The renormalization group flow in 2D N=2 SUSY Landau-Ginsburg models

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Abstract

We investigate the renormalization of N=2 SUSY L-G models with central charge $c = 3p/(2 + p)$ perturbed by an almost marginal chiral operator. We calculate the renormalization of the chiral fields up to $gg^*$ order and of nonchiral fields up to $g(g^*)$ order. We propose a formulation of the nonrenormalization theorem and show that it holds in the lowest nontrivial order. It turns out that, in this approximation, the chiral fields can not get renormalized $\Phi^k = \Phi^k_0$. The $\beta$ function then remains unchanged $\beta = \epsilon g$.

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1 Introduction

A better understanding of the structure of N=2 SUSY theories in two dimensions is important from two points of view. First the N=2 SUSY conformal theories can be regarded as building blocks for the background “spacetime” in string theories with N=1 SUSY in four dimensions [1], [2]. Second they are of great interest from the point of view of statistical systems where N=2 conformal field theories can be understood as two-dimensional self-dual critical points in $Z_N$-symmetric statistical systems [4]. The content of the minimal model series with central charge $c = 3p/(2 + p)$, $p = 2, 3, \ldots$, i.e. the classification of primary fields and fusion rules, was extensively derived by several authors [4], [5], [6]. There are also interesting connections between the N=2 minimal models and singularity theory which have been discovered recently [12], [13].

The structure beyond the conformally invariant point of the theory still remains open for investigation.

In the space of all two dimensional N=2 SUSY theories, minimal models are fixed points of the renormalization group flow with $c < 3$ [2], [3], [4]. In our paper we try to describe the structure of this space by investigating the RG-flow near the fixed point. Different N=2 minimal models have a different value of the supersymmetry index, so passing from one theory to the other cannot occur by a smooth and finite change of parameters as it can for N=0,1 SUSY [3], [7], [8], [9]. It was pointed out by Cvetič and Kutasov that the fixed points of N=2 SUSY are infinitely apart from each other in the space of coupling constants where the metric tensor is defined by the Zamolodchikov [8] formula $G_{ij}(g) = \langle \Phi_i(g) \Phi_j(g) \rangle_g$. We confirm this result by a perturbative calculation.

Another interesting issue is the nonrenormalization conjecture [10], [11] - the absence of nontrivial renormalization of the superpotential. Ordinarily this conjecture is proven in the loop expansion, and it is not clear whether it survives the infrared regularization; renormalization and resummations necessary to define the nontrivial Landau-Ginsburg fixed point. Existing proofs of the nonrenormalization theorem [10], [11] are based on the structure of the graphical expansion about free field theory. This expansion is badly infrared divergent in a massless Landau-Ginsburg model; these divergences must be resummed and the effect on the nonrenormalization theorem is unclear. One can expect that nonrenormalization should enforce the existence of only one renormalization constant (the wave function renormalization) for all fields in the theory. Stating it differently we could expect that renormalized fields $\tilde{\Phi}^k$ would be again powers of some basic field $\tilde{\Phi}$.

Indeed it is not entirely clear what the precise statement of nonrenormalization is. For instance there always exist local coordinates near a fixed point in which the $\beta$ function is linear [17], moreover any statement about the form of the $\beta$ function can be changed by redefinition of couplings. Such coordinates are usually singular (they are essentially the bare coordinates) and put all nontrivial fixed points at coordinate infinity (although they might be a finite physical distance in the Zamolodchikov
metric; in any case for our situation the result of [7] indicates that there are no nearby fixed points). One might say that nonrenormalization is the statement that there are no divergent counterterms needed to renormalise the chiral fields sector, so by a finite renormalization (nonsingular coordinate transformation) one can bring the chiral fields $\beta$ function into a linearized form. We will calculate this $\beta$ function in the composite operator (conformal) perturbation theory scheme of Zamolodchikov, and show that it is the scheme for which this form of nonrenormalization holds to leading nontrivial order.

In this paper we look closely at N=2 SUSY minimal theories described in the Landau-Ginsburg formulation [2], [3] by one chiral field $\Phi$ where the lagrangian is

$$ S_0 = \int d^2 z d^2 \theta d^2 \bar{\theta} \Phi \Phi^* + \int d^2 z d^2 \theta \Phi^{p+2} + \int d^2 z d^2 \bar{\theta} \Phi^{*p+2} $$

at the conformally invariant point with the central charge value $c = 3p/(2 + p)$. We perturb the L-G model (1) by the most marginal operator $\Phi^p$. Solving the perturbed theory up to $gg^*$ order for chiral fields we prove, in leading nontrivial order, that the nonrenormalization theorem holds for chiral fields and their coupling constants in a trivial way $\tilde{\Phi}^k = \Phi^k$. The renormalization of nonchiral fields does not obey a simple relation.

We organize our paper as follows. In the first section we recall the LG model together with its field content. In section 3 we describe the geometrical approach to the RG-flow and point out features specific to a chiral theory. In section 4 we present perturbative results for nonchiral fields and in section 5 we discuss the renormalization of chiral fields and couplings and calculate the $\beta$ function in this approximation.

While the present work was being completed, we received a preprint by W.A.Leafer-Hermann [25] investigating substantially the same problem, we comment on his results in the discussion.

## 2 N=2 SUSY Landau-Ginsburg models

The conformally invariant point of N=2 SUSY discrete series with the central charge value $c = 3p/(2 + p)$ $p = 2, 3 \ldots$ can be regarded as a L-G model depending on the single chiral field $\Phi$ and its hermitian conjugate $\Phi^*$ [4], [3]. A general complex SUSY field is a function on the N=2 superspace with local coordinates $Z = (z, \theta, \theta^*)$, $\bar{Z} = (\bar{z}, \bar{\theta}, \bar{\theta}^*)$. The charge of super variables $\theta, \theta^*$ is $q = \pm 1/2$. The covariant derivatives in these coordinates are

$$ D = \partial + \theta \frac{\partial}{\partial z} $$

$$ D^* = \frac{\partial}{\partial \bar{\theta}} + \theta^* \frac{\partial}{\partial \bar{z}} $$

(2)
The N=2 (anti)chiral fields $\Phi, \Phi^*$ are defined by the conditions

$$D^*\Phi^* = \overline{D^*}\Phi^* = 0$$
$$D\Phi = \overline{D}\Phi = 0$$ (3)

The conformally invariant L-G model is described by the action (1), where for the dependence of the chiral field on the superspace parameters we use the shorthand notation $\Phi(z, \theta, \bar{z}, \bar{\theta}) = \Phi(Z, \bar{Z})$ and respectively $\Phi^*(z, \theta^*, \bar{z}, \bar{\theta}^*) = \Phi^*(Z^*, \bar{Z}^*)$. Following the Zamolodchikov prescription [2], [3], [14] we identify different composite fields of $\Phi, \Phi^*$ with the primary fields of N=2 SUSY theories already classified [4], [5], [6].

For our calculations we find useful the identification of N=2 with para fermion models (PF) [4], [6]. In terms of parafermion fields $\phi$ and free field $\varphi$ lowest components of the holomorphic part of (anti)chiral fields $(\Phi^k, \Phi^*)$ look like

$$\Phi^k(z) \approx \phi_k^k(z) : \exp \left( i \frac{k\varphi(z)}{\sqrt{2p(p+2)}} \right) :$$

$$\Phi^{*k}(z) \approx \phi_{-k}^k(z) : \exp \left( i \frac{-k\varphi(z)}{\sqrt{2p(p+2)}} \right) :$$ (4)

where the conformal dimension and charge are given by

$$\Delta = q = \frac{k}{2(p+2)}$$

$$\Delta = -q = \frac{k}{2(p+2)} \quad k = 1, \ldots, p$$ (5)

The antiholomorphic part dependent on $\bar{z}$ has the same form as above. As a consequence of the equation of motion $D\overline{D}\Phi^* = (p+2)\Phi^{p+1}$, the $(p+1)$ power of the chiral field $\Phi$ is not primary. Various composites of the chiral-antichiral fields are expressed in terms of parafermions by the identification of the lowest component

$$\Phi^k\Phi^{*l}(z) \approx \phi_m^j(z) : \exp \left( i \frac{m\varphi(z)}{\sqrt{2p(p+2)}} \right) :$$ (6)

where $m = k - l$ and $j = \min(k+m, 2p-(k+m))$ and the conformal dimension and charge are given by

$$\Delta = \frac{j(j+2) - m^2}{4(p+2)}, \quad q = \frac{m}{2(p+2)}$$ (7)

The highest components of the field are defined by the operator product expansion of lowest components with the supersymmetry generators which are expressed in terms of parafermionic currents $\psi_1^\pm$ by

$$G_{1/2}^\pm(z) = \sqrt{\frac{2p}{p+2}} \psi_1^\pm(z) : \exp \left( \pm i \frac{(p+2)\varphi(z)}{\sqrt{2p(p+2)}} \right) :$$ (8)

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To complete the set of N=2 SUSY generators we recall that the U(1) current is given by

\[ J(z) = \frac{i}{2} \sqrt{2p(2+p)} \partial \varphi(z) \] (9)

and the stress energy tensor is

\[ T(z) = T_{PF} - \frac{1}{4} \partial \varphi^2(z) : \] (10)

The fusion rules of the N=2 SUSY and the PF model are in ref. [4], [6], as well as N=2 SUSY Ward identities and the general form of the three point function that we use in the perturbative calculation. The normalization of the fields is set by the normalization of the two point function which we choose to be

\[ \langle \Phi_{m_1}^{l_1}(Z_1) \Phi_{m_2}^{l_2}(Z_2) \rangle = Z_{12}^{-2\Delta_1}(1 + 2q_1 \frac{\theta^*_{l_2} \theta_{l_1}}{Z_{12}})\delta_{l_1,l_2}\delta_{m_1,m_2} \] (11)

for nonchiral fields and

\[ \langle \Phi_{l_1}^{l_1}(Z_1) \Phi_{-l_2}^{l_2}(Z_2^*) \rangle = Z_{12}^{-2\Delta_1}(1 + 2q_1 \frac{\theta_{l_1} \theta^*_{l_2}}{Z_{12}})\delta_{l_1,l_2} \] (12)

for chiral fields, where \( Z_{12} = z_{12} - \theta_1 \theta^*_2 - \theta_2 \theta^*_1 \) for nonchiral two point function and \( Z_{12} = z_{12} - \theta_1 \theta^*_2 \) for chiral fields. In this model the relevant perturbations are those by chiral fields \( \Phi^k \). All nonchiral fields are irrelevant since their dimensions are always greater than zero and so the dimensions of the highest component are always \( \Delta > 1 \). This is also the reason why we can treat the coupling constant space as finite dimensional.

To have some grasp on the RG-flow in these theories we investigate the model (4) perturbed by the almost marginal (for large \( p \)) operator \( \Phi^p \). The complete action we consider is

\[ S = \frac{1}{2\pi} \left( S_0 + g \int d^2zd^2\theta \Phi^p + g^* \int d^2zd^2\theta^* \Phi^{*p} \right) \] (13)

where \( 1/2\pi \) is the normalization of the action needed to define correctly the stress-energy tensor (see [3], [8], [9], [16]). The dimension of the \( \Phi^p \) field is \( \Delta_p = 1/2 - \epsilon \), where \( \epsilon = 1/(p+2) \). Since integrated fields are charged the coupling constant charge (left plus right) can be identified as \( q(g) = 2/(p+2) = 2\epsilon \).

3 Geometrical description of RG-flow in N=2 SUSY theories

In this section we recall the notation of the geometrical approach to the RG-flow developed in recent years [8], [16]. We point out some facts which are specific to N=2 SUSY chiral theories.
The field theory can be considered as a set of (connected) correlation functions
\[
\langle \Phi_1(z_1) \ldots \Phi_N(z_N) \rangle ,
\]
where the local fields are elements of the infinite dimensional vector space and form a closed operator algebra. N-point correlation functions can be regarded as maps \( \otimes_n A \to R \), i.e. tensors of rank \((0, N)\). The correlation functions depend on a number of coupling constants \( g^i \). Different descriptions of a system, corresponding for example to different choices of renormalization scales, are related to each other by a continuous change of \( g^i \)'s. The \( g^i \) can be regarded as coordinates on the state manifold \( M \) of the given system and equivalent coordinate systems are related by diffeomorphisms.

The correlation function (14) can be derived from the Lagrangian formulation of the theory by the functional integral
\[
\int D\Phi \Phi_1(z_1) \ldots \Phi_N(z_N) e^{-S(\Phi)}
\]
The decomposition \( S = S_0 + S_{prt} \), as in (13), corresponds to the choice of bare coordinates \( g_0^i \) in which \( S_0 \) is the unperturbed action at the point \( g_0^i = 0 \) and \( S_{prt} = g_0^i \int d^2Z \Phi_{0i} + h.c. \). In this formulation the space of integrated fields \( \Phi_{0i} = \int d^2Z \Phi_{0i}(Z, \bar{Z}) \) can be identified with the space tangent to the manifold \( M \). For example, in the bare coordinate system the derivatives \( \frac{\partial}{\partial g_0^i} \) acting on the correlation function are
\[
\frac{\partial}{\partial g_0^i} \langle \Phi_{01}(z_1) \ldots \Phi_{0N}(z_N) \rangle = \langle \Phi_{0i} \Phi_{01}(z_1) \ldots \Phi_{0N}(z_N) \rangle .
\]

The RG-flow is a one-parameter group of the action space diffeomorphisms with group parameter \( t = \ln(\frac{a}{R}) \), where \( a \) is the scale of the system (i.e. lattice spacing) and \( R \) is the scale of the observer. The generating vector field is \( \Theta = \beta^i \Phi_i \), the integrated trace of stress-energy tensor, and the components \( \beta^i = \frac{d}{dt} g_i \) are called beta functions; \( \beta^i \) depend on \( g \) - the choice of coordinate system. In the neighborhood of a fixed point \( g^i^* \) one can introduce the bare coordinates which linearize the beta functions
\[
\beta^i_0 = g_0^j \gamma^i_j ,
\]
where \( \gamma^i_j \) is the matrix of anomalous dimensions at the fixed point. The RG-flow is described in generic coordinates \( g \) by
\[
g^i = Z^i_j(g) g_0^j , \quad \Phi_i = \Phi_{0j} \tilde{Z}^i_j(g) ,
\]
where \( g^i \) and \( \Phi_i \) are renormalized couplings and fields. The \( Z^i_j(g) \), \( \tilde{Z}^i_j(g) \) functions determine beta functions and the anomalous dimension matrix.
The requirement that the RG-flow leaves the theory invariant imposes constraints on correlation functions which can be written down as the Callan-Szymanzik equation \[8, 9\]

\[
\left[\sum_{a=1}^{N} \frac{1}{2} Z_i^a \frac{\partial}{\partial Z_i^a} + \hat{\Gamma}(g) - \sum_{i=1}^{p} \beta^j(g) \frac{\partial}{\partial g^j}\right] \langle \Phi_1(z_1) \ldots \Phi_N(z_N) \rangle = 0 \tag{19}\]

\(\hat{\Gamma}(g)\) is the matrix of anomalous dimensions acting on fields as \(\hat{\Gamma}(g) \Phi_i = \gamma_i^j \Phi_j\) and

\[
Z_i^a \frac{\partial}{\partial Z_i^a} = z^a \frac{\partial}{\partial z^a} + \frac{1}{2} \theta^a \frac{\partial}{\partial \theta^a} + c.c. \tag{20}\]

for chiral fields and respectively \(Z^*\) for antichiral fields and

\[
Z_i^a \frac{\partial}{\partial Z_i^a} = z^a \frac{\partial}{\partial z^a} + \frac{1}{2} \theta^a \frac{\partial}{\partial \theta^a} + \frac{1}{2} \theta^* \frac{\partial}{\partial \theta^*} + c.c. \tag{21}\]

for nonchiral fields.

The most general relevant perturbation of (11) is

\[S_{pr} = g^k \int d^2 Z \Phi_k + h.c. \tag{22}\]

with coupling constants having charges \(q(g^k) = -q(g^*k) = \frac{(p+2-k)}{2(p+2)} = 2\epsilon_k\) where \(\epsilon_k \ll 1\) for \(1 \ll k\). At \(g^i = 0\) the space tangent to the manifold of chiral coupling constants has a basis of integrated chiral fields \(\Phi_i = \int d^2 Z \Phi_i(Z, \bar{Z})\) with the hermitian scalar product

\[(\Phi_i | \Phi_j^*) = \delta_{ij}. \tag{23}\]

The metric on the manifold \(\mathcal{M}\) is \(G_{ij} dg^i d\bar{g}^j = d^2 s\). As a consequence of the hermitian structure, the renormalization transformations (18) should be also hermitian.

Looking at the N=2 SUSY fusion rules \[4, 8\], we can find that in general fields renormalize as follows

\[
\tilde{\Phi}^k = \Phi^k + O(gg^*) \Phi^k + O(g^3) \tag{24}
\]

\[
\tilde{\Phi}_m^l = \Phi_m^l + O(g) \Phi_{m-2}^l + O(g^*) \Phi_{m+2}^l + O(g^2) \tag{25}
\]

The condition (24) is justified as the correct renormalization procedure for this perturbation if one considers the three point functions of chiral fields which includes \(\Phi^p\) field. From the N=2 SUSY Ward identities one can easily find that the only nonvanishing three point function could be \(\langle \Phi^k \Phi^{(k-2)} \Phi^p \rangle = CF(\theta_i, z_i)\) [6]. The fact that the lowest component of the \(\Phi^p\) is free scalar field allows us to calculate the \(C\) constant easily and one can check that it is \(C \approx \langle \bar{\phi}_k \phi_{(k-2)} \rangle = 0\). Therefore all three point functions involving the \(\Phi^p\) field vanish. This leads us to set the normalization
condition for this perturbation theory as in equation (24) if we choose the coordinate system with \( \partial_k G_{ij} = 0 \). Than the metric for chiral fields will have the form

\[
G_{ij} = \delta_{ij} + O(gg^*) \delta_{ij}
\]  

(26)

Since the second derivatives of the metric (at \( g = 0 \)) determine the curvature tensor in the frame in which \( \partial_k G_{ij} = 0 \), we have to look closely at the renormalization prescription for the chiral field in the next paragraphs. For nonchiral fields we use the condition of renormalization set down by the requirement that the renormalized metric is

\[
G_{ij} = \delta_{ij} + O(gg^*).
\]  

(27)

4 Renormalization of nonchiral fields

It follows from the N=2 SUSY Ward identities [3] that the only nonvanishing three point functions of \( \Phi_1, \Phi_2, \Phi_3 \) are the ones for which either \( q_1 + q_2 + q_3 = 0 \) or \( q_1 + q_2 + q_3 = \pm \frac{1}{2} \). This fact coincides with the soft breaking of the U(1)-chiral current symmetry [13]. According to these properties renormalized fields should have exactly the same charge as the unrenormalized fields, when expressed in terms of fixed point operators and the charge of the couplings is accounted for.

Using these two facts we get the general formula (in the perturbed theory (13)) for the renormalization of nonchiral fields in the first order in \( g(g^*) \)

\[
\tilde{\Phi}_m^l = \Phi_m^l + g A_{m,m-2}^l \Phi_{m-2}^l + g^* A_{m,m+2}^l \Phi_{m+2}^l + O(g^2). 
\]  

(28)

All fields in the above formula have to be nonchiral. For notational convenience we set \( \Phi^p = \Phi_1, \Phi_{-1}^l = \Phi_2, \Phi_{-2}^l = \Phi_3 \).

According to fusion rules the chiral fields \( \Phi^l_m \) could renormalize by mixing with \( D\bar{D}\Phi_{l \pm 2} \), but these fields have dimensions differing by a number close to \( \frac{1}{2} \) and so such mixing is not possible [8]. Fields \( \Phi^l_{\pm 1} \) do not mix either, as it is easy seen from the parafermion fusion rules and the N=2 SUSY Ward identities [4], [6]. The requirement that the metric has the form (27) is equivalent to the condition

\[
\frac{\partial}{\partial g} \langle \Phi_2(1)\Phi_3(0) \rangle \big|_{g=0} = 0
\]  

(30)

It is enough to consider the above equation only for two nonchiral fields, because fields \( \Phi_{m+2}^l, \Phi_{m-2}^l \) do not mix with each other. Following perturbation theory methods we get that

\[
A_{23} = \frac{1}{2} C (\Gamma_{13}^2 + \Gamma_{13}^1 - \Gamma_{13}^3)
\]

\[
A_{32} = \frac{1}{2} C (\Gamma_{12}^3 + \Gamma_{23}^1 - \Gamma_{13}^2)
\]  

(31)
where

\[ \Gamma_{23}^1 = \frac{1}{2\pi} |z_2 - z_3|^{2(\Delta_2 + \Delta_3 - 2 - \epsilon)} \int d^4Z_1 d^4\theta_2 d^4\theta_3 \langle \Phi_1(Z_1)\Phi_2(Z_2)\Phi_3(Z_3) \rangle \]

\[ = -\frac{\Gamma(1 - 2\epsilon)\Gamma(\epsilon + \Delta_3 - \Delta_2)\Gamma(\epsilon + \Delta_2 - \Delta_3)}{2\Gamma(2\epsilon)\Gamma(1 - \epsilon + \Delta_3 - \Delta_2)\Gamma(1 - \epsilon + \Delta_2 - \Delta_3)} \times \]

\[ \times \left[ (1 + \Delta_2 + \Delta_3 - \epsilon)\Delta_2 + \Delta_3 - \epsilon \right]^2 \]

\[ \Gamma_{13}^2 = -\frac{2}{\Gamma(1 - \epsilon + \Delta_2 - \Delta_3)\Gamma(\epsilon + \Delta_3 + \Delta_2)\Gamma(1 - 2\Delta_2)} \times \]

\[ \times \left[ (1 + \Delta_2 + \Delta_3 - \epsilon)(\Delta_2 - q_2) \right]^2. \tag{32} \]

\(\Gamma_{13}^2\) is obtained in a similar way as \(\Gamma_{23}^1\) by integration over all variables corresponding to the field \(\Phi_2 = \Phi_m^l\) and \(\Gamma_{12}^2\) is obtained by the exchange of indices 2 \& 3 in \(\Gamma_{13}^2\). \(\Delta, q\) are respectively dimensions and charges of fields. \(C\) is a constant derived from fusion rules and the normalization of fields (11), (12) of fields \(\Phi\) is defined up to a multiplicative constant. Under the perturbation (13) the space of three point function derived directly from N=2 SUSY Ward identities [6] which is defined up to a multiplicative constant. Under the perturbation (13) of the space of fields \(\Phi_m^l\) (for each \(l\)) is divided into two invariant subspaces of \(m\) odd or even. Within each subspace one can write the formula for the anomalous dimension matrix \(\Gamma\) (fields \(\Phi^l, \Phi^l\bar{\ell}, \) being chiral, are excluded)

\[ \gamma_j^i(g) = \begin{pmatrix}
\cdot & \cdot & \Delta_{m+2} & 0 & 0 & 0 \\
\cdot & \Delta_{m+2} & \gamma_{m+2,m} & 0 & 0 & 0 \\
0 & g^*\gamma_{m+2,m} & \Delta_m & g\gamma_{m,m-2} & 0 & 0 \\
0 & 0 & g^*\gamma_{m-2,m} & \Delta_{m-2} & \cdot & \cdot \end{pmatrix} \tag{33} \]

where \(\gamma_{m,m-2} = \gamma_{m-2,m} = \frac{1}{2} C[\epsilon\Gamma_{23} + (\Delta_3 - \Delta_2)(\Gamma_{13}^2 - \Gamma_{12}^2)]\).

In the general case it is not easy to find the eigenvalues of this matrix, but one can see that it is hermitian and that the eigenvalues will be of order \(gg^*\), \(\Delta_m = \Delta_m + O(gg^*)\), as could be expected.

5 Renormalization of chiral fields

Considering the U(1) charge conservation we can expect that the chiral field \(\Phi^k\) would get renormalized according to

\[ \tilde{\Phi}^k = \Phi^k + A_k^p g^k g^p + A_k^{k+2} g^k g^{k+2} + A_k^{p+k} g^k g^p + O(g^3). \tag{34} \]

where the barred indicies correspond to the antichiral coupling constants and fields. The requirement that \(\partial_k G_{ij} = 0\) and the fact that the three point function of chiral fields \(\langle \Phi^k \Phi^{k'} \Phi^p \rangle = 0\) for any \(k, k'\), sets the constants \(A_k^p, A_k^{k+2}\) to zero.
We can restrict our attention to the second order derivatives of metric tensor which for the perturbed theory (13) is given by

\[
\frac{1}{(2\pi)^2} \int_{|z_1 - w_1|^2 = 1} d^2 Z_2 d^2 W_2^* \left( \Phi^k(Z_1) \Phi^{*k}(W_1^*) \Phi^p(Z_2) \Phi^{*p}(W_2^*) \right)^2
\]

where we set also \(\theta_{z_1} \theta_{w_1}^{*} = 0\), which allows us to keep the normalization of the Zamolodchikov metric [8] at the fixed point independent of the dimension of the field \(\Phi^k\).

For the Riemannian geometry the curvature tensor can be expressed in this case as

\[
R_{abcd} = -\frac{1}{2} (G_{ac,bd} + G_{bd,ac} - G_{ad,bc} - G_{bc,ad})
\]

For example the curvature component \(R_{kp\bar{p}\bar{k}}\) is given by

\[
R_{kp\bar{p}\bar{k}} = -\frac{1}{2} (G_{kp,\bar{k}\bar{p}} + G_{\bar{k}\bar{p},kp} - G_{p\bar{p},k\bar{k}} - G_{k\bar{k},p\bar{p}}).
\]

In the bare coordinate system this would read

\[
R_{kp\bar{p}\bar{k}} = \frac{1}{2} (I_0 + \tilde{I}_0 - I_2)
\]

where

\[
I_0 = \int_{|z_1 - w_1|^2 = 1} d^2 Z_2 d^2 W_2^* \left( \Phi^k(Z_1) \Phi^{*k}(W_1^*) \Phi^p(Z_2) \Phi^{*p}(W_2^*) \right)^2
\]

\[
\tilde{I}_0 = I(p, k)
\]

\[
I_2 = \int_{|z_1 - w_1|^2 = 1} d^2 Z_2 d^2 W_2^* \left( \Phi^k(Z_1) \Phi^{*k}(Z_2^*) \Phi^p(W_1) \Phi^{*p}(W_2^*) \right)^2
\]

The four point functions of equation (39) and their integrals \(I_0\) are calculated in the Appendix.

The fact that the integral \(I_2 = 0\) follows from considering the four point function of \(I_2\) with lowest components of \(\Phi^p, \Phi^{*k}\) fields. It is equal to total derivative in \(w_2\) as can be easily checked using the supersymmetry (as in the \(I_0\) case) and its integral is zero. In fact one could expect such result because the two point functions of chiral-chiral
field vanish due to N=2 SUSY and so there could not be a supersymmetric result of integral $I_2$.

Let's further consider what limits are imposed by the U(1) symmetry on the curvature tensor at $g = 0$ point i.e. how can it be changed by the allowed coordinate transformation. Because of the U(1) symmetry the components of the curvature tensor cannot be changed by a linear transformation of coordinates as one cannot freely rotate one coordinate into the other. Since in our perturbation theory (13) we are limited only to the third order in $g$ coordinate transformation it means that the curvature tensor is restricted to have the value it has in the bare coordinate system as calculated in equation (38). It means that if we allow the second order in $gg^*$ renormalization of chiral fields the curvature tensor would have to be

$$R_{kppk} = \frac{1}{2}(I_0 + \tilde{I}_0 + 2A_{ppk}^k + 2A_{kkp}^p)$$

...(42)

where constants are related to the renormalization of the fields by the relation

$$\Phi^p = \phi^p + A_{kkp}^k g^k \Phi^p$$

...(43)

$$\Phi^k = \phi^k + A_{ppk}^p g^{*k} \Phi^k$$

...(44)

This for $k = p$ means that the second order renormalization of the $\phi^p$ field must be zero (if there are no first order terms) because the curvature is a tensor and must transform homogeneously. Therefore we need to look more closely at what form of chiral fields renormalization is allowed.

It is possible to better understand and confirm this result if we adopt the method of looking at renormalization of the conformal fields as the coupling constant dependent contact terms in the operators product expansion of chiral fields which was developed by Kutasov [27].

Since we are only interested in the four point function of $\langle \Phi_k \Phi_{*k} \Phi_p \Phi_{*p} \rangle$ or in other words the second order derivatives of $G_{kk,pp}$ or $G_{pp,kk}$ we can restrict our attention to the contact terms which arise from the expansion of two chiral fields

$$\Phi_k(Z_1)\Phi_p(Z_2) = (conformal \ theory) + \delta^2(Z_{12}) C_{kkp}^{*k} \Phi_{*k}(Z_2) + \delta^2(Z_{12}) C_{kp}^{*k} g^{*k} \Phi_p(Z_2)$$

...(45)

where $\delta^2(Z_{12}) = \delta^2(\theta_1 - \theta_2)\delta^2(z_{12})$ is the SUSY delta function and the $C_{jl}^i$ constant can depend on $gg^*$, $g^k g^{*k}$. In principle we could have also the contact terms from the OPE of chiral-antichiral field but it is easy to check that such contact terms could contribute to the second order derivatives of $G_{ij}$ by zeroth order in $g$ components and these have to be zero. We can read it from the expansion $\Phi_k(Z_1)\Phi^{*p}(Z_2) \approx \delta^2(z_{12}) C(g = 0) \phi_{l}^{k-p}(z_2, \theta_1, \theta_2^*)$ (where $\phi_{l}^{k-p}$ is a nonchiral field). At $g = 0$ conformally invariant point the left and right hand side of this relation should have the same
dimensions and since the $\frac{k+p}{2(p+2)} < 1$ for any $k$ there can not be such a term as always the dimension of the right hand side would be greater than 1.

We can reexpress the second order derivatives of the metric tensor by the contact terms as

$$
\partial_p \partial_p \langle \Phi_k(Z) \Phi_k^*(0) \rangle = I_0 + |C_{kp}^{k-2}|^2 \langle \Phi_{k-2}(Z) \Phi_{k-2}^*(0) \rangle + \left( \frac{C_{kp}^k + C_{kp}^{k-2}}{C_{kp}} \right) \langle \Phi_k(Z) \Phi_k^*(0) \rangle
$$

so that we can identify (at $g = 0$) the first order derivatives of the contact term constants with the second order in $g$ renormalization constants for the fields ($C_{kp}^k g^* \bar{g} = C_{kp}^k = A_{pjk}$ and $C_{kp}^{k-2} = 0$).

Following the method developed by Kutasov [27] the first order derivatives of the contact term of the expansion (14) can be determined by the four point function in the frame in which the first order derivatives of the metric are set to zero. More precisely they are equal to the terms which are proportional to the delta type singularities arising from the once integrated four point function [27]. In our case we have to consider the contact term coming from

$$
\frac{1}{(2\pi)^2} \int d^2 W_2 \bigg| \langle \Phi^k(Z_1) \Phi^{*k}(W_1^*) \Phi^p(Z_2) \Phi^{*p}(W_2^*) \rangle \bigg|^2 = \left( \text{finite term} \right) + \delta^2(Z_{12}) C_{kp}^k \langle \Phi_k(Z_1) \Phi_k^*(W_1^*) \rangle
$$

Let’s consider the lowest components of $\Phi_k$ field four point function which is calculated in the Appendix

$$
\langle \phi^k(z_1) \phi^{*k}(w_1) \psi^p(z_2) \psi^{*p}(w_2) \rangle = \frac{4}{|z_{12}|^2} [2\epsilon \langle \phi^4 \rangle]^2 + \partial_{w_2}(2\epsilon \langle \phi^4 \rangle)[w_2 - z_1] \langle \phi^4 \rangle + \text{c.c}
\begin{equation}
+ |\partial_{w_2}(w_2 - z_1) \langle \phi^4 \rangle|^2 + \text{tot.derivatives}
\end{equation}

where now the total derivatives terms include the dependence $\partial_{w_2} \frac{1}{w_2 - z_i} = i\pi \delta^2(w_2 - z_i)$ [19], and $\langle \phi^4 \rangle = \langle \phi^k(z_1) \phi^{*k}(w_1) \phi^p(z_2) \phi^{*p}(w_2) \rangle$. Integrating over the $w_2$ function (18) we find that the contact term dependence gives the zero coefficient in front of the $\delta^2(Z_{12})$ term. The contribution from integrating the $\langle \phi^4 \rangle$ term is also finite as in the limit $z_{12} \to 0$. The last statement can be checked using the known formula [22].

\begin{align}
&\int d^2 z |z|^{2a} |1 - z|^{2b} |\zeta - z|^{2c} = \frac{s(a + b + c) s(b)}{s(a + c)} |J_1(a, b, c, \zeta)|^2 + \\
&+ \frac{s(a) s(c)}{s(a + c)} |J_2(a, b, c, \zeta)|^2
\end{align}
Here

\[ J_1(a, b, c, \zeta) = \frac{\Gamma(-a - b - c - 1)\Gamma(b + 1)}{\Gamma(-a - c)} \times F(-c, -a - b - c - 1, -a - c, \zeta) \]

\[ J_2(a, b, c, \zeta) = z^{1 + a + c} \frac{\Gamma(a + 1)\Gamma(c + 1)}{\Gamma(a + c + 2)} F(-b, a + 1, a + c + 2, \zeta) \]

(50)

\( F \) is the hypergeometric function, and \( s(a) = \sin(\pi a) \).

We find that the integration gives (with \( z_1 = 1 \) and \( w_1 = 0 \) for clarity)

\[
(4\epsilon)^2 |z_2 - 1|^{-2(1-k/(p+2))} |z|^{2k/(p+2)} \frac{s\left(\frac{p}{p+2}\right)s\left(\frac{k}{p+2}\right)}{s\left(\frac{p+k}{p+2}\right)} \times \\
\times \left[ \frac{\Gamma(-1 + \frac{p}{p+2})\Gamma(1 + \frac{k}{p+2})}{\Gamma\left(\frac{p+k}{p+2}\right)} F\left(\frac{p}{p+2}, -1 + \frac{p}{p+2}, \frac{p+k}{p+2}, 1 - z\right) \right]^2 - \\
\frac{\Gamma(1 - \frac{p}{p+2})\Gamma(1 - \frac{k}{p+2})}{\Gamma(2 - \frac{p+k}{p+2})} z^{1-\frac{p+k}{p+2}} F\left(-\frac{k}{p+2}, 1 - \frac{k}{p+2}, 2 - \frac{p+k}{p+2}, 1 - z\right) \right]^2
\]

(51)

which behaves as \( |z - 1|^{-2(1-k/(p+2))} \) when \( z \to 1 \) and is no singular enough to give contribution to delta type singularity (integral over a small neighborhood around 1 goes to zero).

From the contact terms consideration we find the same result we concluded before. The SUSY demands that the second order in \( |g| \) renormalization of the chiral fields is zero.

One more way of understanding this result is to consider how the metric tensor components \( G_{pp} \) would have to change under the allowed coordinate transformation \( \tilde{g} = g(1 + a g^* g) \). This kind of transformation would have to produce the nonzero component \( \tilde{G}_{pp} = -ag^* g \) which is not compatible with the chiral renormalization of the fields (34). This means that such a coordinate transformation have to be excluded.

From the Callan-Szymanzik equation (13) we find that the anomalous dimensions matrix and the beta function remain unchanged.

\[ \gamma_k^i = \Delta_k \delta_k^i \]  \( \beta(g) = \epsilon g \)  \( (52) \)  \( (53) \)
6 Discussion

Our calculation shows that there is no perturbative fixed point near the $A_p$ fixed point theory [2], [3], [12], [13]. As we have mentioned in the introduction such result is well understood on the consideration of the Witten index $Tr(-)^F = p$ of N=2 SUSY theory, which can not be changed perturbatively. We know [2], [12], [13] that in the infrared limit the theory given by the superpotential $W(\Phi) = \Phi^{p+2} + g\Phi^p$ will flow to the fixed point theory $W(\Phi) = \Phi^p$ so these two fixed points should be infinitely apart as was argued by Cvetič and Kutasov [7]. Our result for the $\beta$ function $\beta = g\epsilon$ confirms that to the leading order in $g$. It states that the nonrenormalization theorem holds in this perturbation theory in a simple way $\tilde{\Phi} = \Phi$. The most important content of it is that the chiral renormalization of the fields is not compatible with the allowed (by U(1) charge conservation) redefinition of the coupling constants $\tilde{g} = g(1 + agg^*)$.

This information can be also read off from the expansion of the Zamoldichikov metric $G_{ij}$ in the bare coordinates $G_{kk}^0 = \delta_{kk}(1 + I(k, p))$ where $I(k, p) = 1 + O(\epsilon)$. There are no singular terms in the expansion of the metric as $\epsilon \to 0$. The chiral bare couplings coordinate system gives a well defined metric along the renormalization group flow given by the perturbed potential $W(\Phi) = \Phi^{p+2} + g\Phi^p$. There is no need for the singular coupling redefinitions to keep the metric regular in the neighborhood of the fixed point as it is the case in N=0, N=1 SUSY L-G models [14], [3].

Our result is also in agreement with the recent work by West and Howe [28]. It was shown there that as long as the SUSY is preserved by the perturbation the effective potential does not get renormalized. However there is the possibility that the nonrenormalization theorem would not hold in the more general renormalization prescription scheme [28]. The present work states the nonrenormalization theorem even more restrictively, showing that there are no compatible with N=2 SUSY renormalization constant terms of order $gg^*$ other than zero.

We feel also obliged to comment on the result obtained by Leaf-Hermann [25] for the integration of the four point function $\langle \psi^p\psi^p\psi^p\psi^p \rangle$ over two variables. Clearly his integral is the special case of our integration formulas when $k = p$ ($\epsilon_k = \epsilon$). The discrepancy between our $I_{hh} \approx 4\pi^2$ and his result $I_{hh}(L - H) \approx 4\pi^2\epsilon$ is quite substantial and so deserves a full comment. The four point functions we integrate are identical (however obtained by different methods) as can be easily checked, so the source of the difference lies in the method of integration. The problem is caused by the expansion of the $2F_1(2\epsilon, 2\epsilon; 4\epsilon; z)$ and $2F_1(1 - 2\epsilon, 1 - 2\epsilon; 1 - 4\epsilon; z)$ hypergeometric functions of [25]. In general any function with the parameters $2F_1(\alpha, \alpha + m; \gamma; z)$ where $m$ is natural number, will have a very nonuniform expansion [26], and additionally when $2\alpha + m = \gamma$ there will be also a logarithmic divergence at $z = 1$ [28]. Consequently for the above set of parameters the hypergeometric function contains the infinite series of logarithms [26], and the approximation of $2F_1$ functions for small $\epsilon$, which we believe was made in the [25] is not valid. Neglecting the fact of highly ununiform behaviour of this type of hypergeometric functions would probably always
lead to the wrong answer. Analytic continuation of the complete 2D integrals will lead to the correct answer free of divergences for every term in $\epsilon$ expansion as we have shown in the appendix.

I would like to thank Emil Martinec for bringing this problem to my attention and spending considerable amount of time on discussions. I would like also to acknowledge the referee for constructive comments (specially pointing out the nonuniform behaviour of some hypergeometric functions) which led us to correct some major flaws in an early version of this paper. This paper is submitted in partial fulfillment of the requirements for a Ph.D. degree in physics at the University of Chicago.

7 Appendix

The four point function for the highest components of $\langle \Phi^k \Phi^* \Phi^P \Phi^*P \rangle$ can be easily calculated in terms of the lowest components using the OPE with the supercurrents $G_1^\pm$ [3, 18]. For clarity in the formulas of this section we introduce the decomposition of (anti)chiral fields in terms of the lowest $\varphi^k$ and highest $\psi^k$ components,

$$\Phi^k(Z) = \varphi^k(z) + \theta \psi^k(z)$$

$$\Phi^*k(Z^*) = \varphi^*k(z) + \theta^* \psi^*k(z)$$

and mostly suppress antiholomorphic dependence.

The relative normalization of $\varphi^k$ and $\psi^k$ is given by the two point function [12]. Then the four point function of highest components is equal to

$$\langle \psi^k(z_1) \psi^*k(w_1) \psi^p(z_2) \psi^*p(w_2) \rangle =$$

$$= \frac{4}{(z_1 - z_2)} \left[ \frac{p}{p + 2} \partial_{w_1} + \frac{k}{p + 2} \partial_{w_2} + (w_1 - w_2) \partial_{w_1} \partial_{w_2} \right] \times$$

$$\times \langle \varphi^k(z_1) \varphi^*k(w_1) \varphi^p(z_2) \varphi^*p(w_2) \rangle$$

Using the fact that $\varphi^p = : \exp \left( \frac{i p \varphi}{\sqrt{2 p(p + 2)}} \right)$: one can easily find the formula for the four point function up to contact terms

$$\langle \psi^k(z_1) \psi^*k(w_1) \psi^p(z_2) \psi^*p(w_2) \rangle =$$

$$\frac{4 k}{p + 2} \left( \frac{z_1 - w_1}{z_2 - w_2} \right)^{\frac{k}{p + 2}} \left( \frac{z_1 - z_2}{w_1 - w_2} \right)^{\frac{k}{p + 2}} \times$$

$$\times \left[ 1 + \frac{p}{p + 2} \left( \frac{z_1 - w_2}{z_2 - w_2} \right) \left( \frac{z_1 - z_2}{w_1 - w_2} \right) +$$

$$+ \frac{k}{p + 2} \left( \frac{w_2 - z_2}{z_1 - z_2} \right) \left( \frac{w_2 - w_1}{z_1 - w_1} \right) \right]$$

Analogously we can calculate the contribution for the supersymmetric four point function from the lowest components of chiral fields.
\[
\langle \phi^k(z_1) \phi^k(w_1) \psi^p(z_2) \psi^p(w_2) \rangle = \\
= \frac{2}{(z_1 - z_2)} \left[ -\frac{p}{p + 2} + (z_1 - w_2) \partial_{w_2} \right] \times \\
\times \langle \varphi^k(z_1) \varphi^k(w_1) \varphi^p(z_2) \varphi^p(w_2) \rangle = \\
= \frac{2(z_1 - w_1)}{p + 2} \left( \frac{z_2 - w_2}{z_1 - z_2} \right)^\frac{p}{p + 2} \left[ \frac{(z_1 - z_2)(w_1 - w_2)}{(z_1 - w_2)(w_1 - z_2)} \right] \times \\
\times \left[ -\frac{p}{p + 2} \frac{(z_1 - z_2)}{z_1 - z_2} + \frac{k}{p + 2} \frac{(w_1 - z_1)}{w_2 - w_1} \right]
\]
\[
(57)
\]

The integral of (56) or (57) over \(Z_2\) and \(W_2\) has various poles of order close to 2 and logarithmic ones. These poles are the consequence of the OPE of any two chiral operators which produces the pole under the integration whenever two (or more) operators come close together. Logarithmic poles are the consequence of the presence of the U(1) current in the OPE of chiral-antichiral fields. Due to SUSY there are contact terms [19] which exactly cancel these poles. This comes from a more careful consideration of the OPE of the holomorphic and antiholomorphic part of chiral fields.

The OPE of the holomorphic part of the chiral field, given in terms of the lowest and highest components, normalized by the two point function (12), is

\[
\varphi^k(z_1) \varphi^k(z_2) = \frac{1}{z_1^{2\Delta}} \left[ 1 + z_1 \frac{2q}{3c} J(z_2) + \cdots \right]
\]

\[
\psi^k(z_1) \psi^k(z_2) = \frac{4\Delta}{z^{2\Delta + 1}} \left[ 1 + z_1 \frac{2q - 1}{3c} J(z_2) + \cdots \right]
\]

The antiholomorphic parts have the same OPE. In expansion 58 we write down only terms which produce poles under the integral. A logarithmic pole arises from the OPE of the charge current \(J(z)\) with a third operator present in the four point function \(J\). \(J\) is the lowest component of a superfield; therefore a supersymmetric renormalization procedure should cancel these divergences. Analytic continuation of the hypergeometric function which arises from integration of (56) automatically subtracts these divergences and is compatible with N=2 supersymmetry, as our calculation shows.

To evaluate the integral (59) we calculate the contribution given by different components of superchiral field \(\Phi^k\). All the formulas involved can be written more clearly if we use the translational invariance and choose the scale by putting down \(z_1 = 1, w_1 = 0\). It is also convenient to introduce new coordinates

\[
\zeta = \frac{z_2(w_2 - 1)}{w_2(z_2 - 1)}, \quad \eta = \frac{w_2 - 1}{w_2}.
\]

With these simplifications the contributing parts of integral (59) look like
The most general form of this expression is formula (3.26) in a simple way using the generalized hypergeometric functions. The integrals. After a simple change of integration variables we can express the KLT [20] the formula for the five point function which has exactly the same form as the above and Tye [20] to calculate closed string amplitudes. Here we can almost directly use the components of the chiral superfields \( \Phi^k, \Phi^{\ast k} \).

To evaluate these integrals we follow the method developed by Kawai, Lewellen and Tye [20] to calculate closed string amplitudes. Here we can almost directly use the formula for the five point function which has exactly the same form as the above integrals. After a simple change of integration variables we can express the KLT [20] formula (3.26) in a simple way using the generalized hypergeometric functions. The most general form of this expression is

\[
I_{\hbar} &= \int d^2\zeta d^2\eta \left| 4^k \frac{k}{p+2} (\zeta - \eta)^{-2k} (1 - \zeta)^{-1 + 2k} (1 - \eta)^{-1 + 2k} \right|^2 \\
&\times \left| 1 + \frac{p}{p+2} \frac{\zeta}{(1 - \zeta)} + \frac{k}{p+2} \left[ -1 + \frac{(1 - \zeta)}{\zeta} \right] \right|^2 \\
I_{\h\ell} &= \int d^2\zeta d^2\eta \left| 2^k \frac{k}{p+2} (\zeta - \eta)^{-2k} (1 - \zeta)^{-1 + 2k} (1 - \eta)^{-1 + 2k} \right|^2 \\
&\times \left( 1 + \frac{p}{p+2} \frac{\zeta}{(1 - \zeta)} + \frac{k}{p+2} \left[ -1 + \frac{(1 - \zeta)}{\zeta} \right] \right) \\
&\times \left( \frac{p}{p+2} \frac{1}{(1 - \zeta)} - \frac{k}{p+2} \right) \\
I_{\ell\bar{l}} &= \int d^2\zeta d^2\eta \left| 2^{2k} (\zeta - \eta)^{-2k} (1 - \zeta)^{-1 + 2k} (1 - \eta)^{-1 + 2k} \right|^2 \\
&\times \left| \frac{p}{p+2} \frac{1}{(1 - \zeta)} - \frac{k}{p+2} \right|^2
\] (60) (61) (62)

The indexes \( l, (\bar{l}) \) and \( h, (\tilde{h}) \) refer respectively to the lowest and highest left (right) components of the chiral superfields \( \Phi^k, \Phi^{\ast k} \).

To evaluate these integrals we follow the method developed by Kawai, Lewellen and Tye [20] to calculate closed string amplitudes. Here we can almost directly use the formula for the five point function which has exactly the same form as the above integrals. After a simple change of integration variables we can express the KLT [20] formula (3.26) in a simple way using the generalized hypergeometric functions. The most general form of this expression is

\[
J &= \int d^2z_1 d^2z_2 |z_1|^{2a_1} |1 - z_1|^{2b_1} |z_1 - z_2|^{2c} |z_2|^{2a_2} |1 - z_2|^{2b_2} \\
&\times z_1^{m_1} (1 - z_1)^{n_1} z_2^{m_2} (1 - z_2)^{n_2} \\
&= s(a_1) s(b_2) \int d\zeta \int d\zeta |1 - \zeta|^{\tilde{a}_1} |\zeta|^{\tilde{b}_1} |1 - \zeta| c |\zeta|^{a_2} |\zeta|^{b_2} \\
&\times \int d\eta_1 \int d\eta_2 |1 - \eta_1|^{\tilde{a}_2} |\eta_1 - \eta_2|^{\tilde{b}_2} |\eta_2|^{a_1} |\eta_2|^{b_1} |1 - \eta_2|^{b_1} \\
&\times (1 \leftrightarrow 2)
\] (63)

Where

\[
m_i, n_i \in Z \ i = 1, 2 \\
\tilde{a}_i = a_i + m_i, \quad \tilde{b}_i = b_i + n_i, \quad (1 \leftrightarrow 2) \text{ means exchange of indexes } 1, 2 \text{ and } s(a) = \sin(\pi a).
\] (64)
After a change of integration variables

$$\eta_1 \rightarrow \frac{1}{1 - \eta_1}$$ (65)

we can express the double integral in $\zeta_i$ and $\eta_i$ as $\mathbf{3F}_2$ generalized hypergeometric functions [21], [22]. Using the integral definitions of hypergeometric functions [21], [22] and standard transformation formulas of them [21], [22] we find the result for $J$ to be

$$J = \pi^2 \frac{\Gamma(a_1 + 1)\Gamma(b_2 + 1)\Gamma(a_2 + a_1 + c + 2)\Gamma(c + 1)}{\Gamma(a_1 + c + 2)\Gamma(b_2 + a_1 + c + 3)} \times$$

$$\times \mathbf{3F}_2(-b_1, a_1 + 1, a_2 + a_1 + c + 2; a_1 + c + 2, b_2 + a_1 + c + 3; 1) \times$$

$$\times \frac{\Gamma(-1 - b_1 - a_1 - c)\Gamma(-1 - b_2 - a_2 - c)}{\Gamma(-b_1 - c)\Gamma(-a_2 - c)\Gamma(-b_2)\Gamma(-a_1)} \times$$

$$\times \mathbf{3F}_2(-c, 1 + a_1, 1 + b_2; -c - a_2, -c - b_1; 1) + (1 \leftrightarrow 2)$$ (66)

After plugging in the values of the exponents from (60), (61), (62) into the equation (63) we get a not very appealing expresion containing values of $\mathbf{3F}_2$ type hypergeometric functions [23] at one (which we will call further $\mathbf{3F}_2$ series). The general summation formula for $\mathbf{3F}_2$ series analogous to the Gauss summation formula for "regular" hypergeometric functions, $\mathbf{2F}_1(a, b; c; 1) = \frac{\Gamma(a)\Gamma(c-a-b)}{\Gamma(a+b-c)}$, is clearly full-filed for the $\mathbf{3F}_2$ series appearing here. For the $\epsilon \ll 1$ we have that $I(k, p) = 1 + \mathbf{O}(\epsilon^2)$

$$I_{k,l} = I(k, p) = \Gamma^2(1 + 2\epsilon)\Gamma^2(1 - 2\epsilon) \times$$

$$\times [\mathbf{3F}_2^2(2\epsilon, -2\epsilon, 2\epsilon_k; 1, 1; 1) +$$

$$+ (2\epsilon)^2(1 - 2\epsilon_k)^2 \mathbf{3F}_2^2(1 + 2\epsilon, 1 - 2\epsilon, 2 - 2\epsilon_k; 2, 2; 1)]$$ (67)

$$I_{h,l} = I_{l,h} = I(k, p) \times 4(\Delta_k - \epsilon)$$ (68)

$$I_{h,h} = I(k, p) \times [4(\Delta_k - \epsilon)]^2$$ (69)

The above expressions are finite since a generalized hypegeometric function $\mathbf{3F}_2(d_1, d_2, d_3; e_1, e_2; z)$ is absolutely convergent at $z = 1$ if $(e_1 + e_2 - d_1 - d_2 - d_3) > 0$ [23], which is clearly full-filed for the $\mathbf{3F}_2$ series appearing here. For the $\epsilon \ll 1$ we have that $I(k, p) = 1 + \mathbf{O}(\epsilon^2)$
because the \( 3F_2(2\epsilon, -2\epsilon, 2\epsilon k; 1, 1; 1) \) is finite for any \( \epsilon \) and is 1 at \( \epsilon = 0 \) and also \( 3F_2(1 + 2\epsilon, 1 - 2\epsilon, 2 - 2\epsilon k; 2, 2; 1) \) is finite for all \( \epsilon \) as well as for \( \epsilon = 0 \). The last statement is not obvious but can be easily checked using the formula given by Luke

\[
3F_2(d, 1, 1; 2, e; 1) = \frac{e - 1}{d - 1} (\Psi(e - 1) - \Psi(e - d))
\]

(70)

where \( \Psi(x) = \frac{d}{dx} \ln \Gamma(x) \).

Unfortunately we were unable to find generally valid transformation of the expression (66) which would lead us more directly to the above answers. However it is possible to do a couple of checks of the results we obtained.

First it can be easily seen that the four point functions (56) and (57) contain a total divergence with respect to \( w_2 \) pieces. So following common sense the integration of the four point function with and without these divergencies should be the same. This check is nontrivial because the total divergence terms are in fact divergent as \( w_2 \to \infty \) like \( w_2^{2\epsilon} \) and if true confirms that the analytic continuation really holds.

For the lowest and highest components of chiral fields we have the following four point functions, defined up to total divergence term

\[
\langle \phi^k(z_1) \phi^*k(w_1) \psi^p(z_2) \psi^*p(w_2) \rangle = \frac{2\epsilon(1 - w_2)^{-\frac{1}{p+2}} (z_2 - w_2)^{-\frac{\epsilon}{p+2}}}{(z_1 - z_2)^{\frac{1}{p+2}} (z_1 - w_2)(w_1 - z_2)} + \partial w_2 (\ldots)
\]

(71)

\[
\langle \psi^k(z_1) \psi^*k(w_1) \psi^p(z_2) \psi^*p(w_2) \rangle = \frac{4\epsilon (z_1 - w_1)^{-\frac{1}{p+2}} (z_2 - w_2)^{-\frac{\epsilon}{p+2}}}{(z_1 - z_2)^{\frac{1}{p+2}} (z_1 - w_2)(w_1 - z_2)} \times \left[ 1 - \frac{(w_2 - z_2)(w_1 - z_1)}{(w_1 - z_2)(w_1 - w_2)} \right] + \partial w_2 (\ldots)
\]

(72)

As a check for our results we did calculate the four point function, with total divergence term included and without it. In either case we obtain the same result (67), (68), (69). This way we also prove that total divergences do not have to be included in analytic continuation procedure even if they are possibly contributing a divergent term.

Another prove that our calculation is correct is the fact that we obtain a supersymmetric answer. The integrated supersymmetric four point function has a compact
\[ \int d^2z_2 d^2w_2 \langle \Phi^k(z_1, \bar{z}_1) \Phi^{*k}(w_1, \bar{w}_1) \Phi^p(z_2, \bar{z}_2) \Phi^{*p}(w_2, \bar{w}_2) \rangle = 
\]
\[ = 4\pi^2 |z_1 - w_1|^{-4(\Delta_k - \epsilon)} \left( 1 + 4(\Delta_k - \epsilon) \frac{\theta_{z_1} \theta_{w_1}^{*}}{|z_1 - w_1|} \right)^2 \times I(k, p) \] 

which is exactly a N=2 SUSY two point function of a chiral field with a dimension and charge equal to \((\Delta_k - \epsilon)\). After the above checks we can believe that the result we got is correct.

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