Emergent statistical-mechanical structure in the dynamics along the period-doubling route to chaos

Alvaro Diaz-Ruelas and Alberto Robledo

1 Instituto de Física, Universidad Nacional Autónoma de México - Apartado Postal 20-364, México 01000 D.F., Mexico
2 Centro de Ciencias de la Complejidad, Universidad Nacional Autónoma de México - Apartado Postal 20-364, México 01000 D.F., Mexico

received 14 November 2013; accepted in final form 11 February 2014
published online 4 March 2014

PACS 05.45.Ac – Low-dimensional chaos
PACS 05.20.Gg – Classical ensemble theory
PACS 05.45.Df – Fractals

Abstract – We consider both the dynamics within and towards the supercycle attractors along the period-doubling route to chaos to analyze the development of a statistical-mechanical structure. In this structure the partition function consists of the sum of the attractor position distances known as supercycle diameters and the associated thermodynamic potential measures the rate of approach of trajectories to the attractor. The configurational weights for finite \( N \), and infinite \( N \to \infty \), periods can be expressed as power laws or deformed exponentials. For a finite period the structure is undeveloped in the sense that there is no true configurational degeneracy, but in the limit \( N \to \infty \) this is realized together with the analog property of a Legendre transform linking entropies of two ensembles. We also study the partition functions for all \( N \) and the action of the central limit theorem via a binomial approximation.

Copyright © EPLA, 2014

Introduction. – For thermal systems formed by particles interacting via standard forces the limit of validity of equilibrium statistical mechanics is, trivially, non-equilibrium. Thermal systems constitute the normal realm of the Boltzmann-Gibbs (BG) formalism, but there are other types of systems for which it has been known for some time that they accept a statistical-mechanical description of the BG type. These are multifractals and chaotic nonlinear dynamical systems [1], among which one-dimensional unimodal iterated maps, represented by the quadratic logistic map, are familiar model systems [2,3] that exhibit such properties. The chaotic attractors generated by this class of maps have ergodic and mixing properties and not surprisingly they can be described by a thermodynamic formalism compatible with BG statistics [1]. But at the transition to chaos, the period-doubling accumulation point, the so-called Feigenbaum point, these two properties are lost and this suggests the possibility of exploring the limit of validity of the BG structure in a precise but simple enough setting.

Recently a comprehensive description has been given [4,5] of the elaborate dynamics that takes place both inside and towards the Feigenbaum attractor. Amongst several conclusions, these studies established that the two types of dynamics are related to each other in a statistical-mechanical way, i.e. the dynamics at the attractor provides the “microscopic configurations” in a partition function while the approach to the attractor is efficiently described by an entropy obtained from it. As we show below, this property conforms to \( q \)-deformations [4–7] of the ordinary exponential weight of BG statistics. This novel statistical-mechanical feature arises in relation to a multifractal attractor with vanishing Lyapunov exponent. Here we explore in more detail this property with focus on how the statistical-mechanical structure develops along the period-doubling bifurcation cascade [2,3], i.e. out of chaos.

Deformed exponentials appear in the studies of many physical systems. For instance, simulated velocity distributions of statistical-mechanical models resemble closely the so-called \( q \)-Gaussian expression [8,9], suggesting the occurrence of generalized statistical-mechanical structures under nonequilibrium conditions. Here, as an effort to provide a firm basis to a wider content discussion, we chose
to study a nontrivial archetypal system under ergodicity and mixing failure and precisely determine its properties independently of any method that assumes a statistical-mechanical formalism. After that, the results obtained can be analyzed in relation to generalized entropy expressions or properties derived from them.

**Brief recall of the dynamics within and towards the Feigenbaum attractor.** – The trajectories associated with the period-doubling route to chaos in unimodal maps exhibit elaborate dynamical properties that follow concerted patterns. At the period-doubling accumulation points, periodic attractors become multifractal before turning chaotic. At these points the Lyapunov exponent $\lambda$ vanishes as it changes sign [2,3]. There are two sets of properties associated with the attractors involved: those of the dynamics inside the attractors and those of the dynamics towards the attractors. These properties have been characterized in detail, the organization of trajectories and also that of the sensitivity to initial conditions at the Feigenbaum attractor are described in ref. [4], while the features of the rate of approach of an ensemble of trajectories to this attractor has been explained in ref. [5].

We recall some of the basic features of the bifurcation forks that form the period-doubling cascade sequence in unimodal maps, often illustrated by the logistic map $f_\mu(x) = 1 - \mu x^2$, $-1 \leq x \leq 1$, $0 \leq \mu \leq 2$ [2,3]. The knowledge of the dynamics towards a particular family of periodic attractors, the so-called superstables [2,3], facilitates the understanding of the rate of approach of trajectories to the Feigenbaum attractor, located at $\mu = \mu_\infty = 1.401155189092 \ldots$, and highlights the source of the discrete scale-invariant property of this rate [5]. The family of trajectories associated with these superstables—also called supercycles—of periods $2^N$, $N = 1, 2, 3, \ldots$, are located along the bifurcation forks. The positions (or phases) of the $2^N$-attractor are given by $x_j = f_j^{(1)}(0)$, $j = 1, 2, \ldots, 2^N$. Associated with the $2^N$-attractor at $\mu = \mu_N$ there is a $(2^N-1)$-repellor consisting of $2^N - 1$ positions $y_k$, $k = 0, 1, 2, \ldots, 2^N - 1$. These positions are the unstable solutions, $|df_f^{(2\nu-1)}(y)/dy| > 1$, of $y = f_f^{(2\nu-1)}(y)$, $y = 1, 2, \ldots, N$. The first, $n = 1$, originates at the initial period-doubling bifurcation, the next two, $n = 2$, start at the second bifurcation, and so on, with the last group of $2^{N-1}$, $n = N$, setting out from the $N-1$ bifurcation. The diameters $d_{N,m}$ are defined as $d_{N,m} = \max_{x_{m}} |f_N^{(2\nu-1)}(x_{m})| [2,3]$. The sums of diameters $d_{N,m}$ defined from the supercycle orbits with $\mu_N < \mu_\infty$. See fig. 1 in ref. [5]. This property has been basic in obtaining rigorous results for the sensitivity to initial conditions for the Feigenbaum attractor [4], and for the dynamics of approach to this attractor [5]. Other families of periodic attractors share most of the properties of supercycles.

The organization of the total set of trajectories as generated by all possible initial conditions as they flow towards a period $2^N$-attractor has been determined in detail [5,10]. It was found that the paths taken by the full set of trajectories in their way to the supercycle attractors (or to their complementary repellors) are exceptionally structured. The dynamics associated to families of trajectories always displays a characteristically concerted order in which positions are visited, and this is reflected in the dynamics of the supercycities of periods $2^N$ via the successive formation of gaps in phase space (the interval $-1 \leq x \leq 1$) that finally give rise to the attractor and repellor multifractal sets. To observe explicitly this process an ensemble of initial conditions $x_0$ distributed uniformly across phase space was considered and their positions were recorded at subsequent times [5,10]. This set of gaps develops in time beginning with the largest one associated with the first repellor position, then followed by a set of two gaps associated with the next two repellor positions, next a set of four gaps associated with the four next repellor positions, and so forth. The gaps that form consecutively all have the same width in the logarithmic scales [5], and therefore their actual widths decrease as a power law, the same power law followed, for instance, by the position sequence $x_0 = \alpha^{-N}$, $\tau = 2^N$, $N = 0, 1, 2, \ldots$, for the trajectory inside the attractor starting at $x_0 = 0$ (and where $\alpha \approx 2.50291$ is the absolute value of Feigenbaum’s universal constant). The locations of this specific family of consecutive gaps advance monotonically toward the sparsest region of the multifractal attractor located at $x = 0$. See refs. [4,5,10].

**Sums of diameters as partition functions.** – The rate of convergence $W_t$ of an ensemble of trajectories towards any attractor/repellor pair along the period-doubling cascade is a convenient single-time quantity that has a straightforward definition and is practical to implement numerically. A partition of phase space is made of $N_0$ equally sized boxes or bins and a uniform distribution of $N_0$ initial conditions is placed along the interval $-1 \leq x \leq 1$. The ratio $N_0/N_t$ can be adjusted to achieve optimal numerical results [5]. The quantity of interest is the number of boxes $W_t$ that contain trajectories at time $t$. This rate has been determined for the supercycles $\mu_N$, $N = 1, 2, 3, \ldots$, and its accumulation point $\mu_\infty$ [5]. See fig. 19 in that reference where $W_t$ is shown in logarithmic scales for the first five supercycles of periods $2^1$ to $2^5$ where we can observe the following features: In all cases $W_t$ shows a similar initial and nearly constant plateau $W_t \approx \Delta$, $1 \leq t \leq t_0$, $t_0 = O(1)$, and a final
Emergent statistical-mechanical structure at the onset of chaos

well-defined decay to zero. As it can be observed in the left panel of fig. 19 in [5], the duration of the final decay grows approximately proportionally to the period $2N$ of the supercycle. There is an intermediate slow decay of $W_t$ that develops as $N$ increases with duration also just about proportional to $2^N$. For the shortest period $2^1$, there is no intermediate feature in $W_t$; this appears first for period $2^2$ as a single dip and expands with one undulation every time $N$ increases by one unit. The expanding intermediate regime exhibits the development of a power-law decay with logarithmic oscillations (characteristic of discrete scale invariance). In the limit $N \to \infty$ the rate takes the form $W_t \simeq h((\ln \tau/\ln 2) \tau^{-B}, \tau = t - t_0$, where $h(x)$ is a periodic function with $h(1) = 1$ and $B \simeq 0.8001 \ [5]$. The rate $W_t$, at the values of time for period doubling, $\tau = 2^n, n = 1, 2, 3, \ldots < N$, can be obtained quantitatively from the supercycle diameters $d_{n,m}$. Specifically,

$$Z_{\tau} \equiv \frac{W_t}{\Delta} = \sum_{m=0}^{2^{n-1}-1} d_{n,m}$$

In the above expression, $\tau = t - t_0 = 2^{n-1}, n = 1, 2, 3, \ldots < N$. Equation (1) expresses the numerical procedure followed in [11] to evaluate the exponent $B$ but it also suggests a statistical-mechanical structure if $Z_{\tau}$ is identified as a partition function where the diameters $d_{n,m}$ play the role of configurational terms [5]. The diameters $d_{N,m}$ scale with $N$ for fixed $m$ as $d_{N,m} \simeq \alpha^{-N-1}_y$, $N$ large, where the $\alpha_y$ are universal constants obtained from the finite discontinuities of Feigenbaum’s trajectory scaling function $\sigma(y) = \lim_{N \to -\infty} (d_{N+1,m}/d_{N,m}), y = \lim_{N \to -\infty} (m/2^N) [2,5]$. The largest two discontinuities of $\sigma(y)$ correspond to the sparsest and denser regions of the multifractal attractor at $\mu_\infty$, for which we have, respectively, $d_{N,0} \simeq \alpha^{-N-1}_y$ and $d_{N,1} \simeq \alpha^{-2(N-1)}_y (d_{1,0} = 1)$.

A closer analysis of the partition functions for the supercycles. – We proceed now to study in more detail the diameters $d_{N,m}$ so that we can evaluate the soundness of their association with configurational terms in a partition function. With this in mind we determined their values for the supercycles of periods $2^N$ from $N = 1$ to $N = 12$, that is, starting with the case of a single diameter $d_{1,0} = 1$ and following successively up to a set of 2048 diameters $d_{12,m}, m = 0, 1, \ldots , 2^{12} - 1$. This task required the precise evaluation of the control parameter values $\mu_N, N = 1, \ldots , 12$.

In fig. 1 we show the lengths of these sets when arranged with decreasing values, namely, we present the $d_{N,m}$ as a function of their rank $r$, the size-rank distributions, in logarithmic scales, as it is often done for these type of distributions that exhibit frequently power law behavior. We observe in fig. 1 that the distributions have a downhill terraced (or multiple-plateau) structure, the diameters form well-defined size groups and these sizes decrease on average a fixed amount (in the logarithmic scales shown) equal to $\log_{10} \alpha \simeq 0.39844$ from group to group. This amount reflects the well-known [2,3] power-law scaling of diameters sizes via the universal constant $\alpha$. Their size-rank distributions satisfy a piecewise Zipf-like law. For example, for the largest diameter we have $d_{N,0}/d_{N+1,0} \simeq \alpha^1$, whereas for the smallest we have $d_{N,1}/d_{N+1,1} \simeq \alpha^2$. Therefore $d_{N,0} \simeq \alpha^{-N}$ and $d_{N,1} \simeq \alpha^{-2N}$.

We observe clearly in figs. 1 and 3 that the diameter lengths within each group are not equal, so that there is no degeneracy in them. However the differences in lengths within groups diminishes rapidly as $N$ increases. There are two groups with only one member, the largest and the shortest diameters, and the numbers within each group grow monotonically from each end towards the middle-sized length group. The numbers of diameters forming these groups can be neatly arranged into a Pascal Triangle (see fig. 2), and therefore we anticipate the action of the Central Limit Theorem, in a form reminiscent of the De Moivre-Laplace theorem, so that in the limit $N \to \infty$, the middle-sized-length group of diameters dominates the partition function $Z_{\tau}$ and a situation similar to the saddlepoint approximation occurs. Also, in the limit $N \to \infty$ the lengths of the dominant group (as well as those of all other groups of diameters with smaller lengths) become closer in size (see the trend in fig. 1), so that in the limit $N \to \infty$ there appears a true degeneracy in the dominant partition function configurations that gives the statistical-mechanical structure the required characteristics for ensemble equivalence and the Legendre transform property central to statistical mechanics.

A comparison of our results with those of another approach of local ordering, that of Pomeau and manifolds, does not seem to provide more specific information than the former subindex $m$. Subindex $e = 0, 1, \ldots , N - 1$ designates the group terrace (as in fig. 1), with decreasing size as $l$ increases, and the subindex $i = 1, 2, \ldots , (N - 1)$ identifies the individual diameter within the group, again

![Fig. 1: (Colour on-line) Length-rank distributions of diameters $d_{N,m}$ for the first 12 supercycles in logarithmic scales. The distributions have a downhill terraced or multiple-plateau structure and the diameters form well-defined length groups. Their values for fixed period $2^N$ decrease on average as $\log_{10} \alpha \simeq 0.39844$ from group to group with $N$ fixed or from $N$ to $N + 1$ for the same kind of group.](image-url)
with decreasing size as \( i \) increases. The diameters \( d_{N,1,i} \) are written as \( d_{N,1,i} = A_{N,1,i} \alpha^{-(N-1)/2} \) where the scaling factors \( \alpha^{-(N-1)/2} \) and \( \alpha^{-2l} \) give the size of the group terrace (see fig. 2) and the amplitude \( A_{N,1,i} \) fixes the value of the individual diameter. (Due to the De Moivre-Laplace theorem, in the limit \( N \to \infty \) the amplitude \( A_{N,2,i} \) can be obtained from an inverse complementary error function, but we do not expand on this here). The scaling factor \( \alpha^{-(N-1)/2} = \alpha^{-(N-1)/2} \) can be rewritten exactly as a \( q \)-exponential, defined as \( \exp_q(x) \equiv [1 + (1 - q)x]^{1/(1-q)} \), via use of the identity \( \alpha^{-(N-1)/2} \equiv (1 + \epsilon_l)^{-\ln \alpha/\ln 2} = 2^{-N-1/2} \). That is, \( d_{N,1,i} \cong A_{N,1,i} \exp_q(-\epsilon_l) \), where \( q = 1 + \beta^{-1} = \ln \alpha/\ln 2 \). Similarly, \( Z_\tau \cong \exp_q\left(-B\epsilon_l\right) \), where \( Q = 1 + B^{-1} \) and \( \epsilon_l = \tau - 1 = 2^{-N-1} - 1 \). Therefore, taking the above into account in eq. (1) we have

\[
\exp_q(-B\epsilon_l) \cong \sum_{l,N} A_{N,1,i} \exp_q(-\epsilon_l),
\]

Equation (2) resembles a basic statistical-mechanical expression except for the presence of the amplitudes \( A_{N,1,i} \) and the fact that \( q \)-deformed exponential weights appear in place of ordinary exponential weights (that are recovered when \( Q = q = 1 \)).

To explore further we use a binomial approximation for \( Z_\tau \) [5]. That is, we adopt the approximation of considering the diameter lengths in each group to actually have equal length \( (A_{N,1,i} = 1) \) and assume that this common lengths are given by the binomial combination of the scale factors of those diameters that converge to the most crowded and most sparse regions of the multifractal attractor. Namely, the \( 2^{N-1} \) diameters at the \( N \)-th supercycle have lengths equal to \( \alpha^{-(N-1)/2} \) and occur with multiplicities \( \binom{N-1}{l} \) where \( l = 0, 1, \ldots, N - 1 \). See fig. 2. The imposed degeneracy within groups in the diameter lengths complete the Pascal Triangle structure across the bifurcation cascade. This feature significantly simplifies the evaluation of the partition function in eq. (1) and directly yields

\[
Z_\tau = \sum_{l=0}^{N-1} \left( \frac{N-1}{l} \right) \alpha^{-(N-1)/2} \alpha^{-2l} = (\alpha + \alpha^{-2})^{N-1},
\]

where \( \tau = 2^{N-1} \). We obtain \( B = 0.8386 \), and \( Q = 2.1924 \), a surprisingly good approximation when compared to the numerical estimates \( B = 0.8001 \) and \( Q = 2.2498 \) of the exact values [5]. Equation (2) reads now

\[
\exp_q(-\beta F) = \sum_{l=0}^{N-1} \Omega(N-1,l) \exp_q(-\epsilon_l),
\]

where \( F/\epsilon = (1 - q)/(1 - Q) \), and \( \Omega(N-1,l) = \binom{N-1}{l} \alpha^{-2l} = 2^{(N-1)/2} \). In the language of thermal systems eq. (4) reads as follows: There are \( N - 1 \) degrees of freedom that generate \( 2^{N-1} \) configurations, and these occupy \( N \) energy levels with degeneracies \( \Omega(N-1,l), l = 0, 1, \ldots, N - 1 \). Under the binomial approximation the energies of the \( 2^{N-1} \) configurations become confined into the energy values \( \epsilon_l = 2^{(N-1)/2}, l = 1, 2, \ldots, N \). In the generalized canonical partition function all the \( q \)-exponential weights acquire a fixed inverse temperature \( \beta = \ln \alpha/\ln 2 \). When we extend the study of the quadratic map to the infinite family of unimodal maps with extremum of nonlinearity \( 1 < z < \infty \) the inverse temperature \( \beta \) can be varied continuously, as the universal constant \( \alpha(z) \) varies monotonically with \( z \) [12].

**A limiting statistical-mechanical structure for the dynamics at the Feigenbaum point.** – According to our scheme, for finite \( N \) (the supercycle of period \( 2^N \) at \( \mu_N \)) we can form \( N - 1 \) partition functions \( Z_\tau, \tau = 2^{n-1}, n = 1, 2, 3, \ldots, N - 1 \). The number of terms in these partition functions range from a single term, \( d_{1,0} \), to \( 2^{N-1} \) terms, \( d_{N,m}, m = 0, 1, 2, \ldots, 2^{N-1} - 1 \). As explained, for uniform distributions of initial conditions \(-1 \leq x_0 \leq 1 \) at \( \mu = \mu_N \), the partition functions \( Z_\tau \) measure the fraction of ensemble trajectories still away from the attractor at times \( \tau = 2^{n-1}, n = 1, 2, 3, \ldots, N - 1 \). These times coincide with the sequential process of phase-space gap formation by the trajectories [5]. The gaps correspond to the intervals in \(-1 \leq x \leq 1 \) located between the bifurcation forks in the period-doubling cascade, when \( \mu = \mu_N \), that is, the gap intervals are placed between consecutive diameters. As \( N \) grows new smaller gaps proliferate while the new diameters grow in number and each of them decreases in value. See fig. 3 where the numbers of diameters are shown for each group formed for the case of the 12th supercycle. The number of groups into which the diameters distribute increases as it does the number of diameters within each group. As we have indicated these increments obey the entries in the Pascal Triangle generated by a binomial. Although the diameters within each group are
never equal their differences decrease rapidly. The dominant term in \( Z_r \) is that associated with \( \Omega(N-1, (N-1)/2, N) \), odd, and in the limit \( N \to \infty \) we have that \( Z_\infty = \Omega(N \to \infty, l = N/2 \to \infty) \). We interpret this last equality as ensemble equivalence in the thermodynamic limit (here \( N \to \infty \) is the attractor at the transition to chaos).

It is more convenient to describe the ensemble equivalence in terms of the binomial approximation of the partition function \( Z_r \) given by eqs. (3) and (4), where \( \Omega(N-1, l) \) plays the role of a “microcanonical” partition function representing the system configurations with fixed diameter length \( \alpha^{-N/2}\alpha^{-2l} \) and \( Z_r \) stands for the “canonical” partition function that is formed by weighting the degenerate configurations \( \Omega(N-1, l) \) for each length group by the factor \( \alpha^{-N/2}\alpha^{-2l} = \exp(-\beta \epsilon_l) \). According to the De Moivre-Laplace early form of the central limit theorem the growth of \( N \) drives the binomial distribution towards a Gaussian distribution

\[
\delta^N P_{l, N\to l} \simeq \frac{1}{\sqrt{2\pi N \rho \sigma}} \exp \left( -\frac{x^2}{2N \rho \sigma} \right),
\]

(5)

where \( \delta = \alpha^{-1} + \alpha^{-2}, \rho \sim \alpha^{-2}, \sigma \sim \alpha^{-1} \) and \( x = l - N \rho \). For large \( N \) the midpoint terms in the expansion of the binomial dominate, \( Z_r = (\alpha^{-1} + \alpha^{-2})^{N-1} \simeq (N/2)^{N-1} \simeq 2^{N-1} \alpha^{-3N/2} \), and in the limit \( N \to \infty \) we have that \( Z_\infty = \Omega(N \to \infty, l = N/2 \to \infty) \). For \( N \) fixed the “energies” \( \epsilon_l \) range from \( 2^{N-1} - 1 \) to \( 2^{2(N-1)} - 1 \). For large \( N \) the ‘energy’ that corresponds to the “microcanonical” partition function that becomes the dominant term in \( Z_r \) is \( \epsilon_{N/2} \simeq 2^{3/2N} \).

A crossover to ordinary BG-type statistics takes place when \( \mu \gtrsim \mu_c \) and the attractor becomes chaotic. For \( \Delta \mu \equiv \mu - \mu_c > 0 \) the attractors are made up of \( 2^N \), \( N = 1, 2, 3, \ldots \), bands, with \( N \) larger for \( \Delta \mu \) smaller, while the Lyapunov exponent scales as \( \lambda \sim 2^{-N} \). The trajectories consist of an interband periodic motion of period \( 2^N \) and an intraband chaotic motion. As explained in ref. [5] the consideration of backward iterations in unimodal maps, together with the expansion of separation of trajectories when \( \lambda > 0 \), can be invoked to write a partition function similar to that in eq. (2) but now with ordinary exponentials as configurational weights.

As is well known [1], the so-called thermodynamic formalism for the description of the geometric properties of multifractal sets is built around a statistical-mechanical framework of the BG type. The partition function formulated to study multifractal properties, like the spectrum of singularities \( f(\alpha) \), is written as

\[
Z(\tilde{\tau}, \tilde{\beta}) = \sum_m^{M} p_m^{\tilde{\tau}_m^{\tilde{\beta}_m}},
\]

(6)

where the \( l_m \) in one-dimensional systems are \( M \) disjoint interval lengths that cover the multifractal set and the \( p_m \) are probabilities given to these intervals. The standard practice consists of demanding that \( Z(\tilde{\tau}, \tilde{\beta}) \) neither vanishes nor diverges in the limit \( l_m \to 0 \) for all \( m \) (notice that in this limit \( M \to \infty \)). Under this condition the exponents \( \tilde{\tau} \) and \( \tilde{\beta} \) define a function \( \tilde{\tau}(\tilde{\beta}) \) from which \( f(\alpha) \) is obtained via Legendre transformation [1]. When the multifractal is an attractor its elements are ordered dynamically, and for the Feigenbaum attractor the trajectory with initial condition \( x_0 = 0 \) generates in succession the positions that form the diameters, generating the entire set of diameters \( d_{N,m} \), \( m = 0, 1, 2, \ldots, 2^{(N-1)} - 1, N = 1, 2, \ldots \). Because the diameters cover the attractor it is natural to choose the covering lengths at stage \( N \) to be \( d_{N,m} = d_{N,m} \) and to assign to each of them the same probability \( p_{m}^{(N)} = (1/2)^{N-1} \), and the condition \( Z(\tilde{\tau}, \tilde{\beta}) = 1 \) reproduces eq. (1) when \( p_{m}^{(N)} = \tau^{-1} = (1/2)^{N-1} \) with \( \tilde{\tau} = -\beta \) and \( \tilde{\beta} = 1 \).

**Summary and discussion.** – The items we studied are the following: i) The partition function we considered is the sum of attractor position distances (the so-called diameters of the supercycles [2,3]) for each period \( 2^N \) along the bifurcation cascade that leads to the transition to chaos. ii) For uniformly distributed sets of initial conditions \( x_0 \) the partition function is equal to the number of bins that still contain trajectories en route to the attractor at time \( \tau = 2^n, n = 1, 2, 3, \ldots \ll N \), where the supercycle period is \( \tau = 2^N, N > 1 \). iii) For \( N \) fixed the values of the diameters distribute into well-defined groups with a size-rank structure that develops into a power law as \( N \) increases. These groups can be arranged into a Pascal Triangle when considering all \( N \) up to \( N \to \infty \), but the diameters within each group are not equal. Nevertheless, their differences diminish rapidly as \( N \) increases, so that a binomial approximation can be introduced such that the diameters within each group are considered equal for all \( N \). iv) In the limit \( N \to \infty \) the diameter-group degeneracy imparts the partition function the required structure to observe ensemble equivalence, and other familiar features of statistical mechanics, even though the configurational weights...
are not exponential. v) The visible or “macroscopic”
manifestation of the statistical-mechanical structure, the
emergence of a power law with log-periodic modulation as-
associated with the rate of approach of trajectories towards
the Feigenbaum attractor, is linked to the sequential pro-
cess of phase-space gap formation. vi) Beyond the transi-
tion to chaos, when the attractors become sets of chaotic
bands, the configurational weights are converted into or-
dinary exponentials and the usual BG form is recovered.

The main advance presented here with respect to ref. [5]
is the determination of the terrace structure displayed by
the diameters for finite $N$ shown in figs. 1 and 3. This fact
allowed us to write the partition function in eq. (1) explic-
tively as eq. (2). Therefore we were able to study how the
lack of configurational degeneracy gradually disappears as
$N \to \infty$ leading to ensemble equivalence.

Chaotic dynamics in nonlinear systems accepts
statistical-mechanical descriptions [1]. Unimodal maps,
usually represented by the logistic map, offer a simple but
nontrivial model system in which to explore the develop-
ment of such a statistical-mechanical structure, to examine
the gradual fulfilment of basic elements and eventually the
display of the full ordinary features of the BG formalism.
A unimodal map is a well-defined and controllable numeri-
cal laboratory for the observation of the limit of validity of
the BG formalism when the ergodic and mixing properties
of chaotic dynamics break down. As it as long been known
unimodal maps display two bifurcation cascades that take
place in opposite directions in control parameter space,
one for $\mu < \mu_\infty$ when periodic attractors double their pe-
riods, and the other for $\mu > \mu_\infty$ when chaotic-band attrac-
tors attractors split doubling their number of bands. The
two cascades meet at $\mu = \mu_\infty$. Infinitely many reproduc-
tions of these inverse cascades appear within the windows
of periodicity that interrupt the chaotic-band attractors
for $\mu > \mu_\infty$ [2,3].

As we have mentioned, the ergodic and mixing trajec-
tories of chaotic-band attractors conform to a statistical-
mechanical structure of the BG type [1]. We have
described that the positions of periodic attractors can
be used to define partition functions and that these cap-
ture information on the dynamics towards the attrac-
tors [5]. However, as we have explained, these partition
functions lack some standard properties required in a ther-
modynamic formalism, such as the degeneracy of config-
urational states that manifests as ensemble equivalence
and the correspondence of their respective thermodynamic
potentials in the thermodynamic limit (that in the unimi-
modal map model is the limit $N \to \infty$ of infinite period).
For finite $N$ the configurational terms (diameters $d_{N,m}$)
separate into well-defined magnitude (length) groups but
they are not equal within each group. These groups of
diameters are the prototypes of “microcanonical” ensem-les while the consideration of all groups, all diameters
for a given supercycle of period $2^N$, is the candidate ver-
dition of the “canonical” ensemble. As we have seen, when
$N \to \infty$ the diameters seem to fulfill a binomial approxi-
mation such that the (vanishing) lengths within the dom-
inant diameter groups (with divergent numbers) become
equal and the De Moivre-Laplace theorem establishes the
equivalence between the “microcanonical” and “canoni-
cal” ensembles. The binomial approximation we presented
for finite $N$ allows for a conventional interpretation in the
language of thermal systems.

***

Support by DGAPA-UNAM-IN100311 and CONACyT-
CB-2011-167978 (Mexican Agencies) is acknowledged.

REFERENCES

[1] Beck C. and Schlogl F., Thermodynamics of Chaotic
Systems (Cambridge University Press, Cambridge) 1993.
[2] Schuster R. C., Deterministic Chaos. An Introduction
(VCH Publishers, Weinheim) 1988.
[3] Hilborn R. G. and Just W., Chaos and Nonlinear
Dynamics (Oxford University Press, New York) 2000.
[4] Mayoral E. and Robledo A., Phys. Rev. E, 72 (2005)
026209.
[5] Robledo A. and Moyano L. G., Phys. Rev. E, 77 (2008)
036213.
[6] Tsallis C., J. Stat. Phys., 52 (1988) 479.
[7] Tsallis C., Introduction to Nonextensive Statistical
Mechanics: Approaching a Complex World (Springer)
2009.
[8] Ma W.-J., Hu C.-K., J. Phys. Soc. Jpn., 79 (2010)
024005.
[9] Ma W.-J., Hu C.-K., J. Phys. Soc. Jpn., 79 (2010)
024006.
[10] Moyano L. G., Silva D. and Robledo A., Cent. Eur.
J. Phys., 7 (2009) 591.
[11] Grassberger R., Phys. Rev. Lett., 95 (2005) 140601.
[12] van der Weele J. P., Capel H. W. and Kluiving R.,
Physica A, 145 (1987) 425.