A SINGLE FINITE-TIME SYNCHRONIZATION SCHEME OF
TIME-DELAY CHAOTIC SYSTEM WITH EXTERNAL
PERIODIC DISTURBANCE

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Abstract. In this paper, dynamical behaviors of three-dimensional chaotic system with time-delay and external periodic disturbance are investigated. When the periodic perturbation term and time-delay are added, the system presents more abundant dynamic behaviors, which can be switched between periodic state and chaotic state. Based on Lyapunov stability theory, a sufficient condition for finite-time synchronization is given. A single controller is proposed to realize finite-time synchronization of time-delay chaotic system with external periodic disturbance. The addressed scheme is provided in the form of linear inequality which is simple and easy to be realized. At the same time, it also displays that when delay term \( \tau \) takes different values, the time of synchronization shows certain difference. The feasibility and effectiveness of the finite-time synchronization method is verified by theoretical analysis and numerical simulation.

1. Introduction. In recent years, chaotic system has attracted much attention because of its wide use in various fields and great progress has been achieved in the research of nonlinear dynamics. In 1963, Lorenz found the first chaotic system [1] when he studied atmospheric convection. Since then, chaos and hyperchaos generation have been extensively studied due to its theoretical and practical applications, such as secure communications, biological engineering, chemical processing, neural work, mathematics [2]-[7], and the references therein. The dynamic behavior of chaotic system is sensitive to initial values. When the parameters are fixed, the dynamics of the system can be switched between stable state, periodic state and chaotic state by selecting appropriate initial value, see Ref [6]. Based on the chaotic dynamics of artificial neural networks, a novel model for human memory was presented [7]. An auto-switched chaotic system was proposed and its FPGA implementation was verified [8].

Synchronization is the complete reconstruction of the chaotic state of the two systems. In 1990, Pecora and Carroll [9] introduced a method to synchronize two

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identical chaotic systems with different initial conditions. Chaos synchronization was studied in neural work, secure communication, information processing and so on [10]-[12]. Recently, synchronization has been applied into complex dynamical networks [13]-[14]. A novel method for synchronization of fractional order complex dynamical networks using a new fractional Proportional-Integral pinning control scheme was presented in [15]. The synchronization of chaotic systems has been one of the most interesting topics in nonlinear science, meanwhile, many theoretical and experimental results have been achieved. Up to now, a variety of synchronization methods have been presented in dynamical systems such as projective synchronization, lag synchronization, impulsive synchronization, adaptive synchronization, phase synchronization, complete synchronization [16]-[25]. For example, the alternating between complete synchronization and hybrid synchronization of hyperchaotic Lorenz system with time delay was discussed in [18]. Zhang, et al. [20] considered the lag projective synchronization of fractional-order delayed chaotic systems. Chaotic bursting lag synchronization of Hindmarsh-Rose system via a single controller was discussed in [21]. Adaptive anti-synchronization and adaptive feedback control and synchronization of non-identical chaotic were investigated respectively in [23] and [24].

Most of these results about chaos synchronization are derived based on the asymptotic stability of the chaotic systems. It means that the chaos synchronization can only be realized with infinite settling time. From the point of practical application, synchronization time is expected as short as possible. As time goes on, people began to realize the importance of synchronization time. To improve the speed of convergence, many effective methods have been discussed and finite-time synchronization is addressed. Moreover, research shows that the finite-time synchronization technique has demonstrated better robustness and disturbance rejection properties than that of asymptotic methods. For example, Mei et al. [26] investigated finite-time synchronization of complex dynamical networks with time-delay. Finite-time chaos synchronization of unified chaotic system with uncertain parameters was studied [27]. Finite-time synchronization of cyclic switched complex networks under feedback control is discussed [28]. It is well known that external disturbance always occurs in the system, it is difficult to know the parameters of a chaotic system precisely. Therefore, the synchronization of chaotic system in the case of external disturbances and parameter uncertainties is important in practical applications.

Simultaneously, in chaotic systems, time-delay often exists in physics, biology, economy, and other disciplines [29]-[31]. The dynamical behaviors of chaotic system with time-delay have been widely studied, especially the synchronization and stability. Accordingly, many schemes have been proposed to realize the chaos synchronization of time-delay with external disturbance. However, external disturbance and time-delay often coexist in many chaotic systems, which causes more complex dynamic behaviors and it is more difficult to synchronize. As far as the authors know, there are few results in the sense of synchronization of a chaotic system with time-delay and external disturbance.

Motivated by above discussion, the dynamic behaviors of time-delay Lü system with periodic perturbation are investigated in this paper. Furthermore, a finite-time synchronization method is proposed to realize the synchronization of the proposed system. The rest of the paper is arranged as follows. In Section 2, definition of synchronization and some relative lemmas are presented. In Section 3, some
dynamical behaviors of time-delay chaotic system with external disturbance are discussed. In Section 4, based on Lyapunov stability theory, a systematic design procedure for the finite-time synchronization of the chaotic system is proposed, numerical simulations are demonstrated to verify the theoretical results. Finally, some conclusions and future investigation directions are given in Section 5.

2. Preliminaries. Finite-time synchronization means that the state of the slave system can be synchronized the state of the master system in finite time. To prove the finite-time stability of the error system, some relative definitions and lemmas are given in this section. The definition of finite-time synchronization is described as follows.

**Definition 1.** [32] Two chaotic systems

\[ \dot{x} = F(t, x), \]
\[ \dot{y} = G(t, y) + U, \]

are considered, where \( x, y \) are two \( n \)-dimensional state vectors, \( t \) is time, \( F : \mathbb{R}^n \to \mathbb{R}^n \) and \( G : \mathbb{R}^n \to \mathbb{R}^n \) are nonlinear functions of the system, respectively, \( U \) is a synchronous controller. If there is a constant \( t_0 > 0 \), such that

\[ \lim_{t \to t_0} \|y - x\| = 0, \]

along with \( \|y - x\| = 0 \) for \( t \geq t_0 \), then synchronization of two chaotic systems is realized in a finite-time.

**Lemma 2.1.** [33] Assume that a continuous, positive-definite function \( V(t) \) satisfies the following differential inequality:

\[ \dot{V}(t) \leq -kV^\beta(t), \quad \forall t \geq t_0, \quad V(t_0) \geq 0, \]  \( \tag{1} \)

where \( k > 0 \), \( 0 < \beta < 1 \) are constants. Then, for any given \( t_0 \), \( V(t) \) satisfies inequality

\[ V^{1-\beta}(t) \leq V^{1-\beta}(t_0) - k(1 - \beta)(t - t_0), \quad t_0 \leq t \leq t_1, \]  \( \tag{2} \)

and

\[ V(t) \equiv 0, \quad \forall t \geq t_1, \]  \( \tag{3} \)

with \( t_1 \) given by

\[ t_1 = t_0 + \frac{V^{1-\beta}(t_0)}{k(1 - \beta)}. \]  \( \tag{4} \)

**Proof.** Considering differential equation

\[ \dot{X}(t) = -kX^\beta(t), \quad X(t_0) = V(t_0), \]  \( \tag{5} \)

although it is not suitable for the global Lipschitz condition, the unique solution of Eq.(5) can be solved as

\[ X^{1-\beta}(t) = X^{1-\beta}(t_0) - k(1 - \beta)(t - t_0), \]  \( \tag{6} \)

From the comparison Lemma 2.1, we can obtain

\[ V^{1-\beta}(t) \leq V^{1-\beta}(t_0) - k(1 - \beta)(t - t_0), \quad t_0 \leq t \leq t_1, \]  \( \tag{7} \)

and

\[ V(t) \equiv 0, \quad \forall t \geq t_1 \]

with \( t_1 \) given in (7). \( \square \)
Lemma 2.2. [34] For any real numbers $x_1$ and $x_2$, if $\alpha \geq \left(\frac{3}{4}\right)^{\frac{1}{2}}$, then
\[ |x_1x_2| \leq (\alpha|x_1| + \frac{1}{2}x_2^2)^{\frac{1}{2}} \] (8)
holds.

3. **System description.** The investigation is carried on the Lü model[35], which are described as
\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1), \\
\dot{x}_2 &= cx_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3,
\end{align*}
\] (9)
where $a, b, c$ are system parameters, $x = (x_1, x_2, x_3)^T \in R^{3\times 1}$ is the state vector of system (9). By setting different parameters, the rich dynamical behaviors of system (9) are plotted in Figs 1-4, which confirmed that chaotic attractors, limit circle, stable state can be generated under appropriate parameters even the initial values keep the same.

**Figure 1.** Phase trajectory and the time series of Eq.(9) with $a = 15, b = 3, c = 7$ (a) $(x_1, x_2)$ (b) $(t, x_1)$

**Figure 2.** Phase trajectory and the time series of Eq.(9) with $a = 15, b = 0.91, c = 7$ (a) $(x_1, x_2)$ (b) $(t, x_1)$
Finite-time synchronization scheme

Periodic disturbance often exists in electrical and biochemical systems. To consider the effect of periodic disturbance on the dynamic behavior of the system, periodic perturbation term is added to the second equation of (9), and it can be rewritten as follows

\[
\begin{align*}
    \dot{x}_1 &= a(x_2 - x_1), \\
    \dot{x}_2 &= cx_2 - x_1x_3 + Ax_1 \cos \omega t, \\
    \dot{x}_3 &= x_1x_2 - bx_3,
\end{align*}
\]

where \( x = (x_1, x_2, x_3)^T \in R^{3 \times 1} \) is the state vector of system (10), \( Ax_1 \cos \omega t \) indicates the periodic disturbance. When the parameters are selected as \( a = 15, b = 0.91, c = 7, A = 10, \omega = 0.001 \), Fig.5 shows that the system (10) has a chaotic attractor. Compared with Fig.2 and Fig.5, it can be seen that when the parameters \( a, b, c \) of the system are taken the same value, the system changes from the periodic-2 state to the chaotic state only by adding periodic perturbation term to the second equation. When the parameters are selected as \( a = 15, b = 3, c = 7, A = 100, \omega = 0.001 \), the periodic state is presented in Fig.6. Comparing Fig.1 and Fig.6, it is observed that the system transforms from chaotic state into periodic state.
Time-delay phenomenon often occurs in physics, biology and economy. External disturbance and time-delay often coexist in many systems, which cause more complex dynamic behaviors. Therefore periodic perturbation term and time-delay term are considered to add to the second and third equations of (9), respectively, and it can be written as follows

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1), \\
\dot{x}_2 &= cx_2 - x_1x_3 + Ax_1 \cos \omega t, \\
\dot{x}_3 &= x_1x_2 - bx_3(t - \tau),
\end{align*}
\]

(11)

where \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^{3 \times 1} \) is the state vector of system (11), \( \tau > 0 \) is an unknown time-delay. The time-delay chaotic system is an infinite dimensional system, it usually has more complex dynamic behavior. When the parameters are selected as \( a = 15, b = 0.91, c = 7, A = 10, \omega = 0.001, \tau = 0.3 \), Fig.7 shows that the system has a chaotic attractor. From Fig.2 and Fig.7, it is easy to find that the dynamics of system (11) can change from periodic-2 into chaotic. On the contrary, for appropriate \( \tau \), the dynamics of system (11) can also transform from chaotic state into periodic-1 or periodic-2. These results are confirmed in Fig.8 and Fig.9.
In this section, it is found that the system with external perturbation and time-delay term can be transformed between stable state, periodic state and chaotic state.

4. **Synchronization scheme.** The existing method is only to realize the synchronization of general three-dimensional chaotic system. However, the time-delay and external periodic disturbance often occur in chaotic system, a new finite-time synchronization method of time-delay chaotic system with external disturbance is presented in this section. To gain the main result, the system (11) is taken as the master system, the corresponding slave system is considered as

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1), \\
\dot{y}_2 &= cy_2 - y_1y_3 + Ay_1 \cos \omega t + u, \\
\dot{y}_3 &= y_1y_2 - by_3(t - \tau),
\end{align*}
\]  

(12)
where $y = (y_1, y_2, y_3)^T \in \mathbb{R}^{3 \times 1}$ is the state vector of system (12), $u$ is a control signal to be designed.

To investigate synchronization of system (11) and (12), we can denote $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, $e_3 = y_3(t - \tau) - x_3(t - \tau)$. Subtracting Eq.(11) from Eq.(12), one can get the error system

\[
\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1), \\
\dot{e}_2 &= ce_2 - e_1x_3 - e_3x_1 - e_1e_3 + Ae_1 \cos \omega t + u, \\
\dot{e}_3 &= e_1x_2 + e_2x_1 + e_1e_2 - be_3.
\end{aligned}
\]

By designing a controller, synchronization between drive system (11) and slave system (12) can be achieved in a finite-time $t_0$.

The problem is transformed into designing a controller to stabilize the error system (13) in a finite-time $t_0$. The procedure is designed as follows, i.e. it exists a constant $t_0$, when $t > t_0$, $\lim e_1 = \lim e_2 = \lim e_3 = 0$.

Now, some assumptions to be used are given as follows.

**Assumption 1.** Suppose there is a positive number $d$, so that $|Ae_1 \cos \omega t| \leq d$.

**Assumption 2.** Suppose there is a positive number $\delta$, so that $|e_2| \leq \delta$.

**Theorem 4.1.** Suppose Assumptions 1-2 are satisfied, if the controller is given as

\[ u = -\alpha e_2 + e_1x_3 + e_3x_1 + e_1e_3 - k_1 \text{sign}(e_1) - k_2 \text{sign}(e_2) - k_3 \text{sign}(e_3), \]

where $k_1, k_2, k_3$ are to be estimated such that

\[
\begin{aligned}
\alpha a - k_1 &\geq 0, \\
k_2 - (\alpha a - k_1) - k_3 - d - 2\sqrt{2}a &\geq \mu, \\
k_1 - k_2 - k_3 - d - a\delta &\geq \nu,
\end{aligned}
\]

then the finite-time synchronization between systems (11) and (12) can be achieved, i.e. there exists constant $t_0$ such that $\lim_{t \to t_0} e_1 = \lim_{t \to t_0} e_2 = \lim_{t \to t_0} e_3 = 0$ for all $t > t_0$, where $\mu > 0$, $\nu > 0$, $t_0 = \frac{2}{\beta} V(0)\frac{1}{2}$, $\beta = \frac{m}{2\mu}$, $m = \min\{\frac{V}{\alpha}, \frac{3\mu}{\sqrt{2}}\}$ and $V = (\alpha|e_1| + \frac{1}{2}e_2^2)^{\frac{1}{2}} + e_1e_2$. 

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**Figure 9.** Phase trajectory and the time series of Eq.(11) with $a = 15, b = 3, c = 7, A = 0.1, \omega = 0.001, \tau = 0.3$

(a) $(x_1, x_2)$  
(b) $(t, x_1)$
Proof. According to the second and third equations of system (13), we can deduce that if \( \lim_{t \to t_0} e_1 = \lim_{t \to t_0} e_2 = 0 \), then \( \lim_{t \to t_0} e_3 = 0 \). Therefore, in the following, we only need to prove \( \lim_{t \to t_0} e_1 = \lim_{t \to t_0} e_2 = 0 \).

Since \( \alpha > \left( \frac{2}{3} \right)^2 \), according to Lemma 2.2, the Lyapunov function is chosen as

\[
V = (\alpha|e_1| + \frac{1}{2}e_2^2) \frac{3}{2} + e_1e_2.
\]

Along the trajectories of system (13), the derivative of Eq.(16) is calculated and yields

\[
\dot{V} = \frac{3}{2}(\alpha|e_1| + \frac{1}{2}e_2^2) \frac{3}{2} [\text{asign}(e_1)e_1 + e_2\dot{e}_2] + \dot{e}_1e_2 + e_1\dot{e}_2
\]

\[
= \frac{3}{2}(\alpha|e_1| + \frac{1}{2}e_2^2) \frac{3}{2} [\text{asign}(e_1)(e_2 - e_1) + e_2(-k_1\text{sign}(e_1)
- k_2\text{sign}(e_2) - k_3\text{sign}(e_3) + Ae_1\cos\omega t)] + e_2(\alpha e_2 - ae_1)
+ e_1(-k_1\text{sign}(e_1) - k_2\text{sign}(e_2) - k_3\text{sign}(e_3) + Ae_1\cos\omega t)
\]

If \( a \geq 0 \) is fulfilled, then \( \alpha\text{asign}(e_1)e_1 \geq 0 \). Thus, one gets

\[
\dot{V} \leq -\frac{3}{2\sqrt{2}}e_2^2(k_2 + (k_1 - \alpha)a)\text{sign}(e_1)e_2 + k_3\text{sign}(e_2)e_3
- Ae_1\cos\omega t\text{sign}(e_2)) + ae_2 - |e_1|(|k_1 + k_2\text{sign}(e_1)e_2)
+ k_3\text{sign}(e_1)e_3 - Ae_1\cos\omega t\text{sign}(e_1) + ae_2\text{sign}(e_1))
\]

Using inequality (15), we have \( \alpha a - k_1 \geq 0 \). Thus, one gets

\[
\dot{V} \leq -\frac{3}{2\sqrt{2}}e_2^2[k_2 - (\alpha a - k_1) - k_3 - d - \frac{2\sqrt{2}}{3}a]
- |e_1|(|k_1 - k_2 - k_3 - d - |a|\delta]
\leq -\frac{3}{2\sqrt{2}}\mu e_2^2 - \nu |e_1|
= -\frac{\nu}{\alpha}|e_1| - \frac{3\mu}{2\sqrt{2}}e_2^2
\leq -m(\alpha|e_1| + \frac{1}{2}e_2^2)
= -m[\alpha|e_1| + \frac{1}{2}e_2^2] \frac{3}{2},
\]

where \( m = \min\left(\frac{3\mu}{2\sqrt{2}}, \frac{\nu}{\alpha}\right) \).

Based on Lemma 2.2, we have

\[
e_1e_2 \leq (\alpha|e_1| + \frac{1}{2}e_2^2) \frac{3}{2}
\]

Substituting (18) into (17) leads to the inequality

\[
\dot{V} \leq -m\left(\frac{1}{2}((\alpha|e_1| + \frac{1}{2}e_2^2) + e_1e_2)\right) \frac{3}{2} = -\left(\frac{1}{2}\right)^{\frac{3}{2}}mV^{\frac{3}{2}} = -\gamma V^{\frac{3}{2}},
\]
where \((\frac{1}{2})^2m = \gamma\).

By solving the above inequality, according to Lemma 2.1, one gets

\[ V(t) \leq (V_0^{\frac{1}{3}} - \frac{\gamma t}{3})^3. \]

Since \(V(t) \geq 0\), it can be obtained that \(\frac{2\gamma t}{3} \leq V_0^{\frac{1}{3}}\), which means that \(t \leq \frac{3V_0^{\frac{1}{3}}}{\gamma}\). Therefore, there exists constant \(t_0 = \frac{3V_0^{\frac{1}{3}}}{\gamma}\), such that \(\lim_{t \to t_0} e_1 = \lim_{t \to t_0} e_2 = 0\). The proof of Theorem 4.1 is completed.

Based on systems (11)-(12) and formula (15), we can take
\[ k_1 = 0.0021, k_2 = 0.001, k_3 = 0.001. \]
Obviously, the conditions of Theorem 4.1 are satisfied. According to Theorem 4.1, if the controller \(u\) is taken as Eq.(14), the master system and the slave system will be synchronized in finite time.

**Figure 10.** 2D overview chaotic attractor and the chaotic time series of Eq.(11) with \(\tau = 0.005\) (a) \((x_1, x_2)\) (b) \((t, x_1)\)

**Figure 11.** 2D overview chaotic attractor and the chaotic time series of Eq.(11) with \(\tau = 0.3\) (a) \((x_1, x_2)\) (b) \((t, x_1)\)

In the simulation process, Matlab software is used to solve differential equations. When \(a = 15, b = 3, c = 10, A = 0.1, \omega = 0.9\), the system is in the chaotic state with
\( \tau = 0.005 \) and \( \tau = 0.3 \), which are given in Fig.10 and Fig.11, respectively. Initial values of the master system and the slave system are chosen as \((-0.01, -0.9, -0.8)\) and \((-0.3, -0.92, -0.7)\). The dynamics of error system is shown in Fig.12. From Fig.12, it is easy to see that the master system (11) can synchronize the slave system (12) in finite time. In addition, to investigate the effect of time-delay \( \tau \) on the synchronization time, we select different values of \( \tau \). Numerical simulation displays that, when \( \tau \) is taken different values, the time of synchronization shows a certain difference in Fig.13 -15.

**Figure 12.** The error dynamics between systems (11) and (12) with \( \tau = 0.3 \) (a) \( e_1 \) (b) \( e_2 \) (b) \( e_3 \)

**Figure 13.** The error states \( e_1 \) between systems (11) and (12) with \( \tau = 0.005 \), \( \tau = 0.05 \) and \( \tau = 0.3 \)

**Figure 14.** The error states \( e_2 \) between systems (11) and (12) with \( \tau = 0.005 \), \( \tau = 0.05 \) and \( \tau = 0.3 \)
Figure 15. The error states $e_3$ between systems (11) and (12) with $\tau = 0.005$, $\tau = 0.05$ and $\tau = 0.3$

5. Conclusion and future works. In this paper, dynamical behaviors of chaotic system with periodic disturbance and time-delay are investigated. It is found that, in improved Lü system with external periodic disturbance $A_H \cos \omega t$ and time-delay, stable state, periodic solution and chaotic state can be controlled and selected by choosing appropriate parameters. Based on Lyapunov stability theory, a single controller for chaotic system with external disturbance is designed to achieve synchronization in finite time. The designed synchronization schemes possess the following advantages.

(i) The method is universal, it can be used not only for some special chaotic systems but also for general chaotic systems.

(ii) The scheme is suitable for systems with time-delay and periodic disturbance, and can achieve synchronization in a relatively short time.

(iii) The designed synchronization is new, simple and easy to realize in experiment compared with those where complex control functions are used. These results are not only obtained by theoretical analysis, but also can be verified by numerical simulations.

In our future work, we hope that our approach could be extended to other systems with more complex structure and hyperchaotic systems. Furthermore, we expect to achieve finite-time synchronization of two chaotic systems with different orders via this method.

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