Thermodynamics of Rotating Lovelock-Lifshitz Black Branes

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Abstract

We investigate the thermodynamics of rotating Lovelock-Lifshitz black branes. We calculate the conserved and thermodynamic quantities of the solutions and obtain a relation between energy, entropy and angular momentum densities with temperature and angular velocity. We also obtain a Smarr-type formula for the energy density as a function of entropy and angular momentum densities, and we show that the thermodynamic quantities calculated in this paper satisfy the first law of thermodynamics. Finally, we investigate the stability of black brane solutions in both canonical and grand-canonical ensemble. We find that the solutions are thermally stable for $z \leq n - 1$, while they can be unstable for $z > n - 1$. 

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I. INTRODUCTION

Holography presents a general tool for studying many kinds of physical systems. It is known that finite temperature in the field theory often corresponds to the presence of a horizon in its gravitational dual. Although the idea of holography has been first used in the AdS/CFT correspondence, recently these techniques have been brought to bear on other types of systems \[1\]. Most notably, the holographic techniques are also useful to study condensed matter systems at strong coupling \[2\]. Indeed, the Lifshitz black holes have been found to emerge as gravity duals of some condensed matter systems with anisotropic scaling symmetry,

\[ t \to \lambda^z t, \quad x \to \lambda x, \]

where \( z \) is the dynamical exponent.

From a holographic point of view, this suggests the following asymptotic form for the spacetime metric

\[
\begin{aligned}
ds^2 &= -\frac{r^{2z}}{l^{2z}} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\varphi^2 + r^2 \sum_{i=1}^{n-2} dx_i^2,
\end{aligned}
\]

which is known as Lifshitz spacetime and obeys the scale invariance

\[ t \to \lambda^z t, \quad r \to \lambda^{-1} r, \quad x \to \lambda x. \]

The metric (1) is the well-known AdS metric for \( z = 1 \). This spacetime possesses a timelike Killing vector, but is not a space of constant curvature. In order to have solutions with Lifshitz asymptotic, one may use an action involving a 2-form and a 3-form field with a Chern-Simons coupling or a massive vector field \[3\]. Asymptotic Lifshitz solutions has been investigated by many authors. Only a few exact solutions have been found, and most of the solutions have been obtained numerically \[4\].

Because of the fact that corrections from higher powers of the curvature must be considered on the gravity side of the correspondence in order to investigate CFTs with different values of their central charges, the holography of gravity theories including higher powers of the curvature have attracted increased attention \[5, 6\]. Asymptotic Lifshitz solutions in the vacuum of higher-derivative gravity have been investigated and it is shown that the higher-curvature terms with suitable coupling constant may play the role of the desired matter \[7\]. Recently, one of us introduced some exact and numerical Lifshitz solutions in higher-curvature gravity with cubic-curvature terms \[8, 9\]. From the point of view of holography,
thermodynamics of Lifshitz black holes has been investigated by many authors in different theories of gravity [10].

In this paper, we would like to investigate the thermodynamics of rotating Lifshitz black branes in third-order Lovelock gravity. We use the counterterm method introduced in [11] in order to compute the conserved quantities of the spacetime. The motivation for considering rotating Lifshitz black branes is to investigate the effect of rotation parameter on the properties of black branes. Specially, we would like to find the effect of rotation parameter on the stability of rotating Lifshitz black branes. Although asymptotically AdS black branes with flat horizon are stable [12, 13], we find that asymptotic Lifshitz black branes with a flat horizon can be unstable.

The outline of our paper is as follows. In Sec. II we give a brief review of general formalism of calculating the conserved quantities of Lifshitz black branes. In Sec. III, we introduce rotating Lifshitz black branes. Section IV is devoted to the investigation of the thermodynamics of rotating Lifshitz black branes. We also perform a local stability analysis of the black branes in the canonical and grand-canonical ensembles. We finish our paper with some concluding remarks.

II. GENERAL FORMALISM

Here, we give a brief review of the general formalism of calculating the conserved quantities of asymptotically Lifshitz black branes in third-order Lovelock gravity. The action of third-order Lovelock gravity may be written as

\[ I_g = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left( \sum_{i=1}^{[n/2]} \alpha_i \mathcal{L}_i - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right), \tag{2} \]

where \([x]\) is the integer part of \(x\), \(\Lambda\) is the cosmological constant, and \(\alpha_i\)'s are Lovelock coefficients with \(\alpha_1 = 1\). In the above action, \(F_{\mu\nu} = \partial_{[\mu} A_{\nu]}\) where \(A_\mu\) representing a massive gauge field with mass \(m\), and \(\mathcal{L}_i\) is the \(i\)th order Lovelock Lagrangian given as

\[ \mathcal{L}_i = \frac{1}{2^i \delta^{\mu_1 \mu_2 \cdots \mu_{2i}}_{\nu_1 \nu_2 \cdots \nu_{2i}}} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \cdots R_{\nu_{2i-1} \nu_{2i}}^{\mu_{2i-1} \mu_{2i}}. \tag{3} \]

In Lovelock gravity only terms with order less than or equal \([n/2]\) contribute to the field equations, the rest being total derivatives in the action. Here, we consider Lovelock gravity up to a third-order term and therefore we consider \((n + 1)\)-dimensional spacetimes with
n ≥ 6 (though in situations where we set α_2 = α_3 = 0 our solutions will be valid for n ≥ 2). The explicit form of the variational principle, when the spacetime has a boundary. In order to have a well-defined variational principle, one should add the following boundary term to the above action

\[ I_b = \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-h} (K + 2\alpha_2 J + 3\alpha_3 P), \]  

where the boundary \( \partial M \) is the hypersurface at some constant \( r \), \( h_{mn} \) is the induced metric, \( K \) is the trace of the extrinsic curvature, \( K_{pq} = \nabla (n_p n_q) \) of the boundary (where the unit vector \( n^\mu \) is orthogonal to the boundary and outward-directed) and \( J \) and \( P \) are \[ J = \frac{1}{3} (3KR - 6KR_{mn}R^{mn} + 3KK_{mn}K^{mn} - 2K_{mn}K^{np}K^{mn} - K^3), \]  

\[ P = KR^2 - 4KR_{mn}R^{mn} + K^{mnop}R_{mnop} - 4K_{mn}R_{mn} + 12K_{mn}R_{mn} + 2K_{mn}K_{mn}R - 12K_{mn}K^{np}R_{np} - 12K_{mn}K_{mn}K^3 \]  

In general, the total action \( I_g + I_b \) is not finite when evaluated on the solution, as is the Hamiltonian and other associated thermodynamic quantities. In order to have a finite action, we must add some counterterms to the action (2). Here, we restrict ourselves to the case of rotating Lifshitz black branes with a flat horizon. For this case, as in the case of static solutions of Lovelock gravity [11], the following counterterms make the action finite:

\[ I_{ct} = -\frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-h} \left( \frac{(n-1)l^4}{l^5} + \frac{zq}{2l} \sqrt{-A_m A^m} \right). \]  

This is due to the fact that on the boundary \( A_m A^m = -q^2 \) is constant for rotating Lifshitz solutions. The variation of total action \( I = I_g + I_b + I_{ct} \) about a solution of the equation of
motion is

\[ \delta I = \int d^n x (S_{mn} \delta h^{mn} + S_m \delta A^m), \]

where

\[ S_{mn} = \frac{\sqrt{-h}}{16\pi} \left[ \Pi_{mn} + \frac{zq}{2l} (-A_p A^p)^{-1/2} (A_m A_n - A_p A^p h_{mn}) \right], \]

\[ S_m = \frac{\sqrt{-h}}{16\pi} \left[ \Pi^\mu n_{\mu m} + \frac{zq}{2l} (-A_p A^p)^{-1/2} A_m \right], \]

with

\[ \Pi_{mn} = K_{mn} - Kh_{mn} + 2\alpha_2 (3J_{mn} - Jh_{mn}) + 3\alpha_3 (5P_{mn} - Ph_{mn}) + \frac{(n - 1) L^4}{l^5} h_{mn}. \] (9)

In Eq. (9) \( J_{mn} \) and \( P_{mn} \) are

\[ J_{mn} = \frac{1}{3} (2KK_{mp} K_n^p + K_{pq} K_{pq} K_{mn} - 2K_{mp} K_{pq} K_{qn} - K^2 K_{mn}), \] (10)

\[ P_{mn} = \frac{1}{3} \left\{ [K^4 - 6K^2 K_{pq} K_{pq} + 8KK_{pq} K_{pq} K_{pq} - 6K_{pq} K_{pq} K_{rs} K_{rp} + 3(K_{pq} K_{pq})^2] K_{mn} \right. \\
- (4K^2 - 12K K_{pq} K_{pq} + 8K_{pq} K_{pq} K_{pq} - 24K K_{mp} K_{pq} K_{qn} - 24K_{mp} K_{pq} K_{qr} K_{rs} K_{pn}) \\
+ (12K^2 - 12K K_{rs} K_{rs}) K_{mp} K_{pq} K_{qn} + 24K K_{mp} K_{pq} K_{qr} K_{rs} K_{ns} \right\}. \] (11)

The dual field theory is nonrelativistic for asymptotically Lifshitz spacetimes, and therefore it will not have a covariant relativistic stress tensor. However, one can define a stress tensor complex, consisting of the energy density \( \mathcal{E} \), energy flux \( \mathcal{E}_i \), momentum density \( \mathcal{P}_i \), and spatial stress tensor \( \mathcal{P}_{ij} \) as [16]

\[ \mathcal{E} = 2S_t^t - S_t^i A_i, \quad \mathcal{E}_i = 2S^i_t - S^i_A_i, \] (12)

\[ \mathcal{P}_i = -2S^t_i + S^t A_i, \quad \mathcal{P}_i^j = -2S^j_i + S^j A_i, \] (13)

which satisfy the conservation equations

\[ \partial_t \mathcal{E} + \partial_i \mathcal{E}_i = 0, \quad \partial_t \mathcal{P}_j + \partial_i \mathcal{P}_i^j = 0. \] (14)

III. ROTATING LIFSHITZ BLACK BRANES WITH ONE ROTATION PARAMETER

First, we introduce rotating Lifshitz spacetime in third-order Lovelock gravity. The Lifshitz metric with one rotation parameter in \( (n+1) \) dimensions may be written as [8]

\[ ds^2 = -\frac{r^{2z}}{l^{2z}} (\Xi dt - ad\varphi)^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^4} (adt - \Xi l^2 d\varphi)^2 + r^2 \sum_{i=1}^{n-2} dx_i^2, \] (15)
where \( a \) is the rotation parameter and
\[
\Xi^2 = 1 + \frac{a^2}{l^2}.
\]

The metric (15) is a generalization of the asymptotically rotating AdS metric introduced in [17]. Although the metric (1) may be transformed to the rotating Lifshitz metric (15) by the transformation
\[
t \mapsto \Xi t - a\phi, \quad \phi \mapsto \Xi \phi - \frac{a}{l^2} t,
\]
the metric (15) generates a new spacetime if \( \phi \) is periodically identified. Indeed, the periodic nature of \( \phi \) allows the metrics (1) and (15) to be locally mapped into each other but not globally, and so they are distinct [18]. We will see that the rotating solution is not stable for a large rotation parameter, and therefore the properties of the metrics (1) and (15) are not the same.

Using the ansatz
\[
A = q r^2 \frac{\Xi}{l^2} (\Xi dt - ad\phi),
\]
for the gauge field and defining \( \hat{\alpha}_2 \equiv (n-2)(n-3)\alpha_2 \) and \( \hat{\alpha}_3 \equiv (n-2)...(n-5)\alpha_3 \) for convenience, it is straightforward to show that the action (2) supports the Lifshitz metric (15) provided
\[
m^2 = \frac{(n-1)z}{l^2},
\]
\[
q^2 = \frac{2(z-1)L^4}{zL^4},
\]
\[
\Lambda = -\frac{[(z-1)^2 + n(z-2) + n^2]L^4 + n(n-1)(\hat{\alpha}_2 l^2 - 2\hat{\alpha}_3)}{2l^6},
\]
where we define
\[
L^4 \equiv l^4 - 2l^2 \hat{\alpha}_2 + 3\hat{\alpha}_3.
\]

for simplicity and we use this definition throughout the paper.

Second, we consider the black brane solution, which asymptotes to the metric (15). The metric of such a solution may be written as
\[
ds^2 = -\frac{r^{2z}}{l^{2z}} f(r)(\Xi dt - a d\phi)^2 + \frac{l^2}{r^2 g(r)} dr^2 + \frac{r^2}{l^4} (adt - \Xi l^2 d\phi)^2 + r^2 \sum_{i=1}^{n-2} dx_i^2.
\]
Using the ansatz
\[ A = q \frac{r^z}{l^z} h(r) [\Xi dt - ad\varphi] \]
for the gauge field, the action (2) reduces to
\[ I_g = \frac{n - 1}{16\pi} \int d^n x dr r^{n-1} \sqrt{g} \left\{ r^n \left( -\frac{\Lambda}{n(n-1)} l^2 - g + \frac{\hat{\alpha}_l}{l^2} g^2 - \frac{\hat{\alpha}_3}{l^4} g^3 \right) \right\}^{'} + \frac{q^2 r^{n-1}}{2(n-1)f} \left\{ (zh + rh')^2 g + m^2 l^2 h^2 \right\}. \] (21)

Functionally varying (21) with respect to \( f(r) \), \( g(r) \), and \( h(r) \) yields upon simplification
\[
\hat{\alpha}_3 \left[ (n + 6z - 6) f + 3rf' \right] g^3 + \hat{\alpha}_2 l^2 \left[ (n + 4z - 4) f + 2rf' \right] g^2 \\
- l^4 \left[ (n + 2z - 2) f + rf' \right] g - \frac{\Lambda l^6}{n-1} f = \frac{q^2 l^4}{2(n-1)} \left\{ (zh + rh')^2 g - m^2 l^2 h^2 \right\},
\]
\[
\left\{ r^n \left( -\frac{\Lambda l^2}{n(n-1)} g + \frac{\hat{\alpha}_l}{l^2} g^2 - \frac{\hat{\alpha}_3}{l^4} g^3 \right) \right\}^{'} = \frac{q^2 r^{n-1}}{2(n-1)f} \left\{ (zh + rh')^2 g + m^2 l^2 h^2 \right\},
\]
\[
2r^2 h'' - r \left[ (lnf)' - (lng)' \right] (rh' + zh) + 2(z + n)rh' + 2(n - 1)zh = 2m^2 l^2 h / g. \] (22)

The numerical solutions of the above field equations (22) are the same as those given in [8]. This is due to the fact that the rotating metric (20) and static metric (\( \Xi = 1 \)) are locally the same and therefore the metric functions are the same in both cases.

### IV. THERMODYNAMICS OF ROTATING LISHTZ BLACK BRANES

Now, we investigate the thermodynamics of rotating black branes. The temperature of the event horizon is given by
\[ T = \frac{1}{2\pi} \left( -\frac{1}{2} \nabla_b \xi_a \nabla^b \xi^a \right) \left. ^{1/2} \right|_{r=r_0}, \] (23)
where \( \xi \) is the Killing vector
\[ \xi = \partial_t + \Omega \partial_\varphi \] (24)
and \( \Omega \) is the angular velocity of the Killing horizon given as
\[ \Omega = - \left[ g_{t\varphi} \right] \left. ^{1/2} \right|_{r=r_0} = \frac{\sqrt{\Xi^2 - 1}}{\Xi l}. \] (25)
Using Eq. (23) and the expansion of the metric functions given in the appendix (see Eq. (41)), the temperature can be obtained as
\[ T = \frac{r_0^{z+1}}{4\pi l} \left( f'g' \right) \left. ^{1/2} \right|_{r=r_0} = \frac{r_0^{z+1} \sqrt{f_1 g_1}}{4\pi l^{z+1} \Xi}. \] (26)
The form of the near-horizon expansion given in the appendix suggests that \( f_1 \) and \( g_1 \) are proportional to \( r_0^{-1} \), and therefore

\[
T = \frac{\eta}{4\pi \Xi} r_0^z, \quad (27)
\]

where \( \eta \) is a proportionality constant.

The entropy can be calculated through the use of [19]

\[
S = \frac{1}{4} \sum_{i=1}^{p} k\alpha_i \int d_{m-1} x \sqrt{\tilde{g}} \tilde{L}_{i-1},
\]

where the integration is done on the \((n-1)\)-dimensional spacelike hypersurface of the Killing horizon with induced metric \( \tilde{g}_{ab} \) (whose determinant is \( \tilde{g} \)), and \( \tilde{L}_i \) is the \( i \)th order Lovelock Lagrangian of \( \tilde{g}_{ab} \). Since we are dealing with flat horizon, \( \tilde{L}_1 = \tilde{L}_2 = 0 \) and therefore the entropy density is

\[
S = \frac{1}{4} \Xi r_0^{n-1}. \quad (28)
\]

The conserved quantity along the radial coordinate \( r \), which relates to the coefficients of the expansion of the metric function at the horizon \((r = r_0)\) and at infinity is [11]

\[
C_0 = \{(1 - 2 \hat{\alpha}_2 l_2^2 + 3 \hat{\alpha}_3 l_3^2) [rf' + 2(z - 1)f] - q^2(zh + rh'h) \sqrt{g} \} \frac{r^{n+z-1}}{l^{z+1}} (\frac{f}{g})^{1/2}. \quad (29)
\]

Using the expansion of metric functions at the horizon given in the appendix and Eqs. (26-28), the constant \( C_0 \) (29) at \( r = r_0 \) can be written as

\[
C_0 = \frac{r_0^{n+z} \sqrt{f_1 g_1}}{l^{z+1}} = 16\pi TS = \eta r_0^{n+z-1}. \quad (30)
\]

On the other side, by use of the large \( r \) expansions given in the appendix for the metric functions, we obtain

\[
C_0 = FC_1, \quad (31)
\]

where

\[
F = \frac{2(z - 1)(z + n - 1)^2 \{(z - n + 1)l^4 + 2(n - 2)\hat{\alpha}_2 l^2 - 3(n + z - 3)\hat{\alpha}_3\}}{zl^{z+5} K}. \quad (32)
\]

for Lovelock gravity and

\[
F = \frac{4(n - 1)}{(2n - 1)l^n} \quad (33)
\]

in the case of Einstein gravity with \( z = n - 1 \).
Using Eqs. (12) and (13), one can compute the energy and angular momentum densities of the black brane for \( z \neq n - 1 \) as

\[
E = \frac{(n + z - 1)\Xi^2 - z}{16\pi(z + n - 1)} F C_1 = \frac{(n + z - 1)\Xi^2 - z}{16\pi(z + n - 1)} \eta r_0^{n+z-1},
\]

(34)

\[
J = \frac{l}{16\pi} \Xi \sqrt{\Xi^2 - 1} F C_1 = \frac{l}{16\pi} \Xi \sqrt{\Xi^2 - 1} \eta r_0^{n+z-1},
\]

(35)

where \( F \) is given in Eq. (32). The above expressions are valid for the case of \( z = n - 1 \) provided \( F \) reads from Eq. (33). One may note that Eq. (34) reduces to that given in [11] for the case of the static Lifshitz solution (\( \Xi = 1 \)), and the angular momentum density vanishes as expected.

Now, using Eqs. (30), (31), (34) and (35), we find

\[
E = \frac{n - 1}{n - 1 + z} TS + \Omega J.
\]

(36)

Again for \( \Xi = 1 \), Eq. (36) reduces to that of static case since both \( \Omega \) and \( J \) vanish. Also, it is worth noting that Eq. (36) reduces to

\[
E = \frac{n - 1}{n} TS + \Omega J.
\]

(37)

for asymptotic AdS black branes [12]. Recently, it has been shown that the Komar-conserved quantity corresponding to the null Killing vector is also proportional to the Hawking temperature [20].

Dividing Eq. (34) by Eq. (35) gives us a Smarr-type formula as

\[
E(S, J) = \frac{(n + z - 1)\Xi^2 - z}{(z + n - 1)l\Xi \sqrt{\Xi^2 - 1}} J,
\]

where \( \Xi \) depends on \( S \) and \( J \) through the following equation

\[
16\pi J - \eta l \left( \frac{4S}{\Xi} \right)^{(n+z-1)/(n-1)} \Xi \sqrt{\Xi^2 - 1} = 0.
\]

One may then regard the parameters \( S \) and \( J \) as a complete set of extensive parameters for the energy density \( E(S, J) \) and define the intensive parameters conjugate to \( S \) and \( J \). These quantities are the temperature and the angular velocity

\[
T = \left( \frac{\partial E}{\partial S} \right)_J, \quad \Omega = \left( \frac{\partial E}{\partial J} \right)_S.
\]

(38)

It is a matter of straightforward calculation to show that the intensive quantities calculated by Eq. (38) coincide with Eqs. (25) and (27) found in Sec. IV. Thus, the thermodynamic quantities calculated in this paper satisfy the first law of thermodynamics

\[
dE = TdS + \Omega dJ.
\]
V. STABILITY OF ROTATING LIFSHITZ BLACK BRANES

Now, we investigate the stability of rotating Lifshitz black branes. The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates can be carried out by finding the determinant of the Hessian matrix of the energy with respect to its extensive parameters $X_i$, $H_{X_iX_j} = \left[ \partial^2 E / \partial X_i \partial X_j \right]$. In our case the energy density is a function of the entropy and angular momenta densities. The number of thermodynamic variables depends on the ensemble which is used. In the canonical ensemble, the angular momenta is a fixed parameter, and therefore the positivity of heat capacity, $C_J = T(\partial T / \partial S)_J$, is sufficient to assure the local stability. This quantity can be calculated as

$$C_J = \frac{S[(n-z-1)\Xi^2 + z]}{(n+2z-1)(\Xi^2 - 1) + z\Xi^2}.$$  

Since the denominator is positive, the heat capacity is positive provided

$$(n-z-1)\Xi^2 + z > 0. \quad (39)$$

In the grand-canonical ensemble, the determinant of the Hessian matrix of the energy with respect to $S$ and $J$ can be obtained as

$$\left| H_{S,J}^\varepsilon \right| = \frac{z}{\Xi^2l^2S^2[(n-z-1)\Xi^2 + z]},$$

which shows that the Lifshitz black branes are stable provided the condition (39) holds.

Thus, the condition of stability of Lifshitz black branes is the same in both canonical and grand-canonical ensembles. One may note that the stability analysis is also the same in Einstein and Lovelock gravity for Lifshitz black branes with a flat boundary. For $z \leq n-1$, the condition (39) is satisfied and therefore Lifshitz black branes are stable. But, they can be unstable for $z > n-1$ provided the rotation parameter $a$ satisfies

$$a > \left( \frac{n-1}{z - (n-1)} \right)^{1/2}l. \quad (40)$$

VI. CONCLUSIONS

In this paper, we investigated the thermodynamics of rotating Lifshitz black branes in Lovelock gravity. We obtained the conserved quantities of the solutions through the use of counterterm method introduced in [11]. We showed that the energy density $\mathcal{E}$, entropy
density \( S \), temperature \( T \), angular velocity \( \Omega \) and angular momentum density \( J \) are related by Eq. (36). This relation is the generalization of Eq. (37) for asymptotic AdS black branes. Also, one may note that the relation (36) reduces to the relation between the energy density \( \mathcal{E} \), entropy density \( S \), and temperature \( T \) for static Lifshitz solutions. We also obtain a Smarr-type formula for the energy density \( \mathcal{E}(S,J) \) as a function of extensive quantities \( S \) and \( J \). We showed that the thermodynamic quantities calculated in this paper satisfy the first law of thermodynamics. Finally, we investigated the stability of the black brane solutions in both canonical and grand-canonical ensemble. We found that the solutions are thermally stable for the solutions with \( z \leq n-1 \), while they can be unstable for \( z > n-1 \) provided the rotation parameter satisfies Eq. (40). It is worth noting that, for \( z > 2(n-1) \), the rotation parameter can be less than \( l \).

One may generalize rotating Lifshitz black brane solutions with one rotating parameter to the case of rotating solutions with more rotation parameters. The rotation group in \( n+1 \) dimensions is \( SO(n) \) and therefore the number of independent rotation parameters is \([n/2]\), where \([x]\) is the integer part of \( x \). Equation (36) for Lifshitz black branes with \( k \leq [n/2] \) rotation parameters generalizes to

\[
\mathcal{E} = \frac{n-1}{n-1+z}TS + \sum_{i=1}^{k} \Omega_i J_i,
\]

where \( \Omega_i \) and \( J_i \)'s are the \( i \)th component of angular velocity and angular momentum, respectively. Also, one may generalize the above equation to the case of charged rotating Lifshitz black branes in the presence of Maxwell field.

VII. APPENDIX

In this appendix, we consider the near-horizon and large-\( r \) behavior of rotating Lifshitz solutions. Since we consider nonextreme black brane solutions, the functions \( f(r) \) and \( g(r) \) go to zero linearly, that is

\[
f(r) = f_1(r-r_0) + f_2(r-r_0)^2 + f_3(r-r_0)^3 + f_4(r-r_0)^4 + \ldots
\]

\[
g(r) = g_1(r-r_0) + g_2(r-r_0)^2 + g_3(r-r_0)^3 + g_4(r-r_0)^4 + \ldots
\]

\[
h(r) = f_1^{1/2}\{h_1(r-r_0) + h_2(r-r_0)^2 + h_3(r-r_0)^3 + h_4(r-r_0)^4 + \ldots\},
\]

(41)
where $f_i$'s, $g_i$'s and $h_i$'s are constants which can be obtained through the use of field equations [11].

In order to have the appropriate asymptotic behavior for the metric functions, one may use the straightforward perturbation theory:

\[
\begin{align*}
    f(r) &= 1 + \epsilon w_f(r), \\
    g(r) &= 1 + \epsilon w_g(r), \\
    h(r) &= 1 + \epsilon w_h(r),
\end{align*}
\]

where $\epsilon$ is an infinitesimal parameter. It is a matter of calculations to show that the metric functions at large $r$ are [11]

\[
\begin{align*}
    h(r) &= 1 - \epsilon \left\{ \frac{C_1}{r^{n+z-1}} + \frac{C_2}{r^{(n+z-1+\gamma)/2}} + \frac{C_3 \Theta(n + z - 1 - \gamma)}{r^{(n+z-1-\gamma)/2}} \right\}, \\
    f(r) &= 1 - \epsilon \left\{ \frac{C_1 F_1}{r^{n+z-1}} + \frac{C_2 F_2}{r^{(n+z-1+\gamma)/2}} + \frac{C_3 F_3 \Theta(n + z - 1 - \gamma)}{r^{(n+z-1-\gamma)/2}} \right\}, \\
    g(r) &= 1 - \epsilon \left\{ \frac{C_1 G_1}{r^{n+z-1}} + \frac{C_2 G_2}{r^{(n+z-1+\gamma)/2}} + \frac{C_3 G_3 \Theta(n + z - 1 - \gamma)}{r^{(n+z-1-\gamma)/2}} \right\},
\end{align*}
\]

(42)
where $C_i$'s are arbitrary constants,

\[
\gamma = \left\{ (17 - 8B)z^2 - 2(3n + 9 - 8B)z + n^2 + 6n + 1 - 8B \right\}^{1/2},
\]

\[
F_1 = 2 (z - 1) (z - n + 1) K^{-1},
\]

\[
F_2 = (F_1 - F_2) \left\{ 8z K [(z - 1) B + 2n + z - 3] \right\}^{-1},
\]

\[
F_3 = (F_1 + F_2) \left\{ 8z K [(z - 1) B + 2n + z - 3] \right\}^{-1},
\]

\[
G_1 = 2 (z - 1) (n + z - 1) K^{-1},
\]

\[
G_2 = (G_1 + G_2) \left\{ 8z K [(z - 1) B + 2n + z - 3] \right\}^{-1},
\]

\[
G_3 = (G_1 - G_2) \left\{ 8z K [(z - 1) B + 2n + z - 3] \right\}^{-1},
\]

\[
K = (z - 1)(n + z - 1) B + z(z - 1) + n(n - 1),
\]

\[
F_1 = 8(z - 1)[(z - 1)(n + z - 1) B + z(z - 1) + n(n - 1)]
\times [(z - 1)(n + 3z - 3) B - 2z^2 + (n + 3)z + n(n - 2) - 1],
\]

\[
F_2 = \gamma[n - 1 + (z - 1) B] \left\{ 8(1 + B)(z - 1)^3 + (17n - 9 + 8B)(z - 1)^2 + 2(n + 8)(n - 1)(z - 1) + n^2(n - 1) - (n - 1) \gamma^2 \right\},
\]

\[
G_1 = 8(z - 1)[2(z - 1) B - 3z + 3n - 1][(z - 1)(n + z - 1) B + z(z - 1) + n(n - 1)],
\]

\[
G_2 = \left\{ 8(1 + B)(z - 1)^3 + (17n - 9 + 8B)(z - 1)^2 + 2(n + 8)(n - 1)(z - 1) + n^2(n - 1) \right\} \gamma - (n - 1) \gamma^3,
\]

\[
B = (l^4 - 4\phi_2 l^2 + 9\phi_3)/l^4, \text{ and }
\]

\[
\Theta(n + z - 1 - \gamma) = \begin{cases} 
1 \quad n + z - 1 > \gamma \quad (z < \frac{n - B}{2 - B}) \\
0 \quad n + z - 1 \leq \gamma \quad (z \geq \frac{n - B}{2 - B}) 
\end{cases}
\]

In the case of $z = n - 1$ in Einstein gravity, the metric functions at large $r$ are

\[
h(r) = 1 - \epsilon \frac{C_1 \ln r + C_2}{r^{2(n - 1)}},
\]

\[
f(r) = 1 - \epsilon \frac{3n - 4}{(n - 1)(2n - 3) r^{2(n - 1)}},
\]

\[
g(r) = 1 - \epsilon \left\{ \frac{2(n - 2)(C_1 \ln r + C_2)}{(2n - 3) r^{2(n - 1)}} + \frac{(n^2 - 2) C_1}{(n - 1)(2n - 3)^2 r^{2(n - 1)}} \right\}.
\]

(43)
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[1] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. 101, 061601 (2008).
[2] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009); S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, Phys. Rev. Lett. 101, 031601 (2008); S. A. Hartnoll and P. K. Kovtun, Phys. Rev. D 76, 066001 (2007); C. P. Herzog, J. Phys. A 42, 343001 (2009).
[3] S. Kachru, X. Liu and M. Mulligan, Phys. Rev. D 78, 106005 (2008).
[4] R. B. Mann, J. High Energy Phys. 06 (2009) 075; G. Bertoldi, B. A. Burrington and A. W. Peet, Phys. Rev. D 80, 126003 (2009); K. Balasubramanian and J. McGreevy, Phys. Rev. D 80, 104039 (2009); U. H. Danielsson, L. Thorlacius, J. High Energy Phys. 03 (2009) 070; H. Maeda and G. Giribet, [arXiv:1105.1331] [gr-qc]; K. Copsey and R. B. Mann, J. High Energy Phys. 03 (2011) 039.
[5] J. de Boer, M. Kulaxizia, and A. Parnachev, J. High Energy Phys. 06 (2010) 008; X. O. Camanho and J. D. Edelstein, ibid. 06 (2010) 099.
[6] R. C. Myers, M. F. Paulosb and A. Sinha, J. High Energy Phys. 08, 035 (2010).
[7] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D 80, 104029 (2009); E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, J. High Energy Phys. 04 (2010) 030; R. G. Cai, Y. Liu and Y. W. Sun, ibid. 10 (2009) 080; D. W. Pang, ibid. 01 (2010) 116.
[8] M. H. Dehghani and R. B. Mann, J. High Energy Phys. 07 (2010) 019.
[9] W. G. Brenna, M. H. Dehghani, and R. B. Mann, Phys. Rev. D 84, 024012 (2011).
[10] G. Bertoldi, B. A. Burrington and A. W. Peet, Phys. Rev. D 80 126004 (2009); G. Bertoldi, B. A. Burrington and A. W. Peet, Phys. Rev. D 82 106013 (2010); G. Bertoldi, B. A. Burrington, A. W. Peet, I. G. Zadeh, Phys. Rev. D 83:, 126006 (2011); M. H. Dehghani, R. B. Mann and R. Pourhasan, Phys. Rev. D 84, 046002 (2011).
[11] M. H. Dehghani and R. B. Mann, Phys. Rev. D 82, 064019 (2010).
[12] M. H. Dehghani and A. Khodam-Mohammadi, Phys. Rev. D 67, 084006 (2003).
[13] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505.
[14] M. H. Dehghani and R. B. Mann, Phys. Rev. D 73, 104003 (2006); M. H. Dehghani, N. Bostani and A. Sheykhi, *ibid.* 73, 104013 (2006).

[15] A. Yale, arXiv:1107.1250.

[16] S. F. Ross and O. Saremi, J. High Energy Phys. 09 (2009) 009.

[17] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D 54, 3840 (1996); A. M. Awad, Class. Quant. Grav. 20, 2827 (2003).

[18] J. Stachel, Phys. Rev. D 26, 1281 (1982).

[19] T. Jacobson and R. C. Myers, Phys. Rev. Lett. 70, 3684 (1993).

[20] R. Banerjee and B. R. Majhi, Phys. Rev. D 81, 124006 (2010); R. Banerjee, B. R. Majhi, S. K. Modak and S. Samanta, *ibid.* 82, 124002 (2010).