Gauge Theory Wilson Loops and Conformal Toda Field Theory

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Abstract

The partition function of a family of four dimensional $\mathcal{N} = 2$ gauge theories has been recently related to correlation functions of two dimensional conformal Toda field theories. For $SU(2)$ gauge theories, the associated two dimensional theory is $A_1$ conformal Toda field theory, i.e. Liouville theory. For this case the relation has been extended showing that the expectation value of gauge theory loop operators can be reproduced in Liouville theory inserting in the correlators the monodromy of chiral degenerate fields. In this paper we study Wilson loops in $SU(N)$ gauge theories in the fundamental and anti-fundamental representation of the gauge group and show that they are associated to monodromies of a certain chiral degenerate operator of $A_{N-1}$ Toda field theory. The orientation of the curve along which the monodromy is evaluated selects between fundamental and anti-fundamental representation. The analysis is performed using properties of the monodromy group of the generalized hypergeometric equation, the differential equation satisfied by a class of four point functions relevant for our computation.
1 Introduction and Discussion

In a recent paper [1], Gaiotto has constructed a large class of four dimensional $\mathcal{N} = 2$ gauge theories that describe the low energy dynamics of a stack of $N$ M5-branes compactified on a punctured Riemann surface $C_{(f_n), g}$. The surface is characterized by the genus $g$ and the number of punctures ($f_n$). There are different types of puncture and each type is labeled by a Young tableaux with $N$ boxes. Following the construction in [1], it is possible to associate to any surface $C_{(f_n), g}$ a four dimensional gauge theory $\mathcal{T}_{(f_n), g}$ characterized by the same data labeling the surface. The different S-duality frames of the gauge theory correspond to the different ways of sewing the Riemann surface from pairs of pants.

In [2], Alday, Gaiotto and Tachikawa (AGT) have related the four dimensional theory $\mathcal{T}_{(f,g)}$ when $N = 2$ to two dimensional Liouville theory defined on the Riemann surface $C_{(f,g)}$. An important ingredient of the analysis is given by the the Nekrasov partition function $Z_{\text{Nekrasov}}$ [3][4], that is computed considering a two parameters deformation of the gauge theory and is written as $Z_{\text{Nekrasov}} = Z_{\text{classical}}Z_{1\text{-loop}}Z_{\text{instanton}}$, where the three factors are the classical, the quantum and the instanton contributions. Given a certain S-duality frame, $Z_{\text{instanton}}$ is identified with the BPZ [5] conformal block $F_{\alpha\beta}$ associated to the pants decomposition corresponding to the chosen S-duality frame. The $Z_{\text{instanton}}(\epsilon, \alpha, m)$ depends on the deformation parameters ($\epsilon_1, \epsilon_2$), the Coulomb branch coordinates $\alpha$ and the mass parameters $m$. These quantities are related respectively to the Liouville coupling constant $b$ and the momenta of the internal states $\alpha$ and external states $\beta$.

Using the results of Pestun [6], the partition function of the gauge theory defined on $S^4$ was identified with a Liouville correlation function, considering Liouville theory on the Riemann surface $C_{f,g}$ with coupling constant $b = 1$. It results

$$Z_{\mathcal{T}_{f,g}} = \int [da] Z_{\text{Nekrasov}} Z_{\text{Nekrasov}}^{-1} = \langle V_{\beta_1} \ldots V_{\beta_f} \rangle$$

(1)

where $V_{\beta_1} \ldots V_{\beta_f}$ are Liouville primary fields associated to the $f$ punctures of $C_{f,g}$ with momenta related to the mass parameters of the gauge theory. The relation (1) is satisfied because, besides the correspondence between the conformal blocks and $Z_{\text{instanton}}$ that we have already mentioned, it is possible to show that the factors $Z_{\text{classical}}$ and $Z_{1\text{-loop}}$ of the $Z_{\text{Nekrasov}}$ reproduce the product of three point functions of Liouville primary fields given by the DOZZ formula [7][8][9].

The AGT proposal was soon extended by Wyllard [10] to the $N > 2$ case. In this configuration the Liouville theory is replaced by the conformal $A_{N-1}$ Toda field theory.

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1In the $N = 2$ case, there is only one type on puncture and the gauge theory includes only $SU(2)$ gauge groups and $SU(2)$ flavor groups.

2See [2] for the precise dictionary.

3See for instance [11]. We quickly review Toda field theory in the next section.
a generalization of Liouville theory that is invariant under the $\mathcal{W}_N$ algebra, an extension of the Virasoro algebra. Given a generic punctured surface, the partition function of the corresponding gauge theory is given by a correlation function of the Toda primary fields associated to the punctures. Since in the $N > 2$ case there are many types of punctures, it is necessary to consider different types of primaries [10][12]. Despite the fact that Liouville theory is a $A_1$ Toda theory, the generalization from $N = 2$ to $N > 2$ is not straightforward because the higher rank Toda theories are much less understood than Liouville theory. First, the three point function of primary fields is known exactly only when one of the insertion is a certain degenerate field [13]. Furthermore, differently from Liouville ($A_1$ Toda), in the general case it is not possible to decompose the higher point correlation functions in terms of three point functions of $\mathcal{W}_N$ primary fields and $\mathcal{W}_N$ conformal blocks [14]. Due to these complications, it is not possible to write a differential equation for the four point function with only one degenerate insertion, like for the Liouville case. It is however possible to write a differential equation for a four point function with a degenerate and a semidegenerate field [13]. Other recent results concerning the relation between Toda theories and gauge theories are given in [15][16][17][18][19][20][21][22][23][24][25].

In [6], Pestun considers also the case where a circular supersymmetric Wilson loop is inserted in the partition function. It results that the expectation value of the supersymmetric circular Wilson loop in $\mathcal{N} = 2$ gauge theories is given by the following matrix model

\begin{equation}
\langle W_R \rangle = \int [da] Z_{\text{Nekrasov}} Z_{\text{Nekrasov}} \text{Tr}_R e^{2\pi ia},
\end{equation}

where $R$ is a representation of the gauge group. The Liouville theory description of line operators for theories with $SU(2)$ gauge groups was analyzed in [29][30]. The authors show that the insertion of a supersymmetric loop operator in the partition function of a $\mathcal{N} = 2$ gauge theory correspond in the Liouville theory to the monodromy of a chiral degenerate operator. With this prescription t’Hooft operators, dyonic operators and Wilson operators are treated in the same framework and the nature of the operators is completely encoded by the curve on $C_{f,g}$ along which the monodromy is computed. A precise dictionary between charges of loop operators and unoriented closed curves on $C_{f,g}$ was given in [31].

In this paper we study the two dimensional realization of circular supersymmetric Wilson loops in $SU(N)$ gauge theories with $N > 2$. In particular, we generalize the proposal of [29][30] showing that the fundamental and anti-fundamental Wilson loops in $SU(N)$ $T_{(f_h),g}$ theories are associated to monodromies of a particular chiral degenerate operators of $A_{N-1}$.

\footnote{In [19], Pestun derives also the matrix model for the supersymmetric circular Wilson loop in $\mathcal{N} = 4$ theory, proving the conjectures of [26][27].}

\footnote{Other aspects of Wilson loops in $\mathcal{N} = 2$ theories have been recently analyzed in [28].}

\footnote{For these theories $SU(N)$ is the gauge group with the highest rank but lower rank groups are also admitted in the quiver.}
Toda field theory. The analysis is performed using properties of the monodromy group of the generalized hypergeometric equation, that is the differential equation satisfied by Toda four point functions with two degenerate fields, i.e. the class of correlation function relevant for our computation. Our result is in agreement with Pestun’s results [2] when we set to one the Toda coupling constant. We show that Wilson loops in the fundamental and anti-fundamental representation of an SU(N) gauge group are obtained from the same degenerate operator evaluating the monodromy along curves with opposite orientation. This implies that loop operators in SU(N) \( T_{(f_a),g} \) gauge theories are associated to oriented curves on the Riemann surface \( C_{(f_a),g} \). The orientation for the curves is a new feature of the \( N > 2 \) case, since for the SU(2) gauge theories the orientation of the monodromy is not relevant because the representations of this group are real.

The rest of the paper is organized as follows. In section 2 we review \( A_{N-1} \) conformal Toda field theory, focusing in particular on the four point function with two degenerate insertions. We describe the generalized hypergeometric differential equation satisfied by this correlation function and review basic properties of its monodromy group. In section 3 we show that SU(N) Wilson loops in the fundamental and anti-fundamental representations are associated to monodromies of a particular chiral degenerate Toda field. Appendix A reviews basic properties of Lie algebras and Appendix B presents an explicit representation of the hypergeometric monodromy group.

Note: The Toda field theory description of SU(N) Wilson loops in any representation of the gauge group has been recently obtained in [32] using a different approach. We thank the authors for informing us of their results before publication.

### 2 \( A_{N-1} \) Conformal Toda Field Theory

We collect in this section some basic and known facts about \( A_{N-1} \) Toda field theory, following mostly [11]. The dynamical fields in the theory are a set of \( N - 1 \) scalars \( \varphi_i \) (\( i = 1, \ldots, N - 1 \)) that propagate on a two dimensional Riemann surface. The set of scalars form the components of an \( N - 1 \) vector \( \varphi \) defined in the root space of the \( A_{N-1} \) Lie algebra, i.e. \( \varphi = \sum_{k=1}^{N-1} \varphi_k e_k \) where \( e_k \) is a simple root.\(^7\) The action is

\[
S_{A_{N-1}} = \int dt \sqrt{g} \left( \frac{1}{8\pi} g^{\alpha\beta} (\partial_\alpha \varphi, \partial_\beta \varphi) + \frac{\langle Q, \varphi \rangle}{4\pi} R + \mu \sum_{k=1}^{N-1} e^{b(e_k, \varphi)} \right)
\]

where \( b \) is the dimensionless coupling constant, \( \mu \) is a constant called cosmological constant, \( g_{\alpha\beta} \) and \( R \) are the non-dynamical metric and curvature of the Riemann surface. The scalar product in the root space is defined such that \( \langle e_i, e_j \rangle = K_{ij} \) where \( K_{ij} \) is the Cartan matrix.

\(^7\)We review few aspects of Lie algebra theory in Appendix A.
of the $A_{N-1}$ algebra \[35\]. Conformal invariance of the theory requires the background charge $Q$ to be related to the coupling constant $b$ as

$$Q = q\rho$$

where $q = (b + \frac{1}{b})$ and $\rho$ is the Weyl vector. In the following we will consider $g_{\alpha\beta} = \delta_{\alpha\beta}$ and we will use the complex notation for the two dimensional coordinates, i.e. $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$.

Besides conformal invariance, $A_{N-1}$ conformal Toda field theory enjoys also higher spin symmetries. In total there are $N-1$ holomorphic currents $W^{(i+1)}$ $(i = 1, \ldots, N-1)$ with conformal dimension $(i+1)$, where $W^{(2)} = T$ is the usual stress tensor with conformal dimension 2. The $N-1$ symmetry currents form a $\mathcal{W}_N$ algebra, that is a consistent extension of the Virasoro symmetry. The Laurent expansions of the currents are

$$W^{(i+1)}(z) = \sum_n \frac{W^{(i+1)}_n}{z^{n+1}}$$

and the system includes also antiholomorphic currents with analogous properties so that the total symmetry is $\mathcal{W}_N \times \bar{\mathcal{W}}_N$, i.e. a product of an holomorphic and an anti-holomorphic $\mathcal{W}$-algebra. We note that when $N = 2$, the only holomorphic current is the stress tensor, thus the $W_2$ algebra is the Virasoro algebra. Indeed the $A_1$ Toda field theory is the well studied Liouville theory, a theory that posses Virasoro invariance.

The $\mathcal{W}_N$ primary fields $V$, are defined such that

$$W^{(i+1)}_0 V = w^{(i+1)} V, \quad W^{(i+1)}_n V = 0 \quad \text{for} \quad n > 0$$

and like for the Virasoro algebra, the descendant fields are obtained acting on the primaries with the operators $W^{(i+1)}_{-n}$ where $n > 0$. In Toda field theory the primaries are realized as exponential fields parameterized by $\alpha$, a vector in the root space of the $A_{N-1}$ algebra

$$V_\alpha = e^{\langle \alpha, \varphi \rangle}$$

and their conformal dimension $\Delta(\alpha) \equiv w^{(2)}(\alpha)$ is given by

$$\Delta(\alpha) \equiv w^{(2)}(\alpha) = \frac{1}{2} \langle \alpha, 2Q - \alpha \rangle.$$  

An important set of primaries is given by the completely degenerate fields. For these fields, the vector $\alpha$ is given by

$$\alpha = -b\Omega_1 - \frac{1}{b}\Omega_2$$

where $\Omega_1$ and $\Omega_2$ are two highest weights of finite dimensional representations of the algebra $A_{N-1}$. In the OPE of the degenerate fields with a generic primary appear only a finite set of primaries \[36\]

$$V_{-b\Omega_1 - \frac{1}{b}\Omega_2} \cdot V_\alpha = \sum_{s,t} [V_{\alpha_{s,t}}]$$
where \( \alpha'_{s,t} = \alpha - bh_{s}^{\Omega_1} - \frac{1}{2} h_{t}^{\Omega_2} \) and \( h_{s}^{\Omega} \) are the weights of the representation of \( A_{N-1} \) that has \( \Omega \) as highest weight. \([V_{\alpha'_{s,t}}]\) represent the family of operators that are descendants of \( V_{\alpha'_{s,t}} \).

We omit numerical factors on the right hand side of the formula (10).

### 2.1 Four Point Correlation Function

In [13] the four point correlation function with two degenerate insertions \( V_{-b\omega_1} \) and \( V_{-b\omega_{N-1}} \) has been computed\(^8\). It results

\[
\langle V_{\alpha_1}(0) V_{-b\omega_1}(z, \bar{z}) V_{-b\omega_{N-1}}(1) V_{\alpha_2}(\infty) \rangle = |z|^{2b(\alpha_1, h_1)} |1 - z|^{-\frac{2b^2}{N}} G(z, \bar{z})
\] (11)

where \( G(z, \bar{z}) \) satisfies the generalized hypergeometric differential equation in each of the two complex variables \( z \) and \( \bar{z} \). In details

\[
D(A_1, \ldots, A_N; B_1, \ldots, B_N) G(z, \bar{z}) = 0
\] (12)

\[
\bar{D}(A_1, \ldots, A_N; B_1, \ldots, B_N) G(z, \bar{z}) = 0
\] (13)

where

\[
D(A_1, \ldots, A_N; B_1, \ldots, B_N) = z(z\partial + A_1) \ldots (z\partial + A_N) - (z\partial + B_1 - 1) \ldots (z\partial + B_N - 1)
\] (14)

and \( \partial = \frac{\partial}{\partial z} \). The parameters \( A_k \) and \( B_k \) are related to the Toda momenta as

\[
A_k = -b^2 + b(\alpha_1 - Q, h_1) + b(\alpha_2 - Q, h_k)
\]

\[
B_k = 1 + b(\alpha_1 - Q, h_1) - b(\alpha_1 - Q, h_{k+1})
\] (15)

where we take \( h_{N+1} = h_1 \) so that \( B_N = 1 \). \( h_k \) are the \( N \) weights of the fundamental representation of the \( A_{N-1} \) algebra\(^9\). The \( \bar{D} \) operator is obtained from \( D \) replacing \( z \) with \( \bar{z} \).

The solutions of the differential equation (12) are defined on the Riemann sphere and have three singular points, namely 0, 1, \( \infty \), that are the positions where we have located three of the fields. In each punctured neighborhood of the singularities is possible to define \( N \) linearly independent solutions. We denote these solutions as

\[
\Lambda^{(s)} = (\Lambda^{(s)}_1, \ldots, \Lambda^{(s)}_N) \text{ defined in a neighborhood of 0,}
\]

\[
\Lambda^{(t)} = (\Lambda^{(t)}_1, \ldots, \Lambda^{(t)}_N) \text{ defined in a neighborhood of 1,}
\]

\[
\Lambda^{(u)} = (\Lambda^{(u)}_1, \ldots, \Lambda^{(u)}_N) \text{ defined in a neighborhood of } \infty.
\] (16)

\(^8\)The authors analyzed a more general configuration with one degenerate and one semidegenerate insertion. We specify to the two degenerate insertions case because this is the configuration that we will need in the following.

\(^9\)See Appendix A, formula (36).
For an explicit expression of these functions, see for instance \[33\]. Through analytical continuation it is possible to extend these solutions outside their domain of definition and it is thus possible to consider analytical continuations along closed paths. If the closed path encircles one or more singularities, the vector of solution is linearly transformed by an element of $GL(N, \mathbb{C})$, i.e. a monodromy matrix. Given a vector of $N$ linearly independent solutions, it is possible to compute the monodromy matrices around all the three singularities. These matrices, denoted as $M_{(0)}$, $M_{(1)}$ and $M(\infty)$, represent the monodromies computed around the three homotopy classes of the three punctured sphere. The monodromy matrices form a subgroup of $GL(N, \mathbb{C})$, the monodromy group, defined by the following relation

$$M(\infty)M_{(1)}M_{(0)} = 1.$$  

(17)

This group is a representation on the linear space of solutions of the first homotopy group of the Riemann sphere with three punctures. We note that the monodromy group is invariant under conjugation inside $GL(N, \mathbb{C})$. That is, given three matrices satisfying the relation (17) and given $X \in GL(N, \mathbb{C})$, also the conjugated matrices $\tilde{M} = XMX^{-1}$ satisfies the relation (17). The conjugation relates monodromy matrices that are associated to vector solutions related by the linear transformation $\tilde{\Lambda} = X\Lambda$. This implies that given a representation of the group for a certain basis of independent solutions, it is possible to know the representation of the group for a different set of solutions through a simple conjugation. It results that the three vector solutions in (16) have diagonal monodromy matrices respect to the singularity where they are nearby defined.

The conformal blocks in the $s, t, u$ channel are given by

$$F^{(s)}_k = z^{b(\alpha_1, h_1)}(1-z)^{-\frac{k^2}{N}}\Lambda^{(s)}_k,$$

$$F^{(t)}_k = z^{b(\alpha_1, h_1)}(1-z)^{-\frac{k^2}{N}}\Lambda^{(t)}_k,$$

$$F^{(u)}_k = z^{b(\alpha_1, h_1)}(1-z)^{-\frac{k^2}{N}}\Lambda^{(u)}_k$$

(18)

where $k = 1, \ldots, N$. The four point function is obtained considering bilinear combinations of $F(z)$ and $\bar{F}(\bar{z})$ that give a single valued function. More precisely

$$\langle V_{\alpha_1}(0)V_{-b\omega_1}(z, \bar{z})V_{-b\omega_{N-1}}(1)V_{\alpha_2}(\infty) \rangle = \sum_{k,r} C^{(s)}_{kr} F^{(s)}_k \bar{F}^{(s)}_r = \sum_{k,r} C^{(t)}_{kr} F^{(t)}_k \bar{F}^{(t)}_r = \sum_{k,r} C^{(u)}_{kr} F^{(u)}_k \bar{F}^{(u)}_r$$

(19)

where $C^{(s,t,u)}_{kr}$ are diagonal matrices whose entries are related to the three point functions in the $s, t, u$-channel\[10\].

\[10\]This relation was used in [9][11] to obtain equations for three point correlation functions.
3 Wilson Loops in Conformal Toda Field Theory

In \cite{29} \cite{30}, SU(2) gauge theory loop operators have been associated to monodromies of chiral degenerate operators in Liouville theory, i.e. $A_1$ conformal Toda field theory. The curve on the Riemann surface along which the monodromy is computed depends on the charge of the loop operator in the way described in \cite{31}. In this paper we focus on electrically charged loops, i.e. Wilson loops. The simplest example of such an operator is given by a loop that has fundamental charge respect to only one of the gauge groups in the theory. According to \cite{29} \cite{30} \cite{31}, the two dimensional representation of this Wilson loop is given by the monodromy of a chiral degenerate operator evaluated along a closed curve encircling the tube associated to the relevant gauge group, see Figure 1.

![Figure 1: A sphere with four punctures in a given pair of pants decomposition. The tube in between the two couples of punctures is associated to a gauge group and the closed curve is associated to a Wilson loop operator.](image)

The prescription for computing the monodromy around this curve was given in \cite{29} \cite{30} in terms of fusion moves and braiding moves. In the following we review the prescription in a way that can be easily generalized to the SU($N$) case.

\[
(t) \rightarrow (s) \quad M_{(0)} \quad (s) \rightarrow (t)
\]

![Figure 2: The prescription to compute the Wilson loop include a change of basis from the $t$-channel to the $s$-channel, a monodromy around $z = 0$ and finally a change of basis from $s$-channel to the $t$-channel. Dashed lines represent identity states.](image)

First, one inserts in the correlator two degenerate chiral fields $V_{\frac{1}{2}}(z), V_{\frac{1}{2}}(1)$ and fuse them to the identity. The starting point is thus the conformal block in the $t$-channel with the identity operator as internal state, we denote it as $\mathcal{F}_2^{(t)}$. Then, $\mathcal{F}_2^{(t)}$ is written in \footnote{Since we are now looking at the $A_1$ Toda (Liouville) theory, there are 2 conformal blocks in the $t$-channel but only one of them has the identity as internal state. In the following, we will consider $A_{N-1}$ Toda and the conformal block with the identity as internal state will be denoted as $\mathcal{F}_N^{(t)}$.}
terms of the conformal blocks in the $s$-channel $\mathcal{F}^{(s)}_k$ using the inverse fusion matrix $F^{−1}_{kr}$, i.e. $\mathcal{F}^{(t)}_2 = F^{−1}_{2r} \mathcal{F}^{(s)}_r$. The $F^{−1}$ describes the linear relation between two different basis of the hypergeometric functions $\Lambda$. In particular $\Lambda^{(t)} = F^{−1} \Lambda^{(s)}$ and $\Lambda^{(s)} = F \Lambda^{(t)}$. In the next step, the degenerate field $V_{−bω}$ is moved around the operator $V_{α}(0)$. This represents a monodromy around the $z = 0$ singular point. In the $s$-channel this monodromy acts diagonally on the conformal blocks and is given by a diagonal matrix $e^{2πibα}F^{(0)(s)}_{kr}$. Note that $M^{(s)}_{(0)kr}$ is the monodromy associated to hypergeometric functions $\Lambda^{(s)}_1$ and $e^{2πibα}$ is the monodromy of the factor $z^{b(α,h_1)}(1 − z)^{-\frac{k^2}{N}}$. The result is then rotated back to the $t$-channel using the fusion matrices $F_{kr}$ and projected on the $\mathcal{F}^{(t)}_2$ conformal block, see Figure 2.

More precisely, denoting with $\hat{\mathcal{L}}$ the set of operations we have just described, we have

$$\hat{\mathcal{L}} \cdot \mathcal{F}^{(t)}_2 = e^{2πibα}F^{−1}_{kr}M^{(s)}_{(0)kr}F_{kr} \mathcal{F}^{(t)}_2$$

and thus the conformal block is an eigenstate of $\hat{\mathcal{L}}$. It follows that the Wilson loop is described inserting in the correlator

$$\mathcal{L} = e^{2πibα}F^{−1}_{kr}M^{(s)}_{(0)kr}F_{kr} = e^{2πibα}M^{(t)}_{(0)22}.$$ 

The Wilson loop operator is thus given by the $(2, 2)$ component of the hypergeometric monodromy matrix around $z = 0$ expressed in the $t$-channel basis multiplied by the monodromy of the factor $z^{b(α,h_1)}(1 − z)^{-\frac{k^2}{N}}$ that appears in $\Lambda^{(t)}$. We now proceed to generalize this result to the $A_{N−1}$ Toda field theory. We are interested in Wilson loops in the fundamental representation, we propose that these operators are associated to the monodromy of a degenerate field $V_{−bω_1}$. Since $ω_1$ is the first of the fundamental weights, this operator is naturally associated to the fundamental representation of $SU(N)$. Generalizing the procedure in [29] [30], besides $V_{−bω_1}(z)$ we introduce also the degenerate field $V_{−bω_{N−1}}(1)$ and consider the conformal block where the two fields fuse to the identity. Indeed it results that

$$V_{−bω_1} \cdot V_{−bω_{N−1}} = [V_0] + [V_{−b(ω_1 + ω_{N−1})}]$$

and the identity operator is included in the OPE of the two fields. The channel where the two degenerate fields are fused each other is associated to the conformal blocks $\mathcal{F}^{(t)}_k = z^{b(α_1,h_1)}(1 − z)^{-\frac{k^2}{N}}\Lambda^{(t)}_k$ defined in the previous section. In order to understand which one of the conformal blocks describes the state where the degenerate fields are fused to the identity, we compute the monodromy around the $z = 1$ singularity. The monodromy is computed considering the contribution of the $\Lambda^{(t)}_k$ solutions and the contribution of the $z^{b(α_1,h_1)}(1 − z)^{-\frac{k^2}{N}}$ factor. It results $\mathcal{F}^{(t)} \rightarrow e^{−2πik^2}M^{(t)}_{(t)2} \mathcal{F}^{(t)}$ where $M^{(t)}_{(t)}$ is the monodromy
matrix around $z = 1$ of the solutions $\Lambda_{t_k}^{(t)}$. More explicitly

$$
\begin{pmatrix}
\mathcal{F}_1^{(t)} \\
\mathcal{F}_2^{(t)} \\
\vdots \\
\mathcal{F}_N^{(t)}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
B_{-b(\omega_1 + \omega_{N-1})}^{-b\omega_1, -b\omega_{N-1}} & 0 & 0 \\
0 & B_{-b(\omega_1 + \omega_{N-1})}^{-b\omega_1, -b\omega_{N-1}} & 0 \\
0 & 0 & \ldots
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_1^{(t)} \\
\mathcal{F}_2^{(t)} \\
\vdots \\
\mathcal{F}_N^{(t)}
\end{pmatrix}
$$

(23)

where

$$B_{\alpha_2, \alpha_3}^{\alpha_1} = e^{i\pi(\Delta(\alpha_1) - \Delta(\alpha_2) - \Delta(\alpha_3))}$$

(24)

and $\Delta(\alpha) = \frac{1}{2}(\alpha, 2Q - \alpha)$ is the conformal dimension of the Toda primary operators. We thus identify $\mathcal{F}_N^{(t)}$ as the conformal block with the identity in the internal channel, while $\mathcal{F}_1^{(t)}, \ldots, \mathcal{F}_{N-1}^{(t)}$ are the conformal blocks with the internal field in the adjoint representation, in agreement with (22).

Generalizing straightforwardly the prescription of [29] [30], we have that an $SU(N)$ Wilson loop in the fundamental representation is associated to

$$\mathcal{L} = e^{2\pi i b(\alpha, h_1)} M_{(0)NN}^{(t)}$$

(25)

where $M_{(0)NN}^{(t)}$ is the $(N, N)$ component of the monodromy matrix around $z = 0$, expressed in the $t$-channel basis. To compute this quantity, it is possible to use the matrices that relate the different set of solutions (16). We will take however a different route, using properties of the monodromy group of the generalized hypergeometric equation.

We have already mentioned that the monodromy group is invariant under conjugation. An element in the conjugacy class corresponds to a particular set of solutions and through conjugation, it is possible to know the form of the monodromy matrices for other sets of solutions. An explicit realization of the monodromy group for the generalized hypergeometric equation was given in [34] [35], we review this construction in Appendix B. Considering a certain conjugation, it is possible to obtain the monodromy matrices for the basis of functions $\Lambda^{(1)}$ where $M_{(1)}^{(t)}$ is diagonal. In this basis it results

$$M_{(0)NN}^{(t)} = e^{-2\pi ib(\alpha, h_1)} \frac{e^{i\pi bq} - e^{-i\pi bq}}{e^{i\pi bq N} - e^{-i\pi bq N}} \sum_k e^{2\pi ib(\alpha - Q, h_k)}$$

(26)

where $h_k$ are the $N$ weights of the fundamental representation of $SU(N)$ and $q = (b + \frac{1}{2})$. Following [10] we take $\alpha = \tilde{a} + Q$ where $\tilde{a}$ is an imaginary $(N - 1)$ vector that parameterizes

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12 See for instance [33].

13 See Appendix B for details.
the Cartan subalgebra of $SU(N)$. Expanding $\tilde{a}$ on the basis of simple roots as $\tilde{a} = \sum_{i=1}^{N-1} a_i e_i$ we get

$$\sum_k e^{2\pi i b (\alpha - Q, h_k)} = \left(e^{i2\pi b a_1} + e^{i2\pi b(a_2-a_1)} + \ldots + e^{i2\pi b(a_{N-1} - a_{N-2})} + e^{i2\pi b(-a_{N-1})}\right)$$

$$= \text{Tr}_F e^{i2\pi ba}$$

(27)

where $a$ is an anti-hermitian traceless $N \times N$ matrix and the trace is evaluated in the fundamental representation $F$. We thus conclude that a Wilson loop in the fundamental representation of the gauge group $SU(N)$, in the dual $A_{N-1}$ Toda theory is described by

$$\mathcal{L} = \frac{1}{[N]_{e^{i\pi bq}}} \text{Tr}_F e^{i2\pi ba},$$

(28)

where $[N]_{e^{i\pi bq}} = \frac{e^{i\pi bq N} - e^{-i\pi bq N}}{e^{i\pi bq} - e^{-i\pi bq}}$ is the quantum deformed number $N$ where the parameter of the deformation is $e^{i\pi bq}$. It follows immediately that in the limit $b \to 1$, the Wilson loop reduces to

$$\mathcal{L} = \frac{1}{N} \text{Tr}_F e^{i2\pi a},$$

(29)

in agreement with the result of Pestun [2]. Considering $N = 2$, we have the Wilson loop in $A_1$ Toda theory, i.e. Liouville theory. It results

$$\mathcal{L} = \frac{\cos (2\pi ba)}{\cos (\pi bq)}$$

(30)

in agreement with [29] [30] [15]. Repeating the same procedure considering the inverse matrix $M_{(0)}^{-1}$ instead of $M_{(0)}$, we compute the loop operator associated to the monodromy defined around the same curve but with opposite orientation [16]. We denote this operator with $\tilde{\mathcal{L}}$. It results

$$\tilde{\mathcal{L}} = \frac{1}{[N]_{e^{i\pi bq}}} \text{Tr}_F e^{i2\pi ba},$$

(31)

where the trace is evaluated in the anti-fundamental representation $\bar{F}$. We conclude that in the $N > 2$ case, to completely characterize a monodromy operator, it is necessary to specify also the orientation of the curve along which the monodromy is evaluated, see Figure 3.

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[14] In this formula $a$ is one of the eigenvalues of the traceless anti-hermitian matrix.
[15] In [30], the authors consider a different normalization for loop operators.
[16] See Appendix B for details.
A Wilson loop in the fundamental representation $\mathcal{L}$ and a Wilson loop in the anti-fundamental representations $\bar{\mathcal{L}}$ are associated to curves with opposite orientation. In this example we consider the gauge theory associated to the sphere with two simple punctures and two full punctures.

### A Some Lie Algebra Notions

Given a finite dimensional semisimple Lie algebra with rank $r$ (see for instance [37]), we denote $H^i$ ($i = 1, \ldots, r$) the elements in the Cartan subalgebra and $E^\alpha$ the remaining operators. It is possible to consider a basis of operators such that

$$
[H^i, E^\alpha] = \alpha^i E^\alpha \quad \quad [E^\alpha, E^{-\alpha}] = \frac{2}{\alpha \cdot \alpha} \alpha \cdot H
$$

where the $r$-dimensional vectors $\alpha$ are the roots of the algebra and the scalar product is $\alpha \cdot \beta = \sum_{i=1}^r \alpha_i \beta_i$. It is useful to define the root space product $\langle \beta, \alpha \rangle$ as

$$
\langle \beta, \alpha \rangle = \frac{2}{\alpha \cdot \alpha} \beta \cdot \alpha
$$

We denote with $\Phi$ the set of roots of the algebra. The simple roots $e_i$ ($i = 1, \ldots, r$) form a subset of roots $\Delta \subset \Phi$ that are a basis for the full root space. It results that any root in $\Phi$ can be written as linear combination of simple roots, i.e. $\alpha = \sum_{i=1}^r k_i e_i$ where $k_i$ are all positive integers or all negative integers. The matrix defined by $\langle e_i, e_j \rangle = K_{ij}$ is the Cartan matrix and its entries are integer numbers. For an arbitrary representation of the algebra, it is possible to define a basis $|\lambda\rangle$ such that $H^i |\lambda\rangle = \lambda^i |\lambda\rangle$. The $r$-dimensional vectors $\lambda$ are called weights and satisfy the relation $\langle \lambda, \alpha \rangle \in \mathbb{Z}$. They can be expanded on a basis of fundamental weights $\omega_i$ such that $\lambda = \sum_{i=1}^r \delta_i \omega_i$ and the coefficient of the expansion $\delta_i$ are integer numbers called Dynkin labels. The fundamental weights are defined such that $\langle \omega_i, e_j \rangle = \delta_{ij}$. The highest weight of a representation is the weight for which the sum of the Dynkin labels is the highest. It can be shown that the highest weight completely characterize the representation. Roots and simple roots are the weights and fundamental weights of the adjoint representation. The Weyl vector $\rho$ is defined as the sum of all the fundamental weights, i.e. $\rho = \sum_{i=1}^r \omega_i$. It follows that $\langle \rho, e_i \rangle = 1$. 

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Figure 3: A Wilson loop in the fundamental representation $\mathcal{L}$ and a Wilson loop in the anti-fundamental representations $\bar{\mathcal{L}}$ are associated to curves with opposite orientation. In this example we consider the gauge theory associated to the sphere with two simple punctures and two full punctures.
A.1 $A_{N-1}$ Algebra

A linear representation of the $A_{N-1}$ algebra is given by $N \times N$ traceless matrices. Denoting $a_{\mu\nu}$ the matrix that have entry 1 at position $(\mu, \nu)$ and zero elsewhere, we can construct the generators $E^{\alpha_{\mu\nu}} = a_{\mu\nu}$ for $(\mu \neq \nu)$ and $H^i = \sum_{\mu=1}^{N} \epsilon^i_{\mu} a_{\mu\mu}$, where $\epsilon^i_{\mu} = (\epsilon^i_1, \ldots, \epsilon^i_N)$ satisfy $\sum_{\mu=1}^{N} \epsilon^i_{\mu} = 0$. We have

$$[H^i, E^{\alpha_{\mu\nu}}] = (\epsilon^i_{\mu} - \epsilon^i_{\nu}) E^{\alpha_{\mu\nu}}$$

(34)

thus the positive roots are given by $\alpha^i_{\mu\nu} = (\epsilon^i_{\mu} - \epsilon^i_{\nu})$ where $1 < \mu < \nu < N$ and the simple roots are $e_j = (\epsilon^j_j - \epsilon^j_{j+1})$ where $1 < j < N - 1$. In the main text we omit indices labeling the components of roots or weights. The Cartan matrix $K_{ij} = \langle e_i, e_j \rangle$ is given by

$$K_{ij} = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}$$

(35)

The fundamental representation $F$ of the $SU(N)$ algebra has the first fundamental weight as highest weight, i.e. $h_1 = \omega_1$. The set of $N$ weights of $F$ is given by

$$h_k = \omega_1 - e_1 - \ldots - e_{k-1}$$

(36)

where $k = 1, \ldots N$ and we assumed $e_0 = 0$. The anti-fundamental representation $\bar{F}$ of the $SU(N)$ algebra has the last fundamental weight $\omega_{N-1}$ as highest weight. $\bar{h}_k$ are the $N$ weights of $\bar{F}$. Simple roots and fundamental weights are related by the Cartan matrix as $e_i = \sum_j K_{ij} \omega_j$. Other useful relations are

$$\langle \rho, h_1 \rangle = \frac{N - 1}{2}, \quad \langle h_1, h_1 \rangle = \frac{N - 1}{N}, \quad \langle \rho, \sum_{k=1}^{N} h_k \rangle = 0.$$ 

(37)

B The Hypergeometric Monodromy Group

The monodromy group of the generalized hypergeometric equation (12) was analyzed in details in [34] and [35]. An explicit representation of the group is given by

$$M(\infty) = A, \quad M(0) = B^{-1}, \quad M(1) = A^{-1}B$$

(38)
where

\[
A = \begin{pmatrix}
0 & 0 & \ldots & 0 & -c_N \\
1 & 0 & \ldots & 0 & -c_{N-1} \\
0 & 1 & \ldots & 0 & -c_{N-2} \\
\vdots & & & & \ddots \\
0 & 0 & \ldots & 1 & -c_1
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 & 0 & \ldots & 0 & -b_N \\
1 & 0 & \ldots & 0 & -b_{N-1} \\
0 & 1 & \ldots & 0 & -b_{N-2} \\
\vdots & & & & \ddots \\
0 & 0 & \ldots & 1 & -b_1
\end{pmatrix}
\]

and the entries \(c_k\) and \(b_k\) are defined by

\[
\prod_{k=1}^{N}(t - e^{2\pi i A_k}) = t^N + c_1 t^{N-1} + \ldots + c_N, \quad \prod_{k=1}^{N}(t - e^{2\pi i B_k}) = t^N + b_1 t^{N-1} + \ldots + b_N
\]

The representation of the monodromy group given in (38), produces the monodromy matrices associated to a certain basis of independent solutions. Considering a different basis correspond to a conjugation of the monodromy matrices. In particular, we are interested in the expression of \(M(0)\) and \(M^{-1}(0)\) in a basis where \(M(1)\) is diagonal, i.e. in the basis of solutions \(\Lambda(t)\). It results that given a certain \(D \in GL(N, \mathbb{C})\), we have

\[
M^{(t)}_{(1)} = D^{-1} M_{(1)} D = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & & & \ddots \\
0 & 0 & \ldots & 1 \frac{b_N}{c_N}
\end{pmatrix}
\]

\[
M^{-(t)}_{(0)N} = D^{-1}_{Nk} M^{-1}_{(0)k} D_{rN} = \frac{-c_N b_{N-1} + b_N c_{N-1}}{(c_N - b_N) b_N}
\]

\[
M^{-(t)}_{(0)NN} = D^{-1}_{Nk} M^{-1}_{(0)k} D_{rN} = \frac{(-c_1 + b_1) b_N}{c_N - b_N}.
\]

where \(M^{(t)}_{(0)NN}\) and \(M^{-(t)}_{(0)NN}\) are the \((N, N)\) components of the matrices \(M^{(t)}_{(0)}\) and \(M^{-(t)}_{(0)}\). The parameter \(c_N\), \(c_{N-1}\) and \(c_1\) are related to the \(A_k\) parameters as

\[
c_N = (-1)^N \prod_{k=1}^{N} e^{2\pi i A_k},
\]

\[
c_{N-1} = (-1)^{N-1} \sum_{k_1 > k_2 > \ldots > k_{N-1} \geq 1} e^{2\pi i A_{k_1}} e^{2\pi i A_{k_2}} \ldots e^{2\pi i A_{k_{N-1}}},
\]

\[
c_1 = - \sum_{k} e^{2\pi i A_k}
\]

(44)
and in analogous way $b_N$, $b_{N-1}$ and $b_1$ are related to $B_k$. Expressing $A_k$ and $B_k$ in terms of the Toda momenta using (13) and considering $\alpha_1 = \alpha$ and $\alpha_2 = 2Q - \alpha$, applying formulas like the (37), we have

\[
M^{(t)}_{(1)} = D^{-1}M_{(1)}D = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
& & \ddots & & \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & e^{2\pi ib^2N}
\end{pmatrix},
\]

(45)

\[
M^{(t)}_{(0)NN} = e^{-2\pi ib(\alpha,h_1)} \frac{e^{i\pi bq} - e^{-i\pi bq}}{e^{i\pi bqN} - e^{-i\pi bqN}} \sum_k e^{2\pi ib(\alpha-Q,h_k)},
\]

(46)

\[
M^{-1(t)}_{(0)NN} = e^{2\pi ib(\alpha,h_1)} \frac{e^{i\pi bq} - e^{-i\pi bq}}{e^{i\pi bqN} - e^{-i\pi bqN}} \sum_k e^{-2\pi ib(\alpha-Q,h_k)}.
\]

(47)

References

[1] D. Gaiotto, “N=2 dualities,” arXiv:0904.2715 [hep-th].

[2] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91, 167 (2010) arXiv:0906.3219 [hep-th].

[3] N. A. Nekrasov, “Seiberg-Witten Prepotential From Instanton Counting,” Adv. Theor. Math. Phys. 7, 831 (2004) arXiv:hep-th/0206161.

[4] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” arXiv:hep-th/0306238.

[5] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” Nucl. Phys. B 241, 333 (1984).

[6] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824 [hep-th].

[7] H. Dorn and H. J. Otto, “Two and three point functions in Liouville theory,” Nucl. Phys. B 429, 375 (1994) arXiv:hep-th/9403141.

[8] A. B. Zamolodchikov and A. B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory,” Nucl. Phys. B 477, 577 (1996) arXiv:hep-th/9506136.

[9] J. Teschner, “On the Liouville three point function,” Phys. Lett. B 363, 65 (1995) arXiv:hep-th/9507109.
[10] N. Wyllard, “$A_{N-1}$ conformal Toda field theory correlation functions from conformal $N=2$ SU(N) quiver gauge theories,” JHEP 0911, 002 (2009) [arXiv:0907.2189 [hep-th]].

[11] V. A. Fateev and A. V. Litvinov, “Correlation functions in conformal Toda field theory I,” JHEP 0711, 002 (2007) [arXiv:0709.3806 [hep-th]].

[12] S. Kanno, Y. Matsuo, S. Shiba and Y. Tachikawa, “$N=2$ gauge theories and degenerate fields of Toda theory,” arXiv:0911.4787 [hep-th].

[13] V. A. Fateev and A. V. Litvinov, “On differential equation on four-point correlation function in the conformal Toda field theory,” JETP Lett. 81, 594 (2005) [Pisma Zh. Eksp. Teor. Fiz. 81, 728 (2005)] [arXiv:hep-th/0505120].

[14] P. Bowcock and G. M. T. Watts, “Null vectors, three point and four point functions in conformal field theory,” Theor. Math. Phys. 98, 350 (1994) [Teor. Mat. Fiz. 98, 500 (1994)] [arXiv:hep-th/9309146].

[15] A. Mironov and A. Morozov, “On AGT relation in the case of U(3),” Nucl. Phys. B 825, 1 (2010) [arXiv:0908.2569 [hep-th]].

[16] G. Bonelli and A. Tanzini, “Hitchin systems, $N=2$ gauge theories and W-gravity,” arXiv:0909.4031 [hep-th].

[17] L. F. Alday, F. Benini and Y. Tachikawa, “Liouville/Toda central charges from M5-branes,” arXiv:0909.4776 [hep-th].

[18] D. Nanopoulos and D. Xie, “Hitchin Equation, Singularity, and $N=2$ Superconformal Field Theories,” arXiv:0911.1990 [hep-th].

[19] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and $N=2$ Gauge Systems,” arXiv:0909.2453 [hep-th].

[20] H. Itoyama, K. Maruyoshi and T. Oota, “Notes on the Quiver Matrix Model and 2d-4d Conformal Connection,” arXiv:0911.4244 [hep-th].

[21] T. Eguchi and K. Maruyoshi, “Penner Type Matrix Model and Seiberg-Witten Theory,” JHEP 1002, 022 (2010) [arXiv:0911.4797 [hep-th]].

[22] R. Schiappa and N. Wyllard, “An $A_r$ threesome: Matrix models, 2d CFTs and 4d $N=2$ gauge theories,” arXiv:0911.5337 [hep-th].

[23] A. Mironov, A. Morozov and S. Shakirov, “Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions,” JHEP 1002, 030 (2010) [arXiv:0911.5721 [hep-th]].

[24] M. Fujita, Y. Hatsuda and T. S. Tai, “Genus-one correction to asymptotically free Seiberg-Witten prepotential from Dijkgraaf-Vafa matrix model,” arXiv:0912.2988 [hep-th].

[25] A. Mironov, A. Morozov and S. Shakirov, “Conformal blocks as Dotsenko-Fateev Integral Discriminants,” arXiv:1001.0563 [hep-th].
[26] J. K. Erickson, G. W. Semenoff and K. Zarembo, “Wilson loops in $N = 4$ supersymmetric Yang-Mills theory,” Nucl. Phys. B 582, 155 (2000) [arXiv:hep-th/0003055].

[27] N. Drukker and D. J. Gross, “An exact prediction of $N = 4$ SUSYM theory for string theory,” J. Math. Phys. 42, 2896 (2001) [arXiv:hep-th/0010274].

[28] S. J. Rey and T. Suyama, “Exact Results and Holography of Wilson Loops in $N=2$ Superconformal (Quiver) Gauge Theories,” arXiv:1001.0016 [hep-th].

[29] N. Drukker, J. Gomis, T. Okuda and J. Teschner, “Gauge Theory Loop Operators and Liouville Theory,” arXiv:0909.1105 [hep-th].

[30] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, “Loop and surface operators in $N=2$ gauge theory and Liouville modular geometry,” arXiv:0909.0945 [hep-th].

[31] N. Drukker, D. R. Morrison and T. Okuda, “Loop operators and S-duality from curves on Riemann surfaces,” JHEP 0909, 031 (2009) [arXiv:0907.2593 [hep-th]].

[32] N. Drukker, D. Gaiotto and J. Gomis, “The Virtue of Defects in 4D Gauge Theories and 2D CFTs,” arXiv:1003.1112 [hep-th].

[33] N. E. Noerlund, “Hypergeometric Functions,” Acta Math. 94 (1955), 289-349.

[34] F. Beukers, G. Heckman, “Monodromy for the hypergeometric function $\,F_{n-1}$”, Inventiones Mathematicae 95 (1989), 325-354

[35] K. Okubo, “On the group of Fuchsian equations”, Seminar Reports of Tokyo Metropolitan University, 1987

[36] V. A. Fateev and S. L. Lukyanov, “The Models of Two-Dimensional Conformal Quantum Field Theory with $Z(n)$ Symmetry,” Int. J. Mod. Phys. A 3, 507 (1988).

[37] J. E. Humphreys, “Introduction to Lie Algebras and Representation Theory,” Springer-Verlag (1972).