INTERPOLATION IN FRAGMENTS OF INTUITIONISTIC PROPOSITIONAL LOGIC

Gerard R. Renardel de Lavalette

Department of Philosophy

University of Utrecht
INTERPOLATION IN FRAGMENTS OF
INTUITIONISTIC PROPOSITIONAL LOGIC

Gerard R. Renardel de Lavalette

Department of Philosophy, University of Utrecht
Heidelbergaan 2, 3584 CS Utrecht, The Netherlands

Abstract
We show in this paper that all fragments of intuitionistic
propositional logic based on a subset of the connectives \( \land, \lor, \to, \neg \)
satisfy interpolation. Fragments containing \( \leftrightarrow \) or \( \rightarrow \) are briefly
considered.

This is a revised and extended version of 'Interpolation in natural
fragments of intuitionistic propositional logic', Logic Group
Preprint Series nr. 15, University of Utrecht, 1986.

1980 Mathematical Subject Classification: 03B20, 03C40

Key words and phrases: intuitionistic propositional logic, fragment,
interpolation.

October 1987
§1. Introduction.

1.1. A fragment \([C]\) of IpL (intuitionistic propositional logic) is a subset of the set of formulae of IpL built up from the propositional variables and constants (\(T\) and \(\bot\)) by means of connectives from the set \(C\) only. If \(C=\{c_1,c_2,\ldots\}\), then we write \([c_1,c_2,\ldots]\) for \([C]\). In this note, we mainly consider the primitive connectives \(\land, \lor\) and \(\rightarrow\); one can, however, also think at defined connectives \(c_A\) where \(A=A(P_1,\ldots,P_n)\) is some formula, with \(c_A(B_1,\ldots,B_n) := A(B_1,\ldots,B_n)\). Examples of defined connectives are\(\neg (\neg A = A\rightarrow \bot)\) and \((A \leftrightarrow B = (A\rightarrow B)\land (B\rightarrow A))\). So e.g. \([\land,\lor,\rightarrow]\) contains all formulae of IpL, and \([\leftrightarrow]\) is the fragment containing all formulae built up with \(\leftrightarrow\) only.

1.2. The interpolation theorem for IpL reads:

Let \(A,B\) be formulae of IpL such that \(A \vdash B\). Then there is an interpolant \(I\) for \(A \vdash B\), i.e.

i) \(A \vdash I \land I \vdash B\);

ii) all propositional variables of \(I\) occur both in \(A\) and in \(B\).

This theorem is a consequence of the interpolation theorem for intuitionistic predicate logic, first proved by Schütte in [S62].

1.3. In this paper, we consider relativizations of the interpolation theorem to elementary fragments (fragments based on primitive connectives or \(\neg\)), and we show interpolation holds in all elementary fragments.

There are many fragments of intuitionistic propositional logic for which interpolation fails, e.g. \([\land,\rightarrow,\neg,\delta]\) with \(\delta(A,B,C) = (A\lor\neg A)\land (A\rightarrow B)\lor (\neg A\rightarrow C)\) (first proved by J. Zucker in [Z78]; see also [R81]). For intermediate logics the situation is the same (see [M85]). In classical logic, however, interpolation holds in all propositional fragments (proved by F. Ville: see [KK71], [KK72], Ch. 1, Exercises).

Another notion of fragment is considered in [R8?], where a strong version of interpolation is proved for the subset NNIL (No Nestings of Implication to the Left) of formulae of IpL, defined inductively by

all propositional variables and constants are in NNIL,

NNIL is closed under \(\land\) and \(\lor\),

if \(A \in NNIL\) and \(P\) is a propositional variable, then \(P\rightarrow A \in NNIL\).

1.4. The rest of this paper is organized as follows: in §2 we fix the notation and present a sequent
calculus for IpL, §3 consists of three lemmata about elementary fragments, §4 contains Schütte's method to prove the interpolation theorem for IpL, which is used subsequently to show interpolation for all elementary fragments, in §5 we discuss the consequences of not adding the constants T and ⊥ to the fragments. §6 is rather tentative: it reports on unsuccessful attempts to prove interpolation for some fragments containing the connectives ↔ and ¬¬.

1.5. Acknowledgements.
The author is indebted to M.H. Lüb, who pointed out to him an error in a previous version of theorem 4.5.

§2. Preliminaries.

2.1. Notation.
All formulae are in intuitionistic propositional logic, with ∧, ∨, →, ¬ as connectives and the constants T and ⊥. P,Q,... are propositional variables; together with T, ⊥ we call them atoms. A,B,C,... are formulae; Γ,Δ,Γ',... are finite (possibly empty) sets of formulae. We write Γ,Δ for the union of Γ and Δ; Γ,A stands for Γ∪{A}.

For sets of formulae F, G we define

\[ F = G := \forall A \in F \exists B \in G (A=B) \text{ and } \forall B \in G \exists A \in F (B=A), \]

where A=B stands for A→B and B→A. We also put

\[ \wedge F := \{ A_1 \wedge \ldots \wedge A_n | A_1,\ldots,A_n \in F \}, \]
\[ F \rightarrow G := \{ A \rightarrow B | A \in F, B \in G \}, \]
\[ \neg F := \{ \neg A | A \in F \}. \]

We define a(A) [a⁺(A)], the set of all [strictly positively occurring] atoms in A, by

\[ a(T) = a⁺(T) = \{ T \}, \]
\[ a(⊥) = a⁺(⊥) = \{ ⊥ \}, \]
\[ a(P) = a⁺(P) = \{ P \}, \]
\[ a(A \land B) = a(A \lor B) = a(A \rightarrow B) = a(A) \cup a(B), \]
\[ a(¬A) = a(A), \]
\[ a⁺(A \land B) = a⁺(A \lor B) = a⁺(A) \cup a⁺(B), \]
\[ a⁺(A \rightarrow B) = a⁺(B), \]
\[ a⁺(¬A) = \emptyset; \]
p(A) [p^+(A)], the set of all [strictly positively occurring] propositional variables in A, is defined by

\[
p(A) = a(A) - \{T, \bot\},
\]

\[
p^+(A) = a^+(A) - \{T, \bot\}.
\]

2.2. The derivation system.

We use the following sequent calculus, denoted by SC:

- **(P)** \( \Gamma, \Gamma \vdash P \)
- **(T)** \( \Gamma \vdash \top \)
- **(⊥)** \( \Gamma, \bot \vdash C \)

**∧R** \[
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B}
\]

**∧L** \[
\frac{\Gamma, \Lambda \vdash C}{\Gamma, \Lambda \land B \vdash C}
\]

**∨R** \[
\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \lor A_2}
\quad (i=1,2)
\]

**∨L** \[
\frac{\Gamma, \Lambda \vdash C}{\Gamma, \Lambda \lor B \vdash C}
\]

**→R** \[
\frac{\Gamma, \Lambda \vdash B}{\Gamma \vdash \Lambda \rightarrow B}
\]

**→L** \[
\frac{\Gamma \vdash \Lambda \quad \Gamma, \Lambda \vdash C}{\Gamma, \Lambda \rightarrow B \vdash C}
\]

**¬R** \[
\frac{\Gamma, \Lambda \vdash \bot}{\Gamma \vdash \neg \Lambda}
\]

**¬L** \[
\frac{\Gamma \vdash \Lambda}{\Gamma, \neg \Lambda \vdash B}
\]

The main formula of a rule is the newly formed formula of the conclusion: \( A \land B \) for \((\land R)\) and \((\land L)\), \( A_1 \lor A_2 \) for \((\lor R)\), \( A \lor B \) for \((\lor L)\), \( A \rightarrow B \) for \((\rightarrow R)\) and \((\rightarrow L)\), \( \neg A \) for \((\neg R)\) and \((\neg L)\).

SC has the following derived rules:

- **(CUT)** cut elimination: if \( \Gamma \vdash A \) and \( \Gamma, \Lambda \vdash B \) then \( \Gamma \vdash B \);
- **(WEAK)** weakening: if \( \Gamma \vdash A \) then \( \Gamma, \Lambda \vdash A \);
- **(SUB)** substitution: if \( \Gamma \vdash A \) then \( \Gamma[P:=B] \vdash A[P:=B] \).
The proofs are standard (as for related systems, e.g. in [S62] and [T75]).

Note that the Subformula Property only holds in the following version:

if $B$ occurs in a cut-free derivation of $\Gamma \vdash A$, then $B=\bot$ or $B$ is
a subformula of $\Gamma, A$;

the addition $B=\bot$ is made necessary by the presence of $(\neg R)$, in which $\bot$ is eliminated.

The following consequence is important in the context of this paper:

\begin{itemize}
  \item[i)] if $B$ is a formula occurring in $\Pi$, then all connectives in $B$ occur in $\Gamma, A$;
  \item[ii)] if $c \in \{\land, \lor, \rightarrow, \neg\}$ and $(cR)$ or $(cL)$ is a rule applied in $\Pi$,
  then $c$ occurs in $\Gamma, A$.
\end{itemize}

For later use (4.5, 4.6), we define a variant $SC^*$ of $SC$ and prove it equivalent to $SC$.

2.3. $SC^*$ is $SC$ with $(\lor L)$ and $(\rightarrow L)$ replaced by:

\[
\begin{array}{c}
(\lor L)^* & \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} & (\rightarrow L)^* & \frac{\Gamma, A \rightarrow B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}
\end{array}
\]

We write $\Gamma \vdash^* A$ for: $\Gamma \vdash A$ is derivable in $SC^*$.

2.4. Lemma. $\Gamma \vdash A$ if and only if $\Gamma \vdash^* A$.

Proof. We write $\Gamma \vdash_n A$ for: $\Gamma \vdash A$ has a derivation with length at most $n$; idem for $\Gamma \vdash_n^* A$. With induction over $n$ one easily proves:

\begin{itemize}
  \item[(1)] if $\Gamma, A \lor B, A \vdash_n C$, then $\Gamma, A \vdash_n C$;
  \item[(2)] if $\Gamma, A \lor B, B \vdash_n C$, then $\Gamma, B \vdash_n C$;
  \item[(3)] if $\Gamma, A \rightarrow B, B \vdash_n C$, then $\Gamma, B \vdash_n C$;
  \item[(4)] if $\Gamma \vdash_n C$, then $\Gamma, \Delta \vdash_n C$;
  \item[(5)] if $\Gamma \vdash_n^* C$, then $\Gamma, \Delta \vdash_n^* C$.
\end{itemize}

We turn to the 'if' part of the lemma. Assume $\Gamma \vdash^* A$, i.e. $\Gamma \vdash_n^* A$ for some $n$; we show $\Gamma \vdash_n A$ with induction over $n$. If $n=1$ then $\Gamma \vdash^* A$ is an axiom, hence $\Gamma \vdash A$; if $n>1$ and $\Gamma \vdash^* A$ is (an axiom or) the conclusion of $(\lor L)^*$ or a rule different from $(\rightarrow L)^*$, then the result directly follows from the induction hypothesis (using that every instance of $(\lor L)^*$ is an instance of $(\lor L)$). If $\Gamma \vdash^* A$ is the conclusion of $(\rightarrow L)^*$, then the premises are of the form $\Gamma, B \rightarrow C \vdash^* B$ and $\Gamma', C \vdash A$ where $\Gamma := \Gamma - \{B \rightarrow C\}$. Now apply (5) to obtain $\Gamma', B \rightarrow C, C \vdash_{n-1}^* A$, and
then the induction hypothesis.

Finally we prove the 'only if' part, with induction over the length of a derivation of $\Gamma \vdash A$.
Assume $\Gamma \vdash A$, so $\Gamma \vdash_n A$ for some $n$. If $n=1$ then $\Gamma \vdash A$ is an axiom, hence $\Gamma \vdash^* A$; if $n>1$ and $\Gamma \vdash A$ is (an axiom or) the conclusion of an instance of $(\lor L)^*$ or a rule different from $(\lor L)$, $(\rightarrow L)$, then the result directly follows from the induction hypothesis. There are three cases left:
i) $\Gamma \vdash A$ is the conclusion of $(\lor L)$ with premises of the form $\Gamma', B \lor C, B \vdash A$ and $\Gamma', B \lor C, C \vdash A$ where $\Gamma' := \Gamma - \{B \lor C\}$: apply (1), (2), the induction hypothesis and $(\lor L)^*$.
ii) $\Gamma \vdash A$ is the conclusion of $(\rightarrow L)$ with premises of the form $\Gamma' \vdash B$ and $\Gamma', C \vdash A$ where $\Gamma' := \Gamma - \{B \rightarrow C\}$: apply (4) to obtain $\Gamma', B \rightarrow C \vdash_{n-1} B$, then the induction hypothesis and $(\rightarrow L)^*$.
iii) $\Gamma \vdash A$ is the conclusion of $(\rightarrow L)$ with premises of the form $\Gamma', B \rightarrow C \vdash B$ and $\Gamma', B \rightarrow C, C \vdash A$ where $\Gamma' := \Gamma - \{B \rightarrow C\}$: apply (3) to obtain $\Gamma', C \vdash_{n-1} A$, then the induction hypothesis and $(\rightarrow L)^*$.

§3. Elementary fragments.

Before turning to interpolation, we derive some properties of elementary fragments.

3.1. Lemma.
i) $[\land, \rightarrow] \equiv \land [\rightarrow]$.
ii) Let $A \in [\rightarrow]$. Then $a^+(A) = \{X\}$ for some atom $X$, and $A \equiv (A \rightarrow X) \rightarrow X$.
iii) $\{A \in [\land, \rightarrow] | a^+(A) \text{ is a singleton} \} \equiv [\rightarrow]$.

Proof.
i) Formula induction, using

$$(A \land B) \rightarrow (C \land D) \equiv (A \rightarrow (B \rightarrow C)) \land (A \rightarrow (B \rightarrow D)).$$

ii) We have $A = A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow X) \ldots) \ (n \geq 0)$ for some atom $X$, so $a^+(A) = \{X\}$. Also $A \equiv (A_1 \land \ldots \land A_n) \rightarrow X \equiv (((A_1 \land \ldots \land A_n) \rightarrow X) \rightarrow X) \rightarrow X \equiv (A \rightarrow X) \rightarrow X$.

iii) Formula induction, using (i), (ii) and

$$(A \rightarrow X) \rightarrow X \land ((B \rightarrow X) \rightarrow X) \equiv (A \rightarrow (B \rightarrow X)) \rightarrow X.$$
3.2. Lemma.
i) \([\land, \lor, \to] \equiv \land[\lor, \to]\).

ii) There is a mapping \(\Rightarrow\) satisfying: if \(A \in [\land, \lor, \to], B \in [\lor, \to]\), then

\[
A \Rightarrow B \in [\lor, \to], \quad A \Rightarrow B \equiv (A \to B) \text{ and } p(A \Rightarrow B) = p(A \to B);
\]

as a consequence we have \([\land, \lor, \to] \to [\lor, \to] \equiv [\lor, \to]\).

Proof.
i) Formula induction, using the following equivalences:

\[
(A \land B) \lor (C \land D) \equiv (A \lor C) \land (B \lor D),
\]
\[
(A \land B) \to (C \land D) \equiv (A \to C) \land (B \to D).
\]

ii) Let \(A \in [\land, \lor, \to], B \in [\lor, \to]\). By (i) we have \(A \equiv A_1 \land \ldots \land A_n\) with \(A_i \in [\lor, \to]\) \((i=1,\ldots,n)\); from the proof of (i) it follows that \(p(A)=p(A_1 \land \ldots \land A_n)\). Now put

\[
A \Rightarrow B := A_1 \Rightarrow (A_2 \Rightarrow \ldots \Rightarrow (A_n \Rightarrow B)\ldots),
\]

and one easily sees that (ii) is satisfied. 

3.3. Lemma.
i) \([\land, \lor, \neg] \equiv \land[\lor, \neg]\).

ii) There is a mapping \(\downarrow\) satisfying: if \(A \in [\land, \lor, \neg]\), then

\[
\downarrow A \in [\lor, \neg], \quad \downarrow A \equiv \neg A \text{ and } p(\downarrow A) = p(\neg A);
\]

as a consequence we have \(\neg[\land, \lor, \neg] \equiv [\lor, \neg]\).

Proof. Analogous to that of 3.2, using the equivalence

\[
\neg(A \land B) \equiv \neg \neg (\neg A \lor \neg B)
\]

and the definition

\[
\downarrow A := \neg \neg (\neg A_1 \lor \ldots \lor \neg A_n)
\]

for \(A \in [\land, \lor, \neg]\) with \(A \equiv A_1 \land \ldots \land A_n\) and \(A_i \in [\lor, \neg]\) \((i=1,\ldots,n)\).
§4. Interpolation in elementary fragments.

4.1. Lemma (Schütte [S62]). Interpolation holds in \([\land,\lor,\rightarrow,\neg]\).

Proof. Let \(A \vdash B\). Then there is a derivation in \(SC\) of \(A \vdash B\). With induction over the length of the derivation it is shown that any partition \(\Gamma,\Delta \vdash C\) of a sequent in the derivation has an interpolant \(I\), i.e.

\[
\begin{align*}
\Gamma & \vdash I; \\
I,\Delta & \vdash C; \\
p(\Gamma) \cap p(\Delta,C) & \subseteq p(I).
\end{align*}
\]

From this the lemma follows (take \(\Gamma=\{A\},\Delta=\emptyset, C=B\).

The method to obtain the interpolant \(I\) for \(\Gamma,\Delta \vdash A\) can be rendered as follows:

| Rule   | premises                                                                 | conclusion                                   |
|--------|---------------------------------------------------------------------------|---------------------------------------------|
| \((iP1)\) | \(\Gamma [T] \Delta, P \vdash P\)                                       | \(\Gamma, P [P] \Delta \vdash P\)           |
| \((iT)\)  | \(\Gamma [T] \Delta \vdash T\)                                           |                                            |
| \((i \perp 1)\) | \(\Gamma [T] \Delta, \perp \vdash C\)                                   | \(\Gamma, \perp [\perp] \Delta \vdash C\)  |
| \((i \land R)\) | \(\Gamma [I_1] \Delta \vdash A\) \quad \Gamma [I_2] \Delta \vdash B\)  | \(\Gamma [I_1 \land I_2] \Delta \vdash A \land B\) |
| \((i \lor L 1)\) | \(\Gamma [I_1] A, \Delta \vdash C\) \quad \Gamma [I_2] B, \Delta \vdash C\) | \(\Gamma [I_1 \lor I_2] A \lor B, \Delta \vdash C\) |
| \((i \lor L 2)\) | \(\Gamma, A [I_1] \Delta \vdash C\) \quad \Gamma, B [I_2] \Delta \vdash C\) | \(\Gamma, A \lor B [I_1 \lor I_2] \Delta \vdash C\) |
| \((i \rightarrow L 1)\) | \(\Gamma [I_1] \Delta \vdash A\) \quad \Gamma [I_2] B, \Delta \vdash C\) | \(\Gamma [I_1 \rightarrow I_2] A \rightarrow B, \Delta \vdash C\) |
| \((i \rightarrow L 2)\) | \(\Gamma [I_1] \Delta \vdash A\) \quad \Gamma, B [I_2] \Delta \vdash C\) | \(\Gamma, A \rightarrow B [I_1 \rightarrow I_2] \Delta \vdash C\) |
We explain this notation with an example.

(i\land R) means:

\[
\text{if } \Gamma \vdash I_1 \text{ and } I_1,\Delta \vdash A \text{ and } \Gamma \vdash I_2 \text{ and } I_2,\Delta \vdash B \text{ then } \Gamma \vdash I_1 \land I_2 \text{ and } I_1 \land I_2,\Delta \vdash A \land B; 
\]

so (i\land R) indicates how an interpolant for \( \Gamma, \Delta \vdash A \land B \) can be obtained from interpolants for \( \Gamma, \Delta \vdash A \) and \( \Gamma, \Delta \vdash B \).

For rules not mentioned here ((\land L), (\lor R), (\rightarrow R), (\neg R)), the interpolant for the conclusion is the same as for the premise.

4.2. With Schütte’s method (i.e. the method used in the proof of Lemma 4.1), it is easy to prove interpolation for [\land] and for the fragments containing [\land, [\land,\lor], [\land,\rightarrow], [\land,\neg], [\land,\lor,\rightarrow], [\land,\lor,\neg], [\land,\rightarrow,\neg], [\land,\lor,\rightarrow,\neg], [\lor,\lor,\rightarrow,\neg], [\rightarrow,\rightarrow,\rightarrow,\neg]).

This is not evident for fragments containing \lor or \rightarrow, but not \land (\lor, [\lor], [\land,\lor], [\land,\rightarrow], [\land,\neg], [\lor,\lor,\rightarrow,\neg], [\rightarrow,\rightarrow,\rightarrow,\neg]), as (i\lor L1) and (i\rightarrow L1) introduce \land in the definition of the interpolant. To illustrate this, we present the following example where Schütte’s method is applied to a proof of \((P \lor Q) \rightarrow R \vdash (P \lor Q) \rightarrow R :\)

\[
\begin{align*}
P,(P \lor Q) \rightarrow R & \vdash P \\
P,(P \lor Q) \rightarrow R & \vdash P \lor Q \\
(P \lor Q) \rightarrow R & \vdash [P \rightarrow R] P \vdash R \\
(P \lor Q) \rightarrow R & \vdash [(P \rightarrow R) \land (Q \rightarrow R)] P \lor Q \vdash R \\
(P \lor Q) \rightarrow R & \vdash [(P \rightarrow R) \land (Q \rightarrow R)] (P \lor Q) \rightarrow R
\end{align*}
\]

Since \lor is definable in fragments containing \rightarrow (using the constant \bot), we have \([\lor,\rightarrow,\rightarrow] \equiv [\lor,\rightarrow]\) and \([\lor,\rightarrow,\rightarrow] \equiv [\lor,\rightarrow]\), so the only fragments for which interpolation still has to be shown are \([\lor], [\lor,\rightarrow], [\lor,\rightarrow,\rightarrow]\) and \([\lor,\rightarrow,\rightarrow]\). We have a closer look at these four fragments in the rest of this section.

4.3. Interpolation in [\lor] is trivial: if A,B \in [\lor] and A \vdash B, then A=P_1 \lor \ldots \lor P_m, B=Q_1 \lor \ldots \lor Q_n with \forall i \leq m,j \leq n (P_i=Q_j), so p(A) \subseteq p(B) and A is an interpolant. (By a similar argument, interpolation in [\land] is trivial.)
4.4. **Theorem** (Zucker [Z78]). Interpolation holds in $[\to]$.

**Proof** (somewhat different from Zucker's). Let $A,B \in [\to]$ with $A \vdash B$. Schütte's method gives us an interpolant $I$ in the fragment $[\land,\to]$. With induction over the length of the derivation of $A \vdash B$ one can show that $p^+(I) \subseteq p^+(A)$, so by Lemma 3.1(iii) there is an $I'$ equivalent to $I$ in $[\to]$. []

4.5. **Theorem.** Interpolation holds in $[\lor,\to]$.

**Proof.** Let $A,B \in [\lor,\to]$ with $A \vdash B$. By Lemma 2.4 there is a derivation $\Pi$ in $\mathcal{S}\mathcal{C}^*$ of $A \vdash^* B$. We define inductively the subtree $\Pi^+$ of $\Pi$:

- a) $A \vdash^* B$ is in $\Pi^+$;
- b) if the conclusion of a rule is in $\Pi^+$, then so is (are) the premise(s), with the following exception:

(*) if the rule involved is an instance of $(\to L)^*$ with a subformula occurrence of $A$ as main formula, then only the right hand premise is in $\Pi^+$.

With induction over the structure of $\Pi^+$ one easily observes (using the peculiar definition of $(\lor L)^*$ and $(\to L)^*$):

1. every sequent in $\Pi^+$ is of the form $A_0, \Gamma \vdash C$, where $A_0$ is a strictly positive subformula occurrence of $A$ and $\Gamma \cup \{C\}$ a collection of subformula occurrences of $B$.

In other words: the sequents in $\Pi^+$ contain no negative subformula occurrences of $A$ and exactly one subformula occurrence, namely in the premise.

Now we prove, with formula induction:

2. for every strictly positive subformula occurrence $A_0$ of $A$, there is an $I = I(A_0, \Pi^+)$ with

   - i) $I \in [\lor,\to]$,
   - ii) $p(I) \subseteq p(A) \cap p(B)$,
   - iii) $A_0 \vdash I$,
   - iv) for every sequent $A_0, \Gamma \vdash^* C$ ($A_0 \notin \Gamma$) in $\Pi^+$ we have $I, \Gamma \vdash C$;

from this the theorem follows (take $A$ for $A_0$).
I) \( A_0 = P, P \) an atom: take

\[
I(P, \Pi^+) := \begin{cases} P & \text{if } P \in p(B) \\ T & \text{otherwise,} \end{cases}
\]

then (i), (ii), (iii) are trivial. For (iv), we argue as follows: if \( P \in p(B) \), then \( I(P, \Pi^+) = P = A_0 \) and (iv) holds trivially; if \( P \notin p(B) \), then \( P \notin p(\Gamma, C) \) by (1), and with \( A_0, \Gamma \vdash C \) we get (using \( A_0 = P \) and (SUB)) \( T, \Gamma \vdash C \).

II) \( A_0 = A_1 \lor A_2 \): now put

\[
I(A_1 \lor A_2, \Pi^+) := I(A_1, \Pi^+) \lor I(A_2, \Pi^+);
\]

(i), (ii) and (iii) are evident, (iv) is proved as follows with induction over the length of a subderivation of \( \Pi^+ \) containing only sequents of the form \( A_1 \lor A_2, \Gamma \vdash^* C \) (\( A_1 \lor A_2 \notin \Gamma \)). We distinguish three cases, writing \( I \) for \( I(A_1 \lor A_2, \Pi^+) \):

a) The sequent is an axiom: then so is \( I, \Gamma \vdash C \).

b) \( A_1 \lor A_2, \Gamma \vdash^* C \) is the conclusion of a \((\lor)\)-rule:

\[
\frac{A_1, \Gamma \vdash^* C \quad A_2, \Gamma \vdash^* C}{A_1 \lor A_2, \Gamma \vdash^* C}
\]

by the induction hypothesis (2), we have \( I(A_1, \Pi^+), \Gamma \vdash C \) and \( I(A_2, \Pi^+), \Gamma \vdash C \), hence we have \( I, \Gamma \vdash C \) by \((\lor)^*\).

c) \( A_1 \lor A_2, \Gamma \vdash^* C \) is the conclusion of a rule with premise(s) containing \( A_1 \lor A_2 \), e.g. \((\lor)^*\):

\[
\frac{A_1 \lor A_2, \Gamma, D \vdash^* C \quad A_1 \lor A_2, \Gamma, E \vdash^* C}{A_1 \lor A_2, \Gamma, D \lor E \vdash^* C}
\]

by induction, we now have \( I, \Gamma, D \vdash C \) and \( I, \Gamma, E \vdash C \), hence \( I, \Gamma, D \lor E \vdash C \) by \((\lor)\). Similar for other rules.

III) \( A_0 = A_1 \rightarrow A_2 \): let

\[
(\rightarrow L_i)^* \quad \frac{\Gamma_i, A_1 \rightarrow A_2 \vdash^* A_1 \quad A_2, \Gamma_i \vdash^* C_i}{\Gamma_i, A_1 \rightarrow A_2 \vdash^* C_i}
\]

\((i=1, \ldots, n, n \geq 0)\) be all the instances of \((\rightarrow L)^*\) in \( \Pi^+ \) with \( A_1 \rightarrow A_2 \) as main formula. By 2.4 we have \( \Gamma_i, A_1 \rightarrow A_2 \vdash A_1 \) \((i=1, \ldots, n)\), and with 4.2 we find interpolants \( J_i \) \((i=1, \ldots, n)\) in \([\land, \lor, \rightarrow]\) with
Now put

\[ I(A_1 \rightarrow A_2, \Pi^+) := (J_1 \vee \ldots \vee J_n) \Rightarrow I(A_2, \Pi^+). \]

We show (i) - (iv) of (2), writing \( I \) for \( I(A_1 \rightarrow A_2, \Pi^+) \).

(i): by Lemma 3.2(ii).

(ii): by (5) and (1) we have \( p(J_i) \subseteq p(B) \cap p(A) \) (i=1,\ldots,n), so \( p(I) \subseteq p(A) \cap p(B) \) by the induction hypothesis (2) and Lemma 3.2(ii).

(iii): by (4) we have \( J_i, A_1 \rightarrow A_2 \Gamma A_1 \) (i=1,\ldots,n), hence \( J_1 \vee \ldots \vee J_n A_1 \rightarrow A_2 \Gamma A_1 \); also \( A_2 \Gamma I(A_2, \Pi^+) \) by the induction hypothesis (2). This gives \( A_1 \rightarrow A_2 \Gamma I(J_1 \vee \ldots \vee J_n) \Rightarrow I(A_2, \Pi) \), so with Lemma 3.2(ii) we get \( A_1 \rightarrow A_2 \Gamma I \).

(iv): let \( A_1 \rightarrow A_2, \Gamma \vdash^* C \) be a sequent in \( \Pi^+ \). Three cases:

a) it is an axiom: then so is \( I, \Gamma \vdash^* C \).

b) it is the conclusion of one of the \((\rightarrow L_i)^*\), so \( \Gamma = \Gamma_i, C = C_i \). Now \( \Gamma_i \vdash J_i \) by (3), so \( \Gamma_i \vdash J_1 \vee \ldots \vee J_n \); also, by induction hypothesis (2), \( I(A_2, \Pi^+), \Gamma_i \vdash C_i \), hence we have \( (J_1 \vee \ldots \vee J_n) \Rightarrow I(A_2, \Pi), \Gamma_i \vdash C_i \), i.e. (by Lemma 3.2(ii)) \( I, \Gamma_i \vdash C_i \).

c) it is the conclusion of another instance of a rule, with \( A_1 \rightarrow A_2 \) as main formula: now (iv) follows with induction, as under (IIc) above. 

4.6. Theorem. Interpolation holds in \([\vee, \neg]\).

Proof. Analogous to that of Theorem 4.5 above. We only present the differences.

\( \Pi^+ \) is defined in 4.5, but now with the exception

\[ (*') \quad \text{if the rule involved is an instance of } (\neg L)^* \text{ with a subformula occurrence of } A \text{ as main formula, then the premise is not } \Pi^+. \]

This \( \Pi^+ \) satisfies (1) of 4.5, and we prove the following analogue of (2):

(6) for every strictly positive subformula occurrence \( A_0 \) of \( A \), there is an \( I = I(A_0, \Pi^+) \) with

i) \( I \in [\vee, \neg] \),

ii) \( p(I) \subseteq p(A) \cap p(B) \),

iii) \( A_0 \vdash I \),

iv) for every sequent \( A_0, \Gamma \vdash^* C \) (\( A_0 \not\in \Gamma \)) in \( \Pi^+ \) we have \( I, \Gamma \vdash C \).
$A_0 = P$ or $A_0 = A_1 \lor A_2$: as in 4.5.

$A_0 = A_1 \lor A_2$: let

$$
\frac{\Gamma_i \vdash A_1}{\Gamma_i, \neg A_1 \vdash C_i}
$$

(i=1,...,n, n\geq 0) be all the instances of $(\neg L_i)$ in $\Pi$ with $\neg A_1$ as main formula. By 4.2 we find interpolants $J_i$ (i=1,...,n) in $\{\land, \lor, \neg\}$ with

1. $\Gamma_i \vdash J_i$,  
2. $J_i \vdash A_1$,  
3. $p(J_i) \subseteq p(\Gamma_i) \cap p(A_1)$.

Now put

$$I(\neg A_1, \Pi^+) := \bigcup (J_1 \lor ... \lor J_n).$$

We show (i) - (iv) of (6), writing $I$ for $I(\neg A_1, \Pi^+)$.

(i): by Lemma 3.3(ii).

(ii): by (9) and (1) we have $p(J_i) \subseteq p(B) \cap p(A)$ (i=1,...,n), so $p(I) \subseteq p(A) \cap p(B)$ by the induction hypothesis (6) and Lemma 3.3(ii).

(iii): by (8) we have $J_i \vdash A_1$ (i=1,...,n), hence $J_1 \lor ... \lor J_n \vdash A_1$. This gives $\neg A_1 \vdash (J_1 \lor ... \lor J_n)$, so with Lemma 3.3(ii) we get $\neg A_1 \vdash I$.

(iv): let $\neg A_1, \Gamma \vdash C$ be a sequent in $\Pi^+$. Three cases:

a) it is an axiom: then so is $I, \Gamma \vdash C$.

b) it is the conclusion of one of the $(\neg L_i)^*$, so $\Gamma = \Gamma_i, C = C_i$. Now $\Gamma_i \vdash I$ by (7), so $\Gamma_i \vdash J_1 \lor ... \lor J_n$, hence we have $(J_1 \lor ... \lor J_n), \Gamma_i \vdash C_i$, i.e. (by Lemma 3.3.(iii)) $I, \Gamma_i \vdash C_i$.

c) it is the conclusion of an instance of another rule, with $\neg A_1$ as main formula: now (iv) follows with induction, as under (IIc) in 4.5.

§5. Fragments without $\top$ or $\bot$.

To make life simpler, we considered $\top$ and $\bot$ as constants which are present in every fragment. If we do not choose to do so, we have to be slightly more careful, as we shall now explain.

5.1. $\top$ is definable in fragments containing $\rightarrow$ (by $P \rightarrow P$), $\land$ and $\neg$ (by $\neg(P \land \neg P)$), or $\lor$ and $\neg$ (by $\neg(P \lor \neg P)$); similarly, $\bot$ is definable in fragments containing $\rightarrow$ and $\land$, $\neg$ and $\lor$, or $\neg$ and $\rightarrow$. In such fragments we have e.g. the following derivable sequents:
\[ P \rightarrow P \vdash Q \rightarrow Q, \]
\[ P \land \neg P \vdash Q \land \neg Q, \]

and an interpolant $I$ for any of these must satisfy $p(I) = \emptyset$, which is impossible without constants. The obvious remedy is to strengthen the premise in the formulation in the Interpolation Theorem by adding any of the following conclusions:

\[
\begin{align*}
\text{p(A) \land p(B) \neq \emptyset, or} \\
\text{not (A \vdash \bot) and not (\vdash B).}
\end{align*}
\]

§6. Other fragments, open problems.

In this last section we consider fragments containing $\leftrightarrow$ and $\rightarrow$, and sketch some attempts to prove interpolation.

6.1. The only fragments based on $\land$, $\lor$, $\rightarrow$ and $\leftrightarrow$ are

\[
\begin{align*}
[A, \rightarrow, \land, \lor, \leftrightarrow], \\
[\land, \rightarrow, \land, \lor, \leftrightarrow] (= [\land, \rightarrow, \lor, \land, \lor, \leftrightarrow]) \text{ and} \\
[\land, \lor, \rightarrow, \land, \lor, \leftrightarrow] (= [\land, \lor, \rightarrow, \land, \lor, \leftrightarrow]),
\end{align*}
\]

for

\[
\begin{align*}
A \leftrightarrow (A \rightarrow B) & \equiv A \land B, \\
(A \land B) \leftrightarrow B & \equiv A \rightarrow B, \\
A \leftrightarrow (A \lor B) & \equiv A \rightarrow B, \\
(A \leftrightarrow B) \leftrightarrow (A \lor B) & \equiv A \land B.
\end{align*}
\]

So $[\leftrightarrow]$ is the only new fragment. We conjecture that interpolation holds, but a proof has not been found. We sketch two approaches.

6.2. The sequent calculus $\mathcal{S}\mathcal{C}(\leftrightarrow)$ for $[\leftrightarrow]$ is defined as the axioms (P), (T) and (⊥), plus the following rules:

\[
\begin{align*}
(\leftrightarrow R) & & (\leftrightarrow L) \\
& \frac{\Gamma, A \vdash B & \Gamma, B \vdash A \quad \Gamma, A \rightarrow B \vdash C} & \quad \frac{\Gamma \vdash A & \Gamma, B \vdash C \quad \Gamma, A \rightarrow B \vdash C} \\
& & \frac{\Gamma \vdash A \rightarrow B \vdash C} & \quad \frac{\Gamma \vdash B & \Gamma, A \rightarrow B \vdash C} \\
& & \frac{\Gamma, A \rightarrow B \vdash C} & \quad \frac{\Gamma, A \leftrightarrow B \vdash C}
\end{align*}
\]
To see that a formula $A \in [\leftrightarrow]$ is derivable in $SC(\leftrightarrow)$ if and only if it is derivable in $SC$ (considering $A \leftrightarrow B$ as an abbreviation of $(A \rightarrow B) \land (B \rightarrow A)$), one argues as follows. Define $SC^+$ as the union of $SC$ and $SC(\leftrightarrow)$, show that the cut rule is a derived rule in $SC^+$, observe that

$$(A \rightarrow B) \land (B \rightarrow A) \vdash A \leftrightarrow B \text{ and } A \leftrightarrow B \vdash (A \rightarrow B) \land (B \rightarrow A)$$

are derivable in $SC^+$ and conclude, for sequents $\Gamma \vdash A$ in $[\leftrightarrow]$: $\Gamma \vdash A$ derivable in $SC(\leftrightarrow)$ if and only if $\Gamma \vdash A$ derivable in $SC^+$ if and only if $\Gamma \vdash A$ derivable in $SC$.

It is not immediately clear how to prove interpolation for $[\leftrightarrow]$ with $SC(\leftrightarrow)$: the 'interpolation rules' to be used in Schütte's method are e.g.

(i-$\leftrightarrow$L1)

$$\frac{\Gamma \vdash [I_1] \Delta \vdash A \quad \Gamma \vdash [I_2] B, \Delta \vdash C}{\Gamma; [I_1 \land I_2] \Delta \vdash A \leftrightarrow B, C}$$

(i-$\leftrightarrow$L2)

$$\frac{\Delta \vdash [I_1] \Gamma \vdash A \quad \Gamma; \Delta \vdash [I_2] B \vdash C}{\Gamma; [I_1 \rightarrow I_2] \vdash A \leftrightarrow B, \Delta \vdash C}$$

so we get interpolants containing $\land$ and $\rightarrow$.

6.3. Another candidate method to prove interpolation for $[\leftrightarrow]$, which works in classical logic (see [Z78]), is: show $A(p) \vdash A(A(\overline{T}))$. Unfortunately, this does not hold for all $A \in [\leftrightarrow]$: to see this, take $A(p) := (p \leftrightarrow q) \leftrightarrow (p \leftrightarrow r)$, then $A(A(\overline{T})) \equiv A(q \leftrightarrow r) \equiv q \leftrightarrow r$, but

$$(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow r) \vdash q \leftrightarrow r \text{ is not derivable;}$$

to see this, take $p := \perp, q := \neg \neg r$.

Despite these unsuccessful attempts, we state the following conjecture.

interpolation holds for $[\leftrightarrow]$.

6.5. Fragments with $\neg \rightarrow$.

We introduce a new connective $\neg$ for double negation. The sequent calculus $SC(\neg)$ is based on sequents $\Gamma \vdash A$ or $\Gamma \vdash \bot$ (think of this last sequent as being equivalent to $\Gamma \vdash \perp$), and contains the axioms and rules of $SC$ (possibly with sequents $\Gamma \vdash \bot$), together with
The cut rule is a derived rule of $\mathbb{SC}(\neg)$, and it is easy to see that

\[ \neg A \vdash \neg \neg A \quad \text{and} \quad \neg \neg A \vdash A \]

are derivable in $\mathbb{SC}(\neg)$. For fragments containing $\neg$ but not $\neg$, the rule $(-R)$ can be skipped, but $(-L)$ is still needed because of $(-R)$. To extend Schütte's method, the following 'interpolation rules' are needed:

\[
\begin{align*}
(i-L1) & \quad \Gamma [I] \Delta \vdash A \\
& \quad \Gamma [I] \neg A, \Delta \vdash

(i-L2) & \quad \Delta [I] \Gamma \vdash A \\
& \quad \Gamma, \neg A \vdash \neg [I] \Delta \vdash

(i-L1) & \quad \Gamma [I] \Delta, A \vdash \\
& \quad \Gamma [I] \Delta, \neg A \vdash

(i-L2) & \quad \Gamma, A [I] \Delta \vdash \\
& \quad \Gamma, \neg A \vdash \neg [I] \Delta \vdash
\end{align*}
\]

Unfortunately, this extension of Schütte's method may (by $i-L2$) introduce $\neg$ in the definition of an interpolant for $A \vdash B$ with $A, B$ in some fragment containing $\neg$, but not $\neg$. Closer inspection learns that $i-L2$ is only needed in fragments containing $\neg$ or $\rightarrow$, so interpolation holds in the rather trivial fragments $[\neg]$, $[\neg, \wedge]$ and $[\neg, \vee]$ and $[\neg, \wedge, \vee]$. For the other fragments, the question arises:

which fragments containing $\neg$ and $\rightarrow$ satisfy interpolation?

6.5. **Uniform interpolation.**

Finally we state the following open problem: does $\mathbb{IPC}$ satisfy uniform interpolation, i.e. are there, for every formula $A$ and every $P \subseteq a(A)$, a uniform right interpolant $I_R = I_R(A, P)$ and a uniform left interpolant $I_L = I_L(A, P)$ such that
A ⊩ I_R, I_L ⊩ A,
a(I_R), a(I_L) ⊆ P,
for all B with A ⊩ B and a(A) ∩ a(B) ⊆ P we have I_R ⊩ B, and
for all B with B ⊩ A and a(A) ∩ a(B) ⊆ P we have B ⊩ I_L.

In classical logic, the left and right variant are equivalent. Uniform interpolation holds for classical
propositional logic, but not for classical predicate logic (see [H63]), and hence not for intuitionistic
predicate logic.
References.

[H63] L. Henkin, An extension of the Craig-Lyndon interpolation theorem, The Journal of Symbolic Logic 28 (1963) 201-216.

[KK71] G. Kreisel & J.L. Krivine, Elements of Mathematical logic (Model Theory) (North-Holland Publishing Company, Amsterdam, 1971).

[KK72] G. Kreisel & J.L. Krivine, Modelltheorie (Springer-Verlag, Berlin, 1972).

[P85] M. Porebska, Interpolation for fragments of intermediate logics, Bulletin of the Section of Logic, Polish Academy of Sciences 14 (1985) 79-83.

[R81] G.R. Renardel de Lavalette, The interpolation theorem in fragments of logics, Indagationes Mathematicae 43 (1981) 71-86.

[R8?] G.R. Renardel de Lavalette, Appendix to A. Visser’s “Evaluation”, to appear.

[S62] K. Schütte, Der Interpolationssatz der intuitionistischen Prädikatenlogik, Mathematische Annalen 148 (1962) 192-200.

[T77] G. Takeuti, Proof Theory (North-Holland Publishing Company, Amsterdam, 1975).

[Z78] J.I. Zucker, Interpolation for fragments of the propositional calculus (preprint ZW 116/78, Mathematisch Centrum, Amsterdam, 1978).