THE METHOD OF SHIFTED PARTIAL DERIVATIVES CANNOT SEPARATE THE PERMANENT FROM THE DETERMINANT

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Abstract. The method of shifted partial derivatives introduced in [9, 7] was used to prove a super-polynomial lower bound on the size of depth four circuits needed to compute the permanent. We show that this method alone cannot prove that the padded permanent \( \ell^n - m \text{perm}_m \) cannot be realized inside the \( GL_{n^2} \)-orbit closure of the determinant \( \text{det}_n \) when \( n > 2m^2 + 2m \). Our proof relies on several simple degenerations of the determinant polynomial, Macaulay's theorem that gives a lower bound on the growth of an ideal, and a lower bound estimate from [7] regarding the shifted partial derivatives of the determinant.

1. Introduction

Let \( S_m \) denote the permutation group on \( m \) elements and let \( y^i_j \) be linear coordinates on \( \mathbb{C}^{m^2} \). The permanent polynomial is

\[
\text{perm}_m(y^i_j) = \sum_{\sigma \in S_m} y_{\sigma(1)}^1 \cdots y_{\sigma(m)}^m.
\]

Valiant’s famous conjecture \( \text{VP} \neq \text{VNP} \) may be phrased as:

**Conjecture 1.1.** [15] There does not exist a polynomial size circuit computing the permanent.

Let \( W = \mathbb{C}^N \) with linear coordinates \( x_1, \ldots, x_N \), let \( W^* \) denote the dual vector space, let \( S^n W \) denote the space of degree \( n \) homogeneous polynomials on \( W^* \), and let \( \text{Sym}(W) = \bigoplus_n S^n W \). Let \( \text{End}(W) \) denote the space of endomorphisms of \( W \), so in particular if \( P \in S^n W \), \( \text{End}(W) \cdot P \subset S^n W \) is the set of homogeneous degree \( n \) polynomials obtainable by linear specializations of the variables \( x_1, \ldots, x_N \) in \( P(x_1, \ldots, x_N) \).

Since the determinant \( \text{det}_n \in S^n \mathbb{C}^{n^2} \) is in \( \text{VP} \), Conjecture 1.1 would imply the following conjecture:

**Conjecture 1.2.** [15] Let \( \ell \) be a linear coordinate on \( \mathbb{C} \) and consider any linear inclusion \( \mathbb{C} \oplus \mathbb{C}^{m^2} \rightarrow W = \mathbb{C}^{n^2} \), so in particular \( \ell^{n-m} \text{perm}_m \in S^n W \). Let \( n(m) \) be a polynomial. Then for all sufficiently large \( m \),

\[
[\ell^{n-m} \text{perm}_m] \not\in \text{End}(W) \cdot [\text{det}_{n(m)}].
\]

The polynomial \( \ell^{n-m} \text{perm}_m \) is called the **padded permanent**.

Instead of arbitrary circuits, by [2, 8, 14, 10, 1] one could prove Valiant’s conjecture by restricting to depth-four circuits and proving a stronger lower bound: If \( P_n \in S^n \mathbb{C}^N \) is a sequence of polynomials that can be computed by a circuit of size \( s = s(n) \), then \( P_n \) is computable by a homogeneous \( \Sigma \Pi \Sigma \Pi \) circuit of size \( 2^{O(\sqrt{n \log(n s) \log(N)})} \). So to prove \( \text{VP} \neq \text{VNP} \), it would be sufficient to show the permanent \( \text{perm}_m \) is not computable by a size \( 2^{O(\sqrt{m \log(\text{poly}(m))})} \)
homogeneous $\Sigma \Pi \Sigma \Pi$ circuit. Work of Gupta, Kamath, Kayal, and Saptharishi [7] generated considerable excitement, because it came tantalizingly close to proving Valiant’s conjecture by showing that the permanent does not admit a size $2^{o(\sqrt{m})}$ homogeneous $\Sigma \Pi \Sigma \Pi$ circuit with bottom fanin bounded by $\sqrt{m}$.

Any method of proof that separates $\text{VP}$ from $\text{VNP}$ would also have to separate the determinant from the permanent. We show that this cannot be done with the method of proof in [7], the method of shifted partial derivatives. This method builds upon the method of partial derivatives (see, e.g., [12]), which dates back to Sylvester [13].

1.1. The method of shifted partial derivatives. The space $S^k W^*$ may be interpreted as the space of homogeneous differential operators on $\text{Sym}(W)$ of order $k$. Given a homogeneous polynomial $P \in S^n W$, consider the linear map

$$P_{k,n-k} : S^k W^* \to S^{n-k} W$$

$$D \mapsto D(P).$$

In coordinates the map is $\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} \mapsto \frac{\partial^k p}{\partial x_{i_1} \cdots \partial x_{i_k}}$.

Given polynomials $P, Q \in S^n W$, and $k < n$, $P \in \text{End}(W) \cdot Q$ implies that rank$(P_{k,n-k}) \leq$ rank$(Q_{k,n-k})$. The method of partial derivatives is to find a $k$ such that rank$(P_{k,n-k}) >$ rank$(Q_{k,n-k})$ to prove $P \notin \text{End}(W) \cdot Q$.

Now consider $P_{k,n-k} \otimes \text{Id}_{S^\tau W} : S^k W^* \otimes S^\tau W \to S^{n-k} W \otimes S^\tau W$. Project (multiply) the image to $S^{n-k+\tau} W$ to obtain a map

$$P_{(k,n-k)\{\tau\}} : S^k W^* \otimes S^\tau W \to S^{n-k+\tau} W$$

$$D \otimes R \mapsto D(P) R$$

Again $P \in \text{End}(W) \cdot Q$ implies that rank$(P_{(k,n-k)\{\tau\}}) \leq$ rank$(Q_{(k,n-k)\{\tau\}})$. The method of shifted partial derivatives is to find $k, \tau$ such that rank$(P_{(k,n-k)\{\tau\}}) >$ rank$(Q_{(k,n-k)\{\tau\}})$ to prove $P \notin \text{End}(W) \cdot Q$.

Remark 1.3. Both these methods are algebraic in the sense that they actually prove $P \notin \text{End}(W) \cdot Q$ where the overline denotes Zariski closure. Most known lower bound techniques for Valiant’s conjecture have this property, see [6].

Remark 1.4. These methods may be viewed as special cases of the Young-flattenings introduced in [11].

From the perspective of algebraic geometry, the method of shifted partial derivatives compares growth of Jacobian ideals: For $P \in S^n W$, consider the ideal in $\text{Sym}(W)$ generated by the partial derivatives of $P$ of order $k$. Call this the $k$-th Jacobian ideal of $P$, and denote it by $I_P^{k}$. It is generated in degree $n-k$. The method is comparing the dimensions of the Jacobian ideals in degree $n-k+\tau$, i.e., the Hilbert functions of the Jacobian ideals.

1.2. Statement of the result. We prove this method cannot give better than a quadratic separation of the permanent from the determinant:

Theorem 1.5. There exists a constant $M$ such that for all $m > M$ and every $n > 2m^2 + 2m$, any $\tau$ and any $k < n$,

$$\text{rank}((\ell_{n-m} \text{perm}_{m})_{(k,n-k)\{\tau\}}) < \text{rank}((\text{det}_{n})_{(k,n-k)\{\tau\}}).$$

Despite this, it may be possible that a more general Young flattening is able to prove, e.g. a $\omega(m^2)$ lower bound on $n$. This motivated the companion paper [3] where we study Jacobian ideals and their minimal free resolutions.
1.3. **Overview of the proof.** The proof of Theorem 1.5 splits into four cases:

- (C1) Case $k > n - \frac{n}{m+1}$,
- (C2) Case $2m \leq k \leq n - 2m$,
- (C3) Case $k < 2m$ and $\tau > \frac{3}{4}n^2m$,
- (C4) Case $k < 2m$ and $\tau < 6\frac{n^4}{m}$.

Note that C1,C2 overlap when $n > 2m^2 + 2m$ and C3,C4 overlap when $n > \frac{m^2}{4}$, so it suffices to take $n > 2m^2 + 2m$.

In the first case, the proof has nothing to do with the padded permanent or its derivatives: it is valid for any polynomial in $m^2 + 1$ variables. Cases 2,3 only use that we have a padded polynomial. In the fourth case, the only property of the permanent that is used is an estimate on the size of the space of its partial derivatives. Case C1 is proved by showing that in this range the partials of the determinant can be degenerated into the space of all polynomials of degree $n - k$ in $m^2 + 1$ variables. Cases C2,C3 use that when $k < n - m$, the Jacobian ideal of any padded polynomial $\ell^{n-m}P \in S^nW$ is contained in the ideal generated in degree $n - m - k$ by $\ell^{n-m-k}$, which has slowest possible growth by Macaulay’s theorem as explained below. Case C2 compares that ideal with the Jacobian ideal of the determinant; it is smaller in degree $n - k$ and therefore smaller in all higher degrees by Macaulay’s theorem. Case C3 compares that ideal with an ideal with just two generators in degree $n - k$. Case C4 uses a lower bound for the determinant used in [7] and compares it with a very crude upper bound for the dimension of the space of shifted partial derivatives for the permanent.

We first review Macaulay’s theorem and then prove each case.

We will use the notation:

\[ t = \tau + n - k. \]

Fix index ranges $1 \leq s, t, i, j \leq n$. If $\mathcal{I} \subset \text{Sym}(W)$ is an ideal, we let $\mathcal{I}_d \subset S^dW$ denote its component in degree $d$.

We make repeated use of the estimate

\[ \ln(q!) = q \ln(q) - q + \Theta(\ln(q)). \]

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2. **Macaulay’s Theorem**

We only use Corollary 2.4 from this section in the proof of Theorem 1.5.

**Theorem 2.1** (Macaulay, see, e.g., [5]). Let $\mathcal{I} \subset \text{Sym}(W)$ be an ideal, let $d$ be a natural number, and set $Q = \text{dim } S^dV/\mathcal{I}_d$. Write

\[ Q = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_\delta}{\delta} \]

with $a_d > a_{d-1} > \cdots > a_\delta$ (such an expression exists and is unique). Then

\[ \text{dim } \mathcal{I}_{d+\tau} \geq \binom{N + d + \tau - 1}{d + \tau} - \left( \binom{a_d + \tau}{d + \tau} + \binom{a_{d-1} + \tau}{d + \tau - 1} + \cdots + \binom{a_\delta + \tau}{\delta + \tau}. \right) \]
Remark 2.2. Gotzmann [4] showed that if \( \mathcal{I} \) is generated in degree at most \( d \), then equality is achieved for all \( \tau \) in (3) if equality holds for \( \tau = 1 \). Ideals satisfying this minimal growth exist, they are lex-segment ideals, see [5].

Remark 2.3. Usually Macaulay’s theorem is stated in terms of the coordinate ring \( \mathbb{C}[X] := \text{Sym}(W)/\mathcal{I} \) of the variety (scheme) \( X \subset W^* \) that is the zero set of \( \mathcal{I} \), namely

\[
\dim \mathbb{C}[X]_{d+\tau} \leq \sum_{\sigma = d}^{a_d + \tau} \binom{a_d + \tau}{\sigma} + \sum_{\sigma = d + 1}^{a_{d-1} + \tau} \binom{a_{d-1} + \tau}{\sigma} + \cdots + \binom{a_{\delta} + \tau}{\delta + \tau}.
\]

Corollary 2.4. Let \( W = \mathbb{C}^N \). Let \( \mathcal{I} \) be an ideal such that \( \dim \mathcal{I}_d \geq \dim S^{d-q}W = \binom{N+\delta-d-1}{d-q} \) for some \( q < d \). Then \( \dim \mathcal{I}_{d+\tau} \geq \dim S^{d-q+\tau}W = \binom{N+\tau+d-q-1}{\tau+d-q} \).

**Proof.** First use the identity

\[
(a + b)_d = \sum_{j=1}^{q} \binom{a + b - j}{b - j + 1} + \binom{a + b - q}{b - q}
\]

with \( a = N - 1 \), \( b = d \). Write this as

\[
\binom{N-1+d}{d} = Q_d + \binom{N-1+d-q}{d-q}.
\]

Set

\[
Q_{d+\tau} := \sum_{j=1}^{q} \binom{N-1+d+\tau-j}{d+\tau-j+1}.
\]

By Macaulay’s theorem, any ideal \( \mathcal{I} \) with

\[
\dim \mathcal{I}_d \geq \binom{N-1+d-q}{d-q}
\]

must satisfy

\[
\dim \mathcal{I}_{d+\tau} \geq \binom{N-1+d}{d+\tau} - Q_{d+\tau} = \binom{N-1+d-q+\tau}{d-q+\tau}.
\]

\( \square \)

We will use Corollary 2.4 with \( N = n^2 \), \( d = n-k \), and \( d - q = m \).

3. Case C1

Our assumption is \((m+1)(n-k) < n\). It will be sufficient to show that some \( R \in \text{End}(W) \cdot \det_n \) satisfies \( \text{rank}(\ell_{m,n-k}(\text{perm}_n)_{(k,n-k)[\tau]}) < \text{rank}(R_{k,n-k}(\text{perm}_n)_{(k,n-k)[\tau]}) \). Block the matrix \( x = (x_i^j) \in \mathbb{C}^{n^2} \) as a union of \( n-k \) \( m \times m \) blocks in the upper-left corner plus the remainder, which by our assumption includes at least \( n-k \) elements on the diagonal. Set each diagonal block to the matrix \( (y_i^j) \) (there are \( n-k \) such blocks), fill the remainder of the diagonal with \( \ell \) (there are at least \( n-k \) such terms), and fill the remainder of the matrix with zeros. Let \( R \) be the restriction of the determinant to this subspace. Then the space of partials of \( R \) of degree \( n-k \), \( R_{k,n-k}(\text{perm}_n^* \subset S^{n-k} \mathbb{C}^{m^2}) \) contains a space isomorphic to \( S^{n-k} \mathbb{C}^{m^2+1} \), and \( \mathcal{I}_{n-k} \text{perm}_n^* \subset S^{n-k} \mathbb{C}^{m^2+1} \) so we conclude.
Example 3.1. Let \( m = 2, \ n = 6 \). The matrix is

\[
\begin{pmatrix}
y_1^1 & y_2^1 \\
y_1^2 & y_2^2 \\
y_1^3 & y_2^3 \\
y_1^4 & y_2^4 \\
y_1^5 & y_2^5 \\
y_1^6 & y_2^6 \\
\ell & \ell
\end{pmatrix}.
\]

The polynomial \((y_1^1)^2\) is the image of \( \frac{\partial^2}{\partial x_1^2 \partial x_2^2 \partial x_3 \partial x_6} \) and the polynomial \( y_2^1 y_2^2 \) is the image of \( \frac{\partial^4}{\partial x_1^3 \partial x_2^3 \partial x_3 \partial x_6} \).

4. Case C2

As long as \( k < n - m, \dim \mathcal{I}_{n-k}^{\ell - m} \perm_{m,k} \subset \ell - m - k \cdot S^m W \), so

\[
(5) \quad \dim \mathcal{I}_{n-k+\tau}^{\ell - m} \perm_{m,k} \leq \binom{n^2 + m + \tau - 1}{m + \tau}.
\]

By Corollary 2.4, it will be sufficient to show that

\[
(6) \quad \dim \mathcal{I}_{n-k}^{\det_{n,k}} = \binom{n}{k}^2 \geq \dim S^m W = \binom{n^2 + m - 1}{m}.
\]

In the range \( 2m \leq k \leq n - 2m \), the quantity \( \binom{n}{k}^2 \) is minimized at \( k = 2m \) and \( k = n - 2m \), so it is enough to show that

\[
(7) \quad \binom{n}{2m}^2 \geq \binom{n^2 + m - 1}{m}.
\]

Using (1)

\[
\ln \left( \frac{n}{2m} \right)^2 = 2\left[ n \ln(n) - 2m \ln(2m) - (n - 2m) \ln(n - 2m) - \Theta(\ln(n)) \right]
\]

\[
= 2\left[ n \ln\left( \frac{n}{n - 2m} \right) + 2m \ln\left( \frac{n - 2m}{2m} \right) \right] - \Theta(\ln(n))
\]

\[
= -4m + m \ln\left( \frac{n}{2m} - 1 \right)^4 - \Theta(\ln(n)),
\]

where to obtain the last line we used \( (1 - \frac{2m}{n})^n > e^{-2m} e^{\Theta(\frac{m^2}{n})} \), and

\[
\ln \left( \frac{n^2 + m - 1}{m} \right) = (n^2 + m - 1) \ln(n^2 + m - 1) - m \ln(m) - (n^2 - 1) \ln(n^2 - 1) - \Theta(\ln(n))
\]

\[
= (n^2 - 1) \ln\left( \frac{n^2 + m - 1}{n^2 - 1} \right) + m \ln\left( \frac{n^2 + m - 1}{m} \right) - \Theta(\ln(n))
\]

\[
= m \ln\left( \frac{n^2}{m} - \frac{m - 1}{m} \right) + m - \Theta(\ln(n)).
\]

So (7) will hold when \( 4m \ln\left( \frac{n}{2m} - 1 \right)^4 > \ln\left( \frac{n^2}{m} - \frac{m - 1}{m} \right) \) which holds for all sufficiently large \( m \) when \( n > m^2 \).
Here we simply degenerate \( \det_n \) to \( R = \ell_1^n + \ell_2^n \) by e.g., setting all diagonal elements to \( \ell_1 \), all the sub-diagonal elements to \( \ell_2 \) as well as the \((1,n)\)-entry, and setting all other elements of the matrix to zero. Then \( \mathcal{I}^{R,k}_{n-k} = \text{span}\{\ell_1^{n-k}, \ell_2^{n-k}\} \). In degree \( n-k+\tau \), this ideal consists of all polynomials of the form \( \ell_1^{n-k}Q_1 + \ell_2^{n-k}Q_2 \) with \( Q_1, Q_2 \in S^\tau \mathbb{C}^n \), which has dimension \( 2\dim S^\tau \mathbb{C}^n - \dim S^{\tau-(n-k)} \mathbb{C}^n \) because the polynomials of the form \( \ell_1^{n-k} \ell_2^{n-k} Q_3 \) with \( Q_3 \in S^{\tau-(n-k)} \mathbb{C}^n \) appear in both terms. By this discussion, or simply because this is a complete intersection ideal, we have

\[
\dim \mathcal{I}^{R,k}_{n-k+\tau} = 2 \left( \frac{n^2 + \tau - 1}{\tau} \right) - \left( \frac{n^2 + \tau - (n-k) - 1}{\tau - (n-k)} \right).
\]

We again use the estimate (5) from Case C2, so we need to show

\[ 2 \left( \frac{n^2 + \tau - 1}{\tau} \right) - \left( \frac{n^2 + \tau + m - 1}{\tau + m} \right) - \left( \frac{n^2 + \tau - (n-k) - 1}{\tau - (n-k)} \right) > 0. \]

Divide by \( \left( \frac{n^2 + \tau - 1}{\tau} \right) \). We need

\[
2 > \Pi_{j=1}^m \frac{n^2 + \tau + m - j}{\tau + m - j} + \Pi_{j=1}^{n-k} \frac{\tau - j}{n^2 + \tau - j}
\]

\[
= \Pi_{j=1}^m \left( 1 + \frac{n^2}{\tau + m - j} \right) + \Pi_{j=1}^{n-k} \left( 1 - \frac{n^2}{n^2 + \tau - j} \right)
\]

The second line is less than

\[
(1 + \frac{n^2}{\tau})^m + (1 - \frac{n^2}{n^2 + \tau - 1})^{n-k}.
\]

We analyze (11) as a function of \( \tau \). Write \( \tau = n^2 \delta \), for some constant \( \delta \). Then (11) is bounded above by

\[
e^{\frac{1}{3} + \frac{2}{3} \cdot \frac{m}{n^2} \delta}.
\]

The second term goes to zero for large \( m \), so we just need the first term to be less than 2, so we take, e.g. \( \delta = \frac{3}{2} \).

6. Case C4

We use a lower bound on \( \mathcal{I}^{\det,n,k}_{n-k+\tau} \) from [7]: Given a polynomial \( f \) given in coordinates, its \textit{leading monomial} in some order, is the monomial in its expression that is highest in the order. If an ideal is generated by \( f_1, \ldots, f_q \) in degree \( n-k \), then in degree \( n-k+\tau \), its dimension is at least the number of monomials in degree \( n-k+\tau \) that contain a leading monomial from one of the \( f_j \).

If we order the variables in \( \mathbb{C}^n \) by \( x_1^n > x_2^n > \cdots > x_1^n > x_2^n > \cdots > x_n^n \), then the leading monomial of any minor is the product of the elements on the principal diagonal. Even this is difficult to estimate, so in [7] they restrict further to only look at leading monomials among the variables on the diagonal and super diagonal: \( \{x_1^n, \ldots, x_n^n, x_1^n, x_2^n, \ldots, x_{n-1}^{n-1}\} \). Among these, they compute that the number of leading monomials of degree \( n-k \) is \( \binom{n+k}{2k} \). Then then show that in degree \( n-k+\tau \) the dimension of this ideal is bounded below by \( \binom{n+k}{2k} \left( \frac{n^2 + \tau - 2k}{\tau} \right) \) so we conclude

\[
\dim \mathcal{I}^{\det,n,k}_{n-k+\tau} \geq \binom{n+k}{2k} \left( \frac{n^2 + \tau - 2k}{\tau} \right).
\]
We compare this with the very crude estimate

$$\dim I_{n-k+\tau}^{\ell_{n-m} \text{perm}_m} \leq \sum_{j=0}^{k} \binom{m}{j}^2 \left( \frac{n^2 + \tau - 1}{\tau} \right),$$

where \( \sum_{j=0}^{k} \binom{m}{j}^2 \) is the dimension of the space of partials of order \( k \) of \( \ell_{n-m} \text{perm}_m \), and the \( \left( \frac{n^2 + \tau}{\tau} \right) \) is what one would have if there were no syzygies (relations among the products).

We have

(13) \[ \ln \left( \frac{n + k}{2k} \right) = n \ln \frac{n + k}{n - k} + k \ln \frac{n^2 - k^2}{4k^2} + \Theta(\ln(n)) \]

\[ = k \ln \frac{n^2 - k^2}{4k^2} + \Theta(\ln(n)) \]

(14) \[ \ln \left( \frac{n^2 + \tau - 2k}{n^2 + \tau - 1} \right) = n^2 \ln \left( \frac{n^2 + \tau - 2k}{n^2 - 2k} \right) \frac{n^2 - 1}{n^2 + \tau - 1} \]

\[ + \tau \ln \frac{n^2 + \tau - 2k}{n^2 + \tau - 1} + 2k \ln \frac{n^2 - 2k}{n^2 + \tau - 2k} + \Theta(\ln(n)) \]

\[ = -2k \ln \left( \frac{\tau}{n^2 + 1} \right) + \Theta(\ln(n)), \]

where the second lines of expressions (13),(14) hold because \( k < 2m \). We split this into two sub-cases: \( k \geq \frac{m}{2} \) and \( k < \frac{m}{2} \).

6.1. **Subcase** \( k \geq \frac{m}{2} \). In this case we have \( \sum_{j=0}^{k} \binom{m}{j}^2 < \binom{2m}{m} \). We show the ratio

(15) \[ \frac{n^2 + \tau - 2k}{n^2 + \tau - 1} \]

is greater than one. Now

(16) \[ \ln \left( \frac{2m}{m} \right) = m \ln 4 + \Theta(\ln(m)). \]

\[ = k \ln 4 \frac{m}{4} + \Theta(\ln(m)). \]

Then (15) is greater than one if

\[ k \ln \left( \frac{n^2 - k^2}{4k^2} \left( \frac{\tau}{n^2 + 1} \right) \frac{1}{4} \right) + \Theta(\ln(n)) \]

is positive. This will occur if

\[ \frac{n^2 - k^2}{4k^2} \left( \frac{\tau}{n^2 + 1} \right) \frac{1}{4} > 1 \]

i.e., if

\[ \tau < n^2 \left( \frac{\sqrt{n^2 - k^2}}{2k4^{\frac{m}{2}}} - 1 \right). \]

Write this as

(17) \[ \tau < n^2 \left( \frac{n}{2\epsilon m4^{\frac{m}{2}}} - 1 \right). \]

The worst case is \( \epsilon = 2 \) where it suffices to take \( \tau < \frac{n^4}{6m} \).
6.2. **Subcase** $k < \frac{m}{2}$. Here we use that $\sum_{j=0}^{k} \binom{m}{j}^2 < k \binom{m}{k}^2$ and the same argument gives that it suffices to have

$$\tau < n^2 \left( \frac{\sqrt{n^2 - k^2}}{2k} - \frac{1}{\sqrt{k} - 1} \right).$$

The worst case is $k = \frac{m}{2}$, where the estimate easily holds when $\tau < \frac{n^3}{6m}$.

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