NOTES ON DISPERSIONFUL AND DISPERSIONLESS VORTEX FILAMENT EQUATIONS IN 1+1 AND 2+1 DIMENSIONS

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Abstract

The vortex filament equations (VFE) in 1+1 and 2+1 dimensions are considered. Some of these equations are integrable. Also the VFE with potentials and with self-consistent potentials are presented. Finally several examples of integrable dispersionless VFE (dVFE) are considered.

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1 Introduction

The vortex filament equation (VFE) has the form

$$\gamma_{st} = \gamma_s \times \gamma_{sss}, \quad (1a)$$

where $\gamma(s,t)$ denotes the position of the vortex filament in $R^3$ with $t$ and $s$ being the time and the arclength parameter respectively. Sometimes we use the following standard form of the VFE

$$\gamma_t = \gamma_s \times \gamma_{ss} \quad (1b)$$

which follows from (1a). Hence we obtain

$$\gamma_{tt} = -\frac{1}{2}(\gamma_{ss})^2 \gamma_s - (\gamma_{ss})^2 \gamma_{ss}. \quad (2)$$
Hasimoto [13] introduced a map $h : \gamma \rightarrow q = ke^{\int^s \tau(z) dz}$, in order to transform the VFE into the nonlinear Schrodinger equation (NLSE) for $q$

$$i q_t + q_{ss} + 2|q|^2 q = 0.$$  \hspace{1cm} (3)

Here $k$ and $\tau$ respectively denote the curvature and the torsion along $\gamma$. In this paper we consider some dispersionful and dispersionless VFE in 1+1 and 2+1. Some of these equations are integrable. Some properties of the VFE from the the various points of view were studied in [1-42].

## 2 Integrable VFE in 1+1

First in this section we consider the some well-known (1+1)-dimensional isotropic and anisotropic VFE. Some examples as follows.

a) The anisotropic VFE. It has the form

$$\gamma_t = \gamma_s \times \gamma_{ss} + V,$$  \hspace{1cm} (4)

where $V$ is the vector function. As well-known this equation is integrable in the following cases:

1) For the case $V = 0$.

$$\gamma_t = \gamma_s \times \gamma_{ss} + V, \hspace{1cm} (5a)$$

$$V_s = \alpha (\gamma_{ss}) \gamma_{ss}. \hspace{1cm} (5b)$$

2)

$$\gamma_t = \gamma_s \times \gamma_{ss} + V, \hspace{1cm} (6a)$$

$$V_s = \gamma_s \times A \gamma_s, \hspace{1cm} (6b)$$

where $A = \text{diag}(A_1, A_2, A_3)$, $A_k = \text{const}$.

b) The isotropic VFE. It looks like

$$\gamma_{st} = \gamma_s \times \gamma_{sss}, \hspace{1cm} (7)$$

or

$$\gamma_t = \gamma_s \times \gamma_{ss} + f(t). \hspace{1cm} (8)$$

Hence as $f = 0$ we obtain (1b).

c) Next well-known example can be written as [26-27]

$$\gamma_{st} = \gamma_{sss} + \frac{3}{2} (\gamma_{ss})^2 \gamma_{ss}. \hspace{1cm} (9)$$

d) One of the interesting example is the following VFE [1]

$$\gamma_{st} = \alpha (\gamma_s \times \gamma_{ss}) + \beta (\gamma_{sss} + \frac{3}{2} (\gamma_{ss})^2 \gamma_{ss}). \hspace{1cm} (10)$$

e) Finally we present the following known generalization of the VFE

$$\gamma_{st} = \gamma_s \times \gamma_{ss} + \frac{1}{s} \gamma_s \times \gamma_{ss} + \gamma_s \times A \gamma_s. \hspace{1cm} (11)$$

In the isotropic case we have

$$\gamma_{st} = \gamma_s \times \gamma_{ss} + \frac{1}{s} \gamma_s \times \gamma_{ss} \hspace{1cm} (12)$$

and so on.
3 VFE with the potentials

One of interesting generalizations of the VFE (1) are the VFE with potentials. May be the simplest example of the such equations is the following anisotropic Myrzakulov LV (M-LV) equation

\[
\gamma_{st} = (\alpha \gamma_{ss}^2 + \beta u + \delta) \gamma_s \times \gamma_{ss} + \gamma_s \times A \gamma_s, \tag{13}
\]

where \( u \) is the scalar real function (potential). In the isotropic case, the M-LV equation (13) takes the form

\[
\gamma_s = (\alpha \gamma_{ss}^2 + \beta u + \delta) \gamma_s \times \gamma_{ss}. \tag{14}
\]

In Table 1 we presented some examples the VFE with potentials. Here and below \( \alpha, \beta, \delta = \text{consts}, \text{[\(\cdot\)],} \) is commutator,

\[
g = \mu \gamma_{ss}^2 - u + \nu, \quad \dot{\gamma} = \gamma \cdot \sigma, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3). \tag{15}
\]

Table 1. The VFE with potentials

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LVII equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u [\hat{\gamma}_s, \sigma_3]\) |
| The M-LVI equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u \gamma_{ss} [\hat{\gamma}_s, \sigma_3]\) |
| The M-LV equation | \(\gamma_s = (\mu \gamma_{ss}^2 - u + \nu) \gamma_s \times \gamma_{ss}\) |
| The M-LIV equation | \(2i\dot{\gamma}_{st} = n [\hat{\gamma}_s, \gamma_{ssssss}] + 2g [\hat{\gamma}_s, \gamma_{ss}] \gamma_s\) |
| The M-LIII equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + 2i \nu \gamma_{ss}\) |
| The M-XCII equation | \(\dot{\gamma}_{st} = (\alpha \gamma_{ss}^2 + \beta u + \delta) \gamma_{ss}\) |
| The M-XCIII equation | \(\dot{\gamma}_{st} = (\alpha \sqrt{\gamma_{ss}^2 + \beta u + \delta}) \gamma_{ss}\) |

4 VFE with the self-consistent potentials

The typical representative of the VFE with the self-consistent potentials is the Myrzakulov XLII equation having the form

\[
\gamma_{st} = \{ (\mu \gamma_{ss}^2 - u + m) \gamma_s \times \gamma_{ss} \} \gamma_s + \gamma_s \times A \gamma_s, \tag{16a}
\]

\[
u_t + u_s + \lambda (\gamma_{ss}^2)_s = 0. \tag{16b}
\]

As \( A = 0 \), hence we get the isotropic M-XLII equation

\[
\gamma_s = (\mu \gamma_{ss}^2 - u + \nu) \gamma_s \times \gamma_{ss}, \tag{17a}
\]

\[
u_t + u_s + \lambda (\gamma_{ss}^2)_s = 0. \tag{17b}
\]

In this section we present some VFE with the self-consistent potentials. Some of these equations are integrable, e.g. the Myrzakulov XXXIV equation, shortly, the M-XXXIV equation (about our notations, see e.g., Refs. [43-52] and also Refs. [53-59]).

Table 2.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LII equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u [\hat{\gamma}_s, \sigma_3]\) |
|                  | \(\rho u_{tt} = \nu_0^2 u_{ss} + \lambda (\gamma_{ss})_{ss}\) |
| The M-LI equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u [\hat{\gamma}_s, \sigma_3]\) |
|                  | \(\rho u_{tt} = \nu_0^2 u_{ss} + \alpha (u_s^2)_{ss} + \beta u_{ssss} + \lambda (\gamma_{ss})_{ss}\) |
| The M-L equation  | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u [\hat{\gamma}_s, \sigma_3]\) |
|                  | \(u_t + u_s + \lambda (\gamma_{ss})_s = 0\) |
| The M-XIX equation | \(2i\dot{\gamma}_{st} = [\hat{\gamma}_s, \gamma_{ssss}] + u [\hat{\gamma}_s, \sigma_3]\) |
|                  | \(u_t + u_s + \alpha (u_s^2)_{ss} + \beta u_{ssss} + \lambda (\gamma_{ss})_s = 0\) |
### Table 3.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-XLVIII equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ss}] + u\gamma_{3s}[\gamma_s, \sigma_3] + \rho u = v_0^2 u_{ss} + \lambda(\gamma_{ss}^2)_{ss}\) |
| The M-XLVII equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ss}] + u\gamma_{3s}[\gamma_s, \sigma_3] + \rho u = v_0^2 u_{ss} + \alpha(u^2)_{ss} + \beta u_{ssss} + \lambda(\gamma_{ss}^2)_{ss}\) |
| The M-XLVI equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ss}] + u\gamma_{3s}[\gamma_s, \sigma_3] + u_t + u_s + \lambda(\gamma_{ss}^2)_{s} = 0\) |
| The M-XLV equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ss}] + u\gamma_{3s}[\gamma_s, \sigma_3] + u_t + u_s + \alpha(u^2)_{s} + \beta u_{ssss} + \lambda(\gamma_{ss}^2)_{s} = 0\) |

### Table 4.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-XLIV equation | \(\gamma_t = (\mu \gamma_{ss}^2 - u + m)\gamma_s \times \gamma_{ss}\) |
| The M-XLIII equation | \(\gamma_t = (\mu \gamma_{ss}^2 - u + m)\gamma_s \times \gamma_{ss}\) |
| The M-XLII equation | \(\gamma_t = (\mu \gamma_{ss}^2 - u + m)\gamma_s \times \gamma_{ss}\) |
| The M-XLI equation | \(\gamma_t = (\mu \gamma_{ss}^2 - u + m)\gamma_s \times \gamma_{ss}\) |

### Table 5.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-XL equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2[\mu \gamma_{ss}^2 - u + m][\gamma_s, \gamma_{ss}]_{s}\) |
| The M-XXXIX equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2[\mu \gamma_{ss}^2 - u + m][\gamma_s, \gamma_{ss}]_{s}\) |
| The M-XXXVIII equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2[\mu \gamma_{ss}^2 - u + m][\gamma_s, \gamma_{ss}]_{s}\) |
| The M-XXXVII equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2[\mu \gamma_{ss}^2 - u + m][\gamma_s, \gamma_{ss}]_{s}\) |

### Table 6.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-XXXVI equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2iu\gamma_{ss}\) |
| The M-XXXV equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2iu\gamma_{ss}\) |
| The M-XXXIV equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2iu\gamma_{ss}\) |
| The M-XXXIII equation | \(2i\gamma_{st} = [\gamma_s, \gamma_{ssss}] + 2iu\gamma_{ss}\) |

### Table 7.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LXIX equation | \(\gamma_{st} = \frac{1}{\sqrt{\gamma_{ss}^2}}[-\sqrt{\gamma_{ss}^2} - u^2]_{ss} + u\gamma_s \times \gamma_{ss}\) |
| The M-LXXX equation | \(u_s = v\sqrt{\gamma_{ss}^2} - u^2\) |
| The M-LXXVI equation | \(v_t = -\gamma_s \cdot (\gamma_{st} \times \gamma_{ss})\) |
Table 8.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-V equation | \( \gamma_t = \frac{1}{2} [\gamma_s, \gamma_{ss}] + \frac{1}{4} \gamma_s^2, (\gamma_{ss})_{ss}, \) \( \gamma_s \in osp(2|1) \) |

5 VFE with the electromagnetic interaction

One of interesting problem is the interaction between the vortex filament and the electromagnetic field. In theory, this interaction describes by the coupled system of the VFE and the Maxwell equations. In the soliton limit, hence, we get the coupled system of the VFE and the Schrodinger-type equation. As example, we consider the following system of the coupled equations

\[
\begin{align*}
\gamma_{st} &= [(\alpha|\phi|^2 + \beta \gamma_{ss}^2 + \delta)\gamma_s \times \gamma_{ss}] + \gamma_s \times A_{\gamma_s}, \\
\frac{d\phi_t}{dt} + \phi_{ss} + (\mu|\phi|^2 + \nu \gamma_{ss}^2 + \lambda)\phi &= 0.
\end{align*}
\]  

Hence in the isotropic case we have

\[
\begin{align*}
\gamma_t &= (\alpha|\phi|^2 + \beta \gamma_{ss}^2 + \delta)\gamma_s \times \gamma_{ss}, \\
\frac{d\phi_t}{dt} + \phi_{ss} + (\mu|\phi|^2 + \nu \gamma_{ss}^2 + \lambda)\phi &= 0.
\end{align*}
\]

In this section we present some systems of equations which describe interaction between the vortex filament and electromagnetic fields.

Table 9.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LXXI equation | \( \gamma_{st} = \gamma_s \times \gamma_{ss} + \alpha|\phi|^2 \gamma_{ss} + \gamma_s \times A_{\gamma_s} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + \lambda \gamma_{ss}^2 \phi = 0 \) |
| The M-LXXII equation | \( \gamma_{st} = \gamma_s \times \gamma_{ss} + \alpha|\phi|^2 \gamma_{ss} + \gamma_s \times A_{\gamma_s} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda (\gamma_{ss}^2 \phi)_s = 0 \) |
| The M-LXXIII equation | \( \gamma_{st} = \gamma_s \times \gamma_{ss} + \alpha|\phi|^2 \gamma_{ss} + \gamma_s \times A_{\gamma_s} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda \gamma_{ss}^2 \phi_s = 0 \) |

Table 10.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LXXIV equation | \( \gamma_t = (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + \lambda \gamma_{ss}^2 \phi = 0 \) |
| The M-LXXV equation | \( \gamma_t = (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda (\gamma_{ss}^2 \phi)_s = 0 \) |
| The M-LXXVI equation | \( \gamma_t = (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda \gamma_{ss}^2 \phi_s = 0 \) |

Table 11.

| Name of equation | Equation of motion |
|------------------|--------------------|
| The M-LXXVII equation | \( \gamma_t = \alpha \gamma_s \times \gamma_{ssssss} + (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + \lambda \gamma_{ss}^2 \phi = 0 \) |
| The M-LXXVIII equation | \( \gamma_t = \alpha \gamma_s \times \gamma_{ssssss} + (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda (\gamma_{ss}^2 \phi)_s = 0 \) |
| The M-LXXIX equation | \( \gamma_t = \alpha \gamma_s \times \gamma_{ssssss} + (\mu|\phi|^2 + \nu)\gamma_s \times \gamma_{ss} \) \\
|                  | \( \frac{d\phi_t}{dt} + \phi_{ss} + i\lambda \gamma_{ss}^2 \phi_s = 0 \) |
6 Integrable VFE in 2+1

It is well-known that each (1+1)-dimensional integrable systems admits several (not one) integrable (and not integrable) systems in 2+1 dimensions. In the previous sections we presented some examples integrable and nonintegrable VFE in 1+1 dimensions. In this section we consider the several VFE in 2+1 dimensions which are the (2+1)-dimensional integrable extensions of the VFE (1) or (4). Some examples as follows.

i) The anisotropic (2+1)-dimensional VFE.

\[
\gamma_{st} = \gamma_s \times (\gamma_{sss} + \alpha^2 \gamma_{sy}) + u_s \gamma_{sy} + u_y \gamma_{ss} + W_s, \tag{20a}
\]
\[
u_s - \alpha^2 u_y = -2\alpha^2 \gamma_s (\gamma_{ss} \times \gamma_{sy}), \tag{20b}
\]
\[W_y = F_s. \tag{20c}\]

Hence we obtain the well-known isotropic version which has the form

\[
\gamma_{st} = \gamma_s \times (\gamma_{sss} + \alpha^2 \gamma_{sy}) + u_s \gamma_{sy} + u_y \gamma_{ss}, \tag{21a}
\]
\[
u_s - \alpha^2 u_y = -2\alpha^2 \gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}). \tag{21b}\]

ii) The anisotropic Myrzakulov I equation (about our notations, see e.g., Refs [43-52] and also [53-59]). It reads as [44]

\[
\gamma_{st} = (\gamma_s \times \gamma_{sy} + u \gamma_s)_s + \gamma_s \times V, \tag{22a}
\]
\[
u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \tag{22b}
\]
\[V_y = A \gamma_{sy}. \tag{22c}\]

In the isotropic case we get (the isotropic M-I equation)

\[
\gamma_{st} = (\gamma_s \times \gamma_{sy} + u \gamma_s)_s, \tag{23a}
\]
\[
u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \tag{23b}\]

or

\[
\gamma_t = \gamma_s \times \gamma_{sy} + u \gamma_s, \tag{24a}
\]
\[
u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}). \tag{24b}\]

iii) The Myrzakulov II equation [44]

\[
\gamma_{st} = (\gamma_s \times \gamma_{sy} + u \gamma_s)_s + 2b^2 \gamma_{sy} - 4cv \gamma_{ss}, \tag{25a}
\]
\[
u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \tag{25b}
\]
\[\nu_s = \frac{1}{16b^2c^2}(\gamma_{ss})^2. \tag{25c}\]

iv) The Myrzakulov III equation [44]

\[
\gamma_{st} = (\gamma_s \times \gamma_{sy} + u \gamma_s)_s + 2b(cb + d) \gamma_{sy} - 4cv \gamma_{ss}, \tag{26a}
\]
\[
u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \tag{26b}
\]
\[\nu_s = \frac{1}{4(2bc + d)^2}(\gamma_{ss})^2. \tag{26c}\]

v) The Myrzakulov XXII equation [44]

\[-i \gamma_{st} = \frac{1}{2}([\gamma_s, \gamma_{sy}] + 2iu \gamma_s)_s + i^2 \gamma_{ss} - 2ia^2 \gamma_{sy}, \tag{27a}\]
\[ u_s = -\gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \quad (27b) \]
\[ v_s = \frac{1}{4a^2} (\gamma_{ss})_y. \quad (27c) \]

vi) The Myrzakulov VIII equation [44]

\[ \gamma_{st} = \gamma_s \times \gamma_{sss} + u\gamma_{ss} + W_s, \quad (28a) \]
\[ u_y = \gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}), \quad (28b) \]
\[ W_y = F_x. \quad (28c) \]

In the isotropic case, we obtain

\[ \gamma_{st} = \gamma_s \times \gamma_{sss} + u\gamma_{ss}, \quad (29a) \]
\[ u_y = \gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}). \quad (29b) \]

vii) The Myrzakulov XX equation [44]

\[ \gamma_{st} + \gamma_s \times \{ (b + 1)\gamma_{sss} - b\gamma_{syy} + bu\gamma_{sy} + (b + 1)u_s\gamma_{ss} = 0, \quad (30a) \]
\[ u_{sy} = \gamma_s \cdot (\gamma_{ss} \times \gamma_{sy}). \quad (30b) \]

viii) The Myrzakulov IX equation [44]

\[ i\dot{\gamma}_{st} + \frac{1}{2}[\hat{\gamma}_s, M_1 \hat{\gamma}_s] + A_2 \dot{\gamma}_{ss} + A_1 \dot{\gamma}_{sy} = 0, \quad (31a) \]
\[ M_2 u = \frac{\alpha^2}{2l} tr(\gamma_s[\gamma_{ss}, \gamma_{sy}]). \quad (31b) \]

Finally we note that all of these (2+1)-dimensional VFE are integrable. And in 1+1 dimensions they reduce to the VFE (1) or (4). Of course, there are also nonintegrable (2+1)-dimensional extensions of the VFE (1) or (4). One of such extensions has the form

\[ \gamma_{st} = \gamma_s \times (\gamma_{sss} + b\gamma_{syy}) + \gamma_s \times A\gamma_s. \quad (32) \]

The isotropic version of the (32) has the form

\[ \gamma_{st} = \gamma_s \times (\gamma_{sss} + b\gamma_{syy}). \quad (33) \]

7 Integrable planar VFE

In this section we present some planar filament equations. Here \( \gamma(s, t) \) denotes an evolving planar curve, parametrized by arclength \( s \), \( k \) is its curvature. Such equations have been studied from the different point of views (see, for example, Ref. [27]).

Example 1. First we consider the following planar VFE [26-27]

\[ \gamma_{st} = \gamma_{sss} + a\gamma_{ss} + b\gamma_s, \quad (34) \]

where

\[ a = \gamma_{ss}^2 + \frac{3}{4} \sqrt{\gamma_{ss}^2}, \quad b = \frac{3}{2} (\gamma_{ss}^2)_s. \quad (35) \]

Example 2. The Myrzakulov X equation. It is integrable and has the form [44]

\[ \gamma_{st} + \gamma_{sss} + 3 \sqrt{\gamma_{ss}^2} \gamma_{ss} - 3\alpha^2 \gamma_{yy} = 0. \quad (36) \]

Finally we note that the equations (34) and (36) are integrable.
8 Integrable dispersionless VFE

A considerable interest has been paid recently to dispersionless or quasi-classical limits of integrable equations and hierarchies. Study of dispersionless hierarchies is of great importance since they arise in the analysis of various problems in physics, mathematics and applied mathematics from the theory of quantum fields and strings to the theory of conformal maps on the complex plane.

Above we presented some dispersionful VFE. Now we want present some examples integrable dispersionless VFE (dVFE) in 1+1 and 2+1 dimensions. For simplicity, we consider only the planar dVFE.

Example 1. Simplest example integrable dVFE reads as
\[
\gamma_{st} = \frac{3}{4} \sqrt{\gamma_{ss}^2} \gamma_{ss},
\]  
(37)

It is the Myrzakulov XCVIII equation [44]. As well-known it is L-equivalent to the dispersionless KdV (dKdV) equation (or the Riemann equation)
\[
k_t = \frac{3}{2} k k_s,
\]  
(38)

where \( k \) is the curvature of the plane curve.

Example 2. The Myrzakulov XCVII equation. It is integrable and has the form [44]
\[
(\gamma_{st} - \frac{3}{4} \sqrt{\gamma_{ss}^2} \gamma_{ss})_s = - \frac{3}{4i} (\gamma_{syy} \cdot \sigma_2 \gamma_s),
\]  
(39)

where
\[
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]  
(40)

Example 3. The Myrzakulov XCVI equation. This equation is also integrable. It looks like [44]
\[
\gamma_{st} = [W - 3 \partial_z^{-1}(\sqrt{\gamma_{ss}^2})_z] \gamma_{sz},
\]  
(41a)
\[
W_z = -3(\sqrt{\gamma_{zz}^2 \partial_z^{-1}(\sqrt{\gamma_{ss}^2})_z} z, 
\]  
(41b)

where \( z = s + iy \).

Example 4. The Myrzakulov XCV equation, which is integrable and reads as [44]
\[
\gamma_{st} = (\frac{3}{4} V - \frac{1}{2} \gamma_{ss}^2 + W) \gamma_{ss},
\]  
(42a)
\[
V_s = (\sqrt{\gamma_{ss}^2})_y,
\]  
(42b)
\[
(W \sqrt{\gamma_{ss}^2})_s = (\frac{3}{4} V_y - \frac{3}{2} \gamma_{ss}^2)_y.
\]  
(42c)

Example 5. The Myrzakulov C equation which reads as [44, 60]
\[
\gamma_{st} = f_1 \gamma_s \times \gamma_{ss} + f_2 \gamma_{ss} + f_3 \gamma \times \gamma_s,
\]  
(43)

where \( f_k(\gamma, \gamma_s, ...) \) is some scalar functions of the arguments. Note that the M-C equation (43) is L-equivalent to the Benney equation. Note that these dVFE are related with the integrable dispersionless spin systems (see, e.g. Ref. [43]).

9 Conclusion

In this paper we have presented some dispersionful and dispersionless VFE in 1+1 and 2+1 dimensions. Some of these equations are integrable. All of these equations admit different types exact solutions like solitons, knotes, breaking waves, etc. It is of great interest to study such solutions of the VFE in 1+1 and 2+1 dimensions and their integrability. We are currently investigating this issue and our finding will appear in a future paper.
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