Selection principles in function spaces with the compact-open topology

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Abstract

For a Tychonoff space $X$, we denote by $C_k(X)$ the space of all real-valued continuous functions on $X$ with the compact-open topology. A subset $A \subset X$ is said to be sequentially dense in $X$ if every point of $X$ is the limit of a convergent sequence in $A$. In this paper, the following properties for $C_k(X)$ are considered.

$$S_1(S, S) \Rightarrow S_{fin}(S, S) \Rightarrow S_1(S, D) \Rightarrow S_{fin}(S, D)$$

$$\uparrow \uparrow \uparrow \uparrow$$

$$S_1(D, S) \Rightarrow S_{fin}(D, S) \Rightarrow S_1(D, D) \Rightarrow S_{fin}(D, D)$$

For example, a space $C_k(X)$ satisfies $S_1(S, D)$ (resp., $S_{fin}(S, D)$) if whenever $(S_n : n \in \mathbb{N})$ is a sequence of sequentially dense subsets of $C_k(X)$, one can take points $f_n \in S_n$ (resp., finite $F_n \subset S_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup \{F_n : n \in \mathbb{N}\}$) is dense in $C_k(X)$. Other properties are defined similarly.

In [22], we obtained characterizations these selection properties for $C_p(X)$. In this paper, we have gave characterizations for $C_k(X)$.

Keywords: compact-open topology, function space, $R$-separable, $M$-separable, $\gamma_k$-set, sequentially separable, strongly sequentially separable, selectively sequentially separable, selection principles

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1. Introduction

For a Tychonoff space $X$, we denote by $C_k(X)$ the space of all real-valued continuous functions on $X$ with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A, U] = \{ f \in C(X) : f(A) \subset U \}$, where $A$ is a compact subset of $X$ and $U$ is a non-empty open subset of $\mathbb{R}$. Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of $X$, we can represent a basic neighborhood of the point $f \in C_k(X)$ as $< f, A, \epsilon >$ where $< f, A, \epsilon > := \{ g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall \ x \in A \}$, $A$ is a compact subset of $X$ and $\epsilon > 0$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $A$ and $B$ be sets consisting of families of subsets of an infinite set $X$. Then:

$S_1(A, B)$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $A$ there is a sequence $\{b_n\}_{n \in \mathbb{N}}$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $B$.

$S_{fin}(A, B)$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $A$ there is a sequence $\{B_n\}_{n \in \mathbb{N}}$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in B$.

$U_{fin}(A, B)$ is the selection hypothesis: whenever $U_1, U_2, ... \in A$ and none contains a finite subcover, there are finite sets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\{\bigcup F_n : n \in \mathbb{N}\} \in B$.

The following prototype of many classical properties is called ”$A$ choose $B$” in [28].

$(A \choose B)$ : For each $U \in A$ there exists $V \subseteq U$ such that $V \in B$.

Then $S_{fin}(A, B)$ implies $(A \choose B)$.

In this paper, by a cover we mean a nontrivial one, that is, $U$ is a cover of $X$ if $X = \bigcup U$ and $X \notin U$.

An open cover $U$ of a space $X$ is called:

• an $\omega$-cover (a $k$-cover) if each finite (compact) subset $C$ of $X$ is contained in an element of $U$ and $X \notin U$ (i.e. $U$ is a non-trivial cover);

• a $\gamma$-cover (a $\gamma_k$-cover) if $U$ is infinite, $X \notin U$, and for each finite (compact) subset $C$ of $X$ the set $\{U \in U : C \notin U\}$ is finite.

A space $X$ is said to be a $\gamma_k$-set if each $k$-cover $U$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$.

In a series of papers it was demonstrated that $\gamma$-covers and $k$-covers play a key role in function spaces [8, 9, 10, 13, 16, 22, 23, 24, 25, 27] and many
others. We continue to investigate applications of $k$-covers in function spaces with the compact-open topology.

2. Main definitions and notation

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space $X$ is called sequentially separable if it has a countable sequentially dense set. Call $X$ strongly sequentially separable, if $X$ is separable and every countable dense subset of $X$ is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

For a topological space $X$ we denote:

- $\mathcal{O}$ — the family of open covers of $X$;
- $\Gamma$ — the family of open $\gamma$-covers of $X$;
- $\Gamma_k$ — the family of open $\gamma_k$-covers of $X$;
- $\Omega$ — the family of open $\omega$-covers of $X$;
- $\mathcal{K}$ — the family of open $k$-covers of $X$;
- $\mathcal{K}_{cz}$ — the family of countable co-zero $k$-covers of $X$;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{D}^\omega$ — the family of countable dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathbb{K}(X)$ — the family of all non-empty compact subsets of $X$.

- A space $X$ is $R$-separable, if $X$ satisfies $S_1(\mathcal{D}, \mathcal{D})$ (Def. 47, [2]).
- A space $X$ is $M$-separable (selective separability), if $X$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
- A space $X$ is selectively sequentially separable, if $X$ satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ (Def. 1.2, [3]).

For a topological space $X$ we have the next relations of selectors for sequences of dense sets of $X$.

\[
\begin{align*}
S_1(\mathcal{S}, \mathcal{S}) & \Rightarrow S_{fin}(\mathcal{S}, \mathcal{S}) \Rightarrow S_1(\mathcal{S}, \mathcal{D}) \Rightarrow S_{fin}(\mathcal{S}, \mathcal{D}) \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
S_1(\mathcal{D}, \mathcal{S}) & \Rightarrow S_{fin}(\mathcal{D}, \mathcal{S}) \Rightarrow S_1(\mathcal{D}, \mathcal{D}) \Rightarrow S_{fin}(\mathcal{D}, \mathcal{D})
\end{align*}
\]
Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ converges to $x$, $x = \lim A$, if $A$ is infinite, $x \notin A$, and for each neighborhood $U$ of $x$, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{ A \subseteq X : x \in \overline{A \setminus A} \}$;
- $\Gamma_x = \{ A \subseteq X : x = \lim A \}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to $x$. So, simply $\Gamma_x$ may be the set of non-trivial convergent sequences to $x$.

We write $\Pi(A_x, B_x)$ without specifying $x$, we mean $(\forall x)\Pi(A_x, B_x)$.

So we have three types of topological properties described through the selection principles:

- local properties of the form $S_*(\Phi_x, \Psi_x)$;
- global properties of the form $S_*(\Phi, \Psi)$;
- semi-local of the form $S_*(\Phi, \Psi_x)$.

Our main goal is to describe the topological properties for sequences of dense sets of $C_k(X)$ in terms of selection principles of $X$.

3. $S_1(\mathcal{D}, \mathcal{S})$

Recall that $X$ a $\gamma'_k$-set if it satisfies the selection hypothesis $S_1(\mathcal{K}, \Gamma_k)$ [9].

**Theorem 3.1.** ([11]) For a Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(\Omega_0, \Gamma_0)$ (i.e., $C_k(X)$ is strongly Fréchet-Urysohn);
2. $X$ is a $\gamma'_k$-set.

Recall that the $i$-weight $iw(X)$ of a space $X$ is the smallest infinite cardinal number $\tau$ such that $X$ can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than $\tau$.

**Theorem 3.2.** (Noble [13]) A space $C_k(X)$ is separable iff $iw(X) = \aleph_0$.

**Theorem 3.3.** For a Tychonoff space $X$ with $iw(X) = \aleph_0$ the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(\mathcal{D}, \mathcal{S})$;
2. Every dense subset of \( C_k(X) \) is sequentially dense;
3. \( X \) satisfies \( S_1(K, \Gamma_k) \) (\( X \) is a \( \gamma'_k \)-set);
4. \( X \) is a \( \gamma_k \)-set;
5. \( C_k(X) \) is Fréchet-Urysohn;
6. \( C_k(X) \) satisfies \( S_{fin}(\mathcal{D}, \mathcal{S}) \);
7. \( X \) satisfies \( S_{fin}(\mathcal{K}, \Gamma_k) \);
8. Each finite power of \( X \) satisfies \( S_{fin}(\mathcal{K}, \Gamma_k) \);
9. \( C_k(X) \) satisfies \( S_1(\Omega_0, \Gamma_0) \);
10. \( C_k(X) \) satisfies \( S_1(\mathcal{D}, \Gamma_0) \).

**Proof.** (1) \( \Rightarrow \) (6) is immediate.

(4) \( \Leftrightarrow \) (5). By Theorem 4.7.4 in [17].

(3) \( \Leftrightarrow \) (4). By Theorem 18 in [4].

(3) \( \Leftrightarrow \) (7). By Theorem 5 in [9].

(3) \( \Leftrightarrow \) (8). By Theorem 7 in [9].

(3) \( \Leftrightarrow \) (9). By Theorem 3.1.

(9) \( \Rightarrow \) (10) is immediate.

(6) \( \Rightarrow \) (2). Let \( D \) be a dense subset of \( C_k(X) \). By \( S_{fin}(\mathcal{D}, \mathcal{S}) \), for sequence \( (D_i : D_i = D \) and \( i \in \mathbb{N} \) \) there is a sequence \( (K_i : i \in \mathbb{N}) \) such that for each \( i, K_i \) is finite, \( K_i \subset D_i \), and \( \bigcup_{i \in \mathbb{N}} K_i \) is a countable sequentially dense subset of \( C_k(X) \). It follows that \( D \) is a sequentially dense subset of \( C_k(X) \).

(2) \( \Rightarrow \) (4). Let \( \mathcal{U} \) be an open \( k \)-cover of \( X \). Note that the set \( \mathcal{D} := \{ f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1 \text{ for some } U \in \mathcal{U} \} \) is dense in \( C_k(X) \), hence, it is sequentially dense. Take \( f_n \in D \) such that \( f_n \mapsto 0 \). Let \( f_n \upharpoonright (X \setminus U_n) \equiv 1 \) for some \( U_n \in \mathcal{U} \). Then \( \{ U_n : n \in \mathbb{N} \} \) is a \( \gamma_k \)-subcover of \( \mathcal{U} \), because of \( f_n \mapsto 0 \).

Hence, \( X \) is a \( \gamma_k \)-set.

(3) \( \Rightarrow \) (1). Let \( (D_{i,j} : i, j \in \mathbb{N}) \) be a sequence of dense subsets of \( C_k(X) \) and let \( D = \{ f_i : i \in \mathbb{N} \} \) be a countable dense subset of \( C_k(X) \).

For every \( i, j \in \mathbb{N} \) consider \( \mathcal{U}_{i,j} = \{ U_{h,i,j} : U_{h,i,j} = (f_i - h)^{-1}(-\frac{1}{j}, \frac{1}{j}) \text{ for } h \in D_{i,j} \} \). Note that \( \mathcal{U}_{i,j} \) is an \( k \)-cover of \( X \) for every \( i, j \in \mathbb{N} \). Since \( X \) a \( \gamma_k \)-set, there is a sequence \( (U_{h(i,j),i,j} : i, j \in \mathbb{N}) \) such that \( U_{h(i,j),i,j} \subset \mathcal{U}_{i,j} \) and \( \{ U_{h(i,j),i,j} : i, j \in \mathbb{N} \} \) is an element of \( \Gamma_k \). Claim that \( \{ h(i,j) : i, j \in \mathbb{N} \} \) is a dense subset of \( C_k(X) \). Fix \( g \in C(X) \) and a base neighborhood \( W = g, A, \epsilon > 0 \) of \( g \), where \( A \) is a compact subset of \( X \) and \( \epsilon > 0 \). There are \( f_i \in D \) and \( j \in \mathbb{N} \) such that \( f_i, A, \frac{1}{j} > \subseteq W \). Since \( \{ U_{h(i,j),i,j} : i, j \in \mathbb{N} \} \) is an element of \( \Gamma_k \), there is \( j' > j \) such that \( A \subset U_{h(i,j'),i,j'} \), hence, \( h(i,j') \in f_i, A, \frac{1}{j'} > \subseteq < f_i, A, \frac{1}{j} > \subseteq W \).
Since $C_k(X)$ is Fréchet-Urysohn, every dense subset of $C_k(X)$ is sequentially dense. It follows that $\{h(i,j) : i,j \in \mathbb{N}\}$ is sequentially dense.

(10) $\Rightarrow$ (3). Let $\{U_i : i \in \mathbb{N}\} \subset \mathcal{K}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K,U,i,j} \in C(X) : \text{such that } f_{K,U,i,j} \upharpoonright K \equiv d_j, f_{K,U,i,j} \upharpoonright (X \setminus U) \equiv 1\}$ where $K \in \mathbb{K}(X), K \subset U \in U_i$ for every $i \in \mathbb{N}$. Since $D$ is a dense subset of $C_k(X)$, then $D_i$ is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (10), there is a set $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\}$ such that $f_{K(i),U(i),i,j(i)} \in D_i$ and $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_0$. Claim that a set $\{U(i) : i \in \mathbb{N}\} \in \Gamma_k$. Let $K \in \mathbb{K}(X)$ and let $W = [K, (-\frac{1}{2}, \frac{1}{2})]$ be a base neighborhood of 0. Since $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_0$, there is $i' \in \mathbb{N}$ such that $f_{K(i),U(i),i,j(i)} \in W$ for every $i > i'$. It follows that $K \subset U(i)$ for every $i > i'$ and, hence, $\{U(i) : i \in \mathbb{N}\} \in \Gamma_k$. 

\[\square\]

Let $S \subset \mathbb{K}(X)$. An open cover $U$ of a space $X$ will be call:

- a $s$-cover if each $C \in S$ is contained in an element of $U$ and $X \notin U$;
- a $\gamma_s$-cover if $U$ is infinite, $X \notin U$, and for each $C \in S$ the set $\{U \in U : C \notin U\}$ is finite.

**Definition 3.4.** Let $S \subset \mathbb{K}(X)$. A space $X$ is called a $\gamma_s$-set if each $s$-cover of $X$ contains a sequence which is a $\gamma_s$-cover of $X$.

**Definition 3.5.** A space $X$ will be call a $\gamma_k^{\infty}$-set if each countable cozero $k$-cover $U$ of $X$ contains a set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$.

For a mapping $f : X \mapsto Y$ we will denote by $f(k) = \{f(K) : K \in \mathbb{K}(X)\}$.

**Theorem 3.6.** For a space $X$ with $iw(X) = \aleph_0$, the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(\mathcal{D}^\omega, S)$;
2. $C_k(X)$ is strongly sequentially separable;
3. $X$ is a $\gamma_k^{\infty}$-set;
4. $X$ satisfies $S_1(\mathcal{K}^{\omega}_{\omega}, \Gamma_k)$;
5. for every a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space $X$ on a separable metric space $Y$, the space $Y$ is $\gamma_f(k)$-set.
Proof. (3) $\Rightarrow$ (5). Let $f$ be a condensation $f : X \hookrightarrow Y$ from the space $X$ on a separable metric space $Y$. If $\mu$ is a $f(k)$-cover of $Y$, then there is $\mu' \subseteq \mu$ such that $\mu'$ is a $f(k)$-cover of $Y$ and $|\mu'| = \aleph_0$. The family $f^{-1}(\mu') = \{f^{-1}(V) : V \in \mu'\}$ is a countable co-zero $k$-cover of $X$. By the argument that $X$ is a $\gamma_{f(k)}$-set, we have that $Y$ is $\gamma_{f(k)}$-set.

The remaining implications follow from the proofs of Theorem 3.3 and Theorem 18 in [4].

Corollary 3.7. For a separable metrizable space $X$, the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(D, S)$;
2. Every dense subset of $C_k(X)$ is sequentially dense;
3. $C_k(X)$ is strongly sequentially separable;
4. $C_k(X)$ is a Fréchet-Urysohn;
5. $C_k(X)$ is metrizable and separable;
6. $X$ satisfies $S_1(K, \Gamma_k)$;
7. $X$ satisfies $S_1(K, \mathcal{K})$;
8. $X$ satisfies $S_{fin}(K, \mathcal{K})$;
9. $X$ is a hemicompact.

Proof. By Theorem 3.3 and Theorem 6 in [4].

A space $X$ is called a $k$-Lindelöf space if for each open $k$-cover $U$ of $X$ there is a $V \subseteq U$ such that $V$ is countable and $V \in \mathcal{K}$. Each $k$-Lindelöf space is Lindelöf, so normal, too.

Lemma 3.8. ([17]) $C_k(X)$ has countable tightness iff $X$ is $k$-Lindelöf.

By Theorem 3.3, Theorem 3.6 and Lemma 3.8 we have

Theorem 3.9. For a Tychonoff space $X$ with $iw(X) = \aleph_0$ the following statements are equivalent:

1. $C_k(X)$ is Fréchet-Urysohn;
2. $C_k(X)$ is strongly sequentially separable and has countable tightness;
3. $X$ satisfies $S_1(K^\omega, \Gamma_k)$ and is $k$-Lindelöf;
4. Every dense subset of $C_k(X)$ contains a countable sequentially dense subset of $C_k(X)$.

In Doctoral Dissertation, A.J. March considered the following problem (Problem 117 in [15]): Is it possible to find a space $X$ such that $C_k(X)$ is strongly sequentially separable but $C_k(X)^2$ is not strongly sequentially separable?

We get a negative answer to this question.

**Proposition 3.10.** Suppose $X$ has property $S_1(K_{cz}, \Gamma_k)$. Then $X \sqcup X$ has property $S_1(K_{cz}, \Gamma_k)$.

**Proof.** Let $U = \{U_i : i \in \mathbb{N}\}$ be a countable $k$-cover of $X \sqcup X$ by cozero sets. Let $X \sqcup X = X_1 \sqcup X_2$ where $X_i = X$ for $i = 1, 2$. Consider $\mathcal{V}_1 = \{U_1^i = U_i \cap X_1 : X_1 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ and $\mathcal{V}_2 = \{U_2^i = U_i \cap X_2 : X_2 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ as families of subsets of the space $X$. Define $\mathcal{V} := \{U_1^i \cap U_2^i : U_1^i \in \mathcal{V}_1$ and $U_2^i \in \mathcal{V}_2\}$. Note that $\mathcal{V}$ is a countable $k$-cover of $X$ by cozero sets. By Theorem 18 in [4], there is $\{U_{i_n}^1 \cap U_{i_n}^2 : n \in \mathbb{N}\} \subset \mathcal{V}$ such that $\{U_{i_n}^1 \cap U_{i_n}^2 : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X$. It follows that $\{U_{i_n} : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X \sqcup X$.

**Theorem 3.11.** For Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ is strongly sequentially separable;
2. $(C_k(X))^n$ is strongly sequentially separable for each $n \in \mathbb{N}$.

**Proof.** By Theorem 3.6, Proposition 9.1 and the argument that $C_k(X \sqcup X) = C_k(X) \times C_k(X)$.

A.J. March considered the problem (Problem 116 in [15]): Is it possible to find spaces $X$, $Y$ such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable but $C_k(X) \times C_k(Y)$ is not strongly sequentially separable?

A.Miller constructed the following example [18].

**Example 3.12.** There exist disjoint subsets of the plane $X$ and $Y$ such that both $X$ and $Y$ are $\gamma_k$-sets but $X \cup Y$ is not. Let $X$ be the open disk of radius one, i.e., $X = \{(x, y) : x^2 + y^2 < 1\}$, and $Y$ be any singleton on the boundary of $X$, e.g., $Y = \{(1, 0)\}$. 

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Thus, we have the example of the subsets of the plane $X$ and $Y$ such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable, but $C_k(X \cup Y)$ is not.

Note that (in contrast to the $C_p$-theory) $C_k(X \cup Y) \neq C_k(X) \times C_k(Y)$.

In [4], the authors considered the next problem (Problem 21 in [4]) : Is the class of $\gamma_k$-sets closed for finite unions ?

A particular answer to this problem and March’s problem is the following

**Theorem 3.13.** Suppose that $X$ and $Y$ are $\gamma_k$-sets, $i\omega(X) = i\omega(Y) = \aleph_0$ and $Y$ is first-countable. Then $X \bigcup Y$ is a $\gamma_k$-set.

**Proof.** By Theorem 3.6 $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable. Notice that each hemicompact space belong to the class $S_1(K, \Gamma_k)$, and the converse holds for first countable spaces [16]. It follows that $C_k(Y)$ is separable metrizable (first countable) space. By Theorem 9 in [6], $C_k(X) \times C_k(Y)$ is a strongly sequentially separable. Since $C_k(X) \times C_k(Y) = C_k(X \bigcup Y)$ and, by Theorem 3.6 we have that $X \bigcup Y$ is a $\gamma_k$-set.

**Corollary 3.14.** The product $C_k(X) \times C_k(Y)$ of strongly sequentially separable space $C_k(X)$ and strongly sequentially separable first-countable space $C_k(Y)$ belongs to the class of strongly sequentially separable spaces.

4. $S_1(D, D)$

In [10] it was shown that a Tychonoff space $X$ belongs to the class $S_1(K, K)$ if and only if $C_k(X)$ has countable strong fan tightness (i.e. for each $f \in C_k(X)$, $S_1(\Omega_f, \Omega_f)$ holds [26]).

Lj.D.R. Kočinac proved the next

**Theorem 4.1.** (Theorem 6 in [4]) For a first countable Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ is first countable;
2. $C_k(X)$ has countable strong fan tightness;
3. $C_k(X)$ has countable fan tightness;
4. $X$ is locally compact Lindelöf space;
5. $X$ satisfies $S_1(K, K)$;
6. $X$ satisfies $S_{fin}(K, K)$;
We consider the generalizations (Theorem 4.2 and Theorem 5.3) of the Theorem 4.1 to the class of Tychonoff spaces with \( iw(X) = \aleph_0 \).

**Theorem 4.2.** For a Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C_k(X) \) satisfies \( S_1(D, D) \);
2. \( X \) satisfies \( S_1(K, K) \);
3. Each finite power of \( X \) satisfies \( S_1(K, K) \);
4. \( C_k(X) \) satisfies \( S_1(\Omega_0, \Omega_0) \) [countable strong fan tightness];
5. \( C_k(X) \) satisfies \( S_1(D, \Omega_0) \).

**Proof.** (2) \( \equiv \) (3). By Theorem 5 in [14].

(2) \( \equiv \) (4). By Theorem 2.2 in [10].

(1) \( \Rightarrow \) (2). Let \( K_i \subseteq K \) for every \( i \in \mathbb{N} \) and \( D \) be a countable dense subset of \( C_k(X) \). Consider \( D_i = \{ f_{K_i, d} \in C(X) : f|_X = 1 \text{ and } |f(K) - d| < 1 \} \) and \( K_{i, j} = \{ K_{i, j, f} : f \in D_i \} \). Claim that \( K_{i, j} \subseteq K \) for every \( (i, j) \in \mathbb{N} \). Fix \( K_i \in \mathbb{K}(X) \) and \( < d_j, K, \frac{1}{i} > \) a base neighborhood of \( d_j \). Since \( D_{i, j} \) is a dense subset of \( C_k(X) \), there is \( f \in D_{i, j} \) such that \( f \in < d_j, K, \frac{1}{i} > \), hence, \( K \subseteq K_{i, j, f} \). Fix \( j \in \mathbb{N} \), by (2), there is a family \( \{ K_{i, j, f} : i \in \mathbb{N} \} \) such that \( K_{i, j, f(i, j)} \subseteq K_{i, j} \) and \( \{ K_{i, j, f(i, j)} : i \in \mathbb{N} \} \subseteq K \). So \( f(i, j) \in D_{i, j} \) for \( i, j \in \mathbb{N} \). Claim that \( \{ f(i, j) : i, j \in \mathbb{N} \} \) is dense in \( C_k(X) \). Let \( p \in C(X) \), \( K \subseteq \mathbb{K}(X) \), \( \epsilon > 0 \) and let \( < p, K, \epsilon > \) be a base neighborhood of \( p \). There is \( j' \in \mathbb{N} \) such that \( d_{j'} \in < p, K, \frac{\epsilon}{2} > \). Since \( \{ K_{i, j, f(i, j')} : i \in \mathbb{N} \} \subseteq K \), there is \( i' \in \mathbb{N} \) such that \( K \subseteq K_{i', j', f(i', j')} \) and \( \frac{\epsilon}{2} < \frac{\epsilon}{2} \). It follows that \( |f(i', j')(x) - p(x)| < |f(i', j')(x) - d_{j'}(x)| + |d_{j'}(x) - p(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \) for every \( x \in K \). Hence, \( f(i', j') \in < p, K, \epsilon > \) and \( \{ f(i, j) : i, j \in \mathbb{N} \} \) is dense in \( C_k(X) \).

(4) \( \Rightarrow \) (5) is immediate.

(5) \( \Rightarrow \) (1). Let \( (D_{i, j} : i \in \mathbb{N}) \) be a sequence of dense subsets of \( C_k(X) \) for each \( j \in \mathbb{N} \) and let \( D = \{ d_j : j \in \mathbb{N} \} \) be a countable dense subset of \( C_k(X) \).
By (5), for every \( j \in \mathbb{N} \) there is a family \( \{d^i_j : i \in \mathbb{N}\} \) such that \( d^i_j \in D_{i,j} \) and \( \{d^i_j : i \in \mathbb{N}\} \in \Omega_{d_j} \). Note that \( \{d^i_j : i, j \in \mathbb{N}\} \in \mathcal{D} \).

\[
5. \quad S_{\text{fin}}(\mathcal{D}, \mathcal{D})
\]

According to [11] \( X \) belongs to \( S_{\text{fin}}(\mathcal{K}, \mathcal{K}) \) if and only if \( C_k(X) \) has countable fan tightness (i.e., for each \( f \in C_k(X) \), \( S_{\text{fin}}(\Omega_f, \Omega_f) \) holds [1]).

**Theorem 5.1.** For a Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C_k(X) \) satisfies \( S_{\text{fin}}(\mathcal{D}, \mathcal{D}) \);
2. \( X \) satisfies \( S_{\text{fin}}(\mathcal{K}, \mathcal{K}) \);
3. Each finite power of \( X \) satisfies \( S_{\text{fin}}(\mathcal{K}, \mathcal{K}) \).
4. \( C_k(X) \) satisfies \( S_{\text{fin}}(\Omega_0, \Omega_0) \) [countable fan tightness];
5. \( C_k(X) \) satisfies \( S_{\text{fin}}(\mathcal{D}, \Omega_0) \).

**Proof.** (2) \( \iff \) (3). By Theorem 6 in [14].

(2) \( \iff \) (4) see in [11].

The remaining implications are proved similarly to the proof of Theorem 4.2.

**Remark 5.2.** It is easy to see that every hemicompact space is in the class \( S_1(\mathcal{K}, \mathcal{K}) \) and, thus, in \( S_{\text{fin}}(\mathcal{K}, \mathcal{K}) \). By Proposition 5 in [1], the converse is also true in the class of first countable spaces.

**Corollary 5.3.** For a first countable Tychonoff space \( X \) the following statements are equivalent:

1. \( C_k(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{D}) \);
2. \( C_k(X) \) satisfies \( S_{\text{fin}}(\mathcal{D}, \mathcal{D}) \);
3. \( X \) satisfies \( S_1(\mathcal{K}, \mathcal{K}) \).
6. $S_1(\mathcal{S}, \mathcal{D})$

**Definition 6.1.** A $\gamma_k$-cover $U$ of co-zero sets of $X$ is $\gamma_k$-shrinkable if there exists a $\gamma_k$-cover $\{F(U) : U \in U\}$ of zero-sets of $X$ with $F(U) \subset U$ for every $U \in U$.

For a topological space $X$ we denote:

- $\Gamma^sh_k$ — the family of $\gamma_k$-shrinkable $\gamma_k$-covers of $X$.

**Theorem 6.2.** For a Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(\Gamma_0, \Omega_0)$;
2. $X$ satisfies $S_1(\Gamma^sh_k, \mathcal{K})$.

**Proof.** (1) $\Rightarrow$ (2). Let $C_k(X)$ satisfies $S_1(\Gamma_0, \Omega_0)$ and $\{F_i : i \in \mathbb{N}\} \subset \Gamma^sh_k$.

For each $i \in \mathbb{N}$ we consider a set $D_i = \{f_{F(U),U,i} \in C(X) : f_{F(U),U,i} \upharpoonright F(U) = 0 \text{ and } f_{F(U),U,i} \upharpoonright (X \setminus U) = 1 \text{ for } U \in F_i\}$.

Since $\{F(U) : U \in F_i\}$ is a $\gamma_k$-cover of $X$, we have that $D_i$ converge to $f \equiv 0$ for each $i \in \mathbb{N}$.

Since $C_k(X)$ satisfies $S_1(\Gamma_0, \Omega_0)$, there is a sequence $\{f_{F(U),U,i} : i \in \mathbb{N}\}$ such that for each $i$, $f_{F(U),U,i} \in D_i$, and $\{f_{F(U),U,i} : i \in \mathbb{N}\}$ is an element of $\Omega_0$.

Consider $\{U_i : i \in \mathbb{N}\}$.

(a). $U_i \in F_i$.

(b). $\{U_i : i \in \mathbb{N}\}$ is a $k$-cover of $X$.

Let $K$ be a non-empty compact subset of $X$ and $U = < f, K, \frac{1}{2} >$ be a base neighborhood of $f$, then there is $f_{F(U),U,i} \in D_i$. It follows that $K \subset U_i$. We thus get $X$ satisfies $S_1(\Gamma^sh_k, \mathcal{K})$.

(2) $\Rightarrow$ (1). Let $\{f_{k,i} : k \in \mathbb{N}\}$ be a sequence converge to $f$ for each $i \in \mathbb{N}$. Without loss of generality we can assume that $f = 0$, a set $W^i_k = \{x \in X : \frac{1}{i} < f_{k,i}(x) < \frac{1}{i} \}
eq X$ for any $i \in \mathbb{N}$ and $S^i_k = \{x \in X : \frac{1}{i} \leq f_{k,i}(x) \leq \frac{1}{i} \} 
eq X$ for any $i \in \mathbb{N}$.

Consider $V_i = \{W^i_k : k \in \mathbb{N}\}$ and $\mathcal{S}_i = \{S^i_k : k \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Claim that $V_i$ is a $\gamma_k$-cover of $X$. Since $\{f_{k,i} : k \in \mathbb{N}\}$ converge to $f$, for each compact subset $K \subset X$ there is $k_0 \in \mathbb{N}$ such that $f_{k,i} \in < f, K, \frac{1}{i} >$ for $k > k_0$. It follows that $K \subset W^i_k$ for any $k > k_0$. Since $V_{i+1}$ is a $\gamma_k$-cover, $\mathcal{S}_{i+1}$ is a $\gamma_k$-cover, too. $\mathcal{S}_{i+1}$ is a refinement of the family $V_i$, hence, $V_i \in \Gamma^sh_k$.

By $X$ satisfies $S_1(\Gamma^sh_k, \mathcal{K})$, there is a sequence $\{W^i_{k(i)} : i \in \mathbb{N}\}$ such that for each $i$, $W^i_{k(i)} \in V_i$, and $\{W^i_{k(i)} : i \in \mathbb{N}\}$ is an element of $\mathcal{K}$.
We claim that \( f \in \{ f_{k(i),i} : i \in \mathbb{N} \} \). Let \( U = < f, K, \epsilon > \) be a base neighborhood of \( f \) where \( \epsilon > 0 \) and \( K \in \mathbb{K}(X) \), then there are \( i_0, i_1 \in \mathbb{N} \) such that \( \frac{1}{i_0} < \epsilon, i_1 > i_0 \) and \( W_{k(i)}^{i_1} \supseteq K \). It follows that \( f_{k(i),i_1} \in < f, K, \epsilon > \) and, hence, \( f \in \{ f_{k(i),i} : i \in \mathbb{N} \} \).

\[ \square \]

**Lemma 6.3.** Let \( \mathcal{U} = \{ U_n : n \in \mathbb{N} \} \) be a \( \gamma_k \)-shrinkable co-zero cover of a space \( X \). Then the set \( S = \{ f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1 \text{ for some } n \in \mathbb{N} \} \) is sequentially dense in \( C_k(X) \).

**Proof.** Let \( h \in C(X) \). For each \( n \in \mathbb{N} \), take \( f_n \in C(X) \) such that \( f_n \upharpoonright F(U_n) = h \upharpoonright F(U_n) \) and \( f_n \upharpoonright (X \setminus U_n) \equiv 1 \). Then obviously \( f_n \in S \), and \( f_n \mapsto h \), because \( \{ F(U_n) : n \in \mathbb{N} \} \) is a \( \gamma_k \)-cover.

\[ \square \]

**Theorem 6.4.** For a Tychonoff space \( X \) with \( iw(X) = \aleph_0 \) the following statements are equivalent:

1. \( C_k(X) \) satisfies \( S_1(S,D) \);
2. \( C_k(X) \) satisfies \( S_1(S,\Omega_0) \);
3. \( C_k(X) \) satisfies \( S_1(\Gamma_0,\Omega_0) \);
4. \( X \) satisfies \( S_1(\Gamma_k^{sh},K) \).

**Proof.** (1) \( \Rightarrow \) (4). Let \( \{ \mathcal{F}_i : i \in \mathbb{N} \} \subset \Gamma_k^{sh} \). By Lemma 6.3, \( S_i = \{ f \in C(X) : f \upharpoonright (X \setminus F_n^{i}) \equiv 1 \text{ for some } F_n^{i} \in \mathcal{F}_i \} \) is a sequentially dense subset of \( C_k(X) \) for each \( i \in \mathbb{N} \).

By (1), there is \( \{ f_i : i \in \mathbb{N} \} \) such that \( f_i \in S_i \) and \( \{ f_i : i \in \mathbb{N} \} \in D \).

Consider a sequence \( \{ F_{n(i)}^{i} : i \in \mathbb{N} \} \).

(a). \( F_{n(i)}^{i} \in \mathcal{F}_i \) for \( i \in \mathbb{N} \).

(b). \( \{ F_{n(i)}^{i} : i \in \mathbb{N} \} \) is a \( k \)-cover of \( X \).

Let \( K \in \mathbb{K}(X) \) and let \( U = < 0, K, \frac{1}{2} > \) be a base neighborhood of \( 0 \), then there is \( f_{i'} \in \{ f_i : i \in \mathbb{N} \} \) such that \( f_{i'} \in U \). It follows that \( K \subset F_{n(i')}^{i'} \).

(4) \( \Rightarrow \) (3). Let \( X \) satisfies \( S_1(\Gamma_k^{sh},K) \) and let \( \{ f_{i,m} \}_{m \in \mathbb{N}} \) converge to \( 0 \) for each \( i \in \mathbb{N} \).

Consider \( \mathcal{F}_i = \{ F_{i,m} : m \in \mathbb{N} \} = \{ f_{i,m}^{-1}(\frac{1}{2},\frac{1}{2}) : m \in \mathbb{N} \} \) for each \( i \in \mathbb{N} \).

Without loss of generality we can assume that a set \( F_{i,m} \neq X \) for any \( i,m \in \mathbb{N} \). Otherwise there is sequence \( \{ f_{i,k,m_k} \}_{k \in \mathbb{N}} \) such that \( \{ f_{i,k,m_k} \}_{k \in \mathbb{N}} \) uniform converge to \( 0 \) and \( \{ f_{i,k,m_k} : k \in \mathbb{N} \} \in \Omega_0 \).

Note that \( \mathcal{F}_i \) is a \( \gamma_k \)-shrinkable co-zero cover of \( X \) for each \( i \in \mathbb{N} \).
By (4), there is a sequence \((F_{i,m(i)} : i \in \mathbb{N})\) such that for each \(i\), \(F_{i,m(i)} \in \mathcal{F}_i\), and \(\{F_{i,m(i)} : i \in \mathbb{N}\}\) is an element of \(\mathcal{K}\).

We claim that \(0 \in \{F_{i,m(i)}(i) : i \in \mathbb{N}\}\). Let \(W = (0, K, \epsilon)\) be a base neighborhood of \(0\) where \(\epsilon > 0\) and \(K \in \mathcal{K}(X)\), then there are \(i_0, i_1 \in \mathbb{N}\) such that \(\frac{1}{i_0} < \epsilon\), \(i_1 > i_0\) and \(F_{i_1,m(i_1)} \supset K\). It follows that \(F_{i_1,m(i_1)}(i) \in (0, K, \epsilon)\) and, hence, \(0 \in \{F_{i,m(i)}(i) : i \in \mathbb{N}\}\) and \(C_k(X)\) satisfies \(S_1(\Gamma_0, \Omega_0)\).

(3) \(\Rightarrow\) (2) is immediate.

(2) \(\Rightarrow\) (1). Suppose that \(C_k(X)\) satisfies \(S_1(\mathcal{S}, \Omega_0)\). Let \(D = \{d_n : n \in \mathbb{N}\}\) be a dense subspace of \(C_k(X)\). Given a sequence of sequentially dense subspaces of \(C_k(X)\), enumerate it as \(\{S_{n,m} : n, m \in \mathbb{N}\}\). For each \(n \in \mathbb{N}\), pick \(d_{n,m} \in S_{n,m}\) so that \(d_n \in \{d_{n,m} : m \in \mathbb{N}\}\). Then \(\{d_{n,m} : m, n \in \mathbb{N}\}\) is dense in \(C_k(X)\).

\[\square\]

7. \(S_{\text{fin}}(\mathcal{S}, \mathcal{D})\)

The following Theorems are proved similarly to Theorems 6.2 and 6.4.

**Theorem 7.1.** For a Tychonoff space \(X\) the following statements are equivalent:

1. \(C_k(X)\) satisfies \(S_{\text{fin}}(\Gamma_0, \Omega_0)\);
2. \(X\) satisfies \(S_{\text{fin}}(\Gamma_k^\text{sh}, \mathcal{K})\).

**Theorem 7.2.** For a Tychonoff space \(X\) with \(iw(X) = \aleph_0\) the following statements are equivalent:

1. \(C_k(X)\) satisfies \(S_{\text{fin}}(\mathcal{S}, \mathcal{D})\);
2. \(C_k(X)\) satisfies \(S_{\text{fin}}(\mathcal{S}, \Omega_0)\);
3. \(C_k(X)\) satisfies \(S_{\text{fin}}(\Gamma_0, \Omega_0)\);
4. \(X\) satisfies \(S_{\text{fin}}(\Gamma_k^\text{sh}, \mathcal{K})\).

8. \(S_1(\mathcal{S}, \mathcal{S})\)

In [22], we proved the following theorems.

**Theorem 8.1.** (Theorem 3.3 in [22]) For a Tychonoff space \(X\) the following statements are equivalent:

1. \(C_k(X)\) satisfies \(S_1(\Gamma_0, \Gamma_0)\);
2. $X$ satisfies $S_1(\Gamma^*_k, \Gamma_k)$.

**Theorem 8.2.** (Theorem 3.5 in [22]) For a Tychonoff space $X$ such that $C_k(X)$ is sequentially separable the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(S, S)$;
2. $C_k(X)$ satisfies $S_1(S, \Gamma_0)$;
3. $C_k(X)$ satisfies $S_1(\Gamma_0, \Gamma_0)$;
4. $X$ satisfies $S_1(\Gamma^*_k, \Gamma_k)$;
5. $C_k(X)$ satisfies $S_{\text{fin}}(S, S)$;
6. $C_k(X)$ satisfies $S_{\text{fin}}(S, \Gamma_0)$;
7. $C_k(X)$ satisfies $S_{\text{fin}}(\Gamma_0, \Gamma_0)$;
8. $X$ satisfies $S_{\text{fin}}(\Gamma^*_k, \Gamma_k)$.

We can summarize the relationships between considered notions in next diagrams.

\[
\begin{align*}
S_1(S, S) & \iff S_{\text{fin}}(S, S) \Rightarrow S_1(S, D) \Rightarrow S_{\text{fin}}(S, D) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{align*}
\]

Diagram 1. The Diagram of selectors for sequences of dense sets of $C_k(X)$.

\[
\begin{align*}
S_1(\Gamma^*_k, \Gamma_k) & \iff S_{\text{fin}}(\Gamma^*_k, \Gamma_k) \Rightarrow S_1(\Gamma^*_k, \mathcal{K}) \Rightarrow S_{\text{fin}}(\Gamma^*_k, \mathcal{K}) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{align*}
\]

Diagram 2. The Diagram of selection principles for a space $X$ corresponding to selectors for sequences of dense sets of $C_k(X)$. 

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9. On the particular solution to one problem

Recall that Arens’s space $S_2$ is the set $\{(0,0),(\frac{1}{n},0),(\frac{1}{n},\frac{1}{nm}) : n, m \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0,0),(\frac{1}{n},0) : n > 0\}$ and $C_n = \{(\frac{1}{n},0),(\frac{1}{n},\frac{1}{nm}) : m > 0\}, n > 0$. The sequential fan is the quotient space $S_\omega = S_2/C_0$ obtained from the Arens’s space by identifying the points of the sequence $C_0$.[12]

Proposition 9.1. If $C_k(X)$ satisfies $S_{\text{fin}}(\Gamma_0,\Omega_0)$, then $S_\omega$ cannot be embedded into $C_k(X)$.

The following problem was posed in the paper [4].

Problem 30. Does a first countable (separable metrizable) space belong to the class $S_1(\Gamma_k,\mathcal{K})$ if and only if it is hemicompact?

A particular answer to this problem is the following

Theorem 9.2. Suppose that $X$ is first countable stratifiable space and $iw(X) = \aleph_0$. Then following the statements are equivalent:

1. $X$ satisfies $S_{\text{fin}}(\Gamma_{sh},\mathcal{K})$;
2. $X$ satisfies $S_{\text{fin}}(\Gamma_k,\mathcal{K})$;
3. $X$ satisfies $S_1(\mathcal{K},\Gamma_k)$;
4. $X$ is hemicompact.

Proof. (1) $\Rightarrow$ (4). Since $X$ is first countable stratifiable space and, by Proposition 9.1, $S_\omega$ cannot be embedded into $C_k(X)$, then, by Theorem 2.2 (+ Remark) in [7], $X$ is a locally compact. A locally compact stratifiable space is metrizable [5]. A well-known that a locally compact metrizable space can be represented as $X = \bigsqcup_{\alpha<\tau} X_\alpha$ where $X_\alpha$ is a $\sigma$-compact for each $\alpha < \tau$.

Since $iw(X) = \aleph_0$, then $\tau \leq c$. Claim that $|\tau| < \omega_1$.

Assume that $|\tau| \geq \omega_1$. Then there is a continuous mapping $f : X \mapsto D$ from $X$ onto a discrete space $D$ where $|D| \geq \omega_1$. It follows that $D$ satisfies $S_{\text{fin}}(\Gamma_{sh},\mathcal{K})$ ($S_{\text{fin}}(\Gamma,\Omega)$) and, hence, $D$ is Lindelöf, a contradiction.

It follows that $X$ is a locally compact and Lindelöf, and, hence, $X$ is a hemicompact.

(4) $\Rightarrow$ (3). Since $X$ is hemicompact and $iw(X) = \aleph_0$, then $C_k(X)$ is a separable metrizable space [17]. Hence, $C_k(X)$ satisfies $S_1(\mathcal{D},\mathcal{S})$, and, by Theorem 3.3, $X$ satisfies $S_1(\mathcal{K},\Gamma_k)$. 

\qed
**Corollary 9.3.** Suppose that $X$ is separable metrizable space. Then $X$ satisfies $S_{\text{fin}}(\Gamma_k, K)$ if and only if $X$ is hemicompact.

**Remark 9.4.** In class of first countable stratifiable spaces with $iw(X) = \aleph_0$ (in particular, in class of separable metrizable spaces) all properties in Diagram 1 (and, hence, Diagram 2) coincide.

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