Present status of the Penrose inequality

Marc Mars

Facultad de Ciencias, Universidad de Salamanca, Plaza de la Merced s/n, 37008 Salamanca, Spain

Received 30 July 2009
Published 22 September 2009
Online at stacks.iop.org/CQG/26/193001

Abstract

The Penrose inequality gives a lower bound for the total mass of a spacetime in terms of the area of suitable surfaces that represent black holes. Its validity is supported by the cosmic censorship conjecture, and therefore its proof (or disproof) is an important problem in relation with gravitational collapse. The Penrose inequality is a very challenging problem in mathematical relativity and it has received continuous attention since its formulation by Penrose in the early seventies. Important breakthroughs have been made in the last decade or so, with the complete resolution of the so-called Riemannian Penrose inequality and a very interesting proposal to address the general case by Bray and Khuri. In this review, the most important results on this field will be discussed and the main ideas behind their proofs will be summarized, with the aim of presenting what is the status of our present knowledge in this topic.

PACS numbers: 04.20.−q, 04.20.Cv, 04.20.Dw, 04.70.Bw, 02.40.—k, 02.40.Ft, 02.40.Ma, 02.40.Vh

1. Introduction

In the early seventies, Penrose [135, 136] showed that by combining several ingredients of the ‘establishment viewpoint’ of gravitational collapse, an inequality of the form

\[ M \geq \sqrt{\frac{A}{16\pi}} \] (1)

follows, where \( M \) is the total mass and \( A \) is the area of a black hole. Cosmic censorship is one of the fundamental ingredients of the argument, and by far the weakest one. Thus, finding a counterexample to (1) would very likely involve a spacetime for which cosmic censorship fails to hold. In fact, this was Penrose’s original motivation to study the inequality. On the other hand, a proof of a suitable version of (1) would give indirect support to cosmic censorship. Inequalities of this type are collectively termed ‘Penrose inequalities’ (sometimes also ‘isoperimetric inequality for black holes’ [62]), and finding suitable versions thereof and trying to prove them have become a major task in mathematical relativity. After a first period of heuristic proofs and partial results, important breakthroughs have been made in the last
10 years. The aim of this review is to try to explain the problem and describe the main approaches that have been followed.

The first observation to be made is the necessity of replacing the area of the black hole in (1) by the area of a suitable alternative surface. This is because in order to determine whether a spacetime is a black hole, detailed knowledge of its global future behaviour is required. On the other hand, cosmic censorship is precisely a statement on the global future evolution of a spacetime. In order to have an inequality logically independent of cosmic censorship (although motivated by it), the area on the black hole must be replaced by the area of a surface which can be located independently of the global future behaviour of the spacetime (for instance, directly in terms of the initial data) and which is guaranteed (or at least expected) to have an area less than or equal to the event horizon that may eventually form during the evolution.

The global set-up which supports the validity of (1) is well known and, in rough terms, goes as follows. Assume a spacetime \((M, g)\) which is asymptotically flat in the sense of being strongly asymptotically predictable, admitting a complete future null infinity \(\mathcal{I}^+\) and satisfying \(J^- (\mathcal{I}^+) \neq M\) (see [161] for definitions). The event horizon \(\mathcal{H}\) is the boundary of \(J^- (\mathcal{I}^+)\) and it is, therefore, a null hypersurface at least Lipschitz continuous. Assume, moreover, that the spacetime admits an asymptotically flat partial Cauchy surface with total ADM energy \(E_{\text{ADM}}\) and which intersects \(\mathcal{H}\) on a cut \(S\). If \(\mathcal{H}\) is a smooth hypersurface, then this cut is a smooth embedded surface which has a well-defined area \(|S|\). For general event horizons, the area \(|S|\) still makes sense provided it is interpreted as its two-dimensional Hausdorff measure (the Hausdorff measurability of \(S\) is demonstrated in [37]). Now consider any cut \(S_1\) to the causal future of \(S\) along the event horizon. The black hole area law [78, 79] states \(|S_1| \geq |S|\) provided the null energy condition holds. This area theorem was proven for general event horizons in [37] under much milder asymptotic conditions. From physical principles, the spacetime is expected to settle down to some equilibrium configuration. Also assuming that all the matter fields are swallowed by the black hole in the process (an external electromagnetic field would not alter the conclusions; see section 8), the uniqueness theorems for stationary black holes (see e.g. [85]) imply that the spacetime must approach the Kerr metric (modulo several technical conditions that still remain open; see [44] for a recent account). For the Kerr metric, the area of the event horizon \(A_{\text{Kerr}}\) is independent of the cut (as for any Killing horizon) and takes the value (in units \(G = c = 1\)) \(A_{\text{Kerr}} = 8\pi M (M + \sqrt{M^2 - L^2/M^2}) \leq 16\pi M^2\), where \(M\) and \(L\) are respectively the total mass and total angular momentum of the spacetime (we do not use the more common term ‘\(J\)’ for the angular momentum to avoid confusion with the energy flux used later). In particular, \(M\) should be the asymptotic value of the Bondi mass along \(\mathcal{I}^+\). Since gravitational waves carry positive energy, the Bondi mass cannot increase to the future [31, 139]. Provided the Bondi mass approaches the ADM mass \(M_{\text{ADM}}\) of the initial slice (which is only known under additional assumptions; see [11, 74, 107, 169]), the inequality \(M_{\text{ADM}} \geq \sqrt{|S|/16\pi}\) follows. This inequality is still global in the sense that the cut \(S\) of the event horizon cannot be determined directly in terms of the initial data. Penrose’s idea was to consider situations in which one could estimate the area of the cut from below in terms of the area of some surface that could be located without having to solve the whole future evolution.

One such situation occurs when the initial data set is asymptotically Euclidean and contains a future trapped surface (see below for definitions). Then, the singularity theorems of Penrose [133], Hawking [76] and others (see [148] for a review) state that, provided the strong energy condition holds (in some cases, the null energy condition suffices), the maximal globally hyperbolic development of these data must contain a singularity, i.e. an incomplete inextendible causal geodesic. Very little is known in general about the nature of
the singularity that forms. More specifically, it is not known whether the future development admits a complete $\mathcal{I}^+$ and therefore defines a black hole spacetime. The weak cosmic censorship conjecture, first proposed by Penrose [134], asserts that all singularities lie behind an event horizon and therefore are invisible to an observer at infinity. Not much is known about the general validity of this conjecture, which remains a fundamental open problem in gravitational collapse physics (see [162]). Rigorous results are available only in the case of spherical symmetry, where the conjecture has been proven for several matter models [36, 50].

Under cosmic censorship, the initial data containing a future trapped surface $S$ develop a black hole spacetime. A general result on black hole spacetimes [47, 80] is that no future trapped surface can enter into the causal past of $\mathcal{I}^+$. Therefore, the intersection of the event horizon and the initial data set $\Sigma$ defines a spacelike surface $\mathcal{H}_\Sigma$ that separates $S$ from the asymptotic region. In general, the location of $\mathcal{H}_\Sigma$ cannot be determined directly from the initial data. Moreover, $\mathcal{H}_\Sigma$ can have a smaller area than $S$, even though it lies in its exterior. Nevertheless, under suitable restrictions on $S$ (details will be given below) it makes sense to consider all surfaces in $\Sigma$ enclosing $S$. The infimum of the areas of all such surfaces, denoted by $A_{\text{min}}(S)$, has the obvious property that $|\mathcal{H}_\Sigma| \geq A_{\text{min}}(S)$. Consequently, the inequality

$$M_{\text{ADM}} \geq \sqrt{\frac{A_{\text{min}}(S)}{16\pi}}$$

follows from Penrose’s heuristic argument. The need of using the minimum area enclosure of $S$ in (2) was first noted by Jang and Wald [94], and the first example showing that $S$ may have a greater area than surfaces enclosing it is due to Horowitz [86].

Inequality (2) involves objects defined solely in terms of the local geometry of the initial data set and its validity can therefore be addressed independently of whether weak cosmic censorship (or any other of the ingredients entering into the argument) holds or not. It is clear that, since $A_{\text{min}}(S)$ depends on the hypersurface $\Sigma$ containing $S$, we can still take the supremum of the right-hand side with respect to all asymptotically flat hypersurfaces containing $S$, and the resulting inequality still follows from Penrose’s heuristic argument. This gives a clearly stronger inequality. However, it depends on the piece of spacetime available and therefore loses the desirable property of depending solely on objects defined on the initial data set.

If the initial data set is asymptotically hyperbolic instead of asymptotically flat (i.e. such that it intersects future null infinity in its Cauchy development), the same heuristic argument gives (2) with the Bondi mass replacing the ADM mass on the left-hand side.

Another set-up where a local geometric inequality is implied by the global heuristic argument of gravitational collapse appears in the seminal paper by Penrose [136] and consists of a null shell $\mathcal{N}$ (with compact cross sections) of collapsing dust in the Minkowski spacetime. As described in more detail below, for any given shape of the shell (restricted to be convex at each instant of time in order to avoid shell crossings in the past) and any chosen cross section $S$ on the shell, the energy density of the collapsing dust can be adjusted so that $S$ is a marginally trapped surface with respect to the geometry of the spacetime left after the shell has passed. Since a singularity will definitely form in this set-up (the shell has self-intersections in its future), cosmic censorship predicts the formation of a black hole. Similarly as before, the marginally trapped surface $S$ cannot enter the causal past of $\mathcal{I}^+$ [45]. The intersection of the event horizon with the shell $\mathcal{N}$ must therefore lie in the causal past of $S$. However, since the shell is convex and collapsing, the event horizon cut automatically has at least the same area as $S$. Consequently, the heuristic collapse argument implies $M \geq \sqrt{|S|/16\pi}$, where $M$ is
the mass of the shell. By energy conservation, this can be computed directly on $S$ in terms of its geometry as a surface in Minkowski. Thus, a geometric inequality is obtained for a class of spacelike surfaces in the Minkowski spacetime. The status of this version of the Penrose inequality will be discussed in detail later on.

All versions of the Penrose inequality have the structure of a lower bound of the total mass of the spacetime in terms of the area of suitably chosen surfaces. Therefore, they can be regarded as strengthenings of the positive mass theorem, which says that the total mass of an asymptotically flat spacetime cannot be negative, under suitable energy and completeness conditions. The positive mass theorem also has a rigidity part, namely that the total mass vanishes only for the Minkowski spacetime. The Penrose inequality conjecture also has a rigidity part, which in rough terms states that equality will only happen for the Schwarzschild spacetime. Heuristically, this can be understood because, in the case of equality, the final mass of the spacetime must coincide with the starting one. Therefore, no gravitational waves are emitted in the process. This suggests that the whole configuration should be stationary. However, the only stationary, vacuum black hole is the Kerr spacetime, and equality happens for this metric only if the angular momentum is zero, i.e. if the metric is, in fact, the Schwarzschild spacetime.

Since the original proposal by Penrose, this topic has become an active area of research. However, the problem has proven to be a difficult one and relatively little progress was made during the first few decades. The most important contribution in this period is due to Geroch [61], who observed that a suitable functional defined on surfaces embedded in a spacelike hypersurface $(\Sigma, \gamma)$ is monotonically increasing if the curvature scalar of $\gamma$ is non-negative and the surfaces are moved outwards at a speed which is inversely proportional to the mean curvature of the surface at each point. This is the so-called inverse mean curvature flow (IMCF). This functional (now called the Geroch mass) has the property of approaching the ADM energy of the hypersurface (provided this is asymptotically Euclidean) if the surfaces become sufficiently spherical at infinity. Geroch’s original idea was to prove the positive mass theorem by starting the inverse mean curvature flow at a point (where the Geroch mass vanishes). This idea was then adapted by Jang and Wald [94], who noted that the Geroch mass coincides exactly with the right-hand side of (1) if the starting surface is connected, of spherical topology and minimal (i.e. with vanishing mean curvature). If the flow existed globally and the leaves could be seen to approach round spheres at infinity, then the monotonicity of the Geroch mass would imply the Penrose inequality in the particular case of time-symmetric initial data sets. The flow, however, generically develops singularities and therefore the argument could not be made rigorous at the time. The only case where the Penrose inequality could be proven to hold was in spherical symmetry (irrespective of whether the initial data set was time symmetric or not) using the so-called Hawking mass, which is a generalization of the Geroch mass when the second fundamental form is not zero.

In the late nineties, however, two important breakthroughs were made. First, Huisken and Ilmanen [89] were able to prove that Geroch’s heuristic derivation could be turned into a rigorous proof. This required deep results in geometric analysis and geometric measure theory. These authors therefore established the validity of the Penrose inequality for asymptotically Euclidean Riemannian manifolds of non-negative curvature scalar and having a boundary consisting of an outermost minimal surface (this is now called the Riemannian Penrose inequality). Although this boundary was allowed to be disconnected, the Penrose inequality could only be established for the area of any of its connected components. Shortly afterwards, Bray [22] was able to prove the Riemannian Penrose inequality in full generality (i.e. in terms of the total area of the outermost minimal surface, independently of whether this is connected or not). Bray’s argument is completely different to the previous one and uses a deformation
of the given metric in such a way that all the non-trivial geometry gets swallowed inside
the minimal surface while the total mass of the space does not increase and the area of the
horizon does not decrease. This process settles down into an equilibrium state given by the
Schwarzschild metric. Since the Penrose inequality is fulfilled in the final state (in fact, with
equality), the Penrose inequality also holds for the original space. These two fundamental
results have boosted tremendously the interest in the Penrose inequality, which has become
a very important topic in mathematical relativity. Although the general case is still open,
several ideas have been proposed to approach it. In particular, an important step forward
has been made very recently by Bray and Khuri [29]. Despite the fact that several issues
remain still open regarding this proposal (in particular, the existence of solutions of certain
PDE problems), the idea is indeed very promising.

The aim of this review is to present the most important developments in this field. I have
chosen not to follow a historical order, and rather I have tried to organize the presentation
in a way which I consider logically convenient. Since several approaches share some of the
techniques, I have collected many preliminary results in one section (section 2). While this
has the potential disadvantage that, in a first reading, it is not clear why and where such results
are needed, it has the advantage of simplifying the location of the required tools. Readers
wishing to enter straight into the topic of the Penrose inequality may skip this section and refer
back to it whenever necessary.

An important warning is in order. The topic ‘Penrose inequalities’ is vast and has many
ramifications. Although I will try to cover the most important results in the field, I make no
claim of exhaustivity. There are several other review papers on this topic in the literature.
For the early results, the reader is advised to look at [119]. The breakthroughs of Huisken
and Ilmanen, and Bray triggered the publication of a number of interesting reviews, where the
proofs of the Riemannian Penrose inequality were treated in detail (see [24–26, 83, 87, 88,
124, 145]).

The structure of the review is as follows. In section 2, the basic facts that will be needed
later are introduced. This section is divided into five subsections. In subsection 2.1, the
geometry of \((n - 2)\)-dimensional surfaces as submanifolds of an \(n\)-dimensional spacetime
is described, and several types of surfaces are defined. In subsection 2.2, some relevant
variational formulae for the null expansions are summarized. In subsection 2.3, the geometry
of codimension-2 surfaces as submanifolds of spacelike hypersurfaces is discussed and two
important existence theorems for outermost surfaces with suitable properties are reviewed. In
subsection 2.4, the Hawking and Geroch quasi-local masses for surfaces in four-dimensional
spacetimes and their general variation formulae are discussed. Subsection 2.5 recalls the
notion of asymptotically Euclidean. Section 3 is devoted to the discussion of the various
formulations of the Penrose inequality that have been proposed. Section 4 deals with the
spherically symmetric case. In section 5, the so-called Riemannian Penrose inequality is
treated. The main ideas behind the remarkable proofs of Huisken and Ilmanen and of Bray are
discussed in subsections 5.3 and 5.4, respectively. However, previous approaches based on
spinors and isoperimetric methods which admit interesting generalizations are also covered
in subsections 5.1 and 5.2. Section 6 discusses the Penrose inequality when the initial data
set is asymptotically hyperbolic instead of asymptotically Euclidean. Section 7 is devoted
to describing different attempts that have been proposed to address the Penrose inequality in
the non-time-symmetric case. Particular attention is paid to the very promising recent ideas
put forward by Bray and Khuri [29]. Section 8 treats several strengthenings of the Penrose
inequality when particular matter fields or symmetries are present. Section 9 is devoted
to describing some applications of the Penrose inequality. The review finishes with some
concluding remarks in section 10.
2. Basic facts and definitions

A spacetime \((\mathcal{M}, g)\) is a connected Hausdorff manifold endowed with a smooth metric of the Lorentzian signature (with sign convention \([-+\cdots]\)). Smoothness is assumed for simplicity; many of the results below hold under weaker differentiability assumptions. We further take \(\mathcal{M}\) orientable and \((\mathcal{M}, g)\) time orientable and that an orientation for both has been chosen. In this review we will be mainly concerned with four-dimensional spacetimes, although higher dimensional results will be mentioned at some places. \(\mathcal{M}\) will always be four dimensional unless explicitly stated.

The Penrose inequality involves the area of codimension-2 surfaces. Let us therefore start with some basic properties concerning their geometry.

2.1. Geometry of codimension-2 surfaces

\(S\) will denote a compact, embedded, oriented, codimension-2 surface in an \(n\)-dimensional spacetime \((\mathcal{M}, g)\) defined via an embedding \(\Phi: S \to \mathcal{M}\) (\(S\) will usually be identified with its image). The induced first fundamental form on \(S\), denoted by \(h\), is assumed to be positive definite, i.e. \(S\) is spacelike. Such an object will simply be called ‘surface’. The area of \(S\) is denoted by \(|S|\).

At \(p \in S\), we denote by \(T_pS\) and \(N_pS\) the tangent and normal spaces of \(S\), respectively. This implies \(T_p\mathcal{M} = T_pS \oplus N_pS\). For any vector \(\bar{V}\) at \(p\), this decomposition defines a parallel and a normal vector according to \(\bar{V} = \bar{V}^\parallel + \bar{V}^\perp\). The second fundamental form vector \(\bar{K}\) of \(S\) is defined, as usual, as \(\bar{K}(\bar{X}, \bar{Y}) = -(\nabla^g_{\bar{X}}\bar{Y})^\perp\), where \(\bar{X}\) and \(\bar{Y}\) are tangent to \(S\) and \(\nabla^g\) is the covariant derivative on \((\mathcal{M}, g)\). This tensor is symmetric and its trace \(\bar{h} = \text{tr}_h\bar{K}\) defines the mean curvature vector. The trace-free part of \(\bar{K}\) will be called \(\bar{H}\) in the following.

\(S\) being oriented and the ambient spacetime being oriented and time oriented, it easily follows that there exist (globally on \(S\)) two linearly independent null vector fields \(\bar{l}^+\) and \(\bar{l}^−\) orthogonal to \(S\). These vectors, which span the normal bundle \(NS = \bigcup_pN_pS\), will always be chosen to be future directed and satisfying \((\bar{l}^+ \cdot \bar{l}^−) = -2\) (the dot stands for a scalar product with the spacetime metric \(g\)). They are uniquely defined up to rescalings \(\bar{l}^+ \to F\bar{l}^+, \bar{l}^− \to F^{-1}\bar{l}^−\) \((F > 0)\), plus interchange \(\bar{l}^+ \leftrightarrow \bar{l}^−\). The null expansions \(S\) are defined as \(\theta_\pm = (\bar{H} \cdot \bar{l}^\pm)\) and they contain the same information as \(\bar{H}\) since \(\bar{H} = -\frac{1}{2}(\theta_+ + \theta_-)\).

A surface \(S\) can be classified according to the causal character of \(\bar{H}\). Unfortunately, there is no unique and generally accepted agreement on how to call the various types of surfaces which arise. In this review, the following notation will be used (in our convention, the zero vector is both future null and past null):

Future trapped. \(\bar{H}\) is timelike and future directed everywhere (equivalently, \(\theta_+ < 0\) and \(\theta_- < 0\)).

Weakly future trapped. \(\bar{H}\) is causal and future directed everywhere \((\theta_+ \leq 0, \theta_- < 0)\).

Marginally future trapped. \(\bar{H}\) is proportional to one of the null normals \(\bar{l}^\pm\) with a non-negative proportionality factor \((\theta_+ = 0\) and \(\theta_- \leq 0\) or vice versa).

Past trapped, weakly past trapped and marginally past trapped are defined by reversing all inequalities.

All these types of surfaces have a mean curvature vector of a definite causal character and time orientation. For brevity, when no indication to future or past is given, then future should be understood (i.e. a ‘trapped surface’ is meant to be a future trapped surface and similarly for the other cases).
It is often useful to consider surfaces for which one of the null normals can be geometrically
selected. This preferred normal will be called ‘outer null normal’ and will always be denoted
as $l^+$. Note that ‘outer’ here does not necessarily refer to any notion of exterior to $S$, so
this definition must be used with special care. Nevertheless, when the null normals can be
geometrically distinguished, the corresponding null expansions are also geometrically distinct
and the following definitions (which place no restriction on $\theta_-$) become of interest.

Weakly outer trapped. $\theta_+ \leq 0$.

Marginally outer trapped or MOTS. $\theta_+ = 0$ (equivalently, $\vec{H}$ points along the outer null
normal).

Weakly outer untrapped. $\theta_+ \geq 0$.

Outer untrapped. $\theta_+ > 0$.

These definitions depend not only on the choice of outer direction, but also on the time
orientation of the spacetime. If the time orientation is reversed (without modifying the outer
direction), then $-l^-$ becomes the future outer null direction. Therefore by adding the word
‘past’ to any of the above four definitions, the surface is meant to satisfy the same inequalities
with $\theta_+$ replaced by $-\theta_-$ For instance, a past weakly outer trapped surface satisfies $\theta_- \geq 0$.

Very recently, Bray and Khuri [29] have proposed a version of the Penrose inequality
which involves a type of surfaces whose definition is insensitive to the reversals of time
direction. However, they still require a preferred outer direction.

Generalized trapped surface. At each point, either $\theta_+ \leq 0$ or $\theta_- \geq 0$ (or both).

Generalized apparent horizon. At each point, either $\theta_+ = 0$ and $\theta_- \leq 0$ or $\theta_- = 0$ and
$\theta_+ \geq 0$. Note that generalized apparent horizons are in particular generalized trapped
surfaces.

In the detailed classification of surfaces according to their mean curvature vector presented
in [147], generalized apparent horizons are termed null untrapped, which, in my opinion, has
the advantage of being more descriptive. In fact, my preferred term for these surfaces would
be null outer untrapped because this emphasizes the fact that they have a privileged ‘outer’
direction. However, notation is always a very personal matter and, in this review, I will stick to
the name generalized apparent horizon put forward by Bray and Khuri in their new proposal
of the Penrose inequality.

In terms of the mean curvature vector, a generalized trapped surface is one where $\vec{H}$ is
allowed to point everywhere except along a spacelike outer direction (defined to be a spacelike
direction with a positive scalar product with $l^+$). A generalized apparent horizon has a mean
curvature vector which is null everywhere and moreover has a non-negative scalar product
with any outer spacelike direction. Note that the second condition is ensured provided the
scalar product with one outer spacelike direction is non-negative.

Generalized trapped surfaces are indeed generalizations of the previous concepts. In
particular, weakly future trapped and weakly outer trapped surfaces are automatically
generalized trapped surfaces, and the same is true for marginally outer trapped surfaces.

A table summarizing all these definitions is given in table 1 below.

2.2. First-order variations of the null expansions

An important technical tool that will be often used below is the first-order change of the null
expansions $\theta_\pm$ under variations of the surface. In this section we write down the variations
of $\theta_\pm$ along the null normals $l^\pm$, which we write as $\mathcal{L}_{l^\pm} \theta_\pm$. For the sake of completeness, let
us first briefly recall how variations of a geometric object $F$ are defined. Any vector field
$\xi$ in a neighbourhood of $S$ defines a variation of the surface as follows. For small enough
\( \lambda \in \mathbb{R} \), let \( \Phi_\lambda : S \rightarrow M \) be the embedding defined by moving \( p \in S \) a parametric amount \( \lambda \) along the integral curve of \( \xi \) starting at \( p \). We write \( S_\lambda = \Phi_\lambda(S) \). Assume, for definiteness, that \( F \) is a covariant tensor geometrically defined on any surface and take a variation \( \Phi_\lambda \) of \( S \). Define \( F_\lambda(p) = \Phi_\lambda^*(F|_{\Phi_\lambda^*(p)}) \), i.e. the pull-back of the tensor \( F \) attached to \( S_\lambda \) at the point \( \Phi_\lambda(p) \). This defines a curve of tensors at each point \( p \in S \). The geometric variation is simply \( \overline{\xi}_\lambda F = \partial_\lambda F_\lambda(p) \). This derivative only depends on the values of \( \xi \) on \( S \) for truly geometric objects. However, when \( F \) requires additional structure for its definition, then derivatives of \( \xi \) on \( S \) may also arise. This behaviour occurs for instance in \( \overline{\xi}_\lambda \theta_\lambda \) which requires making a specific choice of \( \tilde{I}_\lambda \) on each surface \( S_\lambda \).

Applying this procedure to the area of \( S \) gives (see e.g [101]) the \textit{first variation of area} \( \frac{d\text{Area}(S)}{dt}(t, p) \big|_{t=0} = \int_S (\mathbf{H} \cdot \xi) \eta_S \), where \( \eta_S \) is the metric volume form of \( S \). In particular, the change in area of \( S \) for variations along the null directions \( \tilde{t}^\pm \) is determined by the integral of the null expansions \( \theta_\pm \).

Let us next write down \( \overline{\xi}_\lambda \theta_\pm \). Since \( \tilde{t}^+ \) and \( \tilde{t}^- \) are interchangeable, the corresponding expression will also hold for variations along \( \tilde{t}^- \) after making the substitution \( + \leftrightarrow - \) everywhere. We start with the variation of the induced metric, which is well known to be

\[
\tilde{I}_\lambda(h_{AB}) = 2\tilde{I}_\lambda \cdot \tilde{K}_{AB},
\]

where \( A, B, \) etc, are tensorial indices on \( S \). This implies

\[
\tilde{I}_\lambda(\eta_S) = \theta_\lambda \eta_S
\]

for the variation of the volume form on \( S \). For the variation of \( \theta_\lambda \) along \( \tilde{t}^\lambda \), it is necessary to relate the null normal \( \tilde{I}_\lambda \) on each one of the varied surfaces \( S_\lambda \) to the corresponding null normal on the original surface. As already stressed, this introduces a dependence on the first derivative of \( \tilde{t}^\lambda \) via \( Q^\lambda = -1/2(\tilde{I}^- \cdot \nabla^\lambda \tilde{I}^\pm)|_{t=0} \) in the resulting expression, which is the well-known Raychaudhuri equation:

\[
\overline{\xi}_\lambda \theta_\lambda = Q^\lambda \theta_\lambda - K_{AB}^\lambda K^{AB} - \text{Ric}(\tilde{I}_\lambda, \tilde{I}_\lambda),
\]

where \( K_{AB}^\lambda = (\tilde{I}^\pm \cdot \tilde{K}_{AB}) \) and Ric is the Ricci tensor of \( (M, g) \). A more involved calculation gives the derivative of \( \theta_- \) along \( \tilde{I}_\lambda \) (see e.g. [2]):

\[
\overline{\xi}_\lambda \theta_- = -Q^\lambda \theta_- - K_{AB}^\lambda K^{AB} - \text{Ric}(\tilde{I}_\lambda, \tilde{I}_\lambda) + \frac{1}{2} \text{Riem}(\tilde{I}_\lambda, \tilde{I}_\lambda, \tilde{I}_\lambda, \tilde{I}_\lambda) + 2(D_A S^A + S_A S^A),
\]

(5)

where Riem is the Riemann tensor of \( (M, g) \), \( D \) denotes the Lévi-Civita-covariant derivative of \( (S, h) \) and the 1-form \( S_A \) is defined by

\[
S(X) = -\frac{1}{2} (\tilde{I}_\lambda \cdot \nabla^\lambda \tilde{I}_\lambda).
\]

### Table 1. Types of surfaces according to their null expansion(s).

| Name of the surface                  | Null expansions                                      |
|--------------------------------------|------------------------------------------------------|
| Future trapped                      | \( \theta_+ < 0 \) and \( \theta_- < 0 \)            |
| Weakly future trapped               | \( \theta_+ < 0 \) and \( \theta_- < 0 \)            |
| Marginally future trapped           | \( \theta_+ = 0 \) or \( \theta_- < 0 \) or \( \theta_+ < 0 \) and \( \theta_- = 0 \) |
| Weakly outer trapped                | \( \theta_+ < 0 \)                                   |
| Marginally outer trapped (MOTS)     | \( \theta_+ = 0 \)                                   |
| Weakly outer untrapped              | \( \theta_+ > 0 \)                                   |
| Outer untrapped                     | \( \theta_+ > 0 \)                                   |
| Generalized trapped surface         | At no point, \( \theta_+ < 0 \) and \( \theta_- < 0 \) |
| Generalized apparent horizon        | At each point, \( \theta_+ = 0 \) and \( \theta_- > 0 \) |

Future trapped

- \( \theta_+ < 0 \) \( \theta_- < 0 \)

Weakly future trapped

- \( \theta_+ < 0 \) \( \theta_- < 0 \)

Marginally future trapped

- \( \theta_+ = 0 \) \( \theta_- < 0 \) or \( \theta_+ < 0 \) \( \theta_- = 0 \)

Weakly outer trapped

- \( \theta_+ < 0 \)

Marginally outer trapped (MOTS)

- \( \theta_+ = 0 \)

Weakly outer untrapped

- \( \theta_+ > 0 \)

Outer untrapped

- \( \theta_+ > 0 \)

Generalized trapped surface

- At no point, \( \theta_+ < 0 \) \( \theta_- < 0 \)

Generalized apparent horizon

- At each point, \( \theta_+ = 0 \) \( \theta_- > 0 \)
where $\bar{X}$ is any tangent vector to $S$. The Gauss identity for $S$ as a submanifold of $(\mathcal{M}, g)$ implies

$$R(g) = R(h) - (\bar{H} \cdot \bar{H}) + \bar{K}_{AB} \bar{K}^{AB} - 2 \text{Ric}(\bar{\iota}, \bar{\iota}) + \frac{1}{2} \text{Riem}(\bar{\iota}, \bar{\iota}, \bar{\iota}, \bar{\iota}),$$

where $R(g)$ is the scalar curvature of $(\mathcal{M}, g)$. Thus, (5) can be rewritten as

$$\varepsilon_{i}^{\theta} = -Q^{\theta} + \text{Ein}(\bar{\iota}, \bar{\iota}) - (R(h) - (\bar{H} \cdot \bar{H})) + 2(\Delta S^{A} + S_{A} S^{A}),$$

where Ein is the Einstein tensor of $g$. These expressions are valid in any dimension. The expression for $\varepsilon_{i}^{\theta}$ follows from (7) by interchanging $+$ and $-$ and substituting $S_{A} \rightarrow -S_{A}$ (see formula (6)).

2.3. Surfaces embedded in a spacelike hypersurface

Codimension-2 surfaces usually arise as codimension-1 surfaces embedded in a spacelike hypersurface $\Sigma$ of the $n$-dimensional spacetime $\mathcal{M}$. The induced metric on $\Sigma$ will be denoted by $\gamma_{ij}$ (Latin, lower case indices run from 1 to $n-1$) and the second fundamental form with respect to the unit future normal $\bar{n}$ will be denoted by $A_{ij}$. The constraint equations relate the geometry of $\Sigma$ to some components of the Einstein tensor:

$$R(\gamma) = A_{ij} A^{ij} + (\text{tr}_{\gamma} A)^{2} = 16\pi \rho, \tag{8}$$

$$\nabla_{j} A_{ij}^{l} - \nabla_{i} \text{tr}_{\gamma} A = -8\pi J_{i}, \tag{9}$$

where $R(\gamma)$ is the curvature scalar of $\gamma$, $\text{tr}_{\gamma} A$ is the trace of the second fundamental form, $\nabla_{i}$ is the covariant derivative in $(\Sigma, \gamma)$, the total energy density is defined as $8\pi \rho \equiv \text{Ein}(\bar{n}, \bar{n})$ and the energy flux 1-form $J_{i}$ is defined as $8\pi J_{i}\gamma^{\nu} = -\text{Ein}(\bar{X}, \bar{n})$ on any vector $\bar{X}$ tangent to $\Sigma$. The initial data set satisfies the dominant energy condition provided $\rho \geq |J|$, where $|J|$ is the norm of $J^{i}$ with respect to the metric $\gamma_{ij}$. Note that, despite their names, $\rho$ and $J$ are defined directly in terms of the Einstein tensor. No field equations are therefore assumed either here or elsewhere in this review. The initial data set is said to be time symmetric whenever $A_{ij} = 0$.

Assume, as before, that $S$ is orientable and that a preferred unit normal $\bar{m}$ tangent to $\Sigma$ can be selected. In this situation, the null normals will always be uniquely chosen as $\bar{\iota} = \bar{n} \pm \bar{m}$. As a submanifold of $\Sigma$, $S$ has a second fundamental form $\kappa_{AB}$ with respect to $\bar{m}$. Its trace $p$ is the mean curvature of $S$ in $\Sigma$. The trace-free part of $\kappa_{AB}$ will be written as $\Pi_{AB}^{m}$. The decomposition of the second fundamental form $A_{ij}$ into tangential and normal components to $S$ will also play a role. We will denote by $A_{AB}^{m}$ the restriction of $A_{ij}$ on $S$, $\Pi_{AB}^{m}$ the trace-free part of this tensor and $q$ its trace. Then (see e.g. [101])

$$\bar{K}_{AB} = -A_{AB}^{m} \bar{n} + \kappa_{AB} m, \quad \Pi_{AB} = -\Pi_{AB}^{m} \bar{n} + \Pi_{AB}^{n} \bar{m} \quad \text{and} \quad \bar{H} = -q \bar{n} + p \bar{m},$$

which implies $\theta_{A} = \pm p + q$.

The three remaining independent components of $A_{ij}$ can be encoded in the trace $\text{tr}_{\gamma}(A)$ (or in $A_{ij} m^{i} m^{j}$) and in the normal-tangential components:

$$S_{A} \equiv A_{ij} m^{i} e_{A}^{j}. \tag{10}$$

This definition of $S_{A}$ is consistent with (6) once the choice of $\bar{\iota}$ described above is made. The general variation formulae (4) and (7) can be rewritten in this context, after a straightforward calculation which uses the constraint equation (8), as

$$\varepsilon_{i}^{\theta} = -\frac{1}{2} \Pi_{AB}^{m} \Pi_{AB}^{n} - \frac{3}{4} p^{2} - \frac{1}{2} R(\gamma) + \frac{1}{2} R(h), \tag{11}$$

$$\varepsilon_{i}^{\theta} = p A_{ij} m^{i} m^{j} - A_{AB}^{m} K_{AB} + D_{A} S^{A} + 8\pi (\bar{J} \cdot \bar{m}). \tag{12}$$
In this $2 + 1 + 1$ context, MOTS satisfy $p+q = 0$, generalized trapped surfaces satisfy $p \leq |q|$ and generalized apparent horizons $p = |q|$. Note that the latter always have non-negative mean curvature, which means that outward variations do not decrease the area. This property in fact generalizes to a global statement for the so-called outermost generalized horizons, as follows.

Consider an initial data set $(\Sigma, \gamma_{ij}, A_{ij})$ with $\Sigma$ being a compact manifold with boundary and assume that the boundary can be split into two disjoint components $\partial^- \Sigma$ and $\partial^+ \Sigma$, respectively (neither of which is necessarily connected). We denote by $\Sigma^\circ$ the interior of $\Sigma$, so that $\Sigma = \Sigma^\circ \cup \partial^- \Sigma \cup \partial^+ \Sigma$. We want to think of $\partial^- \Sigma$ as the ‘inner’ boundary, which means that we will endow it with a unit normal $\vec{m}^-$ pointing towards $\Sigma^\circ$. Similarly $\partial^+ \Sigma$ is the ‘outer’ boundary, which we endow with the unit normal $\vec{m}^+$ pointing outside $\Sigma$ (see figure 1).

In many cases, $\Sigma$ will contain plenty of weakly outer trapped surfaces. An important question then arises: is there an outermost weakly outer trapped surface $S$? Intuitively, this means a surface which encloses all others or equivalently, such that no weakly outer trapped surface can penetrate outside $S$. In order to make this precise, it is necessary to have a well-defined notion of ‘outside’ of $S$. This can be achieved by restricting the class of surfaces to those which are homologous to the outer boundary. More explicitly, a surface (recall that all surfaces are compact in this review) $S'$ is called bounding if it is contained in $\Sigma^\circ \cup \partial^- \Sigma$ and, together with $\partial^+ \Sigma$ it bounds an open domain $\Omega^\circ$. For such surfaces, the mean curvature $p$ and the outer expansion $\theta^+$ are calculated with respect to the normal pointing towards $\Omega^\circ$. A surface $S$ is outermost in some class if no other surface in the class enters the domain $\Omega$ bounded by $S$ and $\partial^+ \Sigma$. The existence of outermost MOTS in this context has been proven by Andersson and Metzger [5] assuming the inner boundary to be weakly outer trapped ($\theta^- \leq 0$) and the outer boundary to be outer untrapped ($\theta^+ > 0$), so that they play the role of barriers. The proof uses the Gauss–Bonnet theorem, so the result is valid in the $(3+1)$-dimensional setting. The precise statement is as follows.

**Theorem 1** [5]. Let $(\Sigma, \gamma_{ij}, A_{ij})$ be three dimensional and have an inner and an outer boundary as described in the previous paragraph. Assume that the inner boundary has $\theta^- (\partial^- \Sigma) \leq 0$ and the outer boundary $\theta^+ (\partial^+ \Sigma) > 0$. Then there exists a unique smooth embedded bounding MOTS $S$, i.e. $S \cup \partial^+ \Sigma = \partial \Omega$ and $\theta^+(S) = 0$, which is outermost: any other weakly outer trapped surface $S'$ which together with $\partial^+ \Sigma$ bounds a domain $\Omega'$ satisfies $\Omega \subset \Omega'$.

An important issue refers to the topology of the outermost MOTS $S$. A classic result by Hawking [80] states that the topology of each connected component of the outermost MOTS has to be either toroidal or spherical, provided the initial data set satisfies the dominant energy condition. Recently, this result has been extended to higher dimensions by Galloway and
Schoen [59], where it is proven that, leaving aside some exceptional cases, the outermost MOTS in any spacelike hypersurface of a spacetime satisfying the dominant energy condition must be of a positive Yamabe type. The exceptional cases have been ruled out by Galloway in [60], thus leaving only the positive Yamabe case. In four spacetime dimensions, this implies that each connected component of the outermost MOTS must be a sphere.

An existence result involving generalized trapped surfaces and generalized apparent horizons instead of weakly trapped surfaces and MOTS has been recently proven by Eichmair [56]. This result does not use the Gauss–Bonnet theorem, so it is not restricted to (3+1) dimensions. However, it relies on regularity of minimal surfaces, which restricts the dimension of \( \Sigma \) to be at most 7. Outermost generalized apparent horizons have one fundamental advantage over MOTS; they are area outer minimizing, i.e. they have an area less than or equal to any other bounding surface fully contained in the closure of the exterior region. This area minimizing property makes these surfaces potentially very interesting for the Penrose inequality, as first discussed by Bray and Khuri [29]. This theorem of Eichmair answers in the affirmative a conjecture due to Bray and Ilmanen (cited in [29]) on the existence of outermost generalized apparent horizons and its area outer minimizing property.

**Theorem 2** [56]. Let \((\Sigma, \gamma_{ij}, A_{ij})\) be \(m\)-dimensional, with \(3 \leq m \leq 7\) and have an inner and an outer boundary as described above. Assume that the inner boundary is a generalized trapped surface \(p \leq |q|\) and the outer boundary satisfies \(p > |q|\). Then there exists a unique \(C^2\)-embedded generalized apparent horizon \(S (p = |q|)\) which is bounding, \(S \cup \partial^+ \Sigma = \partial \Omega\), and outermost: any other generalized apparent horizon \(S'\) which together with \(\partial^+ \Sigma\) bounds a domain \(\Omega'\) satisfies \(\Omega \subset \Omega'\). Moreover, \(S\) is area outer minimizing with respect to variations in \(\Omega\).

In the purely Riemannian case \((A_{ij} = 0)\), MOTS and generalized apparent horizons are simply minimal surfaces (i.e. \(p = 0\)) and the above two theorems become statements on the existence of an outermost minimal surface on a compact domain with barrier boundaries. This particular case was already known before (see section 4 of [89] and references therein). In fact, the existence of an outermost minimal surface played an important role in the proof of the Riemannian Penrose inequality by Huisken and Ilmanen [89].

### 2.4. Hawking mass, Geroch mass and their variation formulae

Huisken and Ilmanen’s proof of the Riemannian Penrose inequality is based on a very interesting result by Geroch [61] who found a certain functional (now called the Geroch mass) which is monotonic under the so-called inverse mean curvature flows. This functional is defined for surfaces embedded in a Riemannian manifold \((\Sigma, \gamma)\). For codimension-2 surfaces embedded in a spacetime, the analogue to the Geroch mass is the so-called Hawking mass [77] (both coincide when the surface is embedded in a time-symmetric initial data set). Since these two masses have similar properties, it is conceivable that the Hawking mass may also be useful to tackle the general Penrose inequality (i.e. for non-time-symmetric initial data sets). A natural first step is to analyse whether the Hawking mass is also monotonic under suitable flows. Monotonicity along null directions was first studied by Hayward [71]. More recently, the rate of change of the Hawking mass along an IMCF in an arbitrary initial data set \((\Sigma, \gamma_{ij}, A_{ij})\) was studied in [122]. The null and initial data set approaches can be unified and extended by using a spacetime flow formulation, where the two-dimensional surface is varied in spacetime. This is discussed in detail in [28]. Let us briefly describe the main points and how previous results fit into this framework.
For any closed spacelike surface \( S \) embedded in a four-dimensional spacetime \((\mathcal{M}, g)\), the Hawking mass \([77]\) is defined by
\[
M_{\mathcal{H}}(S) = \sqrt{\frac{|S|}{16\pi}} \left( \frac{\chi(S)}{2} - \frac{1}{16\pi} \int_S \left( \tilde{H} \cdot \tilde{H} + \frac{4}{3} C \right) \eta_S \right),
\]
where \( \chi(S) \) is the Euler characteristic of \( S \). Hence, \( \chi(S) = 2 \) for a connected surface with spherical topology. For an arbitrary compact surface, \( \chi(S) = \sum 2(1 - g) \) where the sum is over the connected components and \( g \) is the genus. At this point, the parameter \( C \) in (13) is a completely arbitrary constant. Its inclusion is relevant for settings where the spacetime has a cosmological constant or when dealing with hyperbolic initial data sets in an asymptotically flat spacetime (see section 6). This constant was first introduced in \([33]\) in the time-symmetric context.

Let \( \tilde{V}^* \) denote the Hodge dual operation on the normal space \( N_pS \) (which, under our assumptions, has a Lorentzian induced metric and it is orientable, so it admits a canonical volume element once an orientation is chosen). This operation is idempotent and transforms any vector into an orthogonal vector with opposite norm. Whenever \( S \) admits a notion of ‘outer’, the orientation of the normal bundle will be chosen so that \( \tilde{l}^* = \tilde{l}^\perp \) holds (this implies \( \tilde{l}^\perp = -\tilde{l}^\perp \)).

Let us also denote by \( \nabla^\perp \) be the connection on the normal bundle (i.e. if \( \tilde{V} \) is normal to \( S \) and \( \tilde{X} \) is tangent to \( S \), \( \nabla^\perp \tilde{X} = \langle (\nabla \tilde{X}) \tilde{V} \rangle \)). For any orthogonal variation vector \( \tilde{\xi} \), the derivative of the Hawking mass along \( \tilde{\xi} \) reads as \([28]\)
\[
\frac{dM_{\mathcal{H}}(S_\lambda)}{d\lambda} \bigg|_{\lambda=0} = \frac{1}{8\pi} \sqrt{\frac{|S|}{16\pi}} \int_S \left[ (\text{Ein}(\tilde{H}^*, \tilde{\xi})^* + C(\tilde{H}^* \cdot \tilde{\xi}^*)) + 8\pi \Theta^T(\tilde{H}^*, \tilde{\xi}^*) \right.
\]
\[+ \text{tr}_S (\tilde{H} \cdot \nabla^\perp \nabla^\perp \tilde{\xi}) - \left( \frac{1}{2} R(h) - \frac{1}{4} (\tilde{H} \cdot \tilde{H}) \right) \left[ (\tilde{\xi} \cdot \tilde{H}) - a \right] \eta_S, \tag{14}\]
where \( a \) is the constant obtained by averaging \( (\tilde{\xi} \cdot \tilde{H}) \) on \( S \), i.e.
\[
a = \frac{\int_S (\tilde{\xi} \cdot \tilde{H}) \eta_S}{|S|},
\]
and \( 8\pi \Theta^T(\tilde{H}^*, \tilde{\xi}^*) \equiv (\tilde{\Pi}_{AB} \cdot \tilde{H}^*)(\tilde{\xi}^* \cdot \tilde{H}^*) - \frac{1}{2}(\tilde{\Pi}_{AB} \cdot \tilde{H}^*)^2(\tilde{\xi}^* \cdot \tilde{H}^*) \) is the ‘transverse part of the gravitational energy’. This object has good positivity properties \([28]\). In particular, it is non-negative if \( \tilde{H} \) and \( \tilde{\xi} \) are both non-timelike and have a positive inner product.

From expression (14), it is straightforward to show that \( dM_{\mathcal{H}}(S_\lambda)/d\lambda \geq 0 \) whenever \( (\text{Ein} + C g) \) satisfies the dominant energy condition (recall that a covariant tensor satisfies this property when it is non-negative when acting on any combination of causal future-directed vectors). \( \tilde{H} \) is spacelike and the variation vector takes the form
\[
\tilde{\xi} = \frac{1}{(\tilde{H} \cdot \tilde{H})}(a \tilde{H} + c \tilde{H}^*) \iff (\tilde{\xi} \cdot \tilde{H}) = a, \quad (\tilde{\xi} \cdot \tilde{H}^*) = -c, \tag{15}\]
where \(|c| \leq a \) is an arbitrary constant. Following Hayward \([71]\) in the null case, these flows have been termed uniformly expanding in \([28]\) since the mean curvature along \( \tilde{\xi} \) and its dual \( \tilde{\xi}^* \) are both constant.

The case with \( c = \pm a \) corresponds to a null variation vector and corresponds exactly to the flow studied by Hayward \([71]\). The case \( c = 0 \) can be naturally called an inverse mean curvature flow vector (because \( \xi = -\frac{1}{(\tilde{H} \cdot \tilde{H})} \tilde{H} \), and monotonicity of the Hawking mass in this case was first mentioned in \([89]\) and studied in detail by Frauendiener \([57]\). The case \(|c| < a \) corresponds to a spacelike flow vector and therefore can be rephrased as variations within an initial data set, as follows.
For a surface $S$ embedded in the initial data set $(\Sigma, \gamma_{ij}, A_{ij})$, the mean curvature vector of $S$ reads as $\vec{H} = -q\vec{n} + p\vec{m}$ and the Hawking mass takes the form

$$M_H(S) = \sqrt{\frac{|S|}{16\pi}} \left( \frac{X(S)}{2} - \frac{1}{16\pi} \int_S \left( p^2 - q^2 + \frac{4}{3} C \right) nS \right).$$

For an outwards pointing variation vector $\xi$ tangent to $\Sigma$, i.e. $\vec{\xi} = e^\psi \vec{m}$, the general variation formula (14) can be rewritten in terms of the initial data geometry as

$$\frac{dM_H(S_\lambda)}{d\lambda} \bigg|_{\lambda=0} = \frac{1}{8\pi} \sqrt{\frac{|S|}{16\pi}} \int_S \left[ e^\psi \left( p(8\pi \rho - C) + 8\pi q J_{h} m' \right) + \frac{1}{2} e^\psi \left( \Pi_{AB} \cdot \Pi_{\bar{A}\bar{B}} \right) 
+ e^\psi (S_A \cdot D^A \psi) + e^\psi q (D_A S^A) 
+ \left[ \Delta S \psi - \frac{1}{2} R(h) + \frac{1}{4} (p^2 - q^2) \right] (pe^\psi - a) \right] nS,$$

where the $\cdot$ operation acts on two tensors of the same class $X_{A_1...A_t}Y^{A_1...A_t}$, and gives $(X_{A_1...A_t}Y^{A_1...A_t}) = pX_{A_1...A_t}X^{A_1...A_t} - 2qX_{A_1...A_t}Y^{A_1...A_t} + pY_{A_1...A_t}Y^{A_1...A_t}$.

This quadratic expression is non-negative provided $|q| \leq p$ or equivalently when $\theta_+ \geq 0$, $\theta_- \leq 0$ at each point, which is an ‘untrappedness’ condition (note that this condition is basically the complementary at each point of the generalized trapped surface condition). In the particular case of $C = 0$ and IMCF ($pe^\psi = a$), the variation formula (16) was obtained in [122].

The first two terms in (16) are non-negative provided $8\pi |\vec{J}| \leq 8\pi \rho - C$ holds (this is automatically true if Ein + $C_g$ satisfies the dominant energy condition) and $|q| \leq p$. The last two terms have no sign in general. However, the factor in round brackets on each of them integrates to zero. Thus, monotonicity can be ensured [28] provided the following conditions hold simultaneously: (i) $qe^\psi = c$ or $S^A$ is divergence free and (ii) $\Delta S \psi - \frac{1}{2} R(h) + \frac{1}{4} (p^2 - q^2) = a$ or $e^\psi p = a$, where $c$ and $a$ are constants. This leads to four different alternatives for which the Hawking mass is monotonic. The uniformly expanding flow condition (15) with $|a| < a$ corresponds to the case $青海^\psi = c$ and $pe^\psi = a$. This follows easily from the expressions $\vec{H} = -q\vec{n} + p\vec{m}$ and its dual $\vec{H}^* = -q\vec{m} + p\vec{n}$. How restrictive is any of these four alternatives on a given spacetime remains an open problem.

While the Hawking mass is intrinsically a functional on surfaces in spacetime (although it obviously admits a rewriting in (3+1) language), the Geroch mass is directly a functional on surfaces in a spacelike slice $(\Sigma, \gamma)$. Its definition is

$$M_G(S) = \sqrt{\frac{|S|}{16\pi}} \left( \frac{X(S)}{2} - \frac{1}{16\pi} \int_S \left( p^2 + \frac{4}{3} C \right) nS \right).$$

The Geroch mass is fully insensitive to the second fundamental form on the slice (in this sense, it is a purely Riemannian object) and always satisfies $M_G(S) \leq M_H(S)$. Its variation can be obtained formally from expression (16) simply by putting all terms depending on the second fundamental form equal to zero. In particular, this requires substituting $8\pi \rho \to 1/2R(\gamma)$, which follows from (8) when the second fundamental form vanishes. Explicitly,

$$\frac{dM_G(S_\lambda)}{d\lambda} \bigg|_{\lambda=0} = \frac{1}{8\pi} \sqrt{\frac{|S|}{16\pi}} \int_S \left[ e^\psi p \left( \frac{1}{2} R(\gamma) - C \right) + \frac{1}{2} e^\psi p \Pi_{AB} \Pi_{\bar{A}\bar{B}} + e^\psi p D_A \psi D^A \psi 
+ \left[ \Delta S \psi - \frac{1}{2} R(h) + \frac{p^2}{4} \right] (pe^\psi - a) \right] nS.$$

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We emphasize that this result is valid for any spacelike hypersurface of the spacetime, not only for time-symmetric ones. For vanishing $C$, this expression was first obtained by Geroch [61]. For arbitrary $C$ and an imposing IMCF, this expression was derived in [33].

2.5. Asymptotic flatness

An initial data set $(\Sigma, \gamma_{ij}, A_{ij})$ is asymptotically Euclidean provided $\Sigma$ is the disjoint union of a compact set $K$ and a finite union of asymptotic ends $\Sigma^\infty_i$, each of which is diffeomorphic to $\mathbb{R}^3 \setminus B$, where $B$ is a closed ball (we restrict to $(3+1)$ dimensions for definiteness). Moreover, in Cartesian coordinates $x^i$ in $\Sigma^\infty_i$ induced by this diffeomorphism, the metric and second fundamental forms have the following asymptotic behaviour:

$$\gamma_{ij} = \delta_{ij} + O\left(\frac{1}{r}\right), \quad \partial \gamma_{ij} = O\left(\frac{1}{r^2}\right), \quad \partial^2 \gamma_{ij} = O\left(\frac{1}{r^3}\right),$$

$$A_{ij} = O\left(\frac{1}{r^2}\right), \quad \partial A_{ij} = O\left(\frac{1}{r^3}\right),$$

where $r = \sqrt{\delta_{ij}x^ix^j}$. The Penrose inequality involves the total mass of a spacetime. In the asymptotically Euclidean setting, the total ADM energy–momentum vector [6] is defined through the coordinate expressions

$$E_{ADM} = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (\partial_j \gamma_{ij} - \partial_i \gamma_{jj}) \, dS^i,$$

$$P_{ADM} = \lim_{r \to \infty} \frac{1}{8\pi} \int_{S_r} (A_{ij} - \gamma_{ij} \nabla_r A) \, dS^i,$$

where $S_r$ is the surface at constant $r$ and $dS^i = n^i \, dS$ with $\vec{n}$ being the outward unit normal to $S_r$ and $dS$ being the surface element. This definition depends a priori on the choice of coordinates $x^i$. However, they can be shown to define geometric quantities on $(\Sigma, \gamma_{ij}, A_{ij})$ provided the constraints satisfy the additional decay properties

$$R(\gamma) = O\left(\frac{1}{r^4}\right), \quad \nabla_i A_j^i - \nabla_j A = O\left(\frac{1}{r^3}\right).$$

In fact, the ADM energy–momentum is shown in [19], generalizing results in [14], to be well defined under much weaker conditions, where the metric and second fundamental forms belong to appropriate weighted Sobolev spaces involving just two derivatives for $\gamma$ and one for $A_{ij}$, and the fact that the left-hand sides of (20) are integrable on $\Sigma$. The total ADM mass of an asymptotically Euclidean initial data set is defined as $M_{ADM} = \sqrt{E_{ADM}^2 - \delta^{ij} P_{ADM}^i P_{ADM}^j}$.

The positive mass theorem of Schoen and Yau [141, 143] states that provided the initial data satisfy the dominant energy condition, $\rho \geq |\mathcal{J}|$, then $M_{ADM}$ is real and in fact strictly positive except whenever $(\Sigma, \gamma_{ij}, A_{ij})$ corresponds to a slice of the Minkowski spacetime. So far, this theorem has been proven in any spacetime dimension up to $n = 8$. Furthermore, it also holds for spin manifolds of any dimension, for which Witten’s spinorial proof [132, 168] of the positive mass theorem can be applied.

3. Formulations of the Penrose inequality

The heuristic derivation of the Penrose inequality for a surface $S$ relies on two fundamental assumptions: first, the ‘establishment viewpoint’ of gravitational collapse holds and, second, $S$ is known to lie behind the event horizon in any situation where a black hole does indeed
form. This second aspect makes it useful to consider the class of weakly outer trapped surfaces \( S \) in a given asymptotically Euclidean slice. In order to have a useful notion of outer direction in this case, one restricts the surfaces to be boundaries of domains \( D^{+} \) containing the asymptotically Euclidean end (if there is more than one end, one should be selected from the outset). The outer normal is chosen to point inside this domain. Such surfaces will be called weakly outer trapped boundaries in the following. Given two surfaces \( S_{1} \) and \( S_{2} \) (not necessarily weakly outer trapped) that respectively bound exterior domains \( D_{1}^{+} \) and \( D_{2}^{+} \), we shall say that \( S_{2} \) encloses \( S_{1} \) provided \( D_{1}^{+} \) contains \( D_{2}^{+} \). The complementary of the outer domain \( D^{+} \) in \( \Sigma \) is called an ‘interior region of \( S \)’ and denoted by \( D^{-} \). It is also useful to define the outer trapped region \( T_{S}^{+} \) in \( \Sigma \) to be the union of interior regions of all weakly outer trapped boundaries. An important fact is that in strongly asymptotically predictable spacetimes satisfying the null convergence condition (i.e. \( \text{Ric}(l, l) \geq 0 \) for each null vector \( l \)), the interior region of any weakly outer trapped boundary is contained in the black hole region \( M\setminus \mathcal{J}^{-}(\mathcal{I}^{\infty}) \) (see proposition 12.2.4 in [161] and theorem 6.1 in [45]). Consequently, the same holds for \( T_{S}^{+} \). Thus, this type of surfaces is good candidates for a Penrose inequality. As mentioned in section 1, the area of the event horizon cut \( \mathcal{H}_{\Sigma} = \mathcal{H} \cap \Sigma \) may be smaller than the area of any such surface \( S \). However, for spacetime dimensions \( n \leq 8 \), any surface \( S \) that bounds an exterior domain always has a minimal area enclosure, i.e. the outermost of all surfaces which enclose \( S \) and have an area less than or equal to any other surface enclosing \( S \). We will denote by \( A_{\text{min}}(S) \) the area of the minimal surface enclosure of \( S \). For a weakly outer trapped boundary in a black hole spacetime it follows that \( A_{\text{min}}(S) \leq |\mathcal{H}_{\Sigma}| \), because \( \mathcal{H}_{\Sigma} \) encloses \( S \). Hence, the heuristic argument by Penrose implies \( M_{\text{ADM}} \geq \sqrt{A_{\text{min}}(S)/16\pi} \) where \( S \) is any weakly outer trapped boundary. Therefore, it also follows that

\[
M_{\text{ADM}} \geq \sup_{S} \sqrt{\frac{A_{\text{min}}(S)}{16\pi}},
\]

where the supremum is taken with respect to all weakly outer trapped boundaries \( S \). This inequality can be rewritten in a much simpler way as follows.

Since a sufficiently large coordinate sphere \( S_{r} \) in the asymptotically Euclidean region has positive outer expansion, theorem 1 can be applied [5] to conclude that \( \partial T_{S}^{+} \) is a smooth marginally outer trapped boundary, which by construction encloses all other weakly outer trapped boundaries. It immediately follows that \( A_{\text{min}}(\partial T_{S}^{+}) \geq A_{\text{min}}(S) \) for any weakly outer trapped boundary \( S \), and hence also \( A_{\text{min}}(\partial T_{S}^{+}) \geq \sup_{S} A_{\text{min}}(S) \). On the other hand, \( \partial T_{S}^{+} \) is itself a weakly outer trapped boundary and hence \( A_{\text{min}}(\partial T_{S}^{+}) \leq \sup_{S} A_{\text{min}}(S) \). Thus, they necessarily coincide and the Penrose inequality (21) can be written in the simpler form

\[
M_{\text{ADM}} \geq \sqrt{\frac{A_{\text{min}}(\partial T_{S}^{+})}{16\pi}}.
\]

Although the Penrose inequality, seen as a consequence of cosmic censorship, should be expected to hold only for the minimum area enclosure of \( \partial T_{S}^{+} \), one can often find in the literature a version of the Penrose inequality involving the area of the outermost MOTS:

\[
M_{\text{ADM}} \geq \sqrt{\frac{|\partial T_{\Sigma}^{+}|}{16\pi}}.
\]

This is of course a stronger and simpler looking inequality. However, it is presently known not to be true. A counterexample has been found by Ben-Dov [20] by considering a spherically symmetric spacetime composed of four regions. The innermost region is a portion of a dust-filled closed FLRW spacetime. It is followed by a portion of the Kruskal spacetime of mass \( M \) which is then joined to another region made of dust, closed FLRW in such a
way that the outermost region, which is again vacuum, has less mass than the intermediate Kruskal region. In this context, there exists a slice for which the outermost MOTS lies in the intermediate Kruskal portion, which directly leads to a violation of (23). This example is also a counterexample to a version of the Penrose inequality suggested by Penrose himself in [137], which goes as follows. Suppose that an asymptotically Euclidean initial data set contains a weakly future trapped boundary \( S \) and the spacetime evolves according to weak cosmic censorship. Then \( S \) is contained in the black hole region that forms. Consider the null hypersurface \( \mathcal{N} \) consisting of past-directed null geodesics emanating orthogonally from \( S \) towards the outer direction, and let \( \theta_k \) be the corresponding null tangent vector. Since \( S \) is weakly future trapped, we have \( \theta_k \geq 0 \) and the area of \( S \) does not decrease initially along \( \mathcal{N} \), and this remains to be true as long as \( \theta_k \) stays non-negative along the null hypersurface. Since the collapse takes place in the future and \( \mathcal{N} \) extends towards the past and exterior region, one expects from physical grounds that \( \theta_k \) does not change sign (this is because \( \mathcal{N} \) approaches weaker gravitational fields and hence outer past-directed expansions should be positive). Under these circumstances, the intersection of \( \mathcal{N} \) with the event horizon has at least the same area as \( S \), and hence the usual heuristic argument implies \( M_{\text{ADM}} \geq \sqrt{|S|/16\pi} \). The counterexample by Ben-Dov shows that this inequality is not generally true either. The difficulty with the argument is double. First, the assumption that \( \theta_k \) never becomes negative along \( \mathcal{N} \) need not be true (note that such a property is not implied by the Raychaudhuri equation (4)). In the example by Ben-Dov, this property is not fulfilled if the starting surface \( S \) is any of the spherically symmetric future-trapped surfaces lying either in the intermediate Kruskal region or in the innermost closed dust FLRW region (see figure 6 in [20]). Second, the above argument also relies on the implicit assumption that \( \mathcal{N} \) intersects the black hole event horizon. In the example in [20] these hypersurfaces end up in the white hole singularity and do not intersect the event horizon anywhere.

Weakly outer trapped surfaces on a spacelike hypersurface satisfy \( \theta_\pm = p + q \leq 0 \) and the outermost such surface \( \partial T_\Sigma^+ \) is a MOTS (i.e. \( \theta_\pm \equiv p + q = 0 \)). Given an initial data set \((\Sigma, \gamma_{ij}, A_{ij})\), a new one \((\Sigma, \gamma'_{ij}, A'_{ij})\) can be obtained simply by changing the time orientation, i.e. by setting \( \gamma'_{ij} = \gamma_{ij} \) and \( A'_{ij} = -A_{ij} \). The heuristic argument to prove the Penrose inequality obviously applies equally well to this new initial data set. For surfaces which are boundaries of exterior domains, the future outer null direction is now \( \vec{l}'_\pm = -\vec{l}_\pm \), and therefore the weakly outer trapped surfaces satisfy \( \theta'_\pm = -\theta_\pm \leq 0 \), or equivalently \( p - q \leq 0 \) (this is, in fact, obvious since the change in time orientation changes \( q \rightarrow -q \)). Similarly as before, one can construct the past trapped region \( \partial T^-\Sigma \) as the union of interiors of all such boundaries. Andersson and Metzger’s result (theorem 1) again implies the existence of a unique outermost surface \( \partial T^-\Sigma \) which satisfies \( p - q = 0 \) (i.e. a past marginally outer trapped surface). Thus, Penrose’s heuristic argument also supports the inequality

\[
M_{\text{ADM}} \geq \sqrt{\frac{A_{\text{min}}(\partial T^-\Sigma)}{16\pi}}. \tag{24}
\]

In a given initial data set, neither \( T_\Sigma^+ \) contains \( T^-\Sigma \) nor vice versa, in general. Although not supported by any heuristic argument, a version of the Penrose inequality along the lines of (23) which has also been proposed is

\[
M_{\text{ADM}} \geq \sqrt{\frac{|\partial(T_\Sigma^+ \cup T^-\Sigma)|}{16\pi}}. \tag{25}
\]

Since \( T_\Sigma^+ \cup T^-\Sigma \) contains both trapped sets, the area of its boundary is not smaller than the minimal area enclosures \( A_{\text{min}}(\partial T_\Sigma^+) \) or \( A_{\text{min}}(\partial T^-\Sigma) \). Thus, (25) is a stronger inequality.
than (22) or (24). So far, no counterexample to (25) has been found and it is not clear whether the inequality should hold or not. In the time-symmetric case, this inequality reduces to either (22) or (24) and its validity has been proven in general, as discussed in section 5. Inequality (25) also reduces to either (22) or (24) in any situation where \( q \) is known to be identically zero on the outermost future and past MOTS as well as on their minimal area enclosures. An example of this behaviour is given by a class of axially symmetric and conformally flat initial data sets presented in [52]. For a subclass thereof, the validity of inequality (25) has been verified numerically in [97].

In the spherically symmetric case, both \( T^+_{\Sigma_1} \) and \( T^-_{\Sigma_1} \) are obviously spherically symmetric and one of them necessarily contains the other. The Penrose inequality (25) is known to be true in that case, as we discuss in the following section. Since the example by Ben-Dov deals with a spherically symmetric spacetime and a spherically symmetric slice, it cannot provide a counterexample to (25). On the other hand, very few explicit examples are known where (25) has been verified in situations where it does not reduce to either (22) or (24). The only cases outside spherical symmetry that I am aware of involve a numerical analysis of three special classes of initial data sets [104]. In all these examples, inequality (25) was numerically confirmed.

Still another version of the Penrose inequality has been very recently proposed by Bray and Khuri. The fundamental idea behind this proposal is to use generalized trapped surfaces instead of weakly outer trapped surfaces. Recall that, on an initial data set, a surface that bounds an exterior domain is a generalized trapped surface provided \( p \leq |q| \). This class of surfaces has two main advantages over weakly outer trapped surfaces. First, its definition is insensitive to time reversals, so that one can get rid of the complications of dealing with several sets, such as \( T^+_{\Sigma_1} \), \( T^-_{\Sigma_1} \) or their union. There is only one generalized trapped set \( T_{\Sigma_1} \) (in a given initial data set) and this contains both \( T^+_{\Sigma_1} \) and \( T^-_{\Sigma_1} \) (and, obviously, its union). Moreover, the boundary \( \partial T_{\Sigma_1} \) is a smooth generalized apparent horizon, due to Eichmair’s result (theorem 2). The second advantage is that \( \partial T_{\Sigma_1} \) is area outer minimizing, i.e. \( |\partial T_{\Sigma_1}| = A_{\text{min}}(\partial T_{\Sigma_1}) \). Consequently, a Penrose inequality involving generalized trapped surfaces can be stated in a much simpler form in terms of the area of \( \partial T_{\Sigma_1} \), without the need of invoking minimal area enclosures. The Penrose inequality proposed by Bray and Khuri thus reads as

\[
M_{\text{ADM}} \geq \sqrt{\frac{|\partial T_{\Sigma_1}|}{16\pi}}.
\]

This version is automatically stronger than (22) or (24) because \( T_{\Sigma_1} \) encloses the other trapped sets. Arguments in favour of this version will be given below, where we discuss Bray and Khuri’s approach in more detail. We should emphasize, however, that the heuristic argument by Penrose does not support this version. The reason is that one of the key ingredients for the heuristics to go through is that the surfaces under consideration lie behind the event horizon provided a black hole does indeed form. However, this property is not generally true for generalized trapped surfaces. An example can be constructed as follows. Consider a Cauchy slice \( \Sigma \) of the Kruskal spacetime. The portion of \( \Sigma \) lying in the domain of outer communications (i.e. in the exterior Schwarzschild region) has a boundary which necessarily lies in the union of the black hole event horizon, the bifurcation surface and the white hole event horizons. Select \( \Sigma \) such that this boundary has a non-empty intersection with both the black hole and the white hole event horizons. Since we consider the Kruskal spacetime, the intersection of \( \Sigma \) with the black hole event horizon is a smooth MOTS and the intersection with the white hole event horizon is a smooth past MOTS, it follows that the outermost generalized apparent horizon \( \partial T_{\Sigma} \) must contain both of them. However, \( \Sigma \) can be easily chosen so that the former intersects transversally (see figure 2). Consequently, some portion of \( \partial T_{\Sigma} \) (which
must be $C^3$) must lie strictly inside the domain of outer communications, i.e. outside the black hole region.

4. The Penrose inequality in spherical symmetry

The Penrose inequality has been proven to hold under the assumption of spherical symmetry, i.e. when $(\Sigma, \gamma_{ij}, A_{ij})$ is invariant under an $SO(3)$ action with (generically) $S^2$ orbits. This inequality was first established by Malec and Ó Murchadha [120] assuming that the initial data are maximal, i.e. $\text{tr}_r A = 0$ and in full generality by Hayward [72]. In either case, the inequality bounds the ADM energy (not the ADM mass) in terms of the area of the most external among the two surfaces between $\partial T_+^{\Sigma}$ (the outermost MOTS) and $\partial T_+^{\Sigma}$ (the outermost past MOTS).

Both arguments are ultimately based on properties of the Misner–Sharp quasi-local energy [129], although this is made fully explicit only in [72]. This quasi-local energy is exactly the Hawking mass specialized to spherical symmetry. Thus, the inequality in this case can be easily derived from the general expressions in subsection 2.4.

For a spherically symmetric asymptotically Euclidean initial data set, the outer trapped region $T_+^{\Sigma}$, being a geometrically defined set, is necessarily spherically symmetric. Its boundary $\partial T_+^{\Sigma}$ is therefore a metric sphere (if non-empty, which we assume from now on). Now, two different possibilities arise. The region outside $\partial T_+^{\Sigma}$ either contains minimal spheres (case (i)) or does not (case (ii)). In case (i), there is an outermost minimal surface $S_m$ lying outside $\partial T_+^{\Sigma}$. This surface is necessarily a 2-sphere and therefore encloses $\partial T_+^{\Sigma}$. Let us consider the region outside $S_m$ in case (i) and the region outside $\partial T_+^{\Sigma}$ in case (ii). We denote this region by $\Sigma_{\text{ext}}$ and the corresponding first and second fundamental forms by $\gamma_{\text{ext}}$ and $A_{\text{ext}}$, respectively. This region is free of minimal spheres, so that the metric can be written as $\gamma_{\text{ext}} = \frac{dr^2}{1 - 2m(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$, with $2m(r) > r$. At infinity, $\lim_{r \to \infty} m(r) = E_{\text{ADM}}$. From spherical symmetry, the second fundamental form can be written as $A_{\text{ext}} = W(r)dr^2 + Z(r)(d\theta^2 + \sin^2 \theta d\phi^2)$.

A direct calculation shows that the mean curvature $p$ and the trace $q$ of the second fundamental forms on the surfaces $\{r = \text{const}\}$ are $p = \frac{2}{r} \sqrt{1 - \frac{2m}{r}} > 0$, $q = 2Z/r^2$. 

Figure 2. Spacelike hypersurface $\Sigma$ in the Kruskal spacetime which intersects both the black hole and white hole event horizons in such a way that the two surfaces defined by these intersections meet transversally. In the figure, only the portion of $\Sigma$ lying outside the black hole and the white hole regions is shown. However, $\Sigma$ is a Cauchy hypersurface, so the intersection with the black hole event horizon is a compact surface without a boundary, and the same holds for the intersection with the white hole event horizon. Since both surfaces are generalized trapped surfaces, the boundary $\partial T_+^{\Sigma}$ must enter the shaded region somewhere.
Consequently, the Hawking mass (13) (with \( C = 0 \)) of these spheres reads as \( M_H(r) = m(r) + Z^2/(2r) \). Its radial derivative can be immediately obtained from (16), using the fact that \( \partial_t = (1 - 2m(r)/r)^{-1/2} \dot{m} \), which implies \( e^\psi p = 2/r \). This gives

\[
\frac{dM_H}{dr} = 4\pi r^2 \left( \rho + \frac{Z}{r} J_r \right),
\]

where \( J_r = (\vec{J} \cdot \hat{r}) \). The dominant energy condition \( \rho \geq |\vec{J}| \) becomes in this case \( \rho \geq |J_r| \sqrt{1 - 2m(r)/r} \). Furthermore, asymptotically Euclidean demands \( W(r) = O(1/r^3) \), \( Z(r) = O(1) \) so that \( \lim_{r \to \infty} M_H(r) = E_{\text{ADM}} \).

Let us first deal with case (i): since \( S_m \) is a minimal surface and cannot be weakly outer trapped (it lies outside \( \partial T^+ \)), it must have \( q > 0 \) and hence it is past weakly outer trapped (\( \theta_- > 0 \)). The spheres in the asymptotic region have \( \theta_- = q - p < 0 \), and consequently there must exist an outermost sphere \( \partial T^- \) with vanishing \( \theta_- \) (i.e., an outermost past MOTS). In the exterior of \( \partial T^- \), we have \( \theta_+ = p + q > 0 \) and \( \theta_- = -p + q < 0 \). This implies \( |q| < p \) and thus the bound \( |Zr^{-1}(1 - 2m/r)^{-1/2}| < 1 \) outside \( \partial T^- \). This, together with the dominant energy condition, implies that \( M_H(r) \) is non-decreasing outside \( \partial T^- \) (this is just a particular case of the monotonicity properties of \( M_H \) discussed above). Being \( \partial T^- \) a past MOTS, we have \( M_H(\partial T^-) = \frac{\gamma}{2} = \sqrt{|\partial T^-|/16\pi} \). The monotonicity of \( M_H(r) \) from this surface to infinity and the fact that \( S_m \) is area outer minimizing establish the Penrose inequality:

\[
E_{\text{ADM}} \geq \sqrt{\frac{|\partial T^-|}{16\pi}} \geq \sqrt{\frac{A_{\text{min}}(\partial T^-)}{16\pi}}
\]

for case (i). For case (ii), we have that \( \partial T^+ \) is automatically area outer minimizing and a similar argument applies: if there is an outermost past MOTS \( \partial T^- \) outside \( \partial T^+ \), apply the monotonicity of \( M_H \) from \( \partial T^- \) to infinity. If there is none, apply monotonicity of \( M_H \) from \( \partial T^+ \) to infinity. In either case, one concludes (since \( \partial T^+ \) is area outer minimizing) that

\[
E_{\text{ADM}} \geq \sqrt{\frac{|\partial T^+|}{16\pi}}.
\]

Note that the Penrose inequality does not state (27) in case (i). There, the minimum area needed to enclose \( \partial T^+ \) must be used. This agrees with the discussion in section 3. As already mentioned there, Ben-Dov [20] has found an explicit example in spherical symmetry where inequality (27) is violated. On the other hand, the above argument in fact proves the Penrose inequality for the outermost of the two surfaces \( \partial T^+ \) and \( \partial T^- \) and thus also for \( \partial (T^+ \cup T^-) \), due to spherical symmetry.

Also note that the spherically symmetric Penrose inequality above involves the total energy of the slice. This is weaker than the expected Penrose inequality in terms of the total ADM mass. If the slice \((\Sigma, \gamma_{ij}, A_i)\) is such that one can generate a piece of spacetime which admits another slice with vanishing total momentum, then the Penrose inequality in terms of the ADM mass also follows. However, the existence of this piece of spacetime is not always obvious. In any case, it would be of interest to find a proof of the Penrose inequality in terms of the total ADM mass in spherical symmetry directly in terms of the given data. This might give some new clues on how the general Penrose inequality can be addressed.

5. Riemannian Penrose inequality

As already mentioned, the field has experienced a fundamental breakthrough in the last decade or so with the complete proof of the Penrose inequality in the time-symmetric case, first by
Huisken and Ilmanen [89] for a connected horizon and then by Bray [22] for an arbitrary horizon. Both papers dealt with the four-dimensional case. However, while Huisken and Ilmanen’s proof is very specific to four dimensions (because of the use of the Geroch mass), Bray’s approach can be generalized to any spacetime dimension not bigger than 8, as recently shown by Bray and Lee [27].

By definition, an initial data set is called time symmetric whenever $A_{ij} = 0$. This has two immediate consequences, namely that the ADM 3-momentum vanishes identically, so that $M_{\text{ADM}} = E_{\text{ADM}}$, and that only one constraint equation remains,

$$R(\gamma) = 16\pi \rho,$$

which gives $R(\gamma) \geq 0$ provided the dominant energy condition holds. In fact, the weak energy condition (defined as $\text{Ein}(\ddot{u}, \ddot{u}) \geq 0$ for all causal vectors) suffices in this case.

Another immediate consequence is that $\theta_+ = -\theta_- = p$ and, hence, the trapped region $T^+_\Sigma$, the past trapped region $T^-_\Sigma$ and the generalized trapped region $T_\Sigma$ all coincide in this case. Its boundary $S_m$ is the outermost minimal surface, which is non-empty as soon as there is a bounding surface with negative mean curvature (with respect to the normal pointing towards the chosen asymptotically Euclidean end). This is a corollary of theorems 1 and 2. However, in this context this result is known to hold even in more generality. As discussed in [89], following classic results on minimal surfaces [127], it is sufficient to define the trapped region $K$ as the image of all immersed minimal surfaces in $\Sigma$ together with the bounded connected components of its complementary. It follows that the boundary of this set is a collection of smooth, embedded, minimal 2-spheres and that any connected component of $\Sigma \setminus K$ is an ‘exterior’ region, i.e. an asymptotically Euclidean manifold free of minimal surfaces (even immersed) and with a compact and minimal boundary composed of a finite union of 2-spheres. The Penrose inequality therefore becomes an inequality relating the total mass and the area of the outermost minimal surface $S_m$ in $(\Sigma, \gamma)$ with respect to the chosen asymptotically Euclidean end. The corresponding inequality

$$M_{\text{ADM}} \geq \sqrt{\frac{|S_m|}{16\pi}}$$

is usually termed the ‘Riemannian Penrose inequality’ since it directly involves Riemannian manifolds, with no further structure coming from the ambient Lorentzian manifold.

As already mentioned, the Penrose inequality has a rigidity part, namely that equality is achieved only for slices of the Kruskal extension of the Schwarzschild metric. The time-symmetric slices of this spacetime define a manifold $(\Sigma_{\text{Sch}} = (\mathbb{R}^3 \setminus 0, \gamma_{\text{Sch}}))$ with induced metric

$$\gamma_{\text{Sch}} = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2).$$

The surface $r = m/2$ is minimal and separates the manifold into two isometric pieces (corresponding to the two asymptotic ends of the Kruskal metric). The rigidity part of the Riemannian Penrose inequality states that if equality is achieved in (28), then the region in $(\Sigma, \gamma)$ outside its outermost minimal surface $S_m$ is isometric to the domain $r > m/2$ of $(\Sigma_{\text{Sch}}, \gamma_{\text{Sch}})$ with $m = M_{\text{ADM}}$.

Before discussing the breakthroughs of Huisken and Ilmanen, and Bray, let us discuss some previous attempts to address the Riemannian Penrose inequality.

### 5.1. Spinor methods

Shortly after Schoen and Yau proved the positive mass theorem, Witten [168] proposed a completely different method using spinors (see [132] for a rigorous version of Witten’s ideas).
It is natural to ask whether spinorial techniques can be also applied to the Penrose inequality. After all, the spinor techniques had been successfully extended to prove the positive mass theorem in the presence of black holes (more precisely, marginally trapped surfaces) in the initial data [64] (see [82] for a rigorous proof). The main difficulty lies in finding suitable boundary conditions on Witten’s equation on the boundary of the black holes so that the boundary term arising by integrating the Schrödinger–Lichnerowicz [111, 146] identity can be related to the area of the black hole. An interesting attempt to achieve this in the Riemannian case is due to Herzlich [81], who obtained a Penrose-like inequality involving not just the total mass and area of the minimal surface but also a Sobolev-type constant of the manifold. Depending on the space under consideration, Herzlich’s inequality may turn out to be stronger or weaker than the Penrose inequality (see [121] for a limiting case where this inequality reduces simply to a positive mass statement). Nevertheless, the inequality is still optimal in the sense that equality is achieved only for the Schwarzschild manifold.

The class of manifolds considered in [81] consists of asymptotically Euclidean three-dimensional, orientable Riemannian manifolds \((\Sigma, \gamma)\) having an inner boundary \(\partial \Sigma\) which is topologically an \(S^2\) and geometrically a minimal surface. No assumption is made on whether this surface is the outermost minimal surface in \((\Sigma, \gamma)\) or not. This already indicates that the inequality to be proven cannot be the standard Penrose inequality because minimal surfaces with a large area can be shielded from an asymptotically Euclidean region with small mass by an outermost minimal surface with a sufficiently small area, while maintaining non-negative Ricci curvature everywhere. This is often called a ‘shielding effect’, and explicit examples are easily constructed by cutting the Schwarzschild manifold (29) on a sphere at \(r < m/2\) and attaching a three-sphere of large radius with a cap near the north pole removed (this manifold belongs to the class of initial data leading to the so-called Oppenheimer–Snyder spherical dust collapse [131]).

The inequality proven by Herzlich reads as [81]

\[
M_{\text{ADM}} \geq \frac{\sigma}{2(1 + \sigma)} \sqrt{\frac{[\partial \Sigma]}{\pi}},
\]

where \(\sigma\) is a geometric, scale-invariant, quantity on \((\Sigma, \gamma)\) defined as

\[
\sigma = \sqrt{\frac{[\partial \Sigma]}{\pi}} \inf_{f \in C_\infty^\infty, f \neq 0} \frac{\int_{\Sigma} (df, df) \gamma}{\int_{\partial \Sigma} f^2 \eta_{\partial \Sigma}},
\]

where \(C_\infty^\infty\) denotes, as usual, the collection of smooth functions with compact support. Equality in (30) occurs if and only if \((\Sigma, \gamma)\) is isometric to the exterior of the Schwarzschild manifold (29) outside the minimal surface \(r = m/2\).

The proof is based on an improvement of the positive mass theorem (also proven in [81]) valid for asymptotically Euclidean Riemannian manifolds \((\Sigma, \bar{\gamma})\) of non-negative scalar curvature having an inner boundary \(\partial \Sigma\) which is topologically a sphere and which may have positive, but not too large, mean curvature (with respect to the normal pointing towards infinity). More precisely, the mean curvature \(p\) must satisfy the upper bound

\[
p \leq 4 \sqrt{\frac{\pi}{[\partial \Sigma]}}.
\]

This positive mass theorem also has a rigidity part that states that equality is achieved if and only if \((\Sigma, \bar{\gamma})\) is the exterior of a ball in the Euclidean space (this implies, in particular, that the only surface \(S\) in \(\mathbb{R}^3\) which is topologically an \(S^2\) and which satisfies \(p \leq 4 \sqrt{\pi/[\Sigma]}\) is in fact a sphere). The proof of this theorem involves finding appropriate boundary conditions for the Witten spinor on \(\partial \Sigma\) so that the boundary term in the Schrödinger–Lichnerowicz identity on the inner boundary gives a non-negative contribution.
The Penrose-like inequality (30) is then proven by finding a conformally rescaled metric $\tilde{g} = f^4 g$, with $f \to 1$ at infinity in such a way that $\tilde{g}$ has a vanishing scalar curvature and the mean curvature of $\partial \Sigma$ with respect to the metric $\tilde{g}$ saturates inequality (31). Moreover, the conformal rescaling is shown to decrease the mass at least by the amount given on the right-hand side of (30). Here is where the quantity $\sigma$ arises. Since the mass after the rescaling is non-negative due to the positive mass theorem above, the Penrose-like inequality follows.

The rigidity part holds because, in the case of equality in (30), $(\Sigma, g)$ is conformal to the flat metric outside a ball and, moreover, its curvature scalar vanishes (if this was non-zero, then the decrease in mass due to the conformal rescaling would be larger than the right-hand side of (30), which cannot occur in the equality case). Since the only conformally flat, scalar flat and asymptotically Euclidean Riemannian manifold with a minimal surface is the Schwarzschild space (29), the rigidity part follows.

This Penrose-like inequality has been generalized in three different ways. First, Herzlich [84] extended the results to spin manifolds of arbitrary dimension $n$. In this case, the boundary $\partial \Sigma$ is assumed to be a compact and connected minimal surface of a positive Yamabe type (i.e. such that it admits a metric of positive constant scalar curvature $R_0$). Denoting by $\mathcal{Y}$ the Yamabe constant (i.e. $\mathcal{Y} = R_0 |\partial \Sigma|^\frac{2}{n-2}$, where $|\partial \Sigma|$ is the $(n-1)$-dimensional volume of the boundary), the Penrose-like inequality reads as

$$M_{\text{ADM}} \geq \frac{1}{8\pi} \sqrt{\mathcal{Y}(n-1) \frac{\sigma}{n-2} |\partial \Sigma|^\frac{2}{n-1}-\frac{1}{\sigma+1} |\partial \Sigma|^\frac{2}{n-1}},$$

(32)

where $\sigma$ is the analogous scale-invariant quantity in higher dimensions:

$$\sigma = \sqrt{\frac{4(n-1)}{(n-2)\mathcal{Y}}} |\partial \Sigma|^{\frac{1}{n-1}} \inf_{f \in C^\infty_0, f \neq 0} \int_{\partial \Sigma} (df, df)_\gamma \eta_\gamma.$$

The idea of the proof is similar to the three-dimensional case.

The second generalization [118] involves maximal initial data sets $(\Sigma, \gamma_{ij}, A_{ij})$. Although this result is not a Riemannian Penrose inequality, the methods used are very similar to the previous ones and it is therefore natural to include it here. The idea of the proof involves, again, a positive mass theorem for manifolds with a suitable boundary and finding an appropriate conformal factor which transforms the data so that the previous mass theorem can be applied while decreasing the mass by a certain amount.

More precisely, the class of manifolds $(\Sigma, \gamma_{ij}, A_{ij})$ under consideration is asymptotically Euclidean spin manifolds of arbitrary dimension $n$ satisfying the dominant energy condition $\rho \geq |\mathcal{J}|$. As before, the boundary $\partial \Sigma$ is connected, compact and of a positive Yamabe type. The condition of being minimal is replaced by three conditions, namely (i) the mean curvature $p$ (with respect to the direction pointing towards infinity) is non-positive, (ii) $q \geq |p|$ (or alternatively $-q \geq |p|$) everywhere and

$$(iii) \quad \Theta = \left(\frac{n-2}{(n-1)\mathcal{Y}} \right)^\frac{1}{n-1} |\partial \Sigma|^{\frac{1}{n-1}} \sup_{\Sigma}(p + \sqrt{q^2 + S_\Lambda S^\Lambda}) < 1.$$

($S_\Lambda$ is defined in (10)). Under these circumstances, the total energy satisfies the bound [118]

$$E_{\text{ADM}} \geq \frac{1}{8\pi} \sqrt{\mathcal{Y}(n-1) \frac{\sigma(1-\Theta)}{n-2} |\partial \Sigma|^{\frac{2}{n-1}}-\frac{1}{\sigma+1-\Theta} |\partial \Sigma|^{\frac{2}{n-1}}},$$

(33)

The constant $\Theta$ is always non-negative under assumptions (i)–(ii). Thus, inequality (33) is weaker than the corresponding one in the time-symmetric case (32). The conditions on the boundary $\partial \Sigma$ are somewhat surprising. Conditions (i) and (ii) state that the mean curvature vector of the surface points inwards (in the sense that its product with the outer normal
\( \bar{m} \) is non-positive) and is causal past (future) everywhere. Thus, the boundary is indeed weakly past (future) trapped. However, these types of surfaces are necessarily not area outer minimizing (except in the very special case that their mean curvature vector vanishes identically). Consequently, the minimal area enclosure of the boundary lies, in general, inside \( \Sigma \). Thus, this Penrose-like inequality is obtained for a surface for which the original Penrose inequality is not expected to hold. It is therefore interesting that such an inequality exists.

The third generalization is due to Khuri [108] and again involves a non-vanishing second fundamental form. Since the method uses the Jang equation in a fundamental way, we postpone its discussion to the end of subsection 7.4.

### 5.2. Isoperimetric surfaces

An interesting attempt to proof the Riemannian Penrose inequality is discussed in [30]. The idea is to consider a special class of surfaces which interpolate between the outermost minimal surface \( S_m \) and infinity, in such a way that the Geroch mass is non-increasing and the surfaces approach large round spheres in the asymptotic region. Although this sounds familiar with the inverse curvature flow argument of Geroch, the idea exploited by Bray is in fact very different. Indeed, instead of using flows of surfaces, Bray considers, for any given volume, an area minimization problem (i.e. an isoperimetric problem). Assume that the outermost minimal surface \( S_m \) is connected and consider the class of surfaces in the same homology class as \( S_m \).

One can associate with each surface \( S \) in this class the volume bounded between \( S_m \) and \( S \) (counted negatively in the portion where \( S \) lies inside \( S_m \) and positively where it lies outside).

For a given value of \( V \geq 0 \), consider the collection of surfaces which bound, together with \( S_m \), precisely a volume \( V \) and define \( A(V) \) as the infimum of the corresponding areas. If the infimum is attained on a surface \( S_V \), then this surface is obviously of constant mean curvature \( p(V) \) because it is the solution of an isoperimetric problem. Bray’s idea is to show that the Geroch mass \( M_G(V) \) simply becomes

\[
M_G(V) = \sqrt{\frac{A(V)}{16\pi}} \left( 1 - \frac{1}{16\pi} A(V)'(V)^2 \right),
\]

provided the surface \( S_V \) is connected and is of spherical topology. Now consider a variation of \( S_V \) along its outer unit normal \( \bar{m} \) with unit speed. If we denote by \( S_V(t) \) the corresponding flow of surfaces (with \( S_V(0) = S_V \)), the second variation of area gives

\[
\frac{d^2|S_V(t)|}{dt^2} \bigg|_{t=0} = \int_{S_V} (\mathcal{L}_{\bar{m}} p + p^2) \eta_{S_V} = \int_{S_V} \left( -\frac{1}{2} \Pi_{AB}' \Pi_{AB} + \frac{1}{4} p^2 - \frac{1}{2} R(\gamma) + \frac{1}{2} R(h) \right) \eta_{S_V}
\]

where we have used (11) in the second equality. Using now the Gauss–Bonnet theorem and the non-negativity of the curvature scalar of \( (\Sigma, \gamma) \), we conclude \( \frac{d^2|S_V(t)|}{dt^2} \bigg|_{t=0} \leq 4\pi + \frac{1}{2} p(V)^2 A(V) \).

Denoting by \( V(t) \) the volume bounded by \( S_V(t) \), it follows that \( A(V(t)) \leq |S_V(t)| \) because \( A(V) \) is the infimum of all areas bounding a volume \( V \). Since these two functions touch at \( t = 0 \), it follows that \( \frac{d^2|S_V(t)|}{dt^2} \bigg|_{t=0} \leq \frac{d^2|S_V(t)|}{dt^2} \bigg|_{t=0} \leq 4\pi + \frac{1}{2} p(V)^2 A(V) \). Performing a change of variables \( t \to V \), one concludes

\[
A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)} \implies M_G(V)' \geq 0.
\]
Although the argument just described assumes that $A(V)$ is differentiable, the conclusion is still valid if a suitable weak (distributional) derivative is taken [30]. The other two assumptions that enter into the argument are that (i) $S_V$ is connected and is of spherical topology and (ii) the infimum of $A(V)$ is in fact attained, i.e. the surface $S_V$ exists. The first condition turns out to be crucial and needs to be imposed as an assumption. More precisely, Bray demands the weaker assumption that for each $V > 0$, if one or more minimizers of $A(V)$ exists, then at least one of them is connected; this is termed ‘Condition 1’ in [30]. Most of the technical work in [30] consists in showing that condition (ii) (i.e. the existence of a minimizer for all $V \geq 0$) imposes no extra restriction. To that aim, Bray first argues that the class of metrics $(\Sigma, \gamma)$ can be chosen to be exactly Schwarzschild at infinity (a similar, but weaker, reduction is also used in Bray’s full proof of the Riemannian Penrose inequality summarized in subsection 5.4 and will be discussed in more detail there). The heart of the proof of existence of the minimizer $S_V$ consists in proving that the isoperimetric surfaces in the Schwarzschild spacetime are given by the spherical orbits of the $SO(3)$ isometry group. In contrast to what one may expect, proving this fact is not a trivial matter. The fundamental idea behind the construction is to use a comparison metric (in this case, the flat metric on $\mathbb{R}^3$) for which one knows that the spheres are the solutions of the isometric problem. This argument, which was tailored for the Schwarzschild metric, has been simplified and substantially extended in [23], where simple conditions are found on a given spherically symmetric metric which ensure that the spheres are the minimizers of the isoperimetric problem. Summarizing, Bray proves in [30] that the Penrose inequality (28) holds for each asymptotically Euclidean Riemannian manifold which has a connected outermost minimal surface $S_m$ and satisfies ‘Condition 1’.

### 5.3. Huisken and Ilmanen’s proof

The heuristic idea behind the proof of Huisken and Ilmanen was first proposed by Geroch [61] and is based on the observation that the Geroch mass (17) (with $C = 0$) is monotonically increasing if the surfaces are moved by inverse mean curvature (i.e. $p e^\psi = 1$) provided the scalar curvature of $(\Sigma, \gamma)$ is non-negative. This fact is clear from (18) since $p e^\psi = a = 1$ and the derivative of $M_G(S_\lambda)$ is then a sum of non-negative terms. Another immediate property of $M_G$ is that its value on any connected, topologically $S^2$ minimal surface is $M_G = \sqrt{|S|/(16\pi)}$. For surfaces $S_r = \{ r = \text{const} \}$ in the asymptotically Euclidean end $\Sigma^\infty$, the asymptotic decay of $\gamma$ implies $\lim_{r \to \infty} M_G(S_r) = M_{\text{ADM}}$. Thus, $M_G$ indeed interpolates between the left- and right-hand sides of the Penrose inequality.

Geroch’s original idea was to prove the positive mass theorem by starting the inverse mean curvature flow from a point so that $M_G = 0$ initially (the point can be approached as the limit of very small spheres). The positivity of mass would follow provided the inverse mean curvature flow remained smooth all the way to infinity and the flow approached large coordinate spheres in the asymptotically Euclidean end. Later on, Jang and Wald [94] realized that the same argument could be used to prove the Penrose inequality by starting from the outermost minimal surface. In fact, $p$ vanishes on this minimal surface, and hence the velocity $e^\psi$ diverges there. However, by starting the flow from surfaces which approximate the outermost minimal surface from outside, the Penrose inequality would follow. However, it was immediately realized that the flow will not remain smooth in general and that singularities will develop. The first variation of area implies that, as long as the flow remains smooth, the area of the surfaces increases exponentially. Now consider, as a very simple example, two disjoint balls in the Euclidean space. By symmetry, each surface will evolve under an IMCF to a larger sphere with exponentially increasing radius. Thus, the two surfaces will necessarily touch in finite time and the flow cannot remain smooth forever. A less trivial example, discussed in
[87], consists of a thin torus in the Euclidean space. The differential equation satisfied by \( p \) under an IMCF is of parabolic type and the maximum principle implies that \( p \) stays bounded above in terms of its initial value (and the background geometry) as long as the flow remains smooth. Thus, the torus will flow outwards at positive speed bounded away from zero and it will thicken. However, a sufficiently thick torus has vanishing mean curvature at points on its inner rim. The speed becomes infinity there and the flows necessarily stop being smooth.

The presence of singularities made this idea dormant for decades. The only case where the method was made rigorous involved a particular case of metrics called quasi-spherical metrics with divergence-free shear [17], where the flow was seen to remain smooth all the way to infinity. Huisken and Ilmanen’s fundamental contribution was to define the flow in a suitably weak sense so that the singularities could be treated (and, in fact, basically avoided).

An important ingredient in Huisken and Ilmanen’s approach is the use of a level set formulation for the IMCF (which is a geometric parabolic flow). This means describing the leaves of the flow as the level sets of a real function \( u \) on \( \Sigma \). The IMCF condition translates directly into the following degenerate elliptic equation for \( u \):

\[
\text{div}_\gamma \left( \frac{\nabla u}{|\nabla u|} \right) = |\nabla u|, \tag{35}
\]

where \( \text{div}_\gamma V \) is the divergence of \( V \) in \((\Sigma, \gamma)\). In principle it is possible that \( u \) remains constant on open sets, which has the immediate consequence that the flow may jump across regions with positive measure. The fundamental idea is to use these jumps precisely to avoid the singularities that the smooth IMCF would otherwise have. In order to achieve this, Huisken and Ilmanen find a variational formulation for (35). This equation is not the Euler–Lagrange equation of any functional. However, by freezing the right-hand side to \(|\nabla u|\), the authors write down the functional (which now depends on \( u \))

\[
J_u(v) = \int_{\Sigma} (|\nabla v| + v|\nabla u|) \eta, \tag{36}
\]

The critical points of this functional with respect to compactly supported variations of \( v \) give (35) with \( u \) being replaced by \( v \) on the left-hand side. One then looks for functions \( u \) which minimize their own functional, thus giving (35).

This variational formulation has a geometric counterpart which, in rough terms, implies that each of the level sets of \( u \) (defined as \( \partial \{ u < t \} \) for positive \( t \)) is area outer minimizing. Thus, if we start from a smooth surface \( S_0 \) which is area outer minimizing and with positive mean curvature, the IMCF (which enjoys short time existence, thanks to its geometric parabolic character) will evolve the surface smoothly for some “time” \( \lambda \). For small enough values of \( \lambda \), each level set will be outer area minimizing. However, there may well exist a value for which \( S_{\lambda} \) ceases to be area outer minimizing. Take the smallest of such values, \( \lambda_1 \). This means that there exists a surface \( S'_{\lambda_1} \) which encloses \( S_{\lambda_1} \) and has less or equal area. In fact, it must have equal area because if it had a strictly less area, it would also have an area less than some close enough previous leaf (the flow is smooth up to \( \lambda_1 \) and the area changes smoothly). The surface \( S'_{\lambda_1} \) may have common points with \( S_{\lambda_1} \). However, the surface \( S'_{\lambda_1} \setminus S_{\lambda_1} \) must be a minimal surface because otherwise it would admit a compactly supported variation that would reduce its area while still remaining outside \( S_{\lambda_1} \) and enclosing it. However, this varied surface would (i) enclose all surfaces in the flow from \( S_0 \) up to \( S_{\lambda_1} \), (ii) have a larger area than \( S_0 \) (because the latter is area outer minimizing) and (iii) have less area than \( S_{\lambda_1} \). So, for some value of the flow parameter smaller than \( \lambda_1 \) it would have the same area, and it would enclose it. This would contradict the definition of \( \lambda_1 \) as being the smallest value with such a property. Note that, \textit{a priori}, this outer minimizing prescription does not exclude the fact that the flow may become singular already in the interval \( 0 < \lambda < \lambda_1 \), where all the leaves remain area outer
minimizing. The fact that this cannot happen is one of the several statements that Huisken and Ilmanen had to show, and which makes their proof technically difficult.

In the torus example before, the smooth flow would thicken the torus (in an axially symmetric way) until a surface is obtained which has exactly the same area as the surface obtained by closing its hole by two horizontal planes. It is clear in this example that the new pieces have vanishing mean curvature.

Since the level set function $u$ is such that each level set is outer area minimizing, this means that the IMCF evolves smoothly for as long as the leaves remain area outer minimizing until a surface which is not area outer minimizing is eventually reached. At this point, the surface jumps (meaning that $u$ remains constant between the two surfaces). The flow should be continued from the new surface outwards. Since the new surface has pieces with $p = 0$, the IMCF cannot be defined there in a classical sense. However, the weak formulation in terms of level sets also takes care of this.

A crucial condition for the monotonicity argument to go through in this formulation is that the Geroch mass does not increase in the jumps. We know that the total area does not change. Moreover, the new pieces of the surface have $p = 0$, while the deleted pieces have $p > 0$ because the flow was smooth up to and including $S_\lambda$. Thus, across the jump the new pieces of the surface have $p = 0$, while the deleted pieces have $p > 0$ because the flow was smooth up to and including $S_\lambda$. Thus, across the jump the Geroch mass is non-decreasing provided the Euler characteristic of the surface does not decrease. The example with the torus shows that in general the topology of the surface may change across the jump, in that case changing from a surface with genus 1 to a topological sphere. In this particular case, the Euler characteristic increases and the Geroch mass would be monotonically increasing. The question is therefore whether this is a general property or not. The example with 2-spheres in the Euclidean space shows that this cannot be true in general. There, the value of the Euler characteristic is initially 4 and after the jump a topological sphere forms, for which $\chi = 2$. A general result by Huisken and Ilmanen [89] is that a topologically $S^2$ surface in an asymptotically Euclidean 3-manifold $(\Sigma, \gamma)$ outside the outermost minimal surface cannot jump, via the weak formulation of the IMCF, to another surface with a smaller Euler characteristic, i.e. no holes can appear on the surface after the jump. Nevertheless, the outermost minimal surface $S_m$ in $(\Sigma, \gamma)$ need not be connected. Consequently, for the Geroch mass to remain monotonic under the weak IMCF, a connected component $S_i$ of the outermost minimal surface must be chosen as an initial surface. The Penrose inequality proven by Huisken and Ilmanen is, therefore,

$$M_{\text{ADM}} \geq \max_i \sqrt{\frac{|S_i|}{16\pi}},$$

(37)

where $i$ runs over the connected components of the outermost minimal surface.

As mentioned before, the Riemannian Penrose inequality also has a rigidity part. In this setting, this states that equality is achieved in (37) if and only if the region in $(\Sigma, \gamma)$ outside its outermost minimal surface $S_m$ is isometric to the domain $r > m/2$ of $(\Sigma_{\text{Sch}}, \gamma_{\text{Sch}})$ (29) with $m = M_{\text{ADM}}$. Heuristically, it is clear that some rigidity is to be expected already from the monotonicity property of the Geroch mass. If equality is achieved, then the derivative of the Geroch mass is zero everywhere. In particular, in the smooth part of the flow, $R(\gamma) = 0, \psi = \text{const}$ and each leaf is totally umbilical (i.e. $\Pi_{AB} = 0$) at each point. The IMCF condition gives $p = \text{const}$ on each leaf. Since $\partial_\lambda = e^\psi \hat{m} = \frac{1}{p} \hat{m}$, the variation formula (11) implies $\frac{dp}{d\lambda} = -\frac{3}{4} p + \frac{R(h)}{2p^2}$. This implies that each leaf is a metric sphere (because $R(h) = \text{const}$). The exponential growth of the area gives $|S_i| = |S_0| e^{\lambda}$. Defining a new variable $\hat{r} = \sqrt{|S_i|/4\pi}$, the metric $\gamma$ takes the form $\gamma = \frac{2}{p^2} \hat{r}^2 + \hat{r}^2 d\Omega$. The expression for the Geroch mass and the fact that it takes a value $m$ independent of $\lambda$ gives $\hat{r}^2 p^2 = 4(1 - 2m)$. and hence $\gamma$ is the Schwarzschild metric outside the minimal surface $\hat{r} = 2m$. This is, of
Huisken and Ilmanen’s proof has interesting side consequences. First, the argument does not use the positive mass theorem anywhere and therefore the method gives an independent proof of this important result. In the presence of an outermost minimal surface, positivity of \( M_{\text{ADM}} \) is an obvious consequence of the theorem. If \( \Sigma \) is free of minimal surfaces, it suffices to start the weak inverse mean curvature flow at one point, making rigorous the original idea of Geroch.

Another interesting consequence deals with the so-called Bartnik mass (or capacity) [15], defined as follows. A three-dimensional Riemannian manifold \((\Omega, \gamma)\) with no boundary and compact metric closure is called admissible if it can be isometrically embedded into a complete and connected asymptotically Euclidean three-dimensional manifold \((\Sigma, \gamma)\) satisfying the following three properties: (a) the curvature scalar is non-negative, (b) the boundary of \( \Sigma \) is either empty or a minimal surface and (c) \( \Sigma \) is free of any other minimal surfaces (the original definition excludes minimal surfaces even at the boundary of \( \Sigma \); this modification is due to Huisken and Ilmanen [89]). \((\Sigma, \gamma)\) is called an admissible extension. The Bartnik capacity is the infimum of the ADM masses of all possible extensions of \((\Omega, \gamma)\). By the positive mass theorem, it is immediately non-negative and it was conjectured in [16–18] to be positive if \((\Omega, \gamma)\) is not a subset of the Euclidean space. Huisken and Ilmanen used the weak IMCF to reach a slightly weaker conclusion, namely that if the Bartnik capacity is zero, then \((\Omega, \gamma)\) is locally isometric to the Euclidean space. The idea of the proof is to take a point \( p \) in \( \Omega \) where the metric is non-flat and choose any admissible extension \((\Sigma, \gamma)\). Consider the weak IMCF starting at \( p \). For some small enough value of the flow parameter, the corresponding leaf \( S_\lambda \) can be proven to have a positive Geroch mass and to lie within \( \Omega \) independently of the extension (this means in particular that \( S_\lambda \) is area outer minimizing independently of the extension). Thus, monotonicity of the IMCF gives the lower bound \( M_{\text{ADM}} \geq M_G(S_\lambda) \). Since this is independent of the extension, the result is established.

Another consequence of Huisken and Ilmanen’s construction is the so-called exhaustion property of the Bartnik mass. This can be formulated as follows. Consider an asymptotically Euclidean Riemannian manifold \((\Sigma, \gamma)\) and any increasing collection of bounded sets \( \Omega_i \subset \Sigma \) satisfying \( \cup_i \Omega_i = \Sigma \). Then the Bartnik mass \( m_B(\Omega_i) \) tends to the ADM mass when \( i \to \infty \). This is proven in [89] by taking suitable coordinate balls within each \( \Omega_i \) in such a way that the coordinate radius \( R_i \) tends to \( \infty \). It is shown, by using the IMCF, that the Bartnik mass of each \( \Omega_i \) is bounded below by the Geroch mass of the corresponding coordinate balls. The Geroch mass of the coordinate balls tends to the ADM mass when \( R_i \to \infty \). Since the Bartnik mass of \( \Omega_i \) is obviously bounded above by \( M_{\text{ADM}} \) of \( \Sigma \), the exhaustion property \( m_B(\Omega_i) \to M_{\text{ADM}} \) follows.

5.4. Bray’s proof

Bray [22] was able to prove the Riemannian Penrose inequality for arbitrary outermost minimal surfaces, with no restriction of connectedness. Bray’s proof also uses geometric flows in an essential way, but instead of flowing surfaces in a fixed Riemannian manifold, as in Huisken and Ilmanen’s proof, it uses a flow of metrics. The idea is to modify the metric with a one-parameter family of conformal factors so that the curvature scalar remains non-negative, the new horizon with respect to the conformal metric has the same area as the original one and the total ADM mass does not increase during the process. If, moreover, the flow can be defined in such a way that, as \( t \to \infty \), the Riemannian manifold outside its horizon converges to the \( r > m/2 \) portion of the Schwarzschild manifold (29) in a suitable sense, then the Penrose
inequality for the original manifold follows because the inequality (in fact, equality) holds in the limit. This conformal flow also allows Bray to show the rigidity part of the Penrose conjecture, namely that equality holds if and only if the starting Riemannian manifold is isometric to the exterior region of Schwarzschild.

Bray’s statement of the Penrose inequality deals with three-dimensional asymptotically Euclidean Riemannian manifolds \((\Sigma, \gamma)\) with one or several asymptotic ends (in the latter case, one of them is selected and the mass and the concept of ‘outer’ are taken with respect to this end) and which contain an area outer minimizing horizon \(S_0\). By definition, a horizon is a smooth, compact, minimal surface which is a boundary of an open set \(\Omega^-\) containing all possible asymptotic ends on \((\Sigma, \gamma)\) except the one that has been selected. In the case of several ends, the existence of such a horizon is guaranteed, otherwise \((\Sigma, \gamma)\) is assumed to have one. A horizon is area outer minimizing if no other surface containing it has strictly a larger area. An area outer minimizing horizon need not be outermost, although the outermost one (which always exists) will have at least the same area as \(S_0\) (by the area outer minimizing property of the former). As already said, \(S_0\) need not be connected and Bray’s theorem states 

\[ M_{\text{ADM}} \geq \sqrt{\frac{\text{Area}(S_0)}{16\pi}} \]

with equality if and only if \((\Sigma, \gamma)\) is isometric to the exterior region of the Schwarzschild manifold.

The first step in Bray’s proof is to reduce the class of metrics under consideration. To that aim, Bray uses an interesting result due to Schoen and Yau [142] (see theorem 7 in [30]) which states that given an asymptotically Euclidean, three-dimensional Riemannian manifold \((\Sigma, \gamma)\) (see proposition 4.1 in [144] for a similar statement in any dimension) with \(R(\gamma) \geq 0\), and any positive number \(\epsilon\), there always exists an asymptotically Euclidean metric \(\gamma_\epsilon\) on \(\Sigma\), with \(R(\gamma_\epsilon) \geq 0\) everywhere and identically zero outside a compact set \(K\), which is conformally flat outside \(K\), i.e. \(\gamma_\epsilon = s_\epsilon^4 \delta\) on \(\Sigma^\infty = \Sigma \setminus K\), where \(s_\epsilon : \Sigma^\infty \to \mathbb{R}\) is a positive function approaching a positive constant at infinity (on each one of the asymptotic ends). Moreover, the following conditions are fulfilled:

\[
\left| M_{\text{ADM}}(\gamma) - M_{\text{ADM}}(\gamma_\epsilon) \right| \leq \epsilon, \\
\left| \frac{\gamma_\epsilon(\vec{X}, \vec{X})}{\gamma(\vec{X}, \vec{X})} - 1 \right| \leq \epsilon, \quad \forall \vec{X} \in T_x\Sigma \quad \text{and} \quad \forall x \in \Sigma,
\]

(38)

i.e. the total mass changes within \(\epsilon\) and the metric itself changes point-wise at most by \(\epsilon\) (in the sense that unit vectors change their norm at most by \(\epsilon\)). The condition that \(\gamma_\epsilon\) has vanishing curvature scalar outside \(K\) implies that \(s_\epsilon\) is harmonic (with respect to the flat metric). Therefore, the metric \(\gamma_\epsilon\) is called harmonically flat. This perturbation result has been strengthened in [30] to show that \(s_\epsilon\) can in fact be taken to be spherically symmetric, in which case \(\gamma_\epsilon\) is the Schwarzschild metric outside \(K\). Such a metric is called Schwarzschild at infinity. This stronger version was needed to apply the isoperimetric methods described in subsection 5.2.

Schoen and Yau’s perturbation result implies that, in order to prove the Riemannian Penrose inequality, it is sufficient to consider harmonically flat metrics. Indeed, if there existed a counterexample to the Penrose inequality, i.e. a metric with total ADM mass strictly smaller than the Geroch mass of its outermost minimal surface, then we would be able to find a harmonically flat metric \(\gamma_\epsilon\) with the ADM mass within \(\epsilon\) of the original one. Since the area of the outermost minimal surface also depends continuously on \(\epsilon\), it is clear that \(\epsilon\) can be arranged so that the Penrose inequality would also be violated for the harmonically flat metric \(\gamma_\epsilon\).

For this class of manifolds, Bray defines a flow of metrics by a conformal rescaling \(\gamma_t = u_t^4\gamma\) which depends on a parameter \(t \geq 0\). The function \(u_t\) is defined via an elliptic
equation for \(v_t \equiv \frac{du}{dt}\) together with the initial value \(u_0 = 1\) (so that \(\gamma_0 = \gamma\)). \(v_t\) is defined by solving the Dirichlet problem:

\[
\begin{aligned}
\Delta_x v_t &= 0 & \text{on } \Sigma_t, \\
v_t &= 0 & \text{on } \Sigma \setminus \Sigma_t, \\
v_t|_{\Sigma_t} &= 0, \\
v_t &\to -e^{-\gamma} & \text{at } \infty,
\end{aligned}
\tag{39}
\]

where \(S_t\) is the outermost minimal area surface enclosing \(S_0\) with respect to the metric \(\gamma_t\) and \(\Sigma_t\) is the exterior of \(S_t\). Since \(S_t\) is defined using the metric one is trying to construct, it is not obvious \textit{a priori} that such problem admits a solution. Bray uses a discretization argument whereby \(u_t(x)\) as a function of \(t\) (i.e. for a fixed value of \(x\)) is discretized as a continuous piecewise linear function \(u^\epsilon_t(x)\) (in \(t\)), where the jumps in the derivatives occur at fixed intervals of length \(\epsilon\). The slope of \(v^\epsilon_t = \frac{du^\epsilon_t}{dt}\) on the \(k\)th interval, i.e. on \(t \in [(k-1)\epsilon, k\epsilon), k \in \mathbb{N}\), is defined inductively in \(k\). At \(k = 1\), \(v^\epsilon_t\) is zero inside \(S_0\) and solves the Laplace equation \(\Delta_{\gamma_0} v^\epsilon_1 = 0\) with boundary data \(v^\epsilon_1 = 0\) on \(S_0\) and \(v^\epsilon_1 \to -1\) at infinity. \(S^\epsilon_1\) is then defined as the outermost minimal area enclosure of \(S_0\) with respect to the metric \((u^\epsilon_{\text{outer}})^2 \gamma_0\) (the function \(u^\epsilon_{\text{outer}}(x)\) is known by continuity). Note that since \(S_0\) was not assumed to be the outermost minimal surface, \(S^\epsilon_1\) may be far away from \(S_0\) even for arbitrarily small \(\epsilon\). Assume that the construction has been carried out for \(k \in \mathbb{N}\); then the slope \(v^\epsilon_t\) on the interval \(t \in [(k\epsilon, (k+1)\epsilon)\) is defined as being zero inside \(S_k\) and solving the Laplace equation \(\Delta_{\gamma_k} v^\epsilon_k = 0\) which approaches the constant \(-1\) at infinity and vanishes on \(S_k\). The surface \(S_{k+1}\) is defined as the outermost minimal area enclosure of \(S_k\) with respect to the metric \((u^\epsilon_{\text{outer}})^2 \gamma_k\) (again, the function \(u^\epsilon_{\text{outer}}(x)\) is known by continuity).

Thus, while the metric is changed continuously in \(t\), the boundaries \(S^\epsilon_t\) change only at discrete values. Existence of \(u^\epsilon_t\) follows from the inductive construction. The conformal flow of metrics depends on \(\epsilon\). It is an important ingredient of Bray’s proof to show that a suitable limit \(\epsilon \to 0\) exists, thus defining the flow of metrics in (39). The limit is performed in two ways as follows. First, the conformal factors are seen to converge when \(\epsilon \to 0\) to a locally Lipschitz function \(u_t\). This already defines the metric \(\gamma_t\) and the surfaces \(S_t\) as before. Extending the discrete collection of \(S^\epsilon_k\) to a one-parameter family via \(S^\epsilon(t) \equiv S^\epsilon_k\) for \(t \in [k\epsilon, (k+1)\epsilon)\), the convergence of these surfaces when \(\epsilon \to 0\) (for fixed \(t\)) is then considered. It turns out that the limit may depend on the subsequence \(\epsilon_t \to 0\). Bray denotes the collection of all limit surfaces as \([S_\alpha(t)]\), where \(\alpha\) identifies the element in the collection. Convergence is seen to be in the Hausdorff distance sense and each element in \([S_\alpha]\) is proven to be a smooth surface.

The surfaces \(S_t\) and \([S_\alpha(t)]\) need not always coincide. However, they are closely related: for any \(t_2 > t_1 \geq 0\), \(S_{t_1}\) encloses \(S_{t_1}(t_1)\) (for any \(\alpha_1\)) and \(S_{t_2}(t_2)\) encloses \(S_{t_2}\) (for any \(\alpha_2\)). Thus, they must coincide for any value where \(S_t\) depends continuously on \(t\). Bray then proves that \(S_t\) is continuous except for \(t\) in a countable set \(J\) (which may be empty). Furthermore, \(S_t\) is everywhere upper semicontinuous and the right limit always encloses the left limit. Moreover, for \(t_2 > t_1\), \(S_{t_1}\) encloses \(S_{t_2}\). Thus, the flow of surfaces \(S_t\) is everywhere outwards and may jump, also outwards, at a countable number of places.

In addition, this is crucial for the proof of the Penrose inequality, the area of \(S_t\) is constant for all \(t\), even at the jumps. Outside the jumps, this constancy can be understood heuristically because \(S_t\) is a closed minimal surface and hence its area does not change to first order with respect to any variation, in particular the variation which transforms \(S_t\) into \(S_{t+\delta}\) (\(\delta\) small) in the metric \(\gamma_t\). The area also changes because the metric depends on \(t\). However, since \(v_t = 0\) on \(S_t\) by construction, the metric \(\gamma_{t+\delta}\) coincides with \(\gamma_t\) on \(S_t\) to first order and therefore the area of \(S_t\) with the metric \(\gamma_t\) coincides to first order with its area with respect to
This implies that $|S_0|$ is constant. At the jumps, the statement requires a careful analysis of the various limiting procedures involved.

The second crucial ingredient in Bray’s proof is the fact that the total mass $m(t)$ of the conformally rescaled metric $\gamma_t$ is a non-increasing function of $t$. The proof relies on two facts. First, the definition of $v_t$ in (39) seems to depend on $\gamma$ and on $t$. However, as a simple consequence of how the Laplacian changes under conformal rescalings, it follows that $v_t$ depends only on $\gamma_t$ (i.e. $v_t$ solves a suitable Dirichlet problem with respect to the metric $\gamma_t$, with no reference to the original metric $\gamma_0$). Therefore, proving that $\frac{dm(t)}{dt} \leq 0$ is equivalent to showing $\frac{dm(t)}{dt}|_{t=0} \leq 0$ because there is nothing that distinguishes $\gamma_0$ from any other metric $\gamma_t$ in the flow. For $t = 0$, the function $v_0$ restricted to $\Sigma_0$ (the exterior of $S_0$ in $\Sigma$) is just minus the Green function of $\gamma_0$, defined as

$$\begin{align*}
\Delta_{\gamma_0} G &= 0 \quad \text{on } \Sigma_0, \\
G|_{S_0} &= 0, \\
G &\to 1 \quad \text{at } \infty.
\end{align*}$$

The total mass for a harmonically flat metric $u^4 \delta$ can be computed directly from the behaviour of $u$ at infinity. Namely, if $u = a + b/(2r) + O(1/r^2)$, then the ADM mass is $m = ab$. The metric $\gamma_t$ is of the form $\gamma_t = (1 - tG + O(t^2))\gamma_0 = (1 - tG + O(t^2))u^4 \delta$, with $U = 1 - \frac{M_{\text{ADM}}}{2} + O(r^{-2})$, where $M_{\text{ADM}}$ is the ADM mass of $\gamma_0$. Expanding the Green function $G$ in the asymptotic region as $G = 1 - c/(2r) + O(1/r^2)$, it follows from a direct calculation that $m(t) = M_{\text{ADM}} + t(c - 2M_{\text{ADM}}) + o(t)$. Thus, the mass will not increase provided $c \leq 2M_{\text{ADM}}$. This turns out to be a general property of the Green function $G$ of any asymptotically flat metric on a manifold with a compact minimal boundary $S_0$. This result, also due to Bray [22], has independent interest and has been already used in another context by Miao [117] to characterize the Schwarzschild initial data as the only static initial data with a non-empty minimal boundary.

The proof of $c \leq 2M_{\text{ADM}}$ relies on the positive mass theorem and uses a technique first introduced by Bunting and Masood-ul-Alam [34] to prove uniqueness of the Schwarzschild black hole. The idea is to double $(\Sigma_0, \gamma_0)$ across its boundary $S_0$ (i.e. take two copies and identify the boundaries) to define a new manifold $\hat{\Sigma}$. Also define a function $\Phi(x) = \frac{1 + G(x)}{2}$ on one of the copies and $\Phi(x) = \frac{1 - G(x)}{2}$ on the other copy. This defines a function which is harmonic away from $S_0$ and which approaches 1 on one asymptotic end and zero on the other asymptotic end. This function is also $C^1$ everywhere. Consider the conformally rescaled metric $\hat{\gamma} = \Phi^4 \gamma_0$ on $\hat{\Sigma}$. The asymptotic behaviour of $G$ near the infinity where it vanishes shows that $(\hat{\Sigma}, \hat{\gamma})$ admits a one-point compactification there, thus defining a complete asymptotically Euclidean Riemannian manifold. Furthermore, the behaviour of the curvature scalar under conformal rescalings and the fact that $\Phi$ is harmonic imply that $\gamma_0$ has non-negative curvature scalar away from $S_0$. If $(\hat{\Sigma}, \hat{\gamma})$ were smooth, the positive mass theorem could be invoked to conclude that the mass of $\hat{\gamma}$ is non-negative, and zero if and only if $(\hat{\Sigma}, \hat{\gamma})$ is the Euclidean space. However, the metric $\hat{\gamma}$ is not smooth across $S_0$ and Bray needs to use an approximation argument to reach the same conclusion. Consequently, the ADM mass of $(\hat{\Sigma}, \hat{\gamma})$ is non-negative. The mass of this manifold is just $M_{\text{ADM}} - c/2$, from which $c \leq 2M_{\text{ADM}}$ follows.

Using related techniques, Miao [116] has extended these results and has proven that the positive mass theorem holds for complete, asymptotically Euclidean Riemannian manifolds $(\Omega, \gamma)$ of dimension $n \leq 7$ and non-negative curvature scalar, with a metric which is allowed to be non-smooth across a compact hypersurface $S$ provided (i) this hypersurface separates $\Omega$ into two pieces $\Omega^\pm$, with $\Omega^+$ containing the chosen asymptotically Euclidean end, (ii) the induced metric on $S$ from both sides coincides and (iii) the mean curvature with respect to the
The limit. Here is where the asymptotic condition

\[ \lim_{r \to \infty} \frac{\gamma'}{\gamma} = 0. \]

is left out for a sufficiently large \( t \). Thus, the shrinking manifold \( S_t \) disappears in the limit. Here is where the asymptotic condition \( u_i \to e^{-t} \) at infinity comes into play. For very large \( t \), the asymptotic value of the conformal factor \( u_t \) also tends to zero. Thus, vectors that were unit at large distances in the original metric \( \gamma_0 \) have increasingly smaller lengths in the metric \( \gamma_t \). However, this metric is also asymptotically Euclidean, so for a suitable choice of coordinates, the metric must approach the flat metric in Cartesian coordinates. In these coordinates, the region \( \Sigma_t \) shrinks more slowly (or it may even expand). In order to see why this is so, consider the metric defined by

\[ g_t^{\text{Euc}} = e^{-4t} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \] (41)
on the domain \( r > R_0(t) \) with \( R_0(t) \to \infty \) as \( t \to \infty \). The domain is shrinking in the coordinates \( \{r, \theta, \phi\} \). However, (41) is the Euclidean metric for any \( t \). The transformation to the standard spherical coordinates \( \{r', \theta, \phi\} \) is given by \( r' = e^{-2t} r \). In the new coordinates, the domain is \( r' > e^{-2t} R_0(t) \) which is not shrinking if \( R_0(t) \) grows at most as \( e^t \). This is the type of behaviour that occurs to the conformal flow of metrics introduced by Bray. This can be seen explicitly by considering the flow in the particular case when the starting metric is the Schwarzschild manifold (29). Rotational symmetry allows one to integrate the equations easily. The flow of metrics \( \gamma_t \) in the exterior domain \( r \geq m/2 \) is, for \( t \geq 0 \), [22]

\[ \gamma_t = \begin{cases} \frac{4n^2}{r^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) & m \leq r \leq \frac{m}{2} e^{2t} \text{ interior domain}, \\
\left(\frac{m}{2} e^t + e^{-t}\right)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) & r \geq \frac{m}{2} e^{2t} \text{ exterior domain}. \end{cases} \] (42)

The exterior domain indeed shrinks in these coordinates. However, performing the coordinate transformation (or the diffeomorphism if an active point of view is preferred) \( r' = r e^{-2t} \), with \( \theta \) and \( \phi \) unchanged, the exterior region is transformed back into the original exterior part of the Schwarzschild manifold. The metric (42) in the region already swapped by the surfaces \( \{S_t\} \) is cylinder-like in the sense that each surface \( r = \text{const} \) has the same area. This is a general property of Bray’s flow, since each of the surfaces \( S_t \) has the same area as the starting horizon. The behaviour in the exterior part is also general. The idea is that since the surface \( S_t \) grows unboundedly, eventually it swallows all the interior geometric features of \( (\Sigma_0, \gamma_0) \) and leaves only the asymptotic geometry. Since the mass does not increase along the flow of metrics, its limit exists and is non-negative due to the positive mass theorem. Bray proves that the limit mass is, in fact, positive and hence the geometry in the exterior approximates Schwarzschild
with increasing accuracy. The asymptotic behaviour of the conformal factor $u_t \to e^{-t}$ has the effect that this exterior remains unbounded, in the sense that after performing the appropriate diffeomorphism, all the surfaces $S_t$ stay within a bounded set. More precisely, Bray shows that for each $\epsilon > 0$ there exists $T$ so that for $t > T$ there exists a diffeomorphism $\Phi_t$ between $(\Sigma_t, \gamma_t)$ and the exterior region $r \geq m_F/2$ of a fixed Schwarzschild manifold $(\Sigma_{Sch}, \gamma_{Sch})$ of mass $m_F$, such that both metrics are $\epsilon$-close to each other (in the sense that the length of any unit vector of the metric $\gamma_t$ has length in the interval $(1 - \epsilon, 1 + \epsilon)$ for the metric $\Phi_t^* \gamma_{Sch}$). Moreover, $m(t)$ and $m_F$ also differ at most by $\epsilon$. This concludes the proof of the inequality

$$M_{ADM} \geq \sqrt{\frac{|S_0|}{16\pi}}$$

for the original metric. As Bray points out, no property from $(\Sigma, \gamma_0)$ inside $S_0$ is used in the proof, therefore establishing (43) also for manifolds with boundary $S_0$, provided this is an area outer minimizing horizon.

While Huisken and Ilmanen’s proof is strongly dependent on the dimension of the manifold, Bray’s approach can be generalized to any dimension $n \leq 7$, as proven by Bray and Lee in [27]. The limitation $n \leq 7$ comes basically from two facts. First, regularity of minimal surfaces holds only for dimensions $n \leq 7$, and the outermost minimal surface enclosure of $S_0$, which is a crucial ingredient in the method, needs to be smooth for the argument to go through. Second, the positive mass theorem for general asymptotically Euclidean manifolds (not spin) is only known to hold so far for dimensions $n \leq 7$ (the underlying reason is again regularity of minimal surfaces, as Schoen and Yau’s proof uses this type of surfaces in a fundamental way). The statement of the Riemannian Penrose inequality in higher dimensions is, of course, different to (43), since the physical dimension of mass is not length in higher dimensions. The proper statement in (space) dimension $n$ is

$$M_{ADM} \geq \frac{1}{2} \left( \frac{|S_0|}{\omega_{n-1}} \right)^{\frac{1}{n-1}},$$

where $\omega_{n-1}$ is the area of an $(n-1)$-dimensional sphere of unit radius. The rigidity statement says that equality in (44) occurs only for the higher dimensional Schwarzschild metric (first discussed by Tangherlini [152]):

$$\gamma_{Sch}^{(n)} = \left( 1 + \frac{m}{|x|^{n-2}} \right)^{\frac{n-2}{2}} \delta_{ij} \, dx^i \, dx^j.$$

As discussed in [27], several of the steps in Bray’s proof extend easily to higher dimensions: the definition and existence of the conformal flow, the proof that the area of $S_t$ remains constant under the flow and the fact that the ADM mass $m(t)$ does not increase. However, the arguments required to show that $S_t$ eventually encloses all bounded sets, and that the flow of metrics converges to a Schwarzschild metric after a suitable $t$-dependent diffeomorphism, do in fact depend strongly on the dimension. This is because the original argument is based on the Gauss–Bonnet theorem and a Harnack-type inequality which is valid only in three dimensions. Moreover, the convergence to Schwarzschild requires a refinement of the rigidity part of the positive mass theorem, which roughly speaking gives a quantitative description of how the metric approaches the flat metric when the mass tends to zero. In Bray [22], this is proven using spinor techniques. This result is extended by Lee [110] to arbitrary higher dimensional manifolds for which the positive mass theorem holds (in particular, for dimensions $n \leq 7$).

The main technical work to establish (44) in [27] is therefore devoted to prove that the conformal flow converges in a suitable sense to the metric (45). Besides being more general, the argument presented in [27] in fact simplifies and streamlines some of the original steps in the three-dimensional case.
6. Penrose inequality for asymptotically hyperbolic Riemannian manifolds

Despite its difficulty, the Riemannian Penrose inequality is considerably simpler than the general Penrose inequality because all the complications arising from the second fundamental form disappear. The time-symmetric case, however, is not the only one where this kind of simplification occurs. Another example is given by initial data sets \((\Sigma, \gamma_{ij}, A_{ij})\) which are umbilical, i.e. when the second fundamental form is proportional to the metric with a constant proportionality factor. Writing \(A_{ij} = \lambda \gamma_{ij}\), the Hamiltonian constraint (8) becomes

\[
R(\gamma) = -6\lambda^2 + 16\pi \rho \quad \text{(or } R(\gamma) = -n(n-1)\lambda^2 + 16\pi \rho \text{ if } \Sigma \text{ is } n\text{-dimensional}).
\]

If the energy condition \(\rho \geq 0\) holds, then \(R(\gamma) > -6\lambda^2\).

In an asymptotically flat spacetime, slices with this property and \(\lambda \neq 0\) cannot reach spacelike infinity because the decay (19) is obviously not satisfied. This type of initial data is usually called ‘asymptotically hyperbolic’ (also ‘hyperboloidal’) since the simplest example is given by the hyperboloid \(t = \sqrt{r^2 + \lambda^{-2}}\) in the Minkowski spacetime. As a Riemannian manifold, this hypersurface is just the standard three-dimensional hyperbolic space of radius \(1/\lambda\) (i.e. \(\mathbb{R}^3\) endowed with the metric of constant negative curvature \(-\lambda^2\)). Asymptotically hyperbolic initial data sets approach null infinity provided \(\gamma_{ij}\) satisfies suitable asymptotic conditions.

In the context of the Penrose inequality, the model example consists of the spherically symmetric slices \(\Sigma\) of the Kruskal spacetime satisfying \(A_{ij} = \lambda \gamma_{ij}\). It turns out that for \(\lambda > 0\), \(\Sigma\) is fully contained in the advanced Eddington–Finkelstein portion of the spacetime and satisfies the equation (in advanced coordinates \((v, r, \theta, \phi)\))

\[
\frac{dv}{dr} = \frac{\lambda r - \sqrt{1 - \frac{2m}{r} + \lambda^2 r^2}}{(1 - \frac{2m}{r}) \sqrt{1 - \frac{2m}{r} + \lambda^2 r^2}}.
\]

These hypersurfaces approach future null infinity for \(r \to \infty\) and intersects the white hole event horizon on the surface \(S_t \subset \{r = 2m\}\), which has mean curvature \(p = 2\lambda\) (because \(q = 2\lambda\) on any surface of \(\Sigma\) and \(\theta_\text{e} = -p + q\) vanishes on the white hole event horizon). The induced metric on \(\Sigma\) is

\[
dt^2 = \frac{dr^2}{1 - \frac{2m}{r} + \lambda^2 r^2} + r^2 d\Omega^2
\]

and obviously satisfies \(R(\gamma) = -6\lambda^2\). This model space arises not only as an umbilic hypersurface of Kruskal, but also as time-symmetric hypersurfaces in spherically symmetric solutions of the vacuum Einstein field equations with a cosmological constant, i.e. satisfying \(\text{Ein}(g) = -\Lambda g\). The general spacetime solution is given by the so-called Kottler metric [106] (discovered independently by Weyl [166]) and reads as

\[
ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2.
\]

This metric is often called Schwarzschild–de Sitter (\(\Lambda > 0\)) or Schwarzschild–anti de Sitter (\(\Lambda < 0\)) since it contains both the Schwarzschild (when \(\Lambda = 0\)) and the de Sitter metrics (when \(m = 0\)). The Kottler metric admits an interesting generalization (still satisfying the vacuum Einstein equations with a cosmological constant) whereby the spheres \(t = \text{const}\) and \(r = \text{const}\) are replaced by any compact connected manifold \(N^2\) without boundary, and the metric is

\[
ds^2 = -\left(k - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{k - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2.
\]
where \( k = 1, 0, -1 \) depending, respectively, on whether the genus of \( N^2 \) is 0, 1 or higher. Here, \( d\Omega_k \) stands for the two-dimensional metric of constant curvature \( k \). The Kottler metric obviously corresponds to the case \( k = 1 \). For \( \Lambda < 0 \) and \( m > 0 \), let \( r_s \) be the only positive root of \( k - 2m/r - \Lambda r^2/3 = 0 \). The hypersurface \( \{ t = 0 \} \) of (48) restricted to the range \( r \geq r_s > 0 \) is a manifold with boundary which admits a smooth conformal compactification (see [38, 40]) to a manifold with boundary consisting of two copies of \( N^2 \). One of the copies corresponds to the inner boundary \( t = r_s \) and the other one to the ‘surface at infinity’, usually denoted by \( \partial \Sigma \).

In the case \( k = 1 \), the induced metric on \( \{ t = 0 \} \) is exactly (46) with \( \Lambda = -3\lambda^2 \). In this context, the surface \( S_1 = \{ r = 2m \} \) plays no special role regarding the Penrose inequality (because the surface is outer untrapped) and instead one has to look for minimal surfaces (similarly as in any time-symmetric case). Assuming again \( m > 0 \), the outermost minimal surface of (46) is given by \( S_2 = \{ r = r_s \} \). Since \( r_s < 2m \), we have that \( S_1 \) encloses \( S_2 \).

It follows from this discussion that the Penrose inequality in this context has two flavours, one for which the surface of interest has \( p = 2|\lambda| \) and another one where the surface to consider is minimal. The Riemannian manifolds \( (\Sigma, g) \) relevant for this setting satisfy \( R(\gamma) > -6\lambda^2 \), which is a consequence of (8) both when \( A_{ij} = \lambda \gamma_{ij} \) and \( \rho \geq 0 \) and when the spacetime satisfies \( \text{Ein}(g) = -\Delta g + 8\pi T \) provided \( \Lambda \equiv -3\lambda^2 \) and the energy–momentum \( T \) satisfies the weak energy condition.

The metric \( \gamma \) has to satisfy appropriate asymptotics so that a useful concept of mass can be defined. Both the appropriate asymptotic behaviour and the definition of mass are considerably more difficult than the corresponding definition in the asymptotically Euclidean case. In the case when the boundary at infinity \( \partial \Sigma \) is a 2-sphere, the definition was first given in [164]. The definition for other topologies at infinity (and admitting weaker asymptotic conditions) appears in [40] (see also [39] for a definition of mass from a spacetime viewpoint). For the \( \{ t = 0 \} \) slice of (48), the mass according to this definition is \( m \).

The statement of the Penrose inequality in this setting should be expected to be different when it involves outermost minimal surfaces or outermost surfaces with mean curvature \( p = 2|\lambda| \). In the case of boundaries with several connected components or when the topology of \( \partial \Sigma \) is not spherical, it is not clear how the precise statement of the inequality should read. However, when \( \partial \Sigma = S^2 \) and the inner boundary \( S \) is connected, a natural version of the inequality reads as

\[
M \geq (1 - g_S) \sqrt{\frac{|S|}{4\pi}} + \frac{\lambda^2}{2} \left( 1 - \epsilon \right) \left( \frac{|S|}{4\pi} \right)^{\frac{1}{2}},
\]

(49)

where \( M \) is the total mass, \( g_S \) is the genus of \( S \) and \( \epsilon = 0, 1 \) depending, respectively, on whether \( S \) is outermost minimal or outermost with mean curvature \( p = 2|\lambda| \). In the minimal case, this inequality has been proposed in [67] for the case \( g_S = 0 \) and in [38] for arbitrary genus (in this reference, the Penrose inequality in terms of the asymptotic value of the Geroch mass under a smooth inverse mean curvature flow is also discussed for \( \partial \Sigma \) of arbitrary genus). In [67], an inequality for \( \partial \Sigma \) of arbitrary genus also appears. However, as noted in [38], this version fails for the slice \( \{ t = 0 \} \) of the generalized Kottler metric when \( N^2 \) has at least genus 3. In the case of surfaces with \( p = 2|\lambda| \), inequality (49) with \( \epsilon = 1 \) was conjectured in [164]. Support for the validity of (49) comes from the fact that the \( \{ t = 0 \} \) slice of the Kottler metric gives equality. Moreover, by choosing the value \( C = -3\lambda^2 \), the Geroch mass (17) evaluated on a surface \( S \) with \( p = 2|\lambda| \) gives precisely the right-hand side of (49). This value of the constant \( C \) is adapted to the inequality \( R(\gamma) \geq -6\lambda^2 \) because then the Geroch mass is monotonic under a smooth inverse mean curvature flow [33, 67]. This can be seen explicitly from the general formula (18).
Thus, it is tempting to try and adapt Huisken and Ilmanen’s proof to the hyperbolic case, at least when the boundary $S$ is connected. However, a recent result by Neves [130] shows that this is not possible in general. The difficulty lies in the fact that the Geroch mass of a flow of surfaces moved by inverse mean curvature does not necessarily approach the total mass of the asymptotically hyperbolic manifold. This is proven in [130] by showing that in the manifold $\{r \geq 2m\} \times S^2$ with metric (46), there exist surfaces with the Geroch mass larger than $m$ and which can be flowed smoothly by inverse mean curvature all the way to infinity. Consequently, the limit of the Geroch mass under the flow is still larger than $m$ due to the monotonicity of $M_G$. In this example, the inner boundary does not satisfy $p = 2|\lambda|$. However, the author is able to modify the construction so that the flow starts on a horizon $(p = 2|\lambda|)$ and remains smooth for all values of the parameter in such a way that the leaves of the flow do not become rounder (in a precise sense) at infinity. Thus, according to the author, it becomes impossible to compare the limit value of the Geroch mass with the value of the total mass of the manifold, which makes the inverse mean curvature flow inconclusive for the Penrose inequality. Despite this failure of the inverse mean curvature method to prove inequality (49), its validity is still open.

Using isoperimetric surface methods [49] (see subsection 5.2), inequality (49) (with $\epsilon = 1$) has been proven in the special case that $(\Sigma, \gamma)$ outside a compact set is isometric to the Kottler metric (46) outside a sphere $r = \text{const}$ and the following two conditions are satisfied: (i) $(\Sigma, \gamma)$ contains a unique closed and connected surface $S$ with $p = 2|\lambda|$ and (ii) the isoperimetric surfaces $S_V$ (with respect to $S$) are connected and coincide with the spheres $r = \text{const}$ in the asymptotic region for $V$ large enough.

It is also interesting to note that the Penrose inequality in the hyperbolic setting may be a powerful tool to address the uniqueness problem of asymptotically hyperbolic static initial data sets [38]. More precisely, denoting by $U$ the square norm of the static Killing vector and by $M_G(U)$ the Geroch mass of the level sets of $U$, it is proven in [38] that the validity of the Penrose inequality

$$M_G(U) \geq (1 - g_S)\sqrt{\frac{|S|}{16\pi}} + \frac{\lambda^2}{2}\left(\frac{|S|}{4\pi}\right)^{\frac{3}{2}}$$

implies a uniqueness theorem for the generalized Kottler metric in the case that $\partial^\infty \Sigma$ if of genus larger than 1 and $S$ is connected. The reason is that in the static setting, the Geroch mass $M_G(U)$ can be bounded above in terms of the mass of the generalized Kottler solution with the same surface gravity $\kappa$ (provided this satisfies the inequality $0 < \kappa \leq |\lambda|$). The area $|S|$ is also bounded below by the area of the Killing horizon of the corresponding Kottler solution. Combining these inequalities with (50), it follows that equality in (50) is in fact the only possibility. However, the only static initial data set which saturates (50) turns out to be the $\{t = 0\}$ slice of the Kottler metric. Thus, a proof of the Penrose inequality for $M_G(U)$ would establish a uniqueness result for static initial data sets in the hyperbolic setting.

7. The general Penrose inequality

The validity of the Penrose inequality for arbitrary initial data sets (with an arbitrary second fundamental form and without the assumption of spherical symmetry) is still open. The proofs by Huisken and Ilmanen and Bray of the Riemannian Penrose inequality involve manifolds $\Sigma$ with a positive definite metric of non-negative curvature scalar admitting an outermost minimal surface. Although primarily intended to cover the time-symmetric case, the proofs only require the presence of a minimal surface and a Riemannian metric with non-negative
scalar of curvature. Thus, the method also gives useful results in the case of maximal hypersurfaces, \(\nabla^A A = 0\) (assuming the energy density \(\rho\) in (8) to be non-negative), provided the Riemannian manifold contains at least a bounding minimal surface. The Riemannian arguments in the previous section, however, do not settle the general Penrose inequality in the maximal hypersurface case for two reasons: first because they would give a lower bound for the ADM energy instead of the ADM mass and second because the outermost minimal surface does not coincide, in general, with any of the minimal area enclosures arising in any of the versions of the Penrose inequalities discussed in section 3 (except in the time-symmetric case, of course).

Although the general Penrose inequality remains open, several methods have been proposed to address it. We devote this section to discuss them.

7.1. Null shells of dust

As described in section 1, Penrose’s original set-up [136] to test the validity of cosmic censorship consisted of a shell of matter moving inwards at the speed of light in a flat spacetime. The shell is assumed to have closed and connected cross sections and the matter within the shell is made of null dust (meaning that the particles defining the shell are massless and that all pressures vanish). In order to have a flat metric inside the imploding shell, all points in their interior must be causally disconnected (to the past) with all points on the shell. Choosing a Minkowskian time \(t\) inside the shell, this demands that, to the past of some \(t = t_0\), the null hypersurface \(\mathcal{N}\) swept by the incoming shell of matter must be free of self-intersections. Since the matter within the shell moves at the speed of light, the cross section \(S_t \equiv \mathcal{N} \cap \{t = \text{const}\}\) can be viewed, after the natural identification of points in different instants of time, as the surface lying at distance \(t_0 - t\) from \(S_{t_0}\), where distance is positive to the exterior and negative to the interior (the fact that \(S_{t_0}\) separates the Euclidean space into an interior and an exterior is always true as a consequence of the Jordan–Brower separation theorem for smooth hypersurfaces in \(\mathbb{R}^n\); see e.g. [112]). The distance level function from a given closed surface in the Euclidean space is smooth everywhere in its exterior if and only if the surface is convex (i.e. it has non-negative principal curvatures). The set-up, therefore, requires that the null hypersurface has one cross section \(S_{t_0}\) which is convex. This property is then true for all \(t \leq t_0\). Towards the future, \(\mathcal{N}\) will become singular at the first focal point of the incoming null geodesics. A spacetime singularity will form there. Outside the null shell, the metric is no longer flat, in particular because gravitational waves may be emitted by the collapsing dust.

Penrose devised this physical process as a potential counterexample to inequality (1). The fundamental simplification of this problem is that the inequality can be translated into an inequality directly in the Minkowski space, as follows [66, 136, 156, 158].

Let \(\vec{l}^-\) be the future-directed null tangent to \(\mathcal{N}\) normalized to satisfy \(\vec{l}^- (t) = 1\), where, as above, \(t\) is a Minkowskian time in the interior part of the shell. Take any closed, spacelike surface embedded in \(\mathcal{N}\) and let \(\vec{l}^+\) be its future null normal satisfying \((\vec{l}^+ \cdot \vec{l}^-) = -2\). The energy–momentum of the spacetime is a distribution supported on \(\mathcal{N}\) which reads as

\[ T_{\alpha \beta} = 8\pi \mu l_{\alpha}^- l_{\beta}^- \delta, \]

where \(\mu\) is the energy density of the shell and the Dirac \(\delta\) is defined with respect to the volume form \(d\sigma\) induced by the normal \(l_{\alpha}^\mu\) to \(\mathcal{N}\), i.e. \(l_{\mu}^- d\sigma = \eta_{\mu\nu\rho\sigma} e_{\nu}^0 e_{\rho}^1 e_{\sigma}^2\), where \(e_{\alpha}^\mu = \frac{\partial x^\alpha}{\partial \eta^\mu}\) and \(\eta^\mu \to x^\alpha (\eta^\mu)\) is a coordinate expression for the embedding of \(\mathcal{N}\) (see e.g. [125] for details). The null expansion \(\theta_+\) jumps across \(\mathcal{N}\). The jump can be determined using the Raychaudhuri equation (4). One way of doing this is by extending the null vector \(l_{\pm}\) to a geodesic null congruence and taking its divergence on each side of the shell. This defines the null expansion \(\theta_+\) as a discontinuous function on the spacetime. A distribution can then...
be introduced as \( \theta_+ = \delta^E \Theta + \theta^t_1(1 - \Theta) \), where \( \Theta \) is the standard Heaviside distribution (it acts on test functions by integration on the domain outside the shell) and the superscript \( I(E) \) stands for interior (exterior) of the shell. Since \( \partial_\nu \Theta = -l_{\mu} \delta \), the derivative of \( \theta_+ \) along \( \tilde{\nu}_+ \) gives

\[
\tilde{l}^\mu \partial_\nu \theta_+ = \tilde{l}^\mu \partial_\nu \delta^E \Theta + \tilde{l}^\mu \partial_\nu \theta^t_1(1 - \Theta) + 2[\theta_+ \delta],
\]

where the jump \( [\theta_+] = \{(\delta^E \Theta - \delta^t_1)\} \). The Raychaudhuri equation (4), which in this case is a distributional equation, has a singular part supported on \( \mathcal{N} \) only through the term \(-\text{Ric}(\tilde{\nu}_+, \tilde{\nu}_+) = -32\pi \mu \delta \). The jump must therefore satisfy \( \theta^E|_{\mathcal{N}} = \theta^t_1|_{\mathcal{N}} - 16\pi \mu \).

For an arbitrary surface embedded within a null hypersurface, the expansion along the null direction tangent to the hypersurface coincides with the null expansion of the hypersurface. This means, in particular, that it only depends on the point where it is calculated but not on the specific surface passing through that point. This has as an immediate consequence the fact that the incoming null expansion \( \theta_- \) of the shell is continuous across the shell. On the surface \( S_0 \), the null expansion \( \theta_- \) coincides with the mean curvature of \( S_0 \) as a surface of the Euclidean space with respect to the inner normal and it is therefore non-positive (since \( S_0 \) is convex) and not everywhere zero (since \( S_0 \) is closed). It follows from the Raychaudhuri equation that \( \theta_- \leq 0 \) everywhere on \( \mathcal{N} \). Consequently, right after the shell has passed, a spacelike surface \( S \subset \mathcal{N} \) is marginally outer trapped (i.e. \( \theta^E = 0 \)) if and only if it is marginally future trapped.

Assume that such an \( S \) exists along the shell. We want to show that the Penrose heuristic argument based on cosmic censorship then implies \( M_B \geq \sqrt{\frac{16\pi}{16\pi}} \), where \( M_B \) is the Bondi mass (see e.g. [161]) for its definition) of past null infinity at the cut defined by \( \mathcal{N} \). Indeed, under cosmic censorship the singularity that necessarily forms in the future is shielded from infinity by an event horizon. Since \( S \) must be contained in the black hole region, the intersection \( \mathcal{H} \) of the event horizon with \( \mathcal{N} \) must lie completely in the causal past of \( S \). Using the fact that the null expansion \( \theta_- \) is non-positive, this implies \( |\mathcal{H}| \geq |S| \). Since the standard heuristic argument gives \( M_B \geq \sqrt{\frac{16\pi}{16\pi}} \), the above claim follows. Note that, in this case, the inequality is expected to hold for \( S \) irrespective of whether or not this surface is area outer minimizing with respect to any spacelike slice. Now using the conservation equation \( \nabla \alpha T^a_\alpha = 0 \) (which holds in a distributional sense; see e.g. [125]), it follows that the integral \( \int_S \mu \eta_S \) does not depend on the cut \( \hat{S} \) of \( \mathcal{N} \). Evaluating this integral at past null infinity gives precisely the Bondi mass (this can be easily shown, for instance, using the Hawking mass introduced in subsection 2.4). Now using the fact that \( S \) is marginally outer trapped from the exterior, we have, after defining \( \theta_+ \equiv \theta^E_+ \),

\[
\int_S \theta_+ \eta_S = \int_S \theta^t_1 \eta_S = \int_S 16\pi \mu \eta_S = 16\pi M_B \geq \sqrt{16\pi |S|},
\]

where the last step is precisely the Penrose inequality in this setting. Thus, the Penrose inequality for incoming shells can be rewritten as

\[
\int_S \theta_+ \eta_S \geq \sqrt{16\pi |S|}, \tag{51}
\]

which has the remarkable property of making no reference to the exterior geometry at all. Since the density \( \mu \) is freely specifiable, this inequality should hold for any closed spacelike surface \( S \) in the Minkowski spacetime for which the null hypersurface generated by past-directed and outer null geodesics orthogonal to \( S \) remains regular everywhere. A similar inequality can be derived in any spacetime dimension [66].

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In the particular case of a surface lying on the past null cone of a point \( p \), the surface \( S \) can be described by a single positive function \( r \) which measures the distance to \( p \) (after projection to a constant time hypersurface). A straightforward calculation gives the outer null expansion \( \theta_\ast \) and inequality (51) becomes

\[
\int_{S^2} \left( r + \frac{|dr|^2}{r} \right) \eta_{S^2} \geq -4\pi \int_{S^2} r^2 \eta_{S^2},
\]  

(52)

where all geometric objects refer to the standard metric of unit radius on the sphere. This inequality already appears in [136] and a more detailed derivation can be found in [12]. Its validity (in fact of a stronger version) was proven by Tod [156] as a consequence of the Sobolev inequality applied to a suitable class of functions on \( \mathbb{R}^3 \) (see [157] for a different proof which gives an even stronger inequality).

If \( S \) lies in a spacelike hyperplane in Minkowski, then \( \theta_\ast \) is the mean curvature \( p \) of \( S \) as a surface of the Euclidean space and the inequality becomes

\[
\int_{S} p \eta_{S} \geq \sqrt{16\pi|S|}.
\]  

(53)

As first noted by Gibbons in his PhD thesis, this inequality is exactly the Minkowski inequality for convex bodies in the Euclidean space (see e.g. [35]). This settles the Penrose inequality when \( S \) lies on a constant time hyperplane [66]. The range of validity of the Minkowski inequality (53) has been extended by Trudinger [160], to cover all mean convex bodies in the Euclidean space, i.e. all closed surfaces with \( p \geq 0 \). Gibbons has used this result to claim the validity of the Penrose inequality (51) in the general case. The idea of the argument was to project \( S \) orthogonally onto a constant time hyperplane. The projection \( \hat{S} \) can be seen to have at least the same area as \( S \), i.e. \( |\hat{S}| \geq |S| \). Furthermore, by direct calculation, the author finds that the mean curvature \( \hat{p} \) of the projected surface is non-negative and that \( \int_{\hat{S}} \theta_{\ast} \eta_{S} = \int_{\hat{S}} \hat{p} \eta_{\hat{S}} \). Thus, the Penrose inequality would follow from Trudinger’s version of the Minkowski inequality. Unfortunately, the calculation leading to \( \hat{p} \geq 0 \) and \( \int_{\hat{S}} \theta_{\ast} \eta_{S} = \int_{\hat{S}} \hat{p} \eta_{\hat{S}} \) contains an error which invalidates the argument. Instead of going into the details of the derivation, it is simpler to just note that one can easily construct a surface on a null hypersurface \( \mathcal{N} \) in the Minkowski spacetime which has a projection \( \hat{S} \) which is not mean convex. Consider the past null cone of a point at \( t = 1 \), and consider the sphere obtained as the cross section of \( \mathcal{N} \) with \( t = 0 \). A function \( s \) on \( S^2 \) taking values in \([0,1]\) defines a surface \( \hat{S} \) on the hyperplane \( \{t = 0\} \) simply by moving radially inwards each point of the sphere a distance \( s \). It is easy to construct surfaces \( \hat{S} \) which are not mean convex. Consider, for instance, a surface of revolution defined by \( s(\theta) \) with equatorial symmetry and with a neck on the equator (i.e. such that \( s(\theta) \) is symmetric under \( \theta \rightarrow \pi - \theta \) and \( s_0 = s(\theta = \pi/2) \) is a local maximum). The principal curvatures on a point on the equator is simply \( 1/(1-s_0) \) and \( 1/(1-s_0) + s''_0/(1-s_0)^2 \), where \( s''_0 \) is the second derivative of \( s(\theta) \) at the equator. Thus, if \( s''_0 < -2(1-s_0) \) (i.e. the second derivative on the equator is negative and sufficiently large in absolute value on the equator), then the surface is not mean convex. But \( \hat{S} \) is obviously the orthogonal projection on \( t = 0 \) of the surface on \( \mathcal{N} \) constructed by lifting each of its points a temporal amount \( s \) to the future (see figure 3). This example shows that the projection performed in [66] is not correct. The validity of the Penrose inequality for null shells is therefore still open (in any spacetime dimension).

Analogously as in (52), inequality (51) can be rewritten in terms of the geometry of an arbitrary closed and convex surface \( S_0 \) in \( \mathbb{R}^3 \) and a function \( s \) defined on \( S_0 \), as
\[ \int_{S_0} \left( P_0 - 2s K_0 \right) \left( 1 + g_0^{AB} \partial_A s \partial_B s \frac{1 - 2s P_0 + s^2 (P_0^2 - K_0)}{(1 - s P_0 + s^2 K_0)^2} \right. \\
\left. + \kappa_0^{AB} \partial_A s \partial_B s \frac{s (2 - s P_0)}{(1 - s P_0 + s^2 K_0)^2} \right) \eta S_0 \geq \sqrt{16\pi \int_{S_0} (1 - s P_0 + s^2 K_0) \eta S_0}, \quad (54) \]

where \( g_0^{AB} \) and \( \kappa_0^{AB} \) are, respectively, the induced metric and second fundamental form of \( S_0 \) with respect to the outer normal, \( P_0 \) and \( K_0 \) are the mean curvature and the Gauss curvature of \( S_0 \) or, in terms of the principal curvatures \( \lambda_1, \lambda_2 \) of the surface, \( P_0 = \lambda_1 + \lambda_2 \) and \( K_0 = \lambda_1 \lambda_2 \). The smooth function \( s \) must satisfy the inequalities \( s < \frac{1}{\lambda_1} \) and \( s < \frac{1}{\lambda_2} \), but is otherwise arbitrary. It defines the surface \( \hat{S} \) (and hence \( S \)) in a similar way as before. Details of the derivation of (54) and some of its consequences will be given elsewhere [115].

7.2. Spinor techniques on null hypersurfaces: Ludvigsen–Vickers and Bergqvist approaches

Witten’s proof of the positive mass theorem is based on the properties of spinors satisfying a suitable elliptic equation and which approach a constant spinor at spatial infinity. The same ideas have been applied to prove positivity of the Bondi mass using asymptotically hyperbolic initial data sets (see [41] and references therein). Another possibility of approaching the positivity of the Bondi mass is to use null hypersurfaces. This allowed Ludvigsen and Vickers [113] to replace the elliptic equations for the spinor by much simpler transport equations. Similar ideas (with different transport equations) allowed the same authors [114] to argue that, in spacetimes satisfying the dominant energy condition, the inequality \( M_B \geq \sqrt{|S|/16\pi} \) holds for any weakly future trapped (in particular, marginally future trapped) surface \( S \) which has the property that one of the two null hypersurfaces generated by past-directed null geodesics normal to \( S \) can be extended to past null infinity while remaining smooth everywhere. Although this is a global assumption on the spacetime, it makes no reference to the future evolution of the spacetime and hence it is logically independent of cosmic censorship. For instance, this assumption is automatically satisfied for the incoming null shell of dust discussed in the previous section. However, Bergqvist [21] found a gap in the proof and the range of validity of the argument remains, at present, open. We discuss this next.

Bergqvist reformulated Ludvigsen and Vickers’ argument so that all spinors could be completely dispensed of. The idea is, in some sense, analogous to the inverse mean curvature.
flow for the Geroch mass and is based on using a functional on spacelike closed surfaces $S_\mu$ constructed as follows: start with a closed spacelike surface $S$ and let $\mathbf{l}$ and $\mathbf{k}$ be past-directed null normals to $S$ partially fixed by the normalization $(\mathbf{l} \cdot \mathbf{k}) = -2$. Also assume that $\mathbf{l}$ points outwards of $S$ in the sense that the null geodesics starting on $S$ with tangent vector $\mathbf{l}$ extend to infinite values of the affine parameter and intersect $I^-$. Let $\mu$ be the affine parameter of this geodesic with $\mu = \mu_0$ on $S$, where $\mu_0$ is a constant to be chosen later and assume that the surfaces $\{\mu = \text{const}\}$ are smooth for all $\mu \geq \mu_0$. The collection of all these surfaces defines a null hypersurface $N$ (the ‘outer past null cone’ of $S$) which is assumed to intersect $I^-$ on a smooth cut. The Bergqvist mass is defined on each leaf $\{S_\mu\}$ of this foliation as

$$M_b(S_\mu) = \frac{1}{16\pi} \left( \int_{S_\mu} \theta_k \eta_{S_\mu} + 4\pi \chi(S_\mu) \mu \right),$$

where $\theta_k$ is the null expansion along $\mathbf{k}$. Using expressions (3) and (7), it is easy to obtain

$$\mathbf{l} \left( \int_{S_\mu} \theta_k \eta_{S_\mu} \right) = \int_{S_\mu} (\text{Ein}(\mathbf{l}, \mathbf{k}) + 2S_A^S A^4) \eta_{S_\mu} - 4\pi \chi(S_\mu),$$

where the Gauss–Bonnet theorem $\int_{S} R(h) \eta_{S} = 4\pi \chi(S)$ has been used. We therefore get

$$\frac{dM_b(S_\mu)}{d\mu} = \frac{1}{16\pi} \int_{S_\mu} (\text{Ein}(\mathbf{l}, \mathbf{k}) + 2S_A^S A^4) \eta_{S_\mu} \geq 0,$$

provided the dominant energy condition is satisfied. Assuming that $\mu_0$ can be chosen so that asymptotically $\theta_k = -2/\mu + O(1/\mu^2)$, i.e. without a term in $\mu^{-2}$, then the Bergqvist mass can be seen to approach the Bondi mass when $\mu \to \infty$. Assuming now that the initial surface $S$ is a marginally trapped surface, if the value of $M_b$ could be somehow related to the area of $S$, a Penrose inequality would follow. Bergqvist [21] shows that this can indeed be done but only under the assumption that the induced metric $g_{AB}^{S_\mu}$ of the surface $S_\mu$ becomes round at infinity in the sense that $\lim_{\mu \to \infty} \mu^{-2}\ g_{AB}^{S_\mu} \to g_{AB}^S$, where $g_{AB}^S$ is the standard unit metric on the sphere. At present, it is not clear how restrictive is this requirement. The idea however remains interesting and deserves further investigation.

7.3. Uniformly expanding flows

The success of proving the Riemannian Penrose inequality in the connected horizon case using the monotonicity of the Geroch mass under the (weak) inverse mean curvature flow suggests a possible scenario for approaching the Penrose inequality in the general case. As discussed in subsection 2.4, the Hawking mass (13) is a functional on surfaces which coincides with the Geroch mass in the time-symmetric context. Moreover, the Hawking mass (with $C = 0$) takes the value $\sqrt{3\pi/(16\pi)}$ on any topological 2-sphere which is either a marginally outer trapped surface, a past marginally outer trapped surface or a generalized apparent horizon (all of which have a null mean curvature vector). Moreover, as discussed in subsection 2.4, under suitable spacetime variations of a given surface, the Hawking mass is monotonically increasing [28]. Since its numerical value for large coordinate spheres in the asymptotically Euclidean region approaches the ADM energy, a flow which interpolates between the horizon and infinity and falls into any of the four monotonicity cases discussed in subsection 2.4 would imply the general Penrose inequality between the ADM energy and the area of the horizon. Among the four cases, the closest one to the Riemannian inverse mean curvature flow is the so-called uniformly expanding flow, defined by (15).

With these monotonicity properties, the situation regarding the general Penrose inequality can be compared to the status of the Riemannian Penrose inequality after Geroch’s heuristic
It is conceivable that the uniformly expanding flows might be useful for the proof of the general Penrose inequality. However, many issues remain open. For instance, in a spacetime formulation, the jumps that occurred in the Riemannian setting will remain, but it is unclear how and when the jumps should take place, even from a purely heuristic point of view. Moreover, the inverse mean curvature flow is a parabolic equation in the Riemannian setting, so that local existence is guaranteed, but the situation is quite different for the uniformly expanding flows.

Local existence in this case has been proven only for null flows, i.e. $|c| = a$. A null flow will obviously not reach spacelike infinity. Nevertheless, assuming that a sufficiently large portion of the spacetime is at hand, this null flow could be used to study the Penrose inequality involving the Bondi mass. The difficulty, however, is that the uniformly expanding null flow seems to have the tendency to deform the surface in such a way that the mean curvature vector becomes causal at some places, even if one starts with a surface with spacelike mean curvature everywhere. This behaviour is observed in explicit examples in the Minkowski spacetime (provided the surfaces do not lie on a constant time hyperplane and in fact cover a sufficiently large time interval). One alternative that may still give interesting results is to use a flow which is null and future directed for some interval and then continue with a null past-directed uniformly expanding flow, then with a future-directed null flow and so on. This flow can be constructed in a tubular neighbourhood of the given initial data set, and therefore does not need strong global assumptions on the spacetime. Moreover, it is conceivable that this broken flow can approach spacelike infinity. However, it is not clear which criterion should be used to stop the future flow and continue with the past one (and vice versa). Moreover, the construction must be such that the mean curvature vector remains spacelike everywhere in order to ensure monotonicity of the Hawking mass.

Regarding the non-null case, $|c| < a$, the flow equations form a so-called forward–backward parabolic system, for which no local existence theory is known (this was noted by Huisken and Ilmanen [89] in the case $c = 0$ and was extended to arbitrary $c$ in [28]). This is, of course, a major difficulty and addressing it would require a much better understanding of this type of partial differential equations. In Huisken and Ilmanen’s work, a fundamental part of the analysis dealt with the level set formulation, which gives a degenerate elliptic equation. More specifically, the existence of a variational formulation turned out to be a fundamental ingredient to solve the problem of existence and to study the jumps. Remarkably, the uniformly expanding flows also admit a variational formulation [28]. The new basic ingredient is that the field to be varied is not just the level set function $v$ (see (36)) as in the Riemannian case, but also the spacelike hypersurface $\Sigma_1$ where this function is defined. Whether this variational formulation can give useful hints on how to define the jumps remains an open and difficult problem.

It should be remarked in this context that the simplest spacelike uniformly expanding flow corresponds to $c = 0$, i.e. $\xi = \frac{1}{(H H)} H (\alpha = 1$ can be chosen without loss of generality$)$. The flow vector is therefore the inverse mean curvature vector. In terms of initial data information $\Sigma, g_{ij}, A_{ij}$, this corresponds to the case $q = 0$ (this condition has been often termed ‘polar gauge’ in the literature). Assuming this gauge condition and a second restriction which turns out to coincide with the inverse mean curvature flow condition $pe^\psi = \text{const}$, Jezierski was able to prove [99, 100] the Penrose inequality for small (but nonlinear) electrovacuum perturbations of the Reissner–Nordström time-symmetric initial data outside the black hole event horizon. His method was based on writing the Hamiltonian constraint as a divergence term plus a non-positive reminder. The Penrose inequality is established by integrating this equation between the event horizon and infinity after using the Gauss theorem to transform the divergence into a surface integral at infinity and on the horizon (the former gives the ADM
energy and the latter the area term in the inequality). Although Jezierski’s argument does not use the monotonicity of the Hawking mass, the calculation does in fact correspond to the general identity (16) specialized to the case at hand. With hindsight, one can therefore conclude that this monotonicity property of the Hawking mass is the underlying reason why the argument works. Jezierski supports the plausibility of the gauge conditions \( q = 0 \) and \( p e^\psi = 1 \) by studying linear perturbations of Reissner–Nordström, where he finds that the two equations decouple, one giving a parabolic equation that needs to be integrated radially outwards and another one also parabolic but which needs to be integrated radially inwards. This is, of course, a manifestation of the forward–backward parabolic nature of the full system. A similar existence result for linear, axial (i.e. odd) perturbations of maximal slices of the Schwarzschild spacetime has been obtained in [138].

Another observation worth mentioning regarding the inverse mean curvature vector flow (i.e. \( c = 0 \)) is that the monotonicity formula turns out to be insensitive to the value of the energy flux \( \vec{J} \). It follows that the weak energy condition is already sufficient to ensure monotonicity of the Hawking mass in this context. In principle, this opens up the possibility that the full Penrose inequality might be true for spacetimes satisfying just the weak energy condition. This is not so, however. An explicit counterexample for scalar field initial data has been constructed by Husain [91].

7.4. Jang equation

Schoen and Yau’s proof of the positive mass theorem proceeded in two steps. First, the purely Riemannian case (i.e. vanishing second fundamental form) was solved by using minimal surface techniques. In a second step, the general case \((\Sigma, \gamma_{ij}, A_{ij})\) was treated by modifying the metric \( \gamma_{ij} \) with a transformation introduced by Jang [95], namely

\[
\hat{\gamma}_{ij} = \gamma_{ij} + \partial_i f \partial_j f, \tag{55}
\]

where \( f \) solves the so-called Jang equation

\[
\left( \gamma^{ij} \frac{\nabla^i f \nabla^j f}{1 + |df|^2_{\gamma}} \right) \left( \frac{\nabla_i \nabla_j f}{\sqrt{1 + |df|^2_{\gamma}}} - A_{ij} \right) = 0. \tag{56}
\]

This transformation has the property that the curvature scalar of \( \hat{\gamma} \) has a lower bound that allows one to prove the existence of a conformal factor \( \Omega > 0 \) such that the conformally rescaled metric \( \Omega^2 \hat{\gamma}_{ij} \) has vanishing scalar curvature. This metric is still asymptotically Euclidean and has at most the same mass as the original metric. The Riemannian positive mass theorem then gives the desired result. The proof is however involved because the Jang equation does not admit regular solutions when the initial data set \((\Sigma, \gamma_{ij}, A_{ij})\) contains marginally trapped surfaces, but the idea can nevertheless be made to work [143].

A natural question, already asked in [22], is whether a similar idea can be applied to prove the general Penrose inequality. The Penrose inequality has already been proven in full generality in the Riemannian setting so the present status is similar to the situation of the proof of the positive mass theorem after its Riemannian proof. This idea has been analysed by Malec and Ó Murchadha [123]. Their argument is based on the observation that if the Jang equation could be used to prove the general Penrose inequality, it should be able to do so in the particular case of spherical symmetry. Restricting to spherically symmetric functions \( f \), the Jang equation becomes a simple ODE and existence of regular solutions in the exterior region outside the full trapped region \( T^+_\Sigma \cup T^-_\Sigma \) can be easily shown. The difficulty of the method is that along the process (Jang’s transformation and subsequent conformal rescaling), the mass of the manifold should not increase and the area of the outermost horizon \( S = \partial (T^+_\Sigma \cup T^-_\Sigma) \)
should not decrease. This is because one wants to use the Riemannian Penrose inequality for the final manifold and conclude that the same inequality holds in the original initial data. It is also clear that $S$ must be transformed into a minimal surface after the conformal rescaling. The simplest situation where this can be achieved is by demanding that $S$ become minimal for the Jang-transformed metric $\hat{\gamma}_{ij}$. This requires that the outer normal derivative of $f$ diverges to either $+\infty$ or $-\infty$ on $S$. In the first case, it follows that the metric $\hat{\gamma}$ has a cylindrical end near $S$. The conformal transformation is expected to compactify this end with one point (this behaviour was found in [143], and this was important in order to apply the Riemannian positive mass theorem). Hence, the area of $S$ decreases in the process (it vanishes in the final manifold) and nothing can be concluded. In the second case (outer normal derivative of $f$ diverging to $-\infty$), the conformal factor $\Omega$ is expected to have an interior local minimum. This would imply that the outer normal derivative of $\Omega$ is negative on $S$, and hence that this surface is not the outermost minimal surface in the final manifold. Thus, the Riemannian Penrose inequality applied to the final manifold is again inconclusive for the original one. Although these arguments are not definitive in discarding the Jang equation method for the Penrose inequality, they indeed show that difficulties should be expected for the method to work, at least when $S$ is required to transform to a minimal surface by the Jang transformation. The situation where $S$ is minimal only after the final conformal transformation is not considered in [123] and should be further investigated.

Very recently, the Jang equation has been successfully applied to prove a Penrose-like inequality in the spirit of Herzlich’s inequality discussed in subsection 5.1, but allowing a non-vanishing second fundamental form. The class of initial data $(\Sigma, \gamma_{ij}, A_{ij})$ under consideration is such that $\Sigma = K \cup \Sigma^\infty$ with $K$ being compact and $\Sigma^\infty$ being an asymptotically Euclidean end. The boundary $\partial \Sigma$ is compact and consists of a finite collection of future MOTS (with respect to the normal pointing towards $\Sigma$). Moreover, no weakly future or past trapped surface strictly enclosing $\partial \Sigma$ is allowed to exist in $\Sigma$. In other words, $\partial \Sigma$ is the outermost MOTS in $\Sigma$ and, moreover, no past weakly outer trapped boundary is allowed to exist in $\Sigma$ except possibly $\partial \Sigma$ itself. Also assuming that $(\Sigma, \gamma_{ij}, A_{ij})$ satisfies the dominant energy condition $\rho \geq |\vec{J}|$, Khuri has recently proven [108] that the Penrose-like inequality

$$E_{\text{ADM}} \geq \frac{\hat{\sigma}}{2(1 + \hat{\sigma})} \sum_{a=1}^{k} \frac{[\partial_a \Sigma]}{\pi}$$

(57)

holds, where $k$ is the number of connected components of $\partial \Sigma$ and the scale-invariant quantity $\hat{\sigma}$ is defined as

$$\hat{\sigma} = \frac{1}{\sum_{a=1}^{k} \sqrt{4\pi |\partial_a \Sigma|}} \inf_{\eta \in C^\infty} \int_{\Sigma} (dv, dv)_{\eta}.$$ 

The infimum is taken with respect to functions $v$ with approach 1 at infinity and 0 on $\partial \Sigma$. The metric $\bar{\gamma}$ is constructed using the Jang transformation (55) and $f$ is a solution to the Jang equation approaching zero at infinity and blowing up to $+\infty$ on $\partial \Sigma$. The existence of such an $f$ has been established by Metzger in [128]. The asymptotic behaviour of $f$ at infinity is such that $(\Sigma, \bar{\gamma})$ is asymptotically Euclidean and $E_{\text{ADM}}(\bar{\gamma}) = E_{\text{ADM}}(\gamma)$. Since $f \to \infty$ on $\partial \Sigma$, the level sets $S_T \equiv \{ f = T \}$, for $T$ large enough, form a foliation near $\partial \Sigma$ which converge to $\partial \Sigma$. The idea is to conformally transform $\bar{\gamma}$ outside $S_T$ in such a way that the conformally rescaled metric $\gamma^T_{ij} = u^T_{\bar{\gamma}} \bar{\gamma}_{ij}$ is still asymptotically Euclidean, with vanishing curvature scalar and such that each connected component $S_{T,a}$ of $S_T$ satisfies $p^T_a = 4\sqrt{\pi/|S_{T,a}|}$, where $p^T_a$ is the mean curvature of $S_{T,a}$ with respect to $\gamma^T_{ij}$ and the area is also calculated with this metric. Existence of $u^T_T$ is established as a consequence of the positivity properties of $R(\bar{\gamma})$ and the blowing up behaviour of $f$ at $\partial \Sigma$. The conformal rescaling is such that the ADM energy
decreases by an amount which is at least equal to the right-hand side of (57) except for terms that vanish in the limit $T \to \infty$. The only remaining step to conclude (57) is that the ADM energy of $\gamma^T$ is non-negative. But this is precisely the content of the positive mass theorem proven by Herzlich [81] for asymptotically Euclidean manifolds with a connected boundary of spherical topology and satisfying (31). Khuri notes that this positive mass theorem still holds if the boundary has a finite number of connected components of spherical topology, each one satisfying bound (31). In the case at hand, this inequality is satisfied (in fact, saturated) by construction and the spherical topology is guaranteed by Galloway and Schoen’s results [59, 60] on the topology of outermost MOTS.

Comparing this Penrose-like inequality with the difficulties described by Malec and Ó Murchadha to use the Jang transformation to prove the general Penrose inequality in spherical symmetry, the main difference is that this method ultimately relies on a positive mass theorem instead of on the Riemannian Penrose inequality. Thus, there is no need to obtain an outermost minimal boundary after the metric is modified by the Jang transformation and the subsequent conformal rescaling.

7.5. Bray and Khuri approach

Very recently, Bray and Khuri have made an important step forward towards establishing the general Penrose inequality. As mentioned in section 3, these authors propose to use generalized apparent horizons as the appropriate surfaces for which the Penrose inequality should hold. An important guiding principle that led Bray and Khuri to make this conjecture is related to the fact that, independently of which method for proving the inequality is used, the estimates involved must all give equality whenever $(\Sigma, g_{ij}, A_{ij})$ is any of the slices of the Kruskal spacetime for which equality holds. Therefore, a preliminary issue of importance is: for which slices of the Kruskal spacetime should equality be expected? Any spacelike Cauchy slice $\Sigma$ of the Kruskal spacetime must intersect both the black hole and the white hole event horizons. In Kruskal coordinates, the metric reads as (see [161])

$$ds^2 = \frac{32m^3}{r} e^{-\frac{\nu}{2}} du \, dv + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2),$$

where $uv > -1$ and $r(uv)$ is defined by $uv = e^{-\nu} \left( \frac{1}{2} - 1 \right)$ (our sign convention for $u$ is different to that in [161]). The null vectors $\partial_v$ and $-\partial_u$ are future directed, the black hole event horizon is located at $u = 0$, the white hole event horizon at $v = 0$ and the domain of outer communications $U$ is located at $[u > 0, v > 0]$. If the boundary of $\Sigma^{DOC} \equiv \Sigma \cap U$ satisfies $u = 0$ everywhere, then $\partial \Sigma^{DOC}$ is an area outer minimizing MOTS. Moreover, since its area is $16\pi m^2$ and there are no other weakly outer trapped surfaces (future or past) in $\Sigma^{DOC}$, it follows that this slice satisfies any of the versions of the Penrose inequality (22), (24) or (25). It also satisfies the inequality involving generalized apparent horizons (26) provided there are no generalized trapped surfaces in $\Sigma^{DOC}$ (this is plausible although not yet proven, as far as I know). Similar things happen if $v = 0$ everywhere on the boundary of $\Sigma^{DOC}$. However, if neither $u$ nor $v$ are identically zero on $\partial \Sigma^{DOC}$, then the situation is quite different. In this case, the boundary $\partial \Sigma^{DOC}$ is neither a future or past MOTS and, in most cases, it is not even smooth. Moreover, the intersection of $\Sigma$ with the $\{u = 0\}$ hypersurface (which is always a MOTS and has area $16\pi m^2$) is not area outer minimizing because its mean curvature $p$ is negative whenever $v < 0$ (i.e. on the points lying in the white hole event horizon with respect to the second asymptotically flat spacetime region). Thus, its minimal area enclosure has a strictly less area. Consequently, the version (22) of the Penrose inequality holds but not with equality (this statement assumes that $\{u = 0\} \cap \Sigma$ is the outermost MOTS of the slice, which is again plausible but not yet proven, as far as I know). It follows that not all
slices of the Kruskal spacetime are automatically equality cases, at least for the version (22). Obviously, the more the slices of Kruskal satisfy equality, the sharper is the version of the Penrose inequality, in the sense of being capable of identifying the Kruskal spacetime in a larger number of cases. A version that gives equality for any slice of the Kruskal spacetime is (25), even when the boundary of $\Sigma^{DOC}$ is non-smooth. Although, as already mentioned, no counterexample to this version has been found, its validity would however come as a surprise because the minimal area enclosure of $\partial (T^+ \cup T^-)$ may a priori have a much smaller area than itself. The alternative put forward by Bray and Khuri involves generalized trapped surfaces. This has the advantage that, as soon as $\Sigma^{DOC}$ has a smooth boundary, this is a generalized trapped surface (in fact, a generalized apparent horizon). Eichmair’s result (see subsection 2.3) implies that an outermost generalized apparent horizon must exist on $\Sigma$. It is highly plausible that $\partial \Sigma^{DOC}$ is in fact the outermost apparent horizon in this case. Since its area is $16\pi m^2$, any slice with smooth $\partial \Sigma^{DOC}$ would belong to the equality case of the Penrose inequality (26). If the boundary $\partial \Sigma^{DOC}$ is not smooth, then it cannot be its own minimal area enclosure, and hence it cannot give equality neither in (22) nor in (24).

These considerations led Bray and Khuri to conjecture the following version of the Penrose inequality. (Recall that our definition of asymptotically Euclidean includes the condition that $(\Sigma, \gamma_{ij})$ be complete and recall also that an initial data set is called Schwarzschild at infinity if outside a compact set the metric $\gamma$ is exactly Schwarzschild. See the discussion after (38).)

**Conjecture 1**[29]. Suppose that the initial data set $(\Sigma, \gamma_{ij}, A_{ij})$ is asymptotically Euclidean and Schwarzschild at infinity, with total mass $M_{ADM}$ (in a chosen end) and satisfying the dominant energy condition $\rho \geq |\vec{J}|$. Let $S$ be a closed surface which bounds an open set $U$ containing all the asymptotically Euclidean ends except the chosen one. Assume that $S$ is a generalized trapped surface (with respect to the normal pointing towards the chosen end). Then

$$M_{ADM} \geq \sqrt{\frac{|S_{min}|}{16\pi}},$$

where $S_{min} = \partial U_{min}$ is the minimal area enclosure of $S$ (i.e. $U \subset U_{min}$ and $S_{min}$ has least area among all surfaces with this property). Furthermore, equality occurs if and only if $(\Sigma \setminus U_{min}, \gamma_{ij}, A_{ij})$ is the induced data of an embedding of $\Sigma \setminus U_{min}$ into the Kruskal spacetime such that $S_{min}$ is mapped to a generalized apparent horizon.

The use of generalized apparent horizons is indeed a completely new idea for the Penrose inequality. This version is not supported by Penrose’s heuristic argument of gravitational collapse because it is not true that all generalized apparent horizons lie inside the event horizon in a black hole spacetime. In fact, little is known in general about this type of surfaces in general spacetimes. An important question regarding the plausibility of conjecture 1 was posed by Wald [163] who asked whether there are any generalized apparent horizons in the Minkowski spacetime (the question of whether surfaces with causal mean curvature can exist in the Minkowski spacetime was already asked in [126] in a somewhat different context). The existence of any generalized apparent horizon embedded in a spacelike Cauchy slice of Minkowski and bounding a compact set would immediately falsify conjecture 1. Khuri has been able to prove [109] that no such surfaces are present in the Minkowski spacetime. In fact, he proves much more by showing that any asymptotically Euclidean initial data set satisfying the dominant energy condition and with a compact (non-empty) boundary consisting of finitely many generalized trapped surfaces satisfies a strict positive mass theorem $E_{ADM} > |\vec{P}_{ADM}|$. The proof is based on Witten’s spinorial method for the positive mass.
The strategy that Bray and Khuri propose to address conjecture 1 is related to Schoen and Yau’s reduction of the general positive mass theorem to the time-symmetric case. Recall that this was based on the Jang transformation (55) of the metric, where the function \( f \) solves the Jang equation. This equation is specifically tailored so that it always admits a solution for any initial data set in the Minkowski spacetime (the solution is the height function of the slice over any constant time hyperplane). This is relevant because slices of Minkowski immediately give equality in the positive mass theorem. A fundamental observation of Bray and Khuri is that, since the equality case of the Penrose inequality should correspond to the Schwarzschild metric, the Jang equation should be accordingly modified so that it holds identically for slices in this spacetime. More generally, Bray and Khuri consider static spacetimes \((M, g) = (\mathbb{R} \times \Sigma, ds^2 = -\phi^2 dt^2 + \gamma)\), where \( \phi > 0 \) and \( \gamma \) is a Riemannian metric. The idea is to consider spacelike graphs \((t = f(x), x)\) for \( x \in \Sigma \) and derive an equation for the graph function which holds identically in this case and which still makes sense for an arbitrary initial data set \((\Sigma, \gamma_{ij}, A_{ij})\).

The induced metric on the graph is

\[
\gamma_{ij} = \gamma_{ij} - \phi^2 \partial_i f \partial_j f,
\]

which is Riemannian provided the gradient of \( f \) is not too large with respect to \( \gamma \), namely if \( \phi \partial_i |df|_\gamma < 1 \). However, this expression can also be used to define \( \gamma_{ij} \) given an initial data set \((\Sigma, \gamma_{ij}, A_{ij})\) and two arbitrary functions \( \phi \) and \( f \). This implies that the Riemannian metric \( \gamma \) of any initial data set can be obtained as the induced metric of a hypersurface in a suitably constructed static spacetime. With this point of view, the function \( f \) becomes arbitrary (with no restriction on its gradient) due to the identity

\[
(1 - \phi^2 |df|_\gamma^2) (1 + \phi^2 |df|_\gamma^2) = 1.
\]

The inverse metric is

\[
\gamma^{ij} = \gamma^{ij} - v^i v^j \quad \text{(58)}
\]

where indices are lowered and raised with the metric \( \gamma_{ij} \) and its inverse. The second fundamental form on the graph is [29]

\[
\overline{A}_{ij} = \phi \nabla_i \nabla_j f + \nabla_i \phi \nabla_j f + \nabla_j \phi \nabla_i f \quad \text{(59)}
\]

Of course, this second fundamental form has nothing to do \textit{a priori} with the second fundamental form \( A_{ij} \) of the given initial data set. Nevertheless, \((\Sigma, \gamma_{ij}, \overline{A}_{ij})\) is the induced geometry of a spacelike hypersurface in a static spacetime. The existence of the isometry generated by \( \partial_t \) can be used to relate the geometry of this slice to the geometry of the corresponding \( \{t = 0\} \) slice, i.e. of \((\Sigma, \overline{\gamma})\). After an involved calculation, this observation leads to the following remarkable identity for the curvature scalar \( R(\overline{\gamma}) \) [29], called the \textit{generalized Schoen–Yau identity}:

\[
R(\overline{\gamma}) = 16\pi (\rho - J_i v^i) + ||A - A||^2 + 2|\overline{z}|^2 - 2 \phi \text{ div}_\gamma (\phi \overline{z})
\]

\[
+ \text{tr}_\gamma(A - A)[\text{tr}_\gamma(A + A) + 2A_{ij} v^i v^j] + 2 v^i \partial_i (\text{tr}_\gamma(A - A)), \quad \text{(60)}
\]

where \( \overline{z}_i = (A_{ij} - \overline{A}_{ij}) v^j \) and \( \overline{z} \) is obtained after raising its index with the inverse of the metric \( \overline{\gamma}_{ij} \). In the case \( \phi = 1 \), this identity was obtained by Schoen and Yau [143] and was used to show that any solution \( f \) to the Jang equation defines a metric \( \overline{\gamma}_{ij} \) which admits a conformal rescaling with non-negative curvature scalar.

Bray and Khuri’s approach aims at finding appropriate functions \( \phi \) and \( f \) so that the Penrose inequality for \((\Sigma, \gamma_{ij}, A_{ij})\) follows as a consequence of the Riemannian Penrose
admit regular solutions when \( \partial/\Sigma_1 \). The hope is that this singular behaviour on the boundary can be adjusted so that the surface \( \partial/\Sigma_1 \) is area outer minimizing in \( \Sigma \). Thus, an upper bound for \( \tilde{M}_{\text{ADM}} \) implies an upper bound for \( |\delta \Sigma|_\gamma \), which is the type of information relevant for the full Penrose inequality.

Showing the existence of solutions of the generalized Jang equation (61) which satisfy this criterion is a fundamental open issue in this approach. Nevertheless, by construction there is an interesting particular case where (61) admits solutions, namely when the initial data set \( (\Sigma, \gamma_{ij}, A_{ij}) \) is in fact a slice of a static spacetime, i.e. when there exist functions \( \phi > 0 \) and \( f \) such that \( A_{ij} = \tilde{A}_{ij} \). In this case not only the generalized Jang equation holds trivially, but also \( \tilde{z} = 0 \). Using (60), this implies \( R(|\tilde{\varphi}|) \geq 0 \). If \( f \) decays fast enough at infinity (for inequality applied to the transformed data \( (\Sigma, \tilde{\varphi}) \). In order to use the Riemannian Penrose inequality, it is necessary that \( R(|\tilde{\varphi}|) \geq 0 \). The dominant energy condition \( \rho \geq |J|_\gamma \), together with the fact that the vector \( \tilde{v} \) satisfies \( |\tilde{v}|_\gamma < 1 \) (see (58)) implies that the first three terms in (60) are non-negative. The remaining terms have no sign in general. Motivated by the structure of (60), Bray and Khuri introduce the generalized Jang equation \( \text{tr}_\varphi(\tilde{A} - A) = 0 \) or, explicitly,

\[
\phi^2\partial_i f \partial^j f \left( \phi \nabla_i \nabla_j f + \nabla_i \phi \nabla_j f + \nabla_i \phi \nabla_j f \right) - A_{ij} = 0. \tag{61}
\]

This obviously reduces to the original Jang equation (56) when \( \phi = 1 \). In the present case, however, this equation involves two unknowns, \( \phi \) and \( f \). The Jang equation is known not to admit regular solutions when \( \Sigma \) contains a future or past MOTS. In a similar fashion, Bray and Khuri observe that the generalized Jang equation may blow up on surfaces satisfying \( |\rho| = |q| \), which are generalized trapped surfaces. This, combined with the existence of an outermost generalized apparent horizon on \( (\Sigma, \gamma_{ij}, A_{ij}) \), led the authors to study conjecture 1 in the particular case that \( \partial \Sigma \) is a generalized apparent horizon and that no further generalized trapped surfaces exist in \( \Sigma \). The conjecture in this setting is

Conjecture 2 [29]. Suppose that the initial data set \( (\tilde{\Sigma}, \gamma_{ij}, A_{ij}) \) is asymptotically Euclidean and Schwarzschild at infinity, with total mass \( M_{\text{ADM}} \) (in a chosen end) and satisfying the dominant energy condition \( \rho \geq |J| \). Let \( S \) be a closed surface which bounds an open set \( \Sigma^\text{int} \) containing all the asymptotically Euclidean ends except the chosen one and that \( S \) is an outermost generalized trapped surface with respect to the normal pointing towards the chosen end, i.e. \( \Sigma \equiv \tilde{\Sigma} \setminus \Sigma^\text{int} \) contains no generalized trapped surfaces. Then

\[
M_{\text{ADM}} \geq \sqrt{\frac{|\partial \Sigma|}{16\pi}}. \tag{62}
\]

Furthermore, equality occurs if and only if \( (\Sigma, \gamma_{ij}, A_{ij}) \) is the induced data of an embedding of \( \Sigma \) into the Kruskal spacetime such that \( \partial \Sigma \) is mapped to a generalized apparent horizon.

Under the conditions of this conjecture, it is plausible that (61) admits regular solutions in \( \Sigma \). The boundary behaviour is typically singular, as examples in the Kruskal spacetime show. The hope is that this singular behaviour on the boundary can be adjusted so that the surface \( \partial \Sigma \) has non-positive mean curvature with respect to the transformed metric \( \tilde{\varphi}_{ij} \). This is useful because the area of any surface never decreases under the transformation \( \gamma_{ij} \to \tilde{\varphi}_{ij} \) (due to the fact that the volume form of \( \tilde{\varphi}_{ij} \) is \( \eta_\varphi = (1 + \phi^2|f|^2_\gamma)\eta_\gamma \)). Consequently, the minimal area enclosure \( \tilde{S}_{\text{min}} \) of \( \partial \Sigma \) in \( (\Sigma, \tilde{\varphi}) \) satisfies

\[
|\tilde{S}_{\text{min}}|_{\tilde{\varphi}} \geq |\tilde{S}_{\text{min}}|_\gamma \geq |\partial \Sigma|_\gamma. \tag{63}
\]

where the subindex denotes which metric is used to calculate the area and the second inequality holds because \( \partial \Sigma \) is area outer minimizing in \( (\Sigma, \gamma) \). Thus, an upper bound for \( |\tilde{S}_{\text{min}}|_{\tilde{\varphi}} \) (via the Riemannian Penrose inequality) implies an upper bound for \( |\partial \Sigma|_\gamma \), which is the type of information relevant for the full Penrose inequality.
instance, if \( f \) is of compact support) then \( M_{\text{ADM}}(\gamma) = M_{\text{ADM}}(\mathcal{P}) \). Assuming that \( \partial \Sigma \) is a generalized apparent horizon, it follows that \( \partial \Sigma \) is a minimal surface in (\( \Sigma, \mathcal{P}_\Sigma \)) provided \( \phi \) and \( f \) are smooth up to the boundary and \( \phi|_\Sigma = 0 \) (it should be remarked that these two conditions are quite restrictive, for instance they hold for slices of the Kruskal spacetime only if they intersect the black hole event horizon precisely at the bifurcation surface \( [u = v = 0] \)).

The Riemannian Penrose inequality then gives \( M_{\text{ADM}}(\mathcal{P}) \geq \sqrt{|S_{\text{min}}|/16\pi} \), and hence the Penrose inequality \( M_{\text{ADM}}(\gamma) \geq \sqrt{|\partial \Sigma|/16\pi} \) follows from (63).

Returning to the general case, whenever the generalized Jang equation is satisfied, the curvature scalar of \( \mathcal{P}_\Sigma \) reduces to

\[
R(\mathcal{P}) = 16\pi (\rho - J_i v^i) + ||A - \phi||^2_\mathcal{P} - 2|\mathbf{\tilde z}|^2_\mathcal{P} - \frac{2}{\phi} \operatorname{div}_\mathcal{P}(\phi \mathbf{\tilde z}).
\]  

(64)

This expression has no sign in general, so the Riemannian Penrose inequality cannot be applied directly. However, the generalized Jang equation involves two unknowns, so it must be supplemented by a second condition in order to have a determined problem. Bray and Khuri discuss two possibilities.

The simplest one consists in putting equal to zero the last summand in (64), i.e.

\[
\operatorname{div}_\mathcal{P}(\phi \mathbf{\tilde z}) = 0.
\]  

(65)

The two equations (61) and (65) are called the *Jang-zero divergence equations* in [29]. Equation (65) is third order in \( f \). However, after subtracting suitable derivatives of (61) it can be converted into a second-order equation for \( f \) (with quadratic second derivatives). The resulting system is degenerate elliptic. Bray and Khuri conjecture that the system admits solutions with appropriate behaviour at infinity and such that \( \partial \Sigma \) is a minimal surface with respect to \( \mathcal{P}_\Sigma \). Under this conjecture, the Penrose inequality as stated in (2) follows (modulo a technical point regarding the equality case; see Conjecture 7 in [29]).

The second possibility is based on the observation that, while (61) does not imply \( R(\mathcal{P}) \geq 0 \), the integrated inequality \( \int_{\Sigma} \phi R(\mathcal{P}) \eta_\mathcal{P} \geq 0 \) follows from (64) under suitable decay conditions at infinity and restrictions on \( \partial \Sigma \). Bray and Khuri make the interesting observation that in any situation (irrespective of whether \( R(\mathcal{P}) \geq 0 \) or not) where the mass \( M_{\text{ADM}}(\mathcal{P}) \) can be shown to satisfy

\[
M_{\text{ADM}}(\mathcal{P}) = \sqrt{|S_{\text{min}}|/16\pi} \geq \int_{\Sigma} Q(\lambda) R(\mathcal{P}) \eta_\mathcal{P}
\]  

(66)

with some \( Q(\lambda) \geq 0 \), then the prescription \( \phi = Q \) implies \( M_{\text{ADM}}(\gamma) \geq \sqrt{|S_{\text{min}}|/16\pi} \) and hence the Penrose inequality (62) provided \( M_{\text{ADM}}(\gamma) \geq M_{\text{ADM}}(\mathcal{P}) \). The question is, therefore, under which circumstances a general type inequality of form (66) holds. As the authors point out, any such inequality would imply the Riemannian Penrose inequality as a particular case. It is therefore natural to study whether the known proofs of the Riemannian Penrose inequality are capable of establishing the validity of (66) for some \( Q \geq 0 \). The authors show explicitly that this is indeed the case for the Huisken and Ilmanen method, provided the second homology class of \( \Sigma \) is trivial and \( \partial \Sigma \) is connected. The idea is to integrate (18) with respect to \( \lambda \) and convert the double integral (in \( \lambda \) and on \( \Sigma \)) as a volume integral. By using the weak formulation in terms of level sets, this can be accomplished even along the jumps. The result is that \( Q = |u|\mathcal{P}\sqrt{\varepsilon^2|S_{\text{min}}|/16\pi}^{1/2} \), where \( u \) is the weak solution of the inverse mean curvature flow equal to zero on \( \partial \Sigma \). Thus, existence of appropriate solutions for the pair of equations (61) and \( \phi = Q(\lambda) \) implies the general Penrose inequality (62) for connected \( \partial \Sigma \) (assuming that \( \Sigma \) has a trivial second homology class). As noted by the authors, existence in this case looks harder than for the Jang-zero divergence system because \( Q \) vanishes identically.
wherever the inverse mean curvature flow jumps. The equation therefore implies $\phi = 0$ on the jumps. However, if $f$ stays smooth, then $\overline{A}_{ij} = 0$ there (see (59)). But then, the generalized Jang equation (61) requires $\text{tr} A = 0$ along the jumps, which is a condition on the initial data and not an equation. Thus, existence of classical solutions of the system $\phi = Q$ and (61) should not be expected in general. It may be, however, that existence can be granted if $f$ is allowed to be unbounded (or even undefined) in suitable regions.

The other existing method to prove the Riemannian Penrose inequality is the conformal flow of metrics due to Bray [22]. As discussed in [29], this method is also capable of producing an inequality of form (66). In this case, $Q$ is expected to be continuous and strictly positive outside $S_{\text{min}}$. On the other hand, the resulting equation $\phi = Q$ is not local, in the sense that it does not define a local PDE at each point. The existence of the coupled system with the generalized Jang equations appears to be difficult in this case as well.

8. Stronger versions of the Penrose inequality

The Penrose inequality can be strengthened under some circumstances, in particular when suitable matter fields are present in the spacetime (see section 13.2 in [151] for a recent discussion). In order to understand heuristically why this is to be expected, let us return to the original argument by Penrose based on cosmic censorship. Assume that the collapsing matter is electrically charged and that the end-state of the collapse is a black hole in equilibrium. In this situation, all the matter sources of the electromagnetic field are expected to lie within the event horizon and the black hole is therefore electrovacuum in its exterior. According to the black hole uniqueness result, the end-state is therefore a Kerr–Newman black hole. The area of any cut of the event horizon in this spacetime is given by

$$|S| = 4\pi \left( 2M^2 - Q^2 + 2M \sqrt{M^2 - L^2/M^2} - Q^2 \right),$$

(67)

where $L$ is the total angular momentum of the final state and $Q$ is the total electric charge. From (67), it immediately follows that $|S| \leq 4\pi (M + \sqrt{M^2 - Q^2})^2$ which makes no reference to the angular momentum. Since the total electric charge of the spacetime is conserved provided no charged matter escapes to infinity, the Penrose heuristic argument implies that any asymptotically Euclidean electrovacuum initial data set should satisfy the inequality

$$\sqrt{A_{\text{min}}(\partial T^2_s)}_{16\pi} \leq \frac{1}{2} (M_{\text{ADM}} + \sqrt{M_{\text{ADM}}^2 - Q^2}).$$

(68)

In the time-symmetric case, the electrovacuum initial data reduce to the triple $(\Sigma, \gamma_{ij}, E_i)$, where the electric field $\tilde{E}$ satisfies $\text{div}_\gamma \tilde{E} = 0$. The total charge is defined as $4\pi Q = \int_{\Sigma} (\tilde{E} \cdot \tilde{m}) n_{\Sigma}$, where $S$ is homologous to any large sphere in the asymptotically Euclidean end. Inequality (68) simplifies to

$$\sqrt{[S_m]}_{16\pi} \leq \frac{1}{2} (M_{\text{ADM}} + \sqrt{M_{\text{ADM}}^2 - Q^2}),$$

(69)

where $S_m$ is the outermost minimal surface. For this inequality to make sense, it is necessary that $M_{\text{ADM}} \geq |Q|$ for all electrovacuum initial data. This is a strengthening of the positive mass theorem in the presence of electromagnetic fields and was first proven in [64] (see [63] for a generalization including matter and [42] for a rigorous statement). The minimum value of the right-hand side of (69) is $|Q|/2$. Thus, for horizons of a small area ($|S_m| < 4\pi Q^2$) the Penrose inequality (69) reduces to the positive mass theorem $M_{\text{ADM}} \geq |Q|$, with no reference to the area of the horizon. This implies, in particular, that (69) does not admit an equality case (i.e. a rigidity statement) for horizons of a small area. On the other hand, the equality
case in the Penrose inequality $|S_m| \leq 16\pi M_{\text{ADM}}^2$ has the interesting consequence of providing a variational characterization of the Schwarzschild metric (29) as the absolute minimum of total mass among asymptotically Euclidean Riemannian manifolds of non-negative curvature scalar and with an outermost minimal surface of a given area $|S_m|$. This is similar to the variational characterization of the Euclidean space as the absolute minimum of total mass among asymptotically Euclidean Riemannian manifolds with $R(\gamma) \geq 0$.

A natural question is whether there exists another version of the charged Riemannian Penrose inequality which is able to give a variational characterization (among metrics of a fixed charge and fixed area of the outermost minimal surface) of the Reissner–Nordström and Papapetrou–Majumdar metrics, which are the only static and electrovacuum black holes (see [43] and references therein). A strengthening of (69) that has been proposed is

$$M_{\text{ADM}} \geq \frac{1}{2} \left( \sqrt{\frac{|S_m|}{4\pi}} + Q^2 \sqrt{\frac{4\pi}{|S_m|}} \right).$$

(70)

This inequality was first discussed and proven by Jang [96] in the case of asymptotically Euclidean, electrovacuum initial data sets $(\Sigma, \gamma_{ij}, E_i)$ with a connected outermost minimal surface $S_m$ provided the inverse mean curvature flow starting on $S_m$ remains smooth. This last requirement is, in fact, unnecessary, thanks to the weak formulation of the flow introduced by Huisken and Ilmanen. This establishes (70) for connected $S_m$.

Inequality (70) is, however, not generally true when the outermost minimal surface is allowed to have several connected components. A counterexample has been found by Weinstein and Yamada [165]. Their basic idea was to realize that the Papapetrou–Majumdar spacetime, which represents a static configuration of $N$ black holes of masses $m_i > 0$ and charges $Q_i = \epsilon m_i$, with $\epsilon = \pm 1$, has the property that the total area $|S|$ of the event horizon violates inequality (70). Indeed, using $Q = M_{\text{ADM}}$, it follows that

$$M_{\text{ADM}} - \frac{1}{2} \left( \sqrt{\frac{|S|}{4\pi}} + Q^2 \sqrt{\frac{4\pi}{|S|}} \right) = -\frac{1}{2} \sqrt{\frac{4\pi}{|S|}} \left( M_{\text{ADM}} - \sqrt{\frac{|S|}{4\pi}} \right)^2 \leq 0,$$

irrespective of the value of $|S|$. The only way how (70) could hold is $|S| = 4\pi M_{\text{ADM}}^2$. However, a simple computation gives $|S| = 4\pi \sum (m_i)^2$. Since the ADM mass is $M_{\text{ADM}} = \sum m_i$, equality can only happen where there is only one black hole (i.e. when the spacetime is the extreme Reissner–Nordström black hole). For any configuration with two or more black holes, (70) is violated for the area of the event horizon. This argument is, however, not a proof that (70) fails to hold because the static initial data set (i.e. the hypersurface orthogonal to the static Killing vector) in the Papapetrou–Majumdar spacetime does not contain any minimal surface. Each connected component of the event horizon corresponds to an asymptotic cylinder. Thus, some engineering is required to construct an electrovacuum initial data set with a minimal surface and which violates (70). The method followed in [165] consists in an adaptation of the gluing technique developed in [93]. More specifically, it consists in taking two copies of a static initial data set of the Papapetrou–Majumdar spacetime with two black holes of equal mass $m_1 = m_2 = m$, such that one of the copies has positive charges and the other one negative charges. By modifying the geometry far enough along the cylindrical ends, the two copies can be glued together to construct an initial data set with two asymptotically Euclidean ends and a minimal surface with two connected components. The final step is to conformally transform the data so that the curvature scalar vanishes. By taking $m$ small enough, the resulting manifold violates inequality (70). As the authors stress, this is not a counterexample of (69), which is the inequality that follows from Penrose’s heuristic argument.

Returning to the question of whether the charged Riemannian Penrose inequality provides a variational characterization of the electrovacuum static black holes (see also [65] for a related
discussion), we note that inequality (69) can be written in the following equivalent way (cf [151]):

\[
\begin{cases}
    M_{\text{ADM}} \geq |Q| & \text{if } |S_m| \leq 4\pi Q^2 \quad \text{(case (i))} \\
    M_{\text{ADM}} \geq \frac{1}{2} \left( \sqrt{\frac{|S_m|}{4\pi}} + \sqrt{\frac{4\pi}{|S_m|}} Q^2 \right) & \text{if } |S_m| \geq 4\pi Q^2 \quad \text{(case (ii))}
\end{cases}
\] (71)

The Reissner–Nordström black holes have event horizons (or equivalently outermost minimal surfaces in their static initial data) which satisfy \( |S_m| \geq 4\pi Q^2 \) (see (67) with \( L = 0 \), with equality only for the extreme Reissner–Nordström case \( (M = |Q|) \). So, these metrics belong to case (ii) above and, in fact, saturate the corresponding inequality. Similarly, the event horizon of the Papapetrou–Majumdar spacetime has area \( |S| = 4\pi \sum m_i^2 \leq 4\pi \left( \sum m_i \right)^2 = 4\pi Q^2 \), with equality only if there is only one black hole (i.e. the metric is extreme Reissner–Nordström again). So, this case belongs to case (i) and obviously the inequality is again saturated. Moreover, the Papapetrou–Majumdar static initial data are the only asymptotically Euclidean, electrovacuum initial data \((\Sigma, g_{ij}, E_i)\) satisfying \( M = |Q| \) (see Theorem 1.2 in [42] as well as the related previous work [63, 155]). Thus, the formulation (71) would in fact provide a variational characterization of all static charged black holes provided the inequality can be proven in case (ii) with equality only for the Reissner–Nordström initial data.

In the non-time-symmetric case, Gibbons conjectured [65] inequality (70) for connected and outermost future (or past) marginally outer trapped surfaces \( S \). In the non-connected case, the corresponding conjecture involves the sum over each connected component of the right-hand side of (70). Although no counterexample is explicitly known, such an inequality would reduce in vacuum to a stronger statement than the standard Penrose inequality. As noted by Weinstein and Yamada [165], it seems that initial data representing two Schwarzschild black holes sufficiently far apart should violate this version of the inequality. In the particular case of spherical symmetry (where \( S \) is automatically connected), Gibbons’s conjecture has been proven by Malec and Ó Murchadha [120] for maximal initial data sets and by Hayward [73] in the general case.

In the above discussion, we have dropped the angular momentum term in (67) and have retained the charge. It is natural to ask what is the situation in the reverse case, i.e. when the charge is dropped and the angular momentum in kept. The inequality that results is \( |S| \leq 8\pi M(M + \sqrt{M^2 - L^2/M^2}) \). However, in contrast to the electromagnetic case, the total angular momentum of the final end-state after the collapse has been completed need not coincide with the initial one since gravitational waves carry angular momentum and this can be radiated away. As first discussed in [58] (see also [86]), there is one important situation where angular momentum must be conserved along the evolution, namely in the axially symmetric case. Under this restriction, the Penrose heuristic argument implies [79, 52]

\[
A_{\text{max}}(\partial T_T^\mu) \leq 8\pi M_{\text{ADM}} \left( M_{\text{ADM}} + \sqrt{M^2_{\text{ADM}} - L^2/M^2_{\text{ADM}}} \right).
\] (72)

Similarly as before, this inequality only makes sense provided one can show that any asymptotically Euclidean and axially symmetric initial data set satisfying the dominant energy condition \( \rho \geq |\vec{J}| \) satisfies the inequality \( M_{\text{ADM}} \geq \sqrt{|L|} \). Again, this is strengthening of the positive mass theorem. This inequality is supported by a heuristic argument based on cosmic censorship and the conservation of angular momentum in the axially symmetric case [54] and its validity has been rigorously proven in [53] for any initial data set \((\Sigma, g_{ij}, A_{ij})\) which is vacuum, maximal \((\text{tr} A = 0)\), contains one or more asymptotically Euclidean ends as well as possibly additional asymptotically cylindrical ends (which correspond to degenerate horizons) and such that the outermost MOTS is connected (see also [46] for an extension
which furthermore admits non-negative energy density and relaxes some of the technical requirements in [53]). Moreover, the case of equality $M_{\text{ADM}} = |L|$ occurs if and only if the initial data are a slice of the extreme Kerr black hole. This provides a variational characterization of extreme Kerr. The inequality in the case with an outermost MOTS with several connected components remains still open. Numerical evidences for its validity have been recently given in [55].

The situation for the Penrose inequality with angular momentum is therefore similar to the charged case. Inequality (72) is equivalent to (cf [151])

$$\begin{cases} 
M_{\text{ADM}}^2 \geq |L| & \text{if } |S| \leq 8\pi |L| \quad (\text{case (i)}) \\
M_{\text{ADM}}^2 \geq \frac{|S|}{16\pi} + \frac{4\pi L^2}{|S|} & \text{if } |S| \geq 8\pi |L| \quad (\text{case (ii)}),
\end{cases}$$

(73)

where $|S| = A_{\text{min}}(\partial T_{\Sigma})$. This version of the Penrose conjecture (for axially symmetric initial data sets) admits a rigidity case which states that equality in case (ii) can only occur if the initial data are a slice of the Kerr black hole. Again, this would provide a variational characterization of the Kerr metric.

9. Some applications of the Penrose inequality

In this section, we briefly mention some recent situations where the Penrose inequality has been exploited to derive new results. The list is probably not exhaustive, but it gives a hint on the potential power of the Penrose inequality as a geometric tool for addressing, a priori, completely unrelated problems.

We have already mentioned in subsection 5.3 that the Riemannian Penrose inequality has interesting consequences for the quasi-local definition of mass due to Bartnik. The Riemannian Penrose inequality has also allowed for a dual definition of quasi-local mass due to Bray [22] (see also [26]). Here, an asymptotically Euclidean domain with non-negative curvature scalar and whose boundary $S$ is area outer minimizing is kept fixed, and all possible ‘fill in’s’ (with non-negative curvature scalar) are considered. The inner mass is defined as the supremum of $\sqrt{|S|/(16\pi)}$, where $S$ is the minimal area needed to enclose all the asymptotically Euclidean ends, except the given one. As a consequence of the Riemannian Penrose inequality, the inner mass is always bounded above by the ADM mass of the given region. The definition can also be extended to the case of a non-zero second fundamental form.

In section 6, we have also mentioned another application of the Penrose inequality for the uniqueness problem of static black holes with a negative cosmological constant and topology at infinity of genus larger than 1.

The Riemannian Penrose inequality (in fact, its proof using an inverse mean curvature flow) has been applied recently to obtain lower bounds of the so-called Brown–York energy for simply connected, compact, three-dimensional domains $(\Omega, \gamma)$ with non-negative curvature scalar and with smooth boundary $\partial \Omega$ of positive Gauss curvature and positive mean curvature $p$ (with respect to the outer direction). The Brown–York mass is defined as

$$M_{\text{BY}}(\partial \Omega) = \frac{1}{8\pi} \int_{\partial \Omega} (p_0 - p) \eta_{\partial \Omega},$$

where $p_0$ is the mean curvature of $\partial \Omega$ when this surface is embedded isometrically in $\mathbb{R}^3$. This mass is proven to be non-negative in [149]. Using the inverse mean curvature flow, Shi and Tam prove [150] (among other things) that the inequality $M_{\text{BY}}(\partial \Omega) \geq M_G(\partial \Omega)$ holds with equality if only if $\Omega$ is a standard ball in $\mathbb{R}^3$.
Still another application of the Riemann–Penrose inequality is due to Corvino [48], who has shown that asymptotically Euclidean, three-dimensional, Riemannian manifolds with non-negative curvature scalar and small mass cannot contain any minimal surface and must be diffeomorphic to $\mathbb{R}^3$. The required ‘small mass’ condition reads as $2M_{\text{ADM}}\sqrt{K} \leq 1$, where $K$ is a positive upper bound for all the sectional curvatures of the manifold. The proof is based on the fact that any outermost minimal surface $S$ must satisfy $|S| \geq \frac{4\pi}{K}$ as a consequence of the Gauss–Bonnet theorem. Therefore, the Penrose inequality implies that no minimal surface can exist under these circumstances.

10. Concluding remarks

In this review, I have discussed the present status of the Penrose inequality. The emphasis has been put on trying to describe the techniques involved in the various approaches to prove it and trying to place the results into the right context so that a clear picture emerges of how impressive the body of work in this field has already been and what are the open problems that still remain. Although I have tried to cover the main results in this field, some aspects have been touched upon in less detail. For instance, I have concentrated mostly on the four-dimensional case, although several results in higher dimensions have been discussed at various places. Some further results in higher dimensions can be found in [13, 51, 68, 92].

Numerical work has also been important for a better understanding of the Penrose inequality. Outermost marginally outer trapped surfaces (i.e. the boundary of the outer trapped region $\partial T_{\text{out}}$) are located routinely in numerical black hole evolutions in order to track the boundaries of the black holes. The numerical routines to do this job are collectively termed apparent horizon finders (see [153] for a review) and obviously they can also be used to test the validity of the Penrose conjecture. They have been used to check whether the Penrose inequality is fulfilled in explicit numerical examples, as well as for looking for counterexamples to some of its versions. They can also serve as a test to make sure that the MOTS being located is, in fact, the outermost one [97]. In this review, numerical results have been mentioned only very tangentially. Further details can be found in [52, 90, 97, 98, 102–105, 154].

In a (3+1) evolution of a spacetime, the outermost MOTS generates a tube of surfaces which is generally smooth but may jump from time to time [3]. In the smooth part, this tube has been called a marginally outer trapped tube [32]. If the foliation is by marginally trapped surfaces (instead of MOTS), the tube is usually called a trapping horizon [70] or dynamical horizon [10] (with slight differences in their definitions). A proper study of the evolution of these tubes, specially their late-time behaviour, is potentially a powerful method for establishing the Penrose inequality [4]. This is because it is expected that, at late times, the tube approaches the event horizon of the spacetime. Furthermore, if the MOTS foliating the tube are in fact marginally trapped surfaces, then their area increases with time [10]. Since this area is believed to approach that of the event horizon of the final black hole that forms (and this is greater than the initial ADM mass, as usual), the Penrose inequality would follow from a detailed understanding of the late-time evolution of the spacetime and of the outermost marginally outer trapped tube. This approach however, is very different in spirit to those discussed above because understanding the late-time behaviour of the tube goes a long way towards establishing weak cosmic censorship. Thus, in essence, the Penrose inequality would follow because cosmic censorship would hold. So, instead of looking at the Penrose inequality as an indirect test of cosmic censorship, as originally envisaged by Penrose, it would become a remarkable corollary of a much stronger theorem establishing weak cosmic censorship (or something very close to it). In any case, studying the evolution of the outermost tube is
an active area of research, which combines physical properties, numerical simulations and rigorous geometric results. The interested reader is referred to [1–4, 7–9, 69, 75, 98, 140, 167] and references therein.

To conclude, as I have tried to show in this review, the Penrose conjecture is a very challenging problem that requires techniques from geometric analysis, partial differential equations, Riemannian and Lorentzian geometry as well as physical intuition. The recent advances in this field have been impressive and our understanding of the problem is now better than ever. Nevertheless, many open problems remain and their study is likely to uncover new and unexpected features in the future.

Acknowledgments

I am indebted to Hugh L Bray, Alberto Carrasco, José Luis Jaramillo, Markus Khuri, Miguel Sánchez, José M M Senovilla, Juan Valiente Kroon, László B Szabados and Raúl Vera for useful comments on a previous version of this review. Financial support under projects FIS2006-05319 of the Spanish MEC, SA010CO5 of the Junta de Castilla y León and P06-FQM-01951 of the Junta de Andalucía is gratefully acknowledged.

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