ON THE EXTENDED $Z_N$-TODA HIERARCHY

CHUANZHONG LI†, JINGSONG HE‡

Department of Mathematics, Ningbo University, Ningbo, 315211, China
†lichuanzhong@nbu.edu.cn
‡hejingsong@nbu.edu.cn

Abstract. The extended flow equations of a new $Z_N$-Toda hierarchy which takes values in a commutative subalgebra $Z_N$ of $gl(N,\mathbb{C})$ is constructed. Meanwhile we give the Hirota bilinear equations and tau function of this new extended $Z_N$-Toda hierarchy (EZTH). Because of logarithm terms, some extended Vertex operators are constructed in generalized Hirota bilinear equations which might be useful in topological field theory and Gromov-Witten theory. Meanwhile the Darboux transformation and bi-hamiltonian structure of this hierarchy are given. From hamiltonian tau symmetry, we give another different tau function of this hierarchy with some unknown mysterious connections with the one defined from the point of Sato theory.

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1. Introduction
The KP hierarchy and Toda lattice hierarchy as completely integrable systems have many important applications in mathematics and physics including the theory of Lie algebra representation, orthogonal polynomials and random matrix model [1–5]. KP and Toda systems have many kinds of reduction or extension, for example BKP, CKP hierarchy, extended Toda hierarchy (ETH) [6, 7], bigraded Toda hierarchy (BTH) [8]–[14] and so on. There are another kinds of generalization called multi-component KP [15, 16] or multi-component Toda systems.
which attract more and more attention because its widely use in many fields such as the fields of multiple orthogonal polynomials and non-intersecting Brownian motions.

The multicomponent KP hierarchy was discussed with application on representation theory and random matrix model in [15,16]. In [3], it was noticed that τ functions of a 2N-multicomponent KP provide solutions of the N-multicomponent 2D Toda hierarchy. The multicomponent 2D Toda hierarchy was considered from the point of view of the Gauss-Borel factorization problem, the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions and matrix Riemann-Hilber problem [17]-[20]. In fact the multicomponent 2D Toda hierarchy in [18] is a periodic reduction of bi-infinite matrix-formed two dimensional Toda hierarchy. The coefficients(or dynamic variables) of the multicomponent 2D Toda hierarchy take values in complex finite-sized matrix. The multicomponent 2D Toda hierarchy contains matrix-formed Toda equation as the first flow equation.

Adding additional logarithm flows to the Toda lattice hierarchy, it becomes the extended Toda hierarchy [6] which governs the Gromov-Witten invariant of CP1. The extended bigraded Toda hierarchy(EBTH) is the extension of the bigraded Toda hierarchy (BTH) which includes additional logarithm flows [8][10]. The Hirota bilinear equation of EBTH was equivalently constructed in our early paper [9] and a very recent paper [21], because of the equivalence of \( t_{1,N} \) flow and \( t_{0,N} \) flow of EBTH. Meanwhile it was proved to govern Gromov-Witten invariant of the total descendent potential of \( \mathbb{P}^1 \) orbifolds [21]. A nature question is what about the corresponding extended multi-component Toda hierarchy( as a matrix-formed generalization of extended Toda hierarchy [6]) and extended multicomponent bigraded Toda hierarchy. There is a class of orbifolds which should be governed by some logarithmic hierarchies. That is why we think this kind of new logarithmic hierarchy might be useful in Gromov-Witten invariants theory governed by these two new hierarchies. With this motivation, our paper [22] will be devoted to construct a kind of Hirota quadratic equation taking values in a differential matrix algebra set. This kind of Hirota bilinear equation might be useful in Gromov-Witten theory. In [23], a new hierarchy called as \( Z_m \)-KP hierarchy which take values in a maximal commutative subalgebra of \( gl(m, \mathbb{C}) \) was constructed, meanwhile the relation between Frobenius manifold and dispersionless reduced \( Z_m \)-KP hierarchy was discussed. This inspires us to consider the Hirota quadratic equation of the commutative version of extended multi-component Toda hierarchy which might be useful in Frobenius manifold theory in this paper.

This paper is arranged as follows. In the next section we recall factorization problem and construct the logarithm matrix operators using which we define the extended flow of the multicomponent \( Z_N \)-Toda hierarchy. In Section 3, we will give the Lax equations of the extended \( Z_N \)-Toda hierarchy (EZTH), meanwhile the multicomponent \( Z_N \)-Toda equations and the extended equations are introduced in this hierarchy. By Sato equations, Hirota bilinear equations of the EZTH are proved in Section 4. The tau function of the EZTH will be defined in Section 5 which lead to the formalism of the generalized matrix Vertex operators and Hirota quadratic equations in Section 6. In section 7, multi-fold transformations of the EZTH will be constructed using determinant technique in [25]. To prove the integrability of this new hierarchy, Bi-hamiltonian structure and tau symmetry of the EZTH are constructed in Section 8. Section 9 will be devoted to a short conclusions and discussions.

2. Factorization and Logarithm operators

Suppose that \( G \) is a group which contains linear invertible elements of complex \( N \times N \) complex matrices and its Lie algebra \( g \) denotes the associative algebra of complex \( N \times N \)
complex matrices $M_N(\mathbb{C})$. Now we will consider the linear space of function $g : \mathbb{R} \to M_N(\mathbb{C})$ with the shift operator $\Lambda$ acting on these functions as $(\Lambda g)(x) := g(x + \epsilon)$. A Left multiplication by $X : \mathbb{R} \to M_N(\mathbb{C})$ is as $X \Lambda^j, (X \Lambda^j)(g) := X(x) \circ g(x + j\epsilon)$ with defining the product $(X(x) \Lambda^i) \circ (Y(x) \Lambda^j) := X(x)Y(x + i\epsilon) \Lambda^{i+j}$. Then the set $\mathfrak{g}$ of Laurent series in $\Lambda$ as an associative algebra is a Lie algebra under the standard commutator.

This Lie algebra has the following important splitting

\begin{equation}
(2.1) \quad \mathfrak{g} = \mathfrak{g}_+ \circ \mathfrak{g}_-,
\end{equation}

where

\begin{equation*}
\mathfrak{g}_+ = \left\{ \sum_{j \geq 0} X_j(x) \Lambda^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}, \quad \mathfrak{g}_- = \left\{ \sum_{j < 0} X_j(x) \Lambda^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}.
\end{equation*}

The splitting (2.1) leads us to consider the following factorization of $g \in G$

\begin{equation}
(2.2) \quad g = g_-^{-1} \circ g_+, \quad g_{\pm} \in G_{\pm}
\end{equation}

where $G_{\pm}$ have $\mathfrak{g}_{\pm}$ as their Lie algebras. $G_+$ is the set of invertible linear operators of the form $\sum_{j \geq 0} g_j(x) \Lambda^j$; while $G_-$ is the set of invertible linear operators of the form $1 + \sum_{j < 0} g_j(x) \Lambda^j$. This algebra has a maximal commutative subalgebra $Z_N = \mathbb{C}[\Gamma]/(\Gamma^N)$ and $\Gamma = (\delta_{i,j+1})_{ij} \in gl(N, \mathbb{C})$. Denote $Z_N(\Lambda) := g_c$, then we have the following splitting

\begin{equation}
(2.3) \quad \mathfrak{g}_c = \mathfrak{g}_{c+} \circ \mathfrak{g}_{c-},
\end{equation}

where

\begin{equation*}
\mathfrak{g}_{c+} = \left\{ \sum_{j \geq 0} X_j(x) \Lambda^j, \quad X_j(x) \in Z_N \right\}, \quad \mathfrak{g}_{c-} = \left\{ \sum_{j < 0} X_j(x) \Lambda^j, \quad X_j(x) \in Z_N \right\}.
\end{equation*}

The splitting (2.3) leads us to consider the following factorization of $g_c \in G_c$

\begin{equation}
(2.4) \quad g_c = g_{c-1} \circ g_{c+}, \quad g_{c\pm} \in G_{c\pm},
\end{equation}

where $G_{c\pm}$ have $\mathfrak{g}_{c\pm}$ as their Lie algebras. $G_{c+}$ is the set of invertible linear operators of the form $\sum_{j \geq 0} g_j(x) \Lambda^j$; while $G_{c-}$ is the set of invertible linear operators of the form $1 + \sum_{j < 0} g_j(x) \Lambda^j$.

Now we introduce the following free operators $W_0, \bar{W}_0 \in G_c$

\begin{equation}
(2.5) \quad W_0 := e^{\sum_{j=0}^{\infty} t_j \frac{\Lambda^j}{\epsilon^j} + s_j \frac{\Lambda^j}{\epsilon^j}(\epsilon^j - \epsilon)},
\end{equation}

\begin{equation}
(2.6) \quad \bar{W}_0 := e^{\sum_{j=0}^{\infty} t_j \frac{\Lambda^{-j}}{\epsilon^j} + s_j \frac{\Lambda^{-j}}{\epsilon^j}(\epsilon^j - \epsilon)},
\end{equation}

where $t_j, s_j \in \mathbb{C}$ will play the role of continuous times.

We define the dressing operators $W, \bar{W}$ as follows

\begin{equation}
(2.7) \quad W := S \circ W_0, \quad \bar{W} := \bar{S} \circ \bar{W}_0, \quad S \in G_{c-}, \quad \bar{S} \in G_{c+}.
\end{equation}

Given an element $g \in G_c$ and denote $t = (t_j), s = (s_j); j \in \mathbb{N}$, one can consider the factorization problem in $G_c$ similarly as [18]

\begin{equation}
(2.8) \quad W \circ g = \bar{W},
\end{equation}

i.e. the factorization problem

\begin{equation}
(2.9) \quad S(t, s) \circ W_0 \circ g = \bar{S}(t, s) \circ \bar{W}_0.
\end{equation}
Observe that $S$, $\bar{S}$ have expansions of the form
\begin{equation}
S = \mathbb{I}_N + \omega_1(x)\Lambda^{-1} + \omega_2(x)\Lambda^{-2} + \cdots \in G_{c-},
\end{equation}
\begin{equation}
\bar{S} = \bar{\omega}_0(x) + \bar{\omega}_1(x)\Lambda + \bar{\omega}_2(x)\Lambda^2 + \cdots \in G_{c+}.
\end{equation}

Also we define the symbols of $S$, $\bar{S}$ as $\mathbb{S}, \bar{\mathbb{S}}$
\begin{equation}
\mathbb{S} = \mathbb{I}_N + \omega_1(x)\lambda^{-1} + \omega_2(x)\lambda^{-2} + \cdots ,
\end{equation}
\begin{equation}
\bar{\mathbb{S}} = \bar{\omega}_0(x) + \bar{\omega}_1(x)\lambda + \bar{\omega}_2(x)\lambda^2 + \cdots .
\end{equation}

The inverse operators $S^{-1}, \bar{S}^{-1}$ of operators $S, \bar{S}$ have expansions of the form
\begin{equation}
S^{-1} = \mathbb{I}_N + \omega'_1(x)\Lambda^{-1} + \omega'_2(x)\Lambda^{-2} + \cdots \in G_{c-},
\end{equation}
\begin{equation}
\bar{S}^{-1} = \bar{\omega}'_0(x) + \bar{\omega}'_1(x)\Lambda + \bar{\omega}'_2(x)\Lambda^2 + \cdots \in G_{c+}.
\end{equation}

Also we define the symbols of $S^{-1}, \bar{S}^{-1}$ as $\mathbb{S}^{-1}, \bar{\mathbb{S}}^{-1}$
\begin{equation}
\mathbb{S}^{-1} = \mathbb{I}_N + \omega'_1(x)\lambda^{-1} + \omega'_2(x)\lambda^{-2} + \cdots ,
\end{equation}
\begin{equation}
\bar{\mathbb{S}}^{-1} = \bar{\omega}'_0(x) + \bar{\omega}'_1(x)\lambda + \bar{\omega}'_2(x)\lambda^2 + \cdots .
\end{equation}

The Lax operators $\mathcal{L} \in G_c$ are defined by
\begin{equation}
\mathcal{L} := W \circ \Lambda \circ W^{-1} = \bar{W} \circ \Lambda^{-1} \circ \bar{W}^{-1},
\end{equation}
and have the following expansions
\begin{equation}
\mathcal{L} = \Lambda + u_1(x) + u_2(x)\Lambda^{-1}.
\end{equation}

In fact the Lax operators $\mathcal{L} \in G_c$ are also be equivalently defined by
\begin{equation}
\mathcal{L} := S \circ \Lambda \circ S^{-1} = \bar{S} \circ \Lambda^{-1} \circ \bar{S}^{-1}.
\end{equation}

These definitions are continuous interpolated version of the multi-component commutative Toda hierarchy, i.e. a continuous spatial parameter $x$ was brought into this hierarchy. Under this meaning, the continuous flow $\frac{\partial}{\partial x}$ is missing. To make these flows complete, we define the following logarithm matrix
\begin{equation}
\log_+ \mathcal{L} = W \circ \epsilon \partial \circ W^{-1} = S \circ \epsilon \partial \circ S^{-1},
\end{equation}
\begin{equation}
\log_- \mathcal{L} = -\bar{W} \circ \epsilon \partial \circ W^{-1} = -\bar{S} \circ \epsilon \partial \circ \bar{S}^{-1},
\end{equation}
where $\partial$ is the derivative about spatial variable $x$.

Combining these above logarithm operators together can derive following important logarithm matrix
\begin{equation}
\log \mathcal{L} := \frac{1}{2}(\log_+ \mathcal{L} + \log_- \mathcal{L}) = \frac{1}{2}(S \circ \epsilon \partial \circ S^{-1} - \bar{S} \circ \epsilon \partial \circ \bar{S}^{-1}) := \sum_{i=-\infty}^{+\infty} W_i \Lambda^i \in G_c,
\end{equation}
which will generate a series of flow equations which contain the spatial flow in later defined Lax equations.
3. Lax equations of EZTH

In this section we will use the factorization problem (2.8) to derive Lax equations. Let us first introduce some convenient notation

Definition 1. Matrix operators $B_j, D_j$ are defined as follows

\begin{equation}
B_j := \frac{\mathcal{L}^j}{(j+1)!}, \quad D_j := \frac{2\mathcal{L}^j}{j!}(\log \mathcal{L} - c_j), \quad c_j = \sum_{i=1}^{j} \frac{1}{i}, \quad j \geq 0.
\end{equation}

Now we give the definition of the extended $Z_N$-Toda hierarchy (EZTH).

Definition 2. The extended $Z_N$-Toda hierarchy is a hierarchy in which the dressing operators $S, \bar{S}$ satisfy following Sato equations

\begin{align}
(3.2) & \quad \epsilon \partial_t j S = -(B_j)_- S, \quad \epsilon \partial_t j \bar{S} = (B_j)_+ \bar{S}, \\
(3.3) & \quad \epsilon \partial_s j S = -(D_j)_- S, \quad \epsilon \partial_s j \bar{S} = (D_j)_+ \bar{S}.
\end{align}

Then one can easily get the following proposition about $W, \bar{W}$.

Proposition 1. The dressing operators $W, \bar{W}$ are subject to following Sato equations

\begin{align}
(3.4) & \quad \epsilon \partial_t j W = (B_j)_+ W, \quad \epsilon \partial_t j \bar{W} = (B_j)_+ \bar{W}, \\
(3.5) & \quad \epsilon \partial_s j W = \left(\frac{\mathcal{L}^j}{j!}(\log_+ \mathcal{L} - c_j) - (D_j)_- \right)W, \quad \epsilon \partial_s j \bar{W} = \left(-\frac{\mathcal{L}^j}{j!}(\log_- \mathcal{L} - c_j) + (D_j)_+ \right)\bar{W}.
\end{align}

From the previous proposition we derive the following Lax equations for the Lax operators.

Proposition 2. The Lax equations of the EZTH are as follows

\begin{align}
(3.6) & \quad \epsilon \partial_t j \mathcal{L} = [(B_j)_+, \mathcal{L}], \quad \epsilon \partial_s j \mathcal{L} = [(D_j)_+, \mathcal{L}], \quad \epsilon \partial_t j \log \mathcal{L} = [(B_j)_+, \log \mathcal{L}], \\
(3.7) & \quad \epsilon (\log \mathcal{L}) j = -((D_j)_-, \log_+ \mathcal{L}) + [(D_j)_+, \log_- \mathcal{L}].
\end{align}

To see this kind of hierarchy more clearly, the $Z_N$-Toda equations as the $t_0$ flow equations will be given in next subsection.

3.1. The extended $Z_N$-Toda equations. As a consequence of the factorization problem (2.8) and Sato equations, after taking into account that $S \in G_{c-}$ and $\bar{S} \in G_{c+}$, the $t_0$ flow of $\mathcal{L}$ in the form of $\mathcal{L} = \Lambda + U + V\Lambda^{-1}$ is as

\begin{equation}
(3.8) \quad \epsilon \partial_{t_0} \mathcal{L} = [\Lambda + U, V\Lambda^{-1}],
\end{equation}

which lead to matrix-Toda equation

\begin{align}
(3.9) & \quad \epsilon \partial_{t_0} U = V(x + \epsilon) - V(x), \\
(3.10) & \quad \epsilon \partial_{t_0} V = U(x)V(x) - V(x)U(x - \epsilon).
\end{align}

Of course, one can switch the order of the matrices because of the commutativity of $Z_N$. Suppose

\begin{equation}
U = \begin{bmatrix} u_0 & 0 \\ u_1 & u_0 \end{bmatrix}, \quad V = \begin{bmatrix} v_0 & 0 \\ v_1 & v_0 \end{bmatrix},
\end{equation}

\begin{equation}
(3.11) \quad U = \begin{bmatrix} u_0 & 0 \\ u_1 & u_0 \end{bmatrix}, \quad V = \begin{bmatrix} v_0 & 0 \\ v_1 & v_0 \end{bmatrix},
\end{equation}
then the specific coupled Toda equation is
\begin{align}
(3.12) \quad \epsilon \partial_{t_0} u_0 &= v_0(x + \epsilon) - v_0(x), \\
(3.13) \quad \epsilon \partial_{t_0} u_1 &= v_1(x + \epsilon) - v_1(x), \\
(3.14) \quad \epsilon \partial_{t_0} v_0 &= u_0(x)v_0(x) - v_0(x)v_0(x - \epsilon), \\
(3.15) \quad \epsilon \partial_{t_0} v_1 &= (u_1(x) - u_1(x - \epsilon))v_0(x) - v_1(x)(u_0(x) - u_0(x - \epsilon)).
\end{align}

To get the standard matrix-Toda equation, one need to use the alternative expressions
\begin{align}
U := \omega_1(x) - \omega_1(x + \epsilon) = \epsilon \partial_{t_1} \phi(x), \\
V = e^{\phi(x)} e^{-\phi(x-\epsilon)} = -\epsilon \partial_{t_1} \omega_1(x).
\end{align}

From Sato equation we deduce the following set of nonlinear partial differential-difference equations
\begin{align}
(3.16) \quad \left\{ \begin{array}{l}
\omega_1(x) - \omega_1(x + \epsilon) = \epsilon \partial_{t_1} \left( e^{\phi(x)} \cdot e^{-\phi(x)} \right), \\
\epsilon \partial_{t_1} \omega_1(x) = - e^{\phi(x)} e^{-\phi(x-\epsilon)}.
\end{array} \right.
\end{align}

Observe that if we cross the two first equations, then we get
\[ \epsilon^2 \partial_{t_1}^2 \phi(x) = e^{\phi(x+\epsilon)} e^{-\phi(x)} - e^{\phi(x)} e^{-\phi(x-\epsilon)} \]
which is the \( N \times N \) matrix-valued extension of the Toda equation, from which the original Toda equation appears for \( N = 1 \). When \( N = 2 \), the equation for \( \phi \) is the following coupled Toda system
\[ \epsilon^2 \partial_{t_1}^2 \phi_0(x) = e^{\phi_0(x+\epsilon)-\phi_0(x)} - e^{\phi(x)} e^{\phi_0(x)-\phi_0(x-\epsilon)}, \]
\[ \epsilon^2 \partial_{t_1}^2 \phi_1(x) = (\phi_1(x + \epsilon) - \phi_1(x)) e^{\phi_0(x+\epsilon)-\phi_0(x)} - (\phi_1(x) - \phi_1(x - \epsilon)) e^{\phi_0(x)-\phi_0(x-\epsilon)}. \]

In the calculation, the identity
\[ \left[ \begin{array}{cc}
\phi_0 & 0 \\
\phi_1 & \phi_0
\end{array} \right] = \left[ \begin{array}{cc}
e^{\phi_0} & 0 \\
\phi_1 e^{\phi_0} & e^{\phi_0}
\end{array} \right]\]
is used. Besides above \( Z_N \)-Toda equations, with logarithm flows the EZTH also contains some extended flow equations in the next part. Here we consider the extended flow equations in the simplest case, i.e. the \( s_0 \) flow for \( \mathcal{L} = \Lambda + u_0 + u_1\Lambda^{-1} \),
\begin{align}
(3.18) \quad \epsilon \partial_{s_0} \mathcal{L} &= [(Se_{\delta x}S^{-1})_+ , \mathcal{L}] \\
(3.19) \quad \epsilon \partial_{s_0} \mathcal{L} &= [\epsilon_{\delta x}SS^{-1}, \mathcal{L}] \\
(3.20) \quad \epsilon \mathcal{L}_x,
\end{align}

which leads to the following specific equation
\begin{align}
(3.21) \quad \partial_{s_0} U &= U_x, \quad \partial_{s_0} V = V_x.
\end{align}

To see the extended equations clearly, one need to rewrite the extended flows in the Lax equations of the EZTH as in the following lemma.

**Lemma 1.** The extended flows in Lax formulation of the EZTH can be equivalently given by
\begin{align}
(3.22) \quad \frac{\partial \mathcal{L}}{\partial s_j} = [D_j, \mathcal{L}],
\end{align}
(3.23) \[ D_j = \left( \frac{\mathcal{L}_j}{j!}(\log_+ \mathcal{L} - c_j) \right)_+ - \left( \frac{\mathcal{L}_j}{j!}(\log_- \mathcal{L} - c_j) \right)_-, \]

which can also be rewritten in the form

(3.24) \[ e \frac{\partial \mathcal{L}}{\partial s_n} = [D_n, \mathcal{L}], \]

\[ \tilde{D}_j = \frac{\mathcal{L}_j}{j!} \varepsilon \partial + \left[ \frac{\mathcal{L}_j}{j!} \left( \sum_{k<0} W_k(x) \Lambda^k - c_j \right) \right]_+ - \left[ \frac{\mathcal{L}_j}{j!} \left( \sum_{k \geq 0} W_k(x) \Lambda^k - c_j \right) \right]_- \]

Then one can derive the \( s_1 \) flow equation of the EZTH as

\[ \epsilon U_{s_1} = (1 - \Lambda)(V(\Lambda^{-1} - 1)^{-1} \epsilon (\log V)_x) - 2(\Lambda - 1)V + \frac{\epsilon^2}{2} U_x^2 + \epsilon V_x, \]
\[ \epsilon V_{s_1} = ((\Lambda^{-1} - 1)^{-1} \epsilon V_x V^{-1} + 2)(U(x - \epsilon) - U(x))V + \epsilon V_x U(x - \epsilon) + \epsilon(U_x(x - \epsilon) + U_x(x))V, \]

where \( U, V \) without bracket behind them means \( U(x), V(x) \) respectively. To give a linear description of the EZTH, we introduce matrix wave functions \( \psi, \bar{\psi} \) in the following part.

The matrix wave functions of the EZTH are defined by

(3.25) \[ \psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \bar{\chi}, \]

where

(3.26) \[ \chi(z) := z^{\frac{\epsilon}{\epsilon_0}} I_N, \quad \bar{\chi}(z) := z^{-\frac{\epsilon}{\epsilon_0}} I_N, \]

and the “.” means the action of an operator on a function. Note that \( \Lambda \cdot \chi = z \chi \) and the following asymptotic expansions can be defined

(3.27) \[ \psi = z^{\frac{\epsilon}{\epsilon_0}} (I_N + \omega_1(x) z^{-1} + \cdots) \psi_0(z), \quad \psi_0 := e^{\sum_{j=1}^{\infty} \frac{z^j \sum_{s_j \geq 0} \frac{(\log z - c_j)}{s_j}}{j!} z^j}, \quad z \to \infty, \]
\[ \bar{\psi} = z^{-\frac{\epsilon}{\epsilon_0}} (\bar{\omega}_0(x) + \bar{\omega}_1(x) z + \cdots) \bar{\psi}_0(z), \quad \bar{\psi}_0 := e^{\sum_{j=0}^{\infty} \frac{z^{-j} \sum_{s_j \geq 0} \frac{(\log z - c_j)}{s_j}}{j!} z^j}, \quad z \to 0. \]

We can further get linear equations in the following proposition.

**Proposition 3.** The matrix wave functions \( \psi, \bar{\psi} \) are subject to following Sato equations

(3.28) \[ \mathcal{L} \cdot \psi = z \psi, \quad \mathcal{L} \cdot \bar{\psi} = z \bar{\psi}, \]
(3.29) \[ \epsilon \partial_s \psi = (B_j)_+ \cdot \psi, \quad \epsilon \partial_s \bar{\psi} = (B_j)_+ \cdot \bar{\psi}, \]
(3.30) \[ \epsilon \partial_s \psi = \left( \frac{\mathcal{L}_j}{\epsilon j!} (\log_+ \mathcal{L} - c_j) - (D_j)_- \right) \cdot \psi, \quad \epsilon \partial_s \bar{\psi} = \left( -\frac{\mathcal{L}_j}{\epsilon j!} (\log_- \mathcal{L} - c_j) + (D_j)_+ \right) \cdot \bar{\psi}. \]

4. **HIROTA BILINEAR EQUATIONS**

From Lax equations, one can find the \( s_0 \) flow is equivalent to spatial flow \( \partial_z \). Basing on this fact, Hirota bilinear equations which are equivalent to Lax equations of the EZTH can be derived in following proposition.

**Proposition 4.** \( W \) and \( \bar{W} \) are matrix-valued wave operators of the extended \( Z_N \)-Toda hierarchy if and only the following Hirota bilinear equations hold

(4.1) \[ W \Lambda^r W^{-1} = \bar{W} \Lambda^{-r} \bar{W}^{-1}, \quad r \in \mathbb{N}. \]
Proof. ⇒ Set
\[(4.2) \quad \alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots), \quad \beta = (\beta_1, \beta_2, \ldots),\]
be a multi index and
\[(4.3) \quad \partial^\alpha := \partial_{t_0}^{\alpha_0} \partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2} \ldots, \quad \partial^\beta := \partial_{s_1}^{\beta_1} \partial_{s_2}^{\beta_2} \ldots .\]
Suppose \(\partial^\theta = \partial^\alpha \partial^\beta\). Firstly we shall prove the left statement leads to
\[(4.4) \quad W(x, t, \Lambda) \Lambda^r W^{-1}(x, t', \Lambda) = W(x, t, \Lambda) \Lambda^{-r} W^{-1}(x, t', \Lambda)\]
for all integers \(r \geq 0\). Using the same as the method used in [7,9], by induction on \(\alpha\), we shall prove that
\[(4.5) \quad W(x, t, \Lambda) \Lambda^r (\partial^\theta W^{-1}(x, t, \Lambda)) = W(x, t, \Lambda) \Lambda^{-r} (\partial^\theta W^{-1}(x, t, \Lambda)).\]
When \(\theta = 0\), it is obviously true according to the definition of matrix-valued wave operators. Suppose eq.\((4.5)\) is true in the case of \(\theta \neq 0\). Note that
\[
eq \left\{ \begin{array}{ll}
[(\partial_{t_j} S) S^{-1} + S \Lambda^j S^{-1}] W, & p_j = t_j, \\
[(\partial_{s_j} S) S^{-1} + S \Lambda^j \partial_x S^{-1}] W, & p_j = s_j,
\end{array} \right.
\]
and
\[
eq \left\{ \begin{array}{ll}
(B_j)_+ W, & p_j = t_j, \\
[-(D_j)_- + \frac{\omega_j}{\delta_j}(\log_+ \mathcal{L} - c_j)] W, & p_j = s_j,
\end{array} \right.
\]
which further lead to
\[
eq \left\{ \begin{array}{ll}
(B_j)_+ \bar{W}, & p_j = t_j, \\
[(D_j)_+ - \frac{\omega_j}{\delta_j}(\log_- \mathcal{L} - c_j)] \bar{W}, & p_j = s_j,
\end{array} \right.
\]
This further implies
\[(\partial_{p_j} W) \Lambda^r (\partial^\theta W^{-1}) = (\partial_{p_j} \bar{W}) \Lambda^{-r} (\partial^\theta \bar{W}^{-1})\]
by considering \((4.5)\) and furthermore we get
\[W \Lambda^r (\partial_{p_j} \partial^\theta W^{-1}) = W \Lambda^{-r} (\partial_{p_j} \partial^\theta W^{-1}).\]
Thus if we increase the power of \(\partial_{p_j}\) by 1, eq.\((4.5)\) still holds. The induction is completed.

\(\Leftrightarrow\) Vice versa, by separating the negative and the positive part of the equation, we can prove \(S, \bar{S}\) are a pair of matrix-valued wave operators.

To give a description in terms of matrix-valued wave functions, following symbolic definitions are needed.

If the series have forms
\[W(x, t, s, \Lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t, s, \partial_x) \Lambda^i \text{ and } \bar{W}(x, t, s, \Lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t, s, \partial_x) \Lambda^i,
\]

8
Proposition 5. Let \( \lambda \) denote their corresponding left symbols \( \mathcal{W} \), \( \mathcal{W} \) and right symbols \( \mathcal{W}^{-1}, \mathcal{W}^{-1} \) as following

\[
\mathcal{W}(x, t, s, \lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t, s, \partial_x)\lambda^i, \quad \mathcal{W}^{-1}(x, t, s, \lambda) = \sum_{i \in \mathbb{Z}} a_i'(x, t, s, \partial_x)\lambda^i,
\]

\[
\mathcal{W}(x, t, s, \lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t, s, \partial_x)\lambda^i, \quad \mathcal{W}^{-1}(x, t, s, \lambda) = \sum_{i \in \mathbb{Z}} b_i'(x, t, s, \partial_x)\lambda^i.
\]

With above preparation, it is time to give another form of Hirota bilinear equation (see following proposition) after defining residue as \( \text{Res}_\lambda \sum_{n \in \mathbb{Z}} a_n \lambda^n = a_{-1} \) using the similar proof as \([3, 7, 9]\).

Proposition 5. Let \( s_0 = s'_0 \), \( S \) and \( \bar{S} \) are matrix-valued wave operators of the \( Z_N \)-Toda hierarchy if and only if for all \( m \in \mathbb{Z}, r \in \mathbb{N} \), the following Hirota bilinear identity hold

\[
\text{Res}_\lambda \left\{ \lambda^{r+m-1} \mathcal{W}(x, t, s, \epsilon \partial_x, \lambda) \mathcal{W}^{-1}(x - me, t', \bar{p}, s, \epsilon \partial_x, \lambda) \right\} = \text{Res}_\lambda \left\{ \lambda^{-r-m-1} \bar{W}(x, t, s, \epsilon \partial_x, \lambda) \bar{W}^{-1}(x - me, t', \bar{p}, s', \epsilon \partial_x, \lambda) \right\}. \tag{4.6}
\]

Proof. Let \( m \in \mathbb{Z}, r \in \mathbb{N} \) and \( s_0 = s'_0 \). Put

\[
W(x, t, s, \Lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t, s, \partial_x)\Lambda^i \quad \text{and} \quad W(x, t, s, \Lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t, s, \partial_x)\Lambda^i,
\]

\[
W^{-1}(x, t, s, \Lambda) = \sum_{i \in \mathbb{Z}} \Lambda^i a_i'(x, t, s, \partial_x) \quad \text{and} \quad \bar{W}^{-1}(x, t, \Lambda) = \sum_{j \in \mathbb{Z}} \Lambda^j b_j'(x, t, s, \partial_x),
\]

and compare the coefficients in front of \( \Lambda^{-m} \) in eq. (4.4):

\[
\sum_{i+j=-m-r} a_i(x, t, s, \partial_x)a_j'(x - me, t', \bar{p}, s', \partial_x) = \sum_{i+j=-m+r} b_i(x, t, s, \partial_x)b_j'(x - me, t', \bar{p}, s', \partial_x).
\]

This equality can be written also as eq. (4.6).

To give Hirota quadratic function in terms of tau functions, we need to define and prove the existence of tau function of the EZTH firstly in the next section.

5. Tau-functions of EZTH

Introduce the following sequences:

\[
t - [\lambda] := (t_j - \epsilon(j - 1)!\lambda^j, 0 \leq j \leq \infty). \tag{5.1}
\]

A matrix-valued function \( \tau \in Z_N \) depending only on the dynamical variables \( t \) and \( \epsilon \) is called the Matrix tau-function of the EZTH if it provides symbols related to matrix-valued wave operators as following,
Lemma 2. The following equations hold

\begin{align}
(5.2) & \quad S : = \frac{\tau(s_0 + x - \frac{\xi}{2}, t_j - \frac{(j-1)!}{\lambda^j}, s; \epsilon)}{\tau(s_0 + x - \frac{\xi}{2}, t, s; \epsilon)}, \\
(5.3) & \quad S^{-1} : = \frac{\tau(s_0 + x + \frac{\xi}{2}, t_j + \frac{(j-1)!}{\lambda^j}, s; \epsilon)}{\tau(s_0 + x + \frac{\xi}{2}, t, s; \epsilon)}, \\
(5.4) & \quad \bar{S} : = \frac{\tau(s_0 + x + \frac{\xi}{2}, t_j + \epsilon(j-1)!\lambda^j, s; \epsilon)}{\tau(s_0 + x + \frac{\xi}{2}, t, s; \epsilon)}, \\
(5.5) & \quad \bar{S}^{-1} : = \frac{\tau(s_0 + x - \frac{\xi}{2}, t_j - \epsilon(j-1)!\lambda^j, s; \epsilon)}{\tau(s_0 + x + \frac{\xi}{2}, t, s; \epsilon)}.
\end{align}

Here the division means the multiplication of numerator matrix by the inverse of the denominator matrix. One can get the solution $U, V$ in terms of tau functions as

\begin{align}
(5.6) & \quad U = (\log \tau)_{xx}, \quad V = \log \frac{\tau(x + \epsilon)\tau(x - \epsilon)}{\tau^2(x)}
\end{align}

When $N = 2$

\begin{align}
(5.7) & \quad \begin{bmatrix} u_0 & 0 \\ u_1 & u_0 \end{bmatrix} = \begin{bmatrix} \frac{\log \tau_0}{\tau_0} & 0 \\ (\frac{\tau_0}{\tau_0})_x & (\log \tau_0)_{xx} \end{bmatrix}, \\
(5.8) & \quad e^{v_0} \begin{bmatrix} v_0 & 0 \\ v_1 e^{v_0} & v_0 \end{bmatrix} = \begin{bmatrix} e^{v_0} & 0 \\ v_1 e^{v_0} & e^{v_0} \end{bmatrix}, \\
(5.9) & \quad = \begin{bmatrix} \tau_0(x + \epsilon)\tau_0(x - \epsilon) \\ \tau_0 & \tau_0 \end{bmatrix}^{-1} \begin{bmatrix} \tau_0(x + \epsilon)\tau_0(x - \epsilon) \\ \tau_0 & \tau_0 \end{bmatrix}^{-1}.
\end{align}

This implies

\begin{align}
(5.10) & \quad u_0 = (\log \tau_0)_{xx}, \quad u_1 = \left(\frac{\tau_1}{\tau_0}\right)_x, \\
(5.11) & \quad v_0 = \log \frac{\tau_0(x + \epsilon)\tau_0(x - \epsilon)}{\tau_0}, \quad v_1 = (\Lambda - 1 + \Lambda^{-1})\frac{\tau_1(x)}{\tau_0(x)}.
\end{align}

By Proposition 5 one can prove the following lemma

**Lemma 2.** The following equations hold

\begin{align}
(5.12) & \quad \sum_{k=1}^{N} S(x, t, \lambda_1)_{ik} S^{-1}(x + \epsilon, t + [\lambda_2], \lambda_1)_{kj} = \sum_{k=1}^{N} \bar{S}(x, t, \lambda_2)_{ik} \bar{S}^{-1}(x, t - [\lambda_1^{-1}], \lambda_2)_{kj}, \\
(5.13) & \quad \sum_{k=1}^{N} S(x, t, \lambda_1)_{ik} \bar{S}^{-1}(x, t - [\lambda_2^{-1}], \lambda_1)_{kj} = \sum_{k=1}^{N} S(x, t, \lambda_2)_{ik} \bar{S}^{-1}(x, t - [\lambda_1^{-1}], \lambda_2)_{kj}, \\
& \quad \sum_{k=1}^{N} \bar{S}(x, t, \lambda_1)_{ik} \bar{S}^{-1}(x + \epsilon, t + [\lambda_2], \lambda_1)_{kj} = \sum_{k=1}^{N} \bar{S}(x, t, \lambda_2)_{ik} \bar{S}^{-1}(x + \epsilon, t + [\lambda_1], \lambda_2)_{kj}.
\end{align}

Using Lemma 2 we can prove the following important proposition which gives the existence of matrix-valued tau functions.
Proposition 6. Given a pair of wave operators $S$ and $\bar{S}$ of the METH there exists corresponding matrix-valued invertible tau-functions $\tau \in Z_N$, which is unique up to multiplication by a non-vanishing function independent of $t_j, j \geq 1$.

Proof. Here, we shall note that the $Z_N$-valued tau function $\tau(x, t)$ corresponding to the wave operators $S$ and $\bar{S}$ is in fact $\tau(x - \epsilon/2, t)$. The system is equivalent to:

$$\log S = \left( \exp \left( -\epsilon \sum_{j=0}^{\infty} j! \lambda^{-j+1} \partial_{t_j} \right) - 1 \right) \log \tau,$$

$$\log \bar{S} = \left( \exp \left( \epsilon \partial_x + \epsilon \sum_{j=0}^{\infty} j! \lambda^{j+1} \partial_{t_j} \right) - 1 \right) \log \tau,$$

$$\partial_{s_0} \log \tau(x, t) = \partial_x \log \tau(x, t).$$

Then using Lemma 2 will help us to derive the existence of tau function of this hierarchy. 

After giving tau functions of the EZTH, what is the Hirota bilinear equation in form of tau function becomes a natural question which will be answered in the next section with the help of generalized Vertex operators.

6. Generalized matrix Vertex operators and Hirota quadratic equations

In this section we continue to discuss on the fundamental properties of the tau function of the EZTH, i.e., the Hirota quadratic equations of the EZTH. So we introduce the following vertex operators

$$\Gamma^{\pm a} = \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda^{j+1}}{(j+1)!} + s_j \frac{\lambda^j}{j!} (\log \lambda - c_j) \right) \times \exp \left( \mp \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda]_{\partial} \right),$$

$$\Gamma^{\pm b} = \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda^{-j-1}}{(j+1)!} - s_j \frac{\lambda^{-j}}{j!} (\log \lambda - c_j) \right) \times \exp \left( \mp \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda]_{\partial} \right).$$

where

$$[\lambda]_{\partial} = \epsilon \sum_{j=0}^{\infty} j! \lambda^{j+1} \partial_{t_j}.$$

Because of the logarithm $\log \lambda$, the vertex operators $\Gamma^{\pm a} \otimes \Gamma^{\pm a}$ and $\Gamma^{\pm b} \otimes \Gamma^{\pm b}$ are multi-valued function. There are monodromy factors $M^a$ and $M^b$ respectively as following among different branches around $\lambda = \infty$

\begin{align*}
(6.1) \quad M^a &= \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (s_j \otimes 1 - 1 \otimes s_j) \right\}, \\
(6.2) \quad M^b &= \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j=0}^{\infty} \frac{\lambda^{-j}}{j!} (s_j \otimes 1 - 1 \otimes s_j) \right\}.
\end{align*}
In order to offset the complication, we need to generalize the concept of Vertex operators which leads it to be not scalar-valued any more but take values in a differential operator algebra in $Z_N$. So we introduce the following vertex operators

\begin{align}
\Gamma_\alpha^\delta &= \exp \left( - \sum_{j > 0} \frac{j! \lambda^{j+1}}{\epsilon} (\epsilon \partial_x) s_j \right) \exp(x \partial_{s_0}), \\
\Gamma_\beta^\delta &= \exp \left( - \sum_{j > 0} \frac{j! \lambda^{-(j+1)}}{\epsilon} (\epsilon \partial_x) s_j \right) \exp(x \partial_{s_0}),
\end{align}

Then

\begin{align}
\Gamma_\alpha^\delta \otimes \Gamma_\alpha^\delta &= \exp(x \partial_{s_0}) \exp \left( \sum_{j > 0} \frac{j! \lambda^{j+1}}{\epsilon} (\epsilon \partial_x)(s_j - s'_j) \right) \exp(x \partial_{s'_0}), \\
\Gamma_\beta^\delta \otimes \Gamma_\beta^\delta &= \exp(x \partial_{s_0}) \exp \left( \sum_{j > 0} \frac{j! \lambda^{-(j+1)}}{\epsilon} (\epsilon \partial_x)(s_j - s'_j) \right) \exp(x \partial_{s'_0}).
\end{align}

After computation we get

\begin{align}
\left( \Gamma_\alpha^\delta \otimes \Gamma_\alpha^\delta \right) M^\alpha &= \exp \left\{ \pm \frac{2 \pi i}{\epsilon} \sum_{j > 0} \frac{\lambda^j}{j!} (s_j - s'_j) \right\} \\
\exp \left( \pm \frac{2 \pi i}{\epsilon} \left( (s_0 + x) - (s'_0 + x + \sum_{j > 0} \frac{\lambda^j}{j!} (s_j - s'_j)) \right) \right) \left( \Gamma_\alpha^\delta \otimes \Gamma_\alpha^\delta \right) \\
&= \exp \left( \pm \frac{2 \pi i}{\epsilon} (s_0 - s'_0) \right) \left( \Gamma_\alpha^\delta \otimes \Gamma_\alpha^\delta \right),
\end{align}

\begin{align}
\left( \Gamma_\beta^\delta \otimes \Gamma_\beta^\delta \right) M^b &= \exp \left\{ \pm \frac{2 \pi i}{\epsilon} \sum_{j > 0} \frac{\lambda^{-j}}{j!} (s_j - s'_j) \right\} \\
\exp \left( \pm \frac{2 \pi i}{\epsilon} \left( (s_0 + x) - (s'_0 + x + \sum_{j > 0} \frac{\lambda^{-j}}{j!} (s_j - s'_j)) \right) \right) \left( \Gamma_\beta^\delta \otimes \Gamma_\beta^\delta \right) \\
&= \exp \left( \pm \frac{2 \pi i}{\epsilon} (s_0 - s'_0) \right) \left( \Gamma_\beta^\delta \otimes \Gamma_\beta^\delta \right).
\end{align}

Thus when $s_0 - s'_0 \in \mathbb{Z} \epsilon$, $\left( \Gamma_\alpha^\delta \otimes \Gamma_\alpha^\delta \right) (\Gamma^a \otimes \Gamma^{-a})$ and $\left( \Gamma_\beta^\delta \otimes \Gamma_\beta^\delta \right) (\Gamma^{-b} \otimes \Gamma^b)$ are all single-valued near $\lambda = \infty$.

Now we should note that the above vertex operators take value in $Z_N$-valued differential operator algebra $\mathbb{C}[\partial, x, t, s, \epsilon] := \{ f(x, t, \epsilon) | f(x, t, s, \epsilon) = \sum_{i \geq 0} \sum_{k \geq 0} c_{ik}(x, t, s, \epsilon) \Gamma^k \partial^i \}$. 

12
Theorem 1. The invertible $\mathbb{Z}_N$-valued matrix $\tau(t,s,\epsilon)$ is a tau-function of the EZTH if and only if it satisfies the following Hirota quadratic equation of the EZTH.

\begin{equation}
\text{Res}_\lambda \lambda^{r-1} \left( (\Gamma^a_a \otimes \Gamma^a_a)(\Gamma^a_a \otimes \Gamma^{-a}_a) \right) (\tau \otimes \tau) = \text{Res}_\lambda \lambda^{r-1} \left( (\Gamma^b_b \otimes \Gamma^b_b)(\Gamma^{-b}_b \otimes \Gamma^b_b) \right) (\tau \otimes \tau)
\end{equation}

computed at $s_0 - s'_0 = l\epsilon$ for each $l \in \mathbb{Z}$, $r \in \mathbb{N}$.

Proof. We just need to prove that the HBEs are equivalent to the right side in Proposition 5. By a straightforward computation we can get the following four identities

\begin{align}
\Gamma^a_a \Gamma^a_a \tau &= \tau(s_0 + x - \epsilon/2, t, s)\lambda^{s_0/\epsilon}W(x,t,s,\epsilon \partial_x, \lambda)\lambda^{1_Nx/\epsilon}, \\
\Gamma^a_a \Gamma^{-a}_a \tau &= \lambda^{-(s_0+s)/\epsilon}W^{-1}(x,t,s,\epsilon \partial_x, \lambda)\tau(x + s_0 + \epsilon/2, t, s), \\
\Gamma^b_b \Gamma^{-b}_b \tau &= \tau(x + s_0 - \epsilon/2, t, s)\lambda^{s_0/\epsilon}W(x,t,s,\epsilon \partial_x, \lambda)\lambda^{2Nx/\epsilon}, \\
\Gamma^b_b \Gamma^b_b \tau &= \lambda^{-s_0/\epsilon}W^{-1}(x,t,s,\epsilon \partial_x, \lambda)\tau(x + s_0 + \epsilon/2, t, s).
\end{align}

The proof of four equations eq.(6.10)-eq.(6.13) can be derived by similar method as in \[7,9\]. By substituting four equations eq.(6.10)-eq.(6.13) into the HBEs eq.(6.9), eq.(4.6) is derived. \hfill \Box

Doing a transformation on the eq.(6.9) by $\lambda \to \lambda^{-1}$, then the eq.(6.9) becomes

\begin{equation}
\text{Res}_\lambda \lambda^{r-1} \left( (\Gamma^a_a \otimes \Gamma^a_a)(\Gamma^a_a \otimes \Gamma^{-a}_a - \Gamma^{-a}_a \otimes \Gamma^a_a) \right) (\tau \otimes \tau) = 0
\end{equation}

computed at $s_0 - s'_0 = l\epsilon$ for each $l \in \mathbb{Z}$, $r \in \mathbb{N}$. That means

\begin{equation}
\frac{d\lambda}{\lambda} \left( (\Gamma^a_a \otimes \Gamma^a_a)(\Gamma^a_a \otimes \Gamma^{-a}_a - \Gamma^{-a}_a \otimes \Gamma^a_a) \right) (\tau \otimes \tau)
\end{equation}

is regular in $\lambda$ computed at $s_0 - s'_0 = l\epsilon$ for each $l \in \mathbb{Z}$. The eq.(6.15) in the case when $N = 1$ is exactly the Hirota quadratic equation of the extended Toda hierarchy in \[7\]. As we know, the Vertex operator in fact gives one special Backlund transformation of the EZTH. To give more information on the relations among different solutions of the EZTH, the Darboux transformation of the EZTH will be constructed using kernel determinant technique as \[24,25\] in the next section.

7. Darboux transformation of the EZTH

In this section, we will consider the Darboux transformation of the EZTH on Lax operator

\begin{equation}
\mathcal{L} = \Lambda + U + V\Lambda^{-1},
\end{equation}

i.e.

\begin{equation}
\mathcal{L}^{[1]} = \Lambda + U^{[1]} + V^{[1]}\Lambda^{-1} = W\mathcal{L}W^{-1},
\end{equation}

where $W$ is the Darboux transformation operator. That means after Darboux transformation, the spectral problem about spectral $N \times N$ matrix-valued $\phi$

\begin{equation}
\mathcal{L}\phi = \Lambda\phi + U\phi + V\Lambda^{-1}\phi = \lambda\phi,
\end{equation}

will become

\begin{equation}
\mathcal{L}^{[1]}\phi^{[1]} = \lambda\phi^{[1]}.
\end{equation}
To keep the Lax pair of the EZTH invariant in Proposition 2, i.e.

\[ \partial_t L = [(B_j)_+, L], \partial_s L = [(D_j)_+, L], \]

\[ B_j := B_j(L), D_j := D_j(L), \]

the dressing operator \( W \) should satisfy the following dressing equation

\[ W_{t_j} = -W(B_j)_+ + (WB_jW^{-1})_+ W, \quad j \geq 0. \]

where \( W_{t_j} \) means the derivative of \( W \) by \( t_j \). To give the Darboux transformation, we need the following lemma.

Lemma 3. The operator \( B := \sum_{n=0}^{\infty} b_n \Lambda^n \) is a non-negative difference operator; \( C := \sum_{n=1}^{\infty} c_n \Lambda^{-n} \)

is a negative difference operator and \( f, g \) (short for \( f(x), g(x) \)) are two functions of spatial parameter \( x \), following identities hold

\[ \left( Bf \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g \right)_- = B(f) \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g, \quad \left( f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g B \right)_- = f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} B^*(g), \]

\[ \left( Cf \frac{1}{1-\Lambda} g \right)_+ = C(f) \frac{1}{1-\Lambda} g, \quad \left( f \frac{1}{1-\Lambda} g C \right)_+ = f \frac{1}{1-\Lambda} C^*(g). \]

Proof. Here we only give the proof of the eq.\((7.8)\) by direct calculation

\[
\left( Bf \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g \right)_- = \sum_{m=0}^{\infty} b_m (f(x + me) \Lambda^m \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g)_- \\
= \sum_{m=0}^{\infty} b_m f(x + me) (\frac{\Lambda^{-1}}{1-\Lambda^{-1}})_- g \\
= \sum_{m=0}^{\infty} b_m f(x + me) \Lambda^{-1} (1-\Lambda^{-1}) g \\
= B(f) \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g, \tag{7.10}
\]

\[
\left( f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g B \right)_- = \sum_{m=0}^{\infty} (f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g b_m \Lambda^m)_- \\
= \sum_{m=0}^{\infty} (f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g(x - me) b_m (x - me))_+ \\
= \sum_{m=0}^{\infty} f(\frac{\Lambda^{-1}}{1-\Lambda^{-1}} g(x - me) b_m (x - me) \\
= \sum_{m=0}^{\infty} f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} b_m (x - me) g(x - me) \\
= f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} B^*(g). \tag{7.11}
\]

Similar proof for the eq.\((7.9)\) can be got easily.
Now, we will give the following important theorem which will be used to generate new solutions.

**Theorem 2.** If \( \phi \) is the first wave function of the EZTH, the Darboux transformation operator of the EZTH

\[
W(\lambda) = (1 - \phi(\phi(x - \epsilon))^{-1} \Lambda^{-1}) = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1},
\]

will generate new solutions \( U^{[1]}, V^{[1]} \) from seed solutions \( U, V \)

\[
U^{[1]} = u + (\Lambda - 1)\phi(\phi(x - \epsilon))^{-1},
\]

\[
V^{[1]} = \Lambda^{-1}v \frac{\phi \Lambda^{-2} \phi}{\Lambda^{-1} \phi^2}.
\]

**Proof.** In the following proof, using eq. (7.8) in Lemma 3, a direct computation will lead to the following

\[
W_{t_n} W^{-1} = (\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1})_{t_n} \phi \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1}
\]

\[= (((B_n) + \phi) \circ (1 - \Lambda^{-1}) \circ \phi^{-1}) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1}
\]

\[= (B_n) \circ \phi \circ (1 - \Lambda^{-1}) \circ ((B_n) + \phi) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1}
\]

\[= -(\phi \circ ((1 - \Lambda^{-1}) \cdot \phi^{-1}(x) ((B_n(x) + \phi(x))) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_{-}
\]

\[= -(\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (B_n(x) + \phi(x)) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_{-}
\]

\[= -(\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (B_n(x)) + \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_{-}
\]

\[= -(\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (B_n(x)) + (\phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_{+}
\]

\[= -W(B_n) W^{-1} + (WB_n W^{-1})_{+}.
\]

Therefore

\[
W = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1},
\]

can be as a Darboux transformation of the EZTH. Eqs. (7.13)-(7.14) can be directly got from eq.(7.2). □

Define \( \phi_i = \phi_i^{[0]} := \phi|_{\lambda = \lambda_i} \), then one can choose the specific one-fold Darboux transformation of the EZTH as following

\[
W_1(\lambda_1) = \mathbb{I}_N - \phi_1(\phi_1(x - \epsilon))^{-1}\Lambda^{-1}.
\]

Meanwhile, we can also get Darboux transformation on wave function \( \phi \) as following

\[
\phi^{[1]} = (\mathbb{I}_N - \phi_1(x)(\phi_1(x - \epsilon))^{-1}\Lambda^{-1})\phi.
\]

Then using iteration on Darboux transformation, the \( j \)-th Darboux transformation from the \( (j - 1) \)-th solution is as
After iteration on Darboux transformations, the following theorem about the two-fold Darboux transformation can be derived by direct calculation.

**Theorem 3.** The two-fold Darboux transformation operator $W_2 \in G_c$ of the EZTH is as following

\[(7.21) \quad W_2 = \mathbb{I}_N + t_1^{[2]} \Lambda^{-1} + t_2^{[2]} \Lambda^{-2},\]

where

\[(7.22) \quad t_1^{[2]} = \frac{\phi_1(x)\phi_2(x-2\epsilon) - \phi_2(x)\phi_1(x-2\epsilon)}{\phi_1(x-\epsilon)\phi_2(x-2\epsilon) - \phi_2(x-\epsilon)\phi_1(x-2\epsilon)},\]

\[(7.23) \quad t_2^{[2]} = \frac{\phi_1(x)\phi_2(x-\epsilon) - \phi_2(x)\phi_1(x-\epsilon)}{\phi_2(x-2\epsilon)\phi_1(x-\epsilon) - \phi_1(x-2\epsilon)\phi_2(x-\epsilon)}.\]

The Darboux transformation leads to new solutions from seed solutions

\[(7.24) \quad U^{[2]} = U + (\Lambda - 1)t_1^{[2]},\]

\[(7.25) \quad V^{[2]} = t_2^{[2]}(x)(\Lambda^{-2}V)t_2^{[2]-1}(x-\epsilon).\]

In fact the Darboux operator $W_2$ can be further written as

\[(7.26) \Delta_2 = \frac{1}{\Delta_2} \left| \begin{array}{cccccccccc} \delta_{ij} & 0 & \cdots & 0 & \Lambda^{-1} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \phi_{1,11}(x-\epsilon) & \cdots & \phi_{1,13}(x-\epsilon) & \phi_{1,14}(x-\epsilon) & \phi_{1,15}(x-2\epsilon) & \cdots & \phi_{1,12}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) \\ 0 & \phi_{1,12}(x-\epsilon) & \cdots & \phi_{1,13}(x-\epsilon) & \phi_{1,14}(x-\epsilon) & \phi_{1,15}(x-2\epsilon) & \cdots & \phi_{1,12}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) \\ -\phi_{1,11}(x-\epsilon) & \phi_{1,11}(x-\epsilon) & \cdots & \phi_{1,12}(x-\epsilon) & \phi_{1,12}(x-\epsilon) & \phi_{1,13}(x-\epsilon) & \phi_{1,13}(x-\epsilon) & \phi_{1,12}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,1}(x-2\epsilon) \\ 0 & \phi_{1,1N}(x-\epsilon) & \cdots & \phi_{1,1N}(x-\epsilon) & \phi_{1,1N}(x-\epsilon) & \phi_{1,1N}(x-2\epsilon) & \cdots & \phi_{1,1N}(x-2\epsilon) & \cdots & \phi_{1,1N}(x-2\epsilon) & \cdots & \phi_{1,1N}(x-2\epsilon) \\ 0 & \phi_{2,11}(x-\epsilon) & \cdots & \phi_{2,12}(x-\epsilon) & \phi_{2,12}(x-\epsilon) & \phi_{2,13}(x-\epsilon) & \phi_{2,13}(x-\epsilon) & \phi_{2,12}(x-2\epsilon) & \cdots & \phi_{2,1}(x-2\epsilon) & \cdots & \phi_{2,1}(x-2\epsilon) \\ -\phi_{2,11}(x) & \phi_{2,11}(x) & \cdots & \phi_{2,12}(x) & \phi_{2,12}(x) & \phi_{2,13}(x) & \phi_{2,13}(x) & \phi_{2,12}(x) & \cdots & \phi_{2,1}(x) & \cdots & \phi_{2,1}(x) \\ 0 & \phi_{2,1N}(x-\epsilon) & \cdots & \phi_{2,1N}(x-\epsilon) & \phi_{2,1N}(x-\epsilon) & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) \\ 0 & \phi_{2,1N}(x-\epsilon) & \cdots & \phi_{2,1N}(x-\epsilon) & \phi_{2,1N}(x-\epsilon) & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) & \cdots & \phi_{2,1N}(x-2\epsilon) \\ \end{array} \right|,
Similarly, we can generalize the Darboux transformation to n-fold case which is contained in the following theorem.

**Theorem 4.** The n-fold Darboux transformation of EZTH equation is as follows

\[
W_n = \mathbb{I}_N + t_1^{[n]} \Lambda^{-1} + t_2^{[n]} \Lambda^{-2} + \cdots + t_n^{[n]} \Lambda^{-n}
\]

where

\[
W_n \cdot \phi_i |_{i \leq n} = 0.
\]

The Darboux transformation leads to new solutions form seed solutions

\[
U^{[n]} = U + (\Lambda - 1) t_n^{[n]},
\]

\[
V^{[n]} = t_n^{[n]}(x)(\Lambda^{-n} V) t_n^{[n]-1}(x - \epsilon),
\]

where

\[
(W_n)_{ij} = \frac{1}{\Delta_n} \begin{bmatrix}
\delta_{ij} & 0 & \cdots & \Lambda^{-1} & \cdots & 0 & \cdots & 0 & \cdots & \Lambda^{-n} & \cdots & 0 \\
0 & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
0 & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
0 & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
0 & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
0 & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) & \phi_{n,1}(x-2\epsilon) & \cdots & \phi_{n,N}(x-2\epsilon) & \cdots & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) \\
0 & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) & \phi_{n,1}(x-2\epsilon) & \cdots & \phi_{n,N}(x-2\epsilon) & \cdots & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) \\
\end{bmatrix}
\]

\[
\Delta_n = \begin{bmatrix}
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) & \phi_{1,1}(x-2\epsilon) & \cdots & \phi_{1,N}(x-2\epsilon) & \cdots & \phi_{1,1}(x-\epsilon) & \cdots & \phi_{1,N}(x-\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) & \phi_{n,1}(x-2\epsilon) & \cdots & \phi_{n,N}(x-2\epsilon) & \cdots & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) \\
\phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) & \phi_{n,1}(x-2\epsilon) & \cdots & \phi_{n,N}(x-2\epsilon) & \cdots & \phi_{n,1}(x-\epsilon) & \cdots & \phi_{n,N}(x-\epsilon) \\
\end{bmatrix}
\]

It can be easily checked that \(W_n \phi_i = 0, \ i = 1, 2, \ldots, n\).

Taking seed solution \(U = (0)_{N \times N}, V = \mathbb{I}_N\), then using Theorem 4 one can get the \(n\)-th new solution of the EZTH as

\[
U^{[n]} = (0 - \Lambda^{-1}) \partial_\theta \log \tilde{W}_r(\phi_1, \phi_2, \ldots, \phi_n),
\]

\[
V^{[n]} = e^{(1-\Lambda^{-1})(1-\Lambda^{-1}) \log \tilde{W}_r(\phi_1, \phi_2, \ldots, \phi_n)},
\]
where $W_r(\phi_1, \phi_2, \ldots, \phi_n)$ is the discrete “Wronskian” in terms of $\Gamma$
\begin{equation}
W_r(\phi_1, \phi_2, \ldots, \phi_n) = \det(\Lambda^{-j+1}\phi_{n+1-i})_{1 \leq i, j \leq n}.
\end{equation}

“Wronskian” in terms of $\Gamma$ means in the process of calculation, we treat every element $\phi_{n+1-i}$ not directly in matrix form but as a scalar polynomial of element $\Gamma$. After getting the values of $U^{[n]}, V^{[n]}$ in terms of $\Gamma$, we then rewrite it in matrix form.

7.1. Soliton solutions. After above preparation over the first Darboux transformation, in this section, we will use the first Darboux transformation of the EZTH to generate new solutions from trivial seed solutions. In particular, some matrix-valued soliton solutions will be shown using the first Darboux transformation.

For $N = 2$, one can take seed solution $U = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then the initial wave function $\phi_i$ satisfies
\begin{equation}
\Lambda \phi + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1} \phi = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix} \phi, \quad 1 \leq i \leq n.
\end{equation}

\begin{equation}
\phi = \exp\left( \frac{x}{\epsilon} \log \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix} \right), \quad z_1 \neq 0,
\end{equation}

with $z_1 + z_1^{-1} = \lambda_1, z_2 + z_1^{-1} - \frac{2}{z_1^2} = \lambda_2$.

\begin{equation}
S = E + \omega_1 \Lambda^{-1} + \nu \frac{x}{\epsilon} \Lambda^{-1} - \frac{x}{\omega_1} \epsilon \Lambda^{-2} + \ldots, \quad \omega_1 = \text{cons}.
\end{equation}

Under this initial equation, the operator $A_1$ in above Lemma 1 is in form of
\begin{equation}
A_1 = (\Lambda + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}) \epsilon \partial - (\Lambda - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}).
\end{equation}

\begin{equation}
\frac{\partial \phi}{\partial s_1} = \left( [(\Lambda + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1}) \epsilon \partial - (\Lambda - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda^{-1})] \phi, \right.
\end{equation}

Then solution $\phi$ in terms of $x, s_1$ can be chosen in the form
\begin{equation}
\phi = \exp\left( \frac{x + \lambda s_1}{\epsilon} \log Z + \frac{s_1}{\epsilon} (-Z + Z^{-1}) \right), \quad Z = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix},
\end{equation}

where $Z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z^{-1} = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1^{-1} & 0 \\ -z_2 z_1^{-1} & z_1^{-1} \end{bmatrix} = \lambda + \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix}.
\end{equation}

\begin{equation}
U^{[1]} = (\Lambda - 1) \frac{\cosh\left( \frac{x + \lambda s_1}{\epsilon} \log Z + \frac{s_1}{\epsilon} (-Z + Z^{-1}) \right)}{\cosh\left( \frac{x + \lambda s_1}{\epsilon} \log Z + \frac{s_1}{\epsilon} (-Z + Z^{-1}) \right) - \log Z},
\end{equation}

\begin{equation}
V^{[1]} = (1 - \Lambda^{-1}) \log[2 \cosh\left( \frac{x + \lambda s_1}{\epsilon} \log Z + \frac{s_1}{\epsilon} (-Z + Z^{-1}) \right)].
\end{equation}

Using $Z = \begin{bmatrix} \log z_1 & 0 \\ \frac{z_2}{z_1} & \log z_1 \end{bmatrix}, \cosh\left( \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \right) = \begin{bmatrix} \cosh a & 0 \\ b \cosh a & \cosh a \end{bmatrix}$, one can derive the specific elements in new solutions $U^{[1]}, V^{[1]}$ as
(7.40) \[ u_0^{[1]} = (\Lambda - 1) \frac{\cosh(x) \log z_1 + \frac{a}{e} \left( \lambda_1 - z_1 + z_1^{-1} \right)}{\cosh(\frac{z}{e} \log z_1 + \frac{a}{e} (\lambda_1 - z_1 + z_1^{-1}))}. \]

(7.41) \[ u_1^{[1]} = \left( \frac{x}{\epsilon} z_1 + \frac{s_1}{\epsilon} (\lambda_2 - z_2 - z_1^2) \right) \frac{\cosh(x) \log z_1 + \frac{a}{e} (\lambda_1 - z_1 + z_1^{-1})}{\cosh(\frac{z}{e} \log z_1 + \frac{a}{e} (\lambda_1 - z_1 + z_1^{-1}))}. \]

(7.42) \[ v_0^{[1]} = (1 - \Lambda^{-1})^2 \log[2 \cosh(x) \log z_1 + \frac{s_1}{\epsilon} (\lambda_1 - z_1 + z_1^{-1})], \]

(7.43) \[ v_1^{[1]} = (1 - \Lambda^{-1})^2 \log(\frac{x}{\epsilon} z_1 + \frac{s_1}{\epsilon} (\lambda_2 - z_2 - z_2^2)). \]

8. Bi-Hamiltonian structure and tau symmetry

To describe the integrability of the EZTH, we will construct the Bi-Hamiltonian structure and tau symmetry of the EZTH in this section as [23]. For a matrix \( A = (a_{ij}) = \sum_{i=0}^{N-1} a_i \Gamma^i \), the vector field \( \partial_A \) over EZTH is defined by

\[ \partial_A = \sum_{i=0}^{N-1} \sum_{k \geq 0} a_i^{(k)} \left( \frac{\partial}{\partial u_i^{(k)}} + \frac{\partial}{\partial v_i^{(k)}} \right). \]

For a function \( f = \int f dx \), we have

\[ \partial_A f = \int \sum_{i=0}^{N-1} \sum_{k \geq 0} a_i^{(k)} \left( \frac{\partial f}{\partial u_i^{(k)}} + \frac{\partial f}{\partial v_i^{(k)}} \right) dx = \int Tr_N \sum_{k \geq 0} A \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \right) dx, \]

where

\[ \left( \frac{\partial}{\partial u} \right)_{ij} = \frac{\delta}{\delta u_{ij}}, \quad \left( \frac{\partial}{\partial v} \right)_{ij} = \frac{\delta}{\delta v_{ij}}, \]

and

\[ Tr_N A = \text{the trace of} \begin{bmatrix} \frac{1}{N} & \frac{1}{N-1} & \cdots & \frac{1}{2} \\ 0 & \frac{1}{N-1} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{1} \end{bmatrix} A. \]

In this section, we will consider the EZTH on Lax operator

\[ \mathcal{L} = \Lambda + u + e^v \Lambda^{-1}, \]

Then we can define the hamiltonian bracket as

\[ \{ \tilde{f}, \tilde{g} \} = \int Tr_N \sum_{w, w'} \frac{\delta f}{\delta w} \left( w, w' \right) \frac{\delta g}{\delta w'} dx, \quad w, w' = u_i \text{ or } v_j, \quad 0 \leq i, j \leq N - 1. \]

For \( u(x) = \sum_{i=0}^{N-1} u_i(x) \Gamma^i \), \( v(x) = \sum_{i=0}^{N-1} v_i(x) \Gamma^i \), the bi-Hamiltonian structure for the EMTH can be given by the following two compatible Poisson brackets which is a generalization in matrix form of the extended Toda hierarchy in [6].
\{v_i(x), v_j(y)\}_1 = \{u_i(x), u_j(y)\}_1 = 0,
(8.7)
\{u_i(x), v_j(y)\}_1 = \frac{1}{\epsilon} \delta_{ij} \left[ e^{\epsilon \partial_x} - 1 \right] \delta(x - y),
\{u_i(x), u_j(y)\}_2 = \frac{1}{\epsilon} \delta_{ij} e^{\epsilon \partial_x} \left[ e^{v(x)} - e^{v(x) e^{-\epsilon \partial_x}} \right] \delta(x - y),
(8.8)
\{v_i(x), v_j(y)\}_2 = \frac{1}{\epsilon} \delta_{ij} e^{\epsilon \partial_x} - e^{-\epsilon \partial_x} \delta(x - y).
\]

For any difference operator \(A = \sum_k A_k \Lambda^k\), define residue \(\text{Res}A = A_0\). In the following theorem, we will prove the above Poisson structure can be as the the Hamiltonian structure of the METH.

**Theorem 5.** The flows of the EZTH are Hamiltonian systems of the form

\[
\frac{\partial u_i}{\partial t_{k,j}} = \{u_i, H_{k,j}\}_1, \quad \frac{\partial v_i}{\partial t_{k,j}} = \{v_i, H_{k,j}\}_1, \quad k = 0, 1; \quad j \geq 0,
\]

with \(t_{0,j} = t_j, t_{1,j} = s_j\). They satisfy the following bi-Hamiltonian recursion relation

\[
\{\cdot, H_{1,n-1}\}_2 = n\{\cdot, H_{1,n}\}_1 + 2\{\cdot, H_{0,n-1}\}_1, \quad \{\cdot, H_{0,n-1}\}_2 = (n + 1)\{\cdot, H_{0,n}\}_1.
\]

Here the Hamiltonians have the form

\[
H_{k,j} = \int h_{k,j}(u, v; u_x, v_x; \ldots; \epsilon) dx, \quad k = 0, 1; \quad j \geq 0,
\]

with

\[
h_{0,j} = \frac{1}{(j + 1)!} \text{Tr}_N \text{Res} \mathcal{L}^{j+1}, \quad \tilde{H}_{1,j} = \frac{2}{j!} \text{Tr}_N \text{Res} \left[ \mathcal{L}^j (\log \mathcal{L} - c_j) \right].
\]

**Proof.** For \(\beta = 0\), i.e. the original Toda hierarchy, the proof is same as the proof in [6].

Here we will prove that the flows \(\frac{\partial}{\partial t_{1,n}}\) are also Hamiltonian systems with respect to the first Poisson bracket. In [3], the following identity has been proved

\[
\text{Tr}_N \text{Res} \left[ \mathcal{L}^n d(S \epsilon \partial_x S^{-1}) \right] \sim \text{Tr}_N \text{Res} \mathcal{L}^{n-1} d\mathcal{L},
\]

which show the validity of the following equivalence relation:

\[
\text{Tr}_N \text{Res} \left( \mathcal{L}^n d \log_+ \mathcal{L} \right) \sim \text{Tr}_N \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

Here the equivalent relation \(\sim\) is up to a \(x\)-derivative of another 1-form.

In a similar way as eq. (8.12), we obtain the following equivalence relation

\[
\text{Tr}_N \text{Res} \left[ \mathcal{L}^n d(S \epsilon \partial_x S^{-1}) \right] \sim -\text{Tr}_N \text{Res} \mathcal{L}^{n-1} d\mathcal{L},
\]

i.e.

\[
\text{Tr}_N \text{Res} \left( \mathcal{L}^n d \log_- \mathcal{L} \right) \sim \text{Tr}_N \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

Combining (8.13) with (8.15) together can lead to

\[
\text{Tr}_N \text{Res} \left( \mathcal{L}^n d \log \mathcal{L} \right) \sim \text{Tr}_N \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

Suppose

\[
A_{\alpha,n} = \sum_k a_{\alpha,n+1,k} \Lambda^k,
\]

\[
A_{\alpha,n} = \sum_k a_{\alpha,n+1,k} \Lambda^k,
\]
Then from
\( (8.18) \frac{\partial L}{\partial t_{k,n}} = [(B_{k,n})_+, L] = [-(B_{k,n})_-, L], \) \( B_{0,n} = B_n, B_{1,n} = D_n, \)
we can derive the equation
\( \epsilon \frac{\partial u}{\partial t_{\beta,n}} = a_{\beta,n;1}(x + \epsilon) - a_{\beta,n;1}(x) \in Z_N, \) \( \beta = 0, 1, \)
\( \epsilon \frac{\partial v}{\partial t_{\beta,n}} = a_{\beta,n;0}(x - \epsilon)e^{v(x)} - a_{\beta,n;0}(x)e^{v(x+\epsilon)} \in Z_N. \)

The equivalence relation \( (8.13) \) now readily follows from the above two equations. By using
\( (8.13) \) we obtain
\[ d\tilde{h}_n = \frac{2}{n!} dTr_N \text{Res} \left[ L^n (\log L - c_n) \right] \]
\( \sim \frac{2}{(n - 1)!} Tr_N \text{Res} \left[ L^{n-1} (\log L - c_n) dL \right] + \frac{2}{n!} Tr_N \text{Res} \left[ L^{n-1} dL \right] \)
\( = \frac{2}{(n - 1)!} Tr_N \text{Res} \left[ L^{n-1} (\log L - c_{n-1}) dL \right] \)
\( = Tr_N \text{Res} \left[ a_{1,n;0}(x) du + a_{1,n;1}(x - \epsilon)e^{v(x)} dv \right]. \)

It yields the following identities
\( \frac{\partial H_{1,n}}{\partial u} = a_{1,n;0}(x), \) \( \frac{\partial H_{1,n}}{\partial v} = a_{1,n;1}(x - \epsilon)e^{v(x)}. \)

This agrees with the Lax equation
\( \frac{\partial u_i}{\partial t_{1,n}} = \{u_i, H_{1,n}\} = \frac{1}{\epsilon} \left[ e^{\epsilon \partial_x} - 1 \right] \frac{\partial H_{1,n}}{\partial v_i} = \frac{1}{\epsilon} \left[ a_{1,n;1}(x + \epsilon) - a_{1,n;1}(x) \right]_i, \)
\( \frac{\partial v_i}{\partial t_{1,n}} = \{v_i, H_{1,n}\} = \frac{1}{\epsilon} \left[ 1 - e^{\epsilon \partial_x} \right] \frac{\partial H_{1,n}}{\partial u_i} = \frac{1}{\epsilon} \left[ a_{1,n;0}(x - \epsilon)e^{v(x)} - a_{1,n;0}(x)e^{v(x+\epsilon)} \right]_i. \)

From the above identities we see that the flows \( \frac{\partial}{\partial t_{1,n}} \) are Hamiltonian systems of the form
\( (8.9). \) For the case of \( \beta = 1 \) the recursion relation follows from the following trivial identities
\[ n \frac{2}{n!} L^n (\log L - c_n) = L \frac{2}{(n - 1)!} L^{n-1} (\log L - c_{n-1}) - 2 \frac{1}{n!} L^n \]
\[ = \frac{2}{(n - 1)!} L^{n-1} (\log L - c_{n-1}) L - 2 \frac{1}{n!} L^n. \]

Then we get, for \( \beta = 1, \)
\[ na_{1,n+1;1}(x) = a_{1,n;0}(x + \epsilon) + ua_{1,n;1}(x) + e^{v}a_{1,n;2}(x - \epsilon) - 2a_{0,n+1;1}(x) \]
\[ = a_{1,n;0}(x) + u(x + \epsilon)a_{1,n;1}(x) + e^{v(x+2\epsilon)}a_{1,n;2}(x) - 2a_{0,n+1;1}(x). \]

This further leads to
\[ \{u_i, H_{1,n-1}\} = \left\{ \left[ L e^{v(x)} - e^{v(x)} \Lambda^{-1} \right] a_{1,n;0}(x) + u(x) \left[ \Lambda - 1 \right] a_{1,n;1}(x - \epsilon)e^{v(x)} \right\}_i \]
\[ = n \left[ a_{1,n+1;1}(x)e^{v(x+\epsilon)} - a_{1,n+1;1}(x - \epsilon)e^{v(x)} \right] + 2 \left[ a_{0,n+1;0}(x)e^{v(x+\epsilon)} - a_{0,n+1;0}(x - \epsilon)e^{v(x)} \right]_i. \]
This is exactly the recursion relation on flows for $u$. The similar recursion flow on $v$ can be similarly derived. Theorem is proved till now.

Similarly as [6], the tau symmetry of the MZTH can be proved in the following theorem.

**Theorem 6.** The MZTH has the following tau-symmetry property:

$$\frac{\partial h_{\alpha,m}}{\partial t_{\beta,n}} = \frac{\partial h_{\beta,n}}{\partial t_{\alpha,m}}, \quad \alpha, \beta = 0, 1, \ m, n \geq 0.$$ (8.26)

**Proof.** Let us prove the theorem for the case when $\alpha = 1, \beta = 0$, other cases are proved in a similar way

$$\frac{\partial h_{1,m}}{\partial t_{0,n}} = \frac{2}{m!(n+1)!} Tr_N Res[-(L^{n+1})_-, L^m (\log_+ L - c_m)]$$

$$= \frac{2}{m!(n+1)!} Tr_N Res[(L^m (\log_+ L - c_m))_+, (L^{n+1})_-]$$

$$= \frac{2}{m!(n+1)!} Tr_N Res[(L^m (\log_+ L - c_m))_+, L^{n+1}] = \frac{\partial h_{0,n}}{\partial t_{1,m}}.$$ (8.27)

Theorem is proved.

This property justifies the following alternative definition of another kind of tau function for the EZTH.

**Definition 3.** The the tau function in $Z_N$ of the EZTH can be defined by the following expressions in terms of the densities of the Hamiltonians:

$$h_{\beta,n} = \epsilon (\Lambda - 1) \frac{\partial \log \tau}{\partial t_{\beta,n}}, \quad \beta = 0, 1; \ n \geq 0,$$ (8.28)

with $t_{0,j} = t_j, t_{1,j} = s_j$.

With above two different definitions tau functions of this hierarchy, some mysterious connections between these two kinds of tau functions become an open question. One is from Sato theory and another is from hamiltonian tau symmetry.

**9. Conclusions and Discussions**

In this paper, we constructed a new hierarchy called EZTH and extended Sato theory to this hierarchy including Sato equations, matrix wave operators, Hirota quadratic equations, the existence of the tau function. Similarly as extended Toda hierarchy and extended bigraded Toda hierarchy in Gromov-Witten theory of $\mathbb{C}P^1$ and orbifold respectively, this hierarchy deserves further studying and exploring because of its potential applications in topological quantum fields and Gromov-Witten theory. Basing on above two different definitions tau functions of this hierarchy, to derive the mysterious deep connections between these two kinds of tau functions with one defined from Sato theory and another from hamiltonian tau symmetry become an interesting question. This is not easy and will be included in our future work.

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