LARGE DIMENSIONAL HOMOMORPHISM SPACES BETWEEN WEYL MODULES AND SPECHT MODULES

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ABSTRACT. We give a family of pairs of Weyl modules for which the corresponding homomorphism space is at least 2-dimensional. Using this result we show that for fixed parameters $e > 0$ and $p \geq 0$ there exist arbitrarily large homomorphism spaces between pairs of Weyl modules.

1. INTRODUCTION

Let $F$ be a field of characteristic $p \geq 0$. Take $q \in F^\times$ with the property that $1 + q + \ldots + q^{f-1} = 0 \in F$ for some integer $2 \leq f < \infty$ and let $e \geq 2$ be minimal with this property. For $n \geq 0$, we write $H_n = \mathcal{H}_{F,q}(S_n)$ to denote the Hecke algebra of the symmetric group $S_n$ and $\mathcal{S}_n = \mathcal{S}_{F,q}(S_n)$ to denote the corresponding $q$-Schur algebra. For each partition $\mu$ of $n$, we may define a $\mathcal{H}_n$-module $S^\mu$, known as a Specht module, and an $\mathcal{S}_n$-module $\Delta(\mu)$, known as a Weyl module. Recall that if $\mu$ and $\lambda$ are partitions of $n$ then

$$\dim(\text{Hom}_{\mathcal{H}_n}(S^\mu, S^\lambda)) \geq \dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda)))$$

with equality if $q \neq -1$ [2]. Despite much investigation, there are few known examples of Weyl modules $\Delta(\mu)$ and $\Delta(\lambda)$ such that $\dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) > 1$. The first such pairs were recently exhibited by Dodge [3].

Working in the symmetric group algebra and using results of Chuang and Tan [1] on the radical filtrations of Specht modules belonging to Rouquier blocks, he showed that for any $k$ satisfying $k(k+1)/2+1 < p$ there exist partitions $\mu$ and $\lambda$ of some integer $n$ such that $\dim(\text{Hom}_{\mathcal{F}_{avail}}(S^\mu, S^\lambda)) = k$. In particular, for $p \geq 5$ there exist Specht modules, and hence Weyl modules, such that the corresponding homomorphism space is at least 2-dimensional. Using Lemma 1.1 below, Dodge’s result proves the following: Let $F$ be a field of characteristic $p \geq 5$. Then given any integer $l \geq 0$ there exist partitions $\alpha$ and $\beta$ of some integer $m$ such that $\dim(\text{Hom}_{\mathcal{F}_{avail}}(S^\alpha, S^\beta)) \geq l$.

Lemma 1.1. Suppose $\mu$ and $\lambda$ are partitions of an integer $n$ such that $\dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) = k$. Then there exist partitions $\alpha$ and $\beta$ of some integer $m$ such that $\dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\alpha), \Delta(\beta))) = k^2$.

Proof. We may assume $k \geq 1$. If $\mu = (\mu_1, \ldots, \mu_a)$ and $\lambda = (\lambda_1, \ldots, \lambda_b)$ then, since $\dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) \neq 0$, we have $\lambda \succeq \mu$ so that $a \geq b$ and $\lambda_i \geq \mu_i$. Define partitions $\alpha$ and $\beta$ by

$$\alpha = \begin{cases} \mu + \lambda_1, & 1 \leq i \leq a, \\ \mu_{i-a}, & a+1 \leq i \leq 2a, \end{cases}$$

$$\beta = \begin{cases} \lambda_i + \lambda_1, & 1 \leq i \leq a, \\ \lambda_{i-b}, & a+1 \leq i \leq 2a, \end{cases}$$

so that

Then $\dim(\text{Hom}_{\mathcal{S}_n}(\Delta(\alpha), \Delta(\beta))) = k^2$ by the generalized row and column removal theorems [5] Theorem 3.1 or [4] Prop. 10.4.

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Lemma 2.1. The number of entries of $T$.

Let $[0]! = 1$ and for $m \geq 0$ define
$$\binom{m}{j} = \frac{[m]!}{[j]![m-j]!}.$$ For integers $m$ and $j$, if any of the conditions $m \geq j \geq 0$ fail we define $\binom{m}{j} = 0$. Using Lemma 2.4 below or otherwise, it is straightforward to show that $\binom{m}{k}$ is then well-defined for any $m, k \in \mathbb{Z}$, and may be considered as an element of $\mathbb{Z}[q]$.

2. Proof of Theorem 1.2

In this section, we give the proof of the main result. Fix a field $F$ and an element $q \in F^\times$ such that $e = \min\{f \geq 2 \mid 1 + q + \ldots + q^{f-1} = 0\}$ exists. For $n \geq 0$ let $S_n = S_{F,q}(\mathbb{S}_n)$ and $H_n = H_{F,q}(\mathbb{S}_n)$. The characteristic of the field plays no further role in this paper. We first recall a method to determine the dimension of the homomorphism space between a pair of Weyl modules. For full details, we refer the reader to [5] Section 2.2.

2.1. Homomorphism spaces. Fix partitions $\lambda$ and $\mu$ of $n$. For every composition $\nu$ of $n$, we define $m_{\nu} \in H_n$ and a cyclic right $H_n$-module $M^\nu = m_{\nu}H$. Let $T_\nu(\lambda, \mu)$ denote the set of row-standard $\lambda$-tableaux of type $\nu$, with $T_0(\lambda, \nu) \subseteq T_\nu(\lambda, \nu)$ the subset of semistandard tableaux. For each $\lambda \in T_\nu(\lambda, \nu)$ we define an $H_n$-homomorphism $\Theta_\lambda : M^\nu \rightarrow S^\lambda$ such that $\{\Theta_\lambda \mid \lambda \in T_0(\lambda, \nu)\}$ are linearly independent.

Let $\ell(\nu)$ denote the number of parts of any composition $\nu$. For $1 \leq d < \ell(\mu)$ and $1 \leq t \leq \mu_{d+1}$ we define an element $h_{d,t} \in H_n$. Let $\text{EHom}_{H_n}(M^\mu, S^\lambda)$ be the space spanned by $\{\Theta_\lambda \mid \lambda \in T_\nu(\lambda, \nu)\}$ and let
$$\Psi(\mu, \lambda) = \{\Theta \in \text{EHom}_{H_n}(M^\mu, S^\lambda) \mid \Theta(m_{\mu}h_{d,t}) = 0 \text{ for all } 1 \leq d < \ell(\mu), 1 \leq t \leq \mu_{d+1}\}.$$ This definition is motivated by the following result which follows from [5] Theorem 2.2 and the remark following [5] Corollary 2.4.

Lemma 2.1.

$$\Psi(\mu, \lambda) \cong _F \text{Hom}_{S_n}(\Delta(\mu), \Delta(\lambda)).$$

We therefore want to determine $\Psi(\mu, \lambda)$. First we set up some notation. If $\lambda \in T_\nu(\lambda, \nu)$, let $T_j^\nu$ denote the number of entries of $T$ which lie in row $j$ and which are equal to $i$. We extend this definition by setting $T_j^{\nu^*} = \sum_{k \geq i} T_j^k$, and similarly for other definitions. If $m \geq 0$ define
$$[m] = 1 + q + \ldots + q^{m-1} \in F.$$ Let $[0]! = 1$ and for $m \geq 1$, set $[m]! = [m][m-1]!$. If $m \geq j \geq 0$, set
$$\binom{m}{j} = \frac{[m]!}{[j]![m-j]!}.$$
Lemma 2.2 ([5] Proposition 2.7). Suppose that $T \in T_r(\lambda, \mu)$. Choose $d$ with $1 \leq d < \ell(\mu)$ and $t$ with $1 \leq t \leq \mu_{d+1}$. Let $S$ be the set of row-standard tableaux obtained by replacing $t$ of the entries in $T$ which are equal to $d+1$ with $d$. Each tableau $S \in S$ will be of type $\nu(t, d)$ where

$$
\nu(t, d) = \begin{cases} 
\mu_j + t, & j = d, \\
\mu_j - t, & j = d + 1, \\
\mu_j, & \text{otherwise.}
\end{cases}
$$

Recall that $\Theta_T : M^{\mu} \to S^{\lambda}$ and $\Theta_S : M^{\nu(t, d)} \to S^{\lambda}$. Then

$$
\Theta_T(m_{\mu}h_{d, t}) = \sum_{S \in S} \left( \prod_{j=1}^{\ell(\lambda)} q^{T^2_{\nu,j} - T^2_{\nu,j}} \left[ \frac{S^2_j}{T^2_{\nu,j}} \right] \right) \Theta_S(m_{\nu(t, d)}).
$$

Lemma 2.3 ([5] Proposition 2.9). Suppose $\lambda$ is a partition of $n$ and $\nu$ is a composition of $n$. Let $S \in T_r(\lambda, \nu)$. Suppose $1 \leq r \leq \ell(\lambda) - 1$ and that $1 \leq d \leq \ell(\nu)$. Let

$$
\mathcal{G} = \left\{ g = (g_1, g_2, \ldots, g_{\ell(\nu)}) \mid g_{d} = 0, \sum_{i=1}^{\ell(\nu)} g_i = S^d_{r+1} \text{ and } g_i \leq S^i_r \text{ for } 1 \leq i \leq \ell(\nu) \right\}.
$$

For $g \in \mathcal{G}$, let $g_{d-1} = \sum_{i=1}^{d-1} g_i$ and let $U_g$ be the row-standard tableau formed from $S$ by moving all entries equal to $d$ from row $r$ to row $r + 1$ and that $1 \neq d$ moving $g_i$ entries equal to $i$ from row $r$ to row $r + 1$. Then

$$
\Theta_S = (-1)^{S^d_{r+d}} q^{-(d^d_{r+1})} q^{-S^d_{r+d}S^d_{r+d}} \sum_{g \in \mathcal{G}} \prod_{i=1}^{\ell(\nu)} q^{g_i S^i_{r+1}} \left[ \frac{S^i_{r+1} + g_i}{g_i} \right] \Theta_{U_g}.
$$

In the following section, we apply these two lemmas to find elements of $\Psi(\mu, \lambda)$.

Example. Let $e = 2$. Take $\lambda = (7, 5, 3)$ and $\mu = (5, 5, 3, 1, 1)$. We identify a $\lambda$-tableau $T$ of type $\nu \trianglerighteq \mu$ with the image $\Theta_T(m_{\nu}) \in S^\lambda$. Recall that if $\lambda \trianglerighteq \nu$ then $T_0(\lambda, \nu) = \emptyset$ so that we immediately have $\Theta(m_{\mu}h_{1, t}) = 0$ for $t = 3, 4, 5$ and $\Theta(m_{\mu}h_{2, 3}) = 0$.

1. Let $\Theta(m_{\mu}) = \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array}$. Then

$$
\Theta(m_{\mu}h_{4, 1}) = \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = 0,
$$

$$
\Theta(m_{\mu}h_{3, 1}) = \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = 0,
$$

$$
\Theta(m_{\mu}h_{2, 1}) = q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + [5] \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + [5] \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = 0,
$$

$$
\Theta(m_{\mu}h_{2, 2}) = q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + [5] \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + q^4 \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} + [5] \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array} = 0,
$$

2. Let $\Theta(m_{\mu}) = \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array}$. Then
\[\Theta(m_\mu h_{1,1}) = [6] + [4] - q^3[2] = 0,\]
\[\Theta(m_\mu h_{1,2}) = [6] + [3] + q^3[2] = 0,\]
so that \(\Theta \in \Psi(\mu, \lambda).\)

(2) Let
\[\Phi = \Phi_{11} + \Phi_{12} + \Phi_{13} + \Phi_{14} + \Phi_{15} + \Phi_{16}.\]
Then \(\Phi \in \Psi(\mu, \lambda).\)

2.2. Gaussian Polynomials. In order to tell if a homomorphism \(\Theta\) lies in \(\Psi(\mu, \lambda)\) we record some results about the Gaussian polynomials \([\begin{array}{c} m \\ j \end{array}]\). The first is well-known.

**Lemma 2.4.** Suppose \(m, j \geq 0\). Then
\[\begin{bmatrix} m+j \\ j \end{bmatrix} = \begin{bmatrix} m \\ j \end{bmatrix} + q^{j} \begin{bmatrix} m \\ j \end{bmatrix}
= \begin{bmatrix} m \\ j \end{bmatrix} + q^{m-j+1} \begin{bmatrix} m \\ j-1 \end{bmatrix}.\]

**Lemma 2.5** (Lemma 2.6). Suppose \(m, k \geq l \geq 0\). Then,
\[\sum_{j \geq 0} (-1)^j q^j \begin{bmatrix} m-j \\ k \end{bmatrix} = q^{m-k} \begin{bmatrix} m-l \\ k-l \end{bmatrix}.\]

**Lemma 2.6.** Suppose that \(m \geq 0\) and write \(m = m^*e + m'\) where \(0 \leq m' < e\). If \(m' < j \leq e-1\) then \([\begin{array}{c} m \\ j \end{array}] = 0.\)

**Proof.** Write
\[\begin{bmatrix} m \\ j \end{bmatrix} = \frac{m|m-1| \ldots |m-j+1|}{|j| |j-1| \ldots |1|}\]
so that one of the terms in the numerator and none of the terms in the denominator are equal to zero. \(\square\)

The next lemma follows immediately.

**Lemma 2.7.** Suppose \(1 \leq j \leq e-1.\) Then
\[\begin{bmatrix} ae-1+j \\ j \end{bmatrix} = 0\]
for all \(a \geq 0.\)

**Lemma 2.8.** Suppose \(m \geq l \geq 0,\) that \(k \geq 1\) and that \(a_1, \ldots, a_k \geq 0\) are such that \(\sum_{i=1}^{k} a_i = m.\) Then
\[\sum_{c_1+\ldots+c_k=l} \prod_{i=1}^{k} q^{a_i-c_i}(c_{i+1}+\ldots+c_k) [\begin{array}{c} a_i \\ c_i \end{array}] = [\begin{array}{c} m \\ l \end{array}].\]
Proof. The result is true for \( m = 0 \) so suppose that \( m \geq 1 \) and that the lemma holds for \( m - 1 \). Using Lemma 2.4 and the inductive hypothesis,

\[
\sum_{c_1 + \ldots + c_k = l} \prod_{i=1}^{k} q^{(a_i - c_i)(c_{i+1} + \ldots + c_k)} \left[ \begin{array}{c} a_i \\ c_i \end{array} \right]
\]

\[
= \sum_{c_1 + \ldots + c_k = l} \left( \prod_{i=1}^{k-1} q^{a_i - c_i}(c_{i+1} + \ldots + c_k) \right) \left[ \begin{array}{c} a_k - 1 \\ c_k \end{array} \right] + q^{a_k - c_k} \left[ \begin{array}{c} a_k - 1 \\ c_k - 1 \end{array} \right]
\]

\[
= \sum_{c_1 + \ldots + c_k = l} \left( \prod_{i=1}^{k-1} q^{a_i - c_i}(c_{i+1} + \ldots + c_k) \right) \left[ \begin{array}{c} a_k \\ c_k \end{array} \right] + q^{a_k - c_k} \sum_{c_1 + \ldots + c_k = l-1} \left( \prod_{i=1}^{k-1} q^{a_i - c_i}(c_{i+1} + \ldots + c_k) \right) \left[ \begin{array}{c} a_k - 1 \\ c_k \end{array} \right]
\]

\[
= \left[ \begin{array}{c} m - 1 \\ l \end{array} \right] + q^{l - 1} \left[ \begin{array}{c} m - 1 \\ l - 1 \end{array} \right]
\]

as required.

An alternative proof may be constructed by counting the number of \( l \)-dimensional vector spaces of an \( m \)-dimensional vector space over the finite fields. \( \square \)

2.3. Elements of \( \Psi(\mu, \lambda) \). We are now ready to prove Theorem 1.2. Fix \( a \geq b \geq c + 1 \geq 4 \) and define partitions

\[
\mu = \mu(a, b, c, e) = (ac - 3, be - 3, ce - 3, e - 1, e - 1),
\]

\[
\lambda = \lambda(a, b, c, e) = ((a + 2)e - 5, be - 3, ce - 3)),
\]

of some integer \( n \). If \( T \in T_\ell(\lambda, \nu) \) for some \( \nu \geq \mu \), recall that \( T^i_j \) is the number of entries equal to \( i \) in row \( j \) of \( T \). We denote \( T \) by

\[
T = 1^{T^1_1} 2^{T^1_2} 3^{T^1_3} 4^{T^2_1} 5^{T^2_2},
\]

where we omit terms if \( T^i_j = 0 \). Our strategy is to define linearly independent elements \( \Theta \) and \( \Phi \) in \( \text{EHom}_{\mathbb{F}_q}(S^\mu, S^\lambda) \) and use Lemmas 2.2 and Lemma 2.3 to show that \( \Theta(m_\mu h_{d,t}) = \Phi(m_\mu h_{d,t}) = 0 \) for all \( 1 \leq d \leq 4 \) and \( 1 \leq t \leq \mu_{d+1} \). Theorem 1.2 then follows by Lemma 2.1.

Lemma 2.9. Suppose that \( T \in T_0(\lambda, \mu) \) has the form

\[
T = \begin{cases}
1^{a-3} 2^{b-1} 3^{c-1} 4^{T^1_1} 5^{T^2_1} \\
3^{T^1_1} 4^{T^2_1} 5^{T^2_2}
\end{cases}
\]

Then the following results hold.

1. Suppose \( 1 \leq t \leq e - 1 \). Write \( T \xrightarrow{t} S \) if \( S \) is a row-standard \( \nu \)-tableau formed from \( T \) by changing \( t \) entries equal to 5 in \( T \) into 4s. If \( T \xrightarrow{t} S \) then \( S \) is semistandard and

\[
\Theta_T(m_\mu h_{d,t}) = \sum_{T \xrightarrow{t} S} q^{(T^1_1 + T^2_2)(S^1_1 - T^1_1)} S^1_1 \left[ \begin{array}{c} S^1_1 \\ T^1_1 \end{array} \right] q^{(S^2_2 - T^1_1)} S^2_2 \left[ \begin{array}{c} S^2_2 \\ T^2_2 \end{array} \right] \Theta_S(m_\nu).
\]
(2) Suppose $1 \leq t \leq e - 1$. Write $T \xrightarrow{3,t} S$ if $S$ is a row-standard $\nu$-tableau formed from $T$ by changing $t$ entries equal to 4 in $T$ into 3s. If $T \xrightarrow{3,t} S$ then $S$ is semistandard and

$$\Theta_T(m_\mu h_{3,t}) = \sum_{T \xrightarrow{3,t} S} q^{(T_2^3 + T_2^3)(S_2^4 - T_1^3)} \frac{S_1^3}{T_1^3} q^{(S_2^4 - T_2^3)} \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \Theta_S(m_\nu).$$

(3) Suppose $1 \leq t \leq \mu_3 - 1$. Write $T \xrightarrow{2,t} S$ if $S$ is a row-standard $\nu$-tableau formed from $T$ by first changing $t$ entries equal to 3 in $T$ into 2s in the second and third rows and then exchanging all entries equal to 2 in row 3 with entries not equal to 2 in row 2. If $T \xrightarrow{2,t} S$ then $S$ is semistandard and

$$\Theta_T(m_\mu h_{2,t}) = \sum_{T \xrightarrow{2,t} S} (-1)^{T_2^2 - S_2^2} q^{(T_2^2 - S_2^2)} + S_2^t \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right] q^{T_2^3(S_2^4 - T_2^3)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \Theta_S(m_\nu) \right].$$

In particular, $\Theta_T(m_\mu h_{2,t}) = 0$ for $t > e - 1$.

(4) Suppose $1 \leq t \leq \mu_2 - 1$. Write $T \xrightarrow{1,t} S$ if $S$ is a row-standard $\nu$-tableau formed from $T$ by first changing $t$ entries equal to 2 in $T$ into 1s and then exchanging all entries equal to 1 in row 2 with entries not equal to 1 in row 1. If $T \xrightarrow{1,t} S$ then $S$ is semistandard and

$$\Theta_T(m_\mu h_{1,t}) = \sum_{T \xrightarrow{1,t} S} (-1)^{T_2^2 - S_2^2} q^{(T_2^2 - S_2^2)} + S_2^t \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right] q^{T_2^3(S_2^4 - T_2^3)} q^{(T_2^3 + T_3^4)(S_2^4 - T_2^3)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \Theta_S(m_\nu) \right].$$

where $T_2^2 = (b - 1)e - 2$. In particular, $\Theta_T(m_\mu h_{1,t}) = 0$ for $t > 2e - 2$.

Proof. To check the tableaux $S$ are semistandard, observe that $ae - 3 \geq be - 3$ and that $(b - 1)e - 2 \geq ce - 3$. Parts (1) and (2) are then just restatements of Lemma 2.2. Now consider (3). Use Lemma 2.2 to write $\Theta_T$ as a linear combination of terms $\Theta_R(m_\nu)$ where $R$ is formed from $T$ by changing entries equal to 3 into 2s. If $s > 0$ entries are changed in the first row then the term occurs with coefficient a multiple of $[e - 1 + s] = 0$ by Lemma 2.7, so we may assume all entries changed are in the last two rows. It then follows from Lemma 2.8 that $\Theta_T(m_\mu h_{2,t}) = \sum_{T \xrightarrow{2,t} S} b(S) \Theta_S(m_\nu)$ where

$$b(S) = \sum_{j \geq 0} (-1)^{j} q^{(-\frac{j}{2} + 1)} q^{j(2 - T_2^3 + S_2^3)} \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right] q^{T_2^3(S_2^4 - T_2^3)} q^{(T_2^3 + T_3^4)(S_2^4 - T_2^3)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \right].$$

Changing the limits of the sum and applying Lemma 2.9, we obtain

$$b(S) = (-1)^{T_2^3 - S_2^3} q^{T_3^4(S_2^4 - T_2^3)} q^{(T_2^3 + T_3^4)(S_2^4 - T_2^3)} q^{(-\frac{T_2^3 - S_2^3}{2} + 1)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \right]$$

$$\sum_{j \geq 0} (-1)^{j} q^{j(\ell)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \right] \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right] (b - 1)e - 2$$

$$= (-1)^{T_2^3 - S_2^3} q^{T_3^4(S_2^4 - T_2^3)} q^{(T_2^3 + T_3^4)(S_2^4 - T_2^3)} q^{(-\frac{T_2^3 - S_2^3}{2} + 1)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \right] q^{S_3^3(T_2^3 + S_3^3)} \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right]$$

$$= (-1)^{T_2^3 - S_2^3} q^{(T_2^3 - S_2^3)} + S_2^t \left[ \frac{b}{a} - 1 + 1 + \frac{c}{d} - 3 - S_2^2 \right] q^{T_3^4(S_2^4 - T_2^3)} \left[ \frac{S_3^3}{T_3^3} \frac{S_3^3}{T_3^3} \right]$$

as required.

The proof of part (4) of the lemma follows on identical lines. □
Proposition 2.10. Define a tableau $\mathbf{T} \in \mathcal{T}_0(\lambda, \mu)$ by

$$
\mathbf{T} = \begin{cases}
1^{ae-3}2^{e-1}3^{e-1} \\
2^{(b-1)e-2}3^{e-1} \\
3^{(e-2)e-1}4^{e-1}5^{e-1}
\end{cases}
$$

and let $\Theta = \Theta_\mathbf{T}$. Then $\Theta \in \Psi(\mu, \lambda)$.

Proof. Note that $\mathbf{T}$ has the form described in Lemma 2.9. Suppose $1 \leq t \leq e - 1$. Then applying Lemma 2.9 and Lemma 2.7

$$
\Theta(m_\mu h_{1,t}) = \left[ \begin{array}{c}
(c+1) e - 1 + t \\
2b - 1
\end{array} \right] = 0;
$$

$$
\Theta(m_\mu h_{3,t}) = \left[ \begin{array}{c}
(c-2) e - 1 + t \\
2b - 1
\end{array} \right] = 0;
$$

$$
\Theta(m_\mu h_{2,t}) = q^t \left[ \begin{array}{c}
(b - 1) e - 1 + t \\
2b - 1
\end{array} \right] = 0.
$$

Now suppose $1 \leq t \leq 2e - 2$. Then

$$
\Theta(m_\mu h_{1,t}) = \sum_{\{\sum_{j=1}^5 t_j = e - 1\}} \left[ \begin{array}{c}
(a - b + 1) e - 1 + t \\
2b - 1
\end{array} \right] = 0.
$$

But if $\mathbf{T} \frac{\rightarrow}{d} \mathbf{S}$ then $S^t_{j+1} = e - 1$ and if $S^t_j = e - 1$ then $1 \leq t \leq e - 1$ and then

$$
\left[ \begin{array}{c}
(a - b + 1) e - 1 + t \\
2b - 1
\end{array} \right] = 0.
$$

by Lemma 2.7. Hence $\Theta(m_\mu h_{d,t}) = 0$ for all $1 \leq d \leq 4$ and all $1 \leq t \leq \mu_{d+1}$ as required.

Proposition 2.11. Let $\mathcal{A}$ denote the set of $\lambda$-tableaux $\mathbf{A}$ of type $\mu$ which have the form

$$
1^{ae-3}2^{e-1}3^{A_1^1}4^{A_1^2}5^{A_1^3}
$$

and $\mathcal{B}$ denote the set of $\lambda$-tableaux $\mathbf{B}$ of type $\mu$ which have the form

$$
1^{ae-3}2^{e-1}3^{B_1^1}4^{B_1^2}5^{B_1^3}
$$

so that all $\mathbf{A} \in \mathcal{A} \cup \mathcal{B}$ are semistandard.

Set

$$
\Phi = \sum_{\mathbf{A} \in \mathcal{A}} \Theta_{\mathbf{A}} - q \sum_{\mathbf{B} \in \mathcal{B}} \Theta_{\mathbf{B}}.
$$

Then $\Phi \in \Psi(\mu, \lambda)$.

Proof. Note that all tableaux $\mathbf{A} \in \mathcal{A} \cup \mathcal{B}$ have the form described in Lemma 2.9 and use the notation of that lemma. For $1 \leq d \leq 4$ and $1 \leq t \leq \mu_{d+1}$, let

$$
D(d,t) = \{ \mathbf{S} \in \mathcal{T}_0(\lambda, \nu) \mid \mathbf{A} \frac{\rightarrow}{d} \mathbf{S} \text{ for some } \mathbf{A} \in \mathcal{A} \cup \mathcal{B} \}.
$$

For $\mathbf{S} \in D(d,t)$ define $b_\mathcal{A}(\mathbf{S})$ to be the coefficient of $\Theta_\mathbf{S}(m_\mu)$ in $\sum_{\mathbf{A} \in \mathcal{A}} \Theta_\mathbf{A}(m_\mu h_{d,t})$, define $b_\mathcal{B}(\mathbf{S})$ to be its coefficient in $\sum_{\mathbf{B} \in \mathcal{B}} \Theta_\mathbf{B}(m_\mu h_{d,t})$ and set $b(\mathbf{S}) = b_\mathcal{A}(\mathbf{S}) - q b_\mathcal{B}(\mathbf{S})$ to be its coefficient in $\Phi(m_\mu h_{d,t})$. 

□
Take \( d = 4 \) and \( 1 \leq t \leq e - 1 \) and suppose that \( S \in \mathcal{D}(d, t) \). Using Lemma 2.39 and applying Lemma 2.8 and Lemma 2.4 we have

\[
b_A(S) = \sum_{A \in A, d,t} q^{(A_1^d + A_2^d)(S_1^d - A_1^d)} A_1^d q^{A_3^d(S_2^d - A_2^d)} A_2^d S_3^d A_4^d = \sum_{A_1^d + A_2^d + A_3^d = e - 1} q^{A_1^d(S_1^d - A_1^d)} A_1^d q^{A_3^d(S_2^d - A_2^d)} A_2^d S_3^d A_4^d = [S_1^d + S_2^d + S_3^d] A_1^d + A_2^d + A_3^d] = [e - 1 + t] t = 0.
\]

An identical argument shows that \( b_B(S) \) is also zero.

Now take \( d = 3 \) and \( 1 \leq t \leq e - 1 \). Suppose that \( S \in \mathcal{D}(d, t) \). Then

\[
b(S) = \sum_{A \in A, d,t} q^{(A_1^d + A_2^d)(S_1^d - A_1^d)} q^{A_3^d(S_2^d - A_2^d)} S_3^d A_1^d A_2^d A_3^d = q \sum_{b \in B, d,t} q^{(B_1^d + B_2^d)(S_1^d - B_1^d)} q^{(S_2^d - B_2^d)} S_3^d B_1^d B_2^d B_3^d = q^{((c-1)e-2)((c-1)e-2-S_3^d)} \sum_{A_1^d + A_2^d = e-1} q^{A_1^d(S_1^d - A_1^d)} A_1^d \sum_{B_1^d + B_2^d = e-2} q^{B_2^d(S_1^d - B_1^d)} B_2^d S_3^d = q^{((c-1)e-2)((c-1)e-2-S_3^d)} \sum_{(c-1)e-2} q^{((c-1)e-1)((c-1)e-1-S_3^d)} S_3^d = q^{((c-1)e-2)((c-1)e-2-S_3^d)} \sum_{(c-1)e-2} q^{((c-1)e-1)((c-1)e-1-S_3^d)} e - 2
\]

where, by Lemma 2.39 \([S_3^d]_{(c-1)e-2}\) and \([S_3^d]_{(c-1)e-1}\) are zero unless \(S_3^d = (c-1)e-2\) or \(S_3^d = (c-1)e-1\). If \(S_3^d = (c-1)e-2\) then \(S_1^d + S_2^d = e - 1 + t\) and

\[
b(S) = q^{((c-1)e-1)t} [e - 1 + t] t = 0
\]

by Lemma 2.4. If \(S_3^d = (c-1)e-1\) then \(S_1^d + S_2^d = e - 2 + t\). Note that \([(c-1)e-1] = -q^{(c-1)e-1} \). Applying Lemma 2.34 and Lemma 2.4 we have

\[
b(S) = q^{((c-1)e-2)(t-1)} [(c-1)e-1] [e - 2 + t] e - 1 - q^{((c-1)e-1)t+1} [e - 2 + t] e - 2 \]

\[
= -q^{((c-1)e-2)t+1} \left[ \left[ e - 2 + t \right] e - 1 + q \left[ e - 2 + t \right] e - 2 \right]
\]

\[
= -q^{((c-1)e-2)t+1} \left[ e - 1 + t \right] t
\]

as required.
Now take $d = 2$ and $1 \leq t \leq c - 1$ and suppose that $S \in \mathcal{D}(2, t)$. If $A \in \mathcal{A}$ note that $A^d_1 = (c - 1)e - 2$. Then
\begin{align*}
b_A(S) &= \sum_{A \xrightarrow{2} S} (-1)^{A_2^d - 3_2} q^{(3^d_2 - 3^d_2) + 2^d_2} \left[ \frac{(b - 1)e - 2 + t - A_2^d}{a} \right] \left[ \frac{(b - 1)e - 2 - S_2^d}{A^d_2} \right] A_2^d \left[ \frac{S_2^d + S_2^d}{A^d_2} \right] \left[ \frac{S_2^d}{A^d_2} \right] \\
&= (-1)^{(c - 1)e - 2} q^{(e - 2 - S_2^d) + 2^d_2} \left[ \frac{(b - c)e + t}{(b - 1)e - 2 - S_2^d} \right] q^{A_2^d} \left[ \frac{S_2^d}{A_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \\
&= (-1)^{(c - 1)e - 2} q^{(e - 2 - S_2^d) + 2^d_2} \left[ \frac{(b - c)e + t}{(b - 1)e - 2 - S_2^d} \right] \left[ \frac{S_2^d + S_2^d}{e - 1} \right]
\end{align*}
and the same argument shows that
\begin{align*}
b_B(S) &= (-1)^{(c - 1)e - 1} q^{(e - 1) - (e - 1)} + 2^d_2 \left[ \frac{(b - c)e - 1 + t}{t} \right] \left[ \frac{S_2^d + S_2^d}{e - 2} \right].
\end{align*}
Note that if $A \xrightarrow{2} S$ for some $A \in \mathcal{A}$ then $e - 1 \leq S_2^d + S_2^d \leq 2e - 2$ and if $B \xrightarrow{2} S$ for some $B \in \mathcal{B}$ then $e - 2 \leq S_2^d + S_2^d \leq 2e - 3$. So by Lemma 2.7, $b(S) = 0$ unless $S_2^d + S_2^d = e - 2$ or $S_2^d + S_2^d = e - 1$. If $S_2^d + S_2^d = e - 2$ then
\begin{align*}
b(S) &= (-q)q^{S_2^d} \left[ \frac{(b - c)e - 1 + t}{t} \right] = 0
\end{align*}
by Lemma 2.7. If $S_2^d + S_2^d = e - 1$ then recall that $[e - 1] = q^{-e} = -q^3$. Then
\begin{align*}
b(S) &= q^{S_2^d} \left[ \frac{(b - c)e + t}{t} \right] + (q)q^{S_2^d} \left[ \frac{(b - c)e - 1 + t}{t - 1} \right] [e - 1] \\
&= q^{S_2^d} \left[ \frac{(b - c)e + t}{t} \right] - q^3 \left[ \frac{(b - c)e - 1}{t - 1} \right] \\
&= q^{S_2^d} \left[ \frac{(b - c)e + t}{t} \right] = 0
\end{align*}
by Lemma 2.4 and Lemma 2.7.

Finally take $d = 1$ and $1 \leq t \leq 2e - 2$ and suppose that $S \in \mathcal{D}(1, t)$. By Lemma 2.4
\begin{align*}
b_A(S) &= \sum_{A \xrightarrow{2} S} (-1)^{A_2^d - 3_2} q^{(3^d_2 - 3^d_2) + 2^d_2} \left[ \frac{(a - b + 1)e - 1 + t}{a - 3 - S_2^d} \right] q^{A_2^d} \left[ \frac{S_2^d}{A_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \\
&= (-1)^{A_2^d - 3_2} q^{(e - 2 - S_2^d) + 2^d_2} \left[ \frac{(a - b + 1)e - 1 + t}{a - 3 - S_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \\
&= (-1)^{A_2^d - 3_2} q^{(e - 2 - S_2^d) + 2^d_2} \left[ \frac{(a - b + 1)e - 1 + t}{a - 3 - S_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \left[ \frac{S_2^d}{A_2^d} \right] \\
&= (-1)^{A_2^d - 3_2} q^{(e - 2 - S_2^d) + 2^d_2} \left[ \frac{(a - b + 1)e - 1 + t}{a - 3 - S_2^d} \right] \left[ \frac{S_2^d + S_2^d}{e - 1} \right].
\end{align*}

Since $e - 1 \leq S_2^d + S_2^d \leq 2e - 2$, Lemma 2.7 shows that the last term is zero unless $S_2^d + S_2^d = e - 1$. In this case $1 \leq t \leq e - 1$ and $S_2^d = (b - 1)e - 2$ and so $b_A(S)$ has a factor
\begin{align*}
\left[ \frac{(a - b + 1)e - 1 + t}{t} \right] = 0
\end{align*}
by Lemma 2.7. An identical argument shows that $b_B(S) = 0$.

This completes the proof that $\Phi(m_d h_{d,t}) = 0$ for all $1 \leq d \leq 4$ and all $1 \leq t \leq \mu_{d+1}$.

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