A local form for the automorphisms of the spectral unit ball

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ABSTRACT

If $F$ is an automorphism of $\Omega_n$, the $n^2$-dimensional spectral unit ball, we show that, in a neighborhood of any cyclic matrix of $\Omega_n$, the map $F$ can be written as conjugation by a holomorphically varying non singular matrix. This provides a shorter proof of a theorem of J. Rostand, with a slightly stronger result.

1. Background

Let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $sp(A)$ the spectrum of $A$. The spectral ball $\Omega_n$ is the set

$$\Omega_n := \{ A \in \mathcal{M}_n : \forall \lambda \in sp(A), |\lambda| < 1 \}.$$ 

Let $F$ be an automorphism of $\Omega_n$, that is to say, a biholomorphic map of the spectral ball into itself. Ransford and White [6] proved that, by composing with a natural lifting of a Möbius map of the disk, one could reduce oneself to the case where $F(0) = 0$, and that in that case the linear map $F'(0)$ was a linear automorphism of $\Omega_n$, so that by composing with its inverse, one is reduced to the case $F(0) = 0$, $F'(0) = I$ (the identity map). We then say that the automorphism if normalized. Ransford and White [6] proved that such automorphisms preserve the spectrum of matrices.

We say that two matrices $X, Y$ are conjugate if there exists $Q \in \mathcal{M}_n^{-1}$ such that $X = Q^{-1}YQ$.

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Baribeau and Ransford [1] (see also [2] for a more elementary proof) proved that every spectrum-preserving $C^1$-diffeomorphism of an open subset of $\mathcal{M}_n$, and thus every normalized automorphism of the spectral ball is a pointwise conjugation:

$$F(X) = Q(X)^{-1}XQ(X).$$

(1)

Rostand’s contribution [7] was to show that $Q(X)$ could be chosen locally holomorphically in a neighborhood of every $X$ admitting $n$ distinct eigenvalues.

We will give a short proof of a slightly stronger result: the exceptional set of matrices where the local holomorphic choice cannot be guaranteed will be of complex codimension 2 instead of 1.

The motivation for this result was a conjecture formulated in [6] about the automorphisms of the spectral ball, which reduces to asking whether any normalized automorphisms can be written in the form (1), where $Q$ would be globally holomorphic on $\mathcal{O}_n$, and depend only on the conjugacy class of $X$. Notice that a recent result of Zwonek [8] shows that any proper map of the spectral ball to itself is actually an automorphism of it, so that the proof of the Ransford-White conjecture would yield a description of all the proper maps of the spectral ball into itself.

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2. Statement

**Definition 1**  We say that a matrix $M$ is cyclic (or non-derogatory) if there exists a cyclic vector for $M$, i.e. $v \in \mathbb{C}^n$ such that $(v, Mv, \ldots, M^{k-1}v, \ldots)$ spans $\mathbb{C}^n$, which is equivalent to the fact that $(v, Mv, \ldots, M^{n-1}v)$ is a basis of $\mathbb{C}^n$.

Many equivalent definitions of this notion can be found, for instance in [3, 4], or [5, Proposition 3]. We point one out: $M$ is cyclic if and only if for any $\lambda \in \mathbb{C}$, \(\dim \ker(M - \lambda I_n) \leq 1\). In particular, any matrix with $n$ distinct eigenvalues is cyclic, and for any given spectrum $\lambda_1, \ldots, \lambda_n$, the set of non-cyclic matrices with that spectrum is the algebraic set

$$\{ M : \exists j : \dim \ker(M - \lambda_j I_n) \geq 2 \}.$$

Hence the set of non-cyclic matrices is of codimension 1 in the set of matrices which admit at least one multiple eigenvalue, itself of codimension 1 in $\mathcal{M}_n$.

**Theorem 2**

Let $F$ be a spectrum-preserving holomorphic map of $\Omega_n$. Let $X_0 \in \Omega_n$ be a cyclic matrix. Then there exists a neighborhood $V_{X_0}$ of $X_0$ and a map $Q$ holomorphic from $V_{X_0}$ to $\mathcal{M}_n^{-1}$ such that for any $X \in V_{X_0}$, $F(X) = Q(X)^{-1}XQ(X)$. 

