Some Rarita-Schwinger Type Operators

Charles F. Dunkl, Junxia Li, John Ryan and Peter Van Lancker

Abstract

In this paper we study a generalization of the classical Rarita-Schwinger type operators and construct their fundamental solutions. We give some basic integral formulas related to these operators. We also establish that the projection operators appearing in the Rarita-Schwinger operators and the Rarita-Schwinger equations are conformally invariant. We further obtain the intertwining operators for other operators related to the Rarita-Schwinger operators under actions of the conformal group.

Keywords: Clifford algebra, Almansi-Fischer decomposition, conformal transformations, inner product.

Classification: Primary 30G35; Secondary 53C27

1 Introduction

In representation theory for $O(n)$ and $SO(n)$, one considers functions $f : U \to \mathcal{H}_k$ where $U$ is a domain in $\mathbb{R}^n$ and $\mathcal{H}_k$ is the space of harmonic polynomials homogeneous of degree $k$. Such spaces are invariant under actions of $O(n)$. If one refines to the covering group $Spin(n)$ of $SO(n)$, one replaces spaces of harmonic polynomials with spaces of homogeneous polynomial solutions to the Euclidean Dirac equation arising in Clifford analysis. See [BDS]. Clifford analysis is the study of and applications of Dirac type operators. In this context the Rarita-Schwinger operators arise. See [BSSV1] [BSSV2] [Va1] [Va2] [LRV1] [LRV2] [LR]. The Rarita-Schwinger operators are generalizations of the Dirac operator which in turn is a natural generalization of the Cauchy-Riemann operator. Rarita-Schwinger operators are also known as Stein-Weiss operators after [SW]. We denote a Rarita-Schwinger operator by $R_k$, where $k = 0, 1, \cdots, m, \cdots$. When $k = 0$ we have the Dirac operator.

Here we start by constructing the Rarita-Schwinger operators and their fundamental solutions. This is based on the fundamental solution of the Dirac operator. Next, we give a summary of results on Rarita-Schwinger operators appearing in [BSSV1], giving detailed proofs and extending some of those results. We present a more detailed and alternative approach to that given in [BSSV1]. This includes a Stokes’ Theorem, Borel-Pompeiu Formula, Cauchy’s Integral Formula and a Cauchy Transform. We also obtain intertwining operator for $R_k$ under actions of the conformal group, together with intertwining operators for the kernels to the Rarita-Schwinger operators, and the conformal invariance of Cauchy’s Theorem and Cauchy’s Integral Formula.
All of this ultimately helps to build the basics of Rarita-Schwinger type operators, including a theory of Rarita-Schwinger operators on examples of conformally flat manifolds. See for instance [LRV1, LRV2, LR].

2 Preliminaries

A Clifford algebra, $Cl_n$, can be generated from $\mathbb{R}^n$ by considering the relationship

$$x^2 = -\|x\|^2$$

for each $x \in \mathbb{R}^n$. We have $\mathbb{R}^n \subseteq Cl_n$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$, then $x^2 = -\|x\|^2$ tells us that $e_ie_j + e_je_i = -2\delta_{ij}$. Let $A = \{j_1, \ldots, j_r\} \subseteq \{1, 2, \ldots, n\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq n$. An arbitrary element of the basis of the Clifford algebra can be written as $e_{j_1} \cdots e_{j_r}$. Hence for any element $a \in Cl_n$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$. For $a \in Cl_n$, we will need the following anti-involutions:

- **Reversion:**
  $$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$
  where $|A|$ is the cardinality of $A$. In particular, $e_{j_1} \cdots e_{j_r} = e_{j_r} \cdots e_{j_1}$. Also $\tilde{ab} = \tilde{b}\tilde{a}$ for $a, b \in Cl_n$.

- **Clifford conjugation:**
  $$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A$$
  satisfying $\overline{e_{j_1} \cdots e_{j_r}} = (-1)^r e_{j_r} \cdots e_{j_1}$ and $\overline{ab} = \overline{b}\overline{a}$ for $a, b \in Cl_n$.

For each $a = a_0 + \cdots + a_1 \cdots e_1 \cdots e_n \in Cl_n$ the scalar part of $\bar{a}a$ gives the square of the norm of $a$, namely $a_0^2 + \cdots + a_i^2$. The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(n) := \{a \in Cl_n : a = y_1 \cdots y_p : y_1, \ldots, y_p \in S^{n-1}, p \in \mathbb{N}\}$$

and is clearly a group under multiplication in $Cl_n$.

Now suppose that $y \in S^{n-1} \subseteq \mathbb{R}^n$. Look at $yxy = yx^\parallel y + yx^\perp y = -x^\parallel y + x^\perp y$ where $x^\parallel y$ is the projection of $x$ onto $y$ and $x^\perp y$ is perpendicular to $y$. So $yxy$ gives a reflection of $x$ in the $y$ direction. By the Cartan–Dieudonné Theorem each $O \in O(n)$ is the composition of a finite number of reflections. If $a = y_1 \cdots y_p \in Cl_n$, then

$$Oa = \tilde{a} = \tilde{y_1} \cdots \tilde{y_p} \circ \cdots \circ \tilde{y_p} \circ \cdots \circ \tilde{y_1}$$
\(Pin(n)\), then \(\tilde{a} := y_p \ldots y_1\) and \(ax\tilde{a} = O_a(x)\) for some \(O_a \in O(n)\). Choosing \(y_1, \ldots, y_p\) arbitrarily in \(S^{n-1}\), we see that the group homomorphism
\[
\theta : Pin(n) \rightarrow O(n) : a \mapsto O_a
\]
with \(a = y_1 \ldots y_p\) and \(O_a(x) = ax\tilde{a}\) is surjective. Further \(-ax(-\tilde{a}) = ax\tilde{a}\), so \(1, -1 \in \ker(\theta)\). In fact \(\ker(\theta) = \{\pm 1\}\). The Spin group is defined as
\[
Spin(n) := \{a \in Pin(n) : a = y_1 \ldots y_p \text{ and } p \text{ even}\}
\]
and is a subgroup of \(Pin(n)\). There is a group homomorphism
\[
\theta : Spin(n) \rightarrow SO(n)
\]
which is surjective with kernel \(\{1, -1\}\). See [P] for details.

The Dirac Operator in \(\mathbb{R}^n\) is defined to be
\[
D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.
\]
Note \(D^2 = -\Delta_n\), where \(\Delta_n\) is the Laplacian in \(\mathbb{R}^n\).

Let \(\mathcal{M}_k\) denote the space of \(Cl_n\)-valued polynomials, homogeneous of degree \(k\) and such that if \(p_k \in \mathcal{M}_k\) then \(Dp_k = 0\). Such a polynomial is called a left monogenic polynomial homogeneous of degree \(k\). Note if \(h_k \in \mathcal{H}_k\), the space of \(Cl_n\)-valued harmonic polynomials homogeneous of degree \(k\), then \(Dh_k \in \mathcal{M}_{k-1}\).

But \(D(p_k - 1)(u) = (-n - 2k + 2)p_k(u)\), so
\[
\mathcal{H}_k = \mathcal{M}_k \bigoplus u\mathcal{M}_{k-1}, h_k = p_k + up_{k-1}.
\]
This is the so-called Almansi-Fischer decomposition of \(\mathcal{H}_k\). See [BDS], [R2].

Note that if \(Df(u) = 0\) then \(\tilde{f}(u)\tilde{D} = -\tilde{f}(u)D = 0\). So we can talk of right monogenic polynomials, homogeneous of degree \(k\) and we obtain by conjugation a right Almansi-Fischer decomposition,
\[
\mathcal{H}_k = \overline{\mathcal{M}_k} \bigoplus \overline{\mathcal{M}_{k-1}}u,
\]
where \(\overline{\mathcal{M}_k}\) stands for the space of right monogenic polynomials homogeneous of degree \(k\).

### 3 The Rarita-Schwinger Operator \(R_k\)

Suppose \(U\) is a domain in \(\mathbb{R}^n\). Consider a function of two variables
\[
f : U \times \mathbb{R}^n \rightarrow Cl_n
\]
such that for each \( x \in U, f(x, u) \) is a left monogenic polynomial homogeneous of degree \( k \) in \( u \). Consider the action of the Dirac operator:

\[
D_x f(x, u).
\]

As \( \text{Cl}_n \) is not commutative then \( D_x f(x, u) \) is no longer monogenic in \( u \) but it is still harmonic and homogeneous of degree \( k \) in \( u \). So by the Almansi-Fischer decomposition, \( D_x f(x, u) = f_{1,k}(x, u) + u f_{2,k-1}(x, u) \) where \( f_{1,k}(x, u) \) is a left monogenic polynomial homogeneous of degree \( k \) in \( u \) and \( f_{2,k-1}(x, u) \) is a left monogenic polynomial homogeneous of degree \( k - 1 \) in \( u \). Let \( P_k \) be the left projection map

\[
P_k : \mathcal{H}_k \rightarrow \mathcal{M}_k,
\]

then \( R_k f(x, u) \) is defined to be \( P_k D_x f(x, u) \). The left Rarita-Schwinger equation is defined to be (see [BSSVI])

\[
R_k f(x, u) = 0.
\]

We also have a right projection \( P_{k,r} : \mathcal{H}_k \rightarrow \overline{\mathcal{M}}_k \), and a right Rarita-Schwinger equation \( f(x, u) D_x P_{k,r} = f(x, u) R_k = 0 \). Since

\[
D_x f(x, u) = p_k(x, u) + u p_{k-1}(x, u) \quad \text{and} \quad D_u u p_{k-1}(x, u) = -(n + 2k - 2) p_{k-1}(x, u),
\]

we have \( u p_{k-1}(x, u) = -\frac{1}{n + 2k - 2} u D_u D_x f(x, u) \). Thus \( (1 - P_k) D_x f(x, u) = u p_{k-1}(x, u) = -\frac{1}{n + 2k - 2} u D_u D_x f(x, u) \). Hence

\[
P_k D_x f(x, u) = \frac{1}{n + 2k - 2} u D_u D_x f(x, u) + D_x f(x, u) = \left( \frac{u D_u}{n + 2k - 2} + 1 \right) D_x f(x, u).
\]

So we obtain that \( P_k = \left( \frac{u D_u}{n + 2k - 2} + 1 \right) \) and \( R_k = \left( \frac{u D_u}{n + 2k - 2} + 1 \right) D_x \). See [BSSVI].

It is crucial to ask if there are any non-trivial solutions to this equation. First for any \( k \)-monogenic polynomial \( p_k(u) \) we have trivially \( R_k p_k(u) = 0 \). In particular the reproducing kernel of \( \mathcal{M}_k \) is annihilated by \( R_k \). We now produce a representation of this reproducing kernel. Consider the fundamental solution \( G(u) = \frac{1}{\omega_n \| u \|^n} \) to the Dirac operator \( D \), where \( \omega_n \) is the surface area of the unit sphere, \( \mathbb{S}^{n-1} \).

Consider the Taylor series expansion of \( G(v - u) \) and restrict to the \( k \)th order terms in \( u_1, \ldots, u_n \) \( (u = u_1 e_1 + \ldots + u_n e_n) \). These terms have as vector valued coefficients

\[
\frac{\partial^k}{\partial v_1^{k_1} \ldots \partial v_n^{k_n}} G(v) \quad (k_1 + \ldots + k_n = k).
\]
As $G(v)$ is a solution to the Dirac equation, $DG(v) = \sum_{i=1}^{n} e_j \frac{\partial G(v)}{\partial v_j} = 0$, we can replace $\frac{\partial}{\partial v_1}$ by $-\sum_{j=2}^{n} e_j \frac{\partial}{\partial v_j}$. Doing this each time $\frac{\partial}{\partial v_1}$ occurs and collecting like terms we obtain a finite series of polynomials homogeneous of degree $k$ in $u$

$$\sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v)$$

where the summation is taken over all permutations of monogenic polynomials $(u_2 - u_1 e_1^{-1} e_2), \ldots, (u_n - u_1 e_1^{-1} e_n)$, each term in the summation contains $j_2$ copies of $(u_2 - u_1 e_1^{-1} e_2), \ldots, j_n$ copies of $(u_n - u_1 e_1^{-1} e_n)$, and

$$P_{\sigma}(u) = \frac{1}{k!} \Sigma (u_{i_1} - u_1 e_1^{-1} e_{i_1}) \ldots (u_{i_k} - u_1 e_1^{-1} e_{i_k}), V_{\sigma}(v) = \frac{\partial^k G(v)}{\partial v_2^{j_2} \ldots \partial v_n^{j_n}}$$

$j_2 + \ldots + j_n = k$, and $i_k \in \{2, \ldots, n\}$. Here summation is taken over all permutations of the monomials without repetition. See [BDS]. Note that this series is the sum of the $k$-th order terms in the Taylor expansion of $G(v - u)$ and consequently it is a vector.

Now $\int_{S^{n-1}} V_{\sigma}(u) u P_{\mu}(u) dS(u) = \delta_{\sigma,\mu}$ where $\delta_{\sigma,\mu}$ is the Kronecker delta and $\mu$ is a set of $n - 1$ non-negative integers summing to $k$. See [BDS]. Following [BDS] it can be seen that the polynomial $P_{\sigma}$ is left monogenic and the set of all such polynomials, homogeneous of degree $k$, forms a basis for the right $\text{Cl}_n$ module $\mathcal{M}_k$. Consequently, the expression

$$Z_k(u, v) := \sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v) v$$

is the reproducing kernel of $\mathcal{M}_k$ with respect to integration over $S^{n-1}$ (see [BDS]). Further as $Z_k(u, v)$ does not depend on $x$,

$$R_k Z_k = 0.$$

Note that $V_{\sigma}(v) v$ on $S^{n-1}$ extends to $V_{\sigma}(-v^{-1}) G(v)$ on $\mathbb{R}^n$ and this function is a right monogenic polynomial in $v$ and it is homogeneous of degree $k$. See [BDS] and elsewhere.

We may ask if there are any solutions to $R_k f = 0$ that depend on $x$. To do this we look at the interaction of the operator $R_k$ with conformal transformations.

### 4 Conformal transformations

We first establish the invariance properties of the projection operator $P_k$. 

4.1 Conformal invariance of the projection $P_k$

Let $P_{k,w}$ and $P_{k,u}$ be the projections with respect to $w$ and $u$ respectively.

4.1.1 Orthogonal transformations

Let $x = ay_\bar{a}$, and $u = aw_\bar{a}$.

Lemma 1. \[ P_{k,w}a f(ay_\bar{a}, aw_\bar{a}) = a P_{k,u} f(x, u), \text{ where } a \in \text{Pin}(n). \]

Proof Let $f(x, u) = f_1(x, u) + u f_2(x, u)$, where $f_1(x, u)$ and $f_2(x, u)$ are monogenic polynomials homogeneous of degree $k$ and $k - 1$ in $u$. So $P_{k,u} f(x, u) = f_1(x, u) = f_1(ay_\bar{a}, aw_\bar{a})$ and $a f(ay_\bar{a}, aw_\bar{a}) = a f_1(ay_\bar{a}, aw_\bar{a}) + a aw_\bar{a} f_2(ay_\bar{a}, aw_\bar{a}) = a f_1(ay_\bar{a}, aw_\bar{a}) + w a f_2(ay_\bar{a}, aw_\bar{a})$.

Further as $a f_1(ay_\bar{a}, aw_\bar{a})$ and $a f_2(ay_\bar{a}, aw_\bar{a})$ are monogenic polynomials homogeneous of degree $k$ and $k - 1$ in $w$ respectively, we have $P_{k,w} a f(ay_\bar{a}, aw_\bar{a}) = a f_1(ay_\bar{a}, aw_\bar{a}) = a P_{k,u} f(x, u).$ $\blacksquare$

4.1.2 Inversion

Let $x = y^{-1}, u = \frac{y w y}{\|y\|^2}$.

Lemma 2. \[ P_{k,w} \frac{y}{\|y\|^n} f(y^{-1}, \frac{y w y}{\|y\|^2}) = \frac{y}{\|y\|^n} P_{k,u} f(x, u). \]

Proof Since $f(x, u) = f_1(x, u) + u f_2(x, u)$, by substitution we have

$$f(y^{-1}, \frac{y w y}{\|y\|^2}) = f_1(y^{-1}, \frac{y w y}{\|y\|^2}) + \frac{y w y}{\|y\|^2} f_2(y^{-1}, \frac{y w y}{\|y\|^2}).$$

Now multiplying both sides of the above equation by $\frac{y}{\|y\|^n}$, one gets

$$\frac{y}{\|y\|^n} f(y^{-1}, \frac{y w y}{\|y\|^2}) = \frac{y}{\|y\|^n} f_1(y^{-1}, \frac{y w y}{\|y\|^2}) + \frac{y w y}{\|y\|^n \|y\|^2} f_2(y^{-1}, \frac{y w y}{\|y\|^2})$$

$$= \frac{y}{\|y\|^n} f_1(y^{-1}, \frac{y w y}{\|y\|^2}) - w \frac{y}{\|y\|^n} f_2(y^{-1}, \frac{y w y}{\|y\|^2}).$$

Now Let $P_{k,w}$ act on the previous equation. We have

$$P_{k,w} \frac{y}{\|y\|^n} f(y^{-1}, \frac{y w y}{\|y\|^2}) = \frac{y}{\|y\|^n} f_1(y^{-1}, \frac{y w y}{\|y\|^2}) = \frac{y}{\|y\|^n} f_1(x, u) = \frac{y}{\|y\|^n} P_{k,u} f(x, u),$$

which follows from the facts that $\frac{y}{\|y\|^n} f_1(y^{-1}, \frac{y w y}{\|y\|^2})$ and $\frac{y}{\|y\|^n} f_2(y^{-1}, \frac{y w y}{\|y\|^2})$ are monogenic and homogeneous of degree $k$ and $k - 1$ in $w$. $\blacksquare$
4.1.3 Translations

Let \( x = y + a, a \in \mathbb{R}^n \). In order to keep the homogeneity of \( f(x, u) \) in \( u \), \( u \) does not change under translation. So we have

**Lemma 3.** \( P_k f(x, u) = P_k f(y + a, u) \), where \( x = y + a \).

4.1.4 Dilations

Let \( x = \lambda y \), where \( \lambda \in \mathbb{R}^+ \). It is obvious to observe that \( P_k \) is invariant under dilation.

**Lemma 4.** \( P_k f(x, u) = P_k f(\lambda y, u) \), where \( x = \lambda y \).

Hence \( P_k \) is conformally invariant.

Ahlfors [A] and Vahlen [V] show that given a Möbius transformation \( y = \phi(x) \) on \( \mathbb{R}^n \cup \{\infty\} \) it can be expressed as \( y = (ax + b)(cx + d)^{-1} \) where \( a, b, c, d \in Cl_n \) and satisfy the following conditions:

i. \( a, b, c, d \) are all products of vectors in \( \mathbb{R}^n \).

ii. \( ab, cd, bc, da \in \mathbb{R}^n \).

iii. \( a \bar{d} - b \bar{c} = \pm 1 \).

When \( c = 0 \), \( \phi(x) = (ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1} = \pm ax\bar{a} + bd^{-1} \). Now assume \( c \neq 0 \), then \( \phi(x) = (ax + b)(cx + d)^{-1} = ace^{-1} \pm (cx\bar{c} + d\bar{c})^{-1} \), this is the so-called Iwasawa decomposition. Using this notation and the conformal weights, \( f(\phi(x)) \) is changed to \( J(\phi, x)f(\phi(x)) \), where \( J(\phi, x) = \frac{cx + d}{\|cx + d\|^n} \). Note when \( \phi(x) = x + a \) then \( J(\phi, x) \equiv 1 \). Now using the Iwasawa decomposition, we get the following result:

**Theorem 1.** \( P_{k,w} J(\phi, x)f(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}) = J(\phi, x)P_{k,u} f(\phi(x), u) \), where \( u = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2} \) and where \( P_{k,w} \) and \( P_{k,u} \) are the projections with respect to \( w \) and \( u \) respectively.

Note that if the Möbius transformation is either translation or dilation then \( u = w \). This explains why in Lemma 4 the term \( u \) is not multiplied by \( \lambda \).

Lemmas 1 and 2 and Theorem 1 establish intertwining relationships for the projection operator, \( P_k \), under actions of the conformal group.
4.2 Conformal invariance of the Rarita-Schwinger operator $R_k$

Now let us establish the intertwining operators for $R_k$ and the conformal invariance of the equation $R_k f = 0$. Let $R_{k,u}$ and $R_{k,w}$ be the Rarita-Schwinger operators with respect to $u$ and $w$ respectively.

We will need the following. If we have the Möbius transformation $y = \phi(x)$ and $D_x$ is the Dirac operator with respect to $x$ and $D_y$ is the Dirac operator with respect to $y$ then $D_x = J_{-1}(\phi, x)^{-1}D_y J_1(\phi, x)$, where $J_{-1}(\phi, x) = \frac{cx + d}{\|cx + d\|^{n+2}}$ and $J_1(\phi, x) = J(\phi, x) = \frac{cx + d}{\|cx + d\|^n}$. See [R1].

4.2.1 Orthogonal transformations $O \in O(n), a \in Pin(n)$

**Theorem 2.** If $x = ay\bar{a}$, $u = aw\bar{a}$, then $aR_{k,u} f(x, u) = R_{k,w} \bar{a} f(ay\bar{a}, aw\bar{a})$.

**Proof**

$$R_{k,u} f(x, u) = P_{k,u} D_x f(x, u) = P_{k,u} a^{-1} D_y \bar{a} f(ay\bar{a}, u)$$

Therefore, by Lemma 1

$$aP_{k,u} a^{-1} D_y \bar{a} f(ay\bar{a}, u) = P_{k,w} a a^{-1} D_y \bar{a} f(ay\bar{a}, aw\bar{a}) = R_{k,w} \bar{a} f(ay\bar{a}, aw\bar{a}) . \ $$

In fact, Theorem 2 tells us that if $R_k f(x, u) = 0$ then $R_k \bar{a} f(ay\bar{a}, aw\bar{a}) = 0$.

4.2.2 Inversion

Let $x = y^{-1}, (=\frac{-y}{\|y\|^2})$.

**Theorem 3.** Set $u = \frac{yw y}{\|y\|^2}$, then $\frac{y}{\|y\|^n+2} R_{k,u} f(x, u) = R_{k,w} G(y) f(y^{-1}, \frac{yw y}{\|y\|^2})$.

**Proof**

$$R_{k,u} f(x, u) = P_{k,u} D_x f(x, u) = P_{k,u} G_{-1}(y)^{-1} D_y G(y) f(y^{-1}, u),$$

where $G_{-1}(y) = y\|y\|^n$.

Therefore by Lemma 2,

$$G_{-1}(y) P_{k,u} G_{-1}(y)^{-1} D_y G(y) f(y^{-1}, u)$$

$$= P_{k,w} G_{-1}(y) G_{-1}(y)^{-1} D_y G(y) f(y^{-1}, \frac{yw y}{\|y\|^2}) = R_{k,w} G(y) f(y^{-1}, \frac{yw y}{\|y\|^2}). \ $$

Consequently, if $R_k f(x, u) = 0$, then $R_k G(y) f(y^{-1}, \frac{yw y}{\|y\|^2}) = 0$. 

8
4.2.3 Dilations

Let $x = \lambda y, \lambda \in \mathbb{R}^+$. $R_k f(x, u) = R_k f(\lambda y, u)$ and if $R_k f(x, u) = 0$ then $R_k f(\lambda y, u) = 0$.

4.2.4 Translations

Let $x = y + a, a \in \mathbb{R}^n$. In order to preserve homogeneity of polynomials in $u$, $f(x, u)$ is transformed under a translation by $a$ to $f(y + a, u)$ (Note: otherwise the action of the Vahlen matrices is not correct). So $R_k f(x, u) = 0$ implies $R_k f(y + a, u) = 0$, where $x = y + a$.

Now using the Iwasawa decomposition of $(ax + b)(cx + d)^{-1}$, we obtain intertwining operators for $R_k$:

Theorem 4.

$$R_{k,x,w} J_1(\phi, x) \psi(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}) = J_{-1}(\phi, x) R_{k,y,u} \psi(y, u),$$

where $y = \phi(x)$, $u = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}$, $R_{k,x,w} = P_{k,w} D_x$ and $R_{k,y,u} = P_{k,u} D_y$.

Consequently, we obtain that $R_k f(x, u) = 0$ implies

$$R_k J(\phi, x) f(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}) = 0,$$

where $u = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}$. For this last formula see also [BSSV1].

5 A Kernel for $R_k$ and Some Basic Integral Formulas

Now applying inversion from the left we obtain that if $R_k Z_k(u, v) = 0$ then $R_k G(x) Z_k(\frac{ux}{\|x\|^2}, v) = 0$. That is,

$$F_k(x, u, v) = \frac{x}{\|x\|^n} Z_k(\frac{ux}{\|x\|^2}, v) = \frac{x}{\|x\|^{n+2k}} Z_k(xux, v)$$

is a non-trivial solution to $R_k f(x, u) = 0$ on $\mathbb{R}^n \setminus \{0\}$. Note that this function is monogenic in $u$.  


Similarly, applying inversion from the right we obtain that

\[ Z_k(u, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|^n} = Z_k(u, xvx) \frac{x}{\|x\|^n + 2k} \]

is a non-trivial solution to \( f(x, v)R_k = 0 \) on \( \mathbb{R}^n \setminus \{0\} \). In fact, \([\text{BSSV1}]\), this function is \( F_k(x, u, v) \), and, up to a constant, \( F_k(x, u, v) \) is the fundamental solution of \( R_k \). One proof of this statement is given in the following. Let \( M_k \) denote the space of left monogenic polynomials homogeneous of degree \( k \) and suppose \( a \in \text{Pin}(n) \), then

\[ l_\tilde{a} : M_k \rightarrow M_k : f(u) \rightarrow \tilde{a}f(au) \]

is an isomorphism. Similarly, let \( \overline{M}_k \) denote the space of right monogenic polynomials homogeneous of degree \( k \) then

\[ r_\tilde{a} : \overline{M}_k \rightarrow \overline{M}_k : f(u) \rightarrow f(au) \tilde{a} \]

is also an isomorphism. Using these isomorphisms it may be seen that for each \( a \in \text{Pin}(n) \) then \( \pm \tilde{a}Z_k(au, av) \tilde{a} \) is also the reproducing kernel for \( M_k \). We choose the plus sign when \( a \in \text{Spin}(n) \) and the minus sign when \( a \in \text{Pin}(n) \setminus \text{Spin}(n) \). Now consider a non-zero vector \( x \in \mathbb{R}^n \), then \( \frac{x}{\|x\|} \in \text{Pin}(n) \). So we have

\[ Z_k(u, v) = -\frac{x}{\|x\|}Z_k(\frac{xux}{\|x\|^2}, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|} \]

that is,

\[ -\frac{x}{\|x\|}Z_k(\frac{xux}{\|x\|^2}, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|} \]

is also the reproducing kernel for \( M_k \). Now we look at the fundamental solution of \( R_k \) which has the representation \( Z_k(u, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|^n} \).

Then using the previous equality, we get

\[ Z_k(u, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|^n} = -\frac{x}{\|x\|}Z_k(\frac{xux}{\|x\|^2}, \frac{xvx}{\|x\|^2}) \frac{x}{\|x\|} \frac{x}{\|x\|^n} = -\frac{x}{\|x\|^n}Z_k(\frac{xux}{\|x\|^2}, v) \]

Further suppose \( \mu \) is a \( Cl_n \) valued measure on \( \mathbb{R}^n \) with compact support, \( [\mu] \). It follows for suitable choices of \( \mu \) the integral \( \int_{[\mu]} F_k(x, u, v)d\mu \) defines a solution to \( R_kf = 0 \) on \( (\mathbb{R}^n \setminus [\mu]) \times \mathbb{R}^n \).

### 5.1 Stokes’ Theorem

We first build Stokes’ Theorem for the Rarita-Schwinger operator. This is based on Stokes’ Theorem for the Dirac operator.

**Theorem 5.** *(Stokes’ Theorem for the Dirac operator, [BDS]*)

Let \( \Omega \) and \( \Omega' \) be domains in \( \mathbb{R}^n \) and suppose the closure of \( \Omega \) lies in \( \Omega' \). Further
suppose the closure of $\Omega$ is compact and $\partial \Omega$ is piecewise smooth. Let $f,g \in C^1(\Omega', Cl_n)$. Then
\[
\int_{\partial \Omega} g(x,u) d\sigma_x f(x,u) = \int_{\Omega} [(g(x,u)D_x f(x,u) + g(x,u)(D_x f(x,u))] dx^n,
\]
where $dx^n = dx_1 \wedge \cdots \wedge dx_n$, $d\sigma_x = n(x) d\sigma(x)$, $\sigma$ is scalar Lebesgue measure on $\partial \Omega$ and $n(x)$ is unit outer normal vector to $\partial \Omega$. We may write $n(x)$ as $\sum_{i=1}^n n_i(x)e_i$, where $n_i(x)$ are scalar-valued functions.

$g(x,u)D_x$ means $D_x$ acts from the right on $g(u,x)$.

**Definition 1.** For any $Cl_n$-valued polynomials $P(u), Q(u)$, the inner product $(P(u), Q(u))_u$ with respect to $u$ is given by
\[
(P(u), Q(u))_u = \int_{S^{n-1}} P(u)Q(u) dS(u).
\]

This inner product differs slightly from the Fischer inner product in [BSSV1]. There the inner product is $\int_{S^{n-1}} R(u)Q(u) dS(u)$ for a $Cl_n$ valued polynomial $R(u)$. If we place $R(u) = \overline{P(u)}$ we see that, as the conjugation $-$ is an isomorphism, the two inner products are equivalent. For any $p_k \in M_k$, one obtains (see [BDS])
\[
p_k(u) = (Z_k(u,v), p_k(v))_v = \int_{S^{n-1}} Z_k(u,v)p_k(v) dS(v).
\]

Using Stokes’ Theorem for the Dirac operator, we can obtain the basic formulas related to the Rarita-Schwinger operators.

**Lemma 5.** Suppose $p_k$ is a left monogenic polynomial homogeneous of degree $k$ and $p_{k-1}$ is a left monogenic polynomial homogeneous of degree $k-1$ then
\[
\int_{S^{n-1}} \tilde{p}_{k-1}(u)up_k(u) dS(u) = 0.
\]

Outline Proof: As we are integrating over the unit sphere the previous integral can be written as
\[
\int_{S^{n-1}} \tilde{p}_{k-1}(u)n(u)p_k(u) dS(u).
\]
By the Clifford-Cauchy Theorem [BDS] this integral vanishes. □

We now have
Theorem 6. (Rarita-Schwinger Stokes’ Theorem) [BSSV1] Let \( \Omega' \) and \( \Omega \) be as in Theorem 5. Then for \( f, g \in C^1(\Omega', \mathcal{M}_k) \), we have

\[
\int_{\partial \Omega} (g(x,u)d\sigma_x f(x,u))_u = \int_{\Omega} (g(x,u)R_k, f(x,u))_u dx^n + \int_{\Omega} (g(x,u), R_k f(x,u))_u dx^n.
\]

Further

\[
\int_{\partial \Omega} (g(x,u)d\sigma_x f(x,u))_u = \int_{\partial \Omega} (g(x,u), P_k d\sigma_x f(x,u))_u
\]

\[
= \int_{\partial \Omega} (g(x,u) d\sigma_x P_k, f(x,u))_u.
\]

Outline Proof: The first identity is obtained by first applying Stokes’ Theorem to the integral \( \int_{\partial \Omega} (g(x,u)d\sigma_x f(x,u))_u \) to obtain

\[
\int_{\Omega} (g(x,u)D, f(x,u))_u + (g(x,u), Df(x,u))_u dx^n.
\]

Both \( g(x,u)D \) and \( Df(x,u) \) have an Almansi-Fischer decomposition with respect to \( u \). So applying Lemma 5 with respect to \( u \) and Definition 1 and these Almansi-Fischer decompositions give the result.

The second collection of identities again arise by applying the Almansi-Fischer decomposition \( d\sigma_x f(x,u) \) and \( g(x,u)d\sigma_x \) respectively, and then applying Definition 1 and Lemma 5 with respect to \( u \).

Now if both \( f(x,u) \) and \( g(x,u) \) are solutions of \( R_k \), then we have the following result:

Corollary 1. (Cauchy’s Theorem)
If \( R_k f(x,u) = 0 \) and \( g(x,u)R_k = 0 \) for \( f, g \in C^1(\Omega', \mathcal{M}_k) \), then

\[
\int_{\partial \Omega} (g(x,u), P_k d\sigma_x f(x,u))_u = 0.
\]

Let \( S \) be a hypersurface in \( \mathbb{R}^n \) and \( y = \phi(x) = (ax + b)(cx + d)^{-1} \). Now look at Cauchy’s Theorem:

\[
0 = \int_S (g(y,u), P_k d\sigma_y f(y,u))_u = \int_S (g(y,u), P_k n(y)f(y,u))_u d\sigma(y)
\]

\[
= \int_{\phi^{-1}(S)} \left( g(\phi(x), u), P_k \bar{J} (\phi, x) n(x) J(\phi, x)f(\phi(x), u) \right)_u d\sigma(x)
\]

\[
= \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} g(\phi(x), u) P_k, w \bar{J} (\phi, x) n(x) J(\phi, x)f(\phi(x), u) dS(u) d\sigma(x).
\]
Set \( u = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2} \), since \( P_{k,u} \) can interchange with \( \tilde{J}(\phi, x) \), the previous equation equals

\[
0 = \int_{\phi^{-1}(S)} g(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}) \tilde{J}(\phi, x) P_{k,u} n(x) J(\phi, x) f(\phi(x)),
\]

\[
\frac{(cx + d)w(cx + d)}{\|cx + d\|^2} dS(w) d\sigma(x)
\]

\[
= \int_{\phi^{-1}(S)} (g(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}) \tilde{J}(\phi, x), P_{k,u} d\sigma_x J(\phi, x)
\]

\[
f(\phi(x), \frac{(cx + d)w(cx + d)}{\|cx + d\|^2})) w.
\]

Therefore, Cauchy’s Theorem is conformally invariant under Möbius transformations.

We now wish to introduce the Borel-Pompeiu Theorem from [BSSV1]. First we will need:

**Lemma 6.** Suppose \( h_k : \mathbb{R}^n \rightarrow Cl_n \) is a harmonic polynomial homogeneous of degree \( k \) and \( n > 2 \). Suppose \( u \in S^{n-1} \) then

\[
\frac{1}{\omega_n} \int_{S^{n-1}} h_k(xux) dS(x) = c_k h_k(u),
\]

where \( c_k = \frac{n - 2}{n - 2 + 2k} \).

The proof follows from [DX] (Proposition 2.2.3 on page 34).

**Proof** Fix \( u \in S^{n-1} \), suppose \( h_k : \mathbb{R}^n \rightarrow Cl_n \) is a harmonic polynomial homogeneous of degree \( k \). Now compute

\[
I(h_k) = \frac{2}{\omega_n} \int_{(x,u) > 0} h_k(u - 2 \langle x, u \rangle x) dS(x),
\]

note \( xux = u - 2 \langle x, u \rangle x \) is the reflection of \( u \) in the mirror \( x^\perp \) (where \( \int_{(x,u) > 0} dS(x) = \frac{1}{2} \int_{S^{n-1}} dS(x) = \frac{\omega_n}{2} \), and \( \|x\| = 1 \)). Since \( xux = u - 2 \langle x, u \rangle x \) is invariant under \( x \rightarrow -x \) we have

\[
I(h_k) = \frac{1}{\omega_n} \int_{S^{n-1}} h_k(u - 2 \langle x, u \rangle x) dS(x).
\]
By rotation invariance assume \( u = (0, \cdots, 0, 1) \). Any harmonic homogeneous polynomial of degree \( k \) has an expression

\[
h_k(y) = \sum_{j=0}^{k} \|y\|^{k-j} P_{k-j}^{j+n/2-1} \left( \frac{y_n}{\|y\|} \right) h_{j}(y_1, \cdots, y_{n-1}),
\]

where \( h_j \) is harmonic and homogeneous of degree \( j \) and the normalized Gegenbauer polynomial is

\[
P_{m}^{\lambda}(t) = \sum_{i=0}^{m} \frac{(-m)_i (m+2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} \left( \frac{1-t}{2} \right)^i.
\]

In the coordinate system \( y = (y' \sin \theta, \cos \theta) \) with \( 0 \leq \theta \leq \pi \) and \( y' \in S^{n-2} \) the integral is

\[
\int_{S^{n-1}} h_k(y) dS(x) = c' \int_0^\pi \sin^{n-2} \theta d\theta \int_{S^{n-2}} h_k(y' \sin \theta, \cos \theta) dS_{n-2}(y').
\]

Set \( u = (0, \cdots, 0, 1) \) and \( x = (x' \sin \theta, \cos \theta) \) with \( x' \in S^{n-2} \), then

\[
u = 2 \langle x, u \rangle x = (-2x' \cos \theta \sin \theta, 1 - 2 \cos^2 \theta).
\]

Thus

\[
\int_{S^{n-1}} h_k(u-2 \langle x, u \rangle x) dS(x)
\]

\[
= \sum_{j=0}^{k} c' \int_0^\pi P_{k-j}^{j+n/2-1} (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta \int_{S^{n-2}} h_k(-2x' \cos \theta \sin \theta) dS_{n-2}(x')
\]

\[
= \sum_{j=0}^{k} c' \int_0^\pi P_{k-j}^{j+n/2-1} (1 - 2 \cos^2 \theta)(-2 \cos \theta \sin \theta)^j \sin^{n-2} \theta d\theta \int_{S^{n-2}} h_k(x') dS_{n-2}(x')
\]

\[
= h_{k_0} c' \int_0^\pi P_{k}^{n/2-1} (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta.
\]

The integrals equal zero for \( j > 0 \), by the orthogonality property of harmonics. Next set \( t = \cos \theta \) and \( dt = -\sin \theta d\theta \), then

\[
\int_0^\pi P_k^\lambda (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta = \int_{-1}^1 P_k^\lambda (1 - 2t^2)(1 - t^2)^{\lambda - \frac{1}{2}} dt
\]

\[
= \sum_{i=0}^{k} \frac{(-k)_i (k + 2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} \int_{-1}^1 t^{2i} (1 - t^2)^{\lambda - \frac{1}{2}} dt
\]

\[
= \sum_{i=0}^{k} \frac{(-k)_i (k + 2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} B(i + \frac{1}{2}, \lambda + \frac{1}{2})
\]

\[
= \frac{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \sum_{i=0}^{k} \frac{(-k)_i (k + 2\lambda)_i (\frac{1}{2})_i}{(\lambda + \frac{1}{2})_i i! (\lambda + 1)i}.
\]
The Saalschütz summation formula (for \(-k + a + b + 1 = c + d\)) is
\[
\binom{3F_2}{-k, a, b; c, d; 1} = \frac{(c - a)_k(d - a)_k}{(c)_k(d)_k}.
\]

\[
\int_0^{\pi} P_k^\lambda(1 - 2\cos^2 \theta) \sin^{n-2} \theta d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\lambda + \frac{1}{2}}{2}\right)}{\Gamma(\lambda + 1)} \frac{(\lambda)_k(\lambda + 1)_k}{\Gamma(\lambda + 1)} \frac{\lambda}{\lambda + k}.
\]

The normalizing constant is (now \(\lambda = \frac{n}{2}\))
\[
c_n' = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \text{ and }
1 \omega_n \int_{\mathbb{S}^{n-1}} h_k(u - 2\langle x, u \rangle x) dS(x) = \frac{n}{n-1+k} h_{k_0} = c_k h_{k_0},
\]

and \(h_k(0, 0, \ldots, 1) = h_{k_0}\), where \(c_k = \frac{n - 2}{n - 2 + 2k}\). ■

We now introduce
\[
E_k(x, u, v) := \frac{1}{\omega_n c_k} F_k(x, u, v).
\]

**Theorem 7.** (Borel-Pompeiu Theorem) Let \(\Omega'\) and \(\Omega\) be as in Theorem 5 and \(y \in \Omega\). Then for \(f \in C^1(\Omega', \mathcal{M}_k)\)
\[
f(y, u) = \int_{\partial \Omega} \left(E_k(x - y, u, v), P_{k, v}d\sigma_x f(x, v)\right)_v - \int_{\Omega} \left(E_k(x - y, u, v), R_{k, v}f(x, v)\right)_v dx^n.
\]

**Proof** Here we will use the representation
\[
E_k(x - y, u, v) = \frac{1}{\omega_n c_k} Z_k(u, (x - y)v(x - y)) \frac{x - y}{\|x - y\|^n},
\]

and \(R_{k, v}\) is the Rarita-Schwinger operator with respect to \(v\). Consider a ball \(B(y, r)\) centered at \(y\) with radius \(r\) such that \(B(y, r) \subset \Omega\). By Stokes’ Theorem, we have
\[
\int_{\Omega} \left(E_k(x - y, u, v), R_{k, v}f(x, v)\right)_v dx^n
\]
\[
= \int_{\Omega \setminus B(y, r)} \left(E_k(x - y, u, v), R_{k, v}f(x, v)\right)_v dx^n + \int_{B(y, r)} \left(E_k(x - y, u, v), R_{k, v}f(x, v)\right)_v dx^n.
\]

The second integral tends to zero as \(r\) tends to zero. This follows from the degree of homogeneity of \(E_k(x - y, u, v)\). Now applying Stokes’ Theorem to the first
integral, one gets
\[ \int_{\Omega \setminus B(y,r)} (E_k(x - y, u, v), R_{k,v} f(x, v))_v dx^n \]
\[ = \int_{\partial \Omega} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(x, v))_v - \int_{\partial B(y,r)} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(x, v))_v. \]

Now let us look at
\[ \int_{\partial B(y,r)} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(x, v))_v dx^n = \int_{\partial B(y,r)} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(y, v))_v \]
\[ + \int_{\partial B(y,r)} (E_k(x - y, u, v), P_{k,v} d\sigma_x [f(x, v) - f(y, v)])_v. \]

Since the second integral on the right hand side tends to zero as \( r \) goes to zero, we only need to deal with the first integral.
\[ \int_{\partial B(y,r)} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(y, v))_v \]
\[ = \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} E_k(x - y, u, v) P_{k,v} d\sigma_x f(y, v) dS(v) \]
\[ = \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k \left( u, \frac{(x - y)v(x - y)}{\|x - y\|^2} \right) \frac{x - y}{\|x - y\|} P_{k,v} \frac{y - x}{\|x - y\|^n} f(y, v) dS(v) d\sigma(x), \]
where \( n(x) \) is the unit outer normal vector and \( d\sigma(x) \) is the scalar measure on \( \partial B(y, r) \). Now \( n(x) \) here is \( \frac{y - x}{\|y - x\|} \). Hence the previous integral becomes
\[ \frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x - y)v(x - y)}{\|x - y\|^2} \right) \frac{x - y}{\|x - y\|} P_{k,v} \frac{y - x}{\|x - y\|^n} f(y, v) dS(v) d\sigma(x). \]

By equation (1) this integral becomes
\[ \frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x - y)v(x - y)}{\|x - y\|^2} \right) \frac{x - y}{\|x - y\|} P_{k,v} \frac{y - x}{\|x - y\|^n} f(y, v) dS(v) d\sigma(x) \]
\[ = \frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} Z_k(u, \frac{(x - y)v(x - y)}{\|x - y\|^2}) f(y, v) d\sigma(x) dS(v) \]
\[ \int_{\mathbb{S}^{n-1}} Z_k(u, v) f(y, v) dS(v) = f(y, u). \]
Theorem 8. ([BSSV1]) (Cauchy’s Integral Formula) If $R_k f(x, v) = 0$, then for $y \in \Omega$,

$$f(y, u) = \int_{\partial \Omega} (E_k(x - y, u, v), P_k d\sigma_x f(x, v))_v$$

$$= \int_{\partial \Omega} (E_k(x - y, u, v)d\sigma_x P_k f(x, v))_v.$$

We now show the conformal invariance of Cauchy’s Integral Formula. We start with inversion.

Since

$$x^{-1} - y^{-1} = -y^{-1}(x - y)x^{-1} = -x^{-1}(x - y)y^{-1}$$

$$= \frac{-x}{\|x\|^2}(x - y) \frac{y}{\|y\|^2} = \frac{-y}{\|y\|^2}(x - y) \frac{x}{\|x\|^2},$$

$$E_k(x^{-1} - y^{-1}, u, v) = G(x^{-1} - y^{-1})Z_k \left( \frac{(x^{-1} - y^{-1})u(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1}\|^2}, v \right)$$

$$= -G(y)^{-1}G(x - y)G(x)^{-1}Z_k \left( \frac{(x - y)yuy(x - y)x}{\|x\|^2\|y\|^2\|x - y\|^2}, v \right)$$

$$= -G(y)^{-1}G(x - y)G(x)^{-1} \frac{-x}{\|x\|}Z_k \left( \frac{(x - y)yuy(x - y)}{\|y\|^2\|x - y\|^2}, \frac{xvx}{\|x\|^2} \right) \frac{x}{\|x\|}$$

$$= G(y)^{-1}G(x - y)Z_k \left( \frac{(x - y)u'(x - y)}{\|x - y\|^2}, \frac{xvx}{\|x\|^2} \right) x\|x\|^{n-2}, \text{ set } u' = \frac{yuy}{\|y\|^2}$$

$$= -G(y)^{-1}G(x - y)Z_k \left( \frac{(x - y)u'(x - y)}{\|x - y\|^2}, \frac{xvx}{\|x\|^2} \right) G(x)^{-1}$$

$$= -G(y)^{-1}E_k(x - y, u', v')G(x)^{-1},$$

where $u' = \frac{yuy}{\|y\|^2}$ and $v' = \frac{xvx}{\|x\|^2}$.

Now consider

$$E_k(axa - aya, u, v) = G(a(x - y)â)Z_k \left( \frac{a(x - y)âua(x - y)â}{\|a(x - y)â\|^2}, v \right)$$

$$= aG(x - y)âZ_k \left( \frac{a(x - y)âua(x - y)â}{\|x - y\|^2}, v \right)$$

$$= \pm aG(x - y)âaZ_k \left( \frac{âa(x - y)âua(x - y)âa}{\|x - y\|^2}, âva \right) â$$

17
Multiplying both sides of the previous equation by $J_k \left(\frac{(x - y)u'(x - y)}{\|x - y\|^2}, \tilde{a}v \right)$, set $u' = \tilde{a}u$.

Thus, by the fact that $\sigma = 0$, we get

$$f(y', u) = \int_S \left( E_k(x' - y', u, v), P_k n(x') f(x', v) \right) d\sigma(x')$$

$$= \int_S \int_{\mathbb{S}^{n-1}} E_k(x' - y', u, v) P_k n(x') f(x', v) dS(v) d\sigma(x').$$

Using the Iwasawa decomposition, one gets

$$E_k(\phi(x) - \phi(y), u, v) = J(\phi, y)^{-1} E_k(x - y, u', v') J(\phi, x)^{-1},$$

where $u' = \frac{(cy + d)u(cy + d)}{\|cy + d\|^2}$, $v' = \frac{(cx + d)v(cx + d)}{\|cx + d\|^2}$, and $\phi$ is the Möbius transformation.

Suppose $S$ is a smooth hypersurface lying in $\mathbb{R}^n$. Let $x' = \phi(x)$ and $y' = \phi(y)$, now let us consider Cauchy’s Integral Formula

$$f(\phi(x), u) = \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} J(\phi, y)^{-1} E_k(x - y, u', v') J(\phi, x)^{-1} P_k n(x') J(\phi, x) f(x', v) dS(v) d\sigma(x')$$

Multiplying both sides of the previous equation by $J(\phi, y)$, we obtain

$$J(\phi, y) f(\phi(y), \frac{(cy + d)u'(cy + d)}{\|cy + d\|^2}) = \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} E_k(x - y, u', v') P_k n(x) J(\phi, x) f(\phi(x), \frac{(cx + d)v'(cx + d)}{\|cx + d\|^2}) dS(v) d\sigma(x).$$
where \( u = \frac{(cy + d)u'(cy + d)}{\|cy + d\|^2} \) and \( v = \frac{(cx + d)v'(cx + d)}{\|cx + d\|^2} \).

Therefore, Cauchy’s Integral Formula is conformally invariant.

Now if the function \( \psi \) has compact support in \( \Omega \), then by the Borel-Pompeiu Theorem we have the following formula:

**Theorem 9.** \[ \int \int_{\mathbb{R}^n} -(E_k(x - y, u, v), R_k \psi(x, v))_u dx^n = \psi(y, u) \] for each \( \psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k) \).

Similarly, we get a Cauchy transform for the Rarita-Schwinger operator \( R_k : \)

**Definition 2.** For a domain \( \Omega \subset \mathbb{R}^n \) and a function \( f : \Omega \times \mathbb{R}^n \to Cl_n \), where \( f(x, u) \) is monogenic in \( u \), the Cauchy (or \( T_k \)-transform) of \( f \) is formally defined to be

\[
(T_k f)(y, v) = - \int \int_{\Omega} (E_k(x - y, u, v), f(x, u))_u dx^n, \quad y \in \Omega.
\]

**Theorem 10.** \( R_k T_k \psi = \psi \) for \( \psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k) \). i.e

\[
R_k \int \int_{\mathbb{R}^n} (E_k(x - y, u, v), \psi(x, u))_u dx^n = \psi(y, v).
\]

**Proof** For each fixed \( y \in \mathbb{R}^n \), let \( R(y) \) be a bounded rectangle in \( \mathbb{R}^n \) centered at \( y \). Then

\[
R_k \int \int_{\mathbb{R}^n \setminus R(y)} (E_k(x - y, u, v), \psi(x, u))_u dx^n = 0.
\]

Now consider

\[
\frac{\partial}{\partial y_i} \int \int_{R(y)} (E_k(x - y, u, v), \psi(x, u))_u dx^n
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \int_{R(y)} (E_k(x - y, u, v) - E_k(x - y - \varepsilon e_i, u, v), \psi(x, u))_u dx^n
\]

If we translate the rectangle by \( \varepsilon \) in \(-e_i\) direction, then the derivative will be shifted from \( E_k \) to \( \psi \). Hence the previous integral becomes

\[
\int \int_{R(y)} (E_k(x - y, u, v), \psi(x, u) - \psi(x + \varepsilon e_i, u))_u dx^n + \frac{1}{\varepsilon} \int \int_{(R(y + \varepsilon e_i) \setminus R(y)) \cup (R(y) \setminus R(y + \varepsilon e_i))} (E_k(x - y, u, v), \psi(x, u) - \psi(x + \varepsilon e_i, u))_u dx^n
\]
When $\varepsilon$ tends to zero, the integral is equal to
\[
\int \int_{R(y)} (E_k(x - y, u, v), \frac{\partial \psi(x, u)}{\partial x_i})_u dx^n + \int_{\partial R_1(y) \cup \partial R_2(y)} (E_k(x - y, u, v), \psi(x, u))_u d\sigma(x)
\]
where $\partial R_1(y)$ and $\partial R_2(y)$ are the two faces of $R(y)$ with normal vectors $\pm e_i$. So
\[
D_y \int \int_{R(y)} (E_k(x - y, u, v), \psi(x, u))_u dx^n = \int \int_{R(y)} \sum_{i=1}^{n} e_i(E_k(x - y, u, v), \frac{\partial \psi(x, u)}{\partial x_i})_u dx^n + \int_{\partial R(y)} n(x)(E_k(x - y, u, v), \psi(x, u))_u d\sigma(x).
\]
When the volume of $R(y)$ tends to zero, the first integral tends to zero by the homogeneity of the kernel $E_k$. So we shall concentrate attention on the integral
\[
P_k \int_{\partial R(y)} n(x)(E_k(x - y, u, v), \psi(x, u))_u d\sigma(x).
\]
This is equal to
\[
P_k \int_{\partial R(y)} \int_{S^{n-1}} n(x)E_k(x - y, u, v)\psi(x, u)dS(u)d\sigma(x),
\]
which in turn is equal to
\[
P_k \int_{\partial R(y)} \int_{S^{n-1}} n(x)E_k(x - y, u, v)\psi(y, u)dS(u)d\sigma(x)
\]
\[+P_k \int_{\partial R(y)} \int_{S^{n-1}} n(x)E_k(x - y, u, v)(\psi(x, u) - \psi(y, u))dS(u)d\sigma(x).
\]
But the last integral on the right side of the above formula tends to zero as the surface area of $\partial R(y)$ tends to zero. Hence we are left with
\[
P_k \int_{\partial R(y)} \int_{S^{n-1}} n(x)E_k(x - y, u, v)\psi(y, u)dS(u)d\sigma(x).
\]
By Stokes’ Theorem this is equal to
\[
P_k \int_{\partial B(y, r)} \int_{S^{n-1}} n(x)E_k(x - y, u, v)\psi(y, u)dS(u)d\sigma(x).
\]
In turn this is equal to
\[
    P_k \int_{\partial B(y,r)} \frac{1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{y - x}{\|x - y\|} \frac{x - y}{\|x - y\|^2} Z_k \left( \frac{(x - y)u(x - y)}{\|x - y\|^2}, v \right) \psi(y, u) dS(u)d\sigma(x)
\]
\[
= P_k \int_{\partial B(y,r)} \frac{1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{1}{r^{n-1}} Z_k \left( \frac{(x - y)u(x - y)}{\|x - y\|^2}, v \right) \psi(y, u) dS(u)d\sigma(x).
\]
By Lemma 6, the integral becomes \( P_k \int_{\mathbb{S}^{n-1}} Z_k(u, v) \psi(y, u) dS(u) = P_k \psi(y, v) = \psi(y, v). \)

Now we may establish the intertwining operators for the convolution operator \( E_k \ast . \) More precisely we shall show that:

**Theorem 11.** If \( \psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k), \) then
\[
    J_1(\phi, y) \int_{\mathbb{R}^n} \left( E_k(x' - y', u, v), \psi(x', v) \right)_v d(x')^n
\]
\[
= \int_{\mathbb{R}^n} \left( E_k(x - y, u', w), \tilde{J}_1(\phi, x), \psi(\phi(x), w) \right)_w dx^n,
\]
where \( x' = \phi(x), y' = \phi(y), u = \frac{(cy + d)u'(cy + d)}{\|cy + d\|^2} \) and \( v = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}. \)

Alternatively,
\[
    J_1(\phi, -)E_k \ast \psi = E_k \tilde{J}_1(\phi, -) \ast \psi.
\]

**Proof** First consider inversion, let \( \phi(x) = x^{-1}, \phi(y) = y^{-1}. \) Then
\[
\int_{\mathbb{R}^n} \left( E_k(x^{-1} - y^{-1}, u, v), \psi(x^{-1}, v) \right)_v d(x^{-1})^n
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} E_k(x^{-1} - y^{-1}, u, v) \psi(x^{-1}, v) dS(v) d(x^{-1})^n
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x^{-1} - y^{-1})v(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1}\|^2} \right) \frac{x^{-1} - y^{-1}}{\|x^{-1} - y^{-1}\|^2} \psi(x^{-1}, v) dS(v) d(x^{-1})^n.
\]
Since
\[
x^{-1} - y^{-1} = -y^{-1}(x - y)x^{-1} = -x^{-1}(x - y)y^{-1} = \frac{-x}{\|x\|^2} \frac{y}{\|y\|^2} = \frac{-y}{\|y\|^2} \frac{x}{\|x\|^2},
\]
\[
Z_k \left( u, \frac{(x^{-1} - y^{-1})v(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1\|^2}} \right) = Z_k \left( u, \frac{y(x - y)xvx(x - y)y}{\|x\|^2\|y\|^2\|x - y\|^2} \right)
\]
\[
= -\frac{y}{\|y\|} Z_k \left( \frac{yuv}{\|y\|^2}, \frac{(x - y)w(x - y)}{\|x - y\|^2} \right) \frac{y}{\|y\|}, \text{ set } w = \frac{xvx}{\|x\|^2}.
\]
Now the previous integral becomes
\[
\int \int_{\mathbb{R}^n} \int_{S^{n-1}} \frac{y}{\|y\|} Z_k \left( \frac{yuy}{\|y\|^2}, \frac{(x-y)w(x-y)}{\|x-y\|^2} \right) \frac{y}{\|y\|} y \|y\|^{n-2} G(x-y)x \|x\|^{n-2} \\
\psi(\phi(x), v) \frac{1}{\|x\|^{2n}} \text{d}S(v) \text{d}x^n
\]
\[
= \int \int_{\mathbb{R}^n} \int_{S^{n-1}} -y \|y\|^{n-2} Z_k \left( \frac{yuy}{\|y\|^2}, \frac{(x-y)w(x-y)}{\|x-y\|^2} \right) G(x-y) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), v) \text{d}S(v) \text{d}x^n
\]
\[
= \int \int_{\mathbb{R}^n} \int_{S^{n-1}} -y \|y\|^{n-2} E_k(x-y, u', w)) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), v) \text{d}S(v) \text{d}x^n,
\]
where \( u' = \frac{yuy}{\|y\|^2}, w = \frac{xvx}{\|x\|^2} \). Then the previous integral is
\[
\int \int_{\mathbb{R}^n} \int_{S^{n-1}} -y \|y\|^{n-2} E_k(x-y, u', w)) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), \frac{xwx}{\|x\|^2}) \text{d}S(w) \text{d}x^n
\]
Now multiplying both sides of the equation by \( \frac{y^{-1}}{\|y\|^{n-2}} \), we obtain
\[
\frac{y}{\|y\|^n} \int \int_{\mathbb{R}^n} (E_k(x^{-1} - y^{-1}, u, v), \psi(x^{-1}, v)) \text{d}(x^{-1})^n
\]
\[
= \int \int_{\mathbb{R}^n} \left( E_k(x - y, u', w) \frac{x}{\|x\|^{n+2}}, \psi(\phi(x), \frac{xwx}{\|x\|^2}) \right) \text{d}x^n,
\]
where \( u = \frac{yuy}{\|y\|^2} \) and \( v = \frac{xwx}{\|x\|^2} \).

Next, consider orthogonal transformations. We will apply similar arguments used to establish the equation under inversion. Let \( \phi(x) = ax\tilde{a} \) and \( \phi(y) = ay\tilde{a} \), where \( a \in Pin(n) \). Then
\[
\int \int_{\mathbb{R}^n} (E_k(ax\tilde{a} - ay\tilde{a}, u, v), \psi(\phi(x), v)) \text{d}(ax\tilde{a})^n
\]
\[
= \int \int_{\mathbb{R}^n} (E_k(a(x - y)\tilde{a}, u, v), \psi(\phi(x), v)) \text{d}(ax\tilde{a})^n
\]
\[
= \int \int_{\mathbb{R}^n} \int_{S^{n-1}} Z_k(u, \frac{a(x-y)\tilde{a}v(x-y)\tilde{a}}{\|x-y\|^2}) \frac{a(x-y)\tilde{a}}{\|x-y\|^n} \psi(\phi(x), v) \text{d}S(v) \text{d}x^n
\]
\[
= \pm \int \int_{\mathbb{R}^n} \int_{S^{n-1}} aZ_k(\tilde{a}u, \frac{(x-y)\tilde{a}v(x-y)}{\|x-y\|^2}) \tilde{a} \frac{a(x-y)\tilde{a}}{\|x-y\|^n} \psi(\phi(x), v) \text{d}S(v) \text{d}x^n.
\]
Set $w = \bar{a}v$, then $v = aw\bar{a}$. Hence the integral becomes
\[
\int\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} aZ_k(\bar{a}ua, \frac{(x-y)w(x-y)}{\|x-y\|^2}) \frac{(x-y)\bar{a}}{\|x-y\|^n} \psi(\phi(x), v) dS(v) dx^n
\]
\[
= \int\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} aE_k(x-y, u', w)\bar{a}\psi(\phi(x), aw\bar{a}) dS(aw\bar{a}) dx^n,
\]
where $u' = \bar{a}ua$ and $v = aw\bar{a}$. By the Iwasawa decomposition of $\phi(x) = (ax + b)(cx + d)^{-1}$, we obtain
\[
J_1(\phi, y) \int\int_{\mathbb{R}^n} (E_k(x'-y', u, v), \psi(x', v))_v d(x')^n
\]
\[
= \int\int_{\mathbb{R}^n} (E_k(x-y, u', w)\bar{J}_{-1}(\phi, x), \psi(\phi(x), w))_w dx^n,
\]
where $J_1(\phi, x) = J(\phi, x) = \frac{cx + d}{\|cx + d\|^n}$, $J_{-1}(\phi, x) = \frac{cx + d}{\|cx + d\|^{n+2}}$, $x' = \phi(x)$, $y' = \phi(y)$, $u = \frac{(cy + d)u'(cy + d)}{\|cy + d\|^2}$ and $v = \frac{(cx + d)w(cx + d)}{\|cx + d\|^2}$. Alternatively,
\[
J_1(\phi, -)E_k \psi = E_k\bar{J}_{-1}(\phi, -) \psi. \quad \blacksquare
\]

References

[A] L. V. Ahlfors, Old and new in Möbius groups, Ann. Acad. Sci. Fenn. Ser. A I Math., 9 (1984) 93-105.

[BDS] F. Brackx, R. Delanghe, and F. Sommen, Clifford Analysis, Pitman, London, 1982.

[BSSV1] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, Rarita-Schwinger Type Operators in Clifford Analysis, J. Funct. Anal. 185 (2001), No.2, 425-455.
[BSSV2] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, *Symmetric Analogues of Rarita-Schwinger Equations*, Annals of Global Analysis and Geometry 21 (2002), 215-240.

[DSS] R. Delanghe, F. Sommen, and V. Souček, *Clifford Analysis and Spinor Valued Functions*, Kluwer Academic, Dordrecht, 1992.

[DX] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge University Press, Cambridge, 2001.

[LRV1] J. Li, Carmen Vanegas and John Ryan, *Rarita-Shwinger Type Operators on Cylinders*, Advances in Applied Clifford Algebra, 22, 2012, 771-788.

[LRV2] J. Li, Carmen Vanegas and John Ryan, *Rarita-Shwinger Type Operators on the Sphere*, to appear.

[LR] J. Li and John Ryan, *Some operators associated to Rarita-Shwinger type operators*, accepted for publication in Complex Variables and Elliptic Equations.

[P] I. Porteous, *Clifford algebra and the classical groups*, Cambridge University Press, Cambridge, 1995.

[R1] J. Ryan, *Conformally coinvariant operators in Clifford analysis*, Z. Anal. Anwendungen, 14, 1995, 677-704.

[R2] J. Ryan, *Iterated Dirac operators in \( C^n \)*, Z. Anal. Anwendungen, 9, 1990, 385-401.

[S] V. Souček, *Conformal invariance of higher spin equations*, in "Proc. Symp. Analytical and Numerical methods in Clifford Analysis, Seiffen 1996," 175-186.

[Su] A. Sudbery, *Quaternionic Analysis*, Mathematical Procedings of the Cambridge Phil. Soc., (1979), 85, 199-225.

[SW] E. M. Stein, G. Weiss, *Generalizations of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math, 90 (1968), 163-196.

[Va1] P. Van Lancker, *Higher spin fields on smooth domains*, in Clifford Analysis and Its Applications, Eds. F. Brackx, J.S.R. Chisholm and V. Souček, Kluwer, Dordrecht 2001, 389-398.

[Va2] P. Van Lancker, *Rarita-Schwinger fields in the half Space*, Complex Variables and Elliptic Equations, 51, 2006, 563-579.
[V] K. Th. Vahlen, *Über Bewegungen und komplexe Zahlen*, (German) Math. Ann., 55(1902), No.4 585-593.

Charles F. Dunkl
Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA. Email:cfd5z@virginia.edu

Junxia Li
Department of Mathematics, University of Arkansas, Fayetteville, AR 72701, USA. Email:jxl004@uark.edu

John Ryan
Department of Mathematics, University of Arkansas, Fayetteville, AR 72701, USA. Email:jryan@uark.edu

Peter Van Lancker
Faculty of Applied Engineering Sciences, University College of Gent, Member of Gent University, Schoonmeerstaat 52, 9000 Gent, Belgium. Email:Peter.VanLancker@hogent.be