Duality in non-commutative gauge theories as a non-perturbative Seiberg–Witten map

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Abstract: We study the equivalence/duality between various non-commutative gauge models at the classical and quantum level. The duality is realised by a linear Seiberg–Witten-like map. The infinitesimal form of this map is analysed in more details.

Keywords: Non-commutative theory, gauge symmetry, duality.
1. Introduction

Quantum field theories over non-commutative spaces (for a review see [1]) seem to be relevant to the non-perturbative description of the string theory [2]–[6].

Moreover, the non-commutative gauge models mimic some features expected in the string field theory introduced by E. Witten [7].

Thus, classical localised solutions called non-commutative solitons found in the framework of non-commutative theory [8]–[11] in the string picture were shown to correspond to branes [12, 13].

In the non-commutative case, one can formally develop a perturbation expansion similar to the commutative one. It was found that a phenomenon called IR/UV mixing occurs due to non-commutativity [14]–[16]. This again reflects a feature of open/closed duality in string theory. In the case of non-commutative field theories it prevents us from giving a direct proof of renormalisability of such theories.\(^1\)

On the other hand, string theory possesses a number of dualities relating various string backgrounds [18] (for reviews see [19]), \textit{M-theory} being the name of the embracing non-perturbative model. In this context one naturally may ask if there

\(^1\)An exception may occur, however, when in commutative theory no UV counterterms are needed at least at one loop e.g., like in Wess–Zumino model. In this case the IR/UV mixing is trivial and no problems with renormalisability arise [17].
are similar duality relations in the non-commutative field theory models. Indeed, in [5, 20] it was found that gauge models on different but dual non-commutative tori are the same due to Morita equivalence.

In the case of non-commutative planes, it was shown in a $p$-dimensional non-commutative gauge model one can recover a set of $p \pm 2k$ dimensional gauge models as expansions around different non-perturbative solutions in the above model [21, 22]. Depending on whether $p$ is even or odd the models belong to the universality class of $N = \infty$ IKKT [23], or BFSS [24] M(atrix) models. The above matrix models arise as reductions of ordinary SU($N$) Yang–Mills model to, respectively, zero and one dimension. Such a relation between large $N$ reduced model and the noncommutative model appears to be an avatar of the old idea by Eguchi–Kawai (see [25]).

From a different point of view, the above map between different models can be interpreted as a non-perturbative Seiberg–Witten map since it is an explicit solution to the Seiberg–Witten equation [6], which depends non-perturbatively on the gauge coupling and the non-commutativity parameter.

Traditionally, attempts to solve and classify the solutions to Seiberg–Witten equations were made using the power expansion of non-commutativity parameter $\theta$ originally introduced in [6]. Although, considerable progress was achieved since that time [26]–[31], this approach remains extremely complicated.

Here we propose an alternative way to find the Seiberg–Witten map starting from background independence. The background independent formulation in our context is the formulation of the non-commutative gauge model in terms of Hermitian (or as appropriately required) operators acting on a separable infinite-dimensional Hilbert space with no explicit dependence on the space coordinates or derivatives. The non-commutative spaces in this approach appear as classical background solutions to these models, which break (spontaneously) the background invariance. Seiberg–Witten map in this picture relates formulations of the model around different backgrounds. The particular property of the map we describe is that it is linear in fields.

In the present work we explore the duality realised by the map both in the non-perturbative case, when it relates the models in non-commutative spaces of different dimensionality [21, 32], or models with different gauge groups, and in the perturbative case it corresponds to a smooth change of the non-commutativity parameter [33]. We extend the analysis to the quantum case when duality is realised as a quantum symmetry in the path integral approach.

The plan of the paper is as follows. In section 2 we review the non-commutative gauge field theory. We introduce the gauge model of Yang–Mills interacting with a Higgs scalar multiplet as an expansion of the the bosonic part of $N = \infty$ matrix model around a classical solution of the model.\textsuperscript{2} In section 3 we describe the dualities

\textsuperscript{2}This approach is similar in its main lines to the one used in [34]. Note, however, the difference
in the non-commutative gauge model first in their most general form and afterwards specialising to particular cases of dualities relating models with different gauge groups as well as relating models in different dimensions. In Section 4 we focus on the infinitesimal form of proposed transformation which appears to be a linear variant of the Seiberg–Witten map. Further, in Section 5 we analyse quantum implications of the symmetry realised by the map. Finally section six contains our conclusions and further remarks.

2. The Model

Large $N$ BFSS and IKKT matrix models were proposed to describe non-perturbative string theory [23, 24]. They are reductions of the ten dimensional $SU(N)$ Yang–Mills (YM) models to respectively zero and one dimensions. On the other hand, it can be shown that infinite-dimensional Hilbert space ($N = \infty$) models are again YM-like, but living either on commutative or non-commutative spaces. In the limit $N \to \infty$, such models were shown to have the infinite-dimensional gauge group of area preserving diffeomorphisms [36].

In spite of an apparent dissimilarity, these models are related to the “usual” non-commutative $U(1)$ Yang–Mills model by an appropriate redefinition of the gauge fields.

Here, we will analyse the situation where Euclidean “space-time” coordinates are all non-commutative,

$$[[x^\mu, x^\nu] = i \theta^{\mu\nu}, \quad \det \|\theta^{\mu\nu}\| \neq 0. \quad (2.1)$$

This can be easily generalised to degenerate $\theta^{\mu\nu}$. In such a context, the commutative variable are treated as parameters.

The model we consider here corresponds to (the bosonic part of) the IKKT model. This model is given by the background invariant action of the following form,

$$S = -\frac{1}{4g^2} \text{tr}[X_i, X_j]^2, \quad (2.2)$$

where the gauge fields $X_i$ are Hermitian operators acting on the Hilbert space $\mathcal{H}$. In this form, the model is formulated entirely in terms of an abstract Hilbert space and Hermitian operators acting on it. It contains no explicit space-time data which we introduce later as particular solutions breaking this background invariance.

Equations of motion corresponding to the action (2.2) look as follows,

$$[X_i, [X_i, X_j]] = 0. \quad (2.3)$$
Picking up a particular solution to the equations of motion in the form \( X_i^{(0)} = \Lambda_i^\mu p_\mu \), where \( p_\mu \) is a complete irreducible set of operators, (i.e. ones satisfying, \[
[p_\mu, p_\nu] = -i\theta^{-1}_{\mu\nu}, \quad \forall F : [p_i, F] \Rightarrow F \sim I,
\] (2.4)
where the last expression is Schur’s lemma implying the irreducibility of the representation of the algebra generated by \( p_\mu \), one can expand operators \( X_i \) around this background as follows
\[
X_i = \Lambda_i^\mu (p_\mu + A_\mu) + \Phi_i,
\] (2.5)
where \( \Phi_i \Lambda_i^\mu = 0 \).

By an appropriate linear transformation depending on \( \Lambda_i^\mu \) one can make \( X_i \) to acquire the following form,
\[
X_\mu = p_\mu + A_\mu, \quad \mu = 1, \ldots, p; \quad X_a = \Phi_a, \quad a = p + 1, \ldots, D.
\] (2.6)

Now, taking Weyl ordering with respect to operators \( x_\mu = -\theta^{\mu\nu} p_\nu \) will mean that in the chosen background, the model describes a gauge field \( A_\mu(x) \) interacting with a “multiplet” of scalars \( \Phi_a(x) \).

Indeed, the Weyl transformation maps an operator \( \phi \) to its symbol according to the following rule
\[
\phi(x) = \sqrt{\det \theta} \int \frac{d^p k}{(2\pi)^p/2} e^{ik \cdot x} \text{tr} e^{ik \times p} \phi,
\] (2.7)
where \( k \times p = k_\mu \theta^{\mu\nu} p_\nu \) and \( k \cdot x = k_\mu x^\mu \). Under this map, the operator products are transformed into star products of symbols given by,
\[
\phi \ast \chi(x) = e^{2\theta^{\mu\nu}\partial_\mu\partial_\nu\phi(x)\chi(x')}|_{x'=x},
\] (2.8)
where \( \partial_\mu' \) denotes derivatives with respect to \( x'^\mu \). The derivatives can be expressed algebraically in terms of the star product as well,
\[
\partial_\mu \phi = i[p_\mu, \phi](x), \quad \nabla_\mu \phi = i[X_\mu, \phi](x) = i[(p_\mu + A_\mu), \phi](x).
\] (2.9)

The Weyl map (2.7) is invertible. The inverse is given by
\[
\phi = \int \frac{d^p k}{(2\pi)^p} dx e^{-ik \times p - ik \cdot x} \phi(x).
\] (2.10)

The action can be written as a functional over non-commutative functions,
\[
S = \int d^p x \left(-\frac{1}{4g^2} (F_{\mu\nu} - \theta^{-1}_{\mu\nu})^2 + \frac{1}{2g^2} \nabla_\mu \Phi_a \ast \nabla_\mu \Phi_a - \frac{1}{4g^2} [\Phi_a, \Phi_b]^2\right),
\] (2.11)
where,

\[ \mathcal{F}_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \]  

\[(2.12a)\]

\[ \nabla_{\mu} \Phi = \partial_{\mu} \Phi + [A_{\mu}, \Phi] \]  

\[(2.12b)\]

and the gauge coupling \( g^2 \) is defined by,

\[ g^2 = g'^2 \sqrt{\det \theta}. \]  

\[(2.12c)\]

The products in eq. (2.11) should be taken as star products, however, one star product in each monomial can be substituted by the usual product following the identity,

\[ \int d^p x \phi \ast \chi = \int d^p x \phi(x) \chi(x). \]  

\[(2.13)\]

Let us note that the system possesses a “local” U(1) gauge symmetry which in the background invariant formulation is the unitary symmetry of the Hilbert space.

Another difference from the “standard” Yang–Mills model which can be observed in the gauge field part of the action (2.11) is that the field strength \( \mathcal{F}_{\mu \nu} \) comes shifted by the quantity \( \theta_{\mu \nu}^{-1} \). Although, for constant \( \theta_{\mu \nu}^{-1} \) this shift has no effect on the equations of motion, it is in conformity with the presumed string theory origin of the model, where the gauge field appears in combination \( \mathcal{F}_{\mu \nu} + B_{\mu \nu} \) with the stringy \( B_{\mu \nu} \)-field.

Two comments are in order:

- If in the solution, the set of operators \( p_\mu \) is reducible in the sense that the second condition of (2.4) is not fulfilled, one should complete the set by appropriate operators to make it irreducible. Operators \( q_\alpha \alpha = p + 1, \ldots, p' \) commuting with \( p_\mu \),

\[ [p_\mu, q_\alpha] = 0, \]  

\[(2.14)\]

form a closed algebra,

\[ [q_\alpha, q_\beta] = ic_{\alpha \beta}(q). \]  

\[(2.15)\]

In particular, this can be a finite dimensional Lie algebra or another piece of the Heisenberg algebra. In the last case, we have new additional non-commutative coordinates which the fields should depend on, while in the first case one has a non-abelian gauge symmetry corresponding to representations of the algebra (2.15). We don’t know to which extent the representation of the algebra (2.14) can depend on the space “points” \( x^\mu = -\theta^{\mu \nu} p_\nu \) in general, but the simplest case is when the Hilbert space is split as \( \mathcal{H}' \otimes V \), where \( \mathcal{H}' \) is the infinite
dimensional subspace of the Hilbert space on which restriction of \( p_\mu \) generate an irreducible representation while \( V \) realises an irreducible representation of the algebra (2.14).

For example, for \( p_\mu = I_{(n)} \otimes p'_\mu \), where \( p'_\mu \) form an irreducible set and \( I_{(n)} \) is the \( n \times n \) unity matrix one has the set of matrices \( \sigma_\alpha, \alpha = 1, \ldots, n^2 - 1 \) commuting with \( p_\mu \). In this case, an arbitrary fluctuation of the field \( X_i \) around this background can be expanded in terms \( n \times n \)-matrix valued functions, the Weyl map generalises to,

\[
\phi(x) = \sigma^\alpha \sqrt{\det \theta} \int \frac{dp}{(2\pi)^p/2} e^{ikx} \frac{tr(\sigma^\alpha \otimes e^{ikx})}{2} \cdot \phi,
\]

where \( \sigma^\alpha \) are generators of \( su(n) \) with the normalisation given by the \( n \times n \)-matrix trace,

\[
tr_{(n)} \sigma^\alpha \sigma^\beta = \delta^{\alpha\beta}.
\]

• Beyond the basic set of fields in (2.2) one can also try to add some “mater” fields in the fundamental representation. Let us note, however, that any real field in the background invariant representation corresponds to a Hermitian operator which realises the adjoint representation of the Hilbert space unitary symmetry. In the star form such fields belong to adjoint representation of the gauge group. When trying to add complex fields in the fundamental representation of the gauge group, one finds only fields in the bi-fundamental representation of \( G \times G \) where \( G \) is the gauge group. Indeed, the complex field \( \phi \) in the fundamental representation should satisfy \( \partial_\mu \phi(x) = ip_\mu \ast \phi(x) \). Therefore, its action in the background form should look like follows,

\[
S_{\text{fund}} = tr \left( \frac{1}{2} X_\mu \phi \phi^\dagger X_\mu - V(\phi \ast \phi^\dagger) \right).
\]

Obviously, beyond the desired symmetry,

\[
\phi \rightarrow U^{-1} \phi, \quad \phi^\dagger \rightarrow \phi^\dagger U, \quad U \in G,
\]

there is another one

\[
\phi \rightarrow \phi V, \quad \phi^\dagger \rightarrow V^{-1} \phi^\dagger, \quad V \in G.
\]

3. The Hilbert Space Picture

As we have seen in the previous section, the model (2.2) may look like a Yang–Mills model with scalar multiplet in various dimensions or with various gauge groups depending on the background solution chosen.
The common origin of these models leads to duality relations among them. The roots of this duality are as follows. As we know algebras of functions on different non-commutative spaces and with different gauge groups are all isomorphic to the algebra of operators on the infinite dimensional separable Hilbert space and, therefore isomorphic among themselves. In particular, they are Morita equivalent. What is important is that this isomorphism can relate smooth functions to smooth i.e. preserves the topology of the algebra of functions. This is in contrast to what one has in the commutative case.

In this section we are going to consider maps relating such models. We will consider maps relating models with different gauge groups and maps between models in different dimensions. In the next section we consider even more particular case of map changing only the non-commutativity parameter $\theta^{\mu\nu}$.

Before going to particular cases let us discuss some general aspects of the map.

Consider two different backgrounds $p^{(1)}_{\mu_1}$ and $p^{(2)}_{\mu_2}$ each having the form
\[ p^{(l)}_{\mu_l} = p'_\mu \otimes I_{n_l}, \] (3.1)
where $p'_\mu$, satisfy,
\[ [p'_\mu, p'_{\nu}] = -i \theta^{-1}_{(l)\mu\nu}, \quad \det \theta_{(l)} \neq 0, \] (3.2)
and form a complete irreducible set of operators on $H'_l$, $l = 1, 2$.

The ranges $I_{(\mu_1)}$ and $I_{(\mu_2)}$ of indices $\mu_1$ and $\mu_2$ are two subsets of orders respectively $p^{(1)}$ and $p^{(2)}$ of the sequence $1, \ldots, D$.

Operators $p^{(l)}$, as given by eq. (3.1) fail to form a complete irreducible set due to their degeneracies when $n_l > 1$. These degeneracies are solved according to the previous section by sets of $n_l \times n_l$-dimensional Pauli matrices $\sigma^{(l)}_{\alpha_l}$, $\alpha_l = 1, \ldots, n^2_l - 1$, which together with $\sigma_0$ generate the algebra $u(n_l)$. Correspondingly, the Hilbert space is split into,
\[ H \sim H'_l \otimes V_{(l)}, \quad l = 1, 2. \] (3.3)

Now, using the definitions for the Weyl map and inverse Weyl map (2.16) and (2.10), one can write down the formula for passing from one background to another. Thus, for a non-commutative function $\phi^{(1)}(x^{(1)})$ with respect to the first background one has a unique image $\phi^{(2)}(x^{(2)})$ with respect to the second one, which is given by the following covariant map,
\[
\phi^{(2)}(x^{(2)}) = \sigma^{(2)}_{\alpha_2} \sqrt{\det \theta^{(2)}} \int \frac{dk^{(2)}}{(2\pi)^{p^{(2)}/2}} e^{ik^{(2)}x^{(2)}} \int dx^{(1)} \text{tr}_{V_{(1)}} E^{\alpha_2}\left(k^{(2)}; x^{(1)} \right) \phi^{(1)}(x^{(1)}) 
\equiv \left( SW_{21} \phi^{(1)} \right)(x^{(2)}), \] (3.4)
where,

\[ E^{\alpha_2}(k(2); x(1)) = \sigma^{(1)}_{\alpha_1} \int \frac{dk_{(1)}}{(2\pi)^{p_{(1)}}} e^{i k_{(1)} \cdot x_{(1)}} \text{tr} \left( \sigma^{(1)}_{\alpha_1} \otimes e^{i k_{(1)} \cdot p_{(1)}} \sigma^{(2)}_{\alpha_2} \otimes e^{i k_{(2)} \cdot p_{(2)}} \right) \]  

(3.5)

\[ \text{tr}_{V(l)} \] is the trace taken over \( V(l) \) indices. Trace with no label is performed over the Hilbert space \( \mathcal{H} \).

This map defines transformation rules for all fields but gauge ones. The gauge field in different backgrounds, in fact, corresponds to Weyl symbols of different operators,

\[ A^{(l)}_{\mu_1} = X_{\mu_1} - p^{(l)}_{\mu_1}. \]  

(3.6)

Therefore, beyond the “covariant” part of transformation given by formula analogous to eq. (3.4) there is also an inhomogeneous part.

Also, due to the fact that some indices which in the first background correspond to the gauge field in the second background may correspond to the scalar field and vice versa if \( \mu_1 \) and \( \mu_2 \) run different ranges (\( I(\mu_1) \neq I(\mu_2) \)), the map may interchange the gauge field the scalar field.

Taking this into account one has for the map of the gauge field,

\[ A^{(2)}_\alpha = \text{SW}_{21}(p^{(1)}_\alpha + A^{(1)}_\alpha) - p^{(2)}_\alpha, \quad \alpha \in I(\mu_2) \cap I(\mu_1), \]  

(3.7a)

\[ A^{(2)}_\alpha = \text{SW}_{21} \Phi^{(1)}_\alpha - p^{(2)}_\alpha, \quad \alpha \in I(\mu_2) \cap \bar{I}(\mu_1), \]  

(3.7b)

\[ \Phi^{(2)}_\alpha = \text{SW}_{21}(p^{(1)}_\alpha + A^{(1)}_\alpha), \quad \alpha \in \bar{I}(\mu_2) \cap I(\mu_1), \]  

(3.7c)

\[ \Phi^{(2)}_\alpha = \text{SW}_{21} \Phi^{(1)}_\alpha, \quad \alpha \in \bar{I}(\mu_2) \cap \bar{I}(\mu_1), \]  

(3.7d)

where \( \text{SW}_{21} \) is given by eq. (3.4). The notations in the above equations are as follows, \( \alpha \in I(\mu_1) \cap I(\mu_2) \) means that index \( \alpha \) belongs to both range of \( \mu_1 \) and ones of \( \mu_2 \), \( \alpha \in I(\mu_1) \cap \bar{I}(\mu_2) \) means that \( \alpha \) belongs to the range of \( \mu_1 \) but not to one of \( \mu_2 \), and so on. In the equations above we have not written the explicit dependence on \( x(l) \), but assume that fields with \( (l) \) label depend on \( x(l) \), where \( l = 1, 2 \).

It is not difficult to verify that, under this map, gauge equivalent configurations are mapped into gauge equivalent ones. In particular, one has for the gauge fields,

\[ \left\{ \begin{array}{c}
g^{-1}_{(2)}(p_{(2)} + A_{(2)})g_{(2)} \\
g^{-1}_{(2)}\Phi_{(2)}g_{(2)}
\end{array} \right\} = \text{SW}_{21} \left\{ \begin{array}{c}
g^{-1}_{(1)}(p_{(1)} + A_{(1)})g_{(1)} \\
g^{-1}_{(1)}\Phi_{(1)}g_{(1)}
\end{array} \right\}, \]  

(3.8)

where, \( (A_{(1)}, \Phi_{(1)}) \) map into \( (A_{(2)}, \Phi_{(2)}) \) according to (3.7) while \( g_{(1)} \) maps into \( g_{(2)} \) according to (3.4). This means that the map we have obtained is a Seiberg–Witten map.

In the following subsections we consider the particular examples realising either \( \text{U}(1) \)-\( \text{U}(2) \) duality or duality between models in different dimensions.
3.1 U(1) – U(n) duality

Let us present the explicit construction for the map from U(1) to U(2) gauge model in the case of two-dimensional non-commutative space. The map we are going to discuss can be straightforwardly generalised to the case of arbitrary even dimensions as well as to the case of arbitrary U(n) group.

The two-dimensional non-commutative coordinates are,

\[ [x^1, x^2] = i \theta. \] (3.9)

The non-commutative analog of complex coordinates is given by oscillator rising and lowering operators,

\[ a = \sqrt{\frac{1}{2\theta}} (x^1 + ix^2), \quad \bar{a} = \sqrt{\frac{1}{2\theta}} (x^1 - ix^2) \] (3.10)

where \( |n\rangle \) is the so-called oscillator basis formed by eigenvectors of \( N = \bar{a}a \),

\[ N |n\rangle = n |n\rangle. \] (3.12)

The gauge symmetry in this background is non-commutative U(1).

We will now construct the non-commutative U(2) gauge model. For this, consider the U(2) basis which is given by following vectors,

\[ |n', a\rangle = |n'\rangle \otimes e_a, \quad a = 0, 1 \] (3.13)

\[ e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (3.14)

where \( \{|n'\rangle\} \) is the oscillator basis and \( \{e_a\} \) is the “isotopic” space basis.

The one-to-one correspondence between U(1) and U(2) bases can be established in the following way [35],

\[ |n'\rangle \otimes e_a \sim |n\rangle = |2n' + a\rangle, \] (3.15)

where \( |n\rangle \) is a basis element of the U(1)-Hilbert space and \( |n'\rangle \otimes e_a \) is a basis element of the Hilbert space of U(2)-theory. (Note, that they are two bases of the same Hilbert space.)

Let us note that the identification (3.15) is not unique. For example, one can put an arbitrary unitary matrix in front of \( |n\rangle \) in the r.h.s. of (3.15). This in fact describes all possible identifications and respectively maps from U(1) to U(2) model.

Under this map, the U(2) valued functions can be represented as scalar functions in U(1) theory. For example, constant U(2) matrices are mapped to particular functions in U(1) space. To find these functions, it suffices to find the map of the basis of the \( u(2) \) algebra given by Pauli matrices \( \sigma_\alpha, \alpha = 0, 1, \ldots, 3 \).
In the U(1) basis Pauli matrices look as follows,

\begin{align*}
\sigma_0 &= \sum_{n=0}^{\infty} (|2n\rangle \langle 2n| + |2n+1\rangle \langle 2n+1|) \equiv I, \quad (3.16a) \\
\sigma_1 &= \sum_{n=0}^{\infty} (|2n\rangle \langle 2n+1| + |2n+1\rangle \langle 2n|), \quad (3.16b) \\
\sigma_2 &= -i \sum_{n=0}^{\infty} (|2n\rangle \langle 2n+1| - |2n+1\rangle \langle 2n|), \quad (3.16c) \\
\sigma_3 &= \sum_{n=0}^{\infty} (|2n\rangle \langle 2n| - |2n+1\rangle \langle 2n+1|), \quad (3.16d)
\end{align*}

while the “complex” coordinates \(a'\) and \(\bar{a}'\) of the U(2) invariant space are given by the following,

\begin{align*}
a' &= \sum_{n=0}^{\infty} \sqrt{n} \left( |2n-2\rangle \langle 2n| + |2n-1\rangle \langle 2n+1| \right), \quad (3.17a) \\
\bar{a}' &= \sum_{n=0}^{\infty} \sqrt{n+1} \left( |2n+2\rangle \langle 2n| + |2n+3\rangle \langle 2n+1| \right). \quad (3.17b)
\end{align*}

One can see that when trying to find the Weyl symbols for operators given by (3.16), (3.17), one faces the problem that the integrals defining the Weyl symbols diverge. This happens because the respective functions (operators) do not belong to the non-commutative analog of \(L^2\) space (are not square-trace).

Let us give an alternative way to compute the functions corresponding to operators (3.16) and (3.17). To do this let us observe that operators

\begin{align*}
\Pi_+ &= \sum_{n=0}^{\infty} |2n\rangle \langle 2n|, \quad (3.18) \\
\Pi_- &= \sum_{n=0}^{\infty} |2n+1\rangle \langle 2n+1|, \quad (3.19)
\end{align*}

can be expressed as\(^3\)

\begin{equation}
\Pi_+ = \frac{1}{2} \sum_{n=0}^{\infty} \left( 1 + \sin \frac{\pi}{2} \left( n + \frac{1}{2} \right) \right) |n\rangle \langle n| \rightarrow \frac{1}{2} \left( 1 + \sin \pi \left( \bar{z} \ast z + \frac{1}{2} \right) \right), \quad (3.20)
\end{equation}

\(^3\)Weyl symbols of \(a\) and \(\bar{a}\) are denoted, respectively, as \(z\) and \(\bar{z}\). The same rule applies also to primed variables.
and,

\[ \Pi_- = \mathbb{I} - \Pi_+ = \frac{1}{2} \left( 1 - \sin_\pi \left( \bar{z} \ast z + \frac{1}{2} \right) \right) = \frac{1}{2} \left( 1 - \sin_\pi |z|^2 \right), \]  

(3.21)

where \( \sin_\pi \) is the “star” \( \sin \) function defined by the star Taylor series,

\[ \sin_\pi f = f - \frac{1}{3!} f \ast f \ast f + \frac{1}{5!} f \ast f \ast f \ast f \ast f - \cdots, \]

(3.22)

with the star product defined in variables \( z, \bar{z} \) as follows,

\[ f \ast g(\bar{a}, a) = e^{\partial \bar{a} - \partial a} f(\bar{z}, z) g(\bar{z}', z') |z' = z, \]

(3.23)

where \( \partial = \partial/\partial z, \bar{\partial} = \partial/\partial \bar{z} \) and analogously for primed \( z' \) and \( \bar{z}' \). For convenience we denoted Weyl symbols of \( a \) and \( \bar{a} \) as \( z \) and \( \bar{z} \).

The easiest way to compute (3.20) and (3.21) is to find the Weyl symbol of the operator,

\[ I^\pm_k = \frac{1 \pm \sin (\bar{a}a + \frac{1}{2})}{(\bar{a}a + \gamma)^k}, \]

(3.24)

were \( \gamma \) is some constant, mainly \( \pm 1/2 \).

For sufficiently large \( k \), the operator \( I^\pm_k \) becomes square trace for which the formula (2.7) defining the Weyl map is applicable. The Weyl symbol for smaller values of \( k \) can be obtained using the following recurrence relation,

\[ I^\pm_{k-m}(\bar{z}, z) = \left( |z|^2 + \gamma - \frac{1}{2} \right) \ast \cdots \ast \left( |z|^2 + \gamma - \frac{1}{2} \right) \ast I^\pm_k(\bar{z}, z). \]

(3.25)

The last equation requires computation of only finite number of derivatives of \( I^\pm_k(\bar{z}, z) \) arising from the star product with polynomials in \( \bar{z}, z \).

From the viewpoint of the gauge theory (2.2), the configuration (3.17) can be seen as a solution to equations of motion in the U(1) theory, while operators (3.16) are the ones commuting with this solution, i.e. generators of its symmetry algebra.

### 3.2 Duality between models in different dimensions

Consider the situation when \( V \) is another Hilbert space or product of Hilbert spaces. This topic was considered in [21, 32] here we only shortly review this.

Consider the Hilbert space \( \mathcal{H} \) corresponding to two-dimensional non-commutative space (3.9), and \( \mathcal{H} \otimes \mathcal{H} \) which corresponds to four-dimensional non-commutative space generated by

\[ [x^1, x^2] = i\theta_{(1)}, \quad [x^3, x^4] = i\theta_{(2)}. \]

(3.26)
In the last case non-commutative complex coordinates correspond to two sets of oscillator operators, \( a_1, a_2 \) and \( \bar{a}_1, \bar{a}_2 \), where,

\[
\begin{align*}
    a_1 &= \sqrt{\frac{1}{2\theta(1)}} (x^1 + ix^2), & \bar{a}_1 &= \sqrt{\frac{1}{2\theta(1)}} (x^1 - ix^2) \quad (3.27a) \\
    a_1 |n_1\rangle &= \sqrt{n_1} |n_1 - 1\rangle, & \bar{a}_1 |n_1\rangle &= \sqrt{n_1 + 1} |n_1 + 1\rangle, \quad (3.27b) \\
    a_2 &= \sqrt{\frac{1}{2\theta(2)}} (x^3 + ix^4), & \bar{a}_2 &= \sqrt{\frac{1}{2\theta(2)}} (x^3 - ix^4) \quad (3.27c) \\
    a |n\rangle_2 &= \sqrt{n_2} |n_2 - 1\rangle, & \bar{a}_2 |n\rangle_2 &= \sqrt{n_2 + 1} |n_2 + 1\rangle, \quad (3.27d)
\end{align*}
\]

and the basis elements of the “four-dimensional” Hilbert space \( \mathcal{H} \otimes \mathcal{H} \) are \( |n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle \).

The isomorphic map \( \sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \) is given by assigning a unique number \( n \) to each element \( |n_1, n_2\rangle \) and putting it into correspondence to \( |n\rangle \in \mathcal{H} \) [21, 32].

As we discussed earlier, this map induces an isomorphic map of connections and non-commutative functions from two to four dimensional non-commutative spaces.

This can be easily generalised to the case with arbitrary number of factors \( \mathcal{H} \otimes \cdots \otimes \mathcal{H} \) corresponding to \( p/2 \) “two-dimensional” non-commutative spaces. In this way, one obtains the isomorphism \( \sigma \) which relates two-dimensional non-commutative function algebra with a \( p \)-dimensional one, for \( p \) even.

It is worthwhile to note that in the case of the map which relates different dimensions, the flat connection is never mapped to flat one, since the tensor \( \theta_{\mu\nu} \) is different in \( \mathcal{H}(2) \) and \( \mathcal{H}(2) \otimes \mathcal{H}(2) \). (Obviously in the first case it is two-dimensional while in the second one it is four-dimensional.) This map again solves the Seiberg–Witten condition that smooth bounded functions are mapped to smooth and bounded and gauge equivalent configurations are mapped to gauge equivalent ones, and therefore it is a non-perturbative Seiberg–Witten map. This map is singular when \( \theta = 0 \).

In fact, the arguments above indicate that the model (2.2) is non-perturbatively independent on \( \theta_{\mu\nu} \) or the dimensionality of the non-commutative Yang–Mills model. This fact is called background independence [37], also a presumable feature of string field theory [38].

### 4. Perturbative Seiberg–Witten Map

So far, we have considered maps which relate algebras of non-commutative functions in different dimensions or at least taking values in different Lie algebras. Due to the fact that they change considerably the geometry, these maps could not be deformed smoothly into the unit map. (At least it is not obvious that it can be done.) In this section we consider a more restricted class of maps which do not change either
dimensionality or the gauge group but only the non-commutativity parameter. Obviously, this can be smoothly deformed into identity map, therefore one may consider infinitesimal transformations.

In the approach of the second section the non-commutativity parameter is given by the solution to the equations of motion. In this framework, the SW map is given by the change of the background solution $p_\mu$ to a slightly different one $p_\mu + \delta p_\mu$. Then, a solution with the constant field strength $F^{(\delta p)}_{\mu\nu}$ will change the non-commutativity parameter as follows,

$$\theta^{\mu\nu} + \delta \theta^{\mu\nu} \equiv (\theta^{-1} + \delta \theta^{-1})^{-1} = (\theta^{-1} + F_{\mu\nu})^{-1}. \tag{4.1}$$

Note, that the above equation does not require $\delta \theta$ to be infinitesimal.

Since we are considering solutions to the gauge field equations of motion $A_\mu = \delta p_\mu$ one should fix the gauge for it. A convenient choice would be e.g. the Lorentz gauge, $\partial_\mu \delta p_\mu = 0$. Then, the solution with

$$A^{(\delta p)}_\mu \equiv \delta p_\mu = (1/2)\epsilon^{\mu\alpha\nu} p_\alpha \tag{4.2}$$

with antisymmetric $\epsilon^{\mu\nu}$ has the constant field strength

$$F^{(\delta p)}_{\mu\nu} \equiv \delta \theta^{-1} = \epsilon_{\mu\nu} + (1/4)\epsilon_{\mu\alpha\beta} \epsilon_{\beta\nu} = \epsilon_{\mu\nu} + O(\epsilon^2). \tag{4.3}$$

This corresponds to the following variation of the non-commutativity parameter,

$$\delta \theta^{\mu\nu} = -\theta^{\mu\alpha} \epsilon_{\alpha\beta} \theta^{\beta\nu} - \frac{1}{4} \theta^{\mu\alpha} \epsilon_{\alpha\gamma} \theta^{\nu\rho} \epsilon_{\rho\beta} \theta^{\beta\nu} = -\theta^{\mu\alpha} \delta \theta^{-1}_{\alpha\beta} \theta^{\beta\nu} + O(\epsilon^2). \tag{4.4}$$

Let us note that such kind of infinitesimal transformations were considered in a slightly different context in [39].

Let us find how non-commutative functions are changed with respect to this transformation. In order to do this, let us consider how the Weyl symbol (2.7) transforms under the variation of background (4.2). For an arbitrary operator $\phi$ after short calculation we have,

$$\delta \phi(x) = \frac{1}{4} \delta \theta^{\alpha\beta} (\partial_\alpha \phi * p_\beta(x) + p_\beta * \partial_\alpha \phi(x)). \tag{4.5}$$

In obtaining this equation we had to take into consideration variation of $p_\mu$ and of the factor $\sqrt{\det \theta}$ in the definition of the Weyl symbol (2.7).

By construction, this variation satisfies the “star-Leibnitz rule”,

$$\delta (\phi * \chi)(x) = \delta \phi * \chi(x) + \phi * \delta \chi(x) + \phi(\delta *) \chi(x), \tag{4.6}$$

where $\delta \phi(x)$ and $\delta \chi(x)$ are defined according to (4.5) and variation of the star-product is given by,

$$\phi(\delta *) \chi(x) = \frac{1}{2} \delta \theta^{\alpha\beta} \partial_\alpha \phi * \partial_\beta \chi(x). \tag{4.7}$$
The property (4.6) implies that \( \delta \) provides an homomorphism (which is in fact an isomorphism) of star algebras of functions.

The above transformation (4.5) do not apply, however, to the gauge field \( A_\mu(x) \) and gauge field strength \( F_{\mu\nu}(x) \). This is the case because the respective operators are not background independent. Indeed, according to (2.6) \( A_\mu = X_\mu - p_\mu \), where \( X_\mu \) is background independent. Therefore the gauge field \( A_\mu(x) \) transforms inhomogeneously,

\[
\delta A_\mu(x) = \frac{1}{4} \delta \theta^{\alpha\beta} (\partial_\alpha A_\mu * p_\beta + p_\beta * \partial_\alpha A_\mu) + \frac{1}{2} \theta_{\mu\alpha} \delta \theta^{\alpha\beta} p_\beta. \tag{4.8}
\]

The transformation law for \( F_{\mu\nu}(x) \) can be computed using (2.12a) and the “star-Leibnitz rule” (4.6) as well as the fact that it is the Weyl symbol of the operator,

\[
F_{\mu\nu} = i[X_\mu, X_\nu] - \theta_{\mu\nu}. \tag{4.9}
\]

Of course, both approaches give the same result,

\[
\delta F_{\mu\nu}(x) = \frac{1}{4} \delta \theta^{\alpha\beta} (\partial_\alpha F_{\mu\nu} * p_\beta + p_\beta * \partial_\alpha F_{\mu\nu})(x) - \delta \theta^{-1}_{\mu\nu}. \tag{4.10}
\]

The infinitesimal map described above has the following properties:

\( i \). It maps gauge equivalent configurations to gauge equivalent ones, therefore it satisfies the Seiberg–Witten equation,

\[
U^{-1} * A * U + U^{-1} * dU \rightarrow U'^{-1} * A' * U' + U'^{-1} * dU'. \tag{4.11}
\]

\( ii \). It is linear in the fields.

\( iii \). Any background independent functional is invariant under this transformation. In particular, any gauge invariant functional whose dependence on gauge fields enters through the combination \( X_{\mu\nu}(x) = F_{\mu\nu} + \theta_{\mu\nu}^{-1} \) is invariant with respect to (4.5)–(4.10). This is also the symmetry of the action provided that the gauge coupling transforms according to (2.12c).

\( iv \). Formally, the transformation (4.5) can be represented in the form,

\[
\delta \phi(x) = \delta x^\alpha \partial_\alpha \phi(x) = \phi(x + \delta x) - \phi(x),
\]

where \( \delta x^\alpha = -\theta^{\alpha\beta} \delta p_\beta \) and no star product is assumed. This looks exactly like a coordinate transformation.

One may naturally raise the question: how is this connected with the “standard” SW map found in [6]?

In (4.2) we have chosen \( \delta p_\mu \) independent of gauge field background. (In fact the gauge field background was switched-on later, after the transformation.) An
alternative way would be to have nontrivial field $A_{\mu}(x)$ from the very beginning and to chose $\delta p_{\mu}$ to be of the form,

$$\delta_{SW}p_{\mu} = -\frac{1}{2} \epsilon_{\mu\nu}^\alpha \theta^\alpha A_\alpha. \quad (4.13)$$

Then, the transformation laws corresponding to such a transformation of the background coincide exactly with the standard SW map. The substitution (4.13) is possible because the function $p_{\mu} = -\theta^{-1}_{\mu\nu} x^\nu$ has the same gauge transformation properties as $-A_{\mu}(x),$

$$p_{\mu} \rightarrow U^{-1} * p_{\mu} * U(x) - U^{-1} * \partial_{\mu} U(x). \quad (4.14)$$

5. The Quantum Theory

In the previous section we found that the map (4.2) changes the fields leaving the gauge invariant functionals including the action unchanged. To be a symmetry in the “quantum field theory” e.g. the path integral formulation of “quantum” theory, one should check the invariance of the quantum measure as well.

Consider the partition function corresponding to the model (2.11). In the traditional approach, the path integral representation for the partition function is obtained via canonical quantisation. This approach can be easily generalised also to models with spatial non-commutativity. In the case of non-degenerate space-time non-commutativity one can use path integral as the definition for the partition function and, generally, of the non-commutative “quantum” theory.

So, consider the path integral representation for the partition function

$$Z = \int [dA][d\Phi] \det M e^{iS + iS_{g.f.}}. \quad (5.1)$$

where $[dA]$ and $[d\Phi]$ are usual functional measures for Weyl symbols, $S_{g.f.}$ is the gauge fixing term and $\det M$ Faddeev–Popov determinant. For example generalised Lorentz gauge correspond to the choice,

$$S_{g.f.}^{\text{Lorentz}} = \int \frac{1}{2\alpha} (\partial_{\mu} A_{\mu})^2; \quad (5.2)$$

and Faddeev–Popov determinant,

$$\det M = \det \partial_{\mu} \nabla_{\mu}. \quad (5.3)$$

Let us analyse the background invariance properties of the quantum theory as given by the formal path integral (5.1). As established in the previous section, the classical action is invariant under changes of the background (4.2) provided respective rules for transformation of fields, products and couplings are applied. What remains
to be established to generalise the “classical” results is the invariance of the functional measure.

Another source of trouble would be the gauge fixing term which was added to the theory during “quantisation” and which may spoil the background invariance. Apparently, it is difficult to find a gauge fixing term which would be background invariant. This happens because background invariant functionals are all gauge invariant. However, we claim that the way the gauge fixing term destroys gauge invariance is an in-offensive one, and indicates only that the gauge fixing prescription does depend on the background. Moreover at finite volume (finite Hilbert space) it is not necessary.

To establish the transformation properties of the measure, let us recall that the measure can be seen as the determinant of the invariant functional quadratic form,

\[ \| \delta A_\mu \|^2 = \int dx \ (\delta A_\mu (x))^2, \]
\[ \| \delta \Phi \|^2 = \int dx \ (\delta \Phi_a (x))^2, \]

(5.4)

where \( \delta A_\mu (x) \) and \( \delta \Phi_a (x) \) are independent variations of field \( A_\mu (x) \) and \( \Phi_a (x) \). The equations (5.4) can be rewritten as follows,

\[ \| \delta A_\mu \|^2 = (2\pi)^{p/2} \sqrt{\det \theta} \tr (\delta X_\mu)^2, \]
\[ \| \delta \Phi \|^2 = (2\pi)^{p/2} \sqrt{\det \theta} (\delta \Phi_a)^2, \]

(5.5)

where in the last equations, the Hilbert space operators are used and \( \delta A_\mu \) is replaced by the equivalent \( \delta X_\mu \).

As one can see from the equations (5.5), the dependence of functional norm from the background enters only through factors \( (2\pi)^{p/2} \sqrt{\det \theta} \). (The first factor \( (2\pi)^{p/2} \) is important only in the case when dimensionality changes, which is not the case for infinitesimal background transformations.) Therefore, variation of norms with respect to change of background (4.2) is as follows,

\[ \delta \| \delta A_\mu \|^2 = \frac{1}{2} \delta (\ln \det \theta) \| \delta A_\mu \|^2 = -\frac{1}{2} \theta^{-1}_{\alpha\beta} \delta \theta^\alpha\beta \| \delta A_\mu \|^2 \]
\[ \delta \| \delta \Phi_a \|^2 = \frac{1}{2} \delta (\ln \det \theta) \| \delta \Phi_a \|^2 = -\frac{1}{2} \theta^{-1}_{\alpha\beta} \delta \theta^\alpha\beta \| \delta \Phi_a \|^2. \]

(5.6a)

(5.6b)

According, to Eqs. (5.6) the functional measure changes as follows,

\[ \delta [dA] = (p\theta^{-1}_{\alpha\beta} \delta \theta^\alpha\beta \int dx) [dA], \]
\[ \delta [d\Phi] = ((D - p)\theta^{-1}_{\alpha\beta} \delta \theta^\alpha\beta \int dx) [d\Phi], \]
\[ \delta [dA][d\Phi] = (D\theta^{-1}_{\alpha\beta} \delta \theta^\alpha\beta \int dx) [dA][d\Phi]. \]

(5.7a)

(5.7b)

(5.7c)

Thus, we end up with an “anomaly”:

\[ \frac{\delta Z}{\delta \theta^\alpha\beta} = (D\theta^{-1}_{\alpha\beta} V) Z, \]

(5.8)
where $\mathcal{V}$ is the (regularised) volume of space-time.

Since the “anomaly”, we obtained in such a way is a constant proportional to the
the volume of the space-time, it can be absorbed in a ($\theta$-dependent) renormalisation
of the vacuum energy.

Let us note, that the “anomaly” is proportional to the factor $D$ which is the
number of bosonic fields. Fermions, while transforming in the same way as the
bosonic fields under the Seiberg–Witten map, the fermionic measure contributes
with an opposite sign. Then, in supersymmetric models, or at least in models with
equal numbers of bosons and fermions the “anomaly” (5.8) vanishes.

5.1 Remarks on Regularisation

All these results were obtained in the naive approach when no regularisation and
renormalisation is taken into consideration. The regularisation and renormalisation
may change drastically some conclusions concerning the fate of the classical symme-
tries of the theory and we will discuss this issue here.

For the case of pure non-commutative Yang–Mills–scalar model one can write
down regularisation schemes satisfying all necessary criteria. Higher covariant deriva-
tive or dimensional regularisation schemes seem to satisfy the background invariance.
A particularly interesting regularisation of the non-commutative Yang–Mills model
would be the finite $N$ IKKT matrix model [23] obtained by the truncation of the
Hilbert space to $N$ dimensions. This regularisation is non-perturbative, beyond this
it is suitable for the numeric computations. The required properties follow from the
explicit background invariant form of the action.

Although, the algebra (2.4) for finite $N$ is altered this model approaches the
non-commutative Yang–Mills model in the background invariant form (2.2) in the
limit $N \to \infty$. This method is good enough if we deal with a purely bosonic theory.
For fermion containing models there can appear problems connected with fermionic
spectrum doubling [40]. Another potential problem of this method can be identified
with the existence of non-compact directions in the path integral associated with flat
directions in the action. For IKKT-type matrix models these are associated with flat
directions in the potential. Such potential infinities have been investigated [44], [45]
both at finite $N$. It turns out that for sufficiently large $N$ (typically larger than four)
the measure of the path integral is convergent. Thus, finite-$N$ truncation is a valid
background invariant non-perturbative regularisation.

In continuum perturbation theory a momentum cutoff regularisation is the pop-
ular choice. For finite cutoff $\Lambda$ IR-UV mixing implies the existence of IR singularities
[16] reflecting the UV divergences of the commutative theory. No renormalisation
procedure is yet known which deals with such singularities. Such singularities are log-
arithmic at one-loop in supersymmetric theories and as such, amenable to standard
renormalisation. However, it was shown that power singularities appear at higher
loops, and their resummation seems not possible [46]. It has been argued [47] that
such power IR singularities reflect linear potentials among constituent $D_0$ branes in a matrix model realisation of non-commutative theories indicating an instability for non-supersymmetric theories.

It is thus fair to say that in perturbation theory, a momentum cutoff regularisation and renormalisation of non-commutative theories is an open problem.

The use of finite-$N$ truncation of the Hilbert space in the perturbative expansion is not popular. One reason is that it is not convenient in the continuum formulation of perturbation theory. Also, it seems to modify the “classical background” around which we do perturbation theory: the commutation relations $[x^i, x^j] = \theta^{ij}$ with constant $\theta$ cannot be satisfied in a finite dimensional Hilbert space. We do believe however, that their slight modification by a projection operator, at large $N$, where $N$ is the dimension of the Hilbert space is innocuous for the regularisation although it can contribute to finite quantities after renormalisation. From the other hand, it allows non-perturbative analysis in which the system may visibly choose the “preferred” background.

Combined with the aforementioned result, that Hilbert space truncation is a regularisation at large $N$, it seems to be the right scheme in order to discuss the fate of the classical dualities we have presented.

We have shown above that up to a constant renormalisation of the path integral, classical duality is a symmetry of the path integral and the unregulated measure. Thus, any “anomaly” to this symmetries will appear as a non-invariance of the cutoff.

In the finite-$N$ truncation regularisation scheme it is straightforward to answer this question. We will consider as a motivating example the simplest classical map between a U(1) and an U($n$) non-commutative gauge theory with one non-commutative plane. Let us define in the U(1) case the regularised theory by truncating to the first $nN$ states of the harmonic oscillator Hilbert space. The Hilbert space of the U($n$) gauge theory is the tensor product of a $n$ dimensional vector space and the harmonic oscillator Hilbert space. We regularise the theory by keeping the first $N$ states of this Hilbert space. Then the classical map is one-to-one on the remaining finite vector spaces. This map introduces the correspondence between the cutoff parameters of these theories. Thus, the cutoff $\Lambda$ in U(1) theory corresponds to the cutoff of order $\Lambda/n^{2/p}$ in the U($n$) model.

This type of cutoff respects the duality symmetry. Similar cutoff procedures exist for the other duality maps. It is obvious however that singularities are expected if one considers the large $n$-limit of non-commutative U($n$) gauge theory.

In general the cutoffs in different backgrounds can be related via background invariant quantities. Thus, in the large $N$ limit one has background invariant trace,

$$N = \text{tr} \mathbb{I} = \frac{n}{(2\pi)^{p/2} \sqrt{\det \theta}} \int_R d^p x = \frac{n R^p}{(2\pi)^{p/2} \sqrt{\det \theta}},$$

(5.9)
where $R$ is the IR cutoff in the theory. This implies that under the equivalence maps,

$$
\frac{nR^p}{(2\pi)^{p/2}\sqrt{\det \theta}} = \frac{n\sqrt{\det \theta} \Lambda^p}{(2\pi)^{p/2}} = \text{invariant},
$$

where $\Lambda$ is the UV cutoff and we also used the explicit relation between the IR and UV cutoffs, $R^p = (\det \theta) \Lambda^p$, which follows, e.g. from the fact that $x^\mu = -\theta^{\mu\nu} p_\nu$.

Perturbation theory in a given dual version breaks explicitly the duality invariance since different perturbation theories correspond to expansions around different backgrounds of the universal theory. It is thus, not surprising that different perturbation theories have different physics. What we claim here is that there is a non-perturbative definition of the theory which accommodates the classical duality symmetries with no anomalies. This heavily relies on the renormalisation procedure (that so far is not well understood) respecting the symmetry. It is in principle possible however that the symmetry be broken spontaneously due to dynamics. Whether this is realized is beyond our tools at the moment.

It seems that were it for the duality to survive a properly regularised theory, it would imply non-perturbative equivalence of models in different number of non-commutative dimensions. In perturbation theory such theories are plagued by (generically) power IR singularities that render the physics of the theory obscure. Quantum validity of duality will imply that such divergences are artifacts of perturbation theory. They should not be there in a non-perturbative treatment. On the other hand, resummation of subsectors of perturbation theory are not expected to lead to substantial improvement.

6. Conclusion

In the present work we have studied dualities in the non-commutative gauge models. Classically, such dualities relate gauge models in different dimensions as well as models with different “local” gauge groups. “Smaller” transformations relate the same model on spaces with different non-commutativity parameter $\theta$. This is similar to the situation with the Seiberg–Witten map. Indeed, the map satisfies the condition that it maps gauge equivalent configurations to gauge equivalent ones and satisfies an appropriate differential equation.

Nevertheless, in the case of “small” maps when one can compare our solution with the Seiberg–Witten ansatz it appears to be different. The difference consists in the fact that our solution is linear in fields and has a different structure of singularities.

The above classical duality symmetries can be extended to the quantum theory. In the path integral approach this is equivalent to the invariance of the measure. Naively, i.e. neglecting the issues with IR/UV divergences, the functional measure is always duality invariant. For the consistency one should regularise and renormalise the theory. In the purely bosonic theory it is possible to present such a regularisation,
and respective renormalisation. In the case of chiral fermions, however, problems can appear. In general since the duality symmetry is intrinsically connected with the gauge invariance we believe that gauge anomaly-free theories should possess also background invariance at the quantum level.

In the opposite case, quantum breaking of this symmetry (an anomaly) may not be fatal for the consistency of the theory. It may signal the appearance of inequivalent perturbative vacua. Non-perturbatively, the theory may favour some of these vacua relative to others.

Another interesting feature of the maps described here is that they relate models with different couplings and different cutoffs. This may reveal interesting information about various regimes of the gauge models. Unfortunately, so far there is no reliable description of the Quantum Non-perturbative Field theories even in the weak coupling regime due to problems related with IR/UV mixing.

It is an important open problem to understand better a background invariant renormalisation of non-commutative theories. Even in perturbation theory (which breaks background invariance) such a procedure is not understood. This will send light to the role duality maps play in the physics of non-commutativity.

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References

[1] A. Connes, A short survey of noncommutative geometry, hep-th/0003006.

[2] P.-M. Ho and Y.-S. Wu, Noncommutative geometry and d-branes, Phys. Lett. B398 (1997) 52–60, [hep-th/9611233].

[3] P.-M. Ho, Y.-Y. Wu, and Y.-S. Wu, Towards a noncommutative geometric approach to matrix compactification, Phys. Rev. D58 (1998) 026006, [hep-th/9712201].

[4] C.-S. Chu and P.-M. Ho, Noncommutative open string and d-brane, Nucl. Phys. B550 (1999) 151–168, [hep-th/9812219].

[5] A. Connes, M. R. Douglas, and A. Schwarz, Noncommutative geometry and matrix theory: Compactification on tori, JHEP 02 (1998) 003, [hep-th/9711162].

[6] N. Seiberg and E. Witten, String theory and noncommutative geometry, JHEP 09 (1999) 032, [hep-th/9908142].
[7] E. Witten, *Noncommutative geometry and string field theory*, Nucl. Phys. **B268** (1986) 253.

[8] R. Gopakumar, S. Minwalla, and A. Strominger, *Noncommutative solitons*, JHEP **05** (2000) 020, [hep-th/0003160].

[9] A. P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, Phys. Lett. B **495** (2000) 407 [arXiv:hep-th/0007043].

[10] C. Sochichiu, *Noncommutative tachyonic solitons: Interaction with gauge field*, JHEP **08** (2000) 026, [hep-th/0007217].

[11] M. Aganagic, R. Gopakumar, S. Minwalla, and A. Strominger, *Unstable solitons in noncommutative gauge theory*, hep-th/0009142.

[12] J. A. Harvey, P. Kraus, F. Larsen, and E. J. Martinec, *D-branes and strings as non-commutative solitons*, JHEP **07** (2000) 042, [hep-th/0005031].

[13] J. A. Harvey, P. Kraus, and F. Larsen, *Exact noncommutative solitons*, JHEP **12** (2000) 024, [hep-th/0010060].

[14] T. Filk, *Divergencies In A Field Theory On Quantum Space*, Phys. Lett. B **376** (1996) 53.

[15] M. V. Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, JHEP **03** (2000) 035, [hep-th/0002186].

[16] S. Minwalla, M. V. Raamsdonk, and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **02** (2000) 020, [hep-th/9912072].

[17] H. O. Girotti, M. Gomes, V. O. Rivelles and A. J. da Silva, *A consistent noncommutative field theory: The Wess-Zumino model*, Nucl. Phys. B **587** (2000) 299 [arXiv:hep-th/0005272].

[18] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B **443** (1995) 85–126, [hep-th/9503124].

[19] J. H. Schwarz, Nucl. Phys. Proc. Suppl. **55B** (1997) 1 [hep-th/9607201].
    S. Forste and J. Louis, Nucl. Phys. Proc. Suppl. **61A** (1998) 3 [hep-th/9612192].
    C. Vafa, *Lectures on strings and dualities*, [hep-th/9702201].
    E. Kiritsis, *Introduction to non-perturbative string theory*, [hep-th/9708130].
    B. de Wit and J. Louis, *Supersymmetry and dualities in various dimensions*, [hep-th/9801132].
    A. Sen, *An introduction to non-perturbative string theory*, [hep-th/9802051].
    E. Kiritsis, *Supersymmetry and duality in field theory and string theory*, [hep-ph/9911525].

[20] A. Schwarz, *Morita equivalence and duality*, Nucl. Phys. B **534** (1998) 720–738, [hep-th/9805034].
[21] C. Sochichiu, *On the equivalence of noncommutative models in various dimensions*, hep-th/0007127.

[22] C. Sochichiu, *Some notes concerning the dynamics of noncommutative solitons in the m(atrix) theory as well as in the noncommutative yang-mills model*, hep-th/0104076.

[23] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *A large-N reduced model as superstring*, Nucl. Phys. B498 (1997) 467, [hep-th/9612115].

[24] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. D55 (1997) 5112–5128, [hep-th/9610043].

[25] T. Eguchi and H. Kawai, *Reduction Of Dynamical Degrees Of Freedom In The Large N Gauge Theory*, Phys. Rev. Lett. 48 (1982) 1063.

[26] A. Dimakis and F. Muller-Hoissen, *Moyal deformation, seiberg-witten-map, and integrable models*, hep-th/0007160.

[27] B. Jurco, L. Moller, S. Schraml, P. Schupp, and J. Wess, *Construction of non-abelian gauge theories on noncommutative spaces*, hep-th/0104153.

[28] A. A. Bichl, J. M. Grinstein, L. Popp, M. Schweda, and R. Wulkenhaar, *Perturbative analysis of the seiberg-witten map*, hep-th/0102044.

[29] D. Brace, B. L. Cerchiai, and B. Zumino, *Nonabelian gauge theories on noncommutative spaces*, hep-th/0107225.

[30] T. Asakawa and I. Kishimoto, *Comments on gauge equivalence in noncommutative geometry*, JHEP 11 (1999) 024, [hep-th/9909139].

[31] S. Goto and H. Hata, *Noncommutative monopole at the second order in theta*, Phys. Rev. D62 (2000) 085022, [hep-th/0005101].

[32] C. Sochichiu, *Exercising in K-theory: Brane condensation without tachyon*, hep-th/0012262.

[33] A. P. Polychronakos, *Noncommutative Chern-Simons terms and the noncommutative vacuum*, JHEP 0011 (2000) 008 [arXiv:hep-th/0010264].

[34] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Prog. Theor. Phys. 99 (1998) 713 [hep-th/9802085]. H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya and T. Tada, Prog. Theor. Phys. Suppl. 134 (1999) 47 [hep-th/9908038]. H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Nucl. Phys. B 565 (2000) 176 [hep-th/9908141].

[35] V. P. Nair and A. P. Polychronakos, *On level quantization for the noncommutative Chern-Simons theory*, Phys. Rev. Lett. 87 (2001) 030403 [arXiv:hep-th/0102181].

[36] C. Sochichiu, *M(any) vacua of IIB*, JHEP 05 (2000) 026, [hep-th/0004062].
[37] N. Seiberg, *A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation*, JHEP 09 (2000) 003, [hep-th/0008013].

[38] E. Witten, *On background independent open string field theory*, Phys. Rev. D46 (1992) 5467–5473, [hep-th/9208027].

[39] T. Ishikawa, S.-I. Kuroki, and A. Sako, *Noncommutative cohomological field theory and GMS soliton*, hep-th/0107033.

[40] C. Sochichiu, *Matrix models: Fermion doubling vs. anomaly*, Phys. Lett. B 485 (2000) 202 [arXiv:hep-th/0005156].

[41] T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, Int. J. Mod. Phys. A15 (2000) 1011–1030, [hep-th/9903187].

[42] L. Faddeev and A. Slavnov, *Gauge Fields: Introduction to Quantum Theory*, vol. 50 of Frontiers in Physics Series. Benjamin/Cummings, 1980.

[43] T. D. Bakeyev, Theor. Math. Phys. 122 (2000) 355 [Teor. Mat. Fiz. 122 (2000) 426] [hep-th/9812208]. T. D. Bakeyev and A. A. Slavnov, “Higher covariant derivative regularization revisited,” Mod. Phys. Lett. A 11 (1996) 1539 [hep-th/9601092].

[44] W. Krauth and M. Staudacher, *Finite Yang-Mills integrals*, Phys. Lett. B 435 (1998) 350 [arXiv:hep-th/9804199].

[45] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, JHEP 0007 (2000) 013 [arXiv:hep-th/0003208]. J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, Nucl. Phys. Proc. Suppl. 94 (2001) 685 [arXiv:hep-lat/0009030].

[46] A. Koshelev, private communication.

[47] M. Van Raamsdonk, *The meaning of infrared singularities in noncommutative gauge theories*, JHEP 0111 (2001) 006 [arXiv:hep-th/0110093].