Thermodynamically compatible model for wavefields simulation in deformed porous medium saturated by a compressible viscous fluid

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Abstract. A computational model for the small amplitude wave propagation in an elastic porous medium saturated by the viscous compressible fluid is discussed. The presented model is an extension of the model [1] and its derivation is based on the symmetric hyperbolic thermodynamically compatible system for two-phase solid-fluid mixture with finite deformations of the solid phase. In the present consideration, the fluid viscosity is taken into account via the unified hyperbolic model for viscous flows with shear stress relaxation. The governing equations form a hyperbolic system written in terms of the mixture velocity, relative velocity of phase motion, pressure and shear stress of the mixture that allows to apply an efficient finite difference method for numerical solution. Some numerical examples are presented, showing physically correct results, and, in particular, the frequency dependence of the shear wave velocity.

1. Introduction

Modeling of dynamic processes in deformed porous media filled with a fluid is of interest for many industrial and environmental applications. Of particular interest is the modeling of wavefields in saturated porous media, a theoretical study of which was pioneered by M. Biot in the middle of the last century. In the subsequent development of Biot’s theory, special attention was paid to modeling wave fields for seismic applications, see, for example [2] and references therein. In recent time a new actual problems such as CO2 storage, medical applications, hydrofracking, etc., require a new advanced models and methods for studying flows in deformed porous medium.

For the development of new advanced models, the application of the continuum approach using the theory of multiphase media looks promising. In [1], the solid-fluid two-phase model of porous medium based on the extension of the unified model of continuum [3] is proposed and its application to simulation of small amplitude wave propagation is considered. In the present paper we generalize the model for small amplitude wave propagation from [1] to the case when a fluid saturating the elastic porous medium is viscous. It turns out that the fluid viscosity has a significant influence on shear waves behavior. Some numerical examples showing physically correct characteristics of wavefields are presented.
2. Governing equations of the model

In this section, we present the system of linear partial differential equations (PDE) for the description of small amplitude wave propagation in an initially unstrressed porous medium saturated by a compressible viscous fluid. These governing equations is an extension of the PDE's derived in [1] from the general nonlinear two-phase model of compressible fluid flow in elasto-plastic porous medium. The derivation is based on the assumptions that the initially unstressed medium undergoes small perturbations, and the linearization of the nonlinear system near the unstressed state can be done. Let us assume that the initially unstressed two-phase medium is characterized by fluid and solid volume fractions $\alpha_1^0$ and $\alpha_2^0$ ($\alpha_1^0 + \alpha_2^0 = 1$) with corresponding mass densities $\rho_1^0, \rho_2^0$. Then the total density of the mixture is $\rho = \alpha_1^0 \rho_1^0 + \alpha_2^0 \rho_2^0$ and porosity is $\phi = \alpha_1$ (index “1” denotes the fluid state). The resulting governing equations for small amplitude wave propagation read as:

$$\frac{\partial v^i}{\partial t} + \frac{\partial P}{\partial x_i} - \frac{\partial s_{ik}}{\partial x_k} = 0, \quad (1a)$$

$$\frac{\partial w^k}{\partial t} + \left( \frac{1}{\rho_1^0} - \frac{1}{\rho_2^0} \right) \frac{\partial P}{\partial x_i} = -\frac{c_1^0 c_2^0}{\theta} w^k, \quad (1b)$$

$$\frac{\partial P}{\partial t} + K \frac{\partial w^k}{\partial x_k} + \frac{\alpha_1^0 \alpha_2^0}{\theta} (\rho_2^0 - \rho_1^0) K \frac{\partial w^k}{\partial x_k} = 0, \quad (1c)$$

$$\frac{\partial s_{ik}}{\partial t} - \mu \left( \frac{\partial v^i}{\partial x_k} + \frac{\partial v^k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v^j}{\partial x_j} \right) = -s_{ik} \frac{\tau}{\tau}. \quad (1d)$$

Here $v^i = c_1^0 v_1^i + c_2^0 v_2^i$ is the mixture velocity of the medium, $v_1^i, v_2^i$ are the fluid and solid velocities respectively, $c_1^0 = \alpha_1^0 \rho_1^0 / \rho, c_2^0 = \alpha_2^0 \rho_2^0 / \rho$ are the mass fractions of initially unstressed medium ($c_1^0 + c_2^0 = 1$), $w^i = v_1^i - v_2^i$ is the phase velocity, $s_{ik}$ is the shear stress tensor (deviator of the total stress), $P$ is the pressure (common for both phases), $\theta$ and $\tau$ are the shear relaxation time and shear stress relaxation time, respectively.

There are four material constants in the above system: elastic moduli $K, \mu$ and relaxation parameters $\theta, \tau$. Before determining them, it should be noted that this system describes processes in a porous medium for the entire range of porosity $0 \leq \phi \leq 1$. Indeed, in the pure fluid limit ($\phi = \alpha_1^0 = 1, \alpha_2^0 = 0$), system (1) transforms into the visco-elastic Maxwell-type hyperbolic system for acoustics in viscous flow, which can be obtained as a linearization of governing equations of the unified model of continuum [3] with a specific choice of shear strain relaxation term:

$$\frac{\partial v^i}{\partial t} + \frac{\partial P}{\partial x_i} - \frac{\partial s_{ik}}{\partial x_k} = 0, \quad (2a)$$

$$\frac{\partial P}{\partial t} + K \frac{\partial w^k}{\partial x_k} = 0, \quad (2b)$$

$$\frac{\partial s_{ik}}{\partial t} - \mu \left( \frac{\partial v^i}{\partial x_k} + \frac{\partial v^k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v^j}{\partial x_j} \right) = -s_{ik} \frac{\tau}{\tau}. \quad (2c)$$

Note that equation (1c) with $c_2^0 = 0$ can be neglected because the relative velocity has no influence on the wavefield. One can see that for small $\tau$, we can consider an asymptotic expansion with respect to $\tau$ which results in the Navier-Stokes shear stress tensor in the leading terms:

$$s_{ij} = \mu \tau \left( \frac{\partial v^i}{\partial x_k} + \frac{\partial v^k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v^j}{\partial x_j} \right) + O(\tau^2), \quad (3)$$

where $\eta = \mu \tau$ is the effective shear viscosity. Stress tensor (3) together with equations (2a), (2b) form the PDE’s system for the propagation of acoustic waves in Newtonian viscous fluids.
For the pure solid limit \((\phi = \alpha_1^0 = 0, \alpha_2^0 = 1)\), one can also neglect the equation for the relative velocity. The resulting governing equations transform to the same system \((2)\) where, however, the relaxation time should be taken as \(\tau = \infty\), see \([3]\), and thus, we arrive at the linear elasticity equations:

\[
\begin{align*}
\rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial P}{\partial x_i} - \frac{\partial s_{ik}}{\partial x_k} &= 0, \\
\frac{\partial P}{\partial t} + K \frac{\partial v_i}{\partial x_i} &= 0, \\
\frac{\partial s_{ik}}{\partial t} - \mu \left( \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3} \delta_{ik} \frac{\partial v_i}{\partial x_j} \right) &= 0.
\end{align*}
\]

All of the above means that for arbitrary porosity \((0 \leq \phi \leq 1)\), equations \((1)\) can serve as a two-phase model for describing a porous medium saturated with a viscous liquid, with an appropriate specification of the modulus \(K(\phi)\), \(\mu(\phi)\) and relaxation parameters \(\tau(\phi), \theta(\phi)\) which is discussed in the following.

The wave propagation velocities are characterized by the bulk modulus \(K(\phi) = (\alpha_1^0/K_1^{-1} + \alpha_2^0/K_2^{-1})^{-1}\) and shear modulus \(\mu(\phi) = \alpha_1^0\mu_1 + \alpha_2^0\mu_2\), where \(K_1, K_2\) and \(\mu_1, \mu_2\) are the bulk moduli and shear moduli of the fluid and solid phase respectively. The formula for \(K\) is derived in \([1]\) under the assumption that the pressures of fluid and solid are equal due to the small scale of pore space. The shear modulus of the mixture is taken in this form based on the standard averaging assumption for the mixture modulus. Note, that in \([1]\) the mixture shear modulus was taken as \(\mu = \alpha_1^0\mu_2\) which is equivalent to our present definition if to assume \(\mu_1 = 0\).

The relative velocity relaxation time \(\theta\) characterizes the interfacial friction and depends on the viscosity of fluid and permeability \([2]\). As for the shear stress relaxation time \(\tau\) of real media, this is, as a rule, a strongly nonlinear function of the state parameters, and it should be taken in the form obtained theoretically or indirectly, in the form of an empirical function that better matches the available experimental data. Up to our knowledge, there is no available experimental data on shear wave propagation in porous media. That is why we take \(\tau\) as an empiric function allowing us to study the shear wave propagation and attenuation qualitatively. We define the relaxation time as:

\[
\tau = \left( \frac{\alpha_1^0}{\tau_1} + \frac{\alpha_2^0}{\tau_2} \right)^{-1},
\]

where \(\tau_1, \tau_2\) are relaxation times of fluid and solid, respectively, and \(n\) is a constant. One can see that if the skeleton is pure elastic and thus, \(\tau_2 = \infty\), the relaxation time of the mixture is computed as \(\tau = \tau_1 / (\alpha_1^0)^n\).

Recall that the fluid shear modulus \(\mu_1\), relaxation time \(\tau_1\) and fluid dynamic viscosity \(\eta_1\) are connected by the relation \(\eta_1 = \mu_1 \tau_1\).

3. Numerical examples

In this section, we verify the correctness of the obtained theoretical results and numerically check the dependence of the compressional and shear wave velocities on frequency. To this end, let us consider a homogeneous medium with the material parameters from table 1.

In the pure fluid and solid states, the bulk characteristic speeds \(\sigma_{0,i}, \sigma_{b,2}\), the bulk modulus \(K_1, K_2\), and shear modulus \(\mu_1, \mu_2\) can be uniquely determined from the seismic velocities of longitudinal \(\sigma_{0,i}\) and shear \(\sigma_{s,i}\) waves by the formulas:

\[
\sigma_{0,i}^2 = \frac{\rho_0 c_{0,i}^2}{\rho_i c_{s,i}^2}, \quad \mu_i = \rho_i c_{s,i}^2, \quad K_i = \rho_i c_{b,i}^2, \quad i = 1, 2.
\]
For numerical simulations, we use finite difference schemes on staggered grids, in details described in [1]. Note, that this approach is well suited for approximating symmetric hyperbolic...
systems and is easy to use. We consider a two-dimensional square physical domain $\Omega$ covered by a numerical grid with $N$ grid points in each direction and sampling step $\Delta x$ so that $\Omega = [N\Delta x]^2$ with centering at the origin. To suppress artificial reflections from non-physical computational boundaries, we apply the perfectly matched layer (PML) technique in the original split-field formulation [4].

To generate a seismic waves, we use vertical-type source by adding a source term in the right-hand side of the equation for $s_{11}$ in system (1). We define the source term $F(x, y, t)$ at point $(x_0, y_0)$ as:

$$F(x, y, t) = \delta(x_0, y_0) f(t),$$

where $\delta$ is Dirac’s delta function. Ricker’s wavelet $f(t)$ is defined in the time domain as:

$$f(t) = (1 - \omega^2(t - t_0)^2/2)\exp[-\omega^2(t - t_0)^2/4],$$

where $\omega$ is the angular frequency and $t_0$ s is the time wavelet delay.

For each selected frequency $\omega$ we scale the spatial step $\Delta x$ and recording time $T$ by some scaling parameter $\varepsilon$, leaving the number of nodes $N$ constant. This is done because the seismic wavelength depends on the frequency and the size of the computational domain $\Omega$ must be equal at least several wavelengths.

Let us consider three typical frequencies $\omega = 10$, $10^3$, and $10^5$ [rad/s], and take into account both relaxation mechanisms for relative velocity and shear stress relaxation times. Other parameters such as $\phi = \alpha^0 = 0.2$, power index $n = 8$, source location $(x_0, y_0) = (0, 0)$, grid size $N = 10^4$, $\Delta x_0 = 5\text{m}$, registration time $T_0 = 10\text{s}$, time delay $t_0 = 2\pi/\omega$, and scale parameter $\varepsilon = 10^{-2}$ are unchanged. All seismograms are recorded in the receivers located along the main diagonal of computational domain $\Omega$ with uniform spacing.

![Figure 2: Phase velocities (top row) of the sound waves and attenuation factor per wavelength $\alpha_\lambda$ (bottom row) for all three modes: fast P-mode (left column), slow P-mode (middle column), and shear mode (right column). Porosity $\phi = 0.2$. Other material parameters are given in table 1. The dashed horizontal lines denote the characteristic velocities of system (1) for the given porosity when $\tau = \theta_2 = \infty$ (no dissipation).](image)
Figure 3 (top) shows the seismogram of the vertical mixture velocity $v^2$ in case $\omega = 10$ recorded for time $T_0$ with grid spacing $\Delta x_0$. Figure 3 (middle) shows the seismogram of the vertical mixture velocity $v^2$ in case $\omega = 10^3$ recorded for time $T = \varepsilon T_0$ with $\Delta x = \varepsilon \Delta x_0$. Finally, figure 3 (bottom) shows the seismogram of the vertical mixture velocity $v^2$ in case $\omega = 10^5$ recorded for time $T = \varepsilon^2 T_0$ with $\Delta x = \varepsilon^2 \Delta x_0$. One can notice a strong frequency dependence of the velocity of all three types of waves. To identify the different types of waves on the seismograms, we marked the fast compressional wave with the letter “P”, the slow compressional wave by “Pb” (“b” stands for Biot wave), and the shear wave by letter “S”.

The presented seismograms make it possible to estimate the propagation velocities of all types of waves and compare them with the theoretical sound velocities $V(\omega)$ (phase velocities) depicted in figure 3 for the given porosity $\phi = 0.2$ which can be obtained from the dispersion relation of system (1), e.g. see [1]. One can observe a good agreement between theoretical and numerical results. Moreover, we remark a rather strong wavefield attenuation in seismogram figure 3 (middle) for $\omega = 10^3$ that corresponds to strong attenuation factors in figure 3.

4. Conclusion

The computational model for small amplitude wave propagation in an elastic porous medium filled with a compressible viscous fluid for the entire range of porosity $0 \leq \phi \leq 1$ is proposed. Governing equations of the model are derived on the base of hyperbolic thermodynamically compatible system theory applied to a two-phase solid-fluid mixture. They form a hyperbolic first order PDE’s system written in terms of mixture velocity, mixture pressure, mixture stress deviator and relative velocity of phase motion. This allows an application of high-order finite difference numerical method on staggered grid. Some numerical test problems with special attention to shear wave propagation were solved showing physically correct behavior of numerically obtained solution.

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