ON THE GEOMETRY OF MULTILINEAR FORMS

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Abstract. We develop a constructive process which determines all extreme points of the unit ball of the space of $m$-linear forms, $m \geq 1$. Our method provides a full characterization of the geometry of that space through finitely many elementary steps, and thus it can be extensively applied in both computational and theoretical problems.

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1. Introduction

Mathematical models involving multilinear forms are abundant in applied sciences, in particular multivariable polynomials represent an endless source of examples of such matter.

It is often that intrinsic difficulties in understanding multilinear problems are manifestations of the geometry complexity of the space of multilinear forms. As a way of example, we mention the problem of finding sharp constants in classical multilinear, convex inequalities. Routine applications of the Krein-Milman Theorem often reduces the candidate set to the extreme points, thus, genuine difficulties
in determining sharp constants heavily rely on the lack of understanding upon the geometry of the space of multilinear operators.

This is a critical issue resting in the core of pure and applied mathematical analysis. Previous works on this theme include [4, 9, 10]; however up-to-date, only problems involving low dimensions and/or low degrees have been successfully investigated; see also [3, 8, 13] for related issues. In this article we tackle the problem in full generality.

Let $B_{\mathbb{R}^n}$ denote the closed unit ball of $\mathbb{R}^n$, endowed with the sup norm. We denote the space of all $m$–linear forms $T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ by $L^m(\mathbb{R}^n)$. As usual, we equip this vector space with the norm

\[
\|T\| := \sup_{\|x_1\|,\ldots,\|x_n\| \leq 1} |T(x_1,\ldots,x_n)|.
\]

The closed unit ball of $L^m(\mathbb{R}^n)$ will be denoted by $K$, i.e.,

\[
K := \left\{ T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} : T \text{ is } m\text{-linear and } \|T\| \leq 1 \right\}.
\]

The key objective of this paper is to thoroughly characterize the geometry of $K$, by establishing all of its extreme points, henceforth denoted by $C_{m,n}$ or simply by $C$. We describe a procedure involving only finite elementary steps to determine $C$. A particularly interesting inference from this process is that the coordinates of $C$ are all rational points. In the sequel, we investigate optimization problems in classical real inequalities with the aid of our main characterization theorem. The examples included here have been influenced by the authors’ personal taste; however it is clear that our approach can be applied to a very large class of optimization problems.

The paper is organized as follows. In Section 2 we gather some preliminary tools and discuss notations to be used throughout the whole article. In Section 3 we obtain the main results of the paper, namely Theorem 13 and Theorem 15 which determine all extreme points of the closed unit ball of the space of $m$-linear forms in arbitrary dimensions. In Section 4 we discuss the algorithm inferred from the proofs delivered in the previous Section. Applications of the main results in the investigation of sharp constants in classical real inequalities are discussed in the last Section 5.

2. Preliminary results and notations

As previously commented, throughout the paper, $\mathbb{R}^n$ will always be equipped with the sup norm, unless mentioned otherwise. Following classical notations, given a matrix $M$, its transpose is denoted by $M^t$. The set $\{1,\ldots,n\}$ will be denoted by $[n]$. For $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ and $j = (j_1,\ldots,j_m) \in [n]^m$, we define

\[
x^j := \prod_{i=1}^m x_i^{(j_i)} \in \mathbb{R},
\]

where $x_i^{(j_i)}$ denotes the $j_i$-th coordinate of vector $x_i$. We also define

\[
\omega(x) := (x^j)_{j \in [n]^m} \in \mathbb{R}^{n^m},
\]

using the lexicographic order.

If $m, n$ are positive integers, let us set

\[
V_m^n := \{ \omega(x) : x = (x_1, x_2, \ldots, x_m) \text{ and } x_i \in \text{ext } (B_{\mathbb{R}^n}) \text{ for all } i \in [m] \}.
\]
Finally, we recall that given a vector space \( E \) and a convex set \( A \subseteq E \), a vector \( x \in A \) is said to be an extreme point of \( A \) if \( y, z \in A \) with \( x = (y + z)/2 \) implies \( y = z \). From now on \( \text{ext} (A) \) denotes the set of extreme points of \( A \).

2.1. **Bases of vertices of hypercubes.** We start off by proving some basic facts about \( \text{ext} (B_{\mathbb{R}^n}) \) that will be useful later.

**Lemma 1.** (Minkowski/Krein-Milman) If \( E \) is a locally convex space and \( K \) is a nonempty convex and compact subset of \( E \), then \( K \) has at least one extreme point and \( K = \text{conv}(\text{ext} K) \), where \( \text{ext} K \) is the set of all extreme points of \( K \) and \( \text{conv}(A) \) denotes the closed convex hull of \( A \).

**Lemma 2.** There exists a basis of \( \mathbb{R}^n \) composed by vectors from \( \text{ext} (B_{\mathbb{R}^n}) \).

**Proof.** This is a direct consequence of Krein-Milman Theorem. \( \square \)

**Lemma 3.** Let \( v_1, \ldots, v_m, u_1, \ldots, u_m \in \mathbb{R}^n \). Then
\[
\langle \omega(v), \omega(u) \rangle = \prod_{i=1}^{m} \langle v_i, u_i \rangle,
\]
where \( v = (v_1, \ldots, v_m) \) and \( u = (u_1, \ldots, u_m) \).

**Proof.** One simply notices that
\[
\langle \omega(v), \omega(u) \rangle = \sum_{k \in [n]^m} v^k u^k
= \sum_{k \in [n]^m} (v_1^{(k_1)} \cdots v_m^{(k_m)})(u_1^{(k_1)} \cdots u_m^{(k_m)})
= \sum_{k \in [n]^m} (v_1^{(k_1)} u_1^{(k_1)}) \cdots (v_m^{(k_m)} u_m^{(k_m)})
= \left( \sum_{k_1 \in [n]} v_1^{(k_1)} u_1^{(k_1)} \right) \cdots \left( \sum_{k_m \in [n]} v_m^{(k_m)} u_m^{(k_m)} \right)
= \prod_{i=1}^{m} \langle v_i, u_i \rangle.
\]

\( \square \)

**Proposition 4.** For all \( i \in [m] \), let
\[
\beta_i = \{v_{i,1}, \ldots, v_{i,n}\}
\]
be a set of non-null vectors in \( \mathbb{R}^n \). The following assertions are equivalent:

(i) \( \beta_i \) is a basis of \( \mathbb{R}^n \) for all \( i \in [m] \).

(ii) \( \Lambda_m(\beta_1, \ldots, \beta_m) := \{ \omega(x) : x = (x_i)_{i=1}^{m} \in \Pi_{i \in [m]} \beta_i \} \) is a basis of \( \mathbb{R}^m \).

**Proof.** (i) \( \Rightarrow \) (ii) For all \( i \in [m] \) there is a basis \( \gamma_i = \{u_{i,1}, \ldots, u_{i,n}\} \) of \( \mathbb{R}^n \) satisfying
\[
\langle v_{i,r}, u_{i,s} \rangle = \delta_{r,s}^i,
\]
where \( \delta_{r,s}^i \) is the Kronecker’s delta and \( r, s \in [n] \). Given \( i = (i_1, \ldots, i_m) \) and \( j = (j_1, \ldots, j_m) \in [n]^m \), consider
\[
v_j = (v_{1,j_1}, \ldots, v_{m,j_m})
\]
Proposition 7. Let \( \mathbb{R}^n \) be a basis of \( \mathbb{R}^n \).

By Lemma 3, we have

\[
\phi(u) := \prod_{j \in [m]} \delta_{j,i} \cdot \Pi_{i \in [m]} (v_{s,j_i}, u_{s,i}) = \langle \omega(v_j), \omega(u_i) \rangle.
\]

(ii) \( \Rightarrow \) (i) Suppose that (i) is not valid. Thus, there is \( j_0 \in [m] \) such that \( \beta_{j_0} \) is not a basis of \( \mathbb{R}^n \), i.e., there is a \( k_0 \in [n] \) such that

\[
v_{j_0,k_0} = \sum_{i \neq k_0} \alpha_i v_{j_0,i}
\]

for certain scalars \( \alpha_i, i \neq k_0 \). Therefore it is immediate that \( \Lambda_m(\beta_1, \ldots, \beta_m) \) is not composed by linearly independent vectors.

\[\square\]

Corollary 5. If \( \beta \) is a basis of \( \mathbb{R}^n \), then

\[\Lambda_m(\beta) := \Lambda_m(\beta, \ldots, \beta)\]

is a basis of \( \mathbb{R}^{mn} \).

From Lemma 2 and Corollary 5 we have the following consequence:

Corollary 6. There exists a basis of \( \mathbb{R}^{mn} \) contained in \( V_m^n \).

2.2. Some algebraic tools. We denote by \( O(n^m) \) the set of all orthogonal \( n^m \times n^m \) matrices. Given \( (c_i)_{i=1}^m \in \mathbb{R}^m \), we define \( \text{diag}(c_i)_{i=1}^m \) to be the \( m \times m \) diagonal matrix whose entries are \( c_i \). Let

\[G_m^n := \left\{ \text{diag}(x^j)_{j \in [n]^m} \in O(n^m) : x = (x_i)_{i=1}^m \text{ and } x_i \in \text{ext}(B_{R^n}) \text{ for all } i \in [m] \right\},\]

where \( x^j \) is as in \([2.2]\), and we still use the lexicographic order.

Proposition 7. Let \( m, n \) be positive integers. Then

(i) \( G_m^n \) is a subgroup of \( O(n^m) \);

(ii) The map \( \phi : G_m^n \times V_m^n \to V_m^n \) given by

\[\phi(g, \omega(x)) := \omega(x) \cdot g,\]

where \( x = (x_1, \ldots, x_m) \) and \( x_1, \ldots, x_m \in \text{ext}(B_{R^n}) \) is well defined;

(iii) \( \phi \) is a free (left) group action.

Proof. (i) The identity belongs to \( G_m^n \); in fact, we just need to consider \( x_1 = \cdots = x_m = (1, 1, \ldots, 1) \in \text{ext}(B_{R^n}) \). If \( g \in G_m^n \), then

\[g^{-1} = g^t = g \in G_m^n.\]

Now, let us show that \( G_m^n \) is closed under multiplication. Given \( g, h \in G_m^n \), there are \( x_1, \ldots, x_m, y_1, \ldots, y_m \in \text{ext}(B_{R^n}) \) such that

\[g = \text{diag}(x^j)_{j \in [n]^m} \text{ and } h = \text{diag}(y^j)_{j \in [n]^m}.\]

Define \( z_i = (x_i^s y_i^s)_{s \in [n]} \in \text{ext}(B_{R^n}) \), \( i = 1, \ldots, m \). Then

\[g \cdot h = \text{diag}(x^j y^j)_{j \in [n]^m} = \text{diag}(z^j)_{j \in [n]^m} \in G_m^n.\]

(ii) Now let us show that \( \phi \) is well defined, i.e., \( \phi \) does not depend on the representatives and \( \phi(G_m^n, V_m^n) \) is contained in \( V_m^n \).

Let us first show that \( \phi \) does not depend on the representatives. Suppose that \( g \in G_m^n \) is represented by

\[\text{diag}(a^j)_{j \in [n]^m} = \text{diag}(b^j)_{j \in [n]^m}\]
where \( a_1, \ldots, a_m, b_1, \ldots, b_m \in ext(B_{\mathbb{R}^n}) \) and \( x_1, \ldots, x_m, y_1, \ldots, y_m \in ext(B_{\mathbb{R}^n}) \) are such that
\[
\omega(x) = \omega(y).
\]
Then
\[
a^j_i = b^j_i \quad \text{and} \quad x^j_i = y^j_i
\]
for all \( j \in [n]^m \). Thus,
\[
\omega(x) \cdot \text{diag}(a^j_i)_{j \in [n]^m} = (x^j_i a^j_i)_{j \in [n]^m} = (y^j_i b^j_i)_{j \in [n]^m} = \omega(y) \cdot \text{diag}(b^j_i)_{j \in [n]^m}.
\]
We conclude that \( \phi \) does not depend on the representatives.

We will show that \( \omega(x) \cdot g \in V^n_m \), where \( x = (x_1, \ldots, x_m) \) and \( x_1, \ldots, x_m \in ext(B_{\mathbb{R}^n}) \). If
\[
g = \text{diag}(a^j_i)_{j \in [n]^m}
\]
with \( a_1, \ldots, a_m \in ext(B_{\mathbb{R}^n}) \), then
\[
\omega(x) \cdot g = (x^j_i a^j_i)_{j \in [n]^m} = \omega(y),
\]
with \( y_i = (a^j_i x^j_i)_{j \in [n]} \in ext(B_{\mathbb{R}^n}) \), and thus \( \omega(x) \cdot g \in V^n_m \).

(iii) Let us show that \( \phi \) is a group action. Let \( I \) be the identity of \( G^m_n \). Then
\[
\omega(x) \cdot I = \omega(x).
\]
Moreover, given \( g, h \in G^m_n \), then
\[
\phi(g \cdot h, \omega(x)) = \omega(x) \cdot (g \cdot h)
\]
\[= (\omega(x) \cdot g) \cdot h \]
\[= \phi(h, \omega(x) \cdot g) \]
\[= \phi(h, \phi(g, \omega(x))).
\]
Now let us show that \( \phi \) is a free action. Given \( x_1, \ldots, x_m, y_1, \ldots, y_m \in ext(B_{\mathbb{R}^n}) \), define
\[
g = \text{diag}(z^j_i)_{j \in [n]^m},
\]
where \( z_i = (x_i^{(s)} y_i^{(s)})_{s \in [n]} \). Thus
\[
\phi(g, \omega(x)) = \omega(x) \cdot g = \omega(y).
\]
\[\square\]

**Corollary 8.** Given \( u \in V^n_m \), there is a basis of \( \mathbb{R}^m_n \), \( \beta \subseteq V^m_n \), such that \( u \in \beta \).
More precisely, if \( \gamma \) is a basis of \( \mathbb{R}^m_n \) contained in \( V^m_n \), then
\[
\eta = \{ \phi(g, v) : v \in \gamma \}
\]
is a basis of \( \mathbb{R}^m_n \) contained in \( V^m_n \) for all \( g \in G^m_n \).

**Proof.** By Corollary 3 there is a basis \( g = \{v_1, \ldots, v_m\} \) of \( \mathbb{R}^m_n \) such that \( g \subseteq V^m_n \).
Since \( \phi : G^m_n \times V^m_n \to V^m_n \) is a free (left) group action, there exists a \( g_1 \in G^m_n \) such that
\[
v_1 \cdot g_1 = \phi(g_1, v_1) = u.
\]
Since
\[
v \cdot g = \phi(g, v) \in V^m_n
\]
for all \( g \in G^m_n \) and all \( v \in V^m_n \) and since \( g_1 \) is invertible,
\[
\beta := \{ v \cdot g_1 : v \in g \} = \{ \phi(g_1, v) : v \in g \} \subseteq V^m_n
\]
is a basis of \( \mathbb{R}^m_n \) and obviously contains \( u \). \[\square\]
3. The geometry of $\mathcal{L}^{(m \mathbb{R}^n)}$

The main results of this section are Theorem 13 and Theorem 15. They provide an elementary constructive characterization of the extreme points of the closed unit ball of $\mathcal{L}^{(m \mathbb{R}^n)}$.

3.1. The first main result. Given a multilinear form $T \in \mathcal{L}^{(m \mathbb{R}^n)}$, we can represent it as

$$T(y) = \sum_{i \in [n]^m} a_i y^i,$$

and thus

$$T(y) = \langle a^T, \omega(y) \rangle,$$

where

$$a^T = (a_i)_{i \in [n]^m}.$$

For the sake of simplicity we shall sometimes denote $T$ just by $a^T$. The following result is a straightforward consequence of the Krein-Milman Theorem:

**Proposition 9.** If $a^T \in \mathcal{L}^{(m \mathbb{R}^n)}$, then

$$\|a^T\| = \max \{|\langle a^T, \omega(x) \rangle| : x = (x_1, \ldots, x_m) \text{ and } x_1, \ldots, x_m \in \text{ext}(B_{\mathbb{R}^n})\}.$$

The following lemmata can be easily verified and thus its proof omitted.

**Lemma 10.** Let $V$ be a vector space of dimension $m < \infty$. If $\alpha = \{v_1, \ldots, v_k\}$ is a linearly independent set of $V$ with $k < m$ and $\beta = \{u_1, \ldots, u_m\}$ is a basis of $V$, then there exists a basis $\gamma$ of $V$ such that $\alpha \subseteq \gamma$ and $\gamma \setminus \alpha \subseteq \beta$.

**Lemma 11.** Let $V$ be a vector space. If $\Omega = \{v_1, \ldots, v_s\}$ is a set of non-null vectors in $V$, then there exists $\alpha \subseteq \Omega$ such that $\alpha$ is a maximal linearly independent set and

$$\Omega \subseteq \text{span}(\alpha).$$

We will also use the following observation, which we announce as a lemma for future reference:

**Lemma 12.** Let $v = (v_1, \ldots, v_n)$, $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. If

$$w = \frac{1}{2}(u + v),$$

then there is an $\alpha \in \mathbb{R}^n$ such that $u = w + \alpha$ and $v = w - \alpha$.

Next is our first main result, which gives an instrumental characterization of extreme points of the closed unit ball of the space of $m$-linear forms:

**Theorem 13.** Let $a^T \in B_{\mathcal{L}^{(m \mathbb{R}^n)}}$. The following assertions are equivalent:

(i) $a^T \in \text{ext} \left( B_{\mathcal{L}^{(m \mathbb{R}^n)}} \right)$

(ii) There exists $\beta \subseteq V_m^n$, basis of $\mathbb{R}^{n^m}$, such that $|\langle a^T, u \rangle| = 1$ for all $u \in \beta$.

**Proof.** We start off by proving (ii) implies (i). Let $\beta \subseteq V_m^n$ be a basis of $\mathbb{R}^{n^m}$, such that $|\langle a^T, u \rangle| = 1$ for all $u \in \beta$. From Lemma 12 it suffices to prove that given $b \in \mathbb{R}^{n^m}$ such that $a^T + b, a^T - b \in B_{\mathcal{L}^{(m \mathbb{R}^n)}}$, we have $b = 0$. If $a^T + b, a^T - b \in B_{\mathcal{L}^{(m \mathbb{R}^n)}}$, we have

$$|\langle a^T + b, u \rangle| \leq 1$$
and

\[ |\langle a^T - b, u \rangle| \leq 1 \]

for all \( u \in \beta \). Since \( |\langle a^T, u \rangle| = 1 \) for all \( u \in \beta \), we have

\[ \langle b, u \rangle = 0 \]

for all \( u \in \beta \); therefore \( b = 0 \).

Now let us prove that (i) implies (ii). Let us suppose, for the sake of contradiction, that for all \( \beta \subseteq V_m^n \), basis of \( \mathbb{R}^{n^m} \), there is \( u_\beta \in \beta \) such that \( |\langle a^T, u_\beta \rangle| < 1 \).

Note that \( \Omega := \{ u \in V_m^n : |\langle a^T, u \rangle| = 1 \} \) does not contain any basis \( \beta \subseteq V_m^n \) of \( \mathbb{R}^{n^m} \) and \( \text{card}(\Omega) < \infty \). Suppose that \( \Omega \neq \emptyset \).

Then, by Lemma 11 there is \( \alpha = \{ \eta_1, \ldots, \eta_k \} \subseteq \Omega \), a maximal linearly independent set, such that

\[ \Omega \subseteq \text{span} (\alpha) . \]

By Lemma 10 and Corollary 6 there is a basis \( \gamma = \{ \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_{n^m-k} \} \) of \( \mathbb{R}^{n^m} \) contained in \( V_m^n \), such that

\[ \xi_j \in V_m^n \setminus \Omega \]

for all \( j = 1, \ldots, n^m-k \). In fact, if there were \( \xi_j \in \Omega \), then \( \xi_j \in \text{span}(\alpha) \) and \( \gamma \) would be linearly dependent. Let

\[ V_m^n \setminus \Omega := \{ \zeta_1, \ldots, \zeta_s \} . \]

For all \( i \in [s] \), there is a \( r_i > 0 \) such that

\[ -1 < \langle a^T, \zeta_i \rangle - r_i < \langle a^T, \zeta_i \rangle < \langle a^T, \zeta_i \rangle + r_i < 1 . \]

Defining \( r = \min \{ r_i : i \in [s] \} \), we have

\[ -1 < \langle a^T, \zeta_i \rangle - r < \langle a^T, \zeta_i \rangle < \langle a^T, \zeta_i \rangle + r < 1 . \]

For all \( i \in [s] \), there exist unique real scalars \( p_j^\zeta_i \) and \( l_j^\zeta_i \) such that

\[ \zeta_i = \sum_{j=1}^{n^m-k} p_j^\zeta_i \xi_j + \sum_{j=1}^k l_j^\zeta_i \eta_j . \]

Define

\[ p := \max \{ |p_j^\zeta_i| : i = 1, \ldots, s \} . \]

Since \( \xi_1 \in V_m^n \setminus \Omega \), it follows that \( p \geq 1 \). Since \( \gamma \) is a basis, there is a \( 0 \neq b \in \mathbb{R}^{n^m} \) such that

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_{n^m-k} \\
\eta_1 \\
\vdots \\
\eta_k
\end{pmatrix}
= 
\begin{pmatrix}
\frac{r}{p} \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]
Thus
\[ \langle a^T + b, \zeta_i \rangle = \langle a^T, \zeta_i \rangle \pm \langle b, \zeta_i \rangle = \langle a^T, \zeta_i \rangle \pm \langle b, \sum_{j=1}^{p} \zeta_i_j \eta_j + \sum_{j=1}^{k} \zeta_i_j \xi_j \rangle = \langle a^T, \zeta_i \rangle \pm \langle b, \sum_{j=1}^{p} \zeta_i_j \xi_j \rangle \]
\[ = \langle a^T, \zeta_i \rangle \pm \langle b, \sum_{j=1}^{l} \zeta_i_j \eta_j \rangle = \langle a^T, \zeta_i \rangle \pm \langle b, \gamma_i \rangle = \langle a^T, \zeta_i \rangle \pm p \gamma_i \eta_i / p. \]

Therefore
\[ |\langle a^T + b, \zeta_i \rangle| = |\langle a^T, \zeta_i \rangle | \leq |\langle a^T, \gamma_i \rangle | < 1. \]

If \( \nu \in \Omega \), then
\[ |\langle a^T + b, \nu \rangle | = |\langle a^T, \nu \rangle | = 1, \]
because \( \nu \in \text{span}(\alpha) \). We thus conclude that \( a^T + b, a^T - b \in B_{L(m \mathbb{R}^n)}. \) Since
\[ a^T = \frac{1}{2} (a^T + b + a^T - b), \]
it follows that \( a^T \) is not an extreme of \( B_{L(m \mathbb{R}^n)}. \)

If \( \Omega = \emptyset \), then \( a^T \) is an interior point of \( B_{L(m \mathbb{R}^n)}, \) and the proof is complete. \( \square \)

3.2. The second main result. Let
\[ \mathcal{B} = \{ \beta_1, \ldots, \beta_s \} \]
be the set of all basis of \( \mathbb{R}^{n_m} \) such that \( \beta_j \subseteq V_m^n \) for all \( j \) and \( \omega(e, e, \ldots, e) \in \beta_j \) for all \( j \), where \( e = e_1 + \ldots + e_n \). By Corollary we have \( \mathcal{B} \neq \emptyset \). For all \( i \in [s] \), define the matrix \( H_{\beta_i} \) whose lines are the vectors of \( \beta_i \). For instance, if
\[ \beta_i = \{ v_{i,1}, \ldots, v_{i,n_m} \}, \]
then
\[ H_{\beta_i} = \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,n_m} \end{array} \right) \]
is an \( n_m \times n_m \) matrix. Consider, for all \( i \in [s] \) and all \( f \in \text{ext} (B_{L(m \mathbb{R}^n)}) \), the sets
\[ \mathcal{A}_{i,f} = \{ a^T : H_{\beta_i}(a^T)^t = f^t \} \]
and
\[ \mathcal{A} = \bigcup_{i \in [s], f \in \text{ext}(B_{L^n})} \mathcal{A}_{i,f}. \]

Note that
\[ \text{card} (\mathcal{A}) \leq \text{card} (\mathcal{B}) \cdot 2^{n_m} < \infty. \]

Define, for all \( g \in C_{m}^{n_m}, \)
\[ \mathcal{C}_g = \{ a^T \cdot g : a^T \in \mathcal{A}, \ |\langle a^T, \nu \rangle | \leq 1 \ \forall \nu \in V_m^n \} \]
and

\[ C = \bigcup_{g \in G^n_m} C_g. \]

Note also that

\[ \text{card}(C) \leq \text{card}(A) \cdot \text{card}(G^n_m) < \infty. \]

**Lemma 14.** Let \( a^T = (a_i)_{i \in [n]^m}, \ g \in G^n_m \) and \( \omega(x) \) with \( x = (x_1, \ldots, x_m) \) and \( x_1, \ldots, x_m \in \text{ext}(B_{\mathbb{R}^n}) \). Then

\[ \langle a^T \cdot g, \omega(x) \rangle = \langle a^T, \phi(g, \omega(x)) \rangle. \]

**Proof.** Since \( g \in G^n_m \), there are \( b_1, \ldots, b_m \in \text{ext}(B_{\mathbb{R}^n}) \) such that

\[ g = \text{diag}(b_j)_{j \in [n]^m}. \]

Thus,

\[ a^T \cdot g = (a_i)_{i \in [n]^m} \cdot \text{diag}(b_j)_{j \in [n]^m} = (a_j b_j)_{j \in [n]^m}. \]

Therefore

\[ \langle a^T \cdot g, \omega(x) \rangle = \sum_{j \in [n]^m} (a_j b_j) x_j = \sum_{j \in [n]^m} a_j (x_j b_j) = \langle (a_j)_{j \in [n]^m}, (x_j b_j)_{j \in [n]^m} \rangle = \langle a^T, \omega(x) \cdot g \rangle = \langle a^T, \phi(g, \omega(x)) \rangle. \]

Next theorem is our second main result of this section:

**Theorem 15.** \( \text{ext}(B_{\mathbb{L}^m_{\mathbb{R}^n}}) = C. \)

**Proof.** Let us first show that \( \text{ext}(B_{\mathbb{L}^m_{\mathbb{R}^n}}) \subseteq C \). If \( a^T \in \text{ext}(B_{\mathbb{L}^m_{\mathbb{R}^n}}) \), then by Theorem 13 there exists \( \beta = \{v_1, \ldots, v_n^m\} \subseteq V^n_m \), basis of \( \mathbb{R}^n_m \), such that

\[ |\langle a^T, v \rangle| = 1 \ \forall \ v \in \beta. \]

Let \( H \) be the matrix whose lines are the vectors of \( \beta \) and let

\[ f = (\langle a^T, v_1 \rangle, \ldots, \langle a^T, v_n^m \rangle) \in \text{ext}(B_{\mathbb{R}^n^m}). \]

Then

\[ H \cdot (a^T)^t = f^t. \]

Since \( \phi : G^n_m \times V^n_m \to V^n_m \) is a free action, there is a \( g \in G^n_m \) such that \( \phi(g, v_1) = \omega(e, e, \ldots, e) \). Then, still using the notation introduced in (3.1), by Corollary 8 we have

\[ \{\phi(g, v) : v \in \beta\} = \beta_j \]

for a certain \( j \). Therefore

\[ H_{\beta_j} = \begin{pmatrix} \phi(g, v_1) \\ \vdots \\ \phi(g, v_n^m) \end{pmatrix} = H \cdot g. \]
Let $x$ be solution of
\[ H_{\beta_j} \cdot x^t = f^t. \]
Hence
\[ x^t = H_{\beta_j}^{-1} \cdot f^t \]
\[ = g^{-1} \cdot (H^{-1} \cdot f^t) \]
\[ = g \cdot (a^T)^t. \]
Therefore,
\[ (3.4) \quad a^T \cdot g = x \in A_{j,f} \subseteq A. \]
Given $y = (y_1, \ldots, y_m)$ with $y_1, \ldots, y_m \in \text{ext}(B_{\mathbb{R}^n})$, by Lemma 14, we have
\[ (3.5) \quad |\langle x, \omega(y) \rangle| = |\langle a^T \cdot g, \omega(y) \rangle| = |\langle a^T, \phi(g, \omega(y)) \rangle| \leq 1. \]
By Proposition 9 we have $x \in B_{\mathbb{L}(\mathbb{R}^n)}$ and finally, by (3.4) and (3.5) we get
\[ a^T = x \cdot g \in C_g \subseteq C, \]
i.e.,
\[ \text{ext} (B_{\mathbb{L}(\mathbb{R}^n)}) \subseteq C. \]
Now, let us show that $C \subseteq \text{ext} (B_{\mathbb{L}(\mathbb{R}^n)})$. If $x \in C$, then $x = a^T \cdot g$ with $a^T \in A$ and $g \in G^n_m$, where $|\langle a^T, v \rangle| \leq 1$ for all $v \in V^n_m$. There are $j$ and $f$ such that $a^T \in A_{j,f}$ and there is $\beta_j \in B$ such that $|\langle a^T, u \rangle| = 1$ for all $u \in \beta_j$. Since $g$ is invertible then
\[ \beta := \{ \phi(g, u) : u \in \beta_j \} \]
is a basis of $\mathbb{R}^{n^m}$. Then, for all $u \in \beta_j$, by Lemma 14, we have
\[ |\langle x, \phi(g, u) \rangle| = |\langle (a^T \cdot g), \phi(g, u) \rangle| = |\langle a^T, \phi(g \cdot g, u) \rangle| = |\langle a^T, u \rangle| = 1. \]
Since $|\langle a^T, v \rangle| \leq 1$ for all $v \in V^n_m$, using the same argument, by Lemma 14 we have
\[ |\langle x, \phi(g, v) \rangle| = |\langle (a^T \cdot g), \phi(g, v) \rangle| = |\langle a^T, \phi(g \cdot g, v) \rangle| = |\langle a^T, v \rangle| \leq 1 \]
for all $v \in V^n_m$ and hence $x \in B_{\mathbb{L}(\mathbb{R}^n)}$. By Theorem 13 we conclude that $x \in \text{ext} (B_{\mathbb{L}(\mathbb{R}^n)})$. □

**Corollary 16.** For all positive integers $m, n$, the coefficients of the extreme points $T \in B_{\mathbb{L}(\mathbb{R}^n)}$ are rational numbers.

**Proof.** Note that we start off with an $n^m \times n^m$ matrix whose entries are 1 or $-1$. We solve a linear system whose independent terms are 1 or $-1$. The extreme points are found among these solutions, and obviously all of its coordinates are rational numbers. □

### 4. Constructive process

An easily implemented algorithm can be extracted from the proofs delivered in the previous two sections. Below we summarize how to find all extreme points of the closed unit ball of $B_{\mathbb{L}(\mathbb{R}^n)}$:...
Step 1: Determinate all \( n^m \times n^m \) invertible matrices whose lines belong to \( V_m^n \), that contain \( \omega(e, \ldots, e) \). Note that using the notations from (3.1) and (3.2) the set of such matrices is

\[
\mathcal{M} = \{ H_{\beta_i} : \beta_i \in \mathcal{D} \subseteq B \}
\]

for a certain \( \mathcal{D} \).

Step 2: For all choices of \( f \in \text{ext}(B_{\mathbb{R}^n}) \) and each matrix \( H_{\beta_i} \) collected in Step 1, solve the linear system

\[
H_{\beta_i} \cdot (a^T)^t = f^t.
\]

Step 3: Among all solutions given by the second step, verify which solutions also satisfy

\[
|\langle a^T, v \rangle| \leq 1
\]

for all \( v \in V_m^n \).

Step 4: Among all solutions given by the third step, calculate

\[
a^T \cdot g
\]

for all \( g \in G_m^n \). The set of all such \( a^T \cdot g \) is precisely the set of all extreme points of \( B_{\mathcal{L}(\mathbb{R}^n)} \).

4.1. **Examples.** As mentioned earlier, previous knowledge on extreme points of the unit ball in the space of multilinear forms were limited to low dimensions and/or low degrees. The simplest case, \( n = m = 2 \), appears in the work of S.G. Kim, [14], and accordingly can be obtained by our method.

**Example 17.** All extreme points of \( B_{\mathcal{L}(\mathbb{R}^2)} \) are:

\[
\pm (0,0,0,1), \pm \frac{1}{2}(1,1,1,-1), \pm \frac{1}{2}(1,1,-1,1), \pm \frac{1}{2}(1,-1,1,1), \\
\pm \frac{1}{2}(-1,1,1,1), \pm (0,0,1,0), \pm (0,1,0,0), \pm (1,0,0,0).
\]

For 3-forms and 4-forms, though, very little, if anything, were previously known, even restricted to the plane. Here are some illustrative examples:

**Example 18.** The following vectors are extreme points of \( B_{\mathcal{L}(\mathbb{R}^3)} \):

\[
\pm (1,0,0,0,0,0,0,0), \pm \frac{1}{2}(1,-1,1,-1,1,1,1), \pm \frac{1}{2}(0,0,0,0,-1,1,1,1).
\]

All extreme points of \( B_{\mathcal{L}(\mathbb{R}^3)} \) can be found through the algorithm above described.

**Example 19.** Here are some extreme points of \( B_{\mathcal{L}(\mathbb{R}^4)} \):

\[
\pm (1,0,0,0,0,0,0,0,0,0,0,0,0,0), \\
\pm \frac{1}{8}(-1,1,1,-1,1,1,1,1,1,1,1,1,1,1), \\
\pm \frac{1}{4}(0,1,0,1,0,-1,0,1,0,-1,0,1,0,1,0), \\
\pm \frac{1}{8}(1,1,1,-3,-1,-1,-1,3,1,1,1,1,1,1,1,5), \\
\pm \frac{1}{2}(0,0,0,-1,0,0,0,1,0,0,1,0,0,0,1), \\
\pm \frac{1}{4}(0,0,-1,-1,1,-1,0,2,0,0,1,1,-1,1,0,2).
\]

Again, the complete list of extreme points of \( B_{\mathcal{L}(\mathbb{R}^4)} \) can be found through the algorithm above described.
4.2. The planar case. In the special case, \( n = 2 \), we have
\[
\|a^T\| = \max \left\{ |\langle a^T, \omega(x_1, \ldots, x_m) \rangle| : x_1, \ldots, x_m \in \{ (1, 1), (-1, 1) \} \right\},
\]
for any arbitrary integer \( m \). Let \( x = (x_i)_{i=1}^m \) and \( y = (y_i)_{i=1}^m \) be such that
\[
x_i, y_i \in \{ (1, 1), (-1, 1) \}
\]
for all \( i \in [m] \). Since \( x \neq y \), there exists a \( j_0 \in [m] \) such that \( x_{j_0} \neq y_{j_0} \). Thus, by Lemma 1, we have
\[
\langle \omega(x), \omega(y) \rangle = \Pi_{i=1}^m \langle x_i, y_i \rangle = 0.
\]
So, we can determinate the extreme points of \( B_{L^2_{\mathbb{R}^2}} \) as follows:

**Step 1:** Build the matrix \( H \) such that the lines are the values of \( \Lambda_2(\beta) \), where \( \beta = \{ (1, 1), (-1, 1) \} \).

**Step 2:** For each matrix \( f \in \text{ext}(B_{L^2_{\mathbb{R}^n}}) \), solve the linear system
\[
Hx^t = f^t.
\]

From the above routine we have the following result:

**Proposition 20.** For all positive integer \( m \) we have
\[
\text{card}(\text{ext}(B_{L^2_{\mathbb{R}^2}})) = 2^{(2^m)}.
\]

5. Applications: optimization problems in classical inequalities

In this section we briefly discuss the fit of our main characterization theorems within investigations pertaining to classical inequalities. Of particular interest, we formally solve the open problem of determining all optimal constants of the \( m \)-linear Bohnenblust–Hille inequalities for real scalars.

We start off with two observations, which we state as propositions for future references. The former is a straightforward consequence of the Krein-Milman Theorem (Lemma 1) and Theorem 15:

**Proposition 21.** Let \( f : B_{L^2_{\mathbb{R}^n}} \to \mathbb{R} \) be a convex and continuous function. Then
\[
\max_{T \in B_{L^2_{\mathbb{R}^n}}} f(T) = \max \{ f(T) : T \in C \}.
\]

The next result is also useful for computational purposes:

**Proposition 22.** Let \( 1 \leq \lambda < \infty \). If \( f_\lambda : B_{L^2_{\mathbb{R}^n}} \to \mathbb{R} \) is
\[
f_\lambda(T) = \left( \sum_{i \in [n]^m} |T(e_{i_1}, \ldots, e_{i_m})|^\lambda \right)^{1/\lambda},
\]
then
\[
\max_{T \in B_{L^2_{\mathbb{R}^n}}} f_\lambda(T) = \max \{ f_\lambda(T) : T \in A \cap B_{L^2_{\mathbb{R}^n}} \},
\]
where \( A \) is given by (3.3).

**Proof.** By Proposition 21 we know that
\[
\max_{T \in B_{L^2_{\mathbb{R}^n}}} f_\lambda(T) = \max \{ f_\lambda(T) : T \in C \},
\]
i.e., there is a \( T_0 \in C \) such that
\[
\max_{T \in B_{L^2_{\mathbb{R}^n}}} f_\lambda(T) = f_\lambda(T_0).
\]
Since \( T_0 = a^S \cdot g \) for some \( a^S \in A \) and \( g \in G^n_m \), and
\[
|\langle a^S, v \rangle| \leq 1
\]
for all \( v \in V^n_m \). We have
\[
\max \{ f_\lambda(T) : T \in A \cap B_{L(R^n)} \} \leq \max \{ f_\lambda(T) : T \in C \}
\]
\[
= f_\lambda(T_0)
\]
\[
= f_\lambda(a^S \cdot g)
\]
\[
= f_\lambda(a^S)
\]
\[
\leq \max \{ f_\lambda(T) : T \in A \cap B_{L(R^n)} \}.
\]
\[\Box\]

5.1. Classical multilinear inequalities: sharp values. Let \( K = \mathbb{R} \) or \( \mathbb{C} \). The (classical) Bohnenblust–Hille inequality, [2], asserts that for all \( m \)-linear forms \( T: \mathbb{K}^n \times \cdots \times \mathbb{K}^n \to \mathbb{K} \) and all positive integers \( n \),
\[
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{2m} \right)^{\frac{m+1}{2m}} \leq B^\mathbb{K}_m(n) \| T \|,
\]
for an optimal constant \( B^\mathbb{K}_m(n) \geq 1 \), and
\[
B^\mathbb{K}_m(\infty) := \sup_n B^\mathbb{K}_m(n) < \infty.
\]

From Proposition 21 we have the following formula for the optimal constants \( B^\mathbb{R}_m(n) \):
\[
B^\mathbb{R}_m(n) = \max \{ T \in C_{m,n} \},
\]
where \( C_{m,n} \) is the (finite) set created by the elementary constructive process of Section 4.

When \( m = 2 \), inequality (5.1) recovers the famous Littlewood’s 4/3 inequality, and it is well known that \( B^\mathbb{R}_2(\infty) = \sqrt{2} \). For \( m \geq 3 \), the precise values of sharp constants \( B^\mathbb{R}_m(\infty) \) remain unknown, despite of their intrinsic applications in the case of real scalars, see [14].

It follows from (5.3), however, that given two positive integers \( m, n \) the precise value of \( B^\mathbb{R}_m(n) \) can be fully determined and formally computed by the constructive method earlier described after a finite number of elementary steps. The same can be done for any similar inequalities, like the mixed Littlewood-type inequalities.

5.2. Classical multilinear inequalities: algebraic properties. It is appealing to observe that, since the coordinates of extreme points of \( B_{L(\infty)} \) are rational numbers, we can easily conclude that:

**Proposition 23.** For all positive integers \( m, n \), the optimal constants \( B^\mathbb{R}_m(n) \) are algebraic numbers.

The above result cannot be straightforwardly extended to the case \( n = \infty \), i.e., we cannot conclude \( B^\mathbb{R}_m(\infty) \) are algebraic numbers. In what follows though, we will show that when considering \( m \)-linear forms defined over \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \) with \( n \geq 2^{m-1} \) the sharp constants are indeed algebraic, and equal \( 2^{1 - \frac{m}{2}} \). This result provides a partial solution to the question of whether the constants \( B^\mathbb{R}_m(\infty) \)
are algebraic or not. Our proof is based on a different set of tools, which includes
the Mixed Littlewood inequality and the Khinchin inequality; we recall them here
for the sake of the readers:

**Mixed Littlewood inequality.** For all continuous \((m + 1)\)-linear forms \(T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}\) and for all positive integers \(n\), we have

\[
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{1}{2}} \right)^{\frac{m+1}{2m}} \leq M_{m+1}(n) \|T\|
\]

(5.4)

and

\[
M_{m+1} := \sup_n M_{m+1}(n) < \infty.
\]

**Khinchin inequality.** (see [5]). For any \(0 < q < \infty\), there are positive constants
\(A_q, B_q\) such that

\[
A_q \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left| \sum_{j=1}^{n} a_j r_j(t) \right|^q dt \right)^{\frac{1}{q}} \leq B_q \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}},
\]

for any positive integer \(n\) and sequence of scalars \((a_j)_{j=1}^{n}\). Here \(r_j\) denote the
Rademacher functions. The best constants \(A_q\) are (see [5]):

\[
A_q = \sqrt{2} \left( \frac{\Gamma \left( \frac{1+q}{2} \right)}{\sqrt{\pi}} \right)^\frac{1}{q} \quad \text{if } 2 > q \geq q_0 \cong 1.8474;
\]

\[
A_q = 2^{\frac{1}{2}-\frac{1}{q}} \quad \text{if } q < q_0.
\]

The number \(q_0\) above is the unique real scalar satisfying \(\Gamma \left( \frac{m+1}{2} \right) = \sqrt{\pi} \).

**Lemma 24.** Let \(m \geq 1\) and \(n \geq 2\) be positive integers. For all continuous \((m + 1)\)-
linear forms \(T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}\) we have

\[
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{1}{2}} \right)^{\frac{m+1}{2m}} \leq 2^{\frac{1}{2m}} B_m(n) \|T\|
\]

(5.5)

and the constant \(2^{\frac{1}{2m}} B_m(n)\) is sharp.

**Proof.** The inequality

\[
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{}{m}} \right)^{\frac{m+1}{2m}} \leq A^{-1}_{m+1} B_m(n) \|T\|
\]

is a straightforward consequence of the Khinchin inequality; here \(A_{m+1}\) are the
associated constants of the Khinchin inequality. Since for any \(1 \leq p \leq 2\) the
maximum of

\[
f(a, b) = \frac{(a^2 + b^2)^{1/2}}{(\frac{1}{p}|a + b|^p + \frac{1}{q}|a - b|^q)^{1/p}}
\]

is
is $2^{\frac{1}{n}-\frac{1}{2}}$, in our case the constants of the Khinchin inequality can be taken as $\frac{\log m + 1}{2}$, i.e., $2^{\frac{1}{n}}$ (recall that we are dealing with continuous $m + 1$–linear forms $T : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$). Thus

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} \left( \sum_{j_{m+1}=1}^{2} |T(e_{j_1, \ldots, e_{j_m})|^2 \right)^{\frac{1}{m+1}} \right)^{\frac{m+1}{m}} \leq 2^{\frac{1}{m}} B_m(n) \|T\|.$$ 

We just need to prove that the constant $2^{\frac{1}{m}} B_m(n)$ is sharp.

From now on, for any continuous $m$–linear form $T_m : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ we define

$$\widetilde{T}_m(x^{(1)}, \ldots, x^{(m)}) = T_m(z^{(1)}, \ldots, z^{(m)}),$$

$$\widetilde{T}^*_m(x^{(1)}, \ldots, x^{(m)}) = T_m(u^{(1)}, \ldots, u^{(m)}),$$

where, for all $k = 1, \ldots, m$, we consider

$$z^{(k)} = (x_1^{(k)}, x_3^{(k)}, x_5^{(k)}, \ldots),$$

$$u^{(k)} = (x_2^{(k)}, x_4^{(k)}, x_6^{(k)}, \ldots).$$

Note that

$$\|\widetilde{T}_m\| = \|\widetilde{T}^*_m\| = \|T_m\|.$$

Let $\varepsilon > 0$ and $T_m : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be such that

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} |T_m(e_{j_1, \ldots, e_{j_m})|^2 \right)^{\frac{1}{m+1}} > (B_m^\mathbb{R}(n) - \varepsilon) \|T_m\|. \tag{5.7}$$

Define the $m + 1$–linear operator $R_{m+1} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ by

$$R_{m+1}(x^{(1)}, \ldots, x^{(m+1)}) = \left( x_2^{(m+1)} - x_1^{(m+1)} \right) \widetilde{T}_m(x^{(1)}, \ldots, x^{(m)}) + \left( x_2^{(m+1)} + x_1^{(m+1)} \right) \widetilde{T}^*_m(x^{(1)}, \ldots, x^{(m)}).$$

By the definition of $R_{m+1}$, we have

$$\|R_{m+1}\| = \|2T_m\|$$

and we can also note that for all $e_{j_1}, \ldots, e_{j_m}$, we have

$$|R_{m+1}(e_{j_1}, \ldots, e_{j_m}, e_1)| = |R_{m+1}(e_{j_1}, \ldots, e_{j_m}, e_2)|.$$

Since we are using just two coordinates of the last variable and since, for any $1 \leq p \leq 2$, the maximum of

$$f(a, b) = \frac{(a^2 + b^2)^{1/2}}{(\frac{1}{p} |a + b|^p + \frac{1}{p} |a - b|^p)^{1/p}}$$
is $2^{\frac{1}{2n}-\frac{1}{2n}}$ (it is attained when $|a| = |b| > 0$), we have

$$
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{2} |R_{m+1}(e_{j_1}, \ldots, e_{j_{m+1}})|^2 \right)^{\frac{1}{2}} \right)^{\frac{m+1}{2m}}
$$

$$
= 2^{\frac{m+1}{2m}} \left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{2} |R_{m+1}(e_{j_1}, \ldots, e_{j_{m+1}}, \varepsilon_1 \varepsilon_{1} + \varepsilon_2 e_2)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \right)\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{2} |R_{m+1}(e_{j_1}, \ldots, e_{j_{m+1}}, \varepsilon_1 \varepsilon_{1} + \varepsilon_2 e_2)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \right)^{-1}.
$$

It is obvious that both $2\widehat{T}_m$ and $2\widecheck{T}_m$ also satisfy \eqref{5.7}. Thus

$$
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{2} |R_{m+1}(e_{j_1}, \ldots, e_{j_{m+1}})|^2 \right)^{\frac{1}{2}} \right)^{\frac{m+1}{2m}}
$$

$$
> 2^{\frac{1}{2m}} \left( \frac{1}{2} (\|B^R_m(n) - \varepsilon\|2T_m) \right)^{\frac{2m}{m+1}} + \frac{1}{2} \left( (\|B^R_m(n) - \varepsilon\|2T_m) \right)^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}}
$$

$$
= 2^{\frac{1}{2m}} \left( \frac{1}{2} (\|B^R_m(n) - \varepsilon\|2T_m) \right)^{\frac{2m}{m+1}} + \frac{1}{2} \left( (\|B^R_m(n) - \varepsilon\|2T_m) \right)^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}}
$$

$$
= 2^{\frac{1}{2m}} (\|B^R_m(n) - \varepsilon\|2T_m).
$$

Letting $\varepsilon \to 0$ we thus conclude that $2^{\frac{1}{2m}}B^R_m(n)$ is sharp.

Suppose that now we have $m$-linear forms defined in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ with $n \geq 2m-1$. The proof that the sharp constants are $2^{\frac{1}{2n}-\frac{1}{2n}}$ is now a straightforward consequence of the Hölder inequality for mixed sums combined with \eqref{5.5} and the following simple inequality:

\begin{equation}
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left( \sum_{j_{m+1}=1}^{2} |T(e_{j_1}, \ldots, e_{j_{m+1}})|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq \sqrt{2} \|T\|
\end{equation}

for all $T : \mathbb{R}^n \times \cdots \times \mathbb{R}^{n-1} \times \mathbb{R}^2 \to \mathbb{R}$. Considering the strongly non-symmetric $m$-linear forms used in the proof of \cite[Theorem 4.1]{LS} we easily prove that the estimates are sharp.

### 5.3. The case of complex scalars

The case of the optimal Bohnenblust–Hille constants for complex scalars is obviously not encompassed by the previous techniques. The main point is that the geometry of the closed unit ball $B_{\ell^{(m C^n)}}$ is rather different and essentially unknown. In this subsection, however, we tackle R. Blei’s problem concerning sharp estimates for complex inequalities; more precisely, Orlicz’s, Littlewood’s $(\ell_1, \ell_2)$, and Littlewood’s $4/3$ inequalities:
For each positive integer \( n \), following the notation used by [1], let \( \kappa_{O}^{C}(n) \), \( \kappa_{L}^{C}(n) \), \( \kappa_{4/3}^{C}(n) \) be extrema constants for the following inequalities:

\[
\left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |T(e_i, e_j)| \right)^{2} \right)^{1/2} \leq \kappa_{O}^{C}(n) \|T\|,
\]

\[
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} |T(e_i, e_j)|^{2} \right)^{1/2} \leq \kappa_{L}^{C}(n) \|T\|,
\]

and

\[
\left( \sum_{i,j=1}^{n} |T(e_i, e_j)|^{4/3} \right)^{3/4} \leq \kappa_{4/3}^{C}(n) \|T\|
\]

for all bilinear forms \( T : \mathbb{C}^{n} \times \mathbb{C}^{n} \to \mathbb{C} \). Classical inequalities, see [11,13,17], due to Orlicz and Littlewood assert that

\[
\kappa_{O}^{C}(\infty) := \lim_{n \to \infty} \kappa_{O}^{C}(n) < \infty,
\]

\[
\kappa_{L}^{C}(\infty) := \lim_{n \to \infty} \kappa_{L}^{C}(n) < \infty,
\]

\[
\kappa_{4/3}^{C}(\infty) := \lim_{n \to \infty} \kappa_{4/3}^{C}(n) < \infty.
\]

The exact values of \( \kappa_{O}^{C}(n) \) and \( \kappa_{L}^{C}(n) \) are stated as an open problem in [11, Page 31]. We solve this problem here for \( n = 2 \), with the aid of techniques introduced by Jameson, [12], concerning unital bilinear forms when dealing with a specific form of two-dimensional Grothendieck’s inequality. We will ultimately prove:

**Theorem 25.** \( \kappa_{O}^{C}(2) = \kappa_{L}^{C}(2) = \kappa_{4/3}^{C}(2) = 1 \).

**Proof.** Let \( A, B \) be complex \( C^* \)-algebras with identities \( e_A, e_B \). According to [12] we say that a bilinear form \( V : A \times B \to \mathbb{C} \) is unital if

\[
V(e_A, e_B) = \|V\| = 1.
\]

Note that if \( A, B \) are finite-dimensional spaces and \( T \) is any bilinear form with \( \|T\| = 1 \), then there will be unitary elements \( x_0 \in A, y_0 \in B \) such that \( T(x_0, y_0) = 1 \), and then a unital form \( V \) is obtained by defining

\[
V(x, y) = T(x_0 x, y_0 y).
\]

In fact, we have

\[
V(e_A, e_B) = T(x_0 e_A, y_0 e_B) = T(x_0, y_0) = 1,
\]

\[
\|V(x, y)\| \leq \|T\| \|x_0 x\| \|y_0 y\| \leq \|T\| \|x\| \|y\|
\]

and thus \( V(e_A, e_B) = \|V\| = 1 \).

Recall that \( \mathbb{C}^2 \) is a \( C^* \)-algebra with product \( xy = (x_1 x_2, y_1 y_2) \) and unit \( e = e_1 + e_2 \). Let \( T : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \) be a bilinear form with \( \|T\| = 1 \). Then, by the Krein–Milman theorem there are extreme elements of the closed unit ball of \( \ell^2_{\infty} \), denoted by \( x_0 = (\alpha_1, \alpha_2) \) and \( y_0 = (\beta_1, \beta_2) \in \ell^2_{\infty} \) such that

\[
T(x_0, y_0) = \|T\| = 1.
\]
It is well known that the extrema elements of the closed unit ball of $\mathbb{C}^2$ have all coordinates with modulo 1, see for instance [6, page 434]. Hence $|\alpha_i| = |\beta_j| = 1$, for all $i, j \in \{1, 2\}$. Let is define the unital bilinear form $V$

\[ V(x, y) = T(x_0x, y_0y). \]

One notes that

\begin{equation}
(5.10) \quad \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |V(e_i, e_j)|^2 \right)^{1/2} = \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |T(e_i, e_j)|^2 \right)^{1/2} \;
\end{equation}

indeed

\[ \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |V(e_i, e_j)|^2 \right)^{1/2} = \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |T(x_0e_i, y_0e_j)|^2 \right)^{1/2} \]

\[ = \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |\alpha_i\beta_jT(e_i, e_j)|^2 \right)^{1/2} = \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |T(e_i, e_j)|^2 \right)^{1/2} \].

Equality (5.10), combined with the previous arguments, yields

\[ \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |T(e_i, e_j)|^2 \right)^{1/2} \leq C \|T\|, \]

for all bilinear forms $T: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ with $\|T\| = 1$ if, and only if,

\[ \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |V(e_i, e_j)|^2 \right)^{1/2} \leq C \|V\| \]

for all unital bilinear forms $V: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by the method (5.3). In conclusion, as to understand the sharp constant problem – objective of current study – it suffices to restrict the analysis to unital bilinear forms. Next we recall two important pieces of information, namely [12, Lemma 2.3] and [12, Theorem 1], listed below for the readers’ convenience:

1. Any unital bilinear form $T: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ is of the form

\[ T(x, y) = (a + ih) x_1y_1 + (b - ih) x_1y_2 + (c - hi)x_2y_1 + (d + hi)x_2y_2, \]

where each of $a + b, c + d, a + c, b + d, a + d, b + c$ is non-negative and $a + b + c + d = 1$;

2. Let $T: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ be a bilinear form given by

\[ T(x, y) = (a + ih) x_1y_1 + (b - ih) x_1y_2 + (c - hi)x_2y_1 + (d + hi)x_2y_2. \]

Then $T$ is unital if and only if the following conditions hold:

(i) $a + b + c + d = 1$;

(ii) each of $a + b, c + d, a + c, b + d, a + d, b + c$ is non-negative;

(iii) $h^2 \leq bcd + acd + abd + abc$.

The above results allow us to re-state Blei’s problem of finding $L^2_\mathbb{C}(2)$ as an optimization problem:

Maximize the function $f: \mathbb{R}^4 \to [0, \infty)$ given by

\[ f(a, b, c, d, h) = (a^2 + b^2 + 2h^2)^{1/2} + (c^2 + d^2 + 2h^2)^{1/2} \]
when subject to the constrains
\[
\begin{align*}
g_1(a, b, c, d, h) &= -a - b \\ g_2(a, b, c, d, h) &= -c - d \\ g_3(a, b, c, d, h) &= -a - c \\ g_4(a, b, c, d, h) &= -b - d \\ g_5(a, b, c, d, h) &= -a - d \\ g_6(a, b, c, d, h) &= -b - c \\ g_7(a, b, c, d, h) &= h^2 - (bcd + acd + abd + abc) \\ t(a, b, c, d, h) &= a + b + c + d - 1
\end{align*}
\]

Applying Karush–Kuhn–Tucker Theorem one finds the maximum of \( f \) over that set is precisely 1, and hence we have proven
\[
L_2^c(2) = 1.
\]

Since
\[
\left( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |T(e_i, e_j)| \right)^2 \right)^{1/2} \leq \sum_{i=1}^{2} \left( \sum_{j=1}^{2} |T(e_i, e_j)|^2 \right)^{1/2},
\]
by (5.11) and symmetry we have
\[
1 \leq \kappa_0^c(2) \leq \kappa_L^c(2) \leq 1.
\]
The Hölder inequality combined with (5.12) gives us
\[
\kappa_{4/3}^c(2) = 1,
\]
which finally concludes the proof of Theorem 25.

5.4. **Grothendieck’s constants.** Let \( K_G^{(m)}(d) \) be the optimal constant such that
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \langle x_i, y_j \rangle \leq K_G^{(m)}(d) \max \left\{ \left| \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} s_i t_j \right| : |s_i| \leq 1, |t_j| \leq 1 \right\}
\]
for all \( d \)-dimensional real Hilbert spaces \( H \), all unit vectors \( x_1, ..., x_m, y_1, ..., y_m \in H \) and all \( m \times m \) scalar matrices \( a_{ij} \). Denoting
\[
K_G(d) := \sup_m K_G^{(m)}(d),
\]
Grothendieck’s theorem asserts that
\[
K_G := \sup_d K_G(d) < \infty.
\]

For a detailed survey on the Grothendieck theorem we refer to [19]. The constants \( K_G, K_G(d) \) and \( K_G^{(m)}(d) \) are, in general, unknown (see, for instance, [8]) and important in physical problems (see [11] and the references therein).

The problem of finding truncated sharp constants can be re-written as
\[
\max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset S} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \langle x_i, y_j \rangle \right\} \leq K_G^{(m)}(d) \max_{|s_i| \leq 1, |t_j| \leq 1} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} s_i t_j \right\},
\]
where \( S = \{ x \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 = 1 \} \) where \( d \) is the dimension of the Hilbert space.
Another way to interpret \((5.13)\) is by saying that for any positive integers \(m, d\) and any bilinear form \(T : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}\) there holds
\[
(5.14) \quad \max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) \langle x_i, y_j \rangle \right| \leq K_G^{(m)}(d) \|T\|,
\]
as
\[
\|T\| = \max_{|s_i| \leq 1, |t_j| \leq 1} \left| T \left( \sum_{i=1}^{m} t_i e_i, \sum_{j=1}^{m} s_j e_j \right) \right| = \max_{|s_i| \leq 1, |t_j| \leq 1} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) t_i s_j \right|.
\]
By \((5.14)\) it is obvious that
\[
K_G^{(m)}(d) = \sup_{\|T\| \leq 1} \left( \max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) \langle x_i, y_j \rangle \right| \right).
\]
Thus, finding the sharp values of \(K_G^{(m)}(d)\) is equivalent to finding the maximum of the function
\[
f_{m,d} : B_{\mathcal{C}(\mathbb{R}^m, \mathbb{R})} \to \mathbb{R}
\]
\[
f_{m,d}(T) = \max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) \langle x_i, y_j \rangle \right|
\]
where \(B_{\mathcal{C}(\mathbb{R}^m, \mathbb{R})}\) denotes the closed unit ball of the space of bilinear forms \(T : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}\).

The following lemma is straightforward:

**Lemma 26.** Let \(m, d\) be positive integers. The function \(f_{m,d}\) is continuous and convex.

Since \(f_{m,n}\) is continuous and convex and \(B_{\mathcal{C}(\mathbb{R}^m, \mathbb{R})}\) is convex and compact we have the following result:

**Proposition 27.** For all positive integers \(m, d\) we have
\[
K_G^{(m)}(d) = \max_{T \in \mathcal{C}} \left( \max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) \langle x_i, y_j \rangle \right| \right)
\]
and
\[
K_G(d) = \sup_{m} \left( \max_{T \in \mathcal{C}} \left( \max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T(e_i, e_j) \langle x_i, y_j \rangle \right| \right) \right).
\]

Since \(\mathcal{C}\) is finite and fully determined, the task reduces to calculate
\[
\max_{\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m} \subset \mathbb{S}^{d-1}} \left| \sum_{i=1}^{m} \sum_{j=1}^{m} T_0(e_i, e_j) \langle x_i, y_j \rangle \right|
\]
for all \(T_0 \in \mathcal{C}\) and this can be easily calculated by the Lagrange Multipliers method.
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