A DIAMETER BOUND FOR SASAKI MANIFOLDS
WITH APPLICATION TO UNIQUENESS FOR
SASAKI-EINSTEIN STRUCTURE

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Abstract. In this paper we give a diameter bound for Sasaki manifolds with positive transverse Ricci curvature. As an application, we obtain the uniqueness of Sasaki-Einstein metrics on compact Sasaki manifolds modulo the action of the identity component of the automorphism group for the transverse holomorphic structure.

1. Introduction

A Sasaki manifold is a Riemannian manifold \((S, g)\) whose cone metric \(\bar{g} = dr^2 + r^2 g\) on \(C(S) = S \times \mathbb{R}_+\) is Kähler. Then Sasakian geometry sits naturally in two aspects of Kähler geometry, since for one thing, \((S, g)\) is the base of the cone manifold \((C(S), \bar{g})\) which is Kähler, and for another thing any Sasaki manifold is contact, and the one dimensional foliation associated to the characteristic Reeb vector field admits a transverse Kähler structure.

The main purpose of this paper is to prove a Myers’ type theorem for Sasaki manifolds and give a diameter bound for complete Sasaki manifolds with positive transverse Ricci curvature. Our main result is stated as follows.

**Theorem A.** Let \((S, g)\) be a \((2n + 1)\) dimensional complete Sasaki manifold with Sasakian structure \(S = \{g, \xi, \eta, \Phi\}\). Suppose \(\text{Ric}^T \geq \tau g^T\) for some constant \(\tau > 0\). Then

\[
\text{diam}(S, g) \leq 2\pi \sqrt{\frac{2n - 1}{\tau}}.
\]

As an application of Theorem A, we have uniqueness of Sasaki-Einstein metrics up to the action of the identity component of the automorphism group for the transverse holomorphic structure. For toric cases, the uniqueness of Sasaki-Einstein metrics was recently obtained by Cho, Futaki and Ono [4] by showing that the argument of Guan [8] is valid also for the space of Kähler potentials for the transverse Kähler structure.

In this paper, we shall prove such uniqueness without toric assumption by applying Theorem A and the argument of Bando and Mabuchi in [2].

**Theorem B.** Let \((S, g)\) be a compact Sasaki manifold with Sasakian structure \(S = \{g, \xi, \eta, \Phi\}\). Assume that the set \(E\) of all Sasaki-Einstein metrics which is compatible with \(g\) is non-empty. Then the identity component of the automorphism group for the transverse holomorphic structure acts transitively on \(E\).

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This paper is organized as follows: In Section 2, we give a brief review of Sasakian geometry and transverse Kähler geometry. In Section 3, by showing a Myers’ type theorem on complete Sasaki manifolds, we give a proof of Theorem A. Our proof is then based on a variational formula for a minimizing normal geodesic in the sense of sub-Riemannian geometry (see [13] for example). Finally in Section 4, we shall show that an argument similar to Bando and Mabuchi [2] allows us to obtain a proof of Theorem B.

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2. Brief review of Sasakian geometry

2.1. Sasaki manifolds. We recall the basic theory of Sasaki manifolds. For the details, see [1] and [7]. Throughout this paper, we assume that all manifolds are connected. Let \((S,g)\) be a Riemannian manifold and \((C(S), \bar{g}) = (S \times \mathbb{R}_+, dr^2 + r^2 g)\) be its cone manifold, where \(\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}\) and \(r\) is the standard coordinate on \(\mathbb{R}_+\).

**Definition 2.1.** \((S,g)\) is called a Sasaki manifold if the cone manifold \((C(S), \bar{g})\) is a Kähler manifold.

A Sasaki manifold \(S\) is often identified with the submanifold \(\{r = 1\} \subset (C(S), \bar{g})\) and hence the dimension of \(S\) is odd. Let \(\dim S = 2n + 1\). Then, of course, \(\dim C(S) = n + 1\). Let \(J\) be the complex structure of the cone \((C(S), \bar{g})\) and define \(\tilde{\xi} := J(r \frac{\partial}{\partial r})\). The restriction \(\xi := \tilde{\xi}|_{\{r = 1\}}\) of \(\tilde{\xi}\) to the submanifold \(\{r = 1\}\) gives a vector field on \(S\). The vector field \(\xi\) is called the Reeb vector field. The 1-dimensional foliation \(\mathcal{F}_\xi\) generated by \(\xi\) is called the Reeb foliation. Define a differential 1-form \(\eta\) on \(S\) by \(\eta := g(\xi, \cdot)\). Then, one can see that

1. \(\tilde{\xi}\) is a Killing vector field and satisfies \(L_{\tilde{\xi}} J = 0\),
2. \(\nabla_{\xi} \xi = 0\),
3. \(\eta(\xi) = 1, \iota_\xi d\eta = 0\).

In particular \(\xi\) is a Killing vector field on \(S\). The 1-form \(\eta\) gives a \(2_n\)-dimensional subbundle \(D\) of the tangent bundle \(TS\) by

\[D = \ker \eta.\]

The subbundle \(D\) is a contact structure of \(S\) and there is an orthogonal decomposition

\[TS = D \oplus L_\xi,\]

where \(L_\xi\) is the 1-dimensional trivial bundle generated by the Reeb vector field \(\xi\).

Next we define a section \(\Phi\) of the endomorphism bundle \(\text{End}(TS)\) of the tangent bundle \(TS\) by \(\Phi = \nabla \xi\). Then it satisfies that

\[\Phi^2 = -\text{id} + \eta \otimes \xi\]

and \(g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)\). Furthermore, \(\Phi|_D = J|_D\) and \(\Phi|_{L_\xi} = 0\), and this shows that \(\Phi\) gives a complex structure of \(D\). We call the quadruple \(S = (g, \xi, \eta, \Phi)\) a Sasakian structure of \(S\). From these description, the restriction \(g_D := g|_{D \times D}\) of the metric \(g\) to \(D\) is an Hermitian metric on \(D\) and the associated 2-form of the Hermitian metric is equal to \(\frac{i}{2} d\eta|_{D \times D}\):

\[d\eta(X, Y) = 2g(\Phi X, Y)\]
for each $X, Y \in D$. Since $\eta$ is a contact form, $\frac{1}{m}(\frac{1}{2}d\eta)^n \wedge \eta$ is a non-vanishing $(2n + 1)$-form and coincides with the Riemannian volume form $dV_g$. The covariant differentiation of $\Phi$ can be written as a language of the curvature;

$$
(\nabla_X \Phi)(Y) = R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi
$$

for any $X, Y \in TS$.

2.2. **Transverse holomorphic structures and transverse Kähler structures.**

As we saw in the last subsection, $\tilde{\xi} - \sqrt{-1}J\tilde{\xi}$ is a holomorphic vector field on $C(S)$. Hence there is a $C^\infty$-action generated by $\tilde{\xi} - \sqrt{-1}J\tilde{\xi}$. The local orbits of this action defines a transverse holomorphic structure on the Reeb foliation $F_\xi$ in the following sense; There is an open covering $\{U_\alpha\}_{\alpha \in A}$ of $S$ and submersions $\pi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$ such that when $U_\alpha \cap U_\beta \neq \emptyset$,

$$
\pi_\alpha \circ \pi_\beta^{-1} : (U_\alpha \cap U_\beta) \rightarrow (U_\alpha \cap U_\beta)
$$
is biholomorphic. On each open set $V_\alpha \subset \mathbb{C}^n$ we can give a Kähler structure as follows. First note that there is a canonical isomorphism $(\pi_\alpha|_B) : D_p \rightarrow T_{\pi_\alpha(p)}V_\alpha$ for any $p \in U_\alpha$. Since $\xi$ generates isometries of $(S, g)$, the restriction $g_D$ of the Sasaki metric $g$ to $D$ gives a well-defined Hermitian metric $g^D_T$ on $V_\alpha$. This Hermitian structure is in fact Kähler. The fundamental 2-form $\omega_\alpha^T$ of $g^T_\alpha$ is the same as the restriction of $\frac{1}{2}d\eta$ to $U_\alpha$. Hence we see that $\pi_\alpha \circ \pi_\beta^{-1} : (U_\alpha \cap U_\beta) \rightarrow (U_\alpha \cap U_\beta)$ gives an isometry of Kähler manifolds. The collection of Kähler metrics $\{g^T_\alpha\}_{\alpha \in A}$ on $\{V_\alpha\}_{\alpha \in A}$ is called a **transverse Kähler metric**. Since they are isometric over the overlaps we simply denote by $g^T$. We also write $\nabla^T, R^T, \text{Ric}^T, s^T$ for its Levi-Civita connection, the curvature, the Ricci tensor and the scalar curvature. By identifying $D_p$ and $T_{\pi_\alpha(p)}V_\alpha$, we have the following formulas for curvature;

1. $$R(X, Y, Z, W) = R^T(X, Y, Z, W) + g(\Phi(X), Z)g(\Phi(Y), W) - g(\Phi(X), W)g(\Phi(Y), Z) + 2g(\Phi(X), Y)g(\nabla_\xi Z, W),$$

2. $$\text{Ric}^T(X, Y) = \text{Ric}(X, Y) + 2g(X, Y)$$

for any local sections $X, Y, Z, W$ of $D$. For the detail, see [1].

2.3. **Basic forms.** In this section we assume that the Sasaki manifold $(S, g)$ is compact.

**Definition 2.2.** A $k$-form $\alpha$ on $S$ is called **basic** if

$$
eq \xi \alpha = L_\xi \alpha = 0.
$$

Let $\Lambda^k_B$ be the sheaf of germs of basic $k$-forms and $\Omega^k_B$ be the set of all basic $k$-forms.

Let $(x, z^1, \ldots, z^n)$ be a foliation chart on $U_\alpha$. Consider a complex basic form $\alpha$ which can be written as

$$
\alpha = \alpha_{i_1, \ldots, i_p, j_1, \ldots, j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_q},
$$

We call such $\alpha$ a **basic $(p, q)$-form**. It is easy that the definition of basic $(p, q)$-forms is independent of choice of foliation chart. Let $\Lambda^p,q_B$ be the sheaf of germs of basic $(p, q)$-forms and $\Omega^p,q_B$ be the set of all basic $(p, q)$-forms. Then for each $k$, $\Lambda^k_B \otimes \mathbb{C}$ (resp. $\Omega^k_B \otimes \mathbb{C}$) can be decomposed as

$$
\Lambda^k_B \otimes \mathbb{C} = \oplus_{p+q=k} \Lambda^p,q_B, \quad (\text{resp. } \Omega^k_B \otimes \mathbb{C} = \oplus_{p+q=k} \Omega^p,q_B).
$$
Since the exterior derivative $d$ preserves the basic forms, its restriction $d_B$ to the space of basic forms can be decomposed into $d_B = \partial_B + \bar{\partial}_B$ by well-defined operators

$$\partial_B : \Lambda_B^{p,q} \to \Lambda_B^{p+1,q}$$

and

$$\bar{\partial}_B : \Lambda_B^{p,q} \to \Lambda_B^{p,q+1}.$$ 

Let $d_B^*, \partial_B^*$ and $\bar{\partial}_B^*$ be the formal adjoint operators of $d_B$, $\partial_B$ and $\bar{\partial}_B$ and define

$$\Delta_B := d_B^* d_B + \partial_B^* \partial_B, \quad \Box_B := \partial_B^* \partial_B + \partial_B \bar{\partial}_B^*.$$ 

As in the cases of compact Kähler manifolds, both $\Box_B$ and $\Box_B$ are real operators and satisfy $\Delta_B = \frac{1}{2} \Box_B = \frac{1}{2} \Box_B$ (See [5]). Moreover, as shown later $\Delta_B$ coincides with Riemannian Laplacian $\Delta$ on the space of basic functions. Now we can consider the basic de Rham complex $(\Omega_B, d_B)$ and the basic Dolbeault complex $(\Omega^{p,q}_B, \partial_B)$. Their cohomology group is called the basic cohomology group. Similarly, we can consider the basic harmonic forms. El-Kacimi-Alaoui shows in [5] that there is an isomorphism between basic cohomology groups and the space of basic harmonic forms.

We denote by $C_c^\infty(S)$ the set of smooth all basic functions on $S$. For arbitrary basic function $\varphi \in C_c^\infty(S)$, define

$$\eta_\varphi := \eta + 2d_B^* \varphi,$$

where $d_B^* = \frac{\sqrt{-1}}{2}(\partial_B - \bar{\partial}_B)$. Then we have

$$\frac{1}{2} d \eta_\varphi = \frac{1}{2} d \eta + d_B d_B^* \varphi = \frac{1}{2} d \eta + \sqrt{-1} \partial_B \bar{\partial}_B \varphi.$$ 

Thus, for small $\varphi$, $\eta_\varphi \wedge (\frac{1}{2} d \eta_\varphi)^n$ is nowhere vanishing and the 1-form $\eta_\varphi$ gives a new Sasaki structure $\mathcal{S}_\varphi = (g_\varphi, \xi, \eta_\varphi, \Phi)$. By construction, $\mathcal{S}_\varphi$ defines the same transverse holomorphic structure with that of $S$ (see [7] for the detail). Under such a deformation, the transverse Kähler form is deformed in the same basic $(1,1)$ class $[\frac{1}{2} d \eta]_B$. We call this class the basic Kähler class. Note that the contact bundle $D$ may be changed under the deformation.

As we saw in the last subsection, the transverse Kähler form $\{\omega_\alpha^T\}_{\alpha \in A}$ of a Sasaki manifold $(S, g)$ satisfies

$$\pi_\alpha^* \omega_\alpha^T = \frac{1}{2} d \eta |_{\nu_\alpha}.$$ 

Thus they are glued together and give a $d_B$-closed basic $(1,1)$-form $d \eta$ on $S$. We also call $\omega^T = \frac{1}{2} d \eta$ the transverse Kähler form. Similarly we see that the Ricci forms of the transverse Kähler metric $\{\rho_\alpha^T\}_{\alpha \in A}$,

$$\rho_\alpha^T = -\sqrt{-1} \partial \bar{\partial} \log \det(g_\alpha^T),$$ 

are glued together and give a $d_B$-closed basic $(1,1)$-form $\rho^T$ on $S$. $\rho^T$ is called the transverse Ricci form. Of course, the transverse Ricci form $\rho^T$ depends on Sasaki metrics $g$. Nevertheless its basic de Rham cohomology class is invariant under deformations of the Sasaki structure by basic functions. The basic de Rham cohomology class $[\rho^T/2\pi]_B$ is called the basic first Chern class and denoted by $c_1^B(S)$.

2.4. Basic first Chern class and Monge-Ampère equations. Let $(S, g)$ be a $(2n + 1)$-dimensional compact Sasaki manifold.

**Definition 2.3.** A Sasaki-Einstein manifold is a Sasaki manifold $(S, g)$ with $Ric = 2ng$. 
The Einstein condition of a Sasaki manifold is translated into Einstein conditions of the Riemannian cone \((C(S), \tilde{g})\) or the transverse Kähler structure. In short, these conditions are equivalent:

1. \(g\) is a Sasaki-Einstein metric.
2. The Riemannian cone \((C(S), \tilde{g})\) is a Ricci-flat Kähler manifold.
3. The transverse Kähler metric \(g^T\) satisfies \(\text{Ric}^T = (2n + 2)g^T\).

We say that the basic first Chern class \(c_1^B(S)\) of \(S\) is positive if \(c_1^B(S) > 0\) and \(c_1^B(S) = (2n + 2)\frac{1}{2}d\eta\) (in particular \(c_1(D) = 0\)). Then by a result of El Kacimi-Alaoui [5], there is a unique basic function \(h \in \mathbb{C}^\infty(S)\) such that

\[
\rho^T -(2n + 2)\frac{1}{2}d\eta = \sqrt{-1}\partial_B \bar{\partial}_B h, \quad \int_S (e^h - 1)(\frac{1}{2}d\eta)^n \wedge \eta = 0.
\]

Suppose that we can get a Sasaki-Einstein metric by a form \(g_\varphi\) for some basic function \(\varphi\). Then associated transverse Kähler form \(\omega_\varphi^T = \frac{1}{2}d\eta + \sqrt{-1}\partial_B \partial_B \varphi\) satisfies

\[
\rho_\varphi^T = (2n + 2)\omega_\varphi^T.
\]

This leads the transverse Kähler-Einstein (or equivalently Sasaki-Einstein) equation

\[
\frac{\det(g_\varphi^T + \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}} {\det(g_\varphi^T)} = \exp(-(2n + 2)\varphi + h)
\]

with \(g_\varphi^T + \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}\) positive definite.

In [4] and [7], the existence and uniqueness of Sasaki-Einstein metrics on compact toric Sasaki manifold is studied. In [7], the authors proved that for any compact toric Sasaki manifold \((S, g)\) with \(c_1^B(S) > 0\) and \(c_1(D) = 0\), we can get a Sasaki-Einstein metric by deforming the Sasakian structure varying the Reeb vector field (cf. Theorem 1.2. in [7]). Uniqueness of such Einstein metrics up to a connected group action is proved in [4]. Given a Sasaki manifold \((S, g)\), we say that another Sasaki metric \(g'\) on \(S\) is compatible with \(g\) if \(g\) and \(g'\) have the same Reeb vector field and the transverse holomorphic structure. Note that \(g\) and \(g'\) has the same basic Kähler class. Indeed, for corresponding Sasakian structure \(S' = (g', \xi', \eta', \Phi')\), it satisfies that \(\xi := \eta - \eta'\) is basic because \(\xi = \xi'\). This shows that \(d\eta - d\eta' = d\zeta\) and in particular \(\frac{1}{2}d\eta = \frac{1}{2}d\eta' = \frac{1}{2}d\xi\). Hence by transverse \(\bar{\partial}\)-Lemma (see [5]), there exists a basic function \(\varphi \in \mathbb{C}^\infty(S)\) such that \(\frac{1}{2}d\varphi = \frac{1}{2}d\eta + \sqrt{-1}\partial_B \partial_B \varphi\).

**Definition 2.5.** The automorphism group of the transverse holomorphic structure of \((S, g)\) is the biholomorphic automorphisms of \(C(S)\) which commute with the holomorphic flow generated by \(\tilde{\xi} - \sqrt{-1}J\tilde{\xi}\).
We denote by $\text{Aut}(C(S),\tilde{\xi})$ the group of the automorphisms of transverse holomorphic structure and by $G := \text{Aut}(C(S),\tilde{\xi})_0$ its identity component. It is known that the action of $\text{Aut}(C(S),\tilde{\xi})$ on $C(S)$ descends to an action on $S$ preserving the Reeb vector field and the transverse holomorphic structure of the Reeb foliation. In particular, $G$ acts on the space of all Sasaki metrics on $S$ which is compatible with $g$. The Lie algebra of $\text{Aut}(C(S),\tilde{\xi})$ is explained as follows.

**Definition 2.6** (Futaki-Ono-Wang, [7]). A complex vector field $X$ on $S$ is called a Hamiltonian holomorphic vector field if

1. $(\pi_\alpha)_*X$ is a holomorphic vector field on $V_\alpha$ for each $\alpha \in A$,
2. the complex valued function $u_X := \sqrt{-1}\eta(X)$ satisfies

$$\tilde{\partial}_B u_X = -\frac{1}{2} t_X \eta.$$

By definition, every Hamiltonian holomorphic vector field is supposed to commute with $\xi$. We denote by $\mathfrak{h}$ the set of all Hamiltonian holomorphic vector fields. One can check easily that $\mathfrak{h}$ is in fact a Lie algebra. Then it is proved in [4] that the Lie algebra of $\text{Aut}(C(S),\tilde{\xi})$ is isomorphic to $\mathfrak{h}$. (For detailed descriptions, see also [7]). Under the notations and conventions, they proved that, for toric cases, $G$ acts transitively on the space of all Sasaki-Einstein metrics compatible with $g$.

### 2.5. Basic Laplacians for Sasaki manifolds.

In the previous subsection, we introduced the notion of basic Laplacian, which is defined on the space of basic forms. Here we shall show that the basic Laplacian $\Delta_B$ coincides with the restriction $\Delta|_{C^\infty_B(S)}$ of the Riemannian Laplacian $\Delta$ to $C^\infty_B(S)$. Let $T := T^\xi \subset \text{Isom}(S,g)$ be the compact subgroup of $\text{Isom}(S,g)$ generated by the Reeb vector field $\xi$ and $dt$ be the normalized Haar measure on $T$. For any smooth function $\varphi \in C^\infty(S)$ define

$$B(\varphi) := \int_T t^* \varphi dt.$$ 

Then $B$ defines a linear operator on $C^\infty(S)$. It is clear that $B(\varphi) \in C^\infty_B(S)$ for any $\varphi \in C^\infty(S)$ and $B(\varphi) = \varphi$ if and only if $\varphi \in C^\infty_B(S)$. Furthermore one can show that $B$ is symmetric with respect to the $L^2$-inner product on $C^\infty(S)$ by Fubini theorem and the symmetry of $T$. Hence we obtain a orthogonal decomposition

$$C^\infty(S) = C^\infty_B(S) \oplus C^\infty_B(S)^\perp,$$

where $C^\infty_B(S)^\perp$ is the orthogonal complement of $C^\infty_B(S)$ with respect to $L^2$-inner product and $B$ is the orthogonal projection from $C^\infty(S)$ onto $C^\infty_B(S)$.

We denote $d^*$ the formal adjoint operator of $d$. For each $\varphi \in C^\infty_B(S)$ and $\alpha \in \Omega^1_B(S)$, we have

$$(d_B \varphi, \alpha) = (d \varphi, \alpha) = (\varphi, d^* \alpha) = (B(\varphi), d^* \alpha) = (\varphi, Bd^* \alpha),$$

where $(\cdot, \cdot)$ is the $L^2$-inner product on the space of smooth differential forms. This shows that $d_B^* = B \circ d^*$ and hence we obtain

$$\Delta_B \varphi = d_B^* d_B \varphi = Bd^* d \varphi = B \Delta \varphi.$$
Furthermore, for each $\varphi \in C^\infty_B(S)$ and $t \in T$, $t^* \Delta \varphi = \Delta t^* \varphi = \Delta \varphi$ since $t$ acts on $(S, g)$ as an isometry. Therefore we obtain

$$B \Delta \varphi = \int_T t^* \Delta \varphi dt = \int_T \Delta \varphi dt = \Delta \varphi.$$

By combining the equalities (3) and (4), we have the following

**Proposition 2.7.** For each $\varphi \in C^\infty_B(S)$ we have $\Delta_B \varphi = \Delta \varphi$.

Using the foliation chart, we can get an explicit formula for the basic complex Laplacian $\Box_B = -\frac{1}{2} \Delta_B$ by a similar calculation in Kähler geometry.

**Proposition 2.8.** For a foliation chart $(x, z^1, \cdots, z^n)$, we have

$$\Box_B \varphi = -(g^T)^{ij} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$$

for each $\varphi \in C^\infty_B(S)$.

### 3. A diameter bound for complete Sasaki manifolds

In this section, we assume that the Sasaki manifold $(S, g)$ is complete. A piecewise smooth curve $\gamma : [0, l] \to S$ is called horizontal if the differential $\dot{\gamma}(t)$ tangents to $D_{\gamma(t)}$ for all $t \in [0, l]$. For each $p, q \in S$, put

$$L_D(\gamma) := \int_0^l |\dot{\gamma}(t)| dt,$$

and define

$$d_D(p, q) := \inf \{L(\gamma) \mid \gamma \in \Omega(p, q, D)\},$$

where $\Omega(p, q, D)$ is the set of all piecewise smooth horizontal curves joining $p$ to $q$. The function $L_D : \Omega(p, q, D) \to \mathbb{R}$ is called the length of horizontal curves and the function $d_D$ on $S \times S$ is called the Carnot-Carathéodory metric of $S$. For the Riemannian distance function $d$ of $(S, g)$, it is clear that $d \leq d_D$. Since the contact distribution $D$ is bracket generating (i.e., brackets of local sections of $D$ generates all local sections of $TS$), the classical theorem of Chow tells us that the function $d_D$ gives a distance of $S$ and the topology induced by the distance coincides with the original topology of $S$ (For the proof, see [13] for example). The main result of this section is stated as follows. We say that the transverse Ricci curvature is bounded from below if there exist a constant $\tau \in \mathbb{R}$ such that $\text{Ric}^T(X, X) \geq \tau g(X, X)$ for each $X \in D$. We express the condition by $\text{Ric}^T \geq \tau g^T$. Hasegawa and Seino shows in [9] that a complete Sasaki manifold with $\text{Ric}^T \geq \tau g^T$ for a positive constant $\tau > 0$ is compact with finite fundamental group. Then we shall show the following stronger result.

**Theorem 3.1.** Let $(S, g)$ be a $(2n + 1)$ dimensional complete Sasaki manifold with Sasakian structure $S = \{g, \xi, \eta, \Phi\}$. Suppose that $\text{Ric}^T \geq \tau g^T$ for some constant $\tau > 0$. Then

$$\text{diam}(S, d_D) \leq 2\pi \sqrt{\frac{2n - 1}{\tau}}.$$

Then we can obtain Theorem A immediately because $d \leq d_D$. Our proof of Theorem 3.1 is based on a variational formula of the energy of normal geodesics on the space of horizontal curves.
3.1. Normal geodesics. A notion of normal geodesics is defined in sub-Riemannian geometry as the projection on $S$ of solutions of the “Hamiltonian equation”, which is defined below.

A sub-Riemannian manifold is a triple $(S, E, g_E)$ of a smooth manifold $S$, a subbundle $E$ of the tangent bundle $TS$ and a metric $g_E$ on $E$. For a Sasaki manifold $(S, g)$, the pair of the contact structure $D \subset TS$ and the restriction $g_D$ of the Sasaki metric $g$ to $D$ defines a sub-Riemannian structure of $S$, that is, $(S, D, g_D)$ is a sub-Riemannian manifold. Hence we can apply the notions of sub-Riemannian geometry to Sasakian geometry. The detailed description can be seen in [13] and [16] for example. Let $T^*S$ be the cotangent bundle of $S$ and $H_D : T^*S \to \mathbb{R}$ the function on $T^*S$ defined by

$$H_D(p, \alpha) := \frac{1}{2}(g_D)^{-1}(\alpha|_D, \alpha|_D) = \frac{1}{2}g^{-1}(\alpha, \alpha) - \frac{1}{2}\alpha(\xi)^2$$

for each $(p, \alpha) \in T^*S$. We call the function $H_D$ the Hamiltonian function. For any foliation chart $(x_0, \cdots, x_{2n})$ with $\frac{\partial}{\partial x_0} = \xi$ and the canonical coordinates $(x_0, \cdots, x_{2n}, \alpha_0, \cdots, \alpha_{2n})$ on $T^*S$, consider the following ordinary differential equation:

\begin{align}
\dot{x}_i &= \frac{\partial H_D}{\partial \alpha_i}, \\
\dot{\alpha}_i &= -\frac{\partial H_D}{\partial x_i}.
\end{align}

We call it the Hamiltonian equation.

**Definition 3.2.** A smooth curve $\gamma : [0, l] \to S$ is called a normal geodesic if there exists a cotangent lift $\Gamma(t) = (\gamma(t), \alpha(t)) : [0, l] \to T^*S$ which satisfies the Hamiltonian equation (5).

By existence and uniqueness of solutions of ordinary differential equations, the Hamiltonian equation (5) has unique solution determined by initial value $\Gamma(0) = (p, \alpha) \in T^*_pS$. For a normal geodesic $\gamma(t)$ with the cotangent lift $\Gamma(t) = (\gamma(t), \alpha(t))$, the Hamiltonian equation can be rewritten as

\begin{align}
\dot{\gamma}(t) &= g^{-1}(\alpha) - \alpha(\xi)\xi, \\
\dot{\alpha}_i &= -\frac{1}{2}\frac{\partial g^{ij}}{\partial x_k}\alpha_k\alpha_j,
\end{align}

where $g^{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is the component of the Sasaki metric $g$ with respect to the local coordinate $(x_0, \cdots, x_{2n})$ and $(g^{ij})$ is the inverse matrix of $(g_{kj})$. This shows that a normal geodesic is always horizontal. Furthermore, the equation (6) implies

\begin{align}
\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) &= -2\alpha_0\Phi(\dot{\gamma}(t)),
\end{align}

where $\alpha_0 = \alpha(\xi)$ is constant by (6) and by that $\xi$ is a Killing vector field. In particular we see that $\gamma(t)$ is constant speed.

Note that, for a smooth curve $\gamma : [0, l] \to S$ which satisfies the equation (7) for some constant $\alpha_0 \in \mathbb{R}$, we have

\begin{align}
\frac{d}{dt}(g(\dot{\gamma}(t), \xi)) &= g(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \xi) + g(\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}\xi) \\
&= -2\alpha_0 g(\Phi(\dot{\gamma}(t)), \xi) + g(\dot{\gamma}(t), \Phi(\dot{\gamma}(t))) \\
&= -2\alpha_0 g(\Phi(\dot{\gamma}(t)), \xi) + \frac{1}{2}d\eta(\dot{\gamma}(t), \dot{\gamma}(t)) = 0.
\end{align}
Hence we see that $\gamma$ is horizontal if and only if $\dot{\gamma}(t) \in D_{\gamma(0)}$. Now for each smooth horizontal curve $\gamma : [0,l] \to S$ which satisfies the equation (7), define $\alpha(t) := g(\dot{\gamma}(t) + \alpha_0 \xi)$ and $\Gamma(t) := (\gamma(t), \alpha(t))$. Then we can easily check that the curve $\Gamma(t)$ satisfies the equation (5), that is, $\gamma$ is a normal geodesic. This shows the following

**Proposition 3.3.** A smooth curve $\gamma : [0,l] \to S$ is a normal geodesic if and only if it satisfies the equation (7) for some constant $\alpha_0 \in \mathbb{R}$ and $\dot{\gamma}(0) \in D_{\gamma(0)}$.

A subbundle $E \subset TS$ of the tangent bundle of $S$ is called *strong bracket generating* if for each $p \in S$ and each nonzero local section $X$ of $E$ around $p$ we have $E_p + [X,E]_p = T_pS$. For a Sasaki manifold $(S,g)$, the corresponding contact structure $D$ is strong bracket generating. Indeed, for each $p \in S$ and nonzero local section $X$ of $D$ around $p$ we have

$$g([X,\Phi(X)],\xi) = g(\nabla \Phi(X),\xi) - g(\Phi(X),\xi) = -g(\Phi(X),\Phi(X)) + g(X,\Phi^2(X)) = -2g(X,X) \neq 0.$$  

This shows $\xi_p \in D_p + [X,D]_p$ and hence we obtain $T_pS = D_p + [X,D]_p$.

As in the case of Riemannian geometry, every normal geodesic is locally a unique length minimizing curve. By the fact that $D$ is strong bracket generating, Strichartz proved that Hopf-Rinow type theorem for the sub-Riemannian manifold $(S,D,g_D)$ still holds, i.e., any two points on a complete Sasaki manifold can be joined by a length minimizing normal geodesic (See [16] and [17]).

**Remark 3.4.** The assumption that $D$ is strong bracket generating is essential. Indeed, for a sub-Riemannian manifold $(S,D,g_D)$ such that $D$ is not strong bracket generating, the Hopf-Rinow type theorem does not hold in general. There is some examples of length minimizing horizontal curves which are not normal geodesics. These examples can be seen in [13].

### 3.2. Second Variational formula.

For each $p,q \in S$, consider a functional $E_D : \Omega(p,q,D) \to \mathbb{R}$ defined by

$$E_D(\gamma) := \frac{1}{2} \int_0^l g(\dot{\gamma}(t),\dot{\gamma}(t)) dt,$$

which is called the *energy* of a horizontal curve $\gamma$. It is well known in Riemannian geometry, for a constant speed horizontal curve $\gamma$, $\gamma$ minimizes the length functional $L_D : \Omega(p,q,D) \to \mathbb{R}$ if and only if it minimizes the energy functional. In particular, a length minimizing normal geodesic joining $p$ to $q$ is an energy minimizing curve. We shall give a second variational formula of the energy functional on $\Omega(p,q,D)$ for a normal geodesic. In this subsection, we assume that every curve $\gamma$ is regular, that is, $\gamma$ is smooth and $|\dot{\gamma}(t)| \neq 0$ for all $t \in [0,l]$.

Recall that a *variation* of a smooth curve $\gamma : [0,l] \to S$ is a smooth mapping $f : (-\varepsilon,\varepsilon) \times [0,l] \to S$ which satisfies $f(s,0) = \gamma(0)$, $f(s,l) = \gamma(l)$ and $f(0,t) = \gamma(t)$. A smooth vector field $V(t)$ along $\gamma(t)$ is called a *variation vector field* of $\gamma$ if it satisfies $V(0) = V(l) = 0$. Given a variation $f(s,t)$ of $\gamma$, we can construct a variation vector field $V(t)$ by $V(t) := \frac{\partial f}{\partial t}(s,t)|_{s=0}$. Conversely, for each variation vector field $V(t)$ of $\gamma$, there exists a variation $f(s,t)$ of $\gamma$ whose associated variation vector field is $V(t)$.

For a horizontal curve $\gamma \in \Omega(p,q,D)$, let $f(s,t) : (-\varepsilon,\varepsilon) \times [0,l] \to S$ be a variation of $\gamma$. A variation $f(s,t)$ is said to be *admissible* if $\frac{\partial f}{\partial t} \in D$ for each
(s, t) ∈ (−ε, ε) × [0, l]. Similarly, a variation vector field $V(t)$ of $γ$ is said to be admissible if there exists an admissible variation $f(s, t)$ whose variation vector field is $V(t)$. A similar argument of Ritoré and Rosales in [15] tells us that the set $T_γΩ(p, q, D)$ of all admissible variation vector fields of $γ$ is given by

$$T_γΩ(p, q, D) = \left\{ V(t) ∈ T_γΩ(p, q) \mid \frac{d}{dt}g(V(t), ξ) = 2g(V(t), Φ(γ(t))) \right\},$$

where $T_γΩ(p, q)$ is the set of all variation vector fields of $γ$.

**Proposition 3.5.** Let $γ : [0, l] → S$ be a normal geodesic. For each admissible variation $f(s, t)$ of $γ$, define $E_D(s) := E_D(f(s, t))$. Then

$$E_D''(0) = -\int_0^l g(V, \nabla_{γ(t)}V + R(γ(t))γ(t)) dt$$

$$+ 2\alpha_0 \int_0^l \{ η(V)g(V, γ(t)) + g(\nabla_{γ(t)}V, Φ(V)) \} dt.$$ 

**Proof.** At first, we have

$$E_D''(s) = \frac{1}{2} \frac{d^2}{ds^2} \int_0^l g \left( \frac{∂f}{dt}, \frac{∂f}{dt} \right) dt$$

$$= \frac{d}{ds} \int_0^l g \left( \frac{Df}{dt}, \frac{Df}{dt} \right) dt$$

$$= \frac{d}{ds} \int_0^l \left\{ \frac{d}{dt}g \left( \frac{∂f}{ds}, \frac{∂f}{dt} \right) - g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂t} \right) \right\} dt$$

$$= \int_0^l \frac{d^2}{dsdt}g \left( \frac{∂f}{ds}, \frac{∂f}{dt} \right) dt$$

$$- \int_0^l g \left( \frac{∂f}{ds}, \frac{D}{dt} \frac{∂f}{∂s} + \frac{D}{dt} \frac{∂f}{∂t} \right) dt - \int_0^l g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂t} \right) dt.$$ 

By summing the first and third terms, we obtain

$$\int_0^l \frac{d^2}{dsdt}g \left( \frac{∂f}{ds}, \frac{∂f}{dt} \right) dt - \int_0^l g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂t} \right) dt$$

$$= - \int_0^l g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂s} + \frac{D}{dt} \frac{∂f}{∂t} \right) dt.$$

and hence

$$E_D''(s) = -\int_0^l g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂s} + \frac{D}{dt} \frac{∂f}{∂t} \right) dt$$

$$- \int_0^l g \left( \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂t} \right) dt.$$ 

We shall now calculate the second term of (10). Because $γ$ is a normal geodesic, for the integrand we have

$$g\left( \frac{D}{ds} \frac{∂f}{∂s}, \frac{D}{dt} \frac{∂f}{∂t} \right) |_{s=0} = g(∇_V V, ∇_{γ(t)}γ(t)) = -2\alpha_0 g(∇_V V, Φ(γ(t)))$$
by substituting 0 to $s$. To integrate both sides, notice that
\[
\frac{d^2}{dsdt} \left( \eta \frac{\partial f}{\partial s} \right) = \frac{d}{ds} \left\{ g \left( \frac{D}{dt} \xi, \frac{\partial f}{\partial s} \right) + \eta \left( \frac{D}{dt} \frac{\partial f}{\partial s} \right) \right\} = \frac{d}{ds} \left\{ g \left( \frac{D}{dt} \frac{\partial f}{\partial s} \right) \right\} = 2 \frac{d}{ds} \left\{ g \left( \frac{D}{ds} \frac{\partial f}{\partial s} \right) + g \left( \frac{D}{dt} \xi, \frac{\partial f}{\partial s} \right) \right\}.
\]
Furthermore, by
\[
g \left( \frac{D}{ds} \frac{\partial f}{\partial s} \right) = g \left( \frac{D}{dt} \xi, \frac{\partial f}{\partial s} \right) = g \left( \frac{D}{ds} \Phi \left( \frac{\partial f}{\partial t} \right), \frac{\partial f}{\partial s} \right) = g \left( \frac{D}{ds} \Phi \left( \frac{\partial f}{\partial t} \right), \frac{\partial f}{\partial s} \right) + g \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right) - \eta \left( \frac{\partial f}{\partial s} \right) g \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right) + \frac{D}{dt} \frac{\partial f}{\partial s} \frac{\partial f}{\partial s},
\]
we obtain the following equality;
\[
\frac{d^2}{dsdt} \left( \eta \frac{\partial f}{\partial s} \right) = -\eta \left( \frac{\partial f}{\partial s} \right) g \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right) + \frac{D}{dt} \frac{\partial f}{\partial s} \frac{\partial f}{\partial s}. \quad (12)
\]
Since $\frac{\partial f}{\partial s}(s, 0) = \frac{\partial f}{\partial s}(s, l) = 0$, the integration of both sides of the equality (12) with respect to $t$ leads us the following equality:
\[
\int_0^l g \left( \frac{D}{dt} \xi, \frac{\partial f}{\partial s} \right) dt = \int_0^l \eta \left( \frac{\partial f}{\partial s} \right) g \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right) + \frac{D}{dt} \frac{\partial f}{\partial s} \frac{\partial f}{\partial s} dt.
\]
In particular, by substituting 0 to $s$, we have
\[
\int_0^l g \left( \Phi (\dot{\gamma}(t)), \nabla_{\gamma(t)} V \right) dt = \int_0^l \eta(V) g \left( V, \dot{\gamma}(t) \right) - g \left( \nabla_{\gamma(t)} V, V \right) dt = \int_0^l \eta(V) g \left( V, \dot{\gamma}(t) \right) + g \left( \nabla_{\gamma(t)} V, \Phi(V) \right) dt. \quad (13)
\]
Combine the equality (10), (11) and (13), we obtain
\[
E_D'(0) = -\int_0^l g \left( V, \nabla_{\gamma(t)} V + R(V, \dot{\gamma}(t)) \dot{\gamma}(t) \right) dt + 2\alpha_0 \int_0^l \{ \eta(V) g \left( V, \dot{\gamma}(t) \right) + \eta \left( \nabla_{\gamma(t)} V, \Phi(V) \right) \} dt,
\]
which is the desired formula.
3.3. A proof of Theorem 3.1. Our proof of Theorem 3.1 is based on the classical proof of Myers’ theorem.

Let $p, q$ be an arbitrary pair of points of $S$ and $\gamma : [0, l] \to S$ be a minimizing normal geodesic joining $p$ to $q$ (Since $(S, g)$ is complete, such a normal geodesic always exists). We may assume that $|\dot{\gamma}(t)| = 1$ for all $t \in [0, l]$. Then $\gamma$ is a minimizer of $E_D : \Omega(p, q, D) \to \mathbb{R}$ and hence $E_D(0) \geq 0$ for each admissible variation $f(s, t) : (-\varepsilon, \varepsilon) \times [0, l] \to S$ of $\gamma$. Choose tangent vectors $X_1, \ldots, X_{2(n-1)} \in T_p S$ such that $\{X_1, \ldots, X_{2(n-1)}, \dot{\gamma}(0), \Phi(\dot{\gamma}(0))\}$ is an orthonormal basis of $D$. For each $i = 1, \ldots, 2(n-1)$, consider the following linear differential equation for $X_i(t) \in D_{\dot{\gamma}(t)}$ defined by

$$\nabla^T_{\dot{\gamma}(t)}X_i(t) = \nabla_{\dot{\gamma}(t)}X_i(t) - g(\nabla_{\dot{\gamma}(t)}X_i(t), \xi)\xi = 0$$

and $X_i(0) = X_i$.

By existence and uniqueness theorem for linear ordinary differential equations, there is a unique global solution $X_i(t) \in D$, $t \in [0, l]$.

**Lemma 3.6.** $\{X_1(t), \ldots, X_{2(n-1)}, \dot{\gamma}(t), \Phi(\dot{\gamma}(t))\}$ is an orthonormal basis of $T_{\dot{\gamma}(t)}S$ for all $t \in [0, l]$.

**Proof.** First note that $g(\dot{\gamma}(t), \dot{\gamma}(t)) = g(\Phi(\dot{\gamma}(t)), \Phi(\dot{\gamma}(t))) = 1$ and $g(\Phi(\dot{\gamma}(t)), \dot{\gamma}(t)) = 0$. Furthermore, since $g(X_i, X_j) = \delta_{ij}$ and

$$\frac{d}{dt}g(X_i(t), X_j(t)) = g(D\frac{d}{dt}X_i(t), X_j(t)) + g(X_i(t), D\frac{d}{dt}X_j(t))$$

$$= g(\nabla^T_{\dot{\gamma}(t)}X_i(t), X_j(t)) + g(X_i(t), \nabla^T_{\dot{\gamma}(t)}X_j(t)) = 0,$$

we see that $g(X_i(t), X_j(t)) = \delta_{ij}$ for each $t \in [0, l]$. Hence it is sufficient to show that $X_i(t)$ is perpendicular to both $\dot{\gamma}(t)$ and $\Phi(\dot{\gamma}(t))$.

Define

$$f(t) = t(f_1(t), f_2(t)) = t(g(X_i(t), \dot{\gamma}(t)), g(X_i(t), \Phi(\dot{\gamma}(t))))$$

Then, we have

$$\frac{df_1}{dt} = \frac{d}{dt}g(X_i(t), \dot{\gamma}(t)) = g(\nabla_{\dot{\gamma}(t)}X_i(t), \dot{\gamma}(t)) + g(X_i(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t))$$

$$= -2\alpha_0 g(X_i(t), \Phi(\dot{\gamma}(t))) = -2\alpha_0 f_2(t),$$

and

$$\frac{df_2}{dt} = \frac{d}{dt}g(X_i(t), \Phi(\dot{\gamma}(t))) = g(\nabla_{\dot{\gamma}(t)}X_i(t), \Phi(\dot{\gamma}(t))) + g(X_i(t), \nabla_{\dot{\gamma}(t)}\Phi(\dot{\gamma}(t)))$$

$$= 2\alpha_0 g(X_i(t), \dot{\gamma}(t)) = 2\alpha_0 f_1(t).$$

Thus the function $f(t) = t(f_1(t), f_2(t))$ satisfies the following ordinary differential equation:

$$\begin{cases}
\frac{df}{dt} = \begin{pmatrix} 0 & -2\alpha_0 \\ 2\alpha_0 & 0 \end{pmatrix} f(t), \\
\quad f(0) = 1(0, 0).
\end{cases}$$

This shows that $f(t) = 0$ for all $t \in [0, l]$ and hence we obtain the desired result. □

For each $i = 1, \ldots, 2(n-1)$, define $h(t) := \sin \left(\frac{2\pi i t}{2(n-1)}\right)$ and $V_i(t) := h(t)X_i(t)$. Since $X_i(t) \in D$ is perpendicular to $\Phi(\dot{\gamma}(t))$, we see that $V_i(t) \in T_{\dot{\gamma}(t)}\Omega(p, q, D)$. Let $f_i(s, t)$ be an admissible variation of $\gamma$ whose variation vector field is $V_i(t)$. Let us
calculate the second variation $E''_D(0) = \frac{d^2}{dt} E_D(f_i(s, t)) \geq 0$ explicitly. Note that since $X_i(t) \in D$ and $\nabla^T_{\dot{\gamma}}(t) X_i(t) = 0$ we have $\eta(V_i(t)) = 0$ and

$$g(\nabla_{\dot{\gamma}}(t) V_i(t), \Phi(V_i(t))) = h(t)g(h'(t) X_i(t) + h(t)\nabla_{\dot{\gamma}}(t) X_i(t))$$

$$= h(t)h'(t)g(X_i(t), \Phi(X_i(t))) + h^2(t)g(\nabla_{\dot{\gamma}}(t) X_i(t), \Phi(X_i(t)))$$

$$= h(t)h'(t)\eta(X_i(t), X_i(t)) + h^2(t)g(\nabla^T_{\dot{\gamma}}(t) X_i(t), \Phi(X_i(t)))$$

$$= 0.$$

Hence for $V_i(t)$ we have

$$(14) \quad E''_D(0) = -\int_0^l g(V_i, \nabla_{\dot{\gamma}}(t) \nabla_{\dot{\gamma}}(t) V_i + R(V_i, \dot{\gamma}(t)) \dot{\gamma}(t)) dt.$$

Now we can calculate easily

$$\nabla_{\dot{\gamma}}(t) \nabla_{\dot{\gamma}}(t) V_i(t) = \nabla_{\dot{\gamma}}(t) \left( h'(t) X_i(t) + h(t)\nabla_{\dot{\gamma}}(t) X_i(t) \right)$$

$$= \nabla_{\dot{\gamma}}(t) \left( h'(t) X_i(t) + h(t)g(\nabla_{\dot{\gamma}}(t) X_i(t), \xi) \right)$$

$$= h''(t) X_i(t) + h'g(\nabla_{\dot{\gamma}}(t) X_i(t), \xi)$$

$$\quad + \frac{d}{dt} \left( h(t)g(\nabla_{\dot{\gamma}}(t) X_i(t), \xi) \right) \xi + h(t)g(\nabla_{\dot{\gamma}}(t) X_i(t), \xi)\Phi(\dot{\gamma}(t)).$$

Since $V_i(t)$ is perpendicular to both $\xi$ and $\Phi(\dot{\gamma}(t))$, we obtain

$$(15) \quad g(V_i(t), \nabla_{\dot{\gamma}}(t) \nabla_{\dot{\gamma}}(t) V_i(t)) = h(t)h''(t) = -\left( \frac{2\pi}{l} \right)^2 \sin\left( \frac{2\pi t}{l} \right).$$

Similarly we have

$$(16) \quad g(V_i(t), R(V_i(t), \dot{\gamma}(t)) \dot{\gamma}(t)) = \sin^2\left( \frac{2\pi t}{l} \right) g(X_i(t), R(X_i(t), \dot{\gamma}(t)) \dot{\gamma}(t))$$

$$= \sin^2\left( \frac{2\pi t}{l} \right) R(X_i(t), \dot{\gamma}(t), \dot{\gamma}(t), X_i(t))$$

$$= \sin^2\left( \frac{2\pi t}{l} \right) R^T(X_i(t), \dot{\gamma}(t), \dot{\gamma}(t), X_i(t))$$

by equation (1). By substituting (15) and (16) to (14) we obtain the following inequality;

$$(17) \quad 0 \leq E''_D(0) = \int_0^l \sin^2\left( \frac{2\pi t}{l} \right) \left\{ \left( \frac{2\pi}{l} \right)^2 - R^T(X_i(t), \dot{\gamma}(t), \dot{\gamma}(t), X_i(t)) \right\} dt.$$

Next define

$$V(t) := h(t)\Phi(\dot{\gamma}(t)) + k(t)\xi$$

for smooth functions $h(t), k(t) : [0, l] \to \mathbb{R}$ with $h(0) = h(l) = k(0) = k(l) = 0$. Then the condition (8) implies that $V(t) \in T, \Omega(p, q, D)$ if and only if $k'(t) = 2h(t)$.

We suppose that $h(t) := \sin\left( \frac{2\pi t}{l} \right)$ and $k(t) := \frac{l}{4}\left( 1 - \cos\left( \frac{4\pi t}{l} \right) \right)$. Then we can easily check that $k'(t) = 2h(t)$. At first we have

$$\nabla_{\dot{\gamma}}(t) V(t) = \nabla_{\dot{\gamma}}(t) h(t)\Phi(\dot{\gamma}(t)) + \nabla_{\dot{\gamma}}(t) k(t)\xi$$

$$= h'(t)\Phi(\dot{\gamma}(t)) + h(t) \left\{ (\nabla_{\dot{\gamma}}(t) \Phi)(\dot{\gamma}(t)) + \Phi(\nabla_{\dot{\gamma}}(t) \dot{\gamma}(t)) \right\}$$

$$\quad + k'(t)\xi + k(t)\Phi(\dot{\gamma}(t))$$

$$= h'(t)\Phi(\dot{\gamma}(t)) + h(t) (-\xi + 2\alpha_0\dot{\gamma}(t)) + k'(t)\xi + k(t)\Phi(\dot{\gamma}(t))$$

$$= (h'(t) + k(t))\Phi(\dot{\gamma}(t)) + h(t)\xi + 2\alpha_0 h(t)\dot{\gamma}(t).$$
In addition, by differentiating again we have
\[
\nabla \hat{\gamma}(t) \left( (h'(t) + k(t))\Phi(\hat{\gamma}(t)) \right) = (h''(t) + k'(t))\Phi(\hat{\gamma}(t)) + (h'(t) + k(t))\nabla \hat{\gamma}(t)\Phi(\hat{\gamma}(t)) \\
= (h''(t) + k'(t))\Phi(\hat{\gamma}(t)) \\
+ (h'(t) + k(t))(-\xi + 2\alpha_0\hat{\gamma}(t)) \\
= (h''(t) + k'(t))\Phi(\hat{\gamma}(t)) \\
- (h'(t) + k(t))\xi + 2\alpha_0(h'(t) + k(t))\hat{\gamma}(t),
\]
and
\[
\nabla \hat{\gamma}(t)2\alpha_0h(t)\hat{\gamma}(t) = 2\alpha_0 \left( h'(t)\hat{\gamma}(t) + h(t)\nabla \hat{\gamma}(t)\hat{\gamma}(t) \right) \\
= 2\alpha_0h'(t)\hat{\gamma}(t) - (2\alpha_0)^2h(t)\Phi(\hat{\gamma}(t)).
\]
By combining them we obtain
\[
\nabla \hat{\gamma}(t)\nabla \hat{\gamma}(t)V(t) = (h''(t) + 3h(t) - (2\alpha_0)^2h(t))\Phi(\hat{\gamma}(t)) \\
+ 2\alpha_0(2h'(t) + k(t))\hat{\gamma}(t) - k(t)\xi
\]
and hence
\[
(18) \quad g(V(t), \nabla \hat{\gamma}(t)\nabla \hat{\gamma}(t)V(t)) = h(t) \left( h''(t) + 3h(t) - (2\alpha_0)^2h(t) \right) - k^2(t).
\]
For the curvatures we have
\[
(19) \quad g(V(t), R(V(t), \hat{\gamma}(t))\hat{\gamma}(t)) = h^2(t)g(\Phi(\hat{\gamma}(t)), R(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t))\hat{\gamma}(t)) \\
+ 2h(t)k(t)g(\Phi(\hat{\gamma}(t)), R(\xi, \hat{\gamma}(t))\hat{\gamma}(t)) \\
+ k^2(t)g(\xi, R(\xi, \hat{\gamma}(t))\hat{\gamma}(t)) \\
= h^2(t)R(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t), \hat{\gamma}(t), \Phi(\hat{\gamma}(t))) + k^2(t). \\
= h^2(t)R^T(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t), \hat{\gamma}(t), \Phi(\hat{\gamma}(t))) - 3h^2(t) + k^2(t).
\]
By substituting (18) and (19) to (9), we obtain
\[
(20) \quad 0 \leq E''_\theta(0) \\
= -\int_0^T \left\{ h(t) \left( h''(t) + 3h(t) - (2\alpha_0)^2h(t) \right) - k^2(t) \right\} dt \\
- \int_0^T \left\{ h^2(t)R^T(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t), \hat{\gamma}(t), \Phi(\hat{\gamma}(t))) - 3h^2(t) + k^2(t) \right\} dt \\
- (2\alpha_0)^2 \int_0^T h^2(t) dt \\
= -\int_0^T \left\{ h(t)h''(t) + h^2(t)R^T(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t), \hat{\gamma}(t), \Phi(\hat{\gamma}(t))) \right\} dt \\
= \int_0^T \sin^2 \left( \frac{2\pi t}{T} \right) \left\{ (\frac{2\pi}{T})^2 - R^T(\Phi(\hat{\gamma}(t)), \hat{\gamma}(t), \hat{\gamma}(t), \Phi(\hat{\gamma}(t))) \right\} dt.
\]
Finally, by summing (17) and (20), we obtain
\[
0 \leq \int_0^T \sin^2 \left( \frac{2\pi t}{T} \right) \left\{ (\frac{2\pi}{T})^2(2n - 1) - Ric^T(\hat{\gamma}(t), \hat{\gamma}(t)) \right\} dt.
\]
Furthermore, by assumption $Ric^T \geq \tau g^T$, 
\[ 0 \leq \int_0^t \sin^2 \left( \frac{2\pi t}{I} \right) \left\{ \left( \frac{2\pi}{I} \right)^2 (2n-1) - \tau \right\} dt. \]
This shows that $0 \leq \left( \frac{2\pi}{I} \right)^2 (2n-1) - \tau$ and hence 
\[ d_D(p, q) = t \leq 2\pi \sqrt{\frac{2n-1}{\tau}}. \]
Hence we obtain $\text{diam}(S, d_D) \leq 2\pi \sqrt{\frac{2n-1}{\tau}}$ and this completes the proof of Theorem 3.1.

4. A PROOF OF THEOREM B

4.1. Generalized Aubin’s equation. In this section we give a proof of Theorem B. Our proof is based on the arguments of Bando and Mabuchi in [2]. Throughout this section, we denote $(S, g)$ by a $(2n + 1)$-dimensional compact Sasaki manifold with $c_1^2(S) > 0$ and $c_1(D) = 0$ and $S = (g, \xi, \eta, \Phi)$ by the associated Sasakian structure. By assumption, we may assume $[\rho^T]_B = (2n + 2)[\omega^T]_B$. Put $\mathcal{J}(g)$ to be the set of all Sasaki-Einstein metrics on $S$ which is compatible with $g$ and $\mathcal{H} := \{ \varphi \in C^\infty_B(S) \mid (g^T_{ij} + \frac{\partial^2 \varphi}{\partial x^i \partial x^j})$ is positive definite $\}$. Clearly $g_\varphi \in \mathcal{J}(g)$ for each $\varphi \in \mathcal{H}$. We denote by $\mathcal{E}$ the set of all Sasaki-Einstein metrics in $\mathcal{J}(g)$. Throughout this section, we assume $\mathcal{E} \not= \emptyset$.

Let $V := \int_S (\frac{1}{2} d\eta)^n \wedge \eta$ and define the functionals $L_\eta$, $M_\eta$, $I_\eta$ and $J_\eta$ on $\mathcal{H}$ by

\[
L_\eta(\varphi) := \frac{1}{V} \int_a^b dt \int_S \varphi_t \left( \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
\[
M_\eta(\varphi) := -\frac{1}{V} \int_a^b dt \int_S \varphi_t (s^T(\varphi) - n(2n + 2)) \left( \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
\[
I_\eta(\varphi) := \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta \right)^n \wedge \eta - \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
\[
J_\eta(\varphi) := \frac{1}{V} \int_a^b dt \int_S \varphi_t \left( \frac{1}{2} d\eta \right)^n \wedge \eta - \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
where $\{ \varphi_t \mid t \in [a, b] \}$ is an arbitrary piecewise smooth path in $\mathcal{H}$ such that $\varphi_a = 0$ and $\varphi_b = \varphi$. These are the “Sasaki version” of the functionals defined on the space of Kähler potentials in [2] and have the similar properties to those. The precise definitions and basic properties can be seen in the Appendix.

Since $[\rho^T]_B = (2n + 2)[\omega^T]_B$, there exists a unique basic function $h \in C^\infty_B(S)$ which satisfies $\rho^T - (2n + 2) \omega^T = \sqrt{-1} \partial \bar{\partial} g h$ and $\int_S (e^h - 1)(\frac{1}{2} d\eta)^n \wedge \eta = 0$. Consider the following one-parameter families of equations;

\[
\text{det}(g^T_{ij} + \frac{\partial^2 \varphi}{\partial x^i \partial x^j}) = \exp(-t(2n + 2) \psi_t + h); \quad t \in [0, 1],
\]
\[
\text{det}(g^T_{ij} + \frac{\partial^2 \varphi}{\partial x^i \partial x^j}) = \exp(-t(2n + 2) \varphi_t - L_\eta(\varphi_t) + h); \quad t \in [0, 1],
\]
where solutions $\psi_t$ and $\varphi_t$ are both required to belong to $\mathcal{H}$. Note that, for both equations, these are just the transverse Kähler-Einstein equation at $t = 1$. As a
remark in [2], there is no difference between (21) and (22) in finding solutions for $t \neq 0$.

**Remark 4.1.** Choose an arbitrary $t \in [0, 1]$. Let $\psi_t$ (resp. $\varphi_t$) be a solution of (21) (resp. (22)) and $g_t$ be the Sasaki metric corresponding to the Sasaki structure $\eta_{\psi_t}$ (resp. $\eta_{\varphi_t}$). Then $g_t$ satisfies $\rho_t^T = t(2n+2)\omega_t^T + (1-t)(2n+2)\omega^T$, and in particular we have $\rho_t^T - t(2n+2)\omega_t^T \geq 0$. Furthermore if $t \neq 0$, then $\rho_t^T - t(2n+2)\omega_t^T$ is strictly positive.

We first consider the existence of the equation (22) at $t = 0$. For the equation (21), a result of El-Kacimi-Alaoui [5] guarantees the existence of a solution at $t = 0$. Then the existence and uniqueness of a solution of the equation (22) follows immediately.

**Theorem 4.2** (El Kacimi-Alaoui, [5]). If $t = 0$, then the equation (21) has a solution which is unique up to an additive constant.

**Corollary 4.3.** The equation (22) has a unique solution $\varphi_0$ at $t = 0$. The solution $\varphi_0$ satisfies $L_\eta(\varphi_0) = 0$.

**Proof.** Take any solution $\psi_0 \in \mathcal{H}$ of the equation (21) at $t = 0$ and define $\varphi_0 := \psi_0 - L_\eta(\psi_0)$. Then it is easy to check that $\varphi_0$ is a solution of the equation (22). This proves the existence of a solution of (22). Furthermore, for any solution $\varphi_0$ of (22) we have

$$
\int_S \left( \frac{1}{2} d\eta \right)^n \wedge \eta = \int_S \left( \frac{1}{2} d\eta \right)^n \wedge \eta \varphi
= \int_S \exp(-L_\eta(\varphi_0) + h) \left( \frac{1}{2} d\eta \right)^n \wedge \eta
= \exp(-L_\eta(\varphi_0)) \int_S e^h \left( \frac{1}{2} d\eta \right)^n \wedge \eta
= \exp(-L_\eta(\varphi_0)) \int_S \left( \frac{1}{2} d\eta \right)^n \wedge \eta.
$$

This shows that $L_\eta(\varphi_0) = 0$. Therefore, $\varphi_0$ is also a solution of equation (21) at $t = 0$. Now the required uniqueness now follows from Theorem 4.2 and that $L_\eta(\varphi_0) = 0$. 

For each $\varphi \in \mathcal{H}$, we denote by $\Box_\varphi := \Box_{B, g_\varphi}$ the basic complex Laplacian with respect to the Sasaki metric $g_\varphi$. The following proposition shows the local extension property of solutions of (22) for $t \in (0, 1)$ (see also [18]).

**Proposition 4.4.** Let $0 < \tau < 1$. Suppose that the equation (22) has a solution $\varphi_\tau$ at $t = \tau$. Then for some $\varepsilon > 0$, $\varphi_\tau$ uniquely extends to a smooth one parameter family $\{ \varphi_t \mid t \in [0, 1] \cap [\tau - \varepsilon, \tau + \varepsilon] \}$ of solutions of (22).

**Proof.** Let $2 \leq k \in \mathbb{Z}$ and fix $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. Let $C^{k, \alpha}_B(S)$ be the set of all basic functions which belong to $C^{k, \alpha}(S)$, and $\mathcal{H}^{k, \alpha}$ be the open set of all functions $\varphi \in C^{k, \alpha}_B(S)$ satisfying that $(g_{ij}^T + \frac{\partial^2 \varphi}{\partial z_i \partial z_j})$ is positive definite. Define $\Gamma : \mathcal{H}^{k, \alpha} \times \mathbb{R} \to C^{k-2, \alpha}_B(S)$ by

$$
\Gamma(\varphi, t) := \log \left( \frac{\det(g_{ij}^T + \frac{\partial^2 \varphi}{\partial z_i \partial z_j})}{\det(g_{ij}^T)} \right) + t(2n+2)\varphi + L_\eta(\varphi) - h.
$$
Then its Fréchet derivative $D_ϕΓ$ with respect to the first factor at $(ϕ, t)$ is given by

$$
D_ϕΓ(ψ) = (-□ϕ + t(2n + 2))ψ + \frac{1}{V} \int_S ψ(\frac{1}{2} dηϕ)^n \wedge ηϕ
$$

for each $ψ ∈ C^∞_B(S)$. Note that, by the well-known regularity theorem, we have $ψ ∈ C^∞_B(S)$ for every $(ϕ, t) ∈ \mathcal{H}^{k, α} × \mathbb{R}$ whenever $Γ(ϕ, t) = 0$. Since $Γ(ϕ_t, τ) = 0$, an application of the implicit function theorem now reduces the proof to showing that $D_ϕΓ$ is invertible at $(ϕ_t, τ)$. There are the following cases.

**Case 1:** $τ = 0$. Then $D_ϕΓ$ at $(ϕ_0, 0)$ is given by

$$
D_ϕΓ(ψ) = -□ϕ_0ψ + \frac{1}{V} \int_S ψ(\frac{1}{2} dηϕ_0)^n \wedge ηϕ_0,
$$

which is invertible.

**Case 2:** $τ ≠ 0$. First note $ρ^T_ϕ > τ(2n + 2)ω^T_ϕ$ by Remark 4.1. Then the similar argument of Lichnerowicz [10] tells us that the first positive eigenvalue of $□ϕ_ϕ$ is greater than $τ(2n + 2)$ (see also the proof of Theorem 2.4.3 in [6]). This shows that $D_ϕΓ|_{(ϕ_t, τ)}$ is invertible.

Remark 4.5. A Hamiltonian holomorphic vector field $X$ is said to be normalized if the Hamiltonian function $u_X$ satisfies that

$$
\int_S u_X e^{\frac{h}{2} |dη|^2} = 0.
$$

For any $X ∈ \mathfrak{h}$, there exists a constant $c$ such that $X + cξ$ is normalized Hamiltonian holomorphic vector field. We denote by $\mathfrak{h}_0$ the set of all normalized Hamiltonian holomorphic vector fields. If $\mathfrak{h}_0 = \{0\}$ and $τ = 1$, the result of Futaki, Ono and Wang (cf. Theorem 5.1 in [7]) tells us that $\ker(□ϕ_ϕ - (2n + 2)) ≅ \mathfrak{h}_0 = \{0\}$ and the first positive eigenvalue of $□ϕ_ϕ$ is greater than $2n + 2$. This shows that $D_ϕΓ|_{(ϕ_0, 1)}$ is invertible. Hence we obtain that Proposition 4.4 still holds for the case that $\mathfrak{h}_0 = \{0\}$ and $τ = 1$.

Next we shall give a bound for solutions of (22). By El Kacimi-Alaou’s generalization of Yau’s estimate [19] for transverse Monge-Ampère equations, the $C^0$-estimate for solutions $ϕ$ of (22) implies the $C^{2, α}$-estimate for them. First of all, we give a bound for the oscillation

$$
osc_Sϕ = \sup_S ϕ - \inf_S ϕ
$$

for $ϕ ∈ \mathcal{H}$. The following proposition is proved by the same way as Kähler geometry.

**Proposition 4.6.** Let $ϕ ∈ \mathcal{H}$. We assume that there exists real constants $A, δ > 0$ such that

$$
\|ψ\|_{L_2^{2m/(m-1)}} ≤ A\|dψ\|_{L_2}, \quad δ\|ψ\|^2_{L_2} ≤ \|dψ\|^2_{L_2}
$$

for every basic function $ψ ∈ C^∞_B(S)$ which satisfies $\int_S ψ dV_g = 0$. Moreover, suppose that

$$
\sup_S \frac{\text{det}(g^T_1 + \frac{∂^2 ϕ}{∂x_i ∂x_j})}{\text{det}(g_1)} ≤ B
$$

for some constant $B > 0$. Then there exists a real constant $C > 0$ depending only $A, δ$ and $B$ which satisfies

$$
osc_Sϕ ≤ C.
$$

Proposition 4.6 has the following important implication. In our proof, Theorem A is essential to obtain a bound for the infimum of basic functions $ϕ ∈ \mathcal{H}$. 


Proposition 4.7. Let $G = G_g$ be the Green function of the initial metric $g$ and $K$ be the real constant which satisfies $\inf G \geq -K$. For $\varphi \in \mathcal{H}$, assume that $\rho_T^T \geq t(2n + 2)T^T$ for some $t \in (0, 1]$. Then there exists a positive constant $\gamma > 0$ such that

$$\text{osc}_S \varphi \leq I_\eta(\varphi) + 2n \left( KV_0 + \frac{(2\pi)^2(2n - 1)}{t(2n + 2)} \right),$$

where $V_0 := V/n!$ is the volume of $(S, g)$.

Proof. First we observe that, by the identity $\frac{1}{2} d\varphi = \frac{1}{2} d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \varphi$, we have $\Box_0 \varphi \leq n$ and $\Box_0 \varphi \geq -n$.

Since the basic Laplacian coincides with the restriction of the Riemannian Laplacian to $C^\infty_\mathcal{B}(S)$ (cf. Proposition 2.7), we have

$$\varphi(p) = \frac{1}{V_0} \int_S \varphi dV_g + \int_S (G(p, q) + K)(\Delta_0 \varphi)(q) dV_g(q)$$

$$= \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta \right)^n \wedge \eta + \int_S (G(p, q) + K)(2 \Box_0 \varphi)(q) dV_g(q)$$

$$\leq \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta \right)^n \wedge \eta + 2nKV_0.$$

This leads the following estimate for $\varphi$;

$$\sup_S \varphi \leq \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta \right)^n \wedge \eta + 2nKV_0.$$  

(23)

On the other hand, by using the Green function $G_\varphi$ of $g_\varphi$ we have

$$\varphi(p) = \frac{1}{V_0} \int_S \varphi dV_{g_\varphi} + \int_S (G_\varphi(p, q) + K_\varphi)(\Delta_\varphi \varphi)(q) dV_{g_\varphi}(q)$$

$$= \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta_\varphi \right)^n \wedge \eta_\varphi + \int_S (G_\varphi(p, q) + K_\varphi)(2 \Box_\varphi \varphi)(q) dV_{g_\varphi}(q),$$

where $K_\varphi = \sup(-G_\varphi)$. Since $\text{Ric}^T \geq t(2n + 2)$ by assumption, we have $\text{Ric} \geq t(2n + 2) - 2 \geq -2$. Then Theorem 3.2 in [2] tells us that there exists a positive constant $\gamma > 0$ which depends only $n$ and satisfies

$$\varphi(p) \geq \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta_\varphi \right)^n \wedge \eta_\varphi - 2n\gamma \text{diam}(S, g_\varphi)^2.$$

Moreover, by Theorem A we have

$$\text{diam}(S, g_\varphi) \leq 2\pi \sqrt{\frac{2n - 1}{t}}.$$

Hence we obtain

$$\inf_S \varphi \geq \frac{1}{V} \int_S \varphi \left( \frac{1}{2} d\eta_\varphi \right)^n \wedge \eta_\varphi - 2n\gamma \frac{(2\pi)^2(2n - 1)}{t}$$

(24)

and hence

$$\text{osc}_S \varphi = \sup_S \varphi - \inf_S \varphi \leq I_\eta(\varphi) + 2n \left( KV_0 + (2n - 1)\gamma \frac{(2\pi)^2}{t} \right)$$

(25)

by inequalities (23) and (24).

We then see that a bound for $I_\eta$ on solutions of (22) implies a priori $C^0$-estimate for solutions.
Proposition 4.8. Let \( \varphi_t \) be a solution of (22) at \( t \) and \( A > 0 \) be a constant which satisfies

\[
I_\eta(\varphi_t) \leq A.
\]

Then there exists a real constant \( C > 0 \) depending only \( A, n \) and the initial metric \( g \) which satisfies

\[
\sup_S \varphi_t \leq C.
\]

Proof. By Proposition 4.7, there exists \( C_1 > 0 \) which depends only \( A, n \) and the initial metric \( g \) such that

\[
tosc_S \varphi_t \leq t \left( I_\eta(\varphi_t) + 2n \left( K_0 V_0 + \gamma \frac{(2\pi)^2(2n-1)}{t(2n+2)} \right) \right) \leq C_1,
\]

where \( K_0 \) is a constant which satisfies \( \inf G_\eta \geq -K_0 \). By integrating both sides of (22) we have

\[
\int_S \exp(-t(2n+2)\varphi_t - L_\eta(\varphi_t) + h) \left( \frac{1}{2} d\eta \right)^n \wedge \eta = \int_S \left( \frac{1}{2} d\eta \right)^n \wedge \eta
\]

and hence

\[
(26) \quad -t(2n+2)\varphi_t \left( p_t \right) - L_\eta(\varphi_t) + h \left( p_t \right) = 0
\]

for some \( p_t \in S \). Then there exists a constant \( C_2 > 0 \) which depends only \( A, n \) and the initial metric \( g \) such that

\[
| -t(2n+2)\varphi_t \left( p \right) - L_\eta(\varphi_t) + h \left( p \right) | \leq |t(2n+2)\varphi_t \left( p_t \right) - t(2n+2)\varphi_t \left( p \right) |
\]

\[
+ \ |h \left( p \right) - h \left( p_t \right) |
\]

\[
\leq t(2n+2)osc_S \varphi_t + 2 \sup_S |h| \leq C_2
\]

for each \( p \in S \). This shows that

\[
\sup_S \left| \log \frac{\det \left( g^{ij}_+ \right)}{\det \left( g_{ij}^- \right)} \right| \leq \left| \exp(-t(2n+2)\varphi_t - L_\eta(\varphi_t) + h) \right| \leq C_2.
\]

Hence by Proposition 4.6 there exists a constant \( C_3 > 0 \) such that

\[
osc_S \varphi \leq C_3.
\]

For \( p_t \) defined above we have

\[
|L_\eta(\varphi_t) - \varphi_t \left( p_t \right) | = \left| \frac{1}{V} \int_0^1 ds \int_S (\varphi_t - \varphi_t \left( p_t \right)) \left( \frac{1}{2} d\eta \right)^n \wedge \eta \right|
\]

\[
\leq \frac{1}{V} \int_0^1 ds \int_S osc_S \varphi_t \left( \frac{1}{2} d\eta \right)^n \wedge \eta
\]

\[
= osc_S \varphi_t \leq C_3.
\]

Then by combining (26), we obtain

\[
\{ 1 + t(2n+2) \} | \varphi_t \left( p_t \right) | \leq | \varphi_t \left( p_t \right) - L_\eta(\varphi_t) + h \left( p_t \right) |
\]

\[
\leq | \varphi_t \left( p_t \right) - L_\eta(\varphi_t) | + | h \left( p_t \right) |
\]

\[
\leq C_3 + \sup_S | h |
\]
and hence
\[ \sup_{S} \varphi_t = \text{osc}_S \varphi_t + \inf_{S} \varphi_t \leq \text{osc}_S \varphi + \varphi_t(p_1) \leq 2C_3 + \sup_{S} |h| \]
If we put \( C := 2C_3 + \sup_{S} |h| \), then it depends only \( A, n \) and the initial metric \( g \), and satisfies \( \sup_{S} \varphi_t \leq C \). This completes the theorem. \( \square \)

To obtain a bound for \( I_\eta \), we need to see the behavior of \( M_\eta \) along the solutions of (22). The following lemma asserts that \( M_\eta \) is non-increasing along the solutions, whose proof can be given as in [2].

**Lemma 4.9.** Let \( \{ \varphi_t \mid t \in [0, 1] \} \) be an arbitrary smooth family of solution of (22). Then
\[ \frac{dM_\eta(\varphi_t)}{dt} = -(1-t)(2n+2) \frac{d}{dt} (I_\eta(\varphi_t) - J_\eta(\varphi_t)) \leq 0. \]

Combining Proposition 4.8, Lemma 4.9 and Proposition A.3, we obtain the following result.

**Theorem 4.10.** Let \( 0 < \tau < 1 \). Then any solution \( \varphi_\tau \) of (22) at \( t = \tau \) uniquely extends to a smooth family \( \{ \varphi_t \mid t \in [0, \tau] \} \) of solutions of (22). In particular the equation (22) admits at most one solution at \( t = \tau \).

In particular, if \( \eta_0 = \{ 0 \} \) then there exists at most one Sasaki-Einstein metric of \( S \) which is compatible with \( g \) by Remark 4.5 and Theorem 4.10.

**proof of Theorem 4.10.** First note that a smooth family \( \{ \varphi_t \mid t \in [0, \tau] \} \) of solutions of (22) is unique if it exists because of the implicit function theorem and the uniqueness of solutions of (22) at \( t = 0 \). Hence it is sufficient to show that a solution \( \varphi_\tau \) of (22) at \( t = \tau \) can be extended to a smooth family \( \{ \varphi_t \mid t \in [0, \tau] \} \) of solutions of (22). We therefore assume, for contradiction, that any such extension is impossible. Then by Proposition 4.4 we have a maximal smooth family \( \{ \varphi_t \mid t \in [\sigma, \tau] \} \) of solutions of (22) for some \( 0 \leq \sigma \). In this proof we always denote by \( t \in \mathbb{R} \) a real number satisfying \( \sigma < t \leq \tau \). For arbitrary solution \( \varphi_t \) we have
\[ I_\eta(\varphi_t) \leq (n+1) (I_\eta(\varphi_t) - J_\eta(\varphi_t)) \leq (n+1) (I_\eta(\varphi_\tau) - J_\eta(\varphi_\tau)) \]
by Lemma 4.9 and Proposition A.3. In particular, there exists a constant \( A > 0 \) which is independent of \( t \) such that \( I_\eta(\varphi_t) \leq A \). Hence by Lemma 4.8, there exists a constant \( C > 0 \) which depends only \( A, n \) and the initial metric \( g \) such that \( \sup_{S} \varphi_t \leq C \). By El Kacimi-Alaoui’s generalization of Yau’s estimate, we can find a constant \( C_1 > 0 \) such that \( ||\varphi_t||_{C^{2,\alpha}} \leq C_1 \) for all \( t \in (\sigma, \tau) \) and fixed \( \alpha \in (0, 1) \). We now choose an arbitrary decreasing sequence \( \{ t_j \}_{j=1}^\infty \subset (\sigma, \tau) \) such that \( \lim_{j \to \infty} t_j = \sigma \). Then by Arzela-Ascoli’s theorem, there exists a convergent subsequence of \( \{ \varphi_t \}_{j=1}^\infty \), which leads to a contradiction to the maximality of \( \{ \varphi_t \mid t \in (\sigma, \tau) \} \). \( \square \)

4.2. Solutions at \( t = 1 \). Next we mention at \( t = 1 \). By assumption, \( \varepsilon \neq 0 \) and hence the equation (21) has a solution at \( t = 1 \). We begin the following lemma.

**Lemma 4.11.** Let \( \{ \varphi_t \}_{t \in [0, 1]} \) be a smooth family of solutions of equation (21). Put \( \varphi := \varphi_1 \) and \( \eta_{SE} := \eta_\varphi \). Then
\[ \int_S \varphi \left( \frac{1}{2} d\eta_{SE} \right)^n \wedge \eta_{SE} = 0 \]
for each \( \psi \in \ker(\Box_{SE} - (2n + 2)) \), where \( \Box_{SE} \) is the basic complex Laplacian for the Sasaki-Einstein metric \( g_{SE} = g_\varphi \).
Proof. By differentiating the logarithms of both sides of equality (21) at \( t = 1 \), we obtain

\[
\left( \Box_{SE} - (2n + 2) \right) \dot{\varphi}_1 |_{t=1} = (2n + 2) \varphi.
\]

Then the lemma follows immediately. \( \square \)

Consider the \( G \)-action on \( \mathcal{E} \). Let \( O \) be an arbitrary \( G \)-orbit in \( \mathcal{E} \). For each \( g_{SE} \in \mathcal{E} \), we can uniquely associate a function \( \varphi = \varphi(g_{SE}) \in \mathcal{H} \) such that \( g_{SE} = g_{\varphi} \) and \( \varphi \) satisfies the equation (21) at \( t = 1 \). Hence we can regard \( O \) as a subset of the all solutions of the equation (21) at \( t = 1 \) in \( \mathcal{H} \). By the identification, we endow \( O \) with the topology induced from the \( C^{2,\alpha} \)-norm on \( C^\infty_B(S) \). Then the \( G \)-action on \( O \) is clearly continuous. Hence the topology on \( O \) coincides with the natural topology of the homogeneous space \( O \cong G/K_{g_{SE}} \), where \( K_{g_{SE}} \) is the isotropic subgroup of \( G \) at \( g_{SE} \). For each \( \psi \in \ker(\Box_{SE} - (2n + 2)) \) we have associated normalized Hamiltonian holomorphic vector field \( X_\psi \):

\[
X_\psi = \psi \xi + \nabla^\psi \frac{\partial}{\partial z^1} - \eta(\nabla^\psi \frac{\partial}{\partial z^2}) \xi
\]

for a foliation coordinate \((x_0, z_1, \ldots, z_n)\) (see Theorem 5.1 of [7]). Let \( f_{\psi, t} \) be a corresponding one-parameter group; \( f_{\psi, t} = \exp(tX^{SE}_\psi) \), where \( X^{SE}_\psi \) is the real part of \( X_\psi \). We put \( g_{SE}(t) := f_{\psi, t}g_{SE} \) and \( \varphi(t) := \varphi(g_{SE}(t)) \). Then we can check easily that \( \varphi(0) = \psi + C \) for some \( C \in \mathbb{R} \). On the other hand, since \( \varphi(t) \) satisfies the equation (21) we have \( \Box_{SE} \dot{\varphi}(0) = (2n + 2) \dot{\varphi}(0) \) by differentiating the equality (21). This shows that \( C = 0 \) and hence \( \dot{\varphi}(0) = \psi \).

Conversely, for each smooth curve \( g(t) \in O \) with \( g(0) = g_{SE} \), take the corresponding smooth functions \( \varphi(t) \in \mathcal{H} \). Then we have \( \Box_{SE} \dot{\varphi}(0) = (2n + 2) \dot{\varphi}(0) \) by differentiating the identity (21). Thus we obtain

\[ T_{g_{SE}} O \cong \ker(\Box_{SE} - (2n + 2)). \]

Define \( \iota := (I_\eta - J_\eta)|_O \geq 0 : O \to \mathbb{R} \). The basic properties of \( \iota \) are as follows.

Lemma 4.12. Let \( g_{SE} \in O \). Then the followings are equivalent.

1. \( g_{SE} \) is a critical point of \( \iota \).
2. \( \varphi(g_{SE}) \) satisfies the condition (28).

This is immediately from (43). The following lemma shows the existence of a minimizer of \( \iota \).

Lemma 4.13. The functional \( \iota \) is proper. In particular, its minimum is always attained at some point of the orbit \( O \).

Proof. Let \( g_{SE} \in O \) with \(|\iota(g_{SE})| \leq r \) for some \( r > 0 \). Then by Proposition A.3 we have \( I_\eta(\varphi) \leq (n + 1)r \) for \( \varphi := \varphi(g_{SE}) \). Since \( \rho^T_\varphi = (2n + 2)\omega^T_\varphi \), we have

\[
\mathrm{osc}_S \varphi \leq C_r
\]

by Proposition 4.7, where \( K := \sup(-G_\eta) \) and

\[
C_r = (n + 1)r + 2n \left( KV_0 + \gamma \frac{(2\pi)^2(2n - 1)}{6n} \right).
\]

On the other hand, from (21) we obtain

\[
\int_S \left( \frac{1}{2} d\eta \right)^n \wedge \eta = \int_S \left( \frac{1}{2} d\eta_\varphi \right)^n \wedge \eta_\varphi = \int_S \exp(-(2n + 2)\varphi + h)\left( \frac{1}{2} d\eta \right)^n \wedge \eta
\]
Lemma 4.14. Let \( \iota \) of Hermitian pairings on basic forms induced from the transverse Kähler metric Kähler geometry (see [2] and Theorem 5.1 in [7]); for any \( (29) \)

\[
\text{need the following formula for } g
\]

the transverse Kähler metric \( g \).

Thus if we put \( C_r^1 := C_r + \frac{1}{2n+2} \sup_S |h| \) we have \( \sup_S \varphi \leq C_r^1 \). Then the result follows from El Kacimi-Alaoui’s generalization of Yau’s estimate [19]. \( \square \) \( \square \)

Then we shall calculate the Hessian of \( \iota \) at a critical point. For the proof, we need the following formula for \( g_{SE} \in \mathcal{E} \), which is shown by the same calculation as Kähler geometry (see [2] and Theorem 5.1 in [7]);

\[
(29) \quad \Box_{SE}(\bar{\partial}_B \psi, \bar{\partial}_B \varphi') = -\langle \bar{\partial}_B \bar{\partial}_B \psi, \bar{\partial}_B \bar{\partial}_B \varphi' \rangle + \langle \bar{\partial}_B(\Box_{SE} \psi), \bar{\partial}_B \varphi' \rangle,
\]

for any \( \varphi' \in \ker(\Box_{SE} - (2n + 2)) \) and \( \psi \in C^\infty_\mathcal{E}(S) \), where \( \langle \cdot, \cdot \rangle \) is the natural Hermitian pairings on basic forms induced from the transverse Kähler metric \( g^T_{SE} \).

**Lemma 4.14.** Let \( g_{SE} \in O \) be a critical point of \( \iota \). Then the Hessian \( (\text{Hess } \iota)_{g_{SE}} \) of \( \iota \) at \( g_{SE} \) is given by

\[
(\text{Hess } \iota)_{g_{SE}}(\varphi', \varphi'') = \frac{2n+2}{V} \int_S \left( 1 - \frac{1}{2} \Box_{SE} \varphi' \right) \varphi' \varphi'' \left( \frac{1}{2} \eta_{\varphi'} \right)^n \land \eta_{\varphi''}
\]

for each \( \varphi', \varphi'' \in \ker(\Box_{SE} - (2n + 2)) \cong T_{g_{SE}} O \), where \( \varphi := \varphi(g_{SE}) \).

**Proof.** Let \( \{ \varphi_{s,t} \mid (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \} \) be a smooth family of functions satisfying the following conditions;

\[
g_{\varphi_{s,t}} \in O, \quad \varphi_{0,0} = \varphi, \quad \frac{\partial \varphi_{s,t}}{\partial s}|_{s,t=0} = \varphi', \quad \frac{\partial \varphi_{s,t}}{\partial t}|_{s,t=0} = \varphi''.
\]

We shall denote \( \Box_{\varphi_{s,t}} \) by \( \Box_{s,t} \) for brevity. Since \( \varphi_{s,t} \) satisfies (21), we obtain

\[
(30) \quad (-\Box_{s,t} + (2n + 2)) \frac{\partial \varphi_{s,t}}{\partial t} = 0
\]

by differentiating the equation (21) with respect to \( t \). Further differentiation with respect to \( s \) yields

\[
(31) \quad -\left( \frac{\partial_{\bar{\partial}_B} \bar{\partial}_B \varphi_{s,t}}{\partial s}, \frac{\partial_{\bar{\partial}_B} \bar{\partial}_B \varphi_{s,t}}{\partial t} \right)_{s,t} + (-\Box_{s,t} + (2n + 2)) \left( \frac{\partial^2 \varphi_{s,t}}{\partial s \partial t} \right) = 0,
\]

where \( \langle \cdot, \cdot \rangle_{s,t} \) is the natural Hermitian pairing on complex basic forms induced from the transverse Kähler metric \( g^T_{\varphi_{s,t}} \). By evaluating this at \( (s, t) = (0, 0) \), we obtain

\[
(-\Box_{SE} + (2n + 2)) \left( \frac{\partial^2 \varphi_{s,t}}{\partial s \partial t} \right)|_{s,t=0} = \langle \bar{\partial}_B \bar{\partial}_B \varphi', \bar{\partial}_B \bar{\partial}_B \varphi'' \rangle_{0,0}
\]

\[
= -\Box_{SE} \langle \bar{\partial}_B \varphi', \bar{\partial}_B \varphi'' \rangle_{0,0} + \langle \bar{\partial}_B \Box_{SE} \varphi', \bar{\partial}_B \varphi'' \rangle_{0,0}
\]

\[
= (-\Box_{SE} + (2n + 2)) \langle \bar{\partial}_B \varphi', \bar{\partial}_B \varphi'' \rangle_{0,0}.
\]

This shows that

\[
\frac{\partial^2 \varphi_{s,t}}{\partial s \partial t}|_{s,t=0} \equiv \langle \bar{\partial}_B \varphi', \bar{\partial}_B \varphi'' \rangle_{0,0} (\text{mod } \ker(-\Box_{SE} + (2n + 2)) \otimes \mathbb{C})
\]

\[
\equiv \langle \bar{\partial}_B \varphi'', \bar{\partial}_B \varphi' \rangle_{0,0} (\text{mod } \ker(\Box_{SE} + (2n + 2)) \otimes \mathbb{C}).
\]
Now we can calculate the Hessian of $\iota$;

$$(\text{Hess } \iota)_{g_{SE}}(\varphi', \varphi'') = \frac{\partial^2}{\partial s \partial t} (J_1(\varphi_{s,t}) - J_1(\varphi_{t,\varphi})) |_{s,t=0} = \frac{\partial}{\partial s} \left\{ \frac{1}{V} \int \varphi_{s,t} \square_{s,t} \frac{\partial^2 \varphi_{s,t}}{\partial t^2} \left( \frac{1}{2} d\eta_{\varphi_{s,t}} \right) n \wedge \eta_{\varphi_{s,t}} \right\} |_{s,t=0} = \frac{\partial}{\partial s} \left\{ \frac{2n+2}{V} \int \varphi_{s,t} \frac{\partial^2 \varphi_{s,t}}{\partial t^2} \left( \frac{1}{2} d\eta_{\varphi_{s,t}} \right) n \wedge \eta_{\varphi_{s,t}} \right\} |_{s,t=0}$$

(cf. (30))

$$= \frac{2n+2}{V} \int \varphi' \varphi'' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} + \frac{2n+2}{V} \int \varphi' \frac{\partial^2 \varphi_{s,t}}{\partial s \partial t} |_{s,t=0} \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} - \frac{2n+2}{V} \int \varphi' \square_{SE} \varphi' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi}$$

$$= \frac{2n+2}{V} \int \varphi' \varphi'' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} + \frac{2n+2}{V} \int \varphi' \frac{\partial^2 \varphi_{s,t}}{\partial s \partial t} |_{s,t=0} \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} - \frac{2n+2}{V} \int \varphi' \square_{SE} \varphi' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi}$$

$$= \frac{2n+2}{V} \int \left\{ \varphi' \varphi'' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} \right\} + \frac{2n+2}{V} \int \left\{ \frac{1}{2} \square_{SE} \varphi' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi} \right\} = \frac{2n+2}{V} \int \left( 1 - \frac{1}{2} \square_{SE} \varphi \right) \varphi' \varphi'' \wedge \left( \frac{1}{2} d\eta_{\varphi} \right) n \wedge \eta_{\varphi}$$

□□

The following proposition is crucial for our proof of Theorem B.

**Proposition 4.15.** For every critical point $g_{SE} \in O$ of $\iota$ with non-degenerate Hessian, $\varphi_1 := \varphi(g_{SE})$ can be extended to a smooth family $\{\varphi_t \mid t \in [1-\varepsilon, 1]\}$ of solutions of (21) for some $\varepsilon > 0$.

**Proof.** Put $L^2_B(S;g_{SE})$ to be the closure of $C^\infty_{SE}(S)$ in $L^2(S;g_{SE})$. Let $W := \ker(\square_{\varphi_1} - (2n + 2))$ and $P$ be the orthogonal projection from $L^2_B(S;g_{SE})$ to $W$. Fixing $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, define $W_{k,\alpha}^\perp$ to be the intersection of the orthogonal complement of $W$ and $C^k_{B,\alpha}(S)$, for $k = 0, 1, 2, \cdots$. Recall that $\varphi_1$ belongs to $W_{k,\alpha}^\perp$. Let $k \geq 2$, and consider the mapping

$$\Psi : \mathbb{R} \times \mathcal{H}^{k,\alpha}(S) \to C^{k-2,\alpha}_{B}(S), \quad \Psi(t, \varphi) := \log \left( \frac{\det(g_{ij}^B) + \frac{\partial^2 \varphi}{\partial x_i \partial x_j}}{\det(g_{ij}^B)} \right) + t(2n+2) \varphi - h.$$

Note that, by the well-known regularity theorem, any $\varphi \in \mathcal{H}^{k,\alpha}$ satisfying $\Psi(t, \varphi) = 0$ is automatically smooth. For each $\varphi \in \mathcal{H}^{k,\alpha}(S)$, we write

$$\varphi = \varphi_1 + \psi + \theta,$$

where $\psi := P(\varphi - \varphi_1) \in W$ and $\theta := (1 - P)(\varphi - \varphi_1) \in W_{k,\alpha}^\perp$. Now the equation

$$(32) \quad \Psi(t, \varphi) = 0$$
is written in the form
\[ P\Psi(t, \varphi_1 + \psi + \theta) = 0, \quad \Psi_0(t, \psi, \theta) = 0, \]
where \( \Psi_0 \) is defined by
\[ \Psi_0(t, \psi, \theta) := (1 - P)\Psi(t, \varphi_1 + \psi + \theta). \]
Then clearly \( \Psi_0(1, 0, 0) = 0 \) and the Fréchet derivative \( D_\theta \Psi_0|_{(1,0,0)} \) of \( \Psi_0 \) with respect to \( \theta \) at \( (t, \psi, \theta) = (1, 0, 0) \) is
\[ D_\theta \Psi_0|_{(1,0,0)}(\theta') = (\Box_{SE} + (2n + 2))\theta', \]
which is invertible. Hence by the implicit function theorem we obtain a smooth mapping \( U \ni (t, \psi) \mapsto \theta_{t, \psi} \in W_{k,\alpha}^\perp \) of a neighborhood \( U \) of \( (1, 0) \in \mathbb{R} \times W \) to \( W_{k,\alpha}^\perp \) such that
\[ \begin{align*}
(1) & \quad \theta_{1,0} = 0, \\
(2) & \quad \|\theta_{t,\psi}\|_{C^{k,\alpha}} \leq \delta \text{ on } U \text{ for some } \delta > 0 \quad \text{and} \\
(3) & \quad \Psi_0(t, \psi, \theta) = 0 \text{ (where } \|\theta\|_{C^{k,\alpha}} \leq \delta \text{) is, as an equation in } \theta \in C_k^{\perp}(S), \text{ uniquely solvable in the form } \theta = \theta_{t,\psi} \text{ on } U. 
\end{align*} \]
By differentiating the identity \( \Psi_0(t, \psi, \theta_{t,\psi}) = 0 \) at \( (1, 0) \) we obtain
\[ \begin{align*}
& (\Box_{SE} + (2n + 2)) \left( \frac{\partial}{\partial t} \theta_{t,\psi}|_{(1,0)} \right) = -(2n + 2)\varphi_1, \\
& (D_\psi \theta_{t,\psi})|_{(1,0)}(\psi') = 0 \quad \text{for all } \psi' \in W.
\end{align*} \]
Then the equation (32), on a small neighborhood of \( \varphi_1 \), reduces to
\[ \Psi_1(t, \psi) = 0, \]
where \( \Psi_1(t, \psi) := P\Psi(t, \varphi_1 + \psi + \theta_{t,\psi}) \) for \( (t, \psi) \in U \). Recall that \( \Psi(1, \varphi) = 0 \) for all \( \varphi \in O \). Hence \( \Psi_1 = 0 \) on \( \{ t = 1 \} \) and therefore the mapping
\[ U|_{t \neq 1} \ni (t, \psi) \mapsto \Psi_2(t, \psi) := \frac{\Psi_1(t, \psi)}{t - 1} \]
naturally extends to a smooth map on \( U \) to \( W \) (denoted by the same \( \Psi_2 \)). Note that, for \( t = 1 \), we have
\[ \Psi_2(1, 0) = \frac{\partial \Psi_1(1, 0)}{\partial t}|_{t=1} = 0. \]
Hence, if the Fréchet derivative \( D_\psi \Psi_2|_{(1,0)} \) is invertible, we obtain the desired result. The Fréchet derivative \( D_\psi \Psi_2|_{(1,0)} \) is written in the following form, whose proof is given later.

**Lemma 4.16.** For each \( \psi', \psi'' \in W \),
\[ \int_S D_\psi \Psi_2|_{(1,0)}(\psi') \cdot \psi''(\frac{1}{2}d\eta_1)^n \wedge \eta_1 = (2n + 2) \int_S \left( 1 - \frac{1}{2} \Box_{SE} \right) \psi' \psi''(\frac{1}{2}d\eta_1)^n \wedge \eta_1 = V(\text{Hess } \psi', \psi''). \]

Then by this lemma, \( D_\psi \Psi_2|_{(1,0)} \) is invertible. Hence the implicit function theorem shows that the equation \( \Psi_2(t, \psi) = 0 \) in \( \psi \) is uniquely solvable in a neighborhood of \( (1, 0) \) to produce a smooth curve \( \{ \psi(t) \mid t \in (1 - \varepsilon, 1) \} \) in \( \ker(\Box_{SE} - 2n + 2) \) such that \( \psi(1) = 0 \) and \( \Psi_2(t, \psi(t)) = 0 \). Therefore, we have \( \Psi(t, \varphi_1 + \psi(t) + \theta_{t,\psi(t)}) = 0 \) for \( t \in (1 - \varepsilon, 1) \) and hence \( \{ \varphi_1 + \psi(t) + \theta_{t,\psi(t)} \mid t \in (1 - \varepsilon, 1) \} \) is a one parameter family of solutions of (21).
Finally, we shall prove Lemma 4.16. First we shall show the following formula:

\[(35) \quad -\int_S \varphi' (\partial_B \bar{\partial}_B \varphi, \partial_B \bar{\partial}_B \varphi', \partial_B \varphi') \varphi_1 (\frac{1}{2} \eta) \land \eta\]

\[= \int_S (2n + 2) \varphi' \varphi'' - \langle \partial_B \varphi', \partial_B \varphi'' \rangle \varphi_1 ((-\Box \varphi_1 + (2n + 2)) \psi)(\frac{1}{2} \eta) \land \eta\]

for each \(\varphi', \varphi'' \in W\) and \(\psi \in C^\infty(S)\). For (35), put \(\zeta := (-\Box \varphi_1 + (2n + 2)) \psi\). Then we have

\[\int_S (2n + 2) \varphi' \varphi'' - \langle \partial_B \varphi', \partial_B \varphi'' \rangle \varphi_1 (\frac{1}{2} \eta) \land \eta\]

\[= -\sqrt{-1} \int_S n \zeta (\varphi' \partial_B \bar{\partial}_B \varphi'' + \partial_B \varphi' \bar{\partial}_B \varphi'') \land (\frac{1}{2} \eta) \land \eta\]

\[= -\sqrt{-1} \int_S n \zeta \partial_B (\varphi' \bar{\partial}_B \varphi'') \land (\frac{1}{2} \eta) \land \eta\]

\[= \sqrt{-1} \int_S n \varphi' \partial_B \zeta \land \partial_B \varphi'' \land (\frac{1}{2} \eta) \land \eta\]

\[= \int_S \varphi' (\partial_B \zeta, \partial_B \varphi'') \varphi_1 (\frac{1}{2} \eta) \land \eta\]

\[= \int_S \varphi' (\partial_B(-\Box \varphi_1 + (2n + 2)) \psi, \partial_B \varphi'') \varphi_1 (\frac{1}{2} \eta) \land \eta\]

\[= \int_S \varphi' (-\Box \varphi_1 (\partial_B \psi, \partial_B \varphi'') \varphi_1 + (2n + 2)(\partial_B \psi, \partial_B \varphi'') \varphi_1) (\frac{1}{2} \eta) \land \eta\]

\[- \int_S \varphi' (\partial_B \bar{\partial}_B \psi, \partial_B \bar{\partial}_B \varphi'') \varphi_1 (\frac{1}{2} \eta) \land \eta\]

\[= \int_S \varphi' (\partial_B \bar{\partial}_B \psi, \partial_B \bar{\partial}_B \varphi'') \varphi_1 (\frac{1}{2} \eta) \land \eta.\]

This shows (35). Then, by (34) we have

\[D_\psi \Psi_2 \mid_{(1,0)}(\psi') = D_\psi \partial \Psi_1 \mid_{(1,0)}(\psi')\]

\[= (2n + 2) \psi' - P \left\langle \partial_B \bar{\partial}_B \left( \frac{\partial \theta_\psi}{\partial t} \right) \mid_{(1,0)}, \partial_B \bar{\partial}_B \psi \right\rangle \varphi_1\]

for each \(\psi' \in W\). Hence, it follows that

\[\int_S D_\psi \Psi_2 \mid_{(1,0)}(\psi') \cdot \psi'' (\frac{1}{2} \eta) \land \eta \]

\[= \int_S \left\{ (2n + 2) \psi' \psi'' - \psi'' \left\langle \partial_B \bar{\partial}_B \left( \frac{\partial \theta_\psi}{\partial t} \right) \mid_{(1,0)}, \partial_B \bar{\partial}_B \psi \right\rangle \varphi_1 \right\} (\frac{1}{2} \eta) \land \eta\]

\[= (2n + 2) \int_S (\psi' \psi'' - ((2n + 2) \psi' \psi'' - \langle \partial_B \psi', \partial_B \psi'' \rangle \varphi_1) \psi \right\} (\frac{1}{2} \eta) \land \eta\]

\[= (2n + 2) \int_S \left( 1 - \frac{1}{2} \Box S E \varphi \right) \psi' \psi'' (\frac{1}{2} \eta) \land \eta\]

\[= V(\text{Hess } \nu)_{g_E}(\psi', \psi'').\]

This proves the lemma. □
Remark 4.17. Fix a $G$-orbit $\mathcal{O}$ in $\mathcal{E}$ arbitrary and take a minimizer $g_{SE}$ of $\iota : O \to \mathbb{R}$. Then $g_{SE}$ is a critical point of $\iota$ and the Hessian is automatically positive semi-definite. We shall realize a critical point for $\iota$ with positive definite Hessian by a small change of the initial metric $g$.

For sufficient small $\delta \in (0, 1)$, define $g^\delta := g_{SE}^\delta$, where $\varphi_1 = \varphi(g_{SE})$. The associated transverse Kähler form is given by $(\omega^\delta)^T = (1 - \delta)\omega^T + \delta \omega^T = \omega^T + \delta \sqrt{-1}\partial_B \partial_B \varphi_1$. When the role of the initial metric $g$ is played by the new Sasaki metric $g^\delta$, the Sasaki-Einstein metric $g_{SE} = g_{SE}^1$ corresponds to $g_{SE}^\delta$ for a basic function

$$\varphi_1^\delta = (1 - \delta)\varphi_1 + C_\delta,$$

where $C_\delta$ is a constant. Then $g_{SE}$ is a critical point of $\iota^\delta$ with positive definite Hessian, where $\iota^\delta$ denotes the one corresponding to $\iota$. Indeed, for each $\psi \in \ker(\Box_{SE})$ we have

$$\int_S \psi \varphi_1^\delta \left(\frac{1}{2}d\eta_{SE}\right)^n \wedge \eta_{SE} = (1 - \delta) \int_S \psi \varphi_1 \left(\frac{1}{2}d\eta_{SE}\right)^n \wedge \eta_{SE} = 0$$

and hence $\varphi_1^\delta$ is a critical point of $\iota^\delta$ by Lemma 4.12. Moreover, by Lemma 4.14 we have

$$(\text{Hess } \iota^\delta)_{SE}(\psi', \psi'') = \frac{2n + 2}{V} \int_S \left(1 - \frac{1}{2}\Box_{SE}\varphi_1^\delta \right) \psi' \psi'' \left(\frac{1}{2}d\eta_{SE}\right)^n \wedge \eta_{SE}$$

$$= (1 - \delta)(\text{Hess } \iota)_{SE}(\psi', \psi'')$$

$$+ \delta \frac{2n + 2}{V} \int_S \psi' \psi'' \left(\frac{1}{2}d\eta_{SE}\right)^n \wedge \eta_{SE},$$

where $\Box_{SE}$ is the basic complex Laplacian with respect to the Sasaki metric $g_{SE}$. This shows that $(\text{Hess } \iota^\delta)_{SE}$ is positive definite. Hence by the argument in the last subsection and Proposition 4.15, $\varphi_1^\delta$ can be uniquely extended to a smooth family $\{\varphi_t^\delta \mid t \in [-1, 1]\}$ of the equation (21) with respect to the initial metric $g^\delta$.

4.3. Proof of Theorem B. Let $O^\prime$ and $O''$ be arbitrary $G$-orbits in $\mathcal{E}$. Then by the argument of Remark 4.17, for a suitable choice of the initial metric $g_0$, the function $\iota' : O' \ni g' \mapsto I(g_0, g') - J(g_0, g') \in \mathbb{R}$ has a critical point $g_{SE}^0 \in O'$ with positive definite Hessian. Recall that the function $\iota'' : O'' \ni g'' \mapsto I(g_0, g'') - J(g_0, g'') \in \mathbb{R}$ takes its minimum at some point $g_{SE}^{0'} \in O''$. We now put $g_0^\delta := (1 - \delta)g_0 + \delta g_{SE}^{0'}$ for $\delta \in [0, 1]$. Again by the argument of Remark 4.17 applied to $O''$, $g_{SE}^\delta$ is a critical point of $(\iota'')^\delta$ with positive definite Hessian whenever $\delta \in (0, 1)$. Hence by Proposition 4.15, $\varphi_t' := \varphi(g_{SE}^\delta)$ (with respect to the initial metric $g_0^\delta$) can be extended uniquely to a smooth family $\{\varphi_t' \mid t \in (1 - \varepsilon, 1]\}$ of (21) with respect to the initial metric $g_0^\delta$.

We finally consider the functional $\iota'_\delta : O' \ni g' \mapsto I(g_0, g') - J(g_0, g') \in \mathbb{R}$. Note that $\iota'_\delta$ converges to $\iota'$ as $\delta$ tends to 0. Then for a sufficiently small $\delta > 0$, $g_{SE}^\delta$ is a critical point of $\iota'_\delta$ with positive definite Hessian. Hence $\varphi_t' := \varphi(g_{SE}^\delta)$ (with respect to the initial metric $g_0^\delta$) can be extended uniquely to a smooth family $\{\varphi_t' \mid t \in (1 - \varepsilon, 1]\}$ of (21) with respect to the initial metric $g_0^\delta$. By Theorem 4.10 and the equivalence between the equations (21) and (22) for $t \in (0, 1]$, we conclude $g' = g''$. Thus, $O' = O''$ and the proof is now complete.
5. Concluding remarks

Theorem A plays a central role to obtain a priori $C^0$-estimate for solutions of the equation (22) (cf. Proposition 4.7). We remark that a similar estimate can be obtained without the diameter bound in the following way. Let $(S, g)$ be a $(2n+1)$-dimensional Sasaki manifold with Sasaki structure $S = \{g, \xi, \eta, \Phi\}$. Consider a solution $\varphi_t \in \mathcal{H}$ of the equation (22) at $t$. Note that, as shown in Remark 4.1, $\varphi_t$ satisfies

$\rho^T_{\varphi_t} \geq t(2n+2)\omega^T_{\varphi_t}$.

We introduce a family of contact structures by multiplication of positive constant $\mu$,

\begin{align}
\eta_{\varphi_t, \mu} &= \mu^{-1}\eta_{\varphi_t}, \\
\xi_{\mu} &= \mu \xi.
\end{align}

Then we see that $(\eta_{\varphi_t, \mu}, \xi_{\mu})$ gives a Sasakian structure with the metric $g_{\varphi_t, \mu}$ on $S$. The transversal metric $g^T_{\varphi_t, \mu}$ is given by $g^T_{\varphi_t, \mu} = \mu^{-1}g^T_{\varphi_t}$, and the volume form of $g_{\varphi_t, \mu}$ is given by

\begin{align}
\eta_{\varphi_t, \mu} \wedge (d\eta_{\varphi_t, \mu})^n &= \mu^{-(n+1)}\eta_{\varphi_t} \wedge (d\eta_{\varphi_t})^n.
\end{align}

Let $\Delta_{\varphi_t, \mu}$ be the Laplacian with the Green function $G_{\varphi_t, \mu}$ and $\text{Ric}_{\varphi_t, \mu}$ the Ricci tensor with respect to $g_{\varphi_t, \mu}$. The following is a well-known fact on the Green function of compact Riemannian manifolds.

**Fact 5.1** ([2]). Let $(S, g)$ be a $(2n+1)$-dimensional compact Riemannian manifold with the Green function $G(p, q)$. We assume

\begin{align}
\text{diam}(S, g)^2 \text{Ric} \geq -\varepsilon^2 g
\end{align}

for a constant $\varepsilon \geq 0$. Then there exists a constant $\gamma(n, \varepsilon) > 0$ which depends only on $m$ and $\varepsilon$ and we have

\begin{align}
G(p, q) \geq -\gamma(n, \varepsilon)\frac{\text{diam}(S, g)^2}{\text{Vol}(S, g)}
\end{align}

for the Green function of $(S, g)$.

Fact 5.1 has the following implication on the volume and the diameter of $(S, g_{\varphi_t, \mu})$.

**Proposition 5.2.** Let $(S, g)$ be a $(2n+1)$-compact Sasakian manifold and $\varphi_t$ a solution of (22) at $t$. If we set $\mu = t^{-1}$, then we have estimates of the volume and the diameter with respect to the metric $g_{\varphi_t, \mu}$,

\begin{align}
\text{Vol}(S, g_{\varphi_t, \mu}) = t^{n+1}V_0, \\
\text{diam}(S, g_{\varphi_t, \mu}) \leq \pi,
\end{align}

where $V_0 = \int_S \frac{1}{n!}(d\eta)^n \wedge \eta$.

**Proof.** Since $\mu = t^{-1}$, we have

\begin{align}
\text{Vol}(S, g_{\varphi_t, \mu}) &= \int_S \frac{1}{n!}(d\eta_{\varphi_t, \mu})^n \wedge \eta_{\varphi_t, \mu} = \mu^{-(n+1)} \int_S \frac{1}{n!}(d\eta_{\varphi_t})^n \wedge \eta_{\varphi_t} \\
&= t^{n+1} \int_S \frac{1}{n!}(d\eta)^n \wedge \eta \\
&= t^{n+1}V_0.
\end{align}
Furthermore, we have
\[ \text{Ric}_{\varphi_t,\mu}(X,Y) = \text{Ric}_{\varphi_t}^T(X,Y) - 2g_{\varphi_t,\mu}(X,Y) \]
and
\[ \mu g_{\varphi_t,\mu}(X,Y) = g_{\varphi_t}(X,Y) \]
for all \( X, Y \in \ker \eta_{\varphi_t,\mu} \). Since the transversal Ricci curvature is invariant under the multiplication by positive constant of a transversal metric, thus \( \text{Ric}_{\varphi_t,\mu}^T = \text{Ric}_{\varphi_t}^T \),
for all \( X, Y \in \ker \eta_{\varphi_t,\mu} \). Then we have
\[ \text{Ric}_{\varphi_t,\mu}^T(X,Y) = \text{Ric}_{\varphi_t}^T(X,Y) \]
\[ \geq t(2n+2)\mu g_{\varphi_t,\mu}(X,Y) \]
\[ = t(2n+2)\mu g_{\varphi_t,\mu}(X,Y) \]
\[ = t(2n+2)\mu g_{\varphi_t,\mu}(X,Y). \]

Therefore we have
\[ \text{Ric}_{\varphi_t,\mu}(X,Y) \geq t(2n+2)\mu g_{\varphi_t,\mu}(X,Y) - 2g_{\varphi_t,\mu}(X,Y). \]

Since \( \mu = t^{-1} \), we have \( \text{Ric}_{\varphi_t,\mu}(X,Y) \geq 2ng_{\varphi_t,\mu}(X,Y) \). It follows that
\[ \text{Ric}_{\varphi_t,\mu}(X,\xi_{\mu}) = 2\eta_{\varphi_t,\mu}(X) \]
\[ = 2ng_{\varphi_t,\mu}(X,\xi_{\mu}) \]
for all \( X \in TS \). Therefore we obtain
\[ \text{Ric}_{\varphi_t,\mu} \geq 2ng_{\varphi_t,\mu}. \]

Finally, we have \( \text{diam}(S, g_{\varphi_t,\mu}) \leq \pi \) by Myers’ theorem.

Now we consider about the oscillation \( \text{osc}_{S} \varphi_t \) of \( \varphi_t \). As shown in Proposition 4.7, we have
\[ \sup_{S} \varphi_t \leq \frac{1}{V} \int_{S} \varphi_t(d\eta)^n \wedge \eta - 2nKV_0, \]
where \(-K\) is the infimum of the Green function \( G \) with respect to the metric \( g \). We shall give an estimate for the infimum of \( \varphi_t \). Let \( \Delta_{t,\mu} \) be the Laplacian and \( G_{t,\mu} \) the Green function with respect to the Sasaki metric \( g_{\varphi_t,\mu} \). By Proposition 5.2 and Fact 5.1, we have
\[ G_{t,\mu} \geq -\gamma(n,0)\frac{\pi^2}{t(n+1)V}, \]
where \( \mu = t^{-1} \) as in Proposition 5.2. We denote by \( \square_{t,\mu} \) the basic complex Laplacian with respect to the transversal Kähler form \( \frac{1}{t}d\eta_{\varphi_t,\mu} \). Then it follows that \( \Delta_{t,\mu} \varphi_t = 2\square_{t,\mu} \varphi_t \).

By \( d\eta_{\varphi_t,\mu} = \mu^{-1}(d\eta + \sqrt{-1}\partial \bar{\partial} \varphi_t) \), we have
\[ \square_{t,\mu} \varphi_t = \mu \square_{t,\mu} \mu^{-1} \varphi_t = \mu \text{tr}_{\eta_{\varphi_t,\mu}}(d\eta_{\mu} - d\eta_{\varphi_t,\mu}) \geq -nt^{-1}, \]
where \( \eta_{\mu} = \mu^{-1}\eta \) and \( \mu = t^{-1} \). By applying the Fact 5.1 to \((S, g_{\varphi_t,\mu})\), we have
\[ \varphi_t(p) = \frac{1}{t(n+1)V} \int_{S} \varphi_t(d\eta_{\varphi_t,\mu})^n \wedge \eta_{\varphi_t,\mu} \]
\[ + \int_{S} (G_{t,\mu}(p, q) + \gamma(n,0)\frac{\pi^2}{t(n+1)V}) (\Delta_{t,\mu} \varphi_t) dV_{g_{\varphi_t,\mu}}(q). \]
By \((38)\), the first term is given by
\[
\frac{1}{t^{n+1}V} \int_S \varphi_t(d\eta_{\varphi_t}, \eta)^n \wedge \eta = \frac{1}{V} \int_S \varphi_t(d\eta_{\varphi_t})^n \wedge \eta.
\]
By using \((39)\) and \((40)\), we have
\[
\int_S \left( G_{t, \mu}(p, q) + \gamma(n, 0) \frac{\pi^2}{t^{n+1}V} \right) \left( \Delta_{t, \mu} \varphi_t \right) dV_{g_{\varphi_t}, \mu}(q)
\geq -2n \gamma(n, 0) \frac{\pi^2}{t^{n+1}V} \frac{n!}{t(n!)}
= -2n \gamma(n, 0) \frac{\pi^2}{t(n!)}.
\]
Thus we obtain
\[
\varphi_t(x) \geq \frac{1}{V} \int_S \varphi_t(d\eta_{\varphi_t})^n \wedge \eta - 2n \gamma(n, 0) \frac{\pi^2}{t(n!)}.
\]
This gives the desired estimate
\[
\text{osc}_S \varphi_t = \sup_S \varphi_t - \inf_S \varphi_t \leq I(0, \varphi_t) + 2n \left( K\frac{V_0}{V} + \gamma(n, 0) \frac{\pi^2}{t(n!)} \right).
\]
By applying the inequality \((42)\), we can prove directly the uniqueness of Sasaki-Einstein metrics up to the action of the identity component of the automorphism group for the transverse holomorphic structure.

The deformation of a Sasakian structure defined by \((36)\) and \((37)\) is called a \(D\)-homothetic deformation. By applying \(D\)-homothetic deformations to complete Sasaki manifolds with positive transverse Ricci curvature, Hasegawa and Seino shows in \([9]\) that such Sasaki manifolds are compact with finite fundamental group. Although this method does not lead to a diameter bound, it is applicable to \(C^0\) estimates for solutions of \((22)\).

**Appendix A.** Some functionals on the space of Sasakian metrics

In this appendix, we introduce some functionals on the space of Kähler potentials for the transverse Kähler structure. Let \((S, g)\) be a \((2n+1)\)-dimensional Sasaki manifold with Sasakian structure \(S = (g, \xi, \eta, \Phi)\). We assume that \(c_1(B) > 0\) and \(c_1(D) = 0\). Let \(V_0 = \frac{1}{n!} \int_S (\frac{1}{2} d\eta)^n \wedge \eta\) be the volume of \((S, g)\) and put \(V := n!V_0\).

**A.1. Functionals** \(L_\eta\) and \(M_\eta\). For each \(\varphi', \varphi'' \in \mathcal{H}\), we put
\[
L(\varphi', \varphi'') := \frac{1}{V} \int_a^b dt \int_S \varphi_t \left( \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
\[
M(\varphi', \varphi'') := -\frac{1}{V} \int_a^b dt \int_S \varphi_t \left( s^T(\varphi_t) - n(2n+2) \right) \left( \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge \eta_{\varphi_t},
\]
where \(\{ \varphi_t \mid t \in [a, b] \}\) is an arbitrary piecewise smooth path in \(\mathcal{H}\) such that \(\varphi_a = \varphi'\) and \(\varphi_b = \varphi''\) and \(s^T(\varphi_t)\) is the transverse scalar curvature for the transverse Kähler
metric $\omega_T^\varphi = \omega^T + \sqrt{-1}\partial_B \bar{\partial}_B \varphi_t$. The functionals were defined by Futaki-Ono-Wang [7], and proved that the definition of the functions is independent of choice of the path $\{\varphi_t \mid t \in [a, b]\}$. Put $L_\varphi(\varphi) := L(0, \varphi)$ and $M_\varphi(\varphi) := M(0, \varphi)$. Then the critical points of $M_\varphi$ give Sasaki-Einstein metrics which are compatible with the initial metric $g$. As in the Kähler geometry, the functionals $L$ and $M$ have the following properties. The proofs are given by the same arguments in Kähler geometry. A functional $H : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to satisfy the 1-cocycle condition if
\begin{align*}
(1) \quad & H(\varphi', \varphi'') + H(\varphi''', \varphi') = 0, \\
(2) \quad & H(\varphi', \varphi'') + H(\varphi''', \varphi'') + H(\varphi'''', \varphi') = 0
\end{align*}
for each $\varphi', \varphi'', \varphi''' \in \mathcal{H}$.

**Proposition A.1.**

1. The functionals $L$ and $M$ satisfy the 1-cocycle condition.
2. $L(\varphi', \varphi'' + C) = L(\varphi' - C, \varphi'') = L(\varphi', \varphi'') + C$ (resp. $M(\varphi' + C, \varphi'' + C) = M(\varphi', \varphi'')$) for each $\varphi', \varphi'' \in \mathcal{H}$ and $C \in \mathbb{R}$.

Hence we can define the mapping $M$ (denoted by the same $M$) on $\mathcal{S}(g)$ by

$M(g', g'') := M(\varphi', \varphi')$,

where $\varphi', \varphi''$ are basic functions such that $g_{\varphi'} = g'$ and $g_{\varphi''} = g''$. We call the functional $M_\varphi$ on the space of Sasaki metrics which have the same basic Kähler class as the initial metric $g$ defined by

$M_\varphi(g') := M(g, g')$

the transverse K-energy map of the Sasaki manifold $(S, g)$.

**A.2. The functionals $I_\varphi$ and $J_\varphi$.** For each $\varphi', \varphi'' \in \mathcal{H}$, we put

\begin{align*}
I(\varphi', \varphi'') & := \frac{1}{V} \int_S (\varphi'' - \varphi') \left( \frac{1}{2} d\eta_{\varphi'} \wedge \eta_{\varphi'} - \frac{1}{2} d\eta_{\varphi''} \wedge \eta_{\varphi''} \right), \\
J(\varphi', \varphi'') & := \frac{1}{V} \int_a^b dt \int_S \varphi_t \left( \frac{1}{2} d\eta_{\varphi'} \wedge \eta_{\varphi'} - \frac{1}{2} d\eta_{\varphi''} \wedge \eta_{\varphi''} \right),
\end{align*}

where $\{\varphi_t \mid t \in [a, b]\}$ is an arbitrary piecewise smooth path in $\mathcal{H}$ such that $\varphi_a = \varphi'$ and $\varphi_b = \varphi''$. The following lemma is proved by direct calculations.

**Proposition A.2.**

1. $J(\varphi', \varphi'') = \frac{1}{V} \int_B (\varphi'' - \varphi') \left( \frac{1}{2} d\eta_{\varphi'} \wedge \eta_{\varphi'} - L(\varphi', \varphi'') \right)$. In particular, the definition of $J$ is independent of choice of the path $\{\varphi_t \mid t \in [a, b]\}$.
2. $I(\varphi' + C, \varphi'' + C) = I(\varphi', \varphi'')$ and $J(\varphi' + C, \varphi'' + C) = J(\varphi', \varphi'')$ for each $\varphi', \varphi'' \in \mathcal{H}$ and constant $C \in \mathbb{R}$.

By Proposition A.2, we can define the mappings $I$ and $J$ (denoted by the same notations) on $\mathcal{S}(g)$ by

$\begin{align*}
I(g', g'') & := I(\varphi', \varphi'') \text{ and } J(g', g'') := J(\varphi', \varphi''), \\
I_\varphi(g') & := I(g, g'), \quad J_\varphi(g') := J(g, g')
\end{align*}$

where $\varphi', \varphi''$ are basic functions such that $g_{\varphi'} = g'$ and $g_{\varphi''} = g''$. Put $I_{\varphi}(g') := I(g, g')$, $J_{\varphi}(g') := J(g, g')$. 
for each $g' \in \mathcal{S}(g)$. The functional $J$ does not satisfy the 1-cocycle condition in general, but it satisfies the following equality:

$$J(\varphi', \varphi'') + J(\varphi'', \varphi''') = J(\varphi', \varphi''') - \frac{1}{V} \int_S (\varphi''' - \varphi'') \left( \left( \frac{1}{2} d\eta_{\varphi'} \right)^n \wedge \eta_{\varphi'} - \left( \frac{1}{2} d\eta_{\varphi''} \right)^n \wedge \eta_{\varphi''} \right).$$

Put $I_\eta(\varphi) := I(0, \varphi)$ and $J_\eta(\varphi) := J(0, \varphi)$ for each $\varphi \in \mathcal{H}$. We now take an arbitrary smooth path $\{ \varphi_t | t \in [a, b] \}$ in $\mathcal{H}$. Then by a simple calculation we have

$$(43) \quad \frac{d}{dt} (I_\eta(\varphi_t) - J_\eta(\varphi_t)) = \frac{1}{V} \int_S \varphi_t \triangle_{B,t} \varphi_t \left( \frac{1}{2} d\eta_{\varphi_t} \right)^n \wedge (\eta_{\varphi_t}),$$

where $\square_t = \frac{1}{2} \Delta_{B,t}$ is the basic complex Laplacian with respect to the Sasakian metric $g_{\varphi_t}$. The following properties of $I_\eta$ and $J_\eta$ are essential to obtain the $C^0$-estimate for the solutions of equation (22).

**Proposition A.3.** $I_\eta, I_\eta - J_\eta, J_\eta$ are non negative functionals and satisfies the following inequality:

$$0 \leq I_\eta(\varphi) \leq (n + 1) (I_\eta(\varphi) - J_\eta(\varphi)) \leq nI_\eta(\varphi).$$

Propositions A.1, A.2 and A.3 can be obtained by a similar way as in Kähler cases (see [12] for example).

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