SCHWARZ LEMMAS FOR MAPPINGS WITH BOUNDED LAPLACIAN

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Abstract. We establish some Schwarz type Lemmas for mappings defined on the unit disk with bounded Laplacian. Then we apply these results to obtain boundary versions of the Schwarz lemma.

1. Introduction and preliminaries

Motivated by the role of the Schwarz lemma in complex analysis and numerous fundamental results, see for instance [15, 16] and references therein, in 2016, the first author [20](a) has posted on ResearchGate the project “Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis”\footnote{Motivated by S. G. Krantz paper [13].}. Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question “What are the most recent versions of the Schwarz lemma ?”\footnote{The subject has been presented at Belgrade analysis seminar [21].} In this project and in [15], cf. also [11], we developed the method related to holomorphic mappings with strip codomain (we refer to this method as the approach via the Schwarz-Pick lemma for holomorphic maps from the unit disc into a strip). It is worth mentioning that the Schwarz Lemma has been generalized in various directions, see [1] [11] [12] [14] [19].

Recently X. Wang, J.-F. Zhu [18] and Chen and Kalaj [5] have studied boundary Schwarz lemma for solutions to Poisson’s equation. They improved Heinz’s theorem [8] and Theorem A below. We found that Theorem A is a forgotten result of H. Hethcote [9], published in 1977.

Note that previously B. Burgeth [2] improved the above result of Heinz and Theorem for real valued functions (it is easy to extend his result for complex valued functions, see below) by removing the assumption $f(0) = 0$ but it is overlooked in the literature. Recently, M. Mateljević and M. Svetlik [14] proved a Schwarz lemma for real harmonic functions with values in $(-1, 1)$ using a completely different approach than B. Burgeth [2]. In this paper, we further develop the method initiated in [14]. More precisely, we show if $f : \mathbb{U} \rightarrow (-1, 1)$, $f \in C^2(\mathbb{U})$ and continuous on...
If we consider the function

$$u(z) := P[\phi](z) - G[g](z),$$
then $u$ satisfies the Poisson’s equation

$$\begin{cases}
\Delta u = g & \text{on the disk } \mathbb{U}, \\
\lim_{r \to 1} u(re^{i\theta}) = \phi(e^{i\theta}) & \text{a.e. on the circle}.
\end{cases}$$

One can easily see that the previous equation has non-unique solution. Indeed, the Poisson kernel $P(z) = \frac{1 - |z|^2}{|1 - z|^2}$ is a harmonic function on the unit disk and $\lim_{r \to 1} P(re^{i\theta}) = 0$ a.e., but $P \neq 0$.

It is well known that if $\phi$ is continuous on the unit circle, then the harmonic function $P[\phi]$ extends continuously on $\mathbb{T}$ and equals to $\phi$ on $\mathbb{T}$, see Hörmander [10].

The following is a consequence of the maximum principle for subharmonic functions.

**Theorem** (Harmonic Majoration). *Let $u$ be a subharmonic function in $\mathcal{C}^2(\mathbb{U}) \cap \mathcal{C}(\overline{\mathbb{U}})$. Then

$$u \leq P[u|_T] \text{ on } \mathbb{U}.$$*

**Notations:** For $b \in (-1, 1)$, and $r \in [0, 1]$ let us denote

$$A_b(r) := \frac{1 - r^2}{1 + r^2} b + \frac{4}{\pi} \arctan r,$$

$$B_b(r) := \frac{1 - r^2}{1 + r^2} b - \frac{4}{\pi} \arctan r,$$

$$M_b(r) := \frac{4}{\pi} \arctan \frac{a + r}{1 + ar}$$

and

$$m_b(r) := M_b(-r) = \frac{4}{\pi} \arctan \frac{a - r}{1 - ar},$$

where

$$a = \tan \frac{b\pi}{4}.$$  

Finally, if $f \in \mathcal{C}^2(\mathbb{U}) \cap \mathcal{C}(\overline{\mathbb{U}})$, then $f^* := f|_{\mathbb{T}}$.

2. **THE SCHWARZ LEMMA FOR HARMONIC FUNCTIONS FROM $\mathbb{U}$ INTO $\mathbb{U}$**

The classical Schwarz Lemma (Heinz's theorem [8]) for a harmonic mapping $f$ from the unit disc to the unit such that $f(0) = 0$ is given by

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|.$$  

Moreover, this inequality is sharp for each point $z \in \mathbb{U}$. 
Later, in 1977, H. Hethcote [9] improved the above result of Heinz by removing the assumption $f(0) = 0$ and showed the following:

**Theorem A.** [9, 17] If $f$ is a harmonic mapping from the unit disc to the unit disc, then
\[
|f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0)| \leq \frac{4}{\pi} \arctan |z|.
\]

Then B. Burgeth [2] improved the above result of Heinz and Theorem A for real valued functions (it is easy to extend his result for complex valued functions, see below) by removing the assumption $f(0) = 0$.

First, we remark that the estimate of Theorem A cannot be sharp for all values $z$ in the unit disc. Indeed, let $f$ be a real harmonic functions with codomain $(-1, 1)$ and let $b = f(0)$. A simple study of the function $A_b$, defined by $A_b = \frac{1 - r^2}{1 + r^2}b + \frac{4}{\pi} \arctan r$, shows that if $b > \frac{2}{\pi}$, then there exists $r_0$ unique in $(0, 1)$ such that $A_b(r_0) = 1$. Moreover, $\max_{r \in [0,1]} A_b(r) > 1$. Similarly, for $b < -\frac{2}{\pi}$, there exists $r_1$ unique in $(0, 1)$ such that $B_b(r_1) = -1$ and $\min_{r \in [0,1]} B_b(r) < -1$, where $B_b(r) := \frac{1 - r^2}{1 + r^2}b - \frac{4}{\pi} \arctan r$.

Recently, M. Mateljević and M. Svetlik [14] proved a Schwarz lemma for real harmonic functions with values in $(-1, 1)$ using completely different approach than B. Burgeth [2].

**Theorem B.** [14] Let $u : U \to (-1, 1)$ be a harmonic function such that $u(0) = b$. Then
\[
m_b(|z|) \leq u(z) \leq M_b(|z|) \quad \text{for all} \quad z \in U.
\]
This inequality is sharp for each point $z \in U$.

Let us recall that, $M_b(r) = \frac{4}{\pi} \arctan \frac{a + r}{1 + ar}$, $m_b(r) = \frac{4}{\pi} \arctan \frac{a - r}{1 - ar}$ and,
\[a = \tan \frac{b\pi}{4}.
\]

Clearly this result improves Theorem A for real harmonic functions, since

**Proposition 2.1.** Let $b$ be a real number in $(-1, 1)$. Then
\[
(2.1) \quad M_b \leq A_b \quad \text{and} \quad B_b \leq m_b \quad \text{on} \quad [0,1].
\]

That is for any $r \in [0,1]$ and any $b \in (-1, 1)$, we have
\[
\frac{4}{\pi} \arctan \frac{a + r}{1 + ar} \leq \frac{1 - r^2}{1 + r^2}b + \frac{4}{\pi} \arctan r,
\]
and,
\[
\frac{1 - r^2}{1 + r^2} b - \frac{4}{\pi} \arctan r \leq \frac{4}{\pi} \arctan \frac{a - r}{1 - ar},
\]
where \(a = \tan \frac{b\pi}{4}\).

**Proof.** Let us show the first inequality
\[
\frac{4}{\pi} \arctan \frac{a + r}{1 + ar} \leq \frac{1 - r^2}{1 + r^2} b + \frac{4}{\pi} \arctan r,
\]
that is:
\[
\arctan \frac{a + r}{1 + ar} - \arctan r \leq \frac{1 - r^2 (b\pi)}{4}
\]
which is equivalent to
\[
\arctan \left( \frac{a(1 - r^2)}{r^2 + 2ar + 1} \right) \leq \frac{1 - r^2}{1 + r^2} \arctan(a).
\]

Let fix \(r \in (0, 1)\) and consider the mapping
\[
\phi(a) = \arctan \left( \frac{a(1 - r^2)}{r^2 + 2ar + 1} \right) - \frac{1 - r^2}{1 + r^2} \arctan(a).
\]
The derivative of \(\phi\) is equal to
\[
\phi'(a) = \frac{4r (r^2 - 1) a}{(r^2 + 1)(a^2 + 1)((r^2 + 1) a^2 + 4ra + r^2 + 1)}.
\]
Thus \(\phi\) has a maximum at \(a = 0\). As \(\phi(0) = 0\), we get \(\phi(a) \leq 0\). \(\square\)

Combining Theorem B and Proposition 2.1, we get

**Corollary 1.** If \(u : U \to (-1, 1)\) is harmonic, then

\[
u(z) = \frac{1 - |z|^2}{1 + |z|^2} b \leq M_{\delta}(|z|) - \frac{1 - |z|^2}{1 + |z|^2} b \leq \frac{4}{\pi} \arctan |z|.
\]

and

\[
u(z) = \frac{1 - |z|^2}{1 + |z|^2} b \geq m_{\delta}(|z|) - \frac{1 - |z|^2}{1 + |z|^2} b \geq -\frac{4}{\pi} \arctan |z|.
\]

where \(b = u(0)\).

**Remark 1.** One may obtain Proposition 2.1 using Theorem B.
Indeed, for \(b \in (-1, 1)\), let Har\((b)\) denote the family of all real valued harmonic maps \(u\) from \(U\) into \((-1, 1)\) with \(u(0) = b\). For \(z \in U\), set \(M_{\delta}^*(z) = \sup\{u(z) : u \in \text{Har}(b)\}\).

If \(v \in \text{Har}(b)\), then we have \(v(r) \leq A_{\delta}(r)\), we call \(A_{\delta}\) a majorant for \(\text{Har}(b)\). It is clear that \(M_{\delta}^* \leq A_{\delta}\).

From the proof of Theorem 5 [14], it follows that \(M_{\delta}^*(r) = M_{\delta}(r)\) on \([0, 1)\). Hence \(M_{\delta} \leq A_{\delta}\) on \([0, 1)\).
Our first aim is to extend Theorem B for complex harmonic functions from the unit disc to the unit disc, and to recover Theorem A.

**Theorem 1.** Let \( f : \mathbb{U} \to \mathbb{U} \) be a harmonic function from the unit disc to the unit disc. Then
\[
|f(z)| \leq M_b(|z|), \text{ where } b = |f(0)|.
\]

**Proof.** (1) Fix \( z_0 \) in the unit disc and choose unimodular \( \lambda = e^{i\alpha} \) such that \( \lambda f(z_0) > 0 \).

Define \( u(z) = \Re(\lambda f(z)); u(z_0) = |f(z_0)|. \)

Hence, using Theorem B, we get
\[
|f(z_0)| \leq u(z_0) \leq M_{u(0)}(|z_0|) \leq M_{f(0)}(|z_0|).
\]

As the mapping \( b \mapsto A_b(|z_0|) \) is increasing. \( \square \)

**Remark 2.**

One can easily recover the estimate
\[
\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \leq \frac{4 \pi}{\arctan |z|}
\]
obtained in Theorem A.

Indeed, let \( u(z) = \Re(\lambda f(z)) \). By equations 2.2 and 2.3, we have
\[
\left| u(z) - \frac{1 - |z|^2}{1 + |z|^2} u(0) \right| \leq \frac{4 \pi}{\arctan |z|}.
\]

But
\[
u(z) - \frac{1 - |z|^2}{1 + |z|^2} u(0) = \Re \left( \lambda \left( f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right) \right).
\]

Now, one can choose \( \lambda \), such that \( u(z) - \frac{1 - |z|^2}{1 + |z|^2} u(0) = \left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \).

3. **Schwarz Lemma for mappings with bounded Laplacian**

In [5], for a given continuous function \( g : G \to \mathbb{C} \), Chen and Kalaj establish some Schwarz type Lemmas for mappings \( f \) in \( G \) satisfying the Poisson’s equation \( \Delta f = g \), where \( G \) is a subset of the complex plane \( \mathbb{C} \). Then they apply these results to obtain a Landau type theorem, which is a partial answer to the open problem in [6]. Set \( A(z) = \frac{1 - |z|^2}{1 + |z|^2} \).

We claim that Theorem 1 in Chen and Kalaj paper [5] holds under additional conditions. Here we suggest a modification of this result.

**Theorem C.** Let \( g \in C(\overline{\mathbb{U}}) \) and \( \phi \in L^\infty(\mathbb{T}) \). Then the function
\[
f := P[\phi] - G[g]
\]
is $C^2(\mathbb{U})$ and satisfies the Poisson’s equation

\[
\begin{cases}
\Delta f = g, \\
f|_{S^1} = \phi.
\end{cases}
\]

In addition, we have

\[
|f(z) - P[\phi](0)A(z)| \leq \frac{4}{\pi}||P[\phi]||_{\infty} \arctan |z| + \frac{1}{4}||g||_{\infty}(1 - |z|^2),
\]

where $||P[\phi]||_{\infty} = \sup_{z \in \mathbb{U}}|P[\phi](z)|$ and $||g||_{\infty} = \sup_{z \in \mathbb{D}}|g(z)|$. By $f|_{S^1} = \phi$, we mean that $\lim_{r \to 1^{-}}f(re^{i\theta}) = \phi(e^{i\theta})$ a.e.

**Remark 3.** Let $g \in C(\mathbb{U})$ and $\phi \in C(\mathbb{T})$. Then the solutions of the Poisson’s equation

\[
\begin{cases}
\Delta f = g, \\
f|_{S^1} = \phi,
\end{cases}
\]

are given by $f := P[\phi] - G[g]$, see [10] p. 118-120, and we have the estimate (3.1). Here the boundary condition means that $\lim_{z \in \mathbb{U}, z \to \xi}f(\xi) = \phi(\xi)$, for all $\xi \in \mathbb{T}$. Thus the version stated in Chen and Kalaj (Theorem 1 in [5]) holds in this setting.

Now we show that Theorem 4 (the next result below) implies Theorem C.

Let $f$ be a real valued $C^2(\mathbb{U})$ continuous on $\overline{\mathbb{U}}$, such that $\Delta f = g$ and $f^* = f|_{\mathbb{T}}$. Let $K := ||P[f^*]||_{\infty}$. By Theorem 4 we have

\[
m_b/K(|z|)K - ||g||_{\infty} \frac{(1 - |z|^2)}{4} \leq f(z) \leq M_b/K(|z|)K + ||g||_{\infty} \frac{(1 - |z|^2)}{4},
\]

where $b = P[f^*](0)$. Using Proposition 2.1 we get

\[
M_b/K(|z|)K \leq \frac{1 - |z|^2}{1 + |z|^2}b + \frac{4K}{\pi} \arctan |z|,
\]

\[
m_b/K(|z|)K \geq \frac{1 - |z|^2}{1 + |z|^2}b - \frac{4K}{\pi} \arctan |z|.
\]

Hence

\[
\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2}P[f^*](0) \right| \leq \frac{4}{\pi}||P[f^*]||_{\infty} \arctan |z| + \frac{1}{4}||g||_{\infty}(1 - |z|^2)
\]

If $f$ is complex valued function, we consider $u = \Re(\lambda f)$, where $\lambda$ is a complex number of modulus 1.
Theorem 4. Let $f$ be $C^2(\mathbb{U})$ real-valued function, continuous on $\mathbb{U}$. Let $b = P[f^*](0)$, $c \in \mathbb{R}$ and $K$ is a positive number such that $K \geq ||P[f^*]||_{\infty}$, where $f^* = f|_{S^1}$.

(i) If $f$ satisfies $\Delta f \geq -c$, then
$$f(z) \leq KM_{b/K}(|z|) + \frac{c}{4}(1 - |z|^2).$$

(ii) If $f$ satisfies $\Delta f \leq c$, then
$$f(z) \geq Km_{b/K}(|z|) - \frac{c}{4}(1 - |z|^2).$$

Proof. (i) Define $f^0(z) = f(z) + \frac{c}{4}(|z|^2 - 1)$, and set $P[f^*](0) = b$. Then $f^0$ is subharmonic and $f^0 \leq P[f^*]$. As $\frac{1}{K}P[f^*]$ is a real harmonic function with codomain $(-1, 1)$, by Theorem 5 in [14], we obtain $P[f^*](z) \leq KM_{b/K}(|z|)$. Thus
$$f(z) \leq KM_{b/K}(|z|) + \frac{c}{4}(1 - |z|^2), \text{ for all } z \in \mathbb{U}.$$

(ii) If $f$ satisfies $\Delta f \leq c$, then define $f_0(z) = f(z) - \frac{c}{4}(|z|^2 - 1)$, and set $P[f^*](0) = b$. In a similar way, using Theorem 5 [14], we show that for all $z \in \mathbb{U}$, we get
$$f(z) \geq Km_{b/K}(|z|) - \frac{c}{4}(1 - |z|^2).$$

\[\square\]

Corollary 2. Suppose that $f : \mathbb{U} \to \mathbb{U}$, $f \in C^2(\mathbb{U})$ and continuous on $\mathbb{U}$, and $|\Delta f| \leq c$ on $\mathbb{U}$ for some $c > 0$. Then
$$|f(z)| \leq M_b(|z|) + \frac{c}{4}(1 - |z|^2),$$
where $b = |P[f^*](0)|$.

Proof. Fix $z_0$ in the unit disc and choose $\lambda$ (depends on $z_0$) such that $\lambda f(z_0) = |f(z_0)|$. Define $u(z) = \Re(\lambda f(z))$ (we say that $u$ is real valued harmonic associated to complex valued harmonic $f$ at $z_0$). We have $\Delta u = \Re(\lambda \Delta f)$. As $u$ is a real function with codomain $(-1, 1)$ satisfying $|\Delta u| \leq c$, by Theorem [1] we get
$$|u(z)| \leq M_{b_1}(|z|) + \frac{c}{4}(1 - |z|^2), \text{ where } b_1 = P[u^*](0).$$
We have $b_1 = P[u^*](0) = \Re(\lambda P[f^*](0)) \leq |P[f^*](0)|$. Hence
$$|f(z_0)| \leq M_{b_1}(|z_0|) + \frac{c}{4}(1 - |z|^2),$$
where $b = |P[f^*](0)|$, as the mapping $b \mapsto A_b(|z_0|)$ is increasing. \[\square\]

Similarly, under the conditions of the previous theorem, we obtain
\begin{equation}
|f(z) - P[f^*](0)A(z)| \leq \frac{4}{\pi} \arctan |z| + \frac{c}{4}(1 - |z|^2)
\end{equation}
4. Boundary Schwarz Lemmas

We establish Schwarz Lemmas at the boundary for solutions to $|\Delta f| \leq c$. Our results are generalizations of Theorem 1.1 [18] and Theorem 2 [5].

**Theorem 5.** Suppose $f \in C^2(U)$, continuous on $\overline{U}$ with codomain $(-1,1)$, such that $\Delta f \geq -c$. If $f$ is differentiable at $z = 1$ with $f(1) = 1$, then the following inequality holds:

$$f_x(1) \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2},$$

where $b = P[f^*](0)$.

Before giving the proof, one can easily show that

$$M'_b(r) = \frac{4}{\pi} \left[ \frac{1 - a^2}{(a^2 + 1) r^2 + 4ar + a^2 + 1} \right].$$

Hence

$$M'_b(1) = \frac{2}{\pi} \left[ \frac{1 - a}{1 + a} \right] = \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b).$$

As $a = \tan \frac{b\pi}{4}$.

**Proof.** Since $f$ is differentiable at $z = 1$, we know that

$$f(z) = 1 + f_z(1)(z - 1) + f_{\bar{z}}(1)(\bar{z} - 1) + o(|z - 1|).$$

That is

$$f_x(1) = \lim_{r \to 1^-} \frac{f(r) - 1}{r - 1}.$$

On the other hand, Theorem [4(i)] leads to

$$1 - M_b(r) - \frac{c}{4}(1 - r^2) \leq 1 - f(r).$$

Dividing by $(1 - r)$ and letting $r \to 1^-$, we get

$$f_x(1) \geq M'_b(1) - \frac{c}{2}. \tag{4.1}$$

Thus

$$f_x(1) \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2}.$$ 

$\square$
Corollary 3. Suppose $f \in C^2(\mathbb{U})$, continuous on $\mathbb{U}$ with codomain $(-1, 1)$ and differentiable at $z = 1$ with $f(1) = 1$.

(i) If $\Delta f \geq -c$ then,
$$f_x(1) \geq -b + \frac{2}{\pi} - \frac{c}{2}.$$

(ii) If $|\Delta f| \leq c$ and $f(0) = 0$, then $|b| \leq \frac{c}{4}$ and
$$f_x(1) \geq -\frac{3}{4} c + \frac{2}{\pi}.$$

Proof. (ii) Using $M_b \leq A_b$ and $M_b(1) = A_b(1) = 1$, we get
$$M'_b(1) \geq A'_b(1) = -b + \frac{2}{\pi}.$$

(ii) The estimate $|b| \leq \frac{c}{4}$ follows directly from Theorem 4.

Remark 6. One can also prove directly that $M'_b(1) \geq A'_b(1)$, that is
\begin{equation}
\frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \geq -b + \frac{2}{\pi} \quad \text{for } b \in [0, 1).
\end{equation}

Using the convexity of the tangent function, we get
$$\tan x \geq 2(x - \frac{\pi}{4}) + 1 \quad \text{for } x \in [0, \pi/2).$$

For $b \in [0, 1)$, let us substitute $x$ by $\frac{\pi}{4} (1 - b)$, we obtain
$$\frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \geq -b + \frac{2}{\pi}.$$

The following theorem is a generalization on Theorem 2 in [5] where the authors proved a Schwarz Lemma on the boundary for functions satisfying $\Delta f = g$ and under the assumption $f(0) = 0$.

Theorem 7. Suppose $f \in C^2(\mathbb{U}) \cap C(\mathbb{U})$ is a function of $\mathbb{U}$ into $\mathbb{U}$ satisfying
$$|\Delta f| \leq c,$$
where $0 \leq c < \frac{4}{\pi} \tan \frac{\pi}{4} (1 - b)$. If for some $\xi \in \mathbb{T}$, $\lim_{r \to 1^-} |f(r\xi)| = 1$, then
$$\liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2}.$$

where $b = |P[f^*](0)|$.

If, in addition, we assume that $f(0) = 0$, then,
$$|b| \leq \frac{c}{4}$$
and
\[ \liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq -\frac{3}{4} c + \frac{2}{\pi}. \]

**Proof.** Using Corollary 2, we have
\[ |f(\xi) - f(r\xi)| \geq 1 - |f(r\xi)| \geq 1 - M_b(r) - \frac{c}{4}(1 - r^2). \]
Thus
\[ \liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \lim_{r \to 1^-} \frac{1 - M_b(r) - \frac{c}{4}(1 - r^2)}{1 - r} = M'_b(1) - \frac{c}{2}. \]
The conclusion follows as \( M'_b(1) = \frac{2}{\pi} \tan \frac{\pi}{4}(1 - b) \).

If in addition, we assume that \( f(0) = 0 \), using Equation 3.2, we obtain \( |b| < \frac{c}{4} \).
Hence
\[ \liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \frac{2}{\pi} \tan \frac{\pi}{4}(1 - b) - \frac{c}{2} \geq -b + \frac{2}{\pi} - \frac{c}{2} \geq -\frac{3}{4} c + \frac{2}{\pi}. \]
The second estimates follows from inequality (4.2). \qed

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