Nonperturbative renormalization of the lattice Sommerfield vector model

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The lattice Sommerfield model, describing a massive vector gauge field coupled to a light fermion in 2d, is an ideal candidate to verify perturbative conclusions. In contrast with continuum exact solutions, we prove that there is no infinite field renormalization, implying the reduction of the degree of the ultraviolet divergence, and that the anomalies are non renormalized. Such features are the counterpart of analogue properties at the basis of the Standard Model perturbative renormalizability. The results are non-perturbative, in the sense that the averages of invariant observables are expressed in terms of convergent expansions uniformly in the lattice and volume.

I. INTRODUCTION.

Most properties of the Standard Model are known only at a perturbative level with series expansions expected to be generically diverging; in particular its perturbative renormalizability [1],[2] relies on two crucial properties, the reduction of the degree of divergence with respect to power counting and the cancellation of the anomalies [3] ensured by the Adler-Bardeen theorem [4]. Such properties are essential to maintain the renormalizability present with massless bosons. The phenomenon of the reduction of the degree of divergence can be already seen in a $U(1)$ gauge theory like QED. Adding a mass to the photon breaks gauge invariance and produces a propagator of the form\[ \frac{1}{k^2 + M^2}; \] due to the lack of decay of the second term the theory becomes dimensionally non renormalizable. However the transition in a $U(1)$ gauge theory like QED from a $M = 0$ to a $M \neq 0$ case is soft and the theory remains perturbatively renormalizable [5]; the photons are coupled to a conserved current $k_\mu j_\mu = 0$ so that the contribution of the non-decaying part of the propagator is vanishing. A similar reduction happens in the electroweak sector, but the fermion mass violates the chiral symmetry and leads to the Higgs introduction; again the renormalizability proof relies on the fact that the $k_\mu k_\nu$ term in the propagator does not contribute [2]. The chiral symmetry is generically violated by anomalies which need to cancel out, and such cancellation is based on the Adler-Bardeen property.

All the above arguments are valid in perturbation theory and non-perturbative effects could be missed. This issue would be solved by a non-perturbative lattice anomaly-free formulation of gauge theory, which is still out of reach, see for instance [6]-[8]. In particular one needs to get high values of cut-off, exponential in the inverse coupling, a property which is the non-perturbative analogue of renormalizability. The implementation of the Adler-Bardeen theorem and of the reduction of the degree of divergence in a non-perturbative context is however a non trivial issue, as their perturbative derivation uses dimensional regularizations, and functional integral derivations [9] are essentially one loop results [10].

It is convenient therefore to investigate such properties in a simpler context, and the Sommerfield model [11], describing a massive vector gauge field coupled to a light fermion in 2d, appears to be the ideal candidate, see also [12],[13],[14]. More exactly, we consider a version of this model with non zero fermionic mass, but our results are uniform in the mass. The model can be seen as a $d = 2$ QED with a massive photon; as in 4d, at the level of perturbation theory the transition from $M = 0$ to a $M \neq 0$ is soft and the theory remains super-renormalizable. Again this follows from the conservation of currents, which is ensured at the level of correlations by dimensional regularization; the same regularization provides the anomaly renormalizability [12]. In this case however we have access to non-perturbative information and we can check such conclusions. Exact solutions are known in the continuum version of the Sommerfield model \[15\], [11], [16], [17]. Remarkably the above perturbative features are not verified; there is an infinite wave function renormalization incompatible with the superrenormalizability, and anomalies have a value depending on the regularization.

In this paper we consider the Sommerfield model on the lattice, and we analyze it using the methods of constructive renormalization. The lattice preserves a number of symmetries, in the form of Ward Identities. Our main result is that there is no infinite field renormalization, which is the counterpart of superrenormalizability, and that the Adler-Bardeen theorem holds with finite lattice. Non perturbative violation of the above perturbative conclusions is therefore excluded. Other 2d models previously rigorously constructed, see [18]-[25], lack of these features. Quantum simulations of 2d models [26]-[28] have been also considered in the literature, but they regard mostly the Schwinger model, to which the Sommerfield model reduces when the boson and fermion mass is vanishing. Our results are non-perturbative, in the sense that the averages of gauge invariant observables are expressed in terms of convergent expansions uniformly in the lattice and volume.

The paper is organized in the following way. In §II we define a lattice version of the Sommerfield model. In §III we derive exact Ward Identities for the model. In §IV we integrate the boson field and in §V we perform a non-perturbative multiscale analysis for the fermionic fields. In §VI we prove the validity of the Adler-Bardeen theorem and in §VII the conclusions are presented.
II. THE LATTICE SOMMERFIELD MODEL

If \( \gamma_0 = \sigma_1 \), \( \gamma_1 = \sigma_2 \), we define

\[
\langle O \rangle = \frac{1}{Z} \int \prod_x d\bar{\psi}_x d\psi_x \int_{\mathbb{R}^{nA}} \prod_x dA_{\mu,x} e^{-S(A,\psi)} O
\]

where \( Z \) the normalization, \( x \in A \), with \( A \) a square lattice with step \( a \) with antiperiodic boundary conditions and

\[
S(A,\psi) = S_A(A) + S_\psi(A,\psi)
\]

with

\[
S_A(A) = a^2 \sum_x \left[ \frac{1}{4} F_{\mu,\nu,x} F_{\mu,\nu,x} + \frac{M^2}{2} A_{\mu,x} A_{\mu,x} \right]
\]

\[
S_\psi(A,\psi) = a^2 \sum_x \left[ \bar{\psi}_x \gamma_\mu \psi_x + \right.
\]

\[
a^{-1} Z_{\psi}(\bar{\psi}_x \gamma_\mu + e^{ie\alpha} A_{\mu,x} \psi_x + a_{\mu} - \bar{\psi}_x + a_{\mu} \gamma_\mu - e^{ie\alpha} A_{\mu,x} \psi_x)]
\]

with \( a_{\mu} = ae_{\mu} \), \( c_0 = (1,0), c_1 = (0,1), \gamma_\mu = \gamma_\mu + r \)

\( F_{\mu,\nu} = d_{\mu,\nu} A_{\mu} - d_{\nu,\mu} A_{\nu} \) and \( d_{\mu,\nu} = a^{-1}(A_{\mu,x+e_{\nu}} - A_{\mu,x}) \),

\( \bar{m} = (m + 4r/a) \) and \( r = 1 \) is the Wilson term. Note that if \( 1/a \) and \( L \) are the integral is finite dimensional.

We generalize the model adding a term \( (1 - \xi)x^2 \sum_x (d_{\mu,\nu})^2, \xi \leq 1 \) so that the bosonic action is given by \( \frac{1}{a^2} \sum_x (\sum_{\mu}(d_{\mu,\nu})^2 + \xi \sum_{\mu}(d_{\mu,\nu})^2) \). The original model is recovered with \( \xi = 1 \).

The correlations can be written as derivatives of the generating function,

\[
e^{W_x(J,B,\psi)} = \int P(\psi) P(dA) e^{-V(A,J,\psi)} + (\psi,\phi) + a^2 \sum_x B_x O
\]

with \( O = O(A + J, \psi) \) an observable, and \( P(dA) \) the gaussian measure with covariance

\[
\hat{g}_{\mu,\nu}(k) = \frac{1}{|\sigma|^2 + M^2} (\delta_{\mu,\nu} + \frac{\xi \sigma_{\mu} \sigma_{\nu}}{1 - (1/|\sigma|^2) + M^2})
\]

with \( \sigma_{\mu}(k) = (e^{ik_a} - 1)a^{-1} \). \( P(d\psi) \) is the fermionic propagator

\[
\bar{g}(\psi) = Z^{-1}(\hat{k}_\mu \gamma_\mu + a^{-1} m(k) I)^{-1} \quad \text{with} \quad \hat{k}_\mu = \sin(k_a/a)
\]

\( m(k) = m + ra^{-1}(\cos ak_0 + \cos ak_2 - 2) \); finally

\[
V(A,\psi) = a^2 \sum_x \left[ O_{\mu,x}^+ G_{\mu,x}(A) + O_{\mu,x}^- G_{\mu,x}(A) \right]
\]

with \( O_{\mu} = Z \bar{\psi}_x (\gamma_{\mu} - r) \psi_x + a_{\mu} \) and

\[
O_{\mu}^- = -Z \bar{\psi}_x + a_{\mu} (\gamma_{\mu} + r) \psi_x
\]

\( G_{\mu}^+ = a^{-1}(\psi \bar{\psi} A_{\mu,x} + 1) \).

If \( M = 0 \) the model (1) invariant under the gauge transformation \( A_{\mu,x} \to A_{\mu,x} + d_{\alpha,x} \) and \( \psi_x \to \psi e^{-ieA_{\mu,x}} \); if \( M \neq 0 \) the invariance is lost.

III. WARD IDENTITIES AND \( \xi \)-INDEPENDENCE

If we restrict to observables such that \( O(A,\psi) = O(A + d_{\alpha}, \psi e^{-ieA_{\mu,x}}) \) (which we call invariant observables) there is gauge invariance in the external fields also for \( M \neq 0 \), that is

\[
W_\xi(J + da, e^{-ie\alpha} \psi, B) = W_\xi(J, \psi, B)
\]

This follows by performing in (5) the change of variables \( \psi_x \to \psi e^{ieA_{\mu,x}} \), with Jacobian equal to 1 (the integral is finite-dimensional) and noting that \( (e^{-ie\alpha} \psi, \psi e^{-ie\alpha}) \) and

\[
S_\psi(A + J, \psi e^{-ie\alpha}) = S_\psi(A + J + da, \psi)
\]

(8) implies that \( \partial_\alpha W_\xi(J + da, e^{-ie\alpha} \psi, B) = 0 \). We define \( \Gamma_{\mu_1,...,\mu_n,\nu_1,...,\nu_m} \) as the derivatives of \( W_\xi \) with respect to \( J_{\mu_1,x_1}, ..., B_{\nu_n,x_n} \). By performing in (8) derivatives with respect to \( \alpha \) and the external fields we get the Ward Identities (expressing current conservation)

\[
\sum_{\mu_1} \sigma_{\mu}(p) \Gamma_{\mu_1,...,\nu_n}(p_1,...,p_{n-1}) = 0
\]

and

\[
\sigma_{\mu}(p) \Gamma_{\mu}(p,k) = \hat{S}(k) - \hat{S}(k + p)
\]

where \( \hat{\Gamma}_{\mu}(p,k) = \frac{\partial_{\psi,\phi}}{\partial_{\psi,\phi}} |_{\psi,\phi} \) is the vertex function and \( \hat{S}(k) = \frac{\partial_{\psi,\phi}}{\partial_{\psi,\phi}} |_{\psi,\phi} \) the 2-point function.

The conservation of current expressed by the above WI implies that for invariant observables

\[
\partial_\xi W_\xi(J,0,B) = 0
\]

that is the averages are \( \xi \) independent. This follows from

\[
\partial_\xi \int P(dA) \int \prod_x d\bar{\psi}_x d\psi_x O = 0, \quad \text{with} \quad (A,\psi) \text{ invariant};
\]

\[
\partial_\xi \int P(dA) \int \prod_x d\bar{\psi}_x d\psi_x O = \int \frac{1}{L^2} \sum_p \partial_\xi \hat{g}^A_{\mu}(p) \int P(dA) A_{\mu,p} A_{\mu,-p} \int \prod_x d\bar{\psi}_x d\psi_x O
\]

from which we get, using that \( A_{\mu,p} = \hat{g}^A_{\mu}(p) \hat{A}^A_{\mu}(p) \)

\[
\partial_\xi \hat{g}^A_{\mu}(p) \partial_\xi \hat{A}^A_{\mu}(p) \int P(dA) \int \prod_x d\bar{\psi}_x d\psi_x O(A + J, \psi)|_0
\]

By noting that

\[
\partial (\hat{g}^A)^{-1} = - (\hat{g}^A)^{-1} \partial \hat{g}^A (\hat{g}^A)^{-1}
\]

and \( \partial \hat{g}^A \) is proportional to \( \hat{g}^A \partial_\xi \), by using

\[
\partial_\alpha \int P(dA) \int \prod_x d\bar{\psi}_x d\psi_x O(A + da, \psi)|_0 = 0
\]
then (13) is vanishing.

(12) ensures that the averages does not depend on \( \xi \), so that one can set \( \xi = 0 \) in the boson propagator, that is the non decaying part of the propagator does not contribute. In perturbation theory the scaling dimension with \( \xi = 0 \) (\( z = 2 \)) and \( \xi = 1 \) (\( z = 0 \)) is, if \( n \) is the order, \( n_A \) the number of \( A \) fields and \( n_\psi \) the number of \( \psi \) fields

\[
d + (d - z - 2)n_\psi/2 - (d - 1)n_\psi/2 - (d - z)n_A/2
\]

hence in \( d = 2 \) the theory is dimensionally renormalizable with \( \xi = 1 \) and superrenormalizable with \( \xi = 0 \) (in \( d = 4 \) one pass from non-renormalizability to renormalizability). The lattice regularization ensures that the theory remains perturbatively superrenormalizable, as with dimensional regularization. We will investigate the validity of this property at a non-perturbative level.

Finally, we define the axial current as \( j_5^\mu = Z_5^\mu \bar{\psi}_x \gamma_5 x \nu \psi_x \), where \( Z_5^\mu \) is a constant to be chosen so that the electric charge of the chiral and e.m. current are the same, defined as the amputated part of the 3-point correlation at zero momenta (see [10]), that is

\[
\lim_{k,p \to 0} \frac{\partial^3 W}{\partial B_\mu^c \partial \phi_k \partial \phi_p} = 0
\]

where the source term is \( (B_5^\mu, j_5^\mu) \). The axial current is conserved even for \( m = 0 \), due to the presence of Wilson term, and one has

\[
\sigma_\mu(p) \tilde{\Gamma}_5_{\nu,\mu}(p) = H_\nu(p)
\]

with \( \tilde{\Gamma}_5_{\nu,\mu} \), the derivative of \( W \) with respect to \( B_\mu, J_{\nu,\nu} \). \( H_\nu(p) \) is called the anomaly and in the non-interacting case \( V = 0 \) one gets if \( m = 0 \)

\[
H_\mu = \int \frac{d \phi}{2} \epsilon_{\nu\rho} \phi_\nu \phi_\rho + O(a^2)
\]

(lattice or dimensional regularization [12] produce the same result) and \( Z_5^\mu = 1 \). In the interacting case \( H_\mu(p) \) is a series in \( e \) and the non renormalization property means that all higher orders corrections vanishes.

IV. INTEGRATION OF THE BOSON FIELDS

We can integrate the boson field

\[
\int P(dA) e^{-V} =\left[1 + \sum_{\gamma = 0}^{\infty} a^{\gamma + 1} \epsilon^{-2} C^\gamma(V, n) \right] e^{V(N)}
\]

where \( \epsilon^{V(N)}(\psi, J) \) is the truncated expectation, that is the sum of connected diagrams, and \( V(N) = a^2 \sum_x \sum_\xi \epsilon^{-1} C^\xi g_\mu(x, x) \epsilon^{-1} g_\nu(x, x) O_\mu^\nu \]

\[
\sum_{n,m} a^{2n+m} \sum_{x,y} \left[ \prod_{j=1}^n O_{j,1}(x, x) \right] \left[ \prod_{k=1}^m O_{j,2}(y, x) \right] |W_{n,m}(x, y)|
\]

Note that \( a^2 g_\mu(x, x) \leq C \). We call \( a = \gamma^{-N} \), where \( \gamma > 1 \) is a scaling parameter.

Theorem 1 The kernels in (21) for \( n \geq 2 \) verify,

\[
|W_{n,m}| \leq C^{n+m} e^{2(n-1)} \gamma(N(2-n-m)) \left( |g_\mu|_1 \right)^n
\]

Proof. A convenient representation for \( \epsilon^{V(N)}_\mu \) is given by the following formula [29]

\[
\epsilon^{V(N)}_\mu = \sum_{T \in \mathcal{T}} \prod_{i,j \in T} V_{i,j} \int d p_T e^{-V_T(x)}
\]

where \( V_{i,j} = e^{2a} \epsilon^2 \mathcal{A}_n(x_i, x_j) \), \( \mathcal{T} \) is the set of tree graphs \( T \) on \( X = (\ldots, n), \) \( s \in (0, 1) \) is an interpolation parameter, \( V_T(s) \) is a convex linear combination of \( V(Y) = \sum_{i,j \in Y} \epsilon_{i,j} V_{i,j}, Y \) subsets of \( X \) and \( d p_T \) is a probability measure. The crucial point is that \( V(Y) \) is stable, that is

\[
V(Y) = \sum_{i,j \in Y} V_{i,j} = a^2 e^{2\epsilon} \mathcal{A}_n \left( \sum_{i,Y} \epsilon_{i,1} A_{i,x_i} \right) \geq 0
\]

Therefore one can bound the exponential \( e^{-V_T(s)} \leq 1 \) finding

\[
|W_{n,m}| \leq C^n a^{-n-1} \sum_{\mathcal{T}} \prod_{i,j \in \mathcal{T}} a^2 e^{2\epsilon} |g_{\mu}|_1 \leq C^n a^{2N(n-2)-n} |g_{\mu}|_1^n
\]

With \( m \neq 0 \) we get an extra \( a^{2N}, \) so that one recovers the dimensional factor \( \gamma(N(2-n)) \).

For \( \xi = 0 \) \( |g_{\mu}|_1 \leq CM^{-2} \) and \( |g_{\mu}|_\infty \leq C |\log a| \) while for \( \xi = 1 \) \( |g_{\mu}|_1 \leq C |\log a| \) and \( |g_{\mu}|_\infty \leq C a^{-2} \). We write \( W_{n,m} = \lambda^{-1} W_n \) with \( \lambda = a \), \( J \) independent; the same therefore is true for \( Z_1 \), and as the numerator (1) is intere, than (1) is analytic in \( |\lambda| < \lambda_0 \) and equal to the \( \xi = 0 \) case. It remains to prove that the correlations with \( \xi = 0 \) are analytic for \( |\lambda| < \lambda_0 \) and \( Z_0 = 1 + O(\lambda) \).

The factor \( D = 2 - n - m \) is the scaling dimension, and the terms with \( D < 0 \) are irrelevant. The marginal term for \( \xi = 0 \) is \( \epsilon^{V(N)}_\mu (V; 2) = \sum_{x_1, x_2} \sum_{x_1, x_2} e^{-2a} \epsilon_{x_1} A_{x_1} O_{x_1}^{x_1} e^{-2a} A_{x_2} O_{x_2}^{x_2} \lambda v_{\mu, \xi, x_2} (x_2, x_1) = e^{-2a} \epsilon_{x_1} g_{\mu, \xi, x_2} (x_2, x_1)
\]

which can be rewritten as

\[
\int_0^1 dt g_{\mu, \xi, x_1} (x_1, x_2) e^{-V(t)}
\]
with
\[ 2\tilde{V}(t) = a^2 t \left( (\varepsilon_1 A_\mu(x_1) + \varepsilon_2 A_\mu(x_2))^2 \right) \]
\[ + a^2 (1 - t) (g_{A,\mu}(x_1, x_1) + g_{A,\mu}(x_2, x_2)) \]
in agreement with (23). For definiteness we keep only the dimensionally non irrelevant terms considering
\[ e^{W_1(J,B,\phi)} = \int P(d\psi) e^{V + G(B)} + (\psi, \phi) \]
with \( G(B) \) is a generic source term for gauge invariant observables and \( V = \)
\[ a^2 \sum_x \sum_{a,b} a^{-1} (e^{-\frac{1}{2}x^a_{\mu}(x,v)} e^{\beta_{\mu}(x),v} - 1) O_\mu + \tilde{E}^T(V; 2) \]
Note that \( a^2 g_{A,\mu} \) vanishes as \( a \to 0 \). In the case of the chiral current
\[ G(B) = a^2 \sum_x Z^B_{x,\mu} \chi_{\mu} \gamma_5 \psi_{\mu} \]

\section{V. INTEGRATION OF THE FERMIONIC FIELDS.}

Our main result is the following

**Theorem 2** For \( |\lambda| \leq \lambda_0 M^2 \), with \( \lambda_0 \) independent on \( a, m \) and \( Z_\psi = 1 \) the correlations of (30) are analytic in \( \lambda \): when the fermion mass is vanishing the anomaly is
\[ H_\mu = \frac{\zeta_{\lambda,0} P_0}{2x} + \theta (a^2 \psi^2) \]
In order to integrate the fermionic fields we introduce a decomposition of the propagator
\[ g^0(x) \equiv \sum_{h=-\infty}^N g^{(h)}(x) \]
\[ \tilde{g}^{(h)}(k) = \int f^{(h)}(k \int f^{(h)}(k) \quad \text{with support in } \gamma^{h-1} \leq |k| \leq \gamma^{h+1} \quad \text{one has to distinguish two regimes, the ultraviolet high energy scales } h \geq h_M \text{ with } h_M = \log M \text{ the mass scale, and the infrared regime } h \leq h_M \quad \text{in the kernels one uses the non locality of the quartic interaction} \quad [19],[30],[31] \quad \text{After the integration of the fields } \psi^N_0, \psi^{N-1}, ... \psi^h \quad h \geq h_M \text{ one gets an effective potential with kernels } W_{\text{kin}} \text{ with } \text{finite fields } (l = 2n) \text{ similar to (21), which can be written as an expansion in } \lambda \text{ and in the kernels } W_{2,0}, W_{4,0} W_{2,1} \text{ with } k \geq h + 1 \text{. Assuming that, for } k \geq h + 1 \text{ one has } |W_{2,0}^k| \leq \lambda^h M^2 \text{, } |W_{4,0}^k - v\lambda|_1 \leq \lambda^2 M^2 \text{ and } |W_{2,1}^k - v\lambda|_1 \leq \lambda^2 M^2 \text{ then we get}
\[ |W_{l,m}^h| \leq C^{l+m} (\lambda / M^2)^{d_l} \gamma^{(h+2-l)} \]
for \( d_l = \max(l/2 - 1, 1) \) if \( m = 0 \), and \( d_l = \max(l/2 - 1, 0) \) if \( m = 1 \). The proof of (34) is based on the analogous of formula (23) for Grassmann expectations
\[ \mathcal{E}_{\psi}(\prod_{k=1}^n \tilde{\psi}(P_i)) = \sum_{T \in T} \prod_{i,j \in T} V_{i,j} \int d\phi(t) (s) \text{ det } G \]
and the use of Gram bounds for get an estimate on det \( G \); in addition one uses that \( |v|_1 \leq C M^{-2} \), \( |g_{A,\mu}| \leq C \gamma^{-h} \), \( |g_{A,\mu}| \leq C \gamma^{-h} \). We proceed by induction to prove the assumption. One needs to show that there is an improvement in the bounds due to the non locality of the boson propagator. The kernel of the 2-point function \( \tilde{W}_{2,0}(x,y) \)
which can be written as sum over \( n \) of truncated expectations and, if \( E_{\tilde{h},N} \) is the truncated expectation with respect to \( P(d\psi^{(h,N)}) \)
\[ \lambda \frac{\partial}{\partial \phi^2} (n - 1)! \sum_{x_{1,2}} v_{\mu,\mu} \tilde{E}_{\tilde{h},N}^{(T)}(\frac{\partial}{\partial \phi_{\mu}} \tilde{O}_{\mu,\mu} \tilde{O}_{\mu,\mu}; V; ...) \]
By using the property, if \( \tilde{\psi}(P) = \prod_{f \in P} \tilde{\psi}_{x_f} \)
\[ E_{\tilde{h},N}(\tilde{\psi}(P_1 \cap P_2) ... \tilde{\psi}(P_n)) = E_{\tilde{h},N}(\tilde{\psi}(P_1) \tilde{\psi}(P_2) ... \tilde{\psi}(P_n)) + \sum_{j \in K_1} \tilde{E}_{\tilde{h},N}^{(T)}(\tilde{\psi}(P_1)) \sum_{j \in K_2} \tilde{E}_{\tilde{h},N}^{(T)}(\tilde{\psi}(P_2)) \]
we get, omitting the \( \mu \) dependence, \( W_{2,0}(x,y) = \)
\[ \lambda^h \sum_{z_{1,2}} v(y, z_1) g^{(h,N)}(y + a_1, z_2) W_{2,0}(z_2, x) W_{1,0}(z_1) \]
\[ \lambda a^{-1} (e^{-\frac{1}{2}x^a_{\mu}(x,v)} - 1) a^4 \sum_{z_{1,2}} g^{(h,N)}(x, z) W_{2,0}(z_2, y) \]
\[ + \lambda a^4 \sum_{z_{1,2}} v(y, z_2) g^{(h,N)}(y + a_1, z_1) W_{2,1}(z_1, z_1, x) \]
The second term is bounded by
\[ C \lambda^h \gamma^{-h} \log a \leq \lambda / M^2 \gamma^{-h} / 2 \] for a small enough. The first term contains \( \tilde{W}_{1,0}(0,0) = 0 \). Regarding the last term we get a bound
\[ 0 \leq \lambda^{h} \gamma^{-h} \log a \]
\[ \sup_{z_{1,2}} |a^2 \sum_{y} v(y, z_2) g^{(h,N)}(y + a_1, z_1) | a^2 \sum_{z_{1,2}} |W_{2,1}(z_1, z_1, 0)| \]
By using the inductive hypothesis
\[ \gamma^{(h+2-l)} \leq C \lambda M^{-2} C_2 \gamma^{-h} \]
\[ \lambda C_1 |a^2 \sum_{y} |g^{(h,N)}|^2 \leq \lambda M^{-2} C_2 \gamma^{-h} \]
\[ [a^2 \sum_{y} (g^{(h,N)})^2] \leq \lambda M^{-2} C_2 \gamma^{-h} \]

**FIG. 1.** Graphical representation of (36)
\[ a^2 \sum_{z_{1,2}} |W_{2,1}(z_1, z_1, 0)| \leq C \] we get for (38) the bound
\[ \lambda C_1 |a^2 \sum_{y} \langle g^{(h,N)} \rangle | \leq \lambda C_1 |a^2 \sum_{y} |v|^3 | \times \]
\[ [a^2 \sum_{y} (g^{(h,N)})^2] \leq \lambda M^{-2} C_2 \gamma^{-h} \leq \lambda M^{-2} \gamma^{-h} / 2 \]
for \( h \geq h_M \), for \( h_M = C \log M \) and \( C \) large enough. Note that the above estimates uses crucially that \( \xi = 0 \); for \( \xi = 1 \) \([v^2 \sum_y |v|^2]^{\frac{1}{2}}\) would be non bounded uniformly in \( N \).

A similar computation can be repeated for \( W_{2,1} \); in particular for the quartic term one uses that the bubble graph is finite \( A = \int \text{d}k \text{Tr}(\gamma(k)\gamma_{\mu}(k)\gamma_{\nu}) \leq C\lambda M^2(\lambda h^{-1} + A)|W_{1,2}|. \) The above estimates work for \( h \geq h_M \) and it says that the theory is superrenormalizable up to that scale.

In the infrared regime \( h_m \leq h \leq h_M \), where \( h_m = \log_\gamma m \) is the fermion mass scale, the multiscale integration procedure is the same as in the Thirring model with a finite cut-off \([23]\). The theory is renormalizable in this regime and there is wave function renormalization at each scale \( Z_h \sim \gamma^{\eta h} \), \( \eta = O(\lambda^2) > 0 \) and an effective coupling with asymptotically vanishing function. The expansions converge therefore uniformly in \( a, L, M \) and the limit \( a \to 0, L \to \infty \) can be taken.

\[
(1 \pm \tau) P_\nu = \begin{array}{c}
\begin{array}{c}
\Downarrow
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\Downarrow
\end{array}
\end{array}
\]

FIG. 2. Graphical representation of (44)

VI. ANOMALY NON-RENORMALIZATION

The average of the chiral current \( \Gamma^5_{\mu,\nu} = \frac{\partial^2 W}{\partial J_\mu \partial J_\nu} \bigg|_0 \) for \( m = 0 \) is expressed by a series in \( \lambda \). It is convenient to introduce a continuum relativistic model \( e^{\tilde{W}(J,B,\phi)} = \int P_\tilde{Z}(d\tilde{\psi})e^{-\tilde{V}+\tilde{Z}^+(J,J)+\tilde{Z}^-(B,B)+\langle\phi,\phi\rangle} \) (40)

where \( P_{\tilde{Z}}(d\tilde{\psi}) \) has propagator \( \frac{1}{\tilde{Z}^{-\gamma_{\mu,\nu}}} \), with \( \chi \) a momentum cut-off selecting momenta \( \leq \gamma \tilde{N} \), and

\[
V = \tilde{Z}^2 \tilde{\lambda} \int dxdyv(x,y)\tilde{j}_{\mu,x}\tilde{j}_{\mu,y}
\]

with \( v \) exponentially decaying with rate \( M^{-1} \) with quartic coupling \( \tilde{\lambda} \); finally \( \tilde{j}_{\mu,x}^+ \equiv \tilde{j}_{\mu,x} = \tilde{\psi}_x \gamma_{\mu} \tilde{\psi}_x \) and \( j^+ = \tilde{\psi}_x \gamma_{\mu} \gamma_5 \tilde{\psi}_x \).

The infrared scales \( h \leq h_M \) of the two models differs by irrelevant terms and one can choose \( \tilde{\lambda} \) and \( \tilde{Z}, \tilde{Z}^- \) as function of \( \lambda \) so that the corresponding running couplings flow to the same fixed point for \( h \to -\infty \). As a result, defining

\[
\tilde{\Gamma}^5_{\mu,\nu} = \frac{\partial^2 \tilde{W}}{\partial E_\mu \partial J_\nu} \bigg|_0
\]

we get

\[
\tilde{\Gamma}_{\mu,\nu}^5(p) = Z_5 \tilde{\Gamma}_{\mu,\nu}^5(p) + R_{\mu,\nu}(p)
\]

where \( R_{\mu,\nu}(p) \) is a continuous function at \( p = 0 \), while \( \tilde{\Gamma}_{\mu,\nu}(p) \) is not; this provide a relation between the lattice and the continuum model.

The model (40) has two global symmetries, that is \( \psi \to e^{i\alpha} \psi \) and \( \psi \to e^{i\alpha \gamma_5} \psi \), but the WI acquires extra terms associated with the momentum regularization [30]. In particular, if \( \tau = \tilde{\lambda} \tilde{\psi}(0)/4\pi \), in the limit of removed cut-off \( \tilde{N} \to \infty \)

\[
(1 \mp \tau)p_\mu \tilde{\Gamma}^\pm_{\mu,\nu}(k,p) = \frac{\tilde{Z}^\pm}{\tilde{Z}} \gamma^\pm(\tilde{S}(k) - \tilde{S}(k + p))
\]

where \( \tilde{\Gamma}^\pm_{\mu,\nu} \) is the vertex function of are the vertex correlations of (40) of the current (+) and chiral current (–) and \( \gamma^+ = I, \gamma^- = \gamma_5 \). In the same way the WI for the current is

\[
p_\mu \tilde{\Gamma}^5_{\mu,\nu} = \frac{\tilde{Z}^+ - \gamma_{\nu\mu}p_\nu}{4\pi \tilde{Z}^2} (1 + \tau) \quad p_\nu \tilde{\Gamma}^5_{\mu,\nu} = \frac{\tilde{Z}^+ - \gamma_{\nu\mu}p_\nu}{4\pi \tilde{Z}^2} (1 - \tau)
\]

By comparing (44) with the Ward Identity (11), and using that the vertex and the 2-point correlations of lattice and continuum model coincide up to subleading term in the momentum, we get a relation between the parameters \( \tau, \tilde{Z}^+, \tilde{Z}^- \)

\[
\frac{\tilde{Z}^+}{\tilde{Z}(1 - \tau)} = 1
\]

Moreover the condition on \( Z_5 \) (18) and (44) imply

\[
\frac{\tilde{Z}^+}{\tilde{Z}(1 - \tau)} = Z_5 \frac{\tilde{Z}^-}{\tilde{Z}(1 + \tau)} = 1
\]

from which \( Z_5 = 1 + (1 + \tau) \frac{\tilde{Z}^-}{\tilde{Z}} \). By the Ward Identity (10) we get

\[
p_\mu \tilde{\Gamma}^5_{\mu,\nu}(p) = \frac{\tilde{Z}^+ - \gamma_{\nu\mu}p_\nu}{2\pi \tilde{Z}^2} (1 - \tau) + p_\nu R_{\mu,\nu}(p) = 0
\]

so that

\[
R_{\mu,\nu}(0) = \frac{\tilde{Z}^+ - \gamma_{\nu\mu}p_\nu}{2\pi \tilde{Z}^2 (1 - \tau)} = -(1 + \tau)\varepsilon_{\nu\mu}/Z_5
\]

Finally

\[
p_\mu \tilde{\Gamma}^5_{\mu,\nu}(p) = Z_5 p_\mu [\tilde{\Gamma}^5_{\mu,\nu}(p) + R_{\mu,\nu}(p)] = [(1 - \tau)\varepsilon_{\nu\mu} - (1 + \tau)\varepsilon_{\nu\mu}]/4\pi = 1/2\pi \varepsilon_{\nu\mu}p_\nu
\]

that is all the dependence of the coupling disappears.
VII. CONCLUSIONS

We have analyzed a lattice version of the Sommerfield model. Both the reduction of the degree of ultraviolet divergence, manifesting in the finiteness of the field renormalization, and the Adler-Bardeen theorem hold at a non-perturbative level, in contrast with exact solutions in the continuum. Non-perturbative violation of perturbative results are therefore excluded. This provides support to the possibility of a rigorous lattice formulation of the electroweak sector of the Standard Model with step exponentially small in the inverse coupling, which requires an analogous reduction of degree of divergence. New problems include the fact that a multiscale analysis is necessary also for the boson sector, and the fact that the symmetry is chiral and anomaly cancellation is required; Adler-Bardeen theorem on a lattice is exact for non chiral theories [33] but has subdominant corrections for chiral ones [34].

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