THE ASYMPTOTICS OF THE LAPLACIAN 
ON A MANIFOLD WITH BOUNDARY II

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Abstract. We study the fifth term in the asymptotic expansion of the heat operator trace for a partial differential operator of Laplace type on a compact Riemannian manifold with Dirichlet or Neumann boundary conditions.

§1 Statement of results

Let $M$ be a smooth compact Riemannian manifold of dimension $m$ with smooth boundary $\partial M$. Let $V$ be a smooth vector bundle over $M$ and let $D$ be a partial differential operator of Laplace type on $C^\infty(V)$. There exists a unique connection $\nabla$ on $V$ and a unique endomorphism $E$ of $V$ so that

$$D = -(\text{Tr}(\nabla^2) + E).$$

If $\phi \in C^\infty(V)$, let $\phi_m$ be the covariant derivative of $\phi$ with respect to the inward unit normal. Let $S$ be an endomorphism of $V|_{\partial M}$.

We define the modified Neumann boundary operator $B^+_S$ and the Dirichlet boundary operator $B^-$ by

$$B^+_S \phi := (\phi_m + S\phi)|_{\partial M} \quad \text{and} \quad B^- \phi := \phi|_{\partial M}.$$

Let $B = B^+_S$ denote either boundary operator; we set $S = 0$ for Dirichlet boundary conditions to have a uniform notation. Let $D_B$ be the operator defined by the appropriate boundary condition. If $F$ is a smooth function on $M$, there is an asymptotic series as $t \downarrow 0$ of the form

$$\text{Tr}_{L^2}(F e^{-tD_B}) \cong \Sigma_{n \geq 0} t^{(n-m)/2} a_n(F, D, B)$$

where the $a_n(F, D, B)$ are locally computable; see [6] for details. We computed $a_n$ for $n \leq 4$ in [3]; we have changed notation slightly from that paper. In this paper, we compute $a_5$ if the boundary is totally geodesic. If the boundary is not totally geodesic, the second fundamental form $L$ enters and the combinatorial complexity becomes formidable. In this setting, we also obtain a result by restricting attention.
to regions in $\mathbb{R}^m$. In particular, we determine the coefficient of $|\nabla L|^2$ in $a_5(\Delta_0, B_0^\pm)$ for the Laplacian $\Delta_0$ on functions with pure Neumann or Dirichlet boundary conditions. In the Neumann case, this coefficient plays an important role in the work of Gursky [7] on compactness of isospectral sets in planar domains. The determination of this coefficient was, in large part, the original motivation for the present paper.

We adopt the following notational conventions. Let indices $i, j, ...$ range from 1 through $m$ and index a local orthonormal frame $\{e_1, ..., e_m\}$ for the tangent bundle of $M$. On the boundary, let indices $a, b, ...$ range from 1 through $m - 1$ and index a local orthonormal frame for the tangent bundle of the boundary; let $e_m$ be the inward unit normal. We adopt the Einstein convention and sum over repeated indices. Let $R_{ijkl}$ be the components of the curvature tensor of the Levi-Civita connection on $M$. With our sign convention, $R_{1212}$ is negative on the standard sphere in Euclidean space. The Ricci tensor $\rho$ and the scalar curvature $\tau$ are given by

$$\rho_{ij} := R_{ikkj} \quad \text{and} \quad \tau = \rho_{ii} = R_{kk}.$$ 

Let $\rho^2 := \rho_{ij}\rho_{ij}$ and $R^2 := R_{ijkl} R_{ijkl}$ be the norm of the Ricci and full curvature tensors. Let $\Omega_{ij}$ be the endomorphism valued components of the curvature of the connection on $V$, and let $L_{ab}$ be the components of the second fundamental form;

$$L_{ab} := (\nabla_{e_a} e_b, e_m).$$

Let ‘;’ denote multiple covariant differentiation with respect to the Levi-Civita connection of $M$ and let ‘;’ denote multiple tangential covariant differentiation on the boundary with respect to the Levi-Civita connection of $\partial M$; the difference between these two is measured by the second fundamental form. When sections of bundles built from $V$ are involved, ‘;’ will mean $\nabla^M \otimes 1 + 1 \otimes \nabla^V$ and ‘;’ will mean $\nabla^{\partial M} \otimes 1 + 1 \otimes \nabla^{V|\partial M}$. Thus $E_{ia} = E_{ia}$ since there are no tangential indices in $E$ to be differentiated, while $E_{ab}$ and $E_{ab}$ do not agree since the index $a$ is also being differentiated. Since $L$ and $S$ are only defined on the boundary, these tensors can only be differentiated tangentially. Let $dx$ and $dy$ be the Riemannian volume elements on $M$ and on $\partial M$. If $f_1 \in C^\infty(M)$ and if $f_2 \in C^\infty(\partial M)$, define

$$f_1[M] := \int_M f_1(x) \, dx \quad \text{and} \quad f_2[\partial M] = \int_{\partial M} f_2(y) \, dy.$$ 

**Theorem 1.1.** Suppose the boundary of $M$ is totally geodesic. Then

$$a_5(F, D, B_0^\pm) = \pm 5760^{-1}(4\pi)^{(m-1)/2} \text{Tr}\{F(360E_{mm} + 1440E_m S + 720E^2 + 240E_{aa} + 240\tau E + 120\Omega_{ab}\Omega_{ab} + 48\tau_{ii} + 20\tau^2 - 8\rho^2 + 8R^2 - 120\rho_{mm}E - 20\rho_{mm}\tau + 480\tau S^2 + (90^+, -360^+)\Omega_{am}\Omega_{am} + 12\tau_{mm} + 48\rho_{mm,aa} + 15\rho_{mm:mm} + 270\tau_{mm}S + 120\rho_{mm}S^2 + 960S_{aa}S + 600S_{aa}S + 16\rho_{mm,ab} - 17\rho_{mm,mm} - 10\rho_{ammb}R_{ammb} + 2880E_{mm}^{(1, 4)}  + 1440S^4 \} + F_m((195/2^+, 60^-)\tau_m + 20\tau S - 90\rho_{mm}S + 270S_{aa} + (630^+, 450^-)E_{mm} + 1440ES + 720S^3) + F_{mm}(60\tau - 90\rho_{mm} + 360E + 360S^2 + 180SF_{mm:mm} + 45F_{mm:mm:mm})[\partial M].$$

As mentioned above, if $\partial M$ is not totally geodesic, the number of invariants becomes unmanageable; the general formula involves 63 additional terms; see Lemma 5.1 for details. However, if we restrict to the Laplacian $\Delta_0$ on functions with Neumann or Dirichlet boundary conditions, assume $M$ is a domain in $\mathbb{R}^m$, and set $F = 1$, a great simplification occurs. Recall that $L_{bec} - L_{acb} = R_{abcm}$, see for
example [3, Lemma A.1 (b)]. Consequently, if the curvature tensor $R$ vanishes, then $\nabla L$ is a symmetric 3-tensor. Thus there exist universal constants such that

$$a_5(\Delta_0, \mathcal{E}_5) = \pm (4\pi)^{(m-1)/2} \frac{5760}{\pi} \int \left\{ \frac{3}{4} L_{abcd} L_{abcc} + \frac{2}{3} L_{aabb} L_{bddd} \right. \right.$$

$$+ e_3^+ L_{ab} L_{acc} L_{cc} L_{dd} + e_4^+ L_{ab} L_{ac} L_{cd} + e_5^+ L_{ab} L_{bc} L_{cd} L_{dd} \right.

$$+ e_6^- L_{ab} L_{bc} L_{cd} L_{da} \left\} |\partial M|.$$ 

**Theorem 1.2.** $e_1^+ = -19 \cdot 45/16$ and $e_1^- = 45/16$.

**Remark.** The constant $e_1^+$ is of particular interest as it controls certain compactness estimates for Neumann boundary conditions and plays an important role in theorems of Gursky [7]; it was not previously known and its determination was one of the primary goals of this study. The value of $e_1^-$ was previously known; see [11,12]. Levitin [10] has used the algorithm of Kennedy [8, 9] to compute $R$ in the ball in $\mathbb{R}^n$ for any $m$; this result has been used by van den Berg [1] to determine $e_2^+, e_3^+$, $(e_4 + e_5)^\pm$, and $e_6^-$. See also related work by Bord et al [2].

Here is a brief outline to the paper. In §2, we recall some functorial properties of the invariants $a_n(\cdot)$, and recall the calculation of $a_n$ for $n \leq 4$. In §3, we compute $a_5$ for the special case $m = 1$; this is an important step in the general case. In §4, we prove Theorem 1.1; we shall omit the calculation of the coefficient of $\Omega_{am} \Omega_{nm}$ as it is fairly lengthy; details are available from the authors. In §5, we prove Theorem 1.2. §6 is an appendix in which we give some variational formulas used elsewhere in the paper.

## §2 Functorial properties

We summarize below properties established in [3]:

**Lemma 2.1.**

1) Let $N^\nu(F) = F_{m,\ldots,m}$ be the $\nu^{th}$ normal covariant derivative. There exist invariant local formulae $a_n(x, D)$ and $a_{n,\nu}(y, D)$ so that:

$$a_n(F, D, B) = \left\{ F a_n(x, D) \right\} [M] + \left\{ \sum_{0 \leq \nu \leq n-1} N^\nu(F) a_{n,\nu}(y, D, B) \right\} |\partial M|.$$ 

The interior invariant $a_n(x, D)$ is homogeneous of order $n$ in the jets of the symbol of $D$ and vanishes for $n$ even; it is independent of the boundary condition chosen. The boundary invariants $a_{n,\nu}$ are homogeneous of order $n - \nu - 1$.

2) Let $M = M_1 \times M_2$ and $D = D_1 \otimes 1 + 1 \otimes D_2$. Let $\partial M_1 = \emptyset$. Then

$$a_n(x, D) = \sum_{p+q=n} a_{p,1}(x_1, D_1) a_q(x_2, D_2)$$

$$a_n,\nu(y, D) = \sum_{p+q=n} a_{p,\nu}(x_1, D_1) a_q(y_2, D_2, B).$$

3) If we expand $a_n(x, D)$ or $a_{n,\nu}(y, D, B)$ with respect to a Weyl basis, the coefficients depend on the dimension $m$ only through a normalizing constant. They are independent of the dimension of $V$.

4) If $D(\epsilon) = e^{-2\epsilon F_1}$, then $\frac{d}{d\epsilon}|_{\epsilon=0} a_n(1, D, B) = (m-n) a_n(F, D, B)$.

5) If $D(\epsilon) = D - \epsilon F \cdot I_2$, then $\frac{d}{d\epsilon}|_{\epsilon=0} a_n(1, D, B) = a_n-2(F, D, B)$.

6) If $D(\epsilon) = \frac{d}{d\epsilon}|_{\epsilon=0} a_n(F, D, B) = a_{n-2}(F, D, B)$.

7) If $m = n + 2$, then $\frac{d}{d\epsilon}|_{\epsilon=0} a_n(e^{-2\epsilon F}, e^{-2\epsilon F}, D, B) = 0$. 

8) If $m > n$, then $\frac{d}{d\epsilon}|_{\epsilon=0} a_n(e^{-2\epsilon F}, e^{-2\epsilon F}, D, B)$.
Let $m = 1$ and let $b \in C^\infty[0,1]$ be real. Let $A = \partial_x - b$, $A^* = -\partial_x - b$, $D_1 = A^* A$, $D_2 = A A^*$, and $S = b$. Then

$$(n - 1)(a_n(F, D_1, B^-) - a_n(F, D_2, B^+_0)) = a_{n-2}(\partial^2_x F + 2b\partial_x F, D_1, B^-).$$

\textbf{Remark.} Assertion (3) is a bit formal; it is best illustrated by reference to Theorems 2.4 and 2.5 below which express $a_n$ for $n \leq 4$ in terms of a Weyl basis.

There are two additional properties which we shall need which were not discussed in [3]. The first relates certain coefficients in a Weyl basis for Neumann boundary conditions to the corresponding coefficients for Dirichlet boundary conditions. Suppose that $M = N \times S^1$ and that $V$ is the trivial line bundle. Let $T(y, \theta) = (y, -\theta)$ define an involution of $M$ where $\theta \in \mathbb{R}/2\pi\mathbb{Z} = S^1$ is the usual periodic parameter. Let $M = N \times [0, \pi] \subset \tilde{M}$ and let $F$ be a smooth function on $\tilde{M}$ which is preserved by $T$. Let $e_m$ be the inward unit normal on $M$; $e_m = \partial_y$ at $\theta = 0$ and $e_m = -\partial_\theta$ at $\theta = \pi$. We assume $D$ is preserved by $T$; this means that if $D = -\{g^{ij}\partial_i + A^k \partial_k + B\}$, then

$g^{ab}(y, \theta) = g^{ab}(y, -\theta)$, $g^{mm}(y, \theta) = g^{mm}(y, -\theta)$,

$g^{am}(y, \theta) = -g^{am}(y, -\theta)$, $A^a(y, \theta) = -A^a(y, -\theta)$,

$A^a(y, \theta) = A^a(y, -\theta)$, $B(y, \theta) = B(y, -\theta)$.

Let $B^\pm$ denote pure Neumann and Dirichlet boundary conditions so that $S = 0$.

\textbf{Lemma 2.2.} We adopt the notation established above. Let $n$ be arbitrary and let $\nu$ be even. Then

$$a_{n,\nu}(y, D, B^+_0) + a_{n,\nu}(y, D, B^-) = 0.$$

\textbf{Proof.} Let $F$ be an even function. Since $D$ is invariant under the involution $T$, we may decompose the eigenfunctions of $D$ on $M$ into the even and odd eigenfunctions. The even eigenfunctions satisfy pure Neumann boundary conditions on $M$; the odd eigenfunctions satisfy Dirichlet boundary conditions on $M$. Thus

$$Tr_{L^2_M}(Fe^{-tD^-}) + Tr_{L^2_M}(Fe^{-tD^+}) = Tr_{L^2_M}(Fe^{-tD}).$$

We equate coefficients of $t$ in the asymptotic expansions of both sides of this equation. The interior integrals cancel since on the left hand side we are integrating over $[0, \pi]$ twice and on the right hand side we are integrating over $[-\pi, \pi]$. There are no boundary integrals on the right hand side so the boundary integrals on the left hand side must cancel. This shows

$$\Sigma_\nu \{N^\nu(F)(a_{n,\nu}(y, D, B^+_0) + a_{n,\nu}(y, D, B^-))\}[\partial M] = 0.$$

Since we may specify the $N^\nu(F)$ arbitrarily for $\nu$ even, the desired vanishing theorem now follows. \hfill \Box

In the next Lemma, we shall restrict to the case $n = 5$ in the interests of clarity and note that there is a more general principal applicable.
Lemma 2.3. When \(a_5\) is expanded in a Weyl basis, the coefficients of the following terms are zero: \(F_{im} \text{Tr}(\Omega_{am;a}), \ F \text{Tr}(\Omega_{am;ma}), \ F \text{Tr}(S\Omega_{am}), \ F \text{Tr}(S_a\Omega_{am}), \ FL_{aa} \text{Tr}(\Omega_{mm;ab})\).

Proof. We adapt an argument given in [4]. Let \(F\) be real. If \(D\) and \(S\) are real, then \(\text{Tr}_{L^2}(Fe^{-iD\theta})\) is real so the coefficients of the terms listed above must be real. On the other hand, if \(V\) is unitary with respect to some fiber metric on \(V\) and if \(E\) and \(S\) are self-adjoint, then \(D\) is self-adjoint so again \(\text{Tr}_{L^2}(Fe^{-iD\theta})\) is real. We take \(V\) to be a line bundle; \(\Omega\) is pure imaginary in this context. Thus the coefficients must be pure imaginary as well and hence must vanish. \(\square\)

We refer to [3] for the computation of \(a_n\) for \(n \leq 4\).

Theorem 2.4 (Dirichlet).
1) \(a_0(F, D, B^-) = (4\pi)^{-m/2} \text{Tr}(F)[M]\).
2) \(a_1(F, D, B^-) = -4^{-1}(4\pi)^{-(m-1)/2} \text{Tr}(F)[\partial M]\).
3) \(a_2(F, D, B^-) = (4\pi)^{-m/2} \{ \text{Tr}(6FE + F\tau)[M] + \text{Tr}(2FL_{aa} - 3F_m)[\partial M] \}\)
4) \(a_3(F, D, B^-) = -384^{-1}(4\pi)^{-(m-1)/2} \{ \text{Tr}(96E + 16\tau - 8\rho_{mm} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}) - 30F_mL_{aa} + 24F_{mm}[\partial M]\}\)
5) \(a_4(F, D, B^-) = (4\pi)^{-m/2} \{ \text{Tr}(F(60E_{kk} + 60\tau E + 180E^2 + 30\Omega^2 + 12\tau_{kk} + 5\tau^2 - 2\rho^2 + 2R^2))[M] + \text{Tr}(F(-120E_{mm} - 12\tau_{mm} + 120E_{aa} + 20L_{aa} + 4R_{amam}L_{bb} - 12L_{amam}L_{ab} + 4R_{abab}L_{ac} + 24L_{aa}L_{bb} + 0L_{abab} + 40/21L_{aa}L_{bb}L_{cc} + 88/7L_{ab}L_{cc} + 32/21L_{ab}L_{bc}L_{ac} + F_{mm}(-180E - 30\tau + 0R_{amam} - 180/7L_{aa}L_{bb} + 120L_{ab}L_{ab} + 24F_{mm}L_{aa} + 30F_{iiim})[\partial M]\}\)

Theorem 2.5 (Neumann).
1) \(a_0(F, D, B_\pm^\pm) = (4\pi)^{-m/2} \text{Tr}(F)[M]\).
2) \(a_1(F, D, B_\pm^\pm) = 4^{-1}(4\pi)^{-(m-1)/2} \text{Tr}(F)[\partial M]\).
3) \(a_2(F, D, B_\pm^\pm) = 384^{-1}(4\pi)^{-(m-1)/2} \{ \text{Tr}(F(96E + 16\tau - 8\rho_{mm} + 13L_{aa}L_{bb} + 2L_{ab} + 96SL_{aa} + 192S^2) + F_{im}(6L_{aa} + 96S)M]\}\)
4) \(a_3(F, D, B_\pm^\pm) = 384^{-1}(4\pi)^{-(m-1)/2} \{ \text{Tr}(F(240E_{mm} + 42\tau_{mm} + 24L_{aa}L_{bb} - 12R_{amam}L_{ab} + 4R_{abab}L_{ac} + 24L_{aa}L_{cc} + 40L_{ab}L_{ac} + 8L_{ab}L_{ab}L_{cc} + 32/3L_{ab}L_{bc}L_{ac} + 0L_{mm} + 720E + 120S\tau + 0R_{amam} + 144S_{aa}L_{bb} + 48S_{ab}L_{ab} + 12S_{aa}L_{bb} + 12S_{ab}L_{ab} + 320S_{aa}L_{bb} + 120S + F_{im}(180E + 30\tau + 0R_{amam} + 12L_{aa}L_{bb} + 12L_{ab}L_{ab} + 72S_{aa}L_{bb} + 240S^2) + F_{mm}(24L_{aa} + 120S) + 30F_{iiim})[\partial M]\}\)

§3 Dimension One

This section is devoted to the calculation of \(a_5\) in the one dimensional case; this forms the foundation for our later computations. We restrict to scalar operators; recall \(S = 0 \text{ if } B = B^\pm\).

Theorem 3.1. \(a_5(F, D, B_\pm^\pm) = 384^{-1}\{ (24F_{mm} + 96E_{mm}S + 48E^2 + 192E^2 + 96S^2) + F + (42S - 30\tau)E_{im} + 96ES + 48S^2)F_{im} + (24 + 24S^2)F_{mm} + 12SF_{mm} + 3F_{iimmn} \}[\partial M]\).
Remark. If we set $F = 1$, this agrees with Theorem 3.4 of [5].

Proof. We use Lemma 2.1 (1) to see there exist universal constants so that

$$a_5(F, D, B^2_S) = \pm 384^{-1}\{(b_1^+ E_{mm} + b_2^+ E_m S + b_3^+ E^2 + b_4^+ E S^2$$
$$+ b_5^+ S^4) F + (b_6^+ E_m + b_7^+ ES + b_8^+ S^3) F_m + (b_9^+ E + b_{10}^+ S^2) F_{mm}$$
$$+ b_{11}^+ S F_{mmm} + b_{12}^+ F_{mmm m})\} \partial M.$$ 

Let $M$ be the interval $[0, \pi]$, and set $S = 0$. Double the interval to obtain a copy of the circle, and assume $E$ and $F$ are even on the double. We apply Lemma 2.2 to see

$$a_5(F, D, B^2_S) + a_5(F, D, B^-) = 0.$$ 

This shows that $b_1^+ = b_1^\rightarrow$, $b_2^+ = b_3^\rightarrow$, $b_3^+ = b_5^\rightarrow$, and $b_{12}^+ = b_{12}^\rightarrow$. No conclusion can be drawn about the terms involving $S$ since we have set $S = 0$. Also, no conclusion can be drawn about the term involving $E_{mm}$, since this will vanish for an even function.

This shows that the general formula for $a_5$ in the one-dimensional case is

$$a_5(F, D, B^2_S) = \pm 384^{-1}\{(b_1 E_{mm} + b_2^+ E_m S + b_3 E^2 + b_4^+ E S^2$$
$$+ b_5^+ S^4) F + (b_6^+ E_m + b_7^+ ES + b_8^+ S^3) F_m + (b_9^+ E + b_{10}^+ S^2) F_{mm}$$
$$+ b_{11}^+ S F_{mmm} + b_{12}^+ F_{mmm m})\} \partial M.$$ 

By Theorems 2.4 and 2.5,

$$a_3(F, D, B^2_S) = \pm 384^{-1}\{(96 E + 192 S^2) F + 96 S F_m + 24 F_{mm}\} \partial M.$$ 

Let $D(\epsilon) = D - \epsilon I$. Then $E(\epsilon) = E + \epsilon I$. By Lemma 2.1 (6)

$$\kappa |_{\epsilon = 0} a_5(F, D, B^2_S) = a_5(F, D, B^2_S).$$ 

This shows $b_3 = 48, b_4 = 192, b_7^\rightarrow = 96$, and $b_9 = 24$. Next let $D(\epsilon) = D - \epsilon F$. Then $E(\epsilon) = E + \epsilon F$. By Lemma 2.1 (5)

$$\kappa |_{\epsilon = 0} a_5(1, D, B^2_S) = a_5(F, D, B^2_S).$$ 

It now follows that $b_1 = 24$ and $b_5^\rightarrow = 96$. This shows that

$$a_5(F, D, B^2_S) = \pm 384^{-1}\{(24 E_{mm} + 96 E_m S + 48 E^2 + 192 ES^2 + b_5^+ S^4) F$$
$$+ (b_6^+ E_m + 96 ES + b_8^+ S^3) F_m + (24 E + b_{10}^+ S^2) F_{mm}$$
$$+ b_{11}^+ S F_{mmm} + b_{12}^+ F_{mmm m})\} \partial M.$$ 

We now adopt the notation of Lemma 2.1 (8). Let $M = [0, 1]$ and let $b \in C^\infty[0, 1]$ be real. Let $A = \partial_x - b$, $D_1 = A^* A$, and $D_2 = AA^*$. Then $E_1 = -b_x - b^2$ and $E_2 = b_x - b^2$. We take Dirichlet boundary conditions for $D_1$ and modified Neumann boundary conditions given by $A^*$ for $D_2$; this means $S = b$. We assume $F$ vanishes to infinite order at the endpoint $x = 1$. We expand $a_3$ with Dirichlet boundary conditions,

$$a_3(F_{xx} + 2b F_x, D_1, B^-)$$
$$= -384^{-1}\{96(-b_x - b^2)(F_{xx} + 2b F_x) + 24(F_{xx} + 2b F_x)_{xx}\}(0)$$
$$= -384^{-1}\{(-96b_x F_{xx} - 96b^2 F_{xx} - 192b_x b F_x - 192b^3 F_x$$
$$+ 24 F_{xxxx} + 48b_{xx} F_x + 96b_x F_{xx} + 48b F_{xxx}\}(0)$$
$$= -384^{-1}\{24 F_{xxxx} + 48b F_{xxx} - 96b^2 F_{xx}$$
$$+(48b_{xx} - 192b_x b - 192b^3) F_x\}(0),$$
\( a_5 \) with Neumann boundary conditions,

\[
-a_5(F, D_2, B_5^+) = -384^{-1}\{(24(b_x - b^2)_{xx} + 96(b_x - b^2)_x b + 48(b_x - b^2)_x^2 + 192(b_x - b^2)b^2 + b^+_0 b^1)F + (b^+_0 (b_x - b^2)_x + 96(b_x - b^2)b + b^+_0 b^1)F_x + (24(b_x - b^2)_x + b^+_0 b^2)F_{xx} + b^+_1 b^F_{xxx} + b^+_2 F_{xxxx}\}(0),
\]

and \( a_5 \) with Dirichlet boundary conditions,

\[
a_5(F, D, B^-) = -384^{-1}\{(24(-b_x - b^2)_{xx} + 48(-b_x - b^2)^2 F + b^+_6 (-b_x - b^2)_x F_x + 24(-b_x - b^2)_x F_{xx} + b^+_2 F_{xxxx}\}(0).
\]

By Lemma 2.1 (8),

\[
a_3(F_{xx} + 2bF_x, D_1, B^-) = 4\{a_5(F, D_1, B^-) - a_5(F, D_2, B_5^+)\}.
\]

This leads to the equations

| Term   | Equation       | Term   | Equation       |
|--------|----------------|--------|----------------|
| \( b^4F \) | \( 0 = 48 - 192 + b^+_0 + 48 \) | \( b^3F_x \) | \( -192 = 4(-96 + b^+_0) \) |
| \( b_x bF_x \) | \( -192 = 4(-2b^+_0 + 96 - 2b^+_6) \) | \( b_{xx} F_x \) | \( 48 = 4(b^+_0 - b^-_6) \) |
| \( b^2F_{xx} \) | \( -96 = 4(-24 + b^+_0 - 24) \) | \( b_{F_{xx}} \) | \( 48 = 4b^+_1 \) |
| \( F_{xxxx} \) | \( 24 = 4(b_{12} + b_{12}) \) |

We solve this system of equations and complete the proof by computing:

\[
\begin{align*}
b^+_0 &= 96, & b^+_1 &= 42, & b^-_6 &= 30, & b^+_8 &= 48, \\
b^+_0 &= 24, & b^+_1 &= 12, & b_{12} &= 3. & \quad \square
\end{align*}
\]

§4 Totally geodesic boundary

This section is devoted to the proof of Theorem 1.1. We assume for the remainder of this section that the boundary of \( M \) is totally geodesic. We begin the proof of Theorem 1.1 by expressing \( a_5 \) in terms of universal expressions with undetermined coefficients. Note that \( 5760 = 15 \cdot 384 = 16 \cdot 360 \).

**Lemma 4.1.** Suppose the boundary of \( M \) is totally geodesic. There exist universal constants so that \( a_5(F, D, B_5^+) = \pm (4\pi)^{(m-1)/2}25760^{-1}\text{Tr}\{F(360E_{mm} + 1440E_{mS} + 720E^2 + 2880ES^2 + 1440S^3) + 240E_{aa} + 240rE + 120\Omega_{ab}\Omega_{ab} + 48\tau_{ui} + 20r^2 - 8p^2 + 8R^2 - 120\rho_{mm}E - 20\rho_{mm}\tau + 480\tau S^2 + c^+_5\Omega_{am}S_{am} + c^+_2\Omega_{am}S_{am} + c^+_3\Omega_{mm}S_{mm} + c^+_5\tau_{mm}S + c^+_6\rho_{mm}S^2 + c^+_7S_{S_{aa}} + c^+_8S_{S_{aa}} + c^+_9R_{mmmb}p_{ab} + c^+_0\rho_{mm}\rho_{mm} + c^+_1\rho_{mm}\rho_{mm} + F_m((630^+, 450^+)E_{mm} + 1440ES + 720S^3 + c^+_5\tau_{mm} + 240\tau S + c^+_1\rho_{mm}S + c^+_4\rho_{mm}S + c^+_1\rho_{mm}S) + F_{mm}(360E + 360S^2 + 60\tau + c^+_5\rho_{mm}) + 180SF_{mm} + 45F_{mm}\}}[\partial M].
\]

**Proof.** Since the boundary is totally geodesic, we may interchange ‘;’ and ‘,:’ as convenient. We write down a basis for the space of invariants using H. Weyl’s theorem. The only tricky part of this analysis is the study of terms of order 4 in the jets of the metric; these terms are studied at the end of the proof. We omit the curvature terms \( R_{mabc}R_{mabc} \) and \( \rho_{mcp}\rho_{nc} \) since these vanish on a totally geodesic boundary. Since \( \tau_{mm} = 2\rho_{mm;m} \), we also omit the terms \( \rho_{mm;m}S \) and \( F_m\rho_{mm;m} \) from our list. Finally, we use Lemma 2.3 to omit the following invariants from our list:

\[
\text{Tr}(F\Omega_{am;am}), \text{Tr}(F\Omega_{am;am}S), \text{Tr}(F\Omega_{am;am}S), \text{Tr}(F_{mm}\Omega_{am;am}).
\]
We can use previous results to determine many of the coefficients in a Weyl spanning set:

1. We use Theorem 3.1 to determine the coefficients of the following expressions which do not involve $\Omega$, $R$, or the tangential derivatives of $S$ and $E$:

\[
FE_{,mm}, FE_{,mS}, FE^2, FES^2, FS^4, F_{,mE_{,m}}, F_{,mES},
F_{,mS^3}, F_{,mmE}, F_{,mmS^2}, F_{,mmmm}, F_{,mmmmm}.
\]

2. We use Lemma 2.1 (2) to see $\sum_{r=0}^{\infty} a_5(1, D - \epsilon I, B) = a_3(1, D, B)$; this enables us to determine the coefficient of $\tau E$ and $\rho_{mm} E$ in $a_5$ from the expression for $a_3$ given in Theorems 2.4 and 2.5.

3. We apply Lemma 2.1 (2) to a product $M = M_1 \times M_2$ where $\partial M_1 = \emptyset$ to see

\[
a_{5,\nu}(x_1, y_2, D_1 \otimes 1 + 1 \otimes D_2, \mathcal{B}^+_S) = \sum_{p+q=5} a_{p,q}(x_1, D_1) a_{q,\nu}(y_2, D_2, \mathcal{B}^+_S).
\]

(i) We take $(m_1, m_2) = (4, 1)$ and study the terms in $a_{5,0}$ which are independent of $M_2$. These terms arise when $(p, q) = (4, 1)$ so

\[
a_{5,0}(\cdot, \mathcal{B}^+_S) = \pm(4\pi)^{(m-1)/2} \left\{ 60E_D^{D_1} + 30(\Omega^{D_1})^2 + 12\tau_{,aa}^M \\
+ 5(\tau^{M_1})^2 - 2(\rho^{M_2})^2 + 2(R^{M_1})^2 + \ldots \right\}/(4 \cdot 384).
\]

This determines the coefficients of the invariants $E_{,aa}, \Omega_{,ab}, \tau_{,ii}, \tau^2, \rho^2,$ and $R^2$ in $a_{5,0}$; still to be determined, of course, is the coefficient of $\tau_{,mm}$. Similarly, we study the cross terms to evaluate the coefficient of $\tau S^2$ in $a_{5,0}$, to evaluate the coefficient of $\tau S$ in $a_{5,1}$, and to evaluate the coefficient of $\tau$ in $a_{5,2}$; the cross terms arise when $(p, q) = (2, 3)$.

(ii) We take $(m_1, m_2) = (2, 2)$ and compute the cross terms involving the curvature tensor in $a_{5,0}$. These arise when $(p, q) = (2, 3)$ so

\[
a_{5,0}(\cdot, \mathcal{B}^+_S) = \pm(4\pi)^{(m-1)/2} \tau^{M_1}(16\tau^{M_2} - 8\rho^{M_2})/(6 \cdot 384) + \ldots.
\]

Since $\tau^2 = 2\tau^{M_1}\tau^{M_2} + \ldots$ and $\tau\rho_{mm} = \tau^{M_1}\rho^{M_2} + \ldots$, the coefficient of $\tau^2$ in $a_{5,0}$ is $16 \cdot 5760/(2 \cdot 6 \cdot 384) = 20$ and the coefficient of $\tau\rho_{mm}$ in $a_{5,0}$ is $-8 \cdot 5760/(6 \cdot 384) = -20$.

We use Lemma 2.2 to see that $c^+_\mu = c^-_\mu$ with our normalizing sign convention for many of the remaining coefficients; a notable exception is the coefficient of $\Omega_{,am}\Omega_{,am}$ since this term vanishes under the hypothesis of Lemma 2.2.

We conclude the proof of Lemma 3.1 by studying the terms which are linear in the second covariant derivative $\nabla^2 R$ of the curvature tensor. We must set some of the indices equal to $m$ and contract the remaining indices in pairs in an expression $R^{i_1i_2i_3i_4;i_5i_6}$. If none of the indices $i_\mu$ are equal to $m$, we must contract 3 pairs of indices. The usual argument then shows this result is a multiple of $\tau_{,ii}$. Thus we can restrict our attention to the case where either 2 or 4 of the indices $i_\mu$ are equal to $m$. If 4 of the indices are equal to $m$, we get a multiple of $\rho_{mm:mm}$. In the remaining case, 2 of the indices are $m$ and modulo lower order terms we must consider the invariants

\[
R_{abba:mm}, R_{amma:bb}, R_{abam:bm}, \text{ and } R_{ambm:ab}.
\]

By the second Bianchi identity, $0 = R_{ambm:ab} + R_{amma:bb} + R_{abam:mb}$; this eliminates $R_{ambm:ab}$ from our list. Modulo lower order terms, we may replace $R_{abam:mb}$ by $R_{ambm:ab}$. By the second Bianchi identity,

\[
0 = R_{abam:bm} + R_{abmb:am} + R_{abba:mm} = 2R_{abam:bm} + R_{abba:mm};
\]
this eliminate $R_{abam:bm}$ from our list as well. We then note

$$\text{span}\{R_{abba:mm}, R_{amma:bb}, \tau_{iii}, \tau_{mm}\} = \text{span}\{\tau_{iii}, \tau_{mm}, \rho_{mm:aa}, \rho_{mm:mm}\}. \quad \square$$

We complete the proof of Theorem 1.1 by evaluating the constants which appear in Lemma 4.1. We will not compute $c_1^\pm$ as this requires somewhat different techniques; details are available from the authors upon request. This term vanishes for the scalar Laplacian in any event.

Let $D(\epsilon) = e^{-2\epsilon f} D, S(\epsilon) = e^{-\epsilon f} S, F(\epsilon) = e^{-2\epsilon f} F$. We assume $f_m|_{\partial M} = 0$ to preserve the condition that the boundary is totally geodesic. If $m = 7$, we use Lemma 2.1 (7) to see

$$\frac{d}{d\epsilon}|_{\epsilon = 0} a_5(F, D) = 0.$$  

We use Lemma 4.1 and results from §6 to compute $\frac{d}{d\epsilon}|_{\epsilon = 0} a_5(F, D)$. We integrate by parts and combine terms to express $\frac{d}{d\epsilon}|_{\epsilon = 0} a_5(F, D)$ in terms of independent integrands and derive the system of equations

| Term          | Equation | Term   | Equation       |
|---------------|----------|--------|----------------|
| $F f_{mmmm}$  | $39 = 2c_2 + c_4$ | $F f_{mm} S^2$ | $120 = c_6^+$ |
| $F f_{mam}$   | $303 = 6c_3 + 12c_2 + c_4$ | $F f_{mm} \tau$ | $40 = 2c_2 + c_4$ |
| $F f_{aa\rho}$ | $60 = -2c_2 - c_9 - 2c_{10} - c_{15}$ | $F f_{a\tau_a}$ | $12 = c_2$ |
| $F f_{a\rho}$  | $324 = -c_{15} + 12c_2 + 6c_4$ | $F f_{a\tau_a} S_a$ | $270 = c_{14}^+$ |
| $F f_{a\rho}$  | $63 = 4c_2 + c_4$ | $F f_{a\rho \rho_{ab}}$ | $16 = c_9$ |
| $F f_{a\rho}$  | $(195^+, 120^-) = 2c_{12}^+$ | $F f_{mm} f_{aa}$ | $90 = -c_{15}$ |
| $F f_{a\rho}$  | $1080 = c_6^+ + c_{12}^-$ | $F f_{a\rho} f_{mm} S$ | $-90 = c_{13}^+$ |
| $F f_{a\rho}$  | $360 = c_{12}^+ - c_{13}^-$ | $F f_{mm} S_{aa} S$ | $270 = c_{13}^+$ |
| $F f_{a\rho}$  | $378 = 5c_3 + 2c_{11} + 24c_2 + 2c_4$ | $F f_{mm} \rho_{mm}$ | $-c_{15} - 6c_{10} - c_{11} + 12c_2$ |

We solve this system of equations and complete the proof by computing:

$$c_2 = 12, \quad c_3 = 24, \quad c_4 = 15, \quad c_5^+ = 270, \quad c_6^+ = 120, \quad c_7^+ = 960, \quad c_8^+ = 600, \quad c_9 = 16, \quad c_{10} = -17, \quad c_{11} = -10, \quad c_{12} = 195/2, \quad c_{12}^- = 60, \quad c_{13}^- = -90, \quad c_{14}^+ = 270, \quad c_{15} = -90. \quad \square$$

§5 Domains in flat space

In this section, we complete the proof of Theorem 1.2. We drop the assumption that the boundary of $M$ is totally geodesic; thus “a” and “b,” differ. Lemma 4.1 generalizes to this context to become

**Lemma 5.1.** Suppose the boundary of $M$ is smooth but not necessarily totally geodesic.

1) There exist universal constants so that

$$a_5(F, D, B)^2 = \pm 5760^{-1}(4\pi)^{(m-1)/2} \text{Tr} \{F \{360E_{mm} + 1440E_{mm} S + 720E^2 + 240E_{aa} + 240rE + 120r\Omega_{ab} + 48\tau_{ii} + 20r^2 - 8\rho^2 + 8R^2 - 120\rho_{mm} E - 20\rho_{mm} \tau + 480rS^2 + (90^+, -360^-)\Omega_{am} \Omega_{am} + 12\tau_{mm} + 24\rho_{mm:aa} + 15\rho_{mm:mm} + 270\tau_{mm} S +$$
This completes the proof of 1).

2) $d_{43} = 151.875$, $d_{44} = -11.25$, and $d_{55} = -30$.

3) $a_{43} = 39.375$, $a_{44} = -11.25$, and $a_{55} = -105$.

Proof. We write down a basis for the space of invariants using H. Weyl's theorem. We use Theorem 1.1 to evaluate the coefficients of the expressions not involving the second fundamental form $L$; we also Lemma 2.1 (2) to evaluate other coefficients using suitable product formulas. The following invariants can be expressed in terms of invariants already appearing in the list above and are therefore omitted: $FL_{aa}R_{cmcb}$, $FL_{ab}R_{acmb}$, $FL_{ab}R_{acbm}$, $F_{mm}L_{aa}$, $F_{mm}mL_{aa}$, $F_{mm}mS$, $F_{im}p_{mm}$. We can also replace ‘;’ by ‘;’ in some expressions at the cost of introducing additional lower order terms involving the second fundamental form. Thus, for example, by Lemma 6.1, we may express

$$
\rho_{mm;m} = (\tau_{m} - R_{abbb;m})/2 = (\tau_{m} - 2R_{abbbm})/2
$$

This completes the proof of 1).

To prove 2) and 3), we suppress the coefficients of $F$ and $F_{mm}$ and we suppress terms of length greater than 1 in the coefficients of $F_{m}$. This permits us to express

\[ a_{5}(F, D, B^{±}(S)) = \pm 5760^{-1}(4\pi)^{(m-1)/2} \text{Tr} F^{(s)} + \ldots \]

Here, $m \leq 5$.

We have to be a bit careful since we have not used linearly independent monomials in the variational formulas in §6. Also, and equally importantly, we must integrate by parts in computing certain integral invariants. We apply Lemma 2.1 (7) to see that if $m = 7$, then

\[ 0 = \frac{d}{d\epsilon}|_{\epsilon=0}a_{5}(e^{-2\epsilon f}F, e^{-2\epsilon f}D, B^{±}(S_{\epsilon})) \]

The information in the tables below uses information from §3 and from the appendix in §6; the sum of each column multiplied by the appropriate entry and coefficient is
zero; this yields the equations necessary to compute $d_{43}^\pm$, $d_{44}^\pm$, and $d_{55}^\pm$ and complete the proof.  □

Neumann Boundary Conditions $m = 7$

| Term | Coeff | $f_{3m}F_{3mm}$ | $f_{6m}F_{6mm}L_{bb,a}$ | $f_{6a}F_{6m}L_{ab,b}$ | $f_{3m}F_{3mm}F_{3m}$ | $f_{aam}F_{3m}$ |
|------|-------|-----------------|--------------------------|------------------------|------------------------|----------------|
| $F_{3m}E_{3m}$ | 630 | 0 | 0 | 0 | 2.5 | 2.5 |
| $F_{3m}^2\pi$ | 97.5 | 0 | 0 | 0 | −12 | −12 |
| $F_{3m}S_{3a,a}$ | 270 | 0 | 2.5 | −5 | 0 | 2.5 |
| $F_{3mm}S$ | 45 | −14 | −4 | −8 | −9 | −4 |
| $F_{3mm}L_{a,a,b}$ | 180 | 2.5 | 0 | 0 | 0 | 0 |
| $F_{3m}L_{aa,bb}$ | $d_{43}^\pm$ | 0 | −4 | 12 | 0 | −6 |
| $F_{3m}L_{ab,ab}$ | $d_{44}^\pm$ | 0 | −2 | 10 | 0 | −1 |
| $F_{3mm}L_{aa}$ | $d_{55}^\pm$ | −6 | −3 | 0 | 0 | 0 |

Dirichlet Boundary Conditions $m = 7$

| Term | Coeff | $f_{3m}F_{3mm}$ | $f_{6m}F_{6mm}L_{bb,a}$ | $f_{6a}F_{6m}L_{ab,b}$ | $f_{3m}F_{3mm}F_{3m}$ | $f_{aam}F_{3m}$ |
|------|-------|-----------------|--------------------------|------------------------|------------------------|----------------|
| $F_{3m}E_{3m}$ | 450 | 0 | 0 | 0 | 2.5 | 2.5 |
| $F_{3m}^2\pi$ | 60 | 0 | 0 | 0 | −12 | −12 |
| $F_{3mm}S$ | 45 | −14 | −4 | −8 | −9 | −4 |
| $F_{3mm}L_{a,a,b}$ | $d_{43}^\pm$ | 0 | −4 | 12 | 0 | −6 |
| $F_{3m}L_{ab,ab}$ | $d_{44}^\pm$ | 0 | −2 | 10 | 0 | −1 |
| $F_{3mm}L_{aa}$ | $d_{55}^\pm$ | −6 | −3 | 0 | 0 | 0 |

We can now proceed with our study. We set $m = 2$, set $F = 1$, and suppress all terms of length greater than 2 to see

\[
a_5(1, D, B^\pm_S) = \pm 5760^{-1}(4\pi)^{(-m-1)/2} \text{Tr}((135/2)\tau_{mm} + 270\tau_{mm}S - 360S_{1:1} + d_{43}^\pm L_{11}E_{3m} + e_{71}^\pm L_{11:1}L_{11:1} + e_{78}^\pm L_{11:1}S + e_{78}^\pm L_{11}(\tau + \ldots)).
\]

Lemma 5.2. $e_{71}^\pm = -53.4375$, $e_{77}^\pm = 270$, $e_{78}^\pm = 22.5$, $e_{1}^- = 2.8125$, $e_{8}^- = -90$.

Proof. We consider the variation $D(\epsilon) = e^{-2\epsilon f} D$ with $\mathcal{B}_S^\pm(\epsilon)$ defined appropriately and with $E(0) = \tau(0) = 0$. By Lemma 2.1 (4),

\[
-3a_5(f, D, B_{S}^\pm) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0} a_5(1, e^{-2\epsilon f} D, B_{S}^\pm(\epsilon)).
\]

We use this relation to evaluate $d_{2,1}^\pm$, $e_{1}^\pm$, and $e_{77}^\pm$ from the tables given below; the variational formulas used to create these tables are contained in §6. We also apply the results of §3 to evaluate certain of the coefficients. □

Neumann Boundary Conditions $m = 2$
In the appendix, we present some identities used in the previous sections. In Lemma 6.1, we recall the basic equations of structure of [3] and some other preliminary results. In Lemma 6.2, we compute certain integral formulas modulo lower order terms. In Lemma 6.3, we give some variational formulas. Additional formulas useful in studying manifolds with non totally geodesic boundaries are available from the authors upon request as are the proofs are entirely elementary if somewhat tedious.

**Lemma 6.1.**
1) $\text{devol}_m = g^{1/2}dx_1 \cdots dx_m$, and $\Gamma_{\nu\mu\gamma} = \frac{1}{2}g^{\sigma\gamma}(\partial_\mu g_{\nu\sigma} + \partial_\sigma g_{\nu\mu} - \partial_\nu g_{\mu\sigma}).$
2) If $D = -(g^{\mu\nu}\partial_\mu \partial_\nu + P_a^\mu \partial_a + Q)$, then $\omega_\nu = \frac{1}{2}g_{\mu\nu}(P_\nu + g^{\sigma\mu} \Gamma_{\sigma\gamma}^\rho), \quad E = Q - g^{\mu\nu}(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \omega_\nu \Gamma_{\mu\nu\gamma}), \quad \Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu.$
3) $L_{\alpha\beta} = (\partial_m, \nabla_\alpha)_{\beta} = \Gamma_{\alpha\beta}^m = -\frac{1}{2}g^{mm}(\partial_m \nabla_\alpha)_{\beta}.$
4) $\rho_{am} = L_{bb;a} - L_{ab;cb}, \quad \rho_{am;m} = -R_{amm;c;e} + \rho_{mm;ae}, \quad R_{abck;e} = 2R_{ab;bk;e}, \quad \rho_{ij;j} = \frac{1}{2}\tau_{ij}, \quad \rho_{am;mc} = \frac{1}{2}\tau_{im} - \rho_{mm;m} = \tau_{m} - R_{ab;bm}, \quad R_{ab;cm} = L_{bc;a} - L_{ac;b},$
5) $f_a = f_a, f_{am} = f_{ma} = f_{ma} + L_{ab}f_{b}, f_{ab} = f_{ab} - L_{ab}f_{m}, f_{ii} = f_{mm} + f_{mm} - L_{aa}f_{m}, f_{am} = f_{ma} - \rho_{mm}f_{m} = f_{ma} = f_{ma} + L_{aba}f_{b} + 2L_{abc}f_{ac} - L_{ac}f_{b;m} - L_{ab}f_{m;im}, f_{aam} = f_{maam} + 2L_{ab}f_{b;m} + 2R_{amm};f_{m;e} - 2\rho_{mm};f_{m;e} + 2R_{ammbf_{ab};a} + \rho_{mm};f_{m;e}.$

We must integrate by parts in computing certain integral invariants. The following formulas are useful; we suppress lower order terms of length greater than 3.

**Lemma 6.2.**
1) $\int_{\partial M} f_{maa}F_{;m} = \int_{\partial M}(f_{maa}F_{;m} + F_{;m}f_{a}L_{bb} - 2F_{;m}f_{a}L_{ab});$
2) $\int_{\partial M} f_{am}F_{;am} = \int_{\partial M}(-f_{maa}F_{;m} - F_{;m}f_{a}L_{bb} + F_{;m}f_{a}L_{ab} + ...);$ 
3) $\int_{\partial M} f_{a}F_{;mma} = \int_{\partial M}(-2f_{a}F_{;m}L_{ab});$
4) \( \int_{\partial M} f_{;aamm} = \int_{\partial M} (2 f_{;ami} L_{a}^{ab} - f_{;amm} L_{aa} - \frac{1}{2} \tau_{;m} f_{;m} + \ldots) \).

We consider the variation
\[
D(\epsilon) = e^{-2\epsilon f} D, \quad g(\epsilon) = e^{2\epsilon f} g, \quad F(\epsilon) = e^{-2\epsilon f} F.
\]

To keep the boundary conditions constant, we set
\[
S(\epsilon) = e^{-\epsilon f} \{ \omega_{m}(0) - \omega_{m}(\epsilon) + S \}.
\]

Let \( \epsilon \), be an orthonormal frame for the tangent and cotangent bundles of \( M \) with respect to the reference metric \( g(0) \). Let \( \epsilon_{i}(\epsilon) = e^{-\epsilon f} \epsilon_{i} \) and \( \epsilon^{i}(\epsilon) = e^{\epsilon f} \epsilon^{i} \) be the corresponding frames for the metric \( g(\epsilon) \). We remark that contraction and differentiation do not commute;
\[
\frac{d}{d\epsilon}|_{\epsilon=0}(\Phi_{ij}) = (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{ij} - 2f\Phi_{ii}
\]
for example. Although \( \Gamma \) is not tensorial, its variation is tensorial. Let
\[
\Gamma_{ij}^{k} := (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{ij}^{k}
\]
we keep the distinction between lower and upper indices since
\[
(\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{ij}^{k} \neq (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{ijk}; \quad \text{(we shall not need \( \Gamma_{ijk} \)).}
\]

**Lemma 6.3.**

1. \( \Gamma_{ij}^{k} := (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{ij}^{k} = \delta_{ik} f_{;j} + \delta_{jk} f_{;i} - \delta_{ij} f_{;k} \).
2. \( (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{mm} = -f_{;a}, \quad \text{and} \quad (\frac{d}{d\epsilon}|_{\epsilon=0}\Gamma)_{mm}^{m} = f_{;m} \).
3. \( (\frac{d}{d\epsilon}|_{\epsilon=0}L)_{ab} = -\delta_{ab} f_{;m} + f L_{ab} \).
4. \( (\frac{d}{d\epsilon}|_{\epsilon=0}S) = -f S + \frac{1}{2}(m-2) f_{;m} \).
5. \( (\frac{d}{d\epsilon}|_{\epsilon=0}E) = -2f E + \frac{1}{2}(m-2) f_{;ii} \).
6. \( (\frac{d}{d\epsilon}|_{\epsilon=0}R)_{ijkl} = \delta_{ik} f_{;jl} + \delta_{jk} f_{;il} - \delta_{ij} f_{;lk} + 2f R_{ijkl} \).
7. \( (\frac{d}{d\epsilon}|_{\epsilon=0}R)_{mabm} = -\delta_{ab} f_{;mm} - f_{;ab} + 2f R_{mabm} \).
8. \( (\frac{d}{d\epsilon}|_{\epsilon=0}P)_{ij} = (2-m) f_{;ij} - g_{ij} f_{;kk} \).
9. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\rho_{m})_{m} = -2f \rho_{mm} - f_{;aa} + (1-m) f_{;m} \).
10. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\tau) = -2f \tau + 2(1-m) f_{;ii} \).
11. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(E_{;m}) = -3f E_{;m} + \frac{1}{2}(m-2) f_{;ii} f_{;jm} - 2f_{;m} E \).
12. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\tau_{;m}) = -3f \tau_{;m} + 2(1-m) f_{;i} f_{;mm} - 2f_{;m} \tau \).
13. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\phi_{;i}) = -2f \phi_{;i} + (\frac{d}{d\epsilon}|_{\epsilon=0} \phi_{;i} f_{;i} + (m-2) f_{;ii} \phi_{;i} \).
14. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\phi_{;mm}) = -2f \phi_{;mm} + (\frac{d}{d\epsilon}|_{\epsilon=0} \phi_{;mm} f_{;mm} - f_{;m} \phi_{;m} + f_{;a} \phi_{;a} \).
15. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(F_{;mm}) = -4f F_{;mm} - 2f_{;m} F_{;m} + f_{;a} F_{;a} \).
16. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(F_{;m}) = -5f F_{;m} - 9f_{;m} F_{;m} + 3f_{;a} F_{;ai} \).
17. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(F_{;a}) = -f F_{;a} + 4f_{;a} F_{;ai} - 7f_{;m} F_{;m} + f_{;a} F_{;a} \).
18. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(E_{;aa}) = -4f E_{;aa} + \frac{1}{2}(m-2) f_{;i} f_{;iaa} - 2f_{;aa} E + (m-7) f_{;a} E_{;a} + (m-1) f_{;m} E_{;m} \).
19. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(E_{;a}) = -4f E_{;a} + \frac{1}{2}(m-2) f_{;i} f_{;iaa} - 2f_{;aa} E + (m-7) f_{;a} E_{;a} \).
20. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(E_{;mm}) = -4f E_{;mm} + \frac{1}{2}(m-2) f_{;i} f_{;immm} - 2f_{;mm} E + f_{;a} E_{;a} - 5f_{;m} E_{;m} \).
21. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(E_{;ii}) = -4f E_{;ii} + \frac{1}{2}(m-2) f_{;i} f_{;iijj} - 2f_{;ii} E + (m-6) f_{;a} E_{;a} + (m-6) f_{;m} E_{;m} \).
22. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\tau_{;a}) = -4f \tau_{;a} + 2(1-m) f_{;i} f_{;iaa} - 2f_{;aa} \tau + (m-7) f_{;a} \tau_{;a} + (m-1) f_{;m} \tau_{;m} \).
23. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\tau_{;m}) = -4f \tau_{;m} + 2(1-m) f_{;i} f_{;immm} - 2f_{;mm} \tau + f_{;a} \tau_{;a} - 5f_{;m} \tau_{;m} \).
24. \( (\frac{d}{d\epsilon}|_{\epsilon=0}(\tau_{;ii}) = -4f \tau_{;ii} + 2(1-m) f_{;i} f_{;iijj} - 2f_{;ii} \tau + (m-6) f_{;i} \tau_{;i} \).
\[ \frac{\partial}{\partial t} = 0(\rho_{mn;aa}) = -4f_{\rho_{mm;aa}} + (1 - m)f_{fmm;aa} - f_{fbb;aa} - 2f_{f;am\rho_{mm} - 2f_{aa\rho_{mm} - 4f_{f;pm;aa} + (m - 1)f_{f;am\rho_{mm} + (m - 1)f_{f;mm;mm}}.} \]

\[ \frac{\partial}{\partial t} = 0(\rho_{mm;mm}) = -4f_{\rho_{mm;mm} - f_{aa;bb} + (1 - m)f_{fmm;mm} - 2f_{f;am\rho_{mm} + f_{aa\rho_{mm} + 4f_{f;am\rho_{mm} + 2f_{f;am\rho_{mm} - 5f_{f;mm;mm}}.} \]

\[ \frac{\partial}{\partial t} = 0(\tau E) = -4f_{\tau E} + \frac{1}{4}(m - 1)f_{\tau E} + 2(1 - m)f_{f;ii\tau}. \]

\[ \frac{\partial}{\partial t} = 0(\tau^2) = -4f_{\tau^2} + 4(1 - m)f_{\tau E}. \]

\[ \frac{\partial}{\partial t} = 0(R^2) = -4f_{R^2 - 8f_{f;ij\rho_{ij}}.} \]

\[ \frac{\partial}{\partial t} = 0(\Omega^2) = -4f_{\Omega^2.} \]

\[ \frac{\partial}{\partial t} = 0(\rho_{mm}E) = -4f_{\rho_{mm}E} + (1 - m)f_{fmm;mm} + \frac{1}{2}(m - 2)f_{f;ii\rho_{mm}.} \]

\[ \frac{\partial}{\partial t} = 0(\rho_{mm}\tau) = -4f_{\rho_{mm}\tau} + (1 - m)f_{fmm;mm} + 2(1 - m)f_{f;ii\rho_{mm}.} \]

\[ \frac{\partial}{\partial t} = 0(R_{amm;ba} = -4f_{R_{amm;ba} - 2f_{f;ab}R_{amm;bb} - 2f_{f;mm;mm}.} \]

\[ \frac{\partial}{\partial t} = 0(\Omega_{am}\Omega_{am}) = -4f_{\Omega_{am}\Omega_{am}.} \]

\[ \frac{\partial}{\partial t} = 0(E;mm;S) = -4f_{E;mm;S} + \frac{1}{2}(m - 2)f_{f;iiim;S} - 2f_{f;mm;ES} + \frac{1}{2}(m - 2)f_{f;mm;E;mm}. \]

\[ \frac{\partial}{\partial t} = 0(ES^2) = -4f_{ES^2} + \frac{1}{2}(m - 2)f_{f;iiim;S^2} + (m - 2)f_{f;mm;ES}. \]

\[ \frac{\partial}{\partial t} = 0(S^2) = -4f_{S^2 + 2(m - 2)f_{f;mm;S^2}.} \]

\[ \frac{\partial}{\partial t} = 0(\tau_{mm}S) = -4f_{\tau_{mm}S} + 2(1 - m)f_{f;iiim;S} + \frac{1}{2}(m - 2)f_{f;mm;S} - 2f_{f;mm;S}. \]

\[ \frac{\partial}{\partial t} = 0(\rho_{mm}S^2) = -4f_{\rho_{mm}S^2} - f_{aa;bb} + (1 - m)f_{fmm;mm} + 2(1 - m)f_{f;mm;S}. \]

\[ \frac{\partial}{\partial t} = 0(\tau S^2) = -4f_{\tau S^2 + 2(1 - m)f_{f;iiim;S^2} + (m - 2)f_{f;mm;S}.} \]

\[ \frac{\partial}{\partial t} = 0(S_{aa}S) = -4f_{S_{aa}S} - f_{aa;bb} + (m - 5)f_{f;aa;S_{aa}S} + \frac{1}{2}(m - 2)f_{f;mm;S_{aa}S}. \]

\[ \frac{\partial}{\partial t} = 0(S_{aa}S_{aa}) = -4f_{S_{aa}S_{aa} - 2f_{f;aa;S_{aa}S_{aa}} + (m - 2)f_{f;mm;S_{aa}S_{aa}}.} \]

\[ \frac{\partial}{\partial t} = 0(F_{m}E_{mm}) = -6f_{F_{m}E_{mm}} + 2f_{f;im;F_{m}E_{mm} - \frac{1}{2}(m - 2)f_{f;iiim;F_{m}E_{mm}.} \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}ES) = -6f_{F_{mm}ES} - 2f_{f;mm;F_{m}ES} + \frac{1}{2}(m - 2)f_{f;iiim;F_{mm}S}. \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}S) = -6f_{F_{mm}S} - 2f_{f;mm;F_{mm}S} + \frac{1}{2}(m - 2)f_{f;mm;F_{mm}S}. \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}F_{mm}) = -6f_{F_{mm}F_{mm}S} - 2f_{f;mm;F_{mm}F_{mm}S} - f_{aa;F_{mm}S} + (1 - m)f_{f;mm;F_{mm}S} + \frac{1}{2}(m - 2)f_{f;mm;F_{mm}S}. \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}E) = -6f_{F_{mm}E} + f_{a;F_{a}E} - 2f_{f;mm;F_{mm}E} - 5f_{f;mm;F_{mm}E}. \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}S^2) = -6f_{F_{mm}S^2} + f_{aa;S_{aa}S} - 2f_{f;mm;F_{mm}S^2} - 5f_{f;mm;F_{mm}S^2}. \]

\[ \frac{\partial}{\partial t} = 0(F_{mm}\tau) = -6f_{F_{mm}\tau} + f_{aa;\tau} - 2f_{f;mm;F_{mm}\tau} - 5f_{f;mm;F_{mm}\tau} + 2(1 - m)f_{f;iiim;F_{mm}.} \]

\[ \frac{\partial}{\partial t} = 0(F_{mm};\rho_{mm}) = -6f_{F_{mm};\rho_{mm} + f_{aa;\rho_{mm} - 2f_{f;mm;\rho_{mm}} - 5f_{f;mm;\rho_{mm}} - f_{aa;F_{mm}S} + (1 - m)f_{f;mm;F_{mm}}.}} \]
61) \( \frac{d}{dx}(F_{mm} S) = -6 f_{mm} S + 3 f_{,a} F_{,am} S - 7 f_{m} S - 2 f_{mm} F S - 9 f_{m} S + f_{,a} F_{,am} S + \frac{1}{2} (m - 2) f_{,m} F_{mm} \).

62) \( \frac{d}{dx}(F_{mmm} \gamma) = -6 f_{mmm} \gamma - 9 f_{mm} F_{m} + f_{m} F_{mm} + 16 f_{m} F_{mm} + 4 f_{am} F_{am} - 14 f_{m} F_{mm} + 6 f_{,a} F_{mma} + 6 f_{a} R_{mmm} F_{b} - 2 f_{mm} F_{m} \).

63) \( \frac{d}{dx}(L_{aa} \gamma_m) = -4 f_{aa} \gamma_m + (1 - m) f_{,m} \gamma_m + 2(1 - m) f_{,im} L_{aa} - 2 f_{,m} L_{aa} \gamma_m \).

64) \( \frac{d}{dx}(L_{aa} \gamma_{bb}) = -4 f_{aa} \gamma_{bb} + (1 - m) f_{m,aa} S + (m - 5) f_{b} L_{aa,bb} - f_{bb} L_{aa} S + \frac{1}{2} (m - 2) f_{im} L_{aa,bb} \).

65) \( \frac{d}{dx}(L_{abc} L_{ab} \gamma_{bc}) = -4 f_{abc} L_{ab} \gamma_{bc} - 2 f_{m,ac} L_{aa,cc} + 4 f_{a} L_{cb} L_{ab} \gamma_{bc} - 2 f_{,c} L_{ab} L_{abc} - 4 f_{a} L_{ab} L_{abc} \).

66) \( \frac{d}{dx}(F_{m} L_{aa,bb} - 6 f_{m} F_{la,bb} + (1 - m) f_{m,bb} F_{m} + (m - 5) f_{b} F_{m} L_{aa,bb} - 6 f_{m} F_{la,bb} - 2 f_{m} F L_{ab} \gamma_{bc} - f_{m,bb} F_{m} + (m - 2) f_{cb} F_{m} L_{eb} + 2(m - 3) f_{e} F_{m} L_{aa,ec} - f_{bb} F_{m} L_{aa} - f_{e} F_{m} L_{ae,cc} \).

67) \( \frac{d}{dx}(F_{mm,mm} \gamma_{aa}) = -6 f_{mm,mm} \gamma_{aa} - 9 f_{mm} F_{mm} L_{aa} + 3 f_{bb} F_{bm} L_{aa} - 7 f_{mm,mm} \gamma_{aa} + 3 f_{am} F_{mm} L_{aa} \gamma_{a} - 2 f_{mm} F_{mm} L_{aa} + (1 - m) f_{mm} F_{mm} \).

**Remark.** Branson and Gilkey [3, Theorem 7.2] also computed \( a_{4} \) for mixed boundary conditions. Their formula for \( a_{4} \) gave incorrectly the values \( \beta_{3} = -42 \) and \( \beta_{4} = 6 \). Vassilevich [13] showed the correct values were \( \beta_{3} = -12 \) and \( \beta_{4} = -24 \). The error in Branson-Gilkey arose from incorrectly applying the variational formula contained in Lemma 6.3 (4), which is valid for Neumann boundary conditions, to the more general context of mixed boundary conditions. If one were to consider mixed boundary conditions, the relevant variational formula for \( S \) would become \( \frac{d}{dx}(S) = -f_{,m} S + (m - 2) f_{m} \Pi^{+} / 2 \) and the other variational formulas involving \( S \) in Lemma 6.3 would then need to be changed accordingly. We wish to caution the reader concerning this point.

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