AN ELLIPTIC HARNACK INEQUALITY FOR
RANDOM WALK IN BALANCED ENVIRONMENTS

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Abstract. We prove a Harnack inequality for the solutions of a difference equation with non-elliptic balanced i.i.d. coefficients. Along the way we prove a (weak) quantitative homogenisation result, which we believe is of interest too.

1. Introduction

This paper deals with a Random Walk in Random Environment (RWRE) on $\mathbb{Z}^d$ which is defined as follows: Let $\mathcal{M}_d$ denote the space of all probability measures on the nearest neighbors of the origin $\{\pm e_i\}_{i=1}^d$, with the $\sigma$-algebra which is inherited from the finite-dimensional space in which it is embedded, and let $\Omega = (\mathcal{M}_d)^{\mathbb{Z}^d}$ with the product $\sigma$-algebra. An environment is a point $\omega \in \Omega$, we denote by $P$ the distribution of the environment on $\Omega$. For the purposes of this paper, we assume that $P$ is an i.i.d. measure, i.e. for a given environment $\omega \in \Omega$, the Random Walk on $\omega$ is a time-homogenous Markov chain jumping to the nearest neighbors with transition kernel

$$P_\omega (X_{n+1} = z + e | X_n = z) = \omega(z, e) \geq 0, \quad \sum_e \omega(z, e) = 1.$$

The quenched law $P_\omega^z$ is defined to be the law on $(\mathbb{Z}^d)^\mathbb{N}$ induced by the kernel $P_\omega$ and $P_\omega^z(X_0 = z) = 1$. We let $P^z = P \otimes P_\omega^z$ be the joint law of the environment and the walk, and the annealed law is defined to be its marginal

$$P^z = \int_{\Omega} P_\omega^z dP(\omega).$$

A comprehensive account of the results and the remaining challenges in the understanding of RWRE can be found in Zeitouni’s Saint Flour lecture notes [12] and in the more recent survey paper by Drewitz and Ramírez [5].

In this paper we will focus on a special class of environments: the balanced environment. In particular, we solve the challenge of adapting the methods that were developed for the elliptic case in [9] and [7] to non-elliptic cases.

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**Definition 1.1** An environment $\omega$ is said to be **balanced** if for every $z \in \mathbb{Z}^d$ and neighbor $e$ of the origin, $\omega(z,e) = \omega(z,-e)$.

Of course we want to make sure that the walk really spans $\mathbb{Z}^d$:

**Definition 1.2** An environment $\omega$ is said to be **genuinely $d$-dimensional** if for every neighbor $e$ of the origin, there exists $z \in \mathbb{Z}^d$ such that $\omega(z,e) > 0$.

Throughout this paper we make the following assumption.

**Assumption 1** $P$-almost surely, $\omega$ is balanced and genuinely $d$-dimensional.

Note that whenever the distribution is ergodic, the above assumption is equivalent with

$$P[\omega(z,e) = \omega(z,-e)] = 1, \quad \text{and} \quad P[\omega(z,e) > 0] > 0$$

for every $z \in \mathbb{Z}^d$ and a neighbor $e$ of the origin.

Note that unlike [7] we do not allow holding times in our model. We do this for the sake of simplicity. Holding times in our case could be handled exactly as they are handled in [7].

**Example 1** Take $P = \nu_{\mathbb{Z}^d}$ as above with

$$\nu \left[ \omega(z,e_i) = \omega(z,-e_i) = \frac{1}{2}, \ \omega(z,e_j) = \omega(z,-e_j) = 0, \ \forall j \neq i \right] = \frac{1}{d}, \ i = 1, ..., d.$$  

In this model, the environment chooses at random one of the $\pm e_i$ direction, see Figure 1.

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**Figure 1.** An illustration of Example 1 restricted to a small box.
1.1 Main results.

We have two main results in this paper. The first result is a sort of quantitative homogenization. In [3] the following quenched invariance principle was proved. Let
\[ \bar{\omega}_n = \tau - X_n \omega, \quad n \in \mathbb{N} \]
where \( \tau \) is the shift on \( \Omega \), be the environment viewed from the point of view of the particle.

**Theorem 1.3** ([3]) Assume that the environment is i.i.d., balanced and genuinely d-dimensional, then there exists a unique invariant distribution \( Q \) on \( \Omega \) for the process \( \{ \bar{\omega}_n \}, n \in \mathbb{N} \) which is absolutely continuous with respect to \( P \), moreover the following quenched invariance principle holds: for \( P \) almost \( \omega \), the rescaled random walk \( \{ X^N(t) = N^{-1/2}X_{[Nt]}, t \geq 0 \} \) converges weakly under \( P_0^\omega \) to a Brownian motion with deterministic non-degenerate diagonal covariance matrix \( \Sigma \), \( \Sigma_{i,i} = 2E_Q[\omega(0,e_i)], i = 1, ..., d \).

Then the content of Theorem 1.3 is that on the large scale, the RWRE behaves like a Brownian Motion with covariance matrix \( \Sigma \). The next theorem gives a quantitative bound on how much time it takes until this behavior is seen.

Namely, we will provide a quantitative estimate (that holds with high probability) for the difference between the discrete-harmonic function and the corresponding homogenized function, where the two functions have the same (up to discretization) boundary conditions.

To be specific, for any discrete finite set \( E \subset \mathbb{Z}^d \), we say that a function \( u : \bar{E} \to \mathbb{R} \) is \( \omega \)-harmonic in \( E \) if for any \( x \in E \),
\[ L_\omega u(x) := \sum_{y \sim x} \omega(x,y)[u(y) - u(x)] = 0. \]

Let \( B_R = \{ z \in \mathbb{R}^d : \| z \|_2 < R \} \) be the ball of radius \( R \) in \( \mathbb{R}^d \). For \( x \in \mathbb{R}^d \), let \( B_R(x) = x + B_R \) and let \( B_R^{\text{dis}}(x) = B_R(x) \cap \mathbb{Z}^d \).

Let \( O = B_1 \) be the unit ball. Let \( \Sigma \) be the covariance matrix of the limiting Brownian Motion as in Theorem 1.3. Let \( F : B_1 \to \mathbb{R} \) be a function which is continuous in \( B_1 \), smooth in \( B_1 \) and satisfies
\[ \sum_{i,j=1}^d \Sigma_{ij} \partial_{ij} F(x) = 0 \quad \text{for all } x \in B_1. \]

For given \( R > 0 \), we denote by
\[ O_R^{\text{dis}} := \{ x \in B_R^{\text{dis}} : y \in B_R^{\text{dis}} \text{ for all } y \sim x \} \]
the biggest subset of \( B_R^{\text{dis}} \) such that \( \overline{O_R^{\text{dis}}} = B_R^{\text{dis}} \). For any \( k > 0 \), we let
\[ I_k(O_R^{\text{dis}}) := \{ x \in O_R^{\text{dis}} : \text{dist}(x, \partial O_R^{\text{dis}}) > k \} \quad (1.1) \]
denote the subset of \( O_R^{\text{dis}} \) that has distance more than \( k \) away from \( \partial O_R^{\text{dis}} \).
We define the function $F_R: \overline{\mathcal{O}^\text{dis}_R} \to \mathbb{R}$ by

$$F_R(x) = F(x/R) \quad \text{for every } x \in \overline{\mathcal{O}^\text{dis}_R},$$

so that $F_R$ is the function obtained by “stretching” the domain of $F$. Let $G = G_{R,\omega}: \overline{\mathcal{O}^\text{dis}_R} \to \mathbb{R}$ be the solution of the Dirichlet problem

$$\begin{cases}
L_\omega G = 0 & \text{on } \overline{\mathcal{O}^\text{dis}_R} \\
G = F_R & \text{on } \partial \overline{\mathcal{O}^\text{dis}_R}.
\end{cases}$$

In other words, $G$ is the $\omega$-harmonic function on $\overline{\mathcal{O}^\text{dis}_R}$ whose boundary data on $\partial \overline{\mathcal{O}^\text{dis}_R}$ agrees with that of $F_R$.

For $i \in \mathbb{N}$, let $M^i_F$ denote the supremum (of the absolute values) of all $i$-th order partial derivatives of $F$ on $B_1$.

We can thus state the following quantitative estimate.

**Theorem 1.4** For any $\epsilon > 0$, there exists $n_1 = n_1(\epsilon, P) > \epsilon^{-2}$ and $\delta = \delta(P) > 0$ such that for any $R > n_1$, with probability greater than or equal to $1 - C \exp(-R^4)$,

$$\max_{x \in \overline{\mathcal{O}^\text{dis}_R}} |F_R(x) - G_{R,\omega}(x)| \leq \epsilon (M^2_F + M^3_F).$$

As a consequence of the above Theorem we get the following result on the exit distribution from a large ball for the random walk: Let $A \subseteq \partial B_1$ be open in the relative topology of $\partial B_1$. We write $\beta(A)$ for the probability that a Brownian Motion starting at the origin with covariance matrix $\Sigma$ leaves the ball $B_1$ through $A$. Fix $\epsilon > 0$ and $R$, and we say the the environment $\omega$ is $(A, R, \epsilon)$-good if the probability that the RWRE in environment $\omega$ leaves $B_R$ through $RA$ is within $\epsilon$ from $\beta(A)$.

**Corollary 1.5** Assume that the environment is i.i.d., balanced and genuinely $d$-dimensional. There exists $\alpha > 0$ such that for every $A$ and $\epsilon$ there are finite and positive constants $C_1$ and $C_2$ such that the probability that $\omega$ is $(A, R, \epsilon)$-good is greater than or equal to $1 - C_1 \exp(-C_2 R^\alpha)$.

Our second main result is a Harnack inequality for functions that are $\omega$-harmonic. Then we prove the following which is our main result.

**Theorem 1.6** Under Assumption 1, there exists a constant $C$ and a random $R_0$ satisfying $P(R_0 < \infty) = 1$ such that for every $R > R_0$ and every non-negative $\omega$-harmonic function $f: B_{2R} \to \mathbb{R}$, we have

$$\sup\{f(x) : x \in B_R\} \leq C \inf\{f(x) : x \in B_R\}.$$

Furthermore, there exist $\beta > 0$ and $\gamma > 0$ such that for every $M$,

$$P(R_0 > M) < \exp(-\gamma M^\beta).$$

In addition, the constant $C$ can be taken arbitrarily close to the constant in the classical Harnack inequality in $\mathbb{R}^d$ for the corresponding Brownian motion with covariance $\Sigma$.

From Theorem 1.6 we get the following weak Liouville type estimate.
Corollary 1.7 Let $\delta < 1$. For $P$-almost $\omega$ and every $\omega$-harmonic function $f : \mathbb{Z}^d \to \mathbb{R}$, if
\[ \sup_{0 < |x|} \frac{|f(x)|}{|x|^\delta} < \infty \]
then $f$ is a constant function.

We note that Harnack inequalities for balanced environments in the elliptic setting have been proven before, see [8,10,12], and for other non-elliptic cases, see [2]. However, we believe that our result is the first Harnack inequality in the context of RWRE which is valid in a case which is non-reversible and non-elliptic. It is also to the best of our knowledge the first Harnack inequality in the context of RWRE where the optimal Harnack constant is established.

1.2 Structure of the paper.

Many of our arguments are based on results established in [3]. In Section 1.3 we collect those results so that we can use them later in the paper. In Section 2 we prove Theorem 1.4. In Section 3 we discuss the connectivity structure of the balanced directed percolation. While proving some results of independent interest, the main goal of this section is to provide percolation estimates necessary for the proof of Theorem 1.6. Then, finally, in Sections 4 and 5 we collect results from previous sections and prove Theorem 1.6. In particular, the proof combines ideas for Harnack inequalities that we learned from [8] and [2], some of which go back to [6].

1.3 Input from [3].

In this section we review some useful definitions and results from [3].

We first define the rescaled walk, which is a useful notion in the study of non-elliptic balanced RWRE, and recall some basic facts about it.

Let $\{X_n\}_{n=0}^\infty$ be a nearest neighbor walk in $\mathbb{Z}^d$, i.e. a sequence in $\mathbb{Z}^d$ such that $\|X_{n+1} - X_n\|_1 = 1$ for every $n$. Let $\alpha_n, n \geq 1$ be the coordinate that changes between $X_n - 1$ and $X_n$, i.e. $\alpha(n) = i$ whenever $X_n - X_{n-1} = e_i$ or $X_n - X_{n-1} = -e_i$.

The following is [3, Definition 3].

Definition 1.8 The stopping times $T_k, k \geq 0$ are defined as follows: $T_0 = 0$. Then
\[ T_{k+1} = \min\{t > T_k : \{\alpha(T_k + 1), \ldots, \alpha(t)\} = \{1, \ldots, d\}\} \leq \infty. \]

We then define the rescaled random walk to be the sequence (no longer a nearest neighbor walk) $Y_n = X_{T_n}$. $(Y_n)$ is defined as long as $T_n$ is finite.

The following estimates both annealed and quenched have been derived in [3], cf. Lemma 2.1, 2.2, 2.3 and 2.4:

Lemma 1.9 $P$-almost surely, $T_k < \infty$ for every $k$. There exists a constant $C$ such that for every $n$,
\[ P(T_1 > n) < e^{-Cn^{\frac{1}{d}}}. \]
and

\[ P(\omega : E_\omega(T_1) > k) \leq e^{-Ck^{1/2}}. \]

Moreover for every \( 0 < p < \infty \),

\[ E[E_\omega(T_1^p)] < \infty. \]

The maximum principle, originally due to Alexandrov-Bakelman-Pucci in the continuum and adapted to balanced random walks by Kuo and Trudinger, \([3]\) is one key analytical tool in the proof of the existence of the stationary measure for the walk \( Q \). In our non-elliptic setting it can be restated in terms of the rescaled process as follows, cf Theorem 3.1 of \([3]\):

For \( N \in \mathbb{N} \) and \( k = k(N) \in (0, N) \cap \mathbb{Z} \), let \( T_1^{(N)} = T_1^{(N,k)} = \min \{ T_1, k \} \). Let \( h : \mathbb{Z}^d \to \mathbb{R} \) be a real valued function, and for every \( z \in \mathbb{Z}^d \), let \( L_\omega^{(N)} h(z) := h(z) - E_\omega^z[h(X_{T_1^{(N)}})] \).

Let \( Q \subseteq \mathbb{Z}^d \) be finite and connected, and let \( \partial^{(k)} Q = \{ z \in \mathbb{Z}^d - Q : \exists x \in Q \parallel z - x\parallel_\infty < k \} \).

We say that a point \( z \) in \( Q \) is exposed if there exists \( \beta = \beta(z, h) \in \mathbb{R}^d \) such that \( h(z) - \langle \beta, z \rangle \geq h(x) - \langle \beta, x \rangle \) for every \( x \in Q \cup \partial^{(k)} Q \). We let \( D_h \) be the set of exposed points. Further, we define the angle of vision \( I_h(z) \) as follows:

\[ I_h(z) = \{ \beta \in \mathbb{R}^d : \forall x \in Q \cup \partial^{(k)} Q h(x) \leq h(z) + \langle \beta, x-z \rangle \}. \quad (1.2) \]

This is the set of hyperplanes that touch the graph of \( h \) at \((z, h(z))\) and are above the graph of \( h \) all over \( Q \cup \partial^{(k)} Q \). A point \( z \) is exposed if and only if \( I_h(z) \) is not empty.

**Theorem 1.10** (Maximum Principle, \([3]\) Theorem 3.1) There exists \( N_0 \) such that for every \( N > N_0 \) and every \( 0 < k < N \), every balanced environment \( \omega \) and every \( Q \) of diameter \( N \), if for every \( z \in Q \)

\[ P_\omega^z(T_1 > (\log N)^k) < e^{-(\log N)^2} E_\omega^z(T_1 \cdot 1_{T_1 > k}) < e^{-(\log N)^2} P_\omega^z(T_1 > k) < e^{-(\log N)^3} \quad (1.3) \]

then

\[ \max_{z \in Q} h(z) - \max_{z \in \partial^{(k)} Q} h(z) \leq 6N \left( \sum_{z \in Q} 1_{z \in D_h} |L_\omega^{(N)} h(z)|^d \right)^{1/d} \quad (1.4) \]

We now turn to some definitions and results pertaining to Percolation.

**Definition 1.11** For \( \omega \in \Omega \) and \( x, y \in \mathbb{Z}^d \), we denote by \( x \sim y \) the occurrence

\[ P_\omega^z(\exists_n X_n = y) > 0. \]

We say that a set \( A \subseteq \mathbb{Z}^d \) is strongly connected w.r.t. \( \omega \) if for every \( x \) and \( y \) in \( A \), \( x \sim y \). A set \( A \subseteq \mathbb{Z}^d \) is called a sink w.r.t. \( \omega \) if it is strongly connected and \( x \not\sim y \) for every \( x \in A \) and \( y \not\in A \).

For every two neighbors \( x \) and \( y \), we draw a directed edge from \( x \) to \( y \), denoted \( xyz \), whenever \( \omega(x, y-x) > 0 \).
For a measure $Q$ which is invariant w.r.t. the point of view of the particle and is absolutely continuous w.r.t. $P$, we define

$$\text{supp } Q = \{ \omega : \frac{dQ}{dP}(\omega) > 0 \},$$

where the derivative is the Radon-Nykodim derivative. This is well define up to a set of $P$-measure zero.

We define $\text{supp}_{\omega} Q = \{ z \in \mathbb{Z}^d : \tau_{-z}(\omega) \in \text{supp } Q \}$. In view of Corollary 4.12 and Lemma 5.6 of [3] we have the following lower bound on the density of a sink:

**Lemma 1.12**

1. There exists $\Phi > 0$ such that for $P$-almost every $\omega$, every sink has lower density at least $\Phi$.

2. For every ergodic $Q$ which is invariant w.r.t. the point of view of the particle and is absolutely continuous w.r.t. $P$, $P$-a.s. there are only finitely many sinks contained in $\text{supp}_{\omega} Q$.

3. $P$-a.s., every point in $\text{supp}_{\omega} Q$ is contained in a sink.

In other words, the lemma says that a.s. $\text{supp}_{\omega} Q$ is a finite union of sinks, each of which has lower density at least $\Phi$.

As announced in Remark 3 of [3] let us now state

**Proposition 1.13** There exists a unique sink.

**Proof.** We use an adaptation of the easy part of the percolation argument of Burton and Keane [4], easy since we already know that there are finitely many sinks. Even though the finite energy condition is not satisfied, a very similar yet slightly weaker condition holds. Let $S_1(\omega), S_2(\omega), \ldots, S_k(\omega)$ be distinct sinks, $k \geq 2$. Note that by ergodic theorem the number of sinks is $P$ a.s. constant. Define

$$\text{dist}(P) := \min (|z - w| : z \text{ and } w \text{ are in two distinct sinks}).$$

Note that due to shift invariance and ergodicity $\text{dist}(P)$ is a $P$-almost sure constant, and therefore we treat it as a natural number rather than as a random variable. Let $z$ and $w$ be two points such that $|z - w| = \text{dist}(P)$, and such that the event $U = U(z, w) = \{ z \text{ and } w \text{ are in two distinct sinks} \}$ has a positive $P$ probability. Let $i$ be a direction s.t. $\langle e_i, z - w \rangle \neq 0$. Let $R$ be the following measure on $\Omega \times \Omega$: we sample $\omega$ and $\omega'$. For all $x \neq z$, we take $\omega(x) = \omega'(x)$ to be sampled i.i.d. according to $\nu$. We then take $\omega(z) \sim (\nu|\omega(e_i) = 0)$ and $\omega'(z) \sim (\nu|\omega(e_i) \neq 0)$. Again, everything is independent. Let $P_1$ be the distribution of $\omega$ and $P_2$ be the distribution of $\omega'$. Note that $P_1$ and $P_2$ are both absolutely continuous w.r.t. $P$, and that $P_1(U) > 0$ and $P_2(U) = 0$. We now condition on the (positive probability) event $U(\omega) \setminus U(\omega')$.

Call $S_1$ the sink containing $z$ in $\omega$, call $S_2$ the sink containing $w$ in $\omega$, and $S_3, \ldots, S_k$ all the other sinks in $\omega$. For $j = 2, \ldots, k$, the environments $\omega$ and $\omega'$ agree on $S_j$, so $S_2, \ldots, S_k$ are all sinks in $\omega'$.

Assume w.l.o.g. that $\langle e_i, z - w \rangle > 0$, and let $z' = z + e_i$. Then $z'$ is in no sink in $\omega'$ b/c it is too close to $S_2$. Note that $x \leq z'$ for every $x \in S_1$. Thus no point in $S_1$ is in a sink in $\omega'$. Further, no point outside of $S = \cup_{j=1}^k S_j$ is in a sink in $\omega'$. Indeed, if $y \in \mathbb{Z}^d \setminus S$ is in
a sink $W$, then $W$ cannot intersect $S$, and thus $W$ is a sink in $\omega$ other than $S_1, \ldots, S_k$. However, there is no such sink, thus $W$ does not exist.

Thus there are only $k-1$ sinks in $\omega'$, in contradiction to $P_2$ being absolutely continuous with respect to $P$.

□

The last result that we need is the following lemma, which follows immediately from [3, Proposition 5.9] and Proposition 1.13.

**Lemma 1.14** Let $S$ be the (a.s. unique) sink. Then for $P$-a.e. $\omega$ and every $z \in \mathbb{Z}^d$, 

$$P_\omega^z(\exists N \text{ s.t. } \forall n>N X_n \in S) = 1.$$ 



## 2. Quantitative estimates for the invariance principle

In this section we prove a quantitative homogenization bound. Namely, we will provide a quantitative estimate (that holds with high probability) for the difference between the discrete-harmonic function and the corresponding homogenized function, where the two functions have the same (up to discretization) boundary conditions.

To be specific, for any discrete finite set $E \subset \mathbb{Z}^d$, we say that a function $u : \Bar{E} \to \mathbb{R}$ is $\omega$-harmonic in $E$ if for any $x \in E$,

$$L_\omega u(x) := \sum_{y \sim x} \omega(x, y) \left[ u(y) - u(x) \right] = 0.$$ 

Let $B_R = \{ z \in \mathbb{R}^d : \|z\|_2 < R \}$ be the ball of radius $R$ in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, let $B_R(x) = x + B_R$ and let $B_{R}^\text{dis}(x) = B_R(x) \cap \mathbb{Z}^d$.

Let $\mathcal{O} = B_1$ be the unit ball. Let $\Sigma$ be the covariance matrix of the limiting Brownian Motion as in Theorem 1.3. Let $F : B_1 \to \mathbb{R}$ be a function which is continuous in $B_1$, smooth in $B_1$ and satisfies

$$\sum_{i,j=1}^{d} \Sigma_{ij} \partial_{ij} F(x) = 0 \quad \text{for all } x \in B_1.$$ 

For given $R > 0$, we denote by

$$\mathcal{O}_R^\text{dis} := \{ x \in B_R^\text{dis} : y \in B_R^\text{dis} \text{ for all } y \sim x \}$$

the biggest subset of $B_R^\text{dis}$ such that $\overline{\mathcal{O}_R^\text{dis}} = B_R^\text{dis}$. For any $k > 0$, we let

$$I_k(\mathcal{O}_R^\text{dis}) := \{ x \in \mathcal{O}_R^\text{dis} : \text{dist}(x, \partial \mathcal{O}_R^\text{dis}) > k \} \quad (2.1)$$

denote the subset of $\mathcal{O}_R^\text{dis}$ that has distance more than $k$ away from $\partial \mathcal{O}_R^\text{dis}$.

We define the function $F_R : \overline{\mathcal{O}_R^\text{dis}} \to \mathbb{R}$ by

$$F_R(x) = F(x/R) \quad \text{for every } x \in \overline{\mathcal{O}_R^\text{dis}},$$
so that $F_R$ is the function obtained by “stretching” the domain of $F$. Let $G = G_{R,\omega} : \mathcal{O}^{\text{dis}}_R \to \mathbb{R}$ be the solution of the Dirichlet problem
\[
\begin{cases}
L_\omega G = 0 & \text{on } \mathcal{O}^{\text{dis}}_R \\
G = F_R & \text{on } \partial \mathcal{O}^{\text{dis}}_R.
\end{cases}
\]
In other words, $G$ is the $\omega$-harmonic function on $\mathcal{O}^{\text{dis}}_R$ whose boundary data on $\partial \mathcal{O}^{\text{dis}}_R$ agrees with that of $F_R$.

For $i \in \mathbb{N}$, let $M^i_F$ denote the supremum (of the absolute values) of all $i$-th order partial derivatives of $F$ on $B_1$.

Our goal in this section is to obtain Theorem 1.4, namely that for any $\epsilon > 0$, there exists $n_1 = n_1(\epsilon, P) > \epsilon^{-2}$ and $\delta = \delta(P) > 0$ such that for any $R > n_1$, with probability greater than or equal to $1 - C \exp(-R^3)$,
\[
\max_{x \in \mathcal{O}^{\text{dis}}_R} |F_R(x) - G_{R,\omega}(x)| \leq \epsilon (M^2_F + M^3_F). \tag{2.2}
\]

First, note that the “stretched” version $F_R$ of the function $F$ is very “flat” when $R$ is large. Indeed, for $x, y \in \mathcal{O}^{\text{dis}}_R \subset R\mathcal{O}$, by Taylor expansion,
\[
F_R(y) - F_R(x) = \frac{1}{R} \langle \nabla F(\frac{x}{R}), y - x \rangle + \frac{1}{2R^2} (y - x)^t D^2 F(\frac{x}{R})(y - x) + \rho_y \|y - x\|^3, \tag{2.3}
\]
where the error term $\rho_y$ is bounded by $R^{-3}M^3_F$. Hence, we conclude that
\[
|L_\omega F_R(x)| \leq \frac{M^2_F}{R^2} + \frac{2dM^3_F}{R^3} \leq \frac{C_F}{R^2} \quad \text{for all } x \in \mathcal{O}^{\text{dis}}_R, \tag{2.4}
\]
where $C_F = C(M^2_F + 2dM^3_F)$. Note that the constant $C$ may differ from line to line and so may $C_F$.

Our next observation is that since (by the quenched CLT) the diffusion matrix of $X_n/\sqrt{n}$ converges to $\Sigma$, the function $F_R$ should be “approximately $\omega$-harmonic” in a sense that will be made precise in (2.7) below. Indeed, for $x \in \mathbb{Z}^d, n \in \mathbb{N}$, let $M^{(n)}_\omega(x)$ be the $d \times d$ covariance matrix with entries
\[(M^{(n)}_\omega(x))_{ij} := E^x_\omega[(X_n(i) - x)(X_n(j) - x)]/n, \quad 1 \leq i, j \leq n,
\]
where $X_n(i)$ denotes the $i$-th coordinate of $X_n$. For any fixed $\epsilon > 0$ and $\alpha > 0$, by Theorem 1.3 there exists $n_0 = n_0(\epsilon, \alpha, P)$ such that for any $n \geq n_0$,
\[
P \left[ \|M^{(n)}_\omega(x) - \Sigma\|_1 < \epsilon \right] > 1 - \alpha. \tag{2.5}
\]
Moreover, for any $x \in I_{n_0}(\mathcal{O}^{\text{dis}}_R)$, by (2.3) we have
\[
E^x_\omega[F_R(X_{n_0}) - F_R(x)] = \frac{1}{R^2} E^x_\omega[(X_{n_0} - x)^t D^2 F(\frac{x}{R})(X_{n_0} - x)] + E^x_\omega[\rho_{X_{n_0}} \|X_{n_0} - x\|^3]
\[
= \frac{n_0}{R^2} \sum_{i,j=1}^d \partial_{i,j} F(\frac{x}{R})M^{(n_0)}_\omega(x)_{ij} + E^x_\omega[\rho_{X_{n_0}} \|X_{n_0} - x\|^3].
\]
Here $D_2F(x)$ denotes the Hessian matrix of $F$ at $x$. Under the event
\[
A_{n_0}(x) = \{ \omega : \| M_{\omega}^{(n_0)}(x) - \Sigma \|_1 < \epsilon \},
\]
recalling that $\sum_{i,j=1}^{d} \partial_{ij} F(\frac{\tau}{R}) \Sigma_{ij} = 0$, we see that
\[
\left| \sum_{i,j=1}^{d} \partial_{ij} F(\frac{\tau}{R}) M_{\omega}^{(n_0)}(x)_{ij} \right| \leq \epsilon d^2 M^2.
\]
Hence for $x \in I_{n_0}(\mathcal{O}_R^{\text{dis}})$, when $A_{n_0}(x)$ occurs we obtain
\[
|E_{\omega}^{x}[F_\tau(X_{n_0})] - F_\tau(x)| \leq \epsilon n_0 d^2 R^{-2} M^2 + C M^3 n_0 \frac{4}{3} R^{-3} \leq \epsilon n_0 R^{-2} C_F
\]
for $R > \sqrt{\frac{\epsilon n_0}{\epsilon}} \vee n_0$.

Recall the definition of $n_0$ and $A_{n_0}$ in (2.5) and (2.6). We claim that taking $k = \sqrt{R}$, (1.3) also happens with high probability. For fixed $\epsilon, \alpha > 0$, we define the events (Here for any discrete set $Q$, $|Q|$ denotes the cardinality of $Q$)
\[
A_R^{(1)} = \left\{ \omega : \sum_{x \in B_R^{\text{dis}}} 1_{\omega \in A_{n_0}(x)} \frac{1}{|B_R^{\text{dis}}|} > 1 - 2\alpha \right\},
\]
\[
A_R^{(2)} = \left\{ \omega : P_{\omega}^{(x)}(T_1 > R^{1/2}) < e^{-(\log R)^3} \text{ for all } x \in \mathcal{O}_R^{\text{dis}} \right\},
\]
where the stopping time $T_1$ in the definition of $A_R^{(2)}$ is as in Definition 1.8.

**Lemma 2.1** Let $\epsilon \in (0, 1), \alpha > 0, n_0 = n_0(\epsilon, \alpha, P)$ be the same as in (2.5). There exist $C = C(n_0, P)$ and $c = c(n_0, P)$ such that $P(A_R^{(1)} \cap A_R^{(2)}) > 1 - C e^{-c R^{1/7}}$.

We postpone the proof of Lemma 2.1 until after the proof of Theorem 1.4. Throughout this section we always take $k = k(R) := \sqrt{R}$.

To prove Theorem 1.4 we consider the error
\[
H(x) = H_{R, \omega}(x) := F_\tau(x) - G_\tau(x), \quad x \in \overline{\mathcal{O}_R^{\text{dis}}}.
\]
Then $H$ is the solution of the discrete Dirichlet problem
\[
\begin{cases}
L_\omega H = L_\omega F_\tau & \text{on } \mathcal{O}_R^{\text{dis}} \\
H = 0 & \text{on } \partial \mathcal{O}_R^{\text{dis}}.
\end{cases}
\]
However, $H$ is not defined on $\partial (k) \mathcal{O}_R^{\text{dis}}$. To apply Theorem 1.10 we define an auxiliary function $H' : B_{R^{k+1}}^{\text{dis}} \to \mathbb{R}$ to be the solution of the Dirichlet problem $H'|_{\partial B_{R^{k+1}}^{\text{dis}}} = 0$ and
\[
L_\omega H' = \begin{cases}
L_\omega F_\tau & \text{on } \mathcal{O}_R^{\text{dis}} \\
0 & \text{on } B_{R^{k+1}}^{\text{dis}} \setminus \mathcal{O}_R^{\text{dis}}.
\end{cases}
\]
Notice that by the definitions of $H$ and $H'$,
\[
H'(x) = E_\omega^x \left[ \sum_{i=0}^{\tau_{R^{k+1}}-1} -L_\omega F_\tau(X_i) 1_{X_i \in \mathcal{O}_R^{\text{dis}}} \right] \quad \text{for all } x \in \overline{B_{R^{k+1}}^{\text{dis}}}, \quad (2.8)
\]
Hence
\[\tau (Q) := \inf \{ n \geq 0 : X_n \notin Q \} \] denotes the exit time from \( Q \subset \mathbb{Z}^d \) and \( \tau_R := \tau (B_{R}^{\text{dis}}) \).

Let \( \omega \) be any fixed constant and let \( \omega > X \). Let \( R > \frac{\sqrt{n_0}}{\epsilon} \) be outside of \( D_h \) (Lemma 2.2), and then controlling the \( \omega \)-Laplacian in the remaining points (Lemma 2.3 below).

**Lemma 2.2** Let \( \epsilon, \alpha > 0 \) be any fixed constants and let \( n_0 = n_0(\epsilon, \alpha, P) \) be as in (2.5). Let \( R > \frac{\sqrt{n_0}}{\epsilon} \vee n_0 \). For any \( x \in I_{n_0}(O_{R}^{\text{dis}}) \), if \( \omega \in A_{n_0}(x) \), then \( x \notin D_h \).

From Lemma 2.2 we see that each \( x \) is not counted in the sum in (1.4).

**Proof of Lemma 2.2.** When \( R > \frac{\sqrt{n_0}}{\epsilon} \vee n_0 \), \( x \in I_{n_0}(O_{R}^{\text{dis}}) \) and \( \omega \in A_{n_0}(x) \), by (2.7) and our definition of \( \gamma \),

\[ E_\omega^x [h(X_{n_0}) - h(x)] = E_\omega^x [H'(X_{n_0}) - H'(x)] + \gamma E_\omega^x [\|X_{n_0}\|^2 - \|x\|^2] \]

\[ = E_\omega^x [F_R(X_{n_0}) - F_R(x)] + \gamma E_\omega^x [\|X_{n_0}\|^2 - \|x\|^2] \]

\[ \geq -2\epsilon n_0 R^{-2} M_R^2 + \gamma n_0 > 0. \]

Thus \( E_\omega^x [h(X_{n_0})] > h(x) \). In particular, since \( E_\omega^x [X_{n_0}] = x \), for every \( \alpha \in \mathbb{R}^d \) we get \( E_\omega^x [h(X_{n_0}) + \langle \alpha, X_{n_0} - x \rangle] > h(x) \). So for every \( \alpha \in \mathbb{R}^d \) there exists \( y \) in the support of \( X_{n_0} \) with \( h(y) + \langle \alpha, y - x \rangle > h(x) \), which implies \( x \notin D_h \).

Our next step is to control the \( \omega \)-Laplacian of the function \( h \). By definition, on \( B_{R+k} \),

\[ |L_\omega h| \leq |\gamma - 1_{O_{R}^{\text{dis}}} C_F / R^2| \leq C_F / R^2. \]

Hence

\[ |L^{(R)}_\omega h(x)| \leq C_F E_\omega^x [T_1] / R^2. \] (2.10)
For $p > 0$ and $K > 0$, we define the event

$$A_R^{(3)}(p, K) := \left\{ \omega : \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_x^\omega [T_1])^p \leq K \right\}.$$ 

**Lemma 2.3** Let $p > d$. There exist positive constants $\delta$ and $C$ depending only on $p$ and the environment measure $P$ such that

$$P \left( A_R^{(3)}(p, C) \right) > 1 - Ce^{-R^d}.$$ 

We postpone the proof of Lemma 2.3 until after the proof of Theorem 1.4.

We are now ready to bound the function $H$ on $O^{\text{dis}}_R$.

**Proof of Theorem 1.4.** Let $\alpha = \alpha(\epsilon) > 0$ be a small constant to be determined later. Let $n_0 = n_0(\epsilon, \alpha, P)$ be as in (2.5) and $R > \sqrt{n_0 / \epsilon} \vee n_0$. We only need to prove (2.2) for $\omega \in A_R^{(1)} \cap A_R^{(2)} \cap A_R^{(3)}$.

First we estimate $\max_{x \in B_R^{\text{dis}}} h(x)$. By Lemma 2.2, $\max_{x \in \partial(k) B_R^{\text{dis}}} h(x) \neq 0$ only if $\omega \notin A_n^{(x)}(x)$ or $x \in J_k := B_R^{\text{dis}} \setminus B_{R-k}^{\text{dis}}$. Note that $|J_k| \leq CkR^{d-1}$.

By Theorem 1.10, on the event $A_R^{(2)}$,

$$\max_{x \in B_R^{\text{dis}}} h(x) - \max_{x \in \partial(k) B_R^{\text{dis}}} h(x) \leq 6R \left( \sum_{x \in B_R^{\text{dis}}} 1_{x \in D_h} (-L^{(R)}_\omega h(x))_+ \right)^{\frac{1}{d}} \leq CR^2 \left( \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_n^{(x)}(x)} \right)^{\frac{1}{d+1}} + \left( \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} [L^{(R)}_\omega h(x)]^{d+1} \right)^{\frac{1}{d}}.$$ 

On the event $A_R^{(1)}$,

$$\sum_{x \in B_R^{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_n^{(x)}(x)} \leq |J_k| + \sum_{x \in B_R^{\text{dis}}} 1_{\omega \notin A_n^{(x)}(x)} \leq (C \frac{k}{R} + 2\alpha) |B_R^{\text{dis}}|.$$ 

Recall that $k = \sqrt{R}$. Hence, for $R > C/\alpha^2$,

$$\frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} 1_{x \in J_k \text{ or } \omega \notin A_n^{(x)}(x)} \leq 3\alpha.$$  

(2.12)
Applying (2.10) and Lemma 2.3 on the event $A_R^{(3)}$, by (2.12) and (2.11),
\[ \max_{x \in B_R^{(3)}} h(x) = \max_{x \in O_R^{(3)}} h(x) \leq CR^2(3\alpha)^{1/(d+1)}(C_F R^{-2}) = C_F \alpha^{1/(d+1)}d. \] (2.13)

Moreover,
\[ \max_{\partial(\kappa) B_R^{(3)}} h \leq \gamma(R + \kappa)^2 + \max_{\partial(\kappa) B_R^{(3)}} H' \]
\[ \leq \gamma(R + \kappa)^2 + \max_{\partial O_R^{(3)}} H' \leq C_F(\epsilon + \frac{\kappa}{R}), \]

where in the second inequality we used the fact that $H'$ is $\omega$-harmonic in $B_{R+k} \setminus B_R$ and that an $\omega$-harmonic function achieves its maximum in the boundary.

Therefore, from (2.11), (2.12) and (2.13), we have
\[ \max_{\partial O_R^{(3)}} H \leq \max_{\partial O_R^{(3)}} H' + \frac{C_F}{R} \leq \max h + \epsilon C_F \leq C_F(\epsilon + \frac{\kappa}{R}) + C_F \alpha^{1/qd}. \]

Replacing $F$ by $-F$, similar arguments also give the same upper bound for $\max_{\partial O_R^{(3)}} (-H)$.

Theorem 1.4 now follows by taking $\alpha = e^{(d+1)d}$ and $R \geq e^{-2} + n_0$. \[\square\]

We now prove Lemmas 2.1 and 2.3

**Proof of Lemma 2.1.** We start with estimating the probability of $A_R^{(2)}$. By Lemma 1.9 for every $x \in O_R^{(2)}$,
\[ Pr^x(T_1 > R^{1/2}) \leq e^{-CR^{1/6}}, \]
and thus by Markov’s inequality
\[ P \left( \{ \omega : P^\omega(T_1 > R^{1/2}) \geq e^{-(\log R)^3} \} \right) \leq e^{(\log R)^3 - CR^{1/6}}, \]
and a union bound over all possible values of $x$ yields
\[ P(A_R^{(2)}) \geq 1 - Ce^{-R^{1/7}}. \] (2.14)

Next we estimate the probability of the event $A_R^{(1)}$. Note that the event $A_{n_0}(x)$ is determined by $\omega_{|B||y - x| \leq n_0}$, and therefore $(A_{n_0}(x_i))_{i \in I}$ are independent events whenever \[ \forall i, j \in I \|x_i - x_j\| > 2n_0. \] For every $z \in [-n_0, n_0]^d$ we write $I_z = (z + (2n_0 + 1)Z^d) \cap O_R^{(2)}$. Then for every $z$ the events $(A_{n_0}(x_i))_{i \in I_z}$ are independent, and each happens with probability greater than or equal to $1 - \alpha$. Therefore by Chernoff’s inequality, for every $z \in [-n_0, n_0]^d$
\[ P \left( \sum_{x \in I_z} 1_{\omega \in A_{n_0}(x)} |I_z| > 1 - 2\alpha \right) > 1 - e^{-C|I_z|}. \]

remembering that we have only finitely many choices of $z$, and that $|I_z| = \Theta(R^d)$ for every $z$, a union bound gives us
\[ P(A_R^{(1)}) > 1 - e^{-CR^d}. \] (2.15)

The lemma now follows from (2.14) and (2.15).
Proof of Lemma 2.3. For $W = R^{1/2}$ we write $E_{\omega}^{x}[T_1] = E_1(x) + E_2(x)$ with $E_1(x) = E_{\omega}^{x}[T_1 \cdot 1_{T_1 \leq W}]$ and $E_2(x) = E_{\omega}^{x}[T_1 \cdot 1_{T_1 > W}]$. We separately control

$$\frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_1(x))^p \quad \text{and} \quad \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_2(x))^p.$$ 

To control the empirical $L^p$ norm of $E_1(x)$ we note that $E_1(x) \leq E_{\omega}^{x}[T_1]$, and that, as in the proof of Lemma 2.1, $E_1(x_1)$ and $E_1(x_2)$ are independent whenever $\|x_1 - x_2\| > 2W$. Therefore, following the same decomposition as in the proof of Lemma 2.1, for $K_1 = 2E \left[ \left( E_{\omega}^{x}(T_1) \right)^p \right]$ we get

$$P \left[ \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_1(x))^p > K_1 \right] < e^{-CR^{\delta'}} \quad (2.16)$$

with $\delta' > 0$ determined by $p$, by the choice of $W$ and by the dimension. We now turn to control $E_2(x)$. We note that by Lemma 1.9 $E \left[ E_{\omega}^{x}(T_1^2) \right] < \infty$, and that $E \left[ E_{\omega}^{x}(1_{T_1 > W}) \right] < e^{-cW^{1/3}} = e^{-cR^{1/6}}$. Thus by Cauchy-Schwarz,

$$E \left[ E_2(x) \right] \leq Ce^{-cR^{1/6}}. \quad (2.17)$$

From (2.17) we learn that

$$P \left[ \exists x \in B_R^{\text{dis}} \ E_2(x) > 1 \right] \leq |B_R^{\text{dis}}| \cdot e^{-cR^{1/6}} \leq e^{-R^{1/7}}. \quad (2.18)$$

From (2.18) we get that

$$P \left[ \frac{1}{|B_R^{\text{dis}}|} \sum_{x \in B_R^{\text{dis}}} (E_2(x))^p > 1 \right] \leq e^{-R^{1/7}}. \quad (2.19)$$

The lemma now follows from (2.16) and (2.19) once we remember that $E_{\omega}^{x}(T_1)^p \leq p(E_1(x)^p + E_2(x)^p)$ for all $x$. \hfill \square

One corollary of Theorem 1.4 is particularly useful for us. This is Corollary 2.4 below, which is a very slight generalization of Corollary 1.5.

For any $A \subset \partial B_1$ and $R \geq 1$, we define

$$\tilde{A}_R := \{ x \in \partial B_R^{\text{dis}} : x/|x|_2 \in A \}. \quad (2.20)$$

For $x \in \mathbb{R}^d$, we let $P_{\text{BM}}^{x}$ denote the law of the Brownian motion with limiting covariance matrix $\Sigma$ and starting point $x$. We may describe an event without mentioning the underlying Brownian motion. E.g., it should be clear what $P_{\text{BM}}^{x}(\text{exits the ball } B_1 \text{ through } A)$ means.
Corollary 2.4 Let \( A \subseteq \partial B_1 \) be open in the relative topology of \( \partial B_1 \). Assume also that the boundary of \( A \) w.r.t. the topology of \( \partial B_1 \) has measure zero w.r.t. the \((d - 1)\) dimensional Lebesgue measure on \( \partial B_1 \). For \( x \in \overline{B_1} \), let

\[
\chi_A(x) = P_{BM}^x(\text{exits } B_1 \text{ through } A)
\]

For \( \epsilon, r \in (0, 1) \) and \( R > 0 \), let

\[
G(A, R, r, \epsilon) = \left\{ \omega : \max_{x \in B_{VR}^\epsilon} |\chi_A(x) - P_{\omega}^x(X. \text{ exits } B_R^\text{dis} \text{ through } \tilde{A}_R)| \leq \epsilon \right\}.
\]

Then, there are constants \( c, C \) depending on \( A, r \) and \( \epsilon \), such that for every \( R \),

\[
P(G(A, R, r, \epsilon)) \geq 1 - C \exp(-cR^\delta).
\]

Proof. We fix \( \epsilon > 0 \). Recall the constant \( n_1 \) in Theorem 1.4. If suffices to prove the lemma for all \( R \geq n_1(\epsilon^4, P) \).

Our proof consists of several steps.

Step 1. First, we will define two functions \( F^{(1)}, F^{(2)} \) with smooth boundary data such that \( F^{(1)} \leq \chi_A \leq F^{(2)} \). For \( A \subseteq \partial B_1 \) and \( \epsilon \in (0, 1) \), we define subsets \( A^-_\epsilon, A^+_\epsilon \) of \( \partial B_1 \) as

\[
A^-_\epsilon = \{ x \in A : \text{dist}(x, \partial B_1 \setminus A) \geq \epsilon \},
\]

\[
A^+_\epsilon = \{ x \in \partial B_1 : \text{dist}(x, A) \leq \epsilon \}.
\]

Clearly, \( A^-_{2\epsilon} \subset A^- \subset A^+_\epsilon \subset A^+_2 \). We can construct two smooth functions \( f^{(\ell)} : \partial B_1 \to [0, 1], \ell = 1, 2 \) such that all of their \( i \)-th order partial derivatives have absolute values less than \( \epsilon^{-i} \) for \( i = 1, 2, 3 \), and

\[
\left\{ \begin{array}{l}
f^{(1)}|_{A^-_{2\epsilon}} = 1 \\
\sigma^{(1)}|_{\partial B_1 \setminus A^-} = 0
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
f^{(2)}|_{A^+_2} = 1 \\
\sigma^{(2)}|_{\partial B_1 \setminus A^+_2} = 0
\end{array} \right.
\]

Then \( f^{(1)} \) is supported on \( A^-_\epsilon \) and \( f^{(2)} \) is supported on \( A^+_2 \). Now for \( \ell = 1, 2 \), let \( F^{(\ell)} : \overline{B_1} \to [0, 1] \) be the solution of the Dirichlet problem

\[
\left\{ \begin{array}{l}
\sum_{i,j=1}^d \Sigma_{ij} \partial_{ij} F^{(\ell)} = 0 \\
F^{(\ell)} = f^{(\ell)}
\end{array} \right. \quad \text{in } B_1
\]

\[
\left\{ \begin{array}{l}
F^{(\ell)} = 0 \\
\text{on } \partial B_1.
\end{array} \right.
\]

Note that \( f^{(1)} \leq 1_A \leq f^{(2)} \) on \( \partial B_1 \) and so

\[
F^{(1)} \leq \chi_A \leq F^{(2)} \quad \text{in } \overline{B_1}.
\]

Note also that by the definitions of \( f^{(\ell)} \), we have for \( i = 1, 2, 3 \) and \( \ell = 1, 2 \),

\[
M^{i}_{F^{(\ell)}} \leq C/\epsilon^i,
\]

where \( M^{i}_{F^{(\ell)}} \) denotes the supremum of the absolute values of all \( i \)-th order derivatives of \( F \) over \( B_1 \). Moreover, for \( \ell = 1, 2 \),

\[
\sup_{B_r} |\chi_A - F^{(\ell)}| \leq \sup_{x \in B_r} P_{BM}^x(\text{exits } \partial B_1 \text{ from } A^+_2 \setminus A^-_{2\epsilon}) \xrightarrow{r \to 0} 0 \quad (2.21)
\]
Step 2. Next, we will define two ω-harmonic functions on $B_{R+1}^{\text{disc}}$ whose boundary values agree with that of $F^{(\ell)}$, $\ell = 1, 2$. Let $\mathcal{O}_{R+1}^{\text{disc}} = \{ x \in B_{R+1}^{\text{disc}} : y \in B_{R+1}^{\text{disc}} \text{ for all } y \sim x \}$. Note that $\mathcal{O}_{R+1}^{\text{disc}} = B_{R+1}^{\text{disc}}$. For $\ell = 1, 2$, let $G^{(\ell)}_{R+1} : B_{R+1}^{\text{disc}} \to [0, 1]$ be the solution of the Dirichlet problem

$$
\begin{cases}
    L_\omega G^{(\ell)}_{R+1} = 0 & \text{in } \mathcal{O}_{R+1}^{\text{disc}} \\
    G^{(\ell)}_{R+1} = F^{(\ell)}_{R+1} & \text{in } \partial \mathcal{O}_{R+1}^{\text{disc}}.
\end{cases}
$$

Recall that $F^{(\ell)}_{R+1}(x) := F^{(\ell)}(x/(R + 1))$ for $x \in B_{R+1}$. Then, for any $R \geq n_1(\epsilon^4, P)$, by Theorem 1.4, with probability at least $1 - C \exp(CR^6)$ we have

$$
\max_{B_{R+1}^{\text{disc}}} |F^{(\ell)}_{R+1} - G^{(\ell)}_{R+1}| \leq \epsilon^4(M^2_{F^{(\ell)}} + M^3_{F^{(\ell)}}) \leq C\epsilon, \quad \ell = 1, 2. \quad (2.22)
$$

Step 3. We will show for all $R \geq n_1(\epsilon^4, P) \geq \epsilon^{-8}$,

$$
F^{(1)}_{R+1} - \epsilon \leq 1 - \tilde{A}_R \leq F^{(2)}_{R+1} + C\epsilon \quad \text{on } \partial B_{R}^{\text{disc}}. \quad (2.23)
$$

First, for any $x \in \partial B_{R}^{\text{disc}} \setminus \tilde{A}_R$, since $R \geq \epsilon^{-8}$, we have $\text{dist}(\frac{x}{R+1}, \partial B_1 \setminus A^-_\epsilon) \leq 1/R \leq \epsilon^8$ and $\text{dist}(\frac{x}{R+1}, A^+_\epsilon) \geq C\epsilon$. Hence for $x \in \partial B_{R}^{\text{disc}} \setminus \tilde{A}_R$,

$$
1 - F^{(1)}_{R+1}(x) \geq F_{BM}^{x/R+1}(\text{exits } \partial B_1 \setminus A^-_\epsilon) \geq 1 - \epsilon^7,
$$

which implies that $F^{(1)}_{R+1} - \epsilon \leq 1 - \tilde{A}_R$ on $\partial B_{R}^{\text{disc}}$. The first inequality of (2.23) is proved.

Similarly, to obtain the second inequality of (2.23), notice that for any $x \in \tilde{A}_R$, $\text{dist}(\frac{x}{R+1}, A^+_\epsilon) \leq 1/R \leq \epsilon^8$ and $\text{dist}(\frac{x}{R+1}, \partial B_1 \setminus A^-_\epsilon) \geq C\epsilon$. Hence for $x \in \tilde{A}_R$,

$$
F^{(2)}_{R+1}(x) \geq F_{BM}^{x/R+1}(\text{exits } \partial A^+_\epsilon) \geq 1 - C\epsilon^7,
$$

which implies that $F^{(2)}_{R+1} + C\epsilon \leq 1 - \tilde{A}_R$ on $\partial B_{R}^{\text{disc}}$. Our proof of (2.23) is now complete.

Step 4. With (2.23), assuming (2.22) we have

$$
G^{(1)}_{R+1} - \epsilon \leq 1 - \tilde{A}_R \leq G^{(2)}_{R+1} + C\epsilon \quad \text{on } \partial B_{R}^{\text{disc}}
$$

and so on $B_{R}^{\text{disc}}$,

$$
G^{(1)}_{R+1} - \epsilon \leq P^x_{\omega}(X \text{ exits } B_{R}^{\text{disc}} \text{ through } \tilde{A}_R) \leq G^{(2)}_{R+1} + C\epsilon.
$$

This inequality, together with (2.21) and (2.22), yields

$$
\max_{x \in B_{R}^{\text{disc}}} |\chi_{A}(\frac{x}{R}) - P^x_{\omega}(X \text{ exits } B_{R}^{\text{disc}} \text{ through } \tilde{A}_R)| \leq C\epsilon.
$$

Recalling that (2.22) occurs with probability at least $1 - C \exp(CR^6)$, the corollary is proved.
3. Percolation estimates

In this section we study connectivity properties of the balanced directed percolation at $\omega$.

The main results of this section are the following two propositions.

**Proposition 3.1** There exists $\alpha > 0$ depending only on the dimension and a constant $C < \infty$ depending on $P$, s.t.

$$\mathbb{P}\{\text{there exists } x \text{ s.t. } 0=x, x \notin C \text{ and } \|x\| = k\} < Ck^d e^{-k\alpha}. \quad (3.1)$$

**Proposition 3.2** For $x, y \in \mathbb{Z}^d$ we define the distance $\text{dist}_{\omega}(x, y) \leq \infty$ as the length of a shortest $\omega$-path from $x$ to $y$. Note that in general $\text{dist}_{\omega}(x, y)$ may be different from $\text{dist}_{\omega}(y, x)$. Then for the same $\alpha > 0$ as in Proposition 3.1, and some constant $C < \infty$ depending on $P$, we have that for every $x$ and $y$

$$\mathbb{P}\{\text{dist}_{\omega}(x, y) > C\|x - y\| ; x, y \in C\} < C\|x - y\| e^{-\|x-y\|^\alpha}. \quad (3.2)$$

Unfortunately we need to provide different proofs for Proposition 3.1 in two dimensions and in larger dimensions. The reason is that our high-dimensional proof uses the fact that Bernoulli percolation in $d-1$ dimensions has a non-trivial critical value, whereas our 2-dimensional proof relies heavily on planarity.

### 3.1 Proof of Propositions 3.1 and 3.2 in three or more dimensions.

**Claim 3.3** Let $A(n)$ be the number of sinks in $[-n, n]^d$. Then a.s. there exists some $n_0$ such that $A(n+1) \leq A(n)$ for all $n > n_0$. In particular, the limit $A = \lim_{n \to \infty} A(n)$ exists a.s.

**Proof.** Let $n_0$ be so large that

1. For all $n > n_0$, every sink in $[-n, n]^d$ has density at least $\Phi$.
2. For all $n > n_0$,

$$\frac{|[-n - 1, n + 1]^d \setminus [-n, n]^d|}{|[-n - 1, n + 1]^d|} < \frac{\Phi}{2}. \quad (3.3)$$

Lemma 1.12 guarantees that almost surely $n_0 < \infty$.

Fix $n > n_0$. Then every sink in $[-n - 1, n + 1]^d$ intersects $[-n, n]^d$, and thus contains (at least one) sink in $[-n, n]^d$. Therefore $A(n+1) \leq A(n)$. \qed

We can also identify the value of the limit $A = \lim_{n \to \infty} A(n)$.

**Claim 3.4** $\lim_{n \to \infty} A(n) = 1$.

**Proof.** By Proposition 1.13 the infinite sink is unique. Assume for contradiction that $A > 1$. Then there exists some $N > n_0$ (where $n_0$ is from the proof of Claim 3.3) such that $A(n) = A > 1$ for all $n > N$. By induction on the previous argument, for every $n \geq N$, every sink in $[-n, n]^d$ intersects $[-N, N]^d$. Furthermore, every sink in $[N, N]^d$ is contained in a sink in $[-n, n]^d$, because $A(n) = A(N)$. Let $x_1$ and $x_2$ belong to two
distinct sinks in $[-N, N]^d$. Then the set of points (in $\mathbb{Z}^d$) reachable from $x_1$ is disjoint of the set of points reachable from $x_3$. This stands in contradiction to Lemma 1.14 which implies that every point in the (infinite) sink is reachable from all points in $\mathbb{Z}^d$, and is thus reachable from both $x_1$ and $x_2$. □

As an immediate consequence of Claim 3.4 we get the following Grimmett-Marstrand type lemma.

**Lemma 3.5**

$$\lim_{n \to \infty} P(A(n) = 1) = 1.$$  

Typically, the sink is ubiquitous in the cube. We make precise a weak sense of this statement.

**Lemma 3.6** Fix $k$ and $\epsilon$. For all $n$ large enough, with probability greater than $1 - \epsilon$ the following happens:

1. There is a unique sink in $[-n, n]^d$.
2. The sink intersects every sub-cube of side length $n/k$.

**Proof.** We already know that Item 1 holds w.h.p. for large enough $n$. To see that Item 2 holds too, we note that it is enough to intersect $(4k)^d$ cubes of side length $n/2k$. If $n$ is large enough then each of them has a unique sink, and by the same induction as in the proof of Claim 3.3, the sink in $[-n, n]^d$ intersects each of them (note that their number does not grow with $n$). □

Another important fact is the following.

**Claim 3.7** There exist $N_0 > N$ and $\xi > 0$ such that for all $n > N_0$, for every $x \in \partial[-n, n]^d$, the probability that $x$ is in the sink of $[-n, n]^d$ is greater than or equal to $\xi$.

To prove Claim 3.7, we first need some definitions.

For $z \in \mathbb{Z}^d$ and $n > 0$, we write $Q_n(z)$ for the cube $[-n, n]^d + (2n + 1)z$.

Now let $N$ be large enough for Lemma 3.6 to hold with $k = 10$ and let $\epsilon = \epsilon(d)$ be small enough for what follows, and let $n^* = 10N$. Then for every $z$, with probability greater than $1 - 3d\epsilon$, we have that $Q_{n^*}(z)$ contains a unique sink, that so does every nearest neighbour of $z$, and that the sinks are connected to each other. We call a cube satisfying these conditions good. Note that the event of goodness is independent beyond distance 2. Therefore, from the Liggett-Schonmann-Stacey theorem [11], we Corollary 3.8 below.

We write $C(Q_{n^*}(z))$ for the sink in the cube $Q_{n^*}(z)$. If there is more than one, we use some arbitrary scheme to choose one (in fact, we will only use this definition for good cubes, for which there is anyway only one sink).

**Corollary 3.8** If $n^*$ is large enough, the good cubes dominate Bernoulli site percolation which is supercritical for dimension $d - 1$ (remember that $d \geq 3$).

We write $\mathcal{D}$ for the infinite cluster of the percolation of good cubes. We write

$$\hat{\mathcal{C}} = \bigcup_{z \in \mathcal{D}} \mathcal{C}(Q_n^*(z)),$$

and note that $\hat{\mathcal{C}} \subseteq \mathcal{C}$.

**Proof of Claim 3.8** Let $n^*$ be as in Corollary 3.8 and assume w.l.o.g. that $(2n^* + 1) \mid (2n + 1)$. Break $Q_n(0)$ into cubes $(Q_n^*(z))_{z \in \mathcal{N}}$, and let $z_0$ be such that $x \in Q_n^*(z_0)$. Then with probability bounded away from zero in $n$, the good cubes in $\mathcal{N}$ have a giant component $\hat{\mathcal{C}}$ and conditioned on this event, with probability greater than or equal to $\kappa(2N+1)^d$ there is a path from this component to the point $x$. Under event, whose probability is bounded away from zero in $n$, the point $x$ is in the sink. □

We now advance towards proving Proposition 3.1. We start by proving the bound for connection in the other direction.

**Lemma 3.9** For $z \in \mathbb{Z}^d$, write $Ψ_\omega(z) := \{x : x \leq z\}$. Then there exists $\alpha > 0$ such that

$$P \left[ |Ψ_\omega(0)| > k \text{ and } Ψ_\omega(z) \cap \hat{\mathcal{C}} = \emptyset \right] < Ce^{-k^\alpha}. \quad (3.4)$$

**Proof.** Let $Ψ_\omega^{(n^*)} := \{z : Ψ_\omega(0) \cap Q_n^*(z) \neq \emptyset\}$.

We enumerate the set $Ψ_\omega^{(n^*)}$ using a breadth-first-search algorithm, as follows. First we arbitrarily enumerate $\mathbb{Z}^d$, i.e. take a bijection $\pi : \mathbb{N} \to \mathbb{Z}^d$ with $\pi(0) = 0$. Then we define a sequence $(a_n)$ in $\mathbb{Z}^d \cup \{\infty\}$ inductively as follows.

We begin by writing $a_0 = 0$, and $Φ_0 := \{x \leq 0 \text{ in } Q_n^*(0)\}$, i.e. the set of points $x$ s.t. there is a directed path from $x$ to $0$ which is contained in $Q_n^*(0)$.

Given $a_0, \ldots, a_n$ and $Φ_n$, we define $a_{n+1}$ and $Φ_{n+1}$ as follows. If $a_n = \infty$ then $a_{n+1} = \infty$ and $Φ_{n+1} = Φ_n$. Otherwise, take

$$k_{n+1} = \inf \left\{ k : \pi(k) \notin \{a_0, \ldots, a_n\} \text{ s.t. } \exists x \in Q_n^*(\pi(k)) \ x \leq 0 \text{ in } Q_n^*(\pi(k)) \cup \left( \bigcup_{j=0}^{n} Q_n^*(a_j) \right) \leq \infty, \right\}$$

and $a_{n+1} = \left\{ \begin{array}{ll} \pi(k_{n+1}) & k_{n+1} < \infty \\ \infty & k_{n+1} = \infty. \end{array} \right.$

We then take

$$Φ_{n+1} = \left\{ x : x \leq 0 \text{ in } \bigcup_{j=0}^{n+1} Q_n^*(a_j) \right\}$$

if $a_{n+1} \neq \infty$ and $Φ_{n+1} = Φ_n$ if $a_{n+1} = \infty$.

We define a filtration $(F_n)_{n \geq 0}$ by

$$F_n = σ(a_0, \ldots, a_n, \omega|_{\bigcup_{j=0}^{n} Q_n^*(a_j)}).$$

Fix a coordinate $i = 1, \ldots, d$, and define the stopping times

$$T_0 = 0 ; \ T_{n+1} = \inf \{ j > T_n : (a_j, i) \notin \{(a_k, i) : k = 0, \ldots, T_n\} \}.$$ 

We write $W_\omega^{(i)} = \sup\{ n : T_n < \infty \} \leq \infty.$
Let $F_n$ be the event that $Q_n(a_n)$ is a good cube, $\Phi_n \cap C(Q_n(a_n)) \neq \emptyset$ and that $a_n$ belongs to the infinite component in the hyperplane $\{z : \langle z, i \rangle = \langle a_n, i \rangle \}$. Note that if there exists $n$ such that $F_n$ occurs then $\Psi_\omega(z) \cap  \hat{C} \neq \emptyset$.

Next we note that by Claim 3.7 and Corollary 3.8, $P[F_{T_n}|F_n; W^{(i)}_\omega \geq n] \geq \rho$ for some $\rho > 0$.

Therefore for all $i$ and $n$,

$$P\left(W^{(i)}_\omega \geq n \text{ and } 0 \notin C\right) < \rho^n.$$

Note that if $\Psi^{(n^*)}_\omega > n$ then there exists $i$ such that $W^{(i)}_\omega > n^{1/d}$, which implies the lemma with $\alpha < 1/d$.

□

Using the fact that $\hat{C} \subseteq C$, and that if $x \in C$ and $x \neq 0$ then $0 \in C$, we get the following corollary to Lemma 3.9.

**Corollary 3.10** there exists $\alpha > 0$ such that

$$P \left[|x : x \neq 0| > k \text{ and } 0 \notin C\right] < Ce^{-k\alpha}. \quad (3.5)$$

**Proof of Proposition 3.1.** Using Corollary 3.10

$$P \left[\text{there exists } x \text{ s.t. } 0 \leq x, \ x \notin C \text{ and } |x| = k\right] \leq \sum_{x:|x|=k} P\left[0 \leq x, \ x \notin C\right] = |\partial[-k,k]^d|P\left[|y : y \neq 0| > k \text{ and } 0 \notin C\right] \leq Cke^{-k\alpha}. \quad (3.6)$$

**Proof of Proposition 3.2.** First note that by The Antal-Pisztora theorem (Theorem 1.1 of [1]) there exists $C$ and $\gamma$ such that

$$P\left[\text{dist}_\omega(z,w) > C|z-w| \ ; \ z,w \in \hat{C}\right] < Ce^{-\gamma|z-w|}. \quad (3.7)$$

Next we note that $|\Psi_\omega(y)| = \infty$ for $y \in C$, and thus by Lemma 3.9 we get that

$$P\left[\exists w \in \hat{C}\text{dist}_\omega(w,y) > k \ ; \ y \in C\right] < Ck^d e^{-k\alpha}. \quad (3.8)$$

We need to show that for all $x$,

$$P\left[\exists z \in \hat{C}\text{dist}_\omega(x,z) > k \ ; \ x \in C\right] < e^{-\gamma k}. \quad (3.9)$$

To this end we write $\mathbf{1}$ for the vector $(1,1,\ldots,1) \in \mathbb{Z}^d$ and then define a sequence $(x_n)_{n \geq 0}$ as follows. $x_0 = x$. Given $x_n$ we choose $x_{n+1}$ as a nearest neighbor of $x$ satisfying

1. $\langle x_{n+1}, \mathbf{1} \rangle > \langle x_n, \mathbf{1} \rangle$, and
2. $x_n \neq x_{n+1}$.

If there is more than one such neighbor, we apply some arbitrary scheme to choose one. The existence of $x_n$ is guaranteed by the balancedness of the environment. Write $D(j)$ for the union of the sinks in the boxes in the connected component of the Bernoulli
percolation on \( \{ z : |⟨z, 1⟩ - j| \leq 1 \} \). Then, the events \( (E_j = x_{3jn^*} \in D(j))_{j \geq 1} \) are independent and of positive probability. (3.8) follows.

Proposition 3.2 now follows from (3.6), (3.7) and (3.8).

3.2 Proof of Propositions 3.1 and 3.2 in two dimensions.

We write \( x \sim y \) if \( |x - y|_1 = 1 \). A sequence \( (x_i)_{i=0}^n \) is called a path if \( x_0 \sim x_1 \ldots \sim x_n \). A subset \( S \subset \mathbb{Z}^d \) is said to be connected if for any \( x, y \in S \), there is a path \( (x_i)_{i=0}^n \subset S \) with \( x_0 = x \) and \( x_n = y \).

**Proof of Proposition 3.1:** First, we will show that the “holes” outside of the sink are rectangles.

Let \( C \) be a connected (in the sense of \( \sim \)) component of \( \mathbb{Z}^2 \setminus C \). For the sake of clarity, we color the unit square centered at \( x \) (with sides parallel to the lattice) with white color if \( x \notin C \), and with blue color if \( x \in C \). Now consider the interface between the white and blue areas. The border of the blue area may consist of straight lines and angles with degrees 90° or 270°. See Figure 2. However, case (C) in Figure 2 is impossible, since every point in the sink has at least two neighbors in opposite sides that are in \( C \).

Therefore, the border of \( C \) consists of only straight lines and right angles. In other words, it is a rectangle.

The rectangle is finite w.p. 1, because the probability of an infinite line with no bond orthogonal emanating from it is zero, and there are countably many such lines.

The probability for a given rectangle \( L \) to be a connected component of \( \mathbb{Z}^2 \setminus C \) is exponentially small in the length of the boundary of \( L \). This proves Proposition 3.1 in two dimensions with \( \alpha = 1/2 \).

**Proof of Proposition 3.2:** When the dimension \( d = 2 \), our proof of Proposition 3.2 consists of several steps. In Steps 1–3, we define and estimate several geometric quantities in \( \omega \). In Step 4 we will prove Proposition 3.2 using these geometric estimates.

**Step 1.** First, we define a few terms. For a fixed environment \( \omega \), the ES-stair (ES stands for east-south) is defined to be the infinite path starting from \( o \) which goes first vertically upwards until it has the possibility to move right, and then it takes every opportunity to move right and moves downwards if a step to the right is not possible. The part of the ES-stair above the horizontal line is called the ES-path. See Figure 3. We estimate the length \( L^{ES}(\omega) \) of the ES-path. To do this,
we set $V_0 = V_0(\omega) = \inf\{n \geq 0 : \omega(ne_2, e_1) > 0\}$, $H_0 := 1$, and define recursively for $j \geq 0$,

$$H_j := \inf \{n > 0 : \omega((n + H_0 + \ldots + H_{j-1})e_1 + (V_0 - j)e_2, e_2) > 0\}.$$ 

We have the length of the ES-path

$$L^{ES}(\omega) = 2V_0 + \sum_{j=0}^{V_0-1} H_j.$$ 

Observe that $V_0, H_1, H_2, \ldots$ are independent (under $P$) geometric random variables, and $(H_j)_{j=1}^\infty$ are identically distributed. Hence we conclude that $L^{ES}$ has exponential tail. That is, for $x \geq 0$,

$$P(L^{ES} > x) \leq Ce^{-cx}.$$ 

Similarly, we can define the EN-stair, EN-path and $L^{EN}$. The shorter one among the EN- and ES-path (If $L^{ES} = L^{EN}$ then take the EN-path) is simply called the E-path. The E-path has length

$$L^E = L^E(\omega) := L^{EN} \wedge L^{ES},$$

which also has exponential tail

$$P(L^E > x) \leq Ce^{-cx}, \quad \forall x \geq 0. \quad (3.9)$$

**Step 2.** The set of vertices that lie below (or on) the ES-stair and above (or on) the EN-stair is called the E-bubble, denoted by $B^E(\omega)$. In other words, $B^E(\omega)$ is the area enclosed by the EN- and ES-stairs. Clearly, $\#B^E(\omega) \leq (L^{ES} + L^{EN})^2$ and so it has stretched exponential tail. That is, for $x \geq 0$,

$$P(\#B^E > x) \leq Ce^{-c\sqrt{x}}. \quad (3.10)$$

Here $\#B^E$ denotes the cardinality of the set $B^E$.

**Step 3.** Now we will define the east-tadpole. We denote the E-path by $p^E(\omega)$ and its end-point by $R^E(\omega) \in Ne_1$. Set $R^E_0 = 0$ and define for $j \geq 0$

$$R^E_{j+1} := R^E_j + E(\theta_{R^E_j}\omega).$$

In other words, $R^E_j$ is end-point of the concatenation $p^E_j(\omega) := \bigcup_{i=0}^{j-1} p^E(\theta_{R^E_i}\omega)$ of $j$ consecutive E-paths. For any $n \geq 0$, let

$$M(n) := \inf\{i \geq 1 : R^E_i \cdot e_1 \geq n\}$$

and define the east-tadpole $T^E(n) = T^E_m(n)$ to be the union of the first $M(n) - 1$ E-paths and the $M(n)$-th E-bubble. Namely,

$$T^E(n) := p^{E}_{M(n)-1}(\omega) \bigcup B^E_j(\theta_{R^E_j}\omega).$$

Since $M(n) \leq n$, we have

$$\#T^E(n) \leq \sum_{i=0}^{n-1} L^E(\theta_{R^E_i}\omega) + \max_{j=0,\ldots,n-1} \#B^E_j(\theta_{R^E_j}\omega),$$
Figure 3. The ES-(EN-) path is marked with a solid red (blue) line. The shaded region is an E-bubble.

where the right side is a sum of 1-dependent random variables. It then follows from (3.9) and (3.10) that for \( n \in \mathbb{N} \),

\[
P(\#T^E(n) > Cn) \leq Ce^{-c\sqrt{n}}.
\]

As the most important property of the tadpole, notice that for any \( x \in T^E_\omega(n) \), if \( o \preceq x \), then we can find a \( \omega \)-path in \( T^E(n) \) from \( o \) to \( x \).

Step 4. Finally, we are ready to prove the theorem. Without loss of generality, assume that \( x = o \) and \( y = (y_1, y_2) \) with \( y_1, y_2 \geq 0 \). Let \( z = (y_1, 0) \). We define the north-tadpole \( T^N(z) \) similarly as in Step 3. By the remark at the end of Step 3 if \( o \preceq y \), then we can find a \( \omega \)-path in \( T^E(y_1) \cup T^N(y_2) \) from \( o \) to \( y \). Therefore,

\[
P[\text{dist}_\omega(o, y) > C\|y\| : o \preceq y] \\
\leq P(\#T^E(y_1) + \#T^N_\theta(y_2) \geq C(|y_1| + |y_2|)] \\
\leq Ce^{-c\sqrt{n}}.
\]

\[\Box\]

3.3 Relation to Harmonic functions.

From Propositions 3.1 and 3.2 we can now prove a useful corollary regarding harmonic functions. Note that in uniformly elliptic cases the following statement is trivial. We note that in the non uniformly elliptic setting there are genuinely \( d \)-dimensional, finite-range dependent counter examples to the following corollary.

Corollary 3.11 For every \( \epsilon > 0 \) there exists \( m^* \) and a constant \( C' \) such that with probability greater than or equal to \( 1 - \epsilon \), for every non-negative \( \omega \)-harmonic function \( h \) on \([-2m^*, 2m^*]^d\), we have

\[
\max\{h(x) : x \in [-m^*, m^*]^d\} \leq C' \min\{h(x) : x \in [-m^*, m^*]^d\}. \tag{3.11}
\]

In particular, if we denote by \( P^x_\omega \) the hitting probability of the quenched random walk in \( \partial[-2m^*, 2m^*]^d \), namely \( P^x_\omega(z) := P^x_\omega(X_{T^\partial[-2m^*, 2m^*]^d} = z) \) for all \( z \in \partial[-2m^*, 2m^*]^d \), then with probability greater than \( 1 - \epsilon \),

\[
\max\{\|P^x_\omega - P^y_\omega\|_{TV} : x, y \in [-m^*, m^*]^d\} \leq 1 - 1/C'. \tag{3.12}
\]
Proof. Fix $m$. Let $\xi = \xi(m)$ be so small that $P[\exists z \in [-m,m]^d, \omega(z,e) \in (0,\xi)] < m^{2d}e^{-m^\alpha}$.
By Proposition 3.2 with probability at least $1 - m^{2d}e^{-m^\alpha}$, for every $x$ and $y$ in $[-3m/2, 3m/2]^d$ we have $\text{dist}_\omega(x,y) < Cm$, and therefore
\[
\max\{h(x) : x \in [-3m/2, 3m/2]^d \cap C\} \leq \xi^{-Cm} \min\{h(x) : x \in [-3m/2, 3m/2]^d \cap C\}.
\]
For a random walk in the random environment $\omega$, let $T_1$ be the first time it is in $C$, and $T_2$ the first time it is outside $[-3m/2, 3m/2]^d$. We note that by Lemma 3.9 with probability at least $1 - m^{2d}e^{-m^\alpha}$, for every $x \in [-m, m]^d$, we have $P_\omega(T_1 < T_2) = 1$ and thus for all $x \in [-m, m]^d$,
\[
\min\{h(x) : x \in [-3m/2, 3m/2]^d \cap C\} \leq h(x) \leq \min\{h(x) : x \in [-3m/2, 3m/2]^d \cap C\}.
\]
Equation (3.11) now follows if we take $m^\alpha$ such that $3m^{2d}e^{-m^\alpha} < \epsilon$ and $C' = \xi^{-Cm^\alpha}$. (3.12) follows from (3.11) and the fact that $P_\omega^\gamma$ is a harmonic function of $x$.

\[\square\]

4. An oscillation inequality

The goal of this section is to obtain an oscillation estimate (Theorem 4.1) for $\omega$-harmonic functions.

For any finite subset $E \subset \mathbb{Z}^d$ and any function $u : E \to \mathbb{R}$, we define the oscillation of $u$ over the set $E$ by
\[
\text{osc}_E u := \max_{x,y \in E} \{u(x) - u(y)\}.
\]
For $z \in \mathbb{Z}^d$, $0 < \alpha < 1 < \gamma < \infty$ and $R > 0$, we let $H_{z,R}^{\alpha,\gamma}$ denote the event that every $\omega$-harmonic function $f : B_{\gamma R}^{\text{dis}}(z) \to \mathbb{R}$ satisfies
\[
\frac{\text{osc}_{B_{\gamma R}^{\text{dis}}(z)} f}{\text{osc}_{B_{\gamma R}^{\text{dis}}(z)} f} \leq \alpha.
\]

Theorem 4.1 There exist constants $\Psi$ and $\upsilon < 1$ such that $1 - P(H_{0,R}^{\upsilon,\Psi})$ decays stretched exponentially with $R$.

The proof uses techniques of probability coupling. For any $R > 0$, $z \in \mathbb{Z}^d$, define the hitting time of the inner-boundary of the ball $B_{\gamma R}^{\text{dis}}(z)$ as
\[
\tau_{R,z} = \tau_{R,z}(X) = \inf\{n \geq 0 : X_n \in \partial(\mathbb{Z}^d \setminus B_{\gamma R}^{\text{dis}}(z))\}.
\]
The underlying process of the stopping time $\tau_{R,z}$ should be understood from the context. For instance, the subscripts of $X_{\tau_{R,z}}$ and $Y_{\tau_{R,z}}$ represent two different stopping times $\tau_{R,z}(X)$ and $\tau_{R,z}(Y)$, respectively.

Our observation is that the oscillation estimate (4.1) will follow if for every $\omega \in \Omega$ and any $x, y \in B_{\gamma R}^{\text{dis}}(z)$, there is a coupling of two paths $(X_n), (Y_n)$ in $B_{\gamma R}^{\text{dis}}(z)$ such that

(a) The marginal distributions of $(X_n)$ and $(Y_n)$ are $P_x^{\omega}$ and $P_y^{\omega}$, respectively. With abuse of notation, we use $P_x^{\omega,y}$ to denote the joint law of $(X_n, Y_n)_{n \geq 0}$.

(b) $P_x^{\omega,y}(X_{\tau_{R,z}} = Y_{\tau_{R,z}}) > 1 - \alpha$. 

Indeed, for any $\omega$-harmonic function $f$, $f(X_n)$ is a martingale under the quenched law $P^x_\omega$. Hence, by the optional stopping theorem, for any $x, y, z \in B^\text{dis}_R (z)$,
\[
\begin{aligned}
f(x) - f(y) &= E^x_\omega [f(X_{\tau_{R, z}}) - f(Y_{\tau_{R, z}})] \\
&\leq P^x_\omega (X_{\tau_{R, z}} \neq Y_{\tau_{R, z}}) \text{ osc } f_{B^\text{dis}_R (z)} \\
&\leq \alpha \text{ osc } f_{B^\text{dis}_R (z)}.
\end{aligned}
\]

We start by describing a multi scale structure.

4.1 Multi scale structure.

We fix three (large) parameters $R_0, M$ and $K$ and one (small) parameter $\tilde{\epsilon}$ whose values will be determined later. We now say that $K$ needs to be an even number, and require that $M > 100$. Further requirements will come later.

Let $\Gamma = \{A_1, A_2, \ldots, A_k\}$ be a covering of $\partial B_1 (0)$ by closed sets intersecting only in their boundaries (in the $\partial B_1 (0)$ topology) and having relative boundaries with measure zero, such that the diameter of each of them is smaller than $1/M^2$.

For a ball $B = B^\text{dis}_R (x)$ and a point $y \in B^\text{dis}_R (x)$, we denote by $\mathcal{D}_\omega (B, y)$ the distribution on the set $\{1, \ldots, k\}$ with $\mathcal{D}_\omega (B, y) (j) = P^y_\omega (T_{\partial B^\text{dis}_R (x)} = T_{RMA_j})$. We write $\mathcal{D}_\Sigma (B, y)$ for the distribution on the set $\{1, \ldots, k\}$ with $\mathcal{D}_\Sigma (B, y) (j) = P^y_{BM} (T_{\partial B^\text{dis}_R (x)} = T_{RMA_j})$ where $P_{BM}$ is the distribution of Brownian Motion with covariance matrix $\Sigma$.

We now define the notion of goodness of a ball $B^\text{dis}_R (x)$.

**Definition 4.2**

1. If $R \leq R_0$, we say that the ball $B = B^\text{dis}_R (x)$ is good if it satisfies the event in (3.12).
2. If $R > R_0$, we say that the ball $B = B^\text{dis}_R (x)$ is good if for every $y \in B^\text{dis}_R (x)$,
\[
\|\mathcal{D}_\omega (B, y) - \mathcal{D}_\Sigma (B, y)\|_{TV} < \tilde{\epsilon}.
\]

The following claim follows from Corollaries 3.11 and 2.4.

**Claim 4.3** There exists $\delta$ such that for every $x$ and $R$, the probability that the ball $B^\text{dis}_R (x)$ is good is at least $1 - \exp(-R^6)$.

We can now define our multi scale structure. We will recursively define the notion of an admissible ball. Claim 4.3 will then help us estimate the probability that a ball is admissible. We start with setting the scales. $R_0$ is given to us. We then define
\[
R_k := R^K_{k-1}
\]
for $k \geq 1$.

For now, we only define admissibility for balls of radius $R_k, k = 0, 1, 2, \ldots$. To define admissibility, we first choose a parameter $\nu < \delta/K$, where $\delta$ is as in Claim 4.3.

**Definition 4.4**

1. A ball of radius $R_0$ is called admissible if it is good.
2. A ball of radius $R_k, k \geq 1$ is called admissible if
(a) Every sub ball of radius larger than $R_{k-1}$ is good, and
(b) there are at most $R_k^\nu$ non-admissible sub balls of radius $R_{k-1}$.

We now estimate the probability that a ball of radius $R_k$ is not admissible. We denote by $\mathcal{A}(x, k)$ the event that $B_{R_k}^{\text{dis}}(x)$ is admissible.

**Lemma 4.5** For every $x$ and $k$, the probability that $B_{R_k}^{\text{dis}}(x)$ is not admissible is bounded by $e^{-R_k^\nu/2}$.

**Proof.** For $k = 0$ this follows from the fact that $\nu < \delta$. For $k \geq 1$ we prove the lemma by induction. Let $A$ be the event that there exists a sub ball of $B_{R_k}^{\text{dis}}(x)$ of radius larger than $R_{k-1}$ which is not good, and let $B$ be the event that there are more than $R_k^\nu$ non-admissible sub balls of radius $R_{k-1}$. We estimate the probabilities of $A$ and $B$.

We start with estimating the probability of $A$. There are less than $(2R_k)^d + 1$ sub balls of size greater than $R_{k-1}$ in $B_{R_k}^{\text{dis}}(x)$, and each of them is bad with probability less than $e^{-R_{k-1}^\nu}$. Thus

$$\Pr(A) \leq (2R_k)^d e^{-R_k^\nu}.$$  

We continue with estimating the probability of $B$. To this end we partition $B_{R_k}^{\text{dis}}(x)$ into $(MR_{k-1})^d$ subsets $L_1, L_2, \ldots, L_{(MR_{k-1})^d}$ such that for every $j$ and every $z, y \in L_j$, the events $\mathcal{A}(z, k-1)$ and $\mathcal{A}(y, k-1)$ are independent. For given $j$, we write

$$U(j) = \sum_{z \in L_j} (1 - \mathbf{1}_{\mathcal{A}(z, k-1)}).$$

Then $U(j)$ is a binomial $(|L_j|, \Pr(\mathcal{A}(0, k-1)))$ random variable, and by the induction hypothesis is dominated by a binomial $((2R_k)^d, e^{-R_{k-1}^\nu/2})$ variable. Let $\nu - \nu/2K < \nu' < \nu$ Then, For every $j$,

$$\Pr(U(j) > R_k^\nu') \leq (2R_k)^d R_k^{\nu'} e^{-R_k^\nu/R_{k-1}^\nu/2} \leq e^{-0.9R_k^\nu},$$

and so

$$\Pr(B) \leq (MR_{k-1})^d e^{-0.9R_k^\nu}.$$  

Thus,

$$\Pr(\mathcal{A}^c(x, k)) \leq \Pr(A) + \Pr(B) \leq (2R_k)^d+1 e^{-R_k^\nu} + (MR_{k-1})^d e^{-0.9R_k^\nu} \leq e^{-R_k^\nu/2}.$$  

\[\square\]

Before we continue with the proof of the oscillation inequality, we define admissibility also for balls whose radius is not exactly $R_k$ for some $k$.

**Definition 4.6** Let $R > R_0$. Let $k$ be such that $R_k < R < R_{k+1}$. A ball of radius $R$ is called **admissible** if

1. every sub ball of radius $R_k$ is admissible according to definition 4.4, and
2. every sub ball of radius greater than $R_k$ is good.

As a corollary of Lemma 4.5 we get the following corollary.
Corollary 4.7  For every $x$ and $R \geq R_0$, the probability that $B_{R}^{\text{dis}}(x)$ is not admissible is bounded by $e^{-R^2/2K}$.

4.2 The coupling.

In this subsection we define a coupling that will be the main tool in proving the oscillation inequality. We start with a notion of a basic coupling, and will afterwards compose the coupling from many basic couplings.

Definition 4.8  Let $R \geq R_0$ and let $y$ and $z$ be points in $B_{R}^{\text{dis}}(x)$. The basic coupling $\mu^{(x,R;z,y)}$ is a joint distribution of two times (i.e. natural numbers) $\tilde{T}_y$ and $\tilde{T}_z$ and two paths $(\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{T}_y})$ starting at $y$ and $(\tilde{Z}_1, \ldots, \tilde{Z}_{\tilde{T}_z})$ started at $z$ sampled as follows.

(1) If $R > R_0$, then $(\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{T}_y})$ and $(\tilde{Z}_1, \ldots, \tilde{Z}_{\tilde{T}_z})$ are sampled as random walks in the environment $\omega$, starting respectively at $y$ and $z$, with $\tilde{T}_y$ and $\tilde{T}_z$ being the respective stopping times of reaching $\partial B_{R}^{\text{dis}}(x)$, where the two walks are coupled in a way that maximizes the probability that $\tilde{Y}_{\tilde{T}_y}$ and $\tilde{Z}_{\tilde{T}_z}$ are in the same element of $\{x + RMA_1, x + RMA_2, \ldots, x + RMA_k\}$.

(2) If $R = R_0$ then $(\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{T}_y})$ and $(\tilde{Z}_1, \ldots, \tilde{Z}_{\tilde{T}_z})$ are sampled as random walks in the environment $\omega$, starting respectively at $y$ and $z$, with $\tilde{T}_y$ and $\tilde{T}_z$ being the stopping times of reaching $\partial B_{R}^{\text{dis}}(x)$, where the two walks are coupled in a way that maximizes the probability that $\tilde{Y}_{\tilde{T}_y} = \tilde{Z}_{\tilde{T}_z}$.

On good balls, the basic coupling has a relatively good success probability, as evident by the following lemma, which follows immediately from the definition of good balls.

Lemma 4.9  Let $B_{R}^{\text{dis}}(x)$ be a good ball, and let $y, z \in B_{R}^{\text{dis}}(x)$.

(1) If $R > R_0$ then

$$\mu^{(x,R;z,y)}\left(||\tilde{Y}_{\tilde{T}_y} - \tilde{Z}_{\tilde{T}_z}|| < R/M\right) > 1 - \frac{1}{M} - 2\bar{c}.$$  

(2) If $R = R_0$ then

$$\mu^{(x,R;z,y)}\left(\tilde{Y}_{\tilde{T}_y} = \tilde{Z}_{\tilde{T}_z}\right) > C',$$ 

where $C'$ is as in Corollary 4.7.

We now concatenate basic couplings, and get the following construction. Let $R$ be $R_0$ multiplied by a power of $M$, and let $y, z \in B_{R}^{\text{dis}}(x)$. We will define $\mu^{(x,R;z,y)}$ as a joint distribution of a random walk $(Y_n)$ starting at $y$, a random walk $(Z_n)$ starting at $z$, two sequences of stopping times $T_{y}^{(m)}$ and $T_{y}^{(m)}$, two sequences of points $(y_m)$ and $(z_m)$, and a sequence of balls $(B_{R}^{\text{dis}}(x_m))$.

To define $\mu^{(x,R;z,y)}$, we first construct a coupling, and then take $\mu^{(x,R;z,y)}$ to be its distribution. We write $x_0 = x$; $y_0 = y$; $z_0 = z$ and $R^{(0)} = R$, and also $T_{y}^{(0)} = T_{z}^{(0)} = 0$. We also start the two random walks at the points $Y_0 = y$ and $Z_0 = z$. 
Inductively, for \( m = 1, 2, \ldots \), we now sample \( y_m, z_m, x_m, R^{(m)} = T_y^{(m)}, T_z^{(m)} \) and 
\[
\left( Y_n \right)_{n = T_y^{(m-1)}}^{T_y^{(m)}} \quad \text{and} \quad \left( Z_n \right)_{n = T_z^{(m-1)}}^{T_z^{(m-1)}},
\]
assuming that we already sampled \( y_{m-1}, z_{m-1}, x_{m-1}, \) 
\( R^{(m-1)}, T_y^{(m-1)}, T_z^{(m-1)} \) and \( \left( Y_n \right)_{n=0}^{T_y^{(m-1)}} \) and \( \left( Z_n \right)_{n=0}^{T_z^{(m-1)}} \).

We sample \( \tilde{T}_y, \tilde{T}_z \) and the two paths \( (\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{T}_y}) \) and \( (\tilde{Z}_1, \ldots, \tilde{Z}_{\tilde{T}_z}) \) according to 
\[
\tilde{\mu}(x_{m-1}, R^{(m-1)}; y_{m-1}, y_{m-1}) \quad \text{We then assign:}
\]
\[
T_y^{(m)} := T_y^{(m-1)} + \tilde{T_y}; \quad T_z^{(m)} := T_z^{(m-1)} + \tilde{T_z} \quad \text{for} \quad y_m = \tilde{Y}_{\tilde{T}_y}; \quad z_m = \tilde{Z}_{\tilde{T}_z},
\]
as well as 
\[
Y_n = \tilde{Y}_{n - T_y^{(m-1)}} \quad \text{for} \quad n = T_y^{(m-1)}, \ldots, T_y^{(m)} \quad \text{and} \quad Z_n = \tilde{Z}_{n - T_z^{(m-1)}} \quad \text{for} \quad n = T_z^{(m-1)}, \ldots, T_z^{(m)}.
\]

To determine \( x_m \) and \( R^{(m)} \), we need to consider two different cases.

1. If \( R^{(m-1)} > R_0 \), then if \( \|y_m - z_m\| < R^{(m-1)}/M \) then we take \( R^{(m)} = R^{(m-1)}/M \), 
   and \( x_m \) such that \( y_m, z_m \in B^{\text{dis}}_{R^{(m)}}(x_m) \). Else we take \( R^{(m)} = MR^{(m-1)} \) and 
   \( x_m = x_{m-1} \).

2. If \( R^{(m-1)} \leq R_0 \), we stop the process (i.e. we stop the process one step after we reached a radius smaller than or equal \( R_0 \)).

Write \( F_m \) for the \( \sigma \)-algebra generated by the environment \( \omega \) and by \( (y_k, z_k, x_k, R^{(k)}, T_y^{(k)}, T_z^{(k)})_{k \leq m} \)

as well as \( (Y_n)_{n \leq T_y^{(m)}} \) and \( (Z_n)_{n \leq T_z^{(m)}} \).

We define 
\[
L_m := \frac{\log (R^{(m)}/R_0)}{\log M}.
\]

Note that \( (L_m)_m \) is a random process whose step size is 1. If \( R^{(m)} > R_0 \), and the ball 
\( B^{\text{dis}}_{R^{(m)}}(x_m) \) is good, then by Lemma 4.9
\[
\mu^{(x, R; z, y)} \left( L_{m+1} = L_m - 1 \middle| F_m ; B^{\text{dis}}_{R^{(m)}}(x_m) \text{ is good and } R^{(m)} > R_0 \right) > 1 - \frac{1}{M} - 2\bar{\epsilon}. \tag{4.2}
\]

At this point we choose \( M \) and \( \bar{\epsilon} \) so that \( 1 - \frac{1}{M} - 2\bar{\epsilon} > 2/3 \). For any \( l > 0 \) we let \( \mathbb{T}(l) \) be the stopping time 
\[
\mathbb{T}(l) := \inf \{ m : L_m \leq l \}.
\]

Then from (4.2) we get domination by a biased one dimensional random walk which gives us the following estimate.

**Lemma 4.10** Let \( z, y, x, R \) be so that \( z, y \in B^{\text{dis}}_R(x) \), and let \( l < \log R/\log M \). For every 
\( j \), 
\[
\mu^{(x, R; z, y)} \left( \sup \{ L_m : m \leq \mathbb{T}(l) \} > L_0 + j ; \forall m \leq \mathbb{T}(l) \} B^{\text{dis}}_{R^{(m)}}(x_m) \text{ is good } \right| \omega \right) \leq 2^{-j} \tag{4.3}
\]

Before we define the third and last coupling, we need an estimate regarding the hitting points of the two random walks in the the coupling \( \mu^{(x, R; z, y)} \).

Lemma 4.11 Fix $k$, and let $R = R^{(0)} = R_k^{K/2}$. Let $x, y, z$ be such that $y, z \in B^{\text{dis}}_R(x)$ and let $\omega$ be such that $B^{\text{dis}}_{R_k+1}(x)$ is admissible. Let

$$m := \inf\{m : R^{(m)} = R_k/M^k\}.$$ 

Then for every $w \in B^{\text{dis}}_{R_k+1}(x)$,

$$\mu(x, R, z, y)(\|x_m - w\| < 10R_k) < R^{-\rho K}$$

for some $\rho > 0$ which is determined only by $P$ and $M$.

Note that using this lemma, once $P$ and $M$ are chosen, we can choose $K$ so that the exponent in (4.4) is as small as we like.

Proof. Let

$$\tilde{m} = \inf\left\{m : R^{(m)} \geq R_k \text{ or } x_m \notin B^{\text{dis}}_{R_k+1/2}(x)\right\}.$$ 

First we show that

$$\mu(x, R, z, y)(\tilde{m} < x_m) < 2R^{-\rho_2 K}$$

for $\rho_2 = \rho_2(P, M) > 0$ which is specified below.

To see (4.5), we first note that by Lemma 4.11 for $\rho_1 = \frac{\log 2}{2\log M}$,

$$\mu(x, R, z, y)\left(\sup_{m < \tilde{m}} R^{(m)} \geq R_{k+1}\right) < 2^{-\frac{k}{2} \frac{\log R_k}{\log m}} = R^{-\rho_1 K}.$$ 

To estimate the probability that $x_m \notin B^{\text{dis}}_{R_k+1/2}(x)$, we estimate $E\mu(x, R, z, y)\|x_m - x\|$ and use Markov’s inequality. To estimate the expectation, we note that $(L_m)$ is dominated by a random walk with a $(2/3, 1/3)$ bias, and thus, for every $l$,

$$E\mu(x, R, z, y)(\#m : L_m = l) \leq \begin{cases} 2, & l \leq L_0 \\ 2^{L_0 - l}, & l \geq L_0, \end{cases}$$

and

$$\|x_{m+1} - x_m\| \leq MR^{(m)}.$$ 

Thus, noting that $M > 3$, we get

$$E\mu(x, R, z, y)(\|x_m - x\|; \max\{R^{(m)} : m \leq \tilde{m}\} \leq R^{\frac{3}{4} K}_k) \leq R^{\frac{3}{4} K}_k,$$

and we get

$$\mu(x, R, z, y)(x_m \notin B^{\text{dis}}_{R_k+1/2}(x)) = \mu(x, R, z, y)(\|x_m - x\| > R_{k+1}/2) \leq \mu(x, R, z, y)\left(\max\{R^{(m)} : m \leq \tilde{m}\} > R^{\frac{3}{4} K}_k\right)
\leq E\mu(x, R, z, y)(\|x_m - x\|; \max\{R^{(m)} : m \leq \tilde{m}\} \leq R^{\frac{3}{4} K}_k)
\leq \frac{R^{\rho_1 K}_k + R^{-K/4}_k < R^{-\rho_2 K}_k}{R_{k+1}/2}.$$
for appropriate $\rho_2 > 0$. Thus (4.5) holds.

We now prove the statement of the lemma. By (4.5), with probability at least $1 - 2R_k^{-\rho_2 K}$, until time $\tilde{m}$ the coupling only sees good balls. Therefore we can couple the random walk $(L_m)$ with a random walk $(\xi_m)$ such that

1. $(\xi_{m+1} - \xi_m)$ is an iid sequence satisfying $P(\xi_{m+1} - \xi_m = 1) = 1/3$ and $P(\xi_{m+1} - \xi_m = -1) = 2/3$, and
2. For every $m$ we have $L_{m+1} - L_m \leq \xi_{m+1} - \xi_m$.

We call $m$ a regeneration if

$$\sum_{n > m} M^{\xi_n - \xi_m} < 1.$$  

(Note that our definition of regenerations is quite different from the standard definitions in the literature). We use $K(m)$ to denote the event that $m$ is a regeneration. We need to use the following fact, which says that there are plenty of regenerations. Note that Claim 4.12 below is a statement regarding biased simple random walks.

**Claim 4.12** There exist $\kappa > 0$ and $\nu > 0$ such that for every $N$,

$$P\left( \sum_{m=1}^{N} 1_{K(m)} < \kappa N \right) < e^{-\nu N}.$$  

Using Claim 4.12 we now finish the proof of Lemma 4.11. Note that if $m$ is a regeneration, then

$$\|x_{\tilde{m}} - x_{m+1}\| < R_k^{(m)}/M.$$  

Therefore $\|w - x_{\tilde{m}}\| < R_k$ only if $\|w - x_{m+1}\| < 2R_k^{(m)}/M$ for every regeneration $m$, and this happens with probability bounded above by $C/M$ for every such $m$. The lemma follows. \qed

Using Lemma 4.11 we estimate the probability that $B_{\text{dis}}^{x_{\tilde{m}}}(x_{\tilde{m}})$ is not admissible, and see that if $K$ is large enough, then this probability is quite small. Indeed, this probability is bounded by the number of non-admissible balls of radius $R_k(\tilde{m})$ with the uniform bound, obtained in Lemma 4.11 on the hitting probability of every ball. We get that

$$\mu(x,R;\cdot,y)(\neg A(x_{\tilde{m}}, k)) \leq R_k^{-\rho K} R_k^\delta.$$  

(4.6)

For $K$ large enough, the power $\delta - \rho K$ is negative, which gives us a probability that is a negative power of $R_k$ to hit a non-admissible ball.

We can now proceed with the proof of Theorem 4.1.

**Proof of Theorem 4.1** In light of Lemma 4.11 and of (4.6), we write $\kappa := \rho K - \delta$. Then, there exists a choice of our parameters such that $\kappa > 0$ and, (4.6) says that

$$\mu(x,R;\cdot,y)(\neg A(x_{\tilde{m}}, k)) \leq R_k^{-\kappa}.$$  

We take $\Psi = 2M$.

Let $R > 0$ and assume that $R > R_0$ and that $\log(R/R_0)/\log M$ is an integer number. We later explain why these assumptions on $R$ do not limit the generality. Let $f : B^{\text{dis}}_{\Psi R}(0) \to \mathbb{R}$ be $\omega$-harmonic function. Let $k_1$ be the largest number such that
Let $y, z \in B_R^{\text{dis}}(0)$. Let $(Y_n), (Z_n), (T^m_y), (T^m_z), (x_m), (R(m))$ be sampled according to the coupling $\mu^{(x,R;z,y)}$. We say that the coupling is successful if the following conditions are satisfied.

1. We require that
   $$\{Y_n : n > 0\} \cup \{Z_n : n > 0\} \subseteq B^{\text{dis}}_{\Psi R}(0).$$

2. For every $j = 0, \ldots, k_1$, we write
   $$\tilde{m}(j) := \inf \{m : R(m) = R_j/M^j\}.$$ We require that for every $j$, the ball $B^{\text{dis}}_{R_j}(x_{\tilde{m}(j)})$ is admissible.

3. We require that for every $j$ and every $m \geq \tilde{m}(j)$, we have $R(m) < R_j$, and
   $$\{Y_n : n > T^m_y\} \cup \{Z_n : n > T^m_z\} \subseteq B^{\text{dis}}_{R_j}(x_{\tilde{m}(j)}).$$

4. We require that
   $$Z_{T^{\tilde{m}(0)+1}_y} = Y_{T^{\tilde{m}(0)+1}_y}.$$

We call the event in Item 1 $A_1$, that in item 2 $A_2$, that in item 3 $A_3$ and that in item 4 $A_4$. We write $A = A_1 \cap A_2 \cap A_3 \cap A_4$. The main step in proving Theorem 4.1 is the following claim.

**Claim 4.13** There exists $\varrho > 0$ such that $\mu^{(x,R;z,y)}(A|A(x,R)) > \varrho$ uniformly in $x, R; z, y$.

We now see how Theorem 4.1 follows from Claim 4.13 and then we prove the claim. For every point $p \in \partial B^{\text{dis}}_R(0)$ let $P_y(p) = P^y_\omega(Y_{T^{\partial \text{dis}}_0(0)} = p)$ and $P_z(p) = P^z_\omega(Y_{T^{\partial \text{dis}}_0(0)} = p)$.

By Claim 4.13 if $B^{\text{dis}}_R(0)$ is admissible then

$$\sum_{p \in \partial B^{\text{dis}}_R(0)} |P_y(p) - P_z(p)| \leq 1 - \varrho,$$

or equivalently

$$\sum_{p \in \partial B^{\text{dis}}_R(0)} \min \{P_y(p), P_z(p)\} \geq \varrho.$$

We write $m_p = \min \{P_y(p), P_z(p)\}$, and then

$$\sum_{p \in \partial B^{\text{dis}}_R(0)} (P_y(p) - m_p) = \sum_{p \in \partial B^{\text{dis}}_R(0)} (P_z(p) - m_p) \leq 1 - \varrho.$$

Remembering that

$$f(y) = \sum_{p \in \partial B^{\text{dis}}_R(0)} P_y(p)f(p)$$

$$f(z) = \sum_{p \in \partial B^{\text{dis}}_R(0)} P_z(p)f(p)$$

we conclude that

$$f(y) - f(z) \leq (1 - \varrho)|f(y)| + (1 - \varrho)|f(z)| \leq \varrho(|f(y)| + |f(z)|).$$
and equivalently for \( z \), we get

\[
\begin{align*}
    f(z) - f(y) &= \sum_{p \in \partial B_R^{\text{dis}}(0)} f(p)P_z(p) - \sum_{p \in \partial B_R^{\text{dis}}(0)} f(p)P_y(p) \\
    &= \sum_{p \in \partial B_R^{\text{dis}}(0)} f(p)(P_z(p) - m) - \sum_{p \in \partial B_R^{\text{dis}}(0)} f(p)(P_y(p) - m) \\
    &\leq \max_{p \in \partial B_R^{\text{dis}}(0)} f(p) \sum_{p \in \partial B_R^{\text{dis}}(0)} (P_z(p) - m) \\
    &\quad - \min_{p \in \partial B_R^{\text{dis}}(0)} f(p) \sum_{p \in \partial B_R^{\text{dis}}(0)} (P_y(p) - m) \\
    &\leq (1 - \varrho) \left[ \max_{p \in \partial B_R^{\text{dis}}(0)} f(p) - \min_{p \in \partial B_R^{\text{dis}}(0)} f(p) \right]
\end{align*}
\]

which, since \( y \) and \( z \) are arbitrary, proves the proposition with \( \varrho = 1 - \varrho \).

We still need to prove Claim 4.13.

**Proof of Claim 4.13.** We bound the probability of \( A_1^c \) exactly the same way (4.5) is shown. By (4.6),

\[
\mu^{(x,R,z,y)} \left( A(B_{R_j}^{\text{dis}}(x_{\tilde{m}(j)})) \cap \bigcap_{h>j} A(B_{R_k}^{\text{dis}}(x_{\tilde{m}(h)})) \right) > 1 - R_j^{-K},
\]

and as \( R_k \) grows faster than exponentially in \( k \), we get

\[
\mu^{(x,R;z,y)}(A_2) = \mu^{(x,R;z,y)} \left( \bigcap_{j} A(B_{R_j}^{\text{dis}}(x_{\tilde{m}(j)})) \right) \geq 1 - \sum_{j=1}^{\infty} R_j^{-K},
\]

which is, by the copice of \( R_0 \), as close as we want to 1.

By (4.5),

\[
\mu^{(x,R;z,y)}(A_3) \geq 1 - \sum_{j} R_j^{-\varrho_2 K}
\]

which, again, is close to 1.

By Corollary 3.11

\[
\mu^{(x,R;z,y)}(A_4 | A_1 \cap A_2 \cap A_3) > C'.
\]

Therefore \( \mu^{(x,R;z,y)}(A | \omega) \) is bounded away from zero in \( R \) and in \( \Omega \) satisfying \( A(B_R^{\text{dis}}(x)) \).

Finally we provide a proof of Claim 4.12.

**Proof of Claim 4.12.** We call \( n \) a renewal if \( \xi_m > \xi_n \) for all \( m < n \) and \( \xi_m < \xi_n \) for all \( m > n \). Denote by \( \tau_k \) the \( k \)th renewal. Then \((\tau_{k+1} - \tau_k)_{k \geq 1}\) is an i.i.d. sequence and \( \tau_1 \), as well as \( \tau_2 - \tau_1 \) have exponential tails. write \( U_k = \xi_{\tau_k} \) and \( V_k = \tau_{k-1} - \tau_k \). Then \((V_k)_{k \geq 1}\)
is an i.i.d. sequence and there exists $\nu > 0$ such that $P(V_1 > l) < e^{-\nu l}$ for every $l$. In addition, $U_k - U_{k-1} \geq 1$ for every $k$. For $n = \tau_k$, we have

$$\sum_{n>m} M^\xi_{n-\xi_m} = \sum_{j=k}^\infty M^\xi_j \left[ \sum_{m=\tau_j+1}^{\tau_{j+1}} M^{-\xi_m} \right]$$

$$\leq \sum_{j=k}^\infty M^{U_k} [V_j M^{U_j-1}] \leq \sum_{j=k}^\infty V_j M^{k-j-1}.$$

So, in particular, $\tau_k$ is a regeneration if $V_j M^{k-j-1} < 2^{k-j-1}$ for every $j \geq k$. For $j \geq k$, we say that $j$ influences $k$, and denote it by $I[j \to k]$, if $V_j M^{k-j-1} \geq 2^{k-j-1}$. So $\tau_k$ is a regeneration if it is not influenced by any $j \geq k$. Let

$$I(j) = \sum_{k \leq j} 1_{I[j \to k]}.$$ 

Then $(I(j))_{j \geq 1}$ is an i.i.d. sequence with exponential tails, and for $M$ large enough we have $E(I(j)) < 1$. Thus by a large deviation estimate with probability exponentially close to 1, $\sum_{j=1}^J I(j) < CJ$ for some $C < 1$, and under this event there are at least $(1 - C)J$ regenerations.

5. Proof of the Harnack inequality

In this section we prove Theorem 1.6. We start with some preparation and notation.

5.1 Preparation and notation.

Let $\nu > 0$ and let $\Gamma = \{A_1, A_2, \ldots, A_k\}$ be a covering of $\partial B_1(0)$ as in Subsection 4.1 (Page 25), except that the diameter of the sets $A_1, A_2, \ldots, A_k$ is bounded by $\nu$. The exact value of $\nu$ will be specified later. Let $\hat{\epsilon} > 0$, and for $j = 1, \ldots, k$ let $D_\omega(B, y)(j)$ and $D_\Sigma(B, y)(j)$ be as in Subsection 4.1. For $0 < \rho < 1$ we write

$$\mathbf{U}(z, R, \rho, \Gamma, \hat{\epsilon}) := \left\{ \forall y \in B^-_{\rho R}(z) \forall j = 1, \ldots, k \left[ \frac{D_\omega(B^{\text{dis}}_{\rho R}(z), y)(j) - D_\Sigma(B^{\text{dis}}_{R}(z), y)(j)}{D_\Sigma(B^{\text{dis}}_{R}(z), y)(j)} < \hat{\epsilon} \right] \right\}.$$

Let $\nu$ and $\overline{\nu}$ be such that by Theorem 4.1 the probability of $H_{x, \overline{\nu}}^{x, \overline{\nu}}$ decays stretched exponentially. Let $0 < \Xi < 1/4$, and for $z \in \mathbb{Z}^d$ and $R > 0$ let $\mathcal{G}_\Xi(z, R)$ be the following event:

$$\mathcal{G}_\Xi(z, R) := \bigcap_{x \in B^{\text{dis}}_{\rho R}(z), R_\Xi < r < R} \left( H_{x, \overline{\nu}}^{x, \overline{\nu}} \cap \mathbf{U}(z, R, \rho, \Gamma, \hat{\epsilon}) \right) \quad (5.1)$$

Claim 5.1 $1 - P(\mathcal{G}_\Xi(z, R))$ decays stretched exponentially with $R$.

Proof. This follows from Corollary 2.4 and Theorem 4.1. \qed
We will prove that a Harnack inequality for \( \omega \)-harmonic functions holds for every ball \( B^\text{dis}_R(z) \) satisfying \( \mathcal{G}_z(z, R) \).

### 5.2 Main lemma.

Let \( \tilde{H} \) be the Harnack constant for harmonic functions in \( \mathbb{R}^d \), and let \( H > \tilde{H} + 10\bar{\epsilon} \).

**Lemma 5.2** Let \( z \) and \( R \) be such that the ball \( B^\text{dis}_R(z) \) satisfies \( \mathcal{G}_z(z, 2R) \). Let \( f : B^\text{dis}_R(z) \to \mathbb{R} \) be non-negative and \( \omega \)-harmonic. Then

\[
\max_{x \in B^\text{dis}_R(z)} f(x) \leq H \min_{x \in B^\text{dis}_R(z)} f(x).
\]

**Proof.** Assume for contradiction that there exist \( x, y \in B^\text{dis}_R(z) \) such that \( f(x) > H f(y) \). Let \( \bar{\alpha} < 2 \) be such that for every non-negative harmonic function \( f \) on \( B_\delta(0) \), we have \( \max_{B_\delta(0)} f \leq (\tilde{H} + \bar{\epsilon}) \min_{B_\delta(0)} f \). We define a sequence of radii \( r_0, r_1, r_2, \ldots, r_k \) as follows: \( r_0 = R, r_1 = \bar{\epsilon}R/2, \) and then \( r_j = r_1/j^2 \). We take \( k \) to be the largest s.t. \( r_k > R/3 \). Note that \( k = k(R) > CR^{(1-\bar{\epsilon})/2} > R^{1/3} \).

We now define a sequence of pairs of points \( (x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k) \). We will always have \( \|x_j - y_j\| < 2r_j \) and that the distance of both \( x_j \) and \( y_j \) from \( \partial B^\text{dis}_R(0) \) is less than \( 4r_j \).

We set a constant \( D = H/(\tilde{H} - 2\bar{\epsilon}) - 1 > 0 \).

At this point we can determine \( \nu \), the mesh of the partition \( \Gamma \). We take \( \nu \) so that

\[
\nu \cdot \frac{\log 2(\tilde{H} + 2\bar{\epsilon}D^{-1})}{\log \nu} < \min_{j=1,\ldots} r_j/r_{j-1}
\]

where \( \nu \) and \( \nu \) are as in Theorem 4.1.

We start by choosing \( x_0 = x \) and \( y_0 = y \). Then we need to explain how \( x_{j+1} \) and \( y_{j+1} \) are chosen, provided that we know \( x_j \) and \( y_j \). This explanation is postponed to after Claim 5.3 below.

We can find a point \( z_j \) (for \( j = 0 \) we take \( z_0 = 0 \)) such that \( x_j \) and \( y_j \) are both in \( B^\text{dis}_{r_j}(z_j) \). Then for every set \( A \in \Gamma \), we have

\[
(\tilde{H} + \bar{\epsilon})^{-1} P^z_{\omega} T_{\partial B^\text{dis}_{\bar{\alpha}r_j}(z_j)} = T_{z_j + \bar{\alpha}r_j A}
\]

\[
\leq P^y_{\omega} T_{\partial B^\text{dis}_{\bar{\alpha}r_j}(z_j)} = T_{z_j + \bar{\alpha}r_j A}
\]

\[
\leq (\tilde{H} + \bar{\epsilon}) P^z_{\omega} T_{\partial B^\text{dis}_{\bar{\alpha}r_j}(z_j)} = T_{z_j + \bar{\alpha}r_j A}
\]

**Claim 5.3** If \( f(x_j)/f(y_j) > \tilde{H} + 2\bar{\epsilon} \) then there exists \( A \in \Gamma \) such that

\[
\max_{z_j + \bar{\alpha}r_j A} f > \frac{1}{\tilde{H} + 2\bar{\epsilon}} \min_{z_j + \bar{\alpha}r_j A} f
\]

We postpone the proof of Claim 5.3.

We now explain how \( x_{j+1} \) and \( y_{j+1} \) are chosen, provided that we know \( x_j \) and \( y_j \).
By Claim 5.3 there is $A \in \Gamma$ such that (5.4) holds. Note that the diameter of $z_j + \tilde{\alpha} r_j A$ is bounded by $\nu r_j$. Thus we can find a point $z_{j+1}$ such that $z_j + \tilde{\alpha} r_j A \subseteq B_{\nu r_j}^{\text{dis}}(z_{j+1})$. In particular, we get that

$$\max_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f > \frac{1}{\hat{H} + 2\hat{\epsilon}} \min_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f.$$ 

Now note that by the choice of $\nu$ (5.2), we get that

$$r_{j+1} > \nu r_j \cdot \frac{\log 2(\hat{H} + 2\hat{\epsilon})}{\log v}.$$ 

Thus, since the event $\mathcal{G}_\Xi(z, R)$ occurs, we get

$$\text{osc}_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f \geq \frac{\log 2(\hat{H} + 2\hat{\epsilon})^{2}}{\log v} \cdot \text{osc}_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f = 2(\hat{H} + 2\hat{\epsilon}) \cdot \text{osc}_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f$$

We can then calculate

$$\frac{\max_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f}{\min_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f} \geq 1 + \frac{\text{osc}_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f}{\min_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f} \geq 1 + 2(\hat{H} + 2\hat{\epsilon})^{2} D^{-1} \frac{\text{osc}_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f}{\min_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f}$$

$$= 1 + 2(\hat{H} + 2\hat{\epsilon})^{2} D^{-1} \left[ \frac{\max_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f}{\min_{B_{\nu r_j}^{\text{dis}}(z_{j+1})} f} - 1 \right] \geq 1 + 2(\hat{H} + 2\hat{\epsilon}) D^{-1} \left[ \frac{f(x_j)}{f(y_j)} - (\hat{H} + 2\hat{\epsilon}) \right]$$

(5.5)

We now take $x_{j+1}$ and $y_{j+1}$ to be, respectively, points where the maximum and the minimum of $f$ in $B_{\nu r_j}^{\text{dis}}(z_{j+1})$ are obtained. It is easy to verify that the pairs $(x_j, y_j)$ satisfy the requirements above, namely that $\|x_j - y_j\| < 2r_j$ and that the distance of both $x_j$ and $y_j$ from $\partial B_{2r_j}^{\text{dis}}(0)$ is less than $4r_j$ for every $j$.

Write $\alpha_j = f(x_j)/f(y_j) - (\hat{H} + 2\hat{\epsilon})$. Then by our assumption, $\alpha_0 > D$, and we can show that $\alpha_j > D$ for every $j \geq 1$. Indeed, inductively using (5.5),

$$\alpha_{j+1} + (\hat{H} + 2\hat{\epsilon}) \geq 1 + 2(\hat{H} + 2\hat{\epsilon}) D^{-1} \alpha_j \geq 1 + 2(\hat{H} + 2\hat{\epsilon}) \geq (\hat{H} + 2\hat{\epsilon}) + D.$$ 

Once we know that $\alpha_j > D$, again using (5.5), we get that

$$\alpha_{j+1} + (\hat{H} + 2\hat{\epsilon}) \geq 1 + 2(\hat{H} + 2\hat{\epsilon}) D^{-1} \alpha_j \geq (\hat{H} + 2\hat{\epsilon}) + (\hat{H} + 2\hat{\epsilon}) D^{-1} \alpha_j$$

which means that $\alpha_{j+1} > (\hat{H} + 2\hat{\epsilon}) D^{-1} \alpha_j > 2\alpha_j$, and inductively, $\alpha_j > C 2^j$. For $k$ as defined in the beginning of this proof, this means that $f(x_k)/f(y_k)$ grows super-exponential with $\|y_k - x_k\|$, which contradicts Corollary 3.11.

Proof of Claim 5.3. Write $C_1 := \frac{1}{\hat{H} + 2\hat{\epsilon}} \frac{f(x_j)}{f(y_j)}$. Assume for contradiction that

$$\max_{z_j + \tilde{\alpha} r_j A} f \leq C_1 \min_{z_j + \tilde{\alpha} r_j A} f$$

□
for every $A \in \Gamma$.

Then

$$f(x_j) \leq \sum_{A \in \Gamma} P^{x_j}_\omega \left( T_{\partial B^{\text{dis}}_{\alpha r_j}}(z_j) = T_{z_j + \alpha r_j A} \right) \max_{z_j + \alpha r_j A} f$$

$$\leq C_1 \sum_{A \in \Gamma} P^{x_j}_\omega \left( T_{\partial B^{\text{dis}}_{\alpha r_j}}(z_j) = T_{z_j + \alpha r_j A} \right) \min_{z_j + \alpha r_j A} f$$

$$\leq (\bar{H} + \bar{\epsilon}) C_1 \sum_{A \in \Gamma} P^{y_j}_\omega \left( T_{\partial B^{\text{dis}}_{\alpha r_j}}(z_j) = T_{z_j + \alpha r_j A} \right) \min_{z_j + \alpha r_j A} f$$

and we reach a contradiction. □

**Proof of Theorem 1.6** The theorem follows from Lemma 5.2 and Claim 5.1 □

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