SMOOTH RATIONAL CURVES ON RATIONAL SURFACES

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ABSTRACT. Consider the scheme parametrizing non-constant morphisms from a fixed projective curve to a projective surface. There is a rational map between this scheme and the Chow variety of 1-cycles on the surface. We prove that, if the curve is non-singular, then this rational map is a morphism. As a consequence, we obtain that, if the surface is rational and we fix a divisor class containing a non-singular rational curve, then the scheme parametrizing rational curves on this class is irreducible. Further, if the class has non-negative self-intersection, then the scheme of rational curves has expected dimension.

1. Introduction

There are many ways of viewing curves on an algebraic variety: as effective 1-cycles, closed subschemes or morphisms from a fixed curve. Each of them gives rise to different parameter spaces of curves: Chow schemes, Hilbert schemes or the scheme of morphisms from a fixed curve, respectively. These parameter spaces are related to each other via canonical morphisms or rational maps; see [Kol96, Chapter I] for an overview.

The aim of this paper is to study the relation between the schemes of morphisms from a fixed curve to a projective surface and the schemes parametrizing 1-cycles of this surface.

Throughout the text we let \( C \) be a projective curve and \( X \) be a projective surface over an algebraically closed field \( k \) of arbitrary characteristic. Let \( \text{Mor}_{>0}(C, X) \) be the scheme parametrizing non-constant morphisms from \( C \) to \( X \). Recall that Chow functors of 1-cycles are introduced in [Kol96]. These functors are represented by a scheme if \( \text{char} \, k = 0 \), but, in general, only coarsely represented by a scheme when \( \text{char} \, k > 0 \), see [Kol96, Theorems I.3.21 and I.4.13]. We denote this scheme \( \text{Chow}_1(X) \). Furthermore, there is a morphism

\[ \Theta : \text{Mor}_{>0}^n(C, X) \to \text{Chow}_1(X), \]

where \( \text{Mor}_{>0}^n(C, X) \) stands for the normalization of \( \text{Mor}_{>0}(C, X) \), see [Kol96, Corollary I.6.9]. This morphism is described by taking any \( k \)-point \([f]\) corresponding to a morphism \( f : C \to X \) to the cycle associated to the proper pushforward \( f_*[C] \). In particular, \( \Theta \) defines a rational map from \( \text{Mor}_{>0}(C, X) \) to \( \text{Chow}_1(X) \).

On the other hand, since \( X \) is a surface, 1-cycles are just divisors. Hence, they can be parametrized by a functor taking each \( k \)-scheme \( S \) to the set of relative effective Cartier divisors on \( X \times S \) over \( S \). This functor is representable in arbitrary characteristic and its representing scheme is the disjoint union of complete linear systems of divisors on \( X \), which we denote \( \text{LinSys}(X) \), see Section 2.

It is natural to ask whether we can obtain a morphism from \( \text{Mor}_{>0}(C, X) \) to \( \text{LinSys}(X) \) such as \( \Theta \). Indeed, the first result of this paper tells us that when \( C \) is

2010 Mathematics Subject Classification. Primary 14H10. Secondary 14J26.
nonsingular, then we have such a morphism without the need to use normalization. More precisely, we have the following.

**Theorem 1.1.** Suppose $C$ is a nonsingular and irreducible projective curve and $X$ is a projective surface over $k$. Then there is a natural morphism

$$\Xi : \text{Mor}_{>0}(C, X) \rightarrow \text{LinSys}(X)$$

defined on $k$-points as $\Xi([f]) = f_*[C]$, which is invariant under the $\text{Aut}(C)$-action

$$\text{Aut}(C) \times \text{Mor}_{>0}(C, X) \rightarrow \text{Mor}_{>0}(C, X)$$

$$(\alpha, [f]) \mapsto [f \circ \alpha].$$

Next, consider $C = \mathbb{P}^1$. Let $\beta$ be a class in $\text{Pic} X$ and $|\beta|$ be the corresponding complete linear system. Notice that the preimage of $|\beta|$ under $\Xi$ is a subscheme $\text{Mor}(\mathbb{P}^1, X, \beta) \subset \text{Mor}_{>0}(\mathbb{P}^1, X)$ parametrizing morphisms $f : \mathbb{P}^1 \rightarrow X$ such that the divisor $f_*[\mathbb{P}^1]$ belongs to the class $\beta$.

Let $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta) \subset \text{Mor}(\mathbb{P}^1, X, \beta)$ be the open subscheme parametrizing morphisms which are birational onto their images, and let $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta)$ be the union of components of $\text{Mor}(\mathbb{P}^1, X, \beta)$ such that $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta) \cap \text{Mor}(\mathbb{P}^1, X, \beta) \neq \emptyset$, that is, the closure of $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta)$ in $\text{Mor}(\mathbb{P}^1, X, \beta)$. It is natural to ask under which conditions on $X$ and on the classes $\beta$ we can determine whether $\text{Mor}(\mathbb{P}^1, X, \beta)$ or $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta)$ are irreducible. In [Tes09], it is proved that if $X$ is a Del Pezzo surface of degree greater or equal to 2, then for any class $\beta$ in $\text{Pic} X$, $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta)$ is either irreducible or empty.

It is also natural to ask whether we can determine the dimension of the components of $\text{Mor}(\mathbb{P}^1, X, \beta)$. We have that for every irreducible component $M$ in $\text{Mor}(\mathbb{P}^1, X, \beta)$, there is a bound for its dimension

$$\dim M \geq -K_X \cdot \beta + 2,$$

where $K_X$ denotes the canonical class of $X$, see [Deb13, p.45]. We call this lower bound the **expected dimension** of $M$.

The main result of this paper is a criterion to determine when $\text{Mor}(\mathbb{P}^1, X, \beta)$ is irreducible, and has expected dimension.

**Theorem 1.2.** Let $X$ be a rational surface and let $\beta \in \text{Pic} X$ be a class such that $|\beta|$ has a non-singular rational curve. Then, $\text{Mor}(\mathbb{P}^1, X, \beta)$ is irreducible and

$$\dim \text{Mor}(\mathbb{P}^1, X, \beta) = \begin{cases} 3, & \text{if } \beta^2 < 0, \\ -K_X \cdot \beta + 2, & \text{if } \beta^2 \geq 0. \end{cases}$$

Under these conditions we see that $\text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta) \neq \emptyset$. Hence, it is straightforward to deduce the following.

**Corollary 1.3.** Let $X$ and $\beta$ satisfy the hypotheses of Theorem 1.2. Then

$$\text{Mor}(\mathbb{P}^1, X, \beta) = \text{Mor}_{\text{bir}}(\mathbb{P}^1, X, \beta).$$

**Structure of the paper.** In Section 2 we recall the definition of complete linear systems $|\beta|$ in terms of their functor of points, as well as the results needed to prove Theorem 1.1. In Section 3 we prove Theorem 1.2 and, as an example, we give a complete list of the divisor classes satisfying the hypotheses of Theorem 1.2 on a smooth cubic surface in $\mathbb{P}^3$. 
Acknowledgements. I would like to express my gratitude to Eduardo Esteves and Vladimir Guletskiĭ for enlightening discussions. I am also thankful to Carolina Araujo, Thomas Eckl, and Roy Skjelnes for helpful suggestions.

This work was supported by CNPq, National Council for Scientific and Technological Development under the grant [159845/2019-0].

2. Curves on surfaces

We recall that LinSys($X$) is the scheme parametrizing effective divisors on $X$. More precisely, denote Noe/$k$ to be the category of locally noetherian schemes over $k$, and Set to be the category of sets. Then, LinSys($X$) is the scheme representing the functor $L\in\text{Sys}(X) : \text{(Noe/k)}^{\text{op}} \rightarrow \text{Set}$ taking each $k$-scheme $f : S \rightarrow \text{Spec } k$ to $L$ in $\text{Sys}(X)(S) = \{\text{Relative effective Cartier divisors of } X_S \text{ over } S\}$, where $X_S = X \times S$. Furthermore, for each $\beta \in \text{Pic } X$, let $L$ be an invertible sheaf in the class $\beta$. Hence, we can define the subfunctor $L_{\beta}(X) : \text{(Noe/k)}^{\text{op}} \rightarrow \text{Set}$ defined as

\[
L_{\beta}(X)(S) = \left\{ \begin{array}{l}
\text{Relative effective divisors } D \subset X_S \text{ over } S \\
\text{such that } O_{X_S}(D) \cong L_S \otimes f_S^* M \text{ for some } M \text{ invertible on } S
\end{array} \right\},
\]

where $L_S$ and $f_S$ denote the base changes of $L$ and $f$. If $X$ is integral, then $L_{\beta}(X)$ is represented by a projective space, see [Kle05, Theorem 9.3.13]. We call this projective space the complete linear system of divisors of $\beta$ and denote it $|\beta|$. It follows that

\[\text{LinSys}(X) = \coprod_{\beta \in \text{Pic } X} |\beta|.\]

We state two of the results used in the proof of Theorem 1.1. In order to do so, we use the following notation: for any noetherian scheme $Y$ and any closed subscheme $W$ of $Y$ we denote $[W]$ to be the fundamental cycle on $Y$ associated to it. We remark that both results come from much more general and detailed theories. The versions stated here have been simplified for our purposes.

Lemma 2.1 ([KLU96]). Let $S$ be a noetherian scheme, and let $f : Y_1 \rightarrow Y_2$ be a $S$-morphism of noetherian schemes. Suppose that

1. $f$ is finite;
2. $f$ is a local complete intersection morphism;
3. $\dim O_{Y_2,f(p)} - \dim O_{Y_1,p} = 1$ for all $p \in Y_1$ and;
4. $f$ is curvilinear, i.e. for each $p \in Y_1$, the rank of the fiber $O_{Y_1,Y_2}(p)$ is at most one.

Then, the proper pushforward $f_*[Y_1]$ is the fundamental cycle $[W]$ associated to a relative effective Cartier divisor $W \rightarrow Y_2$ over $S$.

Proof. This can be deduced from the aforementioned reference in the following way: from [KLU96, Proposition 2.10] we have that $f$ is locally of flat dimension 1, and by [KLU96, Corollary 2.5] we have that $f_*[Y_1] = [W]$ for an effective divisor $W$. 

of $Y_2$. This divisor is defined by a zeroth Fitting ideal by [KLU96, Theorem 3.5], hence it follows by the local criterion of flatness that it is $S$-flat. \hfill $\square$

**Lemma 2.2 ([SV00]).** Let $S$ be a connected noetherian scheme and $Y_1$ be a flat $S$-scheme. Let $f : Y_1 \to Y_2$ be a proper $S$-morphism of schemes of finite type and $S'$ be a noetherian $S$-scheme. Let $f_{S'} : Y_{1,S'} \to Y_{2,S'}$ be the base change of $f$. Suppose that $f_*(Y_1) = [W]$, where $W$ is a $S$-flat closed subscheme of $Y_2$. Then $(f_{S'})_*(Y_{1,S'}) = [W_{S'}].$

**Proof.** This is a particular case of [SV00, Theorem 3.6.1]. \hfill $\square$

**Proof of Theorem 1.1.** It suffices to define a morphism between the functors of points of $\text{Mor}_{>0}(C,X)$ and $\text{LinSys}(X)$. Notice that since $C$ is geometrically irreducible the functor of points of the former scheme evaluated at an irreducible noetherian $k$-scheme $S$ is given by

$$\text{Mor}_{>0}(C,X)(S) := \left\{ \text{S-morphisms } g : C_S \to X_S \text{ which do not factor through } S \right\}.$$ 

We check that every morphism in this set satisfies the hypotheses of Lemma 2.1.

For each point $s \in S$, let $g_s : C_s \to X_s$ be the induced morphism between fibers and $D$ be the scheme-theoretic image of $g_s$, then either $D$ is a point or a curve. If $D$ was a point, then by the Rigidity Lemma [MK94, Proposition 6.1, pg 115], we have that $C_S$ factors through $S$, which is a contradiction, therefore $D$ is a curve and $g_s$ factors through a surjective morphism $C_s \to D$. Since $C_s$ is irreducible this surjective morphism is finite, see [Stacks, 0CCL]. In particular, for every $q \in X$ over $s$, $g^{-1}(q) \subset C_s$ is either empty or consists of finitely many points and since $g$ is proper we conclude $g$ is also a finite morphism. Also notice that $g$ factors through the smooth morphism $C_S \times_S X_S \to X_S$ via the graph morphism, therefore it is a local complete intersection.

Further, since both $C_S$ and $X_S$ are flat over $S$, for any point $p \in C_S$ over $s \in S$, let $q = g(p) \in X_S$. We have that

$$\dim \mathcal{O}_{X_S,q} - \dim \mathcal{O}_{C_S,p} = \dim \mathcal{O}_{X_S,q} - \dim \mathcal{O}_{C_s,p},$$ 

see [Har77, Prop. III.9.5]. Hence, we have

$$\dim \mathcal{O}_{X_S,q} - \dim \mathcal{O}_{C_S,p} = \text{codim}([q], X_s) - \text{codim}([p], C_s).$$ 

Notice that we can have two situations:

1. the closure of both $p$ and $q$ on the fibers are of maximal codimension;
2. $p$ is the generic point of $C_s$ and $[q] \subset X_s$ is of dimension 1 (since if $[q]$ was of dimension 0 it would be a closed point, in this case $C_s$ would be contracted to a point by $g_s$ and, again by the Rigidity Lemma, $g$ would factor through $S$).

In both situations we obtain $\dim \mathcal{O}_{X_S,q} - \dim \mathcal{O}_{C_S,p} = 1$.

Finally, notice that $\Omega_{C_S/X_S}(p) \cong \Omega_{C_s/X_s}(p)$ and we have the exact sequence

$$g^*_s \Omega_{X_s/\kappa(s)} \to \Omega_{C_s/\kappa(s)} \to \Omega_{C_s/X_s} \to 0.$$ 

It follows that we have a surjective map $\Omega_{C_s/\kappa(s)}(p) \to \Omega_{C_s/X_s}(p)$. Since $C_s$ is smooth over $\text{Spec} \kappa(s)$, we have $\dim \Omega_{C_s/\kappa(s)}(p) = 1$, hence $\dim \Omega_{C_s/X_s}(p) \leq 1$. In other words, $g$ is curvilinear.
Thus, by Lemma 2.1 we have that the proper pushforward $g_\ast [C_S]$ is the fundamental cycle associated to a relative effective Cartier divisor $W \hookrightarrow X_S$ over $S$. Hence, for each irreducible noetherian $k$-scheme $S$ we have a well defined map
\[(2) \Xi_S : \text{Mor}_{>0}(C, X)(S) \to \text{LinSys}(X)(S)\]
taking a morphism $g$ to $g_\ast [C_S]$. By Lemma 2.2, we have that the maps $\Xi_S$ are natural on $S$. In other words, we have a natural transformation between the presheaves $\text{Mor}_{>0}(C, X)$ and $\text{LinSys}(X)$ restricted to the category of irreducible noetherian schemes over $k$.

Since both presheaves are representable, they are Zariski sheaves. And since every locally noetherian scheme over $k$ is covered by irreducible noetherian schemes this natural transformation extends to the category $\text{Noe}/k$. Yoneda Lemma implies this natural transformation corresponds uniquely to the morphism on the statement, which is $\text{Aut}(C)$-invariant by definition. $\square$

3. RATIONAL CURVES ON RATIONAL SURFACES

Before the proof of Theorem 1.2, we recall two results. The first one is the theorem on dimension of fibers in the convenient presentation below. For a detailed proof see [Mus17, Proof of Proposition 5.5.1].

**Lemma 3.1 (Dimension of fibers).** Let $f : Y_1 \to Y_2$ be a surjective morphism of algebraic varieties over $k$. Suppose that $Y_2$ is irreducible and that all (closed) fibers of $f$ are irreducible and of the same dimension $m$. Then:

1. there is a unique irreducible component $Y_1^0$ of $Y_1$ that dominates $Y_2$ and;
2. every irreducible component $Z$ of $Y_1$ is a union of fibers of $f$ with
\[\dim Z = \dim f(Z) + m.\]

In particular, $\dim Y_1^0 = \dim Y_2 + m$.

**Proposition 3.2.** Let $\beta \in \text{Pic} X$. Suppose that there exists a point $[C] \in |\beta|$ corresponding to a non-singular irreducible projective curve $C \subset X$. Then the general member of $|\beta|$ is irreducible and nonsingular.

Furthermore, if $C$ is rational, then the general member is also rational.

**Proof.** Since $|\beta|$ represents the functor $\text{LinSys}_\beta(X)$ there exists a universal relative effective divisor $D \subset X \times |\beta|$, given by a flat morphism
\[\varphi : D \hookrightarrow X \times |\beta| \to |\beta|\]
such that for every effective divisor $D \subset X$ in the class $\beta$ we have $\varphi^{-1}([D]) \cong D$.

Since $C$ is nonsingular, the morphism $\varphi$ has a nonsingular fiber and since it is flat there is an open subset $U \subset |\beta|$ such that $\varphi|_{\varphi^{-1}(U)}$ is smooth, in particular, every fiber over $U$ is non-singular.

Moreover, by Stein factorization [Har77, Corollary III.11.5] we have that $\varphi$ factors as
\[\varphi : D \xrightarrow{\varphi_1} \text{Spec} \mathcal{O}_D \xrightarrow{\varphi_2} |\beta|,\]
where the fibers of $\varphi_1$ are connected and $\varphi_2$ is a finite morphism. Since $H^0(C, \mathcal{O}_C) \cong k$, it follows from [Gro63, Corollaire 7.8.8] that there is a neighbourhood $V \subset |\beta|$ of $[C]$ such that $\varphi_2|_{\varphi_2^{-1}(V)}$ is an isomorphism. We conclude that the fibres of $\varphi$ over the open subset $U \cap V \subset |\beta|$ are nonsingular and connected, and hence irreducible.
For the last assertion, if we assume that $C$ is rational, then we have that its arithmetic genus is $p_a(C) = 0$. Since $\varphi$ is flat, all of its fibers have the same Hilbert polynomial, and hence, the same arithmetic genus. In particular every fiber over $U \cap V$ has arithmetic genus 0 and, thus, it is a rational curve. \hfill \Box

**Proof of Theorem 1.2.** Denote $M_3 := \text{Mor}(\mathbb{P}^1, X, \beta)$ for simplicity and let $\Xi$ be the morphism defined in Theorem 1.1. Then, we have $M_3 \cong \Xi^{-1}(|\beta|)$. Notice that the image of the morphism

$$\Xi_\beta := \Xi|_{M_\beta} : M_\beta \to |\beta|$$

consists of effective divisors supported on rational curves.

The fiber of $\Xi_\beta$ over a point in $\text{im}(\Xi_\beta)$ consists of an $\text{Aut}(\mathbb{P}^1)$-orbit of a point $[f]$ corresponding to a morphism $f : \mathbb{P}^1 \to X$ such that $\phi(f) \mathbb{P}^1$ is in the class $\beta$. Since there are only finitely many automorphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi \circ f = f$ (see for instance [KV07, Lemma 2.1.12]), we have that the stabilizers $\text{Aut}(\mathbb{P}^1)|_{[f]}$ are of dimension zero and since $\text{Aut}(\mathbb{P}^1)$ is irreducible we have that the $\text{Aut}(\mathbb{P}^1)$-orbit of $[f]$ is irreducible of dimension $\dim \text{Aut}(\mathbb{P}^1) = 3$.

If $\beta^2 < 0$, then any effective divisor $D$ in $|\beta|$ is linearly equivalent to $C$ if and only if $D = C$. Hence, $|\beta|$ is a point and $\dim |\beta| = 0$, thus $M_\beta$ is a unique $\text{Aut}(\mathbb{P}^1)$-orbit of dimension 3.

If $\beta^2 \geq 0$, since $|\beta|$ contains a nonsingular irreducible curve $C$, Proposition 3.2 implies that $\text{im} (\Xi_\beta)$ contains a Zariski dense open subset $U \subset |\beta|$.

Define $M_U := \Xi^{-1}_\beta(U)$. By Lemma 3.1 (applied to the reduction of $M_U$), there exists a unique irreducible component $M_U^0 \subset M_U$ such that the image of $M_U^0$ dominates $U$ and

$$\dim M_U^0 = \dim U + 3 = \dim |\beta| + 3.$$ 

Let $M_\beta^0$ be the irreducible component of $M_\beta$ whose underlying topological space is closure of $M_U^0$ in $M_\beta$, so that $\dim M_\beta^0 = \dim M_U^0$.

Let $[C] \in |\beta|$ be a non-singular rational curve, by Riemann-Roch and the adjunction formula we obtain

$$\dim |\beta| = [C]^2 + 1 = -K_X \cdot \beta - 1.$$ 

And thus $M_\beta^0$ has expected dimension $-K_X \cdot \beta + 2$.

**Claim.** Suppose that $M_\beta' \subset M_\beta$ is an irreducible component such that $M_\beta' \neq M_\beta^0$. Then, $\Xi_\beta(M_\beta')$ is contained in a proper closed subset of $|\beta|$.

**Proof of claim.** Suppose that $\Xi_\beta(M_\beta')$ is not contained in a proper closed subset of $|\beta|$, in other words $\Xi_\beta(M_\beta')$ is dense on $|\beta|$. In particular, $U \cap \Xi_\beta(M_\beta')$ is dense in $U$ and thus $M_\beta' \cap M_U$ is an irreducible open subset of $M_\beta'$ dominating $U$. By definition of $M_U^0$ we have $M_\beta' \cap M_U \subseteq M_U^0$ and, thus, $M_\beta' \subset M_\beta^0$. Since $M_\beta'$ is an irreducible component we have $M_\beta' = M_\beta^0$. \hfill \Box

Hence, if $M_\beta'$ is an irreducible component of $M_\beta$ distinct from $M_\beta^0$, by the second part of Lemma 3.1 it follows that

$$\dim M_\beta' = \dim \Xi_\beta(M_\beta') + 3 < \dim |\beta| + 3 = -K_X \cdot \beta + 2$$

which is impossible since we have the lower bound (1). Hence, $M_\beta = M_\beta^0$. \hfill \Box
Example 3.3. Let $\sigma : X \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at $r \leq 8$ points in general position \( \{p_1, \ldots, p_r\} \). Then $X$ is a Del Pezzo surface of degree $9 - r$. The classes $\beta \in \text{Pic} X$ such that $|\beta|$ contains nonsingular rational curves have been completely classified in [GHI13, Theorem 3.6] for $r \leq 7$. In other words, we have a complete list of classes satisfying the conditions of Theorem 1.2 for $X$.

On Table 1 we list such classes and the dimension of the irreducible $M_\beta := \text{Mor}(\mathbb{P}^1, X, \beta)$ when $r = 6$, that is, when $X$ is isomorphic to a nonsingular cubic surface in $\mathbb{P}^3$. In particular, we note that there exist linear systems $|\beta|$ with nonsingular rational curves for all values of $\beta^2$ greater or equal to $-1$. We refer the reader to the aforementioned reference for the complete list for all $r \leq 7$.

| $\beta^2$ | $d$ | $(m_1, \ldots, m_6)$ up to permutation | $\dim M_\beta$ |
|-----------|-----|--------------------------------------|----------------|
| $-1$      | 0   | $(-1, 0, 0, 0, 0, 0)$                 | 3              |
|           | 1   | $(1, 1, 0, 0, 0, 0)$                 |                |
|           | 2   | $(1, 1, 1, 1, 1, 1)$                 |                |
| 0         | 1   | $(1, 0, 0, 0, 0, 0)$                 | 4              |
|           | 2   | $(1, 1, 1, 1, 0, 0)$                 |                |
|           | 3   | $(2, 1, 1, 1, 1, 1)$                 |                |
| $1 + 2t$  | $1 + t$ | $(t, 0, 0, 0, 0, 0)$ | $5 + 2t$ |
|           | $2 + t$ | $(1 + t, 1, 1, 0, 0, 0)$ |                |
|           | $2 + 2t$ | $(1 + t, 1 + t, 1 + t, 1 + t, 1 + t, 0, 0)$ |                |
|           | $3 + t$ | $(2 + t, 1, 1, 1, 1, 0)$ |                |
|           | $3 + 2t$ | $(2 + t, 1 + t, 1 + t, 1 + t, 1 + t, 1 + t, 1 + t, 1 + t)$ |                |
|           | $3 + 3t$ | $(2 + 2t, 1 + t, 1 + t, 1 + t, 1 + t, t)$ |                |
|           | $4 + 2t$ | $(2 + t, 2 + t, 2 + t, 1 + t, 1 + t)$ |                |
|           | $4 + 3t$ | $(2 + 2t, 2 + t, 2 + t, 1 + t, 1 + t, 1 + t)$ |                |
| $2 + 2t$  | $2 + t$ | $(1 + t, 1, 0, 0, 0, 0)$ | $6 + 2t$ |
|           | $3 + t$ | $(2 + t, 1, 1, 1, 0, 0)$ |                |
|           | $3 + 2t$ | $(2 + t, 1 + t, 1 + t, 1 + t, 1 + t, 0, 0)$ |                |
|           | $4 + 2t$ | $(2 + t, 2 + t, 2 + t, 1 + t, 1 + t, 1 + t)$ |                |
|           | $4 + t$ | $(3 + t, 1, 1, 1, 1, 1)$ |                |
|           | $4 + 3t$ | $(3 + 2t, 1 + t, 1 + t, 1 + t, 1 + t, 1 + t)$ |                |
|           | $5 + 2t$ | $(3 + t, 2 + t, 2 + t, 2 + t, 1 + t)$ |                |
|           | $5 + 3t$ | $(3 + 2t, 2 + t, 2 + t, 2 + t, 1 + t, 1 + t)$ |                |
|           | $6 + 3t$ | $(3 + 2t, 3 + t, 2 + t, 2 + t, 2 + t, 2 + t)$ |                |
| 4         | 4   | $(2, 2, 2, 0, 0, 0)$                 | 8              |
|           | 6   | $(4, 2, 2, 2, 2, 0)$                 |                |
|           | 8   | $(4, 4, 4, 2, 2, 2)$                 |                |
|           | 10  | $(4, 4, 4, 4, 4, 4)$                 |                |

Table 1. List of classes $\beta$ satisfying hypothesis of Theorem 1.2 on a smooth cubic surface in $\mathbb{P}^3$.

The information on Table 1 is organized as follows: for any $\beta \in \text{Pic} X$ recall we can write $\beta = d\alpha - \sum_{i=1}^r m_i \varepsilon_i$, where $\alpha$ is the pullback of the class of a line in $\mathbb{P}^2$ and $\varepsilon_i$ is the class of the exceptional divisor $\sigma^{-1}(p_i)$. For each non-negative integer $t$, we list each $\beta$ in terms of $d := d(t)$ and the tuples $(m_1, \ldots, m_6) := (m_1(t), \ldots, m_6(t))$. Moreover, the classes are listed up to permutation of the $m_i$. 
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