On the Isomorphic Description of Chiral Symmetry Breaking by Non-Unitary Lie Groups

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It is well-known that chiral symmetry breaking (χSB) in QCD with \( N_f = 2 \) light quark flavours can be described by orthogonal groups as \( O(4) \to O(3) \), due to local isomorphisms. Here we discuss the question how specific this property is. We consider generalised forms of χSB involving an arbitrary number of light flavours of continuum or lattice fermions, in various representations. We search systematically for isomorphic descriptions by non-unitary, compact Lie groups. It turns out that there are a few alternative options in terms of orthogonal groups, while we did not find any description entirely based on symplectic or exceptional Lie groups. If we adapt such an alternative as the symmetry breaking pattern for a generalised Higgs mechanism, we may consider a Higgs particle composed of bound fermions and trace back the mass generation to χSB. In fact, some of the patterns that we encounter appear in technicolour models. In particular if one observes a Higgs mechanism that can be expressed in terms of orthogonal groups, we specify in which cases it could also represent some kind of χSB of techniquarks.

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1 Chiral flavour symmetry breaking

We start by briefly reviewing the process of $\chi$SB in the physically relevant case of QCD with $N_f = 2$ quark flavours, which have masses far below the intrinsic scale $\Lambda_{\text{QCD}}$. In a low energy picture restricted to these two flavours, the QCD Lagrangian can be written as

$$
\mathcal{L} = i\bar{q}_L D q_L + i\bar{q}_R D q_R + m_u (\bar{u}_L u_R + \bar{u}_R u_L) + m_d (\bar{d}_L d_R + \bar{d}_R d_L) + \mathcal{L}_{\text{pure gauge}}.
$$

(1.1)

In this notation, $u$ and $d$ are spinor fields for the two quark flavours, which are decomposed into left- and right-handed components by the chiral projectors,

$$
\begin{align*}
  u_{L,R} &= \frac{1}{2} (\mathbb{1} \pm \gamma_5) u, & \bar{u}_{L,R} &= \frac{1}{2} \bar{u} (\mathbb{1} \mp \gamma_5) & \text{etc.}
\end{align*}
$$

(1.2)

We further used the short-hand notations $q_L = \left( \begin{array}{c} u_L \\ d_L \end{array} \right)$, $q_R = \left( \begin{array}{c} u_R \\ d_R \end{array} \right)$, $\bar{q}_L = (\bar{u}_L, \bar{d}_L)$, $\bar{q}_R = (\bar{u}_R, \bar{d}_R)$, and $D = \left( \begin{array}{cc} D & 0 \\ 0 & D \end{array} \right)$, where $D$ is the (massless) Dirac operator. In the chiral limit of vanishing quark masses, $m_u, m_d \to 0$, the left- and right-handed components decouple. Therefore $\bar{q}_L, q_L$ on one hand, and $\bar{q}_R, q_R$ on the other hand, can be rotated by arbitrary $U(2)$ transformations, keeping $\mathcal{L}$ invariant. Thus the Lagrangian has the global symmetry

$$
U(2)_L \otimes U(2)_R = SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A.
$$

(1.3)

On the right-hand-side of eq. (1.3) we split off the vectorial subgroup $U(1)_V$ of simultaneous (L and R) phase rotations, which is related to the conservation of the baryon number. The additionally separated subgroup $U(1)_A$ for opposite (L vs. R) phase rotations is the axial symmetry, which is explicitly broken in QCD through an anomaly.

We focus on the remaining chiral flavour symmetry, which takes for $N_f$ massless quark flavours the form $SU(N_f)_L \otimes SU(N_f)_R$. One generally assumes that QCD in the chiral limit (and infinite volume) would perform

1 This property represents an interesting hierarchy problem. The existence of this problem is sometimes denied, based on the argument that light fermions are protected from strong mass renormalisation by approximate chiral symmetry. In a non-perturbative framework, however, it is difficult to implement (approximate) chiral symmetry. This has been achieved in sophisticated ways [1, 2], but they do still not make light fermions appear natural. An attempt to arrange for this in a brane world model is discussed in Ref. [3].
spontaneous $\chi_{\text{SB}}$,

$$SU(N_f)_L \otimes SU(N_f)_R \rightarrow SU(N_f)_V ,$$

(1.4)

where the vectorial group $SU(N_f)_V$ corresponds again to simultaneous L and R transformations. According to the Vafa-Witten Theorem, this remaining flavour symmetry cannot break spontaneously $[4]$. Hence this process yields $N^2_f - 1$ Nambu-Goldstone bosons (NGBs).

In Nature the light quarks are not exactly massless; a small explicit symmetry breaking is superimposed, so that the NGBs turn into light quasi-NGBs. For $N_f = 2$ they are identified with the pion triplet $\pi^+, \pi^0, \pi^-$. If one further includes the (somewhat heavier) $s$-quark, the quasi-NGBs also embrace the kaons and the $\eta$-particle. Chiral perturbation theory $[5]$ deals with these quasi-NGBs as an effective approach to low energy QCD.

This formalism works best for $N_f = 2$. In this case, $\chi_{\text{SB}}$ can be described alternatively as

$$O(4) \rightarrow O(3)$$

(1.5)
due to the local isomorphisms $SU(2) \otimes SU(2) \sim O(4)$, and $SU(2) \sim O(3)$. Thus the orthogonal groups provide an equivalent effective picture of soft pion physics, which is often more convenient. Discussions of quasi-spontaneous symmetry breaking $O(N) \rightarrow O(N - 1)$, where a weak external “magnetic field” provides a small mass to the $N - 1$ quasi-NGBs, can be found for instance in Refs. $[6]$.

Also studies of generalised forms of $\chi_{\text{SB}}$ have a long history, for early versions see e.g. Refs. $[7]$. A later motivation — closer to our work — departs from the fact that quarks also interact weakly, hence $\chi_{\text{SB}}$ in QCD “breaks” the electroweak gauge symmetry and generates a small contribution to the $W^\pm$- and $Z^0$-mass, without involving the Higgs field.

The concept of technicolour models is to replace the Higgs sector completely by a mechanism of this kind at high energy: new fermions (techniquarks) are added to the Standard Model. They are confined by a gauge group beyond the Standard Model. At low energy they build condensates, which induce $m_W, m_Z$, while the hierarchy problem is controlled due to asymptotic freedom (for reviews, see Refs. $[9]$). In this approach the Higgs particle consists of tightly bound fermions, in some analogy to the Cooper

\footnote{We adapt here a wide-spread terminology, with inverted commas, however, because strictly speaking a gauge symmetry can never break, see e.g. Ref. $[8]$.}
pairs in superconductors, or even more in superfluids, since the broken symmetry is global.

The Higgs sector of the Standard Model (before gauging) follows the symmetry breaking pattern (1.5), so we have also there the choice between the use of special unitary or orthogonal groups. Hence it is indeed tempting to try to interpret the Higgs particle as an object composed of tightly bound fermions with \( N_f = 2 \). The question if this works out explicitly is debated in the literature, but it is not the concern of this work. Here we discuss the question if an analogous interpretation is still conceivable if one observes — up to moderate energy — some Higgs mechanism following a non-standard pattern, involving compact Lie groups different from the transition (1.5). Our consideration leads to a list which specifies in which cases such a pattern is isomorphic to any kind of \( \chi_{SB} \). If such an isomorphic \( \chi_{SB} \) process exists, the door is open for speculations that the Higgs particle is composed of bound fermions, which might be manifest at very high energy.

The group theoretical properties that we employ are certainly encoded in the comprehensive mathematical literature on Lie groups, see e.g. Refs. [10]. This note is physics-oriented and focuses on conceivable \( \chi_{SB} \) processes.

## 2 A very general perspective on chirality

We first adapt a very general perspective, where chirality just means a global symmetry in the form of two equal but independent groups \( G \), which breaks down to one such symmetry group. (We are not yet concerned with corresponding fermion representations.) Schematically we could write

\[
G_L \otimes G_R \rightarrow G_V ,
\]

but all we really use at this point is the property that the number of group generators — the order of the symmetry group — is divided by 2. In fact this allows for even more general options than scheme (2.1). We want to check if such a transition could be described by orthogonal groups, according to

\[
O(N) \rightarrow O(n) \quad (N > n) .
\]

\[^3\text{In particular the book by R. Gilmore is useful in the present context.}\]
The group orders imply the condition \( N(N - 1) = 2n(n - 1) \). With the ansatz \( k := N - n \) we obtain

\[
N = \frac{1}{2} \left[ 4k + 1 \pm \sqrt{8k^2 + 1} \right].
\]  

(2.3)

The argument of the square root must be an odd square number, which we write as \((2\ell + 1)^2\). This takes the condition to the form

\[
k^2 = \frac{1}{2} \ell(\ell + 1).
\]  

(2.4)

So we are looking for numbers, which are doubly figurative, namely the square triangular numbers \( F_i \). This is a classical problem in number theory [11]. There is an infinite string of (rapidly growing) solutions, which can be written iteratively as

\[
F_0 = 0, \quad F_1 = 1, \quad F_{i+2} = 34F_{i+1} - F_i + 2 \quad \text{(for } i \geq 0). \]

(2.5)

Inserting these numbers into eqs. (2.4) and (2.3) leads to

\[
\left( \begin{array}{c} N \\ n \end{array} \right) = \left( \begin{array}{c} 4 \\ 3 \end{array} \right), \quad \left( \begin{array}{c} 21 \\ 15 \end{array} \right), \quad \left( \begin{array}{c} 120 \\ 85 \end{array} \right) \ldots
\]

(2.6)

where the first solution is the physical one that we mentioned in Section 1.

\[\text{4}\]  

3 Chiral fermions in the complex representation

Let us now be more specific and consider the case of \( \chi_{\text{SB}} \) as it occurs in QCD. The quarks are in the complex, fundamental representation of the colour gauge group, and \( \chi_{\text{SB}} \) follows the pattern anticipated in eq. (1.4).\[\text{5}\]

It turns out that an isomorphic description in the form (2.2) has solely the well-known solution \( N_f = 2, N = 4, n = 3 \).

\[\text{[11]}\]

\[\text{4}\] The negative sign in eq. (2.3) never contributes any sensible solution, since it always corresponds to \( n \leq 0 \).

\[\text{5}\] Different patterns will be addressed in Sections 5 to 7.
To demonstrate this, it is sufficient to compare the order before $\chi_{SB}$,

$$2(N_f^2 - 1) = \frac{1}{2}N(N - 1) \quad \text{(with } N_f \geq 2),$$

which means

$$N = \frac{1}{2} \left[ 1 + \sqrt{(4N_f)^2 - 15} \right]. \quad (3.1)$$

It is easy to see that the square root is integer only for $N_f = 2$.

As an extension we also consider the (hypothetical) case where $\chi_{SB}$ involves the full unitary groups,

$$U(N_f)_L \otimes U(N_f)_R \to U(N_f)_V.$$

Now condition (3.1) is modified to

$$N = \frac{1}{2} \left[ 1 + \sqrt{1 + (4N_f)^2} \right]. \quad (3.2)$$

Here the square root is only integer for the physically pointless case $N_f = 0$.

So without splitting off the phase factors, we would not find any solution for a description in the form (2.2).

That pattern was originally considered in QCD. For $N_f = 2$ it would require a light meson quartet, where the $\eta$-particle is added to the pion triplet. For $N_f = 3$ one would have to add the $\eta'$-particle to extend the light meson octet to a nonet. However, in both cases the additional meson is too heavy to fit into the multiplet (this is a facet of the “$U(1)$ problem”). Therefore that pattern was dismissed in favour of the scheme sketched in Section 1.

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6 The minimal assumption to single out $N_f = 2$, $N = 4$, $n = 3$, is even more modest: it would have been sufficient to start from the ansatz in Section 2 and add the condition that the rank (cf. Table 1) is also divided by 2 under $\chi_{SB}$, as scheme (2.1) suggests. This leaves transition (1.5) as the only solution of the form (2.2).

7 Lattice simulations in the “quenched approximation” generate configurations based on the probability weight given by the Euclidean gauge action alone. Generally the contribution of degenerate fermion flavours to this weight is given by the fermion determinant to the power $N_f$, hence the quenched approximation corresponds technically to $N_f = 0$. It has been used extensively in lattice QCD because it speeds up the simulations drastically; in this sense, or in the limit $m \to \infty$ which renders the fermion determinant constant, $N_f = 0$ is not completely academic, though still not physical.
4 Involving a product of orthogonal groups

Of course we can ease the conditions for a description of $\chi_{SB}$ by orthogonal groups if we allow for the ansatz

$$O(n) \otimes O(n) \rightarrow O(n)$$ (4.1)

instead of scheme (2.2). Then the only condition is a local isomorphism

$$SU(N_f) \sim O(n).$$ (4.2)

Counting once more the generators leads to the Diophantine equation

$$N_f^2 - 1 = \frac{1}{2}n(n - 1),$$ (4.3)

which is not as simple as the cases that we encountered in Section 3. The general formula for inductive solutions leads to

$$\begin{pmatrix} N_f^{(i+1)} \\ n^{(i+1)} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} N_f^{(i)} \\ n^{(i)} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ (4.4)

It can be obtained conveniently from D. Alpern’s Online Calculator [12], and Refs. [11] review its number theoretical background. We arrive at solutions with $N_f > 1$ by starting from $\begin{pmatrix} N_f^{(0)} \\ n^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This yields two strings of solutions,

$$\begin{pmatrix} N_f \\ n \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 11 \\ 16 \end{pmatrix}, \begin{pmatrix} 64 \\ 91 \end{pmatrix} \ldots$$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 23 \\ 33 \end{pmatrix}, \begin{pmatrix} 134 \\ 190 \end{pmatrix} \ldots$$ (4.5)

However, so far we have only considered the necessary condition for the orders to match. Of course, an isomorphism requires more than that. Now that we have a set of solution candidates, we compare as a further criterion the rank, i.e. the number of simultaneously diagonalisable generators. It amounts to $N_f - 1$ for $SU(N_f)$, and for $O(n)$ it is $[n/2]$, which means $n/2$ if $n$ is even, and $(n - 1)/2$ if $n$ is odd, cf. Table [1].
We combine this condition with eq. (4.3), eliminate \( n \) and solve for \( N_f \). This yields two solutions with \( N_f > 1 \),

\[
N_f = 2 \ , \ n = 3 \quad \text{or} \quad N_f = 4 \ , \ n = 6 .
\] (4.6)

Indeed these are the two cases where the isomorphism (4.2) is known to work [10]. The former case is once more equivalent to the solution anticipated in Section 1, if we add \( O(3) \otimes O(3) \sim O(4) \). The second case,

\[
SU(4)_L \otimes SU(4)_R \rightarrow SU(4)_V \sim O(6)_L \otimes O(6)_R \rightarrow O(6)_V
\] (4.7)

can be viewed as the only alternative description of \( \chi_{SB} \) in the complex representation in terms of orthogonal groups. In QCD it would mean to include even the \( c \)-quark into the \( \chi_{SB} \) scheme. However, its mass of \( m_c \approx 1.3 \text{ GeV} \) is too heavy to be captured by chiral perturbation theory.

5 \( \chi_{SB} \) in the real or pseudo-real representation

The literature refers additionally to another two forms of \( \chi_{SB} \), which we have not covered yet. Studies of technicolour models pointed out that chiral fermions in four dimensions, interaction through a Yang-Mills gauge field, can perform exactly three types of spontaneous \( \chi_{SB} \), depending on the representation of the fermion field [13, 14]. In this work we also consider further variants, which may occur in explicit \( \chi_{SB} \) (through an asymmetric term in the Lagrangian, like an explicit fermion mass in a vector theory), or through an anomaly (as in the 1-flavour Schwinger model) or through the regularisation (as in the case of staggered lattice fermions, see Section 7).

The current section, however, does address the three \( \chi_{SB} \) patterns that Refs. [13, 14] referred to. To present them, we consider the Dirac matrices

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8Generally the chiral condensate \( \Sigma = -\lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\Psi} \Psi \rangle \) is the order parameter for spontaneous \( \chi_{SB} \), where \( \bar{\Psi}, \Psi \) incorporate all fermion components (\( m \) is the fermion mass and \( V \) the volume). In the chiral limit of the Schwinger model (2d QED) it is ill-defined in the quenched case \( (N_f = 0, \text{cf. footnote 7}) \) [15], but it takes a finite value for \( N_f = 1 \) [16] (here chiral symmetry simply means invariance under \( \psi \to \exp(i\alpha\gamma_5)\psi, \bar{\psi} \to \bar{\psi}\exp(i\alpha\gamma_5) \) \((\alpha \in \mathbb{R})\)). There the \( \chi_{SB} \) pattern agrees with QCD. For \( N_f > 1 \), \( \Sigma \) vanishes; at finite \( m \) it is an example for \( \chi_{SB} \) not matching any of the three patterns established in Refs. [13, 14]. This can be seen from the microscopic Dirac spectrum [17], since no Banks-Casher plateau [18] emerges (cf. subsequent remarks on Random Matrix Theory).
in the Weyl representation, where the chiral projectors of eq. \[1.2\] are diagonal. Then the Dirac operator (in a Yang-Mills gauge background) has a off-diagonal block structure, which takes in Euclidean space the form

\[ D = \gamma_\mu (i\partial_\mu + gA_\mu) = \begin{pmatrix} 0 & d \\ d^\dagger & 0 \end{pmatrix}, \quad \text{with} \quad A_\mu = \sum_{a=1}^{N_c} A^a_\mu T_a, \]  

(5.1)

where \(T_a\) are the generators of the gauge group, and \(g\) is the gauge coupling.

We have considered so far the case \(N_c \geq 3\) and \(A_\mu\) in the fundamental representation, with matrix elements \(d_{ij} \in \mathbb{C}\). In this case the irreducible Dirac fermion representation of the gauge group is complex (complex representation). One alternative is the case \(N_c = 2\), still in the fundamental representation, where an additional symmetry ensures \(d_{ij} \in \mathbb{R}\) (real representation). Finally, for \(N_c \geq 3\) but \(A_\mu\) in the adjoint representation, the matrix elements \(d_{ij}\) are real quaternionic (pseudo-real representation), as summarised in Refs. \[19\]. Ref. \[20\] discussed the Dirac spectra in these three classes. They match the spectra in three distinct types of Random Matrices, which had been identified by F.J. Dyson \[21\]. Numerical simulations with chiral lattice quarks \[2\] (without doubling) confirm that QCD obeys the predictions for the complex representation \[22\]; this also captures the distinction between the Random Matrix formulae for different topological sectors \[23\].

In the other two cases occurring in 4d Yang-Mills theory, quark and antiquark representations are equivalent, hence the (unbroken) chiral symmetry group is enlarged to \(SU(2N_f)\), and the \(\chi\mathrm{SB}\) patterns are

\[
\begin{align*}
\text{real} & : \quad SU(2N_f) \rightarrow SO(2N_f) \quad 2N_f^2 + N_f - 1 \, \text{NGBs} \\
\text{pseudo-real} & : \quad SU(2N_f) \rightarrow Sp(2N_f) \quad 2N_f^2 - N_f - 1 \, \text{NGBs}.
\end{align*}
\]

(The ansatz in Section 2 was a broad generalisation, but still restricted to an (extended) framework of the complex representation.)

These schemes can be illustrated \[14\] by writing the fermion fields as vectors consisting of \(2N_f\) (2-component) spinors, \(\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\), \(\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\). \(\Psi\) and \(\Phi\) are in the fundamental representation of \(SU(2N_f)\) (the anomalous phase being split off), and \(\psi_i\), \(\phi_i\) are composed of \(N_f\) spinors.

With the definition \(\psi^\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)\) (and \(\phi^\pm\) analogous) the scalar product can be written as \(\Psi^\dagger \chi = (\psi^+, \psi^-)^\dagger \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix}\), which shows its invari-
ance under the transformations
\[
\psi^+ \rightarrow U\psi^+ \quad U \in U(N_f) \quad \text{and} \quad \psi^- \rightarrow U^*\psi^- \quad V \in SU(N_f) \quad (5.2)
\]
(and the same for \(\phi^\pm\)). Together they build the \(SU(N_f) \otimes SU(N_f) \otimes U(1)\) subgroup of \(SU(2N_f)\), which is relevant for \(\chi_{\text{SB}}\) in the complex representation.

To capture the other two options, note that the following bilinear forms are invariant under further subgroups of \(SU(2N_f)\),
\[
\begin{align*}
s &= \Psi^T \Phi & \text{preserved under } O(2N_f) \\
a &= \Psi^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \Phi & \text{preserved under } Sp(2N_f) .
\end{align*}
\]
The properties \(2(\psi^+)^T \phi^- = s - ia\), \(2(\psi^-)^T \phi^+ = s + ia\), show that the transformation matrix \(U\) in (5.2) preserves \(s\) and \(a\), hence \(U(N_f)\) is a subgroup of both, \(O(2N_f)\) and \(Sp(2N_f)\) \([10]\). Now the generators for the various subgroups of \(SU(2N_f)\) can be extracted \([14]\), confirming the orders displayed in Table \([\dagger]\).

We recall that \(Sp(2N)\) is the group of symplectic transformations, which can be represented by real \(2N \times 2N\) matrices \([10]\). We have used above the property that \(Sp(2N)\) has order \(N(2N + 1)\), and its rank is \(N\), as indicated in Table \([\dagger]\). Counting the number of NGBs, we note that neither of these options — real or pseudo-real — could explain the pion triplet, or the light meson octet, which are observed in Nature. This confirms once more that only the complex representation is relevant for the strong interaction.

Let us nevertheless check the possibilities of a description by orthogonal groups also for these additional types of \(\chi_{\text{SB}}\):

- In the real representation the issue is only to find an isomorphism of \(SU(2N_f)\) to some orthogonal group \(O(N)\). We saw in Section 4 that this works out in two cases: \(N_f = 1, n = 3\), or \(N_f = 2, n = 6\).

- In the pseudo-real representation we have to match in addition \(Sp(2N_f)\) with some group \(O(n)\) (or \(SO(n)\)). If we set order and rank equal, we see that \(n\) must be odd, and we are left with two solutions: \(N_f = 1, n = 3\) and \(N_f = 2, n = 5\).
In the first case no $\chi$SB takes place, since $SU(2) \sim Sp(2) \sim O(3)$, in agreement with the vanishing number of NGBs being generated.

Regarding the second case, the isomorphism $Sp(4) \sim O(5)$ does in fact hold \cite{ref10}, so the pseudo-real $\chi$SB for $N_f = 2$ can be described as $O(6) \rightarrow O(5)$. This transition is also covered by Refs. \cite{ref6}.

6 Probing symplectic and exceptional Lie groups

Next we address the question if $\chi$SB could be described isomorphically in terms of symplectic or exceptional Lie groups, as an alternative to the orthogonal groups that we have considered so far. Note that these alternatives are compact as well. For the very general perspective of Section 2, the ansatz $Sp(2N) \rightarrow Sp(2n)$ leads again to a non-trivial Diophantine equation, $N(2N + 1) = 2n(2n + 1)$, with the recursive solution \cite{ref11, ref12}

\[
\begin{pmatrix} N^{(i+1)} \\ n^{(i+1)} \end{pmatrix} = \begin{pmatrix} 17 & 24 \\ 12 & 17 \end{pmatrix} \begin{pmatrix} N^{(i)} \\ n^{(i)} \end{pmatrix} + \begin{pmatrix} 10 \\ 7 \end{pmatrix}.
\]

We may start from the trivial solution $N = n = 0$, which yields the sequence

\[
\begin{pmatrix} N \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}, \begin{pmatrix} 348 \\ 246 \end{pmatrix}, \ldots
\]

Let us now include the exceptional Lie groups $G_2, F_4, E_6, E_7$ and $E_8$ into the consideration (see e.g. Ref. \cite{ref24}), and we also allow for transitions between different types of groups. As long as we solely require the order to be divided by 2, we find further options, such as $Sp(10) \rightarrow O(15), O(21) \rightarrow Sp(14)$ or $O(32) \rightarrow E_8$. Moreover, in the solution $O(8) \rightarrow G_2$ also the rank is divided by 2. However, it does still not match the pattern \cite{ref21}, and in none of these cases the unbroken symmetry can be identified with some kind of chiral symmetry in the usual sense.

For the options that occurred in Sections 3 to 5, such an identification with chiral symmetry holds. However, in all these cases we would need some

\footnote{If we require also the rank to be divided by 2, as in footnote \cite{ref8} no non-trivial solution persists.}
| Group | $SU(N)$ | $O(N)$ | $Sp(2N)$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-------|---------|--------|---------|------|------|------|------|------|
| order $\Omega$ | $N^2 - 1$ | $\frac{1}{2}N(N - 1)$ | $N(2N + 1)$ | 14 | 52 | 78 | 133 | 248 |
| rank $r$ | $N - 1$ | $[N/2]$ | $N$ | 2 | 4 | 6 | 7 | 8 |

Table 1: The order and rank of various Lie groups that we considered. ([N/2] means the integer among N/2 and (N − 1)/2.)

isomorphism of the group type that we focus on to $SU(N_f)$. For the latter the order $\Omega$ and the rank $r$ are related as

$$\Omega = r(r + 2).$$

(6.3)

For the symmetry groups under consideration here, we display $\Omega$ and $r$ in Table 1. We see that the relation $Sp(2) \sim SU(2)$, which we encountered before in Section 5, is the only solution to eq. (6.3) among the symplectic or exceptional groups. This relation is actually an identity [10], so one should not regard it as an alternative description.

If we reconsider unitary (instead of special unitary) $\chi_{SB}$, relation (6.3) turns into $\Omega = r^2$, which cannot be matched by any symplectic or exceptional group, see Table 1.

If we also take $SU(N)$ and $O(N)$ into account (and exclude the trivial group $SU(1)$), the only group of Table 1 obeying $\Omega = r^2$ is $O(2)$. However, we will not include the transition $U(1)_L \otimes U(1)_R \rightarrow U(1)_V \sim O(2)_L \otimes O(2)_R \rightarrow O(2)_V$ in our concluding list in Section 8, because it does not agree with the usual notion of $\chi_{SB}$ (in the continuum).

### 7 $\chi_{SB}$ for lattice fermions

Let us finally address further $\chi_{SB}$ patterns, which occur in non-perturbative studies by means of numerical simulations of vector theories on the lattice. Traditionally two types of lattice fermion formulations have usually been applied in Monte Carlo simulations. In one of them, the Wilson fermion [25], a discrete Laplacian term is added to the naive discretisation in order to avoid the fermion doubling. This term breaks the chiral symmetry explicitly, and the chiral limit can only be attained by fine-tuning the bare fermion mass. The pattern of $\chi_{SB}$, however, for Wilson fermions (and variants thereof) is the same as in the continuum.
The same holds for Ginsparg-Wilson fermions [2], which gained importance only recently: they obey an exact, lattice modified chiral symmetry [26], which turns into the standard chiral symmetry in the continuum limit. This modified chiral symmetry prevents additive mass renormalisation, so the chiral limit does not require fine tuning, and also at finite lattice spacing the flavour chiral symmetry breaking follows the same pattern as in the continuum (although the symmetry transformation is local in this case).

The situation is different, however, for the second traditional standard lattice fermion, denoted as the staggered fermion [1]. Unlike the Wilson fermion it does not suffer from additive mass renormalisation. In this respect it is an alternative to the Ginsparg-Wilson fermion; the staggered fermion is much simpler to simulate, but plagued with unpleasant constraints on $N_f$ (if one insists on locality in order to assure a controlled continuum limit [27]).

In its construction one starts from the naive lattice fermion, which is doubled in each direction. By means of a lattice site dependent transformation, the $\gamma$-matrix structure can be removed, so that one only needs to keep track of 1 out of the original 4 spinor components (for $N_f = 1$ in 4 dimensions). One distributes its 16 copies over the sites of unit hypercubes on the lattice. At this point, one distinguishes an even and an odd sub-lattice (it consists of the sites where the sum of the coordinates in lattice units, $x_1 + x_2 + x_3 + x_4$, is even resp. odd). In the chiral limit, the staggered fermion components on these two sub-lattices can be rotated independently in $\mathbb{C}$ without altering the action, which amounts to a global $U(1)_e \otimes U(1)_o$ symmetry. It contains the axial $U(1)$ symmetry, along with a $U(1)$ remnant chiral symmetry, whereas the vectorial $U(1)$ group is redundant in this formulation (see e.g. Ref. [28]).

In the continuum limit 4 flavours can be assembled, and the lattice $U(1)$ invariance is remnant of the corresponding $SU(4) \otimes SU(4)$ chiral flavour symmetry. This would again correspond to the (inappropriate) inclusion of the $c$-quark, as in eq. (4.7). At finite lattice spacing, the transition $U(1) \sim O(2) \rightarrow 1$ is the numerically observed $\chi_{SB}$. Hence the simple property referred to in the last paragraph of Section 6 has some kind of application on the lattice.

For $N_f$ staggered fermions in the complex representation, the global symmetry is extended to $U(N_f)_e \otimes U(N_f)_o$. This is the setting that we addressed in the last paragraphs of Sections 3 and 6. Now the $\chi_{SB}$ pattern yields the
coset space

\[ SU(N_f) \otimes SU(N_f) \otimes U(1)/SU(N_f) = U(N_f) . \]  

(7.1)

This does not allow for any alternative description by non-unitary Lie groups (without building direct products), except for the \(N_f = 1\) case that we mentioned before.

In the real or pseudo-real representation the chiral symmetry group is enlarged to \(U(2N_f)\), similar to the \(\chi_{SB}\) patterns of Section 5. However, compared to the continuum situation that we addressed before, for the staggered fermions the non-breaking symmetry is interchanged, \(i.e.\) the coset space reads \(U(2N_f)/Sp(2N_f) \ (U(2N_f)/SO(2N_f))\) in the real (pseudo-real) representation \[19\]. Here the search for an isomorphic description by non-unitary Lie groups (again without direct products) fails because there is no isomorphism at all to \(U(2N_f)\).

At last we mention that simulations are also possible in a Hamiltonian formulation, though this approach is tedious and therefore not popular. At strong coupling it leads to the \(\chi_{SB}\) pattern \[28\]

\[ U(4N_f) \to U(2N_f) \otimes U(2N_f) . \]  

(7.2)

In this case the order is divided by 2 (which is the minimal requirement that we postulated in Section 2), but the rank remains unchanged. The second property does not hold for any of the non-unitary isomorphic descriptions that we found to obey the first property, hence there is no alternative description for that type of \(\chi_{SB}\). (The invariance of the rank and the conclusion still persists if we split off an axial phase symmetry.)

8 Conclusions

The well-known description of Chiral Symmetry Breaking (\(\chi_{SB}\)) in \(N_f = 2\) QCD by means of orthogonal groups is indeed quite specific. We have considered broad generalisations of \(\chi_{SB}\) and studied the question if there are further isomorphic descriptions by non-unitary Lie groups. We only found a few possibilities in terms of orthogonal groups, but none with symplectic or exceptional Lie groups. The alternatives are summarised in Table \[2\].

In addition the simple \(\chi_{SB} \ U(1) \sim O(2) \to 1\) occurs for \(N_f = 1\) staggered fermion at finite lattice spacing \(i.e.\) on the regularised level.)
Table 2: The list of the $\chi$SB breaking patterns, which are isomorphic to some transition that does not involve unitary groups. We have demonstrated that this list is in fact complete.

We mentioned before that there are no applications of the transitions in Table 2 to QCD with the quark masses observed in Nature. Possibly conceivable applications of the right-most-side of Table 2 could be non-standard variants of the Higgs mechanism (cf. Section 1). However, the case $O(3) \rightarrow SO(2)$ does not provide a sufficient number of NGBs to generate massive gauge bosons $W^\pm, Z^0$. In the other cases there would be an abundance of 2...12 NGBs, and one would have to explain why they have not been manifest in low energy phenomenology. This suggests that additional symmetry breaking would be required to render these particles heavy.

Proceeding to the third column of Table 2, we are led to scenarios without a fundamental scalar particle, where the particle masses are generated by the chiral condensate of (non-standard) fermions. This takes us back to the technicolour models [9] that we already referred to in Section 1. Early versions were based on a direct product of the Standard Model gauge groups with a new (technicolour) gauge group, but since the standard fermions are then technigauge singlets they cannot become massive.

Extended versions (e.g. in Refs. [13, 14]) use a unified gauge group, which contains those of the Standard Model (that it breaks down to), thus coupling techniquarks to standard fermions. This concept matches our discussion, and fermion masses can be generated, though the heavy top quark is still uneasy to explain, and flavour-changing neutral currents emerge, which are not observed. Minimal Walking Models [29] are versions, which are fashion now, and which try to avoid this problem by a near-conformal gauge dynamics so that the coupling “walks” instead of “running”. There are numerous recent attempts to identify a suitable theory with a “walking” gauge coupling by means of lattice simulations [30].
The issue in these studies is the search for an adequate strongly interacting model with a conformal window, i.e. with an IR fixed point. Indicators of this property could be that the ratio $m_{\text{pseudoscalar}}/m_{\text{fermion}} \propto \text{chiral condensate}$, the string tension and the pseudoscalar decay constant vanish in the chiral limit, in contrast to QCD. So far, the numerical studies suffer from difficulties to attain the chiral regime.

Let us finally summarise the scenario that we mentioned before. We assume phenomenology at moderate energy to be consistent with a (possibly non-standard) Higgs mechanism in terms of orthogonal groups. This suggests a multi-component scalar Higgs field.

Now we wonder if the corresponding Higgs particle could still have a fermionic substructure, which may be manifest at very high energy (and which could help for instance to overcome the hierarchy problem). If the orthogonal symmetry breaking pattern can be identified isomorphically with some kind of $\chi_{\text{SB}}$, this scenario is conceivable. Table 2 presents a list of the patterns where this is the case. If, on the other hand, we observe some orthogonal pattern that is not included in this list — or if we observe a pattern based on other Lie groups, which do not match any kind of $\chi_{\text{SB}}$ — such an interpretation is hard to advocate. Hence Table 2 distinguishes whether or not an obvious techniquark candidate exists. If this is the case, the actual viability of such an underlying description is still to be investigated.

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