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Fractal geometry, information growth and nonextensive thermodynamics

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Abstract

This is a study of the information evolution of complex systems by a geometrical consideration. We look at chaotic systems evolving in fractal phase space. The entropy change in time due to the fractal geometry is assimilated to the information growth through the scale refinement. Due to the incompleteness of the state number counting at any scale on fractal support, the incomplete normalization $\sum_i p_i^q = 1$ is applied throughout the paper, where $q$ is the fractal dimension divided by the dimension of the smooth Euclidean space in which the fractal structure of the phase space is embedded. It is shown that the information growth is nonadditive and is proportional to the trace-form $\sum_i p_i - \sum_i p_i^q$ which can be connected to several nonadditive entropies. This information growth can be extremized to give power law distributions for these non-equilibrium systems. It can also lead to a nonextensive thermodynamics for heterogeneous systems which contain subsystems each having its own $q$. It is shown that, within this thermodynamics, the Stefan-Boltzmann law of blackbody radiation can be preserved.

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1 Introduction

The aim of this work is to study the information evolution and the thermodynamics of non-equilibrium complex systems whose phase space volume gradually maps into fractals or multifractals at long time $t \to \infty$[1].

The motivation of this work is relevant to the generalization of Shannon information entropy which are in general connected to dense phase space ($\Gamma$-space). One notices that the generalized entropies[2] are just posited or postulated as such and recover Shannon entropy as the generalization indexes (parameters) take special values. Some of these generalized entropies, e.g., Havrda-Charvat-Tsallis one[3, 4] and Rényi one[5], are believed to have some connections with fractal geometry of phase space and with chaotic behavior of non-equilibrium systems at stationary states, and have been used to develop generalized statistics whose mathematical structure mimics the formal system of the conventional statistical mechanics of Boltzmann-Gibbs[4, 6, 7].

The idea of this work is to study the information or entropy from the outside of the aforementioned formal systems based on the postulated information measures or entropies. We just look at the geometry of the phase space of a non-equilibrium system to calculate the information evolution and to see whether it has something to do with the generalized entropies. We suppose an ensemble of non-equilibrium systems moving in a fractal $\Gamma$-space, the volume of its initial condition gradually mapping into fractal structure. If one looks at the trajectory of a system of the ensemble as it runs over the permitted phase points, one will see that the phase volume is covered more and more and that a fractal emerges from the covered regions. In this case, as the system evolves in time, a scale refinement would be necessary to calculate the heterogeneously occupied volume. So the long time effect of the system of interest can be likened to its ensemble effect: the time behavior of the trajectory of a system is replaced by the iteration of the scale refinement from the initial phase volume of the ensemble. As a consequence, the information evolution in time can be estimated by the ensemble average of the information growth during the scale refinement, i.e.,

$$I(T) = \lim_{T \to \infty} \frac{1}{T} \int_0^T i(t) dt \implies \lim_{v_T \to \infty} \sum_{i=1}^{v_T} p_i \int_{s(i)} I_i ds - \sum_{j=1}^{v_0} p_j \int_{s(j)} I_j ds \quad (1)$$
where $I(T)$ is the average information change during the time $T$, $i(t)$ is the information change per unity of time, $p_i$ is the probability that the system is found on the element $i$ of the phase space, $I_i$ is the density of information on the element $i$ of volume $s(i)$, $v_T$ and $v_0$ are the total numbers of the elements of phase space accessible to the system and visited by the trajectories at time $t = T$ and $t = 0$, respectively. According to our assumption, the $v_T$ elements visited by the system form a fractal structure which can be reproduced by the $v_k$ elements yielded from the $v_0$ elements by certain map (scale refinement) of $k$ iterations. So we can put $v_k = v_T$.

Due to the incompleteness of the counting of state points or the calculation of geometrical elements at any given scale in fractal structures[8], the discussion will be made on the basis of the normalization of incomplete probability distribution[5] proposed for the complex systems having physical states which are accessible to the systems but inaccessible to mathematical treatment[7, 8, 9, 10].

2 Incomplete normalization

Now we look at a fractal phase space of dimension $d_f$ embedded in a smooth Euclidean space of dimension $d$. We know that any counting or calculation of state number must be carried out at certain scale of the phase space. In fractal phase space, the state number and the phase volume change from scale to scale. So the state counting is never complete for a given scale or partition of the phase space. In other words, the calculated states or phase volume is incomplete. Our method consists in taking this incompleteness into account in the probability calculation. We have assumed[7, 8, 9, 10]:

$$\sum_{i_k=1}^{v_k} p_{i_k}^q = 1,$$

where $v_k$ is only the number of the states or phase elements accessible to the summation at the $k^{th}$ iteration, $q$ is given by $q = d_f/d$, $p_{i_k} = s_{i_k}/S_0$ is the probability that the system visit the elements $i$ of volume $s_{i_k}$ of the fractal at the $k^{th}$ iteration, and $S_0$ the volume of the phase space containing the fractal.

The probability defined above is different from the usual frequency or time definition. Here the probability $p_{i_k}$ does not sum to one because it is
the ratio of non-differentiable fractal elements to an integrable and differentiable smooth space volume. So this definition allows one to carry out calculations of fractal or hierarchical probability distributions by using the usual mathematical tricks defined for smooth Euclidean space. It is analogous to the proposition in [1] to define probability $p_i$ for the system to visit the phase element $i$ by the ratio of the number of trajectories ($\propto$ volume $s_i$) on the element $i$ to the total number of trajectories ($\propto$ total volume $S_0$) in the initial conditions uniformly distributed in a Euclidean space.

Eq. (2) has been called incomplete normalization [7, 9]. Its incompleteness lies in the fact that the sum over all the $v_k$ elements at the $k^{th}$ iteration does not mean the sum over all the possible states of the system under consideration. In other words, the volume $s_{ik}$ does not represent the real number of states or trajectories on the element $i_k$ which, as expected for any fractal and hierarchical structure, evolves with iteration or phase space partition.

3 Information growth due to fractal geometry

The evolution of the accessible phase volume of a system during the scale refinement is calculated as follows. The extra state points

$$
\Delta_{ik} = \sum_{j_{k+1}=1}^{n_{ik}} s_{jk+1} - s_{ik}
$$

acquired from certain element $i_k$ at the iterate of $(k+1)^{th}$ order are just the number of unaccessible states at $k^{th}$ order with respect to $(k+1)^{th}$ order, where $n_{ik}$ is the number of elements $s_{jk+1}$ replacing, at $(k+1)^{th}$ iterate, the element $s_{ik}$ and $j_{k+1}$ is the index of these elements.

What is the information change in that case? At the iterate of order $k$, the information content on $s_{ik}$ is given by $I_k(i) = \int s_{ik} I(\rho) ds$ where $I(\rho)$ is the information density as a function of the state density $\rho$ supposed scale invariant. This scale invariance, according to our assumption in Eq. (1), implies constant $\rho$ in time. So at $k+1$ order, we have $I_{k+1}(i) = \sum_{j_{k+1}=1}^{n_{ik}} \int s_{jk+1} I(\rho) ds$.

Remember that the definition of the probability $p_{ik} \propto s_{ik}$ implies constant $\rho$ over all the occupied elements of the phase space. In the case where every phase point is visited with equal probability, constant $\rho$ means constant
state density over these elements. This is a natural result of the uniformly
distributed states in the initial condition phase volume $S_0$ if we consider the
time and scale invariance of $\rho$ mentioned above.

Then the information growth on certain phase element $i_k$ from $k^{th}$ to
$(k+1)^{th}$ iteration reads

$$
\Delta I_k(i) = I_{k+1}(i) - I_k(i) = \int_{\Delta i_k} I(\rho) ds = I(\rho) \Delta i_k
$$

(4)

The relative information growth is then given by

$$
R_{k \rightarrow (k+1)}(i_k) = \Delta I_k(i_k)/I_k(i) = \sum_{j_{k+1}=1}^{n_{i_k}} \frac{p_{j_{k+1}}}{p_{i_k}} - 1.
$$

(5)

The expectation of this relative information growth over all the fractal can
be calculated by using the unnormalized expectation as follows

$$
\bar{R}_{k \rightarrow (k+1)} = \sum_{i_k=1}^{v_k} p_{i_k} R_{k \rightarrow (k+1)}(i_k) = \sum_{i_{k+1}=1}^{v_{k+1}} p_{i_{k+1}} - \sum_{i_k=1}^{v_k} p_{i_k}.
$$

(6)

The total relative information growth from $0^{th}$ to certain order, say, $\lambda$, of the
iteration is then given by

$$
R_\lambda = \bar{R}_{0 \rightarrow \lambda} = \sum_{k=0}^{\lambda-1} \bar{R}_{k \rightarrow (k+1)} = \sum_{i_\lambda=1}^{v_\lambda} p_{i_\lambda} - 1 = \sum_{i_\lambda=1}^{v_\lambda} (p_{i_\lambda} - p_{i_\lambda}^q)
$$

(7)

since $\sum_{i_0=1}^{v_0} p_{i_0} = S_0/S_0 = 1$. We would like to mention that these calculations
can also be formally carried out under the formalism of complete probability
distribution if we suppose $\varphi_i = p_i^q$. In this case, $R_k$ to the $k^{th}$ iteration reads

$$
R_k = \sum_{i_k=1}^{v_k} \varphi_i^{1/q} - 1 \text{ with } \sum_{i_k=1}^{v_k} \varphi_i = 1. \hspace{1cm} \varphi_i \text{ can be of course regarded as a probability distribution on a complete ensemble of } v_k \text{ states.}
$$

Following are some properties of $R_\lambda$ (the index $\lambda$ will be dropped from
now on, i.e. $R = R_\lambda$):

1. Nonadditivity : for a fractal of dimension $d_f$ composed of two sub-
fractals $A$ and $B$ of dimension $d_{f_A}$ and $d_{f_B}$ satisfying product joint
probability $p_{i_Ai_B} = p_{i_A}p_{i_B}$, it is easy to show the following nonadditivity :

$$
R(A + B) = R(A) + R(B) + R(A)R(B).
$$

(8)
2. $R$ is positive and concave for $q > 1$, and negative and convex for $q < 1$. If $q = 1$ or $d = d_f$, the fractal structure does not exist any more, so $R = 0$.

3. $R$ is an information growth attributed to the dimension difference $(d_f - d)$ and calculated from the actual probability distribution $p_i$. An interesting feature of $R$ is that the ratio $\frac{R}{d_f - d}$ leads to the Havrda-Charvat-Tsallis entropy $S = \frac{\sum_i p_i^q - \sum_i p_i^q}{q - 1}[3, 4]$. The asymptote of this ratio for $d_f \to d$ leads to Gibbs-Shannon entropy $S = -\sum_i p_i \ln p_i$. Other nonadditive entropies in the long list given by [2] can be obtained in similar way. For example, the ratio $\frac{\ln(R+1)}{d_f - d}$ gives Rényi entropy $S^r = \ln(\sum_i p_i^q) (\alpha = 1/q)$ for complete distribution[5] or $S^r = \frac{\ln\sum_i p_i}{q - 1}$ for incomplete distribution[11], and the ratio $\frac{(R+1)^{d_f/d} - 1}{d_f - d}$ gives Arimoto entropy $S^a = (\sum_i p_i^q)^{q - 1}[12]$.

4. It is straightforward to prove from the connections of $R$ with different entropies that the maximum of these entropies is (mathematically) equivalent to the extremization of the information growth $R$. So it is expected that the $R$-extremum can be used to obtain probability distributions similar to those obtained from maximum entropy. Here are some examples of the power law probability distributions yielded by the extremization of $R$ ($q \neq 1$) for some chaotic systems.

The extremization of $R = \int_0^1 \rho(x)dx - 1$ ($0 < x < 1$) of the random variable of the continued fraction map[13] $x_{n+1} = 1/x_n - [1/x_n]$) under the constraints associated with the normalization $\int_0^1 \rho^q(x)dx = 1$ and the unnormalized expectation $\overline{x} = \int_0^1 \rho(x)xdx$ gives

$$\rho(x) = \frac{1}{Z} \frac{1}{(1 - \gamma x)^{1/(1-q)}}$$

where $q = 1/2$ for the continued fraction map, $\gamma = -1$ is the Lagrange multiplier associated with $\overline{x}$ and the constant $Z = \sqrt{\ln 2}$ is determined by the incomplete normalization. The Zipf-Mandelbrot’s law $\rho(x) = \frac{A}{(1 - \gamma x)^\alpha}[14]$ can be derived with $1/q = 1 + 1/\alpha$ ($A$ is the normalization constant). The distribution of the Ulam maps $\rho(x) = \frac{1}{\pi(1 - x^2)^{1/2}} (-1 < x < 1)$ can be obtained with $q = 1/3$ if $\overline{x^2}$ is used as a constraint.
It should be emphasized that, although the same kind of distribution functions as mentioned above can also be derived from the maximization of Tsallis entropy $S$ or of Rényi one $S^r$ under the same constraints, the extremization of $R$ are physically different from the entropy maximisation derived from the second law of thermodynamics. $R$ is intrinsically connected with non-equilibrium evolving system. So extremizing $R$ implies looking for the probability distribution that, among many other possible ones, maximizes the information change of the system in evolution. For this kind of systems, it is impossible to talk about maximizing entropy in the sense of the second law because entropy is still in constant variation.

On the other hand, if the entropy of the system of interest is defined such that the $R$-extremization is equivalent to the maximization of the entropy, then the thermostatistics based on the maximum entropy principle may be discussed in connection with the $R$-extremization. This is what we are doing in the following for the statistical thermodynamics derived from maximizing Tsallis entropy.

4 A thermodynamics of non-equilibrium systems with different $q$’s

An important question about the nonextensive statistical thermodynamics based on Tsallis entropy [4] concerns its validity for the systems having different $q$’s for which the thermodynamics must be formulated in a more general way than the thermodynamics for the same $q$-systems [15, 16]. This formulation is crucial for nonextensive statistics because a composite system containing different $q$-systems is a general case in nature. We are showing here that a possible formulation can be made for systems having different $q \neq 1$ on the basis of the nonadditivity given by Eq. (8). We suppose that the non-equilibrium system is at some stationary state which maximizes Tsallis entropy for the total system $A + B$. As mentioned above, this maximization is equivalent to the extremization of $R$ for $A + B$, i.e., $dR(A + B) = 0$. This leads to $\frac{dR(A)}{1+R(A)} + \frac{dR(B)}{1+R(B)} = 0$ which means

$$\frac{(q_A - 1)dS(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)dS(B)}{\sum_i p_i(B)} = 0$$

(10)

where $S$ is the aforementioned Tsallis entropy. Now using the product joint probability and the relationship $\sum_i p_i = Z^{q-1} + (q-1)\beta U$ connected with
the energy distribution function \( p_i = \frac{1}{Z}[1 - (q - 1)\beta E_i]^{1/(q-1)} \) given by the maximization of \( S \), where \( E_i \) is the energy of the system at state \( i \) and \( U = \sum_i p_i E_i \) is the internal energy, we get \( \frac{(q_A - 1)dU(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)dU(B)}{\sum_i p_i(B)} = 0 \) which suggests following energy nonadditivity

\[
\frac{(q_A - 1)dU(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)dU(B)}{\sum_i p_i(B)} = 0. \tag{11}
\]

This relationship should be considered as the analog of the additive energy rule \( dU(A) + dU(B) = 0 \) of Boltzmann-Gibbs statistical thermodynamics. Eq.(10) and Eq.(11) lead to

\[
\beta(A) = \beta(B) \tag{12}
\]

where \( \beta = \frac{\partial S}{\partial U} \) is the inverse temperature.

For the definition of pressure, as discussed in [17], Eq.(10) finally leads to, at stationarity,

\[
\beta \left[ P(A) \frac{(q_A - 1)dV(A)}{\sum_i p_i(A)} + P(B) \frac{(q_B - 1)dV(B)}{\sum_i p_i(B)} \right] = 0 \tag{13}
\]

where \( P = \left( \frac{\partial U}{\partial V} \right)_S \) is the pressure and \( V \) is the volume. The intensive pressure, i.e. \( P(A) = P(B) \), implies

\[
\frac{(q_A - 1)dV(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)dV(B)}{\sum_i p_i(B)} = 0 \tag{14}
\]

This volume can be interpreted as an effective volume allowing one to write the first law as \( dU = TdS - PdV \) in the case where the interface/surface effects cannot be neglected compared to the volume effect. An example of this kind of systems with nonadditive effective volume is discussed in [17].

**5 Thermodynamics of nonadditive blackbody**

Now let us suppose a nonadditive blackbody obeying the above statistical thermodynamics with the volume nonadditivity indicated by Eq.(14). As mentioned above, this case is possible when emission body is small (for example, the thermal emission of nanoparticles or of small optical cavity) such that the surface/interface effect may be important. We have seen in the
above paragraph that $dU$, $dS$, and $dV$ should be proportional to each other. This can be satisfied by $U = f(T)V$ and $S = g(T)V$. In addition, we admit the photon pressure given by $P = \frac{U}{3V} = \frac{1}{3} f(T)$. From the first law $dU = TdS - PdV$, we obtain

$$V \frac{\partial f}{\partial T} dT + f dV = T(V \frac{\partial g}{\partial T} dT + g dV) - \frac{1}{3} f dV, \quad (15)$$

which means $\frac{\partial f}{\partial T} = T \frac{\partial g}{\partial T}$ and $\frac{4}{3} f = T g$ leading to $\frac{1}{3} \frac{\partial f}{\partial T} = \frac{4}{3} f$. We finally get

$$f(T) = c T^4 \quad (16)$$

where $c$ is a constant. This is the Stefan-Boltzmann law. On the other hand, from the relationship $(\frac{\partial S}{\partial V})_T = (\frac{\partial P}{\partial T})_V$, we obtain $g = \frac{1}{3} \frac{\partial f}{\partial T}$ and $g(T) = b T^3$ where $b$ is a constant. Notice that the above calculation is similar to that in the conventional thermodynamics. This is because the thermodynamic functions here, though nonadditive, are nevertheless “extensive” with respect to the effective volume. This result contradicts what has been claimed for blackbody radiation on the basis of non intensive pressure[15], and is valid as far as the pressure is intensive.

Following analysis shows what happens if one supposes additive volume $V$, i.e. $dV(A) + dV(B) = 0$. From Eq. (13), one gets

$$P(A) \frac{(q_A - 1)}{\sum_i p_i(A)} = P(B) \frac{(q_B - 1)}{\sum_i p_i(B)}. \quad (17)$$

So the conventional pressure $P = (\frac{\partial U}{\partial V})_S$ becomes non intensive. If one still supposes $P = \frac{U}{3V}$, this will lead to a deviation from the conventional Stefan-Boltzmann law as shown in [15]. There are two questions here to be noticed. 1) Is non-intensive pressure possible? 2) The relationship $P = \frac{U}{3V}$ was established within the conventional thermodynamics for additive photon gas. Is it still true for nonextensive or nonadditive photon gas having non intensive pressure? Obviously, the final answers to these questions require experimental proofs (which are still missing as far as we know).

6 Conclusion

In summarizing, we have studied the information growth during long time evolution of chaotic systems having fractal phase space through the scale refinement in the phase space. This information growth turns out to take the
trace form $\sum_i p_i - \sum_i p_i^q$ which can be connected with several entropies generalizing Shannon one. It is shown that the power law probability distributions of several chaotic systems can be obtained by extremizing this information growth. However, this work leaves open the questions as to whether in general one can maximize entropy to get the probability distributions for non-equilibrium systems in (stationary or not) evolution and whether one should extremize the entropy or information change. In any case, if the information-entropy of the system is of Tsallis type, these two methods turn out to be mathematically equivalent.

On the basis of the information growth, we discussed the thermodynamics of non-equilibrium systems in stationary state which maximizes Tsallis entropy. Intensive variables like temperature and pressure can be defined for an ensemble of systems having different $q$’s. It is argued that Stefan-Boltzmann law for blackbody radiation can be preserved within this thermodynamics. We would like to emphasize that this work is carried out by using incomplete probability distribution and the corresponding unnormalized expectation. We have noticed that the normalized $q$-expectation $U = \sum_i p_i^q E_i$ for incomplete distribution could not be used due to the product joint probability $p_{ij}(A + B) = p_i(A)p_i(B)$ connected with the nonadditivity given by Eq.(8) and Eq.(10). In this case, the unnormalized expectation allows one to split the thermodynamics of the composite system into those of the subsystems, a necessary condition for the establishment of zeroth law. This constraint on the nonextensive thermodynamics favours the use of the unnormalized expectations $U = \sum_i p_i E_i$ at least for the systems having different $q$’s. However, if the complete probability distribution $\sum_i p_i = 1$ is employed in the framework of the thermodynamics based on Tsallis entropy $S$, we have $p_i^{q_A+B} (A + B) = p_i^{q_A}(A)p_i^{q_B}(B)$ as generalized product joint probability derived from Eq.(8). In this case, the unnormalized expectation $U = \sum_i p_i^q E_i$ or its normalized variation $U = \sum_i p_i^q E_i / \sum_i p_i^q$ should be used in order to split the thermodynamics and to establish the zeroth law as discussed in [18].

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