Non-perturbative vacuum polarization effects in two-dimensional supercritical Dirac-Coulomb system. I. Vacuum charge density

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Based on the original combination of analytical methods, computer algebra tools and numerical calculations, proposed recently in Refs. [1]-[3], the non-perturbative vacuum polarization effects in the 2+1 D supercritical Dirac-Coulomb system with $Z > Z_{cr}$ are explored. Both the vacuum charge density $\rho_{VP}(\vec{r})$ and vacuum energy $E_{VP}$ are considered. The main result of the work is that in the overcritical region $E_{VP}$ turns out to be a rapidly decreasing function $\sim -\eta_{eff} Z^3/R$ with $\eta_{eff} > 0$ and $R$ being the size of the external Coulomb source. Due to a lot of details of calculation the whole work is divided into two parts I and II. In the present part I we consider the evaluation and behavior of the vacuum density $\rho_{VP}$, which further is used in the part II for evaluation of the vacuum energy, with emphasis on the renormalization, convergence of the partial expansion for $\rho_{VP}$ and behavior of the integral induced charge $Q_{VP}$ in the overcritical region.

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1. INTRODUCTION

There is now a lot of interest to the non-perturbative effects in the QED vacuum, caused by diving of discrete levels into the lower continuum in the supercritical static or adiabatically slowly varying Coulomb fields, that are created by localized extended sources with $Z > Z_{cr}$ (see Refs. [4]-[8] and refs. therein). Such effects have attracted a considerable amount of theoretical and experimental activity, since in 3+1 QED for $Z > Z_{cr,1} \approx 170$ a non-perturbative reconstruction of the vacuum state is predicted, which should be accompanied by a number of nontrivial effects including the vacuum positron emission (see Refs. [4],[9]-[12] and refs. therein). Similar in essence effects are expected to come out also both in 2+1 D (planar graphene-based hetero-structures [13–20]) and in 1+1 D (one-dimensional “hydrogen ion” [21–28]). Moreover, in the graphene case the role of the effective QED vacuum is played by the graphene itself, the electrons and holes are the analogues of virtual pairs, while the effective fine-structure coupling constant $\alpha_g \sim 1$.

Recently, in Refs. [1]-[3] an original combination of analytical methods, computer algebra tools and numerical calculations has shown that for a wide range of the system parameters in the one-dimensional Dirac-Coulomb problem the nonlinear effects could lead in the supercritical region to the behavior of the vacuum energy, substantially different from the perturbative quadratic growth, up to (almost) quadratic decrease into the negative region $\sim -|\eta| Z^2$. In the present paper, which is divided into two parts I and II, these methods are applied to the study of analogous vacuum polarization effects for a 2+1 Dirac-Coulomb system in the overcritical region. More concretely, in the present part I we consider the behavior of the vacuum charge density $\rho_{VP}$, while in the subsequent part II with account of results, obtained in I, the behavior of the vacuum energy $E_{VP}$ is explored.

The external Coulomb field $A_0^{ext}(\vec{r})$ is chosen in the form of a projection onto a plane of the potential of the uniformly charged sphere with radius $R$

$$A_0^{ext}(\vec{r}) = Z|e| \left[ \frac{1}{R} \theta (R-r) + \frac{1}{r} \theta (r-R) \right],$$

what leads to the potential energy

$$V(r) = -Z\alpha \left[ \frac{1}{R} \theta (R-r) + \frac{1}{r} \theta (r-R) \right].$$

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Compared to the model of the uniformly charged ball such potential turns out to be more preferable, since it allows to perform the most part of calculations in the analytical form, while the evaluation of critical charges shows that in both cases the final answers should be quite close. And although with the standard choice of the fine-structure coupling $\alpha \simeq 1/137$ and without special selection of the Coulomb field cut-off parameters such a system could be treated only as a toy-model of 3+1 D problem, its study for $Z > Z_{cr}$ should be of considerable interest, since it allows to reproduce almost all the properties of the realistic 3+1 D problem of vacuum polarization by superheavy nuclei or nuclear quasi-molecule, but with certain substantial simplifications due to a smaller number of rotational degrees of freedom. For these reasons the radius of the external source is chosen as in 3+1 D for the case of superheavy nuclei

$$R = R(Z) \simeq 1.2 (2.5 Z)^{1/3} \text{fm} .$$

(3)

As it was shown in Refs. [1]-[3], the actual calculation of the vacuum energy, based on the UV-renormalization of the fermionic loop, could be performed without resorting to the vacuum charge density and shell effects. However, in fact the decrease of $\varepsilon_{VP}^{ren}$ in the overcritical region is caused primarily by the non-perturbative changes in the vacuum density for $Z > Z_{cr,1}$ due to the discrete levels diving into the lower continuum (“the shell effect”). In 1+1 D, due to the specific of one-dimensional Dirac-Coulomb problem, the total vacuum shells number depends on $Z$ as $\sim Z^s$, $1 < s < 2$, at least within the considered in Refs. [1]-[3] range of external parameters. Therefore, they turn out to be able just to reduce the growth rate of the non-renormalized $\varepsilon_{VP}$ in the overcritical region up to $\sim Z^\nu$, $1 < \nu < 2$, and so the dominant contribution comes from the renormalization term $\eta Z^2$. In 2+1 and 3+1 D the shell effect shows up much more pronouncedly. As a consequence, in this case $\varepsilon_{VP}^{ren}(Z)$ behaves in the overcritical region remarkably more nonlinearly, what will be demonstrated in the part II of the present work for the same 2+1 Dirac-Coulomb system with the external field (1).

It is also worth-while noticing that our work is aimed mainly at the study of vacuum effects, caused by extended supercritical Coulomb sources with non-zero size $R$ like superheavy nuclei or charged impurities in graphene, which provide a physically clear problem statement. At the same time, certain aspects of the physics, associated with supercritical point-like charges, have been studied by many authors for a long time. In particular, in Ref. [29] this phenomenon was investigated with regard to the concept of the vacuum charge. It turned out that when the radius of a supercritical nucleus ($Z > 137$) tends to zero, the vacuum charge screens the nuclear charge to 137 that prevents a further diving of the electron states. This result leads to the conclusion that the interaction with a point charge in QED cannot effectively have the coupling strength greater than 1 [10, 29]. In Ref. [30] it was explored, how the previous statement may alter in the presence of nuclear recoil and vacuum polarization operators. Purely mathematical aspects of supercritical point-like charges have been also extensively discussed in terms of possible self-adjoint extensions of the Dirac hamiltonian. A quite rigorous and comprehensive treatment of this problem, based on the theory of self-adjoint extensions of symmetric operators combined with the Krein’s method of directional functionals, was presented in Refs. [31],[32] (see also refs. therein). However, despite the fact that a rigorous and consistent theory of self-adjoint operators was applied to the problem, the electronic states cannot be still completely determined for $Z > 137$. There remains an ambiguity in choosing the distinguished one between the possible self-adjoint extensions. Our work deals with extended sources only and so is free from such ambiguities.

As in other works on vacuum polarization in the strong Coulomb field, radiative corrections from virtual photons are neglected. Henceforth, if it is not stipulated separately, relativistic units $\hbar = m_c = c = 1$ are used. Thence the coupling constant $\alpha = e^2$ is also dimensionless, what significantly simplifies the subsequent analysis, while the numerical calculations, illustrating the general picture, are performed for $\alpha = 1/137.036$.

2. PERTURBATION THEORY FOR THE VACUUM DENSITY IN 2+1 QED

In the 2+1 QED within the perturbation theory (PT) the leading order vacuum charge density $\rho_{VP}^{(1)}(\vec{r})$ is defined by means of the relation

$$\rho_{VP}^{(1)}(\vec{r}) = -\frac{1}{4\pi} \Delta_2 A_{VP,0}^{(1)}(\vec{r}) ,$$

(4)

where $\Delta_2$ is the two-dimensional Laplace operator. In (4) the Uehling potential $A_{VP,0}^{(1)}(\vec{r})$ is expressed via the polarization operator $\Pi_R(-\vec{q}^2)$ and the Fouriet-transform of the external potential $\tilde{A}_0(\vec{q})$ [12]

$$A_{VP,0}^{(1)}(\vec{r}) = \frac{1}{(2\pi)^2} \int d^2 q \, e^{i\vec{q} \cdot \vec{r}} \Pi_R(-\vec{q}^2) \tilde{A}_0(\vec{q}) ,$$

$$\tilde{A}_0(\vec{q}) = \frac{1}{(2\pi)^2} \int d^2 r' \, e^{-i\vec{q} \cdot \vec{r}'} A_{VP,0}^{ext}(\vec{r}') ,$$

(5)
where
\[ \Pi_R(-q^2) = \frac{\alpha}{2q} \left[ \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \right]. \] (6)

Here the polarization operator is defined through relation \( \Pi_R^{(2)}(q) = (q^2 - q^2) \Pi_R(q^2) \) and so is dimensionless. Calculation of the explicit form of the polarization function (6) is based on the two-dimensional representation of Dirac matrices (concerning the latter choice see below in Section 3). From (5), (6) for the external source (1) there follows the explicit expression for the corresponding Uehling potential in the form of an axial-symmetric function (the details of calculations are given in Appendix A)

\[ A_{VP,0}^{(1)}(r) = \frac{Z \alpha |e|}{4} \int_0^\infty dq \frac{J_q(qr)}{q} \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \times \]
\[ \times \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) \mathbf{H}_1(qR) - J_1(qR) \mathbf{H}_0(qR) \right] \right], \] (7)

with \( J_\nu(z) \) being the Bessel function and \( \mathbf{H}_\nu(z) \) the Struve one.

Proceeding further, in the next step by means of (4) one finds the axial-symmetric vacuum charge density \( \rho_{VP}^{(1)}(r) \). To clarify the question of the possibility of inserting the Laplace operator under the sign of the integral over \( dq \) in (7), let us consider the asymptotics of the integrand for large \( q \). The leading term of the asymptotics takes the form

\[ \frac{\sin(q(r + R)) + \cos(q(r - R))}{\sqrt{r^3/2} q^3}. \] (8)

The action of the Laplace operator in (8) yields the term, which behaves \( \sim 1/q \) at \( r = R \), namely

\[ \Delta_2 \frac{\sin(q(r + R)) + \cos(q(r - R))}{\sqrt{r^3/2} q^3} = \frac{\sin(q(r + R)) + \cos(q(r - R))}{\sqrt{r^3/2} q} + O(1/q^2). \] (9)

Therefore, at \( r = R \) the possibility of inserting the Laplace operator under the sign of the integral in (7) is absent, since in this case, according to (9), the integral over \( dq \) diverges logarithmically.

Let us also mention that the factor \( 1/\sqrt{r} \) in (9) does not produce any singularity in the behavior of \( A_{VP,0}^{(1)}(r) \) for \( r \to 0 \), since the asymptotics (8) of the integrand in (7) is achieved by taking the limit \( qr \to \infty \) in the argument of the Bessel function \( J_0(qr) \). The correct behavior of \( A_{VP,0}^{(1)}(r) \) for \( r \to 0 \) should be found from (7) by taking \( qr \to 0 \) in \( J_0(qr) \), which leads to the converging integral over \( dq \), hence, \( A_{VP,0}^{(1)}(0) \), as well as \( \rho_{VP}^{(1)}(0) \), is finite.

So the vacuum density, obtained from (4) with account of (9), equals to

\[ \rho_{VP}^{(1)}(r) = \frac{Z \alpha |e|}{16\pi} \int_0^\infty dq q J_0(qr) \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \times \]
\[ \times \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) \mathbf{H}_1(qR) - J_1(qR) \mathbf{H}_0(qR) \right] \right], \] (10)

and is finite for all \( r \neq R \) with the logarithmic singularity at \( r \to R \).

From (10) by an explicit calculation one obtains that to the leading order of PT the integral vacuum charge vanishes exactly

\[ \int d^2r \rho_{VP}^{(1)}(r) = 0. \] (11)

And although in this case the relation (11) turns out to be the direct consequence of the renormalization condition \( \Pi_R(q^2) \sim q^2 \) for \( q \to 0 \) (see Appendix B), actually it should be considered as an additional indication in favor of the assumption, that in presence of the external field, uniformly vanishing at the spatial infinity, and without any special boundary conditions and/or nontrivial topology of the field manifold, one should expect that in the undercritical region with \( Z < Z_{cr} \) the correctly renormalized integral vacuum charge should vanish, while the vacuum polarization could only distort its spatial distribution [12, 33]. It should be noted, however, that this is not a theorem, but just a plausible statement, which in any concrete case should be verified via direct calculation. In the case under consideration the direct check shows (see Appendix B) that upon renormalization the induced vacuum charge turns out to be non-vanishing only for \( Z > Z_{cr} \) due to the non-perturbative effects, caused by diving of discrete levels into the lower continuum in accordance with Refs. [4],[9]-[12]. And in the part II it will be shown, how the latter circumstance shows up in the behavior of the vacuum energy in the overcritical region.
3. THE WICHMANN-KROLL APPROACH FOR 2+1 QED

The most efficient non-perturbative approach to evaluation of the vacuum density $\rho_{VP}(\vec{r})$ is based on the method of Wichmann and Kroll (WK) [34] (see also Ref. [33] and refs. therein). The starting point of the WK approach is the following expression for the induced charge density

$$\rho_{VP}(\vec{r}) = -\frac{|e|}{2} \left( \sum_{\epsilon_n < \epsilon_F} \psi_n(\vec{r})^\dagger \psi_n(\vec{r}) - \sum_{\epsilon_n \geq \epsilon_F} \psi_n(\vec{r})^\dagger \psi_n(\vec{r}) \right),$$  \hspace{1cm} (12)

where in such problems with external Coulomb source $\epsilon_F$ should be chosen at the threshold of the lower continuum, i.e. $\epsilon_F = -1$, while $\epsilon_n$ and $\psi_n(\vec{r})$ are the eigenvalues and eigenfunctions of the corresponding Dirac-Coulomb spectral problem (DC).

The essence of the WK-method is that the vacuum density (12) can be represented via contour integration of the trace of the Green function of the spectral DC-problem in the complex energy plane. By definition, the Green function satisfies the equation

$$(-i\vec{\alpha} \vec{\nabla} + V(\vec{r}) + \beta - \epsilon) G(\vec{r}, \vec{r}'; \epsilon) = \delta(\vec{r} - \vec{r}').$$  \hspace{1cm} (13)

Here it should be noted that in 2+1 D there are two types of spinors, responsible for two possible choices of the signature of two-dimensional Dirac matrices [35, 36]. At the same time, in 2+1 D the Dirac matrices could also be taken as four-dimensional ones. In the latter case, the DC spectral problem in the external field (1) for the four-component wavefunction splits into two independent subsystems, which transform into each other under the change of the sign of the total angular momentum $m_j \rightarrow -m_j$. Hence, the degeneracy of states with definite value of $m_j$ equals to 2, and in what follows this factor will be explicitly shown in all the expressions for $E_{VP}$ and $\rho_{VP}$, while the DC spectral problem without any loss of generality will be considered within the two-dimensional representation with $\alpha_i = \sigma_i$, $\beta = \sigma_3$.

The formal solution of (13) reads

$$G(\vec{r}, \vec{r}'; \epsilon) = \sum_n \frac{\psi_n(\vec{r}) \psi_n(\vec{r}')}{\epsilon_n - \epsilon}. \hspace{1cm} (14)$$

Following Ref. [34], in the next step the vacuum charge density is expressed via the integrals along the contours $P(R_0)$ and $E(R_0)$ on the first sheet of the Riemann energy surface (Fig.1)

$$\rho_{VP}(\vec{r}) = -\frac{|e|}{2} \lim_{R_0 \rightarrow \infty} \left( \frac{1}{2\pi i} \int_{P(R_0)} de \text{Tr}G(\vec{r}, \vec{r}; \epsilon) + \frac{1}{2\pi i} \int_{E(R_0)} de \text{Tr}G(\vec{r}, \vec{r}; \epsilon) \right). \hspace{1cm} (15)$$

![Fig.1. (Color online) Special contours in the complex energy plane, used for representation of the vacuum charge density via contour integrals. The direction of contour integration is chosen in correspondence with (14).](image-url)
Taking account of that the DC spectral problem in the external field (1) allows for separation of radial and angular variables in the form

$$\psi(r) = \frac{1}{\sqrt{2\pi}} \left( i \psi_1(r) e^{i(m_j - 1/2)\phi} \right),$$

(16)

where $m_j$ is the half-integer total angular momentum, one finds for the trace of the Green function (14)

$$\text{Tr} G(r, r; \epsilon) = \frac{1}{2\pi} \text{Tr} G(r, r; \epsilon) = \frac{2}{2\pi} \sum_{m_j=\pm 1/2, \pm 3/2, \ldots} \text{Tr} G_{m_j}(r, r; \epsilon),$$

(17)

where $\psi^{in}_{m_j}(r; \epsilon)$ and $\psi^{out}_{m_j}(r; \epsilon)$ are the solutions of the radial spectral DC problem with the same $m_j$, which are regular for $r \to 0$ and $r \to +\infty$ correspondingly, while $J_{m_j}(\epsilon)$ is their Wronskian

$$J_{m_j}(\epsilon) = [\psi^{in}_{m_j}(r; \epsilon), \psi^{out}_{m_j}(r; \epsilon)].$$

In (18), a convenient for what follows denotation

$$[f, g]_a = a(f_2(a)g_1(a) - f_1(a)g_2(a))$$

is introduced.

Defined in such a way, $\text{Tr} G_{m_j}$ possesses the required normalization. It should be noted also that actually $J_{m_1}(\epsilon)$ is nothing else, but the Jost function of the DC problem for the given $m_1$: the real-valued zeros of $J_{m_j}(\epsilon)$ lie on the first sheet in the interval $-1 \leq \epsilon < 1$ and coincide with discrete levels $\epsilon_{n,m_1}$, while the complex ones reside on the second sheet with negative imaginary part of the wavenumber $k = \sqrt{\epsilon^2 - 1}$, and for $\text{Re} k > 0$ correspond to elastic resonances.

For the external potential (1) $\text{Tr} G_{m_j}$ can be easily found in the analytical form. For the fixed $m_j$, the radial DC spectral problem takes the form of the system

\[
\begin{aligned}
\frac{d}{dr} \psi_1(r) + \frac{1/2 - m_j}{r} \psi_1(r) &= (\epsilon - V(r) + 1)\psi_2(r), \\
\frac{d}{dr} \psi_2(r) + \frac{1/2 + m_j}{r} \psi_2(r) &= -(\epsilon - V(r) - 1)\psi_1(r).
\end{aligned}
\]

(19)

For $r \leq R$ it is convenient to choose the linearly independent solutions of the system (19) as the following ones

- for $\psi_1(r; \epsilon)$: $\mathcal{I}_1(r) = \xi I_{|m_j - 1/2|}(\xi r)$, $\mathcal{K}_1(r) = -\xi K_{|m_j - 1/2|}(\xi r)$;
- for $\psi_2(r; \epsilon)$: $\mathcal{I}_2(r) = (1 - \epsilon - V_0) I_{|m_j + 1/2|}(\xi r)$, $\mathcal{K}_2(r) = (1 - \epsilon - V_0) K_{|m_j + 1/2|}(\xi r)$;

with $I_\nu(z)$ and $K_\nu(z)$ being the Infeld and Macdonald functions correspondingly,

$$V_0 = Z\alpha/R, \quad \xi = \sqrt{1 - (\epsilon + V_0)^2}, \quad \text{Re} \xi \geq 0.$$ 

(20)

(21)

For $r > R$ the fundamental pair of solutions of (19) should be taken in the form

- for $\psi_1(r; \epsilon)$: $\mathcal{M}_1(r) = \frac{1 + \epsilon}{\gamma} \left[ (s - \nu) M_{\nu - 1/2,s}(2\gamma r) + \left( m_j + \frac{Q}{\gamma} \right) M_{\nu + 1/2,s}(2\gamma r) \right]$,
  $W_1(r) = \frac{1 + \epsilon}{\gamma} \left[ \left( m_j + \frac{Q}{\gamma} \right) W_{\nu - 1/2,s}(2\gamma r) - W_{\nu + 1/2,s}(2\gamma r) \right]$;
- for $\psi_2(r; \epsilon)$: $\mathcal{M}_2(r) = \frac{\gamma}{\nu} \left[ (s - \nu) M_{\nu - 1/2,s}(2\gamma r) - \left( m_j + \frac{Q}{\gamma} \right) M_{\nu + 1/2,s}(2\gamma r) \right]$,
  $W_2(r) = \frac{\gamma}{\nu} \left[ \left( m_j + \frac{Q}{\gamma} \right) W_{\nu - 1/2,s}(2\gamma r) + W_{\nu + 1/2,s}(2\gamma r) \right]$;

(22)

with $M_{b,c}(z)$, $W_{b,c}(z)$ being the Whittaker functions [37],

$$Q = Z\alpha, \quad s = \sqrt{m_j^2 - Q^2}, \quad \nu = \frac{\epsilon Q}{\gamma}, \quad \gamma = \sqrt{1 - \epsilon^2}, \quad \text{Re} \gamma \geq 0.$$ 

(23)
The functions $\psi_{m_j}^\text{in}(r; \epsilon)$ and $\psi_{m_j}^\text{out}(r; \epsilon)$, which enter into $\operatorname{Tr}G_{m_j}$, are constructed now as such linear combinations of solutions (20) and (22), that satisfy the conditions of regularity for $r \to 0$ and $r \to +\infty$ correspondingly. As a result, the final expression for $\operatorname{Tr}G_{m_j}$ takes the form

$$\operatorname{Tr}G_{m_j}(r, r; \epsilon) = \left\{ \begin{array}{ll}
\frac{1}{[\mathcal{I}, \mathcal{K}]} \left( \mathcal{I}_0 \mathcal{K}_1 + \mathcal{I}_2 \mathcal{K}_2 - \frac{[\mathcal{K}, \mathcal{W}]_R}{[\mathcal{I}, \mathcal{W}]_R} \left( \mathcal{I}_0^2 + \mathcal{I}_2^2 \right) \right), & r \leq R, \\
\frac{1}{[\mathcal{M}, \mathcal{W}]} \left( \mathcal{M}_0 \mathcal{W}_1 + \mathcal{M}_2 \mathcal{W}_2 - \frac{[\mathcal{M}, \mathcal{R}]_R}{[\mathcal{I}, \mathcal{W}]_R} \left( \mathcal{W}_1^2 + \mathcal{W}_2^2 \right) \right), & r > R,
\end{array} \right. \quad (24)$$

where

$$[\mathcal{I}, \mathcal{K}] = \epsilon + V_0 - 1, \quad [\mathcal{M}, \mathcal{W}] = -4\gamma^2(1+\epsilon) \frac{\Gamma(2s+1)}{\Gamma(s-\nu)} , \quad (25)$$

while the Wronskian, which enters into the expression for $\operatorname{Tr}G_{m_j}$ (17), equals to

$$J_{m_j}(\epsilon) = [\mathcal{I}, \mathcal{W}]_R . \quad (26)$$

Proceeding further, in the next step one finds the asymptotics of $\operatorname{Tr}G_{m_j}$ on the arcs $C_1(R_0)$ and $C_2(R_0)$ in the upper half-plane (Fig.1) for $|\epsilon| \to \infty$, $0 < \text{Arg} \epsilon < \pi$:

$$\operatorname{Tr}G_{m_j}(r, r; \epsilon) \to \left\{ \begin{array}{ll}
\frac{i}{r} + \frac{i}{2r^2} \left( \frac{m_j^2}{r^2} + 1 \right) - \frac{i}{r^2 \epsilon^3} \left( \frac{m_j^2}{r} V_0 + \frac{m_j}{2r} + r V_0 \right) + O \left( |\epsilon|^{-4} \right), & r < R, \\
\frac{i}{r} + \frac{i}{2r^2} \left( \frac{m_j^2}{r^2} + 1 \right) - \frac{i}{r^2 \epsilon^3} \left( \frac{m_j^2}{r} Q_0 + \frac{m_j}{2r} + Q \right) + O \left( |\epsilon|^{-4} \right), & r > R, \quad (27)
\end{array} \right.$$ 

and on the arcs $C_3(R_0)$ and $C_4(R_0)$ in the lower half-plane $|\epsilon| \to \infty$, $-\pi < \text{Arg} \epsilon < 0$:

$$\operatorname{Tr}G_{m_j}(r, r; \epsilon) \to \left\{ \begin{array}{ll}
-\frac{i}{r} - \frac{i}{2r^2} \left( \frac{m_j^2}{r^2} + 1 \right) + \frac{i}{r^2 \epsilon^3} \left( \frac{m_j^2}{r} V_0 + \frac{m_j}{2r} + r V_0 \right) + O \left( |\epsilon|^{-4} \right), & r < R, \\
-\frac{i}{r} - \frac{i}{2r^2} \left( \frac{m_j^2}{r^2} + 1 \right) + \frac{i}{r^2 \epsilon^3} \left( \frac{m_j^2}{r} Q_0 + \frac{m_j}{2r} + Q \right) + O \left( |\epsilon|^{-4} \right), & r > R. \quad (28)
\end{array} \right.$$ 

Upon integration over the arcs, the leading terms in the asymptotics (27,28) give $(i/r)(-2i R_0)$ for the contribution from $C_1 + C_2$ and $(i/r)(+2i R_0)$ from $C_3 + C_4$, which cancel each other. At the same time, the integration of the next-to-leading terms in (27,28) contains always inverse powers of $R_0$ in the answer, hence, their contribution vanishes for $R_0 \to \infty$.

So there follows from (27) and (28), that the integration along the contours $P(R_0)$ and $E(R_0)$ (Fig.1) in (15) could be reduced to the imaginary axis, whence one obtains the final expression for the vacuum charge density

$$\rho_{VP}(r) = 2 \sum_{m_j=1/2,3/2,...} \rho_{VP|m_j}(r) , \quad (29)$$

where

$$\rho_{VP|m_j}(r) = \frac{|\epsilon|}{(2\pi)^2} \int_{-\infty}^{\infty} dy \operatorname{Tr}G_{m_j}(r, r; iy) , \quad (30)$$

$$\operatorname{Tr}G_{m_j}(r, r; iy) = \operatorname{Tr}G_{m_j}(r, r; iy) + \operatorname{Tr}G_{-m_j}(r, r; iy) .$$

In the case, when there exist a set of negative discrete levels with $-1 \leq \epsilon_n < 0$, instead of (30) one gets by analogy with Ref. [38]

$$\rho_{VP|m_j}(r) = \frac{|\epsilon|}{2\pi} \left[ \sum_{m_j=\pm|m_j|} \sum_{-1 \leq \epsilon_n < 0} \psi_{n,m_j}(r) \psi_{n,m_j}(r) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \operatorname{Tr}G_{m_j}(r, r; iy) \right] . \quad (31)$$
Proceeding further, let us mention the general properties of $\text{Tr}G_{m_j}$ under the change of the sign of external field ($Q \rightarrow -Q$) and complex conjugation

$$
\text{Tr}G_{-m_j}(Q; r, r; \epsilon) = -\text{Tr}G_{m_j}(-Q; r, r; -\epsilon),
$$

and their direct consequence

$$
\text{Tr}G_{m_j}(Q; r, r; iy)^* = -\text{Tr}G_{-m_j}(-Q; r, r; iy).
$$

There follows from (32) and (33) that actually $\rho_{VP}(|m_j|)(r)$ is determined by $\text{Re}\text{Tr}G_{m_j}(Q; r, r; iy)$ and so is definitely a real quantity, odd in $Q$ in accordance with the Furry theorem. In the purely perturbative region, the representation of $\rho_{VP}(|m_j|)(r)$ as an odd series in powers of external field (2) follows directly from the Born series for the Green function $G_{m_j} = G_{m_j}^{(0)} + G_{m_j}^{(0)}(-V)G_{m_j}^{(0)} + G_{m_j}^{(0)}(-V)G_{m_j}^{(0)}(-V)G_{m_j}^{(0)} + \ldots$, whence

$$
\text{Re} \text{Tr} G_{m_j}(r, r; iy) = \sum_{k=0}^{\infty} \text{Re} \text{Tr} \left[ G_{m_j}^{(0)} (-V G_{m_j})^{2k+1} (r, r; iy) \right],
$$

where $G_{m_j}^{(0)}$ is the Green function of the free radial Dirac equation with the same $m_j$. At the same time, in presence of negative discrete levels and, moreover, in the overcritical region with $Z > Z_{cr,1}$, the oddness in $Q$ property of $\rho_{VP}$ maintains [38], but now the dependence on the external field cannot be described by a power series (34) any more, since there appear in $\rho_{VP}$ certain essentially non-perturbative and so non-analytic in $Q$ components.

4. THE DIVERGENCIES AND RENORMALIZATION OF THE VACUUM DENSITY IN 2+1 D

The expressions (29)-(31) for the vacuum density require renormalization, since there follows from the asymptotics of $\text{Tr}G_{m_j}$ for $r \rightarrow \infty$

$$
\text{Tr}G_{m_j}(r, r; iy) \rightarrow \frac{iy}{\sqrt{1 + y^2/r}} \frac{1}{r} - \frac{Q}{(1 + y^2/r)^{3/2}} \frac{1}{r^2} + O(1/r^3), \quad r \rightarrow \infty,
$$

that the non-renormalized density $\rho_{VP}(r)$ decreases for $r \rightarrow \infty$ as $1/r^2$, hence, the non-renormalized induced charge turns out to be logarithmically divergent.

The general result, obtained in Ref. [38] within the expansion of $\rho_{VP}$ in powers of $Q$, which is valid for $1+1$ and $2+1$ D always, and for a spherically-symmetric external potential in the three-dimensional case, is that all the divergencies of $\rho_{VP}$ originate solely from the fermionic loop with two external photon lines, which gives rise to the Uehling potential $A_{VP}^{(1)}$, whereas all the next-to-leading orders of expansion are already free from divergencies (see also Ref. [33] and refs. therein). In $3+1$ D there diverges (logarithmically) also the fermionic loop with 4 external lines, but in the spherically-symmetric case this divergence is not relevant for calculation of the vacuum density and vacuum energy, since within the partial expansion each term with definite momentum and parity of order $O(Q^3)$ for the density and of order $O(Q^4)$ for the energy turns out to be automatically gauge-invariant and finite without any additional regularization. Hence, in this case the renormalization procedure for the vacuum density is actually the same for all the three spatial dimensions and reduces to the diagram with two external lines.

Thus, in order to find the renormalized density $\rho_{ren}^{VP}(r)$, the linear in $Q$ terms in the expression for $\text{Tr}G_{m_j}$ in (24) should be extracted and replaced by the renormalized perturbative density $\rho_{VP}^{(1)}(r)$, determined in (10), which corresponds to the first-order PT and does not vanish only for $|m_j| = 1/2$ (see Appendix A). For these purposes let us obtain first the component of the vacuum density $\rho_{VP}^{(3+)}_{m_j}(r)$, that is defined in the next way

$$
\rho_{VP}^{(3+)}_{m_j}(r) = \frac{|e|}{2\pi} \left[ \sum_{m_j = \pm |m_j| - 1 \leq \epsilon_n < 0} \psi_{n,m_j}(r) \tilde{\psi}_{n,m_j}(r) \right]^{(3+)} + \frac{1}{\pi} \int_0^{\infty} dy \text{Re} \left[ \text{Tr}G_{m_j}(r, r; iy) - 2 \text{Tr}G_{m_j}^{(1)}(r; iy) \right],
$$

(36)
where $G_{m_j}^{(1)}(r; iy)$ is the linear in $Q$ component of the partial Green function $G_{m_j}(r, r; iy)$, which is found from the first Born approximation for $G_{m_j}$:

$$G_{m_j}^{(1)} = Q \left( \partial G_{m_j}/\partial Q \right)_{Q=0} = G_{m_j}^{(0)}(-V)G_{m_j}^{(0)}.$$  

(37)

For the external potential (2) the explicit expression of $G_{m_j}^{(1)}(r; iy)$ for $r \leq R$ reads

$$\text{Tr}G_{m_j}^{(1)}(r; iy) = \frac{Q}{(iy-1)^2} \left[ (\gamma^2 K_{[m_j-1/2]}^{2}(\gamma r) + (1 - iy)^2 K_{[m_j+1/2]}^{2}(\gamma r)) \right.\right.$$

$$\left. \times \int_{0}^{r} dr' \frac{r'}{R} \left( \hat{\gamma}^2 I_{[m_j-1/2]}^{2}(\gamma r') + (1 - iy)^2 I_{[m_j+1/2]}^{2}(\gamma r') \right) \right.\right.$$

$$\left. + \left( \hat{\gamma}^2 I_{[m_j-1/2]}^{2}(\gamma r) + (1 - iy)^2 I_{[m_j+1/2]}^{2}(\gamma r) \right) \right.\right.$$

$$\left. \times \left\{ \int_{0}^{R} dr' \frac{r'}{R} \left( \hat{\gamma}^2 I_{[m_j-1/2]}^{2}(\gamma r') + (1 - iy)^2 I_{[m_j+1/2]}^{2}(\gamma r') \right) \right.\right.$$

$$\left. + \int_{R}^{\infty} dr' \left( \hat{\gamma}^2 K_{[m_j-1/2]}^{2}(\gamma r') + (1 - iy)^2 K_{[m_j+1/2]}^{2}(\gamma r') \right) \right\},$$

(38)

while for $r > R$

$$\text{Tr}G_{m_j}^{(1)}(r; iy) = \frac{Q}{(iy-1)^2} \left[ (\gamma^2 K_{[m_j-1/2]}^{2}(\gamma r) + (1 - iy)^2 K_{[m_j+1/2]}^{2}(\gamma r)) \right.\right.$$

$$\left. \times \left\{ \int_{0}^{R} dr' \frac{r'}{R} \left( \hat{\gamma}^2 I_{[m_j-1/2]}^{2}(\gamma r') + (1 - iy)^2 I_{[m_j+1/2]}^{2}(\gamma r') \right) \right.\right.$$

$$\left. + \int_{R}^{\infty} dr' \left( \hat{\gamma}^2 K_{[m_j-1/2]}^{2}(\gamma r') + (1 - iy)^2 K_{[m_j+1/2]}^{2}(\gamma r') \right) \right\},$$

(39)

where $\hat{\gamma} = \sqrt{1 + y^2}$. The integrals, entering into the expressions (38) and (39) for $\text{Tr}G_{m_j}^{(1)}$, could also be in principle evaluated analytically, but they are not given here explicitly because of their cumbersome form.

So the renormalized vacuum density takes the form

$$\rho_{VP}^{ren}(r) = 2 \left[ \rho_{VP}^{(1)}(r) + \sum_{m_j=1/2,3/2,...} \rho_{VP|m_j}^{(3+)}(r) \right],$$

(40)

where $\rho_{VP}^{(1)}(r)$ is the perturbative vacuum density (4), evaluated by means of the polarization function (6) in the first order of PT. Such expression for $\rho_{VP}^{ren}(r)$ guarantees the vanishing total vacuum charge $Q_{VP}^{ren} = \int d^2 r \rho_{VP}^{ren}(r)$ for $Z < Z_{cr,1}$, since $Q_{VP}^{(1)}$ is zero by construction, while the subsequent direct check confirms that the contribution of $\rho_{VP|m_j}^{(3+)}(r)$ to $Q_{VP}^{ren}$ for $Z < Z_{cr,1}$ vanishes too. Unlike 1+1 D, in 2+1 D such a check cannot be performed in the purely analytical form any more due to complexity of expressions, entering into $\rho_{VP|m_j}^{(3+)}(r)$. Nevertheless, it could be quite reliably performed via combination of analytical and numerical methods (see Appendix B). Moreover, it suffices to verify the disappearance of the total charge $Q_{VP}^{ren}$ not for the entire subcritical region, but only in absence of negative discrete levels. In presence of the latter, the vanishing total charge for $Z < Z_{cr,1}$ follows from model-independent arguments, which are based on the starting expression for the vacuum density (12). There follows from
Moreover, with increasing external parameters (see Appendix C), there follows from (44) that disappears. Since the integral over \( dy \) in (36) converges uniformly with respect to \( |m_j| \) and \( r \), considered as the external parameters (see Appendix C), there follows from (44) that \( \rho_{VP}^{(3+)}(r) \) for \( |m_j| \to \infty \) behaves like \( O(|m_j|^{-3}) \). So the partial series in (40) converges and the renormalized vacuum density \( \rho_{VP}^{ren}(r) \) turns out to be finite everywhere besides the logarithmic singularity at \( r = R \), originating from \( \rho_{VP}^{(1)}(r) \).

In the Fig. 2 for the external potential (1) with the parameters \( \alpha \) and \( R = R(Z) \), specified above, the renormalized vacuum density \( \rho_{VP}^{ren}(r) \) is shown for the purely perturbative region with \( Z = 10 \), thereafter for \( Z = 108 \), when the first critical \( Z_{cr,1} \approx 108.1 \) is not reached yet, then for \( Z = 109 \), when the first discrete level has just dived into the lower continuum, further for \( Z = 133 \), when the second critical \( Z_{cr,2} \approx 133.2 \) is not reached yet, and finally for \( Z = 134 \), i.e. just after the second discrete level diving into the lower continuum. The critical charges are found from the transcendental equation, which follows from matching of regular solutions for \( r < R \) (20) and \( r > R \) (22) at \( \epsilon = -1 \):

\[
I_{|m_j|,1/2}^1(\xi R) \left[ K_{2s-1} \left( \sqrt{8QR} \right) + K_{2s+1} \left( \sqrt{8QR} \right) + \sqrt{\frac{2}{QR}} m_j K_{2s} \left( \sqrt{8QR} \right) + \sqrt{2(2-V_0)} I_{|m_j|,1/2}^1(\xi R) K_{2s} \left( \sqrt{8QR} \right) \right] = 0.
\]
In (45), $K_{\nu}(z)$ is the Macdonald function, into which the Whittaker functions transform for $\epsilon \to -1$. The numerical integration confirms that the total vacuum charge for $Z = 10, 108$ equals to zero, for $Z = 109, 133$ due to above-mentioned twofold degeneracy it equals to $(-2|e|)$, while for $Z = 134$ to $(-4|e|)$ correspondingly (for details of calculations see Appendix B). At $r = R \rho_{FP}^{(3+)}(r)$ reveals a logarithmic singularity, which is caused by the contribution of $\rho_{FP}^{(1)}(r)$ (Fig. 3). At the same time, $\rho_{FP}^{(3+)}(r) = \sum_{m_j} \rho_{FP}^{(3+)}(r)$ turns out to be a continuous function everywhere, what is shown in Fig. 4. Because it is difficult to see from the Fig. 4 that $\int r dr \rho_{FP}^{(3+)}(r)$ for $Z = 10$ and $Z = 108$ vanishes, in the Fig. 4a we demonstrate that $\rho_{FP}^{(3+)}(r)$ is actually a sign-alternating function of $r$ for these $Z$. Let us also mention that in the overcritical region with increasing $Z$ the change of $\rho_{FP}^{(3+)}(r)$ proceeds not only in a discrete manner due to vacuum shells formation from the discrete levels diving into the lower continuum, but also continuously, due to the changes in the density of states of the continuous spectrum and evolution of discrete levels with growing $Z$.

![Fig. 2. $\rho_{FP}^{(3+)}(r)$ for $Z = 10, 108, 109, 133, 134.$](image1)

![Fig. 3. $\rho_{FP}^{(1)}(r)$ for $Z = 10, 108, 109, 133, 134.$](image2)
It should be also noted that for undercritical $Z$ the main contribution to the partial expansion (40) for $\rho^{(3+)}_{VP}(r)$ is given by the term $\rho^{(3+)}_{VP,|m_j|=1/2}$ (see Fig. 5). At the same time, for overcritical $Z$ the sum (40) depends first of all on the contribution from those $|m_j|$ for which at least one level has already sunk into the lower continuum.
Thus, the correct approach to evaluation of $\rho^{\text{ren}}_{VP}(r)$ for all regions of $Z$ consists in making use of relations (36) and (40) with subsequent check of the expected integer value of the induced charge via direct numerical integration of $\rho^{\text{ren}}_{VP}(r)$. The appearing general picture for the renormalized vacuum density $\rho^{\text{ren}}_{VP}(r)$ and its components in the range $0 < Z < 1000$ is shown in the Figs.6-8 for the four most representative values $Z = 100, 200, 500, 1000$. The almost vertical pics on the curves $\rho^{\text{ren}}_{VP}(r)$ in the Fig.6 correspond to the logarithmic singularities in the vacuum density, caused by $\rho^{(1)}_{VP}(r)$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig6}
\caption{$\rho^{\text{ren}}_{VP}(r)$ for $Z = 100, 200, 500, 1000$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{$\rho^{(1)}_{VP}(r)$ for $Z = 100, 200, 500, 1000$.}
\end{figure}
Moreover, it should be clear that the main contribution to the vacuum density for the given $Z$ is produced by those partial channels, in which the discrete levels have already reached the lower continuum. The histograms, demonstrating the contribution from different partial channels to the total induced vacuum charge $Q_{VP}(Z)$, are shown in the Fig.9 for $Z = 500$ and $Z = 1000$. In the Fig.10a both the dependence on $Z$ for the total number $N(Z)$ of vacuum shells (from all $m_j$), formed from levels, which have already sunk into the lower continuum, and its power-like approximation $\tilde{N}(Z) = 6.69 \times 10^{-5} \times Z^{2.17}$, are shown. As it follows from the Fig.10a, in 2+1 D the growth rate of the shells total number for $Z \gg Z_{cr,1}$ becomes already more than quadratic. Moreover, the power $Z^{2.17}$ estimates the growth rate of $N(Z)$ from below, since only the range $0 < Z < 1000$ is used for this approximation. By extending the approximation range in $Z$ to the right the growth rate increases, but such $Z$ in this work are not considered by default. In any
case, however, the growth rate of shells number for $Z \gg Z_{cr,1}$ more than quadratic one significantly influences on the behavior of the vacuum energy in the overcritical region in 2+1 D, since now the shell effect becomes the dominant factor in $\mathcal{E}_{\mathcal{V}P}^{ren}(Z)$ for $Z \gg Z_{cr,1}$. In the Fig.10b the dependence of the shells number in the lowest channel $N_{1/2}(Z)$ on $Z$ due to levels with $|m_j|=1/2$, which have sunk into the lower continuum, and its power-like approximation $\tilde{N}_{1/2}(Z) = -3.53 + 0.046Z$, are presented. The last result shows that in its structure the partial channel with $|m_j|=1/2$ is practically identical with $\rho^{\mathcal{V}P}_{1/cr}(Z)$ for the one-dimensional case, while the growth rate of the vacuum shells number in this channel, as in 1+1 D, is close to the linear one, at least in the considered range of external parameters.

5. CONCLUSION

Thus, in the present part I of our work the vacuum polarization effects in the 2+1-dimensional QED for a model potential (1) have been considered at the level of the renormalized vacuum density $\rho^{\mathcal{V}P}_{ren}(Z)$. The 2+1-dimensional case differs significantly from the one-dimensional one first of all in that $\rho^{\mathcal{V}P}_{1/cr}(Z)$ is represented now by an infinite series in the rotational quantum number $m_j$, and so there appears an additional problem of its convergence. As it was shown in this paper, this problem could be also solved successfully by renormalization of the vacuum density within PT, i.e. via regularization of the solely divergent Feynman graph in the form of the fermionic loop with two external lines. Simultaneously, the integral vacuum charge vanishes automatically in the subcritical region, and is changed by $(-2|e|)$ upon diving of each subsequent doubly degenerate discrete level into the lower continuum. Such behavior of $\rho^{\mathcal{V}P}_{1/cr}(Z)$ in the overcritical region confirms once more the assumption of the neutral vacuum transmutation into the charged one under such conditions [4, 9–12], and thereby of the spontaneous positron emission, accompanying the emergence of the next vacuum shell due to the total charge conservation.

It would be also worth to note that in its structure the partial channel with $|m_j|=1/2$ is almost identical with $\rho^{\mathcal{V}P}_{1/cr}(Z)$ for the one-dimensional case, and the growth rate of the vacuum shells number in this channel, as in 1+1 D, does not exceed $\sim Z^s$, $1 < s < 2$, at least in the considered range of the external parameters (see Fig.10b). However, due to the contribution from the whole partial series for $\rho^{\mathcal{V}P}_{1/cr}(Z)$, in the overcritical region the growth rate of the total vacuum shells number turns out to be sufficiently higher (see Fig.10a). The estimate given above is not less than $\sim Z^{2.17}$, since for $Z \gg Z_{cr,1}$ the levels with $|m_j| > 1/2$ reach the lower continuum too. And in the following part II of our work we will show that the origin of the essentially nonlinear behavior of the vacuum energy $\mathcal{E}_{\mathcal{V}P}^{ren}(Z)$ for $Z > Z_{cr,1}$ is indeed the non-perturbative changes in the vacuum charge density due to the increasing with $Z$ total number of discrete levels from different partial channels, diving into the lower continuum ("the shell effect"), and so in 2+1 D the rate of decrease of $\mathcal{E}_{\mathcal{V}P}^{ren}(Z)$ turns out to be sufficiently larger than $\sim -\eta Z^2$.

Appendix A: The Uehling Potential in 2+1 D

Here we consider in detail the derivation of the Uehling potential in 2+1 D for a static external field $A_0^{ext}(r)$ in the form (1). The starting point is the expression (5)

$$A_{\mathcal{V}P,0}^{(1)}(\vec{r}) = \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\vec{r}} \Pi_R (-q^2) \tilde{A}_0^{ext}(\vec{q}), \quad \tilde{A}_0(\vec{q}) = \int d^2y e^{-i\vec{q}\vec{y}} A_0^{ext}(y). \quad (A.1)$$

Proceeding further, we recall the plane wave expansion in cylindrical functions

$$e^{ikr \cos \varphi} = \sum_{m_1=-\infty}^{\infty} e^{im_1 \varphi} J_{m_1}(kr), \quad (A.2)$$
whence

\[
\tilde{A}_0(\bar{q}) = \int_0^{2\pi} d\varphi_y \int_0^{\infty} dy \ y e^{-iq_y\cos\varphi_y} A_0^{\text{ext}}(y)
\]

\[
= \sum_{m_i=-\infty}^{+\infty} (-i)^{m_i} \int_0^{2\pi} d\varphi_y e^{im_i\varphi_y} J_{m_i}(qy) A_0^{\text{ext}}(y)
\]

\[
= \sum_{m_i=-\infty}^{+\infty} (-i)^{m_i} 2\pi \delta_{m_i,0} \int_0^{\infty} dy \ J_{m_i}(qy) A_0^{\text{ext}}(y) = 2\pi \int_0^{\infty} dy \ J_0(qy) A_0^{\text{ext}}(y)
\]

\[
= 2\pi Z|e| \left[ \frac{1}{R} \int_0^R dy \ J_0(qy) + \int_R^{\infty} dy \ J_0(qy) \right]
\]

\[
= \frac{\pi Z|e|}{q} \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) H_1(qR) - J_1(qR) H_0(qR) \right] \right],
\]

where \( H_{\nu}(z) \) is the Struve function. As a result,

\[
A_0^{(1)}_{VP,0}(R) = \frac{Z|e|}{8\pi} \sum_{m_i=-\infty}^{+\infty} \int_0^{\infty} dq \ \frac{1}{q} \int_0^{2\pi} d\varphi_q e^{iq \cos\varphi_q} \left[ 2 \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \right]
\]

\[
\times \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) H_1(qR) - J_1(qR) H_0(qR) \right] \right]
\]

\[
\times \left[ \frac{2\pi}{q} \delta_{m_i,0} \int_0^{\infty} dq \ J_{m_i}(qr) \right]
\]

\[
= \frac{Z|e|}{8\pi} \sum_{m_i=-\infty}^{+\infty} \int_0^{\infty} dq \ \frac{1}{q} \int_0^{2\pi} d\varphi_q e^{im_i\varphi_q} \left[ 2 \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \right]
\]

\[
\times \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) H_1(qR) - J_1(qR) H_0(qR) \right] \right]
\]

\[
= \frac{Z|e|}{4} \int_0^{\infty} dq \ J_0(qr) \left[ 2 \frac{2}{q} + \left( 1 - \frac{4}{q^2} \right) \arctg \left( \frac{q}{2} \right) \right]
\]

\[
\times \left[ 2 \left[ 1 + J_1(qR) - qR J_0(qR) \right] + \pi qR \left[ J_0(qR) H_1(qR) - J_1(qR) H_0(qR) \right] \right],
\]

that is the final answer for the Uehling potential (7) in the case of the external field (1). It should be specially mentioned that there are only the terms with \( m_i = 0 \) in the expansion of \( A_0^{(1)}_{VP,0}(r) \) (hence, of both \( \rho_{VP}(r) \) and \( \mathcal{E}_{VP} \)) in cylindrical waves, what should be quite clear, since \( A_0^{(1)}_{VP,0}(r) \) is assembled exclusively from two axially symmetric functions — the external field (1) and the polarization function (6). So within the partial expansion for \( \rho_{VP} \) and \( \mathcal{E}_{VP} \) in \( m_j = m_i + m_s \) the perturbative contributions \( \rho^{(1)}_{VP} \) and \( \mathcal{E}^{(1)}_{VP} \) correspond always only to \( |m_j| = 1/2 \).

Appendix B: Explicit Calculation of \( Q_{VP}^{ren} \)

Here we will show that in the problem under consideration with the external field (1) the renormalized induced charge \( Q_{VP}^{ren} \) vanishes in the subcritical region \( Z < Z_{cr,1} \). The initial expression for \( Q_{VP}^{ren} \) reads

\[
Q_{VP}^{ren} = \int d^2r \rho_{VP}^{ren}(r) = Q_{VP}^{(1)} + Q_{VP}^{(3+)},
\]

(B.1)
First of all, it should be noted that vanishing of $Q_{V}^{(1)}$ in the problems of such type with decreasing in the spatial infinity Coulomb field follows from sufficiently more general considerations (under the same conditions of absence of any special boundary conditions and/or nontrivial topology of the field manifold), and so does not require for direct integration of the expression (10).

For simplicity let us consider the 1+1 D problem, although such analysis could be easily performed in 2+1 and 3+1 D in presence of a static external field too. In the momentum space (up to factors like 2π and general signs) the static equation for the potential \( A_0(q) \), generated by the charge density \( \rho(q) \), takes the form

\[
(\vec{q}^2 - \Pi_R(q))A_0(q) = \rho(q) ,
\]

where \( \Pi_R(q) \) is the polarization operator in the standard form with dimension \([q^2]\), which is introduced via relation

\[
\Pi_R^{\mu\nu}(q) = (\delta^{\mu\nu} - q^\mu q^\nu/q^2) \Pi_R(q).
\]

Within PT one should propose that \( \Pi_R(q) \ll q^2 \), while the potential \( A_0 \) should be considered within the expansion \( A_0(q) = A^{(0)}(q) + A^{(1)}(q) + \ldots \). The classical part of the potential is determined from the equation

\[
q^2A^{(0)}(q) = \rho(q) ,
\]

while its first quantum correction — from equation

\[
q^2A^{(1)}(q) = \Pi_R(q)A^{(0)}(q) .
\]

The r.h.s. in (B.5) is just the induced vacuum charge density \( \rho_{V}^{(1)}(q) \) to the first order of PT. Then in the coordinate representation (up to multipliers like 2π)

\[
\rho_{V}^{(1)}(x) = \int dy \, \Pi_R(x-y)A^{(0)}(y) = \int dq \, \Pi_R(q)A^{(0)}(q)e^{iqx} .
\]

Upon integrating this equality over all \( x \), one obtains the total induced charge \( Q_{V}^{(1)} \) to the first order of PT

\[
Q_{V}^{(1)} = \int dq \, \delta(q)\Pi_R(q)A^{(0)}(q) .
\]

And since in PT there holds the renormalization condition \( \Pi_R(q) \sim q^2 \) for \( q^2 \to 0 \), there follows from (B.7) that \( Q_{V}^{(1)} = 0 \) subject to condition, that the singularity of the Fourier-transform of the external potential \( A^{(0)}(q) \) for \( q \to 0 \) is weaker, than \( 1/q^2 \). For the Coulomb potentials like (2) in 1+1 D the singularity of \( A^{(0)}(q) \) is just a logarithmic one, hence \( Q_{V}^{(1)} = 0 \). In 2+1 and 3+1 D the singularity of \( A^{(0)}(q) \) is already stronger, \( 1/|q| \) and \( 1/q^2 \) correspondingly, but there appears an additional factor \( |q|^{D-1} \) from the integration measure, which compensates these singularities already by itself. Therefore for the Coulomb potentials like (1) in all the three spatial dimensions there follows \( Q_{V}^{(1)} = 0 \) to the first order of PT. It is indeed this simple reasoning that was implied in Section 2 under the statement “in this case the relation (11) is the direct consequence of the renormalization condition”.

However, beyond the first-order PT and especially in the whole subcritical region \( Z < Z_{cr,1} \), when in presence of negative discrete levels the dependence of \( \rho_{V} \) on the external field cannot be described by the power series (34) any more, the check of the zero value of \( Q_{V}^{(1)} \) requires a sufficiently more detailed consideration, that in 1+1 D, where it could be performed almost completely in a purely analytical form [1], but in 2+1 and 3+1 D requires for an additional numerical doings.

In 2+1 D this check could be most efficiently performed in the next way. In the first step, let us take into account the uniform convergence of the partial series in \( m_j \) with respect to \( r \), considered as an external parameter in (B.2) (see Appendix C). Then it is possible to insert the integral under the sign of the sum in the expression for \( Q_{V}^{(3+)} \)

\[
Q_{V}^{(3+)} = \sum_{m_j=1/2, 3/2, \ldots} Q_{V,P[m_j]}^{(3+)} , \quad Q_{V,P[m_j]}^{(3+)} = 4\pi \int_0^\infty r \, dr \, \rho_{V,P[m_j]}^{(3+)}(r) .
\]
Proceeding further, from the explicit form of the vacuum density \( \rho_{VP,m_j}(r) \), given in (36), for \( Q^{(3+)}_{VP,m_j} \) one obtains the following expression

\[
Q^{(3+)}_{VP,m_j} = 2|e| \left[ \sum_{m_j = \pm m_j} \sum_{-1 \leq \epsilon_m, m_j < 0} 1 \right. \\
+ \frac{1}{\pi} \int_0^\infty r \, dr \int_0^\infty dy \left( \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G^{(1)}_{m_j}(r, r; iy) \right),
\]  

(B.9)

Now let us show that in the subcritical region \( Q^{(3+)}_{VP,m_j} \) for all \( m_j \) vanish exactly, and so the total induced charge \( Q^{(3+)\beta}_{VP} \) does the same. The most direct method here is the straightforward numerical calculation of the double integral in (B.9). However, it turns out to be a sufficiently time-consuming numerical task. As an alternative way, it is possible to take advantage from the following circumstances. First, as it was stated above in the Section 4, it suffices to verify the disappearance of \( Q^{(3+)\beta} \) not for the whole subcritical region, but only in absence of negative discrete levels. The second point is the observation, that if the integrals over \( dr \) in (B.9) become the internal ones, they could be calculated analytically by means of Ref. [40]. For these purposes let us exchange the sequence of integrations in (B.9), inserting the intermediate regularization of \( \int dr \) at the upper limit, otherwise the exchange of integrations and especially inserting \( \int dr \) under the sign of derivative with respect to \( Q \) is not allowed, since \( \int r \, dr \, Tr G_{m_j}(r, r; iy) \) is logarithmically divergent for large \( r \) due to the asymptotics (35) of \( Tr G_{m_j}(r, r; iy) \).

With account of these circumstances it suffices to deal with \( Q^{(3+)}_{VP,m_j} \) in the following form

\[
Q^{(3+)}_{VP,m_j} = \frac{2|e|}{\pi} \int_0^\infty dy \lim_{R_1 \to \infty} \Re \left[ \int_0^{R_1} r \, dr \, Tr G_{m_j}(r, r; iy) \right] \\
- 2Q \left( \frac{\partial}{\partial Q} \int_0^{R_1} r \, dr \, Tr G_{m_j}(r, r; iy) \right)_{Q=0}.
\]  

(B.10)

Let us consider now \( \int_0^{R_1} r \, dr \, Tr G_{m_j}(r, r; iy) \) in (B.10). Taking into account the following indefinite integrals [40]

\[
\int \frac{1}{x} U_{\rho,\sigma}(x) U_{\rho,\sigma}(x) dx = -U_{\rho,\sigma}(x) \frac{\partial}{\partial \rho} U'_{\rho,\sigma}(x) + U'_{\rho,\sigma}(x) \frac{\partial}{\partial \rho} U''_{\rho,\sigma}(x),
\]

\[
\int \frac{1}{x} W_{\rho,\sigma}(x) W_{\rho,\sigma}(x) dx = \frac{1}{\mu - \rho} \left( W'_{\rho,\sigma}(x) W_{\mu,\sigma}(x) - W_{\rho,\sigma}(x) W'_{\mu,\sigma}(x) \right),
\]

\[
\int \left( \frac{b^2 - a^2}{4} + \frac{\mu a - \rho b}{x} + \frac{\nu^2 - \sigma^2}{x^2} \right) U_{\mu,\nu}(ax) U_{\rho,\sigma}(bx) dx = U_{\mu,\nu}(ax) U'_{\rho,\sigma}(bx) - U'_{\mu,\nu}(ax) U_{\rho,\sigma}(bx),
\]

where \( U_{\rho,\sigma}(x), U_{\rho,\sigma}(x) = M_{\rho,\sigma}(x) \) or \( W_{\rho,\sigma}(x), \) the prime stands for the derivative with respect to the argument, and the explicit form of \( Tr G_{m_j}(r, r; iy) \), which is given in (24), one obtains

\[
\int_0^{R_1} r \, dr \, Tr G_{m_j}(r, r; iy) = \frac{1}{[I, K]} \left( J_1 - \frac{[K, W]_R}{[I, W]_R} J_2 \right) |_{\epsilon = iy} + \frac{1}{[M, W]} \left( J_3 - \frac{[M, W]_R}{[I, W]_R} J_4 \right) |_{\epsilon = iy},
\]  

(B.12)
where

\[ J_1 = \int_0^R r \, dr \left( T_1 K_1 + T_2 K_2 \right) \]

\[ = (1 - \epsilon - V_0) \left\{ \frac{R^2}{2} \left[ \frac{(m_j + 1/2)^2}{(\xi R)^2} + 1 \right] I_{[m_j + 1/2]}(\xi R) K_{[m_j + 1/2]}(\xi R) \right. \]

\[ + \frac{1}{4} \left( I_{[m_j + 1/2] - 1}(\xi R) + I_{[m_j + 1/2] + 1}(\xi R) \right) \left( K_{[m_j + 1/2] - 1}(\xi R) + K_{[m_j + 1/2] + 1}(\xi R) \right) \]

\[ - \frac{m_j + 1/2}{2\xi^2} \right\} - \xi^2 \left\{ \frac{R^2}{2} \left[ \frac{(m_j - 1/2)^2}{(\xi R)^2} + 1 \right] I_{[m_j - 1/2]}(\xi R) K_{[m_j - 1/2]}(\xi R) \right. \]

\[ + \frac{1}{4} \left( I_{[m_j - 1/2] - 1}(\xi R) + I_{[m_j - 1/2] + 1}(\xi R) \right) \left( K_{[m_j - 1/2] - 1}(\xi R) + K_{[m_j - 1/2] + 1}(\xi R) \right) \]

\[ - \frac{m_j - 1/2}{2\xi^2} \left. \right\} , \tag{B.13} \]

\[ J_2 = \int_0^R r \, dr \left( T_1^2 + T_2^2 \right) \]

\[ = \frac{r^2}{2} \left\{ (V_0 + (\epsilon - 1))^2 \left[ I_{[m_j + 1/2]}(\xi R)^2 - I_{[m_j + 1/2] - 1}(\xi R) I_{[m_j + 1/2] + 1}(\xi R) \right] \right. \]

\[ + \xi^2 \left[ I_{[m_j - 1/2]}(\xi R)^2 - I_{[m_j - 1/2] - 1}(\xi R) I_{[m_j - 1/2] + 1}(\xi R) \right] \left. \right\} , \tag{B.14} \]

\[ J_3 = \int_0^R r \, dr \left( M_1 W_1 + M_2 W_2 \right) = \mathcal{J}_3(R_1) - \mathcal{J}_3(R) , \]

\[ \mathcal{J}_3(r) = -2(\epsilon + 1) \left\{ \left( m_j + \frac{Q}{r} \right) \right. \]

\[ \times \left[ \left( \frac{1}{2} - \frac{\nu + 1/2}{2r} \right) M_{\nu+1/2,s}(2\gamma r) + \frac{s + \nu + 1}{2\gamma r} M_{\nu+3/2,s}(2\gamma r) \right] \frac{\partial}{\partial \mu_1} W_{\mu_1,s}(2\gamma r) \bigg|_{\mu_1 = \nu + 1/2} \]

\[ - M_{\nu+1/2,s}(2\gamma r) \left. \right\} + (s - \nu) \left\{ \left( \frac{Q}{r} - m_j \right) \right. \]

\[ \times \left[ \left( \frac{1}{2} - \frac{\nu - 1/2}{2r} \right) M_{\nu-1/2,s}(2\gamma r) + \frac{s + \nu}{2\gamma r} M_{\nu+1/2,s}(2\gamma r) \right] \frac{\partial}{\partial \mu_2} W_{\mu_2,s}(2\gamma r) \bigg|_{\mu_2 = \nu - 1/2} \]

\[ - M_{\nu-1/2,s}(2\gamma r) \left. \right\} + \epsilon \left[ \left( \frac{1}{2} - \frac{\nu + 1/2}{2r} \right) M_{\nu+1/2,s}(2\gamma r) + \frac{s + \nu}{2\gamma r} M_{\nu+3/2,s}(2\gamma r) \right] \]

\[ - \left( \frac{1}{2} - \frac{\nu - 1/2}{2r} \right) M_{\nu-1/2,s}(2\gamma r) + \frac{s + \nu}{2\gamma r} M_{\nu+1/2,s}(2\gamma r) \right] \right. \]

\[ = \epsilon \left[ \left( \frac{Q^2}{r^2} - m_j^2 \right) \left[ M_{\nu+1/2,s}(2\gamma r) \left( \frac{1}{2} - \frac{\nu - 1/2}{2r} \right) W_{\nu-1/2,s}(2\gamma r) - \frac{W_{\nu+3/2,s}(2\gamma r)}{2\gamma r} \right) \right. \]

\[ + \epsilon \left( \frac{Q^2}{r^2} - m_j^2 \right) \left[ M_{\nu+1/2,s}(2\gamma r) \left( \frac{1}{2} - \frac{\nu + 1/2}{2r} \right) W_{\nu-1/2,s}(2\gamma r) + \frac{W_{\nu+1/2,s}(2\gamma r)}{2\gamma r} \right] \right. \]

\[ - \left( \frac{1}{2} - \frac{\nu - 1/2}{2r} \right) M_{\nu+1/2,s}(2\gamma r) + \frac{s + \nu + 1}{2\gamma r} M_{\nu+3/2,s}(2\gamma r) \right] W_{\nu-1/2,s}(2\gamma r) \right\} , \tag{B.15} \]
\[ J_4 = \int_{R}^{R_1} r \, dr \, (W_1^2 + W_2^2) = J_{41}(R_1) - J_{41}(R) + J_{42}(R_1) - J_{42}(R) , \]

\[ \frac{J_{41}(r)}{J_{42}(r)} = \left[ \frac{1 + \gamma^2}{\gamma^2} \right] \left\{ (m_j - \frac{Q}{\gamma})^2 \right\} \]

\[ \times \left[ \left( \frac{1}{2} \frac{\mu_1}{2\gamma^r} W_{\mu_1,s}(2\gamma r) - \frac{W_{\mu_1+1,s}(2\gamma r)}{2\gamma r} \frac{\partial}{\partial \mu_1} W_{\mu_1,s}(2\gamma r) \right) \right|_{\mu_1=\nu-1/2} \]

\[ + \left[ \left( \frac{1}{2} \frac{\mu_2}{2\gamma^r} W_{\mu_2,s}(2\gamma r) - \frac{W_{\mu_2+1,s}(2\gamma r)}{2\gamma r} \frac{\partial}{\partial \mu_2} W_{\mu_2,s}(2\gamma r) \right) \right|_{\mu_2=\nu+1/2} \]

\[ \times \left[ \left( \frac{1}{2} - \frac{\nu - 1/2}{2\gamma r} W_{\nu-1/2,s}(2\gamma r) - \frac{W_{\nu+1/2,s}(2\gamma r)}{2\gamma r} \right) \right] \]

\[ \left( \frac{1}{2} - \frac{\mu_1}{2\gamma^r} W_{\nu+1/2,s}(2\gamma r) - \frac{W_{\nu+3/2,s}(2\gamma r)}{2\gamma r} \right) \]}

\[ \text{In (B.13)-(B.16) the notations are the same as introduced earlier in Section 3. In the next step, the limit } R_1 \to \infty \text{ is calculated. In this limit the linear in } Q \text{ logarithmically divergent terms, originating from the asymptotics (35), cancel each other, whence } \]

\[ \lim_{R_1 \to \infty} \text{Re} \left[ \int_{0}^{R_1} r \, dr \, \text{Tr} \, G_{m_j}(r, r; iy) - 2Q \left( \frac{\partial}{\partial Q} \int_{0}^{R_1} r \, dr \, \text{Tr} \, G_{m_j}(r, r; iy) \right) \right]_{Q=0} \]

\[ = \text{Re} \left[ \frac{1}{[\mathcal{I}, \mathcal{K}]} \left( J_1 - \left[ \mathcal{K}, \mathcal{W} \right] R_{m_j} J_2 \right) - \frac{1}{[\mathcal{M}, \mathcal{W}]} \left( J_3(R) - \left[ \mathcal{M}, \mathcal{W} \right] R_{m_j} (J_{41}(R) + J_{42}(R)) \right) \right]_{m_j \to -m_j} \]

\[ - 2Q \left[ \frac{\partial}{\partial Q} \left( \frac{1}{[\mathcal{I}, \mathcal{K}]} \left( J_1 - \left[ \mathcal{K}, \mathcal{W} \right] R_{m_j} J_2 \right) \right) - \frac{1}{[\mathcal{M}, \mathcal{W}]} \left( J_3(R) - \left[ \mathcal{M}, \mathcal{W} \right] R_{m_j} (J_{41}(R) + J_{42}(R)) \right) \right]_{Q=0} \] \]

\[ \text{In (B.17) the derivatives with respect to } Q \text{ are not shown explicitly due to their cumbersome form.} \]

\[ \text{As a result, there remains only a single numerical integration over } dy, \text{ which despite the complexity of the integrand does not already pose any problems, since the integral is definitely convergent. Namely, the asymptotical behavior of the integrand is estimated as } \sim 1/|y|^5 \text{ (see Appendix C below). Therefore such integration can be performed via standard numerical recipes, and in this way by means of (B.10)-(B.17) one can verify that in the subcritical region in absence of negative levels the total induced charge vanishes. More concretely, we have checked by explicit calculations with WorkingPrecision } \to 100 \text{ and PrecisionGoal } \to 15 \text{ that for } Z \geq 50, \text{ when there are no negative discrete levels yet, } Q_{VP,1/2}^{(3+)} = 2.5582 \times 10^{-30}|e|, Q_{VP,3/2}^{(3+)} = 2.4007 \times 10^{-41}|e|, \text{ while the other partial charges } Q_{VP,|m_j|}^{(3+)} \text{ with higher } |m_j| \text{ decrease further according to the law } |m_j|^{-3}. \text{ These results look quite convincing for the assertion that in the subcritical region } Q_{VP}^{(3+)} = 0. \text{ In the same way it is possible to verify that in the overcritical } Z > Z_{cr,1} \text{ the total vacuum charge } Q_{VP}^{(3+)} \text{ is equal to an integer number of } (-2|e|) \text{ in dependence on the number of levels, which have dived into the lower continuum, and with account of their degeneracy.} \]
Appendix C: Verifying the Uniform Convergence of the Integral $\int dy \, \text{Re} \left[ \text{Tr} G_{m_j}^{(i)}(r; r; iy) - 2 \text{Tr} G_{m_j}^{(i)}(r; iy) \right]$

Here it will be shown that the integral $\int dy \, \text{Re} \left[ \text{Tr} G_{m_j}^{(i)}(r; r; iy) - 2 \text{Tr} G_{m_j}^{(i)}(r; iy) \right]$, that defines the main component of $\rho_{VP,m_j}(r)$ in (36), converges uniformly with respect to $m_j$ and $r$. For these purposes $\text{Tr} G_{m_j}$ should be represented as follows:

$$\text{Tr} G_{m_j}(r, r; \epsilon) = \theta(R - r) \text{Tr} G_{m_j}^{in}(r, r; \epsilon) + \theta(r - R) \text{Tr} G_{m_j}^{out}(r, r; \epsilon), \quad (C.1)$$

where

$$\text{Tr} G_{m_j}^{in}(r, r; \epsilon) = \text{Tr} G_{m_j}^{0,in}(r, r; \epsilon) + \text{Tr} \Delta G_{m_j}^{in}(r, r; \epsilon),$$

$$\text{Tr} G_{m_j}^{0,in}(r, r; \epsilon) = \frac{1}{[\mathcal{J}, \mathcal{K}]} (I_0 K_1 + I_0 K_2), \quad (C.2)$$

and

$$\text{Tr} G_{m_j}^{out}(r, r; \epsilon) = \text{Tr} G_{m_j}^{0,out}(r, r; \epsilon) + \text{Tr} \Delta G_{m_j}^{out}(r, r; \epsilon),$$

$$\text{Tr} G_{m_j}^{0,out}(r, r; \epsilon) = \frac{1}{[\mathcal{M}, \mathcal{W}]} (M_1 W_1 + M_2 W_2), \quad (C.3)$$

In (C.2)-(C.3) the notations, introduced earlier in the main text (24), are used.

Now let us consider more thoroughly, up to $O\left(1/\epsilon^6\right)$ inclusively, the asymptotics of $\text{Tr} G_{m_j}$ for $\epsilon$ on the arcs of the large circle in the upper half-plane $C_1$ and $C_2$ (Fig. 1) ($|\epsilon| \to \infty, 0 < \arg \epsilon < \pi$). The corresponding asymptotics on the arcs of the large circle in the lower half-plane could be obtained then from general properties of $\text{Tr} G_{m_j}$ (33).

The asymptotics of $\text{Tr} G_{m_j}$ for $0 < r < R$ has the following form (about the vicinity of the point $r = 0$ see below)

$$\text{Tr} G_{m_j}^{in}(r, r; \epsilon) \to C_0^{in}(r) + \frac{C_2^{in}(r)}{\epsilon^2} + \frac{C_3^{in}(r)}{\epsilon^3} + \frac{C_4^{in}(r)}{\epsilon^4} + \frac{C_5^{in}(r)}{\epsilon^5} + O \left(\frac{1}{|\epsilon|^6}\right),$$

$$C_0^{in}(r) = \frac{i}{r}, \quad C_2^{in}(r) = \frac{i}{2r} \left(\frac{m_j^2}{r^2} + 1\right),$$

$$C_3^{in}(r) = -\frac{i}{r^2} \left(\frac{m_j^2}{r} + \frac{m_j}{r^2} + r V_0\right),$$

$$C_4^{in}(r) = \frac{3i}{2r^3} \left(\frac{(m_j^2 - 1) m_j^2}{4r^2} + \frac{m_j^2}{2} + m_j^2 V_0^2 + m_j V_0 + r^2 V_0^2 + \frac{r^3}{4}\right),$$

$$C_5^{in}(r) = \frac{3i}{r^4} \left(-2m_j^2 V_0^2 - m_j^2 + 2m_j^2 V_0 + m_j\right)$$

$$-r \left(\frac{2}{3} m_j^2 V_0^2 + m_j V_0 + m_j V_0^2 + \frac{m_j}{4}\right) - r^3 \left(\frac{2}{3} V_0^3 + \frac{V_0}{2}\right).$$
\[
\text{Tr\Delta}G_m^{in}(r, r; \epsilon) \rightarrow e^{-2\xi(R-r)} \left[ \frac{D^{in}_4(r)}{c^4} + \frac{D^{in}_5(r)}{c^5} + O \left( \frac{1}{|\epsilon|^6} \right) \right], \\
D^{in}_4(r) = -\frac{1}{4rR^2} \left( \frac{m^2 Q}{r R} - \frac{im_j Q}{r} + \frac{im_j Q}{R} + Q \right), \\
D^{in}_5(r) = -\frac{1}{r R^2} \left( -\frac{im^3 Q}{4r R^2} + \frac{im^3 Q}{4r^2} + \frac{m^3 Q}{2r R} - \frac{m^3 Q}{2r^2} + \frac{m^2 Q^2}{4r^2 R} + \frac{im^3 Q}{8r^2 R} \right)
\]

\[(C.5)\]

It follows from (C.4) and (C.5) with account of \( \text{Re}\xi > 0 \) (22) that the asymptotics of \( \text{Tr}G_m \) for \( r < R \) is defined via asymptotics of \( \text{Tr}C_m^{in} \). At the same time, for \( r \rightarrow R \) one should take into account that the contribution from \( \text{Tr}\Delta G_m^{in} \) becomes non-zero.

The asymptotics of \( \text{Tr}G_m \) for \( r > R \) reveals the same structure:

\[
\text{Tr}G_m^{out}(r, r; \epsilon) \rightarrow C^{out}_0(r) + C^{out}_1(r) + C^{out}_2(r) + C^{out}_3(r) + C^{out}_4(r) + C^{out}_5(r) + O \left( \frac{1}{|\epsilon|^6} \right), \\
C^{out}_0(r) = \frac{i}{r}, \quad C^{out}_1(r) = \frac{i}{2r} \left( \frac{m^2}{r^2} + 1 \right), \\
C^{out}_2(r) = -\frac{i}{r^2} \left( \frac{m^2 Q}{r} + \frac{m_j}{2r} + Q \right), \\
C^{out}_3(r) = \frac{3i}{2r^3} \left( \frac{m^2 - 1}{4r^2} + \frac{1}{2} + m^2 \left( \frac{Q}{r} \right)^2 + m_j \frac{Q}{r} + Q^2 + \frac{r^2}{4} \right), \\
C^{out}_4(r) = \frac{3i}{2r^4} \left[ -2m^2 Q/r - m^2 + 10/3m^2 Q/r + m_j - \right. \\
\left. -mr \left( \frac{2}{3} \right) \left( \frac{Q}{r} \right)^3 + m^2 Q/r + m_j \left( \frac{Q}{r} \right)^2 + \frac{m_j}{4} - \frac{Q}{6r} \right] - r^3 \left( \frac{2}{3} \right) \left( \frac{Q}{r} \right)^3 + \frac{Q}{2r} \right].
\]

\[(C.6)\]

\[
\text{Tr}\Delta G_m^{out}(r, r; \epsilon) \rightarrow e^{-2\gamma(r-R)} \left( \frac{R}{r} \right)^{\epsilon_2 Q} \left[ \frac{D^{out}_4(r)}{c^4} + \frac{D^{out}_5(r)}{c^5} + O \left( \frac{1}{|\epsilon|^6} \right) \right], \\
D^{out}_4(r) = -\frac{1}{4rR^2} \left( \frac{m^2 Q}{r R} - \frac{im_j Q}{r} - \frac{im_j Q}{R} + Q \right), \\
D^{out}_5(r) = -\frac{1}{r R^2} \left( \frac{im^3 Q}{4r R^2} - \frac{im^3 Q}{4r^2} + \frac{m^3 Q}{2r R} - \frac{m^3 Q}{2r^2} + \frac{m^2 Q^2}{4r^2 R} + \frac{im^3 Q}{8r^2 R} \right)
\]

\[(C.7)\]

From (C.6) and (C.7) with account of \( \text{Re}\gamma > 0 \) (24) one finds that the asymptotics of \( \text{Tr}G_m \) for \( r > R \) is defined by the asymptotics of \( \text{Tr}C_m^{out} \), whereas for \( r \rightarrow R \), on the contrary, the contribution of \( \text{Tr}\Delta G_m^{out} \) cannot be neglected.

Now let us verify that with account of the contributions of \( \text{Tr}\Delta G_m^{in, out} \) the asymptotics of \( \text{Tr}G_m^{in}(R, R; \epsilon) \) and
TrG_{m_j}^{\text{out}}(R, R; \epsilon) actually coincide. Indeed, from (C.4)-(C.7) at $r = R$ one obtains

$$
\text{Tr}G_{m_j}^{\text{in}}(R, R; \epsilon) \to C_0^{\text{in}}(R) + \frac{C_2^{\text{in}}(R)}{\epsilon^2} + \frac{C_4^{\text{in}}(R)}{\epsilon^4} + \frac{C_6^{\text{in}}(R) + D_4^{\text{in}}(R)}{\epsilon^6} + O \left( \frac{1}{|\epsilon|^6} \right) ,
$$

(C.8)

and the following relations between the coefficients of the in- and out-expansions

$$
C_0^{\text{in}}(R) = C_0^{\text{out}}(R), \quad C_2^{\text{in}}(R) = C_2^{\text{out}}(R), \quad C_4^{\text{in}}(R) = C_4^{\text{out}}(R), \quad C_6^{\text{in}}(R) = C_6^{\text{out}}(R),
$$

(C.9)

From (C.8) and (C.9) there follows that at $r = R$ the asymptotics of $\text{Tr}G_{m_j}$ is continuous and takes the form (C.8).

Proceeding further and making use of the general properties of $\text{Tr}G_{m_j}$ (33), one finds the asymptotics of $\text{Tr}G_{m_j}$ for $\epsilon$ on the arcs of the large circle in the lower half-plane $C_3$ and $C_4$ (Fig. 1) ($|\epsilon| \to \infty, -\pi < \text{Arg} \epsilon < 0$). As a result, the asymptotics of $\text{Re} \text{Tr}G_{m_j}^{\text{in}}(r, r; iy)$ for $|y| \to \infty$, considered in Section 3 in terms of $\text{Tr}G_{m_j}^{\text{in}}(r, r; \epsilon)$ up to $O(1/\epsilon^3)$ only, takes now the following form:

for $r < R$

$$
\frac{2}{r^2|y|^3} \left( \frac{m^2}{r} V_0 + r V_0 \right) + \frac{6}{r^4|y|^6} \left[ -\frac{m^4}{2r} V_0 + \frac{m^2}{2r} V_0 - r \left( \frac{2}{3} m^2 V_0 + m^2 V_0 \right) \right] - r^3 \left( \frac{2}{3} V_0 + \frac{V_0}{2} \right) + O \left( \frac{1}{|y|^7} \right) ;
$$

(C.10)

at $r = R$

$$
\frac{2}{R^2|y|^3} \left( \frac{m^2}{R} V_0 + Q \right) - \frac{1}{2R^4|y|^4} \left( \frac{m^2}{2R} V_0 + Q \right) + \frac{6}{R^4|y|^6} \left[ -\frac{m^4}{2R} V_0 + \frac{2m^2}{3R} V_0 - r \left( \frac{2}{3} m^2 V_0 + \frac{V_0}{12} \right) \right] - R^3 \left( \frac{2}{3} V_0 + \frac{V_0}{2} \right) + O \left( \frac{1}{|y|^6} \right) ;
$$

(C.11)

for $r > R$

$$
\frac{2}{r^2|y|^3} \left( \frac{m^2}{r} Q + Q \right) + \frac{6}{r^4|y|^6} \left[ -\frac{m^4}{2r} Q + \frac{5m^2}{6r} Q - r \left( \frac{2}{3} m^2 \left( \frac{Q}{r} \right)^3 + \frac{m^2}{r} Q - \frac{Q}{6r} \right) \right] - r^3 \left( \frac{2}{3} \left( \frac{Q}{r} \right)^3 + \frac{Q}{2r} \right) + O \left( \frac{1}{|y|^7} \right) .
$$

(C.12)

Let us specially note that from (C.10)-(C.12) there might appear an impression, that they do not provide the continuity at $r = R$. However, as it has been already mentioned above, for $r \to R$ one should take into account in the asymptotics of $\text{Tr}G_{m_j}^{\text{in}}(r, r; iy)$ and $\text{Tr}G_{m_j}^{\text{out}}(r, r; iy)$ the non-vanishing contributions from $\text{Tr}\Delta G_{m_j}^{\text{in}}(r)$ and $\text{Tr}\Delta G_{m_j}^{\text{out}}$. Namely, on account of them the asymptotics $\text{Re} \text{Tr}G_{m_j}^{\text{in}}(r, r; iy)$ for $|y| \to \infty$ can be represented as follows

$$
\text{Re} \text{Tr}G_{m_j}^{\text{in}}(r, r; iy) \to \frac{2}{r^2|y|^3} \left( \frac{m^2}{r} V_0 + r V_0 \right) + \frac{6}{r^4|y|^6} \left[ -\frac{m^4}{2r} V_0 + \frac{m^2}{2r} V_0 - r \left( \frac{2}{3} m^2 V_0 + \frac{V_0}{2} \right) \right] + O \left( \frac{1}{|y|^7} \right) + \text{Re} \left( e^{-2\xi(R-r)} \right) \bigg|_{\epsilon=iy} + \frac{D_4^{\text{in}}(r) + D_4^{\text{in}}(r)|_{m_j \to -m_j} + D_5^{\text{in}}(r) + D_5^{\text{in}}(r)|_{m_j \to -m_j}}{iy^3} + O \left( \frac{1}{|y|^7} \right) .
$$

(C.13)
Then from (C.13) for the asymptotics of \( \text{Re} \text{Tr} G_{[m,j]}^{in}(R, R; iy) \) for \( |y| \to \infty \) one obtains

\[
\text{Re} \text{Tr} G_{[m,j]}^{in}(R, R; iy) \to \frac{2}{R^2 |y|^3} \left( \frac{m_j^2}{R} V_0 + Q \right) + \frac{6}{R^1 |y|^5} \left[ \frac{m_j^2}{2R} V_0 + \frac{m_j^2}{2R} V_0 \right] - R \left( \frac{2}{3} m_j^2 V_0^3 + m_j^2 V_0 \right) - R^3 \left( \frac{2}{3} V_0^3 + \frac{V_0}{2} \right) \to - \frac{1}{2R^3 y^4} \left( \frac{m_j^2 Q}{R^2} + Q \right) + \frac{1}{2R^3 y^4} \left( \frac{m_j^2 Q}{R^2} + Q \right) + O \left( \frac{1}{|y|^6} \right).
\]

(C.14)

Now let us study the order \( O \left( 1/|y|^5 \right) \) in (C.14) in more detail

\[
\frac{6}{R^1 |y|^5} \left[ \frac{m_j^2}{2R} V_0 + \frac{m_j^2}{2R} V_0 - R \left( \frac{2}{3} m_j^2 V_0^3 + m_j^2 V_0 \right) - R^3 \left( \frac{2}{3} V_0^3 + \frac{V_0}{2} \right) \right] + \frac{6}{R^1 |y|^5} \left[ \frac{2m_j^2}{3R} V_0 - R \left( \frac{2}{3} m_j^2 V_0^3 + m_j^2 V_0 - \frac{V_0}{12} \right) - R^3 \left( \frac{2}{3} V_0^3 + \frac{V_0}{2} \right) \right] = 0.
\]

(C.15)

As a result, there follows from (C.13)-(C.15) precisely the asymptotics of \( \text{Re} \text{Tr} G_{[m,j]}^{out}(r, r; iy) \) (C.11).

Quite analogously, the general form of the asymptotics of \( \text{Re} \text{Tr} G_{[m,j]}^{out}(r, r; iy) \) for \( |y| \to \infty \) takes the form

\[
\text{Re} \text{Tr} G_{[m,j]}^{out}(r, r; iy) \to \frac{2}{r^2 |y|^3} \left( \frac{m_j^2}{r} Q + Q \right) + \frac{6}{r^1 |y|^5} \left[ \frac{m_j^2}{2r} Q + \frac{m_j^2}{2r} Q \right] - r \left( \frac{2}{3} m_j^2 \left( \frac{Q}{r} \right)^3 + m_j^2 \left( \frac{Q}{r} \right)^3 - r^3 \left( \frac{2}{3} \left( \frac{Q}{r} \right)^3 + \frac{Q}{2r} \right) \right] + O \left( \frac{1}{|y|^7} \right) + \text{Re} \left( e^{-2\gamma (r-R)} \right) \left( \frac{R}{r} \right)^{\text{vQ}} \times \left[ \frac{D_{4}^{out}(r) + D_{4}^{out}(r)|_{m_j \to -m_j} + D_{5}^{out}(r) + D_{5}^{out}(r)|_{m_j \to -m_j} + O \left( \frac{1}{|y|^5} \right) \right].
\]

(C.16)

Then from (C.16) for the asymptotics of \( \text{Re} \text{Tr} G_{[m,j]}^{out}(R, R; iy) \) for \( |y| \to \infty \) there follows

\[
\text{Re} \text{Tr} G_{[m,j]}^{out}(R, R; iy) \to \frac{2}{R^2 |y|^3} \left( \frac{m_j^2}{R} Q + Q \right) + \frac{6}{R^1 |y|^5} \left[ \frac{m_j^2}{2R} Q + \frac{m_j^2}{2R} Q \right] - R \left( \frac{2}{3} m_j^2 \left( \frac{Q}{R} \right)^3 + m_j^2 \left( \frac{Q}{R} \right)^3 - R^3 \left( \frac{2}{3} \left( \frac{Q}{R} \right)^3 + \frac{Q}{2R} \right) \right] - \frac{1}{2R^3 y^4} \left( \frac{m_j^2 Q}{R^2} + Q \right) + \frac{1}{2R^3 y^4} \left( \frac{m_j^2 Q}{R^2} + Q \right) + O \left( \frac{1}{|y|^6} \right).
\]

(C.17)
From (C.19) there follows that the first non-vanishing term in the asymptotics of $\text{Re} \rho$ whence it follows that (C.16)-(C.18) precisely reproduce the asymptotics of $\text{Re} \text{Tr} r$ of the point $y \to \infty$.

Proceeding further, by taking into account that the subtraction of $2 \text{Tr} G_{m_j}^{(1)}(r; iy)$ removes all the linear in $Q$ and $V_0$ terms, from (C.10)-(C.12) one finds the asymptotics of $\text{Re} \left[ \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G_{m_j}^{(1)}(r; iy) \right]$ turns out to be proportional to $Q^3/|y|^5 \times$ multiplier, depending only on $m_j$ and $r$. The factor $Q^3$ underlines that by construction $\rho^{(3+)}_{V_F}(r)$ does not contain any linear in $Q$ terms, while the asymptotical behavior $\sim |y|^{-5}$ by itself guarantees the uniform convergence of the integral $\int dy \text{Re} \left[ \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G_{m_j}^{(1)}(r; iy) \right]$ with respect to $m_j$ and $r$.

It would be also worth-while noticing that the asymptotics (C.10)-(C.12) cannot be used in the infinitesimal vicinity of the point $r = 0$, since $r$ enters into $\text{Tr} G_{m_j}(r, r; iy)$ via combinations $r (1 + y^2)$ and $r (1 - (iy + V_0)^2)$, which for $|y| \to \infty$ in the vicinity of $r = 0$ might remain finite. In the case under consideration, however, when the external potential (1) is regular at the origin, there follows from the obvious physical reasons that the induced vacuum density should also be finite and continuous at the origin. Therefore it can be obtained by means of a limit transition by continuity from the region, where $r$ is non-zero. Hence, the uniform convergence of the integral $\int dy \text{Re} \left[ \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G_{m_j}^{(1)}(r; iy) \right]$ holds also for $r = 0$.

At the next stage let us use the asymptotics of $\text{Tr} G_{m_j}(r, r; iy)$ for $r \to \infty$, substantially specified compared to (35), namely

$\text{Tr} G_{m_j}(r, r; iy) \to \frac{iy}{\sqrt{1 + y^2} r} + \frac{Q}{(1 + y^2)^{3/2} r^2}$  
\[ + \frac{1}{2} \frac{1}{(1 + y^2)^{5/2} r^3} (-im_j^2 y^3 - im_j^2 y + m_j y^2 + m_j + 3iQ^2 y^3) \]  
\[ + \frac{1}{2} \frac{1}{(1 + y^2)^{7/2} r^4} (2m_j^2 Q y + m_j^2 Q y^2 - m_j^2 Q + 3m_j Q y^3) \]  
\[ + 3im_j Q y - 4Q^2 y^2 + Q^3 + Q^2 y^2 + Q) + O \left( \frac{1}{r^7} \right). \]  

From (C.19) there follows that the first non-vanishing term in the asymptotics of $\text{Re} \left[ \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G_{m_j}^{(1)}(r; iy) \right]$ for $r \to \infty$ should be proportional to $Q^3/r^4 \times$ multiplier, depending on $m_j$ and $y$ only. The factor $Q^3$ underlines once more that $\rho^{(3+)}_{V_F}(r)$ by construction does not contain any linear in $Q$ terms. In turn, due to the uniform convergence of the integral $\int dy \text{Re} \left[ \text{Tr} G_{m_j}(r, r; iy) - 2 \text{Tr} G_{m_j}^{(1)}(r; iy) \right]$ with respect to $m_j$ and $r$, established above, the asymptotics of $\rho^{(3+)}_{V_F}(r)$ for $r \to \infty$ turns out to be $\sim r^{-4}$ uniformly with respect to $m_j$, what in turn provides the possibility of...
permutation of summation over \( m_j \) and integration over \( dr \) in (B.8).

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