COSETS OF AFFINE VERTEX ALGEBRAS INSIDE LARGER STRUCTURES

THOMAS CREUTZIG AND ANDREW R. LINSHAW

ABSTRACT. Given a finite-dimensional reductive Lie algebra $\mathfrak{g}$ equipped with a nondegenerate, invariant, symmetric bilinear form $B$, let $V_k(\mathfrak{g}, B)$ denote the universal affine vertex algebra associated to $\mathfrak{g}$ and $B$ at level $k$. Let $A_k$ be a vertex (super)algebra admitting a homomorphism $V_k(\mathfrak{g}, B) \to A_k$. Under some technical conditions on $A_k$, we characterize the commutant $C_k = \text{Com}(V_k(\mathfrak{g}, B), A_k)$ for generic values of $k$. We establish the strong finite generation of $C_k$ in the following cases: $A_k = V_k(g', B')$, $A_k = V_{k-l}(g', B') \otimes F$, and $A_k = V_{k-l}(g', B') \otimes V_l(g'', B'')$. Here $g'$ and $g''$ are finite-dimensional Lie (super)algebras containing $\mathfrak{g}$, equipped with nondegenerate, invariant, (super)symmetric bilinear forms $B'$ and $B''$ which extend $B$, $l \in \mathbb{C}$ is fixed, and $F$ is a free field algebra admitting a homomorphism $V_l(g, B) \to F$. Our approach is essentially constructive and leads to minimal strong finite generating sets for many interesting examples.

1. INTRODUCTION

Vertex algebras are a fundamental class of algebraic structures that arose out of conformal field theory and have applications in a diverse range of subjects. The coset or commutant construction is a standard way to construct new vertex algebras from old ones. Given a vertex algebra $\mathcal{V}$ and a subalgebra $\mathcal{A} \subset \mathcal{V}$, $\text{Com}(\mathcal{A}, \mathcal{V})$ is the subalgebra of $\mathcal{V}$ which commutes with $\mathcal{A}$. This was introduced by Frenkel and Zhu in [FZ], generalizing earlier constructions in representation theory [KP] and physics [GKO], where it was used to construct the unitary discrete series representations of the Virasoro algebra. Many examples have been studied in both the physics and mathematics literature; for a partial list see [AP, B-H, BFH, D-X, DLY, HLY, JLI, JLI]. Although it is widely believed that $\text{Com}(\mathcal{A}, \mathcal{V})$ will inherit properties of $\mathcal{A}$ and $\mathcal{V}$ such as rationality, $C_2$-cofiniteness, and strong finite generation, no general results of this kind are known.

In this paper we study cosets of the universal affine vertex algebra $V_k(\mathfrak{g}, B)$ associated to a reductive Lie algebra $\mathfrak{g}$ with a nondegenerate bilinear form $B$, inside a larger vertex algebra $A_k$ whose structure constants depend continuously on $k$. The main examples we have in mind are the following.

1. $A_k = V_k(g', B')$ where $g'$ is a finite-dimensional Lie (super)algebra containing $\mathfrak{g}$, and $B'$ is a nondegenerate, invariant (super)symmetric bilinear form on $\mathfrak{g}'$ extending $B$.

2. $A_k = V_{k-l}(g', B') \otimes F$ where $F$ is a free field algebra admitting a homomorphism $V_l(g, B) \to F$ for some fixed $l \in \mathbb{C}$ satisfying some mild restrictions. By a free field algebra, we mean any vertex algebra obtained as a tensor product of a Heisenberg algebra $\mathcal{H}(n)$, a free fermion algebra $\mathcal{F}(n)$, a $\beta\gamma$-system $\mathcal{S}(n)$ or a symplectic fermion algebra $\mathcal{A}(n)$.

3. $A_k = V_{k-l}(g', B') \otimes V_l(g'', B'')$. Here $g''$ is another finite-dimensional Lie (super)algebra containing $\mathfrak{g}$, equipped with a nondegenerate, invariant, (super)symmetric bilinear

Key words and phrases. affine vertex algebra; coset construction; commutant construction; orbifold construction; invariant theory; finite generation; $W$-algebra.
form $B''$ extending $B$. If $V_l(g'', B'')$ is not simple, we may replace $V_l(g'', B'')$ with its quotient by any ideal; of particular interest is the irreducible quotient $L_l(g'', B'')$.

We shall prove that $C_k$ is strongly finitely generated in cases (1) and (2) above for generic values of $k$, and in case (3) when $A_k = V_{k-l}(g', B') \otimes V_l(g'', B'')$, and both $k$ and $l$ are generic. We shall also prove this when $A_k = V_{k-l}(g', B') \otimes L_l(g'', B'')$ for certain non-generic values of $l$ in some interesting examples. These are the first general results on the structure of cosets, and our proof is essentially constructive. The key ingredient is a notion of deformable family of vertex algebras that was introduced by the authors in [CLII]. A deformable family $B$ is a vertex algebra defined over a certain ring of rational functions in a formal variable $\kappa$, and $B_\infty = \lim_{\kappa \to \infty}$ has a well-defined vertex algebra structure. A strong generating set for $B_\infty$ gives rise to a strong generating set for $B$ with the same cardinality. In the above examples, $C_k$ is a quotient of a deformable family $C$, and a strong generating set for $C$ gives rise to a strong generating set for $C_k$ for generic values of $k$. We will show that

$$C_\infty = \lim_{k \to \infty} \operatorname{Com}(V_k(g, B), A_k) \cong \mathcal{V}^G, \quad \mathcal{V} = \operatorname{Com}\left(\lim_{\kappa \to \infty} V_k(g, B), \lim_{k \to \infty} A_k\right),$$

where $G$ is a connected Lie group with Lie algebra $g$. Moreover, $\mathcal{V}$ is a tensor product of free field and affine vertex algebras and $G$ preserves each tensor factor of $\mathcal{V}$. The description of $C_k$ for generic values of $k$ therefore boils down to a description of the orbifold $\mathcal{V}^G$. This is an easier problem because $\mathcal{V}$ decomposes into a direct sum of finite-dimensional $G$-modules, whereas $C_k$ is generally not completely reducible as a $V_k(g, B)$-module.

Building on our previous work on orbifolds of free field and affine vertex algebras [LII, LIII, LIV, CLII], we will prove that for any vertex algebra $V$ which is a tensor product of free field and affine vertex algebras and any reductive group $G \subset \text{Aut}(V)$ preserving the tensor factors, $V^G$ is strongly finitely generated. The proof depends on a classical theorem of Weyl (Theorem 2.5A of [W]), a result on infinite-dimensional dual reductive pairs (see Section 1 of [KR] as well as related results in [DLM, Wal, Wall]), and the structure and representation theory of the vertex algebras $B^{\text{Aut}(B)}$ for $B = \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), A(n)$. Finally, we apply our general result to find minimal strong finite generating sets for $C_k$ in some concrete examples which have been studied previously in the physics literature.

In physics language, the tensor product of two copies of $C_k$ is the symmetry algebra of a two-dimensional coset conformal field theory of a Wess-Zumino-Novikov-Witten model. Minimal strong generating sets for many examples of coset theories have been suggested in the physics literature; see especially [B-H]. For example, such generating sets for $\operatorname{Com}(V_k(sp_{2n}), V_{k+1/2}(sp_{2n}) \otimes L_{-1/2}(sp_{2n}))$ and $\operatorname{Com}(V_k(so_n), V_{k-1}(so_n) \otimes L_1(so_n))^{Z/2Z}$ were conjectured in [B-H], and we prove these conjectures in Examples 7.1 and 7.8. In this notation, $L_l(g)$ denotes the simple affine vertex algebra of $g$ at level $l$. The problem of finding minimal strong generators is presently of interest in the conjectured duality of families of two-dimensional conformal field theories with higher spin gravity on three-dimensional Anti-de-Sitter space. Strong generators of the symmetry algebra of the conformal field theory correspond to higher spin fields, where the conformal dimension becomes the spin. The original higher spin duality [CG] involves cosets $\operatorname{Com}(V_k(sl_n), A_k)$, where $A_k = V_{k-1}(sl_n) \otimes L_1(sl_n)$. This case is discussed in Example 7.7. Example 7.9 is the algebra appearing in the $N = 1$ superconformal version of the higher spin duality [CHR1], and Example 7.3 proves a conjecture of that article on the structure of $\operatorname{Com}(V_k(sp_{2n}), V_{k-1/2}(osp(1|2n) \otimes S(n)))$. Example 7.6 is the symmetry algebra of the $N = 2$
supersymmetric Kazama-Suzuki coset theory on complex projective space [KS]. This family of coset theories is the conjectured dual to the full $N = 2$ higher spin supergravity [CHR2]. Finally, in Example 7.10 we find a minimal strong generating set consisting of 30 fields for the $W$-algebra associated to the $sl_3$ parafermion algebra.

2. VERTEX ALGEBRAS

We will assume that the reader is familiar with the basic notions in vertex algebra theory, which has been discussed from various points of view in the literature (see for example [B FBZ FLM FHL LZ K]). We will follow the notation in [CLII]. Let $\mathfrak{g}$ be a finite-dimensional, Lie (super)algebra, equipped with a (super)symmetric, invariant bilinear form $B$. The universal affine vertex (super)algebra $V_k(\mathfrak{g}, B)$ associated to $\mathfrak{g}$ and $B$ is freely generated by elements $X^\xi, \xi \in \mathfrak{g}$, satisfying the operator product expansions

$$X^\xi(z)X^\eta(w) \sim kB(\xi, \eta)(z - w)^{-2} + X^{[\xi, \eta]}(w)(z - w)^{-1}.$$ 

The automorphism group $\text{Aut}(V_k(\mathfrak{g}, B))$ is the same as $\text{Aut}(\mathfrak{g})$; each automorphism acts linearly on the generators $X^\xi$. If $B$ is the standardly normalized supertrace in the adjoint representation of $\mathfrak{g}$, and $B$ is nondegenerate, we denote $V_k(\mathfrak{g}, B)$ by $V_k(\mathfrak{g})$. We recall the Sugawara construction, following [KRW]. If $\mathfrak{g}$ is simple and $B$ is nondegenerate, we may choose dual bases $\{\xi\}$ and $\{\xi'\}$ of $\mathfrak{g}$, satisfying $B(\xi', \eta) = \delta_{\xi, \eta}$. The Casimir operator is $C_2 = \sum_\xi \xi \xi'$, and the dual Coxeter number $h'$ with respect to $B$ is one-half the eigenvalue of $C_2$ in the adjoint representation of $\mathfrak{g}$. If $k + h' \neq 0$, there is a Virasoro element

$$(2.1) \quad L^g = \frac{1}{2(k + h')} \sum_\xi :X^\xi X^{\xi'}:$$

of central charge $c = \frac{k - \text{dim}_{\mathfrak{g}}}{k + h'}$. This is known as the Sugawara conformal vector, and each $X^\xi$ is primary of weight one.

The Heisenberg algebra $\mathcal{H}(n)$ has even generators $\alpha^i, i = 1, \ldots, n$, satisfying

$$(2.2) \quad \alpha^i(z)\alpha^j(w) \sim \delta_{i,j}(z - w)^{-2}.$$ 

It has the Virasoro element $L^\mathcal{H} = \frac{1}{2} \sum_{i=1}^n :\alpha^i \alpha^i:$ of central charge $n$, under which $\alpha^i$ is primary of weight one. The automorphism group $\text{Aut}(\mathcal{H}(n))$ is isomorphic to the orthogonal group $O(n)$ and acts linearly on the generators.

The free fermion algebra $\mathcal{F}(n)$ has odd generators $\phi^i, i = 1, \ldots, n$, satisfying

$$(2.3) \quad \phi^i(z)\phi^j(w) \sim \delta_{i,j}(z - w)^{-1}.$$ 

It has the Virasoro element $L^\mathcal{F} = -\frac{1}{2} \sum_{i=1}^n :\phi^i \partial \phi^i:$ of central charge $-\frac{n}{2}$, under which $\phi^i$ is primary of weight $\frac{1}{2}$. We have $\text{Aut}(\mathcal{F}(n)) \cong O(n)$, and it acts linearly on the generators. Note that $\mathcal{F}(2n)$ is isomorphic to the bc-system $\mathcal{E}(n)$, which has odd generators $b^i, c^i, i = 1, \ldots, n$, satisfying

$$(2.4) \quad b^i(z)b^j(w) \sim \delta_{i,j}(z - w)^{-1}, \quad c^i(z)c^j(w) \sim \delta_{i,j}(z - w)^{-1},$$

$$b^i(z)c^j(w) \sim 0, \quad c^i(z)b^j(w) \sim 0.$$ 

The $\beta \gamma$-system $S(n)$ has even generators $\beta^i, \gamma^i, i = 1, \ldots, n$, satisfying

$$(2.5) \quad \beta^i(z)\gamma^j(w) \sim \delta_{i,j}(z - w)^{-1}, \quad \gamma^i(z)\beta^j(w) \sim -\delta_{i,j}(z - w)^{-1},$$

$$\beta^i(z)\beta^j(w) \sim 0, \quad \gamma^i(z)\gamma^j(w) \sim 0.$$
It has the Virasoro element $L^S = \frac{1}{2} \sum_{i=1}^n (\beta^i \partial \gamma^i : - : \beta^i \gamma^i : )$ of central charge $-n$, under which $\beta^i, \gamma^i$ are primary of weight $\frac{1}{2}$. The automorphism group $\text{Aut}(S(n))$ is isomorphic to the symplectic group $Sp(2n)$ and acts linearly on the generators.

The symmetric fermion algebra $\mathcal{A}(n)$ has odd generators $e^i, f^i, i = 1, \ldots, n$, satisfying

$$
e^i(z) f^j(w) \sim \delta_{i,j} (z - w)^{-2}, \quad f^j(z) e^i(w) \sim -\delta_{i,j} (z - w)^{-2},$$

$$e^i(z) e^j(w) \sim 0, \quad f^i(z) f^j(w) \sim 0.
$$

It has the Virasoro element $L^A = -\sum_{i=1}^n : e^i f^i :$ of central charge $-2n$, under which $e^i, f^i$ are primary of weight one. We have $\text{Aut}(\mathcal{A}(n)) \cong Sp(2n)$, and it acts linearly on the generators.

As a matter of terminology, we say that a vertex algebra $A$ is of type $W(d_1, \ldots, d_k)$ if $A$ has a minimal strong generating set consisting of an element in each weight $d_1, \ldots, d_k$.

**Filtrations.** A filtration $\mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots$ on a vertex algebra $A$ such that $A = \bigcup_{k \geq 0} \mathcal{A}_{(k)}$ is a called a good increasing filtration $\mathcal{L}$ if for all $a \in \mathcal{A}_{(k)}, b \in \mathcal{A}_{(l)}$, we have

$$a \circ_n b \in \begin{cases} \mathcal{A}_{(k+l)} & n < 0 \\ \mathcal{A}_{(k+l-1)} & n \geq 0. \end{cases}$$

Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $\text{gr}(A) = \bigoplus_{k \geq 0} \mathcal{A}_{(k)}/\mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$-graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on $A$. Moreover, $\text{gr}(A)$ has a derivation $\partial$ of degree zero, and we call such a ring a $\partial$-ring. For each $r \geq 1$ we have the projection

$$\phi_r : \mathcal{A}_{(r)} \to \mathcal{A}_{(r)}/\mathcal{A}_{(r-1)} \subset \text{gr}(A).$$

The key feature of $\mathcal{R}$ is the following reconstruction property $\mathcal{L}$. Let $\{a_i : i \in I\}$ be a set of generators for $\text{gr}(A)$ as a $\partial$-ring, where $a_i$ is homogeneous of degree $d_i$. In other words, $\{\partial^k a_i : i \in I, k \geq 0\}$ generates $\text{gr}(A)$ as a ring. If $a_i(z) \in \mathcal{A}_{(d_i)}$ satisfies $\phi_{d_i}(a_i(z)) = a_i$ for each $i$, then $A$ is strongly generated as a vertex algebra by $\{a_i(z) : i \in I\}$.

For any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and bilinear form $B$, $V_k(\mathfrak{g}, B)$ admits a good increasing filtration

$$V_k(\mathfrak{g}, B) \circ_0 \subset V_k(\mathfrak{g}, B) \circ_1 \subset \cdots, \quad V_k(\mathfrak{g}, B) = \bigcup_{j \geq 0} V_k(\mathfrak{g}, B) \circ_j,$$

where $V_k(\mathfrak{g}, B) \circ_j$ is spanned by iterated Wick products of the generators $X^{\xi_i}$ and their derivatives, of length at most $j$. We have a linear isomorphism $V_k(\mathfrak{g}, B) \cong \text{gr}(V_k(\mathfrak{g}, B))$, and an isomorphism of graded $\partial$-rings

$$\text{gr}(V_k(\mathfrak{g}, B)) \cong (\text{Sym} \bigoplus_{j \geq 0} V_j) \otimes (\bigwedge \bigoplus_{j \geq 0} W_j), \quad V_j \cong \mathfrak{g}_0, \quad W_j \cong \mathfrak{g}_1.$$

The $\partial$-ring structure on $(\text{Sym} \bigoplus_{j \geq 0} V_j) \otimes (\bigwedge \bigoplus_{j \geq 0} W_j)$ is given by $\partial x_j = x_{j+1}$ for $x \in V_j$ or $x \in W_j$, and the weight grading on $V_k(\mathfrak{g}, B)$ is inherited by $\text{gr}(V_k(\mathfrak{g}, B))$.

For $\mathcal{V} = \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), \mathcal{A}(n)$ we have good increasing filtrations $\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \cdots$, where $\mathcal{V}_{(j)}$ is spanned by iterated Wick products of the generators and their derivatives of length at most $j$. We have linear isomorphisms

$$\mathcal{H}(n) \cong \text{gr}(\mathcal{H}(n)), \quad \mathcal{F}(n) \cong \text{gr}(\mathcal{F}(n)), \quad \mathcal{S}(n) \cong \text{gr}(\mathcal{S}(n)), \quad \mathcal{A}(n) \cong \text{gr}(\mathcal{A}(n)).$$
and isomorphism of $\partial$-rings
\[
\text{gr}(\mathcal{H}(n)) \cong \text{Sym} \bigoplus_{j \geq 0} V_j, \quad \text{gr}(\mathcal{F}(n)) \cong \bigwedge \bigoplus_{j \geq 0} V_j
\]
(2.11)
\[
\text{gr}(\mathcal{S}(n)) \cong \text{Sym} \bigoplus_{j \geq 0} (V_j \oplus V_j^*), \quad \text{gr}(\mathcal{A}(n)) \cong \bigwedge \bigoplus_{j \geq 0} (V_j \oplus V_j^*),
\]
where $V_j \cong \mathbb{C}^n$ and $V_j^* \cong (\mathbb{C}^n)^*$. As above, the $\partial$-ring structure is given by $\partial x_j = x_{j+1}$ for $x \in V_j$ or $V_j^*$, and $\text{gr}(\mathcal{V})$ inherits the weight grading on $\mathcal{V}$.

Finally, for all the vertex algebras $\mathcal{V} = \mathcal{V}_k(g, B), \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), \mathcal{A}(n)$ these filtrations are $\text{Aut}(\mathcal{V})$-invariant. For any reductive group $G \subset \text{Aut}(\mathcal{V})$, we have linear isomorphisms $\mathcal{V}^G \cong \text{gr}(\mathcal{V}^G)$ and isomorphisms of $\partial$-rings $\text{gr}(\mathcal{V})^G \cong \text{gr}(\mathcal{V}^G)$.

3. DEFORMABLE FAMILIES

Following [CLI], we recall the notion of a deformable family of vertex algebras. Let $K \subset \mathbb{C}$ be a subset which is at most countable, and let $F_K$ denote the $\mathbb{C}$-algebra of rational functions in a formal variable $\kappa$ of the form $\frac{p(\kappa)}{q(\kappa)}$ where $\deg(p) \leq \deg(q)$ and the roots of $q$ lie in $K$. A deformable family will be a free $F_K$-module $B$ with the structure of a vertex algebra with coefficients in $F_K$. Vertex algebras over $F_K$ are defined in the same way as ordinary vertex algebras over $\mathbb{C}$. We assume that $B$ possesses a $\mathbb{Z}_{\geq 0}$-grading $B = \bigoplus_{m \geq 0} B[m]$ by conformal weight where each $B[m]$ is free $F_K$-module of finite rank. For $k \not\in K$, we have a vertex algebra
\[
B_k = B/(\kappa - k),
\]
where $(\kappa - k)$ is the ideal generated by $\kappa - k$. Clearly $\dim_{\mathbb{C}}(B_k[m]) = \text{rank}_{F_K}(B[m])$ for all $k \not\in K$ and $m \geq 0$. We have a vertex algebra $B_\infty = \lim_{\kappa \to \infty} B$ with basis $\{a_i | i \in I\}$, where $\{a_i | i \in I\}$ is any basis of $B$ over $F_K$, and $\alpha_i = \lim_{\kappa \to \infty} a_i$. By construction, $\dim_{\mathbb{C}}(B_\infty[m]) = \text{rank}_{F_K}(B[m])$ for all $m \geq 0$. The vertex algebra structure on $B_\infty$ is defined by
\[
\alpha_i \circ_n \alpha_j = \lim_{\kappa \to \infty} a_i \circ_n a_j, \quad i, j \in I, \quad n \in \mathbb{Z}.
\]
The $F_K$-linear map $\phi : B \to B_\infty$ sending $a_i \mapsto \alpha_i$ satisfies
\[
\phi(\omega \circ_n \nu) = \phi(\omega) \circ_n \phi(\nu), \quad \omega, \nu \in B, \quad n \in \mathbb{Z}.
\]
Moreover, all normally ordered polynomial relations $P(\alpha_i)$ among the generators $\alpha_i$ and their derivatives are of the form
\[
\lim_{\kappa \to \infty} \tilde{P}(a_i),
\]
where $\tilde{P}(a_i)$ is a normally ordered polynomial relation among the $a_i$’s and their derivatives, which converges termwise to $P(\alpha_i)$. In other words, suppose that
\[
P(\alpha_i) = \sum_j c_j m_j(\alpha_i)
\]
is a normally ordered relation of weight $d$, where the sum runs over all normally ordered monomials $m_j(\alpha_i)$ of weight $d$, and the coefficients $c_j$ lie in $\mathbb{C}$. Then there exists a relation
\[
\tilde{P}(a_i) = \sum_j c_j(\kappa) m_j(a_i)
\]
where $\lim_{\kappa \to \infty} c_j(\kappa) = c_j$ and $m_j(a_i)$ is obtained from $m_j(\alpha_i)$ by replacing $\alpha_i$ with $a_i$. 
Example 3.1 (Affine vertex superalgebras). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$, where $\dim(\mathfrak{g}_0) = n$ and $\dim(\mathfrak{g}_1) = 2m$. Suppose that $\mathfrak{g}$ is equipped with a nondegenerate, invariant, supersymmetric bilinear form $B$. Fix a basis $\{\xi_1, \ldots, \xi_n\}$ for $\mathfrak{g}_0$ and $\{\eta_1^\pm, \ldots, \eta_m^\pm\}$ for $\mathfrak{g}_1$, so the generators $X^{\xi_i}, X^{\eta_j^\pm}$ of $V_k(\mathfrak{g}, B)$ satisfy

$$
X^{\xi_i}(z)X^{\xi_j}(w) \sim \delta_{i,j} k(z-w)^{-2} + X^{[\xi_i, \xi_j]}(w)(z-w)^{-1};
X^{\eta_j^+}(z)X^{\eta_j^-}(w) \sim \delta_{i,j} k(z-w)^{-2} + X^{[\eta_j^+, \eta_j^-]}(w)(z-w)^{-1},
$$

\begin{equation}
X^{\xi_i}(z)X^{\eta_j^\pm}(w) \sim X^{[\xi_i, \eta_j^\pm]}(w)(z-w)^{-1},
\end{equation}

\begin{equation}
X^{\eta_j^\pm}(z)X^{\eta_j^\pm}(w) \sim X^{[\eta_j^\pm, \eta_j^\pm]}(w)(z-w)^{-1}.
\end{equation}

Let $\kappa$ be a formal variable satisfying $\kappa^2 = k$, and let $F = F_\kappa$ for $K = \{0\}$. Let $\mathcal{V}$ be the vertex algebra with coefficients in $F$ which is freely generated by $\{a^{\xi_i}, a^{\eta_j^\pm} | i = 1, \ldots, n, j = 1, \ldots, m\}$, satisfying

$$
a^{\xi_i}(z)a^{\xi_j}(w) \sim \delta_{i,j} (z-w)^{-2} + \frac{1}{\kappa} a^{[\xi_i, \xi_j]}(w)(z-w)^{-1},
$$

$$
a^{\eta_j^+}(z)a^{\eta_j^-}(w) \sim \delta_{i,j} (z-w)^{-2} + \frac{1}{\kappa} a^{[\eta_j^+, \eta_j^-]}(w)(z-w)^{-1},
\end{equation}

$$
a^{\xi_i}(z)a^{\eta_j^\pm}(w) \sim \frac{1}{\kappa} a^{[\xi_i, \eta_j^\pm]}(w)(z-w)^{-1},
$$

$$
a^{\eta_j^\pm}(z)a^{\eta_j^\pm}(w) \sim \frac{1}{\kappa} a^{[\eta_j^\pm, \eta_j^\pm]}(w)(z-w)^{-1}.
\end{equation}

For $k \neq 0$, we have a surjective vertex algebra homomorphism

$$
\mathcal{V} \rightarrow V_k(\mathfrak{g}, B), \quad a^{\xi_i} \mapsto \frac{1}{\sqrt{k}} X^{\xi_i}, \quad a^{\eta_j^\pm} \mapsto \frac{1}{\sqrt{k}} a^{\eta_j^\pm},
$$

whose kernel is the ideal $(\kappa - \sqrt{k})$, so $V_k(\mathfrak{g}, B) \cong \mathcal{V}/(\kappa - \sqrt{k})$. Then

\begin{equation}
\mathcal{V}_\infty = \lim_{k \rightarrow \infty} \mathcal{V} \cong \mathcal{H}(n) \otimes \mathcal{F}(m),
\end{equation}

and has even generators $a^{\xi_i}$ for $i = 1, \ldots, n$, and odd generators $e^{\eta_j^+}, e^{\eta_j^-}$ for $j = 1, \ldots, m$, satisfying

$$
a^{\xi_i}(z)a^{\xi_j}(w) \sim \delta_{i,j} (z-w)^{-2},
$$

$$
e^{\eta_j^+}(z)e^{\eta_j^-}(w) \sim \delta_{i,j} (z-w)^{-2}.
\end{equation}

Lemma 3.2 ([CL], Lemma 8.1). Let $K \subset \mathbb{C}$ be at most countable, and let $\mathcal{B}$ be a vertex algebra over $F_K$ as above. Let $U = \{a_{i} | i \in I\}$ be a strong generating set for $\mathcal{B}_\infty$, and let $T = \{a_{i} | i \in I\}$ be the corresponding subset of $\mathcal{B}$, so that $\phi(a_{i}) = a_{i}$. There exists a subset $S \subset \mathbb{C}$ containing $K$ which is at most countable, such that $F_S \otimes F_K \mathcal{B}$ is strongly generated by $T$. Here we have identified $T$ with the set $\{1 \otimes a_{i} | i \in I\} \subset F_S \otimes F_K \mathcal{B}$.

Corollary 3.3. For $k \notin S$, the vertex algebra $B_k = \mathcal{B}/(\kappa - k)$ is strongly generated by the image of $T$ under the map $\mathcal{B} \rightarrow B_k$.

If $U$ is a minimal strong generating set for $\mathcal{B}_\infty$ it is not clear in general that $T$ is a minimal strong generating set for $\mathcal{B}$, since there may exist relations of the form $\lambda(k)a_{j} = P$, where $P$ is a normally ordered polynomial in $\{a_{i} | i \in I, i \neq k\}$ and $\lim_{k \rightarrow \infty} \lambda(k) = 0$, although
\[ \lim_{k \to \infty} P \] is a nontrivial. However, there is one condition which holds in many examples, under which \( T \) is a minimal strong generating set for \( B \).

**Proposition 3.4.** Suppose that \( U = \{ \alpha_i | i \in I \} \) is a minimal strong generating set for \( B_\infty \) such that \( \text{wt}(\alpha_i) < N \) for all \( i \in I \). If there are no normally ordered polynomial relations among \( \{ \alpha_i | i \in I \} \) and their derivatives of weight less than \( N \), the corresponding set \( T = \{ a_i | i \in I \} \) is a minimal strong generating set for \( B \).

**Proof.** If \( T \) is not minimal, there exists a decoupling relation \( \lambda(k)a_j = P \) for some \( j \in I \) of weight \( \text{wt}(a_j) < N \). By rescaling if necessary, we can assume that either \( \lambda(k) \) or \( P \) is nontrivial in the limit \( k \to \infty \). We therefore obtain a nontrivial relation among \( \{ \alpha_i | i \in I \} \) and their derivatives of the same weight, which is impossible. \( \square \)

In our main examples, the fact that relations among the elements of \( U \) and their derivatives do not exist below a certain weight is a consequence of Weyl’s second fundamental theorem of invariant theory for the classical groups [W].

### 4. Orbifolds of Free Field Algebras

By a **free field algebra**, we mean any vertex algebra \( V = \mathcal{H}(n) \otimes \mathcal{F}(m) \otimes S(r) \otimes \mathcal{A}(s) \) for integers \( m, n, r, s \geq 0 \), where \( B(0) \) is declared to be \( \mathbb{C} \) for \( B = \mathcal{H}, S, \mathcal{F}, \mathcal{A} \). Building on our previous work, we establish the strong finite generation of \( V^G \) for any reductive group \( G \subset \text{Aut}(V) \) which preserves the tensor factors of \( V \). Our description of these orbifolds is ultimately based on a classical theorem of Weyl (Theorem 2.5A of [W]). Let \( V_k = \mathbb{C}^n \) for \( k \geq 1 \), and let \( G \subset GL_n \), which acts on the ring \( \text{Sym} \bigoplus_{k \geq 1} V_k \). For all \( p \geq 1 \), \( GL(p) \) acts on \( \bigoplus_{k=1}^p U_k \) and commutes with the action of \( G \). There is an induced action of \( GL(\infty) = \lim_{p \to \infty} GL(p) \) on \( \bigoplus_{k \geq 1} V_k \), so \( GL(\infty) \) acts on \( \text{Sym} \bigoplus_{k \geq 1} V_k \) and commutes with the action of \( G \). Therefore \( GL(\infty) \) acts on \( R = (\text{Sym} \bigoplus_{k \geq 1} V_k)^G \) as well. Elements \( \sigma \in GL(\infty) \) are known as polarization operators, and given \( f \in R \), \( \sigma f \) is known as a polarization of \( f \).

**Theorem 4.1.** \( R \) is generated by the polarizations of any set of generators for \( (\text{Sym} \bigoplus_{k \geq 1} V_k)^G \). Since \( G \) is reductive, \( (\text{Sym} \bigoplus_{k \geq 1} V_k)^G \) is finitely generated, so there exists a finite set \( \{ f_1, \ldots, f_r \} \), whose polarizations generate \( R \).

As shown in [CLII] (see Theorem 6.4) there is an analogue of this result for exterior algebras. Let \( S = (\bigwedge \bigoplus_{k \geq 1} V_k)^G \) and let \( d \) be the maximal degree of the generators of \( (\text{Sym} \bigoplus_{k \geq 1} V_k)^G \). Then \( S \) is generated by the polarizations of any set of generators for \( (\bigwedge \bigoplus_{k \geq 1} V_k)^G \). In particular, \( S \) is generated by a finite number of elements together with their polarizations. By a similar argument, the same holds for rings of the form

\[ T = ((\text{Sym} \bigoplus_{k \geq 1} V_k) \otimes (\bigwedge \bigoplus_{k \geq 1} W_k))^G, \]

where \( V_k = \mathbb{C}^n, W_k = \mathbb{C}^m \), and \( G \subset GL_n \times GL_m \) is any reductive group.

**Theorem 4.2.** Let \( V = \mathcal{H}(m) \otimes \mathcal{F}(n) \otimes S(r) \otimes \mathcal{A}(s) \) for integers \( m, n, r, s \geq 0 \), and let \( G \subset O(m) \times O(n) \times Sp(2r) \times Sp(2s) \) be a reductive group of automorphisms of \( V \) that preserves the factors \( \mathcal{H}(m), \mathcal{F}(n), S(r), \) and \( \mathcal{A}(s) \). Then \( V^G \) is strongly finitely generated.
Proof. Note that $\mathcal{V} \cong \text{gr}(\mathcal{V})$ as $G$-modules, and
\[
\text{gr}(\mathcal{V}^G) \cong \text{gr}(\mathcal{V})^G \cong \left( (\text{Sym} \bigoplus_{j \geq 0} V_j) \otimes (\bigwedge j \geq 0 V_j) \right) \otimes (\text{Sym} \bigoplus_{j \geq 0} W_j) \otimes (\bigwedge j \geq 0 W_j))^G,
\]
as supercommutative rings. Here $V_j \cong \mathbb{C}^m$, $V_j^* \cong \mathbb{C}^n$, $W_j \cong \mathbb{C}^{2r}$, $W_j^* \cong \mathbb{C}^{2s}$.

By a general theorem of Kac and Radul [KR] (see also [DLM] for the case of compact $G$), for each of the vertex algebras $B = \mathcal{H}(m), \mathcal{S}(n), \mathcal{F}(r), \mathcal{A}(s)$, we have a dual reductive pair decomposition
\[
B \cong \bigoplus_{\nu \in H} L(\nu) \otimes M^\nu,
\]
where $H$ indexes the irreducible, finite-dimensional representations $L(\nu)$ of $\text{Aut}(B)$, and the $M^\nu$’s are inequivalent, irreducible, highest-weight $B^{\text{Aut}(B)}$-modules. Therefore
\[
\mathcal{V} \cong \bigoplus_{\nu, \mu, \gamma, \delta} L(\nu) \otimes L(\mu) \otimes L(\gamma) \otimes L(\delta) \otimes M^\nu \otimes M^\mu \otimes M^\gamma \otimes M^\delta,
\]
where $L(\nu)$, $L(\mu)$, $L(\gamma)$, and $L(\delta)$ are irreducible, finite-dimensional modules over $O(m)$, $O(n)$, $Sp(2r)$ and $Sp(2s)$, respectively, and $M^\nu$, $N^\mu$, $M^\gamma$ and $M^\delta$ are irreducible, highest-weight modules over $\mathcal{H}(m)^{O(m)}, \mathcal{F}(n)^{O(n)}, \mathcal{S}(r)^{Sp(2r)},$ and $\mathcal{A}(s)^{Sp(2s)}$, respectively. An immediate consequence whose proof is the same as the proof of Lemma 14.2 of [LV] is that $\mathcal{V}^G$ has a strong generating set which lies in the direct sum of finitely many irreducible modules over $\mathcal{H}(m)^{O(m)} \otimes \mathcal{F}(n)^{O(n)} \otimes \mathcal{S}(r)^{Sp(2r)} \otimes \mathcal{A}(s)^{Sp(2s)}$.

By Theorem 9.4 of [LV], $\mathcal{S}(r)$ is of type $\mathcal{W}(2, 4, \ldots, 2r^2 + 4r)$ and has strong generators
\[
\tilde{w}^{2k+1} = \frac{1}{2} \sum_{i=1}^{r} \left( : \beta^i \partial^{2k+1} \gamma^i : - : (\partial^{2k+1} \beta^i ) \gamma^i : \right), \quad k = 0, 1, \ldots, r^2 + 2r - 1.
\]
By Theorem 11.1 of [LV], $\mathcal{F}(n)$ is of type $\mathcal{W}(2, 4, \ldots, 2n)$ and has strong generators
\[
\tilde{f}^{2k+1} = -\frac{1}{2} \sum_{i=1}^{n} : \phi^i \partial^{2k+1} \phi^i :, \quad k = 0, 1, \ldots, n - 1.
\]
By Theorem 3.11 of [CLII], $\mathcal{A}(s)$ is of type $\mathcal{W}(2, 4, \ldots, 2s)$ and has strong generators
\[
w^{2k} = \frac{1}{2} \sum_{i=1}^{s} \left( : e^i \partial^{2k} f^i : + : (\partial^{2k} e^i ) f^i : \right), \quad k = 0, 1, \ldots, s - 1.
\]
In [LV], it was conjectured that $\mathcal{H}(m)$ is of type $\mathcal{W}(2, 4, \ldots, m^2 + 3m)$, and has strong generators
\[
\tilde{j}^{2k} = \sum_{i=1}^{m} a^i \partial^{2k} a^i :, \quad k = 0, 1, \ldots, \frac{1}{2}(m^2 + 3m - 2).
\]
In [LV] we proved this for $m \leq 6$, and although we did not prove it for $m > 6$, we showed that $\mathcal{H}(m)^{O(m)}$ has strong generators $\{ j^{2k} | 0 \leq k \leq K \}$ for some $K$.

For any irreducible $\mathcal{H}(m)^{O(m)} \otimes \mathcal{F}(n)^{O(n)} \otimes \mathcal{S}(r)^{Sp(2r)} \otimes \mathcal{A}(s)^{Sp(2s)}$-submodule $\mathcal{M}$ of $\mathcal{V}$ with highest-weight vector $f = f(z)$, and any subset $S \subset \mathcal{M}$, define $\mathcal{M}_S$ to be the subspace spanned by the elements
\[
\omega_i \in \mathcal{H}(m)^{O(m)}, \quad v_i \in \mathcal{F}(n)^{O(n)}, \quad \mu_i \in \mathcal{S}(r)^{Sp(2r)}, \quad \zeta_i \in \mathcal{A}(s)^{Sp(2s)}, \quad \alpha \in S.
\]
By the same argument as Lemma 9 of [LII], there is a finite set \( S \) of vertex operators of the form
\[
j^{2a_i} (h_1) \cdots j^{2a_t} (h_t) j^{2b_{i1}} (j_1) \cdots j^{2b_{u1}} (j_u) w^{2c_{i1}} (k_1) \cdots w^{2c_{v1}} (k_v) w^{2d_1} (l_1) \cdots w^{2d_w} (l_w) f,
\]
such that \( M = M_S \). In this notation
\[
\begin{align*}
j^{2a_i} &\in \mathcal{H}(m)^{O(m)}, \quad 0 \leq h_i \leq 2a_i \leq K, \\
j^{2b_{i1}} &\in \mathcal{F}(n)^{O(n)}, \quad 0 \leq j_i < 2b_{i1} + 1 \leq 2n - 1, \\
w^{2c_{i1}} &\in \mathcal{S}(r)^{Sp(2r)}, \quad 0 \leq k_i < 2c_{i1} + 1 \leq 2r^2 + 4r - 1, \\
w^{2d_1} &\in \mathcal{A}(s)^{Sp(2s)}, \quad 0 \leq l_i \leq 2d_i \leq 2s - 2.
\end{align*}
\]
Combining this with the strong finite generation of each of the vertex algebras \( \mathcal{H}(m)^{O(m)} \), \( \mathcal{F}(n)^{O(n)} \), \( \mathcal{S}(r)^{Sp(2r)} \), and \( \mathcal{A}(s)^{Sp(2s)} \), completes the proof. \( \square \)

5. Orbifolds of affine vertex superalgebras

In [LIII] it was shown that for any Lie algebra \( \mathfrak{g} \) with a nondegenerate form \( B \), and any reductive group \( G \) of automorphisms of \( V_k(g, B) \), \( V_k(g, B)^G \) is strongly finitely generated for generic values of \( k \). Here we extend this result to the case of affine vertex superalgebras. Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a finite-dimensional Lie superalgebra over \( \mathbb{C} \), where \( \dim(\mathfrak{g}_0) = n \) and \( \dim(\mathfrak{g}_1) = 2m \), and let \( B \) be a nondegenerate form on \( \mathfrak{g} \). Let \( \mathcal{V} \) be the deformable vertex algebra from Example 3.1, such that \( V_k(g, B) \cong \mathcal{V}/(\kappa - \sqrt{k}) \), and \( \mathcal{V}\infty = \lim_{k \to \infty} \mathcal{V} \cong \mathcal{H}(n) \otimes \mathcal{F}(m) \). Define an \( F \)-linear map \( \psi : \mathcal{V} \to \mathcal{V}_\infty \) by
\[
\psi \left( \sum_r c_r(\kappa)m_r(\alpha^{\xi_i}) \right) = \sum_r c_r(\kappa)m_r(\alpha^{\xi_i}), \quad c_r = \lim_{\kappa \to \infty} c_r(\kappa).
\]

In this notation, \( m_r(\alpha^{\xi_i}) \) is a normally ordered monomial in \( \partial^i a^{\xi_i} \), and \( m_r(\alpha^{\xi_i}) \) is obtained from \( m_r(\alpha^{\xi_i}) \) by replacing each \( a^{\xi_i} \) with \( \alpha^{\xi_i} \). This map is easily seen to satisfy \( \psi(\omega \circ_n \nu) = \psi(\omega) \circ_n \psi(\nu) \) for all \( \omega, \nu \in \mathcal{V} \) and \( n \in \mathbb{Z} \).

Note that \( \mathcal{V} \) has a good increasing filtration, where \( \mathcal{V}(d) \) is spanned by normally ordered monomials in \( \partial^i a^{\xi_i} \) and \( \partial^i a^{\xi_i} \) of degree at most \( d \). We have isomorphisms of \( \partial \)-rings
\[
\text{gr}(\mathcal{V}) \cong F \otimes_{\mathbb{C}} \text{(Sym} \bigoplus_{j \geq 0} V_j \bigotimes_{j \geq 0} W_j) \cong F \otimes_{\mathbb{C}} \text{gr}(\mathcal{V}_\infty), \quad V_j \cong \mathfrak{g}_0, \quad W_j \cong \mathfrak{g}_1.
\]
The action of \( G \) on \( \mathcal{V} \) preserves the formal variable \( \kappa \), and we have
\[
\text{gr}(\mathcal{V}^G) \cong \text{gr}(\mathcal{V})^G \cong F \otimes_{\mathbb{C}} R \cong F \otimes_{\mathbb{C}} \text{gr}(\mathcal{V}_\infty)^G \cong F \otimes_{\mathbb{C}} \text{gr}(\mathcal{V}_\infty)^G,
\]
where \( R = ((\text{Sym} \bigoplus_{j \geq 0} V_j) \otimes (\bigwedge \bigoplus_{j \geq 0} W_j))^G \). Finally, \( \mathcal{V}^G[w] \) is a free \( F \)-module and
\[
\text{rank}_F(\mathcal{V}^G[w]) = \dim_{\mathbb{C}}((\mathcal{V}_\infty)^G[w]) = \dim_{\mathbb{C}}(V_k(g, B)^G[w])
\]
for all \( w \geq 0 \) and \( k \in \mathbb{C} \).

Fix a basis \( \{ \xi_1, \ldots, \xi_m \} \) for \( V_i \), which corresponds to
\[
\{ \partial^i a^{\xi_1}, \ldots, \partial^i a^{\xi_m} \} \subset \mathcal{V}, \quad \{ \partial^i \alpha^{\xi_1}, \ldots, \partial^i \alpha^{\xi_m} \} \subset \mathcal{V}_\infty,
\]
respectively. Similarly, fix a basis \( \{ \eta_1, \ldots, \eta_m \} \) for \( W_i \) corresponding to
\[
\{ \partial^i a^{\eta_1}, \ldots, \partial^i a^{\eta_m} \} \subset \mathcal{V}, \quad \{ \partial^i \alpha^{\eta_1}, \ldots, \partial^i \alpha^{\eta_m} \} \subset \mathcal{V}_\infty,
\]
respectively.
respectively. The ring $R$ is graded by degree and weight, where $\xi_{1,\ell}, \ldots, \xi_{n,\ell}, \eta_{1,\ell}, \ldots, \eta_{m,\ell}$ have degree 1 and weight $l + 1$. Choose a generating set $S = \{s_i \mid i \in I\}$ for $R$ as a $\partial$-ring, where $s_i$ is homogeneous of degree $d_i$ and weight $w_i$. We may assume that $S$ contains finitely many generators in each weight. We can find a corresponding strong generating set $T = \{t_i \mid i \in I\}$ for $V^G$, where

$$t_i \in (V^G)_{(d_i)} \quad \phi_{d_i}(t_i) = 1 \otimes s_i \in F \otimes_{\mathbb{C}} R.$$ 

Here $\phi_{d_i} : (V^G)_{(d_i)} \to (V^G)_{(d_i)}/(V^G)_{(d_i-1)} \subset \text{gr}(V^G)$ is the usual projection. In particular, the leading term of $t_i$ is a sum of normally ordered monomials of degree $d_i$ in the variables $a^{\xi_1}, \ldots, a^{\xi_n}$ and their derivatives, and the coefficient of each such monomial is independent of $\kappa$. Let $u_i = \psi(t_i)$ where $\psi$ is given by (5.1), and define

$$(V^G)_\infty = \langle U \rangle \subset (V_\infty)^G,$$

where $\langle U \rangle$ is the vertex algebra generated by $\{u_i \mid i \in I\}$.

Fix $w \geq 0$, and let $\{m_1, \ldots, m_r\}$ be a set of normally ordered monomials in $t_i$ and their derivatives, which spans the subspace $V^G[w]$ of weight $w$. Then $(V^G)_\infty[w]$ is spanned by the corresponding monomials $\mu_l = \psi(m_l)$ for $l = 1, \ldots, r$, which are obtained from $m_l$ by replacing $t_i$ with $u_i$. Given normally ordered polynomials

$$P(u_i) = \sum_{l=1}^r c_l \mu_l \in (V^G)_\infty[w], \quad \tilde{P}(t_i) = \sum_{l=1}^r c_l(\kappa)m_l \in V^G[w],$$

with $c_l \in \mathbb{C}$ and $c_l(\kappa) \in F$, we say that $\tilde{P}(t_i)$ converges termwise to $P(u_i)$ if

$$\lim_{\kappa \to \infty} c_l(\kappa) = c_l, \quad l = 1, \ldots, r.$$ 

In particular, $\tilde{P}(t_i)$ converges termwise to zero if and only if $\lim_{\kappa \to \infty} c_l(\kappa) = 0$ for $l = 1, \ldots, r$.

**Lemma 5.1.** For each normally ordered polynomial relation $P(u_i)$ in $(V^G)_\infty$ of weight $m$ and leading degree $d$, there exists a relation $\tilde{P}(t_i) \in V^G$ of weight $m$ and leading degree $d$ which converges termwise to $P(u_i)$.

**Proof.** We may write $P(u_i) = \sum_{a=1}^d P^a(u_i)$, where $P^a(u_i)$ is a sum of normally ordered monomials $\mu = \partial^{d_1} u_{i_1} \cdots \partial^{d_r} u_{i_r}$ of degree $a = d_{i_1} + \cdots + d_{i_r}$. The leading term $P^d(u_i)$ corresponds to a relation in $R$ among the generators $s_i$ and their derivatives, i.e., $P^d(s_i) = 0$. It follows that $P^d(t_i) \in (V^G)_{(d-1)}$. Since $P^a(u_i) \in ((V^G)_\infty)^{a}$ for $a = 1, \ldots, d - 1$, we have $P(t_i) \in (V^G)_{(d-1)}$. Since $\{t_i \mid i \in I\}$ strongly generates $V^G$, we can express $P(t_i)$ as a normally ordered polynomial $P_0(t_i)$ of degree at most $d - 1$. Let $Q(t_i) = P(t_i) - P_0(t_i)$, which is therefore a relation in $V^G$ with leading term $P^d(t_i)$.

If $P_0(t_i)$ converges termwise to zero, we can take $\tilde{P}(t_i) = Q(t_i)$ since $P(t_i)$ converges termwise to $P(u_i)$. Otherwise, $P_0(t_i)$ converges termwise to a nontrivial relation $P_1(u_i)$ in $(V^G)_\infty$ of degree at most $d - 1$. By induction on the degree, there is a relation $\tilde{P}_1(t_i)$ of leading degree at most $d - 1$, which converges termwise to $P_1(u_i)$. Finally, $\tilde{P}(t_i) = P(t_i) - P_0(t_i) - \tilde{P}_1(t_i)$ has the desired properties. \(\square\)

**Corollary 5.2.** $(V^G)_\infty = (V_\infty)^G = (\mathcal{H}(n) \otimes \mathcal{F}(m))^G$. 

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Proof. Recall that \( \text{rank}_F(\mathcal{V}^G[w]) = \dim_{\mathbb{C}}((\mathcal{V}_\infty)^G[w]) \) for all \( w \geq 0 \). Since \( (\mathcal{V}^G)_\infty \subset (\mathcal{V}_\infty)^G \), it suffices to show that \( \text{rank}_F(\mathcal{V}^G[w]) = \dim_{\mathbb{C}}((\mathcal{V}^G)_\infty[w]) \) for all \( w \geq 0 \). Let \( \{m_1, \ldots, m_r\} \) be a basis for \( \mathcal{V}^G[w] \) as an \( F \)-module, consisting of normally ordered monomials in \( t_i \) and their derivatives. The corresponding elements \( \mu_l = \psi(m_l) \) for \( l = 1, \ldots, r \) span \( (\mathcal{V}^G)_\infty[w] \), and by Lemma 5.1 they are linearly independent. Otherwise, a nontrivial relation among \( \mu_1, \ldots, \mu_r \) would give rise to a nontrivial relation among \( m_1, \ldots, m_r \). \( \square \)

Theorem 5.3. \( V_k(\mathfrak{g}, B)^G \) is strongly finitely generated for generic values of \( k \).

Proof. This is immediate from Theorem 4.2 applied to \( \mathcal{V} = \mathcal{H}(n) \otimes \mathcal{F}(m) \) and Corollaries 3.3 and 5.2. \( \square \)

Theorem 5.4. Let \( \mathcal{V} = \mathcal{H}(m) \otimes \mathcal{F}(n) \otimes \mathcal{S}(r) \otimes \mathcal{A}(s) \) be a free field algebra and let \( \mathfrak{g} \) be a Lie superalgebra equipped with a nondegenerate form \( \mathcal{B} \). Let \( G \) be a reductive group of automorphisms of \( \mathcal{V} \otimes V_k(\mathfrak{g}, B) \) which preserves each tensor factor. Then \( (\mathcal{V} \otimes V_k(\mathfrak{g}, B))^G \) is strongly finitely generated for generic values of \( k \).

Proof. We have \( \lim_{k \to \infty} \mathcal{V} \otimes V_k(\mathfrak{g}, B) \cong \mathcal{V} \otimes \mathcal{H}(n) \otimes \mathcal{A}(m) \) where \( n = \dim(\mathfrak{g}_0) \) and \( m = \frac{1}{2} \dim(\mathfrak{g}_1) \), and \( \lim_{k \to \infty} (\mathcal{V} \otimes V_k(\mathfrak{g}, B))^G \cong (\mathcal{V} \otimes \mathcal{H}(n) \otimes \mathcal{A}(m))^G \). Clearly \( G \) preserves the tensor factors, so the claim follows from Theorem 4.2 and Corollary 3.3. \( \square \)

6. COSETS OF \( V_k(\mathfrak{g}, B) \) INSIDE LARGER STRUCTURES

Let \( \mathfrak{g} \) be a finite-dimensional reductive Lie algebra, equipped with a nondegenerate symmetric, invariant bilinear form \( \mathcal{B} \). Let \( A_k \) be a vertex (super)algebra whose structure constants depend continuously on \( k \), admitting a homomorphism \( V_k(\mathfrak{g}, B) \to A_k \), and let \( C_k = \text{Com}(V_k(\mathfrak{g}, B), A_k) \). Many cosets of this form have been studied in both the physics and mathematics literature. One class of examples is \( A_k = V_k(\mathfrak{g}', B') \), where \( \mathfrak{g}' \) is a Lie (super)algebra containing \( \mathfrak{g} \), and \( B \) is a nondegenerate, (super)symmetric invariant form on \( \mathfrak{g}' \) extending \( B \). Another class of examples is \( A_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F} \) where \( \mathfrak{g}' \) and \( B' \) are as above and \( \mathcal{F} \) is a free field algebra admitting a map \( \phi : V_l(\mathfrak{g}, B) \to \mathcal{F} \) for some fixed \( l \in \mathbb{C} \). We require that the action of \( \mathfrak{g} \) on \( \mathcal{F} \) integrates to an action of a group \( G \) whose Lie algebra is \( \mathfrak{g} \), and that \( G \) preserves the tensor factors of \( \mathcal{F} \). The map \( V_k(\mathfrak{g}, B) \to A_k \) is just the diagonal map \( X^\xi : X^\xi_0 \otimes 1 + 1 \otimes \phi(X^\xi_i) \).

To construct examples of this kind, we recall a well-known vertex algebra homomorphism \( \tau : V_{-1/2}(\mathfrak{sp}_{2n}) \to \mathcal{S}(n) \). In terms of the basis \( \{e_{i,j}\} | 1 \leq i \leq 2n, 1 \leq j \leq 2n \} \) for \( \mathfrak{gl}_{2n} \), a standard basis for \( \mathfrak{sp}_{2n} \) consists of

\[
e_{j,k+n} + e_{k,j+n}, \quad -e_{j+n,k} - e_{k+n,j}, \quad e_{j,k} - e_{n+k,n+j}, \quad 1 \leq j, k \leq n.
\]

Define \( \tau \) by

\[
X^{e_{j,k+n}+e_{k,j+n}} \mapsto \gamma^j \gamma^k ; \quad X^{-e_{j+n,k} - e_{k+n,j}} \mapsto \beta^j \beta^k ; \quad X^{e_{j,k} - e_{n+k,n+j}} \mapsto : \gamma^j \beta^k :.
\]

It is easily checked that the \( \mathfrak{sp}_{2n} \)-action coming from the zero modes \( \{X^\xi(0) | \xi \in \mathfrak{sp}_{2n}\} \) integrates to the usual action of \( \mathcal{S}p(2n) \) on \( \mathcal{S}(n) \). There is a similar homomorphism \( \sigma : V_1(\mathfrak{so}_m) \to \mathcal{F}(m) \) whose zero mode action integrates to \( SO(m) \). If \( \mathfrak{g} \) is any reductive Lie algebra which embeds in \( \mathfrak{sp}_{2n} \), and \( B_1 \) is the restriction of the form on \( \mathfrak{sp}_{2n} \) to \( \mathfrak{g} \), we obtain a restriction map \( \tau_\mathfrak{g} : V_1(\mathfrak{g}, B_1) \to \mathcal{S}(n) \). Similarly, if \( \mathfrak{g} \) embeds in \( \mathfrak{so}_m \) we obtain a restriction
map $\sigma_g : V_1(g, B_2) \to \mathcal{F}(m)$, where $B_2$ is the restriction of the form on $\mathfrak{so}_m$ to $g$. Finally, we have the diagonal embedding
\[ V_1(g, B_1 + B_2) \to S(n) \otimes \mathcal{F}(m), \quad X^\xi \mapsto \tau_g(X^\xi) \otimes 1 + 1 \otimes \sigma_g(X^\xi). \]

The action of $g$ coming from the zero modes integrates to an action of a connected Lie group $G$ with Lie algebra $g$, which preserves both $S(n)$ and $\mathcal{F}(m)$.

Finally, we mention one more class of examples $\mathcal{A}_k = V_{k-1}(g', B') \otimes V_{l}(g'', B'')$. Here $g''$ is another finite-dimensional Lie (super)algebra containing $g$, equipped with a nondegenerate, invariant, (super)symmetric bilinear form $B''$ extending $B$. As usual, the map $V_k(g, B) \to \mathcal{A}_k$ is the diagonal map $X^\xi \mapsto X^\xi \otimes 1 + 1 \otimes X^\xi$. If $V_l(g'', B'')$ is not simple, we may replace $V_l(g'', B'')$ with its quotient by any nontrivial ideal in the above definition.

In order to study all the above cosets $\mathcal{C}_k$ from a unified point of view it is useful to axiomatize $\mathcal{A}_k$. A vertex algebra $\mathcal{A}_k$ with structure constants depending continuously on $k$, which admits a map $V_k(g, B) \to \mathcal{A}_k$ will be called good if the following conditions hold.

(1) There exists a deformable family $\mathcal{A}$ defined over $F_K$ for some (at most countable) subset $K \subset \mathbb{C}$ containing zero, such that $\mathcal{A}_k = \mathcal{A}/(\kappa - \sqrt{k})$. Letting $V$ be as in Example 3.1 there is a homomorphism $V \to \mathcal{A}$ inducing the map $V_k(g, B) \to \mathcal{A}_k$ for each $k$ with $\sqrt{k} \notin K$.

(2) For generic values of $k$, $\mathcal{A}_k$ admits a Virasoro element $L^A$ and a conformal weight grading $\mathcal{A}_k = \bigoplus_{d \in \mathbb{N}} \mathcal{A}_k[d]$. For all $d$, $\dim(\mathcal{A}_k[d])$ is finite and independent of $k$.

(3) For generic values of $k$, $\mathcal{A}_k$ decomposes into finite-dimensional $g$-modules, so the action of $g$ integrates to an action of a connected Lie group $G$ having $g$ as Lie algebra.

(4) We have a vertex algebra isomorphism
\[ \mathcal{A}_\infty \equiv \lim_{k \to \infty} \mathcal{A}_k \cong \mathcal{H}(d) \otimes \tilde{\mathcal{A}}, \quad d = \dim(g). \]

Here $\tilde{\mathcal{A}}$ is a vertex subalgebra of $\lim_{k \to \infty} \mathcal{A}_k$ with Virasoro element $L^\tilde{A}$ and $\mathbb{N}$-grading by conformal weight, with finite-dimensional graded components.

(5) Although $L^g$ need not act diagonalizably on $\mathcal{A}_k$, it induces a grading on $\mathcal{A}_k$ into generalized eigenspaces corresponding to the Jordan blocks of each eigenvalue. In general, these generalized eigenspaces can be infinite-dimensional. However, any highest-weight $V_k(g, B)$-submodule of $\mathcal{A}_k$ has finite-dimensional components with respect to this grading for generic values of $k$.

Note that for generic values of $k$, $\mathcal{C}_k$ has the Virasoro element $L^C = L^A - L^g$, where $L^g$ is the Virasoro element in $V_k(g, B)$. Note that $\lim_{k \to \infty} L^g = L^H = \frac{1}{2} \sum_{i=1}^{d} \alpha^i \otimes \alpha^{g_i} = L^H + L^A$. It is not difficult to check that all the above examples $\mathcal{A}_k = V_k(g', B')$, $\mathcal{A}_k = V_{k-1}(g', B') \otimes \mathcal{F}$, and $\mathcal{A}_k = V_{k-1}(g', B') \otimes V_{l}(g'', B'')$ are good. Also, $V_{k-1}(g', B') \otimes V_{l}(g'', B'')$ remains good if we replace $V_l(g'', B'')$ by its quotient by any ideal. Suppose that $\dim(g) = d$ and $g' = g'_{0} \oplus g'_{1}$ where $\dim(g'_{0}) = n$ and $\dim(g'_{1}) = m$. For $\mathcal{A}_k = V_k(g', B')$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n - d) \otimes \mathcal{A}(m)$. Similarly, for $\mathcal{A}_k = V_{k-1}(g', B') \otimes \mathcal{F}$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n - d) \otimes \mathcal{A}(m) \otimes \mathcal{F}$. Finally, for $\mathcal{A}_k = V_{k-1}(g', B') \otimes V_{l}(g'', B'')$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n - d) \otimes \mathcal{A}(m) \otimes V_{l}(g'', B'')$. The same holds if we replace $V_l(g'', B'')$ with its quotient by any ideal.

**Lemma 6.1.** Let $g$ be reductive and $B$ nondegenerate, and suppose that $\mathcal{A}_k$ is good. For generic values of $k$, $\mathcal{C}_k = \ker(L^g_0) \cap (A_k)^G$. 

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Theorem 6.2. If \( g \) is completely reducible on \( A \) algebra \( g \), the number of simple and abelian summands that the lemma holds for any reductive Lie algebra \( g \) have a vertex algebra isomorphism \( (6.2) \).

Proof. Suppose first that \( g \) is simple. Clearly any \( \omega \in C_k \) is annihilated by \( L_0^g \) and is \( G \)-invariant. Conversely, suppose that \( \omega \in \text{Ker}(L_0^g) \cap (\mathcal{A}_k)^G \). If \( \omega \notin C_k \), \( X_{\xi_i} \circ \omega \neq 0 \) for each \( i = 1, \ldots, d \). We show that such an \( \omega \) cannot exist for generic level \( k \).

Recall that \( \mathcal{A}_k \) is \( \mathbb{N} \)-graded by conformal weight (i.e., \( L_0^A \)-eigenvalue). Write \( \omega \) as a sum of terms of homogeneous weight, and let \( m \) be the maximum value which appears. Let \( g_+ \subset \hat{g} \) be the Lie subalgebra generated by the positive modes \( \{ X_{\xi}(k) | \xi \in g , k > 0 \} \). Note that each element of \( U(g_+) \) lowers the weight by some \( k > 0 \), and the conformal weight grading on \( U(g_+) \) is the same as the grading by \( L_0^g \)-eigenvalue. An element \( x \in U(g_+) \) of weight \( -k \) satisfies \( x(\omega) \in \mathcal{A}_{m-k} \). Also, \( x(\omega) \) lies in the generalized eigenspace of \( L_0^g \) of eigenvalue \( -k \), and \( x(\omega) = 0 \) if \( k > m \).

It follows that \( U(g_+) \omega \) is a finite-dimensional vector space graded by conformal weight. In particular, the subspace \( M \subset U(g_+) \omega \) of minimal weight \( m \) is finite-dimensional. Hence it is a finite-dimensional \( g \)-module, and is thus a direct sum of finite-dimensional highest-weight \( g \)-modules. Moreover, \( U(g_+) \) acts trivially on \( M \). Since \( g \) is simple and \( L^2 \) is the Sugawara vector at level \( k \), the eigenvalue of \( L_0^g \) on \( M \) is given by

\[
(L_0^g|_M = \frac{\text{Cas}(M)}{k + h^\vee},
\]

where \( \text{Cas}(M) \) is the Casimir eigenvalue on \( M \). In fact, each irreducible summand of \( M \) must have the same \( L_0^g \) eigenvalue and hence the same Casimir eigenvalue. This is a rational number. The \( L_0^g \) eigenvalue on \( M \) must actually be a negative integer \( r \), with \(-m \leq r \leq -1\). This statement and \( (6.2) \) can only be true for special rational values of \( k \).

Similarly, if \( g = \mathbb{C} \), so that \( G = \mathbb{C}^* \) and \( V_k(g, B) \cong \mathcal{H}(1) \) for \( k \neq 0 \), we have \( C_k = \text{Ker}(L_0^H) \cap (A_k)^{\mathbb{C}^*} \) for generic values of \( k \). This is immediate from the fact that \( \mathcal{H}(1) \) acts completely reducibly on \( \mathcal{A}_k \), so that \( (A_k)^{\mathbb{C}^*} \cong \mathcal{H}(1) \otimes C_k \). Now it follows by induction on the number of simple and abelian summands that the lemma holds for any reductive Lie algebra \( g \).

\[ \square \]

Theorem 6.2. Let \( g \) be reductive and \( B \) nondegenerate, and suppose that \( A_k \) is good. Then we have a vertex algebra isomorphism

\[
\lim_{k \to \infty} C_k \cong \mathcal{A}^G.
\]

Proof. The operator \( L_0^g \) acts on the (finite-dimensional) spaces \( A_k[n] \) of weight \( k \) and commutes with \( G \), so it maps \( A_k^n \) to itself. By the preceding lemma, \( C_k[n] \) is the kernel of this map. Let \( \phi : A_k[n] \to A_{\infty}[n] = (\mathcal{H}(d) \otimes \mathcal{A})[n] \) be the map sending \( \omega \mapsto \lim_{k \to \infty} \omega \), which is injective and maps \( C_k[n] \) into \( \mathcal{A}[n] \). Then \( \Phi = \phi \circ L_0^g : A_k[n] \to (\mathcal{H}(d) \otimes \mathcal{A})^G[n] \) also has kernel equal to \( C_k[n] \). It is enough to show that \( \dim(\text{Ker}(\Phi)) \geq \dim(\mathcal{A}^G[n]) \). Equivalently, we need to show that \( \dim(\text{Coker}(\Phi)) \geq \dim(\mathcal{A}^G[n]) \). To see this, note that any element in the image of \( L_0^g \) is a linear combination of elements of the form : \((\partial^i \alpha_{\xi}) | \nu : \) for \( \xi_i \in g \) and \( i \geq 0 \). Under \( \phi \) these get mapped to : \((\partial^i \alpha_{\xi}) \circ \Phi | \nu \). In particular, each term has weight at least one under \( L_0^H \), so \( \mathcal{A}^G[n] \) injects into \( \text{Coker}(\Phi) \).

\[ \square \]

Corollary 6.3. Let \( g \) be reductive and \( B \) nondegenerate, and suppose that \( A_k \) is good. Suppose that \( \mathcal{A} \) is a tensor product of free field and affine vertex algebras, and the induced action of \( G \) preserves each tensor factor. Then \( C_k \) is strongly finitely generated for generic values of \( k \).

Corollary 6.4. Let \( g \) be reductive and \( B \) nondegenerate. Let \( g' \) be a Lie (super)algebra containing \( g \), equipped with a nondegenerate (super)symmetric form \( B' \) extending \( B \), and let \( A_k = V_k(g', B') \).
Then

\[ C_k = \text{Com}(V_k(\mathfrak{g}, B), V_k(\mathfrak{g}', B')) \]

is strongly finitely generated for generic values of \( k \).

Let \( F \) be a free field algebra admitting a map \( V_l(\mathfrak{g}, B) \to F \) for some fixed \( l \), and let \( A_k = V_{k-l}(\mathfrak{g}', B') \otimes F \). If the induced action of \( G \) preserves the tensor factors of \( F \),

\[ C_k = \text{Com}(V_k(\mathfrak{g}, B), V_{k-l}(\mathfrak{g}', B') \otimes F) \]

is strongly finitely generated for generic values of \( k \).

Finally, let \( \mathfrak{g}'' \) be another Lie (super)algebra containing \( \mathfrak{g} \), equipped with a nondegenerate (super)symmetric form \( B'' \) extending \( B \), and let \( A_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'') \). Then

\[ C_k = \text{Com}(V_k(\mathfrak{g}, B), V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')) \]

is strongly finitely generated for generic values of both \( k \) and \( l \).

7. SOME EXAMPLES

To illustrate the constructive nature of our results, the rest of this paper is devoted to finding minimal strong finite generating sets for \( C_k \) in some concrete examples.

**Example 7.1.** Let \( \mathfrak{g} = \mathfrak{sp}_{2n} \) and let \( A_k = V_{k+1/2}(\mathfrak{sp}_{2n}) \otimes S(n) \). Using the map \( V_{-1/2}(\mathfrak{sp}_{2n}) \to S(n) \) given by (6.1), we have the diagonal map \( V_k(\mathfrak{sp}_{2n}) \to A_k \). Clearly

\[ C_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), A_k) \]

satisfies \( C_\infty \cong S(n)^{Sp(2n)} \) which is of type \( \mathcal{W}(2, 4, \ldots, 2n^2 + 4n) \) by Theorem 9.4 of [LV]. It follows that for generic values of \( k \), \( C_k \) is of type \( \mathcal{W}(2, 4, \ldots, 2n^2 + 4n) \).

It is well known [KYY] that \( S(n)^{Sp(2n)} \cong L_{-1/2}(\mathfrak{sp}_{2n})^{Sp(2n)} \) where \( L_{-1/2}(\mathfrak{sp}_{2n}) \) denotes the irreducible quotient of \( V_{-1/2}(\mathfrak{sp}_{2n}) \). It follows that for \( A_k = V_{k+1/2}(\mathfrak{sp}_{2n}) \otimes L_{-1/2}(\mathfrak{sp}_{2n}), \)

\[ C_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), A_k) \]

is also of type \( \mathcal{W}(2, 4, \ldots, 2n^2 + 4n) \) for generic values of \( k \). This was conjectured in the physics literature by Blumenhagen, Eholzer, Honecker, Hornfeck, and Hubel (see Table 7 of [B-H]).

**Example 7.2.** Let \( \mathfrak{g} = \mathfrak{sp}_{2n} \) and \( A_k = V_k(\mathfrak{osp}(1|2n)) \). Then

\[ C_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), V_k(\mathfrak{osp}(1|2n))) \]

satisfies \( \lim_{k \to \infty} C_k \cong A(n)^{Sp(2n)} \). Since \( A(n)^{Sp(2n)} \) is of type \( \mathcal{W}(2, 4, \ldots, 2n) \) by Theorem 3.11 of [CLII], \( C_k \) is of type \( \mathcal{W}(2, 4, \ldots, 2n) \) for generic values of \( k \). In fact, by Corollary 5.7 of [CLII], \( A(n)^{Sp(2n)} \) is freely generated; there are no nontrivial normally ordered polynomial relations among the generators and their derivatives. It follows that \( C_k \) is freely generated for generic values of \( k \).

**Example 7.3.** Let \( \mathfrak{g} = \mathfrak{sp}_{2n} \) and \( A = V_{k+1/2}(\mathfrak{osp}(1|2n)) \otimes S(n) \). Then

\[ C_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), V_{k-1/2}(\mathfrak{osp}(1|2n) \otimes S(n))) \]

satisfies \( \lim_{k \to \infty} C_k \cong (A(n) \otimes S(n))^{Sp(2n)} \). Using classical invariant theory and an argument similar to the proof of Theorem 7.1 of [CLII], one can show that \( (A(n) \otimes S(n))^{Sp(2n)} \).
has the following minimal strong generating set:

\[
j^{2k} = \frac{1}{2} \left( \sum_{i=1}^{n} : e^{i} \partial^{2k} f^{i} : + : (\partial^{2k} e^{i}) f^{i} : \right), \quad 0 \leq k \leq n - 1,
\]

(7.1)

\[
w^{2k+1} = \frac{1}{2} \left( \sum_{i=1}^{n} : \beta^{i} \partial^{2k+1} \gamma^{i} - : (\partial^{2k+1} \beta^{i}) \gamma^{i} : \right), \quad 0 \leq k \leq n - 1,
\]

\[
\mu^{k} = \frac{1}{2} \left( \sum_{i=1}^{n} : \beta^{i} \partial^{k} f^{i} - \gamma^{i} \partial^{k} e^{i} : \right), \quad 0 \leq k \leq 2n - 1.
\]

In particular, \((A(n) \otimes S(n))^{Sp(2n)}\) has a minimal strong generating set consisting of even generators in weights 2, 2, 4, 4, ..., 2n, 2n and odd generators in weights \(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots, \frac{4n+1}{2}\), and so does \(C_{k}\) for generic values of \(k\). Let \(L = -j^{0} + w^{1}\) denote the Virasoro element of \((A(n) \otimes S(n))^{Sp(2n)}\), which has central charge \(-3n\). Then \(L\) and \(\mu^{0}\) generate a copy of the \(N = 1\) superconformal algebra with \(c = -3n\). Similarly, for noncritical values of \(k\), \(L = L^{\text{sp}(1|2n)} - L^{\text{sp}(n)} + w^{1}\) and \(\mu^{0}\) generate a copy of the \(N = 1\) algebra inside \(C_{k}\).

**Example 7.4.** Let \(g = \mathfrak{gl}_{n}\) and \(A_{k} = V_{k}(\mathfrak{gl}(n|1))\). In this case, \(C_{k} = \text{Com}(V_{k}(\mathfrak{gl}_{n}), A_{k})\) satisfies \(\lim_{k \to \infty} C_{k} \cong \mathcal{H}(1) \otimes (A(n)_{GL(n)})\). By Theorem 4.3 of [CLII], \(A(n)_{GL(n)}\) is of type \(W(2, 3, \ldots, 2n + 1)\) so \(C_{k}\) is generically of type \(W(1, 2, 3, \ldots, 2n + 1)\).

**Example 7.5.** Let \(g = \mathfrak{gl}_{n}\) and \(A_{k} = V_{k}(\mathfrak{gl}(n|1)) \otimes S(n)\). In this case, \(C_{k} = \text{Com}(V_{k}(\mathfrak{gl}_{n}), A_{k})\) satisfies \(\lim_{k \to \infty} C_{k} \cong \mathcal{H}(1) \otimes (A(n) \otimes S(n))^{GL(n)}\). As in Example 7.3, \((A(n) \otimes S(n))^{GL(n)}\) has the following minimal strong generating set.

\[
w^{k} = \sum_{i=1}^{n} : e^{i} \partial^{k} f^{i} :, \quad j^{k} = \sum_{i=1}^{n} : \beta^{i} \partial^{k} \gamma^{i} :, \quad 0 \leq k \leq 2n - 1,
\]

\[
\nu^{k} = \sum_{i=1}^{n} : e^{i} \partial^{k} \gamma^{i} :, \quad \mu^{k} = \sum_{i=1}^{n} : \beta^{i} \partial^{k} f^{i} :, \quad 0 \leq k \leq 2n - 1.
\]

The even generators are in weights 1, 2, 2, 3, 3, ..., 2n, 2n, 2n + 1, and the odd generators are in weights \(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \ldots, \frac{4n+1}{2}, \frac{4n+1}{2}\). Note that \((A(n) \otimes S(n))^{GL(n)}\) has the following \(N = 2\) superconformal structure of central charge \(-3n\).

(7.2)

\[
L = j^{1} - \frac{1}{2} \partial j^{0} - w^{0}, \quad F = j^{0}, \quad G^{+} = \nu^{0}, \quad G^{-} = \mu^{0}.
\]

For generic values of \(k\), \(C_{k}\) has a minimal strong generating set with even generators in weights 1, 1, 2, 2, 3, 3, 2, 2, ..., 2n, 2n, 2n + 1, and odd generators in weights \(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \ldots, \frac{4n+1}{2}, \frac{4n+1}{2}\). Finally, for noncritical values of \(k\), \(C_{k}\) has an \(N = 2\) structure given by

\[
L = j^{1} - \frac{1}{2} \partial j^{0} + L^{\text{sp}(n|1)} - L^{\text{sp}(n)}, \quad F = j^{0}, \quad G^{+} = \sum_{i=1}^{n} : X^{i} \gamma^{i} :, \quad G^{-} = \sum_{i=1}^{n} : \beta^{i} X^{i} \gamma^{i} :.
\]

**Example 7.6.** Let \(g = \mathfrak{gl}_{n}\) and \(A_{k} = V_{k-1} (\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)\). There is a map \(V_{k}(\mathfrak{gl}_{n}) \to V_{k-1} (\mathfrak{sl}_{n+1})\) corresponding to the natural embedding \(\mathfrak{gl}_{n} \to \mathfrak{sl}_{n+1}\), and a homomorphism \(V_{1}(\mathfrak{gl}_{n}) \to \mathcal{E}(n)\) appearing in [FKRW], so we have a diagonal homomorphism \(V_{k}(\mathfrak{gl}_{n}) \to A_{k}\). Then \(C_{k} = \text{Com}(V_{k}(\mathfrak{gl}_{n}), A_{k})\) satisfies \(\lim_{k \to \infty} C_{k} = (\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}\), where \(\mathcal{H}(2n)\) is the Heisenberg algebra with generators \(a^{1}, \ldots, a^{n}\) and \(\bar{a}^{1}, \ldots, \bar{a}^{n}\) satisfying

\[
a^{i}(z)\bar{a}^{j}(w) \sim \delta_{i,j}(z - w)^{-2}, \quad a^{i}(z) a^{j}(w) \sim 0, \quad \bar{a}^{i}(z) \bar{a}^{j}(w) \sim 0.
\]
Then \((\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}\) has a minimal strong generating set

\[ j^k = \sum_{i=1}^{n} b^i \partial^k c^i, \quad w^k = \sum_{i=1}^{n} a^i \partial^k \bar{a}^i, \quad 0 \leq k \leq n - 1, \]

\[ \nu^k = \sum_{i=1}^{n} b^i \partial^k \bar{a}^i, \quad \mu^k = \sum_{i=1}^{n} a^i \partial^k c^i, \quad 0 \leq k \leq n - 1, \]

In particular, \((\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}\) has even generators in weights 1, 2, 3, \ldots, \(n, n+1\), and odd generators in weights \(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \ldots, \frac{2n+1}{2}\). Therefore \(C_k\) has a minimal strong generating set in the same weights for generic values of \(k\). Finally, \((\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}\) has an \(N = 2\) structure given by

\[ (7.3) \quad L = -j^1 + \frac{1}{2} \partial j^0 - w^0, \quad F = j^0, \quad G^+ = \nu^0, \quad G^- = \mu^0, \]

which deforms to an \(N = 2\) structure on \(C_k\).

**Example 7.7.** Let \(g = sl_n\) and \(A_k = V_{k-1}(sl_n) \otimes L_1(sl_n)\). Note that \(L_1(sl_n) \cong \text{Com}(\mathcal{H}, \mathcal{E}(n))\) where \(\mathcal{H}\) is the copy of the rank one Heisenberg algebra generated by \(\sum_{i=1}^{n} b^i \partial^i\) and \(\mathcal{E}(n)\) is the rank \(n\) \(bc\)-system. Then \(C_k = \text{Com}(V_k(sl_n), A_k)\) satisfies

\[ \lim_{k \to \infty} C_k \cong L_1(sl_n)^{SL(n)} = \text{Com}(\mathcal{H}, \mathcal{E}(n))^{SL(n)} \cong \text{Com}(\mathcal{H}, \mathcal{E}(n))^{GL(n)} = \text{Com}(\mathcal{H}, \mathcal{E}(n))^{GL(n)}. \]

By \([FKRW]\), \(\mathcal{E}(n)^{GL(n)} \cong \mathcal{W}_1+...+n \cong \mathcal{W}(sl_n)\) so \(\lim_{k \to \infty} C_k \cong \mathcal{W}(sl_n)\) and hence is of type \(\mathcal{W}(2, 3, \ldots, n)\). It follows that \(C_k\) is of type \(\mathcal{W}(2, 3, \ldots, n)\) for generic values of \(k\).

**Example 7.8.** Let \(g = so_n\) and let \(A_k = V_{k-1}(so_n) \otimes L_1(so_n)\). We have a projection \(V_k(so_n) \to L_1(so_n)\), and a diagonal map \(V_k(so_n) \to A_k\). In this case we are interested not in \(C_k = \text{Com}(V_k(so_n), A_k)\) but in the orbifold \((C_k)^{\mathbb{Z}/2\mathbb{Z}}\). Note that \(\mathbb{Z}/2\mathbb{Z}\) acts on each of the vertex algebras \(V_k(so_n), V_{k-1}(so_n)\) and \(L_1(so_n);\) the action is defined on generators by multiplication by \(-1\). There is an induced action of \(\mathbb{Z}/2\mathbb{Z}\) on \(C_k\).

We have isomorphisms

\[ \lim_{k \to \infty} ((C_k)^{\mathbb{Z}/2\mathbb{Z}}) \cong \lim_{k \to \infty} (C_k)^{\mathbb{Z}/2\mathbb{Z}} \cong (L_1(so_n)^{SO(n)})^{\mathbb{Z}/2\mathbb{Z}} \cong L_1(so_n)^{O(n)} \cong F(n)^{O(n)}. \]

This appears as Theorems 14.2 and 14.3 of \([KHW]\) in the cases where \(n\) is even and odd, respectively; in both cases, \(L_1(so_n)^{SO(n)}\) decomposes as the direct sum of \(F(n)^{O(n)}\) and an irreducible, highest-weight \(F(n)^{O(n)}\)-module. Therefore \(\lim_{k \to \infty} (C_k)^{\mathbb{Z}/2\mathbb{Z}}\) is of type \(\mathcal{W}(2, 4, \ldots, 2n)\), so \((C_k)^{\mathbb{Z}/2\mathbb{Z}}\) is of type \(\mathcal{W}(2, 4, \ldots, 2n)\) for generic values of \(k\). This proves the conjecture appearing in Table 7 of \([BH]\).

**Example 7.9.** Let \(g = so_n\) and let \(A_k = V_{k-1}(so_{n+1}) \otimes F(n)\). Recall that we have a map \(V_1(so_n) \to F(n)\), so we have a diagonal map \(V_k(so_n) \to A_k\). As above, there is an action of \(\mathbb{Z}/2\mathbb{Z}\) acts on each of the vertex algebras \(V_k(so_n), V_{k+1}(so_{n+1})\) and \(F(n)\), and therefore on \(C_k\), and we are interested in the orbifold \((C_k)^{\mathbb{Z}/2\mathbb{Z}}\). We have \(C_k = (\mathcal{H}(n) \otimes F(n))^{SO(n)}\), and \((C_k)^{\mathbb{Z}/2\mathbb{Z}} = (\mathcal{H}(n) \otimes F(n))^{O(n)}\). As in Example 7.3, \((C_k)^{\mathbb{Z}/2\mathbb{Z}}\) has the following minimal strong generating set.

\[ w^{k+1} = \sum_{i=1}^{n} : \phi^i \partial^{2k+1} \phi^i :, \quad j^{2k} = \sum_{i=1}^{n} : \alpha^i \partial^{2k} \alpha^i :, \quad 0 \leq k \leq n - 1, \]
\[ \mu^k = \sum_{i=1}^{n} : \alpha^i \partial^k \phi^i : , \quad 0 \leq k \leq 2n - 1. \]

Therefore \((C_k)^{\mathbb{Z}/2^2}\) has even generators in weights 2, 2, 4, 4, \ldots, 2n, 2n and odd generators in weights \(2, 4, 6, \ldots, \frac{2n+2}{2}\), for generic values of \(k\). Moreover, \((\mathcal{H}(n) \otimes \mathcal{F}(n))^O(n)\) has an \(N = 1\) superconformal structure with generators \(L = -w^0 + \frac{1}{2}j^0\) and \(\mu^0\), which deforms to an \(N = 1\) structure on \((C_k)^{\mathbb{Z}/2^2}\).

**Example 7.10.** Let \(\mathfrak{g}\) be a simple, finite-dimensional Lie algebra of rank \(d\), with Cartan subalgebra \(\mathfrak{h}\). Recall that for a positive integer \(k\), the parafermion algebra \(N_k(\mathfrak{g})\) is defined to be \(\text{Com}(\mathcal{H}(d), L_k(\mathfrak{g}))\), where \(\mathcal{H}(d)\) is the Heisenberg algebra corresponding to \(\mathfrak{h}\) and \(L_k(\mathfrak{g})\) is the irreducible affine vertex algebra at level \(k\). The coset \(C_k(\mathfrak{g}) = \text{Com}(\mathcal{H}(d), V_k(\mathfrak{g}))\) is defined for all \(k \in \mathbb{C}\), and for a positive integer \(k\), \(N_k(\mathfrak{g})\) is the irreducible quotient of \(C_k(\mathfrak{g})\) by its maximal proper ideal. In the case \(g = \mathfrak{sl}_2\), it was shown in [DLWY] that \(C_k(\mathfrak{sl}_2)\) is of type \(W(2, 3, 4, 5)\). This was used to establish the \(C_2\)-cofiniteness of \(N_k(\mathfrak{sl}_2)\), and plays an important role in the structure of \(N_k(\mathfrak{g})\) for a general simple \(\mathfrak{g}\) [ALY].

For any simple \(\mathfrak{g}\) of rank \(d\), a choice of simple roots give rise to \(d\) copies of \(\mathfrak{sl}_2\) inside \(\mathfrak{g}\) which generate \(\mathfrak{g}\). We have corresponding embeddings of \(W(2, 3, 4, 5)\) into \(C_k(\mathfrak{g})\), whose images generate \(C_k(\mathfrak{g})\) [DW]. However, these do not strongly generate \(C_k(\mathfrak{g})\). Corollary 6.4 implies that \(C_k(\mathfrak{g})\) is strongly finitely generated for generic values of \(k\) for any simple \(\mathfrak{g}\). We shall construct a minimal strong generating set for \(C_k(\mathfrak{sk}_3)\) consisting of 30 elements.

We work in the usual basis for \(\mathfrak{sl}_3\) consisting of \(\{\xi_{ij} | i \neq j\}\) together with \(\{\xi_{ii} - \xi_{i+1,i+1}\}\) for \(i = 1, 2\). We have \(\lim_{k \to \infty} V_k(\mathfrak{sk}_3) = \mathcal{H}(2) \otimes \tilde{A}\) where \(\tilde{A} \cong \mathcal{H}(6)\) with generators \(\alpha_{i}^{12}, \alpha_{i}^{23}, \alpha_{i}^{13}, \alpha_{i}^{21}, \alpha_{i}^{32}, \alpha_{i}^{31}\). After suitably rescaling, these generators satisfy

\[ \alpha_{12}(z)\alpha_{21}(w) \sim (z-w)^{-2}, \quad \alpha_{23}(z)\alpha_{32}(w) \sim (z-w)^{-2}, \quad \alpha_{13}(z)\alpha_{31}(w) \sim (z-w)^{-2}. \]

By classical invariant theory, \(\mathcal{H}(6)^G\) for \(G = \mathbb{C}^{\times} \times \mathbb{C}^{\times}\) has a strong generating set

\[ q_{i,j}^{12} = : \partial_i \alpha_{12} \partial_j \alpha_{21} :, \quad q_{i,j}^{13} = : \partial_i \alpha_{13} \partial_j \alpha_{31} :, \quad q_{i,j}^{23} = : \partial_i \alpha_{23} \partial_j \alpha_{32} :, \quad i, j \geq 0, \]

\[ c_{i,j,k} = : \partial_i \alpha_{21} \partial_j \alpha_{32} \partial_k \alpha_{13} :, \quad c'_{i,j,k} = : \partial_i \alpha_{21} \partial_j \alpha_{32} \partial_k \alpha_{13} :, \quad i, j, k \geq 0. \]

Not all of these generators are necessary. In fact, \(\{q_{0,i}^{12} | 0 \leq i \leq 3\}, \{q_{0,i}^{23} | 0 \leq i \leq 3\}\), and \(\{q_{0,i}^{13} | 0 \leq i \leq 3\}\) generate three commuting copies of \(W(2, 3, 4, 5)\), and all the above quadratics lie in one of these copies. Similarly, we need at most \(\{c_{i,j,k}, c'_{i,j,k} | i, j, k \leq 2\}\). This follows from the decoupling relations

\[ : q_{i,0}^{12} c_{i,j,k} : = -\frac{i}{2i + 4} c_{i+2,j,k}, \quad i \geq 1, \quad j, k \geq 0, \]

\[ : q_{i,0}^{23} c_{i,j,k} : = -\frac{j}{2j + 4} c_{i,j+2,k}, \quad j \geq 1, \quad i, k \geq 0, \]

\[ : q_{0,i}^{31} c_{i,j,k} : = -\frac{k}{2k + 4} c_{i,j+2,k}, \quad k \geq 1, \quad i, j \geq 0, \]

\[ : q_{0,i}^{21} c'_{i,j,k} : = -\frac{i}{2i + 4} c'_{i+2,j,k}, \quad i \geq 1, \quad j, k \geq 0, \]

\[ : q_{0,i}^{32} c'_{i,j,k} : = -\frac{j}{2j + 4} c'_{i,j+2,k}, \quad j \geq 1, \quad i, k \geq 0, \]

\[ : q_{0,i}^{13} c'_{i,j,k} : = -\frac{k}{2k + 4} c'_{i,j+2,k}, \quad k \geq 1, \quad i, j \geq 0. \]
There are some relations among the above cubics and their derivatives, such as \( \partial c_{0,0,0} = c_{1,0,0} + c_{0,1,0} + c_{0,0,1} \). A minimal strong generating set for \( H(6)^G \) consists of
\[
\{ q_{0,i}^{12}, q_{0,i}^{23}, q_{0,i}^{13} | 0 \leq i \leq 3 \} \cup \{ c_{0,j,k}, c'_{0,j,k} | 0 \leq j,k \leq 2 \}.
\]
We conclude that for generic values of \( k \), \( C_k(\mathfrak{sl}_3) \) is of type \( W(2^3, 3^5, 4^7, 5^9, 6^4, 7^2) \). In other words, a minimal strong generating set consists of 3 fields in weight 2, 5 fields in weight 3, 7 fields in weight 4, 9 fields in weight 5, 4 fields in weight 6, and 2 fields in weight 7.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA
E-mail address: creutzig@ualberta.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER
E-mail address: andrew.linshaw@du.edu