The epic journey from Kepler’s laws to Newton’s law of universal gravitation revisited

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1 Introduction

Historically, three outstanding achievements in the 17th century heralded the grand opening of modern science, namely, the discovery of Kepler’s laws on planet-motions; the study of gravitation force and the formulation of basic principles of mechanics by Galileo, and the mathematical analysis of the above two that enabled Newton to achieve the grand synthesis that naturally led to his far-reaching generalization: The Law of Universal Gravitation. The publication of Philosophiae Naturalis Principia Mathematica in 1687 not only created modern science, but also started a profound revolution on our understanding of the universe (i.e. the civilization of rational mind). In summary, the major results of the mathematical analysis in Newton’s Principia can be stated as the following four theorems, namely

*Theorem A*: Kepler’s second law holds (i.e. $\frac{dA}{dt} = \text{constant}$) if and only if the acceleration (resp. force) is pointing toward the center (i.e. centripetal).

*Theorem I*: The Kepler’s first and second laws imply that the acceleration vector $a$ is pointing toward the sun and with its magnitude equal to

$$\frac{\pi^2 (2a)^3}{2 \ T^2 \ r^2},$$

namely

$$a = \frac{\pi^2 (2a)^3}{2 \ T^2 \ r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} \quad (1)$$

*Theorem II*: (The uniqueness theorem and the converse of Theorem I) Suppose that the acceleration vector is centripetal and with its magnitude inversely proportional to the square of distance, namely

$$a = \frac{K}{r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} \quad (2)$$

Then, the motion satisfies Kepler’s second law and its orbit is a conic section.

*Theorem III*: The gravitation force of a thin spherical shell with uniform (area-wise)}
density exerting on an outside particle \( P \) is equal to

\[ G \frac{Mm}{OP^2} \]  

(3)

where \( M \) (resp. \( m \)) is the total mass of the spherical shell (resp. the mass of the particle at \( P \)) and \( OP \) is the distance between the center \( O \) and \( P \). [The following Theorem III’ is an immediate corollary of Theorem III.]

\textit{Theorem III’}: Let \( \Sigma_1 \) and \( \Sigma_2 \) be a pair of spherical bodies with radially uniform densities (i.e. each of them can be decomposed into the union of thin spherical shells of Theorem III). Then the magnitude of the (total) gravitation force between them is equal to

\[ G \frac{M_1M_2}{O_1O_2^2} \]  

(4)

where \( M_1 \) (resp. \( M_2 \)) are the total masses of \( \Sigma_1 \) (resp. \( \Sigma_2 \)) and \( O_1 \) (resp. \( O_2 \)) are their centers.

However, the mathematical analysis (mainly geometrical) that Newton gave in Principia are rather difficult to understand, although most steps are quite elementary. Currently (i.e. Fall of 2007), the authors are giving a course at National Taiwan University, Taipei entitled ”selected topics on mathematics and civilization.” This paper is an outcome of our preparation of lectures on this topic. In order to make such a tour of revisiting Newton’s epic journey enjoyable we tried to provide alternative proofs of Theorems I, II, III that are elementary, simple and with clean-cut ideas (cf. §3 and §4). We hope the new proofs of §3 and §4 will make the revisiting of this epic journey also understandable, enjoyable, thus inspiring for earnest young students.

2 Kepler’s laws of planet-motions

2.1 Some remarks on the historical background of ”pre-Kepler” astronomy

(1) The fascinating puzzle of planet motions:

Ever since remote ancient times, most civilizations noticed the strange behaviors
of five prominent stars, each of them, wandering among the background of all the other "fixed" stars, each in their unique patterns and with their individual periods. They are called planets (i.e. wanderers) in Greek time, and nowadays called Venus, Mercury, Mars, Jupiter and Saturn. In ancient astronomy of many prominent civilizations, the study of planet motions was naturally the central topic but it remained to be a fascinating puzzle up until the discovery of Kepler’s laws of planet-motions [K-2,K-3].

(2) Ptolemy and Copernicus:

Among various models on planet-motions of the pre-Kepler era which enable us to provide more or less self-consistent explainations of astronomical observations on planet-motions, the Ptolemy model [Pt] and the Copernicus model [Co] are certainly the most outstanding two. We shall only mention here the following two points on their main features, namely.

(i) The Ptolemy model puts the earth at the center (i.e. geocentric), while the Copernicus model puts the sun at the center (i.e. heliocentric.)

(ii) Both models use the method of epicycles to achieve a reasonably adequate fitting to astronomical observations, which were not that accurate anyway.

(3) Tycho de Brahe:

Tycho de Brahe (1546-1601) was a Danish nobleman who was destined to devote his entire life to astronomical observation. His astronomic interest was inspired by the solar eclipse of 1560 (i.e. by the predictability of astronomical events); while the occurrence of a conjunction of Jupiter and Saturn in 1563 led him to realize the lack of accurate astronomical data, which further inspired him to upgrade the accuracy of his astronomical intruments and observations. Anyhow, his striving for accuracy made him well-prepared for the big event of the discovery of a Nova on November 11, 1572; and his book "De nova stella" made him a leading astronomer of the entire Europe, a pride of the Kingdom of Denmark, and earned
him the patronage of Denmark’s king, Frederick II. Frederick gave him the island of Hveen, on which he built the observatory Uraniborg with the financial support of the king and carried out nightly observations for more than twenty years, thus accumulating a treasure of astronomical data on planet motions that the Kepler’s monumental achievement was based upon.

2.2 Johannes Kepler (1571-1630) and Kepler’s three laws of planet-motions

(1) Kepler was one of the many children of a poor family. Young Kepler won a sequence of scholarships that enabled him to attend the University of Tübingen, where he learned the Copernicus system from Michael Mästlin and became a firm believer of the heliocentric theory. He was preparing himself for a career as a Lutheran minister, however, fate intervened to change his destiny. The sudden death of the mathematics teacher of a high school at Gratz and the recommendation of Tübingen Faculty for the substitute of such a post teaching both mathematics and astronomy, thus starting Kepler’s life-time pursuit in astronomy. Of course, we should mention another major intervention of fate that inspired him to embark his life-long journey in the search of laws of planet-motions.

(2) Mysterium Cosmographicum (1596):

In the Copernicus system, there are altogether six planets revolving around the sun. To the pious young Kepler, such a system is a perfect creation of God, the fact that there are exactly six planets (although it is not the case nowadays) must have its profound reason. Anyhow, this underlying ”profound reason” was one of the mysteries of the universe that the young teacher was earnestly searching for. According to Kepler himself, a wonderful revelation occurred to him on July 19th of 1595, namely, the reason must be that there are exactly five Platonic solids (i.e. regular polyhedra) and each of them is placed between the six ”orbital spheres” such that it is the inscribing (resp. circumscribing) polyhedra of one of the six ”orbital spheres”. Thus, such a wonderful geometric structure not only explains
why there are exactly six planets, but it also determines the ratios among the 
radii of the six orbital spheres. This is the origin of Kepler’s first book and 
he devoted his entire life to study planet motions in order to verify his ”wild 
conjecture.”

(3) Astronomia Nova (1609):

Kepler, of course, sent a copy of the above book to Tycho de Brahe, and most 
likely, such a master of astronomy would dismiss such a ”wild conjecture” merely 
as a youthful fantasy. However, he was impressed by the keen intelligence and 
bold originality of this young astronomer. By the time of 1600, Tycho de Brahe 
needed the mathematical talent of young Kepler to ”understand” his life-time 
astronomical observations, while Kepler needed the access to Tycho’s treasure 
of astronomical data to verify his grand ”mystery of universe”. On Jan 1st of 
1600, Kepler set out to join Tycho de Brahe in Prague to be his assistant, up 
until the death of Tycho de Brahe in October of 1601. Subsequently, Kepler 
succeeded Tycho de Brahe to be the imperial mathematician and got hold of the 
superb Tychonic data. The eighteen months of conjunction between the two out-
standing astronomers was actually a personality mismatch, but it miraculously 
accomplished one of the greatest ”relay” in the history of sciences. It took many 
years of Kepler’s superhuman endeavors and superb mathematical talent, only 
after many setbacks, twists and turns and with tremendous perseverance and 
ingenuity, he finally discovered the first law and the second law on the motion of 
Mars [K-2], namely.

*The first law:* Mars moves on an elliptical orbit with the sun situated at one of 
its foci.

*The second law:* The area sweeping across by the interval joining the Mars to-
ward the sun per unit time is a constant, as indicated in Figure 1.
The above two empirical laws, based upon in depth mathematical analysis of the observation data of Tycho de Brahe, heralded the grand opening of modern astronomy.

(4) Harmonica Mundi (1619):

It took Kepler another decade of hard work to verify that the same kind of first law and second law also hold for the other five planets, and moreover, he discovered the following remarkable third law, namely

\[
\frac{(2a)^3}{T^2},
\]

The third law: The ratio between the cube of the major axis and the square of period, i.e. \(\frac{(2a)^3}{T^2}\), remains the same for all the six planets. This was, indeed, a wonderful reward for Kepler’s life-long search of a kind of harmony among planetary orbits. His youthful fantasy was somehow vindicated.

3 On the mathematical analysis of Kepler’s laws

Let us begin with the second law on planet motions, whose discovery actually precedes that of the first law [K-2], and then proceed to the mathematical analysis of both the first and the second laws jointly. As it has been discussed in §1, such a journey was the monumental contribution of Newton (cf. Principia). However, the proof of the latter given in Principia is quite involved and rather difficult to grasp (i.e. understand) his original insight that led to such a proof. In this section, we shall present two simple and
straightforward alternatives of the latter whose underlying geometric ideas are rather clear.

3.1 Mathematical analysis of Kepler’s second law

This is the easy part of the journey but it is a ”good beginning” of basic importance. Historically, the discovery of the second law (cf. [K-2]) not only preceded that of the first law, but it also provided the crucial and advantageous stepping-stone that eventually led him to the discovery of the first law. Moreover, in essentially the same way, the understanding of the mathematical as well as the physical meaning of Kepler’s second law was also the good beginning for Newton’s journey. Anyhow, this step is very simple, straightforward but very important.

(1) First of all, the second law is, by itself, local in nature. Let \((r, \theta)\) be the polar coordinates of \(P\) (i.e., the position of the planet) with the position of the sun as the origin. Then, the second law of Kepler simply asserts that

\[
\frac{dA}{dt} = \frac{1}{2} r^{2} \omega, \quad \omega = \frac{d\theta}{dt} \quad \text{(angular velocity)}
\]

is equal to a constant \(k\), which is in fact equal to the total area divided by the period \(T\), namely.

\[
\text{total area} = \int_{0}^{T} dA = \int_{0}^{T} k \, dt = k \cdot T
\]

Hence, in conjunction with the first law, one has the following powerful simple equation

\[
r^{2} \omega = \frac{2\pi ab}{T} (= 2k)
\]

(2) Let \(v\) (resp. \(n\)) be the velocity (resp. upward unit normal) vector. Then, as indicated in Figure 2.

\[
\overrightarrow{OP} \times v = 2kn
\]

and hence

\[
\frac{d}{dt}(\overrightarrow{OP} \times v) = v \times v + \overrightarrow{OP} \times a = 0, \quad \left( a = \frac{dv}{dt} \right)
\]

which implies that the acceleration vector \(a\) is collinear with \(\overrightarrow{OP}\), namely, the force, \(ma\), is centripetal.
3.2 Mathematical analysis of Kepler’s 1st and 2nd laws

In this subsection, we shall present two alternative proofs of one of the major result of Newton’s Principia, namely.

Theorem I: Let \( \mathbf{a} \) be the acceleration vector of a planet motion as described by the 1st and the 2nd laws of Kepler. Then \( \mathbf{a} \) is always pointing towards the sun and with its magnitude equal to 
\[
\frac{\pi^2 (2a)^3}{2 T^2} \times \text{times of the inverse of the square of distance between the planet and the sun},
\]

\[
|\mathbf{a}| = \frac{\pi^2 (2a)^3}{2 T^2} \frac{1}{r^2} \quad (\ast)
\]

where \( r \) is the distance, \( T \) is the period and \( 2a \) is the length of the major axis of its ellipse-orbit.

Remark: The direction of \( \mathbf{a} \) is always pointing toward the sun; this is exactly the kinematical significance of Kepler’s 2nd law (cf. §3.1). Thus, it suffices to prove the second assertion (i.e. (\ast)) on the magnitude of \( \mathbf{a} \).

First proof of (\ast): As indicated in Figure 3, \( \mathbf{v} \) (resp. \( \mathbf{a} \)) is the velocity (resp. acceleration) vector at \( P \), \( \{F'_1, F'_2\} \) are the reflectionally symmetric points of \( \{F_1, F_2\} \) with respect to the tangent line \( l_p \) and \( \varepsilon = \angle M_1 PF_1 \).

Set \( d_1, d_2 \) to be the distances of \( F_1, F_2 \) toward \( l_p \) and \( h \) to be the height of \( \triangle F_1F'_1F'_2 \).
Then
\[ 4a^2 = \overrightarrow{F_1F_2}^2 = \overrightarrow{F_1H}^2 + \overrightarrow{HF_2}^2 = (d_1 + d_2)^2 + h^2 \]
\[ 4c^2 = \overrightarrow{F_1'F_2'}^2 = \overrightarrow{F_1'H}^2 + \overrightarrow{HF_2'}^2 = (d_1 - d_2)^2 + h^2 \] (10)

Hence
\[ 4b^2 = 4a^2 - 4c^2 = (d_1 + d_2)^2 - (d_1 - d_2)^2 = 4d_1d_2. \] (11)

On the other hand, Kepler’s second law asserts that
\[ |\mathbf{v}|r \sin \varepsilon = \frac{2\pi ab}{T}, \quad r \sin \varepsilon = d_1 = \frac{b^2}{d_1} \]
\[ \Rightarrow |\mathbf{v}| = \frac{2\pi ab}{T d_1} = \frac{2\pi a^2}{bT} \overrightarrow{F_2M_2} \] (12)

Note that
\[ \overrightarrow{F_2M_2} = \overrightarrow{F_2O} + \overrightarrow{OM_2} = \begin{pmatrix} -c \\ 0 \end{pmatrix} + a \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \] (13)

Therefore
\[ \mathbf{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{2\pi a}{bT} \overrightarrow{F_2M_2} = \begin{pmatrix} 0 \\ -c \end{pmatrix} + \frac{2\pi a^2}{bT} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \] (14)

and hence
\[ \mathbf{a} = \frac{d}{dt} \mathbf{v} = \frac{d}{dt} \mathbf{v} \cdot \frac{d}{dt} = \frac{2\pi a^2}{bT} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} + \frac{2\pi ab}{T} \cdot \frac{1}{r^2} \] (15)

\[ = \frac{\pi^2}{2} \cdot \frac{(2a)^3}{T^2} \cdot \frac{1}{r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} \]
Remarks:

(i) The above clean-cut simple proof also reveals the kinematical meaning of the 3rd law.

(ii) In retrospect, the equation (11) already provides a "hand and glove fitting" between the 2nd law and the ellipticity. Thus, it becomes very easy to deduce the simple formula (14) of $v$, from which the formula of $a$ (i.e. (15)) follows immediately.

Second proof of ($\ast$):

As indicated in Figure 4, $(x, y)$ (resp. $(r, \theta)$) are the Cartesian (resp. polar) coordinates of $P$ with the origin $O$ (resp. $O'$) situated at its symmetric center (resp. one of its foci). Thus

$$x = r \cos \theta - c, \ y = s \sin \theta$$

and it is quite simple to deduce the following polar coordinate equation, namely

$$\frac{1}{r} = \frac{a - c \cos \theta}{b^2}$$

First of all, it follows from the 2nd law that $a$ is always pointing toward $O'$, namely

$$a = (\ddot{r} - r \omega^2) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \ \omega = \frac{d\theta}{dt} \text{ (angular velocity)}$$

Thus, it suffices to show that

$$r^2(\ddot{r} - r \omega^2) = -\frac{\pi^2 (2a)^3}{2T^2}$$

($\ast'$)
The following is a rather straightforward sequence of simple computations that deduces \((\star')\) with the repeated help of the 2nd law, namely

\[
\frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{\text{total area}}{T} \quad (= \frac{\pi ab}{T}) \tag{7'}
\]

Step 1 : By the differentiation of (17) with respect to time, one has

\[
-\frac{\dot{r}}{r^2} = \frac{c}{b^2} \sin \theta \cdot \omega \tag{19}
\]

Thus

\[
\dot{r} = -\frac{c}{b^2} \sin \theta (r^2 \omega) = -\frac{2\pi ac}{bT} \sin \theta \tag{19'}
\]

Step 2 : By the differentiation of (19'), one has

\[
\ddot{r} = -\frac{2\pi ac}{bT} \cos \theta \omega \tag{20}
\]

Hence, again using (7')

\[
r^2 \ddot{r} = -\frac{2\pi ac}{bT} \cos \theta (r^2 \omega) = -\frac{4\pi^2 a^2}{T^2} c \cos \theta \tag{21}
\]

Step 3 : By (7') and (17), one has

\[
r^2 (-r \omega^2) = -\frac{1}{r} (r^2 \omega)^2 = -\frac{1}{r} \frac{4\pi^2 a^2 b^2}{T^2}
\]

\[
= -\frac{4\pi^2 a^3}{T^2} + \frac{4\pi^2 a^2}{T^2} c \cos \theta \tag{22}
\]

thus proving

\[
r^2 (\ddot{r} - r \omega^2) = \frac{\pi^2}{2} \frac{(2a)^3}{T^2} \tag{\star'}
\]


Remarks:

(i) In comparison between the above two proofs, the first proof is more geometrical, while the second proof is more computational in nature; and both of them are elementary, clean-cut and very simple.

(ii) In the second proof, one uses the 2nd law four times (namely, in obtaining (18), (19'), (21) and (22)) which enable us to simplify the computations at each step, thus making the computations altogether rather straightforward, elementary and
simple. Of course one still needs to differentiate twice in order to compute the term of $r^2\ddot{r}$ in $(\ast')$. However, the above proof only uses the analytical-geometric fact that

$$\frac{d}{d\theta} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

### 3.3 On the extension of Theorem I to the other two types of conic sections

Historically, Greek geometers first studied plane sections of right circular cylinders, and discovered the remarkable characteristic property of such curves (i.e. ellipses) of having a pair of foci with $F_1P + PF_2$ equal to a constant. Later, they discovered that the geometric proofs could be extended to prove similar results for plane sections of right circular cones, which include two more type of curves, namely hyperbola and parabola. Anyhow, it is again quite natural to seek generalizations of Theorem I for centripetal motions with the other two types of conic sections as their orbits, although such motions can hardly be observed in celestial events simply because they are no longer periodic!

**Theorem I’**: Let $\mathbf{a}$ be the acceleration vector of a centripetal motion with the branch of hyperbola, as indicated in Figure 5, as the orbit. Then

$$\mathbf{a} = \frac{(2k)^2 a}{b^2} \cdot \frac{1}{r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix}$$

(23)

where $2k$ is the constant value of $r^2\frac{d\theta}{dt}$.

**Proof:** As indicated in Figure 5, let $\{F_i^\prime\}$ be the reflection-image of $\{F_i\}$ and $d_i$ be the distances between $\{F_i, F_i^\prime\}$ and the tangent line $l_p$. Then

$$F_1F_2^2 = 4c^2 = (d_1 + d_2)^2 + h^2, \quad F_1F_2^\prime = 4a^2 = (d_2 - d_1)^2 + h^2$$

(24)

$$\Rightarrow d_1d_2 = c^2 - a^2 = b^2$$

On the other hand, by the centripetality

$$r^2\frac{d\theta}{dt} = 2\frac{dA}{dt} = 2k = |\mathbf{v}|r \sin \varepsilon = |\mathbf{v}| \cdot d_1$$

(25)
Therefore
\[ |\mathbf{v}| = \frac{2k}{d_1} = \frac{2k}{b^2} d_2 = \frac{2k}{b^2} \overrightarrow{M_2F_2}, \quad \angle(\overrightarrow{M_2F_2}, \mathbf{v}) = \frac{\pi}{2} \quad (26) \]

Hence
\[
\mathbf{v} = \frac{2k}{b^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\overrightarrow{M_2O} + \overrightarrow{OF_2}) = \frac{2ka}{b^2} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} + \frac{2k}{b^2} \begin{pmatrix} 0 \\ c \end{pmatrix} \quad (27)
\]

Thus having, by (25) and (27)
\[
a = \frac{d\mathbf{v}}{d\theta} \frac{d\theta}{dt} = \frac{(2k)^2a}{b^2} \begin{pmatrix} 1 & 1 \\ -\cos \theta & -\sin \theta \end{pmatrix} \quad (28)
\]

**Remark:** The same kind of proof will show that such a motion along the other branch of hyperbola also satisfies the inverse square law, except it will be repulsive instead of attractive. In fact, such motions occur naturally in the scattering theory [Fa].

**Theorem II**: Let \( \mathbf{a} \) be the acceleration vector of a centripetal motion with the parabola, as indicated in Figure 6, as its orbit. Then
\[
a = \frac{2k^2}{p} \frac{1}{r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} \quad (29)
\]
**Proof:** By the centripetality, one has
\[ r^2 \frac{d\theta}{dt} = 2 \frac{dA}{dt} = 2k = |\mathbf{v}| r \sin \varepsilon = |\mathbf{v}| \cdot d \]

On the other hand, it is easy to see that
\[ \theta + \varepsilon = \theta + \left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\pi}{2} + \frac{\theta}{2} \]
\[ d = r \cos \frac{\theta}{2}, \quad p = d \cos \frac{\theta}{2} = r \cos^2 \frac{\theta}{2} \]

Therefore,
\[ |\mathbf{v}| = \frac{2k}{r \cos \frac{\theta}{2}} = \frac{2k}{p} \cos \frac{\theta}{2} \]

\[ \mathbf{v} = \frac{2k}{p} \cos \frac{\theta}{2} \left( \begin{array}{c} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{array} \right) = \frac{k}{p} \left( \begin{array}{c} -\sin \theta \\ \cos \theta \end{array} \right) + \left( \begin{array}{c} 0 \\ \frac{k}{p} \end{array} \right) \]

Hence
\[ \mathbf{a} = \frac{d\mathbf{v}}{d\theta} \cdot \frac{d\theta}{dt} = \frac{2k^2}{p} \frac{1}{r^2} \left( \begin{array}{c} -\cos \theta \\ -\sin \theta \end{array} \right) \]

\[ \square \]

**Remark:** By a simple coordinate transformation of \((r, \theta)\) to \((r, \tilde{\theta})\) with \(\theta = \tilde{\theta} + \pi\), the equation (17) of ellipse becomes
\[ \frac{1}{r} = \frac{a(1 + e \cos \tilde{\theta})}{b^2}, \quad ae = c \] \hspace{1cm} (17’)

It is straightforward to check that, using the above polar coordinate equation for hyperbola (resp. parabola), i.e. for the cases of \(e > 1\) (resp. \(e = 1\)), the second proof
of Theorem I automatically extend to that of Theorem I’ and Theorem I”, again by straightforward differentiation and stepwise applications of \( r^2 \omega = 2k \) (cf. §3.2). Thus, the second proof of Theorem I actually also provides a proof of both Theorem I’ and Theorem I” without modification, while the first proof of Theorem I can also be extended to similar proofs of Theorem I’ and Theorem I” with some simple modifications, as above.

### 3.4 The uniqueness theorem and the converse of Theorem I

**Theorem II:** Suppose that
\[
a = \frac{K}{r^2} \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix}, \quad K > 0 \quad (\ast')
\]

Then the motion satisfies Kepler’s second law and its orbit is a conic section with the center as one of its foci.

**Proof:** Centripetality implies that there exists a constant \( k \) such that
\[
|v| \cdot r \sin \varepsilon = 2k, \quad r^2 \frac{d\theta}{dt} = 2k
\]

Therefore, it follows directly from (\ast') and (33) that
\[
\frac{d}{d\theta} v(\theta) = a(\theta) \frac{dt}{d\theta} = \frac{K}{2k} \frac{d}{d\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}
\]

Hence, there exists a constant vector \( c \) such that
\[
v(\theta) = \frac{K}{2k} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} + c
\]

Without loss of generality, we may assume that \( r(\theta) \) is minimal at \( \theta = 0 \) and \( v(0) \) is pointing upward, as indicated in Figure 7.

![Figure 7](image-url)
Thus \( c \) is also pointing upward and

\[
\mathbf{v}(\theta) = \frac{K}{2k} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} + \begin{pmatrix} 0 \\ |c| \end{pmatrix}
\]  

(36)

Now, again by the second law, we have

\[
\begin{vmatrix}
\cos \theta & \frac{K}{2k}(-\sin \theta) \\
\sin \theta & \frac{K}{2k} \cos \theta + |c|
\end{vmatrix} = 2k
\]

(37)

Namely

\[
\frac{1}{r} = \frac{K}{(2k)^2}(1 + e \cos \theta), \quad e = \frac{2k|c|}{K}
\]

(38)

This is exactly the polar coordinate equation of a conic section with \( e \) as its eccentricity!

\( \Box \)

Remarks:

(i) Note that it is again the second law that plays the important role in the above very simple straightforward proof. Conceptually, the second law is actually the conservation law of angular momentum, which is rooted in the rotational symmetry of the space.

(ii) In the case of elliptical orbit, one has (cf. (7'))

\[
2k = \frac{2\pi ab}{T}, \quad \frac{K}{2k} = \frac{2\pi a^2}{bT}, \quad |c| = \frac{2\pi ac}{bT}
\]

(39)

thus having

\[
\frac{K}{(2k)^2} = \frac{a}{b^2}, \quad \frac{2k|c|}{K} = e = \frac{c}{a}
\]

(40)

4 The gravitation force of a body with spherically symmetric density exerting on an outside particle

In this section, we shall present an alternative proof of the following theorem which plays a fundamental, decisive rôle in Newton’s discovery of the law of universal gravitation (cf. Principia and the chapter 15 on “the superb theorems” in [Ch].)

Theorem III': The gravitation force of a body with spherically symmetric density and
total mass of M exerting on an outside particle of mass m is equal to that of a particle of mass M situated at its center.

Remark: It is easy to see that the proof of Theorem III′ can be reduced to that of the special case of a thin spherical shell with uniform (area-wise) density, namely.

**Theorem III**: Let Σ be a thin spherical shell with uniform (area-wise) density ρ, radius R and P be an outside particle of mass m. Then the total gravitation force of Σ exerting on P is equal to

\[ \frac{G M m}{OP^2}, \quad M = 4\pi R^2 \rho \]  \hspace{1cm} (41)

Where \( G \) is the gravitation constant and \( OP \) is the distance between the center of Σ and \( P \).

**Proof**: As indicated in Figure 8, \( OP \cdot OP' = R^2 \). Therefore \( \triangle OPQ \) and \( \triangle OQP' \) have a common angle at \( O \) and

\[ \frac{OP}{OQ} = \frac{OQ}{OP'} \]  \hspace{1cm} (42)

thus having

\[ \triangle OPQ \sim \triangle OQP', \quad \triangle OQP : OQP' = \frac{OP}{OQ} : \frac{OP'}{OQ} \]  \hspace{1cm} (43)

Figure 8

Note that, it follows readily from the rotational symmetry of the whole geometric setting with respect to the line of \( OP \), the total gravitation force of Σ exerting on \( P \)
is clearly in the direction of $\overrightarrow{PO}$. Thus, it suffices to find the total sum of

$$|d\mathbf{F}| \cos \theta = G \frac{\rho dA \cdot m}{QP^2} \cos \theta$$

(44)

Set $d\sigma$ to be the solid angle of the cone with $dA$ as its base and $P'$ as its vertex. Then, as indicated in Figure 9

$$dA \cos \theta = \overrightarrow{P'Q} \cdot d\sigma.$$  

(45)

Therefore

$$|d\mathbf{F}| \cos \theta = G \rho \frac{\overrightarrow{P'Q} \cdot d\sigma}{QP^2} = G \rho \frac{R^2}{OP^2} d\sigma$$

(46)

and hence, the total gravitation force is given by

$$\sum G \rho \frac{R^2}{OP^2} d\sigma = G \rho \frac{R^2}{OP^2} \sum d\sigma = G \frac{4\pi R^2 \rho \cdot m}{OP^2} = G \frac{Mm}{OP^2}$$

(47)

Remarks:

(i) Note that a body with spherically symmetric density can be regarded as the non-overlapping union of concentric thin spherical shells with uniform area-wise densities (depending on their radii). Thus Theorem III’ is a direct corollary of Theorem III. Moreover, the same method of the above proof of Theorem III also shows that the total gravitation force between a pair of such bodies is, in fact, equal to

$$G \frac{M_1 M_2}{O_1 O_2^2}$$

(48)

where $M_1$ and $M_2$ are their total masses and $O_1 O_2$ is the distance between their centers (cf. §5 for a brief discussion on the significance of such ”superb theorems” [Ch].)
(ii) In retrospect, it is the advantageous subdivision of the spherical surface Σ, corresponding to the infinitesimal subdivision of the total solid angle at $P'$, that achieves the wonder of such a remarkably simple proof. Geometrically speaking, only such a subdivision can achieve the full extend usage of the "sphericality" of the geometric situation of the pair \( \{ \Sigma, p \} \).

5 Grand synthesis and far-reaching generalization: Newton’s law of universal gravitation

Historically, we may regard the Copernicus’ book of 1543, ”On the revolutions of the Celestial Spheres” as the grand opening salvo of the scientific revolution, while Newton’s book of 1687, ”Philosophiae Naturalis Principia Mathematica” was the triumphant culmination of such a most important revolution. The crowning achievement of Principia is the Newton’s law of universal gravitation, based upon the mathematical synthesis of Kepler’s laws of planet-motion; and Galileo’s study of gravitation force and the basic principles of mechanics.

As it has already been pointed out in the introduction, the major components that enabled Newton to achieve the grand mathematical synthesis in Principia consists of the four theorems as stated in §1, whose proofs in Principia (cf. [Ar] [Ch]) are elementary, geometrical but difficult to comprehend. Now, with the simple, elementary and clean-cut proofs of §3 and §4 at hand, we hope that such simplifications will make the revisiting of Newton’s epic journey from Kepler’s and Galileo’s laws to the universal gravitation law enjoyable and inspiring for common reader, including earnest high school students. Anyhow, the following are some additional crucial ideas and highlights that may also be helpful for appreciating such a journey:

(1) The discovery of Kepler’s three laws on planet-motions is a monumental milestone of the civilization of rational mind which, for the first time, grasp the wonderful organization of the solar system. However, they are empirical laws based upon the astronomical data of Tycho de Brahe which are, themselves, only with the assured accuracy of up to 2 minutes. Moreover, they are only ”verified” to be
fitting for the six planets of Tycho’s era; whether or not such laws still hold for other yet to be discovered planets is another matter.

(2) Note that the sun and the planets are spherical bodies of huge sizes just by themselves. However, their sizes are comparatively much, much smaller than the distances between them. Therefore, in the mathematical analysis of the underlying reasons of Kepler’s first and second laws (cf. §3), it is still reasonable to regard them as mere points. The proof of Theorem I reveals that the ”physical cause” of planet-motions is a kind of attractive force toward the sun whose magnitude is inversely proportionate to the square of distance. In retrospect, one may regard the elliptical orbits as a beautiful ”hint” of the Nature awaiting to inspire some mathematical mind to the discovery of inverse-square law.

(3) After realizing that heavenly motions of planets are, in fact, governed by this kind of attractive force, one nationally proceeds to investigate whether the gravitation force studied by Galileo, or the force that keeps the moon circulating the earth are also the same kind of force? Here, one needs Theorem III (or III’) in order to compute the total gravitation force of the earth exerting on an earthy object or the moon. Historically, it is the difficulty of proving this theorem that delayed Newton’s publication of universal gravitation law. (cf. [Ar] [Ch, p. 12, 13 and 302]).

(4) The importance of Theorem II lies in the vast applications of the law of universal gravitation, rather than contributing to its discovery. We refer to §15 of [Ar] for a discussion whether Newton actually proved Theorem II in Principia; and to [De] [Ha] [Ma] and [Go] for other proofs of Theorems I and/or II.

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