THE FIRST NEGATIVE FOURIER COEFFICIENT OF AN EISENSTEIN SERIES NEWFORM

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Abstract. There have been a number of papers on statistical questions concerning the sign changes of Fourier coefficients of newforms. In one such paper, Linowitz and Thompson gave a conjecture describing when, on average, the first negative sign of the Fourier coefficients of an Eisenstein series newform occurs. In this paper, we correct their conjecture and prove the corrected version.

1. Introduction

For a Dirichlet character $\chi$ and a positive integer $N$, we will denote by $M_k(N,\chi)$ the vector space of modular forms on $\Gamma_0(N)$ of weight $k$, level $N$ and character $\chi$. Let $E_k(N,\chi)$ be the subspace of Eisenstein series and $S_k(N,\chi)$ the subspace of cusp forms. For a prime $p$, we let $T_p$ be the $p$th Hecke operator.

Let $H^*_k(N)$ be the subspace of $S_k(\chi_0,N)$ of newforms with trivial character $\chi_0$. Given a newform $f \in H^*_k(N)$, let $\lambda_f(p)$ be the eigenvalue of $f$ with respect to the Hecke operator $T_p$. The restriction to the trivial character ensures that the sequence $\{\lambda_f(p)\}$ is real. Many authors have studied the sequence of signs of the Hecke eigenvalues of $f$. For example, one could pose questions such as:

(i) Are there infinitely many primes $p$ such that $\lambda_f(p) > 0$ (or $\lambda_f(p) < 0$)?

(ii) What is the first change of sign? More specifically, what is the smallest $n \geq 1$ (or prime $p$) such that $\lambda_f(n) < 0$ (or $\lambda_f(p) < 0$)? This is an analogue of the least quadratic non-residue problem.

(iii) Given an arbitrary sequence of signs $\varepsilon_p \in \{\pm1\}$, what is the number of newforms $f$ (in some family) such that $\text{sgn} \lambda_f(p) = \varepsilon_p$ for all $p \leq x$?

In the cusp form setting, questions (i) and (ii) are answered in [5], [6], and [9]. In this paper, we focus on (iii). Kowalski, Lau, Soundararajan and Wu [6] obtained a lower bound for the proportion of newforms $f \in H^*_k(N)$ whose sequence of eigenvalues $\lambda_f(p)$ has signs coinciding with a prescribed sequence $\{\varepsilon_p\}$:
Theorem 1.1 (Kowalski, Lau, Soundararajan, Wu, 2010). Let $N$ be a squarefree number, $k \geq 2$ an even integer, and $\{\varepsilon_p\}$ a sequence of signs. Then, for any $0 < \varepsilon < \frac{1}{2}$, there exists some $c > 0$ such that

\[
\frac{1}{|H^*_k(N)|} \# \{f \in H^*_k(N) : \text{sgn} \lambda_f(p) = \varepsilon_p \text{ for } p \leq z, \ p \nmid N \} \geq \left( \frac{1}{2} - \varepsilon \right)^{\pi(z)}
\]

for $z = c\sqrt{\log kN \log \log kN}$ provided $kN$ is large enough. Here $\pi(z)$ is the number of primes less than or equal to $z$.

Now, let $\chi_1, \chi_2$ be Dirichlet characters modulo $N_1, N_2$ and for an integer $k > 2$, define the following variant of the sum of divisors function:

\[
(1.1) \quad \sigma_{\chi_1,\chi_2}^{k-1}(n) = \sum_{d|n} \chi_1\left(\frac{n}{d}\right)\chi_2(d)d^{k-1}.
\]

Now assume that $\chi_1$ and $\chi_2$ are not simultaneously principal (mod 1). It is well known (see, for example, [3]) that if $\chi_1\chi_2(-1) = (-1)^k$, then the function

\[
E_k(\chi_1, \chi_2, z) := \frac{\delta(\chi_1)}{2} L(1 - k, \chi_2) + \sum_{n \geq 1} \sigma_{\chi_1,\chi_2}^{k-1}(n)q^n,
\]

is an Eisenstein series of weight $k$, level $N_1N_2$ and character $\chi_1\chi_2$. Here $q = e^{2\pi iz}$ and

\[
\delta(\chi_1) = \begin{cases} 1, & \text{if } \chi_1 \text{ is principal} \\ 0, & \text{otherwise}. \end{cases}
\]

In 1977, Weisinger [11] developed a newform theory for $E_k(N, \chi)$ analogous to the one developed by Atkin and Lehner [1] for cusp forms. In this theory, we have:

- The newforms of $E_k(N, \chi)$ are functions of the form $E_k(\chi_1, \chi_2, z)$ for which $N = N_1N_2$, $\chi = \chi_1\chi_2$, and $\chi_1, \chi_2$ are primitive.
- The eigenvalue of $E_k(\chi_1, \chi_2, z)$ with respect to the Hecke operator $T_p$ is $\sigma_{\chi_1,\chi_2}^{k-1}(p)$. In other words, the eigenvalues of this type of Eisenstein series coincide with its Fourier coefficients.

By exploiting the analytical properties of $\sigma_{\chi_1,\chi_2}^{k-1}(n)$, Linowitz and Thompson [8] answered the three questions mentioned at the beginning of this article for Eisenstein series newforms.

Note that by (1.1), $\sigma_{\chi_1,\chi_2}^{k-1}(n) \in \mathbb{R}$ when $\chi_1, \chi_2$ are real characters. Since we want $E_k(\chi_1, \chi_2, z)$ to be an Eisenstein series, we exclude the case when $\chi_1$ and $\chi_2$ are principal. We call these types of characters quadratic because for every fundamental discriminant $D$, i.e., for each discriminant arising from a quadratic number field, we can associate a real character defined by $\chi_D(m) = (D/m)$. Therefore, counting Eisenstein series newforms of level
$N \leq x$ is equivalent to counting fundamental discriminants $D_1, D_2$ with $|D_1D_2| \leq x$. Let

$$D := \{(D_1, D_2) : |D_1D_2| \leq x\}.$$ 

Taking all of these facts into consideration, Linowitz and Thompson [8] showed:

**Theorem 1.2** (Linowitz, Thompson, 2015). Let $\{p_1, \ldots, p_k\}$ be a sequence of primes and $\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\} \in \{-1, 0, 1\}$ a sequence of signs. Then,

$$\frac{1}{|D|} \# \{(D_1, D_2) \in D : \text{sgn}\sigma_{\chi_1, \chi_2}^{k-1}(p_i) = \varepsilon_{p_i}, \ 1 \leq i \leq k\}$$

$$\xrightarrow{x \to \infty} \prod_{\varepsilon_{p_i} = 0}^{\varepsilon_{p_i} \neq 0} \frac{1}{(p_i + 1)^2} \prod_{1 \leq i \leq k} \frac{p_i(p_i + 2)}{2(p_i + 1)^2}.$$ 

Now, let $\eta(D_1, D_2)$ represent the smallest prime $p$ such that $\text{sgn}(\sigma_{\chi_1, \chi_2}^{k-1}(p)) = -1$. Linowitz and Thompson [8] then conjectured:

**Conjecture 1.1.** We have

$$\frac{\sum_{|D_1D_2| \leq x} \eta(D_1, D_2)}{\sum_{|D_1D_2| \leq x} 1} \xrightarrow{x \to \infty} \theta,$$

where

$$\theta := \sum_{k=1}^{\infty} \frac{p_k^2(p_k + 2)}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{2 + p_j(p_j + 2)}{2(p_j + 1)^2} \approx 3.9750223902 \ldots$$

They gave a heuristic argument as evidence towards their conjecture, showing:

$$\frac{\sum_{|D_1D_2| \leq x} \eta(D_1, D_2)}{\sum_{|D_1D_2| \leq x} 1} \xrightarrow{x \to \infty} \sum_{k=1}^{\infty} p_k \text{Prob}(\eta(D_1, D_2) = p_k)$$

$$= \sum_{k=1}^{\infty} p_k \text{Prob}(\varepsilon_{p_k} = -1) \prod_{i=1}^{k-1} \text{Prob}(\varepsilon_{p_i} = 0 \text{ or } 1)$$

$$= \sum_{k=1}^{\infty} \frac{p_k^2(p_k + 2)}{2(p_k + 1)^2} \prod_{i=1}^{k-1} \left( \frac{1}{(p_i + 1)^2} + \frac{p_i(p_i + 2)}{2(p_i + 1)^2} \right),$$

where the last equality follows from Theorem 1.2. The problem with this argument is that Theorem 1.2 fixes a set of primes and then lets $x \to \infty$. In this argument we need to allow the primes to tend to infinity with $x$. The authors stated: “[W]e have a good understanding of the effect of the small primes, but one would need to argue that the primes after some cutoff point do not make much of an impact on the average. Presumably, this would require using the large sieve”.
The goal of the present article is to correct their conjecture by proving the following result:

**Theorem 1.3.** We have

$$\sum_{|D_1 D_2| \leq x} \eta(D_1, D_2) \xrightarrow{x \to \infty} \Theta \cdot (1 - \beta) + \alpha,$$

where

$$\Theta = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)},$$

$$\alpha = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)},$$

and

$$\beta = \sum_{k=1}^{\infty} \frac{p_k}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)}.$$

Numerically,

$$\Theta \cdot (1 - \beta) + \alpha \approx 4.63255603509332 \ldots$$

The numerical computation was done using Sage. We used RIF for interval arithmetic and we truncated at $k = 1000$.

2. **Main Tools**

First we will need asymptotic estimates for some sets of fundamental discriminants. It is well known (see, for example, [2]) that

$$\sum_{|D| \leq x} \sim \frac{x}{\zeta(2)},$$

where $D$ runs over all fundamental discriminants with $|D| \leq x$. Here $\zeta$ is the Riemann zeta function. Now, let $n_1(m)$ be the smallest integer $n \geq 1$ relatively prime to $m$ such that the congruence $x^2 \equiv n \pmod{m}$ has no solutions. Even though Vinogradov’s conjecture remains open, it is possible to show that large values of $n_1(p)$ are rare. More specifically, using the large sieve, Linnik [7] showed that for all $\varepsilon > 0$, we have

$$\# \{p \leq x : n_1(p) > x^{\varepsilon} \} \ll_{\varepsilon} 1.$$ 

Using similar ideas to the ones from Linnik’s paper, Erdős [4] obtained a result concerning the average of $n_1(p)$ as $p$ varies over prime numbers less than or equal to $x$:

$$\frac{1}{\pi(x)} \sum_{p \leq x} n_1(p) \xrightarrow{x \to \infty} \sum_{k=1}^{\infty} \frac{p_k}{2^k},$$

where

$$\pi(x) = \sum_{p \leq x} 1.$$
where $p_k$ is the $k$th prime and $\pi(x)$ is the prime counting function. In a similar fashion, Pollack [10] considered a variation of (2.2). We summarize his result in the following theorem:

**Theorem 2.1** (Pollack, 2012). For each fundamental discriminant $D$, let $\chi_D$ be the associated Dirichlet character, i.e., $\chi_D(m) := \left(\frac{D}{m}\right)$. For each character $\chi$, let $n_\chi$ denote the least $n$ for which $\chi(n) \notin \{0, 1\}$. Finally, let $n(D) := n_{\chi_D}$. Then

(i) Uniformly in $k$ such that the $k$th prime satisfies $p_k \leq (\log x)^{\frac{1}{3}}$, we have

$$\#\{|D| \leq x : n(D) = p_k\} = \frac{p_k}{2(p_k + 1)} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)} \frac{x}{\zeta(2)} + O(x^{\frac{2}{3}}).$$

(ii) For each $n_{\chi}$, let $\sum_{\substack{|D| \leq x \atop n(D) > (\log x)^{\frac{1}{3}}}} n(D) = o(x)$.

Therefore, using (2.1), we have

$$\lim_{x \to \infty} \frac{1}{\sum_{|D| \leq x} n(D)} \sum_{|D| \leq x} n(D) = \Theta,$$

where

$$\Theta := \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)} \approx 4.9809473396 \ldots$$

We will also need the following lemma from Linowitz and Thompson [8]:

**Lemma 2.1.** Let $P(\varepsilon, p)$ denote the proportion of fundamental discriminants $D$ with $\left(\frac{D}{p}\right) = \varepsilon$. Then, we have

$$P(\varepsilon, p) = \begin{cases} \frac{p}{2p + 2}, & \text{if } \varepsilon \in \{\pm 1\} \\ \frac{1}{p + 1}, & \text{if } \varepsilon = 0. \end{cases}$$

3. Proof of Theorem 1.3

Let $\chi_1, \chi_2$ be Dirichlet characters associated with the fundamental discriminants $D_1$ and $D_2$. For a prime $p$,

$$\sigma_{\chi_1, \chi_2}^{k-1}(p) = \sum_{d|p} \chi_1\left(\frac{p}{d}\right) \chi_2(d) d^{k-1} = \chi_1(p) + \chi_2(p) p^{k-1},$$
so that

\begin{equation}
\operatorname{sgn} \sigma_{\chi_1, \chi_2}^{k-1}(p) = \begin{cases} 
\chi_1(p), & \text{if } p \mid D_2 \\
\chi_2(p), & \text{otherwise.}
\end{cases}
\end{equation}

**Proof of Theorem 1.3.** By (2.1),

\[
\sum_{|D_1D_2| \leq x} 1 = \sum_{|D_1| \leq x} \sum_{D_2 \leq \frac{x}{|D_1|}} 1 \sim \frac{x}{\zeta(2)} \sum_{|D_1| \leq x} \frac{1}{|D_1|}.
\]

Let \(A(x) := \sum_{|D| \leq x} 1\) and \(f(x) := \frac{1}{x}\). Since \(A(x) \sim \frac{x}{\zeta(2)}\), then by partial summation

\[
\sum_{|D_1| \leq x} \frac{1}{|D_1|} = A(x)f(x) - A(1)f(1) - \int_1^x A(t)f'(t) \, dt
\]

\[
\sim \frac{1}{\zeta(2)} - 1 + \int_1^x \frac{dt}{\zeta(2)t}
\]

\begin{equation}
\sim \frac{\log x}{\zeta(2)}.
\end{equation}

Hence

\begin{equation}
\sum_{|D_1D_2| \leq x} 1 \sim \frac{x \log x}{\zeta(2)^2}.
\end{equation}

Now let us estimate the numerator. For the sake of simplicity, let \(\eta := \eta(D_1, D_2)\). Then,

\[
\sum_{|D_1D_2| \leq x} \eta = \sum_{|D_1D_2| \leq x} \eta + \sum_{|D_1D_2| \leq x} \eta.
\]

If \(\eta \mid D_2\), then by (3.1), \(\eta\) is the smallest prime \(p\) such that \(\chi_1(p) \notin \{0, 1\}\), and with the notation of Theorem 2.1, this means that \(\eta = n(D_1)\). Similarly, if \(\eta \nmid D_2\), then \(\eta = n(D_2)\). Therefore,

\[
\sum_{|D_1D_2| \leq x} \eta = \sum_{|D_1D_2| \leq x} n(D_1) + \sum_{|D_1D_2| \leq x} n(D_2).
\]

Now,

\[
\sum_{|D_1D_2| \leq x} n(D_2) = \sum_{|D_1D_2| \leq x} n(D_2) - \sum_{|D_1D_2| \leq x} n(D_2),
\]

so that

\begin{equation}
\sum_{|D_1D_2| \leq x} \eta = \sum_{|D_1D_2| \leq x} n(D_2) + \sum_{\eta \mid D_2} n(D_1) - \sum_{\eta \mid D_2} n(D_2).
\end{equation}
By (2.3), we have
\[ \sum_{|D_1D_2| \leq x} n(D_2) = \sum_{|D_1| \leq x} \sum_{|D_2| \leq \frac{x}{|D_1|}} n(D_2) \]
\[ \sim \Theta \frac{x}{\zeta(2)} \sum_{|D_1| \leq x} \frac{1}{|D_1|} \]

(3.5)
\[ \sim \Theta \frac{x \log x}{\zeta(2)^2}, \]

where the final estimate follows from (3.2). Now, by Lemma 2.1, the proportion of fundamental discriminants such that \( p \mid D \) is \( 1 \) if \( p \) is prime and \( p \) is odd. Hence,
\[ \sum_{|D_1D_2| \leq x \atop n \mid D_2} n(D_1) = \sum_{|D_1| \leq x} n(D_1) \sum_{|D_2| \leq \frac{x}{|D_1|}} \frac{1}{n(D_1)D_2} \]

\[ = \sum_{|D_1| \leq x} \frac{n(D_1)}{n(D_1) + 1} \sum_{|D_2| \leq \frac{x}{|D_1|}} 1 \]

\[ \sim \frac{x}{\zeta(2)} \sum_{|D_1| \leq x} \frac{n(D_1)}{|D_1|(|n(D_1)| + 1)}. \]

To find an asymptotic for the last sum we again use partial summation. Let
\[ B(x) := \sum_{|D_1| \leq x} \frac{n(D_1)}{n(D_1) + 1}. \]

Then, by (i) of Theorem 2.1,
\[ \sum_{|D_1| \leq x \atop n(D_1) \leq (\log x)^{1/2}} \frac{n(D_1)}{n(D_1) + 1} = \sum_{k=1}^{\infty} \frac{p_k}{p_k + 1} \#\{|D_1| \leq x : n(D_1) = p_k\} \]
\[ \sim \alpha \frac{x}{\zeta(2)} \]

where
\[ \alpha = \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)}. \]

Now, by (ii) of Theorem 2.1,
\[ \sum_{|D_1| \leq x \atop n(D_1) > (\log x)^{1/2}} \frac{n(D_1)}{n(D_1) + 1} \leq \sum_{|D_1| \leq x \atop n(D_1) > (\log x)^{1/2}} n(D_1) = o(x). \]
Hence,
\[ B(x) = \sum_{|D_1| \leq x, n(D_1) \leq (\log x)^{1/3}} \frac{n(D_1)}{n(D_1) + 1} + \sum_{|D_1| \leq x, n(D_1) > (\log x)^{1/3}} \frac{n(D_1)}{n(D_1) + 1} \sim c \frac{x}{\zeta(2)}. \]

Therefore,
\[ \sum_{|D_1| \leq x, |D_2| \leq x, |D_1| |D_2| \leq x} \frac{n(D_1)}{|D_2| (n(D_1) + 1)} = B(x) f(x) - B(1) f(1) - \int_1^x B(t) f'(t) \, dt \sim c \frac{\log x}{\zeta(2)}, \]

so that
\[ (3.6) \quad \sum_{|D_1| \leq x, \eta \mid D_2} n(D_1) \sim c \frac{x \log x}{\zeta(2)^2}. \]

Finally,
\[ \sum_{|D_1| \leq x, \eta \mid D_2} n(D_1) = \sum_{|D_2| \leq x} n(D_2) \sum_{|D_1| \leq \frac{x}{|D_2|}} \frac{1}{n(D_1) + 1}. \]

To get an estimate for the inner sum, let
\[ C(x) := \sum_{|D_1| \leq \frac{x}{|D_2|}} \frac{1}{n(D_1) + 1}. \]

Then, by (i) of Theorem 2.1,
\[ \sum_{|D_1| \leq \frac{x}{|D_2|}, n(D_1) \leq (\log x)^{1/3}, p_k \leq (\log x)^{1/3}} \frac{1}{n(D_1) + 1} = \sum_{k=1}^\infty \frac{1}{p_k + 1} \# \{ |D_1| \leq \frac{x}{|D_2|} : n(D_1) = p_k \} \]
\[ \sim c \frac{x}{\zeta(2)|D_2|}, \]

where
\[ \beta = \sum_{k=1}^\infty \frac{p_k}{2(p_k + 1)^2} \prod_{j=1}^{k-1} \frac{p_j + 2}{2(p_j + 1)}. \]
On the other hand,
\[
\sum_{\substack{|D_1| \leq \frac{x}{\log x}^\frac{1}{3} \\ n(D_1) > (\log x)\frac{1}{3} }} \frac{1}{n(D_1) + 1} \leq \sum_{\substack{|D_1| \leq \frac{x}{\log x}^\frac{1}{3} \\ n(D_1) > (\log x)\frac{1}{3} }} \frac{1}{n(D_1) + 1} \sim \frac{x}{|D_2|\zeta(2)((\log x)\frac{1}{3} + 1)} \leq \frac{x}{(\log x)\frac{1}{3} + 1} = o(x).
\]

Hence,
\[
C(x) = \sum_{\substack{|D_1| \leq \frac{x}{\log x}^\frac{1}{3} \\ n(D_1) \leq (\log x)\frac{1}{3} }} \frac{1}{n(D_1) + 1} + \sum_{\substack{|D_1| \leq \frac{x}{\log x}^\frac{1}{3} \\ n(D_1) > (\log x)\frac{1}{3} }} \frac{1}{n(D_1) + 1} \sim \frac{x}{\zeta(2)|D_2|}.
\]

From this we see that
\[
(3.7) \quad \sum_{\substack{|D_1 D_2| \leq x \\ \eta|D_2|}} n(D_2) \sim \sum_{|D_2| \leq x} \beta \frac{n(D_2)x}{\zeta(2)|D_2|} \sim \Theta \beta \frac{x \log x}{\zeta(2)^2},
\]
where the last estimate follows from partial summation and applying Theorem 2.1. Therefore, plugging (3.5), (3.6) and (3.7) into (3.4) shows that
\[
\sum_{|D_1 D_2| \leq x} \eta \sim (\Theta + \alpha - \Theta \beta) \frac{x \log x}{\zeta(2)^2}.
\]
This together with (3.3) completes the proof. \qed

Remark. We can give the following explanation of why Linowitz and Thompson’s Conjecture 1.1 was slightly off from the correct number: the result from Theorem 1.2 is not uniform in \( k \) for the choice of the \( p_k \) (we fix a set of primes beforehand), while the result from Theorem 2.1 is uniform in \( k \) satisfying \( p_k \leq (\log x)^{\frac{1}{4}} \). In order to make Linowitz and Thompson’s heuristic argument rigorous we would first need to show that Theorem 1.2 holds uniformly in \( k \) such that \( p_k \leq f(x) \) for some function \( f(x) \) with \( f(x) \xrightarrow{x \to \infty} \infty \). Then,
\[
\frac{\sum_{|D_1 D_2| \leq x} \eta(D_1, D_2)}{\sum_{|D_1 D_2| \leq x} 1} = \sum_{p_k \leq f(x)} p_k \text{Prob}(\eta(D_1, D_2) = p_k) + \sum_{p_k > f(x)} p_k \text{Prob}(\eta(D_1, D_2) = p_k) \xrightarrow{x \to \infty} \theta + \mu,
\]
where $\theta$ is the conjectured constant (1.2) and

$$\mu = \lim_{x \to \infty} \sum_{p_k > f(x)} p_k \text{Prob}(\eta(D_1, D_2) = p_k).$$

Linowitz and Thompson conjectured that $\mu = 0$, but according to Theorem 1.3, $\mu$ does make a small contribution.

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