Can one restore Lorentz invariance in quantum N=2 string?

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Abstract

We consider quantum $N = 2$ string embedded into the $N = 4$ topological framework from the perspective of the old covariant quantisation. Making use of the causality and cyclic symmetry of tree amplitudes we argue that no Lorentz covariant boson emission vertex can be constructed within the $N = 4$ topological formalism.

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1. Introduction

Critical \(N = 2\) string provides a conventional framework for describing self–dual gauge theory or self–dual gravity in two spatial and two temporal dimensions. Apart from a number of salient features characterising the model like the absence of massive excitations in the spectrum of physical states, continuous family of sectors interpolating between \(R\) and \(NS\) and connected by spectral flow, the vanishing of scattering amplitudes with more than three external legs to all orders in perturbation theory, there is a fundamental drawback intrinsic to the model. It lacks manifest Lorentz covariance. Technically, in order to construct the \(U(1)\) current entering the \(N = 2\) superconformal algebra (SCA) out of the fermionic fields at hand one has to introduce a complex structure in the target which breaks the full Lorentz group \(SO(2, 2)\) down to the \(U(1, 1)\) subgroup.

A way out of the problem has been observed by Siegel \([1]\) and elaborated in much more detail by Berkovits and Vafa \([2]\) (see also the related works \([3, 4]\)). The idea is that one can embed the \(N = 2\) string in a larger \(N = 4\) topological framework by adding to the theory two more fermionic currents and two more bosonic ones which extend the \(N = 2\) SCA to a small \(N = 4\) SCA. Classically the new constraints prove to be functionally dependent on those forming the \(N = 2\) SCA \([1, 5]\). The crucial observation, however, is that extending the \(N = 2\) algebra to a small \(N = 4\) algebra one raises also the group of external global automorphisms (\(U(1, 1)\) in the case on the \(N = 2\) string) to the full Lorentz group. In other words, within the \(N = 4\) topological framework one reveals a tempting possibility to restore the Lorentz invariance missing in the \(N = 2\) formalism. Curiously enough, this resembles the Green–Schwarz superstring, for which the extracting of a functionally independent set of fermionic first class constraints is known to break the manifest Lorentz covariance.

A classical action for the \(N = 4\) topological extension of the \(N = 2\) string has been constructed recently in Refs. \([5, 6]\). In order to provide the higher global symmetry, on the world–sheet of the string some extra fields are to be introduced, these complementing the \(d = 2, N = 2\) conformal supergravity multiplet to \(d = 2, N = 4\) one. The global limit of the four local supersymmetry transformations corresponds to a twisted version of the \(N = 4\) supersymmetry algebra \([7]\). Interestingly enough, being classically equivalent on a flat background, the theories lead to different geometries when put in a curved space. In contrast to the Kähler geometry characterising the \(N = 2\) case, for the \(N = 4\) model a manifold has to admit a covariantly constant holomorphic two–form in order to support the \(N = 4\) twisted supersymmetry \([7]\). This restricts the holonomy group to be a subgroup of \(SU(1, 1)\) and leads to a Ricci–flat manifold already at the classical level.

Turning to the quantum description for the \(N = 4\) model, the reducibility of the constraints causes a serious complication of the BRST procedure \([1]\) and a rigorous BRST quantisation of the \(N = 4\) topological string is unknown. To bypass the problem, in Ref. \([2]\) (see also \([3, 4]\)) a set of reasonable prescriptions to calculate scattering amplitudes has

\[^3\]In contrast to the Green–Schwarz superstring, the constraints intrinsic to the \(N = 4\) model are scalars. Hence they do not seem to require infinitely many ghost fields.
been proposed and shown to reproduce the results known for the $N = 2$ string including an elegant proof of the vanishing theorems. Notice, however, that the prescription essentially relies upon a specific topological twist which does not treat all the currents on equal footing and breaks the Lorentz group down to the $U(1, 1)$ subgroup.

The purpose of this paper is to reconsider the issue of the Lorentz invariance in the quantum $N = 4$ topological string from the perspective of the old covariant quantisation. Avoiding problematic BRST analysis this still maintains one within a conventional framework. In the next section we briefly discuss the extension of the $N = 2$ SCA to a small $N = 4$ SCA and give an explicit form of the Lorentz transformations arising within the $N = 4$ topological framework. Sect. 3 contains the discussion of vertex operators. In particular, we show that, although the vertex operators describing asymptotic physical states do respect the full Lorentz group, the causality and cyclic symmetry of tree amplitudes prevent one from constructing a Lorentz invariant boson emission vertex. Thus, the conclusion we draw is that, although giving an efficient key to the restoration of the Lorentz invariance at the classical level, the $N = 4$ formalism fails to do so at the quantum level.

2. $N=2$ SCA, small $N=4$ SCA and Lorentz symmetry

Let us consider a $c = 6$ CFT corresponding to the critical closed $N = 2$ string. The matter sector involves two complex bosons $x^a(z, \bar{z}), a = 0, 1,$ and four complex fermions $\psi^a(z), \varphi^a(\bar{z})$, the latter belonging to the right and left movers, respectively. The fields are assigned with the standard propagators

\[
\langle x^a(z, \bar{z}) \bar{x}^\bar{a}(z', \bar{z}') \rangle = \langle \bar{x}^\bar{a}(z, \bar{z}) x^a(z', \bar{z}') \rangle = -\eta^{\bar{a}a}(\ln(z - z') + \ln(\bar{z} - \bar{z}')),
\]

\[
\langle \psi^a(z) \bar{\psi}^\bar{a}(z') \rangle = \langle \bar{\psi}^\bar{a}(z) \psi^a(z') \rangle = -\frac{\eta^{\bar{a}a}}{z - z'},
\]

\[
\langle \varphi^a(z) \varphi^a(\bar{z'}) \rangle = \langle \varphi^a(\bar{z}) \bar{\varphi}^{\bar{a}}(z') \rangle = -\frac{\eta^{\bar{a}a}}{\bar{z} - \bar{z}'},
\]

where $\eta^{\bar{a}a} = \text{diag} (-, +)$ stands for the Minkowski metric in the target.

Given the matter fields, one can readily construct the $N = 2$ superconformal currents (in what follows we discuss only the right movers and use the abbreviation $\varphi^a = \varphi^a \eta^{\bar{a}a} \bar{x}^\bar{a}$)

\[
T(z) = -\partial x \partial \bar{x} + \frac{1}{2}(\psi \partial \bar{\psi} + \bar{\psi} \partial \psi),
\]

\[
G(z) = \partial x \bar{\psi}, \quad \bar{G}(z) = \partial \bar{x} \psi,
\]

\[
J(z) = \bar{\psi} \psi.
\]

The corresponding OPE’s are well known

\[
T(z) T(z') \sim \frac{3}{(z - z')^2} + \frac{2T(z')}{(z - z')^2} + \frac{\partial T(z')}{z - z'},
\]

\[\text{On the cylinder we conjugate as } (x^a)^* = \bar{x}^\bar{a}, (\psi^a)^* = \bar{\psi}^\bar{a}. \text{ The target metric is Hermitian } \eta^{\bar{a}b*} = \eta^{\bar{a}a}.\]
which also imply that the fermionic currents $G$ and $\bar{G}$ carry conformal spin 3/2 while the bosonic $U(1)$ current $J$ has conformal spin 1. Notice that with respect to the latter the generators $G(z)$ and $\bar{G}(z)$ are charged with the charges $-1$ and $+1$, respectively.

A striking point about the $N = 2$ algebra is that it admits a continuous automorphism [8] with a local parameter $\alpha(z)$

$$
T' = T - i\partial\alpha J + (i\partial\alpha)^2, \quad J' = J - 2i\partial\alpha, \\
G' = e^{i\alpha}G, \quad \bar{G}' = e^{-i\alpha}\bar{G},
$$

(4)

which relates R and NS sectors (spectral flow) and allows one to stick with a preferred representation. For the rest of the paper we choose to work in the NS representation. Since, by the very construction, each current in the $N = 2$ SCA holds invariant under the action of the $U(1, 1)$ group, the latter provides a global automorphism of the $N = 2$ algebra which also coincides with the global symmetry group intrinsic to the $N = 2$ string.

With a closer inspection one can further discover that some extra currents can be constructed out of the matter fields at hand [1, 2]

$$
H(z) = \partial x \psi, \quad \bar{H}(z) = \partial \bar{x} \bar{\psi}, \\
J^{(1)}(z) = \psi \bar{\psi}, \quad J^{(2)} = \bar{\psi} \psi,
$$

(5)

where we denoted $\varphi \bar{\psi} = \varphi^a \epsilon_{ab} \psi^b$, $\varphi \bar{\psi} = \varphi^a \epsilon_{ab} \bar{\psi}^b$ and $\epsilon_{ab} = (\epsilon^a_{ab})^*$, $\epsilon_{01} = -1$, is the Levi-Civita totally antisymmetric tensor. These extend the $N = 2$ SCA to a small $N = 4$ SCA

$$
T(z) \ H(z') \sim \frac{3}{2} \frac{H(z')}{(z - z')^2} + \frac{\partial H(z')}{z - z'}, \\
T(z) \ \bar{H}(z') \sim \frac{3}{2} \frac{\bar{H}(z')}{(z - z')^2} + \frac{\partial \bar{H}(z')}{z - z'},
$$
In checking the OPE’s the identities prove to be helpful. 

Thus, altogether there are four fermionic currents of conformal spin $3/2$. The bosonic triplet $J, J^{(1)}, J^{(2)}$ (spin 1) forms an $su(1, 1)$ Kac–Moody subalgebra. It is noteworthy that viewed as constraints at the classical level the extra currents prove to be functionally dependent on those forming the $N = 2$ algebra [1, 2]. A remarkable fact, however, is that extending the algebra that way, one raises the global automorphism group to the $N = 4$ algebra invariant, these provide an apparent automorphism. A less obvious point is that within the extended framework the $U(1)$ automorphism one had in

\begin{align*}
T(z) \ J^{(1,2)}(z') &\sim \frac{J^{(1,2)}(z')}{(z - z')^2} + \frac{\partial J^{(1,2)}(z')}{z - z'}, \\
G(z) \ \bar{H}(z') &\sim -\frac{J^{(2)}(z')}{(z - z')^2} - \frac{1}{2} \frac{\partial J^{(2)}(z')}{z - z'}, \\
\bar{G}(z) \ H(z') &\sim -\frac{J^{(1)}(z')}{(z - z')^2} - \frac{1}{2} \frac{\partial J^{(1)}(z')}{z - z'}, \\
G(z) \ J^{(1)}(z') &\sim -\frac{2H(z')}{z - z'}, \quad \bar{G}(z) \ J^{(2)}(z') = -\frac{2\bar{H}(z')}{z - z'}, \\
J(z) \ H'(z') &\sim \frac{H(z')}{z - z'}, \quad J(z) \ \bar{H}(z') \sim \frac{\bar{H}(z')}{z - z'}, \\
J(z) \ J^{(1)}(z') &\sim \frac{2J^{(1)}(z')}{z - z'}, \quad J(z) \ J^{(2)}(z') \sim -\frac{2J^{(2)}(z')}{z - z'}, \\
H(z) \ \bar{H}(z') &\sim -\frac{2}{(z - z')^3} - \frac{J(z')}{(z - z')^2} - \frac{1}{2} \frac{\partial J(z')}{z - z'} - \frac{T(z')}{z - z'}, \\
H(z) \ J^{(2)}(z') &\sim -\frac{2G(z')}{z - z'}, \quad \bar{H}(z) \ J^{(1)}(z') \sim -\frac{2\bar{G}(z')}{z - z'}, \\
J^{(1)}(z) \ J^{(2)}(z') &\sim \frac{4}{(z - z')^2} + \frac{4J(z')}{z - z'}. \tag{6}
\end{align*}

In checking the OPE’s the identities

\begin{align*}
\epsilon_{ab}\eta^{\dot{b}\dot{a}}\epsilon_{\dot{b}\dot{a}} = \eta_{a\dot{a}}, \quad \eta^{\dot{a}\dot{a}}\eta^{\dot{b}\dot{b}}\epsilon_{\dot{a}\dot{b}} = \epsilon^{\dot{a}\dot{b}}, \tag{7}
\end{align*}

prove to be helpful.

Thus, altogether there are four fermionic currents of conformal spin $3/2$. The bosonic triplet $J, J^{(1)}, J^{(2)}$ (spin 1) forms an $su(1, 1)$ Kac–Moody subalgebra. It is noteworthy that viewed as constraints at the classical level the extra currents prove to be functionally dependent on those forming the $N = 2$ algebra [1, 2]. A remarkable fact, however, is that extending the algebra that way, one raises the global automorphism group to the full Lorentz group $SO(2, 2)$ and restores the Lorentz invariance in the classical $N = 2$ string [3, 4].

Let us dwell on the issue for the case of the quantum $N = 4$ algebra. First of all, it is straightforward to verify that the continuous automorphism (4) remains to hold in the extended version, provided the new currents transform in accord with

\begin{align*}
H' = e^{-i\alpha}H, \quad \bar{H}' = e^{i\alpha}\bar{H}, \quad J^{(1)'} = e^{-2i\alpha}J^{(1)}, \quad J^{(2)'} = e^{2i\alpha}J^{(2)}. \tag{8}
\end{align*}

Hence, one can continue to work in a preferred (NS) representation. Turning to global automorphisms, as transformations from the conventional $SU(1, 1)$ group leave each current of the $N = 4$ algebra invariant, these provide an apparent automorphism. A less obvious point is that within the extended framework the $U(1)$ automorphism one had in
the \( N = 2 \) case is accompanied by two extra transformations, altogether forming another \( SU(1, 1)_{outer} \) group (we stick with the terminology of Ref. [4]). For the elementary field combinations \( \psi \bar{\varphi}, \varphi \bar{\psi}, \bar{\psi} \bar{\varphi} \), these read (independently of the statistics of the fields involved)

\[
\begin{array}{|c|c|c|}
\hline
SU(1, 1)_{outer} & \delta_{\omega} & \delta_{\beta} & \delta_{\lambda} \\
\hline
\psi \bar{\varphi} & \omega(\psi \varphi - \bar{\psi} \bar{\varphi}) & 0 & i\lambda(\psi \varphi + \bar{\psi} \bar{\varphi}) \\
\varphi \bar{\psi} & \omega(\psi \bar{\varphi} - \bar{\psi} \varphi) & -2i\beta(\psi \bar{\varphi}) & -i\lambda(\psi \bar{\varphi} - \bar{\psi} \varphi) \\
\bar{\psi} \bar{\varphi} & -\omega(\psi \bar{\varphi} - \bar{\psi} \varphi) & 2i\beta(\bar{\psi} \bar{\varphi}) & -i\lambda(\psi \bar{\varphi} - \bar{\psi} \varphi) \\
\hline
\end{array}
\]

where \( \omega, \beta \) and \( \lambda \) are real infinitesimal constant parameters. We have put the \( U(1) \) automorphism mentioned above in the second column. Being applied to the \( N = 4 \) superconformal currents the \( SU(1, 1)_{outer} \) transformations amount to (we give them in an infinitesimal form)

\[
T' = T, \quad G' = G - \omega \bar{H} + \omega H, \quad \bar{G}' = \bar{G} + \omega \bar{H} - \omega H, \\
H' = H - \omega \bar{G} + \omega G, \quad \bar{H}' = \bar{H} + \omega G - \omega G, \quad J' = J + \omega J^{(2)} - \omega J^{(1)}, \\
J^{(1)'} = J^{(1)} - 2i\omega J, \quad J^{(2)'} = J^{(2)} + 2i\omega J, \quad (9)
\]

and

\[
T' = T, \quad G' = G + \lambda \bar{H} + \lambda H, \quad \bar{G}' = \bar{G} - \lambda \bar{H} - \lambda H, \\
H' = H + \lambda \bar{G} - \lambda G, \quad \bar{H}' = \bar{H} + \lambda G - \lambda G, \quad J' = J - i\lambda J^{(2)} - i\lambda J^{(1)}, \\
J^{(1)'} = J^{(1)} + 2i\lambda J, \quad J^{(2)'} = J^{(2)} + 2i\lambda J. \quad (11)
\]

It is straightforward, although a bit tedious, to verify that Eqs. (9)–(11) do provide an automorphism of the \( N = 4 \) SCA.

Since in two spatial and two temporal dimensions the Lorentz group factorizes as \( SO(2, 2) = SU(1, 1) \times SU(1, 1)' \) one reveals a tempting possibility to restore the full Lorentz symmetry for the \( N = 2 \) string treated in the enlarged \( N = 4 \) topological formalism. At the classical level this has been achieved [3, 5] by means of enlarging the world sheet \( d = 2 \), \( N = 2 \) conformal supergravity multiplet to the \( d = 2, N = 4 \) conformal supergravity multiplet. The extra fields transform nontrivially under \( SU(1, 1)_{outer} \) and render the full action \( SO(2, 2) \) invariant. At the same time, as classically the constraints \( H = 0, \bar{H} = 0, J^{(1)} = 0, J^{(2)} = 0 \) prove to be functionally dependent on those forming the \( N = 2 \) SCA \([1, 5]\) one still has the classical equivalence between the two formalisms. In
the next section we turn to discuss how far one can get in restoring the Lorentz invariance at the quantum level.

3. Vertex operators

Our main concern in this section will be the structure of tree amplitudes. In particular, we shall study the constraints imposed on physical vertices by causality and cyclic symmetry of the amplitudes. To this end we decompose in modes (as we mentioned above one can choose the NS representation due to the spectral flow; in the relations below $n$ is an integer and $r$ is a half integer)

$$L_n = \frac{1}{2\pi i} \oint dz \ z^{n+1} \ : T(z) : , \quad G_r = \frac{1}{2\pi i} \oint dz \ z^{r+1/2} : G(z) : ,$$

$$\bar{G}_r = \frac{1}{2\pi i} \oint dz \ z^{r+1/2} : \bar{G}(z) :, \quad J_n = \frac{1}{2\pi i} \oint dz \ z^n : J(z) :,$$

$$H_r = \frac{1}{2\pi i} \oint dz \ z^{r+1/2} : H(z) :, \quad \bar{H}_r = \frac{1}{2\pi i} \oint dz \ z^{r+1/2} : \bar{H}(z) : ,$$

$$J_n^{(1)} = \frac{1}{2\pi i} \oint dz \ z^n : J^{(1)}(z) :, \quad J_n^{(2)} = \frac{1}{2\pi i} \oint dz \ z^n : J^{(2)}(z) : ,$$

(12)

check the unitarity of the representation

$$L_n^+ = L_{-n}, \quad J_n^+ = J_{-n}, \quad G_r^+ = \bar{G}_{-r}, \quad H_r^+ = \bar{H}_{-r}, \quad J_n^{(1)+} = -J_n^{(2)},$$

(13)

and work out the full algebra

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{6}{n}\pi(n+1)n(n-1)\delta_{n+m,0};$$

$$[L_n, G_r] = (\frac{1}{2}n - r) G_{n+r}, \quad [L_n, \bar{G}_r] = (\frac{1}{2}n - r) \bar{G}_{n+r},$$

$$[L_n, J_m] = -m J_{n+m}, \quad [G_r, J_n] = G_{n+r},$$

$$\{G_r, \bar{G}_q\} = L_{r+q} - \frac{1}{2}(r-q) J_{r+q} + (r + \frac{1}{2})(r - \frac{1}{2})\delta_{r+q,0};$$

$$[\bar{G}_r, J_n] = -\bar{G}_{n+r}, \quad [J_n, J_m] = 2n\delta_{n+m,0};$$

$$[L_n, H_r] = (\frac{1}{2}n - r) H_{n+r}, \quad [L_n, \bar{H}_r] = (\frac{1}{2}n - r) \bar{H}_{n+r},$$

$$[L_n, J_n^{(1)}] = -m J_n^{(1)+}, \quad [L_n, J_n^{(2)}] = -m J_n^{(2)+},$$

$$\{G_r, \bar{H}_q\} = \frac{1}{2}(q - r) J_{r+q}, \quad \{\bar{G}_r, H_q\} = \frac{1}{2}(q - r) J_{r+q},$$

$$[G_r, J_n^{(1)}] = -2H_{r+n}, \quad [G_r, J_n^{(2)}] = -2\bar{H}_{r+n},$$

$$[J_n, H_r] = H_{r+n}, \quad [J_n, \bar{H}_r] = -\bar{H}_{r+n},$$

$$[J_n, J_n^{(1)}] = 2J_n^{(1)+}, \quad [J_n, J_n^{(2)}] = -2J_n^{(2)+},$$

$$\{H_r, \bar{H}_q\} = -L_{r+q} + \frac{1}{2}(q - r) J_{r+q} - (r + \frac{1}{2})(r - \frac{1}{2})\delta_{r+q,0};$$

$$[H_r, J_n^{(2)}] = -2G_{r+n}, \quad [H_r, J_n^{(1)}] = -2\bar{G}_{r+n},$$

$$[J_n^{(1)}, J_n^{(2)}] = 4J_{n+m} + 4n\delta_{n+m,0};$$

(14)
with other (anti) commutators vanishing. With respect to the \( N = 2 \) algebra physical states are defined to obey the standard relations

\[
L_n \mid \text{phys} \rangle = G_r \mid \text{phys} \rangle = \bar{G}_r \mid \text{phys} \rangle = J_n \mid \text{phys} \rangle = 0 \quad n, r > 0
\]

\[
L_0 \mid \text{phys} \rangle = J_0 \mid \text{phys} \rangle = 0,
\]

where the absence of the normal ordering constants in the last two relations follows from a rigorous BRST analysis\(^5\). A simple inspection of the full \( N = 4 \) algebra shows then that the modes \( H_r, \bar{H}_r, J^{(1)}_n, J^{(2)}_n \) for \( r, n > 0 \) automatically annihilate the physical states \((15)\) if so do the zero modes of \( J^{(1)} \) and \( J^{(2)} \). Thus the only extra conditions coming with the new generators are

\[
J^{(1)}_0 \mid \text{phys} \rangle = 0, \quad J^{(2)}_0 \mid \text{phys} \rangle = 0.
\]

Notice that this seems to be more restrictive than the situation at the classical level, where the constraints \( H = 0, \bar{H} = 0, J^{(1)} = 0, J^{(2)} = 0 \) prove to be functionally dependent and do not imply any new information as compared to that encoded into the \( N = 2 \) superconformal currents. The analysis of the relations \((15)\) is well known \([10]\) which has also been confirmed by the calculation of the one–loop partition function \([10, 11]\). The spectrum of physical states consists of a single ground state, this being a massless scalar. As the operators \( J^{(1)}_0 \) and \( J^{(2)}_0 \) give zero when acting on that state, the \( N = 2 \) and \( N = 4 \) spectra coincide.

We now turn to discuss vertex operators. As the critical intercept equals zero (\( L_0 = 0 \) on physical states), a vertex operator valid for describing an asymptotic physical state must carry conformal spin zero. On account of Eqs. \((15),(16)\) the natural guess is

\[
V_0(k, \bar{k}) = :e^{ik\bar{x} + i\bar{k}x}:, \quad k\bar{k} = 0.
\]

Worth noting is that the vertex holds invariant with respect to both the conventional \( SU(1, 1) \) and \( SU(1, 1)_{outer} \) (see the first line in the table above), thus exhibiting the full Lorentz invariance. To construct a vertex operator \( V(k, \bar{k}, z, \bar{z}) \) which would be capable of describing the emission of the bosonic state it is instructive to analyse the causality of the corresponding amplitudes. This implies, in particular, that spurious physical states decouple from physical processes

\[
\langle \text{spur} | V(k, 1) | \text{phys} \rangle = 0,
\]

where for simplicity we restricted ourselves to the three point function which proves to be sufficient for our purposes. As is well known, the decoupling of spurious physical states of the form \( \sum_{n>0} \langle \chi_n | L_n \) with

\[
\sum_{n>0} \langle \chi_n | L_n L_0 = 0 \rightarrow \langle \chi_n | (n + L_0) = 0,
\]

\(^5\) In the \( N = 4 \) framework the vanishing of the normal ordering constant for the operator \( J_0 \) is also required by the last commutator in Eq. \((14)\).
requires $V$ to carry conformal spin 1

$$T(z) V(z') \sim \frac{V(z')}{(z-z')^2} + \frac{\partial V(z')}{z-z'}.$$ (20)

Actually, from Eqs. (20) and (12) one finds

$$(L_n - n - L_0)V(k, 1) = V(k, 1)(L_n - L_0),$$ (21)

which when combined with Eq. (19) renders the amplitude (18) vanishing. Given the matter fields, the most general ansatz for the boson emission vertex is

$$V(k, z, \bar{z}) =: \{ \alpha \bar{k} \partial x + \beta k \partial \bar{x} + \lambda (\bar{k} \psi)(k \bar{\psi}) + \mu k \partial x + \gamma (\bar{k} \psi)(k \bar{\psi}) + \rho (\bar{k} \psi)(k \bar{\psi}) + \omega (k \psi)(k \bar{\psi}) \} e^{ik\bar{x} + ikx},$$ (22)

with $\alpha, \beta, \lambda, \mu, \gamma, \rho, \omega$ some complex constants to be determined below (here $\alpha, \beta, \lambda, \omega$ not to be confused with the infinitesimal parameters involved in the global $SU(1,1)$ outer automorphism we discussed in the previous section). The term like $(k \psi)(k \bar{\psi})$ one could try to include into the ansatz above proves to be redundant due to the identity

$$\epsilon_{cp} \epsilon_{\bar{p}k} = -\eta_{cs} \eta_{\bar{p}k} + \eta_{ck} \eta_{\bar{s}k}.$$ (23)

At this stage we refrain from requiring the vertex to respect the full Lorentz symmetry, while by the very construction the ansatz does respect the conventional $SU(1,1)$.

Consider further the spurious physical states generated by the fermionic currents $\sum_{r>0} \langle \chi_r | G_r \rangle$ with

$$\sum_{r>0} \langle \chi_r | G_r L_0 = 0 \rightarrow \langle \chi_r | (r + L_0) = 0.$$ (24)

Taking into account the explicit form of $G(z)$ and the fact that $V$ has conformal spin 1, one infers that the corresponding OPE can involve poles no higher than of the second order

$$G(z) V(z') \sim \frac{U_1(z')}{(z-z')^2} + \frac{U_2(z')}{z-z'},$$ (25)

with $U_1(z)$ and $U_2(z)$ to be determined below. The amplitude (18) then acquires the form

$$\sum_{r>0} \langle \chi_r | U_1(1)(r + \frac{1}{2}) + U_2(1) | \text{phys}. \rangle.$$ (26)

Assuming further that $U_1$ carries conformal spin $h$

$$T(z) U_1(z') \sim \frac{hU_1(z')}{(z-z')^2} + \frac{\partial U_1(z')}{z-z'},$$ (27)

and taking into account Eq. (24) one immediately reveals that Eq. (26) vanishes provided

$$h = \frac{1}{2}, \quad \text{and} \quad U_2 = \partial U_1.$$ (28)
Thus the OPE of the fermionic current $G(z)$ with the boson emission vertex must be of the form

$$G(z) \ V(z') \sim \frac{U_1(z')}{(z - z')^2} + \frac{\partial U_1(z')}{z - z'}, \quad (29)$$

with $U_1$ having conformal spin $1/2$.

The current $\bar{G}(z)$ can be treated in the same way yielding the OPE

$$\bar{G}(z) \ V(z') \sim \frac{\bar{U}_1(z')}{(z - z')^2} + \frac{\partial \bar{U}_1(z')}{z - z'}, \quad (30)$$

with $\bar{U}_1$ being a conformal field of spin $1/2$.

So far we have considered spurious (bra) states generated by $G_r, \bar{G}_r$ and $L_n$ independently. It is instructive then to recall the anticommutator in the algebra $\{G_r, \bar{G}_r\} = L_{2r}, \ r > 0$, which relates the states and provides further information on the structure of $V$. Actually, starting with the spurious state of the form $\sum_{r>0} \langle \chi_{2r} | L_{2r} \rangle$ with $\langle \chi_{2r} | (2r + L_0) = 0$, one readily finds the identity

$$\sum_{r>0} \langle \chi_{2r} | (2r + 1) V(1) + \partial V(1) |\text{phys}\rangle = \sum_{r>0} \langle \chi_{2r} | G_r \{(r + \frac{1}{2}) \bar{U}_1(1) + \partial \bar{U}_1(1)\} |\text{phys}\rangle$$

$$+ \sum_{r>0} \langle \chi_{2r} | \bar{G}_r \{(r + \frac{1}{2}) U_1(1) + \partial U_1(1)\} |\text{phys}\rangle. \quad (31)$$

Now we have to guess the OPE of $G(z)$ with $\bar{U}_1(z')$ and $\bar{G}(z)$ with $U_1(z')$. Since these have to match the left hand side of the identity (31) the poles of the second order or higher are not allowed (this is also prompted by the conformal spin which carry the fields $U_1(z)$ and $\bar{U}_1(z)$) and one finally has to set

$$G(z) \ \bar{U}_1(z') \sim \frac{\bar{W}(z')}{z - z'}, \quad \bar{G}(z) \ U_1(z') \sim \frac{W(z')}{z - z'}. \quad (32)$$

Substitutions of this in Eq. (31) yields

$$V = W + \bar{W}, \quad (33)$$

which together with Eqs. (29),(30), (32) gives a closed set of equations to fix $V$. Finally, given the explicit form of the ansatz (22) it is easy to verify the OPE's

$$J(z) \ V(z') \sim 0, \quad J^{(1,2)}(z') \sim 0, \quad (34)$$

which hold independently of the value of the constant coefficients entering the ansatz. This seems reasonable since, for instance, the $U(1)$ charge is then conserved in the amplitude. The relations above imply also the decoupling of the spurious physical (bra) states generated by $J_n, J^{(1,2)}_n$ with $n > 0$. In a similar way, making use of the algebra $[G_r, J_0^{(1)}] = -2H_r, [\bar{G}_r, J_0^{(2)}] = -2\bar{H}_r, [L_0, J^{(1,2)}_0] = 0$ one can show the decoupling of those generated by $H_r, \bar{H}_r$ with $r > 0$. Thus the decoupling of spurious physical states
generated by the currents extending the \( N = 2 \) SCA to the \( N = 4 \) SCA proves to be automatic and do not impose further restrictions on the form of the emission vertex.

In the \( N = 2 \) framework the \( U(1,1) \) invariance of the formalism selects only the first three terms in the ansatz \((22)\). Then a simple inspection of Eqs. \((24),(30),(32),(33)\) shows that the only essential restriction coming along with the causality conditions reads
\[
\lambda = i(\beta - \alpha).
\]  

(35)

This leaves two unspecified constants in the \( U(1,1) \) invariant ansatz. Calculating further the three point tree amplitude with physical in and out states
\[
A_{\text{tree}}^{\text{right}}(1,2,3) = \langle V_0(k_1,\infty)V(k_2,1)V_0(k_3,0) \rangle = -i(\alpha \bar{k}_2 k_3 + \beta k_2 \bar{k}_3),
\]  

(36)

one finds this to possess cyclic symmetry only when
\[
\beta = -\alpha.
\]  

(37)

Up to an irrelevant number coefficient this reproduces the vertex found by Ooguri and Vafa \([11]\) within the alternative superfield approach
\[
V(k,z,\bar{z}) =: \{ i k \partial \bar{x} - i \bar{k} \partial x - 2(\bar{k} \psi)(k \psi) \} e^{i k \bar{x} + i \bar{k} x} \quad , \quad k \bar{k} = 0.
\]  

(38)

Given the vertex, tree amplitudes involving more than three legs prove to vanish \([11]\) which persists also to higher orders in perturbation expansion \([2]\).

Turning to the \( N = 4 \) formalism, one is to consider the full ansatz \((23)\). The causality arguments now enforce the conditions
\[
\lambda = i(\beta - \alpha), \quad \rho = i\gamma, \quad \omega = -i\mu,
\]  

(39)

where one has to use the identity
\[
\bar{k} \psi(\bar{k} \partial x) - k \partial x(\bar{k} \psi) = -(k \bar{k}) \bar{\psi} \partial x,
\]  

(40)

which is a consequence of Eqs. \((7)\) and \((23)\). The new terms entering the ansatz bring the extra contribution to the three point function
\[
- i(\alpha \bar{k}_2 k_3 - \alpha k_2 \bar{k}_3 + \mu k_2 \bar{k}_3 + \gamma \bar{k}_2 k_3),
\]  

(41)

and the last two terms reveal cyclic symmetry without any restrictions on the coefficients \( \gamma, \mu \). Then a simple inspection of the table displaying the action of the \( SU(1,1)_{\text{outer}} \) group on the elementary field combinations shows that the correlator above is not \( SU(1,1)_{\text{outer}} \) invariant at any value of \( \alpha, \mu, \gamma \). A way out could be to allow the constants to transform nontrivially under \( SU(1,1)_{\text{outer}} \). This type of reasoning has been advocated in Ref. \([12]\) where two of the constants were identified with the gravitational and Maxwell string couplings. In our opinion, however, this does not fit well neither the field theory framework nor the string theory framework.
Thus, we conclude that no Lorentz invariant boson emission vertex can be constructed within the $N = 4$ topological formalism. The maximal subgroup one can retain proves to be $U(1, 1)$ which brings one back to the Ooguri–Vafa vertex (38). Ultimately, our conclusion based on the old covariant quantisation proves to be in line with the analysis of Refs. [2, 3, 4] which relied upon the topological prescription for calculating the scattering amplitudes.

4. Concluding remarks

To summarize, in this paper we reconsidered the issue of restoring the Lorentz invariance in the quantum $N = 2$ string. We used the passage to the equivalent $N = 4$ topological formalism which raises the global automorphism group of the underlying superconformal algebra to the full Lorentz group. Being efficient in restoring the invariance at the classical level, the $N = 4$ formalism, however, is not powerful enough to do so in the quantum theory. In particular, the causality and cyclic symmetry of tree amplitudes prevent one from the construction of a Lorentz invariant boson emission vertex. Although asymptotic states still can be described by the Lorentz invariant vertices.

An interesting continuation of this work could be a rigorous BRST analysis, although in the BRST framework we do not see means which could impact the conclusion drawn in this paper.

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