On the asymptotics of supremum distribution for some iterated processes

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Abstract

In this paper, we study the asymptotic behavior of supremum distribution of some classes of iterated stochastic processes \( \{X(Y(t)) : t \in [0, \infty)\} \), where \( \{X(t) : t \in \mathbb{R}\} \) is a centered Gaussian process and \( \{Y(t) : t \in [0, \infty)\} \) is an independent of \( \{X(t)\} \) stochastic process with a.s. continuous sample paths. In particular, the asymptotic behavior of \( \mathbb{P}\left( \sup_{s \in [0,T]} X(Y(s)) > u \right) \) as \( u \to \infty \), where \( T > 0 \), as well as \( \lim_{u \to \infty} \mathbb{P}\left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right) \), for some suitably chosen function \( h(u) \) are analyzed. As an illustration, we study the asymptotic behavior of the supremum distribution of iterated fractional Brownian motion process.

Key words: exact asymptotics, supremum distribution, iterated process, iterated fractional Brownian motion, Gaussian process.

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1 Introduction

Let \( \{X(t) : t \in \mathbb{R}\} \) and \( \{Y(t) : t \in [0, \infty)\} \) be two independent stochastic processes. This contribution is devoted to the analysis of asymptotic behavior of supremum distribution of iterated process \( \{X(Y(t)) : t \in [0, \infty)\} \).

Originated by Burdzy [7, 8] for the case of iterated Brownian motion, the problem of analyzing the properties of iterated processes was intensively studied in recent years. Motivation for the analysis of the process \( \{X(Y(t))\} \) in case of \( \{X(t)\} \) and \( \{Y(t)\} \) being independent Brownian motions was delivered by its connections to the 4th order PDE’s (see, e.g., [14, 2, 25]). A vast literature is devoted to the analysis of many interesting probabilistic properties of iterated Brownian motions (see, e.g., [9, 15, 28, 6, 16, 13, 17]). We also refer to [10] where convergence of finite dimensional distributions of \( n \)th iterated Brownian motion is studied and [31] where infinite iterations of i.i.d. random walks are analyzed.

Recent studies also focus on properties of \( \{X(Y(t)) : t \in [0, \infty)\} \) for the case of more general Gaussian processes \( \{X(t)\} \). One of interesting example of such processes is fractional Laplace motion \( \{B_H(\Gamma(t)) : t \in [0, \infty)\} \), where \( \{\Gamma(t) : t \in [0, \infty)\} \) is a Gamma process. Motivation for analyzing fractional Laplace motions stems from hydrodynamic models (see, e.g., [18]). This kind of processes were described in
Section 2.2 is devoted to the special case of the process \(\{B_H(t)\}\) as \(u\) distribution, that is, \(T\) reduces to \(\{\text{some general conditions on the process}\}\) process with stationary increments, a.s. continuous sample paths.

We start in Section 2.1 by giving general result for the case of \(\{X(t)\}\) for general classes of stochastic processes \(\{X(t)\}, \{Y(t)\}\) with a.s. continuous sample paths.

**Notation and organization of the paper:**

In Section 2 we study the asymptotic behavior of

\[
P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) \quad \text{as } u \to \infty,
\]

where \(T > 0\) and \(\{X(t): t \in \mathbb{R}\}, \{Y(t): t \in [0, \infty)\}\) are independent stochastic processes. This problem is closely related to the analysis of asymptotic behavior of the supremum distribution of the process \(\{X(t)\}\) over a random time interval (see, e.g., [12, 3, 4, 30, 11]).

We start in Section 2.1 by giving general result for the case of \(\{X(t)\}\) being Gaussian process with stationary increments and convex variance function (see Section 2.1 assumptions A1 – A3). In this case, under some general conditions on the process \(\{Y(t)\}\) (see Section 2.1 assumptions L1, L2), we show that (1) reduces to

\[
P \left( \sup_{s \in [0,T]} X(s) > u \right) \quad \text{as } u \to \infty,
\]

where \(T\) is a non-negative random variable independent of \(\{X(t)\}\) with asymptotically Weibullian tail distribution, that is,

\[
P (T > u) = Cu^\gamma \exp(-\beta u^\alpha)(1 + o(1))
\]

as \(u \to \infty\), where \(\alpha, \beta, C > 0, \gamma \in \mathbb{R}\) (see, e.g., [3] for details). We write \(T \in \mathcal{W}(\alpha, \beta, \gamma, C)\) if \(T\) satisfies (3).

Section 2.2 is devoted to the special case of the process \(\{B_H(Y(t))\}\), where \(\{B_H(t): t \in \mathbb{R}\}\) is a fractional Brownian motion (fBm) with Hurst parameter \(H \in (0, 1)\), that is, a centered Gaussian process with stationary increments, a.s. continuous sample paths, \(B_H(0) = 0\), and covariance function \(\text{Cov}(B_H(t), B_H(s)) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |t - s|^{2H})\). Due to self-similarity of the process \(\{B_H(t)\}\), we are able to provide the exact asymptotics of (1) for the whole range of Hurst parameters \(H \in (0, 1)\). As an illustration, in Proposition 2.3 we work out the exact asymptotics of the supremum distribution of iterated fractional Brownian motion \(\{B_{H_2}(B_{H_1}(t)) : t \in [0, \infty)\}\), where \(\{B_{H_1}(t)\}, \{B_{H_2}(t)\}\) are independent.
fractional Brownian motions with Hurst parameters $H_1, H_2$ respectively. Note that small deviation counterpart of this problem was recently studied in [5].

In Section 2.3, the case of $\{X(t)\}$ being a stationary Gaussian process is analyzed (see Section 2.3 assumptions D1, D2). In this case the exact asymptotics of (1) can be achieved under a general condition of finite average span of the process $\{Y(t)\}$ (see Section 2.3 assumption S1). This problem is strongly related to the analysis of (2) in case of $T$ being a random variable with finite mean. In this case the asymptotics of (2) has the form (see [4], Theorem 3.1, and also [26] for the classical result of Pickands’ on deterministic time interval)

$$
P \left( \sup_{s \in [0,T]} X(s) > u \right) = E T C^{1/\alpha} H_\alpha u^{2/\alpha} \Psi(u)(1 + o(1))$$

as $u \to \infty$, where $H_\alpha$ is the Pickands’ constant defined by the limit

$$H_\alpha = \lim_{T \to \infty} \frac{1}{T} E \exp \left( \sup_{t \in [0,T]} \sqrt{2} B_{\alpha/2}(t) - t^\alpha \right),$$

and $\Psi(u) := P(N > u)$ with $N$ denoting the standard normal random variable.

In the second part of the paper, we study

$$\lim_{u \to \infty} P \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right),$$

for some suitably chosen function $h(u)$.

First, in Theorem 3.1 we investigate limiting behavior of (4) for the case of $\{X(t)\}$ and $\{Y(t)\}$ being independent Gaussian processes with stationary increments that satisfy some general regularity conditions (see Section 3, assumptions B1 – B3). Then, in Theorem 3.2 and Proposition 3.3 the case of $\{X(t)\}$ being stationary Gaussian process is studied. We analyze $\{X(Y(t))\}$ for both weakly and strongly dependent stationary Gaussian processes $\{X(t)\}$ (see Section 3, assumptions D1 – D3). In these settings we provide (4) in the case of $\{Y(t)\}$ being a centered Gaussian process with stationary increments, as well as for self-similar process $\{Y(t)\}$ that is not necessarily Gaussian.

### 2 Short timescale case

In this section, we study the asymptotic behavior of

$$P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) \text{ as } u \to \infty,$$

where $T > 0$, for the case of $\{X(t) : t \in \mathbb{R}\}$ being a centered Gaussian process with a.s. continuous sample paths. We focus on two important classes of Gaussian processes. First, processes $\{X(t)\}$ with stationary increments are studied. Then, we analyze the case of stationary processes $\{X(t)\}$. 

3
2.1 The stationary increments case

Let \( \{X(t) : t \in \mathbb{R}\} \) be a centered Gaussian process with stationary increments, a.s. continuous sample paths, \( X(0) = 0 \) a.s., and variance function \( \sigma_X^2(t) := \text{Var}(X(t)) \) that satisfies the following assumptions

- **A1** \( \sigma_X^2(t) \in C^1([0, \infty)) \) is convex;
- **A2** \( \sigma_X^2(t) \) is regularly varying at \( \infty \) with parameter \( \alpha_\infty \in (1, 2) \);
- **A3** there exists \( D > 0 \) such that \( \sigma_X^2(t) \leq Dt^{\alpha_\infty} \) for each \( t \geq 0 \).

To provide a general result for (5) we assume that \( \{Y(t) : t \in [0, \infty)\} \) is a stochastic process with a.s. continuous sample paths, which is independent of \( \{X(t)\} \) and its extremal distributions belong to the Weibullian class of random variables, that is,

- **L1** \( M := \sup_{s \in [0, T]} Y(s) \in W(\alpha_1, \beta_1, \gamma_1, C_1), \) with \( \alpha_1, \beta_1, C_1 > 0, \gamma_1 \in \mathbb{R}; \)
- **L2** \( K := -\inf_{s \in [0, T]} Y(s) \in W(\alpha_2, \beta_2, \gamma_2, C_2), \) with \( \alpha_2, \beta_2, C_2 > 0, \gamma_2 \in \mathbb{R}. \)

**Remark 2.1** Note that assumptions L1, L2 cover, e.g., a class of general Gaussian processes.

In the following theorem we present structural form of the asymptotics. The explicit asymptotic expansion is presented in Corollary 2.2.

**Theorem 2.1** Let \( \{X(t) : t \in \mathbb{R}\} \) be a centered Gaussian process with stationary increments and variance function \( \sigma_X^2(t) \) that satisfies assumptions A1 – A3 and \( \{Y(t) : t \in [0, \infty)\} \) be an independent of \( \{X(t)\} \) stochastic process with a.s. continuous sample paths that satisfies L1, L2. If

(i) \( \mathbb{P}(K > u) = o(\mathbb{P}(M > u)) \) as \( u \to \infty \), then

\[
\mathbb{P}\left( \sup_{s \in [0, T]} X(Y(s)) > u \right) = \mathbb{P}(X(M) > u) (1 + o(1)) \quad \text{as } u \to \infty;
\]

(ii) \( \mathbb{P}(M > u) = o(\mathbb{P}(K > u)) \) as \( u \to \infty \), then

\[
\mathbb{P}\left( \sup_{s \in [0, T]} X(Y(s)) > u \right) = \mathbb{P}(X(K) > u) (1 + o(1)) \quad \text{as } u \to \infty;
\]

(iii) \( \mathbb{P}(K > u) = C_2 \mathbb{P}(M > u)(1 + o(1)), \) as \( u \to \infty \), then

\[
\mathbb{P}\left( \sup_{s \in [0, T]} X(Y(s)) > u \right) = (\mathbb{P}(X(M) > u) + \mathbb{P}(X(K) > u)) (1 + o(1)) \quad \text{as } u \to \infty.
\]

The proof of Theorem 2.1 is presented in Section 4.1.

If the variance function of \( \{X(t)\} \) is regular enough, then the straightforward application of Corollary 3.2 in [3] enables us to give the exact form of the asymptotics.
Corollary 2.2 Let \( \{X(t) : t \in \mathbb{R}\} \) be a centered Gaussian process with stationary increments and variance function that satisfies \( A_1 \) and \( \{Y(t) : t \in [0, \infty)\} \) be an independent of \( \{X(t)\} \) stochastic process with a.s. continuous sample paths that satisfies \( L_1, L_2 \). Additionally, if \( \sigma^2_X(t) = D t^{2\alpha} + o(t^{\alpha+\alpha_\alpha}), \) as \( t \to \infty \), with \( \alpha_\alpha \in (1, 2) \), \( D > 0 \), and \( \alpha = \min(\alpha_1, \alpha_2) \), then

\[
\sup_{s \in [0,T]} X(Y(s)) \in \mathcal{W}(\tilde{\alpha}, \gamma, \tilde{\gamma}, \tilde{C}),
\]

where

\[
\tilde{\alpha} = \frac{2\alpha}{\alpha + \alpha_\alpha}, \quad \tilde{\beta} = \beta^{\alpha/\alpha_\alpha} \left(\frac{D}{2}\right)^{\alpha/\alpha_\alpha} \left(\left(\frac{\alpha}{\alpha_\alpha}\right)^{\alpha/\alpha_\alpha} + \left(\frac{\alpha_\alpha}{\alpha}\right)^{\alpha/\alpha_\alpha}\right),
\]

\[
\tilde{\gamma} = \frac{2\gamma}{\alpha + \alpha_\alpha}, \quad \tilde{C} = CD^{-1/\alpha_\alpha} \sqrt{2(\alpha + \alpha_\alpha)} \left(\frac{\alpha_\alpha}{2\alpha}\beta D^{\alpha_\alpha/\alpha}\right)^{\gamma/\alpha_\alpha},
\]

with

\[
(\beta, \gamma, C) = \begin{cases} 
(\beta_1, \gamma_1, C_1) & \text{for } \mathbb{P}(K > u) = o(\mathbb{P}(M > u)) \text{ as } u \to \infty, \\
(\beta_2, \gamma_2, C_2) & \text{for } \mathbb{P}(M > u) = o(\mathbb{P}(K > u)) \text{ as } u \to \infty, \\
(\beta_1, \gamma_1, C_1 + C_2) & \text{for } \mathbb{P}(K > u) = \frac{C_1}{C_2}\mathbb{P}(M > u)(1 + o(1)) \text{ as } u \to \infty.
\end{cases}
\]

2.2 The case of fBm

Let \( \{B_H(t) : t \in \mathbb{R}\} \) be a fractional Brownian motion with Hurst parameter \( H \in (0, 1] \). In this section, we analyze the asymptotic behavior of

\[
\mathbb{P}\left(\sup_{s \in [0,T]} B_H(Y(s)) > u\right) \quad \text{as } u \to \infty,
\]

where \( T > 0 \) and \( \{Y(t) : t \in [0, \infty)\} \) is an independent of \( \{B_H(t)\} \) stochastic process with a.s. continuous sample paths that satisfies assumptions \( L_1, L_2 \). Due to self-similarity of the process \( \{B_H(t)\} \), we are able to provide the exact asymptotics of (6) for the whole range of Hurst parameters \( H \in (0, 1] \), which includes cases of both convex and concave variance functions.

Proposition 2.3 Let \( \{B_H(t) : t \in \mathbb{R}\} \) be a fractional Brownian motion with Hurst parameter \( H \in (0, 1] \) and \( \{Y(t) : t \in [0, \infty)\} \) be an independent of \( \{B_H(t)\} \) stochastic process with a.s. continuous sample paths that satisfies \( L_1, L_2 \). If:

\( H \in (0, 1/2), \) then

\[
\sup_{s \in [0,T]} B_H(Y(s)) \in \mathcal{W}\left(\frac{2\alpha}{\alpha + 2H}, \tilde{\beta}, \frac{2\alpha - 3\alpha H + 2\gamma}{\alpha + 2H}, \tilde{C}_1\right),
\]

\( H = 1/2, \) then

\[
\sup_{s \in [0,T]} B_H(Y(s)) \in \mathcal{W}\left(\frac{2\alpha}{\alpha + 2H}, \tilde{\beta}, \frac{2\gamma}{\alpha + 2H}, 2\tilde{C}_2\right),
\]

\]

5
\( H \in (1/2, 1) \), then

\[
\sup_{s \in [0,T]} B_H(Y(s)) \in \mathcal{W}\left( \frac{2\alpha}{\alpha + 2H}, \tilde{\beta}, \frac{2\gamma}{\alpha + 2H}, C_2 \right),
\]

where

\[
\alpha = \min(\alpha_1, \alpha_2), \quad \tilde{\beta} = \beta^{\frac{2H}{\alpha + 2H}} \left( \frac{1}{2} \right)^{\frac{2H}{\alpha + 2H}} + \left( \frac{H}{\alpha} \right)^{\frac{\gamma}{\alpha + 2H}},
\]

\[
\tilde{C}_1 = \mathcal{H}_H \left( \frac{1}{2} \right)^{\frac{1}{2H}} \frac{C}{\sqrt{\alpha + 2H}} H^{\frac{\alpha + 6H + 2\gamma - 2}{2\alpha + 2H}} (\alpha \beta)^{\frac{1 - 2H - \gamma}{\alpha + 2H}}, \quad \tilde{C}_2 = \frac{C \sqrt{\Pi}}{\sqrt{\alpha + 2H}} \left( \frac{H}{\alpha \beta} \right)^{\frac{\gamma}{\alpha + 2H}},
\]

with \( \beta, \gamma, C \) as in Corollary 2.2.

The proof of Proposition 2.3 is presented in Section 4.2.

We now apply Proposition 2.3 to calculate the exact asymptotics for the special case of iterated fractional Brownian motion process \( \{B_{H_2}(B_{H_1}(t))\} \).

**Proposition 2.4** Let \( \{B_{H_1}(t) : t \in \mathbb{R}\} \) and \( \{B_{H_2}(t) : t \in [0, \infty)\} \) be independent fractional Brownian motions with Hurst parameters \( H_1, H_2 \in [0, 1) \) respectively. Then

\[
\sup_{s \in [0,T]} B_{H_2}(B_{H_1}(s)) \in \mathcal{W}(\alpha, \beta, \gamma, C),
\]

where

\[
\alpha = \frac{2}{H_2 + 1}, \quad \beta = \left( \frac{1}{T} \right)^{\frac{2H_1 H_2}{H_2 + H_2}} \left( \frac{1}{2} \right)^{\frac{H_1 H_2}{2H_2}} \left( \frac{1}{2} \right)^{\frac{H_1 H_2}{H_2}} + \left( \frac{H_2}{2} \right)^{\frac{1}{H_2 + H_2}},
\]

and

\[
(\gamma, C) = \begin{cases} 
(\gamma_1, C_1) & \text{for } H_1 \in (0, 1/2), H_2 \in (0, 1/2), \\
(\gamma_2, 2C_2) & \text{for } H_1 \in (0, 1/2), H_2 = 1/2, \\
(\gamma_2, C_2) & \text{for } H_1 \in (0, 1/2), H_2 \in (1/2, 1], \\
(\gamma_3, 2C_3) & \text{for } H_1 = 1/2, H_2 \in (0, 1/2), \\
(\gamma_4, 4C_4) & \text{for } H_1 = 1/2, H_2 = 1/2, \\
(\gamma_4, 4C_4) & \text{for } H_1 = 1/2, H_2 \in (1/2, 1], \\
(\gamma_3, C_3) & \text{for } H_1 \in (1/2, 1], H_2 \in (0, 1/2), \\
(\gamma_4, 2C_4) & \text{for } H_1 \in (1/2, 1], H_2 = 1/2, \\
(\gamma_4, C_4) & \text{for } H_1 \in (1/2, 1], H_2 \in (1/2, 1], 
\end{cases}
\]
Due to self-similarity of fBm

\[ C_1 = \left( \frac{1}{T} \right)^{H_2 - 3H_1 H_2} \frac{H_{H_1} H_{H_2}}{H_1 \sqrt{\pi (1 + H_2)}} \left( \frac{1}{2} \right)^{3H_1 + 1} H_2^{\frac{1}{2(1 + H_2)}}. \]

Additionally, by stationarity of the increments of fBm

\[ P \left( \sup_{s \in [0, T]} B_{H_2}(B_{H_1}(s)) > u \right) = P \left( \sup_{s \in [0, 1]} B_{H_2}(B_{H_1}(s)) > \frac{u}{T^{H_1 H_2}} \right). \]

Moreover, due to Lemma 4.2 in [3] (see also [27], Theorem D3)

\[ \sup_{s \in [0, 1]} B_{H_1}(s) \in \mathcal{W} \left( 2, \frac{1}{2}, \frac{1}{H_1} - 3, \frac{H_{H_1}}{H_1 \sqrt{\pi}} \right) \text{ for } H_1 \in (0, 1/2); \]

\[ \sup_{s \in [0, 1]} B_{H_1}(s) \in \mathcal{W} \left( 2, \frac{1}{2}, -1, \frac{2}{\sqrt{2\pi}} \right) \text{ for } H_1 = 1/2; \]

\[ \sup_{s \in [0, 1]} B_{H_1}(s) \in \mathcal{W} \left( 2, \frac{1}{2}, -1, \frac{1}{\sqrt{2\pi}} \right) \text{ for } H_1 \in (1/2, 1]. \]

Additionally, by stationarity of the increments of fBm

\[ - \inf_{s \in [0, 1]} B_{H_1}(s) \overset{d}{=} \sup_{s \in [0, 1]} B_{H_1}(s). \]

Now, in order to complete the proof it suffices to apply Proposition 2.3

\[ \square \]

2.3 The stationary case

In this section, we analyze the asymptotic behavior of \((5)\) for the case of \(\{X(t) : t \in [0, \infty)\}\) being a centered stationary Gaussian process with a.s. continuous sample paths and covariance function \(r(t) := \operatorname{Cov}(X(s), X(s + t))\). We impose the following assumptions on \(r(t)\) (see, e.g., [27]):

\[ \textbf{D1} \quad r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0, \text{ with } \alpha \in (0, 2] \text{ and } C > 0; \]

\[ \textbf{D2} \quad r(t) < 1 \text{ for all } t > 0. \]

In this case, we are able to give the exact form of the asymptotics for general class of stochastic processes \(\{Y(t) : t \in [0, \infty)\}\) that are independent of \(\{X(t)\}\), have a.s. continuous sample paths and finite average span over interval \([0, T]\). Therefore, we assume that
\[ \mathbb{E} \left[ \sup_{s \in [0,T]} Y(s) - \inf_{s \in [0,T]} Y(s) \right] < \infty. \]

**Proposition 2.5** Let \( \{X(t) : t \in \mathbb{R}\} \) be a centered stationary Gaussian process with covariance function \( r(t) \) that satisfies D1, D2 and \( \{Y(t) : t \in [0, \infty)\} \) be an independent of \( \{X(t)\} \) stochastic process with a.s. continuous sample paths that satisfies S1. Then

\[
P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) = \mathbb{E}(T) C^\perp \mathcal{H}_\alpha u^{2\alpha} \Psi(u)(1 + o(1))
\]

as \( u \to \infty \), where \( T = \sup_{s \in [0,T]} Y(s) - \inf_{s \in [0,T]} Y(s) \).

**Proof.** Due to stationarity of the process \( \{X(t)\} \), we have

\[
P \left( \sup_{t \in [0,T]} X(Y(t)) > u \right) = P \left( \sup_{t \in [\inf_{s \in [0,T]} Y(s), \sup_{s \in [0,T]} Y(s)]} X(t) > u \right) = P \left( \sup_{t \in [0,T]} X(t) > u \right).
\]

Now, in order to complete the proof it suffices to apply Theorem 3.1 in [4].

**Remark 2.2** Equivalently, Proposition 2.5 states that

\[
P \left( \sup_{t \in [0,T]} X(Y(t)) > u \right) = \mathbb{E}(T) P \left( \sup_{t \in [0,1]} X(t) > u \right) (1 + o(1))
\]

as \( u \to \infty \), where \( T = \sup_{s \in [0,T]} Y(s) - \inf_{s \in [0,T]} Y(s) \).

### 3 Long timescale case

In this section, we investigate

\[
\lim_{u \to \infty} P \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right)
\]

for a suitably chosen function \( h(u) \).

In order to formulate the results, it is convenient to introduce the notation

\[
\sigma^{-1}(t) := \inf\{y \in [0, \infty) : \sigma(y) > t\}
\]

for the generalized inverse of the function \( \sigma(t) \).

We start with the observation that (7) can be straightforwardly obtained for any independent, self-similar processes \( \{X(t)\} \) and \( \{Y(t)\} \) with a.s. continuous sample paths.
Remark 3.1 Let \( \{X(t) : t \in \mathbb{R}\} \) and \( \{Y(t) : t \in [0, \infty)\} \) be independent, self-similar stochastic processes with a.s. continuous sample paths and self-similarity indexes \( \lambda_X \) and \( \lambda_Y \) respectively. Then, for \( h(u) = u^{1/\lambda_X \lambda_Y} (1 + o(1)) \) as \( u \to \infty \), we have

\[
\lim_{u \to \infty} \mathbb{P} \left( \sup_{t \in [0,h(u)]} X(Y(t)) > u \right) = \mathbb{P} \left( \sup_{t \in [\inf s \in [0,1] Y(s), \sup s \in [0,1] Y(s)]} X(t) > 1 \right).
\]

In the next theorem, we extend this observation to the case of \( \{X(t) : t \in \mathbb{R}\} \) and \( \{Y(t) : t \in [0, \infty)\} \) being two independent, centered Gaussian processes with stationary increments, a.s. continuous sample paths, \( X(0) = 0 \) and \( Y(0) = 0 \) a.s., and variance functions \( \sigma_X^2(t) := \text{Var}(X(t)) \) and \( \sigma_Y^2(t) := \text{Var}(Y(t)) \) respectively. We assume that variance functions of both processes satisfy the following assumptions

\begin{itemize}
  \item [B1] \( \sigma^2(t) \in C([0, \infty)) \) is ultimately strictly increasing;
  \item [B2] \( \sigma^2(t) \) is regularly varying at \( \infty \) with parameter \( \alpha \in (0, 2] \);
  \item [B3] \( \sigma^2(t) \) is regularly varying at \( 0 \) with parameter \( \beta \in (0, 2] \).
\end{itemize}

In order to formulate the result, it is convenient to introduce the notation

\[
\mathcal{L}(\alpha_X, \alpha_Y) = \mathbb{P} \left( \sup_{t \in [\inf s \in [0,1] B_{\alpha_Y/2}(s), \sup s \in [0,1] B_{\alpha_Y/2}(s)]} B_{\alpha_X/2}(t) > 1 \right),
\]

where \( \{B_{\alpha_X/2}(t)\} \) and \( \{B_{\alpha_Y/2}(t)\} \) are independent fractional Brownian motions with Hurst parameters \( \frac{\alpha_X}{2} \) and \( \frac{\alpha_Y}{2} \) respectively.

Theorem 3.1 Let \( \{X(t) : t \in \mathbb{R}\} \) and \( \{Y(t) : t \in [0, \infty)\} \) be independent, centered Gaussian processes with stationary increments that satisfy B1 – B3 with parameters \( \alpha_X, \beta_X, \alpha_Y, \beta_Y \) respectively. Then, for \( h(u) = \sigma_Y^{-1}(\sigma_X^{-1}(u) / (1 + o(1))) \) as \( u \to \infty \), we have

\[
\lim_{u \to \infty} \mathbb{P} \left( \sup_{t \in [0,h(u)]} X(Y(t)) > u \right) = \mathcal{L}(\alpha_X, \alpha_Y).
\]

The proof of Theorem 3.1 is presented in Section 4.3.

The second part of this section focuses on the analysis of limiting behavior of (7) in the case of \( \{X(t) : t \in \mathbb{R}\} \) being a centered stationary Gaussian process with a.s. continuous sample paths and covariance function \( r(t) := \text{Cov}(X(s), X(s + t)) \) that satisfies

\begin{itemize}
  \item [D1] \( r(t) = 1 - C |t|^{a} + o(|t|^{a}) \) as \( t \to 0 \), with \( a \in (0, 2] \) and \( C > 0 \);
  \item [D2] \( r(t) < 1 \) for all \( t > 0 \);
  \item [D3] \( r(t) \log(t) \to r \) as \( t \to \infty \), with \( r \in [0, \infty) \).
\end{itemize}
We study (7) for both weakly and strongly dependent stationary Gaussian processes, i.e., for \( r = 0 \) and \( r > 0 \) respectively. We refer to [29][30] for recent results on asymptotic behavior of suprema of strongly dependent Gaussian processes.

In this settings, in Theorem 3.2, we provide (7) in the case of \( \{Y(t) : t \in [0, \infty)\} \) being a centered Gaussian process with stationary increments and variance function \( \sigma_Y^2(t) \) that satisfies conditions B1 – B3. Moreover, in Proposition 3.3 we analyze (7) for self-similar process \( \{Y(t)\} \) that is not necessarily Gaussian.

**Theorem 3.2** Let \( \{X(t) : t \in \mathbb{R}\} \) be a centered stationary Gaussian process with covariance function that satisfies D1 – D3 and \( \{Y(t) : t \in [0, \infty)\} \) be an independent of \( \{X(t)\} \) Gaussian process with a.s. continuous sample paths, stationary increments, and variance function \( \sigma_Y^2(t) \) that satisfies B1 – B3 with parameters \( \alpha_Y, \beta_Y \). Then, for \( h(u) = \sigma_Y^{-1} \left( C \frac{1}{\alpha} H_\alpha u^\frac{2}{\alpha} \Psi(u) \right)^{-1} (1 + o(1)) \) as \( u \to \infty \), we have

\[
\lim_{u \to \infty} \mathbb{P} \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right) = 1 - \mathbb{E} \exp \left( -t \exp(-r + \sqrt{2r}N) \right),
\]

where \( T = \sup_{s \in [0,1]} B_{\alpha_Y/2}(s) - \inf_{s \in [0,1]} B_{\alpha_Y/2}(s) \) and \( \mathcal{N} \) is a normal random variable independent of \( T \).

The proof of Theorem 3.2 is given in Section 4.4.

**Proposition 3.3** Let \( \{X(t) : t \in \mathbb{R}\} \) be a stationary Gaussian process with covariance function that satisfies D1 – D3 and \( \{Y(t) : t \in [0, \infty)\} \) be a self-similar stochastic process with parameter \( \lambda_Y \), independent of the process \( \{X(t)\} \). Then, for \( h(u) = \left( C \frac{1}{\alpha} H_\alpha u^\frac{2}{\alpha} \Psi(u) \right)^{-1/\lambda_Y} (1 + o(1)) \) as \( u \to \infty \), we have

\[
\lim_{u \to \infty} \mathbb{P} \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right) = 1 - \mathbb{E} \exp \left( -t \exp \left( -r + \sqrt{2r}N \right) \right),
\]

where \( T = \sup_{s \in [0,1]} Y(s) - \inf_{s \in [0,1]} Y(s) \) and \( \mathcal{N} \) is a normal random variable independent of \( T \).

**Proof.** Due to stationarity of the process \( \{X(t)\} \) and self-similarity of the process \( \{Y(t)\} \), we have

\[
\mathbb{P} \left( \sup_{t \in [0,h(u)]} X(Y(t)) > u \right) = \mathbb{P} \left( \sup_{t \in [0,\sup s \in [0,h(u)]} \sup_{s \in [0,h(u)]} X(t) > u \right)
\]

\[
= \mathbb{P} \left( \sup_{t \in [0,(h(u))^\lambda_Y T]} X(t) > u \right)
\]

\[
= 1 - \mathbb{E} \exp \left( -t \exp \left( -r + \sqrt{2r}N \right) \right), \quad (8)
\]

where (8) follows by the reasoning as in the proof of Theorem 3.2.

\( \square \)

**Remark 3.2** Note, that in the case of weakly dependent stationary Gaussian process \( \{X(t)\} \), that is, if \( r = 0 \) in D3, we obtain the following result

\[
\lim_{u \to \infty} \mathbb{P} \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right) = 1 - \mathbb{E}e^{-T},
\]

10
where in the setting of Theorem 3.2, 
\[ h(u) = \sigma^{-1}_Y \left( C^{1/\alpha} H_{\alpha \Psi(u)} \right) (1 + o(1)) \] as \( u \to \infty \), and \( T = \sup_{s \in [0,1]} B_{\alpha Y/Y} - \inf_{s \in [0,1]} B_{\alpha Y/Y} \);

and in the setting of Proposition 3.3, 
\[ h(u) = \left[ C^{1/\alpha} H_{\alpha \Psi(u)} \right]^{-1/\lambda Y} (1 + o(1)) \] as \( u \to \infty \), and \( T = \sup_{s \in [0,1]} Y(s) - \inf_{s \in [0,1]} Y(s) \).

4 Proofs

In this section, we present detailed proofs of Theorem 2.1, Proposition 2.3, Theorem 3.1 and Theorem 3.2.

4.1 Proof of Theorem 2.1

In view of inclusion – exclusion principle

\[ P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) = P_1(u) + P_2(u) - P_3(u), \] (9)

where

\[ P_1(u) = P \left( \sup_{s \in [-K,0]} X(s) > u \right), \quad P_2(u) = P \left( \sup_{s \in [0,M]} X(s) > u \right), \]

\[ P_3(u) = P \left( \sup_{s \in [-K,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u \right). \]

Observe that by definition of the process \( \{X(t)\} \),

\[ P_1(u) = P \left( \sup_{s \in [0,K]} X(s) > u \right). \] (10)

The case (i) is a consequence of the fact that, by (10) and Theorem 3.1 in [3], \( P(K > u) = o(P(M > u)) \) implies \( P_1(u) = o(P_2(u)) \). Thus,

\[ P_2(u) \leq P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) \leq P_1(u) + P_2(u) = P_2(u)(1 + o(1)) \]

as \( u \to \infty \), which in view of Theorem 3.1 in [3], completes the proof for the case (i). A similar reasoning implies that for the case (ii), we have

\[ P_1(u) \leq P \left( \sup_{s \in [0,T]} X(Y(s)) > u \right) \leq P_1(u) + P_2(u) = P_1(u)(1 + o(1)) \]

as \( u \to \infty \), which in view of Theorem 3.1 in [3], completes the proof for the case (ii).

In order to prove (iii), without loss of generality, we assume that

\[ P(M > u) \geq P(K > u)(1 + o(1)) \] (11)
as \( u \to \infty \). Due to (9) combined with (10) and Theorem 3.1 in [3], it suffices to show that \( P_3(u) \) is negligible. We distinguish the case \( K \leq M \) and the case \( K > M \) and obtain

\[
P_3(u) = P\left( \sup_{s \in [-K,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u, K \leq M \right) \\
+ P\left( \sup_{s \in [-K,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u, K > M \right) \\
\leq P\left( \sup_{s \in [-M,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u \right) \\
+ P\left( \sup_{s \in [-K,0]} X(s) > u, \sup_{s \in [0,K]} X(s) > u \right) \\
\leq 2P\left( \sup_{s \in [-M,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u \right) (1 + o(1)) \tag{12}
\]

as \( u \to \infty \), where (12) is due to the assumption (11).

To find an upper bound of (12) it is convenient to make the following decomposition

\[
P\left( \sup_{s \in [-M,0]} X(s) > u, \sup_{s \in [0,M]} X(s) > u \right) \\
= \int_0^{a(u)} + \int_{a(u)}^{A(u)} + \int_{A(u)}^{\infty} P\left( \sup_{s \in [-w,0]} X(s) > u, \sup_{s \in [0,w]} X(s) > u \right) dF_M(w) \\
= I_1 + I_2 + I_3,
\]

where

\[
a(u) = u^{\frac{2}{\alpha \infty + 2 \alpha_1}}, A(u) = u^{\frac{4}{2 \alpha \infty + \alpha_1}}.
\]  \tag{13}

Let \( \varepsilon > 0 \). We analyze each of the integrals \( I_1, I_2, I_3 \) separately.

**Integral \( I_1 \):**

\[
I_1 \leq \int_0^{a(u)} P\left( \sup_{s \in [0,w]} X(s) > u \right) dF_M(w) \\
\leq P\left( \sup_{s \in [0,a(u)]} X(s) > u \right) \\
\leq \text{Const} \ a(u) \left( \frac{u}{\sigma_X(a(u))} \right)^{\frac{2}{\alpha \infty}} \Psi\left( \frac{u}{\sigma_X(a(u))} \right) \\
\leq \exp\left( -u^{\frac{2 \alpha_1}{\alpha_\infty + \alpha_1} + \varepsilon} \right) (1 + o(1)) \tag{14}
\]

as \( u \to \infty \), where (14) is due to (16) in [3] (see also the proof of Lemma 6.3 in [3]).
Integral $I_3$:

$$I_3 \leq \int_{A(u)}^{\infty} \mathbb{P} \left( \sup_{s \in [0, w]} X(s) > u \right) dF_M(w)$$

\[ \leq \mathbb{P}(M > A(u)) \]

\[ = C_1(A(u))^\gamma \exp \left( -\beta_1(A(u))^{0_1} \right) (1 + o(1)) \]

\[ \leq \exp \left( -u^{\frac{2\alpha_1}{\alpha_1 + \alpha\infty}} + \varepsilon \right) (1 + o(1)) \]

as $u \to \infty$.

The above, combined with the observation that for each $\eta > 0$ and sufficiently large $u$,

$$\mathbb{P}(X(M) > u) = \mathbb{P}(\sigma_X(M), \mathcal{N} > u) \geq \mathbb{P}(\sigma_X(M) > u^{\frac{\alpha\infty}{\alpha_1 + \alpha\infty}}) \mathbb{P}(\mathcal{N} > u^{\frac{\alpha_1}{\alpha_1 + \alpha\infty}})$$

\[ \geq \exp \left( -u^{\frac{2\alpha_1}{\alpha_1 + \alpha\infty}} + \eta \right), \]

leads to the conclusion that $I_1$ and $I_3$ are negligible.

Integral $I_2$: Observe that, due to A1, $\sigma_X^2(|t|) \leq \sigma_X^2(|t - s|)$, for each $(s, t) \in [-w, 0] \times [0, w]$. Hence

$$\text{Var}(X(s) + X(t)) = 2\sigma_X^2(|s|) + 2\sigma_X^2(|t|) - \sigma_X^2(|t - s|) \leq 3\sigma_X^2(w), \quad (15)$$

for $(s, t) \in [-w, 0] \times [0, w]$. Thus, according to the Borell inequality (see, e.g., [1], Theorem 2.1), combined with (15), $I_2$ is bounded by

$$\int_{A(u)}^{\infty} \mathbb{P} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] > 2u \right) dF_M(w)$$

\[ \leq 2 \int_{A(u)}^{\infty} \exp \left( -\frac{2u^2}{3\sigma_X^2(w)} \left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] \right) \right) \right)^2 dF_M(w). \quad (16) \]

Moreover,

$$0 \leq \mathbb{E} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] \right) \leq 2 \mathbb{E} \left( \sup_{s \in [0,w]} X(t) \right). \quad (17)$$

To find the upper bound of $\mathbb{E} \sup_{t \in [0,w]} X(t)$, we use metric entropy method (see, e.g., [20], Chapter 10). At the beginning, for any $T \subseteq \mathbb{R}$ define the semimetric

$$d(s,t) := \sqrt{\mathbb{E}|X(t) - X(s)|^2} = \sigma_X(|t - s|).$$

The metric entropy $\mathbb{H}_d(T, \varepsilon)$ is defined as log $N_d(T, \varepsilon)$, where $N_d(T, \varepsilon)$ denotes the minimal number of points in an $\varepsilon$-net in $T$ with respect to the semimetric $d$.

Observe that for $T = [0, w],$

$$N_d(T, \varepsilon) \leq \frac{2w}{\sigma_X^{-1}(\varepsilon)}. \quad (18)$$

13
which, in view of Theorem 10.1 in [20], implies that
\[
\mathbb{E} \sup_{t \in [0, w]} X(t) \leq 4\sqrt{2} \int_0^{\sigma_X(w)} \sqrt{\log \frac{2w}{\sigma_X^{-1}(\epsilon)}} \, d\epsilon \\
\leq 4\sqrt{2} \int_0^{\frac{\sqrt{D}w}{\epsilon}} \sqrt{\log \frac{2Dw}{\epsilon}} \, d\epsilon \\
= 4\sqrt{2} \int_{\frac{1}{w}}^{\infty} \frac{\sqrt{D}w}{2} x^{-\frac{\alpha_\infty}{2}} - 1 \sqrt{\log 2w} \, dx \\
\leq 4\alpha_\infty \sqrt{Dw} \int_{\frac{1}{w}}^{\infty} x^{-\frac{\alpha_\infty + 1}{2}} \, dx \\
\leq B w^{\frac{\alpha_\infty}{2}} ,
\]
where \(B = \frac{8\sqrt{D}w}{\alpha_\infty \epsilon - 1}\). (18) is due to A3, and (19) is by substitution \(x := D^{1/\alpha_\infty} \epsilon^{-2/\alpha_\infty}\).

Finally, due to (17) combined with (20) and (13)
\[
0 \leq \frac{1}{2u} \mathbb{E} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] \right) \leq B w^{\frac{\alpha_\infty}{u}} \leq B u^{-\frac{\alpha_\infty}{\alpha_1 + \alpha_1}} ,
\]
for each \(w \in [a(u), A(u)]\), which implies that
\[
\left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] \right) \right)^2 \to 1
\]
as \(u \to \infty\), uniformly for \(w \in [a(u), A(u)]\), and hence
\[
\exp \left( - \frac{2u^2}{3\sigma^2_X(w)} \left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s,t) \in [-w,0] \times [0,w]} [X(s) + X(t)] \right) \right)^2 \right) = o \left( \frac{u}{\sigma_X(w)} \right) 
\]
as \(u \to \infty\), uniformly for \(w \in [a(u), A(u)]\). Thus, combining (16) with (21), we obtain, for sufficiently large \(u\), the following upper bound,
\[
I_2 \leq 2\varepsilon \int_{a(u)}^{A(u)} \Psi \left( \frac{u}{\sigma_X(w)} \right) \, dF_M(w) \leq 2\varepsilon \mathbb{P} \left( \sup_{s \in [0,M]} X(s) > u \right) .
\]
which in view of Theorem 3.1 in [3], implies that
\[
\limsup_{u \to \infty} \frac{I_2}{\mathbb{P}(X(M) > u)} \leq 2\varepsilon .
\]
In order to complete the proof it suffices to pass with \(\varepsilon \to 0\). \(\square\)

4.2 Proof of Proposition 2.3

The idea of the proof is analogous to the proof of Theorem 2.1, thus we present only main steps of the argumentation. In view of inclusion – exclusion principle
\[
\mathbb{P} \left( \sup_{s \in [0,T]} B_H(Y(s)) > u \right) = P_1(u) + P_2(u) - P_3(u) ,
\]
(22)
where \( P_1(u) = \mathbb{P} \left( \sup_{s \in [-\kappa,0]} B_H(s) > u \right) \), \( P_2(u) = \mathbb{P} \left( \sup_{s \in [0,M]} B_H(s) > u \right) \), 
\( P_3(u) = \mathbb{P} \left( \sup_{s \in [-\kappa,0]} B_H(s) > u, \sup_{s \in [0,M]} B_H(s) > u \right) \).

Moreover observe that

\[
P_1(u) = \mathbb{P} \left( \sup_{s \in [0,K]} B_H(s) > u \right). \tag{23}
\]

Since the arguments for the cases \( \mathbb{P}(K > u) = o(\mathbb{P}(M > u)) \) as \( u \to \infty \), and \( \mathbb{P}(M > u) = o(\mathbb{P}(K > u)) \) as \( u \to \infty \) are similar to those in the proof of Theorem 2.1, then we focus on the case \( \mathbb{P}(K > u) = \frac{C_2}{C_1} \mathbb{P}(M > u)(1 + o(1)) \) as \( u \to \infty \).

Without loss of generality, we assume that

\[ \mathbb{P}(M > u) \geq \mathbb{P}(K > u)(1 + o(1)) \]

as \( u \to \infty \). Due to (22) combined with (23) and Theorem 4.1 in [3], it suffices to show that \( P_3(u) \) is negligible. In an analogous way to (12), we obtain the following upper bound

\[ P_3(u) \leq 2 \mathbb{P} \left( \sup_{s \in [-M,0]} B_H(s) > u, \sup_{s \in [0,M]} B_H(s) > u \right)(1 + o(1)) \]

as \( u \to \infty \). Then, we consider decomposition

\[
\mathbb{P} \left( \sup_{s \in [-M,0]} B_H(s) > u, \sup_{s \in [0,M]} B_H(s) > u \right) \nonumber \\
= \left( \int_0^{a(u)} + \int_{a(u)}^{A(u)} + \int_{A(u)}^{\infty} \right) \mathbb{P} \left( \sup_{s \in [-w,0]} B_H(s) > u, \sup_{s \in [0,w]} B_H(s) > u \right) dF_M(w) \\
= I_1 + I_2 + I_3,
\]

where

\[
a(u) = u^{\frac{H}{2+\alpha}}, \quad A(u) = u^{\frac{4}{3H+\alpha}}. \tag{24}
\]

Let \( \varepsilon > 0 \). We investigate the asymptotic behavior of each of the integrals.

**Integral I₁:** Due to self-similarity of \( \{B_H(t)\} \) combined with Lemma 4.2 in [3], we have, as \( u \to \infty \),

\[
I_1 \leq \mathbb{P} \left( \sup_{s \in [0,a(u)]} B_H(s) > u \right) = \mathbb{P} \left( \sup_{s \in [0,1]} B_H(s) > \frac{u}{(a(u))^H} \right) \leq \exp \left( -u^{\frac{2\alpha}{\alpha+H}+\varepsilon} \right)(1 + o(1)).
\]

**Integral I₂:** We have, as \( u \to \infty \),

\[
I_2 \leq \mathbb{P}(M > A(u)) \leq \exp \left( -u^{\frac{2\alpha}{\alpha+H}+\varepsilon} \right)(1 + o(1)).
\]

Observe that, due to Theorem 4.1 in [3], for each \( \eta > 0 \) and sufficiently large \( u \),

\[
\mathbb{P} \left( \sup_{s \in [0,M]} B_H(s) > u \right) \geq \exp \left( -u^{\frac{2\alpha}{\alpha+H}+\eta} \right)(1 + o(1))
\]

15
as \( u \to \infty \). Thus, we conclude that \( I_1 \) and \( I_3 \) are negligible.

**Integral I_2:** Observe that \(|t|^{2H} \leq |t-s|^{2H} \), for each \((s, t) \in [-w, 0] \times [0, w]\). Hence

\[
\forall \epsilon > 0, \quad \lim_{u \to \infty} \epsilon \int_{a(u)}^{A(u)} \left( \sup_{(s, t) \in [-w, 0] \times [0, w]} [B_H(s) + B_H(t)] \right) dF_M(w) \leq 2 \epsilon \int_{a(u)}^{A(u)} \exp \left( -\frac{2u^2}{3w^{2H}} \left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s, t) \in [-w, 0] \times [0, w]} [B_H(s) + B_H(t)] \right) \right)^2 \right) dF_M(w).
\]

Moreover, due to self-similarity of \( \{B_H(t)\} \)

\[
0 \leq \mathbb{E} \left( \sup_{(s, t) \in [-w, 0] \times [0, w]} [B_H(s) + B_H(t)] \right) \leq 2 \mathbb{E} \left( \sup_{s \in [0, w]} B_H(s) \right) = B w^H,
\]

where \( B = 2 \mathbb{E} \sup_{s \in [0, w]} B_H(s) \), which due to (24), implies that

\[
\left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s, t) \in [-w, 0] \times [0, w]} [B_H(s) + B_H(t)] \right) \right)^2 \to 1
\]
as \( u \to \infty \), uniformly for \( w \in [a(u), A(u)] \), and hence

\[
\exp \left( -\frac{2u^2}{3w^{2H}} \left( 1 - \frac{1}{2u} \mathbb{E} \left( \sup_{(s, t) \in [-w, 0] \times [0, w]} [B_H(s) + B_H(t)] \right) \right)^2 \right) = o \left( \frac{\Psi \left( \frac{u}{w^{H}} \right)}{w^{H}} \right)
\]
as \( u \to \infty \), uniformly for \( w \in [a(u), A(u)] \). Thus, combining (26) with (27), we obtain, for sufficiently large \( u \), the following upper bound,

\[
I_2 \leq 2 \epsilon \int_{a(u)}^{A(u)} \frac{dF_M(w)}{\Psi \left( \frac{u}{\sigma_X(w)} \right)} \leq 2 \epsilon \mathbb{P} \left( \sup_{s \in \mathcal{M}} X(s) > u \right).
\]

which in view of Theorem 4.1 in [3], implies that

\[
\limsup_{u \to \infty} \frac{I_2}{\mathbb{P} (X(\mathcal{M}) > u)} \leq 2 \epsilon.
\]

In order to complete the proof it suffices to pass with \( \epsilon \to 0 \). \( \square \)

### 4.3 Proof of Theorem 3.1

In further analysis we use the following notation

\[
X_{\sigma_Y(h(u))}(s) := \frac{X(\sigma_Y(h(u))s)}{\sigma_X(\sigma_Y(h(u)))} \quad \text{and} \quad Y_{h(u)}(s) := \frac{Y(h(u)s)}{\sigma_Y(h(u))}.
\]
Moreover, we denote 
\[
V_u := \inf_{s \in [0,1]} Y_{h(u)}(s), \quad W_u := \sup_{s \in [0,1]} Y_{h(u)}(s),
\]
\[
V := \inf_{s \in [0,1]} B_{\alpha Y/2}(s), \quad W := \sup_{s \in [0,1]} B_{\alpha Y/2}(s).
\]
Let \( \varepsilon > 0 \) and \( 0 < A_\infty < \infty \). We start with the observation that \( \lim_{u \to \infty} h(u) = \infty \), which also implies that \( \lim_{u \to \infty} \sigma_Y(h(u)) = \infty \). Hence, due to Lemma 5.2 in [12]
\[
(V_u, W_u) \Rightarrow (V, W) \quad \text{as} \quad u \to \infty
\] (28)
and
\[
\sup_{s \in [v,w]} X_{\sigma_Y(h(u))}(s) \Rightarrow \sup_{s \in [v,w]} B_{\alpha X/2}(s) \quad \text{as} \quad u \to \infty,
\] (29)
uniformly for \((v, w) \in [-A_\infty, 0) \times [0, A_\infty] \), where \( \Rightarrow \) denotes convergence in distribution.

By continuity of the sample paths of the processes \( \{X(t)\} \) and \( \{Y(t)\} \),
\[
P \left( \sup_{t \in [0,h(u)]} X(Y(t)) > u \right) = P \left( \sup_{t \in [\inf_{s \in [0,h(u)]} Y(s), \sup_{s \in [0,h(u)]} Y(s)]} X(t) > u \right) \]
\[
= P \left( \sup_{t \in [V_u, W_u]} X(\sigma_Y(h(u)) t) > u \right) \]
\[
= P \left( \sup_{t \in [V_u, W_u]} X_{\sigma_Y(h(u))}(t) > \frac{u}{\sigma_X(\sigma_Y(h(u)))} \right).
\] (30)

To find an upper bound of (30) we consider the following decomposition
\[
P \left( \sup_{t \in [V_u, W_u]} X_{\sigma_Y(h(u))}(t) > \frac{u}{\sigma_X(\sigma_Y(h(u)))} \right) \leq \left( \int_{-A_\infty}^{0} \int_{0}^{A_\infty} \int_{0}^{A_\infty} + \int_{-\infty}^{0} \int_{0}^{A_\infty} \right) P \left( \sup_{t \in [v,w]} X_{\sigma_Y(h(u))}(t) > \frac{u}{\sigma_X(\sigma_Y(h(u)))} \right) d(V_u, W_u) (v, w) \]
\[
= I_1 + I_2 + I_3.
\]

We analyze each of the integrals \( I_1, I_2, I_3 \) separately.

\textit{Integral I₁}: Due to (28), for sufficiently large \( u \),
\[
I_1 \leq P(V_u \leq -A_\infty) \leq (1 + \varepsilon) P(V \leq -A_\infty).
\]

\textit{Integral I₃}: Due to (28), for sufficiently large \( u \),
\[
I_3 \leq P(W_u > A_\infty) \leq (1 + \varepsilon) P(W > A_\infty).
\]
In further analysis we use the following notation

\[ I_2 = \int_{-A_\infty}^0 \int_{-A_\infty}^{A_\infty} \mathbb{P} \left( \sup_{t \in [v,w]} X_{\sigma_Y(h(u))}(t) > \frac{u}{\sigma_X(\sigma_Y(h(u)))} \right) d_{(V_u,W_u)}(v, w) \]

\[ \leq (1 + \varepsilon) \int_{-A_\infty}^0 \int_{-A_\infty}^{A_\infty} \mathbb{P} \left( \sup_{t \in [v,w]} B_{\alpha_X/2}(t) > 1 \right) d_{(V_u,W_u)}(v, w) \]

\[ \leq (1 + \varepsilon)^2 \int_{-A_\infty}^0 \int_{-A_\infty}^{A_\infty} \mathbb{P} \left( \sup_{t \in [v,w]} B_{\alpha_X/2}(t) > 1 \right) d_{(V,W)}(v, w) \]

\[ \leq (1 + \varepsilon)^2 \mathbb{P} \left( \sup_{t \in [v,w]} B_{\alpha_X/2}(t) > 1 \right), \]

where (31) is due to (29) and the fact that \( \lim_{u \to \infty} \frac{u}{\sigma_X(\sigma_Y(h(u)))} = 1 \), and (32) is due to (28), and the observation that \( \mathbb{P} \left( \sup_{t \in [v,w]} B_{\alpha_X/2}(t) > 1 \right) \) is bounded and continuous function with respect to \((v, w)\). Thus, for each \( \varepsilon > 0, A_\infty > A_0 > 0 \),

\[ \limsup_{u \to \infty} \mathbb{P} \left( \sup_{t \in [0,h(u)]} X(Y(t)) > u \right) \leq (1 + \varepsilon)^2 \mathbb{P} \left( \sup_{t \in [V,W]} B_{\alpha_X/2}(t) > 1 \right) \]

\[ \geq (1 - \varepsilon)^2 \mathbb{P} \left( \sup_{t \in [V,W]} B_{\alpha_X/2}(t) > 1 \right). \]

In order to complete the proof it suffices to pass with \( A_0 \to 0, A_\infty \to \infty \), and \( \varepsilon \to 0 \). \[ \square \]

### 4.4 Proof of Theorem 3.2

In further analysis we use the following notation

\[ T_u := \sup_{s \in [0,1]} Y_{h(u)}(s) - \inf_{s \in [0,1]} Y_{h(u)}(s), \quad \text{where} \quad Y_{h(u)}(s) := \frac{Y(h(u)s)}{\sigma_Y(h(u))} \]

Let \( \varepsilon > 0 \) and \( 0 < A_0 < A_\infty < \infty \). Note that due to Lemma 5.2 in [12]

\[ T_u \Rightarrow T \quad \text{as} \quad u \to \infty, \]

where \( \Rightarrow \) denotes convergence in distribution.

It is convenient to consider the following decomposition

\[ \mathbb{P} \left( \sup_{s \in [0,h(u)]} X(Y(s)) > u \right) = \mathbb{P} \left( \sup_{s \in [0,T_u \sigma_Y(h(u))]} X(s) > u \right) \]

\[ = \left( \int_0^{A_0} + \int_{A_0}^{A_\infty} + \int_{A_\infty}^{\infty} \right) \mathbb{P} \left( \sup_{s \in [0,T_u \sigma_Y(h(u))]} X(s) > u \right) dF_{T_u}(t) \]

\[ = I_1 + I_2 + I_3. \]
We analyze each of the integrals $I_1, I_2, I_3$ separately.

**Integral $I_1$:** Due to Lemma 3.3 in [30], for sufficiently large $u$, 

\[
I_1 \leq \mathbb{P}\left(\sup_{s \in [0, A_0 \sigma Y(h(u))]} X(s) > u\right)
\]

\[
\leq (1 + \varepsilon) \left[1 - \mathbb{E}\exp\left(-A_0 \exp(-r + \sqrt{2rN})\right)\right]
\]

as $u \to \infty$.

**Integral $I_3$:** Due to (33), for sufficiently large $u$, 

\[
I_3 \leq \mathbb{P}(T_u > A_\infty) \leq (1 + \varepsilon)\mathbb{P}(T > A_\infty).
\]

**Integral $I_2$:** 

\[
I_2 = \int_{A_0}^{A_\infty} \mathbb{P}\left(\sup_{s \in [0, t \sigma Y(h(u))]} X(s) > u\right) dF_{T_u}(t)
\]

\[
\leq (1 + \varepsilon) \int_{A_0}^{A_\infty} \left(1 - \mathbb{E}\exp\left(-t \exp\left(-r + \sqrt{2rN}\right)\right)\right) dF_{T_u}(t) \tag{34}
\]

\[
\leq (1 + \varepsilon)^2 \int_{A_0}^{A_\infty} \left(1 - \mathbb{E}\exp\left(-t \exp\left(-r + \sqrt{2rN}\right)\right)\right) dF_T(t), \tag{35}
\]

where (34) is by Lemma 3.3 in [30] and (35) is due to (33), and the observation that $1 - \mathbb{E}\exp\left(-t \exp\left(-r + \sqrt{2rN}\right)\right)$ is bounded and continuous function with respect to $t \in [A_0, A_\infty]$. Thus, for each $\varepsilon > 0$, $A_\infty > A_0 > 0$, 

\[
\limsup_{u \to \infty} \mathbb{P}\left(\sup_{s \in [0, h(u)]} X(Y(s)) > u\right) \leq (1 + \varepsilon)^2 \int_{A_0}^{A_\infty} \left(1 - \mathbb{E}\exp\left(-t \exp\left(-r + \sqrt{2rN}\right)\right)\right) dF_T(t)
\]

\[
+ (1 + \varepsilon) \left[1 - \mathbb{E}\exp\left(-A_0 \exp(-r + \sqrt{2rN})\right)\right] + (1 + \varepsilon)\mathbb{P}(T > A_\infty).
\]

Analogously,

\[
\liminf_{u \to \infty} \mathbb{P}\left(\sup_{s \in [0, h(u)]} X(Y(s)) > u\right) \geq (1 - \varepsilon)^2 \int_{A_0}^{A_\infty} \left(1 - \mathbb{E}\exp\left(-t \exp\left(-r + \sqrt{2rN}\right)\right)\right) dF_T(t).
\]

In order to complete the proof it suffices to pass with $A_0 \to 0, A_\infty \to \infty$, and $\varepsilon \to 0$. 

\[\square\]

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