A Structural Analysis of Field/Circuit Coupled Problems Based on a Generalised Circuit Element

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Abstract In some applications there arises the need of a spatially distributed description of a physical quantity inside a device coupled to a circuit. Then, the in-space discretised system of partial differential equations is coupled to the system of equations describing the circuit (Modified Nodal Analysis) which yields a system of Differential Algebraic Equations (DAEs). This paper deals with the differential index analysis of such coupled systems. For that, a new generalised inductance-like element is defined. The index of the DAEs obtained from a circuit containing such an element is then related to the topological characteristics of the circuit’s underlying graph. Field/circuit coupling is performed when circuits are simulated containing elements described by Maxwell’s equations. The index of such systems with two different types of magnetostatic formulations (A* and T-Ω) is then deduced by showing that the spatial discretisations in both cases lead to an inductance-like element.

Keywords Differential Algebraic Equations · Differential Index · Modified Nodal Analysis · Eddy Currents · T-Omega Formulation

1 Introduction

Classically in applications with electrical circuitry, they are modelled as networks of which elements are described by lumped models. Those elements idealise the behaviour of the spatially distributed devices to simple algebraic or differential relations between currents and voltages. Most circuit solvers use Modified Nodal Analysis (MNA) to describe the circuit topology [17], which
together with the device models leads to a system of Differential Algebraic Equations (DAEs) \[20,16,15\]. In many applications standard MNA lumped-element models are not precise enough. This happens when some of the devices can only be treated by detailed field models, e.g. to resolve nonlinear wave propagation effects or frequency-dependent current distributions. An example of such an application is an electrical drive, consisting of a machine, a power converter and a control device.

Spatially distributed electromagnetic fields are described by Maxwell’s equations \[21,18\]. Those are a set of Partial Differential Equations that can be numerically simulated by techniques such as the Finite Element Method \[24\] or the Finite Integration Technique \[38\]. However, simulating all devices and the interconnecting circuit with such a method is prohibitively costly. A better approach is to couple the system of equations describing the circuit with the spatially discretised systems of equations describing the fields inside some selected devices \[36,5,8\]. This is sometimes called refined modelling \[6\].

Under low frequency assumptions, Maxwell’s equations can be simplified to a magnetoquasistatic setting. Typically, the remaining equations are combined into a formulation by defining appropriate potentials. Depending on the choice of potentials, different formulations are obtained \[9,37\], such as the A* and the T-Ω formulations. After spatial discretisation of the magnetoquasistatic approximation, a system of DAEs is obtained \[13\].

The coupling of the field and circuit systems leads to a coupled system of DAEs whose numerical (and analytical) complexity can be described by the notion of its index \[28\]. There are several index definitions \[22\]. This paper focuses on the field/circuit coupled system’s differential index and develops theoretical results that allow to deduce the index by studying the properties of the field’s subsystem of equations and the topological features of the circuit, only. For that purpose a new generalised inductance-like element is defined and index results of a circuit containing these elements are deduced. Both common magnetoquasistatic formulations are coupled to a circuit described by MNA and are shown to be important examples of such a type of element. The new simplified analysis for the A* formulation confirms known results from literature (see e.g. \[5,25\]), while the analysis of the T-Omega case is new. Related works, e.g. \[35\], are neither considering MNA nor 3D models.

The paper is structured as follows: Section 2 is a brief introduction into DAEs and defines the differential index. Section 3 introduces the MNA, important concepts for its index study and a new generalised definition of a circuit element with novel theoretical results. The field equations, different formulations thereof, field-circuit couplings and spatial discretisation are presented in Section 4. Section 5 states the index results for the coupled system. Finally, in Section 6 some numerical results are shown and Section 7 draws the conclusions.
2 Differential Algebraic Equations

A system of differential algebraic equations has the form

\[ F(x', x, t) = 0, \quad (1) \]

with \( x' = \frac{dx}{dt}, \) \( \det \left( \frac{\partial F}{\partial x'} \right) = 0 \) and \( x : I = [t_0, t_{\text{end}}] \to \mathbb{R}^n. \)

The index (see [11]) allows to classify a system of DAEs according to its numerical and analytical complexity. Even though there are several different index types (e.g. the perturbation, nilpotency or tractability index), they all coincide in the case of linear DAEs [11]. For the analysis in the paper, the differential index concept is used. It can be intuitively thought of as a way of measuring how far away the system is of an ordinary differential equation (ODE) in terms of differentiation. The higher the index of a DAE is, the more difficulties arise when treating the system numerically or analytically. Higher index DAEs are for example more difficult to initialise or have a higher sensitivity towards small perturbations.

**Definition 1 (Differential index [11])** The system of DAEs (1) has differential index \( m \), if \( m \) is the minimal number of differentiations

\[ \frac{d}{dt} F(x', x, t), \ldots, \frac{d^{(m)}}{dt^{(m)}} F(x', x, t) \]

needed, such that one can write a system of ordinary differential equations

\[ x' = \Phi(x, t), \]

with \( \Phi \) being a continuous function in \( x \) and \( t \), only with algebraic manipulations of equations (1)-(2).

**Assumption 21 (Smoothness)** In our differential index analysis we assume that all the functions involved in the studied system are sufficiently differentiable.

For a relaxation of Assumption 21 other index definitions should be used, such as the tractability index (see e.g. [15]).

3 Circuit System

Modern electrical circuit simulators use Modified Nodal Analysis [29], where the circuit is modelled as a directed graph with an incidence matrix \( A \). We consider circuits containing capacitors (C), inductors (L), resistors (R) and voltage (V) and current (I) sources. Using Kirchhoff’s Current Law and the lumped parameter models describing the devices at the circuit branches by
algebraic or differential equations [29], the system of DAEs of the conventional MNA [15] is obtained,

\[
A_C \frac{d}{dt} \mathbf{q}(A_C^T \mathbf{e}, t) + A_R \mathbf{g}(A_R^T \mathbf{e}, t) + A_L \mathbf{i}_L + A_V \mathbf{i}_V + A_I \mathbf{i}_{\text{src}}(t) = 0
\]

\[
\frac{d}{dt} \phi_L(\mathbf{i}_L, t) - A_L^T \mathbf{e} = 0
\]

(3)

\[
A_V^T \mathbf{e} - \mathbf{v}_{\text{src}}(t) = 0,
\]

for \( t \in I = [t_0, t_{\text{end}}] \subset \mathbb{R} \). Here, \( \mathbf{A}_s \) represents the columns of the incidence matrix attributed to branches that contain a specific device and \( \mathbf{A} = [A_C \ A_R \ A_L \ A_V \ A_I]^T \), \( \mathbf{e} : I \to \mathbb{R}^n \) is the vector of node potentials, \( \mathbf{i}_s : I \to \mathbb{R}^n \), the vector of currents through branches containing element \( \mathbf{i} \) and \( \mathbf{q}(\cdot), \mathbf{g}(\cdot), \phi_L(\cdot), \mathbf{i}_{\text{src}}(\cdot), \mathbf{v}_{\text{src}}(\cdot) \) are functions of the lumped parameter models (C, R, L, I, V). The voltage across a branch can be extracted with \( \mathbf{v}_s = A_V^T \mathbf{e} \).

When discussing the characteristics of circuits, two concepts that describe the topological properties of the underlying graph are used [10,29]:

**Definition 2 (Cutset, loop) [34, Appendix A.1]** Given a graph \( G = (V, E) \), with \( V \) being the set of all nodes and \( E \) the set of all edges, we define

- a **cutset** as a set of branches \( E_c \) such that its deletion from graph \( G \), \( G'(V, E \setminus E_c) \) results in a disconnected graph and adding any branch \( e_c \in E_c \) to \( G' \), again leads to a connected graph.

- a **loop** as a subgraph \( G_l \) such that it is connected and every node \( v_l \in G_l \) connects exactly two edges of \( G_l \) with each other.

As the coupling of a field model with a circuit is studied, we introduce a new notation for the branches representing generalised elements that will be defined later and that describes our field models. From now on, columns of the incidence matrix, currents and voltages corresponding to generalised elements will be denoted by the subscript \( \lambda \).

In order to ensure existence and uniqueness of the circuit’s solution, the following properties for its topology and functions are assumed (see [15]).

**Assumption 31 (Well posedness)** The MNA circuit equations fulfil

- there are no cutsets containing only current sources, that is,

\[
\ker(A_R A_C A_V A_L A_\lambda)^T = \{0\}.
\]

- there are no loops of voltage sources

\[
\ker A_V = \{0\}.
\]

- the functions describing conductances, inductances and capacitances

\[
G(\mathbf{v}_R, t) = \frac{\partial \mathbf{g}(\mathbf{v}_R, t)}{\partial \mathbf{v}_R}, \quad L(\mathbf{i}_L, t) = \frac{\partial \phi(\mathbf{i}_L, t)}{\partial \mathbf{i}_L} \quad \text{and} \quad C(\mathbf{v}_C, t) = \frac{\partial \mathbf{q}(\mathbf{v}_C, t)}{\partial \mathbf{v}_C}
\]

are positive definite.

The index study of system [3] under Assumption 31 has already been carried out e.g. in [15]. However, a new generalised element is now introduced, which admits the coupling of more complex element-types.
3.1 Generalised Element

The following section presents the definition of the generalised element and concludes with index results of the system of DAEs that results when describing a circuit that contains such elements. This allows to give index statements about circuit systems coupled to DAEs arising from refined models without having the need of analysing the overall coupled system.

**Definition 3 (Inductance-like element)** We define an inductance-like element as one described by a DAE

\[
F \left( \frac{d}{dt} x_\lambda, \frac{d}{dt} i_\lambda, x_\lambda, i_\lambda, v_\lambda, t \right) = 0,
\]

with \( x_\lambda : \mathcal{I} \to \mathbb{R}^{n_{dof}} \) and \( i_\lambda, v_\lambda : \mathcal{I} \to \mathbb{R}^{n_s} \), such that at most one differentiation

\[
\frac{d}{dt} F \left( \frac{d}{dt} x_\lambda, \frac{d}{dt} i_\lambda, x_\lambda, i_\lambda, v_\lambda, t \right) = 0
\]

is needed to obtain from equations (4)-(5) a system of the form

\[
\frac{d}{dt} x_\lambda = f_x(x_\lambda, i_\lambda, v_\lambda, t),
\]

\[
\frac{d}{dt} \phi(i_\lambda, x_\lambda, t) = f_\phi(x_\lambda, i_\lambda, v_\lambda, t),
\]

with the properties

- \( \frac{\partial}{\partial i_\lambda} \phi(i_\lambda, x_\lambda, t) \) is regular.
- \( \frac{\partial}{\partial v_\lambda} \left( \left( \frac{\partial \phi}{\partial x_\lambda} \right)^{-1} \left( -\frac{\partial \phi}{\partial x_\lambda} f_x - \frac{\partial \phi}{\partial t} + f_\phi \right) \right) \) is positive definite.

**Remark 1** Crucial in (4) is that the time derivative of the branch voltage \( v_\lambda \) does not appear in the expression.

**Example 1** Two examples for inductance-like devices are

(a) classical inductances written as

\[
v_\lambda(t) = L \frac{d}{dt} i_\lambda(t),
\]

with \( L \) positive definite. Here \( x_\lambda = \{ \} \), \( f_x = \{ \} \), \( \phi(i_\lambda, t) = L i_\lambda(t) \) and \( f_\phi = v_\lambda(t) \).

(b) flux-formulated inductances with

\[
\phi(t) = \phi_L(i_\lambda, t)
\]

\[
v_\lambda(t) = \frac{d}{dt} \phi(t),
\]

where \( L(i_\lambda, t) := \frac{\partial}{\partial i_\lambda} \phi_L(i_\lambda, t) \) is positive definite. Here

\[
x_\lambda = \phi(t) \quad f_x(v_\lambda) = v_\lambda(t)
\]

\[
\phi(i_\lambda, t) = \phi_L(i_\lambda, t) \quad f_\phi(x_\lambda) = v_\lambda(t).
\]
Like in [15], we define for the index study the projector $Q_*$ onto the kernel of $A_\lambda^\top$ and its complementary $P_*=I-Q_*$, which projects onto the support of $A_\lambda^\top$.

**Theorem 1 (Circuit index)** Given an inductance-like element $\lambda$ following Definition 3 which is coupled to a circuit fulfilling assumptions [12] with $v_\lambda = A_\lambda^\top e$, then the entire system has differential index

(i) 1 if there are no cutsets containing only inductors, current sources and inductance-like elements (LI- cutsets), that is, $\ker(A_R A_C A_V)^\top = \{0\}$

nor loops of voltage sources and capacitances only (CV-loops), that is, $\ker Q_V^\top A_V = \{0\}$.

(ii) 2, otherwise.

**Proof** The proof is analogous to the differential index proof in [15], by taking into account the new terms in the system

\[
\begin{align*}
A_C & \frac{d}{dt} q(A^\top e, t) + A_R g(A^\top e, t) + A_L i_L + A_V i_V + \left[ A_\lambda i_\lambda \right] + A_i^\top i_{src}(t) = 0 \\
\frac{d}{dt} \phi_\lambda(i_L, t) - A_L^\top e &= 0 \\
A_V^\top e - v_{src}(t) &= 0 \\
F \left( \frac{d}{dt} x_\lambda, \frac{d}{dt} i_\lambda, x_\lambda, i_\lambda, A_\lambda^\top e \right) &= 0
\end{align*}
\]

accounting for the inductance-like elements.

**Remark 2** The index results are valid for circuits containing multiple inductance-like elements

\[
F_1 \left( \frac{d}{dt} x_{\lambda,1}, \frac{d}{dt} i_{\lambda,1}, x_{\lambda,1}, i_{\lambda,1}, A_{\lambda,1}^\top e \right), \ldots, F_n \left( \frac{d}{dt} x_{\lambda,n}, \frac{d}{dt} i_{\lambda,n}, x_{\lambda,n}, i_{\lambda,n}, A_{\lambda,n}^\top e \right).
\]

**Proposition 31 (Linear index-2 components)** Let the DAE of the inductance-like element of Definition 3 have the structure

\[
0 = F \left( \frac{d}{dt} x_\lambda, \frac{d}{dt} i_\lambda, x_\lambda, i_\lambda, v_\lambda, t \right) = \tilde{F} \left( \frac{d}{dt} x_\lambda, \frac{d}{dt} i_\lambda, x_\lambda, i_\lambda, t \right) + B v_\lambda,
\]

with $B \in \mathbb{R}^{(n_{dof}+n_\lambda) \times n_\lambda}$, that is, the voltage term in the original DAE system is linear, whenever the inductance-like element is contained in an LI-cuts system, then, the coupled system of Theorem 1 has linear index-2 components.

**Proof** Analogous to the differential index proof in [15], one can see that the index-2 components are the node potentials in $LI\lambda$ cutsets, that is, $Q_{CRV} e$, where $Q_{CRV}$ is a projector onto $\ker (A_C A_R A_V)^\top$, and the currents in branches containing voltage sources in CV-loops, that is $Q_{V-C} e$, with $Q_{V-C}$ a projector onto $\ker Q_V^\top A_V$. If the voltage $A_\lambda^\top e$ in the original DAE of the inductance-like device is thus linear, then the possible index-2 component $A_\lambda^\top Q_{CRV} e$ is linear. The other possible index-2 components are part of the original MNA equations and it has already been shown previously (see e.g. [7]) that they are linear.
Now that we know inductance-like elements behave (from the index point-of-view) like an inductance in a circuit, two more complex examples of such type of elements with practical relevance will be presented in the following: the spatially discretised magnetoquasistatic models in $A^*$ and $T$-$\Omega$ formulations.

4 Refined Systems

The electromagnetic field in a magnetoquasistatic approximation is defined by Maxwell’s equations for the eddy current problem [18]:

$$\nabla \times E = -\frac{\partial}{\partial t} B$$  \hspace{1cm} (8a)

$$\nabla \times H = J$$  \hspace{1cm} (8b)

$$\nabla \cdot B = 0,$$  \hspace{1cm} (8c)

where the time derivative of the electric flux density is disregarded with respect to $J$ ($\frac{\partial}{\partial t} D = 0$) in Maxwell-Ampère’s equation (8b). Here, $E$ is the electric field strength, $B$ the magnetic flux density, $H$ the magnetic field strength and $J$ the electric current density. All quantities are vector fields $I \times \Omega \rightarrow \mathbb{R}^3$ depending on time and space, where $\Omega \subset \mathbb{R}^3$. The quantities are related via the material equations

$$J = \sigma E + J_s$$

$$H = \nu B.$$

The nonnegative conductivity $\sigma$ and the positive reluctivity $\nu = \mu^{-1}$ depend on space and their dependence on the fields is for now disregarded for simplicity of notation. $J_s$ is the source current density.

**Assumption 41 (Domain see Figure [1])** The domain $\Omega \subset \mathbb{R}^3$ has two types of subdomains, the source domains $\Omega^s_r$, $r = 1, \ldots, n_s$ and the conducting domain $\Omega_c$. They fulfil the following properties.

- $\Omega$ is contractible (see [10]).
- All subdomains are disjoint, that is,

$$\Omega^s_i \cap \Omega^s_j = \emptyset, \text{ for } i \neq j \quad \text{and} \quad \Omega_c \cap \Omega^s_j = \emptyset \quad \forall j.$$

- The conductivity $\sigma$ is positive in $\Omega_c$ and zero everywhere else.
- The source current density $J_s$ is only nonzero in $\Omega_s = \bigcup_i \Omega^s_i$. 
4.1 A* and T-Ω Formulations

Typically, Maxwell's equations are formulated by defining potentials [9]. In the T-Ω formulation [137], an electric vector potential $T : \mathcal{I} \times \Omega_c \to \mathbb{R}^3$ only on the conducting domain $\Omega_c$ and a magnetic scalar potential $\psi : \mathcal{I} \times \Omega \to \mathbb{R}$ on the entire domain $\Omega$ are defined, such that

$$J_c = \nabla \times T \quad \text{and} \quad H = H_s + T - \nabla \psi,$$

where $J_c = \sigma E$ is the conduction current density and $H_s$ can be thought of as a source magnetic field with $\nabla \times H_s = J_s$. The following system of partial differential equations (PDEs) arises

$$\nabla \times \rho \nabla \times T + \mu \frac{\partial}{\partial t} T - \mu \frac{\partial}{\partial t} \nabla \psi + \mu \frac{\partial}{\partial t} H_s = 0 \quad \text{in} \quad \Omega_c$$
$$\nabla \cdot \mu T - \nabla \cdot (\mu \nabla \psi) + \nabla \cdot \mu H_s = 0 \quad \text{in} \quad \Omega,$$

(9)

where $\rho : \mathcal{I} \times \Omega_c \to \mathbb{R}^3$ is the specific resistance $\sigma^{-1}$.

Another possibility is the A-ϕ formulation. Here, a magnetic vector potential $A : \mathcal{I} \times \Omega \to \mathbb{R}^3$ and an electric scalar potential $\varphi : \mathcal{I} \times \Omega \to \mathbb{R}$ are introduced (see [19,31]), such that

$$B = \nabla \times A \quad \text{and} \quad E = -\frac{\partial}{\partial t} A - \nabla \varphi.$$

The gauge freedom allows to choose a magnetic vector potential which leads to the A* formulation, where $E = -\frac{\partial}{\partial t} A$. This yields the following PDE

$$\sigma \frac{\partial}{\partial t} A + \nabla \times (\nu \nabla \times A) = J_s.$$

(10)

In the simplest case electric boundary conditions are set at $\partial \Omega$, that is, the tangential component of the electric field is zero $E_t = 0$. For $n$ the unit vector
normal to $\partial \Omega$ this translates into setting zero Neumann boundary conditions for the magnetic scalar potential $\psi$

$$
\mu \nabla \psi \cdot n = 0,
$$
as long as $\Omega_c \cap \partial \Omega = \emptyset$ and the magnetic source function is chosen to be $\mu \mathbf{H}_s \cdot n = 0$, and zero Dirichlet boundary conditions for the magnetic vector potential $\mathbf{A}$

$$
\mathbf{n} \times \mathbf{A} = 0.
$$

Also the tangential component of the electric vector potential $\mathbf{T}$ is set to zero at $\partial \Omega_c$.

### 4.2 Circuit Coupling

In order to couple the three dimensional system of field equations with the zero dimensional circuit’s equations, winding functions \[33\] are introduced. They distribute the currents or voltages of the circuit on the field’s domain $\Omega$.

There are different types of conductor models that lead to winding functions with different properties \[8,33\]. We will consider the stranded conductor model, where a divergence-free winding function $\chi_s : \Omega_s \rightarrow \mathbb{R}^{3 \times n_s}$ is constructed, such that for each coil $j$

$$
\mathbf{J}_s^{(j)} = \chi_s^{(j)} i_s^{(j)},
$$
where $i_s^{(j)}$ is the current through the coil, $\sup(\mathbf{J}_s^{(j)}) = \Omega_s^{(j)}$ and

$$
\mathbf{J}_s = \sum_j \chi_s^{(j)} i_s^{(j)} = \chi_s \mathbf{i}_s.
$$

In the case of the T-$\Omega$ formulation, a function $\zeta_s : \Omega \rightarrow \mathbb{R}^{3 \times n_s}$ is defined with $\nabla \times \zeta_s = \chi_s$ and thus

$$
\mathbf{H}_s = \zeta_s \mathbf{i}_s.
$$

We start by deriving the coupling equation \[40\] with the definition of voltage as

$$
\mathbf{v}_s = -\int_{\Omega} \mathbf{E} \cdot d\Omega.
$$

Using Gauss’s theorem and Faraday-Lenz’s law \[3a\], we obtain

$$
\mathbf{v}_s = \frac{d}{dt} \int_{\Omega} \zeta_s \cdot \mathbf{B} \, d\Omega - \int_{\partial \Omega} (\zeta_s \times \mathbf{E}) \cdot d\mathbf{S},
$$
which, due to the electric boundary conditions leads to the coupling equation

$$
\mathbf{v}_s = \int_{\Omega} \zeta_s \cdot \frac{d}{dt}(\mathbf{T} - \nabla \psi + \zeta_s \mathbf{i}_s) \, d\Omega.
$$
In the A* formulation, the voltage of the circuit can be related to the field quantities (see [33]) via

$$v_s = \frac{d}{dt} \int_{\Omega} \mathbf{\chi}_s \cdot \mathbf{A} \, d\Omega.$$

The degrees of freedom of both formulations are on dual sides of the diagram in Figure 2 and therefore it is said that both formulations are complementary. Those type of systems can be used for an error approximation of the discretisation method [1].

![Maxwell House diagram](image.png)

**Fig. 2** Maxwell House diagram, based on [33].

### 4.3 Discretised Systems

First the field model is discretised in space with a suitable method, such as the Finite Element Method (FEM) [23] with appropriate basis functions that fulfil the exact discrete de Rham sequence [10] or the Finite Integration Technique (FIT) [38].

For the Finite Element discretisation with H(curl)-conforming basis functions $\boldsymbol{\nu}_i : \Omega \to \mathbb{R}^3$, ($i = 1, \ldots, n_{\text{dof}}$), the weak formulation in the case of the
magnetic vector potential is \[4\]

\[
\int_\Omega \sigma \frac{\partial A}{\partial t} \cdot \nu_i + \left( \nu \nabla \times A \right) \cdot \left( \nabla \times \nu_i \right) d\Omega = \int_\Omega \chi_s \cdot \nu_i d\Omega
\]

\[
\frac{d}{dt} \int_\Omega \chi_s \cdot A d\Omega = v_s,
\]

for all \(i\). With the Ritz-Galerkin method the magnetic vector potential is approximated by

\[
A = \sum_{i=1}^{n_{\text{dof}}} a_i(t) \nu_i
\]

and the conductivity matrix is for example constructed as

\[
(M_\sigma)_{i,j} = \int_\Omega \sigma \nu_i \cdot \nu_j d\Omega.
\]

The T-\(\Omega\) weak formulation as well as the rest of the material matrices are built analogously with the appropriate basis functions.

Eventually, the semi-discrete T-\(\Omega\) formulation

\[
C^T M_\mu C t + M_\mu \frac{d}{dt} t + M_\mu \tilde{S}^T \frac{d}{dt} \Psi + M_\mu Y_s \frac{d}{dt} i_s = 0
\]

\[
\tilde{S} M_\mu t + \tilde{S} M_\mu \tilde{S}^T \Psi + \tilde{S} M_\mu Y_s i_s = 0
\]

\[
Y_s^T M_\mu \frac{d}{dt} t + Y_s^T M_\mu \tilde{S}^T \frac{d}{dt} \Psi + Y_s^T M_\mu Y_s \frac{d}{dt} i_s - v_s = 0
\]

lead to two different systems of DAEs describing the same physical phenomenon. Here, \(M_i\) are the material matrices that describe the material relations between the discrete quantities, \(C\) and \(\tilde{S}^T\) and \(S\) are the discrete curl, gradient and divergence operators. \(X_s\) and \(Y_s\) are the discretisations of the winding functions \(\chi_s\) and \(\zeta_s\) respectively. The in the FEM notation not very common matrix factorisation of systems (12) and (13) borrowed from the Finite Integration Technique [39] is used for convenience in the analysis below.

In order to solve both systems, consistent initial conditions are imposed for \(a(t_0) = a_0\), \(t(t_0) = t_0\), \(\Psi(t_0) = \Psi_0\) and either \(v_s(t_0) = v_{s,0}\) or \(i_s(t_0) = i_{s,0}\), depending on which lumped quantity is given as an excitation to the system. To ensure uniqueness of solution, also discrete gauging conditions are inserted (see e.g. [12][39]).

Proposition 41 (System matrices) The discretisation matrices have the following properties.
The material matrices $M_\star$ are symmetric positive definite for $\star = \{\mu, \nu\}$ and symmetric positive semidefinite for $\star = \{\sigma\}$.

The discrete gradient matrix $-\tilde{S}^\top$ is assumed to be projected to a subspace where the boundary conditions are imposed to the degrees of freedom and thus has full column rank, that is, $\ker \tilde{S}^\top = \{0\}$.

For the mentioned suitable discretisations, $CS^\top = 0$.

For FIT all three properties are classical results (see e.g. [39]). In the case of FEM, the first property follows directly from how the material matrices are constructed analogously to equation (11). Both the second and third properties follow from the fact that the spaces spanned by the basis functions including boundary conditions fulfil the de Rham sequence.

5 DAE Index Analysis

Before heading to the index results of both field-circuit coupled systems, a specific structured inductance-like element, that eases the later analysis, is introduced.

Proposition 51 (Inductance-like element) A device described by a DAE

$$F\left( \frac{d}{dt}x_\lambda, \frac{d}{dt}i_\lambda, x_\lambda, i_\lambda, v_\lambda, t \right) = 0,$$

where at most one differentiation $\frac{d}{dt} F\left( \frac{d}{dt}x_\lambda, \frac{d}{dt}i_\lambda, x_\lambda, i_\lambda, v_\lambda, t \right) = 0$ is needed, such that one can write

$$\frac{d}{dt}x_\lambda = f_x(x_\lambda, i_\lambda, v_\lambda, t) \quad (14)$$

$$\frac{d}{dt}i_\lambda = L_\lambda(x_\lambda)^{-1}v_\lambda + f_i(x_\lambda, i_\lambda, t), \quad (15)$$

with $L_\lambda(x_\lambda)$ being positive definite, is an inductance-like device.

Proof We define $\phi(i_\lambda, x_\lambda, t) = L_\lambda(x_\lambda)i_\lambda$. The first property is thus fulfilled, as

$$\frac{\partial}{\partial i_\lambda} \phi(x_\lambda, i_\lambda, t) = L_\lambda(x_\lambda),$$

which is positive definite and therefore regular. Also, setting

$$f_\phi(x_\lambda, i_\lambda, v_\lambda, t) = v_\lambda + \frac{\partial}{\partial x_\lambda} (L_\lambda(x_\lambda)i_\lambda)f_x(x_\lambda, i_\lambda, v_\lambda, t) + L_\lambda(x_\lambda)f_i(x_\lambda, i_\lambda, t)$$

leads to equation (15), where $\frac{\partial}{\partial v_\lambda} \left( \left( \frac{\partial \phi}{\partial x_\lambda} \right)^{-1} \left( - \frac{\partial \phi}{\partial x_\lambda} f_x - \frac{\partial \phi}{\partial t} + f_\phi \right) \right) = L_\lambda^{-1}(x_\lambda)$ is again positive definite and fulfils the second property of an inductance-like element.
5.1 DAE Index of the T-Ω Formulation

Let the tree-cotree gauge \([2]\) be introduced. For a simply connected region \(Ω\), the values of the degrees of freedom \(t\) are set to zero on a tree \(T_c\) of the mesh inside the conducting region \(Ω_c\) that adequately takes care of the boundary conditions. For that, a projector \(\bar{P}\) is defined that projects onto the edges of the cotree of \(T_c\). \(\bar{P}\) is the reduction of the projection matrix \(\bar{P}\) by deleting all the zero columns. In case of a multiply-connected region, cuts have to be defined in \(Ω\) to ensure a correct gauging condition (see \([40]\)).

Assumption 51 (Gauged T-Ω formulation) The discretised T-Ω system \([12]\) is gauged and thus rewritten as

\[
P^\top C^\top \mathbf{M}_\mu C P t_{\text{red}} + P^\top M_\mu \left( P \frac{d}{dt} t_{\text{red}} + \tilde{S}^\top \frac{d}{dt} \psi + Y_s \frac{d}{dt} i_s \right) = 0 \quad (16a)
\]

\[
\tilde{S} \mathbf{M}_\mu \left( P t_{\text{red}} + \tilde{S}^\top \psi + Y_s i_s \right) = 0 \quad (16b)
\]

\[
Y_s^\top M_\mu \left( P \frac{d}{dt} t_{\text{red}} + \tilde{S}^\top \frac{d}{dt} \psi + Y_s \frac{d}{dt} i_s \right) - v_s = 0 \quad (16c)
\]

such that the matrix \(K_\rho = P^\top C^\top \mathbf{M}_\rho C P\) has full rank, i.e. \(\det(K_\rho) \neq 0\).

Proposition 52 The discrete field \(P x\) is not a gradient field, i.e. \(P x \neq \tilde{S}^\top y\) for \(x, y \neq 0\).

Proof The Proposition follows directly from Assumption 51 and Property 41.

Proposition 53 (Discrete Helmholtz split) Every \(x \in \mathbb{R}^n\) can be written as \(x = \tilde{S}^\top x_1 + M_\mu^{-1} C^\top x_2\), where \(x_1 \in \mathbb{R}^m\), \(x_2 \in \mathbb{R}^n\), with \(\tilde{S}^\top \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{n \times n}\) being the matrices defined in Section 4.3.

Proof As \(M_\mu\) is positive definite and \(\ker(\tilde{S} \mathbf{M}_\mu) = \text{im}(M_\mu^{-\frac{1}{2}} C^\top)\), we have

\[
y = M_\mu^{-\frac{1}{2}} \tilde{S}^\top y_1 + M_\mu^{-\frac{1}{2}} C^\top y_2,
\]

for all \(y \in \mathbb{R}^n\). Furthermore, there exists a \(y_0\) such that \(x = M_\mu^{-\frac{1}{2}} y_0 = \tilde{S}^\top x_1 + M_\mu^{-\frac{1}{2}} C^\top x_2\).

Assumption 52 (Discrete current densities) It is assumed that

\[
0 \neq CY_s x \neq CP y, \quad \text{for } x, y \neq 0,
\]

where we recall that \(Y_s\) is the discrete winding function of the T-Ω formulation.

The previous assumption imposes that the curl of the discretised magnetic source field, which is the discretised source current density \(j^{(s)}_\mu\) associated with \(\Omega_s^{(s)}\), is different from the curl of the discretised electric vector potential \(t_{\text{red}}\), which corresponds to the conduction current density \(j_c\) associated with \(\Omega_c\). As \(\Omega_s^{(s)} \cap \Omega_c = \emptyset\) for all \(i\), the assumption is reasonable.
Proposition 54 (T-Ω inductance-like element) The discrete (gauged) system of equations of the T-Ω formulation with circuit coupling equation (16) is an inductance-like element.

Proof By differentiating equation (16b) once, one can extract
\[
\frac{d}{dt} \psi = -L_\mu^{-1} S M_\mu P \frac{dt_{\text{red}}}{dt} - L_\mu^{-1} S M_\mu Y_s \frac{di_s}{dt},
\]
with \( L_\mu = \tilde{S} M_\mu \tilde{S}^T \) positive definite due to Property 41.

In order to obtain an expression \( \frac{dt_{\text{red}}}{dt} \), first the positive definiteness of the matrix \( P^T W P \), with \( W = M_\mu - M_\mu \tilde{S}^T L_\mu^{-1} \tilde{S} M_\mu \) is shown.

\[
W = M_\mu^2 (I - M_\mu^2 \tilde{S}^T L_\mu^{-1} \tilde{S} M_\mu^2) M_\mu^2 = M_\mu^2 Q_\rho M_\mu^2,
\]
where \( Q_\rho \) is a projector and thus positive semidefinite. Let us assume there exists an \( x \neq 0 \) with \( x^T P^T W P x = 0 \). Then \( W \tilde{z} P x = 0 \) and \( W P x = 0 \) and thus \( P x = \tilde{S}^T L_\mu^{-1} \tilde{S} M_\mu P x \), which cannot be due to Proposition 52.

Secondly, inserting (17) into (16a), yields
\[
\frac{dt_{\text{red}}}{dt} = -(P^T W)^{-1} P_\sigma^T W Y_s \frac{di_s}{dt} - (P^T W)^{-1} P^T K_\rho P t_{\text{red}}.
\]
Finally, using equations (17), (18) and (21c), one obtains
\[
v_s = L_\lambda \frac{di_s}{dt} = Y_s^T W (P^T W)^{-1} K_\rho t_{\text{red}},
\]
with
\[
L_\lambda = Y_s^T (W - WP (P^T W)^{-1} P^T W) Y_s := Y_s^T W P Y_s.
\]
If \( L_\lambda \) is positive definite, then, applying Proposition 51 concludes the proof.

Let us verify that \( x^T L_\lambda x > 0 \), for \( x \neq 0 \). Analogously to the case of \( P^T W P \), one can show that \( W_p = W^\frac{1}{2} Q_W W^\frac{1}{2} \), with \( Q_W \) being a projector and again we just need to show that \( (W - W P (P^T W)^{-1} P^T W) Y_s x \neq 0 \). Let us assume \( (W - W P (P^T W)^{-1} P^T W) Y_s x = 0 \), then
\[
W Y_s x = W P (P^T W)^{-1} P^T W Y_s x =: W P y.
\]
Using Proposition 53, we can write
\[
Y_s x = \tilde{S}^T x_1 + M_\mu^{-1} C^T x_2 \quad \text{and} \quad P y = \tilde{S}^T y_1 + M_\mu^{-1} C^T y_2
\]
and
\[
W Y_s x = C^T x_2 = W P y = C^T y_2.
\]
But, according to Assumption 52,
\[
C M_\mu^{-1} C^T x_2 = C Y_s x \neq C P y = C M_\mu^{-1} C^T y_2,
\]
thus \( C^T x_2 \neq C^T y_2 \) and \( W Y_s x \neq W P y \). Therefore, \( L_\lambda \) is positive definite which concludes the proof.
We have proven that the T-$\Omega$ formulation embedded in a circuit behaves like an inductance from the index point of view. Exciting the field model either with a current source or a voltage source can be seen as a particular circuit coupling and this yields the following Corollary.

**Corollary 1 (Excitation index of the T-$\Omega$ formulation)** *The discrete (gauged) system of equations of the T-$\Omega$ formulation with circuit coupling equation (16) has differentiation index*

- 1, if the voltage $v_s$ is prescribed.
- 2, if the current $i_s$ is prescribed.

Therefore, a voltage excitation leads to a system with a lower index and, hence, can be numerically handled in an easier way. Now, the same analysis will be done for the $A^*$ formulation in order to compare both cases.

### 5.2 DAE Index of the $A^*$ Formulation

Again a tree-cotree gauge is introduced, albeit this time in the non-conducting domain $\Omega_c$. Notice that the reduction of the projection matrix $P$ that deletes the necessary degrees of freedom is different from the one in the T-$\Omega$ formulation (16).

**Assumption 53 (Gauged $A^*$ formulation)** *The discrete system (13) is gauged and thus rewritten as*

\[
P^\top M_\sigma P \frac{d}{dt} a_{\text{red}} + P^\top C^\top M_\nu C P a_{\text{red}} - P^\top X_s i_s = 0
\]

\[
\frac{d}{dt} X_s^\top P a_{\text{red}} - v_s = 0
\]

*such that the matrix pencil $\lambda \bar{M}_\sigma + K_\nu$ is positive definite for $\lambda > 0$ with $\bar{M}_\sigma = P^\top M_\sigma P$ and $K_\nu = P^\top C^\top M_\nu C P$.*

For simplicity of notation, we introduce the matrix $\bar{X}_s = P^\top X_s$.

**Assumption 54 (Discrete winding function)** *The discrete gauged winding function matrix $\bar{X}_s$ fulfils*

- it has full column rank.
- $\text{im} \bar{X}_s \perp \text{im} \bar{M}_\sigma$.

This last assumption states properties of the discrete matrices, inspired by the properties of the domains and the continuous functions stated in Assumption 41. The first property is motivated by the fact that each column of matrix $\bar{X}_s$ is the discretisation of a different winding function $\chi_s^{(j)}$ which all have disjoint supports. Similarly, the conducting domain and the source domain are disjoint, which inspires the second property that could be relaxed but is kept for simplicity.
**Proposition 55 (A* inductance-like element)** The discrete (gauged) system of equations of the A* formulation with coupling equation (20) is an inductance-like element.

**Proof** First, a projector $Q_{\sigma}$ onto $\ker \bar{M}_{\sigma}$ is defined and $P_{\sigma} = I - Q_{\sigma}$. System (20) can now be rewritten as

\[
\bar{M}_{\sigma} \frac{d}{dt} a_{\text{red}} + P_{\sigma}^{\top} K_{\nu} a_{\text{red}} - P_{\sigma}^{\top} \bar{X}_{s} i_{s} = 0 \quad (21a)
\]

\[
Q_{\sigma}^{\top} K_{\nu} a_{\text{red}} - Q_{\sigma}^{\top} \bar{X}_{s} i_{s} = 0 \quad (21b)
\]

\[
\frac{d}{dt} \bar{X}_{s} a_{\text{red}} - v_{s} = 0. \quad (21c)
\]

Equation (21a) allows to extract $P_{\sigma} \frac{d}{dt} a_{\text{red}} = f_{\sigma}(a_{\text{red}}, i_{s})$, where $f_{\sigma}$ can directly be computed from the equation. After one time differentiation of (21b) and inserting into (21c)

\[
\frac{d}{dt} i_{s} = L_{\lambda}^{-1} v_{s} + f_{\nu}(a_{\text{red}}, i_{s}),
\]

is obtained, with $f_{\nu}$ being a result of inserting $f_{\sigma}$ into (21c). Here $L_{\lambda} = \bar{X}_{s}^{\top} Q_{\sigma} (Q_{\sigma}^{\top} K_{\nu} Q_{\sigma} + P_{\sigma}^{\top} P_{\sigma})^{-1} Q_{\sigma}^{\top} \bar{X}_{s}$ positive definite, as $Q_{\sigma} \bar{X}_{s}$ has full column rank due to Assumption 54 and $Q_{\sigma}^{\top} K_{\nu} Q_{\sigma} + P_{\sigma}^{\top} P_{\sigma}$ is positive definite due to Assumption 53.

Again, the particular circuit case of a voltage or current source is considered and the same Corollary as far the T-$\Omega$ coupling follows from the previous Proposition.

**Corollary 2 (Excitation index of the A* formulation)** The discrete (gauged) system of equations of the A* formulation with circuit coupling equation (21) has differentiation index

- 1, if the voltage $v_{s}$ is prescribed.
- 2, if the current $i_{s}$ is prescribed.

This result was already proven in [5] for a different gauge and follows now as a Corollary from Proposition 55 and Theorem 1.

5.3 Field-Dependent Materials

In case of having field-dependent materials, the material matrices $M_{\mu}(h)$ for the T-$\Omega$ formulation and $M_{\nu}(b)$ for the A* formulation, depend on the discretised field quantities $h = C t + S^{\top} \psi + Y_{s} i_{s}$ and $b = C a$, respectively.

Taking the time derivative of an equation leads to systems like (20) and (16) with differential material matrices $M_{\mu,d}$ and $M_{\nu,d}$ (see [30] Chapter 3, [32] Appendix A.3) instead of the regular material matrices $M_{\mu}$ and $M_{\nu}$.
whenever there is a time derivative. For example in the first equation of the T-Ω formulation
\[ C^T M_\rho C t + \frac{d}{dt} (M_\rho(h)h) \]
applying the chain rule we have
\[ C^T M_\rho C t + M_{\mu,d}(h) \frac{d}{dt} h. \]
The differential material matrices are built with the differential permeability \( \mu_d(H) \) (respectively the differential reluctivity \( \nu_d(B) \)), where \( || \cdot || \) is the Euclidean norm, \( H = ||H|| \) and \( B = ||B|| \),
\[ \begin{align*}
\mu_d(s) &= \mu(||s||)I + \frac{1}{s} \frac{\partial \mu(s)}{\partial s} ss^T, \\
\nu_d(s) &= \nu(s)I + \frac{1}{s} \frac{\partial \nu(s)}{\partial s} ss^T,
\end{align*} \]
with \( s \in \mathbb{R}^3, s = ||s|| \) and \( I \in \mathbb{R}_{3 \times 3} \) the identity tensor of second rang. Those tensors are positive definite (see [26, Chapter 2], [30, Chapter 3]) under the following natural physical assumptions for the B-H curve \( B = f_{BH}(H) \).

**Assumption 55 (B-H curve [27])** The B-H curve
\[ f_{BH}(H) = \mu(H)H : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \]
fulfils
\[ \begin{align*}
&- f_{BH}(s) \text{ is continuously differentiable}. \\
&- f_{BH}(0) = 0. \\
&- f_{BH}(s) \geq \mu_0, \forall s > 0. \\
&- \lim_{s \rightarrow \infty} f'(s) = \mu_0.
\end{align*} \]
with \( \mu_0 > 0 \) being the vacuum permeability.

Analogously to the regular material matrices, we have that the differential material matrices are positive definite provided that the differential material tensor is positive definite, which is the case. Therefore, the index analysis for the \( A^* \) and T-Ω formulations can be analogously transferred to field-dependent materials under Assumption 55.

**Proposition 56 (Linear index-2 components)** Both the gauged T-Ω system (16) as well as the \( A^* \) system (20) with field-dependent materials have the structure described in Proposition 31 and thus lead to a DAE with linear index-2 components when coupled to a circuit.

The \( A^* \) and T-Ω formulations are complementary, that is, the potentials defined on them live on spaces dual to each other. Also the excitations imposed on the formulations (either current or voltage excited) live on dual spaces (see Figure 2). Therefore, it could be thought that, whereas for one formulation
it is more convenient to impose a current excitation, for the other a voltage excitation is better. However, the analysis shows that both behave equally from the index point of view, as they are inductance-like elements. In both cases a voltage excitation leads to an index-1 system as for the case of an inductor and the index results are more connected to the physics described by the DAE rather than to the formulation.

6 Numerical Results

Numerical examples for a field/circuit coupled system using the A* formulation and comparing an index-1 coupled case with an index-2 one have already been demonstrated previously (see e.g. [5]).

For the numerical simulations in this paper, the T-Ω system of equations is solved for a model consisting of a square coil and an aluminium core (see Figure 3) and coupled to a circuit. The discretisation is carried out with the Finite Integration Technique and a tree-cotree gauge is applied to the discretised electric vector potential in the conducting region (see Section 5.1).

Two different coupling scenarios are considered. An index-1 case with the magnetoquasistatic device coupled to a voltage source \( v_s(t) = \sin(2\pi f_s t) \) (see Figure 4(a)) is compared to an index-2 setting (Figure 4(b)) where the device is coupled to a current source \( i_s(t) = \sin(2\pi f_s t) \), with \( f_s = 2\pi \). In both cases, the simulation is performed first with the given excitation \( v_s(t) \) (respectively \( i_s(t) \)) and afterwards with a slightly perturbed excitation \( \tilde{v}_s(t) \) (respectively \( \tilde{i}_s(t) \)).
\tilde{v}_s(t), \text{ i.e.}
\tilde{v}_s(t) = v_s(t) + p(t), \quad \tilde{i}_s(t) = i_s(t) + p(t),
with perturbation
\begin{align*}
p(t) &= \varepsilon_p \sin(2\pi f_s \cdot 10^9 t) \\
\varepsilon_p &= 10^{-4}
\end{align*}
and $\varepsilon_p = 10^{-4}$. For both cases, an implicit Euler scheme is performed at

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{field_circuit_coupling}
\caption{Different field/circuit coupling schemes.}
\end{figure}

\begin{enumerate}
\item Index 1: Inductance-like element coupled to voltage source $v_s = \sin(2\pi f_s t)$.
\item Index 2: Inductance-like element coupled to current source $i_s = \sin(2\pi f_s t)$.
\item Index 1: Inductance-like element coupled to voltage source $v_s = \sin(2\pi f_s t)$ with perturbation $v_p = \varepsilon_p \sin(2\pi f_p t)$.
\item Index 2: Inductance-like element coupled to current source $i_s = \sin(2\pi f_s t)$ with perturbation $i_p = \varepsilon_p \sin(2\pi f_p t)$.
\end{enumerate}

time interval $I = [0, 0.5]$ with varying step sizes $\delta t = \{8 \cdot 10^{-5}, 4 \cdot 10^{-5}, 2 \cdot 10^{-5}, 10^{-5}\}$.

Consistent initial conditions on the degrees of freedom $x(t_0) = 0$ are set for the index-1 simulation. In the index-2 case, it has been shown that for DAEs with linear index-2 components, which is the case for our system (see
Proposition 56), a consistent initial value is obtained after two implicit Euler iterations [7,14]. Here, the starting point $x(t_0 - 8 \cdot 10^{-5}) = 0$ is selected.

Figure 5 shows the simulation results for both index-1 and index-2 with non-perturbed and perturbed excitations. It can be seen that the perturbed index-2 case (Figure 5(d)) oscillates due to the higher sensitivity of index-2 DAE systems to small perturbations, whereas the index-1 simulation (Figure 5(b)) is not significantly affected by the small perturbation on the excitation.

7 Conclusions

The paper discusses index results for field/circuit coupled systems for different formulations. The case of the $A^*$ formulation was already studied previously (see [5]). However, the index for the field/circuit coupled system of DAEs obtained with a T-$\Omega$ formulation had not been analysed before. In order to study the index of the coupled systems, a new generalised element type is introduced that from the index point of view behaves like an inductor in the MNA formulation. This eases the later index analysis, as now local properties of the DAE system describing only the element have to be verified together with topological characteristics of the circuit in order to obtain the index of the entire coupled system.

Both the $A^*$ as well as the T-$\Omega$ formulation field/circuit coupling indexes have been shown to behave like inductances. This yields to the conclusion.
that even though the degrees of freedom of the two formulations and the voltage (respectively the current) excitations live on dual spaces, prescribing the voltage results in a lower index system in both formulations.

A further study could define a generalised capacitance-like element and derive under which approximations Maxwell’s equations embedded as a generalised element to a circuit correspond to a capacitance-like element.

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