Abstract

We describe the quantization of 2-plectic manifolds as they arise in the context of the quantum geometry of M-branes and non-geometric flux compactifications of closed string theory. We review the groupoid approach to quantizing Poisson manifolds in detail, and then extend it to the loop spaces of 2-plectic manifolds, which are naturally symplectic manifolds. In particular, we discuss the groupoid quantization of the loop spaces of $\mathbb{R}^3$, $\mathbb{T}^3$ and $S^3$, and derive some interesting implications which match physical expectations from string theory and M-theory.
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1. Introduction and summary

Higher quantization. The quantization of general physical systems is a long-standing problem. It is the inverse operation of taking the classical limit of a quantum mechanical system. As the classical limit cannot be expected to yield an injective map between quantum and classical systems, it is not surprising that quantization is not unique. Correspondingly, there is a wealth of different approaches to quantization. In this paper we will be interested in the groupoid approach, which also employs the techniques of geometric quantization. We will limit ourselves to the kinematical problem of quantization, i.e., the construction of an algebra of quantum operators from a classical phase space.

Aside from their quantum physics origin, there are many reasons for studying quantized phase spaces. In particular, the description of certain string theory backgrounds often requires generalized notions of geometry that can be regarded as quantized symplectic manifolds. As we will review below, both string theory as well as recent developments in M-theory also suggest to consider analogous extensions for multisymplectic manifolds. These are manifolds endowed with closed, non-degenerate differential forms of degrees larger than two. In the context of Nambu mechanics, multisymplectic manifolds serve as multiphase spaces leading to dynamical systems with time evolution governed by more than one Hamiltonian.

The development of quantization methods for multisymplectic manifolds is still only in its infancy; only the issue of prequantization has been explored in detail. Recall that to prequantize a symplectic manifold one constructs a principal circle bundle (or an associated line bundle) over the manifold which is endowed with a connection whose curvature is proportional to the symplectic form. These structures are used in the definition of the Atiyah algebroid, whose sections yield a faithful representation of the Poisson algebra of the symplectic manifold. In the simplest multisymplectic setting, the case of 2-plectic manifolds, the prequantum circle bundle is replaced by a prequantum gerbe. Furthermore, these manifolds come with a Lie 2-algebra that essentially takes over the role of the Poisson algebra [2]. A faithful representation of this Lie 2-algebra can be constructed on sections of a Courant algebroid that is naturally derived from the multisymplectic structure [3, 4]. While the prequantization of 2-plectic manifolds seems clear, the incorporation of a polarization turning the prequantization into an actual quantization is, however, far from obvious.

Here we will exploit the proposal of Hawkins [5] of how to include a polarization into the groupoid approach to quantization. In the general groupoid approach, see e.g. [6, 7, 8], one starts from the observation that any Poisson manifold naturally gives rise to a Lie algebroid. There is a standard technique for quantizing duals of Lie algebras as convolution algebras of the Lie groups obtained from integrating the Lie algebras. This technique can be extended to Lie algebroids, which yields convolution algebras of their integrating Lie groupoids. Quantization in this sense therefore corresponds to integration. The Lie algebroid of a Poisson manifold, however, has twice the dimension of the original manifold. To correct for this, one needs to introduce a polarization and in [5] it was shown how to do this. By using the resulting twisted polarized convolution algebras, one can use the groupoid approach to obtain Moyal planes, noncommutative tori, as well as any geometrically quantized Kähler
The groupoid approach is particularly interesting for the quantization of multisymplectic manifolds for two reasons. Firstly, multisymplectic manifolds are in a certain sense “categorified” symplectic manifolds, and the groupoid approach seems very suitable for a categorification. Secondly, Hawkins’ version of the groupoid approach directly constructs an algebra of operators, avoiding the introduction of a Hilbert space. This is useful as the Lie 2-algebra of a 2-plectic manifold should yield some nonassociative algebra of quantum operators, because its Jacobiator (or associator) is in general non-vanishing. Therefore a model of quantum operators as endomorphisms on some linear Hilbert space seems to be excluded.

Instead of categorifying Hawkins’ approach, we will follow a different route in this paper which goes back to ideas due to Gawędzki [9] and Brylinski [10]. The loop space of any 2-plectic manifold carries a natural symplectic structure. In the case of three-dimensional 2-plectic manifolds, this symplectic structure can moreover be extended to a Kähler structure. So instead of quantizing the 2-plectic manifold directly, we can apply the prescriptions of groupoid quantization to the loop space of the manifold. However, it should be stressed that the difficulties of dealing with multisymplectic manifolds and prequantum gerbes is traded for the difficulties of working with the corresponding infinite-dimensional loop spaces.

There is yet a fourth way of considering the problem of quantizing multisymplectic manifolds. As stated above, multisymplectic manifolds may give rise to multiphase spaces in Nambu mechanics. Just as a symplectic form yields a Poisson structure, certain multisymplectic forms encode a Nambu-Poisson structure. It has been suggested to use Nambu-Poisson structures as starting points for quantization and to quantize them in terms of Filippov’s $n$-Lie algebras [11]. This approach, however, has not delivered complete results so far; see [12] and references therein.

**Physical context.** The treatment of this paper is mostly technical, so let us briefly summarize some of the physical settings in which the implications of our loop space quantization are applicable. For a more detailed discussion of our motivation from the perspective of string theory, see [13].

Our approach to the quantization of 2-plectic manifolds from the perspective of their corresponding loop spaces is natural from several different points of view. In the context of M-theory, it was realized some time ago by [14, 15] that canonical quantization of an open membrane boundary on an M5-brane in a constant $C$-field background leads to a noncommutative loop space. This is a functional analog of the noncommutative geometry that arises in canonical quantization of open strings ending on D-branes in a constant $B$-field background; as the M2-brane boundaries are strings, this describes a noncommutative closed string geometry. Further motivation for this approach comes from the fact that loop spaces have been recently used very successfully in lifting the Nahm construction of monopoles to self-dual strings, in a manner that is closely related to certain quantized 2-plectic manifolds [16, 17, 18].

A separate context in which our approach is applicable is to the description of non-


geometric three-form flux compactifications of closed string theory. Loop space quantization is very natural from the point of view of closed string sigma-models, and it yields a tractable framework in which to analyze nonassociative structures which have been found recently. In certain duality frames, our results reproduce the noncommutative geometry probed by closed strings on a three-torus $\mathbb{T}^3$ which wind around one of the cycles, as derived in [19] through canonical quantization and closed string mode expansions; in particular, we nicely reproduce the picture of [20] for the quantized $\mathbb{T}^3$ with $Q$-flux (T-fold) as a fibration of noncommutative two-tori over the circle $S^1$. The Lie 2-algebraic nature of the nonassociative geometry found through closed string vertex operators in flat space in [21] is contained in our quantization of the loop space of $\mathbb{R}^3$, while our quantization of the three-sphere $S^3$ should naturally contain the nonassociative structures which have been found in the SU(2) WZW model [22].

Some of the categorified ingredients of higher quantization also have natural physical relevance. Given a gerbe with integral curvature $H$ on a manifold $M$, the generalized tangent bundle $E = TM \oplus T^*M$ has the structure of an exact Courant algebroid with the $H$-twisted Courant-Dorfman bracket. This relationship between abelian gerbes and Courant algebroids is useful for understanding prequantization conditions. Thus Courant algebroids offer a unifying picture of approaches to quantization of 2-plectic manifolds; recall that they are the natural analogs of the Atiyah Lie algebroid for a U(1)-gerbe over $M$.

Courant algebroids also naturally appear in string phase spaces. This exemplifies the relation between exact Courant algebroids and gerbes, since a gerbe on $M$ with connective structure naturally determines a $\text{U}(1)$-valued 2-holonomy over surfaces $S \subset M$. Dimensional reduction over torus bundles of exact Courant algebroids yields non-exact Courant algebroids with additional symmetries that can be understood in terms of T-duality. In particular, T-duality (including the Büscher rules) can be understood as an isomorphism of Courant algebroids over T-dual spaces. In this way Courant algebroids offer a geometric description of non-geometric fluxes and T-folds; see [23] for further details. The relevance of Courant algebroids for the geometricization and quantization of non-geometric $R$-flux backgrounds in terms of nonassociative geometry is also stressed in [24].

Lie 2-algebras have further recently proven useful as gauge structures in certain M-brane models [25, 26]. As explained in [13], these M2-brane models are closely related to the quantization of 2-plectic manifolds, in particular of $S^3$. Lie 2-algebras and their integrating Lie 2-groups also play an important role in the nonassociative quantum geometry of closed string $R$-flux compactifications [24] in the same manner as the quantizations spelled out in the present paper.

**Summary.** We now come to a summary of the results of this paper. Preliminary accounts of the material in this article can be found in the papers [27, 28, 13], to which the reader can turn for a more concise and relaxed presentation of our ideas and results, as well as further physical background and applications.

In the following we review all ingredients in the combination of the loop space and the groupoid approaches to quantization in detail. As an explicit example for the groupoid ap-
proach, we work out the quantization which yields the standard $\kappa$-Minkowski space. When it comes to loop spaces, the difficulty in using groupoids is that the convolution algebra requires the introduction of a measure. Unfortunately, there is no natural reparametrization invariant measure on loop space and we therefore cannot expect to arrive at a complete picture. We can, however, glean some interesting consequences from following the groupoid approach as far as possible. We will focus on three particular examples: three-dimensional Euclidean space $\mathbb{R}^3$, the three-torus $\mathbb{T}^3$ and the three-sphere $S^3$. In each case we demonstrate that the loop space quantization reduces to the standard geometric quantizations of $\mathbb{R}^2$, $\mathbb{T}^2$ and $S^2$, respectively, via an M-theory type dimensional reduction with the loops winding around the compactified direction.

In the case of the real affine space $\mathbb{R}^3$, we find that the groupoid structure receives corrections in powers of the Poisson tensor that are unexpected from the corresponding structures on $\mathbb{R}^2$. They combine into a twist element that yields an operator product which agrees with the noncommutative deformations of loop space derived directly from M-theory in [14, 15, 29].

In the case of the torus $\mathbb{T}^3$, the loop space decomposes into different winding sectors. It is therefore necessary to introduce a Bohr-Sommerfeld quantization condition that essentially agrees with results expect from a closed string sigma-model analysis. Moreover, the Bohr-Sommerfeld variety exhibits an interesting non-trivial structure which reproduces exactly the $Q$-space noncommutative geometry of closed string zero modes from [19] as well as the noncommutative torus fibration of [20].

Finally, the case of the sphere $S^3$ follows more closely that of the groupoid quantization of $S^2$, where one identifies the space of global holomorphic sections of the prequantum line bundle over the pair groupoid with the Hilbert space. Just like line bundles over $S^2$, abelian (bundle) gerbes over $S^3$ are characterized up to isomorphism by an integer. When transgressed to loop space, they yield a special class of line bundles. One can readily define the global holomorphic sections of these line bundles. The explicit description of the space of such sections, however, is highly non-trivial. Using twistor techniques we manage to identify a special class of sections that reduce to the sections of the prequantum line bundles over $S^2$ when following the projection $\pi : S^3 \to S^2$ of the Hopf fibration.

Our analysis leaves a number of open problems. First of all, it would be interesting to describe the full space of global holomorphic sections of the transgressed gerbes over $S^3$. Defining some notion of square integrability on these sections would require revisiting the issue of constructing suitable measures on loop spaces. If such measures were found and proven to be natural, one could also resolve the issue of how to define the groupoid convolution algebras that should appear in our quantization of loop spaces. Most importantly, however, one should explore a more direct approach to the quantization of 2-plectic manifolds. Just as the quantization of a symplectic manifold yields a prequantum line bundle and eventually a Hilbert space, the quantization of a 2-plectic manifold yields a prequantum gerbe and presumably a categorified or 2-Hilbert space. Correspondingly, the Lie algebra of quantum operators that forms a deformed representation of the Poisson algebra of a symplectic manifold should be replaced by a Lie 2-algebra of quantum operators forming a
deformed representation of the Lie 2-algebra replacing the Poisson algebra in the case of a
2-plectic manifold.

Outline. This paper is structured as follows: In §2 we review the groupoid approach
to quantization in some detail, in particular Hawkins’ proposal which incorporates polar-
ization. In §3 we explain the extension of Poisson and symplectic geometry as well as
prequantization to the categorified context. We review the Kähler geometry of certain
loop spaces and show how they allow us to reduce categorified prequantization to ordi-
nary prequantization. In §4–§6 we apply Hawkins’ version of the groupoid approach to
the quantization of the loop spaces of $\mathbb{R}^3$, $\mathbb{T}^3$ and $S^3$, respectively. Three appendices at
the end of the paper collect relevant definitions and some technical details for the reader’s
convenience. In Appendix A we give all necessary details concerning Lie groupoids, Lie
algebroids and integrating symplectic groupoids. In Appendix B we collect various features
of gerbes in the different contexts that we need them in the main text of the paper. In
Appendix C we present details on Courant algebroids within the context of categorified
geometric quantization.

2. Quantization of Poisson manifolds and symplectic groupoids

The groupoid approach to quantization was introduced by Weinstein as a means of extend-
ing the framework of geometric quantization of symplectic manifolds to generic Poisson
manifolds; in a certain sense, explained below and in Appendix A, Poisson manifolds are
the infinitesimal objects associated with symplectic groupoids. In this section we will re-
view the quantization of Poisson manifolds in the framework of Lie groupoids as presented
by Hawkins in [5], who clarified the notion of polarization in this context. We are mainly
interested in the application of this quantization procedure to $\mathbb{R}^d$, $\mathbb{T}^d$, $S^2$, and $\kappa$-deformed
spaces. These are spaces for which many notions like that of groupoid polarization and
the derivation of a twist element for the convolution algebra simplify. The review here
will therefore be rather concise on the technical side. Instead, we will present a number of
explicit examples in detail.

2.1. Quantization of Lie algebra duals

The motivation behind the groupoid approach to quantization is the observation that the
quantization of the dual of a Lie algebra yields a noncommutative algebra of functions
which can be identified with the twisted convolution $C^*$-algebra of the integrating Lie
group. Consider the two Kirillov-Kostant-Souriau (KKS) Poisson structures on $M = g^*$,
the dual of a Lie algebra $g$. Interpreting elements of $g$ as linear functions on $M$, one defines
$\{g_1, g_2\}_\pm(x) := \pm \langle x, [g_1, g_2] \rangle$ for $g_1, g_2 \in g$, $x \in M$. These brackets extend to polynomial
functions via the Leibniz identity for the Lie bracket. To obtain all smooth functions on
$g^*$, we consider the completion of the polynomial algebra with respect to a suitable norm.
The resulting linear Poisson structures on $C^\infty(g^*)$ are invariant under the coadjoint action
of the corresponding connected Lie group $G$. 

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Quantization corresponds to a map from (a subset of) the algebra of smooth functions $C^\infty(g^*)$ on the dual Lie algebra to the reduced convolution $C^*$-algebra $C^*_r(G)$ of the integrating Lie group which is $G$-invariant. Roughly speaking, the steps are as follows: We start from the functions in $C^\infty(g^*)$, which we Fourier transform to $C^\infty(g)$ and subsequently exponentiate to a group $G$. There, we can convolve and finally follow the steps back to $C^\infty(g^*)$. Below we flush out some details of this construction, following [30, 31].

The basic idea is to identify the generators $g^i$ of $g$ as coordinates for $g^*$ via duality. For small $t \in \mathbb{R}$, this gives a one-to-one correspondence between group elements $\exp(t g) \in G$, $g \in g$, and functions of the form $e^{tx}$ on $g^*$. Pulling back the group multiplication to these functions thus yields a quantization of $C^\infty(g^*)$. The group multiplication is computed via the usual Baker-Campbell-Hausdorff formula

$$\exp(g) \exp(h) = \exp \left( H(g, h) \right) \quad (2.1)$$

with $g, h \in g$. Here

$$H(g, h) = g + h + \frac{1}{2} [g, h] + \frac{1}{12} \left( [g, [g, h]] + [[g, h], h] \right) + \ldots \quad (2.2)$$

is the Hausdorff series whose terms lie in $g$. For nilpotent Lie algebras, this construction is equivalent to Kontsevich’s quantization of the linear Poisson structure on $g^*$ and to the quantization by the universal enveloping algebra of $g$; in this case the exponential map is a diffeomorphism from the whole of $g$ to $G$.

At the level of Schwartz functions, the Fourier transform is a linear map $S(g^*) \to S(g)$ which sends $f \in S(g^*)$ to $\tilde{f} \in S(g)$ with

$$\tilde{f}(g) = \int_{g^*} dx \, e^{-2\pi i \langle x, g \rangle} \, f(x) \quad (2.3)$$

for $g \in g$, where $dx$ is the translationally invariant Haar measure for the additive group structure on $g^*$. For the inverse Fourier transform, we use the exponential map to “identify” $g$ with $G$. Clearly, $\exp : g \to G$ is not a bijection in general, so this can only be done on an open set $U$ in $g$ where the exponential map is a diffeomorphism. To circumvent this problem, we restrict to a dense subset of functions $S_c(g^*) \subset S(g^*)$ which under Fourier transformation maps to the space $C^\infty_c(g)$ of smooth functions with compact support on $g$.

This also ensures integrability with respect to the left-invariant Haar measure

$$d\mu(s) = \lambda(g) \, dg \quad \text{with} \quad \lambda(g) := | \det(d \exp)_g | \quad , \quad (2.4)$$

where $s = \exp(g)$ and $dg$ is the translationally invariant Haar measure on $g$. For nilpotent algebras, one can take the set $U$ to be all of $g$ and replace $C^\infty_c(g) \subset L^1(G, d\mu)$ with the larger Schwartz space $S(g)$; in this case $\det(d \exp)_g = 1$ for all $g \in g$.

In this picture, the inverse Fourier transform of $\tilde{f} \in C^\infty_c(g)$ is defined by

$$W(\tilde{f}) := \int_G d\mu(s) \, W(s) \, \tilde{f}(s) \quad , \quad (2.5)$$
where \( W(s) = \exp \left(2\pi i \langle x, g \rangle\right) \) for \( s = \exp(g) \) obeys \( W(s) W(t) = W(st) \); it thus represents \( G \) with the various identifications made above. The \( C^* \)-algebra generated by \( W(\tilde{f}) \) for \( \tilde{f} \in C_c^\infty(g) \) is the image of the representation induced by \( W \) of the reduced \( C^* \)-algebra \( C_r^*(G) \). This can be seen by explicitly computing

\[
W(\tilde{f}) W(\tilde{f}') = \int_G d\mu(s) W(s) \left( \int_G d\mu(t) \tilde{f}(t) \tilde{f}'(t^{-1} s) \right) = W(\tilde{f} \circ \tilde{f}') ,
\]

where

\[
(\tilde{f} \circ \tilde{f}')(s) := \int_G d\mu(t) \tilde{f}(t) \tilde{f}'(t^{-1} s)
\]

is the usual convolution product on \( C_r^*(G) \). It follows that \( W \) defines an algebra homomorphism.

Rewriting all of this in terms of the identifications made above, the deformed product \( f \ast f' := W(\tilde{f} \circ \tilde{f}') \) of functions \( f, f' \in S_c(g^*) \) thereby reads

\[
(f \ast f')(x) = \int_G d\lambda(g) \int_G d\lambda'(g') e^{2\pi i \langle x, g' \rangle} \tilde{f}(g) \tilde{f}'(H(-g, g'))
\]

\[
= \int_G d\lambda(g) \int_G d\lambda'(g') e^{2\pi i \langle x, g + g' \rangle} e^{2\pi i \langle x, H(-g, g') + g - g' \rangle} \tilde{f}(g) \tilde{f}'(g') .
\]

**Example.** The \( \kappa \)-Minkowski algebra is the solvable Lie algebra \( g_\kappa \) of dimension \( d \) with generators \( g^0, g^1, \ldots, g^{d-1} \) obeying

\[
[g^i, g^j] = i \kappa^{-1} g^i \quad \text{and} \quad [g^i, g^j] = 0
\]

for \( i, j = 1, \ldots, d - 1 \) and \( \kappa > 0 \). Generally, if \( [g, h] = zh \) for \( z \in \mathbb{C} \), then the Hausdorff series can be summed explicitly to give the braiding identity

\[
\exp(g) \exp(h) = \exp(e^z h) \exp(g) .
\]

It follows that

\[
\exp(i c_\mu g^\mu) = \exp(i c_0 g^0) \exp(i c_i^l g^i) \quad \text{with} \quad c_i^l := \frac{\kappa}{c_0} (1 - e^{-\kappa^{-1} c_0}) c_i .
\]

For \( p_0, p_1, \ldots, p_{d-1} \in \mathbb{R} \), we write

\[
W(p_0, \vec{p}) = V_{\vec{p}} U_{p_0} ,
\]

where \( \vec{p} = (p_1, \ldots, p_{d-1}) \), while

\[
U_{p_0} = \exp \left( i p_0 g^0 \right) \quad \text{and} \quad V_{\vec{p}} = \exp \left( - i p_i g^i \right) .
\]

These elements generate the non-abelian \( \kappa \)-Minkowski group \( G_\kappa \) [32] with the group law

\[
W(p_0, \vec{p}) W(p_0', \vec{p}') = W(p_0 + p_0', \vec{p} + e^{-\kappa^{-1} p_0} \vec{p}') .
\]
This group law is that of the crossed product $G_\kappa \cong \mathbb{R}^{d-1} \rtimes \kappa \mathbb{R}$ with the twisting group isomorphism $\alpha_\kappa(p_0)\vec{p} := e^{-\kappa^{-1}p_0} \vec{p}$ for $\vec{p} \in \mathbb{R}^{d-1}$. It is connected and simply-connected, hence $G_\kappa$ is uniquely determined by its Lie algebra $\mathfrak{g}_\kappa$. The Jacobian in the Haar measure (2.4) is $\lambda(p_0,\vec{p}) = e^{\kappa^{-1}p_0}$, and the $C^*$-algebra generated by

$$W(\tilde{f}) := \int_{G_\kappa} dp_0 \ dp\vec{p} \ e^{\kappa^{-1}p_0} \tilde{f}(p_0,\vec{p}) W(p_0,\vec{p}) , \quad (2.15)$$

for $\tilde{f} \in C^\infty_c(G_\kappa)$, is the representation of $C^*_r(G_\kappa)$ by $W$. The convolution product of two functions $\tilde{f}, \tilde{f}' \in C^\infty_c(G_\kappa)$ takes the explicit form

$$(\tilde{f} \star \tilde{f}')(p_0,\vec{p}) = \int_{G_\kappa} dp_0' \ dp\vec{p'} \ e^{\kappa^{-1}p_0'} \tilde{f}(p_0,p') \tilde{f}(p_0-p_0',e^{\kappa^{-1}p_0'}(\vec{p}-\vec{p}')) . \quad (2.16)$$

**Example.** The Moyal product is also a special case of this quantization. Consider the real vector space $V = \mathbb{R}^d$ with constant Poisson bivector $\pi = \frac{1}{2} \pi^{ij} \partial_i \wedge \partial_j \in \Lambda^2 V^*$, $\pi^{ij} = -\pi^{ji} \in \mathbb{R}$. The Heisenberg algebra is the 2-step nilpotent Lie algebra $\mathfrak{g}_{\pi}$ of dimension $d+1$ with basis $g^1,\ldots,g^d,h$ and relations

$$[g^i,g^j] = i \pi^{ij} h \quad \text{and} \quad [h,g^i] = 0 . \quad (2.17)$$

In this case, one defines the usual Weyl operators

$$W(p) = \exp \left( i p_i g^i \right) \quad (2.18)$$

for any $p \in V^*$, which satisfy

$$W(p) W(p') = \exp \left( -\frac{i}{2} p_i \pi^{ij} p_j' h \right) W(p+p') . \quad (2.19)$$

Here $\lambda = 1$, so the associated $C^*$-algebra is generated by

$$W(\tilde{f}) := \int_{V^*} dp W(p) \tilde{f}(p) . \quad (2.20)$$

for $\tilde{f} \in C^\infty_c(V^*)$. Because (2.19) is not a group representation, this $C^*$-algebra is not related to the group $C^*$-algebra; in fact, this is a projective representation of the abelian group structure on $V^*$. One can transform it into a representation of the 2-step nilpotent Heisenberg group $H^{d+1}_\pi$ which is a central extension of $V^*$ by a fixed element $\pi \in \Lambda^2 V^*$, in which case the $C^*$-algebra associated to the relations (2.17) is related to the $C^*$-algebra of $H^{d+1}_\pi$. Since $h$ commutes with everything we can set it to a scalar $h = h$; this means that the representation generated by $W$ is irreducible. Then the twisting factor in (2.8) is given by

$$H(-p',p) = p-p' - \frac{h}{2} p_i \pi^{ij} p_j' , \quad (2.21)$$

and so in this case the convolution product on the Heisenberg group $H^{d+1}_\pi$ induces the usual Moyal product on $\mathcal{S}_c(V)$. This defines the *twisted* convolution $C^*$-algebra $C^*_r(V^*,\pi,h)$ where the twisting group two-cocycle $\sigma_{\pi,h} : V^* \times V^* \to U(1)$ is given by

$$\sigma_{\pi,h}(p,p') := e^{-\frac{i}{2} p_i \pi^{ij} p_j'} . \quad (2.22)$$

The cocycle condition here is equivalent to antisymmetry of $\pi^{ij}$ and it ensures that the algebra of Weyl operators (2.19) is associative.
2.2. Poisson geometry, Lie algebroids and integrating Lie groupoids

The quantization of duals of Lie algebras can be lifted to a quantization of Lie algebroids.\(^1\) As we briefly review below, Poisson manifolds come with a natural Lie algebroid structure. We can therefore generalize the above quantization procedure to this class of manifolds.

Recall that the Poisson bivector field $\pi$ which defines the Poisson bracket

$$\{f, g\}_\pi := \pi(df, dg) \quad (2.23)$$

has vanishing Schouten bracket $[\pi, \pi]_M = 0$. This is equivalent to the the Leibniz rule and the Jacobi identity

$$\{f, g, h\}_\pi = f \{g, h\}_\pi + \{f, h\}_\pi g \quad \text{and} \quad \{f, \{g, h\}_\pi\} = \{\{f, g\}_\pi, h\}_\pi + \{g, \{f, h\}_\pi\}_\pi \quad (2.24)$$

being satisfied for all $f, g, h \in C^\infty(M)$. The pair $(C^\infty(M), \{-, -\}_\pi)$ is called a Poisson algebra; it has in particular the structure of a Lie algebra. If $M$ is a symplectic manifold with symplectic structure given by a closed non-degenerate two-form $\omega \in \Omega^2(M)$, then the inverse of $\omega$ gives rise to a Poisson bivector field $\pi = \omega^{-1}$ and thus any symplectic manifold is also a Poisson manifold.

On the other hand, a bivector field $\pi$ on $M$ induces a map $\pi^\sharp : T^*M \to TM$ via contraction together with a bracket on $C^\infty(M, T^*M) = \Omega^1(M)$,

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta) \quad (2.25)$$

for $\alpha, \beta \in \Omega^1(M)$, where $\mathcal{L}$ denotes the Lie derivative. The objects $(T^*M, \pi^\sharp, [-, -]_\pi)$ form a Lie algebroid if and only if the Schouten bracket of $\pi$ vanishes. Thus, the cotangent bundle of a Poisson manifold is naturally a Lie algebroid. See [33] for further aspects of the connection between Poisson structures and Lie algebroids.

We would now like to generalize the quantization of Lie algebra duals to Lie algebroids $E = T^*M$ describing the Poisson manifolds $(M, \pi)$. The dual bundle $E^*$ of such a Lie algebroid $E$ is naturally a Poisson manifold with Poisson structure inherited from the Lie bracket $[-, -]_E$ of $E$ [34]; explicitly, the Poisson bracket $\{-, -\}_{E^*}$ on $C^\infty(E^*)$ is given by:

(i) $\{f \circ p_{E^*}, g \circ p_{E^*}\}_{E^*} = 0$ for $f, g \in C^\infty(M)$, where $p_{E^*} : E^* \to M$ is the bundle projection.

(ii) $\{X, f \circ p_{E^*}\}_{E^*} = (\mathcal{L}_f)_{\rho(X)} \circ p_{E^*}$ for $f \in C^\infty(M)$ and fiberwise linear functions $X \in C^\infty(M, E) \subset C^\infty(E^*)$.

(iii) $\{X, Y\}_{E^*} = [X, Y]_E$ for fiberwise linear functions $X, Y \in C^\infty(M, E) \subset C^\infty(E^*)$.

Note that the Poisson structure on $E^*$ is completely determined by that on $M$.

By attempting to quantize the cotangent Lie algebroid $E = T^*M$ instead of $M$, we have doubled the dimension of the space to be quantized. This has to be undone by introducing a polarization. The next step in the quantization procedure is therefore to

\(^1\)See Appendix A for the definition and examples of Lie algebroids.
derive a twisted polarized convolution algebra of an integrating Lie groupoid, generalizing that of a Lie group; see Appendix A for the relevant definitions. In this paper, we will follow Hawkins’ approach [5], in which the elements of geometric quantization are lifted to groupoids. Moreover, the notion of integration of the Lie algebroid $T^*M$ of a Poisson manifold $(M, \pi)$ is shifted to the equivalent one of finding an integrating symplectic groupoid $\Sigma$ of the Poisson manifold $(M, \pi)$. The essential steps of his procedure are:

1. Find an integrating symplectic groupoid $\Sigma$ of the Poisson manifold $(M, \pi)$.

2. Construct a prequantization of $\Sigma$ with data $(E, \nabla, \sigma)$, where $E$ is the prequantum line bundle with connection $\nabla$ and $\sigma$ is a cocycle twist.

3. Endow $\Sigma$ with a groupoid polarization $P$.

4. Construct the $C^*$-algebra of $\Sigma/P$ as a twisted polarized convolution algebra, either through half-densities or through a Haar system.

One issue with this prescription is that not every Poisson manifold allows for an integrating symplectic groupoid, see Appendix A for more details. This, however, is merely a groupoid version of the statement that not every manifold is quantizable. Note that integrability of $T^*M$ is equivalent to integrability of the brackets (2.25) on $\Omega^1(M)$. In the classes of examples that we consider in this paper, the integrating symplectic groupoid will always be taken to be a suitable variant of the pair groupoid $\Sigma = \text{Pair}(M) = M \times M$, which integrates the tangent Lie algebroid $TM$.

In the ensuing subsections, we will review Hawkins’ approach and its ingredients to the extent necessary for our further constructions. We shall also give the groupoid quantization yielding $\kappa$-Minkowski space as a novel example.

2.3. Geometric, Berezin and Berezin-Toeplitz quantization

Before coming to the groupoid formalism, let us briefly review the pertinent ingredients of geometric quantization. In the formulation of Kostant, a prequantization of a symplectic manifold $(M, \omega)$ consists of a hermitian line bundle $E$ over $M$ with connection $\nabla$ and curvature two-form $F_\nabla = \nabla^2 = -2\pi i \omega$; this imposes the prequantization condition that the symplectic form must define an integer cohomology class $[\omega] \in H^2(M, \mathbb{Z})$, i.e., $\omega \in \Omega^2_{\text{cl}, \mathbb{Z}}(M)$, where generally $\Omega^k_{\text{cl}, \mathbb{Z}}(M)$ denotes the group of closed $k$-forms (i.e., de Rham $k$-cocycles) on $M$ with integer periods. The prequantum Hilbert space is identified with the space of $L^2$-sections of $E$.\footnote{The prequantization condition ensures that this space is sufficiently large.} 2 It is well-known that this Hilbert space is too large, and it is necessary to reduce it by choosing “half a canonical coordinate system” [35]. This can be done by introducing a polarization, i.e., a foliation of $(M, \omega)$ by Lagrangian submanifolds or, equivalently, a smooth distribution $\mathcal{P} \subset T_c^*M$ which is integrable and Lagrangian. Locally, the cotangent bundle $T^*N$ of a Lagrangian submanifold $N \subset M$ can be naturally identified with the full symplectic manifold $M$; a real polarization then corresponds to a choice of...
“momentum space/position space” representation. A section \( \psi \in L^2(M, E) \) is polarized if it is covariantly constant along the leaves of the polarization \( \mathcal{P} \), i.e., \( \nabla_X \psi = 0 \) for all \( X \in \mathcal{C}^\infty(M, \mathcal{P}) \).

The construction of the actual Hilbert space \( \mathcal{H} \) from the prequantum Hilbert space and a polarization is in general quite complicated. Here we are merely interested in the quantization of Kähler manifolds using Kähler polarization: We identify \( \mathcal{H} \) with (a completion of) \( H^0(M, E) \), the space of global holomorphic sections of \( E \). This construction of the Hilbert space is common to geometric quantization [35] as well as to Berezin and Berezin-Toeplitz quantization [36, 37], see also [38]. The inner product on \( \mathcal{H} \) is obtained from the hermitian structure \( h \) on \( E \) together with a suitable measure \( d\mu \) on \( M \) such as, e.g., the Liouville measure \( \frac{\omega^n}{n!} \) for a compact Kähler manifold \( (M, \omega) \) of dimension \( \dim_{\mathbb{C}} M = n \); it is given by

\[
\langle \psi_1 | \psi_2 \rangle := \int_M d\mu(x) \ h(\psi_1(x), \psi_2(x)) \quad \text{for} \quad \psi_1, \psi_2 \in H^0(M, E) \ . \tag{2.26}
\]

To complete the quantization, we need a map from a quantizable subset of functions \( \mathcal{C}_q^\infty(M) \subseteq \mathcal{C}^\infty(M) \) to endomorphisms on \( \mathcal{H} \). In Berezin and Berezin-Toeplitz quantization, one uses the Rawnsley coherent states [39] \( |x\rangle \in \mathcal{H}, \ x \in M \), to construct the coherent state projector

\[
\hat{P}_x := |x\rangle \langle x| (x|x) , \tag{2.27}
\]

which is simultaneously a function on \( M \) and an operator on \( \mathcal{H} \), and hence bridges the classical and quantum worlds. It induces the maps

\[
B : \text{End} (\mathcal{H}) \rightarrow \mathcal{C}_q^\infty(M), \quad B(\hat{f}) (x) := \text{tr}_{\mathcal{H}} (\hat{f} \hat{P}_x) ,
\]

\[
T : \mathcal{C}_q^\infty(M) \rightarrow \text{End} (\mathcal{H}) , \quad T(f) := \int_M d\mu(x) \ f(x) \hat{P}_x \ . \tag{2.28}
\]

The image of the map \( B \) defines the set of quantizable functions \( \mathcal{C}_q^\infty(M) \). Since \( B \) is injective, we can define its inverse \( B^{-1} : \mathcal{C}_q^\infty(M) \rightarrow \text{End} (\mathcal{H}) \), and this is the quantization map in Berezin quantization. The map \( T \) is used in Berezin-Toeplitz quantization. The classic example is the fuzzy sphere, which is the Berezin or Berezin-Toeplitz quantization of the Kähler manifold \( \mathbb{C}P^1 \).

Instead of working with a prequantum line bundle, we can also follow Souriau and work with the corresponding principal \( U(1) \)-bundle \( p : P \rightarrow M \) endowed with a connection \( A \in \Omega^1(P) \) such that \( dA = p^* \omega \). The line bundle \( E \) is then recovered as the associated bundle \( E = P \times_{\rho} \mathbb{C} \), where \( \rho(e^{i\phi}) \cdot z = e^{i\phi} z \) for all \( z \in \mathbb{C} \). The KKS prequantization gives a faithful unitary representation of the Poisson algebra \( (\mathcal{C}^\infty(M), \{-,-\}_{\omega^{-1}}) \) on a Hilbert space, which can be constructed by using the Atiyah Lie algebroid associated to \( P \). The Atiyah Lie algebroid sequence is the exact sequence of vector bundles

\[
\text{ad}(P) \rightarrow E_{\text{Ad}}(P) := TP / U(1) \rightarrow TM \ , \tag{2.29}
\]

\[\text{while geometric quantization employs the same Hilbert space, the quantization map differs from the ensuing ones.}\]
where $TM$ is the tangent Lie algebroid of $M$ and $\text{ad}(P)$, being a bundle of Lie algebras over $M$, is naturally a Lie algebroid with the zero anchor map and fiberwise Lie bracket. The vector bundle $E_A(P) := TP/U(1) \to M$ naturally inherits from $TP$ the structure of a Lie algebroid, called the Atiyah Lie algebroid; its smooth sections are the $U(1)$-invariant vector fields on $P$, and it integrates to the Atiyah Lie groupoid $G_A(P) = P \times_{U(1)} P$ which is the quotient of the pair groupoid $\text{Pair}(P) = P \times P$ by $U(1)$. In this sense, consideration of the Atiyah Lie algebroid over $M$ is equivalent to considering the cotangent bundle $T^*M$ as a Lie algebroid.

A connection $A$ on $P$ is equivalent to a splitting of the exact sequence (2.29),\(^4\) which gives a monomorphism of Lie algebras

$$
(C^\infty(M), \{-,-\}_\omega) \longrightarrow (C^\infty(M, E_A(P)), [-,-]_{TP/U(1)}).
$$

This yields a faithful representation of the Poisson algebra of $C^\infty(M)$. In particular, the Poisson algebra of $C^\infty(M)$ acts as linear differential operators on $U(1)$-equivariant functions $P \to \mathbb{C}$, which correspond to (global) $L^2$-sections of the hermitian line bundle $E$ associated to $P$. The Poisson algebra $C^\infty(M, T^*P/U(1))$ of the Lie algebroid $E_A(P)$ is quantized to the $C^*$-algebra of the groupoid $G_A(P)$, which is isomorphic to the $C^*$-algebra $\mathcal{K}(L^2(M)) \otimes C^*_r(U(1))$ of $U(1)$-invariant compact operators on $L^2(P)$ (see §2.7 below).

This prequantization is only the first step, and to complete the picture one should introduce a polarization and restrict to a polarized Hilbert space. There are however no faithful representations of the Poisson algebra on the polarized Hilbert space and one has to use approximate Lie algebra homomorphisms. As we will be working in the Kostant picture using prequantum line bundles, we refrain from going into further details.

### 2.4. Prequantization and polarization of symplectic groupoids

Prequantization of groupoids dates back to work of Weinstein and Xu [6], see also [7, 8] for more recent accounts. Consider a symplectic groupoid $(\Sigma, \omega)$ integrating a Poisson manifold $(M, \pi)$. This means that $\Sigma \rightrightarrows M$ is a Lie groupoid, $(\Sigma, \omega)$ is a symplectic manifold, $\omega$ is a multiplicative two-form and the target map\(^5\) of the groupoid $t : \Sigma \to M$ is a Poisson map, see Appendix A. A prequantization of $(\Sigma, \omega)$ is given by the prequantization of $\Sigma$ as a symplectic manifold, i.e., a hermitian line bundle $E \to \Sigma$ endowed with a connection $\nabla$ such that $F_\nabla = -2\pi \text{i} \omega$, together with a two-cocycle $\sigma$.

The cocycle $\sigma$ provides an associative multiplication on fibers of $E$ at different composable points of $\Sigma$. It is a section of the hermitian coboundary line bundle $\partial^*E^* := \text{pr}_1^*E^* \otimes m^*E \otimes \text{pr}_2^*E^*$ over the 2-nerve $\Sigma_{(2)}$, where $E^*$ denotes the line bundle dual to $E$. The fiber multiplication is defined on $\psi_{g_1} \in E_{g_1}$ and $\psi_{g_2} \in E_{g_2}$ by

$$
\psi_{g_1} \cdot_\sigma \psi_{g_2} = (\sigma(g_1, g_2), \psi_{g_1} \otimes \psi_{g_2}),
$$

\(^4\)This splitting occurs in the category of vector bundles, not of Lie algebroids, or else the corresponding curvature vanishes. Alternatively, we can get non-trivial Chern classes by passing to $L_\infty$-algebroids.

\(^5\)We use $(s, t, m, 1)$ to denote the structure maps of the groupoid. Moreover, $\text{pr}_1$ and $\text{pr}_2$ denote the obvious projections from the “set of composable arrows” $\Sigma_{(2)}$, i.e. the 2-nerve of the simplicial manifold underlying $\Sigma$, to $\Sigma$. 

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which induces a multiplication $\bullet_\sigma : C^\infty(\Sigma, E) \otimes C^\infty(\Sigma, E) \to C^\infty(\Sigma^{\otimes 2}, m^*E)$ on sections of $E$. Associativity is ensured by the multiplicative cocycle property

$$\sigma(g_1, m(g_2, g_3)) \otimes \sigma(g_2, g_3) = \sigma(g_1, g_2) \otimes \sigma(m(g_1, g_2), g_3)$$

(2.32)

for $g_1, g_2, g_3 \in \Sigma$. We further demand that $\sigma$ has unit norm, and that it is covariantly constant with respect to the (local) symplectic potential $\theta$ for $\omega$:

$$\nabla \sigma = d\sigma - i(\partial^*\theta)\sigma = 0 \, .$$

(2.33)

The condition on $\Sigma$ to be prequantizable is identical to that of $M$ being prequantizable in the sense of Kostant [7, 8]: All the periods of $M$ have to be integer multiples of $2\pi$; in that case, the prequantization of $\Sigma$ is unique (up to isomorphism).

A polarization of a symplectic groupoid $\Sigma$ [5] is a polarization $\mathcal{P}$ of $\Sigma$ as a symplectic manifold which is multiplicative, i.e., the distribution $\mathcal{P} \subset T_{\Sigma}\Sigma$ is compatible with and closed under the groupoid multiplication: $m^*(\mathcal{P}_2)|_{(g_1, g_2)} = \mathcal{P}_m(g_1, g_2)$, where $\mathcal{P}_2 := (\mathcal{P} \times \mathcal{P}) \cap T_{\Sigma}\Sigma$; see [5] for further details. The polarizations $\mathcal{P} \subset T_{\Sigma}\Sigma$ that we will be mostly interested in are given in terms of foliations of a fibration of groupoids $p : \Sigma \to \Sigma/\mathcal{P}$, which give rise to strongly admissible polarizations. If the leaves of $\mathcal{P}$ are simply connected, then $E$ is trivial along these leaves and there is a canonical identification $E \cong p^*E_\mathcal{P}$ where $E_\mathcal{P}$ is a line bundle over $\Sigma/\mathcal{P}$. Abusing notation, we denote the corresponding “reduced” cocycle with coefficients in $E_\mathcal{P}$ by the same symbol $\sigma$. Moreover, even when the polarization $\mathcal{P}$ is not the kernel foliation of a fibration of groupoids, we continue to use the same notation $\Sigma/\mathcal{P}$ to indicate the polarized symplectic groupoid.

2.5. Twisted polarized convolution algebra

To construct the twist element $\sigma$, we will always start from a symplectic potential $\theta$ with $d\theta = \omega$ which is adapted, i.e., compatible with the polarization $\mathcal{P}$ in the sense that $\theta \in C^\infty(\Sigma, \mathcal{P}^\perp)$. In these cases, the line bundle $E_\mathcal{P}$ is trivial on $\Sigma/\mathcal{P}$, and $\sigma \in C^\infty((\Sigma/\mathcal{P})_2, U(1))$. To satisfy the multiplicativity condition $0 = \partial^*\omega = \partial^*d\theta = d\partial^*\theta$, we have to demand that $\partial^*\theta$ is closed. The cocycle $\sigma$ is then constructed from the projection $p : \Sigma \to \Sigma/\mathcal{P}$ via

$$p^*(\sigma^{-1}d\sigma) = i\partial^*\theta \, ,$$

(2.34)

where we have used (2.33).

The convolution product on $C^\infty_c(\Sigma/\mathcal{P})$ is now defined as follows. Introduce a left Haar system of measures $\{d\mu^x| x \in (\Sigma/\mathcal{P})_0\}$ on $(\Sigma/\mathcal{P})^x := \{g \in \Sigma/\mathcal{P} | t(g) = x\} = t^{-1}(x)$, i.e., $d\mu^x(g^{-1}h) = d\mu^x(h)$, and set

$$\left(\tilde{f} \otimes_\sigma \tilde{f}'\right)(g) = \int_{kk'=g} \sigma(k, k') \tilde{f}(k) \tilde{f}'(k')$$

$$:= \int_{(\Sigma/\mathcal{P})^x(g)} d\mu^x(h) \sigma(g h, h^{-1}) \tilde{f}(g h) \tilde{f}'(h^{-1})$$

(2.35)

This condition is necessary for it to be compatible with the polarization later on.

7See Appendix A.
for $\tilde{f}, \tilde{f}' \in C^\infty_c(\Sigma/P)$. Then the appropriate completion defines the $\sigma$-twisted reduced convolution $C^*$-algebra $C^*_r(\Sigma/P, \sigma)$ of the polarized symplectic groupoid $\Sigma/P$; it is the convolution algebra of polarized sections of $E$.

A more canonical definition uses half-densities and sections of the associated complex line bundle $\Omega^{1/2}_{\Sigma/P} \to \Sigma/P$ to define $C^*_r(\Sigma/P, \sigma)$, which ensures that the integrand used in (2.35) is always a density on each fiber $(\Sigma/P)^x$ of the map $t$; for the most part the above definition will suffice for the examples we consider.

2.6. Generalized Bohr-Sommerfeld quantization condition

If the leaves of the polarization $P$ are not simply connected, then the construction above requires some further truncation. In this case, the leaves only admit parallel sections when the holonomy of the connection $\nabla$ is trivial; if not, then the symplectic potential $\theta$ is not adapted. This condition on the holonomy can be regarded as a generalized Bohr-Sommerfeld quantization condition. We define the Bohr-Sommerfeld groupoid as the subvariety $\Sigma_0 \subseteq \Sigma$ such that the holonomy $^8$

$$\text{hol}_\gamma(\theta) := \exp \left( 2\pi i \oint_{\gamma} \theta \right) \tag{2.36}$$

is equal to 1 for all conjugacy classes of loops $[\gamma] \in \pi_1(\Sigma_0)^\sim$, i.e., $\oint_{\gamma} \theta \in \mathbb{Z}$. This in turn trivializes the line bundle $E_P$ over $\Sigma_0/P$, and so the previous construction can now be applied to the reduced groupoid $\Sigma_0/P$.

2.7. Quantization map

Lagrangian submanifolds of $\Sigma$ with a section that is covariantly constant along each leaf of the polarization $P$ (and which vanishes on leaves not intersecting $E$) correspond to elements of $A := C^*_r(\Sigma/P, \sigma)$. From the definition of a symplectic groupoid, the graph of the multiplication $m$ is a Lagrangian submanifold of $\Sigma \times \Sigma \times \Sigma$ and so quantizes to an element of $A^* \otimes A^* \otimes A$, i.e., a map $A^* \otimes A \to A$, the unit section $\mathbb{1} : M \hookrightarrow \Sigma$ is a Lagrangian submanifold which quantizes to a unit element in $A$, and the graph of the inversion $g \mapsto g^{-1}$ is a Lagrangian submanifold of $\Sigma \times \Sigma$ which quantizes to an element of $A^* \otimes A$. These elements define the multiplication, identity and inversion in the noncommutative algebra $A$.

The procedure just outlined yields a quantization of the dual $E^*$ of the associated Lie algebroid $E = A(\Sigma/P)$. The quantization map $E^* \to C^*_r(\Sigma/P, \sigma)$ is given by the composition of the extension of Fourier transform to vector bundles $E \to E^*$, described in [40, §7] using suitable left Haar systems on $E$ and $E^*$, with the extension of the exponential map $E \to \Sigma/P$, described in [40, §2] in the case that the Lie algebroid $E$ is endowed with a connection. Here we regard the twisted polarized convolution algebra $C^*_r(\Sigma/P, \sigma)$ as a

---

8The formula (2.36) for the holonomy as written is symbolic in general; for topologically non-trivial symplectic potentials $\theta$, a precise expression is given in Appendix B.

9For a symplectic manifold $(M, \omega)$, we denote $\overline{M} = (M, -\omega)$. 

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quantization of the Poisson manifold \( M \), in the sense that the diagram

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{t} & M \\
& \searrow & \swarrow \\
& \Sigma/\mathcal{P} & \\
\end{array}
\]

(2.37)

can be used to “pullback” the groupoid multiplication on \( \Sigma/\mathcal{P} \) to \( C^\infty(M) \); however, even though the target map \( t : \Sigma \to M \) is a Poisson map and \( \Sigma \) is \( s \)-connected, the pullback \( t^*(f) \) of a function \( f \in C^\infty(M) \) is generically not quantizable, and in general the explicit quantization maps \( C^\infty(M) \to \mathcal{C}_r(\Sigma/\mathcal{P}, \sigma) \) are not known.

**Example.** When \( M = pt \) is a one-point space and the Lie groupoid \( \Sigma \) is a Lie group \( G \), the left-invariant Haar measure on \( G \) induces a left Haar system and the convolution algebra \( \mathcal{C}_r(G) \) is the reduced group \( C^* \)-algebra. The Lie algebra \( \mathfrak{g} \) of \( G \) is the associated Lie algebroid \( A(G) \), and the Poisson structure on the dual \( A^*(G) = \mathfrak{g}^* \) is the usual linear KKS Poisson structure. In this case the quantization map \( \mathcal{C}_r^\infty(\mathfrak{g}^*) \to \mathcal{C}_r^*(G) \) is essentially the composition of Fourier transformation on the affine space \( \mathfrak{g}^* \to \mathfrak{g} \) with the exponential map \( \exp : \mathfrak{g} \to G \), as constructed in §2.1.

**Example.** When \( \Sigma \) is the pair groupoid \( \text{Pair}(M) = M \times M \) of an oriented manifold \( M \), the 2-nerve is \( \Sigma(2) = M \times M \times M \) with multiplication map \( m(x, y, z) = (x, z) \) and projections \( \text{pr}_1(x, y, z) = (x, y) \), \( \text{pr}_2(x, y, z) = (y, z) \). The product in \( \mathcal{C}_r^*(M \times M) \) is the convolution of kernels

\[
(\tilde{f} \ast \tilde{f}')(x, z) = \int_M \text{d}\text{vol}_M(y) \tilde{f}(x, y) \tilde{f}'(y, z),
\]

(2.38)

and the \( C^* \)-algebra \( \mathcal{C}_r^*(M \times M) \) acts as integral kernels on \( L^2(M) \), so that \( \mathcal{C}_r^*(M \times M) \) is isomorphic to the algebra \( \mathcal{K}(L^2(M)) \) of compact operators. In this case the associated Lie algebroid \( A(\text{Pair}(M)) \) is the tangent bundle \( TM \), with the standard symplectic Poisson structure on the dual \( A^*(\text{Pair}(M)) = T^*M \); the quantization map is thus \( \mathcal{C}_r^\infty(T^*M) \to \mathcal{K}(L^2(M)) \). If \( M \) is additionally a symplectic manifold of dimension \( 2k \) with symplectic two-form \( \omega_M = d\theta_M \), then it is easy to check that the symplectic structure \( \omega := \ast^s \omega_M - t^s \omega_M \) on \( \Sigma \) is multiplicative: If we write \( \omega(x, y) = \omega_M(x) - \omega_M(y) \) for \( x, y \in M \), then one has

\[
\text{pr}_1^* \omega(x, y, z) = \omega_M(x) - \omega_M(y),
\]

\[
\text{pr}_2^* \omega(x, y, z) = \omega_M(y) - \omega_M(z),
\]

\[
m^* \omega(x, y, z) = \omega_M(x) - \omega_M(z),
\]

(2.39)

and hence \( \partial^s \omega = \text{pr}_1^* \omega - m^* \omega + \text{pr}_2^* \omega = 0 \). If \( E_M \) is a prequantum line bundle on \( M \), then a prequantum line bundle on \( \Sigma = M \times M \) is \( E = E_M \boxtimes \overline{E_M} \); if \( \mathcal{P}_M \) is a polarization on \( M \), then a symplectic groupoid polarization on \( \Sigma \) is given by \( \mathcal{P} = \mathcal{P}_M \times \overline{\mathcal{P}_M} \). The local symplectic potential \( \theta := \ast^s \theta_M - t^s \theta_M \) is also multiplicative, \( \partial^s \theta = 0 \), whence the
prequantization cocycle is \( \sigma = 1 \). The ensuing restriction to the algebra \( C^*_r(\Sigma/P) \) with the convolution product (2.38) may be constructed from polarized sections of the line bundle \( E \otimes \Omega^{1/2}_P \), where \( \Omega_P := \det(P^\perp_M) \otimes \det(\overline{P^\perp_M}) \) [5]. Then \( C^*_r(\Sigma/P) \) is isomorphic to the algebra of compact operators on the space of \( P_M \)-polarized sections of the complex line bundle \( E_M \to M \). In (2.38) the product of polarized sections \( \bar{f}(x,y) \tilde{f}(y,z) \) lives in the square root of \( (\Omega_P(x,y) \otimes (\Omega_P)(y,z)) \). In order to integrate this over \( t^{-1}(y) = (\Sigma/P)^y \), we have to tensor with non-vanishing sections of the square root of \( \det T_{(x,y)}^* t^{-1}(y) \) and of \( \det T_{(y,z)}^* s^{-1}(y) \) to get a top form. By [5, Thm. 5.3], this can be done by contraction with the Liouville form \( d\mu_M = (\omega_M)^k = \det \omega_M s^{-1}(y) \) on \( M \). See [41] for a derivation of the Moyal product on \( M = \mathbb{R}^2 \) within this framework.

2.8. Groupoid quantization and \( \kappa \)-Minkowski space

Let us now study a non-trivial example for the groupoid quantization of the dual of a Lie algebra in detail: the \( \kappa \)-Minkowski algebra from §2.1. We start from the real vector space \( V = \mathbb{R}^d \) with the +-KKS Poisson structure induced by the \( \kappa \)-Minkowski Lie algebra (2.9).

For \( d = 2 \) the corresponding \( \kappa \)-Minkowski group \( G_\kappa \cong \mathbb{R} \times \mathbb{R} > 0 \) is isomorphic to a connected affine group on the real line, also known as an \( a + b \)-group (here with \( a = e^{-\kappa^{-1} p_0}, b = p_1 \)). A symplectic groupoid over an affine group on \( \mathbb{R} \) has been constructed in [42]; in the following we work in a different parameterization for arbitrary \( d \geq 2 \) and with a different affine group. This example is a special case of that studied in [5, §6.3].

As a manifold, the symplectic groupoid over \( V \) is the cotangent bundle \( \Sigma = T^* G_\kappa \); we will use coordinates \( x = (x^0, \vec{x}) \) on the affine space \( g^*_x \cong T^*_x G_\kappa \) and coordinates \( p = (p_0, \vec{p}) \) on the group manifold \( G_\kappa \). We will follow now the general construction of groupoid structure maps on the cotangent bundle of a Lie group, see Appendix A. The unit embedding is trivially given by \( 1 : g^*_x \to T^*_x G_\kappa \), or explicitly by \( 1_{(x^0, \vec{x})} = ((x^0, \vec{x}), (0, \vec{0})) \).

To derive the source and target maps, note that the left and right actions of \( G_\kappa \) on itself read

\[
L_p(q) = (p_0 + q_0, \vec{p} + e^{-\kappa^{-1} p_0} \vec{q}) \quad \text{and} \quad R_p(q) = (q_0 + p_0, \vec{q} + e^{-\kappa^{-1} q_0} \vec{p}) \tag{2.40}
\]

with inverses

\[
L_p^{-1}(q) = (q_0 - p_0, e^{\kappa^{-1} p_0} (\vec{q} - \vec{p})) \quad \text{and} \quad R_p^{-1}(q) = (q_0 - p_0, \vec{q} - e^{-\kappa^{-1} q_0} \vec{p}) \tag{2.41}
\]

where \( p^{-1} = (-p_0, e^{\kappa^{-1} p_0} \vec{p}) \) for \( p, q \in G_\kappa \). These actions induce derivative maps \( dL_p = L_{p^*} : T_q G_\kappa \to T_{pq} G_\kappa \) and \( dR_p = R_{p^*} : T_q G_\kappa \to T_{qp} G_\kappa \). The source and target maps \( s, t : T^* G_\kappa \to g^*_x \cong T^*_x G_\kappa \) are then given by

\[
\langle s(x, p) , (y^0, \vec{y}) \rangle := (x^0, \vec{x}) \circ dR_p(y^0, \vec{y}) = (x^0 - \kappa^{-1} p_1 x^i \vec{x}) \left( \begin{array}{c} y^0 \\ \vec{y} \end{array} \right) \tag{2.42}
\]

and

\[
\langle t(x, p) , (y^0, \vec{y}) \rangle := (x^0, \vec{x}) \circ dL_p(y^0, \vec{y}) = (x^0 - e^{-\kappa^{-1} p_0} \vec{x}) \left( \begin{array}{c} y^0 \\ \vec{y} \end{array} \right) \tag{2.43}
\]
where we identify $V$ with its dual $V^*$.

Multiplication of two points $(x, p)$ and $(y, q)$ in $T^*G_\kappa$ is defined if $s(y, q) = t(x, p)$ or

$$y^0 = x^0 + \kappa^{-1} e^{-\kappa^{-1} p_0} q_i x^i \quad \text{and} \quad \vec{y} = e^{-\kappa^{-1} p_0} \vec{x}.$$  \hfill (2.44)

As $y \in T_q^*G_\kappa$ is hence determined by $(x, p)$ and the fiber point $q$, the 2-nerve of the groupoid $\Sigma$ is $\Sigma(2) = T^*G_\kappa \times G_\kappa$. We will use coordinates $((x^0, \vec{x}), (p_0, \vec{p}), (q_0, \vec{q}))$ on $\Sigma(2)$. In these coordinates the product $m$ and the projections $pr_1, pr_2$ read

$$m(x, p, q) := ((x^0, \vec{x}), (p_0 + q_0, \vec{p} + e^{-\kappa^{-1} p_0} \vec{q})),$$

$$pr_1(x, p, q) := ((x^0, \vec{x}), (p_0)),$$

$$pr_2(x, p, q) := ((x^0 + \kappa^{-1} e^{-\kappa^{-1} p_0} q_i x^i, e^{-\kappa^{-1} p_0} \vec{x}), (q_0, \vec{q})).$$  \hfill (2.45)

A polarization $\mathcal{P}$ of the groupoid $\Sigma$ is given by the kernel of the tangent map to the bundle projection $p : T^*G_\kappa \to G_\kappa$, regarded as a fibration of groupoids, i.e., $\Sigma/\mathcal{P} = G_\kappa$.

Define the adapted symplectic potential $\theta = x^\mu dp_\mu$, which gives rise to the canonical symplectic structure $\omega = d\theta = dx^\mu \wedge dp_\mu$ on $T^*G_\kappa$. Note that $1^*\theta = 0$ and $\theta$ is conormal to the groupoid polarization $T^*G_\kappa \to G_\kappa$. Moreover, one has

$$\partial^*\theta = x^\mu dp_\mu + (x^0 + \kappa^{-1} e^{-\kappa^{-1} p_0} q_i x^i) dq_0 + e^{-\kappa^{-1} q_0} x^i dq_i$$

$$- (x^0 + \kappa^{-1} e^{-\kappa^{-1} p_0} q_i x^i) (dp_0 + dq_0)$$

$$- x^i (dp_i + e^{-\kappa^{-1} p_0} dq_i - \kappa^{-1} e^{-\kappa^{-1} p_0} q_i dp_0) = 0.$$  \hfill (2.46)

Thus the twist element $\sigma = 1$ is trivial and we obtain the untwisted convolution algebra (2.16). This is a more general feature of the groupoid quantization of duals of Lie algebras [5, §6.3].

3. Quantization of 2-plectic manifolds and loop spaces

In this section we will discuss the use of loop spaces and knot spaces in the quantization of 2-plectic manifolds. We start with an outline of our approach, and then review all necessary notions from loop space geometry before discussing the loop space extension of the groupoid approach to quantization.

3.1. Overview

Just as a symplectic structure on a manifold $M$ is defined in terms of a closed, non-degenerate two-form, a 2-plectic structure is given by a closed, non-degenerate three-form. As we saw in §2.3, a symplectic form with integer periods represents the first Chern class of a prequantum line bundle in geometric quantization, fixing it up to isomorphism. Correspondingly, a 2-plectic form with integer periods specifies the Dixmier-Douady class of an abelian prequantum gerbe. Pushing the analogy further, we recall that the Hilbert space in geometric quantization is derived from the space of polarized global sections of the prequantum line bundle. The corresponding notions for a gerbe, however, seem to be still unclear.
One can in principle work with the realization of gerbes as principal $\text{PU}(\mathcal{H})$-bundles $P \to M$ where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space (see e.g. [43, 44]), and consider the space of sections thereof; indeed, $H^3(M, \mathbb{Z})$ is the group of isomorphism classes of projective bundles $P$ (with infinite-dimensional separable fibers) while the automorphism group of $P$ is the group $H^2(M, \mathbb{Z})$ of complex line bundles on $M$, see e.g. [45]. However, the appropriate definition of a polarization still remains to be found. But the lesson here is that attempting to resort to conventional geometric settings thus inevitably leads us into the realm of infinite-dimensional analysis, which will be a recurrent theme below.

We can circumvent these quantization issues by using a trick that goes back to Gawędzki [9], see also [10]. The Dixmier-Douady class of the abelian prequantum gerbe on $M$ gives rise to the first Chern class of a prequantum line bundle on the loop space $LM$ of $M$ via the cohomological transgression homomorphism. A representative of the first Chern class yields a symplectic structure on $LM$. We will focus on three-dimensional Riemannian 2-plectic manifolds. Their knot spaces, i.e. the loop spaces factored by the group of reparameterizations of the loops, comes with a natural complex structure. This complex structure can be combined with the symplectic structure obtained from the transgressed volume form to a Kähler structure. In principle, we can then follow the usual recipe of geometric quantization to construct a Hilbert space as the space of global sections of the corresponding prequantum line bundle on knot space. The problem with the definition of a polarization is solved e.g. by using ordinary Kähler polarization on the global sections.

This approach is also well-motivated from a different perspective: While a symplectic form on $M$ induces a Lie algebra provided by a Poisson bracket on smooth functions on $M$, a 2-plectic form induces a Lie 2-algebra consisting of smooth functions together with a subspace of one-forms on $M$. The only non-trivial binary bracket in this Lie 2-algebra is a Poisson-like bracket on the one-forms. As we will see below, the prequantization of this Lie 2-algebra is rather clear. One can construct a representation of this Lie 2-algebra, just as the KKS prequantization gave a representation of the usual Poisson algebra. However, it is not known how to implement a polarization into this picture. If we apply the transgression map to switch to the loop space setting, the situation becomes much nicer: The transgression map embeds the subspace of one-forms on $M$ into the space of smooth functions on the loop space of $M$ and it maps the 2-plectic form on $M$ to a symplectic form on $LM$. On loop space, the transgression of the Poisson-like brackets of two one-forms is precisely the loop space Poisson bracket of the transgression of the one-forms.

To actually quantize loop space, we then apply Hawkins’ groupoid approach. The reason for this is the following: Ultimately, we hope to establish a more direct quantization procedure for multisymplectic manifolds including polarization. It seems that the latter will involve nonassociative structures, and a model based on a Hilbert space and its endomorphisms seems no longer suitable. Hawkins’ approach, however, circumvents the construction of a Hilbert space. Moreover, it is much more suitable for categorification. It thus seems more likely that the groupoid approach will allow us to compare the loop space quantization with a yet to be developed direct quantization of 2-plectic manifolds.
3.2. Nambu-Poisson geometry

Consider a manifold $M$ of dimension $d$ together with its algebra of smooth functions $\mathcal{A} = C^\infty(M)$. A Nambu-Poisson structure of order $n$ on $M$ \cite{1, 46} is an $n$-ary, totally antisymmetric and multi-linear map $\{-,\ldots,-\} : \mathcal{A}^n \to \mathcal{A}$ which satisfies the Leibniz rule

$$\{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1 \{f_2, \ldots, f_{n+1}\} + \{f_1, \ldots, f_{n+1}\} f_2 ,$$

(3.1)

and the fundamental identity

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\}$$

$$+ \ldots + \{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}$$

(3.2)

for $f_i, g_i \in C^\infty(M)$. As special cases, we have the usual Poisson structures for $n = 2$ and a derivation $\{-\}$ on $\mathcal{A}$ for $n = 1$.

Nambu-Poisson manifolds can be used as multiphase spaces in Nambu mechanics, akin to the way in which Poisson structures on manifolds define phase spaces in Hamiltonian mechanics; such multiphase spaces are a starting point for higher quantization. A Nambu-Poisson bracket gives the vector space $C^\infty(M)$ of smooth functions on $M$ the structure of an $n$-Lie algebra \cite{11} called a Nambu-Poisson algebra. An $n$-Lie algebra is a vector space $\mathcal{A}$ with an $n$-ary, totally antisymmetric multilinear map which satisfies the fundamental identity. It has often been suggested that a Nambu-Poisson structure of order $n$ should turn into an $n$-Lie algebra under quantization, just as a Poisson algebra turns into a Lie algebra of endomorphisms on a Hilbert space, see e.g. \cite{48, 12} and references therein. This suggestion, however, does not seem to yield quantum spaces with all the desired features.

Recall that a Poisson structure on $M$ can be defined in terms of a bivector field $\pi \in C^\infty(M, TM \wedge TM)$ via (2.23) for which the Schouten bracket vanishes. For general Nambu-Poisson structures, we consider a multivector field $\pi \in C^\infty(M, \wedge^n TM)$, which reads in local coordinates $x^i, i = 1, \ldots, d$, as

$$\pi = \pi_{i_1 \ldots i_n} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_n}} .$$

(3.3)

The higher analogue of the vanishing of the Schouten bracket reads as

$$\pi_{i_1 \ldots i_{n-1} l} \frac{\partial}{\partial x^l} \pi_{j_1 \ldots j_n} - \pi_{i_1 j_2 \ldots j_n} \frac{\partial}{\partial x^{i_1}} \pi_{j_1 \ldots i_{n-1} j_1} - \ldots - \pi_{j_1 \ldots j_n-1 l} \frac{\partial}{\partial x^{i_1}} \pi_{i_1 \ldots i_{n-1} j_n} = 0$$

(3.4)

together with

$$N_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_n} + N_{j_1 j_2 \ldots i_n i_1 j_2 \ldots j_n} = 0 ,$$

(3.5)

where

$$N_{i_1 \ldots i_n j_1 \ldots j_n} := \pi_{i_1 \ldots i_n} \pi_{j_1 \ldots j_n} + \pi_{j_1 \ldots i_n} \pi_{i_1 \ldots j_n}$$

$$+ \cdots + \pi_{j_n i_2 \ldots i_{n-1} i_1} \pi_{j_1 \ldots j_{n-1} i_n} - \pi_{j_n i_2 \ldots i_{n-1} i_1} \pi_{j_1 \ldots j_{n-1} i_n} .$$

(3.6)

Now the $n$-vector field $\pi$ defines a Nambu-Poisson structure on $M$ via

$$\{f_1, \ldots, f_n\}_\pi := \pi(df_1, \ldots, df_n)$$

(3.7)

if and only if $\pi$ satisfies both (3.4) and (3.5), cf. \cite{46}. Note that (3.5) is automatically satisfied for $n = 2$. For further details on Nambu-Poisson structures, see \cite{49, 50}.

\textsuperscript{10}see also \cite{47} for a very detailed review
3.3. Multisymplectic geometry

A Poisson structure is often derived from a symplectic structure on $M$: A symplectic form $\omega$ defines a map $TM \to T^* M$. As $\omega$ is non-degenerate, this map is invertible and its inverse is a bivector field $\pi = \omega^{-1} : T^* M \to TM$. In particular, there is a Hamiltonian vector field $X_f$ for every function $f \in C^\infty(M)$ which is defined through $df = \iota_{X_f} \omega := \omega(X_f, -)$. The Poisson structure is then given by $\{f,g\}_\pi = \omega(X_f, X_g) = \mathcal{L}_{X_f} f = df(X_g)$. The Hamiltonian vector fields $X_f$ satisfy $L_{X_f} \omega = 0$ and therefore generate symplectomorphisms of the manifold $M$. Since $[X_f, X_g] = X_{\{f,g\}}$, the map $f \mapsto X_f$ provides an embedding of the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\}_\pi)$ into the Lie algebra of symplectomorphisms. On two-dimensional manifolds these are the area-preserving diffeomorphisms, and in the cases $M = \mathbb{R}^2$ and $M = S^2$ the Lie algebra of Hamiltonian vector fields coincides with the Lie algebra of symplectomorphisms.

To a certain extent, we can generalize this construction to Nambu-Poisson structures by introducing multisymplectic forms. An $n$-form $\varpi$ is called a multisymplectic $n$-form or an $n-1$-plectic form\(^\text{11}\) on $M$ if it is closed and non-degenerate, i.e., $\iota_X \varpi = 0$ if and only if $X = 0$ for $X \in C^\infty(M, TM)$. Simple examples of $n$-plectic forms are volume forms on orientable $n$-dimensional manifolds.

A multisymplectic $n$-form $\varpi$ defines a map $TM \to \bigwedge^{n-1} T^* M$. This map is only invertible for $d = n$, in which case the multisymplectic structure is given by a volume form on $M$. Volume forms $\varpi$ define a Nambu-Poisson structure via

$$\{f_1, \ldots, f_n\}_\varpi = \varpi^{-1}(df_1, \ldots, df_n). \quad (3.8)$$

In this paper, we will be mostly interested in the case $n = d = 3$, and in particular we will consider the 2-plectic Nambu-Poisson manifolds $\mathbb{R}^3$, $\mathbb{T}^3$ and $S^3$.

Multisymplectic $p+1$-forms also naturally give rise to Poisson-like brackets on certain $p-1$-forms: Assume that the manifold $M$ is endowed with a $p$-plectic structure $\varpi$. The space of Hamiltonian $p-1$-forms on $M$, $\mathfrak{sy}^{p-1}(M, \varpi)$, is the space of $p-1$-forms $\alpha$ for which there is an associated Hamiltonian vector field $X_\alpha \in C^\infty(M, TM)$ such that $d\alpha = \iota_{X_\alpha} \varpi$. Again, these Hamiltonian vector fields form a Lie algebra and satisfy $\mathcal{L}_{X_\alpha} \varpi = 0$; hence they generate multisymplectomorphisms of $M$. There are now two obvious generalizations of the Poisson bracket on $\mathfrak{sy}^{p-1}(M, \varpi)$ [2]: The hemi-bracket and the semi-bracket are defined on $\alpha, \beta \in \mathfrak{sy}^{p-1}(M, \varpi)$ as

$$\{\alpha, \beta\}_{h, \varpi} := \mathcal{L}_{X_\alpha} \beta \quad \text{and} \quad \{\alpha, \beta\}_{s, \varpi} := \iota_{X_\alpha} \iota_{X_\beta} \varpi, \quad (3.9)$$

respectively. Both brackets yield maps $\mathfrak{sy}^{p-1}(M, \varpi) \times \mathfrak{sy}^{p-1}(M, \varpi) \to \mathfrak{sy}^{p-1}(M, \varpi)$. Note that

$$\{\alpha, \beta\}_{h, \varpi} - \{\alpha, \beta\}_{s, \varpi} = d \iota_{X_\alpha} \beta. \quad (3.10)$$

Furthermore, the hemi-bracket satisfies the Jacobi identity but is not antisymmetric, while

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\(^{11}\)The number $n-1$ here indicates the degree of categorification of the notion of symplectic structure; in particular 1-plectic amounts to symplectic.
the semi-bracket is antisymmetric but does not satisfy the Jacobi identity. Due to (3.10), the failure of the brackets to be antisymmetric or to fulfill the Jacobi identity is always an exact form.

3.4. Prequantization of 2-plectic manifolds

We will now briefly review the prequantization of 2-plectic manifolds following [4], see also [2]. We will see that the Poisson Lie algebra $\Pi_\omega = C^\infty(M)$ of smooth functions on a symplectic manifold $(M,\omega)$ is replaced by the Lie 2-algebra $\Pi_\varpi = C^\infty(M) \oplus \mathfrak{h}^1(M,\varpi)$ on a 2-plectic manifold $(M,\varpi)$. Moreover, just as the Atiyah Lie algebroid allowed us to construct a faithful representation of the Poisson algebra on its space of sections, i.e., a prequantization, an exact Courant algebroid allows us to define a representation of the Lie 2-algebra. As argued in [2, 4], this should be regarded as a prequantization of $(M,\varpi)$.

The semi-bracket on a $p$-plectic manifold is well-defined on the quotient of the space $\mathfrak{h}^{p-1}(M,\varpi)$ by the group of closed $p-1$-forms on $M$; in particular, the quotient by exact $p-1$-forms is a Lie algebra and if $M$ is contractible then it can be resolved by the augmented de Rham complex

$$\mathbb{R} \hookrightarrow C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-2}(M) \xrightarrow{d} \mathfrak{h}^{p-1}(M,\varpi).$$

As shown in [52], such a resolution gives rise to an $L_\infty$-algebra. Here, however, we will use a truncated form of this homotopy Lie algebra.

Recall that a Lie 2-algebra is a two-term $L_\infty$-algebra $L_{-1} \xrightarrow{\mu_1} L_0$ endowed with unary, binary and ternary maps $\mu_1, \mu_2, \mu_3$ with grading $1,0,-1$ that satisfy homotopy Jacobi identities, see [54]. On a 2-plectic manifold $(M,\varpi)$ we define the Lie 2-algebra $\Pi_\varpi$ as the vector space

$$\Pi_\varpi = L_{-1} \oplus L_0 \quad \text{with} \quad L_0 = \mathfrak{h}^1(M,\varpi) \quad \text{and} \quad L_{-1} = C^\infty(M)$$

(3.12)

together with the maps

$$\mu_1(f + \alpha) = df,$$

$$\mu_2(f + \alpha, g + \beta) = \{\alpha,\beta\}_{s,\varpi} = \varpi(X_\alpha, X_\beta, -),$$

$$\mu_3(f + \alpha, g + \beta, h + \gamma) = \varpi(X_\gamma, X_\beta, X_\alpha),$$

(3.13)

where $f,g,h \in L_{-1}, \alpha,\beta,\gamma \in L_0$ and $X_\alpha$ denotes again the Hamiltonian vector field corresponding to the one-form $\alpha$.

Consider now the exact Courant algebroid $C$ associated with the 2-plectic manifold $(M,\varpi)$ together with its associated strong homotopy Lie algebra $L_\infty(C)$ as reviewed in

12For “exact” multisymplectic manifolds, a further bracket can be constructed that is both antisymmetric and satisfies the Jacobi identity [51].

13See Appendix C for the definitions related to Courant algebroids.

14In this paper, we are interested in semistrict Lie 2-algebras. More general, weak Lie 2-algebras have been defined in [53]. Semistrict Lie 2-algebras (as well as the hemistrict ones derived from the hemi-bracket) are special cases of these.
Appendix C. By [4, §7] there exists a morphism of Lie 2-algebras, given by \( \alpha \mapsto \lambda(X_\alpha) + \alpha \), which embeds \( \Pi_{2\mathbb{R}} \) into \( L_\infty(C) \). In this sense, the Courant algebroid \( C \) allows us to construct a representation of the Lie 2-algebra \( \Pi_{2\mathbb{R}} \), just as the Atiyah Lie algebroid enabled us to construct a representation of the Poisson algebra on a symplectic manifold.

Altogether, the problem of prequantizing 2-plectic manifolds seems to be solved. To obtain a complete quantization, however, we have to restrict to polarized sections. A possible generalization of polarizations to multisymplectic manifolds is explored in [55]: For a \( p \)-plectic vector space \( (V, \varpi) \) the \( k \)-orthogonal complement of a subspace \( W \subseteq V \) is the subspace

\[
W_{⊥,k} = \{ v \in V \mid \varpi(v, w_1, \ldots, w_k) = 0 \text{ for all } w_1, \ldots, w_k \in W \}. \tag{3.14}
\]

The subspace \( W \) is called \( k \)-Lagrangian if \( W = W_{⊥,k} \). Correspondingly, a submanifold \( N \) of a \( p \)-plectic manifold \( (M, \varpi) \) is called \( k \)-Lagrangian if the tangent space \( T_xN \) is a \( k \)-Lagrangian subspace of \( T_xM \) for all \( x \in N \). Despite this notion, it seems still unclear how to properly incorporate polarization into the quantization of multisymplectic manifolds. This is why we will now turn our attention to loop spaces, where this problem is under control.

### 3.5. Differential geometry of loop spaces

The free loop space \( \mathcal{L}M \) of a \( d \)-dimensional manifold \( M \) is the space of smooth maps \( S^1 \to M \), \( \mathcal{L}M := \mathcal{C}^\infty(S^1, M) \); it is the configuration space of a bosonic string sigma-model on \( S^1 \times \mathbb{R} \) with target space \( M \). We can turn \( \mathcal{L}M \) into a smooth manifold modeled on the loop space of \( \mathbb{R}^d \) [56], see also [57]. In particular, there are open covers of \( \mathcal{L}M \) given by \( U = (LU_a) \), where \( U = (U_a) \) is a cover of \( M \). For example, the loop space \( \mathcal{L}S^d \) of the \( d \)-dimensional sphere can be covered by the patches \( U_a = LU_a = \mathcal{L}(S^d \setminus \{ a \}) \), \( a \in S^d \).

We will describe a loop by a map \( x : S^1 \to M \), and use local coordinates \( (x^i) \) on some patch \( U_a \) of \( M \) yielding parametrized loops \( (x^i(\tau)) : S^1 \to U_a \). To streamline notation, we will usually combine the continuous loop parameter \( \tau \in S^1 \) with the discrete index \( i = 1, \ldots, d \) into a multi-index and write \( x^{i\tau} := x^i(\tau) \).

The infinite-dimensional tangent bundle \( T\mathcal{L}M \) over the free loop space \( \mathcal{L}M \) has fibers given by \( T_x\mathcal{L}M := \mathcal{C}^\infty(S^1, x^*TM) \), i.e., a tangent vector to a loop \( x \in \mathcal{L}M \) is a vector field along the map \( x(\tau) \). In local coordinates we can describe a section \( X \) of \( T\mathcal{L}M \) as a map \( (X^i(\tau)) : S^1 \to x^*TM \). Together with the functional derivatives \( \frac{\delta}{\delta x^i(\tau)} \), this map combines into a derivation acting on functionals on \( \mathcal{L}M \) given by

\[
X = \oint \mathrm{d}\tau \ X^i(\tau) \frac{\delta}{\delta x^i(\tau)} = \oint \mathrm{d}\tau \ X^{i\tau} \frac{\delta}{\delta x^{i\tau}} , \tag{3.15}
\]

where throughout we denote integration over the circle \( S^1 \) by \( \oint \). There is a natural diffeomorphism of the manifolds \( T\mathcal{L}M \) and \( \mathcal{L}TM \). Note that each loop \( x(\tau) : S^1 \to M \) comes with a natural tangent vector \( \dot{x}(\tau) := \frac{dx(\tau)}{d\tau} \in \mathcal{L}TM \cong T\mathcal{L}M \).

We define the cotangent bundle \( T^*\mathcal{L}M \) as the vector bundle dual to \( T\mathcal{L}M \); it is the phase space of a closed string sigma-model on the cylinder \( S^1 \times \mathbb{R} \) with target space \( M \). Unlike \( T\mathcal{L}M \), this definition is more subtle: While \( \mathcal{L}TM \) is modeled on the vector space \( \mathcal{L}\mathbb{R}^d \),
the model space for $T^*\mathcal{LM}$ is the dual space $(\mathcal{L}\mathbb{R}^d)^*$ consisting of $\mathbb{R}^d$-valued distributions on the circle. In analogy with the tangent bundle, we hence define a restricted cotangent bundle $\hat{T}^*\mathcal{LM}$ by the union of pullbacks of the cotangent spaces of $M$, i.e., its fibers are $\hat{T}^*_x\mathcal{LM} := C^\infty(S^1, x^*T^*M)$. Clearly $T^*\mathcal{LM} \supset \hat{T}^*\mathcal{LM}$ by the map $f \mapsto \int d\tau f(\tau)$ on $\mathcal{L}\mathbb{R} \to \mathbb{R}$, and we have $\hat{T}^*\mathcal{LM} \cong L^*T^*M$. In either case, we write a section of the (restricted) cotangent bundle $\alpha$ as

$$\alpha = \int d\tau \alpha_{i\tau} \delta x^i\tau,$$

(3.16)

where

$$\left(\delta x^i\tau, \frac{\delta}{\delta x^j\sigma}\right) := \delta^i_j \delta(\tau - \sigma).$$

(3.17)

For $\alpha \in C^\infty(\mathcal{LM}, \hat{T}^*\mathcal{LM})$ the quantity $\alpha_{i\tau}$ is a function of $\tau$, while for $\alpha \in C^\infty(\mathcal{LM}, T^*\mathcal{LM})$ the map

$$\alpha_{i\tau} : T\mathcal{LM} \longrightarrow \mathbb{R}, \quad X = \int d\tau X^i\tau \frac{\delta}{\delta x^i\tau} \longmapsto \int d\tau \alpha_{i\tau} X^i\tau$$

(3.18)

is generally a distribution.

The definition of $T^*\mathcal{LM}$ gives rise to sections of the antisymmetric tensor products of this bundle, i.e., the loop space $n$-forms $\Omega^n(\mathcal{LM})$. We will call such differential forms local if they can be written in local coordinates as a single integral over the loop as

$$\alpha = \int d\tau \frac{1}{n!} \alpha_{i_1...i_n\tau} \delta x^{i_1\tau} \wedge \cdots \wedge \delta x^{i_n\tau}.$$  

(3.19)

Note that all one-forms on loop space are therefore local. Using the total differential $\delta$, which in local coordinates reads as

$$\delta = \int d\tau \delta x^i\tau \frac{\delta}{\delta x^i\tau},$$

(3.20)

we can define closed and exact forms, and therefore the de Rham cohomology groups $H^n(\mathcal{LM}, \mathbb{R})$.

The differential geometry of $\mathcal{LM}$ can be made rigorous by regarding loop spaces e.g. as diffeological spaces, where the diffeology on $\mathcal{LM}$ is equivalent to a smooth Fréchet manifold structure. If $(M, g)$ is a Riemannian manifold, then there is also a natural Riemannian structure on $\mathcal{LM}$ induced from that on $M$ by

$$\langle X_1, X_2 \rangle := \int d\tau |\dot{x}(\tau)| g_{x(\tau)}(X_1(\tau), X_2(\tau))$$

(3.21)

for $X_1, X_2 \in T_x\mathcal{LM}$, together with a Banach manifold completion of $T\mathcal{LM}$ with respect to the $L^\infty$-norm on $T_x\mathcal{LM}$ given by

$$\|X\| := \sup_{\tau \in S^1} |X(\tau)|.$$  

(3.22)
3.6. Knot spaces

For our purposes, the free loop space $\mathcal{LM}$ is too big and we have to impose various restrictions. First of all, we can demand that the maps $x : S^1 \to M$ have nice properties. We thus introduce the loop space of immersions $\mathcal{L}_i M \subset \mathcal{LM}$ with the additional condition that up to isolated points, the map $x$ is an embedding. The images of the maps contained in $\mathcal{L}_i M$ are called singular knots [10]. Moreover, we can restrict to the space of actual embeddings $\mathcal{L}_e M \subset \mathcal{L}_i M \subset \mathcal{LM}$; here the images of the maps $x \in \mathcal{L}_e M$ are knots, and the maps themselves give rise to parametrizations of knots. We would like our description to be invariant under reparameterizations of these knots, which are given by the smooth action of the group of orientation-preserving diffeomorphisms of the circle $\mathcal{R} = \text{Diff}^+(S^1)$ via precomposition. Under such a coordinate change $\tau \mapsto \tilde{\tau} = R(\tau)$, we have e.g. the relations

$$\tilde{x}(\tilde{\tau}) = x(\tau), \quad \dot{x}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} x(\tau) = \tilde{R}(\tau)^{-1} \dot{x}(\tau) \quad \text{and} \quad d\tilde{\tau} = \tilde{R}(\tau) d\tau. \quad (3.23)$$

The $\mathcal{R}$-fixed points define a natural embedding $M \hookrightarrow \mathcal{LM}$ given by the constant paths (zero modes) $x(\tau) = x_0 \in M$. The quotient of $\mathcal{LM}$ by $\mathcal{R}$ is singular. But if we divide $\mathcal{L}_i M$ and $\mathcal{L}_e M$ by the action of $\mathcal{R}$, we obtain the space of singular oriented knots, $\mathcal{K}_i M$, and the space of oriented knots, $\mathcal{K}_e M$ [10]. Our constructions work on both types of knot spaces, and we will therefore simply write $\mathcal{KM}$, leaving the choice of the type of knots to the reader. However, when referring to “knots” and “knot spaces” we will always mean oriented knots.

In our description of knot spaces, we will still use the local coordinates from the description of loop space. To do this, we will have to make sure that all formulas are invariant under reparameterizations. At infinitesimal level, reparameterizations are generated by the vector fields

$$R = \oint d\tau \ R(\tau) \dot{x}^{i\tau} \frac{\delta}{\delta x^{i\tau}}. \quad (3.24)$$

A smooth functional $f$ on $\mathcal{KM}$ therefore satisfies

$$\dot{x}^{i\tau} \frac{\delta}{\delta x^{i\tau}} f = 0. \quad (3.25)$$

This restricts the tangent bundles of the knot spaces, if their local sections are regarded as derivations acting on functionals. In particular, we impose the relations

$$\dot{x}^{i\tau} \frac{\delta}{\delta x^{i\tau}} = 0 \quad \text{and} \quad \delta x^{i\tau} \dot{x}_{i\tau} = 0. \quad (3.26)$$

For the second relation, we assumed a metric on knot space induced from the target manifold $M$ to raise and lower indices. Because of these relations, differential forms on knot spaces are of a special form. For example, a one-form $\alpha$ can be written as

$$\alpha(x) = \oint d\tau \ \alpha_{[i_1 i_2]}(x) \dot{x}^{i_1\tau} \delta x^{i_2\tau}, \quad (3.27)$$

where $\alpha_{[i_1 i_2]}(x) = -\alpha_{[i_2 i_1]}(x)$. Note that the identification of $T\mathcal{KM}$ with $\mathcal{KTM}$ does not hold for knot spaces, as reparametrization transformations act differently on both spaces.
3.7. Transgression

The transgression map

\[ \mathcal{T} : \Omega^{k+1}(M) \rightarrow \Omega^k(\mathcal{L}M) \]  
(3.28)

is defined via the correspondence

\[ \begin{array}{ccc}
\mathcal{L}M \times S^1 & \xrightarrow{ev} & \mathcal{L}M \\
\downarrow pr & & \downarrow pr \\
M & \xrightarrow{ev} & \mathcal{L}M 
\end{array} \]  
(3.29)

between \( M \) and its free loop space \( \mathcal{L}M \). Here \( ev \) is the evaluation map of the loop \( x(\tau_0) \in M \) at the given angle \( \tau_0 \in S^1 \), and \( pr \) is the obvious projection. The transgression map is the composition of the pullback along \( ev \) with the pushforward \( pr \) given by integration over the fiber \( S^1 \) of \( pr \). It produces the loop space \( k \)-form

\[ (\mathcal{T} \alpha)_x = \int d\tau \ i_\mathcal{L}(ev^* \alpha) \quad \text{for} \quad \alpha \in \Omega^{k+1}(M), \]  
(3.30)

where \( i_\mathcal{L} \) is contraction with the natural tangent vector \( \dot{x}(\tau) \) to a loop \( x : S^1 \rightarrow M \). Explicitly, in local coordinates \( x^i(\tau) \) we have

\[ \alpha = \frac{1}{(k+1)!} \alpha_{i_1...i_{k+1}}(x) \ dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} \]
\[ \mapsto \mathcal{T} \alpha := \int d\tau \frac{1}{k!} \alpha_{i_1...i_{k+1}}(x(\tau)) \dot{x}^{i_{k+1}} \delta x^{i_1} \tau \wedge \cdots \wedge \delta x^{i_k} \tau. \]  
(3.31)

Differential forms on loop space which are in the image of the transgression map and therefore of the above form are called \textit{ultra-local}.

The image of a transgression is invariant under the group of loop reparameterizations \( \mathcal{R} \). Moreover, the transgression map commutes with the exterior derivative, i.e., \( \delta(\mathcal{T} \alpha) = \mathcal{T}(d\alpha) \). In fact, it descends on integer cohomology to a homomorphism

\[ \mathcal{T} : H^{k+1}(M, \mathbb{Z}) \rightarrow H^k(\mathcal{L}M, \mathbb{Z}). \]  
(3.32)

For \( k = 1 \) there is a natural transgression map which computes the holonomy of a connection on a line bundle over \( M \). For \( k = 2 \) transgression takes a Dixmier-Douady class on a manifold \( M \) to a first Chern class on the free loop space \( \mathcal{L}M \); put differently, a bundle gerbe \( \mathcal{G} \) on \( M \) gives rise to a line bundle \( \mathcal{T} \mathcal{G} \) on \( \mathcal{L}M \).\[ ^{15} \] In particular, transgression can also be refined to a homomorphism on differential cohomology \( \tilde{H}^3(M) \rightarrow \tilde{H}^2(\mathcal{L}M) \). We can thus take a prequantum gerbe on \( M \) and map it to a prequantum line bundle on \( \mathcal{L}M \), which can be quantized in the usual way.

We can restrict the image of the transgression to the open subsets \( \mathcal{L}_i M \) and \( \mathcal{L}_e M \) of \( \mathcal{L}M \) consisting of restricted immersions and embeddings. As the transgression map is invariant

\[ ^{15} \text{See Appendix B for the pertinent definitions regarding gerbes.} \]
under actions of the group of reparameterizations $\mathcal{R} = \text{Diff}^+(S^1)$, the restricted image of a transgression descends to a differential form on the corresponding knot spaces.

It should be stressed, however, that the transgression map is not surjective. In particular, one-forms on knot space are of the form $(3.27)$ but this does not imply they are in the image of the transgression map: $\alpha_{[i_1 i_2]}(x) / (x(\tau))$ is not necessarily of the form $\alpha_{[i_1 i_2]}(x(\tau))$.

3.8. Poisson algebras on knot spaces

Every loop space $\mathcal{L}M$ is naturally a symplectic manifold with local symplectic potential $\vartheta = \oint d\tau \dot{x}_i \delta x^{i\tau}$. The corresponding two-form $\omega = \delta \vartheta$ is closed and it is weakly symplectic in the sense that it defines an injection $T\mathcal{L}M \to T^*\mathcal{L}M$ with dense image.

On the knot space $\mathcal{K}M$, however, the symplectic potential $\vartheta$ vanishes and we have to find a different source for a symplectic structure. The most natural way is to assume a 2-plectic structure on $M$, that we can transgress to a symplectic structure on the knot space. Given a closed non-degenerate three-form $\varpi$ on $M$, we define a symplectic structure $\omega$ on $\mathcal{K}M$ via the transgression

$$\omega = T\varpi = \oint d\tau \iota_x (ev^*\varpi).$$

(3.33)

As the transgression map commutes with the exterior derivative, this form is closed. Moreover, its kernel consists of vector fields

$$X = \oint d\tau \chi(\tau) \dot{x}^{i\tau} \frac{\delta}{\delta x^{i\tau}}$$

(3.34)

which generate reparameterizations and are therefore excluded from the tangent bundle on knot space. On $T\mathcal{K}M$, the two-form $\omega$ is therefore non-degenerate and defines a symplectic structure.

We can now invert $T\varpi$ to define a Poisson bracket on knot space $\mathcal{K}M$. The result is a global bivector field $\Pi \in C^\infty(\mathcal{L}M, \wedge^2 \mathcal{L}TM)$ with vanishing Schouten bracket. Locally, we have

$$\oint d\tau \Pi^{i\sigma,j\tau} \omega_{j\tau,k\rho} = \delta^i_k \delta(\sigma - \rho),$$

(3.35)

where $\Pi^{i\sigma,j\tau}$ and $\omega_{j\tau,k\rho}$ are the components of the bivector field $\Pi$ and of the two-form $\omega = T\varpi$. In these components, the Poisson bracket reads as

$$\{f, g\}_\Pi := \oint d\sigma \oint d\tau \Pi^{i\sigma,j\tau} \left( \frac{\delta}{\delta x^{i\sigma}} f \right) \left( \frac{\delta}{\delta x^{j\tau}} g \right)$$

(3.36)

for loop space functionals $f, g \in C^\infty(\mathcal{L}M)$.

The hemi-bracket and semi-bracket that we defined in §3.3 are mapped to ordinary Poisson brackets on loop space under transgression. Recall that the 2-plectic structure yields Poisson-like brackets (3.9) of one-forms on $M$. The transgression map now converts these to the Poisson bracket on loop space as

$$T(\{\alpha, \beta\}_h, \varpi) = T(\{\alpha, \beta\}_s, \varpi) = \{T\alpha, T\beta\}_{(T\varpi)^{-1}},$$

(3.37)

27
which follows from the observation $T(\iota_{X_\alpha} \varpi) = \iota_{X_{\alpha T}} (T \varpi)$ and that $T(df) = \delta(Tf) = 0$ as $Tf = 0$ for $f \in \mathcal{C}^\infty(M)$. This feature can be regarded as further motivation for using loop spaces or knot spaces.

With these structures inherited on the loop space $L \mathcal{M}$, we may formally follow the recipes of §2. Alternatively, we could have started from a Nambu-Poisson structure induced by a trivector field $\pi$ on $M$. The key feature here is that the local quantity $\Pi^{ij} = \pi^{ijk} \dot{x}_k$ behaves like an ordinary Poisson structure.

### 3.9. Kähler geometry of knot spaces of three-dimensional 2-plectic manifolds

Let us now restrict to Riemannian three-dimensional manifolds $M$. In these cases there is a natural complex structure on the corresponding knot space $K \mathcal{M}$ [58]: The tangent space $T_x K \mathcal{M}$ restricted to a point $x(\tau)$ on the loop $x$ is the orthogonal complement of the tangent vector $\dot{x}(\tau)$ in $T_x(x(\tau)) M$ and therefore isomorphic to the plane $\mathbb{R}^2$. The tangent vector to the loop provides an axis of rotation on this plane, and we define the action of an almost complex structure $J$ as the rotation of the restricted vector around this axis by angle $\pi / 2$. This action clearly extends smoothly to the whole of $T_x K \mathcal{M}$ and we have $J^2 = -\text{id}$.

In local coordinates $x^{i\tau}$, we have
\[
J X = J \oint d\tau \; X_i^{i\tau} \frac{\delta}{\delta x^{i\tau}} = \oint d\tau \; \tilde{\varepsilon}^{ijk} X_j \frac{\delta}{\delta x^{i\tau}} x^{j\tau} \frac{\delta}{\delta x^{i\tau}}, \tag{3.38}
\]
where indices are raised and lowered with the metric $g$ on $M$, e.g., $X_j := g_{ij}(x(\tau)) X^{i\tau}$, and $\tilde{\varepsilon}^{ijk}$ is the “metric-corrected” Levi-Civita symbol given by
\[
\tilde{\varepsilon}_{ijk} = \det(g)^{1/2} \varepsilon_{ijk} \quad \text{and} \quad \tilde{\varepsilon}^{ijk} = \det(g)^{-1/2} \varepsilon^{ijk} = \det(g)^{-1} \varepsilon^{ijk}. \tag{3.39}
\]

A straightforward calculation confirms $J^2 = -\text{id}$. It should be stressed that this almost complex structure is not integrable, in the sense that no non-empty subset of a complex Fréchet space is biholomorphic to an open subset. However, it is formally integrable in the sense that the corresponding Nijenhuis tensor vanishes. Formal integrability is in fact sufficient for the introduction of a Dolbeault differential and the definition of holomorphic functions.

One can now show that $\omega = T \varpi$ is of type $(1, 1)$ with respect to the complex structure $J$, i.e., $\omega(J X_1, J X_2) = \omega(X_1, X_2)$. Furthermore, the induced inner product $\langle -, - \rangle$ on $TK \mathcal{M}$ defined in (3.21) satisfies
\[
\langle X_1, X_2 \rangle = \omega(X_1, J X_2), \tag{3.40}
\]
because the volume form $\varpi$ on $M$ originates from the Riemannian metric $g$ on $M$. Altogether, we have turned the knot space $K \mathcal{M}$ into a Kähler manifold, up to technical issues associated with the notions of integrability of the complex structure. For further details on symplectic and complex structures on loop spaces and knot spaces, see [10] and references therein.
3.10. Prequantization of symplectic groupoids on knot spaces

Our aim is to map the problem of quantizing a 2-plectic manifold \((M, \varpi)\) onto a geometric quantization problem on the knot space \(K\mathcal{M}\) of \(M\). Thus far we have established the transgression map \([10]\) that takes a prequantum gerbe over a manifold \(M\) to a prequantum line bundle over its knot space \(K\mathcal{M}\). Let us discuss this point in some more detail.

The 2-plectic structure on \(M\) is specified by a closed, non-degenerate globally defined three-form \(\varpi\); if \(\varpi \in \Omega^3_{\text{cl}}(M)\), then \(\varpi\) arises as the curvature of the connective structure of a gerbe \(\mathcal{G}\) represented by a smooth Deligne two-cocycle \((g_{abc}, A_{ab}, B_a)\) with respect to an open cover \(U = (U_a)\) of \(M\) (see Appendix B). We assume that the cover \(U\) is fine enough for \(U = (U_a) := (\mathcal{K}U_a)\) to cover the free knot space \(\mathcal{K}M\). The transgression map then yields the closed symplectic two-form \(\omega := \mathcal{T}\varpi\) on knot space \(\mathcal{K}M\). The corresponding smooth hermitian line bundle \(\mathcal{T}\mathcal{G}\) over \(\mathcal{K}M\) has fiber \(\mathcal{T}\mathcal{G}_x\) over a circle \(x : S^1 \hookrightarrow M\) equal to the set of isomorphism classes of flat trivializations of the pullback \(x^*\mathcal{G}\) of the gerbe \(\mathcal{G}\) to \(S^1\). It comes with a product, inherited from the bundle gerbe product: If two knots \(x_1\) and \(x_2\) are composable, then \(\mathcal{T}\mathcal{G}_{x_1x_2} \cong \mathcal{T}\mathcal{G}_{x_1} \otimes \mathcal{T}\mathcal{G}_{x_2}\). The connection \(\vartheta_\mathcal{G} \in \Omega^1(\mathcal{T}\mathcal{G})\) whose curvature is \(\mathcal{T}\varpi\) can be written in the patch \(\mathcal{K}U_a\) as

\[
\vartheta_\mathcal{G} = \delta z_a + \mathcal{T}B_a \quad \text{on} \quad \mathcal{K}U_a,
\]

where \(z_a \in \mathbb{C}\) are coordinates on the fibers. Since \(B_a - B_b = dA_{ab}\) on the intersection \(U_{ab} = U_a \cap U_b\), one has

\[
z_a - z_b = \mathcal{T}A_{ab} \quad \text{on} \quad \mathcal{K}U_{ab}.
\]

The \(U(1)\)-action on the fibers is given by the gauge group of the gerbe, described in Appendix B, i.e., the Cheeger-Simons cohomology group \(\check{H}^2(M)\) of line bundles with connection \((E, a)\) modulo gauge transformations; from (3.41) it follows that the fiber coordinate \(z \in \mathbb{C}\) shifts under this action as \(z \mapsto z + \mathcal{T}a\). Since \(\check{H}^2(S^1) \cong U(1)\), the pullback \(x^*E\) of the line bundle \(E \to M\) to a knot \(x \cong S^1\) is a trivial line bundle with flat connection, i.e., the pullback \(x^*\mathcal{G}\) of the gerbe \(\mathcal{G}\) over \(M\) to a knot is a principal homogeneous space for \(U(1)\), and \(\mathcal{T}\mathcal{G}\) becomes a principal \(U(1)\)-bundle on \(\mathcal{K}M\).

Instead of directly quantizing the prequantum gerbe \(\mathcal{G}\) over \(M\), we will use the Poisson structure on \(\mathcal{K}M\) induced by \(\mathcal{T}\varpi\) and apply Hawkins’ groupoid approach to quantization. We will make no attempt here to develop a proper rigorous theory of symplectic groupoids over knot spaces and their quantization, paralleling the description of the previous section; this is a highly involved technical task that could presumably invoke the setting of diffeological spaces. Our main concern here will be to understand what sort of noncommutative knot spaces arise, and how they capture the quantum geometry of closed strings and M-branes in background fluxes. In all examples that we treat later on the groupoids over knot space \(\mathcal{K}M\) will be of the form \(\mathcal{K}\Sigma\) for a (not necessarily symplectic) groupoid \(\Sigma \to M\); this will simplify the development somewhat as the appropriate groupoid structures on \(\mathcal{K}\Sigma \to \mathcal{K}M\) can be obtained by using functoriality of the transgression map. But even then, the complete construction of appropriate analogs of entities like a convolution...
$C^*$-algebra appear to be prohibitively complicated. In particular, it is well-known that there are no reparametrization invariant measures on the loop space $LM$, and suitable convolution products should presumably be constructed in some way from the quasi-invariant Wiener measure $Dx(\tau) \exp \left( -\frac{1}{2} \oint d\tau |\dot{x}(\tau)|^2 \right)$ for the quantum mechanics of free geodesic motion on $M$.

Our choice of symplectic groupoid will be partly dictated by the existence of suitable co-isotropic restrictions of the velocity vectors $\dot{x}(\tau)$ to one-dimensional sub-bundles of $TM \subset KTM$ for which our quantization will reduce to a known quantization of the induced Poisson structure on a corresponding codimension one submanifold $N$ of $M$. We may then mimic the symplectic groupoid quantization of $N$ by using transgression techniques.

### 3.11. Regression

Let $(KS, \omega)$ be a symplectic groupoid integrating the Poisson manifold $(KM, \Pi)$. We will investigate what the symplectic groupoid quantization corresponds to on the original groupoid $\Sigma \rightrightarrows M$. As a start, one can ask to what corresponds the prequantization data of $(KS, \omega)$; naturally, the answer would be a particular prequantum bundle gerbe on $\Sigma$ with connective structure. For this, one requires an inverse to the transgression map $T$, called a regression map, whose details have been recently worked out by Waldorf [59] (see also [61]). Note that because the map $T$ is not surjective, the inverse of the transgression exists only on the image of $T$. We will briefly sketch the construction of the connective structure $T^{-1}\omega$ on the bundle gerbe, glossing over many of the intricate technical details. For simplicity, we perform the inversion on loop space.

For this, we embed loop space $\mathcal{L}\Sigma$ into path space $\mathcal{P}\Sigma$; after factoring out reparametrizations and thin homotopy, the path space is naturally a groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$ with source and target fibrations $\text{ev}_0, \text{ev}_1 : \mathcal{P}\Sigma \to \Sigma$ which evaluate a path $g(\tau) \in C^\infty([0, 1], \Sigma)$ at its initial and final endpoints $g(0), g(1) \in \Sigma$, and groupoid multiplication given by concatenation of paths. In what follows we take $\mathcal{P}\Sigma$ to be the space of smooth oriented paths with the same initial points $g(0) = g_0 \in \Sigma$. It can be shown that the resulting regression map is independent of the choice of $g_0$. The fibered product $\mathcal{P}\Sigma^{[2]}$ over $g \in \Sigma$ consists of pairs of paths between $g_0$ and $g$ in $\Sigma$. By reversing the orientation of one of the paths, such a pair may be identified with a loop based at $g_0$; to simplify notation, in the ensuing construction we will take $\mathcal{L}\Sigma$ to be the space of smooth loops in $\Sigma$ based at $g_0$. Then we can identify $\mathcal{P}\Sigma^{[2]}$ with $\mathcal{L}\Sigma$. A technical subtlety here is that the loop resulting from concatenation of two paths may not be smooth at the two points where the paths are composed. This problem is rectified by reparametrizing the paths $g(\tau)$ in $\mathcal{P}\Sigma$ such that they have “sitting instants” at these points, i.e., they are locally constant in neighborhoods of $\tau = 0$ and $\tau = 1$. On $\mathcal{L}\Sigma$ this structure is defined in terms of an integral over $[0, 1]$ which is invariant under such reparametrizations. In particular, differential forms $\eta \in \Omega^k(\mathcal{L}\Sigma)$ which are compatible with the multiplication in the smooth path groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$ respect composability of paths; if in addition they are reparametrization invariant then we may define the regression
form $T^{-1} \eta \in \Omega^{k+1}(\Sigma)$ at the point $g_0 \in \Sigma$ by

$$(T^{-1} \eta)_{g_0}(v_0, v_1, \ldots, v_k) := \lim_{\sigma \to 0^+} \frac{1}{\sigma} \eta_{g(\tau)|_{[-1,0]}}(\tilde{v}_1, \ldots, \tilde{v}_k) |_{g(0) = g_0, \dot{g}(0) = v_0}$$  \hspace{2cm} (3.43)$$

where $v_i \in T_{g_0} \Sigma$, $g(\tau) \in \mathcal{P} \Sigma$ is any smooth path such that $g(\tau)|_{[-1,0]}$ is thin-homotopic to the constant path $g_0$, and $\tilde{v}_i \in \mathcal{L}T_{g_0} \Sigma$ are extensions of the tangent vectors $v_i$ along the path $g(\tau)$ such that $\eta_{g(\tau)|_{[-1,0]}}(\tilde{v}_1, \ldots, \tilde{v}_k) = 0$, see e.g. [17]. This map also commutes with the exterior derivative, i.e., $d(T^{-1} \eta) = T^{-1} (\delta \eta)$, where the exterior derivative $\delta$ on $\mathcal{P} \Sigma$ respects the groupoid structure by including boundary contributions from evaluation at the endpoints.

Let $\mathcal{L}$ be a hermitian line bundle over $L \Sigma$ with an associative fiberwise product [59]. Regression then maps the line bundle $\mathcal{L} \to L \Sigma$ to the bundle gerbe $T^{-1} \mathcal{L}$ on $\Sigma$ given by

$$\begin{align*}
\begin{array}{c}
\mathcal{L} \\
\downarrow \\
L \Sigma
\end{array}
\begin{array}{c}
\downarrow pr_1 \\
\downarrow pr_2
\end{array}
\begin{array}{c}
\mathcal{P} \Sigma \\
\downarrow ev_1 \\
\downarrow \Sigma
\end{array}
\end{align*}$$  \hspace{2cm} (3.44)$$

Note that this construction holds without the need of introducing a connection [60]; however, in this case the gerbe is unique only up to thin homotopies of paths and there is no distinguished inverse map in general.

Let $\nabla = \delta + \vartheta$ be a reparametrization invariant connection on $\mathcal{L} \to L \Sigma$ of curvature $F_\nabla = -2\pi i \omega$ which commutes with the multiplication on $\mathcal{L}$. Then we may apply (3.43) to $F_\nabla \in \Omega^2(L \Sigma)$ to get a three-form $H := T^{-1} F_\nabla \in \Omega^3(\Sigma)$. We define a curving $B \in \Omega^2(\mathcal{P} \Sigma)$ by pulling $H$ back along the evaluation map $ev : \mathcal{P} \Sigma \times [0,1] \to \Sigma$ and integrating over the fiber to get

$$B = \int_0^1 d\tau \; \iota_\dot{x} ev^*(T^{-1} F_\nabla) \; .$$  \hspace{2cm} (3.45)$$

One readily checks the requisite compatibility equations

$$pr_1^*(B) - pr_2^*(B) = F_\nabla \quad \text{and} \quad \delta B = ev_1^*(T^{-1} F_\nabla)$$  \hspace{2cm} (3.46)$$

over $L \Sigma$ and $\mathcal{P} \Sigma$, respectively. Note that the two-form $B$ depends only on the curvature of the line bundle $\mathcal{L}$; see [59] for further technical details and requirements of this construction.

The regression map $T^{-1}$ provides a functorial isomorphism between the differential cohomology group $\hat{H}^3(\Sigma)$ and the subgroup of $\hat{H}^2(L \Sigma)$ consisting of classes of line bundles on $L \Sigma$ with fiberwise product and reparametrization invariant compatible connection. It gives a functorial equivalence between flat bundles on $L \Sigma$ and flat gerbes on $\Sigma$. It also restricts to a bijection between stable isomorphism classes of trivial gerbes with connection and isomorphism classes of trivial bundles with fiberwise product and reparametrization invariant compatible connection; the transgression map (3.30) in this case induces a group
isomorphism between the gauge equivalence classes $\Omega^2(\Sigma)/\Omega^2_{\text{cl,cl}}(\Sigma)$ of topologically trivial $B$-fields $b$ and the subgroup of $\Omega^1(\mathcal{L}\Sigma)/\delta C^\infty(\mathcal{L}\Sigma, \mathbb{R})$ consisting of reparametrization invariant compatible topologically trivial gauge fields $a$ modulo compatible gauge transformations, with the inverse map given by (3.43). Note that here we implement the distinction stable isomorphism (as opposed to actual isomorphism) in order to account for the fact that gauge transformations of gerbes consist themselves of line bundles and so are naturally compared via bundle morphisms, see Appendix B; put differently, bundle gerbes are objects of a 2-category, in contrast to line bundles which are objects of a category.

3.12. A generic example

A representative class of symplectic groupoids that we shall frequently encounter are cotangent groupoids with arrow set the cotangent bundle $\Sigma = T^*M$ over a manifold $M$ of dimension $d$, the canonical symplectic structure, and various choices for the structure maps $(s, t, \mathbb{1}, m)$. The cotangent bundle $T^*KM$ is the phase space of a closed string sigma-model on $S^1 \times \mathbb{R}$, but in the following we will restrict ourselves to the sub-bundle $\hat{T}^*KM \subset T^*KM$ which is defined analogously to the tangent bundle, see §3.5. The cotangent bundle over the knot space of $M$ is naturally an infinite-dimensional symplectic manifold with symplectic two-form written in local canonical coordinates $(x, p)$ on $\Sigma = T^*M$ as

$$\omega = \oint d\tau \, \delta x^{ir} \wedge \delta p_{ir}$$

at a given knot $x : S^1 \hookrightarrow M$. The corresponding Poisson brackets should be evaluated on smooth global sections of the (trivializable) infinite jet bundle $J^\infty(\Sigma \times S^1) \to S^1$, i.e., equivalence classes (or “jets”) of smooth sections of the trivial fibration $\Sigma \times S^1 \to S^1$ whose Taylor coefficients coincide to all orders, with induced coordinates $(\tau, x, p, \dot{x}, \dot{p}, \ldots)$; if we wish to consider only ultra-local functionals that do not depend explicitly on the loop parameter $\tau$, then we should restrict to functionals on the infinite jet space $J^\infty(\mathcal{L}\Sigma) \subset J^\infty(\Sigma \times S^1)$ consisting of jets of smooth maps $S^1 \to \Sigma$. There is a symplectic potential $\vartheta \in \Omega^1(\mathcal{L}\Sigma)$, with $\omega = \delta \vartheta \in \Omega^2_{\text{cl,cl}}(\mathcal{L}\Sigma)$, given by

$$\vartheta = \oint d\tau \, p(\tau) ,$$

where the globally defined momentum

$$p(\tau) = p_{ir} \, \delta x^{ir}$$

is a normal covector field on the knot $x$, i.e., a section $p \in \hat{T}^*_x \mathcal{L}M = \mathcal{C}^\infty(S^1, x^*T^*M)$. Since $\omega$ is exact, it is the curvature of the trivial prequantum line bundle $\mathcal{L} = \hat{T}^*KM \times \mathbb{C}$, which naturally has a trivial fiberwise product (induced by multiplication in $\mathbb{C}$); the wave functionals of the quantum sigma-model are sections of this bundle.
The corresponding bundle gerbe is also trivial and up to thin homotopy it is given by

\[
\begin{array}{ccc}
\hat{T}^*KM \times \mathbb{C} & \rightarrow & \hat{T}^*PM \times \mathbb{C} \\
\downarrow & & \downarrow \\
\hat{T}^*KM & \xrightarrow{\text{pr}_1} & \hat{T}^*PM \\
\downarrow & & \downarrow \text{ev}_1 \\
T^*M & & \\
\end{array}
\]

(3.50)

Even though the symplectic potential (3.48) is not in the image of the transgression map, let us apply the regression formula (3.43) to it. This yields a quadratic form \( \tilde{b} := \mathcal{T}^{-1} \vartheta \) given at a point \((x,p) \in T^*M\) by

\[
\tilde{b}_{(x,p)}(v, \frac{\partial}{\partial p_i}) = 0 \quad \text{and} \quad \tilde{b}_{(x,p)}(v, \frac{\partial}{\partial x^i}) = p_i
\]

(3.51)

for all tangent vectors \( v \in T_{(x,p)}\Sigma \). Note that \( \tilde{b} \) only defines a section of \( T^*\Sigma \otimes T^*\Sigma \): Although \( \vartheta \) commutes with the multiplication on \( \mathcal{L} \), it is not a reparametrization invariant one-form on \( \hat{T}^*KM \) and so its regression \( \mathcal{T}^{-1} \vartheta \) does not automatically antisymmetrize to give an element of \( \Omega^2(\Sigma) \). On the other hand, as a section of \( \mathcal{K}\Sigma \) the momentum \( p(\tau) \) transforms as in (3.23) and the symplectic potential (3.48) is reparametrization invariant; when viewed in this way the antisymmetrization of the section \( \tilde{b} \) yields the two-form

\[
b = - \sum_{i,j=1}^{d} \left( \frac{1}{2} (p_i - p_j) \, dx^i \wedge dx^j + p_i \, dx^i \wedge dp_j \right)
\]

(3.52)

with curvature

\[
H = db = \sum_{i,j=1}^{d} \left( dx^i \wedge dx^j \wedge dp_i + dx^i \wedge dp_i \wedge dp_j \right).
\]

(3.53)

Note that we may shift \( b \) by any closed two-form \( \beta \in \Omega^2(\Sigma) \), \( db = 0 \), which transgresses to a gauge transformation of the symplectic potential

\[
\vartheta_{\beta} = \oint d\tau \left( p(\tau) + \iota_{\omega} ev^* \beta \right).
\]

(3.54)

This modification generally affects the quantization of \( \hat{T}^*KM \); it depends on the cohomology class of a flat gerbe 1-connection in \( H^2(M, \mathbb{U}(1)) \).

### 3.13. Winding sectors and Bohr-Sommerfeld 2-leaves

There are two main modifications which must be made when the 2-plectic manifold \( M \) is multiply connected. They can be deduced by noting that the homotopy groups of the loop space are given by

\[
\pi_n(\mathcal{L}M) \cong \pi_{n+1}(M) \times \pi_n(M)
\]

(3.55)
whenever \( M \) is connected.

Firstly, setting \( n = 0 \) in (3.55) we note that in this case the knot space \( \mathcal{K}M \) (and the underlying loop space \( \mathcal{L}M \)) is disconnected with connected components \( \mathcal{K}_wM \), called “winding sectors”, parametrized by the set \( w \in \pi_1(M)^\sim \) of conjugacy classes of the fundamental group of \( M \). These winding modes appear as zero modes for the velocity vectors such that \( \oint d\tau \dot{x}(\tau) = w \). Formally, the twisted convolution \( C^* \)-algebra of the polarized symplectic groupoid on \( \mathcal{K}M \) then decomposes as a graded vector space

\[
C^*_r(\mathcal{K}_w\Sigma/\mathcal{P}, \sigma) = \bigoplus_{w \in \pi_1(M)^\sim} C^*_r(\mathcal{K}_w\Sigma/\mathcal{P}, \sigma),
\]

where \( C^*_r(\mathcal{K}_w\Sigma/\mathcal{P}, \sigma) \) is the space of polarized sections of a line bundle \( \mathcal{L}_w \to \mathcal{K}_w\Sigma \) in the topological sector \( w \in \pi_1(M)^\sim \). The collection of these line bundles \( \mathcal{L} = (\mathcal{L}_w)_{w \in \pi_1(M)^\sim} \) is regarded as a single line bundle on \( \mathcal{K}\Sigma \). Hence on loop space one quantizes not just a functional \( f \) but rather a pair \( (f, w) \in C^\infty(\mathcal{K}M) \times \pi_1(M)^\sim \). In general the multiplication in \( C^*_r(\mathcal{K}_w\Sigma/\mathcal{P}, \sigma) \) does not preserve the grading; instead, the group multiplication in \( \pi_1(M) \) gives \( C^*_r(\mathcal{K}_w\Sigma/\mathcal{P}, \sigma) \) the structure of an algebra over \( C^*_r(\mathcal{K}_0\Sigma/\mathcal{P}, \sigma) \).

Secondly, setting \( n = 1 \) in (3.55) we see that we must impose a Bohr-Sommerfeld quantization condition over the leaves of the polarization as we did in \( \$2.6 \), but now parametrized by conjugacy classes \( [S] \in \pi_2(M)^\sim \) of the second homotopy group of \( M \) in each winding sector \( w \in \pi_1(M)^\sim \). For this, we construct a subvariety \( \Sigma_0 \subseteq \Sigma \) by demanding that the holonomy \( \text{hol}_{\gamma_w}(\vartheta) = 1 \) for all loops \( \gamma_w \) on \( \mathcal{K}_w\Sigma_0 \) and for all \( w \in \pi_1(M)^\sim \). If the symplectic potential \( \vartheta \) happens to lie in the image of the transgression map, then this condition should be compatible with the regressed picture: Under regression, the trivialization of \( \mathcal{L}_w \) over \( \mathcal{K}_w\Sigma_0/\mathcal{P} \) determines a gauge equivalence class of trivial gerbes with connective structure over \( \Sigma_0/\mathcal{P} \). To fix a trivialization, we note that if \( \gamma_w \) is the image of the smooth map \( \ell_w(\rho) : S^1 \to \mathcal{K}_w\Sigma_0 \), then the image of the adjoint map \( \ell_w'(\tau, \rho) := \ell_w(\rho)(\tau) \) is a torus \( T^{2}_{\gamma_w} \subseteq S^1 \times S^1 \), or “2-loop”, in \( \Sigma_0 \subseteq \Sigma \); conversely, any smooth 2-loop \( T^2 \subseteq \Sigma_0 \) defines a loop on \( \mathcal{K}_w\Sigma_0 \) for each winding mode \( w \in \pi_1(M)^\sim \). Then the holonomy of the symplectic potential around the loop \( \gamma_w \) is equal to the 2-holonomy of the regressed connective structure on the gerbe \( \mathcal{T}^{-1}L_w \) around the associated 2-loop \( [59],^16 \)

\[
\text{hol}_{\gamma_w}(\vartheta) = \text{hol}_{T^{2}_{\gamma_w}}(B_w) := \exp \left( 2\pi i \oint_{T^{2}_{\gamma_w}} B_w \right),
\]

and hence Bohr-Sommerfeld quantization on \( \mathcal{K}_w\Sigma \) requires that \( \oint_{T^2} B_w \in \mathbb{Z} \) for all 2-loops \( T^2 \subseteq \Sigma_0 \).

4. Quantization of vector spaces

In these remaining sections we will consider three explicit examples of the formalism described in the preceding sections. Here we will derive the Moyal product on the Euclidean space \( \mathbb{R}^d \), which was reviewed in \( \$2.1 \), using the groupoid approach to quantization. We shall then extend it to the 2-plectic manifold \( \mathbb{R}^3 \) endowed with its natural volume form.

---

^16See Appendix B for the precise expression for the 2-holonomy of a topologically non-trivial \( B \)-field.
### 4.1. Symplectic geometry

The following construction is an extended version of that presented in [5, §6.2]. As we will draw parallels to our new construction in the next subsection, we will be very explicit in our discussion. We consider the Poisson manifold given by the real vector space $V = \mathbb{R}^d$ with constant Poisson structure $\pi^{ij} = -\pi^{ji}$, $i,j = 1, \ldots, d$. As discussed in §2.2, the Poisson structure on $V$ turns the cotangent bundle $T^*V$ into a Lie algebroid. This Lie algebroid can be integrated to a Lie groupoid if and only if there is an integrating symplectic groupoid for $V$ as defined in Appendix A [62].

The preferred choice for such an integrating symplectic groupoid is $(\Sigma, \omega)$, where $\Sigma$ is the cotangent bundle $T^*V \cong V \times V^*$. We will use dual Cartesian coordinates $x^i$ on $V$ and $p_i$ on $V^*$. In these coordinates, the symplectic two-form $\omega$ is given by $\omega = -(dp_i dx^j) = dx^i \wedge dp_i$. The groupoid structure on $\Sigma$ depends on the matrix $\pi = (\pi^{ij})$, regarded as a linear map $\pi : V^* \to V$. The object inclusion map of the groupoid $\Sigma$ is trivially given by $\mathbb{1}_x : (x^i) \mapsto (x^i, x^i)$, where we used the isomorphism between $V$ and its dual $V^*$. We can encode an arrow $\alpha \in \Sigma$ by a pair $(x^i, p_i)$ using the Bopp shifts

$$\begin{align*}
(x^i + \frac{1}{2} \pi^{ij} p_j) & \quad \mapsto \quad (x^i - \frac{1}{2} \pi^{ij} p_j).
\end{align*} \tag{4.1}
$$

The source and target maps on the groupoid $\Sigma$ are thus

$$s(\alpha) = s(x, p) = (x^i + \frac{1}{2} \pi^{ij} p_j) \quad \text{and} \quad t(\alpha) = t(x, p) = (x^i - \frac{1}{2} \pi^{ij} p_j). \tag{4.2}
$$

Recall that one condition for $\Sigma$ to be an integrating groupoid is that $t : \Sigma \to V$ is a Poisson map. That this is indeed the case follows from a straightforward computation

$$\{t^* f, t^* g\}_{\omega^{-1}} = \left( \frac{\partial t^i}{\partial x^k} \frac{\partial f(y)}{\partial y^i} \frac{\partial y^j}{\partial p_k} + \frac{\partial g(y)}{\partial y^i} \frac{\partial t^j}{\partial x^k} \frac{\partial f(y)}{\partial y^j} \frac{\partial y^i}{\partial p_k} \right) \bigg|_{y = t(x, p)} = t^* \{f, g\}_\pi. \tag{4.3}
$$

The set of composable arrows (2-nerve) $\Sigma_{(2)}$ can be identified with $V \times V^* \times V^*$: An element $(x, p, p')$ represents the concatenation $\alpha_1 \circ \alpha_2$ of two arrows $\alpha_1$ and $\alpha_2$ as

$$\begin{align*}
x^i + \pi^{ij} (p_j + p'_j) & \quad \mapsto \quad x^i + \pi^{ij} (p_j - p'_j) \quad \mapsto \quad x^i - \pi^{ij} (p_j + p'_j).
\end{align*} \tag{4.4}
$$

In coordinates, the projections $\text{pr}_1(\alpha_1 \circ \alpha_2) = \alpha_1$ and $\text{pr}_2(\alpha_1 \circ \alpha_2) = \alpha_2$ are given by

$$\begin{align*}
\text{pr}_1(x, p, p') &= \left( x^i + \frac{1}{2} \pi^{ij} p_j, p'_j \right) \quad \text{and} \quad \text{pr}_2(x, p, p') &= \left( x^i - \frac{1}{2} \pi^{ij} p'_j, p_i \right), \tag{4.5}
\end{align*}
$$

and the groupoid multiplication $m(\alpha_1, \alpha_2) = \alpha_1 \circ \alpha_2$ reads

$$m(x, p, p') = \left( x^i, p_i + p'_i \right). \tag{4.6}$$

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This completes the construction of the relevant structures on the groupoid \( \Sigma \). To verify that \( \Sigma \) integrates \( V \), we have to check that \( \omega = dx^i \wedge dp_i \) is multiplicative for this groupoid structure, i.e., \( \partial^* \omega = 0 \). One readily computes

\[
\begin{align*}
pr_1^* \omega &= dx^i \wedge dp_i + \frac{1}{2} \pi^{ij} dp_i \wedge dp_j, \\
pr_2^* \omega &= dx^i \wedge dp_i - \frac{1}{2} \pi^{ij} dp_i \wedge dp_j, \\
m^* \omega &= dx^i \wedge dp_i + dx^i \wedge dp'_i, \\
\end{align*}
\]

and thus \( \partial^* \omega := pr_1^* \omega - m^* \omega + pr_2^* \omega = 0 \).

The prequantization and polarization of \( \Sigma \) are straightforward: Since \( \omega \) is exact, the prequantum line bundle is the trivial bundle \( \Sigma \times \mathbb{C} \), which we endow with a connection \( \nabla \) of curvature \( F_{\nabla} = -2\pi i \omega \). The projection \( p : \Sigma \to \Sigma/P \), \( p = s - t \) with \( P = \text{span} \{ \partial/\partial x^i \} \) maps \( \Sigma \) to \( V^* \). Note that \( p \) is a fibration of groupoids if we regard \( V^* \) as an additive group.

The twist element \( \sigma = \sigma_\pi \) used in the construction of the twisted convolution algebra can be obtained from an adapted symplectic potential \( \theta \) as described in §2.4 and §2.5. The simplest choice is \( \theta = x^i dp_i \), for which \( \partial^* \theta \) is closed. Moreover, this symplectic potential is perpendicular to the polarization \( P \) and therefore adapted. The twist element \( \sigma_\pi \) on \( (\Sigma/P)_{(2)} = V^* \times V^* \) is obtained from the equation

\[
\sigma_\pi^{-1} d\sigma_\pi = i \partial^* \theta = d \left( -\frac{1}{2} p_i \pi^{ij} p'_j \right),
\]

and up to an irrelevant constant phase we arrive at the group cocycle

\[
\sigma_\pi(p, p') = e^{-\frac{1}{2} p_i \pi^{ij} p'_j}.
\]

A Haar system is given by the translationally invariant Haar measure \( dp \) on \( V^* \). Alternatively, the relevant half-form bundles are

\[
\Omega^{1/2}_\Sigma = \sqrt{\bigwedge^d T^*(V \times V^*)} \quad \text{and} \quad \Omega^{1/2}_P = \sqrt{\bigwedge^d T^*V}.
\]

Altogether, the twisted polarized convolution algebra on \( \Sigma/P \cong V^* \) is then given by functions on \( V^* \) with the convolution product

\[
(\tilde{f} \rtimes_{\pi} \tilde{g})(p) = \int_{V^*} dp' \sigma_\pi(p', p - p') \tilde{f}(p') \tilde{g}(p - p'),
\]

since in this case \( (\Sigma/P)^{pt} = V^* = t^{-1}(t(p)) \) for all \( p \in V^* \). We therefore recover the algebra of functions on the Moyal plane \( V_\pi \) as a quantization of the associated dual Lie algebroid \( A^*(\Sigma/P) = V \).

### 4.2. 2-plectic geometry

We will now generalize the above discussion to the case of the 2-plectic manifold \( (\mathbb{R}^3, \omega) \). In usual Cartesian coordinates, the 2-plectic form reads as \( \omega = \pi^{-1} dx^1 \wedge dx^2 \wedge dx^3 \). To
quantize this space, we switch to its knot space $K\mathbb{R}^3$ which is a symplectic manifold with symplectic form obtained by transgressing the 2-plectic form to get

$$\omega := T\omega = \oint d\tau \omega_{ijk} \dot{x}^{k\tau} \delta x^{i\tau} \wedge \delta x^{j\tau} = \oint d\rho \oint d\tau \omega_{ijk} \dot{x}^{k\tau} \delta(\tau - \rho) \delta x^{i\rho} \wedge \delta x^{j\tau},$$

as explained in §3.8. We define a Poisson structure from the symplectic structure via the formula

$$\{f,g\}_{\omega^{-1}} := \iota_{X_f} \iota_{X_g} \omega$$

for $f, g \in C^\infty(K\mathbb{R}^3)$, where $X_f$ denotes the Hamiltonian vector field defined through $\iota_{X_f} \omega = df$. Here we find

$$\{f,g\}_\pi := \oint d\rho \oint d\tau \delta(\tau - \rho) \pi^{ijk} \delta x^{i\tau} \wedge \delta p^{j\rho},$$

where $\pi^{ijk} = \pi_{\epsilon}^{ijk}$. One readily checks that this Poisson bracket indeed satisfies the Jacobi identity. We shall now use the groupoid approach to quantize the underlying Poisson algebra on knot space.

First, we need an integrating symplectic groupoid. The groupoid quantization of $V = \mathbb{R}^3$ suggests using the (restricted) cotangent groupoid $\hat{T}^*K\mathbb{R}^3$. As before, we describe this space by the standard coordinates $(x^{i\tau}, p^{i\tau})$ on the (restricted) loop space $\hat{T}^*\mathbb{R}^3 = L_1\Sigma$. This is possible as long as all expressions obtained are invariant under the appropriate action of the group of reparametrization transformations $\mathcal{R} = \text{Diff}^+(S^1)$. In particular, the loop space coordinates $x^{i\tau}$ are invariant under actions of $\mathcal{R}$ and the $p^{i\tau}$ transform in the same representation of $\mathcal{R}$ as $\dot{x}^{i\tau}$. In these coordinates the canonical symplectic two-form $\omega$ on $L_1\Sigma$ is given by

$$\omega = \oint d\tau \oint d\rho \delta(\tau - \rho) \delta x^{i\tau} \wedge \delta p^{i\rho},$$

and it can be derived from the symplectic potential

$$\vartheta = \oint d\tau x^{i\tau} \delta p^{i\tau}.$$  

Both of these differential forms are invariant under the group of reparametrization transformations $\mathcal{R}$.

On $L_1\Sigma$, we choose source and target maps invariant under the action of $\mathcal{R}$ which are given by

$$s(x^{i\tau}, p^{i\tau}) = x^{i\tau} + \frac{1}{2} \pi^{ijk} p^{j\tau} \frac{\dot{x}^{k\tau}}{|\dot{x}^{i\tau}|^2},$$

$$t(x^{i\tau}, p^{i\tau}) = x^{i\tau} - \frac{1}{2} \pi^{ijk} p^{j\tau} \frac{\dot{x}^{k\tau}}{|\dot{x}^{i\tau}|^2},$$

where we used the Euclidean metric on $\mathbb{R}^3$ to lower indices as $\dot{x}^{i\tau} := \delta_{ij} \dot{x}^{j\tau}$. Here the target map $t$ is a Poisson map only to lowest order in $\pi$, i.e.,

$$\{t^*f, t^*g\}_{\pi^{-1}} := \oint d\tau \left( \left( \frac{\delta}{\delta x^{i\tau}} t^*f \right) \left( \frac{\delta}{\delta p^{i\tau}} t^*g \right) - \left( \frac{\delta}{\delta x^{i\tau}} t^*g \right) \left( \frac{\delta}{\delta p^{i\tau}} t^*f \right) \right)$$

$$= t^*\{f,g\}_{\pi} + \mathcal{O}(\pi^2) \quad \text{for} \quad f, g \in C^\infty(L_1\mathbb{R}^3).$$
For the moment, let us nevertheless continue and complete the groupoid structure first to lowest order in \( \pi \). The 2-nerve of \( L_3 \Sigma \) is \( K_3 \Sigma(2) = L_3(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)/\mathcal{R} \), and comparison with the case \( V = \mathbb{R}^3 \) suggests using the projections and multiplication maps given by

\[
\begin{align*}
\text{pr}_1(x^{i\tau}, p_{i\tau}, p_{i\tau}') &:= \left( x^{i\tau} + \frac{1}{2} \pi^{ijk} p_{j\tau} \frac{\dot{x}_{k\tau}}{|\dot{x}_{\tau}|^2}, p_{i\tau}' \right), \\
\text{pr}_2(x^{i\tau}, p_{i\tau}, p_{i\tau}') &:= \left( x^{i\tau} - \frac{1}{2} \pi^{ijk} p_{j\tau} \frac{\dot{x}_{k\tau}}{|\dot{x}_{\tau}|^2}, p_{i\tau} \right), \\
m(x^{i\tau}, p_{i\tau}, p_{i\tau}') &:= \left( x^{i\tau}, p_{i\tau} + p_{i\tau}' \right).
\end{align*}
\]

(4.18)

The canonical symplectic structure \( \omega \) on \( L_3 \Sigma \) is also multiplicative only to lowest order in \( \pi \), i.e., \( \partial^* \omega = O(\pi) \). We have therefore arrived at an integrating symplectic groupoid for \( (L_3 \mathbb{R}^3, \pi) \) to lowest order in \( \pi \), and we can now attempt to correct these structures order by order. We will demonstrate this procedure to the first non-trivial order in \( \pi \).

The first step is to ensure that \( t \) is a Poisson map to order \( O(\pi^2) \). To preserve the groupoid structures, we will attempt to do so by replacing \( p_{i\tau} \) in all groupoid maps and the symplectic form by a more general expression \( \tilde{p}_{i\tau} = p_{i\tau} + \pi P^{(1)}_{i\tau} \). Such an expression can indeed be found and it reads as

\[
\tilde{p}_{i\tau} = p_{i\tau} + \pi \left( p_{i\tau} \frac{\ddot{x}_{j\tau}}{3|\dot{x}_{\tau}|^4} + \frac{\epsilon_{ijk}}{|\dot{x}_{\tau}|^4} \left( \frac{1}{2} \frac{\ddot{x}_{j\tau}}{3} p_{k\tau} \frac{\dot{x}_{l\tau}}{3|\dot{x}_{\tau}|^4} + \frac{4}{3} \frac{\ddot{x}_{j\tau}}{3} p_{l\tau} p_{i\tau} \right) \right).
\]

(4.19)

To lowest order in \( \pi \), we can invert this formula to get

\[
p_{i\tau} = \tilde{p}_{i\tau} - \pi \left( \frac{\tilde{p}_{i\tau} \ddot{x}_{j\tau}}{3|\dot{x}_{\tau}|^4} + \ldots \right),
\]

(4.20)

and therefore replacing \( p_{i\tau} \) by \( \tilde{p}_{i\tau} \) corresponds to a mere coordinate change.

The next step requires us to modify the groupoid structure so that \( \omega \) is multiplicative to lowest order in \( \pi \). We have

\[
\partial^* \omega = \text{pr}_1^* \omega + \text{pr}_2^* \omega - m^* \omega = \frac{1}{2} \oint \pi^{ijk} \delta \left( \frac{\dot{x}_{k\tau}}{|\dot{x}_{\tau}|^2} \right) \wedge \delta (\tilde{p}_{i\tau} \tilde{p}_{j\tau}) = \oint \text{d}\tau \, \delta x^{i\tau} \wedge R_{i\tau},
\]

(4.21)

where \( R_{i\tau} \) can be chosen to be a closed one-form of order \( O(\pi) \) which transforms like \( p_{i\tau} \) under actions of the group \( \mathcal{R} \). For closed \( R_{i\tau} \), there is a potential \( R_{i\tau} = \delta r_{i\tau} \) and we can use this to modify the groupoid multiplication according to

\[
m(x, \tilde{p}, \tilde{p}') = (x^{i\tau}, \tilde{p}_{i\tau} + \tilde{p}_{i\tau}' + r_{i\tau}).
\]

(4.22)

Note that both \( s(m(x, \tilde{p}, \tilde{p}')) \) and \( t(m(x, \tilde{p}, \tilde{p}')) \) are modified to second order in \( \pi \) and therefore the groupoid structure is still preserved to first order in \( \pi \). Moreover, this modification ensures that \( \omega \) is multiplicative, and therefore there is a symplectic potential with

\[
\partial^* \vartheta = \delta \left( - \frac{1}{2} \oint \pi^{ijk} \tilde{p}_{i\tau} \tilde{p}_{j\tau}' \frac{\dot{x}_{k\tau}}{|\dot{x}_{\tau}|^2} \right).
\]

(4.23)
This form is clear to first order in $\pi$ by computing $\partial^* \partial$ for $r_i^\tau = 0$. The contributions from $r_i^\tau$ add the missing terms of the form $- \frac{1}{2} \oint d\tau \pi^{ijk} \tilde{p}_i^\tau \tilde{p}_{j'}^\tau \delta(\frac{\dot{x}_k^\tau}{|\dot{x}|^2})$. This algorithm can presumably be iterated to compute the corrections to arbitrary order in $\pi$.

The group cocycle corresponding to (4.23) reads as

$$\sigma_\pi(x, \tilde{p}, \tilde{p}') = \exp \left( -\frac{i}{2} \oint d\tau \pi^{ijk} \tilde{p}_i^\tau \tilde{p}_{j'}^\tau \frac{\dot{x}_k^\tau}{|\dot{x}|^2} \right).$$

(4.24)

As discussed in §3.10, we will not attempt a precise definition of the corresponding convolution algebra, as e.g. the choice of measure on loop space is unclear to us. We can nevertheless glean a few features of the quantization from the form of the twisting cocycle (4.24). Recall that we worked with the symplectic two-form $\omega = \oint d\tau \delta x_i^\tau \wedge \delta \tilde{p}_i^\tau$. Therefore the variable $\tilde{p}$ is canonically conjugate to $x$ and the form of $\sigma_\pi(x, \tilde{p}, \tilde{p}')$ implies a relation between quantized coordinate functions given by

$$[\hat{x}_i^\tau, \hat{x}_j^\rho] = i \pi^{ijk} \frac{\dot{x}_k^\tau}{|\dot{x}|^2} \delta(\tau - \rho).$$

(4.25)

To first order in $\pi$, this relation is identical to those of the noncommutative loop spaces found in [14, 15, 29] describing the quantum geometry of open membranes ending on M5-branes in a constant supergravity C-field background. The higher order terms in $\pi$ computed by [15, 29] could presumably come from higher order corrections to the groupoid structure above.

This loop space construction can be reduced to that of the symplectic space $(\mathbb{R}^2, \omega)$ by applying a simple dimensional reduction prescription from M-theory to string theory: We choose a particular M-theory direction, say $x^3$, and compactify $\mathbb{R}^3 \rightarrow \mathbb{R}^2 \times S^1$ along this direction on a circle $S^1$ of radius $r$. Imposing the condition that loops extend around this direction yields

$$x^i(\tau) = x_0^i + \frac{\tau}{r} \delta^{i,3}, \quad \dot{x}^i(\tau) = \frac{1}{r} \delta^{i,3} \quad \text{and} \quad p^3(\tau) = 0$$

(4.26)

with $x_0^i \in \mathbb{R}$ constant zero modes. The Nambu-Poisson and multisymplectic structures on $\mathbb{R}^3$ reduce to Poisson and symplectic structures on $\mathbb{R}^2$ given by $\pi^{ij} = r \pi^{ij3}$ and $\omega_{ij} = r^{-1} \omega_{ij3}$. In general, the loop space groupoid structure at a given $\tau_0 \in S^1$ reduces to that of $\mathbb{R}^2$ after integrating over the loop parameter.

5. Noncommutative tori

In this section we extend the groupoid quantization of §4 to tori. The underlying groupoid structures turn out to be the essentially the same; the novelty in this class of examples is that the generalized Bohr-Sommerfeld quantizations of §2.6 and §3.13 come into play. Hence we shall only highlight the topological aspects which crucially affect the quantization.
5.1. Symplectic geometry

Geometric quantization on the symplectic groupoid associated to the Poisson $d$-torus leads rather naturally to the noncommutative torus. The groupoid quantization of the torus $\mathbb{T}^2$ with its standard constant symplectic structure was originally done by Weinstein \[63\], who was the first to apply Bohr-Sommerfeld quantization to a symplectic groupoid with polarization having multiply connected leaves. Here we follow again [5, §6.5] and work on tori of arbitrary dimension. Let $V = \mathbb{R}^d$ be a real vector space of dimension $d$, and let $\Lambda = \mathbb{Z}^d \subset V$ be a lattice of rank $d$. Then the quotient $\mathbb{C}^d = V/\Lambda$ is a $d$-dimensional torus.

We endow $\mathbb{C}^d$ with a constant Poisson structure given as before by an antisymmetric $d \times d$ matrix $\pi$ on $V$. As in the case of vector spaces from §4.1, we take the symplectic groupoid $(\Sigma, \omega)$ to be the cotangent groupoid $\Sigma = T^*\mathbb{C}^d \simeq \mathbb{C}^d \times V^*$ with the canonical symplectic structure $\omega = dx^i \wedge dp_i$ for $x = (x^i) \in \mathbb{C}^d$ and $p = (p_i) \in V^*$. The structure maps of the groupoid are exactly those described in §4.1. Likewise, the prequantization is given by the trivial line bundle $E = \Sigma \times \mathbb{C}$ with connection $\nabla = d + \theta$, where now the global symplectic potential is

$$\theta = p_i \, dx^i. \quad (5.1)$$

The polarization $\mathcal{P} = \text{span}\{ \frac{\partial}{\partial x^i} \}$ is the kernel of the fibration of groupoids $\mathbb{T}^d \times V^* \to V^*$, so that again $\Sigma/\mathcal{P} \cong V^*$.

The crucial distinction now from the quantization of the universal covering space $V$ is that the symplectic potential (5.1) is not adapted to $\mathcal{P}$; this is also reflected in the fact that the leaves of $\mathcal{P}$, being $d$-tori, are not simply connected. There are natural identifications $H_1(\mathbb{T}^d, \mathbb{Z}) = \Lambda$ and $H^1(\mathbb{T}^d, \mathbb{Z}) = \Lambda^*$. Let $\gamma_i \subset \mathbb{T}^d$ for $i = 1, \ldots, d$ be a basis of one-cycles dual to the one-forms $dx^i$ in this sense. Given an arbitrary cycle $\gamma = m^i \gamma_i$, with winding numbers $m = (m^i) \in \pi_1(\mathbb{T}^d) = \pi_0(\Lambda) = \Lambda$, the holonomy of the connection given by (5.1) is

$$\text{hol}_\gamma(\theta) = e^{2\pi i \langle p, m \rangle}. \quad (5.2)$$

The Bohr-Sommerfeld quantization condition $\text{hol}_\gamma(\theta) = 1$ thus holds if and only if $\langle p, m \rangle \in \mathbb{Z}$ for all $m \in \Lambda$, i.e., $p \in \Lambda^* \subset V^*$. The Bohr-Sommerfeld groupoid is then $\Sigma_0 = \mathbb{T}^d \times \Lambda^*$, which is a disconnected union

$$\Sigma_0 = \bigsqcup_{p \in \Lambda^*} \mathbb{T}^d_p \quad (5.3)$$

of the Bohr-Sommerfeld leaves $\mathbb{T}^d_p \cong \mathbb{T}^d$; its quantization is given by the twisted convolution algebra of $\Sigma_0/\mathcal{P} = \Lambda^*$. Note that while the locations of the Bohr-Sommerfeld leaves depend on the particular choice of symplectic potential $\theta$, any two such choices are intertwined by the action of the group $\text{SL}(d, \mathbb{Z})$ of automorphisms of the lattice $\Lambda$, which preserve the symplectic two-form $\omega$.

The computation of the group two-cocycle $\sigma_\pi : \Lambda^* \times \Lambda^* \to U(1)$ proceeds exactly as before and results in the skew bicharacter (4.9), but now with $p, p' \in \Lambda^*$. A left Haar system
is given by discrete measure on the abelian group $\Lambda^*$, and the twisted convolution product is
\[
(f \otimes_\pi g)(p) = \sum_{p' \in \Lambda^*} \sigma_\pi(p', p - p') \tilde{f}(p') \tilde{g}(p - p').
\] (5.4)

Thus for $\tilde{f}, \tilde{g} \in S(\Lambda^*)$, the twisted group convolution algebra $C^*_r(\Lambda^*, \sigma_\pi)$ is the usual algebra of functions on the noncommutative torus $\mathbb{T}^d_\pi$, regarded as a deformation of the algebra of Fourier series on $\mathbb{T}^d$; the quantization map is achieved via pullback along the Fourier transformation $C^\infty(\mathbb{T}^d) \to S(\Lambda^*)$.

5.2. 2-plectic geometry

Now let us endow $\mathbb{T}^3$ with a constant Nambu-Poisson structure specified by an antisymmetric tensor $\pi^{ijk}$ of rank 3. Equivalently, we can consider the three-torus $\mathbb{T}^3$ with its standard 2-plectic structure $\pi = \pi^{12} d x^1 \wedge d x^2$. A loop $x(\tau) : S^1 \to \mathbb{T}^3$ induces a homomorphism of homology groups $H_1(S^1, \mathbb{Z}) \to H_1(\mathbb{T}^3, \mathbb{Z})$. Since $H_1(S^1, \mathbb{Z}) = \mathbb{Z}$ and $H_1(\mathbb{T}^3, \mathbb{Z}) = \Lambda$, it follows that to any map $x(\tau) \in \mathcal{KT}^3$ we may assign an element $w = w(x(\tau)) = (w^j) \in \Lambda$ whose components are the winding numbers of the loop. The Fourier mode expansion of any knot $x$ is therefore given by
\[
x(\tau) = x_0 + \tau w + \sum_{n \neq 0} e^{2\pi i n \tau} \xi_n
\] (5.5)
with $x_0 \in \mathbb{T}^3$ and $\xi_n = \overline{\xi_{-n}}$ complex numbers, so that the tangent vectors $\dot{x}(\tau)$ now have zero modes, i.e., $\int d\tau \dot{x}(\tau) = w$. This defines a decomposition of the knot space
\[
\mathcal{KT}^3 \cong \mathbb{T}^3 \times \Lambda \times \mathcal{V}
\] (5.6)
where $\mathcal{V}$ is the infinite-dimensional real vector space of “oscillators”; in particular the knot space is Pontryagin self-dual, i.e., $\mathcal{KT}^3 \cong \mathcal{KT}^3$, and hence respects the T-duality symmetry of closed string theory with target space $\mathbb{T}^3$. Thus the cotangent groupoid is a disconnected union
\[
\mathcal{KS} = \bigsqcup_{w \in \Lambda} \mathcal{K}_w \Sigma,
\] (5.7)
where each connected component is an identical copy $\mathcal{K}_w \Sigma \cong \mathbb{T}^3 \times V^* \times \mathcal{V}$; this decomposition nicely parallels the decomposition of the prequantum bundle gerbe on $\mathbb{T}^3$ given in Appendix B. On each component we construct the analogous groupoid structures to the ones in the case of $\mathcal{KBT}^3$ from §4.2, and then take the direct sum over all winding sectors $w \in \Lambda$.

To lowest order in $\pi$, the symplectic two-form on $\mathcal{K}_w \Sigma$ is given by (3.47). The global symplectic potential is given by (3.48), and its holonomy can be computed as the 2-holonomy
around the 2-loop $\mathbb{T}^2_{w,\gamma} \subset \mathbb{T}^3$ associated to a cycle $\gamma = m^i \gamma_i$ with $m = (m^i) \in \Lambda$ in the topological sector $w \in \Lambda$. The 2-loop is parametrized by the adjoint map
\[
\ell^\vee_w(\tau, \rho) = \ell_0 + m \rho + w \tau + \sum_{l,n \neq 0} e^{2\pi i (l \rho + n \tau)} \lambda_{l,n}
\]
with $\ell_0 \in \mathbb{T}^3$, while the global momentum $p(\tau) \in C^\infty(S^1, x^* T^* M)$ has mode expansion
\[
p(\tau) = p_{0,i} \, dx^i_0 + \sum_{n \neq 0} e^{2\pi i n \tau} \rho_{n,i} \, dx^i_0
\]
with $p_0 = (p_{0,i}) \in V^*$. The 2-holonomy is then given by
\[
\text{hol}_{w,\gamma}(\vartheta) = \exp \left( 2\pi i \oint d\rho \oint d\tau \frac{\partial \ell^\vee_w(\tau, \rho)}{\partial \rho} p_i(\tau) \right) = e^{2\pi i (p_0, m)}.
\]
The Bohr-Sommerfeld 2-leaves are thus parametrized by quantized momentum zero modes $p_0 \in \Lambda^* \subset V^*$, and hence to lowest order the Bohr-Sommerfeld variety is
\[
\mathcal{K}_{w,0} = \mathbb{T}^3 \times \Lambda^* \times V^*,
\]
independently of the winding mode $w \in \Lambda$. The locations of the Bohr-Sommerfeld 2-leaves do however depend on the gauge orbits of the symplectic potential through the winding sectors. Given a constant two-form $\beta$ on $\mathbb{T}^3$, representing the cohomology class of a flat gerbe 1-connection in $H^2(\mathbb{T}^3, U(1))$, consider a gauge transformation (3.54) of the symplectic potential. Then the Bohr-Sommerfeld quantization condition is generalized to
\[
p_0 + \iota_w \beta \in \Lambda^*;
\]
which defines a lattice $\Lambda^*_{w,\beta} \cong \Lambda^*$ with $\Lambda^*_{0,\beta} = \Lambda^* = \Lambda^*_{w,0}$.

At higher order in $\pi$, we can redefine the momentum $p(\tau)$ to $\tilde{p}(\tau)$ as described in §4.2, and the Bohr-Sommerfeld quantization condition now reads $\tilde{p}_0 \in \Lambda^*$; as above, this modification only affects the precise locations of the Bohr-Sommerfeld 2-leaves. It is easy to see that, in each fixed topological sector $w \in \Lambda$, the corrected groupoid structures follow exactly as described in §4.2; although additional terms do arise through $\oint d\tau \dot{x}(\tau) = w$ from integrating by parts over the loop, they are always hit by the loop space exterior derivative $\delta$ and therefore vanish. We thus arrive again at the twisting cocycle (4.24), but now with the quantized zero mode condition $\tilde{p}_0, \tilde{p}'_0 \in \Lambda^*$.

We can see the effect of the winding sectors $w \in \Lambda$ on the quantization of loop space explicitly at the level of zero modes: Integrating the commutation relations (4.25) over the loop parameters $\tau, \rho$ we arrive at
\[
[x^i_0, x^j_0] = i \pi^{ijk} w_k.
\]
This relation reproduces the expected noncommutative geometry of closed strings in $\mathbb{T}^3$ with constant $H$-flux in the non-geometric $Q$-space duality frame [19, 64]: Only closed strings which extend around cycles of the torus can probe noncommutativity, in which case
the geometry is that of the noncommutative torus $T^3_{\pi_w}$, where $\pi_{ij}^w = \pi^{ijk} w_k$. On the other hand, after summing over all winding sectors $w \in \Lambda$ as in (5.7) we obtain the anticipated quantum geometry of the corresponding T-fold obtained via T-duality from the torus $T^3$ with $H$-flux [20]. Since $\pi_{ij}$ has rank 2, the full quantized loop space contains a fibration of noncommutative two-tori.

6. Fuzzy spheres

For our final example, we demonstrate how to recover the standard fuzzy two-sphere from the groupoid quantization of $S^2$; this is a special instance of the class of examples given in §2.7. Then we describe how to extend this quantization to the loop space of the 2-plectic manifold $S^3$.

6.1. Symplectic geometry

Let us start with a detailed discussion of the groupoid quantization of $S^2$; we will refer to it later on when discussing the groupoid quantization of $L_i S^3$. The canonical symplectic structure on $S^2$ is given by its area form

$$\omega_{S^2} = \frac{1}{4\pi} \sin \theta^1 \, d\theta^1 \wedge d\theta^2$$

(6.1)

written in standard angular coordinates $0 \leq \theta^1 \leq \pi$, $0 \leq \theta^2 \leq 2\pi$. The sphere has unit area with respect to this area form, $\int_{S^2} \omega_{S^2} = 1$. Over $S^2$, we can define a family of prequantum line bundles $E_k, k \in \mathbb{Z}$, with connection $\nabla$ and associated curvature $F_{\nabla}$ such that $F_{\nabla} := \nabla^2 = -2\pi i k \omega$. The first Chern number of $E_k$ is computed as

$$c_1(E_k) = \int_{S^2} \frac{i}{2\pi} F = k.$$  

(6.2)

To construct a Kähler polarization on the sections of $E_k$, we introduce local gauge potentials for the connection $\nabla$ and consider the natural complex structure on $S^2$. We will cover $S^2$ by the two patches $U_+$, for which $\theta^1 \neq \pi$, and $U_-$, for which $\theta^1 \neq 0$. On these patches we define the respective potentials

$$A_+ = \frac{i k}{2} \left( \cos \theta^1 - 1 \right) d\theta^2 \quad \text{and} \quad A_- = \frac{i k}{2} \left( \cos \theta^1 + 1 \right) d\theta^2.$$  

(6.3)

The complex structure on $S^2$ is given by the linear map $J$ acting on the vector fields $\frac{\partial}{\partial \theta^i}$ that span $C^\infty(S^2, TS^2)$ as

$$J \frac{\partial}{\partial \theta^1} = \frac{1}{\sin \theta^1} \frac{\partial}{\partial \theta^2} \quad \text{and} \quad J \frac{\partial}{\partial \theta^2} = -\sin \theta^1 \frac{\partial}{\partial \theta^1}$$  

(6.4)

with $J^2 = -\text{id}$. The usual projection of a one-form onto its antiholomorphic part is given by

$$\Pi^{0,1} : \xi \mapsto \xi^{0,1} = \frac{1}{2} (\xi + i \xi \circ J),$$

(6.5)

\footnote{See [24] for a discussion of the relation between these two perspectives on the noncommutative Q-space geometry.}
and thus the antiholomorphic parts of the connections read as

\[ \nabla_{\pm}^{0,1} = \Pi^{0,1} \circ (d_\pm + A_\pm) \]

\[ = \frac{1}{2} \left( \frac{\partial}{\partial \theta^1} + \frac{i}{\sin \theta^1} \frac{\partial}{\partial \theta^2} - \frac{k (\cos \theta^1 \pm 1)}{2 \sin \theta^1} \right) d\theta^1 \]

\[ + \frac{1}{2} \left( \frac{\partial}{\partial \theta^2} - i \sin \theta^1 \frac{\partial}{\partial \theta^1} + \frac{i k}{2} (\cos \theta^1 \pm 1) \right) d\theta^2 \]

on \( U_\pm \). We will now construct covariantly constant \( L^2 \)-sections \( \psi \) of the line bundle \( E_k \). For example, on \( U_+ \) we look for solutions to \( \nabla_+^{0,1} \psi_+ = 0 \) or equivalently

\[ \left( \frac{\partial}{\partial \theta^2} - i \sin \theta^1 \frac{\partial}{\partial \theta^1} + \frac{i k}{2} (\cos \theta^1 - 1) \right) \psi_+ = 0 . \]

The general solution is the product of a particular solution and solutions \( s_+ \) to \( \bar{\partial}_+ s_+ := \Pi^{0,1} \circ d_+, s_+ = 0 \), giving

\[ \psi_+(\theta^1, \theta^2) = \left( \frac{1}{1 + \tan^2 \left( \frac{\theta^2}{2} \right)} \right)^{k/2} s_+ \left( \tan \left( \frac{\theta^1}{2} \right) e^{i \theta^2} \right) . \]

Square integrability with respect to the area form (6.1),

\[ \frac{1}{4\pi} \int_0^\pi d\theta^1 \int_0^{2\pi} d\theta^2 \sin \theta^1 \bar{\psi}_+ \psi_+ < \infty , \]

implies that \( s_+ \) is a polynomial of degree \( p \leq k \). By introducing complex stereographic coordinates \( z_+ = \tan \left( \frac{\theta^1}{2} \right) e^{i \theta^2} \), we arrive at a more familiar formulation: Then \( s_+ \) are polynomials of maximal degree \( k \) in \( z_+ \) and therefore determine sections of the holomorphic line bundle \( \mathcal{O}_{\mathbb{C}P^1}(k) \) of degree \( k \) over the complex projective line \( \mathbb{C}P^1 \cong S^2 \), which is a Kähler manifold with Kähler two-form

\[ \omega_{\mathbb{C}P^1} = \omega_{S^2} , \quad \omega_{\mathbb{C}P^1} \big|_{U_+} = \frac{i}{2\pi} \partial_+ \bar{\partial}_+ \log(1 + z_+ \bar{z}_+) = \frac{i}{2\pi} \frac{dz_+ + d\bar{z}_+}{(1 + z_+ \bar{z}_+)^2} . \]

For the integrating symplectic groupoid we take the pair groupoid \( \Sigma = \text{Pair}(\mathbb{C}P^1) = \mathbb{C}P^1 \times \mathbb{C}P^1 \) with multiplicative symplectic two-form \( \omega = s^* \omega_{\mathbb{C}P^1} - t^* \omega_{\mathbb{C}P^1} \). The prequantum line bundle on \( \Sigma \) is \( E = \mathcal{O}_\Sigma(k,0) \otimes \mathcal{O}_\Sigma(0,k) \), endowed with connection of curvature \(-2\pi i k \omega\), while the prequantization cocycle is trivial, \( \sigma = 1 \). The polarization on \( \Sigma \) is given by the generalized Kähler polarization \( \mathcal{P} = T_{\mathbb{C}}^{0,1} \mathbb{C}P^1 \times T_{\mathbb{C}}^{1,0} \mathbb{C}P^1 \); in local complex coordinates \((z, \bar{z}; w, \bar{w}) \in \mathbb{C} \times \mathbb{C}\) one has \( \mathcal{P} = \text{span}\{ \partial, \partial_\bar{z} \} \), and a basis of covariantly constant sections is given by \( \xi = (dz, d\bar{w}) \).

The algebra \( \mathcal{C}^*_\sigma(\Sigma/\mathcal{P}) \) is constructed from polarized sections \( \psi \otimes \sqrt{\xi} \in C^\infty(\Sigma, E \otimes \Omega^{1/2}_\mathcal{P}) \), where \( \Omega_\mathcal{P} = \det \left( T_{\mathbb{C}}^{0,1} \mathbb{C}P^1 \right) \otimes \det \left( T_{\mathbb{C}}^{1,0} \mathbb{C}P^1 \right) \). This identifies

\[ \mathcal{C}^*_\sigma(\Sigma/\mathcal{P}) \cong H^0(\mathcal{O}_{\mathbb{C}P^1}, \mathcal{O}_{\mathbb{C}P^1}(k)) \otimes H^0(\mathcal{O}_{\mathbb{C}P^1}, \mathcal{O}_{\mathbb{C}P^1}(k)) , \]

which is the operator algebra \( \mathcal{K}(H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(k))) \) anticipated from §2.3. The Hilbert space \( \mathcal{H}_k := H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(k)) \) is finite-dimensional, and in terms of homogeneous coordinates \((z_0, z_1) \in \mathbb{C}^2\) on \( \mathbb{C}P^1 \) it is spanned by homogeneous polynomials of degree \( k \). It
follows that a basis of $C^*(\Sigma/\mathcal{P})$ is provided by the sections

$$
\psi_{p,q}(z, \zeta; w, \overline{w}) = \frac{z_0^p \overline{z}_1^q w^k \overline{w}^{k-q}}{|z|^k |w|^k}, \quad p, q = 0, 1, \ldots, k
$$

(6.12)

where $|z|^2 := \overline{z}_0 z_0 + \overline{z}_1 z_1$. Hence any section $\tilde{f} \in C^*(\Sigma/\mathcal{P})$ can be expanded as a finite linear combination

$$
\tilde{f}(z, \zeta; w, \overline{w}) = \sum_{p,q=0}^k F_{pq} \psi_{p,q}(z, \zeta; w, \overline{w}), \quad F_{pq} \in \mathbb{C},
$$

(6.13)

which under the mapping $\tilde{f} \mapsto F := (F_{pq})$ identifies $C^*(\Sigma/\mathcal{P})$ with the linear space $\text{Mat}_{\mathbb{C}}(k+1)$ of $(k+1) \times (k+1)$ complex matrices. The convolution product (2.38) is given by

$$
(f \ast \tilde{f}')(z, \zeta; w, \overline{w}) = \int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^1}(y, \overline{y}) \tilde{f}(z, \zeta; y, \overline{y}) \tilde{f}'(y, \overline{y}; w, \overline{w})
$$

$$
= \sum_{p,q,p',q'=0}^k F_{pq} F'_{pq'} \psi_{p,q}(z, \zeta; w, \overline{w})
$$

$$
\times \int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^1}(y, \overline{y}) \psi_{p',q'}(y, \overline{y}; y, \overline{y})
$$

(6.14)

for polarized sections $\tilde{f} = \sum_{p,q} F_{pq} \psi_{p,q}$ and $\tilde{f}' = \sum_{p,q} F'_{pq} \psi_{p,q}$. That this formula induces the matrix product $F \cdot F'$ now follows from the identity

$$
\int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^1}(y, \overline{y}) \psi_{p,q}(y, \overline{y}; y, \overline{y}) = \delta_{p,q}
$$

(6.15)

expressing orthonormality of the basis sections of $\mathcal{H}_k = H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(k))$, and therefore

$$
(f \ast \tilde{f}')(z, \zeta; w, \overline{w}) = \sum_{p,q=0}^k (F \cdot F')_{pq} \psi_{p,q}(z, \zeta; w, \overline{w})
$$

(6.16)

with $(F \cdot F')_{pq} = \sum_r F_{pr} F'_{rq}$. Hence $C^*(\Sigma/\mathcal{P}) \cong \text{Mat}_{\mathbb{C}}(k+1)$ as $C^*$-algebras.

Note that the "diagonal" subalgebra of $C^*(\Sigma/\mathcal{P})$ consists of functions on $\mathbb{C}P^1$. The explicit quantization map can then be given by identifying the space of homogeneous polynomials of degree $k$ with the $k$-particle Hilbert space in the Fock space of two harmonic oscillators as

$$
H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(k)) \cong \text{span}\{|p| \mid p = 0, 1, \ldots, k\}, \quad |p| := \frac{(\hat{a}_0^\dagger)^p (\hat{a}_1^\dagger)^{k-p}}{\sqrt{p!(k-p)!}} |0\rangle
$$

(6.17)

where $|p\rangle$ are orthonormal number basis states in the Fock space generated by creation and annihilation operators obeying $[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$ and $\hat{a}_\alpha |0\rangle = 0$ for $\alpha, \beta = 0, 1$. The Berezin quantization map of §2.3 in this case is given by [38]

$$
B^{-1} : C^\infty(\mathbb{C}P^1) \longrightarrow C^*(\Sigma/\mathcal{P}), \quad B^{-1}(\psi_{p,q}(z, \zeta; z, \overline{\zeta})) = |p\rangle \langle q|.
$$

(6.18)
The orthonormality relation \((6.15)\) then follows from
\[
\int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^1}(z, \overline{z}) \psi_{p,q}(z, \overline{z}, z, \overline{z}) = \text{tr} \mathcal{A}_k (B^{-1}(\psi_{p,q}))
\]
\[
= \sum_{r=0}^{k} \langle r | p \rangle \langle q | r \rangle = \delta_{p,q} . \tag{6.19}
\]

This construction can be straightforwardly generalized to all complex projective spaces \(\mathbb{C}P^n\) with \(n \geq 1\), and in fact to arbitrary compact Kähler manifolds with integral symplectic two-forms. See \([65]\) for the symplectic groupoid quantization of \(S^2\) as a coset space, which yields the standard Podleś quantum sphere.

### 6.2. 2-plectic geometry

Let us now come to the quantization of \(S^3\); as previously we closely parallel the quantization of \(S^2\). Following our strategy, the notion of a prequantum line bundle is replaced by that of a prequantum gerbe. To simplify the analysis involved, we then transgress the gerbe to a line bundle over loop space and from this derive a Hilbert space.

Just as the line bundles over \(S^2\) are characterized by their Chern class \(k \in H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}\) up to isomorphism, the gerbes over \(S^3\) are characterized by their Dixmier-Douady class \(k \in H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}\). The canonical 2-plectic form on \(S^3\) is given by its volume form which reads as
\[
\omega_{S^3} = \frac{1}{2\pi} \sin^2 \theta^1 \sin \theta^2 \, d\theta^1 \wedge d\theta^2 \wedge d\theta^3 \tag{6.20}
\]
in standard angular coordinates \(0 \leq \theta^1, \theta^2 \leq \pi, 0 \leq \theta^3 \leq 2\pi\), and the volume of the three-sphere with respect to \(\omega_{S^3}\) is \(\int_{S^3} \omega_{S^3} = 1\). This volume form is determined by a contact structure on \(S^3\) which is specified by a connection one-form \(\kappa\) on the Hopf fibration \(\pi : S^3 \rightarrow S^2\) defined via the pullback \(d\kappa = \pi^* (\omega_{S^2})\). The global contact form \(\kappa\) determines a splitting \(T^* S^3 = E^* \oplus E^0\) of the co-oriented contact distribution \(E \rightarrow S^3\), a trivialization \(E^0 = S^3 \times \mathbb{R}\) and an orientation on \(S^3\) with volume form \(\omega_{S^3} = \kappa \wedge d\kappa\). Moreover, it makes \(S^3\) locally isomorphic to the first jet bundle \(T^* N \times \mathbb{R}\) of real-valued functions on a Legendrian submanifold \(N\), i.e., a one-dimensional submanifold whose tangent bundle lies in the hyperplane field \(\ker \kappa\). The pushforward \(\pi_1 : S^1(S^3, \omega_{S^3}) \rightarrow C^\infty(S^2)\) of Hamiltonian one-forms on \(S^3\) by integration over the \(S^1\)-fibers of the Hopf fibration then yields the Poisson algebra of functions on \(S^2\).

The prequantum gerbes \(\mathcal{G}_k, k \in \mathbb{Z}\), have Dixmier-Douady class \([k \omega_{S^3}]\), curvature \(H = -2\pi^2 i k \omega_{S^3}\) and topological charge
\[
dd(\mathcal{G}_k) = \int_{S^3} \frac{iH}{2\pi^2} = k \tag{6.21}
\]
We cover \(S^3\) by the two patches \(U_+\) and \(U_-\), for which \(\theta^1 \neq \pi\) and \(\theta^1 \neq 0\) respectively. The
connective structure with respect to these patches reads as

\[ B_+ = -\frac{i k}{2} \left( \theta^1 - \frac{1}{2} \sin(2 \theta^1) \right) \sin \theta^2 \, d\theta^2 \wedge d\theta^3, \]
\[ B_- = -\frac{i k}{2} \left( \theta^1 - \frac{1}{2} \sin(2 \theta^1) - \pi \right) \sin \theta^2 \, d\theta^2 \wedge d\theta^3, \]
\[ A_{+-} = \frac{\pi i k}{2} \cos \theta^2 \, d\theta^3. \] (6.22)

Note that the intersection \( U_{+-} = U_+ \cap U_- \) is homotopic to \( S^2 \) and the gauge potential \( A_{+-} \) coincides (up to an exact term) with the standard gauge potential on the line bundle \( E_k \to S^2 \) from §6.1.

We now transgress the prequantum gerbe \( \mathcal{G}_k \) to a line bundle \( \mathcal{T} \mathcal{G}_k \) on knot space \( \mathcal{K}S^3 \). As we want a local description in terms of components of a connective structure, we have to introduce a cover of \( \mathcal{K}S^3 \). For this, we start from the cover \( (U_a), \, a \in S^3 \), with \( U_a := S^3 \setminus \{a\} \) and consider the knot spaces of the open sets \( \mathcal{K}U_a \). We thus cover \( \mathcal{K}S^3 \) by the patches \( U_a := \mathcal{K}U_a \). Working with angular coordinates has the advantage that different patches \( U_a \) merely correspond to different ranges of the angles.

The gauge potentials of the prequantum line bundle on the patches \( U_a \) can be obtained from a transgression of the \( B \)-fields on the patch \( U_a \). For example, over \( U_+ \) we have

\[ A_+ := \mathcal{T}B_+ = -\frac{i k}{2} \oint d\tau \left( \theta^1 - \frac{1}{2} \sin(2 \theta^1) \right) \sin \theta^2 \, (\dot{\theta}^2 \delta \theta^3 - \dot{\theta}^3 \delta \theta^2) \] (6.23)

The canonical complex structure \( J_{S^3} \) on \( \mathcal{K}S^3 \) acts in local coordinates on the vector fields \( \frac{\delta}{\delta \theta^i} \) as

\[ J_{S^3} \frac{\delta}{\delta \theta^i} = \tilde{\varepsilon}_{ijk} \frac{\dot{\theta}^k}{|\theta^i|} g^{jl} \frac{\delta}{\delta \theta^l}, \] (6.24)

where \( g \) is the metric on \( S^3 \) defined by the line element

\[ ds^2 = (d\theta^1)^2 + \sin^2 \theta^1 \left((d\theta^2)^2 + \sin^2 \theta^2 (d\theta^3)^2\right), \] (6.25)

while

\[ \tilde{\varepsilon}_{ijk} := \sqrt{g} \varepsilon_{ijk}, \quad \varepsilon_{ijk} := \frac{1}{\sqrt{g}} \varepsilon_{ijk} \quad \text{and} \quad \sqrt{g} = \sin^2 \theta^1 \sin \theta^2. \] (6.26)

One easily verifies that \( J_{S^3}^2 = -\text{id} \).

Let us now analyze the covariantly constant sections with respect to the antiholomorphic part of the connection \( \nabla^{0,1} \) on the line bundle \( \mathcal{T} \mathcal{G}_k \) defined by the gauge potentials \( A_a \) on the patches \( U_a \). Using (6.5), we have for example on \( U_+ \) the formula

\[ \nabla_+^{0,1} = \frac{1}{2} \oint d\tau \delta \theta^i \left( \frac{\delta}{\delta \theta^i} + i \tilde{\varepsilon}_{ijk} \frac{\dot{\theta}^k}{|\theta^i|} g^{jl} \frac{\delta}{\delta \theta^l} \right) + A_+^{0,1}, \] (6.27)

and a corresponding formula involving \( A_+^{1,0} \). From comparison with the analysis of covariantly constant sections of the prequantum line bundles \( E_k \) on \( S^2 \), the interesting part of
covariantly constant sections $\psi_a$ of $\mathcal{T}G_k$ over $\mathcal{U}_a$ are the holomorphic functionals $s_a$ satisfying $\delta_a s_a = 0$. The construction of these sections is surprisingly complicated, and we will use results of [66] for this. Instead of reviewing these constructions in generality, we present the concrete application to the case at hand.

Let us restrict to the patch $\mathcal{U}_a \cong \mathcal{K}\mathbb{R}^3$. In Cartesian coordinates $x^i, i = 1, 2, 3$, the metric $g$ can be read off from the line element

$$ds^2 = \left(\frac{2}{1 + (x^1)^2 + (x^2)^2 + (x^3)^2}\right)^2 ((dx^1)^2 + (dx^2)^2 + (dx^3)^2). \tag{6.28}$$

The Dolbeault operator on loop space is rewritten as

$$\bar{\delta}_a = \frac{1}{2} \oint d\tau \delta x^{i\tau} \left( \frac{\delta}{\delta x^{i\tau}} + \iota \tilde{\varepsilon}_{ijk} \frac{\dot{x}^k\tau}{|\dot{x}^\tau|} g^{jl} \frac{\delta}{\delta x^{l\tau}} \right). \tag{6.29}$$

Plugging in (6.28), the Dolbeault operator reduces to that on the knot space $\mathcal{K}\mathbb{R}^3$ of Euclidean space $\mathbb{R}^3$ given by

$$\bar{\delta}_a = \frac{1}{2} \oint d\tau \delta x^{i\tau} \left( \frac{\delta}{\delta x^{i\tau}} + \iota \varepsilon_{ijk} \frac{\dot{x}^k\tau}{|\dot{x}^\tau|} \frac{\delta}{\delta x^{j\tau}} \right), \tag{6.30}$$

where in this formula $|\dot{x}^\tau|$ denotes the norm of the vector $\dot{x}^\tau$ with respect to the Euclidean metric on $\mathbb{R}^3$. Constructing the holomorphic local sections $s_a$ thus amounts to finding holomorphic functionals on $\mathcal{K}\mathbb{R}^3$.

Recall that the tangent space at a loop $x \in \mathcal{K}\mathbb{R}^3$ consists of the two-dimensional orthogonal complements to the tangent vector to the loop at every point $x(\tau)$. To describe these complements, it is helpful to switch to a twistor formulation as in [66]. This idea goes back to Drinfeld and LeBrun. The twistor description we will use was developed in full generality in [67]; see also [68] for a detailed discussion in a notation closely related to ours here. The connection between loop spaces and twistor geometry arises because oriented lines in a Riemannian three-manifold, like the tangent space to a loop at a point $x(\tau)$, are parametrized by LeBrun’s twistor Cauchy-Riemann manifolds. At the heart of any twistor geometry is the double fibration, which for $\mathbb{R}^3$ reads as

$$\begin{array}{c}
\Pi_1 \\
P \\
\Pi_2
\end{array} \xleftarrow{F} \xrightarrow{\mathbb{R}^3}$$

The space $F = \mathbb{R}^3 \times \mathbb{C}P^1$ is called the correspondence space, while $P$ is the (mini)twistor space which can be identified with the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{C}P^1}(2)$ or equivalently with the holomorphic tangent bundle over $\mathbb{C}P^1$. The map $\Pi_1$ is given by

$$\Pi_1(x, \lambda) = x^{\alpha \beta} \lambda_\alpha \lambda_\beta,$$  

where $\lambda_\alpha, \alpha = 1, 2$, are homogeneous coordinates on $\mathbb{C}P^1$ and

$$(x^{\dot{\alpha} \dot{\beta}}) = (x^{\dot{\beta} \dot{\alpha}}) = \begin{pmatrix}
x^1 + i x^2 & -x^3 \\
-x^3 & -x^1 + i x^2
\end{pmatrix}.$$
The map $\Pi_2$ is the trivial projection from $F \cong \mathbb{R}^3 \times \mathbb{C}P^1$ to $\mathbb{R}^3$. It is easy to see that this double fibration establishes two correspondences: Between points in $\mathbb{R}^3$ and sections of $P \cong \mathcal{O}_{\mathbb{C}P^1}(2)$, and between points in $P$ and oriented lines in $\mathbb{R}^3$.

The correspondence space $F$ is a Cauchy-Riemann manifold: it comes with an integrable distribution $\mathcal{D} \subset TF$ that is almost Lagrangian, i.e., $[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}$ and $\mathcal{D} \cap \mathcal{D} = \emptyset$. As with many Cauchy-Riemann manifolds, the distribution $\mathcal{D}$ is induced by an embedding into a complex manifold. Here we can embed $F$ into Penrose’s twistor space $T \cong \mathbb{R}^4 \times S^2$, which is biholomorphic to the total space of the rank two holomorphic vector bundle $\mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1)$, cf. [68]. Explicitly, we have

$$\mathcal{D} = \text{span}\left\{ \lambda^\alpha \lambda^\beta \frac{\partial}{\partial x^\sigma} \right\} \oplus T^0\mathbb{C} \mathbb{C}P^1.$$

Using $\mathcal{D}$, we can define the smooth sub-bundle $HF = TF \cap (\mathcal{D} \oplus \overline{\mathcal{D}})$. On this sub-bundle, there is an endomorphism $J_F$ mapping $\text{Re}(X) \in HF$ to $-\text{Im}(X) \in HF$ for every $X \in \overline{\mathcal{D}}$; it satisfies $J_F^2 = -\text{id}$ and the bundle $HF \to F$ has a canonical orientation.

At every point $x(\tau)$ of the loop, we can now write the tangent vector as

$$\lambda^\alpha \lambda^\beta x^{\dot{\alpha}} x^{\dot{\beta}} = \lambda^{\dot{\alpha}} \lambda^{\dot{\beta}} + \dot{\lambda}^{\dot{\alpha}} \lambda^{\dot{\beta}} + \lambda^{\dot{\alpha}} \dot{\lambda}^{\dot{\beta}},$$

where $\lambda^{\dot{\alpha}} := \varepsilon^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta}}$ and $\dot{\lambda}^{\dot{\alpha}} := \varepsilon^{\dot{\alpha}\dot{\beta}} \dot{\lambda}^{\dot{\beta}}$. The ambiguity in the choice of $\lambda^{\dot{\alpha}}$ is a phase, which is factored out because we consider them as homogeneous coordinates on $\mathbb{C}P^1$. We thus obtain a map $x(\tau) \mapsto \lambda^{\dot{\alpha}}(\tau) \in \mathbb{C}P^1$, and we can lift a loop $x \in K\mathbb{R}^3$ to a loop $\tilde{x} = (x, \lambda) \in K_T F$; the loop space $K_T F$ is the space of transverse loops in $F$ [66], i.e., the loops form equivalence classes of smooth immersions $\tilde{x} : S^1 \to F$ under reparametrization transformations such that for every $\tau \in S^1$, $\tilde{x}(\tau)$ is transverse to $H_{\tilde{x}(\tau)} F$. One can show that the tangent space to $K_T F$ at a loop $\tilde{x}$ is isomorphic to $C^\infty(S^1, \tilde{x}^* HF)$, and the endomorphism $J_F$ on $HF$ induces a complex structure $J_F$ on $TK_T F$ at each point of the loop [66].

The interesting point now is that the map $x \mapsto \tilde{x}$, which embeds $K\mathbb{R}^3$ into $K_T F$, intertwines the complex structure $J_F$ with the complex structure $J_{\mathbb{R}^3}$ on $K\mathbb{R}^3$. The (complexified) orthogonal complement to $\dot{x}^{\dot{\alpha} x^{\dot{\beta}}}$, which is the tangent space at $x$ restricted to the point $x(\tau)$, is spanned by the vectors

$$v^{\dot{\alpha} x^{\dot{\beta}}} := \lambda^{\dot{\alpha}} \lambda^{\dot{\beta}} \quad \text{and} \quad \overline{v}^{\dot{\alpha} x^{\dot{\beta}}} := \dot{\lambda}^{\dot{\alpha}} \dot{\lambda}^{\dot{\beta}},$$

because the Euclidean inner product reads explicitly as

$$(v, \tilde{x}) = -\frac{i}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\gamma}\dot{\delta}} v^{\dot{\alpha} x^{\dot{\beta}}} \dot{x}^{\dot{\gamma} x^{\dot{\delta}}} \tag{6.37}$$

and $\lambda^{\dot{\alpha}} \lambda_{\dot{\alpha}} = \dot{\lambda}^{\dot{\alpha}} \dot{\lambda}_{\dot{\alpha}} = 0$. We rewrite the loop space Dolbeault operator as

$$\overline{\delta}_a = \frac{1}{2} \int d\tau \delta x^{\dot{\alpha} x^{\dot{\beta}}} \delta^{\dot{\alpha} x^{\dot{\beta}}} \tag{6.38}$$

and examine the vector fields

$$\lambda^{\dot{\alpha}} \lambda^{\dot{\beta}} \delta^{\dot{\alpha} x^{\dot{\beta}}} \quad \lambda^{\dot{\alpha}} \dot{\lambda}^{\dot{\beta}} \delta^{\dot{\alpha} x^{\dot{\beta}}} \quad \text{and} \quad \dot{\lambda}^{\dot{\alpha}} \dot{\lambda}^{\dot{\beta}} \delta^{\dot{\alpha} x^{\dot{\beta}}} \tag{6.39}$$
We find that \( \lambda^{\dot{\alpha} \tau} \lambda^{\dot{\beta} \tau} \delta_{\dot{\alpha} \dot{\beta} \tau} f = 0 \) is equivalent to \( \lambda^{\dot{\alpha} \tau} \lambda^{\dot{\beta} \tau} \frac{\delta}{\delta x^{\dot{\alpha} \dot{\beta} \tau}} f = 0 \) for any \( f \in C^\infty(K \mathbb{R}^3) \). Moreover, we have \( \lambda^{\dot{\alpha} \tau} \lambda^{\dot{\beta} \tau} \delta_{\dot{\alpha} \dot{\beta} \tau} = 0 \), and \( \lambda^{\dot{\alpha} \tau} \lambda^{\dot{\beta} \tau} \delta_{\dot{\alpha} \dot{\beta} \tau} \) contains the vector field \( \dot{x}^{\dot{\alpha} \dot{\beta} \tau} \frac{\delta}{\delta x^{\dot{\alpha} \dot{\beta} \tau}} \) which generates loop reparameterizations and therefore vanishes. We thus saw the intertwining of the complex structures explicitly.

It follows that we can pullback any holomorphic function on \( K_T F \) along the map \( x \mapsto \hat{x} \) to obtain a holomorphic function on \( K \mathbb{R}^3 \). Lempert gives a nice way of constructing holomorphic functions on \( K_T F \) in [66]: Take a \((1,0)\)-form \( \alpha \) on \( T \cong \mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1) \supset F \). Then the map

\[
y \mapsto f(y) = (T \alpha)_y = \oint \tau \cdot i_y (ev^* \alpha) , \quad y \in K_T F \tag{6.40}
\]
defines a holomorphic function \( K_T F \rightarrow \mathbb{C} \). This formula can be generalized to expressions of multiple integrals, as products of holomorphic functions are holomorphic. Lempert explains this generalization, which follows in the spirit of Chen’s iterated integrals, and conjectures that the space of functions thus obtained is locally dense in the space of holomorphic functions on \( K_T F \).

However, it seems rather clear to us that the corresponding set of holomorphic functions obtained via pullback is not even dense in the space of holomorphic functions on \( K \mathbb{R}^3 \) (although we have no proof). Let us nevertheless push the analysis a little further. For example, consider the \((1,0)\)-form

\[
\alpha = \frac{x^{1\dot{\alpha}} \lambda^{\dot{\alpha}} d(x^{2\dot{\beta}} \lambda^{\dot{\beta}})}{(\lambda_{1\tau})^2} \tag{6.41}
\]
on \( T \), yielding the function

\[
f(x) = \oint \tau \cdot x^{1\dot{\alpha} \tau} \lambda_{1\tau} (x^{2\dot{\beta} \tau} \lambda^{\dot{\beta} \tau} + x^{2\dot{\beta} \tau} \lambda^{\dot{\beta} \tau}) \quad \text{with} \quad \dot{x}^{\dot{\alpha} \dot{\beta} \tau} = \lambda^{\dot{\alpha} \tau} \lambda^{\dot{\beta} \tau} . \tag{6.42}
\]

If we restrict ourselves to loops with \( \dot{x}^{1\tau} = \dot{x}^{2\tau} = 0 \neq \dot{x}^{3\tau} \) and \( \lambda^1 = \dot{\lambda}^1 = 0 = \dot{\lambda}^3 \), then this function restricts to

\[
f(x) = (x^1 + i x^2) \oint \tau \cdot \lambda^{1\tau} \lambda^{2\tau} = (x^1 + i x^2) V , \tag{6.43}
\]
where \( V \) is some constant volume factor (possibly requiring regularization). This restriction can roughly be interpreted as a reduction\(^\text{18}\) of \( K \mathbb{S}^3 \rightarrow S^2 \cong \mathbb{CP}^1 \), and the function \( f \) correspondingly restricts to the function \( z_+ := x^1 + i x^2 \) which is a building block for global holomorphic sections of the line bundle \( \mathcal{O}_{\mathbb{CP}^1}(1) \rightarrow \mathbb{CP}^1 \). Powers of \( f \) will correspondingly reduce to functions yielding sections of the bundles \( \mathcal{O}_{\mathbb{CP}^1}(k) \).

We have thus far described the space of holomorphic sections of the line bundles \( T \mathcal{G}_k \) that are transgressions of the prequantum gerbes \( \mathcal{G}_k \) over \( S^3 \). From here, one can in principle

\(^{18}\)Note that more appropriately, one should use the embedding of \( \mathbb{CP}^1 \) into \( K \mathbb{S}^3 \) which is induced by the preimage of the Hopf fibration \( \pi : \mathbb{S}^3 \rightarrow \mathbb{CP}^1 \). A reduction of the function \( f \) on \( K \mathbb{S}^3 \) to \( \mathbb{CP}^1 \) would then correspond to the pullback of \( f \) along the embedding map. However, the corresponding formulas for \((1,0)\)-forms are complicated and resisted our attempts of an analytical treatment.
follow the steps in the quantization of $S^2$: We define the Hilbert space $\mathcal{H}$ to be the (infinite-dimensional) space of holomorphic sections of $\mathcal{T}_G^k$ with some restrictive condition imposed. On this space, we define some natural inner product, pair “elementary sections” with a suitable set of creation and annihilation operators, and use the outer tensor product $\mathcal{H} \otimes \mathcal{H}^*$ as a quantized algebra of functions on $KS^3$. Altogether, we have shown that a quantization of $S^3$ using knot space is fruitful and exhibits some of the desired features, such as the reduction to the geometric quantization of $S^2$. The access to a detailed description, however, seems to be obstructed by technical difficulties and, most importantly, the lack of suitable measures on knot space.

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Appendix

A. Lie groupoids and Lie algebroids

In this appendix we collect a number of definitions and examples concerning groupoids as they appear for example in [5]. For a book on groupoids see [69], and for a review of the use of groupoids in quantization see e.g. [70].

Groupoids. Recall that a group is a small category\(^{19}\) with one object, in which all morphisms are invertible. The single object corresponds to the unit in the group, while the morphisms correspond to the group elements. Removing the condition that there is a single object, we arrive at the notion of a groupoid: A groupoid is a small category in which every morphism is an isomorphism.

More explicitly, a groupoid consists of a base set $G(0) = M$ and a set of arrows $G(1) = G$ between elements of $M$. Each arrow can be projected onto its source and target, yielding two maps $s, t : G \to M$. We write $G \rightrightarrows M$ and say that $G$ is a groupoid over $M$. Each arrow $g \in G$ has an inverse arrow $g^{-1} \in G$ in the opposite direction. Two arrows with matching head and tail can be concatenated leading to a partial multiplication on $G \times G$: $(g, h) \mapsto m(g, h) = g h \in G$. There is an inclusion map $\mathbb{I} : M \to G, x \mapsto \mathbb{1}_x$ of an object

\(^{19}\)“Small” means that both the objects and morphisms in the category form sets.
x to an arrow loop x → x. The picture of arrows between elements of the base yields the following consistency relations between the maps s, t, m, 1:

(i) The concatenation of arrows via the multiplication m is associative when defined;
(ii) s(m(g, h)) = s(g) and t(m(g, h)) = t(h);
(iii) s(1x) = t(1x) = x;
(iv) g 1t(g) = 1s(g) g = g;
(v) The inverse g⁻¹ of g is two-sided: s(g⁻¹) = t(g), t(g⁻¹) = s(g), g⁻¹ g = 1t(g),
g g⁻¹ = 1s(g);
where g, h ∈ G, x ∈ M.

Lie groupoids. A Lie groupoid is a groupoid where both sets of objects and morphisms are smooth manifolds, the source and target maps are surjective submersions, and all operations are smooth. In this paper, all groupoids are Lie groupoids.

Note that every Lie group G is a Lie groupoid G ⇒ pt over a one-point manifold pt. The source and target maps are both s(g) = t(g) = pt and the unit map is given by the group identity element 1pt = e.

A non-trivial example of a Lie groupoid important for our discussion is the pair groupoid Pair(M) = M × M over a smooth manifold M. Here an element (x, y) of M × M corresponds to an arrow with source x and target y: s(x, y) = x and t(x, y) = y. This interpretation implies that the inverse is given by (x, y)⁻¹ = (y, x), that the multiplication law is (x, y) · (y, z) = (x, z) for x, y, z ∈ M and that the unit map is 1x = (x, x).

Lie algebroids. A Lie algebroid is a smooth vector bundle E → M over a manifold M endowed with a Lie bracket on the C∞(M)-module of smooth sections [−, −]E : C∞(M, E) ⊗ C∞(M, E) → C∞(M, E) and a vector bundle map ρ : E → TM, ψ ↦ ρψ called an anchor map such that:

(i) The map ρ is a Lie algebra homomorphism:

ρ[ψ₁, ψ₂]E = [ρψ₁, ρψ₂]TM , \quad ψ₁, ψ₂ ∈ C∞(M, E) . \quad (A.1)

(ii) The Leibniz rule is satisfied:

[ψ₁, f ψ₂]E = f [ψ₁, ψ₂]E + ρψ₁(f) ψ₂ , \quad ψ₁, ψ₂ ∈ C∞(M, E) , \quad f ∈ C∞(M) . \quad (A.2)

The tangent bundle E = TM is trivially a Lie algebroid, with the usual commutator Lie bracket on vector fields over M and the identity anchor map. Note also that every Lie algebra g is a Lie algebroid g → pt over a one-point manifold pt with trivial anchor map ρ = 0.

We are mostly interested in the Lie algebroid arising from the cotangent bundle T∗M of a Poisson manifold (M, π), where the anchor map ρ : T∗M → TM is given by contraction with the Poisson bivector field π, see §2.2.
Differentiation. Just as a Lie algebra is the linearization of a Lie group at the unit element, a Lie algebroid is obtained by linearizing a Lie groupoid at the units, i.e. the objects. The Lie algebroid $A(G) \to M$ of a Lie groupoid $G \rightrightarrows M$ is given by setting:

1. The vector bundle
\[
A(G) = \ker \text{d}t|_M = 1^* \ker \text{d}t = \bigsqcup_{x \in M} T_x t^{-1}(x) \subset TG
\] is the normal bundle to the embedding $1 : M \hookrightarrow G$, with bundle projection given by $s$ or $t$ (which coincide on $M$).
2. The anchor map is given by $\rho = ds : \ker \text{d}t \to TM$, restricted to $A(G)$.
3. The Lie bracket $[-, -]_{A(G)}$ is inherited from the usual commutator on $C^\infty(G,TG)$, restricted to left-invariant vector fields.

This construction defines a functor from the category of Lie groupoids (and smooth homomorphisms) to the category of Lie algebroids, which generalizes the classical Lie-functor from Lie groups to Lie algebras: Let $\mathfrak{g}$ be the Lie algebra of a Lie group $G$. Then $\mathfrak{g}$ is the Lie algebroid of the Lie groupoid $G$.

The tangent bundle $TM$ is the Lie algebroid of the pair groupoid $\text{Pair}(M) = M \times M$, as $A(M \times M) = \bigsqcup_{x \in M} x \times T_x M = TM$.

Integration. We say that a Lie groupoid $G \rightrightarrows M$ is an integration of a Lie algebroid $E \to M$ if $A(G) \cong E$ as Lie algebroids. While every Lie algebra can be integrated to a Lie group by the exponential map, the same is not true in general for Lie groupoids. For a discussion of the corresponding obstructions, see e.g. [71]. However, if a Lie algebroid $E \to M$ does integrate to a Lie groupoid, then there exists a unique Lie groupoid $G \rightrightarrows M$ (up to isomorphism) with connected and simply connected $s$-fibers whose Lie algebroid is $A(G) = E$. See [40, §2] for the construction of an exponential map $A(G) \to G$ for a general Lie groupoid $G$ whose Lie algebroid $A(G)$ is endowed with a connection; this generalizes the exponential map on the tangent bundle $\exp : TM \to M$ of a manifold $M$ with connection.

Symplectic groupoids. A Lie groupoid $G \rightrightarrows M$ naturally has the structure of a simplicial manifold with 0-nerve $M$ and 1-nerve $G$. We define the 2-nerve of a Lie groupoid $G$ as the set of composable arrows $G(2) \subset G \times G$; whence $(g_1,g_2) \in G(2)$ if and only if $t(g_1) = s(g_2)$. On this set we have the multiplication map $m : G(2) \to G$ satisfying $s(m(g_1,g_2)) = s(g_1)$ and $t(m(g_1,g_2)) = t(g_2)$ for $g_1,g_2 \in G$. There are also projections $pr_1 : G(2) \to G$ and $pr_2 : G(2) \to G$ onto the first and second arrow, respectively: $pr_1(g_1,g_2) = g_1$ and $pr_2(g_1,g_2) = g_2$. The maps $s$, $t$, $m$, $pr_1$, and $pr_2$ are face maps for this simplicial structure. This leads to the definition of the simplicial coboundary operator
\[
\partial^* := pr_1^* - m^* + pr_2^* : \Omega^k(G) \longrightarrow \Omega^k(G(2))
\] which commutes with the exterior derivative.

A symplectic groupoid $\Sigma$ is a Lie groupoid endowed with a symplectic form $\omega \in \Omega^2(\Sigma)$ which is multiplicative, i.e., $\partial^* \omega := pr_1^* \omega - m^* \omega + pr_2^* \omega = 0$. For a detailed discussion of
multiplicative two-forms, see e.g. [72]. We say that a symplectic groupoid \( \Sigma \) over a Poisson
manifold \( M \) integrates \( M \) if \( t : \Sigma \to M \) is a Poisson map\(^{20}\) and the fibers of \( s : \Sigma \to M \) are
connected (\( \Sigma \) is “\( s \)-connected”); in fact, if \( \Sigma \rightrightarrows M \) is a symplectic groupoid then its base
\( M \) is always a Poisson manifold. When it exists, an integrating symplectic groupoid for a
Poisson manifold \( M \) is unique up to isomorphism.

It is shown in [62] that a Poisson manifold is integrable in this sense if and only if the Lie
algebroid \( T^*M \) is integrable to a Lie groupoid; in this case there is a canonical isomorphism
\( A(\Sigma) \cong T^*M \) of Lie algebroids.

Note that on a symplectic groupoid \( \Sigma \), there is a unique Poisson structure on the base
such that \( t \) and \( s \) are respectively Poisson and anti-Poisson maps [73].

**Example.** The cotangent bundle \( T^*G \) of a Lie group \( G \) can be extended to a symplectic
groupoid over \( \mathfrak{g}^* \) where \( \mathfrak{g} \) is the Lie algebra of \( G \) [34], see [74] for a review. The embedding
\( 1 \) is trivially given by \( 1 : \mathfrak{g}^* \to T^*_eG \). Recall that the left and right \( G \)-actions on \( G \),
\( L_h(g) := h g \) and \( R_h(g) := g h \) with \( g, h \in G \), have derivative maps \( dL_h, dR_h : \mathfrak{g} \to T_hG \).
Given an element \((x, h) \in T^*G\) with \( h \in G \) and \( x \in T_h^*G \cong \mathfrak{g}^* \), we define the source and
target maps

\[
\begin{aligned}
s(x, h) &:= x \circ dR_h \\
t(x, h) &:= x \circ dL_h ,
\end{aligned}
\]  

(A.5)

which live in the dual space of \( \mathfrak{g} \). The product \((x, h) := (x_1, h_1) \cdot (x_2, h_2)\) of two elements
of \( T^*G \) is defined if \( t(x_1, h_1) = (x_2, h_2) \), i.e., \( x_1 \circ dL_{h_1} = x_2 \circ dR_{h_2} \). We then put \((x, h) := x_1 \circ dR_{h_2^{-1}} = x_2 \circ dL_{h_1^{-1}}\), and one has \( s(x, h) = s(x_1, h_1) \) and \( t(x, h) = t(x_2, h_2) \). Moreover,
the target map \( t \) is a Poisson map with respect to the canonical symplectic Poisson structure
on \( T^*G \) and the +KKS Poisson structure on \( \mathfrak{g}^* \).

**Fibrations of groupoids.** A fibration of Lie groupoids is a smooth homomorphism \( p : G \to G' \) between Lie groupoids \( G \) and \( G' \), such that the base map \( p(0) : G(0) \to G'(0) \) and the
map \( F^* : G \to p(0)^*G' \) are surjective submersions; here \( p(0)^*G' \) denotes the pullback bundle
of \( G' \) along \( p(0) \).

**B. \( p \)-gerbes with connective structures**

Below we summarize some definitions concerning gerbes with connective structure. In
this paper we use both the concrete description of a gerbe provided by Čech cohomology,
following the approach of Hitchin and Chatterjee (see [75, §1.2] and [76]), and also Murray’s
description in terms of bundle gerbes [77, 78]. See [79] for a comprehensive review with
further details and additional references. Smooth functions, line bundles and gerbes are
the first three elements in a sequence of \( p \)-gerbes with \( p = -1, 0, 1 \). As it is effortless to
generalize most of the notions introduced below to \( p \)-gerbes, we will do so immediately.

We start by reviewing Cheeger-Simons differential characters and how they capture a
generalized notion of holonomy for \( p \)-gerbes. We then come to Deligne cohomology, which

\(^{20}\)This means that \((\Sigma, t)\) forms a symplectic realization of the Poisson manifold \( M \).
gives a cochain model for Cheeger-Simons differential characters and classifies abelian p-gerbes with connective structure. Finally, we look at the description for $p = 1$ in terms of bundle gerbes and its relation to differential cohomology, with the d-torus serving as an illustrative example which makes contact with some of our loop space constructions from §5.2.

**Cheeger-Simons cohomology.** Let $M$ be a smooth manifold. Let $Z_k(M)$ denote the group of smooth $k$-cycles on $M$, and $\Omega^k_{\text{cl},\mathbb{Z}}(M)$ the closed $k$-forms on $M$ with integer periods. Recall [80, 10] that the degree $k$ Cheeger-Simons cohomology group of differential characters $\tilde{H}^k(M)$ is the infinite-dimensional abelian group which can described in terms of two exact sequences. Firstly, one has

$$0 \to H^{k-1}(M, \mathbb{U}(1)) \to \tilde{H}^k(M) \xrightarrow{\omega} \Omega^k_{\text{cl},\mathbb{Z}}(M) \to 0,$$

where the field strength map $\chi \mapsto \omega_\chi$ defines the curvature of the differential character $\chi$; its kernel is the group of flat fields on $M$. Secondly, one has

$$0 \to \Omega^{k-1}(M) / \Omega^1_{\text{cl},\mathbb{Z}}(M) \to \tilde{H}^k(M) \xrightarrow{c} H^k(M, \mathbb{Z}) \to 0,$$

where the map $\chi \mapsto c(\chi)$ defines the characteristic class of the differential character which obeys the compatibility condition $c(\chi) = [\omega_\chi]$; the kernel of this map is the torus of topologically trivial fields whose classes $[\theta]$ have curvature $d\theta$.

A differential character $\chi \in \tilde{H}^k(M)$ defines a holonomy

$$\text{hol}_S(\chi) := \exp \left( 2\pi i \oint_S \theta_\chi \right) \in \mathbb{U}(1)$$

for any $k-1$-cycle $S \in Z_{k-1}(M)$, where the potential $\theta_\chi \in \Omega^{k-1}(S)$ is defined by $\omega_\chi|_S = d\theta_\chi$ and we have used $H^k(S, \mathbb{Z}) = 0$. For flat fields $\omega_\chi = 0$, the holonomy defines an element $[\text{hol}(\chi)] \in H^{k-1}(M, \mathbb{U}(1))$.

The Cheeger-Simons cohomology $\tilde{H}^1(M)$ is the space of differentiable maps $g : M \to \mathbb{U}(1)$; the characteristic class map is $c(g) = g^*([d\phi]) \in H^1(M, \mathbb{Z})$ where $[d\phi]$ is the fundamental class of $S^1$, the curvature is the one-form $\omega_g = d\log g$, and the holonomy is the evaluation $\text{hol}_x(g) = g(x)$ of $g$ at $x \in M$.

The differential cohomology $\tilde{H}^2(M)$ is the group of gauge equivalence classes of line bundles with connection $(E, \nabla)$ on $M$ and gauge group generated by $\tilde{H}^1(M)$; the characteristic class map in this case computes the first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$, while the connection $\nabla$ determines a curvature $F_\nabla$ and a holonomy $\text{hol}_\nabla(\gamma)$ for $\gamma \in Z_1(M)$ which coincide with the curvature and holonomy of the corresponding differential character.

The abelian group $\tilde{H}^3(M)$ consists of differential characters in degree three, which are gauge equivalence classes of gerbes $\mathcal{G}$ over $M$ with connective structure $(A, B)$ and gauge group generated by the differential cohomology $\tilde{H}^2(M)$.

Following this pattern, the gauge equivalence classes of $p$-gerbes with connective structure over $M$ are given by the differential cohomology $\tilde{H}^{p+2}(M)$ with gauge group generated by $\tilde{H}^{p+1}(M)$. 

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**Deligne cohomology.** An explicit cochain model for the Cheeger-Simons groups is provided by Deligne cohomology. Recall [10] that the degree \( k \) smooth Deligne cohomology is the \( k \)-th Čech hypercohomology of the truncated sheaf complex \( D(k) \)

\[
0 \rightarrow U(1)_M \xrightarrow{\text{d log}} \Omega^1_M \xrightarrow{\text{d}} \Omega^2_M \xrightarrow{\text{d}} \cdots \xrightarrow{\text{d}} \Omega^k_M ,
\]

where \( U(1)_M \) is the sheaf of smooth \( U(1) \)-valued functions on \( M \) and \( \Omega^k_M \) is the sheaf of differential \( k \)-forms on \( M \). Consider thus the double complex with respect to a Stein cover \( U = (U_a)_{a \in I} \) of \( M \) given by

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\delta & \delta & \delta & \\
C^2(U, U(1)_M) & \xrightarrow{\text{d log}} & C^2(U, \Omega^1_M) & \xrightarrow{\text{d}} C^2(U, \Omega^2_M) \\
\delta & \delta & \delta & \\
C^1(U, U(1)_M) & \xrightarrow{\text{d log}} & C^1(U, \Omega^1_M) & \xrightarrow{\text{d}} C^1(U, \Omega^2_M) \\
\delta & \delta & \delta & \\
C^0(U, U(1)_M) & \xrightarrow{\text{d log}} & C^0(U, \Omega^1_M) & \xrightarrow{\text{d}} C^0(U, \Omega^2_M)
\end{array}
\]

(B.5)

Here \( \delta \) is the usual Čech coboundary operator, and \( C^p(U, U(1)_M) \) and \( C^p(U, \Omega^1_M) \) denote the Čech \( p \)-cochains. The degree \( n \) Deligne cohomology group \( H^n_D(M, D(k)) \) is computed from the ‘diagonal complex’

\[
\begin{array}{c}
C^0(U, U(1)_M) \xrightarrow{\text{d}} C^1(U, U(1)_M) \oplus C^0(U, \Omega^1_M) \\
\xrightarrow{\text{d}} C^2(U, U(1)_M) \oplus C^1(U, \Omega^1_M) \oplus C^0(U, \Omega^2_M) \xrightarrow{\text{d}} \cdots
\end{array}
\]

(B.6)

truncated to sheaves in \( D(k) \). The differentials \( \text{d} \) are defined as

\[
\text{d} \theta := \begin{cases} 
\delta \theta - (-1)^p \frac{1}{2 \pi i} \text{d log } \theta , & \theta \in C^p(U, U(1)_M) , \ k > 0 \\
0 , & \theta \in C^p(U, U(1)_M) , \ k = 0 \\
\delta \theta - (-1)^p \text{d } \theta , & \theta \in C^p(U, \Omega^1_M) , \ n < k \\
0 , & \theta \in C^p(U, \Omega^1_M) , \ n \geq k.
\end{cases}
\]

(B.7)

where we inserted a convenient factor \( \frac{1}{2 \pi i} \) into the logarithmic differential. Note that the Čech coboundary operator acting on Čech cochains with values in \( U(1)_M \) has to be regarded as the multiplicative one.

**Examples.** Below we denote contractible \( n \)-fold intersections of open sets of the cover \( U \) by \( U_{a_1 \cdots a_n} := U_{a_1} \cap \cdots \cap U_{a_n} \).

A degree 0 smooth Deligne class \( g \in H^0_D(M, D(0)) \) is a Čech cochain \( (g_a) \in C^0(U, U(1)_M) \) satisfying \( \text{d}(g_a) = 1 \) or, equivalently, \( \delta(g_a) = g_a^{-1}g_b = 1 \). Therefore \( (g_a) \) defines a smooth map \( g : M \rightarrow U(1) \).
A Deligne class \((g_a, A_{ab}) \in H^1_M(M, D(1))\) is a pair
\[
(g_a, A_{ab}) \in C^1(U, U(1)_M) \oplus C^0(U, \Omega^1_M)
\] (B.8)
satisfying the cocycle conditions \(d(g_a, A_{ab}) = (1, 0)\), which explicitly reads as
\[
g_{ab} g_{bc} g_{ca} = 1 \quad \text{on} \quad U_{abc},
A_a - A_b = \frac{1}{2\pi i} d \log g_{ab} \quad \text{on} \quad U_{ab} .
\] (B.9)
The equivalence relations, including gauge transformations, are captured by a Deligne one-coboundary \(d(h_a)\) with \((h_a) \in C^0(U, U(1)_M)\). In formulas, \((g_a, A_{ab}) \sim (g_a, A_{ab}) + d(h_a)\) reads as
\[
g_{ab} \sim g_{ab} (\delta h)_{ab} = g_{ab} h_a^{-1} h_b = h_a^{-1} g_{ab} h_b ,
A_a \sim A_a + \frac{1}{2\pi i} d \log h_a .
\] (B.10)
The \(U(1)\) Čech one-cocycle \(g_{ab} : U_{ab} \to U(1)\) determines smooth transition functions on overlaps for a hermitian line bundle \(E \to M\). This cocycle represents the first Chern class \(c_1(L) = [g_{ab}] \in H^1(\Sigma, U(1)) \cong H^2(\Sigma, \mathbb{Z})\), where the canonical isomorphism follows from the exponential sequence
\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp} U(1) \longrightarrow 1 ;
\] (B.11)
it is the obstruction to triviality of the line bundle \(E \to M\). The local one-forms \(A_a \in \Omega^1(U_a)\) define a unitary connection \(\nabla = d + A\) on \(E\). When restricted to \(U_a\), its curvature \(F\) is given by \(dA_a\).

A Deligne class \((g_{abc}, A_{ab}, B_a) \in H^2_M(M, D(2))\) is a triple
\[
(g_{abc}, A_{ab}, B_a) \in C^2(U, U(1)_M) \oplus C^1(U, \Omega^1_M) \oplus C^0(U, \Omega^2_M)
\] (B.12)
satisfying the cocycle conditions \(d(g_{abc}, A_{ab}, B_a) = (1, 0, 0)\). That is,
\[
g_{abc} g_{bcd} g_{cda} g_{dab}^{-1} = 1 \quad \text{on} \quad U_{abcd} ,
A_{ab} + A_{bc} + A_{ca} = \frac{1}{2\pi i} d \log g_{abc} \quad \text{on} \quad U_{abc} ,
B_a - B_b = dA_{ab} \quad \text{on} \quad U_{ab} .
\] (B.13)
A Deligne two-coboundary \(d(h_{ab}, a_a)\) with \((h_{ab}, a_a) \in C^1(U, U(1)_M) \oplus C^0(U, \Omega^1_M)\) defines a gauge transformation \((g_{abc}, A_{ab}, B_a) \sim (g_{abc}, A_{ab}, B_a) + d(h_{ab}, a_a)\), or
\[
g_{abc} \sim g_{abc} (\delta h)_{ab} = g_{abc} h_a h_{bc} h_{ca} ,
A_{ab} \sim A_{ab} + \frac{1}{2\pi i} d \log h_{ab} + a_a - a_b ,
B_a \sim dA_b .
\] (B.14)
The \(U(1)\) Čech two-cocycle \(g_{abc} : U_{abc} \to U(1)\) specifies a hermitian “transition” line bundle \(E_{ab}\) over each overlap \(U_{ab}\), an isomorphism \(E_{ab} \cong E^*_{ba}\), and a trivialization of the line bundle

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$E_{ab} \otimes E_{bc} \otimes E_{ca}$ on each triple overlap $U_{abc}$; the pair $G = (E_{ab}, g_{abc})$ is called a local gerbe on $M$. This cocycle represents the Dixmier-Douady class $dd(G) \in H^3(M, U(1)) \cong H^3(M, \mathbb{Z})$; it is the obstruction to triviality of the gerbe. The Čech one-cochain $A_{ab}$ defines connection one-forms on each line bundle $E_{ab} \rightarrow U_{ab}$ such that the section $g_{abc}$ is covariantly constant with respect to the induced connection on $E_{ab} \otimes E_{bc} \otimes E_{ca}$. The collection of two-forms $B_a \in \Omega^2(U_a)$ are called the curvature of the Čech one-cochain $A_{ab}$. Together, the pair $(A_{ab}, B_a)$ defines a connective structure on the gerbe $G = (E_{ab}, g_{abc})$. When restricted to $U_a$, its curvature is given by $H = dB_a$. The gauge group of the gerbe is generated by line bundles $\eta \rightarrow M$ with connection $\nabla = d + A$ and curvature $f_\nabla = da$ with $\omega = -\frac{1}{2\pi i} f_\nabla \in \Omega^2_{cl, \mathbb{Z}}(M)$ through the gauge transformations

$$E \rightarrow E_{ab} \otimes \eta|_{U_{ab}}, \quad A_{ab} \rightarrow A_{ab} + a|_{U_{ab}} \quad \text{and} \quad B_a \rightarrow B_a + f_\nabla. \quad (B.15)$$

In general, a Deligne $p+1$-cocycle defines the connective structure of a $p$-gerbe. The details of the construction are evident from the above examples.

**Holonomy and curvature.** The construction of holonomy and curvature of a Deligne class defines an isomorphism $H^{p+1}_S(M) \rightarrow H^{p+2}(M)$.

Given a degree 1 smooth Deligne class $[(g_{ab}, A_a)]$ represented by a hermitian line bundle $E \rightarrow M$ with unitary connection $\nabla = d + A$, the curvature is the globally defined two-form given by $F_\nabla = dA_a$ on $U_a$ with $\omega = -\frac{1}{2\pi i} F_\nabla \in \Omega^2_{cl, \mathbb{Z}}(M)$. By Stokes’ theorem, the holonomy of $\nabla$ around any one-cycle $\gamma \subset M$ is then obtained from the product formula [9, 81]

$$\text{hol}_\gamma(A) = \prod_{a \in I} \exp(2\pi i \int_{\gamma_a} A_a) \prod_{a,b \in I} g_{ab}(\gamma_{ab}) , \quad (B.16)$$

where $\gamma_a \subset U_a$ is a path in a subdivision of the loop $\gamma$ into segments and $\gamma_{ab} = \gamma_a \cap \gamma_b$ is a point in $U_{ab}$. This definition agrees with the general definition of holonomy in terms of differential characters.

Given a degree 2 smooth Deligne class $[(g_{abc}, A_{ab}, B_a)]$ represented by a gerbe $G$ over $M$ with connective structure $(A, B)$, the curvature of the corresponding differential character is the globally defined closed three-form $H = H_{(A,B)}$ given by $H = dB_a$ on $U_a$ with $\varpi = -\frac{1}{2\pi i} H \in \Omega^3_{cl, \mathbb{Z}}(M)$, while its characteristic class is the Dixmier-Douady class $dd(G) \in H^3(M, \mathbb{Z})$ of the gerbe $G$. Its holonomy around a two-cycle $S \subset M$ is obtained by choosing a triangulation $\{S_a\}_{a \in I}$ of $S$ subordinate to the open cover $S \cap U$. Keeping careful track of orientations, by repeated application of Stokes’ theorem one arrives at the product formula [9, 10, 82, 81]

$$\text{hol}_S(B) = \prod_{a \in I} \exp(2\pi i \int_{S_a} B_a) \prod_{a,b \in I} \exp(2\pi i \int_{S_{ab}} A_{ab}) \prod_{a,b,c \in I} g_{abc}(S_{abc}) , \quad (B.17)$$

where $S_{ab}$ is the common boundary edge of the surfaces $S_a$ and $S_b$, and $S_{abc} = S_{ab} \cap S_{bc} \cap S_{ca}$ are vertices of the triangulation of $S$. The coincidence between this expression and the general formula in terms of differential characters is shown explicitly in [83].
**Bundle gerbes.** A bundle gerbe on $M$ [77] is a pair $\mathcal{G} = (E, X)$, where $\phi : X \to M$ is a surjective submersion and $E$ is a hermitian line bundle\(^{21}\) over the fiber product $X^{[2]}$ equipped with an associative fiber multiplication

$$E_{(x_1,x_2)} \otimes E_{(x_2,x_3)} \to E_{(x_1,x_3)} \quad (B.18)$$

for all $(x_1, x_2), (x_2, x_3) \in X^{[2]}$. Recall that the fiber product $X^{[2]} := X \times_M X = \{(x_1, x_2) \in X \times X \mid \phi(x_1) = \phi(x_2)\}$ is a pair groupoid $X^{[2]} \to X$ with base $X$, and source and target maps $\text{pr}_1 : (x_1, x_2) \mapsto x_1$ and $\text{pr}_2 : (x_1, x_2) \mapsto x_2$, respectively. Note that the map $\phi$ admits local sections, and hence defines a “quasi-cover” of $M$. For each $k$ there is a linear map

$$\delta := \text{pr}_1^* - \text{pr}_2^* : \Omega^k(X) \to \Omega^k(X^{[2]}) \quad (B.19)$$

which commutes with the exterior derivative. The kernel of $\delta$ is the image of the injection $\phi^* : \Omega^k(M) \hookrightarrow \Omega^k(X)$, while its image is the subspace of forms in $\Omega^k(X^{[2]})$ which commute with the multiplication in the groupoid $X^{[2]} \to X$. Due to the bundle gerbe product, the bundle $E$ yields a $\text{U}(1)$-groupoid extension of $X^{[2]}$ and there are isomorphisms $E_{(x,x)} \cong \mathbb{C}$ and $E_{(x_1,x_2)} \cong E^*_{(x_2,x_1)}$.

A connective structure $(\nabla, B)$ on a bundle gerbe

\[
\begin{array}{ccc}
E & \xrightarrow{\text{pr}_1} & X^{[2]} \\
\downarrow & \nearrow \quad \downarrow \text{pr}_2 & \quad \downarrow \phi \\
X & \xrightarrow{\phi} & M
\end{array}
\]

is given by a unitary connection $\nabla = d + A$ on $E$ which commutes with the multiplication on $E$. Then there exists a (not unique) curving $B \in \Omega^2(X)$ such that

$$F_{\nabla} = \delta(B) = \text{pr}_1^*(B) - \text{pr}_2^*(B) \in \Omega^2(X^{[2]}) \quad (B.21)$$

Since $\delta(dB) = d\delta(B) = dB_{\nabla} = 0$, it satisfies $dB = \phi^*(H)$ for a unique three-form $H = H_{(\nabla, B)}$ with $\varpi = - \frac{1}{2\pi i} H \in \Omega^3_{cl, \mathbb{Z}}(M)$. The curvature $\varpi$ is again the de Rham representative of the Dixmier-Douady class $dd(\mathcal{G}) \in H^3(M, \mathbb{Z})$, which is the obstruction to triviality of the bundle gerbe.

For the Deligne cohomology of bundle gerbes with connective structure, choose a Stein cover $U = \{U_a\}_{a \in I}$ of $M$ with local sections $\psi_a : U_a \to X$ of $\phi : X \to M$, and let $X_U := \bigsqcup_a U_a$ be the nerve of $U$ with the obvious surjective submersion $X_U \to M$. Then $X_U^{[2]} = \bigsqcup_{a,b} U_{ab}$, and the sections $\psi_a$ define a fiber preserving map $\psi : X_U \to X$. We use this map to pullback the bundle gerbe $(E, X)$ to a stably isomorphic\(^{22}\) bundle gerbe $(\psi^*(E), X_U)$, which is just a collection of line bundles $\psi^*(E)_{ab} \to U_{ab}$ satisfying the cocycle

\[\text{\footnotesize\textsuperscript{21}}\text{In the case of a bundle } p\text{-gerbe, } E \text{ would be a bundle } p - 1\text{-gerbe. For simplicity, we restrict ourselves here to bundle (1-)gerbes.} \]

\[\text{\footnotesize\textsuperscript{22}}\text{The notion of stable isomorphism of bundle gerbes is explained below.} \]

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conditions for a local gerbe. We can trivialize the line bundles by unit norm sections \( \sigma_{ab} : U_{ab} \to \psi^*(E)_{ab} \) over each overlap. Then we may multiply \( \sigma_{ab} \) and \( \sigma_{bc} \) together using the bundle gerbe product; on \( U_{abc} \) we have \( \sigma_{ab} \sigma_{bc} = g_{abc} \sigma_{ac} \), and associativity implies that \( g_{abc} : U_{abc} \to U(1) \) is a Čech two-cocycle. We also define \( A_{ab} \in \Omega^1(U_{ab}) \) by \( A_{ab} = (\psi_a, \psi_b)^*(A) \), so that \( \nabla \sigma_{ab} = A_{ab} \otimes \sigma_{ab} \), and \( B_a \in \Omega^2(U_a) \) by \( B_a = \psi_a^*(B) \). From (B.21) the cocycle conditions (B.13) then easily follow.

**Example.** Let \( M = \mathbb{T}^d = V/\Lambda \) be a \( d \)-dimensional torus, where \( V = \mathbb{R}^d \) is a real vector space of dimension \( d \) and \( \Lambda = \mathbb{Z}^d \) a lattice in \( V \) of maximal rank. Let \( X = V \), and let \( \phi : V \to \mathbb{T}^d \) be the universal cover of \( \mathbb{T}^d \); it has disconnected fibers whose connected components are labeled by “winding numbers” \( w = (w^1, \ldots, w^d) \in \Lambda \). We write \( v = (v^1, \ldots, v^d) \) for a vector in \( V \). Then \((u, v) \in V^2 \) if and only if \( w := u - v \in \Lambda \), so there is a disconnected union

\[ V^2 = \bigsqcup_{w \in \Lambda} V_w \]  

(B.22)

for all \( w \in \Lambda \). Define a line bundle \( E \to V^2 \) by \( E := \bigsqcup_{w \in \Lambda} E_w \), with \( E_w \cong V \times \mathbb{C} \) the trivial line bundle over each connected component endowed with trivial fiberwise product induced by the multiplication in \( \mathbb{C} \). Then \( G(\mathbb{T}^d) = (E, V) \) is a bundle gerbe on \( \mathbb{T}^d \). If \( dx^i, i = 1, \ldots, d \), denotes the basis for \( H^1(\mathbb{T}^d, \mathbb{Z}) = \Lambda^* \) dual to a canonical basis of one-cycles \( \gamma_i \) for \( H_1(\mathbb{T}^d, \mathbb{Z}) = \Lambda \), i.e., \( \oint_{\gamma_i} dx^j = \delta^j_i \), then the pullback by \( \phi \) of the projections \( pr_1, pr_2 : V^2 \to V \) of the basis forms \( dv^i := \phi^*(dx^i) \) will also be denoted \( dv^i \) on each connected component \( V_w \) (note that \( dv^i = dx^i \) for \((u, v) \in V^2 \)). For each winding sector \( w \in \Lambda \), endow the line bundle \( E_w \to V \) with connection \( \nabla_w = d + A_w \) given by

\[ A_w = \frac{1}{3!} \varepsilon_{ijk} w^i v^j d v^k \]  

(B.23)

where \( \varepsilon_{ijk} \) is a totally anti-symmetric constant three-tensor. Then the collection of one-forms \( A = (A_w)_{w \in \Lambda} \) are part of a connective structure for the bundle gerbe \( G(\mathbb{T}^d) \). One has \( F_{\nabla_w} = dA_w = \delta_w(B) \), where \( \delta_w \) is the restriction of the image of the map \( \delta \) to \( \Omega^2(V) \to \Omega^2(V_w) \), and the curving \( B \in \Omega^2(V) \) is given by

\[ B = \frac{1}{3!} \varepsilon_{ijk} v^j dv^k \wedge dv^k . \]  

(B.24)

Since \( dB = \frac{1}{3!} \varepsilon_{ijk} dv^i \wedge dv^j \wedge dv^k \), the curvature \( H \in \Omega^3(\mathbb{T}^d) \) reads as

\[ H = \frac{1}{3!} \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k . \]  

(B.25)

When \( d = 3 \) and \( \varepsilon_{ijk} = \varepsilon_{ijk} \), then \( H \in \Omega^3_{cl, \mathbb{Z}}(\mathbb{T}^3) \) and the Dixmier-Douady class of the bundle gerbe \( G(\mathbb{T}^3) \) is the natural generator of \( H^3(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z} \).
Stable isomorphism. For any line bundle $T \to X$ we define a bundle gerbe

$$
\begin{array}{ccc}
\delta(T) & \to & T \\
\downarrow & & \downarrow \\
X^{[2]} & \xrightarrow{\text{pr}_1} & X \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi} & M
\end{array}
$$

where $\delta(T) = \text{pr}_1^*(T) \otimes \text{pr}_2^*(T^*)$, i.e., $\delta(T)_{(x_1, x_2)} = T_{x_2} \otimes T^*_{x_1}$. A bundle gerbe $G$ over $M$ which is isomorphic to a bundle gerbe of the form $(\delta(T), X)$ has vanishing Dixmier-Douady class $\dd(G) = 0$ and is said to be trivial. The gerbe $(\delta(T), X)$ has a natural connective structure provided by choosing a connection $\nabla$ on $T \to X$, and taking the induced connection $\nabla' := \delta(\nabla) = \text{pr}_1^*(\nabla) - \text{pr}_2^*(\nabla)$ on $\delta(T) \to X^{[2]}$ with curving $B = F_{\nabla} \in \Omega^2(X)$; then $dB = 0$ so $(\delta(T), X)$ is a flat gerbe, i.e., $H = 0$. More generally, a connective structure on a trivial gerbe is specified by a two-form $b \in \Omega^2(M)$ such that

$$
B = F_{\nabla} + \pi^*(b) .
$$

Its curvature is

$$
H = db ,
$$

and the Deligne two-cocycle $(g_{abc}, A_{ab}, B_a)$ of this connective structure is a gauge transform of the trivial two-cocycle $(1, 0, b)$ by a Deligne two-coboundary $d(h_{ab}, a_a)$, cf. (B.14),

$$
g_{abc} = h_{ab} h_{bc} h_{ca} , \quad A_{ab} = a_{a} - a_{b} + \frac{1}{2\pi i} d \log h_{ab} \quad \text{and} \quad B_a = b + da_a ,
$$

where $(h, a)$ is the Deligne one-cocycle of the line bundle $(T, \nabla)$. The gauge equivalence classes of connective structures on the trivial gerbe thus form the group $\Omega^2(M) / \Omega^2_{\text{cl}, \mathbb{Z}}(M)$ of topologically trivial $B$-fields on $M$.

A stable isomorphism between bundle gerbes $G = (E, X)$ and $G' = (E', X')$ with connective structure is a trivialization of the product $G^* \otimes G' := (E^* \otimes E', X \times_M X')$ as a gerbe with connective structure; their connections coincide $\nabla' = \nabla$, while their $B$-fields and $H$-fluxes are related by $B' = B + F_{\nabla'} + \phi^*(b)$ and $H' = H + db$ for some two-form $b \in \Omega^2(M)$. The set of stable isomorphism classes of bundle gerbes with connective structure is the Cheeger-Simons differential cohomology group $\hat{H}^3(M)$.

C. Courant algebroids

In this appendix we collect relevant definitions and results concerning Courant algebroids, which are the appropriate generalizations of Lie algebroids arising in higher quantization. A more detailed overview of this material can be found e.g. in [84].
**Courant algebroids.** Courant algebroids are symplectic Lie 2-algebroids [53]. Explicitly, consider a vector bundle $E \to M$ over a smooth manifold $M$ equipped with a non-degenerate symmetric bilinear form $\langle -,- \rangle$ and a skew-symmetric bracket
\[
[-,-]_E : C^\infty(M,E) \otimes C^\infty(M,E) \to C^\infty(M,E),
\]
together with an anchor map $\rho : E \to TM$. We define further the *Jacobiator* $J : C^\infty(M,E) \otimes C^\infty(M,E) \otimes C^\infty(M,E) \to C^\infty(M,E)$ by
\[
J(\psi_1,\psi_2,\psi_3) := \left[[\psi_1,\psi_2]_E,\psi_3\right]_E + \left[[\psi_2,\psi_3]_E,\psi_1\right]_E + \left[[\psi_3,\psi_1]_E,\psi_2\right]_E,
\]
a ternary map $T : C^\infty(M,E) \otimes C^\infty(M,E) \otimes C^\infty(M,E) \to C^\infty(M)$ by
\[
T(\psi_1,\psi_2,\psi_3) = \left[[\psi_1,\psi_2]_E,\psi_3\right]_E + \left[[\psi_2,\psi_3]_E,\psi_1\right]_E + \left[[\psi_3,\psi_1]_E,\psi_2\right]_E,
\]
and the pullback $\mathcal{D} : C^\infty(M) \to C^\infty(M,E)$ of the exterior derivative $d$ via the adjoint map $\rho^*$ by
\[
\langle \mathcal{D}f, \psi \rangle = \rho^*_\psi(f),
\]
where $f \in C^\infty(M)$ and $\psi, \psi_i \in C^\infty(M,E)$.

Such a vector bundle is called a *Courant algebroid* [85] if the following conditions are satisfied:

(i) The Jacobi identity holds up to an exact expression: $J(\psi_1,\psi_2,\psi_3) = DT(\psi_1,\psi_2,\psi_3)$;
(ii) The anchor map $\rho$ is compatible with the bracket: $\rho_{[\psi_1,\psi_2]_E} = [\rho_{\psi_1},\rho_{\psi_2}]_{TM}$;
(iii) There is a Leibniz rule: $[\psi_1,f\psi_2]_E = f[\psi_1,\psi_2]_E + \rho_{\psi_1}(f)\psi_2 - \frac{1}{2}\langle \psi_1,\psi_2 \rangle \mathcal{D}f$;
(iv) $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0$;
(v) $\rho_{\psi}(\langle \psi,\psi_i \rangle_E) = \langle [\psi,\psi_i]_E + \frac{1}{2}D\langle \psi,\psi_i \rangle,\psi_2 \rangle + \langle \psi_1, [\psi,\psi_2]_E + \frac{1}{2}D\langle \psi,\psi_2 \rangle \rangle$;

where $\psi, \psi_i \in C^\infty(M,E)$ and $f, g \in C^\infty(M)$.

An involutive Lagrangian sub-bundle $D \subseteq E$, i.e., $D \cong D^\perp$ with respect to the metric $\langle -,- \rangle$, is called a *Dirac structure* along $M$. The integrability condition for a Dirac structure is the requirement that its space of sections $C^\infty(M,D)$ is closed under the bracket $[-,-]_E$ of the Courant algebroid. Dirac structures provide a generalized and unified framework in which to treat symplectic and Poisson structures on manifolds.

**Exact Courant algebroids.** The Courant algebroid we are exclusively interested in is given by the *standard Courant algebroid* structure on the Pontryagin bundle $C = TM \oplus T^*M$. On sections of $C$, we define the *Dorfman bracket*
\[
(X_1,\alpha_1) \circ (X_2,\alpha_2) := [X_1,X_2]_{TM} + \mathcal{L}_{X_1}\alpha_2 - \iota_{X_2}d\alpha_1.
\]
This bracket is convenient, as it satisfies a Leibniz rule: $(\psi_1 \circ (\psi_2 \circ \psi_3) = (\psi_1 \circ \psi_2) \circ \psi_3 + \psi_2 \circ (\psi_1 \circ \psi_3)$ for $\psi_i \in C^\infty(M,C)$. We are, however, more interested in the *Courant(-Dorfman) bracket* which is the skew-symmetrization of the Dorfman bracket given by
\[
\left[(X_1,\alpha_1), (X_2,\alpha_2) \right] := ([X_1,X_2]_{TM}, \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 - \frac{1}{2}d(\iota_{X_1}\alpha_2 - \iota_{X_2}\alpha_1)).
\]
The Courant algebroid structure on $C$ is given by the Courant bracket, the metric induced by the natural pairing between $TM$ and $T^*M$,

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle = \iota_{X_1} \alpha_2 + \iota_{X_2} \alpha_1 ,$$  \hspace{1cm} (C.7)

and the anchor map is the trivial projection $\rho : C \to TM$; the map $\mathcal{D} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M, C)$ is given by $\mathcal{D} f = \frac{1}{2} df$. This is an exact Courant algebroid, i.e., it fits into the short exact sequence

$$0 \to T^*M \xrightarrow{\rho^*} C \xrightarrow{\rho} TM \to 0 .$$  \hspace{1cm} (C.8)

The bundle metric on $C$ is of split signature, and the cotangent bundle $T^*M$ is a Lagrangian sub-bundle defining a Dirac structure along $M$.

Conversely, any closed three-form $\varpi$ on $M$ yields an exact Courant algebroid with a Lagrangian splitting: We endow $C = TM \oplus T^*M$ with the structure of a Courant algebroid with the $\varpi$-twisted Courant bracket given by [86]

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{\varpi} := \left( [X_1, X_2]_{TM}, \mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 - \frac{1}{2} d(\iota_{X_1} \alpha_2 - \iota_{X_2} \alpha_1) + \iota_{X_1} \iota_{X_2} \varpi \right) ,$$  \hspace{1cm} (C.9)

and the remaining structure maps given as above. Every exact Courant algebroid on $M$ is isomorphic to one of this form; given a Lagrangian splitting $\lambda : TM \to C$, the closed three-form $\varpi \in Z^3(M)$ is obtained as

$$\varpi(X_1, X_2, X_3) = \langle [\lambda(X_1), \lambda(X_2)]_C, \lambda(X_3) \rangle .$$  \hspace{1cm} (C.10)

Given two closed three-forms $\varpi$ and $\varpi'$, the associated exact Courant algebroids are isomorphic if and only if $[\varpi] = [\varpi']$ in $H^3(M, \mathbb{R})$. Thus exact Courant algebroids over a manifold $M$ are classified by the degree three cohomology $H^3(M, \mathbb{R})$. In particular, given a bundle gerbe with connective structure encoded in a Deligne two-cocycle $[(g_{abc}, A_{ab}, B_a)]$ with respect to a Stein cover $U = \{ U_a \to M \}$ of $M$, we put the standard Courant algebroid $C_a := TU_a \oplus T^*U_a$ on each patch $U_a$ and glue them together using $dA_{ab}$; this gives an exact Courant algebroid $C \to M$ with a splitting $\lambda : TM \to C$ given by the curving $(B_a)$. Under this correspondence, a Dirac structure $D \subset C$ along $M$ corresponds to a canonical flat $D$-connection on the associated gerbe.

For details on how to integrate exact Courant algebroids, see [87, 88] and references therein. Polarization in this context should be described as a lift of a Dirac structure $D$ of the standard Courant algebroid $C = TM \oplus T^*M$ to an integrating Courant groupoid $TG \oplus T^*G$ over $TM \oplus \mathfrak{A}^*(G)$, where $G \rightrightarrows M$ is a Lie groupoid. Note that the vector bundle $D \to M$ is naturally a Lie algebroid with bracket given by the Courant bracket and anchor map the restriction to $D$ of the canonical projection $E \to TM$; the general problem of integration of Dirac structures was solved in [72].

**Lie 2-algebras.** Given an exact Courant algebroid $C$ associated to a 2-plectic manifold, we can define an associated Lie 2-algebra$^{23}$ $L_\infty(C)$ as follows [89, 53]: The underlying

\footnote{Recall that a (semistrict) Lie 2-algebra is a 2-term $L_\infty$-algebra.}
graded vector space is

\[ L_\infty(C) = L_{-1} \oplus L_0 \quad \text{with} \quad L_0 = C^\infty(M, C) \quad \text{and} \quad L_{-1} = C^\infty(M), \] (C.11)

equipped with maps

\[
\begin{align*}
\mu_1(f + \psi) & := Df, \\
\mu_2(f_1 + \psi_1, f_2 + \psi_2) & := [\psi_1, \psi_2]_E + \frac{1}{2}(\langle \psi_1, Df_2 \rangle - \langle Df_1, \psi_2 \rangle), \\
\mu_3(f_1 + \psi_1, f_2 + \psi_2, f_3 + \psi_3) & := -T(\psi_1, \psi_2, \psi_3),
\end{align*}
\] (C.12)

where \( f, f_i \in C^\infty(M) \) and \( \psi, \psi_i \in C^\infty(M, C) \).

**Example.** Let \( M = \mathbb{T}^d \) be the \( d \)-dimensional torus with the constant \( H \)-flux (B.25). In the local coordinates \((x^i)\) for \( \mathbb{T}^d \) used in (B.25), a natural frame for \( TM \oplus T^*M \) is given by

\[
X_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad P^i = dx^i.
\] (C.13)

Writing \( X_i \) for \((X_i, 0)\) and \( P^i \) for \((0, P^i)\) for simplicity, the metric is given by

\[
\langle X_i, P^j \rangle = \delta^i_j.
\] (C.14)

The corresponding twisted Courant-Dorfman algebra is isomorphic to the \( d \)-dimensional Heisenberg algebra, realized as the 2-step nilpotent Lie algebra of rank \( d \) with the non-trivial brackets

\[
[X_i, X_j]_H = \omega_{ijk} P^k.
\] (C.15)

Together with the remaining non-trivial structure map

\[
T(X_i, X_j, X_k) = 3 \omega_{ijk},
\] (C.16)

this yields the Lie 2-algebra describing the nonassociative deformation of the closed string phase space by a constant background \( H \)-flux in the \( R \)-space duality frame [19, 64, 24].

**Generalized geometry.** In *generalized geometry*, the standard Courant algebroid \( C = TM \oplus T^*M \) is called the *generalized tangent bundle* and it is a replacement for the ordinary tangent bundle. The Courant-Dorfman bracket is similarly thought of as a replacement for the ordinary commutator of vector fields. Just as the group of diffeomorphisms \( \text{Diff}(M) \) acts as bundle maps of \( TM \) preserving the Lie bracket \([-,-]_{TM} \), the group \( \text{Diff}(M) \ltimes \Omega^2_{\text{cl}}(M) \) acts as bundle morphisms of \( C \) preserving the Courant algebroid structure.

More generally, for any Courant algebroid \( E \to M \) we define a *generalized complex structure* to be a vector bundle map \( J : E \to E \) with \( J^2 = -1 \) which preserves the fiberwise metric and satisfies an integrability condition. The map \( J \) can be equivalently encoded by a complex Dirac structure \( D \subset E \otimes \mathbb{C} \) which is transverse to its complex
conjugate $\overline{D}$. The bundle $D$ corresponds to the $+i$-eigenbundle of the complexification of $J$ and $\overline{D}$ to the $-i$-eigenbundle. Then $E \otimes \mathbb{C}$ is a complex Courant algebroid with a splitting $E \otimes \mathbb{C} = D \oplus \overline{D}$ into Dirac sub-bundles. This induces flat $\overline{D}$-connections on the corresponding gerbes which may be thought of as generalized holomorphic structures in prequantization, see [90].

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