Distortion risk measures in random environments: construction and axiomatic characterization

Shuo Gong\textsuperscript{1}, Yijun Hu\textsuperscript{1}, Linxiao Wei\textsuperscript{2}\textsuperscript{*}

\textsuperscript{1} School of Mathematics and Statistics  
Wuhan University  
Wuhan, Hubei 430072  
People’s Republic of China

\textsuperscript{2} College of Science  
Wuhan University of Technology  
Wuhan, Hubei 430070  
People’s Republic of China

Abstract: The risk of a financial position shines through by means of the fluctuation of its market price. The factors affecting the price of a financial position include not only market internal factors, but also other various market external factors. The latter can be understood as sorts of environments to which financial positions have to expose. Motivated by this observation, this paper aims to design a novel axiomatic approach to risk measures in random environments. We construct a new distortion-type risk measure, which can appropriately evaluate the risk of financial positions in the presence of environments. After having studied its fundamental properties, we also axiomatically characterize it by proposing a novel set of axioms. Furthermore, its coherence and dual representation are investigated. The new class of risk measures in random environments is rich enough, for example, it not only can recover some known risk measures such as the common weighted value at risk and range value at risk, but also can induce other new specific risk measures such as risk measures in the presence of background risk. Examples are given to illustrate the new framework of risk measures. This paper gives some theoretical results about risk measures in random environments, helping one to have an insight look at the potential impact of environments on risk measures of positions.

Key words: Distortion risk measure, coherent risk measure, representation, random environment, Choquet integral.

Mathematics Subject Classification (2020): 91G70, 91B05

\textsuperscript{*}Email addresses: shuogong@whu.edu.cn (S. Gong), yjhu.math@whu.edu.cn (Y. Hu), lxwei@whut.edu.cn (L. Wei).

\textsuperscript{*}Corresponding author: Linxiao Wei.

\textsuperscript{*}Supported in part by the National Natural Science Foundation of China (Nos: 12271415, 12001411) and the Fundamental Research Funds for the Central Universities of China (WUT: 2021IVB024).
1 Introduction

Risk measures are used in order to quantitatively evaluate the risk of a financial position or portfolio. Axiomatic approach to risk measures was initiated by Artzner et al. (1999) who introduced the concept of coherent risk measures by proposing a set of axioms, and provided the representations for the coherent risk measures. Later, coherent risk measures were axiomatically extended to the convex risk measures by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). In the above framework of risk measures, a risk measure is defined, in general, as any mapping from some class of random variables defined on a prefixed probability space to the real numbers, where the random variables represent the losses (or gains) of financial positions. The quantity of the risk measure of a financial position is interpreted as the capital requirement that the holder of the position has to safely invest in order to satisfy a regulator, see Artzner et al. (1999). In the context of insurance pricing theory, risk measures that satisfy the coherence axioms have been investigated by Denneberg (1990) and Wang (1996), see also Denneberg (1994) and Wang et al. (1997). These risk measures are now often called distortion risk measures. We refer to above axiomatic framework for risk measures as the classic setting. For more details about risk measures, we also refer to Föllmer and Schied (2016).

By risk of a financial position, we mean its market risk. Thereby, the risk of a financial position shines through by means of the fluctuation of its market price. From the economic point of view, the factors influencing the price of a position include not only market internal factors, but also other various market external factors. The latter is understood to be any factor from the outside of the market itself, as long as it can influence the price of the position. We refer to a market external factor as an environment relative to the price of the position. It is conceivable that when an environment takes different values, the severity of the corresponding fluctuation of the price of a position should not be expected to be the same. In other words, the environments, which are usually random, can notably affect the risk of a position in the market. This consideration suggests that when one evaluates the risk of a position, it should be reasonable to take into account the impacts of environments on the risk of the position as well. Therefore, we naturally come to risk measures in random environments. For a typical instance, consider a financial derivative, for example, say a futures or an option, by the modern finance theory, with no doubt, the price of the derivative is surely affected not only by the market internal factors, but also by the price of its underlying asset, because their prices have an intrinsic relationship between each other. Intuitively, when the price of the underlying asset takes different values (also called levels), the severity of the corresponding fluctuation of the price of the derivative should typically not be the same. In other words, when the price of the underlying asset is at different levels, the corresponding outcomes of the price of the derivative are typically not the same. Thereby intuitively, the risk of the underlying asset should have a tight impact on the risk of its derivative. This observation suggests that the input argument to the measure of risk of the derivative should include not only the outcomes of the random loss from the derivative, but also the risk of the underlying asset. In other words, there should be two input arguments to the risk measure of the derivative: the random loss of the derivative and the risk of its underlying asset. Here, upon the price of the underlying asset is referred to an environment relative to its derivative, it would naturally lead to a risk measure in an environment. For a detailed example demonstrating above intuition, see Subsection 4.2 below, where it will be revealed that the riskiness of the underlying asset can
do be transmitted to its derivative, which is in agreement with above intuition. In a wide sense, a random environment could also be any of risk factors except the position’s price itself, such as interest rate, foreign exchange rate, credit spread and volatility, as pointed out in the Fundamental Review of Trading Book (FRTB) of Basel III & IV (BCBS, 2016) for market risk. Even more, in a more wider sense, a random environment could also be some economic factor, for example, see the economic factor Θ in Example 2.1 and Section 5.1 of Wang and Ziegel (2021).

For another example, law-based risk measures such as value at risk (VaR) and average value at risk (AVaR) (also known as expected shortfall (ES)) have been prominent in banking regulation and financial risk management, for example, in the framework of Basel III & IV (BCBS, 2016, 2019), an ES at level $p = 0.975$ is specified. Assuming that the distribution of a risk is successfully captured, it need not be able to comprehensively describe the nature of the risk. From the perspective of a regulator, a regulator would most likely to be more concerned about the performance of a risk in an adverse environment, e.g. during a catastrophic financial event, see for example Acharya et al. (2012) for related discussions, see also the guideline hidden in the stress adjustment adopted by the FRTB of Basel III & IV (BCBS, 2016). Only the distribution of the risk may be not enough to distinguish a potentially huge loss in a financial crisis from a potentially huge loss in a common economy but no loss in a financial crisis, for example, see Wang and Ziegel (2021) for related discussions. By referring to either a financial crisis or a common economy as a possible value that an economic factor takes, and to the economic factor as a random economic environment, it thereby may be useful to evaluate a risk under different values that a random economic environment can take. Aggregating these evaluations in a single number would mathematically necessarily and naturally lead to a risk measure in a random environment.

For one more concrete example, from the perspective of a financial institution’s risk assessment, suppose that a regulator specifies a risk measure, for example, AVaR (i.e. ES) at level 0.975 as in the Basel Committee on Banking Supervision (BCBS, 2016, 2019), and two financial institutions (or risk analysts) assess the risk of the same portfolio separately. Since their subjective judgements about the potential probability distribution of the random loss from the portfolio are usually not the same, their reported AVaR values are typically not the same. Among various possible factors affecting their judgements, with no doubt, their attitudes towards the risk preference usually take a significant role in their judgement activities. However, the loss random variable from the portfolio is the same for both institutions. Therefore, the risk measure should not be only determined by the loss random variable, but also the risk preference. Here, the risk preference of an institution can be regarded as an environment, when the risk of the loss random variable is assessed by the institution. Consequently, we naturally again come to a risk measure in a random environment.

Motivated by above considerations from three aspects, the main purpose of this paper is to design a new framework, within which the risk of any position exposed to a random environment can be appropriately evaluated. Briefly speaking, in this paper, a novel axiomatic framework and new distortion-type risk measures are presented, which provide measures of risk of positions with respect to a given random environment. To our best knowledge, we have not found reports about risk measures in random environments available in the literature. We believe that it is worth studying.

The class of distortion risk measures is one of the most important classes of risk mea-
sures, since it contains a rich family of risk measures including common VaR and AVaR, for example, see Belles-Sampera et al. (2014), where term tail value-at-risk (TVaR) was used instead of AVaR, Föllmer and Schied (2016), or Wang and Ziegel (2021). Particularly, when the distortion functions are concave, then the distortion risk measures are coherent in the sense of Artzner et al. (1999), for example, see Denneberg (1994), Wang et al. (1997) and Föllmer and Schied (2016). Therefore, this paper strives to present distortion-type risk measures with respect to random environments, where the random environments are described by random variables.

In the present paper, first, we construct distortion-type risk measures with respect to any realization of a random environment, which evaluate the risk of a financial position provided with a realization of the random environment. Their properties are discussed. Moreover, these distortion-type risk measures are axiomatically characterized. Second, we construct distortion-type risk measures with respect to random environments, which evaluate the risk of a financial position with respect to a random environment. After their fundamental properties are discussed, they are also axiomatically characterized. Furthermore, the coherence and dual representation for them are investigated. At last, examples are given to illustrate the proposed distortion risk measures in random environments. On one hand, as one of applications, new risk measures in the presence of background risk are induced. On the other hand, some common risk measures in the literature such as weighted value at risk (WVaR) and range value at risk (RVaR) can be recovered. Furthermore, an explicit expression for the distortion-type risk measure of a financial derivative is given, where the random environment represents the price of the underlying asset, and is assumed to obey the geometric Brownian motion. It is revealed that the riskiness of the underlying asset can be transmitted to its derivative, which is in accordance with intuition.

It is worth mentioning that there have been a number of conditional risk measures in the literature, which are most closely related to the present risk measures in random environments, for example, see Mainik and Schaanming (2014), Adrian and Brunnermeier (2016), Acharya et al. (2017), Kleinow et al. (2017), Dhaene et al. (2022) and the references therein, just name a few. These conditional risk measures were established to study systemic risk, and have a common feature of being defined via conditional probability/distribution. With such a paradigm, there are two random variables involved, one of which usually describes the systemic risk (i.e. the overall financial system risk), and the other describes an individual risk (i.e. a financial institution’s risk). These systemic risk measures mainly focus on the investigation about the impact of each individual risk on the systemic risk. For instance, very recently, Dhaene et al. (2022) established a significant conditional distortion risk measure by means of conditional distribution function of a random variable conditional upon the occurrence of an event determined by the other random variable, see Definition 3.1 and Remark 3.2(d) of Dhaene et al. (2022) for instance. Although the topic we focus on is to evaluate the risk of financial positions in uncertain environments, mathematically, the idea of establishing the distortion risk measure in random environments partially intersects that of establishing conditional risk measures, especially the conditional distortion risk measure proposed by Dhaene et al. (2022). Nevertheless, as an intrinsical difference from these conditional risk measures, we make full use of regular conditional probability to construct distortion risk measures in random environments instead of conditional probability/distribution. An advantage of using regular conditional probability is that it is very helpful for us to develop an axiomatic approach to characterize risk measures in random environments as well, al-
though it also simultaneously makes relevant arguments more difficult and complicated. For instance, by virtue of the regular conditional probability, we can also be able to further axiomatically characterize the proposed distortion risk measure in random environments by proposing a novel set of axioms with sound financial interpretation. Furthermore, its coherence and dual representation can also be suitably investigated accordingly. It is well-known that the axiomatic approach to risk measures is one of the most important and prevalent approaches in the study of risk measures. Therefore, we argue and believe that the present framework of risk measures and techniques used are of own interesting and significance.

It might be helpful to briefly comment on the main contributions of this paper. First, we present a novel axiomatic framework for risk measures in random environments, within which not only the random losses are taken into account, but also random environments. In other words, in the new axiomatic approach, there are two input arguments to the risk measures: the random loss and the random environment. Compared with the classic setting, the new framework is much more structurally complicated. Unlike the classic setting, the new framework reveals that a two-hierarchy structure of axioms for risk measures arises: the axioms in the first hierarchy concern their formulations in the state-by-state sense (i.e. state-wise), while ones in the second hierarchy concern their formulations in the overall sense (i.e. environment-wise), which take those axioms in the first hierarchy as a basis. It indicates that the axioms in the classic setting correspond to those in the first hierarchy (i.e. state-wise hierarchy). Second, mathematically, the candidate for an environment is quite flexible. This feature ensures as much comprehensive applicability of the new framework of risk measures as possible. For example, when the random environment is understood to represent the risk preference of a risk analyst, then with the help of a plausible condition, the risk measures in random environments can recover the common WVaR and RVaR, which are two important risk measures in the literature. When the random environment is related to the so-called background risk, then one can induce new risk measures in the presence of background risk that could not be well handled in classic axiomatic approach, which enriches existing studies in the literature. At last, compared with the classic setting, those relevant methods and techniques used in the classic setting have been modified and updated in the new framework. Since there are now two-hierarchical axioms involved in the new framework, the arguments to the characterizations (i.e. representations) of risk measures in random environments become not only more complicated but also more delicate than the ones in the classic setting as in Denneberg (1994) and Wang et al. (1997). Therefore, besides those methods and techniques used in the classic setting, new methods and techniques need to be designed and incorporated into the arguments to the representations of risk measures in the new framework.

The rest of this paper is organized as follows. In Section 2, we prepare preliminaries including the introduction of distortion risk measures in random environments, notations and a measurability lemma. Section 3 is devoted to the main results: constructions of distortion risk measures in random environments, formulations of various axioms and statements of main results. As one of applications, distortion risk measures in the presence of background risk are introduced. In Section 4, examples are provided to illustrate the proposed distortion risk measures in random environments. It is also pointed out that under a plausible condition, the proposed distortion risk measures in random environments can also recover the common WVaR and RVaR. Concluding remarks are summarized in Section 5. In the Appendix, we provide the proofs of all main results, presented in Section 3, of this paper.
2 Preliminaries

Let \((\Omega, \mathcal{F})\) be a measurable space, and \(P\) a fixed probability measure on it, acting as a reference measure. We denote by \(\mathcal{X}\) the linear space of all bounded measurable functions (also called random variables) on \((\Omega, \mathcal{F})\) equipped with the supremum norm \(\| \cdot \|\), and by \(\mathcal{X}_+\) the subset of \(\mathcal{X}\) consisting of those elements which are non-negative. To be consistent with the literature on risk measures, we work with \(\mathcal{X}\) rather than \(L^\infty(\Omega, \mathcal{F}, P)\) of essentially bounded random variables on \((\Omega, \mathcal{F}, P)\) so that we can avoid tedious integrability considerations in the sequel, for example, see Föllmer and Schied (2016, Chapter 4) or Wang and Ziegel (2021, Section 3). Throughout this paper, we assume that \((\Omega, \mathcal{F}, P)\) is an atomless probability space. When a probability measure \(Q\) on \((\Omega, \mathcal{F})\) is concerned, if needed, the corresponding \(\mathcal{X}\) (or \(\mathcal{X}_+\), respectively) is also denoted by \(\mathcal{X}(Q)\) (or \(\mathcal{X}_+(Q)\), respectively) in order to emphasize the involvement of the probability measure \(Q\). Note that indeed \(\mathcal{X} = \mathcal{X}(Q)\) (or \(\mathcal{X}_+ = \mathcal{X}_+(Q)\), respectively) for any probability measure \(Q\) on \((\Omega, \mathcal{F})\). The random loss of a financial position is described by an element in \(\mathcal{X}\). For any \(X \in \mathcal{X}\), we denote by \(\text{Ran}(X)\) the range of \(X\), by \(P_X := P \circ X^{-1}\) the probability distribution of \(X\) with respect to \(P\), that is, \(P_X(A) := P \circ X^{-1}(A) := P(X \in A)\) for any \(A \in \mathcal{B}(\mathbb{R})\), the Borel algebra of subsets of the real line \(\mathbb{R}\), and by \(F_X(x) := P(X \leq x)\), \(x \in \mathbb{R}\), the distribution function of \(X\) with respect to \(P\). For an integrable random variable \(X\) on \((\Omega, \mathcal{F}, P)\), a random variable \(Z \in \mathcal{X}\) and each \(z \in \text{Ran}(Z)\), we denote by \(E[\cdot|Z]\) and \(E[\cdot|Z = z]\) the conditional expectations of \(X\) with respect to \(Z\) and the event \(\{Z = z\}\) under \(P\), respectively. For \(A \in \mathcal{F}\), \(X, X_n \in \mathcal{X}, n \geq 1\), we say that the sequence \(\{X_n; n \geq 1\}\) increases to \(X\) on \(A\), denoted by \(X_n \uparrow X\) on \(A\), if for all \(n \geq 1\) and every \(\omega \in A\), \(X_n(\omega) \leq X_{n+1}(\omega)\) and \(\lim_{n \to +\infty} X_n(\omega) = X(\omega)\). For \(X, X_n \in \mathcal{X}, n \geq 1\), we call that the sequence \(\{X_n; n \geq 1\}\) eventually increases to \(X\), denoted by \(X_n \uparrow X\) eventually, if for every \(\omega \in \Omega\), there is an integer \(N := N(\omega) \geq 1\), so that for all \(n \geq N\), \(X_n(\omega) \leq X_{n+1}(\omega)\) and \(\lim_{n \to +\infty} X_n(\omega) = X(\omega)\).

By a \(U[0,1]\) random variable on \((\Omega, \mathcal{F}, P)\) we mean a random variable which is uniformly distributed on \([0,1]\). Define the set

\[\mathcal{X}^-(P) := \{ X \in \mathcal{X}(P) : \text{there exists a } U[0,1] \text{ random variable on } (\Omega, \mathcal{F}, P) \text{ independent of } X\}.\]

This set will serve as the random environments. Note that any discrete random variable \(X \in \mathcal{X}(P)\) belongs to \(\mathcal{X}^-(P)\), for example, see Lemma 3 of Liu et al. (2020). Note also that \(\mathcal{X}^-(P)\) may be a subset of \(\mathcal{X}\) which does not coincide with \(\mathcal{X}\). For the case allowing for \(\mathcal{X}^-(P) = \mathcal{X}\), we refer to Example 9 of Liu et al. (2020).

We introduce more notations. For \(a, b \in \mathbb{R}\), \(a \vee b\) stands for \(\max\{a, b\}\), and \(a \wedge b\) means \(\min\{a, b\}\). For \(X, Y \in \mathcal{X}\), \(X \vee Y\) stands for \(\max\{X, Y\}\), and \(X \wedge Y\) means \(\min\{X, Y\}\). For \(Z \in \mathcal{X}^-(P)\), by a \(P_Z\)-null set \(N \in \mathcal{B}(\mathbb{R})\) we mean that \(P_Z(N) := P(Z \in N) = 0\). Let \(D \subset \mathbb{R}\) be a non-empty set, denote by \(\mathcal{B}(D)\) the Borel algebra of Borel subsets of \(D\), that is, \(\mathcal{B}(D) := \mathcal{B}(\mathbb{R}) \cap D := \{ B \cap D : B \in \mathcal{B}(\mathbb{R}) \}\). Thus \((D, \mathcal{B}(D))\) is a measurable space. For a set \(A\), \(A^c\) means the complement set of \(A\), and \(1_A\) stands for the indicator function of \(A\), while \(1_\emptyset = 0\) with convention. \(\mathbb{R}_+ := [0, +\infty)\).

Throughout this paper, we assume that for any random variable \(Z \in \mathcal{X}(P)\), the regular conditional probability \(\{p(z, \cdot) : z \in \mathbb{R}\}\) with respect to \(Z\) exists, that is, for each \(z \in \mathbb{R}\), \(p(z, \cdot)\) is a probability measure on \((\Omega, \mathcal{F})\); for each \(A \in \mathcal{F}\), \(p(\cdot, A)\) is a Borel function on
(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and for any integrable random variable $X$ on $(\Omega, \mathcal{F}, P)$,

$$E[X|Z = z] = \int_{\Omega} X(\omega)p(z, d\omega) \text{ for } P_Z - \text{a.e. } z \in \mathbb{R}. \quad (2.1)$$

For example, see pages 11-15 of Ikeda and Watanabe (1981) for above basic facts about regular conditional probability.

Note that if $(\Omega, \mathcal{F})$ is a standard measurable space, then for any random variable $Z$ on $(\Omega, \mathcal{F}, P)$, the regular conditional probability $\{p(z, \cdot) : z \in \mathbb{R}\}$ with respect to $Z$ exists. Moreover, a Polish space (i.e. a complete separable metric space) with the topological $\sigma$-algebra is a standard measurable space, for example, see Ikeda and Watanabe (1981, pages 13-15).

For example, see pages 11-15 of Ikeda and Watanabe (1981) for above basic facts about regular conditional probability.

For example, see pages 11-15 of Ikeda and Watanabe (1981) for above basic facts about regular conditional probability.

For example, see pages 11-15 of Ikeda and Watanabe (1981) for above basic facts about regular conditional probability.

For example, see pages 11-15 of Ikeda and Watanabe (1981) for above basic facts about regular conditional probability.

Any mapping $\mu : \mathcal{F} \to \mathbb{R}_+$ with $\mu(\emptyset) = 0$ is called a set function on $\mathcal{F}$. A set function $\mu$ on $\mathcal{F}$ is called normalized, if $\mu(\Omega) = 1$, and is called monotone, if $\mu(A) \leq \mu(B)$ for any $A, B \in \mathcal{F}$ with $A \subseteq B$. Given a monotone set function $\mu$ on $\mathcal{F}$, for any $X \in \mathcal{X}$, the Choquet integral of $X$ with respect to $\mu$ is defined as

$$\int Xd\mu := \int X(\omega)\mu(d\omega) := \int_{-\infty}^{0} [\mu(X > x) - \mu(\Omega)]dx + \int_{0}^{+\infty} \mu(X > x)dx.$$

When $\mu$ is taken as a distorted probability $g \circ P$, where $g : [0, 1] \to [0, 1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, the corresponding Choquet integral with respect to $g \circ P$ is known as the distortion risk measure, while $g$ is called a distortion function. For example, see Yaari (1987), Denneberg (1994), Wang (1996), Wang et al. (1997), Acerbi (2002), Belles-Sampera et al. (2014) and Föllmer and Schied (2016). Note also that the requirement of monotonicity of a distortion function is not necessary in general, for example, see Wang et al. (2020).

Given an $\alpha \in (0, 1)$, for any $X \in \mathcal{X}(P)$, the VaR of $X$ at the confidence or tolerance level $\alpha$ is defined by

$$\text{VaR}_\alpha(X) := \inf \{x : F_X(x) \geq \alpha\},$$

and the AVaR of $X$ at the confidence or tolerance level $\alpha$ is defined by

$$\text{AVaR}_\alpha(X) := \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_\theta(X)d\theta.$$

**Definition 2.1 (Regularity for a family of distortion functions)** Let $D \subseteq \mathbb{R}$ be a non-empty set, and $\{g_d : d \in D\}$ be a family of distortion functions. The family $\{g_d : d \in D\}$ is called regular on $D$, if for any fixed $x \in [0, 1]$, the function $d \to g_d(x)$ is $\mathcal{B}(D)$-measurable, that is, it is a Borel function on $D$.

Comonotonicity is an important notion in finance and insurance literature. We introduce the definition of local comonotonicity for random variables.

**Definition 2.2 (Local-comonotonicity)** Let $A \in \mathcal{F}$. For $X, Y \in \mathcal{X}$, $X$ and $Y$ are called local-comonotonic on $A$, if for every $\omega_1, \omega_2 \in A$,

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0.$$
Particularly, we simply say that $X$ and $Y$ are comonotonic on $\Omega$, if they are local-comonotonic on $\Omega$.

We now introduce the definition of risk measures with respect to random environments.

**Definition 2.3 (Risk measures with respect to random environments)** A risk measure with respect to random environments is defined as any functional $\rho(X; Z) : \mathcal{F} \times \mathcal{F}^\perp(P) \to \mathbb{R}$, where the first input argument represents the random losses of financial positions, while the second input argument serves as random environments. Particularly, for any position $X \in \mathcal{F}$ and any random environment $Z \in \mathcal{F}^\perp(P)$, the quantity $\rho(X; Z)$ is called the risk measure of position (or random loss) $X$ with respect to random environment $Z$.

For terminology convenience, the values that a random environment $Z$ takes are also called the states of $Z$. For any random environment $Z \in \mathcal{F}^\perp(P)$, denote by $\{K_Z(z, \cdot) : z \in \mathbb{R}\}$ the regular conditional probability with respect to $Z$. By the properties of regular conditional probability $\{K_Z(z, \cdot) : z \in \mathbb{R}\}$, there exists a $P_Z$-null set $N_0 \in \mathcal{B}(\mathbb{R})$ such that for every $z \in N_0$,

$$K_Z(z, \{Z \in B\}) = 1_{B}(z) \quad \text{for any } B \in \mathcal{B}(\mathbb{R}),$$

(2.2)

for example, see Ikeda and Watanabe (1981, Corollary of Theorem 3.3, page 15). Therefore, for every $z \in N_0^c$, $K_Z(z, \{Z = z\}) = 1$. Consequently, for each $z \in N_0^c \cap \text{Ran}(Z)$, we denote by $K_Z^*(z, \cdot)$ the restriction of probability measure $K_Z(z, \cdot)$ to the $\sigma$-algebra $\mathcal{F} \cap \{Z = z\} := \{A \cap \{Z = z\} : A \in \mathcal{F}\}$ such that $(\{Z = z\}, \mathcal{F} \cap \{Z = z\}, K_Z^*(z, \cdot))$ is a probability space, where for any $A^* := A \cap \{Z = z\} \in \mathcal{F} \cap \{Z = z\}$ with some $A \in \mathcal{F}$, $K_Z^*(z, A^*) := K_Z(z, A)$. Similarly, we denote by $P_Z^*$ the restriction of $P_Z$ to the Borel $\sigma$-algebra $\mathcal{B}(\text{Ran}(Z))$ such that $(\text{Ran}(Z), \mathcal{B}(\text{Ran}(Z)), P_Z^*)$ is a probability space, where for any $B^* := B \cap \text{Ran}(Z) \in \mathcal{B}(\text{Ran}(Z))$ with some $B \in \mathcal{B}(\mathbb{R})$, $P_Z^*(B^*) := P_Z(B)$. By $P_Z - \text{a.e. } z \in \text{Ran}(Z)$ we mean that there is a $P_Z$-null set $B_0 \in \mathcal{B}(\mathbb{R})$ with $N_0 \subseteq B_0$ such that $z \in B_0^c \cap \text{Ran}(Z)$, where the $P_Z$-null set $N_0$ is as in (2.2). Similarly, given a state $z \in \text{Ran}(Z)$, by $K_Z(z, \cdot) - \text{a.e. } \omega \in \{Z = z\}$ we mean that there is an $\Omega_0 \in \mathcal{F}$ with $K_Z(z, \Omega_0) = 0$ such that $\omega \in \Omega_0^c \cap \{Z = z\}$. When the random environment $Z$ is degenerate, that is, $Z$ takes sole possible value, say $z_0$, then $K_Z(z_0, \cdot)$ is just the probability measure $P$, and hence $K_Z(z, \cdot)$ can be chosen to be identical to $P$ for each $z \in \mathbb{R}$.

**Definition 2.4 (Regularity for a family of functionals on $\mathcal{F}(K_Z(z, \cdot))$ or $\mathcal{F}_+(K_Z(z, \cdot))$)** Let $D \subseteq \mathbb{R}$ be a non-empty set. Given a random environment $Z \in \mathcal{F}^\perp(P)$, let $\{\rho_Z(\cdot; z) : z \in D\}$ be a family of functionals, where for each $z \in D$, $\rho_Z(\cdot; z)$ is a functional from $\mathcal{F}(K_Z(z, \cdot))$ (or $\mathcal{F}_+(K_Z(z, \cdot))$, respectively) to $\mathbb{R}$. We call the family $\{\rho_Z(\cdot; z) : z \in D\}$ is regular on $D$, if for any arbitrarily fixed $X \in \mathcal{F}(K_Z(z, \cdot))$ (or $\mathcal{F}_+(K_Z(z, \cdot))$, respectively), the function $z \to \rho_Z(X; z)$ is $\mathcal{B}(D)$-measurable.

Next, we introduce the definition of risk measures provided that the random environment takes a pre-specified state. This notion is crucial to our studying risk measures with respect to random environments.

**Definition 2.5 (Risk measures with respect to environment $Z = z$)** Given a random environment $Z \in \mathcal{F}^\perp(P)$ and a state $z \in \text{Ran}(Z)$, any functional $\rho_Z(\cdot; z) : \mathcal{F}(K_Z(z, \cdot)) \to \mathbb{R}$ (or $\mathcal{F}_+(K_Z(z, \cdot)) \to \mathbb{R}$, respectively) is called a risk measure with respect to (deterministic) environment $Z = z$. Particularly, for any position $X \in \mathcal{F}(K_Z(z, \cdot))$ (or $\mathcal{F}_+(K_Z(z, \cdot))$, respectively).
Definition 2.6 (Environment-wise comonotonicity for functionals) Given any random environment \( Z \in \mathcal{X}^+(P) \), for any \( z \in \text{Ran}(Z) \), let \( \tau_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R} \) be a functional. For any positions \( X_1, X_2 \in \mathcal{X}(K_Z(z, \cdot)) \), the two functions \( \tau_Z(X_1; \cdot), \tau_Z(X_2; \cdot) : \text{Ran}(Z) \to \mathbb{R} \) are called environment-wise comonotonic, if for every \( z_1, z_2 \in \text{Ran}(Z) \),

\[
(\tau_Z(X_1; z_1) - \tau_Z(X_1; z_2)) (\tau_Z(X_2; z_1) - \tau_Z(X_2; z_2)) \geq 0.
\]

We end this section with a measurability lemma. Since we have not found its proof in the literature, for the purpose of relative self-containedness, we provide its proof here.

Lemma 2.1 Let \( g(z, x) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) be a function such that for any \( x \in \mathbb{R}_+ \), \( g(\cdot, x) \) is a Borel function on \( \mathbb{R} \), and for any \( z \in \mathbb{R} \), \( g(z, \cdot) \) is left-continuous. Then for any Borel function \( \phi : \mathbb{R} \to \mathbb{R}_+ \), the function \( z \to g(z, \phi(z)) \) is a Borel function on \( \mathbb{R} \).

Proof Since \( \phi : \mathbb{R} \to \mathbb{R}_+ \) is \( \mathcal{B}(\mathbb{R}) \)-measurable, there exists a sequence of simple functions \( \phi_n(z) \) such that \( \phi_n(z) \uparrow \phi(z) \) for each \( z \in \mathbb{R} \).

For each \( n \geq 1 \), define a function \( g_n : \mathbb{R} \to \mathbb{R} \) by \( g_n(z) := g(z, \phi_n(z)) \), \( z \in \mathbb{R} \). Then for every \( z \in \mathbb{R} \), by the left-continuity of \( g(z, \cdot) \),

\[
\lim_{n \to \infty} g_n(z) = g(z, \phi(z)).
\]

Hence, it is sufficient to prove that for each \( n \geq 1 \), the function \( g_n \) is a Borel function on \( \mathbb{R} \).

For \( n \geq 1 \), denote by \( \text{Ran}(\phi_n) := \{a_{n,i} : 1 \leq i \leq k_n\} \) the finite set of values that \( \phi_n \) takes. Hence, for any \( b \in \mathbb{R} \),

\[
\{z : g_n(z) \leq b\} = \bigcup_{1 \leq i \leq k_n} \{\{z : g(z, a_{n,i}) \leq b\} \cap \{z : \phi_n(z) = a_{n,i}\}\}
\]

is a Borel set. Consequently, \( g(z, \phi(z)) \) is a Borel function on \( \mathbb{R} \). The lemma is proved.

3 Main results

In this section, we present the main results of this paper. First, we begin with introducing some axioms and definitions of distortion risk measures with respect to random environments. Then, we discuss their fundamental properties, and axiomatically characterize them. The coherence and dual representations of distortion risk measures with respect to random environments are investigated. Finally, as an application, we induce two new distortion risk measures in the presence of background risks. All the proofs of the main results of this section will be postponed to Appendix, because they are quite lengthy.

3.1 Distortion risk measures in random environments

In this subsection, first, some axioms for risk measures in random environments are listed. Then distortion risk measures with respect to random environments are constructed. After studying their properties, we axiomatically characterize them as well.

We begin with introducing the axioms for risk measures \( \rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R} \) provided with a random environment \( Z \in \mathcal{X}^+(P) \) taking a pre-specified state \( z \in \text{Ran}(Z) \).
(AI.1) State-wise law invariance: Given a random environment $Z$ and a state $z \in \text{Ran}(Z)$, for any $X \in \mathcal{F}(K_Z(z, \cdot))$, the risk measure $\rho_Z(X; z)$ only depends on the probability distribution $K_Z(z, \cdot) \circ X^{-1}$ of $X$ with respect to $K_Z(z, \cdot)$, that is, $\rho_Z(X; z) = \rho_Z(Y; z)$ for any $X, Y \in \mathcal{F}(K_Z(z, \cdot))$ with $K_Z(z, \cdot) \circ X^{-1} = K_Z(z, \cdot) \circ Y^{-1}$.

(AII.1) Environment-wise law invariance: Given any random environment $Z$ and a state $z \in \text{Ran}(Z)$, for any $X \in \mathcal{F}(K_Z(z, \cdot))$, the risk measure $\rho_Z(X; z)$ only depends on the probability distribution $K_Z(z, \cdot) \circ X^{-1}$ of $X$ with respect to $K_Z(z, \cdot)$, that is, $\rho_Z(X; z) = \rho_Z(Y; z)$ for any $X, Y \in \mathcal{F}(K_Z(z, \cdot))$ with $K_Z(z, \cdot) \circ X^{-1} = K_Z(z, \cdot) \circ Y^{-1}$.

(AII.2) Environment-wise monotonicity: Given any random environment $Z$ and a state $z \in \text{Ran}(Z)$, for positions $X, Y \in \mathcal{F}(K_Z(z, \cdot))$, if $X(\omega) \leq Y(\omega)$ for $K_Z(z, \cdot) \circ X^{-1}$ of $X$ with respect to $K_Z(z, \cdot)$, then $\rho_Z(X; z) \leq \rho_Z(Y; z)$.

(AII.3) Environment-wise comonotonic additivity: Given any random environment $Z$ and a state $z \in \text{Ran}(Z)$, for two positions $X, Y \in \mathcal{F}(K_Z(z, \cdot))$, if $X$ and $Y$ are local-comonotonic on $\{Z = z\}$, then $\rho_Z(X + Y; z) = \rho_Z(X; z) + \rho_Z(Y; z)$.

(AII.4) Environment-wise continuity from below: Given a random environment $Z$, for positions $X, X_n \in \mathcal{F}(K_Z(z, \cdot))$, $n \geq 1$, if $X_n \uparrow X$ on $\{Z = z\}$, then $\lim_{n \to \infty} \rho_Z(X_n; z) = \rho_Z(X; z)$.

Since the state of environment is pre-specified, the financial meanings of the state-wise Axioms (AI.1)-(AI.4) can be interpreted similarly to those as in the classic setting, for example, see Artzner et al. (1999), Wang et al. (1997), Kusuoka (2001) and Föllmer and Schied (2016).

Next, we proceed to introduce axioms for risk measures $\rho(\cdot; \cdot) : \mathcal{F} \times \mathcal{F}^\perp(P) \to \mathbb{R}$ with respect to random environments. For this purpose, we need one more assumption as follows:

**Assumption A** Given any random environment $Z \in \mathcal{F}^\perp(P)$, for each $z \in \text{Ran}(Z)$, there exists a functional $\tau_Z(\cdot; z) : \mathcal{F}(K_Z(z, \cdot)) \to \mathbb{R}$ with $\tau_Z(1; z) = 1$, such that the family $\{\tau_Z(\cdot; z) ; z \in \text{Ran}(Z)\}$ of functionals is regular on $\text{Ran}(Z)$.

Moreover, for $P_Z - \text{a.e. } z \in \text{Ran}(Z)$, the functional $\tau_Z(\cdot; z)$ satisfies Axioms (AI.1)–(AI.4).

Note that the existence of such a family $\{\tau_Z(\cdot; z) ; z \in \text{Ran}(Z)\}$ as in Assumption A will be demonstrated later, see Definition 3.2(1), Remark 3.2(i) and Proposition 3.1(1) below.

Now, we are ready to state the axioms for risk measures $\rho(\cdot; \cdot) : \mathcal{F} \times \mathcal{F}^\perp(P) \to \mathbb{R}$.

(AII.1) Environment-wise law invariance: Given any random environment $Z \in \mathcal{F}^\perp(P)$, for each $z \in \text{Ran}(Z)$, let $\tau_Z(\cdot; z)$ be the functional as in Assumption A. For two positions $X_1, X_2 \in \mathcal{F}$, if $\tau_Z(X_1; \cdot)$ and $\tau_Z(X_2; \cdot)$ have the same probability distribution with respect to $P_Z^*$, that is, $P_Z^* \circ \tau_Z^{-1}(X_1; \cdot) = P_Z^* \circ \tau_Z^{-1}(X_2; \cdot)$, then $\rho(X_1; Z) = \rho(X_2; Z)$.

(AII.2) Environment-wise monotonicity: Given any random environment $Z \in \mathcal{F}^\perp(P)$, for each $z \in \text{Ran}(Z)$, let $\tau_Z(\cdot; z)$ be the functional as in Assumption A. For two positions $X_1, X_2 \in \mathcal{F}$, if $\tau_Z(X_1; z) \leq \tau_Z(X_2; z)$ for $P_Z - \text{a.e. } z \in \text{Ran}(Z)$, then $\rho(X_1; Z) \leq \rho(X_2; Z)$.

(AII.3) Environment-wise comonotonic additivity: Given any random environment $Z \in \mathcal{F}^\perp(P)$, for each $z \in \text{Ran}(Z)$, let $\tau_Z(\cdot; z)$ be the functional as in Assumption A. For two positions $X_1, X_2 \in \mathcal{F}$, if for $P_Z - \text{a.e. } z \in \text{Ran}(Z)$, $X_1$ and $X_2$ are local-comonotonic on $\{Z = z\}$, and the functions $\tau_Z(X_1; \cdot)$ and $\tau_Z(X_2; \cdot)$ are environment-wise comonotonic, then $\rho(X_1 + X_2; Z) = \rho(X_1; Z) + \rho(X_2; Z)$. 
(AII.4) Environment-wise continuity from below: Given any random environment $Z \in \mathcal{X}^\perp(P)$, for each $z \in \text{Ran}(Z)$, let $\tau_Z(\cdot; z)$ be the functional as in Assumption A. For positions $X, X_n \in \mathcal{X}$, $n \geq 1$, if the sequence $\{X_n; n \geq 1\}$ is bounded below by some constant $D \in \mathbb{R}$ (i.e. $D \leq X_n(\omega)$ for each $n \geq 1$ and every $\omega \in \Omega$), $X_n \uparrow X$ eventually, and $\tau_Z(X_n; z) \leq \tau_Z(X; z)$ for each $n \geq 1$ and $P_Z-a.e. z \in \text{Ran}(Z)$, then
\[
\lim_{n \to +\infty} \rho(X_n; Z) = \rho(X; Z).
\]

Axioms (AII.1)-(AII.4) can be interpreted in the context of finance as follows. Note first that $\tau_Z(X; z)$ as in Assumption A exactly represents the risk measure of position $X$ under the condition that the environment $Z$ takes the state $z$, as will be seen in the sequel. Axiom (AII.1) means that for two positions $X_1$ and $X_2$, if their state-wise riskinesses $\tau_Z(X_1; \cdot)$ and $\tau_Z(X_1; \cdot)$ have the same distribution with respect to the environment’s probability distribution, then their overall riskinesses should be the same. This characteristic has some similarity to the law invariance in the classic setting. (AII.2) says that if position $X_1$ is less risky than another position $X_2$ in almost-all-state-wise sense, then the overall riskiness of $X_1$ should not exceed that of $X_2$. (AII.3) means that if two positions $X_1$ and $X_2$ are comonotonic in almost-all-state-wise sense, and moreover the two risk measures $\tau_Z(X_1; z)$ and $\tau_Z(X_2; z)$ are also comonotonic with respect to the state variable $z$, then the overall riskiness of $X_1 + X_2$ should be the superposition of those of $X_1$ and $X_2$, because spreading risk within comonotonic risks cannot reduce the total risk. As for (AII.4), from mathematical point of view, it more or less belongs to a kind of technical requirement. Nevertheless, it can still be interpreted in the financial context as follows. For a sequence of random losses $X_n$, $n \geq 1$, if for almost all states $z \in \text{Ran}(Z)$, the state-wise riskinesses $\tau_Z(X_n; z)$ of $X_n$, $n \geq 1$, have a ceiling that is just the state-wise riskiness $\tau_Z(z)$ of another random loss $X$, meanwhile the sequence of random losses $X_n$, $n \geq 1$, eventually increases to the random loss $X$ regardless of what state the environment takes, then the sequence of overall riskinesses of $X_n$, $n \geq 1$, should also converge to that of $X$.

**Remark 3.1** (i) In the course of introducing above Axioms (AI.1)-(AI.4) and (AII.1)-(AII.4), if the domains $\mathcal{X}’(K_Z(z, \cdot))$ and $\mathcal{X} \times \mathcal{X}^\perp(P)$ of risk measures are replaced by $\mathcal{X}^+_*(K_Z(z, \cdot))$ and $\mathcal{X}^+_* \times \mathcal{X}^\perp(P)$, respectively, then the counterparts of these axioms can be parallel introduced. In the sequel, if needed, those counterparts should be in use. Since there should have no risk of confusion, we do not want to repeat those counterparts almost verbatim.

(ii) Note that the state-wise monotonicity (AI.2) and state-wise comonotonic additivity (AI.3) of $\rho_Z(\cdot; z)$ imply the positive homogeneity of $\rho_Z(\cdot; z)$, that is, $\rho_Z(cX; z) = c \cdot \rho_Z(X; z)$ for any $c > 0$ and any $X \in \mathcal{X}’(K_Z(z, \cdot))$. Similarly, the environment-wise monotonicity (AII.2) and environment-wise comonotonic additivity (AII.3) of $\rho$ imply the positive homogeneity of $\rho(\cdot; Z)$, that is, given a random environment $Z$, $\rho(cX; Z) = c \cdot \rho(X; Z)$ for any $c > 0$ and any position $X \in \mathcal{X}’$. For more details, we refer to Theorem 11.2 and Exercise 11.1 of Denneberg (1994).

**Definition 3.1 (Normalization)** (1) Given a random environment $Z$ and any $z \in \text{Ran}(Z)$, a risk measure $\rho_Z(\cdot; z) : \mathcal{X}’(K_Z(z, \cdot)) \to \mathbb{R}$ with respect to environment $Z = z$ is called normalized, if $\rho_Z(1; z) = 1$.

(2) A risk measure $\rho(\cdot; \cdot) : \mathcal{X} \times \mathcal{X}^\perp(P) \to \mathbb{R}$ with respect to random environments is called normalized, if $\rho(1; Z) = 1$ for any $Z \in \mathcal{X}^\perp(P)$. 

11
The normalization can be interpreted as follows. For a degenerate random loss \( X = 1 \), its capital requirement should reasonably be 1. While in the context of insurance, the normalization is also known as the so-called no unjustified risk loading, for example, see Wang et al. (1997). A normalized risk measure is translation invariant (also known as cash invariant), if it is positively homogeneous and comonotonically additive.

**Definition 3.2 (Distortion risk measures with respect to \( K_Z(z, \cdot) \) and random environments)** Let \( \{g_z; z \in \mathbb{R}\} \) be a family of left-continuous distortion functions, which is regular on \( \mathbb{R} \).

1. Let \( Z \in \mathcal{X}^{-}(P) \) be an arbitrarily fixed random environment. For each \( z \in \mathbb{R} \), the normalized distortion risk measure \( \rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R} \) (or \( \mathcal{X}_+(K_Z(z, \cdot)) \to \mathbb{R} \), respectively) with respect to \( K_Z(z, \cdot) \) is defined by

\[
\rho_Z(X; z) := \int_{-\infty}^{0} [g_z \circ K_Z(z, \{X > \alpha\}) - 1] d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) d\alpha \quad (3.1)
\]

for \( X \in \mathcal{X}(K_Z(z, \cdot)) \) (or \( \mathcal{X}_+(K_Z(z, \cdot)) \), respectively).

2. For each random environment \( Z \in \mathcal{X}^{-}(P) \), let \( h_Z \) be a distortion function associated with \( Z \). The normalized distortion risk measure \( \rho(\cdot; \cdot) : \mathcal{X} \times \mathcal{X}^{-}(P) \to \mathbb{R} \) (or \( \mathcal{X}_+ \times \mathcal{X}^{-}(P) \to \mathbb{R} \), respectively) with respect to random environments is defined by

\[
\rho(X; Z) := \int_{-\infty}^{0} \left[ h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) d\alpha \right\} - 1 \right) \right] d\beta
\]

\[
+ \int_{0}^{\infty} h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) d\alpha \right\} - 1 \right) d\beta
\]

\[
+ \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) d\beta \quad (3.2)
\]

for \( (X, Z) \in \mathcal{X} \times \mathcal{X}^{-}(P) \) (or \( \mathcal{X}_+ \times \mathcal{X}^{-}(P) \), respectively).

**Remark 3.2** (i) For any fixed \( X \in \mathcal{X} \), by Lemma 2.1, we know that the right hand side of (3.1) is a Borel function on \( \mathbb{R} \) with respect to variable \( z \in \mathbb{R} \). Hence, the family \( \{\rho_Z(\cdot; z); z \in \mathbb{R}\} \) of functionals defined by (3.1) is regular on \( \mathbb{R} \), that is, for any position \( X \in \mathcal{X} \), the function \( z \to \rho_Z(X; z) \) is a Borel function on \( \mathbb{R} \). Thus, the family \( \{\rho_Z(\cdot; z); z \in \text{Ran}(Z)\} \) is regular on \( \text{Ran}(Z) \) as well. Note also that the family \( \{\rho_Z(\cdot; z); z \in \text{Ran}(Z)\} \) is also regular on \( \text{Ran}(Z) \), upon the subset \( \{g_z; z \in \text{Ran}(Z)\} \) of \( \{g_z; z \in \mathbb{R}\} \) is regular on \( \text{Ran}(Z) \). Furthermore, when \( z \in \text{Ran}(Z) \), then (3.1) defines a distortion risk measure \( \rho_Z(\cdot; z) \) with respect to environment \( Z = z \) in the sense of Definition 2.5, which is transparently normalized, and it is the case what we will really concern in the sequel.
(ii) Noting (3.1), we can rewrite $\rho(X; Z)$ defined by (3.2) in terms of $\rho_Z(\cdot; z)$ as in (3.1):

$$
\rho(X; Z) = \int_{-\infty}^{0} [h_{z} \circ P_{Z} (\{ z : \rho_{Z}(X; z) > \beta \}) - 1] d\beta + \int_{0}^{\infty} h_{z} \circ P_{Z} (\{ z : \rho_{Z}(X; z) > \beta \}) d\beta
$$

$$
= \int_{-\infty}^{0} [h_{z} \circ P_{Z}^* (\{ z : \rho_{Z}(X; z) > \beta \} \cap \text{Ran}(Z)) - 1] d\beta
$$

$$
+ \int_{0}^{\infty} h_{z} \circ P_{Z}^* (\{ z : \rho_{Z}(X; z) > \beta \} \cap \text{Ran}(Z)) d\beta
$$

$$
= \int \rho_{Z}(X; \cdot) dh_{z} \circ P_{Z}^*,
$$

(3.3)

for $(X, Z) \in \mathfrak{X} \times \mathfrak{X}^\perp (P)$.

Now, we turn to study the properties of distortion risk measures $\rho_{Z}(\cdot; z)$ defined by (3.1) and $\rho$ defined by (3.2).

**Proposition 3.1** Let $\{ g_{z} ; z \in \mathbb{R} \}$ be a family of left-continuous distortion functions, which is regular on $\mathbb{R}$.

1. Let $Z \in \mathfrak{X}^\perp (P)$ be an arbitrarily given random environment, and $N_{0} \in \mathfrak{B}(\mathbb{R})$ the $P_{Z}$-null set as in (2.2). For each $z \in \mathbb{R}$, let the functional $\rho_{Z}(\cdot; z)$ be the normalized distortion risk measure with respect to $K_{Z}(z, \cdot)$ defined by (3.1). Then for every $z \in N_{0} \cap \text{Ran}(Z)$, the normalized distortion risk measure $\rho_{Z}(\cdot; z)$ with respect to environment $Z = z$ satisfies Axioms (AI.1)–(AI.4).

2. For each random environment $Z \in \mathfrak{X}^\perp (P)$, let $h_{z}$ be a distortion function associated with $Z$. For each $z \in \text{Ran}(Z)$, let the functional $\tau_{Z}(\cdot; z) : \mathfrak{X}(K_{Z}(z, \cdot)) \to \mathbb{R}$ in Assumption A be $\rho_{Z}(\cdot; z)$ defined by (3.2) with respect to random environments satisfies Axioms (AII.1)–(AII.3). In addition, if the distortion function $h_{z}$ is left-continuous, and each distortion function in the subset $\{ g_{z} ; z \in \text{Ran}(Z) \}$ of $\{ g_{z} ; z \in \mathbb{R} \}$ is continuous, then the distortion risk measure $\rho$ defined by (3.2) further satisfies Axiom (AII.4).

Proposition 3.1(1) says that for $P_{Z} - \text{a.e. } z \in \text{Ran}(Z)$, the normalized distortion risk measure $\rho_{Z}(\cdot; z)$ defined by (3.1) with respect to environment $Z = z$ satisfies Axioms (AI.1)–(AI.4), when the distortion function $g_{z}$ is left-continuous. An interesting question is whether or not the converse assertion is true. In other words, given a family $\{ \rho_{Z}(\cdot; z) ; z \in \text{Ran}(Z) \}$ of normalized risk measures with respect to environment $Z = z$, if for $P_{Z} - \text{a.e. } z \in \text{Ran}(Z)$, the normalized risk measure $\rho_{Z}(\cdot; z)$ with respect to environment $Z = z$ satisfies Axioms (AI.1)–(AI.4), are they of the form as in (3.1)? This is indeed the issue of the so-called representations for functionals in the risk measure literature. The following propositions positively answer this question. Basically, the approaches to those representations are the same as ones in the classic setting such as in Denneberg (1994) and Wang et al. (1997), which can also be seen from the subsequent proofs of the propositions. Nevertheless, since those representations will be crucial to the subsequent study, and will play important roles in the study of representations for risk measures with respect to random environments, where the state of a random environment is not pre-specified when the risk of a position is evaluated, we think it would be necessary and helpful to clarify them in the name of propositions.

**Proposition 3.2** Given a random environment $Z \in \mathfrak{X}^\perp (P)$, assume that for each state $z \in \text{Ran}(Z)$, there is a normalized risk measure $\rho_{Z}(\cdot; z) : \mathfrak{X}(K_{Z}(z, \cdot)) \to \mathbb{R}$ with respect to
environment $Z = z$. If for $P_Z \text{-- a.e. } z \in \text{Ran}(Z)$, the normalized risk measure $\rho_Z(\cdot; z)$ with respect to environment $Z = z$ satisfies Axioms (AI.2) and (AI.3), then for every such $P_Z \text{-- a.e. } z \in \text{Ran}(Z)$, there exists a monotone and normalized set function $\gamma_z$ on $\mathcal{F}$ depending on $z$ and uniquely determined by $\rho_Z(\cdot; z)$ such that for any $X \in \mathcal{X}_+(K_Z(z, \cdot))$,

$$\rho_Z(X; z) = \int_0^\infty \gamma_z(\{X > \alpha\})d\alpha.$$  

Particularly, for every $A \in \mathcal{F}$, $\gamma_z(A) := \rho_Z(1_A; z)$.

**Proposition 3.3** Given a random environment $Z \in \mathcal{X}^{\pm}(P)$, assume that for each state $z \in \text{Ran}(Z)$, there is a normalized risk measure $\rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R}$ with respect to environment $Z = z$. If for $P_Z \text{-- a.e. } z \in \text{Ran}(Z)$, the normalized risk measure $\rho_Z(\cdot; z)$ with respect to environment $Z = z$ satisfies Axioms (AI.1)--(AI.4), then there is a $P_Z$--null set $N \in \mathcal{B}(\mathbb{R})$, so that for every $z \in N^c \cap \text{Ran}(Z)$, there exists a left-continuous distortion function $g_z$ depending on $z$ such that for any $X \in \mathcal{X}_+(K_Z(z, \cdot))$,

$$\rho_Z(X; z) = \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\})d\alpha.$$  

Particularly, for $u \in [0, 1]$, $g_z(u) := \rho_Z\left(1_{\{U > 1-u\}}; z\right)$, where $U$ is a $U[0, 1]$ random variable on $(\Omega, \mathcal{F}, P)$ independent of $Z$.

Next proposition extends Proposition 3.3 to general random losses.

**Proposition 3.4** Given a random environment $Z \in \mathcal{X}^{\pm}(P)$, assume that for each state $z \in \text{Ran}(Z)$, there is a normalized risk measure $\rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R}$ with respect to environment $Z = z$. If for $P_Z \text{-- a.e. } z \in \text{Ran}(Z)$, the normalized risk measure $\rho_Z(\cdot; z)$ with respect to environment $Z = z$ satisfies Axioms (AI.1)--(AI.4), then there is a $P_Z$--null set $N \in \mathcal{B}(\mathbb{R})$ as in Proposition 3.3, so that for every $z \in N^c \cap \text{Ran}(Z)$, there exists a left-continuous distortion function $g_z$ depending on $z$ such that for any $X \in \mathcal{X}(K_Z(z, \cdot))$,

$$\rho_Z(X; z) = \int_{-\infty}^0 (g_z \circ K_Z(z, \{X > \alpha\}) - 1)d\alpha + \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\})d\alpha,$$

where the distortion function $g_z$ is as in Proposition 3.3.

**Remark 3.3** As pointed out in the previous section, when the random environment $Z$ is degenerate, then for each $z \in \mathbb{R}$, $K_Z(z, \cdot) = P(\cdot)$. Intuitively, in this case, the randomness of the environment disappears. Hence, Propositions 3.3 and 3.4 basically go back to the classic setting, see Wang et al. (1997). The main distinction between this paper and that of Wang et al. (1997) lies in the assumptions employed. Precisely, given a random environment $Z \in \mathcal{X}^{\pm}(P)$, $\mathcal{X}_+$ contains a continuous random variable under probability measure $K_Z(z, \cdot)$ for $P_Z \text{-- a.e. } z \in \text{Ran}(Z)$, whereas Wang et al. (1997) assumed that the collection of risks contains all the Bernoulli$(u)$ random variables, $0 \leq u \leq 1$.

Meanwhile, Proposition 3.1(2) says that the normalized distortion risk measure $\rho$ defined by $\text{[3.2]}$ with respect to random environments satisfies Axioms (AI.1)--(AI.4), upon the distortion function $h_Z$ is left-continuous, and each distortion function in $\{g_z, z \in \text{Ran}(Z)\}$ is continuous. An interesting question is how about the converse assertion. In other words, if a normalized risk measure $\rho$ with respect to random environments satisfies Axioms (AI.1)--(AI.4), is it of the form as in $\text{[3.2]}$? Indeed again, this lies in the issue of representations.
for risk measures. The following theorems positively answer this question, which are three of the main results of this paper.

**Theorem 3.1** Suppose that Assumption A holds. If a normalized risk measure \( \rho : \mathcal{X} \times \mathcal{X}^\perp(P) \to \mathbb{R} \) with respect to random environments satisfies Axioms (AII.2)–(AII.4), then for any random environment \( Z \in \mathcal{X}^\perp(P) \), there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) as in Proposition 3.3, so that there exist a family \( \{g_z : z \in N^c \cap \text{Ran}(Z)\} \) of left-continuous distortion functions depending on the states of \( Z \) and a monotone, normalized set function \( \gamma_Z \) on \( \mathcal{B}(\text{Ran}(Z)) \) depending on \( Z \) such that for any \( X \in \mathcal{X}_+ \),

\[
\rho(X; Z) = \int_0^\infty \gamma_Z \left( \left\{ z \in N^c \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\}) d\alpha > \beta \right\} \right) d\beta,
\]

where the distortion function \( g_z \) is as in Proposition 3.3.

**Theorem 3.2** Suppose that Assumption A holds. If a normalized risk measure \( \rho : \mathcal{X} \times \mathcal{X}^\perp(P) \to \mathbb{R} \) with respect to random environments satisfies Axioms (AII.1)–(AII.4), then for any random environment \( Z \in \mathcal{X}^\perp(P) \), there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) as in Theorem 3.1, so that there exist a family \( \{g_z : z \in N^c \cap \text{Ran}(Z)\} \) of left-continuous distortion functions depending on the states of \( Z \) and a function \( h_Z : [0, 1] \to [0, 1] \) depending on \( Z \) with \( h_Z(0) = 0 \) and \( h_Z(1) = 1 \) such that for any \( X \in \mathcal{X}_+ \),

\[
\rho(X; Z) = \int_0^\infty h_Z \circ P_Z^\ast \left( \left\{ z \in N^c \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\}) d\alpha > \beta \right\} \right) d\beta,
\]

where the distortion function \( g_z \) is as in Theorem 3.1.

Provided with some axioms, Theorems 3.1 and 3.2 give representations for risk measures of non-negative random losses with respect to random environments. Next theorem extends Theorem 3.2 to general random losses.

**Theorem 3.3** Suppose that Assumption A holds. If a normalized risk measure \( \rho : \mathcal{X} \times \mathcal{X}^\perp(P) \to \mathbb{R} \) with respect to random environments satisfies Axioms (AII.1)–(AII.4), then for any random environment \( Z \in \mathcal{X}^\perp(P) \), there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) as in Theorem 3.2, so that there exist a family \( \{g_z : z \in N^c \cap \text{Ran}(Z)\} \) of left-continuous distortion functions depending on the states of \( Z \) and a function \( h_Z : [0, 1] \to [0, 1] \) depending on \( Z \) with \( h_Z(0) = 0 \) and \( h_Z(1) = 1 \) such that for any \( X \in \mathcal{X}_+ \),

\[
\rho(X; Z) = \int_{-\infty}^0 \left[ h_Z \circ P_Z^\ast \left( \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \beta \right\} \right) - 1 \right] d\beta \\
+ \int_0^\infty h_Z \circ P_Z^\ast \left( \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \beta \right\} \right) d\beta,
\]

(3.4)

where the function \( h_Z \) is as in Theorem 3.2, and for \( z \in N^c \cap \text{Ran}(Z) \), \( \tau_Z(\cdot; z) \) as in Assumption A is given by

\[
\tau_Z(X; z) = \int_{-\infty}^0 [g_z \circ K_Z(z, \{X > \alpha\}) - 1] d\alpha + \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\}) d\alpha,
\]

where the distortion function \( g_z \) is as in Theorem 3.2.
3.2 Coherence and dual presentation

In this subsection, we will discuss the coherence of distortion risk measures $\rho$, defined by (3.2), with respect to random environments. It will be shown that when the distortion functions are concave, then the distortion risk measure with respect to a random environment is coherent in the sense of Artzner et al. (1999). Furthermore, dual representations for them are provided.

In general, a risk measure is defined as any mapping $\rho : \mathcal{X} \to \mathbb{R}$. By Artzner et al. (1999), a risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called coherent, if it satisfies the following four properties (or axioms):

(i) Monotonicity: $X \leq Y$ implies $\rho(X) \leq \rho(Y)$ for any $X, Y \in \mathcal{X}$.

(ii) Translation invariance: $\rho(X + a) = \rho(X) + a$ for any $X \in \mathcal{X}$ and $a \in \mathbb{R}$.

(iii) Positive homogeneity: $\rho(cX) = c\rho(X)$ for each $c > 0$ and $X \in \mathcal{X}$.

(iv) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in \mathcal{X}$.

Now, we state one of the main results of this subsection, which concerns the coherence of distortion risk measures with respect to random environments.

**Theorem 3.4** Let $\{g_z; z \in \mathbb{R}\}$ be a family of concave distortion functions, which is regular on $\mathbb{R}$. Let a random environment $Z \in \mathcal{X}^+(P)$ be fixed, and $h_Z$ be a concave distortion function associated with $Z$. Then the risk measure $\rho_Z : \mathcal{X} \to \mathbb{R}$ defined by

$$\rho_Z(X) := \int_{-\infty}^{0} \left[ h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) d\alpha > \beta \right\} - 1 \right) \right] d\beta$$

$$+ \int_{0}^{\infty} h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) d\alpha > \beta \right\} \right) d\beta,$$

$X \in \mathcal{X}$, is coherent.

Theorem 3.4 says that the risk measure $\rho_Z$ defined by (3.5) is coherent. Hence, by Artzner et al. (1999) or Föllmer and Schied (2016), it should also have a dual representation. Next, we turn to discuss the dual representation for $\rho_Z$. It turns out that $\rho_Z$ can be expressed by means of a repeated Choquet integral on some product space with respect to two finitely additive probability measures. For this purpose, we need a little more preparations.

Given a monotone set function $\mu_1 : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_+$ and a set of monotone set functions $\mu_2 := \{\mu_2(z, \cdot); z \in \mathbb{R}\}$ on $\mathcal{F}$ satisfying that $\mu_2(\cdot, A)$ is a Borel function on $\mathbb{R}$ for every $A \in \mathcal{F}$, we define a set function $\mu$ on $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$, the product $\sigma$-algebra of $\mathcal{B}(\mathbb{R})$ and $\mathcal{F}$, through

$$\mu(A) := \int \left( \int 1_A(z, \omega) \mu_2(z, d\omega) \right) \mu_1(dz), \quad A \in \mathcal{B}(\mathbb{R}) \times \mathcal{F},$$

16
where the repeated integral is understood in the sense of Choquet integral. Apparently, \(\mu\) is monotone, see also Proposition 12.1(i) of Denneberg (1994). Such a defined \(\mu\) is called the generalized product of the system \(\{\mu_1, \mu_2(z, \cdot); z \in \mathbb{R}\}\), and is denoted by \(\mu := \mu_1 \otimes \mu_2\).

For a bounded random variable \(Y\) on \((\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \times \mathcal{F})\), the Choquet integral of \(Y\) with respect to \(\mu\) is denoted by \(\int Y d\mu\). If \(\mu_1\) is finitely additive (i.e. \(\mu_1(A \cup B) = \mu_1(A) + \mu_1(B)\) for disjoint sets \(A\) and \(B\)), then an application of Proposition 12.1(iv) of Denneberg (1994) implies that

\[
\int Y d\mu = \int \left( \int Y(z, \omega)\mu_2(z, d\omega) \right) \mu_1(dz).
\]

In the sequel, in order to avoid tedious measurability considerations, it is sometimes convenient and helpful for us to extend a monotone set function \(\nu\) defined on \(\mathcal{B}(\mathbb{R})\) onto the whole power set \(2^{\mathbb{R}}\), the family of all subsets of \(\mathbb{R}\). Let \(\nu: \mathcal{B}(\mathbb{R}) \to \mathbb{R}_+\) be a monotone set function on \(\mathcal{B}(\mathbb{R})\). Define

\[
\tilde{\nu}(A) := \inf \{\nu(B) : A \subseteq B \in \mathcal{B}(\mathbb{R})\}, \quad A \in 2^{\mathbb{R}}.
\]

The set function \(\tilde{\nu}: 2^{\mathbb{R}} \to \mathbb{R}_+\) is called outer set function of \(\nu\). Apparently, \(\nu = \tilde{\nu}\) on \(\mathcal{B}(\mathbb{R})\). Furthermore, it is known that \(\tilde{\nu}\) is monotone on \(2^{\mathbb{R}}\), see Proposition 2.4(i) of Denneberg (1994). Under this point of view, in the course of defining the generalized product \(\hat{\nu} := \hat{\nu}_1 \otimes \hat{\nu}_2\) on \(2^{\mathbb{R}}\) as just described above, we could replace \(\mu_1\) with its outer set function \(\tilde{\mu}_1\), and drop the assumption on \(\mu_2\) that \(\mu_2(\cdot, A)\) is a Borel function on \(\mathbb{R}\) for every \(A \in \mathcal{F}\). Then we would define a generalized product \(\hat{\mu} := \hat{\mu}_1 \otimes \hat{\mu}_2\) on \(2^{\mathbb{R}} \times \mathcal{F}\), the product \(\sigma\)-algebra of \(2^{\mathbb{R}}\) and \(\mathcal{F}\). In this situation, we could similarly define the Choquet integral \(\int Y d\hat{\mu}\) of a bounded random variable \(Y\) on \((\mathbb{R} \times \Omega, 2^{\mathbb{R}} \times \mathcal{F})\) with respect to \(\hat{\mu} := \hat{\mu}_1 \otimes \hat{\mu}_2\), and all relevant conclusions still remain true. Particularly, \(\int Y d\hat{\mu} = \int Y d\mu\) for all bounded random variables \(Y\) on \((\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \times \mathcal{F})\). For more details, we refer to Chapters 2 and 12 of Denneberg (1994).

We introduce more notations. \(\mathcal{M}_{1,f}(\Omega, \mathcal{F})\) denotes the set of all finitely additive normalized set functions \(Q: \mathcal{F} \to [0, 1]\), and \(E_Q(X)\) denotes the integral of \(X \in \mathcal{F}\) with respect to \(Q\), as constructed in Theorem A.54 of Föllmer and Schied (2016). An application of Lemma 4.97 of Föllmer and Schied (2016) yields that the integral \(E_Q(X)\) is equal to the Choquet integral \(\int X dQ\). Similarly, \(\mathcal{M}_{1,f}(\mathbb{R}, \mathcal{B}(\mathbb{R}))\) denotes the set of all finitely additive normalized set functions \(Q: \mathcal{B}(\mathbb{R}) \to [0, 1]\), and \(E_Q(W)\) denotes the integral of a Borel function \(W\) on \(\mathbb{R}\) with respect to \(Q\). Again, applying Lemma 4.97 of Föllmer and Schied (2016) to the measurable space \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) implies that the integral \(E_Q(W)\) is equal to the Choquet integral \(\int W dQ\) for bounded Borel function \(W\) on \(\mathbb{R}\).

Now, we are ready to state the dual representation for \(\rho_Z\) defined by (3.5), which is another main result of this subsection.

**Theorem 3.5** Let \(\{g_z; z \in \mathbb{R}\}\) be a family of concave distortion functions, which is regular on \(\mathbb{R}\). Let a random environment \(Z \in \mathcal{X}^\perp(P)\) be fixed, and \(h_Z\) be a concave distortion function associated with \(Z\). Denote \(\mathcal{D}_1 := \{Q_1 \in \mathcal{M}_{1,f}(\mathbb{R}, \mathcal{B}(\mathbb{R})): Q_1(B) \leq h_Z \circ P_Z(B)\) for all \(B \in \mathcal{B}(\mathbb{R})\}\), \(\mathcal{D}_2 := \{Q_2 := \{Q_2(z, \cdot) \in \mathcal{M}_{1,f}(\Omega, \mathcal{F}); z \in \mathbb{R}\} : \text{ for every } z \in \mathbb{R}, Q_2(z, A) \leq g_z \circ K_Z(z, A) \text{ for all } A \in \mathcal{F}\}\). Let \(\mathcal{C} := \{(Q_1, Q_2): Q_1 \in \mathcal{D}_1, Q_2 \in \mathcal{D}_2\}\). Then for any position \(X \in \mathcal{X}\),

\[
\rho_Z(X) = \sup_{(Q_1, Q_2) \in \mathcal{C}} \int \left( \int X(\omega)Q_2(z, d\omega) \right) \hat{Q}_1(dz), \tag{3.6}
\]
Let $\phi$ where

**Remark 3.4** If the random environment $Z$ is degenerate, say $P(Z = z_0) = 1$ with some $z_0 \in \mathbb{R}$, then for each $z \in \mathbb{R}$, $K_Z(z, \cdot) = P(\cdot)$, and $\mathcal{D}_1$ is the singleton $\{\delta_{z_0}\}$ consisting of the Dirac measure concentrating at $z_0$. Hence, (3.6) becomes

$$\rho_{z_0}(X) = \sup_{Q(z_0, \cdot) \in \mathcal{D}} \int X(\omega) Q(z_0, d\omega) = \sup_{Q(z_0, \cdot) \in \mathcal{D}} E Q(z_0, \cdot)[X],$$

where $\mathcal{D} := \{Q(z_0, \cdot) \in \mathcal{M}_1 f(\Omega, \mathcal{F}) : Q(z_0, A) \leq g_{z_0} \circ P(A) \text{ for all } A \in \mathcal{F}\}$. This is nothing else, but just the classical representation result, for example, see Theorem 4.94 (c) and (d) of Föllmer and Schied (2016).

### 3.3 Distortion risk measures in the presence of background risk

From the point of view of risk contribution, Tsanakas (2008) studied risk measures in the presence of background risk. In this subsection, from the capital requirement perspective, we will introduce new risk measures in the presence of background risk, by making use of the distortion risk measure defined as in (3.2). Note also that for any $(X, Z) \in \mathcal{X} \times \mathcal{X}$, the distortion risk measure defined as in (3.2) is still well-defined, and the properties stated in Proposition 3.2(2) keep true, though in (3.2) the random environment $Z$ is supposed to belong to $\mathcal{X}^+(P)$. Alternatively, we could assume that $\mathcal{X}^+(P) = \mathcal{X}$. Throughout this subsection, let $g : [0, 1] \to [0, 1]$ be an increasing, concave and differentiable function with $g(0) = 0, g(1) = 1$ and bounded first derivative $g'$.

Let $X$ be a random loss and $Y$ a background risk. For the convenience of our discussion and without loss of generality, we assume that $X, Y \in \mathcal{X}$. Define $Z := X + Y$ serving as a random environment, $F_Z(z) := P(Z > z), z \in \mathbb{R}$, and a distribution function $s(z) := 1 - g(F_Z(z)), z \in \mathbb{R}$. Let $L_s$ be the Lebesgue-Stieltjes measure induced by $s(z)$ on $\mathcal{B}(\mathbb{R})$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $E[X|Z] = \phi(Z)$ and $E[X|Z = z] = \phi(z)$ for any $z \in \text{Ran}(Z)$.

Now, making use of (3.2), we turn to construct a distortion risk measure of $X$ with respect to random environment $Z$. For simplicity, for any $z \in \text{Ran}(Z)$, let the distortion function $g_z$ in (3.2) be the identity function, that is, $g_z(x) = x$, and let the distortion function $h := h_Z$ in (3.2) be chosen later. Thanks to (3.2), we define a distortion risk measure $\rho_h(X; Z)$ of $X$ with respect to random environment $Z$ by

$$\rho_h(X; Z) := \int_{-\infty}^{0} \left[ h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (K_Z(z, \{ X > \alpha \}) - 1) \, d\alpha \right\} \right) + \int_{0}^{\infty} K_Z(z, \{ X > \alpha \}) \, d\alpha > \beta \right) - 1 \right] d\beta$$

$$+ \int_{0}^{\infty} \left[ h_Z \circ P_Z \left( \left\{ z : \int_{-\infty}^{0} (K_Z(z, \{ X > \alpha \}) - 1) \, d\alpha \right\} \right) + \int_{0}^{\infty} K_Z(z, \{ X > \alpha \}) \, d\alpha > \beta \right) \, d\beta. \quad (3.7)$$

We call $\rho_h(X; Z)$ the distortion risk measure of $X$ with respect to background risk $Y$, where
$Z := X + Y$. Note that by (2.1),

$$\int_{-\infty}^{0} (K_{Z}(z; \{X > \alpha\}) - 1)d\alpha + \int_{0}^{\infty} K_{Z}(z; \{X > \alpha\})d\alpha$$

$$= \int_{\Omega} X(\omega)K_{Z}(z, d\omega) = E[X|Z = z] = \phi(z) \quad \text{for } P_{Z} - \text{a.e. } z \in \mathbb{R},$$

which, along with (3.7), yields an alternative expression for $\rho_{h}(X; Z)$ :

$$\rho_{h}(X; Z) = \int_{-\infty}^{0} [h_{Z} \circ P_{Z}(\phi > \beta) - 1]d\beta + \int_{0}^{\infty} h_{Z} \circ P_{Z}(\phi > \beta)d\beta. \quad (3.8)$$

Next, we will specify the distortion function $h_{Z}$ as above. For this purpose, we define two functions $v, u: \mathbb{R} \to [0, 1]$ by

$$v(x) := L_{s}(\{z: \phi(z) > x\}), \; x \in \mathbb{R},$$

$$u(x) := P_{Z}(\{z: \phi(z) > x\}), \; x \in \mathbb{R}. \quad \text{Then } v \text{ and } u \text{ are non-increasing and right-continuous functions, and take values in } [0, 1].$$

We denote by $u^{-1}$ and $u^{-1+}$ the left-continuous and right-continuous inverse functions of $u$, respectively, that is, for any $p \in (0, 1)$,

$$u^{-1}(p) := \inf\{x \in \mathbb{R} : u(x) \leq p\},$$

$$u^{-1+}(p) := \sup\{x \in \mathbb{R} : u(x) \geq p\},$$

with $u^{-1}(0) := u^{-1+}(0) := +\infty$ and $u^{-1}(1) := u^{-1+}(1) := -\infty$ by convention. Note that $u^{-1}(p)$ and $u^{-1+}(p)$ are finite for all $p \in (0, 1)$. Also clearly,

$$u^{-1}(p) \leq u^{-1+}(p) \quad \text{for any } p \in (0, 1).$$

Note that let $x \in \mathbb{R}$ be such that $0 < u(x) < 1$, then $u^{-1}(u(x))$ and $u^{-1+}(u(x))$ are finite, and

$$u^{-1}(u(x)) \leq x \leq u^{-1+}(u(x)). \quad (3.9)$$

For more details about above different inverse functions of $u$, we refer to Dhaene et al. (2002) and Föllmer and Schied (2016, Appendix A.3).

Define two functions $h_{L}$ and $h_{R}$ on $[0, 1]$, respectively, by

$$h_{L}(p) := v[u^{-1}(p)], \; p \in [0, 1],$$

$$h_{R}(p) := v[u^{-1+}(p)], \; p \in [0, 1].$$

Clearly, $h_{L}$ and $h_{R}$ are two distortion functions. We denote by $\rho_{L}$ and $\rho_{R}$ the distortion risk measure $\rho_{h}$ as in (3.8), when the distortion function $h_{Z}$ is chosen as $h_{L}$ and $h_{R}$, respectively. Note that $v$ is non-increasing, from (3.9) and the definitions of $h_{L}$ and $h_{R}$, it follows that for any $\beta \in \mathbb{R},$

$$h_{L}(u(\beta)) = v[u^{-1}(u(\beta))] \geq v(\beta) \geq v[u^{-1+}(u(\beta))] = h_{R}(u(\beta)), \quad (3.10)$$

19
which, together with (3.8), results in the following proposition:

**Proposition 3.5** We have that

\[ \rho_R(X; Z) \leq \rho_L(X; Z). \]

Let us end this subsection with a brief review of distortion risk measure in the presence of background risk introduced by Tsanakas (2008). In Tsanakas (2008), the distortion risk measure \( \Gamma(X; Y) \) of risk \( X \) with respect to background risk \( Y \) is defined by

\[ \Gamma(X; Y) := E[X g'(F_Z(Z))], \quad (3.11) \]

see Tsanakas (2008). By the total expectation law,

\[ E[X g'(F_Z(Z))] = E[E(X|Z)g'(F_Z(Z))] = \int_{-\infty}^{\infty} \phi(z)g'(F_Z(z))F_Z(dz). \quad (3.12) \]

By Fubini’s theorem, an elementary calculation yields that

\[ \int_{-\infty}^{\infty} \phi(z)g'(F_Z(z))F_Z(dz) = \int_{-\infty}^{0} [L_s(\phi > \beta) - 1] d\beta + \int_{0}^{\infty} L_s(\phi > \beta) d\beta, \]

which, together with (3.11) and (3.12), implies

\[ \Gamma(X; Y) = \int_{-\infty}^{0} [L_s(\phi > \beta) - 1] d\beta + \int_{0}^{\infty} L_s(\phi > \beta) d\beta. \quad (3.13) \]

Consequently, from (3.8), (3.10) and (3.13), it follows that

\[ \rho_R(X; Z) \leq \Gamma(X; Y) \leq \rho_L(X; Z). \]

### 4 Examples

In this section, by examples, we will illustrate the proposed distortion risk measures with respect to random environments. In the first subsection, we will show that under a mild condition, the distortion risk measures with respect to random environments \( \rho \) defined by (3.2) can recover the common WVaR and RVaR, where the random environment can be interpreted as a financial institution’s risk preference. In the second subsection, we will calculate the distortion risk measure of a financial derivative, where the random environment represents the price of the underlying asset, and is assumed to obey the geometric Brownian motion. It will be revealed that the riskiness of the underlying asset can be transmitted to its derivative, which is in accordance with intuition.

#### 4.1 WVaR and RVaR

Throughout this subsection, for simplicity, we assume that \( \mathcal{X}^{-1}(P) = \mathcal{X} \). Let us briefly recall the definitions of WVaR introduced by Acerbi (2002) and Cherny (2006) and RVaR studied by Cont et al. (2010) and Embrechts et al. (2018).
Given a probability measure $\nu$ on $(0, 1)$, the WVaR of a position $X \in \mathcal{X}(P)$ with respect to $\nu$ is defined by

$$\text{WVaR}_{\nu}(X) := \int_{(0,1)} \text{AVaR}_{\theta}(X) \nu(d\theta).$$

Given two confidence levels $0 < \alpha_1 < \alpha_2 < 1$, the RVaR of a position $X \in \mathcal{X}(P)$ cross the confidence level range $[\alpha_1, \alpha_2]$ is defined by

$$\text{RVaR}_{\alpha_1,\alpha_2}(X) := \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \text{VaR}_{\theta}(X) d\theta.$$

Let $\mu$ be a probability measure on $(0, 1)$. Denote by $F_\mu$ the distribution function induced by $\mu$, that is,

$$F_\mu(x) := \begin{cases} 
\mu((0, 1) \cap (0, x]), & \text{if } x > 0, \\
0, & \text{if } x \leq 0.
\end{cases}$$

Denote by $F_\mu^{-1+}$ the right-continuous inverse function of $F_\mu$, that is $F_\mu^{-1+}(p) := \inf\{x \in \mathbb{R} : F_\mu(x) > p\}$, $p \in (0, 1)$; $F_\mu^{-1+}(0) := 0$, $F_\mu^{-1+}(1) := 1$.

Given a position $X \in \mathcal{X}$, by the assumption that $\mathcal{X}^{-1}(P) = \mathcal{X}$, let $U$ be a $U[0, 1]$ random variable on $(\Omega, \mathcal{F}, P)$ independent of $X$. Define a random environment $Z := F_\mu^{-1+}(U)$, then $Z$ is independent of $X$ and $P_Z = \mu$. Denote by $N_0$ the $P_Z$--null set as in (2.2). Hence, by (2.1) we know that there exists a $P_Z$--null set $N \in \mathcal{B}(\mathbb{R})$ with $N_0 \subseteq N$ such that for every $z \in N^c$ and any $t \in \mathbb{R}$,

$$K_Z(z, \{X > t\}) = E[1_{\{X > t\}} | Z = z] = P(X > t). \quad (4.1)$$

For $Z$ as above, the continuous distortion function $h_Z$ associated with $Z$ is defined as the identity function, that is,

$$h_Z(x) := x, \quad x \in [0, 1]. \quad (4.2)$$

Next, we proceed by two cases:

**Case 1:** For each $z \in (0, 1)$, a concave distortion function $g_z : [0, 1] \rightarrow [0, 1]$ is defined by

$$g_z(x) := \begin{cases} 
\frac{x}{1-z}, & \text{if } 0 \leq x \leq 1 - z, \\
1, & \text{if } 1 - z < x \leq 1.
\end{cases} \quad (4.3)$$

while, for $z \notin (0, 1)$, we define $g_z : [0, 1] \rightarrow [0, 1]$ by $g_z(x) := x$. Clearly, the family $\{g_z; z \in \mathbb{R}\}$ is regular on $\mathbb{R}$, and thus it is also regular on $(0, 1)$. It is well-known that to the distortion function $g_z$ as in (4.3), the corresponding distortion risk measure is exactly the AVaR, for example, see Belles-Sampera et al. (2014), Föllmer and Schied (2016, Example 4.71), or Wang and Ziegel (2021).

For each $z \in [0, 1]$, let the risk measure $\rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \rightarrow \mathbb{R}$ be as in (3.1), and the risk measure $\rho : \mathcal{X} \times \mathcal{X}^{-1}(P) \rightarrow \mathbb{R}$ as in (3.2). Noting (4.1), after plugging distortion functions $g_z$ as above into (3.1), we have that for every $z \in N^c \cap \text{Ran}(Z),$

$$\rho_Z(X; z) = \int_{-\infty}^{0} (g_z \circ P(X > \alpha) - 1) d\alpha + \int_{0}^{\infty} g_z \circ P(X > \alpha) d\alpha = \text{AVaR}_z(X). \quad (4.4)$$
Recall that the Choquet integral of $\rho_Z(X; \cdot)$ with respect to the probability measure $P^*_Z$ is equal to the integral of $\rho_Z(X; \cdot)$ with respect to $P^*_Z$, see Lemma 4.97 of Föllmer and Schied (2016). Consequently, substituting (4.1) into (3.2), by (3.3) and (4.1) we have that

$$\rho(X; Z) = \int \rho_Z(X; z)P^*_Z(dz) = \int_{(0,1)} \mbox{AVaR}_z(X)\mu(dz),$$

which is exactly the WVaR$_\mu(X)$.

**Case 2:** Given two confidence levels $0 < \alpha_1 < \alpha_2 < 1$, suppose that the probability measure $\mu$ has a probability density function

$$f_\mu(x) := \frac{1}{\alpha_2 - \alpha_1} 1_{[\alpha_1, \alpha_2]}(x), \quad x \in (0,1).$$

Define a family $\{g_z; z \in \mathbb{R}\}$ of left-continuous distortion functions by

$$g_z(x) := \begin{cases} 0, & \text{if } 0 \leq x \leq 1 - z, \\ 1, & \text{if } 1 - z < x \leq 1, \end{cases}$$

when $z \in [\alpha_1, \alpha_2]$, while $g_z(x) := x, x \in [0,1]$, when $z \notin [\alpha_1, \alpha_2]$. Apparently, $\{g_z; z \in \mathbb{R}\}$ is regular on $\mathbb{R}$. It is well-known that to the distortion function $g_z, z \in [\alpha_1, \alpha_2]$, the corresponding distortion risk measure is exactly the VaR, for example, see Belles-Sampera et al. (2014) or Wang and Ziegel (2021).

For any $z \in [\alpha_1, \alpha_2]$, again let the risk measure $\rho_Z(\cdot; z) : \mathcal{X}(K_Z(z, \cdot)) \to \mathbb{R}$ be as in (3.1), and the risk measure $\rho : \mathcal{X} \times \mathcal{X}^+(P) \to \mathbb{R}$ as in (3.2). Noting (4.1), after substituting distortion functions $g_z$ as above into (3.1) we know that for every $z \in N^c \cap \mbox{Ran}(Z)$,

$$\rho_Z(X; z) = \int_{-\infty}^0 (g_z \circ P(X > \alpha) - 1)d\alpha + \int_0^\infty g_z \circ P(X > \alpha)d\alpha = \mbox{VaR}_z(X).$$

Consequently, by (3.2), (3.3) and Lemma 4.97 of Föllmer and Schied (2016), we know that

$$\rho(X; Z) = \int \rho_Z(X; z)P^*_Z(dz) = \int_{[\alpha_1, \alpha_2]} \mbox{VaR}_z(X)\mu(dz) = \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \mbox{VaR}_\theta(X)d\theta,$$

which is exactly the RVaR$_{\alpha_1, \alpha_2}(X)$.

### 4.2 Calculation for risk measures of financial derivatives

Note that in the definition of $\rho(X; Z)$ as in (3.2), although there is a boundedness requirement on the position $X$ and the random environment $Z$, it can be well defined for integrable $X$ and $Z$ as well. In this subsection, as a simple example, under some reasonable assumptions, we address the issue how an underlying asset can affect the measures of risk of its derivatives. Consider the measure of risk of a financial derivative, say a call-type derivative, the random loss of which is denoted by $X$, we assume that its underlying asset’s price, say a stock’s price, obeys the geometric Brownian motion, that is, the underlying asset’s price satisfies the Black-Scholes model. Here, the underlying asset’s price serves as a random environment $Z$ relative to its derivative. It will be revealed that the riskiness of the underlying asset can be transmitted to its derivative, which is in accordance with intuition.
Recall that a stochastic process \( \{Z_t; \ 0 \leq t \leq T\}, \ T > 0 \), is called a geometric Brownian motion, if it is governed by the following stochastic differential equation (SDE):

\[
\begin{align*}
dZ_t &= rZ_t dt + \sigma Z_t dB_t, \quad 0 \leq t \leq T, \\
Z_0 &= 1,
\end{align*}
\]

where \( r \in \mathbb{R} \) is the return rate of the stock, \( \sigma > 0 \) is the volatility, and \( \{B_t; \ t \geq 0\} \) is a standard Brownian motion. Then the solution of the above SDE is

\[
Z_t = \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\}, \quad 0 \leq t \leq T.
\]

Consider an one-period setting, that is \( T := 1 \). Write \( \mu := r - \sigma^2 / 2 \), \( \sigma^2 := \sigma^2 \) and \( Z := Z_1 \), then \( \ln Z \) represents the logarithmic return of the stock, and is normally distributed with mean \( \mu \) and variance \( \sigma^2 \), that is, for any \( \lambda > 0 \),

\[
P_Z(\{z : z > \lambda\}) := P(Z > \lambda) := \int_{\ln \lambda}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\} \, dz = 1 - \Phi \left( \frac{\ln \lambda - \mu}{\sigma} \right),
\]

where \( \Phi \) is the distribution function of the standard normal random variable. Note that \( Z - 1 \) is the return of the stock.

For \( z > 0 \), when the stock price \( Z \) is \( z \), we assume that the expected loss of the derivative with respect to the probability measure \( K_Z(z, \cdot) \) is equal to certain multiple of the negative return \( 1 - z \) of the stock. For the sake of statement and without loss of generality, we set the multiple be one. In order to acquire an explicit expression for the distortion risk measure \( \rho(X; Z) \) defined by (3.2), we need to know the specific distribution of the random loss \( X \) of the derivative. From the practical point of view, it should be empirically acquired. Here theoretically, for simplicity as an example, we assume that \( X \) is normally distributed under \( K_Z(z, \cdot) \), considering that there might be many factors affecting the price of the derivative. More precisely, we assume that \( X \) is normally distributed with mean \( 1 - z \) and variance \( \sigma^2_2 \) with respect to \( K_Z(z, \cdot) \), that is, for any \( \alpha \in \mathbb{R} \),

\[
K_Z(z, \{X > \alpha\}) = \int_{\alpha}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - (1 - z))^2}{2\sigma^2} \right\} \, dx = 1 - \Phi \left( \frac{\alpha - (1 - z)}{\sigma_2} \right).
\]

It is conceivable that whatever the specific distribution of \( X \) is of, the return rate \( r \) of the stock would most likely appear in the expression for the distortion risk measure \( \rho(X; Z) \) as long as the distribution of \( X \) involves the negative return \( 1 - z \) of the stock. In return, this feature would indicate that the riskiness of the stock can be transmitted to its derivative.

Next, we calculate the risk measure \( \rho(X; Z) \) defined by (3.2). For simplicity, we take the distortion functions as following: for any \( z \in \mathbb{R} \),

\[
g_z(x) := \begin{cases} 
\frac{x}{1-a}, & \text{if} \quad 0 \leq x \leq 1 - a, \\
1, & \text{if} \quad 1 - a < x \leq 1,
\end{cases}
\]

where \( a \in (0, 1) \), and

\[
h_z(x) := \begin{cases} 
\frac{x}{1-b}, & \text{if} \quad 0 \leq x \leq 1 - b, \\
1, & \text{if} \quad 1 - b < x \leq 1,
\end{cases}
\]

23
where \( b \in (0, 1) \). Then, the risk measure \( \rho(X; Z) \) is given by

\[
\rho(X; Z) = 1 + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) \, dx \\
- e^\mu \left[ e^{\sigma_1 \Phi^{-1}(1-b)} - \frac{\sigma_1}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(x) e^{\sigma_1 x} \, dx \right], \tag{4.5}
\]

where \( \Phi^{-1} \) is the inverse function of \( \Phi \). We call \( \rho(X; Z) \) the distortion risk measure of the derivative \( X \), where \( Z \) represents the price of the underlying asset, and it provides an evaluation of the risk of \( X \).

**Remark 4.1** Recall that \( \mu := r - \frac{\sigma^2}{2} \) and note that

\[
\frac{\sigma_1}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(x) e^{\sigma_1 x} \, dx < \frac{\sigma_1}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(\Phi^{-1}(1-b)) e^{\sigma_1 x} \, dx = e^{\sigma_1 \Phi^{-1}(1-b)}. \tag{4.6}
\]

(4.5) shows that the larger the return rate \( r \) of the stock is, the smaller the risk measure \( \rho(X; Z) \) of the derivative is. In other words, the more risky (i.e. the smaller return rate \( r \)) the underlying asset is, the more risky (i.e. the larger loss) the derivative is. This feature also reveals that the riskiness of the underlying asset can be transmitted to its derivative, which is in accordance with intuition. There appears something interesting as well. Consider the derivative as an European call option, on one hand, the famous Black-Scholes formula for European call option states that the premium of an European call option does not depend on the return rate \( r \) of the stock due to the use of risk-neutral probability. On the other hand, (4.5) states that the risk measure of the derivative can do depend on the return rate \( r \) of the stock under our model.

**Proof of (4.5)** First, we know that for any \( z > 0 \) and any \( \alpha \in \mathbb{R} \),

\[
g_z \circ K_Z(z, \{X > \alpha\}) = \begin{cases} \frac{1}{1 - \alpha} \left( 1 - \Phi \left( \frac{\alpha - (1-z)}{\sigma_2} \right) \right), & \text{if } \alpha \geq 1 - z + \sigma_2 \Phi^{-1}(a), \\
1, & \text{if } \alpha < 1 - z + \sigma_2 \Phi^{-1}(a). \tag{4.6} \end{cases}
\]

We now claim that for any \( z > 0 \),

\[
\rho_Z(X; z) := \int_{-\infty}^{0} [g_z \circ K_Z(z, \{X > \alpha\}) - 1] \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha \\
= 1 - z + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) \, dx. \tag{4.7}
\]

In fact, for \( z > 0 \) with \( 1 - z + \sigma_2 \Phi^{-1}(a) \geq 0 \), by (4.6) and change-of-variable,

\[
\int_{-\infty}^{0} [g_z \circ K_Z(z, \{X > \alpha\}) - 1] \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha \\
= 1 - z + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) \, dx.
\]
Similarly, for $z > 0$ with $1 - z + \sigma_2 \Phi^{-1}(a) < 0$, by (4.6) and change-of-variable,
\[
\int_{-\infty}^{0} [g_z \circ K_z(z; \{X > \alpha\}) - 1] d\alpha + \int_{0}^{\infty} g_z \circ K_z(z; \{X > \alpha\}) d\alpha \\
= 1 - z + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) dx.
\]

In summary, (4.7) holds.

Write
\[
\delta := 1 + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) dx \in \mathbb{R}.
\]

Noting (4.7), it is not hard to check that for any $\beta \in \mathbb{R}$,
\[
P_Z \{z : \rho_Z(X; z) > \beta\} = \begin{cases} 0, & \text{if } \beta \geq \delta, \\ \Phi \left( \frac{\ln(\delta - \beta) - \mu}{\sigma_1} \right), & \text{if } \beta < \delta,
\end{cases}
\]
and thus,
\[
h_Z \circ P_Z \{z : \rho_Z(X; z) > \beta\} = \begin{cases} 0, & \text{if } \beta \geq \delta, \\ \frac{1}{1 - b} \Phi \left( \frac{\ln(\delta - \beta) - \mu}{\sigma_1} \right), & \text{if } \Delta \leq \beta < \delta, \\ 1, & \text{if } \beta < \Delta,
\end{cases}
\]
where $\Delta := \delta - e^{\mu} e^{\sigma_1 \Phi^{-1}(1-b)}$. Therefore, we conclude that the risk measure $\rho(X; Z)$, defined by (3.2), is given by
\[
\rho(X; Z) := \int_{-\infty}^{0} [h_Z \circ P_Z \{z : \rho_Z(X; z) > \delta\} - 1] d\beta + \int_{0}^{\infty} h_Z \circ P_Z \{z : \rho_Z(X; z) > \beta\} d\beta \\
= 1 + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) dx \\
- e^{\mu} \left[ e^{\sigma_1 \Phi^{-1}(1-b)} - \frac{\sigma_1}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(x) e^{\sigma_1 x} dx \right],
\]
which just implies the desired (4.5). Indeed, if $\Delta \geq 0$, then by (4.8) and change-of-variable,
\[
\int_{-\infty}^{0} [h_Z \circ P_Z \{z : \rho_Z(X; z) > \beta\} - 1] d\beta + \int_{0}^{\infty} h_Z \circ P_Z \{z : \rho_Z(X; z) > \beta\} d\beta \\
= \int_{0}^{\infty} h_Z \circ P_Z \{z : \rho_Z(X; z) > \beta\} d\beta \\
= 1 + \sigma_2 \Phi^{-1}(a) + \frac{\sigma_2}{1 - a} \int_{\Phi^{-1}(a)}^{+\infty} (1 - \Phi(x)) dx \\
- e^{\mu} \left[ e^{\sigma_1 \Phi^{-1}(1-b)} - \frac{\sigma_1}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(x) e^{\sigma_1 x} dx \right].
\]
Similarly, if $\Delta < 0$, then by (4.5) and change-of-variable,
\[
\int_{-\infty}^{0} [h \circ P_{Z} \{\{z : \rho_{Z}(X; z) > \beta\}\} - 1] d\beta + \int^{\infty}_{0} h \circ P_{Z} \{\{z : \rho_{Z}(X; z) > \beta\}\} d\beta
\]
\[=
\Delta + \int_{\Delta}^{\infty} h \circ P_{Z} \{\{z : \rho_{Z}(X; z) > \beta\}\} d\beta
\]
\[= 1 + \sigma_{2} \Phi^{-1}(a) + \frac{\sigma_{2}}{1 - a} \int^{+\infty}_{\Phi^{-1}(a)} (1 - \Phi(x)) dx
\]
\[- e^{\mu} \left[ e^{\sigma_{1} \Phi^{-1}(1-b)} - \frac{\sigma_{1}}{1 - b} \int_{-\infty}^{\Phi^{-1}(1-b)} \Phi(x) e^{\sigma_{1} x} dx \right].
\]
In summary, (4.9) holds. The proof of (4.5) is completed.

5 Concluding remarks

In reality, the random loss of a financial portfolio is usually influenced by various environmental factors from the outside of the financial market itself. In this paper, in order to evaluate the risk of positions exposed to environments, we (1) construct a new distortion-type risk measure, and (2) axiomatically characterize it by virtue of a new axiomatic framework. Unlike the classic axiomatic framework for univariate risk measures, the new axiomatic framework is of a two-hierarchy structure: the state-wise (i.e. individual state sense) and environment-wise (i.e. overall sense) axioms, respectively. Such a two-hierarchy axiomatic structure is flexible enough, which mathematically allows for flexibility of the candidate for random environments, and can cover examples that could not be handled in classic axiomatic approaches to univariate risk measures so far.

By checking the proofs of the main results, it is clear that most parts of the main results would be still true if we would assume that the loss random variables are integrable and bounded below. Even more, by a standard truncation and approximation approach, one could deal with more general random losses rather than bounded random losses. Nevertheless, this is no longer the focus of this paper. However, from the mathematical point of view, it would be interesting to see it be worked out where the loss random variables are only assumed to be of finite $p$-th moments, $1 \leq p < \infty$.

Acknowledgements

The authors thank Prof. Ruodu Wang for his very helpful discussions and comments on an earlier version of the present paper.

6 Appendix

In this appendix, we provide the proofs of all main results of this paper.

Proof of Proposition 3.1.

(1) Given an environment $Z \in \mathcal{B}^{-}(P)$, let the $P_{Z}$–null set $N_{0} \in \mathcal{B}(\mathbb{R})$ be as in (2.2). Given $z \in N_{0} \cap \text{Ran}(Z)$, since $K^{*}_{Z}(z, A \cap \{Z = z\}) := K_{Z}(z, A)$ for any $A \in \mathcal{F}$, hence for
any $X \in \mathcal{X}(K_Z(z, \cdot))$, by the definition of $\rho_Z(\cdot; z)$, we have that
\[
\rho_Z(X; z) = \int Xdg_z \circ K_Z(z, \cdot)
= \int_0^0 [g_z \circ K_Z^*(z, \{X > \alpha \} \cap \{Z = z\}) - 1] d\alpha
+ \int_0^\infty g_z \circ K_Z^*(z, \{X > \alpha \} \cap \{Z = z\}) d\alpha.
\]

(6.1)

By (6.1), the state-wise law invariance (AI.1) and the state-wise monotonicity (AI.2) are apparent. The state-wise continuity from below (AI.4) is a direct application of The Monotone Convergence Theorem of Denneberg (1994, Theorem 8.1) by letting the monotone set function $\mu$ and $A \in \mathcal{F}$, and that $\rho$ in Assumption A be the risk measure which is exactly the desired state-wise comonotonic additivity (AI.3).

Clearly, $X^*$ and $Y^*$ belong to $\mathcal{X}$. Moreover, it is not hard to verify that $X^*$ and $Y^*$ are comonotonic on $\Omega$, and that
\[
\int X^*dg_z \circ K_Z(z, \cdot) = \int Xdg_z \circ K_Z(z, \cdot), \quad \int Y^*dg_z \circ K_Z(z, \cdot) = \int Ydg_z \circ K_Z(z, \cdot)
\]
and
\[
\int (X^* + Y^*)dg_z \circ K_Z(z, \cdot) = \int (X + Y)dg_z \circ K_Z(z, \cdot).
\]

Therefore, after setting a monotone set function $\mu$ on $\mathcal{F}$ to be $\mu(A) := g_z \circ K_Z(z, A), A \in \mathcal{F}$, then by applying Denneberg (1994, Proposition 5.1(vi)) or Föllmer and Schied (2016, Theorem 4.88) to the monotone set function $\mu$ on $\mathcal{F}$ and the comonotonic $X^*$ and $Y^*$, we obtain that
\[
\rho_Z(X + Y; z) = \int (X^* + Y^*)dg_z \circ K_Z(z, \cdot)
= \int X^*dg_z \circ K_Z(z, \cdot) + \int Y^*dg_z \circ K_Z(z, \cdot)
= \rho_Z(X; z) + \rho_Z(Y; z),
\]
which is exactly the desired state-wise comonotonic additivity (AI.3).

(2) Given an environment $Z \in \mathcal{X}^+(P)$, for each $z \in \text{Ran}(Z)$, let the functional $\tau(\cdot; z)$ in Assumption A be the risk measure $\rho_Z(\cdot; z)$ defined by (6.1), then by above Proposition 3.1(1) we know that Assumption A holds. Taking (3.3) and previous Proposition 3.2(1) into account, then the environment-wise law invariance (AII.1) and the environment-wise monotonicity (AII.2) are straightforward and clear.
We now show the environment-wise comonotonic additivity (AII.3). Given \( Z \in \mathcal{X}^{-\perp}(P) \), for \( X, Y \in \mathcal{X} \) such that for \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \), \( X \) and \( Y \) are local-comonotonic on \( \{Z = z\} \), and the two functions \( \rho_{Z}(X; \cdot) \) and \( \rho_{Z}(Y; \cdot) \) are environment-wise comonotonic, then by above Proposition 3.2(1), we know that for every such \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \),

\[
\rho_{Z}(X + Y; z) = \int (X + Y)dg_{z} \circ K_{Z}(z, \cdot) = \rho_{Z}(X; z) + \rho_{Z}(Y; z),
\]

which, together with (3.3) and Proposition 5.1(vi) of Denneberg (1994) or Theorem 4.88 of Föllmer and Schied (2016), results to

\[
\rho(X + Y; Z) = \int [\rho_{Z}(X; \cdot) + \rho_{Z}(Y; \cdot)]dh_{Z} \circ P_{Z}^{*} = \int \rho_{Z}(X; \cdot)dh_{Z} \circ P_{Z}^{*} + \int \rho_{Z}(Y; \cdot)dh_{Z} \circ P_{Z}^{*} = \rho(X; Z) + \rho(Y; Z).
\]

Finally, we show the environment-wise continuity from below (AII.4). Given \( Z \in \mathcal{X}^{-\perp}(P) \), for positions \( X, X_{n} \in \mathcal{X} \), \( n \geq 1 \), such that \( D \leq X_{n} \), \( n \geq 1 \), with some constant \( D \in \mathbb{R} \), \( X_{n} \uparrow X \) eventually, and \( \rho_{Z}(X_{n}; z) \leq \rho_{Z}(X; z) \) for each \( n \geq 1 \) and \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \), then for any \( \alpha \in \mathbb{R} \), it is not hard to check that for every \( \omega \in \Omega \),

\[
\lim_{n \to \infty} 1_{(\alpha, +\infty)}(X_{n}(\omega)) = 1_{(\alpha, +\infty)}(X(\omega)).
\]

Hence, by the Lebesgue’s Dominated Convergence Theorem, we know that for any above \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \) and any \( \alpha \in \mathbb{R} \),

\[
\lim_{n \to \infty} K_{Z}(z, \{X_{n} > \alpha\}) = K_{Z}(z, \{X > \alpha\}),
\]

and thus,

\[
\lim_{n \to \infty} g_{z} \circ K_{Z}(z, \{X_{n} > \alpha\}) = g_{z} \circ K_{Z}(z, \{X > \alpha\}),
\]

since \( g_{z} \) is continuous. Therefore, from (6.1) and the Lebesgue’s Dominated Convergence Theorem, it follows that for any above \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \),

\[
\lim_{n \to \infty} \rho_{Z}(X_{n}; z) = \rho_{Z}(X; z). \tag{6.2}
\]

Recall the assumption that \( \rho_{Z}(X_{n}; z) \leq \rho_{Z}(X; z) \) for each \( n \geq 1 \) and \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \), from (6.2) it follows that for any \( \beta \in \mathbb{R} \) and any above \( P_{Z} - a.e. \ z \in \text{Ran}(Z) \),

\[
\lim_{n \to \infty} 1_{(\beta, +\infty)}(\rho_{Z}(X_{n}; z)) = 1_{(\beta, +\infty)}(\rho_{Z}(X; z)),
\]

which, together with the Lebesgue’s Dominated Convergence Theorem, implies that for any \( \beta \in \mathbb{R} \),

\[
\lim_{n \to \infty} P_{Z}^{*}(\{z : \rho_{Z}(X_{n}; z) > \beta\} \cap \text{Ran}(Z)) = P_{Z}^{*}(\{z : \rho_{Z}(X; z) > \beta\} \cap \text{Ran}(Z)). \tag{6.3}
\]
Notice that for any above $P_Z - a.e. \ z \in \text{Ran}(Z)$, $D = \rho_Z(D; z) \leq \rho_Z(X_n; z)$ for each $n \geq 1$, due to the state-wise monotonicity of $\rho_Z(; z)$. Hence, for each $n \geq 1$ and any $\beta \in \mathbb{R}$,

$$P_Z^*(\{z : \rho_Z(D; z) > \beta\} \cap \text{Ran}(Z)) \leq P_Z^*(\{z : \rho_Z(X_n; z) > \beta\} \cap \text{Ran}(Z)) \leq P_Z^*(\{z : \rho_Z(X; z) > \beta\} \cap \text{Ran}(Z)),$$

which, as well as (6.3) and the left-continuity of $h_Z$, yields that for any $\beta \in \mathbb{R}$,

$$\lim_{n \to \infty} h_Z \circ P_Z^*(\{z : \rho_Z(X_n; z) > \beta\} \cap \text{Ran}(Z)) = h_Z \circ P_Z^*(\{z : \rho_Z(X; z) > \beta\} \cap \text{Ran}(Z)), \quad (6.4)$$

and that for each $n \geq 1$ and any $\beta \in \mathbb{R}$,

$$h_Z \circ P_Z^*(\{z : \rho_Z(D; z) > \beta\} \cap \text{Ran}(Z)) \leq h_Z \circ P_Z^*(\{z : \rho_Z(X_n; z) > \beta\} \cap \text{Ran}(Z)) \leq h_Z \circ P_Z^*(\{z : \rho_Z(X; z) > \beta\} \cap \text{Ran}(Z)), \quad (6.5)$$

since $h_Z$ is non-decreasing.

Keeping (3.3) in mind, by (6.4), (6.5) and the Lebesgue’s Dominated Convergence Theorem, we know that

$$\lim_{n \to \infty} \rho(X_n; Z) = \rho(X; Z),$$

which is just the desired assertion. Proposition 3.1 is proved.

**Proof of Proposition 3.2.**

Basically speaking, Proposition 3.2 is an application of Corollary 13.3 of Denneberg (1994), or Schmeidler’s Representation Theorem (for example, see Theorem 11.2 of Denneberg (1994)). However, on one hand, since there will be various $P_Z$–null sets involved in the sequel (for example, see the proof of Proposition 3.3 below), and there is also certain equivalent relation needed to be pointed out (see (6.6) below), and on the other hand, for the sake of relative self-containedness, we sketch the proof here.

Fix arbitrarily an environment $Z \in \mathcal{X}^{-1}(P)$. Let the $P_Z$–null set $N_0 \in \mathcal{B}(\mathbb{R})$ be as in (2.2), and denote by $B_0 \in \mathcal{B}(\mathbb{R})$ the $P_Z$–null set with $N_0 \subseteq B_0$ such that for every $z \in B_0 \cap \text{Ran}(Z)$, the normalized risk measure $\rho_Z(; z)$ satisfies Axioms (AI.2) and (AI.3). Note first that given a state $z \in B_0 \cap \text{Ran}(Z)$, for any position $X \in \mathcal{X}(K_Z(z, \cdot))$, we have that

$$\rho_Z(X; z) = \rho_Z(X1_{\{Z=z\}}; z). \quad (6.6)$$

Indeed, (6.6) is due to the state-wise monotonicity (AI.2), because $X$ and $X1_{\{Z=z\}}$ are equal on $\{Z = z\}$.

Given arbitrarily a state $z \in B_0 \cap \text{Ran}(Z)$, we first claim that the normalized risk measure $\rho_Z(; z)$ is monotone, that is $\rho_Z(X; z) \leq \rho_Z(Y; z)$ for any $X, Y \in \mathcal{X}(K_Z(z, \cdot))$ with $X(\omega) \leq Y(\omega)$ for every $\omega \in \Omega$. In fact, given any $X, Y \in \mathcal{X}(K_Z(z, \cdot))$ with $X(\omega) \leq Y(\omega)$ for every
\( \omega \in \Omega \), then \( X(\omega) \leq Y(\omega) \) for every \( \omega \in \{ Z = z \} \). Hence, from the state-wise monotonicity (AI.2), it follows that \( \rho_Z(X; z) \leq \rho_Z(Y; z) \).

Next, we claim that the normalized risk measure \( \rho_Z(\cdot; z) \) is comonotonic additive, that is
\[
\rho_Z(X + Y; z) = \rho_Z(X; z) + \rho_Z(Y; z)
\]
for any \( X, Y \in \mathcal{X}(K_Z(z, \cdot)) \) so that \( X \) and \( Y \) are comonotonic on \( \Omega \). In fact, given any \( X, Y \in \mathcal{X}(K_Z(z, \cdot)) \) so that \( X \) and \( Y \) are comonotonic on \( \Omega \), then clearly, \( X \) and \( Y \) are local-comonotonic on \( \{ Z = z \} \). Hence, from the state-wise comonotonic additivity (AI.3), it follows that
\[
\rho_Z(X + Y; z) = \rho_Z(X; z) + \rho_Z(Y; z).
\]

Consequently, from the Schmeidler’s Representation Theorem (for example, see Theorem 11.2 of Denneberg (1994)) or Corollary 13.3 of Denneberg (1994), it follows that there is a monotone and normalized set function \( \gamma \) on \( \mathcal{F} \) such that for any \( X \in \mathcal{X}(K_Z(z, \cdot)) \),
\[
\rho_Z(X; z) = \int_0^\infty \gamma_z(\{ X > \alpha \})d\alpha.
\]

Particularly, for every \( A \in \mathcal{F} \), \( \gamma_z(A) := \rho_Z(1_A; z) \). Taking (6.6) into account, we also know that for every \( X \in \mathcal{Y}(K_Z(z, \cdot)) \),
\[
\rho_Z(X; z) = \rho_Z(X1_{\{Z=z\}}; z) = \int_0^\infty \gamma_z(\{ X > \alpha \} \cap \{ Z = z \})d\alpha,
\]
and that for every \( A \in \mathcal{F} \),
\[
\gamma_z(A) := \rho_Z(1_A; z) = \rho_Z(1_{A\cap\{Z=z\}}; z).
\]

Proposition 3.2 is proved.

**Proof of Proposition 3.3.**

The key points of the proof are basically the same as ones of proof of Theorem 3 of Wang et al. (1997) by replacing the probability measure there with probability measure \( K_Z(z, \cdot) \) provided with a random environment \( Z \in \mathcal{X}(P) \) and certain \( z \in \text{Ran}(Z) \). Nevertheless, since there is a little distinction between the assumptions employed as pointed out in Remark 3.3, while we also concern the continuity issue of the distortion function involved, we would like to provide an alternative proof to construct the desired distortion function.

Let a random environment \( Z \in \mathcal{X}(P) \) be fixed, and \( U \) a \( U[0, 1] \) random variable on \( (\Omega, \mathcal{F}, P) \) independent of \( Z \). Let the \( P_Z \)-null set \( N_0 \) be as in (2.22), and denote by \( B_0 \in \mathcal{B}(\mathbb{R}) \) the \( P_Z \)-null set with \( N_0 \subseteq B_0 \) such that for every \( z \in B_0 \cap \text{Ran}(Z) \), the normalized risk measure \( \rho_Z(\cdot; z) \) satisfies Axioms (AI.1)–(AI.4). From (2.11), we know that there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) such that for any \( z \in N^c \cap \text{Ran}(Z) \), \( U \) is also uniformly distributed on \( [0, 1] \) under probability measure \( K_Z(z, \cdot) \). Furthermore, we can choose the \( P_Z \)-null set \( N \) large enough such that \( B_0 \subseteq N \). Therefore, for every \( z \in N^c \cap \text{Ran}(Z) \), from Proposition 3.2 it follows that there exists a monotone and normalized set function \( \gamma_z \) on \( \mathcal{F} \) depending on \( z \) and uniquely determined by \( \rho_Z(\cdot; z) \) such that for any \( X \in \mathcal{X}(K_Z(z, \cdot)) \),
\[
\rho_Z(X; z) = \int_0^\infty \gamma_z(\{ X > \alpha \})d\alpha. \tag{6.7}
\]
Note that for every \( A \in \mathcal{F} \), \( \gamma_z(A) := \rho_Z(1_A; z) \).

For each \( z \in N^c \cap \text{Ran}(Z) \), define a function \( g_z : [0, 1] \to \mathbb{R}_+ \) by
\[
g_z(u) := \rho_Z(1_{\{u > 1-u\}}; z). \tag{6.8}
\]
The state-wise continuity of \( \rho_Z(\cdot; z) \) from below yields that \( g_z \) is left-continuous. Moreover, by the normalization, the state-wise law invariance and the state-wise monotonicity of \( \rho_Z(\cdot; z) \), it is not hard to verify that \( g_z \) has the following three properties: (1) \( g_z(0) = 0 \), (2) \( g_z(1) = 1 \), and (3) \( g_z(u) \leq g_z(v) \) for any \( 0 \leq u \leq v \leq 1 \). Thus, the function \( g_z \) is a left-continuous distortion function.

Given arbitarily a state \( z \in N^c \cap \text{Ran}(Z) \), for any position \( X \in \mathcal{X}^+(K_Z(z, \cdot)) \) and any \( t \geq 0 \), note that
\[
K_Z(z, \{X > t\}) = 1 - [1 - K_Z(z, \{X > t\})] = K_Z(z, \{U > 1 - K_Z(z, \{X > t\})\}),
\]
hence, the Bernoulli random variables \( 1_{\{X > t\}} \) and \( 1_{\{U > 1 - K_Z(z, \{X > t\})\}} \) have the same probability distribution with respect to \( K_Z(z, \cdot) \). Therefore, by the definitions of \( \gamma_z \) and \( g_z \) and the state-wise law invariance of \( \rho_Z(\cdot; z) \), we have that
\[
\gamma_z(\{X > t\}) = \rho_Z(1_{\{X > t\}}; z) = \rho_Z(1_{\{U > 1 - K_Z(z, \{X > t\})\}}; z) = g_z(K_Z(z, \{X > t\})) = g_z(K_Z(z, \{X > t\})) = g_z(K_Z(z, \{X > t\})),
\]
which, together with (6.7), yields
\[
\rho_Z(X; z) = \int_0^\infty \gamma_z(\{X > \alpha\})d\alpha = \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\})d\alpha,
\]
which proves the proposition. Proposition 3.3 is proved.

**Proof of Proposition 3.4.**

By the same approach as that in Appendix of Wang et al. (1997), we can extend Proposition 3.3 to real-valued random variables. For the sake of relative self-containedness, we sketch the proof here.

Let a random environment \( Z \in \mathcal{X}^-(P) \) be fixed. Provided with a position \( X \in \mathcal{X} \), then for any \( m < 0 \), \( X \lor m - m \in \mathcal{X}_+ \). By Proposition 3.3 we know that there exists a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) such that for any \( z \in N^c \cap \text{Ran}(Z) \),
\[
\rho_Z(X \lor m - m; z) = \int_0^\infty g_z \circ K_Z(z, \{X \lor m - m > \alpha\})d\alpha,
\]
where the left-continuous distortion function \( g_z \) is defined by (6.8).

Keeping in mind the fact that a constant (regarded as a degenerate random variable) and any random variable are local-comonotonic on \( \{Z = z\} \), from the state-wise comonotonic additivity of \( \rho_Z(\cdot; z) \) we know that
\[
\rho_Z(X \lor m - m; z) = \rho_Z(X \lor m; z) + \rho_Z(-m; z) = \rho_Z(X \lor m; z) + (-m),
\]
in which that \( \rho_Z(-m; z) = -m \) has been used, which is due to the positive homogeneity and normalization of \( \rho_Z(\cdot; z) \). Thus, by change-of-variable,
\[
\rho_Z(X \lor m; z) = \int_0^\infty g_z \circ K_Z(z, \{X \lor m - m > \alpha\})d\alpha - (-m)
\]
\[
= \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\}) - 1)d\alpha + \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\})d\alpha. \tag{6.9}
\]
Since \( X \) is bounded, we can choose \( m \) small enough, for example less than \(-\|X\|\), so that \( \rho_Z(X \lor m; z) = \rho_Z(X; z) \) and
\[
\int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha \\
= \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha.
\]
Consequently, from (6.9) it follows that
\[
\rho_Z(X; z) = \int_{-\infty}^{0} (g_z \circ K_Z(z, \{X > \alpha\}) - 1) \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha,
\]
which shows the desired assertion. Proposition 3.4 is proved.

**Proof of Theorem 3.1.**

Let a random environment \( Z \in \mathcal{X}^{-\lor}(P) \) be given, and let the family \( \{\tau_Z(\cdot; z) : z \in \text{Ran}(Z)\} \) of functionals be as in Assumption A. Applying Propositions 3.3 and 3.4 to the functionals \( \{\tau_Z(\cdot; z) : z \in \text{Ran}(Z)\} \) implies that there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \), so that for each \( z \in N^c \cap \text{Ran}(Z) \), there exists a left-continuous distortion function \( g_z \) defined by (6.8) such that for any \( X \in \mathcal{X}_+ \),
\[
\tau_Z(X; z) = \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha,
\]and that for any \( X \in \mathcal{X} \),
\[
\tau_Z(X; z) = \int_{-\infty}^{0} [g_z \circ K_Z(z, \{X > \alpha\}) - 1] \, d\alpha + \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha. \tag{6.11}
\]

To show the theorem, it is sufficient for us to show that there exists a monotone and normalized set function \( \gamma_Z \) on \( \mathcal{B}(\text{Ran}(Z)) \) depending on \( Z \) such that for any \( X \in \mathcal{X}_+ \),
\[
\rho(X; Z) = \int_{0}^{\infty} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_{0}^{\infty} g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha > \beta \right\} \, d\beta \\
= \int_{0}^{\infty} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \beta \right\} \, d\beta. \tag{6.12}
\]
Note that in (6.12), for the simplicity of notations, we have dropped the parentheses after the set function \( \gamma_Z \), which should bracket the relevant measurable set, but this should not prevent one from rightly understanding. Therefore, we will also keep dropping this kinds of parentheses accordingly in the subsequent proofs.

To this end, we first define two set functions \( \alpha_Z, \beta_Z : \mathcal{B}(\text{Ran}(Z)) \to \mathbb{R}_+ \), through
\[
\alpha_Z(B^*) := \sup\{\rho(X; Z) : X \in \mathcal{X}, \tau_Z(X; z) \leq 1_{B^*}(z) \text{ for any } z \in N^c \cap \text{Ran}(Z)\}, \]
\[
\beta_Z(B^*) := \inf\{\rho(Y; Z) : Y \in \mathcal{X}, 1_{B^*}(z) \leq \tau_Z(Y; z) \text{ for any } z \in N^c \cap \text{Ran}(Z)\},
\]
\( B^* \in \mathcal{B}(\text{Ran}(Z)) \). Indeed, it is not hard to verify that \( \alpha_Z(\emptyset) = \beta_Z(\emptyset) = 0 \), and that \( \alpha_Z \) and \( \beta_Z \) are monotone. By the environment-wise monotonicity (AII.2) of \( \rho \), \( \alpha_Z \leq \beta_Z \). Thus, \( \alpha_Z \) and \( \beta_Z \) are normalized as well.
Let \( \gamma_z : \mathcal{B}(\text{Ran}(Z)) \to \mathbb{R}_+ \) be a monotone and normalized set function such that

\[
\alpha_Z \leq \gamma_Z \leq \beta_Z,
\]

for example, we can choose \( \gamma_Z = \alpha_Z \).

Given \( X \in \mathcal{F}_i \), for any positive integer \( n \) and each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \), define two random variables \( X_{i,1}, X_{i,2} : (\Omega, \mathcal{F}) \to \mathbb{R} \) as

\[
X_{i,1}(\omega) := \begin{cases} 
\frac{i}{2^n} ; & \text{if } \tau_Z(X; Z(\omega)) > \frac{i+1}{2^n}, \\
X(\omega) - \frac{i}{2^n} ; & \text{if } \frac{i}{2^n} < \tau_Z(X; Z(\omega)) \leq \frac{i+1}{2^n}, \\
0 ; & \text{if } 0 < \tau_Z(X; Z(\omega)) \leq \frac{i}{2^n}, \\
\frac{1}{n \cdot 2^n - 1} \cdot X(\omega) ; & \text{if } \tau_Z(X; Z(\omega)) = 0,
\end{cases}
\]

and

\[
X_{i,2}(\omega) := \begin{cases} 
\frac{i}{2^n} ; & \text{if } \tau_Z(X; Z(\omega)) > \frac{i}{2^n}, \\
X(\omega) - \frac{i-1}{2^n} ; & \text{if } \frac{i-1}{2^n} < \tau_Z(X; Z(\omega)) \leq \frac{i}{2^n}, \\
0 ; & \text{if } 0 < \tau_Z(X; Z(\omega)) \leq \frac{i-1}{2^n}, \\
\frac{1}{n \cdot 2^n - 1} \cdot X(\omega) ; & \text{if } \tau_Z(X; Z(\omega)) = 0.
\end{cases}
\]

In fact, the \( \mathcal{F} \)-measurability of \( X_{i,1} \) and \( X_{i,2} \) follows from the facts that \( \{ \tau_Z(X; z) ; z \in \text{Ran}(Z) \} \) is regular on \( \text{Ran}(Z) \), and that \( Z \) is \( \mathcal{F}/\mathcal{B}(\text{Ran}(Z)) \)-measurable. Particularly, we note that for any \( t \in \mathbb{R}_+ \), \( \{ \omega : \tau_Z(X; Z(\omega)) > t \} \) is a \( \mathcal{F} \)-measurable set. Moreover, it is not hard to verify that for each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \), \( X_{i,1} \) and \( X_{i,2} \) are local-comonotonic on \( \{ Z = z \} \) for each \( z \in \text{Ran}(Z) \).

Next, we will show that (6.12) holds for \( \gamma_Z \) defined as above, and thus complete the proof of the theorem. We divide the proof into three steps.

Step one. We conclude that for any positive integer \( n \geq 1 \) and each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \),

\[
\rho \left( \sum_{i=1}^{n \cdot 2^n - 1} X_{i,1}^* ; Z \right) \leq \frac{1}{2^n} \sum_{i=1}^{n \cdot 2^n - 1} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n} \right\}
\]

\[
\leq \rho \left( \sum_{i=1}^{n \cdot 2^n - 1} X_{i,2}^* ; Z \right). \tag{6.13}
\]

To this end, we first show that for any \( z \in N^c \cap \text{Ran}(Z) \) and each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \),

\[
\tau_Z(2^n \cdot X_{i,1}^* ; z) \leq 1_{\{ \tau_Z(X_{i,1}^* ; z) > \frac{i}{2^n} \}}(z) \leq \tau_Z(2^n \cdot X_{i,2}^* ; z). \tag{6.14}
\]

Given arbitrarily \( z \in N^c \cap \text{Ran}(Z) \), we now calculate \( \tau_Z(X_{i,1}^* ; z) \). Note that \( 0 \leq X_{i,1}^* + \frac{i}{2^n} \), and that \( X_{i,1}^* \) and \( \frac{i}{2^n} \) are local-comonotonic on \( \{ Z = z \} \), from the state-wise comonotonic additivity (AI.3) and Remark 3.1(ii), it follows that

\[
\tau_Z \left( X_{i,1}^* + \frac{i}{2^n} ; z \right) = \tau_Z \left( X_{i,1}^* ; z \right) + \tau_Z \left( \frac{i}{2^n} ; z \right) = \tau_Z(X_{i,1}^* ; z) + \frac{i}{2^n},
\]

for example, we can choose \( \gamma_Z = \alpha_Z \).
which yields that

\[ \tau_Z(X_{i,1}^*; z) = \tau_Z \left( X_{i,1}^* + \frac{i}{2^n}; z \right) - \frac{i}{2^n}. \]

(6.15)

Based on (6.10) and (6.15), we calculate \( \tau_Z(X_{i,1}^*; z) \). Since \( \tau_Z(X; z) \) is non-negative, there are four possibilities for the value of \( \tau_Z(X; z) \).

Case one: \( \tau_Z(X; z) > \frac{i+1}{2^n} \). In this case, by the definition of \( X_{i,1}^* \), we know that \( X_{i,1}^*(\omega) = \frac{1}{2^n} \) for \( \omega \in \{Z = z\} \). Hence, by (6.10) and (6.15),

\[
\begin{align*}
\tau_Z(X_{i,1}^*; z) &= \tau_Z \left( X_{i,1}^* + \frac{i}{2^n}; z \right) - \frac{i}{2^n} \\
&= \int_0^\infty g_z \circ K_Z \left( z, \left\{ \omega : \frac{i}{2^n} > \alpha \right\} \right) d\alpha - \frac{i}{2^n} \\
&= \int_0^\infty g_z \circ K_Z^* \left( z, \left\{ \omega : \frac{i}{2^n} > \alpha, Z(\omega) = z \right\} \right) d\alpha - \frac{i}{2^n} \\
&= \int_0^\infty g_z \circ K_Z^* \left( z, \left\{ \omega : \frac{1}{2^n} + \frac{i}{2^n} > \alpha \right\} \right) d\alpha - \frac{i}{2^n} \\
&= \frac{1}{2^n}. \quad (6.16)
\end{align*}
\]

By the same argument similar to above Case one, we can calculate \( \tau_Z(X_{i,1}^*; z) \) for the other three cases as follows.

Case two: \( \frac{i}{2^n} < \tau_Z(X; z) \leq \frac{i+1}{2^n} \). Then, by (6.10) and (6.15),

\[
\tau_Z(X_{i,1}^*; z) = \tau_Z \left( X_{i,1}^* + \frac{i}{2^n}; z \right) - \frac{i}{2^n} = \tau_Z(X; z) - \frac{i}{2^n}. \quad (6.17)
\]

Case three: \( 0 < \tau_Z(X; z) \leq \frac{i}{2^n} \). Then, by (6.10) and (6.15),

\[
\tau_Z(X_{i,1}^*; z) = \tau_Z \left( X_{i,1}^* + \frac{i}{2^n}; z \right) - \frac{i}{2^n} = 0. \quad (6.18)
\]

Case four: \( \tau_Z(X; z) = 0 \). Then, by (6.10) and (6.15),

\[
\tau_Z(X_{i,1}^*; z) = \tau_Z \left( X_{i,1}^* + \frac{i}{2^n}; z \right) - \frac{i}{2^n} = 0. \quad (6.19)
\]

In summary, for any \( z \in N^c \cap \text{Ran}(Z) \) and each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \),

\[
\tau_Z(X_{i,1}^*; z) \leq \frac{1}{2^n} \cdot \mathbb{1}_{\{\tau_Z(X; z) > \frac{i+1}{2^n}\}}(z). \quad (6.20)
\]

By a similar argumentation as above, we can also steadily show that for any \( z \in N^c \cap \text{Ran}(Z) \) and each \( i \in \{1, \ldots, n \cdot 2^n - 1\} \),

\[
\tau_Z(X_{i,2}^*; z) = \begin{cases} 
\frac{1}{2^n}, & \text{if } \tau_Z(X; z) > \frac{i+1}{2^n}, \\
\tau_Z(X; z) - \frac{i+1}{2^n}, & \text{if } \frac{i}{2^n} < \tau_Z(X; z) \leq \frac{i+1}{2^n}, \\
0, & \text{if } 0 \leq \tau_Z(X; z) \leq \frac{i+1}{2^n}.
\end{cases}
\]

34
Hence
\[
\frac{1}{2^n} \cdot 1_{\{\tau_Z(X; z) > \frac{i}{2^n}\}}(z) \leq \tau_Z(X^*_{i,2}; z).
\]  
(6.21)

Thus, recalling that \(\tau_Z(\cdot; z)\) is positive homogeneous, see Remark 3.1(ii), (6.20) and (6.21) together yield
\[
\tau_Z(2^n \cdot X^*_{i,1}; z) \leq 1_{\{\tau_Z(X; z) > \frac{i}{2^n}\}}(z) \leq \tau_Z(2^n \cdot X^*_{i,2}; z),
\]
which is just (6.14).

Note that \(\{z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n}\} \in \mathcal{B}(\text{Ran}(Z))\), from (6.14) and the definitions of \(\alpha_Z, \gamma_Z\) and \(\beta_Z\), it follows that
\[
\rho(2^n \cdot X^*_{i,1}; Z) \leq \rho(2^n \cdot X^*_{i,2}; Z).
\]
Thus, by the positive homogeneity of \(\rho(\cdot; Z)\) (see Remark 3.1(iii)),
\[
2^n \cdot \rho(X^*_{i,1}; Z) \leq \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n} \right\} \leq 2^n \cdot \rho(X^*_{i,2}; Z),
\]
which implies that for any \(n \geq 1\),
\[
\sum_{i=1}^{n-2^n-1} \rho(X^*_{i,1}; Z) \leq \frac{1}{2^n} \sum_{i=1}^{n-2^n-1} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n} \right\}
\]
\[
\leq \sum_{i=1}^{n-2^n-1} \rho(X^*_{i,2}; Z).
\]  
(6.22)

Observing (6.22), to show (6.13), it is sufficient for us to show
\[
\sum_{i=1}^{n-2^n-1} \rho(X^*_{i,1}; Z) = \rho \left( \sum_{i=1}^{n-2^n-1} X^*_{i,1}; Z \right)
\]  
(6.23)
and
\[
\sum_{i=1}^{n-2^n-1} \rho(X^*_{i,2}; Z) = \rho \left( \sum_{i=1}^{n-2^n-1} X^*_{i,2}; Z \right).
\]  
(6.24)

Next, we claim that for any fixed \(m \in \{1, \ldots, n \cdot 2^n - 2\}\), the two functions \(\tau_Z \left( \sum_{i=1}^{m} X^*_{i,1}; \cdot \right)\) and \(\tau_Z \left( X^*_{m+1,i}; \cdot \right)\) are environment-wise comonotonic. In fact, by the definition of \(X^*_{i,1}\),
\[
\sum_{i=1}^{m} X^*_{i,1}(\omega) = \begin{cases} 
\frac{m}{2^n}, & \text{if } \tau_Z(X; Z(\omega)) > \frac{m+1}{2^n}, \\
X(\omega) - \frac{1}{2^n}, & \text{if } \frac{1}{2^n} < \tau_Z(X; Z(\omega)) \leq \frac{m+1}{2^n}, \\
0, & \text{if } 0 < \tau_Z(X; Z(\omega)) \leq \frac{1}{2^n}, \\
\frac{m}{n \cdot 2^n - 1} \cdot X(\omega), & \text{if } \tau_Z(X; Z(\omega)) = 0,
\end{cases}
\]  
(6.25)
and

\[ X^*_{m+1,1}(\omega) = \begin{cases} 
\frac{1}{n}, & \text{if } \tau(Z; X(\omega)) > \frac{m+2}{2^n}, \\
X(\omega) - \frac{m+1}{2^n}, & \text{if } \frac{m+1}{2^n} < \tau(Z; X(\omega)) \leq \frac{m+2}{2^n}, \\
0, & \text{if } \tau(Z; X(\omega)) \leq \frac{m+1}{2^n}, \\
\frac{1}{n \cdot 2^{n-1}} \cdot X(\omega), & \text{if } \tau(Z; X(\omega)) = 0.
\end{cases} \]  

(6.26)

By an elementary calculation similar to that of \(\tau_Z(X^*_i; z)\), i.e. (6.10)–(6.19), we can obtain that for any \(z \in N^c \cap \text{Ran}(Z)\),

\[ \tau_Z \left( \sum_{i=1}^{m} X^*_{i,1} ; z \right) = \begin{cases} 
\frac{m}{2^n}, & \text{if } \tau(Z; z) > \frac{m+1}{2^n}, \\
\tau_Z(X; z) - \frac{1}{2^n}, & \text{if } \frac{1}{2^n} < \tau(Z; x) \leq \frac{m+1}{2^n}, \\
0, & \text{if } 0 \leq \tau(Z; x) \leq \frac{1}{2^n},
\end{cases} \]  

(6.27)

and

\[ \tau_Z(X^*_{m+1,1}; z) = \begin{cases} 
\frac{1}{2^n}, & \text{if } \tau(Z; z) > \frac{m+2}{2^n}, \\
\tau_Z(X; z) - \frac{m+1}{2^n}, & \text{if } \frac{m+1}{2^n} < \tau(Z; x) \leq \frac{m+2}{2^n}, \\
0, & \text{if } 0 \leq \tau(Z; x) \leq \frac{m+1}{2^n}.
\end{cases} \]  

(6.28)

Hence, we can steadily verify that the two functions \(\tau_Z \left( \sum_{i=1}^{m} X^*_i ; \cdot \right)\) and \(\tau_Z(X^*_{m+1,1}; \cdot)\) are environment-wise comonotonic. Meanwhile, from (6.25) and (6.26), it is clear that given any \(z \in N^c \cap \text{Ran}(Z)\), for \(m \in \{1, \cdots , n \cdot 2^n - 2\}\), \(\sum_{i=1}^{m} X^*_i\) and \(X^*_{m+1,1}\) are local-comonotonic on \(\{Z = z\}\). Therefore, by the environment-wise comonotonic additivity (AII.3),

\[ \sum_{i=1}^{n \cdot 2^n - 1} \rho(X^*_i; Z) = \rho \left( \sum_{i=1}^{n \cdot 2^n - 1} X^*_i ; Z \right), \]

which is just (6.23).

By a similar argumentation as above, we have that for any \(z \in N^c \cap \text{Ran}(Z)\) and each \(m \in \{1, \cdots , n \cdot 2^n - 2\}\),

\[ \sum_{i=1}^{m} X^*_{i,2}(\omega) = \begin{cases} 
\frac{m}{2^n}, & \text{if } \tau(Z; X(\omega)) > \frac{m}{2^n}, \\
X(\omega), & \text{if } 0 < \tau(Z; X(\omega)) \leq \frac{m}{2^n}, \\
\frac{m}{n \cdot 2^{n-1}} \cdot X(\omega), & \text{if } \tau(Z; X(\omega)) = 0,
\end{cases} \]  

(6.29)

\[ X^*_{m+1,2}(\omega) = \begin{cases} 
\frac{1}{2^n}, & \text{if } \tau(Z; X(\omega)) > \frac{m+1}{2^n}, \\
X(\omega) - \frac{m}{2^n}, & \text{if } \frac{m}{2^n} < \tau(Z; X(\omega)) \leq \frac{m+1}{2^n}, \\
0, & \text{if } 0 < \tau(Z; X(\omega)) \leq \frac{m+1}{2^n}, \\
\frac{1}{n \cdot 2^{n-1}} \cdot X(\omega), & \text{if } \tau(Z; X(\omega)) = 0,
\end{cases} \]  

(6.30)

\[ \tau_Z \left( \sum_{i=1}^{m} X^*_{i,2} ; z \right) = \begin{cases} 
\frac{m}{2^n}, & \text{if } \tau(Z; z) > \frac{m}{2^n}, \\
\tau_Z(X; z), & \text{if } 0 \leq \tau(Z; x) \leq \frac{m}{2^n},
\end{cases} \]  

(6.31)
and

$$\tau_Z(X_{m+1,2}; z) = \begin{cases} \frac{1}{2^n}, & \text{if } \tau_Z(X; z) > \frac{m+1}{2^n}, \\ \tau_Z(X; z) - \frac{m}{2^n}, & \text{if } \frac{m}{2^n} < \tau_Z(X; z) \leq \frac{m+1}{2^n}, \\ 0, & \text{if } 0 \leq \tau_Z(X; z) \leq \frac{m}{2^n}. \end{cases} (6.32)$$

Hence, we can steadily verify that the two functions $\tau_Z(\sum_{i=1}^{m} X_{i,2}; \cdot)$ and $\tau_Z(X^*_{m+1,2}; \cdot)$ are environment-wise comonotonic, and that $\sum_{i=1}^{m} X_{i,2}$ and $X^*_{m+1,2}$ are local-comonotonic on $\{Z = z\}$ for each $z \in N^c \cap \text{Ran}(Z)$. Therefore, by the environment-wise comonotonic additivity (AII.3),

$$\sum_{i=1}^{n-2^n-1} \rho(X^*_{i,2}; Z) = \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,2}; Z \right),$$

which is just (6.24). Consequently, (6.13) follows from (6.22), (6.23) and (6.24).

Step two. We continue to conclude that

$$\lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,1}; Z \right) = \lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,2}; Z \right) = \rho(X; Z). (6.33)$$

To this end, we first show that

$$\lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,1}; Z \right) = \rho(X; Z). (6.34)$$

Write $X_{n,1} := \sum_{i=1}^{n-2^n-1} X_{i,1}$. For any positive integer $n \geq 1$, by (6.25)–(6.28), we know that

$$X_{n,1}(\omega) = \begin{cases} n - \frac{1}{2^n}, & \text{if } \tau_Z(X; Z(\omega)) > n, \\ X(\omega) - \frac{1}{2^n}, & \text{if } \frac{1}{2^n} < \tau_Z(X; Z(\omega)) \leq n, \\ 0, & \text{if } 0 < \tau_Z(X; Z(\omega)) \leq \frac{1}{2^n}, \\ X(\omega), & \text{if } \tau_Z(X; Z(\omega)) = 0, \end{cases}$$

and that for any $z \in N^c \cap \text{Ran}(Z)$,

$$\tau_Z(X_{n,1}; z) = \begin{cases} n - \frac{1}{2^n}, & \text{if } \tau_Z(X; z) > n, \\ \tau_Z(X; z), & \text{if } \frac{1}{2^n} < \tau_Z(X; z) \leq n, \\ 0, & \text{if } 0 \leq \tau_Z(X; z) \leq \frac{1}{2^n}. \end{cases}$$

Clearly, $-\frac{1}{2} \leq X_{n,1}$ for each $n \geq 1$, and $\tau_Z(X_{n,1}; z) \leq \tau_Z(X; z)$ for each $n \geq 1$ and each $z \in N^c \cap \text{Ran}(Z)$. Moreover, $X_{n,1} \uparrow X$ eventually. Indeed, for any $\omega \in \Omega$, $\tau_Z(X; Z(\omega)) \geq 0$, since $X \in \mathcal{X}_+$ and $\tau_Z(0; Z(\omega)) \leq \tau_Z(X; Z(\omega))$. If $\tau_Z(X; Z(\omega)) = 0$, then for any $n \geq 1$, $X_{n,1}(\omega) = X(\omega)$. If $\tau_Z(X; Z(\omega)) > 0$, then there exists some integer $N := N(\omega) \geq 1$ such that

$$\frac{1}{2^N} < \tau_Z(X; Z(\omega)) \leq N.$$
Hence, for all \( n \geq N \),
\[
\frac{1}{2^n} \leq \frac{1}{2^N} < \tau_Z(X; Z(\omega)) \leq N \leq n,
\]
and thus \( X_{n,1}(\omega) = X(\omega) - \frac{1}{2^n} \) for all \( n \geq N \). In summary, \( X_{n,1} \uparrow X \) eventually. Therefore, by the environment-wise continuity from below (AII.4), we have that
\[
\lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,1}^*; Z \right) = \rho(X; Z),
\]
which shows that (6.34) holds.

Next, we proceed to show that
\[
\lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,2}^*; Z \right) = \rho(X; Z). \tag{6.35}
\]

Write \( X_{n,2} := \sum_{i=1}^{n-2^n-1} X_{i,2}^* \). For any positive integer \( n \geq 1 \), by (6.29)–(6.32), we have that
\[
X_{n,2}(\omega) = \begin{cases} 
  n - \frac{1}{2^n}, & \text{if } \tau_Z(X; Z(\omega)) > n - \frac{1}{2^n}, \\
  X(\omega), & \text{if } 0 \leq \tau_Z(X; Z(\omega)) \leq n - \frac{1}{2^n}, 
\end{cases}
\]
and that for any \( z \in N^c \cap \text{Ran}(Z) \),
\[
\tau_Z(X_{n,2}; z) = \begin{cases} 
  n - \frac{1}{2^n}, & \text{if } \tau_Z(X; z) > n - \frac{1}{2^n}, \\
  \tau_Z(X; z), & \text{if } 0 \leq \tau_Z(X; z) \leq n - \frac{1}{2^n}. 
\end{cases}
\]

Clearly, \( 0 \leq X_{n,2} \) for each \( n \geq 1 \), \( X_{n,2} \uparrow X \) eventually and \( \tau_Z(X_{n,2}; z) \leq \tau_Z(X; z) \) for each \( n \geq 1 \) and each \( z \in N^c \cap \text{Ran}(Z) \). Hence, by the environment-wise continuity from below (AII.4), we know that
\[
\lim_{n \to \infty} \rho \left( \sum_{i=1}^{n-2^n-1} X_{i,2}^*; Z \right) = \rho(X; Z),
\]
which shows that (6.35) holds. Now, (6.33) follows from (6.34) and (6.35).

Step three. We claim that
\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{i=1}^{n-2^n-1} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n} \right\} \\
= \int_0^\infty \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \alpha \} d\alpha. \tag{6.36}
\]
In fact, since \( \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \) is decreasing with respect to variable \( s \),
\[
\int_0^n \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \, ds \\
= \sum_{k=1}^{n-2^n-1} \int \frac{k}{2^n} \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \, ds \\
\leq \frac{1}{2^n} \cdot \sum_{k=1}^{n-2^n-1} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{k}{2^n} \right\} \\
= \sum_{i=0}^{n-2^n-2} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i+1}{2^n} \right\} \cdot \frac{1}{2^n} \\
\leq \sum_{i=0}^{n-2^n-2} \int \frac{i+1}{2^n} \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \, ds \\
= \int_0^{n-2^n} \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \, ds \\
\leq \int_0^\infty \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > s \} \, ds.
\]

Taking \( n \to \infty \) in both sides of above inequality results in
\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{i=0}^{n-2^n-1} \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \frac{i}{2^n} \right\} \\
= \int_0^\infty \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \beta \} \, d\beta,
\]
which is exactly (6.36).

As a conclusion, by (6.33), (6.33) and (6.36), we know that
\[
\rho(X; Z) = \int_0^\infty \gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > \alpha \} \, d\alpha,
\]
which, together with (6.10), yields that
\[
\rho(X; Z) = \int_0^\infty \gamma_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{ X > \alpha \}) \, d\alpha > \beta \right\} \, d\beta.
\]

Theorem 3.1 is proved.

**Proof of Theorem 3.2.**

Let a random environment \( Z \in \mathcal{X}^{-1}(P) \) be arbitrarily given, and let the family \( \{ \tau_Z(\cdot ; z) : z \in \text{Ran}(Z) \} \) of functionals be as in Assumption A. Recalling (2.2), for every \( z \in N^c_0 \),
\[
K_Z(z, \{ Z \in B \}) = 1_B(z) \quad \text{for any } B \in \mathcal{B}(\mathbb{R}),
\]
where the \( P_Z \)-null set \( N_0 \in \mathcal{B}(\mathbb{R}) \) is as in (2.2). By Theorem 3.1, we know that there is a \( P_Z \)-null set \( N \in \mathcal{B}(\mathbb{R}) \) with \( N_0 \subseteq N \), so that there exist a family \( \{ g_z ; z \in N^c \cap \text{Ran}(Z) \} \)}
of left-continuous distortion functions defined by \(6.8\) and a monotone and normalized set function \(\gamma_Z\) on \(\mathcal{B}(\text{Ran}(Z))\) depending on \(Z\) such that for any \(X \in \mathcal{X}_+,\)

\[
\rho(X; Z) = \int_0^\infty \gamma_Z \{ z \in N \cap \text{Ran}(Z) : \tau_Z(X; z) > \beta \} \, d\beta
\]

\[
= \int_0^\infty \gamma_Z \left\{ z \in N \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{X > \alpha\}) \, d\alpha > \beta \right\} \, d\beta,
\]

(6.38)

where for every \(z \in N \cap \text{Ran}(Z)\), \(\tau_Z(\cdot; z)\) is given by

\[
\tau_Z(Y; z) = \int_0^\infty g_z \circ K_Z(z, \{Y > \alpha\}) \, d\alpha, \quad Y \in \mathcal{X}_+,
\]

(6.39)

and for \(Y \in \mathcal{X}'\),

\[
\tau_Z(Y; z) = \int_{-\infty}^0 \left[ g_z \circ K_Z(z, \{Y > \alpha\}) - 1 \right] \, d\alpha + \int_0^\infty g_z \circ K_Z(z, \{Y > \alpha\}) \, d\alpha.
\]

(6.40)

We first claim that for any \(N \cap B^* := N \cap B \cap \text{Ran}(Z) \in \mathcal{B}(\text{Ran}(Z))\) with some \(B \in \mathcal{B}(\mathbb{R})\),

\[
\gamma_Z(N \cap B^*) := \gamma_Z(N \cap B \cap \text{Ran}(Z)) = \rho(1_B(Z); Z).
\]

(6.41)

In fact, from \(6.37\) and \(6.38\), it follows that

\[
\rho(1_B(Z); Z) = \int_0^\infty \gamma_Z \left\{ z \in N \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{1_B(Z) > \alpha\}) \, d\alpha > \beta \right\} \, d\beta
\]

\[
= \int_0^\infty \gamma_Z \left\{ z \in N \cap \text{Ran}(Z) : \int_0^1 g_z \circ K_Z(z, \{Z \in B\}) \, d\alpha > \beta \right\} \, d\beta
\]

\[
= \int_0^\infty \gamma_Z \left\{ z \in N \cap \text{Ran}(Z) : 1_B(z) > \beta \right\} \, d\beta
\]

\[
= \gamma_Z(N \cap B \cap \text{Ran}(Z)),
\]

which shows that \(6.41\) holds.

To show the theorem, it is sufficient for us to show that there exists a function \(h_Z : [0, 1] \to [0, 1]\) with \(h_Z(0) = 0\) and \(h_Z(1) = 1\), such that for any \(t \geq 0\),

\[
\gamma_Z \{ z \in N \cap \text{Ran}(Z) : \tau_Z(X; z) > t \} = h_Z \circ P_Z^x \{ z \in N \cap \text{Ran}(Z) : \tau_Z(X; z) > t \}.
\]

(6.42)

For this purpose, write \(\text{Ran}(P_Z^x) := \{ P_Z^x(N \cap B^*) : B^* \in \mathcal{B}(\text{Ran}(Z))\}\). Next, we proceed to construct a function \(h_Z : \text{Ran}(P_Z^x) \to [0, 1]\). For every \(u \in \text{Ran}(P_Z^x)\), there is some \(B_u^* := B_u \cap \text{Ran}(Z) \in \mathcal{B}(\text{Ran}(Z))\) with some \(B_u \in \mathcal{B}(\mathbb{R})\) such that \(P_Z^x(N \cap B_u^*) = u\), and thus we define \(h_Z\) as

\[
h_Z(u) := \rho(1_{B_u}(Z); Z).
\]

(6.43)

We first claim that \(h_Z\) is well defined on \(\text{Ran}(P_Z^x)\). In fact, for \(u \in \text{Ran}(P_Z^x)\), let \(B_{u,1} := B_{u,1} \cap \text{Ran}(Z)\), \(B_{u,2} := B_{u,2} \cap \text{Ran}(Z) \in \mathcal{B}(\text{Ran}(Z))\) with some \(B_{u,1}, B_{u,2} \in \mathcal{B}(\mathbb{R})\) be such
that \( P^*_Z(N^c \cap B_{u,1}^*) = P^*_Z(N^c \cap B_{u,2}^*) = u \), then \( \tau_Z(1_{B_{u,1}}(Z); \cdot) \) and \( \tau_Z(1_{B_{u,2}}(Z); \cdot) \) have the same probability distribution with respect to \( P^*_Z \). Indeed, for any \( D \in \mathcal{B}(\mathbb{R}) \), by (6.37) and (6.39),
\[
P^*_Z \circ \tau_Z^{-1}(1_{B_{u,1}}(Z); \cdot)(D) = P^*_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_0^\infty g_z \circ K_Z(z, \{1_{B_{u,1}}(Z) > \alpha\})d\alpha \in D \right\}
\]
\[
= P^*_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_0^1 g_z \circ K_Z(z, \{Z \in B_{u,1}\})d\alpha \in D \right\}
\]
\[
= P^*_Z \{ z \in N^c \cap \text{Ran}(Z) : 1_{B_{u,1}}(z) \in D \}
\]
\[
= P^*_Z \circ 1^{-1}_{B_{u,1}}(D). \quad (6.44)
\]
Similarly,
\[
P^*_Z \circ \tau_Z^{-1}(1_{B_{u,2}}(Z); \cdot)(D) = P^*_Z \circ 1^{-1}_{B_{u,2}}(D). \quad (6.45)
\]
Note that
\[
P^*_Z(1_{B_{u,1}} = 1) = P^*_Z(B_{u,1}) = P^*_Z(N^c \cap B_{u,1}^*) = u
\]
\[
= P^*_Z(N^c \cap B_{u,2}^*) = P^*_Z(B_{u,2}) = P^*_Z(1_{B_{u,2}} = 1),
\]
which implies that \( P^*_Z \circ 1^{-1}_{B_{u,1}} = P^*_Z \circ 1^{-1}_{B_{u,2}} \). Hence, keeping in mind (6.44) and (6.45), we know that \( \tau_Z(1_{B_{u,1}}(Z); \cdot) \) and \( \tau_Z(1_{B_{u,2}}(Z); \cdot) \) have the same probability distribution with respect to \( P^*_Z \). Therefore, by the environment-wise law invariance (AII.1), we conclude that \( \rho(1_{B_{u,1}}(Z); Z) = \rho(1_{B_{u,2}}(Z); Z) \), which means that \( h_Z \) defined by (6.43) is well defined.

We further claim that \( h_Z(0) = 0 \) and \( h_Z(1) = 1 \). In fact, setting \( B_0 := \emptyset \) implies \( B_0^* := B_0 \cap \text{Ran}(Z) = \emptyset \). Thus \( h_Z(0) = \rho(1_{\emptyset}(Z); Z) = \rho(0; Z) = 0 \), where the last equality is guaranteed by the positive homogeneity of \( \rho(\cdot; Z) \), see Remark 3.1(ii). Similarly, setting \( B_1 := \mathbb{R} \) implies \( B_1^* := B_1 \cap \text{Ran}(Z) = \text{Ran}(Z) \). Thus \( h_Z(1) = \rho(1_{\mathbb{R}}(Z); Z) = \rho(1; Z) = 1 \), due to the normalization of \( \rho \). Moreover, it is clear that the range of \( h_Z \) is contained in \([0,1]\).

Now, we can arbitrarily extend \( h_Z \) from \( \text{Ran}(P^*_Z) \) to the whole interval \([0,1]\), say by linear interpolation, because those \( u \notin \text{Ran}(P^*_Z) \) do not matter. We still denote by \( h_Z \) this extension, because it should have no risk of notation confusion.

Next, we proceed to show that (6.42) holds for \( h_Z \) defined by (6.43). Given \( X \in \mathcal{X}_+ \), for any \( t \geq 0 \), denote \( B_t^* := \{ z \in \text{Ran}(Z) : \tau_Z(X; z) > t \} \), then \( B_t^* \in \mathcal{B} \text{Ran}(Z) \), since \( \{\tau_Z(X; z); z \in \text{Ran}(Z)\} \) is regular on \( \text{Ran}(Z) \). Hence, there exists some \( B_t \in \mathcal{B}(\mathbb{R}) \) such that \( B_t^* = B_t \cap \text{Ran}(Z) \). From (6.41) and the definition of \( h_Z \) as in (6.43), it follows that
\[
h_Z \circ P^*_Z(N^c \cap B_t^*) := h_Z(P^*_Z(N^c \cap B_t^*)) = \rho(1_{B_t}(Z); Z) = \gamma_Z(N^c \cap B_t^*),
\]
That is,
\[
\gamma_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > t \} = h_Z \circ P^*_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X; z) > t \},
\]
which just shows that (6.42) holds for \( h_Z \) defined by (6.43). Theorem 3.2 is proved.

Proof of Theorem 3.3.
All what we need to do is to extend Theorem 3.2 to real-valued random losses. Indeed, the argumentation of extending Theorem 3.2 from non-negative random variables \( X \in \mathcal{X}_+ \) to real-valued random variables \( X \in \mathcal{X} \) is basically the same as that of extending Proposition 3.3 from non-negative random variables to real-valued random variables. Nevertheless, to keep the proof relatively self-contained, we sketch the proof here.

Let a random environment \( Z \in \mathcal{X}^\perp(P) \) be fixed. Provided with a position \( X \in \mathcal{X} \), then for any \( m < 0 \), \( X \lor m - m \in \mathcal{X}_+ \). From Theorem 3.2 we know that

\[
\rho(X \lor m - m; Z) = \int_0^\infty h_Z \circ P^*_Z \{ z \in N^c \cap \text{Ran}(Z) : \tau_Z(X \lor m - m; z) > \beta \} d\beta, \tag{6.46}
\]

where the \( P_Z \)-null set \( N \) and the function \( h_Z \) are as in Theorem 3.2, and for every \( z \in N^c \cap \text{Ran}(Z) \), by change-of-variable,

\[
\tau_Z(X \lor m - m; z) = \int_m^\infty g_z \circ K_Z(z, \{ X > \alpha \}) d\alpha, \tag{6.47}
\]

where the distortion function \( g_z \) is as in Theorem 3.2.

Observing the facts that the two functions \( \tau_Z(X \lor m; \cdot) \) and \( \tau_Z(-m; \cdot) = -m \) are apparently environment-wise comonotonic, and that \( X \lor m \) and \(-m\) are local-comonotonic on \( \{ Z = z \} \) for each \( z \in \text{Ran}(Z) \), by the environment-wise comonotonic additivity (AII.3), we have that

\[
\rho(X \lor m - m; Z) = \rho(X \lor m; Z) + \rho(-m; Z) = \rho(X \lor m; Z) + (-m), \tag{6.48}
\]

where the fact that \( \rho(-m; Z) = -m \) has been used, which is due to the positive homogeneity of \( \rho(\cdot; Z) \) and the normalization of \( \rho \). Thus, from (6.46), (6.47), (6.48) and change-of-variable, it follows that

\[
\rho(X \lor m; Z) = \rho(X \lor m - m; Z) - (-m)
\]

\[
= \int_m^0 \left[ h_Z \circ P^*_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_m^\infty g_z \circ K_Z(z, \{ X > \alpha \}) d\alpha + m > \beta \right\} - 1 \right] d\beta \\
+ \int_m^\infty h_Z \circ P^*_Z \left\{ z \in N^c \cap \text{Ran}(Z) : \int_m^\infty g_z \circ K_Z(z, \{ X > \alpha \}) d\alpha + m > \beta \right\} d\beta. \tag{6.49}
\]

Since \( X \) is bounded, we can choose \( m \) small enough, for example less than \(-\|X\|\), so that

\[
\rho(X \lor m; Z) = \rho(X; Z) \tag{6.50}
\]

and

\[
\int_m^\infty g_z \circ K_Z(z, \{ X > \alpha \}) d\alpha + m \\
= \int_m^0 [g_z \circ K_Z(z, \{ X > \alpha \}) - 1] d\alpha + \int_0^\infty g_z \circ K_Z(z, \{ X > \alpha \}) d\alpha \\
= \tau_Z(X; z), \tag{6.51}
\]

42
where the last equality is due to (6.40). After plugging (6.50) and (6.51) into (6.49), then letting $m \to -\infty$ in both sides of (6.49) results in the desired assertion (3.4). Theorem 3.3 is proved.

**Proof of Theorem 3.4.**

Taking (3.2) and (3.3) into account, (3.5) can be rewritten as

$$
\rho_Z(X) = \int \rho_Z(X; \cdot)dh \circ P_Z
$$

(6.52)

$X \in \mathcal{X}$, where $\rho_Z(\cdot; z)$ is defined by (3.1), that is, for every $z \in \mathbb{R},$

$$
\rho_Z(X; z) = \int Xdg_z \circ K_Z(z, \cdot).
$$

(6.53)

Therefore, the monotonicity is straightforward. By change-of-variable, elementary calculations can show the translation invariance and positive homogeneity. Next, we simply check the subadditivity. In fact, since the distortion functions $g_z$ and $h_Z$ are concave, by Example 2.1 of Denneberg (1994) or Proposition 4.75 of Föllmer and Schied (2016), we know that for each $z \in \mathbb{R}$, $g_z \circ K_Z(z, \cdot)$ is a monotone and submodular set function on $\mathcal{F}$, and that $h_Z \circ P_Z$ is a monotone and submodular set function on $\mathcal{B}(\mathbb{R})$. Hence, from (6.53) and the Subadditivity Theorem of Denneberg (1994, Theorem 6.3), it follows that for each $z \in \mathbb{R}$ and $X_1, X_2 \in \mathcal{X},$

$$
\rho_Z(X_1 + X_2; z) \leq \int X_1dg_z \circ K_Z(z, \cdot) + \int X_2dg_z \circ K_Z(z, \cdot) = \rho_Z(X_1; z) + \rho_Z(X_2; z),
$$

which, together with (6.52), the monotonicity of Choquet integral and the Subadditivity Theorem of Denneberg (1994, Theorem 6.3), results in

$$
\rho_Z(X_1 + X_2) \leq \int \rho_Z(X_1; \cdot)dh \circ P_Z + \int \rho_Z(X_2; \cdot)dh \circ P_Z = \rho_Z(X_1) + \rho_Z(X_2).
$$

Theorem 3.4 is proved.

**Proof of Theorem 3.5.**

For $X \in \mathcal{X}$, recall that by (6.52) and (6.53),

$$
\rho_Z(X) = \int \rho_Z(X; \cdot)dh \circ P_Z,
$$

(6.54)

where $\rho_Z(\cdot; z)$ is defined by (3.1), that is, for every $z \in \mathbb{R},$

$$
\rho_Z(X; z) = \int Xdg_z \circ K_Z(z, \cdot).
$$

(6.55)

Since $h_Z$ is concave, by Example 2.1 of Denneberg (1994) or Proposition 4.75 of Föllmer and Schied (2016), we know that $h_Z \circ P_Z$ is a monotone, normalized and submodular set function on $\mathcal{B}(\mathbb{R})$. Applying Theorem 4.94 of Föllmer and Schied (2016) to $h_Z \circ P_Z$ implies that

$$
\int \rho_Z(X; \cdot)dh \circ P_Z = \sup_{Q \in \mathcal{Q}} E_Q(\rho_Z(X; \cdot)),
$$

(6.55)
where \( \mathcal{D}_1 := \{ Q_1 \in \mathcal{M}_1(f(\mathbb{R}, \mathcal{B}(\mathbb{R})) : Q_1(B) \leq h_Z \circ P_Z(B) \text{ for all } B \in \mathcal{B}(\mathbb{R}) \} \), and the supremum taken over \( \mathcal{D}_1 \) can be attained at some \( Q_{1,x} \in \mathcal{D}_1 \), that is,

\[
\int \rho_Z(X; \cdot) dh_Z \circ P_Z = E_{Q_{1,x}}(\rho_Z(X; \cdot)) = \int \rho_Z(X; \cdot) dQ_{1,x}. \tag{6.56}
\]

Similarly, for each \( z \in \mathbb{R} \), since \( g_z \) is concave, then \( g_z \circ K_Z(z, \cdot) \) is a monotone, normalized and submodular set function on \( \mathcal{F} \). Again applying Theorem 4.94 of Föllmer and Schied (2016) to \( g_z \circ K_Z(z, \cdot) \), we have that

\[
\rho_Z(X; z) = \sup_{Q \in \mathcal{D}_{2,z}} E_Q(X), \tag{6.57}
\]

where \( \mathcal{D}_{2,z} := \{ Q_{2,z} \in \mathcal{M}_1(\Omega, \mathcal{F}) : Q_{2,z}(A) \leq g_z \circ K_Z(z, A) \text{ for all } A \in \mathcal{F} \} \), and the supremum taken over \( \mathcal{D}_{2,z} \) can be attained at some \( Q_{2,z,x} \in \mathcal{D}_{2,z} \), that is,

\[
\rho_Z(X; z) = E_{Q_{2,z,x}}(X) = \int X dQ_{2,z,x}. \tag{6.58}
\]

Thanks to Remark 3.2(i), recall that the function \( z \rightarrow \rho_Z(X; z) \) is a Borel function on \( \mathbb{R} \). Hence, repeating integration in both sides of \( \tag{6.58} \) with respect to \( \hat{Q}_{1,x} \), the outer set function of \( Q_{1,x} \), we have that

\[
\int \rho_Z(X; z) \hat{Q}_{1,x}(dz) = \int \rho_Z(X; z) Q_{1,x}(dz) = \int \left( \int X(\omega) Q_{2,z,x}(d\omega) \right) Q_{1,x}(dz),
\]

which, together with \( \tag{6.54} \) and \( \tag{6.56} \), gives rise to

\[
\rho_Z(X) = \int \left( \int X(\omega) Q_{2,z,x}(d\omega) \right) Q_{1,x}(dz). \tag{6.59}
\]

On the other hand, given any \( (Q_1, Q_2) \in \mathcal{C} \), where \( Q_1 \in \mathcal{D}_1, Q_2 := \{ Q_{2,z} \in \mathcal{D}_{2,z}; z \in \mathbb{R} \} \), then for each \( z \in \mathbb{R} \), both \( \tag{6.57} \) and \( \tag{6.58} \) together yield

\[
\int X(\omega) Q_{2,z}(d\omega) = E_{Q_{2,z}}(X) \leq \rho_Z(X; z) = \int X(\omega) Q_{2,z,x}(d\omega). \tag{6.60}
\]

Repeating integration in both sides of \( \tag{6.60} \) with respect to \( \hat{Q}_1 \), the outer set function of \( Q_1 \), from the monotonicity of Choquet integral, \( \tag{6.55} \), \( \tag{6.56} \) and \( \tag{6.60} \), we have that

\[
\int \left( \int X(\omega) Q_{2,z}(d\omega) \right) \hat{Q}_1(dz) \leq \int \rho_Z(X; z) \hat{Q}_1(dz) = \int \rho_Z(X; z) Q_1(dz)
\]

\[
= E_{Q_1}(\rho_Z(X; \cdot)) \leq \int \rho_Z(X; \cdot) dh_Z \circ P_Z = \int \rho_Z(X; \cdot) dQ_{1,x}
\]

\[
= \int \left( \int X(\omega) Q_{2,z,x}(d\omega) \right) Q_{1,x}(dz)
\]

\[
= \int \left( \int X(\omega) Q_{2,z,x}(d\omega) \right) \hat{Q}_{1,x}(dz). \tag{6.61}
\]

Observing that \( (Q_{1,x}, Q_{2,z,x}) \in \mathcal{C} \), the desired assertion \( \tag{3.6} \) follows \( \tag{6.59} \) and \( \tag{6.61} \). Theorem 3.5 is proved.
References

[1] Acerbi, C. (2002), Spectral measures of risk: A coherent representation of subjective risk aversion, Journal of Banking and Finance, 26, 1505-1518.

[2] Acerbi, C. and Tasche, D. (2002), On the coherence of expected shortfall, Journal of Banking and Finance, 26(7), 1487-1503.

[3] Acharya, V.V., Engle, R. and Richardson, M. (2012), Capital shortfall: a new approach to ranking and regulating systemic risks, American Economic Review, 102, 59-64.

[4] Acharya, V.V., Pedersen, L.H., Philippon, T. and Richardson, M. (2017), Measuring systemic risk, The Review of Financial Studies, 30 (1), 2-47.

[5] Adrian, T. and Brunnermeier, M.K. (2016), CoVaR, The American Economic Review, 106 (7), 1705-1741.

[6] Artzner, P., Delbaen, F., Eber J.M. and Heath D. (1999), Coherent measures of risk, Mathematical Finance, 9, 203-228.

[7] BCBS (2016), Standards. Minimum Capital Requirements for Market Risk. January 2016. Basel Committee on Banking Supervision. Document d352, Basel: Bank for International Settlements.

[8] BCBS (2019), Minimum Capital Requirements for Market Risk. February 2019. Basel Committee on Banking Supervision. Document d457, Basel: Bank for International Settlements.

[9] Belles-Sampera, J., Guillén, M. and Santolino, M. (2014), Beyond value-at-risk: Glue VaR distortion risk measures, Risk Analysis, 34(1), 121-134.

[10] Cherny, A. S. (2006), Weighted V@R and its Properties, Finance and Stochastics, 10(3), 367-393.

[11] Cont, R., Deguest, R. and Scandolo, G. (2010), Robustness and sensitivity analysis of risk measurement procedures, Quantitative Finance, 10(6), 593-606.

[12] Denneberg, D. (1990), Distorted probabilities and insurance premiums, Methods of Operations Research, 63, 3-5.

[13] Denneberg, D. (1994). Non-additive Measure and Integral, Dordrecht, Boston, London: Kluwer Academic Publishers.

[14] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D. (2002), The concept of comonotonicity in actuarial science and finance: Theory, Insurance: Mathematics and Economics, 31(1), 3-33.

[15] Dhaene, J., Laeven, R.J.A., Zhang Y. (2022), Systemic risk: Conditional distortion risk measures, Insurance: Mathematics and Economics, 102, 126-145.

[16] Embrechts, P., Liu, H. and Wang, R. (2018), Quantile-based risk sharing, Operations Research, 66(4), 936-949.
[17] Föllmer, H. and Schied, A. (2002), Convex measures of risk and trading constraints, Finance and Stochastics, 6(4), 429-447.

[18] Föllmer, H. and Schied, A. (2016), Stochastic Finance: An Introduction in Discrete Time, 4th ed., De Gruyter Studies in Mathematics, Vol. 27. Berlin: Walter De Gruyter.

[19] Frittelli, M. and Rosazza Gianin, E. (2002), Putting order in risk measures, Journal of Banking and Finance, 26, 1473-1486.

[20] Gollier, C. and Pratt, J.W. (1996), Risk vulnerability and the tempering effect of background risk, Econometrica, 64(5), 1109-1123.

[21] Heaton, J. and Lucas, D. (2000), Portfolio choice in the presence of background risk, The Economic Journal, 110, 1-26.

[22] Ikeda, N. and Watanabe, S. (1981), Stochastic Differential Equations and Diffusion Processes, North-Holland Mathematical Library, Vol. 24, Kadansha Ltd.

[23] Kaas, R., van Heerwaarden, A.E. and Goovaerts, M.J. (1994), Ordering of Actuarial Risks, Education Series 1, CAIRE, Brussels.

[24] Kleinow, J., Moreira, F., Strobl, S. and Vähämaa, S. (2017), Measuring systemic risk: a comparison of alternative market-based approaches, Finance Research Letters, 21, 40-46.

[25] Kusuoka, S. (2001), On law invariant coherent risk measures, Advances in Mathematical Economics, 3, 83-95.

[26] Liu, P., Wang, R. and Wei, L. (2020), Is the inf-convolution of law-invariant preferences law-invariant?, Insurance: Mathematics and Economics, 91, 144-154.

[27] Mainik, G. and Schaanning, E. (2014), On dependence consistency of CoVaR and some other systemic risk measures, Statistics & Risk Modeling, 31 (1), 49-77.

[28] Shiryaev, A.N. and Boas, R.P.(Translator) (1984), Probability, Springer-Verlag, New york.

[29] Tsanakas, A. (2008), Risk measurement in the presence of background risk, Insurance: Mathematics and Economics, 42(2), 520-528.

[30] Wang, R., Wei, Y. and Willmot, Gordon E. (2020), Characterization, robustness, and aggregation of signed Choquet integrals, Mathematics of Operations Research, 45(3), 993-1015.

[31] Wang, R. and Ziegel, J.F. (2021), Scenario-based risk evaluation, Finance and Stochastics, 25, 725-756.

[32] Wang, S.S. (1996), Premium calculation by transforming the layer premium density, ASTIN Bulletin, 26, 71-92.

[33] Wang, S.S., Young, V.R. and Panjer, H.H. (1997), Axiomatic characterization of insurance prices, Insurance: Mathematics and Economics, 21, 173-183.

[34] Yaari, M.E. (1987), The dual theory of choice under risk, Econometrica, 55, 95-115.