Two–Dimensional Integrable Systems and Self–Dual Yang–Mills Equations∗

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Abstract

The relation between two–dimensional integrable systems and four–dimensional self–dual Yang–Mills equations is considered. Within the twistor description and the zero–curvature representation a method is given to associate self–dual Yang-Mills connections with integrable systems of the Korteweg–de Vries and non–linear Schrödinger type or principal chiral models. Examples of self–dual connections are constructed that as points in the moduli do not have two independent conformal symmetries.

1 Introduction

As a system of partial differential equations, the self–dual Yang–Mills (SDYM) equations are invariant under the action of the group of conformal transformations acting on the four space–time coordinates. It is well known that invariant solutions by the action of a subgroup with two conformal generators satisfy a differential equation in two–variables since each one–dimensional subgroup reduces by one the number of independent variables. These solutions are called self–similarity solutions of the original equations. It has been observed that this procedure allows one to describe the corresponding invariant solutions in terms of a two–dimensional integrable system. This is the case for the principal chiral model14,15,16 (for the classical euclidean O(1, 3) non–linear σ–model this was already noticed in Ref.[10]) as well as for the Korteweg–de Vries (KdV) and Nonlinear Schrödinger (NLS) equations.6,7 A review of these results can be found in Ref.[1]. It is a nontrivial aspect of the reduction
problem to describe the resulting two–dimensional equations. For the principal chiral model this description is based in the formulation of SDYM equations in terms of the Yang matrix. An alternative approach consists in describing the SDYM equations as a compatibility condition for two linear differential equations, this allows one to identify the reduced solutions in the KdV and NLS cases.

It seems, however, that any reasonable relation between two integrable equations should be based in their definitions as integrable systems. The SDYM equations describe a connection for a bundle over the Grassmannian of two–dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two–planes in the tangent space of the Grassmannian. As proved by Ward,\textsuperscript{13,17} this allows one to characterize SDYM connections in terms of the splitting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialization of the bundle. For two–dimensional integrable systems the situation is quite analogous. An important feature of equations such as the KdV, NLS or the principal chiral model consists in the possibility of constructing solutions through a factorization problem in the circle. This is equivalent to the splitting problem for the SDYM case if we choose the transition function having the form required by the factorization problem of a two–dimensional integrable system.

It is the main result of this paper the construction of a map taking arbitrary solutions of a two–dimensional integrable system into solutions of the SDYM equations. This map comes from the zero–curvature formulation of a two–dimensional integrable system and represents an extension of the correspondence derived from the factorization problem. A generic property of the relation for a principal chiral model, the KdV, and NLS equations is the appearance of arbitrary functions. In particular, this generalizes the self–similarity solutions under a group generated by two translations. Section 2 is devoted to the analysis of these properties. The analysis of the correspondence for a Yang matrix is given in Section 3 where we study its relation with a chiral field. As a special case one gets Ward’s construction\textsuperscript{13,14,15} we mentioned before. Finally in Section 4 we present examples of SDYM connections derived from two–dimensional integrable systems that can not be obtained by symmetry reduction by two conformal generators.

Let us observe that the present construction does not exhaust all possible relations between SD equations and integrable systems. This is the case for the Nahm’s equations that can be associated to SDYM equations in an alternative way to that followed in this paper.\textsuperscript{5}

\section{Integrable systems and self–dual Yang–Mills equations}

The compatibility condition for the linear system of two first–order differential equations on the vector $\psi$,

$$\psi_x = U(\lambda)\psi$$

\[ \psi_t = V(\lambda)\psi, \]

implies for the two matrix functions \( U(x, t, \lambda) \), \( V(x, t, \lambda) \), depending on a complex variable \( \lambda \), the nonlinear equation

\[ U_t - V_x + [U, V] = 0. \quad (2.1) \]

Under an appropriate choice of the functions \( U(\lambda) \) and \( V(\lambda) \) one can represent through (2.1) a wide class of non-linear partial differential equations in the variables \( x \) and \( t \) known as integrable systems.\(^3\) Concrete examples for these functions that we shall consider below are as follows.

Let \( U \) and \( V \) be given by the expressions

\[
U(\lambda) := \lambda U_1 + U_0, \quad V(\lambda) := \lambda^2 U_1 + \lambda U_0 + V_0.
\]

then Eq. (2.1) implies for the coefficients \( U \) and \( V \) the relations

\[
\begin{align*}
U_{0,t} - V_{0,x} + [U_0, V_0] &= 0, \\
U_{1,t} - U_{0,x} + [U_1, V_0] &= 0, \\
U_{1,x} &= 0.
\end{align*}
\]

In particular if we take

\[
U(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \quad V(\lambda) = \begin{pmatrix} -\frac{1}{4}u_x & \lambda + \frac{1}{2}u \\
\lambda^2 - \frac{1}{2}u\lambda - \frac{1}{3}(u_{xx} + 2u^2) & \frac{1}{4}u_x \end{pmatrix},
\]

we obtain the Korteweg–de Vries (KdV) equation\(^8\) for the scalar function \( u(x, t) \)

\[ 4u_t = u_{xxx} + 6uu_x. \]

Letting now \( U \) and \( V \) be given by

\[
U(\lambda) = \begin{pmatrix} i\lambda & p \\ -p^* & -i\lambda \end{pmatrix}, \quad V(\lambda) = \begin{pmatrix} i\lambda^2 - \frac{i}{2}|p|^2 & \lambda p - \frac{i}{2}p_x \\
-\lambda p^* - \frac{i}{2}p^*_x & -i\lambda^2 + \frac{i}{2}|p|^2 \end{pmatrix},
\]

the system above results in the nonlinear Schödinger (NLS) equation for the complex scalar field \( p(x, t) \)

\[ ip_t = \frac{1}{2}p_{xx} + |p|^2 p. \]

The equations of a principal chiral field for functions \( u(x, y) \), \( v(x, y) \) with values in a Lie algebra \( \mathfrak{g} \)

\[
u_y + \frac{1}{2}[u, v] = 0, \quad v_x - \frac{1}{2}[u, v] = 0 \quad (2.2)
\]

can equally be represented with the aid of (2.1) if we choose in this case

\[
U(\lambda) = \frac{u}{\lambda - 1}, \quad V(\lambda) = \frac{v}{\lambda + 1}
\]
with the substitution \( t \to y \).¹⁹

The preceding formulas can be conveniently represented as the zero curvature condition for a connection

\[
\omega : \mathbb{C} \to \Lambda^1(M, \mathfrak{g})
\]

\[
\lambda \mapsto \omega(\lambda),
\]

where \( \lambda \) is a spectral parameter and \( \omega(\lambda) \) is a \( \mathfrak{g} \)-valued 1–form over the two-dimensional time manifold \( M \) having as local coordinates the variables \( x, t \) (or \( x \) and \( y \)). In terms of local coordinates \( x, y \) (or \( x, t \)) of the manifold \( M \) we can write the connection as

\[
\omega(\lambda) = U(\lambda)dx + V(\lambda)dy,
\]

for which the vanishing of the curvature

\[
\Omega(\lambda) := d\omega(\lambda) - \frac{1}{2}[\omega(\lambda), \omega(\lambda)]
\]

is equivalent to the relation (2.1).

Self–dual Yang–Mills (SDYM) equations constitute another important example of non–linear partial differential equations which are also integrable. Now, we shall be dealing with a four dimensional system instead of the two–dimensional cases previously analyzed. If \( z = (z^{AA'}) \), \( A = 0, 1, A' = 0', 1' \) represents a \( 2 \times 2 \) complex matrix describing the local coordinates of a four dimensional complex manifold and \( \Phi = \Phi_{AA'}dz^{AA'} \) is a \( \mathfrak{g} \)-valued connection described by the set of functions \( \Phi_{AA'}(z) \), then the curvature 2–form \( F = d\Phi - \frac{1}{2}[\Phi, \Phi] \) can be written as

\[
F = F_{AA'BB'}dz^{AA'} \wedge dz^{BB'}.
\]

A self–duality condition¹⁷ for the Yang–Mills fields \( F_{AA'BB'} \) follows from the decomposition \( F_{AA'BB'} = \varphi_{AB} \varepsilon_{A'B'} + \varphi_{A'B'} \varepsilon_{AB} \) where \( \varepsilon \) is the Levi–Civita tensor. Thus, the connection \( \Phi \) is self–dual (SD) if

\[
\varphi_{A'B'} = 0.
\]

Given a SD connection \( \Phi \) any gauge transformation \( \Phi \to \Phi^g = dg \cdot g^{-1} + \text{Ad}_g \Phi \) gives again a SD connection. A conformal transformation of coordinates \( z \mapsto z = (A \cdot z + B) \cdot (C \cdot z + D)^{-1} \), with \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(4, \mathbb{C}) \), generates a local diffeomorphism \( \phi \) such that \( \phi^* \Phi \) is also a SD connection. We say that two SD connections are equivalent if there exists a gauge and/or a conformal transformation taking one into the other. When we introduce certain topological requirements this space of equivalence classes of SD connections is called the moduli space.

The structure of this set of non–linear partial differential equations and its relation with the two–dimensional integrable systems presented before are best understood through the geometry of a complex vector bundle over the Riemann sphere¹³,¹⁷ as we shall explain briefly.
Let us denote by $T$ a four-dimensional complex linear space, the twistor space, and consider its decomposition as a direct sum of a pair of two-dimensional subspaces $S$ and $S'$, $T = S \oplus S'$. We associate to this pair of subspaces a coordinate chart of the Grassmannian manifold of two-dimensional subspaces of $T$ as follows. If $V$ is a subspace of $T$ that does not intersect $S'$, we assign to $V$ the linear map $z : S' \to S$ having as graph $V$. Then, one is allowed to identify the coordinates $z^{AA'}$ on which the connection $\Phi = \Phi_{AA'}d^2z^{AA'}$ depends with the coordinate functions of these points in the Grassmannian. If a point in $T$ has coordinates $(x^A, x_{A'})$ adapted to the representation $T = S \oplus S'$, the linear subspace $V$ determined by the linear transformation $z$ is characterized by the relations $x^A = z^{AA'}x_{A'}$.

Let $\lambda^{A'}$ denote the coordinates of a point in the dual of $S'$ and define the vector fields on the Grassmannian
\[
\partial_A := \partial_{AA'}\lambda^{A'},
\]
given in terms of the local coordinates $z$ by the differential operators $\partial_{AA'} := \partial/\partial z^{AA'}$ that generate the tangent space. The SD equations for the connection $\Phi$ are then equivalent to the requirement that $F$ vanishes over the two-plane $C\{\partial_A\}_{A=0,1}$
\[
F(\partial_A, \partial_B) = 0. \tag{2.3}
\]
One has
\[
F(\partial_A, \partial_B) = \lambda^{A'}\lambda^{B'}F(\partial_{AA'}, \partial_{BB'}) = \lambda^{A'}\lambda^{B'}(\varepsilon_{AB}\varepsilon_{A'B'} + \varepsilon_{A'B'}\varepsilon_{AB})
\]
from which our assertion follows. The connection $\Phi$ satisfies the equation
\[
[\partial_A - \Phi_A, \partial_B - \Phi_B] = \partial_B\Phi_A - \partial_A\Phi_B + [\Phi_A, \Phi_B] = 0
\]
that we get from (2.3) with $\Phi_A := \Phi(\partial_A) = \Phi_{AA'}\lambda^{A'}$ and the SD equations are
\[
F_{00'10'} = F_{00'11'} + F_{01'10'} = F_{01'11'} = 0. \tag{2.4}
\]
All relations involving the coordinates $\lambda^{A'}$ are homogeneous in these variables, they are therefore well defined on the projective $\lambda^A$ plane. With the standard covering by the two charts $C_+, C_-$
\[
C_+ := \{\lambda^{A'}/\lambda^{0'}, \lambda^{0'} \neq 0\}
\]
\[
C_- := \{\lambda^{0'}/\lambda^A, \lambda^A \neq 0\}
\]
the induced equations are
\[
[\partial^{+}_A - \Phi^{+}_A, \partial^{+}_B - \Phi^{+}_B] = 0, \quad \partial^{+}_A := \partial_{A0'} + \lambda\partial_{A1'}, \quad \Phi^{+}_A := \Phi_{A0'} + \lambda\Phi_{A1'}, \quad \lambda \in C_+,
\]
\[
[\partial^{-}_A - \Phi^{-}_A, \partial^{-}_B - \Phi^{-}_B] = 0, \quad \partial^{-}_A := \lambda\partial_{A0'} + \partial_{A1'}, \quad \Phi^{-}_A := \lambda\Phi_{A0'} + \Phi_{A1'}, \quad \lambda \in C_-.
\]
Then, we have trivializations $\psi_\pm : C_\pm \to G$ defined by
\[
\partial^{\pm}_A\psi_+ = \Phi^{\pm}_A\psi_+, \quad \partial^{-}_A\psi_- = \Phi^{-}_A\psi_-.
\]
that on \( \mathbb{C}_+ \cap \mathbb{C}_- \) solve the splitting problem for the transition function \( \psi, \psi_+ = \psi \cdot \psi_- \). This was just the situation for the two-dimensional integrable systems considered before and tells us about the possibility of describing SD connections by means of integrable two-dimensional non-linear partial differential equations. This is precisely the case if we let \( \omega(\lambda) \) be defined on the tangent vectors \( \partial_{AA'}, \partial_{BB'} \), equivalently the coordinates \( x, t \) (or \( x, y \)) are functions of \( z \) and \( \lambda \) belongs to \( \mathbb{C}_+ \) or \( \mathbb{C}_- \). For suitable functions \( x(z), t(z) \) we define the SD connection \( \Phi \) associated to \( \omega \) by the relations

\[
\Phi(\partial_A) = \omega(\partial_A). \tag{2.5}
\]

The zero-curvature condition for \( \omega, d\omega - 1/2[\omega, \omega] = 0 \), on the vectors \( \partial_A, \partial_B \) implies the SD equations for \( \Phi \):

\[
0 = (d\omega - \frac{1}{2}[\omega, \omega])(\partial_A, \partial_B) = [\partial_A - \omega(\partial_A), \partial_B - \omega(\partial_B)] = [\partial_A - \Phi(\partial_A), \partial_B - \Phi(\partial_B)] = F(\partial_A, \partial_B).
\]

The 1-form \( \omega \) induces a self-dual connection if its contraction with \( \partial_A \) depends on \( \lambda \) as \( \Phi(\partial_A) \) does. Explicit expressions for \( \omega \) are obtained in each case by imposing the condition (2.5) upon the \( \lambda \)-dependent 1-form \( \omega \). Now, we are in a position to formulate the relation between self-dual connections and integrable systems of KdV and NLS type.

**Theorem 2.1** Let \( \omega = U(\lambda)dx + V(\lambda)dt \) be the zero-curvature 1-form

\[
\omega(\lambda) = (\lambda U_1 + U_0)dx + (\lambda^2 U_1 + \lambda U_0 + V_0)dt.
\]

Then, there exists a self-dual connection \( \Phi \) associated to \( \omega(\lambda) \) in \( \mathbb{C}_+ \) if

\[
t(z) = m(z^{00'}, z^{10'})
\]

\[
x(z) = -z^{01'}\partial_{00'}m(z) - z^{11'}\partial_{10'}m(z) + n(z^{00'}, z^{10'})
\]

for arbitrary functions \( m \) and \( n \). The coefficients of the self-dual connection are given by

\[
\Phi_{A0'} = U_0\partial_{A0'}x + V_0\partial_{A0'}t
\]

\[
\Phi_{A1'} = U_1\partial_{A1'}x.
\]

**Proof:** On \( \mathbb{C}_+ \) we have \( \partial_A^+ := \partial_{A0'} + \lambda \partial_{A1'} \) and

\[
\omega(\partial_A^+) = U(\lambda)\partial_A^+x + V(\lambda)\partial_A^+t
\]

\[
= (\lambda U_1 + U_0)(\partial_{A0'}x + \lambda \partial_{A1'}x) + (\lambda^2 U_1 + \lambda U_0 + V_0)(\partial_{A0'}t + \lambda \partial_{A1'}t).
\]

Then, Eq. (2.3) implies that the coefficients of \( \lambda^3 \) and \( \lambda^2 \) must vanish and this gives the desired result. \( \square \)

The second class of SD connections arising from two-dimensional integrable systems we shall consider is related to the equations of the principal chiral field. The proof of the following theorem reproduces the preceding one.
Theorem 2.2 Let the 1-form
\[ \omega(\lambda) := -\frac{u}{\lambda - 1} dx + \frac{v}{\lambda + 1} dy \]
have zero-curvature for \(|\lambda| < 1\) in \(\mathbb{C}\) and \(x, y\) be functions of the form
\[
\begin{align*}
  x & = x(z^{00'} - z^{01'}, z^{10'} - z^{11'}) \\
  y & = y(z^{00'} + z^{01'}, z^{10'} + z^{11'}),
\end{align*}
\]
then the associated connection \(\Phi(\partial_A) = \omega(\partial_A)\) with components given by
\[
\begin{align*}
  \Phi_{A0'} &= 0, \quad \Phi_{A1'} = u\partial_{A1'}x + v\partial_{A1'}y
\end{align*}
\]
is self-dual.

We observe that the coordinates \(x, y\) on which a principal chiral field depends determine a SD connection if they are defined on the image under \(z\) of the projective lines \(\mathbb{C}(0, 0, 1, -1)\) and \(\mathbb{C}(0, 0, 1, 1)\) respectively.

Integrable systems of the type considered in Theorem 2.1 form a large family. Let us mention the modifications of KdV and NLS, the Fordy–Kulish NLS equations in homogeneous spaces, or the Burgers equation. Similarly, the principal chiral model contains several integrable systems as those derived from the theory of harmonic maps, not only in Lie groups, but in general in Grassmannians.\(^{11}\) Also it has a number of reductions, such as \(\sigma\)-models, the Gross–Neveu model and others.\(^{20}\) Even in the simplest case \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})\) this model has interesting reductions. Let us mention the sinh–Gordon equation, see Ref.\(^{[12]}\) for an analysis of the relations of this equation with SDYM and harmonic maps, the massive Thirring model, and the self–induced transparency equations. As examples, we shall write down explicitly the details for the two first mentioned integrable systems: the sinh–Gordon equation and the massive Thirring model. Consider the action of the homographic transformation \(\lambda \rightarrow \frac{\lambda - 1}{\lambda + 1}\) on the zero–curvature 1–form \(\omega\) of Theorem 2.2, the result is the zero–curvature representation for harmonic maps used in Ref.\(^{[11]}\). Now an arbitrary gauge transformation gives a new zero–curvature 1–form
\[
\omega(\lambda) = (\lambda L_1 + L_0)dx + (M_0 + \lambda^{-1}M_1)dy.
\]
In the chart obtained by a left Lorentz transformation of coordinates
\[
z \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot z
\]
we have a SD connection
\[
\begin{align*}
  \Phi_{A0'} &= L_0 \partial_{A0'}x - M_1 \partial_{A1'}y, \\
  \Phi_{A1'} &= M_0 \partial_{A1'}y - L_1 \partial_{A0'}x.
\end{align*}
\]
Let \( \{E, H, F\} \) be the standard Cartan–Weyl basis of \( \mathfrak{sl}(2, \mathbb{C}) \). For the sinh-Gordon equation we have

\[
L_0 = \exp(-r)E + \frac{r}{2} H, \quad L_1 = \exp(r)F
\]
\[
M_0 = \exp(-r)F + \frac{r}{2} H, \quad M_1 = \exp(r)E,
\]
where the function \( r \) satisfies the sinh–Gordon equation

\[
r_{xy} = 2 \sinh(2r).
\]

If we take now

\[
L_0 = 2uE - i|u|^2 H, \quad L_1 = iH - 2u^* F
\]
\[
M_0 = -2v^* F - i|v|^2 H, \quad M_1 = iH + 2vE,
\]
we obtain for the functions \( u \) and \( v \) the equations of the massive Thirring model

\[
iu_y = 2u|v|^2 - 2v \]
\[
iu_x = -2v|u|^2 - 2u.
\]

Conversely, if we let \( x \) and \( t \) be two functions of \( z \) as in Theorem 2.1 for which

\[
\Delta := \frac{\partial (x, t)}{\partial (z^{00}', z^{10}')} = \partial_{00}x \partial_{10}t - \partial_{10}x \partial_{00}t \neq 0 \quad (2.6)
\]
and we construct the differential 1-form \( \omega(\lambda) = U(\lambda)dx + V(\lambda)dt \), where we define \( U_0, U_1, V_0 \) according to the formulas

\[
U_0 := \frac{1}{\Delta}(\Phi_{00}' \partial_{10}t - \Phi_{10}' \partial_{00}t)
\]
\[
U_1 := \frac{1}{\Delta}(\Phi_{00}' \partial_{10}x - \Phi_{10}' \partial_{00}x)
\]
\[
V_0 := \frac{1}{\Delta}(\Phi_{01}' \partial_{10}t - \Phi_{11}' \partial_{00}t), \quad (2.7)
\]
then we obtain the inversion of the formulas given in Theorem 2.1 defining the connection \( \Phi \) in terms of the coefficients \( U_0, U_1, V_0 \) of \( \omega \).

**Proposition 2.1** Let \( \omega(\lambda) \) the 1-form given by \( \omega(\lambda) = (\lambda U_1 + U_0)dx + (\lambda^2 U_1 + \lambda U_0 + V_0)dt \), where \( U_0, V_0, U_1 \) are defined in (2.7) and depend on \( z \) through the functions \( x, t \) of Theorem 2.1. Then the curvature \( d\omega - 1/2[\omega, \omega] = 0 \) if \( \Phi \) is self–dual and satisfies the gauge condition

\[
\Phi_{01}' \partial_{10}x - \Phi_{11}' \partial_{00}x = 0.
\]

**Proof:** From the expressions for \( \Phi_{AA}' \) in Theorem 2.1 and the relations satisfied by \( x \) and \( t \), we get the following expressions for the curvature components,

\[
F_{0010'} = \Delta (U_{0,t} - V_{0,x} + [U_0, V_0])
\]
\[
F_{0111'} = \Delta U_{1,x}
\]
\[
F_{0011'} + F_{0110'} = \Delta (U_{1,t} - U_{0,x} - [V_0, U_1]).
\]
The curvature $\Omega = d\omega - 1/2[\omega, \omega]$ becomes
\[
\Omega(\lambda) = dx \wedge dt \left\{ \lambda^2 U_{1,x} + \lambda (U_{0,x} - U_{1,t} - [U_1, V_0]) + V_{0,x} - U_{0,t} - [U_0, V_0] \right\}
\]
that vanishes upon the SD equations. □

For the principal chiral fields one has,

**Proposition 2.2** Let $x$ and $y$ be as in Theorem 2.2 and define
\[
u := -\frac{1}{\Delta} (\Phi_{01'} \partial_{11'} x - \Phi_{11'} \partial_{01'} x)
\]
where
\[
\Delta = \frac{\partial (x, y)}{\partial (z^{01'}, z^{11'})}.
\]
Suppose that $\Phi$ is chosen such that $u$ and $v$ depends on $z$ through the functions $x$ and $y$ defined in Theorem 2.2, then $u$ and $v$ are solutions of the equations of a principal chiral field if $\Phi$ is self–dual and satisfies the gauge condition $\Phi_{A0'} = 0$.

**Proof:** The choice made for $u, v, x$ and $y$ together with the gauge condition $\Phi_{A0'} = 0$ imply the relations
\[
F_{00'10'} = 0
\]
\[
F_{01'11'} = \Delta \left( u_y - v_x + [u, v] \right)
\]
\[
F_{00'11'} + F_{01'10'} = \Delta \left( u_y + v_x \right)
\]
and the result follows. □

All the 2D integrable equations cited above has a common feature, their zero–curvature formulation has a rational dependence in the spectral parameter $\lambda$. But there exists 2D integrable systems with an elliptic dependence in the spectral parameter such as the Landau-Lifshitz and the Krichever–Novikov equations. It is an open question whether they are related to the standard SDYM equations or there exists an elliptic deformed version of the SDYM equations.

### 3 The Yang matrix and chiral fields

The equation of a principal field (2.2) can be equally written as a pair of conditions
\[
u := \frac{1}{\Delta} (\Phi_{01'} \partial_{11'} y - \Phi_{11'} \partial_{01'} y)
\]
\[
F_{00'10'} = 0
\]
\[
F_{01'11'} = \Delta \left( u_y - v_x + [u, v] \right)
\]
\[
F_{00'11'} + F_{01'10'} = \Delta \left( u_y + v_x \right)
\]
and the result follows. □

The first of them represents the vanishing of the curvature for the connection $udx + vdy$ and this allows one to introduce the chiral field $s$ related to its currents $u$ and $v$ by the formulas
\[
u = s_x \cdot s^{-1}, \quad v = s_y \cdot s^{-1}.
\]
Here $s$ represents a function of $x$ and $y$ with values in the Lie group under consideration. Then, upon substitution in the second equation, we get for $s$ the new condition

$$(s_x \cdot s^{-1})_y + (s_y \cdot s^{-1})_x = 0.$$  

The situation we have just described has a precise analogue for the self–duality equations previously considered. As follows from Theorem 2.2, one can construct a SD connection in terms of a principal chiral field with the gauge condition

$$\Phi_{A0'} = 0.$$  

This type of SD connections can be conveniently represented by means of a function $J(z)$, the Yang matrix, with values in the corresponding Lie group. The SD equations $F_{00'10'} = F_{01'11'} = 0$ in terms of the connection $\Phi_{AA'}$ are

$$\begin{align*}
[\partial_{00'} - \Phi_{00'}, \partial_{10'} - \Phi_{10'}] &= 0, \\
[\partial_{01'} - \Phi_{01'}, \partial_{11'} - \Phi_{11'}] &= 0.
\end{align*}$$  

This pair of zero–curvature conditions allows us to find functions $\varphi_{0'}$ and $\varphi_{1'}$ for which

$$\Phi_{A0'} = \partial_{A0'} \varphi_{0'} \cdot \varphi_{0'}^{-1}, \quad \Phi_{A1'} = \partial_{A1'} \varphi_{1'} \cdot \varphi_{1'}^{-1}.$$  

We can write the connection $\Phi$ as

$$\Phi = \Phi_{AA'} dz^{AA'} = \partial_{A0'} \varphi_{0'} \cdot \varphi_{0'}^{-1} dz^{A0'} + \partial_{A1'} \varphi_{1'} \cdot \varphi_{1'}^{-1} dz^{A1'} = d\varphi_{0'} \cdot \varphi_{0'}^{-1} + \text{Ad}_{\varphi_{0'}}(\partial_{A1'} J \cdot J^{-1} dz^{A1'})$$

where we have set

$$J := \varphi_{0'}^{-1} \cdot \varphi_{1'}.$$  

Thus it appears that every SD connection is gauge equivalent to the one given by the formulas

$$\Phi_{A0'} = 0, \quad \Phi_{A1'} = \partial_{A1'} J \cdot J^{-1} dz^{A1'}$$

provided $J$ satisfies the equation

$$\partial_{00'}(\partial_{11'} J \cdot J^{-1}) - \partial_{10'}(\partial_{01'} J \cdot J^{-1}) = 0,$$

which is equivalent to the SD condition $F_{00'11'} + F_{01'10'} = 0$. Moreover, from the relation contained in Theorem 2.2 we obtain

$$\partial_{A1'} J \cdot J^{-1} = s_x \cdot s^{-1} \partial_{A1'} x + s_y \cdot s^{-1} \partial_{A1'} y$$

what tell us that for a given chiral field $s(x, y)$, we can take $J(z) = s(x(z), y(z))$ as a Yang matrix if the coordinate functions $x, y$ are those prescribed by Theorem 2.2. In fact one finds the explicit relation

$$\partial_{00'}(\partial_{11'} J \cdot J^{-1}) - \partial_{10'}(\partial_{01'} J \cdot J^{-1}) = \Delta \left( (s_x \cdot s^{-1})_y + (s_y \cdot s^{-1})_x \right)$$

with

$$\Delta = \frac{\partial (x, y)}{\partial (z^{01'}, z^{11'})}.$$
4 Reduction by symmetries

The conformal group represents the symmetry group for the SDYM equations. Invariant solutions under the action of a subgroup of rank two are described by a system of equations containing two independent variables instead of the variables $z^{AA'}$ appearing in the original equations. Thus, it is possible to describe SD connections possessing two translational symmetries in terms of KdV and NLS equations. These connections correspond to a real form $\mathbb{R}^2$ of the twistor space $T$ and the choice $m(z) = z^{00'}$ and $n(z) = -z^{10'}$ for the two functions in the formula of Theorem 2.1. An analogous result for the principal chiral field follows from Theorem 2.2 when we define $x(z) = z^{00'} - z^{01'}$ and $y(z) = z^{10'} + z^{11'}$ in that case.

In this section we present examples of connections derived from our general construction and not having two independent conformal symmetries.

The group of transformations preserving up to a conformal factor the symmetric bilinear form
$$g = g_{AA'B'B'}dz^{AA'}dz^{BB'} = dz^{10'}dz^{01'} - dz^{11'}dz^{00'}$$
coincides with the conformal group defined in Section 2. If a conformal transformation has as fundamental vector field $X = \sum_{AA'} X^{AA'} \partial_{AA'}$, then the coefficients $X^{AA'}(z)$ are rational functions of $z^{AA'}$ and are characterised by the condition
$$\mathcal{L}_X g = \Lambda g,$$
where $\mathcal{L}_X$ is the Lie derivative operator along $X$ and $\Lambda$ represents the infinitesimal conformal factor. In components, the condition above reads
$$g_{AA'CC'} \partial_{BB'} X^{CC'} + g_{CC'B'B'} \partial_{AA'} X^{CC'} = \Lambda g_{AA'B'B'},$$
or in a more explicit form
$$\partial_{00'} X^{11'} = \partial_{10'} X^{01'} = \partial_{01'} X^{10'} = \partial_{11'} X^{00'} = 0,$$
$$\partial_{00'} X^{01'} = \partial_{10'} X^{11'}, \partial_{00'} X^{10'} = \partial_{01'} X^{11'}, \partial_{11'} X^{01'} = \partial_{01'} X^{00'}, \partial_{10'} X^{10'} = \partial_{01'} X^{00'},$$
$$\partial_{11'} X^{00'} + \partial_{00'} X^{00'} = \partial_{01'} X^{01'} + \partial_{10'} X^{10'} = \Lambda.$$

Suppose that the connection $\Phi$ has a conformal symmetry generated by $X$. Then it follows by that $\Phi$ satisfies an equation of the form
$$\mathcal{L}_X (\Phi) = dW + [W, \Phi],$$
for some $W : T \to g$ that under a gauge transformation generated by $g$ transforms according to $W \mapsto X(g) \cdot g^{-1} + \text{Ad}gW$. For the curvature $F$ we have the condition
$$\mathcal{L}_X F = [W, F],$$
and for any Ad–invariant bilinear form $B$ in the Lie algebra $g$ we obtain
$$\mathcal{L}_X B(F, F) = B([W, F], F) + B(F, [W, F]) = 0. \quad (4.1)$$
This is a necessary condition in order to $X$ generates a conformal symmetry of the connection $\Phi$.

A simple computation proves that

$$B(F, F) = \mathcal{F}\Omega$$

where $\Omega = dz^{00'} \wedge dz^{01'} \wedge dz^{10'} \wedge dz^{11'} \in \Lambda^4(T)$ is the standard volume form on $T$, and therefore

$$\mathcal{L}_X B(F, F) = (X\mathcal{F} + \mathcal{F}\text{div}_\Omega X)\Omega,$$

where $\text{div}_\Omega X = \sum_{AA'} \partial_{AA'} X^{AA'}$ is the standard divergence of a vector field that for a conformal vector field is $\text{div}_\Omega X = 2\Lambda$. Equation (4.1) for the vector field $X$ can be written now as the equivalent condition

$$X\mathcal{F} + \mathcal{F}\text{div}_\Omega X = 0. \quad (4.2)$$

Let us now consider the particular solution to the principal chiral field model given by

$$s(x, y) = \exp((x + y)A) \cdot g.$$  

Here $A$ and $g$ are constants elements in $\mathfrak{g}$ and $G$ respectively, with the normalization condition $B(A, A) = 1$. For the functions $u = s_x \cdot s^{-1}$ and $v = s_y \cdot s^{-1}$ we obtain $u = v = A$, and the associated connection of THEOREM 2.2 becomes

$$\Phi_{A0'} = 0, \quad \Phi_{A1'} = A\partial_{A1'}(x + y). \quad (4.3)$$

**PROPOSITION 4.1** Define

$$x(z) = -\exp(z^{00'} - z^{01'}) - \exp(z^{10'} - z^{11'}),$$

$$y(z) = \exp(z^{00'} + z^{01'}) + \exp(z^{10'} + z^{11'}).$$

Then, the symmetry group of conformal transformations for the SD connection (4.3) is one–dimensional.

**Proof:** For the function $\mathcal{F}$ in (4.2) we find the expression

$$\mathcal{F} = (\exp(z^{00'} - z^{01'}) + \exp(z^{00'} + z^{01'}))(\exp(z^{10'} - z^{11'}) + \exp(z^{10'} + z^{11'})).$$

Upon substitution in (4.2) we obtain for $X$ the conditions

$$X^{A1'} = 0, \quad A = 0, 1,$$

$$X^{00'} + X^{10'} = \text{div}_\Omega X,$$

that follows from the rational character of the coefficients $X^{AA'}(z)$ of the vector field $X$. These conditions and the differential equations satisfied by any conformal field $X$ imply that $X$ is proportional to $\partial_{00'} - \partial_{10'}$. This proves that this SD connection obtained from the principal chiral field model through the map prescribed by THEOREM 2.2 have at most a conformal symmetry. □
Analogous considerations show that the same is true for the KdV and NLS
type equations. For if we let the zero–curvature 1–form be defined as
\( \omega_+(\lambda) = U_1(dx + \lambda dt) \) for a constant element \( U_1 \) in \( g \) such that \( B(U_1, U_1) = 1 \) and take

\[
\begin{align*}
x(z) &= -1/2(z^{01'}(z^{00'})^2 + z^{11'}(z^{10'})^2) + \exp(z^{00'}) + \exp(z^{10'}), \\
t(z) &= 1/6((z^{00'})^3 + (z^{10'})^3),
\end{align*}
\]

we obtain as conformal symmetry group, for the SD connection associated to this
solution of a NLS type equation, the one–dimensional subgroup generated by

\[
\partial_{00'} + \partial_{01'} + z^{01'}\partial_{01'} + z^{11'}\partial_{11'}.
\]

Therefore this SD connection associated to a NLS type equation as in Theorem
2.1 has at most a conformal symmetry.

Finally, in the KdV case, for the constant vectors \( U_0, U_1 \in g \) satisfying \( B(U_0, U_0) = B(U_1, U_1) = 0 \) and \( B([U_0, U_1], [U_0, U_1]) = 1 \), we define the zero–curvature 1–form

\[
\omega_+(\lambda) = (\lambda U_1 + U_0)(dx + \lambda dt)
\]

and the functions

\[
\begin{align*}
x(z) &= -z^{01'} \exp(z^{00'}) - z^{11'} \exp(z^{10'}) + \exp(-z^{00'}) + \exp(3z^{10'}), \\
t(z) &= \exp(z^{00'}) + \exp(z^{10'}),
\end{align*}
\]

The symmetry group of conformal transformations for the associated SD connection
is generated in this case by \( \partial_{01'} - \partial_{11'} \).

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