Revisiting the Set Cover Conjecture

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Abstract

In the Set Cover problem, the input is a ground set of $n$ elements and a collection of $m$ sets, and the goal is to find the smallest sub-collection of sets whose union is the entire ground set. In spite of extensive effort, the fastest algorithm known for the general case runs in time $O(mn^2)$ [Fomin et al., WG 2004]. In 2012, as progress seemed to halt, Cygan et al. [TALG 2016] have put forth the Set Cover Conjecture (SeCoCo), which asserts that for every fixed $\varepsilon > 0$, no algorithm with runtime $2^{(1-\varepsilon)n}\text{poly}(m)$ can solve Set Cover, even if the input sets are of arbitrary large constant size. We propose a weaker conjecture, which we call Log-SeCoCo, that is similar to SeCoCo but allows input sets of size $O(\log n)$.

To support Log-SeCoCo, we show that its failure implies an algorithm that is faster than currently known for the famous Directed Hamiltonicity problem. Even though Directed Hamiltonicity has been studied extensively for over half a century, no algorithm significantly faster than $2^n\text{poly}(n)$ is known for it. In fact, we show a fine-grained reduction to Log-SeCoCo from a generalization of Directed Hamiltonicity, known as the nTree problem, which too can be solved in time $2^n\text{poly}(n)$ [Koutis and Williams, TALG 2016]. We further show an equivalence between solving the parameterized versions of Set Cover and of nTree significantly faster than their current known runtime. Finally, we show that even moderate runtime improvements for Set Cover with bounded-size sets would imply new algorithms for nTree and for Directed Hamiltonicity.

Our technical contribution is to reinforce Log-SeCoCo (and arguably SeCoCo) by reductions from other famous problems with known algorithmic barriers, and hope it will lead to more results in this vein, particularly reinforcing the Strong Exponential-Time Hypothesis (SETH) by reductions from other well-known problems.

1 Introduction

In the Set Cover problem, the input is a ground set $[n] = \{1, ..., n\}$ and a collection of $m$ sets, and the goal is to find the smallest sub-collection of sets whose union is the entire ground set. An exhaustive search takes $O(n2^m)$ time, and a dynamic-programming algorithm has runtime $O(mn2^n)$ [FKW04], which is faster when $m > n$, a common assumption that we will make throughout. In spite of extensive effort, no algorithm that runs in time $O^{*}(2^{(1-\varepsilon)n})$ is known, where throughout, $O^{*}(\cdot)$ hides polynomial factors in the instance size, and unless stated otherwise, $\varepsilon > 0$ denotes a fixed constant (and similarly $\varepsilon'$). As better runtimes were found only for special cases [Koi09, BHK09, Ned16, BHPK17], it was conjectured that the above runtime is optimal [CDL+16], even if the input sets are small. To state this more formally, let $\Delta$-Set Cover denote the Set Cover problem where all sets have size at most $\Delta > 0$.

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Conjecture 1.1 (The Set Cover Conjecture (SeCoCo) [CDL+16]). For every fixed $\varepsilon > 0$ there is $\Delta(\varepsilon) > 0$, such that no algorithm (even randomized) solves $\Delta$-Set Cover in time $O^*(2^{(1-\varepsilon)n})$.

This conjecture clearly implies that for every $\Delta = \omega(1)$, no algorithm solves $\Delta$-Set Cover in time $O^*(2^{(1-\varepsilon)n})$. Several conditional lower bounds were based on this conjecture [CDL+16, BKKK16, BHH15, KL16, KST17], for example for Connected Vertex Cover by reducing Set Cover to it. The authors of SeCoCo asked whether the problems they reduce Set Cover to can be reduced back to Set Cover, so that their runtime complexity would stand and fall with SeCoCo. They believed it would be hard to do, since it would probably provide for those problems an alternative algorithm with runtime that matches the currently fastest one, which is very complex and took decades to achieve for some (e.g., for Steiner Tree).

Connection to SETH. No formal connection is known to date between the SeCoCo conjecture and the Strong Exponential Time Hypothesis (SETH) of [IP01], which asserts that for every $\varepsilon > 0$ there exists $k(\varepsilon)$, such that $k$CNC cannot be solved in time $O^*(2^{(1-\varepsilon)n})$. This question was already raised by Cygan et al. [CDL+16], who provided a partial answer by showing a SETH-based lower bound for a certain variant of Set Cover (that counts the number of solutions). It is known that the weaker assumption ETH implies a $2^{\Omega(n)}$ time lower bound for Set Cover, even if $\Delta = O(1)$, and that SAT can be solved in time $O^*(2^{(1-\varepsilon)n})$ if and only if Set Cover can be solved in time $O^*(2^{(1-\varepsilon)m})$, see [CDL+16]. Some researchers hesitate to rely on SeCoCo as a conjecture, and prefer other, more popular conjectures such as SETH. For example, a runtime lower bound for Subset Sum was recently shown [ABHS17] based on SETH, even though a lower bound based on SeCoCo was already known [CDL+16]. Overall, the necessity of SeCoCo is questioned.

We address this matter by proposing a weaker assumption, and showing an independent justification for it. Our conjecture deals with $\Delta$-Set Cover for $\Delta = O(\log n)$, as follows.

Conjecture 1.2 (The Log-Set Cover Conjecture (Log-SeCoCo)). For every fixed $\varepsilon > 0$, there is $\Delta(\varepsilon,n) = O(1/\varepsilon \cdot \log n)$ such that no algorithm (even randomized) solves $\Delta$-Set Cover in time $O^*(2^{(1-\varepsilon)n})$.

The fastest algorithm known for $\Delta$-Set Cover runs in time $O^*(2^{n\Delta})$ [Koi09] for $\lambda = (2\Delta - 2)/\sqrt{(2\Delta - 1)^2 - 2\ln(2)} \leq 1 + 1/(2\Delta)$, where the inequality assumes $\Delta \geq 2$, hence this runtime is slightly faster than for general Set Cover. It seems that the known hardness results that are based on SeCoCo can be based also on our conjecture, with appropriate adjustments related to the sets sizes in the cases of Set Cover parameterized by the universe size plus the solution size [CDL+16] and Parity of Set Covers [BHH15] (where the second reduction uses techniques from the first one).

Our Results. Our first result connects the conjecture to the nTree problem (see Figure 1), where the input is a directed graph $G$ and a directed tree $T$ (i.e., a tree with arbitrary edges orientations) with the same number of nodes $n'$, and the goal is to determine whether $G$ contains an isomorphic copy of $T$ as a subgraph. This problem includes as a special case the well known Directed Hamiltonicity problem, where given as input a directed graph $G$, the goal is to determine whether $G$ contains a simple path (cycle) that visits all the nodes (the Hamiltonian cycle and path are easily reducible to each other with only small overhead). Next, we show that the failure of Log-SeCoCo has interesting consequences.

Theorem 1.3. Suppose Log-SeCoCo fails, namely, there is $\varepsilon > 0$ such that $\Delta$-Set Cover with every $\Delta = O(1/\varepsilon \cdot \log n)$ can be solved in time $O^*(2^{(1-\varepsilon)n})$. Then for some $\delta(\varepsilon) > 0$, nTree on $n'$ nodes can be solved in time $O^*(2^{(1-\delta)n'})$. 

In the special case of Directed Hamiltonicity, we actually reduce to rather constrained instances of Set Cover.

**Theorem 1.4.** Suppose Log-SeCoCo fails, namely, there is $\varepsilon > 0$ such that $\Delta$-Set Cover with every $\Delta = O(1/\varepsilon \cdot \log n)$ can be solved in time $O^*(2^{(1-\varepsilon)n})$. Then for some $\delta(\varepsilon) > 0$, Directed Hamiltonicity on $n'$ nodes can be solved in time $O^*(2^{(1-\delta)n})$. This holds even when in $\Delta$-Set Cover, all sets are of equal sizes.

It follows from our proof that one can assume that in the $\Delta$-Set Cover instance, every solution of size at most $O(\varepsilon n/\log n)$ (exactly $\varepsilon n/\log n$ in Theorem 1.4) consists of disjoint sets, and that it suffices to find any solution (of arbitrary size) that uses disjoint sets.

In addition, we can show equivalence between the well-known parameterized versions of $n$Tree and Set Cover, which we define next. Let $\text{Set Cover}(p)$ be the parameterized version of Set Cover, where one has to find a sub-collection of sets that covers at least $p$ elements, and let $k$Tree be the parameterized version of $n$Tree, where the goal is to find in a given (host) graph $G$ an isomorphic copy of a given (pattern) tree $T$ on $k$ nodes. It is known that $\text{Set Cover}(p)$ admits an algorithm based on color coding with running time $O^*(2^p)$ [Blä03], and that $k$Tree admits an algorithm with runtime $O^*(2^k)$ [KW16].

We can show that whenever $p$ is not too small, $\text{Set Cover}(p)$ can be solved in time $O^*(2^{(1-\varepsilon)p})$ if and only if $k$Tree can be solved in time $O^*(2^{(1-\varepsilon)k})$. The “if” direction is an easy extension of a result in [KT17], where it is shown that if $k$Tree can be solved in time $O^*(2^{(1-\varepsilon)k})$ then $\text{Set Cover}$ can be solved in time $O^*(2^{(1-\varepsilon^2)\Omega(n)})$. The “only if” direction is shown in the next theorem, by a reduction from $k$Tree back to $\text{Set Cover}(p)$ with sets sizes bounded by $\Delta = O(\log n)$ (Section 3), which for general $\Delta$ we call $\Delta$-Set Cover($p$).

**Theorem 1.5.** Suppose that for some $\varepsilon > 0$, $\Delta$-Set Cover($p$) with $\Delta = O(1/\varepsilon \cdot \log n)$ runs in time $O^*(2^{(1-\varepsilon)p})$. Then for some $\delta(\varepsilon) > 0$, $k$Tree with every $k = \omega(\log^2 n')$ can be solved in time $O^*(2^{(1-\delta)k})$.

By inspecting our proof for this theorem, we can further assume that in the input $\Delta$-Set Cover instances, every solution of size at most $O(\varepsilon k/\log n)$ has disjoint sets, and that it suffices to find an arbitrarily sized solution that uses only disjoint sets.

We can also show that even moderate improvements to the fastest known runtime for $\Delta$-Set Cover, namely, to the $O^*(2^{1-1/2\Delta}n)$ time algorithm of [Koi09], implies improvements for $n$Tree and for Directed Hamiltonicity (Section 4).

**Discussion.** Our first result (Theorem 1.3) supports the validity of Log-SeCoCo based on the $n$Tree problem, which we believe does not admit an algorithm with runtime $O^*(2^{(1-\varepsilon)n})$, for two reasons. First, this problem includes the well-known Directed Hamiltonicity problem, and for the last 50 years no algorithm significantly faster than $O^*(2^n)$ was found for it, despite extensive efforts [Bel60, HK61, Bel62, Woe03], in contrast to the progress on its undirected version [Bjo14]. Second, for the generalization of $n$Tree and $k$Tree, known as the Subgraph Isomorphism problem, where the pattern graph can be any graph and with any number of nodes, a runtime lower bound $n^{\Omega(n)}$ is known assuming ETH [CFG+16], even when the host and pattern graphs have the same number of nodes.

We see it as an evidence that also $k$Tree does not become easier as the size $k$ of the pattern graph increases all the way to $k = n$, which would imply that the conditional lower bound of [KT17], which shows that $k$Tree cannot be solved in time $O^*(2^{(1-\varepsilon)k})$, extends to $k = n$.

The fact that the exponential runtimes of $k$Tree and $\Delta$-Set Cover($p$) with $\Delta = O(\log n)$ are tied together (see the paragraph before Theorem 1.5) suggests that they might be equivalent also in the case where $p = k = n$. If true, then by our results (see Figure 1), solving Set Cover significantly...
A Map of New and Known Reductions

Directed Hamiltonicity $\ll O^*(2^n)$

nTree $\ll O^*(2^n)$

Log-SeCoCo

(O(log n))-Set Cover $\ll O^*(2^n)$

Set Cover $\ll O^*(2^n)$

Steiner Tree $\ll O^*(2^k)$

Set Cover(p) $\ll O^*(2^p)$

kTree $\ll O^*(2^k)$

Subset Sum $\ll O^*(2^k)$

SeCoCo

c_\varepsilon-Set Cover $\ll O^*(2^{(1-\varepsilon)n})$

$\varepsilon^j$-Set Partitioning $\ll O^*(2^{(1-\varepsilon^j)n})$

Figure 1: An overview of new and known reductions, where an arrow from a box with $A \ll O^*(2^n)$ to $B \ll O^*(2^n)$ represents a reduction from problem $A$ to problem $B$, such that if $B$ can be solved in time $O^*(2^{(1-\varepsilon)n})$, then $A$ can be solved in time $O^*(2^{(1-\varepsilon')n})$. We denote by $t$ the size of the solution in Steiner Tree, and by $b$ the number of bits required to represent the integers in Subset Sum. The problems we focus on are drawn in thick frames. Notice that if problem $A$ is above problem $B$ then there is a path, and a reduction, from $A$ to $B$.

faster than $O^*(2^n)$ is equivalent to achieving the same runtime in the special case of $\Delta$-Set Cover with $\Delta = O(\log n)$, which can be seen as an analogue to the SETH sparsification lemma [IPZ01].

**Open Problem 1.6.** Does an algorithm for $\Delta$-Set Cover with $\Delta = O(\log n)$ with runtime $O^*(2^{(1-\varepsilon)n})$ imply an algorithm for Set Cover with runtime $O^*(2^{(1-\varepsilon)(\log n)})$?

Perhaps surprisingly, in the special but common case $m = n^O(1)$, we can resolve this open problem positively.

**Observation 1.7.** If for some constants $\varepsilon > 0$ and $c > 0$, $\Delta$-Set Cover with $\Delta = O(c \log n)$ can be solved in time $O^*(2^{(1-\varepsilon)n})$ then for some $\delta(\varepsilon) > 0$, Set Cover with $m = O(n^c)$ can be solved in time $O^*(2^{(1-\delta\varepsilon)n})$.

To see this, simply guess which sets of size larger than $\Delta$ participate in an optimal solution, using an exhaustive search over at most $n \cdot m \cdot \binom{m}{\lfloor m/\Delta \rfloor}$ choices, and then apply the assumed algorithm for the remaining sets. This leads to a more general observation: if for some increasing function $f$ and all $\Delta = O(f(n))$, $\Delta$-Set Cover can be solved in time $O^*(2^{(1-\varepsilon)f(n)})$ then Set Cover with $m \leq 2^{O(f(n))}$ can be solved in time $O^*(2^{(1-\varepsilon)\log n})$. As also the case $m = 2^{O(n)}$ can be solved in time $O^*(2^{(1-\varepsilon)n})$ (in fact, in time $O^*(1)$), the focus should be on the case where $m$ is both super-polynomial and sub-exponential in $n$. 


We note that the results easily generalize to weighted Directed Hamiltonicity (i.e., TSP) and nTree by using a generalized conjecture about the weighted version of Set Cover, whose input is similar to the Set Cover only with a positive weight for each set, and the goal is to find a minimum-weight sub-collection whose union is the entire ground set. The generalized conjecture then states that for every fixed $\varepsilon > 0$, weighted Set Cover with the cardinality of every set bounded by $O(1/\varepsilon \cdot \log n)$ cannot be solved in time $O^*(2^{1-\varepsilon}n)$.

**Prior Work.** Faster algorithms are known for variants of Set Cover, as follows. Set Cover can be solved in time $(m + 2^n)\text{poly}(n)$ [BHK09], which for $m = n^{o(1)}$ is faster than the aforementioned $O(mn2^n)$ algorithm of [FKW04]. The case where all sets are of size $q$ and the goal is to determine whether $p$ pairwise-disjoint sets can be packed, can be solved in time $O^*(2^{1-\varepsilon}pq)$ for $\varepsilon(q) > 0$ [BHPK17]. Determining whether a Set Cover instance has a solution of size at most $\sigma n$ can be done in time $O^*(2^{(1-\Omega(\sigma^4))n})$ [Ned16]. The fastest known runtime for Directed Hamiltonicity is $O^*(2^{\Theta(\sqrt{n}/\log n)})$ [Bjö16]. Several problems, including Directed Hamiltonicity and Set Cover, were shown to belong to the class EPNL, defined as all problems that can be solved by a non-deterministic turing machine with space $n + O(\log n)$ bits [IV15].

On the hardness front, conditional lower bounds based on SeCoCo are known for Set Partitioning, Connected Vertex Cover, Steiner Tree, Maximum Graph Motif [BKK16], Edge Colorful Path [KL16], and the dynamic, general and connected versions of Dominating Set [KST17].

**Techniques.** To demonstrate our basic technique, let us present an extremely simple reduction from Directed Hamiltonicity to $\Delta$-Set Cover with $\Delta = O(\log n)$. Given a directed graph $G$, first guess (by exhaustive search) a relatively small set of nodes (“representatives”), and an ordering for them $z_1, z_2, \ldots$ in a potential Hamiltonian cycle. Then construct a Set Cover instance whose ground set is the nodes of $G$ and has the following sets: for every possible path of length $\Delta$ in $G$ from some $z_i$ to $z_{i+1}$ without visiting any representative in between, there is a set that with all the path nodes except for $z_{i+1}$. A Hamiltonian cycle in $G$ clearly corresponds to a set cover using exactly $n/\Delta$ sets, and vice versa.

## 2 Main Reduction

In this section we prove Theorem 1.3. The heart of the proof is actually the following lemma.

**Lemma 2.1.** For every integer $\Delta \in [N]$, nTree on $N$ nodes can be reduced to $O(N^{4N/\Delta} \sqrt{N})$ instances of $\Delta$-Set Cover with $n \leq N + 4N/\Delta$ elements in time $O(N^\Delta + N^{4N/\Delta} \sqrt{N} f(n, m, \Delta))$, where $f(n, m, \Delta)$ is the time of solving a single $\Delta$-Set Cover instance.

**Proof of Theorem 1.3.** Given an algorithm with runtime $O^*(2^{1-\varepsilon}n)$ for $\Delta$-Set Cover on $n$ elements and $\Delta = O(1/\varepsilon \cdot \log n)$, we reduce the nTree instance by applying Lemma 2.1 with

$$\Delta = 16/\varepsilon \cdot \log n' = O(1/\varepsilon \cdot \log n),$$

and then solve each of the $O(n^{4n'/\Delta} \sqrt{n'}) = O^*(2^{en'/4})$ instances in the assumed time of

$$O^*(2^{(1-\varepsilon)n}) \leq O^*(2^{(1-\varepsilon)(n'+4n'/(16/\varepsilon \cdot \log n'))}) \leq O^*(2^{(1-\varepsilon)(n'+\varepsilon n'/4 \cdot \log n')},$$

to a total runtime of

$$O^*(2^{16/\varepsilon \cdot \log^2 n'+\varepsilon n'/4+(1-\varepsilon)(n'+\varepsilon n'/4 \cdot \log n')) = O^*(2^{n'-\varepsilon n'/2}),$$

which concludes the proof for $\delta(\varepsilon) = \varepsilon/2$. \qed
Let us outline the proof of Lemma 2.1. Consider an instance \((G, T)\) of \(n\text{Tree}\), and for ease of exposition, assume that the tree \(T\) is rooted at some node \(r\), and all edges are directed away from it; the general case of an arbitrarily directed tree is similar, where subtrees are defined with respect to a designated root vertex, and the edge orientations are taken into account when comparing subtrees. The idea is to create at most \(2^{N/\Delta \log(e\Delta)} \cdot (N/\Delta)!\) instances of \(\Delta\text{-Set Cover}\) on \(n \leq N + 4N/\Delta\) elements, such that if at least one of them has a \(t \leq 4N/\Delta\) sized solution then the answer to \(n\text{Tree}\) is positive. The first step is to process the tree \(T\) to create \(t\) subtrees of it, each of size at most \(\Delta\), such that the union of their nodes is \(T\) and they may intersect only at their roots, which are labeled according to the subtrees they are in. Then, we guess which nodes in \(G\) are the images of the roots \(r\) of our subtrees of \(T\). For every such choice, we enumerate over every possibility for the intended subtree and add a set to the \(\Delta\text{-Set Cover}\) instance being constructed, but only for those that do not violate the global structure of the tree, taking into account the edges between the subtrees. On the one hand, if there exists a solution of size \(t\) to the \(\Delta\text{-Set Cover}\) instance, then there is a one to one correspondence between the sets in the solution and the labeled roots. Hence, such a solution corresponds to a copy of the tree \(T\) in \(G\). On the other hand, if the \(n\text{Tree}\) instance has a solution, then after the tree \(T\) is partitioned into subtrees as above, we can make all the guesses in \(G\) correctly.

We begin with an auxiliary lemma that will be used to prove Lemma 2.1.

**Lemma 2.2.** There is a polynomial-time algorithm (see Algorithm 1) that given a tree \(T\) on \(N\) nodes and an integer \(l'\), finds a collection \(S\) of rooted subtrees of \(T\) such that:

a. The number of rooted subtrees is \(|S| \leq \frac{2^N}{l' + 1}\).

b. The number of nodes in each subtree is at most \(2(l' - 1)\).

c. Every node in \(T\) is in some subtree.

d. Two subtrees may only intersect in their roots.

Note that in the collection \(S\) returned from Algorithm 1, for every \(s \in S\) the subgraph of \(T\) that corresponds to \(s\), which we also refer to as \(s\), is a connected subtree of \(T\) and so it has a root which we call \(r(s)\).

**Lemma 2.3.** The set \(S\) returned by Algorithm 1 is of size \(|S| \leq \frac{2^N}{l' + 1}\) (and item a is satisfied).

**Proof.** Denote the collection of sets of size smaller than \(l'\) in \(S\) (that were added in lines 10 and 13) \(S_s\), and \(S_b\) otherwise. Sets in \(S_s\) were created only if \(r(s)\) at the time of their creation was the root of at least one (other) set in \(S_b\) (line 10) or was the last traversed node in the DFS search 13. Together with the fact that each root has at most one set from \(S_s\), we conclude that each set \(s' \in S_s\) excluding at most one, can be associated with a unique set in \(S_b\). (e.g. with one of those containing \(r(s')\)).

Hence, \(|S_s| - 1\) is bounded by the \(|S_b|\). Each set \(s \in S\) has at least one node, and at most one of its nodes is contained in other sets. We have reached the following equation regarding unique nodes:

\[
N - |S_s| - |S_b| = \sum_{s_s \in S_s} (|s_s| - 1) + \sum_{s_b \in S_b} (|s_b| - 1) \geq |S_b|l',
\]

and so

\[
N \geq |S_s| + |S_b|(l' + 1),
\]

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Algorithm 1

Input: a tree $T$ rooted at $r$, a size parameter $l'$
Output: a partition of $T$ into subtrees of size at most $2(l' - 1)$, two of which may intersect only in their roots.

1: initialize $S := \emptyset$, and for all $v \in V$ let $s(u) := \{u\}$.
2: traverse the tree $T$ using a DFS search starting from $r$, and whenever returning from a node $v$ with a parent $p$ in $T$, do the following:
3:   let $s(p) := s(p) \cup s(v)$.
4:   if $|s(p)| \geq l'$ then
5:     add $s(p)$ to $S$.
6:     if $p$ has uncovered children then
7:       let $s(p) := \{u\}$.
8:   else let $s(p) := \emptyset$.
9:   else if $|s(p)| < l'$ and $p$ does not have uncovered children and $p \in s$ for some $s \in S$ then
10:      add $s(p)$ to $S$.
11:     let $s(p) := \emptyset$.
12:   else if $|s(p)| < l'$ and $p$ does not have uncovered children and $p \notin s$ for every $s \in S$ then
13:      add $s(p)$ to $S$.  \(\triangleright p\) is the last node traversed in the tree
14: return $S$

which implies

$$|S_b| \leq \frac{N - |S_s|}{l' + 1} \leq \frac{N}{l' + 1}.$$  

Thus, we get

$$|S| = |S_s| + |S_b| \leq 2|S_b| + 1 \leq \frac{2N}{l' + 1} + 1,$$

and the proof of Lemma 2.3 is completed.  \(\square\)

Lemma 2.4. Items b–d are satisfied by the output of Algorithm 1.

Proof. Since in the worst case we add a subtree in the first time the accumulated number of nodes crossed $l'$, the number of nodes of each subtree is bounded by $2(l' - 1)$. In addition, every node is in some subtree since for every node $v$, $s(v)$ is initiated to $\{v\}$, and will be the child node at some point, where it will go up the tree until added to $S$. To see why the last requirement holds, consider the fact that whenever an accumulated set goes up the tree and encounters an existing root, it will be added to $S$ rather than continue its path up the tree. \(\square\)

Proof of Lemma 2.1. We first apply the aforementioned Algorithm 1 for partition $T$ into subtrees that satisfy the conditions in Lemma 2.2. By picking $l' = \Delta/2 - 1$, we obtain that each set is bounded by $\Delta$ and that $|S| \leq 4N/\Delta$. Hence, the cardinality of $R := \{r(s)\}_{s \in S}$ is bounded by $4N/\Delta$. For $S$ returned by Algorithm 1, let $R_T = \{r(s) : s \in S\}$ (note that $|R_T|$ may be smaller than $|S|$). In the next step, we guess $|R_T|$ nodes in $G$ that will function as the image of the nodes in $R_T$ in a potential subgraph isomorphism function and denote them by $R_G$, and then guess a bijection $f$ from $R_T$ to $R_G$. The guessing is done by exhaustive search over $\binom{N}{|R_T|}$ choices of nodes, and together with the number of ways to choose a bijection it can be done in time $\binom{N}{|R_T|} |R_T|!$. Then we enumerate all sets $s'$ of nodes of size at most $\Delta$ in $G$, and denote by $G(s')$ the graph induced from each on $G$. For every subtree $s \in S$, we look for an isomorphic copy of $s$ in subgraphs $G(s')$. 7
that contain \( f(r(s)) \) as a root and no other node in \( R_G \), and that satisfy \(|s'| = |s|\). For each one that we found, we add to the constructed \textit{Set Cover} instance a set \( s'_{G'} \) with the root \( r' \) labeled \( r'_s \) where \( s \) corresponds to the subtree \( s \) of \( T \) whose copy found to be in \( G(s') \). Note that the number of elements in the \textit{Set Cover} instance is exactly \( N - |R_T| + |S| \).

Now we show that the size constraints follow. We bound the number of instances using a factorial bound that is derived from Stirling approximation. As \( |R_T| \leq 4N/\Delta \), the number of \textit{Set Cover} instances is bounded by

\[
\binom{N}{4N/\Delta} (4N/\Delta)! \leq \left( \frac{Ne}{4N/\Delta} \right)^{4N/\Delta} e^{4N/\Delta} \left( \frac{4N/\Delta}{\Delta e} \right)^{4N/\Delta} \leq N^{4N/\Delta} e^{4N/\Delta} \leq O(N^{4N/\Delta} \sqrt{N})
\]

as required.

We now prove that at least one of the \textit{Set Cover} instances has solution of size at most \(|S|\) (in fact exactly \(|S|\) as no smaller solutions available) if and only if the \textit{nTree} instance is a yes instance. For the first direction, assume that the \textit{nTree} instance is a yes instance. Considering the isomorphic copy of \( T \) in \( G \), its \(|S|\) subtrees as Algorithm 1 outputs on \( T \) will be sets in the \textit{Set Cover} instance the reduction outputs, and so the it has a solution of size at most \(|S|\). For the second direction, if a \textit{Set Cover} instance has a solution \( I \) of size at most \(|S|\) and since the number of labeled roots is \(|S|\), it must be that for each subtree \( s \in S \) its labeled root is in exactly one set in \( I \), and so \(|I| = |S|\). Since \( I \) is a legal solution and \( S \) covers all the nodes, no node in \( V(G) \setminus R_G \) appears twice in \( I \). The conclusion is that these sets together form the required tree, concluding the proof of Lemma 2.1.

We note that in the case of Theorem 1.4 for \textit{Directed Hamiltonicity}, we do not have to use Algorithm 1, but simply guess \( n/\Delta \) representative nodes in \( G \) and their ordering in the potential cycle, and then enumerate all paths of size \( \Delta \) to represent paths between consecutive representatives. Hence we obtain a \( \Delta\text{-Set Cover} \) instance with the additional limitation that all sets are of equal sizes.

**Lemma 2.5.** For every integer \( \Delta \in [N] \), \textit{Directed Hamiltonicity} on \( N \) nodes can be reduced to \( O(N^{N/\Delta} \sqrt{N}) \) instances of \( \Delta\text{-Set Cover} \) with \( n = N \) elements in time \( O(N^\Delta + N^{N/\Delta} \sqrt{N} f(n, m, \Delta)) \), where \( f(n, m, \Delta) \) is the time of solving a single \( \Delta\text{-Set Cover} \) instance, and the sets are of equal sizes.

### 3 Equivalence of the Parameterized Versions

In this section we prove Theorem 1.5. The reduction is almost the same as in Lemma 2.1, however now Algorithm 1 gets as an input a tree of size \( k \), and so the number of nodes whose identity and ordering we guess in the graph \( G \) is \( 4k/\Delta \). Similar to the previous section, we get the following lemma.

**Lemma 3.1.** For every integer \( \Delta \in [N] \), \textit{kTree} with \( |V(G)| = N \) can be reduced to \( O(N^{4k/\Delta} \sqrt{N}) \) instances of \( \Delta\text{-Set Cover}(p) \) with \( n \leq N + 4k/\Delta \) total elements and \( p \leq k + 4k/\Delta \) in time \( O(N^\Delta + N^{4k/\Delta} \sqrt{N}) \).

**Proof of Theorem 1.5.** Similar calculation as before with appropriate adjustments lead to a total runtime of

\[
O^*(2^{16/\epsilon \log^2 n + \epsilon k/4 + (1-\epsilon)(k+\epsilon k/4)}) \leq O^*(2^{k-\epsilon k/2})
\]

where the last inequality is due to the assumption that \( k = \omega(\log^2 n') \).
4 Moderate Improvements Imply New Algorithms

In this section we show how moderate improvements for variants of \textit{Set Cover} imply new algorithms for \textit{Directed Hamiltonicity}, and similar results can be achieved for \textit{nTree}. Given any algorithm for $\Delta$-\textit{Set Cover} with runtime $f(n,m,\Delta)$, by Lemma 2.5 \textit{Directed Hamiltonicity} admits an algorithm with runtime $O(N^{\Delta} + N^{\Delta/N} f(n,m,\Delta))$. We now demonstrate how this algorithm behaves with different regimes of $\Delta$.

If there exists $\varepsilon > 0$ such that for every $\Delta = \text{poly}(\log n), f(n,m,\Delta) = O^*(2^{(1-\frac{1}{\Delta}\varepsilon)n})$ then by considering $\Delta = \log (1+\varepsilon')/\varepsilon n = \text{poly}(\log n)$ for $\varepsilon' > 0$, \textit{Directed Hamiltonicity} has an algorithm with runtime

$$O(2^{\log(1+\varepsilon')/\varepsilon+1}N) + O^*(2^{N/\log(1+\varepsilon')/\varepsilon-1}N : 2^{1-1/(\log(1+\varepsilon')/\varepsilon N)^{1-\varepsilon}}N) = O^*(2^{1-1/(\log(1+\varepsilon')/\varepsilon-2)N})N$$

Considering larger regimes, if for some fixed $\varepsilon > 0$, $\delta \in (0,1/2)$, and $\Delta = O(n^\delta)$, $f(n,m,\Delta) = O^*(2^{(1-(1+\varepsilon)\log \Delta/n)}N)$ then \textit{Directed Hamiltonicity} can be solved in time

$$2^{N^\delta \log N} + 2^{N^{1-\delta}} \cdot O^*(2^{1-(1+\varepsilon)\log \Delta/N^{1-\delta}}N) = 2^{N-\Theta((1-\varepsilon)N^{\delta})}$$

Note that to break the fastest known $2^{N-\Theta(\sqrt{\log N})}$ algorithm for \textit{Directed Hamiltonicity} by [Bjö16], it is enough to have either $f(n,m,\Delta) = O^*(2^{(1-(2+\varepsilon)\log \Delta/n)}N)$ for $\Delta = n^{1/2-\delta'}$ with every fixed $\delta' > 0$, or $f(n,m,\Delta) = O(m \cdot 2^{1-(1+\varepsilon)\log \Delta/n})$ for $\Delta = \sqrt{n}$, taking into account that most algorithms for variants of \textit{Set Cover} that have the factor $m$ in their runtime, do not have it with higher power than one. Similarly, by Lemma 2.1 \textit{nTree} admits an algorithm with runtime $N^\Delta + N^{\Delta/N} f(N + 4N/\Delta, N^\Delta, \Delta)$ and similar bounds can be achieved.

References

[ABHS17] A. Abboud, K. Bringmann, D. Hermelin, and D. Shabtay. SETH-based lower bounds for subset sum and bicriteria path. CoRR, 2017. Available from: http://arxiv.org/abs/1704.04546.

[Bel60] R. Bellman. Combinatorial processes and dynamic programming. In Combinatorial analysis, Proceedings of symposia in applied mathematics, pages 217–249. American mathematical society, 1960. doi:10.1090/psapm/010.

[Bel62] R. Bellman. Dynamic programming treatment of the travelling salesman problem. J. ACM, 9(1):61–63, 1962. doi:10.1145/321105.321111.

[BHH15] A. Björklund, D. Holger, and T. Husfeldt. The parity of set systems under random restrictions with applications to exponential time problems. In 42nd International Colloquium on Automata, Languages and Programming (ICALP 2015), volume 9134, pages 231–242. Springer, 2015. doi:10.1007/978-3-662-47672-7_19.

[BHK09] A. Björklund, T. Husfeldt, and M. Koivisto. Set partitioning via inclusion-exclusion. SIAM J. Comput., 39(2):546–563, July 2009. doi:10.1137/070683933.

[BHPK17] A. Björklund, T. Husfeldt, K. Ptteri, and M. Koivisto. Narrow sieves for parameterized paths and packings. Journal of Computer and System Sciences, 87:119 – 139, 2017. doi:10.1016/j.jcss.2017.03.003.

[Bjo14] A. Björklund. Determinant sums for undirected hamiltonicity. SIAM Journal on Computing, 43(1):280–299, 2014. doi:10.1137/110839229.

[Bjö16] A. Björklund. Below All Subsets for Some Permutational Counting Problems . In 15th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2016), volume 53 of Leibniz International Proceedings in Informatics (LIPIcs), pages 17:1–17:11, 2016. doi:10.4230/LIPIcs.SWAT.2016.17.
[BKK16] A. Björklund, P. Kaski, and Ł. Kowalik. Constrained multilinear detection and generalized graph motifs. *Algorithmica*, 74(2):947–967, 2016. doi:10.1007/s00453-015-9981-1.

[Blä03] M. Bläser. Computing small partial coverings. *Information Processing Letters*, 85(6):327–331, 2003. doi:10.1016/S0020-0190(02)00434-9.

[CDL+16] M. Cygan, H. Dell, D. Lokshtanov, D. Marx, J. Nederlof, Y. Okamoto, R. Paturi, S. Saurabh, and M. Wahlström. On problems as hard as CNF-SAT. *ACM Transactions on Algorithms*, 12(3):41:1–41:24, 2016. doi:10.1145/2925416.

[CFG+16] M. Cygan, F. V. Fomin, A. Golovnev, A. S. Kulikov, I. Mihajlin, J. Pachocki, and A. Socała. Tight bounds for graph homomorphism and subgraph isomorphism. In 27th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’16, pages 1643–1649. SIAM, 2016. doi:10.1137/1.9781611974331.ch112.

[FKW04] F. V. Fomin, D. Kratsch, and G. J. Woeginger. Exact (exponential) algorithms for the dominating set problem. In 30th International Conference on Graph-Theoretic Concepts in Computer Science, WG’04, pages 245–256. Springer-Verlag, 2004. doi:10.1007/978-3-540-30559-0_21.

[HK61] M. Held and R. M. Karp. A dynamic programming approach to sequencing problems. In *Proceedings of 16th ACM National Meeting*, ACM ’61, pages 71.201–71.204. ACM, 1961. doi:10.1145/800029.808532.

[IP01] R. Impagliazzo and R. Paturi. On the complexity of k-SAT. *Journal of Computer and System Sciences*, 62(2):367–375, March 2001. doi:10.1006/jcss.2000.1727.

[IPZ01] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001. doi:10.1006/jcss.2001.1774.

[IY15] Y. Iwata and Y. Yoshida. On the equivalence among problems of bounded width. In 23rd Annual European Symposium on Algorithms (ESA 2015), pages 754–765. Springer, 2015. doi:10.1007/978-3-662-48350-3_63.

[KL16] Ł. Kowalik and J. Lauri. On finding rainbow and colorful paths. *Theoretical Computer Science*, 628(C):110–114, 2016. doi:10.1016/j.tcs.2016.03.017.

[Koi09] M. Koivisto. Partitioning into sets of bounded cardinality. In *Parameterized and Exact Computation (IWPEC 2009)*, volume 5917 of Lecture Notes in Computer Science, pages 258–263. Springer-Verlag, 2009. doi:10.1007/978-3-642-11269-0_21.

[KST17] R. Krithika, A. Sahu, and P. Tale. Dynamic parameterized problems. In *11th International Symposium on Parameterized and Exact Computation (IPEC 2016)*, volume 63 of Leibniz International Proceedings in Informatics (LIPIcs), pages 19:1–19:14. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017. doi:10.4230/LIPIcs.IPEC.2016.19.

[KT17] R. Krauthgamer and O. Trabelsi. Conditional lower bound for subgraph isomorphism with a tree pattern. *CoRR*, 2017. Available from: http://arxiv.org/abs/1708.07591.

[KW16] I. Koutis and R. Williams. LIMITS and applications of group algebras for parameterized problems. *ACM Trans. Algorithms*, 12(3):31:1–31:18, May 2016. doi:10.1145/2885499.

[Ned16] J. Nederlof. Finding large set covers faster via the representation method. In 24th Annual European Symposium on Algorithms (ESA 2016), volume 57 of Leibniz International Proceedings in Informatics (LIPIcs), pages 69:1–69:15. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPIcs.ESA.2016.69.

[Woe03] G. J. Woeginger. Exact algorithms for NP-hard problems: A survey. In M. Jünger, G. Reinelt, and G. Rinaldi, editors, *Combinatorial Optimization - Eureka, You Shrink!,* pages 185–207. Springer-Verlag, 2003. doi:10.1007/3-540-36478-1.