SUSY Coherents States and Classical Trajectories

*Musongela Lubo , *Kikunga Kasenda Ivan , †Likwolo Katambwa Stanislas

*Physics Department, Faculty of Sciences, University of Kinshasa, P.O.Box Kin 190, Kinshasa, D.R.Congo
†Faculty of Agronomy, University of Uele, P.O.Box Isiro 670, Isiro, D.R.Congo

Email : musongela.lubo@unikin.ac.cd, ivan.kikunga@unikin.ac.cd, likwolostanislas@yahoo.fr

Abstract

A generalization of coherent states has been developed in the context of supersymmetric quantum mechanics. For most cases, no link has been made with the corresponding classical system. In this work, we consider a very simple superpotential and compare the classical and quantum trajectories.

Keywords : Coherent States, SUSYQM.

1 Introduction

Coherent states and their applications in theoretical and technological fields have been the subject of many studies recently [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19] and [20]. Coherent states play an important role in quantum physics. Among their important properties is the fact that the mean value of the position operator in these states perfectly reproduces the classical behavior in the case of the harmonic oscillator. These states also saturate the Heisenberg inequality. Many generalizations have been proposed for the notion of coherent states for systems which are more complex than the harmonic oscillator. The path taken by supersymmetric quantum mechanics is a very simple one. When a Hamiltonian can be written as a product of an operator and its adjoint (plus a constant), one is basically the situation of the harmonic oscillator with a creation and a destruction operator. The coherent states in this context are defined as the eigenvectors of the destruction operator. It has been shown that these states saturate a generalized uncertainty relation which is not the Heisenberg one [1]. For some potentials which play an important role in Physics and Chemistry (For example the Morse potential, ... ), the SUSY coherent states have been computed using a clever trick which bypasses the resolution of the Riccati equation for the determination of the superpotential [11]. But when it comes to the study of the mean value of the position or the momentum
operator, one is confronted by the fact that these coherent states are not normalizable. One may try an approach using wave packets but the analysis become cumbersome.

The path taken here is the following. For the harmonic oscillator, the coherent states are normalizable. The superpotential is a one degree polynomial. To get an insight into the subject, we study an ad hoc system whose superpotential is polynomial. For simplicity and to ensure normalizability, we take it to be of third degree. The physical potential is then a sixth degree polynomial whose classical trajectories can be computed. The coherent states can be obtained in closed form. We then proceed to the calculation of the mean value of the position operator for these states. Comparison is then made with the classical trajectories.

The paper is organized as follows. The second section gives a quick reminder of the formalism of SUSY coherent states that we need. The third section deals with the calculation of the mean value of the position operator for our toy model. The treatment of the harmonic oscillator is put in the Appendix and shows that our analysis captures what is known by other methods.

2 Our Approach

The solution of the classical equations of motion for the harmonic oscillator is given by

\[ q_{\text{class}}(t) = F \cos \omega t + G \sin \omega t, \tag{1} \]

where \( F \) and \( G \) are constants.

For a small interval of time this can be written a power series

\[ q_{\text{class}}(t) = A_0 + A_1 \omega t + A_2(\omega t)^2 + \ldots \tag{2} \]

where one has

\[ \frac{A_2}{A_0} = -\frac{1}{2}, \quad \frac{A_3}{A_1} = -\frac{1}{6}, \ldots \tag{3} \]

We can write the "quantum trajectory" as

\[ \langle \psi_S, t | \hat{q} | \psi_S, t \rangle = \sum_{l=0}^{+\infty} \Omega_l(\omega t)^l \tag{4} \]

We are studying a one dimensional system with a rescaled undimensionalized coordinate \( q \). Suppose one can write the physical potential \( V(q) \) of such a system in terms of a function \( x(q) \) by the relation

\[ V(q) - E_0 = \frac{1}{2} [x^2(q) + \frac{dx(q)}{dq}]. \tag{5} \]
Then one can factorize the Hamiltonian

\[ \hat{H} = \hat{A}^+ \hat{A} + E_0, \]  

(6)

where the operators \( \hat{A} \) and \( \hat{A}^+ \) have the form

\[ \hat{A} = \frac{1}{\sqrt{2}} \left[ \frac{d}{dq} - x(q) \right]; \quad \hat{A}^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dq} - x(q) \right]. \]  

(7)

The function \( x(q) \) is the superpotential. Eq.(7) are reminiscent of the harmonic oscillator. Using that similarity, the coherent states are defined as the eigenfunctions of the "annihilation operator" \( \hat{A} \). Using Eq.(7) one is led to an equation of separable variables. The solution reads

\[ \psi_H(q, \alpha) = N \exp \left[ \sqrt{2\alpha} q + \int_0^q x(\xi) d\xi \right]. \]  

(8)

In Eq.(8), \( \alpha \) is a complex variable which caracterizes the coherent states. When trying to implement the SUSY treatment to a system whose physical potential is known, the difficult part is the resolution of the Riccati equation given in Eq.(5).

For the harmonic oscillator, one introduces the rescaled variables

\[ \hat{Q} = \sqrt{\frac{m\omega}{\hbar}} \hat{q}; \quad \hat{P} = \sqrt{m\omega\hbar} \hat{p}, \]  

(9)

and the rescaled potential

\[ \hat{U}(Q) = \hbar \omega \hat{V}(q). \]  

(10)

The superpotential is given by

\[ x(q) = -q, \]  

(11)

so that the coherent state takes the form

\[ \psi_H(q, \alpha) = N \exp \left( \sqrt{2\alpha} q - \frac{1}{2} q^2 \right). \]  

(12)

For the Morse potential, the superpotential is given by [1]

\[ x(q) = -s + \exp \left( -q \sqrt{2\chi_e} \right) + \sqrt{\frac{\chi_e}{2}}, \]  

(13)

the parameters \( s \) and \( \chi_e \) are related to the Morse potential’s energy levels : 

\[ E_n = s(n + \frac{1}{2}) - \chi_e(n + \frac{1}{2})^2. \]  

(14)

So, The coherent states takes the form
\[ \psi_H(q, \alpha) = \exp \left[ -\frac{1}{2 \chi_e} \exp \left( -\sqrt{2 \chi_e} q \right) - \frac{s - \chi_e}{\sqrt{2 \chi_e}} q + \sqrt{2} \alpha q \right] \]  \hspace{1cm} (15)

Clearly, such a function in not square integrable and the mean values can not be computed.

The aim of this paper is to choose a superpotential such that the corresponding coherent states are normalizable and the computation of the mean values not too complicated. The quantity we are interested in is the mean value

\[ \langle \hat{q} \rangle = \frac{\langle \psi_S, t | \hat{q} | \psi_S, t \rangle}{\langle \psi_S, t | \psi_S, t \rangle} \]  \hspace{1cm} (16)

One has to realize that the states given in Eq.(8) were obtained in the Heisenberg picture : the wave function does not depend on time. To pass to the Schrödinger picture one has to use the evolution operator

\[ | \psi_S, t \rangle = \exp \left( -\frac{i}{\hbar} \hat{H} t \right) | \psi_H \rangle \]  \hspace{1cm} (17)

where the Hamiltonian takes the form

\[ \hat{H} = \hbar \omega \left( \hat{p}^2/2 + V(q) \right) \]  \hspace{1cm} (18)

The factorisation of the Hamiltonian and the evolution operator can be used to obtain the wave function in the Schrödinger picture as a power series

\[ | \psi_S, t \rangle = \exp \left( -\frac{i}{\hbar} E_0 t \right) \sum_{k=0}^{n} \left( -\frac{\omega t}{k!} \right)^k (\hat{A}^+ \hat{A})^k | \psi_H \rangle \]  \hspace{1cm} (19)

The mean value then reads

\[ \langle \psi_S, t | \hat{q} | \psi_S, t \rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} \omega^{m+n}}{m! n!} t^{m+n} \langle \psi_H | \hat{H}^m \hat{q} \hat{H}^n | \psi_H \rangle \]  \hspace{1cm} (20)

The relation

\[ \hat{A}^+ = -\hat{A} - \sqrt{2} x(q) \]  \hspace{1cm} (21)

is obtained by adding the two relations given in Eq.(7). With the fact that the wave function we are studying is an eigenvector of the "annihilation operator", we find

\[ \hat{A}^+ \hat{A} | \psi_H \rangle = \alpha [-\alpha - \sqrt{2} x(q)] | \psi_H \rangle \]  \hspace{1cm} (22)

Subtracting the relations in Eq.(7), we similarly find
\[
\frac{d}{dq} |\psi_H> = [\sqrt{2} \alpha + x(q)] |\psi_H> .
\] (23)

This allows us to obtain the second order contribution to the wave function

\[
(\hat{A}^+ \hat{A})^2 |\psi_H> = \left[ \frac{\sqrt{2}}{2} d^2 x(q) + (2 \alpha^2 + \alpha \sqrt{2} x(q)) \frac{dx(q)}{dq} + (\sqrt{2} \alpha x(q) + \alpha^2)^2 \right] |\psi_H> .
\] (24)

This suggests the introduction of special functions \( f_n \) such that

\[
(\hat{A}^+ \hat{A})^n |\psi_H> = f_n(q, \alpha) |\psi_H> .
\] (25)

From the previous considerations, one derives the recurrence formula

\[
f_{n+1} = - \frac{1}{2} \frac{\partial^2 f_n}{\partial q^2} - [\sqrt{2} \alpha + x(q)] \frac{\partial f_n}{\partial q} - [\sqrt{2} \alpha x(q) + \alpha^2] f_n .
\] (26)

Note that

\[
f_0(q, \alpha) = 1.
\] (27)

Working in the position representation, the time dependent wave function

\[
\psi_S(q, t, \alpha) = \exp \left( - \frac{i}{\hbar} E_0 t \right) \sum_{m=0}^{\infty} \frac{(-i \omega t)^m}{m!} [(\hat{A}^+ \hat{A})^m \psi_H(q, \alpha)]
\] (28)

takes the form

\[
\psi_S(q, t, \alpha) = \exp \left( - \frac{i}{\hbar} E_0 t \right) \sum_{m=0}^{\infty} \frac{(-i \omega t)^m}{m!} f_m(q, \alpha) \psi_H(q, \alpha).
\] (29)

The position mean value is then given by an infinite double sum

\[
S <\psi|\hat{q}|\psi> = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\omega t)^{m+n} \frac{i^{m+n}(-1)^n}{m!n!} C_{m,n}
\] (30)

where the coefficients \( C_{m,n} \) are the integrals

\[
C_{m,n} = \int_{-\infty}^{+\infty} dq q f^*_m(q, \alpha) f_n(q, \alpha) \psi^*_H(q, \alpha) \psi_H(q, \alpha).
\] (31)

The double summation can be rewritten as a power series in the time parameter

\[
S <\psi|\hat{q}|\psi> = \sum_{l=0}^{\infty} (\omega t)^l \Omega_l ,
\] (32)

with
\[ \Omega_l = i^l (-1)^l \sum_{m=0}^{l} \frac{(-1)^m}{m!(l-m)!} C_{m,l-m}. \]  

(33)

Eq. (32) is written with the fact that the quantity \( S < \psi | \psi >_S \) is an unessential constant.

One can consider that Eq. (32) and Eq. (33) gives the answer to our question about the mean value of the position operator in this context. For any practical case, one has to compute the integrals of Eq. (31) and makes the appropriate summation.

At this point one has to point technical difficulties. The first one is that the recursion relations of Eq. (26) can quickly leads to large formulas even for simple superpotentials. The second one is that it is not always possible to have an analytical expression for the integrals appearing in Eq. (31). Third, one has to be careful about the order at which one can stop to the series of Eq. (32) to obtain a reliable estimate.

To test our approach, we first used it on the harmonic oscillator. The results are quite good and reproduce some important characteristics of the coherent states as known in the literature. To keep our presentation light, we have put this treatment in the Appendix.

3 A Toy Model

The generalisation of coherent states considered here was introduced in [11]. For many potentials used in theoretical chemistry, the corresponding states are not normalizable and this leads to technical difficulties when one is interested in mean values. We want to restrict ourselves to systems with square integrable generalised coherent states. In the context of quantum supersymmetry, the most important ingredient is the superpotential. For the harmonic oscillator, the superpotential in linear in the position. The next non trivial thing is to study a polynomial superpotential. A second degree superpotential is readily seen to lead to a non normalisable generalised coherent state. This leads us to study a toy model whose superpotential is given by

\[ x(q) = x_1 q - \frac{1}{3} x_1^2 q^3, \]  

(34)

with

\[ x_1 < 0, \]  

(35)

where \( x_1 \) is a free parameter. The corresponding physical potential is a sixth degree polynomial (See Eq. (5))

\[ U(Q) = \hbar \omega (U_0 + U_2 Q^2 + U_4 Q^4 + U_6 Q^6). \]  

(36)
Its coefficients are related to those of the rescaled superpotential by

\[ U_0 = 0; \quad U_2 = 0; \quad U_4 = -\left( \frac{\hbar}{m\omega} \right)^2 \frac{x_1^3}{3}; \quad U_6 = \left( \frac{\hbar}{m\omega} \right)^3 \frac{x_1^4}{18}. \]  

(37)

It has to be noted that if one begins with the potential, the characteristic frequency is given by

\[ \hbar \omega = \frac{U_4^3}{U_6}. \]  

(38)

This potential has the form

\[ \text{Figure 1 – A Potential for the Toy Model for the values } U_4 = 1 \text{ and } U_6 = 1. \]

Classically, all the trajectories are bounded and periodic, with the period

\[ T = 4 \sqrt{\frac{2}{m}} \int_{Q^+}^{Q} \frac{1}{\sqrt{E - U(Q)}} dQ \]  

(39)

which depends on the energy of the system because \( Q_+ \) is such that \( E - U(Q_+) = 0 \).

We now analyse the mean value of the position operator for the corresponding coherent states and compare them with the classical trajectories. From Eq.(26) one easily sees that the functions \( f_n \) will be polynomial in the variable \( q \). This will highly simplify the computations. We thus introduce the coefficients \( \tilde{f}_n(\alpha) \) by

\[ f_n(q, \alpha) = \sum_{k=0}^{n} \tilde{f}_{n,k}(\alpha)q^k, \]  

(40)

with

\[ \tilde{f}_{0,0} = \tilde{f}_{0,0}^* = 1. \]  

(41)

We know now to compute
\[ C_{m,n} = \sum_{k=0}^{m} \sum_{p=0}^{n} \int_{-\infty}^{+\infty} dq \cdot q^{k+p+1} \tilde{f}_{m,k}(\alpha) f_{n,p}(\alpha) \exp(2\beta q + x_1 q^2 - \frac{1}{6} x_1^2 q^4) \]

where \(2\beta = \sqrt{2} (\alpha + \alpha^*)\).

The coefficients we have introduced obey the following recursion relations which naturally come from Eq. (26):

\[ \tilde{f}_{n+1,l} = -\frac{1}{2} (l+1)(l+2) \tilde{f}_{n,l+2} - \sqrt{2} \alpha (l+1) \tilde{f}_{n,l+1} - (x_1 + \alpha^2) \tilde{f}_{n,l} \]

\[-\sqrt{2} \alpha x_1 \tilde{f}_{n,l-1} + \frac{1}{3} x_1^2 (l-2) \tilde{f}_{n,l-2} + \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,l-3} \quad (3 < l < 3n-2),\]

\[ \tilde{f}_{n+1,0} = -\tilde{f}_{n,2} - \sqrt{2} \alpha \tilde{f}_{n,1} - \alpha^2 \tilde{f}_{n,0}, \]

\[ \tilde{f}_{n+1,1} = -3 \tilde{f}_{n,3} - 2 \sqrt{2} \alpha \tilde{f}_{n,2} - (x_1 + \alpha^2) \tilde{f}_{n,1} - \sqrt{2} \alpha x_1 \tilde{f}_{n,0}, \]

\[ \tilde{f}_{n+1,2} = -6 \tilde{f}_{n,4} - 3 \sqrt{2} \alpha \tilde{f}_{n,3} - (2x_1 + \alpha^2) \tilde{f}_{n,2} - \sqrt{2} \alpha x_1 \tilde{f}_{n,1}, \]

\[ \tilde{f}_{n+1,3n-1} = -3n \sqrt{2} \alpha \tilde{f}_{n,3n} - [(3n-1)x_1 + \alpha^2] \tilde{f}_{n,3n-1} \]

\[-\sqrt{2} \alpha x_1 \tilde{f}_{n,3n-2} + \frac{1}{3} x_1^2 (3n-3) \tilde{f}_{n,3n-3} + \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,3n-4}, \]

\[ \tilde{f}_{n+1,3n} = -(3n x_1 + \alpha^2) \tilde{f}_{n,3n} - \sqrt{2} \alpha x_1 \tilde{f}_{n,3n-1} + \frac{1}{3} x_1^2 (3n-2) \tilde{f}_{n,3n-2} + \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,3n-3}, \]

\[ \tilde{f}_{n+1,3n+1} = -\sqrt{2} \alpha x_1 \tilde{f}_{n,3n} + \frac{1}{3} x_1^2 (3n-1) \tilde{f}_{n,3n-2} + \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,3n-2}, \]

\[ \tilde{f}_{n+1,3n+2} = nx_1^2 \tilde{f}_{n,3n} + \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,3n-1}, \]

\[ \tilde{f}_{n+1,3n+3} = \frac{\sqrt{2}}{3} \alpha x_1^2 \tilde{f}_{n,3n}. \]  (43)

From Eq. (42), we have

\[ C_{0,0} = \int_{-\infty}^{+\infty} dq \cdot q \exp(2\beta q + x_1 q^2 - \frac{1}{6} x_1^2 q^4). \]  (44)

The integrals in Eq. (42) have the generic form

\[ J(n) = \int_{-\infty}^{+\infty} dq \cdot q^n \exp(2\beta q + x_1 q^2 - \frac{1}{6} x_1^2 q^4). \]  (45)

In fact, one simply has to compute the value corresponding to \(n = 0\). In fact, posing
\[ J(0) = K(\beta), \]  

(46)

one has

\[ J(n) = \frac{1}{2^n} \frac{d^n}{d\beta^n} K(\beta). \]  

(47)

To evaluate \( J(0) \), we shall use the saddle point approximation. This is due to the fact that the exponential is a very rapidly decaying function, due to the fourth degree term with a negative sign. The extrema of our function satisfy the equation

\[ q^3 - \frac{3}{x_1} q - \frac{3\beta}{x_1^2} = 0. \]  

(48)

This equation is a particular case of the following

\[ q^3 + a_1 q^2 + a_2 q + a_3 = 0. \]  

(49)

The solution to such an equation can be recast using the following formula. One introduces the intermediate quantities

\[ Q = -\frac{1}{x_1}, R = \frac{3\beta}{2x_1^2}, \]  

(50)

\[ D = Q^3 + R^2; S = \frac{3}{R + \sqrt{D}}; T = \frac{3}{R - \sqrt{D}}. \]  

(51)

The only real extremum is found at

\[ q_0 = S + T - \frac{1}{3} a_1. \]  

(52)

One finally arrives at the following expression

\[ q_0 = \sqrt[3]{\frac{3\beta}{2x_1^2}} + \sqrt[3]{\frac{9\beta^2}{4x_1^4} - \frac{1}{x_1^2}} + \sqrt[3]{\frac{3\beta}{2x_1^2} - \sqrt[3]{\frac{9\beta^2}{4x_1^4} - \frac{1}{x_1^2}}} \]  

(53)

For convenience we introduce the function

\[ g(q) = 2\beta q + x_1 q^2 - \frac{1}{6} x_1^2 q^4. \]  

(54)

Our integral then becomes

\[ K(\beta) \simeq \exp g(q_0) \int_{-\infty}^{+\infty} dq. \exp[-\frac{1}{2} |g''(q_0)|(q - q_0)^2], \]  

(55)

so that
At this point, we will slightly change our point of view. We shall consider that the parameter \( x_1 \) is fixed once and for all. The formula given in Eq. (53) can be seen as giving a link between the number \( \beta \) which specifies the coherent state and the point \( q_0 \) where the function \( g(q) \) attains its maximum. The two quantities vehiculate the same information.

We need to compute the derivatives of the function \( K(\beta) \) with respect to \( \beta \). Looking at Eq. (53) shows it to be a cumbersome task.

The link between \( \beta \) and \( q_0 \) can be expressed by the formula

\[
\beta = -x_1 q_0 + \frac{1}{3} x_1^2 q_0^3,
\]

which can be used to find

\[
g(q_0) = -x_1 q_0^2 + \frac{1}{2} x_1^2 q_0^4,
\]

so that one finds the approximation

\[
K(\beta) = \sqrt{\pi} \exp \left( -x_1 q_0^2 + \frac{1}{2} x_1^2 q_0^4 \right). (-x_1 + x_1^2 q_0)^{-\frac{1}{2}}.
\]

The derivative of \( K(\beta) \) with respect to \( \beta \) is found using the chain rule:

\[
\frac{d}{\beta} \frac{\partial \beta}{q_0} = -x_1 + x_1^2 q_0^2,
\]

\[
\frac{dK}{\beta} = (-x_1 + x_1^2 q_0)^{-1} \frac{dK}{q_0}.
\]

With the help of Eq. (59), it is readily seen that the result can be summarized in the following form

\[
\frac{d}{d\beta} K(\beta) = K(\beta) h_1(q_0),
\]

with

\[
h_1(q_0) = 2q_0 - q_0 x_1^2 (x_1^2 q_0^2 - x_1)^{-2}.
\]

In a similar fashion, this generalizes to the relation

\[
\frac{d^n}{d\beta^n} K(\beta) = K(\beta) h_n(q_0),
\]

supplied by the following recursion relations and initial values

\[
h_{n+1}(q_0) = h_1(q_0) h_n(q_0) + (x_1^2 q_0^2 - x_1)^{-1} \frac{d}{dq_0} h_n(q_0),
\]

\[
\frac{d^n}{d\beta^n} K(\beta) = K(\beta) h_n(q_0),
\]
\[ h_0(q_0) = 1; \quad h_1(q_0) = 2q_0 - q_0 x_1^2 (x_1^2 q_0^2 - x_1)^{-2}. \]  
(66)

Our coefficients then assume the form

\[ C_{m,n} = K(\beta) \sum_{k=0}^{3n} \sum_{p=0}^{3n} \frac{1}{2k+p+1} \tilde{f}^s_{m,k}(\alpha) f_{n,p}(\alpha) h_{k+p+1}(q_0). \]  
(67)

They are used to generate the quantum trajectories. In other hand, the classical trajectories are analytical functions of time. Rather than relying on their periodic character, we shall consider the equation of motion

\[ q''(t) - \mu q^3(t) - \sigma q^5(t) = 0, \]  
(68)

with \( \mu = -4\omega^2 x_1 x_3 \) and \( \sigma = -3\omega^2 x_3^2 \) where \( x_3 = -\frac{1}{3} x_1 \).

One sees that a power series solution of the form

\[ q(t) = \sum_{i=0}^{\infty} A(i) (\omega t)^i \]  
(69)

leads to the following relation between the coefficients

\[ A(2) = \frac{1}{6} (-A^5(0) - 4A^3(0)); \]  
(70)

\[ A(3) = \frac{1}{18} (-5A(1)A^4(0) - 12A(1)A^2(0)); \]  

\[ A(4) = \frac{1}{216} (5A(0) + 32A^7(0) + 48A^5(0) - 60A^2(1)A^3(0) - 72A^2(1)A(0)); \]  

\[ A(5) = \frac{1}{1080} (85A(1)A^8(0) + 432A(1)A^6(0) + 432A(1)A^4(0) - 180A^3(1)A^2(0) - 72A^3(1)). \]

To see how close the quantum trajectory is to the classical one, we will see if the relations given in Eq. (70) are more or less satisfied when one replaces \( A(i) \) by \( \Omega_i \) given in Eq. (33). Taking for numerical illustration \( \alpha = 1 - i, x_1 = -1 \),

\[ \{\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6\} = \{6.10465, -931.073, -9.84843, 1228.23, -92550, 4.48253 \times 10^6, -2.7263 \times 10^8\}. \]  
(71)

The trajectories will be close if the equations Eq. (70) is still satisfied when the constants \( A(i) \) are replaced by the coefficients \( \Omega_i \). For our case, this amount to the vanishing of the following quantities

\[ \{-3.78717 \times 10^6 t^2 \omega^4, 1.80966 \times 10^9 t^4 \omega^6, \ldots\}. \]  
(72)
Figure 2 – Plot of $q(t)$ for the Toy for the values $\alpha = \frac{2\sqrt{2}}{3} - \frac{i}{10000}$, $x_1 = -1$ and $\beta = \frac{4}{3}$. The calculation was done to the 4th order.

This will hold true for times very small compared to the intrinsic frequency of the system.

Conclusions

In this paper, we devised a general method for the computation of the quantum trajectories of generalized coherent states in the supersymmetric context. Our approach was successfully applied to the harmonic oscillator. We then applied it to one of simplest superpotentials whose coherent states are normalizable. We found for this specific case the timescale after which the classical and quantum trajectories begin having significant differences. Although technical difficulties arise, our approach can be used to analyse the product of the uncertainties in physical position and momentum operators and see how it evolves in view of the absolute bound given by the Heisenberg inequality.

Some mathematical issues have not been addressed here. One of them is the convergence of the series obtained for the position operator mean value. This is a very difficult point since we do not have the general formula for the coefficient $\Omega_l$. But in principle, since the problem is well posed, one can hope the results are meaningful. The method we followed here gives the position as an analytic series in time. One has to sum a finite number of terms to obtain an approximation. Such a finite sum is polynomial. This explains why after some time the graph goes to infinity; this simply means we are out of the domain where the approximation is valid.
The analysis done here relies on the saddle point approximation. For the numerical values used here, the function we integrate is very rapidly decreasing. For a more profound study, one will need an explicit evaluation of the domain of validity of the method.

To finish, our work can be seen as a particular illustration of the Ehrenfest theorem i.e in most general cases, the classical and quantum trajectories are not the same.

Appendix : The Harmonic Oscillator

We give here the results obtained by our treatment when applied to the harmonic oscillator. The functions \( f_n \) are polynomial

\[
 f_n(q, \alpha) = \sum_{k=0}^{n} \tilde{f}_{n,k}(\alpha) q^k; \quad \tilde{f}_{0,0} = \tilde{f}_{0}^* = 1. \tag{73}
\]

The coefficients we need are given by the following integrals

\[
 C_{m,n} = \sum_{k=0}^{m} \sum_{p=0}^{n} \int_{-\infty}^{+\infty} dq \cdot q^{k+p+1} \tilde{f}_{m,k}(\alpha) f_{n,p}(\alpha) \psi_{H}^*(q, \alpha) \psi_{H}(q, \alpha). \tag{74}
\]

The wave functions given in Eq. (12) lead to

\[
 C_{m,n} = \sum_{k=0}^{m} \sum_{p=0}^{n} \int_{-\infty}^{+\infty} dq \cdot q^{k+p+1} \tilde{f}_{m,k}(\alpha) f_{n,p}(\alpha) \exp(2\beta q - q^2) \tag{75}
\]

where, by definition, the coefficient \( \beta \) is given by

\[
 2\beta = \sqrt{2}(\alpha + \alpha^*). \tag{76}
\]

We have fewer recursion relations

\[
\begin{align*}
\tilde{f}_{n+1,0} &= -\tilde{f}_{n,2} - \sqrt{2}\alpha \tilde{f}_{n,1} - \alpha^2 \tilde{f}_{n,0}, \tag{77} \\
\tilde{f}_{n+1,n-1} &= -\sqrt{2}\alpha n \tilde{f}_{n,n} + (n - 1 - \alpha^2) \tilde{f}_{n,n-1} + \sqrt{2}\alpha \tilde{f}_{n,n-2}, \\
\tilde{f}_{n+1,n} &= (n - \alpha^2) \tilde{f}_{n,n} + \sqrt{2}\alpha \tilde{f}_{n,n-1}, \\
\tilde{f}_{n+1,n+1} &= \sqrt{2}\alpha \tilde{f}_{n,n}, \\
\tilde{f}_{n+1,l} &= -\frac{1}{2}(l+1)(l+2) \tilde{f}_{n,l+2} - \sqrt{2}\alpha(l+1) \tilde{f}_{n,l+1} + (l - \alpha^2) \tilde{f}_{n,l} + \sqrt{2}\alpha \tilde{f}_{n,l-1}.
\end{align*}
\]

Our coefficients \( C_{m,n} \) are given by integrals of the product of an exponential and a power of the variable \( q \) (See Eq. (74)). Actually, the only thing one needs is
\[ I(\beta) = \int_{-\infty}^{+\infty} dq \exp\left(-q^2 + 2\beta q\right) \]  

(78)

and one know that

\[ I(\beta) = \frac{\sqrt{\pi}}{2} \exp(\beta^2). \]  

(79)

It easily to see that

\[ \int_{-\infty}^{+\infty} dq q^n \exp(2\beta q - q^2) = \frac{1}{2^n} \frac{d^n}{d\beta^n} I(\beta), \]  

(80)

so

\[ \int_{-\infty}^{+\infty} dq q^{k+p+1} \exp(2\beta q - q^2) = \sqrt{\pi} \frac{1}{2^{k+p+1}} \frac{d^{k+p+1}}{d\beta^{k+p+1}}[\exp(\beta^2)]. \]  

(81)

We use the polynomials \( P_s \), defined by the property

\[ \frac{d^s}{d\beta^s} \exp(\beta^2) = \exp(\beta^2) P_s(\beta) \]  

(82)

and one readily finds that they obey the recursion relations

\[ P_{s+1}(\beta) = 2\beta P_s(\beta) + \frac{d}{d\beta} P_s(\beta); \]  

(83)

\[ P_0(\beta) = 1. \]

The coefficients can now be recast in the form

\[ C_{m,n} = \sqrt{\pi} \exp(\beta^2) \sum_{k=0}^{m} \sum_{p=0}^{n} f_{m,k}(\alpha) f_{n,p}(\alpha) \frac{1}{2^{k+p+1}} P_{k+p+1}(\beta). \]  

(84)

Let us now compare the quantum and the classical trajectories for the harmonic oscillator using our approach.

For the quantum behavior on the other side, one finds

\[ \Omega_0 = C_{0,0} = |N|^2 \beta \sqrt{\pi} \exp(\beta^2), \]  

\[ \Omega_1 = 0. \]  

(85)

The second non null coefficient is given by

\[ \Omega_2 = |N|^2 \sqrt{\pi} \exp(\beta^2) \left[ A \frac{1}{2} P_1(\beta) + B \frac{1}{22} P_2(\beta) + C \frac{1}{23} P_1(\beta) \right], \]  

(86)
where the coefficients $A, B, C$ are given by the following expressions:

\[
A = -\frac{1}{2} \tilde{f}_{2,0} + \tilde{f}_{1,0} \tilde{f}^*_{1,0} - \frac{1}{2} \tilde{f}^*_{2,0}, \\
B = -\frac{1}{2} \tilde{f}_{2,1} + \tilde{f}_{1,0} \tilde{f}^*_{1,1} + \tilde{f}_{1,1} \tilde{f}^*_{1,0} - \frac{1}{2} \tilde{f}^*_{2,1}, \\
C = -\frac{1}{2} \tilde{f}_{2,2} + \tilde{f}_{1,1} \tilde{f}^*_{1,1} - \frac{1}{2} \tilde{f}^*_{2,2}.
\] (87)

So, we have

\[
A = -2\beta^4 + 2\beta^2 + 4\alpha (\alpha^*)^2 \beta^2 - 2\alpha \alpha^*, \\
B = -\beta + 4\beta^3 - 8\alpha \alpha^* \beta; \\
C = -2\beta^2 + 4\alpha \alpha^*.
\] (88)

One then finds

\[
\Omega_2 = |N|^2 \sqrt{\pi} \exp (\beta^2) \left( -\frac{1}{2} \right). \\
\] (89)

The following ratio

\[
\frac{\Omega_2}{\Omega_0} = -\frac{1}{2} \\
\] (90)

drives us to conclude that the behavior of the classical trajectory given in Eq.(1) is recovered, at least in the lowest order. One can go to higher order and verify this still works.

**Figure 3** – Plot of $q(t)$ for the Harmonic Oscillator, an approximation to the 8th order.
Références

[1] Molski, M., J. Phys. A : Math. Theor. 42 165301 (2009).
[2] Mikulski, D., Konarski, J., Krzysztof, E., Molski, M., Kabaciński, S. J., Math. Chem. (2015) 53 : 2018.
[3] Sabi Takou D., Avossevou G. Y. H., Kounouhewa. 10.1515/Phys-2015-0021.
[4] Mikulski, D., Molski, M., Konarski, J., Krzysztof, E., Journal of Mathematical Chemistry 52(1) - January 2014. J. Math. Chem. (2014).
[5] Mikulski, D., Konarski, J., Krzysztof, E., Molsky, M., Annals of Physics 339 : 122-134- December 2013.
[6] Pawlowski, J., Szumniak, P., Bednarek, S., Phys. Rev. B 94, 155407-2016.
[7] Clark, L.A., Stokes, A., Beige, A., Phys. Rev. A94, 023840 (2016).
[8] Popov, D., Shi-Hai Dong, Pop, N., Sajfert, V., Šimon, S., Annals of Physics, Volume 39, December 2013, Pages 122-134.
[9] Drăgănescu, G. E., Physica Scripta, Volume 2013, T153, March 23rd, 2013.
[10] Yahiaoui, S. A., Bentaiba, M., Journal of Physics A : Mathematical and Theoretical, Volume 45(44), January 2012.
[11] Ruby, V. C., Senthilvelan, M., J. Math. 51, 052106 (2010).
[12] Malkiewicz, P., Affine Coherent States in Quantum Cosmology. ArXiv : 1512.04304v1[gr-qc].
[13] De Lima Rodrigues, R., De Lima, A. F., De Araújo Ferreira, K., Vaidya, A. N., ArXiv : [hep-th/0205175v7].
[14] Bergeron, H., Gazeau, J. P., Youssef, A. ArXiv : quant-ph/1007.3876.
[15] Balondo Iyela, D., Govaerts, J., Hounkonnou, M. N., Journal of Mathematical Physics 54, 093502 (2013).
[16] Antoine, J. P., Gazeau, J. P., Monceau, P., Klauder, J. R., Penson, K. A., Journal of Mathematical Physics, Volume 42, 2349(2001).
[17] Aremua, I., Gazeau, J. P., Hounkonnou, M. N., Journal of Physics A : Mathematical and Theoretical 45(33). November 2011.
[18] Gazeau, J. P., delOlmo, M. A., Annals of Physics Volume 330, March 2013, Pages 220-245.
[19] Bergeron, H., Gazeau, J. P., Siegl, P., Youssef, A., EPL (Europhysics Letters), Volume 92, Number 6.
[20] El Baz, M., Fresneda, R., Gazeau, J. P., Hassouni, Y., Journal of Physics A : Mathematical and Theoretical Volume 43, Number 38.