Optimization and self-organized criticality in a magnetic system

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We propose a kind of Bak-Sneppen dynamics as a general optimization technique to treat magnetic systems. The resulting dynamics shows self-organized criticality with power law scaling of the spatial and temporal correlations. An alternative method of the extremal optimization is also analyzed here. We provided a numerical confirmation that, for any possible value of its free parameter \( \tau \), the extremal optimization dynamics exhibits a non-critical behavior with an infinite spatial range and exponential decay of the avalanches. Using the chiral clock model as our test system, we compare the efficiency of the two dynamics with regard to their abilities to find the system’s ground state.

Keywords: Self-organized criticality, optimization, Bak-Sneppen model, chiral clock model

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I. INTRODUCTION

Nature which is composed of dead and living things is a system far out of equilibrium. This can be evidenced by measuring nature’s evolution with the appropriate scales of space and time. In particular, the dynamics of the living things is basically governed by the theory of natural selection first proposed by Charles Darwin [1]. This theory is certainly one of the greatest scientific achievements of the nineteenth century. Very briefly, this theory simply says that evolution occurs not by breeding species best adapted to their environment but by driven to extinction those less or poorly adapted. This mechanism leads to the emergence of highly specialized structures. If we also consider the astonishing variability of the species, we then can say that nature is a complex system. Indeed, for all we know, nature operates at the self-organized critical state [2].

One of the most fundamental characteristics of a system with self-organized criticality (SOC) is to exhibit a stationary state with a long-range power law decay of the spatial and temporal correlations [3]. Power law is a very abundant behavior found either in natural phenomena such as the light emitted from quasars, the earthquake’s intensity, the water level of the Nile or as a direct result of human activities like the distribution of cities by size, the repetition of words in the Bible and in traffic jams.

Self-organized critical systems evolve to the complex critical state without the interference of any external agent. Differently from what happens in the usual critical phase transition, in SOC there is not a tuning parameter. The prototypical example of SOC is a pile of sand [3]. Another common property of SOC is that the self-organization state only takes place after a very long transient period. Last but not least, a minor change in the system can cause colossal instabilities called avalanches. Intermittent bursts of activity separating long periods of quiescence is called punctuated equilibrium. Gould has conjectured that the biological evolution itself shows, in fact, this kind of equilibrium [4].

Now, if nature can really be found in a critical self-organized state, is this state optimal or extremal in some sense? Through the systematic rejection of the worst, the Darwinian evolutionary theory is a born optimization structure. By adding interactions and competition between the species we will witness the emergence of co-evolution.

One model specially tailored to represent the co-evolutionary activities of the species is the Bak-Sneppen model [5]. In this model, each species is located on the sites of a lattice and has associated a fitness value between 0 and 1 (randomly drawn from an uniform distribution). At each time step, the species with the smallest associated value as well as its nearest neighbors are selected to replace their fitness with new random numbers. In one dimension, after a long transient time, almost all species have fitness larger than the critical value 0.67 [5].

Recently, inspired by natural processes, some heuristic optimization techniques have been proposed: genetic algorithms [6], simulated annealing [7] and extremal optimization [8]. The extremal optimization (EO) method seems to be the most efficient of them since it brings the system faster and closer to its ground-state [8]. In a few words, this method consists of the following rules: i) a discrete dynamical variable \( S_i \) (initially chosen at random) is associated to each site \( i \) of a lattice with \( N \) points; ii) a fitness \( \lambda_i \) is attributed to that site using some given prescription; iii) all lattice sites are then increasingly ranked according to their fitness (the site with the worst fitness is of rank 1); iv) a site of rank \( k (1 \leq k \leq N) \) is selected with probability \( P(k) \propto k^{-\tau} \) (\( \tau \) is an arbitrary real positive number) and its dynamical variable \( S_i \) is changed to \( S_i' \) (\( S_i' \neq S_i \), compulsorily); v) repeat at step iii) as long as desired.

In this paper, a kind of Bak-Sneppen dynamics is ap-
plied to a spin system which possesses a complex ground state structure in one dimension. The discrete fitness variability is identified as being responsible for the absence of a critical self-organized state. By introducing a noise in the spin configuration space, we show that the system can now reach a critical self-organized state with power law correlations. This simple trick can easily be extended to other types of discrete systems. On the other hand, through an analysis of the spatial and temporal correlations, we provided a numerical confirmation that the extremal optimization method is not a SOC dynamics but a non-critical algorithm with an infinite spatial dimension, the competition between the applied magnetic commensurate and incommensurate phases. In one dimension, this system exhibits a complex phase transition diagram with our experimental system. In higher dimensions, this system can now reach a critical self-organized state with the extremal optimization method is not a SOC dynamics but a non-critical algorithm with an infinite spatial dimension, the extremal optimization (EO) are then investigated for the universality of the BSN dynamics.

II. THE P-STATES CHIRAL CLOCK MODEL

To explain our main ideas, we have chosen the one dimensional p-states chiral clock model ($CC_p$) as our experimental system. In higher dimensions, this system exhibits a complex phase transition diagram with commensurate and incommensurate phases. In one dimension, the competition between the applied magnetic field, which tries to align the spins, and the chirality, which tries to flip them, gives rise to a rich ground-state diagram in the space of the parameters. The hamiltonian is given by

$$H = \sum_{i=1}^{N} \left\{ 1 - \cos(2\pi(S_{i+1} - S_i)/p) - \Delta \right\} + h[1 - \cos(2\pi(S_i - 1)/p))] = -\sum_{i=1}^{N} \lambda_i$$ (1)

where $p$ is a positive integer number, $S_i = 1, \ldots, p$ is the spin variable at the site $i$ (with periodic boundary conditions applied) and $h$ and $\Delta$ are the magnetic field and chirality, respectively. By symmetry arguments, $h$ and $\Delta$ may be restricted to the intervals $[0, \infty]$ and $[0, 1/2]$.

For $p = 3$, the chiral clock model has 3 regions with different ground-states (see Fig.1). We denote by $<1>$, $<12−13>$ and $<123>$ the regions with periods 1, 2 and 3, respectively. The numbers correspond directly to the sequence of the spin states. The region $<12−13>$ is super degenerate, i.e., the spin state sequence 12 may be equally followed by 12 or 13. For a finite ring, this means that the degenerescence of this region grows exponentially with the lattice size.

III. SEARCHING THE GROUND-STATE WITH A BAK-SNEPPEN DYNAMICS

We choose the following form for the fitness $\lambda_i$

$$-\lambda_i = \frac{1}{2} \left\{ 1 - \cos(2\pi(S_{i+1} - S_i)/p) + \Delta \right\} + \frac{1}{2} \left\{ 1 - \cos(2\pi(S_i - S_{i-1})/p) + \Delta \right\} + \frac{1}{3} h[1 - \cos(2\pi(S_{i+1} - 1)/p)] + \frac{1}{3} h[1 - \cos(2\pi(S_i - 1)/p)]$$

(2)

There is a certain art and arbitrariness to select the fitness form. For instance, one could equally well have chosen a pre-factor $1/4$ for the magnetic fields at the sites $i + 1$ and $i - 1$ and $1/2$ for the site $i$. Different fitness forms may lead to quite different values of some physical quantities determined by the dynamics, e.g., the average energy in the stationary state. The fitness defined by Equation (2) is the form with the highest symmetry - the magnetic field is equally distributed between the sites $i - 1$, $i$ and $i + 1$ as well as is the chirality between the pair of sites $[i, i - 1]$ and $[i + 1, i]$. It represents the best result we got for the average energy in a bunch of trials.

Let us now describe our procedure. Initially, to each site $i$ of the ring, we attribute by chance one of the $p$ possible values of the spin variable $S_i$. Using equation (2), we calculate all the corresponding fitness $\lambda_i$ and find the

![Fig. 1: The magnetic field h versus the chirality Δ for the ground state diagram of the p = 3 chiral clock model. The regions <1>, <12−13> and <123> have periods 1, 2 and 3, respectively. The marked points I, II, III and IV were tested for the universality of the BSN dynamics.](image-url)
smaller (worst) one $\lambda_j$. New spin variables are then randomly assigned to the sites $j$, $j+1$ and $j−1$. So although the dynamics involves 3 spins, due to our fitness definition, it affects (changes) 5 consecutive fitness (from the site $j−2$ to $j+2$). Observe that here the Bak-Sneppen dynamics is being applied to the spin configuration space not to the fitness space itself (as it was originally done in the evolutionary context [2]). In the final step, a new site with the smallest fitness is searched and the whole process continues as long as we wish. However, even for a modest lattice size, this dynamics is hampered by the discrete fitness variability. Let us explain why. If $p = 3$ ($h$ and $\Delta$ fixed) there are at most 27 possible values of the fitness (besides, some of these may be degenerate). This means that, for a lattice with a reasonable size, we will generally find not just one but a large number of sites with exactly the same (smallest) fitness value. The simplest solution to the problem seems to put all those sites in a list and to choose one of them at random. We will call this dynamics as the Bak-Sneppen with draw (BSD).

As we shall see later, neither from the point of view of the spatial correlations nor from the time correlations is this dynamics a true SOC.

IV. THE NOISE

It is the discrete variability of the fitness (or, equivalently, of the spin variable) which precludes the establishment of a self-organized critical state, so we searched a simple way to solve the problem. Suppose that a noise $\eta$ is added to the original spin variable $S_i$ (an integer in the interval $[1,p]$), i.e.

$$S_i' = S_i + \eta (1-2r)$$

where $r \in [0,1]$ is a random number generated from a flat distribution. The new spin variable $S_i'$ as well as the corresponding fitness have now the desired continuous characteristic. By choosing the noise $\eta$ sufficiently small the relevant physical properties (like the energy per site or the magnetization per site) will not be affected. In this paper we set $\eta = 10^{-3}$. Using double numerical precision, we varied $\eta$ from $10^{-12}$ up to $10^{-3}$ with no important difference for the physical quantities. With this trick, any previously discrete fitness $\lambda$ is turned into a continuous variable inside some interval $\Delta \lambda$ controlled by the noise $\eta$, the chirality $\Delta$ and the magnetic field $h$. We will call this dynamics as the Bak-Sneppen with noise (BSN). Our next objective is to compare the 3 dynamics EO, BSD and BSN with respect to their efficiencies.

In order to guarantee that the stationary state had already been reached and to get good averages for the physical quantities, a huge amount of computation was performed. Using the same fitness definition (equation (2)) for all the three algorithms BSD, BSN and EO, we simulated each one of them over $2 \times 10^6$ runs on a ring of 4001 sites. Discarding the first $2 \times 10^5$ runs (10% of the total) as the transient time, averages were then taken over the remaining steps. The results for the energy histograms are shown in Figure 2.

The simulations were done at the point III of the region $< 123 >$ (see Figure 1) where the ground state energy per spin is equal to 0.8055. The best performance was obtained by the EO algorithm with $\tau = 2.1$, followed by the EO with $\tau = 1.2$, BSN and BSD. Their respective mean energies were $0.828 \pm 0.003$, $0.969 \pm 0.009$, $0.973 \pm 0.004$ and $0.983 \pm 0.005$.

V. THE SPATIAL AND TEMPORAL DISTRIBUTIONS

To study the spatial correlation, we measured the distribution $D(x)$, of the distance $x$ between two subsequent activated sites [3]. Recall that for the three algorithms we activate a site $i$ by changing its spin state $S_i$ and the corresponding fitness $\lambda_i$. We plotted $D(x)$ in the Figure 3.

Clearly, EO is an algorithm with an infinite spatial range - it doesn’t matter how far one site is of the other, the activation probability is constant. On the other hand, the BSN algorithm exhibits a power law dependence $D(x) \sim x^{-3.19 \pm 0.02}$ with the exponent being compatible with that of the Bak-Sneppen model ($3.23 \pm 0.02$ [11]). Moreover, we got at the points I, II, III and IV (see Figure 1) the same exponent value, indicating that the universality principle holds in the space of the parameters $h$ and $\Delta$. Finally, from the Figure 3 we conclude that the BSD algorithm is of a mixed kind, having the BSN behavior for small distances and the EO for large distances.
In the stationary regime, the BSN dynamics shows an intermittent time evolution with long periods of passivity interrupted by sudden bursts of activity, i.e., it exhibits a punctuated equilibrium. This abrupt change of activity is called an avalanche. We say that an avalanche is happening if there exist one site of the ring whose fitness is smaller than a certain threshold $\lambda_c$. The size $A$ of an avalanche is defined as the number of subsequent time steps with at least one fitness below that threshold. For each one of the three dynamics we calculated the probability distribution of the avalanches $P(A)$. The results are illustrated in the Figure 4.

The Figures 4(a), 4(b) and 4(c) correspond to the BSN, EO and BSD algorithms, respectively. Before we analyze $P(A)$, let us first discuss how the critical thresholds were obtained. At the time step $2 \times 10^8$ (the end of the transient period) and with the magnetic field and chirality fixed at the values $0.8$ and $0.35$, respectively, we determined the fitness histograms of the whole ring. They are shown as insets of the Figure 4. Since in the EO and the BSD algorithms there are only some few possible values for the fitness $\lambda$ (a maximum of $27$ when $p = 3$ and many of them are degenerate), we may vary the (discrete) $\lambda$ values until the avalanches are founded. Remember that if $\lambda < \lambda_c$ there will be no avalanches while if $\lambda > \lambda_c$ only one (infinite) avalanche is present. Due to the discreteness of $\lambda$ values, it is very improbable to find the system in a sub or supercritical regime. We found $\lambda_c = -0.8055$ and $\lambda_c = -1.5061$ for the EO and BSD dynamics, respectively. For the BSN algorithm, however, the fitness is continuous. There is a fitness fluctuation $\Delta \lambda$ due to the presence of the noise $\eta$. In our case, the maximum value of the relative fluctuation $\Delta \lambda / \lambda$ is $1.86 \times 10^{-3}$. To find the critical threshold, we wrote a program which varies the fitness values (inside the range $\Delta \lambda$) until the resulting avalanches have sizes between 500 and 2000 (which seems reasonable to avoid the sub and supercritical regimes). We determined $\lambda_c = -1.5059$.

We can now go back to the probability distribution of the avalanches $P(A)$. They were all calculated at their respective thresholds $\lambda_c$. The EO algorithm (Figure 4(b)) shows an exponential decay $P(A) \sim A^{-0.87 \pm 0.01}$. This value is somewhat discrepant from that of the evolutionary Bak-Sneppen model ($1.07 \pm 0.01$ [1]), but we should remember that, for our system, the exact determination of the critical threshold position is much more difficult and one can easily be found in a sub or supercritical regime. For the BSD algorithm (Figure 4(c)), we found once more a mixed behavior described by $P(A) \sim A^{-0.68} e^{-0.002 A}$.
VI. CONCLUSIONS

Based on the Bak-Sneppen ideas, we proposed an optimization algorithm whose dynamics, in the steady state regime, exhibits self-organized criticality. It has been successfully applied to find the ground states of the clock chiral model and it can be easily extended to any other magnetic system. From the three dynamics studied in this paper, we conclude that only the BSN algorithm conducts the system to a self-organized critical state. The essential point was to recognize that the discrete fitness values, present in the EO and BSD algorithms, preclude the system to have a power law decay of the spatial correlation. To overcome this problem we introduced a small noise in the spin configuration space. Another important conclusion we arrived is that to possess SOC characteristics does not guarantee a dynamics to be optimal. This statement was proved here for the EO algorithm. The EO algorithm, which is not SOC since it has an infinite spatial range and the avalanches decay exponentially, has a peak of the energy histogram (with $\tau = 2.1$) which is only 2.7% further from the exact ground state value. This is a stunning result which cannot be beaten by the BSN dynamics. At $\tau = 1.2$, the EO and BSN are almost equivalent. On the other hand, the BSD algorithm is of a mixed type. It can be obtained by taking the limit $\eta \to 0$ in the BSN.

Finally, it is important to observe that, contrary to what happens with BSN, the EO algorithm has one arbitrary and free parameter $\tau$. If $\tau \to 0$, EO becomes a random walk and if $\tau \to \infty$, the system may stack in a dead end. In both limits, the EO efficiency is completely spoiled and an optimal value of $\tau$ should be found in somewhere between. In the BSN dynamics there is no tuning parameter. Using the ideas described here, we wish to develop a SOC optimization algorithm to some classical combinatorial problems like the bipartition of graphs.

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