Research Article

Weak Solutions and Optimal Control of Hemivariational Evolutionary Navier-Stokes Equations under Rauch Condition

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In this paper, we consider the evolutionary Navier-Stokes equations subject to the nonslip boundary condition together with a Clarke subdifferential relation between the dynamic pressure and the normal component of the velocity. Under the Rauch condition, we use the Galerkin approximation method and a weak precompactness criterion to ensure the convergence to a desired solution. Moreover, a control problem associated with such system of equations is studied with the help of a stability result with respect to the external forces. At the end of this paper, a more general condition due to Z. Naniewicz, namely the directional growth condition, is considered and all the results are reexamined.

1. Introduction

In many engineering situations, one deals with fluid flow problems in tubes or channels, or for semipermeable walls and membranes. In practice, hydraulic control devices are used as a mechanism allowing the adjustment of orifice dimensions so that the normal velocity on the boundary of the tube is regulated to reduce the dynamic pressure. The model that usually describes this situation is represented by the Navier-Stokes equations for incompressible viscous fluids with the nonslip boundary conditions together with a Clarke subdifferential relation between the dynamic pressure and the normal component of the velocity. The resulting multivalued subdifferential boundary condition leads, after a standard variational transformation, to the so-called hemivariational inequality.

The theory of hemivariational inequalities was introduced for the first time by Panagiotopoulos [1–5] for the sake of generalization of the classical convex variational theory to a nonconvex one. The main tool in this effort is the generalized gradient of Clarke and Rockafellar [6–8]. From this perspective, the literature has seen a fast emergence of applications in a mathematical and mechanical point of view, see [3, 4, 9–13] for more details. Among the main applications of this theory, we mention the Newtonian and non-Newtonian Navier-Stokes equations and their variants (the Oseen model, heat-conducting fluids, miscible liquids, etc.) with nonstandard boundary conditions ensuing from the multivalued nonmonotone friction law with leak, slip, or nonslip conditions. For recent directions on the hemivariational theory, we refer to [14–17].

Over the last two decades, intensive research has been conducted on hemivariational inequalities for the stationary and nonstationary Navier-Stokes equations. For convex functionals, the problem has been studied essentially by Chebotarev [18–20]. We mention also [21] for stationary Boussinesq equations and [22] by Konovolova for nonstationary Boussinesq equations. In all these papers, the considered problems were formulated as variational inequalities. In the nonconvex case, the stationary case was considered by Migoński and Ochal [23] and Migoński [24], and the nonstationary case was considered by Migoński and Ochal in [25]; see also [26]. For an equilibrium approach, one can see for example [27]. On the other hand, the optimal control problem involving hemivariational inequalities attracts more and more attention from researchers in recent years. We
refer to the introductions of [28, 29] for a short review on the subject.

There are two main conditions that one can impose on the locally Lipschitz function under a subdifferential effect, namely the classical growth condition or the Rauch condition due to J. Rauch [30]. The last one is less popular even if it was the main assumption in the beginning of the theory of hemivariational inequalities. The Rauch condition expresses actually the ultimate increase of the graph of a certain locally bounded function and is, in fact, a special case of another unpopular condition, namely the directional growth condition due to Namiwicz [31]. An advantage of the Rauch condition is that it allows avoiding smallness conditions (i.e., the relationship between the constants of the problem) brought by the classical growth condition. In the case of the Navier-Stokes equations, the smallness condition links the growth condition constant, the coercivity constant, and the norm of the trace operator. It is, however, not clear how it can be checked in a concrete situation. Another advantage is that it allows us to consider the “Stranger” functions at infinity. In fact, the only thing we require from the function is for the essential supremum of the function on the left side to be greater than the essential infimum on the right side.

Among the disadvantages of the Rauch condition is that although it ensures the existence of a solution, it does not allow the conclusion that the nonconvex functional is locally Lipschitz or even finite on the whole space. The Aubin-Clarke formula cannot be used, and a slight change in the definition of a solution has to be made. On the other hand, we are looking for the dynamical pressure in a larger space, which makes the question of uniqueness more difficult without a classical growth condition even if a monotonicity type assumption is acquired [32]. Finally, it is worth mentioning that there is no direct link between the Rauch condition and the classical growth condition, and the choice depends mainly on the concrete situation.

The present paper represents a continuation of our previous paper [32], where existence and optimal control questions involving the stationary Navier-Stokes problem with the multivalued nonmonotone boundary condition are studied. In this paper, we tackle the nonstationary problem. Always under the Rauch condition, we use the Faedo-Galerkin approximation to regularize the system at the level of the multivalued boundary condition and we use the fact that the approximation sequence so obtained is weakly precompact in the space of integrable functions. We also take advantage of the techniques used in [25] at the level of the nonlinear term to ensure the convergence of the approximate sequence to the desired solution. This study can be also done with the directional growth condition as a generalization. The question of the existence of an optimal control is important in applications. We tackle this subject in the spirit of the works of Barbu [33] and Migórski [34].

The outline of this paper is as follows. In section 2, we state the problem and give its hemivariational form by using the Lamb formulation. In section 3, we regularize our problem by using the Faedo-Galerkin approximation method and prove the existence of solutions to the regularized problem. By combining techniques from [25, 32], we will provide an existence result in section 4. Section 5 is devoted to the optimal control problem subjected to our evolutionary hemivariational inequality, while section 6 is dedicated to the directional growth condition as a generalization of the Rauch condition.

2. Problem Statement

Let $\Omega$ be a bounded simply connected domain in $\mathbb{R}^d$ with $d = 2, 3$ with connected boundary $\partial \Omega$ of class $C^2$ and $\Omega_T = (0, T) \times \Omega$ where $T > 0$. We consider the following evolution Navier-Stokes system:

\begin{equation}
\begin{aligned}
\nu \Delta u + (u \cdot \nabla)u + \nabla p &= f_{\text{ext}}, & \text{in } \Omega_T, \\
\text{div } u &= 0, & \text{in } \Omega_T, \\
u &\frac{\partial u}{\partial n} = u_0, & \text{in } \Omega. \\
\end{aligned}
\end{equation}

This system describes the flux of an incompressible viscous fluid in a domain $\Omega$ subjected to an external force $f_{\text{ext}} = \{f_{\text{ext}, k}\}_{k=1}^d$ and $u = \{u_k\}_{k=1}^d$, $p$ and $\nu$ denote, respectively, the velocity, the pressure, and the kinematic viscosity of the fluid. The nonlinear term $(\nu \cdot \nabla)u$ (called the convective term) is the symbolic notation of the vector $\sum_{k=1}^d u_k (\partial u_k / \partial x_j)$. As usual, we use the Lamb formulation ([35], chapter 1) to rewrite the evolution Navier-Stokes system as follows:

\begin{equation}
\begin{aligned}
u' + \nu \text{rot } u + \nu \text{rot } u \times u + \nabla \tilde{p} &= f_{\text{ext}}, & \text{in } \Omega_T, \\
\text{div } u &= 0, & \text{in } \Omega_T, \\
u &\frac{\partial u}{\partial n} = u_0, & \text{in } \Omega, \\
\end{aligned}
\end{equation}

where $\tilde{p} = p + (1/\nu)|u|^2$ is the total head of the fluid, or “total pressure.”

We suppose that on boundary $\partial \Omega$, the tangential components of the velocity vector are known, and without loss of generality, we put them equal to zero (the nonslip condition):

\begin{equation}
u(x, t) := u(t, x) - u_N (t, x) n_x, \quad \text{on } \partial \Omega_T = (0, T) \times \partial \Omega,
\end{equation}

where $n = \{n_k\}_{k=1}^d$ is the unit outward normal on the boundary $\partial \Omega$ and $u_N (t, x) = u(t, x) \cdot n$, denotes the normal component of the vector. Moreover, we assume the following subdifferential boundary condition:

\begin{equation}p(t, x) = \partial j(u_N(t, x)) \text{ on } \partial \Omega_T,
\end{equation}

where $\partial j(\xi)$ is the Clarke subdifferential of $j$ at $\xi$ and is given by

\begin{equation}\partial j(\xi) = \{\xi^* \in V^*: \langle \xi^*, h \rangle \geq \langle \xi, h \rangle_{V, V^*} \text{ for all } h \in V\},
\end{equation}

and $\langle \cdot, \cdot \rangle_{V, V^*}$ is the generalized derivative of a locally Lipschitz function $j$ at $\xi \in V$ in the direction $h \in V$ defined by
\[ f^0 (\xi ; v) = \limsup_{\nu \to \lambda \to 0} \frac{f(v + \lambda v) - f(v)}{\lambda}. \] (8)

To work conveniently on problems (2), (3), (4), (5), and (6), we need the following functional spaces:

\[ C = \left\{ u \in C^0(\Omega ; \mathbb{R}^d) : \text{div} \ u = 0 \in \Omega, u = 0 \text{ on } \partial \Omega \right\}, \]
\[ V = \text{the closure of } C \text{ in the norm of } H^1(\Omega ; \mathbb{R}^d), \]
\[ H = \text{the closure of } C \text{ in the norm of } L^2(\Omega ; \mathbb{R}^d). \] (9)

Then, we have \( V \subset H = H^* \subset V^* \), with all the embedding being continuous and compact. Moreover, for an interval time \([0, T]\), we introduce the following spaces:

\[ \mathcal{V} = L^2(0, T ; V), \]
\[ \mathcal{H} = L^2(0, T ; H), \]
\[ \mathcal{W} = \left\{ u \in \mathcal{V} : u^* \in \mathcal{V}^* \right\}. \] (10)

Then, we also have the following continuous embedding, \( \mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^* \).

We consider the operators \( \mathcal{A} : V \to V^* \) and \( \mathcal{B} : V \times V \to V^* \) defined by

\[ \langle \mathcal{A} u, v \rangle = \nu \int_{\Omega} \text{rot} \ u \cdot \text{rot} \ v \, dx, \]
\[ \langle \mathcal{B} (u, v), w \rangle = \int_{\Omega} (\text{rot} \ u \times v) \cdot w \, dx, \] (11)

for all \( u, v, w \in V \). As usual, we will use the notation \( \mathcal{B}(...) = \mathcal{B}(\ldots) \). It is well known (cf. [36]) that if the domain \( \Omega \) is simply connected, the bilinear form

\[ ((u, v)) = \int_{\Omega} \text{rot} \ u \cdot \text{rot} \ v \, dx, \] (12)

generates a norm in \( V, \| u \|_V = ((u, v))^{1/2} \), which is equivalent to the \( H^1(\Omega, \mathbb{R}^d) \)-norm. Hence, it is clear that the operator \( \mathcal{A} \) is coercive.

In order to give the weak formulation to problems (2), (3), (4), (5), and (6), we multiply it by a certain \( v \in V \) and apply the Green formula. We obtain

\[ \langle u'(t) + \mathcal{A} u(t) + \mathcal{B}[u(t)], v \rangle + \int_{\Omega} \tilde{p}(t, x) v_N \, dx = \langle f(t), v \rangle, \] (13)

where \( \langle f(t), v \rangle = \int_{\Omega} f_{\text{ext}}(t) \cdot v \, dx \). From relation (6), by using the definition of the Clarke subdifferential, we have

\[ \int_{\partial \Omega} \tilde{p}(t, x) v_N(x) \, d\sigma(x) \leq \int_{\partial \Omega} \tilde{p}(t, x) u_N(x) \, d\sigma(x). \] (14)

The relations (13) and (14) yield to the following weak formulation:

\[ \left\{ \begin{array}{l}
\text{Find } u \in \mathcal{W} \text{ such that for all } v \in \mathcal{V} \text{ and a.e. } t \in (0, T),
\langle u'(t) + \mathcal{A} u(t) + \mathcal{B}[u(t)], v \rangle + \int_{\Omega} \tilde{p}(t, x) v_N(x) \, dx = \langle f(t), v \rangle,
\end{array} \right. \] (15)

The equation above is called an hemivariational inequality.

We have already mentioned in Introduction that the Rauch assumption is not sufficient to make the functional \( f(u) = \int_{\Omega} j(u) \, dx \) locally lipschitz or even finite in the whole space \( \mathcal{V} \). Because of this reason, a slight modified definition of being a solution should be adopted. Define the functional space

\[ L^2_{V^*}(\partial \Omega) = \left\{ u, u_N = \gamma(u) \cdot n \in L^2(\partial \Omega ; \mathbb{R}) \right\}, \] (16)

where \( \gamma \) is the trace operator from \( V \) in \( L^2(\partial \Omega ; \mathbb{R}^d) \). Now, we are able to give what we mean by a solution to the problem (EHVI).

**Definition 1.** A function \( u \in \mathcal{W} \) is said to be the solution of (EHVI) if there exists \( \kappa \in L^1((0, T) \times \partial \Omega, \mathbb{R}) \) such that for a.e. \( t \in (0, T) \)

\[ \left\{ \begin{array}{l}
\langle u'(t) + \mathcal{A} u(t) + \mathcal{B}[u(t)], v \rangle + \int_{\Omega} \kappa(t, x) v_N(x) \, dx = \langle f(t), v \rangle, \quad \forall v \in \mathcal{V},
\kappa(t, x) \in \partial \gamma(u(t, x)), \quad \text{for a.e. } (t, x) \in (0, T) \times \partial \Omega,
\end{array} \right. \] (17)

Note that since \( \mathcal{W} \subset C(0, T ; H) \) continuously, the initial condition \( u(0) = u_0 \) makes sense in \( H \). To justify the above definition, we refer to [31, 32].

### 3. Regularized Problem

In what follows, we restrict our study to superpotentials \( j \), which are independent of \( x \) and which subdifferentiable is obtained by "filling in the gaps" procedure (cf. [30]). Let \( \theta \in L^\infty(\mathbb{R}) \), for \( \varepsilon > 0 \) and \( t \in \mathbb{R} \), we define

\[ \tilde{\theta}_\varepsilon(t) = \text{ess inf}_{|s| \leq \varepsilon} \theta(s), \]
\[ \hat{\theta}_\varepsilon(t) = \text{ess sup}_{|s| \leq \varepsilon} \theta(s). \] (18)

For a fixed \( t \in \mathbb{R} \), the functions \( \tilde{\theta}_\varepsilon, \hat{\theta}_\varepsilon \) are decreasing and increasing in \( \varepsilon \), respectively. Let
\[
\theta(t) = \lim_{\varepsilon \to 0} \theta_\varepsilon(t), \\
\bar{\theta}(t) = \lim_{\varepsilon \to 0} \bar{\theta}_\varepsilon(t),
\]

(19)

and let \( \hat{\theta}(t) : \mathbb{R} \to 2^\mathbb{R} \) be a multifunction defined by

\[
\hat{\theta}(t) = [\theta(t), \bar{\theta}(t)].
\]

(20)

From Chang [37], we know that a locally Lipschitz function \( j : \mathbb{R} \to \mathbb{R} \) can be determined up to an additive constant by the relation

\[
j(t) = \int_0^t \theta(s) ds,
\]

(21)

such that \( \partial j(t) \subset \hat{\theta}(t) \) for all \( t \in \mathbb{R} \). If moreover, the limits \( \theta(t \pm 0) \) exist for every \( t \in \mathbb{R} \), then \( \partial j(t) = \hat{\theta}(t) \).

In order to define the regularized problem, we consider the mollifier

\[
h \in C^\infty_0(-1, 1),
\]

\[
h \geq 0 \text{ with } \int_{-\infty}^{+\infty} h(s) ds = 1,
\]

(22)

and let

\[
\theta_\varepsilon = h_\varepsilon * \theta \text{ with } h_\varepsilon(s) = \frac{1}{\varepsilon} h\left(\frac{s}{\varepsilon}\right),
\]

(23)

where \(*\) denotes the convolution product.

Consider the following auxiliary problem associated to (EHVI):

\[
\begin{cases}
\text{Find } u \in \mathcal{W} \text{ such that for all } v \in V \cap L^2_{\text{loc}}(\partial\Omega), \text{ a.e. } t \in (0, T), \\
\quad \L(t) \left( u'(t) + \mathcal{A} u(t) + \mathcal{B}[u(t)], v \right) + \int_{\Omega} \theta_\varepsilon(u_N(t, x)) \nu \cdot dv = (f(t), v), \\
\quad u(0) = 0,
\end{cases}
\]

(24)

Let \( \{u_m(0)\} \) be an approximation of the given initial value \( u_0 \) such that \( u_m(0) \in V_m \) for \( m \in \mathbb{N} \) and suppose that

\[
\begin{aligned}
\lim_{m \to \infty} u_m & \to u_0 \text{ in } H, \quad \text{as } m \to +\infty, \\
(u_m, m) \text{ is bounded in } V \cap L^2_{\text{loc}}(\partial\Omega).
\end{aligned}
\]

(26)

We consider the following regularized Galerkin system of finite dimensional differential equations associated to (EHVI):

\[
\begin{cases}
\text{Find } u_m \in \mathcal{W}_m \text{ such that for all } v \in \mathcal{W}_m, \text{ a.e. } t \in (0, T), \\
\quad \L(t) \left( u_m'(t) + \mathcal{A} u_m(t) + \mathcal{B}[u_m(t)], v \right) + \int_{\partial\Omega} \theta_\varepsilon(u_N(t)) \nu \cdot dv = (f(t), v), \\
\quad u_m(0) = 0,
\end{cases}
\]

(27)

where \( \mathcal{W}_m = \{ u \in L^2(0, T; V_m) : u'(t) \in L^2(0, T; V_m) \} \).

The generalized derivative \( \mathcal{L} u = u' \) restricted to the subset \( D(D') = \{ u \in \mathcal{V}' : u' \in \mathcal{V}'^{**} \text{ and } u(0) = 0 \} \) defines a linear operator \( \mathcal{L} : D(D') \to \mathcal{V}' \) given by

\[
\llangle \mathcal{L} u, v \gg = \int_0^T \left( u'(t), v(t) \right) dt \text{ for all } v \in \mathcal{V}'.
\]

(28)

For the existence of solutions, we will need the following hypothesis \( H(\theta) \):

1. (Chang assumption) \( \theta \in L^1_{\text{loc}}(\mathbb{R}), \theta(t \pm 0) \) exists for any \( t \in \mathbb{R} \)
2. (Rauch assumption) There is \( \delta_0 > 0 \) such that

\[
\text{ess sup}_{t \in [-\infty, -\delta_0]} \theta(t) \leq 0 \leq \text{ess inf}_{t \in [\delta_0, +\infty]} \theta(t).
\]

(29)

Remark 2. If one assumes more generally that

\[
\text{ess sup}_{t \in [-\infty, -\delta_0]} \theta(t) \leq \alpha \leq \text{ess inf}_{t \in [\delta_0, +\infty]} \theta(t),
\]

(30)

for some real number \( \alpha \), it is possible to come back to the situation where the Rauch assumption is imposed by simply replacing \( \theta \) by \( \theta - \alpha \) and \( f \) by \( f - \alpha \).

Remark 3. We point out that the Rauch and growth conditions are completely independent. Indeed, by taking examples, we show that neither of both conditions implies the other. In fact, consider the function \( \beta : \mathbb{R} \to \mathbb{R} \) defined by

\[
\beta(t) = \begin{cases}
\left[ t^2 \right], & \text{if } |t| \geq 1, \\
-t, & \text{if } |t| < 1,
\end{cases}
\]

(31)

where \( \left[ t \right] \) stands for the integer part of \( t \). One can prove easily (eventually by a contradiction argument) that the function \( \beta \) satisfies the Rauch condition while the growth condition cannot be satisfied. Conversely, one can take a function
\[ \beta : \mathbb{R} \to \mathbb{R} \] defined by \[ \beta(t) = 1 + \sqrt{|t|} \]; it is clear that it satisfies the growth condition but not the Rauch condition as \( \beta \) is positive for negative values.

**Lemma 4.** Suppose that \( H(\theta) \) holds. Then we can determine \( \rho_1, \rho_2 > 0 \), such that for every \( u \in V_m \)

\[
\int_{(0,T) \times \partial \Omega} \theta(u_N(t, z)) u_N(t, z) d\sigma(z) dt \geq -\rho_1 \rho_2 T \cdot \sigma(\partial \Omega).
\]

(32)

**Proof.** This is a classical result in the stationary case (cf. [32], Lemma 3.2). It suffices to integrate over \( t \in (0, T) \) to obtain the result.

**Proposition 5.** The sequence \( (\theta_{c_i}(u_m))_{m \in \mathbb{N}} \) is weakly compact in \( L^1((0, T) \times \partial \Omega) \).

**Proof.** The proof is similar to ([32], Proposition 3.7) with minor changes consisting mainly in replacing \( \partial \Omega \) by \( (0, T) \times \partial \Omega \) and remarking that \( (u_m(t), u_m(t)) \geq 0 \) for a.e. \( t \in [0, T] \).

**Proposition 6.** The regularized problem \( (\mathcal{P}_m^u) \) has at least one solution \( u_m \).

**Proof.** We substitute \( u_m(t) = \sum_{k=1}^m c_{km}(t) z_k \) in \( (\mathcal{P}_m^u) \) to obtain

\[
\sum_{i=1}^m c'_{k}(t) z_i(t) + \sum_{i=1}^m c_{km}(t) \partial_t z_i(t) + \sum_{i,j=1}^m c_{km}(t) c_{jm}(t) b(\partial_t z_i(t), \partial_t z_j(t))
\]

\[
+ \int_{\partial \Omega} \theta_{c_i}(u_N(t, x)) u_N(t, x) d\sigma(t), \quad i = 1, \ldots, m, \text{ a.e. } t \in (0, T],
\]

(33)

\[ c_{km}(0) = \alpha_{km}, \quad k = 1, \ldots, m. \] (34)

The matrix with elements \( (z_k, z_i), \) \( 1 \leq i, k \leq m \) is non-singular (i.e., \( \det \{(z_k, z_i)\}_{k,i=1}^m \neq 0 \)), we invert the matrix, then equation (33) can be written in the usual form:

\[
c'_{km}(t) = \sum_{i=1}^m \beta_{ki} \langle f(t), z_i \rangle - \sum_{i=1}^m \alpha_{ki} c_{im}(t) - \sum_{i,j=1}^m \xi_{ij} c_{km}(t) c_{jm}(t)
\]

\[
- \sum_{i=1}^m \theta_{k} \int_{\partial \Omega} \theta_{c_i}(u_N(t, x)) u_N(t, x) d\sigma(t),
\]

(35)

where the initial values \( c_{km}(0), \) \( k = 1, \ldots, m \) are given, i.e., \( u_{0m} = \sum_{k=1}^m c_{km}(0) z_k \).

\( c_{km}(0) \) is the \( k \)th component of \( u_{0m} \). (36)

The differential system (35) with the initial condition (36) define uniquely the scalar \( c_{km} \) on the interval \( [0, t_m] \).

Then, the solution \( u_m \) exists on \( [0, t_m] \), and we can extend it on the closed interval \( [0, T] \) by using a priori estimates in Lemma 7. Since the scalar function \( t \mapsto \langle f(t), z_i \rangle \) in equation (33) are square integrable, so are the functions \( c_{km} \); therefore, for each \( m \) we have:

\[
u_m \in L^2(0, T; V),
\]

\[
u'_m \in L^2(0, T; V^*).
\]

(37)

**4. Existence Result**

In this section, we will prove the existence of solutions to the problem (EHVI) by analysing the convergence of the sequence \( (u_m) \) to \( \mathcal{P}_m^u \). To do so, we need some a priori estimates.

**Lemma 7.** The solution \( (u_m)_m \) is bounded in \( L^2(0, T; V) \cap L^\infty(0, T; H) \).

**Proof.** From Proposition 5, the regularized problem \( \mathcal{P}_m^u \) has at least one solution \( u_m \). By replacing \( v \) by \( u_m(t) \) in \( \mathcal{P}_m^u \), we get for a.e. \( t \in (0, T) \)

\[
\left\langle u'_m(t) + A u_m(t), v(t) \right\rangle + \int_{\partial \Omega} \theta_{c_i}(u_N(t, x)) u_N(t, x) d\sigma(t) = \langle f(t), v(t) \rangle.
\]

(38)

Because of (37) we have

\[
\left\langle u'_m(t), u_m(t) \right\rangle = \frac{1}{2} \frac{d}{dt} |u_m(t)|^2.
\]

(39)

Then, equation (38) becomes

\[
\frac{d}{dt} |u_m(t)|^2 + 2 \langle A u_m(t), u_m(t) \rangle + 2 \int_{\partial \Omega} \theta_{c_i}(u_N(t, x)) u_N(t, x) d\sigma(t) = 2 \langle f(t), u_m(t) \rangle, \text{ for a.e. } t \in (0, T].
\]

(40)

By the coerciveness of \( A \), the Cauchy-Schwartz inequality, and the Young inequality, we obtain

\[
\frac{d}{dt} |u_m(t)|^2 + M |u_m(t)|^2 + 2 \int_{\partial \Omega} \theta_{c_i}(u_N(t, x)) u_N(t, x) d\sigma(t)
\]

\[
\leq \frac{2}{M} |f(t)|^2_{V^*},
\]

(41)

for a.e. \( t \in (0, T) \) (\( M \) is the constant of coercivity). Integrating equation (40) from 0 to \( s \), \( 0 \leq s \leq T \), and using Lemma 4, one has

\[
|u_m(s)|^2 + M \int_0^s |u_m(t)|^2 dt + 2 \rho_1 \rho_2 s \cdot \sigma(\partial \Omega) \leq \frac{2}{M} \int_0^s |f(t)|^2_{V^*} dt + 2 \rho_1 \rho_2 s \cdot \sigma(\partial \Omega) + |u_m(0)|^2.
\]

(42)
Hence
\[ \sup_{s \in [0, T]} |u_m(s)|^2 \leq \frac{2}{M} \|f\|_{X}^2 + 2 elastic(T \cdot \sigma) + |u_m(0)|^2. \] (43)

The right-hand side of the previous inequality is finite and independent of \( m \). We deduce that \( \{u_m\}_m \) is bounded in \( L^\infty(0, T; H) \).

Again, from (42) we have
\[ M \int_0^T \|u_m(t)\|_V^2 \, dt \leq \frac{2}{M} \|f\|_{X}^2 + 2 elastic(T \cdot \sigma) + |u_m(0)|^2. \] (44)

Then
\[ \|u_m\|_V^2 \leq \frac{2}{M} \|f\|_{X}^2 + 2 elastic(T \cdot \sigma) + \frac{1}{M} |u_m(0)|^2. \] (45)

Then, \( \{u_m\}_m \) remains in a bounded subset of \( \mathcal{Y} \).

**Theorem 8.** Under assumption \( H(\theta) \), the problem (EHVI) has at least one solution.

**Proof.** From Proposition 5 and Proposition 6, we get
\[ u_m \to u \text{ weakly in } \mathcal{Y}, \]
\[ u_m \to u \text{ weakly -- star in } L^\infty(0, T; H), \]
\[ \theta_m(u_m) \to k \text{ weakly in } L^1((0, T) \times \partial \Omega). \] (46)

Now, we focus on the weak convergence of the nonlinear term \( B[u_m] \) by using exactly the same procedure as in [25]. For the case \( d = 2 \), we obtain from Temam [38] the following:
\[ \|B[u_m]\|_V \leq c \|u_m\|_{L^\infty(0, T; H)} \|u_m\|_H \text{ with } c > 0. \] (47)

Moreover, operator \( A \) is continuous. Hence \( \{u_m'\} \) is bounded in \( \mathcal{Y}^* \). Thus, by passing to a next subsequence, if necessary, we have the following:
\[ u'_m \to u' \text{ weakly in } \mathcal{H}. \] (48)

Using the facts that \( \mathcal{H} \subset C(0, T; H) \) continuously, \( \mathcal{Y} \subset \mathcal{H} \) compactly, and \( \mathcal{Y} \subset L^2(0, T; L^2(\Gamma, \mathbb{R}^n)) \) compactly, we have \( u \in C(0, T; H) \) and
\[ u_m \to u \text{ in } \mathcal{H}, \]
\[ y(u_m) \to y(u) \text{ in } L^2(0, T; L^2(\Gamma, \mathbb{R}^n)). \] (49)

Since \( u_m \to u \text{ weakly in } \mathcal{Y} \) and in \( \mathcal{H} \), analogously as in Ahmed [39], we have \( B[u_m] \to B[u] \text{ weakly in } \mathcal{Y}^* \). We remark that if \( d = 3 \), we also have the convergence of \( B[u_m] \to B[u] \text{ weakly in } \mathcal{Y}^* \) by a compactness embedding theorem as in [39].

Let \( \phi \in C^\infty_0(0, T) \) and \( v \in V \cap L^\infty(0, T; H). \) Then, there exists \( \{v_m\}_{m \in \mathbb{N}} \) such that \( v_m \in V \) and \( v_m \to v \text{ in } \mathcal{Y} \), as \( m \to \infty \).

Denoting \( v_m(x, t) = \phi(t)v_m(x) \) and \( g(x, t) = \phi(t)g(x) \), we have \( v_m \to v \text{ in } \mathcal{Y} \). From (36), we have the following:
\[ \int_0^T \left\langle u_m'(t) + A u_m(t) + B[u_m(t)], v \right\rangle dt + \int_{(0, T) \times \partial \Omega} \theta_m(u_m(t)) \psi_{m} \, d\sigma dt = \int_0^T \langle f(t), \psi_m(t) \rangle dt. \] (50)

Using the above convergences, letting \( m \to +\infty \), we obtain
\[ \int_0^T \left\langle u'(t) + A u(t) + B[u(t)], v \right\rangle dt + \int_{(0, T) \times \partial \Omega} \kappa \cdot v_m \, d\sigma dt = \int_0^T \langle f(t), v \rangle dt. \] (51)

Since \( \phi \) is arbitrary, we deduce that
\[ \left\langle u'(t) + A u(t) + B[u(t)], v \right\rangle + \int_{(0, T) \times \partial \Omega} \kappa \cdot v_m \, d\sigma = \langle f(t), v \rangle, \] (52)
for a.e. \( t \in (0, T) \) and for all \( v \in V \cap L^\infty(0, T; H) \).

In order to complete the proof, it will be shown that
\[ \kappa(t, z) \in \partial \{u_m(t, z)\} = \partial \phi(u_m), \text{ for } \text{a.e. } (t, z) \in [0, T] \times \partial \Omega. \] (53)

Since \( \gamma(u_m) \to \gamma(u) \text{ in } L^2(0, T; L^2(\partial \Omega)) \), we obtain \( u_m \to u \) in \( L^2(0, T; L^2(\partial \Omega)) \), and consequently, \( u_m \to u \) for a.e. \( (t, x) \in [0, T] \times \partial \Omega \); then, by applying Egoroff’s theorem, we can find that for any \( a > 0 \), we can determine \( \omega \subset [0, T] \times \partial \Omega \) with \( \sigma(\omega) < a \), such that
\[ u_m \to u \text{ uniformly on } [0, T] \times \partial \Omega \setminus \omega, \] (54)
with \( u \in L^\infty([0, T] \times \partial \Omega \setminus \omega) \). Thus, for any \( a > 0 \), we can find \( \omega \subset [0, T] \times \partial \Omega \) with \( \sigma(\omega) < a \), such that for any \( \mu > 0 \) and for \( e < e_0 < \mu/2 \) and \( n > n_0 > 2/\mu \), we have
\[ |u_m - u|_{\omega} < \frac{\mu}{2}, \text{ on } [0, T] \times \partial \Omega \setminus \omega. \] (55)

Consequently, one obtains
\[ \theta_e(u_m) \leq \text{ess sup } \theta(\xi) \leq \text{ess sup } \theta(\xi) \leq \text{ess sup } \theta(\xi) = \bar{\theta}_e(u_m). \] (56)

Analogously, we prove the inequality
\[ \bar{\theta}_e(u_m) = \text{ess inf } \theta(\xi) \leq \theta_e(u_m). \] (57)
We now take \( v \geq 0 \) a.e. on \([0, T] \times \partial \Omega \setminus \omega\) with \( v \in L^\infty([0, T] \times \partial \Omega \setminus \omega).\) This implies

\[
\int_{[0, T] \times \partial \Omega \setminus \omega} \theta_{\mu}(u_N) v d\sigma \leq \int_{[0, T] \times \partial \Omega \setminus \omega} \theta_{\mu}(u_N) v d\sigma \leq \int_{[0, T] \times \partial \Omega \setminus \omega} \theta_{\mu}(u_N) v d\sigma.
\]

(58)

Taking the limits as \( \varepsilon \to 0 \) and \( m \to \infty, \) we obtain

\[
\int_{[0, T] \times \partial \Omega \setminus \omega} \theta_{\mu}(u_N) v d\sigma \leq \int_{[0, T] \times \partial \Omega \setminus \omega} \kappa v d\sigma \leq \int_{\partial [0, T] \times \partial \Omega} \theta_{\mu}(u_N) v d\sigma,
\]

and as \( \mu \to 0^+ , \) we obtain

\[
\int_{[0, T] \times \partial \Omega \setminus \omega} \theta(u_N) v d\sigma \leq \int_{[0, T] \times \partial \Omega \setminus \omega} \kappa v d\sigma \leq \int_{\partial [0, T] \times \partial \Omega} \bar{\theta}(u_N) v d\sigma.
\]

(60)

Since \( v \) is arbitrary, we have

\[
\kappa \in \left[ \bar{\theta}(u_N), \bar{\theta}(u_N) \right] = \bar{\theta}(u_N),
\]

(61)

where \( \sigma(\partial \Omega) < \alpha \). For \( \alpha \) as small as possible, we obtain the result.

5. Optimal Control

In this section, we provide a result on the dependence of solutions with respect to the density of the external forces and use it to study the distributed parameter optimal control problem corresponding to it.

Let \( f \in L^2(0, T; V^*) \). Under \( H(\theta), \) we denote by \( S_{u_0}^f(f) \subset \mathcal{Y} \) the solution set corresponding to \( f \) of the problem (EHVI). That is, \( u \in \mathcal{W} \) and there exists \( \kappa \in \mathcal{L} \) such that \( u(0) = u_0 \in H, \kappa \in \partial j(u_N) = \bar{\theta}(u_N) \) and

\[
\left( u'(t) + \mathcal{A}u(t) + \mathcal{B}[u(t)], v \right) + \int_{\partial \Omega} \kappa \cdot v_N(z) d\sigma(z) = \langle f(t), v \rangle,
\]

(62)

for a.e. \( t \in [0, T] \) and all \( v \in V \cap L^\infty(\partial \Omega). \)

Lemma 9. Let \( f \in L^2(0, T; V^*). \) For every \( u \in S_{u_0}^f(f), \) there exist \( \delta_0, \delta_1 > 0 \) such that

\[
\int_{[0, T] \times \partial \Omega} \kappa \cdot u_N d\sigma dt \geq -\delta_0, \delta_1 T \sigma(\partial \Omega),
\]

(63)

where \( \delta_0 \) and \( \delta_1 \) are independent from \( u.\)

Proof. By definition of \( \bar{\theta}(u_N) \) and \( \theta_{\mu}(u_N), \) we have for every \( \varepsilon > 0, \) there exists \( \delta \) with \( |\mu| < \delta \) such that

\[
\theta_{\mu}(u_N) - \varepsilon \leq \bar{\theta}(u_N) \leq \theta_{\mu}(u_N) + \varepsilon,
\]

(64)

and there exists \( \tilde{\delta} \) with \( |\mu| < \tilde{\delta} \), such that

\[
\theta_{\mu}(u_N) - \varepsilon \leq \tilde{\theta}(u_N) \leq \theta_{\mu}(u_N) + \varepsilon.
\]

(65)

It follows that

\[
\theta_{\mu}(u_N) - \varepsilon \leq \kappa \leq \theta_{\mu}(u_N) + \varepsilon.
\]

(66)

That is,

\[
\text{ess inf } \theta(s) - \varepsilon \leq \kappa \leq \text{ess sup } \theta(s) + \varepsilon.
\]

(67)

Consistently,

\[
\sup_{u_0, \varepsilon} \kappa \leq \text{ess sup } \theta(s) + \varepsilon,
\]

\[
\inf_{u_0, \varepsilon} \kappa \geq \text{ess inf } \theta(s) - \varepsilon,
\]

(70)

where \( \delta_0 \) is defined in \( H(\theta). \) Thus, from \( H(\theta) \) we obtain

\[
\sup_{u_0, \varepsilon} \kappa \leq \varepsilon,
\]

(71)

It results in \( \kappa < \varepsilon \) if \( u_N < -\delta_0 - \delta_0 - \delta \) and \( \kappa \geq \varepsilon \) if \( u_N > \delta_0 + \delta \). Let \( \varepsilon \to 0^+ \) to arrive at \( \kappa \leq 0 \) if \( u_N < -\delta_0 \) and \( \kappa \geq 0 \) if \( u_N > \delta_0 \). Consequently, as \( \theta \in L^\infty(\mathbb{R}) \) and \( \kappa \in \partial (u_N), \) then in the case of \( |u_N| \leq \delta_0, \) we have

\[
\sup_{|u_N| \leq \delta_0} |\kappa| \leq \text{ess sup } |\theta(s)| = \delta_1.
\]

(72)

It follows that

\[
\int_{[0, T] \times \partial \Omega} \kappa \cdot u_N d\sigma dt = \int_{|u_0| < \delta_1} \kappa \cdot u_N d\sigma dt + \int_{|u_0| > \delta_1} \kappa \cdot u_N d\sigma dt
\]

\[
+ \int_{|u_0| < \delta_0} \kappa \cdot u_N d\sigma dt \geq -\delta_0 \int_{|u_0| < \delta_0} \kappa \cdot u_N d\sigma dt
\]

\[
\cdot |\kappa| d\sigma dt \geq -\delta_0 \delta_1 T \sigma(\partial \Omega).
\]

(73)

Theorem 10. Under \( H(\theta), \) assume that \( f_m \in L^2(0, T; V^*) \) such that \( f_m \to f \) weakly in \( \mathcal{Y} \). Let \( \{u_m\}_m \subset \mathcal{W} \) be a
sequence such that \( u_m \in S^0(u_m) \) for each \( m \in \mathbb{N} \); then, we can find a subsequence (still denoted with the same symbol) such that \( u_m \rightharpoonup u \) weakly in \( V \) and \( u \in S^0(u) \).

**Proof.** Let \( f_m, f \in V^* \) with \( f_m \rightharpoonup f \) weakly in \( V^* \). Let \( \{u_m\}_m \) be a sequence such that \( u_m \in S^0(u_m) \) for each \( m \in \mathbb{N} \); then, by Theorem 8, there exists \( \kappa_m \in L^1([0, T] \times \partial \Omega) \), such that \( \kappa_m(t, x) \in \partial j(u_m(t, x)) \) for a.e. \( (t, x) \in [0, T] \times \partial \Omega \) and

\[
\left\langle u'_m(t) + \partial A u_m(t) + \partial B[u_m(t), v] \right\rangle + \int_{\partial \Omega} \kappa_m v_N \, d\sigma = \left\langle f_m(t), v \right\rangle,
\]

for a.e. \( t \in [0, T] \) and all \( v \in V \cap L^2_N(\partial \Omega) \). With the same calculations as in the last section, one obtains

\[
\frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + M \|u_m(t)\|^2 \leq \frac{2}{M} \|f_m(t)\|_V^2 + \frac{M}{2} \|u(t)\|^2 - 2 \int_{\partial \Omega} \kappa_m u_N \, d\sigma.
\]

(74)

It follows that

\[
\frac{d}{dt} |u_m(t)|^2 + M \|u_m(t)\|^2 \leq \frac{4}{M} \|f_m(t)\|_V^2 - 2 \int_{\partial \Omega} \kappa_m u_N \, d\sigma.
\]

(75)

Integrating over \((0, t)\), we get

\[
|u_m(t)|^2 + M \int_0^t \|u_m(s)\|^2 \, ds \leq \frac{4}{M} \|f_m(t)\|_V^2 + |u(0)|^2 + 2\delta \delta'_1 T \sigma(\partial \Omega).
\]

(76)

It follows that \( \{u_m\}_m \) is bounded in \( L^\infty(0, T; H) \cap L^2(0, T; V) \). Hence, by passing to a subsequence if necessary, there exists \( u \) such that \( \{u_m\}_m \) converges to \( u \) weakly in \( L^2(0, T; V) \) and weakly \( \ast \) in \( L^\infty(0, T; V) \). Using the compactness of the trace operator \( \gamma \), we may assume that \( \gamma u_m \rightharpoonup \gamma u \) in \( L^2(0, T; H) \) and then \( \gamma u_m(t, z) \rightharpoonup \gamma u(t, z) \) for a.e. \( (t, z) \in [0, T] \times \partial \Omega \). Consequently, \( u_m(t, z) \rightharpoonup u(t, z) \) for a.e. \( (t, z) \in [0, T] \times \partial \Omega \). Let us show that there exists \( m_0 \in \mathbb{N} \) such that

\[
\partial j(u_m) \subset \partial j(u_N), \quad \text{for all } m \geq m_0.
\]

(77)

As \( u_m \rightarrow u_N \) for a.e. \( (t, z) \in [0, T] \times \partial \Omega \), we can find by Egoroff’s theorem that for any \( \alpha > 0 \), we can determine \( \omega \subset [0, T] \times \partial \Omega \) with \((dt \times \sigma)(\omega) < \alpha\), such that

\[
u_m \rightharpoonup \nu_N, \quad \text{uniformly on } [0, T] \times \partial \Omega \setminus \omega,
\]

(79)

with \( \nu_N \in L^\infty([0, T] \times \partial \Omega) \setminus \omega \). Thus, for any \( \mu > 0 \), there exists \( m_0 \), such that for all \( m > m_0 \), we have

\[
|u_m - \nu_N| < \frac{\mu}{2}, \quad \text{a.e. on } [0, T] \times \partial \Omega \setminus \omega.
\]

(80)

By using the triangle inequality, we have

\[
\overline{\theta}(u_m) = \sup_{|u_m - \nu_N| < \frac{\mu}{2}} \theta(\xi) \leq \sup_{|u_m - \nu_N| < \frac{\mu}{2}} \theta(\xi) = \overline{\theta}(u_N).
\]

(81)

Analogously, we prove the inequality

\[
\overline{\theta}(u_m) \leq \overline{\theta}(u_N).
\]

(82)

Taking the limit \( \mu \rightarrow 0^+ \), we obtain for each \( m \geq m_0 \)

\[
\kappa_m \in \partial \theta(u_m) \subset \partial \theta(u_N), \quad \text{a.e. on } [0, T] \times \partial \Omega \setminus \omega,
\]

(83)

where \((dt \times \sigma)(\omega) < \alpha\). For \( \alpha \) as small as possible, we obtain

\[
\kappa_m \in \partial \theta(u_m) \subset \partial \theta(u_N), \quad \text{a.e. on } [0, T] \times \partial \Omega \setminus \omega,
\]

(84)

from which we can conclude that the \( \{\kappa_m\}_m \) that follows is bounded. By the Dunford-Pettis theorem ([40], p. 239), we will show that the sequence \( \{\kappa_m\}_m \) is weakly precompact in \( L^1([0, T] \times \partial \Omega) \). For this end, we show that for each \( \mu > 0 \), there exists \( \delta > 0 \), such that for \( \omega \subset [0, T] \times \partial \Omega \), \((dt \times \sigma)(\omega) < \delta\):

\[
\int_\omega |\kappa_m| \, d\sigma < \mu.
\]

(85)

For some \( a > 0 \) and remarking that in \( \{|u_m| > a\} \), 1 < \( |u_m|/a \), one has

\[
\int_\omega |\kappa_m| \, d\sigma d\tau = \int_\omega \left| \kappa_m \right| 1_{\{|u_m| > a\}} \, d\sigma d\tau + \int_\omega \left| \kappa_m \right| 1_{\{|u_m| \leq a\}} \, d\sigma d\tau \\
\leq \frac{1}{a} \int_{[0, T] \times \partial \Omega} |\kappa_m| u_N \, d\sigma d\tau + \int_\omega \left| \kappa_m \right| 1_{\{|u_m| \leq a\}} \, d\sigma d\tau.
\]

(86)

From one hand, one has

\[
\int_{[0, T] \times \partial \Omega} |\kappa_m u_m| \, d\sigma d\tau = \int_{|u_m| > a} |\kappa_m u_m| \, d\sigma d\tau + \int_{|u_m| \leq a} |\kappa_m u_m| \, d\sigma d\tau
\]

\[
= \int_{|u_m| > a} |\kappa_m u_m| \, d\sigma d\tau - \int_{|u_m| < a} |\kappa_m u_m| \, d\sigma d\tau + 2 \int_{|u_m| \leq a} |\kappa_m u_m| \, d\sigma d\tau
\]

\[
\leq \int_{|u_m| > a} |\kappa_m u_m| \, d\sigma d\tau + \int_{|u_m| \leq a} |\kappa_m u_m| \, d\sigma d\tau + 2 \int_{|u_m| \leq a} |\kappa_m u_m| \, d\sigma d\tau
\]

\[
= \int_{[0, T] \times \partial \Omega} |\kappa_m u_m| \, d\sigma d\tau + 2 \int_{|u_m| \leq a} |\kappa_m u_m| \, d\sigma d\tau.
\]

(87)
From equation (74) with \( v = u_m(t) \)

\[
\int_{[0,T] \times \Omega} |\kappa_m u_{mN}| \sigma \, dt \leq \int_0^T \left( f(m(t), u_{m}(t)) \right) \, dt - \int_0^T \left( u_m'(t), u_{m}(t) \right) \, dt \\
- \int_0^T \left( \mathcal{A} u_m(t), u_{m}(t) \right) \, dt \\
+ 2 \int_{\{u_{mN} \leq 0\}} |\kappa_m u_{mN}| \sigma \, dt \\
\leq \int_0^T \left( f(m(t)) \, v - \|u_{m}(t)\|_V \right) \, dt - \frac{1}{2} |\mu(T)|^2 \\
+ \frac{1}{2} |\mu(0)|^2 - M \int_0^T \|u_{m}(t)\|^2 \, dt \, d\tau_\Omega \\
+ 2 \int_{\{u_{mN} \leq 0\}} |\kappa_m| \sigma \, dt \\
\leq c + 2 \delta_0 \sigma_1 T \sigma(\partial \Omega).
\]

(88)

On the other hand, for each \( \varepsilon > 0 \) there is \( a_\varepsilon > 0 \), such that for \( |v| < a_\varepsilon \) one has

\[
|\kappa_m| \leq \text{ess sup}_{|v|} |\theta(s)| + \varepsilon \leq \text{ess sup}_{|v|} |\theta(s)| + \varepsilon.
\]

(89)

This implies

\[
\sup_{|u_{mN}| \leq 0} |\kappa_m| \leq \text{ess sup}_{|v|} |\theta(s)| + \varepsilon.
\]

(90)

We choose for example \( \varepsilon = 1 \), which leads to

\[
\sup_{|u_{mN}| \leq 0} |\kappa_m| \leq \text{ess sup}_{|v|} |\theta(s)| + 1.
\]

(91)

Now we choose \( a \) such that

\[
\frac{1}{a} \int_{[0,T] \times \partial \Omega} |\kappa_m u_{mN}| \sigma \, dt \leq \frac{1}{a} (c + 2 \delta_0 \sigma_1 T \sigma(\partial \Omega)) < \frac{\mu}{2}.
\]

(92)

and \( \delta \) such that

\[
\text{ess sup}_{|v|} |\theta(s)| + 1 < \frac{\mu}{2\delta}.
\]

(93)

With this choice of \( \delta \) one have

\[
\int_\omega |\kappa_m| \frac{1}{|u_{mN}|} |\sigma| \, d\sigma \leq \sup_{|u_{mN}| \leq 0} |\kappa_m| (|\sigma| \times \sigma)(\omega) \\
\leq \text{ess sup}_{|v|} |\theta(s)| + 1 (|\sigma| \times \sigma)(\omega) \\
< \frac{\mu}{2\delta} \delta = \frac{\mu}{2}.
\]

(94)

It follows

\[
\int_\omega |\kappa_m| \sigma \, d\sigma \leq \frac{1}{a} \int_{\partial \Omega} |\kappa_m u_{mN}| \sigma + \int_\omega |\kappa_m| \frac{1}{|u_{mN}|} |\sigma| \, d\sigma < \frac{\mu}{2} + \frac{\mu}{2} = \mu.
\]

(95)

Consequently, we can extract from \( \{ \kappa_m \}_m \) a subsequence (denoted with the same symbol) that converges in \( L^1((0,T) \times \partial \Omega) \) to some \( \kappa \in L^1((0,T) \times \partial \Omega) \). By passing the limit in (74), we get

\[
\left( u'(t) + \mathcal{A} u(t) + \mathcal{B}[u(t)], v \right) + \int_{\partial \Omega} \kappa \nu \, d\sigma = \left( f(t), v \right),
\]

(96)

with

\[
\kappa \in \text{coinc} \big( \tilde{\theta}(u_N) \big) = \tilde{\theta}(u_N), \quad \text{a.e. on } [0,T] \times \partial \Omega.
\]

(97)

**Remark 11.** We will need Theorem 10 just for external forces in \( L^2(0,T;H) \). As in this situation, the duality between \( V \) and \( V^* \) coincides with the one on \( H \), and this will bring no more difficulties.

**Remark 12.** One can prove in the same way as in ([32], Theorem 5.1) that the solutions of (EHVI) are stable under the perturbation of \( \theta \).

In the remaining of this section, we will use the notation \( S(f) \) instead of \( S^0(f) \). We follow Migórski [34], and we let \( \mathcal{U} = L^2(0,T;H) \) be the space of controls and \( \mathcal{U}_{ad} \) a non-empty subset of \( \mathcal{U} \) consisting of admissible controls. Let \( \mathcal{F} : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R} \) be the objective functional we want to minimize. The control problem reads as follows:

\[
\left\{ \begin{array}{l}
\text{Find a control } \tilde{f} \in \mathcal{U}_{ad} \text{ and a state } \tilde{u} \in S(\tilde{f}) \text{ such that :} \\
\mathcal{F}(\tilde{f}, \tilde{u}) = \inf \{ \mathcal{F}(f,u) : f \in \mathcal{U}_{ad}, u \in S(f) \}.
\end{array} \right.
\]

(98)

A pair \( (\tilde{f}, \tilde{u}) \) which solves (98) is called an optimal solution. The existence of such optimal solutions can be proved by using Theorem 10. To do so, we need the following additional hypotheses:
(1) $H(\mathcal{U}_{ad}): \mathcal{U}_{ad}$ is a bounded and weakly closed subset of $\mathcal{U}$

(2) $H(\mathcal{F})$: $\mathcal{F}$ is lower semicontinuous with respect to $\mathcal{U} \times \mathcal{V}$ endowed with weak topology.

**Theorem 13.** Assume that $H(\theta)$, $H(\mathcal{U}_{ad})$, and $H(\mathcal{F})$ are fulfilled. Then problem (98) has an optimal solution.

**Proof.** Let $(f_m, u_m)$ be a minimizing sequence for problem (98), i.e., $f_m \in \mathcal{U}_{ad}$ and $u_m \in S(f_m)$, such that

$$\lim_{m \to \infty} \mathcal{F}(f_m, u_m) = \inf \{ \mathcal{F}(f, u): f \in \mathcal{U}_{ad}, u \in S(f) \} = 0.$$  \hfill (99)

It follows that the sequence $f_m$ belongs to a bounded subset of the reflexive Banach space $\mathcal{V}$. We may then assume that $f_m \to \tilde{f}$ weakly in $\mathcal{V}$ (by passing to a subsequence if necessary). By $H(\mathcal{U}_{ad})$, we have $\tilde{f} \in \mathcal{U}_{ad}$. From Theorem 10, we obtain, by again passing to a subsequence if necessary, that $u_m \to \tilde{u}$ weakly in $\mathcal{V}'$ with $\tilde{u} \in S(\tilde{f})$. By $H(\mathcal{F})$, we have $\theta \leq \mathcal{F}(\tilde{f}, \tilde{u}) \leq \liminf_{m \to \infty} \mathcal{F}(f_m, u_m) = 0$. Which completes the proof.

Next we apply Theorem 13 in a concrete example. Let $X$ be another Hilbert space, $\mathcal{X} = L^2(0, T; X)$, $\mathcal{U}_{ad}$ the set of admissible controls, and $C \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ a bounded linear operator from $\mathcal{X}$ to $\mathcal{U}$. Let $f \in L^2(0, T; V^*)$, as we aim to study the following optimal control problem

$$\begin{cases} 
\text{Find a control } w \in \mathcal{U}_{ad} \text{ and a state } \tilde{u} \in S(C\tilde{w} + f) \text{ such that: } \\
\mathcal{J}(\tilde{w}, \tilde{u}) = \inf \{ \mathcal{J}(w, u): w \in \mathcal{U}_{ad}, u \in S(Cw + f) \},
\end{cases}$$  \hfill (100)

where the objective functional is given by

$$\mathcal{J}(w, u) = \int_0^T \int_\Omega (u(t, x) - z(t, x))^2 \, dx \, dt + \int_0^T h(w(t)) \, dt,$$  \hfill (101)

for some function $h: X \to \mathbb{R}$ and $z \in L^2(0, T; \mathcal{H})$. Such optimal control problems arise in a wide range of applications, particularly in fluid flow control. More specifically, one tries to act on the flow in such a way that a certain flow profile is stabilized or enforced by devices like actuators. Also sensors are used to provide necessary information for the actuation measured here by the control input operator $C$.

Our goal is to minimize the discrepancy between the ideal velocity profile $z$ and the actual flow $u$. Moreover, the cost related to the actuators and the sensors should be also minimized. A more sophisticated example of this framework is the blood flow in an artificial heart. The goal will be to avoid, among other things, the stagnation causing some serious hydromechanical problems.

Let us first announce the following corollaries of Theorem 10.

**Corollary 14.** Under $H(\theta)$ assume that $\varphi_m$, $\varphi$, and $f \in \mathcal{V}^*$, such that $\varphi_m \to \varphi$ weakly in $\mathcal{V}^*$. Then, for every $u_m \in S(\varphi_m + f)$, we can find a subsequence (still denoted with the same symbol), such that $u_m \to u$ in $\mathcal{V}$ and $u \in S(\varphi + f)$.

**Proof.** It suffices to take $f_m = \varphi_m + f$ in Theorem 10.

**Corollary 15.** Under $H(\theta)$, assume that $f \in \mathcal{V}^*$ and $u_m$, $w \in \mathcal{X}$ are such that $u_m$ converges weakly to $w$ in $\mathcal{X}$. Then, for every sequence $(u_m)_m$, such that $u_m \in S(Cw_m + f)$, we can find a subsequence that converges weakly in $L^2(0, T; V)$ to $w \in S(Cw + f)$.

**Proof.** It suffices to take $\varphi_m = Cw_m$ in Corollary 14.

Assume the following:

(i) $f \in L^2(0, T, V^*)$ and $z \in L^2(0, T; H)$

(ii) $\mathcal{X}_{ad}$ is a weakly compact subset of $\mathcal{X}$

(iii) The function $h: X \to \mathbb{R}$ is convex, lower semicontinuous, and satisfies the coercivity condition

$$|h(w)| \geq \alpha |w|^2_X + \beta,$$  \hfill (102)

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. $|\cdot|_X$ stands for the norm of the Hilbert space $X$.

**Theorem 16.** If hypotheses (i)-(iii) and $H(\theta)$ hold, then problem (100) has an optimal solution.

**Proof.** Let $(u_m, f_m)$ be a minimizing sequence to problem (100), i.e., $u_m \in \mathcal{X}_{ad}$ and $u_m \in S(Cu_m + f)$. Then

$$\lim_{m \to \infty} \mathcal{J}(u_m, f_m) = \inf \{ \mathcal{J}(w, u): w \in \mathcal{X}_{ad}, u \in S(Cw + f) \}.$$  \hfill (103)

Denote $f_m = Cu_m + f$, $\mathcal{F}(f_m, u_m) = \mathcal{J}(u_m, f_m)$, and $\mathcal{U}_{ad} = C\mathcal{X}_{ad}$. It suffices now to apply Theorem 13 for $\mathcal{U}_{ad}$ and $\mathcal{F}$.

### 6. Directional Growth Condition

As mentioned in the Introduction, the Rauch condition is a particular case of the directional growth condition due to Naniewicz [31]. It is of common knowledge that the foregoing mentioned conditions are sufficient to establish the existence of solution without any additional growth hypothesis on $j$. The notion of being a solution needs only to be modified. Here, we will reconsider the same problem of the evolutionary hemivariational Navier-Stokes equations but with the more general condition of directional growth.

Let $j: \partial \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function with respect to the first argument and locally Lipschitz with respect to the second argument. We assume the following:
(1) $H(j)$: there exists $\beta: \partial \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ integrable with respect to the first argument and nondecreasing with respect to the second argument such that

$$|j(x, \xi) - j(x, \eta)| \leq \beta(x, r)|\xi - \eta|, \quad \forall \xi, \eta \in B(0, r), \ r \geq 0.$$  \hfill (104)

(2) $H(\beta')$: there exists a function $\alpha: \partial \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ square-integrable with respect to the first argument and nondecreasing with respect to the second argument such that the following estimate holds

$$j'(x; \xi, \eta - \xi) \leq \alpha(x, r)(1 + |\eta|),$$  \hfill (105)

for almost every $x \in \partial \Omega$ and for any $\xi, \eta \in \mathbb{R}$ with $-r \leq \eta \leq r$, $r \geq 0$.

Remark that if $j$ does not depend on $x \in \partial \Omega$, then it satisfies $H(j)$ automatically. The hypothesis $H(\beta')$ is called the directional growth condition.

For $x \in \partial \Omega$ and $\xi \in \mathbb{R}$, define $j_j(x, \xi) = h_j + j(x_0, \xi)$ and denote by $j_j'$ the derivative of $j_j$ with respect to the second argument. As usual, let $\{\varphi_1, \varphi_2, \cdots\}$ be a basis in $V \cap L_0^{\infty}(\partial \Omega)$ and $V_m = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$. We then consider the following regularized problem of the Galerkin type associated with (EHVI), noted $(\mathcal{P}_m^\varepsilon)$: find $u_m \in \mathcal{W}^\varepsilon$ such that $u_m(0) = u_{0m}$ and

$$\left\langle u_m'(t) + \partial \Omega u_m(t) + \mathcal{B}[u_m(t)], \nu \right\rangle + \int_{\partial \Omega} j_j'(u_m(x), \nu_x) \cdot \nu_x \, d\sigma = (f(t), \nu),$$  \hfill (106)

for all $t \in (0, T)$ and all $\nu \in V_m$.

Note that due to the integrability of $\beta$ with respect to the first argument and $H(j)$, the integral above is finite for each $u_m, \nu \in V_m$. In fact, we have

$$\left|j_j'(x; u_m(x), \nu_x) \cdot \nu_x \right| \leq \beta(x, \|u_m\|_{L^{\infty}(\partial \Omega)} + 1) \|\nu_x\|_{L^{\infty}(\partial \Omega)}.$$  \hfill (107)

Since $\beta(\|u\|_{L^{\infty}(\partial \Omega)} + 1) \in L^1(\partial \Omega)$, the integrability of $j_j'(u_m(x), \nu_x)$ over $\partial \Omega$ follows immediately for any $\nu \in V_m$. We have the following lemma (cf. [28], Lemma 3.1).

**Lemma 17.** Suppose that $H(\beta')$ holds. Then the estimate

$$j_j'(x; \xi, \eta - \xi) \leq \alpha(x, r)(1 + |\xi|), \quad 0 < \alpha < 1,$$  \hfill (108)

is valid for any $\xi, \eta \in \mathbb{R}$ with $-r \leq \eta \leq r$, $r \geq 0$, and almost all $x \in \partial \Omega$, where $\alpha(x, r) = 2\alpha(x, r + 1)$.

The problem $(\mathcal{P}_m^\varepsilon)$ has at least one solution in $V_m$. In fact, substitution of $u_m(t) = \sum_{k=1}^m c_{km}(t)\varphi_k$ gives an initial value problem for a system of first order ordinary differential equations for $c_{km}(\cdot)$, $k = 1, 2, \cdots, m$. Its solvability on some interval $(0, t_m)$ follows from the Carathéodory theorem. This solution can be extended on the closed interval $[0, T]$ by using the a priori estimates below.

Using the coercivity of $\mathcal{A}$, the properties of $\mathcal{B}[\cdot]$, and the Young inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{H} + M \|u_m(t)\|^2_{V} + \int_{\partial \Omega} j_j'(u_m(x), \nu_x) \cdot u_m(x) \, d\sigma \leq \frac{M}{2} \|u_m(t)\|^2_{V} + \frac{2}{M} \|f(t)\|^2_{V^*}. $$  \hfill (109)

From Lemma 17, we have

$$j_j'(u_m(x), \nu_x) \cdot u_m(x) \leq -\alpha(x, 0)(1 + |u_m(x)|).$$  \hfill (110)

Then

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{H} + M \|u_m(t)\|^2_{V} \leq \frac{2}{M} \|f(t)\|^2_{V^*} + \int_{\partial \Omega} \alpha(x, 0)(1 + |u_m(x)|) \, d\sigma \leq \frac{2}{M} \|f(t)\|^2_{V^*} + \sigma(\partial \Omega)^{\frac{1}{2}} \|\alpha(0)\|_{L^\infty(\partial \Omega)}.$$  \hfill (111)

Integrating over $(0, t)$, we get

$$\frac{1}{2} \|u_m(t)\|_{H} - \frac{1}{2} \|u_m(0)\|_{H} + M \int_0^t \|u_m(s)\|^2_{V} \, ds \leq \frac{2}{M} \|f(t)\|^2_{V^*} + \sigma(\partial \Omega)^{\frac{1}{2}} \|\alpha(0)\|_{L^\infty(\partial \Omega)}.$$  \hfill (112)

It follows

$$\frac{1}{2} \|u_m(t)\|_{H} \leq \frac{M}{2} \|u_m\|^2_{V} + \frac{2}{M} \|f\|^2_{V^*} - \sigma(\partial \Omega)^{\frac{1}{2}} \|\alpha(0)\|_{L^\infty(\partial \Omega)}.$$  \hfill (113)

It follows that $\{u_m\}_m$ is a bounded subset of $\mathcal{W}^\varepsilon$ and $L^{\infty}(0, T; H)$, so passing to subsequence, if necessary, we have

$$u_m \rightharpoonup u \text{ weakly in } \mathcal{W}^\varepsilon \text{ and weakly }^* \text{ in } L^{\infty}(0, T; H).$$  \hfill (114)

Following the same procedure as in section 4, see also [25], we have $u_m' \rightharpoonup u' \text{ weakly in } \mathcal{W}$ and $\mathcal{B}[u_m] \rightharpoonup \mathcal{B}[u] \text{ weakly in } \mathcal{W}^\varepsilon$. Using the same proof as in ([31], Lemma
exists.\[\delta\epsilon_m\] to the limit as usual to obtain

\[\int j_{\epsilon_m}^T(u_{mN}) \rightarrow \kappa \text{ weakly in } L^1((0, T) \times \partial\Omega). \quad (115)\]

Moreover, the following equality holds

\[\left\langle u_m'(t) + \partial u_m(t) + \mathcal{B}[u_m(t)], v \right\rangle + \int_{\partial\Omega} j_{\epsilon_m}^T(u_{mN}) \cdot v_N d\sigma = (f(t), v), \quad (116)\]

for almost every \( t \in (0, T) \) and \( v \in V \cap L^\infty(\partial\Omega) \). We pass to the limit as usual to obtain

\[\left\langle u_t'(t) + \partial u_t(t) + \mathcal{B}[u_t(t)], v \right\rangle + \int_{\partial\Omega} \kappa(t) \cdot v_N d\sigma = (f(t), v), \quad (117)\]

for almost every \( t \in [0, T] \) and \( v \in V \cap L^\infty(\partial\Omega) \).

We still need to prove that \( \kappa(t, x) \in \partial j(x, u_t(t, x)) \) for almost every \( t \in (0, T) \) and \( x \in \partial\Omega \). Since \( \gamma u_m \rightarrow \gamma u \) in \( L^2(0, T; L^2(\partial\Omega; \mathbb{R}^d)) \), we obtain that \( u_{mN} \rightarrow u_N \) in \( L^2(0, T; L^2(\partial\Omega)) \) and consequently for almost every \( t \in (0, T) \):

\[u_{mN}(t, x) \rightarrow u_N(t, x) \quad \text{a.e. } x \in \partial\Omega. \quad (118)\]

By Egoroff’s theorem, with respect to \( x \in \partial\Omega \), we have for any \( \rho > 0 \), a subset \( \omega \) of \( \Gamma \) with \( u_{mN} \rightarrow u_N \) uniformly on \( \partial\Omega \setminus \omega \),

\[u_{mN} \rightarrow u_N \quad (119)\]

with \( u_N \in L^\infty(\partial\Omega \setminus \omega) \). Let \( v \in L^\infty(\partial\Omega \setminus \omega) \) be arbitrarily given. Due to Fatou’s lemma, for any positive \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) and \( N_\epsilon \) such that

\[
\int_{\partial\Omega \setminus \omega} \frac{j(x, u_{mN}(x) - \tau + \gamma v_N(x)) - j(x, u_m(x) - \tau)}{\nu} d\sigma \\
\leq \int_{\partial\Omega \setminus \omega} \frac{\tilde{j}(x, u_N(x) ; v_N(x))}{\nu} d\sigma + \epsilon, \quad (120)
\]

provided \( m > N_\epsilon, |\tau| < \delta_\epsilon, \) and \( 0 < \nu < \delta_\epsilon \). This inequality multiplied by \( h_{\epsilon_m} \) and integrated over \( \mathbb{R} \) yields

\[
\int_{\partial\Omega \setminus \omega} j_{\epsilon_m}^T(x, u_{mN}(x) + \gamma v_N(x)) - j_{\epsilon_m}^T(x, u_m(x)) d\sigma (x) \\
= \int_{\mathbb{R}} h_{\epsilon_m}(\tau) \int_{\partial\Omega \setminus \omega} \frac{j(x, u_{mN}(x) - \tau + \gamma v_N(x)) - j(x, u_m(x) - \tau)}{\nu} d\sigma d\tau \\
\leq \int_{\partial\Omega \setminus \omega} \tilde{j}(x, u_N(x) ; v_N(x)) d\sigma + \epsilon. \quad (121)
\]

But as \( \nu \rightarrow 0 \), we get

\[
\int_{\partial\Omega \setminus \omega} j_{\epsilon_m}^T(x, u_N(x)) \cdot v_N(x) d\sigma \leq \int_{\partial\Omega \setminus \omega} \tilde{j}(x, u_N(x) ; v_N(x)) d\sigma + \epsilon, \quad (122)
\]

which is valid for \( m > N_\epsilon \). Now letting \( m \rightarrow \infty \), we are led to

\[
\int_{\partial\Omega \setminus \omega} \kappa \cdot v_N d\sigma \leq \int_{\partial\Omega \setminus \omega} \tilde{j}(x, u_N(x) ; v_N(x)) d\sigma + \epsilon. \quad (123)
\]

Since \( \epsilon > 0 \) was chosen arbitrarily

\[
\int_{\partial\Omega \setminus \omega} \kappa \cdot v_N d\sigma \leq \int_{\partial\Omega \setminus \omega} \tilde{j}(x, u_N(x) ; v_N(x)) d\sigma, \quad \text{for all } v \in L^\infty(\partial\Omega \setminus \omega). \quad (124)
\]

But the last inequality easily implies that

\[
\kappa(x) \in \partial j(x, u_N(x)), \quad \text{for a.e. } x \in \partial\Omega \setminus \omega, \quad (125)
\]

where \( \sigma(x) < \rho \). Now since \( \rho \) was chosen arbitrarily

\[
\kappa(x) \in \partial j(x, u_N(x)), \quad \text{for a.e. } x \in \partial\Omega, \quad (126)
\]

which completes the proof.

**Remark 18.** The directional growth condition is meant to study problems involving vector valued functions, i.e., functions on \( \mathbb{R}^N \). Our situation is simpler as \( N = 1 \). In this case, the directional growth condition can be simplified to the following condition

\[
\tilde{j}(x, \xi, -\xi) \leq k(x)|\xi|, \quad \forall \xi \in \mathbb{R}, \text{ for a.e. } x \in \partial\Omega, \quad (127)
\]

for some nonnegative function \( k \in L^2(\partial\Omega) \) (cf. [31], Remark 4.1). Moreover, the Rauch condition and sign condition also fulfill the estimate (127) (cf. [31], Remark 4.7).

**Remark 19.** It is an easy task to check that the results in section 5, regarding optimal solution, are also valid if one replaces the assumption \( H(\theta) \) by the more general assumption \( H(\tilde{j}) \).

**Data Availability**

There is no data needed in our manuscript.

**Conflicts of Interest**

The authors declare that they have no conflict of interest.

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