Induced planet formation in stellar clusters: a parameter study of star–disc encounters

Ingo Thies,1,2,3⋆ Pavel Kroupa1,2,3† and Christian Theis2,4

1Argelander Institut für Astronomie (Sternwarte), Universität Bonn, Auf dem Hügel 71, D-53121 Bonn, Germany
2Institut für Theoretische Physik und Astrophysik der Universität Kiel, Leibnizstraße 15, D-24098 Kiel, Germany
3The Rhine Stellar Dynamical Network
4Institut für Astronomie der Universität Wien, Türkenschanzstraße 17, A-1180 Wien, Austria

Accepted 2005 September 19. Received 2005 August 30; in original form 2004 October 26

ABSTRACT

We present a parameter study of the possibility of tidally triggered disc instability. Using a restricted N-body model that allows for a survey of an extended parameter space, we show that a passing dwarf star with a mass between 0.1 and 1 M⊙ can probably induce gravitational instabilities (GIs) in the pre-planetary solar disc for prograde passages with minimum separations below 80–170 au for isothermal or adiabatic discs. Inclined and retrograde encounters lead to similar results but require slightly closer passages. Such encounter distances are quite likely in young moderately massive star clusters. The induced GIs may lead to enhanced planetesimal formation in the outer regions of the protoplanetary disc and could therefore be relevant for the existence of Uranus and Neptune, whose formation time-scale of about 100 Myr is inconsistent with the disc lifetimes of about a few Myr according to observational data by Haisch, Lada & Lada. The relatively small gas/solid ratio in Uranus and Neptune can be matched if the perturbing fly-by occurred after early gas depletion of the solar system, i.e. when the solar system was older than about 5 Myr.

We also confirm earlier results by Heller that the observed 7° tilt of the solar equatorial plane relative to the ecliptic plane could be the consequence of such a close encounter.

Key words: stellar dynamics – methods: N-body simulations – Kuiper Belt – minor planets, asteroids – Solar system: formation – open clusters and associations: general.

1 INTRODUCTION

1.1 Limits of the coagulation model

In the classical model of planet formation, dust grains collide and stick together forming larger grains and clumps. These clumps also collide with each other forming progressively larger planetesimal bodies. At some point the mass of a planetesimal provides sufficient gravitation to attract surrounding particles: the accretion process begins. While the coagulation process takes several Myr for Jupiter and Saturn (Thommes, Duncan & Levison 2002), it takes at least 100 Myr for Neptune (Hollenbach, Yorke & Johnstone 2000) and probably even longer for objects in the Edgeworth–Kuiper Belt. The time-scale \( T_{\text{coll}} \) of the coagulation phase is mainly determined by the density \( \varphi \) of the surrounding protoplanetary dust. It further depends on the epicyclic frequency \( \kappa \) because the epicyclic motion is an important factor for the collision rate of planetesimals, \( \kappa \) is given by

\[
\kappa^2 = \left( \frac{d\varpi^2}{dr} + 4\varpi^2 \right), \tag{1}
\]

where \( \varpi \) is the orbital angular frequency and \( r \) the radius of the particle orbit, i.e. the distance from the central star. The dust density \( \varphi \) is approximately proportional to the surface density \( \Sigma \), which is expected to fit an \( r^{-3/2} \) law (Weidenschilling 1977; Mayer et al. 2004). Thus, for low mass discs,

\[
\kappa \approx \Omega \approx \Omega_{\text{Kepler}} \propto r^{-3/2}, \tag{2}
\]

where \( \Omega_{\text{Kepler}} \) is the Keplerian frequency. The relationship between \( T_{\text{coll}} \) and \( r \) in a gaseous disc containing fine dust with negligible gravitational interaction between the dust grains can be written as (Safronov 1969; see also Rice & Armitage 2003)

\[
T_{\text{coll}} \propto \frac{1}{\Omega \Sigma} \propto r^3. \tag{3}
\]

The relation may be somewhat less steep due to the smaller end masses of the outer planets and gravitational focusing for larger dust particles. This is in agreement with Pollack et al. (1996) and Kokubo & Ida (2000) who found \( T_{\text{form}} \propto r^\eta \) with \( \eta = 2 \) and \( \eta \approx 2.6 \),
respectively. Assuming a typical formation time-scale for the core of Jupiter of 3 Myr ($r_{\text{H}} = 5.2$ au), this relation leads to prohibitively long formation time-scales up to 600 Myr. Even the conservative $r^2$ law yields time-scales of 40 Myr for Uranus and 100 Myr for Neptune. Thus, we find good agreement of the formation time-scaling given by equation (3) with the more detailed investigations discussed at the beginning of this section.

Recent work by Rafikov (2004) predicts shorter formation times in the order of $\sim 10^7$ yr for the core of an ice giant at 30 au from the Sun. Although this is only one-sixth of the conventional value, it is also near the disc lifetime discussed below and even beyond, if the accretion time of the envelope and the precursory depletion of large planetesimals ($\sim 0.1$–1 km, Rafikov 2004) is taken into account. Goldreich, Lithwick & Sari (2004) also predict similarly short formation time-scales but require a 6 times higher surface density (compared to the minimum mass solar nebula, MMSN; see Weidenschilling 1977; Hayashi 1981) in the region of Uranus and Neptune to allow them to collect most of their final mass until the end of oligarchy. This would also imply a significantly higher total disc mass.

The lifetime of the protoplanetary disc is most probably less than 10 Myr. Based on $JHKL$-excess observations, Haisch, Lada & Lada (2001) found lifetimes less 6 Myr before the disc material is dispersed by the stellar wind, photo-evaporation and gravitational scattering by protoplanets. The masses of observed discs range from 0.003 to 0.3 $M_\odot$, with the bulk below 0.03 $M_\odot$ (Natta 2004). This is comparable to the MMSN, which is between 0.01 and 0.07 $M_\odot$. Thus, unless an unusual high disc mass or surface density profile is assumed, there is a serious problem to explain the existence of gas (Wuchterl et al. 2000), this does not seem to be a satisfactory attempt either.

Recent work by Rafikov (2004) found lifetimes less than 6 Myr before the disc material is dispersed by the Toomre instability within 500 yr. Such encounters are expected to occur with a realistic probability within the first 6 Myr in the Orion Nebula Cluster (ONC; Scally & Clarke 2001; Bonnell et al. 2001). The tilt of the solar rotational axis against the normal direction of the ecliptic plane of about 7° (Eggers et al. 1997; Heller 1993) is interesting in this respect. We confirm the results presented by Heller (1993), who used smoothed particle hydrodynamics (SPH) instead of our restricted $N$-body calculation, that such an inclined encounter leads to the observed tilt.

In this paper, we study if GIs, as a possible originator of planet formation, can be triggered due to stellar fly-bys in a parameter survey of the stellar mass, encounter distance and eccentricity, which are $m$, $r_0$ and $e$, respectively. The aim of this survey is to identify those regions in the $m$-$r_0$-$e$ parameter space where growing local instabilities can be triggered. Due to the extended parameter space such a study can only be performed by fast numerical methods. Though detailed hydrodynamical calculations would be preferable for such models, they are too expensive in terms of CPU costs for scanning a large set of parameter space. Therefore, we applied a simple numerical Ansatz based on the fast restricted $N$-body method as was already used by Ida, Larwood & Burkert (2000). In the future, we will use our present survey to perform hydrodynamical calculations for a few interesting $m$, $r_0$ and $e$ in order to investigate the scenario of tidally triggered GIs in more detail.

In Section 2, we describe our numerical model. Section 3 describes the general effects of star–star encounters on circumstellar discs for an individual example. Section 4 gives a plan of the parameter study and shows the results for coplanar and non-coplanar encounters. Orbits of protoplanet candidates and the tilt of the circumstellar disc due to inclined fly-bys are also described. In Section 5, the results are discussed, and the conclusions are presented in Section 6.

### 1.2 Gravitational instabilities

Fragmentation due to gravitational instabilities (GIs) is another possible solution to this problem. The dynamical free-fall time-scale of a GI in a cloud with a density excess $\varrho_{\text{exc}}$ (relative to the ambient medium) can be estimated as

$$T_{\text{ff}} = \sqrt{\frac{3\pi}{32 G \varrho_{\text{exc}}}}$$  \hspace{1cm} (4)

(Binney & Tremaine 1987), where $G$ is the gravitational constant. For an overdensity $\varrho_{\text{exc}} = 10^{-12}$ g cm$^{-3}$ (which would result by doubling the ambient density locally in the disc at 30 au, the orbital radius of Neptune today) $T_{\text{ff}} \approx 100$ yr, which is six orders of magnitude shorter than the classical formation time-scale and can therefore be neglected compared with the subsequent accretion phase.

There are two major ways for the development of GIs. A very massive disc of more than 0.1 $M_\odot$ becomes unstable to axisymmetric perturbations according to the disc stability criterion $Q$ developed by Toomre (1964),

$$Q = \frac{c_s}{\pi G \Sigma},$$  \hspace{1cm} (5)

where $c_s$ is the speed of sound in the gas, $\kappa$ is the epicyclic frequency and $\Sigma$ is the surface density.

A disc can become unstable if it continues to accrete material from the protostellar cloud, so that $Q$ becomes less than 1. For non-axisymmetric perturbations in gaseous discs, the limit is higher at $Q \simeq \sqrt{3} \approx 1.7$ (Polyachenko, Polyachenko & Strel’Nikov 1997).

Another way towards local instabilities is via tidal perturbations. With this contribution, we argue that also initially stable discs can become locally unstable if an external perturbation on the surface density of the disc causes $Q$ to fall below 1 locally, so that fragmentation only occurs in such regions. For example and as shown below, a star with mass $m = 0.5 M_\odot$ on a hyperbolic orbit with eccentricity $e = 1.5$ and pericentre distance $r_0 = 120$ au may cause a local Toomre instability within 500 yr. Such encounters are expected to occur with a realistic probability within the first 6 Myr in the Orion Nebula Cluster (ONC; Scally & Clarke 2001; Bonnell et al. 2001).

The tilt of the solar rotational axis against the normal direction of the ecliptic plane of about 7° (Eggers et al. 1997; Heller 1993) is interesting in this respect. We confirm the results presented by Heller (1993), who used smoothed particle hydrodynamics (SPH) instead of our restricted $N$-body calculation, that such an inclined encounter leads to the observed tilt.

In this paper, we study if GIs, as a possible originator of planet formation, can be triggered due to stellar fly-bys in a parameter survey of the stellar mass, encounter distance and eccentricity, which are $m$, $r_0$ and $e$, respectively. The aim of this survey is to identify those regions in the $m$-$r_0$-$e$ parameter space where growing local instabilities can be triggered. Due to the extended parameter space such a study can only be performed by fast numerical methods. Though detailed hydrodynamical calculations would be preferable for such models, they are too expensive in terms of CPU costs for scanning a large set of parameter space. Therefore, we applied a simple numerical Ansatz based on the fast restricted $N$-body method as was already used by Ida, Larwood & Burkert (2000). In the future, we will use our present survey to perform hydrodynamical calculations for a few interesting $m$, $r_0$ and $e$ in order to investigate the scenario of tidally triggered GIs in more detail.

In Section 2, we describe our numerical model. Section 3 describes the general effects of star–star encounters on circumstellar discs for an individual example. Section 4 gives a plan of the parameter study and shows the results for coplanar and non-coplanar encounters. Orbits of protoplanet candidates and the tilt of the circumstellar disc due to inclined fly-bys are also described. In Section 5, the results are discussed, and the conclusions are presented in Section 6.

### 1.3 Encounter probability

Tidally induced GIs can only be a viable mechanism if encounters are of sufficient likelihood. To estimate the encounter probability, we need to consider a model of a young cluster.
Assuming a Plummer star-cluster model with given half-mass radius \( R_{0.5} = R_{Pl} / \sqrt{2^{2/3} - 1} \approx 1.305 \, R_{Pl} \) (\( R_{Pl} \) is the Plummer radius), mass \( M_{cl} \) and number of stars \( N_{cl} \), we can calculate the crossing time \( T_{cr} \) from the characteristic velocity dispersion and compute the expected number of encounters \( n_{enc} \) within a given impact parameter \( b \) and therefore the mean time between encounters. This gives us a useful guideline as to how likely the interesting encounters may be.

The relationship between \( b \), the fly-by eccentricity \( e \) and the peri-centre distance \( r_0 \) is

\[
r_0 = \frac{e - 1}{e + 1} \times b.
\]

(6)

The one-dimensional characteristic velocity dispersion \( \sigma_{1D} \) in the Plummer model, derived from the virial theorem, is given by

\[
\sigma_{1D}^2 = \frac{\pi}{32} \frac{G M_{cl}}{R_{Pl}} = \frac{\pi \sqrt{2^{2/3} - 1}}{32} \frac{G M_{cl}}{R_{Pl}}
\]

(7)

and the crossing time is

\[
T_{cr} = \frac{2 \, R_{0.5}}{\sigma_{1D}} = \frac{128 R_{Pl}^3}{\pi (2^{2/3} - 1) \times G M_{cl}}.
\]

(8)

This leads to the mean time between encounters

\[
T_{enc} = \frac{T_{cr}}{n_{enc}} = \frac{T_{cr} \, R_{0.5}^2}{N_{cl} \, b^2}.
\]

(9)

The probability \( P_{enc} \) for at least one event within the disc lifetime \( T_d \) can be derived from the Poisson probability for \( x \) encounters,

\[
P_{enc}(x) = \frac{v}{x!} \exp(-v), \quad \text{where} \quad v = \frac{T_d}{T_{enc}}.
\]

(10)

For at least one event,

\[
P_{enc}(x \geq 1) = 1 - P_{enc}(0) = 1 - \exp\left(-\frac{T_d}{T_{enc}}\right).
\]

(11)

For a young cluster with \( M_{cl} = 500 \, M_{\odot}, R_{0.5} = 0.3 \, pc \) and a mean stellar mass of \( 0.5 \, M_{\odot} \), we get \( 10 < P_{enc} < 32 \) per cent within \( T_d = 6 \, Myr \) for \( 60 \, au < r_0 < 200 \, au \), as shown in Fig. 1. This means that such close encounters are indeed likely in young stellar clusters. The chances are even greater, up to 50 per cent, for a cluster of \( 5000 \, M_{\odot} \) and \( R_{0.5} = 0.5 \, pc \), similar to the ONC (Kroupa, Aarseth & Hurley 2001). Scally & Clarke (2001) propose an encounter probability of only 4 per cent for encounters closer than 100 au in the ONC at its present density and age. This corresponds to a probability of 9 per cent for \( r_0 < 150 \, au \) and is still a reasonably high likelihood. We note that the probabilities for encounters in a younger and denser ONC would be accordingly higher.

2 THE MODEL

We investigate a large region of parameter space in order to study in which fraction of it tidally induced GI may be relevant. Our parameter space is defined by the distance of closest approach \( r_0 \), the eccentricity of the relative orbit \( e \) and the mass of the perturbing star \( m \). While an exact description of GIs requires a 3D stellar-hydrodynamical calculation, we apply a simpler, but numerically much faster restricted \( N \)-body approach. This Ansatz is valid as long as the self-gravity of the disc, the pressure forces or the inelastic collisions between the constituents of the disc can be neglected. For example, simulations of encounters between galaxies have demonstrated the validity of this approach (Toomre & Toomre 1972). Because 3D hydrodynamical calculations are by far too CPU intensive at the moment, such simplified dynamical modelling is the only way to scan the parameter space in detail. Of course, we cannot follow the dynamical evolution of substructures when their self-gravity becomes important or even dominant. However, we can identify regions that are prone to such instabilities.

In our computations, the disc is represented by test particles on initially circular and coplanar orbits. There is no explicit interaction (self-gravity, coagulation, pressure forces) between the particles themselves and also no gravitational feedback of the disc to the Sun or the perturbing star. We assume both stars to move on hyperbolic orbits. This should be typical for young star clusters. Our numerical method requires low computational effort. Therefore, it allows for the requested in-depth parameter study of the main orbital parameters \( m, e \) and \( r_0 \) providing an estimate of the order of magnitude of the perturbation and its probable effects on disc stability. We expect this approach to give useful insights into those regions of the encounter-parameter space where the physics will be of interest for triggered GI, because the primary mechanism driving the fluctuations in disc density within the first 500 yr comes from the changes of the overall potential due to the relative orbit of the two stars.

2.1 Disc properties

In our model, we suppose an initially flat disc with an outer radius \( r_{max} = 100 \, au \), an inner border \( r_{min} = 3 \, au \) and a total mass \( M_{\Delta} = 0.073 M_{\odot} \), which is close to the MMSN. The surface density \( \Sigma \) is then given by

\[
\Sigma(r) = 6300 \, g \, cm^{-2} \times \left(\frac{r}{au}\right)^{-3/2}.
\]

(12)

For later analysis (Section 4.1), the disc is assumed to contain a dust fraction of the order of 1 per cent (Natta 2004). Thus, the surface density of the dust component is approximately \( 1/100 \)th of the total \( \Sigma \). \( \Sigma_{dust}(r) \approx 0.01 \Sigma(r) \). The density profile is needed for the calculation of the local Toomre parameter \( Q \), which we use as an indicator of local GI. As shown in equation (5), \( Q \) depends on the speed of sound in the gas fraction of the disc, which mainly depends

---

Figure 1. Probability \( P_{enc} \) of close encounters within \( T_d = 6 \, Myr \) as a function of the minimum encounter distance \( r_0 \) (equation 6) for three different Plummer clusters: a young open cluster of high density (solid curve), a more expanded cluster of the same mass (thick dotted curve) and a young ONC-type cluster (thin dotted curve; Kroupa et al. 2001). \( P_{enc} \) increases with increasing cluster mass \( M_{cl} \) and decreasing half-mass radius.
on the radial temperature profile. Following a recommendation by Rafikov (2004), we use
\[ c_\text{s} \approx 1200 \text{ m s}^{-1} \times \left( \frac{r}{\text{au}} \right)^{-0.25}. \tag{13} \]

Thus, the initial \( Q \) is approximately given by
\[ Q_0 \approx 18 \left( \frac{r}{\text{au}} \right)^{-0.25}. \tag{14} \]

The changes of \( Q \) also depend on the adopted polytropic index \( \gamma \), which is given with respect to the cooling efficiency of the disc. Large cooling rates prevent tidally compressed clouds from significant heating, i.e. we have an isothermal gas, \( \gamma = 1 \). If the cooling is negligible, the gas is compressed adiabatically. For a diatomic gas, \( \gamma = 1.4 \), which is a good approximation for the protoplanetary gas because it mainly consists of molecular hydrogen. Because \( p \propto \rho^\gamma \), \( c_\text{s} \propto \sqrt{\rho/\rho} \) and \( \Sigma \propto \rho \), the dependence of \( Q \) on the dynamic compression factor \( K(=\Sigma/\Sigma_0) \) is given by
\[ Q(K) = Q_0 K^{-\frac{3}{\gamma}}. \tag{15} \]

The relations above are valid for gaseous discs. For discs that have already been depleted of much of their gas, e.g. through photo-evaporation by their central star or from a nearby O star, the GIs may develop in the dust component in a similar way. The stability of a dust particle disc is then given by
\[ Q_{\text{dust}} = \frac{\sigma_{\text{dust}} c_\text{s}}{\pi G \Sigma_{\text{dust}}}, \tag{16} \]
where \( \sigma_{\text{dust}} \) is the velocity dispersion of the dust particles. If the disc particles and gas molecules have reached thermodynamical energy equipartition, \( \sigma_{\text{dust}} \) can be estimated via the ratio of the dust particle mass \( m_\text{p} \) compared to the molecular mass \( \mu \) of the gas:
\[ \sigma_{\text{dust}} \sim c_\text{s} \times \sqrt{\frac{\mu}{m_\text{p}}}. \tag{17} \]

For micron-sized or larger particles this is negligible compared with other causes of velocity dispersion (e.g. gas turbulences that excite the dust component via gas drag). However, it is reasonable to assume the dust component to be dynamically much cooler for particles of a certain size than the gas component. The dynamical friction damps the velocity dispersion of the dust grains over a long time but the grains remain inert to short-time gas perturbations and turbulence. Barrière-Fouchet et al. (2005) show that intermediate-sized dust grains (between about 1 mm and 1 m) tend to settle efficiently into a thin dust layer in the mid-plane of the disc. If we assume that the disc is already at an intermediate age, i.e. a few Myr, so that the dust has partly coagulated to at least centimeter-sized grains, a significant fraction of the dust will be concentrated in or close to the mid-plane with a low-velocity dispersion. A partial depletion of the gas component will lower the particle size required for most efficient settling. Therefore, we expect \( Q_{\text{dust}} < Q \) and thus a higher likelihood for GIs within the dust layer once the gas has been largely depleted. However, the extension of the simulation to this case has a more qualitative than quantitative character and needs to be ascertained with hydrodynamical models. The role of the dust is discussed in more detail in Section 5.

### 2.2 Numerical methods

Because we neglect particle–particle interactions and the disc-to-star influence, the gravitational acceleration for each test particle depends only on the position of the particle and both stars at a given time \( t \), whereas the star–star encounter is reduced to a two-body problem. This allows to split the motion of the bodies into two parts:

(i) the hyperbolic (or parabolic) encounter of the stars; and
(ii) the motion of the individual particles in the time-dependent gravitational field of both stars.

For each particle, the equations of motion are solved numerically. In each time-step, the acceleration is computed from the time-dependent stellar positions.

The time-dependent positions of the two stars are given by a (semi-)analytical calculation of the orbits. This is reduced to finding the roots of an ordinary equation. In case of parabolic orbits, it can be solved analytically, whereas in the case of hyperbolic orbits, a numerical solution is required. For the latter, we use a Newton–Raphson method. In order to reduce the computational costs, we interpolate the orbits by a cubic spline making use of pre-tabulated orbital positions. The equation of motion of the test particles is solved by a Bulirsch–Stoer method with adaptive step size. To avoid singularities, a gravitational softening with a length of 0.001 au has been applied.

To obtain the local Toomre parameter, the surface density is measured by ‘probe particles’ (probes), which sum up within their probe volume the masses assigned to the test particles. The method is described in more detail in Appendix A.

### 3 SIMULATION OF INDIVIDUAL ENCOUNTERS

To demonstrate the general effects of star–star encounters, we simulated the fly-by of a star with 0.5 \( M_\odot \) with \( r_0 = 120 \) au pericentre distance and an eccentricity of \( e = 1.5 \). For comparison, we reran the fly-by with a larger encounter distance of \( r_0 = 150 \) au.

#### 3.1 Initial conditions

The disc particles are initially set on circular coplanar orbits around the Sun. Depending on the kind of simulation, the particle positions are set up as a polar grid arrangement (regular distribution) or as a random distribution preserving the mean density distribution. For density evaluations, the regular distribution yields less noisy numerical results. On the other hand, the calculation of the mean angular momentum (for the estimate of the tilt of the disc) requires the radial mass distribution to be as smooth as possible. Thus, we used a random reshuffling of the initial particle positions for calculating the tilt in Section 4.3, employing the Mersenne Twister algorithm (Matsumoto & Nishimura 1998).

The initial distance of the Sun and the perturber is set larger than 1000 au to prevent any significant tidal effects before the encounter. For the same reason, the simulation ends after the distance has reached at least the initial value. For higher resolution, the computed population of 250 000 particles and 1000 probes is limited to the area between 25 and 30 au. Such a selection for increasing the spatial resolution is possible because there are no interactions between the particles. The absence of further interactions also allows for a simple scaling to other disc radii and densities, so the results can be scaled up or down for the orbits of Uranus or Pluto and the larger Kuiper Belt objects (KBs). In a special run, which is described in Section 5.2, we extend the simulated area to 100 au, the known region of the Edgeworth–Kuiper Belt. We also extended the disc area to 50 au but reduced the particle number to 10 000 for the plots shown in Fig. 2.
3.2 Passage of a 0.5-M\(_{\odot}\) star at 120 au

A close encounter of a 0.5-M\(_{\odot}\) star at 120 au with \(e = 1.5\) gives a good illustration of the fundamental effects of tidal perturbations on a circumstellar disc. Fig. 2 shows the overall development of the perturbed disc within 50 au during and after an encounter at a 120-au periasteron distance. Within only 400 yr, strong density fluctuations appear as thin tidal arms of highly compressed material. These tidal arms persist a few \(10^2\) yr up to \(10^3\) yr, depending on the mass and the periasteron distance. Within the spiral arms, material can be carried far outwards and even become unbound. The time-dependent surface density is shown in Fig. 3 for encounter distances of 120 au and, for comparison, 150 au. It can be seen that the fly-by distance has a strong influence on the magnitude of the \(\Sigma\) peaks; while a 120-au fly-by causes an increase of more than a factor of 10, the 150-au encounter does little more than doubling \(\Sigma\).

As the fly-by proceeds, we follow \(Q\) in each counting volume and at each time-step the minimum \(Q\) is selected. A plot of the \(Q_{\text{min}}\) values within the computed disc area is shown in Fig. 4 for the isothermal (adiabatic index \(\gamma = 1\)) and the H\(_2/\)He adiabatic case (\(\gamma = 1.4\)). A disc with a locally isothermal equation of state and an initial \(Q = 8.5\) exhibits regions with \(Q < 0.8\) within less than 500 yr after a perihelion passage closer than about 120 au, satisfying the Toomre instability criterion (\(Q < 1\)). For the adiabatic border case, \(Q\) locally falls to 1.2, still below the upper limit for Toomre instability (\(Q < 1.7\)). According to Boss (2004), the combined convective and radiative cooling is sufficient to form clumps similarly to the isothermal case. Cooling times are of the order the orbital period even for optically thick discs and therefore fulfil the criterion for fragmentation investigated by Gammie (2001). Thus, an isothermal equation of state is expected to be a reasonable approximation.

### 4 RESULTS OF THE PARAMETER STUDY

We surveyed the parameter space \((m, e, r_0)\) in order to find the regions where \(Q < 1\) locally occurs. The parameters have been varied successively with appropriate step sizes (Table 1).

![Figure 2](https://academic.oup.com/mnras/article-abstract/364/3/961/1188753)

Figure 2. Disc perturbations during hyperbolic fly-by \((e = 1.5)\) of a star with 0.5 M\(_{\odot}\) at \(r_0 = 120\) au. Only the regions within an initial radius of 50 au are shown.

![Figure 3](https://academic.oup.com/mnras/article-abstract/364/3/961/1188753)

Figure 3. Surface density \(\Sigma\) versus time for \(r_0 = 120\) and 150 au measured in one probe at a radius of 30 au. In the closer fly-by, \(\Sigma\) reaches easily 10 times the initial value about 500 yr after the periastron. The wider fly-by leads only to slight density variations, which illustrates the strong dependence on \(r_0\). The peaks in \(\Sigma\) are due to the density waves moving with a different velocity than the Keplerian orbital motion.

![Figure 4](https://academic.oup.com/mnras/article-abstract/364/3/961/1188753)

Figure 4. Minimum values of \(Q\) within the region 25 au \(\leq r \leq 30\) au (initial conditions) during a 120-au encounter of a 0.5-M\(_{\odot}\) perturber for the isotheermal and the adiabatic case (\(\gamma = 1.0\) and 1.4, respectively). Even in the adiabatic case, \(Q\) grazes the zone of Toomre instability (\(Q_{\text{min}} \approx 1.7\)), whereas the more realistic isothermal case almost ensures the Toomre instability criterion (\(Q_{\text{min}} < 1\)).

| Parameter | From | To   | With step |
|-----------|------|------|-----------|
| \(m\)     | 0.1 M\(_{\odot}\) | 1.0 M\(_{\odot}\) | 0.1–0.2 M\(_{\odot}\) |
| \(e\)     | 1.0  | 10   | 0.5–1.0   |
| \(r_0\)   | 60 au | 200 au | 5 au      |

© 2005 The Authors. Journal compilation © 2005 RAS, MNRAS 364, 961–970

Downloaded from https://academic.oup.com/mnras/article-abstract/364/3/961/1188753
by guest on 29 July 2018
Here, we describe the major results of our parameter study dealing with the triggering of GIs, the masses and orbits of seed candidates for coplanar prograde passages, and the angular momentum tilt due to inclined passages. The possibility of triggering for inclined passages has been studied for isolated cases.

4.1 Triggered Toomre instability for prograde passages

Our parameter study for prograde coplanar passages shows that the Toomre instability criterion $Q < 1$ is locally satisfied for all stellar masses above $0.1 \, M_\odot$ and all eccentricities between $1$ and $10$ for encounter distances $r_0 \leq 80$ au. For perturber stars with $1 \, M_\odot$ and $e = 10$, we found $Q < 1$ occurrence even for fly-bys with $r_0 \leq 170$ au. This critical encounter radius $r_{0, crit}$, the upper limit for $r_0$ up to which the Toomre instability criterion is satisfied, depends on both the perturber mass and the encounter eccentricity in a monotonically increasing manner, as can be seen in Fig. 5. This reveals two major trends: increasing the mass leads to an increased maximum perihelion distance $r_{0, crit}$, and also a higher eccentricity of the perturber orbit results in larger $r_{0, crit}$ (Table 1).

The mass dependency is simply caused by an enhanced perturbation with increasing mass. The eccentricity dependence, on the other hand, is less obvious. Unless the encounter velocity is extremely high or the periapsis distance extremely small (i.e. close to the particle orbits), the angular velocity of the perturber is normally significantly smaller than that of the disc particles. Thus, an increment of the encounter velocity for a given periapsis radius yields a larger $e$ and a smaller particle-perturber relative velocity, i.e. longer exposure time.

An important question is how much mass is included in a typical probe with observed $Q < 1$. A typical probe in our simulation has a radius $h = 0.1$ au (Appendix A), but the total region of a local Toomre instability could be larger than that. In an unperturbed disc, $\Sigma$ has a value of about $40 \, g \, cm^{-2}$ or $1.5 \, M_\odot \, au^{-2}$ at $30$ au, which results in about $0.03$ Earth masses ($M_\oplus$) within $h$. Within a tidal arm, $\Sigma$ can reach to $10$ to $20$ times this value, yielding $0.3$ to $0.6 \, M_\oplus$ per probe.

Because the probe contains only a fraction of the Toomre-unstable material of the tidal arm, the total mass of the unstable region is likely to be much larger. A typical compressed arm is about $0.5$ au wide. A spherical region of $0.5$ au contains $10$–$20 \, M_\oplus$ or about one Neptune mass. If $1$ per cent of this is dust (Natta 2004), a body of about $0.1$–$0.2 \, M_\oplus$ can form out of this seed. This would skip much of the coagulation time otherwise needed and is about $100$ times more massive than the most massive KBOs.

4.2 Triggered Toomre instability for retrograde and inclined passages

In a few examples, we also studied the triggering effects of an inclined prograde and a retrograde passage of a $0.5 \, M_\odot$ perturber. We varied the inclinations between $45^\circ$ and $180^\circ$ in steps of $45^\circ$. As one would expect, the perturbation in both cases is smaller than that of a coplanar prograde passage being due to the larger distance between the perturber and that edge of the disc with the lowest angular velocity relative to the perturber. The results are summarized in Fig. 6.

For the coplanar retrograde passage (inclination $= 180^\circ$), we found $r_{0, crit} = 65$ au for $e = 1.5$ and $r_{0, crit} = 75$ au for $e = 10$. Both are about half the $r_{0, crit}$ in the coplanar prograde case, i.e. the reduction of $r_{0, crit}$ due to inclination is apparently independent of the eccentricity. For other inclinations, the decrease of $r_{0, crit}$ depends on the angle $\psi$ between the pericentre and the ascending node, where $\psi = 90^\circ$ leads to significantly higher $r_{0, crit}$ than $\psi = 0$.

4.3 Tilt of the angular momentum vector

In inclined encounters, there is the possibility for tilting the plane of the disc due to the transfer of angular momentum to the orbit of the perturber. We investigate the possible origin of the obliquity of the rotational axis of the Sun relative to the ecliptic for inclinations $\iota$ between $5^\circ$ and $175^\circ$ and $\psi = 0$. Passages with inclination less than

---

**Table 2.** Coupling radius $a_0$ in units of $r_0$, depending on the mass and orbit of the perturber.

| $m/M_\odot$ | $1$ | $1.5$ | $3$ | $5$ | $10$ | $50$ |
|------------|-----|------|----|----|-----|-----|
| $0.1$      | $0.77$ | $0.71$ | $0.61$ | $0.53$ | $0.44$ | $0.26$ |
| $0.3$      | $0.73$ | $0.68$ | $0.58$ | $0.50$ | $0.41$ | $0.25$ |
| $0.5$      | $0.69$ | $0.64$ | $0.55$ | $0.48$ | $0.39$ | $0.24$ |
| $0.7$      | $0.67$ | $0.62$ | $0.53$ | $0.46$ | $0.38$ | $0.23$ |
| $1.0$      | $0.63$ | $0.58$ | $0.50$ | $0.44$ | $0.36$ | $0.21$ |

---

**Figure 5.** Upper border $r_{0, crit}$ of the regime of $Q < 1$ occurrence in the $(m, e, r_0)$ parameter space for the isothermal equation of state. $Q < 1$ occurs for values of $r_0$ below the border indicated by the grid surface. Clearly visible is the increasing $r_0$ trend for increasing perturber mass and eccentricity. The latter trend is caused by the decreasing relative angular velocity between perturber and test particles. The key on the right indicates the $r_0$-contour levels plotted in the $m, e$ plane.

**Figure 6.** Critical upper pericentre distance $r_{0, crit}$ for a sample of encounters with non-zero inclination $\iota$ compared to the corresponding coplanar prograde fly-by. The perturber mass is $0.5 \, M_\odot$ in all cases. Retrograde passages result in $r_{0, crit}$ about 50 per cent of that for zero inclination independently of the eccentricity. $\psi$ is the angle between the pericentre and ascending node.
Induced planet formation in stellar clusters 967

5 DISCUSSION

5.1 Cooling and fragmentation of discs

Based on an analytical estimate, Rafikov (2005) found that the fragmentation of protoplanetary discs requires unusual high surface densities and temperatures. According to his paper, a local Toomre instability does not necessarily lead to fragmentation. It is crucial that the instability can be achieved despite temperatures that are high enough to provide efficient radiative cooling via the Stefan–Boltzmann law. According to Rafikov (2005), the temperature \( T_{\text{crit}} \) required for efficient cooling at 10 au ranges between 200 and more than 1000 K depending on the disc model. Only if the disc mass is a factor of 10 higher than that used by Boss (2004) is a Toomre instability possible. Because, even if the initial temperature of the condensed region is less than \( T_{\text{crit}} \), the cloud being more unstable, the rapid increase of temperature during gravitational contraction eventually compensates the increasing pressure.

The minimum disc surface density \( \Sigma_{\text{min}} \) and the critical temperature \( T_{\text{crit}} \) are given by (Rafikov 2005)

\[
\Sigma \geq \Sigma_{\text{min}} = \frac{1}{\Omega^2} \left[ \frac{k}{\mu} \right]^{3/5} \left[ f(\tau) \right]^{1/5} \quad (18)
\]

and

\[
T \geq T_{\text{crit}} = \frac{1}{\Omega^2} \left[ \frac{k}{\mu} \right]^{3/5} \left[ f(\tau) \right]^{2/5}, \quad (19)
\]

where \( Q \approx 1 \) is the upper Toomre limit for disc instability, \( \zeta = 2\xi(y - 1) \) is a parameter to be satisfied by the cooling time \( t_{\text{cool}} \) and the orbital frequency \( \Omega \), \( T_{\text{crit}} < \xi \sim 1 \). Note that our \( T_{\text{crit}} \) equals \( T_{\text{min}} \) in Rafikov (2005). \( \sigma \) is the Stefan–Boltzmann constant, \( k \) the Boltzmann constant, \( \mu \approx 2.3 m_H \) the molecular mass of the disc gas and

\[
f(\tau) = \tau + \frac{1}{\tau}, \quad (20)
\]

where \( \tau \) is the optical depth, which we estimate to be in the order of several tens in a compressed region. The value of \( \tau \) depends on the opacity and the surface density of the particular disc region. Given the model parameters used in this paper, an estimate of \( \tau = 60 \) of the compressed disc region and \( \xi = 1 \), conditions (18) and (19) require \( \Sigma \geq 630 \text{ g cm}^{-2} \) with \( T_{\text{crit}} = 330 \text{ K} \) at 30 au. Numerical simulations suggest \( \xi \approx 3 \) (Gammie 2001); in that case, the limits are \( \Sigma_{\text{min}} = 510 \text{ g cm}^{-2} \) and \( T_{\text{crit}} = 210 \text{ K} \). Assuming a tidal compression by a factor of 15 (which is a reasonable value in our simulations), the surface density condition is easily achieved in the compressed area, even with significantly lower disc masses than proposed by Boss (2004) or Mayer et al. (2004). The assumed optical thickness in these cases corresponds to a mean opacity \( \omega = \tau/\Sigma \approx 0.1 \text{ cm}^2 \text{ g}^{-1} \) of the disc material, which is a reasonable value for condensed regions in protoplanetary discs (Podolak 2004; Pollack, McKay & Christoferson 1985, for centimeter-sized dust grains).
Another problem that we do not treat in this paper is the transformation of the presumably unstable regions into (proto-)planets. In case of the formation of Neptune or Uranus, this includes the question of how to get rid of a large amount of gas with respect to the dust, by this ending up with the observed high mass fraction in solid material in these planets.

In principle, the following scenarios for the evolution of a gas-rich dust disc are possible.

A dust phase can partly decouple from the gas, resulting in a comparatively thin dust layer (Natta 2004; Barrièrè-Fouchet et al. 2005). Because the dust evolves basically pressure-free, it is much more prone to GI than the gas and, hence, instabilities in the dust phase could grow faster and form denser regions than the corresponding instabilities in the dynamically hotter gas. A plausible way of increasing the global solid-to-gas ratio is as follows: Jupiter and Saturn form traditionally within a few Myr. Meanwhile, the disc is partially photo-evaporated such that the dust-to-gas ratio increases significantly, at which time the fly-by perturbation occurs. From Fig. 5 we see that GIs can be induced for \( r_{\text{crit}} \leq 200 \) au, and from Fig. 1 it can be seen that the probability of such an encounter lies between 10 and 60 per cent if the Sun was born in a cluster with 1000 \(< N \leq 10 000\) low-mass stars (one encounter within \( \sim 10 \) Myr). In this scenario, the GIs would be induced in a dust-dominated disc (see Section 2.1).

A preferential loss of gas locally might also be possible when the clumps survive sufficiently long and mass segregation by gravity brings the dust to the centre of the clumps, thereby shielding the dust from a later loss by photo-evaporation. Boss (2004) suggested that the clumps can survive for a few dynamical periods, which might be sufficient for the required mass segregation. Moreover, mass segregation can be possibly sped up by vortices, as Klahr & Bodenheimer (2004) show. The solid component tends to settle rapidly in the centre of a vortex due to dynamical friction. Klahr & Bodenheimer (2003) describe the formation of vortices from baroclinic instabilities. However, we expect vortices to occur also in GIs if there is an initial spin angular momentum. The contracting GI will then be spun up by the Coriolis forces.

In a border case, the gas fraction may have been lowered due to normal depletion mechanisms during disc evolution. In the case that the dust may have already settled in or near the mid-plane, the horizontal centring due to vortices will result in an extreme concentration of dust within a small volume, thus skipping the otherwise necessary vertical contraction due to self-gravity.

We want to stress that the discussed possibilities for getting high dust mass fractions in the clumps (i.e. an initial value problem or an evolutionary process) cannot be decided on the basis of our simple simulations. A more refined multicomponent treatment of dust and gas as separate phases (e.g. as done by Theis & Orlova 2004, in the case of dusty galactic discs) would allow for investigating these in detail, but this would be part of a future investigation. However, we do believe that the regions we found unstable would be excellent large-scale seeds acting as birth places of (proto-)planets.

### 5.2 Orbits of candidate planetesimals

On a more speculative note, our simulations allow us to study the likely initial shapes of the orbits of candidate protoplanetary seeds, which are chosen here as the probes where \( Q < 1 \), indicating the possible formation of a planetary core. The fly-by distance is 100 au and two cases, a coplanar and an inclined orbit (\( \iota = 45^\circ \)), are investigated. The perturbations have their greatest impact on the region beyond 30 au, i.e. the Kuiper Belt region.

The distribution of the eccentricities \( e \) and semimajor axis \( a \) is shown in Fig. 9, where, from 10 000 test volumes initially, at radii between 20 and 100 au, only those are plotted for which \( Q < 1 \) occurred at least once.

It can be seen that the orbit of the KBO Sedna is not far outside the bulk of candidate seeds and is well contained in a less-dense peripheral region of candidate seeds for the coplanar case. It is still close to it in a 45° encounter. Also, Pluto is contained in this population, despite its resonance with Neptune, which most probably is the result of orbital evolution after the formation. However, Neptune lies below the candidate seed population but close to a trail of low-eccentric particles. Also due to its low eccentricity, Quaoar lies outside the population. The eccentricities in this plot range from 0.02 (nearly circular) to 1 (parabolic ejection) and would even exceed 1 if plotted over the perihelion distance rather than the semimajor axis. Even the recently announced large KBO 2003 UB313 fits this plot. However, the longitude of its ascending node deviates more from those of the other objects shown. Note that, if the present proposition is realistic, we expect our solar system to be void of KBOs below and above the region populated in Fig. 9.

We note that Ida et al. (2000) propose that the eccentric orbits of the KBOs may have formed as a result of tidal deformation of a pre-existing planetesimal disc. Our alternative notion is that the eccentric outer-solar-system objects may be a result of an encounter that lead to the formation of Uranus, Neptune and the KBOs.
5.3 Earlier SPH simulations of tilted discs

A more extensive study of the disc tilt due to stellar encounters than ours (Section 4.3) has been made by (Heller 1993), who used an SPH model with disc–star and disc–perturber interaction instead of restricted $N$-body computation. His results are essentially the same as in our study. Minor differences due to the absence of viscosity are possible. According to (Heller 1993), the ability of the disc to deform, which is maximal for non-interacting test particles, enhances the resulting disc tilt. The perturbation of the perturber–star motion due to disc–star interactions also alters the results, but because the effects are small in most cases (Heller 1993), they can be neglected here. The important factor causing Heller’s results on the tilt to be somewhat larger than ours is due to his large particle removal radius of 800 au, whereas we neglected particles with a semimajor axis larger than 150 au.

5.4 Limits and perspectives

The restricted $N$-body method has its great advantage in low computational efforts. This allows us to perform a wide parameter study within an acceptable computing time, but on the other hand this also limits the accuracy of the answers.

Because we do not model self-gravity, our approach is only reasonable if the mass of the disc is small compared with the solar mass, making the latter the dominant source of gravity. The same applies for the dynamical feedback on the orbit of the stars and the wobbling of the Sun (which would alter the density evolution of the disc) due to asymmetric disc perturbations.

The absence of hydrodynamical interactions limits the simulation time within which reasonable results for the density evolution can be expected. Furthermore, the effect of cooling mechanisms as mentioned by Boss (2002) and Boss (2004) can only be approximated by adjusting the adiabatic exponent $γ$.

Due to these restrictions, this work provides only a rough estimate of the expected magnitude of triggered instabilities. The evolution of the strong initial perturbations is determined mainly kinematically. So the results are reliable if only the immediate effects are of interest. As time proceeds, the evolution will be more and more determined by hydrodynamical and other interactions within the disc. However, this uncertainty mainly affects the borders of the candidate parameter subset, rather than the gravitational-triggering scenario as a whole.

Despite these limitations, our study gives us an important guideline for a more tightly focussed but, computationally, extremely expensive hydrodynamic research.

6 CONCLUSIONS

Using a simple, but fast numerical method, a parameter study has been made to test the possibility of gravitationally triggered planet formation. This scenario is supported by analytical estimates of the relatively high encounter probability in a typical young cluster (Fig. 1) as well as by the observed obliquity of the solar rotational axis being a possible consequence of an inclined fly-by.

As the main outcome of this study, we summarize the following results.

(i) Prograde passages of stars with masses of 0.1–1 $M_\odot$ can trigger Toomre instabilities within a 30-au radius in a disc of 0.07 $M_\odot$ if an approximately isothermal behaviour of compression is assumed. Toomre instabilities are also expected for adiabatic compression. Higher perturber masses and eccentricities increase the density perturbations and therefore allow for larger areas to become unstable.

(ii) Retrograde passages cause local $Q < 1$ for smaller $r_0$ (about half the value of the prograde case). Inclined passages yield $Q < 1$ for immediately close fly-bys.

(iii) A fly-by of a 0.5-$M_\odot$ star with an inclination between $30^\circ$ and $60^\circ$ and $r_0 \leq 100$ au tilts the disc plane by 3$^\circ$ to 6$^\circ$. This is very similar to the observed solar obliquity.

The duration of the locally triggered GI is in the range of centuries to millennia. Therefore, if these Toomre instabilities can lead to fragmentation and enhanced core formation, they may solve the discrepancy between the short disc lifetime and the long (classical) formation times of the outer planets and maybe even the KBOs. Even if only a fraction of the mass of Neptune today was assembled by triggering, this might be the required seed for the ongoing coagulation and accretion to be finished before the disc dissolves. The high solid-to-gas ratio of Uranus and Neptune may be explained in this scenario if their formation was induced by a fly-by at a solar-system age $\geq 5$ Myr when the disc was depleted of gas. In this situation, $Q_{\text{dust}} < Q$ such that the dust disc should be even more unstable towards GIs than suggested above. This sets possible constraints on the physical properties of the birth cluster of the Sun.

Our assumed disc mass is comparable to the minimum solar nebula mass. This avoids implausible assumptions on the disc mass as they are required for pure gravitational fragmentation of unperturbed discs.

ACKNOWLEDGMENTS

PK acknowledges support through DFG grant 1635/4–1. We thank Simon Goodwin for very useful discussions.

REFERENCES

Barrière-Fouchet L., Gonzalez J.-F., Murray J. R., Humble R. J., Maddison S. T., 2005, A&A, in press
Binney J., Tremaine S., 1987, Galactic dynamics. Princeton Univ. Press, Princeton, p. 184
Bonelll I. A., Smith K. W., Davies M. B., Horne K., 2001, MNRAS, 322, 859
Boss A. P., 2002, ApJ, 576, 462
Boss A. P., 2004, ApJ, 610, 456
Eggers S., Keller H. U., Kroupa P., Markiewicz W. J., 1997, Planet. Space Sci., 45, 1099
Gammie C. F., 2001, ApJ, 553, 174
Goldreich P., Lithwick Y., Sari R., 2004, ApJ, 614, 497
Haïns K. E., Lada E. A., Lada C. J., 2001, ApJ, 553, L153
Hayashi C., 1981, Prog. Theor. Phys. Supp., 70, 35
Heller C. H., 1993, ApJ, 408, 337
Hollenbach D. J., Yorke H. W., Johnstone D., 2000, in Mannings V., Boss A. P., Russell S. S., Protostars and Planets IV. Univ. Arizona Press, Tucson, p. 401
Ida S., Larwood J., Burkert A., 2000, ApJ, 528, 351
Klahr H., Bodenheimer P., 2003, ApJ, 582, 869
Klahr H., Bodenheimer P., 2004, in Garcia-Segura G., Tenorio-Tagle G., Franco J., Yorke H. W., eds, Rev. Mex. Astron. Astrofis. Conf. Ser. Vol. 22, Gravitational Collapse: From Massive Stars to Planets. Instituto de Astronomía Universidad Nacional Autónoma de México, Mexico City, p. 87
Kokubo E., Ida S., 2000, Icarus, 143, 15
Kroupa P., Aarseth S., Hurley J., 2001, MNRAS, 321, 699
Matsumoto M., Nishimura T., 1998, ACM Trans. Model. Comput. Simul., 8, 3
Mayer L., Quinn T., Wadsley J., Stadel J., 2004, ApJ, 609, 1045

© 2005 The Authors. Journal compilation © 2005 RAS, MNRAS 364, 961–970

Downloaded from https://academic.oup.com/mnras/article-abstract/364/3/961/1188753
by guest
on 29 July 2018
APPENDIX A: DENSITY EVALUATION

Calculation of the local Toomre disc stability requires the evaluation of the local surface density. At first, all disc particles are weighted by an individual mass \( m_b \) which corresponds to its initial orbit radius \( r_b \) in such a way that

\[
\sum_{i=1}^{N} m_b = M_d(r_1 \leq r \leq r_2),
\]

(A1)

where \( r_1 \) is the inner and \( r_2 \) the outer border of the simulated disc region. Because the \( N \) particles are initially positioned on a polar grid consisting of \( n_r \) rings with \( n_p \) particles each, which results in an \( r^{-1} \) law of the surface density \( \Sigma \), the mass weighting must be proportional to \( r^{-1/2} \) to fit the needed \( r^{-3/2} \) law (equation 12). Together with equation (A1), this leads to

\[
m(r_b) = \frac{r_b^{-1/2}}{\sqrt{r_2 - r_1}} \times \frac{\sqrt{r_2} - \sqrt{r_1}}{\sqrt{r_{\text{max}} - r_{\text{min}}}} \times M_d.
\]

(A2)

where \( r_{\text{min}} \) and \( r_{\text{max}} \) are the inner and outer radius of the real disc. Here, we use \( r_{\text{min}} = 3 \) au and \( r_{\text{max}} = 100 \) au.

The density evaluation is achieved by simply adding the \( m_b \) within 'counting volumes' surrounding massless probe particles that move along with the disc particles on ballistic (Keplerian) orbits. Within the counting volume, the density is calculated in SPH style using

\[
\langle \rho \rangle_w = \sum_{b=1}^{N} m_b W(d_b, h),
\]

(A3)

where \( d_b \) is the distance of the \( b \)th mass-weighted test particle from the centre of the probe particle. The kernel function \( W \) provides a smooth distance weighting of the test particles for reduction of numerical noise. In our model, the compact spline kernel from Monaghan & Lattanzio (1985) is used. The choice of the kernel radius \( h \) determines the mean number of particles \( n_b \) within the probe volume. For a random distribution, \( n_b \geq 100 \) is required to keep the noise low enough. A regular distribution, as we used here, shows appropriate results even for \( n_b \approx 10 \). For a given (average) \( n_b \) and disc population parameters \( r_1, r_2 \) and \( N \), we get \( h \) from

\[
h = \frac{n_b (r_1 + r_2)(r_2 - r_1)}{N}.
\]

(A4)

Thus, typically, \( h = 0.1 \) au for \( r_1 = 25 \) au, \( r_2 = 30 \) au and \( N = 250000 \).

Because we want to identify unstable regions, i.e. regions where \( Q < 1 \) locally, we applied about 1000 of these test volumes, which concentrate at the areas of the highest density. Their dynamic behaviour is the same as that of the disc test particles. Thus, a high spatial resolution at the areas of interest is automatically provided.

This paper has been typeset from a \TeX/\LaTeX\ file prepared by the author.

© 2005 The Authors. Journal compilation © 2005 RAS, MNRAS 364, 961–970