Nonhermitian Supersymmetric Partition Functions: the case of one bosonic flavor

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We discuss the supersymmetric formulation of the nonhermitian $\beta = 2$ random matrix partition function with one bosonic flavor. This partition function is regularized by adding one conjugate boson and fermion each. A supersymmetric nonlinear $\sigma$-model for the resulting Goldstone degrees of freedom is obtained using symmetry arguments only. For a Gaussian probability distribution the same results are derived using superbosonization and the complex orthogonal polynomial method. The symmetry arguments apply to any model with the same symmetries and a mass gap, and demonstrate the universality of the nonlinear $\sigma$-model.

I. INTRODUCTION

There exists a vast literature [1] showing that the spectra of many physical systems on the scale of the average level spacing (the microscopic scale) are correlated according to universal laws given by random matrix theory. They can be classified according to their (anti-)unitary symmetries and invariant bilinear forms [2, 3]. The reason for this universal behavior is that many physical systems and random matrix models alike can be reduced to field theories with only Goldstone degrees of freedom. On general grounds, such a theory is a nonlinear $\sigma$-model and is determined uniquely by the pattern of symmetry breaking and convergence requirements. One can therefore obtain the microscopic correlation functions from symmetry arguments alone without ever referring to a random matrix model. The construction of the nonlinear $\sigma$-models from symmetries is standard and well known for Hermitian systems [2, 4].

For nonhermitian problems, which appear in open quantum systems, e.g., in the theory of $S$-matrix fluctuations, or Euclidean QCD at nonzero chemical potential, the situation has been investigated to a much lesser extent. Here the formulation of $\sigma$-models based on symmetries has been studied in detail only for the case of partition functions given by products of determinants (i.e., fermionic theories) [5, 6, 7, 9]. A major difference between fermionic theories and bosonic or supersymmetric ones is that there are no convergence problems for Grassmann integrals. For bosonic nonhermitian partition functions (with inverse determinants) only very few derivations of a $\sigma$-model from the underlying symmetries can be found in the literature; in fact, the only works known to us are [3, 10]. In the supersymmetric case (i.e., with both fermions and bosons) we are aware of only one model — the generating function of the spectral density of a Hermitian ensemble deformed by an antihermitian ensemble — where a $\sigma$-model for the Goldstone degrees of freedom [11, 12, 13] has been obtained. However, that $\sigma$-model was derived by direct calculation, not by symmetry arguments. The main objective of the present paper is to show that a $\sigma$-model with exclusively Goldstone degrees of freedom can be obtained from symmetry arguments alone also in supersymmetric cases.

In this paper we study the symmetry class whose simplest representative is a model where a complex Hermitian Gaussian random matrix ensemble is deformed by a complex antihermitian Gaussian random matrix ensemble. This model was introduced by Fyodorov, Khoruzhenko and Sommers [11] and for this reason it will be called the FKS model. Its introduction was motivated by the study of the distribution of resonance poles for systems with broken time-reversal invariance [14]. More recently it was used to describe the Hatano-Nelson model [15] and, in its unquenched form, QCD in three dimensions at nonzero chemical potential [7, 16].

In this paper we study the FKS model [11] for one bosonic flavor, the partition function of which is defined by

$$Z_{-1}(z; a) = \left\langle \frac{1}{\text{Det}(z + H + A)} \right\rangle. \quad (1)$$

The average is over the Gaussian probability distribution with distribution function

$$P(H, A) = e^{-\frac{1}{2} \text{Tr} H^2 - \frac{a}{2} \text{Tr} A^2}, \quad (2)$$

with $H$ a Hermitian $N \times N$ matrix and $A$ an antihermitian $N \times N$ matrix. Note that $P(H, A)$ is invariant by a unitary change of basis, $H \rightarrow g H g^{-1}$, $A \rightarrow g A g^{-1}$, $g \in U(N)$. As was argued in [11], there is a major difference between fermionic and bosonic partition functions: in the large-$N$ microscopic limit, the partition function with one fermionic flavor does not depend on the nonhermiticity parameter, whereas the bosonic partition function [11] does. This behavior was found in [11] using the method of complex orthogonal polynomials for a random matrix model of QCD at nonzero chemical potential [16]. Its explanation was based on the observation that the partition function [11]...
needs to be regularized. In the context of a $\sigma$-model formulation, the technical reason is that an inverse determinant of a nonhermitian matrix cannot be written as a Gaussian integral in general. The regularized partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) = \lim_{\epsilon \to 0} \left( \text{Det}(z^*_f + H - A) \text{Det}^{-1} \left( \begin{array}{cc} i\epsilon & z + H + A \\ z^* + H - A & -i\epsilon \end{array} \right) \right) \] reduces to the partition function for one bosonic flavor for $z^*_f \to z^*$. It has flavors with opposite charges resulting in a ground state which rotates as a function of the nonhermiticity parameter. For one fermionic flavor no regularization is necessary and the ground state does not rotate, so that the free energy does not depend on the nonhermiticity parameter. The regularization procedure of the inverse determinant is known as Hermitization [13, 17, 18].

The partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] is well understood for $z^*_f \to \infty$ in which case it is the two-flavor phase quenched bosonic partition function. In that case, because of a complex conjugated singularity, it diverges logarithmically with $\epsilon$ [19]. In $\sigma$-model language, the singularity is due to a Goldstone boson with a mass that vanishes as $\epsilon \to 0$ [10]. Also, the partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] for $\epsilon \to 0$ acquires a Goldstone fermion of mass $z^* - z^*_f$. We therefore expect the behavior

\[ Z^{-1}(z^*_f|z,z^*;\sigma) \sim (z^* - z^*_f) \log \epsilon + O(\epsilon^0). \] (4)

In this present paper we will derive this result from the $\sigma$-model for the microscopic limit of (3). We will also show that the $O(\epsilon^0)$ term agrees with the partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] evaluated by the method of complex orthogonal polynomials.

A major issue with $\sigma$-models for nonhermitian random matrix models is the proper choice of integration manifold. There are two important developments which have made this choice much less ad hoc. The first of these was the introduction of the so-called Ingham-Siegel integral [4] as an alternative of the Hubbard-Stratonovich transformation. This work provided a simple explanation of the structure of the integration manifold for inverse determinants when convergence arguments are essential. The second development was the introduction of superbosonization which extends the Ingham-Siegel approach to include fermions in a unified fashion [20, 21]. Earlier versions of superbosonization appeared in [22, 23, 24, 25, 26], but the method was put on a mathematically rigorous footing only in [20, 21].

Although it is straightforward to obtain the correct $\sigma$-model using the superbosonization formula of [20, 21], it can be a technically challenging task to evaluate the integrals for more than a few degrees of freedom. The superbosonization method was applied to nonhermitian chiral random matrix ensembles in [27]. However, the construction of a nonlinear $\sigma$-model containing exclusively Goldstone degrees of freedom was not given in that paper.

In this paper we will derive universal results for the symmetry class of the FKS model relying on symmetry arguments only. This makes it manifest that our results apply to all models in the same symmetry class and a mass gap. Analytical results for finite $N$ cannot be obtained from general arguments and require a detailed calculation. Such results will be derived for the FKS model with Gaussian probability distributions using two independent methods, the superbosonization method and the complex orthogonal polynomial method. Each method has its own merits and both deserve to be discussed. In particular, relations between partition functions with different degrees of freedom appear naturally in the complex orthogonal polynomial method. Using superbosonization the universal $\sigma$-model can be recovered from the finite-$N$ results by taking the microscopic limit and eliminating the massive modes.

Finally, let us mention that the quenched spectral density of the FKS model has also been derived [9] by means of the replica limit of the Toda lattice equations [28]. It was shown that the quenched spectral density is the product of a fermionic and a bosonic partition function. To derive the spectral density for one fermionic flavor using the Toda lattice hierarchy, one needs precisely the partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] introduced in [20, 14].

In this paper we will first derive the $\sigma$-model for the microscopic limit of (3) using symmetry arguments only (Section II). Results for finite $N$ will be derived using the superbosonization formula (Section III) and the complex orthogonal polynomial method (Section V). In Section IV we will recover the universal $\sigma$-model from the finite-$N$ results for the FKS model. Concluding remarks are made in Section VI.

II. SYMMETRIES AND SUPERSYMMETRIC $\sigma$-MODEL

In this section we will derive the universal supersymmetric $\sigma$-model for the symmetry class of which the partition function \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] is the simplest representative. The derivation is based on the symmetries of \[ Z^{-1}(z^*_f|z,z^*;\sigma) \] only, and is valid for any other model in the same symmetry class. We will first consider the bosonic sector with one bosonic flavor $\phi_+$ and one conjugate bosonic flavor $\phi_-$. An earlier study of this sector was made in [3].
A. The Phase Quenched Bosonic Partition Function

The regularized phase quenched bosonic partition function can be written as

\[ Z_{pq-bos}(z, z^*; a) = \left( \text{Det}^{-1} \left( \begin{array}{cc} i & z + H + A \\ \frac{1}{i} & i \end{array} \right) \right). \]  

(5)

We will evaluate this partition function in the microscopic limit, keeping \( N \text{Im}z \) and \( Na^2 \) fixed as \( N \to \infty \). To start the argument, we cast the inverse determinant in the form of a Gaussian integral:

\[ Z_{pq-bos}(z, z^*; a) = \int \prod_{k=1}^{N} d\phi^k_+ d\phi^k_- d\phi^*_k d\phi^*_k \exp i \left( \phi^*_+ \phi^-_+ \right) \left( \frac{i}{z + H + A} \right) \left( \phi^*_+ \phi^-_+ \right) \]  

(6)

After averaging over \( H \) and \( A \) the partition function can be expressed as an integral over the positive Hermitian \( 2 \times 2 \) matrix \( Q \) of \( U(N) \)-invariant variables

\[ Q = \sum_{k=1}^{N} \left( \phi^k_+ \phi^k_- \right) \otimes (\phi^k_+ \phi^k_-) = \left( \begin{array}{cc} \phi^*_+ \phi^-_+ & \phi^*_+ \phi^-_- \\ \phi^-_+ \phi^*_+ & \phi^-_- \phi^*_- \end{array} \right). \]  

(7)

If we were dealing with fermions and compact symmetries, we could now consider a form of ‘maximum flavor symmetry’ using the theoretical arguments of Peskin [3,4]. A nonzero expectation value of such a form would signal spontaneous symmetry breaking in the thermodynamic limit. Our analysis must be somewhat different, however, as we are facing the case of noncompact bosons. To express the microscopic limit of the partition function [3] in terms of the Goldstone degrees of freedom residing in \( Q \), we will require that the resulting integration measure and Lagrangian have the same transformation properties as the corresponding objects of the original partition function.

The complex group \( \text{GL}(2) \) acts on the matrix \( Q \) and the vector variables \( \phi \equiv (\phi_+ \phi_-) \) and \( \phi^* = \left( \begin{array}{c} \phi^*_+ \\ \phi^*_- \end{array} \right) \) as

\[ \phi^* \mapsto g \phi^*, \quad \phi \mapsto \phi g^+, \quad Q \mapsto g Q g^+, \quad g \in \text{GL}(2). \]  

(8)

Under such transformations the integration measure \( \prod_{k=1}^{N} d\phi^k_+ d\phi^k_- d\phi^*_k d\phi^*_k \) gets multiplied by the Jacobi determinant \( |\text{Det} g|^{2N} \). Thus, arguing by symmetry and by the transformation behaviors, the corresponding measure in the \( Q \)-variables is inferred to be \( \text{Det}^N(Q) dQ \) where \( dQ \) denotes a \( \text{GL}(2) \)-invariant measure for \( Q \). Note also that the transformation law \( Q \mapsto g Q g^+ \) preserves the properties of Hermiticity and positivity of the matrix \( Q \).

For \( z = z^*, \ a \neq 0 \), and \( \epsilon \to 0 \) we see that the partition function [5] is invariant under the subgroup \( G \subset \text{GL}(2) \) of flavor transformations \( Q \mapsto T Q T^\dagger \) which preserve the Hermitian quadratic form

\[ \phi^*_+ \phi^-_+ + \phi^*_- \phi^-_+ = \text{Tr} Q \sigma_1, \quad \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \]  

(9)

of the bosonic degrees of freedom \( \phi \). Equivalently, the matrices \( T \in G \) are subject to the condition \( T^\dagger \sigma_1 T = \sigma_1 \). Since the Hermitian quadratic form determined by \( \sigma_1 \) is of signature \( (1, 1) \) — to see this, one makes a unitary conjugation transforming \( \sigma_1 \) into \( \sigma_3 = \text{diag}(1, -1) \) — the symmetry group \( G \) of our problem is identified as \( G = U(1, 1) \).

Next we search for the manifold of Goldstone degrees of freedom (or the target space of the nonlinear \( \sigma \)-model) inside the space of matrices \( Q \). For that purpose, we make a temporary change of variables from \( Q \) to \( X := iQ \sigma_1 \). The new matrices \( X \) satisfy \( X = -\sigma_1 X^\dagger \sigma_1 \) and thus lie in the Lie algebra \( \text{Lie}(G) \) of \( G = U(1, 1) \). Now, using \( Q \mapsto TQ T^\dagger \) and the relation \( T^\dagger \sigma_1 = \sigma_1 T^{-1} \) we see that \( G \) acts on \( X \in \text{Lie}(G) \) by the adjoint representation \( X \mapsto TXT^{-1} \).

The space of \( 2 \times 2 \) matrices \( Q \) subject to the conditions \( Q = Q^\dagger > 0 \) is a cone of real dimension four. Writing

\[ Q = \left( \begin{array}{cc} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{array} \right), \]  

(10)

this cone is given by the inequalities \( Q_{++} > 0, Q_{--} > 0 \), and \( Q_{+-} Q_{-+} = |Q_{+-}|^2 < Q_{++} Q_{--} \). Thus our new matrices \( X = iQ \sigma_1 \) do not occupy the entire Lie algebra of \( G \) but lie in the cone \( C^+ \subset \text{Lie}(G) \) which arises as the corresponding image by the map \( Q \mapsto iQ \sigma_1 \). The cone \( C^+ \) may be viewed as the ‘space of states’ of our problem.

Let us look at the positive cone \( C^+ \) in a little bit of detail. First, notice that \( \text{Lie}(G) \) is generated as a Lie algebra over the real numbers by the four generators \( i1, i\sigma_1, i\sigma_2 \), and \( i\sigma_3 \). Let \( K \subset G \) be the maximal compact subgroup which is generated by the first two, \( i1 \) and \( i\sigma_1 \). Thus \( K \) is the group \( U(1) \times U(1) \) of elements \( e^{i(\alpha+\beta \sigma_1)} \) with \( \alpha, \beta \in [0, 2\pi] \).
Now consider \( \mathfrak{t}^+ := C^+ \cap \text{Lie}(K) \), the intersection of the cone \( C^+ \) with the Lie algebra of \( K \). One may ask whether the elements \( \xi \in \text{Lie}(G) \) can be conjugated into \( \mathfrak{t}^+ \) by the adjoint action \( \xi \mapsto T \xi T^{-1} \) of \( G \). The answer is that this is not possible in general, because \( \text{Lie}(G) \) is the Lie algebra of a noncompact group. Nevertheless, the cone \( C^+ \) does have the special property that each of its elements is conjugate to some \( \lambda \in \mathfrak{t}^+ \) by the adjoint action of \( G \). (This follows from basic principles of linear algebra and Lie theory and can be easily verified by direct calculation for our simple case of \( 2 \times 2 \) matrices.) Thus each \( \xi \in C^+ \) can be presented in the ‘diagonalized’ form

\[
\xi = T \lambda T^{-1}, \quad T \in G, \quad \lambda \in \mathfrak{t}^+. \tag{11}
\]

To introduce the proper mathematical language, we say that each orbit of the adjoint \( G \)-action on \( C^+ \) hits the slice \( \mathfrak{t}^+ \subset \text{Lie}(K) \) at least once (actually, exactly once). It follows from the diagonalization (11) and the abelian nature of \( K = U(1) \times U(1) \) that \( C^+ \) has the structure of a direct product \( (G/K) \times \mathfrak{t}^+ \). (More generally, in the case of a nonabelian group \( K \), the cone \( C^+ \) is an associated bundle \( G \times_K \mathfrak{t}^+ \).) Moreover, since \( K \subset G \) is a maximal compact subgroup, the quotient \( G/K \) is a symmetric space of noncompact type. In the case at hand we get the identification

\[
G/K = U(1, 1)/U(1) \times U(1) = H^2 \tag{12}
\]

with a two-dimensional hyperboloid \( H^2 \). To summarize the present discussion: the space of states of our problem, the positive cone \( C^+ \), has a decomposition (mathematically speaking, a ‘fibration’) \( C^+ \cong (G/K) \times \mathfrak{t}^+ \) by adjoint \( G \)-orbits all of which are isomorphic to the same noncompact symmetric space \( G/K \).

Now, by some dynamical principle beyond the reach of symmetry arguments, the system selects one of the \( G \)-orbits of the fibration \( C^+ \equiv (G/K) \times \mathfrak{t}^+ \) for its Goldstone manifold or vacuum orbit. This \( G \)-orbit will in general be specified by an element \( \lambda \in \mathfrak{t}^+ \otimes \mathbb{C} \) of the complexification of \( \mathfrak{t}^+ \). In the case under consideration we have

\[
\lambda = i \lambda_0 \mathbf{1} + i \lambda_1 \mathbf{1}, \tag{13}
\]

where \( \lambda_0 \) and \( \lambda_1 \) would have to be real numbers (with \( |\lambda_0| < \lambda_1 \)) in order for \( \lambda \) to be in \( \mathfrak{t}^+ \), but in view of the principle of steepest descent (or deformation of the integration contour into the complex plane) we should be prepared for \( \lambda_0 \) and/or \( \lambda_1 \) to deviate from the real axis. For example, in the case of the Gaussian ensemble (2) one finds that the \( Q \)-integral for \( z = 0, a = 0 \), and \( \epsilon \to 0 \) has a saddle point (a maximum of the integrand) at \( \lambda_0 = 0 \) and \( \lambda_1 = 1 \), with \( \lambda_0 \) becoming imaginary as \( \text{Re} \, z \) moves away from zero. In general, \( \lambda_0 \) and \( \lambda_1 \) take some other values. While these may be hard to compute, we will see that the universal results emerging in the microscopic limit do not depend on them.

Returning to our original notation, we have identified a Goldstone or saddle-point manifold of matrices \( Q \):

\[
Q = T Q_0 T^\dagger, \quad Q_0 = \lambda_1 \mathbf{1} + \lambda_0 \mathbf{1}, \quad T \in G. \tag{14}
\]

From the discussion above, we know that this \( G \)-orbit \( Q = T Q_0 T^\dagger \) is always isomorphic to the quotient \( G/K \) of the noncompact group \( G \) by a maximal compact subgroup \( K \subset G \).

There exist very many ways of parameterizing the \( G \)-orbit \( Q = T Q_0 T^\dagger \). One possible choice is by a diffeomorphism \( H^2 \cong \mathbb{R}^2 \), exponentiating the real plane \( \mathbb{R}^2 \) spanned by the generators \( \sigma_2 \) and \( \sigma_3 \) as follows:

\[
T = e^{u \sigma_3/2} e^{\sigma_2/2}, \quad Q = T Q_0 T^\dagger = \lambda_1 e^{u \sigma_3/2} e^{\sigma_2} e^{u \sigma_3/2} + \lambda_0 \mathbf{1}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad u, s \in \mathbb{R}. \tag{15}
\]

Let us briefly pause to mention the following heuristic confirming the present scenario. Suppose we were to go beyond the microscopic limit and construct a nonlinear \( \sigma \)-model of spatially fluctuating Goldstone modes with target space \( G/K \). To give a sensible definition of the functional integral of such a field theory, we need the target space to be Riemannian. Now, for the case of a semisimple noncompact Lie group \( G \) it is a fact of differential geometry that there is only one way to get a Riemannian manifold with a \( G \)-invariant geometry: divide \( G \) by a maximal compact subgroup \( K \). In contrast, the situation for fermions with compact symmetries is very different. There, the fibration of the state space by orbits of the symmetry group typically contains orbits of several types, corresponding to a variety of nonisomorphic compact Riemannian symmetric spaces. In that situation, unlike what we are facing here, one has to appeal to a postulate of ‘maximum flavor symmetry’ \( [3] \) to select the proper type of vacuum orbit.

We are now getting ready to switch on the perturbations \( a, \epsilon \), and \( z \neq z^* \) breaking \( G \)-symmetry. Using the transformation law \( Q \mapsto T Q T^\dagger \) for the Goldstone degrees of freedom, the partition function \( \mathcal{Z} \) in the presence of the symmetry-breaking terms remains unchanged if we simultaneously transform

\[
\zeta \mapsto T^{-1} \zeta T^{-1}, \quad A \mapsto T^{-1} A T^{-1}, \tag{16}
\]
We thus find that the microscopic limit of the phase quenched bosonic partition function is given by \[9\]

\[
\zeta = \begin{pmatrix} i\epsilon & i\text{Im}z \\ -i\text{Im}z & i\epsilon \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & -ia \\ ia & 0 \end{pmatrix} = a\sigma_2. \tag{17}
\]

Of course the low-energy limit of the partition function \[9\] must inherit the invariance under the transformation \[16\].

Here, to proceed, we make the assumption that the low-energy measure which is induced on the \(G\)-orbit \(G/K\) of the global mode (or zero mode) converges to the \(G\)-invariant measure, \(d\mu(Q)\), when the regularization parameter \(\epsilon\) is taken to zero. In the case of a compact symmetry group \(G\) this assumption always holds true. However, in the present case of a noncompact symmetry \(G\) (more precisely: a ‘nonmenable’ symmetry \(G\), see \[29\]), the \(G\)-invariant measure is intrinsically unstable with respect to interactions of the Goldstone modes. This circumstance causes a breakdown \[29\] of the standard scenario of spontaneous symmetry breaking, zero mode approximation, and universality.

Nevertheless, in the microscopic limit, i.e., for weakly interacting Goldstone modes in a small enough volume, the said assumption does hold true, and the integration measure on \(G/K\) in the limit of \(a = 0\), \(\text{Im}z = 0\), and \(\epsilon \to 0\), is the \(G\)-invariant measure \(d\mu(Q)\). The \(G\)-invariance of the low-energy partition function then forces the low-energy Lagrangian to be \(G\)-invariant as well. For the mass term there exists only a single invariant to lowest order in \(\zeta\):

\[
\text{Tr}(\zeta Q) = \lambda_1 \text{Tr}(\zeta TT^\dagger). \tag{18}
\]

After averaging, there are no terms linear in \(a\). To order \(O(a^2)\) there are two possible invariants:

\[
\text{Tr}(AQ A Q) \quad \text{and} \quad \text{Tr}(AQ) \text{Tr}(AQ). \tag{19}
\]

While these invariants are independent in general, it so happens in the present case of a single flavor that they are accidentally the same. To verify this, one may exploit the parametrization \[15\] and the relation \(e^{u \sigma_3} \sigma_2 = \sigma_2 e^{-u \sigma_3}\) to find the expressions

\[
\text{Tr}(\sigma_2 Q) = 2\lambda_1 \sinh s, \quad \text{Tr}(\sigma_2 Q \sigma_2 Q) = 2\lambda_1^2 (1 + 2 \sinh^2 s) - 2\lambda_0^2, \tag{20}
\]

which show that \(\text{Tr}(\sigma_2 Q \sigma_2 Q) - \text{Tr}(\sigma_2 Q) \text{Tr}(\sigma_2 Q)\) is a constant independent of \(u\) and \(s\).

Thus we need only include the first invariant in the expression for the partition function. Terms of higher order in \(\zeta\) and \(a^2\) do not contribute in the microscopic limit and will not be considered here. We also see that the unknown parameter \(\lambda_0\) just adds to the low-energy Lagrangian an inessential constant, which will not be considered any further here (i.e., we set \(\lambda_0 = 0\)). The remaining unknown \(\lambda_1\) is determined by the eigenvalue density of the system, and we may take its value to be \(\lambda_1 = 1\) by an appropriate choice of units. Note also that \(\text{Det}^N(Q) = |\text{Det} T|^2 = 1\).

We thus find that the microscopic limit of the phase quenched bosonic partition function is given by \[5\]

\[
Z_{\text{bos}}(z, z^*; a) = \int d\mu(Q) e^{iN\text{Tr}(\zeta Q - \frac{1}{2} Na^2 \text{Tr}(Q \sigma_2 Q \sigma_2))}, \tag{21}
\]

which is an integral over the coset space \(G/K\) of matrices \(Q = TT^\dagger\) with \(G\)-invariant measure \(d\mu(Q)\).

Now the two-hyperboloid \(G/K = U(1,1)/U(1) \times U(1)\) is the simplest member of a certain family – the Hermitian symmetric spaces – with many wonderful properties. In particular, Hermitian symmetric spaces are Kähler manifolds and come with a \(G\)-invariant, closed and non-degenerate two-form, \(\omega\) (the Kähler form). In the case at hand,

\[
\omega = -i \text{Tr}(\sigma_1 T^{-1}dT \wedge T^{-1}dT), \tag{22}
\]

which is clearly invariant under left translations \(T \mapsto gT\) corresponding to the \(G\)-action \(Q \mapsto gQ g^\dagger\), and also pushes down to a well-defined form on the quotient \(G/K\). Using \(d^2 = 0\) and \(d(T^{-1}) = -T^{-1}(dT)T^{-1}\) the expression for \(\omega\) simplifies to

\[
\omega = i d\text{Tr}(\sigma_1 T^{-1}dT). \tag{23}
\]

We will shortly use this formula to compute the expression of our \(G\)-invariant measure \(d\mu(Q)\) in suitable coordinates.

To calculate the integral \[21\] we use the parametrization \[15\], and we note that \(\text{Tr}(\zeta Q) = i\epsilon \text{Tr} Q + \text{Im}(z) \text{Tr} \sigma_2 Q\). From \[20\] we already have the expressions for \(\text{Tr} \sigma_2 Q\) and \(\text{Tr}(\sigma_2 Q)^2\), and for the remaining term in the exponent we find \(\epsilon \text{Tr} Q = 2\epsilon \cosh u \cosh s\). To express the measure \(d\mu(Q)\) of integration we insert the parametrization \[15\] for \(T\) into \[20\] to obtain

\[
\omega = i d\text{Tr}(\sigma_1 T^{-1}dT) = \frac{i}{2} d\text{Tr}(\sigma_1 e^{-s\sigma_2 \sigma_3}) \wedge du = d(\sinh s) \wedge du. \tag{24}
\]
From this result we can say immediately how the measure \( d\mu(Q) \) looks in the present coordinates. Indeed, since the form \( \omega \) is \( G \)-invariant, so is the integration measure \( d(\sinh s) \ du \) corresponding to it. Because the measure \( d\mu(Q) \) is determined uniquely (up to multiplication by a constant) by \( G \)-invariance, we conclude that \( d\mu(Q) \propto d(\sinh s) \ du \).

Assembling terms, the phase quenched partition function \([24]\) becomes

\[
Z_{pq-\text{bos}}(z, z^*; a) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} e^{2N \Im z \ sinh s - Na^2 (1 + 2 \sinh^2 s)} \left( \int_{\mathbb{R}} e^{-2N \epsilon \cosh u \cosh s} du \right) d(\sinh s). \tag{25}
\]

The inner integral over \( u \) diverges as \( |\log \epsilon| \) for \( \epsilon \to 0 \). The outer integral over \( s \) is then a Gaussian integral in \( \sinh s \) which is easily done by completing the square. Thus our final result \([9]\) for the partition function is

\[
Z_{pq-\text{bos}}(z, z^*; a) = |\log \epsilon| \sqrt{\frac{\pi}{2Na^2}} e^{-Na^2 - \frac{N}{2} \text{Im}^2(z/a)} \quad (\epsilon \to 0). \tag{26}
\]

**B. The Partition Function for one Boson**

In this subsection we analyze the partition function \([3]\). To that end, we express the determinant in the numerator of \([3]\) as a Gaussian integral over a \( U(N) \) fundamental vector \( \psi \) of Grassmann variables \( \psi^k \). We then combine \( \psi \) with the boson flavors \( \phi_{\pm} \) to form a supervector \( \Phi = (\phi_+ \phi_- \psi) \) with adjoint

\[
\Phi^* = \begin{pmatrix} \phi_+^* \\ \phi_-^* \\ \psi^* \end{pmatrix}. \tag{27}
\]

The low-energy effective degrees of freedom will emerge from a supermatrix \( Q \) of \( U(N) \)-invariants,

\[
Q = \Phi^* \Phi = \begin{pmatrix} \phi_+^* \phi_+ & \phi_+^* \phi_- & \phi_+^* \psi \\ \phi_-^* \phi_+ & \phi_-^* \phi_- & \phi_-^* \psi \\ \psi \phi_+ & \psi \phi_- & \psi \psi \end{pmatrix} \quad (\bar{\psi} \psi \equiv \sum_k \bar{\psi}^k \psi^k, \text{etc.}) \tag{28}
\]

Note that the boson-boson block of \( Q \) is Hermitian and positive as before. The matrix entry \( Q_{ff} \equiv \bar{\psi} \psi \) of the fermion-fermion block will acquire a nonzero vacuum expectation value and is treated hence as a complex number.

Guided by the symmetries of the microscopic theory, we are now going to identify the low-energy degrees of freedom and the structure of the low-energy Lagrangian. On general field-theoretic grounds, we expect the low-energy theory to be a nonlinear \( \sigma \)-model of interacting Goldstone modes where the target manifold is a symmetric space.

To see why the target space has to be symmetric — we briefly recall the argument here — one may invoke Friedan’s work \([3]\) on the renormalization of nonlinear models and \( \sigma \)-models, which shows that the quantum loop corrections to the target space metric are given by contractions of the Riemann curvature tensor; the one-loop correction, in particular, is given by the Ricci curvature. (These results, while derived in the classical setting, remain valid in the supersymmetric context.) Therefore, in a low-energy fixed point theory the Ricci curvature of the target space must be proportional to the metric tensor. It follows that the curvature has to have the property of being covariantly constant which, in turn, is the condition for a Riemannian manifold to be a symmetric space. This result, which is fundamental for the renormalization theory of nonlinear models and \( \sigma \)-models, will presently be used.

Under the most general linear transformation of the supervector

\[
\Phi^* \mapsto g_L \Phi^*, \quad \Phi \mapsto \Phi (g_R)^{-1}, \tag{29}
\]

the (a priori) superintegration form \( D\Phi D\Phi^* \equiv \prod_{k=1}^N D\phi^k D\phi^{k*} \) transforms as

\[
D\Phi D\Phi^* \mapsto D\Phi D\Phi^* \text{SDet}^N(g_L) \text{SDet}^{-N}(g_R). \tag{30}
\]

To match this transformation behavior, the space of composite variables \( Q \) has to be equipped with the Berezin measure (or superintegration form) \( DQ \text{SDet}^N(Q) \) where \( DQ \) by definition is invariant under \( Q \mapsto g_L Q (g_R)^{-1} \).

The Hermitian quadratic form \([3]\) is replaced by the boson-fermion mixed form

\[
\phi_+^* \phi_- + \phi_-^* \phi_+ - \bar{\psi} \psi = \text{STr} \, Q \Sigma_1, \quad \Sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{31}
\]
The symmetry group of this extended Hermitian form is the pseudo-unitary Lie supergroup $G = U(1,1|1)$ (a close variant $U(1,1|2)$ of which was discussed in detail in [31]).

Given the symmetry group $G = U(1,1|1)$, we now ask again about the fibration of the space of states by $G$-orbits. For that, we temporarily switch from the supermatrices $Q$ to the related supermatrices $X = Q\Sigma_1$, on which the symmetry group $G$ acts by conjugation:

$$Q\Sigma_1 \mapsto T(Q\Sigma_1)T^{-1}.$$  \hfill (32)

We know from Section 4.4.1 that by this action the number part of every matrix $X$ can be brought to diagonal form.

Consider first the generic case of $3 \times 3$ supermatrices $X = Q\Sigma_1$ with three eigenvalues that all differ from one another. The orbit of the $G$-action on such a matrix is a flag supermanifold $U(1,1|1)/U(1) \times U(1) \times U(1)$. Such a space is not symmetric (indeed, the Riemannian curvature is not covariantly constant but varies) and by the renormalizability criterion reviewed above, it can be ruled out as a candidate for the Goldstone manifold of vacuum states.

There exists, however, the possibility for another type of $G$-orbit, which is realized when the fermion-fermion part of $X$ becomes degenerate with an eigenvalue of the boson-boson part. Supermatrices $X$ on such orbits are of the form

$$X = T \begin{pmatrix} \lambda_0 & \lambda_1 & 0 \\ \lambda_1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \pm \lambda_1 \end{pmatrix} T^{-1}, \quad T \in G \quad \text{and} \quad |\lambda_0| < \lambda_1.$$ \hfill (33)

Thus the degeneration occurs in one of two different ways: the boson-boson part of $X$ has eigenvalues $\lambda_0 \pm \lambda_1$ and the vacuum expectation value of $X_{ff} = \bar{\psi}\psi$ may hit either one of these. In both cases our generic $G$-orbit degenerates to

$$G/K \equiv U(1,1|1)/U(1) \times U(1|1),$$ \hfill (34)

where $K \subset G$ is defined for $\langle \bar{\psi}\psi \rangle = \lambda_0 + \lambda_1$ by the equation $k\Sigma_1 k^{-1} = \Sigma_1$, and for $\langle \bar{\psi}\psi \rangle = \lambda_0 - \lambda_1$ by

$$k\Sigma_1' k^{-1} = \Sigma_1', \quad \Sigma_1' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ \hfill (35)

The quotient $G/K$ is a symmetric superspace, and thus satisfies the renormalizability criterion, in both cases. There exists no dynamical or other reason (not in the microscopic limit anyway) why one of the two $G$-orbits corresponding to the two vacuum expectation values $\langle \bar{\psi}\psi \rangle = \lambda_0 \pm \lambda_1$ should be preferred over the other. We also note that these two $G$-orbits are disjoint. The low-energy theory is therefore expected to be a nonlinear $\sigma$-model with a two-component target space, i.e., with one connected component for each of the two vevs.

Without loss, we now simplify the discussion by setting $\lambda_0 = 0$ and $\lambda_1 = 1$ as before. The low-energy degrees of freedom are then represented by two supermatrices $Q$ and $Q'$,

$$Q = T\Sigma_1 T^{-1}\Sigma_1, \quad Q' = T\Sigma_1' T^{-1}\Sigma_1, \quad T \in U(1,1|1).$$ \hfill (36)

Both $Q$ and $Q'$ run through a symmetric superspace $G/K = U(1,1|1)/U(1) \times U(1|1)$, which has the property of being Hermitian. This fact will be of great help in expressing the $G$-invariant integration measures $DQ$ and $DQ'$ in coordinates. Here we just note that $\text{SDet}^N(Q) = \text{SDet}^N(T\Sigma_1 T^{-1}\Sigma_1) = 1$ and

$$\text{SDet}^N(Q') = \text{SDet}^N(T\Sigma_1' T^{-1}\Sigma_1) = (-1)^N.$$ \hfill (37)

To write the result for the partition function in a concise manner, we introduce a superscript $\sigma = \pm 1$ and let $Q^\sigma \equiv Q$ for $\sigma = +1$ and $Q^\sigma \equiv Q'$ for $\sigma = -1$, and we denote the $G$-invariant superintegration form by $D\mu(Q^\sigma)$. The invariance arguments of the previous section still apply. In the microscopic limit we thus find

$$Z_{-1}(z^*_i z, z^*; a) = \sum_{\sigma = \pm 1}^{\sigma^N} \int D\mu(Q^\sigma) e^{i\text{NSTr} \zeta Q^\sigma - 1/2Na^2 \text{STr} Q^\sigma \Sigma_2 Q^\sigma \Sigma_2},$$ \hfill (38)

where the mass matrix $\zeta$ and the extended Pauli matrix $\Sigma_2$ are now given by

$$\zeta = \begin{pmatrix} i & z^* & 0 \\ z & i & 0 \\ 0 & 0 & z^*_f \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ \hfill (39)
To summarize, we have expressed the microscopic limit of \( Z_{-1}(z^*_\tilde{f}; a, a') \) as an integral over Goldstone degrees of freedom only. (Of course, the overall normalization factor cannot be fixed by the arguments in this section.) In the next section we will rederive the result \((\S\S)\) using superbosonization.

The calculation of the integral \((\S\S)\) requires an explicit parametrization of the Goldstone degrees of freedom. For this purpose we choose a fermion-boson factorized generalization of the parametrization \((16)\):

\[
T = T_f T_b, \quad T_f = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}, \quad V = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}, \quad T_b = \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix}, \quad W = e^{u\sigma_3}/2 e^{e\sigma_3/2} .
\]

To motivate this choice, let us observe that \(J\) commutes with \(\omega\) and \(\Sigma\) anti-commutes with \(\sigma\). For our purposes, the form \(\left(\Omega = \Sigma\right)\) in any coordinate system of our choice.

Given the (Kähler) metric \(g\) and the complex structure \(J\) one defines a two-form \(\omega\) (the Kähler form) by the equation \(\omega(u,v) = g(u,Jv)\) for any two tangent vector fields \(u, v\). In our case the induced action of \(J\) on the one-form \((T^{-1}dT)^p\) is given by

\[
(T^{-1}dT)^p \to i\Sigma_1^p (T^{-1}dT)^p = -i(T^{-1}dT)^p \Sigma_1^p .
\]

Given the (Kähler) metric \(g\) and the complex structure \(J\) one defines a two-form \(\omega\) (the Kähler form) by the equation \(\omega(u,v) = g(u,Jv)\). In the present case we find

\[
\omega \equiv \omega^\sigma = -i\text{Str}(\Sigma_1^p (T^{-1}dT)^p \wedge (T^{-1}dT)^p) = i d\text{Str}(\Sigma_1^p T^{-1}dT) .
\]

For our purposes, the form \(\omega\) is a useful object to introduce because \(\omega\) is easier to express than the metric \(g\) and yet carries enough information to construct the Berezin measure \(\Omega = D\mu(Q^\sigma)\). In fact, if \(\omega\) is expressed in coordinates as

\[
\omega = \frac{1}{2} \left( \tilde{A}_{ij} dx^i \wedge dx^j + \tilde{B}_{ik} dx^i \wedge d\xi^k + \tilde{C}_{ij} dx^i \wedge d\xi^j + \tilde{D}_{kl} d\xi^k \wedge d\xi^l \right) ,
\]
where \( \tilde{A}_{ij} = -\tilde{A}_{ji}, \tilde{D}_{kl} = \tilde{D}_{lk}, \) and \( \tilde{B}_{ik} = -\tilde{C}_{ki} \) (due to skewness of the wedge product), then we have
\[
\text{SDet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{SDet} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}
\] (51)
as a consequence of the properties of the complex structure \( J \) relating the metric \( g \) with the Kähler form \( \omega \).

We are now in a position to compute \( D\mu(Q^x) \) with ease. Inserting the parametrization \([10]\) into \([9]\) we obtain
\[
\omega^\sigma = i d \text{STr}(\Sigma_1^x T_b^{-1} d T_b) + i d \text{STr}(T_b \Sigma_1^x T_b^{-1} T_f^{-1} d T_f).
\] (52)
The term \( i d \text{STr}(\Sigma_1^x T_b^{-1} d T_b) = i d \text{Tr}(\sigma_1 W^{-1} d W) = d(\sinh s) \wedge du \) was already computed in \([21]\). The new term is
\[
i d \text{STr}(T_b \Sigma_1^x T_b^{-1} T_f^{-1} d T_f) = d(\sinh s) \alpha \wedge d\beta + i \sigma \beta d\alpha = (i\sigma + \sinh s) d\alpha \wedge d\beta + \alpha d(\sinh s) \wedge d\beta.
\] (53)
The last summand makes no contribution to the superdeterminant of the metric tensor (since \( \sigma = 0 \)) and therefore can be dropped for the purpose of constructing \( D\mu(Q^x) \). The term proportional to \( d\alpha \wedge d\beta = d\beta \wedge d\alpha \) contributes the reciprocal of the analytic square root of \( -(i\sigma + \sinh s)^2 \). Thus from Eqs. \([51]\) and \([55]\) we have
\[
D\mu(Q^x) = \frac{d(\sinh s) d\alpha}{1 - i\sigma \sinh s}.
\] (54)

Please be advised that the symbol \( d\alpha \) in this expression means the derivative \( d\alpha \equiv \partial / \partial \alpha \), although its meaning in the previous equation was that of a differential. Thus we are using the same symbol \( d\alpha \) for two very different objects.

### III. MICROSCOPIC LIMIT OF THE PARTITION FUNCTION

In this section we evaluate the partition function \([38]\) in two different limits. First, we extract the contribution that diverges as \( \log \epsilon \) for \( z_f^\sigma \neq z^* \), and second, we compute the result for \( z_f^\sigma = z^* \) which is regular for \( \epsilon \to 0 \).

#### A. Contribution of order \( \log \epsilon \)

Using \([43]\) for the mass term and the expression \([42]\) for \( \text{STr}(Q \Sigma_2)^2 \), the partition function \([38]\) becomes
\[
Z_{-1}(z_f^\sigma | z, z^*; a) \sim i^{-N} e^{-\frac{2}{3} N a^2} \sum_{\sigma = \pm 1} a^N e^{-iN \sigma z_f^\sigma} \int d(\sinh s) d\alpha \wedge d\beta \times e^{-2 N \epsilon \cosh s \cosh u + N \epsilon (z^* - z^\sigma_f) \sinh s - 2 N a^2 \sinh^2 s + i N \alpha \beta (z^* - z^\sigma_f)(i \sinh s - \sigma) + i \epsilon e^s \cosh s}. \]
(55)
Here we display only the factors of alternating phase; the full overall normalization factor will be inserted below.

The integral over the Grassmann variables \( \alpha, \beta \) yields a factor
\[
N \epsilon e^{u \cosh s} + N \sigma (z^* - z^\sigma_f)(1 - i \sigma \sinh s).
\] (56)
Since the rest of the integrand is even in \( u \) we may replace \( e^u \) in this expression by \( \cosh u \). The \( u \)-integral with the resulting term \( N \epsilon \cosh u \cosh s \) is finite in the limit \( \epsilon \to 0 \) (see next subsection). Therefore we may drop this term here, as we are after the singular contribution \( \propto \log \epsilon \). The \( u \)-integral over the remaining term has the asymptotics
\[
\int du e^{-2 N \epsilon \cosh s \cosh u} = 2 |\log \epsilon| + \mathcal{O}(\epsilon^0).
\] (57)
Doing finally the \( s \)-integral by completing the square we obtain the leading term
\[
Z_{-1}(z_f^\sigma | z, z^*; a) = c_{N} |\log \epsilon| a^{-1} (z^* - z_f^\sigma) e^{-\frac{2}{3} N a^2 \sinh(N z_f^\sigma + N \pi/2)} + \mathcal{O}(\epsilon^0).
\] (58)
The normalization constant \( c_{N} \) is found by keeping track of all constants in the calculations above. Using the formula \( \text{vol} U(N)/\text{vol} U(N - 1) = (2\pi)^N/(N - 1)! \) and Stirling’s approximation for the factorial, we find
\[
c_{N} = e^{N/2}(N/\pi)^{1/2}.
\] (59)
This result \([58]\) will be verified by taking the microscopic limit of the exact finite-\( N \) results that will derived in Section \([IV]\) by means of superbosonization and in Section \([VI]\) using the method of complex orthogonal polynomials.
B. Contribution of order $\epsilon^0$

For $z_j^* = z^*$ the logarithmic singularity $|\log \epsilon|$ vanishes. Indeed, doing the Grassmann integrals over $\alpha, \beta$ and the $u$-integral and then sending $\epsilon \to 0$ we obtain a finite limit

$$
\lim_{\epsilon \to 0^+} \epsilon N \cosh s \int du \, e^{u - 2\epsilon N \cosh s \cosh u} = 1.
$$

(60)

With the substitution $q \equiv \sinh s$ the expression for the partition function now becomes

$$
Z_{-1}(z^*|z, z^*; a) \sim e^{-2N\alpha^2} \sum_{\sigma = \pm 1} e^{-i\sigma N(z^* + \pi/2)} \int \frac{dq}{1 - i\sigma q} e^{N(z - z^*)q - 2Na^2q^2}.
$$

(61)

Introducing an auxiliary integration by $(1 - i\sigma q)^{-1} = \int_0^\infty e^{-t(1 - i\sigma q)} dt$ we can do the Gaussian integral over $q$ by completing the square. The result of this step is immediately expressed in terms of the complementary error function:

$$
\int \frac{dq}{1 - i\sigma q} e^{N(z - z^*)q - 2Na^2q^2} = \pi e^{2N(a^2 + \sigma \text{Im} z)} \text{erfc} \left( a\sqrt{2N} + \frac{\sigma N \text{Im} z}{a\sqrt{2N}} \right),
$$

(62)

which is defined by $\text{erfc}(x) = 1 - \text{erf}(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-t^2} dt$. Our final result reads

$$
Z_{-1}(z^*|z, z^*; a) = e^{\frac{N}{2}(1 + a^2)} 2^{-3/2} \sum_{\sigma = \pm 1} e^{-i\sigma N(z + \pi/2)} \text{erfc} \left( a\sqrt{2N} + \frac{\sigma N \text{Im} z}{a\sqrt{2N}} \right).
$$

(63)

This expression agrees with the result obtained from the microscopic limit of the finite-$N$ results that will be derived by means of superbosonization in the next section, and by using the Cauchy transform of orthogonal polynomials in Section VI.

IV. SUPERBOSONIZATION

As discussed in the introduction, symmetry arguments alone are not sufficient to derive results for finite $N$. This requires an exact evaluation of the partition function which, in this section, is achieved by the method of superbosonization. In the next section we will obtain exact finite-$N$ results by means of complex orthogonal polynomials.

The superbosonization method was introduced to address problems with non-Gaussian disorder [21, 22, 23, 24]. Its main idea is to reduce an integral with symmetries to a lower-dimensional integral. To give a simple example illustrating this general idea, consider a function $f$ of complex variables $z^1, \ldots, z^N$. If $f$ depends only on $x = \sum_{k=1}^N |z_k|^2$ then the integral of $f$ over a $U(N)$-invariant domain in $\mathbb{C}^N$ can be reduced to an integral over just $x$. Similarly, the Grassmann integral of a function $f(\sum_{k=1}^N \bar{\psi}^k \psi^k)$ of anti-commuting variables $\psi^k$ and $\bar{\psi}^k$ is known to be expressible as an integral of $f(y)$ over $y \in U(1)$.

Based on results from invariant theory, superbosonization extends this reduction idea to the general case of invariant functions of supervectors. In the bosonic sector, the method is equivalent to the one introduced in [3]. However, in [3] the fermionic degrees of freedom were bosonized in the usual way by means of a Hubbard-Stratonovich transformation (we will refer to this procedure as the hybrid method), whereas in the superbosonization approach the fermionic and bosonic variables are treated on equal footing. From our perspective, a major advantage of the superbosonization method is that the integration measure is given by a general formula which can be easily applied to a specific case such as the bosonic partition function considered in this paper. In order to execute the integrals, it is essential that the parametrization be chosen judiciously.

The present partition function has also been worked out in a straightforward way using the hybrid method of [3]. That calculation is not more complicated than the superbosonization method, but since it does not provide us with any additional insights we will not discuss the hybrid method any further.

The starting point for superbosonization of the regularized FKS partition function [4] is the representation of the inverse determinant as

$$
\text{Det}^{-1} \left( \begin{array}{c} \phi^* \phi \\ z^* + H - A \end{array} \right) = \int d\phi \, d\phi^* \, e^{-\varepsilon(z^* + H - A)\phi^* \phi} e^{i\phi_+ (z_0 \delta_{kl} + H_{kl} + A_{kl}) \phi_-^* + i\phi_- (z^*_0 \delta_{kl} + H_{kl} - A_{kl}) \phi_+^*}
$$

(64)

and the fermion determinant as

$$
\text{Det}(z^*_0 + H - A) = \int d\psi \, d\bar{\psi} \, e^{-\varepsilon(z^*_0 \delta_{kl} + H_{kl} - A_{kl}) \psi^\dagger}.
$$

(65)
Taking the average over the Gaussian distribution \( | \) of \( H \) and \( A \) we obtain
\[
\left\langle e^{iH_{ab}(b^+ b_- + b_- b^+) - \tilde{\psi} b^+ - b^+ \tilde{\psi}} \right\rangle_H = e^{-(1/2N) \text{Str} \ Q \Sigma_1 Q \Sigma_1},
\]
(66)
\[
\left\langle e^{iA_{ab}(b^+ b_- + b_- b^+) + \tilde{\psi} b^+ - b^+ \tilde{\psi}} \right\rangle_A = e^{-(a^2/2N) \text{Str} \ Q \Sigma_2 Q \Sigma_2},
\]
(67)
where \( Q \) is the supermatrix \( [18] \) of \( U(N) \)-invariant bilinears, and the matrices \( \Sigma_1 \) and \( \Sigma_2 \) were defined in \( [31] \) and \( [39] \). The quadratic terms in the exponents of (64) and (65) combine to make up the mass term:
\[
e^{-\phi_{+}^2 - \phi_{-}^2 + \phi_{+}^* \phi_- + i \phi_{+}^* \phi_- + i \phi_{-}^* \phi_+ - i \phi_{-}^* \phi_+} = e^{i \text{Str} \ Q \zeta Q}.
\]
(68)
The method of superbosonization allows us now to introduce the matrix elements of \( Q \) directly as the new variables of integration. Using a formula proved in \( [20] \) the partition function (after rescaling \( Q \rightarrow NQ \)) reduces to
\[
Z_{\rightarrow \bar{\chi} z^* | z, z^*; a} = (2\pi i)^{-N} \int_{U(N)} \frac{\text{vol} U(N)}{\text{vol} U(N-1)} DQ \text{SDet}^N(NQ) e^{iN \text{Str} \ z_{\bar{\chi}} - (N/2) \text{Str} Q \Sigma_1 Q \Sigma_1 - (Na^2/2) \text{Str} Q \Sigma_2 Q \Sigma_2}.
\]
(69)
By the superbosonization step of passing to (69), the precise meaning of \( Q \) has been transformed: \( Q \) is now the supermatrix
\[
Q = \begin{pmatrix} Q_{bb} & Q_{fb} \\ Q_{fb} & Q_{ff} \end{pmatrix} \equiv \begin{pmatrix} X_{++} & X_{+-} & \xi_+ \\ X_{-+} & X_{--} & \xi_- \\ \eta_+ & \eta_- & y \end{pmatrix},
\]
(70)
where the boson-boson block \( Q_{bb} \equiv X \) is a positive Hermitian matrix, \( Q_{ff} \equiv y \in U(1) \) is a unitary number, and the components of \( Q_{fb} \equiv \eta = (\eta_+ \eta_-) \) and \( Q_{bf} \equiv \xi = (\xi_+ \xi_-) \) are Grassmann variables. The Berezin measure is \( [20, 21] \)
\[
DQ = (4\pi i)^{-1} d^4 X d\eta d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta.
\]
(71)
Following the conventions of \( [20] \) we normalize the flat measure \( d^4 X \) so that \( \lim_{\delta \to 0} \delta^{-2} \int_{X > 0} e^{-\delta/2 (X - 1)^2} d^4 X = 1 \). Note that \( DQ \) is scale-invariant; it is also invariant under the transformation \( Q \Sigma_1 \mapsto Q \Sigma_1' \) for \( \Sigma_1' \) given in (35).

The expression (69) for \( Z_{\rightarrow} \) is suitable for saddle-point analysis in the limit \( N \rightarrow \infty \). Since we are considering the microscopic limit where \( N \zeta \) and \( Na^2 \) are held fixed as \( N \) goes to infinity, the symmetry-breaking terms are subleading in \( 1/N \) and can be temporarily neglected for the purpose of finding the saddle-point manifold. If we set \( \zeta = 0 \) and \( a^2 = 0 \), the logarithm of the integrand becomes
\[
N \text{Str} \ \log Q = \frac{N}{2} \text{Str} Q \Sigma_1 Q \Sigma_1,
\]
(72)
variation of which gives the saddle-point equation \( -1 = \Sigma_1 Q \Sigma_1 \). The solutions of this equation form the two disjoint supermanifolds \( Q = T \Sigma_1 T^{-1} \Sigma_1 \) and \( Q = T \Sigma_1 T^{-1} \Sigma_1 \) which were described in (39). Note that the signs of the solution in the boson-boson sector are fixed by the condition \( Q_{bb} > 0 \).

Next we perform the integration over the massive modes in the large-\( N \) limit. To handle both saddle-point manifolds at once, we recall our notation \( \Sigma_1 = \Sigma_1 \) for \( \sigma = +1 \) and \( \Sigma_1 = \Sigma_1 \) for \( \sigma = -1 \), and we set
\[
Q = T e^{p^i \Sigma_1 T^{-1} \Sigma_1},
\]
(73)
where the matrix \( P \) parameterizes the massive modes. By the very definition of what it means to be a massive mode, \( P \) commutes with \( \Sigma_1 \) in both cases. To do the integral over \( P \) to leading order in \( 1/N \) we may put \( P \) equal to zero in the symmetry-breaking terms with parameters \( a^2 \) and \( \zeta \) (we remind the reader that both \( a^2 \) and \( \zeta \) are of order \( 1/N \)). Thus we need to integrate
\[
\text{SDet}^N(Q) e^{-(N/2) \text{Str} (\Sigma_1)^2} = \sigma^N e^{N \text{Str} P - (N/2) \text{Str} e^{2P}} = \sigma^N e^{(N/2) - 2N \text{Str} P^2 + \ldots}.
\]
(74)
We see that the fluctuations of the massive modes are of the order \( P \sim 1/\sqrt{N} \). Because of the smallness of these fluctuations we may replace the nonlinear Berezin measure for the \( P \)-variables by the flat Berezin measure \( D_P \) (i.e., the product of differentials for the commuting variables and derivatives for the anti-commuting variables). Thus we have \( DQ \sim D_P D_P D_P(Q^*) \) where \( Q^* = T \Sigma_1 T^{-1} \Sigma_1 \) and \( D_P(Q^*) \) is the Berezin measure which is invariant under the transformation \( Q_1 \mapsto Q_1 T Q_1 T^{-1} \). The integral over the massive modes \( P \) then is a simple Gaussian integral \( D_P \). Doing it we immediately arrive at the result (38) of the previous section. Moreover, we are now in principle able to determine the precise normalization constant. We will insert the correct overall normalization when evaluating the partition function below.
V. EXACT CALCULATION USING SUPERBOSONIZATION

We now use the result \((69)\) from superbosonization to derive an exact expression for finite \(N\). To that end we start from the formula for the superdeterminant,

\[
\text{SDet } Q = \text{SDet} \begin{pmatrix} X & \xi \\ \eta & y \end{pmatrix} = \frac{\text{Det}(X)}{y - \eta X^{-1} \xi} ,
\]

(75)

where \(\eta X^{-1} \xi\) means the scalar which is obtained by sandwiching the matrix \(X^{-1}\) between the row vector \(\eta\) and the column vector \(\xi\). In view of Eq. \((76)\), the factor \(\text{SDet}^N(Q)\) of the integrand of \((69)\) is much simplified by making a shift \(y \rightarrow y + \eta X^{-1} \xi\). Such a shift leaves the integral over \(y \in U(1)\) invariant: \(\int_{U(1)} f(y) \, dy = \int_{U(1)} f(y + \eta X^{-1} \xi) \, dy\). After this shift, our integrand depends on the anti-commuting variables only through the following factor:

\[
\Phi = e^{N \eta(A+(1-a^2)\eta)X^{-1} \xi + \frac{\eta^2}{2}(1-a^2)(\eta X^{-1} \xi)^2} , \quad A \equiv A(X) = \sigma_1 X - ia^2 \sigma_2 X - iz_1^2 .
\]

(76)

Using the relation \(\frac{1}{2}(\eta X^{-1} \xi)^2 = \text{Det}^{-1}(X) \eta_+ \xi_+ - \xi_- \) we now carry out the integral over the anti-commuting variables to obtain \(\int d\xi_+ d\eta_+ d\xi_- d\eta_- \Phi = \text{Det}^{-1}(X) F(X, y)\) where

\[
F(X, y) = N(1 - a^2) + N^2 \left( \text{Det } A(X) + (1 - a^2) y \text{ Tr } A(X) + (1 - a^2) y^2 \right) .
\]

(77)

We insert this into the integral representation \((69)\) of the partition function to get

\[
Z_{-1}(z^*; z, z^*; a) = \frac{i^{-N-1} N^N}{4\pi^2(N-1)!} \int_{X>0} d^4 X \text{Det}^{N-2}(X) \int_{U(1)} dy \left. y^{-N+1} F(X, y) \right|_{X_0} \times e^{-c \text{ Tr } X+iNY \text{ Tr } (X\sigma_1 \text{ Re } z+X\sigma_2 \text{ Im } z)-\frac{\eta}{2} \text{ Tr } (X\sigma_1 X\sigma_1 +a^2 X\sigma_2 X\sigma_2)+\frac{1}{2}(1-a^2)y^2} .
\]

(78)

Next we do the \(y\)-integral. For this purpose we introduce the \(a\)-dependent functions

\[
\tilde{h}_{N,k}(\tau) := \frac{(1 - a^2)^k}{2\pi} \int_{U(1)} dy \left. (iy)^{-N+1+k} e^{-i\tau y + \frac{\eta}{2}(1-a^2)y^2} \right|_{X_0} .
\]

(79)

which will be shown presently to be scaled Hermite polynomials. With this definition, what remains to be done is an integral over the positive Hermitian \(2 \times 2\) matrices \(X\):

\[
Z_{-1} = \frac{N^N + 2}{2\pi(N-1)!} \int_{X>0} d^4 X e^{-c N \text{ Tr } X+iNY \text{ Tr } (X\sigma_1 \text{ Re } z+X\sigma_2 \text{ Im } z)-\frac{\eta}{2} \text{ Tr } (X\sigma_1 X\sigma_1 +a^2 X\sigma_2 X\sigma_2)} \times \text{Det}^{N-2}(X) \left( \left. (\tilde{h}_{N,2}(Nz^*_1) + i \text{ Tr } A(X) \tilde{h}_{N,1}(Nz^*_1) - (\text{Det } A(X) + N^{-1}(1-a^2)) \tilde{h}_{N,0}(Nz^*_1) \right) \right) .
\]

(80)

To compute the \(X\)-integral one may use the parametrization

\[
X = \begin{pmatrix} e^u \sqrt{p+v^2+w^2} & v-iw \\ v+iw & e^{-u} \sqrt{p+v^2+w^2} \end{pmatrix} , \quad (u, v, w \in \mathbb{R}, p \in \mathbb{R}+) .
\]

(81)

The integration measure in these coordinates is expressed by

\[
d^4 X = 2 dp du dv dw ,
\]

(82)

and some traces appearing in the exponent of the integrand are

\[
\frac{1}{2} \text{Tr } (X\sigma_1 X\sigma_1) = p + 2 v^2 , \quad \frac{1}{2} \text{Tr } (X\sigma_2 X\sigma_2) = p + 2 w^2 .
\]

(83)

A notable feature here is that the variable \(u\) occurs only in the factor \(e^{-c N \text{ Tr } X} = e^{-2c N \sqrt{p+v^2+w^2} \cosh u}\). Thus the integral over \(u\) for fixed \(\beta := 2c N \sqrt{p+v^2+w^2} \neq 0\) can be carried out and yields the hyperbolic Bessel function

\[
\int_0^\infty e^{-\beta \cosh u} du = K_0(\beta) .
\]

(84)
Comparing the integrals (86) and (79) we read off the relation
\[ H_n(x) = \left( \frac{d^n}{dy^n} e^{-y^2+2xy} \right) \bigg|_{y=0} = \frac{i^n n!}{2\pi i} \int_{U(1)} dy \, y^{-n-1} e^{-2ixy+y^2}, \]
where we have used Cauchy’s formula \( (d^n f/dy^n)(0) = (2\pi i)^{-1} n! \int_{U(1)} f(y) y^{-n-1} dy \). We ultimately want to take the limit \( N \to \infty \). To get a good view of the large-\( N \) asymptotics we introduce the scaled Hermite polynomials
\[ h_n(\tau) = (-1)^n (2e/n)^{-n/2} n!^{-1} H_n(\tau/\sqrt{2n}) = \frac{i^{-n}}{2\pi i} \int_{U(1)} dy \, y^{-n-1} e^{-i\tau y + (n/2)(y^2-1)}. \]
By a saddle-point computation of the \( U(1) \) integral, these polynomials have the large-\( n \) behavior
\[ h_n(\tau) \simeq (n\pi)^{-1/2} \cos(\tau + n\pi/2). \]
Comparing the integrals (80) and (79) we read off the relation
\[ \hat{h}_{N,k}(\tau) = (1 - a^2)^{k+n/2} (e\nu/n)^{n/2} h_n \left( \tau \sqrt{\frac{n}{N(1-a^2)}} \right), \quad n = N - 2 - k. \]
A further simplification of Eq. (80) is now achieved by the 3-term recursion formula
\[ N \hat{h}_{N,2}(\tau) + \tau \hat{h}_{N,1}(\tau) + (N - 2)(1 - a^2) \hat{h}_{N,0}(\tau) = 0, \]
which results from partially integrating \((N(1-a^2) - d/dy) e^{(1-a^2)y^2} = 0\) against \( e^{-i\tau y} y^{-N + 2} dy \). Using the identity (80) to eliminate the \( \hat{h}_{N,2} \) term from (80) we arrive at
\[ Z_{-1} = \frac{2N^{N+2}}{\pi(N-1)!} \int_0^\infty dp \, p^{N-2} e^{-N(1+a^2)p} \int_{\mathbb{R}} dv \int_{\mathbb{R}} dw \, e^{2iN(v\Re w + w \Im v) - 2N(v^2 + a^2w^2)} K_0(2\nu \sqrt{p + v^2 + w^2}) \times \left( 2iv + 2a^2w + z_\tau^* \right) \hat{h}_{N,1}(Nz_\tau^*) + \left( z_\tau^*(2iv + 2a^2w + z_\tau^*) + p + 1/a^4 - 1 + N^{-1} \right) \hat{h}_{N,0}(Nz_\tau^*). \]
This expression for the partition function is exact for all matrix dimensions \( N \geq 2 \). (It is, however, false for \( N = 1 \) because the superbosonization formula fails in that case; see the discussion in [20, 21].)

### A. Calculation of the \( \log \epsilon \) term

We now extract from the integral representation (80) the term which is singular in the limit \( \epsilon \to 0 \). For \( \epsilon \to 0 \) we may replace the hyperbolic Bessel function (84) by its leading logarithm,
\[ K_0(\beta) \simeq -\log(\beta/2) = -\log \epsilon - \log \left( N \sqrt{p + v^2 + w^2} \right), \]
where we keep only the singular term \( \log \epsilon \) for now. The integrals over the variables \( v \) and \( w \) then become Gaussian with mean values \( \langle v \rangle = \frac{1}{2} \Re z \) and \( \langle w \rangle = \frac{1}{2} \Im z \) and variances \( \text{var}(v) = (4N)^{-1} \) and \( \text{var}(w) = (4Na^2)^{-1} \). The remaining integral over \( p \) after scaling \( p \to N^{-1} (1 + a^2)^{-1} p \) yields the gamma function \( \Gamma(N - 1) = \int_0^\infty p^{N-2} e^{-p} dp \) and a similar term with \( N - 1 \) replaced by \( N \). Altogether we obtain
\[ \lim_{\epsilon \to 0} |\log \epsilon|^{-1} Z_{-1} = \frac{N^2 e^{-\frac{2i}{N} \Re z + \frac{2i}{N} \Im z/a}}{(N-1)(1 + a^2)^{N+1}} \left( z_\tau^* - z^* \right) \left( \hat{h}_{N,1}(Nz_\tau^*) + z_\tau^* \hat{h}_{N,0}(Nz_\tau^*) \right). \]
In view of the asymptotic behavior (77) it is clear that this will tend to a good limit for \( N \to \infty \) when the product \( Nz_\tau^* \) is kept fixed. Inserting the definition of the polynomials \( \hat{h}_{n,k} \) and using the recursion relation
\[ H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \]
with \( n = N - 2 \), we find the simplified expression
\[ \lim_{\epsilon \to 0} |\log \epsilon|^{-1} Z_{-1} = a^{-1} (z_\tau^* - z^*) e^{\frac{2i}{N} \Re z - \frac{2i}{N} \Im z/a} C_N(a) (-1)^N H_{N-1}(bz_\tau^*), \]
where the normalization constant \( C_N(a) \) and scale factor \( b \equiv b(a) \) are given by

\[
C_N(a) = (2b(1 + a^2))^{-N+1} \frac{N^N}{(N-1)!}, \quad b = \sqrt{\frac{N}{2(1-a^2)}}.
\]  

(95)

This closed-form expression for the \( \log \epsilon \) contribution is still exact for all matrix dimensions \( N \geq 2 \).

Let us check that this result is consistent with the expression \( 65 \) obtained in the large-\( N \) limit. For that we observe that the Hermite polynomials \( H_n(x) \) for \( \sqrt{n}x \) fixed and \( n \to \infty \) are asymptotic to

\[
(-1)^n H_n(x) \approx \sqrt{2}(2n/e)^{n/2} \cos(\sqrt{2n}x + n\pi/2).
\]  

(96)

Recalling that in the microscopic limit we send \( N \to \infty \) while keeping \( Na^2, Nz^* \) and \( Nz_f^* \) fixed, we then see that the microscopic limit given in \( 65 \) is precisely reproduced.

### B. Contribution of order \( \epsilon^0 \)

We now set \( z_f^* = z^* \) and compute the \( \epsilon^0 \) contribution to the partition function. This contribution is given by the \( \log(p + v^2 + w^2) \) term in the expansion \( 91 \) of the hyperbolic Bessel function. (From the preceding section we know that the constant terms in the expansion of \( K_0(\beta) \) yield zero for \( z_f^* = z^* \).) To facilitate the computation, we write

\[
-2 \log \sqrt{p + v^2 + w^2} = \lim_{\delta \to 0} \left( \int_0^\infty \frac{dr}{r} e^{-r(p+v^2+w^2)} + \log \delta + \gamma \right),
\]  

(97)

where \( \gamma \) is Euler’s constant. The singular constant \( \log \delta \) and \( \gamma \) can be dropped as they make no contribution for \( z_f^* = z^* \). The integrals over \( v, w, \) and \( p \) can then be carried out as before, and the resulting limit \( \delta \to 0 \) exists. The order \( \epsilon^0 \) contribution is thus given by

\[
Z_{-1}(z^*|z, z^*; a) = \left( \frac{N}{2}(1 - a^2) \right)^{N-1} \int_0^\infty dr \frac{e^{-(2N+r)^{-1}Re^2(Nz) - (2Na^2+r)^{-1}Im^2(Nz)}}{(1+a^2 + N^{-1}r)^{N-1} \sqrt{(2N+r)(2Na^2+r)}} \times (-1)^N \frac{N^3}{N!} \left( \frac{N}{2b} \frac{Re z}{2N+r} - \frac{i \Im z}{2Na^2+r} \right) H_{N-1}(bz^*) - \frac{(N-1)(1-a^2)}{N(1+a^2+r)} H_{N-2}(bz^*)).
\]  

(98)

This is the exact finite-\( N \) result for the bosonic partition function. Using the orthogonal polynomial approach it can also be obtained from the Cauchy transform of the fermionic partition function \( 33 \).

Let us take once again the microscopic limit \( (N \to \infty, \text{with } Nz^*, Nz \text{ and } Na^2 \text{ fixed}) \). Doing so in Eq. \( 98 \) we get

\[
Z_{-1} = \frac{e^{N^2/2Na^2}}{\sqrt{2\pi}} \int_0^\infty dr e^{-2Na^2+r} \left( \frac{1}{2N} - \frac{1}{2Na^2+r} \right) \frac{\Im(Nz^*)}{\sin(Nz^* + N\pi/2)}.
\]  

(99)

To check this result we invoke the following identity:

\[
\sqrt{\pi} \int_0^\infty dr e^{-2Na^2+r} = \int_{\mathbb{R}} dq \frac{e^{-2Na^2q^2+2iNq\Im z}}{1+q^2},
\]  

(100)

and a second identity of the same kind which is obtained by differentiating both sides of \( 100 \) with respect to \( \Im z \). Using Euler’s formula \( \cos \theta + i\sin \theta = e^{i\theta} \) we then immediately recover Eq. \( 111 \).

### VI. CALCULATION OF \( Z_{-1} \) USING COMPLEX ORTHOGONAL POLYNOMIALS

In this section we first derive the partition function with one bosonic quark from the Cauchy transform of orthogonal polynomials. In this approach no regularization procedure is required. For comparison with the \( \sigma \)-model we also compute the partition function with one fermionic quark and two conjugate bosonic quarks, which diverges as \( \log \epsilon \).

To apply the method of complex orthogonal polynomials to the FKS model we first express the Gaussian probability distribution for \( H \) and \( A \) given in Eq. \( 2 \) in terms of the eigenvalues \( z_k \) of \( H+A \) and \( z_k^* \) of \( H-A \). The joint
distribution for the eigenvalues was calculated in [12]. Including the expression for the exponent, which follows from
the decomposition
\begin{equation}
\text{Tr} HH^\dagger + \frac{1}{a^2} \text{Tr} AA^\dagger = \frac{1 + 1/a^2}{2} \text{Tr} (H + A)(H^\dagger + A^\dagger) + \frac{1 - 1/a^2}{4} \text{Tr}((H + A)^2 + (H^\dagger + A^\dagger)^2),
\end{equation}
the joint eigenvalue distribution function is given by [12]
\begin{equation}
P(\{z_k, z^*_k\}) = C \prod_{k<l} |z_k - z_l|^2 e^{-\frac{1}{4}(1 + 1/a^2) \sum_k |z_k|^2 - \frac{1}{4}(1 - 1/a^2) \sum_k (z_k^2 + z_k^* z_l^2}).
\end{equation}

The partition functions (and correlation functions) of the FKS model can now be derived by means of the method of
complex orthogonal polynomials with polynomials \(p_n(z)\) defined through
\begin{equation}
\int d^2 z \ w(z, z^*; a) p_k(z) p_l(z^*) = r_k \delta_{kl}
\end{equation}
and weight function given by [40]
\begin{equation}
w(z, z^*; a) = e^{-\frac{1}{4}(1 + 1/a^2)|z|^2 - \frac{1}{4}(1 - 1/a^2)(z^2 + z^* z^2)}.
\end{equation}

These polynomials, which have been known for some time [34, 35, 36], are given by
\begin{equation}
p_n(z) \sim H_n(bz), \quad b = \sqrt{\frac{N}{2(1 - a^2)}},
\end{equation}
where \(H_n\) are the Hermite polynomials. The \(p_k\) are in monic normalization with respect to \(z\). The leading \(a\)-dependence of \(r_k\) for large \(k\) is given by
\begin{equation}
r_k \sim e^{\frac{1}{2} b a^2}.
\end{equation}

In this section, we will not keep track of numerical and \(a\)-dependent prefactors.

General expressions for partition functions in terms of complex orthogonal polynomials have been given in [37, 38, 39]. Below we derive the explicit expressions for the microscopic limit of \(Z_{-1}(z^* | z, z^*; a)\) and \(Z_{-1}(z^*_f | z, z^*; a)\).

A. The partition function \(Z_{-1}(z^* | z, z^*; a)\)

The partition function with one boson, \(Z_{-1}(z^* | z, z^*; a)\), can be expressed as a Cauchy transform [37, 38, 39]
\begin{equation}
Z_{-1}(z^* | z, z^*; a) = \frac{1}{r N_{z-1}} \int d^2 z' w(z', z^*; a) p_{N-1}(z'^*) \frac{1}{z - z'}.
\end{equation}

In the microscopic limit where \(Nz\) and \(Na^2\) are kept fixed for \(N \to \infty\), the Hermite polynomials can be replaced by
their asymptotic limit [40] and the weight function reduces to
\begin{equation}
w(z, z^*; a) \sim e^{-\frac{N \text{Im}^2(z)}{2a^2}}.
\end{equation}

We thus find
\begin{equation}
Z_{-1}(z^* | z, z^*; a) \sim (-i)^{N-1} e^{-\frac{N a^2}{2}} \int dx' dy' e^{-\frac{N a^2}{2} \frac{i N z'^*}{z' - x' - iy'}}.
\end{equation}
The integral over \(x'\) can be performed by a contour integration, whereas the remaining integral over \(y'\) can be expressed
in terms of the complementary error function. This leads to the expression
\begin{equation}
Z_{-1}(z^* | z, z^*; a) \sim (-i)^N e^{-\frac{N a^2}{2}} \int dy' e^{-\frac{N a^2}{2z}} \left(e^{i N z \theta}(e^{2 N y' \theta}(-y' + \text{Im} z) + (-1)^N e^{-i N z} e^{-2 N y' \theta}(y' - \text{Im} z))\right) \sim e^{\frac{1}{2} N a^2} e^{i N (z + \pi/2)} \text{erfc} \left(\frac{2 N a^2 - N \text{Im} z}{\sqrt{2N a}}\right) + e^{\frac{1}{2} N a^2} e^{-i N (z + \pi/2)} \text{erfc} \left(\frac{2 N a^2 + N \text{Im} z}{\sqrt{2N a}}\right),
\end{equation}
in agreement with [63].
B. The partition function $Z_{-1}(z^*_j|z, z^*; a)$

In [19] it was shown that the singular part of the chiral random matrix partition function with a pair of conjugate bosonic quarks and $N_f$ fermionic flavors factorizes. The same reasoning can be applied to the FKS model resulting in

$$Z_{-1}^{(N)}(z^*_j|z, z^*; a) \sim (z^*_j - z^*) Z_1^{(N-1)}(z^*_j; a) Z_{pq-\text{bos}}(z, z^*; a) + O(\epsilon^0),$$

(111)

where the phase quenched bosonic partition function, $Z_{pq-\text{bos}}^{(N)}$, is defined in [3] and $Z_1^{(N-1)}$ is the partition function with one fermionic flavor. $Z_{pq-\text{bos}}^{(N)}$ is given by the weight function times $\log \epsilon$ [3, 19]

$$Z_{pq-\text{bos}}(z, z^*; a) = |\log \epsilon| \frac{w(z, z^*; a)}{r_{N-1}},$$

(112)

and the $N_f = 1$ theory can be expressed in terms of the orthogonal polynomials [15]

$$Z_1^{(N-1)}(z^*_j; a) \sim p_{N-1}(z^*_j).$$

(113)

Inserting this result and (112) into (111) we obtain

$$Z_{-1}^{(N)}(z^*_j|z, z^*; a) \sim (z^*_j - z^*) w(z, z^*; a) e^{-\frac{1}{2} N \alpha^2} H_{N-1}(b z^*_j) \log \epsilon + O(\epsilon^0),$$

(114)

in agreement with the exact finite-$N$ result in [21]. The factor $\exp(-3N\alpha^2/2)$ results from a factor $\exp(-N\alpha^2)$ from $r_{N-1}$ and a factor $\exp(-N\alpha^2/2)$ from the ratio of $p_{N-1}$ (in monic normalization) and $H_{N-1}$. In the microscopic limit this results in

$$Z_{-1}(z^*_j|z, z^*; a) \sim (z^*_j - z^*) e^{-\frac{1}{2} N \alpha^2} e^{-N \text{Im}(z)/(2\alpha^2)} \sin(N z^*_j + N \pi/2) |\log \epsilon| + O(\epsilon^0),$$

(115)

in agreement with the result [18] obtained earlier in this paper.

VII. CONCLUSIONS

We have analyzed the bosonic partition function of a Hermitian random matrix model deformed by a nonhermitian random matrix model. We have shown that the microscopic limit of the partition function can be obtained by essentially using symmetry arguments only. There are, however, several subtleties that deserve attention. First, the partition function has to be regularized by multiplication with a conjugate bosonic and conjugate fermionic determinant. Second, because fermionic degrees of freedom are present, two inequivalent saddle-point manifolds have to be taken into account. Third, convergence of the partition function leads to the boson-boson block of the manifold of the Goldstone degrees of freedom being a noncompact subset of the set of positive definite matrices.

The main advantage of the symmetry approach is that it gives a clear view at universality. Goldstone modes, which are separated from the rest of the excitation spectrum by a mass gap, decouple in the microscopic limit, and their mutual interactions are completely determined by the symmetries and the pattern of symmetry breaking of the microscopic partition function. This means that our results for the FKS model are valid for the microscopic limit of any model with the same symmetries and a mass gap.

To obtain results for finite-size matrices one has to perform a detailed calculation. We have presented results using two different methods: the superbosonization method and the complex orthogonal polynomial method. The disadvantage of the orthogonal polynomial method is that universality is not manifest at all stages of the calculation. In the superbosonization method the universal partition function is obtained after integrating out the massive modes, which is a trivial step when the proper coordinates are used. The orthogonal polynomial approach, which is applicable to invariant random matrix models, has as its main advantage that it can be generalized in a straightforward way to any number of flavors. We have also performed the calculation using a hybrid method where the four-fermion term is decoupled by means of the Hubbard-Stratonovich transformation. Since this calculation did not provide additional insights, we have refrained from presenting it in this paper.

For the present problem, the superbosonization approach does not have a clear advantage over the hybrid method, but in general we expect that it will be simpler to integrate out the massive modes if fermions and bosons are treated in a unified way. We also wish to stress that a major advantage of the superbosonization method is that it can deal with nongaussian probability distributions. Such distributions have important applications in, e.g., quantum gravity and growth phenomena. However, nongaussian perturbations do not affect the universal results obtained in
the microscopic limit. These are determined by symmetries and can be derived from symmetry arguments alone as we have shown in this paper.

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