Rational Hypergeometric Ramanujan Identities for $1/\pi^c$: Survey and Generalizations

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Abstract

We give a simple unified proof for all existing rational hypergeometric ramanujan identities for $1/\pi$, and give a complete survey (without proof) of several generalizations: rational hypergeometric identities for $1/\pi^c$, Taylor expansions, upside-down formulas, and supercongruences.

1 Introduction

In a famous paper [20], S. Ramanujan gave 17 formulas for $1/\pi$. These formulas were proved and generalized much later by numerous authors. In the present paper, which is mainly a survey and does not claim originality, we have several goals. First, we want to show that the 36 rational hypergeometric formulas for $1/\pi$ follow by specialization from a single general formula. Second, we give a list of all known rational hypergeometric formulas for $1/\pi^c$ with $c \geq 2$, many unproved. Third, we will give Taylor expansions of which the $1/\pi^c$ formulas are only the constant term. Fourth, we give a list of what can be called upside-down formulas. Finally, we give the supercongruences corresponding to all the $1/\pi^c$ formulas. A large part of the formulas of this paper apart from the initial $1/\pi$ formulas are due to the second author.

This survey is meant to be exhaustive, which means that we would appreciate feedback from readers who are aware of formulas that are not in our list (and evidently of errors). Note that we do not list formulas involving algebraic (as opposed to rational) parameters, nor do we list the second author’s two-sided formulas where the sums are over $n \in \mathbb{Z}$ instead of $n \geq 0$, or other generalizations.

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We recall that the Pochhammer symbol $(x)_n$ is defined by $(x)_n = x(x + 1) \cdots (x + n - 1) = \Gamma(x + n)/\Gamma(x)$, this last formula allowing us to define it also when $n$ is not an integer.
Definition 1.1  

(1) Let $d \geq 2$ be an integer. For all $n \geq 0$ we define

$$R_n(d) = \prod_{1 \leq i \leq d \atop \gcd(i,d) = 1} \frac{(i/d)_n}{n!}.$$ 

(2) A rational hypergeometric product $H$ is a sequence of the form $H_n = \prod_{d \in I} R_n(d)^{v_d}$ for some finite index set $I$ and positive exponents $v_d$. We define the degree of $H$ by $\deg(H) = \sum_{d \in I} \phi(d)$.

(3) A rational hypergeometric Ramanujan series is a series of the form

$$S(H, a, P) = \sum_{n \geq 0} P(n) \frac{H_n}{a^n},$$

with $P \in \mathbb{Z}[X]$ and $a \in \mathbb{Q}^*$.

Some comments are in order:

(1) We only consider coefficients which are hypergeometric products (as opposed to quotients). We could allow $v_d < 0$, or more general coefficients, and indeed there is a vast amount of formulas for such general series, but we will restrict to those, although we will later give “upside-down” formulas where all the $v_d$ are negative. In the tables that we give below, we will abbreviate $\prod_{d \in I} R_n(d)^{v_d}$ as $\prod d^{v_d}$.

(2) In addition, we restrict to such products which are rational in the hypergeometric motive sense, in other words such that if some irreducible fraction $i/d$ occurs, then all irreducible fractions in $[0,1]$ with denominator $d$ occur.

(3) We only consider $P \in \mathbb{Z}[X]$ (or $P \in \mathbb{Q}[X]$, which is the same up to a multiplicative constant). Ramanujan himself gave formulas where $P$ has coefficients in a quadratic extension of $\mathbb{Q}$, but we will not consider those. Similarly, we restrict to $a \in \mathbb{Q}^*$.

(4) It is immediate to see that $H_n \sim C/n^{\deg(H)/2}$ for some constant $C$, so the series $S$ converges for $|a| > 1$, also for $a = -1$ if $\deg(P) < \deg(H)/2$, and for $a = 1$ if $\deg(P) < \deg(H)/2 - 1$. In fact, in all the examples that we will see we have $\deg(P) = (\deg(H) - 1)/2$ (and in particular $\deg(H)$ is odd).

(5) The hypergeometric function $\sum_{n \geq 0} H_n z^n$ satisfies a linear differential equation of degree $\deg(H)$, in other words there exists a polynomial $Q(n, z)$ of degree $\deg(H)$ in $n$ such that $\sum_{n \geq 0} Q(n, z) H_n z^n = 0$. We may thus restrict to polynomials $P$ such that $\deg(P) < \deg(H)$, since formulas with $P$ differing by a multiple of $Q$ are trivially equivalent. We will give examples of this below.
In fact, the only hypergeometric products that we will consider in this paper are products of the following $R_n(d)$:

$$R_n(p) = p^{-np} \frac{(pn)!}{n^p} \quad \text{for } p = 2, 3, 5,$$

$$R_n(4) = 2^{-6n} \frac{(4n)!}{(2n)!n!^2}, \quad \text{and} \quad R_n(6) = 2^{-4n} 3^{-3n} \frac{(6n)!}{(3n)!(2n)!n!}.$$

**Definition 1.2** A (convergent) hypergeometric Ramanujan series will be called a $1/\pi^c$-formula if its sum $S(H, a, P)$ is equal to an algebraic number divided by $\pi^c$.

A search through the (abundant) literature (together with additional personal investigations) shows that the only algebraic numbers that occur are of the form $\sqrt{k}$ for some $k \in \mathbb{Q}^*$, and we have been able to find exactly 36 series whose sum is of the form $\sqrt{k}/\pi$, 10 whose sum is of the form $\sqrt{k}/\pi^2$, plus a single example for $1/\pi^3$ due to B. Gourevitch and two examples for $1/\pi^4$, one due to J. Cullen the other to Y. Zhao. In addition, there are a number of “divergent” hypergeometric series for $1/\pi^c$ which we will mention.

### 2 Rational Hypergeometric Formulas for $1/\pi$

There are (at least) three methods for proving such formulas. The first, due to Ramanujan, is the use of elliptic functions and generalizations, the second is the use of modular functions, and the third is to use WZ-type summation methods. In the present paper we only use modular functions. In fact, we will show that all the known rational formulas for $1/\pi$ follow from a single general formula giving an identity between complex functions, which can then be specialized to any CM point that we want, and in particular to CM points giving rational formulas.

#### 2.1 A General Identity

An important theorem which can be found for instance in [21] states that any modular form (or function) $F$ of weight $k$ (say on a congruence subgroup of $\Gamma$), expressed locally in terms of a modular function $h$ (of weight 0), is a solution of a linear differential equation of order $k + 1$ with algebraic coefficients, which can be explicitly constructed; if in addition $h$ is a Hauptmodul, the coefficients can be chosen to be polynomials. Equivalently, there exists a sequence $u(n)$ satisfying a polynomial recurrence relation such that for $\Im(\tau)$ sufficiently large we have $F(\tau) = \sum_{n \geq 0} u(n) h(\tau)^n$. We then have the following easy result, where from now on we denote by $D$ the differential operator $D = (1/2\pi i) d/d\tau = q d/dq$:

**Proposition 2.1** As above, let $F$ be a modular function of weight $k$ on some congruence subgroup of $\Gamma$, $h$ a modular function (of weight 0), write $F(\tau) =$
\[ \sum_{n \geq 0} u(n)h(\tau)^n \]  for \( \Im(\tau) \) sufficiently large, and finally set

\[ G^*(\tau) = \frac{D(F)/F}{D(h)/h}(\tau) - \frac{k}{4\pi \Im(\tau) D(h)/h}, \]

which is nonholomorphic but modular of weight 0. We have the following general formula, which can be considered as a formula for \( 1/\pi \):

\[ \sum_{n \geq 0} (n - G^*(\tau))u(n)h(\tau)^n = \frac{k}{4\pi \Im(\tau)} \frac{F}{D(h)/h}(\tau). \]

Proof. First apply \( D \) to the formula expressing \( F \) in terms of \( h \), so that

\[ \frac{D(F(\tau))}{D(h(\tau))/h(\tau)} = \sum_{n \geq 0} nu(n)h(\tau)^n. \]

Thus, if we set \( G = (D(F)/F)/(D(h)/h) \), the left hand side is \( GF = G \sum_{n \geq 0} u(n)h^n \), so we have the identity

\[ \sum_{n \geq 0} (n - G(\tau))u(n)h(\tau)^n = 0. \]

Now \( D(h)/h \) is a modular function of weight 2, but \( D(F)/F \) is only quasi-modular: \( D^*(F)/F = D(F)/F - (k/(4\pi \Im(\tau))) \) is truly modular nonholomorphic of weight 2. Thus, we set \( G^* = (D^*(F)/F)/(D(h)/h) \), and this gives both the formula for \( G^* \) and the desired identity. \( \square \)

Now CM theory tells us that if \( \tau \) is a CM point and \( h(\tau) \) and \( F(\tau) \) have algebraic Fourier coefficients, then both \( h(\tau) \) and \( G^*(\tau) \) will be algebraic numbers, and \( (F)/(D(h)/h)(\tau) \), which has weight \( k-2 \), will be an algebraic number times \( \Omega_\tau^{k-2} \), where \( \Omega_\tau \) is a suitable period, for instance \( \Omega_\tau = \eta(\tau)^2 \). Thus, it will itself be algebraic if \( k = 2 \), otherwise be equal to an algebraic number times a product of values of the gamma function at rational arguments by the Lerch, Chowla–Selberg formula.

The rest of the work consists simply in specializing the above general argument to specific modular functions \( F \) and \( h \) and specific CM points \( \tau \).

Remark. Under this modular interpretation the existence of these \( 1/\pi \) formulas is due exclusively to the existence of the modularity-preserving nonholomorphic modification \( D^* \) of the differential operator \( D \) seen above, which involves \( 1/\pi \).

2.2 First Special Case: Level 1

We first consider modular forms on the full modular group. It is known at least since Klein–Fricke that we have the hypergeometric representation

\[ E_4^{1/4} = {}_2F_1(1/12, 5/12; 1; 1/J_1), \]

\[ 1.1] 1/J_1 \]
where \( E_k = 1 - B_k/(2k) \sum_{n \geq 1} \sigma_{k-1}(n)q^n \) and \( J_1(\tau) = j(\tau)/1728. \) Thanks to the Clausen identity, we deduce that

\[
E_4^{1/2} = \, _3F_2(1/2, 1/6, 5/6; 1, 1/J_1).
\]

This is modular of weight 2, so we apply the above proposition to \( F = E_4^{1/2} \)
and \( h = 1/J_1. \) We compute that \( D(h)/h = -D(j)/j = E_6/E_4, \) \( D(F)/F = (1/6)(E_2 - E_6/E_4), \) hence \( D^*(F)/F = (1/6)(E_2 - E_6/E_4) \) where \( E_2 = E_2 - 3/(\pi 3(\tau)), \) so \( G^* = -(1/6)(1 - E_2^*/E_4/E_6), \) so the general identity specializes to

\[
\sum_{n \geq 0} \left( 6n + 1 - \frac{E_2^*E_4}{E_6}(\tau) \right) \frac{R_n(2)R_n(6)}{J_1(\tau)^n} = \frac{3}{\pi 3(\tau)} \frac{E_4^{3/2}}{E_6}(\tau),
\]

an identity due to the Chudnovsky brothers.

For comparison with higher levels, we set \( s_1 = 1/6, \) so that for instance \( E_4^{1/4} = \, _2F_1(s_1/2, (1 - s_1)/2; 1; 1/J_1). \)

### 2.3 Special Cases: Levels 2 and 3

There is no difference for higher levels compared to level 1, apart from the need to give explicitly the modular functions used and the hypergeometric identities.

As it happens, levels 2 and 3 can be treated together. For \( N = 2 \) and 3 set

\[
F_2(\tau) = \frac{NE_2(N\tau) - E_2(\tau)}{N - 1}, \quad F_4(\tau) = \frac{N^2E_2(N\tau) - E_2(\tau)}{N^2 - 1},
\]

\[
J_N(\tau) = \frac{F_2^4}{F_2^4 - F_4^2}, \quad \text{and} \quad P_2(\tau) = \frac{NE_2(N\tau) + E_2(\tau)}{N + 1}.
\]

The hypergeometric identity is

\[
F_2^{1/2} = \, _2F_1(s_N/2, (1 - s_N)/2; 1; 1/J_N), \quad \text{with} \quad s_N = (N + 1)/12,
\]

so by Clausen

\[
F_2 = 3\, _2F_2(1/2, s_N, 1 - s_N; 1, 1/J_N).
\]

We apply the proposition to \( F = F_2 \) and \( h = 1/J_N. \) We compute that \( D(h)/h = F_4/F_2, \) \( D(F)/F = s_N(P_2 - F_4/F_2), \) hence \( D^*(F)/F = s_N(P_2^* - F_4/F_2) \) with \( P_2^* = P_2 - 6/((N + 1)\pi 3(\tau)), \) so \( G^* = -s_N(1 - P_2^*/F_2/F_4), \) and since \( 6/(N + 1) = 1/(2s_N), \) the general identity specializes to the two identities

\[
\sum_{n \geq 0} \left( 4n + 1 - \frac{P_2^*F_2}{F_4}(\tau) \right) \frac{R_n(2)R_n(4)}{J_2(\tau)^n} = \frac{2}{\pi 3(\tau)} \frac{F_2^2}{F_4}(\tau),
\]

\[
\sum_{n \geq 0} \left( 3n + 1 - \frac{P_2^*F_2}{F_4}(\tau) \right) \frac{R_n(2)R_n(3)}{J_3(\tau)^n} = \frac{3}{2\pi 3(\tau)} \frac{F_2^2}{F_4}(\tau).
\]
2.4 Special Case: Level 4

Here we set
\[ F_2(\tau) = \frac{4E_2(4\tau) - E_2(\tau)}{3}, \quad G_2(\tau) = 4E_2(4\tau) - 4E_2(2\tau) + E_2(\tau) \]
\[ J_4 = \frac{F_2^2}{F_2^2 - G_2^2}, \quad \text{and} \quad P_2(\tau) = E_2(2\tau). \]

The hypergeometric identity is again
\[ F_2^{1/2} = 2F_1(s_N/2, (1 - s_N)/2; 1; 1/J_4), \quad \text{with} \quad s_4 = 1/2, \]
so by Clausen $F_2 = 3F_2(1/2, s_4, 1 - s_4; 1, 1; 1/J_4)$. We apply the proposition to $F = F_2$ and $h = 1/J_4$. We compute that $D(h)/h = G_2, \ D(F)/F = (P_2 - G_2)/3$, hence $D^*(F)/F = (P_2^* - G_2)/3$ with $P_2^* = P_2 - 3/(2\pi\Im(\tau)), \ \text{so} \ G^* = -(1 - P_2^*/G_2)/3$, hence the general identity specializes to
\[ \sum_{n \geq 0} \left(3n + 1 - \frac{P_2^*}{G_2}(\tau)\right) \frac{R_n(2)^3}{J_4(\tau)^n} = \frac{3}{2\pi\Im(\tau)} \frac{F_2}{G_2}(\tau). \]

3 The Basic List of Rational Hypergeometric $1/\pi$ Formulas

From the above four specializations it is now immediate to obtain as many hypergeometric $1/\pi$ formulas as we like. To obtain such formulas which are 
*rational* in the above sense, we first need $J_N(\tau)$ to be rational. This trivially implies that the coefficients of $1/(\pi\Im(\tau))$ on the right-hand side of the formulas are square roots of rational numbers. Thus, if we want the coefficient of $1/\pi$ to be algebraic, we need both $J_N(\tau)$ rational and $\Im(\tau)$ algebraic, and a transcendence theorem (well-known for $N = 1$, but proved similarly for $N > 1$) implies that $\tau$ is a CM point, i.e., of the form $(a + \sqrt{D})/b$ with $D < 0$ and $a, b$ integral. In turn this implies (less trivially) that the other coefficients involved will be square roots of a rational number.

The following table summarizes the results obtained in this way: each formula is of the form $\sum_{n \geq 0} P(n)H_N(n)/a^n = \sqrt{k}/\pi$, where the function $H_N(n) = \frac{1}{3} \frac{(s_N)n(1 - s_N)n}{n^{13}}$ is the coefficient of $x^n$ in $3F_2(1/2, s_N, 1 - s_N; 1, 1; x)$, so that $H_1(n) = R_n(2)R_n(6)$, $H_2(n) = R_n(2)R_n(4)$, $H_3(n) = R_n(2)R_n(3)$, and $H_4(n) = R_n(2)^3$. For uniqueness, we always choose $P$ with content 1 and positive leading coefficient, $k$ is given in factored form, and the square root of $k$ is always the positive one. For future reference, we assign a number from 1 to 36 to each formula.
\[
\sum_{n \geq 0} P(n) \frac{H_N(n)}{a^n} = \frac{\sqrt{k}}{\pi}
\]

| # | \(N\) | \(H_N\) | \(\tau\) | \(a = J_N(\tau)\) | \(P\) | \(k\) |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 \cdot 6 | \(1 + \sqrt{-1}/2\) | \(-2^{3/5}\) | 63x + 8 | \(3 \cdot 5^3\) |
| 2 | 1 | 2 \cdot 6 | \(1 + \sqrt{-11}/2\) | \(-2^{9/3} - 3\) | 154x + 15 | \(2^{11}\) |
| 3 | 1 | 2 \cdot 6 | \(1 + \sqrt{-19}/2\) | \(-2^9\) | 342x + 25 | \(2^{11} \cdot 3\) |
| 4 | 1 | 2 \cdot 6 | \(1 + \sqrt{-27}/2\) | \(-2^{9/3} - 2^{5/3}\) | 506x + 31 | \(2^{11} \cdot 3 \cdot 3^{-3} \cdot 5^3\) |
| 5 | 1 | 2 \cdot 6 | \(1 + \sqrt{-43}/2\) | \(-2^{12} \cdot 5^3\) | 5418x + 263 | \(2^{14} \cdot 3^{-1} \cdot 5^3\) |
| 6 | 1 | 2 \cdot 6 | \(1 + \sqrt{-67}/2\) | \(-2^{9/3} \cdot 3^{11/3}\) | 261702x + 10177 | \(2^{11} \cdot 3 \cdot 5^3 \cdot 11^3\) |
| 7 | 1 | 2 \cdot 6 | \(1 + \sqrt{-163}/2\) | \(-2^{12} \cdot 5^3 \cdot 3^{29/3}\) | 545140134x + 13591409 | \(2^{14} \cdot 3 \cdot 5^3 \cdot 23^3 \cdot 29^3\) |
| 8 | 1 | 2 \cdot 6 | \(\sqrt{2}\) | \(2^{3} \cdot 5^3\) | 28x + 3 | 5^3 |
| 9 | 1 | 2 \cdot 6 | \(-\sqrt{3}\) | \(2^{-2} \cdot 5^3\) | 11x + 1 | \(2^{-2} \cdot 3^{-1} \cdot 5^3\) |
| 10 | 1 | 2 \cdot 6 | \(\sqrt{-4}\) | \(2^{-3} \cdot 11^3\) | 63x + 5 | \(2^{-4} \cdot 3 \cdot 11^3\) |
| 11 | 1 | 2 \cdot 6 | \(\sqrt{-7}\) | \(2^{-6} \cdot 5^3 \cdot 17^3\) | 133x + 8 | \(2^{-2} \cdot 3^{-5} \cdot 5^3 \cdot 17^3\) |
| 12 | 2 | 2 \cdot 4 | \(1 + \sqrt{-5}/2\) | \(-2^2\) | 20x + 3 | \(2^6\) |
| 13 | 2 | 2 \cdot 4 | \(1 + \sqrt{-7}/2\) | \(-2^{-8} \cdot 3^4 \cdot 7^2\) | 65x + 8 | \(3^4 \cdot 7\) |
| 14 | 2 | 2 \cdot 4 | \(1 + \sqrt{-9}/2\) | \(-2^{-6} \cdot 3^2\) | 28x + 3 | \(2^8 \cdot 3^{-1}\) |
| 15 | 2 | 2 \cdot 4 | \(1 + \sqrt{-13}/2\) | \(-2^{-2} \cdot 3^4\) | 260x + 23 | \(2^6 \cdot 3^4\) |
| 16 | 2 | 2 \cdot 4 | \(1 + \sqrt{-25}/2\) | \(-2^{-6} \cdot 3^4 \cdot 5^3\) | 644x + 41 | \(2^{10} \cdot 3^4 \cdot 5^{-1}\) |
| 17 | 2 | 2 \cdot 4 | \(1 + \sqrt{-37}/2\) | \(-2^{-2} \cdot 3^4 \cdot 7^4\) | 21460x + 1123 | \(2^6 \cdot 3^4 \cdot 7^4\) |
| 18 | 2 | 2 \cdot 4 | \(\sqrt{-1}\) | \(2^{-5} \cdot 3^4\) | 7x + 1 | \(2^{-2} \cdot 3^4\) |
| 19 | 2 | 2 \cdot 4 | \(\sqrt{-6}/2\) | \(2\) | 8x + 1 | \(2^2 \cdot 3\) |
| 20 | 2 | 2 \cdot 4 | \(\sqrt{-10}/2\) | \(3^2\) | 10x + 1 | \(2^{-3} \cdot 3^2\) |
| 21 | 2 | 2 \cdot 4 | \(\sqrt{-18}/2\) | \(7^4\) | 40x + 3 | \(3^{-3} \cdot 7^4\) |
| 22 | 2 | 2 \cdot 4 | \(\sqrt{-22}/2\) | \(3^{11/2}\) | 280x + 19 | \(2^3 \cdot 3^4 \cdot 11\) |
| 23 | 2 | 2 \cdot 4 | \(\sqrt{-55}/2\) | \(3^{11/2}\) | 26390x + 1103 | \(2^{-3} \cdot 3^3 \cdot 11^4\) |
| 24 | 3 | 2 \cdot 3 | \((3 + \sqrt{-27})/6\) | \(-2^{12} \cdot 3^2\) | 5x + 1 | \(2^4 \cdot 3^{-1}\) |
| 25 | 3 | 2 \cdot 3 | \((3 + \sqrt{-51})/6\) | \(-2^{6} \cdot 3\) | 51x + 7 | \(2^4 \cdot 3^3\) |
| 26 | 3 | 2 \cdot 3 | \((3 + \sqrt{-75})/6\) | \(-2^{4} \cdot 5\) | 9x + 1 | \(2^4 \cdot 3 \cdot 5^{-1}\) |
| 27 | 3 | 2 \cdot 3 | \((3 + \sqrt{-123})/6\) | \(-2^{10} \cdot 3\) | 615x + 53 | \(2^{10} \cdot 3^3\) |
| 28 | 3 | 2 \cdot 3 | \((3 + \sqrt{-147})/6\) | \(-2^{4} \cdot 3^2 \cdot 7\) | 165x + 13 | \(2^5 \cdot 3^6 \cdot 7^{-1}\) |
| 29 | 3 | 2 \cdot 3 | \((3 + \sqrt{-267})/6\) | \(-2^{12} \cdot 5^6\) | 14151x + 827 | \(2^4 \cdot 3^3 \cdot 5^6\) |
| 30 | 3 | 2 \cdot 3 | \(\sqrt{-6}/3\) | \(2\) | 6x + 1 | \(3^3\) |
| 31 | 3 | 2 \cdot 3 | \(\sqrt{-12}/3\) | \(2^{-1} \cdot 3^3\) | 15x + 2 | \(2^{-4} \cdot 3^6\) |
| 32 | 3 | 2 \cdot 3 | \(\sqrt{-15}/3\) | \(2^{-2} \cdot 5^3\) | 33x + 4 | \(2^{-2} \cdot 3^3 \cdot 5^2\) |
| 33 | 4 | 2^3 | \((1 + \sqrt{-2})/2\) | \(-1\) | 4x + 1 | \(2^3\) |
| 34 | 4 | 2^3 | \((1 + \sqrt{-4})/2\) | \(-2^3\) | 6x + 1 | \(2^3\) |
| 35 | 4 | 2^3 | \(\sqrt{-3}/2\) | \(2^2\) | 6x + 1 | \(2^4\) |
| 36 | 4 | 2^3 | \(\sqrt{-7}/2\) | \(2^6\) | 42x + 5 | \(2^8\) |

Rational hypergeometric formulas for \(1/\pi\)

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Note that a generalization of the proof of the class number 1 problem for imaginary quadratic fields proves that the values for \( a \) listed above (together with the values for the divergent series that we will give below) are the only rational values of \( J_N(\tau) \) at CM arguments \( \tau \), outside of the values 0 and 1 which cannot be used.

Note that we do not claim that we have found all possible rational \( 1/\pi \) formulas, but only that, as far as we can tell, no other such formula exists in the literature, and a rather long search using linear dependence algorithms did not find any additional ones, except from trivial modifications coming from the fact that \( _2 F_1 \) is solution of a linear differential equation of order 2. For instance, we have the following formulas, which are trivially equivalent to the last three formulas of the above list:

\[
\sum_{n \geq 0} \frac{(6n^3 + n^2) R_n(2)^3}{(-8)^n} = -\frac{\sqrt{2}/6}{\pi}, \quad \sum_{n \geq 0} \frac{(2n^3 - n^2) R_n(2)^3}{4^n} = \frac{1}{3} \pi,
\]

\[
\sum_{n \geq 0} \frac{(210n^3 - 5n^2 + n) R_n(2)^3}{64^n} = \frac{4}{3} \pi.
\]

Note that the same method allows us to find divergent series because \( |a| < 1 \):

| # | \( N \) | \( H_N \) | \( \tau \) | \( a = J_N(\tau) \) | \( P \) | \( k \) | sign |
|---|---|---|---|---|---|---|---|
| 37 | 2 | \( 2 \cdot 4 \) | \(-1 + \sqrt{-3}/2\) | \(-2^{-1}3^2\) | \(5x + 1\) | 3 | + |
| 38 | 2 | \( 2 \cdot 4 \) | \(1 + \sqrt{-7}/4\) | \(2^{-8}3^4\) | \(35x + 8\) | \(-2^{3}4^{2}\) | - |
| 39 | 3 | \( 2 \cdot 3 \) | \((2 + \sqrt{-2})/6\) | \(2^{-3}3^{-3}\) | \(10x + 3\) | \(-2^{5}5^{2}\) | - |
| 40 | 3 | \( 2 \cdot 3 \) | \((1 + \sqrt{-11})/6\) | \(2^{3}3^{-3}\) | \(11x + 3\) | \(-2^{3}3^{2}\) | + |
| 41 | 3 | \( 2 \cdot 3 \) | \((3 + \sqrt{-15})/6\) | \(-2^{-2}\) | \(15x + 4\) | \(3^{3}\) | + |
| 42 | 4 | \( 2^3 \) | \((1 + \sqrt{-1})/2\) | \(-2^{-3}\) | \(3x + 1\) | 1 | + |
| 43 | 4 | \( 2^3 \) | \((3 + \sqrt{-7})/8\) | \(2^{-6}\) | \(21x + 8\) | \(-2^{4}\) | - |
| 44 | 4 | \( 2^3 \) | \((1 + \sqrt{-3})/4\) | \(2^{-2}\) | \(3x + 1\) | \(-2^{2}\) | - |

The values of \( k \) given in this table are those coming from the general formula, but correspond to the values obtained from the analytic continuation of the hypergeometric series only when \( a < 0 \). On the other hand, we will see below that they all lead to supercongruences, as well as so-called upside-down formulas. Here the sign of the square root of \( k \) can vary, so is indicated in the last column with respect to the principal determination.

4 Additional Consequences of Proposition 2.1

In the previous section, we have only used Proposition 2.1 for some very specific pairs \( (F, h) \) which lead to rational hypergeometric formulas for \( 1/\pi \). It is evidently possible to use it for other pairs: in particular we could use other subgroups of \( \Gamma \), and in particular the groups \( \Gamma_0(N) \) for \( N > 4 \), or still the same subgroups that we have already considered, but with different \( F \) (since the subgroups \( \Gamma_0(N) \) for \( N \leq 4 \) all have genus 0, there is not much point in changing
the function $h$, since the resulting formulas could also be obtained by standard hypergeometric identities).

Using $\Gamma_5(N)$ for $N > 4$ will not lead to identities involving hypergeometric functions, but for instance to more general functions called Heun functions.

Using different functions $F$ does give additional formulas. We simply give two examples, without proof since they are once again direct applications of Proposition 2.1. These are examples in level 1, so we keep $h = J_1 = j/1728$.

First, we choose $F = E_4^{1/4}$, and we find the general formula

$$\sum_{n \geq 0} \left( 12n + 1 - \frac{E_4^3 E_4}{E_4} (\tau) \right) \left( \frac{1/12}_n (5/12)_n \right)^{n/4} J_1(\tau)^n = \frac{3}{\pi^3(\tau)} E_4(\tau)^{-1/4} \frac{E_4^{3/2}}{E_6(\tau)} .$$

Specializing to $\tau = \sqrt{-3}$ and $\tau = \sqrt{-4}$, and as mentioned above using the Chowla–Selberg formula to compute the values of $E_4(\tau)$, we obtain the following identities:

$$\sum_{n \geq 0} (22n + 1) \left( \frac{1/12}_n (5/12)_n \right)^{n/4} \frac{1}{(125/4)^n} \frac{\Gamma(1/3)}{\Gamma(1/3)^3 B(1/3, 1/3)} = \frac{21/35/43/411/411/4/411/4}{B(1/4, 1/4)},$$

where $B(a, b) = \Gamma(a) \Gamma(b)/\Gamma(a + b)$ is the beta function.

Choosing instead $F = E_6^{1/6}$ leads to the following general formula:

$$\sum_{n \geq 0} \left( 12n + 1 - \frac{E_4^3 E_4}{E_4} (\tau) \right) \left( \frac{1/12}_n (7/12)_n \right)^{n/6} \frac{1}{(1 - J_1(\tau))^n} = \frac{3}{\pi^3(\tau)} E_4(\tau)^{-1/4} \left( \frac{E_4^{3/2}}{E_6(\tau)} \right)^{-7/6},$$

and specializing to the same values of $\tau$ gives

$$\sum_{n \geq 0} (150n + 7) \left( \frac{1/12}_n (7/12)_n \right)^{n/6} \frac{1}{(-121/4)^n} \frac{\Gamma(1/3)}{\Gamma(1/3)^3 B(1/3, 1/3)} = \frac{21/35/43/411/411/4/411/4}{B(1/4, 1/4)},$$

$$\sum_{n \geq 0} (726n + 29) \left( \frac{1/12}_n (7/12)_n \right)^{n/6} \frac{1}{(-1323/8)^n} \frac{\Gamma(1/4)^2}{\Gamma(1/4)^2 B(1/4, 1/4)} = \frac{21/35/43/417/6}{B(1/4, 1/4)} .$$

5 Generalization I: Rational Hypergeometric Formulas for $1/\pi^c$

Finding rational hypergeometric identities for $1/\pi^c$ is easily done using linear dependence algorithms based on the LLL algorithm. Proving them is more difficult: among the methods used is the WZ method, but it is not the only one. In fact some of the identities (and the three known identities for $c \geq 3$) are still conjectural. Explaining them, as we have done for $1/\pi$ formulas has only
started to be done in a recent paper by Dembélé et al. [6]: the $1/\pi^2$ formulas are linked to Asai $L$-functions attached to Hilbert modular forms for real quadratic fields. However, this apparently still does not prove all of them.

In the following table, we list all known formulas, as before coding $H$ as $\prod_{d \in \mathcal{D}}^\mathcal{D}$, meaning that $H_n = \prod_{d \in \mathcal{D}} R_n(d)^{\mathcal{D}_d}$, and the formula is of the form

$$\sum_{n \geq 0} P(n) \frac{H_n}{a^n} = \frac{\sqrt{k}}{\pi^c}.$$ 

| # | $c$ | $H$ | $a$ | $P$ | $k$ |
|---|---|---|---|---|---|
| 1 | 2 | $2^2$ | $-2^2$ | $20x^2 + 8x + 1$ | $2^6$ |
| 2 | 2 | $2^5$ | $-2^{10}$ | $820x^2 + 180x + 13$ | $2^{14}$ |
| 3 | 2 | $2^3 \cdot 3$ | $2^3 \cdot 3$ | $74x^2 + 27x + 3$ | $2^8 \cdot 3^2$ |
| 4 | 2 | $2^3 \cdot 4$ | $2^4$ | $120x^2 + 34x + 3$ | $2^{10}$ |
| 5 | 2 | $2 \cdot 3 \cdot 4$ | $-2^{43}$ | $252x^2 + 63x + 5$ | $2^8 \cdot 3^2$ |
| 6 | 2 | $2 \cdot 3 \cdot 6$ | $-2^{123-6}$ | $1930x^2 + 549x + 45$ | $2^{14} \cdot 3^2$ |
| 7 | 2 | $2 \cdot 3 \cdot 6$ | $-2^{123-6}$ | $5418x^2 + 693x + 29$ | $2^{14} \cdot 5$ |
| 8 | 2 | $2 \cdot 3 \cdot 6$ | $3^{-6}5^6$ | $532x^2 + 126x + 9$ | $2^{-4} \cdot 3^2 \cdot 5^6$ |
| 9 | 2 | $2 \cdot 4 \cdot 6$ | $-2^{10}$ | $1640x^2 + 278x + 15$ | $2^{16} \cdot 3^{-1}$ |
| 10 | 2 | $2 \cdot 8$ | $7^4$ | $1920x^2 + 304x + 15$ | $2^6 \cdot 3^3$ |
| 11 | 3 | $2^2$ | $2^6$ | $168x^4 + 76x^2 + 14x + 1$ | $2^{10}$ |
| 12 | 4 | $2^3 \cdot 3 \cdot 4$ | $-2^{83-3}$ | $4528x^4 + 3180x^3 + 972x^2 + 147x + 9$ | $2^{16} \cdot 3^2$ |
| 13 | 4 | $2^74$ | $2^{12}$ | $43680x^4 + 20632x^3 + 4340x^2 + 466x + 21$ | $2^{22}$ |

Rational hypergeometric formulas for $1/\pi^c$

Most formulas for $1/\pi^2$ were found by the second author, the formula for $1/\pi^4$ was found by B. Gourevitch and the two formulas for $1/\pi^4$ were found by Y. Zhao and J. Cullen respectively.

We can also find in the literature the following divergent formulas for $1/\pi^c$:

| # | $c$ | $H$ | $a$ | $P$ | $k$ | sign |
|---|---|---|---|---|---|---|
| 14 | 2 | $2^5$ | $-2^{10}$ | $10x^2 + 6x + 1$ | $2^8$ | $+$ |
| 15 | 2 | $2^5$ | $-2^{10}$ | $205x^2 + 160x + 32$ | $2^8$ | $+$ |
| 16 | 2 | $2^3 \cdot 3$ | $-3^{-3}$ | $28x^2 + 18x + 3$ | $2^23^2$ | $+$ |
| 17 | 2 | $2 \cdot 5$ | $-2^{85-5}$ | $483x^2 + 245x + 30$ | $2^85^2$ | $+$ |
| 18 | 2 | $2 \cdot 3 \cdot 4$ | $-2^{43-3}$ | $172x^2 + 75x + 9$ | $2^83^2$ | $+$ |
| 19 | 3 | $2^2$ | $-2^6$ | $21x^3 + 22x^2 + 8x + 1$ | $-2^{3}3^2$ | $-$ |
| 20 | 3 | $2^53$ | $2^33$ | $92x^3 + 84x^2 + 27x + 3$ | $-2^83^2$ | $-$ |
| 21 | 4 | $2^5$ | $-2^{105-9}$ | $5532x^4 + 5600x^3 + 2275x^2 + 425x + 30$ | $2^{10}5^2$ | $+$ |
6 Generalization II: Taylor Expansions of $1/\pi^c$ Formulas

6.1 Taylor Expansions of $1/\pi$ Formulas

Following ideas of the second author [11], [14], [15], we are going to generalize the above formulas by considering them as constant terms of Taylor expansions. Note that for any $y$ we can define $(x)_n + y = \Gamma(x + n + y)/\Gamma(x)$, hence since $n! = (1)_n$:

$$R_{n+x}(d) = \prod_{1 \leq i \leq d \atop \text{gcd}(i,d) = 1} \frac{(i/d)_{n+x}}{(1)_{n+x}}.$$  

Thus $H_{n+x}$ makes sense, so we could define the generalized sum as

$$\sum_{n \geq 0} P(n + x) \frac{H_{n+x}}{a^{n+x}}.$$  

However, when $a < 0$ the factor $a^x$ introduces parasitic imaginary terms, so we prefer to define

$$S(H, a, P; x) = \sum_{n \geq 0} P(n + x) \frac{H_{n+x}}{\text{sign}(a)^n |a|^{n+x}} = \sum_{n \geq 0} P(n + x) \frac{H_{n+x}}{a^n |a|^{x}},$$

which is equal to $\text{sign}(a)^x$ times the previous one. Note that this is not the only possible normalization. We could also shift all the Pochhammer indices by $x$ instead of shifting $n$. In all cases, this would give the above series multiplied by a quotient of products of gamma functions involving $x$, so the transformation from one to the other is immediate.

It is clear that $S(H, a, P; x + 1) = \text{sign}(a)(S(H, a, P; x) - P(x)H_x/|a|^x)$, so we may assume if necessary that $x \in [0, 1[$.

In view of the existing literature, we can ask at least two questions: first, give (at least the initial terms of) the power series expansion of $S(H, a, P; x)$ around $x = 0$. Second, give the value of $S(H, a, P; 1/2)$.

One observes that if the value of the sum is $a_0 \sqrt{-D}/\pi$ with $a_0 \in \mathbb{Q}^*$ and $D$ a negative fundamental discriminant, the expansion is always of the form

$$S(H, a, P; x) = a_0 |D| \left( \frac{\sqrt{-D}}{\pi} + 0x - a_2 |D|L(D, 1)x^2 - a_3 D^2 L(D, 2)x^3 + O(x^4) \right),$$

with the $a_i$ rational, and where we write $L(D, m)$ for $\sum_{n \geq 1} (\frac{D}{n})/n^m$ (of course, since $D < 0$ we have $a_2 |D|L(D, 1) = a_2' \sqrt{-D} \pi$ for some rational $a_2'$). Note that, as mentioned above, it is in principle possible to compute the coefficient of $x^4$, but by laziness we have done to so only for cases (33), (35), and (36), see below.

The following table uses the same numbering of the formulas as that given above; the column $C_3$ is related to supercongruences and will be explained below:
\[ S(H, a, P; x) = a_0 |D| \left( \frac{\sqrt{-D}}{\pi} + 0x - a_2 |D| L(D, 1)x^2 - a_3 D^2 L(D, 2)x^3 + O(x^4) \right), \]

\[ S_p(H, a, P) \equiv P(0) \left( \frac{D}{p} \right) p + C_3 L(D, 3 - p)p^3 \pmod{p^4}. \]

| # | D  | a_0 | a_2 | a_3 | \pi^2 S(H, a, P; 1/2)/|a_0|D|\sqrt{-D} | C_3 |
|---|----|-----|-----|-----|----------------------|-----|
| 1 | −15 | 1/3 | 3/4 | 1/2 | \log(3^{4/5}) | 20 |
| 2 | −8 | 2 | 7/2 | 4 | \log(2) | 15 |
| 3 | −24 | 2/3 | 15/4 | 2 | \log(2^{13}/3^4) | 5/2 |
| 4 | −120 | 2/27 | 23/8 | 1/3 | \log(3^{3.5}/7^2) | 5/12 |
| 5 | −15 | 128/9 | 39/4 | 12 | \log(2^{5}/3^2) | 5/64 |
| 6 | −1320 | 2/3 | 63/16 | 1/26 | \log(2^{15}/3^{5}/7^1) | 5/104 |
| 7 | −40020 | 16/3 | 159/128 | 1/1560 | \log(3^{21}/17^2) | 5/36992 |
| 8 | −20 | 1/8 | 1 | 1/2 | 2 \log(3) | −15/2 |
| 9 | −15 | 1/8 | 2 | 3/2 | 2 \log(7) | −5/8 |
| 10 | −132 | 1/96 | 3/2 | 1/8 | 2 \log(11) | −5/4 |
| 11 | −255 | 1/162 | 1 | 1/12 | 2 \log(4207) | −5/81 |
| 12 | −4 | 1 | 3 | 4 | 2 \log(2) | 6 |
| 13 | −7 | 9/7 | 5/2 | 5/2 | 2 \log((88 + 13\sqrt{7})/3^4) | 20/3 |
| 14 | −3 | 16/9 | 21/2 | 20 | (3/2) \log(3^{3}/2^4) | 15/8 |
| 15 | −4 | 9 | 11 | 20 | 2 \log(3^{2}/2^4) | 10/3 |
| 16 | −20 | 36/25 | 23/4 | 4 | \log(2^{18}/3^{5}) | 1/6 |
| 17 | −4 | 441 | 35 | 100 | \log(2/3^{10}/7^6) | 50/147 |
| 18 | −4 | 9/16 | 2 | 5/2 | 2 \log(7/3) | −10/3 |
| 19 | −3 | 2/3 | 6 | 10 | \pi/3 | −15/8 |
| 20 | −8 | 9/64 | 4 | 4 | 2 \log(17/3^4) | −1/3 |
| 21 | −3 | 49/27 | 24 | 10 | 2 \log(173/2^4) | −45/392 |
| 22 | −11 | 18/11 | 10 | 10 | 2 \log(353) | −5/24 |
| 23 | −8 | 9801/64 | 28 | 60 | 2 \log(8668855388657) | −5/1089 |
| 24 | −3 | 4/9 | 5/2 | 10/3 | \log(3^{3}/2^3) | 5/2 |
| 25 | −3 | 4 | 13/2 | 10 | 3 \log(4/3) | 1/4 |
| 26 | −15 | 4/75 | 7/4 | 1 | \log(3^{9}/(2^{2}/5^5)) | 1/4 |
| 27 | −3 | 32 | 37/2 | 40 | 3 \log(2^8/3^5) | 15/4 |
| 28 | −7 | 108/49 | 15/2 | 10 | \log(7^{7}/2^{10}/5^6) | 5/18 |
| 29 | −3 | 500 | 85/2 | 130 | 3 \log(5^6/2^{10}/3^5) | 39/50 |
| 30 | −3 | 1 | 2 | 5/2 | (2/3)(\pi - \log(3^{11}/3^9)) | −15/4 |
| 31 | −4 | 27/32 | 4 | 5 | 2 \log(329/3^6) | −20/9 |
| 32 | −3 | 5/2 | 8 | 13 | 2 \log(239/3^6) | −78/25 |
| 33 | −4 | 1/4 | 1 | 1 | 8L^{(4)/4} | 2 |
| 34 | −8 | 1/8 | 3/2 | 1 | 4L^{(4)/4} | 1 |
| 35 | −4 | 1/2 | 2 | 2 | \pi/2 | −2 |
| 36 | −4 | 2 | 6 | 8 | \pi/6 | −2 |
6.2 Observations for $1/\pi$

Concerning the Taylor expansions around $x = 0$, note that the coefficient $a_1$ of $x^1$ always vanishes, and that the numerator of the first seven $a_2$ are equal to $|D(\tau)| - 4$, where $D(\tau) = -7, -11, -19, -27, -43, -67,$ and $-163$ is the discriminant of the corresponding $\tau$ (not to be confused with the $D$ occurring in the result).

In addition, we also notice a common pattern for the value at $x = 1/2$:

1. In the value of $S(H, a, P; 1/2)$ for $a < 0$: with only a few exceptions listed below, we have $S(H, a, P; 1/2) = c_0|D|\sqrt{-D}/\pi^2$, where $c_0$ and $c_1$ are rational. The exceptions are as follows:

   - In cases (33) and (34) we have $S(H, a, P; 1/2) = c_0|D|\sqrt{-D}(\log(c_1)/\pi^2)$ with $c_0$ rational.
   - In case (13) $c_1$ is not rational but in the quadratic field $\mathbb{Q}(\sqrt{7})$.

2. In the value of $S(H, a, P; 1/2)$ for $a > 0$: in all cases we have $S(H, a, P; 1/2) = c_0|D|\sqrt{-D}\sin(c_1)/\pi^2$ where $c_0$ and $c_1$ are rational (note that $c|D|\sqrt{-D}/\pi$ is of course of this form, for instance by choosing $c_1 = 1$).

It is also possible to guess the coefficient of $x^4$: for instance in cases (33), (35), and (36), set

\[
C_1 = \sum_{n \geq 1} (-1)^n \frac{H_{2n}}{(2n+1)^2} \quad \text{and} \quad C_2 = \sum_{n \geq 1} (-1)^n \frac{H_n}{(2n+1)^2},
\]

where here $H_n = \sum_{1 \leq j \leq n} 1/j$ is the $n$th harmonic sum. Then

\[
S(H, a, P; x) = a_0|D| \left( \frac{\sqrt{-D}}{\pi^2} + 0x - a_2|D|L(D, 1)x^2 \right.
\]
\[
- a_4|x^4 + a_4|x^4 + O(x^5) \right),
\]

with $D = -4$ and

\[
\begin{align*}
    a_4 &= (8/3)(50C_1 - 11C_2 - 22L(-4, 2)\log(2)), \\
    a_4 &= (128/3)(10C_1 - C_2 - 2L(-4, 2)\log(2)), \\
    a_4 &= 128(22C_1 - C_2 - 2L(-4, 2)\log(2))
\end{align*}
\]

for cases (33), (35), and (36) respectively.

6.3 Taylor Expansions of $1/\pi^c$ Formulas for $c \geq 2$

For $c = 2$ one observes that if the value of the sum is $a_0\sqrt{D}/\pi^2$ with $a_0 \in \mathbb{Q}^*$ and $D$ a fundamental discriminant, the expansion is always of the form

\[
S(H, a, P; x) = a_0D \left( \frac{\sqrt{D}}{\pi^2} + 0x - a_2D\sqrt{D}x^2 + 0x^3 \right.
\]
\[
+ a_4D^3L(D, 2)x^4 - a_5a_0D^4L(D, 3)x^5 + O(x^6) \right),
\]

13
with the $a_i$ rational (of course, when $D > 0$ we have $a_4 D^3 L(D, 2) = a_4' \pi^2 D \sqrt{D}$ for some rational $a_4'$).

$$S(H, a, P; x) = a_0 D \left( \frac{\sqrt{D}}{\pi^2} + 0x - a_2 D \sqrt{D} x^2 + 0x^3 ight.$$
$$\left. + a_4 D^3 L(D, 2) x^4 - a_5 a_0 D^4 L(D, 3) x^5 + O(x^6) \right),$$
$$S_p(H, a, P) \equiv P(0) \left( \frac{D}{p} \right) p^2 + C_5 L(D, 4 - p) p^5 \pmod{p^6}. $$

| # | $D$ | $a_0$ | $a_2$ | $a_4$ | $a_5$ | $\pi^3 S(1/2)/(a_0 D \sqrt{D})$ | $C_5$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 8 | 1/2 | 25/4 | 7 | 14$\zeta(3)$/$\pi^2$ | -7/2 |
| 2 | 1 | 128 | 5/2 | 305/4 | 7 | 2$\zeta(3)$/$\pi^2$ | -7/2 |
| 3 | 1 | 48 | 1/3 | 4 | 7/9 | 2$\pi$/3 | 21 |
| 4 | 1 | 32 | 1 | 20 | 7 | $\pi$/3 | 21/2 |
| 5 | 1 | 48 | 3/2 | 157/4 | 91/9 | 4 log($2^5$/3$^3$) | -91/9 |
| 6 | 1 | 384 | 5/6 | 85/4 | 7/9 | 2 log(2) | -315 |
| 7 | 5 | 128/5 | 3/2 | 887/32 | 21/8 | log($2^45^5$/3$^5$) | -35/216 |
| 8 | 1 | 375/4 | 4/3 | 40 | 6944/1125 | 2 asin(164833/5$^4$) | 1953/50 |
| 9 | 12 | 32/9 | 7/24 | 757/1152 | 3/16 | log($3^3$/2$^{14}$) | -15/2 |
| 10 | 28 | 1 | 1/7 | 31/336 | 1/21 | 2 asin(2241857/2$^{25}$) | 35/8 |

The last three (one for $1/\pi^3$ and two for $1/\pi^4$) obey completely similar expansions, but we give them one by one:

11:

$$S(H, a, P; x) = 32 \left( \frac{1}{\pi^3} + 0x - \frac{1}{\pi} x^2 + 0x^3 + (16/3) L(-4, 1) x^4 + 0x^5 
- (8224/45) L(-4, 3) x^6 + 32^2 \cdot 48 L(-4, 4) x^7 + O(x^8) \right),$$
$$S(H, a, P; 1/2) = \frac{8}{\pi^3},$$
$$S_p(H, a, P) \equiv \left( \frac{-4}{p} \right) p^3 - 6 L(-4, 5 - p) p^7 \pmod{p^8}. $$

12:

$$S(H, a, P; x) = 768 \left( \frac{1}{\pi^4} + 0x - \frac{1/2}{\pi^2} x^2 + 0x^3 + (3/8) x^4 + 0x^5 - (147/8) \zeta(2) x^6
+ 0x^7 + (471187/1344) \zeta(4) x^8 - 3968 \zeta(5) x^9 + O(x^{10}) \right),$$
$$S(H, a, P; 1/2) = \frac{9216 \zeta(3)}{\pi^7},$$
$$S_p(H, a, P) \equiv 9p^4 - (837/2) \zeta(6 - p) p^9 \pmod{p^{10}}. $$
\[ S(H, a, P; x) = 2048 \left( \frac{1}{\pi^4} + 0x - \frac{2}{\pi^2}x^2 + 0x^3 + (11/3)x^4 + 0x^5 - (908/15)\zeta(2)x^6 
+ 0x^7 + (53932/7)\zeta(4)x^8 - 95232\zeta(5)x^9 + O(x^{10}) \right), \]

\[ S(H, a, P; 1/2) = \frac{2048/15}{\pi^4}, \]

\[ S_p(H, a, P) \equiv 21p^2 + (279/4)\zeta(6 - p)p^9 \pmod{p^{10}}. \]

The observations for $1/\pi^c$ are essentially identical to the case of $1/\pi$, in particular the coefficients of $x^{2j-1}$ for $1 \leq j \leq c$ vanish.

### 7 Generalization III: Upside-Down Series

For completeness, we list the upside-down series (i.e., with $H_n$ in the denominator) given in [17]. We do not know if all have been proved, but probably not all among those with $c > 1$.

The general recipe is as follows: if \( \sum_{n \geq 0} P(n)H_n/a^n = \sqrt{k}/\pi^c \) is a divergent or semi-convergent series (i.e., with $|a| < 1$ or $a = -1$) then

\[ \sum_{n \geq 1} \frac{P(-n)}{n^{2c+1}H_n(1/a)^n} = A \cdot L(D, c + 1), \]

where $D$ is the fundamental discriminant corresponding to $(-1)^ck$ and $A \in \mathbb{Q}^*$. Thus, the only new value is that of $A$. However, for the reader’s convenience, we give the list explicitly, the numbering corresponding to that of the initial series (which is different for $c = 1$ and $c \geq 2$).
\[ \sum_{n \geq 1} \frac{Q(n)}{n^{2c+1} H_n b^n} = A \cdot L(D, c + 1). \]

| # | c  | H_N | b = 1/a | Q            | D  | A       |
|---|----|-----|---------|--------------|----|---------|
| 37| 1  | 2·4 | -2³3⁻² | 5x - 1       | -3 | -45/2   |
| 38| 1  | 2·4 | 2⁸3⁻⁴  | 35x - 8      | 1  | 72      |
| 39| 1  | 2·3 | 2⁻¹·3³  | 10x - 3      | 1  | 3       |
| 40| 1  | 2·3 | 2⁻⁴·3³  | 11x - 3      | 1  | 48      |
| 41| 1  | 2·3 | -2²    | 15x - 4      | -3 | -27     |
| 42| 1  | 2³   | -2³    | 3x - 1       | -4 | -2      |
| 43| 1  | 2³   | 2⁶     | 21x - 8      | 1  | 1       |
| 44| 1  | 2³   | 2²     | 3x - 1       | 1  | 3       |
| 33| 1  | 2³   | -1     | 4x - 1       | -4 | -16     |
| 14| 2  | 2⁵   | -2⁴    | 10x² - 6x + 1| 1  | -28     |
| 15| 2  | 2⁵   | -2¹⁰  | 205x² - 160x + 32 | 1  | -2     |
| 16| 2  | 2³   | -3³    | 28x² - 18x + 3 | 1  | -14     |
| 17| 2  | 2·5  | -2⁻³⁵  | 483x² - 245x + 30 | 1  | -896   |
| 18| 2  | 2·3·4 | -2⁻⁴·3³ | 172x² - 75x + 9 | 1  | -1792  |
| 19| 3  | 2⁷   | 2⁶     | 21x³ - 22x² + 8x - 1 | 1  | 45/4   |
| 20| 3  | 2³·3 | 2⁻²·3³ | 92x³ - 84x² + 27x - 3 | 1  | 720    |
| 21| 4  | 2⁵   | -2⁻¹⁰·⁵ | 5532x⁴ - 5600x³ + 2275x² - 425x + 30 | 1  | -380928 |

Upside-down series

In particular, note that the last formula gives a series for \( \zeta(5) \).
We finish by giving an example of a nonrational \(1/\pi\) formula, but there exist almost a hundred involving only quadratic irrationals, listed in [1]:

\[
\sum_{n \geq 0} \frac{R_n(2)^3}{(2 + \sqrt{3})^{4n}}(12n + (3 - \sqrt{3})) = \frac{(2 + \sqrt{3})(4/3)^{1/4}}{\pi}.
\]

8 Generalization IV: Supercongruences

It has been noted long ago by several authors that to every \(1/\pi^c\) formula (including divergent ones) corresponds a congruence modulo \(p\) to a higher power than could be expected, what is now called a supercongruence.

The main observation is as follows: if there exists a \(1/\pi^c\)-formula of the form

\[
\sum_{n \geq 0} P(n)H_n/a^n = \sqrt{k}/\pi^c,
\]

then for all primes \(p\) such that \(v_p(a) = v_p(k) = 0\) and not dividing any \(d\) occurring in \(H\) we should have the following precise supercongruence:

\[
\sum_{n=0}^{p-1} P(n)H_n/a^n \equiv P(0)\left(\frac{(-1)^c 4k}{p}\right)p^c \pmod{p^{2c+1}}.
\]

For instance, we have the following supercongruences:

\[
\sum_{n=0}^{p-1} (154n + 15)\frac{R_2(n)R_6(n)}{(-8/3)^{3n}} \equiv 15\left(\frac{-8}{p}\right)p \pmod{p^3}
\]

\[
\sum_{n=0}^{p-1} (5418n^2 + 693n + 29)\frac{R_2(n)R_4(n)R_6(n)}{(-80)^{3n}} \equiv 29\left(\frac{20}{p}\right)p^2 \pmod{p^5}
\]

The same phenomenon is valid for the divergent series for \(1/\pi^c\) that we have given. For instance we have

\[
\sum_{n=0}^{p-1} (35n + 8)\frac{R_2(n)R_4(n)}{(3/4)^{4n}} \equiv 8p \pmod{p^3}
\]

However it has been noticed by several authors that these supercongruences can be refined to a higher power of \(p\) (more precisely to a congruence modulo \(p^{2c+2}\) instead of \(p^{2c+1}\)): the recipe, made precise by the second author, is simply to replace the \(L(D, c + 1)\) occurring in the coefficient of \(x^{2c+1}\) of the Taylor expansions by \(L(D, c + 2 - p)\) times a suitable rational number. In other words,

\[
S_p(H, a, P) := \sum_{n=0}^{p-1} P(n)\frac{H_n}{a^n} \equiv P(0)\left(\frac{(-1)^c 4k}{p}\right)p^c + C_{2c+1}p^{2c+1}L(D, c + 2 - p) \pmod{p^{2c+2}}.
\]
The coefficients $C_{2c+1}$ have been given for all the convergent series in the above tables. The remaining coefficients for the divergent series are as follows:

For the divergent $1/\pi$ formulas,

$$C_3 = (15/4, 0, 0, 0, 12, 2, 0, 0)$$

for formulas (37) to (44).

For the divergent $1/\pi^c$ formulas for $c \geq 2$,

$$C_{2c+1} = (-7/2, -64, -21/2, -210, -63, 0, 0, -1395)$$

for formulas (14) to (21).

The coefficients 0 of course mean that the congruence is valid modulo $p^{2c+2}$ with no correction terms.

We can observe that $C_3$ is almost always divisible by 5, $C_5$ is almost always divisible by 7, and $C_9$ is always divisible by $279 = 3^2 \cdot 31$. We have no explanation for this phenomenon.

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