The number of primitive Vassiliev invariants
top to degree twelve

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Abstract

We present algorithms giving upper and lower bounds for the number of independent primitive rational Vassiliev invariants of degree \( m \) modulo those of degree \( m - 1 \). The values have been calculated for the formerly unknown degrees \( m = 10, 11, 12 \). Upper and lower bounds coincide, which reveals that all Vassiliev invariants of degree \( \leq 12 \) are orientation insensitive and are coming from representations of Lie algebras \( \mathfrak{s} \mathfrak{o} \) and \( \mathfrak{g} \mathfrak{l} \). Furthermore, a conjecture of Vogel is falsified and it is shown that the \( \Lambda \)-module of connected trivalent diagrams (Chinese characters) is not free.

1 Introduction

1.1 Vassiliev invariants

In the year 1990, V. A. Vassiliev introduced \([12]\) a new type of knot invariants that include the information of most of the invariants that followed the celebrated discovery of the Jones polynomial \((8), (7), (9)\).

An immersion of the circle \( S^1 \) in the three-sphere \( S^3 \) having exactly \( m \) double points and no other singularities is called \( m \)-singular. Let \( K_m \) denote the set of ambient isotopy classes of \( m \)-singular immersions. The elements of \( K_0 \) are classes of embeddings, i.e. knots in the classical sense.

Any knot invariant \( v \) with values in an abelian group, can be extended to singular knots inductively, by use of the desingularisation rule:

\[
v(K) := v(K_+) - v(K_-)
\]

where \( K \in K_m, K_+, K_- \in K_{m-1} \) differ only locally like this:

\[
\begin{align*}
K: & \quad \times \\
K_+: & \quad \times \\
K_: & \quad \times
\end{align*}
\]

Definition 1.1 A knot invariant with values in an abelian group is called a Vassiliev invariant of degree \( m \) iff it vanishes on \( K_{m+1} \) and the unknot but not on \( K_m \). A Vassiliev invariant \( v \) is called primitive iff it is a monoid homomorphism, i.e. \( v(K_1 \# K_2) = v(K_1) + v(K_2) \), where \( \# \) denotes the connected sum operation for knots.

For a commutative ring \( k \) let \( V_m^k \) (\( P V_m^k \)) denote the \( k \)-module generated by all \( k \)-valued (primitive) Vassiliev invariants of degree \( \leq m \).

The product of two invariants \( v_1, v_2 \) is given pointwise by \( v_1 \cdot v_2(K) := v_1(K)v_2(K) \) for all \( K \in K_0 \). It is not hard to show that if the degrees of \( v_1, v_2 \) are \( m_1, m_2 \), then the degree of \( v_1 \cdot v_2 \) is \( m_1 + m_2 \). This establishes a graded algebra structure on \( \oplus V_m/V_{m-1} \). It is even a graded Hopf algebra (the coproduct corresponds to the connected sum operation), which explains why we restrict ourselves to primitive Vassiliev invariants.
Remark 1.2 Every Vassiliev invariant can be expressed (uniquely up to invariants of lower degree) as a polynomial in primitive Vassiliev invariants.

Vassiliev invariants have been defined topologically but they are closely related to purely combinatorial objects, which we shall describe now.

1.2 Modules of diagrams

Definition 1.3 1. A free diagram (or Chinese character) of degree \((m,u)\) is a finite abstract graph with \(2m - u\) trivalent and \(u\) univalent vertices. The trivalent vertices are rigid, i.e. a cyclic ordering of the three arriving edges is chosen at every trivalent vertex.

2. A diagram together with a linear ordering of its univalent vertices is called fixed diagram.

3. A diagram of degree \((m,u)\) with \(u > 0\) together with a cyclic ordering of its univalent vertices is called circle diagram of degree \(m\).

In all our pictures the edges are ordered counterclockwise at each vertex. In circle diagrams we depict the cyclic ordering of the univalent vertices by gluing them on an oriented circle. We will need four types of local relations (only the changed parts of the diagrams are shown):

- AS (antisymmetry of vertices):

- IHX relation:

- STU relation:

- FI (framing independence):

AS and IHX are homogenous with respect to \(m\) and \(u\), the STU-relation only with respect to \(m\). The STU and FI relations are defined only for circle diagrams.

Definition 1.4 We have the following \(\mathbb{Z}\)-modules:\[^1\]

\[
\begin{align*}
\mathcal{A}_m &:= \mathbb{Z}\langle \text{circle diagrams of degree } m \rangle / \mathbb{Z}\langle \text{STU relations} \rangle \\
\mathcal{A}'_m &:= \mathcal{A}_m / \mathbb{Z}\langle \text{FI relations} \rangle \\
\mathcal{P}_m &:= \text{submodule of } \mathcal{A}_m \text{ generated by connected circle diagrams} \\
\mathcal{P} &:= \bigoplus_{m=2}^{\infty} \mathcal{P}_m \\
\mathcal{B}_{m,u} &:= \mathbb{Z}\langle \text{connected free diagrams of degree } (m,u) \rangle / \mathbb{Z}\langle \text{AS,IHX relations} \rangle \\
F(u) &:= \mathbb{Z}\langle \text{conn. fixed diagrams with } u \text{ univalent vertices} \rangle / \mathbb{Z}\langle \text{AS,IHX relations} \rangle
\end{align*}
\]

The most important (and highly non-trivial) facts about Vassiliev invariants may be summarized in the following manner.

[^1]: We will denote the free \(k\)-module with basis \(S\) by \(k\langle S \rangle\).
The number of primitive Vassiliev invariants up to degree twelve

\textbf{Theorem 1.5 (Bar-Natan, Birman-Lin, Drinfeld, Kontsevich, Vassiliev)}

\[ \mathcal{V}^\mathbb{Q}_m / \mathcal{V}^\mathbb{Q}_{m-1} \cong \mathcal{A}^\mathbb{Q}_m \otimes \mathbb{Q} \quad \text{for } m \geq 1 \]

\[ \mathcal{P} \mathcal{V}^\mathbb{Q}_m / \mathcal{P} \mathcal{V}^\mathbb{Q}_{m-1} \cong \mathcal{P}_m \otimes \mathbb{Q} \cong \bigoplus_{u=1}^{m} \mathcal{B}_{m,u} \otimes \mathbb{Q} \quad \text{for } m \geq 2 \]

\textbf{Remark 1.6} To be more specific, there is a natural way to define a map \( \mathcal{V}^\mathbb{Q}_m / \mathcal{V}^\mathbb{Q}_{m-1} \to \text{Hom}_\mathbb{Z}(\mathcal{A}^\mathbb{Q}_m, \mathbb{Q}) \), which turns out to be the desired isomorphism. \( \mathcal{A}_m, \mathcal{P}_m, \mathcal{B}_{m,u} \) are finite dimensional, so we use them instead of their duals.

It is very annoying that our knowledge of \( \mathcal{P} \) is so limited. Dror Bar-Natan has computed \( \text{rk } \mathcal{P}_m \) for \( m \leq 9 \). Upper and lower bounds for all degrees have been found ([3], [4], [11]) but they are unacceptably bad. And we know practically nothing about torsion in \( \mathcal{P} \).

The main goal of this paper is to describe two algorithms that give upper bounds for the rank of \( \mathcal{P}_m \). But first, we present a very good lower bound that is due to Dror Bar-Natan and an algebra \( \Lambda \), introduced by Pierre Vogel, that acts on \( \mathcal{P} \).

\section{1.3 Marked surfaces}

If every edge of a free diagram is labeled with exactly one of the symbols "=" or "x", it is called a \textit{marked diagram}. A \textit{marked surface} is a closed compact surface with some points marked on its boundary. At each marked point, an orientation of the boundary component is specified. A marked surface \( F \) is \textit{normalized}, if either \( F \) is orientable and all markings induce the same orientation on \( F \), or \( F \) is non-orientable and the orientations of the markings coincide on each component of \( \partial F \). We call a diagram \textit{embedded} if it is drawn on the 2-sphere \( S^2 \) and the cyclic order given at each vertex is compatible with the orientation of \( S^2 \).

We will thicken the five building blocks of embedded marked diagrams (univalent vertices, trivalent vertices, edges with "=" and "x", crossings of edges) like this:

This assigns to every embedded marked diagram \( D \) a marked surface \( \hat{D} \). If \( D \) has \( u \) univalent vertices then \( \hat{D} \) has \( u \) markings on its boundary. If \( D' \) is the diagram that is obtained after all the markings of a marked diagram \( D \) are forgotten, we say that \( D \) is a \textit{marking} of \( D' \).

Let \( \chi(D) \) denote the number of "x"-marked edges of \( D \).

Now we can define the "thickening map" \( \Phi_m \) from \( \bigoplus_{u=1}^{m} \mathcal{B}_{m,u} \rightarrow \mathbb{Z}\langle \text{marked surfaces} \rangle \):

\textbf{Definition 1.7} If an element \( b \) of \( \mathcal{B}_{m,u} \) is represented by a embedded diagram \( D_b \) then we define \( \Phi_m(b) := \sum_{\text{all markings } D \text{ of } D_b} (-1)^{\chi(D)} \hat{D} \).

\textbf{Remark 1.8} It is easy to show that \( \Phi_m \) is well-defined, e.g. it does not depend on the choice of the embedding and it respects the relations AS and IHX. This implies that \( \text{rk } (\text{im } \Phi_m) \) is a lower bound for \( \text{rk } \mathcal{P}_m \).

Let \( p \) denote the projection from \( \mathbb{Z}\langle \text{marked surfaces} \rangle \) onto \( \mathbb{Z}\langle \text{normalized marked surfaces} \rangle \) and let \( \Phi_m := p \circ \Phi_m \). Of course we then have \( \text{rk } (\text{im } \Phi_m) \leq \text{rk } \mathcal{P}_m \) as well. It can be shown that \( \Phi_m \) contains essentially the same information as \( \Phi_m \).

\textbf{Remark 1.9} On one side, to every marked surface \( s \), there is naturally associated a linear form on \( \mathcal{P}_m \), namely the coefficient of \( s \) in \( \Phi_m \). On the other side, to a finite dimensional Lie algebra, equipped with a symmetric, \( \text{Ad} \)-invariant, non-degenerate, bilinear form and a finite dimensional representation, there is associated a linear form on \( \mathcal{P}_m \), too.
Bar-Natan has shown in \[ \] that all the linear forms obtained via marked surfaces are coming from Lie algebras in the families so and gl and all of their representations. He has also shown that the corresponding Vassiliev invariants contain the same information as the HOMFLY and the (2-variable) Kauffman polynomials and all of their cablings.

1.4 The algebra $\Lambda$

Vogel defined an interesting submodule $\Lambda$ of the module $F(3)$ of fixed diagrams with three univalent vertices. The symmetric group $S_u$ acts on $F(u)$ by permutation of the univalent vertices. There are maps $\phi_i$ (1 \leq i \leq u) from $F(u)$ to $F(u+1)$, given by gluing a trivalent vertex to the $i$-th univalent vertex; the two new univalent vertices get the numbers $i$ and $i + 1$ and the numbers $> i$ are increased by one.

Definition 1.10 $\forall u \in F(3) : u \in \Lambda : \Leftrightarrow \sigma(u) = \epsilon(\sigma)u$ for all $\sigma \in S_3$ and $\phi_1(u) = \phi_2(u)$, where $\epsilon$ is the signature homomorphism.

For a diagram $d$ of $\mathcal{P}$ and a diagram $u$ of $\Lambda$ one makes a simple construction: Delete a trivalent vertex of $d$ (it always has at least one) and the three univalent vertices of $u$. The three remaining open edges of $d$ are glued to those of $u$. The first condition in definition 1.10 together with the AS relation cause that all 6 ways of doing this give the result (in $\mathcal{P}$). The second condition has the effect that it does not matter, at which trivalent vertex $u$ is inserted (here it is essential that $d$ is connected). It is easy to show that the insertion is compatible with the IHX relation, so $\Lambda$ operates on $\mathcal{P}$. $\Lambda$ is even a graded algebra because it acts on itself, and $\mathcal{P}$ is a $\Lambda$-module.

It has been shown in \[ \] that $\Lambda \otimes \mathbb{Q}$ is commutative and that it is contained in $\mathcal{P} \otimes \mathbb{Q}$:

**Proposition 1.11** $\Lambda \otimes \mathbb{Q} \cong \bigoplus_{m=2}^{\infty} \mathfrak{S}_{m,2} \otimes \mathbb{Q}$

Furthermore the following elements $t, x_3, x_4, x_5, \ldots$ of $\Lambda$ are constructed:

\[ \begin{array}{ccc}
\text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} \\
\text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} \\
\text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} & \text{\includegraphics[width=2cm]{triangle}} \\
\end{array} \]

Vogel showed that, in degree $\leq 8$, $\Lambda \otimes \mathbb{Q}$ is generated by $t, x_3, x_5, \ldots$ and isomorphic to $\mathbb{Q}[t, x_3, x_5, \ldots]$. He conjectured that this is true in all degrees, and gave a polynomial in degree 10 for which he could not tell, whether it is trivial or not.

2 Results

We have implemented both algorithms that will be given in section \[ \] and made a program that effectively computes the thickening map $\tilde{\Phi}_m$ described in section \[ \]. The output of these three programs for degree $m$ will be denoted $O_A(m), O_B(m), O_C(m)$, respectively. By corollary \[ \] we have that $O_B(m), O_B(m) \geq \text{rk } \mathcal{P}_m \geq O_C(m)$. First we confirmed for $3 \leq m \leq 9$ that $O_A(m) = O_B(m) = O_C(m) =$ values given in \[ \]. Then we found $O_A(10) = O_B(10) = O_C(10) = 27, O_A(11) = O_B(11) = O_C(11) = 39$. We were astonished that our algorithms are good enough to give the exact values. The complete surprise came with $m = 12$: we had conjectured $O_C(12) \neq \text{rk } \mathcal{P}_{12}$, because of Vogel’s conjecture. After some hundred hours of CPU-time we got the result $O_B(12) = O_C(12) = 55$ (the computation of $O_A(12)$ is too exhaustive and could not be performed).

2.1 Vassiliev invariants

**Theorem 2.1 (computational result)** The sequence of the dimensions of the spaces of rational-valued primitive Vassiliev invariants of degree $m$ modulo those of degree $m-1$ starts

\[ 0, 1, 1, 2, 3, 5, 8, 12, 18, 27, 39, 55, \ldots \]
The number of primitive Vassiliev invariants up to degree twelve

Remark 2.2 For some time only the first seven numbers of this sequence were known and there was some excitement, because it appeared to be the famous Fibonacci sequence. It is a somehow mysterious coincidence that it is again a Fibonacci number in degree twelve.

In view of remark 1.9 we have the following consequence of $O_C(m) = \text{rk } P_m$ for $m \leq 12$.

Corollary 2.3 All rational Vassiliev invariants of degree $\leq 12$ are coming from representations of the classical Lie algebras so and gl.

Remark 2.4 Vogel has shown that a similar statement is false for sufficiently large degrees. Let $m_c$ denote the minimal degree for which not all Vassiliev invariants are coming from semi-simple Lie algebras. Results of Jens Lieberum ([10]) and our calculations together imply that $13 \leq m_c \leq 17$.

Corollary 2.5 Vassiliev invariants up to degree twelve can not distinguish knots from their inverses.

Proof It has been shown that Vassiliev invariants coming from semi-simple Lie algebras are orientation insensitive. Another way is to verify that $\text{rk } B_{m,u} = 0$ for $m \leq 12$ and $u$ odd (see table in section 2.2).

A statement about $A$ similar to remark 1.2 makes it is easy to compute the ranks of $A_m^r$ and $A_m$, corresponding to the numbers of Vassiliev invariants of knots and Vassiliev invariants of framed knots, if $\text{rk } P_m$ is known.

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $\text{rk } P_m$ | 0 | 1 | 1 | 1 | 2 | 3 | 5 | 8 | 12 | 18 | 27 | 39 | 55 |
| $\text{rk } A_m$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 60 | 104 | 184 | 316 | 548 |
| $\text{rk } A_m^r$ | 1 | 0 | 1 | 1 | 3 | 4 | 9 | 14 | 27 | 44 | 80 | 132 | 232 |

2.2 Results about $P$ and $\Lambda$

Our programs work over the field $F_2$, so due to corollary 2.3 we have a little statement about torsion in $P$.

Corollary 2.6 There is no 2-torsion in $P$ in degree $\leq 12$.

By counting the dimensions of the image of the thickening map $\Phi$ for each $B_{m,u}$ separately, we get the following table for $\text{rk } B_{m,u}$.

| $\text{rk } B_{m,u}$ | $u = 2$ | $u = 4$ | $u = 6$ | $u = 8$ | $u = 10$ | $u = 12$ | total |
|---------------------|---------|---------|---------|---------|---------|---------|-------|
| $m = 1$             | 1       |         |         |         |         |         | 1     |
| $m = 2$             |         | 1       |         |         |         |         | 1     |
| $m = 3$             |         |         | 1       |         |         |         | 1     |
| $m = 4$             |         |         |         | 1       |         |         | 2     |
| $m = 5$             |         |         |         |         | 2       |         | 3     |
| $m = 6$             |         |         |         |         |         | 2       | 5     |
| $m = 7$             |         |         |         |         |         |         | 8     |
| $m = 8$             |         |         |         |         |         |         | 12    |
| $m = 9$             |         |         |         |         |         |         | 18    |
| $m = 10$            |         |         |         |         |         |         | 27    |
| $m = 11$            |         |         |         |         |         |         | 39    |
| $m = 12$            |         |         |         |         |         |         | 55    |

Corollary 2.7 The algebra morphism from $\mathbb{Q}[T, X_3, X_5, \ldots]$ to $\Lambda \otimes \mathbb{Q}$ given by $T \rightarrow t, X_i \rightarrow x_i$ is not an isomorphism. In degree $< 11$ it is surjective and has a one dimensional kernel (living in degree 10).
Proof By calculating characters, Vogel has already shown that this algebra morphism is injective in degree \( \leq 9 \) and its kernel in degree 10 has at most dimension one. Because of proposition 1.11, the statements can be verified by counting the number of monomials in each degree \( m \) and comparing it to \( \text{rk} B_{m+2,2} \). In degree 10, for example, there are ten monomials \((t^{10}, t^7x, t^5x^3, t^3x^5, tx^9, t^4x^2, t^2x^3x^5, x^3x^7, x^5, t^2x^3)\), but we have \( \text{rk} B_{12,2} = 9 \).

So one half of Vogel's conjecture is false, but the calculations show that up to degree 10 the other one holds:

**Conjecture** \( \Lambda \otimes Q \) is generated (as algebra over \( Q \)) by the elements \( t, x_3, x_5, \ldots \)

**Corollary 2.8** \( P \otimes Q \) is not a free \( \Lambda \otimes Q \)-module.

Proof Let us assume that \( B_{m,4} \otimes Q \) is a free \( \Lambda \otimes Q \)-module with rank \( \alpha_m \geq 0 \) (\( m \geq 4 \)). Let \( \lambda_m \) denote the dimension of \( \Lambda \otimes Q \) in degree \( m \). Then we have the following formula for the rank of \( B_{m,4} \):

\[
\text{rk} B_{m,4} = \dim_Q B_{m,4} \otimes Q = \sum_{i=4}^{m} \lambda_{m-i} \alpha_i
\]

We have \( \lambda_0, \ldots, \lambda_7 = 1, 1, 1, 2, 3, 4, 5 \), which together with the values \( \text{rk} B_{4,4}, \ldots, \text{rk} B_{11,4} \) implies \( \alpha_4 = \alpha_6 = \alpha_8 = \alpha_{10} = 1, \alpha_5 = \alpha_7 = \alpha_9 = 0, \alpha_{11} = -1 \).

This contradiction shows that — at least in the \( u = 4 \) column of \( P \otimes Q \) — nontrivial relations hold.

We have found another relation which is located in the \( u = 6 \) column in degree 12. Unlike here, its existence can not be shown by simply counting dimensions.

### 2.3 The structure of \( P \otimes Q \) as far as we know it

Using the thickening map, we have built a minimal set of diagrams \( \Omega_{\leq 12} \) that generate \( P \otimes Q \) as \( \Lambda \otimes Q \)-module in degrees up to 12. We were trying to make the elements of \( \Omega_{\leq 12} \) as simple as possible and finally, this lead us to a very special type of diagrams:

**Definition 2.9** Let \( \omega_{i_1i_2\ldots i_k} \) denote the element of \( B_{i_1+i_2+\ldots+i_k+k-1, i_1+i_2+\ldots+i_k} \) that is represented by a "caterpillar" diagram consisting of \( k \) "body segments" with \( i_1, \ldots, i_k \) "legs", respectively.

Here are some examples of caterpillar diagrams for \( \omega_4, \omega_{302}, \omega_{13}, \omega_{02131} \).

![Caterpillar Diagrams]

**Remark 2.10** It is a nice exercise to use the AS and IHX relations to prove that \( \omega_{i_1i_2\ldots i_k} \) is well defined (i.e. for inner segments it makes no difference on which side of the body the legs are drawn). An easy consequence is \( \omega_{i_1\ldots i_k} = \omega_{i_k\ldots i_1} \). The diagrams \( \omega_i \) are also called "wheels with \( i \) spokes".

Let \( \Omega_{\leq 12} \) denote the set consisting of the following 31 elements.
Remark 2.11 At several places in the upper table the choice of a minimal generating set is not unique. We tried to make Ω_{≤12} look as uniform as possible. At first place we were able to renounce on ω’s with odd indices. After this only few choices still had to be done. For example, we preferred ω_{26} over ω_{44} (because of the other entries of the form ω_{2,u−2}) and ω_{2222} over ω_{2204} (because of its symmetry).

Let Pω denote sub-Λ-module of P that is generated by caterpillar diagrams. A glance through the table Ω_{≤12} immediately opens the following two questions:

1. Is Pω already generated by the caterpillar diagrams with even indices?
2. Is Pω = P?

The AS relation and remark 2.10 implies that caterpillar diagrams with an odd number of univalent vertices are always trivial. So if question 2. could be answered positively, it would imply that all Vassiliev invariants are orientation insensitive.

It is tempting to make conjectures about how this table continues (especially for the u = 4 column), but let us just summarize what we know for sure.

Remark 2.12 If Ω = ∪Ω_{m,u} is a minimal set of free diagrams that generate P ⊗ Q as Λ ⊗ Q-module, then

- The u = 2 column is essentially empty: Ω_{m,2} = ∅ for m > 2.
- The u = m = 2i diagonal consists only of wheels: Ω_{2i,2i} = {ω_{2i}}.
- On the first subdiagonal (m − 1 = u = 2i) we have exactly ⌊i/2⌋ elements. A natural choice is Ω_{2i+1,2i} := {ω_{ab} | a > 0; a even; 2a ≤ b; a + b = 2i}.
- On the second subdiagonal (m − 2 = u = 2i) there are exactly #Ω_{2i+2,2i} = ⌈(i+1)^2/12 + 1/2⌉ elements.
- For odd u we know Ω_{m,u} = ∅ if m − u ≤ 5 or u = 1 or m ≤ 12.

Proof The first statement is due to proposition 1.11. The second is obvious and the third and forth follow from results of Oliver Dasbach in [5]. He showed that Ω_{2i+1,2i} ∪ {tω_{2i}} is a basis for B_{2i+1,2i} and dim B_{u+2,u} = ⌈u^2+12u/48⌉ + 1 for u even. It can be verified that {t^2ω_{2i}} ∪ tΩ_{2i+1,2i} are independent in B_{2i+2,2i}, so #Ω_{2i+2,2i} = ⌈(i+1)^2/12 + 1/2⌉. Dasbach has also shown (3) that B_{m,u} = 0 for m − u ≤ 5 and u odd. The same statement for u = 1 is easy to prove.
Remark 2.13 Some time after having found $\Omega_{\leq 12}$, we discovered that caterpillar diagrams have already been used by Chmutov and Duzhin in [1], who call them “baguette diagrams”. The main theorem of [1] states that the elements $\omega_{n_1 \ldots n_k}$ with all $n_i$ even, $\sum_{i=1}^{j-1} n_i < n_j$ for $j < k$ and $\sum_{i=1}^{k-1} n_i < \frac{1}{2} n_k$ are linearly independent. This result is quite striking, but it is useless in our context, because the first interesting diagram in this set is $\omega_{2,4,14}$ and lies in degree 22.

Let $W$ denote the image of $\Omega_{\leq 12}$ under the map $\omega_s \rightarrow w_s$ and make the following abbreviations:

\[ P := 327^{10} - 152T^3X_5 + 252T^3X_5 - 101T^4X_5^2 - 36T^2X_5X_7 - 97X_5 X_7 + 14T X_5^3 + 9X_5 X_7 - 9X_5^2 \]
\[ Q := (327^2 + 357^2 X_5 - 97X_5 X_7 - 4T X_7^2) \otimes w_6 + (-767^3 + 107T^2X_5 + 3X_5) \otimes w_{2002} + (12T^2 - 3X_5) \otimes w_{2002} \]
\[ R := (-16T^6 + 21T^3X_5 - 3T X_5^2 - 2X_5^2) \otimes w_6 + (327^2 - 17T^2X_5 + 3X_5) \otimes w_{2002} + (-36T^4 + 9T X_5) \otimes w_{2002} \]

Theorem 2.14 There is a module morphism
\[ \mu : Q[T, X_3, X_5, \ldots] \otimes Q(W) / Q(P \otimes w_2, Q, TQ, R) \rightarrow \mathcal{P} \otimes Q. \]

It is an isomorphism in degree $\leq 12$.

Proof $\mu$ is given by $\mu(T^a X_3^i X_5^j \ldots \otimes w_s) = t^a x_3^i x_5^j \ldots w_s$. We have seen that Bar-Natan’s thickening map $\Phi$ is injective on $\mathcal{P}_m \otimes Q$ for $m \leq 12$. So we have to calculate $\Phi$ for all diagrams of the form $t^a x_3^i x_5^j \ldots w_s$ ($w_s \in \Omega_{\leq 12}$) with degree $\leq 12$ (there are exactly 175 of them). The program and the results are available via the internet (see section 6.2). It is then easy to verify that the span of the resulting vectors is 171 dimensional and that $P \otimes w_2, Q, TQ$ and $R$ span the kernel of $\Phi \circ \mu$. So $\mu$ is well defined, injective, and because of $\sum_{m=2}^{12} \mathrm{rk} \mathcal{P}_m = 171$ it is also surjective.

3 The principle behind the algorithms

We will now describe a prototype of an algorithm that yields an upper bound for the rank (ubr) of a finitely generated abelian group $A$.

Definition 3.1 A quintuple $(k, S, \varphi, \delta, \rho)$ where $k$ is a field, $S$ is a finite set, $\varphi$ is a mapping $\varphi : S \rightarrow A$ and $\delta, \rho$ are endomorphisms of $k(S)$ shall be called ubr-algorithm for the finitely generated abelian group $A$, iff the following conditions are satisfied:

1. $\varphi(S)$ is a set of generators of $A$,
2. there exists an integer $j$ such that $\delta^{j+1} = \delta^j$,
3. $\varphi \circ \delta = \tilde{\varphi}$,
4. $\varphi \circ \rho = 0$.

Here $\tilde{\varphi}$ denotes the vectorspace homomorphism $\tilde{\varphi} : k(S) \rightarrow A \otimes_{Z} k$ that is induced by $\varphi$.

Definition 3.2 For a given ubr-algorithm let $\Delta := \delta^j$ and $\mathcal{I} := \mathrm{im} \Delta$. Then $\bar{\rho} := \Delta \circ \rho|_{\mathcal{I}}$ is an $\mathcal{I}$-endomorphism. The output of the ubr-algorithm defined as the natural number output$(k, S, \varphi, \delta, \rho) := \dim_k \ker \bar{\rho}$.

Lemma 3.3 Conditions 2 and 3 imply $\Delta(\ker \tilde{\varphi}) = \ker \tilde{\varphi} \cap \mathcal{I}$.

\[ \text{The third item of remark 2.12 and the simple relation } \omega_{0,n_2 \ldots n_k} = 2t \omega_{n_2 \ldots n_k} \text{ should make clear that } \text{only the diagrams } \omega_{n_1 \ldots n_k} \text{ with } n_1 > 0 \text{ and } k > 2 \text{ are of further interest.} \]
Proof: "⊃": 2. ⇒ Δ is a projection onto \( I \).
"⊂": 3. ⇒ \( \hat{\phi} \circ \Delta = \hat{\phi} \) ⇒ \( \Delta(\ker \hat{\phi}) \subset \ker \hat{\phi} \).

**Proposition 3.4** If \((k, S, \varphi, \delta, \rho)\) is an ubr-algorithm for the finitely generated abelian group \( A \) then output\((k, S, \varphi, \delta, \rho)\) ≥ \( \text{rk} \ A \).

**Proof:** The first condition of definition 3.3 implies that \( \hat{\phi} \) is an epimorphism. Due to 4. we have \( \rho(I) \subset \ker \hat{\phi} \). Together with lemma 3.3 and \( \hat{\phi} \circ \Delta = \hat{\phi} \) we get:

\[
\dim_k(A \otimes \mathbb{Z}k) = \dim_k(\hat{\phi}(k(S))) = \dim_k(\hat{\phi}(\Delta(k(S)))) = \dim_k(\hat{\phi}(I)) = \dim_k(I) - \dim_k(\ker(\hat{\phi} \mid I)) = \dim_k(I) - \dim_k(\Delta(\ker(\hat{\phi}))).
\]

Let \( T \) denote the maximum torsion subgroup of \( A \). \( A \) is isomorphic to \( T \times \mathbb{Z}^{\text{rk} \ A} \), and thus \( \dim_k(A \otimes \mathbb{Z}k) \geq \text{rk} \ A \).

**Corollary 3.5** If \((F_p, S, \varphi, \delta, \rho)\) is an ubr-algorithm for \( A \) and \( \text{output}(F_p, S, \varphi, \delta, \rho) = \text{rk} \ A \) then \( A \) has no elements of order \( p \).

**Remark 3.6** To calculate output\((k, S, \varphi, \delta, \rho)\) one has to find a basis of \( I \) and to evaluate the nullity of the corresponding matrix for \( \bar{\rho} \). So the costs (in time and space) of an ubr-algorithm mainly depend on the dimension of \( I \), whereas the quality of the result (the sharpness of the upper bound) depends on the choice of \( \rho \).

## 4 Two algorithms for \( P_m \)

### 4.1 Circle diagrams without loops

Let \( S_n \) denote the set of permutations of \( n \) elements. We will use the standard linear ordering for the permutations:

\[
\pi < \phi :\iff \exists i \in \{1, \ldots, n\} : \pi(i) < \phi(i) \text{ and } \pi(j) = \phi(j) \text{ for all } 1 \leq j < i.
\]

Denote by \( \tau_i \) the elementary transpositions \((i \ i+1)\) and let the product of permutations be defined by \( (\pi \phi)(i) := \phi(\pi(i)) \).

**Lemma 4.1** For any \( \pi \in S_n \) the following two statements hold: If there exists an integer \( i \ (1 \leq i < n) \) with \( \pi(i) \geq \pi(i + 1) \) then \( \pi \tau_i < \pi \). If there exist \( i, j \ (1 \leq i < j \leq n) \) with \( \pi(i) = \pi(j) + 1 \) then \( \pi \tau_{\pi(i)} < \pi \).

**Proof:** Simply identify the \( i \) in the upper definition of \(< \) with this \( i \) here.

The *picture* of a permutation \( \pi \in S_n \) is given by two vertical lines with \( n \) distinct points marked on each, together with \( n \) lines connecting the \( i \)-th and the \( \pi(i) \)-th point (counted upwards). For example the picture of \((1 \ 2 \ 4)(3)\) is

![Picture of (1 2 4)(3)](image)

**Definition 4.2** For any \( \pi \in S_n \) we will denote by \( D^A_\pi \) the element of \( P_{n+1} \) that is obtained by replacing the box in the following figure by the picture of \( \pi \).

![Diagram of Definition 4.2](image)

The map \( \varphi_A : S_n \to P_{n+1} \) is given by \( \pi \to D^A_\pi \).
We will now introduce three types of moves, which replace a permutation by a linear combination of permutations, by showing the parts of their pictures that are concerned. Omitted parts are indicated by dots and are assumed to be identical in a row.

**Type I**

\[ \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \rightarrow \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \]

**Type II**

\[ \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \rightarrow \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} - \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \]

**Type III**

\[ \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \rightarrow \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} - \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} - \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} + \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} + \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} + \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} - \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array} \]

The move I should be interpreted in the following way: when \( \pi(n) = n \) then replace \( \pi \) by \( \tau_1 \cdots \tau_{n-1} \pi \tau_{n-1} \cdots \tau_1 \); the move II reads: when there is some \( i \in \{1, \ldots, n-1\} \) with \( \pi(i) = \pi(i+1) + 1 \) then replace \( \pi \) by \( \tau_i \pi - \tau_1 \pi \tau_{\pi(i)-1} \cdots \tau_1 \); etc.

We call a move reducing, if all the permutations on the right side are smaller than the permutation \( \pi \) on the left, and we then say that \( \pi \) allows a reduction. Permutations that do not allow any reduction are called irreducible.

If a diagram allows more than one reducing move, we have to make a choice. To make our calculations reproducible, we describe how our implementation works. Moves of type II are indexed by the left height of the two crossing lines that are concerned. Moves of type III are indexed by the height of the left endpoint of the line that is “switched” during the move. Now all possible moves of a permutation can be ordered by \( I < II < III \) and \( II_i < II_j \) and \( III_i < III_j \) for \( i < j \).

**Definition 4.3** When \( \pi \in S_n \) allows reductions of type I, II or III then set \( \delta_A(\pi) := \) result of the smallest reducing move. When \( \pi \) is irreducible set \( \delta_A(\pi) := \pi \).

**Remark 4.4** Lemma 4.1 implies that moves I and II are always reducing, and that for moves of type III it suffices to check the first term on the left right.

In the following, we will work with a special element \( \Theta_k \) of \( \mathbb{Z}[S_n] \):

\[ \Theta_k := \prod_{i=1}^{k-1} (1 - \prod_{j=1}^{k-i} \tau_{k-j}) = (1 - \tau_{k-1} \tau_{k-2} \cdots \tau_1)(1 - \tau_{k-1} \cdots \tau_2) \cdots (1 - \tau_{k-1}) \]
Definition 4.5 For $\pi \in S_n$ set $k := n + 1 - \pi^{-1}(1)$ and let $\pi'$ be given by
\[
\pi'(i) = \begin{cases} 
\pi(n + 1 - k + i) - 1 & \text{for } 1 \leq i < k \\
n & \text{for } i = k \\
\pi(n + 1 - i) - 1 & \text{for } k < i \leq n 
\end{cases}
\]
(usually $\pi' \in S_n$).

Now let $\rho_A$ be the $\mathbb{Z}S_n$-endomorphism given via $\rho_A(\pi) := \pi - (-1)^{n-k}\Theta_k\pi' \in S_n$.

This completes the description of the first algorithm $A_m := (k, S_{m-1}, \varphi_A, \delta_A, \rho_A)$.

4.2 Circle diagrams with one loop

To compare permutations of different symmetric groups, we extend the definition of "<" in the following way: $i < j, \pi_1 \in S_i, \pi_2 \in S_j \Rightarrow \pi_1 < \pi_2$.

Definition 4.6 For $\pi \in S_n$ let $D_n^B$ be the element of $\mathcal{P}_n$ that is obtained by replacing the box in the following figure by the picture of $\pi$.

Let $\varphi_{B,m}$ denote mapping $\bigcup_{n=3}^m S_n \to \mathcal{P}_m$ that is given by $S_n \ni \pi \mapsto \varphi_{m-n}(D_n^B)$.

To define $\delta_B$ we have to modify the first two moves. The moves of type III given in section 4.2 and the ordering of moves can be adopted unchanged.

Let $\mu_n, \nu_n \in S_n$ be permutations given by $\mu_n(k) = n + 1 - k$ and $\nu_n = (1 \ 2 \ \ldots \ n)$. $G := \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ acts on $S_n$ by $(a, b, c)\pi := \nu_n^a\mu_n^b\pi\nu_n^c$ for all $\pi \in S_n$. For $\pi \in S_n$ let $(\alpha, \beta, \gamma)$ denote an element of $G$ such that $(\alpha, \beta, \gamma)\pi$ is minimal in the orbit $G\pi$ of $\pi$. The move I’ is to replace $\pi$ by $(-1)^{n\beta}(\alpha, \beta, \gamma)\pi$.

The move II’ is given graphically:

\[
\begin{array}{c}
\vdots \\
\hline \\
\hline \\
\hline \\
\vdots \\
\end{array}
\begin{array}{c}
\uparrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\vdots \\
\end{array}
\begin{array}{c}
\hline \\
\vdots \\
\hline \\
\hline \\
\vdots \\
\end{array}
\begin{array}{c}
\uparrow \\
\downarrow \\
\rightarrow \\
\rightarrow \\
\vdots \\
\end{array}
\begin{array}{c}
\vdots \\
\hline \\
\hline \\
\hline \\
\vdots \\
\end{array}
\]

If the move II’ is applied to an element of $S_n$, the second term in the result lies in $S_{n-1}$. We will use this move only for $n \geq 4$, so we do not have to deal with the symmetric groups $S_1$ or $S_2$.

Definition 4.7 When $\pi \in S_n$ allows reductions of type I’, II’ or III then set $\delta_B(\pi) := \text{result}$ of the smallest reducing move. When $\pi$ is irreducible let $\delta_B(\pi) := \pi$.

For any $\pi \in S_{n-1}$ and $0 \leq k \leq n - 3$ we have the following element of $\mathbb{Z}[S_n]$:

$\Upsilon_{n,k}(\pi) := \chi_{1,n-k-2} \Theta_{n-k-2} \chi_{n-k-2,k+1} (1 - \tau_n) \pi^\#$

Here $\Theta_k$ is the same as in section 4.3 and $\chi_{r,s}, \pi^\# \in S_n$ are given by

$\chi_{r,s}(i) := \begin{cases} 
i+s & \text{if } i \leq r \\
i-r & \text{if } r < i \leq r+s \\
i & \text{if } i > r+s 
\end{cases}$

and $\pi^\#(i) := \begin{cases} \pi(i) & \text{if } \pi(i) < \pi(n-1) \\
\pi(i)+1 & \text{if } \pi(i) > \pi(n-1) \\
\pi(n-1)+1 & \text{if } i = n 
\end{cases}$.

Definition 4.8 For $\pi \in S_n$ with $\pi(1) = 1, \pi \neq \text{id}_n$ set $p := \max\{ j \mid \pi(i) = i \text{ for all } i \leq j \}$ and $q := \pi^{-1}(p + 1)$. If $\pi(n) \neq 2$ and $\pi(n) \neq n$ let $\pi' \in S_{n-1}$ be the permutation given by

$\pi'(i) := \begin{cases} 
i & \text{if } i = 1 \text{ or } n - p < i \leq n - 1 \\
\pi(q+1-i) & \text{if } 2 \leq i \leq q-p \\
\pi(i+p) & \text{if } q-p < i \leq n-p 
\end{cases}$.
Finally for any $\pi \in S_n$ with $3 \leq n \leq m$ set
\[
\rho_{B,m}(\pi) := \begin{cases} 
(1)(2)(3 4 5) + \text{id}_5 - 2\text{id}_4 + \text{id}_3 & \text{if } \pi = \text{id}_m \text{ and } m \geq 5 \\
\pi - \pi\tau_p + (-1)^{q-p}\Upsilon_{n,q-p-1}(\pi') & \text{if } n = m \text{ and } \pi(1) = 1 \text{ and } \pi(n) \neq 2, n \\
\pi - \Upsilon_{n,0}(\pi) & \text{if } 3 \leq n < m \text{ and } \pi(1) = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Now we have our second candidate for an ubr-algorithm $B_m := (k, \bigcup_{n=3}^{m} S_n, \varphi_{B,m}, \delta_B, \rho_{B,m})$.

## 5 Justification of the algorithms A and B

**Theorem 5.1** $A_m$ is an ubr-algorithm for $\mathcal{P}_m$ if $m \geq 2$, $B_m$ is an ubr-algorithm for $\mathcal{P}_m$ if $m \geq 3$.

To prove this, we have to verify the four conditions that we required in section 3. The second one is fulfilled by construction, because in the definition of $\delta$ we used reducing moves. There can be at most $\#S - 1$ reducing steps for elements in a linearly ordered, finite set $S$. So $j = \#S - 1$ is an integer satisfying the condition $\delta^j = \delta^{j+1}$. The remaining parts of the proof of this theorem are given in the rest of this section.

**Corollary 5.2** We have the inequalities
\[
O_A(m) \geq \text{rk } \mathcal{P}_m \geq O_C(m) \quad \text{for } m \geq 2 \quad \text{and} \quad O_B(m) \geq \text{rk } \mathcal{P}_m \quad \text{for } m \geq 3
\]
where $O_A, O_B, O_C$ are given by $O_A(m) := \text{Output}(A_m)$, $O_B(m) := \text{Output}(B_m)$ and $O_C(m) := \dim \Phi_m(\mathcal{Z}(\text{caterpillar diagrams of degree } m))$.

### 5.1 Verification of the first condition

In a first step we show that already the simply connected circle diagrams generate $\mathcal{P}$. One should recall that the circle on which the univalent vertices have been glued is just a means of visualization, not a part of the diagram. By a loop of a diagram we mean a closed path of consecutive edges that meets each edge at most once. Obviously, a loop contains only trivalent vertices and it cannot encounter a vertex twice. A vertex is called bound if there exists a loop going through that vertex, otherwise it is called free.

We have a threefold partition of circle diagrams, given by their degree, the dimension of the first homology and the number of free trivalent vertices:

\[
D_{m,k,n} := \{ D \mid D \text{ has } 2m \text{ vertices, } \dim H_1(D) = k, n \text{ of the trivalent vertices are free} \}
\]

**Lemma 5.3** Any element of $D_{m,k,n}$ with $k > 0$ and $n > 0$ can be expressed (over $\mathcal{P}_m$) in terms of elements of $D_{m,k,n-1}$.

**Proof** Any $D \in D_{m,k,n}$ has by assumption both bound and free trivalent vertices. $D$ is connected, so there exists an edge connecting a free trivalent vertex $f$ with a bound vertex $b$. Then the application of the IHX relation on this edge presents $D$ as difference of two diagrams $\in D_{m,k,n-1}$:

\[
\begin{array}{c}
\text{D}_1
\
\text{D}_2
\end{array}
= \\
\begin{array}{c}
\text{D}_3
\
\text{D}_4
\end{array}
\]

The IHX does not change the homology, but the diagrams on right side have one free trivalent vertex less. \qed
Lemma 5.4 Any element of $D_{m,k,0}$ with $k > 0$ can be expressed (over $P_m$) in terms of elements of $\bigcup_i D_{m,k-1,i}$.

Proof Any circle diagram has at least one free vertex because, by definition, it has one or more univalent vertices. For $k > 0$ any $D \in D_{m,k,0}$ has bound vertices. There must be an edge connecting a free vertex $f$ with a bound vertex $b$. By assumption $f$ has to be a univalent vertex. An application of the STU-relation at $f$ opens the loops going through $b$ without introducing new loops:

The diagrams $D_1, D_2$ on the right satisfy $\dim H_1(D_1) = \dim H_1(D) - 1$. □

Proposition 5.5 $\{D^A_\pi \mid \pi \in S_{m-1}\}$ generates $P_m$.

Proof Lemmas 5.3 and 5.4 imply that the simply connected diagrams $D_{m,0,m-1}$ generate $P_m$ (simply connected diagrams of degree $m$ must have $m-1$ trivalent vertices).

For $D \in D_{m,0,m-1}$ we choose two neighbouring (with respect to the cyclic ordering) univalent vertices $a, b$. Let $P$ denote the (uniquely determined) path in $D$ from $a$ to $b$.

If there are trivalent vertices that do not lie on $P$, we can use exactly the same argument as in lemma 5.3 to increase the number of vertices on $P$. We finally end up with diagrams that have a path $P$ going through all trivalent vertices and that connects two neighbouring univalent vertices. Because of the AS relation, we can suppose that all $m-1$ edges branching off $P$ lie on the left side of $P$. All these diagrams are of the form $D^A_\pi$ with $\pi \in S_{m-1}$. □

Proposition 5.6 $\{D^B_\pi \mid \pi \in S_m\}$ generates $P_m$ for $m \geq 3$.

Proof We want to show that $D_{m,1,0}$ is a set of generators. Because of lemma 5.3 and proposition 5.3, we only have to show that every simply connected circle diagram can be expressed in terms of diagrams having one loop, i.e. elements of $\bigcup_i D_{m,1,i}$. Let $D \in D_{m,0,m-1}$ with $m \geq 3$, then $D$ has a trivalent vertex $t$ that is connected to two univalent vertices $a, b$. We can "throw out" all other univalent vertices between $a$ and $b$ on the circle, by using STU relations:

The second diagram on the right has a loop, so it remains to show that a diagram with a trivalent vertex that is connected to two neighbouring univalent vertices and another trivalent vertex, is equivalent to a diagram with a loop. This is done by the following observation:

We have shown that diagrams having one loop and no free trivalent vertices generate $P_m$ for $m \geq 3$. These diagrams are equivalent by the AS relation to $\pm D^B_\pi$ for some $\pi \in S_m$. □

5.2 Verification of the third condition

Proposition 5.7 The maps $\delta_A$ and $\delta_B$ induce the identity in $P$. 

Proof We verify this for every move separately.

I: If the left and right side of this move are called \( \pi \) and \( \pi' \), then \( D^A_\pi \) and \( D^A_{\pi'} \) have little triangles on the upper and lower end (two of the sides are edges of the diagram, the third is part of the circle). The second picture in the proof of proposition 5.6 together with the fact that \( t \in \Lambda \), implies that one can push the triangle down through the whole diagram, so \( D^A_\pi = D^A_{\pi'} \).

II: This move is an application of the STU relation; the triangle in the second term on the right has been pushed down.

III: The four moves of type III occur, when the following diagram identities are resolved by STU on the right and IHX on the left in two different ways that are indicated by the small arrows.

I’: The diagrams \( D^B_\pi \) are given by putting the permutation \( \pi \in S_n \) between a loop and an oriented circle. The multiplication with \( \nu_n \) on the left or right does not change the diagram, because it just slides vertices on the loop or on the circle. The multiplication with \( \mu_n \) corresponds to a flip of the loop. When \( n \) is odd the AS relation causes the sign to change during the flip.

II’: The relation is of the form \( \pi \to \pi_1 - \pi_2 \) with \( \pi, \pi_1 \in S_n \) and \( \pi_2 \in S_{n-1} \). The picture for type II makes clear that \( D^B_\pi = D^B_{\pi_1} - t D^B_{\pi_2} \).

Since \( \delta_A \) and \( \delta_B \) are defined by these moves, we have hereby shown that \( \hat{\phi}_A \circ \delta_A = \hat{\phi}_A \) and \( \hat{\phi}_{B,m} \circ \delta_B = \hat{\phi}_{B,m} \).

5.3 Verification of the forth condition

First we have to understand \( \Theta_k \). For that purpose, we draw pictures with rectangular boxes named \( \Theta_k \) with \( k \) entries on the left and \( k \) exits on the right. In each such box the pictures of all permutations occurring in \( \Theta_k \) shall be inserted (forgetting the upper \( n - k \) constant strands) and the sum over all resulting diagrams (with the given signs) is taken. In this way a picture with a \( \Theta_k \)-box in fact represents a linear combination of \( 2^k \) diagrams.

Lemma 5.8 In \( A \) the following relation holds:

Proof (by induction on \( k \))
Proposition 5.9 $\hat{\varphi}_A(\pi) = (-1)^{n-k}\hat{\varphi}_A(\Theta_k\pi')$ for any $\pi \in S_n$.

Proof The external vertex on the lower side of $\hat{\varphi}_A(\pi)$ is named A, the one on the upper side B and the lowest on the right side C. We will rotate the circle clockwise, moving $C \to A \to B$. This operation looks like this (in the picture of $\pi$ the line going from $\pi^{-1}(1)$ to 1 has been drawn, the other lines have been omitted):

To get the second equality, one has to pull straight the path from C to A. To do this the $n-k = \pi^{-1}(1)-1$ lowest trivalent vertices are swapped, which is the reason for the factor $(-1)^{n-k}$. The permutation in the box on the right side is $\pi'$ of definition 4.5. So by lemma 5.8 the third diagram in the equation is equivalent to $\hat{\varphi}(\Theta_k\pi')$.

Lemma 5.10 For any $\pi \in S_n$, $0 \leq k \leq n-3$ the element $\hat{\varphi}_{B,n}(\Upsilon_{n,k}(\pi))$ of $P_{n+1}$ is equivalent to the following diagram.

Proof The permutation $\pi^{\#} \in S_{n+1}$, which is used in the definition of $\Upsilon$ in section 4.2 is obtained by doubling the $n$-th string of $\pi$. So the two terms of $(1-\tau_n)\pi^{\#}$ allow a STU relation, after the right endpoints have been glued to the circle. Together with our knowledge of $\Theta_k$, we get the following diagram for $\chi_{1,n-k-2}\Theta_{n-k-2}\chi_{n-k-2,k+1}(1-\tau_n)\pi^{\#}$:

This results in the diagram of the claim.

Proposition 5.11 $\hat{\varphi}_{B,m}(\rho_{B,m}(\pi)) = 0$ for all $\pi \in S_n$ with $3 \leq n \leq m$.

Proof Let $x := (2\ 1)(3\ 5)(4) + (1)(2\ 4)(3)$, then two moves of type I yield

$$\hat{\varphi}_{B,5}(x) = -\hat{\varphi}_{B,5}(\text{id}_5) - \hat{\varphi}_{B,5}(\text{id}_4)$$

Making three moves of type II' we get

$$\hat{\varphi}_{B,5}(x) = \hat{\varphi}_{B,5}((1)(2)(3\ 5)) = \hat{\varphi}_{B,5}((1)(2)(3\ 4\ 5) - \text{id}_4 + \text{id}_3)$$

Both equations together imply $\hat{\varphi}_{B,m}((1)(2)(3\ 4\ 5) + \text{id}_5 - 2\text{id}_4 + \text{id}_3) = 0$ for $m \geq 5$.

For any $\pi \in S_m$ and $1 \leq q < m$, the STU relation allows us to write $\hat{\varphi}(\pi) - \hat{\varphi}(\pi_{\tau_q})$ as a
single diagram $D$ with $\dim H_1(D) = 2$. For the second case in the definition of $\rho_{B,m}$, we keep the newly introduced loop of $D$ and eliminate the old one. We show how this is done in the example $m = 9, p = 3, q = 7$:

$$
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{example.png}
\end{array}
$$

During this operation the diagram $D$ is not changed, except that the $q - p - 1$ trivalent vertices at between $p$ and $q$ are swapped with the AS relation. The permutation on right of the dotted line is $\pi' \in S_{m-1}$ given in definition 5.8. So by lemma 5.10, the last diagram in the equation is $\Upsilon_{n,q-p-1}(\pi')$, and we get $\hat{\varphi}_{B,m}(\pi - \pi \tau_p + (-1)^{q-p} \Upsilon_{n,q-p-1}(\pi')) = 0$.

If $\pi \in S_n$ with $3 \leq n < m$ and $\pi(1) = 1$, let $D_0 := D_\pi$. Let the $D_1$ be the result of inserting a triangle at the lowest trivalent vertex of $D_0$. Then $\varphi(\pi) = t^{m-n} D_0 = t^{m-n-1} D_1$. Lemma 5.10 shows that $D_1 = \Upsilon_{n,0}(\pi)$, and so we have $\hat{\varphi}_{B,m}(\pi - \Upsilon_{n,0}(\pi)) = 0$.

6 Remarks to the implementation

6.1 Dimensions

One reason for the success of the presented algorithms is that the dimension of $I$ (= number of irreducible permutations) is surprisingly small. The moves we are using are very powerful, in the sense that only a small number of permutation survive the reductions. We tried out a large number of additional moves, but no considerable improvement has been achieved this way.

The following table displays the number of permutations and the number of irreducible permutations, for the ubr-algorithms $A$ and $B$:

|   | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|----|----|----|
| $\dim S_A$ | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 | 39916800 |
| $\dim I_A$ | 1 | 1 | 2 | 5 | 16 | 64 | 301 | 1583 | 9145 | 57449 | 389668 |
| $\dim S_B$ | 6 | 30 | 150 | 870 | 5910 | 46230 | 409910 | 4037910 | 43954710 | 522956310 | 522956310 |
| $\dim I_B$ | 1 | 2 | 5 | 10 | 24 | 78 | 331 | 1685 | 9589 | 59782 | 59782 |

The second reason for the success is that $\rho$ is "complicated enough" to reproduce the kernel of $\hat{\varphi}$. It should not surprise that the "correct" $\rho_A$ and $\rho_{B,m}$ have been found by an intensive trial and error process. We did not expect that the calculated upper bounds are sharp; in fact, the algorithms described here are modifications of parts of a much bigger program that computed the "exact" value $\rk P$.

6.2 Hints to the implementations

At first a list of irreducible permutations for the desired degree $m$ should be made. Then $\rho(\pi)$ is calculated for any $\pi$ in this list.

The real difficulty is to evaluate $\bar{\rho}(\pi) = \Delta(\rho(\pi))$. The simplest idea is to consecutively apply $\delta$, until a linear combination of irreducible elements is reached. But this would be much too slow for the interesting degrees $> 9$, because the reduction trees are too nested.

The solution is to do it upside down. After assigning values to the minimal permutations, we go from small permutations to the bigger ones. If we know the values of all permutations smaller than $\pi$, then the value of $\pi$ is given by a single application of $\delta$ and picking at most 7 values out of the table.

One has to assign a $\dim I$-dimensional vector to every $\pi$. A short look at the dimensions shows that keeping this table in memory exceeds the capacity of any computer. But one can
do the calculation component per component. Our implementation does 32 components at a time, yielding 32 rows of the matrix for $\bar{\rho}$ in each run.

Even if each matrix entry uses only one bit, a file in which the matrix for $\bar{\rho}_{B,12}$ is saved contains 426 megabytes. This is one reasons why we are working with $k = \mathbb{F}_2$. The other reason is that, some time ago, we found diagrams $x \in A_{10}$ for which we could show $2x = 0$ but not $x = 0$. To find out, wether $A_{10}$ contains elements of order 2 or not, was the main stimulus to make these computer computations.

The program that computes then rank and nullity of the matrices is a standard Gaussian algorithm, which can of course be implemented very efficiently for $k = \mathbb{F}_2$. By the way, the matrices occurring are not at all sparse: about 40% of the entries are 1 and gzip compresses the files only by factors about 0.9.

It is not a bad idea to add a check sum to each row/column in the data files, because the probability of making an error in reading/writing a bit to hard disc might (on some systems) not be far enough away from $1 : 10^{11}$, which is the approximate number of bits that have to be read (in our implementation).

Acknowledgment

I would like to thank C.-F. Bödigheimer, Michael Eisermann, Christoph Lamm and Jens Lieberum for helpful discussions and reading the manuscript. I am also thankful to the Studienstiftung for financial support and the Graduiertenkolleg for providing the computer on which the computation has been performed.

Computer files

At the following locations in the internet,

- [http://www.uni-bonn.de/~jk/pvi12.html](http://www.uni-bonn.de/~jk/pvi12.html)
- [ftp://ftp.uni-bonn.de/usr/jk/pvi12](ftp://ftp.uni-bonn.de/usr/jk/pvi12)

you will find:

- C/Pascal implementations of the algorithms.
- a basis for $P_{m \leq 12}$ that is produced by the algorithm $B$.
- the values of $\tilde{\Phi}$ of the 175 diagrams that are mentioned in the proof of theorem 2.14.
- a file containing this paper (or a newer version of it).
- a summary of the results of section 2 and possibly some newer data.

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