THE IRREDUCIBLE MODULES FOR THE DERIVATIONS OF THE RATIONAL QUANTUM TORUS

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Abstract. Let $\mathbb{C}_q$ be the quantum torus associated with the $d \times d$ matrix $q = (q_{ij})$, $q_{ii} = 1$, $q_{ij}^{-1} = q_{ji}$, $q_{ij}$ are roots of unity, for all $1 \leq i, j \leq d$. Let $\text{Der}(\mathbb{C}_q)$ be the Lie algebra of all the derivations of $\mathbb{C}_q$. In this paper we define the Lie algebra $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$ and classify its modules which are irreducible and have finite dimensional weight spaces. These modules under certain conditions turn out to be of the form $V \otimes \mathbb{C}_q$, where $V$ is a finite dimensional irreducible $\text{gl}_d$-module.

1. Introduction

Let $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the Laurent polynomial ring in $d$ commuting variables. Let $\text{Der}(A)$ be the Lie algebra of diffeomorphisms of $d$-dimensional torus. It is well known that $\text{Der}(A)$ is isomorphic to the derivations of Laurent polynomial ring $A$ in $d$-commuting variables. Rao has given several irreducible representation of $\text{Der}(A)$ [7]. In a paper by Jiang and Meng [4], it has been proved that classification of irreducible integrable modules of the full toroidal Lie algebra can be reduced to the classification of irreducible $\text{Der}(A) \ltimes A$-modules. Rao has given a classification of irreducible modules for $\text{Der}(A) \ltimes A$ with finite dimensional weight spaces [2]. These modules are of the form $V \otimes A$, where $V$ is a finite dimensional irreducible $\text{gl}_d$-module.

Let $\mathbb{C}_q$ be a quantum torus associated with the $d \times d$ matrix $q = (q_{ij})$ where $q_{ii} = 1$, $q_{ij}^{-1} = q_{ji}$, $q_{ij}$ are roots of unity, for all $1 \leq i, j \leq d$. Let $\text{Der}(\mathbb{C}_q)$ be the Lie algebra of all the derivations of $\mathbb{C}_q$. In [5] W.Lin and S.Tan defined a functor from $\text{gl}_d$-modules to $\text{Der}(\mathbb{C}_q)$-modules. They proved that for a finite dimensional irreducible $\text{gl}_d$-module $V$, $V \otimes \mathbb{C}_q$ is a completely reducible $\text{Der}(\mathbb{C}_q)$-module except finitely many cases. Liu and Zhao [6] simplified Lin and Tan’s proof by proving that the “function $g(s)$” defined by them [5] can be taken as a constant function 1. In this paper we show that the $\text{Der}(\mathbb{C}_q)$-module $V \otimes \mathbb{C}_q$ is an irreducible $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-module (Proposition 2.5). The main purpose of this paper is to prove the converse of Proposition 2.5. We prove that (Theorem 2.6) if $V'$ is an irreducible $\mathbb{Z}^d$-graded $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-module with finite dimensional weight spaces which satisfies some conditions, then $V' \cong V \otimes \mathbb{C}_q$, as a $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-module and

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when restricted to $\text{Der}(C_q)$, it is isomorphic to the $\text{Der}(C_q)$-module defined in [6] and [5]. One of the conditions we put on $V'$ is called as condition (A) (Definition 2.4). The motivation behind the definition of this condition came from the fact that the $\text{Der}(C_q)$-module $V \otimes C_q$ defined in [6] satisfies condition (A) as a $\text{Der}(C_q) \times C_q$-module (Proposition 2.5). We will see that this condition is an important part of the hypothesis as it enables us to use Rao’s results [2] by which our proof simplifies significantly.

In section 2 we begin with the definition and properties of the quantum torus $C_q$. We prove that $C_q$ is a $\text{Der}(C_q)$-module and $\text{Der}(C_q) \times C_q$ is a Lie algebra (Prop 2.2 and Prop 2.3). Section 3 and section 4 are devoted to the proof of the Theorem 2.6. In section 4 we compute the actions of outer derivations of $\text{Der}(C_q)$ and $C_q$ on $V'$. In section 4 we derive the action of inner derivations on $V'$ and complete the proof.

2. Preliminaries

Let $q = (q_{ij})_{d \times d}$ be any $d \times d$ matrix with nonzero complex entries satisfying $q_{ii} = 1, q_{ij}^{-1} = q_{ji}, q_{ij}$ are roots of unity, for all $1 \leq i, j \leq d$. Let $\mathbb{Z}$ consider the non-commutative Laurent polynomial ring $S_{[d]} = \mathbb{C}[t_{1}^{\pm 1}, \cdots, t_{d}^{\pm 1}]$. Let $J_q$ be the ideal of $S_{[d]}$ generated by the elements $\{ t_{i}t_{j} = q_{ij}t_{j}t_{i}, t_{i}t_{i}^{-1} - 1, t_{i}^{-1}t_{i} - 1 \forall 1 \leq i, j \leq d \}$. Let $C_q = S_{[d]} / J_q$. Then $C_q$ is called the quantum torus associated with the matrix $q$. The matrix $q$ is called the quantum torus matrix.

For $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, let $t^n = t_1^{n_1} \cdots t_d^{n_d}$. Define $\sigma, f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ by

$$\sigma(n, m) = \prod_{1 \leq i < j \leq d} q_{ji}^{n_i m_j}, \quad f(n, m) = \sigma(n, m)\sigma(m, n)^{-1}$$

Then one has the following results [1]:

1. $\sigma(n + m, s + r) = \sigma(n, s)\sigma(n, r)\sigma(m, s)\sigma(m, r)$
2. $f(n, m) = f(m, n)^{-1}, \quad f(n, n) = f(n, -n) = 1$
3. $f(n + m, s + r) = f(n, s)f(n, r)f(m, s)f(m, r)$
4. $t^{n+m} = \sigma(n, m)t^{n+m}, \quad [t^n, t^m] = (\sigma(n, m) - \sigma(m, n))t^{n+m}, \quad \forall n, m, s \in \mathbb{Z}^d$.

For $f$, let $\text{rad}(f)$ denote the radical of $f$ which is defined by:

$$\text{rad}(f) = \{ n \in \mathbb{Z}^d : f(n, m) = 1 \forall m \in \mathbb{Z}^d \}$$

It is easy to see that $\text{rad}(f)$ is a subgroup of $\mathbb{Z}^d$. As $C_q$ is $\mathbb{Z}^d$-graded, we define derivations $\partial_1, \partial_2, \cdots, \partial_d$ satisfying

$$\partial_i(t^n) = n_i t^n \text{ for } n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d.$$ 

The inner derivations $\text{ad} t^n (t^m) = (\sigma(n, m) - \sigma(m, n))t^{n+m}$. Note that for $n \in \text{rad}(f)$, $\text{ad} t^n = 0$. For $u = (u_1, u_2, \cdots, u_d) \in \mathbb{C}^d$, define $D(u, r) = t^r \sum_{i=1}^d u_i \partial_i$. 


Let $\text{Der}(\mathbb{C}_q)$, be the space of all derivations of $\mathbb{C}_q$. Let $\text{Der}(\mathbb{C}_q)_n$ denote the set of homogeneous derivations of $\mathbb{C}_q$ with degree $n$. Then we have the following lemma:

**Lemma 2.1** ([1], Lemma 2.48). \(\text{(1)}\) $\text{Der}(\mathbb{C}_q) = \bigoplus_{n=\mathbb{Z}^d} \text{Der}(\mathbb{C}_q)_n$

\[\text{(2)}\]
\[
\text{Der}(\mathbb{C}_q)_n = \begin{cases} 
\text{Cad } t^n, & \text{if } n \notin \text{rad}(f) \\
\bigoplus_{i=1}^d \mathbb{C} t^n \partial_i, & \text{if } n \in \text{rad}(f)
\end{cases}
\]

The space $\text{Der}(\mathbb{C}_q)$ is a Lie algebra with the following bracket operations:

1. $[\text{ad } t^r, \text{ad } t^s] = (\sigma(s, r) - \sigma(r, s)) \text{ad } t^{r+s} \forall r, s \notin \text{rad}(f)$,
2. $[D(u, r), \text{ad } t^s] = (u, s)\sigma(r, s) \text{ad } t^{r+s} \forall r \in \text{rad}(f), s \notin \text{rad}(f), u \in \mathbb{C}_d$,
3. $[D(u, r), D(u', r')] = D(w, r + r') \forall r, r' \in \text{rad}(f), u, u' \in \mathbb{C}_d$ and where $w = \sigma(r, r')(u, r')u' - (u', r)u$.

**Proposition 2.2.** $\mathbb{C}_q$ is a $\text{Der}(\mathbb{C}_q)$-module with the following action:

1. $D(u, r).t^n = (u, n)\sigma(r, n)t^{r+n}, \forall r \in \text{rad}(f), n \in \mathbb{Z}^d, u \in \mathbb{C}_d$,
2. $\text{ad } t^n. t^m = (\sigma(s, n) - \sigma(n, s))t^{s+n}, \forall s \notin \text{rad}(f), n \in \mathbb{Z}^d$.

**Proof.** We have to prove the following:

1. $[D(u, r), D(v, s)]t^n = D(u, r)D(v, s)(t^m) - D(v, s)D(u, r)(t^m)$,
2. $[\text{ad } t^r, \text{ad } t^s]t^m = \text{ad } t^r \text{ad } t^s(t^m) - \text{ad } t^s \text{ad } t^r(t^m)$,
3. $[D(u, r), \text{ad } t^s]t^m = D(u, r)\text{ad } t^s(t^m) - \text{ad } t^s D(u, r)(t^m)$.

For proving (1) consider the L.H.S.

$[D(u, r), D(v, s)]t^n = D(w, r + s)t^m$, where $w = \sigma(r, s)((u, s)v - (v, r)u)$

$= (w, m)\sigma(r + s, m)t^{r+s+m}$

$= \sigma(r, s)\sigma(r + s, m)[(u, s)(v, m) - (v, r)(u, m)]t^{r+s+m}$

Now consider the R.H.S.

$D(u, r)D(v, s)(t^m) - D(v, s)D(u, r)(t^m)$

$= [[(v, m)\sigma(s, m)(u, m + s)\sigma(r, m + s) - (u, m)\sigma(r, m)(v, m + r)\sigma(s, m + r)]t^{m+s+r}$$

$= [(v, m)\sigma(s, m)(u, m)\sigma(r, m + s) - (u, m)\sigma(r, m)(v, m)\sigma(s, m + r)]t^{m+s+r}$

$= \sigma(r, s)\sigma(r + s, m)[(u, s)(v, m) - (v, r)(u, m)]t^{m+s+r}$

The conditions (2) and (3) follow similarly. \(\square\)

Consider the space $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$. We denote its element $(T, t^n)$ by $T + t^n$ where $T \in \text{Der}(\mathbb{C}_q)$ and $t^n \in \mathbb{C}_q$ for $n \in \mathbb{Z}^d$.

**Proposition 2.3.** $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$ is a Lie algebra with the following brackets:
algebra, we only need to check the following Jacobi identities:

1. \([D(u, r), t^n] = (u, n)\sigma(r, n)t^{r+n}, \forall r \in \text{rad}(f), n \in \mathbb{Z}^d, u \in \mathbb{C}^d,\)
2. \([\text{ad} t^s, t^n] = (\sigma(s, n) - \sigma(n, s))t^{s+n}, \forall s \notin \text{rad}(f), n \in \mathbb{Z}^d,\)
3. \([t^n, t^n] = (\sigma(m, n) - \sigma(n, m))t^{m+n}, \forall m, n \in \mathbb{Z}^d.\)

**Proof.** First note that as \(\text{ad} t^s\) acts the same way as \(t^s\) on \(\mathbb{C}_q\) and \(\mathbb{C}_q\) is a Lie algebra, we only need to check the following Jacobi identities:

1. \([D(u, r), [D(v, s), t^m]] + [D(v, s), [t^m, D(u, r)]] + [t^m, [D(u, r), D(v, s)]] = 0\)
2. \([D(u, r), [t^n, t^m]] + [t^n, [t^m, D(u, r)]] + [t^m, [D(u, r), t^n]] = 0\)

(1) follows because

\[
[D(u, r), [D(v, s), t^m]] = [D(u, r), (v, m)\sigma(s, m)t^{s+m}]
= (v, m)\sigma(s, m)(u, s + m)\sigma(r, s + m)t^{r+s+m}
\]

Similarly

\[
[D(v, s), [t^m, D(u, r)]] = -(u, m)\sigma(r, m)(v, m + r)\sigma(s, m + r)t^{r+s+m}
\]

and\n
\[
[t^m, [D(u, r), D(v, s)]] = -\sigma(r, s)((u, s)v - (r, v)u)m\sigma(r + s, m)t^{r+s+m}
\]

Now adding the corresponding terms we get the desired identity. The second Jacobi identity also follows similarly.

\[
\square
\]

The Lie bracket between two elements of \(\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q\) is given by

\[
[D(u, r) + t^m, D(v, s) + t^n] = [D(u, r), D(v, s)] + [D(u, r), t^n]
+ [t^m, D(v, s)] + [t^n, t^n],
\]

\[
[\text{ad} t^s + t^m, \text{ad} t^{s'} + t^n] = [\text{ad} t^s, \text{ad} t^{s'}] + [\text{ad} t^s, t^n] + [t^m, \text{ad} t^{s'}] + [t^m, t^n].
\]

Let \(\mathfrak{h} = \{D(u, 0) : u \in \mathbb{C}^d\} \oplus \mathbb{C}\). Then \(\mathfrak{h}\) is a maximal abelian subalgebra of \(\mathfrak{g}\).

For a finite dimensional irreducible \(gL_d\)-module \(V\), consider the space \(V \otimes \mathbb{C}_q\). Liu and Zhao \[5\], also see \[5\], proved that \(V \otimes \mathbb{C}_q\) is a \(\text{Der}(\mathbb{C}_q)\)-module with the following action:

1. \(\text{ad} t^s v(n) = (\sigma(s, n) - \sigma(n, s))v(n + s)\)
2. \(D(u, r)v(n) = (u, n + \alpha)^{T}v(r + n)\)

where \(ru^T = \sum_{i,j} r_i u_j E_{ij}\) for \(r = (r_1, \cdots, r_d) \in \mathbb{Z}^d\) and \(u = (u_1, \cdots, u_d)^T\), and \(v(n) := v \otimes t^n \in V(n) := V \otimes t^n, n \in \mathbb{Z}^d, s \notin \text{rad}(f), r \in \text{rad}(f), u, \alpha \in \mathbb{C}^d.\)

**Definition 2.4.** Let \(\mathfrak{g} := \text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q\). A \(\mathfrak{g}\)-module \(V'\) is said to satisfy the condition \((\mathcal{A})\) if the following holds:

1. For \(s, r, r+s \notin \text{rad}(f)\), \(\text{ad} t^r \text{ad} t^s - (t^r \text{ad} t^s + t^s \text{ad} t^r + \sigma(s, r)\text{ad} t^{r+s}) = 0\) on \(V'\)
2. \(s, r \notin \text{rad}(f)\), but \(r + s \in \text{rad}(f)\), \(\text{ad} t^r \text{ad} t^s = t^r \text{ad} t^s + t^s \text{ad} t^r\) on \(V'\).
The motivation behind condition (A) came from the $\text{Der}(\mathbb{C}_q)$-modules defined by Liu and Zhao. In the next proposition we will prove that these $\text{Der}(\mathbb{C}_q)$-modules are the $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-modules which satisfy condition (A).

Let $V$ denote an irreducible finite dimensional $\mathfrak{gl}_d$-module. Then we have the following proposition:

**Proposition 2.5.** $V \otimes \mathbb{C}_q$ is an irreducible $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-module which satisfies condition (A) with the following action:

1. $\text{ad } t^s v(n) = (\sigma(s, n) - \sigma(n, s)) v(n + s),$ 
2. $D(u, r) v(n) = \sigma(r, n)((u, n + \alpha) + ru^T) v(r + n),$
3. $t^m v(n) = \sigma(m, n) v(m + n)$ where $v(n) := v \otimes t^n \in V(n) := V \otimes t^n,$

$s \notin \text{rad}(f), r \in \text{rad}(f), u, \alpha \in \mathbb{C}^d, m, n \in \mathbb{Z}^d.$

**Proof.** First we will show that $V \otimes \mathbb{C}_q$ is $\text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$-module. For that we need to check the following conditions:

1. $[D(u, r), t^m] v(n) = (D(u, r) t^m - t^m D(u, r)) v(n),$
2. $[\text{ad } t^s, t^m] v(n) = (\text{ad } t^s t^m - t^m \text{ad } t^s) v(n),$
3. $[t^p, t^q] v(n) = (t^p t^q - t^q t^p) v(n).$

For proving (1) consider the L.H.S.

$$[D(u, r), t^m] v(n) = (u, m) \sigma(r, m) t^{m+r} v(n)$$

$$= (u, m) \sigma(r, m) \sigma(m + r, n) v(n + m + r)$$

Let us compute R.H.S.

$$D(u, r) t^m - t^m D(u, r) v(n)$$

$$= (\sigma(m, n)[\sigma(r, n + m)((u, n + m + \alpha) + ru^T)]$$

$$- \sigma(r, n)((u, n + \alpha) + ru^T) \sigma(m, r + n)) v(n + m + r)$$

$$= (u, m) \sigma(r, m) \sigma(m + r, n) v(n + m + r)$$

Similar calculations will give us the other two conditions. Now we will prove that the above module satisfies condition (A). For consider

$$\text{ad } t^s \text{ad } t^{s'} v(n) = [\sigma(s', n) - \sigma(n, s')] \times$$

$$[\sigma(s, s' + n) - \sigma(s' + n, s)] v(n + s + s'),$$

and

$$t^s \text{ad } t^{s'} v(n) = (\sigma(s', n) - \sigma(n, s')) \sigma(s, n + s') v(n + s + s'),$$

$$t^{s'} \text{ad } t^s v(n) = (\sigma(s, n) - \sigma(n, s)) \sigma(s', n + s) v(n + s + s'),$$

$$\sigma(s', s) \text{ad } t^{s+s'} v(n) = \sigma(s', s) [\sigma(s + s', n) - \sigma(n, s + s')] v(n + s + s'),$$

so we get 8LEJGS-X4TPUT

$$\text{ad } t^s \text{ad } t^{s'} - (t^s \text{ad } t^{s'} + t^{s'} \text{ad } t^s) + \sigma(s', s) \text{ad } t^{s+s'} = 0.$$
Throughout this section the action Proposition 3.1. For $r, s \notin \text{rad}(f)$, then we note that $\sigma(s, s') = \sigma(s', s)$ and $\sigma(s, n)\sigma(s', n) = \sigma(n, s)\sigma(n, s')$. Using these facts one gets the second identity of condition (A).

To prove the irreducibility of $V' = V \otimes \mathbb{C}_q$, let $M$ be a nonzero submodule of $V'$. Then since $M$ is a weight module, we have $M = \bigoplus_{n \in \mathbb{Z}^d} M_n \otimes t^n$, where $M_n = \{v \in V : v \otimes t^n \in M\}$. Let for any nonzero vector $v \in M_n$, consider the $\mathbb{C}_q$ action on it. As we note that $\mathbb{C}_q$ is an irreducible $\mathbb{C}_q$-module with the action $t^n.t^m = \sigma(n, m)t^{n+m}$, we have $v \otimes \mathbb{C}_q \subseteq M$. So it follows that $M_n$ is independent of $n$. So let $M_n = V \subseteq V$. But as $D(u, r)v \otimes t^n \in M$, for $v \in V$, it follows that $V$ is a nonzero $gl_d$-submodule of $V$. So $V = V$ as $V$ is an irreducible $gl_d$-module and hence $M = V'$.

□

We will denote the $\mathfrak{g}$-module in Proposition 2.5 by $F^\alpha(V)$. Our main aim in this paper is to prove the converse of Proposition 2.5 which is as follows:

**Theorem 2.6.** Let $V'$ be an irreducible $\mathbb{Z}^d$-graded $\mathfrak{g}$-module with finite dimensional weight spaces with respect to $\mathfrak{h}$, which satisfies condition (A) and $t^nt^m = \sigma(n, m)t^{n+m}, t^0 = 1$. Then $V' \cong F^\alpha(V)$.

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3. The action of $D(u, r)$ and $\mathbb{C}_q$ on $V'$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g} = \text{Der}(\mathbb{C}_q) \ltimes \mathbb{C}_q$. Let $L(\mathfrak{g})$ be a two sided ideal of $U(\mathfrak{g})$ generated by $\{t^mt^n = \sigma(m, n)t^{m+n} \forall m, n \in \mathbb{Z}^d, t^0 = 1, \text{ad } t^r \text{ad } t^s - (t^r \text{ad } t^s + t^s \text{ad } t^r) + \sigma(s, r) \text{ ad } t^{r+s}, \text{ where } s, r, r+s \notin \text{rad}(f), \text{ and } \text{ad } t^r \text{ad } t^s = t^r \text{ad } t^s + t^s \text{ad } t^r, \text{ for } s, r, r+s \in \text{rad}(f)\}$. Throughout this section $V'$ is as in Theorem 2.6. As $V'$ is a $\mathfrak{g}$-module, and satisfies condition (A), $L(\mathfrak{g})$ acts trivially on $V'$. So $V'$ is a $U(\mathfrak{g})/L(\mathfrak{g})$-module. Let $V' = \bigoplus_{r \in \mathbb{Z}^d} V'_r$ be its weight space decomposition with $V'_r = \{v \in V' : D(u, 0)v = (u, r + \alpha)v, \forall u \in \mathbb{C}^d \text{ and for some } \alpha \in \mathbb{C}^d\}$.

**Proposition 3.1.** For $r, s \notin \text{rad}(f)$, $[t^{-s}\text{ad } t^s, t^{-r}\text{ad } t^r] = 0$ on $V'$. 
Proposition 3.3. have the following proposition:

Proof. In this proof we will use hypothesis that $V'$ satisfies condition (A). Let us consider

\[
[t^{-s}\text{ad} t^s, t^{-r}\text{ad} t^r] = [t^{-s}\text{ad} t^s, t^{-r}]\text{ad} t^r + t^{-r}[t^{-s}\text{ad} t^s, \text{ad} t^r]
= [t^{-s}, t^{-r}]\text{ad} t^s + t^{-s}[\text{ad} t^s, t^{-r}]\text{ad} t^r
+ t^{-r}[t^{-s}, \text{ad} t^r]\text{ad} t^s + t^{-r}t^{-s}[\text{ad} t^s, \text{ad} t^r]
= (\sigma(s, r) - \sigma(r, s))t^{-(s+r)}(s\text{ad} t^r + t^r\text{ad} t^s - \sigma(r, s)\text{ad} t^{r+s})
+ t^{-s}(\sigma(s, -r) - \sigma(-r, s))t^{s-r}\text{ad} t^r
- t^{-r}(\sigma(r, -s) - \sigma(-s, r))t^{r-s}\text{ad} t^s
+ \sigma(r, s)t^{-(s+r)}(\sigma(s, r) - \sigma(r, s))\text{ad} t^{r+s}
= (\sigma(s, r) - \sigma(r, s))(\sigma(-r + s, s)t^{-r}\text{ad} t^r + \sigma(-r + s, r)t^{-s}\text{ad} t^s
- \sigma(r, s)t^{-(r+s)}\text{ad} t^{r+s}) + (\sigma(s, -r) - \sigma(-r, s))\sigma(-s, s - r)t^{-r}\text{ad} t^r
- (\sigma(r, -s) - \sigma(-s, r))\sigma(-r, r - s)t^{-s}\text{ad} t^s
+ \sigma(r, s)(\sigma(s, r) - \sigma(r, s))t^{-(s+r)}\text{ad} t^{r+s}
= 0
\]

We state the following lemma from [3] which will be used later.

Lemma 3.2 ([3, Prop.19.1(b)]). Let $\mathfrak{g}'$ be a Lie algebra which need not be finite dimensional. Let $(V_1, \rho)$ be an irreducible finite dimensional module for $\mathfrak{g}'$. We have a map $\rho: \mathfrak{g}' \to \text{End}(V_1)$. Then $\rho(\mathfrak{g}')$ is a reductive Lie algebra with at most one dimensional center.

Let $U_1 = U(\mathfrak{g})/L(\mathfrak{g})$ and let us define $T'(u, r) = t^{-r}D(u, r) - \sigma(-r, r)D(u, 0)$ as an element of $U_1$ for $r \in \text{rad}(f), u \in \mathbb{C}^d$. Let $T'$ be the subspace generated by $T'(u, r)$ for all $u$ and $r$. Let $\mathfrak{g}'$ be the Lie subalgebra generated by $T'(u, r)$ and $t^{-s}\text{ad} t^s$ for all $u \in \mathbb{C}^d$, $r \in \text{rad}(f)$ and $s \notin \text{rad}(f)$. Let $I$ be a subalgebra of $\mathfrak{g}'$ generated by the elements of the form $t^{-s}\text{ad} t^s$. Then we have the following proposition:

Proposition 3.3. $I$ is an abelian ideal of $\mathfrak{g}'$.

Proof. It follows from lemma [3,1] that $I$ is an abelian subalgebra. To prove that $I$ is an ideal of $\mathfrak{g}'$ we need to prove that $[T'(u, r), t^{-s}\text{ad} t^s] \in I$. So
consider

\[ [T'(u, r), t^{-s} \text{ad} t^s] \]
\[ = [T'(u, r), t^{-s}] \text{ad} t^s + t^{-s}[T'(u, r), \text{ad} t^s] \]
\[ = t^{-s}[T'(u, r), \text{ad} t^s] \text{ (as } [T'(u, r), t^{-s}] = 0, \text{ see prop. } ??(5)) \]
\[ = t^{-s}(t^{-r} D(u, r), t^s) - t^{-s}(r, r) D(u, 0) \]
\[ = t^{-s}(t^{-r}, r) [D(u, r), t^s] + t^{-s}(r, r) D(u, 0) \]
\[ - t^{-s}(r, r) D(u, 0) \]

as \( r \in \text{rad}(f) \), we have \( t^{-r}, \text{ad} t^s \) = 0, so we get

\[ [T'(u, r), t^{-s} \text{ad} t^s] = \sigma(s, r)(u, s) \sigma(r, s) t^{-s} \text{ad} t^s \]
\[ - \sigma(-r, r)(u, s) t^{-s} \text{ad} t^s \in I. \]

This completes the proof.

\[ \square \]

**Proposition 3.4.**

1. \( [T'(u, r), T'(v, s)] = (v, r) \sigma(-s, s) T'(u, r) \]
\[ - (u, s) \sigma(-r, r) T'(v, s) + \sigma(s, r) T'(w, r + s), \]
where \( w = \sigma(r, s)((u, s) v - (v, r) w) \) and therefore \( T' \) is a Lie-subalgebra.
2. \( [D(v, 0), T'(u, r)] = 0. \)
3. Let \( V' = \bigoplus_{r \in \mathbb{Z}} V'_r \) be its weight space decomposition. Then each \( V'_r \) is \( T' \) invariant.
4. Each \( V'_r \) is \( T' \)-irreducible.
5. \( V'_r \cong V'_s \) as \( T' \)-module.

**Proof.** First let us prove (2). Consider

\[ [D(v, 0), T'(u, r)] = [D(v, 0), t^{-r} D(u, r) - \sigma(-r, r) D(u, 0)] \]
\[ = [D(v, 0), t^{-r} D(u, r)] \]
\[ = [D(v, 0), t^{-r} D(u, r) + t^{-r} D(v, 0), D(u, r)] \]
\[ = (v, -r) t^{-r} D(u, r) + t^{-r} D((v, r) u, r) = 0. \]
Now using (2) we have
\[
[T'(u, r), T'(v, s)] = [t^{-r}D(u, r), t^{-s}D(v, s)]
\]
\[
= [t^{-r}D(u, r), t^{-s}]D(v, s) + t^{-s}[t^{-r}D(u, r), D(v, s)]
\]
\[
= [t^{-r}, t^{-s}]D(u, r)D(v, s) + t^{-r}[D(u, r), t^{-s}]D(v, s)
\]
\[
+ t^{-s}[t^{-r}, D(v, s)]D(u, r) + t^{-s}t^{-r}[D(u, r), D(v, s)]
\]
\[
= t^{-r}(u, -s)\sigma(r, -s)t^{-s+r}D(v, s)
\]
\[
+ t^{-s}(v, r)\sigma(s, -r)t^{-r+s}D(u, r) + t^{-s}t^{-r}D(w, r + s),
\]
where \(w = (u, s)v - (v, r)u\)
\[
= - (u, s)\sigma(-r, r)t^{-s}D(v, s) + (v, r)\sigma(-s, s)t^{-r}D(u, r)
\]
\[
+ \sigma(s, r)t^{-(s+r)}D(w, r + s)
\]
but as
\[-(u, s)\sigma(-r, r)\sigma(-s, s)D(v, 0) + (v, r)\sigma(-s, s)\sigma(-r, r)D(u, 0)
\]
\[+ \sigma(s, r)\sigma(-(r + s), (r + s))D(w, 0) = 0,
\]
we get the desired identity. Proof of (3) is clear from (2).

For proving (4), let \(U(\mathfrak{g}) = \oplus_{\alpha \in \mathbb{C}d}U_r\), where \(U_r = \{v \in U(\mathfrak{g}) : [D(u, 0), v] = (u, r + \alpha)v, \forall u \in \mathbb{C}^d\}\). As \(V'\) is an irreducible \(\mathfrak{g}\)-module, for any two elements \(v, w\) of \(V'_r\) there exists an element \(X\) in \(U_0\) such that \(Xv = w\). So \(V'_r\) is an irreducible \(U_0\)-module. The element \(X\) is a linear combination of the elements of the form
\[
D(u, r_1)D(u, r_2) \cdots D(u, r_k)t^{-l}t^{-k}\text{ad} t^{s_1} \cdots \text{ad} t^{s_n},
\]
where \(\sum_{i=1}^{k} r_i = l\) and \(\sum_{i=1}^{s} s_i = k\). But as \(D(u, r)t^{-s} = t^{-s}D(u, r) + (u, -s)\sigma(r, -s)t^{-s}\) and \(t^{-r}\text{ad} t^s = \text{ad} t^s t^{-r} - (\sigma(s, -p) - \sigma(-p, s))t^s - p\), we see that every element of the above form can be written as a linear combination of the elements \(t^{-r_1}D(u, r_1) \cdots t^{-r_k}D(u, r_k)t^{-s_1} \cdots t^{-s_n}\text{ad} t^{s_1} \cdots t^{s_n}\). So \(U_0\) is generated by the elements of the form \(T'(u, r)\) and \(t^{-s}\text{ad} t^s\), where \(r \in \text{rad}(f)\) and \(s \notin \text{rad}(f)\). Now we use lemma 3.2 where we take \(\mathfrak{g}'\) as the Lie algebra generated by \(T'(u, r)\) and \(t^{-s}\text{ad} t^s\) and \(V = V'_r\). As the elements of the form \(t^{-s}\text{ad} t^s\) forms an abelian ideal (Prop.3.3), it follows from the Lemma 3.2 that the elements of the form \(t^{-s}\text{ad} t^s\) must lie in the center of \(\rho(\mathfrak{g}')\) which is at most one dimensional. Consequently it follows that \(t^{-s}\text{ad} t^s\) acts as a scalar on \(V'_r\) and hence \(V'_r\) is an irreducible \(T'\)-module.

Now let us prove (5). As \(t^{(r-s)}V'_r \subseteq V'_s\). But as
\[
V'_r = t^{(r-s)}t^{(s-r)}V'_r \subseteq t^{(r-s)}V'_s \subseteq V'_r.
\]
We get \(V'_r = t^{(r-s)}V'_s\). Define \(\psi : V'_r \rightarrow V'_s\) by \(\psi(v) = t^{(r-s)}v\). Note that \(\psi\) is injective (as it is graded) and surjective, we need to prove that it is a \(T'\)-module homomorphism, i.e., we need to show that \([t^{(s-r)}, T'(u, m)] = 0\).
So consider

\[ [t^{(s-r)}, T^d(u, m)] = [t^{(s-r)}, t^{-m}D(u, m)] - \sigma(-m, m)[t^{(s-r)}, D(u, 0)] \]
\[ = [t^{(s-r)}, t^{-m}]D(u, m) + t^{-m}[t^{(s-r)}, D(u, m)] \]
\[ + \sigma(-m, m)(u, s - r)t^{(s-r)} \]
\[ = -(u, s - r)\sigma(m, s - r)t^{-m}t^{(s-r+m)} \]
\[ + \sigma(-m, m)(u, s - r)t^{(s-r)} \]
\[ = -(u, s - r)\sigma(m, s - r)\sigma(-m, s - r + m)t^{(s-r)} \]
\[ + \sigma(-m, m)(u, s - r)t^{(s-r)} \]
\[ = 0 \]
\[ \square \]

Now by Liu and Zhao [6], \( \text{rad}(f) = m_1\mathbb{Z}e_1 \oplus \cdots \oplus m_d\mathbb{Z}e_d \) for some \( 0 \neq m_i \in \mathbb{Z} \) for \( 1 \leq i \leq d \). Note that this result is true only if all the entries of the matrix \( q \) are roots of unity. Consider the Laurent polynomial ring associated with \( \text{rad}(f) \) which is equal to \( \mathbb{C}[t_1^{\pm m_1}, \ldots, t_d^{\pm m_d}] \) which is equal to \( \mathbb{C}_q \). Let \( A = \mathbb{C}[s_1^{\pm 1}, \ldots, s_d^{\pm 1}] \) be a Laurent polynomial ring, where \( s_i = t_i^{m_i} \) for \( 1 \leq i \leq d \). To avoid notational confusion, we will use the notation \( d(u, r) \) for the derivations of \( \text{Der}(A) \), where \( d(u, r) = t^r \sum_{i=1}^d u_i \partial_i \), \( u = (u_1, u_2, \ldots, u_d) \in \mathbb{C}^d \) and \( r \in \mathbb{Z}^d \). Let \( W \) denote the Lie subalgebra of \( \text{Der}(\mathbb{C}_q) \) generated by the elements \( D(u, r) \), where \( u \in \mathbb{C}^d \), \( r \in \text{rad}(f) \). We have the following proposition:

**Proposition 3.5.** \( \text{Der}(A) \ltimes A \cong W \ltimes \mathbb{C}_q \) with a map \( \phi \) defined as \( \phi(d(u, r) + t^s) = \sqrt{\sigma(r, r)}D(u, r) + \sqrt{\sigma(s, s)}t^s \).

**Proof.** We only need to prove that

\[ \phi[d(u, r) + t^m, d(v, s) + t^n] = [\phi(d(u, r) + t^m), \phi(d(v, s) + t^n)] \]

First consider the L.H.S.

\[ \phi[d(u, r) + t^m, d(v, s) + t^n] = \phi([d(u, r), d(v, s)] + [d(u, r), t^n] + [t^m, d(v, s)]) \]
\[ = \phi(d(w, r + s) + (u, n)t^{r+n} - (v, m)t^{m+s}) \]
where \( w = ((u, s)v - (v, r)u) \)
\[ = \sqrt{\sigma(r + s, r + s)}D(w, r + s) \]
\[ + (u, n)\sqrt{\sigma(r + n, r + n)}t^{r+n} \]
\[ - (v, m)\sqrt{\sigma(m + s, m + s)}t^{m+s}. \]
Now consider the R.H.S.

\[
[\phi(d(u, r) + t^m), \phi(d(v, s) + t^n)]
\]

\[
= [\sqrt{\sigma(r, r)}D(u, r) + \sqrt{\sigma(m, m)}t^m, \sqrt{\sigma(s, s)}D(v, s) + \sqrt{\sigma(n, n)}t^n]
\]

\[
= \sqrt{\sigma(r, r)}\sqrt{\sigma(s, s)}[D(u, r), D(v, s)] + \sqrt{\sigma(s, s)}[t^m, D(v, s)]
\]

\[
= \sqrt{\sigma(r, r)}\sqrt{\sigma(s, s)} D(w', r + s) + \sqrt{\sigma(s, s)}(u, r) t^{r+n}
\]

where \( w' = \sigma(r, s)((u, s)v - (v, r)u) \)

\[
= \sqrt{\sigma(r + s, r + s)}D(w, r + s) + (u, n)\sqrt{\sigma(r + n, r + n)}t^{r+n}
\]

\[- (v, m)\sqrt{\sigma(m + s, m + s)}t^{m+s}.\]

So we get

\[
\phi(d(u, r) + t^m, d(v, s) + t^n) = [\phi(d(u, r) + t^m), \phi(d(v, s) + t^n)].
\]

Using the above proposition we see that from [2], \( T'/I'_2 \cong \mathfrak{gl}_d \), where \( I'_2 \) is the ideal of \( T' \) spanned by the elements \( T'(u, r, n_1, n_2) \) which are defined as follows: \( T'(u, r, n_1, n_2) = T'(u, r) - T'(u, r + n_1) - T'(u, r + n_2) + T'(u, r + n_1 + n_2) \). Again recall from [2] that the Lie subalgebra \( T \) is isomorphic to \( T' \) under the map \( \phi \) where \( T \) is the Lie subalgebra spanned by the elements \( T(u, r) = t^{-r}d(u, r) - d(u, 0) \). Using this isomorphism \( \phi \) we see that \( \phi(T'(u, r)) = \sigma(r, r)T'(u, r) \), so \( \phi^{-1}(T'(u, r)) = \sigma(r, r)^{-1}T(u, r) = \sigma(-r, -r)^{-1}T(u, r) \). As \( T'/I'_2 \cong \mathfrak{gl}_d \), we see that by the same argument as in [2], we have \( V'' \cong V \otimes t' \), where \( V \) is a finite dimensional irreducible representation of \( \mathfrak{gl}_d \). So we have \( V'' \cong V \otimes \mathbb{C}_q \). The isomorphism from \( T'/I'_2 \) to \( \mathfrak{gl}_d \) is given by the map \( \pi' \) defined by \( \pi'(T'(e_i, e_j)) = \sigma(e_j, e_i)^{-1}E_{ji} = E_{ji} \), as \( \sigma(e_i, e_j) = 1 \).

Now let us calculate the action of \( D(u, r) \) on \( V'' = V \otimes t'' := V'(n) \). First consider

\[
T'(u, r)v(n) = \sigma(-r, r)T(u, r)v(n)
\]

\[
= \sigma(-r, r) \sum_{i,j} u_ir_j T(e_i, e_j)v(n)
\]

\[
= \sigma(-r, r) \sum_{i,j} u_ir_j E_{ji} v(n)
\]

So we have

\[
t^{-r}D(u, r)v(n) = \sigma(-r, r)D(u, 0)v(n) + \sigma(-r, r) \sum_{i,j} u_ir_j E_{ji} v(n)
\]
multiply by $t^r$ we get
\[
\sigma(-r,r)D(u,r)v(n) = \sigma(-r,r)(u, n + \alpha)\sigma(r, n)v(n + r) + \sigma(-r,r)(\sum_{i,j}u_i r_j E_{ji})\sigma(r, n)v(n + r).
\]
So we get
\[
D(u,r)v(n) = \sigma(r, n)((u, n + \alpha) + ru^T)v(n + r).
\]
Now as by Proposition 3.4 we have $V'_s \cong V'_s$ as $T'$-module. We identify $V'_s$ as $t^s(V'_0)$, i.e., $t^s(V'_0) = V'_s$ for all $s \in \mathbb{Z}^d$. Now consider
\[
t^m t^n v(0) = t^m v(n) \text{ by identification}
\]
\[
\sigma(m,n) t^{m+n} v(0) = t^m v(n).
\]
So far we have proved the following:

**Proposition 3.6.** Let $V'$ be an irreducible $\mathbb{Z}^d$-graded $\mathfrak{g}$-module with finite dimensional weight spaces with respect to $\mathfrak{h}$, which satisfies condition (A) and $t^m t^n = \sigma(n,m)t^{m+n}$, $t^0 = 1$. Then $V' \cong V \otimes \mathbb{C}_q$, where $V$ is a finite dimensional $\mathfrak{gl}_d$ module. The actions of $D(u,r)$ and $C_q$ on $V'$ are given by the following formula:

1. $D(u,r)v(n) = \sigma(r, n)((u, n + \alpha) + ru^T)v(r + n)$,
2. $t^m v(n) = \sigma(m,n)v(m + n)$, where $u \in \mathbb{C}^d, m,n \in \mathbb{Z}^d, r \in \text{rad}(f)$, and $v(n) = v \otimes t^n$.

4. **ad action on $V'$ and the proof of Theorem 2.6**

To complete the proof of Theorem 2.6 we need to determine the action of $\text{ad}$ on $V'(n)$, which will be done in this section. As by the Lemma 3.2 $t^{-s}\text{ad} t^s$ acts as a scalar on $V'(n)$. Let $t^{-s}\text{ad} t^s v(n) = \lambda(s,n)v(n)$.

**Proposition 4.1.** $\lambda(s,r) = f(r,s)\lambda(s,0) + \sigma(-s,s)[1 - f(r,s)]$.

**Proof.** Consider
\[
t^{-s}\text{ad} t^s t^r v(n) = (t^r t^{-s}\text{ad} t^s + [t^{-s}\text{ad} t^s, t^r])v(n)
\]
\[
\lambda(s,n+r)t^r v(n) = t^r \lambda(s,n)v(n) + (\sigma(-s,r) - \sigma(r,-s))t^{-s+r}\text{ad} t^s v(n) + t^{-s}[\text{ad} t^s, t^r]v(n)
\]
\[
= t^r \lambda(s,n)v(n) + (\sigma(-s,r) - \sigma(r,-s))\sigma(r,s)t^s t^{-s+r}\text{ad} t^s v(n)
\]
\[
= \lambda(s,n)t^r v(n) + (\sigma(-s,r)\sigma(r,s) - 1)\lambda(s,n)t^r v(n)
\]
\[
+ (\sigma(-s,s) - \sigma(-s,s)\sigma(-s,s))t^r v(n)
\]
\[
= \lambda(s,n)t^r v(n) + (f(r,s) - 1)\lambda(s,n)t^r v(n)
\]
\[
+ \sigma(-s,s)(1 - f(r,s))t^r v(n)
\]
Putting \( n = 0 \) in the above equation we get the desired identity for \( \lambda(s, r) \).

**Proposition 4.2.** \( \text{ad} t^s v(n) = (\sigma(s, n) - g(s)\sigma(n, s))v(n + s) \), where \( g(s) = -\sigma(s, s)\lambda(s, 0) + 1 \).

**Proof.** By above proposition we have

\[
   t^{-s}\text{ad} t^s v(n) = [f(n, s)\lambda(s, 0) + \sigma(-s, s)(1 - f(n, s))]v(n)
\]

Multiply by \( t^s \) and using the relation \( t^m t^n = \sigma(m, n) t^{m+n} \), we get

\[
   \text{ad} t^s v(n) = [\sigma(s, f(n, s)\lambda(s, 0) + (1 - f(n, s))]\sigma(s, n)v(n + s)
\]

\[
   = [\sigma(s, n) - g(s)\sigma(n, s)]v(n + s).
\]

**Proposition 4.3.** Let \( g(s) = -\sigma(s, s)\lambda(s, 0) + 1 \) for \( s \notin \text{rad}(f) \) and \( g(s) = 1 \) otherwise. Then \( g(s + r) = g(s)g(r) \forall s, r \in \mathbb{Z}^d \).

**Proof.** First we will prove that for \( s, r, r+s \notin \text{rad}(f) \), \( g(s+r) = g(s)g(r) \). For this we will use the first identity of condition (A), i.e., for \( s, r, r+s \notin \text{rad}(f) \),

\[
   (\text{ad} t^s \text{ad} t^r - (t^s \text{ad} t^r + t^s \text{ad} t^r) + \sigma(s, r)\text{ad} t^{r+s})v(n) = 0
\]

By the definition of ad we have

\[
   \text{ad} t^s \text{ad} t^r v(n) = [\sigma(s, n) - g(s)\sigma(n, s)]\sigma(r, n + s)v(n + s + r).
\]

Similarly by definition,

\[
   -t^r \text{ad} t^s v(n) = -[\sigma(s, n) - g(s)\sigma(n, s)]\sigma(r, n + s)v(n + r + s),
\]

\[
   -t^s \text{ad} t^r v(n) = -[\sigma(r, n) - g(r)\sigma(n, r)]\sigma(s, n + r)v(n + r + s),
\]

and

\[
   \sigma(s, r)\text{ad} t^{r+s} v(n) = \sigma(s, r)[\sigma(r + s, n) - g(r + s)\sigma(n, r + s)]v(n + r + s).
\]

Adding the above terms we get

\[
   \sigma(s, r)\sigma(n, s + r)[g(s)g(r) - g(s + r)] = 0,
\]

So we have

\[
   g(s + r) = g(s)g(r).
\]

Similarly for \( r, s \notin \text{rad}(f) \) and \( s + r \in \text{rad}(f) \), we use the second identity of condition (A). Similar calculations as above and the fact that \( \sigma(s, n)\sigma(r, n) = \sigma(n, s)\sigma(n, r) \) (as \( s + r \in \text{rad}(f) \)) gives us \( g(s)g(r) = 1 \). But as \( s + r \in \text{rad}(f) \) we have \( g(s + r) = 1 \) and hence \( g(s + r) = g(s)g(r) \).

To prove that \( g(s + r) = g(s)g(r) = g(s) \) for \( r \in \text{rad}(f) \) and \( s \notin \text{rad}(f) \), we consider

\[
   [D(u, r), \text{ad} t^s] = (u, s)\sigma(r, s)\text{ad} t^{r+s} \forall r \in \text{rad}(f), s \notin \text{rad}(f), u \in \mathbb{C}^d
\]

So

\[
   [D(u, r), \text{ad} t^s]v(n) = (u, s)\sigma(r, s)[\sigma(s + r, n) - g(s + r)\sigma(n, s + r)]v(n + s + r)
\]
and
\[
D(u, r) \text{ad} t^s v(n) = \left( (\sigma(s, n) - g(s)\sigma(n, s))\sigma(r, n + s) \right) v(n + r + s),
\]
similarly
\[
- \text{ad} t^s D(u, r) v(n) = \left( (\sigma(r, n)\sigma(n + r) - g(s)\sigma(n + r, s))\right) v(n + r + r).
\]

Equating the terms we get following identity
\[-(u, s)\sigma(n, s + r)\sigma(r, s)g(s + r) = -(u, s)\sigma(r, n + s)\sigma(n, s)g(s),\]
Now as \( r \in \text{rad}(f) \) we get
\[g(s + r) = g(s)g(r) \forall s \notin \text{rad}(f) \text{ and } r \in \text{rad}(f)\]

So we have proved the following.

Let \( V' \) be an irreducible \( \mathbb{Z}^d \)-graded \( \mathfrak{g} \)-module with finite dimensional weight spaces with respect to \( \hat{\mathfrak{h}} \), which satisfies condition (A) and \( t^v t^m = \sigma(n, m)t^{n+m} \)
\( t^0 = 1 \). Then \( V' \cong V \otimes \mathbb{C}_q \) and has the following \( \mathfrak{g} \)-action:
\[
\begin{align*}
(1) & \quad D(u, r)v(n) = \sigma(r, n)((u, n + \alpha) + ru^2)v(r + n), \\
(2) & \quad \text{ad} t^s v(n) = (\sigma(s, n) - g(s)\sigma(n, s))v(n + s), \\
(3) & \quad t^m v(n) = \sigma(m, n)v(m + n),
\end{align*}
\]
where \( v(n) := v \otimes t^n \in V(n) := V \otimes t^n, s \notin \text{rad}(f), r \in \text{rad}(f), u, \alpha \in \mathbb{C}^d, m, n \in \mathbb{Z}^d \), and \( g \) is a function satisfying the property that \( g(s + r) = g(s)g(r) \forall s, r \in \mathbb{Z}^d \) and \( g(r) = 1 \forall r \in \text{rad}(f) \).

Now denote the \( \text{Der}(\mathbb{C}_q) \)-module \( V' \) with the above action by \( G^\alpha_q(V) \).

Lin and Tan [5, 2004] proved that \( V \otimes \mathbb{C}_q = V(\psi, b) \otimes \mathbb{C}_q \) is a completely reducible as \( \text{Der}(\mathbb{C}_q) \)-module unless \( (\psi, b) = (\delta_k, k), 1 \leq k \leq d - 1 \) or \( (\psi, b) = (0, b) \) with the following action:
\[
\begin{align*}
(1) & \quad D(u, r)v(n) = \sigma(r, n)((u, n + \alpha) + ru^2)v(r + n), \\
(2) & \quad \text{ad} t^s v(n) = (\sigma(s, n)g(s) - \sigma(n, s))v(n + s),
\end{align*}
\]
where \( g(s + r) = g(s)g(r) \forall s, r \in \mathbb{Z}^d \) and \( g(r) = 1 \forall r \in \text{rad}(f) \). We denote these \( \text{Der}(\mathbb{C}_q) \)-modules by \( F^\alpha_{q^2}(V) \). Then we have the following proposition:

**Proposition 4.4.** \( G^\alpha_{q^2}(V) \cong F^\alpha_{q^2}(V) \) as a \( \text{Der}(\mathbb{C}_q) \)-module.

**Proof.** First we note that \( g(s)^{-1} \neq 0 \) for all \( s \in \mathbb{Z}^d \); for if \( g(s) = 0 \), then
\[
1 = g(0) = g(s - s) = g(s)g(-s) = 0 \text{ a contradiction.}
\]
Now define a map \( \Psi : G^\alpha_{q^2}(V) \to F^\alpha_{q^2}(V) \) by \( \Psi(v(n)) = g(n)^{-1}v(n) \). Then we have to prove the following:
\[
\begin{align*}
(1) & \quad \Psi(D(u, r)v(n)) = D(u, r)\Psi(v(n)), \\
(2) & \quad \Psi(\text{ad} t^s v(n)) = \text{ad} t^s \Psi(v(n)).
\end{align*}
\]
To prove (1) consider
\[ \Psi(D(u,r)v(n)) = \Psi((u, n + \alpha) + ru^T)v(n + r) \]
\[ = g(n + r)^{-1}(\sigma(r, n)((u, n + \alpha) + ru^T)v(n + r)) \]
Now consider the R.H.S.
\[ D(u,r)\Psi(v(n)) = g(n)^{-1}(\sigma(r, n)((u, n + \alpha) + ru^T)v(n + r)), \]
but as \( r \in \text{rad}(f) \) we have \( g(n + r)^{-1} = g(n)^{-1} \) and (1) is proved.

For proving (2) consider
\[ \Psi(\text{ad} t^s v(n)) = \Psi((\sigma(s, n) - g(s)\sigma(n, s))v(n + s)) \]
\[ = g(s + n)^{-1}(\sigma(s, n) - g(s)\sigma(n, s))v(n + s) \]
\[ = (g(s)^{-1}g(n)^{-1}\sigma(s, n) - g(n)^{-1}\sigma(n, s))v(n + s), \]
similarly
\[ \text{ad} t^s \Psi(v(n)) = g(n)^{-1}((\sigma(s, n)g(s)^{-1} - \sigma(n, s))v(n + s) \]
\[ = (g(s)^{-1}g(n)^{-1}\sigma(s, n) - g(n)^{-1}\sigma(n, s))v(n + s) \]
which completes the proof of (2). \( \square \)

To prove Theorem 2.6 we have to prove that \( G_g^\alpha(V) \cong F_1^\beta(V) \), where \( \beta \in \mathbb{C}^d \) and \( l \) is the constant function 1 on \( \mathbb{Z}^d \). To prove this we invoke [6] for the following result:

**Theorem 4.5** (Theorem 3.1,[6]). Let \( g : \mathbb{Z}^d \to \mathbb{C}^* \) be a function satisfying \( g(m)g(n) = g(m + n) \) and \( g(r) = 1 \) for any \( m, n \in \mathbb{Z}^d, r \in \text{rad}(f) \). Let \( V \) be a \( gl_d \) module. Then there exists \( \beta \in \mathbb{C}^d \) such that \( F_g^\alpha(V) \cong F_1^\beta(V) \) as \( \text{Der}(C_q) \)-module, where \( l \) denotes the constant function which maps all the elements of \( \mathbb{Z}^d \) to 1.

So using Theorem 4.5 and Proposition 4.3 we get \( G_g^\alpha(V) \cong F_1^\beta(V) \) and this completes the proof of Theorem 2.6.

**Remark 4.6.** After finishing this work, we came across a paper by Liu and Zhao, Irreducible Harish-Chandra modules over the derivation algebras of rational quantum tori, Glasgow Mathematical Journal Trust 2013, where they consider a smaller Lie algebra.

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