Particle Entrainment in Spherical-Cap Wakes

Norbert G.W. Warncke & René Delfos & Gijs Ooms & Jerry Westerweel
Laboratory for Aero-& Hydrodynamics, Delft University of Technology, NL
E-mail: n.g.w.warncke@tudelft.nl

Abstract. In this work we study the preferential concentration of small particles in the turbulent wake behind a spherical-cap object. We present a model predicting the mean particle concentration in the near-wake as a function of the characteristic Stokes number of the problem, the turbulence level and the Froude number. We compare the model with our experimental results on this flow, measured in a vertical water tunnel.

1. Background

Bubbly flotation is a method widely applied in industry to separate a dispersed solid phase from a liquid (see Nguyen & Schulze (2004)). Rising gas bubbles induce a transport of the dispersed particles, which is attributed to two effects: collision and subsequent attachment of the particles to the gas-liquid interface, and the entrainment of particles from the ambient fluid in the wake of the bubble. This work is about the latter part, the wake entrainment effect. We derive a model which relates the increase in particle concentration in the near wake to the properties of the dispersed phase and the flow. We further show that the predictions of the model agree with our experimental results.

2. Models for Wake Entrainment

2.1. Summary of the derivation

The presented models relate the mean concentration of dispersed particles inside the near-wake relative to that in the outer flow. The dispersed phase is assumed to consist of spherical particles that are reasonably well-described by the equation of motion

\[ \frac{dv_P}{dt} = (1 - \beta)g - \frac{1}{\tau_P}(v_P - u) + \beta \frac{D u}{D_t}, \]  

using only the terms for gravity, Stokes drag, added mass and the acceleration of the fluid (see Maxey & Riley (1983)). The particle response time \( \tau_P = \frac{R^2_p(2\rho_p + \rho_f)}{9\nu_p} \) is assumed to be small, \( \beta = \frac{3\rho_p}{2\rho_p + \rho_f} \). The time-averaged flow around the spherical-cap object and its near wake is described by a potential flow around a sphere and — as a better approximation to the observed geometry — a prolate ellipsoid, the influence of the turbulent velocity fluctuations on the motion of the dispersed phase is described by a particle transport coefficient \( D_P \).
For the derivation of the model we assume a balance of three fluxes over the boundary of the wake: the flux induced by the inertia of the dispersed particles, by gravitational settling, and by turbulent mixing. The flux balance is fulfilled if the system is in a stationary state. With the assumptions stated above, the inertial flux can be derived as

$$\Phi_{\text{inertia}} = -\tau_P (\beta - 1) \frac{3\pi}{2} U_{\infty}^2 R_{\text{wake}} n_{P,\text{exterior}} (2 - \cos^3 \theta_0 + 3 \cos \theta_0)$$

the gravitational flux as

$$\Phi_{\text{grav}} = \pi \tau_P (\beta - 1) g_z R_{\text{wake}}^2 n_{P,\text{ext}} (\cos^2 \theta_0 - 1)$$

and the diffusive flux due to the turbulent mixing as

$$\Phi_{\text{diff}} = C u_{\text{rms}} (n_{P,\text{wake}} - n_{P,\text{ext}}) R_{\text{wake}}^2 2\pi (\cos \theta_0 + 1)$$

Balancing all three terms $\Phi_{\text{inertia}} + \Phi_{\text{grav}} + \Phi_{\text{diff}} = 0$ and rearranging gives

$$\frac{n_{P,\text{wake}}}{n_{P,\text{ext}}} - 1 = \frac{U_{\infty}}{u_{\text{rms}}} \frac{\tau_P (\beta - 1)}{R_{\text{wake}}/U_{\infty}} f(\theta_0, \text{Fr}) \chi,$$

for the increase in the mean particle concentration $n_{P,\text{wake}}$ relative to that in the outer flow $n_{P,\text{ext}}$, with

$$f(\theta_0, \text{Fr}) = \frac{3}{4} (2 - \cos^2 \theta_0 + \cos \theta_0) + \frac{1}{2 \text{Fr}^2} (\cos \theta_0 - 1),$$

the Froude number $\text{Fr}^2 = U_{\infty}^2 g / g R_{\text{wake}}$ and the characteristic Stokes number $\frac{\tau_P (\beta - 1)}{R_{\text{wake}}/U_{\infty}}$.

The model for the ellipsoidal wake is derived in an analogous way as

$$\frac{n_{P,\text{wake}}}{n_{P,\text{ext}}} - 1 = \frac{U_{\infty}}{u_{\text{rms}}} \frac{\tau_P (\beta - 1)}{a/U_{\infty}} f(e, \eta_0, \text{Fr}) \chi,$$
with

$$f(e, \eta_0, \text{Fr}) = \frac{f_{\text{inertia}}(e) + \frac{1}{\text{Fr}^2} f_{\text{grav}}(e)}{f_{\text{diff}}(e)}$$

$$= \sqrt{1 - e^2} e^3 \left((e^2 + 1)(\tanh(e) + \tanh(e \cos \eta_0)) - e \frac{\cos \eta_0 + 1}{e \cos \eta_0 + 1 + \frac{e-1}{e \cos \eta_0 - 1}}\right)$$

$$= \left(\sqrt{1 - e^2} + \frac{\arcsin(e)}{e} + \cos \eta_0 \sqrt{1 - e^2 \cos^2 \eta_0} + \frac{\arcsin(e \cos \eta_0)}{e}\right) (e - (1 - e^2) \tanh(e))^2$$

$$+ \frac{1}{\text{Fr}^2} \sqrt{1 - e^2} + \frac{\arcsin(e)}{e} + \cos \eta_0 \sqrt{1 - e^2 \cos^2 \eta_0} + \frac{\arcsin(e \cos \eta_0)}{e}$$

and the Froude number $\text{Fr}^2 = U_\infty^2 / g z a$ and the characteristic Stokes number $r_p (\beta - 1) / a U_\infty$. The parameter $e = \sqrt{1 - b^2 / a^2}$ is the eccentricity of the ellipse.

3. Methods

3.1. Experimental Setup

The experiments were done in a vertical up-flowing water tunnel with a round test section of $d = 10$ cm (see figure 2). The model used for the gas bubble was a $\theta_0 = 50^\circ$ solid spherical cap, the typical shape for large bubbles rising in a low-viscous liquid Coppus et al. (1977). Both the continuous liquid phase and the dispersed solid phase were measured simultaneously with a two-camera setup, making use of optical filters to separate the light emitted by fluorescent tracers from the directly scattered light from the dispersed particles. The illumination of the measurement region was done with a Nd:YAG laser, creating a light sheet that was aligned with the axis of symmetry of the spherical cap. The flow fields were measured with standard planar PIV. The mean concentration of the dispersed phase was obtained by integrating the scattered light intensity.

We used several types of particles for the dispersed phase to achieve a range of particle response times: both hollow and solid glass spheres as well as ceramic particles of diameter...
The modified particle response time \( \tau_\ast = \tau_P|\beta - 1| = \frac{2R_P^2|\rho_P - \rho_F|}{g\nu_P} \) was in the range from 0.07 ms to 2 ms, resulting in Stokes numbers of \( 3 \cdot 10^{-4} \) to \( 7 \cdot 10^{-2} \).

### 3.2. Concentration Measurements

The average recorded scattered-light intensity of the particles is not constant over the image area, but varies locally due to the divergence angle of the light sheet (linear dependence on the radial distance to the focal point of the cylindrical lens), the particle scattering behaviour and light absorption, hence on the local concentration itself. It is therefore necessary to correct the recorded images for these inhomogeneities. As the distance of the image area to the focal point of the cylindrical lens is large (about 1:10) compared to its width, the light intensity drop due to the divergent light sheet is approximately 10%.

According to the Lambert-Beer law for light absorption by dispersed absorbers, the loss of light intensity is exponential with the absorption coefficient

\[
\alpha = \sigma_A \rho \tag{9}
\]

with the absorption cross section \( \sigma_A \) of the absorber and its number per unit volume \( \rho = N/V \). The term absorption is used somewhat ambiguous here, it is de facto the loss of light intensity in the light sheet plane due to scattering in all directions. Nevertheless, the light absorption model is independent of the underlying mechanism causing the loss in intensity, therefore it can be used here as a phenomenological model.

Using the particle volume load \( \Phi = N \ast V_P/V \) and assuming spherical absorbers yields

\[
\alpha = \frac{\pi/4d_P^2 \Phi}{\pi/6d_P^3} = \frac{3\Phi}{2d_P} \tag{10}
\]

For the conditions used in the experiments, the light intensity loss over the frame width of \( l = 46 \text{mm} \)

\[
1 - \frac{I(l)}{I(0)} = \exp(-\alpha l) = \exp\left( -\frac{3\Phi}{2d_P} \right) \tag{11}
\]

is approximately 12% (\( d_P = 55 \mu m, \Phi = 10^{-4} \)) to 47% (\( d_P = 110 \mu m, \Phi = 10^{-3} \)) for a perfect absorber (isotropic scatterer). As forward scattering is by far the dominant direction, the calculated values are overestimating the real amount of light absorption. The light absorption in the water is neglected.

The third contribution, the anisotropy of Mie-scattering, is shown in (3) for a 55\( \mu m \) gas bubble, \( \lambda = 532 \text{nm} \). For particles with a certain size dispersion and at sufficiently large apertures of the used objective lens, the graphs show a smooth curve with an almost exponential decrease of the scattered light intensity towards larger scattering angles, something that is not necessarily the case for different particles. The intensity drop over the imaging frame (factor 2.6) is considerably larger than the light falloff due to both light absorption and the divergence of the light sheet.

Another source of inhomogeneity is due to the non-constant intensity profile of the laser beam itself. For optimising output intensity, the resonators of the lasers are aligned such to allow for higher modes (TEM\(_{xy}\)) than only the lowest, Gaussian mode (TEM\(_{00}\)). This results in a superposition of the lower modes and in a non-Gaussian beam profile. With an adequately aligned light sheet, the relative deviation from the mean intensity could be kept below 15%.

Mie scattering was found to be the dominant contribution to the inhomogeneity of the recorded light intensity over the frame. However, it is also the most difficult one to correct; there is no simple functional relation, such that must be estimated using a software packages.
Figure 3. Mie-scattering for a 55µm gas bubble (monodisperse, with normal distributed diameter (σ = 5.5µm) and averaged over the entering pupil at F8), field-of-view of a Micro-Nikkor 105 2.8D, 15.4 × 15.4mm sensor, M = 1/3

(MiePlot). An advantage is that Mie scattering is independent of particle concentration, its correction can therefore applied before the correction of other inhomogeneities.

The fluorescence of the tracer particles is isotropic, but influenced by saturation. Close to the beam waist where the photon density is highest, integrating over the light sheet thickness gives lower values than further away from it. This is the result of less illuminated particles (the illuminated volume is smaller), with saturated fluorescence due to the high light intensity. The images of the tracer particles were therefore without any use for the normalisation of the light intensity of the dispersed phase.

The attempts to correct for the inhomogeneity of the illumination had only limited success. It was therefore decided to normalise the measured light intensity with a reference data set which was recorded under identical conditions of illumination, typically recorded at a lower flow speed. This method works best if the absolute mean particle concentration — and therewith the light falloff due to absorption — is identical for both data sets, which is not always fulfilled. The main disadvantage is that due to the normalisation of noisy intensity values with noisy intensity values, the uncertainties in the concentration measurements increase significantly. Subsequent averaging over the wake area does not significantly improve these uncertainties, as the distribution of the sample mean is identical to the distribution of the samples in the case of Cauchy-distributed random variables (the quotient distribution of two normal distributed random variables). Though being noisy, this method was found to be superior to corrections for the inhomogeneous illumination, as it did not require manual adjustments of fitting parameters for each data set.

4. Results and Discussion

Figure 5 shows the agreement of the derived models with our experimental data. The measured increase in solid concentration inside the near-wake is plotted against the quotient of the characteristic Stokes number $St^* = \tau_p (\beta - 1) U_\infty / R_{\text{wake}}$ and the turbulence level $Tl = u_{\text{rms}} / U_\infty$. The concentration values are corrected for the influence of the geometry factors $f(\theta_0, Fr^2)$ and $f(\eta_0, e, Fr^2)$ respectively. In defiance of the high uncertainties in the concentration measurements, a clear trend is visible showing an increase in the concentration ratio towards larger absolute Stokes numbers. The derived models fit the measured concentrations with a rms of the normalised residuals of 1.29 for the spherical wake model, and 1.52 for the ellipsoidal...
The wake size only depends on the properties of the (uniform) flow approaching the spherical cap, as the overall particle volume load of $\Phi < 10^{-3}$ is too small to significantly change the time-averaged velocity field of the flow. The shape of the near-wake is therefore a function of Reynolds number and turbulence level only. Qualitatively, the flow behind the spherical cap consists of an outer flow region, the free shear layers evolving from the rim of the spherical cap, and an inner recirculation area (the near-wake).

For the typical measurement range of $840 \leq Re \leq 6600$, the boundary of the wake (the surface formed by the streamlines separating the outer flow from the inner recirculation area) can be approximated by an ellipsoid, whose size (semi-major and semi-minor axes) is constant within that range of Reynolds numbers, compare figure 4. For Re $< 600$, the wakes are considerably smaller and of non-ellipsoidal shape. Within that range, the size and the aspect ratio of the near-wake only depends on the turbulence level of the incoming flow. For the measurements with the spherical cap mounted closely behind an additional grid (distance $z = 8M$, grid spacing $M$), the approaching flow is strongly turbulent. The momentum transfer over the free shear layers is stronger, resulting in weaker gradients in the time-averaged velocity field and a stronger recirculation. The aspect ratio of the wake is smaller.

The turbulence at the boundary of the near-wake is dominated by the shedding of vortices as

Figure 4. Scaling of the half axes of the wake with Reynolds number, from Warncke et al. (2011). $Re$ is based on the volume-effective diameter of the spherical cap (11mm)
Figure 5. Measured increase in particle concentration in the near-wake relative to the outer flow over the characteristic Stokes number, from Warncke et al. (2011). Shown are the linear fits for the spherical wake model (top) and the ellipsoidal wake model (bottom), from Warncke et al. (2011). $R_{\text{wake}}$ is estimated by the volume-effective radius of the ellipsoid fitted to the wake boundary, $a$ is the semi-major axis of the fit.
Figure 6. Standard deviation of the velocity normal to the wake boundary for different flow parameters: Laminar upstream flow (turbulence level < 3%, left), turbulent upstream flow (turbulence level 15...21%, right); the rear stagnation point has a polar angle of 0, the free shear layers are found at polar angles $|\theta| > \pi/2$, the rim of the spherical cap is at $|\theta_0| > 13\pi/18$, from Warncke et al. (2011)

a result of the break-up of the free shear layers, shortly after the rim of the spherical cap. This can be seen in figure 6: The magnitude of the fluctuating velocity component normal to the wake boundary is proportional to the strength of the mixing. The graph shows similar magnitudes for the wake boundary at $0 \leq |\theta| \leq \pi/2$, for all measured Reynolds numbers. The values for the high-turbulence case ($z/M = 8$) are slightly higher, though showing the same independence of $Re$. The largest differences can be found in the shear layers, where the delayed and weaker vortex shedding at lower mean flow velocities causes weaker momentum exchange.

The fitted slope $\chi$ is the only unknown parameter of the model, containing all the approximations made in the derivation (see chapter 2 of Warncke et al. (2011)). The turbulent transport over the wake boundary is simplified using a mixing length model $D_P \nabla n_P = C u_{\text{rms}} L(n_P, \text{wake} - n_P, \text{ext})/L_{\text{grad}}$ with a constant $C$ of order one. The fitted slope $\chi$ therefore contains $C$, the ratio of $u_{\text{rms}}$ with the mean normal velocity fluctuation over the boundary and the ratio between the mean characteristic length scale of the concentration gradient $L_{\text{grad}}$ with the mean turbulent mixing length $L$ at the wake boundary. Furthermore, the average radial particle acceleration of a real flow around a spherical cap will be different from that of a potential flow, which was used as a model for the time-averaged velocity field around the spherical cap and its near-wake. The fit implies that within the margins of uncertainty, $\chi$ is a constant within the parameter range of the experiments.

5. Conclusion

In this paper, a model was introduced to describe the relative solid holdup of dispersed particles in the near-wake of a spherical-cap body. Within the measurement uncertainties, the model showed a good agreement with the presented experimental data. Furthermore, all simplifications made in the derivation were shown to yield a scaling parameter, which was found to be constant over the selected range of experimental parameters. It can be concluded that the simple model correctly predicts the outcome of the performed experiments.
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