Abstract.
We classify 3-braids up to (2, 2)-move equivalence and, in particular, we show how to adjust the Harikae-Nakanishi-Uchida conjecture so it holds for closed 3-braids. As an important step in classification of 3-braids up to (2, 2)-move equivalence we prove the conjecture for 2-algebraic links and classify (2, 2)-equivalence classes for links up to nine crossings. We also analyze the effect of (2, 2)-move on Kei (involutive quandle) associated to a link. We construct Burnside Kei, $Q(m, n)$, and ask the question, motivated by classical Burnside question: for which values of $m$ and $n$, is $Q(m, n)$ finite?

1 Introduction

A tangle move is a local modification of a link in which a tangle $T_1$ is replaced by a tangle $T_2$. The simplest such a move, that reduces every link in $S^3$ into a trivial link, is a crossing change.

It was believed that there are some nontrivial “tangle moves” with the unknotting property, other than the crossing change. For example, the Montesinos-Nakanishi conjecture stated that every link can be reduced to a trivial link via 3-moves ($\cdots \bigtriangleup \bigtriangleup \bigtriangleup$), the Nakanishi conjecture stated that every knot can be reduced to the trivial knot via 4-moves ($\cdots \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup$), and Harikae-Nakanishi-Uchida conjecture stated that every link can be reduced to a trivial link via (2, 2)-moves ($\cdots \rightarrow \bigtriangleup \bigtriangleup$).

It was proven in [D-P-I] that not every link can be reduced to a trivial link by 3-moves. Therefore, the Montesinos-Nakanishi conjecture does not hold. The Nakanishi 4-move conjecture is still an open problem. In this paper we analyze the Harikae-Nakanishi-Uchida conjecture in detail.
After 2-, 3-, and 4-moves one is tempted to ask about reductions of links by 5-moves. However, it is easy to show that not every link is 5-move equivalent to a trivial link\(^1\). One can show, using the Jones polynomial, that the figure eight knot is not 5-move equivalent to a trivial link\(^2\) (compare [Pr-1]). One can, however, introduce a more delicate move, called \((2,2)\)-move \([\text{compare Figure 1 \[H-U, Pr-3\]}]\) such that a 5-move is a combination of a \((2,2)\)-move and its mirror image, \((-2,-2)\)-move \((\text{compare Figure 1 \[H-U, Pr-3\]}\) as illustrated in Figure 1 [H-U, Pr-3].

We say that two links are \((2,2)\)-move equivalent if one can pass from one to the other by a finite number of \((2,2)\) and \((-2,-2)\)-moves.

**Conjecture 2.1 (Harikae, Nakanishi, Uchida 1992 [Kir])**

*Every link is \((2,2)\)-move equivalent to a trivial link.*

It was shown in [H-U, Pr-2] that the conjecture, as stated, does not hold. The knot 9\(_{49}\) is a counterexample. We will show, however, in this paper, to what extent the conjecture holds, and how can we modify it so it holds in its full generality. In particular, we show that every link up to 9 crossings is \((2,2)\)-move equivalent to a trivial link, 9\(_{40}, 9_{49}\) or their mirror images. The main

\(^1\)We say that two links are 5-move equivalent if one can reach one from the other by a finite number of 5-moves.

\(^2\)If two unoriented links are 5-move equivalent then their Jones polynomials for \(t^5 = -1, t \neq -1\) (with any orientations chosen for links) are equal up to an invertible element in \(\mathbb{Z}[t^{\pm 1}]\). Furthermore, \(V_{9_{49}}(e^{\pi i/5}) = (-t^{1/2} - t^{-1/2})^{n-1} = (\frac{-1 - \sqrt{5}}{2})^{n-1} \neq 0\) but \(V_{9_{40}}(e^{\pi i/5}) = t^2 - t + 1 - t^{-1} + t^{-2} = t^{-2} \frac{t^2 + 1}{t+1} = 0.\)
The result of this paper is a classification of closed 3-braids up to (2,2)-move equivalence. This result motivates the conjecture that every link can be reduced to a trivial link by ±(2,2)-moves and the $(\sigma_1\sigma_2)^6$-moves. (See Figure 2.)

For 3-braids we prove the following classification theorem.

**Theorem 2.2** Every closed 3-braid is (2,2)-move equivalent to a trivial link or to one of the torus links of the type (3,6), (3,12), (3,18) or (3,24), that is to closures of the 3-braids $(\sigma_1\sigma_2)^6$, $(\sigma_1\sigma_2)^{12}$, $(\sigma_1\sigma_2)^{18}$ or $(\sigma_1\sigma_2)^{24}$, respectively.

The braid $(\sigma_1\sigma_2)^{30}$ is 5-move equivalent to the identity 3-braid (Proposition 2.7.(iii)).

**Sketch of the proof of the Theorem 2.2.** First we show that Harikae-Nakanishi-Uchida conjecture holds for algebraic links (Definition 2.3), then we prove the conjecture for links up to 8 crossings and eventually, for links up to 9 crossings with an exception of (2,2)-equivalence classes of 9, 9, 9, and 9 (here $\bar{L}$ denotes the mirror image of $L$). Further, we identify classes of 9, 9, and their mirror images as closures of powers of the center of $B_3$. In particular, the knot $\bar{9}_{49}$ is (2,2)-move equivalent to the closure of the 3-braid $(\sigma_1\sigma_2)^6$.

We define below $n$-algebraic tangles generalizing a concept of an algebraic tangle introduced by J. Conway (his algebraic tangle is 2-algebraic in our definition).

**Definition 2.3** ([P-Ts1]) (i) $n$-algebraic tangles is the smallest family of $n$-tangles which satisfies:

(0) Any $n$-tangle with 0 or 1 crossing is $n$-algebraic.

(1) If $A$ and $B$ are $n$-algebraic tangles then $r^i(A) \ast r^j(B)$ is $n$-algebraic, where $r$ denotes the rotation of a tangle by $\frac{2\pi}{2n}$ angle, and $\ast$ denotes (horizontal) composition of tangles (compare Figure 5).

(ii) If a link $L$ is obtained from an $n$-algebraic tangle by closing its endpoints without introducing any new crossings then $L$ is called an $n$-algebraic link.

**Lemma 2.4** (i) Every 2-algebraic tangle is (2,2)-move equivalent to one of the six 2-tangles that are shown in Figure 8 with possible additional trivial components.

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\[\text{Figure 2: } (\sigma_1\sigma_2)^6\text{-move}\]
(ii) Every 2-algebraic link is $(2,2)$-move equivalent to a trivial link.

Proof: Part (ii) follows immediately from (i) which we prove by induction. Part(i) holds for each tangle with no more than one crossing. Thus we have to show that if it holds for tangles in Figure 3, say $A$ and $B$, then it holds also for $r^i(A) \ast r^j(B)$. All possible compositions involving tangles with a crossing are shown in Figure 4. In Figure 5 the reduction of the “most difficult” case is shown. Other cases are handled in a similar manner. □

Lemma 2.5 Every link up to 8 crossings is $(2,2)$-move equivalent to a trivial link.

Proof: According to Conway [Con], every link up to 8 crossings is 2-algebraic with a possible exception of 8$_{18}$ knot$^4$. The reduction of the 8$_{18}$ knot to a trivial link of two components by my students, Jarek Buczyński and Mike Veve, is illustrated in Figure 6. □

Lemma 2.6 Every link up to 9 crossings is $(2,2)$-move equivalent to a trivial link or one of the following: 9$_{40}$, 9$_{49}$, 9$_{40}$, 9$_{49}$.

Proof: According to Conway [Con] (checked for us by Slavik Jablan) the only possible 9 crossing non-algebraic links (up to mirror images) are: 9$_{34}$, 9$_{39}$, 9$_{40}$, 9$_{41}$, 9$_{47}$, 9$_{49}$, 9$_{40}$, 9$_{41}$, 9$_{42}$, 9$_{52}$, 9$_{61}$. We reduce them one by one as it is required in the theorem. □

We proved in [D-P-2] that the knots 9$_{40}$ and 9$_{49}$ are not $(2,2)$-equivalent to trivial links.

To prove Theorem 2.2, we use the above lemmas and the classical result of Coxeter that the group $B_3/(\sigma_1^5)$ is finite [Cox]. More precisely, the group has 600 elements and there are 45

4To prove that the knot 8$_{18}$ is not 2-algebraic one considers the 2-fold branched cover of $S^3$ branched along the knot $M_{8_{18}}^{(2)}$. Montesinos proved that algebraic knots are covered by Waldhausen graph manifolds [Mo-1]. Bonahon and Siebenmann showed ([B-S], Chapter 5) that $M_{8_{18}}^{(2)}$ is a hyperbolic 3-manifold so it cannot be a graph manifold. The knot 9$_{49}$ is not 2-algebraic neither because its 2-fold branched cover is a hyperbolic 3-manifold. In fact, it is the manifold I suspected from 1983 to have the smallest volume among oriented hyperbolic 3-manifolds [L-M-P-T, Kri, M-A-Y]. Our work on Burnside groups of links allows us to give a simple argument that the knots 9$_{40}$ and 9$_{49}$ are not 2-algebraic: every algebraic link is $(2,2)$-move equivalent to a trivial link but 9$_{40}$ and 9$_{49}$ are not. However, our method does not work for the knot 8$_{18}$.
| * | ![Figure 4](image1.png) | ![Figure 4](image2.png) | ![Figure 4](image3.png) | ![Figure 4](image4.png) | ![Figure 4](image5.png) |
|---|---|---|---|---|---|
| ![Figure 5](image6.png) | ![Figure 5](image7.png) | ![Figure 5](image8.png) | ![Figure 5](image9.png) | ![Figure 5](image10.png) | ![Figure 5](image11.png) |
| ![Figure 5](image12.png) | ![Figure 5](image13.png) | ![Figure 5](image14.png) | ![Figure 5](image15.png) | ![Figure 5](image16.png) | ![Figure 5](image17.png) |
| ![Figure 5](image18.png) | ![Figure 5](image19.png) | ![Figure 5](image20.png) | ![Figure 5](image21.png) | ![Figure 5](image22.png) | ![Figure 5](image23.png) |
| ![Figure 5](image24.png) | ![Figure 5](image25.png) | ![Figure 5](image26.png) | ![Figure 5](image27.png) | ![Figure 5](image28.png) | ![Figure 5](image29.png) |

**Figure 4:**

\[ r(\text{grid}) = \text{twist} \text{ (2,2)move } \text{ untwist} \]

**Figure 5:**

5
Figure 6: Reduction of the $8_{18}$ knot
conjugacy classes. At least 36 of them have representatives of length at most 8 (as checked using
the computer algebra software GAP). By Lemma 2.5, the closures of these braids are (2, 2)-move
equivalent to trivial links. Therefore, it suffices to analyze the remaining nine conjugacy classes of
$B_3/(\sigma_1^3)$ of the length at least 9. Four of them are listed in the Theorem 2.2: $((\sigma_1\sigma_2)^6, (\sigma_1\sigma_2)^{12},
(\sigma_1\sigma_2)^{18}, (\sigma_1\sigma_2)^{24})$. The closure of the remaining 5 braids can be reduced to trivial links with
some effort.

It follows from [D-P-2] that the closures of braids $(\sigma_1\sigma_2)^6, (\sigma_1\sigma_2)^{12}, (\sigma_1\sigma_2)^{18}, (\sigma_1\sigma_2)^{24}$ are
not (2,2)-move equivalent to trivial links. We do not know, however, whether they are (2,2)-move
equivalent among themselves. The method of [D-P-1, D-P-2], which uses Burnside groups of links
does not allow us to separate them (all four links have the same 5th Burnside group).

To connect Theorem 2.2. and Lemma 2.6. we prove the following.

**Proposition 2.7**

(i) The knot $\bar{9}_{49}$ is (2, 2)-move equivalent to the closure of the 3-braid $(\sigma_1\sigma_2)^6$.

(ii) The knot $9_{49}$ is (2, 2)-move equivalent to the closure of the 3-braid $(\sigma_1\sigma_2)^{12}$.

(iii) The 3-braid $(\sigma_1\sigma_2)^{30}$ (considered as a 3-tangle) is 5-move equivalent to the trivial 3-braid.

(iv) The knot $\bar{9}_{40}$ is (2, 2)-move equivalent to the closure of the 3-braid $(\sigma_1\sigma_2)^{18}$.

(v) The knot $9_{49}$ is (2, 2)-move equivalent to the closure of the 3-braid $(\sigma_1\sigma_2)^{24}$.

**Proof:**

(i) The knot $\bar{9}_{49}$ is (2, 2)-move equivalent to the closure of the 3-braid $\alpha_1 = (\sigma_1^2\sigma_2^{-1})^3$ (representing $9_{40}^\circ$) (Figure 7). Furthermore the mirror image of $\alpha_1$ is 5-move equivalent to $(\sigma_1\sigma_2)^6$
which is the square of the center of $B_3$. We have
\[\bar{\alpha}_1 = (\sigma_1^{-1}\sigma_2)^3 = (\sigma_1^{-1}\sigma_2)^2\sigma_2^{-1}\sigma_1^{-1}(\sigma_2^{-1}\sigma_1\sigma_2)\sigma_1^{-1}\sigma_2 = \sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^{-1}(\sigma_1\sigma_2\sigma_1^{-1})\sigma_2 =
\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^{-1} (\sigma_1\sigma_2)^2.\] Further, using three 5-moves we obtain:
\[\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^2 = 5 \quad \sigma_1^2\sigma_2\sigma_1^2\sigma_2^2\sigma_1\sigma_2^2 = (\sigma_1\sigma_2)^6 = \Delta_3^2,\] which is the square of the center of the 3-braid group.

Figure 7: $9_{49}$ (2,2)-move reduced to a 3-braid
(ii) The knot 9_{40} is (2, 2)-move equivalent to the closure of \( \alpha_2 = \sigma_1^2 \sigma_2 \sigma_1^{-2} \sigma_2^2 \sigma_1^{-2} \) as illustrated in Figure 8. This is conjugated to \( (\sigma_1 \sigma_2^3 \sigma_1^{-2} \sigma_2)(\sigma_2 \sigma_1^2 \sigma_1^{-2} \sigma_1) \). Using a 5-move twice we get 
\( (\sigma_1 \sigma_2^3 \sigma_1)(\sigma_2 \sigma_1^2 \sigma_1) \). Now we use expression for a square of the center: \( \Delta_4^4 = (\sigma_1 \sigma_2)^6 = \sigma_1 \sigma_2 \sigma_1^2 \sigma_1 \sigma_2 \sigma_1^3 \sigma_2 = \sigma_2^3 \sigma_1 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1 \) to reduce our braid to \( \Delta_4^4 \sigma_2^{-3} \sigma_1^{-3} \sigma_2^{-2} \Delta_4^4 \) which is reduced further, using two 5-moves, to \( \Delta_4^6 \) as needed.

(iii) We have: \((\sigma_1 \sigma_2)^{15} = \sigma_1^3 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-1} (\sigma_1 \sigma_2)^{12} = \sigma_1^3 \sigma_2 (\sigma_1 \sigma_2)^3 \sigma_1^3 (\sigma_1 \sigma_2)^3 \sigma_2 \sigma_1^{-1} (\sigma_1 \sigma_2)^6 = \sigma_1^3 \sigma_2 (\sigma_1 \sigma_2^3 \sigma_1^{-1} \sigma_2^3 \sigma_1^{-1}) \sigma_1^3 \sigma_2 (\sigma_1 \sigma_2^3 \sigma_1^{-1} \sigma_2^3 \sigma_1^{-1}) \sigma_1 (\sigma_1 \sigma_2 \sigma_1 \sigma_2^3 \sigma_1^{-1} \sigma_2^3 \sigma_1^{-1}) = 5 (\sigma_1^{-2} \sigma_2^6) =_{\text{conj}} (\sigma_1^{-2} \sigma_2^6)^{-3} = 5 (\sigma_1 \sigma_2)^{-15} \) as required.

(iv), (v) It follows from (i),(ii), and (iii).

\( \square \)

Let us stress that all four exceptional braids are powers of \((\sigma_1 \sigma_2)^6\), therefore every closed 3-braids can be reduced to a trivial link by \(\pm(2,2)\)-moves and \((\sigma_1 \sigma_2)^6\)-moves. This motivated us to conjecture that every link can be reduced to a trivial link by \(\pm(2,2)\)-moves and \((\sigma_1 \sigma_2)^6\)-moves.
We are glad to announce that the conjecture has been proven recently (a week before TWCU Conference) in the joint work with T. Tsukamoto [P-Ts2].

The $\pm(2,2)$-move is a rational $\pm\frac{5}{2}$-move (as illustrated in Figure 11) and the $n$-move is an $\mathbf{n}$ rational move. We noted in [Pr-4] that the space of Fox $n$-colorings $\text{Col}_n(L)$ (see Definition 3.1) is preserved by a rational $\frac{n}{q}$-move for any $q$. This can be used to show that different trivial links $U_m$ are not $(2,2)$-move equivalent (because $\text{Col}_n(U_m) = \mathbb{Z}_n^m$). The more sophisticated tool to study rational moves is the non-commutative version of Fox $n$-coloring, the $n$th Burnside group of the link $B_n(L)$ (see Definition 3.1 (ii)). This group, introduced in [D-P-1, D-P-2], is also invariant under rational $\frac{n}{q}$-moves and has been used to prove that knots $9_{40}$ and $9_{49}$ are not $(2,2)$-move equivalent to trivial links.

3 Kei and $(2,2)$-moves

Recall that Kei, $\pm$, called also involutive quandle, was introduced by Mituhisa Takasaki in 1942 [Tak] as an abstract algebra $(Q, \ast)$ with a binary operation $\ast : Q \times Q \rightarrow Q$ satisfying

(i) $a \ast a = a$ for any $a \in Q$ (indecomposability condition)

(ii) $(a \ast b) \ast b = a$ (involutive property)

(iii) $(a \ast b) \ast c = (a \ast c) \ast (b \ast c)$ (right distributivity law).

The above axioms have their correspondence in Reidemeister moves (see Figure 9).

With every group we can associate a core Kei, so that, the Kei operation is given by $a \ast b = ba^{-1}b$. For every unoriented link diagrams of $L$ we can associate the unique Kei, $\bar{Q}(L)$, by assigning to every arc of the diagram a formal variable and taking for every crossing the relation as in Figure 10 [Joy]. $\bar{Q}(L)$ is a link invariant.

We can distinguish links by comparing their associated Kei. For example the Kei of the trivial knot has one element, the Kei of the trefoil knot has three elements, the Kei of the figure-eight
knot has 5 elements and, more generally, the Kei of the rational $\frac{n}{q}$ link has $n$ elements.

The main examples of Kei that we use are:

(i) Dihedral Kei $Z_n$, which is a cyclic group $Z_n$ with the operation $i * j \equiv 2j - i \mod n$, and its direct sums $Z_n \oplus Z_n \oplus \cdots \oplus Z_n$. Notice that the operation $i * j \equiv 2j - i$ is an abelian version of the product $a * b = ba^{-1}b$.

(ii) The free Burnside Kei which is an associated core Kei to the Burnside group $B(m, n)$, where $B(m, n)$ is the free group $F_m$ divided by relations $w^n = 1$ for every $w \in F_m$.

Both of these cases can be used to produce invariants of links.

Definition 3.1  

(i) (a) We say that a link (or a tangle) diagram is $k$-colored if every arc is colored\footnote{We call such a coloring a Fox coloring as R. Fox introduced the construction when teaching undergraduate students at Haverford College in 1956.} by one of the numbers $0, 1, \ldots, k-1$ (forming a group $Z_k$) in such a way that at each crossing the sum of the colors of the undercrossings equals twice the color of the overcrossing modulo $k$.

(b) The set of $k$-colorings forms an abelian group, denoted by $Col_k(D)$.

(ii) (a) (Joy, [12]) The associated core group of an unoriented link diagram $D$, $\Pi^{(2)}_D$, is the group with generators that correspond to arcs of the diagram and any crossing $v_s$ yields the relation $r_s = y_i y_j^{-1} y_k$, where $y_i$ corresponds to the overcrossing and $y_j, y_k$ correspond to the undercrossings at $v_s$ (see Figure 10). The core Kei associated to $\Pi^{(2)}_D$ is the quotient of the Kei $\bar{Q}(D)$ introduced before.

(b) The unreduced $n$th Burnside group of a link $L$ is the quotient of the associated core group of the link by its normal subgroup generated by all relations of the form $w^n = 1$. Succinctly: $\hat{B}_L(n) = \Pi^{(2)}_L/(w^n)$.

(c) The $n$th Burnside group of a link is the quotient of the fundamental group of the double branched cover of $S^3$ with the link as the branch set divided by all relations of the form $w^n = 1$. Succinctly: $B_L(n) = \pi_1(M^{(2)}_L)/(w^n)$.
Both groups, the group of Fox $n$-colorings, $Col_n(L)$, and the Burnside group, $B_L(n)$, are invariant under $n$-moves or, more generally, under rational $\frac{n}{q}$ moves\cite{DP2}.

This motivates us to analyze general “behavior” of Kei under $n$-moves or, more generally, rational moves (including $(2,2)$-moves). It is useful to notice that the free two generator Kei, $Q(2,\infty)$, is isomorphic to dihedral Kei $Z$ with Kei operation, $i \ast j = 2j - i$.

**Lemma 3.2** (Joy) The Kei homomorphism $\phi : Q(2, \infty) \rightarrow Z$, where $\phi$ is given by $\phi(a) = 0$, $\phi(b) = 1$, and $a, b$ are generators of $Q(2, \infty)$, is an isomorphism.

**Sketch of the Proof:** We have $\phi((w * a) * b) = \phi(w) + 2$ and $\phi(w * a) = -\phi(w)$. These properties imply that $\phi$ is an epimorphism, in particular we have (using the left-normed convention: $(...((a*b)*c)*d)... = a*b*c*d...$) that $\phi(b*a) = -1$, $\phi(a*b) = 2$, $\phi(a*b*a) = -2$, $\phi(b*a*b) = 3$, $\phi(b*a*b*a) = -3$, $\phi(a*b*a*b) = 4$, $\phi(a*b*a*b*a) = -4$, $\phi(b*a*b*a*b) = 5$. In general, $\phi(b*a*b*a*b*...a*b) = 2k + 1$ (2k + 1 letters are used), $\phi(a*b*...a*b) = 2k$ (2k letters are used), $\phi(a*b*...a*b*a) = -2k$ (2k + 1 letters are used), and $\phi(b*a*...a*b*a) = -(2k+1)$ (2k + 2 letters are used).

Monomorphism follows from the fact that every element of $Q(2, \infty)$ can be written as a product of elements in the left-normed form (we use the identity $x * (y * z) = x * z * y * z$ to achieve this, compare \cite{Kam}).

Let us denote by $r_n(a, b)$ the relation given by $\phi^{-1}(0) = \phi^{-1}(n)$. For example, $r_5(a, b)$ is given by $a = b * a * b * a * b$. It follows from the above that any Kei that is invariant of $n$-moves must satisfy relations of the form $\phi^{-1}(0) = \phi^{-1}(n)$. In particular, for the $5$-move we get $a = babab$ or equivalently $aba = bab$. A similar relation for Kei is required for the $\pm(2,2)$-move\footnote{One can prove more generally that $\frac{n}{q}$-rational move is preserving Kei satisfying universal relation of the type $\phi^{-1}(0) = \phi^{-1}(n)$. For example, $(2,2)$-move is equivalent to the rational $\frac{5}{2}$-move and the Kei is preserved if $a = b * a * b * a * b$ as illustrated in Figure 11.}

We denote by $Q(m, n)$ the Kei which has $m$ generators and relations $r_n(u, w)$ for any elements $u, w$. We allow $n = \infty$ and then $Q(m, \infty)$ is a free, $m$-generator Kei. $Q(m, n) = Q(m, \infty)/(r_n(u, w))$. We have a Kei homomorphism $q : Q(m, n) \rightarrow B(m, n)$ being the identity on generators. To our knowledge, it is an open problem whether $Q(m, 3)$ is finite or infinite.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Figure 11:}
\end{figure}
Kei indicated that $Q(2,3)$ has $3$ elements and $Q(3,3)$ has $9$ elements. T.Ohtsuki recently (after the TWCU conference) computed that $Q(4,3)$ has $81$ elements. The result of his calculation is equivalent to showing that the Kei homomorphism $Q(4,3) \to B(4,3)$ is a monomorphism. Notice that $Q(m, 3)$ is a commutative Kei because $r_3 = a = b*a*b$ is equivalent to $a*b = b*a$.

For the analysis of $(2,2)$-moves, the Kei $Q(m, 5)$ is very important. Every Kei, which produces an invariant of links which is preserved by $(2,2)$-moves, is the quotient of $Q(m, 5)$. In fact the proof that $940$ and $949$ are not $(2,2)$-move equivalent to trivial links can be formulated in the language of Kei being quotients of $Q(m, 5)$. The 5th Burnside group defined in [DE2] satisfies $a*b*a = b*a*b$. We define the $n$th Burnside Kei of a link $L$, $BQ_n(L)$, as a quotient of a link Kei $\bar{Q}(L)$ by all relations $r_n(u, w)$ (shortly, $BQ_n(L) = \bar{Q}(L)/(r_n(u, w))$). For the trivial link of $m$ components $U_m$, we have $BQ_n(U_m) = Q(m, n)$.

Lemma 3.3 $BQ_n(L)$ is preserved by $n$-moves or, more generally, by rational $\frac{m}{q}$-moves.

Proof: The above lemma follows from Lemma 3.2 and the facts (essentially known to Conway, see [Pr-4]) that the rational $\frac{m}{q}$-move preserves the space of Fox $n$-colorings, $Col_n(L)$. □

On the more philosophical level, Lemma 3.2 can be used to show that many facts involving rational tangle moves and Fox colorings can be formulated in the language of Kei associated to a link.

Motivated by the Burnside’s question about finiteness of $B(m, n)$ groups we can ask a similar question about Kei $Q(m, n)$.

Question 3.4 For which values of $m$ and $n$, is $Q(m, n)$ finite?

$Q(1, n)$ has one element. $Q(2, n)$ has $n$ elements and it is isomorphic to the dihedral Kei $Z_n$. $Q(m, 2)$ has $m$ elements and it is a trivial Kei $\mathbb{K}am$. With support of the knowledge of Burnside groups we could make, maybe too far reaching, prediction that $Q(m, n)$ is finite for $n = 2, 3, 4 \ldots$

7The number of elements of $Q(4,3)$ also has been computed by M.Niebrzydowski. He has also showed that $Q(3,4)$ has $3 \times 2^5 = 96$ elements.

8Burnside groups of links are instances of groups of finite exponents. Our method of analysis of tangle moves rely on the well developed theory of classical Burnside groups and the associated graded Lie rings. A group $G$ is of a finite exponent if there is a finite integer $n$ such that $g^n = e$ for all $g \in G$. If, in addition, there is no positive integer $m < n$ such that $g^m = e$ for all $g \in G$, then we say that $G$ has an exponent $n$. Groups of finite exponents were considered for the first time by Burnside in 1902 [Bur]. In particular, Burnside himself was interested in the case when $G$ is a finitely generated group of a fixed exponent. He asked the question, known as the Burnside Problem, whether there exist infinite and finitely generated groups $G$ of finite exponents.

Let $F_r = \langle x_1, x_2, \ldots, x_r | \rangle$ be the free group of rank $r$ and let $B(r, n) = F_r/N$, where $N$ is the normal subgroup of $F_r$ generated by $\{g^n | g \in F_r\}$. The group $B(r, n)$ is known as the $r$th generator Burnside group of exponent $n$. In this notation, Burnside’s question can be rephrased.
To show that $9_{40}$ and $9_{49}$ are not (2,2)-equivalent to a trivial link we used the 5-th Burnside group $B_L(5)$ of links. Because these groups are Kei’s, we could use $BQ_L(5)$ to show that fact. The Kei $BQ_L(5)$ is at least as good as $B_L(5)$ in analyzing (2,2)-move equivalence. Possibly $BQ_L(5)$ can be used to distinguish (2,2)-move equivalence classes of $9_{40}, \bar{9}_{40}, 9_{49}$ and $\bar{9}_{49}$. The first step in this direction is to get better understanding of Burnside Kei, $BQ_L(5)$. In particular to check whether $q : Q(5, 5) \rightarrow B(5, 5)$ has a nontrivial kernel.

4 Application of (2,2)-moves to tangle embedding

We illustrate our method by one example: The tangle $T_1$ of Figure 12 cannot be embedded in the trivial knot, the Hopf link, and the trefoil knot. To demonstrate this, notice that $T_1$ can be reduced by two $\pm (2,2)$-moves into the tangle $T_2$ which has a trivial component. Therefore, every closure of $T_2$ has a nontrivial Fox 5-coloring. The claim in the example follows now from the fact that a (2,2)-move is not changing the number of 5-colorings and the trivial knot, the Hopf link, and the trefoil knot have only trivial Fox 5-colorings.

as follows. For which values of $r$ and $n$ is the Burnside group $B(r, n)$ finite? $B(1, n)$ is a cyclic group $Z_n$. Burnside proved that $B(r, 3)$ is finite for all $r$ and that $B(2, 4)$ is finite. In 1940 Sanov proved that $B(r, 4)$ is finite for all $r$, and in 1958 M.Hall proved that $B(r, 6)$ is finite for all $r$. However, it was proved by Novikov and Adjan in 1968 that $B(r, n)$ is infinite whenever $r > 1$, $n$ is odd and $n \geq 4381$ (this result was later improved by Adjan, who showed that $B(r, n)$ is infinite if $r > 1$, $n$ is odd and $n \geq 665$). Sergei Ivanov proved that for $k \geq 48$ the group $B(2, 2^k)$ is infinite. Lysënok found that $B(2, 2^k)$ is infinite for $k \geq 13$. It is still an open problem, though, whether, for example, $B(2, 5), B(2, 7)$ or $B(2, 8)$ are infinite or finite [VL, D-P-3].

We have been informed by M.Niebrzydowski that $BQ_L(5)$ is the same 25 element Kei, $Z_5 \oplus Z_5$, for $9_{40}$ and $9_{49}$ knots; e-mail: January 14, 2005.
References

[B-S] F. Bonahon, L. Siebenmann, Geometric splittings of classical knots and the algebraic knots of Conway, to appear(?) in L.M.S. Lecture Notes Series, 75.

[Bur] W. Burnside, On an Unsettled Question in the Theory of Discontinuous Groups, *Quart. J. Pure Appl. Math.* 33, 1902, 230-238.

[Con] J. H. Conway, An enumeration of knots and links and some of their algebraic properties, *Proceedings of the conference on Computational problems in Abstract Algebra held at Oxford in 1967*, J. Leech ed., (First edition 1970), Pergamon Press, 329-358.

[Cox] H.S.M. Coxeter, Factor groups of the braid group, Proc. Fourth Canadian Math. Congress, Banff–1957, 1959, 95-122.

[D-I-P] M. K. Dąbkowski, M. Ishiwata, J. H. Przytycki, Rational moves and tangle embeddings: (2,2)-moves as a case study, *Proceedings of the Conference Topology of Knot VII* (held at TWCU, December 23-26), February, 2005, 37-46, in Japanese.

[D-P-1] M. K. Dąbkowski, J. H. Przytycki, Burnside obstructions to the Montesinos-Nakanishi 3-move conjecture, *Geometry and Topology*, June, 2002, 335-360. [http://front.math.ucdavis.edu/math.GT/0205040](http://front.math.ucdavis.edu/math.GT/0205040)

[D-P-2] M. K. Dąbkowski, J. H. Przytycki, Unexpected connection between Burnside groups and Knot Theory, *Proc. Nat. Acad. Science*, 101, December 2004, 17357-17360. [http://front.math.ucdavis.edu/math.GT/0309140](http://front.math.ucdavis.edu/math.GT/0309140)

[D-P-3] M. K. Dąbkowski, J. H. Przytycki, Burnside groups in knot theory, preprint 2004.

[F-R] R.Fenn, C.Rourke, Racks and links in codimension two, *Journal of Knot Theory and its Ramifications*, 1(4) 1992, 343-406.

[H-U] T. Harikae, Y. Uchida, Irregular dihedral branched coverings of knots, in *Topics in knot theory*, N.A.T.O. A.S.I. series C, 399, (ed. M.Bozhüyük) Kluwer Academic Publisher (1993), 269-276.

[I-MPT] An interview with William Thurston, Warsaw, August 1983, compare [Pr-3](#).

[Joy] D.Joyce, A classifying invariant of knots: the knot quandle, *Jour. Pure Appl. Alg.*, 23, 1982, 37-65.

[Kam] S. Kamada, Knot invariants derived from quandles and racks, *Geometry & Topology Monographs*, Volume 4, 2002-4, 103-117: [http://front.math.ucdavis.edu/math.GT/0211096](http://front.math.ucdavis.edu/math.GT/0211096)
[Kir] R.Kirby, Problems in low-dimensional topology: Geometric Topology (Proceedings of the Georgia International Topology Conference, 1993), Studies in Advanced Mathematics, Volume 2 part 2., Ed. W.Kazez, AMS/IP, 1997, 35-473.

[M-V] A. D. Mednykh, A. Vesnin, Covering properties of small volume hyperbolic 3-manifolds. J. Knot Theory Ramifications 7(3), 1998, 381–392.

[Mo-1] J. M. Montesinos, Variedades de Seifert que son cubiertas ciclicas ramificadas de dos hojas, Bol. Soc. Mat. Mexicana (2), 18, 1973, 1–32.

[Pr-1] J.H.Przytycki, $t_k$-moves on links, In Braids, ed. J.S.Birman and A. Libgober, Contemporary Math. Vol. 78, 1988, 615-656.

[Pr-2] J.H.Przytycki, Skein modules of 3-manifolds, Bull. Ac. Pol.: Math., 39(1-2), 1991, 91-100.

[Pr-3] J. H. Przytycki, From 3-moves to Lagrangian tangles and cubic skein modules, Advances in Topological Quantum Field Theory, Proceedings of the NATO ARW on New Techniques in Topological Quantum Field Theory, Kananaskis Village, Canada from 22 to 26 August 2001; Ed. John M. Bryden, October 2004, 71-125; http://front.math.ucdavis.edu/math.GT/0405248

[Pr-4] J.H.Przytycki, Three talks in Cuautitlan under the general title: Topologia algebraica basada sobre nudos, Proceedings of the First International Workshop on "Graphs – Operads – Logic", Cuautitlan, Mexico, March 12-16, 2001, to appear 2005; http://front.math.ucdavis.edu/math.GT/0109029

[P-Ts1] J. H. Przytycki, T. Tsukamoto, The fourth skein module and the Montesinos-Nakanishi conjecture for 3-algebraic links, J. Knot Theory Ramifications, 10(7), November 2001, 959-982. http://front.math.ucdavis.edu/math.GT/0010282

[P-Ts2] J. H. Przytycki, T.Tsukamoto, Every link can be reduced by $(2,2)$- and $(\sigma_1\sigma_2)^n$-moves, in preparation 2005.

[Tak] M. Takasaki, Abstraction of symmetric transformation, (in Japanese) Tohoku Math. J., 49, 1942/3, 145-207.

[VL] M. Vaughan-Lee, The restricted Burnside problem; Second edition. London Mathematical Society Monographs. New Series, 8. The Clarendon Press, Oxford University Press, New York, 1993. xiv+256 pp.

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