Alternating quotients of right-angled Coxeter groups

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Abstract

Let $W$ be a right-angled Coxeter group corresponding to a finite non-discrete graph $G$. Our main theorem says that $G$ is connected if and only if for any infinite index quasiconvex subgroup $H$ of $W$ and any finite subset $\{\gamma_1, \ldots, \gamma_n\} \subset W \setminus H$ there is a surjection $f$ from $W$ to a finite alternating group such that $f(\gamma_i) \notin f(H)$. A corollary is that a right-angled Artin group splits as a direct product of cyclic groups and groups with many alternating quotients in the above sense.

Similarly, finitely generated subgroups of closed, orientable, hyperbolic surface groups can be separated from finitely many elements in an alternating quotient, answering positively the conjecture of Wilton [Wil12].

1 Introduction

It is often fruitful to study an infinite discrete group via its finite quotients. For this reason, conditions that guarantee many finite quotients can be useful.

One such notion is residual finiteness. A group $G$ is said to be residually finite if for every $g \in G \setminus \{e\}$, there exists a homomorphism $f : G \to F$, where $F$ is a finite group and $f(g) \neq e$.

We could try to strengthen this notion by requiring that any finite set of non-trivial elements is not killed by some map to a finite group. But these two notions are equivalent as we could simply take product of maps, which don’t kill the individual elements.

Another way to modify this is to require that $\gamma$ avoids image of a specified subgroup $H < G$, which does not contain $\gamma$. If this is true for all finitely generated subgroups $H$, we say that $G$ is subgroup separable.

Finitely generated free groups and free abelian groups are subgroup separable. The quotient of a free abelian group is always an abelian group, but in the case of free groups the finite group $F$ could be a priori anything. Wilton proved that we can in fact require $f$ to be a surjection onto a finite alternating group, thus giving us some control over the maps which ‘witness’ subgroup separability [Wil12].

Scott showed that closed, orientable, hyperbolic surface groups are subgroup separable [Sc07].

Extending and combining methods from both papers, our main theorem shows that even in the case of hyperbolic surface groups, we can require the image to be a finite alternating group.
Definition 1.1. Let $H$ be a subgroup of a finitely generated group $G$, let $C$ be a class of groups. We say that $H$ is $C$-separable if for any choice of $\{\gamma_1, \ldots, \gamma_m\} \subset G \setminus H$ there is a surjection $f$ from $G$ to a group in $C$ such that $f(\gamma_i) \notin f(H)$ for all $i$.

Note the difference between this terminology and the one above. We talk about subgroups as $C$-separable in contrast with subgroup separability, which is a property of the entire group.

We will usually take $C$ to be the class of alternating groups or symmetric groups. We will denote these classes by $A$ and $S$, respectively.

In this case, there is a difference between taking a single $\gamma_1$ and multiple group elements as a product of maps surjecting alternating groups is not a map onto an alternating group. In particular, if $G = A_n \times A_m$ then any $\gamma \in G \setminus \{e\}$ does not map to $e$ under at least one of the projections onto factors. However, if we take $\gamma_1, \ldots, \gamma_k$ to be an enumeration of $G \setminus \{e\}$, the unique map is the identity on $G$ and its image is not an alternating group.

The following is our main result.

**Theorem A** (Main Theorem). Let $\mathcal{G}$ be a non-discrete finite simplicial graph of size at least 3. Every infinite index quasiconvex subgroup of a right-angled Coxeter group $W$ associated to $\mathcal{G}$ is $A$-separable (and $S$-separable) if and only if $\mathcal{G}^c$ is connected.

If $\mathcal{G}$ was a discrete graph, there would be difficulties in controlling a permutation parity of the images of generators. It is possible that this can be resolved.

We require infinite index as otherwise the finite quotient by the normal subgroup contained in $H$ could potentially have no alternating quotients.

Quasiconvexity is required as not all finitely generated subgroups of RACG are $C$-separable, where $C$ is the set of finite groups [HW08, Example 10.3].

**Corollary B.** Every finitely generated right-angled Artin group is a direct product of cyclic group and groups whose infinite index quasiconvex subgroups are $A$-separable.

**Corollary C.** Infinite index quasiconvex subgroups of closed, orientable, hyperbolic surface groups are $A$-separable.

## 2 Preliminaries

### 2.1 $A$-separability

We will establish some properties of $A$-separability.

**Lemma 2.1.** Let $A$ and $B$ be finitely generated groups. Then $\{e\} < A \times B$ is not $A$-separable, unless $A = 1$ and $B = A_n$ or vice versa.

**Proof.** There are only finitely many surjections from $A \times B$ onto $A_2, A_3$ and $A_4$. If $A \times B$ is infinite, then there is a non-identity element $g$ in the kernel of all these maps. Consider elements $g, (e, b), (a, e)$, where $a \neq e, b \neq e$. Suppose $f : A \times B \to A_n$ is a surjection, which does not map these elements to $e$.

By the choice of $g$, $n > 4$. The group $f(A \times e)$ is a normal subgroup of $A_n$, so it is $e$ or $A_n$. Similarly for $e \times B$. However $A_n$ is not commutative, so one of $A \times e, e \times B$ is mapped to $e$. 

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If both $A$ and $B$ are finite and $\{e\} < A \times B$ is $\mathcal{A}$-separable, enumerate $A \times B$ as $\gamma_1, \ldots, \gamma_m$. Applying the $\mathcal{A}$-separability condition with respect to this set, we get an isomorphism $f : A \times B \to A_n$. However, $A_n$ is not a direct product, so one of $A, B$ is $A_n$ and the other is trivial.

This implies that passing to a finite degree extension does not in general preserve $\mathcal{A}$-separability of quasiconvex subgroups. However passing to a smaller group does:

**Lemma 2.2.** Let $G$ be a finitely generated group, let $H$ be a finite-index subgroup of $G$, and let $K$ be an infinite index subgroup of $H$. If $K$ is $\mathcal{A}$-separable in $G$, then it is $\mathcal{A}$-separable in $H$.

We need $K$ to be infinite index in $H$, as otherwise it’s possible that $K = N(H)$ in the notation of the proof below. E.g. take $G = A_n$, $H$ a proper subgroup, $K = \{1\}$.

**Proof.** Suppose $\gamma_1, \ldots, \gamma_n \in H \setminus K$.

Let $N(H) = \bigcap_{g \in G} H^g$ be a normal subgroup contained in $H$. Then $N(H)$ is still finite index and let $M = [G : N(H)]$ be this index. Since $G$ is finitely generated, there are only finitely many surjections $f : G \to A_m$ with $m \leq M$. The intersection of preimages of $f(K)$ over such surjections is a finite intersection of finite index subgroups, hence a finite index subgroup. So there exists some $\gamma_0 \in G \setminus K$ such that $f(\gamma_0) \in f(K)$ for all $f : G \to A_m$ with $m \leq M$.

As $K$ is $\mathcal{A}$-separable in $G$, there exists a surjection $f : G \to A_m$, such that $f(\gamma_i) \notin f(K)$ for all $i \in \{0, \ldots, n\}$. By the choice of $\gamma_0$ we have $m > M$. But $[A_m : f(N(H))] \leq M$, so $f(N(H)) = A_m$. In particular, $f(H) = A_m$ and $f|_H$ is the desired surjection. \hfill \Box

### 2.2 Cube complexes

For further details of the definitions from this section, the reader is referred to [HW08].

**Definition 2.3** (Cube, face). An $n$-dimensional cube $C$ is $I^n$, where $I = [-1, 1]$. A face of a cube is a subset $F = \{x \in C : x_i = (-1)\epsilon\}$, where $1 \leq i \leq n$, $\epsilon = 0, 1$.

**Definition 2.4** (Cube complex). Suppose $C$ is a set of cubes and $\mathcal{F}$ is a set of maps between these cubes, each of which is an inclusion of a face. Suppose that every face of a cube in $C$ is an image of at most one inclusion of a face $f \in \mathcal{F}$. Then the cube complex $X$ associated to $(C, \mathcal{F})$ is

$$X = (\bigsqcup_{C \in C} C) / \sim$$

where $\sim$ is the smallest equivalence relation containing $x \sim f(x)$ for every $f \in \mathcal{F}$; $x \in \text{Dom}(f)$.

**Definition 2.5** (Midcube). A midcube $M$ of a cube $I^n$ is a set of the form $\{x \in C : x_i = 0\}$ for some $1 \leq i \leq n$.

If $f : C \to C'$ is an inclusion of a face and $M$ is a midcube of $C$, then $f(M)$ is contained in unique midcube $M'$ of $C'$. Moreover $f|_M : M \to M'$ is an inclusion of a face.
Definition 2.6 (Hyperplane). Let $X$ be a cube complex associated to $(\mathcal{C}, \mathcal{F})$. Let $\mathcal{M}$ be the set of midcubes of cubes of $\mathcal{C}$. Let $\mathcal{F'}$ be the set of restrictions of maps in $\mathcal{F}$ to midcubes.

The pair $(\mathcal{M}, \mathcal{F'})$ satisfies that every face is an image of at most one inclusion of a face, so there is an associated cube complex $X'$. Moreover, inclusions of midcubes descend to a map $\varphi : X' \to X$. A hyperplane $H$ is a connected component of $X'$ together with a map $\varphi|_H$.

Hyperplanes are analogous to codimension-1 submanifolds.

Definition 2.7 (Elementary parallelism, wall). Suppose $X$ is a cube complex. Define a relation of elementary parallelism on oriented edges of $X$ by $\overrightarrow{e_1} \sim \overrightarrow{e_2}$ if they form opposite edges of a square. Extend this to the smallest equivalence relation. The wall $W(\overrightarrow{e})$ is the equivalence class containing $\overrightarrow{e}$. Similarly, we can define an elementary parallelism on unoriented edges and an unoriented wall $W(e)$.

We denote by $\overleftarrow{e}$ the edge $\overrightarrow{e}$ with the opposite orientation.

There is a bijective correspondence between unoriented walls and hyperplanes, where $W(e)$ corresponds to $H(e)$, a hyperplane which contains the unique midcube of $e$. We say $H(e)$ is dual to $e$. By abuse of notation, we sometimes identify $H(e)$ with its image.

Definition 2.8 (Special cube complex). A cube complex is special if the following holds.

1. For all edges $\overrightarrow{e} \not\in W(\overrightarrow{e})$. We say the hyperplanes are 2-sided.
2. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$, then $e_1$ and $e_2$ are not consecutive edges in a square. Equivalently, each hyperplane embeds.
3. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$, $\overrightarrow{e_2} \neq \overrightarrow{e_1}$, then the initial point of $\overrightarrow{e_2}$ is not the initial point of $\overrightarrow{e_1}$. We say that no hyperplane directly self-osculates.
4. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$ and $\overrightarrow{f_2} \in W(\overrightarrow{f_1})$ and $e_1$ and $f_1$ form two consecutive edges of a square and $\overrightarrow{e_2}$ and $\overrightarrow{f_2}$ start at the same vertex, then $\overrightarrow{e_2}$ and $\overrightarrow{f_2}$ are two consecutive edges in some square. We say that no two hyperplanes inter-osculate.

Haglund and Wise have shown that $CAT(0)$ cube complexes are special [HW08, Example 3.3.(3)]. In this paper, we will only ever use specialness of these complexes.

Definition 2.9. A hyperplane bounds a convex cubical subcomplex $Y \subset X$ if it is dual to an edge with endpoints $v \in Y$ and $v' \not\in Y$.

Every special cube complex is contained in a nonpositively curved complex with the same 2-skeleton [HW08, Lemma 3.13]. The hyperplane $H(e)$ separates a $CAT(0)$ cube complex $X$ into two connected components.

Definition 2.10 (Half-space, [Hag08]). Let $X \setminus H$ be the union of cubes disjoint from $H$. If $X$ is $CAT(0)$, $X \setminus H$ has two connected components. Call them half-spaces $H^-$ and $H^+$.

Definition 2.11. Define $N(H)$ to be the union of all cubes intersecting $H$. Let $\partial N(H)$ consist of cubes of $N(H)$ that don’t intersect $H$. In the case of a simply connected special cube complex $\partial N(H)$ has two components; call them $\partial N(H)^+$ and $\partial N(H)^-$. 

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Definition 2.12. A subcomplex $Y$ of a cube complex $X$ is \textit{(combinatorially geodesically) convex} if any geodesic in $X^{(1)}$ with endpoints in $Y$ is contained in $Y$.

The components of the boundary of a hyperplane $\partial N(H)^+, \partial N(H)^-$ and half-spaces are combinatorially geodesically convex \cite[Lemma 2.10]{Hag08}. Any intersection of half-spaces is convex \cite[Corollary 2.16]{Hag08} and a convex subcomplex of a CAT(0) cube complex coincides with the intersection of all half-spaces containing it \cite[Proposition 2.17]{Hag08}.

\section{2.3 Right-angled Coxeter and Artin groups}

Definition 2.13 (Right-angled Coxeter group). Given a graph $\mathcal{G}$ with vertex set $I$, let $S = \{s_i : i \in I\}$. The right-angled Coxeter group associated to $\mathcal{G}$ is the group $C(\mathcal{G})$ given by the presentation $\langle S \mid s_i^2 = 1 \text{ for } i \in I, [s_i, s_j] = 1 \text{ for } (i, j) \in E(\mathcal{G}) \rangle$.

The right-angled Coxeter group $C(\mathcal{G})$ acts on the Davis–Moussong Complex $DM(\mathcal{G})$ \cite{HW08}.

Fix a vertex $v_0 \in DM(\mathcal{G})$. There is a bijection between vertices of $DM(\mathcal{G})$ and elements of $C(\mathcal{G})$ given by $gv_0 \leftrightarrow g$. Vertices $gv_0$ and $gsv_0$ are connected by an edge $ge_s$ labelled $s$. Note that $gs_ig^{-1}$ acts on the left on $DM(\mathcal{G})$ as a reflection in $H(ge_{s_i})$.

There is also a right action of $C(\mathcal{G})$ on $DM(\mathcal{G})^0$, where $s_i$ sends $gv_0$ to $gs_ivi$, the vertex to which $g$ is connected by an edge labeled $s_i$. This action does not extend to $DM(\mathcal{G})$.

More generally, if $\Gamma$ is a subgroup of $C(\mathcal{G})$, the action of $C(\mathcal{G})$ on the right cosets of $\Gamma$ can be realized geometrically as an action of $C(\mathcal{G})$ on $\Gamma \backslash DM(\mathcal{G})^0$. This action is given by $(\Gamma hv_0)g = \Gamma hgv_0$. If $\Gamma$ acts on $DM(\mathcal{G})$ co-compactly, this gives a finite permutation action. We will use this to construct maps from $C(\mathcal{G})$ to $S_n$.

**Definition 2.14.** If $G$ acts on a cube complex $X$, then $H < G$ is quasiconvex if there is a non-empty convex subcomplex $Y \subset X$, which is invariant under $H$ and moreover $H$ acts on $Y$ cocompactly. We say, that $H$ acts on $X$ with core $Y$.

If $G$ is hyperbolic, this coincides with the usual notion of quasiconvexity \cite{Hag08}.

**Definition 2.15 (Right-angled Artin group).** The right-angled Artin group associated to a simplicial graph $\mathcal{G}$ is $A(\mathcal{G}) = \langle V(\mathcal{G}) \mid uv = vu \text{ for } \{u, v\} \in E(\mathcal{G}) \rangle$.

RAAGs are closely related to RACGs by the following lemma.

**Lemma 2.16.** \cite{DJ00} Right-angled Artin groups are finite-index subgroups of with right-angled Coxeter groups.

\section{2.4 Jordan’s Theorem}

**Definition 2.17 (Primitive subgroup).** A subgroup $G < S_n$ is called primitive if it acts transitively on \{1, \ldots, n\} and it does not preserve any nontrivial partition.

If $n$ is a prime and $G$ is transitive, then the action is primitive.

Our main tool is the following.

**Theorem 2.18 (Jordan’s Theorem).** \cite[Theorem 3.3D]{DM96} For each $k$ there exists $n$ such that if $G < S_n$ is a primitive subgroup and there exists $\gamma \in G \backslash \{e\}$, which moves less than $k$ elements, then $G = S_n$ or $A_n$. 

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3 The main theorem and its proof

Our main theorem relates combinatorics of $G$ and $A$-separability of $C(G)$.

**Theorem 3.1 (Main Theorem).** Let $G$ be a non-discrete finite simplicial graph of size at least 3. Then infinite-index quasiconvex subgroups of the right-angled Coxeter group associated to $G$ are $A$-separable and $S$-separable if and only if $G^c$ is connected.

Similar result holds for RAAGs.

**Corollary 3.2.** Let $G$ be a finite simplicial graph of size at least 2. Then the infinite index quasiconvex subgroups of the right-angled Artin group associated to $G$ are $A$-separable if and only if $G^c$ is connected.

**Proof.** $\Rightarrow$: If $H, K$ are components of $G^c$, then $A(G) = A(H^c) \times A(K^c)$ so by Lemma 2.1 $A(G)$ is not $A$-separable.

$\Leftarrow$: Let $G^\pm$ be a graph with vertex set $V(G^\pm) = V(G)^+ \cup V(G)^-$, where $v^\pm$ is connected to $w^\pm$ if and only if $v$ is connected to $w$ in $G$.

If $G^c$ is connected, then so is $(G^\pm)^c$. Indeed if $U$ is a proper subspace and a connected component of $(G^\pm)^c$, then $U \cap (G^-)^c$ or $U \cap (G^+)^c$ is proper component $(G^-)^c$ or $(G^+)^c$, respectively, so $G^c$ is not connected.

The Artin group associated to $G$ is a finite index subgroup of the Coxeter group associated to $G^\pm$, where the inclusion is induced by $g_v \mapsto s_{v^+} s_{v^-}$ [DJ00]. A subgroup of $A(G)$ is quasiconvex in $A(G)$ if and only if it is quasiconvex in $C(G^\pm)$.

By theorem 3.1, infinite index quasiconvex subgroups of the Coxeter group are $A$-separable and by lemma 2.2 the same holds for the Artin group.

**Corollary 3.3.** Finitely generated infinite index subgroups of closed, orientable, hyperbolic surface groups are $A$-separable.

**Proof.** By [Sco78] all infinite index finitely generated subgroups of orientable, hyperbolic surface groups are quasiconvex cocompact.

In [Sco78] Scott further proves that orientable, hyperbolic surface groups are finite index subgroups of the right-angled Coxeter group associated to the graph $C_5$. By the main theorem the quasiconvex subgroups of $C(G)$ are $A$-separable in $C(G)$ and by Lemma 2.2 the same holds for finite-index subgroups of $C(G)$.

**Definition 3.4 (Disjoint hyperplanes, bounding hyperplanes, positive half-space).** Let $X$ be a cube complex, $Y$ a convex subcomplex. Let $D(Y)$ be the set of hyperplanes disjoint from $Y$. Let $B(Y)$ the set of hyperplanes bounding $Y$.

If $H \in D(Y)$, denote by $H^+$ the half-space of $X \setminus \setminus H$ containing $Y$.

Recall that any intersection of half-spaces is convex and conversely any convex subcomplex is an intersection of the half-spaces containing it. Hence it is equivalent to specify a convex subcomplex or the half-spaces in which it is contained (or the set of disjoint hyperplanes if there can be no confusion about the choice of half-spaces, e.g. if only one choice gives a non-empty intersection).
Definition 3.5 (Deletion, vertebra). Suppose $G$ acts on a cube complex $X$ with core $Y$. Define deletion as removing a bounding hyperplane $H_0$ and all its $G$-translates from $D(Y)$. That is the result of deletion of $H_0$ is $Y' = \cap_{H \in D(Y) \setminus \{H_0\}} H^+$.

The cube complex $V = H_0^- \cap Y'$ is called a vertebra. See Figure 1.

A vertebra is an intersection of two combinatorially geodesically convex sets, so it also is combinatorially geodesically convex. In particular, it is connected.

Definition 3.6 (Acting without self-intersections). We say $G$ acts without self-intersections on a cube complex $X$, if $N(gH) \cap N(H) \neq \emptyset$ implies $gH = H$ for all hyperplanes $H$ of $X$ and $g \in G$.

Lemma 3.7. Suppose that $G$ acts without self-intersections on a CAT(0) cube complex $X$ with core $Y$. Then the result $Y'$ of deletion of $H_0$ is also a core.

Moreover, if $C$ is a set of orbit representatives for the action of $G$ on the vertices of $Y$ and $D$ is a set of orbit representatives for the action of $G_{H_0}$ on the vertices of the vertebra $V = H_0^- \cap Y'$, then $C' = C \cup D$ is a set of orbit representatives for the action of $G$ on the vertices of $Y'$.

Proof. Recall that CAT(0) implies special.

First note that $D(Y') = D(Y) \setminus G, \{H_0\}$ by definition and $B(Y) \setminus G, \{H_0\} \subset B(Y')$ as a bounding hyperplane $Y$ still bounds $Y'$ unless it is a translate of $H_0$.

The set of half-spaces containing $Y'$ is invariant under $G$, hence $Y'$ is invariant. The subcomplex $Y'$ is an intersection of half-spaces, hence convex. Suppose $v \in Y' \setminus Y$. Let $v_0, v_1 \ldots v_k$ be a combinatorial geodesic from $v$ to $Y$ of shortest length with edges $e_1, \ldots e_k$ and suppose $k > 1$. Let’s $H_i$ be the hyperplane dual to $e_i$. Then as $v_{k-1} \notin Y$, we have $H_k \in$
Suppose \( G \{ H_0 \} \). Since \( G \) acts on \( X \) without self-intersections \( H_{k-1} \notin G \{ H_0 \} \). And \( H_{k-1} \notin \mathcal{D}(Y') \), because \( v_0, v_k \in Y' \) belong to opposite half-spaces as otherwise the combinatorial geodesic between \( v_0 \) and \( v_k \) could be shortened.

Therefore \( H_{k-1} \notin \mathcal{D}(Y) \). It must intersect \( Y \), so it is not entirely contained in \( H_k' \) and it intersects \( H_k \). Because the cube complex is special, \( H_k \) and \( H_{k-1} \) don’t interosculate. In particular, there is a square with two consecutive sides \( e_{k-1} \) and \( e_k \). Let \( e'_j \) be the edge opposite \( e_j \) in this square. We can now construct a shorter path from \( v_0 \) to \( Y \) with edges \( e_1, \ldots, e_{k-2}, e'_k \). Contradiction.

So \( k \leq 1 \) and \( Y' \) lies in a 1-neighbourhood of \( Y \) and therefore the action is cocompact.

There is a unique edge connecting \( v \in Y' \setminus Y \) to \( Y \) as any path of length 2 is a geodesic or is contained in some square. In the first case by convexity of \( Y \), we have \( v \in Y \). In the second, \( H_0 \notin \mathcal{D}(Y) \).

By invariance of \( Y \), the translates of \( V \) don’t intersect \( Y \). Suppose \( v \in Y' \setminus Y \). There is a unique hyperplane in \( G \{ H_0 \} \) dual to an edge \( e_1 \), which connects \( v \) to \( Y \). Say \( g.H_0 \). Then \( v \) belongs to a unique translate of \( V \), namely \( g.V \).

**Corollary 3.8.** If \( K \) is a subgroup of a Coxeter group \( C(\mathcal{G}) \) and it acts on the Davis-Moussong complex with core \( Y \), then deletion produces another core.

**Proof.** The Davis-Moussong complex \( DM(\mathcal{G}) \) is a \( \text{CAT}(0) \) cube complex, hence simply connected special and the action of \( C(\mathcal{G}) \) on it is without self-intersections and therefore the restriction to \( K \) is also without self-intersections.

**Lemma 3.9.** Suppose \( G \) acts on a \( \text{CAT}(0) \) cube complex \( X \) with core \( Y \). If \( Y' \subset X \) is constructed from \( Y \) using a deletion of \( H = H(e) \), then each edge in \( V = H' \cap Y' \) is dual to a hyperplane intersecting \( H \).

**Proof.** Let \( e' \) be an edge in \( V \) and \( H' \) a hyperplane dual to \( e' \). If \( H' \cap H = \emptyset \), \( H' \) is contained entirely in \( H^- \). But then \( H' \) is disjoint from \( Y \). In particular one of the endpoints of \( e' \) is in the opposite half-space of \( X \setminus H' \) than \( Y \).

Since \( Y' \) is the intersection of all half-space containing \( Y \) with the exception of the \( G \)-translates of \( H' \), the hyperplane \( H' \) is \( g.H \) for some \( g \in G \).

The subcomplex \( Y \) is \( G \)-invariant and \( H \) bounds \( Y \), hence \( H' \) bounds \( Y \). This contradicts \( H' \subset H^- \).

**Corollary 3.10.** Suppose \( G < C(\mathcal{G}) \) acts on \( DM(\mathcal{G}) \) with core \( Y \). If \( Y' \subset X \) is constructed from \( Y \) using a deletion of \( H = H(e) \), then each edge in \( V = H' \cap Y' \) has a label which commutes with the label of \( H \).

**Definition 3.11** (Deletion along a path, deletion with labels). Suppose \( Y \) is a subcomplex of \( X \) and \( p = e_1e_2 \ldots e_n \) is a path in \( X \) and then **deletion of hyperplanes along the path** \( p \) is the deletion of \( H(e_1), H(e_2), \ldots H(e_3) \).

If \( v \in X \), and \( s_1, s_2, \ldots, s_n \) is a sequence of edge labels, then the **deletion with labels** \( s_1, s_2, \ldots, s_n \) at \( v \) is a **deletion of hyperplanes along** \( p \), where \( p \) is a path \( e_1, e_2, \ldots e_3 \) starting at \( v \) with \( e_i \) labelled \( s_i \).

Suppose \( Y_n \) was built from \( Y_0 \) using a series of deletion of hyperplanes \( H_1, \ldots H_n \). We call \( T = Y_n \cap H_1^- \) a tail. Moreover, if \( H_j \) corresponds to \( s_{i_j} \) and the vertex in the vertebra is understood, we say that \( Y_n \) was built from \( Y_0 \) with respect to \( i_1, \ldots i_n \).
Lemma 3.12. Suppose $\mathcal{G}^c$ is connected, $|\mathcal{G}| > 1$ and $H$ acts on $DM(\mathcal{G})$ with a core $Y \subset DM(\mathcal{G})$. Then there exists a core $Y'$ which can be obtained from $Y$ by deletion along a path $e_1, e_2 \ldots e_n$ with the vertebra $Y' \cap H(e_n)^\sim$ a single vertex.

Remark 3.13. The hypothesis that $\mathcal{G}^c$ is connected is necessary. Consider the situation when $\mathcal{G}$ is a square. Then $C(\mathcal{G}) = D_\infty \times D_\infty$ and $DM(\mathcal{G})$ is the standard tiling of $\mathbb{R}^2$. Let $H = D_\infty$ be the subgroup generated by two non-commuting generators of $C(\mathcal{G})$. The invariance of the core and cocompactness of the action imply that any core for $H$ is of the form $\mathbb{R} \times (k,l)$ for some $k, l \in \mathbb{Z}$.

Every hyperplane intersecting such a core divides it into two infinite half-spaces.

Proof of Lemma 3.12. Let $v_0 \in \partial Y$. Let $e_1$ be such that $H(e_1) = H_1$ bounds $Y$. Say the label of $e_1$ is $s_1$. Let $Y_1$ be a cube complex obtained from $Y$ by deletion of $H_1$.

Let $S_1$ be the set of generators labelling the edges of vertebra $V_1$. Then by lemma 3.9, $s_1$ commutes with all generators in $S_1$.

Let $v_1$ be the other endpoint of $e_1$. If $e_2 \notin V_1$ is an edge with endpoint $v_1$ not dual to $H_1$, we can define $H_2, Y_2, V_2$ and $S_2$ similarly as before. Generators in $S_2$ commute with $s_2$ and moreover if $s_1$ and $s_2$ don’t commute, then $S_2$ is a (not necessarily proper) subset of $S_1$.

Indeed, if $s_1$ and $s_2$ are not connected in $\mathcal{G}$, then the hyperplanes $H_1$ and $H_2$ don’t intersect, so $N(H_2) \subset H_1^\sim$. There is an inclusion of $V_2$ into $V_1$ given by sending a vertex of $V_2$ to the unique vertex of $V_1$ to which it is connected by an edge labelled $s_2$. Extending this map to edges and cubes is a label preserving map between cube complexes $V_2$ and $V_1$.

We will now show that, by a series of such operations, we can reach a situation where $S_n = \emptyset$. I.e. the vertebra $V_n$ is a single vertex.

Suppose we have already applied deletion $i$ times and $S_i$ is non-empty. We will use a series of deletions to get $S_{k+1} \subset S_k \subset S_{k-1} \subset \ldots \subset S_{i+1} \subset S_i$.

Since the group does not split as a product, there exists some $a \in S_i$ and $b \notin S_i$ which don’t commute. Since $\mathcal{G}^c$ is connected, there exists a vertex path $s_{i-1}, \ldots s_k = b$ in $\mathcal{G}^c$. Successive generators in this path don’t commute. Apply deletion of hyperplanes labeled $s_i, \ldots, s_k$ starting at some vertex of $v \in V_{i-1}$. Note that the $j$th hyperplane we remove belongs is a subset of $B(Y_{j-1})$ as $s_i \ldots s_{j-1}v \in V_{j-1}$ and $s_j$ does not commute with $s_{j-1}$.

Moreover, $S_j = \{ s \in S_{j-1} : ss_j = s_j s \}$. In particular, $S_{k+1} \subset S_i$ and $a$ does not belong to $S_{k+1}$ as $as_k \neq s_k a$.

Therefore $S_{k+1}$ is a proper subset of $S_i$ and we can continue this process until we get an empty $S_n$. \qed

Remark 3.14. We can even control the label of the hyperplane which was removed last. Indeed, if the last removed hyperplane had label $s_i$, and $b$ is some other generator, pick a vertex path between $s_i$ and $b$ in $\mathcal{G}^c$. Then remove hyperplanes labelled by vertices on this path, starting at the unique vertex of a vertebra.

By lemma 3.7, there is a set of orbit representatives $K$ for the action of $G$ on $Y_n$ with $T \subset K$.

Haglund shows the following [Hag08, Proof of Theorem A].

Lemma 3.15. Suppose $G < C(\mathcal{G})$ acts on $DM(\mathcal{G})$ with a core $Y$ with a set of orbit representatives $K$. Let $\Gamma_0 < C(\mathcal{G})$ be generated by the reflections in the hyperplanes bounding $Y$. 

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Let $\Gamma_1 = \Gamma_1(Y) = \langle G, \Gamma_0 \rangle$. Then $Y$ is a fundamental domain for the action of $\Gamma_0$ on $X$ and $K$ is a a set of orbit representatives for the action of $\Gamma_1$ on $X$.

Let $C(\mathcal{G})$ act on the right cosets of $\Gamma_1 < C(\mathcal{G})$. We have that $s \in S$ sends $\Gamma_1g$ to $\Gamma_1gs = (\Gamma_1gs^{-1})g$. But $gs^{-1}$ is a reflection in the hyperplane $H(g(e_s))$. By definition of $\Gamma_0$ if $H(g(e_s))$ bounds $Y$, $gs^{-1} \in \Gamma_0$ and $\Gamma_1g$ is fixed by $s$.

Moreover, if $K = \{g_1v_0, \ldots, g_nv_0\}$, then $\{g_0, \ldots g_n\}$ is a set of right coset representatives for $\Gamma_1$.

We will first prove that by a suitable sequence of deletions, we can satisfy the conditions of Jordan’s theorem. It follows that we can construct quotients that are either alternating or symmetric.

**Definition 3.16.** If $Y$ is a subset of $X$, then $N_1(Y)$ is union of closed cubes, which have non-empty intersection with $Y$. We define inductively $N_r(Y) = N_1(N_{r-1}(Y))$.

If $Y$ is convex, then so is $N_r(Y)$ (as a neighbourhood is obtained by removing bounding hyperplanes and therefore it is an intersection of convex subcomplexes). And if $H$ acts cocompactly on $Y$, it still acts cocompactly on $N_r(Y)$.

**Proposition 3.17.** Let $C(\mathcal{G})$ be the right-angled Coxeter group associated to $\mathcal{G}$ a finite simplicial graph, $|\mathcal{G}| > 2$, and $H$ acts with a proper core $Y$. Let $\mathcal{C}$ be the class of symmetric and alternating groups. If $\mathcal{G}^c$ is connected $H$ is $\mathcal{C}$-separable.

**Proof.** As $H$ acts with a proper core, there exists a generator of $C(\mathcal{G})$ not contained in $H$. Say $s_0 \notin H$.

Suppose $\gamma_1, \ldots, \gamma_n \notin H$. Fix $v \in Y$. Without loss of generality, we may assume that $Y$ contains $N(v)$ and $\gamma_i v$ for all $i$ (otherwise replace $Y$ with $N_r(Y)$ for a sufficiently large $r$). Moreover, by lemma 3.12 we may assume that there exists a hyperplane $H_0 \notin \mathcal{D}(Y)$ with $|H_0^- \cap Y| = 1$ and by the remark after the proof the label of $H_0$ is $s_0$.

As $\mathcal{G}^c$ is connected, there exists a generator $s_1$ not commuting with $s_0$. Let $v_0$ be the unique vertex of $H_0^+ \cap Y$. Delete $k$ hyperplanes labelled alternately by $s_1$ and $s_0$ starting at $v_0$—delete hyperplanes $H(e_{e_1}), H(s_1e_{e_2}), H(s_1s_2e_{e_3}), \ldots, H(s_1 \ldots s_{k-1} e_{e_k})$ etc. where $k$ is to be specified later and $e_{e_1}$ is the edge labelled $s_1$ starting at $v_0$. Call the resulting core $Y'$.

Let $\Gamma_0$ be the group generated by reflections in hyperplanes bounding $Y'$. Let $\Gamma_1 = \langle \Gamma_0, H \rangle$. Then $[C(\mathcal{G}) : \Gamma_1] = |H \setminus Y'|$, where $|H \setminus Y'|$ denotes the number of vertices of $H \setminus Y'$. A suitable choice of $k$ makes this a prime. As $\Gamma_1 \setminus C(\mathcal{G}) \cong H \setminus Y'$ and $\gamma_i v \notin H.v$, we may choose $\gamma_i$ as one of the coset representatives. In particular, $\gamma_i$ does not fix $\Gamma_1$, so it does not act as an element of $H$.

Let $s_3$ be a generator distinct from $s_1$ and $s_2$.

By the remark after lemma 3.15 $s_3$ fixes the cosets corresponding to the vertices of the tail. So it moves at most $|H \setminus Y'|$ elements. By taking $k$ large enough so that $|H \setminus Y'|$ is still a prime, we may ensure that the conditions of Jordan’s lemma are satisfied.

4 Changing parity

We shall now prove that we may force the action to be alternating (similarly we can force it to be symmetric).
Definition 4.1. The parity of $s_i$ with respect to the core $Y$ is the parity of $s_i$ acting on the right cosets of $\Gamma_1(Y)$.

We will modify the construction of the tail in order to make each $s_i$ act as an even permutation (or we will make at least one of $s_i$ acts as an odd permutation).

Suppose $g.v_0$ is in the tail. If the edge between $g.v_0$ and $gs.v_0$ is in the tail, then $g.v_0$ and $gs.v_0$ map to distinct vertices in $\Gamma_1 \setminus X$, hence $\Gamma_1 g \neq \Gamma_1 gs$.

If $gs.v_0$ is not in the tail, then the hyperplane dual to this edge bounds $Y$ and the reflection in this hyperplane belongs to $\Gamma_1$. Therefore $\Gamma_1 = \Gamma_1 gs g^{-1}$ or equivalently $\Gamma_1 g = \Gamma_1 gs$.

More precisely, suppose $H$ acts with core $Y$ and $Y'$ is the core resulting from deletion of $H_0, \ldots, H_k$, and the label of $H_i$ is $s_i$. Moreover assume $H_0 \cap Y'$ is a single edge.

Then the parity of $s_1$ with respect to $Y'$ is the sum of the parity of $s_1$ with respect to $Y$ and the number of edges labelled $s_1$ in $H_0 \cap Y'$. So we can control parity of $s_1$ by changing the number of edges with label $s_1$ in the tail. Suppose that the conditions of Jordan’s theorem are satisfied with a margin $M$ (i.e. the conditions are satisfied even if $s_3$ moves $|H \setminus Y| + M$ elements). Taking $M = (|\mathcal{G}| - 2)(2d + 1) + 16$, where $d$ is the diameter of $\mathcal{G}^c$ will be sufficient.

First let us show that we can deal with parity of all generators other than $s_1$ and $s_2$.

Lemma 4.2. For any $i \in I \setminus \{1, 2\}$, if the tail of $Y$ is a path with labels $s_1, s_2, \ldots, s_1, s_2, s_1$ of length at least $2d_{\mathcal{G}^c}(v_1, v_i) + 1$ starting at vertex $V$, then there exists a core $Y'$ such that in the associated action the parity of $s_i$ changed and the parities of no $s_j$ changed for $j \in I \setminus \{1, 2, i\}$. Moreover, $|H \setminus Y| = |H \setminus Y'|$ and $Y'$ contains a tail of the same length as $Y$ and the labels of these two paths are the same with the exception of a subpath labeled $s_1, s_2, \ldots, s_1, s_2, s_1$ of length $2d_{\mathcal{G}^c}(v_1, v_i) + 1$.

Proof. Say $v_1 = v_{i_0}, v_{i_1}, \ldots, v_{i_d} = v_i$ is a path in $\mathcal{G}^c$ of the shortest length. Let $Y'$ be a subcomplex built using deletions of hyperplanes $s_{i_0}, s_{i_1}, \ldots, s_{id}, s_{id-1}, \ldots, s_{i_0}, s_2, s_1, \ldots, s_1$ starting at $v$.

Compared to $Y$, the tail of this complex contains two more edges labeled $s_j$, for $0 < j < d$. It also contains an extra edge labeled $s_{id} = s_i$, so the parity of $s_i$ changed and the parity of other generators $s_j$ remains the same for $j \neq 1, 2, i$. \qed

Figure 2: Sketch of the situation in lemma 4.2, where $\Gamma$ is a cycle of length 5 and $i = 5$. Here we've drawn the hyperplanes. The cube complex would be the dual picture. The lower five squares are the old tail and the upper four squares form the end of the new tail.
Figure 3: A sketch of the subgraph of $G$ spanned by $v_1, v_2$ and $v_3$, the segment of the old tail and the new square which replaces this segment in the case 1 of the proof of Lemma 4.3.

Now let’s change the parity of a generator that appears in the tail.

**Lemma 4.3.** If the tail of $Y$ contains a path with labels $s_1, s_2, \ldots, s_1, s_2, s_1$ of length at least 7, then there exists a core $Y'$ such that in the associated action only the parity of $s_1$ changed. Moreover, $|H \setminus Y| = |H \setminus Y'|$ and $Y'$ is built from the same complex as $Y$ using a sequence of deletions, whose labels agree with that of $Y$ with the exception of 5 deletions. (We allow a deletion to be replaced by no deletion.)

**Proof.**

1. Suppose there is some $s_3$ commuting with $s_2$, but not with $s_1$. Then instead of the deletion of the hyperplanes $s_2, s_1, s_2$, delete the hyperplanes labelled $s_2, s_3$. This creates a square. Continue building the tail starting from one of the vertices of the square using the deletions of the hyperplanes with the same labels as before. The new tail contains the same number of $s_2$ labels, two more of $s_3$ and one fewer $s_1$. Hence only the parity of $s_1$ changed.

To be precise, we need to take the path labelled $s_1, s_2, s_1$ which is a subpath of a path labelled $s_2, s_1, s_2, s_1, s_2$ in the tail, as otherwise deleting a hyperplane labelled $s_3$ could potentially introduce more than just a side of a square. Similarly for the other cases in this proof.

2. Suppose there is some $s_3$ commuting with $s_1$, but not $s_2$. Then instead of the deletion of the hyperplanes labelled $s_1, s_2, s_1, s_2, s_1$, delete the hyperplanes labelled $s_1, s_3$ and then delete the hyperplanes labelled $s_2$ at two of the vertices of the square. This creates a square with two spurs. Continue building the tail starting from the remaining vertex of the square. The new tail contains the same number of $s_2$ labels, two more of $s_3$ and one fewer $s_1$. Hence only the parity of $s_1$ changed.

3. Suppose there is no generator commuting with exactly one of $s_1, s_2$. As the graph $G$ is non-discrete, there is a generator commuting with both $s_1$ and $s_2$. The component of $G^c$ containing $v_1$ and $v_2$ is not a proper subgraph so there exist $s_3, s_4$ such that $s_3$ commutes with both $s_1$ and $s_2$, and $s_4$ does not commute with any of the $s_1, s_2, s_3$. Now instead of the deletion of the hyperplanes labelled $s_1, s_2, s_1, s_2, s_1$, delete the hyperplanes labelled $s_4, s_1, s_3$. This creates a square with labels $s_1, s_3, s_1, s_3$. Perform deletion with respect to $s_4$ at one of the vertices. Then continue building the tail.

Let $Y'$ be the new subcomplex. By construction $|H \setminus Y| = |H \setminus Y'|$ and the sequences of labels deleted hyperplanes for the two complexes differ at no more than 5 places. \[\square\]
Using lemmas 4.2 and 4.3, we can now modify segments of the tail to make the parity of all elements even. This completes the proof of the main theorem.

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