Graph Prediction in a Low-Rank and Autoregressive Setting

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Abstract

We study the problem of prediction for evolving graph data. We formulate the problem as the minimization of a convex objective encouraging sparsity and low-rank of the solution, that reflect natural graph properties. The convex formulation allows to obtain oracle inequalities and efficient solvers. We provide empirical results for our algorithm and comparison with competing methods, and point out two open questions related to compressed sensing and algebra of low-rank and sparse matrices.

1. Introduction

We study the prediction problem where the observation is a sequence of graphs adjacency matrices \((A_t)_{0 \leq t \leq T}\) and the goal is to predict \(A_{T+1}\). This type of problem arises in applications such as recommender systems where, given information on purchases made by some users, one would like to predict future purchases. In this context, users and products can be modeled as nodes of a bipartite graph, while purchases or clicks are modeled as edges.

In functional genomics and systems biology, estimating regulatory networks in gene expression can be performed by modeling the data as graphs and fitting predictive models is a natural way for estimating evolving networks in these contexts.

A large variety of methods for link prediction only consider predicting from a single static snapshot of the graph - this includes heuristics (Liben-Nowell & Kleinberg, 2007; Sarkar et al., 2010), matrix factorization (Koren, 2008) or probabilistic methods (Taskar et al., 2003). More recently, some works have investigated using sequences of observations of the graph to improve the prediction, such as using regression on features extracted from the graphs (Richard et al., 2010), using matrix factorization (Koren, 2010), or in some special cases probabilistic techniques. Most techniques, however, do not explicitly take into account the inherently sparse nature of usual sequences of adjacency matrices. In this work, we extend the work of (Richard et al., 2010) to address this and propose in addition a more principled way of predicting using features extracted from the sequence of graph snapshots.

We make the following assumptions about the graph sequence (represented by adjacency matrices \(A_t\)):

1. Low-rank. \(A_t\) has low rank. This reflects the presence of highly connected groups of nodes such as communities in social networks.

2. Autoregressive linear features. We assume given a linear map \(\omega : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^d\) defined by a set of \(n \times n\) matrices \((\Omega_i)_{1 \leq i \leq d}\) such that the vector time series \(\omega(A_t)\) has an autoregressive evolution:

\[
\omega(A_{t+1}) = W_0^T \omega(A_t) + N_t
\]

where \(W_0 \in \mathbb{R}^{d \times d}\) is a sparse matrix such that \((\omega(A_t))_{t \geq 0}\) is stationary. An example of linear features is degrees that is a popularity measure in social and commerce networks.
2. Formulation of an optimization problem

In order to reflect the stationarity assumption on \( \omega(A_t) \) we use a convex loss function

\[
\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+
\]

to penalize the dissimilarity between two feature vectors at successive time steps. Let us introduce

\[
X_{T-1} = \left( \begin{array}{c} \omega(A_0) \ \top \\ \omega(A_1) \ \top \\ \vdots \\ \omega(A_{T-1}) \ \top \end{array} \right) \in \mathbb{R}^{T \times d}
\]

and

\[
X_T = \left( \begin{array}{c} \omega(A_1) \ \top \\ \omega(A_2) \ \top \\ \vdots \\ \omega(A_T) \ \top \end{array} \right) \in \mathbb{R}^{T \times d}
\]

We also use \( \ell \) to design the elementwise extension of \( \ell \) to \( X_s \). In case of quadratic loss, we consider the following \( \ell_1 \) penalized regression objective:

\[
J_1(W) = \frac{1}{d} \|X_{T-1}W - X_T\|_2^2 + \kappa \|W\|_1.
\]

To predict \( A_{T+1} \), we propose a regression term penalized by the sum of \( \ell_1 \) and trace norm in the same fashion as in (Richard et al., 2012) in order to predict the future graph \( A_{T+1} \) given the prediction of its features \( \omega(A_T)^\top W \) should approximate \( \omega(S) \) well:

\[
J_2(S, W) = \frac{1}{d} \|\omega(A_T)^\top W - \omega(S)\|_F^2 + \tau \|S\|_* + \gamma \|S\|_1.
\]

The overall objective function we consider here is the sum of the two partial objectives \( J_1 \) and \( J_2 \), which is convex as \( J_1 \) and \( J_2 \) are both convex.

\[
\mathcal{L}(S, W) = \frac{1}{d} \|X_{T-1}W - X_T\|_2^2 + \kappa \|W\|_1 + \frac{1}{d} \|\omega(A_T)^\top W - \omega(S)\|_F^2 + \tau \|S\|_* + \gamma \|S\|_1.
\]

Let us introduce the linear map \( \Phi \) defined by

\[
\Phi(S, W) = \left( X_{T-1}W, \ \omega(S)\top - \omega(A_T)^\top W \right).
\]

The objective can be written as a penalized least squared regression on the joint variable \((S, W)\):

\[
\mathcal{L}(S, W) = \frac{1}{d} \|\Phi(S, W) - (X_T, 0)\|_2^2 + \gamma \|S\|_1 + \tau \|S\|_* + \|W\|_1.
\]

3. Oracle inequality

Define \( (\delta, \epsilon) = (X_T, 0) - \Phi(A_{T+1}, W_0) \), i.e.,

\[
\delta = X_T - X_{T-1}W_0
\]

and

\[
\epsilon = W_0^\top \omega(A_T) - \omega(A_{T+1})^\top
\]

We define \( M = \sum_{i=1}^d \epsilon_i \Omega_i \), where \( \Omega_i \) are defined in (1) and let

\[
\Xi = X_{T-1}^\top \delta - \omega(A_T) \epsilon^\top.
\]

We defined \( M \) and \( \Xi \) such that they verify

\[
\langle (\delta, \epsilon), \Phi(S, W) \rangle = \langle (M, \Xi), (S, W) \rangle.
\]

The following result can be proved using the tools introduced in (Koltchinskii et al., 2011).

**Proposition 1.** Let \( (\hat{S}, \hat{W}) \) be the minimizers of \( \mathcal{L}(S, W) \) over a convex cone \( S \times W \). Suppose that

1. for some \( \mu > 0 \), and for any \( S_1, S_2 \in S \) and \( W_1, W_2 \in W \),

\[
\frac{1}{d} \|\Phi(S_1 - S_2, W_1 - W_2)\|_2^2 \geq \mu^{-2} \left( \|S_1 - S_2\|_F^2 + \tau \|W_1 - W_2\|_F^2 \right)
\]

2. \( \tau \geq \frac{2\mu}{d} \|M\|_{op}, \ \gamma \geq \frac{2(1-\alpha)}{d} \|M\|_\infty, \ \kappa \geq 2\|\Xi\|_\infty \)

for any real number \( \alpha \in (0, 1) \);

then

\[
\|\hat{S} - A_{T+1}\|_F^2 + T \|\hat{W} - W_0\|_F^2 \leq \mu^2 \min \left\{ \frac{\mu^2}{d} \left( \tau \sqrt{\text{rank}(A_{T+1})} \sqrt{\frac{\sqrt{2} + 1}{2} + \gamma \sqrt{\|A_{T+1}\|_0}} \right)^2 + \frac{\mu^2 \kappa^2}{dT} \|W_0\|_0, \ 2\tau \|A_{T+1}\|_* + 2\gamma \|A_{T+1}\|_1 + 2\kappa \|W_0\|_1 \right\}.
\]

The latter inequality shows how the quality of the solution is bounded by the rank and sparsity of the future graph \( A_{T+1} \), and the interplay between these two prior through the parameter \( \alpha \). The dependence in \( T \) quantifies the improvement of the estimation in terms of the number of observations.
4. Algorithms

4.1. Generalized forward-backward algorithm for minimizing \( \mathcal{L} \)

We use the algorithm designed in (Raguet et al., 2011) for minimizing our objective function. Note that this algorithm outperforms the method introduced in (Richard et al., 2010) as it directly minimizes \( \mathcal{L} \) jointly in \((S,W)\) whereas the previous method first estimates \( \hat{W} \) by minimizing a functional similar to \( J_1 \) and then minimizes \( \mathcal{L}(S,\hat{W}) \).

In addition to this we use the novel joint penalty from (Richard et al., 2012) that is more suited for estimating graphs. The proximal operator for the trace norm is given by the shrinkage operation, if \( Z = U \text{diag}(\sigma_1,\cdots,\sigma_n)V^T \) is the singular value decomposition of \( Z \),

\[
\text{prox}_{\tau \|\cdot\|_*}(Z) = U \text{diag}((\sigma_i - \tau)_+)_i V^T.
\]

Similarly, the proximal operator for the \( \ell_1 \)-norm is the soft thresholding operator defined by using the entry-wise product of matrices denoted by \( \circ \):

\[
\text{prox}_{\gamma \|\cdot\|_1} = \text{sgn}(Z) \circ (|Z| - \gamma)_+.
\]

The algorithm converges under very mild conditions when the step size \( \theta \) is smaller than \( \frac{1}{L} \), where \( L \) is the operator norm of \( \Phi \).

**Algorithm 1** Generalized Forward-Backward to Minimize \( \mathcal{L} \)

**Initialize** \( S, Z_1, Z_2, W, q = 2 \)

**repeat**

- Compute \( (G_S, G_W) = \nabla S,W \Phi(S,W) \)
- Compute \( Z_1 = \text{prox}_q\theta_{\|\cdot\|_*}(2S - Z_2 - \theta G_S) \)
- Compute \( Z_2 = \text{prox}_q\theta_{\|\cdot\|_1}(2S - Z_2 - \theta G_S) \)
- Set \( S = \frac{1}{q} \sum_{k=1}^q Z_k \)
- Set \( W = \text{prox}_{\theta G_{\|\cdot\|_1}}(W - \theta G_W) \)

**until** convergence

**return** \((S,W)\) minimizing \( \mathcal{L} \)

4.2. Non-convex Factorization Method

An alternative method to the estimation of low-rank and sparse matrices by penalizing a mixed penalty of the form \( \tau \|S\|_* + \gamma \|S\|_1 \) as in (Richard et al., 2012) is to factorize \( S = UV^\top \) where \( U, V \in \mathbb{R}^{n \times r} \) are sparse matrices, and penalize \( \gamma (\|U\|_1 + \|V\|_1) \). The objective function to be minimized is

\[
\mathcal{J}(U,V,W) = \frac{1}{d} \|X_T - X_T\|^2_F + \kappa \|W\|_1 + \frac{1}{d} \|\omega(A_T)^\top W - \omega(UV^\top)^\top\|^2_2 + \gamma (\|U\|_1 + \|V\|_1)
\]

which is a non-convex function of the joint variable \((U,V,W)\), making the theoretical analysis more difficult. Given that the objective is convex in a neighborhood of the solution, by initializing the variables adequately, we can write an algorithm inspired by proximal gradient descent for minimizing it.

**Algorithm 2** Minimize \( \mathcal{J} \)

**Initialize** \( U,V,W \)

**repeat**

- Compute \( (G_U, G_V, G_W) = \nabla U,V,W \Phi(UV^\top,W) \)
- Set \( U = \text{prox}_{\theta_G\|\cdot\|_1}(U - \theta G_U) \)
- Set \( V = \text{prox}_{\theta_G\|\cdot\|_1}(V - \theta G_V) \)
- Set \( W = \text{prox}_{\theta_G\|\cdot\|_1}(W - \theta G_W) \)

**until** convergence

**return** \((U,V,W)\) minimizing \( \mathcal{J} \)

5. Numerical Experiments

5.1. A generative model for graphs having linearly autoregressive features

Let \( V_0 \in \mathbb{R}^{n \times r} \) be a sparse matrix, \( V_0^\dagger \) its pseudo-inverse such that \( V_0^\dagger V_0 = V_0^\dagger V_0^\top = I_r \). Fix two sparse matrices \( W_0 \in \mathbb{R}^{r \times r} \) and \( U_0 \in \mathbb{R}^{n \times r} \). Now define the sequence of matrices \((A_t)_{t \geq 0}\) for \( t = 1,2,\cdots \) by

\[
A_t = U_{t-1} W_0 + N_t + M_t
\]

for i.i.d sparse noise matrices \( N_t \) and \( M_t \), which means that for any pair of indices \((i,j)\), with high probability \((N_t)_{i,j} = 0 \) and \((M_t)_{i,j} = 0 \).

If we define the linear feature map \( \omega(A) = AV_0^\dagger \), note that

1. The sequence \( \omega(A_t)^\top \) follows the linear autoregressive relation
   \[
   \omega(A_t)^\top = \omega(A_{t-1})^\top W_0 + N_t + M_t W_0^\top.
   \]
2. For any time index \( t \), the matrix \( A_t \) is close to \( U_t V_0^\top \) that has rank at most \( r \)
3. \( A_t \) is sparse, and furthermore \( U_t \) is sparse
5.2. Results

We tested the presented methods on synthetic data generated as in section (5.1). In our experiments the noise matrices $M_t$ and $N_t$ where built by soft-thresholding iid noise $\mathcal{N}(0, \sigma^2)$, $n = 50, T = 10, r = 5, d = 10, \sigma = .5$. After choosing the parameters $\tau, \gamma, \rho$ by 10-fold cross-validation, we compare our methods to standard baselines in link prediction (Liben-Nowell & Kleinberg, 2007). We use the area under the ROC curve as the measure of performance and report empirical results averaged over 10 runs. Nearest Neighbors (NN) relies on the number of common friends between each pair of nodes, which is given by $A_T = \sum_{t=1}^{T} A_t$ and we denote by Shrink the low-rank approximation of $A_T$. Since $V_0$ is unknown we consider the feature map $\omega(S) = SV$ where $\tilde{A}_T = U\Sigma V^\top$ is the SVD of $A_T$.

| Method     | AUC         |
|------------|-------------|
| NN         | 0.8691 ± 0.0168 |
| Shrink     | 0.8739 ± 0.0169 |
| min $\mathcal{L}$ | 0.9094 ± 0.0176 |
| min $\mathcal{J}$ | 0.9454 ± 0.0087 |

*Table 1. Performance of algorithms in terms of Area Under the ROC Curve.*

6. Discussion

The experiments suggest the empirical superiority of the proposed approaches to the standard baselines. It is very intriguing that the non-convex matrix factorization outperforms the convex rival. A possible explanation is that minimizing the nuclear norm by using the shrinkage operator results in factorizations of the solution by two orthogonal matrices, which conflicts with the sparsity of the solution. The other benefit of the non-convex formulation is its scalability, as the proximal method proposed for the convex formulation scales in $O(n^2)$ in storage and $O(n^3)$ in time. Several questions open perspectives for further investigations.

1. **Choice of the feature map $\omega$**. In the current work we used the projection onto the vector space of the top-$r$ singular vectors of the cumulative adjacency matrix as the linear map $\omega$, and this choice has shown empirical superiority to other choices. The question of choosing the best measurement to summarize graph information as in compress sensing seems to have both theoretical and application potential.

2. **Characterization of sparse and low-rank matrices**. Can all the sparse and low-rank matrices $S$ be written as $S = UV^\top$ where $U, V \in \mathbb{R}^{n \times r}$ are both sparse? Or in other terms, what is the relation between the solution of problems penalized by $\|U\|_1 + \|V\|_1$ -such as $\mathcal{J}$- and those, e.g. $\mathcal{L}$, penalized by $\|S\|_1 + \beta\|S\|_*$?

[Appendix : Proof of proposition (1)]

**A. Preliminary Tools**

**Definition 1** (Orthogonal projections associated with $S$). *Let $S \in \mathbb{R}^{n \times n}$ be a rank $r$ matrix. We can write the SVD of $S$ in two ways: $S = \sum_{j=1}^{r} \sigma_j u_j v_j^\top$ or $S = U\Sigma V^\top$, where $U, V \in \mathbb{R}^{n \times r}$ are orthogonal and $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_r)$. Let $U^\perp$ and $V^\perp$ matrices of size $n \times (n-r)$ ortho-normally completing the bases of $U$ and $V$, and define the projections onto the vector spaces spanned by vectors $u_i$ and $v_i$ for $i = 1, \cdots, r$:

$$P_U = UU^\top, \quad P_U^\perp = U^\perp U^\perp^\top$$
$$P_V = VV^\top, \quad P_V^\perp = V^\perp V^\perp^\top$$

and define the orthogonal projection

$$\mathcal{P}_S(B) = B - P_U^\perp B P_V^\perp$$

We highlight the fact that $\mathcal{P}_S(B)$ can also be written as

$$\mathcal{P}_S(B) = P_U B P_V + P_U B P_V^\perp + P_U^\perp B P_V$$

or

$$\mathcal{P}_S(B) = P_U B + P_U^\perp B P_V$$

We know that $\text{rank}(\mathcal{P}_S(B)) \leq \text{rank}(B)$ and $\text{rank}(P_U B P_V) \leq \text{rank}(B)$. The following inequalities will also be useful:

**Lemma 1** (Rank inequalities). *For any matrix $B$,

1. $\text{rank}(\mathcal{P}_S(B)) \leq 2 \text{rank}(S)$
2. $\text{rank}(P_U B P_V) \leq \text{rank}(S)$

**Lemma 2** (Orthogonality of the decomposition). *For any matrix $B$, with the same notations, we have

$$B = P_U B P_V + P_U B P_V^\perp + P_U^\perp B P_V$$

and the 4 terms are pairwise orthogonal. It follows that
1. We have the identity
\[ \|B\|^2_F = \|P_U B P_V\|^2_F + \|P_{\perp} B P_V\|^2_F + \|P_U B P_{\perp}\|^2_F \]

2. It follows that \( \|P_U B P_V\|_F \leq \|P_{S}(B)\|_F \)

B. Proof

We have for any \((S, W) \in S \times W\), by optimality of \((\hat{S}, \hat{W})\):

\[
\frac{1}{d} \left( \| \Phi(\hat{S} - A_{T+1}, \hat{W} - W_0)\|_F^2 - \| \Phi(S - A_{T+1}, W - W_0)\|_F^2 \right) 
\leq \frac{2}{d} \left( \| \Phi(\hat{S}, \hat{W})\|_F^2 - \| \Phi(S, W)\|_F^2 \right) 
+ \tau(\|S\|_* - \|\hat{S}\|_*) + \gamma(\|S\|_1 - \|\hat{S}\|_1) + \kappa(\|W\|_1 - \|\hat{W}\|_1)
\]

For proving the other bound, we start by setting some notations. Let \( S \in S \), and let \( r = \text{rank}(S) \), \( k = \|S\|_0 \), \( q = \|W\|_0 \). Let \( S = U \text{diag}(\sigma_1, \ldots, \sigma_r) V^T \) be the SVD of \( S \) and let \( S = \Theta_S \circ |S| \), \( W = \Theta_W \circ |W| \), where \( \Theta_S \in \{0, \pm 1\}^{n \times n} \) and \( \Theta_W \in \{0, \pm 1\}^{d \times d} \) are sign matrices of \( S \) and \( W \) such that \( \|\Theta_S\|_0 = k \), \( \|\Theta_W\|_0 = q \) and \( \circ \) is the entry-wise product. Let \( Z = \tau Z_s + \gamma Z_v \)

\[
= \tau \left( \sum_{j=1}^r u_j v_j^T + P_{U \perp} G_s P_{V \perp} \right) + \gamma \left( \Theta_S + G_1 \circ \Theta_S \right)
\]
de note an element of the subgradient of the convex function \( S \mapsto \tau\|S\|_* + \gamma\|S\|_1 \), so \( \|G_s\|_{op} \leq 1 \) and \( \|G_1\|_{\infty} \leq 1 \). There exist \( G_1 \) and \( G_s \) such that

\[
\langle Z, \hat{S} - S \rangle = \tau \left( \sum_{j=1}^r u_j v_j^T \langle \hat{S} - S \rangle \right) + \tau \|P_{U \perp} \hat{S} P_{V \perp} \|_* + \gamma \|\Theta_S + G_1 \circ \Theta_S \|_1
\]

We use the two standard inequalities of convex function subdifferentials \( \langle \partial C(\hat{S}, \hat{W}), (\hat{S} - S, \hat{W} - W) \rangle \leq 0 \) and \( \langle Z - S, Z - Z \rangle \geq 0 \) and a similar inequality on subdifferentials on \( \hat{W} \) and \( W \) of \( W \mapsto \|W\|_1 \), denoted by \( \tilde{Q} \) and \( Q \). We get

\[
\langle \partial C(\hat{S}, \hat{W}), (\hat{S} - S, \hat{W} - W) \rangle 
- \langle Z - Z, \hat{S} - S \rangle - \langle \tilde{Q} - Q, \hat{W} - W \rangle \leq 0 \quad (5)
\]

Therefore we obtain

\[
\langle \nabla_{(S, W)} \Phi(\hat{S}, \hat{W}) - \left( X_T, 0 \right) \|_F^2, (\hat{S} - S, \hat{W} - W) \rangle 
= 2 \langle (\delta, \epsilon), \Phi(\hat{S} - S, \hat{W} - W) \rangle 
- 2 \langle \Phi(\hat{S} - A_{T+1}, \hat{W} - W_0), \Phi(\hat{S} - S, \hat{W} - W) \rangle
\]

The inequality (5) can be written as

\[
\frac{2}{d} \langle \Phi(\hat{S} - A_{T+1}, \hat{W} - W_0), \Phi(\hat{S} - S, \hat{W} - W) \rangle \leq 
- \tau \left( \sum_{j=1}^r u_j v_j^T \langle \hat{S} - S \rangle - \tau \|P_{U \perp} \Phi(\hat{S}) \|_* 
- \gamma \langle \Theta_S, \hat{S} - S \rangle - \gamma \|\Theta_S \|_1 
- \kappa \langle \Theta_W, \hat{W} - W \rangle - \kappa \|\Theta_W \|_1 \right) \quad (6)
\]
Thanks to Cauchy-Schwarz
\[
|\sum_{j=1}^{r} u_j v_j^*, \widehat{S} - S| \leq \sqrt{r} \|P_U(\widehat{S} - S)P_V\|_F
\]
similarly
\[
|\langle \Theta_S, \widehat{S} - S \rangle| \leq \sqrt{K} \|\Theta_S \circ (\widehat{S} - S)\|_F
\]
and
\[
|\langle \Theta_W, \widehat{W} - W \rangle| \leq \sqrt{q} \|\Theta_W \circ (\widehat{W} - W)\|_F
\]
so we have
\[
\begin{align*}
\frac{2}{d} \langle \Phi(\widehat{S} - A_{T+1}, \widehat{W} - W_0), \Phi(\widehat{S} - S, \widehat{W} - W) \rangle & \leq \frac{2}{d} \langle (\delta, \epsilon), \Phi(\widehat{S} - S, \widehat{W} - W) \rangle + \tau \sqrt{r} \|P_U(\widehat{S} - S)P_V\|_F \\
& \quad - \tau \|P_U \bar{S}P_V\|_F + \gamma \sqrt{k} \|\Theta_S \circ (\widehat{S} - S)\|_F - \gamma \|\Theta_S^\perp \circ \bar{S}\|_1 \\
& \quad + \kappa \sqrt{q} \|\Theta_W \circ (\widehat{W} - W)\|_F - \kappa \|\Theta_W^\perp \circ \bar{W}\|_1
\end{align*}
\] (7)

We need to bound \(\langle (\delta, \epsilon), \Phi(\widehat{S} - S, \widehat{W} - W) \rangle\). For this, note that by definition,
\[
\langle (\delta, \epsilon), \Phi(\widehat{S} - S, \widehat{W} - W) \rangle = \langle (M, \Xi), (\widehat{S} - S, \widehat{W} - W) \rangle
\]
and decompose for any \(\alpha \in [0, 1]\)
\[
M = \alpha \left( \mathcal{P}_S(M) + P_U \perp MP_{V\perp} \right) + (1 - \alpha) \left( \Theta_S \circ M + \Theta_S^\perp \circ M \right)
\]
We get by applying triangle inequality, Cauchy-Schwarz, Hölder inequality written for the trace-norm and \(\ell_1\)-norm
\[
\langle M, \widehat{S} - S \rangle
\]
\[
\leq \alpha \left( \|\mathcal{P}_S(M)\|_F \|\mathcal{P}_S(\widehat{S} - S)\|_F \\
+ \|P_U \perp MP_{V\perp}\|_F \|P_U \perp \bar{S}P_{V\perp}\|_F \right) \\
+ (1 - \alpha) \left( \|\Theta_S \circ M\|_F \|\Theta_S \circ (\widehat{S} - S)\|_F \\
+ \|\Theta_S^\perp \circ M\|_F \|\Theta_S^\perp \circ \bar{S}\|_F \right)
\]
by rank and support inequalities obtained again by Cauchy-Schwarz
\[
\langle M, \widehat{S} - S \rangle
\]
\[
\leq \alpha \left( \sqrt{2} \tau \|M\|_{op}\|\mathcal{P}_S(\widehat{S} - S)\|_F \\
+ \|M\|_{op}\|P_U \perp \bar{S}P_{V\perp}\|_F \right) \\
+ (1 - \alpha) \left( \sqrt{K} \|\Theta_S \circ (\widehat{S} - S)\|_F \right) \\
+ \|M\|_{\infty} \|\Theta_S^\perp \circ \bar{S}\|_1
\]
(8)

Now by using
\[
2 \langle \Phi(\widehat{S} - A_{T+1}, \widehat{W} - W_0), \Phi(\widehat{S} - S, \widehat{W} - W) \rangle = \\
\|\Phi(\widehat{S} - A_{T+1}, \widehat{W} - W_0)\|_2^2 + \|\Phi(\widehat{S} - S, \widehat{W} - W)\|_2^2 \\
- \|\Phi(\widehat{S} - A_{T+1}, W - W_0)\|_2^2
\]
we can rewrite the inequality (7) as follows:
\[
\begin{align*}
\frac{1}{d} \left( \|\Phi(\widehat{S} - A_{T+1}, \widehat{W} - W_0)\|_2^2 \\
+ \|\Phi(\widehat{S} - S, \widehat{W} - W)\|_2^2 - \|\Phi(\widehat{S} - A_{T+1})\|_2^2 \right) \\
\leq \frac{2\alpha}{d} \left( \sqrt{2} \tau \|M\|_{op}\|\mathcal{P}_S(\widehat{S} - S)\|_F + \|M\|_{op}\|P_U \perp \bar{S}P_{V\perp}\|_F \right) \\
+ \frac{2(1 - \alpha)}{d} \left( \sqrt{K} \|\Theta_S \circ (\widehat{S} - S)\|_F + \|\Theta_S^\perp \circ \bar{S}\|_1 \right) \\
+ \frac{2}{d} \left( \sqrt{q} \|\Xi\|_{\infty} \|\hat{W} - W\|_F + \|\Xi\|_{\infty} \|\Theta_W^\perp \circ \hat{W}\|_1 \right) \\
+ \tau \sqrt{r} \|P_U(\widehat{S} - S)P_V\|_F - \tau \|P_U \perp \bar{S}P_{V\perp}\|_F \\
+ \gamma \sqrt{k} \|\Theta_S \circ (\widehat{S} - S)\|_F - \gamma \|\Theta_S^\perp \circ \bar{S}\|_1 \\
+ \kappa \sqrt{q} \|\Theta_W \circ (\widehat{W} - W)\|_F - \kappa \|\Theta_W^\perp \circ \bar{W}\|_1 \\
\leq \sqrt{\tau} \left( \frac{2\sqrt{2}}{d} \|M\|_{op} + \tau \right) \|\widehat{S} - S\|_F \\
+ \sqrt{K} \left( \frac{2(1 - \alpha)}{d} \|M\|_{\infty} + \gamma \right) \|\widehat{S} - S\|_F \\
+ \sqrt{q} \left( \frac{2}{d} \|\Xi\|_{\infty} + \kappa \right) \|\hat{W} - W\|_F
\end{align*}
\] (9)

the last inequality being due to the assumptions
\(\tau \geq \frac{2\alpha}{d} \|M\|_{op}, \gamma \geq \frac{2(1 - \alpha)}{d} \|M\|_{\infty}\) and \(\gamma \geq \frac{2}{d} \|\Xi\|_{\infty}\).

So finally, and by using again these assumptions,
\[
\frac{1}{d} \left( \| \Phi(\hat{S} - A_{T+1}, \hat{W} - W_0) \|_F^2 + \| \Phi(\hat{S} - S, \hat{W} - W) \|_F^2 \right) 
\leq
\frac{1}{d} \| \Phi(S - A_{T+1}) \|_F^2 + \left( \sqrt{\tau} (\sqrt{2} + 1) + 2\sqrt{k}\gamma \right) \| \Phi(\hat{S} - S, \hat{W} - W) \|_F^2 + 2\sqrt{\kappa}\| W - \hat{W} \|_F^2 
\leq
\| \Phi(S - A_{T+1}, W - W_0) \|_F^2 + \frac{\mu^2}{\sqrt{d}} \left( \sqrt{\tau} (\sqrt{2} + 1) + 2\sqrt{k}\gamma \right) \| \Phi(\hat{S} - S, \hat{W} - W) \|_F^2 + \frac{2\mu \kappa \sqrt{q}}{dT} \| \Phi(\hat{S} - S, \hat{W} - W) \|_F^2 \quad (10)
\]

and \( bx - x^2 \leq \left( \frac{b}{2} \right)^2 \) gives

\[
\frac{1}{d} \| \Phi(\hat{S} - A_{T+1}, \hat{W} - W_0) \|_F^2 
\leq
\frac{1}{d} \| \Phi(S - A_{T+1}, W - W_0) \|_F^2 + \frac{\mu^2}{4d} \left( 2\sqrt{\kappa} \sqrt{\tau} (\sqrt{2} + 1) + 2\sqrt{k}\gamma \right) + \frac{\mu \kappa \sqrt{q}}{dT} \| \Phi(S - A_{T+1}, W - W_0) \|_F^2 
\]

and the result follows by using

\[
\frac{1}{d} \| \Phi(S_1 - S_2, W_1 - W_2) \|_F^2 
\geq \mu^{-2} \| S_1 - S_2 \|_F^2 + \mu^{-2} \| W_1 - W_2 \|_F^2
\]

and setting \((S, W) = (A_{T+1}, W_0)\):

\[
\| \hat{S} - A_{T+1} \|_F^2 + \| \hat{W} - W_0 \|_F^2 
\leq \frac{\mu^4}{d} \left( \sqrt{\tau} (\sqrt{2} + 1) + 2\sqrt{k}\gamma \right)^2 + \frac{\mu^2 \kappa^2 q}{dT} \]

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