How to break the uniqueness of $W^{1,p}_{loc} (\Omega)$-solutions for very singular elliptic problems by non-local terms

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Abstract. In this paper, we are going to show existence of branches of bifurcation of positive $W^{1,p}_{loc}(\Omega)$-solutions for the very singular non-local $\lambda$-problem

$$- \left( \int_{\Omega} g(x,u) dx \right)^r \Delta_p u = \lambda \left( a(x) u^{-\delta} + b(x) u^\beta \right) \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \quad \text{and } u = 0 \text{ on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\delta > 0$, $0 < \beta < p - 1$, $a$ and $b$ are nonnegative measurable functions and $g$ is a positive continuous function. Our approach is based on sub-supersolutions techniques, fixed point theory, in the study of $W^{1,p}_{loc}(\Omega)$-topology of a solution application and a new comparison principle for sub-supersolutions in $W^{1,p}_{loc}(\Omega)$ to a problem with $p$-Laplacian operator perturbed by a very singular term at zero and sublinear at infinity.

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1. Introduction

In this paper, we will deal with the existence, nonexistence and multiplicity of positive $W^{1,p}_{loc}(\Omega)$-solutions for the singular non-local quasilinear $\lambda$-problem

$$(P_\lambda) \begin{cases} - \left( \int_{\Omega} g(x,u) dx \right)^r \Delta_p u = \lambda \left( a(x) u^{-\delta} + b(x) u^\beta \right) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a smooth bounded domain, $-\Delta_p u = -\text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator, $1 < p < N$, $\delta > 0$, $0 < \beta < p - 1$, $\lambda > 0$ is a real parameter and $a, b, g \geq 0$ are appropriate functions.

An overview about $(P_\lambda)$. This problem is non-local due to the presence of the term $\left( \int_{\Omega} g(x,u) dx \right)^r$, which implies that the equation in $(P_\lambda)$ is no longer pointwise equality. In general, the presence of such terms gives rise some additional difficulties in approaching this kind of problems by classical arguments. For example, many non-local problems are non-variational, in the sense that techniques of variational methods cannot be applied in a direct way. The non-local problems have been extensively studied in recent years, and their applications arise in various contexts, for example, in the study of systems of particles in thermodynamical equilibrium via gravitational potential [2,16], 2-D fully turbulent behavior.

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of real flow [7], thermal runaway in Ohmic heating [4,10], physics of plasmas, population behavior [11], thermodielcic flow in a conductor [19], gravitational equilibrium of polytropic stars [17] and modeling of cell aggregation through interaction with a chemical [28].

Many authors have studied non-local problems, but up to this date there are no results in the literature in the direction of the \( p \)-Laplacian operator with \( p \neq 2 \) in the context of \( W^{1,p}_{\text{loc}}(\Omega) \)-solutions to singular ones. About related problems with weak singularities \((0 < \delta < 1)\) for Laplacian operator, we quote the works [3,23,25,29] that showed existence of positive solutions. We note that just in [3] was considered weak solutions still in \( H^1_0(\Omega) \), while the others ones treated the problem in the context of classical solutions.

Recently, Souza et al. [26] considered

\[
\begin{aligned}
&-g(x, \int_\Omega u^p) \Delta u = \lambda u^\beta \quad \text{in } \Omega, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,
\end{aligned}
\]

where \( 0 < \beta \leq 1 \) and \( g \) satisfies suitable assumptions. They showed how the structure of the branches of bifurcation of the problem is affected by the non-local term both with \( g \) depending on \( x \in \Omega \) and in the autonomous case as well.

Despite García-Melián and Lis [14] not studied neither a singular problem nor a Dirichlet boundary condition problem, we are going to highlight their techniques. They showed existence of solution to the blow-up problem

\[
\left( 1 + \frac{1}{|\Omega|} \int_\Omega g(u)dx \right) \Delta u = \lambda f(u) \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = \infty \text{ on } \partial\Omega, \quad (1.1)
\]

where \( f : [0, \infty) \to (0, \infty) \) is an appropriate continuous function, by decoupling (1.1) in the system

\[
\begin{aligned}
&\Delta u = \alpha f(u) \quad \text{in } \Omega, \quad u = \infty \text{ on } \partial\Omega \\
&\alpha = \lambda \left( 1 + \frac{1}{|\Omega|} \int_\Omega g(u)dx \right)^{-1}
\end{aligned}
\]

and studying the behavior of the pair \((\alpha, u)\) solution of (1.2).

To obtain branches of bifurcation in \((0, \infty) \times \| \cdot \|_\infty, \) we have inspired on the García-Melián and Lis’ strategy by exploring the \((0, \infty) \times W^{1,p}_{\text{loc}}(\Omega)\)-topology of the pair \((\alpha, u)\) by using a new comparison principle for \( W^{1,p}_{\text{loc}}(\Omega) \)-sub and supersolutions that we proved as well. Taking advantage of this approach, we present a complete picture of the bifurcation diagram of problem \((P_\lambda)\). In particular, we show how the presence of the non-local term changes the structure of the bifurcation of the local problem (see problem \((L_\alpha)\) below), which emanates from \((0,0)\) and bifurcates from infinity at infinity.

Before introducing the main results of this work, we need to clarify what we mean by Dirichlet boundary condition and solution to \((P_\lambda)\).

After the remarkable paper of Mckenna [20], in 1991, we know that a solution of the problem \((P_\lambda)\) with \( a = 1, b = 0 \) and \( p = 2 \) still lies in \( H^1_0(\Omega) \) if, and only if, \( 0 < \delta < 3 \). So, for stronger singularities, we need a more general concept of zero-boundary conditions.

**Definition 1.1.** We say that \( u \leq 0 \) on \( \partial\Omega \) if \((u - \epsilon)^+ \in W^{1,p}_0(\Omega) \) for every \( \epsilon > 0 \) given. Furthermore, \( u \geq 0 \) if \(-u \leq 0 \) and \( u = 0 \) on \( \partial\Omega \) if \( u \) is nonnegative and non-positive on \( \partial\Omega \). About solutions.
Definition 1.2. We say that $u$ is a $W_{\text{loc}}^{1,p}(\Omega)$-solution for $(P_\lambda)$ if $u > 0$ in $\Omega$ (for each $K \subset \subset \Omega$ given there exists a positive constant $c_K$ such that $u \geq c_K > 0$ in $K$) and
\[
\left( \int_\Omega g(x,u)dx \right)^r \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_\Omega \left( a(x)u^{-\delta} + b(x)u^\beta \right) \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{1.3}
\]

Our approach is based on issues about existence, uniqueness and $(\alpha,u_\alpha)$-behavior in the $(0,\infty) \times W_{\text{loc}}^{1,p}(\Omega)$-topology for the local problem
\[
(L_\alpha) \ \begin{cases} -\Delta u = \alpha \left( a(x)u^{-\delta} + b(x)u^\beta \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{cases}
\]
where $(L_\alpha)$ is the problem $(P_\lambda)$ with $\lambda = \alpha$ and $r = 0$.

In this sense, we refine the proofs of existence of $W_{\text{loc}}^{1,p}(\Omega)$-solutions found [6,9] and [24] to include both more general potentials $a$ and $b$ and a bigger variation of $p$. The more delicate issue is the uniqueness of solutions in $W_{\text{loc}}^{1,p}(\Omega)$ for the problem $(L_\alpha)$. The main results in [8] and [9] treated about this issue. In [8], by exploring the linearity of the Laplacian operator, they showed uniqueness of solutions to $(P_\lambda)$ with $p = 2$, $b = 0$ and $a \in L^1(\Omega)$, while in [9] the problem $(P_\lambda)$ with $b = 0$ was treated with some restrictions either on the potential $a$ or on the geometry of the domain.

Despite the next result being so classical, it is new even for Laplacian operator both by generality of the potentials $a$ and $b$ and principally by the uniqueness of solution in the context of $W_{\text{loc}}^{1,p}(\Omega)$ for very singular nonlinearities perturbed by $(p-1)$-sublinear ones.

Theorem 1.1. Assume $0 \leq b \in L^{(\frac{p-1}{p})'}(\Omega)$. If one of the assumptions below holds
\begin{align*}
(h_1): & \quad 0 < \delta < 1 \quad \text{and} \quad 0 \leq a \in L^{(\frac{p-1}{p})'}(\Omega), \\
(h_2): & \quad \delta \geq 1 \quad \text{and} \quad 0 \leq a \in L^1(\Omega),
\end{align*}
then there exists a $u = u_\alpha \in W_{\text{loc}}^{1,p}(\Omega)$ solution to the problem $(L_\alpha)$, for each $\alpha > 0$ given. Moreover, if $\delta \leq 1$, then $u \in W_{0,\text{loc}}^{1,p}(\Omega)$. Besides this, the solution is unique if $a + b > 0$ in $\Omega$.

As said above, by using properties of the solution $u_\alpha \in W_{\text{loc}}^{1,p}(\Omega)$ together with a Loc-Schmitt’s theorem [22], we are able to prove the next result.

Before stating it, let us consider
\begin{align*}
(h_3) & \quad a, b \in L^m(\Omega) \quad \text{for some } m > N/p, \\
(h_4) & \quad a, b \in L^m(\Omega) \quad \text{for some } m > N
\end{align*}
and denote by
\[
\Sigma = \left\{ (\lambda,u) \in (0,\infty) \times C(\Omega) : u \in W_{\text{loc}}^{1,p}(\Omega) \cap C(\Omega) \text{ is a solution of } (P_\lambda) \right\}.
\]
So, we have.

Theorem 1.2. Assume $\delta > 0$ and $0 < \beta < p-1$ hold. If:
\begin{align*}
(1) & \quad g \in C(\overline{\Omega} \times [0,\infty),(0,\infty)), \text{ for some } g_\infty \in C(\overline{\Omega}), \\
& \quad \lim_{t \to \infty} g(x,t)t^{\theta_\infty} = g_\infty(x) > 0 \text{ uniformly in } \overline{\Omega}, \tag{1.4}
\end{align*}
and in addition
\begin{align*}
(a) & \quad (h_3) \text{ and } \theta_\infty r < p - 1 - \beta \text{ hold, then } (P_\lambda) \text{ admits at least one solution in } \Sigma, \text{ for each } \lambda > 0 \text{ given. Besides this, we can assume } g_\infty \equiv \infty \text{ if } r \geq 0 \text{ and } g_\infty \equiv 0 \text{ if } r < 0, \\
(b) & \quad (h_4), \theta_\infty r > p - 1 - \beta \text{ and } \theta_\infty < 1 \text{ hold, then there exists } 0 < \lambda^* < \infty \text{ such that } (P_\lambda) \text{ admits at least two } W_{\text{loc}}^{1,p}(\Omega) \cap C(\overline{\Omega})\text{-solutions for each } \lambda \in (0,\lambda^*) \text{ given, at least one solution for } \lambda = \lambda^* \text{ and no solution for } \lambda > \lambda^*. \text{ Moreover, we can admit } g_\infty \equiv 0 \text{ if } r \geq 0 \text{ and } g_\infty \equiv \infty \text{ if } r < 0,
\end{align*}
(2) \( g \in C((0, \infty), (0, \infty)) \), \((h_4)\) is satisfied, for some \( g_\infty \) and \( g_0 \in C(\overline{\Omega}) \)

\[
\lim_{t \to \infty} g(x, t)^{\theta_\infty} = g_\infty(x) > 0 \quad \text{and} \quad \lim_{t \to 0^+} g(x, t)^{\theta_0} = g_0(x) > 0 \quad \text{uniformly in } \overline{\Omega}
\]

(1.5)

and additionally

(a) \( \theta_\infty r < p - 1 - \beta, \; \theta_0 r > p - 1 + \delta \) and \( \theta_0 < 1 \) hold, then there exists a \( 0 < \lambda^* < \infty \) such that \((P_\lambda)\) admits at least two \( W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \)-solutions for \( \lambda > \lambda^* \), at least one for \( \lambda = \lambda^* \) and no solutions for \( 0 < \lambda < \lambda^* \). Besides this, the conclusion is the same if we assume \( g_\infty \equiv g_0 \equiv \infty \) if \( r > 0 \) and \( g_\infty \equiv g_0 \equiv 0 \) if \( r < 0 \),

(b) \( \theta_\infty r > p - 1 - \beta, \; \theta_0 r > p - 1 + \delta \) and \( \theta_\infty, \theta_0 < 1 \) hold, then \((P_\lambda)\) admits at least one \( W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \)-solution for each \( \lambda > 0 \) given. In this case, we can have \( g_\infty \equiv 0 \) and \( g_0 \equiv \infty \) if \( r > 0 \) and \( g_\infty \equiv \infty \) and \( g_0 \equiv 0 \) if \( r < 0 \).

Moreover, in all cases \( \Sigma \) is the continuum of solutions given by a curve which:

(i) emanates from 0 at \( \lambda = 0 \) and bifurcates from infinity at \( \lambda = \infty \) in the case \( 1 - a \),

(ii) emanates from 0 at \( \lambda = 0 \) and bifurcates from infinity at \( \lambda = 0 \) in the case \( 1 - b \),

(iii) emanates from 0 at \( \lambda = \infty \) and bifurcates from infinity at \( \lambda = \infty \) in the case \( 2 - a \),

(iv) emanates from 0 at \( \lambda = \infty \) and bifurcates from infinity at \( \lambda = 0 \) in the case \( 2 - b \),

Below, we draw the \((0, \infty) \times \| \cdot \|_\infty\)-diagram of \( W^{1,p}_{\text{loc}}(\Omega) \)-solutions given by the above theorem.

**Fig. 1.** Theorem 1.2 item 1-(a)

**Fig. 2.** Theorem 1.2 item 1-(b)
Below, we list some of the main contributions of this work to the literature:

1. Our result of uniqueness for the local problem \((L_\alpha)\) improves the main theorems in [8] and [9] by:
   (i) removing any requirement about the geometry of the domain,
   (ii) permitting a perturbation of the very singular term by a sublinear one,
   (iii) including more general potentials \(a\) and \(b\).

2. Singular problems of the type \((P_\lambda)\) involving the \(p\)-Laplacian operator with \(\delta\) assuming any positive value and potentials \(a\) and \(b\) being unbounded have not yet been considered in the literature up to now.

3. Theorem 1.2 complements the principal results in [26] by consider a perturbation of their nonlinearity by a strong singular one.

4. The problem \((P_\lambda)\) with the non-local term \(a(t) = t^r, t > 0\), in the context of singular nonlinearity perturbed by a \((p - 1)\)-sublinear term, has not yet been considered in the literature so far. Our non-local term is not required being bounded either from below by positive constant or from above, and in fact it may be singular at \(t = 0\). See, for instance, [23, 25, 26] and references therein.

This work has the following structure. In Sect. 2, we prove the uniqueness of \(W^{1,p}_\text{loc}(\Omega)\)-solutions to the problem \((L_\alpha)\) inspired on ideas of [13] and [9]. To do this, a comparison principle for \(W^{1,p}_\text{loc}(\Omega)\)-sub and supersolutions is established. As the proof of existence of solution of Theorem 1.1 follows by a refinement of well-known arguments, we will just sketch it in Appendix. In Sect. 3, by exploring the uniqueness of
\(W_{\text{loc}}^{1,p}(\Omega)\)-solutions to problem \((L_\alpha)\), appropriate test functions together with a result of Boccardo and Murat [5], we are able to prove that the operator \(T : (0, \infty) \to W_{\text{loc}}^{1,p}(\Omega)\) (see (1.18)) is well defined and continuous. In Sect. 4, we conclude the proof of Theorem 1.2.

Throughout this paper, we make use of the following notations:

- The norm in \(L^p(\Omega)\) is denoted by \(\| \cdot \|_p\).
- \(W_0^{1,p}(\Omega)\) is the usual Sobolev space endowed with the norm \(\| \nabla u \|_p^p = \int_{\Omega} \| \nabla u \|^p dx\).
- \(|U|\) stands for the Lebesgue measure of measurable set \(U \subset \mathbb{R}^N\).
- \(C_c^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} : u \in C^\infty(\Omega) \text{ and } \text{supp } u \subset \subset \Omega \}\).
- \(L_c^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} : u \in L^\infty(\Omega) \text{ and } \text{supp } u \subset \subset \Omega \}\).
- \(C, C_1, C_2, \ldots\) denote positive constants.

2. Comparison principle for sub- and supersolutions in \(W_{\text{loc}}^{1,p}(\Omega)\)

Below, let us define subsolution and supersolution to the problem

\[
\begin{cases}
-\Delta u = a(x)u^{-\delta} + b(x)u^\beta & \text{in } \Omega, \\
0 & \text{on } \partial\Omega, \quad u > 0 \text{ in } \Omega.
\end{cases}
\]

(1.6)

**Definition 2.1.** A function \(v \in W_{\text{loc}}^{1,p}(\Omega)\) is a subsolution of (1.6) if:

(i) there is a positive constant \(c_K\) such that \(v \geq c_K\) in \(K\) for each \(K \subset \subset \Omega\) given;

(ii) the inequality

\[
\int_{\Omega} \| \nabla v \|^{p-2}\nabla v \nabla \varphi dx \leq \int_{\Omega} \left( \frac{a(x)}{v^\delta} + b(x)v^\beta \right) \varphi dx
\]

(1.7)

holds for all \(0 \leq \varphi \in C_c^\infty(\Omega)\). A function \(v \in W_{\text{loc}}^{1,p}(\Omega)\) satisfying i) and the reversed inequality in (1.7) is called a supersolution to problem (1.6).

**Theorem 2.1.** \((W_{\text{loc}}^{1,p}(\Omega) - \text{Comparison principle})\) Suppose \(b \in L^{\left(\frac{p^*}{p^*+\varepsilon}\right)}(\Omega)\) and \(a + b > 0\) in \(\Omega\). Assume that one of the following hypothesis

\((h_1)\): \(0 < \delta < 1\) and \(a \in L^{\left(\frac{p^*}{p^*+\varepsilon}\right)}(\Omega)\),

\((h'_2)\): \(\delta > 1\) and \(a \in L^1(\Omega)\),

\((h''_2)\): \(\delta = 1\) and \(a \in L^s(\Omega)\) for some \(s > 1\)

holds. If \(v, \bar{v} \in W_{\text{loc}}^{1,p}(\Omega)\) are subsolution and supersolution of (1.6), respectively, with \(v \leq 0\) in \(\partial\Omega\), then \(v \leq \bar{v}\) a.e. in \(\Omega\). Besides this, if in addition \(v, \bar{v} \in W_{0}^{1,p}(\Omega)\) and (1.7) is satisfied for all \(0 \leq \varphi \in W_0^{1,p}(\Omega)\), then the same conclusion holds even for \(a \in L^1(\Omega)\) in \((h''_2)\).

To prove Theorem 2.1, let us consider for each \(\varepsilon > 0\) given, the functional \(J_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}\) defined by

\[
J_\varepsilon(\omega) = \frac{1}{p} \int_{\Omega} \| \omega \|_p^p dx - \int_{\Omega} F_\varepsilon(x, \omega) dx
\]

where \(F_\varepsilon(x, \omega) = \int_0^\omega f_\varepsilon(x, s) ds\), with

\[
f_\varepsilon(x, s) = \begin{cases}
\frac{a(x)(s+\varepsilon)^{-\delta} + b(x)(s+\varepsilon)^\beta}{a(x)\varepsilon^{-\delta} + b(x)\varepsilon^\beta} & \text{if } s \geq 0 \\
\frac{a(x)(s+\varepsilon)^{-\delta} + b(x)(s+\varepsilon)^\beta}{a(x)\varepsilon^{-\delta} + b(x)\varepsilon^\beta} & \text{if } s < 0
\end{cases}
\]
and denote by $\mathcal{C}$ the convex and closed set

$$\mathcal{C} = \left\{ \omega \in W^{1,p}_0(\Omega) : 0 \leq \omega \leq \overline{\omega} \right\},$$

where $\overline{\omega} \in W^{1,p}_0(\Omega)$ is a supersolution to problem (1.6).

**Lemma 2.1.** If $b \in L^{(\frac{p}{p-1})'}(\Omega)$ and one of the hypotheses $(h_1)$, $(h_2')$ or $(h_2'')$ holds, then the functional $J_{\epsilon}$ is coercive and weakly lower semicontinuous on $\mathcal{C}$.

**Proof.** Set $\omega \in \mathcal{C}$. First, we note that if $(h_2'')$ holds, then there exists a $C_\epsilon > 0$ such that $ln|z+\epsilon| \leq C_\epsilon (z+\epsilon)^t$ for all $z \geq 0$ and for $t = \min\{p^*/s', p - 1\} > 0$ fixed. Thus, by using either this fact, $(h_1)$ or $(h_2')$ and Sobolev embedding, we obtain

$$J_{\epsilon}(\omega) = \begin{cases} \frac{1}{p} \|\nabla \omega\|_{p}^p - C \left[ \|a\|_{(\frac{p^*}{p-1})'} \|\omega\|_{p}^{1-\delta} + \|b\|_{(\frac{p^*}{p-1})'} \|\omega\|_{p}^{\beta+1} + 1 \right] & \text{if } 0 < \delta < 1, \\ \frac{1}{p} \|\nabla \omega\|_{p}^p - C \left[ \|a\|_{s'} \|\omega\|_{p}^s + \|b\|_{(\frac{p^*}{p-1})'} \|\omega\|_{p}^{\beta+1} + 1 \right] & \text{if } \delta = 1, \\ \frac{1}{p} \|\nabla \omega\|_{p}^p - C \left[ \|b\|_{(\frac{p^*}{p-1})'} \|\omega\|_{p}^{\beta+1} + 1 \right] & \text{if } \delta > 1, \end{cases}$$

which leads to the coerciveness of $J_{\epsilon}$ in all the cases.

Next, let us show that $J_{\epsilon}$ is weakly lower semicontinuous on $\mathcal{C}$. Let $(\omega_n) \subset \mathcal{C}$ such that $\omega_n \rightharpoonup \omega$ in $W^{1,p}_0(\Omega)$. Suppose first that $0 < \delta < 1$. Take $C_1 > 0$ such that $(\int (\omega_n + \epsilon)^{p^*} dx)^{\frac{1-\delta}{p^*}} \leq C_1$. We claim that

$$\int_0^{\omega_n} \int_0^\infty a(x)(s + \epsilon)^{-\delta} dt \, dx \xrightarrow{n \to \infty} \int_0^{\omega} \int_0^\infty a(x)(s + \epsilon)^{-\delta} dt \, dx. \quad (1.8)$$

In fact, since $a \in L^{(\frac{p^*}{p-1})'}(\Omega)$ it follows from the absolute continuity of the Lebesgue integral that for given $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$\int_A a(x)^{\frac{p^*}{p-1}} \, dx \leq \left( \frac{\epsilon'}{C_1} \right)^{\frac{p^*}{p-1}}$$

for all measurable subset $A$ of $\Omega$ such that $|A| < \delta'$.

Thus, the boundedness of $(\omega_n)$ in $L^{p^*}(\Omega)$ with the above information leads us to

$$\int_A a(x)(\omega_n + \epsilon)^{1-\delta} \, dx \leq \left( \int_A a(x)^{\frac{p^*}{p-1}} \, dx \right)^{\frac{p^*}{p} - \delta} \left( \int_\Omega (\omega_n + \epsilon)^{p^*} \, dx \right)^{1-\delta} \leq \epsilon',$$

that is, $(\omega_n)$ has uniformly absolutely continuous integral over $\Omega$. If $\delta = 1$, we can redo the above arguments. So, in both cases our claim follows by applying Vitali’s convergence theorem. In the case $\delta > 1$, convergence (1.8) follows from the classical Lebesgue’s theorem.

Following close arguments as above, we obtain

$$\int_0^{\omega_n} \int_0^\infty b(x)(s + \epsilon)^\beta \, dt \, dx \xrightarrow{n \to \infty} \int_0^{\omega} \int_0^\infty b(x)(s + \epsilon)^\beta \, dt \, dx,$$

as well. This is enough to finish the proof of the lemma. \qed

Since $\mathcal{C}$ is convex and closed in the $W^{1,p}_0(\Omega)$-topology, it follows from Lemma 2.1 that there exists a $\omega_0 \in \mathcal{C}$ such that

$$J_{\epsilon}(\omega_0) = \inf_{\omega \in \mathcal{C}} J_{\epsilon}(\omega).$$
Lemma 2.2. For all \( \varphi \geq 0 \) in \( C_c^\infty(\Omega) \), we have
\[
\int_\Omega |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \varphi \, dx \geq \int_\Omega \left[ a(\omega_0 + \epsilon)^{-\delta} + b(\omega_0 + \epsilon)^\beta \right] \varphi \, dx.
\]

Proof. First, given a nonnegative \( \varphi \in C_c^\infty(\Omega) \), let us define \( v_t := \min\{\omega_0 + t\varphi, \widetilde{\varphi}\} \) and \( \omega_t := (\omega_0 + t\varphi - \widetilde{\varphi})^+ \) for \( t > 0 \). So, it follows from \( \omega_0 \leq \widetilde{\varphi} \) that \( v_t = \omega_0 \) and \( \omega_t = 0 \) in \( \Omega \setminus \text{supp} \varphi \). From these, we have \( v_t \in C \) because \( \widetilde{\varphi} \in W^{1,p}(\text{supp} \varphi) \) and \( 0 \leq v_t \leq \widetilde{\varphi} \). Besides, since \( \widetilde{\varphi} > 0 \) (see definition 2.1), we can find a \( t > 0 \) enough small such that \( t\varphi \leq 2\widetilde{\varphi} - \omega_0 \), that is, \( \omega_t \in C \) as well.

We define \( \sigma : [0, 1] \rightarrow \mathbb{R} \) by \( \sigma(s) = J_\epsilon (sv_t + (1-s) \omega_0) \). Then,
\[
0 \leq \lim_{s \rightarrow 0^+} \frac{\sigma(s) - \sigma(0)}{s} = \lim_{s \rightarrow 0^+} \frac{J_\epsilon (sv_t + (1-s) \omega_0) - J_\epsilon (0)}{s} = \int_\Omega |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla (v_t - \omega_0) \, dx - \int_\Omega a(\omega_0 + \epsilon)^{-\delta} (v_t - \omega_0) \, dx - \int_\Omega b(\omega_0 + \epsilon)^\beta (v_t - \omega_0) \, dx.
\]

Hence, using \( v_t - \omega_0 = t\varphi - \omega_t \) and the previous inequality, we get
\[
0 \leq t \int_\Omega \left[ \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 \nabla \varphi - a(x) (\omega_0 + \epsilon)^{\delta} \varphi - b(x) (\omega_0 + \epsilon)^\beta \varphi \right] \, dx
\]
\[
- \int_\Omega \left[ \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 \nabla \omega_t - a(x) (\omega_0 + \epsilon)^{\delta} \omega_t - b(x) (\omega_0 + \epsilon)^\beta \omega_t \right] \, dx.
\]
However, since \( \varphi \) is a supersolution of (1.6) and \( 0 \leq \omega_t \in W_0^{1,p}(\Omega) \cap L_\text{loc}^\infty(\Omega) \) (note that \( \omega_t \leq t\varphi \)), by the classical density arguments one obtains
\[
\int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \nabla \omega_t \, dx \geq \int_\Omega \left( a(x) \varphi^{\delta} + b(x) \varphi^\beta \right) \omega_t \, dx.
\]
Dividing both the sides of (1.9) by \( t > 0 \) and using (1.10), we get
\[
0 \leq \frac{1}{t} \int_\Omega \left[ \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 \nabla \varphi - a(x) (\omega_0 + \epsilon)^{\delta} \varphi - b(x) (\omega_0 + \epsilon)^\beta \right] \, dx
\]
\[
+ \frac{1}{t} \int_\Omega \left( \left| \nabla \varphi \right|^{p-2} \nabla \varphi - \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 \right) \nabla \omega_t \, dx
\]
\[
+ \frac{1}{t} \int_\Omega \left[ a(x) \left( (\omega_0 + \epsilon)^{\delta} - \varphi^{\delta} \right) + b(x) \left( (\omega_0 + \epsilon)^\beta - \varphi^\beta \right) \right] \omega_t \, dx. \tag{1.11}
\]
Below, let us estimate the two last integral in (1.11). First, by using \( \omega_t \rightarrow 0 \) a.e. in \( \Omega \) as \( t \rightarrow 0^+ \), the limit \( |\text{supp} \omega_t| \xrightarrow{t \rightarrow 0^+} 0 \) and the monotonicity of the \( p \)-Laplacian operator, we obtain
\[
\frac{1}{t} \int_\Omega \left( \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 - \left| \nabla \varphi \right|^{p-2} \nabla \varphi \right) \nabla \omega_t \, dx \geq \int_{\text{supp } \omega_t} \left( \left| \nabla \omega_0 \right|^{p-2} \nabla \omega_0 - \left| \nabla \varphi \right|^{p-2} \nabla \varphi \right) \nabla \varphi \, dx \xrightarrow{t \rightarrow 0} 0.
\]
To last integral, noting that \( \omega_0 \leq \varphi \), we have
\[
\frac{1}{t} \int_{\text{supp } \omega_t} \left[ a(x) \left( \varphi^{\delta} - (\omega_0 + \epsilon)^{\delta} \right) + b(x) \left( \varphi^\beta - (\omega_0 + \epsilon)^\beta \right) \right] \omega_t \, dx
\]
\[
\geq \int_{\text{supp } \omega_t} \left[ a(x) \left| \varphi^{\delta} - (\omega_0 + \epsilon)^{\delta} \right| + b(x) \left| \varphi^\beta - (\omega_0 + \epsilon)^\beta \right| \right] \varphi \, dx \xrightarrow{t \rightarrow 0} 0.
\]
Hence, by using this information in (1.11), we conclude the proof.

\(\square\)

**Proof of Theorem 2.1.-Conclusion.** Let us set

\[\Omega_\epsilon := \{x \in \Omega : \psi(x) > \omega_0(x) + \epsilon\} \text{ and } \Omega_\epsilon^n = \Omega_\epsilon \cap \{x \in \Omega : \psi(x) < n\}\]

for given \(\epsilon > 0\) and \(n \in \mathbb{N}\). Thus, \(\Omega_\epsilon = \bigcup_{n \in \mathbb{N}} \Omega_\epsilon^n\).

Assume \(|\Omega_\epsilon| > 0\) for some \(\epsilon > 0\). Then, it is clear that \(|\Omega_\epsilon^n| > 0\) for all \(n \geq n_0\), for some \(n_0 \in \mathbb{N}\), because \(\Omega_\epsilon^n \subset \Omega^{n+1}\). Let us fix one of this \(n\). We claim that there exists a ball \(B_R(x_0) \subset \Omega\) such that \(|B_R(x_0) \cap \Omega_\epsilon^n| > 0\). Indeed, from the compactness of \(\Omega\), we can find an open set \(B \subset \mathbb{R}^N\) such that \(|B \cap \Omega_\epsilon^n| > 0\). Denote this measure by \(|B \cap \Omega_\epsilon^n| = 2\delta > 0\). If \(B \cap \partial \Omega \neq \emptyset\), set \(A_{\epsilon_0} = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \epsilon_0\}\), where \(\epsilon_0 > 0\) is taken in such a way that \(|B \cap A_{\epsilon_0}| < \delta\). In this case, \(|B \cap A_{\epsilon_0} \cap \Omega_\epsilon^n| > \delta\). So our claim follows from the fact that \(B \cap A_{\epsilon_0}\) is a compact set.

Set \(\phi \in C_c^\infty(\Omega_\epsilon \cap [0,1])\) such that \(\sup \phi \subset B_{R+\epsilon}(x_0)\), \(\phi = 1\) in \(B_R(x_0)\) and \(|\nabla \phi| \leq Cr^{-t}\) in \(B_{R+\epsilon}(x_0)\backslash B_R(x_0)\) for an appropriate \(t > 0\), which will be determined later. So, it is a consequence of this construction that \(0 = \varphi_1, \varphi_2 \in L^p(\Omega) \cap L_c^\infty(\Omega)\), where

\[\varphi_1 := \phi [v_\epsilon^n - (\omega_0 + \epsilon)^p]^+ v_\epsilon^{1-p} \quad \text{and} \quad \varphi_2 := \phi [v_\epsilon^n - (\omega_0 + \epsilon)^p]^+ (\omega_0 + \epsilon)^{1-p},\]

with \(v_\epsilon := \min\{\psi, n\}\).

Thus,

\[\nabla \varphi_1 = \left[\frac{v_\epsilon^n - (\omega_0 + \epsilon)^p}{v_\epsilon^{1-p}}\right]^+ \nabla \phi + \phi \left[\nabla v_\epsilon - p (\omega_0 + \epsilon)^{p-1} \frac{v_\epsilon^{p-1}}{v_\epsilon^n} \nabla (\omega_0 + \epsilon) + (p - 1) \frac{(\omega_0 + \epsilon)^p}{v_\epsilon^n} \nabla v_\epsilon \right] \chi_{[v_\epsilon \geq \omega_0 + \epsilon]}\]

and

\[\nabla \varphi_2 = \left[\frac{v_\epsilon^n - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}}\right]^+ \nabla \phi + \phi \left[\frac{p v_\epsilon^{p-1}}{(\omega_0 + \epsilon)^p} \nabla v_\epsilon - \nabla (\omega_0 + \epsilon) - (p - 1) \frac{v_\epsilon^n}{(\omega_0 + \epsilon)^p} \nabla (\omega_0 + \epsilon) \right] \chi_{[v_\epsilon \geq \omega_0 + \epsilon]},\]

which lead us to conclude that \(|\nabla \varphi_1|, |\nabla \varphi_2| \in L^p(\Omega)\), because \(0 < c_k \leq v_\epsilon \leq n \in K = \sup \phi\).

Since \(\varphi_1, \varphi_2 \geq 0\) and \(\varphi_1, \varphi_2 \in W^{1,p}_0(\Omega) \cap L_c^\infty(\Omega)\), we get by density arguments that

\[\int_\Omega |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \nabla \varphi_1 \, dx \leq \int_\Omega (a(x)|v_\epsilon|^{-\delta} + b(x)|v_\epsilon|^\beta) \varphi_1 \, dx\]

and

\[\int_\Omega |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \varphi_2 \, dx \geq \int_\Omega \left[a(x)(\omega_0 + \epsilon)^{-\delta} + b(x)(\omega_0 + \epsilon)^\beta\right] \varphi_2 \, dx\]

hold, where \(\omega_0\) is given in Lemma 2.2.

Therefore, by calculating and using the above inequalities, we obtain

\[\int_\Omega |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \nabla \phi \left[\frac{v_\epsilon^n - (\omega_0 + \epsilon)^p}{v_\epsilon^{1-p}}\right]^+ \, dx + \int_\Omega |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \left[\frac{v_\epsilon^n - (\omega_0 + \epsilon)^p}{v_\epsilon^{p-1}}\right]^+ \phi \, dx\]

\[-p \int_\Omega |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \left[\frac{(\omega_0 + \epsilon)^{p-1} \nabla (\omega_0 + \epsilon)}{n^{p-1}}\right] \chi_{[\omega_0 + \epsilon < n]} \phi \, dx \leq \int_\Omega (a(x)|v_\epsilon|^{-\delta} + b(x)|v_\epsilon|^\beta) \varphi_1 \, dx\]
and
\[
\int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] \phi dx
\]
\[+ \int_{\Omega} \int_{\omega < n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx \geq \int_{\Omega} [a(x)(\omega_0 + \epsilon)^{-\delta} + b(x)(\omega_0 + \epsilon)^{\beta}] \varphi_2 dx.
\]

Hence, by combining the previous inequalities we have
\[
\int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right] + \int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx
\]
\[-p \int_{\omega > n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{(\omega_0 + \epsilon)^{p-1} \nabla (\omega_0 + \epsilon)}{n^{p-1}} \right] \chi_{[\omega_0 + \epsilon < n]} \phi dx - \int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx
\]
\[- \int_{\omega \leq n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx \geq \int_{\omega > n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx
\]
\[= \int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \varphi_1 dx - \int_{\Omega} \int |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \varphi_2 dx
\]
\[\leq \int_{\Omega} \left[ \frac{v^{-\delta}}{v_n^{p-1}} - \frac{(\omega_0 + \epsilon)^{-\delta}}{(\omega_0 + \epsilon)^{p-1}} \right] [v_n^p - (\omega_0 + \epsilon)^p] + \phi dx + \int_{\Omega} b(x) \left[ \frac{v_n^{p-1} - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] [v_n^p - (\omega_0 + \epsilon)^p] + \phi dx.
\]

Now, by the previous inequality, the next one
\[- \int_{\omega > n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx = \int_{\omega > n} |\nabla \omega_0|^{p} \left[ 1 + \frac{n^p(p-1)}{(\omega_0 + \epsilon)^p} \right] dx \geq 0
\]
and by the classical Picones’s inequality, we get
\[0 \leq \int_{\omega \leq n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right] + \phi dx - \int_{\omega \leq n} |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + \phi dx
\]
\[\leq \frac{p}{n^{p-1}} \int_{\omega > n} |\nabla \omega_0|^{p-1} |\nabla \omega_0| (\omega_0 + \epsilon)^{p-1} \chi_{[\omega_0 + \epsilon < n]} \phi dx + \int_{\Omega} \int_{B_{R+r} \setminus B_R} |\nabla \omega_0|^{p-1} |\nabla \phi| \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right] + \phi dx
\]
\[+ \int_{B_{R+r} \setminus B_R} |\nabla \omega_0|^{p-1} |\nabla \phi| \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] + dx
\]
\[+ \int_{\Omega} a(x) \left[ \frac{v^{-\delta}}{v_n^{p-1}} - \frac{(\omega_0 + \epsilon)^{-\delta}}{(\omega_0 + \epsilon)^{p-1}} \right] [v_n^p - (\omega_0 + \epsilon)^p] + \phi dx
\]
\[+ \int_{\Omega} b(x) \left[ \frac{v^{\beta}}{v_n^{p-1}} - \frac{(\omega_0 + \epsilon)^{\beta}}{(\omega_0 + \epsilon)^{p-1}} \right] [v_n^p - (\omega_0 + \epsilon)^p] + \phi dx.
\]

(1.12)
Below, let us estimate the integrals in (1.12). For the last two integrals, we can deduce by the assumption $a + b > 0$, the inequality $v^{-\delta} \leq v_n^{-\delta}$ ($n \in \mathbb{N}$) and Lebesgue’s theorem that

$$
\int_{\Omega} a(x) \left[ \frac{v^{-\delta}}{v_n^{-\delta}} - \frac{(\omega_0 + \epsilon)^{-\delta}}{(\omega_0 + \epsilon)^{-p-1}} \right] [v_n^p - (\omega_0 + \epsilon)^p]^+ \phi dx
$$

$$
+ \int_{\Omega} b \left( \frac{v^p}{v_n^p} - \frac{(\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^p-1} \right) [v_n^p - (\omega_0 + \epsilon)^p]^+ \phi dx < -4\epsilon',
$$

holds for some $\epsilon' > 0$ and $n_0 > 1$ large.

Now, let us consider the first integral in the second line. We claim that $\|v > n\|_{n \to \infty} \to 0$. Indeed, otherwise it would exist $\delta' > 0$ and a subsequence $N' \subset \mathbb{N}$ such that $\|v > n\|_{n \to \infty} > \delta'$ for all $n \in N'$. By using that $(v - \epsilon)^+ \in W_0^1(\Omega)$, we would have

$$(n - \epsilon)\delta' < \int_{\{v > n - \epsilon\}} (v - \epsilon)^+ dx \leq \int_{\Omega} (v - \epsilon)^+ dx \leq C\|\nabla(v - \epsilon)^+\|_p < \infty, \quad \forall n \in N',$

which is absurd. As $\|v > n\|_{n \to \infty} \to 0$ and $n_0$ was taking sufficiently large, we obtain

$$
\left| \int_{\{v > n_0\}} |\nabla v|^p \left[ \frac{(\omega_0 + \epsilon)^{p-1} \nabla(\omega_0 + \epsilon)}{n_0^{p-1}} \right] x_{|\omega_0 + \epsilon < n_0|} \phi dx \right| \leq \left( \int_{\{v > n_0\}} |\nabla v|^p \phi^p \right)^{\frac{1}{p}} \|\nabla \omega_0\|_p \leq \epsilon'.
$$

To estimate the first integral on $B_{R+r} \setminus B_R$, we note that the choice of $\phi$ leads us to

$$
\int_{B_{R+r} \setminus B_R} |\nabla v|^{p-1} |\nabla \phi| \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right] dx \leq \int_{B_{R+r} \setminus B_R} |\nabla v|^{p-1} |\nabla \phi| n_0 dx
$$

$$
\leq C n_0 \|\nabla \phi\|_{L^p(B_{R+r} \setminus B_R)}
$$

$$
\leq C n_0 r^{-\frac{1}{p}} |B_{R+r} \setminus B_R|^\frac{1}{p} \leq C n_0 r^{-t+\frac{1}{p}}.
$$

By taking a $t < 1/p$, we can choose $r > 0$ sufficiently small such that

$$
\int_{B_{R+r} \setminus B_R} |\nabla \omega_0|^{p-1} |\nabla \phi| \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right] dx \leq \epsilon'.
$$

In a similar way, we can infer

$$
\int_{B_{R+r} \setminus B_R} |\nabla \omega_0|^{p-1} |\nabla \phi| \left[ \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right] dx \leq \epsilon'
$$

as well. Hence, getting back to inequality (1.12) and using the above information, we get

$$
0 \leq \int_{\{v \leq n_0\}} |\nabla v|^{p-2} |\nabla v| \left( \frac{v_n^p - (\omega_0 + \epsilon)^p}{v_n^{p-1}} \right) \phi dx - \int_{\{v \leq n_0\}} |\nabla \omega_0|^{p-2} |\nabla \omega_0| \left( \frac{v_n^p - (\omega_0 + \epsilon)^p}{(\omega_0 + \epsilon)^{p-1}} \right) \phi dx < 0,
$$

which is an absurd. Therefore, $|\Omega_n| = 0$ for all $n$, which implies $|\Omega_e| = 0$ and so $v \leq \omega_0 + \epsilon \leq \overline{v} + \epsilon$ a.e in $\Omega$ for all $\epsilon > 0$, whence $v \leq \overline{v}$.

To finish the proof, let us assume that $v, \overline{v} \in W_0^{1,p}(\Omega)$ and (1.7) is satisfied for all $0 \leq \varphi \in W_0^{1,p}(\Omega)$. By supposing $(v - \overline{v})^+ \neq 0$, defining $v_n^e(x) := \min\{v(x) + \epsilon, n\}, \overline{v}_n^e(x) := \min\{\overline{v}(x) + \epsilon, n\}$ and testing $v, \overline{v}$ by

$$
\varphi_1 = [(v_n^e)^p - (\overline{v}_n^e)^p]^+ (v_n^e)^{1-p} \text{ and } \varphi_2 = [(v_n^e)^p - (\overline{v}_n^e)^p]^+ (\overline{v}_n^e)^{1-p},
$$
we obtain
\[
\int_{[v+\epsilon>n, \pi+\epsilon\leq n]} \left(-|\nabla v|^p - |\nabla v|^p \frac{(\overline{v} + \epsilon)^{p-1} p}{n^{p-1}} - |\nabla \pi|^p + \frac{(p-1)n^p|\nabla \pi|^p}{(\overline{v} + \epsilon)^p}\right)\ dx
\]
\[+
\int_{[\pi+\epsilon \leq v+\epsilon\leq n]} \left(|\nabla v|^p - p \overline{v} + \epsilon \right)^{p-1}p |\nabla v|^p - p (\overline{v} + \epsilon) \left|\nabla v|^p - 2 \nabla v \nabla \overline{v} + (p-1) \left(\overline{v} + \epsilon\right)^p |\nabla \pi|^p\right)\ dx
\]
\[+ |\nabla \pi|^p - p \left(\overline{v} + \epsilon\right)^{p-1} |\nabla \pi|^p - 2 \nabla \pi \nabla \overline{v} + (p-1) \left(\overline{v} + \epsilon\right)^p |\nabla \pi|^p\right)\ dx.
\]
Denoting by
\[
I = \int_{[v+\epsilon>n, \pi+\epsilon\leq n]} \left(|\nabla v|^p - p \overline{v} + \epsilon \right)^{p-1}p |\nabla v|^p - p (\overline{v} + \epsilon) \left|\nabla v|^p - 2 \nabla v \nabla \overline{v} + (p-1) \left(\overline{v} + \epsilon\right)^p |\nabla \pi|^p\right)\ dx,
\]
and using the previous inequality along with the Picone’s inequality, we have
\[
0 \leq I \leq \int_{[v+\epsilon>n, \pi+\epsilon\leq n]} |\nabla v|^p |\nabla \pi|\ dx \ + \ \int_{[\pi+\epsilon\leq v+\epsilon\leq n]} b \left[ \frac{u^\beta}{n^p-1} - \frac{\overline{u}^\beta}{(\pi^\epsilon)^{p-1}} \right] \ dx
\]
\[+ \ \int_{[\pi+\epsilon\leq v+\epsilon\leq n]} b \left[ \frac{u^\beta}{n^p-1} - \frac{\overline{u}^\beta}{(\pi^\epsilon)^{p-1}} \right] \ dx.
\]
\[\text{(1.13)}
\]
Let us consider each one of the integrals in (1.13). The dominated convergence theorem implies that
\[
\int_{[v+\epsilon>n, \pi+\epsilon\leq n]} |\nabla v|^p |\nabla \pi|\ dx \xrightarrow{n\to\infty} 0.
\]
\[\text{(1.14)}
\]
By manipulating the second integral in (1.13), we obtain
\[
\int_{[v+\epsilon>n, \pi+\epsilon\leq n]} b \left[ \frac{u^\beta}{n^p-1} - \frac{\overline{u}^\beta}{(\pi^\epsilon)^{p-1}} \right] \ dx \xrightarrow{n\to\infty} 0
\]
\[\text{(1.15)}
\]
for all \(n \in \mathbb{N}\) and \(\epsilon > 0\). To the second last one, the dominated convergence theorem implies again
\[
\int_{[v+\epsilon>n, \pi+\epsilon\leq n]} b \left[ \frac{u^\beta}{n^p-1} - \frac{\overline{u}^\beta}{(\pi^\epsilon)^{p-1}} \right] \ dx \xrightarrow{n\to\infty} 0.
\]
\[\text{(1.16)}
\]
Throughout this section, we are going to assume the hypotheses of Theorem 1.1. So, it is well defined the $3.$

Next, let us prove that $(\underline{\nu} - \overline{\nu})^+ = 0$ and this ends the proof.

**Proof of Theorem 1.1 (Uniqueness).** In any case, by Theorem 2.1 we get $u \leq v$ and $v \leq u$, which implies $u = v$.

### 3. $W^{1,p}_\text{loc}(\Omega)$-continuity and a $\alpha$-behavior for a solution application

Throughout this section, we are going to assume the hypotheses of Theorem 1.1. So, it is well defined the solution application $T : (0, \infty) \to W^{1,p}_\text{loc}(\Omega)$ given by

$$T(\alpha) = u_\alpha,$$ 

(1.18)

where $u_\alpha \in W^{1,p}_\text{loc}(\Omega)$ is the unique solution of problem $(L_\alpha)$ given by Theorem 1.1.

Besides this, the below proposition is an immediate consequence of Theorem 2.1.

**Proposition 3.1.** The application $T$ in non-decreasing.

Next, let us prove that $T$ is a “$W^{1,p}_\text{loc}(\Omega)$-continuous application”, i.e.,

if $\alpha_n \to \alpha$ in $\mathbb{R}$, then $T(\alpha_n) \to T(\alpha)$ in $W^{1,p}(U)$ for each $U \subset \subset \Omega$ given.

To do this, let us begin stating a sub-supersolution result whose proof follows close arguments as done in Theorem 2.4 by Nguyen and Schmitt in [22].

**Theorem 3.1.** (Sub and supersolution theorem) Suppose that $\alpha$ and $b$ satisfy $(h_3)$ and $\underline{u}, \overline{u} \in W^{1,p}_\text{loc} \cap C(\Omega) \cap L^\infty(\Omega)$ are subsolution and supersolution of problem $(L_\alpha)$, respectively, with $0 < \underline{u} \leq \overline{u}$ a.e. in $\Omega$. Then, there exists a $u \in W^{1,p}_\text{loc} \cap L^\infty(\Omega)$ satisfying the equation in $(L_\alpha)$ with $u \in [\underline{u}, \overline{u}]$.  

For the last integral, since

$$b \left[ \frac{v^\beta}{(\nu + \epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{\nu} + \epsilon)^{p-1}} \right] \left[ (\nu + \epsilon)^p - (\overline{\nu} + \epsilon)^p \right] \leq \left[ v^\beta (\nu + \epsilon) + \overline{v}^\beta (\overline{\nu} + \epsilon) \right] \in L^1(\Omega),$$

it follows from Fatou’s lemma that

$$\limsup_{\epsilon \to 0} \int_{[\nu + \epsilon \leq \overline{\nu} + \epsilon \leq n]} b \left[ \frac{v^\beta}{(\nu + \epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\nu + \epsilon)^{p-1}} \right] [v^p - \overline{v}^p] \, dx \leq 0, \text{ for all } n \in \mathbb{N}.$$ 

Hence, going back to (1.13) and using (1.14), (1.15), (1.16) and (1.17), we get

$$0 \leq \limsup_{\epsilon \to 0} \liminf_{n \to \infty} I \leq \limsup_{\epsilon \to 0} \liminf_{n \to \infty} \left( \int_{\Omega} a \left[ \frac{v^{-\delta}}{(v^\alpha_n)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}^\alpha_n)^{p-1}} \right] [(v^\alpha_n)^p - (\overline{v}^\alpha_n)^p] \, dx \right) + \int_{[\nu + \epsilon > n, \overline{\nu} + \epsilon \leq n]} b \left[ \frac{v^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\nu + \epsilon)^{p-1}} \right] [n^p - (\overline{\nu} + \epsilon)^p] \, dx \right) \right.$$ 

Since we are assuming $(\nu - \overline{\nu})^+ \neq 0$ and $a + b > 0$ hold, it follows from the previous inequality that

$$0 \leq \limsup_{\epsilon \to 0} \liminf_{n \to \infty} I < 0,$$

which is an absurd. Therefore, $(\nu - \overline{\nu})^+ = 0$ and this ends the proof.
In what follows, $\Phi_1 \in W_0^{1,p}(\Omega)$ will denote the positive normalized eigenfunction associated with
\[ -\Delta_p \Phi_1 = \lambda_1 H_1(x) \Phi_1^{p-1} \text{ in } \Omega, \quad \Phi_1|_{\partial \Omega} = 0 \] (1.19)
where $H_1(x) := \min\{a(x), b(x)\} \geq 0$ and $\lambda_1 > 0$ is the first eigenvalue of (1.19) (see [1] and [12] for more details about (1.19)). If (h3) is satisfied, then by [18] one can conclude that $\Phi_1 \in C(\overline{\Omega})$. Moreover, if (h4) holds, then $\Phi_1$ belongs to the interior of the positive cone in $C_0^1(\Omega)$ (see Corollary 1.1 in [15]), and hence, we conclude by [27] that for some positive constant, one has
\[ C d(x) \leq \Phi_1(x) \text{ in } \Omega, \] (1.20)
where $d(x)$ stands for the distance between $x \in \Omega$ and the boundary $\partial \Omega$.

Similarly, defining $H_2(x) := \max\{a(x), b(x)\} \geq 0$ and denoting the unique positive solution of
\[ -\Delta_p u = H_2(x) \text{ in } \Omega, \quad u|_{\partial \Omega} = 0 \] (1.21)
by $\Phi_2 \in W_0^{1,p}(\Omega)$, it follows from (h3) and [18] that $\Phi_2 \in C(\overline{\Omega})$.

**Lemma 3.1.** $(T(\alpha))$-behavior for small $\alpha > 0$. Suppose that (h3) is satisfied. Then, there exists an $\alpha_0 > 0$ such that $T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]$ for all $\alpha \in (0, \alpha_0)$, where $\underline{u}_\alpha := \alpha^q \Phi_1$ and $\overline{u}_\alpha := \alpha^q \Phi_2$ with $q > \frac{1}{p-1+\delta}$, $l < \frac{1}{p-1+\delta}$ and $t = \frac{p-1}{p-1+\delta}$. In particular, $T(\alpha) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ for all $\alpha \in (0, \alpha_0)$.

**Proof.** Let $\alpha > 0$. Since $q > \frac{p-1}{p-1+\delta}$ hold, we are able to find $\alpha > 0$ such that $\sup_\Omega \Phi_1^{p-1+\delta} \leq \lambda_1^{-1} \alpha^{-1} (p-1+\delta)$ for all $\alpha \in (0, \alpha_1)$. Thus,
\[ -\Delta_p \underline{u}_\alpha \leq \lambda_1 \alpha^q \Phi_1(x) \sup_\Omega \Phi_1^{p-1} \leq \alpha^{1+\delta} \left( \sup_\Omega \Phi_1^q \right) a(x) = \alpha \frac{a(x)}{u_\alpha^q} \leq \alpha \left( \frac{\alpha(x)}{u_\alpha^q} + b(x) \right) \]
holds true.

To the supersolution, define $\overline{u}_\alpha := \alpha^q \Phi_2^t$, where $t = \frac{p-1}{p-1+\delta}$ and $l < \frac{1}{p-1+\delta}$. So, we obtain
\[ \int_\Omega |\nabla \overline{u}_\alpha|^{p-2} \nabla \overline{u}_\alpha \nabla \varphi dx \geq \int_\Omega |\nabla \Phi_2|^{p-2} \nabla \Phi_2 \nabla \left[ \varphi (\alpha^q \Phi_2^{t-1})^{p-1} \right] dx = \int_\Omega H_2(x) \left[ \varphi (\alpha^q \Phi_2^{t-1})^{p-1} \right] dx \]
for all $\varphi \geq 0$ in $C_0^\infty(\Omega)$.

Besides this, by using that $l < \frac{1}{p-1+\delta}$, we can choose $\alpha_2 > 0$ such that
\[ t^{p-1} \alpha^{l(p-1+\delta)-1} \geq 1 + \alpha^{l(\beta+\delta)} \sup_\Omega \Phi_2^{l(\beta+\alpha)} \]
for all $\alpha \in (0, \alpha_2)$. So by using the last inequality and (1.22), we obtain that $\overline{u} \overline{u}$ is a supersolution for $(L_\alpha)$.

Moreover, by taking $\varepsilon > 0$ such that $\varepsilon^{p-1} \lambda_1 \sup_\Omega \Phi_1^{p-1} < 1$, we have $\varepsilon \Phi_1 \leq \Phi_2$. Since $l < q$, there exists $\alpha_3 > 0$ such that $\sup_\Omega \Phi_1^{p-1} \leq \varepsilon^q \alpha^{1-q}$ for all $\alpha \in (0, \alpha_3)$. So by considering $\alpha_0 = \min\{\alpha_1, \alpha_2, \alpha_3\}$, we obtain from the above information that $\underline{u}_\alpha \leq \overline{u}_\alpha$.

Therefore, it follows from Theorem 3.1 that there exists a function $u \in W_0^{1,p} \cap L^\infty(\Omega)$ satisfying the equation in $(L_\alpha)$ with $u \in [\underline{u}_\alpha, \overline{u}_\alpha]$, for all $\alpha > 0$ small enough. Since, $\underline{u}_\alpha \overline{u}_\alpha \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, the function $u$ satisfies the boundary condition given in Definition 1.1. So, by the uniqueness claimed in Theorem 1.1, we conclude $u = \underline{u}_\alpha = T(\alpha)$.

Finally, it follows from the hypothesis (h3), the fact that $T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]$ and Corollary 8.1 in [18] that $u_\alpha \in C(\Omega)$ for $\alpha > 0$ small. As $\underline{u}_\alpha$ and $\overline{u}_\alpha \in C(\overline{\Omega})$ and $\underline{u}_\alpha|_{\partial \Omega} = \overline{u}_\alpha|_{\partial \Omega} = 0$, the required regularity follows.

Following close arguments as done above, we can prove the next lemma.
Lemma 3.2. \((T(\alpha))-\text{behavior for large } \alpha > 0\) Assume \((h_3)\) is satisfied. Then, there exists \(\alpha_\infty > 0\) such that \(T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]\) for all \(\alpha \in (0, \alpha_\infty)\), where \(\underline{u}_\alpha := \alpha^q \Phi_1\) and \(\overline{u}_\alpha := \alpha^q \Phi_2\) with \(q < \frac{1}{p-1-\beta}\), \(t > \frac{1}{p-1-\beta}\) and \(t = \frac{p-1}{p-1-\delta}\). In particular, \(T(\alpha) \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})\) for all \(\alpha \in (\alpha_\infty, \infty)\).

After the above lemmas and Proposition 3.1, we obtain that

\[ T((0, \infty)) \subset W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \]

when \((h_3)\) holds. Now, we are able to prove the continuity of \(T\).

Lemma 3.3. Suppose \((h_3)\) holds. Then, \(T\) is a continuous application in the \(W^{1,p}_{\text{loc}}(\Omega)\) topology as well in \(C(\overline{\Omega})\).

\textbf{Proof.} First let us prove the continuity of \(T\) in the \(W^{1,p}_{\text{loc}}(\Omega)\)-topology. Consider \(\alpha_n \to \alpha > 0\) in \(\mathbb{R}\). Then, it follows from Lemmas 3.1, 3.2 and monotonicity established in Proposition 3.1 that there exist \(0 < \underline{\alpha} < \alpha_0\) and \(\overline{\alpha} > \alpha_\infty\) such that

\[ \underline{\alpha}^q \Phi_1 = u_{\underline{\alpha}} \leq u_{\alpha_n} \leq u_{\overline{\alpha}} = \overline{\alpha}^q \Phi_2 \quad \text{in } \Omega, \text{ for all } n \in \mathbb{N}. \]  

(1.23)

Take an open set \(U \subset \subset \Omega\) and \(\xi \in C^\infty_c(\Omega)\) such that \(0 \leq \xi \leq 1\) and \(\xi = 1\) in \(U\). By using \(u_{\alpha_n} \xi^p\) as a test functions in \((L_{\alpha_n})\), we obtain

\[
\int_{\Omega} |\nabla u_{\alpha_n}|^p \xi^p dx + p \int_{\Omega} |\nabla u_{\alpha_n}|^{p-2} \nabla u_{\alpha_n} \nabla \xi u_{\alpha_n} \xi^{p-1} dx = \alpha_n \int_{\Omega} \left[ a(x) u_{\alpha_n}^{-\delta+1} + b(x) u_{\alpha_n}^{\beta+1} \right] \xi^p dx. \]  

(1.24)

So, it follows from the boundedness of \((u_n)\) in \(L^\infty(\Omega)\) and Young’s inequality that

\[
\int_{\Omega} |\nabla u_{\alpha_n}|^{p-2} \nabla u_{\alpha_n} \nabla \xi u_{\alpha_n} \xi^{p-1} dx \leq \epsilon \int_{\Omega} |\nabla u_{\alpha_n}|^{p-1} |\nabla \xi| u_{\alpha_n} \xi^{p-1} dx \leq \epsilon \int_{\Omega} \left( |\nabla u_{\alpha_n}|^{p-1} \xi^{p-1} \right)^{\frac{p}{p-1}} dx + C(\epsilon) \int_{\Omega} u_{\alpha_n}^{p} |\nabla \xi|^{p} dx \leq \epsilon \int_{\Omega} \xi^{p} |\nabla u_{\alpha_n}|^{p} dx + C(\epsilon), \]  

(1.25)

where \(C(\epsilon)\) is a cumulative positive constant.

Hence, by using (1.23) and (1.25) in (1.24), we get

\[
\int_{U} |\nabla u_{\alpha_n}|^{p} dx \leq \int_{\Omega} |\nabla u_{\alpha_n}|^{p} \xi^p dx \leq C(\epsilon),
\]

which results in \((u_{\alpha_n})\) bounded in \(W^{1,p}_{\text{loc}}(\Omega)\). So, there exists \(u \in W^{1,p}_{\text{loc}}(\Omega)\) such that

\[
\begin{aligned}
  u_{\alpha_n} & \to u \quad \text{in } W^{1,p}(U), \\
  u_{\alpha_n} & \to u \quad \text{in } L^q(U) \text{ for all } 1 \leq q < p^* \\
  u_{\alpha_n}(x) & \to u(x) \quad \text{almost everywhere in } \Omega,
\end{aligned}
\]  

(1.26)

for each \(U \subset \subset \Omega\) given.

By applying [5] (see Theorem 2.1), we obtain

\[ \nabla u_{\alpha_n} \to \nabla u, \quad \text{in } (L^q(\Omega))^N \text{ for any } q < p. \]

As a consequence, for each \(\varphi \in C^\infty_c(\Omega)\) given, one has

\[
\int_{\Omega} \left( |\nabla u_{\alpha_n}|^{p-2} \nabla u_{\alpha_n} - |\nabla u(x)|^{p-2} \nabla u \right) \nabla \varphi dx \to 0.
\]
Moreover, if $K$ denote the support of $\varphi$, we have
\[
\left| \left( \frac{a}{u_\alpha^n} + bu_\alpha^\beta \right) \varphi \right| \leq \left( \frac{a}{(\Omega \inf K)^\delta} + b\bar{\alpha}_2^\beta \sup K \right) \|\varphi\|_\infty \in L^1(K),
\]
whence using the dominated convergence theorem, we get
\[
\alpha_n \int_{\Omega} \left( \frac{a}{u_\alpha^n} + bu_\alpha^\beta \right) \varphi \, dx \to \alpha \int_{\Omega} \left( \frac{a}{u_\alpha} + bu_\alpha^\beta \right) \varphi \, dx.
\]
Hence, we conclude that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \alpha \int_{\Omega} \left( \frac{a}{u_\alpha} + bu_\alpha^\beta \right) \varphi, \quad \forall \varphi \in C_c^\infty(\Omega).
\]

Since $u_{\alpha_n}$ satisfies (1.23), we have $a^\beta \Phi_1 \leq u \leq \bar{\alpha}^\beta \Phi_2$. Thus, as $\Phi_1$ and $\Phi_2 \in C(\bar{\Omega})$, we obtain $0 \leq (u - \epsilon)^+ \leq (\bar{\alpha}^\beta \Phi_2 - \epsilon)^+$, that is, $(u - \epsilon)^+ \in W_{0}^{1,p}(\Omega)$ for each $\epsilon > 0$. Therefore, $u$ satisfies the boundary condition of Definition 1.1. So, by applying the uniqueness of $W_{0}^{1,p}(\Omega)$-solutions to problem $(L_\alpha)$ claimed in Theorem 1.1, we conclude that $u = u_{\alpha}$.

For the $C(\bar{\Omega})$-continuity, it follows from (1.23) that the sequence $(u_{\alpha_n})$ is bounded in $C^\alpha(K)$ for some $\alpha \in (0,1)$ and in each compact $K \subset \Omega$ given. So, it follows from Arzelà–Ascoli’s theorem and (1.26) that $u_{\alpha_n} \to u$ in $C(\Omega)$. Besides this, by using (1.23), we obtain $u \in C(\bar{\Omega})$ and $u_{\alpha_n} \to u$ in $C(\bar{\Omega})$. \hfill $\square$

4. Multiplicity of $W_{0}^{1,p}(\Omega)$-solutions for $(P_\lambda)$

Now we are able to prove Theorem 1.2. Before that, we will introduce the applications $G : D(G) \subset W_{0}^{1,p}(\Omega) \to [0, \infty)$ and $H : (0, \infty) \to (0, \infty)$ defined by
\[
G(u) = \left( \int_{\Omega} g(x,u) \, dx \right)^r \quad \text{and} \quad H(\alpha) = \alpha G(T(\alpha)),
\]
where $D(G) = \{ 0 \leq u \in W_{0}^{1,p}(\Omega) : G(u) < \infty \}$.

In addition, let us consider the system
\[
\begin{cases}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \alpha \int_{\Omega} \left( a(x)u^{-\delta} + b(x)u^\beta \right) \varphi \, dx \\
\alpha G(u) = \lambda,
\end{cases}
\]
(1.28)
remind that
\[
\Sigma = \left\{ (\lambda, u) \in (0, \infty) \times C(\bar{\Omega}) : u \in W_{0}^{1,p}(\Omega) \text{ is solution of } (P_\lambda) \right\}
\]
and set
\[
\Sigma' = \left\{ (H(\alpha), u_\alpha) \in (0, \infty) \times C(\bar{\Omega}) : \alpha \in (0, \infty) \text{ and } u_\alpha \in W_{0}^{1,p}(\Omega) \text{ is a solution of } (L_\alpha) \right\}.
\]

As a consequence of Lemma 3.3, we can prove the next result.

**Lemma 4.1.** Suppose one of the following item holds:

(i) $(h_3)$ is satisfied and $g \in C(\bar{\Omega} \times [0, \infty), (0, \infty))$;
(ii) $(h_4)$, $g \in C(\bar{\Omega} \times (0, \infty), (0, \infty))$ and $\lim_{t \to 0^+} g(x,t)t^{\theta_0} = g_0(x) \geq 0$ uniformly in $\bar{\Omega}$, for some $g_0 \in C(\bar{\Omega})$ and $0 < \theta_0 < 1$.

Then, $T((0, \infty)) \subset D(G)$ and, in particular, $H$ is well defined. Besides this, $H$ is a continuous function.
Proof. Take $\alpha > 0$. It follows from Lemmas 3.1, 3.2 and the monotonicity established in Proposition 3.1 that we can find $0 < \underline{\alpha} = \underline{\alpha}(\alpha) < \alpha_0$ and $\overline{\alpha} = \overline{\alpha}(\alpha) > \alpha_{\infty}$ such that

$$
\underline{\alpha} \Phi_1 \leq u_\alpha \leq \overline{\alpha} \Phi_2 \text{ in } \Omega,
$$

(1.29)

where $\Phi_1$ and $\Phi_2$ are the solutions of problems (1.19) and (1.21), respectively.

First, let us assume $i)$. So, by choosing an $\epsilon, t_0 > 0$ sufficiently small such that $\overline{\alpha} \Phi_2^i < t_0$ for all $x \in \Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \epsilon\}$, we obtain from (1.20), (1.29) and hypothesis $i)$ that $0 < g(x, u_\alpha) \leq C d(x)^{-\theta_0}$ in $\Omega_\epsilon$ for some $C > 0$.

Since $\theta_0 < 1$, it follows from [20] and the previous inequality that $g(x, u_\alpha) \in L^1(\Omega_\epsilon)$. In particular, $H$ is well-defined in this case.

About the case $i)$, the result follows directly from the fact that $\Phi_2$ is a bounded function. So, in both cases, we showed $T(\alpha) \in D(G)$ for each $\alpha > 0$ given.

To show the continuity, consider $\alpha_n \to \alpha > 0$. By an analogous argument as in first part, we can conclude that in any case there exists a $h(x) \in L^1(\Omega)$ such that $g(x, u_{\alpha_n}) \leq h(x)$, for all $x \in \Omega$ and $n \in \mathbb{N}$. Thus, the continuity follows from Lemma 3.3 and convergence dominated theorem. \hfill \Box

After this lemma, by using the uniqueness claimed in Theorem 1.1, we obtain the next one.

Lemma 4.2. Let $\lambda > 0$. Then, problem $(P_\lambda)$ admits a $W^{1,p}_0(\Omega)$-solution if, and only if, there exist $(\alpha, u) = (\alpha_\lambda, u_\lambda) \in (0, \infty) \times W^{1,p}_0(\Omega)$ solution of (1.28). In particular, problem $(P_\lambda)$ admits a $W^{1,p}_0(\Omega)$-solution if, and only if, $\lambda \in H((0, \infty))$.

As a rereading of the above lemma and a consequence of Lemma 3.3, we conclude that

$$
\Sigma = \{ (H(\alpha), u_\alpha) \in (0, \infty) \times C(\overline{\Omega}) : \alpha \in (0, \infty) \text{ and } u_\alpha \in W^{1,p}_0(\Omega) \text{ is a solution of } (L_\alpha) \}
$$

is the continuum of solutions to problem $(P_\lambda)$ given by a curve.

Proof of Theorem 1.2-Completed. Since the additional part in each item follows in analogous way, we will just prove the first part in each one of them.

(1-a) Firstly, note that by the continuity of $g$ and Lemma 3.1, we get $\lim_{\alpha \to 0} H(\alpha) = 0$. Let us split the prove in two cases:

(i) case 1: $r \geq 0$. If $\theta_{\infty} \geq 0$, by taking $U \subset \subset \Omega$ and using (1.4) together with Lemma 3.2, we obtain

$$
\int_\Omega g(x, u_\alpha)dx \geq \int_U g(x, u_\alpha)dx \geq C\alpha^{-\theta_{\infty} l}
$$

for all $\alpha$ sufficiently large. Thus, choosing $l \in \left( \frac{1}{p-1-\beta}, \frac{1}{\theta_{\infty} r} \right)$, we get

$$
H(\alpha) = \alpha \left( \int_\Omega g(x, u_\alpha)dx \right)^r \geq C\alpha^{1-r\theta_{\infty} l} \to \infty \text{ as } \alpha \to \infty.
$$

Suppose now that $\theta_{\infty} < 0$. In this case, by using (1.4) and Lemma 3.2 again, we get

$$
H(\alpha) \geq C\alpha^{1-q\theta_{\infty} r} \to \infty \text{ as } \alpha \to \infty.
$$

So, in both cases we have $H(\alpha) \to \infty$ as $\alpha \to \infty$.

(ii) case 2: $r < 0$. Consider the case $\theta_{\infty} \geq 0$. By hypothesis (1.4) and the continuity of $g$, we obtain

$$
\int_\Omega g(x, u_\alpha)dx \leq C, \text{ that is, } H(\alpha) \geq C\alpha^r \to \infty \text{ as } \alpha \to \infty.
$$
Analogously, when $\theta_\infty < 0$, we obtain by Lemma 3.2 and hypothesis (1.4) that
\[
H(\alpha) \geq C_\alpha \left(1 + \alpha^{\theta_\infty} \right)^r = C \left(\alpha^{\frac{1}{r}} + \alpha^{\frac{1}{r} - \theta_\infty} \right)^r,
\]
showing that $H(\alpha) \to \infty$ as $\alpha \to \infty$ by choosing $l < 1/\theta_\infty r$ in Lemma 3.2. Hence, in all cases we have $H(\alpha) \to 0$ as $\alpha \to 0$ and $H(\alpha) \to \infty$ as $\alpha \to \infty$. Since $H$ is continuous (see Lemma 4.2), our claim follows.

To finish the proof, it just remains to show the behavior of the continuum $\Sigma$ at $\lambda = 0$ and $\lambda = \infty$. For $\lambda = 0$, let us take $\epsilon > 0$ and define $\delta = \inf_{[\epsilon, \infty)} H(\alpha)$. Since $H(\alpha) \to \infty$ as $\alpha \to \infty$, it follows from Lemma 3.3 that $\delta > 0$ and $(0, \delta) \subset H((0, \epsilon))$, that is, for each $\lambda_n \in (0, \delta)$ there exists an $\alpha_n \in (0, \epsilon)$ such that $H(\alpha_n) = \lambda_n$. So, if $\lambda_n \to 0$, then $\alpha_n \to 0$, which implies by Lemma 3.1 that $\|u_{\alpha_n}\|_\infty \to 0$. For $\lambda = \infty$, define $m = \max H(\alpha)$ for each $\alpha > 0$ given, so $m < \infty$ and $(m, \infty) \subset H((M, \infty))$, that is, for each $\lambda_n \in (m, \infty)$, there exists $\alpha_n \in (M, \infty)$ such that $\lambda_n = H(\alpha_n)$. Hence, if $\lambda_n \to \infty$, then $\alpha_n \to \infty$, and so by using Lemma 3.2, we obtain $\|u_{\alpha_n}\|_\infty \to \infty$. See Fig. 1.

(1-b) Initially, suppose $r > 0$. In this case, $\theta_\infty > 0$ because $\theta_\infty r > p - 1 - \beta > 0$. By hypothesis (1.4) and continuity of $g$ in $\overline{\Omega} \times (0, \infty)$, we obtain $g(x, t) \leq C_1 t^{-\theta_\infty}$ for some $C_1 > 0$ and for all $t \geq 0$. Remembering that $Cd(x) \leq \Phi_1(x)$ in $\Omega$ holds because we are assuming $(h_1)$, it follows from Lemma 3.2 that $\int_\Omega g(x, u_n)dx \leq C_2 \alpha^{-\theta_\infty} q$ for $\alpha > 0$ large enough. Thus, by taking $q \in \left(\frac{1}{\theta_\infty q}, \frac{1}{p - 1 - \beta}\right)$, we obtain
\[
H(\alpha) \leq C_3 \alpha^{1 - \theta_\infty q} \to 0 \text{ as } \alpha \to \infty.
\]

Let us now consider the case where $r < 0$. In this case, $\theta_\infty < 0$, because we are assuming $\theta_\infty r > p - 1 - \beta > 0$ again. Hence, in an analogous way as done above, we can prove that $H(\alpha) \leq C_3 \alpha^{-q \theta_\infty r}$ for $\alpha > 0$ sufficiently large. Thus, by taking $q \in \left(\frac{1}{\theta_\infty r}, \frac{1}{p - 1 - \beta}\right)$, we get $H(\alpha) \to 0$ as $\alpha \to \infty$.

In any case, we proved that $\lim_{\alpha \to \infty} H(\alpha) = 0$. On the other hand, $H(\alpha) \to 0$ as $\alpha \to 0$, therefore setting $\lambda^* = \sup_{\mathbb{R}^+} H(\alpha)$, the claims follows.

About the behavior of $\Sigma$. Letting $(\lambda, u) \in \Sigma$, it is clear that $\lambda \leq \lambda^*$. Since $\lim_{\alpha \to 0} H(\alpha) = \lim_{\alpha \to \infty} H(\alpha) = 0$, we get $(0, \delta) \subset H((0, \epsilon) \cap H((M, \infty))$ for each $\epsilon > 0$ small and $M > 0$ large, where $0 < \delta = \min_{[\epsilon, M]} H(\alpha)$. Thus, for each $\lambda_n \in (0, \delta)$ there exists $\alpha_n \in (0, \epsilon)$ and $\alpha_n^2 \in (M, \infty)$ such that $\lambda_n = H(\alpha_n^1) = H(\alpha_n^2)$. So, $\lambda_n \to 0$, $\alpha_n^1 \to 0$, and $\alpha_n^2 \to \infty$, which leads us to conclude that $\|u_{\alpha_n^1}\|_\infty \to 0$ and $\|u_{\alpha_n^2}\|_\infty \to \infty$ after to use Lemmas 3.1 and 3.2 again. See Fig. 2.

(2-a) Initially assume $r > 0$. In this case, $\theta_0 > 0$, because we are considering $\theta r > p - 1 + \delta > 0$. Then, by taking $l < \frac{1}{p - 1 + \delta}$, $\alpha > 0$ sufficiently close to 0, using hypothesis (1.5) and Lemma 3.1, we get
\[
H(\alpha) \geq C_\alpha \left(\int_\Omega \frac{1}{\alpha^{\theta_0} \Phi_2(x)}^{\lambda_0} dx \right)^r = C \alpha^{1 - r \theta_0 l}
\]
for some $C > 0$ constant. By choosing $l > 1/\theta_0 r$ in Lemma 3.1, we obtain from (1.30) that $H(\alpha) \to +\infty$ as $\alpha \to 0^+$. In the same way, in the case where $r, \theta_0 < 0$, by hypotheses (1.5) and Lemma 3.1, we obtain $H(\alpha) \to +\infty$ as $\alpha \to 0^+$.

On the other hand, by following the same idea as used to prove the item (1-a), we can verify that $H(\alpha) \to \infty$ when $\alpha \to \infty$. Thus, by considering $\lambda^* = \inf_{\alpha \in \mathbb{R}^+} H(\alpha)$, the result follows.

(2-b) By the same arguments as used to prove the items (1-b) and (2-a), we can verify that $H(\alpha) \to 0$ and $H(\alpha) \to +\infty$, so the result follows again. These end the proof of Theorem 1.2.
Similarly to the cases $1-a$ and $1-b)$, we are able to verify that the continuum $\Sigma$ behaves as in Fig. 3 (item 2-a) and Fig. 4 (item 2-b), respectively.

**Remark 4.1.** Although our objective in this paper is to present situations of how to break the uniqueness of $W_{l_{oc}}^{1,p}(\Omega)$-solution to local problem $(L_u)$ by the introduction of a non-local term, we note that it is still possible to obtain uniqueness of solutions to non-local problem $(P_{\lambda})$. For instance, when $g(t) = t^\gamma$ for $t > 0$ with either $\{\gamma > 0$ and $r > 0\}$ or $\{-1 < \gamma < 0$ and $r < 0\}$.

5. Appendix

In this section, let us sketch the proof of existence stated in Theorem 1.1. For this, we will consider the following auxiliary problem:

$$
\begin{align*}
-\Delta_p u &= \frac{a_n(x)}{(u+\frac{1}{n})^\delta} + b_n(x)u^\beta \quad \text{in } \Omega, \\
u &= 0 \text{ on } \partial \Omega, \quad u > 0 \quad \text{in } \Omega
\end{align*}
$$

(1.31)

where $a_n(x) = \min\{a(x), n\}$ and $b_n(x) = \{b(x), n\}$, with $n \in \mathbb{N}$.

**Lemma 5.1.** For each $n \in \mathbb{N}$, problem (1.31) admits a solution $u_n \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Furthermore, for each compact set $K \subset \subset \Omega$ there exists $c_K > 0$ such that $u_n \geq c_K > 0$ in $K$, for all $n \in \mathbb{N}$.

**Proof.** For each $v \in L^p(\Omega)$, we claim that there exists a unique function $\omega \in W_0^{1,p}(\Omega)$ solution of

$$
-\Delta_p \omega = \frac{a_n(x)}{(|v|+\frac{1}{n})^\delta} + b_n(x)|v|^\beta.
$$

(1.32)

In fact, consider the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
J(\omega) = \frac{1}{p} \int_\Omega |\nabla \omega|^p dx - \int_\Omega \frac{a_n(x)}{(|v|+\frac{1}{n})^\delta} \omega dx - \int_\Omega b_n(x)|v|^\beta \omega dx.
$$

We can easily verify that $J$ is differentiable, strictly convex and coercive. Hence, $J$ admits a unique critical point, that is, (1.32) admits a solution.

Denoting by $S : L^p(\Omega) \to L^p(\Omega)$ the operator, which associates with each $v \in L^p(\Omega)$ the unique solution $w = S(v) \in L^p(\Omega)$ of (1.32), one can prove that $S$ is a continuous and compact operator. Furthermore, if $\omega = \lambda S(\omega)$ for some $\lambda \in (0,1]$ and $\omega \in W_0^{1,p}(\Omega)$, then by Poincaré’s and Hölder inequalities

$$
\|\omega\|_p^p \leq C_\lambda^p \int_\Omega |\nabla S(\omega)|^p dx = C \lambda^p \int_\Omega \left[\frac{a_n}{(\frac{1}{n} + |\omega|)^\delta} S(\omega) + b_n(x)|\omega|^\beta S(\omega)\right] dx
$$

$$
\leq C \lambda^{p-1} \int_\Omega \left(n^{1+\delta}|\omega| + n|\omega|^\beta + 1\right) dx \leq C \left(\|\omega\|_p + \|\omega\|^\beta_{p+1}\right),
$$

where $C > 0$ is a cumulative constant.

Thus, by the previous inequality, there exists a positive constant $R$, independent of $\lambda$ and $\omega$, such that $\|\omega\|_p \leq R$. So, by the Schaefer fixed point theorem, there exists an $u_n \in W_0^{1,p}(\Omega)$ such that $S(u_n) = u_n$.

Note that $a_n(|t| + \frac{1}{n})^{-\delta} + b_n|t|^\beta \leq C(1 + |t|^{\beta})$, so by [21] we have $u_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Furthermore, as $a_n(|u_n| + \frac{1}{n})^{-\delta} + b_n|u_n|^\beta \geq 0$ we obtain $u_n \geq 0$, which by [27] implies in $u_n > 0$ in $\Omega$. Therefore, $u_n$ is a positive solution of (1.31).

Besides this, suppose that $\tilde{u}_1$ is a solution of

$$
-\Delta_p u = \frac{a_1(x)}{(1+u)^\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \Omega.
$$

(1.33)
Taking \((\tilde{u}_1 - u_n)^+ \in W^{1,p}_0(\Omega)\) as a test function in (1.31) and (1.33) and using that \(-\Delta_p\) is strictly monotonic, we get

\[
0 \leq \int_{\Omega} \left( |\nabla \tilde{u}_1|^{p-2} \nabla \tilde{u}_1 - |\nabla u_n|^{p-2} \nabla u_n \right) \nabla (\tilde{u}_1 - u_n)^+ \, dx \\
\leq C \int_{\Omega} a_1 \left[ \frac{1}{(1 + \tilde{u}_1)^\delta} - \frac{1}{(1 + u_n)^\delta} \right] (\tilde{u}_1 - u_n)^+ \, dx \leq 0.
\]

Hence, \((\tilde{u}_1 - u_n)^+ = 0\), that is, \(\tilde{u}_1 \leq u_n\) in \(\Omega\).

Finally, by [21] we conclude that \(\tilde{u}_1 \in C^{1,\alpha}(\overline{\Omega})\) for some \(\alpha \in (0,1)\). Therefore, using this and the positivity of \(\tilde{u}_1\) in \(\Omega\), the last part of the lemma follows.

\[
\square
\]

**Proof of Theorem 1.1 (Existence–Conclusion).** Consider a sequence \((\Omega_k)\) of smooth open sets in \(\Omega\) such that \(\Omega_k \subset \Omega_{k+1}\), \(\bigcup_k \Omega_k = \Omega\) and define \(\delta_k = \inf_{\Omega_k} \tilde{u}_1 > 0\), where \(\tilde{u}_1\) is the solution of (1.33). Take \(\varphi = (u_n - \delta_1)^+\) as a test function in (1.31). If \((h_1)\) holds, then we have

\[
\int_{u_n > \delta_1} |\nabla u_n|^p \, dx \leq \int_{u_n > \delta_1} \left( \frac{a}{\delta_n^{1-\delta}} + b u_n^{\beta+1} \right) \, dx \\
\leq ||a||_{(p^*_\infty)^\prime} \left( \int_{u_n > \delta_1} u_n^{p^*} \, dx \right)^{\frac{1-\delta}{p}} + ||b||_{(p^*_\infty)^\prime} \left( \int_{u_n > \delta_1} u_n^{p^*} \, dx \right)^{\frac{\beta+1}{p}} \\
\leq C \left[ 1 + \left( \int_{u_n > \delta_1} |\nabla u_n|^p \, dx \right)^{\frac{1-\delta}{p}} + \left( \int_{u_n > \delta_1} |\nabla u_n|^p \, dx \right)^{\frac{\beta+1}{p}} \right].
\]

In a similar way

\[
\int_{u_n > \delta_1} |\nabla u_n|^p \, dx \leq \int_{u_n > \delta_1} \left( \frac{a}{\delta_n^{1-\delta}} + b u_n^{\beta+1} \right) \, dx \leq \delta_1^{1-\delta} \int_{\Omega} \, adx + ||b||_{(p^*_\infty)^\prime} \left( \int_{u_n > \delta_1} u_n^{p^*} \, dx \right)^{\frac{\beta+1}{p}} \\
\leq C \left[ 1 + \left( \int_{u_n > \delta_1} |\nabla u_n|^p \, dx \right)^{\frac{\beta+1}{p}} \right]
\]

is true if \((h_2)\) holds.

Therefore, \(\int_{\Omega_1} |\nabla u_n|^p \, dx\) will be bounded in any case. Furthermore, since \((u_n - \delta_1)^+ \in W^{1,p}_0(\Omega)\) we have

\[
\int_{\Omega_1} u_n^p \, dx \leq \int_{u_n > \delta_1} u_n^p \, dx \leq C \left[ 1 + \int_{\Omega} (u_n - \delta_1)^+ p \, dx \right] \\
\leq C \left[ 1 + \int_{u_n > \delta_1} |\nabla u_n|^p \, dx \right] \leq C.
\]
Thus, we conclude that \((u_n)\) is bounded in \(W^{1,p}(\Omega_1)\). Hence, there exists \(u_{\Omega_1} \in W^{1,p}(\Omega_1)\) and a subsequence \((u_{n_j})\) of \((u_n)\) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
\dfrac{u_{n_j}^\ast}{\varepsilon} \rightharpoonup u_{\Omega_1} & \quad \text{weakly in } W^{1,p}(\Omega_1) \\
\dfrac{u_{n_j}^\ast}{\varepsilon} \rightarrow u_{\Omega_1} & \quad \text{strongly in } L^q(\Omega_1) \text{ for } 1 \leq q < p^* \\
\dfrac{u_{n_j}^\ast}{\varepsilon} \rightarrow u_{\Omega_1} & \quad a.e. \text{ in } \Omega_1. 
\end{array} \right. 
\end{align*}
\]

Proceeding as above, we can obtain subsequences \((u_{n_j})\) of \((u_n)\), where \((u_{n_j+1}) \subset (u_{n_j})\), and functions \(u_{\Omega_k} \in W^{1,p}(\Omega_k)\) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
\dfrac{u_{n_j}}{\varepsilon} \rightharpoonup u_{\Omega_k} & \quad \text{weakly in } W^{1,p}(\Omega_k) \text{ and strongly in } L^p(\Omega_k) \text{ for } 1 \leq q < p^* \\
\dfrac{u_{n_j}}{\varepsilon} \rightarrow u_{\Omega_k} & \quad a.e. \text{ in } \Omega_k. 
\end{array} \right. 
\end{align*}
\]

By construction, \(u_{\Omega_k+1}|_{\Omega_k} = u_{\Omega_k}\). Defining

\[
u = \left\{ \begin{array}{ll}
u_{\Omega_1} & \quad \text{in } \Omega_1, \\
u_{\Omega_k+1} & \quad \text{in } \Omega_{k+1} \setminus \Omega_k,
\end{array} \right.
\]

then \(u \in W^{1,p}_{\text{loc}}(\Omega)\). Furthermore, by following close arguments as done in [24], we are able to show that \(u\) is a positive solution of (1.6).

To finish the proof, let us note that when \(\delta \leq 1\), by taking \(u_n\) as test function in (1.31) and following similar arguments as done above, one can conclude that \((u_n)\) is bounded in \(W^{1,p}_0(\Omega)\). In this case, \(u\) defined as above belongs to \(W^{1,p}_0(\Omega)\).

\[
\square
\]

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