ON ORBITS OF AUTOMORPHISM GROUPS ON HOROSPERICAL VARIETIES

VIKTORIIA BOROVIK, SERGEY GAIFULLIN, AND ANTON SHAFAREVICH

Abstract. In this paper we describe orbits of automorphism group on a horospherical variety in terms of degrees of homogeneous with respect to natural grading locally nilpotent derivations. In case of (may be nonnormal) toric varieties a description of orbits of automorphism group in terms of corresponding weight monoid is obtained.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. If we are given by an affine algebraic variety $X$ we can consider the group of its regular automorphisms $\text{Aut}(X)$. This group naturally acts on $X$. We study orbits of this action.

If the variety $X$ admits an action of an algebraic group $G$, then $\text{Aut}(X)$-orbits are unions of $G$-orbits. So, to describe $\text{Aut}(X)$-orbits we are to obtain a criterium for two $G$-orbits to lie in the same $\text{Aut}(X)$-orbit. This approach is very useful when there are only finite number of $G$-orbits. Often it is more convenient to describe orbits of the neutral component $\text{Aut}(X)^0 \subset \text{Aut}(X)$. Arzhantsev and Bazhov [1] described $\text{Aut}(X)^0$-orbits for normal toric varieties $X$, see also [12]. In this paper we obtain a generalization of this result. We investigate $\text{Aut}(X)^0$-orbits on complexity-zero horospherical varieties $X$. Recall that horospherical variety is an irreducible variety admitting an action of an affine algebraic group such that the stabilizer of a generic point contains a maximal unipotent subgroup in $G$. It is called complexity-zero if $G$-action on $X$ has an open orbit.

The automorphism group of an affine variety usually is not an algebraic group. But we can consider the subgroup $\text{AAut}(X) \subset \text{Aut}(X)$ generated by all algebraic subgroups $H \subset \text{Aut}(X)$. Every algebraic group is generated by subgroups isomorphic to additive and multiplicative group of $\mathbb{K}$. We call such subgroups $\mathbb{G}_a$ and $\mathbb{G}_m$-subgroups respectively. So, $\text{AAut}(X)$ is the subgroup of $\text{Aut}(X)$, generated by all $\mathbb{G}_a$ and $\mathbb{G}_m$-subgroups. Remark that the subgroup $\text{SAut}(X)$ generated by all $\mathbb{G}_a$-subgroups is called the subgroup of special automorphism. In [2] varieties $X$ with transitive action of $\text{SAut}(X)$ on the smooth locus $X^{reg}$ were investigated. Such varieties are called flexible. Arzhantsev, Kujumzhijan and Zaidenberg [3] proved flexibility of normal toric varieties. Boldyrev and Gaifullin [4] give a criterium for a (not nessesary normal) toric variety to be flexible. Shafarevich [11] proved flexibility of horospherical complexity-zero varieties corresponding to a semisimple group $G$. Gaifullin and Shafarevich [8] proved flexibility of normal horospherical
completeness.-zero varieties corresponding to an arbitrary group. So, if we have a normal horospherical complexity-zero variety, all regular points form one $\text{Aut}(X)$-orbit. So in case of normal horospherical variety the goal is to describe singular $\text{Aut}(X)$-orbits.

It is easy to see that $\text{AAut}(X)$ is a subgroup of $\text{Aut}(X)^0$. These groups can be different but we prove that their orbits for horospherical complexity-zero varieties coincide. Then we investigate $\text{AAut}(X)$-orbits. To do this we need two opposite techniques. We need to glue $G$-orbits if they lies in the same $\text{Aut}(X)^0$-orbit and to separate $G$-orbits which are contained in different $\text{Aut}(X)^0$-orbits. Some of $G_a$-orbits we can glue by $T$-normalized $G_a$-actions with respect to the right action of the maximal torus $T \subset G$. We prove that if two $G$-orbits can not be glued by any chain of $T$-normalized $G_a$-actions, then they lies in different $\text{Aut}(X)^0$-orbits.

We obtain a criterium for two $G$-orbits to be contained in one $\text{Aut}(X)^0$-orbit in terms of degrees of $X(T)$-homogeneous locally nilpotent derivations, see Theorem 1. But in arbitrary case the problem of describing these degrees is open. There is completely solved in case of (may be nonnormal) toric varieties. So, in this case we obtain a description of $\text{Aut}(X)^0$-orbits in terms of the weight monoid corresponding to the variety, see Corollary 6.

2. Preliminaries

2.1. Neutral component of automorphism group. Let us define the connected component of $\text{Aut}(X)$ following [10], see also [1].

**Definition 1.** A family $\{\varphi_b, b \in B\}$ of automorphisms of a variety $X$, where the parametrizing set $B$ is an algebraic variety, is an algebraic family if the map $B \times X \to X$ given by $(b, x) \mapsto \varphi_b(x)$ is a morphism.

**Definition 2.** The connected component $\text{Aut}(X)^0$ of the group $\text{Aut}(X)$ is the subgroup of automorphisms that may be included in an algebraic family $\{\varphi_b, b \in B\}$ with an irreducible variety as a base $B$ such that $\varphi_{b_0} = \text{id}_X$ for some $b_0 \in B$.

It is easy to check that $\text{Aut}(X)^0$ is indeed a subgroup, see [10].

If $G$ is an algebraic group and $G \times X \to X$ is a regular action, then we may take $B = G$ and consider the algebraic family $\{\varphi_g, g \in G\}$, where $\varphi_g(x) = gx$. So any automorphism defined by an element of $G$ is included in $\text{Aut}(X)^0$. In particular every $G_a$ and every $G_m$-subgroup is contained in $\text{Aut}(X)^0$. Therefore, $\text{AAut}(X) \subseteq \text{Aut}(X)^0$.

2.2. Cones. Let $M \cong \mathbb{Z}^n$ be a lattice and $P \subset M$ be a finite generated submonoid. Let us consider the following vector space over rational numbers $M_\mathbb{Q} = M \otimes \mathbb{Q}$. The cone in $M_\mathbb{Q}$ spanned by $P$ we denote by $\sigma^\vee = \sigma^\vee(P)$. It is a finitely generated polyhedral cone. The monoid $P$ is called saturated, if $P = \mathbb{Z}P \cap \sigma^\vee$.

Let $N = \hom(M, \mathbb{Z})$ be the dual lattice. Denote by $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ the natural pairing between these lattices. It extends to the pairing $\langle \cdot, \cdot \rangle_\mathbb{Q} : M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}$ between the vector spaces $M_\mathbb{Q}$ and $N_\mathbb{Q} = N \otimes \mathbb{Q}$. Let us denote by $\sigma$ the cone dual to $\sigma^\vee$

$$\sigma = \{u \in N_\mathbb{Q} \mid \langle v, u \rangle \geq 0 \text{ for all } v \in \sigma^\vee\}.$$ 

There is a natural bijection between $k$-dimensional faces of $\sigma$ and $(n - k)$-dimensional faces of $\sigma^\vee$. A face $\tau \subseteq \sigma$ corresponds to the face $\widehat{\tau} = \sigma^\vee \cap (\tau)^\perp$. 


A cone is called pointed if \( \sigma \cap (-\sigma) = \{0\} \). The cone \( \sigma \) is pointed if and only if the cone \( \sigma^\vee \) is of full dimension, i.e. the linear shell \( \langle \sigma^\vee \rangle = M_\Omega \). If \( \sigma^\vee \) is not of full dimension, we replace \( M_\Omega \) by the group generated by \( P \). So, further we assume that \( \sigma \) is pointed.

Let us denote the set of rays of the cone \( \sigma \) by \( \sigma(1) \) and let \( p_\rho \) be the primitive lattice vector on a ray \( \rho \).

**Definition 3.** Let
\[
R_\rho := \{ e \in M \mid \langle e, p_\rho \rangle = -1, \langle e, p_{\rho'} \rangle \geq 0 \; \forall \rho' \neq \rho \in \sigma(1) \}.
\]

Then the elements of the set \( \mathfrak{R} := \bigsqcup_\rho R_\rho \) are called the Demazure roots of the cone \( \sigma \).

We will call ray \( \rho \) the distinguished ray of the Demazure root \( e \) if \( e \in R_\rho \).

**Definition 4.** Let \( \tau \) be a face of \( \sigma^\vee \). Let \( p_1, \ldots, p_k \) be all rays normal to \( \tau \) and \( p_1, \ldots, p_k \) be their primitive vectors. Suppose \( e \) is a Demazure root such that \( \langle e, p_1 \rangle = -1 \) and \( \langle e, p_i \rangle = 0 \) for all \( 2 \leq i \leq k \). Then we say that \( e \) is a \( \tau \)-root.

Let us give some definition according to [14].

**Definition 5.** An element \( p \) of the monoid \( P \) is called saturation point of \( P \), if the moved cone \( p + \sigma^\vee \) has no holes, i.e. \( (p + \sigma^\vee) \cap M \subset P \).

A face \( \tau \) of the cone \( \sigma^\vee \) is called almost saturated, if there is a saturation point of \( P \) in \( \tau \). Otherwise \( \tau \) is called a nowhere saturated face.

The following lemma is known, see for example [4, Lemma 2].

**Lemma 1.** The maximal face, i.e. the whole cone \( \sigma^\vee \), is almost saturated.

### 2.3. Horospherical varieties.

We recall some results on horospherical varieties, all proofs can be found in [9], see also [13].

Let \( G \) be a connected linear algebraic group.

**Definition 6.** An irreducible \( G \)-variety \( X \) is called horospherical, if for a generic point \( x \in X \) the stabilizer of \( x \) contains a maximal unipotent subgroup \( U \subset G \).

If \( X \) contains an open \( G \)-orbit, then \( X \) is called complexity-zero horospherical. In [9] affine complexity-zero horospherical varieties are called \( S \)-varieties.

Suppose that \( X \) is an affine complexity-zero horospherical variety. It is easy to see that the unipotent radical of \( G \) acts trivially on \( X \). Hence we may assume that \( G \) is reductive. Taking a finite covering, we may assume that \( G = T \times G' \), where \( T \) is an algebraic torus and \( G' \) is a semisimple group.

Let \( O \) be the open orbit in \( X \). We have the following sequence of inclusions
\[
\mathbb{K}[X] \hookrightarrow \mathbb{K}[O] \hookrightarrow \mathbb{K}[G].
\]

Let \( B \) be a Borel subgroup of \( G \) and let \( M = \mathfrak{X}(B) \) be the group of characters of \( B \). For an \( \Lambda \in M \) we put
\[
S_\Lambda = \{ f \in \mathbb{K}[G] \mid f(gb) = \Lambda(b)f(g) \text{ for all } g \in G, b \in B \}.
\]

Then
\[
S_\Lambda S_{\Lambda'} = S_{\Lambda + \Lambda'}.
\]
The set $X^+(B)$ of dominant weights consists of all $\Lambda$ such that $S_\Lambda \neq \{0\}$. It is proved in [9] that for an affine complexity-zero horospherical $G$-variety $X$ there is a decomposition

$$\mathbb{K}[X] = \bigoplus_{\Lambda \in P} S_\Lambda$$

for some submonoid $P \in X^+(B)$.

Using notations from the previous section, we denote by $\sigma^\vee$ the cone in $M_\mathbb{Q}$ spanned by $P$. The variety $X$ is normal if and only if $P$ is saturated. There is a one-to-one correspondence between faces of $\sigma$ and $G$-orbits of $X$. More precisely, if $O_\tau \subseteq X$ is the $G$-orbit in $X$ corresponding to a face $\tau$ of the cone $\sigma$, then the ideal of functions vanishing on $O_\tau$ has the form

$$I(O_\tau) = \bigoplus_{\Lambda \in P \setminus \tau} S_\Lambda.$$

This ideal vanishes on the closure $\overline{O_\tau}$. Then

$$O_\tau = \overline{O_\tau} \setminus \left( \bigcup_{\gamma \prec \tau} O_\gamma \right).$$

If $\hat{\xi} \preceq \sigma^\vee$ is the face corresponding to $\xi \preceq \sigma$, we use both denotations: $O_\xi = O_{\hat{\xi}}$.

**Remark 1.** The cone $\sigma^\vee$ can be not of full dimension. But if we replace $M$ by the group generated by $P$, the correspondence between faces and orbits remains.

To obtain a variety $X$ explicitly one should consider generators $\Lambda_1, \ldots, \Lambda_m$ of $P$ and consider the sum of irreducible $G$-representation which are contragradient to ones with highest weights $\Lambda_1, \ldots, \Lambda_m$. In each $V(\Lambda_i)^*$ one need to find the eigenvector $v_i$. Put $v = v_1 + \ldots + v_m$. Then $X \cong \overline{Gv}$. If $m = 1$, then the variety $X$ is the closure of the eigenvector of an irreducible representation. Such varieties are called $HV$-varieties.

An important particular case of horospherical varieties give toric varieties. More information on toric varieties one can find in [5] and [7].

**Definition 7.** A toric variety is a variety $X$ admitting an action of an algebraic torus $T \simeq (\mathbb{K}^\times)^n$ with open orbit.

**Remark 2.** Often by toric variety one mean a normal toric variety. We do not a-priori assume a toric variety to be normal.

So, toric variety is a horospherical complexity-zero variety corresponding to $G \cong (\mathbb{K}^\times)^n$. For a toric variety each nonzero homogeneous component has dimension one. We have

$$A = \bigoplus_{m \in P} \mathbb{K}\chi^m,$$

where $\chi^m = t_1^{m_1} \cdots t_n^{m_n}$ is the character of the torus $T$ corresponding to a point $m = (m_1, \ldots, m_n)$.

**2.4. Derivations.** We recall basic facts from theory of locally nilpotent derivations, see for example [6].

Let $A$ be a commutative associative algebra over $\mathbb{K}$. 
Definition 8. A linear mapping $\partial : A \to A$ is called a derivation if it satisfies the Leibniz rule: $\partial(ab) = a\partial(b) + b\partial(a)$.

A derivation is called locally nilpotent or LND if for any $a \in A$ there is $n \in \mathbb{N}$ such that $\partial^n(a) = 0$.

A derivation is called semisimple if there exists a basis of $A$ consisting of $\partial$-semi-invariants (i.e. $\partial(a) = \lambda a$ for $a \in A$).

Definition 9. A derivation $\partial : A \to A$ is called locally bounded if any element $a \in A$ is contained in a $\partial$-invariant finite-dimensional linear subspace $V \subset A$.

Remark 3. One can see that all semisimple and locally nilpotent derivations are locally bounded.

Exponential mapping gives a correspondence between LND and $\mathbb{G}_a$-subgroups in $\text{Aut}(A)$, and between semisimple derivations and $\mathbb{G}_m$-subgroups in $\text{Aut}(A)$. A derivation $\delta$ corresponds to the subgroup $\{\exp(t\delta)\}$, where $t \in \mathbb{K}$ for $\mathbb{G}_a$-subgroup and $t \in \mathbb{K}^\times$ for $\mathbb{G}_m$-subgroup.

Let $F$ be an abelian group. Consider $F$-grading: $A = \bigoplus_{f \in F} A_f$, $A_f A_g \subseteq A_{f+g}$.

Definition 10. A derivation $\partial : A \to A$ is called $F$-homogeneous of degree $f_0 \in F$ if for all $a \in A_f$ we have $\partial(a) \in A_{f+f_0}$.

Lemma 2. Let $\partial$ be a derivation of $A$. Then $\partial = \sum_{i=1}^{k} \partial_i$, where $\partial_i$ is the homogeneous derivation of degree $i$.

Lemma 3. Let $\partial = \sum_{i=1}^{k} \partial_i$ be a derivation of $A$. Then:

1. If $\partial$ is LND then $\partial_l$ and $\partial_k$ are LNDs.
2. If $\partial$ is locally bounded then if $l \neq 0$, $\partial_l$ is LND, and if $k \neq 0$, $\partial_k$ is LND.

Corollary 1. If $A$ admits an LND, then $A$ admits a $\mathbb{Z}$-homogeneous LND.

Now let $A$ be a finitely generated $\mathbb{Z}$-graded algebra. The following lemmas are known.

Lemma 4. Let $\partial$ be a $\mathbb{Z}$-homogeneous of degree $d \neq 0$ locally bounded derivation. Suppose $\mathbb{Z}$ is included as a subgroup to $\mathbb{Z}^n$. Then among $\mathbb{Z}^n$-homogeneous summands of $\partial$ there is an LND $\delta \neq 0$. And $\mathbb{Z}$-degree of $\delta$ equals to $d$.

Now let $X$ be an affine toric variety. Fix a Demazure root $e \in \mathfrak{R}_\rho$. One can define the following $M$-homogeneous LND $\partial_e$ on the algebra $A = \mathbb{K}[X]$ by the rule $\partial_e(\chi^m) = \langle p_\rho, m \rangle \chi^{e+m}$.

Any homogeneous LND on $A = \mathbb{K}[X]$ has the form $\lambda \partial_e$ for some $\lambda \in \mathbb{K}, e \in \mathfrak{R}$. 
3. Varieties with finite numbers of $G$-orbit

Let $X$ be an irreducible affine variety. Suppose a connected linear algebraic group $G$ acts on $X$ with finite number of $G$-orbits. Then the image of $G$ in $\text{Aut}(X)$ is contained in $\text{AAut}(X) \subset \text{Aut}(X)^0$. Therefore, there is only finite number $\text{AAut}(X)$-orbits on $X$. It is easy to see that $\text{AAut}(X)$ is a normal subgroup of $\text{Aut}(X)$. Therefore, each automorphism permutes $\text{AAut}(X)$-orbits. Hence, the connected group $\text{Aut}(X)^0$ preserves each $\text{AAut}(X)$-orbit. That is $\text{Aut}(X)^0$-orbits coincides with $\text{AAut}(X)$-orbits.

Consider a $G$-orbit $Z$. Denote $\Omega = \{O \text{ is a } G-\text{orbit } | Z \subseteq O \text{ and the closure } O \text{ is } \text{AAut}(X)\text{-invariant}\}$.

**Lemma 5.** There is a $G$-orbit $\Phi(Z) \in \Omega$ such that $\Phi(Z) = \bigcap_{O \in \Omega} O$.

**Proof.** Indeed, the set $\bigcap_{O \in \Omega} O$ is closed $G$-invariant set so it is a finite union of closures of $G$-orbits.

$$\bigcap_{O \in \Omega} O = \overline{O_1} \cup \ldots \cup \overline{O_k}.$$  

In the same time $\text{AAut}(X)$ is a connected group. So each $\overline{O_i}$ is $\text{AAut}(X)$-invariant. But $Z$ is irreducible. So there is a number $j$ such that $Z \subseteq \overline{O_j}$. Then $O_j \in \Omega$ and $\Phi(Z) = O_j$. \hfill $\Box$

**Proposition 1.** Let $Y$ be $\text{AAut}(X)$-orbit such that $Z \subseteq Y$. Then $Y$ contains $\Phi(Z)$.

**Proof.** Otherwise $\text{AAut}(X)$-orbit containing $Z$ is contained in the set $\overline{\Phi(Z)} \setminus \Phi(Z)$. Then $Y$ is a union of $G$-orbits $O_1 \cup \ldots \cup O_k$ where $O_j \subseteq \overline{\Phi(Z)}$ for all $j$. Since the group $\text{AAut}(X)$ is connected, each $O_j$ is $\text{AAut}(X)$-invariant. This implies that $\overline{O_j}$ is $\text{AAut}(X)$-invariant for each $j$. But $Z$ is irreducible so there is a $j$ such that $Z \subseteq \overline{O_j}$. But this contradicts to the definition of $\Phi(Z)$. \hfill $\Box$

Summerizing the precious results we obtain the following corollary.

**Corollary 2.** Let $Y$ be $\text{Aut}(X)^0$-orbit containing $Z$. Denote $$S = \{O \subset \overline{\Phi(Z)} | Z \text{ is not contained in } \overline{\Phi(O)}\}.$$  

Then $Y = \overline{\Phi(Z)} \setminus \bigcup_{O \in S} \overline{O}$.

Note that $S = \{O \subset \overline{\Phi(Z)} | \Phi(O) \neq \Phi(Z)\}$. Therefore, we obtain the following assertion.

**Corollary 3.** Let $O_1$ and $O_2$ be $G$-orbits. Then $\text{Aut}^0(X)$-orbits containing $O_1$ and $O_2$ coincide if and only if $\Phi(O_1) = \Phi(O_2)$.

We assume that we know adjunction of $G$-orbits. Results of this section show that to obtain description of $\text{Aut}^0(X)$-orbits it is sufficient to determine which $G$-orbits have $\text{AAut}(X)$-invariant closures.
4. Automorphisms of a variety with torus action

Let $X$ be an affine algebraic variety admitting an effective $T \cong (\mathbb{K}^{	imes})^n$-action. The group of characters $M = \hat{X}(T)$ is isomorphic to $\mathbb{Z}^n$. The $T$-action corresponds to an $M$-grading on $A = \mathbb{K}[X]$. Let $P$ be the weight monoid of this action i.e. $P = \{ m \in M \mid A_m \neq \{0\} \}$. We use notations from Section 2.2.

**Definition 11.** Let $\tau$ be a face of $\sigma^\vee$. An element $v \in N$ is called $\tau$-bordering if the following conditions are satisfied

- $\langle \omega, v \rangle > 0$ for all $\omega \in P \setminus \tau$.
- if $\partial$ is a nonzero $M$-homogeneous LND of $A$ of degree $e$, then $\langle e, v \rangle \geq 0$.

Let us consider the ideal $I_\tau = \bigoplus_{\omega \in P \setminus \tau} A_\omega$ of $A$. Denote $Z_\tau = \bigcap I_\tau$.

**Proposition 2.** If a face $\tau$ of $\sigma^\vee$ admits a $\tau$-bordering element $v \in N$, then the set $Z_\tau$ is $\AAut(X)$-invariant.

**Proof.** Let us consider the following $\mathbb{Z}$-grading on $A$:

$$A = \bigoplus_{i \in \mathbb{Z}} A_i, \text{ where } A_i = \bigoplus_{\langle \omega, v \rangle = i} A_\omega.$$  

Let $\partial$ be a locally bounded derivation. By Lemma 2 we have $\partial = \sum_{i=1}^k \partial_i$, where $\partial_i$ is homogeneous under $\mathbb{Z}$-grading of degree $i$. If $l < 0$ by Lemma 3 we obtain that $\partial_l$ is LND. Then we can decompose $\partial_l$ into the sum of $M$-homogeneous LNDs:

$$\partial_l = \sum_j \partial_{lj},$$

where $\mathbb{Z}$-degree of $\partial_{lj}$ is equal to $l$. By Lemma 4 there exists an $M$-homogeneous LND $\partial_j$. We obtain a contradiction. Therefore, $l \geq 0$.

If $f \in I$, then it can be decomposed into the sum of $\mathbb{Z}$-homogeneous elements of positive degree $f = f_1 + \ldots + f_r$. Therefore,

$$\partial(f) = \sum_{i \geq 0} \sum_{j \geq l \geq 0} \partial_j(f_i) \in \bigoplus_{p > 0} A_p = I.$$

So, the ideal $I$ is $\partial$-invariant for any semisimple and for any locally nilpotent derivation. This implies that $Z_\tau$ is $\AAut(X)$-invariant.

Let $\rho_1, \ldots, \rho_k$ be the rays of $\sigma$. And let $p_1$ be the primitive vectors on $\rho_i$. Now we suppose that the $M$-grading has the following property:

$$(1) \quad A_\alpha \cdot A_\beta = A_{\alpha + \beta} \quad \text{for all } \alpha, \beta \in P.$$  

The former equality means $A_{\alpha + \beta} = \langle fg \mid f \in A_\alpha, g \in A_\beta \rangle$.

**Lemma 6.** Let $e \in M$ is the degree of a $M$-homogeneous LND $\partial$ such that $e \notin \sigma^\vee$. Then $e$ is a Demazure root of $\sigma$.

**Proof.** If $\omega \in P$ is such an element that $\omega + e \notin P$. Then $\partial(A_\omega) = 0$.

Suppose $\langle e, p_i \rangle = -d \leq -2$. Then there is $\bar{\omega} \in P$ which is an inner point of $\sigma^\vee$ such that $A_{\bar{\omega}} \in \Ker \partial$. Indeed, let us take a saturation point $u$ of $\sigma^\vee$. Let us take an element $v \in u + \sigma^\vee \cap M \subset P$ such that $v$ is an inner point of $\sigma^\vee$ and $\langle v, p_i \rangle$ is not divisible by $d$. Consider the sequence of points $v, v + e, v + 2e, \ldots$ This sequence
leave $\sigma^\vee$. Hence there exists minimal $k$ such that $v + ke \notin P$. Then $v + (k - 1)e$ is an inner point of $\sigma^\vee$ such that $A_{v+(k-1)e} \in \text{Ker } \partial$.

Then there is $l \in \mathbb{N}$ such that $\hat{\omega} - u - \beta \in \sigma^\vee$. Hence $\hat{\omega} - \beta \in u + \sigma^\vee \cap P$. Therefore, there is $\alpha \in P$ such that $A_\beta A_\alpha = A_{\alpha+\beta} = A_{\hat{\omega}} = A_{\hat{\omega}} I \subset \text{Ker } \partial$. By the kernel of any LND is factorially closed. Therefore $A_\beta \subset \text{Ker } \partial$. So, $\partial = 0$.

Suppose $\langle e, p_i \rangle \leq -1$, $\langle e, p_j \rangle \leq -1$. Then each $\alpha \in \langle p_i \rangle \cap P$ and each $\beta \in \langle p_j \rangle \cap P$ are in the kernel of $\partial$. There exist such $\alpha$ and $\beta$ that $\hat{\omega} = \alpha + \beta \in \text{Ker } \partial$ is an inner point of $\sigma^\vee$. Then we again obtain a contradiction.

Thus, there exists unique $i$ such that $\langle e, p_i \rangle \leq -1$ and for all other $j$ we have $\langle e, p_j \rangle \geq 0$. 

**Proposition 3.** Let $\tau$ be a face of $\sigma^\vee$. The subset $Z_\tau$ is not $\text{AAut}(X)$-invariant if and only if there is a nonzero $M$-homogeneous LND with degree $e$, such that $e$ is a $\tau$-root.

**Proof.** Suppose $Z_\tau$ is not $\text{AAut}(X)$-invariant. Let us consider $p = \sum_{p_k \perp \tau} p_k$. By Lemma 2 $p$ is not a $\tau$-bordering element. Hence, there exists a nonzero $M$-homogeneous LND of degree $e$ such that $\langle e, p \rangle < 0$. By Lemma 6 the element $e$ is a Demazure root. Therefore, there is $i$ such that $\langle e, p_i \rangle = -1$ and

$$0 > \langle e, p \rangle = \sum_{p_k \perp \tau} \langle e, p_k \rangle = -1 + \sum_{j \neq i, p_j \perp \tau} \langle e, p_j \rangle,$$

where $\langle e, p_j \rangle \geq 0$ for all $j \neq i$. Hence, $\langle e, p_j \rangle = 0$ for all $j \neq i$.

Now suppose there exists a nonzero $M$-homogeneous LND $\partial$ with degree $e$, where $e$ is a $\tau$-root. Then there is $i$ such that $p_i \perp \tau$ and $\langle e, p_i \rangle = -1$. And for all $i \neq j$ such that $p_j \perp \tau$ we have $\langle e, p_j \rangle = 0$. Denote $\gamma = \langle p_i \rangle \cap \sigma^\vee$. Then for every $u \in \gamma$ we have $\partial(A_u) = \{0\}$. Let $q_1, \ldots, q_s$ be primitive vectors on all rays of $\sigma$ which are not normal to $\tau$. Denote $c_r = \langle e, q_r \rangle$, $1 \leq r \leq s$. There exists $\omega \in P \cap \tau$ such that $\langle \omega, q_r \rangle > c_r$ for all $r$. Therefore, $\omega - e \in P$. Note that $A_\omega \in I_\tau$. If there is $f \in A_\omega$ such that $\partial(f) \neq 0$, then $\partial(f) \notin I_\tau$. Hence, $I_\tau$ is not $\partial$-invariant. That is $Z_\tau$ is not $\text{AAut}(X)$-invariant. Assume $A_\omega \in \text{Ker } \partial$. Then there exists $u \in \gamma$ such that $u + \omega$ is an interior element of $\sigma^\vee$. As in Lemma 9 we obtain $\partial = 0$. 

**Remark 4.** In situation of the proof of Proposition 3 denote $\xi_i = \sigma^\vee \cap \langle p_j \rangle \neq i \perp$. In the proof of Proposition 3 we see that if $Z_\tau$ is not $\text{AAut}$-invariant, then for some $i$ there is $f \in A_\omega$, $\omega \in \xi_i$ such that $\partial(f) \notin I_\tau$. It is easy to see that we can assume $\omega$ to be an interior vector of $\xi_i$. Therefore, if $\partial(f) \in I_\xi$ for some face $\zeta \leq \sigma^\vee$, then $\xi_i$ is contained in $\zeta$. Hence, for some $x \in O_\tau$ and for some $t \in \mathbb{K}$ the point $\exp(t\partial)(x) \in O_\xi$ for some face $\zeta$ containing $\xi_i$.

5. Automorphism orbit on horospherical varieties

Let $X$ be a horospherical complexity-zero variety. In Section 2 we introduce a grading on $A = \mathbb{K}[X]$ and describe $G$-orbits on $X$. So, we can apply to the variety $X$ results of Sections 4 and 5. Combining these results we obtain the following theorem.

**Theorem 1.** Let $X$ be a horospherical complexity-zero variety. Each closure of $\text{Aut}(X)^0$-orbit has the form $\overline{O_\tau}$, where $\tau$ is a face of $\sigma^\vee$ such that there is no any nonzero $M$-homogeneous LND with degree equals a $\tau$-root.
The following corollary follows from this theorem and Remark 3.

**Corollary 4.** Let $H$ be the subgroup generated by $G$ and all exponents of $M$-homogeneous LNDs. Then $\text{Aut}(X)^0$-orbits on $X$ coincides with $H$-orbits.

If we are given by a face $\tau$ of $\sigma^\vee$, it is an easy question if $\tau$ admits a $\tau$-root. But the problem is that for a given $\tau$-root $e$ not always there exists an $M$-homogeneous LND with degree $e$.

**Example 1.** Let $G = \text{SL}_3$ and $P = \mathcal{X}^+(B)$. Then the cone $\sigma^\vee$ in basis of fundamental weights is cone$(e_1,e_2)$. Each face admits an $\tau$-root. For $\tau$ equals the origin, there are two $\tau$-roots: $(-1,0)$ and $(0,-1)$. It is easy to compute that $X \cong V(x_1y_1 + x_2y_2 + x_3y_3) \subset \mathbb{K}^6$. The orbit, corresponding to $\tau$ is the point $q = (0,0,0,0,0,0)$. It is easy to see, that there are no LND with degrees $\tau$-roots. And the point $q$ is $\text{Aut}(X)$-stable since it is the unique singular point.

**Example 2.** We have the similar situation in case of HV-varieties. In this case the cone $\sigma^\vee$ is a ray. Therefore, for $\tau$ equals the origin there is a $\tau$-root. But for all HV-varieties except affine space, $\tau$ corresponds to the unique singular point.

The question for which $\tau$ there exists a $M$-homogeneous LND with the degree equals to a $\tau$-root Let us formulate a conjecture about answer to this question.

**Conjecture 1.** Let $\tau$ be a face of $\sigma^\vee$. Let $p_1,\ldots, p_k$ be all primitive vectors on rays $\rho_1,\ldots, \rho_k$ of $\sigma$ that are normal to $\tau$. For $1 \leq i \leq k$ we denote

$$\xi_i = \sigma^\vee \cap \langle p_1,\ldots, p_{i-1}, p_{i+1}, \ldots, p_k \rangle^\perp.$$ 

The closure of $G$-orbit $O_\tau$ is not $\text{AAut}(X)$-invariant if and only if the following conditions occur

1. there exists $1 \leq a \leq k$ and a $\tau$-root with distinguished ray $\rho_a$.
2. points of $O_\tau$ and $O_\xi$ have equal dimensions of tangent spaces $T_x X$.

**Proof of necessity.** By Proposition 3 if the set $Z_\tau = \overline{O_\tau}$ is not $\text{AAut}(X)$-invariant, then there is a nontrivial $M$-homogeneous LND $\partial$ with the degree $e$, where $e$ is a $\tau$-root. Then the condition (1) occurs. Remark 3 implies that for some $x \in \overline{O_\tau}$ and for some $t \in \mathbb{K}$ the point $\exp(t\partial)(x) \in O_\zeta$ for some face $\zeta$ containing $\xi_a$. Since $O_\tau$ is open in $\overline{O_\tau}$, we can assume $x \in O_\tau$. Therefore, points of $O_\tau$ and $O_\zeta$ have equal dimensions of tangent spaces. Since $O_\tau \subset O_{\xi_a} \subset O_\zeta$, if $x \in O_\tau$, $y \in O_{\xi_a}$, $z \in O_\zeta$, then $\dim T_x X \geq \dim T_y X \geq \dim T_z X$. But $\dim T_x X = \dim T_z X$. Therefore, $\dim T_x X = \dim T_y X$, i.e. condition (2) holds.

But for subclass of toric varieties the answer is known. If $X$ is a normal toric variety, then each Demazure root is the degree of a unique up to multiplicative constant nonzero LND. Therefore, we obtain the following corollary of Theorem 1.

**Corollary 5.** Let $X$ be a normal toric variety. Each closure of $\text{Aut}(X)^0$-orbit has the form $\overline{O_\tau}$, where $\tau$ is a face of $\sigma^\vee$ such that there is no any $\tau$-roots.

This result follows from results of [1] and [12].

Now let $X$ be a nonnormal toric variety. Let $e$ be a Demazure root of $\sigma^\vee$. It is proved in [4][Section 4] that there exists a nonzero $M$-homogeneous LND with degree $e$ if and only if $(e+P) \cap \sigma^\vee \subset P$. Let us call such Demazure roots admissible. Then we obtain the following corollary of Theorem 1.
Corollary 6. Let $X$ be a (may be nonnormal) toric variety. Each closure of \( \text{Aut}(X)^0 \)-orbit has the form $\mathcal{O}_\tau$, where $\tau$ is a face of $\sigma'$ such that there is no any admissible $\tau$-roots.

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