QUADRATIC KILLING STRUCTURE JACOBI OPERATOR FOR REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract. In this paper, we introduce a notion of quadratic Killing structure Jacobi operator (simply, Killing structure Jacobi operator) and its geometric meaning for real hypersurfaces in the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. In addition, we give a classification theorem for Hopf real hypersurfaces with quadratic Killing structure Jacobi operator in complex two-plane Grassmannians.

1. INTRODUCTION

In the class of complex Grassmannians of rank 2, usually we can give the example of a Hermitian symmetric space $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/(U_2U_m)$, which is said to be complex two-plane Grassmannians of compact type. It is viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure $J$ and the quaternionic Kähler structure $J = \text{span}\{J_1, J_2, J_3\}$ (see [7], [22], and [23]).

Recall that a non-zero vector field $X$ of Hermitian symmetric spaces $(\bar{M}, g)$ of rank 2 is called singular if it is tangent to more than one maximal flat in $\bar{M}$. In particular, there are exactly two types of singular tangent vectors $X$ of $G_2(\mathbb{C}^{m+2})$ which are characterized by the geometric properties $JX \in JX$ and $JX \perp JX$ (see [1] and [2]).

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ + \sum_{\nu=1}^{3} \left\{ g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z \right\}$$

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the 3-dimensional distribution (see [1], [2], and [4]).

We have the following two natural geometric conditions: the 1-dimensional distribution $C^\perp = \text{span}\{\xi\}$ and the 3-dimensional distribution $Q^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator $A$ of $M$. Here the almost contact structure vector field $\xi$ defined by $\xi = -JN$ is said to be a Reeb vector field, where $N$ denotes a local unit normal vector field of $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. The almost contact $3$-structure vector fields $\xi_1, \xi_2, \xi_3$ spanning the 3-dimensional distribution $Q^\perp$ of $M$ in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where $J_\nu$ denotes a canonical local basis of the quaternionic Kähler structure $\mathcal{J}$, such that $T_pM = Q \oplus Q^\perp = C \oplus C^\perp$, $p \in M$. By using these invariant conditions for two kinds of distributions $C^\perp$ and $Q^\perp$ in $T_pG_2(\mathbb{C}^{m+2})$, Berndt and Suh gave the complete classification of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ as follows:

**Theorem A** ([2]). Let $M$ be a connected real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $C^\perp$ and $Q^\perp$ are invariant under the shape operator $A$ of $M$ if and only if

1. $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$,

2. $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

On the other hand, the Reeb vector field $\xi$ is said to be Hopf if it is invariant under the shape operator $A$, that is, $A\xi \in C^\perp$. The 1-dimensional foliation of $M$ by the integral curves of the Reeb vector field $\xi$ is said to be a Hopf foliation of $M$. We say that $M$ is a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the almost contact metric structure $(\phi, \xi, \eta, g)$ and the formula $\nabla_X \xi = \phi AX$ for any $X \in TM$, it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf, where the structure tensor $\phi$ and the contact 1-form $\eta$ are defined by

$$JX = \phi X + \eta(X) N, \quad \text{and} \quad \eta(X) = g(X, \xi)$$

respectively. In addition, when the distribution $Q^\perp$ of $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, we say that $M$ is a $Q^\perp$-invariant hypersurface.

Moreover, we say that the Reeb flow of $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is isometric, when the Reeb vector field $\xi$ of $M$ is Killing. It implies that the metric tensor $g$ of $M$ is invariant under the Reeb flow of $\xi$, that is, $\mathcal{L}_\xi g = 0$ where $\mathcal{L}_\xi$ is the Lie derivative along the direction of $\xi$. Related to this notion,
for complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ Berndt and Suh gave a remarkable characterization for real hypersurface of type $(T_A)$ mentioned in Theorem A (see [3]).

Indeed, the notion of isometric Reeb flow is regarded as a typical example of Killing vector fields which are classical objects of differential geometry. As mentioned above, Killing vector fields are defined by vanishing of the Lie derivative of metric tensor $g$ with respect to a vector $X$, that is, $\mathcal{L}_X g = 0$. On the other hand, applying such definition to certain tensors of spacetimes, we can be expressed geometrical symmetries of the spacetimes. In general, the spacetime symmetries are used in the study of exact solutions of Einstein’s field equations in general relativity. As an example of this study, in [26] the spacetime fulfilling Einstein’s field equations with vanishing of pseudo-quasi-conformal curvature tensor was considered and existence of Killing and conformal Killing vectors on such spacetime have been established.

Now let us consider a generalization of such a Killing vector field on $(\tilde{M}, g)$. A symmetric tensor field $\mathcal{K}$ of type $(0, 2)$ on $(\tilde{M}, g)$ is said to be Killing, if it satisfies

$$
(\nabla_X \mathcal{K})(Y, Z) + (\nabla_Y \mathcal{K})(Z, X) + (\nabla_Z \mathcal{K})(X, Y) = 0
$$

for any vector fields $X, Y, \text{ and } Z$. In general, a tensor field $\mathcal{T}$ of type $(0, k)$ on $(\tilde{M}, g)$ is called Killing tensor if the complete symmetrization of $\nabla \mathcal{T}$ vanishes. This is equivalent to $(\nabla_X \mathcal{T})(X, X, \cdots, X) = 0$. It follows again that for such a Killing tensor, the expression $\mathcal{T}(\dot{\gamma}, \dot{\gamma}, \cdots, \dot{\gamma})$ is constant along any geodesic $\gamma$ (see [19]). In particular, the existing literature on symmetric Killing tensors is huge, especially coming from theoretical physics (see [5] and [19]).

As examples of such a symmetric Killing tensor, real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with generalized Killing shape operator were considered by Lee and Suh (see [12]). Recently, in [25] Suh gave a classification for Hopf real hypersurfaces with generalized Killing Ricci tensor in $G_2(\mathbb{C}^{m+2})$. In the study of Generalized Robertson-Walker spacetimes (shortly, GRW spacetimes) in Lorentzian manifolds, the physical meaning of Killing tensor is used to be a perfect fluid spacetime for GRW (see [15], [18], and [20]).

Motivated by such a symmetric Killing tensor, we consider the structure Jacobi operator $R_\xi$, which is a symmetric tensor field of type $(1, 1)$ on $M$ in $G_2(\mathbb{C}^{m+2})$. It is said to be quadratic Killing if the structure Jacobi operator or Killing structure Jacobi operator (see [14], [16] and [20]), if the structure Jacobi operator $R_\xi$ satisfies

$$
g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X) + g((\nabla_Z R_\xi)X, Y) = 0
$$

for any $X, Y, \text{ and } Z \in T_p M, p \in M$. This cyclic parallelism of $R_\xi$ is equivalent to

$$
g((\nabla_X R_\xi)X, X) = 0
$$

for any $X \in T_p M, p \in M$, by virtue of the linearization. Moreover, we give the geometric meaning of quadratic Killing structure Jacobi operator as follows: Let $\gamma$ be any geodesic curve on $M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$ as the initial
conditions. Then the structure Jacobi curvature $R_{\xi}(\dot{\gamma}, \dot{\gamma}) := g(R_{\xi}(\dot{\gamma}), \dot{\gamma})$ is constant along the geodesic $\gamma$ of the vector field $X$. Here we denotes $R_{\xi}$ the structure Jacobi tensor of type $(0,2)$ defined by $R_{\xi}(X, Y) = g(R_{\xi}X, Y)$ for any $X$ and $Y \in T_pM$, $p \in M$, which is symmetric (see Lemma 2.8 in [19]).

From the assumption of the quadratic Killing structure Jacobi operator, first we assert that the unit normal vector field $N$ becomes singular as follows:

**Theorem 1.** Let $M$ be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for $m \geq 3$. If $M$ has a quadratic Killing structure Jacobi operator, then the normal vector field $N$ of $M$ is singular.

Next, by using Theorem 1 we give a classification of Hopf real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with quadratic Killing structure Jacobi operator as follows:

**Theorem 2.** Let $M$ be a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the structure Jacobi operator $R_{\xi}$ of $M$ is quadratic Killing if and only if $M$ is locally congruent to an open part of a tube of $r = \frac{\pi}{4\sqrt{2}}$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

### 2. Preliminaries

As mentioned in the introduction, the complete classifications of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying two invariant conditions for the distributions $C_\perp = \text{span}\{\xi\}$ and $Q_\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ was given in [2].

In fact, in [1] and [2] Berndt and Suh gave the characterizations of the singular unit normal vector $N$ of $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$: There are two types of singular normal vector, those $N$ for which $JN \perp JN$, and those for which $JN \in JN$. In other words, it means that $\xi \in Q$ or $\xi \in Q^\perp$ because $JN = -\xi$, $JN = \text{span}\{\xi_1, \xi_2, \xi_3\} = Q^\perp$, and $TM = Q \oplus Q^\perp$. The following proposition tell us that the normal vector field $N$ on the model spaces of $(\mathcal{T}_A)$ is singular of type of $JN \in JN$, that is, $\xi \in Q^\perp$.

**Proposition A** ([2]). Let $(\mathcal{T}_A)$ be the tube of radius $0 < r < \frac{\pi}{\sqrt{8}}$ around the totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Then the following statements hold:

(i) $(\mathcal{T}_A)$ is a Hopf hypersurface.

(ii) Every unit normal vector field $N$ of $(\mathcal{T}_A)$ is singular and of type $JN \in JN$.

(iii) The eigenvalues and their corresponding eigenspaces and multiplicities are given in Table 7.

(iv) The Reeb flow on $(\mathcal{T}_A)$ is isometric.

On the other hand, by using the notion of isometric Reeb flow, that is, the shape operator $A$ of a Hopf real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ commutes with structure
for any tangent vector fields $X$ and $Y$ on $M$ (see Proposition 4 in [3]). In fact, from (iv) in Proposition A, we see that the shape operator $A$ of $(T_A)$ satisfies $A\phi = \phi A$. Thus, the above equation (2.1) holds on $(T_A)$ and it can be rearranged as

\[(\nabla_X A)Y = -\eta(Y)\phi X + (X\alpha)\eta(Y)\xi + \alpha g(A\phi X, Y)\xi - g(A^2\phi X, Y)\xi \]

\[\quad - \sum_{i=1}^{3} \left\{ \eta_{\nu}(Y)\phi_{\nu}X + g(\phi_{\nu}\xi, Y)\phi_{\nu}Y + 2g(\phi_{\nu}\xi, X)\phi_{\nu}Y \right\} \]

\[\quad + g(\phi_{\nu}\xi, X)\eta_{\nu}(Y)\xi - \eta_{\nu}(\xi)g(\phi_{\nu}X, Y)\xi + g(\phi_{\nu}X, Y)\xi_{\nu} \quad (2.1)\]

\[- \eta(X)\eta_{\nu}(Y)\phi_{\nu}X + g(\phi_{\nu}\phi X, Y)\phi_{\nu}Y \]

for any tangent vector fields $X$ and $Y$ on $T(T_A) = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\mu$.

### 3. Fundamental equations of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

We use some references [9], [17], and [24] to recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let $M$ be a real hypersurface of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ in $G_2(\mathbb{C}^{m+2})$ and $S$ the shape operator of $M$ with respect to $N$, that is, $\nabla_X N = -SN$. The Kähler structure $J$ of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1$, $J_2$, $J_3$ be a canonical local basis of the quaternionic Kähler structure $J$. Then each $J_{\nu}$

| Type   | Eigenvalues       | Eigenspace | Multiplicity |
|--------|-------------------|------------|--------------|
| $(T_A)$ | $\alpha=\sqrt{8}\cot(\sqrt{8}r)$ | $T_\alpha=\mathbb{C}^{3}=\text{span}\{X\}=\text{span}\{\xi\}$ | 1             |
|        | $\beta=\sqrt{2}\cot(\sqrt{2}r)$ | $T_\beta=\mathbb{C}^{3}\oplus\mathbb{Q}=\text{span}\{\xi, \xi_3\}$ | 2             |
|        | $\lambda=-\sqrt{2}\tan(\sqrt{2}r)$ | $T_\lambda=E_{-1}={X\in \mathbb{Q} \mid \phi X=\phi_1 X}$ | $2m-2$        |
|        | $\mu=0$          | $T_\mu=E_{+1}={X\in \mathbb{Q} \mid \phi X=-\phi_1 X}$ | $2m-2$        |
induces an almost contact metric structure \( (\phi, \xi, \eta, g) \) on \( M \). Now let us put
\[
JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N
\]
for any tangent vector \( X \) of a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \), where \( N \) denotes a normal vector of \( M \) in \( G_2(\mathbb{C}^{m+2}) \). Then the following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:
\[
\begin{align*}
\phi_{\nu+1} \xi_\nu &= -\xi_{\nu+2}, & \phi_\nu \xi_{\nu+1} &= \xi_{\nu+2}, & \phi_\nu \xi_\nu &= \phi_\nu \xi, & \eta_\nu(\phi X) &= \eta(\phi_\nu X), \\
\phi_\nu \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X) \xi_\nu, & \phi_{\nu+1} \phi_\nu X &= -\phi_{\nu+2} X + \eta_\nu(X) \xi_{\nu+1},
\end{align*}
\]
where we have used that \( J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu \).

On the other hand, from the parallelism of \( J \) and \( J \), which are defined by
\[
\nabla_X J = 0 \quad \text{and} \quad \nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_\nu \quad (\nu \mod 3),
\]
together with Gauss and Weingarten formulas, it follows that
\[
\begin{align*}
(\nabla_X \phi) Y &= \eta(Y) AX - g(AX, Y) \xi, \quad \nabla_X \xi = \phi AX, & (3.3) \\
\nabla_X \xi_\nu &= q_{\nu+2}(X) \xi_{\nu+1} - q_{\nu+1}(X) \xi_\nu + \phi_\nu AX, & (3.4) \\
(\nabla_X \phi_\nu) Y &= -q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y \\
&\quad + \eta_\nu(Y) AX - g(AX, Y) \xi_\nu. & (3.5)
\end{align*}
\]
Combining these formulas, we find the following
\[
\begin{align*}
\nabla_X (\phi_\nu \xi) &= \nabla_X (\phi \xi_\nu) \\
&= (\nabla_X \phi) \xi_\nu + \phi (\nabla_X \xi_\nu) \\
&= q_{\nu+2}(X) \phi_{\nu+1} \xi - q_{\nu+1}(X) \phi_{\nu+2} \xi + \phi_\nu \phi AX - g(AX, \xi) \xi_\nu + \eta(\xi_\nu) AX.
\end{align*}
\]
Moreover, from \( JJ_\nu = J_\nu J, \nu = 1, 2, 3 \), it follows that
\[
\phi \phi_\nu X = \phi_\nu \phi X + \eta_\nu(X) \xi - \eta(X) \xi_\nu.
\]
Finally, using the explicit expression for the Riemannian curvature tensor \( \bar{R} \) of complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}) \) in the introduction, the Codazzi and Gauss equations of \( M \) in \( G_2(\mathbb{C}^{m+2}) \), are given respectively by
\[
\begin{align*}
(\nabla_X A) Y - (\nabla_Y A) X &= \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi \\
&\quad + \sum_{\nu=1}^{3} \{ \eta_\nu(X) \phi_\nu Y - \eta_\nu(Y) \phi_\nu X - 2g(\phi_\nu X, Y) \xi_\nu \} \\
&\quad + \sum_{\nu=1}^{3} \{ \eta_\nu(\phi X) \phi_\nu Y - \eta_\nu(\phi Y) \phi_\nu X \} \\
&\quad + \sum_{\nu=1}^{3} \{ \eta(X) \eta_\nu(\phi Y) - \eta(Y) \eta_\nu(\phi X) \} \xi_\nu.
\end{align*}
\]
and
\[ R(X,Y)Z - g(AY, Z)AX + g(AX, Z)AY = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \]
\[ + \sum_{\nu=1}^{3} \left\{ g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z \right\} \]
\[ + \sum_{\nu=1}^{3} \left\{ g(\phi_\nu \phi X, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y \right\} \]
\[ + \sum_{\nu=1}^{3} \left\{ \eta(X)\eta_\nu(Z)\phi_\nu \phi Y - \eta(Y)\eta_\nu (Z)\phi_\nu \phi X \right\} \]
\[ + \sum_{\nu=1}^{3} \left\{ \eta(Y)g(\phi_\nu \phi X, Z) - \eta(X)g(\phi_\nu \phi Y, Z) \right\} \xi_\nu \]

(3.9)

for any tangent vector fields \( X, Y \) and \( Z \) on \( M \).

On the other hand, we can derive some important facts from the geometric condition of \( M \) being Hopf, that is, \( A\xi = \alpha \xi \) where \( \alpha = g(A\xi, \xi) \). Among them, we introduce the following formulas which are induced from the Codazzi equation:

**Lemma A** ([3]). If \( M \) is a connected orientable Hopf real hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^m+2), m \geq 3 \), then

\[ \text{grad} \alpha = (\xi \alpha) \xi + 4 \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu \xi \]

(3.10)

and

\[ 2A\phi AX - \alpha A\phi X - \alpha \phi AX = 2\phi X + 2 \sum_{\nu=1}^{3} \left\{ \eta_\nu(X)\phi_\nu \xi - g(\phi_\nu \xi, X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X \right\} \]
\[ - 4 \sum_{\nu=1}^{3} \left\{ \eta(X)\eta_\nu(\xi)\phi_\nu \xi - \eta_\nu(\xi)g(\phi_\nu \xi, X)\xi \right\}; \]

(3.11)

for any tangent vector field \( X \) on \( M \) in \( G_2(\mathbb{C}^m+2) \).

4. **Proof of Theorem 1**

Let \( M \) be a Hopf real hypersurface with quadratic Killing structure Jacobi operator in complex two-plane Grassmannians \( G_2(\mathbb{C}^m+2), m \geq 3 \). From the notion of Killing structure Jacobi operator \( R_\xi \) the equivalent condition, so-called, cyclic parallel structure Jacobi operator \( R_\xi \) can be given by

\[ \Theta_{X,Y,Z} = g((\nabla_X R_\xi)Y, Z) \]
\[ = g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X) + g((\nabla_Z R_\xi)X, Y) = 0, \]
for any tangent vector fields $X$, $Y$, and $Z$ on $M$. The formula (1) is said to be quadratic Killing (or simply, Killing) structure Jacobi operator.

From (3.9) the structure Jacobi operator $R_\xi \in \text{End}(TM)$ is given as follows

$$R_\xi(Y) = R(Y, \xi)\xi$$

$$= Y - \eta(Y)\xi + \alpha AY - \alpha^2 \eta(Y)\xi$$

$$- \sum_{\nu=1}^3 \{ \eta_\nu(Y)\xi_\nu - \eta(Y)\eta_\nu(\xi)\xi_\nu - 3g(\phi_\nu \xi, Y)\phi_\nu \xi + \eta_\nu(\xi)\phi_\nu \phi Y \}$$

(4.1)

for any tangent vector field $Y \in TM$ (see [11] and [13]).

Taking the covariant derivative of (4.1) along the direction of $X$ implies

$$(\nabla_X R_\xi)Y = \nabla_X (R_\xi Y) - R_\xi(\nabla_X Y)$$

$$= -g(\phi A X, Y)\xi - \eta(Y)\phi A X$$

$$- \sum_{\nu=1}^3 \left[ g(\phi_\nu A X, Y)\xi_\nu + 2\eta(Y)g(\phi_\nu, AX)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX \right.$$  

$$+ 3g(\phi_\nu AX, \phi Y)\phi_\nu \xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu \xi$$

$$- 3g(\phi_\nu, Y)\phi_\nu AX + 3\alpha \eta(X)g(\phi_\nu \xi, Y)\xi_\nu$$

$$- 4\eta_\nu(\xi)g(\phi_\nu \xi, Y)AX - 4\eta_\nu(\xi)g(AX, Y)\phi_\nu \xi$$

$$- 2g(\phi_\nu, AX)\phi_\nu \phi Y \right]$$

$$+ g((\nabla X A)\xi, Y)AX + \alpha \eta((\nabla X A)\xi, Y)$$

$$- \alpha \eta(AX, \phi AZ)\xi - \alpha \eta(Y)(\nabla X A)\xi - \alpha \eta(Y)AX$$

(4.2)

for any tangent vector fields $X$ and $Y$ on $M$ (see [11]). From this and using symmetric property of the structure Jacobi operator $R_\xi$ in $G_2(\mathbb{R}^{m+2})$, the quadratic Killing structure Jacobi operator [1] can be rearranged as follows:

$$0 = g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X) + g((\nabla_Z R_\xi)X, Y)$$

$$= g((\nabla_X R_\xi)Y, Z) + g((\nabla_Y R_\xi)Z, X)$$

$$- g(\phi AZ, X)\eta(Y) - \eta(Y)g(\phi AZ, Y) + g((\nabla_Z A)\xi, Y)AX$$

$$+ \alpha g((\nabla Z A)\xi, Y) - \alpha \eta((\nabla Z A)\xi, Y)$$

$$- \alpha g(AX, \phi AZ)\eta(Y) - \alpha \eta(X)g((\nabla Z A)\xi, Y) - \alpha \eta(Y)g(AX, AZ)$$

$$- \alpha \eta(AX, \phi AZ)\xi - \alpha \eta(Y)g(AX, AZ)$$

$$- \sum_{\nu=1}^3 \left[ g(\phi_\nu A Z, X)\eta_\nu(Y) + 2\eta(X)g(\phi_\nu, AZ)\eta_\nu(Y) \right.$$

$$+ \eta_\nu(X)g(\phi_\nu AZ, Y) + 3g(\phi_\nu AZ, \phi X)g(\phi_\nu \xi, Y)$$

$$+ 3\eta(X)\eta_\nu(AZ)g(\phi_\nu \xi, Y) - 3g(\phi_\nu \xi, X)g(\phi_\nu AZ, Y)$$

(4.3)
\[\begin{align*}
&+ 3\alpha \eta(Z) g(\phi_\nu\xi, X) \eta_\nu(Y) - 4\eta_\nu(\xi) g(\phi_\nu\xi, X) g(AZ, Y) \\
&- 4\eta_\nu(\xi) g(AZ, X) g(\phi_\nu\xi, Y) - 2g(\phi_\nu AZ) g(\phi_\nu X, Y) \\
&= g((\nabla_X R_\xi) Y, Z) + g((\nabla_Y R_\xi) X, Z) \\
&+ g(A\phi X, Z) \eta(Y) + \eta(X) g(A\phi Y, Z) + (\xi \alpha) g(A\phi AX, Y) \eta(Z) \\
&- \alpha(\xi \alpha) \eta(X) \eta(Y) \eta(Z) + \alpha^2 \eta(Y) g(A\phi X, Z) - \alpha \eta(Y) g(A\phi AX, Z) \\
&+ \alpha \eta(Y) g(A\phi AX, Z) + \alpha \eta(X) g(A\phi AY, Z) \\
&- \alpha(\xi \alpha) \eta(X) \eta(Y) \eta(Z) + \alpha^2 \eta(X) g(A\phi Y, Z) - \alpha \eta(X) g(A\phi AY, Z) \\
&+ \alpha g((\nabla_X A) Y, Z) + \alpha g(\phi X, Y) \eta(Z) + \alpha \eta(X) g(\phi Y, Z) + 2\alpha \eta(Y) g(\phi X, Z) \\
&+ \sum_{\nu=1}^{3} \left[ \eta_\nu(Y) g(A\phi_\nu X, Z) - 2\eta(X) \eta_\nu(Y) g(A\phi_\nu X, Z) + \eta_\nu(X) g(A\phi_\nu Y, Z) \\
&+ 3g(\phi_\nu \xi, Y) g(A\phi_\nu \phi X, Z) - 3\alpha \eta(X) g(\phi_\nu \xi, Y) g(A_\nu, Z) \\
&+ 3g(\phi_\nu \xi, X) g(A\phi_\nu \phi Y, Z) - 3\alpha g(\phi_\nu \xi, X) \eta_\nu(Y) \eta(Z) \\
&+ 4\eta_\nu(\xi) g(\phi_\nu \xi, X) g(AY, Z) + 4\eta_\nu(\xi) g(\phi_\nu \xi, Y) g(AX, Z) \\
&+ 2g(\phi_\nu \phi Y, X) g(A\phi_\nu \phi Y, Z) + 4g(AX, Y) \eta_\nu(Y) g(\phi_\nu \xi, Z) \\
&- 4\alpha \eta(X) \eta(Y) \eta_\nu(\xi) g(\phi_\nu \xi, Z) - 4\alpha \eta(X) \eta(Y) \eta_\nu(\xi) g(\phi_\nu \xi, Z) \right] \\
&+ \alpha \sum_{\nu=1}^{3} \left[ g(\phi_\nu X, Y) \eta_\nu(Y) + \eta_\nu(X) g(\phi_\nu Y, Z) + 2\eta_\nu(Y) g(\phi_\nu X, Z) \\
&- g(\phi_\nu \phi X, Y) g(\phi_\nu \xi, Z) + g(\phi_\nu \xi, X) g(\phi_\nu Y, Z) \\
&+ \eta_\nu(\phi X) \eta_\nu(Y) \eta(Z) + \eta(X) \eta_\nu(Y) g(\phi_\nu \xi, Z) \right],
\end{align*}\]

where we have used
\[\begin{align*}
g((\nabla_Z A) \xi, X) &= (Z \alpha) \eta(X) - \alpha g(A\phi X, Z) + g(A\phi AX, Z) \\
&= (\xi \alpha) \eta(Z) \eta(Y) + 4 \sum_{\nu=1}^{3} \eta_\nu(\xi) g(\phi_\nu \xi, Z) \eta(X) \\
&- \alpha g(A\phi X, Z) + g(A\phi AX, Z),
\end{align*}\]

and
\[\begin{align*}
g((\nabla_Z A) X, Y) &= g((\nabla_X A) Z, Y) + \eta(Z) g(\phi X, Y) - \eta(X) g(\phi Z, Y) - 2g(\phi Z, X) \eta(Y) \\
&+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(Z) g(\phi_\nu X, Y) - \eta_\nu(X) g(\phi_\nu Z, Y) - 2g(\phi_\nu Z, X) \eta_\nu(Y) \right\} \\
&+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi Z) g(\phi_\nu \phi X, Y) - \eta_\nu(\phi X) g(\phi_\nu \phi Z, Y) \right\}
\end{align*}\]
for any tangent vector fields $X$, $Y$, and $Z$ on $M$. Deleting $Z$ from (1.3) and using (4.2) gives

$$
- g(\phi AX, Y)\xi - \eta(Y)\phi AX - g(\phi AY, X)\xi - \eta(X)\phi AY + \eta(Y)A\phi X \\
+ \eta(X)A\phi Y + (\xi \alpha)g(AX, Y)\xi - 2\alpha(\xi \alpha)\eta(X)\eta(Y)\xi + \alpha^2\eta(Y)A\phi X \\
+ \alpha^2\eta(X)A\phi Y + \alpha(\nabla A)Y + \alpha g(\phi X, Y)\xi + \alpha \eta(X)\phi Y + 2\alpha \eta(Y)\phi X
$$

$$
- \sum_{\nu=1}^{3} \left[ g(\phi_{\nu} AX, Y)\xi_{\nu} + 2\eta(Y)g(\phi_{\nu} \xi, AX)\xi_{\nu} + \eta_{\nu}(Y)\phi_{\nu} AX \\
+ 3g(\phi_{\nu} AX, \phi Y)\phi_{\nu} \xi + 3\eta(\nu)\phi_{\nu}(AX)\phi_{\nu} \xi - 3g(\phi_{\nu} \xi, Y)\phi_{\nu} AX \\
+ 3\alpha \eta(X)g(\phi_{\nu} \xi, Y)\xi_{\nu} - 4\eta_{\nu}(\xi)g(\phi_{\nu} \xi, Y)AX \\
- 4\eta_{\nu}(\xi)g(AX, Y)\phi_{\nu} \xi - 2g(\phi_{\nu} \xi, AX)\phi_{\nu} Y \\
+ g(\phi_{\nu} AY, X)\xi_{\nu} + 2\eta(Y)g(\phi_{\nu} \xi, AY)\xi_{\nu} + \eta_{\nu}(X)\phi_{\nu} AY \\
+ 3g(\phi_{\nu} AY, \phi X)\phi_{\nu} \xi + 3\eta(\nu)\eta_{\nu}(AY)\phi_{\nu} \xi - 3g(\phi_{\nu} \xi, X)\phi_{\nu} AY \\
+ 3\alpha \eta(Y)g(\phi_{\nu} \xi, X)\xi_{\nu} - 4\eta_{\nu}(\xi)g(\phi_{\nu} \xi, X)AY \\
- 4\eta_{\nu}(\xi)g(AY, X)\phi_{\nu} \xi - 2g(\phi_{\nu} \xi, AY)\phi_{\nu} \phi X \right]
$$

$$
+ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(Y)A\phi_{\nu} X - 2\eta(X)\eta_{\nu}(Y)A\phi_{\nu} \xi + \eta_{\nu}(X)A\phi_{\nu} Y \right. \\
+ 3g(\phi_{\nu} \xi, Y)A\phi_{\nu} \phi X - 3\eta(X)g(\phi_{\nu} \xi, Y)A\xi_{\nu} \\
+ 3g(\phi_{\nu} \xi, X)A\phi_{\nu} Y - 3\alpha g(\phi_{\nu} \xi, X)\eta_{\nu}(Y)\xi \\
+ 4\eta_{\nu}(\xi)g(\phi_{\nu} \xi, X)AY + 4\eta_{\nu}(\xi)g(\phi_{\nu} \xi, Y)AX \\
+ 2g(\phi_{\nu} \phi X, Y)A\phi_{\nu} \xi + 4g(AX, Y)\eta_{\nu}(\xi)\phi_{\nu} \xi \\
- 4\alpha \eta(X)\eta(Y)\eta_{\nu}(\xi)\phi_{\nu} \xi - 4\alpha \eta(X)\eta(Y)\eta_{\nu}(\xi)\phi_{\nu} \xi \left. \right]
$$

$$
+ \alpha \sum_{\nu=1}^{3} \left[ g(\phi_{\nu} X, Y)\xi_{\nu} + \eta_{\nu}(X)\phi_{\nu} Y + 2\eta_{\nu}(Y)\phi_{\nu} X - g(\phi_{\nu} \phi X, Y)\phi_{\nu} \xi \\
+ g(\phi_{\nu} \xi, X)\phi_{\nu} Y + \eta_{\nu}(\phi \xi)\eta_{\nu}(Y)\xi + \eta(X)\eta_{\nu}(Y)\phi_{\nu} \xi \right. \\
+ g((\nabla A)Y)\xi_{\nu} + \eta_{\nu}(\phi \xi)\eta_{\nu}(Y)\xi + \eta(X)\eta_{\nu}(Y)\phi_{\nu} \xi \\
- \alpha \eta(Y)A\phi AX + g((\nabla A)X)\xi_{\nu} AX - \alpha g((\nabla A)X, Y)\xi_{\nu} \\
- \alpha g(AX, AY)\xi_{\nu} - \alpha \eta(X)A\phi AY + \alpha(\nabla A)Y - \alpha \eta(Y)(\nabla A)\xi_{\nu} \\
+ \alpha(\nabla A)X - \alpha \eta(X)(\nabla A)\xi_{\nu} = 0.
$$
On the other hand, by the Codazzi equation (3.8) and (3.10) in the latter part of (4.4), we obtain
\[
g((\nabla_X A)\xi, \xi)AY - \alpha g((\nabla_X A)Y, \xi)\xi - \alpha g(AY, \phi AX)\xi - \alpha \eta(Y)A\phi AX
\]
\[
+ g((\nabla_Y A)\xi, \xi)AX - \alpha g((\nabla_Y A)X, \xi)\xi - \alpha g(AX, \phi AY)\xi - \alpha \eta(X)A\phi AY
\]
\[
+ \alpha(\nabla_X A)Y + \alpha(\nabla_Y A)X - \alpha \eta(Y)(\nabla_X A)\xi - \alpha \eta(X)(\nabla_Y A)\xi
\]
\[
= (\xi \alpha)\eta(X)AY + 4 \sum_{\nu=1}^{3} \eta_\nu(\xi)g(\phi_\nu \xi, X)AY - \alpha g(A\phi AX, Y)\xi
\]
\[
- \alpha \eta(Y)A\phi AX - \alpha (\xi \alpha)\eta(X)\eta(Y)\xi - 4\alpha \sum_{\nu=1}^{3} \eta_\nu(\xi)g(\phi_\nu \xi, X)\eta(Y)\xi
\]
\[
- \alpha^2 g(\phi AX, Y)\xi + \alpha g(A\phi AX, Y)\xi + (\xi \alpha)\eta(Y)AX
\]
\[
+ 4 \sum_{\nu=1}^{3} \eta_\nu(\xi)g(\phi_\nu \xi, Y)AX + \alpha g(A\phi AX, Y)\xi
\]
\[
- \alpha \eta(X)A\phi AY - \alpha (\xi \alpha)\eta(X)\eta(Y)\xi
\]
\[
- 4\alpha \sum_{\nu=1}^{3} \eta_\nu(\xi)\eta(X)g(\phi_\nu \xi, Y)\xi - \alpha^2 g(\phi AY, X)\xi + \alpha g(A\phi AY, X)\xi
\]
\[
+ 2\alpha(\nabla_X A)Y + \alpha \eta(Y)\phi X - \alpha \eta(X)\phi Y - 2\alpha g(\phi Y, X)\xi
\]
\[
+ \alpha \sum_{\nu=1}^{3} \{ \eta_\nu(Y)\phi_\nu X - \eta_\nu(X)\phi_\nu Y - 2g(\phi_\nu Y, X)\xi_\nu \eta_\nu(\phi Y)\phi_\nu \phi X \}
\]
\[
+ \alpha \sum_{\nu=1}^{3} \{ - \eta_\nu(\phi X)\phi_\nu \phi Y + \eta(Y)\eta_\nu(\phi X)\xi_\nu - \eta(X)\eta_\nu(\phi Y)\xi_\nu \}
\]
\[
- \alpha \eta(Y)\{(\xi \alpha)\eta(X)\xi + 4 \sum_{\nu=1}^{3} \eta_\nu(\xi)g(\phi_\nu \xi, X)\xi + \alpha \phi AX - A\phi AX \}
\]
\[
- \alpha \eta(X)\{(\xi \alpha)\eta(Y)\xi + 4 \sum_{\nu=1}^{3} \eta_\nu(\xi)g(\phi_\nu \xi, Y)\xi + \alpha \phi AY - A\phi AY \}.
\]

From now on, we want to prove the normal vector field $N$ of a Hopf real hypersurface $M$ in $G_2(C^{m+2})$ is singular. Then by the meaning of singular mentioned in the introduction, we see that either $\xi \in Q$ or $\xi \in Q^\perp$ where $Q$ is the maximal quaternionic subbundle of $TM = Q \oplus Q^\perp$. In order to do this, we may put the Reeb vector field $\xi$ as follows:
\[
\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1
\]

for unit vector fields $X_0 \in Q$ and $\xi_1 \in Q^\perp$ with $\eta(X_0)\eta(\xi_1) \neq 0$. By using the notation (*) we obtain that the Reeb function $\alpha$ is constant along the direction of $\xi$ if and only if the distribution $Q$- or the $Q^\perp$-component of the structure vector field $\xi$ is invariant by the shape operator, that is $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha \xi_1$ (see
and [10]. From this fact, we obtain the following useful formulas for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

**Lemma 4.1.** Let $M$ be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the distribution $Q$ or $Q^\perp$ component of the structure vector field $\xi$ by the shape operator, then the following formulas hold:

(a) $A\phi X_0 = \mu \phi X_0$,
(b) $A\phi_1 = \mu \phi_1$, and
(c) $A\phi_1 X_0 = \mu \phi_1 X_0$

where the function $\mu$ is given by $\mu = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}$.

**Proof.** Putting $X = X_0$ in (3.11) and using $AX_0 = \alpha X_0$, it yields

$$\alpha A\phi X_0 = \alpha^2 \phi X_0 + 2\phi X_0 + 2\eta(\xi_1)\phi_1 X_0 - 4\eta(X_0)\eta(\xi_1)\phi_1 \xi,$$

where we have used $g(\phi_\nu \xi, X_0) = 0$ for $\nu = 1, 2, 3$ and $\eta_2(\xi) = \eta_3(\xi) = 0$.

On the other hand, by (4.7) we obtain

$$\phi_1 \xi = \eta(X_0)\phi_1 X_0 + \eta(\xi_1)\phi_1 \xi_1 = \eta(X_0)\phi_1 X_0.$$  

In addition, from (4.7) and $\phi_1 \xi = \phi_1 \xi$ we have

$$0 = \phi \xi = \eta(X_0)\phi X_0 + \eta(\xi_1)\phi_1 \xi_1 = \eta(X_0)\phi_1 X_0 + \eta(\xi_1)\eta(X_0)\phi_1 X_0,$$

which means

$$\phi X_0 = - \eta(\xi_1)\phi_1 X_0$$

because $\eta(X_0)\eta(\xi_1) \neq 0$. Substituting (4.7) and (4.8) to (4.6), we get

$$\alpha A\phi X_0 = \alpha^2 \phi X_0 + 4\eta^2(X_0)\phi X_0 = (\alpha^2 + 4\eta^2(X_0))\phi X_0.$$  

Since $M$ has non-vanishing geodesic Reeb flow, we see that the vector field $\phi X_0$ is principal with corresponding principal curvature $\mu = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}$.

Similarly, using (4.7) and (4.8), together with $\eta(X_0)\eta(\xi_1) \neq 0$, the formula (4.6) gives (b) and (c). □

When the Reeb function $\alpha$ is vanishing, Pérez and Suh gave the following

**Lemma B ([17]).** Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If $M$ has vanishing geodesic Reeb flow, then the unit normal vector field $N$ of $M$ is singular, that is, either $\xi \in Q$ or $\xi \in Q^\perp$.

**Remark 4.2.** By using the method in the proof of Lemma B, we can assert that if $M$ is a Hopf real hypersurface with constant Reeb curvature, then the unit normal vector field $N$ of $M$ is singular. In fact, since $M$ has constant Reeb function, (3.10) becomes

$$4 \sum_{\nu=1}^3 \eta_\nu(\xi)\phi_\nu \xi = 0$$
By using (4.6), this equation yields $\eta(\xi_1)\phi_1\xi = 0$. From our assumption of $\eta(X)\eta(\xi_1) \neq 0$ and (4.7), it leads to $\phi_1X_0 = 0$. Taking the inner product with $\phi_1X_0$, it implies

$$g(\phi_1X_0, \phi_1X_0) = -g(\phi_1^2X_0, X_0) = g(X_0, X_0) - (\eta_1(X_0))^2 = 1,$$

which gives us a contradiction.

By using Lemma B, in the latter part of this section we prove that the normal vector field $N$ of $M$ is singular, when a Hopf real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ has non-vanishing geodesic Reeb flow $\alpha = g(A\xi, \xi)$.

**Lemma 4.3.** Let $M$ be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the structure Jacobi operator $R_\xi$ of $M$ is quadratic Killing, then the unit normal vector field $N$ of $M$ is singular.

**Proof.** In [8], Lee and Loo show that if $M$ is Hopf, then the Reeb function $\alpha$ is constant along the direction of structure vector field $\xi$, that is, $\xi\alpha = 0$. Then we see that the distribution $Q$- and the $Q^\perp$-component of $\xi$ is invariant by the shape operator $A$, that is, $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha \xi_1$.

Bearing in mind of these facts, putting $X = X_0$ and $Y = \xi_1$ in (4.4) and using (4.5), we obtain

$$-\alpha\eta(X_0)\phi_1\xi + \mu\eta(\xi_1)\phi X_0 + \mu\eta(X_0)\phi_1\xi_1 + 3\alpha(\nabla X_0)\xi_1 + 2\alpha\eta(\xi_1)\phi X_0$$

$$- \alpha^2\eta(\xi_1)\phi X_0 + \mu\alpha^2\eta(\xi_1)\phi X_0 - \alpha^2\eta(X_0)\phi_1\xi_1 + \mu\alpha^2\eta(X_0)\phi_1\xi_1$$

$$+ \sum_{\nu=1}^{3} \left[ \alpha\eta(\xi_1)\phi_\nu X_0 - 3\alpha g(\phi_\nu X_0, \phi_\xi_1)\phi_\nu\xi - 2\alpha\eta(X_0)\eta_\nu(\xi_1)\phi_\nu\xi + \eta_\nu(\xi_1)A\phi_\nu X_0ight. - 2\eta(X_0)\eta_\nu(\xi_1)\phi_\nu\xi - 8\alpha\eta(X_0)\eta(\xi_1)\eta_\nu(\xi_1)\phi_\nu\xi + \alpha\eta_\nu(\xi_1)\phi_\nu X_0 = 0,$$

where we have used $g(\phi_1X_0, \phi_1X_0) = -g(\phi_1^2X_0, \xi_1) = 0$ and

$$g(\phi_\nu X_0, \xi_1) = g(\phi_\nu\xi, X_0) = g(\phi_\nu\xi, \xi_1) = g(\phi_\nu\phi_1 X_0, \xi_1) = 0$$

for all $\nu = 1, 2, 3$. Since $\eta_2(\xi) = \eta_3(\xi) = 0$, together with $g(\phi_1X_0, \phi_1X_0) = 1$, this equation can be rearranged as

$$-\alpha\eta(X_0)\phi_1\xi + \mu\eta(\xi_1)\phi X_0 + \mu\eta(X_0)\phi_1\xi_1 + 3\alpha(\nabla X_0)\xi_1$$

$$+ 2\alpha\eta(\xi_1)\phi X_0 - \alpha^2\eta(\xi_1)\phi X_0 + \mu\alpha^2\eta(\xi_1)\phi X_0$$

$$- \alpha^2\eta(X_0)\phi_1\xi_1 + \mu\alpha^2\eta(X_0)\phi_1\xi_1 + 2\alpha\phi_1 X_0 - 5\alpha\eta(X_0)\phi_1\xi_1$$

$$+ \mu\phi_1 X_0 - 2\mu\eta(X_0)\phi_1\xi - 8\alpha\eta(X_0)(\eta(\xi_1))^2\phi_1\xi = 0.\tag{4.9}$$

From (4.7) and (4.8), (4.9) becomes

$$\eta^2(X_0)\left\{-6\alpha - \mu - \alpha^3 + \mu\alpha^2 - 8\alpha\eta^2(\xi_1)\right\}\phi_1X_0$$

$$- \eta^2(\xi_1)\left\{\mu + 2\alpha - \alpha^3 + \mu\alpha^2\right\}\phi_1X_0$$

$$+ (2\alpha + \mu)\phi_1 X_0 + 3\alpha(\nabla X_0)\xi_1 = 0.\tag{4.10}$$
On the other hand, from (3.4) and (3.10), the assumption \( A\xi_1 = \alpha \xi_1 \) yields
\[
(\nabla_X A)\xi_1 = (X\alpha)\xi_1 + \alpha \nabla_X \xi_1 - A(\nabla_X \xi_1)
\]
\[
= (X\alpha)\xi_1 + \alpha \{ q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX \}
\]
\[
- q_3(X)A\xi_2 + q_2(X)A\xi_3 - A\phi_1 AX
\]
\[
= 4\eta(\xi_1)g(\phi\xi, X)\xi_1 + \alpha \{ q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX \}
\]
\[
- q_3(X)A\xi_2 + q_2(X)A\xi_3 - A\phi_1 AX
\]
for any tangent vector field \( X \) on \( M \). From this, taking the inner product with \( \phi_1 X_0 \) to (4.10) and (3.4), together with \( \alpha \mu = \alpha^2 + 4\eta^2(X_0) \), we get
\[
\eta^2(X_0) \bigg\{ -14\alpha - \mu + 12\alpha \eta^2(X_0) \bigg\} - \eta^2(\xi_1) \bigg\{ \mu + 2\alpha + 4\alpha \eta^2(X_0) \bigg\}
\]
\[
+ 2\alpha + \mu - 12\alpha \eta^2(X_0) = 0,
\]
where we have used \( g(\phi_1 X_0, \phi_1 X_0) = 1 \), \( \eta^2(X_0) + \eta^2(\xi_1) = 1 \), and
\[
g((\nabla_X A)\xi_1, \phi_1 X_0) = \alpha g(\phi_1 AX_0, \phi_1 X_0) - g(A\phi_1 AX_0, \phi_1 X_0)
\]
\[
= \alpha^2 - \alpha \mu = -4\eta^2(X_0).
\]
By using non-vanishing Reeb function \( \alpha \neq 0 \) and \( \alpha \mu = \alpha^2 + 4\eta^2(X_0) \), together with \( \eta^2(\xi_1) = 1 - \eta^2(X_0) \), (4.11) becomes
\[
\eta^2(X_0) \bigg\{ -15\alpha^2 - 4\eta^2(X_0) + 12\alpha^2 \eta^2(X_0) \bigg\}
\]
\[
- \eta^2(\xi_1) \bigg\{ 3\alpha^2 + 4\eta^2(X_0) + 4\alpha^2 \eta^2(X_0) \bigg\} + 3\alpha^2 + 4\eta^2(X_0) - 12\alpha^2 \eta^2(X_0)
\]
\[
= \eta^2(X_0) \bigg\{ -15\alpha^2 - 4\eta^2(X_0) + 12\alpha^2 \eta^2(X_0) \bigg\} - 4\alpha^2 \eta^2(X_0)
\]
\[
+ \eta^2(X_0) \bigg\{ 3\alpha^2 + 4\eta^2(X_0) + 4\alpha^2 \eta^2(X_0) \bigg\} - 12\alpha^2 \eta^2(X_0)
\]
\[
= \eta^2(X_0) \bigg\{ -12\alpha^2 + 16\alpha^2 \eta^2(X_0) \bigg\} - 16\alpha^2 \eta^2(X_0)
\]
\[
= \eta^2(X_0) \bigg\{ -28\alpha^2 + 16\alpha^2 \eta^2(X_0) \bigg\} = 0.
\]
By virtue of \( \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \) in (1) for \( \eta(X_0)\eta(\xi_1) \neq 0 \), and our assumption of non-vanishing geodesic Reeb flow, that is, \( \alpha \neq 0 \), (4.12) implies that \( \eta^2(X_0) = \frac{7}{4} \).
Since the structure vector field \( \xi \) is unit, we should have \( \eta^2(X_0) + \eta^2(\xi_1) = 1 \). From these facts, we obtain \( \eta^2(\xi_1) = -\frac{3}{4} \). It makes a contradiction. This means that either \( \xi = \eta(X_0)X_0 = \pm X_0 \in Q \) or \( \xi = \eta(\xi_1)\xi_1 = \pm \xi_1 \in Q^\bot \), which gives the unit normal tangent \( N \) is singular. □

Summing up Lemmas B and 4.3 we assert that our Theorem 1 in the introduction.

5. QUADRATIC KILLING STRUCTURE JACOBI OPERATOR FOR \( JN \in JN \)

Hereafter, let \( M \) be a Hopf real hypersurface with quadratic Killing structure Jacobi operator in complex two-plane Grassmannians \( G_2(C^{m+2}) \) for \( m \geq 3 \). Then by Theorem 1 our discussions can be divided into two cases according as the Reeb vector field \( \xi \in Q^\bot \) or \( \xi \in Q \).
In this section, we consider the case of $\xi \in Q^\bot$ (i.e. $JN \in JN$ where $N$ is a unit normal vector field on $M$ in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$). Since $Q^\bot$ is 3-dimensional distribution defined by $Q^\bot = \text{span}\{\xi_1, \xi_2, \xi_3\}$, we may put $\xi = \xi_1$. From this, we give an important lemma as follows.

**Lemma 5.1.** Let $M$ be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Let $J_1 \in J$ be the almost Hermitian structure such that $JN = J_1N$ (or $\xi = \xi_1$). Then we obtain

$$\phi AX = 2g(AX, \xi_3)\xi_2 - 2g(AX, \xi_2)\xi_3 + \phi_1 AX$$

for any tangent vector field $X$ on $M$.

**Proof.** Differentiating $\xi = \xi_1$ along any vector field $X \in TM$ and using (3.4), we obtain

$$\phi AX = \nabla_X \xi = \nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX. \quad (5.1)$$

Taking the inner product of (5.1) with $\xi_2$ and $\xi_3$, we obtain

$$g(\phi AX, \xi_2) = q_3(X) + g(\phi_1 A\xi, \xi_2)$$

and

$$g(\phi AX, \xi_3) = -q_2(X) + g(\phi_1 A\xi, \xi_3)$$

respectively. It follows that

$$q_3(X) = 2g(AX, \xi_3) \quad \text{and} \quad q_2(X) = 2g(AX, \xi_2).$$

From this, (5.1) becomes

$$\phi AX = 2g(AX, \xi_3)\xi_2 - 2g(AX, \xi_2)\xi_3 + \phi_1 AX \quad (5.2)$$

for any tangent vector field $X$ on $M$. Moreover, taking the symmetric part of (5.2) we obtain

$$A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X. \quad (5.3)$$

Then, by virtue of Lemma 5.1 we prove the following

**Lemma 5.2.** Let $M$ be a Hopf hypersurface with quadratic Killing structure Jacobi operator in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the Reeb vector field $\xi$ belongs to $Q^\bot$ (i.e. $\xi = \xi_1$), then the distribution $Q^\bot$ is invariant by the shape operator $A$ of $M$, that is, $g(AQ, Q^\bot) = 0$.

**Proof.** By (3.10) we obtain $X\alpha = (\xi\alpha)\eta(X)$ for any $X \in TM$, when the Reeb vector field $\xi$ belongs to the distribution $Q$. From this and taking the inner product of
with $\xi$, we have

$$- g(\phi AX, Y) + g(A\phi X, Y) + (\xi\alpha)g(AX, Y) - \alpha(\xi\alpha)\eta(X)\eta(Y) + 3\alpha^2g(\phi AX, Y)$$

$$- \alpha g(\phi AX, Y) + 3\alpha g(\phi X, Y) + \alpha^2g(\phi AX, Y) - \alpha^2g(\phi AX, Y)$$

$$+ \sum_{\nu=1}^{3} \left[ - \eta_{\nu}(\xi)g(\phi_{\nu} AX, Y) - g(\phi_{\nu} \xi, AX)\eta_{\nu}(Y) - 3g(AX, \xi_{\nu})g(\phi_{\nu} \xi, Y)$$

$$+ 4\alpha\eta_{\nu}(\xi)g(\phi_{\nu} \xi, Y) + \eta_{\nu}(\xi)g(A\phi_{\nu} X, Y) - \eta_{\nu}(\xi)g(\phi_{\nu} \xi, AY)$$

$$- 9\alpha g(\phi_{\nu} \xi, Y)\eta_{\nu}(Y) - 3\alpha\eta_{\nu}(X)g(\phi_{\nu} \xi, Y) + 3\alpha\eta_{\nu}(\xi)g(\phi_{\nu} X, Y) \right] = 0,$$

where we have used

$$g((\nabla X)Y, \xi) = g((\nabla X)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(AX, Y) = (\xi\alpha)\eta(X)\eta(Y),$$

$$g(\phi_{\nu} AX, \xi) = g(\phi_{\nu} AX, \xi) = g(\phi^2\xi_{\nu}, AX) = -g(\xi_{\nu}, AX) + \alpha g(\xi_{\nu}, \eta(X),$$

for any tangent vector fields $X$ and $Y$ on $M$.

On the other hand, from the assumption $\xi = \xi_1 \in Q^1$ we get $\phi_2 \xi = \phi_2 \xi_1 = -\xi_3$ and $\phi_3 \xi = \phi_3 \xi_1 = \xi_2$. By using these formulas into the preceding equation, we get

$$- g(\phi AX, Y) + g(A\phi X, Y) + (\xi\alpha)g(AX, Y) - \alpha(\xi\alpha)\eta(X)\eta(Y)$$

$$+ 2\alpha^2g(\phi AX, Y) - \alpha g(\phi AX, Y) + 3\alpha g(\phi X, Y) + \alpha^2g(\phi AX, Y)$$

$$- g(\phi_1 AX, Y) - 2\eta_3(AX)\eta_2(Y) + 2\eta_2(AX)\eta_3(Y)$$

$$+ g(A\phi_1 X, Y) - 2\eta_2(X)g(A\xi_3, Y) + 2\eta_3(X)g(A\xi_2, Y)$$

$$+ 6\alpha\eta_3(X)\eta_2(Y) - 6\alpha\eta_2(X)\eta_3(Y) + 3\alpha g(\phi_1 X, Y) = 0.$$}

Deleting $Y$ from (5.4), we get

$$- \phi AX + A\phi X + (\xi\alpha)AX - \alpha(\xi\alpha)\eta(X)\xi + 2\alpha^2\phi AX - \alpha A\phi AX + \alpha^2A\phi X$$

$$- \phi_1 AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2$$

$$+ 3\alpha\{2\eta_3(AX)\xi_2 - 2\eta_2(X)\xi_3 + \phi X + \phi_1 X \} = 0.$$}

for any tangent vector field $X$ on $M$.

On the other hand, when $\xi = \xi_1 \in Q$, (3.11) gives us

$$- \phi X + \phi_1 X - 2\eta_2(X)\xi_3 + 2\eta_3(AX)\xi_2 = A\phi AX - \frac{\alpha}{2} A\phi X - \frac{\alpha}{2} \phi AX$$

(5.6)

for any tangent vector field $X$ on $M$. Substituting (5.6) into (5.5), it follows that

$$- \phi AX + A\phi X + (\xi\alpha)AX - \alpha(\xi\alpha)\eta(X)\xi + 2\alpha^2\phi AX - \alpha A\phi AX + \alpha^2A\phi X$$

$$- \phi_1 AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2$$

$$+ 3\alpha\{A\phi AX - \frac{\alpha}{2} A\phi X - \frac{\alpha}{2} \phi AX \} = 0,$$

for any tangent vector field $X$ on $M$. Substituting (5.6) into (5.5), it follows that

$$- \phi AX + A\phi X + (\xi\alpha)AX - \alpha(\xi\alpha)\eta(X)\xi + 2\alpha^2\phi AX - \alpha A\phi AX + \alpha^2A\phi X$$

$$- \phi_1 AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2$$

$$+ 3\alpha\{A\phi AX - \frac{\alpha}{2} A\phi X - \frac{\alpha}{2} \phi AX \} = 0,$$
which implies
\begin{align*}
(-2 + 7\alpha^2)\phi AX + (2 - \alpha^2)A\phi X + 2(\xi\alpha)AX - 2\alpha(\xi\alpha)\eta(X)\xi \\
+ 4\alpha A\phi AX - 2\{\phi_1 AX + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3\} \\
+ 2\{A\phi_1 X - 2\eta_2(X)A\xi_3 + 2\eta_3(X)A\xi_2\} = 0
\end{align*}
(5.7)
for any \(X \in TM\). Bearing in mind of (5.2) and (5.3), the above equation reduces to
\begin{align*}
(-4 + 7\alpha^2)\phi AX + (4 - \alpha^2)A\phi X + 2(\xi\alpha)AX \\
- 2\alpha(\xi\alpha)\eta(X)\xi + 4\alpha A\phi AX = 0.
\end{align*}
(5.8)
From (5.2) and (5.3), we get
\begin{align*}
2\eta_3(AX)\xi_2 - 2\eta_2(AX) = \phi AX - \phi_1 AX 
\end{align*}
(5.9)
and
\begin{align*}
2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 = A\phi X - A\phi_1 X,
\end{align*}
(5.10)
respectively. Substituting (5.9) and (5.10) into (5.7), it becomes
\begin{align*}
(-2 + 7\alpha^2)\phi AX + (2 - \alpha^2)A\phi X + 2(\xi\alpha)AX - 2\alpha(\xi\alpha)\eta(X)\xi \\
- 2\{\phi_1 AX + \phi AX - \phi_1 AX\} + 2\{A\phi_1 X - A\phi X + A\phi_1 X\} = 0,
\end{align*}
which yields
\begin{align*}
(-4 + 7\alpha^2)\phi AX - \alpha^2 A\phi X + 2(\xi\alpha)AX - 2\alpha(\xi\alpha)\eta(X)\xi \\
+ 4\alpha A\phi AX + 4A\phi_1 X = 0.
\end{align*}
(5.11)
Subtracting (5.11) from (5.8), we have \(A\phi X = A\phi_1 X\), which means that \(\phi AX = \phi_1 AX\) for any tangent vector field \(X\) of \(M\). From this, (5.2) becomes
\begin{align*}
g(A\xi_2, X)\xi_2 - g(A\xi_2, X)\xi_3 = 0
\end{align*}
(5.12)
for any tangent vector field \(X\) of \(M\). Taking the inner product of (5.12) with \(\xi_2\) (resp. \(\xi_3\), we get the following for any tangent vector field \(X\) of \(M\)
\begin{align*}
g(A\xi_2, X) = g(AX, \xi_2) = 0 \quad \text{(resp. } g(A\xi_3, X) = g(AX, \xi_3) = 0), \quad (5.13)
\end{align*}
which means that \(g(AQ, Q^\perp) = 0\). It gives a complete proof of Lemma 5.2.

By Theorems A and Propositions A, Lemma 5.2 assure that if a Hopf real hypersurface satisfies all geometric conditions mentioned in Lemma 5.2, then \(M\) is locally congruent to an open part of the model spaces of type \((T_A)\).

From now on, we will check whether a real hypersurface of type \((T_A)\) satisfy our hypothesis given in Lemma 5.2. By Proposition A mentioned in section 2 we see that such real hypersurface is Hopf and its normal vector field satisfies \(JN \in JN\).

In the remained part of this section, we want to check if the structure Jacobi operator \(R_\xi\) for a model space of type \((T_A)\) satisfies the cyclic parallelism. In order to do this, we want to find some necessary and sufficient conditions for structure Jacobi operator \(R_\xi\) of a real hypersurface \((T_A)\) to be quadratic Killing according to each eigenspace including the vector \(Y\).
From such a viewpoint, first, we consider the following case.

**Case A.** $Y \in T_\lambda$

In other words, from (4.4) and (4.5), together with (2.2), the structure Jacobi operator $R_{\xi}$ of a real hypersurface of type $(T_A)$ satisfies the following for any tangent vector field $X \in T(T_A)$

$$
3\alpha(\lambda^2 - \alpha \lambda - 2)g(\phi Y, X)\xi - 2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)\xi_2 - 2(2\alpha - \beta - \lambda)\eta_2(X)\phi_2 Y = 0, \quad (5.14)
$$

where $T(T_A)$ denotes a tangent space of type $(T_A)$ and we have used $\phi_2 Y = \phi_2 \phi Y = -\phi_3 Y \in T_\mu$ and $\phi_3 Y = \phi_3 \phi Y = \phi_2 Y \in T_\mu$ for any $Y \in T_\lambda$.

From now on, we want to check a solution of the equation (5.14) to be satisfied for type $(T_A)$. In fact, the left side of (5.14) depend on the eigenspaces of $(T_A)$ is given as

$$
\text{Left Side of (5.14) = } \begin{cases} 
0 & \text{for } X \in T_\alpha \\
-2(2\alpha - \beta - \lambda)\phi_2 Y & \text{for } X = \xi_2 \in T_\beta \\
-2(2\alpha - \beta - \lambda)\phi_3 Y & \text{for } X = \xi_3 \in T_\beta \\
3\alpha(\lambda^2 - \alpha \lambda - 2)g(\phi Y, X)\xi & \text{for } X \in T_\lambda \\
-2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)(\xi_2 + \xi_3) & \text{for } X \in T_\mu,
\end{cases}
$$

for $Y \in T_\lambda$. By using $\alpha = 2\sqrt{2} \cot(2\sqrt{2}r) = \sqrt{2}(\cot(\sqrt{2}r) - \tan(\sqrt{2}r))$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ with $r \in (0, \frac{\pi}{2\sqrt{2}})$, we get $\lambda^2 - \alpha \lambda - 2 = 0$. From this, the previous formula follows

$$
\text{Left Side of (5.14) = } \begin{cases} 
0 & \text{for } X \in T_\alpha \\
-2(2\alpha - \beta - \lambda)\phi_2 Y & \text{for } X = \xi_2 \in T_\beta \\
-2(2\alpha - \beta - \lambda)\phi_3 Y & \text{for } X = \xi_3 \in T_\beta \\
0 & \text{for } X \in T_\lambda \\
-2(2\alpha - \beta - \lambda)g(\phi_2 Y, X)(\xi_2 + \xi_3) & \text{for } X \in T_\mu,
\end{cases} \quad (5.15)
$$

for $Y \in T_\lambda$.

Bearing in mind of Proposition A, if $r = \frac{\pi}{4\sqrt{2}}$, then $2\alpha - \beta - \lambda = 0$. Hence, when $Y \in T_\lambda$, the structure Jacobi operator $R_{\xi}$ is quadratic Killing if and only the radius $r$ of the tube $(T_A)$ is $\frac{\pi}{4\sqrt{2}}$. It means $\alpha = \mu = 0$, $\beta = \sqrt{2}$, and $\lambda = -\sqrt{2}$.

**Case B.** $Y \in T_\alpha \oplus T_\beta \oplus T_\mu$

Under these situations, we consider our problem for the other cases, that is, $Y \in T_\alpha \oplus T_\beta \oplus T_\mu$. Since $\alpha = 0$, the left side of (4.4) becomes

$$
\text{Left Side of (4.4) = } -g(\phi AX, Y)\xi - \eta(Y)\phi AX - g(\phi AY, X)\xi - \eta(X)\phi AY + \eta(Y)A\phi X + \eta(X)A\phi Y \quad (5.16)
$$
\[
- \sum_{\nu=1}^{3} \left[ g(\phi_\nu AX, Y) \xi_\nu + 2\eta(Y) g(\phi_\nu \xi, AX) \xi_\nu + \eta_\nu(Y) \phi_\nu AX + 3g(\phi_\nu AX, \phi Y) \phi_\nu \xi + 3\eta(Y) \eta_\nu(AX) \phi_\nu \xi - 3g(\phi_\nu, AX) \phi_\nu \phi Y + \eta_\nu(AX) \phi_\nu \phi Y \right.

\]

for any \( X \in T(T_A) \) and \( Y \in T_\alpha \oplus T_\beta \oplus T_\mu \).

**Subcase 1.** \( Y = \xi \in T_\alpha \)

Then, by using \( \alpha = 0 \) the left side of \( (5.16) \) becomes

\[
- \phi AX + A\phi X - \sum_{\nu=1}^{3} \left\{ g(A\phi_\nu \xi, X) \xi_\nu + \eta_\nu(\xi) \phi_\nu AX + 3g(A\xi_\nu, X) \phi_\nu \xi \right\}

\]

\[
+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(X) A\phi_\nu \xi - 3g(\phi_\nu, X) A\xi_\nu + \eta_\nu(\xi) A\phi_\nu X \right\}

\]

\[
= -\phi AX + A\phi X - \phi_1 AX + A\phi_1 X,
\]

where we have used \( \phi_2 \xi = -\xi_3 \), \( \phi_3 \xi = \xi_2 \), and \( \phi \phi_\nu \xi = \phi^2 \xi_\nu = -\xi_\nu + \eta(\xi_\nu) \xi \). According to the composition of the eigenspaces for \( (T_A) \), we see that each eigenspace \( T_\sigma \) of \( (T_A) \) is \( \phi \)- (or \( \phi_1 \) -) invariant, that is, \( \phi T_\sigma = \phi_1 T_\sigma = T_\sigma \). From this, \( (5.17) \) vanishes on all eigenspaces of \( (T_A) \). So, this means that the structure Jacobi operator \( R_\xi \) is quadratic Killing when \( Y \in T_\alpha \).

**Subcase 2.** \( Y \in T_\beta \)

Since \( T_\beta = \text{span}\{\xi_2, \xi_3\} \), we have the following two subcases.

- \( Y = \xi_2 \in T_\beta \)

  Using \( \alpha = 0 \) and \( (5.16) \) can be rearranged as

\[
6\beta_3 Q_3(X) \xi + \beta \eta(X) \xi_3 - \phi_2 AX + 3\phi_3 \phi AX + 2\beta \phi_3 \phi X + A\phi_2 X + 3A\phi_3 \phi X,
\]

for any eigenvector \( X \) on \( (T_A) \). It is well-known that for \( X \in T_\lambda \) (resp. \( X \in T_\mu \)), by the straightforward calculation with \( (3.2) \), we obtain

\[
\phi_2 \phi X \equiv_{X \in T_\lambda} \phi_2 \phi_1 X = -\phi_3 X \in T_\mu
\]

(resp. \( \phi_2 \phi X \equiv_{X \in T_\mu} -\phi_2 \phi_1 X = \phi_3 X \in T_\lambda \)),

\[
\phi_3 \phi X \equiv_{X \in T_\lambda} \phi_3 \phi_1 X = \phi_2 X \in T_\mu
\]

and

\[
\phi_3 \phi X \equiv_{X \in T_\mu} \phi_3 \phi_1 X = \phi_2 X \in T_\lambda.
\]
Subcase 3. Let $X = \xi_3 \in T_\beta$

Similarly, from (5.16) we obtain

\[ -6\beta\eta_2(X)\xi - \beta\eta_2(AX) - 3\phi_2AX \]
\[ -2\beta\phi_2X + A\phi_3X - 3A\phi_2\phi X, \]

for any eigenvector $X$ on $(\mathcal{T}_A)$. More specifically, according to each eigenspace $T_\alpha$, $T_\beta$, $T_\lambda$, and $T_\mu$, it follows

\[ \begin{cases} 
-\beta\xi_2 + A\phi_3 \xi = -\beta\xi_2 + A\xi_2 = 0 & \text{for } X \in T_\alpha \\
-6\beta \xi - \phi_2 A \xi_2 - 3\phi_2AX - 2\beta\phi_2\xi_2 = 0 & \text{for } X = \xi_2 \in T_\beta \\
-3\phi_2AX_3 - 2\beta\phi_3\xi_3 - 3A\phi_2\xi_3 = 0 & \text{for } X = \xi_3 \in T_\beta \\
-\lambda\phi_3X - 3\phi_2AX - 2\beta\phi_2X = 2(\lambda + \beta)\phi_3X - 0 & \text{for } X \in T_\lambda \\
-2\beta\phi_2X + \lambda\phi_3X - 3A\phi_3X = -2\beta(\lambda + \beta)\phi_3X = 0 & \text{for } X \in T_\mu 
\end{cases} \]  

where we have used $\phi_2\phi_2\xi_2 = -\phi_2\xi_2 = -\xi$, $\phi_2\phi_2\xi_2 = \phi_2\xi_2 = 0$, $\beta = \sqrt{2}$ and $\lambda = -\sqrt{2}$.

Subcase 3. $Y \in T_\mu$

Since $Y \in T_\mu$, we see that $\mu = 0$ and $\phi X = \phi_1X \in T_\mu$. From these properties, (5.16) becomes

\[ -\sum_{\nu=1}^{3} \left\{ g(\phi_\nu AX, Y)\xi_\nu + 3g(\phi_\nu AX, Y)\phi_\nu \xi - 2g(\phi_\nu AX, Y)\phi_\nu Y \right\} \]
\[ -\eta_2(X)A\phi_\nu Y - 3g(\phi_\nu X, A\phi_\nu Y - 2g(\phi_\nu Y, X)A\phi_\nu \xi \right\} \]
\[ = -2(\beta + \lambda)\{g(\phi_2X, Y)\xi_3 + g(\phi_3X, Y)\xi_2 + \eta_2(X)\phi_2Y + \eta_3(X)\phi_3Y \}, \]

where we have used $\phi_2\phi Y = \phi_3Y \in T_\lambda$ and $\phi_3\phi Y = -\phi_2Y \in T_\lambda$ for any eigenvector $X$ on $(\mathcal{T}_A)$ and $Y \in T_\mu$. Since $\beta = \sqrt{2}$ and $\lambda = -\sqrt{2}$, (5.20) is identically vanishing for any tangent vector field $X$ on $(\mathcal{T}_A)$.

Summing up these discussions, we assert that the structure Jacobi operator $R_\xi$ of a real hypersurface of type $T_\lambda$ is quadratic Killing if and only if the radius $r$ of the tube around of type $T_\lambda$ is $\frac{\pi}{4\sqrt{2}}$.

6. Quadratic Killing structure Jacobi operator for $JN \perp JN$

Let $M$ be a Hopf real hypersurface with quadratic Killing structure Jacobi operator $R_\xi$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Assume that the unit normal vector field $N$ of $M$ satisfies $JN \perp JN$ (i.e. $\xi \in Q$). Related to the Reeb vector field $\xi$ of $M$ in $G_2(\mathbb{C}^{m+2})$, Lee and Suh gave:
Theorem B \textbf{([9])}. Let $M$ be a connected orientable Hopf real hypersurface in complex two-plane Grassmannians of compact type $G_2(C^{m+2})$, $m \geq 3$. Then the Reeb vector $\xi$ belongs to the distribution $Q$ if and only if $M$ is locally congruent to an open part of $(T_B)$: a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(C^{m+2})$, where $m = 2n$.

By virtue of Theorem 1 and Theorem B, we assert that a Hopf real hypersurface $G$ in complex two-plane Grassmannians $G_2(C^{m+2})$, $m \geq 3$, satisfying the hypothesis in our Theorem 2 is locally congruent to an open part of the model space mentioned in Theorem B. Hereafter, conversely, let us check whether the structure Jacobi operator $R_{\xi}$ of the model space of type $(T_B)$ satisfies our assumption of Killing structure Jacobi operator.

In order to do this, we introduce a proposition given in \textbf{[21]} as follows:

\textbf{Proposition B.} Let $M$ be a connected real hypersurface in complex two-plane Grassmannians $G_2(C^{m+2})$. Suppose that $AQ \subset Q$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $Q$. Then the quaternionic dimension $m$ of $G_2(C^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures

$$
\alpha = -2\tan(2r), \beta = 2\cot(2r), \gamma = 0, \lambda = \cot(r), \mu = -\tan(r),
$$

with some $r \in (0, \frac{\pi}{4})$. The corresponding multiplicities are

$$
m(\alpha) = 1, \ m(\beta) = 3 = m(\gamma) \ m(\lambda) = 4n - 4 = m(\mu)
$$

and the corresponding eigenspaces are

$$
T_\alpha = \mathbb{R}\xi = C^\perp = \text{span}\{\xi\},
$$

$$
T_\beta = J\xi = \text{span}\{\xi_1, \xi_2, \xi_3\},
$$

$$
T_\gamma = J\xi = \text{span}\{\phi_1, \phi_2, \phi_3\},
$$

$$
T_\lambda, \ T_\mu,
$$

where

$$
T_\lambda \oplus T_\mu = TM \ominus (\mathbb{R}\xi \ominus J\xi), \ \ J T_\lambda = T_\lambda, \ \ J T_\mu = T_\mu, \ \ J T_\lambda = T_\mu.
$$

In order to check the converse part, we assume that the structure Jacobi operator $R_{\xi}$ of our model space of type $(T_B)$ satisfies the property of quadratic Killing. Accordingly, by $A\phi\xi = 0$ for $\nu = 1, 2, 3$, the property \textbf{[1]} can be rearranged as

$$
\begin{multline}
g(X, A\phi Y)\xi - \eta(Y)A\phi AX - g(X, A\phi Y)\xi + \eta(X)A\phi Y + \eta(Y)A\phi X + \eta(X)A\phi Y + \alpha^2 \eta(Y)A\phi X + \alpha^2 \eta(X)A\phi Y + 3\alpha(\nabla_X A)Y \\
+ \alpha g(\phi X, Y)\xi + 3\eta g(\phi X, A\phi Y)\xi - \alpha^2 g(\phi AY, X)\xi - 2\alpha g(\phi Y, X)\xi - \alpha^2 \eta(Y)A\phi AX - \alpha^2 \eta(X)A\phi Y \\
+ \sum_{\nu=1}^{3} \left[ - g(\phi_\nu AX, Y)\xi - \eta(Y)\phi_\nu AX - 3g(\phi_\nu AX, Y)\xi \\
- 3\eta(Y)\eta_\nu (AX)\phi_\nu \xi + 3g(\phi_\nu, Y)\phi_\nu AX \\
- 3\alpha \eta_\nu g(\phi_\nu, Y)\xi + 2g(\phi_\nu, AX)\phi_\nu Y \right]
\end{multline}
$$

(6.1)
- \( g(\phi_\nu AY, X)\xi_\nu - 2\eta(X)g(\phi_\nu \xi, AY)\xi_\nu - \eta_\nu(X)\phi_\nu AY \)
- \( 3g(\phi_\nu AY, \phi X)\phi_\nu \xi - 3\eta(X)\eta_\nu(AY)\phi_\nu \xi + 3g(\phi_\nu \xi, X)\phi_\nu AY \)
- \( 3\alpha\eta(Y)g(\phi_\nu \xi, X)\xi_\nu + 2g(\phi_\nu \xi, AY)\phi_\nu \phi_\nu X + \eta_\nu(Y)A\phi_\nu X \)
+ \( \eta_\nu(X)A\phi_\nu Y + 3g(\phi_\nu \xi, Y)A\phi_\nu \phi_\nu X - 3\eta(X)g(\phi_\nu \xi, Y)A\xi_\nu \)
+ \( 3g(\phi_\nu \xi, X)A\phi_\nu \phi_\nu Y - 3\alpha g(\phi_\nu \xi, X)\eta_\nu(Y)\xi_\nu + \alpha g(\phi_\nu, X, Y)\phi_\nu \xi \)
+ \( 2\alpha\eta_\nu(Y)\phi_\nu X + \alpha g(\phi_\nu, X, Y)\phi_\nu \xi + \alpha g(\phi_\nu, X)\phi_\nu Y + \alpha\eta_\nu(\phi X)\eta_\nu(Y)\phi_\nu X \)
+ \( \alpha\eta(X)\eta_\nu(Y)\phi_\nu \xi + \alpha\eta_\nu(Y)\phi_\nu X - 2\alpha g(\phi_\nu, Y, X)\xi_\nu + \alpha\eta_\nu(\phi Y)\phi_\nu \phi_\nu X \)
- \( \alpha\eta_\nu(\phi X)\phi_\nu \phi_\nu Y + \alpha(\eta(Y)\eta_\nu(\phi X)\xi_\nu - \alpha\eta(X)\eta_\nu(\phi Y)\xi_\nu) = 0 \)

for any tangent vector field \( X \) on type \( (T_B) \).

Bearing in mind of our assumption, the structure Jacobi operator \( R_\xi \) for the tube of type \( (T_B) \) is quadratic Killing, taking \( Y \in T_\alpha \) in (6.1) yields

\[
- \phi AX + A\phi X + \alpha^2 A\phi X + 2\alpha^2 \phi AX - 3\alpha A\phi AX + 3\alpha \phi X \\
- 3 \sum_{\nu=1}^{3} \left[ \beta \eta_\nu(X)\phi_\nu \xi + 3\alpha g(\phi_\nu \xi, X)\xi_\nu + \alpha\eta_\nu(X)\phi_\nu \xi + \beta g(\phi_\nu \xi, X)\xi_\nu \right] = 0, \tag{6.2}
\]

where we have used \( (\nabla_X A)\xi = \alpha \phi AX - A\phi AX \) and \( \phi \phi_\nu \xi = \phi^2 \xi_\nu = -\xi_\nu \). Furthermore, taking \( X = \xi_\mu \in T_\beta \) in (6.2) follows

\[
- \phi AX + A\phi X + \alpha^2 A\phi X + 2\alpha^2 \phi AX - 3\alpha A\phi AX + 3\alpha \phi X \\
- 3 \sum_{\nu=1}^{3} \left[ \beta \eta_\nu(\xi_\mu)\phi_\nu \xi + 3\alpha g(\phi_\nu \xi, \xi_\mu)\xi_\nu + \alpha\eta_\nu(\xi_\mu)\phi_\nu \xi + \beta g(\phi_\nu \xi, \xi_\mu)\eta_\nu \xi_\nu \right] \\
= -2\beta \phi_\mu \xi + 2\alpha^2 \beta \phi_\mu \xi + 3\alpha \phi_\mu \xi - 3\beta \phi_\mu \xi - 3\alpha \phi_\mu \xi \\
= -4\beta \phi_\mu \xi + 2\alpha^2 \beta \phi_\mu \xi = 2\beta(\alpha^2 - 2)\phi_\mu \xi = 0,
\]

which implies \( \beta(\alpha^2 - 2) = 0 \). Since \( \beta = 2\cot(2t) \) for \( r \in (0, \frac{\pi}{4}) \), we obtain \( \alpha^2 = 2 \).

On the other hand, taking \( X \in T_\lambda \) in (6.2), together with \( \phi T_\lambda = T_\mu \), provides

\[
- \lambda \phi X + \mu \phi X + \alpha^2 \mu \phi X + 2\alpha^2 \lambda \phi X - 3\alpha \lambda \mu \phi X + 3\alpha \phi X \\
= (-\lambda + \mu + 2\mu + 4\lambda + 3\alpha + 3\alpha)\phi X \\
= (\lambda + \mu + 2\alpha)\phi X = 3(\beta + 2\alpha)\phi X = 0,
\]

where we have used \( \alpha^2 = 2, \lambda \mu = \cot r \cdot (-\tan r) = -1 \), and \( \lambda + \mu = 2\cot(2t) = \beta \).

Applying a method to (6.2) that done above, the left side of (6.2) according to each eigenspace of type \( (T_B) \) is given as

\[
\text{Left Side of (6.2)} = \begin{cases} 
0 & \text{for } X \in T_\alpha \\
2\beta(\alpha^2 - 2)\phi_\mu \xi & \text{for } X = \xi_\mu \in T_\beta \\
-6(\beta + 2\alpha)\xi_\mu & \text{for } X = \phi_\mu \xi \in T_\gamma \\
3(\beta + 2\alpha)\phi X & \text{for } X \in T_\lambda \\
3(\beta + 2\alpha)\phi X & \text{for } X \in T_\mu.
\end{cases}
\]
Now, as the other case we consider the case $Y \in T_{\lambda}$. Then, by using $JT_{\lambda} = T_{\mu}$ and $JT_{\lambda} = T_{\lambda}$, equation (6.1) is rearranged as
\[ g(X, A\phi Y)\xi - g(X, A\phi Y)\xi - \eta(X)A\phi Y + \eta(X)A\phi Y + \alpha^2\eta(X)A\phi Y \\
+ 3\alpha(\nabla_X A)Y + \alpha g(\phi X, Y)\xi + \alpha^2 g(X, A\phi Y)\xi \\
- \alpha^2 g(\phi AY, X)\xi - 2\alpha g(\phi Y, X)\xi - \alpha^2 \eta(X)A\phi Y \\
+ \sum_{\nu=1}^{3} \left[ -g(\phi_{\nu}AX, Y)\xi_{\nu} - 3g(\phi_{\nu}AX, \phi Y)\phi_{\nu}\xi - g(\phi_{\nu}AX, X)\xi_{\nu} \\
- \eta_{\nu}(X)\phi_{\nu}AY - 3g(\phi_{\nu}AY, \phi X)\phi_{\nu}\xi + 3g(\phi_{\nu}\xi, X)\phi_{\nu}\phi AY \\
+ \eta_{\nu}(X)A\phi_{\nu}Y + \alpha g(\phi_{\nu}X, Y)\xi_{\nu} + 3g(\phi_{\nu}\xi, X)A\phi_{\nu}Y \\
- \alpha g(\phi_{\nu}\phi Y, X)\phi_{\nu}\xi + \alpha g(\phi_{\nu}\xi, X)\phi_{\nu}\phi Y \\
- 2\alpha g(\phi_{\nu}Y, X)\xi_{\nu} + \alpha g(\phi_{\nu}\xi, X)\phi_{\nu}\phi Y \right] \\
= (\mu - \lambda - 3\alpha + \alpha^2 \mu - \alpha^2 \lambda)g(X, \phi Y)\xi + (\lambda + \mu + \alpha^2 \mu - \alpha^2 \lambda)\eta(X)\phi Y \\
+ 3\alpha(\nabla_X A)Y + \sum_{\nu=1}^{3} \left[ -3\alpha g(\phi_{\nu}Y, X)\xi_{\nu} + (3\mu + 3\lambda - \alpha)g(\phi_{\nu}Y, X)\phi_{\nu}\xi \right] \\
+ \sum_{\nu=1}^{3} (3\lambda + 3\mu + 2\alpha)g(\phi_{\nu}\xi, X)\phi_{\nu}\phi Y = 0 \\
\]
for any tangent vector field $X$ on type $(T_B)$. Restricting $X \in T_{\alpha}$ in (6.3) provides
\[ (\lambda + \mu + \alpha^2 \mu - \alpha^2 \lambda)\phi Y + 3\alpha(\nabla_{\xi} A)Y = 0 \] (6.4)
for any $Y \in T_{\lambda}$. By the equation of Codazzi (3.8), we get
\[ (\nabla_{\xi} A)Y = (\nabla_{\xi} A)Y + \phi Y + \sum_{\nu=1}^{3} \left\{ -\eta_{\nu}(Y)\phi_{\nu}\xi - 3g(\phi_{\nu}\xi, Y)\xi_{\nu} \right\} \\
= A\phi AY - A\phi AY + \phi Y = (\alpha \lambda - \lambda \mu + 1)\phi Y \]
for any $Y \in T_{\lambda}$. From this, (6.4) becomes
\[ (\lambda + \mu + \alpha^2 \mu - \alpha^2 \lambda + 3\alpha^2 \lambda - 3\alpha \lambda \mu + 3\alpha)\phi Y = 0. \]
Since $\alpha^2 = 2$, $\beta + 2\alpha = 0$, $\lambda + \mu = \beta$ and $\lambda \mu = -1$, the previous equation gives
\[ \beta + 2\mu + 4\lambda + 6\alpha = -2(\beta - \mu - 2\lambda) = 0, \] (6.5)
which gives us a contradiction. In fact, by Proposition B we see that $\beta = 2 \cot(2r)$, $\lambda = \cot(r)$ and $\mu = -\tan(r)$ where $r \in (0, \frac{\pi}{4})$. From this, we get
\[ \beta - \mu - 2\lambda = -\frac{1}{\tan r}, \]
which means that the function $\beta - \mu - 2\lambda$ is non-vanishing for any $r \in (0, \frac{\pi}{4})$.

Summing up those documents in this section, we can assert that there does not exist a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$,
$m \geq 3$, with quadratic Killing structure Jacobi operator when the normal vector field of $M$ is of type $JN \perp JN$.

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