Stationary solutions of the curvature preserving flow on space curves

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Abstract. We study a geometric flow on curves, immersed in $\mathbb{R}^3$, that have strictly positive torsion. The evolution equation is given by

$$X_t = \frac{1}{\sqrt{\tau}} B$$

where $\tau$ is the torsion and $B$ is the unit binormal vector. In the case of constant curvature, we find all of the stationary solutions and linearize the PDE for the torsion around stationary solutions admitting an explicit formula. Afterwards, we prove the $L^2(\mathbb{R})$ linear stability of the stationary solutions corresponding to helices with constant curvature and constant torsion.

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1. Introduction. Substantial work has been done towards understanding geometric flows on curves immersed in Riemannian manifolds. For example, the author of [5] proves that the mean-curvature flow shrinks embedded curves in the plane to a point in finite time, becoming round in the limit. Also, it is found in [6] that the vortex filament flow is equivalent to the non-linear Schrödinger equation, which enables the discovery of explicit soliton solutions. Recently, geometric evolutions that are integrable, in the sense of admitting a Hamiltonian structure, have also been of interest. The authors of [1, 8] analyze the integrability of flows in Euclidean space and Riemannian manifolds, respectively. In this note, we study the following geometric flow for curves in $\mathbb{R}^3$ with strictly positive torsion that preserves arc-length and curvature:

$$X_t = \frac{1}{\sqrt{\tau}} B.$$
Hydrodynamic and magnetodynamic motions related to this geometric evolution equation have been considered, as mentioned in [11]. The case when curvature is constant demonstrates a great deal of structure, as we shall soon see. After rescaling so that the curvature is identically 1, the evolution equation for the torsion is given by:

$$\tau_t = D_s(\tau^{-1/2} - \tau^{3/2} + D_s^2(\tau^{-1/2})).$$

This flow has been studied since at least the publication of [11]. The authors of [11] show that the above evolution equation, which they term the extended Dym equation, is equivalent to the $m^2$KDV equation. In addition, they present auto-Bäcklund transformations and compute explicit soliton solutions. We hope to continue the investigation of the curvature-preserving geometric flow:

- We characterize the flow $X_t = \frac{1}{\sqrt{\tau}} B$ as the unique flow on space curves that is both curvature and arc-length preserving.
- We provide another proof that this flow is equivalent to the $m^2$KDV equation, and by doing so, we find the first two conserved densities of the flow

$$\int \sqrt{\tau} ds \text{ and } \int \tau ds,$$

which are the same as for the KDV equation, and prove the global existence and uniqueness of solutions of the flow (for periodic $C^\infty$ initial data).

- We find all stationary solutions to the geometric flow in the case of constant curvature, including a two-parameter family of explicit solutions.
- We derive the linearization of the evolution equation for the torsion around explicit stationary solutions, and prove the $L^2(\mathbb{R})$ stability of the linearization in the case of constant torsion (or for helices).

A tedious calculation yields the fact that the evolution in the case of non-constant curvature is not integrable, even in the formal sense of [10]. Although we do not pursue this here, it might be interesting to study the case of non-constant curvature, especially for curves with almost constant curvature.

2. Preliminaries. We recall the following standard computation:

**Lemma 2.1** (cf. [1,8]). Let $\gamma(t)$ be a family of smooth curves immersed in $\mathbb{R}^3$ and let $X(s,t)$ be a parametrization of $\gamma(t)$ by arc-length. Consider the following geometric evolution equation:

$$X_t = h_1 T + h_2 N + h_3 B$$  \hspace{1cm} (1)

where $\{T,N,B\}$ is the Frenet-Serret frame and where we denote the curvature and torsion by $\kappa$ and $\tau$, respectively. Let $h_1, h_2, h_3$ be arbitrary smooth functions of $\kappa$ and $\tau$ on $\gamma(t)$. If the evolution is also arc-length preserving, then the evolution equations of $\kappa$ and $\tau$ are

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = P \begin{pmatrix} h_3 \\ h_1 \end{pmatrix}$$
where $P$ is
\[
\begin{pmatrix}
-\tau D_s - D_s \tau & D_s^2 \frac{1}{\kappa} D_s - \frac{\tau^2}{\kappa} D_s + D_s \kappa \\
D_s \frac{1}{\kappa} D_s^2 - D_s \tau^2 + \kappa D_s & D_s (\frac{\tau}{\kappa^2} D_s + D_s \frac{\tau}{\kappa^2}) D_s + \tau D_s + D_s \tau
\end{pmatrix}.
\]

The next theorem follows without difficulty.

**Theorem 2.2.** Up to a rescaling, a geometric evolution of curves immersed in $\mathbb{R}^3$, as in equation (1), is both curvature and arc-length preserving if and only if its evolution equation is equivalent to
\[
X_t = \frac{1}{\sqrt{\tau}} B.
\] (2)

**Proof.** Let $X_t$ be a curvature and arc-length preserving geometric flow. The tangential component of $X_t$ in equation (1) provides no interesting geometric information; it amounts to a re-parametrization of the curve. Thus, we may assume that $h_1 = 0$, so, since $X_t$ is arc-length preserving, $h_2 = 0$ as well. Lemma 2.1 gives us that
\[
\kappa_t = -\tau D_s (h_3) - D_s (\tau h_3) = -2\tau D_s (h_3) - h_3 D_s (\tau).
\]
Since $X_t$ is curvature preserving, we must have $\kappa_t = 0$, or
\[
D_s (\log h_3) = D_s (\log \tau^{-1/2}).
\]
Integrating, we see that
\[
h_3 = \frac{c}{\sqrt{\tau}}
\]
where $c$ is a constant. Therefore, up to a rescaling, the evolution $X_t$ must be precisely as in (2). $\Box$

Unfortunately, the evolution in (2) only makes sense if $\gamma(t)$ has strictly positive torsion (or strictly negative torsion, with the flow $X_t = \frac{1}{\sqrt{-\tau}} B$). This motivates the following definition.

**Definition 1.** We call a smooth curve immersed in $\mathbb{R}^3$ a positive curve if it has strictly positive torsion.

Fortunately, there are many interesting positive curves. For example, some knots admit a parametrization with constant curvature and strictly positive torsion, and there exist closed curves with constant (positive) torsion. Henceforth, we will only consider positive curves.

The partial differential equation governing the evolution of $\tau$ follows below:

**Lemma 2.3.** For the geometric flow given in equation (2), the evolution equations of the curvature and the torsion are $\kappa_t = 0$ and
\[
\tau_t = \kappa D_s (\tau^{-1/2}) + D_s \left( \frac{D_s^2 (\tau^{-1/2}) - \tau^{3/2}}{\kappa} \right).
\] (3)
The equation for the torsion is reminiscent of the Rosenau-Hyman family of equations, which are studied, inter alia, in [7,9]. The authors of [11] call the evolution of the torsion, in the case of constant curvature, the extended Dym equation for its relationship with the Dym equation (a rescaling and limiting process converts the evolution for the torsion to the Dym equation). As discussed in [1], the condition for the flow \( X_t \) from Lemma 2.1 to be the gradient of a functional is for the Frechet derivative of \((h_3, h_1)\) to be self-adjoint. In general, this does not occur for the flow under our consideration in equation (2), so it cannot be integrable in the sense of admitting a Hamiltonian structure (indeed, when \( \kappa \) is not constant, it is not even formally integrable in the sense of [10]). Nevertheless, the case of constant curvature exhibits a great deal of structure, which makes the study of the evolution equation in (3) worthwhile.

3. Constant curvature. When \( \kappa \) is constant, we may rescale the curves \( \gamma(t) \) so that \( \kappa \equiv 1 \). In this way, the evolution of the torsion becomes

\[
\tau_t = D_s \left( \tau^{-1/2} - \tau^{3/2} + D_s^2 \tau^{-1/2} \right).
\]

3.1. Equivalence with the \( m^2 \text{KDV} \) Equation. We recall the notion of “equivalence” of two partial differential equations from [3]:

**Definition 2.** Two partial differential equations are equivalent if one can be obtained from the other by a transformation involving the dependent variables or the introduction of a potential variable.

The authors in [3] discuss a general method of transforming quasilinear partial differential equation, such as the evolution of \( \tau \) in (4), to semi-linear equations. By applying their algorithm, we obtain another proof of the following theorem, first demonstrated in [11].

**Theorem 3.1** ([11]). The evolution equation for the torsion, in the case of constant curvature, which is given by

\[
\tau_t = D_s \left( \tau^{-1/2} - \tau^{3/2} + D_s^2 \tau^{-1/2} \right),
\]

is equivalent to the \( m^2 \text{KDV} \) equation. Thus, it is a completely integrable evolution equation.

**Proof.** First, let \( \tau = v^2 \), so that (4) becomes

\[
2v v_t = D_s \left( \frac{1}{v} - v^3 + D_s^2 \frac{1}{v} \right) = - \frac{v}{v^2} - 3v^2 v_s - \frac{v_{ss}}{v^2} - \frac{6v^3}{v^4} + \frac{6v_s v_{ss}}{v^3}
\]

or, in a simpler form:

\[
v_t = D_s \left( \frac{1}{4v^2} - \frac{3v^2}{4v^4} + \frac{3v_s^2}{4v^4} - \frac{v_{ss}}{2v^5} \right).
\]

A potentiation, \( v = w_s \) followed by a simple change of variables \((t \to -t/2)\) yields

\[
w_t = - \frac{1}{2w_s^2} + \frac{3w_s^2}{2} - \frac{3w_{ss}^2}{2w_s^4} + \frac{w_{sss}}{w_s^3}.
\]
Equation (6) is fecund territory for a pure hodograph transformation, as used, for example, in [3]. Let \( \tilde{t} = t, \xi = u(s,t), \) and \( s = \eta(\xi, \tilde{t}). \) The resulting equation, after a simple computation, is

\[
\eta_{\tilde{t}} = \eta_{\xi\xi\xi} - \frac{3\eta_{\xi}^2}{2\eta_{\xi}} + \frac{\eta_{\xi}^3}{2} - \frac{3}{2\eta_{\xi}}. \tag{7}
\]

We rename the variables to the usual variables of space and time: \( s \) and \( t; \) in addition, we anti-potentiate the equation by letting \( \eta_s = z. \) This makes equation (7) equivalent to

\[
z_t = z_{sss} - \frac{3}{2} \left( \frac{z^2}{z} \right)_s + \frac{3z^2z_s}{2} + \frac{3z_s}{2z^2}. \tag{8}
\]

Lastly, if we let \( q = \sinh(z/2) \) in equation (8), a transformation also discussed in [2], we get the \( m^2 \)KDV equation

\[
q_t = q_{sss} - \frac{3}{2} \left( \frac{qq_s^2}{1 + q^2} \right)_s + 6q^2q_s \tag{9}
\]

which finishes the proof of the theorem. \( \square \)

Equations (4) and (5) give us the first two integrals of motion of this flow:

\[
\int \sqrt{\tau} ds \quad \text{and} \quad \int \tau ds.
\]

The rest can be found by pulling back the \( m^2 \)KDV invariants; we note that these invariants were obtained in a different way by the authors of [11]. Our next step is to prove the long term existence of our geometric flow using Theorem 3.1.

**Corollary 3.2.** Let \( \tau_0 \in C^\infty_{\text{per}}([0,2\pi]) \) be a strictly positive and periodic function. We can solve the evolution equation for the torsion (4) with initial data \( \tau_0 \) and get the unique solution \( \tau(t) \in C^\infty_{\text{per}}([0,2\pi]) \) that is strictly positive for all times \( t \geq 0. \)

**Proof.** In order to prove this corollary, it suffices to show that we can reconstruct the solution of (4) by solving the \( m^2 \)KDV equation (9) instead. The long term existence for solutions to the \( m^2 \)KDV equation will imply the same for (4), so all that remains to prove is that every differential transformation we used in the proof of Theorem 3.1 is truly “invertible” in the sense that no singularities arise. We use the same notation for our differential transformations as before.

For the proof of this corollary, it is more convenient to work with an equivalent form of the \( m^2 \)KDV equation called the Calogero–Degasperis–Fokas (CDF) equation [3], obtained by letting \( z = e^u \) in equation (8):

\[
u_t = u_{sss} - \frac{u_s^3}{2} + \frac{3u_s}{2}(e^{2u} + e^{-2u}).
\]
Beginning with our initial torsion $\tau_0$, we let $v_0 = \sqrt{\tau_0}$ and then

$$w_0(s) = \int_0^s v_0(\tilde{s}) d\tilde{s}.$$ 

By construction, $w_0 \in C^\infty([0, 2\pi])$ is not periodic, but satisfies $w_0'(s) > 0$ for all $s \in [0, 2\pi]$. Thus, $w_0$ has a global inverse $\eta_0(\xi)$ and we also have $\eta_0'(\xi) := z_0(\xi) > 0$ for all $\xi$ in the domain of $\eta_0$, which is some interval of the form $[0, M]$. Finally, let $u_0 = \log(z_0)$, which is well defined and in $C^\infty([0, M])$ because $z_0 > 0$ and $z_0 \in C^\infty([0, M])$. The CDF equation is globally well-posed for smooth functions on a compact interval with periodic boundary conditions (see [2], [7]), so we get a solution $u(\xi, t)$ defined for all time with $u(\xi, 0) = u_0(\xi)$ and $u(0, t) = u(M, t)$ for all $t$. Our next step is to reconstruct the solution for (4) using $u(t)$.

First, let $z(t) = e^{u(t)}$, which will be a strictly positive, periodic, $C^\infty$ function for all times. Then, let

$$\eta(\xi, t) = \int_0^\xi z(\zeta, t) d\zeta \quad \text{for all } \xi \in [0, M]$$

which, since $\int z(\zeta, t) d\zeta$ is a time-independent quantity, satisfies

$$\eta(0, t) = 0 \quad \text{and} \quad \eta(M, t) = 2\pi$$

for all times. Obviously $\eta_t(\xi, t) > 0$ for all $(\xi, t)$, so the hodograph transformation $\tilde{t} = t$, $\tilde{\xi} = w(s, t)$, and $s = \eta(\xi, \tilde{t})$ makes sense. We recover a $C^\infty$ function $w(s, t)$ with the property that

$$w(0, t) = 0 \quad \text{and} \quad w(2\pi, t) = M$$

for all times, and up to a trivial change of coordinates, we have

$$0 < v(s, t) = \frac{\partial w}{\partial s}(s, t_0) \in C^\infty_{\text{per}}([0, 2\pi])$$

for any fixed time $t_0 \geq 0$. Lastly, $\tau(s, t) = v(s, t)^2$ solves equation (4), is in $C^\infty_{\text{per}}([0, 2\pi])$, and satisfies $\tau(s, 0) = \tau_0(s)$. Thus, we have reconstructed our desired solution of (4) using the completely integrable CDF equation instead. $\square$

3.2. Stationary solutions. Helices, with constant curvature and constant torsion, are the obvious stationary solutions. In what follows, we find the rest.

Theorem 3.3. The stationary solutions of

$$\tau_t = D_s(\tau^{-1/2} - \tau^{3/2} + D_s^{2}(\tau^{-1/2}))$$

are given by the following integral formula

$$\int \frac{du}{\sqrt{C + 2Au - u^2 - u^{-2}}} = s + k$$
where $\tau(s) = u(s)^{-2}$ and where $A, k,$ and $C$ are appropriate real constants. When $A = 0$, we get an explicit formula for the solutions:

$$\tau(s) = \frac{2}{C \pm \sqrt{(-4 + C^2)} \cdot \sin(2(s + k))},$$

with $k$ and $C \geq 2$ real constants.

**Proof.** Let $\tau = u^{-2}$, then, after integrating once, we must examine the following ordinary differential equation (where $A$ is a constant):

$$A = u - \frac{1}{u^3} + D_s^2(u). \quad (10)$$

Since equation (10) is autonomous, we may proceed with a reduction of order argument. Let $w(u) = D_s(u)$ so that $D_s^2(u) = wD_u(w)$ by the chain rule. This substitution gives us the first order equation:

$$wD_u(w) = A - u + \frac{1}{u^3} \quad (11)$$

or

$$D_u(w^2) = 2A - 2u + \frac{2}{u^3}.$$

Integrating, we get

$$D_s(u) = w(u) = \sqrt{C + 2A}u - u^2 - u^{-2}$$

which is a separable differential equation. So, the stationary solutions of equation (4) are given by the following integral formula:

$$\int \frac{du}{\sqrt{C + 2Au - u^2 - u^{-2}}} = s + k \quad (12)$$

for appropriate constants $C, A,$ and $k$. It would be pleasant to have explicit solutions, and this occurs in the case when $A = 0$, which is more easily handled. Equation (11) above becomes

$$D_s(u) = w(u) = \sqrt{C - u^2 - u^{-2}}$$

which is a differential equation that can be solved with the aid of Mathematica or another computer algebra system. The result is

$$u(s) = \sqrt{\frac{C \pm \sqrt{(-4 + C^2)} \cdot \sin(2(s + k))}{2}} \quad (13)$$

where $C, k$ are real constants and $C \geq 2$. The corresponding torsion is:

$$\tau(s) = \frac{2}{C \pm \sqrt{(-4 + C^2)} \cdot \sin(2(s + k))}.$$ 

\[\Box\]

Integrating $\tau$ and $\kappa$ as above, using the Frenet-Serret equations, will yield the corresponding stationary curves, up to a choice of the initial Frenet-Serret frame and isometries of $\mathbb{R}^3$. 
3.3. $L^2(\mathbb{R})$ linear stability of helices. First, we derive the linearization of the evolution for the torsion around the stationary solutions corresponding to helices. The linearization of equation (4) at any stationary solution $\tau_0$ is obtained by letting $\tau(s, t) = \tau_0(s, t) + \epsilon w(s, t)$, substituting into equation (4), dividing by $\epsilon$, and then taking the limit as $\epsilon \to 0$. This is nothing more than the Gateaux derivative of our differential operator at $\tau_0$ in the direction of $w$. Alternatively, one may think of $w$ as the first-order approximation for solutions of (4) near the stationary solution. We can perform this operation when $\tau_0$ is given by an explicit formula, but for brevity’s sake, we only mention here the linearization around helices when $\tau_0$ is constant. A short calculation yields

**Proposition 3.4.** The linearization of the evolution equation (4) around the stationary solutions of constant torsion is

$$w_t + 2w_s + \frac{1}{2}w_{sss} = 0.$$  \hspace{1cm} (14)

In what follows, we show the $L^2(\mathbb{R})$ linear stability of the constant torsion stationary solution. First we recall the definition of linear stability:

**Definition 3.** A stationary solution $\phi$ of a nonlinear PDE is called $L^2(\mathbb{R})$ linearly stable when $v = 0$ is a stable solution of the corresponding linearized PDE with respect to the $L^2(\mathbb{R})$ norm and whenever $v_t = 0$ is in $L^2(\mathbb{R})$.

To work towards this, we need to use test functions from the Schwartz space $S(\mathbb{R})$, so we first recall that

$$S(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) \text{ s.t. } \sup_{x \in \mathbb{R}} (1 + |x|)^N |\partial^\alpha f(x)| < \infty \text{ for all } N, \alpha \}.$$

Intuitively, functions in $S(\mathbb{R})$ are smooth and rapidly decreasing. For more on distributions and the Schwartz space, review [4]. The rest of this section is devoted to proving:

**Theorem 3.5.** Helices correspond to $L^2(\mathbb{R})$ linearly stable stationary solutions of

$$\tau_t = D_s (\tau^{-1/2} - \tau^{3/2} + D_s^2 (\tau^{-1/2})).$$

**Proof.** Helices correspond to the constant torsion stationary solutions, so we analyze the linearized PDE in (14):

$$w_t + 2w_s + \frac{1}{2}w_{sss} = 0.$$

Henceforth, $w_0(s)$ denotes the initial data and $w(s, t)$ denotes the respective solution to the above, linear PDE. Thus, to prove the theorem, it suffices to show: for every $\epsilon$, there exists a $\delta$ such that if $w_0 \in L^2(\mathbb{R})$ and $\|w_0\|_2 < \delta$, then $\|w(t)\|_2 < \epsilon$ for all $t \geq 0$.

We follow the standard process of finding weak solutions to linear PDEs via the Fourier transform. Moreover, we know by the Plancherel theorem that we can extend the Fourier transform by density and continuity from $S(\mathbb{R})$ to an isomorphism on $L^2(\mathbb{R})$ with the same properties. Hence, it suffices to prove the desired stability result for initial data in $S(\mathbb{R})$. 


Let $F_t(\xi) = e^{4i\pi^3 \xi^3 t} - 4i\pi \xi t$. We notice that since $F_t$ is a bounded continuous function for all $t \geq 0$, it can be considered as a tempered distribution (or a member of $S'(\mathbb{R})$, the continuous linear functionals on $S(\mathbb{R})$), so its inverse Fourier transform makes sense.

Indeed, we can let $B_t(s) = F^{-1}(F_t(\xi)) \in S'(\mathbb{R})$ and, again, we denote $w_0 \in S(\mathbb{R})$ to be our initial data. Let

$$w(t) = B_t * w_0$$

so that $w(t)$ is a $C^\infty$ function with at most polynomial growth for all of its derivatives (see [4]). Moreover, $w(t)$ satisfies equation (14) in the distributional sense, as can be checked by taking the Fourier transform. Lastly, since the Fourier transform is a unitary isomorphism, it follows that

$$\lim_{t \to 0} w(t) = w_0$$

in the distribution topology of $S'(\mathbb{R})$. Hence, $w(t)$ is the weak solution to equation (14) with initial data $w_0 \in S(\mathbb{R})$. In our final step, we use the Plancherel theorem and the fact that $\mathcal{F}(B_t) = F_t$ is a continuous function with $\|F_t\|_\infty = 1$ for all $t \geq 0$ to get:

$$\|w(t)\|_2 = \|B_t \ast w_0\|_2 = \|F_t \cdot \mathcal{F}(w_0)\|_2 \leq \|F_t\|_\infty \|\mathcal{F}(w_0)\|_2 = \|\mathcal{F}(w_0)\|_2 = \|w_0\|_2.$$ 

From the inequality above, the desired $L^2(\mathbb{R})$ stability for initial data in $S(\mathbb{R})$ follows forthrightly. \hfill \Box

3.4. Numerical rendering of a stationary curve. We provide here the figures obtained from a numerical integration of the Frenet-Serret equations on Mathematica for the following choice of torsion:

$$\tau_1(s) = \frac{2}{3 + \sqrt{5} \sin(2s)}.$$
Figure 2. This is the projection of the curve corresponding to $\tau_1$ into the $xy$ plane. The projections into the other planes look very similar.

Figure 3. This is a top-down view of the same curve, now exhibiting an almost trefoil shape.

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