DIRICHLET DIVISOR PROBLEM
ON GAUSSIAN INTEGERS

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Abstract. We improve existing estimates of moments of the Riemann zeta function. As a consequence, we are able to derive new estimates for the asymptotic behaviour of \( \sum_{N\alpha \leq x} t_k(\alpha) \), where \( N \) stands for the norm of a complex number and \( t_k \) is the \( k \)-dimensional divisor function on Gaussian integers.

1. Introduction

Define the divisor function \( \tau \) and its generalisation, the \( k \)-dimensional divisor function \( \tau_k \) as follows:

\[
\tau(n) = \sum_{d|n, d > 0} 1, \quad \tau_k(n) = \sum_{d_1 \cdots d_k = n, d_1, \ldots, d_k > 0} 1,
\]

where \( n \) is a non-zero integer. That said, \( \tau \equiv \tau_2 \).

One can study asymptotic properties of summatory functions \( \sum_{0<n\leq x} \tau(n) \) and \( \sum_{0<n\leq x} \tau_k(n) \). It can be shown (see [8, (6.27)]) that for growing \( x \to \infty \) we have

\[
\sum_{0<n\leq x} \tau(n) = x \ln x + (2\gamma - 1)x + O(x^{\theta_2}),
\]

\[
\sum_{0<n\leq x} \tau_k(n) = x P_k(\ln x) + O(x^{\theta_k}),
\]

where \( \gamma \approx 0.577 \) denotes the Euler–Mascheroni constant, \( P_k(y) \) is a fixed univariate polynomial of degree \( k - 1 \), and \( \theta_k \) is a positive real. The quest of deriving precise estimates for \( \theta_k \) is one of the central problems of multiplicative number theory, known as Dirichlet divisor problem. It has been widely studied by various authors: Dirichlet has proved that \( \theta_2 \leq 1/2 \), Voronoï improved this estimate to \( \theta_2 \leq 1/3 + \varepsilon \) [14, 15], and more modern developments can be found in [3, 6, 7, 13]. Namely,

\[
\theta_2 \leq 131/416, \quad \theta_3 \leq 43/96, \quad \theta_k \leq \frac{k-1}{k+2}, \quad k \geq 4.
\]

It is natural to extend the notion of divisor functions from integers to other unique factorisation domains such as rings of quadratic integers.
Let $R$ be a principal ring of imaginary quadratic integers. Then it is possible to define divisor functions $t, t_k: R \setminus 0 \to \mathbb{N}$ as

$$t(\alpha) = \sum_{d \mid \alpha, d \in R/U(R)} 1, \quad t_k(\alpha) = \sum_{d_1 \cdots d_k \sim \alpha, d_1, \ldots, d_k \in R/U(R)} 1,$$

where $\alpha \sim \beta$ iff $\alpha | \beta$ and $\beta | \alpha$, and $U(R)$ is a group of units of $R$.

Further, if $R$ is also a Euclidean ring equipped with norm $N: R \to \mathbb{N}$, one can learn asymptotic properties of $\sum_{0 < N\alpha \leq x} t_k(\alpha)$. Lai Dyk Thin [12] has proved that

$$\sum_{0 < N\alpha \leq x} t(\alpha) = l_1 x \ln x + l_2 x + O(x^{3/5+\varepsilon}),$$

$$\sum_{0 < N\alpha \leq x} t_k(\alpha) = x L_k(\ln x) + O(x^{1-1/(k+1)+\varepsilon}),$$

where constants $l_1, l_2$ and polynomial $L_k$ of degree $k - 1$ depend only on $R$.

The aim of following notes is to improve the error term in (1) and (2) in the specific case of Gaussian integers $\mathbb{Z}[i]$. Namely, from now on

$$t(\alpha) = \sum_{\text{arg} d \in [0, \pi/2]} 1, \quad t_k(\alpha) = \sum_{\text{arg} d_1, \ldots, d_k \in [0, \pi/2]} 1,$$

and $N(a + bi) = a^2 + b^2$. In order to obtain this result we improve known estimates of moments of the Riemann zeta function.

2. Moments of $\zeta$

As usual $\zeta(s)$ is the Riemann zeta function. Real and imaginary components of the complex $s$ are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

Denote by $M(A)$ a real function such that

$$\int_1^T |\zeta(1/2 + it)|^A \, dt \ll T^{M(A)+\varepsilon}.$$

Estimates of the Riemann zeta function on critical line $\sigma = 1/2$ are crucial for many applications. The best known result can be found in [5 Th. 8.3]. The following theorem improves it for $A > 12$.

**Theorem 1.** The following choice of $M$ is valid and satisfies (3):

$$M(A) = \begin{cases} 1 + (A - 4)/8, & 4 \leq A \leq 12, \\ 1 + \max\{\frac{32}{205}(A - 6), E(A)\}, & 12 < A \leq C, \\ 1 + \frac{32}{205}(A - 6), & C < A, \end{cases}$$
where

\[ E(A) = \inf_{(k,l) \in P} \left\{ \frac{l}{k} \bigg| (4 - A)k + 4l + 2 \geq 0 \right\}, \]

\[ C = \inf_{(k,l) \in P} \left\{ \frac{4k + 4l + 2}{k} \bigg| 1 - k - \frac{77}{64}l \geq 0 \right\} = 15.782 \ldots \]

and \( P \) denotes the set of exponent pairs, defined in accordance to Krätzel [8, Ch. 2].

**Proof.** The first case follows from the estimates \( \int_1^T |\zeta(1/2 + it)|^4 \, dt \ll T \log^4 T \) by Ingham [4] and \( \int_1^T |\zeta(1/2 + it)|^{12} \, dt \ll T^2 \log^{17} T \) by Heath-Brown [2]. Precisely, let us denote \( \xi(t) = |\zeta(1/2 + it)| \). By Hölder inequality we have

\[
\int_1^T \xi^A(t) \, dt = \int_1^T \xi^{(12-A)/2}(t)\xi^{(3A-12)/2}(t) \, dt \ll \\
\ll \left( \int_1^T \xi^{q_1(12-A)/2}(t) \, dt \right)^{1/q_1} \left( \int_1^T \xi^{q_2(3A-12)/2}(t) \, dt \right)^{1/q_2}
\]

for \( 1/q_1 + 1/q_2 = 1 \). Taking \( q_1 = 8/(12 - A) \) and \( q_2 = 8/(A - 4) \) we get

\[
\int_1^T \xi^A(1/2 + it) \, dt \ll \\
\ll \left( \int_1^T \xi^4(t) \, dt \right)^{(12-A)/8} \left( \int_1^T \xi^{12}(t) \, dt \right)^{(A-4)/8} \ll \\
\ll T^{(12-A)/8} \log^{4(12-A)/8} T \cdot T^{2(A-4)/8} \log^{17(A-4)/8} T = \\
= T^{1+(A-4)/8} \log^{(13A-20)/8} T.
\]

Consider the second case. Denoting \( R \) and \( V \) as in Ivić [5, (8.6)]. It is enough to show that for \( A > 12 \) and for every exponent pair \((k,l) \in P \) such that \( (4 - A)k + 4l + 2 \geq 0 \) we have

\[ R \ll T^{1+\max\left\{ \frac{32}{205} (A-6), l/k \right\} + \epsilon} V^{-A}. \]

But by [5] Th. 8.2] we have

\[ R \ll T^{1+\epsilon} V^{-6} + T^{1+l/k + \epsilon} V^{-2(1+2k+2l)/k}. \]

By condition on \((k,l) \) we know that \( V^{-2(1+2k+2l)/k} \ll V^{-A} \), so

\[ R \ll T^{1+\epsilon} V^{-A} (V^{A-6} + T^{l/k}). \]

But definitely since \( \zeta(1/2 + it) \ll \epsilon^{32/205+\epsilon} \) by Huxley [3] we have \( V \ll T^{62/205+\epsilon} \), which completes the proof of (7).

Now let us investigate the third case. Let \( A > C \). Then by definition of \( C \) there is an exponent pair \((k,l) \) such that \( 1 - k - 77l/64 \geq 0 \) and
where \( c = l/(-2k + 4l + 2) \). One can check that condition \( 1 - k - 77l/64 \geq 0 \) implies \( c \leq 32/205 \).

It is enough to prove that

\[
S := \sum_{r \in R} |\zeta(1/2 + it_r)|^4 \ll T^{1 + \epsilon} 205^{A-6} + \epsilon,
\]

where \(|t_r| \leq T\), \(|t_r - t_s| \geq 1\) for \( 1 \leq r \neq s \leq R \). Cf. [5, (8.58)].

Again since \( \zeta(1/2 + it) \ll t^{32/205 + \epsilon} \) we can split \( \{t_r\}_{r=1}^R \) into \( 1 + \lfloor \frac{32}{205} \log T \rfloor \) disjoint subsets \( \{t_{j,r_j}\}_{r_j=1}^{R_j} \), where for every \( t_{j,r_j} \) we have \( V_j := 2^j \ll |\zeta(1/2 + it_{j,r_j})| \ll 2^{j+1} \). Then

\[
S \ll \sum_{j=0}^{\lfloor \frac{32}{205} \log T \rfloor} \sum_{r \in R_j} 2^{A j} \ll \left( \sum_{j=0}^{\lfloor \frac{32}{205} \log T \rfloor} \sum_{j=c \log T}^{\infty} 2^{(A-6)j} \right) R_j 2^{A j} =: S_2 + S_1.
\]

Here by choice of \( c \) for \( j \geq c \log T \) we have \( R_j \ll T^{1 + \epsilon} 2^{-6j} \), so

\[
S_1 \ll T^{1 + \epsilon} \sum_{j=c \log T}^{\infty} 2^{(A-6)j} \ll T^{1 + \epsilon} 205^{A-6} + \epsilon.
\]

On the other side for \( j \leq c \log T \) we have \( R_j \ll T^{1 + \epsilon} 2^{-6j} \), so

\[
S_2 \ll \sum_{j=0}^{c \log T} 2^{(A-6-2(4k+4l+2)/k)j} \ll T^{1 + l/k + (A-6-2(4k+4l+2)/k)c + \epsilon}.
\]

But

\[
l/k + (A - (4k + 4l + 2)/k)c = l/k + (A - 6 - (-2k + 4l + 2)/k)c =
\]

\[
= l/k + (A - 6)c - l/k = (A - 6)c \leq 32/205 (A - 6),
\]

which completes the proof. \( \square \)

To apply Theorem [10] efficiently, we need a method to evaluate infimums of form (5) and (6) over exponent pairs. We have developed such framework, whose initial version has been described in [9]. Since then the framework has been developed further and released as a \texttt{exp-pairs} package [10]. For instance, an estimate of \( C_r \) given in (6), corresponds to the choice

\[
(k, l) = BA(ABA)^2(A^2BA)^2BAA \cdots
\]
A \( M(A) \) Exponent pair for \( E(A) \)

13 \( 2.134766053 \ldots \)

\( BAA^2(BA)^2A^2BAA(BA)^5A^5BA(A(BA)^2A)^2 \cdot \)

\( BA((BA)^2A)^2(ABA)^3BAA^4(BA)^7A^4BAA^5 \ldots \)

14 \( \frac{1117297289}{491431296} \)

\( BAA^2BA(BAA)^5((BA)^2A)^2A(BA)^3A^3 \cdot \)

\( (ABA)^2(1/6, 2/3) \)

15 \( \frac{61902400787}{25629743097} \)

\( (BAA^2)^2(BA)^2A^4(BA)^4ABA(A(BA)^3A)^2 \cdot \)

\( (BA)^5A(1/6, 2/3) \)

Table 1. Values of \( M(A) \) for \( A = 13, 14, 15 \).

where \( A \) and \( B \) stand for application of \( A \)- and \( B \)-process from Krätzel [8, Ch. 2].

We computed Table I as a reference for estimates of \( M(A) \) provided by Theorem I for integer \( A \in (12, C) \).

3. Moments of \( Z \)

Let us briefly recap key properties of Gaussian integers. The ring \( \mathbb{Z}[i] \) consists of \( \alpha = a + bi \) for integer \( a \) and \( b \).

There are four units of the ring: \( U(\mathbb{Z}[i]) = \{1, i, -1, -i\} \). For any \( \alpha \neq 0 \) its orbit under action of the unit group consists of four elements, one per each quadrant. We will use an element of the orbit from the first quadrant (such that \( \arg \alpha \in [0, \pi/2) \)) as a canonical representative.

The ring is equipped with norm \( N(a + bi) = a^2 + b^2 \), which is a homomorphism of multiplicative group: \( N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta) \). The ring is Euclidean and principal, so it is a unique factorisation domain.

Gaussian integer \( p \) is prime if and only if one of the following cases has place:

- \( p \sim 1 + i \),
- \( p \sim p \), where \( p \equiv 3 \) (mod 4),
- \( N(p) = p \), where \( p \equiv 1 \) (mod 4).

In the last case there are exactly two non-associated \( p_1 \) and \( p_2 \) such that \( N(p_1) = N(p_2) = p \). See [II, §34].

Let \( Z(s) \) denote the Hecke zeta function, which is a Gaussian analogue of the Riemann zeta function. Namely,

\[
Z(s) = \sum_{\alpha \neq 0 \atop \arg \alpha \in [0, \pi/2)} N(\alpha)^{-s}.
\]

Let \( \beta \) be the Dirichlet beta function,

\[
\beta(s) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^{-s}}.
\]
Converting Dirichlet sums to Euler products and back, we have
\[
Z(s) = \prod_{\text{arg } p \in [0, \pi/2]} (1 - N(p)^{-s})^{-1} = \prod_{p \equiv 3 \mod 4} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 1 \mod 4} \frac{1}{1 - p^{-s}}^2 = \prod_p \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \mod 4} \frac{1}{1 - p^{-s} - p^{-2s}} \prod_{p \equiv 1 \mod 4} \frac{1}{1 - p^{-s}}^2 = \zeta(s)\beta(s).
\]

Denote by \(I(A)\) a real function such that
\[
\int_1^T |Z(1/2 + it)|^A dt \ll T^{I(A) + \varepsilon}.
\]

Firstly, \(I(2) = 1\), because
\[
\int_1^T |Z(1/2 + it)|^2 dt \ll \left(\int_1^T |\zeta(1/2 + it)|^4 dt\right)^{1/2} \ll T^{(1+\varepsilon)/2} T^{(1/2+\varepsilon)/2} = T^{1+\varepsilon},
\]
where we applied estimates for fourth moments from Montgomery [11, Th. 10.1].

Secondly, \(I(3) = 5/4\), because
\[
\int_1^T |Z(1/2 + it)|^3 dt \ll \left(\int_1^T |\zeta(1/2 + it)|^{12} dt\right)^{1/4} \left(\int_1^T |\beta(1/2 + it)|^4 dt\right)^{3/4} \ll T^{(2+\varepsilon)/4} T^{3(1+\varepsilon)/4} = T^{5/4+\varepsilon},
\]
where the estimate for twelfth moment is by Ingham [4].

Further, estimates for higher moments of Hecke zeta function are given by the following theorems.

**Theorem 2.** For any \(D > A\)
\[
I(A) = \begin{cases} 
\frac{1}{2} + (M(D) - \frac{77}{205})/D A + \frac{77}{205}, & 4 \leq A \leq D, \\
\frac{32}{205} A + M(A), & D \leq A.
\end{cases}
\]

where \(D = 12.5716 \ldots\)

**Proof.** For brevity below \(Z^A\) means \(|Z(1/2 + it)|^A\), \(\zeta^A\) means \(|\zeta(1/2 + it)|^A\), \(\beta^A\) means \(|\beta(1/2 + it)|^A\), and \(\int \cdot\) stands for \(\int_1^T \cdot dt\).

We start with the first case. Let \(D\) be any real greater than \(A\). Then for \(b = 4(D - A)/D\) we write
\[
\int Z^A = \int \zeta^A \beta^A = \int \zeta^A \beta^b \beta^{A-b}.
\]
Since $\beta \ll T^{32/205+\varepsilon}$ for $t \in [1, T]$ we get
\begin{equation}
\int Z^A \ll T^{32(A-b)/205+\varepsilon} \int \zeta^A \beta^b.
\end{equation}

Now apply Hölder inequality with $q_1 = D/A$, $q_2 = D/(D-A)$, $1/q_1 + 1/q_2 = 1$ to obtain
\begin{equation}
\int \zeta^A \beta^b \ll \left( \int \zeta^D \right)^{A/D} \left( \int \beta^A \right)^{(D-A)/D} \ll T^{AM(D)/D+\varepsilon} T^{(D-A)/D+\varepsilon}.
\end{equation}
Combination of (9) and (10) provides us with the first statement of the theorem.

It remains to choose $D$ to minimize $(M(D) - \frac{T}{205})/D$ and numerical computations by Theorem 1 give us $D = 12.5716\ldots$

The second statement is almost trivial, since
\begin{equation}
\int Z^A = \int \zeta^A \beta^A \ll T^{32A/205+\varepsilon} \int \zeta^A \ll T^{32A/205+M(A)+\varepsilon}.
\end{equation}

\begin{table}[h]
\centering
\begin{tabular}{cccccccccccccccc}
15 & 4 & 2 & 8 & 2 & 12 & 4 & 4 & 3 & 8 & 8 & 4 & 4 & 4 & 2 & 16 & \\
14 & 2 & 12 & 4 & 6 & 4 & 8 & 4 & 12 & 2 & 8 & 2 & 12 & 4 & 8 & 2 & \\
13 & 8 & 2 & 4 & 4 & 4 & 4 & 4 & 2 & 8 & 2 & 8 & 2 & 8 & 4 & 4 & \\
12 & 4 & 6 & 4 & 12 & 3 & 12 & 2 & 10 & 6 & 6 & 4 & 12 & 2 & 12 & 4 & \\
11 & 4 & 4 & 8 & 2 & 4 & 2 & 8 & 4 & 4 & 4 & 4 & 4 & 8 & 2 & 4 & \\
10 & 2 & 8 & 2 & 6 & 6 & 8 & 2 & 6 & 2 & 16 & 4 & 6 & 2 & 8 & 8 & \\
9  & 4 & 4 & 8 & 2 & 4 & 4 & 8 & 4 & 6 & 2 & 4 & 6 & 8 & 2 & 8 & \\
8  & 4 & 6 & 2 & 10 & 2 & 9 & 2 & 8 & 4 & 6 & 4 & 10 & 2 & 12 & 3 & \\
7  & 6 & 2 & 4 & 4 & 4 & 4 & 4 & 2 & 8 & 2 & 8 & 2 & 4 & 4 & 4 & \\
6  & 2 & 8 & 4 & 6 & 2 & 8 & 4 & 9 & 4 & 8 & 2 & 12 & 4 & 8 & 4 & \\
5  & 4 & 2 & 4 & 2 & 8 & 2 & 4 & 2 & 4 & 6 & 4 & 3 & 4 & 4 & 12 & \\
4  & 2 & 6 & 3 & 6 & 2 & 6 & 4 & 10 & 2 & 6 & 2 & 12 & 4 & 6 & 2 & \\
3  & 4 & 2 & 4 & 3 & 4 & 4 & 4 & 2 & 8 & 2 & 8 & 4 & 4 & 4 & 8 & \\
2  & 2 & 4 & 2 & 6 & 2 & 8 & 2 & 6 & 4 & 8 & 4 & 6 & 2 & 12 & 2 & \\
1  & 2 & 2 & 4 & 2 & 4 & 2 & 6 & 4 & 4 & 2 & 4 & 8 & 2 & 4 & \\
0  & 1 & 3 & 2 & 5 & 4 & 6 & 2 & 7 & 3 & 12 & 2 & 10 & 4 & 6 & 8 & \\
\end{tabular}
\caption{Values of $t(a + bi)$ in the first quadrant. Here $a$ increases horizontally and $b$ vertically. Numbers with exactly two divisors are Gaussian primes.}
\end{table}

4. Summatory function of $t_k$

Now we are ready to attack our main aim: the summatory function of $t_k$. Just to get better acquainted we refer readers to Table 2 for the plot of values of $t$. 

\begin{flushright}
\textcircled{1}
\end{flushright}
Similar to real case \( \sum_{n>0} \tau_k(n)n^{-s} = \zeta^k(s) \), one can check that
\[
\sum_{\arg \alpha \in [0, \pi/2]} t_k(\alpha)N(\alpha)^{-s} = Z^k(s).
\]

**Theorem 3.** Let \( S_k(x) = \sum_{\arg \alpha \in [0, \pi/2]} t_k(\alpha) \). Then
\[
S_2(x) = l_1 x \ln x + l_2 x + O(x^{1/2+\varepsilon}),
\]
\[
S_3(x) = xP_3(\log x) + O(x^{3/5+\varepsilon}),
\]
\[
S_k(x) = xP_k(\log x) + O(x^{1-1/2I(k)+\varepsilon}).
\]

where
\[
l_1 = \pi^2/16 = 0.61685\ldots,
\]
\[
l_2 = \pi^2(2\gamma - 1)/16 + \pi\beta'(1)/2 = 0.39827\ldots
\]
\[\deg P_k = k - 1.\]

**Proof.** By Perron formula we have
\[
S_k(x) = \text{res}_{s=1} Z^k(s)x^s/s + O(xT^{-1} + x^{1/2T^{I(k)-1+\varepsilon}}).
\]

Here the residues gives the main term of form \( xP_k(\log x) \). Let us analyse the case of \( k = 2 \) in details. We have
\[
\text{res}_{s=1} Z^2(s)x^s/s = \text{res}_{s=1} (\zeta^2(s)/s \cdot \beta^2(s) \cdot x^s) =
\]
\[
= \text{res}_{s=1} \left( \frac{1}{(s-1)^2} + \frac{2\gamma - 1}{s - 1} + O(1) \right) \times
\]
\[
\times \left( \frac{\pi}{4} + \frac{\pi\beta'(1)}{2} (s - 1) + O(s - 1)^2 \right)^2 \times
\]
\[
\times (x + x \log x(s - 1) + O(s - 1)^2) =
\]
\[
= \frac{\pi^2}{16} x \ln x + \left( \frac{\pi^2(2\gamma - 1)}{16} + \frac{\pi\beta'(1)}{2} \right) x.
\]

With regards to the error term let us choose \( T = x^a \) in order to minimize the magnitude of \( O(xT^{-1} + x^{1/2T^{I(k)-1+\varepsilon}}) \). One can check that \( T = x^{1/2I(k)} \) turns this expression into \( O(x^{1-1/2I(k)+\varepsilon}) \).

Error term for \( k = 2 \) and \( k = 3 \) is a consequence of estimates for \( I(2) \) and \( I(3) \) obtained in the previous section. \( \square \)

Theorem 3 improves results of Thin (1) and (2) for all \( k \). Cases \( k = 2 \) and \( k = 3 \) were given above. For \( 4 \leq k \leq 12 \) by (4) and (8) we have \( 2I(k) \leq 0.5828k + 0.7512 < k < k + 1 \).

For \( 13 \leq k \) we have
\[
2I(k) \leq 2(1 + 32/205(2k - 6)) = 128/205k + 26/205 < k < k + 1.
\]

It is interesting to check the accuracy of our asymptotic estimate on some numerical data. Table 3 shows values of summatory function for
Table 3. Values of $S(x) = \sum_{0<\alpha \leq x} t(\alpha)$ and estimates $\bar{S}(x)$ by (11).

growing $x$, compared against predictions from (11). The last column lists error terms divided by $x^{1/2}$ and fuels our confidence in the correct order of $O$-bound.

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