \partial_x(u^2)(\tau, \cdot) \|_{H^s} d\tau
\]
\[
\leq \int_0^t \| \partial_x(K_\eta(t - \tau, \cdot) \ast u^2(\tau, \cdot)) \|_{H^s} d\tau \leq \int_0^t \| K_\eta(t - \tau, \cdot) \ast u^2(\tau, \cdot) \|_{H^{s+1}} d\tau.
\]
Then, in the estimate (34) we set now \( \phi = (u^2)(\tau, \cdot) \), \( s_1 = s \) and \( s_2 = 1 \); and then we have
\[
\int_0^t \| K_\eta(t - \tau, \cdot) \ast u^2(\tau, \cdot) \|_{H^{s+1}} d\tau \leq \int_0^t \frac{c e^{5\eta(t-\tau)}}{(\eta(t-\tau))^\frac{\eta}{2}} \| u^2(\tau, \cdot) \|_{H^s} d\tau,
\]
where, by the product laws in Sobolev spaces and moreover, by definition of the norm \( \| \cdot \|_{H^s} \) given in (25), we have
\[
\int_0^t \frac{c e^{5\eta(t-\tau)}}{(\eta(t-\tau))^\frac{\eta}{2}} \| u^2(\tau, \cdot) \|_{H^s} d\tau \leq \int_0^t \frac{c e^{5\eta(t-\tau)}}{(t-\tau)^\frac{\eta}{2}} \| u(\tau, \cdot) \|_{H^s}^2 d\tau
\]
\[
\leq \frac{c e^{5\eta T}}{\eta^\frac{\eta}{2}} \left( \sup_{\tau \in [0,T]} \| u(\tau, \cdot) \|_{H^s} \right)^2 \left( \sup_{\tau \in [0,T]} \| u(\tau, \cdot) \|_{H^s} \right) \int_0^t \frac{d\tau}{(t-\tau)^\frac{\eta}{2}} \leq c e^{5\eta T} T^\frac{\eta}{2} \| u \|_{F_T} \| u \|_{F_T}.
\]
Thus, we get the estimate
\[
\sup_{t \in [0,T]} \left\| \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x(u^2)(\tau, \cdot) d\tau \right\|_{H^s} \leq \frac{c e^{5\eta T}}{\eta^\frac{\eta}{2}} T^\frac{\eta}{2} \| u \|_{F_T} \| u \|_{F_T}.
\] (45)

Finally, by estimates (44) and (45) we set the constant \( C_{2,\eta} > 0 \) as
\[
C_{2,\eta} = c_\eta + \frac{c}{\eta^\frac{\eta}{2}},
\] (46)
where \( c_\eta > 0 \) is always the constant given in the formula (22), and the estimate (36) follows. Proposition 4.2 in now proven. 

Once we have the estimates given in Proposition 4.1 and in Proposition 4.2, we fix the time \( T_0 > 0 \) small enough and by the Picard contraction principle we get a solution \( u \in F_{T_0} \) of the integral equation (3).

Now, we prove the uniqueness of this solution \( u \in F_{T_0} \). Let \( u_1, u_2 \in F_{T_0} \) be two solutions of the equation (3) (associated with the same initial datum \( u_0 \)). We define \( v = u_1 - u_2 \) and we will prove that \( v = 0 \). Indeed, recall first that \( v(0, \cdot) = 0 \) and then \( v \) verifies the following integral equation
\[
v(t, \cdot) = -\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \left( \partial_x(u_1^2(\tau, \cdot) - u_2^2(\tau, \cdot)) \right) d\tau.
\]
Since, \( v = u_1 - u_2 \), we write \( u_1^2(\tau, \cdot) - u_2^2(\tau, \cdot) = v(\tau, \cdot)u_1(\tau, \cdot) + u_2(\tau, \cdot)v(\tau, \cdot) \), and thus we have
\[
v(t, \cdot) = -\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \left( \partial_x(v(\tau, \cdot)u_1(\tau, \cdot) + u_2(\tau, \cdot)v(\tau, \cdot)) \right) d\tau.
\] (47)
In this expression we take the norm $\| \cdot \|_{F_T}$ given in (23) and by Proposition 4.2 we have

$$
\|v\|_{F_T} \leq C_{2, \eta} \max(T_0^\frac{2}{3}, T_0^\frac{1}{3}) \|v\|_{F_T} \left( \|u_1\|_{F_T} + \|u_2\|_{F_T} \right).
$$

(48)

From this estimate, the identity $v = 0$ is deduced as follows: let $0 \leq T^* \leq T_0$ be the maximal time such that $v = 0$ at the interval $[0, T^*]$. We will prove that $T^* = T_0$ and by contradiction.

Let us suppose $T^* < T_0$. Let $T_1 \in [T^*, T_0]$ and for the interval in time $[T^*, T_1]$, consider the space $F_{[T_1-T^*]}$ defined in (23) and endowed with the norm $\| \cdot \|_{F_{[T_1-T^*]}}$ given in (23). By estimate (48), we can write

$$
\|v\|_{F_{[T_1-T^*]}} \leq C_{2, \eta} \max \left( (T_1 - T^*)^{\frac{2}{3}}, (T_1 - T^*)^{\frac{1}{3}} \right) \|v\|_{F_{[T_1-T^*]}} \left( \|u_1\|_{F_{[T_1-T^*]}} + \|u_2\|_{F_{[T_1-T^*]}} \right),
$$

and taking $T_1 - T^* > 0$ small enough, then we have $\|v\|_{F_{[T_1-T^*]}} = 0$ and thus we have $v = 0$ in the interval in time $[T^*, T_1]$, which is a contraction with the definition of time $T^*$. Then we have $T^* = T$. Theorem 4.1 is now proven.

4.1.2 Global in time existence and decay in spatial variable

In this section, we prove first that the local in time solution $u \in F_{T_0}$ of the integral equation (3) is extended to the whole interval in time $[0, +\infty[$. Then, we prove the decay in spatial variable given in the formula (5).

**Theorem 4.2** Let $T_0 > 0$ be the time given in Theorem 4.1. Let the Banach space $(F_{T_0}, \| \cdot \|_{F_{T_0}})$ given by formulas (23) and (25) and let $u \in F_{T_0}$ the solution of the integral equation (3) constructed in Theorem 4.1. Then, we have:

1) $u \in C([0, +\infty[, H^s(\mathbb{R}))$.

2) Moreover, for all time $t > 0$, there exists a constant $C = C(t, \eta, u_0, \|u(t)\|_{H^s}) > 0$, which depends on $t > 0$, $\eta > 0$, $u_0$, and the quantity $\|u(t)\|_{H^s}$, such that, for all $x \in \mathbb{R}$, the solution $u(t, x)$ verifies the estimate (7).

**Proof.**

1) Since $u_0 \in H^s(\mathbb{R})$, we get by Theorem 2 of the article [20] that there exists a function $v \in C([0, +\infty[, H^s(\mathbb{R}))$, which is the unique solution of integral equation (3). But, by definition of the Banach space $F_T$, we have the inclusion $F_T \subset C([0, T], H^s(\mathbb{R}))$ and then the solution $u \in F_T$ belongs to the space $C([0, T], H^s(\mathbb{R}))$. Thus, by the uniqueness of solution $v$, we have $u = v$ on the interval of time $[0, T]$ and then

$$
\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^s} = \sup_{t \in [0, T]} \|v(t, \cdot)\|_{H^s}.
$$

In this identity, we can see that $v \in C([0, +\infty[, H^s(\mathbb{R}))$ and thus, the quantity $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^s}$ does not explode in a finite time and thus the solution $u$ extends to the whole interval of time $[0, +\infty[$. Therefore, we have $u \in C([0, +\infty[, H^s(\mathbb{R}))$.

2) In order to prove the property decay of solution $u \in C([0, +\infty[, H^s(\mathbb{R}))$ given in the estimate (5), we will prove that the quantity $\sup_{t \in [0, T]} t^\frac{1}{2} \| (1 + |\cdot|^2)^{1/2} u(t, \cdot) \|_{L^\infty}$ is well-defined for all time $T > 0$. 

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Let $T > 0$. For all $t \in [0, T]$, we write

$$
t^{\frac{3}{2}} \|(1 + | \cdot |^2) u(t, \cdot) \|_{L^\infty} \leq t^{\frac{3}{2}} \left\| (1 + | \cdot |^2) \left( K_\eta(t, \cdot) * u_0 - \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty}
$$

$$
\leq t^{\frac{3}{2}} \left\| (1 + | \cdot |^2) (K_\eta(t, \cdot) * u_0) \right\|_{L^\infty}
$$

$$
+ t^{\frac{3}{2}} \left\| (1 + | \cdot |^2) \left( \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty}
$$

$$
\leq I_1 + I_2,
$$

(49)

We will study the terms $I_1$ and $I_2$ above. For term $I_1$, by Proposition 4.1 we have

$$
I_1 \leq t^{\frac{3}{2}} \|(1 + | \cdot |^2) K_\eta(t, \cdot) * u_0 \|_{L^\infty} \leq C_{1, \eta} e^{5nT} \|(1 + | \cdot |^2) u_0 \|_{L^\infty},
$$

where we set the constant as

$$
\mathcal{C}_0(T, \eta, u_0) = C_{1, \eta} e^{5nT} \|(1 + | \cdot |^2) u_0 \|_{L^\infty} > 0,
$$

(50)

and then, we write

$$
I_1 \leq \mathcal{C}_0(T, \eta, u_0).
$$

(51)

Now, we compute the $I_2$ on the right side of the formula (49). We write

$$
I_2 \leq t^{\frac{3}{2}} \left\| (1 + | \cdot |^2) \left( \int_0^t K_\eta(t - \tau, \cdot) \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty}
$$

$$
\leq t^{\frac{3}{2}} \int_0^t \frac{1}{2} \left\| (1 + | \cdot |^2) \left( \frac{1}{2} K_\eta(t - \tau) * \partial_x (u^2)(\tau, \cdot) \right) \right\|_{L^\infty} d\tau,
$$

(52)

and we will estimate the term $(a)$. Indeed, the first thing to do is to study the quantity

$$
\left| \frac{1}{2} K_\eta(t - \tau, \cdot) * \partial_x (u^2)(\tau, \cdot)(x) \right|,
$$

and by estimates (39) and (40). We have

$$
\left| \frac{1}{2} K_\eta(t - \tau, \cdot) * \partial_x (u^2)(\tau, \cdot)(x) \right| \leq c_\eta \frac{e^{5n(t - \tau)}}{(t - \tau)^{\frac{3}{2}}} \int_\mathbb{R} \frac{|u(\tau, y)||\partial_y u(\tau, y)|}{1 + |x - y|^2} dy,
$$

(53)

where the constant $c_\eta > 0$ is given in (22), and then we write

$$
c_\eta \frac{e^{5n(t - \tau)}}{(t - \tau)^{\frac{3}{2}}} \int_\mathbb{R} \frac{|u(\tau, y)||\partial_y u(\tau, y)|}{1 + |x - y|^2} dy \leq c_\eta \frac{e^{5nT}}{(t - \tau)^{\frac{3}{2}}} \int_\mathbb{R} \frac{|u(\tau, y)||\partial_y u(\tau, y)|}{1 + |x - y|^2} dy
$$

$$
\leq c_\eta \frac{e^{5nT}}{(t - \tau)^{\frac{3}{2}} \tau^\frac{1}{2}} \int_\mathbb{R} \frac{\tau^\frac{1}{2} (1 + |y|^2)|u(\tau, y)||\partial_y u(\tau, y)|}{(1 + |y|^2)(1 + |x - y|^2)} dy
$$

$$
\leq c_\eta \frac{e^{5nT}}{(t - \tau)^{\frac{3}{2}} \tau^\frac{1}{2}} \left( \tau^\frac{1}{2} \|(1 + | \cdot |^2) u(\cdot, \cdot) \|_{L^\infty} \right) \left( \| \partial_x u(\cdot, \cdot) \|_{L^\infty} \right) \int_\mathbb{R} \frac{dy}{(1 + |y|^2)(1 + |x - y|^2)}.
$$

(54)
where we still need to estimate the terms \((a.1)\) and \((a.1)\). For the term \((a.1)\), always with \(s < 1 > \frac{1}{2}\) and thus, we can write \((a.1) \leq \partial\tau u(\tau, \cdot)\|_{H^{s-1}} \leq \|u(\tau, \cdot)\|_{H^s}\). Now, by point 1) of Theorem \([4,2]\), we have \(u \in C([0, +\infty], H^s(\mathbb{R})\) and then, we get \((a.1) \leq \sup_{\tau \in [0, T]} \|u(\tau, \cdot)\|_{H^s}\). Thus, we set the quantity

\[
\mathcal{C}_1(T, u) = \sup_{\tau \in [0, T]} \|u(\tau, \cdot)\|_{H^s} > 0, \tag{55}
\]

and we can write

\[
(a.1) \leq \mathcal{C}_1(T, u). \tag{56}
\]

On the other hand, recall that term \((a.2)\) was estimated in the formula \([31]\) by \((a.2) \leq c\frac{1}{1 + |x|^2}\).

In this way, we substitute estimates \((56)\) and \((31)\) in terms \((a.1)\) and \((a.2)\) respectively given in the formula \([51]\), and we get

\[
c\eta \frac{e^{5\eta T}}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) \left(\|\partial\tau u(\tau, \cdot)\|_{L^\infty}\right) \int_{\mathbb{R}} \left(1 + |y|^2\right) (1 + |x - y|^2) dy \leq c\eta \frac{e^{5\eta t}}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) \mathcal{C}_1(T, u) \frac{1}{1 + |x|^2}, \tag{57}
\]

Then, by formulas \([53], [51]\) and \([57]\), we get the following estimate

\[
\frac{1}{2} \frac{K_n(t - \tau, \cdot) \partial\tau u^2(\tau, \cdot)(x)}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) \mathcal{C}_1(T, u) \frac{1}{1 + |x|^2},
\]

and by this estimate, for term \(a\) given in right side of estimate \([52]\) we can write

\[
\begin{align*}
(a) &= \|1 + | \cdot |^2\| \frac{K_n(t - \tau) \partial\tau u^2(\tau, \cdot)}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) \mathcal{C}_1(T, u) \\
&\leq c\eta \frac{e^{5\eta t}}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) \mathcal{C}_1(T, u) \frac{1}{1 + |x|^2}.
\end{align*}
\]

Now, we get back to estimate \([52]\) and we have

\[
I_2 \leq c\eta \frac{t^\frac{1}{2} e^{5\eta T} \mathcal{C}_1(T, u)}{(t - \tau)^\frac{3}{2} \pi} \left(\tau^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \right) dt \tag{58}
\]

At this point, with the constant \(c\eta > 0\) given in \([22]\) and the constant \(\mathcal{C}_1(T, u)\) given in \([55]\), we set the constant

\[
\mathcal{C}_2(T, \eta, u) = c\eta \frac{T^\frac{1}{2} e^{5\eta T} \mathcal{C}_1(T, u)}{1 + |x|^2} > 0.
\]

and, then we write

\[
I_2 \leq \mathcal{C}_2(T, \eta, u) \frac{t^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \|u(\tau, \cdot)\|_{L^\infty}}{(t - \tau)^\frac{3}{2} \pi} dt. \tag{59}
\]

With estimates \([51]\) and \([59]\), we get back to estimate \([49]\), and then for all \(t \in [0, T]\), we can write

\[
t^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \|u(\tau, \cdot)\|_{L^\infty} \leq \mathcal{C}_0(\eta, T, u_0) + \mathcal{C}_2(\eta, T, u) \frac{t^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \|u(\tau, \cdot)\|_{L^\infty}}{(t - \tau)^\frac{3}{2} \pi} dt. \tag{60}
\]

Now, in order to prove that quantity \(t^\frac{3}{2} \|1 + | \cdot |^2\|_{L^\infty} \|u(\tau, \cdot)\|_{L^\infty}\) does not explode in a finite time, we will use the following Grönwall’s type inequality. For a proof of this result see Lemma 7.1.2 of the book [9].
Lemma 4.1 Let $\beta > 0$ and $\gamma > 0$, such that $\beta + \gamma > 1$. Let $g : [0, T] \rightarrow [0, +\infty]$ a function such that, $g$ verifies:

1) $g \in L^1_{loc}([0, T])$;
2) $t^{\gamma-1}g \in L^1_{loc}([0, T])$, and
3) there exits two constants $a \geq 0$ and $b \geq 0$ such that for almost all $t \in [0, T]$, we have

$$g(t) \leq a + b \int_0^t (t-\tau)^{\beta-1}r^{\gamma-1}g(\tau)d\tau,$$

(61)

then:

a) There exists a continuous and increasing function $\Theta : [0, +\infty[ \rightarrow [0, +\infty[ \text{ defined by}$

$$\Theta(t) = \sum_{k=0}^{+\infty} c_k t^{\sigma k},$$

(62)

where $\sigma = \beta + \gamma - 1 > 0$ and where, for the Gamma function $\Gamma(\cdot)$ the coefficients $c_k > 0$ are given by the recurrence formula:

$$c_0 = 1, \quad \text{and} \quad \frac{c_{k+1}}{c_k} = \frac{\Gamma(k\sigma + 1)}{\Gamma(k\sigma + \beta + \gamma)}, \quad \text{for} \quad k \geq 1.$$

b) For all time $t \in [0, T]$, we have

$$g(t) \leq a\Theta(b^\frac{1}{\sigma} t).$$

(63)

In this lemma, we set $\beta = \frac{2}{3}$, $\gamma = \frac{2}{3}$ (where we have $\beta + \gamma > 1$) and we set the function $g(t) = t^\frac{\sigma}{\sigma} \| (1 + |\cdot|^2) u(t, \cdot) \|_{L^\infty}$, which verifies the points 1), 2) and 3) (with $\gamma - 1 = \frac{1}{3}$).

On the other hand, if for the constant $C_0(T, \eta, u_0) > 0$ given in (69) and for the constant $C_2(T, \eta, u) > 0$ given in (68), we set the parameters $a = C_0(T, \eta, u_0) > 0$, $b = C_2(T, \eta, u) > 0$. Moreover, if we set the parameters $\beta - 1 = -\frac{1}{3}$ and $\gamma - 1 = -\frac{1}{3}$ then, we can see that the point 3) is verified by estimate (69). Also, remark that since $\beta = \frac{2}{3}$ and $\gamma = \frac{2}{3}$, then we have $\sigma = \beta + \gamma - 1 = \frac{1}{3}$ and thus $\frac{1}{\sigma} = 3$.

Then, by estimate (63) of Lemma 4.1 for all time $t \in [0, T]$, we have: for $b^\frac{1}{\sigma} = (C_2(T, \eta, u))^3 > 0$,

$$t^\frac{\sigma}{\sigma} \| (1 + |\cdot|^2) u(t, \cdot) \|_{L^\infty} \leq C_0(T, \eta, u_0)\Theta \left(b^\frac{1}{\sigma} t\right) \leq C_0(T, \eta, u_0)\Theta \left(b^\frac{1}{\sigma} T\right),$$

(64)

Finally, we set the constant

$$C(\eta, t, u_0, u) = \frac{C_0(T, \eta, u_0)\Theta \left((C_2(T, \eta, u))^3 T\right)}{t^\frac{\sigma}{\sigma}} > 0,$$

(65)

and then, we have the estimate given in the formula (5). Theorem 4.2 is now proven.

\begin{flushright}
\begin{tabular}{c}
\textbf{4.1.3 Regularity}\end{tabular}
\end{flushright}

4.1.3 Regularity

In order to finish this proof of Theorem 4.2 we will prove now that the solution $u$ of the equation is smooth enough is spatial variable.

Proposition 4.3 Let $\frac{3}{2} < s \leq 2$ and let $u \in C([0, +\infty[, H^s(\mathbb{R}))$ be the solution of the integral equation (3) given by point 1) of Theorem 4.2. Then, we have $u \in C([0, +\infty[, C^\infty(\mathbb{R}))$.}

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For the first term: the equation (3).

Indeed, for all \( s \in [0, \infty[ \), the function \( \partial^a u(t, \cdot) \) is a Hölder continuous function on \( \mathbb{R} \). Let \( n \in \mathbb{N} \) fix. Then, for \( s \) such that \( \frac{1}{2} < s_1 < \frac{3}{2} \), we set \( \alpha = n + s_1 \) and by (66), we have \( \partial^a u(t, \cdot) \in H^{s_1}(\mathbb{R}) \).

On the other hand, recall that we have the identification \( H^{s_1}(\mathbb{R}) = B_{s_1,2}^{s_1}(\mathbb{R}) \) (where \( B_{s_1,2}^{s_1}(\mathbb{R}) \) denotes a Besov space) and moreover we have the inclusion \( B_{s_1,2}^{s_1}(\mathbb{R}) \subset B_{\infty,\infty}^{s_1-\frac{1}{2}}(\mathbb{R}) \subset B_{\infty,\infty}^{s_1-\frac{1}{2}}(\mathbb{R}) \).

Then, we have \( \partial^a u(t, \cdot) \in B_{\infty,\infty}^{s_1-\frac{1}{2}}(\mathbb{R}) \). But, since \( \frac{1}{2} < s_1 < \frac{3}{2} \), then we have \( 0 < s_1 - \frac{1}{2} < 1 \) and thus \( \partial^a u(t, \cdot) \) is a \( \beta \)-Hölder continuous function with \( \beta = s_1 - \frac{1}{2} \). Theorem 1 is now proven.

4.2 Proof of Theorem 2

Let \( s \in [0, \infty[ \) be the initial datum and suppose that this function verifies the following decay properties: for \( \varepsilon > 0 \),

\[
|u_0(x)| \leq \frac{c}{1 + |x|^{2+\varepsilon}} \quad \text{and} \quad \left| \frac{d}{dx} u_0(x) \right| \leq \frac{c}{1 + |x|^2}.
\]

Let \( u \in C([0, \infty[, C^\infty(\mathbb{R})) \) be the solution of equation (11) associated with the initial datum \( u_0 \) above and given by Theorem 1. In order to prove the asymptotic profile of \( u(t, x) \) given in formula (6), we write the solution \( u(t, x) \) as the integral formulation given in (3) and will study each term on the right-hand side of the equation (3).

For the first term: \( K_\eta(t, \cdot) \ast u_0(x) \), we will prove the following asymptotic development when \( |x| \longrightarrow +\infty \):

\[
K_\eta(t, \cdot) \ast u_0(x) = K_\eta(t, x) \left( \int_\mathbb{R} u_0(y)dy \right) + o(t) \left( \frac{1}{|x|^2} \right).
\]

Indeed, for all \( t > 0 \) and \( x \in \mathbb{R} \) we write:

\[
K_\eta(t, \cdot) \ast u_0(x) = \int_\mathbb{R} K_\eta(t, x - y)u_0(y)dy = \int_\mathbb{R} K_\eta(t, x - y)u_0(y)dy + K_\eta(t, x) \left( \int_\mathbb{R} u_0(y)dy \right) - K_\eta(t, x) \left( \int_\mathbb{R} u_0(y)dy \right) = K_\eta(t, x) \left( \int_\mathbb{R} u_0(y)dy \right) + \int_\mathbb{R} K_\eta(t, x - y)u_0(y)dy - K_\eta(t, x) \left( \int_\mathbb{R} u_0(y)dy \right).
\]

Now, in expression (a) and expression (b) above, first we cut each integral in two parts:

\[
\int_\mathbb{R} (\cdot)dy = \int_{|y|<\frac{|x|}{2}} (\cdot)dy + \int_{|y|>\frac{|x|}{2}} (\cdot)dy.
\]

Proof. Recall that by hypothesis on the initial datum \( u_0 \) given in (23), we have \( u_0 \in H^s(\mathbb{R}) \) for \( \frac{3}{2} < s \leq 2 \) and then by Theorem 1 of the article [20] the solution \( u \in C([0, \infty[, H^s(\mathbb{R})) \) verifies

\[
u \in C \left( \left[0, +\infty[, \bigcap_{\alpha \geq 0} H^\alpha(\mathbb{R}) \right). \]

(66)
and then we arrange the terms in order to write
\[(a) + (b) = \int_{|y| < \frac{|x|}{2}} (K_\eta(t, x - y) - K_\eta(t, x)) \ u_0(y)dy + \int_{|y| > \frac{|x|}{2}} K_\eta(t, x - y)u_0(y)dy - K_\eta(t, x) \left( \int_{|y| > \frac{|x|}{2}} u_0(y)dy \right) = I_1 + I_2 + I_3, \tag{70}\]
and now, in order to prove identity (68) we must prove that
\[I_1 + I_2 + I_3 = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when } |x| \to +\infty. \tag{71}\]

In order to study the term \(I_1\) in identity (70) we need the following technical result.

**Lemma 4.2** Let \(t > 0\) and let \(K_\eta(t, \cdot)\) be the kernel given in (4). Then, we have \(K_\eta(t, \cdot) \in C^1(\mathbb{R})\) moreover, there exists a constant \(C_\eta > 0\), which only depends on \(\eta > 0\), such that we have:

1) for all \(x \neq 0\), \(|\partial_x K_\eta(t, x)| \leq C_\eta e^{6\eta t} \frac{1}{|x|^3}\).

2) \(|\partial_x K_\eta(t, x)| \leq C_\eta e^{6\eta t} \frac{1}{t^{\frac{3}{2}} + |x|^3}\).

The proof of this lemma follows essentially the same lines of the proof of point 1) of Proposition 3.1 and we will postpone this proof for the appendix. Thus, since \(K_\eta(t, \cdot) \in C^1(\mathbb{R})\) then by Taylor expansion of the first order, for \(\theta = \alpha(x - y) + (1 - \alpha)x = x - \alpha y\) and for some \(\alpha \in [0, 1]\), we can write:

\[K_\eta(t, x - y) - K_\eta(t, x) = -y \partial_x K_\eta(t, \theta), \tag{72}\]

and then we have
\[I_1 \leq \int_{|y| \leq \frac{|x|}{2}} |K_\eta(t, x - y) - K_\eta(t, x)| |u_0(y)|dy \leq \int_{|y| \leq \frac{|x|}{2}} |y \partial_x K_\eta(t, \theta)||u_0(y)|dy. \tag{73}\]

We estimate now the last term on the right-hand side. Recall first that by point 1) of Lemma 4.2 we can write \(|\partial_x K_\eta(t, \theta)| \leq C_\eta e^{6\eta t} \frac{1}{|\theta|^3}\), but since we have \(\theta = x - \alpha y\) (with \(\alpha \in [0, 1]\)) then we can write \(|\theta| \geq |x| - |\alpha y| \geq |x| - |y|\) and moreover, since we have \(|y| < \frac{|x|}{2}\) then we write \(|x| - |y| \geq \frac{|x|}{2}\), and thus we get \(|\theta| \geq \frac{|x|}{2}\). Then we have
\[|\partial_x K_\eta(t, \theta)| \leq C_\eta e^{6\eta t} \frac{1}{|x|^3}, \tag{74}\]
and getting back to estimate (73) we get
\[\int_{|y| \leq \frac{|x|}{2}} |y \partial_x K_\eta(t, \theta)||u_0(y)|dy \leq C_\eta e^{6\eta t} \int_{|y| < \frac{|x|}{2}} |y||u_0(y)|dy \leq C_\eta e^{6\eta t} \int_{\mathbb{R}} |y||u_0(y)|dy, \tag{75}\]
where, since the initial datum \(u_0\) verifies \(|u_0(y)| \leq \frac{c}{1 + |y|^{2+\varepsilon}}\) (with \(\varepsilon > 0\)) then the last term on right-hand side converges. Thus, by estimates (68) and (69) we have
\[I_1 \leq (C_\eta e^{6\eta t} ||\cdot||_{L^1}) \frac{1}{|x|^3}, \tag{76}\]

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and then
\[ I_1 = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \] (77)

Now, for term \( I_2 \) in the identity (70) we write
\[ I_2 \leq \int_{|y| > \frac{|x|}{2}} |K_{\eta}(y, x - y)||u_0(y)|dy, \] (78)
and in order to study this term, we have the following estimates: remark that by point 1 of Proposition 3.1 we have
\[ |K_{\eta}(t, x - y)| \leq c \eta e^{\frac{5\eta t}{t^\frac{1}{3}}} \frac{1}{1 + |x - y|^2}, \] (79)
whence, we get
\[ \|K_{\eta}(t, \cdot)\|_{L^1} \leq c \eta e^{\frac{5\eta t}{t^\frac{1}{3}}}. \] (80)

On the other hand, always by the fact that the initial datum \( u_0 \) verifies \( |u_0(y)| \leq c \frac{1}{1 + |y|^{2+\varepsilon}} \) and moreover, since in the term \( I_2 \) we have \( |y| > \frac{|x|}{2} \) then, for \( |x| \) large enough, we get
\[ |u_0(y)| \leq c \frac{1}{1 + |y|^{2+\varepsilon}} \leq c \frac{1}{|x|^{2+\varepsilon}}. \] (81)

With estimates (80) and (81) at hand, we get back to the formula (78) and we write
\[ \int_{|y| > \frac{|x|}{2}} |K_{\eta}(y, x - y)||u_0(y)|dy \leq \frac{c}{|x|^{2+\varepsilon}} \int_{|y| > \frac{|x|}{2}} |K_{\eta}(t, x - y)|dy \leq \frac{c}{|x|^{2+\varepsilon}} \|K_{\eta}(t, \cdot)\|_{L^1} \leq \frac{c}{|x|^{2+\varepsilon}} \left( c \eta e^{\frac{5\eta t}{t^\frac{1}{3}}} \right), \] (82)
and by this estimate and estimate (78) we have:
\[ I_2 = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \] (83)

We study now the term \( I_3 \) in the identity (70). By the estimate (79) and for \( |x| \) large enough we can write
\[ I_3 \leq |K_{\eta}(t, x)| \left( \int_{|y| > \frac{|x|}{2}} |u_0(y)|dy \right) \leq c \eta e^{\frac{5\eta t}{t^\frac{1}{3}}} \frac{1}{|x|^2} \left( \int_{|y| > \frac{|x|}{2}} |u_0(y)|dy \right), \] (84)
but, recall that since we have \( |u_0(y)| \leq \frac{c}{1 + |y|^{2+\varepsilon}} \) then we get \( u_0 \in L^1(\mathbb{R}) \), and thus we have
\[ \lim_{|x| \to +\infty} \left( \int_{|y| > \frac{|x|}{2}} |u_0(y)|dy \right) = 0. \]

Then we can write
\[ I_3 = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \] (85)

Finally, by the estimates (77), (83) and (85) we get the estimate (71).
Now, for the second term on the right-hand side in the integral equation (3): \( \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x (u^2)(x)(\tau, \cdot) d\tau \), we will prove the following asymptotic profile when \( |x| \to +\infty \):

\[
\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) d\tau = \int_0^t K_\eta(t - \tau, x) \left( \int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau + o(t) \left( \frac{1}{|x|^2} \right). \tag{86}
\]

Indeed, for all \( x \in \mathbb{R} \) we write

\[
\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) d\tau = \int_0^t K_\eta(t - \tau, \cdot) \ast (u \partial_x u(\tau, \cdot))(x) d\tau = \int_0^t \int_{\mathbb{R}} K_\eta(t - \tau, x - y) u(\tau, y) \partial_y u(\tau, y) dy \, d\tau, \tag{87}
\]

then, in order to study the term \( (c) \), following the same computations done in the formulas (69), (69) and (70) we write

\[
(c) = K_\eta(t - \tau, x) \left( \int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau + \int_{|y| < \frac{|x|}{t}} (K_\eta(t - \tau, x - y) - K_\eta(t - \tau, x)) (u(\tau, y) \partial_y u(\tau, y)) dy \, d\tau + \int_{|y| > \frac{|x|}{t}} K_\eta(t - \tau, x - y) (u(\tau, y) \partial_y u(\tau, y)) dy \, d\tau - K_\eta(t - \tau, x) \left( \int_{|y| > \frac{|x|}{t}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau,
\]

and getting back to the identity (87) we have:

\[
\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \ast \partial_x (u^2)(\tau, \cdot)(x) d\tau = \int_0^t K_\eta(t - \tau, x) \left( \int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau + \int_0^t \int_{|y| < \frac{|x|}{t}} (K_\eta(t - \tau, x - y) - K_\eta(t - \tau, x)) (u(\tau, y) \partial_y u(\tau, y)) dy \, d\tau + \int_0^t \int_{|y| > \frac{|x|}{t}} K_\eta(t - \tau, x - y) (u(\tau, y) \partial_y u(\tau, y)) dy \, d\tau - \int_0^t K_\eta(t - \tau, x) \left( \int_{|y| > \frac{|x|}{t}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau. \tag{88}
\]

Thus, in order to obtain the asymptotic profile given in (86), we must prove the following estimate:

\[
I_a + I_b + I_c = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \tag{89}
\]

For the term \( I_a \), by the estimates (72) and (74) we can write

\[
I_a \leq \int_0^t \int_{|y| < \frac{|x|}{t}} |K_\eta(t - \tau, x - y) - K_\eta(t - \tau, x)| |y| |u(\tau, y) \partial_y u(\tau, y)| dy \, d\tau 
\leq \int_0^t \left( C_\eta \frac{e^{\delta t(\tau-t)}}{|x|^3} \int_{\mathbb{R}} |y| |u(\tau, y) \partial_y u(\tau, y)| dy \right) d\tau \leq C_\eta \frac{e^{\delta t}}{|x|^3} \int_0^t \int_{\mathbb{R}} |y| |u(\tau, y) \partial_y u(\tau, y)| dy \, d\tau, \tag{90}
\]

where, in order to estimate the last term on the right-hand side we have the following technical result.
Lemma 4.3 Since the initial data $u_0$ verifies $\left| \frac{d}{dx} u_0(x) \right| \leq \frac{c}{1 + |x|^2}$ then there exists a constant $0 < C^* = C^*(t, \eta, u_0, \|u\|_{H^{2}}) < +\infty$, which depends on $t > 0$, $\eta > 0$, the initial data $u_0$ and the solution $u$, such that for all time $\tau \in [0, t]$ and for all $y \in \mathbb{R}$ we have

$$|u(\tau, y)\partial_y u(\tau, y)| \leq \frac{C^*}{\tau^{\frac{3}{2}}(1 + |y|^2)}.$$  

(91)

Proof. The first thing to do is to prove that the function $\partial_y u(\tau, y)$ verifies the following estimate:

$$|\partial_y u(\tau, y)| \leq \frac{C^*_1}{\tau^{\frac{3}{2}}(1 + |y|^2)},$$

(92)

where $C^*_1 > 0$ is a constant which does not depend on the variable $y$. For this we write the solution $u$ as the integral equation (3), then, in each side of this identity (3) we derive respect to the spacial variable $y$ and we have

$$\partial_y u(\tau, y) = K_y(\tau, \cdot) * (\partial_y u_0)(y) - \frac{1}{2} \int^\tau_0 (\partial_y K_y(\tau - \zeta, \cdot)) * \partial_y (u^2)(\zeta, \cdot)(y) d\zeta = I_1 + I_2,$$

and now we must study the terms $I_1$ and $I_2$ above.

In order to study term $I_1$, recall that by the second estimate in formula (67) the initial datum $u_0$ verifies $|\partial_y u_0(y)| \leq \frac{c}{1 + |y|^2}$ and then, in the estimate (32) we can substitute the function $u_0$ by the function $\partial_y u_0$ and thus we write

$$|I_1| \leq |K_y(\tau, \cdot) * (\partial_y u_0)(y)| \leq c_0 \frac{e^{5\eta t}}{\tau^{\frac{3}{2}}} \frac{||1 + \cdot|^2 \partial_y u_0||_{L^\infty}}{1 + |y|^2} \leq c_0 \frac{e^{5\eta t}}{\tau^{\frac{3}{2}}} \frac{||1 + \cdot|^2 \partial_y u_0||_{L^\infty}}{1 + |y|^2},$$

(93)

We study now the term $I_2$ and for this we write

$$|I_2| \leq \left| \frac{1}{2} \int^\tau_0 (\partial_y K_y(\tau - \zeta, \cdot)) * \partial_y (u^2)(\zeta, \cdot)(y) d\zeta \right| \leq \int^\tau_0 \int_\mathbb{R} |\partial_y K_y(\tau - \zeta, y - z)||\partial_z (u^2)(\zeta, z)| dz d\zeta,$$

(a)

where we still need to study the terms (a) and (b). For the term (a) recall that by point 2) of Lemma 4.2 we have

$$|\partial_y K_y(\tau - \zeta, y - z)| \leq C_y \frac{e^{6\eta(\tau - \zeta)}}{\tau^{\frac{3}{2}}(1 + |y - z|^2)},$$

(95)

On the other hand, for the term (b) we have the following estimates

$$|\partial_z (u^2)(\zeta, z)| \leq 2|u(\zeta, z)||\partial_z u(\zeta, z)| = 2\left(1 + |z|^2\right)|u(\zeta, z)||\partial_z u(\zeta, z)| \leq 2\frac{\zeta^{\frac{3}{2}}(1 + |z|^2)|u(\zeta, z)||\partial_z u(\zeta, z)|}{\tau^{\frac{3}{2}}(1 + |z|^2)}$$

(96)

but, using the quantity $\|u\|_{F_1}$ (where the norm $\| \cdot \|_{F_1}$ is given in the formula (25)) we can write

$$\sup_{0 < \zeta < t} \zeta^{\frac{3}{2}}(1 + |\zeta|^2)|u(\zeta, \cdot)||_{L^\infty} \leq \|u\|_{F_1},$$

and moreover, by the estimate (43) we can write $\sup_{0 < \zeta < t} \|\partial_z u(\zeta, \cdot)||_{L^\infty} \leq \|u\|_{F_1}$, and thus, getting back to the estimate (96) we get

$$|\partial_z (u^2)(\zeta, z)| \leq \|u\|^2_{F_1} \frac{1}{\tau^{\frac{3}{2}}(1 + |z|^2)}.$$  

(97)
Once we dispose of the estimates (93) and (97), we get back to estimate (91) and then we write

\[ |I_2| \leq \int_0^t \int_{\mathbb{R}} \left( C_{\eta} e^{6\eta t} \left( \frac{1}{(\tau - \zeta)^2 + 1 + |y - z|^2} \right) \left( ||u||^2_{\tilde{H}^1} \frac{1}{\zeta^4 (1 + |z|^2)} \right) \right) dz \ d\zeta \]

\[ \leq C_{\eta} e^{6\eta t} ||u||^2_{\tilde{H}^1} \left( \int_0^t \frac{dz}{(\tau - \zeta)^2 + 1 + |y - z|^2} \right) \leq C_{\eta} e^{6\eta t} \left( \int_0^t \frac{dz}{(1 + |y|^2)(1 + |z|^2)} \right) \leq C_{\eta} e^{6\eta t} \left( \int_0^t \frac{dz}{\tau^4 (1 + |y|^2)} \right) \]

By the estimates (93) and (98), we set the constant \( C^*_1 \) as \( C^*_1 = \max \left( c_\eta e^{5\eta t} ||(1 + \cdot)^2 \partial_y u_0||_{L^\infty}, C_{\eta t} \frac{1}{\tau^4} e^{6\eta t} \right) > 0 \), and then we can write the estimate (91).

Finally, recall that the by estimate (91) we can write \(|u(\tau, y)| \leq \frac{\mathcal{C}_0(t, \eta, u_0) \Theta \left( \frac{b}{\tau^4} t \right)}{\tau^4 (1 + |y|^2)} \). Thus, we set the constant \( C^* \) as \( C^* = \max \left( \mathcal{C}_0(t, \eta, u_0) \Theta \left( \frac{b}{\tau^4} t \right), C^*_1 \right) > 0 \), and then by the estimate above and the estimate (91) we get the desired estimate (91).

Thus, getting back to the estimate (90), for \(|x| \) large enough we can write

\[ I_a \leq C_{\eta} e^{6\eta t} \left( \int_0^t \int_{\mathbb{R}} \frac{C^*}{\tau^4 (1 + |y|^4)} dy \ d\tau \right) \leq C_{\eta} e^{6\eta t} \left( \int_0^t \frac{dy \ d\tau}{\tau^4 (1 + |y|^4)} \right) \leq C_{\eta} e^{6\eta t} \frac{(C^* t^4)}{|x|^3}, \]

and then we have

\[ I_a = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \]

We study now the term \( I_b \) in the formula (88). By the estimate (91) we get

\[ I_b \leq \int_0^t \int_{|y| > \frac{|\tau|}{2}} |K(t - \tau, x - y)| ||u(\tau, y)\partial_y u(\tau, y)|| dy \ d\tau \leq \int_0^t ||K(t - \tau, x - y)|| dy \ d\tau, \]

but, since in the term \( I_b \) above we have \(|y| > \frac{|\tau|}{2}\) then we can write \( \frac{1}{1 + |y|^2} \leq \frac{c}{|\tau|^4} \), and thus we obtain

\[ \int_0^t \int_{|y| > \frac{|\tau|}{2}} |K(t - \tau, x - y)|| dy \ d\tau \leq \int_0^t \int_{|y| > \frac{|\tau|}{2}} |K(t - \tau, x - y)| dy \ d\tau \]

\[ \leq \frac{C^*}{|x|^4} \int_0^t \||K(t - \tau, \cdot)||_{L^1} \ d\tau, \]

where, by the estimate (88) we write

\[ \frac{C^*}{|x|^4} \int_0^t \||K(t - \tau, \cdot)||_{L^1} \ d\tau \leq \frac{C^*}{|x|^4} \int_0^t \left( c_\eta e^{5\eta (t - \tau)} \right) d\tau \leq \frac{C^*}{|x|^4} \left( c_\eta e^{5\eta t} \frac{1}{\tau^4} \right). \]

Then, for \(|x| \) large enough we have \( I_b \leq \frac{C^*}{|x|^4} \left( c_\eta e^{5\eta t} \frac{1}{\tau^4} \right) \), and thus we can write

\[ I_b = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \to +\infty. \]
We study the term $I_c$ in the equation (88). By the estimates (79) and (91) we have

$$I_c \leq \int_0^t |K_\eta(t-\tau, x)| \left( \int_{|y|>\frac{|x|}{2^t}} |u(\tau, y)\partial_\eta u(\tau, y)|dy \right) d\tau$$

$$\leq \int_0^t \left( \frac{c_\eta e^{5\eta(t-\tau)} 1}{(t-\tau)^\frac{1}{2} |x|^2} \right) \left( \int_{|y|>\frac{|x|}{2^t}} \frac{C^*}{\tau^\frac{3}{2}(1+|y|^2)^2} dy \right) d\tau$$

$$\leq \frac{c_\eta e^{5\eta(t-\tau)} C^*}{|x|^4} \int_0^t \left( \int_{\mathbb{R}} \frac{dy}{\tau^\frac{3}{2}(1+|y|^2)^2} \right) d\tau \leq \frac{c_\eta e^{5\eta C^*}}{|x|^4} \left( \int_0^t \frac{d\tau}{(t-\tau)^\frac{3}{2}} \right) \leq \frac{c_\eta e^{5\eta C^*}}{|x|^4}.$$

but, remark that in the term $I_b$ above we have $|y|>|x|$, then we can write $\frac{1}{1+|y|^2} \leq \frac{c}{|x|^2}$ and thus we get

$$I_c = o(t) \left( \frac{1}{|x|^2} \right), \quad \text{when } |x| \to +\infty. \quad (100)$$

Finally, by the estimates given in formulas (99), (99) and (100), we can write the estimate (89) and the Theorem 2 is now proven. \qed

4.3 Proof of Theorem 3

For $t>0$ we write the solution $u(t, x)$ as the integral formulation (3) and we will start by the following estimates that we shall need later. For the linear term in (3) we write

$$K_\eta(t, \cdot) \ast u_0(x) = \int_{|y| \leq \frac{|x|}{2^t}} (K_\eta(t, x-y) - K_\eta(t, x)) u_0(y) dy + K_\eta(t, x) \int_{|y| > \frac{|x|}{2^t}} u_0(y) dy$$

$$+ \int_{|y| > \frac{|x|}{2^t}} K_\eta(t, x-y) u_0(y) dy$$

$$= I_1 + I_2 + I_3,$$

where we will study the terms $I_1, I_2$ and $I_3$. Recall that the term $I_1$ was already treated in formulas (72) and (76) as follows:

$$I_1 \leq \left( C_\eta \frac{e^{6\eta t}}{t^2} \| u_0 \|_{L^1} \right) \frac{1}{1+|x|^2}. \quad (101)$$

On the other hand, for the term $I_2$ we write

$$I_2 = K_\eta(t, x) \left( \int_{\mathbb{R}} u_0(y) dy \right) - K_\eta(t, x) \left( \int_{|y| > \frac{|x|}{2^t}} u_0(y) dy \right),$$

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where we will observe that the kernel $K_\eta(t, x)$ defined in \([11]\) can be written for $|x|$ large enough as $K_\eta(t, x) = \frac{c_{\eta,t} t}{|x|^2}$, and where the quantity $c_{\eta,t} > 0$ only depends on $\eta > 0$ and $t > 0$. Indeed, by the identity \([16]\) and the identity \([19]\) we can write (for $|x|$ large)

$$K_\eta(t, x) = \frac{2\eta t}{(2\pi i x)^2} + \frac{2t}{(2\pi i x)^2}(I_a + I_b) = \frac{1}{|x|^2} \times \frac{t}{-2\pi^2}(\eta + (I_a + I_b)),$$

where: the quantity $I_a$ is given in expression \([17]\), the quantity $I_b$ is given in expression \([18]\), and moreover, by Lemma \([31]\) we have $|I_a + I_b| \leq c_\eta e^{4\eta t}$. Thus, we define the quantity $c_{\eta,t}$ as follows:

$$c_{\eta,t} = \frac{1}{-2\pi^2}(\eta + (I_a + I_b)),$$

and we have the identity

$$K_\eta(t, x) = \frac{c_{\eta,t} t}{|x|^2}.$$

Once we dispose of this identity, the term $I_2$ is written as:

$$I_2 = \frac{c_{\eta,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y) dy \right) - \frac{c_{\eta,t} t}{|x|^2} \left( \int_{|y| \geq \frac{1}{2}} u_0(y) dy \right). \quad (102)$$

Finally, remark that the term $I_3$ was already studied in formula \([82]\), and recalling this formula we have

$$I_3 \leq \frac{C}{|x|^{2+\varepsilon}} \left( c_\eta e^{5\eta t} \frac{t}{s^\frac{1}{3}} \right). \quad (103)$$

Now, in order to study the nonlinear term in \([3]\), for $0 < s < t$ we will start by studying the expression $\partial_x K_\eta(t - s, \cdot) \ast u^2(s, x)$. Recall that by point 2) of Lemma \([1,2]\) we have

$$|\partial_x K_\eta(t, x)| \leq C_\eta e^{6\eta t} \frac{1}{t^{\frac{2}{3}} 1 + |x|^{\frac{2}{3}}},$$

and moreover, by the estimate \([5]\) given in Theorem \([11]\) we have

$$|u(s, x)| \leq \frac{C(s, \eta, u_0, u)}{1 + |x|^2},$$

where the constant $C(s, \eta, u_0, u) > 0$ (given in the formula \([65]\)) is written as $C(s, \eta, u_0, u) = \frac{\mathcal{C}(T, \eta, u_0, u)}{s^\frac{1}{3}}$.

With these estimates in mind, we write now

$$|\partial_x K_\eta(t - s, \cdot) \ast u^2(s, x)| \leq C_\eta e^{6\eta(t-s)} \frac{\mathcal{C}^2(T, \eta, u_0, u)}{s^\frac{1}{3}} \int_{\mathbb{R}} \frac{dy}{(1 + |x-y|)(1 + |y|^2)} \leq C_\eta e^{6\eta t} \frac{\mathcal{C}^2(T, \eta, u_0, u)}{s^\frac{1}{3}} \frac{1}{1 + |x|^{\frac{2}{3}}}.$$  

Hence, we obtain

$$\left| \int_0^t K_\eta(t - s, \cdot) \partial_x(u^2(s, x)) ds \right| \leq C_\eta e^{6\eta t} \mathcal{C}^2(T, \eta, u_0, u) \left( \int_0^t \frac{ds}{(t-s)^{\frac{2}{3}} s^\frac{1}{3}} \right) \frac{1}{1 + |x|^{\frac{2}{3}}} \leq C_\eta e^{6\eta t} \frac{\mathcal{C}^2(T, \eta, u_0, u)}{t^{\frac{2}{3}}} \frac{1}{1 + |x|^{\frac{2}{3}}}. \quad (104)$$
As mentioned before, these estimates given on the linear and the nonlinear term will be very useful to prove this theorem. We start by getting back to the integral formulation (3) and we write the following profile for the solution:

\[
    u(t, x) = I_1 + \frac{c_{n,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y)dy \right) - \frac{c_{n,t} t}{|x|^2} \left( \int_{|y| \geq \frac{|x|}{2}} u_0(y)dy \right) + I_3 + \int_{0}^{t} \partial_x K_0(t-s, \cdot) \ast u^2(s, x)ds,
\]

where we will consider the following cases:

1) The case \( \int_{\mathbb{R}} u_0(y)dy \neq 0 \). Remark that once we dispose of the estimates for the terms \( I_1, I_2 \) and \( I_3 \), given in formulas (101), (102) and (103) respectively, we can write

\[
    I_1 + \frac{c_{n,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y)dy \right) - \frac{c_{n,t} t}{|x|^2} \left( \int_{|y| \geq \frac{|x|}{2}} u_0(y)dy \right) + I_3 = \frac{c_{n,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y)dy \right) + o(t) \left( \frac{1}{|x|^2} \right), \quad |x| \to +\infty.
\]

On the other hand, for the nonlinear term, by the estimate (104) we can write

\[
    \int_{0}^{t} \partial_x K_0(t-s, \cdot) \ast u^2(s, x)ds = o(t) \left( \frac{1}{|x|^2} \right), \quad |x| \to +\infty.
\]

Thus, getting back to the profile (105), for \( |x| \) large enough we can write:

\[
    |u(t, x)| = \left| \frac{c_{n,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y)dy \right) + o(t) \left( \frac{1}{|x|^2} \right) \right| = \left| \frac{c_{n,t} t}{|x|^2} \left( \int_{\mathbb{R}} u_0(y)dy \right) - \left( o(t) \left( \frac{1}{|x|^2} \right) \right) \right| \\
    \geq \frac{c_{n,t} t}{|x|^2} \int_{\mathbb{R}} u_0(y)dy - \left( o(t) \left( \frac{1}{|x|^2} \right) \right).
\]

Then, recalling the definition of the quantity \( o(t) \left( \frac{1}{|x|^2} \right) \) given in the formula (7), for \( \frac{c_{n,t} t}{2} \int_{\mathbb{R}} u_0(y)dy \geq 0 \) there exists \( M > 0 \) such that for all \( |x| > M \) we have \( o(t) \left( \frac{1}{|x|^2} \right) \leq \frac{c_{n,t} t}{2|x|^2} \int_{\mathbb{R}} u_0(y)dy \). Hence, we have

\[
    -\left( o(t) \left( \frac{1}{|x|^2} \right) \right) \geq -\frac{c_{n,t} t}{2|x|^2} \int_{\mathbb{R}} u_0(y)dy,
\]

and getting back to the estimate from below on the quantity \( |u(t, x)| \) above we obtain

\[
    |u(t, x)| \geq \frac{c_{n,t} t}{2|x|^2} \int_{\mathbb{R}} u_0(y)dy.
\]

2) The case \( \int_{\mathbb{R}} u_0(y)dy = 0 \). Remark that, always by the estimates given on the terms \( I_1, I_2 \) and \( I_3 \) (see...
(107), (102) and (103) we can write now

\[ I_1 + \frac{c_{\eta}t}{|x|^2} \left( \int_{|y| \geq \frac{|x|}{t}} u_0(y)dy \right) - \frac{c_{\eta}t}{|x|^2} \left( \int_{|y| \geq \frac{|x|}{t}} u_0(y)dy \right) + I_3 = I_2 \]

\[ \leq \frac{c_{\eta}t}{|x|^2} \left( \int_{|y| \geq \frac{|x|}{t}} |u_0(y)|dy \right) + \frac{c}{|x|^{2+\varepsilon}} \left( \frac{e^{5\eta t}}{t^\frac{5}{2}} \right) \]

\[ \leq \frac{c_{\eta}t}{|x|^{2+\varepsilon}} \left( \int_{|y| \geq \frac{|x|}{t}} |y||u_0(y)|dy \right) + \frac{c}{|x|^{2+\varepsilon}} \left( \frac{e^{5\eta t}}{t^\frac{5}{2}} \right) \]

\[ \leq \frac{c}{|x|^{2+\varepsilon}} \left( \| \cdot \|_L^1 + \frac{e^{5\eta t}}{t^\frac{5}{2}} \right) \cdot \left( \| \cdot \|_L^1 + \frac{e^{5\eta t}}{t^\frac{5}{2}} \right) \]

With this estimate and the estimate for the nonlinear term given in the formula (104) we can write

\[ \| u(t, x) \| \leq \frac{c}{|x|^{2+\varepsilon}} \left( \| \cdot \|_L^1 + \frac{e^{5\eta t}}{t^\frac{5}{2}} \right) + C_{\eta} e^{5\eta t} \| (T, \eta, u_0, u) \| \frac{1}{1 + |x|^{3}}, \]

and for $|x|$ large enough and $0 < \varepsilon \leq 1$ we have

\[ \| u(t, x) \| \leq \frac{c}{1 + |x|^{2+\varepsilon}} \left( \| \cdot \|_L^1 + \frac{e^{5\eta t}}{t^\frac{5}{2}} + C_{\eta} e^{5\eta t} \| (T, \eta, u_0, u) \| \right) \]

Theorem 3 is proven.

5 The LWP in Lebesgue spaces: proof of Theorem 4

We start by remarking that the kernel $K_\eta(t, \cdot)$ given in (1) and its derivative $\partial_x K_\eta(t, \cdot)$ belong to the space $L^p(\mathbb{R})$ for $1 \leq p \leq +\infty$. Indeed, by point 1) of Proposition 3.1 we have, for all time $t > 0$,

\[ |K_\eta(t, x)| \leq \frac{c_\eta e^{5\eta t}}{t^\frac{5}{2}} \frac{1}{1 + |x|^2} \]

and then, for $1 \leq p \leq +\infty$ we get $\| K_\eta(t, \cdot) \|_{L^p} \leq c_\eta e^{5\eta t} \frac{1}{1 + |x|^2}$, hence, for the sake of simplicity, we write

\[ \| K_\eta(t, \cdot) \|_{L^p} \leq \frac{c_\eta e^{5\eta t}}{t^\frac{5}{2}}. \] (107)

In the same way, recall that by point 2) of Lemma 1.2 we have, for all time $t > 0$, $|\partial_x K_\eta(t, x)| \leq C_\eta e^{6\eta t} \frac{1}{t^\frac{5}{2}} |x|^3$, thereby, for $1 \leq p \leq +\infty$, we obtain

\[ \| \partial_x K_\eta(t, \cdot) \|_{L^p} \leq \frac{C_\eta e^{6\eta t}}{t^\frac{5}{2}}. \] (108)

Estimates (107) and (108) will allow us to study the existence of mild solutions for the Cauchy problem in the framework of Lebesgue spaces and when the initial datum $u_0$ is small enough. Let $T > 0$ fix and consider the Banach space $L^\infty(0, T, L^p(\mathbb{R}))$ with the norm $\sup_{0 < t < T} t^\frac{5}{2} \| u(t, \cdot) \|_{L^p}$. We write

\[ \sup_{0 < t < T} t^\frac{5}{2} \| u(t, \cdot) \|_{L^p} \leq \sup_{0 < t < T} t^\frac{5}{2} \| K_\eta(t, \cdot) \|_{L^p} + \sup_{0 < t < T} t^\frac{5}{2} \left\| \int_0^t K_\eta(t - s, \cdot) \partial_x (u^2(s, \cdot)) ds \right\|_{L^p}, \]

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and we will estimate each term on the right-hand side.

For the first term, by the estimate (107) we can write
\[
\sup_{0 < t < T} t^\frac{1}{4} ||K_\eta(t, \cdot) \ast u_0||_{L^p} \leq \sup_{0 < t < T} t^\frac{1}{4} ||K_\eta(t, \cdot)||_{L^1} ||u_0||_{L^p} \leq \sup_{0 < t < T} t^\frac{1}{4} \left( c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{4}}/} \right) ||u_0||_{L^p} \leq c_\eta e^{5\eta T} ||u_0||_{L^p}. \tag{109}
\]

Now, the second term is estimated as follows: first, for all time \( t \in ]0, T[ \) and for \( 1 \leq q \leq +\infty \) which verifies \( 1 + \frac{1}{p} = \frac{1}{q} + \frac{2}{q} \), we write
\[
\left\| \int_0^t K_\eta(t-s) \ast \partial_x (u^2(s, \cdot)) ds \right\|_{L^p} \leq \int_0^t ||K_\eta(t-s) \ast \partial_x (u^2(s, \cdot))||_{L^p} ds \leq \int_0^t ||\partial_x K_\eta(t-s, \cdot) \ast u^2(s, \cdot)||_{L^p} ds
\]

and then, by the estimate (108) we get
\[
\int_0^t ||\partial_x K_\eta(t-s, \cdot)||_{L^p} ||u^2(s, \cdot)||_{L^q}^2 ds \leq \int_0^t \left( C_\eta \frac{e^{6\eta(t-s)}}{(t-s)^{\frac{1}{4}}} \right) ||u^2(s, \cdot)||_{L^q}^2 ds \leq C_\eta e^{6\eta T} \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} ||u(s, \cdot)||_{L^p}^2 ds
\]

where, the last expression (also known as the Beta function) verifies \( \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} ds \equiv \frac{1}{4c_\eta c_\eta e^{11\eta T}} > 0 \), and thus we can write
\[
\left\| \int_0^t K_\eta(t-s) \ast \partial_x (u^2(s, \cdot)) ds \right\|_{L^p} \leq C_\eta e^{6\eta T} \left( \sup_{0 < t < T} t^\frac{1}{4} ||u(t, \cdot)||_{L^p} \right)^2 \frac{1}{t^{\frac{1}{4}}}. \tag{110}
\]

Once we have this estimate we write
\[
\sup_{0 < t < T} t^\frac{1}{4} \left\| \int_0^t K_\eta(t-s) \ast \partial_x (u^2(s, \cdot)) ds \right\|_{L^p} \leq \sup_{0 < t < T} t^\frac{1}{4} \left( C_\eta e^{6\eta T} \left( \sup_{0 < t < T} t^\frac{1}{4} ||u(t, \cdot)||_{L^p} \right)^2 t^{\frac{1}{4}} \right)
\]

\[
\leq C_\eta e^{6\eta T} \left( \sup_{0 < t < T} t^\frac{1}{4} ||u(t, \cdot)||_{L^p} \right)^2. \tag{110}
\]

Now, with the estimates (109) and (110) we set the quantity \( \delta \) as \( \delta = \frac{1}{4c_\eta c_\eta e^{11\eta T}} > 0 \), and if the initial datum verifies \( ||u_0||_{L^p} < \delta \) then we apply the Picard contraction principle to obtain a solution \( u(t, x) \) of the integral equation (3).

We prove now the uniqueness of this solution and for this we will follow the same ideas given at the end of the proof of Theorem 4.1. Indeed, let us suppose that the integral equation (3) admits two solutions \( u_1 \) and \( u_2 \) (arising from the same initial datum \( u_0 \)) in the space \( L^\infty([0, T[, L^p(\mathbb{R})) \), \( \sup_{0 < t < T} t^\frac{1}{4} \cdot ||\cdot||_{L^p} \). Then, we denote \( v = u_1 - u_2 \) and by recalling the identity (17) we write
\[
v(t, \cdot) = \frac{1}{2} \int_0^t K_\eta(t-s, \cdot) \ast (\partial_x (v(s, \cdot) u_1(s, \cdot) + u_2(s, \cdot) v(s, \cdot))) ds.
\]

Finally, let \( 0 < T^* \leq T \) be the maximal time such that we have \( v = 0 \) on the interval \([0, T^*]\) and we will prove that \( T^* = T \). Indeed, if we suppose \( T^* < T \) then there exists a time \( T^* < T_1 < T \) and we consider
the space $L^\infty([T^*, T_1], L^p(\mathbb{R}))$ endowed with the norm $\| \cdot \|_{(T_1-T^*)} = \sup_{T^* < t < T_1} t^{\frac{1}{2}} \| \cdot \|_{L^p}$. Now, remark that by the estimate (100) we can write

$$\|v\|_{(T_1-T^*)} \leq C_{\eta} e^{6\eta(T_1-T^*)} \|v\|_{(T_1-T^*)} \left(\|u_1\|_{(T_1-T^*)} + \|u_2\|_{(T_1-T^*)}\right),$$

and remark also that for a function $f \in L^\infty([T^*, T_1], L^p(\mathbb{R}))$ we have $\lim_{T \rightarrow T^*} \|f\|_{(T_1-T^*)} = 0$. Therefore, we can take $0 < T_1 - T^*$ small enough such that

$$\|u_1\|_{(T_1-T^*)} + \|u_2\|_{(T_1-T^*)} \leq \frac{1}{2 C_{\eta} e^{6\eta(T_1-T^*)}}.$$

By this inequality and the previous estimate on the quantity $\|v\|_{(T_1-T^*)}$ we obtain $\|v\|_{(T_1-T^*)} = 0$ which contradicts the definition of $T^*$. Theorem 4 is now proven. $\blacksquare$

6 Appendix

Proof of Lemma 3.1

Recall that the term $I_a$ in (17) is given as

$$I_a = \int_{\xi < 0} e^{2\pi i x} \partial_\xi \left( (e^{i t \xi^3 - \eta t (-\xi^3 + \xi)})(3 i t \xi^2 - \eta t (-3 \xi^2 + 1)) \right) d\xi = \int_{\xi < 0} e^{2\pi i x} \partial_\xi \left( \partial_\xi (e^{i t \xi^3 - \eta t (-\xi^3 + \xi)}) \right) d\xi$$

$$= \int_{\xi < 0} e^{2\pi i x} \partial_\xi^2 \left( e^{i t \xi^3 - \eta t (-\xi^3 + \xi)} \right) d\xi = \int_{\xi < 0} e^{2\pi i x} \partial_\xi^2 \tilde{K}_\eta(t, \xi) d\xi.$$

On the other hand, by Lemma 5.1 in [11], we have for all $\xi \neq 0$:

$$\partial_\xi^2 \tilde{K}_\eta(t, \xi) = \tilde{K}_\eta(t, \xi) \xi^2 \left(3 i \xi^2 - \eta \text{sign}(\xi)(3 \xi^2 - 1)\right)^2 + 6 i \xi (i - \eta \text{sign}(\xi)) \tilde{K}_\eta(t, \xi),$$

and then we can write

$$|I_a| \leq \left\| \partial_\xi^2 \tilde{K}_\eta(t, \cdot) \right\|_{L^1([-\infty, 0])} \leq c(1 + \eta^2) t^2 \left\| \tilde{K}_\eta(t, \cdot) (1 + |\cdot|^4) \right\|_{L^1(\mathbb{R})}$$

$$+ c(1 + \eta) t \left\| \tilde{K}_\eta(t, \cdot) (1 + |\cdot|) \right\|_{L^1(\mathbb{R})}. \quad (111)$$

In order to study the term on the right-hand side we have the following estimates: for $m > -1$, by the estimate (13) and denoting by $\Gamma$ the ordinary gamma function, we have:

$$\left\| (1 + |\cdot|^m) \tilde{K}_\eta(t, \cdot) \right\|_{L^1} \leq \left\| \tilde{K}_\eta(t, \cdot) \right\|_{L^1} + \left\| |\xi|^m \tilde{K}_\eta(t, \cdot) \right\|_{L^1}$$

$$\leq C e^{\frac{3|m|}{(\eta t)^{\frac{m}{2}}}} + \int_{|\xi| \leq 2} |\xi|^m e^{-\eta t (|\xi|^3 - |\xi|)} d\xi + \int_{|\xi| \geq 2} |\xi|^m e^{-\eta t \frac{2|m+1|}{m+1}} d\xi$$

$$\leq C e^{\frac{3|m|}{(\eta t)^{\frac{m}{2}}}} + \frac{2m+2}{m+1} e^{2\eta t} + \frac{c_m \Gamma\left(\frac{m+1}{2}\right)}{(\eta t)^{\frac{m}{2}}}$$

$$\leq C e^{\frac{3|m|}{(\eta t)^{\frac{m}{2}}}} + C_m \frac{1}{(\eta t)^{\frac{m+1}{4}}}.$$ \quad (112)
With this estimate (setting first $m = 4$ and then $m = 1$) we get back to (111) and we write

$$|I_a| \leq c(1+\eta)^2 e^{2\eta t} \left( \frac{e^{3\eta t}}{(\eta t)^\frac{3}{2}} + \frac{1}{(\eta t)^\frac{3}{2}} \right) + \frac{c(1+\eta)}{\eta} \left( \frac{e^{3\eta t}}{(\eta t)^\frac{3}{2}} + \frac{1}{(\eta t)^\frac{3}{2}} \right)$$

$$\leq c \left( \frac{1+\eta}{\eta^2} \left( \frac{e^{3\eta t}}{(\eta t)^\frac{3}{2}} + \frac{1}{(\eta t)^\frac{3}{2}} \right) + \frac{c(1+\eta)}{\eta} \left( \frac{e^{3\eta t}}{(\eta t)^\frac{3}{2}} + \frac{1}{(\eta t)^\frac{3}{2}} \right) \right)$$

$$\leq c \left( \frac{1+\eta}{\eta^2} (2e^{4\eta t}) + \left( \frac{1+\eta}{\eta} \right) e^{4\eta t} \right)$$

$$\leq c \left( \frac{1+\eta}{\eta} + 2 \right)^2 e^{4\eta t}.$$  \hspace{1cm} (113)

The term $I_b$ in (17) is treated following the same computations done for the term $I_a$ above.

---

**Proof of Lemma 4.2**

1) Remark first that as we have $K_\eta(t,x) = F^{-1} \left( e^{i(\xi^3 t - \eta |\xi|^3 - |\xi|)} \right)(x)$ and by the identity $\partial_x K_\eta(t,x) = F^{-1} \left( 2\pi i \xi e^{i(\xi^3 t - \eta |\xi|^3 - |\xi|)} \right)(x)$, and moreover, as the function $e^{i(\xi^3 t - \eta |\xi|^3 - |\xi|)}$ and the function $(2\pi i \xi)e^{i(\xi^3 t - \eta |\xi|^3 - |\xi|)}$ belong to the space $L^1(\mathbb{R})$, then by the properties of the inverse Fourier transform we have that $K_\eta(t,x)$ and $\partial_x K_\eta(t,x)$ are continuous functions and thus we have $K_\eta(t,\cdot) \in C^1(\mathbb{R})$.

Now, we write

$$\partial_x K_\eta(t,x) = \int_\mathbb{R} (2\pi i \xi) e^{2\pi i \xi^3} \widehat{K_\eta}(t,\xi) d\xi = \frac{1}{2\pi i x} \int_{\xi < 0} (2\pi i \xi)(2\pi i x) e^{2\pi i \xi^3} \widehat{K_\eta}(t,\xi) d\xi$$

$$+ \frac{1}{2\pi i x} \int_{\xi > 0} (2\pi i \xi)(2\pi i x) e^{2\pi i \xi^3} \widehat{K_\eta}(t,\xi) d\xi,$$

and since $\partial_\xi (e^{2\pi i \xi}) = 2\pi i x e^{2\pi i \xi}$ then we can write

$$\frac{1}{2\pi i x} \int_{\xi < 0} (2\pi i \xi)(2\pi i x) e^{2\pi i \xi^3} \widehat{K_\eta}(t,\xi) d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} (2\pi i \xi)(2\pi i x) e^{2\pi i \xi^3} \widehat{K_\eta}(t,\xi) d\xi$$

$$= \frac{1}{2\pi i x} \int_{\xi < 0} \partial_\xi (e^{2\pi i \xi^3})(2\pi i \xi) e^{i\xi^3 - \eta (\xi^3 - \xi)} d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} \partial_\xi (e^{2\pi i \xi^3})(2\pi i \xi) e^{i\xi^3 - \eta (\xi^3 - \xi)} d\xi,$$
thereafter, we integrate by parts and we get

\[
\frac{1}{2\pi i x} \int_{\xi < 0} \partial_\xi (e^{2\pi i x \xi}) (2\pi i \xi) \tilde{K}_\eta(t, \xi) d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} \partial_\xi (e^{2\pi i x \xi}) (2\pi i \xi) \tilde{K}_\eta(t, \xi) d\xi
\]

\[
= \frac{1}{2\pi i x} \int_{\xi < 0} e^{2\pi i x \xi} (2\pi i \xi) \tilde{K}_\eta(t, \xi) d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} (e^{2\pi i x \xi}) (2\pi i \xi) \tilde{K}_\eta(t, \xi) d\xi
\]

\[
+ \frac{1}{2\pi i x} \int_{\xi < 0} e^{2\pi i x \xi} \partial_\xi \tilde{K}_\eta(t, \xi) d\xi + \frac{1}{2\pi i x} \int_{\xi > 0} (2\pi i \xi) \partial_\xi \tilde{K}_\eta(t, \xi) d\xi
\]

\[
= \frac{1}{x} \left( \int_{\xi < 0} e^{2\pi i x \xi} \tilde{K}_\eta(t, \xi) d\xi + \int_{\xi > 0} (e^{2\pi i x \xi}) \tilde{K}_\eta(t, \xi) d\xi \right)
\]

\[
+ \frac{1}{x} \left( \int_{\xi < 0} e^{2\pi i x \xi} \partial_\xi \tilde{K}_\eta(t, \xi) d\xi + \int_{\xi > 0} e^{2\pi i x \xi} \partial_\xi \tilde{K}_\eta(t, \xi) d\xi \right)
\]

\[
= I_1 + I_2. \tag{114}
\]

In order to study the term \( I_1 \) remark that we have \( I_1 = \frac{1}{x} K_\eta(t, x) \) and by the estimate \((21)\) we obtain

\[
|I_1| \leq C_\eta e^{5\eta t}. \tag{115}
\]

We study now the term \( I_2 \) above. Remark that the have \( \partial^2_\xi (e^{2\pi i x \xi}) = -4\pi^2 x^2 e^{2\pi i x \xi} \), and therefore we write

\[
I_2 = \frac{1}{-4\pi^2 x^3} \left( \int_{\xi < 0} (4\pi^2 x^2) e^{2\pi i x \xi} \partial_\xi \tilde{K}_\eta(t, \xi) d\xi + \int_{\xi > 0} (4\pi^2 x^2) e^{2\pi i x \xi} \partial_\xi \tilde{K}_\eta(t, \xi) d\xi \right)
\]

\[
= \frac{1}{-4\pi^2 x^3} \left( \int_{\xi < 0} \partial^2_\xi \xi \tilde{K}_\eta(t, \xi) d\xi + \int_{\xi > 0} \partial^2_\xi \xi \tilde{K}_\eta(t, \xi) d\xi \right),
\]

then, integrating by parts the last expression we can write

\[
I_2 = \frac{1}{-4\pi^2 x^3} \left( \int_{\xi < 0} e^{2\pi i x \xi} \left( 2\partial^2_\xi \tilde{K}_\eta(t, \xi) + \xi \partial^3_\xi \tilde{K}_\eta(t, \xi) \right) d\xi + \int_{\xi > 0} e^{2\pi i x \xi} \left( 2\partial^2_\xi \tilde{K}_\eta(t, \xi) + \xi \partial^3_\xi \tilde{K}_\eta(t, \xi) \right) d\xi \right),
\]

\[
= (I_{2a}) + (I_{2b}) \tag{116}
\]

and now we will prove the following estimate

\[
|(I_{2a})| + |(I_{2b})| \leq C_\eta e^{5\eta t}. \tag{117}
\]

Indeed, for the term \((I_{2a})\) we write \(|(I_{2a})| \leq C\|\partial^2_\xi \tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])} + C\|\xi \partial^3_\xi \tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])}\), but recall that by the estimates \((111)\) and \((113)\) we have \(\|\tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \leq C_\eta e^{4\eta t}\) and therefore we can write

\[
|(I_{2a})| \leq C_\eta e^{5\eta t} + C\|\xi \partial^3_\xi \tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \leq C_\eta e^{5\eta t} + C\|\xi \partial^3_\xi \tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \tag{118}
\]

Now, we study the term \( c\|\xi \partial^3_\xi \tilde{K}_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \). By Lemma 5.1 in \([1]\) we have for all \(\xi \neq 0\):

\[
\partial^3_\xi \tilde{K}_\eta(t, \xi) = t^3 \tilde{K}_\eta(t, \xi) (3\xi^2 - \eta \text{sign}(\xi)(3\xi^2 - 1)^3
\]

\[
+ t^2 \tilde{K}_\eta(t, \xi) (36\xi^3(\xi^2 - 1) - 72 i \eta \text{sign}(\xi) \xi^3 + 12 i \eta \text{sign}(\xi) \xi - 12\eta^2 \xi)
\]

\[
+ 6t \tilde{K}_\eta(t, \xi) (\xi(i - \eta \text{sign}(\xi))(3\xi^2 - \eta \text{sign}(\xi)(3\xi^2 - 1)) + 6t \tilde{K}_\eta(t, \xi)(i - \eta \text{sign}(\xi)).
\]
Thus, we can write
\[ |\partial^3_t K_\eta(t, \xi)| \leq C_\eta t^3 (1 + |\xi|^6) |\widehat{K}_\eta(t, \xi)| + C_\eta t^2 (1 + |\xi|^5) |\widehat{K}_\eta(t, \xi)| + C_\eta t |\widehat{K}_\eta(t, \xi)|, \]
and thus we get
\[ |\xi||\partial^3_t K_\eta(t, \xi)| \leq C_\eta t^3 (1 + |\xi|^7) |\widehat{K}_\eta(t, \xi)| + C_\eta t^2 (1 + |\xi|^4) |\widehat{K}_\eta(t, \xi)| + C_\eta t (1 + |\xi|) |\widehat{K}_\eta(t, \xi)|. \]

With this estimate we can write
\[
\|\xi\partial^3_t K_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \leq \|\xi\partial^3_t K_\eta(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_\eta t^3 |(1 + |\xi|^7) |\widehat{K}_\eta(t, \cdot)|\|_{L^1(\mathbb{R})}
+ C_\eta t^2 |(1 + |\xi|^4) |\widehat{K}_\eta(t, \cdot)|\|_{L^1(\mathbb{R})} + C_\eta t |(1 + |\xi|) |\widehat{K}_\eta(t, \cdot)|\|_{L^1(\mathbb{R})}
= (a),
\]
and then, by the estimate (112) (setting first \( m = 7 \), thereafter \( m = 4 \) and finally \( m = 1 \)) we have
\[
(a) \leq C_\eta \left( e^{2t_\eta} + t^{-1/3} + t^{-\left(\frac{2}{3}\right)} \right) + C_\eta t^2 \left( e^{2t_\eta} + t^{-1/3} + t^{-\left(\frac{2}{3}\right)} \right) + C_\eta \left( e^{2t_\eta} + t^{-1/3} + t^{-\left(\frac{2}{3}\right)} \right).
\]

and thus we can write \( \|\xi\partial^3_t K_\eta(t, \cdot)\|_{L^1([-\infty, 0])} \leq C_\eta t^5 t \). With this estimate, we get back to the estimate (118) and we write \( |(I_2)_a| \leq C_\eta e^{5t_\eta} \).

The term \( (I_2)_b \) is estimated following the same computations done for the term \( (I_2)_a \) above and thus we have the estimate (117).

Finally, with the estimate (117) we get back to the estimate (116) and we write
\[ |I_2| \leq C_\eta \frac{e^{5t_\eta}}{|x|^3}, \]
and thus, by the estimates (115) and (119) at hand, we get back to the estimate (114) and we can write the desired inequality: \( |\partial_x K_{t,x}| \leq C_\eta \frac{e^{5t_\eta}}{|x|^3} \).

2) We write
\[ |\partial_x K_\eta(t, x)| \leq \int_\mathbb{R} |(2\pi i \xi) e^{2\pi i x \xi} |\widehat{K}_\eta(t, \xi)| d\xi \leq \|(1 + |\xi|) |\widehat{K}_\eta(t, \cdot)|\|_{L^1}, \]
and by the estimate (112) (with \( m = 1 \)) we have
\[
\|(1 + |\xi|) |\widehat{K}_\eta(t, \cdot)|\|_{L^1} \leq C_\eta \left( e^{2t_\eta} + \frac{1}{t^{\frac{1}{3}}} + \frac{1}{t^{\frac{2}{3}}} \right) = C_\eta \left( \frac{t^{\frac{2}{3}} e^{2t_\eta} + t^{\frac{1}{3}} + 1}{t^{\frac{2}{3}}} \right) \leq \frac{C_\eta}{t^{\frac{2}{3}}} e^{5t_\eta}. \]

Then we can write
\[ |\partial_x K_\eta(t, x)| \leq \frac{C_\eta}{t^{\frac{2}{3}}} e^{5t_\eta} \leq \frac{C_\eta}{t^{\frac{2}{3}}} e^{6t_\eta}. \]

Finally, by this estimate and the estimate given in point 1) above: \( |\partial_x K_{t,x}| \leq C_\eta \frac{e^{5t_\eta}}{|x|^3} \), we obtain:
\[ |\partial_x K_\eta(t, x)| \leq C_\eta \frac{e^{6t_\eta}}{t^{\frac{2}{3}} \left( 1 + |x|^3 \right)}. \]
References

[1] B. Alvarez Samaniego. *On the Cauchy problem for a nonlocal perturbation of the KdV equation*. Phd thesis at Instituto de Matematica Pura e Aplicada (IMPA) (2002).

[2] B. Alvarez Samaniego. *On the Cauchy problem for a nonlocal perturbation of the KdV equation*. Differential and Integral Equations, Vol. 16, Number 10: 1249–1280 (2003).

[3] B. Alvarez Samaniego. *Spatial analyticity of solutions of a non local perturbed KdV equation*. Electronic Journal of Qualitative Theory of Differential Equations N°20: 1-21 (2005).

[4] H. Bahouri, J.Y. Chemin & R. Danchin. *Fourier Analysis and nonlinear partial differential equations*. Springer Vol: 343 (2011).

[5] L. Brandolese & G. Karch. *Far field asymptotics of solutions to convection equation with anomalous diffusion*. J. Evolution Equations. 8: 307–326 (2008).

[6] L. Brandolese & M.E. Schonbek. *Large time decay and growth for solutions of a viscous Boussinesq system*. Trans. Amer. Math. Soc. 364: 5057-5090 (2012).

[7] X. Carvajal & M. Scialom. *On the well-posedness for the generalized Ostrovsky, Stepanyams and Tsimring equation*. Nonlinear Analysis. 62: 1277 – 1287 (2005).

[8] Bao-Feng Feng, & T. Kawahara. *Multi-hump stationary waves for a Korteweg-deVries equation with nonlocal perturbations*. Physica D. 137: 237-246 (2000).

[9] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics 840, Springer Verlag, Berlin (1981).

[10] A. Esfahani. *Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation*. Math. Commun. 18: 323–335 (2013).

[11] A. Esfahani. *Well-posedness result for the Ostrovsky, Stepanyams and Tsimring equation at the critical regularity*. Nonlinear Analysis: Real World Applications. 44: 347-364 (2018).

[12] G. Fonseca, F. Linares & G. Ponce. *On persistence problems in fractional weighted spaces*. Proceedings of the AMS, Vol 143, Nº 12: 5353-5367 (2015).

[13] G. Fonseca, R. Pastrán, G. Rodríguez-Blanco *The IVP for a nonlocal perturbation of the Benjamin-Ono equation in classical and weighted Sobolev spaces*. arXiv:1807.10674 (2018).

[14] T. Kato. *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*. Adv. Math. Suppl. Stud. 8: 93–128 (1983).

[15] L. Molinet & F. Ribaud. *The Cauchy Problem for Dissipative Korteweg de Vries Equations in Sobolev Spaces of Negative Order*. Indiana University Mathematics Journal Vol. 50, No. 4: 1745-1776 (2001).

[16] J. Nahas & G. Ponce. *On the persistence properties of solutions of nonlinear dispersive equations in weighted Sobolev spaces*. RIMS Kokyuroku Bessttsatsu, 2336 (RIMS Proc.) (2011).

[17] L.A. Ostrovsky, Y.A. Stepanyams & L.S. Tsimring. *Nonlinear stage of the shearing instability in a stratified liquid of finite depth*. Fluid Dyn. 17: 540-546 (1983).

[18] L.A. Ostrovsky, Y.A. Stepanyams & L.S. Tsimring. *Radiation instability in a stratified shear flow*. Int. J. Nonlinear Mech 19: 151-161 (1984).

[19] L.A. Ostrovsky, S.A. Rybak & L.Sh. Tsimring. *Negative energy waves in hydrodynamics*. Sov. Phys. Usp. 29:1040-1052 (1986).

[20] X. Zhao & S. Cui. *Well-posedness of the Cauchy problem for Ostrovsky, Stepanyams and Tsimring equation with low regularity data*. J. Math. Anal. Appl. 344: 778–787 (2008).

[21] X. Zhao & S. Cui. *Local well-posedness of the Ostrovsky, Stepanyams and Tsimring equation in Sobolev spaces of negative indices*. Nonlinear Analysis 70: 3483–3501 (2009).