Hidden Conformal Symmetry and Quasinormal Modes

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Abstract: We provide an algebraic way to calculate the quasinormal modes of a black hole, which possesses a hidden conformal symmetry. We construct an infinite tower of quasinormal modes from the highest-weight mode, in a simple and elegant way. For the scalar, the hidden conformal symmetry manifests itself in the fact that the scalar Laplacian could be rewritten in terms of the $SL(2,R)$ quadratic Casimir. For the vector and the tensor, the hidden conformal symmetry acts on them through Lie derivatives. We show that for three-dimensional black holes, with an appropriate combination of the components the radial equations of the vector and the tensor could be written in terms of the Lie-induced quadratic Casimir. This makes the algebraic construction of the quasinormal modes feasible. Our results are in good agreement with the previous study.
1. Introduction

Very recently, motivated by the work in [1], it has been found that in many black holes which have holographic 2D CFT descriptions there exists a hidden conformal symmetry. The hidden conformal symmetry is realized by two sets of locally defined vector fields \( \{V_i, \bar{V}_i\} \) satisfying \( SL(2, \mathbb{R}) \) Lie algebra. This symmetry is not globally defined, and is broken by the periodic identification on angular variable. It could not be used to generate new solutions. Nevertheless, it determines the scattering amplitudes by acting on the solution space. More explicitly, the scalar Laplacian could be written as the \( SL(2, \mathbb{R}) \) quadratic Casimir in some limit region. The hidden conformal symmetry was considered to be essential to implement a holographic description of a black hole. It was widely studied in various kinds of black holes, including the 4D Kerr-Newman [2], 4D Kerr-Newman-AdS-dS [3], 3D black holes [19, 21], extremal black holes [4] and others [5].

In retrospect, the appearance of hidden conformal symmetry is not a surprise, considering the fact that the black hole is dual to a 2D CFT. On the CFT side, the conformal symmetry restricts the form of the correlation functions of the operators. Correspondingly, the conformal symmetry acts on the solution space and determines the scattering amplitudes. In the 4D Kerr case, the hidden conformal symmetry only become manifest in the low frequency limit and in the near region, but it determines not only the low frequency scattering amplitudes but also the super-radiant scattering ones. In 3D cases, the hidden conformal symmetry generically manifests itself more clearly, in all regions. Such features
in 3D black holes are due to the fact that the black holes are always locally isomorphic to their covering spaces.

On the other hand, from AdS/CFT correspondence\(^7,\ 9\) the quasinormal modes, which determine the relaxation time of the perturbations about the black hole, are related to the poles of the retarded correlation function in the momentum space in the dual conformal field theory. In a 2D CFT, the retarded Green’s functions for an operator with fixed conformal weights, fixed charges with respect to chemical potentials are determined by the conformal symmetry\(^10\). The poles in the retarded Green’s function could be read easily. On the gravity side, the quasinormal modes could be read from the eigenfunctions satisfying the purely ingoing boundary condition at the black hole horizon and appropriate boundary condition at the asymptotical infinity\(^6\). One has to solve the equations of motion explicitly in order to get the eigenfunctions, whose analytic forms are often out of reach. It is thus interesting to see that the equation of motion with the hidden conformal symmetry acting on could always be solved in terms of hypergeometric functions, due to mathematical fact that the hypergeometric functions could form the representation of the \(SL(2, \mathbb{R})\) group. As a result, the quasinormal modes could be read exactly. Actually one aim of this paper is to show that even without solving the equations of motion explicitly, we can determine the quasinormal modes in an elegant algebraic way.

Another issue on hidden conformal symmetry is how it acts on the vector and tensor fields. In all the studies in the literature, the hidden conformal symmetry has kept being discussed in the scalar equation of motion. As it is an intrinsic property of the black hole, it should also act on the other kinds of perturbations. In this paper, we address this issue. For the locally defined vector fields, they act on the vector and the tensor fields via Lie-derivatives. We show that for 3D black holes, the hidden conformal symmetry acts on the vector and tensor fields in a subtle way. We find that only after an appropriate combination, the equations of motion of the vector and tensor fields could be written as the Lie-induced quadratic Casimir:

\[
(\mathcal{L}^2 + m_i^2) T_+ = 0
\]  

where
\[
\mathcal{L}^2 \equiv -\mathcal{L}_{V_0}\mathcal{L}_{V_0} + \frac{1}{2}(\mathcal{L}_{V_1}\mathcal{L}_{V_{-1}} + \mathcal{L}_{V_{-1}}\mathcal{L}_{V_1})
\]  

is the Casimir commuting with the Lie-derivatives \(\mathcal{L}_{V_i}\), and \(T_+\) is an appropriate superposition of tensor components. Actually the scalar equation of motion could also be cast into the same form (1.2).

The fact that the equations of motion of all perturbations could be written as (1.2) allows us to construct the quasinormal modes in an uniform way. We start from the highest-weight mode, which not only satisfies the equations of motion but also obeys the condition

\[
\mathcal{L}_{V_1} \Psi^{(0)} = 0, \quad \mathcal{L}_{V_0} \Psi^{(0)} = h_R \Psi^{(0)},
\]  

then construct the descendent modes

\[
\Psi^n = (\mathcal{L}_{V_{-1}})^n \Psi^{(0)}. \tag{1.4}
\]
It is nice to find that the descendant modes constitute an infinite tower of quasinormal modes. We will show that all the information of quasinormal modes is encoded in the hidden conformal symmetry. The frequencies of the quasinormal modes take the following form:

\[
\lambda_1 \omega_R^{(n)} = \lambda_2 k + i(h_R + n_R), \quad \tilde{\lambda}_1 \omega_L^{(n)} = \tilde{\lambda}_2 k + i(h_L + n_L)
\]

where \(n_{L,R}\) are non-negative integers and \(\lambda_i, \tilde{\lambda}_i\) are parameters in the hidden conformal symmetry. The spectrum of all kinds of quasinormal modes share the same structure, with the difference being from the conformal weights which are decided by the \(m_t^2\) term.

The way we approach the quasinormal modes is partly motivated by the work in [11]. In this paper, Ivo Sachs and Sergey N. Solodukhin showed that quasinormal modes of the BTZ black hole in topologically massive gravity may be derived from the Killing vector fields. The essential aspect is that the Killing vectors form a \(SL(2,R)\) Lie algebra locally so that they can build an infinite tower of quasinormal modes. Our treatment is in spirit similar to theirs, but differs in detail. In particular, our investigation on the vector and tensor fields has not been presented anywhere else before, to our knowledge. Moreover our discussion includes the warped AdS\(_3\) black hole and self-dual warped AdS\(_3\) black hole of topological massive gravity, whose hidden conformal symmetry is nontrivial, in contrast to the BTZ black hole, which is locally isomorphic to AdS\(_3\) so that the hidden conformal symmetry is not a real surprise. Actually the equations of motion in the warped spacetime are of the form

\[
(\mathcal{L}^2 + b\mathcal{L}_{V_0}^2 + m_t^2)T_+ = 0,
\]

which is slightly different from (1.2) but still allows us to construct the quasinormal modes in the similar way. But now the conformal weight depends not only on the mass but also on the extra quantum numbers.

In the next section, we briefly review the realization of hidden conformal symmetry. In Sec. III, we study the scalar perturbation and determine the quasinormal modes as a warm-up. In Sec. IV, we investigate the action of the hidden conformal symmetry on the vector and gravitational perturbations. In Sec. V, we discuss the quasinormal modes of the BTZ black hole, and reproduce the well-known results. In Sec. VI, we try to generalize the method to the warped AdS\(_3\) and self-dual warped AdS\(_3\) black holes, which need a minor modification of our construction. We will end with discussions in Sec. VII. Some technical details are put into two appendixes.

2. Hidden Conformal Symmetry

In this paper, we will restrict to generic nonextremal black holes which have the hidden conformal symmetry. Generically we may introduce the vector fields

\[
V_0 = \lambda_1 \partial_t + \lambda_2 \partial_\phi,
\]

\[
V_1 = e^{\mu_1 t + \mu_2 \phi}[(A \frac{\Delta'}{\sqrt{\Delta}} + B \frac{1}{\sqrt{\Delta}})\partial_t + (C \frac{\Delta'}{\sqrt{\Delta}} + D \frac{1}{\sqrt{\Delta}})\partial_\phi + \sqrt{\Delta} \partial_r],
\]

\[
V_{-1} = e^{-\mu_1 t - \mu_2 \phi}[(A \frac{\Delta'}{\sqrt{\Delta}} + B \frac{1}{\sqrt{\Delta}})\partial_t + (C \frac{\Delta'}{\sqrt{\Delta}} + D \frac{1}{\sqrt{\Delta}})\partial_\phi - \sqrt{\Delta} \partial_r],
\]
where $\lambda_1, \lambda_2, \mu_1, \mu_2, A, B, C, D$ are all constants satisfying
\begin{align}
\lambda_1 \mu_1 + \lambda_2 \mu_2 &= -1, \\
\lambda_1 &= 2A, \\
\lambda_2 &= 2C, \\
\mu_1 B + \mu_2 D &= 0,
\end{align}
(2.2)
and $\Delta = (r - r_+)(r - r_-)$, $\Delta' = \frac{d^2}{dr^2}$. The above vector fields form a $SL(2, R)$ algebra.

\begin{align}
[V_0, V_{\pm 1}] &= \mp V_{\pm 1}, \\
[V_{+1}, V_{-1}] &= 2V_0
\end{align}
(2.3)
And similarly we can define the left sector $\bar{V}_0, \bar{V}_{\pm 1}$ with parameters $\bar{\mu}_i, \bar{\lambda}_i, \bar{A}, \bar{B}, \bar{C}, \bar{D}$.

The essential aspect is that the scalar Laplacian can be written as the $SL(2, R)$ quadratic Casimir. More explicitly, the radial scalar field equation in a black hole with holographic description is of the form
\begin{align}
(V^2 + m^2_\sigma) \Phi(r) &= 0,
\end{align}
(2.4)
where $V^2 = -V_0^2 + \frac{1}{2}(V_1 V_{-1} + V_{-1} V_1)$ is the $SL(2, R)$ quadratic Casimir operator and $m^2_\sigma$ is a constant. This is true for the 4D Kerr(-Newman) black hole in the low frequency and the near region, and is always true for 3D black holes in the whole region. Actually, one can give the explicit form of the Casimir. But we would not give it here, instead we will derive it in the next section in the general framework of Lie-derivative operation.

3. Scalar Modes

In this section we will derive the scalar equation using the Lie derivatives. This seems useless since we have known the results in Sec. II. But we will see that it is valuable to reproduce it in another way, which could be generalized to discuss the vector and tensor modes.

First we define Lie-induced quadratic Casimir
\begin{align}
\mathcal{L}^2 &\equiv -\mathcal{L}_{V_0} \mathcal{L}_{V_0} + \frac{1}{2}(\mathcal{L}_{V_1} \mathcal{L}_{V_{-1}} + \mathcal{L}_{V_{-1}} \mathcal{L}_{V_1})
\end{align}
(3.1)
where $\mathcal{L}_{V_i}, i = 0, \pm 1$ are the Lie derivatives with respect to the vector fields $V_i$. $\mathcal{L}^2$ is analogue to the $SL(2, R)$ quadratic Casimir $V^2$.

Let $\Phi$ be a scalar field and we immediately have
\begin{align}
\mathcal{L}^2 \Phi &= \Pi^{\rho\sigma} \partial_\rho \partial_\sigma \Phi + \Sigma^\rho \partial_\rho \Phi
\end{align}
(3.2)
where we have defined
\begin{align}
\Pi^{\rho\sigma} &\equiv \frac{1}{2}(V^\rho_1 V^\sigma_{-1} + V^\sigma_{-1} V^\rho_1) - V^\rho_0 V^\sigma_0, \\
\Sigma^\rho &\equiv \frac{1}{2}(V^\rho_1 \partial_\sigma V^\sigma_{-1} + V^\sigma_{-1} \partial_\sigma V^\rho_1) - V^\rho_0 \partial_\sigma V^\sigma_0.
\end{align}
(3.3)
(3.4)
The explicit expressions of Π’s and Σ’s can be found in Appendix A. We use them to find
\[ L^2 \Phi = -\partial_r \Delta \partial_r \Phi + \frac{1}{(r - r_+)(r_+ - r)} \sigma_+^2 - \frac{1}{(r - r_-)(r_+ - r_-)} \sigma_-^2 \Phi \] (3.5)
where \( \sigma_\pm = (\pm A(r_+ - r_-) + B) \partial_t + (\pm C(r_+ - r_-) + D) \partial_\phi \). Since we focus on the black holes which have a hidden conformal symmetry, the scalar equation can be written formally as
\[ (L^2 + m_s^2) \Phi = 0 \] (3.6)
where \( m_s \) is a constant which is related to the conformal weight of the scalar. It varies for different black holes. Certainly for the scalar, (3.6) is exactly the same as (2.4).

To construct the tower of scalar quasinormal modes, we first impose the condition:
\[ L_{V1} \Phi^{(0)} = 0, \quad L_{V0} \Phi^{(0)} = h_R \Phi^{(0)} \] (3.7)
to define the “highest-weight” mode. Since
\[ [L_X, L_Y] = L_{[X,Y]}, \quad L_{aX} = aL_X \] (3.8)
where \( X, Y \) are arbitrary vectors and \( a \) is an arbitrary constant, we get the following relation from the scalar Eq. (3.6):
\[ h_R^2 - h_R - m_s^2 = 0. \] (3.9)
This determines the conformal weight
\[ h_R = \frac{1}{2} (1 + \sqrt{1 + 4m_s^2}). \] (3.10)
We have chosen the “+” root to simplify our discussion. But the other choice can also be considered easily.

From the mode \( \Phi^{(0)} \), we construct an infinite tower of quasinormal scalar modes \( \Phi^{(n)} \) as
\[ \Phi^{(n)} = (L_{V-})^n \Phi^{(0)} , \quad n \in N. \] (3.11)
All the \( \Phi^{(n)} \) are descendents of the mode \( \Phi^{(0)} \). Since the Casimir \( L^2 \) commutes with \( L_{V_i}, i = 0, \pm 1, \Phi^{(n)} \) satisfy the scalar equation as well. To compute the frequency of the quasinormal modes, we may expand the scalar as
\[ \Phi = e^{-i\omega t + ik\phi} \varphi, \] (3.12)
as \( \partial_t \) and \( \partial_\phi \) are always the Killing vectors of the black holes. For the highest-weight mode \( \Phi^{(0)} \), we have
\[ \lambda_1 \omega_0 - \lambda_2 k_0 = ih_R, \] (3.13)
where \( \omega_0 \) and \( k_0 \) are its frequency and angular momentum. In principle, \( k_0 \) could be complex in the solution. Taking the highest mode as quasinormal modes require \( k_0 \) be real. For the descendent mode \( \Phi^{(n)} \), we have
\[ L_{V0} \Phi^{(n)} = (-i\lambda_1 \omega^{(n)}_R + i\lambda_2 k^{(n)}) \Phi^{(n)}, \] (3.14)
where its frequency $\omega_R^{(n)}$ and angular momentum $k_R^{(n)}$ are related to $\omega_0$ and $k_0$ via the relation

$$\omega_R^{(n)} = \omega_0 - in\mu_1, \quad k_R^{(n)} = k_0 + in\mu_2.$$  \hspace{1em} (3.15)

To be a well-defined quasinormal mode, the angular momentum $k_R^{(n)}$ should be real, which requires a choice of complex $k_0$. Note that the real part of the $k_0$ and $k_R^{(n)}$ are always the same, taken as $k$. From the relation (3.15) and the first relation in (2.2), we obtain

$$\lambda_1\omega_R^{(n)} = \lambda_2k + i(h_R + n).$$  \hspace{1em} (3.16)

Alternatively it is more convenient to use just the algebraic relation (3.8) to get this relation.

The relation (3.16) gives the frequencies of the scalar quasinormal modes. We find that the frequencies of the modes only depend on the parameters which appear in the hidden conformal symmetry. Our construction relates the hidden conformal symmetry to the structure of quasinormal modes directly.

Note that we can also determine the left sector modes from the other set of vector fields $\{\bar{V}_i\}$ according to the following rules:

(i) $R \rightarrow L$

(ii) $\lambda_i \rightarrow \bar{\lambda}_i, \mu_i \rightarrow \bar{\mu}_i$, where $i = 1, 2$.

In the next section, we will see that the similar construction could be applied to the vector and gravitational modes, with subtle modifications.

One can solve the highest-weight condition (3.7) explicitly. The solution is just

$$\Phi^{(0)}(0) = C_0(r - r_+)^{a - \frac{b}{r_+ - r_-}}(r - r_-)^{-a + \frac{b}{r_+ - r_-}},$$  \hspace{1em} (3.17)

where $C_0$ is an integration constant and

$$a = -iA\omega + iCk,$$

$$b = -iB\omega + iDk.$$

To satisfy the ingoing boundary condition at the horizon $r = r_+$, we need

$$A + \frac{B}{r_+ + r_-} < 0.$$  \hspace{1em} (3.18)

We will see that this is indeed the case for the black holes studied in this paper. Asymptotically, the solution behaves as

$$\Phi^{(0)} \sim r^{-h_R}.$$  \hspace{1em} (3.19)

So we see that the solution has the right behavior as the quasinormal mode. It is easy to find that the other quasinormal modes have the same asymptotical behavior.

4. Vector and Tensor Modes

Let us first consider the vector modes. Motivated by the impressive result on scalar modes, we try to compute $\mathcal{L}^2 A_\mu$ and expect a similar structure. However it turns out to be more complicated:

$$\mathcal{L}^2 A_\mu = \Pi^{\rho\sigma} \partial_\rho \partial_\sigma A_\mu + \Sigma^\rho \partial_\rho A_\mu + \partial_\mu \Sigma^\sigma A_\sigma + \Theta^{\rho\sigma}_\mu \partial_\rho A_\sigma,$$  \hspace{1em} (4.1)
where \( A_\mu \) is a vector field and \( \Pi^{\rho\sigma}, \Sigma^\sigma \) are defined in (3.3) and (3.4). \( \Upsilon_\mu^{\rho\sigma} \) is defined as

\[
\Upsilon_\mu^{\rho\sigma} \equiv V_1^\rho \partial_\mu V_{-1}^\sigma + V_{-1}^\rho \partial_\mu V_1^\sigma - 2V_0^\rho \partial_\mu V_0^\sigma.
\]

(4.2)

At first glance, (4.1) looks quite different from the scalar Eq. (3.6). Especially the fact that the different components are mixed together make things untractable. Nevertheless, we will show that for 3D black holes, the relation (4.1) could be simplified. The detailed discussion on the vector and the tensor perturbations in 3D black holes could be found in Appendix B.

Notice that the first and the second terms on the right-hand side of (4.1) are similar to the terms that appeared in the scalar modes. The third term vanishes if we only consider the \( A_t \) and \( A_\phi \) components since \( \Sigma^\sigma \) is only a function of \( r \) and independent of \( t \) and \( \phi \). To focus only on the \( A_t \) and \( A_\phi \) components is plausible since the \( A_r \) component can be determined by the other components in 3D. The only trouble comes from the fourth term, which cannot vanish automatically. The trick is that we should consider the superposition of \( A_t \) and \( A_\phi \). Let us define:

\[
A_+ = \kappa_1 A_t + \kappa_2 A_\phi,
\]

(4.3)

where \( \kappa_{1,2} \) are the constants to be determined. We find that a suitable choice of \( \kappa_1 \) and \( \kappa_2 \) can make all the components of \( \Upsilon_\mu^{\rho\sigma} \) vanish. Actually, if

\[
(\kappa_1 \partial_t + \kappa_2 \partial_\phi) V_i^\sigma = 0
\]

(4.4)

where \( i = 0, \pm 1 \) and \( \sigma = t, \phi, r \), then \( \Upsilon_\mu^{\rho\sigma} = 0 \). The above condition can be satisfied if

\[
\kappa_1 : \kappa_2 = -\mu_2 : \mu_1.
\]

(4.5)

Thus, we get

\[
L^2 A_+ = \Pi^{\rho\sigma} \partial_\rho \partial_\sigma A_+ + \Sigma^\rho \partial_\rho A_+.
\]

(4.6)

This shows that \( A_+ \) transform like a scalar. Now the question is if the equation of \( A_+ \) could be written like a scalar:

\[
(L^2 + m^2 v) A_+ = 0.
\]

(4.7)

Certainly \( m_v^2 \) may be different from the scalar case, depending on the backgrounds as well. We will show for the 3D black holes in this paper, (4.7) is always true.

Next we turn to the tensor fields. For the tensor field \( T_{\mu\nu} \), we have

\[
L^2 T_{\mu\nu} = \Pi^{\rho\sigma} \partial_\rho \partial_\sigma T_{\mu\nu} + \Sigma^\rho \partial_\rho T_{\mu\nu} + \partial_\mu \Sigma^\sigma T_{\sigma\nu} + \partial_\nu \Sigma^\sigma T_{\mu\sigma}
\]

\[
+ \Xi_{\mu\nu}^{\rho\sigma} T_{\rho\sigma} + \Upsilon_\mu^{\rho\sigma} \partial_\rho T_{\sigma\nu} + T_{\rho\sigma}^{\rho\sigma} \partial_\rho T_{\mu\nu}
\]

where we have defined

\[
\Xi_{\mu\nu}^{\rho\sigma} \equiv \partial_\rho V_1^\sigma \partial_\sigma V_{-1}^\rho + \partial_\sigma V_1^\rho \partial_\rho V_{-1}^\sigma - 2\partial_\rho V_0^\rho \partial_\sigma V_0^\sigma.
\]

(4.8)

By introducing

\[
T_+ = \kappa_1 T_{tt} + \kappa_2 T_{t\phi} + \kappa_3 T_{\phi t} + \kappa_4 T_{\phi\phi},
\]

(4.9)
we find that when
\[(κ_1 \partial_t + κ_2 \partial_φ)V_1^σ = 0,
(κ_1 \partial_t + κ_3 \partial_φ)V_1^σ = 0,
(κ_2 \partial_t + κ_4 \partial_φ)V_1^σ = 0,
(κ_3 \partial_t + κ_4 \partial_φ)V_1^σ = 0,\]  
(4.10)
all the redundant terms vanish and
\[\mathcal{L}^2 T_+ = \Pi^\rho^σ \partial_ρ \partial_σ T_+ + Σ^ρ \partial_ρ T_+. \]  
(4.11)
The condition (4.10) can be obeyed if the parameters $μ_i, κ_i$ satisfy the relations
\[κ_1 : κ_2 = -μ_2 : μ_1 = κ_3 : κ_4, \quad κ_2 = κ_3. \]  
(4.12)
As the vector case, we expect that the equations of motion of the tensor is
\[(\mathcal{L}^2 + m_t^2)T_+ = 0, \]  
(4.13)
for some constant $m_t$. We will show that for 3D black holes this is the case in the next section.

The above construction may be generalized to the higher-rank tensor fields. In general, for a rank $n$ tensor, we have
\[\mathcal{L}_V T_{l_1 l_2 \cdots l_n} = V^μ \partial_μ T_{l_1 l_2 \cdots l_n} + \partial_1 V^λ T_{λl_2 \cdots l_n} + \cdots + \partial_n V^λ T_{l_1 l_2 \cdots l_n−1 λ}. \]  
(4.14)
We can define a tensor as
\[T_+ = \sum κ_{σ_1 \cdots σ_n} T_{σ_1 \cdots σ_n}, \]  
(4.15)
where the summation is over all $σ_i = t, φ$. Then we can choose the $2^n$ coefficients $κ_\cdots$ such that
\[\mathcal{L}_V T_+ = V^μ_\partial_μ T_+. \]  
(4.16)
with $i = 0, ±1$. Note that this means that $T_+$ transform as a scalar under $SL(2, R)$. This could be satisfied if
\[(κ_{σ_1 \cdots σ_j τ_1 \cdots τ_n} \partial_τ + κ_{σ_1 \cdots σ_j φ_1 \cdots φ_n} \partial_φ) V^λ_1 = 0. \]  
(4.17)
There are $n \cdot 2^{n−1}$ constraints while there are only $2^n$ degrees of freedom. But the above equations are not independent and we can still determine the $2^n$ coefficients. One can begin with $κ_{tt \cdots t}$ and end with $κ_{φφ \cdots φ}$ step by step. Then one finds that
\[\mathcal{L}^2 T_+ = (Π^ρ^σ \partial_ρ \partial_σ + Σ^ρ \partial_ρ) T_+. \]  
(4.18)
and we wish that the equation of motion of $T_+$ could be written as
\[(\mathcal{L}^2 + m_t^2)T_+ = 0 \]  
(4.19)
with \( m_{hs} \) being a constant. In this paper, we just focus on the vector and rank 2 tensor and leave the general case for a future study.

Before we go into the concrete examples, we would like to discuss the physical implications of (4.7) and (4.13) on the quasinormal modes. It is not hard to see that if we have the relations (4.7) and (4.13), all the treatment on the scalar modes could be applied to the vector and tensor modes. That is to say, we can define the "highest-weight" modes \( \Psi^{(0)} \), where \( \Psi^{(0)} \) can be either \( A_{+} \) or \( T_{+} \), as

\[
L_{V_{1}} \Psi^{(0)} = 0, \quad L_{V_{0}} \Psi^{(0)} = h_{R} \Psi^{(0)}.
\]

Moreover, since (3.8) holds for arbitrary tensor fields, we can determine the conformal weight to be

\[
h_{R} = \frac{1}{2} (1 + \sqrt{1 + 4m_{i}^{2}}) \text{ with } m_{i}^{2} = m_{v}^{2} \text{ or } m_{t}^{2}.
\]

Similarly we can construct a tower of quasinormal modes \( \Psi^{(n)} \) as

\[
\Psi^{n} = (L_{V_{-}^{n}}) \Psi^{(0)}.
\]

The frequency of the quasinormal vector and tensor modes share the same structure as the scalar modes (3.16), with the difference coming from the conformal weights. Certainly we can construct the left sector modes in a similar way.

5. Quasinormal Modes in BTZ Black Hole

In this section, we take the BTZ black hole as a typical example to illustrate the above constructions of quasinormal modes. The scalar, vector and spinor quasinormal modes of the BTZ black hole were discussed in [8, 14], while the massive gravitational one in TMG theory was studied in [11] (see also [12]). The metric of a BTZ black hole is [13]

\[
ds^{2} = - \frac{(r^{2} - r_{+}^{2})(r^{2} - r_{-}^{2})}{r^{2}} dt^{2} + \frac{r^{2}}{(r^{2} - r_{+}^{2})(r^{2} - r_{-}^{2})} dr^{2} + r^{2}(d\phi - \frac{r_{+} - r_{-}}{r^{2}} dt)^{2}
\]

The left and right moving temperature are

\[
T_{L} = \frac{r_{+} - r_{-}}{2\pi}, \quad T_{R} = \frac{r_{+} + r_{-}}{2\pi}
\]

From the scalar equation we find the hidden conformal symmetry in the BTZ black hole. In the BTZ case, we should replacing \( r \) to \( r^{2} \) and \( \partial_{r} \) to \( \partial_{r^{2}} \) in the conformal coordinates and the vector fields defined in (2.1). It turns out the parameters in (2.1) should be

\[
\lambda_{1} = -\lambda_{2} = -\frac{1}{4\pi T_{R}}, \quad \mu_{1} = -\mu_{2} = 2\pi T_{R},
\]

\[
A = -C = -\frac{1}{8\pi T_{R}}, \quad B = D = -\frac{\pi T_{R}}{2}
\]

\[
\bar{\lambda}_{1} = \bar{\lambda}_{2} = -\frac{1}{4\pi T_{L}}, \quad \bar{\mu}_{1} = \bar{\mu}_{2} = 2\pi T_{L},
\]

\[
\bar{A} = \bar{C} = -\frac{1}{8\pi T_{L}}, \quad \bar{B} = -\bar{D} = -\frac{\pi T_{L}}{2}
\]
and we can also find that

\[ m^2_v = \frac{1}{4} m^2 \]  

(5.4)

where \( m \) is the scalar mass. By substituting \( \lambda_i \) into (3.16), we find

\[ \omega_R^{(n)} = -k - i4\pi T_R(n_R + h_R), \quad \omega_L^{(n)} = k - i4\pi T_L(n_L + h_L), \quad n_L, n_R \in \mathbb{N} \]  

(5.5)

where \( h_L = h_R = \frac{1}{2}(1 + \sqrt{1 + m^2}) \). This is in complete agreement with [8].

To check the vector modes, we begin with the vector field equation in three-dimensional spacetime:

\[ \epsilon^{\alpha\beta\lambda} \partial_\alpha A_\beta = -mA_\lambda. \]  

(5.6)

We can show that for the BTZ black hole, the above equation can be written as

\[ \tilde{\Delta} A_t = m^2 A_t + 2mA_\phi, \]

\[ \tilde{\Delta} A_\phi = m^2 A_\phi + 2mA_t, \]  

(5.7)

where we have defined the operator \( \tilde{\Delta} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \), which is an analogue to the Laplacian operator acting on the scalar field. See Appendix B.1 for more details. We immediately get

\[ \tilde{\Delta} A_\pm = (m^2 \pm 2m)A_\pm \]  

(5.8)

where \( A_\pm = A_t \pm A_\phi \). Note that this is just what we want. Using the language in the above section, as \( -\mu_2 : \mu_1 = 1 \) and \( -\bar{\mu}_2 : \bar{\mu}_1 = -1 \), we may choose \( \kappa_1 = \kappa_2 = 1 \) and \( \bar{\kappa}_1 = -\bar{\kappa}_2 = 1 \). Hence \( A_\pm \) transform like a scalar mode and

\[ m^2_v = \frac{1}{4} (m^2 + 2m) \]  

(5.9)

for the right-moving sector and

\[ m^2_v = \frac{1}{4} (m^2 - 2m) \]  

(5.10)

for the left-moving sector. Then we get

\[ h_R = \frac{m}{2} + 1, \quad h_L = \frac{m}{2} \]  

(5.11)

which is in agreement with the general result that \( |h_L - h_R| = s \). The frequencies of the quasinormal vector modes are still given by (5.5).

Next, we turn to the gravitational modes. For the standard 3D gravity, there is no propagating gravitational mode. However, for the topological massive gravity, there is a massive graviton, whose equation of motion could be written as a linear equation[11]

\[ \epsilon^{\mu\beta}_\alpha \nabla_\beta h_{\mu\nu} + mh_{\mu\nu} = 0. \]  

(5.12)

Analogue to the vector mode, we can show that for the BTZ black hole,

\[ \tilde{\Delta} h_{tt} = m^2 h_{tt} + 2mh_{t\phi} + 2mh_{t\phi} + h_{tt} + 2h_{\phi\phi}, \]

\[ \tilde{\Delta} h_{t\phi} = m^2 h_{t\phi} + 2mh_{t\phi} + 2mh_{t\phi} + h_{t\phi} + 2h_{\phi\phi}, \]

\[ \tilde{\Delta} h_{\phi\phi} = m^2 h_{\phi\phi} + 2mh_{\phi\phi} + 2mh_{\phi\phi} + h_{\phi\phi} + 2h_{tt}. \]  

(5.13)
See Appendix B for more detail. After defining \( h_\pm = h_{tt} \pm h_{t\phi} \pm h_{\phi t} + h_{\phi\phi} \), we get
\[
\Delta h_\pm = (m^2 \pm 4m + 3)h_\pm
\]  
(5.14)

The above equations are precisely what we expect. Since in this case, we should choose \( \kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 1 \) and \( \bar{\kappa}_1 = -\bar{\kappa}_2 = -\bar{\kappa}_3 = \bar{\kappa}_4 = 1 \). Then \( h_\pm \) are just the \( T_\pm \) we have defined in the previous section. Consequently, we find that
\[
m_t^2 = \frac{1}{4}(m^2 + 4m + 3)
\]
(5.15)
for the right-moving sector and
\[
\bar{m}_t^2 = \frac{1}{4}(m^2 - 4m + 3)
\]
(5.16)
for the left-moving sector. The right and left conformal weight are respectively
\[
h_R = \frac{m + 3}{2}, \quad h_L = \frac{m - 1}{2},
\]
(5.17)
which again is consistent with the fact that \( |h_L - h_R| = s \). The frequencies of the gravitational quasinormal modes take the same form as (5.5).

6. quasinormal Modes in Warped \textit{AdS}$_3$ and Self-dual Warped \textit{AdS}$_3$ Black Hole

In this section, we will generalize the algebraic method to the warped \textit{AdS}$_3$ and the self-dual warped \textit{AdS}$_3$ black holes. For the warped black holes, the scalar equations could not be simply written as (3.6). Actually they take the following form:
\[
(L^2 + b \bar{L}^2_{V_0} + m_s^2)\Phi = 0
\]
(6.1)
with \( b \) and \( m_s^2 \) being constants. This is a little different from the previous discussion due to the presence of the \( b \bar{L}^2_{V_0} \) term. Nevertheless, we can still construct a tower of right-moving modes by imposing the conditions
\[
\mathcal{L}_{V_1}\Phi^{(0)} = 0, \quad \mathcal{L}_{V_0}\Phi^{(0)} = h_R\Phi^{(0)}, \quad \Phi^{(n)} = (\mathcal{L}_{V_{-1}})^n\Phi^{(0)}.
\]
(6.2)
The first two conditions just define the “highest-weight” mode. And the last equation construct the descendent modes. Because of the commutative relation \( [\mathcal{L}_{V_{-1}}, \tilde{\mathcal{L}}_{V_0}] = 0 \), all the modes \( \Phi^{(n)} \) satisfy the scalar equation as well. The following discussion is similar to the one in Sec. III. Here we only give the results:
\[
h_R = \frac{1}{2}(1 + \sqrt{1 + 4(bq^2 + m_s^2)}), \quad \lambda_1\omega_R^{(n)} = \lambda_2k + i(h_R + n)
\]
(6.3)
where \( q \) is defined by \( \tilde{\mathcal{L}}_{V_0}\Phi^{(0)} = q\Phi^{(0)} \). However, we can not construct the left-moving modes due to the noncommutative relation of \( \tilde{\mathcal{L}}_{V_0} \) and \( \tilde{\mathcal{L}}_{V_{-1}} \). Because of the presence of
term in the scalar equation, the conformal weight depends on the quantum number $q$. This fact is in consistency with the known result.

Next, we try to generalize the above discussion to the vector modes. In this case, we find that for any vector $A_{\mu}$

$$\mathcal{L}_{V_0}^2 A_{\mu} = \bar{V}_0^\lambda \partial_\mu V_0^\sigma \partial_\sigma A_{\mu}. \quad (6.4)$$

All the redundant term of $\partial_\mu \bar{V}_0^\lambda$ vanish since $\bar{V}_0^\lambda$ are constant numbers. This implies that we can still define $A_+ = \kappa_1 A_t + \kappa_2 A_\phi$ with $\kappa_1 : \kappa_2 = -\mu_2 : \mu_1$. We still expect that it transforms as a scalar. More explicitly, we wish

$$(\mathcal{L}^2 + b\mathcal{L}_{V_0}^2 + m_v^2)A_+ = 0. \quad (6.5)$$

If this is true, we can discuss the vector modes parallel to the treatment on the scalar modes. We will check this point in the warped $AdS_3$ and self-dual warped $AdS_3$ black hole backgrounds in the next two subsections.

For the warped spacetime, the equation of motion of the gravitational mode could not be written as a linear equation and is much more involved. Here we just assume that there is a massive rank 2 symmetric tensor mode in the warped spacetime. In 3D dimension, its equation of motion is

$$\epsilon^{\alpha\beta}_{\mu} \nabla_\alpha h_{\beta\nu} + mh_{\mu\nu} = 0. \quad (6.6)$$

In this case, we can still define $T_+ = \kappa_1 T_{tt} + \kappa_2 T_{t\phi} + \kappa_3 T_{\phi t} + \kappa_4 T_{\phi\phi}$ with $\kappa_i$’s satisfying (4.12) and wish it to satisfy the equation of the form

$$(\mathcal{L}^2 + b\mathcal{L}_{V_0}^2 + m_t^2)T_+ = 0. \quad (6.7)$$

If this is true, it allows us to construct the tensor quasinormal modes in the similar way.

### 6.1 Warped $AdS_3$ black hole

The metric of the spacelike stretched warped $AdS_3$ black hole is

$$ds^2 = dt^2 + 2M(r)dtd\phi + N(r)d\phi^2 + Q(r)dr^2 \quad (6.8)$$

where

$$M = \nu r - \frac{1}{2} \sqrt{r_+ r_- (\nu^2 + 3)},$$

$$N = \frac{r}{4} [3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r_+ r_- (\nu^2 + 3)}],$$

$$Q = \frac{1}{(\nu^2 + 3)(r - r_+)(r - r_-)}. \quad (6.9)$$

From warped AdS/CFT correspondence, the right- and left-moving temperatures in the dual 2D CFT are

$$T_L = \frac{\nu^2 + 3}{8\pi} (r_+ + r_- - \sqrt{r_+ r_- (\nu^2 + 3)}), \quad T_R = \frac{(\nu^2 + 3)(r_+ - r_-)}{8\pi}. \quad (6.10)$$
The hidden conformal symmetry of the warped AdS$_3$ black hole has been discussed in [19]. From the scalar equation, we find that

$$\lambda_1 = -\frac{2\nu T_L}{(\nu^2 + 3)T_R}, \quad \lambda_2 = \frac{1}{2\pi T_R}, \quad \mu_1 = 0, \quad \mu_2 = -2\pi T_R$$

$$A = -\frac{\nu T_L}{(\nu^2 + 3)T_R}, \quad B = -\frac{\nu(r_+ - r_-)}{\nu^2 + 3}, \quad C = \frac{1}{4\pi T_R}, \quad D = 0$$

$$\bar{\lambda}_1 = -\frac{2\nu}{\nu^2 + 3}, \quad \bar{\lambda}_2 = 0, \quad \bar{\mu}_1 = \frac{\nu^2 + 3}{2\nu}, \quad \bar{\mu}_2 = 2\pi T_L$$

(6.11)

$$\bar{A} = -\frac{\nu}{\nu^2 + 3}, \quad \bar{B} = \frac{\nu 8\pi T_L}{(\nu^2 + 3)^2}, \quad \bar{C} = 0, \quad \bar{D} = \frac{2}{\nu^2 + 3}$$

and $b, q, m^2_s$ are

$$b = \frac{3(\nu^2 - 1)}{4\nu^2}, \quad q = \frac{2\nu\omega}{\nu^2 + 3}, \quad m^2_s = \frac{m^2}{\nu^2 + 3}$$

(6.12)

where $m$ is the scalar mass. Hence the scalar conformal weight is

$$h_R = \frac{1}{2}(1 + \sqrt{1 + 4\frac{m^2(\nu^2 + 3) - 3(\nu^2 - 1)\omega^2}{(\nu^2 + 3)^2}})$$

(6.13)

As emphasized in [18], to compare with the poles of the correlation functions in the dual CFT, we should take the following identification on quantum numbers into account [18]

$$\tilde{\omega} = \frac{2k}{\nu^2 + 3}, \quad \tilde{k} = \frac{2\nu\omega}{\nu^2 + 3},$$

(6.14)

where $\tilde{\omega}, \tilde{k}$ are the quantum numbers of global warped AdS$_3$ spacetime. Then we find the scalar quasinormal modes with the frequencies

$$\omega_R^{(n)} = \frac{1}{\nu^2 + 3}(-4\pi T_L \tilde{k} - i4\pi T_R (n + h_R)).$$

(6.15)

This is in agreement with the result in [17, 18].

Next, we check the vector modes. Since in this case, $\mu_1 = 0$ indicates $\kappa_1 = 1, \kappa_2 = 0$, we should choose $A_+ = A_t$. In Appendix B.1 we show that $A_t$ satisfy

$$\tilde{\Delta} A_t = (m^2_s + 2m\nu)A_t,$$

(6.16)

which could be rewritten as

$$\left(\mathcal{L}^2 + b \tilde{\mathcal{L}}^2_{\nu_0} + m^2_{v}\right)A_+ = 0$$

(6.17)

with $m^2_v = \frac{m^2_s + 2m\nu}{\nu^2 + 3}$ and $b$ has been given in the scalar case. This is in agreement with our expectation. Hence, the vector conformal weight is

$$h_R = \frac{1}{2}(1 + \sqrt{1 + 4\left(\frac{m^2_s + 2m\nu}{\nu^2 + 3} - 3(\nu^2 - 1)\tilde{k}^2}{4\nu^2}\right)})$$

(6.18)

where we have used the identification $\tilde{k} = \frac{2\nu\omega}{\nu^2 + 3}$. The result is in perfect match with the result in [18]. The spectrum of the vector quasinormal modes takes the same form as (6.15).
For the tensor mode, as $\mu_1 = 0, \mu_2 \neq 0$, we may choose
\[ \kappa_1 = 1, \kappa_2 = \kappa_3 = \kappa_4 = 0 \] (6.19)
and have
\[ T_+ = h_{tt}. \] (6.20)
From the equation of motion, we learn that
\[ \tilde{\Delta} h_{tt} = (m^2 + 4m\nu + 3\nu^2)h_{tt} \] (6.21)
which could be rewritten as
\[ (L^2 + b\tilde{L}_t^2 + m_t^2)T_+ = 0 \] (6.22)
with $m_t^2 = \frac{m^2 + 4m\nu + 3\nu^2}{\nu^2 + 3}$. Thus we find the conformal weight of the tensor mode
\[ h_R = \frac{1}{2}(1 + \sqrt{1 + 4\left(\frac{(m^2 + 4m\nu + 3\nu^2)}{\nu^2 + 3} - \frac{3(\nu^2 - 1)\tilde{k}}{4\nu^2}\right)}). \] (6.23)
When $\nu = 1$, the warped black hole reduces to the BTZ black hole and the tensor conformal weight reduces to $h_R$ in (5.17). The spectrum of tensor quasinormal modes takes the same form as (6.15).

6.2 Self-dual warped $AdS_3$ black hole

The self-dual warped $AdS_3$ black hole is a vacuum solution of 3D topological massive gravity. It could be described by the metric
\[ ds^2 = \frac{l^2}{\nu^2 + 3} \left(- (r - r_+)(r - r_-) dt^2 + \frac{1}{(r - r_+)(r - r_-)} dr^2 + \frac{4\nu^2}{\nu^2 + 3} (\alpha d\phi + (r - \frac{r_+ + r_-}{2}) dt)^2 \right), \] (6.24)
where the coordinates range as $t \in [-\infty, \infty], r \in [-\infty, \infty]$ and $\phi \sim \phi + 2\pi$.

The hidden conformal symmetry of this black hole has been discussed in [21]. It turns out that the parameters in the vector fields take the following values
\[ \lambda_1 = -\frac{1}{2\pi T_R}, \quad \lambda_2 = 0, \quad \mu_1 = 2\pi T_R, \quad \mu_2 = 0, \]
\[ A = -\frac{1}{4\pi T_R}, \quad B = 0, \quad C = 0, \quad D = \frac{T_R}{T_L}, \]
\[ \bar{\lambda}_1 = 0, \quad \bar{\lambda}_2 = -\frac{1}{2\pi T_L}, \quad \bar{\mu}_1 = 0, \quad \bar{\mu}_2 = 2\pi T_L, \]
\[ \bar{A} = 0, \quad \bar{B} = 1, \quad \bar{C} = -\frac{1}{4\pi T_L}, \quad \bar{D} = 0, \] (6.25)

---

The scalar quasinormal mode of self-dual black hole has been discussed in [21] in the similar way as the one in [1].
where the right- and left- moving temperature are

\[ T_L = \frac{\alpha}{2\pi}, \quad T_R = \frac{r_+ - r_-}{4\pi}. \]  

(6.27)

The \( b, q, m_s^2 \) are

\[ b = \frac{3(\nu^2 - 1)}{4\nu^2}, \quad q = -\frac{ik}{\alpha}, \quad m_s^2 = \frac{m^2}{\nu^2 + 3}. \]  

(6.28)

The conformal weight for the scalar is just

\[ h_R = \frac{1}{2}(1 + \sqrt{1 - \frac{3(\nu^2 - 1)k^2}{\nu^2\alpha^2} + \frac{4m^2}{\nu^2 + 3}}). \]  

(6.29)

For the vector field, as \( \mu_2 = 0 \), we may choose \( \lambda_4 = 1 \) so that \( A_+ = A_\phi \). In Appendix B.1, we see that \( A_\phi \) satisfy

\[ \tilde{\Delta}A_\phi = (m^2 - 2m\nu)A_\phi, \]  

(6.30)

which could be cast into the form

\[ (\mathcal{L}^2 + b\mathcal{L}_{V_0}^2 + m_v^2)A_+ = 0 \]  

(6.31)

with \( m_v^2 = \frac{m^2 - 2m\nu}{\nu^2 + 3} \). The conformal weight for the vector is then

\[ h_R = \frac{1}{2}(1 + \sqrt{1 - \frac{3(\nu^2 - 1)k^2}{\nu^2\alpha^2} + \frac{4(m^2 - 2m\nu)}{\nu^2 + 3}}). \]  

(6.32)

For the tensor field, as \( \mu_2 = 0, \mu_1 \neq 0 \), we may choose

\[ \kappa_4 = 1, \kappa_1 = \kappa_2 = \kappa_3 = 0 \]  

(6.33)

and have

\[ T_+ = h_{\phi\phi}. \]  

(6.34)

From the equation of motion, we learn that

\[ \tilde{\Delta}h_{\phi\phi} = (m^2 - 4m\nu + 3\nu^2)h_{tt} \]  

(6.35)

which could be rewritten as

\[ (\mathcal{L}^2 + b\mathcal{L}_{V_0}^2 + m_t^2)T_+ = 0 \]  

(6.36)

with \( m_t^2 = \frac{m^2 - 4m\nu + 3\nu^2}{\nu^2 + 3} \). The conformal weight of the tensor field is just

\[ h_R = \frac{1}{2}(1 + \sqrt{1 - \frac{3(\nu^2 - 1)k^2}{\nu^2\alpha^2} + \frac{4(m^2 - 4m\nu + 3\nu^2)}{\nu^2 + 3}}), \]  

(6.37)

which could reduce to \( h_L \) in (5.17) at \( \nu = 1 \).

In all cases, the quasinormal modes can be written as

\[ \omega_n^{(n)} = -i2\pi T_R(n + h_R) \]  

(6.38)

where \( h_R \) can be the scalar, the vector or the tensor conformal weight. The results all in good agreement with [21, 22].
7. Discussions

In this paper, we have studied the relation between the hidden conformal symmetry and the quasinormal modes. We found that the spectrum of the quasinormal modes may be directly read out from the action of the hidden conformal symmetry on various perturbations. Our construction provides a direct rule to find the spectrum. The rule is simple and show clearly that the quasinormal modes are determined completely by the hidden conformal symmetry. We found that in the spectrums,

$$\omega \propto -i2\pi T(h+n),$$

which is in accordance with the structure of the poles of the correlation functions of the dual operators in CFT.

Our construction is based on the relation (3.8) on the Lie-derivatives and the fact that the Lie-induced Casimir $L^2$ defined in (3.1) commutes with the Lie-derivatives. Starting from the highest-weight mode, we can construct its infinite tower of descendent modes. For the scalar, the construction is straightforward, as shown in Sec. III. However, the action of the hidden conformal symmetry on the vector and tensor field is highly nontrivial. We observed that only after some suitable composition the vector and the gravitation modes behaved like the scalar modes. This allowed us to treat the scalar, vector and gravitational modes in a uniform way. From our construction, the spectrum of various kinds of quasinormal modes are in agreement with the CFT prediction and previous study. Moreover, our discussion in Sec. IV suggested that our treatment could be applied to higher-rank tensor fields. It would be nice to have a detailed study on this question. Another interesting issue is to study if the hidden conformal symmetry can determine the fermionic quasinormal modes.

We applied our method to the case of the BTZ black hole and find perfect agreement with the known results. For the warped $\text{AdS}_3$ and self-dual warped $\text{AdS}_3$ black holes, the discussion is subtler. Even for the scalar mode, the scalar equation could not be simply written as the $SL(2,R)$ quadratic Casimir for all quantum numbers. Nevertheless, we can still apply our treatment with a minor modification. For all the scalar, vector and tensor modes, we managed to construct towers of the quasinormal modes, in agreement with the ones found in the literature. Strictly speaking, we only succeeded in finding one set of the quasinormal modes, corresponding to the poles of the correlation functions of the right-moving sector in the dual CFT. It would be nice to find the other set, corresponding to the left-moving ones.

In this paper, we studied the quasinormal modes of the nonextremal black holes. Since the coordinates that are used to implement the hidden conformal symmetry are different in the extremal case, it is interesting to see if the same construction works for the extreme black holes. We expect an similar conclusion.

We discussed the action of the hidden conformal symmetry on the vector and tensor fields in three-dimensional spacetime. It would be interesting to investigate this issue in the Kerr/CFT correspondence in higher dimensions. However in this case, the problem is much more complicated because we have to apply the Newman-Penrose formalism to
obtain the Teukolsky master equation\cite{23, 24} of high spin perturbations. It is not clear how the hidden conformal symmetry is realized in this framework.

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**Appendix A**

The explicit forms of $\Pi^{\rho\sigma}$ and $\Sigma^\sigma$ are the following:

\[
\begin{align*}
\Pi^{rr} &= -\Delta \\
\Pi^{tt} &= (A \frac{\Delta'}{\sqrt{\Delta}} + B \frac{1}{\sqrt{\Delta}})^2 - \lambda_1^2 \\
\Pi^{\phi\phi} &= (C \frac{\Delta'}{\sqrt{\Delta}} + D \frac{1}{\sqrt{\Delta}})^2 - \lambda_2^2 \\
\Pi^{t\phi} &= \Pi^{\phi t} = (A \frac{\Delta'}{\sqrt{\Delta}} B \frac{1}{\sqrt{\Delta}}) (C \frac{\Delta'}{\sqrt{\Delta}} + D \frac{1}{\sqrt{\Delta}}) - \lambda_1 \lambda_2 \\
\Pi^{rt} &= \Pi^{tr} = \Pi^{r\phi} = \Pi^{\phi r} = 0 \\
\end{align*}
\]

and

\[
\Sigma^t = \Sigma^\phi = 0, \quad \Sigma^r = -\Delta'.
\]

**Appendix B: Vector and Tensor Perturbation In (2+1)-dim. Black Holes**

In this section, we give a discussion of the vector and the tensor perturbations in (2+1)-dim. black holes. The discussion is not restricted to the black holes studied in this paper. In fact, we only require the following conditions on the metric:

\[
\partial_t g_{\mu\nu} = 0, \quad \partial_\phi g_{\mu\nu} = 0, \quad g_{rt} = g_{r\phi} = 0.
\]

**B.1 Vector perturbation**

We begin with the vector equation in 3D spacetime

\[
\epsilon_\lambda^{\alpha\beta} \partial_\alpha A_\beta = -mA_\lambda,
\]

which could be written in components

\[
\begin{align*}
A_r &= -\frac{1}{m} \epsilon_r^{\phi}(\partial_r A_\phi - \partial_\phi A_t), \\
\partial_r A_t &= \partial_t A_r - m \frac{(\epsilon_r^{\phi} A_t - \epsilon_t^{\phi} A_r)}{\epsilon_r^{\phi} \epsilon_t^{\phi} - \epsilon_r^{\phi} \epsilon_t^{\phi}}, \\
\partial_r A_\phi &= \partial_\phi A_r - m \frac{(\epsilon_r^t A_\phi - \epsilon_r^t A_t)}{\epsilon_r^t \epsilon_r^\phi - \epsilon_t^t \epsilon_r^\phi}.
\end{align*}
\]
Obviously the $A_r$ component could be decided by $A_t$ and $A_\phi$. Our goal is to find an equation which is analogue to the scalar equation
\[
\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^\mu\nu \partial_\nu \Phi = \cdots. \tag{7.7}
\]
This motivates us to compute $\tilde{\Delta} A_i = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^\mu\nu \partial_\nu A_i$, with $i = t, \phi$. The results are
\[
\begin{align*}
\tilde{\Delta} A_t &= m^2 A_t + m \varepsilon^{tr\phi} \frac{1}{\sqrt{-g}} (g'_{t\phi} A_t - g'_{tt} A_\phi), \\
\tilde{\Delta} A_\phi &= m^2 A_\phi + m \varepsilon^{\phi tr} \frac{1}{\sqrt{-g}} (g'_{t\phi} A_\phi - g'_{\phi\phi} A_t),
\end{align*}
\]
where $\varepsilon^\mu_{\lambda\nu}$ is the Levi-Civita tensor and $\varepsilon^{tr\phi}$ is the Levi-Civita symbol with $\varepsilon^{tr\phi} = 1$.

For the BTZ black hole, all the $r$ coordinates should be replaced by $r^2$, and then
\[
\begin{align*}
g'_{t\phi} &= 0, \quad g'_{tt} = -1, \quad g'_{\phi\phi} = 1, \quad \sqrt{-g} = \frac{1}{2}. \tag{7.8}
\end{align*}
\]
Note that the derivative should be taken with respect to $r^2$. Then we get
\[
\begin{align*}
\tilde{\Delta} A_t &= m^2 A_t + 2mA_\phi, \tag{7.9} \\
\tilde{\Delta} A_\phi &= m^2 A_\phi + 2mA_t.
\end{align*}
\]
For the spacelike stretched warped AdS$_3$ black hole, since $g'_{tt} = \nu, \sqrt{-g} = \frac{1}{2}$, we find that
\[
\tilde{\Delta} A_t = (m^2 + 2m\nu)A_t. \tag{7.10}
\]
For the self-dual warped AdS$_3$ black hole, $g'_{t\phi} = \frac{4\nu^2 \phi}{(\nu^2 + 3)^2}, g'_{\phi\phi} = 0, \sqrt{-g} = \frac{2\nu \phi}{(\nu^2 + 3)^2}$, then $A_\phi$ satisfy
\[
\tilde{\Delta} A_\phi = (m^2 - 2m\nu)A_\phi. \tag{7.11}
\]

**B.2 Tensor perturbation**

In 3D spacetime, the rank 2 symmetric tensor perturbation obeys the equation
\[
\varepsilon^\alpha_{\mu\beta} \nabla_\alpha h_{\beta\nu} + mh_{\mu\nu} = 0. \tag{7.12}
\]
For the BTZ black hole in 3D TMG theory, this is the equation for a massive graviton. However for the warped AdS spacetime, the gravitational perturbation could not be put into such a simple form [16]. Nevertheless we can still assume a massive tensor perturbation in the backgrounds, satisfying this equation.

From this equation, we can easily find
\[
\begin{align*}
h'_{\phi\phi} &= \partial_\phi h_{r\phi} + \Gamma_{(\phi\phi)} + m_{(\phi\phi)}, \\
h'_{t\phi} &= \partial_t h_{r\phi} + \Gamma_{(t\phi)} + m_{(t\phi)}, \\
h'_{\phi t} &= \partial_\phi h_{rt} + \Gamma_{(\phi t)} + m_{(\phi t)}, \\
h'_{tt} &= \partial_t h_{rt} + \Gamma_{(tt)} + m_{(tt)},
\end{align*}
\]
and

\[ h_{rt} = \frac{1}{W}(a_{rt} + m_{(rt)}), \]
\[ h_{r\phi} = \frac{1}{W}(a_{r\phi} + m_{(r\phi)}), \]
\[ h_{rr} = g_{rr}^2(e^{r\phi})^2(g_{tt} h_{r\phi} - 2g_{t\phi} h_{t\phi} + g_{\phi\phi} h_{tt}), \]

where we have defined

\[
\Gamma_{(\phi\phi)} = \Gamma_{r\phi}^\lambda h_{\phi\lambda} - \Gamma_{\phi\phi}^\lambda h_{r\lambda} \\
\Gamma_{(t\phi)} = \Gamma_{r\phi}^\lambda h_{r\lambda} - \Gamma_{t\phi}^\lambda h_{r\lambda} \\
\Gamma_{(\phi t)} = \Gamma_{r\phi}^\lambda h_{\phi\lambda} - \Gamma_{\phi t}^\lambda h_{r\lambda} \\
\Gamma_{(tt)} = \Gamma_{r\phi}^\lambda h_{t\lambda} - \Gamma_{tt}^\lambda h_{r\lambda} \\
m_{(\phi\phi)} = m_{g_{r\phi}} e^{r\phi}(g_{t\phi} h_{\phi\phi} - g_{\phi\phi} h_{t\phi}) \\
m_{(t\phi)} = m_{g_{r\phi}} e^{r\phi}(g_{t\phi} h_{t\phi} - g_{tt} h_{\phi\phi}) \\
m_{(\phi t)} = m_{g_{r\phi}} e^{r\phi}(g_{t\phi} h_{t\phi} - g_{tt} h_{\phi\phi}) \\
m_{(tt)} = m_{g_{r\phi}} e^{r\phi}(g_{t\phi} h_{tt} - g_{tt} h_{t\phi}) \\
m_{(rt)} = -m_{g_{r\phi}} e^{r\phi}(\partial_t h_{\phi\phi} - \partial_\phi h_{tt}) \\
m_{(r\phi)} = -m_{g_{r\phi}} e^{r\phi}(\partial_t h_{t\phi} - \partial_\phi h_{t\phi}) \\
W = m^2 - \frac{1}{4}(e^{r\phi})^2(g_{t\phi} g_{t\phi} - g_{tt} g_{t\phi})
\]

and

\[
a_{rt} = -\frac{1}{2} g_{rt} (e^{r\phi})^2[-g_{tt}(\partial_t h_{\phi\phi} - \partial_\phi h_{t\phi}) + g_{t\phi}(\partial_t h_{t\phi} - \partial_\phi h_{tt})] \\
a_{r\phi} = -\frac{1}{2} g_{rt} (e^{r\phi})^2[-g_{t\phi}(\partial_\phi h_{tt} - \partial_t h_{t\phi}) + g_{t\phi}(\partial_\phi h_{t\phi} - \partial_t h_{\phi\phi})].
\]

Similar to the vector case, we find the following equations:

\[
\Delta h_{\phi\phi} = m^2 h_{\phi\phi} + \frac{m e^{r\phi}}{\sqrt{-g}} \alpha_{\phi\phi} + \frac{\beta}{4(-g)} h_{\phi\phi} + \frac{1}{2(-g)} \gamma_{\phi\phi} + (I)_{\phi\phi} + (II)_{\phi\phi},
\]
\[
\Delta h_{t\phi} = m^2 h_{t\phi} + \frac{m e^{r\phi}}{\sqrt{-g}} \alpha_{t\phi} + \frac{\beta}{4(-g)} h_{t\phi} + \frac{1}{2(-g)} \gamma_{t\phi} + (I)_{t\phi} + (II)_{t\phi},
\]
\[
\Delta h_{\phi t} = m^2 h_{\phi t} + \frac{m e^{r\phi}}{\sqrt{-g}} \alpha_{\phi t} + \frac{\beta}{4(-g)} h_{\phi t} + \frac{1}{2(-g)} \gamma_{\phi t} + (I)_{\phi t} + (II)_{\phi t},
\]
\[
\Delta h_{tt} = m^2 h_{tt} + \frac{m e^{r\phi}}{\sqrt{-g}} \alpha_{tt} + \frac{\beta}{4(-g)} h_{tt} + \frac{1}{2(-g)} \gamma_{tt} + (I)_{tt} + (II)_{tt},
\]

where

\[
\alpha_{\phi\phi} = 2g_{t\phi} h_{\phi\phi} - 2g_{t\phi} h_{t\phi} \\
\alpha_{t\phi} = -g_{t\phi} h_{tt} + g_{tt} h_{\phi\phi} \\
\alpha_{\phi t} = -g_{t\phi} h_{t\phi} + g_{t\phi} h_{tt} \\
\alpha_{\phi\phi} = 2g_{t\phi} h_{tt} - 2g_{tt} h_{t\phi}
\]
\[\gamma_{\phi\phi} = g'_{\phi\phi} g_{\phi\phi} h_{\phi\phi} - g'_{\phi\phi} g_{\phi\phi} h_{\phi\phi} - g'_{\phi\phi} g_{\phi\phi} h_{\phi\phi} + g'_{\phi\phi} g_{\phi\phi} h_{\phi\phi}\]
\[\gamma_{t\phi} = g'_{t\phi} g_{t\phi} h_{\phi\phi} - g'_{t\phi} g_{t\phi} h_{\phi\phi} - g'_{t\phi} g_{t\phi} h_{\phi\phi} + g'_{t\phi} g_{t\phi} h_{\phi\phi}\]
\[\gamma_{t\phi} = g'_{t\phi} g_{t\phi} h_{\phi\phi} - g'_{t\phi} g_{t\phi} h_{\phi\phi} - g'_{t\phi} g_{t\phi} h_{\phi\phi} + g'_{t\phi} g_{t\phi} h_{\phi\phi}\]
\[\gamma_{tt} = g'_{tt} g_{tt} h_{\phi\phi} - g'_{tt} g_{tt} h_{\phi\phi} - g'_{tt} g_{tt} h_{\phi\phi} + g'_{tt} g_{tt} h_{\phi\phi}\]

\[
(I)_{\phi\phi} = \sqrt{-g} \frac{(e^{t\phi})^2}{W} \left[ \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) + \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_{\phi} \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) \right]
\]
\[
(I)_{t\phi} = \sqrt{-g} \frac{(e^{t\phi})^2}{W} \left[ \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) + \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) \right]
\]
\[
(I)_{t\phi} = \sqrt{-g} \frac{(e^{t\phi})^2}{W} \left[ \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) + \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) \right]
\]
\[
(I)_{tt} = \sqrt{-g} \frac{(e^{t\phi})^2}{W} \left[ \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) + \frac{1}{2} \left( \partial_r g'_{\phi\phi} \right) \partial_t \left( \partial_t h_{\phi\phi} - \partial_{\phi\phi} h_{tt} \right) \right]
\]

\[
(II)_{\phi\phi} = \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{\phi\phi} h_{\phi\phi} + g_{\phi\phi} h_{\phi\phi} \right) - \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} - g_{t\phi} h_{t\phi} \right)
\]
\[
(II)_{t\phi} = \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} + g_{t\phi} h_{t\phi} \right) - \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} - g_{t\phi} h_{t\phi} \right)
\]
\[
(II)_{t\phi} = \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} + g_{t\phi} h_{t\phi} \right) - \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} - g_{t\phi} h_{t\phi} \right)
\]
\[
(II)_{tt} = \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} + g_{t\phi} h_{t\phi} \right) - \frac{\sqrt{-g}}{2} \left( \partial_r g'_{\phi\phi} \right) \left( g_{t\phi} h_{t\phi} - g_{t\phi} h_{t\phi} \right)
\]

and \(\beta\) is defined as
\[
\beta = g'_{t\phi} g'_{t\phi} - g'_{tt} g'_{tt}.
\]

The above equations are our main results for the rank 2 tensor perturbations in three-dimensional black hole backgrounds satisfying the conditions \((7.14)\).

For the BTZ black hole, we replace \(r\) to \(r^2\) and \(\partial_r\) to \(\partial_{r^2}\), then we find

\[
(I)_{ij} = (II)_{ij} = 0, \quad \beta = 1, \quad \sqrt{-g} = \frac{1}{2}
\]

\[
\alpha_{\phi\phi} = -2h_{t\phi}, \quad \alpha_{t\phi} = -h_{\phi\phi} - h_{tt}, \quad \alpha_{t\phi} = h_{tt} + h_{\phi\phi}, \quad \alpha_{tt} = 2h_{tt}
\]

such that

\[
\tilde{\Delta} h_{tt} = m^2 h_{tt} + 2m h_{\phi\phi} + 2m h_{t\phi} + h_{tt} + 2h_{\phi\phi},
\]
\[
\tilde{\Delta} h_{t\phi} = m^2 h_{t\phi} + 2m h_{\phi\phi} + 2m h_{tt} + h_{t\phi} + 2h_{\phi\phi},
\]
\[
\tilde{\Delta} h_{tt} = m^2 h_{tt} + 2m h_{\phi\phi} + 2m h_{tt} + h_{tt} + 2h_{tt},
\]
\[
\tilde{\Delta} h_{\phi\phi} = m^2 h_{\phi\phi} + 2m h_{\phi\phi} + 2m h_{tt} + h_{\phi\phi} + 2h_{tt}.
\]
For the warped spacetimes, the discussion is similar but more tedious. It turns out that for the spacelike stretched AdS$_3$ black hole, the component $h_{tt}$ obeys the equation
\[ \tilde{\Delta}h_{tt} = (m^2 + 4m\nu + 3\nu^2)h_{tt}, \tag{7.16} \]
while for the self-dual AdS$_3$ black hole, the component $h_{\phi\phi}$ obeys
\[ \tilde{\Delta}h_{\phi\phi} = (m^2 - 4m\nu + 3\nu^2)h_{\phi\phi}. \tag{7.17} \]

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