Construction of a discrete planar contour by fractional rational Bezier curves of second order

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Abstract. A solution to the problem of formation of a smooth closed curve given an array of points is proposed. For the curve, a spline consisting of fractional rational Bezier curves of second order is taken. It is shown that upon appropriate reparametrization, the standard form of representation of this Bezier curve can be reduced to a more simple form. This form is convenient in construction of a closed spline from said segments, which are connected in the process of formation according to the second order of smoothness. Depending on the calculated value of control parameter in the proposed form of representation of fractional rational Bezier curve, it is possible to construct a closed spline of segments of certain curves of second order.

1. Introduction
Bezier curves find wide application:

- in modern CAD systems for automated development of products of various function with respect for specified functional and aesthetic requirements [1–8, 10];
- in computer graphics in modeling of realistic 3D images and scenes, in 3D animation and dynamic process modeling, and in geometric form transformation [9, 10];
- in geometric form modeling given a discrete data array acquired as a result of experiments or mathematical calculations [11-15].

At that, it is rather often required to solve the problem of constructing a curve that either passes through (problem of interpolation) or passes by (problem of approximation or smoothing) points of a given spatial or planar array. A rather typical problem of interpolation is construction of a composite curve consisting of segments joined with a certain order of smoothness given a planar array of points. Generally, Bezier curves are used as segments of the composite curve [10, 14, 15, 16], while the curve itself constitutes an unclosed contour. At the same time, a number of practical applications require construction of a smooth closed contour given an array of points. These applications include:

- analytic shoe design [17];
- closed contour geometric modeling in formation of a family of offset curves in pocket machining of engineering products [6,11].

Therefore the problem of formation of a sufficiently regular composite closed contour consisting of Bezier curves given a planar array of points is considered urgent. One of the solutions to the problem is considered in the present paper.

2. Problem Definition
The equation of fractional rational Bezier curve of the second order ($BCfr^2$) is of the following standard form [7, 8, 18]:

$$Q(u) = \frac{(1-u)^2 \cdot Q_0 + 2 \cdot q_0 \cdot (1-u) \cdot u \cdot A_0 + u^2 \cdot Q_1}{(1-u)^2 + 2 \cdot q_0 \cdot (1-u) \cdot u + u^2},$$

(1)

where $u = \frac{t}{1+t}$; $t \in [0, \infty)$; $u \in [0,1)$. Reparametrization $u \rightarrow t$ results in the canonical form of fractional rational Bezier curve ($BCfr^2$):

$$Q(u) \rightarrow Q(u(t)) \rightarrow Q^*(t) = \frac{Q_0 + 2 \cdot q_0 \cdot t \cdot A_0 + t^2 \cdot Q_1}{1 + 2 \cdot q_0 \cdot t + t^2}.$$

(2)

Let us prove that parametric representations (1) and (2) describe the same curve ($BCfr^2$): $\{Q(u) | u \in [0,1]\} = \{Q^*(t) | t \in [0, \infty)\}$. There is a homeomorphic correspondence between intervals $I_u(u \in [0,1])$ and $I_t(t \in [0, \infty))$, which is constructively realized in the following way (figure 1):

1. First, the following homeomorphic correspondence is realized through orthogonal projection:

$$G_{ug}(I_u, g \uparrow A_u \leftrightarrow A_g).$$

2. Then the following homeomorphic correspondence is realized through central projection:

$$G_{g}(g, I_t \uparrow A_g \leftrightarrow A_t).$$

3. This results in homeomorphic correspondence

$$G_{ug} = G_{g} \cdot G_{ug}; \quad G_{ug}(I_u, I_t \uparrow A_u \leftrightarrow A_t).$$

Since both curves (1) and (2) are smooth according to second order curve property, on the basis of $G_{ug}(I_u, I_t)$ one can conclude that (1) and (2) describe the same geometric image – a curve ($BCfr^2$) in plane $R^2$. The primary objective is set as follows: given a convex array of points $\{Q_i\}_{i=0}^{n-1}$ in plane $R^2$, it is required to construct a spline ($BCfr^2$) of class $C^2$ with node points $Q_i$, $i=0,1,\ldots,n-1$, with specified tangents $\tau_i$ in said points (figure 2) and specified curvature $k(Q_0)$ in the initial point $Q_0$.

Figure 1. Homeomorphic correspondence between intervals $I_u$ and $I_t$.

Figure 2. A multitude $\{\tau_i\}$ of tangents of the constructed spline consisting of segments ($BCfr^2$).

3. Theory

In order to specify tangents and curvature $k(Q_0)$, let us first solve the auxiliary problem: it is required to construct a spline passing through nodes $Q_0$, $Q_1$, and $Q_2$, consisting of two curves ($BC^2$):

$$Q_{01}(t_i) = (1-t_i^2)Q_0 + 2t_i(1-t_i)A_0 + t_i^2Q_1,$$

(3)
where \( t_1, t_2 \in [0,1] \). At that, curves \( Q_{i_1}(t) \) and \( Q_{i_2}(t) \) have to be connected in the point \( Q_i \) according to the highest order of smoothness.

In order to solve the auxiliary problem, let us acquire the first and the second derivatives from equations (3) and (4):

\[
\begin{align*}
Q'_{00}(t) &= 2(1-t_1) \cdot (A_{i0} - Q_{i0}) + 2t_1 \cdot (Q_{i1} - A_{i1}) ;
Q'_{00}(t) &= 2(Q_{i0} + Q_{i1} - 2A_i) ;
Q'_{i2}(t_2) &= 2Q_{i2} + 2Q_{i1} - 2A_i ;
Q'_{i2}(t_2) &= 2A_{i1} - 2A_i .
\end{align*}
\]

From the conditions \( Q_{00}'(t_1) = Q_{i2}'(t_2) = 0 \) and \( Q_{00}'(t_1) = Q_{i2}'(t_2) = 0 \) follow the equations

\[
\begin{align*}
2Q_i &= A_{i0} + A_{i1} ;
Q_0 - Q_2 &= 2 \cdot (A_{i0} - A_{i1}) ,
\end{align*}
\]

from which the control points are acquired:

\[
A_0 = Q_i - \frac{Q_0 - Q_2}{4} ;
A_1 = Q_i - \frac{Q_0 - Q_2}{4} .
\] (5)

Therefore, we have acquired a spline \( Q_{i2} = Q_{i0}(t_1) \cup Q_{i2}(t_2) \) consisting of two segments connected according to the first order of smoothness with equal curvature at junction point.

The acquired solution to the auxiliary problem allows us to include the following algorithm of distribution of tangents over the given array of points \( \{ Q_i \}_{i=1}^{n-1} \) in the initial data of the main problem:

1. According to formulas (5) the coordinates of the control points are acquired.
2. Points \( A_0, Q_0 \) define the tangent \( \tau_0 \); points \( A_0, A_1 \) define the tangent \( \tau_1 \). In the latter case the equation \( |A_0Q_0| = |Q_0A_i| \) takes place.
3. Points \( A_1, Q_2 \) define the tangent \( \tau_2 \) and a point \( A_2 \) on it that satisfies the equation \( |A_0Q_2| = |Q_2A_{i+1}| \) and so forth.

Therefore, the control point \( A_{i+1} \) and node \( Q_{i+1} \) define the tangent \( \tau_{i+1} \) and the control point \( A_{i+1} \) on it. Let us appoint curvature \( k(Q_i) \) of the constructed spline \((BC)^2\) of class \( C^2 \) equal to curvature of line \( Q_{i0}(t) \) in its point \( t_i = 0 \).

In order to solve the main problem, let us perform modification of the known algorithm [16]. The main point of the proposed modification is stated below. In the mentioned paper, as a segment of a curve of the second order, on the basis of which the formation of an outline passing through the given node points \( \{ Q_i \}_{i=1}^{n-1} \) is performed, it is proposed to apply a curve

\[
Q_{i,i+1}(t) \equiv \frac{Q_i + tA_{i+1} + \gamma t^2 Q_{i+1}}{1 + \gamma t^2} ,
\] (6)

where \( t = \frac{r}{1-r} ; \gamma \in [0,1] \), \( \gamma \) represents a certain numeric parameter. Analysis of acquiring and further utilization of the curve \( Q_{i,i+1}(t) \) given in the paper [16] allows us to draw the following conclusions:

1. Upon substitution of \( t = \frac{r}{1-r} \) into the equation (6) we acquire fractional rational function

\[
Q_i(t) = (1-r)^2 A_i r(1-r) + \frac{\gamma t^2 Q_{i+1}}{(1-r)^2 + r(1-r) + t^2} ,
\] (7)
which does not match function (1) that constitutes the standard form of definition of a fractional rational curve \((BC_{fr})^2\) [7, 8, 18].

2. Attaching parameter \(\gamma_i\) to the third component of numerator in equation (7), unlike attaching it to the second component in the standard form (1), does not allow us to directly specify type and form of the attached segment (parabola, hyperbola, ellipse).

Reasoning from the drawn conclusions 1 and 2, let us express the equation (2) of first segment \(s_{01}\) of the constructed spline in vector form:

\[
\tilde{r}_{01}(t) = \frac{\tilde{r}_{\Theta_0} + 2q_0 \tilde{r}_{A_0} + t^2 \tilde{r}_{\Theta_0}}{1 + 2q_0 t + t^2}.
\]

Let us define the first and the second derivatives of vector function \(\tilde{r}_{01}(t)\):

\[
\begin{align*}
\tilde{r}'_{01}(t) & = 2 \cdot q_0 \cdot (\tilde{r}_{A_0} - \tilde{r}_{\Theta_0}) + t \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{\Theta_0}) + q_0 \cdot t^2 \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{A_0}) \cdot \frac{1}{(1 + 2q_0 t + t^2)^2} \\
\tilde{r}''_{01}(t) & = 2 \cdot \left[\tilde{r}_{\Theta_0} (\tilde{r}_{A_0} - \tilde{r}_{\Theta_0}) + 2q_0 \cdot t \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{A_0}) \right] \cdot \left(1 + 2q_0 t + t^2\right)^2 - \left[ q_0 (\tilde{r}_{A_0} - \tilde{r}_{\Theta_0}) + t \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{\Theta_0}) + q_0 t^2 \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{A_0}) \right] \cdot 4 \cdot \left(1 + 2q_0 t + t^2\right) \cdot (q_0 + t) \cdot \frac{1}{(1 + 2q_0 t + t^2)^4}.
\end{align*}
\]

Let us acquire the values of the acquired derivatives in the initial point \(t=0\) of the first segment \(s_{01}\):

\[
\begin{align*}
\tilde{r}'_{01}(0) & = 2q_0 \cdot (\tilde{r}_{A_0} - \tilde{r}_{\Theta_0}) \\
\tilde{r}''_{01}(0) & = 2q_0 \cdot (\tilde{r}_{\Theta_0} - \tilde{r}_{\Theta_0}) - 8q_0^2 \cdot (\tilde{r}_{A_0} - \tilde{r}_{\Theta_0}).
\end{align*}
\]

Let us acquire the vector product of vectors (10) and (11):

\[
[\tilde{r}_{01}(0), \tilde{r}''_{01}(0)] = 4q_0 \left[ (\tilde{r}_{A_0}, \tilde{r}_{\Theta_0}) - (\tilde{r}_{\Theta_0}, \tilde{r}_{\Theta_0}) \right] + (\tilde{r}_{\Theta_0}, \tilde{r}_{A_0})].
\]

Let us acquire curvature \(k(Q_0)\) of the segment \(s_{01}\) in its initial point \(Q_0\):

\[
k(Q_0) = \frac{[\tilde{r}_{01}(0), \tilde{r}''_{01}(0)]}{[\tilde{r}'_{01}(0)]} = \frac{1}{2 \cdot q_0^3} \cdot \left[ \begin{array}{ccc} 0, 0, x_{\Theta_0} & 1 & 1 \\ 1, x_{A_0}, x_{\Theta_0} & 0 \\ y_{\Theta_0}, y_{\Theta_0}, y_{\Theta_0} & -y_{\Theta_0} \end{array} \right] = \frac{1}{2 \cdot q_0^3} \cdot \frac{2SA_{A_0} A_{Q_1}}{[Q_0, A_0]} = \frac{1}{2 \cdot q_0^3} \cdot \frac{2SA_{A_0} A_{Q_1}}{[Q_0, A_0]}.
\]

As a result, we acquire:

\[
q_0^3 = \frac{SA_{A_0} A_{Q_1}}{k(Q_0) \cdot [Q_0, A_0]}.
\]

According to the algorithm proposed in paper [16], let us construct the following segment \(s_{12}\) of spline \((BC_{fr})^2\). Let us put down the vector equation of this segment:

\[
\tilde{r}_{12}(t) = \frac{\tilde{r}_{\Theta_0} + 2q_1 \cdot t \cdot \tilde{r}_{A_1} + t^2 \cdot \tilde{r}_{\Theta_0}}{1 + 2q_1 \cdot t + t^2}, \quad t = \frac{u}{1 - u}, \quad u \in [0, 1]; \quad t \in [0, \infty).
\]

Since segments \(s_{01}\) and \(s_{12}\) connect in point \(Q_1\) according to the second order of smoothness with continuous curvature,
\[ \{k(Q_i)\}_{S_{12}} = \{k(Q_i)\}_{S_{23}} = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3}, \]

where \( q_0 \) is acquired through formula (14).

We acquire the following formula of curvature:

\[ \{k(Q_i)\}_{S_{12}} = \frac{k(Q_0) \cdot |Q_0 A_0|^3}{S \Delta Q_i A_i A_i |Q_i A_i|^3} = k(Q_0) \cdot \frac{|Q_0 A_0|^3}{|Q_i A_i|^3}. \] (16)

On the other hand, curvature of the curve \( s_{12} \) in the point \( Q_1 \) positioned in the triangle \( Q_1 A_1 Q_2 \) (figure 2) is acquired from the equation:

\[ \{k(Q_i)\}_{S_{12}} = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3}. \] (17)

As follows from equations (16) and (17),

\[ k(Q_0) \cdot \frac{|Q_0 A_0|^3}{|Q_i A_i|^3} = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3}, \] (18)

which allows us to derive the formula

\[ q_i^2 = \frac{S \Delta Q_i A_i A_i}{k(Q_0) \cdot |Q_i A_i|^3} \cdot \frac{|Q_i A_i|^3}{|Q_i A_i|^3}. \] (19)

Next, let us add the following curve – segment \( s_{23} \) – to the constructed spline \((BC, \beta)^2\):

\[ \vec{r}_{23}(t) = \vec{r}_{0} + 2q_2 \cdot t \cdot \vec{r}_{A} + t^2 \cdot \vec{r}_{0} \cdot t, \quad t = \frac{u}{1 - u} ; \quad u \in [0,1); \quad t \in [0, \infty). \] (20)

Curves \( s_{12} \) and \( s_{23} \) connect in point \( Q_2 \) under the same conditions, as in case of the segment \( s_{12} \).

Therefore, the following equation takes place:

\[ \{k(Q_i)\}_{S_{12}} = \{k(Q_i)\}_{S_{23}} = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3} = k(Q_0) \cdot \left( \frac{|Q_0 A_0|^3}{|Q_i A_i|^3} \right)^3. \] (21)

On the other hand, curvature of the curve \( s_{23} \) in the point \( Q_2 \) positioned in the triangle \( Q_2 A_2 Q_3 \) (figure 2) is acquired from the equation:

\[ \{k(Q_i)\}_{S_{23}} = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3}. \] (22)

As follows from equations (21) and (22),

\[ k(Q_0) \cdot \left( \frac{|Q_0 A_0|^3}{|Q_i A_i|^3} \right)^3 = \frac{1}{q_i^2} \cdot \frac{S \Delta Q_i A_i A_i}{|Q_i A_i|^3}, \]

which allows us to derive the formula

\[ q_i^2 = \frac{S \Delta Q_i A_i A_i}{k(Q_0) \cdot |Q_i A_i|^3} \cdot \left( \frac{|Q_i A_i|^3}{|Q_i A_i|^3} \right)^3. \]

Let us generalize by putting down the vector equation of a segment \( s_{j,i+1} \):

\[ \vec{r}_{i,i+1}(t) = \vec{r}_{0} + 2q_{i} \cdot t \cdot \vec{r}_{A} + t^2 \cdot \vec{r}_{Q_{i}}, \quad t = \frac{u}{1 - u}; \quad t \in [0, \infty); \quad u \in [0,1). \] (23)

Generalizing equation (23) in application to segment \( \vec{r}_{i,i+1}(t) \), we acquire the resultant formula
where $i=0, 1, \ldots, n-1$.

Apart from the generalized formula (25), another formula establishing correlation between parameters of neighboring segments of the constructed spline takes place:

$$q_{i+1}^2 = q_i^2 \frac{S \Delta Q_i A_i Q_{i+1} Q_{i+2}}{k(Q_i) |Q_i A_i|^2} \left( \frac{|Q_{i+1} A_i|}{|Q_{i+1} A_{i+1}|} \right)^3,$$

(26)

In paper [18] a classification of fractional rational curves ($BC^2$) in relation to control parameter value $\mu_i = q_i^2$ is provided. In case $\mu_i = 1$, curve ($BC^2$) constitutes a segment of a parabola, in case $\mu_i > 1$ – a segment of hyperbola and in case $\mu_i \in (0,1)$ – a segment of ellipse.

4. Examples of construction of splines ($BC^2$)

4.1. Construction of a spline ($BC^2$) from segments of curves of second order connected according to order of smoothness $C^2$

The initial data is an array of points $\{Q_i\}_0^3 : \{(0, 0), (20, 30), (50, 30), (80, 10)\}$ (figure 3).

![Figure 3](image_url)  
Figure 3. Spline ($BC^2$) constructed through nodes (0, 0), (20, 30), (50, 30), (80, 10).

Using equations (5), let us calculate coordinates of control points $A_0(7.5, 22.5)$ and $A_1(32.5, 37.5)$. Curvature of segment $Q_0Q_1$ of the constructed spline in the point $Q_0$ is calculated by formula (13), where

$$\tilde{r}^2(0) = 2A_0 - 2Q_0; \quad \tilde{r}'(0) = 2Q_0 - 4A_0 + 2Q_1.$$

As a result of the calculations, we acquire $k(Q_0) = 0.00843274042711568$. Let us now calculate the control parameter of segment $Q_0Q_1$ by formula (14): $q_0=1$. The control point of segment $Q_0Q_2$ can be acquired by applying the condition $|A_0Q_2| = |Q_2A_2|$ to the equation $A_2 = 2Q_2 - A_1$. As a result, we acquire its coordinates: $A_2(67.5, 22.5)$. The control parameters of the neighboring segments of the constructed spline are calculated using the formula (26). Their values are $q_1=1$ and $q_2=0.745355992499930$. Types of spline segments ($BC^2$) are defined by the calculated values of control parameters $q_i$: $Q_0Q_1$ is of parabolic type ($q_1^2=1$); $Q_0Q_2$ is of elliptic type ($q_2^2=0.5555555556$).

4.2. Construction of a closed spline ($BC^2$) from segments of curves of second order connected according to order of smoothness $C^2$

The initial data is an array of points $\{Q_i\}_0^3 : \{(0, 0), (0, 1), (1, 1), (1, 0)\}$ (figure 4).

According to the algorithm described earlier for spline ($BC^2$), let us calculate coordinates of control points: $A_0(-0.25, 0.75)$ and $A_1(0.25, 1.25)$, as well as curvature of segment $Q_0Q_1$ in point $Q_0$, $k(Q_0)=0.252982212813470$. Then let us calculate control parameter $q_0=1$ through to formula (14).
By means of formula (26) let us calculate values of control parameters for subsequent segments of the spline: \( q_1 = 1, q_2 = 1.73205080756888 \). In order to close the constructed spline \((BC)_2\), it is required to calculate coordinates of control point \( A_3 \) of the closing segment \( Q_3Q_0 \) and calculate the corresponding control parameter \( q_3 \). Since \( A_1 \in \tau_0 \) and \( A_1 \in \tau_1 \), coordinates of the control point \( A_3 \) of the closing segment \( Q_3Q_0 \) can be calculated as a point of intersection between the tangents \( A_2 = \tau_1 \cap \tau_1 \). The control parameter \( q_3 \) is calculated through formula (26), where \( A_4 = A_3 \); \( A_{i+1} = A_i ; Q_i = Q_3 ; Q_{i+1} = Q_0 \).

Types of segments of the constructed on figure 4 spline \((BC)_2\) are the following: \( Q_1Q_0 \) and \( Q_2Q_3 \) are of parabolic type \( (q_1^2 = 1, q_2^2 = 1) \); \( Q_4Q_5 \) and \( Q_5Q_0 \) are of hyperbolic type \( (q_2^3 = 3, q_3^3 = 3) \).

An example of construction of a closed spline \((BC)_2\) according to the described algorithm for an array of points \( \{Q_i\}^{15} : (0, 0), (20, 30), (50, 30), (80, 40), (110, 40), (140, 50), (200, 50), (240, 0), (200, -50), (140, -40), (110, -40), (80, -40), (50, -30), (20, -30) \) is depicted on figure 5.

5. Consideration of the results
The results of the computational experiments on construction of a spline \((BC)_2\) from segments of second-order curves connected according to the order of smoothness \( C^2 \) have proved simplicity and consistency of the proposed algorithm. The control parameters calculated in the process of construction allow us to directly specify the type of curve of segments of a spline \((BC)_2\). The algorithm of spline construction behaves consistently upon relatively even distribution of nodes of spline \((BC)_2\). However, relatively uneven distribution of nodes results in drastic variations of its shape. Application of the known approaches, as, for example, in paper [15], did not yield positive results in solution to this issue. One of the obstructions to the solution is the fact that segments, within which the values of parameters of connected curve segments vary, are unclosed.

6. Conclusion
In order to construct a smooth closed curve through a given array of points, it is proposed to utilize fractional rational Bezier curves of second order. Representation of such curves in the canonical form by appropriate reparametrization allows us to substantially facilitate the algorithm of construction of a Bezier spline of second order. The acquired construction algorithm is simple in terms of its mathematical model, information and resource intensity. It can be successfully applied in graphic and industrial design, computer animation, machine-building, light industry. Mathematical model of the proposed algorithm of construction of fractional rational Bezier spline of second order can find application in modern CAD systems in preproduction of various goods. For example, it can be applied
in modeling of a closed contour of 2D area in the task of formation of a family of offset curves in pocket machining of machine-building items.

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