A Note on the Completeness of Generalized Eigenfunctions of the Hamiltonian of Reggeon Field Theory in Bargmann Space

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Abstract
The aim of the present paper is to review some old results of Intissar (Commun Math Phys 113:263–297, 1987) and to study some new spectral properties of the operator $H_{\mu,\lambda} = \mu A^*A + i\lambda A^*(A + A^*)A$ which characterizes the Reggeon field theory, where $\mu, \lambda$ are real parameters and $i^2 = -1$. $A\varphi := \frac{d\varphi}{dz}$ and $A^*\varphi := z\varphi; z = x + iy; (x, y) \in \mathbb{R}^2$ are the annihilation and the creation operators, satisfying the commutation relation $[A, A^*] = I$ in Bargmann space $\mathcal{B} = \{\varphi : \mathbb{C} \rightarrow \mathbb{C} \text{ entire function}; \int_{\mathbb{C}} |\varphi(z)|^2 e^{-|z|^2} dxdy < \infty\}$. The Hamiltonian $H_{\mu,\lambda}$ is non-Hermitian with respect to the above standard scalar product. The domain of the adjoint and anti-adjoint parts are not included in each other, nor is the domain of their commutator. Hence the question arises, whether the eigenfunctions of $H_{\mu,\lambda}$ form a complete basis? The main new results of this Note are the determination of the boundary conditions for the eigenvalue problem associated to $H_{\mu,\lambda}$ and the proof of the completeness of the basis mentioned in the question above by transforming $H_{\mu,\lambda}$ in the Hermitian form. In particular, $H_{\mu,\lambda}$ belongs to the class of pseudo-Hermitian Hamiltonians in the Mostafazadeh’s (J Math Phys 43(1):205–214, 2002 or Int J Geom Methods Mod Phys 7:1191–1306, 2010) sense.

Keywords Spectral theory · Gribov operators · Semigroups · Non-self-adjoint operators · Bargmann space · Reggeon field theory
1 Introduction

Let $\mathcal{B}$ be the Bargmann space [6] defined by:

$$\mathcal{B} = \{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire function} ; \int_{\mathbb{C}} |\varphi(z)|^2 e^{-|z|^2} dxdy < \infty \} \quad (1.1)$$

with standard scalar product

$$<\varphi, \psi> = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(z) \overline{\psi(z)} e^{-|z|^2} dxdy \quad (1.2)$$

and with usual basis

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; \ n \in \mathbb{N} \quad (1.3)$$

and

$$\mathcal{P} = \{ p(z), z \in \mathbb{C} \} \text{ is the set of polynomials which is dense in } \mathcal{B}.$$ 

The constant $\frac{1}{\pi}$ in the right (1.2) is chosen as that the norm of the constant function $\varphi \equiv 1$ is one.

Let

$$\mathcal{B}_s = \{ (a_n)_{n=0}^{\infty} \in \mathbb{C}; \sum_{n=0}^{\infty} n! |a_n|^2 < \infty \} \quad \text{(Segal space)} \quad (1.4)$$

$\mathcal{B}_s$ is also a Hilbert space. The scalar product on $\mathcal{B}_s$ is defined by

$$< (a_n), (b_n) >_s = \sum_{n=0}^{\infty} n! a_n \overline{b_n} \quad (1.5)$$

and the associated norm is denoted by $|| \cdot ||_s$.

It is well known that $\mathcal{B}$ is related to $\mathcal{B}_s$ by unitary transform of $\mathcal{B}$ onto $\mathcal{B}_s$, given by the following transform:

$$I : \mathcal{B} \longrightarrow \mathcal{B}_s, \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \longrightarrow I(\varphi) = \left( \frac{1}{n!} \varphi^{(n)}(0) \right)_{n=0}^{\infty} = (a_n)_{n=0}^{\infty} \quad (1.6)$$

In Segal–Bargmann’s representation, we have the annihilation operator $A$ with domain $D(A) = \{ \varphi \in \mathcal{B}; \frac{d}{dz} \varphi \in \mathcal{B} \}$ and the creation operator $A^*$ with domain $D(A^*) = \{ \varphi \in \mathcal{B}; z \varphi \in \mathcal{B} \}$. It is well known that $A^*$ is adjoint of $A$ (with respect above standard scalar product), $D(A)$ is dense in $\mathcal{B}$, $D(A) = D(A^*)$ and the injection $D(A) \hookrightarrow \mathcal{B}$ is a compact mapping (see [19] or lemma 0.2 in [21] for an elementary proof).
The Hamiltonian $H_{\mu,\lambda}$ of Reggeon field theory takes the following form:

$$
\begin{align*}
H_{\mu,\lambda} &= \mu z \frac{d}{dz} + i \lambda (z \frac{d^2}{dz^2} + z^2 \frac{d}{dz}) := \mu N + i \lambda H_I := i \lambda \mu \frac{d^2}{dz^2} + (i \lambda z^2 + \mu z) \frac{d}{dz} \\
\end{align*}
$$

(1.7)

with maximal domain $D(H_{\mu,\lambda}) = \{ \varphi \in B; H_{\mu,\lambda} \varphi \in B \}$

where

$$
N = z \frac{d}{dz} 
$$

(1.8)

with domain $D(N) = \{ \varphi \in B; N \varphi \in B \}$

$$
H_I = z \frac{d^2}{dz^2} + z^2 \frac{d}{dz} 
$$

(1.9)

with maximal domain $D(H_I) = \{ \varphi \in B; H_I \varphi \in B \}$

$(\mu, \lambda) \in \mathbb{R}^2$ and $i^2 = -1$.

The eigenvalue problem associated to $H_{\mu,\lambda}$ is given by

$$
\begin{align*}
\begin{cases}
  i \lambda z \varphi''(z) + (i \lambda z^2 + \mu z) \varphi'(z) = \sigma \varphi(z); & \sigma \in \mathbb{C}.
  \\
  \varphi \in D(H_{\mu,\lambda})
\end{cases}
\end{align*}
$$

(1.10)

where $\varphi'(z)$ denotes the first derivative and $\varphi''(z)$ the second derivative of $\varphi(z)$ with respect to $z$.

Now, we recall some spectral properties of $H_{\mu,\lambda}$:

(i) In 1987, we have given in [19] many spectral properties of $H_{\mu,\lambda}$ for $\mu > 0$ in particular the minimal domain of $H_{\mu,\lambda}$ coincides with its maximal domain, the positiveness of its eigenvalues, the existence of the smallest eigenvalue $\sigma_0 \neq 0$ and an asymptotic expansion of its semigroup $e^{-t H_{\mu,\lambda}}$ as $t \longrightarrow +\infty$.

Where the minimal domain of $H_{\mu,\lambda}$ is:

$$
D(H^{\text{min}}_{\mu,\lambda}) = \{ \varphi \in B, \exists p_n \in \mathcal{P}, p_n \longrightarrow \varphi, \exists \psi \in B; H_{\mu,\lambda} p_n \longrightarrow \psi \}
$$

(1.11)

(ii) In 1984, Ando and Zerner have shown in [5] that the smallest eigenvalue $\sigma_0$ of $H_{\mu,\lambda}$ can be analytically continued in $\mu$ on entire real axis.

(iii) In 1998, we have given in [20] the boundary conditions at infinity for a description of all maximal dissipative extensions in Bargmann space of the minimal Heun’s operator $H_I = z \frac{d^2}{dz^2} + z^2 \frac{d}{dz}$. The characteristic functions of the dissipative extensions have computed and some completeness theorems have obtained for the system of generalized eigenvectors. It is well known that the restriction $H_I^{\text{min}}$ of the closure of $H_I$ on the polynomials set $\mathcal{P}$ is symmetric.
But the minimal domain $D(H_{I}^{\text{min}}) = \{ \varphi \in B; \exists p_n \in P, p_n \rightarrow \varphi, \exists \psi \in B; H_I p_n \rightarrow \psi \}$ of $H_I$ is different of its maximal domain $D(H_I) = \{ \varphi \in B; H_I \varphi \in B \}$.

(iv) It is also well known that $H_I$ is chaotic operator in Devaney’s sense see [11] or [22], in particular its spectrum is $\sigma(H_I) = \mathbb{C}$.

(v) $H_{\mu,\lambda}$ is not self-adjoint operator, nevertheless it has several properties analogous to those of the self-adjoint operators as the following example provides an informative illustration of this fact.

For $c \in \mathbb{R}$ consider the non-normal differential operator

$$H_c = -\frac{d^2}{dx^2} + ix^3 + cx^2$$  \hspace{1cm} (1.12)

on its maximal domain

$$D(H_c) = \{ f \in L_2(\mathbb{R}); H_c f \in L_2(\mathbb{R}) \}$$  \hspace{1cm} (1.13)

It was shown in Ref. [12] that the spectrum of $H_c$ is real and positive and in [10] $H_c$ is closed and has compact resolvent so the spectrum is also discrete, But in [29] Novak recently obtained the following result:

The eigenfunctions of $H_c$ do not form a (Schauder) basis in $L_2(\mathbb{R})$.

It is therefore natural to put the question of the completeness of eigenfunctions of $H_{\mu,\lambda}$ in Bargmann space.

The main difficulty in dealing with this question is that $H_{\mu,\lambda}$ does not satisfy the criteria of the theorems known by the author on the completeness of the eigenvectors of the non-normal operators see for example in [3, 4, 13, 14, 24] and the references therein.

Among other difficulties of analysis of this operator, we can cite that $A^*(A + A^*)A$ is not dominated by $A^*A$, $H_{\mu,\lambda}$ is non-Hermitian with respect to above standard scalar product, the domain of the adjoint and anti-adjoint parts are not included in each other, nor is the domain of their commutator and others difficulties indicated specially in the Refs. [18] and [5].

To circumvent these difficulties and show the completeness of the eigenvectors of $H_{\mu,\lambda}$, we transform the problem of eigenvalues associated to $H_{\mu,\lambda}$ into a Hermitian form in order to apply a classical theorem of completeness of symmetric operators with compact resolvent.

The aim of this Note is the determination of boundary conditions of eigenvalue problem associated to $H_{\mu,\lambda}$ and to give a answer to above question by transform $H_{\mu,\lambda}$ to the Hermitian form.

Now, we give an outline of the content of this paper:

The Sect. 2 deals the action of some elementary operators on Bargmann space useful to study of eigenvalue problem associated to the Hamiltonian $H_{\mu,\lambda}$.

In Sect. 3, we give an asymptotic analysis of the solutions of bi-confluent Heun equation associated to eigenvalue problem of the Hamiltonian $H_{\mu,\lambda}$. 
In Sect. 4, we give on the negative imaginary axis \( z = -iy, y > 0 \) a transformation procedure of \( H_{\mu,\lambda} \) to a symmetric operator with compact resolvent and we show the main result of this paper that the eigenfunctions of \( H_{\mu,\lambda} \) form a complete basis.

2 Action of Some Elementary Operators on Bargmann Space Useful to Study of Eigenvalue Problem Associated to the Hamiltonian \( H_{\mu,\lambda} \)

We introduce the operator \( P \) (changing the sign of \( z \)) defined by

\[
PzP^{-1} = -z \quad \text{and} \quad P \frac{d}{dz} P^{-1} = -\frac{d}{dz},
\]

then we deduce that the formal adjoint of \( H_{\mu,\lambda} \) is given by

\[
PH_{\mu,\lambda}P^{-1} = \mu z \frac{d}{dz} - i\lambda \left( z \frac{d^2}{dz^2} + z^2 \frac{d}{dz} \right) := H_{\mu,-\lambda}
\]

The operator \( P \) is unitary and Hermitian in Bargmann space, since

\[
< P\varphi, P\psi >= < \varphi, \psi > \quad \text{and} \quad < \varphi, P\psi >= < P\varphi, \psi >
\]

Now, let us consider the following eigenvalue problem:

\[
H_{\mu,\lambda}\varphi_n = \sigma_n \varphi_n; \sigma_n \in \mathbb{R}
\]

and

\[
\varphi_n^*(z) = -P\varphi_n(z) = -\varphi_n(-z)
\]

Then we get:

\[
< \varphi_m^*, H_{\mu,\lambda}\varphi_n > = \sigma_n < \varphi_m^*, \varphi_n >
\]

and

\[
H_{\mu,-\lambda}\varphi_m^* = -H_{\mu,-\lambda}P\varphi_m = -PH_{\mu,\lambda}\varphi_m = \sigma_m \varphi_m^*
\]

and

\[
< H_{\mu,-\lambda}\varphi_m^*; \varphi_n > = \sigma_m < \varphi_m^*; \varphi_n >
\]

It follows that

\[
(\sigma_n - \sigma_m) < \varphi_m^*, \varphi_n > = < \varphi_m^*, H_{\mu,\lambda}\varphi_n > - < H_{\mu,-\lambda}\varphi_m^*, \varphi_n >
\]

\[
= < \varphi_m^*, H_{\mu,\lambda}\varphi_n > - < \varphi_m^*, H_{\mu,\lambda}\varphi_n >
\]

\[
= 0
\]
so that
\[ < \varphi_m^*, \varphi_n > = 0 \quad \text{when} \quad \sigma_n \neq \sigma_m \quad (2.7) \]

An orthogonal scalar product in Bargmann space associated to the operator \( P \).

We introduce the operator \( \nu \) (the Hermitian operator of the sign) on the eigenfunctions \( \{ \varphi_n \}, n = 1, 2, \ldots \) of \( H_{\mu, \lambda} \) defined by \( \nu(\varphi_n) = \nu_n \varphi_n \) where
\[
\begin{align*}
\nu_n &= +1, \quad \text{if} \quad < \varphi_n, P \varphi_n > > 0 \\
\nu_n &= -1, \quad \text{if} \quad < \varphi_n, P \varphi_n > < 0
\end{align*}
\]

We introduce the positively defined scalar product \( << \varphi, \psi >> = < \varphi, \nu P \psi > \) with respect to which the eigenstates of \( H_{\mu, \lambda} \) are orthogonal.

3 Asymptotic Analysis of the Solutions of Bi-confluent Heun Equation Associated to Eigenvalue Problem of the Hamiltonian \( H_{\mu, \lambda} \)

The Hamiltonian \( H_{\mu, \lambda} \) has the form \( p(z) \frac{\partial^2}{\partial z^2} + q(z) \frac{\partial}{\partial z} \) with \( p(z) = i \lambda z \) and \( q(z) = i \lambda z^2 + \mu z \) of degree 2 then \( H_{\mu, \lambda} \) is an operator of Heun type.

The eigenvalue problem associated to \( H_{\mu, \lambda} \) does not satisfy the classical ordinary differential equations of the form:
\[ p(z) \frac{\partial^2 \varphi}{\partial z^2} + q(z) \frac{\partial \varphi}{\partial z} = \sigma \varphi \quad (3.1) \]

where \( p(z) \) is a polynomial of degree at most two, \( q(z) \) is a polynomial of the degree exactly one and \( \sigma \) is a constant.

If \( \lambda \neq 0 \), the eigenvalue problem associated to \( H_{\mu, \lambda} \) can be written as follows:
\[ z \frac{\partial^2 \varphi}{\partial z^2} + (z^2 - i \rho z) \frac{\partial \varphi}{\partial z} = -i \frac{\sigma}{\lambda} \varphi; \quad \rho = \frac{\mu}{\lambda}, \quad \sigma \in \mathbb{C} \quad \text{and} \quad (\mu, \lambda) \in \mathbb{R}^2 \quad (3.2) \]

Let \( z = i \sqrt{2} \xi \) and \( \varphi(\sqrt{2} \xi) = \psi(\xi) \) then (3.2) can be transformed to:
\[ \xi \frac{\partial^2 \psi}{\partial \xi^2} + (-2 \xi^2 + \rho \sqrt{2} \xi) \frac{\partial \psi}{\partial \xi} - \frac{\sigma \sqrt{2}}{\lambda} \psi = 0 \quad (3.3) \]

the above equation belongs to bi-confluent Heun equations, denoted by \( BHE(\alpha, \beta, \gamma, \delta) \) which are in the form:
\[ \xi \frac{d^2 u}{d \xi^2} + \left( 1 + \alpha - \beta \xi - 2 \xi^2 \right) \frac{du}{d \xi} + \left( (\gamma - \alpha - 2) \xi - \frac{1}{2}(\delta + (\alpha + 1) \beta) \right) u = 0; \]
\[ (\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4 \quad (3.4) \]
It is well known that

The singular points of the bi-confluent Heun equation (3.4) are 0 and ∞.

At 0, if \((\alpha, \beta, \gamma, \delta) \in (\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3\) then a basis of \(BHE(\alpha, \beta, \gamma, \delta)\) is given by \(\{N(\alpha, \beta, \gamma, \delta, \xi), \xi^{-\alpha}N(\alpha, \beta, \gamma, \delta, \xi)\}\) where \(N(\alpha, \beta, \gamma, \delta, \xi) = \Gamma(\alpha)\sum_{n=0}^{\infty} \frac{A_n(\alpha, \beta, \gamma, \delta)}{\Gamma(\alpha + n)} \xi^n\), the \(A_n\) are polynomials in \(\alpha, \beta, \gamma, \delta\) defined by the following relation

\[
A_{n+2} = \left[\beta(n + 1) + \frac{1}{2}(\delta + (1 + \alpha))\right]A_{n+1} - (\gamma - 2 - \alpha - 2n)(n + 1)(n + 1 + \alpha)A_n
\]

for \(n \geq 0\) and \(A_{-1} = 0, A_0 = 1\), (see proposition 5 of Ref. [30] or Ref. [7])

If \(\alpha \in \mathbb{Z}\) there are logarithmic terms in the bases of solutions of \(BHE(\alpha, \beta, \gamma, \delta)\) at 0 which is a regular singularity (except for some values see [25]).

If \((\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4\) then \(BHE(\alpha, \beta, \gamma, \delta)\) admits as basis of formal solutions at \(\infty\),

\[
\left\{\xi^{-\frac{1}{2}(\gamma - \alpha - 2)} \sum_{n=0}^{\infty} a_n(\alpha, \beta, \gamma, \delta) \xi^{-n}, e^{-\frac{1}{4}(\gamma + \alpha + 2)2} \sum_{n=0}^{\infty} a_n(\alpha, i\beta, -\gamma, -i\delta)(i\xi)^{-n}\right\}
\]

where the complex numbers \(a_n\) are defined by

\[
2(n + 2)a_{n+2} = \left[\frac{1}{2}(\delta + \beta(\gamma - 1) - \beta(n + 1))\right]a_{n+1} - \left(\gamma - \alpha - 2\right)\left(\frac{\gamma + \alpha - 2}{2} - n\right) a_n
\]

for \(n \geq 0\) and \(a_{-1} = 0, a_0 = 1\). (see proposition 8 of Ref. [30])

These equations are more complicated than the equations for classical special functions such as Bessel functions, hypergeometric functions, and confluent hypergeometric functions (see [9] and [1]).

By means of the transformation

\[
u(\xi) = \xi^{-\frac{1}{2} + \gamma} e^{-\frac{\beta\xi + \xi^2}{2}} \phi(\xi)
\]

The BCH equation takes on the following Schrödinger form [25]:

\[
\frac{d^2\phi}{d\xi^2}(\xi) + \left(A\xi^2 + B\xi + C + \frac{D}{\xi} + \frac{E}{\xi^2}\right) \phi(\xi) = 0
\]

With \(A = -1, B = -\beta, C = \gamma - \frac{1}{4}\beta^2, D = -\frac{1}{2}\delta\) and \(E = \frac{1}{4}(1 - \alpha^2)\).
Remark 3.1 (i) If we take $\alpha = -1, \beta = -\rho \sqrt{2}, \gamma = 1$ and $\delta = \frac{2\sigma \sqrt{2}}{\lambda}$ in (3.4) then we get (3.3), so it is $BHE(-1, -\frac{\mu \sqrt{2}}{\lambda}, 1, -\frac{2\sigma \sqrt{2}}{\lambda})$.

By means of the transformation $\psi(\xi) = e^{-\rho \sqrt{2} \xi + \xi^2/2} \phi(\xi)$, the Eq. (3.3) takes the following Schrodinger’s equation form:

$$\frac{d^2 \phi}{d\xi^2}(\xi) + \left(-\xi^2 + \rho \sqrt{2} \xi + 1 - \frac{\rho^2}{2} - \frac{\delta/2}{\xi}\right) \phi(\xi) = 0$$

(3.10)

(ii) As $\alpha = -1 \in \mathbb{Z}$ then the proposition 5 of [30] (see (3.5)) is not applicable.

(iii) We will show there are logarithmic terms in the bases of solutions of $BHE(-1, -\frac{\mu \sqrt{2}}{\lambda}, 1, -\frac{2\sigma \sqrt{2}}{\lambda})$ at 0 which is a regular singularity by using the classical Frobenius-Fuch’s method.

The eigenvalue problem (3.2) associated to $H_{\mu, \lambda}$; $\lambda \neq 0$ can be written as follows

$$\varphi''(z) + p(z)\varphi'(z) + q(z)\varphi(z) = 0; \quad p(z) = (z - i\rho) \quad \text{and} \quad q(z) = \frac{i\sigma}{\lambda z}$$

(3.11)

Equation (3.11) is bi-confluent equation designated in the Ince classification [17] as [0, 1, 14] which has, as in the case of the confluent hypergeometric equation, two singularities located, respectively, at the origin and infinity: one regular, and the second irregular with a singularity rank higher by unity than that for the confluent hypergeometric equation.

We recall that $z_0$ is a regular singular point of (3.11) if $z_0$ is singular and

$$(z - z_0)p(z) \quad \text{and} \quad (z - z_0)^2q(z) \quad \text{are analytic at} \quad z_0.$$

(3.12)

As $\lim_{z \to 0} z p(z) = p_0 = 0$ and $\lim_{z \to 0} z^2 q(z) = q_0 = 0$ then $z_0 = 0$ is regular singular point of (3.11).

Equation (3.11) has two linearly independent solutions which are given by the following proposition.

Proposition 3.2 At $z = 0$ one finds two linearly independent solutions

$$\varphi_1(z) = z \eta(z); \quad \eta \text{ is analytic, } \eta'(0) \neq 0$$

(3.13)

$$\varphi_2(z) = cz \log(z) \xi(z) + \sum_{n=0}^{+\infty} c_n z^n; \quad \xi(z) \text{ is analytic, } \xi'(0) \neq 0$$

(3.14)

Proof The solution of the our bi-confluent Heun equation in a power series of $z$ can be easily constructed by applying the Method of Frobenius to (3.11) whose the basic idea is to look for solutions of the form $\varphi(z) = z^s \sum_{n=0}^{+\infty} a_n z^n$. 

Substitute the expansion $\varphi(z) = z^s \sum_{n=0}^{+\infty} a_n z^n$ into the Eq. (3.11), we found the indicial equation:

$$s(s - 1) = 0 \quad (3.15)$$

To get two linearly independent solutions of (3.11) in the neighborhood of zero, we will use the theorem of Fuchs:

**Theorem 3.3** (Theorem of Fuchs) Consider the equation

$$\varphi''(z) + p(z)\varphi' + q(z)\varphi = 0 \quad (3.16)$$

Let $z_0$ be a regular singular point. That is $(z - z_0)p(z) = \sum_{n=0}^{+\infty} p_n (z - z_0)^n$ and $(z - z_0)^2 q(z) = \sum_{n=0}^{+\infty} q_n (z - z_0)^n$ with certain radii of convergence.

Let $r$ be no bigger than the radius of convergence of either $(z - z_0)p(z)$ or $(z - z_0)^2 q(z)$ and $s_1, s_2$ solve the indicial equation

$$s(s - 1) + p_0 s + q_0 = 0 \quad (3.17)$$

Then

(1) If $s_1 \neq s_2$ and $s_1 - s_2$ is not an integer, then the two linearly independent solutions are given by

$$\varphi_1(z) = (z - z_0)^{s_1} \sum_{n=0}^{+\infty} a_n (z - z_0)^n \quad \text{et} \quad \varphi_2(z) = (z - z_0)^{s_2} \sum_{n=0}^{+\infty} b_n (z - z_0)^n \quad (3.18)$$

(2) If $s_1 = s_2$ then $\varphi_1(z)$ is given by the same formula as above, and $\varphi_2(z)$ is of the form

$$\varphi_2(z) = \varphi_1(z) \log(z - z_0) + (z - z_0)^{s_1} \sum_{n=0}^{+\infty} d_n (z - z_0)^n \quad (3.19)$$

(3) If $s_1 - s_2$ is an integer, then take $s_1$ to be the larger root (More precisely, when $s_1, s_2$ are both complex, take $s_1$ to be the one with larger real part, that is $\Re s_1 > \Re s_2$). Then $\varphi_1(z)$ is still the same, while

$$\varphi_2(z) = c \varphi_1(z) \log(z - z_0) + (z - z_0)^{s_2} \sum_{n=0}^{+\infty} e_n (z - z_0)^n \quad (3.20)$$

(Note that $c$ may be 0).
All the solutions constructed above converge at least for \(0 < |z - z_0| < r\).

The proof of this theorem is through careful estimate of the size of an using the recurrence relation, see Chapter 5 of [2].

In fact the converse of this theorem is also true. That is if all solutions of the equation satisfies \(\lim_{z \to z_0} (z - z_0)^s \varphi(z) = 0\) for some \(s\), then \((z - z_0)^2 p(z)\) and \((z - z_0)^2 q(z)\) are analytic at \(z_0\). This is called Fuchs Theorem. Its proof is a turn of force of complex analysis and can be found in [31] see also, the following references about this topic [16] or [26].

Now, for (3.11), we have \(zp(z) = -i \rho z + z^2\) and \(z^2 q(z) = \frac{i \sigma}{\lambda} z\) then \(p_0 = q_0 = 0\) and from this we deduce that the roots of indicial equation \(s_1 = 1\) and \(s_2 = 0\). It follows that (3.11) satisfy the property (3) of above theorem and consequently at \(z = 0\) (3.13) and (3.14) are two linearly independent solutions of (3.11).

**Lemma 3.3** Let \(\varphi_2\) be the function defined by (3.14) i.e.

\[
\varphi_2(z) = cz \log(z) \xi(z) + \sum_{n=0}^{+\infty} c_n z^n; \quad \xi(z) \text{ is analytic, } \xi'(0) \neq 0
\]  

Then \(\varphi_2(z)\) is not acceptable in Bargmann space and thus it is rejected.

**Proof** From (3.21), if we consider \(\varphi_2(z) = cz \log(z) \xi(z) + \sum_{n=0}^{+\infty} c_n z^n : \xi(z) \text{ is analytic and } \xi'(0) \neq 0\) then as the complex logarithmic function \(f(z) = \log(z)\) is not meromorphic on the whole complex plane and the quotient of two entire functions is meromorphic, we deduce that \(z \log(z) \xi(z)\) is not entire function, in particular \(\varphi_2(z)\) is not entire function. It follows that \(\varphi_2(z)\) is not acceptable in Bargmann space and thus it is rejected.

Now we will look for an asymptotic powers series to the solution \(\varphi(z)\) of (3.11) as \(z\) tends to zero. \(\varphi(z)\) coincides with \(\varphi_1(z)\) because \(\varphi_2\) does not belong to Bargmann space \(\mathcal{B}\).

For two functions \(\varphi(z)\) and \(\psi(z)\), we note \(\varphi(z) \ll \psi(z), z \to z_0\) which is read "\(\varphi(z)\) is much smaller than \(\psi(z)\) as \(z\) tends to \(z_0\) " means \(\lim_{z \to z_0} \frac{\varphi(z)}{\psi(z)} = 0\) and the power series \(\sum_{n=0}^{+\infty} a_n (z - z_0)^n\) is said to be asymptotic to the function \(\varphi(z)\) as \(z \to z_0\) and we write \(\varphi(z) \sim \sum_{n=0}^{+\infty} a_n (z - z_0)^n\) if \(\varphi(z) - \sum_{n=0}^{N} a_n (z - z_0)^n \ll (z - z_0)^N\) as \(z \to z_0\) for every \(N\).

From these notations and from the above lemma, we deduce the following corollary:

**Corollary 3.4**

\[
\varphi(z) \sim z \quad \text{as } z \to 0
\]

Let us now to turn to the point \(z = \infty\), which is the second possible singular point for the solution of (3.11).
Proposition 3.5 At \( z = \infty \) one finds two linearly independent solutions which behave as

\[
\begin{align}
\varphi_1(z) & \sim \text{const} \quad z \to \infty \\
\varphi_2(z) & \sim \frac{1}{z} e^{-\frac{1}{2}z^2 + i\rho z} \quad z \to \infty
\end{align}
\]

(3.23) (3.24)

Proof To analyse the point \( z = \infty \), we can first perform the change of independent variable from \( z \) to \( \xi \), \( z = \frac{1}{\xi} \) and \( \varphi(z) = \varphi(\frac{1}{\xi}) = \phi(\xi) \) and study the behaviour of the transformed equation at \( \xi = 0 \) to get:

\[
\begin{align}
\frac{d}{dz} = -\xi^2 \frac{d}{d\xi} \quad \text{and} \quad \frac{d^2}{dz^2} = \xi^4 \frac{d^2}{d\xi^2} + 2\xi^3 \frac{d}{d\xi} \\
\phi''(\xi) + p(\xi)\phi'(\xi) + q(\xi)\phi(\xi) = 0 \\
\text{where} \quad p(\xi) = -\frac{2\xi^2 - i\rho\xi + 1}{\xi^3} \quad \text{and} \quad q(\xi) = \frac{i\sigma}{\lambda \xi^3}
\end{align}
\]

(3.25)

The origin is regular singular point of (3.25) if \( \xi p(\xi) \) and \( \xi^2 q(\xi) \) are analytic functions in the neighborhood of the origin. As these latter conditions are not fulfilled, then the origin is an irregular singular point and it is of order 3.

Consequently, the theorem of Fauchs is not applicable to (3.25) and this equation cannot be approximated by an Euler equation in the neighborhood of the origin.

If we like to explore some of these techniques, a good starting point would be Chapter 3 of [8] or the book of Kristensson [23].

The case of an irregular singular point is much more difficult to address, although some techniques do exist for obtaining the solution to a homogeneous second-order linear differential equation in the vicinity of an irregular singular point.

In the vicinity of \( z = i\sqrt{2}\xi = \infty \), the difficulty remains even if we use the techniques of the Thom series in Eq. (3.3) by using the proposition 8 of [30] where one can choose as two linearly independent solutions of (3.3) the functions \( \psi_1 \) and \( \psi_2 \), defined by the power series:

\[
\begin{align}
\psi_1(\xi) & = \xi^{\frac{1}{2}(\gamma - \alpha - 2)} \sum_{n=0}^{\infty} a_n(\alpha, \beta, \gamma, \delta) \xi^{-n} \\
\psi_2(\xi) & = \xi^{\frac{1}{2}(\gamma + \alpha + 2)} e^{\xi^2 + \beta \xi} \sum_{n=0}^{\infty} a_n(\alpha, i\beta, -\gamma, -i\delta)(-i\xi)^{-n}
\end{align}
\]

(3.26) (3.27)
where \( \alpha = -1, \beta = -\frac{\mu \sqrt{2}}{\lambda}, \gamma = 1, \delta = \frac{2\sigma \sqrt{2}}{\lambda} \) and the complex numbers \( a_n \) are defined by (3.7).

Now as \( \varphi(z) = \psi(\xi) \) where \( z = i\sqrt{\xi} \), then we deduce from (3.26) and (3.27) that (3.2) at \( \infty \), satisfies:

\[
\varphi_1(z) \sim \text{const. as } z \rightarrow \infty
\]

\[
\varphi_2(z) \sim \frac{1}{z} e^{-\frac{1}{2}z^2 + i\rho z} \text{ as } z \rightarrow \infty
\]

Lemma 3.6 Equation (3.29) is not acceptable in Bargmann space and thus it can be rejected.

Proof Since \( \varphi_2(z) \) in (3.29) has to belong to the Bargmann’s space, it must satisfy the following conditions:

(i) \( \varphi_2(z) \) is an entire function of \( z \).

(ii) \( ||\varphi_2||^2 = \frac{1}{\pi} \int_{C} |\varphi_2|^2 e^{-|z|^2} dx dy < \infty \)

We now show that the normalizability requirement gives a non-trivial boundary condition in some direction of the \( z \)-plane.

In fact for \( \Im mz \rightarrow -\infty \) (\( \rho > 0 \)) the asymptotic behaviour (3.29) is incompatible with (ii) because:

\[
\text{Let } z = \Re ez + i\Im mz, \text{ then } e^{-|z|^2} \left| \frac{1}{z} e^{-\frac{1}{2}z^2 + i\rho z} \right|^2 = \frac{1}{|z|^2} e^{-2(\Im ez)^2} e^{-2\rho \Im mz}; \quad \rho = \frac{\mu}{\lambda}.
\]

This means that an eigenfunction of (1.10) must be asymptotically a constant in the direction of the negative imaginary axis.

The boundary conditions on the eigenvalue problem \( H_{\mu,\lambda}\varphi = \sigma \varphi \) \( \sigma \in \mathbb{C} \) are therefore summarized in the following proposition:

Proposition 3.7 In \( B_0 = \{ \varphi \in B; \varphi(0) = 0 \} \) the Hilbert space orthogonal to the vacuum (the only state of exactly zero energy), the eigenvalue conditions of \( H_{\mu,\lambda}\varphi(z) = \sigma \varphi(z) \) \( \sigma \in \mathbb{C} \) are:

\[
(\ast_1) \varphi(z) \sim z, \quad z \rightarrow 0,
\]

\[
(\ast_2) \varphi(z) \sim \text{const.} \quad z \rightarrow -i\infty
\]

Proof (\( \ast_1 \)) The behaviour (3.14) of the proposition (3.2) is an acceptable one only if \( \sigma = 0 \); in this case on fat \( c_2 = 0 \) and the logarithmic branch point is absent. This implies that the normalized perturbative vacuum in the Bargmann’s space, \( \varphi_0(z) = 1 \).

For \( \sigma \neq 0 \), we must consider functions which vanish at the origin [(3.13) of the proposition (3.2)] let us note that they are automatically orthogonal to the vacuum, then the choose of \( B_0 \).

(\( \ast_2 \)) From the behaviour (3.24) of the proposition (3.5) and the lemma (3.6) we deduce that \( \varphi(z) \sim \text{const.} \quad z \rightarrow -i\infty \).

\( \square \)
As there are many important applications of the closedness of the range of linear operator in the spectral study of differential operators and also in the context of perturbation theory (see e.g. [Goldberg]) [15] then to require an asymptotic behavior of the solutions of the eigenvalue problem associated with $H_{\mu, \lambda}$.

Before to recall some properties of $H_{\mu, \lambda}$, in particular that its range is closed, we need of following lemma:

**Lemma 3.8** In Bargmann space $\mathcal{B}$, we have:

$$\varphi \in \mathcal{B} \iff z \rightarrow g(z) = \frac{\varphi'(z) - \varphi'(0)}{z} \in \mathcal{B} \quad (3.32)$$

**Proof** This lemma is the lemma 1 page 127 in [5] see also [18] or Lemma 0.3 page 338 in [21]. By convenient, we reproduce its proof:

For $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\varphi'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$ and $\frac{\varphi'(z) - \varphi'(0)}{z} = \sum_{n=0}^{\infty} a_{n+2} z^{n+2}$.

By using the isometry between spaces $\mathcal{B}$ and $\mathcal{B}_s$, we deduce that

$$|| \frac{\varphi'(z) - \varphi'(0)}{z} ||^2 = \sum_{n=0}^{\infty} n(n+2)^2 |a_{n+2}|^2.$$  (3.33)

Now, as $\frac{n+2}{n+1} \leq 2$ then $|| \frac{\varphi'(z) - \varphi'(0)}{z} ||^2 \leq 2 \sum_{n=0}^{\infty} (n+2)! |a_{n+2}|^2$ Then if $\varphi \in \mathcal{B}$, then $z \rightarrow \frac{\varphi'(z) - \varphi'(0)}{z} \in \mathcal{B}$.

For the reciprocity, we remark that $\frac{n+2}{n+1} > 1$ and from (3.33) we deduce that

$$|| \varphi || \leq || g ||$$  where $g(z) = \frac{\varphi'(z) - \varphi'(0)}{z}$.

$\square$

**Proposition 3.9** Let $\mathcal{B}_0 = \{ \varphi \in \mathcal{B}; \varphi(0) = 0 \}$ and $H_{\mu, \lambda}^{\text{mim}} = H_{\mu, \lambda}$ with domain $D(H_{\mu, \lambda}^{\text{mim}})$ and $\mathcal{R}(H_{\mu, \lambda}^{\text{mim}})$ denotes the range of $H_{\mu, \lambda}^{\text{mim}}$ then

(i) For $\mu \neq 0$, $|| H_{\mu, \lambda}^{\text{mim}} \varphi || \geq |\mu| || \varphi ||$, $\forall \varphi \in D(H_{\mu, \lambda}^{\text{mim}})$.

(ii) $\mathcal{R}(H_{\mu, \lambda}^{\text{mim}})$ is closed in Bargmann space.

(iii) $\mathcal{R}(H_{\mu, \lambda}^{\text{mim}})$ is dense in Bargmann space.

(iv) For $\mu \neq 0$, $H_{\mu, \lambda}^{\text{mim}}$ is invertible.
Remark 3.10 It is convenient to reproduce the proofs of these properties which serve as a starting point of our study on the completeness of eigenfunctions of $H_{\mu, \lambda}$.

**Proof** (i) For $\lambda \in \mathbb{R}$ we observe that

$$|\Re e < H_{\mu, \lambda}^{\min} \varphi, \varphi > | = |\mu ||| A \varphi |||^2 = |\mu | \sum_{n=0}^{\infty} (n+1) |a_n|^2$$

$$\geq |\mu | \sum_{n=0}^{\infty} |a_n|^2 = |\mu ||| \varphi |||^2$$  \hspace{1cm} (3.34)

Now by using the Cauchy’s inequality, we deduce that

$$(\alpha) || H_{\mu, \lambda}^{\min} \varphi || \geq |\mu ||| A \varphi ||, \forall \varphi \in D(H_{\mu, \lambda}^{\min})  \hspace{1cm} (3.35)$$

In particular

$$(\beta) \text{ the injection of } D(H_{\mu, \lambda}^{\min}) \text{ in } D(A) \text{ is continuous.}  \hspace{1cm} (3.36)$$

and

$$(\gamma) || H_{\mu, \lambda}^{\min} \varphi || \geq |\mu ||| \varphi ||, \forall \varphi \in D(H_{\mu, \lambda}^{\min})  \hspace{1cm} (3.37)$$

(ii) From (3.35) and as the injection of $D(A)$ in $B$ is a compact mapping (see [19] or lemma 0.2 in [21] for an elementary proof), we deduce that:

The injection $D(H_{\mu, \lambda}^{\min}) \longrightarrow B$ is a compact mapping. \hspace{1cm} (3.38)

(iii) From (3.37), it follows that $N(H_{\mu, \lambda}^{\min}) = \{0\}$ and $R(H_{\mu, \lambda}^{\min})$ is closed, where $N(H_{\mu, \lambda}^{\min})$ and $R(H_{\mu, \lambda}^{\min})$ denote the kernel and the range of $H_{\mu, \lambda}^{\min}$ respectively.

(iv) In fact, $< H_{\mu, \lambda}^{\min} \varphi, \psi > = 0 \iff -i\lambda z \psi''(z) + (-i\lambda z^2 + \mu z) \psi'(z) = 0$ \iff $H_{\mu, \lambda} - \mu \psi = 0$.

for $\lambda \neq 0$, we have $\psi'(z) = e^{-\frac{1}{2}z^2 + i\mu z}$.

Now we show that the normalizability requirement for $\psi(z)$ is not verified in some direction of the $z-$plane.

In fact, we will use the lemma (3. 8 for $z \longrightarrow -i\infty$.

Let $z = \Re ez + i \Im mz$, then $e^{-|z|^2} \left| \frac{\psi'(z)}{z} \right|^2 = e^{-|z|^2} \left| \frac{e^{-\frac{1}{2}z^2 + i\mu z} e^{-\frac{1}{2}z^2 + i\mu z} \xi}{z} \right|^2 = \frac{1}{|z|^2} e^{-2(\Re ez)^2} e^{-2\mu \Im mz}$.

Given $\mu$ and $\lambda$, there exists a direction of $z-$plane where $\psi(z) = \int_{0}^{z} e^{-\frac{1}{2}z^2 + i\mu z} d\xi$ cannot be considered in Bargmann space. \qed
Remark 3.11

(1) To sum up in $B_0 = \{ \varphi \in B; \varphi(0) = 0 \}$ the Hilbert space orthogonal to the vacuum (the only state of exactly zero energy), the eigenvalue conditions of $H_{\mu,\lambda} \varphi(z) = \sigma \varphi(z)$ are:

$$\varphi(z) \sim z \text{ for } z \sim 0$$

and

$$\varphi(z) \sim \text{const.} \text{ for } z \sim -i\infty$$

(2) From (iii) and (iv), we deduce that $H_{\mu,\lambda}^{\min}$ is invertible i.e.,

$$0 \in \rho(H_{\mu,\lambda}^{\min}) \text{ where } \rho(H_{\mu,\lambda}^{\min}) \text{ denotes the resolvent set of } H_{\mu,\lambda}^{\min}.$$  

(3) From (3.38) and as $\rho(H_{\mu,\lambda}^{\min}) \neq \emptyset$, we deduce that the resolvent of $H_{\mu,\lambda}^{\min}$ is compact.

(5) From [21] (see theorem 1.2), it is well known that for $\mu \neq 0$ we have $D(H_{\mu,\lambda}^{\min}) = D(H_{\mu,\lambda})$.

(6) From lemmas 9, 10 and 11 of the Ref. [19] (see page 280), it is well known that the eigenvalues of $H_{\mu,\lambda}$ are real.

4 A Transformation Procedure of $H_{\mu,\lambda}$ to a Symmetric Operator with Compact Resolvent on the Negative Imaginary Axis

We begin by recalling an original lemma on Bargmann space with its proof was established in [19]

Lemma 4.1 Let $B$ be Bargmann space, if $\varphi \in B$ then its restriction on $x + i\mathbb{R}$ is square integrable function with measure $e^{-|y|^2}dy$, $y \in \mathbb{R}$ for all fixed $x \in \mathbb{R}$.

Proof Bargmann has built an isometry between the space $B$ and $L_2(\mathbb{R})$ so that, $\forall \varphi \in B$ is uniquely represented by $f \in L_2(\mathbb{R})$ by means of the following integral:

$$\varphi(z) = c \int_{\mathbb{R}} e^{-\frac{x^2}{2} - \frac{q^2}{2} + \sqrt{2}zq} f(q) dq$$  

such that

$$|| \varphi || = || f ||_{L_2(\mathbb{R})}$$

Now, we put $g_z(q) = e^{-\frac{x^2}{2} + \sqrt{2}zq}$, which is function belonging to $L_2(\mathbb{R})$. Then we can write (4.1) in the following form:

$$e^{\frac{z^2}{2}} \varphi(z) = c \int_{\mathbb{R}} g_z(q) f(q) dq$$
As $g_z(q)$ and $f$ are in $L^2(\mathbb{R})$, we can apply the Parseval’s identity at (4.3) to get:

$$e^{\frac{z^2}{2}} \varphi(z) = c_1 \int_{\mathbb{R}} \hat{g}_z(p) \hat{f}(p) dp$$  \hspace{1cm} (4.4)

As $g_z(q)$ is a Gaussian’s function, one knows how to calculate its Fourier transform:

$$\hat{g}_z(p) = \int_{\mathbb{R}} e^{ipq} g_z(q) dq = \int_{\mathbb{R}} e^{-\frac{q^2}{2} + (\sqrt{2\pi} - ip) q} dq = (2\pi)^{\frac{1}{2}} e^{\frac{q^2}{2}(\sqrt{2\pi} - ip)^2}$$  \hspace{1cm} (4.5)

i.e.

$$\hat{g}_z(p) = (2\pi)^{\frac{1}{2}} e^{\frac{q^2}{2}(\sqrt{2\pi} - ip)^2} \hspace{1cm} (4.6)$$

Let $z = x + iy$, then for all fixed $x \in \mathbb{R}$, the function $\hat{g}_z(p)$ can be written under following form:

$$\hat{g}_z(p) = (2\pi)^{\frac{1}{2}} e^{\frac{q^2}{2}(\sqrt{2\pi} - ip)^2} \hspace{1cm} (4.6)$$

If we put $h_x(p) = e^{x^2} e^{-\frac{q}{2}(p^2 + \frac{q}{2}px)}$ then $e^{\frac{z^2}{2}} \varphi(z)$ is the convolution product of $h_x$ with $\hat{f}$ evaluated in $\sqrt{2}y$, i.e.,

$$e^{\frac{(x+iy)^2}{2}} \varphi(x + iy) = C_2 h_x \ast \hat{f}(\sqrt{2}y) \hspace{1cm} (4.7)$$

It follows that $y \rightarrow e^{\frac{(x+iy)^2}{2}} \varphi(x + iy)$ is in $L^2(\mathbb{R})$ for all $x \in \mathbb{R}$.

By applying the Young’s inequality we deduce that:

$$|| e^{\frac{(x+iy)^2}{2}} \varphi(x + i.) ||^2_{L^2(\mathbb{R})} \leq || h_x ||^2_{L^1(\mathbb{R})} \cdot || \hat{f} ||^2_{L^2(\mathbb{R})}$$  \hspace{1cm} (4.8)

where $C_1, C_2$ and $C$ are constants.

As $|| h_x ||^2_{L^1(\mathbb{R})} = C_3 e^{2|x|^2}$ and $|| \hat{f} ||^2_{L^2(\mathbb{R})} = C_4 || f ||^2_{L^2(\mathbb{R})}$, then we deduce that:

$$\int_{\mathbb{R}} e^{-|y|^2} \varphi(x + iy) |^2 dy \leq C_5 e^{-|x|^2} || \varphi ||^2$$  \hspace{1cm} (4.9)

where $C_5$ depends of all the previous constants. \hspace{1cm} $\square$

From the above Lemma, it is possible to restrict ourselves to the negative imaginary axis, $z = -iy$, $y > 0$, where we impose the boundary conditions (3.28) and (3.29).

Let us write therefore $u(y) = \varphi(-iz)$ then we have

$$H_\mu \rho u(y) = -\lambda y u''(y) + (\lambda y^2 + \mu y) u'(y) = \lambda \left[ -y \frac{d^2}{dy^2} + (y^2 + \rho y) \frac{d}{dy} \right] u(y); \hspace{0.5cm} \rho = \frac{\mu}{\lambda}$$  \hspace{1cm} (4.10)
The eigenfunction equation $H_{R}^{\mu,\lambda}$ may be rewritten (for $\lambda \neq 0$) in the form:

$$
\begin{aligned}
H_{R}^{\mu,\lambda} u(y) &= \lambda y \left[ -\frac{d^2}{dy^2} + (y + \rho) \frac{d}{dy} \right] u(y) = \sigma u(y) \\
\left\{ \begin{array}{l}
\quad \quad \quad \quad u(y) \sim y, \quad y \to 0 \\
\quad \quad \quad \quad u(y) \sim \text{const}, \quad y \to +\infty
\end{array} \right. 
\end{aligned}
$$

(4.11)

The term proportional to $u'(y)$ can be eliminated by the transformation

$$
u(y) = T^{-1}(y)u(y) = e^{-\frac{1}{4}(y+\rho)^2} u(y) = e^{\frac{1}{4}(y^2+2\rho y+\rho^2)} u(y); \text{ this implies that}$$

$$v(y) \sim \begin{cases} y & y \to 0 \\ e^{-\frac{1}{4}(\frac{1}{2}+\rho)} & y \to \infty \end{cases}
$$

(4.12)

Then

$$
u(y) = T^{-1}(y)u(y) = e^{-\frac{1}{4}(y+\rho)^2} u(y) = e^{\frac{1}{4}(y^2+2\rho y+\rho^2)} u(y) = e^{-\frac{1}{2}(y+\rho+\frac{\rho^2}{2})} u(y); \text{ this implies that}$$

$$v(y) \sim e^{-\frac{1}{2}(\frac{1}{2}+\rho)}, \quad y \to +\infty
$$

that gives the eigenvalue equation

$$
\begin{aligned}
\lambda y \left[ -\frac{d^2}{dy^2} + \frac{1}{4} (y + \rho)^2 - \frac{1}{2} \right] v(y) &= \sigma v(y) \\
\left\{ \begin{array}{l}
\quad \quad \quad \quad v(y) \sim y, \quad y \to 0 \\
\quad \quad \quad \quad v(y) \sim e^{-\frac{1}{2}(\frac{1}{2}+\rho)}, \quad y \to +\infty
\end{array} \right. 
\end{aligned}
$$

(4.13)

As in vicinity of $y = 0$ we have $v(y) \sim y$ and in vicinity of $y = \infty$ we have $|v(y)|^2 \sim e^{-\left(\frac{1}{2}+\rho y\right)}$ then $v$ vanishes at origin and $v \in L^2[0, \infty[$.

We denote by $L^2_R[0, \infty[$ the space of functions in $L^2[0, \infty[$ satisfying both constraints.

Let

$$
H_{R}^{\mu,\lambda} = T^{-1}(y)H_{R}^{\mu,\lambda} T(y) \quad \text{with} \quad T(y) = e^{\frac{1}{4}(y+\rho)^2}
$$

(4.14)

then

$$
H_{R}^{\mu,\lambda} v(y) = \lambda y \left[ -\frac{d^2}{dy^2} + \frac{1}{4} (y + \rho)^2 - \frac{1}{2} \right] v(y)
$$

(4.15)
Now, if we consider the change of variable $y = x^2$ with $v(y) = x^{1/2}w(x)$ and the similarity transform given by $y^{1/4} = x^{1/2}$ then we get

$$
\tilde{H}_{\mu,\lambda}w(x) = x^{-1/2}H^{\mu,\lambda}x^{1/2}w(x) = x^{-1/2}T^{-1}(x^2)H^{\mu,\lambda}_R T(x^2)x^{1/2}w(x)
\hspace{1cm} = \frac{\lambda}{4} \left[ -\frac{d^2}{dx^2} + \frac{3/4}{x^2} + x^2((x^2 + \rho)^2 - 2) \right] w(x) \quad (4.16)
$$

As $w(x) = x^{-1/2}v(y); \ y = x^2$.

Then

$$
w(x) = x^{-1/2}v(y) \sim x^{3/2}, \ x \longrightarrow 0 \quad (4.17)
$$

and

$$
w(x) \sim x^{-1/2}e^{-1/4(x^2+\rho)^2}, \ x \longrightarrow \infty. \quad (4.18)
$$

Remark 4.2 (1) The similarity transformation from $H_{\mu,\lambda}$ to $\tilde{H}_{\mu,\lambda}$ is non-unitary, but bijective.

(2) On $L^2[0, +\infty[, \ \tilde{H}_{\mu,\lambda}$ with domain $D(\tilde{H}_{\mu,\lambda}) = \{w \in L^2[0, +\infty[; \ \tilde{H}_{\mu,\lambda}w \in L^2[0, +\infty[\}$ is a symmetric operator.

(3) As the injection of maximal domain of $H_{\mu,\lambda}$ in Bargmann space $B$ is compact then we deduce that:

(i) the injection of maximal domain of $\tilde{H}^{\mu,\lambda}_R$ in

$$
B_R = \{u : [0, +\infty[ \longrightarrow \mathbb{C} \text{ analytic}; \ \int_0^{\infty} |u(y)|^2 e^{-y^2} dy < +\infty \} \quad (4.19)
$$

is compact.

and

(ii) the injection of maximal domain of $\tilde{H}_{\mu,\lambda}$ in $L^2[0, +\infty[\$ is also compact, in particular $\tilde{H}_{\mu,\lambda}$ is a symmetric operator with compact resolvent.

(4) The equation $H_{\mu,\lambda}u = \sigma u$ is equivalent to $\tilde{H}_{\mu,\lambda}w = \sigma w,$ where

$$
w(x) = x^{-1/2}e^{-1/4(x^2+\rho)^2}u(y = x^2) \quad (4.20)
$$

is an analytical function in $x$.

We end this section by the main result of this paper

**Theorem 4.3** The eigenfunctions of $H_{\mu,\lambda}; \ \mu > 0$ form a complete basis in Bargmann space.
Proof The above formulated problem of finding the spectrum of the Hamiltonian $H_{\mu,\lambda}$ or of $H_{R,\mu,\lambda}$ acting on Bargmann space or on its restriction functions respectively to negative imaginary axis is equivalent to the problem of finding the spectrum of the symmetric operator $\tilde{H}_{\mu,\lambda}$ in the space $L^2([0, +\infty[)$.

As $\tilde{H}_{\mu,\lambda}$ is symmetric with compact resolvent, then by using the spectral theorem for symmetric operators with compact resolvent, we deduce that the eigenfunctions of $\tilde{H}_{\mu,\lambda}$ form a complete basis:

**Theorem 4.4** (Spectral theorem for symmetric operators with compact resolvent)

1. Let $T$ be a densely defined linear operator in a Hilbert space $\mathcal{H}$ and let $\sigma \in \rho(T) \cap \mathbb{R}$ then $T$ is symmetric iff $R_\sigma = (T - \sigma I)^{-1}$ is self-adjoint.
2. Let $T$ be a symmetric operator in a Hilbert space $\mathcal{H}$ such that $(T - \sigma_0 I)^{-1}$ is compact, then $\sigma_p(T) = \{\sigma_0 + \frac{1}{\gamma}; \gamma \in \sigma_p(T - \sigma_0 I)^{-1}\}$ where $\sigma_p(T)$ denotes the pointual spectrum of $T$.
3. If $T$ is a symmetric operator with compact resolvent then there exists $\sigma_1 \in \rho(T) \cap \mathbb{R}$ such that $(T - \sigma_1 I)^{-1}$ is a compact self-adjoint operator.
4. Let $T$ be a symmetric operator with compact resolvent in a Hilbert space $\mathcal{H}$. For each $\sigma \in \sigma_p(T)$, define $n(\sigma) = \dim \ker(T - \sigma I)$, and choose an orthonormal basis $\{\phi_{\sigma_1,1}, \phi_{\sigma_1,2}, \ldots, \phi_{\sigma_1,n(\sigma)}\}$ of $\ker(T - \sigma I)$. Then $\{\phi_{\sigma_1,i}; \sigma \in \sigma_p(T), i = 1, 2, \ldots, n(\sigma)\}$ forms a complete orthonormal set for $\mathcal{H}$. Moreover, we have the following:

   (i) $\phi \in D(T) \iff \sum_{\sigma \in \sigma_p(T)} \sum_{i=1}^{n(\sigma)} |\sigma|^2 |<\phi, \phi_{\sigma_i}|^2 < \infty$.

   (ii) For any $\phi \in D(T), T\phi = \sum_{\sigma \in \sigma_p(T)} \sum_{i=1}^{n(\sigma)} \sigma |<\phi, \phi_{\sigma_i}| \phi_{\sigma_i}$.

Proof (See for example http://home.iitk.ac.in/~chavan/spectral-th.pdf)

Now, as $\tilde{H}_{\mu,\lambda}$ is symmetric operator with compact resolvent and it is similar to $H_{\mu,\lambda}$ via bounded and boundedly invertible transformations then by using theorem 4.4, we deduce that the eigenfunctions of $\tilde{H}_{\mu,\lambda}$ form a complete basis and it follows that the eigenfunctions of $H_{\mu,\lambda}$; $\mu > 0$ form a complete basis in Bargmann space.

**Definition 4.5** (Mostafazadeh [27])

A linear operator $T : \mathcal{H} \to \mathcal{H}$ acting in a separable Hilbert space $\mathcal{H}$ is said to be pseudo-Hermitian if $D(T)$ is a dense subset of $\mathcal{H}$, and there is an everywhere-defined invertible Hermitian linear operator $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$T^* = \eta T \eta^{-1}$$

(4.21)

By applying the results of Mostafazadeh in [28] (see chapter III on Pseudo-hermitian Hamiltonians with a complete biorthogonal eigenbasis), it can be deduced that

**Corollary 4.6** $H_{\mu,\lambda}$ belongs to the class of pseudo-Hermitian Hamiltonians.
Acknowledgements The author thank the referee for valuable comments and suggestions.

Author Contributions The main new results of this Note are the determination of the boundary conditions for the eigenvalue problem associated to Hamiltonian of Reggeon field Theory and the proof of the completeness of its generalized eigenfunctions. This operator acting on Bargmann space belongs to the class of pseudo-Hermitian Hamiltonians in the Mostafazadeh’s sense. Abdelkader INTISSAR

Declarations

Conflict of interest The authors declare no competing interests.

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