BERGE’S CONJECTURE FOR CUBIC GRAPHS WITH SMALL COLOURING DEFECT

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Abstract. A long-standing conjecture of Berge suggests that every bridgeless cubic graph can be expressed as a union of at most five perfect matchings. This conjecture trivially holds for 3-edge-colourable cubic graphs, but remains widely open for graphs that are not 3-edge-colourable. The aim of this paper is to verify the validity of Berge’s conjecture for cubic graphs that are in a certain sense close to 3-edge-colourable graphs. We measure the closeness by looking at the colouring defect, which is defined as the minimum number of edges left uncovered by any collection of three perfect matchings. While 3-edge-colourable graphs have defect 0, every bridgeless cubic graph with no 3-edge-colouring has defect at least 3. In 2015, Steffen proved that the Berge conjecture holds for cyclically 4-edge-connected cubic graphs with colouring defect 3 or 4. Our aim is to improve Steffen’s result in two ways. First, we show that all bridgeless cubic graphs with defect 3 satisfy Berge’s conjecture irrespectively of their cyclic connectivity. Second, we prove that if the graph in question is cyclically 4-edge-connected, then four perfect matchings suffice, unless the graph is the Petersen graph. The result is best possible as there exists an infinite family of cubic graphs with cyclic connectivity 3 which have defect 3 but cannot be covered with four perfect matchings.

Dedicated to professor Ján Plesník, our teacher and colleague.

1. Introduction

In 1970’s, Claude Berge made a conjecture that every bridgeless cubic graph $G$ can have its edges covered by at most five perfect matchings (see [23]). The corresponding set of matchings is called a Berge cover of $G$.

Berge’s conjecture relies on the fact that every preassigned edge of $G$ belongs to a perfect matching, and hence there is a set of perfect matchings that cover all the edges of $G$. The smallest number of perfect matchings needed for this purpose is the perfect matching index of $G$, denoted by $\pi(G)$. Clearly, $\pi(G) = 3$ if and only if $G$ is 3-edge-colourable, so if $G$ has chromatic index 4, then the value of $\pi(G)$ is believed to be either 4 or 5. In this context it may be worth mentioning that $\pi(Pg) = 5$, where $Pg$ denotes the Petersen graph, and that cubic graphs with $\pi = 5$ are very rare [3] [21].

Berge’s conjecture is closely related to a stronger and arguably more famous conjecture of Fulkerson [9], attributed also to Berge and therefore often referred to as the Berge-Fulkerson conjecture [26]. The latter conjecture suggests that every bridgeless cubic graph contains a collection of six perfect matchings such that each edge belongs to precisely two of them.

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Fulkerson’s conjecture clearly implies Berge’s. Somewhat surprisingly, the converse holds as well, which follows from an ingenious construction due to Mazzuoccolo [23].

Very little is known about the validity of either of these conjectures. Besides the 3-edge-colourable cubic graphs, the conjectures are known to hold only for a few classes of graphs, mostly possessing a very specific structure (see [5, 8, 10, 11, 17, 22]). On the other hand, it has been proved that if Fulkerson’s conjecture is false, then the smallest counterexample would be cyclically 5-edge-connected (Máčajová and Mazzuoccolo [20]).

In this paper we investigate Berge’s conjecture under more general assumptions, not relying on any specific structure of graphs: our aim is to verify the conjecture for all cubic graphs that are, in a certain sense, close to 3-edge-colourable graphs. In our case, the proximity to 3-edge-colourable graphs will be measured by the value of their colouring defect. The colouring defect of a cubic graph, or just defect for short, is the smallest number of edges that are left uncovered by any set of three perfect matchings. This concept was introduced and thoroughly studied by Steffen [28] in 2015 who used the notation \( \mu_3(G) \) but did not coin any term for it. Since a cubic graph has defect 0 if and only if it is 3-edge-colourable, colouring defect can serve as one of measures of uncolourability of cubic graphs, along with resistance, oddness, and other similar invariants recently studied by a number of authors, see for example [1, 2, 7].

In [28, Corollary 2.5] Steffen proved that the defect of every bridgeless cubic graph that cannot be 3-edge-coloured is at least 3, and can be arbitrarily large. He also proved that every cyclically 4-edge-connected cubic graph with defect 3 or 4 satisfies Berge’s conjecture [28, Theorem 2.14].

We strengthen the latter result in two ways.

First, we show that every bridgeless cubic graph with defect 3 satisfies the Berge conjecture irrespectively of its cyclic connectivity.

**Theorem 1.1.** Every bridgeless cubic graph with colouring defect 3 admits a Berge cover.

Second, we prove that if the graph in question is cyclically 4-edge-connected, then four perfect matchings are enough to cover all its edges, except for the Petersen graph. The existence of a cover with four perfect matchings is known to have a number of important consequences. For example, such a graph satisfies the Fan-Raspaud conjecture [6], admits a 5-cycle double cover and has a cycle cover of length \( 4/3 \cdot m \), where \( m \) is the number of edges [28, Theorem 3.1].

**Theorem 1.2.** Let \( G \) be a cyclically 4-edge-connected cubic graph with defect 3. Then \( \pi(G) = 4 \), unless \( G \) is the Petersen graph.

Theorem 1.2 is best possible in the sense that there exist infinitely many 3-connected cubic graphs with colouring defect 3 that cannot be covered with four perfect matchings.

Our proofs use a wide range of methods. Both main results rely on Theorem 4.1 which describes the structure of a subgraph resulting from the removal of a 6-edge-cut from a bridgeless cubic graph. Moreover, the proof
BERGE’S CONJECTURE FOR GRAPHS WITH SMALL DEFECT

of Theorem 1.2 establishes an interesting relationship between the colouring defect and the bipartite index of a cubic graph. We recall that the bipartite index of a graph is the smallest number of edges whose removal yields a bipartite graph. This concept was introduced by Thomassen in [30] and was applied in [30, 31] in a different context.

In closing, we would like to mention that our Theorem 1.1 can be alternatively derived from a recent result published in [29], which states that every bridgeless cubic graph containing two perfect matchings that share at most one edge admits a Berge cover. The two results are connected via an inequality between two uncolourability measures proved in [13, Theorem 2.2] (see also [15, Proposition 4.2]). Although the main result of [29] is important and interesting, its proof is very technical and difficult to comprehend, which is why we have not been able to verify all its details. We believe that more effort is needed to clarify the arguments presented in [29].

Our paper is organised as follows. The next section summarises definitions and results needed for understanding the rest of the paper. Section 3 provides a brief account of the theory surrounding the notion of colouring defect. Specific tools needed for the proofs of Theorems 1.1 and 1.2 are established in Section 4. The two theorems are proved in Sections 5 and 6, respectively. The final section illustrates that the condition on cyclic connectivity in Theorem 1.2 cannot be removed.

2. Preliminaries

2.1. Graphs. Graphs studied in this paper are finite and mostly cubic (that is, 3-valent). Multiple edges and loops are permitted. The order of a graph \( G \), denoted by \( |G| \), is the number of its vertices. A circuit in \( G \) is a connected 2-regular subgraph of \( G \). A \( k \)-cycle is a circuit of length \( k \). The girth of \( G \) is the length of a shortest circuit in \( G \).

An edge cut is a set \( R \) of edges of a graph whose deletion yields a disconnected graph. A common type of an edge cut arises by taking a subset of vertices or an induced subgraph \( H \) of \( G \) and letting \( R \) be the set \( \delta_G(H) \) of all edges with exactly one end in \( H \). We omit the subscript \( G \) whenever \( G \) is clear from the context.

A connected graph \( G \) is said to be cyclically \( k \)-edge-connected for some integer \( k \geq 1 \) if the removal of fewer than \( k \) edges cannot leave a subgraph with at least two components containing circuits. The cyclic connectivity of \( G \) is the largest integer \( k \) not exceeding the cycle rank (Betti number) of \( G \) such that \( G \) is cyclically \( k \)-edge-connected. An edge cut \( R \) in \( G \) that separates two circuits from each other is cycle-separating. It is not difficult to see that the set \( \delta_G(C) \) leaving a shortest circuit \( C \) of a cubic graph \( G \) is cycle-separating unless \( G \) is the complete bipartite graph \( K_{3,3} \), the complete graph \( K_4 \), or the 3-dipole, the graph which consists of two vertices and three parallel edges joining them. Observe that an edge cut formed by a set of independent edges is always cycle-separating. Conversely, a cycle-separating edge cut of minimum size is independent.

2.2. Edge colourings and flows. An edge colouring of a graph \( G \) is a mapping from the edge set of \( G \) to a set of colours. A colouring is proper if any two edge-ends incident with the same vertex receive distinct colours.
A \( k \)-edge-colouring is a proper edge colouring where the set of colours has \( k \) elements. Unless specified otherwise, our colouring will be assumed to be proper and graphs to be subcubic, that is, with vertices of valency 1, 2, or 3.

There is a standard method of transforming a 3-edge-colouring to another 3-edge-colouring: it uses so-called Kempe switches: Let \( G \) be a subcubic graph endowed with a proper 3-edge-colouring \( \sigma \). Take two distinct colours \( i \) and \( j \) from \( \{1, 2, 3\} \). An \((i, j)\)-Kempe chain in \( G \) (with respect to \( \sigma \)) is a non-extendable walk \( L \) that alternates edges coloured \( i \) with those coloured \( j \). It is easy to see that \( L \) is either a bicoloured circuit or path starting and ending at the vertex of valency smaller than 3. The Kempe switch along a Kempe chain produces a new 3-edge-colouring of \( G \) by interchanging the colours on \( L \).

It is often useful to regard 3-edge-colourings of cubic graphs as nowhere-zero flows. To be more precise, one can identify each colour from the set \( \{1, 2, 3\} \) with its binary representation; thus \( 1 = (0, 1) \), \( 2 = (1, 0) \), and \( 3 = (1, 1) \). Having done this, the condition that the three colours meeting at every vertex \( v \) are all distinct becomes equivalent to requiring the sum of the colours at \( v \) to be \( 0 = (0, 0) \). The latter is nothing but the Kirchhoff law for nowhere-zero \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-flows. Recall that an \( A \)-flow on a graph \( G \) is a pair \((D, \phi)\) where \( \phi \) is an assignment of elements of an abelian group \( A \) to the edges of \( G \), and \( D \) is an assignment of one of two directions to each edge in such a way that, for every vertex \( v \) in \( G \), the sum of values flowing into \( v \) equals the sum of values flowing out of \( v \) (Kirchhoff’s law). A nowhere-zero \( A \)-flow is one which does not assign \( 0 \in A \) to any edge of \( G \). If each element \( x \in A \) satisfies \( x = -x \), then \( D \) can be omitted from the definition. It is well known that the latter is satisfied if and only if \( A \cong \mathbb{Z}_2^n \) for some \( n \geq 1 \).

The following well-known statement is a direct consequence of Kirchhoff’s law.

**Lemma 2.1.** (Parity Lemma) Let \( G \) be a cubic graph endowed with a 3-edge-colouring \( \xi \). The following holds for every edge cut \( \delta_G(H) \), where \( H \) is a subgraph of \( G \):

\[
\sum_{e \in \delta_G(H)} \xi(e) = 0.
\]

Equivalently, the number of edges in \( \delta_G(H) \) carrying any fixed colour has the same parity as the size of the cut.

A cubic graph \( G \) is said to be colourable if it admits a 3-edge-colouring. A 2-connected cubic graph that admits no 3-edge-colouring is called a snark. Our definition agrees with that of Cameron et al. \cite{cameron}, Nedela and Skoviera \cite{nedela}, Steffen \cite{steffen}, and others, and leaves the concept of a snark as wide as possible. A more restrictive definition requires a snark to be to be cyclically 4-edge-connected, with girth at least 5, see for example \cite{7}. We call such snarks nontrivial.

**2.3. Perfect matchings.** The classical theorems of Tutte \cite{32} and Plesník \cite{25}, stated below, will be repeatedly used throughout the paper, albeit in a slightly modified form permitting parallel edges and loops. These extensions can be proved easily by using the standard versions of the corresponding theorems.
Let $\text{odd}(G)$ denote the number of odd components of $G$, that is, the components with an odd number of vertices.

**Theorem 2.2.** (Tutte, 1947) A graph $G$, possibly containing parallel edges and loops, has a perfect matching if and only if

$$\text{odd}(G - S) \leq |S| \quad \text{for all } S \subseteq V(G).$$

**Theorem 2.3.** (Plesnık, 1972) Let $G$ be an $(r-1)$-edge-connected $r$-regular graph with $r \geq 1$, and let $A$ be an arbitrary set of $r-1$ edges in $G$. If $G$ has even order, then $G - A$ has a perfect matching.

### 3. Colouring defect of a cubic graph

In this section we discuss a number of structures related to the concept of colouring defect, which will be used in the proofs of Theorems 1.1 and 1.2. More details on this matter can be found in [16] and [28].

For a bridgeless cubic graph $G$ we define a $k$-array of perfect matchings, or briefly a $k$-array of $G$, as an arbitrary collection $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ of $k$ perfect matchings of $G$, not necessarily pairwise distinct. The concept of a $k$-array unifies a number of notions, such as Berge covers, Fulkerson covers, Fan-Raspaud triples, and others. Our main concern here are 3-arrays, which we regard as approximations of 3-edge-colourings.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 3-array of perfect matchings of a cubic graph $G$. An edge of $G$ that belongs to at least one of the perfect matchings of $\mathcal{M}$ will be considered to be covered. An edge will be called uncovered, simply covered, doubly covered, or triply covered if it belongs, respectively, to zero, one, two, or three distinct members of $\mathcal{M}$.

Given a cubic graph $G$, it is natural task to maximise the number of covered edges, or equivalently, to minimise the number of uncovered ones. A 3-array that leaves the minimum number of uncovered edges will be called optimal. The number of edges left uncovered by an optimal 3-array is the colouring defect of $G$, or the defect for short, denoted by $\text{df}(G)$.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 3-array of perfect matchings of a cubic graph $G$. One way to describe $\mathcal{M}$ is based on regarding the indices 1, 2, and 3 as colours. Since the same edge may belong to more than one member of $\mathcal{M}$, an edge of $G$ may receive more than one colour. To each edge $e$ of $G$ we can therefore assign the list $\phi(e)$ of all colours in lexicographic order it receives from $\mathcal{M}$. We let $\omega(e)$ denote the number of colours in the list $\phi(e)$ and call it the weight of $e$ (with respect to $\mathcal{M}$). In this way $\mathcal{M}$ gives rise to a colouring

$$\phi: E(G) \rightarrow \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$$

where $\emptyset$ denotes the empty list. Obviously, a mapping $\sigma$ assigning subsets of $\{1, 2, 3\}$ to the edges of $G$ determines a 3-array if and only if, for each vertex $v$ of $G$, each element of $\{1, 2, 3\}$ occurs precisely once in the subsets of $\{1, 2, 3\}$ assigned by $\sigma$ to the edges incident with $v$. In general, $\phi$ need not be a proper edge-colouring. However, since $M_1$, $M_2$, and $M_3$ are perfect matchings, the only possibility when two edges equally coloured under $\phi$ meet at a vertex is that both of them receive colour $\emptyset$. In this case the third edge incident with the vertex is coloured 123.
A different but equivalent way of representing a 3-array uses a mapping
\[ \chi : E(G) \to \mathbb{Z}_3^2, \quad e \mapsto \chi(e) = (x_1, x_2, x_3) \]
defined by setting \( x_i = 0 \) if and only if \( e \in M_i \), where \( i \in \{1, 2, 3\} \). Since the complement of each \( M_i \) in \( G \) is a 2-factor, it is easy to see that \( \chi \) is a \( \mathbb{Z}_3^2 \)-flow. We call \( \chi \) the characteristic flow for \( \mathcal{M} \). Again, \( \chi \) is a nowhere-zero \( \mathbb{Z}_3^2 \)-flow if and only if \( G \) contains no triply covered edge. In the context of 3-arrays the characteristic flow was introduced in [14, p. 166]. Observe that the characteristic flow \( \chi \) of a 3-array and the colouring \( \phi \) determine each other. In particular, the condition on \( \phi \) requiring all three indices from \( \{1, 2, 3\} \) to occur precisely once in a colour around any vertex is equivalent to Kirchhoff’s law.

The following result characterises 3-arrays with no triply covered edge.

**Proposition 3.1.** Let \( \mathcal{M} \) be a 3-array of perfect matchings of a cubic graph \( G \). The following three statements are equivalent.

(i) \( G \) has no triply covered edge with respect to \( \mathcal{M} \).
(ii) The associated colouring \( \phi : E(G) \to \{\emptyset, 1, 2, 3, 12, 13, 23, 123\} \) is proper.
(iii) The characteristic flow \( \chi \) for \( \mathcal{M} \), with values in \( \mathbb{Z}_3^2 \), is nowhere-zero.

The next important structure associated with a 3-array is its core. The core of a 3-array \( \mathcal{M} = \{M_1, M_2, M_3\} \) of \( G \) is the subgraph of \( G \) induced by all the edges of \( G \) that are not simply covered; we denote it by \( \text{core}(\mathcal{M}) \). The core is called optimal whenever \( \mathcal{M} \) is optimal.

It is worth mentioning that if \( G \) is 3-edge-colourable and \( \mathcal{M} \) consists of three pairwise disjoint perfect matchings, then \( \text{core}(\mathcal{M}) \) is empty. If \( G \) is not 3-edge-colourable, then every core must be nonempty. Figure 1 shows the Petersen graph endowed with a 3-array whose core is its “outer” 6-cycle.

The hexagon is in fact an optimal core.

![Figure 1. An optimal 3-array of the Petersen graph](image_url)

The following proposition, due to Steffen [28, Lemma 2.2], describes the structure of optimal cores in the general case.

**Proposition 3.2.** Let \( \mathcal{M} = \{M_1, M_2, M_3\} \) be an optimal 3-array of perfect matchings of a snark \( G \). Then every component of \( \text{core}(\mathcal{M}) \) is either an even circuit of length at least 6 or a subdivision of a cubic graph. Moreover,
the union of doubly and triply covered edges forms a perfect matching of core($\mathcal{M}$).

The next theorem characterises snarks with minimal possible colouring defect. The lower bound for the defect of a snark – the value 3 – is due to Steffen [28, Corollary 2.5].

**Theorem 3.3.** Every snark $G$ has $df(G) \geq 3$. Furthermore, the following three statements are equivalent.

(i) $df(G) = 3$.
(ii) The core of any optimal 3-array of $G$ is a 6-cycle.
(iii) $G$ contains an induced 6-cycle $C$ such that the subgraph $G - E(C)$ admits a proper 3-edge-colouring under which the six edges of $\delta(C)$ receive colours $1, 1, 2, 2, 3, 3$ with respect to the cyclic order induced by an orientation of $C$.

![Figure 2. The hexagonal core and its vicinity.](image)

If $G$ is an arbitrary snark with $df(G) = 3$, then, by Theorem 3.3 (iii), $G$ contains an induced 6-cycle $C = (e_0e_1e_2e_3e_4e_5)$ such that $G - E(C)$ is 3-edge-colourable. We say that that $C$ is a hexagonal core of $G$.

Let $f_i$ denote the edge of $\delta(C)$ which is incident with $e_{i-1}$ and $e_i$, where $i \in \{0, 1, \ldots, 5\}$ and the indices are reduced modulo 6; see Figure 2. Since $G - E(C)$ is 3-edge-colourable but $G$ is not, it is not difficult to see that for every 3-edge-colouring of $G - E(C)$ the cyclic order of colours around $C$ is $(1, 1, 2, 2, 3, 3)$ up to permutation of colours. Moreover, we can assume that the values of the associated colouring $\phi$ of $G$ induced by $\mathcal{M}$ in the vicinity of $C$ are those as shown in Figure 2 or can be obtained from them by the rotation one step clockwise. Note that the two possibilities only depend on the position of the uncovered edges. In any case, there are no triply covered edges, and so $\phi$ is a proper edge colouring due to Proposition 3.2.

We finish this section by stating the following property of a hexagonal core, which will be needed in Sections 6. We leave the proof to the reader or refer to [16].
Lemma 3.4. Let $G$ be a snark with $df(G) = 3$. If a hexagonal core of $G$ intersects a triangle or a quadrilateral, then the intersection consists of a single uncovered edge.

4. Tools

In this section we establish tools for proving Theorems 1.1 and 1.2. Its main results are stated as Theorems 4.1 and 4.4, both having somewhat bipartite flavour. The first theorem suggests that if a subgraph $H$ of a bridgeless cubic graph is separated from the rest by a 6-edge-cut and has no perfect matching, then the structure of $H$ is – essentially – that of a bipartite cubic graph. Moreover, only one of the two parts is incident with the cut.

**Theorem 4.1.** Let $G$ be a bridgeless cubic graph and let $H \subseteq G$ be a subgraph with $|\delta_G(H)| = 6$. Then $H$ has a perfect matching, or else $H$ contains an independent set $S$ of trivalent vertices such that

(i) every component of $H - S$ is odd,
(ii) $\text{odd}(H - S) = |S| + 2$, and
(iii) $|\delta_G(L)| = 3$ for each component $L$ of $H - S$.

**Proof.** Set $K = G - V(H)$. Assume that $H$ has no perfect matching. Tutte’s Theorem tells us that there exists a set $S \subseteq V(H)$ such that $\text{odd}(H - S) > |S|$. As $H$ has an even number of vertices, the numbers $|S|$ and $\text{odd}(H - S)$ have the same parity, so

$$\text{odd}(H - S) - |S| \geq 2. \quad (1)$$

Set $a = |\delta_G(S) \cap \delta_G(K)|$. Clearly, $a \in \{0, 1, \ldots, 6\}$. Since $|\delta_H(S)| + a = |\delta_G(S)| \leq 3|S|$, we get

$$|\delta_H(S)| \leq 3|S| - a. \quad (2)$$

To bound $|\delta_H(S)|$ from below, we first realise that each odd component of $H - S$ is incident with at least three edges of $\delta_G(H - S)$ because $G$ is bridgeless. Moreover, there are $6 - a$ edges joining $H - S$ to $K$ (see Figure 3). Therefore

$$|\delta_H(S)| = |\delta_H(H - S)| = |\delta_G(H - S)| - (6 - a) \geq 3\text{-odd}(H - S) - (6 - a). \quad (3)$$

If we combine (2) with (3), we get

$$3 \cdot \text{odd}(H - S) - 6 + a \leq |\delta_H(S)| \leq 3|S| - a. \quad (4)$$

Since $3 \cdot \text{odd}(H - S) - 3|S| \geq 6$ according to (1), we can rewrite (4) as

$$-2a \geq 3 \cdot \text{odd}(H - S) - 3|S| - 6 \geq 0,$$

which implies that $a = 0$.

Now we insert $a = 0$ into (1) and get $3 \cdot \text{odd}(H - S) - 3|S| \leq 6$. Together with (1), multiplied by 3, this yields that

$$3 \cdot \text{odd}(H - S) - 6 \leq |\delta_H(S)| \leq 3|S| \leq 3 \cdot \text{odd}(H - S) - 6.$$
Hence,

\[ |\delta_H(S)| = 3|S| = 3 \cdot \text{odd}(H - S) - 6, \tag{5} \]

which implies that \( S \) is an independent set of \( H \), with all its vertices 3-valent, and that each component \( K \) of \( H - S \) is odd with \( |\delta_G(K)| = 3 \). Thus we have proved Statements (i) and (iii). Moreover, Equation (5) implies statement (ii). The proof is complete. \( \square \)

**Example 4.2.** We illustrate Theorem 4.1 in two simple instances. First, if \( G \) is the 3-dimensional cube \( Q_3 \) and \( H = G - V(C) \), where \( C \) is an induced 6-cycle of \( G \), then Theorem 4.1 holds with \( S = \emptyset \). If \( G \) is the Petersen graph \( Pg \) and \( H = G - V(C) \), where \( C \) is again a 6-cycle, then \( H \) is isomorphic to the complete bipartite graph \( K_{1,3} \) and \( S \) is constituted by its central vertex.

We proceed to our second tool, which describes cubic graphs just one step away from being bipartite. Recall that if a cubic graph is bipartite, then it is obviously bridgeless. Moreover, if it is connected, then its bipartition is uniquely determined. On the other hand, if a cubic graph is not bipartite, then, clearly, at least two edges have to be removed in order to produce a bipartite graph. Motivated by these two facts we define a cubic graph \( G \) to be *almost bipartite* if it is bridgeless, not bipartite, and contains two edges \( e \) and \( f \) such that \( G - \{e, f\} \) is a bipartite graph. The edges \( e \) and \( f \) are said to be *surplus edges* of \( G \). If a cubic graph is almost bipartite, then there exists a component \( K \) of \( G \) such that the surplus edges connect vertices within different partite sets of \( K \).

Our aim is to show that every almost bipartite graph is 3-edge-colourable. We start with the following.

**Proposition 4.3.** Every almost bipartite cubic graph has a perfect matching that contains both surplus edges.

**Proof.** Let \( G \) be an almost bipartite cubic graph with surplus edges \( e \) and \( f \). Since \( e \) and \( f \) belong to the same component of \( G \), we may assume that \( G \) is connected. Let \( \{A, B\} \) be the bipartition of \( G' = G - \{e, f\} \). As said
before, the endvertices of one of the surplus edges, say $e$, belong to $A$ and those of $f$ then belong to $B$. In particular, this means that $|A| = |B| = n$ for some positive integer $n$. By Theorem 2.3, there exists a perfect matching $M$ containing the edge $e$. There are $n - 2$ edges of $M$ that match $n - 2$ vertices of $A$ to $n - 2$ vertices of $B$. It follows that $f \in M$. □

We would like to mention that our Proposition 4.3 has been inspired by Lemma 3 (iii) of [19]. In principle, the mentioned lemma could be used to prove Proposition 4.3, however, the proof would not be anything close to being straightforward.

We are now ready for the following theorem.

**Theorem 4.4.** Every almost bipartite cubic graph is $3$-edge-colourable.

**Proof.** Let $G$ be an almost bipartite graph with surplus edges $e$ and $f$. By Proposition 4.3, $G$ has a perfect matching $M$ which includes both $e$ and $f$. Now, $G - M$ is bipartite 2-regular spanning subgraph of $G$, so each component of $G - M$ is an even circuit. It means that $G - M$ can be decomposed into two disjoint perfect matchings $N_1$ and $N_2$. Put together, $M$, $N_1$, and $N_2$ are colour classes of a proper 3-edge-colouring of $G$. □

The previous theorem links the problem of 3-edge-colourability of cubic graphs to an important concept of a bipartite index, which has been introduced to measure how far a graph is from being bipartite. Following [31, Definition 2.4], we define the bipartite index $bi(G)$ of a graph $G$ to be the smallest number of edges that must be deleted in order to make the graph bipartite. In other words, the bipartite index of a graph is the number of edges outside a maximum edge cut.

Our Theorem 4.4 states that all bridgeless cubic graphs with bipartite index at most two are 3-edge colourable. This is, in fact, best possible as there exist infinitely many nontrivial snarks whose bipartite index equals 3: this is true, for example, for the Petersen graph or for all Isaacs flower snarks $J_{2n+1}$; see [12] for the definition and Figure 4 for $J_7$.

![Figure 4. The flower snark $J_7$](image)

Theorem 4.4 thus suggests that, along with oddness, resistance, flow resistance, and other similar invariants extensively studied in [7], bipartite index can serve as another measure of uncolourability of cubic graphs. It
is easy to see, for example, that for a cubic graph \( \text{bi}(G) \geq \omega(G) \), where \( \omega(G) \) denotes the oddness of a cubic graph, that is, the smallest number of odd circuits in a 2-factor of \( G \). Since there exist snarks of arbitrarily large oddness [18], there are snarks of arbitrarily large bipartite index.

5. Proof of Theorem 1.1

The purpose of this section is to establish the existence of a Berge cover for every bridgeless cubic graph of defect 3 (Theorem 1.1) and for every cyclically 4-edge-connected cubic graph of defect 4. We begin with the following auxiliary result.

Lemma 5.1. Let \( G \) be a cubic graph of defect 3 and let \( \mathcal{M} = \{M_1, M_2, M_3\} \) be an optimal 3-array of perfect matchings for \( G \). Then \( G \) has a fourth perfect matching \( M_4 \) which covers at least two of the three edges left uncovered by \( \mathcal{M} \).

Proof. Consider the 6-cycle \( C = (e_0 e_1 \ldots e_5) \) constituting the core of \( \mathcal{M} \). We adopt the notation introduced in Figure 2; in particular, \( e_1, e_3, \) and \( e_5 \) are the three uncovered edges of \( C \). We also assume that the edge colouring \( \phi \) associated with \( \mathcal{M} \) takes values as indicated in Figure 2. By Proposition 3.1, the colouring is proper.

An uncovered edge \( e_i \) of \( C \) will be considered bad if \( G \) has a cycle-separating 3-edge-cut \( R_i \) containing the edges \( e_{i-1} \) and \( e_{i+1} \). We claim that not all of the edges \( e_1, e_3, \) and \( e_5 \) can be bad. Suppose to the contrary that all of them are bad. For \( i \in \{1, 3, 5\} \) let \( h_i \) denote the third edge of the cut \( R_i \). Since the edges \( f_i \) and \( f_{i+1} \) of \( \delta_G(C) \) are both adjacent to \( e_i \), the set \( R_i' = \{f_i, f_{i+1}, h_i\} \) is also a 3-edge-cut. Moreover, all three edges of \( R_i' \) are simply covered. Let \( L_i \) be the component of \( G - R_i' \) that does not contains \( e_i \).

Figure 5. Constructing a 3-edge-colouring of \( G \) provided that \( G \) has three bad edges. The stage preceding Kempe switches.
Recall that the restriction of $\phi$ to $G - E(C)$ is a proper 3-edge-colouring. We show that this colouring can be modified and extended to a 3-edge-colouring $\psi$ of the entire $G$. By Kirchhoff’s law, $\phi(f_1) + \phi(f_2) + \phi(h_1) = 0$. Since $\phi(f_1) = 3$, and $\phi(f_2) = 2$, we have $\phi(h_1) = 1$. Similarly, $\phi(h_3) = 3$ and $\phi(h_5) = 2$. To define $\psi$, we first set $\psi(e_1) = 1$, $\psi(e_3) = 3$, and $\psi(e_5) = 2$, and extend this definition to the uncovered edges of $G$ in such a way that the hexagon $C$ is properly coloured. The values of $\psi$ on $C$ now cause a conflict with the values of $\phi$ on $\delta_G(C)$ at each vertex of $C$. However, this problem can easily be resolved by the use of suitable Kempe switches. Indeed, take the $(3, 2)$-Kempe chain in $G - E(C)$ with respect to $\phi$ that starts with the edge $f_1$ coloured 3; denote the Kempe chain by $B_1$. Clearly, $B_1$ leaves $L_1$ through either $h_1$ or $f_2$. However, $\phi(h_1) = 1$, so $B_1$ ends with $f_2$ (see Figure 5). After switching the colours of $\phi$ on $B_1$ we produce a new 3-edge-colouring of $G - E(C)$, in which the colouring conflicts at the vertices $v_1$ and $v_2$ have been removed. Proceeding similarly with Kempe chains stating at $f_3$ and $f_5$ we resolve the conflicts at the remaining vertices of $C$ and eventually obtain the required proper 3-edge-colouring $\psi$ of $G$. Since $G$ is a snark, this cannot occur. Therefore, at least one uncovered edge of $C$ must not be bad.

Without loss of generality we may assume that the edge $e_1$ is not bad. Take the path $P = e_2e_4e_3 \subseteq C$ and set $H = G - V(P)$. Clearly, $|\delta_G(H)| = 6$, so Theorem 4.1 applies. We claim that $H$ admits a perfect matching. Suppose not. Then, by Theorem 4.1 $H$ contains an independent set $S$ of vertices such that each component $L$ of $H - S$ has $|\delta_G(L)| = 3$ and each edge of $\delta_G(H)$ joins a vertex of $H - S$ to a vertex of $P$. Since the edge $e_1$ belongs to $H - S$, the component containing $e_1$ is nontrivial. It follows that $e_1$ is bad, and we have arrived at a contradiction. Thus $H$ admits a perfect matching, and this matching is readily extended to a perfect matching $M_4$ of $G$ that covers the uncovered edges $e_3$ and $e_5$. The proof is complete. □

Proof of Theorem 4.1 Take an optimal 3-array $\{M_1, M_2, M_3\}$ for $G$. It leaves three uncovered edges. According to Lemma 5.1, there is a perfect matching $M_4$ that covers at least two of them. The remaining uncovered edge (if any) can be covered by a perfect matching $M_5$ guaranteed by Theorem 2.3. Together these perfect matchings a Berge cover of $G$. □

In the rest of this section we prove that every cyclically 4-edge-connected cubic graph with defect 4 admits a Berge cover. Our proof significantly differs from the one provided by Steffen [28, Theorem 2.14] in that it avoids the use of Seymour’s results on p-tuple multicolourings [26] by employing Theorem 4.1 instead.

First we establish the following lemma.

Lemma 5.2. Let $G$ be a cyclically 4-edge-connected cubic graph. If $G$ has an optimal 3-array whose core is a circuit of length $d$, then $\pi(G) \leq 3 + \lfloor d/4 \rfloor$.

Proof. Let $M$ be an optimal 3-array of $G$ whose core is a circuit $C$ of length $d$. By Proposition 3.2, $C$ alternates uncovered and doubly covered edges; in particular, $d$ is even. Note that $d \geq 6$ by Proposition 5.2.

Let $P = efgh \subseteq C$ be an arbitrary path of length 3 whose middle edge is doubly covered. We show that $G$ has a perfect matching containing both uncovered edges of $P$. Set $H = G - V(P)$. If $H$ has a perfect matching,
say $M$, then $M \cup \{e, g\}$ is a perfect matching of $G$ containing both $e$ and $g$, and we are done. Thus we may assume that $H$ has no perfect matching. By Theorem 4.1 there exists an independent set $S \subseteq V(H)$ such that $H - S$ satisfies Items (i)–(iii) of the theorem. In particular, each component $L$ of $H - S$ is odd and has $|\delta_G(L)| = 3$. Since $G$ has no non-trivial 3-cuts, each component is just a vertex, implying that $H$ is a bipartite graph. It follows that $C \cap H$ is a path of even length, and hence $|E(C)| = |E(C \cap H)| + 5$ is an odd number, which is impossible. We have thus proved that there exists a perfect matching containing any two uncovered edges of $C$ joined by a doubly covered edge.

Now we finish the proof. It is easy to see that $C$ contains $[d/4]$ pairwise edge-disjoint paths of length 3 that begin and end with an uncovered edge. At most one uncovered edge remains outside these paths, which means that the uncovered edges of $C$ can be covered by at most $[d/4]$ paths of length 3 that begin and end with an uncovered edge. In other words, there exists a set $S$ of $[d/4]$ perfect matchings that collectively contain all uncovered edges. Hence, $\pi(G) \leq |M \cup S| \leq 3 + [d/4]$, as claimed.

Theorem 5.3. Every cyclically 4-edge-connected cubic graph of defect 4 admits a Berge cover.

Proof. Let $G$ be a cyclically 4-edge-connected cubic graph of defect 4. Proposition 3.2 implies that any core with four uncovered edges must be either an 8-cycle or a subgraph consisting of two disjoint triangles joined by a triply covered edge. Since $G$ is cyclically 4-edge-connected, the latter possibility does not occur. Hence, the core of any optimal 3-array for $G$ is an 8-cycle, and the result follows from Lemma 5.2.

6. Proof of Theorem 1.2

We prove that every cyclically 4-edge-connected cubic graph $G$ of defect 3 has $\pi(G) = 4$ with the only exception of the Petersen graph.

Proof of Theorem 1.2. Let $G$ be a cyclically 4-edge-connected cubic graph with $df(G) = 3$ and $\pi(G) > 4$. Our aim is to show that $G$ is isomorphic to the Petersen graph.

Let $C = (e_0 e_1 \ldots e_5)$ be the hexagonal core of an optimal 3-array $M = \{M_1, M_2, M_3\}$ of $G$, and let $H = G - V(C)$. We may assume that the colouring $\phi$ associated with $M$ and the names of the vertices and edges in the vicinity of $C$ are as in Figure 2. In particular, the endvertices of the edges of the cut $\delta_G(C)$ not lying on $C$ are $u_0, u_1, \ldots, u_5$ and are listed in a cyclic order around $C$. Set $U = \{u_0, \ldots, u_5\}$.

Claim 1. The subgraph $H$ is bipartite.

Proof of Claim 1. If $H$ contains a perfect matching, then the matching can be extended to a perfect matching $M_4$ of $G$ which covers the three uncovered edges of $C$. It follows that $\pi(G) = 4$, contrary to our assumption. Therefore $H$ has no perfect matching. In this situation Theorem 4.1 tells us that there exists a set $S \subseteq V(H)$ such that each component $L$ of $H - S$ has $|\delta_G(L)| = 3$ and each edge of $\delta_G(H)$ joins a vertex of $H - S$ to a vertex of $C$. As $G$ is cyclically 4-edge-connected, each component of $H - S$ is a single vertex.
Consequently, $H$ is a bipartite graph with bipartition $\{S, V(H) - S\}$ and edge set $\delta_G(S)$. This establishes Claim 1.

Next we explore the colouring properties of the subgraph $H^+ = G - E(C) \subseteq G$. Note that $H^+ = H + \{f_0, f_1, \ldots, f_5\}$ and $H^+ \cup C = G$. Set $C^+ = C + \{f_0, f_1, \ldots, f_5\}$. In $C^+$, the endvertices $u_0, u_1, \ldots, u_5$ are assumed as pairwise distinct.

Before proceeding further we need several definitions. A colour vector is any sequence $\alpha = c_0c_1\ldots c_5$ of six colours $c_i \in \{1, 2, 3\}$ satisfying Parity Lemma; that is, $c_0 + \ldots + c_5 = 0$. Every 3-edge-colouring $\sigma$ of $H^+$ thus induces the colour vector $\alpha_\sigma = \sigma(f_0)\sigma(f_1)\ldots\sigma(f_5)$. By permuting the colours 1, 2, and 3 in $\alpha$ we obtain a new colour vector, nevertheless, the difference between the two is insubstantial. Therefore each of the six permutations of colours produces a sequence that encodes essentially 'the same colour vector'. In order to have a canonical representative, we choose from them the lexicographically minimal sequence and call it the tor'. In order to have a canonical representative, we choose from them the lexicographically minimal sequence and call it the tor'.

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Consequently, $H$ is a bipartite graph with bipartition $\{S, V(H) - S\}$ and edge set $\delta_G(S)$. This establishes Claim 1.

Proof of Claim 2. Consider the colour vector induced by a 3-edge-colouring of $H^+$. If there was a colour that does not appear in it, then the corresponding colour class could be extended to a perfect matching of the entire $G$ that covers the three uncovered edges of $C$. Consequently, $G$ could be covered with four perfect matchings, contrary to our assumption. Claim 2 is proved.

There exist exactly fifteen colouring types involving all three colours; they are listed as $\alpha_1, \alpha_2, \ldots, \alpha_{15}$ in the first column of Table 1 following the lexicographic order. Let $A$ denote the set comprising all of them. We now specify three subsets of $A$:

$$B = \{123132, 121323, 123213\},$$

$$C = \{112332, 122133, 123123, 123321\},$$

$$P = \{112233, 112323, 121233, 121332, 122313, 122331, 123231, 123312\}.$$ 

Clearly, $B, C,$ and $P$ form a partition of $A$. The sets $C$ and $P$ have a straightforward interpretation if we choose our graph $G$ to be the Petersen graph $P_G$ and $C$ the hexagonal core of an arbitrary optimal 3-array. In this case, $C$ and $P$ are simply the sets of all colouring types including all three colours that are admissible for $C^+$ and $H^+ = (P_G - V(C))^+$, respectively.

Claim 3. Every colouring type admissible for $H^+$ belongs to $P$.

Proof of Claim 3. To prove the claim it is sufficient to exclude the existence of colourings of $H^+$ whose type belongs to $B \cup C$. We first observe that $H^+$ cannot have a colouring of any type from $C = \{\alpha_3, \alpha_7, \alpha_{10}, \alpha_{13}\}$. Indeed, if it had, then such a colouring could easily be extended to a 3-edge-colouring
BERGE’S CONJECTURE FOR GRAPHS WITH SMALL DEFECT

Table 1. The fifteen colouring types involving all three colours.

| Colouring type | Contained in |
|----------------|--------------|
| \( \alpha_1 = 112233 \) | \( \mathcal{P} \) |
| \( \alpha_2 = 112323 \) | \( \mathcal{P} \) |
| \( \alpha_3 = 112332 \) | \( \mathcal{C} \) |
| \( \alpha_4 = 121233 \) | \( \mathcal{P} \) |
| \( \alpha_5 = 121323 \) | \( \mathcal{B} \) |
| \( \alpha_6 = 121332 \) | \( \mathcal{P} \) |
| \( \alpha_7 = 122133 \) | \( \mathcal{C} \) |
| \( \alpha_8 = 122313 \) | \( \mathcal{P} \) |
| \( \alpha_9 = 122331 \) | \( \mathcal{P} \) |
| \( \alpha_{10} = 123123 \) | \( \mathcal{C} \) |
| \( \alpha_{11} = 123132 \) | \( \mathcal{B} \) |
| \( \alpha_{12} = 123213 \) | \( \mathcal{B} \) |
| \( \alpha_{13} = 123231 \) | \( \mathcal{P} \) |
| \( \alpha_{14} = 123312 \) | \( \mathcal{P} \) |
| \( \alpha_{15} = 123321 \) | \( \mathcal{C} \) |

of the entire \( G \). Since \( G \) is a snark, this cannot happen. Thus it remains to deal with colourings whose type belongs to \( \mathcal{B} = \{ \alpha_5, \alpha_{11}, \alpha_{12} \} \).

Suppose that \( H^+ \) admits a colouring of type \( \alpha_5 = 121323 \). Take the (1,3)-Kempe chain starting at \( f_0 \). It must terminate at \( f_2 \), \( f_3 \), or \( f_5 \), and the corresponding Kempe switches yield the colour vectors 323323, 321123, or 321321, respectively. The first of them has a missing colour, and therefore cannot occur, by Claim 2. The latter two have the type that belongs to \( \mathcal{C} \), which we have already excluded. Therefore \( H^+ \) has no colouring of type \( \alpha_5 \).

Next, consider the colouring type \( \alpha_{11} = 123132 \). Consider the (2,3)-Kempe chain starting at \( f_1 \). The chain necessarily terminates at \( f_2 \), \( f_4 \), or \( f_5 \), and the corresponding switches yield the colour vectors 132132, 133122, or 133133, respectively. The first two colour vectors have the type that belongs to \( \mathcal{C} \), while the last one has a missing colour. Hence, the colouring type \( \alpha_{11} \) is also excluded.

Finally we deal with the colouring type \( \alpha_{12} = 123213 \). This time we consider the (1,2)-Kempe chain starting at \( f_0 \). Its terminal edge must be one of \( f_1 \), \( f_3 \), or \( f_4 \), and the corresponding switches produce the vectors 213213, 223113, or 223223, respectively. Again, the first two colour vectors have the type that belongs to \( \mathcal{C} \), while last one has a missing colour. The colouring type \( \alpha_{12} \) is thus excluded as well.

Summing up, we have shown that \( H^+ \) admits no colouring whose type belongs to \( \mathcal{B} \cup \mathcal{C} \). It follows that \( H^+ \) only admits colourings whose type belongs to \( \mathcal{P} \). Claim 3 is proved.

We continue with the proof by analysing the set \( U = \{ u_0, \ldots, u_5 \} \). In general, the vertices \( u_0, u_1, \ldots, u_5 \) need not be pairwise distinct, and some of them may coincide. Clearly, it cannot happen that \( u_j = u_{j+1} \) for some index
for otherwise \( G \) would contain a triangle \((u_jv_jv_{j+1})\), which is impossible because \( G \) is cyclically 4-edge-connected. If \( u_j = u_{j+2} \) for some index \( j \), then \( G \) contains a quadrangle \((u_jv_jv_{j+1}v_{j+2})\) whose intersection with \( C \) is a path of length two. However, this contradicts Lemma \[5.4\]. Thus, the only possibility is that some of the pairs \( u_i \) and \( u_{i+3} \) coincide. Let \( x \) be the number of such pairs; clearly \( x \in \{0, 1, 2, 3\} \). We prove that \( x = 3 \).

Suppose to the contrary that \( x \leq 2 \). Without loss of generality, we may assume that \( u_2 \neq u_5 \). We now take the graph \( H^+ \) and create from it a new cubic graph \( H^\# \) as follows. First, we remove the edges \( f_2 \) and \( f_5 \) and add an edge \( e \) between \( u_2 \) and \( u_5 \). Then we identify the endvertices \( v_0 \) and \( v_1 \) of \( f_0 \) and \( f_1 \) into a new vertex \( s \). Similarly, we identify the endvertices \( v_3 \) and \( v_4 \) of \( f_3 \) and \( f_4 \) into a vertex \( t \). Finally, we add an edge \( f \) between \( s \) and \( t \); see Figure 6.

We prove that \( H^\# \) is not 3-edge-colourable. If \( H^\# \) had a 3-edge-colouring, then such a colouring would result from a 3-edge-colouring \( \sigma \) of \( H^+ \) whose colour type \( \alpha = c_0c_1 \ldots c_5 \) satisfies all of the following conditions:

(i) \( c_0 \neq c_1 \) \((\sigma \text{ is proper at } s)\),

(ii) \( c_3 \neq c_4 \) \((\sigma \text{ is proper at } t)\), and

(iii) \( c_2 = c_5 \) \((e \text{ is properly coloured})\).

Moreover, Claim 3 tells us that \( \alpha \in \mathcal{P} \). However, by checking the elements of \( \mathcal{P} \) we see that the colouring types \( \alpha_1 \) and \( \alpha_2 \) violate (i), \( \alpha_6 \) and \( \alpha_9 \) violate (ii), and \( \alpha_4, \alpha_8, \alpha_{13}, \text{ and } \alpha_{14} \) violate (iii). In other words, no element of \( \mathcal{P} \) satisfies all the conditions (i)-(iii), and therefore \( H^\# \) admits no 3-edge-colouring.

Observe that \( H^\# - \{e, f\} \) is bipartite with bipartition \( \{s \cup \{s, t\}, (V(H) - S)\} \). Since \( H^\# \) is not 3-edge-colourable, from Theorem \[4.4\] we deduce that \( H^\# \) must have a bridge, say \( b \). A simple counting argument shows that in \( H^\# \) the bridge \( b \) separates \( e \) from \( f \). Let \( Q \) be the component of \( H^\# - b \) containing \( e \). Now, \( Q - e \) is a subgraph of \( G \) separated from the rest of \( G \) by the cut \( \delta_G(Q - e) = \{b, f_2, f_5\} \). Since \( Q \) has at least two vertices and so does the other component of \( G - \delta_G(Q - e) \), we conclude that \( \delta_G(Q - e) \) is a cycle-separating 3-edge-cut in \( G \). This contradicts the assumption that \( G \) is cyclically 4-edge-connected, and proves that \( x = 3 \).

We have thus proved that \( u_0 = u_3, u_1 = u_4, \text{ and } u_2 = u_5 \). Consider the subgraph \( J \subseteq G \) induced by the set \( C \cup \{u_0, u_1, u_2\} \). Each of the vertices \( u_0, \ldots, u_5 \) is connected to each of the vertices \( u_0, \ldots, u_5 \) by a path of length two. Therefore, \( J \) is a cycle-separating 3-edge-cut in \( G \). This contradicts the assumption that \( G \) is cyclically 4-edge-connected, and proves that \( x = 3 \).

We have thus proved that \( u_0 = u_3, u_1 = u_4, \text{ and } u_2 = u_5 \). Consider the subgraph \( J \subseteq G \) induced by the set \( C \cup \{u_0, u_1, u_2\} \). Each of the vertices \( u_0, \ldots, u_5 \) is connected to each of the vertices \( u_0, \ldots, u_5 \) by a path of length two. Therefore, \( J \) is a cycle-separating 3-edge-cut in \( G \). This contradicts the assumption that \( G \) is cyclically 4-edge-connected, and proves that \( x = 3 \).
u_1, and u_2 is 2-valent in J, which means that \( \delta_G(J) \) is a 3-edge-cut. As G is cyclically 4-edge-connected, \( G - (C \cup \{u_0, u_1, u_2\}) \) must be a single vertex. Now it is immediate that G is isomorphic to the Petersen graph.

Summing up, we have proved that every cyclically 4-edge-connected cubic graph \( G \) with \( df(G) = 3 \) and \( \pi(G) \geq 5 \) is isomorphic to the Petersen graph, as required. This completes the proof. \( \square \)

7. Family of snarks with defect 3 and perfect matching index 5

The requirement of cyclic connectivity at least 4 in Theorem 1.2 is essential, because there exist infinitely many 3-edge-connected snarks with defect 3 and perfect matching index 5. They can be constructed as follows:

Take the Petersen graph \( Pg \) and remove a vertex \( v \) from it together with the three incident edges, leaving a graph \( P \) containing three 2-valent vertices. Further, take an arbitrary connected bipartite cubic graph on at least four vertices, and similarly remove a vertex \( u \), leaving a graph \( B \) with tree 2-valent vertices. Create a cubic graph \( H \) by joining every 2-valent vertex of \( P \) to a 2-valent vertex of \( B \). Clearly, the set \( R \) consisting of the newly added edges is a cycle-separating 3-edge-cut in \( H \). We claim that the graph \( H \) has defect 3 and perfect matching index 5.

We first observe that \( df(H) = 3 \). Parity Lemma implies that \( P \) is uncolourable, so \( H \) is a snark, and therefore \( df(H) \geq 3 \). Since \( Pg \) is vertex-transitive, one can choose an optimal array for \( Pg \) in such a way that its hexagonal core avoids the vertex \( v \). As every bipartite graph is 3-edge-colourable, one can easily extend the 3-array of \( Pg \) to a 3-array of \( H \) which retains the original core of \( Pg \). Hence, \( df(H) = 3 \).

Now we want to prove that \( \pi(H) = 5 \). Suppose to the contrary that \( \pi(H) \leq 4 \). Then \( H \) has a covering \( C = \{M_1, M_2, M_3, M_4\} \) with four perfect matchings. Since both \( P \) and \( B \) contain an odd number of vertices, we conclude that each \( M_i \) has an odd number of common edges with \( R \). Moreover, since each edge of \( H \) is in at most two of the four perfect matchings, the weights of edges in the edge cut \( R \) are either 1, 1, 2, or 2, 2, 2. The former possibility is excluded, because contracting \( B \) to a single vertex would produce a covering of \( Pg \) with four perfect matchings, contradicting the fact that \( \pi(Pg) = 5 \). Therefore the weight of each edge in \( R \) equals 2. Now, each vertex of \( B \) is incident with precisely one edge of weight 2, including those in \( R \). It follows that the simply covered edges form a 2-factor of \( B \), say \( F \). However, \( B \) has an odd number of vertices, so at least one circuit of \( F \) is odd, which is impossible because \( B \) is bipartite. Hence, \( \pi(H) \geq 5 \), and from Theorem 1.1 we infer that \( \pi(H) = 5 \).

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References

[1] I. Allie, Oddness to resistance ratios in cubic graphs, Discrete Math. 342 (2019), 387–391.
[2] I. Allie, E. Mácajová, M. Škoviera, Snarks with resistance n and flow resistance 2n, Electron. J. Combin. 29 (2022), #P1.44.
[3] G. Brinkmann, J. Goedegebeur, J. Hägglund, K. Markström, Generation and properties of snarks, J. Combin. Theory Ser. B 103 (2013), 468–488.
[4] P. J. Cameron, A. G. Chetwynd, J. J. Watkins, Decomposition of snarks, J. Graph Theory 11 (1987), 13–19.
[5] F. Chen, G. Fan, Fulkerson-covers of hypohamiltonian graphs, Discrete Appl. Math. 186 (2015), 66–73.
[6] G. Fan, A. Raspaud, Fulkerson’s conjecture and circuit covers, J. Combin. Theory Ser. B 61 (1994), 133–138.
[7] M. A. Fiol, G. Mazzuoccolo, E. Steffen, Measures of edge-uncolorability of cubic graphs, Electron. J. Combin. 25 (2018), #P4.54.
[8] J.-L. Fouquet, J.-M. Vanherpe, On the perfect matching index of bridgeless cubic graphs, arXiv:0904.1296 [cs.DM], (2009).
[9] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, Math. Program. 1 (1971), 16–194.
[10] R.-X. Hao, J. Niu, X. Wang, C.-Q. Zhang, T. Zhang, A note on Berge-Fulkerson coloring, Discrete Math. 309 (2009), 4235–4240.
[11] R.-X. Hao, C.-Q. Zhang, T. Zheng, Berge-Fulkerson coloring for $C_8$-linked graphs, J. Graph Theory 88 (2018), 46–60.
[12] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colorable, Amer. Math. Monthly 82 (1975), 221–239.
[13] L. Jin, E. Steffen, Petersen cores and the oddness of cubic graphs, J. Graph Theory 84 (2017), 109–120.
[14] L. Jin, E. Steffen, G. Mazzuoccolo, Cores, joins and the Fano-flow conjectures, Discuss. Math. Graph Theory 38, 165–175.
[15] J. Karabáš, E. Mácajová, R. Nedela, M. Škoviera, Girth, oddness, and colouring defect of snarks, Discrete Math. 345 (2022), 113040.
[16] J. Karabáš, E. Mácajová, R. Nedela, M. Škoviera, Cubic graphs with colouring defect 3, manuscript.
[17] S. Liu, R.-X. Hao, C.-Q. Zhang, Rotation snark, Berge-Fulkerson conjecture and Catlin’s 4-flow reduction, Appl. Math. Comput. 410 (2021), 126411.
[18] R. Lukoťka, E. Mácajová, J. Mazák, M. Škoviera, Small snarks with large oddness, Electron. J. Combin. 22 (2015), #P1.51.
[19] R. Lukoťka, E. Rollová, Perfect matchings of regular bipartite graphs, J. Graph Theory 85+ (2017) 525–532.
[20] E. Mácajová, G. Mazzuoccolo, Reduction of the Berge-Fulkerson conjecture to cyclically 5-edge-connected snarks, Proc. Amer. Math. Soc. 148 (2020), 464–4652.
[21] E. Mácajová, M. Škoviera, Cubic graphs that cannot be covered with four perfect matchings, J. Combin. Theory Ser. B 150 (2021), 144–176.
[22] P. Manuel, A. S. Shanthi, Berge-Fulkerson conjecture on certain snarks, Math. Comput. Sci. 9 (2015), 209–220.
[23] G. Mazzuoccolo, The equivalence of two conjectures of Berge and Fulkerson, J. Graph Theory 68 (2011), 125–128.
[24] R. Nedela, M. Škoviera, Decompositions and reductions of snarks, J. Graph Theory 22 (1996), 253–279.
[25] J. Plesněk, Connectivity of regular graphs and the existence of 1-factors, Mat. Časopis 22 (1972), 310–318.
[26] P. D. Seymour, On multicouourings of cubic graphs, and conjectures of Fulkerson and Tutte, Proc. London. Math. Soc. 38 (1979), 423–460.
[27] E. Steffen, Classifications and characterizations of snarks, Discrete Math. 188 (1998), 183–203.
[28] E. Steffen, 1-Factor and cycle covers of cubic graphs, J. Graph Theory 78 (2015), 195–206.
[29] W. Sun, F. Wang, A class of cubic graphs satisfying Berge conjecture, Graphs Combin. 38 (2022), Art. No. 66.
[30] C. Thomassen, The Erdös-Pósa property for odd cycles in graphs of large connectivity, Combinatorica 21 (2001), 321–333.
[31] C. Thomassen, Y. Wu, C.-Q. Zhang, The 3-flow conjecture, factors modulo k, and the 1-2-3-conjecture, J. Combin. Theory Ser. B 121 (2016), 308–325.
[32] W. T. Tutte, The factorization of linear graphs, J. London Math. Soc. 22 (1947), 107–111.

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