Interaction of a monopole vortex with an isolated topographic feature in a three-layer geophysical flow

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Abstract. In the frame of a three-layer quasi-geostrophic analytical model of a \( f \)-plane geophysical flow, Lagrangian advection being induced by the interaction of a monopole vortex with an isolated topographic feature is addressed. Two different cases when the monopole locates either within the upper or the middle layer are of our interest. In the bottom layer, there is a delta function topographic feature, which generates a closed recirculation region in its vicinity due to the background flow. This recirculation region extends to the middle and upper layers, and it plays the role of a topographic vortex. The interaction between the monopole and the topographic vortex causes complex, including chaotic, advection of fluid particles. We show that the model’s parameters, namely, the monopole and topographic vortices’ strengths and initial positions, the layers’ depths and densities are responsible for the diverse advection patterns. While the patterns are rather complicated, however, one can single out two major processes, which mostly govern fluid particle advection. The first one is the variation in time of the system’s phase space structure, so that within the closed region of the topographic vortex, there appear periodically unclosed particle pathways by which the particles leave the topographic vortex. The second one is chaotic advection that arises from the nonstationarity of the monopole-topography interaction.

Keywords. Three-layer flow, chaotic advection, monopole-topography interaction

1 Introduction

Generally speaking, topographic vortices are coherent vortical structures appearing as closed recirculation regions over bottom features in the ocean and atmosphere. Topographic vortices play a fundamental role in mass, salinity and temperature advection in the ocean. Moreover, topographic vortices are known to influence the dynamics of different coherent structures, such as unrestrictedly moving vortices (e.g., van Geffen and Davies, 1999; Dewar, 2002; An and McDonald, 2005; Cand on and Marshall, 2012; Zavala Sansón et al., 2012). Such topographic vortices greatly vary in time and size scales (Baines and Smith, 1993; Baines, 1993). In this paper, however, we are only interested in meso- and synoptic scale topographic vortices due to these scales are generally believed to be prevailing in the ocean (Chelton et al., 2011).

The present paper deals with Lagrangian regular and irregular (chaotic) advection being generated by a vortex monopole interacting with a topographic vortex. The topographic vortex under investigation is generated by a regular three-layer \( f \)-plane background flow (e.g., Pedlosky, 1987; Kozlov, 1995) with a delta function bottom irregularity within the lower layer (e.g., Sokolovskiy et al., 1998; Izrailsky et al., 2004; Kozlov et al., 2005). Then we embed a monopole singular vortex (e.g., Gryannik, 1983; Gryannik and Tevs, 1989; Gryannik et al., 2000; Carton, 2001; Reznik, 2010; Reznik and Kizner, 2010) either within the upper or middle layer. So, these singularities move like passive tracers along regular background flow stream-lines (Reznik and Kizner, 2007a,b, 2010), although, generating a complex either periodically (for a time-independent background flow) or quasi-periodically (for a periodically time-dependent background flow) velocity field in the vicinity of themselves. Our main reason for employing such a three-layer model (e.g., Sokolovskiy, 1997; Ryzhov and Koshel, 2011a) is to study Lagrangian advection being induced by the monopole-topography interaction when the monopole being located within either the upper or middle layers. Such monopole positioning can be considered as the simplest models for a surface eddy, and for an interthermocline lens (e.g., Carton et al., 2002; Wang and Dewar, 2003;
2 Model formulation

The simplest way to study a quasi-two-dimensional layered geophysical flow is by exploiting the potential vorticity definition in each layer. For a background three-layer flow under the rigid lid approximation, these definitions read (Pedlosky, 1987)

\[ q_i = \Delta \psi_i + f H_i \zeta_i, \quad q_2 = \Delta \psi_2 + f H_2 (\zeta_2 - \zeta_1), \quad q_3 = \Delta \psi_3 + f H_3 (\zeta_3 - \zeta_2) + f, \]

where \( i = 1, 2, 3 \) corresponds to the upper, middle, and lower layer respectively; \( \zeta_i \) is the layer potential vorticity, \( \Delta \psi_i = \frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y} \) is the two-dimensional relative vorticity with stream-function \( \psi_i \) and two-dimensional velocity field \( u_i, v_i \); \( \zeta_1, \zeta_2 \) are the interface heights between the upper and middle, and middle and lower layers; \( h(x,y) = \tau \delta(r) \) is the Dirac delta function bottom irregularity with effective volume \( \tau \); \( H_i \) is the \( i \)-layer depth; \( f \) is the constant Coriolis parameter. According to the pressure continuity condition, the interface heights can be written in the form (Pedlosky, 1987)

\[ \zeta_1 = \frac{f (\psi_2 - \psi_1) \rho_2}{(g (\rho_2 - \rho_1))}, \quad \zeta_2 = \frac{f (\psi_3 - \psi_2) \rho_3}{(g (\rho_3 - \rho_2))}, \]

where \( \rho_i \) is the \( i \)-layer fluid density; \( g \) is the gravitational acceleration; \( \Delta \rho_1 = \rho_2 - \rho_1 \), and \( \Delta \rho_2 = \rho_3 - \rho_2 \) are the density jumps.

Substituting (2) into (1), one can obtain the detailed potential vorticity,

\[ q_1 = \Delta \psi_1 + k_1 (\psi_2 - \psi_1) + f, \]

\[ q_2 = \Delta \psi_2 + (k_{21} \psi_1 - \psi_2 (k_{21} + k_{22})) + k_{22} \psi_2 + f, \]

\[ q_3 = \Delta \psi_3 + k_3 (\psi_2 - \psi_3) + \frac{f \tau}{H_3} \delta(r) + f, \]

where \( k_1 = \frac{f^2 \rho_2}{n_1 g \Delta \rho_1}, \quad k_3 = \frac{f^2 \rho_3}{n_3 g \Delta \rho_3}, \quad k_{21} = \frac{f^2 \rho_2}{n_2 g \Delta \rho_1}, \quad k_{22} = \frac{f^2 \rho_2}{H_2 g \Delta \rho_2}. \]

Since our study concerns only the cases of the upper- and middle-layer monopole propagation, we set the lower-layer potential vorticity to be always time-independent. However, either the upper or middle layer potential vorticity has one time-dependent singular value moving with the monopole’s center. Hence, we have two sets of singular perturbations of the flow,

\[ q_m = q_m^* + \frac{f}{H_m} \mu_m \delta(|r_i - r_m^*|), \quad q_n = q_n^*, \quad q_3 = q_3^*, \]

where \( m, n = 1, 2, m \neq n, \mu_m \) is the monopole’s strength and \( r_m^* \) is the position of the monopole’s singularity within the \( m \)-layer, \( q_m^* \) is the potential vorticity background value, and \( |r_i - r_m^*| = \sqrt{(x_i - x_m^*)^2 + (y_i - y_m^*)^2} \) with \( x_i, y_i \) being Cartesian coordinates of a fluid particle within the \( i \)-layer.

Potential vorticity \( \Psi \) should satisfy the potential vorticity conservation law in each layer,

\[ \partial_t q_i + J(\psi_i, q_i) = 0. \]

To obtain explicit analytical relations for stream-functions \( \psi_i \), one can split relations (3) by making use of the following procedure (e.g., Gryanik and Tevs, 1989). First, we rewrite (3) in a matrix form,

\[ \Delta \Psi + A \Psi = B, \]

where \( \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad A = \begin{pmatrix} -k_1 & k_1 & 0 \\ 0 & k_21 - (k_{21} + k_{22}) & k_{22} \\ 0 & 0 & -k_3 \end{pmatrix}, \quad B = \begin{pmatrix} f + q_1 \\ f + q_2 \\ f + q_3 \end{pmatrix}. \]

Second, we diagonalize matrix \( A \) through a similarity transformation, \( A = S J S^{-1} \). Matrix \( S \), whose columns are the eigenvectors of matrix \( A \), and diagonal matrix \( J \), whose main diagonal is consisted of the eigenvalues of matrix \( A \), have the form,

\[ S = \begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k_3 (\alpha_2 - 1) & 0 \\ 0 & 0 & -k_3 (\beta_2 - 1) \end{pmatrix}, \]

where

\[ \alpha_1 = -k_{22} / k_{21} - \alpha_2 / k_{21} (-k_{21} - k_{22} + k_3 (\alpha_2 - 1)), \]

\[ \alpha_2 = (k_1 + k_3 + k_{21} + k_{22} + \lambda_0) / (2k_3), \]
\[ \beta_1 = -\frac{k_2}{k_1 + k_2 - \beta_3} \frac{k_2 + k_3 - k_1}{1 - \beta_2}, \]

\[ \beta_2 = \left( k_1 + k_2 + k_3 - \frac{\lambda_0}{2k_3} \right), \]

\[ \lambda_0 = \sqrt{(k_1 - k_3 + k_2 + k_3)^2 - 4(-k_1 - k_3 - k_2 + k_3)}, \]

Third, we introduce vector \( \Phi \) such that \( \mathbf{S} = \mathbf{S} \Phi \), and obtain from (9) new expression \( \Delta \Phi + J \Phi = \mathbf{S}^{-1} \mathbf{B} \), which, taking into consideration relations

\[ \mathbf{S}^{-1} = \frac{1}{2} \begin{pmatrix} \alpha_2 - \beta_2 & \beta_1 - \alpha_1 \\ \beta_2 - 1 & 1 - \beta_1 \end{pmatrix}, \]

and

\[ \gamma = \alpha_2 - \alpha_1 + \beta_2 - \alpha_1 \beta_2 - \alpha_2 \beta_1, \]

has the following detailed form.

\[ \Delta \Phi_1 = -f - \frac{1}{\gamma} \left( (\alpha_2 - \beta_2) q_1 + (\beta_1 - \alpha_1) q_2 + \right) \]

\[ + (\alpha_1 \beta_2 - \alpha_2 \beta_1) q_3, \]

\[ \Delta \Phi_2 = -k_3 (\alpha_2 - 1) \Phi_2 = -\frac{1}{\gamma} \left[ (\beta_2 - 1) q_1 + (1 - \beta_1) q_2 + \right] \]

\[ + (\beta_1 - \beta_2) q_3, \]

\[ \Delta \Phi_3 = -k_3 (\beta_2 - 1) \Phi_3 = -\frac{1}{\gamma} \left[ (1 - \alpha_2) q_1 + (1 - \alpha_1) q_2 + \right] \]

\[ + (\alpha_2 - \alpha_1) q_3. \]

The last step is to obtain explicitly barotropic mode \( \Phi_1 \) and two baroclinic modes \( \Phi_2, \Phi_3 \). As it has been mentioned above, we are interested only in the singular perturbations of form (4) (without losing any generality, we put the background flow velocity to be zero, i.e., \( q_1^* = 0 \) [Kozlov 1992, Izrailev et al. 2004]). Hence, by setting boundary conditions \( \Phi_1 |_{r \to \infty} = 0 \), and \( \partial \Phi_1 / \partial r |_{r \to \infty} = 0 \) to Laplace and Helmholz equations (9), we obtain two sets of Green’s function superpositions satisfying system (9) for the upper \( (m = 1) \) and middle \( (m = 2) \) layer monopole propagation cases.

\[ \Phi_{1m} = \frac{f}{\gamma} \left( -\frac{1}{H_1} \frac{\mu_1}{H_3} \log(r_{1m}^*) + \right) \]

\[ + \left( \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{H_3} \right) \log(r_1), \]

\[ \Phi_{2m} = -\frac{f}{\gamma} \left( -\frac{1}{H_1} \frac{\mu_1}{H_3} K_0 \left( \sqrt{k_3 (\alpha_2 - 1)} r_{1m}^* \right) + \right) \]

\[ + \left( \frac{\beta_1 - \beta_2}{H_3} \right) K_0 \left( \sqrt{k_3 (\beta_1 - 1)} r_{1m}^* \right), \]

\[ \Phi_{3m} = -\frac{f}{\gamma} \left( -\frac{1}{H_1} \frac{\mu_1}{H_3} K_0 \left( \sqrt{k_3 (\beta_2 - 1)} r_{1m}^* \right) + \right) \]

\[ + \left( \frac{\alpha_2 - \alpha_1}{H_3} \right) K_0 \left( \sqrt{k_3 (\beta_1 - 1)} r_{1m}^* \right), \]

where \( r_1 = \sqrt{x_1^2 + y_1^2} \), \( r_{1m}^* = \sqrt{(x_i - x_m^*)^2 + (y_i - y_m^*)^2} \), and \( m, n = 1, 2 \), \( m \neq n \).

Now, introducing a nonvortical plane boundary source flux in the form, \( -U y \), which does not generate any vorticity and is compensated by an analogous drain flux (see a detailed substantiation in, e.g., [Zrailsky et al. 2004]), where \( U \) is a characteristic velocity, one can formulate the final stream-functions of the three-layer model with the monopole moving within the \( m \)-layer,

\[ \psi_{im} = -U y + \Phi_{1m} + \alpha_1 \Phi_{2m} + \beta_1 \Phi_{3m}, \] (11)

where \( \alpha_3 = \beta_3 = 1 \). So, further we will make use of stream-functions (11) with the (10) set of functions \( \Phi_{1m} \) either for the upper \( (m = 1) \) or middle \( (m = 2) \) layer monopole propagation case.

Now, we can introduce certain dimensionless values, which will be used further as parameters governing the different regimes of Lagrangian advection. Introduce length scale \( L = (k_3 (\alpha_2 - 1))^{1/2} \); velocity scale \( U \); the Rossby number, \( \varepsilon = \frac{U}{L} \); and an effective volume of the topography as \( \tau = \pi h_0 L^2 \), where \( h_0, \ L \) are the height and radius of an corresponding cylinder [Sokolovskiy et al. 1998]. Then we introduce the following governing parameters,

\[ \chi = \frac{f \tau}{H_3 UL} = \frac{h_0 \pi}{\varepsilon H_3}, \]

\[ \kappa_m = \frac{f \mu_m}{H_m UL}, \]

which characterize the dimensionless topographic vortex strength the dimensionless monopole vortex strength, respectively. Then, by satisfying the quasi-geostrophic requirement of \( \frac{h_0}{H_3} \sim O(\varepsilon) \), we set \( \chi = \pi \). Thus, choosing the following parameters, \( H_1 = 200 \), \( H_2 = 400 \), \( H_3 = 3000 \), \( \rho_1 = 1026.56 \) kg/m\(^3\), \( \rho_2 = 1027.84 \) kg/m\(^3\), \( \rho_3 = 1028.32 \) kg/m\(^3\), we obtain the characteristic horizontal topographic vortex scale, \( \sim 1.3 \times 10^4 \) m.

3 Equations of motion

Now, by making use of the dimensionless parameters and the geostrophic relations, one can write the equations of motion for the monopole’s center and for a fluid particle, being advected by the monopole-topography interaction velocity field. The monopole motion in the \( m \)-layer is governed by the following equations of motion,

\[ \frac{dx_m}{dt} = -\left( \frac{\partial \psi_{mm}}{\partial y} \right)_{x=x_m^*, y=y_m^*} = W + \chi \frac{y_m^*}{r_m^*} V_m(r_m^*), \] (13)

\[ \frac{dy_m}{dt} = -\left( \frac{\partial \psi_{mm}}{\partial x} \right)_{x=x_m^*, y=y_m^*} = -\frac{x_m^*}{r_m^*} V_m(r_m^*), \]

where \( m = 1, 2 \), and \( W = W(t) \) is the dimensionless background flow velocity:

\[ V_m(\xi) = \frac{1}{\gamma} \left( \alpha_1 \beta_2 - \alpha_2 \beta_1 \right) \frac{1}{\xi} + \alpha_m (\beta_1 - \beta_2) K_1(\xi) + \right) \]

\[ + \beta_m (\alpha_2 - \alpha_1) \sqrt{\frac{(\beta_2 - 1)}{(\alpha_2 - 1)} K_1 \left( \sqrt{\frac{(\beta_2 - 1)}{(\alpha_2 - 1)} \xi} \right)}, \]
and \( r_m^* = \sqrt{(x_m^*)^2 + (y_m^*)^2} \) is the monopole position with \( m = 1,2 \) for the upper- and middle-layer monopole motion, respectively. System (13) is an elaborated system to govern the dynamics of a fluid particle due to the velocity field being generated by an exterior background flow intersecting a delta function bottom irregularity. Lagrangian advection being determined by system (13) has been studied recently in the frame of barotropic (Sokolovskiy et al., 1998; Izrailsky et al., 2004; Koshel and Prants, 2006), two-layer (Kozlov et al., 2005; Ryzhov and Koshel, 2011b) and three-layer baroclinic geophysical flows (Ryzhov and Koshel, 2011a). In our case, however, system (13) governs not a fluid particle’s motion, but a singular vortex’s center motion. So, the upper- and middle-layer monopoles themselves move as fluid particles due to the topographic vortex velocity field. Fluid particles of the monopole-topography interaction system, although, undergo the joint influence of both the monopole and topographic vortex velocity fields.

Motion of a fluid particle being influenced by the cooperate monopole-topography velocity field obeys the relations,

\[
\dot{x}_i = -\frac{\partial \psi_{im}}{\partial y_i} = W + \kappa_m \frac{(y_i - y_m^*)}{r_{im}^*} P_{im}(r_{im}^*) + \chi \frac{y_i}{r_i} V_i(r_i),
\]

\[
\dot{y}_i = \frac{\partial \psi_{im}}{\partial x_i} = -\left( \kappa_m \frac{(x_i - x_m^*)}{r_{im}^*} P_{im}(r_{im}^*) + \chi \frac{x_i}{r_i} V_i(r_i) \right),
\]

where \( r_{im}^* = \sqrt{(x_i - x_m^*)^2 + (y_i - y_m^*)^2} \) is the fluid particle position relatively to the monopole’s center position,

\[
P_{im}(\xi) = \frac{(-1)^n}{\gamma^\frac{1}{2} \xi} \left( (\alpha_n - \beta_n) \left( \frac{1}{\xi} + \alpha_i (\beta_n - 1) K_1(\xi) + \beta_i (1 - \alpha_n) \left( \frac{\beta_2 - 1}{\alpha_2 - 1} K_1 \left( \frac{\beta_2 - 1}{\alpha_2 - 1} \xi \right) \right) \right),
\]

and \( m, n = 1,2, m \neq n \).

4 Monopole motion

First, we briefly analyze system (13) that governs the monopole’s dynamics. An elaborated study of this system has been conducted in (Ryzhov and Koshel, 2011b). If the background exterior flow is constant \( W = W_0 \), system (13) is integrable in the sense of the stream-line-trajectory coincidence (e.g. Zaslavsky, 1998). Due to the bottom topography is singular, any nonzero value of \( W_0 \) always produces a closed Taylor column region called a topographic vortex within the lower layer. To have such closed regions within the middle and upper layers, however, the background velocity should be lower than a critical value. This critical value is the maximal value of the azimuthal velocity in the corresponding layer. So, if one chooses the background flow to satisfy this condition, then three different-size Taylor columns will occur due to the bottom irregularity. These three columns may be thought of as a discrete Taylor cone. Figure 1 depicts azimuthal velocities \( V_i \) depending on distance \( r \) to the topographic vortex elliptic point. We chose \( W_0 = 0.2, \chi = 0.2 \pi \) to ensure the mesoscale closed regions to exist in the all three layers. The points, where the horizontal line intersects the azimuthal velocity curves, correspond to elliptic and hyperbolic critical points of the vortex. Figure 2 demonstrates stream-lines of the resulting topographic vortex in the upper-layer. The red curve indicates the separatrix dividing the flow into the vortical region and the exterior flow. Since we also are interested in the middle-layer monopole propagation case, the vortical region of the middle layer is indicated by the blue dashed curve.

5 Fluid particle advection

5.1 Regular monopole motion

Now we can analyze fluid particle advection being induced by the monopole-topography interaction velocity field. Motion of a fluid particle is governed by system (14), where the right part of the relations comprises the monopole motion solution given by (13). First, we consider the periodic solution of (13). This solution, although cannot be expressed in an analytical form, is time-dependent with a period being equal to the time of the monopole passing a closed tra-
Chaotic advection manifestation is by constructing Poincaré trajectories in a finite time (e.g., Lichtenberg and Lieberman, 1992). Chaotic advection dynamics which is conventionally called chaotic advection freedom, that permits to occur the fluid particle irregular trajectories, and the dashed blue curve corresponds to the middle layer separatrix.

The red curve indicates the separatrix. The dashed blue curve corresponds to the middle layer separatrix.

Fig. 2: Topographic vortex stream-lines for the upper layer. The red curve indicates the separatrix. The dashed blue curve corresponds to the middle layer separatrix.

The Poincaré section analysis is a very useful technique to estimate which part of fluid particles is involved either in regular advection or in chaotic advection, however, this technique fails to show what happens with fluid particles in certain moment of time. So, to address the question, how these fluid particles move during the monopole passing a revolution about the topography, we calculate the number of the flow’s critical points that appear at each instant (Ryzhov and Koshel, 2011b, Ryzhov et al., 2012). The simple idea of this classification is that the more critical points of their initial set survive or, in other worlds, the less topological changes appear during a monopole revolution the more regular system (14) is.

5.2 Diagram of the number of the critical points

As the monopole moves about topography, the number of the flow’s regular critical points changes, that results in the flow topology altering its characteristics in time (e.g., Aref and Brons, 1998). It should be mentioned, that, in the upper layer monopole propagation case, one singular critical point corresponding to the monopole’s center always exists, so, we have excluded it from the consideration. Although, in the middle-layer monopole propagation case, no singular points occur within the upper-layer velocity field due to the singular middle-layer monopole appears as a regular one in the upper layer. So, making use of the introduced classification, we present diagrams of the number of the regular critical points in the upper-layer monopole propagation case and in the middle-layer monopole propagation case, respectively, depending on monopole’s strength \( \kappa \) and initial position \( y \).

These diagrams depict by color how many regular critical points appear at the beginning of the monopole rotation (initial critical points) and at the time the monopole passes a half of its rotation period (half-period critical points). Figure 4a, and fig. 4b correspond to the upper-layer monopole propagation case and to the middle-layer monopole propagation case, respectively.

Now, we offer a detailed explanation for the diagrams. \( \kappa < 0 \) region corresponds to counter-rotation of the monopole and topographic vortex, and \( \kappa > 0 \) corresponds to co-rotation of the monopole and topographic vortex. First, we consider fig. 4a depicting the diagram associated with the upper-layer monopole propagation case. Figure 4a shows the flow’s stream-lines at the initial stage of monopole motion and at the half-period stage. The red curves are the monopole trajectories, and the dashed blue curve corresponds to the unperturbed topographic vortex separatrix.

Also, as a Lagrangian advection measure, we have calculated the escaping time (Kozlov and Koshel, 1999, 2006, Izrailsky et al., 2004), which is determined as the time a fluid particle needs to be carried away by the exterior flow from the unperturbed topographic vortex region. This measure is an analogue for the Lyapunov exponent and it shows where Lagrangian advection progresses faster or slower. Thus, we have uniformly distributed within the separatrix \( 10^4 \) markers, and, then, taken into consideration the time they would need to cross the line far enough out of the vortex interaction (line \( x = 5 \)). The escaping time distributions are shown in...
Fig. 3: Equivalent Poincaré sections of system (14) for the same values of $\kappa = 0.01$ and the same values of perturbation frequency $\omega$ but different monopole initial positions $(x_1^*, y_1^*)$.

(a) $(0, -4)$  
(b) $(-0.8, -3.3107)$  
(c) $(0, -0.6936)$

Fig. 4: Number of the flow’s regular critical points by color. Blue – 3 initial and 3 or 5 half-period points; purple – 5 initial and 5 half-period points, grey – 3 initial and 3 half-period points red – 3 initial and 1 half-period points; green – 1 initial and 1 half-period points. Yellow – 6 initial and 2 half-period points, orange – 4 initial and 2 half-period points, brown – 2 initial and 2 half-period points.

Fig. 6, where unity of the time is equal to the corresponding period of a monopole revolution. A general feature of all the subfigures is the almost circle areas of long-live fluid particles. These areas correspond to the monopole region, which is very intense due to the singularity. Hence, fluid particles within these areas move mostly regular (Ryzhov and Koshel, 2011b), and, therefore, they do not leave the topographic vortex region.

The blue color region corresponds to a strong influence of the monopole motion. At the initial stage of monopole motion, there are three regular critical points to form a heteroclinic structure (see fig. 5a). The topographic vortex cannot be distinctly identified due to no hyperbolic point corresponds to the unperturbed hyperbolic point. So, this case of monopole-topography interaction cannot be considered as a perturbation of the topographic vortex. Moreover, this initial stream-line picture resembles a counter-rotating dipole structure (e.g., Voropayev et al., 2001; Ryzhov, 2011). This structure changing in time results in that, at the half-period stage, there are also three regular critical points to form two homoclinic structures each associated either with the topographic or with the monopole vortices (see fig. 5b). Due to that topological alteration, fluid particle advection is very effective. Most particles are carried away within 5 monopole revolutions (see fig. 6a).

The purple color region corresponds to a moderate influence of the monopole motion. At the initial stage, there are five regular critical points to form both one heteroclinic and
one homoclinic structures (see fig. 5d). This homoclinic structure almost coincides with the unperturbed topographic vortex separatrix, that indicates this case can be considered as a perturbation of the topographic vortex. At the half-period stage, the stream-line picture appears as almost the same as in the aforementioned one of the blue color region (see fig. 5f). Lagrangian advection progresses although less efficient, but still very fast and it extends on the whole separatrix region. No stagnation zone appears within the region (see fig. 6b).

The grey color region corresponds to the least monopole influence. This positive \( \kappa \) region differs topologically from those presented. Due to the vortices are co-rotating, the initial topological structure appears as a co-rotating dipole enveloped by a common topographic vortex separatrix (see fig. 5e). So, this structure can be considered as a topographic vortex with a double center. Although this double center greatly perturbs the fluid particle dynamics, the structure of the topographic vortex can be revealed during a whole monopole revolution (see fig. 5f). Due to the existence of two always unbroken centers, fluid particles in the vicinity of the topographic vortex center move almost regular, however, the surrounding fluid is carried away very fast (see fig. 6f).

The red color region corresponds to a transitional case of the monopole-topography interaction. Initially, the streamline picture appears as a co-rotating dipole structure (see fig. 5e), although, during a monopole revolution, the dipole structure breaks, so, the singular monopole absorbs the topographic vortex elliptic point and becomes a new center of the topographic vortex for a certain time. During this time span, Lagrangian advection within the topographic vortex with the new singular center is rather regular (see fig. 5h). However, during a whole monopole revolution, almost all the fluid from the topographic vortex is carried away (see fig. 6h).

The green color region corresponds to the capturing of the monopole to be as a topographically trapped vortex with the singular monopole’s center playing the role of a new topographic vortex center. Both at the initial and half-period stages, the stream-line portraits comprise only one regular critical point that corresponds to hyperbolic point of the topographic vortex. The initial stream-line portrait is shown in fig. 7a, while the half-period stream-line portrait appears almost the same as that shown in fig. 5f. Thus, this case can be thought of as the topographic trapping of a monopole vortex. Lagrangian advection, in this case, differs insignificantly from the case previously addressed (see fig. 6f).

Now, we consider the diagram shown in fig. 4b for the middle-layer monopole propagation case. The main difference from the upper-layer monopole propagation case is, in this case, no singular points appear within the upper-layer velocity field, so, the monopole vortex appears as a regular one and as the regular topographic vortex also can be broken. Hence, a merger of the vortices can appear due to both vortices being regular. This regularity leads to all the half-period stream-line pictures to appear almost the same with one elliptic point, which is formed by the merger, and one hyperbolic point. Also, the lack of a singular point leads to that Lagrangian advection is much more regular in comparison with that considered above.

The yellow color region corresponds to six initial regular critical points (see fig. 5a) and two half-period critical points (see fig. 5b), that both correspond to the topographic vortex with the monopole vortex being disappeared since the velocity of surrounding flow is too high for a closed circulation region to be formed. The position of the corresponding middle-layer monopole vortex is marked by the cross. Such a half-period stream-line portrait is universal for all the color regions shown in fig. 7a. The corresponding escaping time distribution is shown in fig. 8a. There is a big stagnation region with mostly regular advection corresponding to the lower closed region shown in fig. 7a.

The orange color region is arranged astride the \( \kappa = 0 \) line. This region corresponds to the existence of three initial critical points. The difference between the negative and positive orange color region initial stream-line portraits is shown in fig. 7c,d. Both the corresponding half-period stream-line portraits, however, appear as almost the same as that shown in fig. 7b. Since initially two vortex structures can be reliably identified, and at half-period stage all these structures merge, the escaping time distribution shows very effective and intense advection with no stagnation regions progressing. Figures 8c,d depict the escaping time distribution in the negative and positive \( \kappa \) cases, respectively.

The brown color region corresponds to the existence of two initial and half-period critical points. The middle-layer monopole does not induce a closed region within the upper layer. Despite that, the middle-layer monopole does greatly perturb fluid particle advection. The corresponding stream-line portrait does not change topologically during a monopole revolution and it appears as almost the same as that shown in fig. 7b. However, on both sides of the \( \kappa = 0 \) line, the advection efficiency is very different. In the \( \kappa < 0 \) zone, advection is very irregular (see fig. 8a) due to counter-rotation of the middle-layer monopole and the topographic vortex. On the other hand, in the \( \kappa > 0 \) zone, advection is mostly regular, a big stagnation region appears in region of the topographic vortex (see fig. 8b), due to co-rotation of the middle-layer monopole and the topographic vortex.

Further, we study how irregular motion of the monopole to influence Lagrangian advection.

### 5.3 Irregular monopole motion

In this paragraph, we analyze fluid particle advection being induced by a non-periodic perturbation consisting of periodic background flow oscillation and non-periodic part due to monopole irregular motion within the topographic vortex. That irregular monopole motion is due to the monopole’s singular center moving as a fluid particle in the periodically driven velocity field of the topographic.
vortex, which is known to produce the irregular dynamics (e.g., Sokolovskiy et al., 1998; Kozlov and Koshel, 2001; Izrailsky et al., 2004, 2008; Koshel et al., 2008). Strictly speaking, if one set the background flow to oscillate periodically,

$$W = W_0(1 + \mu_W \cos \nu_W t),$$

(15)

where $\mu_W$ and $\nu_W$ are the magnitude and frequency of the background flow oscillation, then system (13) becomes a system with one and a half degree of freedom permitting the chaotic dynamics to occur. Hence, with such an oscillating background flow, the monopole can start moving out of the topographic vortex, and then it can be trapped temporarily by the topography. And, on the contrary, if the monopole starts moving within the topographic vortex, it now can be carried away by the exterior flow. It should be mentioned that the background flow oscillation also affects the fluid particle dynamics, resulting in certain number of particles to leave the topographic vortex region. However, in our numerical simulation, we chose a very small perturbation magnitude ($\mu_W = 0.01$), so there are very few such particles. So, by making use of such a configuration, we study Lagrangian advection being mostly induced by the short-term monopole-topography interaction.

Figure 9 depicts an example of fluid particle advection being generated by the short-term interaction, while the monopole accomplishes a few revolutions within the topographic vortex. Figure 9 shows the initial configuration of red and green markers corresponding to the topographic and monopole vortex regions, respectively. The unperturbed topographic vortex region is uniformly filled in with $10^4$ red markers. Also, $1.5 \cdot 10^5$ green markers are placed to distinguish the monopole vortex region. The monopole with strength $\kappa = 0.1$ starts moving out of the topographic vortex.
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(a) $\kappa = -0.5$, $y = -3.5$

(b) $\kappa = -0.1$, $y = -3.5$

(c) $\kappa = 0.1$, $y = -5.5$

(d) $\kappa = 0.3$, $y = -4.5$

(e) $\kappa = 0.5$, $y = -4.5$

Fig. 6: Escaping time distribution in the upper-layer monopole propagation case.

Fig. 7: Stream-lines of the flow in the middle-layer monopole propagation case. Red curve corresponds to the monopole motion trajectory. Dashed blue curve is the topographic vortex unperturbed separatrix.

(see fig. 9) at the position with coordinates $x = -2, y = -8.4$. Then, the monopole vortex is captured by the topographic vortex due to chaotic advection (see fig. 9b). Next fig. 9c shows the marker distribution as the monopole has passed a half of rotational period (the black curve points out the trajectory of the monopole’s center). A great deformation caused by the monopole is clearly seen. Figure 9d illustrates the particle distribution after the monopole has made three whole revolutions about the topography. Few red markers from the initial distribution have stayed within the topographic region. Last fig. 9e depicts the monopole leaving the topographic vortex region after four revolutions.

Figure 10 also depicts a series of marker scattering patterns, but for the middle-layer monopole propagation case. In this case, the monopole starts moving at the position with coordinates $x = -1.18, y = -8$ and it appears as a regular vortex within the upper layer. It results in that a closed recirculation region corresponding to the monopole ceases to exist at the half-period stage. Hence, the green markers mostly leave the monopole region (see fig. 10c). However, when the closed recirculation region appears again (see fig. 10d), the monopole captures a great deal of the red markers initially associated with the topographic vortex. Thus, during the topography capturing, the monopole encloses some red markers, then after being carried away from the topographic vortex, it advects them to infinity (see fig. 10e).

Figures 9 and 10 also clearly shows that the particle advection is greatly affected to the number of monopole revolutions about the topography. The longer the monopole revolves about the topography the more effective advection is. To estimate that short-term monopole influence, we have performed a numerical simulation, in which we calculate the number of fluid particles escaping the topographic vortex with respect to the number of the monopole revolutions. Since the monopole motion is irregular, two initially close monopole trajectories wind the topography very differently, with different revolution numbers. Hence, it is impossible to predict how many revolutions complete the monopole starting at a new initial position. Thus, as initial positions for the
monopole, we have chosen two intervals of initial positions
\((x = -2, y \in [-8.42; -8.38])\) for the upper-layer monopole
propagation case, and \((x = -1.18, y \in [-8.02; -7.98])\) for
the middle-layer monopole propagation case.

Then we have followed the evolution of all the monopoles
starting at these initial position, calculating revolution num-
ber \(N\) of each of those monopoles, and obtained the ad-
vection efficiency through expression \(E = n_a/n_i\), where \(n_a\)
is the number of advected out of the topographic vortex
markers, i.e. the markers that have crossed line \(x = 5\), and \(n_i = 10^4\)
is the initial marker distribution number. It is also
worth noting, that although some of these monopoles have
revolved about the topography equal times, Lagrangian ad-
vection being generated by these monopoles is mostly equi-
valent in each case (see fig. 11). Indeed, each point in fig. 11
75 corresponds to one initial position of the monopole. Thus,
if different initial positions correspond to equal number of
monopole revolutions \(N\), then advection efficiency \(E\) is suf-

Figure 11 depicts advection efficiency \(E\) in the upper-
layer monopole propagation case (see fig. 11a,b), and in
the middle-layer monopole propagation case (see fig. 11c,d).
By analyzing these subfigures, one can draw several conclu-
sions. First, \(N = 0.5\) corresponds to the case of monopole
passing very close to the topographic vortex but not being
captured by it. In this case, although, if monopole is very
weak (\(\kappa = 0.01\)), it causes a great deal of fluid particle ad-
vection. A few monopole revolutions are enough for all the
particles from the topographic vortex region to be carried
away. Second, the sign of the monopole self-rotation is not
the main reason of the advection efficiency, but this efficiency
is mostly determined by \(|\kappa|\). Third, evidently, a singular

Fig. 8: Escaping time distribution in the middle-layer monopole propagation case.

Fig. 9: Upper-layer monopole propagation case. Particle scattering at the short-term monopole-topography interaction. Red
and green markers correspond to the topographic and monopole vortex regions, respectively, the blue dashed curve is the
unperturbed topographic vortex separatrix, and the black curve points out the trajectory of the monopole’s center. Subfigures
depict markers distribution at the corresponding instant in time.
monopole (see fig. 11a, b) causes much more efficient advection than a regular one (see fig. 11c, d).

6 Conclusions

In the frame of a three-layer geophysical flow model, Lagrangian advection of fluid particle in the vicinity of a monopole vortex interacting with a topographic vortex has been addressed. Two cases of the monopole propagation have been investigated: the upper-layer monopole propagation, and middle-layer monopole propagation. Such advection has been shown to be determined by two most significant processes. First, chaotic advection due to the nonstationarity of the monopole-topography interaction, and, second, the appearance or disappearance of closed recirculation zones in time. Cooperative influence of these processes causes very effective Lagrangian advection. Two controlling parameters, namely, the monopole’s strength and initial position have been analyzed, and, on the basis of the number of regular critical points assessment, a classification of different regimes of Lagrangian advection has been presented.

By adding a nonstationary term to the background flow, we have analyzed a short-term monopole-topography interaction. If the monopole passes nearly the topographic vortex, it still causes a great deal of particles initially located within the topographic vortex to be carried away. If the monopole is captured by the topographic vortex, then it rotates certain times about the topography, and, finally, is carried away by the background flow. During this passage, the topographic vortex almost completely renews its fluid.

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References

An, B. W. and McDonald, N. R.: Coastal currents and eddies and their interaction with topography, Dyn. Atmos. Oceans, 40, 237–253, doi:10.1016/j.dynatmoce.2005.04.002, 2005.

van Geffen, J. H. G. M. and Davies, P. A.: Interaction of a monopolar vortex with a topographic ridge, Geophys. Astrophys. Fluid Dyn., 90, 1–41, doi:10.1080/03091929908203691, 1999.

Aref, H.: Stirring by chaotic advection, J. Fluid Mech., 143, 1–21, 1984.

Aref, H.: The development of chaotic advection, Phys. Fluids, 14, 1315–25, doi:10.1063/1.1458932, 2002.

Aref, H. and Brons, M.: On stagnation points and streamline topology in vortex flows, J. Fluid Mech., 370, 1–27, doi:10.1017/S0022112098001761, 1998.

Wang, G. H. and Dewar, W. K.: Meddy-seamount interactions: Implications for the Mediterranean salt tongue, J. Phys. Oceanogr., 33, 2446–2461, doi:10.1175/1520-0485(2003)033⟨2446:MIIFTM⟩2.0.CO;2, 2003.

Carton, X.: Hydrodynamical modeling of oceanic vortices, Surv. Geophys., 22, 179–263, 2001.

Carton, X., Chérubin, L., Paulet, J., Morel, Y., Serpette, A., and Cann, B. L.: Meddy coupling with a deep cyclone in the Gulf of Cadiz, J. Mar. Syst., 32, 13–42, doi:10.1016/S0924-7963(02)00028-3, 2002.

Reznik, G. and Kizner, Z.: Two-layer quasi-geostrophic singular vortices embedded in a regular flow. Part 1. Invariants of motion and stability of vortex pairs, J. Fluid Mech., 584, 185–202, doi:10.1017/S0022112007006386, 2007a.

Reznik, G. and Kizner, Z.: Two-layer quasi-geostrophic singular vortices embedded in a regular flow. Part 2. Steady and unsteady drift of individual vortices on a beta-plane, J. Fluid Mech., 584, 185–202, doi:10.1017/S0022112007006404, 2007b.

Candon, S. and Marshall, J. S.: Vortex ring deformation, capture, and entrainment by a columnar vortex, Phys. Fluids, 24, 093 604, doi:http://dx.doi.org/10.1063/1.4753946, 2012.

Baines, P. G.: Topographic effects in stratified flows, Cambridge University Press, 1993.

Kozlov, V. F.: Background currents in geophysical hydrodynamics, Izv. Atmos. Ocean. Phys., 31, 245–250, 1995.

Reznik, G. M.: Dynamics of localized vortices on the beta plane, Izv. Atmos. Ocean. Phys., 46, 784–797, 2010.

Ryzhov, E. A.: On changing the size of the atmosphere of a vortex pair embedded in a periodic external shear flow, Phys. Lett. A,
Fig. 11: Lagrangian advection efficiency $E$ depending on monopole revolution number $N$. 

Kozlov, V. F. and Koshel, K. V.: Some features of chaos development in an oscillatory barotropic flow over an axisymmetric submerged obstacle, Izv. Atmos. Ocean. Phys., 37, 351–361, 2001.

Ryzhov, E. A. and Koshel, K. V.: Estimating the size of the regular region of a topographically trapped vortex, Geophys. Astrophys. Fluid Dyn., 105, 536–551, doi:10.1080/03091929.2010.511205, 2011b.

Zavala Sansón, L., Aguiar, A. C. B., and van Heijst, G. J. F.: Horizontal and vertical motions of barotropic vortices over a submarine mountain, J. Fluid Mech., 695, 173–198, doi:10.1017/jfm.2012.9, 2012.
dominated spreading in localized turbulent convection in a fast rotating stratified fluid, J. Fluid Mech., 423, 71–125, doi:10.1017/S002211200000183X, 2000.

Pedlosky, J.: Geophysical Fluid Dynamics 2 ed., Springer, New York, 1987.

Zaslavsky, G.: Physics of Chaos in Hamiltonian Dynamics, Imperial College Press, London, 1998.

Izrailsky, Y. G., Kozlov, V. F., and Koshel, K. V.: Some specific features of chaotization of the pulsating barotropic flow over elliptic and axisymmetric sea-mounts, Phys. Fluids, 16, 3173–3190, doi:10.1063/1.1767095, 2004.

Izrailsky, Y. G., Koshel, K. V., and Stepanov, D. V.: Determination of the optimal excitation frequency range in background flows, CHAOS, 18, 013107, doi:10.1063/1.2835349, 2008.

Voropayev, S. I., Smirnov, S. A., and Brandt, A.: Dipolar eddies in a stratified shear flow, Phys. Fluids, 13, 3820, doi:10.1063/1.1416499, 2001.

Filyushkin, B. N., Sokolovskiy, M. A., Kozhelupova, N. G., and Vagina, I. M.: Reflection of Intrathermocline Eddies on the Ocean Surface, Doklady Earth Sciences, 439, 986989, 2011.

Filyushkin, B. N. and Sokolovskiy, M. A.: Modeling the evolution of intrathermocline lenses in the Atlantic Ocean, J. Mar. Res., 69, 191–220, 2011.

Lichtenberg, A. and Lieberman, M.: Regular and Stochastic Motion, Springer-Verlag, New York, 1983.

Sokolovskiy, M. A.: Stability of an axisymmetric three-layer vortex, Izv. Atmos. Ocean. Phys., 33, 19–30, 1997.

Sokolovskiy, M. A., Zyryanov, V. N., and Davies, P. A.: On the influence of an isolated submerged obstacle on a barotropic tidal flow, Geophys. Astrophys. Fluid Dyn., 88, 1–30, doi:10.1080/03091929808245466, 1998.