Greedy Algorithm for the Analysis Transform Domain

Raja Giryes
Department of Computer Science, The Technion - Israel Institute of Technology
Haifa, 32000, Israel
raja@cs.technion.ac.il

Abstract

Many signal and image processing applications have benefited remarkably from the theory of sparse representations. In the classical synthesis model, the signal is assumed to have a sparse representation under a given known dictionary. The algorithms developed for this framework mainly operate in the representation domain. Recently, a new model has been introduced, the cosparse analysis one, in which the signal is treated directly. The (co)sparse is expressed by the number of zeros in the coefficients of the signal after applying a given transform, the analysis dictionary, on it. Recently it has been shown that using $\ell_1$-minimization one can stably recover a cosparse signal from a small set of random linear measurements if the operator is a frame. Another effort has provided guarantees for dictionaries that have a near optimal projection procedure using greedy-like algorithms. However, no claims have been given for frames.

In this work we propose a greedy-like technique that operates in the transform domain and provide guarantees that close the gap between greedy and relaxation techniques showing that the proposed method achieves a stable recovery for frames as operators. In addition, we treat the case where the noise is random and provide denoising guarantees for it, closing a gap between the synthesis and analysis frameworks.

I. INTRODUCTION

For more than a decade the idea that signals can be represented sparsely has a great impact on the field of signal and image processing. New sampling theory has been developed [1] together with new tools for handling signals in different types of applications, such as image denoising [2], image deblurring [3], super-resolution [4], radar [5], medical imaging [6] and astronomy [7], to name a few [8]. Remarkably, in most of these fields the sparsity based techniques achieve state-of-the-art results.

The classical sparse model is the synthesis one. In this model the signal $x \in \mathbb{R}^d$ is assumed to have a $k$-sparse representation $\alpha \in \mathbb{R}^n$ under a given dictionary $D \in \mathbb{R}^{d \times n}$. Formally,

$$x = D\alpha, \quad \|\alpha\|_0 \leq k,$$  \hspace{1cm} (1)

where $\|\cdot\|_0$ is the $\ell_0$-pseudo norm that counts the number of non-zero entries in a vector. Notice, that the non-zero elements in $\alpha$ corresponds to a set of columns that creates a low-dimensional subspace in which $x$ resides.

Recently, a new sparsity based model has been introduced: the analysis one [9], [10]. In this framework, we look at the coefficients of $\Omega x$, the coefficients of the signal after applying the transform $\Omega \in \mathbb{R}^{p \times d}$ on it. The sparsity of the signal is measured by the number of zeros in $\Omega x$. We say that a signal is $\ell$-cosparse if $\Omega x$ has $\ell$ zero elements. Formally,

$$\|\Omega x\|_0 \leq p - \ell.$$  \hspace{1cm} (2)

Remark that each zero element in $\Omega x$ corresponds to a row in $\Omega$ to which the signal is orthogonal and all these rows define a subspace to which the signal is orthogonal. Similar to synthesis, when the number of zeros is large the signal’s subspace is low dimensional. Though the zeros are those that define the subspace, in some cases it is more convenient to use the number of non-zeros $k = p - \ell$ as done in [11], [12].

In certain applications, it is more natural and effective to use the analysis framework, as it addresses the signal directly, instead or in addition to the synthesis one. In the denoising problem, a very common strategy is the total variation (TV) denoising [13] which belongs to the analysis framework [10], [14], [15]. For the deblurring problem, a significant improvement over the state-of-the-art has been achieved by the use of the analysis model...
In general, each model implies a different prior on the signal. Thus, the answer to the question which one to use depends heavily on the specific settings of the problem at hand.

The main setup in which the above models have been used is

\[ y = Mx + e, \]

where \( y \in \mathbb{R}^m \) is a given set of measurements, \( M \in \mathbb{R}^{m \times d} \) is the measurement matrix and \( e \in \mathbb{R}^d \) is an additive noise which is assumed to be either adversarial bounded noise [1], [8], [16], [17] or with a certain given distribution such as Gaussian [18], [19], [20]. The goal is to recover \( x \) from \( y \) and this is the focus of our work. For details about other setups, the curious reader may refer to [21], [22].

Clearly, without a prior knowledge it is impossible to recover \( x \) from \( y \) in the case \( m < d \), or have a significant denoising effect when \( e \) is random with a known distribution. Hence, having a prior, such as the sparsity one, is vital for these tasks. Both the synthesis and the analysis models lead to (different) minimization problems that provide estimates for the original signal \( x \).

In synthesis, the signal is recovered by its representation, using

\[
\hat{x}_{S-\ell_0} = \arg\min_{\hat{\alpha} \in \mathbb{R}^n} \|\hat{\alpha}\|_0 \quad s.t \quad \|y - MD\hat{\alpha}\|_2 \leq \lambda_e, \tag{4}
\]

where \( \lambda_e \) is an upper bound for \( \|e\|_2 \) if the noise is bounded and adversarial. Otherwise, it is a scalar dependent on the noise distribution [18], [19], [23]. The recovered signal is simply \( \hat{x}_{S-\ell_0} = D \hat{\alpha}_{S-\ell_0} \). In analysis, we have the following minimization problem.

\[
\hat{x}_{A-\ell_0} = \arg\min_{\hat{x} \in \mathbb{R}^d} \|\Omega \hat{x}\|_0 \quad s.t \quad \|y - MX\|_2 \leq \lambda_e. \tag{5}
\]

The values of \( \lambda_e \) are selected as before depending on the noise properties.

Remark the differences between synthesis and analysis. In the former we have an indirect estimation for the signal as we work with its representation, while in the latter we get a direct estimate since the minimization is done in the signal domain.

Both (4) and (5) are NP-hard problems [10], [24]. Hence, approximation techniques are required. These are divided mainly into two categories: relaxation methods and greedy algorithms. In the first category we have the \( \ell_1 \)-relaxation [9] and the Dantzig selector [18], where the latter has been proposed only for synthesis. The \( \ell_1 \)-relaxation leads to the following minimization problems for synthesis and analysis respectively:

\[
\hat{\alpha}_{S-\ell_1} = \arg\min_{\hat{\alpha} \in \mathbb{R}^n} \|\hat{\alpha}\|_1 \quad s.t \quad \|y - MD\hat{\alpha}\|_2 \leq \lambda_e, \tag{6}
\]

\[
\hat{x}_{A-\ell_1} = \arg\min_{\hat{x} \in \mathbb{R}^d} \|\Omega \hat{x}\|_1 \quad s.t \quad \|y - M\hat{x}\|_2 \leq \lambda_e. \tag{7}
\]

Among the synthesis greedy strategies we mention the thresholding method, orthogonal matching pursuit (OMP) [25], [26], CoSaMP [27], subspace pursuit (SP) [28], iterative hard thresholding [29] and hard thresholding pursuit (HTP) [30]. Their counterparts in analysis are thresholding [31], GAP [10], analysis CoSaMP (ACoSaMP), analysis SP (ASP), analysis IHT (AIHT) and analysis HTP (AHTP) [14].

An important question to ask is what are the recovery guarantees that exist for these methods. Two main tools were used for answering this question in the synthesis context. The first is the mutual coherence which is the maximal inner product between columns in \( M \) after normalization [32] and the second is the restricted isometry property [16]. It has been shown that under some conditions on the mutual coherence or the RIP of \( M \), we have using approximation algorithms a stable recovery in the adversarial noise case [1], [16], [27], [28], [29], [30], [33], [34], and a denoising effect in the random Gaussian case [18], [19], [20], [35].

The advantage of the RIP conditions over the coherence ones is that there exist measurement matrices with \( m = O(k \log(n/k)) \) that satisfy the RIP conditions. However, with the coherence the number of measurements is required to be at least \( O(k^2) \). Hence, in this work we focus on the RIP. It is defined as:

**Definition 1.1 (Restricted Isometry Property (RIP) [16]):** A matrix \( A \in \mathbb{R}^{m \times n} \) has the RIP with a constant \( \delta_k \), if \( \delta_k \) is the smallest constant that satisfies

\[
(1 - \delta_k) \|\hat{\alpha}\|_2^2 \leq \|A\hat{\alpha}\|_2^2 \leq (1 + \delta_k) \|\hat{\alpha}\|_2^2, \tag{8}
\]

whenever \( \hat{\alpha} \in \mathbb{R}^n \) is \( k \)-sparse.
The RIP guarantee for the above techniques in the adversarial noise case reads as if $\delta_{ak} \leq \delta_{alg}$, where $a > 1$ and $\delta_{alg} < 1$ are constants depending on the conditions for each technique, then

$$\|\hat{\alpha}_{alg} - \alpha\|_2^2 \leq C_{alg} \|e\|_2^2,$$

where $\hat{\alpha}_{alg}$ is the recovered representation by one of the methods and $C_{alg} > 2$ is a constant depending on $\delta_{ak}$ which differs for each method.

Similar results have been provided for the case where the noise is random white Gaussian with variance $\sigma^2$. In this case the reconstruction error is guaranteed to be $O(k \log(n)\sigma^2)$ [18], [19], [20]. Unlike the adversarial noise case, here we may have a denoising effect, as the recovery error can be smaller than the initial noise power $d\sigma^2$. Remark that the above results can be extended also to the case where we have a model mismatch and the signal is not exactly $k$-sparse.

In the analysis framework we have similar guarantees for the adversarial noise case. However, since the analysis model treats the signal directly, the guarantees are in terms of the signal and not the representation like in (9). Two extensions for the RIP have been proposed providing guarantees for analysis algorithms. The first is the D-RIP [11]:

**Definition 1.2 (D-RIP [11]):** A matrix $M$ has the D-RIP with a dictionary $D$ and a constant $\delta^{D}_{D,k}$, if $\delta^{D}_{D,k}$ is the smallest constant that satisfies

$$(1 - \delta^{D}_{D,k}) \|D\hat{\alpha}\|_2^2 \leq \|MD\hat{\alpha}\|_2^2 \leq (1 + \delta^{D}_{D,k}) \|D\hat{\alpha}\|_2^2,$$

whenever $\hat{\alpha}$ is $k$-sparse.

The second is the O-RIP [14]:

**Definition 1.3 (O-RIP [14]):** A matrix $M$ has the O-RIP with an operator $\Omega$ and a constant $\delta^{O}_{\Omega,\ell}$, if $\delta^{O}_{\Omega,\ell}$ is the smallest constant that satisfies

$$(1 - \delta^{O}_{\Omega,\ell}) \|v\|_2^2 \leq \|Mv\|_2^2 \leq (1 + \delta^{O}_{\Omega,\ell}) \|v\|_2^2,$$

whenever $\Omega v$ has at least $\ell$ zeroes.

The D-RIP has been used for studying the performance of the analysis $\ell_1$-minimization [11], [36], [37]. It has been shown that if $\Omega$ is a frame with frame constants $A$ and $B$, $D = \Omega^\dagger$ and $\delta^{D}_{D,ak} \leq \delta_{A-\ell_1}(a, A, B)$ then

$$\|\hat{x}_{A-\ell_1} - x\|_2^2 \leq C_{A-\ell_1}(\|e\|_2 + \|\Omega x - [\Omega x]_k\|_1/k),$$

where the operator $[\cdot]_k$ is a hard thresholding operator that keeps the largest $k$ elements in a vector, and $a \geq 1$, $\delta_{A-\ell_1}(a, A, B)$ and $C_{A-\ell_1}$ are some constants. A similar result has been proposed for analysis $\ell_1$-minimization with $\Omega_{2D,DIF}$, the two dimensional finite different operator that corresponds to the discrete gradient in 2D, also known as the anisotropic total variation (TV).

The O-RIP has been used for the study of the greedy-like algorithms ACoSaMP, ASP, AIHT and AHTP with the assumption that there exists a near optimal projection procedure for $\Omega$ with a constant $C_\ell$. More details about this definition can be found in [14]. It has been proven for such operators that if $\delta^{O}_{\Omega,\ell} \leq \delta_{alg}(C_\ell, C_{2\ell-p}, \sigma^2_M)$ then

$$\|\hat{x}_{A-\ell_1} - x\|_2^2 \leq C_{alg}(\|e\|_2 + \|x - x^\ell\|_2^2),$$

where $\sigma^2_M$ is the largest singular value of $M$, $x^\ell$ is the best $\ell$-cosparse approximation for $x$, and $a$, $\delta_{alg}(C_\ell, C_{2\ell-p}, \sigma^2_M)$ and $C_{alg}$ are some constants which differ for each technique.

Notice that the conditions in synthesis imply that no linear dependencies can be allowed within the dictionary as the representation is the focus. Since the analysis model performs in the signal domain, dependencies can be allowed within its dictionaries. A recent series of contributions have shown that high correlations can be allowed in the dictionary also in the synthesis framework if the signal is the target and not the representation [38], [39], [40], [41], [42], [43].
A. Our Contribution

The conditions for greedy-like techniques require the constant $C_\ell$ to be close to 1. Having a general projection scheme with $C_\ell = 1$ is NP-hard [44]. The existence of a program with a constant close to one for a general operator is still an open problem. In particular, it is not known whether there exists a procedure that gives a small constant for frames. Thus, there is a gap between the results for the greedy techniques and the ones for the $\ell_1$-minimization.

In this work we focus on closing this gap for frames. We propose a new greedy program, the transform domain IHT (TDIHT), which is an extension of IHT that operates in the analysis transform domain. We show that it inherits guarantees similar to the ones of analysis $\ell_1$-minimization for frames.

Another gap exists between synthesis and analysis. To the best of our knowledge, no denoising guarantees has been proposed for analysis strategies apart from the work in [31] that studies the performance of thresholding for the case $M = I$. We develop results for Gaussian noise in addition to the ones for adversarial noise, showing that it is possible to have a denoising effect using the analysis model also when $M \neq I$ and for different algorithms other than thresholding.

Our contribution can be summarized by the following theorem:

**Theorem 1.4 (Recovery Guarantees for TDIHT with Frames):** Let $y = Mx + e$ where $\|\Omega x\|_0 \leq k$ and $\Omega$ is a tight frame with frame constants $A$ and $B$, i.e., $A \leq \|\Omega\|_2 \leq B$. For certain selections of $M$ and using only $m = O(\frac{B}{A}k \log(p/k))$ measurements, the recovery result $\hat{x}$ of TDIHT satisfies

$$
\|x - \hat{x}\|_2 \leq O\left(\frac{B}{A}\|e\|_2\right) + O\left(\frac{1 + A}{A^2}\|\Omega_T e x\|_2 + \frac{1}{A^2\sqrt{k}}\|\Omega_T e x\|_1\right),
$$

for the case where $e$ is an adversarial noise, implying that TDIHT leads to a stable recovery. For the case that $e$ is random i.i.d zero-mean Gaussian distributed noise with a known variance $\sigma^2$ we have

$$
E \|x - \hat{x}\|_2^2 \leq O\left(\frac{B^2}{A^2}k \log(p)\sigma^2\right) + O\left(\frac{1 + A}{A^2}\|\Omega_T e x\|_2 + \frac{1}{A^2\sqrt{k}}\|\Omega_T e x\|_1\right)^2,
$$

implying that TDIHT achieves a denoising effect.

**Remark 1.5:** Note that $\Omega_T e x = \Omega x - [\Omega x]_k$.

**Remark 1.6:** Using Remark 2.3 in [27], we can convert the $\ell_2$ norm into an $\ell_1$ norm in the model mismatch terms in (14) and (15), turning it to be more similar to what we have in the bound for analysis $\ell_1$-minimization in (12).

**Remark 1.7:** Theorem 1.4 is a combination of Theorems 4.1 and 4.4, plugging the minimal number of measurements implied by the D-RIP conditions of these theorems. Measurement matrices with sub-Gaussian entries are examples for matrices that satisfy this number of measurements [11].

B. Organization

This paper is organized as follows:

- Section II includes the notations used in this paper together with some preliminary lemmas for the D-RIP.
- Section III presents the transform domain IHT.
- Section IV provides the proof of our main theorem.
- Section V discusses the developed result and concludes the paper.

II. Notations and Preliminaries

We use the following notation in our work:

- We denote by $\|\cdot\|_2$ the euclidian norm for vectors and the spectral $(2 \to 2)$ norm for matrices; by $\|\cdot\|_1$ the $\ell_1$ norm that sums the absolute values of a vector; and by $\|\cdot\|_0$ the $\ell_0$ pseudo-norm which counts the number of non-zero elements in a vector.
• Given a cosupport set $\Lambda$, $\Omega_\Lambda$ is a sub-matrix of $\Omega$ with the rows that belong to $\Lambda$.
• In a similar way, for a support set $T$, $D_T$ is a sub-matrix of $D$ with columns\(^1\) corresponding to the set of indices in $T$.
• supp$(\cdot)$ returns the support of a vector and supp$(\cdot, k)$ returns the set of $k$-largest elements.
• $Q_\Lambda = I - \Omega_\Lambda^T \Omega_\Lambda$ is the orthogonal projection onto the orthogonal complement of range$(\Omega_\Lambda^*)$.
• $P_T = D_T D_T^*$ is the orthogonal projection onto range$(D_T)$.
• Throughout the paper we assume that $n = p$.
• We abuse notation and use $\delta_k$ to denote both the RIP and D-RIP. The use will be clear from the context.
• The original unknown $\ell$-cosparse vector is denoted by $x \in \mathbb{R}^d$, its cosupport by $\Lambda$ and the support of the non-zero entries by $T = \Lambda^C$. By definition $|\Lambda| \geq \ell$ and $|T| \leq k$.
• For a general $\ell$-cosparse vector we use $v \in \mathbb{R}^d$, for a general vector in the signal domain $z \in \mathbb{R}^d$ and for a general vector in the analysis transform or dictionary representation domain $w \in \mathbb{R}^p$.

We now turn to present several key properties of the D-RIP. All of their proofs except of the last one, which we present hereafter, appear in [43].

**Corollary 2.1:** If $M$ satisfies the D-RIP with a constant $\delta_k$ then

$$\|M P_T\|_2^2 \leq 1 + \delta_k$$

for every $T$ such that $|T| \leq k$.

**Lemma 2.2:** For $k \leq \tilde{k}$ it holds that $\delta_k \leq \delta_{\tilde{k}}$.

**Lemma 2.3:** If $M$ satisfies the D-RIP then

$$\|P_T (I - M^* M) P_T\|_2 \leq \delta_k$$

for any $T$ such that $|T| \leq k$.

**Corollary 2.4:** If $M$ satisfies the D-RIP then

$$\|P_{T_1} (I - M^* M) P_{T_2}\|_2 \leq \delta_k,$$

for any $T_1$ and $T_2$ such that $|T_1| \leq k_1, |T_2| \leq k_2, k_1 + k_2 \leq k$.

The last Lemma we present is a generalization of Proposition 3.5 in [27].

**Lemma 2.5:** Suppose that $M$ satisfies the upper inequality of the D-RIP, i.e.,

$$\|M D w\|_2 \leq \sqrt{1 + \delta_k} \|D w\|_2 \quad \forall w, \|w\|_0 \leq k,$$

and that $\|D\|_2 \leq \frac{1}{A} x$. Then for any representation $w$ we have

$$\|M D w\|_2 \leq \frac{\sqrt{1 + \delta_k}}{A} \left( \|w\|_2 + \frac{1}{\sqrt{k}} \|w\|_1 \right).$$

The proof is left to Appendix A. Before we proceed we recall the problem we aim at solving:

**Definition 2.6 (Problem $\mathcal{P}$):** Consider a measurement vector $y \in \mathbb{R}^m$ such that $y = M x + e$, where $x \in \mathbb{R}^d$ is either $\ell$-cosparse under a given and fixed analysis operator $\Omega \in \mathbb{R}^{p \times d}$ or almost $\ell$-cosparse, i.e. $\Omega x$ has $k = p - \ell$ leading elements. The non-zero locations of the $k$ leading elements is denoted by $T$. $M \in \mathbb{R}^{m \times d}$ is a degradation operator and $e \in \mathbb{R}^m$ is an additive noise. Our task is to recover $x$ from $y$. The recovery result is denoted by $\hat{x}$.

### III. Transform Domain Iterative Hard Thresholding

Our goal in this section is to provide a greedy-like approach that provide guarantees similar to the one of analysis $L_1$-minimization. By analyzing the latter we notice that though it operates directly on the signal, it actually minimizes the coefficients in the transform domain. In fact, all the proof techniques utilized for this recovery strategy use the fact that nearness in the analysis dictionary domain implies a nearness in the signal domain [11], [15], [36]. Using this fact, recovery guarantees have been developed for tight frames [11], general frames [36] and the 2D-DIF

\(^1\)By the abuse of notation we use the same notation for the selection sub-matrices of rows and columns. The selection will be clear from the context since in analysis the focus is always on the rows and in synthesis on the columns.
operator which corresponds to total variation (TV) [15]. Working in the transform domain is not a new idea and was used before, especially in the context of dictionary learning [45], [46], [47].

Henceforth, our strategy for extending the results of the $\ell_1$-relaxation is to modify the greedy-like approaches to operate in the transform domain. In this paper we concentrate on iterative hard thresholding (IHT). Before we turn to present the transform domain version of IHT we recall its synthesis and analysis versions.

A. Quick Review of IHT and Analysis IHT

IHT and analysis IHT (AIHT) are assumed to know the cardinalities $k$ and $\ell$ respectively. They aim at approximating variants of (4) and (5):

$$\arg\min_{\alpha} \| y - M \alpha \|_2^2 \quad \text{s.t.} \quad \| \alpha \|_0 \leq k,$$

and

$$\arg\min_{\hat{x}} \| y - M \hat{x} \|_2^2 \quad \text{s.t.} \quad \| \Omega \hat{x} \|_0 \leq p - \ell.$$

IHT aims at recovering the representation and uses only one matrix $A = MD$ in the whole recovery process. AIHT targets the signal and utilizes both $M$ and $\Omega$. For recovering the signal using IHT, one has $\hat{x}_{\text{IHT}} = D \hat{\alpha}_{\text{IHT}}$. IHT [29] and AIHT [14] are presented in Algorithms 1 and 2 respectively.

The iterations of IHT and AIHT are composed of two basic steps. In both of them the first is a gradient step, with a step size $\mu_t$, in the direction of minimizing $\| y - M \hat{x} \|_2^2$. The second step of IHT projects $\alpha_g$ to the closest $k$-sparse subspace by keeping the largest $k$ elements. In AIHT a near-optimal support selection procedure, $S_\ell$, is used for the cosupport selection and then orthogonal projection onto the corresponding orthogonal subspace is performed. In the algorithms’ description we neither specify the stopping criterion, nor the step size selection technique. For exact details we refer the curious reader to [14], [29], [48], [49].

---

**Algorithm 1** Iterative hard thresholding (IHT)

**Require:** $k, A, y$, where $y = A \alpha + e$, $k$ is the cardinality of $\alpha$ and $e$ is an additive noise.

**Ensure:** $\hat{\alpha}_{\text{IHT}}$: $k$-sparse approximation of $\alpha$.

1. Initialize representation $\hat{\alpha}^0 = 0$ and set $t = 0$.
2. **while** halting criterion is not satisfied **do**
   1. Perform a gradient step: $\alpha_g = \hat{\alpha}^{t-1} + \mu_t A^* (y - A \hat{\alpha}^{t-1})$
   2. Find a new support: $T^t = \text{supp}(\alpha_g, k)$
   3. Calculate a new representation: $\hat{\alpha}^t = (\alpha_g)_{T^t}$.
3. **end while**

Form the final solution $\hat{\alpha}_{\text{IHT}} = \hat{\alpha}^t$.

**Algorithm 2** Analysis iterative hard thresholding (AIHT)

**Require:** $\ell, M, \Omega, y$, where $y = M x + e$, $\ell$ is the cosparse of $x$ under $\Omega$ and $e$ is an additive noise.

**Ensure:** $\hat{x}_{\text{AIHT}}$: $\ell$-cosparse approximation of $x$.

1. Initialize estimate $\hat{x}^0 = 0$ and set $t = 0$.
2. **while** halting criterion is not satisfied **do**
   1. Perform a gradient step: $x_g = \hat{x}^{t-1} + \mu_t M^* (y - M \hat{x}^{t-1})$
   2. Find a new cosupport: $\hat{\Lambda}^t = \hat{S}_\ell(x_g)$
   3. Calculate a new estimate: $\hat{x}^t = Q_{\hat{\Lambda}^t} x_g$.
3. **end while**

Form the final solution $\hat{x}_{\text{AIHT}} = \hat{x}^t$. 
B. Transform Domain Analysis Greedy-Like Method

The drawback of AIHT for handling analysis signals is that it assumes the existence of a near optimal cosupport selection scheme $\mathcal{S}$. The type of analysis dictionaries for which a known feasible cosupport selection technique exists is very limited [14], [44]. Note that this limit is not unique only to the analysis framework [38], [39], [42]. Of course, it is possible to use a cosupport selection strategy with no guarantees on its near-optimality constant and it might work well in practice. For instance, simple hard thresholding has been shown to operate reasonably well in several instances where no practical projection is at hand [14]. However, the theoretical performance guarantees depend heavily on the near-optimality constant. Since for many operators there are no known selection schemes with small constants, the existing guarantees for AIHT, as well as the ones of the other greedy-like algorithms, are very limited. In particular, to date, they do not cover frames and the 2D-DIF operator as the analysis dictionary.

To bypass this problem we propose an alternative greedy approach for the analysis framework that operates in the transform domain instead of the signal domain. This strategy aims at finding the closest approximation to $\Omega x$ and not to $x$ using the fact that for many analysis operators proximity in the transform domain implies the same in the signal domain. In some sense, this approach imitates the classical synthesis techniques that recover the signal by putting the representation as the target.

In Algorithm 3 an extension for IHT for the transform domain is proposed. This algorithm makes use of $k$, the number of non-zeros in $\Omega x$, and $D$, a dictionary satisfying $D\Omega = I$. One option for $D$ is $D = \Omega^\dagger$. If $\Omega$ does not have a full row rank, we may compute $D$ by adding to $\Omega$ rows that resides in its rows’ null space and then applying the pseudo-inverse. For example, for the 2D-DIF operator we may calculate $D$ by computing the pseudo inverse of $\Omega_{2D-DIF}$ with an additional row composed of ones divided by $\sqrt{d}$. However, this option is not likely to provide good guarantees. As the 2D-DIF operator is beyond the scope of this paper we refer the reader to [15] for more details on this subject.

$$\textit{Algorithm 3: Transform Domain Iterative hard thresholding (TDIHT)}$$

Require: $k, M \in \mathbb{R}^{m \times d}$, $\Omega \in \mathbb{R}^{d \times d}$, $D \in \mathbb{R}^{d \times p}, y$, where $y = Mx + e$, $D$ satisfies $D\Omega = I$, $x$ is $p - k$ cosparse under $\Omega$ which implies that it has a $k$-sparse representation under $D$, and $e$ is an additive noise.

Ensure: $\hat{x}_{\text{TDIHT}}$: Approximation of $x$.

Initialize estimate $\hat{w}^0 = 0$ and set $t = 0$.

while halting criterion is not satisfied do

$t = t + 1$.

Perform a gradient step: $w_g^t = \Omega D \hat{w}^{t-1} + \mu^t \Omega^* (y - MD \hat{w}^{t-1})$

Find a new transform domain support: $T^t = \text{supp}(w_g^t, k)$

Calculate a new estimate: $\hat{w}^t = (w_g^t)_{T^t}$.

end while

Form the final solution $\hat{x}_{\text{TDIHT}} = D\hat{w}^t$.

IV. FRAME GUARANTEES

We provide theoretical guarantees for the reconstruction performance of the transform domain analysis IHT (TDIHT), with a constant step size $\mu = 1$, for frames. These can be easily extended also to other step-size selection options using the proof technique in [14], [48]. We start with the case that the noise is adversarial.

Theorem 4.1 (Stable Recovery of TDIHT with Frames): Consider the problem $\mathcal{P}$ and apply TDIHT with a constant step-size $\mu = 1$ and $D = \Omega^\dagger$. Suppose that $e$ is a bounded adversarial noise, $\Omega$ is a frame with frame constants $0 < A, B < \infty$ such that $\|\Omega\|_2 \leq B$ and $\|D\|_2 \leq \frac{1}{A}$, and $M$ has the D-RIP with the dictionary $[D, \Omega^*]$ and a constant $\delta_k = \delta^D_{[D, \Omega^*], k}$. If $\rho \triangleq \frac{2k\delta_k B}{A} < 1$ (i.e. $\delta_{uk} \leq \frac{A}{2B}$), then after a finite number of iterations
implying that TDIHT leads to a stable recovery. For tight frames $a = 3$ and for other frames $a = 4$.

The result of this theorem is a generalization of the one presented in [34] for IHT. Its proof follows from the following lemma.

**Lemma 4.2:** Consider the same setup of Theorem 4.1. Then the $t$-th iteration of TDIHT satisfies

\[
\left\| (\Omega x - w_g^t)_{T\cup\hat{T}} \right\|_2 \leq 2\delta_{ak} \frac{B}{A} \left( \left\| (\Omega x - w_g^{t-1})_{T\cup\hat{T}^{-1}} \right\|_2 + \frac{1 + \delta_{2k}}{A} \left( \left\| \Omega_T^{c} x \right\|_2 + \frac{1}{\sqrt{k}} \left\| \Omega_T^{c} x \right\|_1 \right) \right) + \left\| \Omega_{T\cup\hat{T}}^* M^e \right\|_2.
\]

The proof of the above lemma is left to Appendix B. We turn now to prove Theorem 4.1.

**Proof:** [Proof of Theorem 4.1] First notice that $\hat{w}^0 = 0$ implies that $w_g^0 = 0$. Using Lemma 4.2, recursion and the definitions of $\rho$ and $T^e = \arg\max_{|\hat{T}|\leq k} \left\| \Omega_{T\cup\hat{T}}^* M^e \right\|_2$, we have that after $t$ iterations

\[
\left\| (\Omega x - w_g^t)_{T\cup\hat{T}} \right\|_2 \leq \rho^t \left( \left\| (\Omega x - w_g^{t-1})_{T\cup\hat{T}^{-1}} \right\|_2 + \frac{1 + \delta_{2k}}{A} \left( \left\| \Omega_T^{c} x \right\|_2 + \frac{1}{\sqrt{k}} \left\| \Omega_T^{c} x \right\|_1 \right) \right) + \left\| \Omega_{T\cup\hat{T}}^* M^e \right\|_2.
\]

where the last equality is due to the equation of geometric series ($\rho < 1$) and the facts that $w_g^0 = 0$ and $T^0 = \emptyset$. For a given precision factor $\eta$ we have that if $t \geq t^* = \frac{\log(\left\| \Omega_{T\cup\hat{T}}^* M^e \right\|_2 + \eta)}{\log(\rho)}$ then

\[
\rho^t \left\| \Omega_T x \right\|_2 \leq \eta + \left\| \Omega_{T\cup\hat{T}}^* M^e \right\|_2.
\]

As $x = D\Omega x$ and $\hat{x}^t = D\hat{w}^t$, we have using matrix norm inequality that

\[
\left\| x - \hat{x}^t \right\|_2 \leq \left\| D(\Omega x - \hat{w}^t) \right\|_2 \leq \frac{1}{A} \left\| \Omega x - \hat{w}^t \right\|_2.
\]

Using the triangle inequality and the facts that $\hat{w}^t$ is supported on $\hat{T}$ and $\left\| (\Omega_{T\cup\hat{T}}^*)^c x \right\|_2 \leq \left\| \Omega_T^{c} x \right\|_2$, we have

\[
\left\| x - \hat{x}^t \right\|_2 \leq \frac{1}{A} \left\| \Omega_T^{c} x \right\|_2 + \frac{1}{A} \left\| (\Omega x - \hat{w}^t)_{T\cup\hat{T}^{-1}} \right\|_2.
\]

By using again the triangle inequality and the fact that $\hat{w}^t$ is the best $k$-term approximation for $w_g^t$ we get

\[
\left\| x - \hat{x}^t \right\|_2 \leq \frac{1}{A} \left\| \Omega_T^{c} x \right\|_2 + \frac{1}{A} \left\| (\Omega x - w_g^t)_{T\cup\hat{T}} \right\|_2 + \frac{1}{A} \left\| (w_g^t - \hat{w}^t)_{T\cup\hat{T}} \right\|_2
\]

\[
\leq \frac{1}{A} \left\| \Omega_T^{c} x \right\|_2 + \frac{2}{A} \left\| (\Omega x - w_g^t)_{T\cup\hat{T}} \right\|_2.
\]
Plugging (26) and (25) in (29) yields
\[ \|x - \hat{x}\|_2 \leq \frac{2\eta}{A} + 2 \left( 1 + \frac{1 - \rho}{1 - \rho^s} \right) \| \Omega_{T \mathcal{A}} \mathbf{M}^* \mathbf{e} \|_2 \]
\[ + \frac{2(1 + \delta_{2k})}{A^2} \frac{1 - \rho}{1 - \rho^s} \left( 1 + A + \frac{A^2}{2} \right) \| \Omega_{T \mathcal{C}} x \|_2 \]
\[ + \frac{1}{\sqrt{k}} \| \Omega_{T \mathcal{C}} x \|_1. \]

Using the D-RIP and the fact that \( \Omega \) is a frame we have that \( \| \Omega_{T \mathcal{A}} \mathbf{M}^* \mathbf{e} \|_2 \leq B \sqrt{1 + \delta_{2k}} \| \mathbf{e} \|_2 \) and this completes the proof.

Having a results for the adversarial noise case we turn to give a bound for the case where a distribution of the noise is given. We dwell on the white Gaussian noise case. For this type of noise we make use of the following lemma.

Lemma 4.3: If \( \mathbf{e} \) is a zero-mean white Gaussian noise with a variance \( \sigma^2 \) then
\[ E[\max_{|T| \leq k} \| \Omega_{T} \mathbf{M}^* \mathbf{e} \|_2^2] \leq 4 \max_i \| \Omega_i^* \|_2^2 (1 + \delta_1) k \log(p) \sigma^2. \]

Proof: First notice that the \( i \)-th entry in \( \Omega_{T} \mathbf{M}^* \mathbf{e} \) is Gaussian distributed random variable with zero-mean and variance \( \| \mathbf{M} \Omega_i^* \|_2^2 \sigma^2 \). Denoting by \( \mathbf{W} \) a diagonal matrix such that \( \mathbf{W}_{i,i} = \| \mathbf{M} \Omega_i^* \|_2^2 \), we have that each entry in \( \mathbf{W}^{-1} \Omega_{T} \mathbf{M}^* \mathbf{e} \) is Gaussian distributed with variance \( \sigma^2 \). Therefore, using Theorem 2.4 from [20] we have
\[ E[\max_{|T| \leq k} \| \mathbf{W}^{-1} \Omega_{T} \mathbf{M}^* \mathbf{e} \|_2^2] \leq 4k \log(p) \sigma^2. \]
Using the D-RIP we have that \( \| \mathbf{M} \Omega_i^* \|_2^2 \leq (1 + \delta_1) \| \Omega_i^* \|_2^2 \). Since the \( \ell_2 \) matrix norm of a diagonal matrix is the maximal diagonal element we have that \( \| \mathbf{W} \|_2 \leq \max_i (1 + \delta_1) \| \Omega_i^* \|_2 \) and this provides the desired result.

Theorem 4.4 (Denoising Performance of TDIHT with Frames): Consider the problem \( \mathcal{P} \) and apply TDIHT with a constant step-size \( \mu = 1 \). Suppose that \( \mathbf{e} \) is an additive white Gaussian noise with a known variance \( \sigma^2 \) (i.e. for each element \( e_i \sim N(0, \sigma^2) \). \( \Omega \) is a frame with frame constants \( 0 < A, B < \infty \) such that \( \| D \|_2 \leq \frac{1}{A} \), and \( \mathbf{M} \) has the D-RIP with the dictionary \( [\mathbf{D}, \Omega^*] \) and a constant \( \delta_k = \delta_{\| [\mathbf{D}, \Omega^*] \|, k} \). If \( \rho \triangleq \frac{2A \log B}{A^2} < 1 \) (i.e. \( \delta_{ak} \leq \frac{A}{2A} \)), then after a finite number of iterations \( t \geq t^* = \frac{\log((\| \Omega_{T \mathcal{A}} \mathbf{M}^* \mathbf{e} \|_2 + \eta)/\| \Omega_{T \mathcal{C}} x \|_2)}{\log(p)} \)
\[ E \| x - \hat{x} \|_2^2 \leq 32 \left( 1 + \frac{1}{\rho} \right) \frac{B^2}{A^2} \left( 1 + \frac{1 - \rho^s}{1 - \rho} \right) k \log(p) \sigma^2 \]
\[ + \frac{8(1 + \delta_{2k})}{A^4} \left( 1 + \frac{1 - \rho^s}{1 - \rho} \right) \left( 1 + A + \frac{A^2}{2} \right) \| \Omega_{T \mathcal{C}} x \|_2 \]
\[ + \frac{1}{\sqrt{k}} \| \Omega_{T \mathcal{C}} x \|_1, \]
implying that TDIHT has a denoising effect. For tight frames \( a = 3 \) and for other frames \( a = 4 \).

Proof: Using the fact that for tight frames \( \max_i \| \Omega_i^* \|_2^2 \leq B \), we have using Lemma 4.3 that
\[ E \| \Omega_{T \mathcal{A}} \mathbf{M}^* \mathbf{e} \|_2^2 \leq 4B^2(1 + \delta_1) k \log(p) \sigma^2. \]
Plugging this in a squared version of (30) (with \( \eta = 0 \), using the fact that for any two constants \( a, b \) we have \( (a + b)^2 \leq 2a^2 + 2b^2 \), leads to the desired result.

V. DISCUSSION AND CONCLUSION

Notice that in all our conditions, the number of measurements we need is \( O((p - \ell) \log(p)) \) and it is not dependent explicitly on the intrinsic dimension of the signal. Intuitively, we would expect the minimal number of measurements to be rather a function of \( d - r \), where \( r = \text{rank}(\Omega_{\Lambda}) \) is the corank of the cosparse subspace defined by \( \Omega_{\Lambda} \) [10], and henceforth our recovery conditions seems to be sub-optimal.
Indeed, this is the case with the analysis $\ell_0$-minimization problem [10]. However, all the guarantees developed for feasible programs [11, 14, 15, 36, 37] require at least $O((p - \ell) \log(p/(p - \ell)))$ measurements. Apparently, such conditions are too demanding because the corank, which can be much smaller than $p - \ell$, does not play any role in them. However, it seems that it is impossible to robustly reconstruct the signal with fewer measurements [50].

The same argument can be used for the denoising bound we have for TDIHT which is $O((p - \ell) \log(p))$, saying that we would expect it to be $O((d - r) \log(p))$. Interestingly, even the $\ell_0$ solution can not achieve the latter bound but it can achieve the first which is a function of the cosparsity [50].

In this work we developed recovery guarantees for TDIHT with frames. These close a gap between relaxation based techniques and greedy algorithms in the analysis framework and extends the denoising guarantees of synthesis methods to analysis. It is interesting to ask whether these can be extended to other operators such as the 2D-DIF or to other methods such as AIHT or an analysis version of the Dantzig selector. The core idea in this work is the connection between the signal domain and the transform domain. We believe that the relationships used in this work can be developed further, leading to other new results and improving existing techniques.

**APPENDIX A**

**PROOF OF LEMMA 2.5**

*Lemma 2.5.* Suppose that $M$ satisfies the upper inequality of the D-RIP, i.e.,

$$
\|MDw\|_2 \leq \sqrt{1 + \delta_k} \|Dw\|_2 \quad \forall w, \|w\|_0 \leq k,
$$

and that $\|D\|_2 \leq 1/A$. Then for any representation $w$ we have

$$
\|MDw\|_2 \leq \frac{\sqrt{1 + \delta_k}}{A} \left(\|w\|_2 + \frac{1}{\sqrt{k}} \|w\|_1\right).
$$

*Proof:* We follow the proof of Proposition 3.5 in [27]. We define the following two convex bodies

$$
S = \text{conv} \{Dw : w_{T^c} = 0, |T| \leq k, \|Dw\|_2 \leq 1\},
$$

$$
K = \left\{Dw : \frac{1}{A} \left(\|w\|_2 + \frac{1}{\sqrt{k}} \|w\|_1\right) \leq 1\right\}.
$$

Since

$$
\|M\|_{S^{-2}} = \max_{v \in S} \|Mv\|_2 \leq 1 + \delta_k,
$$

it is sufficient to show that $\|M\|_{K^{-2}} = \max_{w \in K} \|Mw\|_2 \leq \|M\|_{S^{-2}}$ which holds if $K \subset S$. For proving the latter, let $Dw \in K$ and $\{T_0, T_1, \ldots, T_J\}$ be a set of distinct sets such that $T_0$ is composed of the indexes of the $k$-largest entries in $w$, $T_1$ of the next $k$-largest entries, and so on. Thus, we can rewrite $Dw = \sum_{i=0}^{J} Dw_{T_i} = \sum_{i=0}^{J} \lambda_i Dw_{T_i}$, where $\lambda_i = \|Dw_{T_i}\|_2$ and $w_{T_i} = Dw_{T_i}/\lambda_i$. Notice that by definition $Dw_{T_i} \in S$. It remains to show that $\sum_{i=0}^{J} \lambda_i \leq 1$ in order to show that $Dw \in S$. It is easy to show that $\|w_{T_i}\|_2 \leq \frac{\|w_{T_{i-1}}\|_1}{\sqrt{k}}$. Combining this with the fact that $\|D\|_2 \leq 1/A$ leads to

$$
\sum_{i=1}^{J} \lambda_i = \sum_{i=1}^{J} \|Dw_{T_i}\|_2 \leq \frac{1}{A} \sum_{i=1}^{J} \|w_{T_i}\|_2 
\leq \frac{1}{A} \sum_{i=0}^{J-1} \|w_{T_i}\|_1 \leq \frac{1}{A} \|w\|_1.
$$

Using the fact that $\lambda_0 = \|Dw_{T_0}\|_2 \leq \frac{1}{A} \|w_{T_0}\|_2 \leq \frac{1}{A} \|w\|_2$, we have

$$
\sum_{i=0}^{J} \lambda_i \leq \frac{1}{A} \left(\|w\|_2 + \frac{1}{\sqrt{k}} \|w\|_1\right) \leq 1,
$$

where the last inequality is due to the fact that $Dw \in K$. □
APPENDIX B
PROOF OF Lemma 4.2

Lemma 4.2. Consider the same setup of Theorem 4.1. Then the $t$-th iteration of TDIHT satisfies

\[ \| (\Omega x - w_g^t)_{T \cup T'} \|_2 \leq 2 \delta_{4k} \frac{B}{A} \| (\Omega x - w_g^{t-1})_{T' \cup T^{-1}} \|_2 \]

\[ + \frac{1 + \delta_{2k}}{A} \left( \left( 1 + A + \frac{A^2}{2} \right) \| \Omega_{T^c} x \|_2 + \frac{1}{\sqrt{k}} \| \Omega_{T^c} x \|_1 \right) \]

\[ + \| \Omega_{T \cup T'} M^e \|_2 . \]

Proof: Our proof technique is based on the one of IHT in [34], utilizing the properties of $\Omega$ and $D$. Denoting $w = \Omega x$ and using the fact that $w_g^t = \Omega D w^{t-1} - \Omega M^e (y - MD \hat{w}^{t-1})$ we have

\[ \| (w - w_g^t)_{T \cup T'} \|_2 = \]

\[ \| (w - \Omega D w^{t-1})_{T \cup T'} - (\Omega M^e (y - MD \hat{w}^{t-1}))_{T \cup T'} \|_2 . \]

By definition $w = \Omega D w$ and $y = MD w + e$. Henceforth

\[ \| (w - w_g^t)_{T \cup T'} \|_2 \]

\[ = \| \Omega_{T \cup T'} D (w - \hat{w}^{t-1}) - \Omega_{T \cup T'} M^e (y - MD \hat{w}^{t-1}) \|_2 \]

\[ = \| \Omega_{T \cup T'} (I - M^e M) D (w - \hat{w}^{t-1}) - \Omega_{T \cup T'} M^e e \|_2 \]

\[ \leq \| \Omega_{T \cup T'} (I - M^e M) D (w_T - \hat{w}^{t-1}) \|_2 \]

\[ + \| \Omega_{T \cup T'} (I - M^e M) Dw_{T^c} \|_2 + \| \Omega_{T \cup T'} M^e e \|_2 , \]

where the last step is due to the triangle inequality. Denote by $P$ the projection onto range($\Omega_{T \cup T'}^* D_{T \cup T'^{-1}}$) which is a subspace of vectors with $4k$-sparse representations. As $w_T - \hat{w}^{t-1}$ is supported on $T \cup T'^{-1}$ and $M$ satisfies the D-RIP for $[\Omega^*, D]$, we have using norm inequalities and Lemma 2.3 that

\[ \| \Omega_{T \cup T'} (I - M^e M) D (w_T - \hat{w}^{t-1}) \|_2 \]

\[ = \| \Omega_{T \cup T'} P (I - M^e M) PD (w_T - \hat{w}^{t-1}) \|_2 \]

\[ \leq \| \Omega_{T \cup T'} \|_2 \| P (I - M^e M) P \|_2 \| D \|_2 \| w_T - \hat{w}^{t-1} \|_2 \]

\[ \leq \delta_{4k} \frac{B}{A} \| w_T - \hat{w}^{t-1} \|_2 , \]

where $A$ and $B$ are the frame constants and we use the fact that $\| D \|_2 \leq \frac{1}{\sqrt{M}} \leq \frac{1}{A}$ and that $\| \Omega_{T \cup T'} \|_2 \leq \| \Omega \|_2 \leq B$. Notice that when $\Omega$ is tight frame, $\Omega^* = D$ and thus $\Omega_{T \cup T'}^* = D_T$. Hence, we have $\delta_{4k}$ instead of $\delta_{4k}$ since range($[\Omega_{T \cup T'}^* D_{T \cup T'^{-1}}]$) is a subspace of vectors with $3k$-sparse representations.

For completing the proof we first notice that $w_T - \hat{w}^{t-1} = (w - \hat{w}^{t-1})_{T \cup T'^{-1}}$ and that $\hat{w}^{t-1}$ is the best $k$-term approximation of $w_g^{t-1}$ in the $\ell_2$ norm sense. In particular it is also the best $k$-term approximation of $(w_g^{t-1})_{T \cup T'^{-1}}$ and therefore $\| (w^{t-1} - w_g^{t-1})_{T \cup T'^{-1}} \|_2 \leq \| (w - w_g^{t-1})_{T \cup T'^{-1}} \|_2$. Starting with the triangle inequality and then applying this fact we have

\[ \| w_T - \hat{w}^{t-1} \|_2 \]

\[ = \| (w - w_g^{t-1} + w_g^{t-1} - \hat{w}^{t-1})_{T \cup T'^{-1}} \|_2 \]

\[ \leq \| (w - w_g^{t-1})_{T \cup T'^{-1}} \|_2 + \| (w^{t-1} - w_g^{t-1})_{T \cup T'^{-1}} \|_2 \]

\[ \leq 2 \| (w - w_g^{t-1})_{T \cup T'^{-1}} \|_2 . \]

Combining (45) and (44) with (43) leads to

\[ \| (w - w_g^t)_{T \cup T'} \|_2 \leq 2 \delta_{4k} \frac{B}{A} \| (w - w_g^{t-1})_{T \cup T'^{-1}} \|_2 \]

\[ + \| \Omega_{T \cup T'} (I - M^e M) Dw_{T^c} \|_2 + \| \Omega_{T \cup T'} M^e e \|_2 . \]
It remains to bound the second term of the rhs. By using the triangle inequality and then the D-RIP with the fact that $\|\Omega_{T \cup T^c} D\|_2 \leq 1$ we have

$$\|\Omega_{T \cup T^c} (I - M^* M) Dw_{T^c}\|_2 \leq \|\Omega_{T \cup T^c} D w_{T^c}\|_2 + \|\Omega_{T \cup T^c} M^* M dw_{T^c}\|_2$$

$$\leq \|w_{T^c}\|_2 + \sqrt{1 + \delta_{2k}} \|MDw_{T^c}\|_2$$

$$\leq \|w_{T^c}\|_2 + \frac{1 + \delta_{2k}}{\alpha} \left(\|w_{T^c}\|_2 + \frac{1}{\sqrt{k}} \|w_{T^c}\|_1\right),$$

where the last inequality is due to Lemmas 2.5 and 2.2. The desired result is achieved by plugging (47) into (46) and using the fact that $1 + \frac{1 + \delta_{2k}}{\alpha} \leq \frac{(1 + \delta_{2k}) (1 + A)}{A}$. \qed

**ACKNOWLEDGMENT**

The author would like to thank Michael Elad, Yoram Bresler and Yaniv Plan for fruitful discussions. R. Giryes is grateful to the Azrieli Foundation for the award of an Azrieli Fellowship.

**REFERENCES**

[1] D. L. Donoho, M. Elad, and V. N. Temlyakov, “Stable recovery of sparse overcomplete representations in the presence of noise,” *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 6–18, Jan. 2006.

[2] K. Dabov, A. Foi, V. Katkovnik, and K. Egiazarian, “Image denoising with block-matching and 3D filtering,” E. R. Dougherty, J. T. Astola, K. O. Egiazarian, N. M. Nasrabadi, and S. A. Rizvi, Eds., vol. 6064, SPIE, 2006, p. 606414.

[3] A. Danielyan, V. Katkovnik, and K. Egiazarian, “BM3D frames and variational image deblurring,” *IEEE Trans. Image Process.*, vol. 21, no. 4, pp. 1715–1728, 2012.

[4] R. Giryes, M. Elad, and M. Proctor, “On single image scale-up using sparse-representations,” in *Proceedings of the 7th international conference on Curves and Surfaces*. Berlin, Heidelberg: Springer-Verlag, 2012, pp. 711–730.

[5] A. Fannjiang, T. Strohmer, and P. Yan, “Compressed remote sensing of sparse objects,” *SIAM Journal on Imaging Sciences*, vol. 3, no. 3, pp. 595–618, 2010.

[6] M. Lustig, D. Donoho, J. Santos, and J. Pauly, “Compressed sensing MRI,” *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 72–82, 2008.

[7] J. Salmon, Z. Harmany, C.-A. Deledalle, and R. Willett, “Poisson noise reduction with non-local PCA,” *Journal of Mathematical Imaging and Vision*, pp. 1–16, 2013.

[8] A. M. Bruckstein, D. L. Donoho, and M. Elad, “From sparse solutions of systems of equations to sparse modeling of signals and images,” *SIAM Review*, vol. 51, no. 1, pp. 34–81, 2009.

[9] M. Elad, P. Milanfar, and R. Rubinstein, “Analysis versus synthesis in signal priors,” *Inverse Problems*, vol. 23, no. 3, pp. 947–968, June 2007.

[10] S. Nam, M. Davies, M. Elad, and R. Gribonval, “The cosem analysis model and algorithms,” *Appl. Comput. Harmon. Anal.*, vol. 34, no. 1, pp. 30 – 56, 2013.

[11] E. J. Candès, Y. C. Eldar, D. Needell, and P. Randall, “Compressed sensing with coherent and redundant dictionaries,” *Appl. Comput. Harmon. Anal.*, vol. 31, no. 1, pp. 59 – 73, 2011.

[12] S. Vaiter, G. Peyré, C. Dossal, and J. Fadili, “Robust sparse analysis regularization,” *IEEE Trans. Inf. Theory*, vol. 59, no. 4, pp. 2001–2016, 2013.

[13] L. I. Rudin, S. Osher, and E. Fatemi, “Nonlinear total variation based noise removal algorithms,” *Phys. D*, vol. 60, no. 1-4, pp. 259–268, Nov. 1992.

[14] R. Giryes, S. Nam, M. Elad, R. Gribonval, and M. E. Davies, “Greedy-like algorithms for the cosem analysis model,” to appear in the Special Issue on Linear Algebra and its Applications on Sparse Approximate Solution of Linear Systems, 2013.

[15] D. Needell and R. Ward, “Stable image reconstruction using total variation minimization,” *SIAM Journal on Imaging Sciences*, vol. 6, no. 2, pp. 1035–1058, 2013.

[16] E. J. Candès and T. Tao, “Near-optimal signal recovery from random projections: Universal encoding strategies?” *IEEE Trans. Inf. Theory*, vol. 52, no. 12, pp. 5406 –5425, Dec. 2006.

[17] E. Candès and T. Tao, “Decoding by linear programming,” *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203 – 4215, Dec. 2005.

[18] ——, “The Dantzig selector: Statistical estimation when p is much larger than n,” *Annals Of Statistics*, vol. 35, p. 2313, 2007.

[19] P. Bickel, Y. Ritov, and A. Tsybakov, “Simultaneous analysis of lasso and dantzig selector,” *Annals of Statistics*, vol. 37, no. 4, pp. 1705–1732, 2009.

[20] R. Giryes and M. Elad, “RIP-based near-oracle performance guarantees for SP, CoSaMP, and IHT,” *IEEE Trans. Signal Process.*, vol. 60, no. 3, pp. 1465–1468, March 2012.

[21] Z. T. Harmany, R. F. Marcia, and R. M. Willett, “This is SPIRAL-TAP: Sparse poisson intensity reconstruction algorithms – theory and practice,” *Trans. Img. Proc.*, vol. 21, no. 3, pp. 1084–1096, Mar. 2012.

[22] M. Soltanolkotabi, E. Elhamifar, and E. J. Candès, “Robust subspace clustering,” *CoRR*, vol. abs/1301.2603, 2013.

[23] E. Candès, “Modern statistical estimation via oracle inequalities,” *Acta Numerica*, vol. 15, pp. 257–325, 2006.
[24] G. Davis, S. Mallat, and M. Avellaneda, “Adaptive greedy approximations,” *Journal of Constructive Approximation*, vol. 50, pp. 57–98, 1997.
[25] S. Chen, S. A. Billings, and W. Luo, “Orthogonal least squares methods and their application to non-linear system identification,” *International Journal of Control*, vol. 50, no. 5, pp. 1873–1896, 1989.
[26] S. Mallat and Z. Zhang, “Matching pursuits with time-frequency dictionaries,” *IEEE Trans. Signal Process.*, vol. 41, pp. 3397–3415, 1993.
[27] D. Needell and J. Tropp, “CoSaMP: Iterative signal recovery from incomplete and inaccurate samples,” *Appl. Comput. Harmon. A.*, vol. 26, no. 3, pp. 301 – 321, May 2009.
[28] W. Dai and O. Milenkovic, “Subspace pursuit for compressive sensing signal reconstruction,” *IEEE Trans. Inf. Theory*, vol. 55, no. 5, pp. 2230 –2249, May 2009.
[29] T. Blumensath and M. Davies, “Iterative hard thresholding for compressed sensing,” *Appl. Comput. Harmon. Anal.*, vol. 27, no. 3, pp. 265 – 274, 2009.
[30] S. Foucart, “Hard thresholding pursuit: an algorithm for compressive sensing,” *SIAM J. Numer. Anal.*, vol. 49, no. 6, pp. 2543–2563, 2011.
[31] T. Peleg and M. Elad, “Performance guarantees of the thresholding algorithm for the cosparse analysis model,” *IEEE Trans. Inf. Theory*, vol. 59, no. 3, pp. 1832–1845, Mar. 2013.
[32] D. L. Donoho and M. Elad, “Optimal sparse representation in general (nonorthogonal) dictionaries via l1 minimization,” *Proceedings of the National Academy of Science*, vol. 100, pp. 2197–2202, Mar 2003.
[33] T. Zhang, “Sparse recovery with orthogonal matching pursuit under RIP,” *IEEE Trans. Inf. Theory*, vol. 57, no. 9, pp. 6215 –6221, Sept. 2011.
[34] S. Foucart, “Sparse recovery algorithms: sufficient conditions in terms of restricted isometry constants,” in *Approximation Theory XIII*. Springer Proceedings in Mathematics, 2010, pp. 65–77.
[35] Z. Ben-Haim, Y. Eldar, and M. Elad, “Coherence-based performance guarantees for estimating a sparse vector under random noise,” *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 5030 –5043, Oct. 2010.
[36] Y. Liu, T. Mi, and S. Li, “Compressed sensing with general frames via optimal-dual-based l1-analysis,” *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4201–4214, 2012.
[37] M. Kabanava and H. Rauhut, “Analysis l1-recovery with frames and Gaussian measurements,” Preprint, 2013.
[38] T. Blumensath, “Sampling and reconstructing signals from a union of linear subspaces,” *IEEE Trans. Inf. Theory*, vol. 57, no. 7, pp. 4660–4671, 2011.
[39] M. Davenport, D. Needell, and M. Wakin, “Signal space CoSaMP for sparse recovery with redundant dictionaries,” *To appear in IEEE Trans. Inf. Theory*, 2013.
[40] R. Giryes and M. Elad, “Can we allow linear dependencies in the dictionary in the synthesis framework?” in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2013.
[41] ——, “Omp with highly coherent dictionaries,” in *10th Int. Conf. on Sampling Theory Appl. (SAMPTA)*, 2013.
[42] ——, “Iterative hard thresholding for signal recovery using near optimal projections?” in *10th Int. Conf. on Sampling Theory Appl. (SAMPTA)*, 2013.
[43] ——, “Recipes on hard thresholding methods,” in *4th Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Dec. 2011, pp. 353 –356.
[44] R. Giryes and D. Needell, “Greedy signal space methods for incoherence and beyond.” *CoRR*, vol. abs/1309.2676, 2013.
[45] R. Gribonval, M. E. Pfetsch, and A. M. Tillmann, “Projection onto the k-cosparse set is NP-hard,” submitted to *IEEE Trans. Inf. Theory*, 2013.
[46] B. Ophir, M. Lustig, and M. Elad, “Multi-scale dictionary learning using wavelets,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 5, no. 5, pp. 1014–1024, 2011.
[47] S. Ravishankar and Y. Bresler, “MR image reconstruction from highly undersampled k-space data by dictionary learning,” *IEEE Trans. Medical Imaging*, vol. 30, no. 5, pp. 1028–1041, 2011.
[48] A. Kyrillidis and V. Cevher, “Recipes on hard thresholding methods,” in *4th Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Dec. 2011, pp. 353 –356.
[49] V. Cevher, “An ALPS view of sparse recovery,” in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2011, pp. 5808–5811.
[50] Y. Plan, R. Vershynin, and R. Giryes, “Limitations of low-dimensional manifold models for the analysis version of compressed sensing,” 2013.