Power Generation and Distribution via Distributed Coordination Control

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Abstract

This paper presents power coordination, power generation, and power flow control schemes for supply-demand balance in distributed grid networks. Consensus schemes using only local information are employed to generate power coordination, power generation and power flow control signals. For the supply-demand balance, it is required to determine the amount of power needed at each distributed power node. Also due to the different power generation capacities of each power node, coordination of power flows among distributed power resources is essentially required. Thus, this paper proposes a decentralized power coordination scheme, a power generation, and a power flow control method considering these constraints based on distributed consensus algorithms. Through numerical simulations, the effectiveness of the proposed approaches is illustrated.

I. INTRODUCTION

In recent years, a smart grid has attracted a tremendous amount of research interest due to its potential benefit to modern civilization. The remarkable development of computer and communication technology has enabled a realization of smart power grid. A key feature of a smart power grid is the change of the power distribution characteristics from a centralized power system to a distributed power system. Though centralized power plants still cover the major portion of power demand, the amount of power demand covered by distributed power resources has been increasing steadily [1].

In distributed power systems, achieving the supply-demand balance, which is one of the fundamental requirements, is a key challenging issue due to its decentralized characteristics. The
supply-demand balance problem has been traditionally considered as the economic dispatch \[2, 3\] which minimizes the total cost of operation of generation systems. However, the traditional dispatch problem is a highly nonlinear optimization problem and thus it has been addressed usually in a centralized manner.

Distributed control of distributed power systems has been considered in \[4\]–\[6\]. In the distributed power system, individual power resources are interconnected with each other through communication and transmission line. Thus, for the coordination of power generation and power flow, sharing information is essential. It is more realistic and economic to use only local information. This kind of problem has been solved in consensus \[7\]–\[10\]. Distributed resource allocation method named “center-free algorithm” has been considered in \[11\]. In the algorithm, since the amount of resource for each node is determined by the sum of weighted difference with its neighbors, we can easily see the equivalence of the algorithm to the consensus. Thus, we can consider that the idea of consensus had been already used in the distributed resource allocation problems.

In this paper, power distribution among distributed power resources is studied. The problem is classified into three subproblems. In the first problem, which is called power coordination, the desired net power\(^1\) is determined. In the second problem and the third problem, we consider power generation and power flow control of distributed power resources with and without power coordination respectively. Note that the power flow control without power coordination is significantly important when the net power and the generation capacities of each power resource are limited. Consensus schemes are employed to generate power coordination, power generation and power flow control signals.

Subsequently, the main contribution of this paper can be summarized as follows. First, the coordination and control of power generation and power flow are precisely defined and formulated. Second, two power distribution schemes with and without power coordination are developed, which may be helpful in coordinating actual power generation and power flow among distributed power plants. The proposed approach deals with the desired net power which is not necessarily an average value as in the distributed averaging problem \[12\]. Third, it is shown that the distributed

\(^1\)The “net power” at a node is the sum of generated power at the node and power flows into it from its neighbor nodes. Detailed explanation on “net power” can be found in the second paragraph of the problem formulation section (Section 2).
consensus algorithms ensure power distribution among distributed nodes that have net power and generation capacity constraints. The proposed approach can be applied even if the desired net power of some power resources does not satisfy the power generation capacity as opposed to those in [6], [13].

This paper is organized as follows. In Section II, power coordination, power generation and power flow control problems are formulated. Main results of this paper are presented in Section III. Illustrative examples are provided in Section IV and the conclusion is given in Section V, respectively.

II. PROBLEM FORMULATION

Fig. 1 shows a graph representing interconnected power systems. Each node denotes a subsystem, i.e., an area that consists of generators and loads, and it has its power demand determined by its loads. Thus, the desired net power for each generator in the subsystem is required to be determined. The power can be transmitted to its neighboring subsystems since they are interconnected with each other through transmission lines. In this paper, we would like to control the power generation and power flow to satisfy supply-demand balance under net power and generation capacity constraints.

Let $p_d^i$ be the desired net power at the $i$-th node; $p_{G_i}$ be the generated power at the $i$-th node; and $p_{F_{ij}}$ be the power flow from the $i$-th node to the $j$-th node, where $p_{F_{ij}} = -p_{F_{ij}}$ and $i, j = 1, \cdots, n$. The net power at a node is determined after the net power exchange of generated powers with its neighbor nodes. Thus, the net power at the $i$-th node can be defined as $p_i \triangleq p_{G_i} + \sum_{j \in N_i} p_{F_{ji}}$, where $N_i$ is the set of neighbor nodes of the $i$-th node. The total generated power and total power demand within the power grid are represented by $p_G = \sum_{i=1}^{n} p_{G_i}$ and $p_D$, respectively.
Throughout this paper, we need the following assumption to represent a reality of power grid network systems.

**Assumption 2.1:**

1) Each node has the net power capacity constraint such as

\[ p_i \leq p_i(k) \leq p_i \] \hspace{1cm} (1)

where \( k \) indicates the sampling instant of the physical power layer. The net power capacity is associated with the amount of power that each node can handle. Thus, the desired net power for each node shall be limited by the net power capacity constraint (1).

2) Each node has the generation capacity constraint such as

\[ p_{G_i} \leq p_{G_i}(k) \leq p_{G_i} \] \hspace{1cm} (2)

The generation capacity is also assumed to be bounded by the net power capacity constraint i.e., \( [p_{G_i}, p_{G_i}] \subseteq [p_i, p_i] \).

3) The graph which represents interconnections among nodes is undirected and connected. This assumption comes from the fact that the power and information flows are bidirectional. However, this assumption can be relaxed to a directed graph.

Note that due to the constraints (1) and (2) in the Assumption 2.1, the power coordination and power flow control problems become non-trivial. Fig. 2 shows a power exchange between pairs of nodes. As shown in Fig. 2, the generated power and the net power at each node can be represented as follows:

\[ p_{G_i}(k) = p_{G_i}(k-1) + \Delta p_{G_i}(k) \] \hspace{1cm} (3)

\[ p_i(k) = p_{G_i}(k) + \sum_{j \in N_i} p_{F_{ji}}(k), i = 1, 2, \ldots, n \] \hspace{1cm} (4)

\[ \sum_{i=1}^{n} p_i(k) = \sum_{i=1}^{n} p_{G_i}(k) \] \hspace{1cm} (5)

where \( \Delta p_{G_i}(k) = p_{G_i}(k) - p_{G_i}(k-1) \) is its own variation in power generation and \( p_{F_{ji}} \) is the interaction term representing the power exchange between two nodes. To make the paper clear, the following problems are formally stated.

**Problem 2.1: (Power coordination)** For a given total power demand \( p_D(k) \) within the network, determine the desired net power \( p_i^d(k) \) of individual node that satisfies the supply-demand balance.
such as

\[ p_{G} (k) = p_{D} (k) = \sum_{i=1}^{n} p_{d}^{i} (k) \]  

under the constraints of Assumption 2.1.

**Problem 2.2:** (Power generation and power flow control with power coordination) For the desired net power given by the power coordination, design power generation and power flow control strategies, i.e, \( \Delta p_{G_i} (k) \) and \( p_{F_{ij}} (k) \), to satisfy \( p_{i} (k) \rightarrow p_{i}^{d} (k) \) under the constraints of Assumption 2.1.

The Problem 2.2 is considered in a subsystem i.e., an area which consists of generators and loads. Thus, it is assumed that the total power demand is given as usually assumed in traditional power dispatch problems. As shown in below, only one node is required to know the total power demand. Thus, the desired net power for each node is determined by power coordination, and power generation is controlled to meet the supply-demand balance. This scenario is formulated under the name of “With power coordination”.

**Problem 2.3:** (Power generation and power flow control without power coordination) For the desired net power which is individually given without power coordination, design power generation and power flow control strategies, i.e, \( \Delta p_{G_i} (k) \) and \( p_{F_{ij}} (k) \), to satisfy \( p_{i} (k) \rightarrow p_{i}^{d} (k) \) under the constraints of Assumption 2.1.

The Problem 2.3 is considered in interconnected subsystems under the assumption that each area has only one collective generator and one collective load. Thus, it is assumed that the desired net power is individually given to each node. Thus, power coordination is not necessary, but the power generation and power flow are controlled to meet the supply-demand balance and given-individual desired net power for each node simultaneously. This scenario is formulated under the name of “Without power coordination”.

**Remark 2.1:** Power system transmission lines have a very high \( X/R \) ratio. Thus, the real power changes are primarily dependent on phase angle differences among generators while they
are relatively not affected by voltages. Further, the phase angle dynamics is generally much faster than the voltage dynamics and thus it is usually assumed that the phase angle dynamics and the voltage dynamics are decoupled \[14\]. In our problem setup, since we deal with the real power, one can see that the phase angle is implicitly considered. Also, in this paper, we mainly concentrate on power distribution, transmission, and coordination issues in operation level. Thus, the lower level signal characteristics are not considered. Also, it is assumed that power flow information can be calculated from the sequence of voltage and current phasors obtained by phasor measurement units (PMUs) \[15\].

Remark 2.2: The problem formulated by \(3\), \(4\), and \(5\) can cover various attribute distribution problems such as gas, water and oil operation, and supply chain management, traffic control, and renewable energy allocation. This paper just focuses on power distribution problem, which has been recently more researched \[4\]–\[6\].

III. Main results

A. Power coordination

In the power coordination, a desired net power is required to be realizable \[16\], i.e., physical constraints such as generation capacity and power flow constraint are required to be satisfied. If the desired net power does not satisfy these constraints, we cannot achieve the control goal with any control input since the desired net power is not realizable. From \(2\) and \(6\), the following condition for a realization (i.e., to make the desired net power realizable) can be obtained:

\[
p_{G} \leq p_{D}(k) \leq \overline{p_{G}}
\]

(7)

Let us consider the power coordination issue more systematically. If the desired net power is both upper- and lower-bounded, then we can make the following rule for the power coordination.

Lemma 3.1: If the desired net power is lower-bounded and upper-bounded, i.e., \(p_{Gi} \leq p_{D}^{i}(k) \leq \overline{p_{Gi}}\) and \(7\) is satisfied, then the following desired net power

\[
p_{D}^{i}(k) = p_{Gi} + \frac{\overline{p_{Gi}} - p_{Gi}}{\overline{p_{G}} - p_{Gi}} (p_{D}(k) - p_{G})
\]

(8)

provides a sufficient condition for the supply-demand balance \(6\).
Proof: If \( p_i^d (k) \) is given by (8), then, under the assumption (7), we have
\[
 p_{G_i} - p_i^d (k) = \frac{p_{G_i} - pG_i}{pG - pG} (p_D (k) - pG) 
\]
which satisfies the left-side inequality \( p_{G_i} \leq p_i^d (k) \). Similarly, if \( p_i^d (k) \) is given by (8), then, under the assumption (7), we obtain
\[
 p_i^d (k) - pG_i = \frac{pG_i}{pG - pG} (p_D (k) - pG) - pG_i 
\]
Thus, the right-side inequality \( p_i^d (k) \leq pG_i \) is also satisfied. Furthermore, summing up the both sides of (8) over \( i \) from 1 to \( n \) yields
\[
 \sum_{i=1}^{n} p_i^d (k) = \sum_{i=1}^{n} p_{G_i} + \frac{\sum_{i=1}^{n} (pG_i - pG_i)}{pG - pG} (p_D (k) - pG) 
\]
\[
 = p_D (k) 
\] (11)

Note that the power coordination (8) requires global information such as \( pG \) and \( pG \). The coordination scheme can be, however, achieved by distributed consensus algorithm using only local information as in [17]. For that, we need the following lemma for a further investigation.

Lemma 3.2: [18] If \( A \in \mathbb{R}^{n \times n} \) is nonnegative and primitive, then
\[
 \lim_{m \to \infty} \left[ \rho(A)^{-1} A \right]^m \to L > 0 
\] (12)
where \( L \triangleq r l^T \), \( Ar = \rho(A) r \), \( A^T l = \rho(A) l \), \( r > 0 \) and \( l > 0 \) in element-wise, and \( r l^T = 1 \) (In fact, \( r \) and \( l \) are the right and left eigenvectors corresponding to the eigenvalue \( \rho(A) \)).

Now, we state one of the main results of this subsection.

Theorem 3.1: The power coordination (8) is achieved if the following consensus scheme is used:
\[
 p_i^d (k) = p_{G_i} + \left( \frac{pG_i - pG_i}{y_{i,ss}} \right) x_{i,ss} 
\] (13)
where $x_{i,ss}$ and $y_{i,ss}$ are steady state solutions of the following equations:

\[
  x_i(t+1) = \frac{1}{1 + |N_i|} x_i(t) + \sum_{j \in N_i \cup \{i\}} \frac{1}{1 + |N_j|} x_j(t)
\]

\[
  y_i(t+1) = \frac{1}{1 + |N_i|} y_i(t) + \sum_{j \in N_i \cup \{i\}} \frac{1}{1 + |N_j|} y_j(t)
\]

with the following initial values

\[
  x_i(0) = \begin{cases} 
p_D(k) - p_{G_i} & \text{if } i \text{ is the leading node} 
  
  -p_{G_i} & \text{otherwise}
  \end{cases}
\]

\[
  y_i(0) = p_{G_i} - p_{G_i}
\]

respectively, where $x_i(t), y_i(t) \in \mathbb{R}$, and $|N_i|$ is the degree of the $i$-th node and the index $t$ represents the sampling instant at the consensus algorithm. The consensus algorithm (13)-(17) is completely decoupled from the physical power layer.

**Proof:** Without loss of generality, it is assumed that the first node is the leading node which knows the total power demand $p_D(k)$. Then, the consensus scheme (14) and (16) can be represented by

\[
  x(t+1) = Q x(t)
\]

\[
  x(0) = \begin{bmatrix} p_D(k) - p_{G_1}, -p_{G_2}, -p_{G_3}, \ldots, -p_{G_n} \end{bmatrix}^T
\]

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$ and $Q = [q_{ij}]$ is nonnegative row stochastic matrix because $q_{ij} = 1/(1 + |N_j|)$ if $j \in N_i \cup \{i\}$ and 0 otherwise. The matrix $Q$ is irreducible because the associated graph for $Q$ is undirected and connected. Also, $Q$ is primitive because it is irreducible and has exactly one eigenvalue of maximum modulus $\lambda = \rho(Q) = 1$ (see [8]) according to Perron-Frobenius theorem [18]. Thus, according to Lemma 3.2, the vector $x$ converges to its steady state solution $x_{ss} = \lim_{m \to \infty} Q^m x(0)$ if there exists a limit of $\lim_{m \to \infty} Q^m$. According to Lemma 3.2, this limit exists for the primitive matrix $Q$ and the steady state solution is given by $x_{ss} = r l^T x(0)$, where $Q r = \rho(Q) r$, $Q^T l = \rho(Q) l$, $r > 0$, $l > 0$ in element-wise, and $r l^T = 1$ with $l = 1$ (here, 1 denotes a vector with ones as its element) and $r = [r_1, r_2, \ldots, r_n]^T$ which satisfies $r l^T = 1$. Thus, this solution is given by

\[
  x_{ss} = r l^T x(0) = 1^T x(0) r = (p_D(k) - p_{G_i}) r
\]
In the similar manner, the steady state solution for

\[ y(t + 1) = Qy(t) \]  

\[ y(0) = [p_{G_1} - p_{G_1}, p_{G_2} - p_{G_2}, \ldots, p_{G_n} - p_{G_n}]^T \]  

is given by

\[ y_{ss} = rl^Ty(0) = 1^Ty(0) r = (\frac{p_{G} - p_{G}}{r}) r \]  

Thus, it follows from (20) and (23) that (13) can be represented by

\[ p^d_i(k) = p_{G_i} + \frac{p_{G_i} - p_{G_i}}{(p_{G} - p_{G})} (p_{D_i}(k) - p_{G_i}) r_i \]  

\[ = p_{G_i} + \frac{p_{G_i} - p_{G_i}}{p_{G} - p_{G}} (p_{D_i}(k) - p_{G_i}) \]  

which is equivalent to (8).

Remark 3.1: The convergence of the algorithm (14)-(17) can be ensured if the sampling of the algorithm is much faster than that of the physical layer. Thus, the power coordination scheme of (13)-(17) is feasible in practice. In more detail, let \( T(k) \) be the time at the \( k \)-th sample instant in the physical power layer. Then, the power coordination (13)-(17) should be completed in \( \Delta k = T(k + 1) - T(k) \). Thus, the time interval \( \Delta t \) should be chosen to \( \Delta t \ll \Delta k \) so that the steady state solutions of (14)-(17) can be obtained in \( \Delta k \).

B. Power generation and power flow control

As mentioned in Problem 2.2 and Problem 2.3, we attempt to design power generation \( \Delta p_{G_i}(k) \) and power flow \( p_{F_{ij}}(k) \) in order to make net power be equal to the desired net power at each node with and without power coordination, respectively. From (3)-(4), the net power of each node can be described as follows:

\[ p_i(k) = p_{G_i}(k - 1) + u_i(k) \]  

\[ u_i(k) = \Delta p_{G_i}(k) + \sum_{j \in N_i} p_{F_{ji}}(k) \]
where $u_i$ is the control input at the $i$-th node.

It is remarkable that, considering the generation capacity, the power generation control input should satisfy the following constraint

$$\Delta p_{G_i}(k) \leq p_{G_i}(k) \leq \Delta p_{G_i}(k)$$

where $\Delta p_{G_i}(k) = p_{G_i} - p_{G_i}(k - 1)$ and $\Delta p_{G_i}(k) = p_{G_i} - p_{G_i}(k - 1)$. Define the net power flow at the $i$-th node as follows

$$p_{F_i}(k) = \sum_{j \in N_i} p_{F_{ji}}(k)$$

and define the coordination error at each node as

$$p_{e,i}(k) = p_i(k) - p_i^d(k)$$

In the sequel, we provide power generation and power flow control schemes with and without power coordination taking account of two different scenarios mentioned in Section II.

1) With power coordination: With the power coordination (13)-(17), the desired net power for each node should satisfy the generation capacity of each node. From (30), the coordination error is given by

$$p_{e,i}(k) = p_i(k - 1) + u_i(k) - p_i^d(k)$$

Our goal is to design $u_i(k)$ such that the coordination error becomes zero.

**Theorem 3.2:** With the power coordination (13)-(17), if control input is given by

$$u_i(k) = \Delta p_{G_i}(k) = p_i^d(k) - p_{G_i}(k - 1)$$

then the constraint (28) and $p_i(k) \rightarrow p_i^d(k)$ can be ensured.

**Proof:** With the power coordination (13)-(17), $p_{G_i} \leq p_{G_i}^d \leq \Delta p_{G_i}$ can be achieved. Thus, we can have

$$\Delta p_{G_i}(k) \leq p_i^d(k) - p_{G_i}(k - 1) \leq \Delta p_{G_i}(k)$$

In (27), choosing $p_{F_{ji}}(k) = 0$ and by the control law of (32), we have $u_i(k) = \Delta p_{G_i}(k)$. Therefore, by (33), the constraint (28) is satisfied. Furthermore, by inserting (32) into (26), we can achieve $p_i(k) \rightarrow p_i^d(k)$. \[\Box\]

Fig. 3 shows the overall power flow control with power coordination. Without loss of generality, it is assumed that node $i$ is a leading node that knows the total demand $p_D(k)$. Only the
information of generation capacity and the degree of node $i$ are exchanged between neighboring nodes. With this information, the desired net power for each node is determined by the power coordination. After the power coordination, power generation control input is given by (32). As a consequence of the control input, $p_i(k) \rightarrow p_{di}^i(k)$ is achieved. In this case, $p_i(k) = p_{Gi}(k)$ because there is no power flow between pairs of nodes.

2) Without power coordination: Let us assume that the desired net power for each node is individually given without power coordination. But it is supposed that the desired net power satisfies the realizability condition (7) and net power capacity constraints (1). In this case, the desired net power for some nodes may not satisfy the generation capacity of their node. Thus, it is not sufficient to provide only the power generation control for each node. Therefore, the power flow control input can be represented as (27), with $p_{Fji}(k)$ enabled in this case. First, we want to design the power generation control to achieve the overall supply-demand balance.
Now, we provide a lemma to derive the main result of this paper.

**Lemma 3.3:** If the power generation control input is designed by the following law

\[
\Delta p_{G_i}(k) = \Delta p_{G_i}(k) + \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k))} \left( p_D(k) - p_G(k - 1) - \sum_{j=1}^{n} \Delta p_{G_j}(k) \right)
\] (34)

then, the control input will satisfy the constraint (28) and the supply-demand balance is achieved.

**Proof:** For the left-side inequality \(\Delta p_{G_i}(k) \leq \Delta p_{G_i}(k)\) of (28), if \(\Delta p_{G_i}(k)\) is given by (34), then we can obtain the following inequality:

\[
\frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k))} \sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k)) \leq 0
\] (35)

For the right-side inequality \(\Delta p_{G_i}(k) \leq \Delta p_{G_i}(k)\), if \(\Delta p_{G_i}(k)\) is given by (34), with the substitution of \(\xi = \sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k)) (\Delta p_{G_j}(k) - \Delta p_{G_j}(k)) + (\Delta p_{G_j}(k) - \Delta p_{G_j}(k)) [p_D(k) - p_G(k - 1) - \sum_{j=1}^{n} \Delta p_{G_j}(k)]\), we can obtain the following inequality:

\[
\Delta p_{G_i}(k) - \Delta p_{G_i}(k) = \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k))} \left( p_D(k) - p_G(k - 1) - \sum_{j=1}^{n} \Delta p_{G_j}(k) \right)
\]

\[
= \frac{\xi}{\sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k))}
\]

\[
= \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k))} \sum_{j=1}^{n} (\Delta p_{G_j}(k) - \Delta p_{G_j}(k)) \leq 0
\] (37)

Furthermore, summing up the both sides of (34) over \(i\) from 1 to \(n\) yields

\[
p_D(k) = p_G(k - 1) + \sum_{i=1}^{n} \Delta p_{G_i}(k) = p_G(k)
\] (38)

Thus, the proof is completed.
However, as in (8), (34) requires global information such as \( \sum_{i=1}^{n} \Delta p_{G_i}(k) \), \( \sum_{i=1}^{n} \Delta p_{G_i}(k) \), and \( p_D(k) - p_G(k - 1) \). For a decentralized power generation control, we now provide the following theorem:

**Theorem 3.3:** The power generation control input (34) can be achieved if the following consensus scheme is used.

\[
\Delta p_{G_i}(k) = \Delta p_{G_i}(k) + \left( \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{w_{i,ss}} \right) z_{i,ss}
\]  

(39)

where \( w_{i,ss}, z_{i,ss} \) are steady state solutions of the following equations:

\[
z_i(t+1) = \frac{1}{1+|N_i|} z_i(t) + \sum_{j \in N_i} \frac{1}{1+|N_j|} z_j(t)
\]  

(40)

\[
z_i(0) = p_i^d(k) - p_{G_i}(k - 1) - \Delta p_{G_i}(k)
\]  

(41)

\[
w_i(t+1) = \frac{1}{1+|N_i|} w_i(t) + \sum_{j \in N_i} \frac{1}{1+|N_j|} w_j(t)
\]  

(42)

\[
w_i(0) = \Delta p_{G_i}(k) - \Delta p_{G_i}(k)
\]  

(43)

**Proof:** The consensus scheme (40) and (41) can be represented by

\[
z(t+1) = Qz(t)
\]  

(44)

\[
z(0) = [z_1(0), z_2(0), \ldots, z_n(0)]^T
\]  

(45)

As in the proof of **Theorem 3.1**, the steady state solution is given by \( z_{ss} = rl^T z(0) \) where \( Qr = \rho(Q) r, Q^T l = \rho(Q) l, r \geq 0, l > 0 \) in element-wise, and \( r^T l = 1 \) with \( l = 1 \) and \( r = [r_1, r_2, \ldots, r_n]^T \) which satisfies \( r^T 1 = 1 \). Thus, this solution is given by

\[
z_{ss} = rl^T z(0) = \left( p_D(k) - p_G(k - 1) - \sum_{i=1}^{n} \Delta p_{G_i}(k) \right) r
\]  

(46)

In the similar manner, the steady state solution for

\[
w(t+1) = Qw(t)
\]  

(47)

\[
w(0) = [w_1(0), w_2(0), \ldots, w_n(0)]^T
\]  

(48)

is given by

\[
w_{ss} = rl^T w(0) = \sum_{i=1}^{n} \left( \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{w_{i,ss}} \right) r
\]  

(49)
Thus, (39) can be represented by the following equation:

\[
\Delta p_{G_i}(k) = \Delta p_{G_i}(k) + \left( \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{w_{i,ss}} \right) z_{i,ss} \\
= \Delta p_{G_i}(k) + \sum_{i=1}^{n} \left( \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)} \right) \left( p_D(k) - p_G(k-1) - \sum_{i=1}^{n} \Delta p_{G_i}(k) \right) r_i \\
= \Delta p_{G_i}(k) + \sum_{i=1}^{n} \left( \frac{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)}{\Delta p_{G_i}(k) - \Delta p_{G_i}(k)} \right) \left( p_D(k) - p_G(k-1) - \sum_{i=1}^{n} \Delta p_{G_i}(k) \right) \\
\]  

(50)

which is equivalent to (34).

Then, from (30), the coordination error after the power generation control input is given by

\[
p_{e,i}(k) = p_{e,i}(k) + \sum_{j \in N_i} p_{F_{ji}}(k) \\
\]  

(51)

where \( p_{e,i}(k) \) is \( p_{G_i}(k) - p_{G_i}(k) \). Now, to make \( p_{e,i}(k) \rightarrow 0 \), we need to design power flow control \( p_{F_{ji}}(k) \), which is summarized in the following theorem.

**Theorem 3.4:** If the power flow control is designed by

\[
p_{F_{ji}}(k) = h_{ij,ss} \\
\]  

(52)

where \( h_{ij,ss} \) is the steady state solution of the following equations:

\[
h_{ij}(t+1) = h_{ij}(t) + a_{ij} (g_j(t) - g_i(t)) \\
h_{ij}(0) = 0 \\
g_i(t+1) = g_i(t) + \sum_{j \in N_i} a_{ij} (g_j(t) - g_i(t)) \\
g_i(0) = p'_{e,i}(k) \\
\]  

(53) \hfill (54) \hfill (55) \hfill (56)

where \( a_{ij} = \frac{1}{1+\max\{|N_i|,|N_j|\}} \) is Metropolis-Hasting weight \( [19] \), then we can have \( p_{e,i}(k) \rightarrow 0 \).

**Proof:** From (55) and (56), we can obtain

\[
g(t+1) = S g(t) \\
g(0) = p'_e(k) \\
\]  

(57) \hfill (58)
where \( g(t) = [g_1(t), g_2(t), \ldots, g_n(t)]^T \) and \( S = [s_{ij}] \) is doubly stochastic, where
\[
s_{ij} = \begin{cases} 
\frac{1}{1+\max(|N_i|,|N_j|)} & \text{if } j \in N_i \\
1 - \sum_{j \in N_i} \frac{1}{1+\max(|N_i|,|N_j|)} & \text{if } i = j \\
0 & \text{otherwise}
\end{cases} \tag{59}
\]
and \( S^T = S \). As in the proof of Theorem 3.1, the steady state solution is given by \( g_{ss} = r^T g(0) \) where \( Sr = \rho(S)r, S^T l = \rho(S)l, r > 0, l > 0 \) in element-wise, and \( r^T l = 1 \) with \( l = 1 \) and \( r \) which satisfies \( r^T 1 = 1 \). Furthermore, \( 1 \) is also the right eigenvector with the associated-eigenvalue \( 1 \) because \( S \) is doubly stochastic. Thus, without loss of generality, let \( r = \frac{1}{n} 1 \), which satisfies \( r^T 1 = 1 \). Then, the solution of (57) with (58) converges to the average as follows:
\[
g_{ss} = r^T g(0) = \frac{1}{n} 1^T p_e'(k) \tag{60}
\]
From \( p_{e,i}'(k) \triangleq p_G_i(k) - p_d_i(k) \), we can obtain \( 1^T p_e'(k) = 0 \) because of \( p_G_i(k) = p_D_i(k) \). Thus, we have \( g_{ss} = 0 \). Hence
\[
g_{i,ss} - g_i(0) = -p_{e,i}'(k) \tag{61}
\]
Also, from (53)-(56), it follows that
\[
g_{i,ss} = g_i(0) + \sum_{j \in N_i} h_{ij,ss} \tag{62}
\]
Thus, from (61) and (62), we have
\[
\sum_{j \in N_i} h_{ij,ss} = -p_{e,i}'(k) \tag{63}
\]
If we choose the interaction control input as (52), it follows from (51) and (63) that
\[
p_{e,i}(k) = p_{e,i}'(k) + \sum_{j \in N_i} h_{ij,ss} = 0 \tag{64}
\]

Fig. 4 shows the power generation and power flow control scheme without power coordination. As previously mentioned, it is assumed that the desired net power for each node \( p_d_i(k) \) is given under (7). The power generation control input is given by (39)-(43). As a consequence of the control input, the supply-demand balance is achieved. Next, the power flow between pairs of nodes is determined by (52)-(56). Then, after the power flow control, \( p_i(k) \rightarrow p_d_i(k) \) is achieved. There are iterations for the power generation control input computation and for the power flow.
Fig. 4: Power generation and power flow control without power coordination

control input computation. Thus, the time interval for iterations (40)-(43) and (53)-(56) should be chosen to $\Delta t \ll \Delta k$ so that the steady state solutions of the iterations can be obtained in $\Delta k$.

Remark 3.2: The constraints of the Assumption 2.1 might be time-varying in renewable power resources such as a wind or a solar system. It is possible for the proposed approach to account for time-varying bounds if the rate of change is not faster than $\Delta k$ and the realization condition (7) is satisfied. This can be easily verified if the time-varying bounds are substituted into the proposed approach instead of the constant bounds.

IV. ILLUSTRATIVE EXAMPLES

In this section, two illustrative examples are provided. The distributed power resources are interconnected as depicted in Fig. 1. The generation capacity and net power capacity of each node are listed in Table 1.

A. Power distribution with power coordination

This example shows the power distribution with power coordination. In this case, without loss of generality, it is assumed that node 1 is the leading node that knows the total power
TABLE I: Generation capacity and net power capacity at each node

| Node(i) | Generation capacity ($p_{G_i}$) | Net power capacity ($p_i$) |
|---------|---------------------------------|----------------------------|
| 1       | [10, 50]                        | [10, 80]                   |
| 2       | [20, 80]                        | [20, 120]                  |
| 3       | [20, 40]                        | [20, 60]                   |
| 4       | [10, 45]                        | [10, 75]                   |
| 5       | [15, 60]                        | [15, 90]                   |
| 6       | [10, 55]                        | [15, 80]                   |

demand $p_D(k)$ of the distributed power system. First, the total power demand $p_D(k)$ satisfying the realization condition (7) is randomly created and it is depicted in Fig. 5 (a). Then, $p^d_i$ is determined by power coordination (13)-(17) as shown in Fig. 5 (b). Next, the power generation control input given by (32) is shown in Fig. 5 (c) and the generated power (i.e., (3)) is depicted in Fig. 5 (d). Consequently, the coordination error given by (30) is zero and the supply-demand balance is also achieved as shown in Fig. 5 (b) and Fig. 5 (d).

B. Power distribution without power coordination

This example shows power distribution without power coordination. In this case, the desired net power for each node is not given by power coordination but they are randomly created under the realization condition (7) as depicted in Fig. 6 (a). First, the power generation control input are given by (39)-(43) as shown in Fig. 6 (b) and the generated power (i.e., (3)) is shown in Fig. 6 (c). After the power generation control, a coordination of power flows between pairs of nodes is necessary to make $p_i$ converge to $p^d_i$. The power flows between pairs of nodes are determined by (52)-(56) and it is depicted in Fig. 6 (d). Next, we can obtain the net power flow at each node from (29) as shown in Fig. 6 (e). Then, the net power at each node is given by (4) as shown in Fig. 6 (f). Then, the coordination error given by (30) is zero and the supply-demand balance is also achieved as shown in Fig. 6 (a) and Fig. 6 (f).
V. CONCLUSION

This paper has addressed power distribution problems in distributed power grid with and without power coordination. First, a power coordination using a consensus scheme under limited net power and power generation capacities was considered. Second, power generation and a power flow control laws with and without power coordination were designed using a consensus
scheme to achieve supply-demand balance.

Since this paper has provided systematic approaches for power distribution among distributed nodes on the basis of consensus algorithms, the results of this paper can be nicely utilized in power dispatch or power flow scheduling. The authors believe that consensus algorithm-based power distribution schemes of this paper have several advantages over typical power dispatch approaches. The first key advantage is that the power coordination and control can be realized via decentralized control scheme without relying upon nonlinear optimization technique. The second advantage is that the framework proposed in this paper can handle power constraint, generation, and flow in a unified framework.

It is noticeable that this paper has considered the power coordination, power generation and power flow control in the higher-level models of grid networks but does not consider lower-level models of grid networks such as current, voltage drop, and line impedance. However, in our future research, it would be meaningful to add links between the higher-level models and the lower-level models. Further, it is desirable to investigate various features such as behavior of self-interested customer, power loss in the transmission line, and structures between the physical and cyber layers such as delays, and mismatches between them.

Remark 5.1: Though the paper has only focused on power distribution, we believe that the proposed approach can be extended to various attribute distribution problems such as traffic control and supply chain management, because we use a fundamental model describing flow of attribute between pairs of distributed resources as well as the amount of attribute in each distributed resource. For example, in a traffic control, each freeway section named “cell” corresponds to each distributed power resource, distribution of vehicle density corresponds to net power of each power resource, desired traffic density corresponds to desired net power, and on-ramp traffic flow corresponds to power generation control input. Thus, the goal of traffic control which satisfies desired traffic density corresponds to that of power distribution which satisfies desired net power.

VI. ACKNOWLEDGEMENT

It is recommended to see ‘Byeong-Yeon Kim, “Coordination and control for energy distribution using consensus algorithms in interconnected grid networks”, Ph.D. Dissertation, School of Information and Mechatronics, Gwangju Institute of Science and Technology, 2013’ for applications
to various engineering problems of the algorithms developed in this paper.

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Fig. 6: Power distribution without power coordination