ONE-STEP SPHERICAL FUNCTIONS OF THE PAIR
(SU(n + 1), U(n))

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Dedicated to our teacher and friend Joe Wolf.

Abstract. The aim of this paper is to determine all irreducible spherical functions of the pair $(G, K) = (SU(n + 1), U(n))$, where the highest weight of their $K$-types $\pi$ are of the form $(m + \ell, \ldots, m + \ell, m, \ldots, m)$. Instead of looking at a spherical function $\Phi$ of type $\pi$ we look at a matrix-valued function $H$ defined on a section of the $K$-orbits in an affine subvariety of $P_n(\mathbb{C})$. The function $H$ diagonalizes, hence it can be identified with a column vector-valued function. The irreducible spherical functions of type $\pi$ turn out to be parameterized by $S = \{(w, r) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq w, 0 \leq r \leq \ell, 0 \leq m + w + r\}$. A key result to characterize the associated function $H_{w,r}$ is the existence of a matrix-valued polynomial function $\Psi$ of degree $\ell$ such that $F_{w,r}(t) = \Psi(t)^{-1}H_{w,r}(t)$ becomes an eigenfunction of a matrix hypergeometric operator with eigenvalue $\lambda(w, r)$, explicitly given. In the last section we assume that $m \geq 0$ and define the matrix polynomial $P_w$ as the $(\ell + 1) \times (\ell + 1)$ matrix whose $r$-row is the polynomial $F_{w,r}$.

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1. Spherical functions

Let $G$ be a locally compact unimodular group and let $K$ be a compact subgroup of $G$. Let $\hat{K}$ denote the set of all equivalence classes of complex finite-dimensional irreducible representations of $K$; for $\delta \in \hat{K}$, let $\xi_\delta$ denote the character of $\delta$, $d(\delta)$ the degree of $\delta$, i.e., the dimension of any representation in the class $\delta$, and $\chi_\delta = d(\delta)\xi_\delta$. We choose the Haar measure $dk$ on $K$ normalized by $\int_K dk = 1$. We shall denote by $V$ a finite-dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\text{End}(V)$ the space of all linear transformations of $V$ into $V$.

A spherical function $\Phi$ on $G$ of type $\delta \in \hat{K}$ is a continuous function on $G$ with values in $\text{End}(V)$ such that

(i) $\Phi(e) = I$ ($I =$ identity transformation).
(ii) $\Phi(xg)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky) dk$, for all $x, y \in G$.

If $\Phi : G \rightarrow \text{End}(V)$ is a spherical function of type $\delta$, then $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$, for all $k, k' \in K$, $g \in G$, and $k \mapsto \Phi(k)$ is a representation of $K$ such that any irreducible subrepresentation belongs to $\delta$. In particular the spherical function $\Phi$ determines its type univocally and let us say that the number of times that $\delta$ occurs in the representation $k \mapsto \Phi(k)$ is called the height of $\Phi$.

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Spherical functions of type \( \delta \) arise in a natural way upon considering representations of \( G \). If \( g \mapsto U(g) \) is a continuous representation of \( G \), say on a finite-dimensional vector space \( E \), then

\[
P(\delta) = \int_K \chi_\delta(k^{-1}) U(k) \, dk
\]

is a projection of \( E \) onto \( P(\delta)E = E(\delta) \); \( E(\delta) \) consists of those vectors in \( E \), the linear span of whose \( K \)-orbit splits into irreducible \( K \)-subrepresentations of type \( \delta \).

The function \( \Phi : G \to \text{End}(E(\delta)) \) defined by

\[
\Phi(g)a = P(\delta)U(g)a, \quad g \in G, \; a \in E(\delta)
\]

is a spherical function of type \( \delta \). In fact, if \( a \in E(\delta) \) we have

\[
\Phi(x)\Phi(y)a = P(\delta)U(x)P(\delta)U(y)a = \int_K \chi_\delta(k^{-1}) P(\delta)U(x)U(k)U(y)a \, dk = \left( \int_K \chi_\delta(k^{-1}) \Phi(xky) \, dk \right) a.
\]

If the representation \( g \mapsto U(g) \) is irreducible, then the associated spherical function \( \Phi \) is also irreducible. Conversely, any irreducible spherical function on a compact group \( G \) arises in this way from a finite-dimensional irreducible representation of \( G \).

If \( G \) is a connected Lie group, it is not difficult to prove that any spherical function \( \Phi : G \to \text{End}(V) \) is differentiable \( (C^\infty) \), and moreover that it is analytic.

Let \( D(G) \) denote the algebra of all left-invariant differential operators on \( G \) and let \( D(G)^K \) denote the subalgebra of all operators in \( D(G) \) that are invariant under all right translations by elements in \( K \).

In the following proposition \((V, \pi)\) will be a finite-dimensional representation of \( K \) such that any irreducible subrepresentation belongs to the same class \( \delta \in \hat{K} \).

**Proposition 1.1.** ([11], [54]) A function \( \Phi : G \to \text{End}(V) \) is a spherical function of type \( \delta \) if and only if

(i) \( \Phi \) is analytic.

(ii) \( \Phi(kgk') = \pi(k)\Phi(g)\pi(k') \), for all \( k, k' \in K \), \( g \in G \), and \( \Phi(e) = I \).

(iii) \[ D\Phi \](\( g \)) = \( \Phi(\( g \)) [D\Phi \](\( e \)) \), for all \( D \in D(G)^K \), \( g \in G \).

The aim of this paper is to determine all irreducible spherical functions of the pair \( (G, K) = (\text{SU}(n+1), \text{SU}(n) \times \text{U}(1)) \), \( n \geq 2 \), whose \( K \) types are of a special kind. This will be done starting from Proposition 1.1.

The irreducible finite-dimensional representations of \( G \) are restrictions of irreducible representations of \( \text{U}(n+1) \), which are parameterized by \((n+1)\)-tuples of integers

\[
m = (m_1, m_2, \ldots, m_{n+1})
\]

such that \( m_1 \geq m_2 \geq \cdots \geq m_{n+1} \).

Different representations of \( \text{U}(n+1) \) can be restricted to the same representation of \( G \). In fact the representations \( m \) and \( p \) of \( \text{U}(n+1) \) restrict to the same representation of \( \text{SU}(n+1) \) if and only if \( m_i = p_i + j \) for all \( i = 1, \ldots, n+1 \) and some \( j \in \mathbb{Z} \).
The closed subgroup $K$ of $G$ is isomorphic to $U(n)$, hence its finite-dimensional irreducible representations are parameterized by the $n$-tuples of integers $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ subject to the conditions $k_1 \geq k_2 \geq \cdots \geq k_n$. We shall say that $\mathbf{k}$ is a one-step representation of $K$ if it is of the following form

$$\mathbf{k} = (m + \ell, \ldots, m + \ell, m, \ldots, m)$$

for $1 \leq k \leq n - 1$.

Let $\mathbf{k}$ be an irreducible finite-dimensional representation of $U(n)$. Then $\mathbf{k}$ is a subrepresentation of $\mathbf{m}$ if and only if the coefficients $k_i$ satisfy the interlacing property

$$m_i \geq k_i \geq m_{i+1}, \quad \text{for all} \quad i = 1, \ldots, n.$$ 

Moreover if $\mathbf{k}$ is a subrepresentation of $\mathbf{m}$ it appears only once. (See [VK]). Therefore the height of any irreducible spherical function $\Phi$ of $(G, K)$ is one. This is equivalent to the commutativity of the algebra $D(G)^K$. (See [GV, TI]).

The representation space $V_\mathbf{k}$ of $\mathbf{k}$ is a subspace of the representation space $V_\mathbf{m}$ of $\mathbf{m}$ and it is also $K$-stable. In fact, if $A \in U(n)$, $a = (\det A)^{-1}$ and $v \in V_\mathbf{k}$ we have

$$\left(\begin{array}{cc} A & 0 \\ 0 & a \end{array}\right) \cdot v = a \left(\begin{array}{cc} a^{-1}A & 0 \\ 0 & 1 \end{array}\right) \cdot v = a^{s_m-s_k} \left(\begin{array}{cc} A & 0 \\ 0 & 1 \end{array}\right) \cdot v,$$

where $s_m = m_1 + \cdots + m_{n+1}$ and $s_k = k_1 + \cdots + k_n$. This means that the representation of $K$ on $V_\mathbf{k}$ obtained from $\mathbf{m}$ by restriction is parameterized by

$$(1) \quad (k_1 + s_k - s_m, \ldots, k_n + s_k - s_m).$$

Let $\Phi^{\mathbf{m}, \mathbf{k}}$ be the spherical function of $(G, K)$ associated to the representation $\mathbf{m}$ of $G$ and to the subrepresentation $\mathbf{k}$ of $U(n)$. Then [TI] says that the $K$-type of $\Phi^{\mathbf{m}, \mathbf{k}}$ is $\mathbf{k} + (s_k - s_m)(1, \ldots, 1)$.

**Proposition 1.2.** The spherical functions $\Phi^{\mathbf{m}, \mathbf{k}}$ and $\Phi^{\mathbf{m}', \mathbf{k}'}$ of the pair $(G, K)$ are equivalent if and only if $\mathbf{m}' = \mathbf{m} + j(1, \ldots, 1)$ and $\mathbf{k}' = \mathbf{k} + j(1, \ldots, 1)$.

**Proof.** The spherical functions $\Phi^{\mathbf{m}, \mathbf{k}}$ and $\Phi^{\mathbf{m}', \mathbf{k}'}$ are equivalent if and only if $\mathbf{m}$ and $\mathbf{m}'$ are equivalent and the $K$-types of both spherical functions are the same, see the discussion in p. 85 of [TI]. We know that $\mathbf{m} \simeq \mathbf{m}'$ if and only if

$$\mathbf{m}' = \mathbf{m} + j(1, \ldots, 1) \quad \text{for some} \ j \in \mathbb{Z}.$$ 

Besides, the $K$ types are the same if and only if

$$k_i + s_k - s_m = k_i' + s_k' - s_m' \quad \text{for all} \ i = 1, \ldots, n.$$ 

Therefore $\mathbf{k}' = \mathbf{k} + p(1, \ldots, 1)$, and now it is easy to see that $p = j$. \hfill \Box

**Remark 1.1.** Given a spherical function $\Phi^{\mathbf{m}, \mathbf{k}}$ we can assume that $s_k - s_m = 0$. In such a case the $K$-type of $\Phi^{\mathbf{m}, \mathbf{k}}$ is $\mathbf{k}$, see [TI].

Our Lie group $G$ has the following polar decomposition $G = KAK$, where the abelian subgroup $A$ of $G$ consists of all matrices of the form

$$a(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & I_{n-1} & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$
(Here $I_{n-1}$ denotes the identity matrix of size $n-1$). Since an irreducible spherical function $\Phi$ of $G$ of type $\delta$ satisfies $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$ for all $k, k' \in K$ and $g \in G$, and $\Phi(k)$ is an irreducible representation of $K$ in the class $\delta$, it follows that $\Phi$ is determined by its restriction to $A$ and its $K$-type. Hence, from now on, we shall consider its restriction to $A$.

Let $M$ be the centralizer of $A$ in $K$. Then $M$ consists of all elements of the form

$$m = \begin{pmatrix} e^{ir} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & e^{ir} \end{pmatrix}, \quad r \in \mathbb{R}, B \in U(n-1), \det B = e^{-2ir}.$$ 

The finite-dimensional irreducible representations of $U(n-1)$ are parameterized by the $(n-1)$-tuples of integers

$$t = (t_1, t_2, \ldots, t_{n-1})$$

such that $t_1 \geq t_2 \geq \cdots \geq t_{n-1}$.

The representation of $U(n)$ in $V_k \subset V_m$, $k = (k_1, \ldots, k_n)$ restricted to $U(n-1)$ decomposes as the following direct sum

$$V_k = \bigoplus_{t \in U(n-1)} V_t,$$

where the sum is over all the representations $t = (t_1, \ldots, t_{n-1}) \in U(n-1)$ such that the coefficients of $t$ interlace the coefficients of $k$: $k_i \geq t_i \geq k_{i+1}$, for all $i = 1, \ldots, n-1$.

If $a \in A$, then $\Phi_{m,k}(a)$ commutes with $\Phi_{m,k}(m)$ for all $m \in M$. In fact we have

$$\Phi_{m,k}(a)\Phi_{m,k}(m) = \Phi_{m,k}(am) = \Phi_{m,k}(ma) = \Phi_{m,k}(m)\Phi_{m,k}(a).$$

But $m = e^{ir} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-ir}B & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $e^{ir}I$ is in the center of $U(n+1)$. Hence $V_t$ is an irreducible $M$-module and $\Phi_{m,k}(a)$ also commutes with the action $U(n-1)$. Since $V_t \subset V_k$ appears only once, by Schur's Lemma it follows that $\Phi_{m,k}(a)|_{V_t} = \phi^{m,k}(a)|_{V_t}$, where $\phi^{m,k}(a) \in \mathbb{C}$ for all $a \in A$.

For $g \in G$, let $A(g)$ denote the $n \times n$ left upper corner of $g$, and let

$$A = \{g \in G : A(g) \text{ is nonsingular} \}.$$ 

Notice that $A$ is an open dense subset of $G$ which is left and right invariant under $K$. The set $A$ can also be described as the set of all $g \in G$ such that $g_{n+1,n+1} \neq 0$. This is a consequence of the following lemma.

**Lemma 1.3.** If $U = (u_{ij}) \in SU(n+1)$, we shall denote by $U_{(ij)}$ the $n \times n$ matrix obtained from $U$ by eliminating the $i$th row and the $j$th column. Then

$$\det U_{(ij)} = (-1)^{i+j} u_{ij}.$$ 

**Proof.** The adjoint of a square matrix $U$ is the matrix whose $ij$-element is defined by $(\text{adj } U)_{ij} = (-1)^{i+j} \det U_{(ji)}$, and it is denoted by $\text{adj } U$. Then $U \text{ adj } U = \det U$. But if $U \in SU(n+1)$ we have $U^{-1} = U^*$ and $\det U = 1$, hence

$$(-1)^{i+j} \det U_{(ji)} = u_{ji},$$

which completes the proof of the lemma. □
As in [GPT1] to determine all irreducible spherical functions of $G$ of type $k$ an auxiliary function $\Phi_k : A \rightarrow \text{End}(V_k)$ is introduced. It is defined by

$$\Phi_k(g) = \pi(A(g)),$$

where $\pi$ stands for the unique holomorphic representation of $\text{GL}(n, \mathbb{C)}}$ corresponding to the parameter $k$. It turns out that if $k_n \geq 0$, then $\Phi_k = \Phi^m_k$ where $m = (k_1, \ldots, k_n, 0)$.

Then instead of looking at a general spherical function $\Phi$ of type $k$, we shall look at the function $H(g) = \Phi(g)\Phi_k(g)^{-1}$ which is well defined on $A$.

2. The differential operators $D$ and $E$

The group $G$ acts in a natural way on the complex projective space $\mathbb{P}_n(\mathbb{C})$. This action is transitive and $K$ is the isotropy subgroup of the point $(0, \ldots, 0, 1) \in \mathbb{P}_n(\mathbb{C})$. Therefore

$$\mathbb{P}_n(\mathbb{C}) \simeq G/K.$$  

Moreover the $G$-action on $\mathbb{P}_n(\mathbb{C})$ corresponds to the action induced by left multiplication on $G/K$. We identify the complex space $\mathbb{C}^n$ with the affine space $\mathbb{C}^n = \{ (z_1, \ldots, z_n, 1) \in \mathbb{P}_n(\mathbb{C}) : (z_1, \ldots, z_n) \in \mathbb{C}^n \}$, and we will take full advantage of the $K$-orbit structure of $\mathbb{P}_n(\mathbb{C})$. The affine space $\mathbb{C}^n$ is $K$-stable and the corresponding space at infinity $L = \mathbb{P}_{n-1}(\mathbb{C})$ is a $K$-orbit. Moreover the $K$-orbits in $\mathbb{C}^n$ are the spheres

$$S_r = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 = r^2 \}.$$  

Thus we can take the points $(r, 0, \ldots, 0) \in S_r$ and $(1, 0, \ldots, 0) \in L$ as representatives of $S_r$ and $L$, respectively. Since $(M, 0, \ldots, 0, 1) = (1, 0, \ldots, \frac{1}{M}) \rightarrow (1, 0, \ldots, 0)$ when $M \rightarrow \infty$, the closed interval $[0, \infty]$ parameterizes the set of $K$-orbits in $\mathbb{P}_n(\mathbb{C})$.

Let us consider on $\mathbb{C}^n$ the $2n$-real linear coordinates $(x_1, y_1, \ldots, x_n, y_n)$ defined by: $x_j(z_1, \ldots, z_n) + iy_j(z_1, \ldots, z_n) = z_j$ for all $(z_1, \ldots, z_n) \in \mathbb{C}^n$. We also introduce the following usual notation:

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right).$$  

From now on any $X \in \mathfrak{sl}(n+1, \mathbb{C})$ will be considered as a left-invariant complex vector field on $G$. We will be interested in the following left-invariant differential operators on $G$,

$$\Delta_P = \sum_{1 \leq j \leq n} E_{n+1,j}E_{j,n+1}, \quad \Delta_Q = \sum_{1 \leq i,j \leq n} E_{n+1,i}E_{j,n+1}E_{ij}.  $$  

Lemma 2.1. The differential operators $\Delta_P$ and $\Delta_Q$ are in $D(G)^K$.

Proof. It is clear from the definitions that $\Delta_P$ and $\Delta_Q$ are elements of weight zero with respect to the Cartan subalgebra of $\mathfrak{t}_C$ of all diagonal matrices. Thus to prove that $\Delta_P$ and $\Delta_Q$ are right invariant under $K$ it suffices to prove, respectively, that

$$[E_{r,r+1}, \Delta_P] = 0 \quad \text{and} \quad [E_{r,r+1}, \Delta_Q] = 0$$  

for $1 \leq r \leq n-1$. We have

$$[E_{r,r+1}, \Delta_P] = \sum_{j=1}^{n} [E_{r,r+1}, E_{n+1,j}]E_{j,n+1} + \sum_{j=1}^{n} E_{n+1,j}[E_{r,r+1}, E_{j,n+1}]$$

$$= -E_{n+1,r+1}E_{r,n+1} + E_{n+1,r+1}E_{r,n+1} = 0.$$  

Similarly,

\[
[E_{r,r+1}, \Delta Q] = \sum_{i,j=1}^{n} [E_{r,r+1}, E_{n+1,i}] E_{j,n+1} E_{ij} + \sum_{i,j=1}^{n} E_{n+1,i} [E_{r,r+1}, E_{j,n+1}] E_{ij} 
+ \sum_{i,j=1}^{n} E_{n+1,i} E_{j,n+1} [E_{r,r+1}, E_{ij}] 
= - \sum_{j=1}^{n} E_{n+1,r+1} E_{j,n+1} E_{rj} + \sum_{i=1}^{n} E_{n+1,i} E_{r,n+1} E_{i,r+1} 
+ \sum_{j=1}^{n} E_{n+1,r+1} E_{j,n+1} E_{rj} - \sum_{i=1}^{n} E_{n+1,i} E_{r,n+1} E_{i,r+1} = 0.
\]

2.1. Reduction to \( P_n(C) \).

According to Proposition [4] if \( \Phi = \Phi^{m,k} \) denotes a generic irreducible spherical function of type \( k \), then \( \Delta_P \Phi = \lambda \Phi \) and \( \Delta_Q \Phi = \mu \Phi \) with \( \lambda, \mu \in \mathbb{C} \), because \( [\Delta_P \Phi](e) = \lambda \Phi \) and \( [\Delta_Q \Phi](e) = \mu \Phi \) since \( V_m \) as a \( K \)-module is multiplicity-free.

We introduced the auxiliary function \( \Phi_k : A \rightarrow \text{End}(V_k) \). It is defined by \( \Phi_k(g) = \pi(A(g)) \) where \( \pi \) stands for the unique holomorphic representation of \( \text{GL}(n, \mathbb{C}) \) of highest weight \( k \). Then instead of looking at a general spherical function \( \Phi \) of type \( k \) we look at the function

\[
H(g) = \Phi(g) \Phi_k(g)^{-1}
\]

which is well defined on \( A \). Then \( H \) satisfies

(i) \( H(e) = I \).

(ii) \( H(gk) = H(g) \), for all \( g \in A, k \in K \).

(iii) \( H(kg) = \pi(k) H(g) \pi(k^{-1}) \), for all \( g \in A, k \in K \).

The projection map \( p : G \rightarrow P_n(C) \) defined by \( p(g) = g \cdot (0, \ldots, 0, 1) \), maps the open set \( A \) onto the affine space \( \mathbb{C}^n \). Thus (ii) says that \( H \) may be considered as a function on \( \mathbb{C}^n \).

The fact that \( \Phi \) is an eigenfunction of \( \Delta_P \) and \( \Delta_Q \) makes \( H \) into an eigenfunction of certain differential operators \( D \) and \( E \) on \( \mathbb{C}^n \), to be determined now.

Since the function \( A \) is the restriction of a holomorphic function defined on an open subset of \( \text{GL}(n+1, \mathbb{C}) \), and \( A(g \exp tE_{j,n+1}) = A(g) \) for all \( t \in \mathbb{R} \) and \( 1 \leq j \leq n \), it follows that \( E_{j,n+1}(\Phi_k) = 0 \). Thus

\[
\Delta_P(H\Phi_k) = \sum_{j=1}^{n} ((E_{n+1,j} E_{j,n+1} H) \Phi_k + (E_{j,n+1} H)(E_{n+1,j} \Phi_k)).
\]

Let \( \hat{\pi} \) be the irreducible representation of \( \mathfrak{gl}(n, \mathbb{C}) \) obtained by derivation of \( \pi \) of highest weight \( k \). Since \( \Phi_k(gk) = \Phi_k(g) \pi(k) \) for all \( g \in A, k \in K \) we have \( E_{i,j} \Phi_k = \Phi_k \hat{\pi}(E_{i,j}) \). Therefore

\[
\Delta_Q(H\Phi_k) = \sum_{i,j=1}^{n} ((E_{n+1,i} E_{j,n+1} H) \Phi_k + (E_{j,n+1} H)(E_{n+1,i} \Phi_k)) \hat{\pi}(E_{ij}).
\]

In the open set \( A \subset G \) let us consider the following differential operators. For \( H \in C^\infty(A) \otimes \text{End}(V_k) \) let
\[ D_1 H = \sum_{j=1}^{n} E_{n+1,j} E_{j,n+1} H, \]

\[ D_2 H = \sum_{j=1}^{n} (E_{j,n+1} H)(E_{n+1,j} \Phi_k) \Phi_k^{-1}, \]

\[ E_1 H = \sum_{i,j=1}^{n} (E_{n+1,i} E_{j,n+1} H) \Phi_k \tilde{\pi}(E_{ij}) \Phi_k^{-1}, \]

\[ E_2 H = \sum_{i,j=1}^{n} (E_{j,n+1} H)(E_{n+1,i} \Phi_k) \tilde{\pi}(E_{ij}) \Phi_k^{-1}. \]

We observe that

(5) \( \Delta_P (H \Phi_k) = (D_1(H) + D_2(H)) \Phi_k \) and \( \Delta_Q (H \Phi_k) = (E_1(H) + E_2(H)) \Phi_k; \)

then it is clear that \( \Phi = H \Phi_k \) is an eigenfunction of \( \Delta_P \) and \( \Delta_Q \) if and only if \( H \) is an eigenfunction of \( D = D_1 + D_2 \) and \( E = E_1 + E_2. \)

**Lemma 2.2.** The differential operators \( D_1, D_2, E_1, \) and \( E_2 \) define differential operators \( D_1, D_2, E_1 \) and \( E_2 \) acting on \( C^\infty(\mathbb{C}^n) \otimes \text{End}(V_k). \)

**Proof.** The only thing we really need to prove is that \( D_1, D_2, E_1, \) and \( E_2 \) preserve the subspace \( C^\infty(\mathcal{A})^K \otimes \text{End}(V_k) \) of all right \( K \)-invariant functions.

For \( H \in C^\infty(\mathcal{A}) \otimes \text{End}(V_k) \) and \( k \in K \) we write \( H^{R(k)}(g) = H(gk) \) for all \( g \in \mathcal{A}. \) Then \( (Ad(k)D)H = (DH^{R(k^{-1})})^{R(k)} \) for any \( D \in D(G). \) In particular any \( D \in D(G)^K \) leaves \( C^\infty(\mathcal{A})^K \otimes \text{End}(V_k) \) invariant.

That \( D_1 \) has this property is a consequence of \( D_1 H = \Delta_P H \) and \( \Delta_P \in D(G)^K. \)

On the other hand, from (5) we get

\[ \Delta_P (H \Phi_k) \Phi_k^{-1} = D_1 H + D_2 H. \]

If \( H \in C^\infty(\mathcal{A})^K \otimes \text{End}(V_k), \) it is easy to verify that \( \Delta_P (H \Phi_k) \Phi_k^{-1} \) belongs to \( C^\infty(\mathcal{A})^K \otimes \text{End}(V_k). \) Therefore \( D_2 \) also preserves \( C^\infty(\mathcal{A})^K \otimes \text{End}(V_k). \)

In a similar way from (5) we obtain

\[ \Delta_Q (H \Phi_k) \Phi_k^{-1} = E_1 H + E_2 H. \]

Since \( \Delta_Q \in D(G)^K \) it follows that \( \Delta_Q (H \Phi_k) \Phi_k^{-1} \in C^\infty(\mathcal{A})^K \otimes \text{End}(V_k) \) for all \( H \in C^\infty(\mathcal{A})^K \otimes \text{End}(V_k). \) Hence to finish the proof of the lemma it suffices to prove that \( E_1 \) leaves invariant \( C^\infty(\mathcal{A})^K \otimes \text{End}(V_k). \)

A computation similar to the one done in the proof of Lemma 2.1 shows that the element

\[ \sum_{i,j=1}^{n} E_{n+1,i} E_{j,n+1} \otimes E_{ij} \in D(G) \otimes D(K) \]

is \( K \)-invariant. Now let us consider the unique linear map

\[ L : D(G) \otimes D(K) \rightarrow \text{End} (C^\infty(\mathcal{A})^K \otimes \text{End}(V_k)) \]

such that \( L(E \otimes F)(H) = (EH)\Phi_k \tilde{\pi}(F)\Phi_k^{-1}. \) Then

\[ \Delta_{Q_1} H = L \left( \sum_{i,j=1}^{n} E_{n+1,i} E_{j,n+1} \otimes E_{ij} \right)(H). \]
Thus if \( k \in K \) and \( H \in C^\infty(A)^K \otimes \text{End}(V_k) \), then

\[
\Delta_{Q_1}H = L \left( \sum_{i,j=1}^{n} Ad(k)(E_{n+1,i}E_{j,n+1}) \otimes Ad(k)E_{ij} \right)(H)
\]

\[
= \sum_{i,j=1}^{n} (Ad(k)(E_{n+1,i}E_{j,n+1})H)\Phi_k \pi(k)\hat{\pi}(E_{ij})\pi(k^{-1})\Phi_k^{-1}
\]

\[
= \sum_{i,j=1}^{n} (E_{n+1,i}E_{j,n+1})H)R(k)\Phi_k R(k)\hat{\pi}(E_{ij})\Phi_k^{-1}R(k) = (\Delta_{Q_1}H)^{R(k)}.
\]

This completes the proof of the lemma. \( \square \)

The proofs of Propositions 2.3 and 2.4 require simple but lengthy computations. Complete details will be given in [PT].

Let \( \hat{\pi} \) be the irreducible representation of \( \mathfrak{gl}(n, \mathbb{C}) \) of highest weight \( k \).

**Proposition 2.3.** For \( H \in C^\infty(C^n) \otimes \text{End}(V_k) \) we have

\[
DH = -\frac{1}{4} \left( 1 + \sum_{1 \leq j \leq n} |z_j|^2 \right) \left( \sum_{1 \leq r \leq n} (H_{x,r}x_r + H_{y,r}y_r)(1 + |z_r|^2) \right)
\]

\[
+ \sum_{1 \leq r \neq k \leq n} (H_{x,r}x_k + H_{y,r}y_k) \text{Re}(z_r \overline{z}_k) - 2 \sum_{1 \leq r \neq k \leq n} H_{x,r}y_k \text{Im}(z_r \overline{z}_k)
\]

\[
+ \sum_{1 \leq r \leq n} \frac{\partial H}{\partial z_r} \hat{\pi} \left( \sum_{1 \leq j \leq n} z_{r,ij} (\delta_{rj} + z_r \overline{z}_j)E_{ij} \right),
\]

**Proposition 2.4.** For \( H \in C^\infty(C^n) \otimes \text{End}(V_k) \) we have

\[
EH = -\frac{1}{4} \left( 1 + \sum_{1 \leq j \leq n} |z_j|^2 \right) \left( \sum_{1 \leq r \leq n} (H_{x,r}x_k + H_{y,r}y_k)\hat{\pi} \left( \sum_{1 \leq j \leq n} (\delta_{rj} + z_k \overline{z}_j)E_{rj} \right) \right)
\]

\[
+ i \sum_{1 \leq r \leq n} H_{x,r}y_k \hat{\pi} \left( \sum_{1 \leq j \leq n} (\delta_{rj} + z_r \overline{z}_j)E_{kj} - (\delta_{rj} + z_k \overline{z}_j)E_{rj} \right)
\]

\[
+ \sum_{1 \leq r \neq k \leq n} \frac{\partial H}{\partial z_r} \hat{\pi} \left( \sum_{1 \leq j \leq n} z_{r,kj} (\delta_{rj} + z_r \overline{z}_j)E_{kj} \right),
\]

2.2. Reduction to one variable.

We are interested in considering the differential operators \( D \) and \( E \) applied to a function \( H \in C^\infty(C^n) \otimes \text{End}(V_n) \) such that \( H(kp) = \pi(k)H(p)\pi(k)^{-1} \), for all \( k \in K \) and \( p \) in the affine complex space \( C^n \). This property of \( H \) allows us to find ordinary differential operators \( \bar{D} \) and \( \bar{E} \) defined on the interval \((0, \infty)\) such that

\[
(DH)(r,0,\ldots,0) = (\bar{D}\bar{H})(r), \quad (EH)(r,0,\ldots,0) = (\bar{E}\bar{H})(r),
\]

where \( \bar{H}(r) = H(r,0,\ldots,0) \). Let \( (r,0) = (r,0,\ldots,0) \in C^n \). We also introduce differential operators \( \bar{D}_1, \bar{D}_2, \bar{E}_1 \) and \( \bar{E}_2 \) in the same way, that is \( \bar{D}_1H(r,0) = (\bar{D}_1\bar{H})(r) \), \( \bar{D}_2H(r,0) = (\bar{D}_2\bar{H})(r) \), \( \bar{E}_1H(r,0) = (\bar{E}_1\bar{H})(r) \) and \( \bar{E}_2H(r,0) = (\bar{E}_2\bar{H})(r) \).

In order to give the explicit expressions of \( \bar{D} \) and \( \bar{E} \) we need to compute a number of second order partial derivatives at the points \((r,0)\) of the function \( H : C^n \rightarrow \text{End}(V_n) \). We recall that \([0, \infty]\) parameterizes the set of \( K \)-orbits in \( P_n(C) \).
Given \( z_1, z_j \in \mathbb{C}, 2 \leq j \leq n \), we need a matrix in \( K \) such that carries the point \((z_1, 0, \ldots, 0, z_j, 0, \ldots, 0) \neq 0 \) to the meridian \( \{(r, 0) : r > 0\} \). A good choice is the following \( n \times n \) matrix

\[
A(z_1, z_j) = \frac{1}{s(z_1, z_j)} \left( z_1 E_{11} - \frac{s_j E_{1j} + z_j E_{jj} + \sum_{k \neq 1, j} E_{kk}}{s(z_1, z_j)} \right) \in SU(n),
\]

where \( s(z_1, z_j) = \sqrt{|z_1|^2 + |z_j|^2} \neq 0 \). Then

\[
(z_1, 0, \ldots, 0, z_j, 0, \ldots, 0)^t = A(z_1, z_j)(s(z_1, z_j), 0, \ldots, 0)^t.
\]

The proofs of the following theorems are similar as in the case of the complex projective plane, considered in [GPT1]. Complete details will appear in the forthcoming paper [PT5].

**Theorem 2.5.** For all \( r > 0 \) we have

\[
\hat{D} \hat{H} = -\frac{1}{4} \left( (1 + r^2) \frac{d^2 \hat{H}}{dr^2}(r) + \frac{1 + r^2}{r} \frac{d \hat{H}}{dr}(r) \right) (2n - 1 + r^2 - 2r^2 \hat{\pi}(E_{11}))
\]

\[
+ \frac{4}{r^2} \sum_{2 \leq j \leq n} \left[ \hat{\pi}(E_{1j}), \hat{H}(r) \right] \hat{\pi}(E_{1j}) + \frac{4(1 + r^2)}{r^2} \sum_{2 \leq j \leq n} \left[ \hat{\pi}(E_{1j}), \hat{H}(r) \right] \hat{\pi}(E_{1j})
\]

**Theorem 2.6.** For all \( r > 0 \) we have

\[
\hat{E} \hat{H} = -\frac{1}{4} \left( (1 + r^2) \frac{d^2 \hat{H}}{dr^2}(r) + \frac{1 + r^2}{r} \frac{d \hat{H}}{dr}(r) \right) \pi(E_{11})
\]

\[
+ \frac{2(1 + r^2)}{r} \sum_{j=2}^{n} \hat{\pi}(E_{1j}) \frac{d \hat{H}}{dr}(r) \hat{\pi}(E_{1j}) - \frac{2(1 + r^2)^2}{r} \sum_{j=2}^{n} \hat{\pi}(E_{1j}) \frac{d \hat{H}}{dr}(r) \hat{\pi}(E_{1j})
\]

\[
- \frac{2(1 + r^2)}{r^2} \sum_{j,k=2}^{n} \left( \left[ \hat{\pi}(E_{1k}), \hat{\pi}(E_{1j}), \hat{H} \right] + \left[ \hat{\pi}(E_{1j}), \hat{\pi}(E_{1k}), \hat{H} \right] \right) \hat{\pi}(E_{jk})
\]

\[
- 4 \sum_{k=1}^{n} \sum_{j=2}^{n} \left[ \hat{\pi}(E_{1j}), \hat{H} \right] \hat{\pi}(E_{1k}) \hat{\pi}(E_{jk}) \right).
\]

### 2.3. The operators \( \hat{D} \) and \( \hat{E} \) in matrix form.

From now on we will assume that the \( K \)-types of our spherical functions will be of the one step kind, i.e., for \( 1 \leq k \leq n - 1 \) let

\[
k = \underbrace{(m + \ell, \ldots, m + \ell, m, \ldots, m)}_{k}\underbrace{(n - k)}_{n - k}.
\]

Therefore, the irreducible spherical functions \( \Phi \) of the pair \((G, K)\), whose \( K \)-type is as in (7), are parameterized by the \( n + 1 \)-tuples \( m \) of the form

\[
m = m(w, r) = (w + m + \ell, m + \ell, \ldots, m + \ell, m + r, m, \ldots, m, -w - r),
\]

where \( 0 \leq w, m \geq -w - r \) and \( 0 \leq r \leq \ell \). See Proposition 1.2 and Remark 1.1.

Thus if we assume \( w \geq w_0 = \max\{0, -m\} \) and \( 0 \leq r \leq \ell \) all the conditions are satisfied.
Therefore all weights of Proposition 2.7. $V_k$ of linear operators of the vector space $V$ of linear differential operators in the submodule $V$ of linear transformations with the action of $U(n-1)$. Thus $\hat{H}(r)$ is a scalar transformation on each $U(n-1)$-submodule $V_{t(s)}$. We denote this scalar by $\hat{h}_s(r)$ and we will identify the function $\hat{H}(r)$ associated to $\Phi$ with the column vector function:

$$\hat{H}(r) = (\hat{h}_0(r), \ldots, \hat{h}_t(r))^t.$$ 

The expression of the differential operator $\hat{D}$ in Theorem 2.3 is given in terms of linear operators of the vector space $V_k$. Now we proceed to give $\hat{D}$ as a system of linear differential operators in the $\ell + 1$ unknowns $\hat{h}_0(r), \ldots, \hat{h}_t(r)$.

**Proposition 2.7.** For all $r > 0$ and $0 \leq s \leq \ell$, we have

$$(\hat{D}\hat{H})_{t(s)}(r) = -\frac{(1 + r^2)^2}{4} h''(r) - \frac{(1 + r^2)}{4r} (2n - 1 + r^2 - 2(m + \ell - s)r^2) h'(r)$$

$$- \frac{1}{r^2} s(\ell + k - s)(\hat{h}_{s-1}(r) - \hat{h}_s(r))$$

$$- \frac{(1 + r^2)}{r^2} (\ell - s)(n - k + s)(\hat{h}_{s+1}(r) - \hat{h}_s(r)).$$

**Proof.** To obtain the proposition from Theorem 2.3 we need to compute in each $V_{t(s)}$, $0 \leq s \leq \ell$, the linear transformations

$$\sum_{2 \leq j \leq n} [\hat{\pi}(E_{j1}), \hat{H}(r)] \hat{\pi}(E_{1j}), \quad \sum_{2 \leq j \leq n} [\hat{\pi}(E_{1j}), \hat{H}(r)] \hat{\pi}(E_{j1}).$$

The highest weight of $V_{t(s)}$ is

$$\mu_s = (m + \ell - s)x_1 + (m + \ell)(x_2 + \cdots + x_k) + (m + s)x_{k+1}$$

$$+ m(x_{k+2} + \cdots + x_n).$$

Therefore all weights of $V_{t(s)}$ are of the form $\mu = \mu_s - \sum_{r=2}^{n-1} n_r \alpha_r$ with $\alpha_r = x_r - x_{r+1}$, thus $\mu = (m + \ell - s)x_1 + \cdots$. Now we observe that for the representations $\pi$ considered here, for all $j = 2, \ldots, n$, we have

$$\hat{\pi}(E_{1j})(V_{t(s)}) \subset V_{t(s-1)} \quad \text{and} \quad \hat{\pi}(E_{j1})(V_{t(s)}) \subset V_{t(s+1)}.$$ 

In fact, if $v \in V_{t(s)}$ is a vector of weight $\mu$, then $\hat{\pi}(E_{1j})v$ is a vector of weight $\mu + x_1 - x_j = (m + \ell - (s - 1))x_1 + \cdots$. Similarly $\hat{\pi}(E_{j1})v$ is a vector of weight $\mu + x_j - x_1 = (m + \ell - (s + 1))x_1 + \cdots$. Therefore

$$\sum_{2 \leq j \leq n} [\hat{\pi}(E_{1j}), \hat{H}(r)] \hat{\pi}(E_{1j})v = \sum_{2 \leq j \leq n} (\hat{\pi}(E_{1j})\hat{H}(r) - \hat{\pi}(E_{j1})\hat{\pi}(E_{1j}))(\hat{\pi}(E_{j1}))$$

$$= (\hat{h}_{s-1}(r) - \hat{h}_s(r)) \sum_{2 \leq j \leq n} \hat{\pi}(E_{j1})\hat{\pi}(E_{1j})v$$
and
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \tilde{H}(r) \hat{\pi}(E_{j1}) v = (\tilde{h}_{s+1}(r) - \tilde{h}_s(r)) \sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v.\]

The Casimir element of \(\text{GL}(n, \mathbb{C})\) is
\[\Delta^{(n)} = \sum_{1 \leq i, j \leq n} E_{ij} E_{ji} = \sum_{1 \leq i \leq n} E_{ii}^2 + \sum_{1 \leq i < j \leq n} (E_{ii} - E_{jj}) + 2 \sum_{1 \leq i < j \leq n} E_{ji} E_{ij}.\]

Similarly, the Casimir operator of \(\text{GL}(n-1, \mathbb{C}) \subset \text{GL}(n, \mathbb{C})\) is
\[\Delta^{(n-1)} = \sum_{2 \leq i, j \leq n} E_{ij} E_{ji} = \sum_{2 \leq i \leq n} E_{ii}^2 + \sum_{2 \leq i < j \leq n} (E_{ii} - E_{jj}) + 2 \sum_{2 \leq i < j \leq n} E_{ji} E_{ij}.\]

Hence
\[\sum_{2 \leq j \leq n} E_{1j} E_{j1} = \frac{1}{2} \left( \Delta^{(n)} - \Delta^{(n-1)} - E_{11}^2 - \sum_{2 \leq j \leq n} (E_{1j} - E_{jj}) \right).\]

We also have
\[\sum_{2 \leq j \leq n} E_{1j} E_{j1} = \sum_{2 \leq j \leq n} E_{j1} E_{1j} + \sum_{2 \leq j \leq n} (E_{11} - E_{jj}).\]

To compute the scalar linear transformation \(\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v\) on \(V_{\ell}(s)\), it is enough to apply it to a highest weight vector \(v_s\) of \(V_j(s)\). The highest weight of \(E_k\) is \(\mu_0\) and the weight of \(v_s\) is \(\mu_s\), see (10). Then we have
\[\hat{\pi}(\Delta^{(n)}) v_s = (k(m + \ell)^2 + (n - k)m^2 + k(n - k)\ell) v_s,\]
\[\hat{\pi}(\Delta^{(n-1)}) v_s = ((k - 1)(m + \ell)^2 + (m + s)^2 + (n - k - 1)m^2 + (\ell - s)(k - 1) + s(n - k - 1) + (\ell - k)(n - k - 1)) v_s,\]
\[\hat{\pi}(E_{11}) v = (m + \ell - s) v_s,\]
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{jj}) v = (mn + \ell k - \ell - m + s) v_s.\]

Therefore, by using (12) and (13), we obtain for all \(v \in V_{\ell}(s)\)
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v = \hat{\pi}\left( \sum_{2 \leq j \leq n} E_{j1} E_{1j} \right) v = s(\ell + k - s) v,\]
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v = (\ell - s)(n - k + s) v.\]

Finally we compute, for all \(v \in V_{\ell}(s)\)
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v = (s(\ell + k - s)(\tilde{h}_{s-1}(r) - \tilde{h}_s(r)) v,\]
\[\sum_{2 \leq j \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{j1}) v = (\ell - s)(n - k + s)(\tilde{h}_{s+1}(r) - \tilde{h}_s(r)) v,\]

and now the proposition follows easily from Theorem 2.6 \(\square\)

To obtain a similar result for the operator \(E\) from Theorem 2.6, we need to compute the linear transformations
\[\sum_{2 \leq j, k \leq n} \hat{\pi}(E_{k1}) \hat{\pi}(E_{1j}) \hat{\pi}(E_{jk})\] and \[\sum_{2 \leq j, k \leq n} \hat{\pi}(E_{1j}) \hat{\pi}(E_{k1}) \hat{\pi}(E_{jk})\]
in each $V_{t(s)}$, $0 \leq s \leq \ell$. This is the content of the following lemma.

**Lemma 2.8.** Let us consider the following elements of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{C})$:

$$F = \sum_{2 \leq j, k \leq n} E_{kj}E_{1j}E_{jk}$$

and

$$F' = \sum_{2 \leq j, k \leq n} E_{1j}E_{kj}E_{jk}.$$  

Then they are $U(n-1)$-invariant and for $0 \leq s \leq \ell$, we have

$$\hat{\pi}(F)|_{V_{t(s)}} = -s(\ell + k - s)(k - s - m - n + 1)I_s,$$

$$\hat{\pi}(F')|_{V_{t(s)}} = -(\ell - s)(n - k + s)(k - s - m - 1)I_s,$$

where $I_s$ stands for the identity linear transformation of $V_{t(s)}$.

**Proof.** It is easy to check that the following elements

$$F^{(n)} = \sum_{1 \leq i, j, k \leq n} E_{ki}E_{ij}E_{jk}$$

and

$$F^{(n-1)} = \sum_{2 \leq i, j, k \leq n} E_{ki}E_{ij}E_{jk}$$

are, respectively, in the centers of the universal enveloping algebras of $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n - 1, \mathbb{C})$. We also have

$$F^{(n)} - F^{(n-1)} = \sum_{1 \leq j, k \leq n} E_{kj}E_{1j}E_{1k} + \sum_{2 \leq i, j, k \leq n} E_{ki}E_{1i}E_{1k} + \sum_{2 \leq i, j \leq n} E_{1i}E_{1j}.$$

Moreover

$$\sum_{1 \leq j, k \leq n} E_{kj}E_{1j}E_{1k} = F + \sum_{1 \leq k \leq n} E_{1k}E_{11}E_{1k} + \sum_{2 \leq k \leq n} E_{11}E_{1j}E_{j1}$$

$$= F + E_{11}^3 + (2E_{11} + 1) \sum_{2 \leq k \leq n} E_{11}E_{1k} + E_{11} \sum_{2 \leq k \leq n} (E_{11} - E_{jj}),$$

$$\sum_{2 \leq i \leq n} \sum_{1 \leq k \leq n} E_{ki}E_{1i}E_{1k} = \sum_{2 \leq i, k \leq n} E_{ki}E_{1i}E_{1k} + \sum_{2 \leq i \leq n} E_{1e}E_{1i}E_{11}$$

$$= F + E_{11} \sum_{2 \leq i \leq n} (E_{11} - E_{ii}) + E_{11} \sum_{2 \leq i \leq n} E_{1e}E_{1i},$$

$$\sum_{2 \leq i, j \leq n} E_{1j}E_{ij}E_{11} = (n - 1) \sum_{2 \leq i \leq n} E_{1j}E_{11} + F'$$

$$= (n - 1) \sum_{2 \leq i \leq n} (E_{11} - E_{ii}) + (n - 1) \sum_{2 \leq i \leq n} E_{1e}E_{1i} + F'.$$

Therefore

$$F^{(n)} - F^{(n-1)} = 2F + F' + (3E_{11} + n) \sum_{2 \leq j \leq n} E_{1j}E_{1j}E_{11} + E_{11}(E_{11} + n - 1)^2$$

$$- (2E_{11} + n - 1) \sum_{2 \leq j \leq n} E_{1j}.$$  

(18)

We also observe that

$$F' = \sum_{2 \leq j, k \leq n} E_{1j}E_{1k}E_{jk} = \sum_{2 \leq j, k \leq n} E_{k1}E_{1j}E_{jk} + \sum_{2 \leq j, k \leq n} (\delta_{jk}E_{11} - E_{kj})E_{jk}$$

$$= F - \Delta^{(n-1)} + E_{11} \sum_{2 \leq j \leq n} E_{1j}.  

(19)$$
From (18) and (19) we obtain

\[
3F = F^{(n)} - F^{(n-1)} + \Delta^{(n-1)} - (3E_{11} + n) \sum_{2 \leq j \leq n} E_{ij} E_{kj}
\]
\[
- E_{11}(E_{11} + n - 1)^2 + (E_{11} + n - 1) \sum_{2 \leq j \leq n} E_{jj}.
\]

(20)

To compute \( \hat{\pi}(F) \) on the \( M \)-module \( V_{t(s)} \) it is enough to know \( \hat{\pi}(F)v_s \), since \( F \) is \( M \)-invariant. Thus we only need to determine \( \hat{\pi}(F^{(n)})v_s \) and \( \hat{\pi}(F^{(n-1)})v_s \) because the other terms have been computed before in (13) and (15).

Since \( F^{(n)} \) is \( K \)-invariant it is enough to compute \( \hat{\pi}(F^{(n)})v_0 \). But \( v_0 \) is a highest weight vector of \( \mathfrak{g}(n, \mathbb{C}) \) in \( V_0 \); hence it is enough to compute \( F^{(n)} \) modulo the left ideal \( U(\mathfrak{g}(n, \mathbb{C}))\mathfrak{t}_+ \) of the universal enveloping algebra of \( \mathfrak{g}(n, \mathbb{C}) \) generated by the set \( \{E_{ij} : 1 \leq i < j \leq n \} \). Similarly, since \( v_0 \) is a highest weight vector of \( \mathfrak{g}(n-1, \mathbb{C}) \) in \( V_0 \) it is enough to compute \( F^{(n-1)} \) modulo the left ideal \( U(\mathfrak{g}(n-1, \mathbb{C}))\mathfrak{m}_+ \) of \( U(\mathfrak{g}(n-1, \mathbb{C})) \) generated by the set \( \{E_{ij} : 2 \leq i < j \leq n \} \).

We start with \( F^{(n)} = \sum_{1 \leq i,j,k \leq n} E_{ki}E_{ij}E_{jk} \). To rewrite \( E_{ki}E_{ij}E_{jk} \) we partition the set of indices into the following subsets, and we use the symbol \( \equiv \) to denote congruence modulo the left ideal \( U(\mathfrak{g}(n, \mathbb{C}))\mathfrak{t}_+ \).

\[
\begin{align*}
& j < k : E_{ki}E_{ij}E_{jk} = 0,
& j = k > i : E_{ki}E_{ij}E_{kk} = E_{ki}(E_{ik} + E_{kk}E_{ik}) \equiv 0,
& j = k = i : E_{kk}E_{kk}E_{kk} = E_{kk}^3,
& j = k < i : E_{ki}E_{ij}E_{kk} = (E_{kk} - E_{ii} + E_{ik}E_{ki})E_{kk}
& \quad = (E_{kk} - E_{ii})E_{kk} + E_{ik}(E_{kk}E_{ki} - E_{kk}E_{ki}) \equiv (E_{kk} - E_{ii})E_{kk},
& i > j > k : E_{ki}E_{ij}E_{jk} = (E_{kj} + E_{ij}E_{kj})E_{jk} = E_{kk} - E_{jj} + E_{jk}E_{kj}
& \quad + E_{ij}(-E_{ji} + E_{jk}E_{ki}) \equiv E_{kk} - E_{jj},
& i = j > k : E_{ki}E_{jj}E_{jk} = (E_{kj} + E_{jj}E_{kj})E_{jk} = E_{kk} - E_{jj} + E_{jk}E_{kj}
& \quad + E_{jj}(E_{kk} - E_{jj} + E_{jk}E_{kj}) \equiv (E_{kk} - E_{jj})(E_{jj} + 1),
& j > i > k : E_{ki}E_{ij}E_{jk} = E_{ki}(E_{ik} + E_{jk}E_{kj}) \equiv E_{kk} - E_{ii},
& j > i = k : E_{kk}E_{kj}E_{jk} = E_{kk}(E_{kk} - E_{jj} + E_{jk}E_{kj}) \equiv E_{kk}(E_{kk} - E_{jj}),
& j > k > i : E_{ki}E_{ij}E_{jk} = E_{ki}(E_{ik} + E_{jk}E_{ij}) \equiv 0.
\end{align*}
\]

Therefore

\[
F^{(n)} = \sum_{1 \leq k \leq n} E_{kk}^3 + \sum_{1 \leq k < j \leq n} 2(E_{kk} - E_{jj})E_{kk}
\]
\[
+ \sum_{1 \leq k < j \leq n} 2(n - j)(E_{kk} - E_{jj}) + \sum_{1 \leq k < j \leq n} (E_{kk} - E_{jj})(E_{jj} + 1)
\]
\[
= \sum_{1 \leq k \leq n} E_{kk}^3 + \sum_{1 \leq k < j \leq n} (E_{kk} - E_{jj})(2E_{kk} + E_{jj} + 2(n - j) + 1).
\]

(22)

In a similar way for \( F^{(n-1)} = \sum_{2 \leq i,j,k \leq n} E_{ki}E_{ij}E_{jk} \in U(\mathfrak{g}(n-1, \mathbb{C})) \), it is enough to work modulo the left ideal \( U(\mathfrak{g}(n-1, \mathbb{C}))\mathfrak{m}_+ \). Then we obtain

\[
F^{(n-1)} = \sum_{2 \leq k \leq n} E_{kk}^3 + \sum_{2 \leq k < j \leq n} (E_{kk} - E_{jj})(2E_{kk} + E_{jj} + 2(n - j) + 1).
\]

(23)
The highest weight of \( V_k \) is \( \mu_0 = (m + \ell)(x_1 + \cdots + x_k) + m(x_{k+1} + \cdots + x_n) \).

Hence, from (22) it is easy to conclude that

\[
\hat{\pi}(F^{(n)}) = (k(m + \ell)^3 + (n - k)m^3 + k\ell(n - k)(2(m + \ell) + m + n - k)) I,
\]

where \( I \) stands for the identity linear transformation of \( V_k \).

The highest weight of \( V_{t(s)} \) is

\[
\mu_s = (m + \ell - s)x_1 + (m + \ell)(x_2 + \cdots + x_k) + (m + s)x_{k+1} + m(x_{k+2} + \cdots + x_n).
\]

Hence, from (23) it is easy to conclude that

\[
\hat{\pi}(F^{(n-1)}) = ((k - 1)(m + \ell)^3 + (m + s)^3 + (n - k - 1)m^3
+ (k - 1)(\ell - s)(2(m + \ell) + m + s + 2(n - k) - 1)
+ (k - 1)(n - k - 1)(2(m + \ell) + m + n - k - 1)
+ s(n - k - 1)(2(m + s) + m + n - k - 1)) I_s.
\]

By taking into account the calculations made in (14) and (15) and replacing them in (19) and (20) we complete the proof of the lemma. \( \square \)

**Proposition 2.9.** For all \( r > 0 \) and \( 0 \leq s \leq \ell \) we have

\[
(\tilde{E}\tilde{H})_s(r) = -\frac{(1 + r^2)^2}{4}(m + \ell - s)\tilde{h}'_s(r)
- \frac{(1 + r^2)}{4r}(m + \ell - s)(2n - 1 + r^2 - 2r^2(m + \ell - s))\tilde{h}'_s(r)
- \frac{(1 + r^2)}{2r}s(\ell + k - s)\tilde{h}'_{s-1}(r) + \frac{(1 + r^2)^2}{2r}(\ell - s)(n - k + s)\tilde{h}'_{s+1}(r)
+ \frac{(1 + r^2)}{r^2}s(\ell + k - s)(k - s - m - n + 1)(\tilde{h}_{s-1}(r) - \tilde{h}_s(r))
+ \frac{(1 + r^2)}{r^2}(\ell - s)(n - k + s)(k - s - m - 1)(\tilde{h}_{s+1}(r) - \tilde{h}_s(r))
+ s(\ell + k - s)(2m + n + \ell - k)(\tilde{h}_{s-1}(r) - \tilde{h}_s(r)).
\]

**Proof.** We need to compute in each \( M \)-module \( V_{t(s)} \), \( 0 \leq s \leq \ell \), the linear transformations appearing in the differential operator \( \tilde{E} \) in Theorem 2.6.

It is important to recall, see (14), that

\[
\hat{\pi}(E_{1j})(V_{t(s)}) \subset V_{t(s-1)} \quad \text{and} \quad \hat{\pi}(E_{1j})(V_{t(s)}) \subset V_{t(s+1)},
\]

for \( 2 \leq j \leq n \).

From the calculations made in the proof of Proposition 2.7 we have

\[
\hat{\pi}(E_{11})v_s = (m + \ell - s)v_s,
\]

\[
\sum_{j=2}^{n} \hat{\pi}(E_{1j}) \frac{d\tilde{H}}{dr} \hat{\pi}(E_{1j})v_s = \tilde{h}'_{s-1} \sum_{j=2}^{n} \hat{\pi}(E_{1j})\hat{\pi}(E_{1j})v_s = \tilde{h}'_{s-1} s(\ell + k - s)v_s,
\]

\[
\sum_{j=2}^{n} \hat{\pi}(E_{1j}) \frac{d\tilde{H}}{dr} \hat{\pi}(E_{1j})v_s = \tilde{h}'_{s+1} \sum_{j=2}^{n} \hat{\pi}(E_{1j})\hat{\pi}(E_{1j})v_s = \tilde{h}'_{s+1} (\ell - s)(n - k + s)v_s.
\]
It is easy to verify that
\[
\sum_{j,k=2}^{n} [\hat{\pi}(E_{k1}), [\hat{\pi}(E_{1j}), \hat{H}]] \hat{\pi}(E_{jk}) = (\tilde{h}_s - \tilde{h}_{s-1}) \hat{\pi}(F) + (\tilde{h}_s - \tilde{h}_{s+1}) \hat{\pi}(F'),
\]
\[
\sum_{j,k=2}^{n} [\hat{\pi}(E_{1j}), [\hat{\pi}(E_{k1}), \hat{H}]] \hat{\pi}(E_{jk}) = (\tilde{h}_s - \tilde{h}_{s-1}) \hat{\pi}(F) + (\tilde{h}_s - \tilde{h}_{s+1}) \hat{\pi}(F').
\]

Then by using Lemma 2.8 we get
\[
\sum_{j,k=2}^{n} \left( [\hat{\pi}(E_{k1}), [\hat{\pi}(E_{1j}), \hat{H}]] + [\hat{\pi}(E_{1j}), [\hat{\pi}(E_{k1}), \hat{H}]] \right) \hat{\pi}(E_{jk}) v_s = 2 \left( (\tilde{h}_{s-1} - \tilde{h}_s) s(\ell + k - s)(k - s - m - n + 1) + (\tilde{h}_{s+1} - \tilde{h}_s)(\ell - s)(n - k + s)(k - s - m - 1) \right) v_s.
\]

On the other hand, we have
\[
\sum_{k=1}^{n} \sum_{j=2}^{n} [\hat{\pi}(E_{j1}), \hat{H}] \hat{\pi}(E_{1k}) \hat{\pi}(E_{kj}) v_s
\]
\[
= \sum_{j=2}^{n} [\hat{\pi}(E_{j1}), \hat{H}] \hat{\pi}(E_{1k}) \hat{\pi}(E_{kj}) v_s + \sum_{j,k=2}^{n} [\hat{\pi}(E_{j1}), \hat{H}] \hat{\pi}(E_{1k}) \hat{\pi}(E_{kj}) v_s
\]
\[
= (m + \ell - s + 1) \sum_{j=2}^{n} [\hat{\pi}(E_{j1}), \hat{H}] \hat{\pi}(E_{1j}) v_s + (\tilde{h}_{s-1} - \tilde{h}_s) \hat{\pi}(F) v_s,
\]
by using (16) and Lemma 2.8 we obtain
\[
= (\tilde{h}_{s-1} - \tilde{h}_s) s(\ell + k - s)(2m + \ell + n - k).
\]

Now the proposition follows easily from Theorem 2.6 \qed

3. Hypergeometrization

Let us introduce the change of variables \( t = (1 + r^2)^{-1} \). Let \( H(t) = \tilde{H}(r) \) and correspondingly put \( \tilde{h}_s(t) = \tilde{h}_s(r) \). Then the differential operator of Proposition 2.7 becomes
\[
(DH)_s(t) = -\left( t(1-t)h'_s(t) + (m + \ell - s + 1 - t(n + m + \ell - s + 1))h''_s(t) \right.
\]
\[
+ \frac{1}{1-t}(\ell - s)(n + s - k)(h_{s+1}(t) - h_s(t)) \right.
\]
\[
+ \frac{t}{1-t} s(\ell - s + k)(h_{s-1}(t) - h_s(t)) \bigg),
\]
for \( t \in (0, 1) \) and \( s = 0, \ldots, \ell \). In matrix notation we have
\[
(25) \quad DH(t) = -\left( t(1-t)H''(t) + (A_0 - t(A_0 + n))H'(t) + \frac{1}{1-t}(B_0 + tB_1)H(t) \right),
\]
where
\[ A_0 = \sum_{s=0}^\ell (m + \ell - s + 1) E_{ss}, \]
\[ B_0 = \sum_{s=0}^\ell (\ell - s)(n + s - k) (E_{s,s+1} - E_{ss}), \]
\[ B_1 = \sum_{s=0}^\ell s(\ell - s + k) (E_{s,s-1} - E_{ss}). \]

Similarly, the differential operator \( E \) of Proposition 2.9 becomes
\[
(\mathcal{H}H)_s(t) = -\left( t(1-t)(m + \ell - s) h_s''(t) \right.
+ (m + \ell - s)(m + \ell - s + 1 - t(m + \ell - s + n + 1)) h_s'(t)
+ (\ell - s)(n - k + s) h_{s+1}'(t) - s(\ell + k - s) t h_{s-1}'(t)
+ \frac{1}{1-t} (\ell - s)(n - k + s)(m + s - k + 1)(h_{s+1}(t) - h_s(t))
+ \frac{1}{1-t} s(\ell - s + k)(m + n + s - k - 1)(h_{s-1}(t) - h_s(t))
- s(\ell - s + k)(2m + n + \ell - k)(h_{s-1}(t) - h_s(t)) \bigg),
\]
for \( t \in (0,1) \) and \( s = 0, \ldots, \ell \). In matrix notation we have
\[
(27) \quad \mathcal{H}H(t) = -\left( t(1-t)MH''(t) + (C_0 - tC_1)H'(t) + \frac{1}{1-t} (D_0 + tD_1)H(t) \right),
\]
where
\[
M = \sum_{s=0}^\ell (m + \ell - s) E_{ss},
\]
\[
C_0 = \sum_{s=0}^\ell (m + \ell - s)(m + \ell - s + 1) E_{ss} + \sum_{s=0}^\ell (\ell - s)(n - k + s) E_{s,s+1},
\]
\[
C_1 = \sum_{s=0}^\ell (m + \ell - s)(m + \ell - s + n + 1) E_{ss} + \sum_{s=0}^\ell s(\ell + k - s)E_{s,s-1},
\]
\[
D_0 = \sum_{s=0}^\ell (\ell - s)(n - k + s)(m + s - k + 1) (E_{s,s+1} - E_{ss})
- \sum_{s=0}^\ell s(\ell + k - s)(m + \ell - s + 1) (E_{s,s-1} - E_{ss}),
\]
\[
D_1 = \sum_{s=0}^\ell s(\ell - s + k)(2m + \ell + n - k) (E_{s,s-1} - E_{ss}).
\]
Let us recall that if \( \Phi \) is an irreducible spherical function of one-step \( K \)-type \( k \) as in (7), then the associated function \( H(t) = (h_0(t), \ldots, h_t(t))^t, \) \( 0 < t < 1 \) is an eigenfunction of the differential operators \( D \) and \( E \).

**Theorem 3.1.** Let \( H(t) = (h_0(t), \ldots, h_t(t))^t \), be the function associated to an irreducible spherical function \( \Phi \) of one-step \( K \)-type \( k \). Then \( H \) is a polynomial eigenfunction of the differential operators \( D \) and \( E \) and \( H(1) = (1, \ldots, 1)^t \).

Moreover if \( m + \ell + 1 \leq s \leq \ell \), the function \( h_s \) is of the form

\[
(28) \quad h_s(t) = t^{s-m-\ell}g_s(t),
\]

with \( g_s \) polynomial and \( g(0) \neq 0 \).

**Proof.** On the open subset \( \mathcal{A} \) of \( G \) defined by the condition \( g_{n+1,n+1} \neq 0 \) we put \( H(g) = \Phi(g)\Phi_k(g)^{-1} \). For any \( -\pi/2 < \theta < \pi/2 \) let us consider the elements

\[
a(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & I_{n-1} & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \in \mathcal{A}
\]

and the left upper \( n \times n \) corner \( A(\theta) \) of \( a(\theta) \).

Let \( \pi \) denotes the irreducible finite-dimensional representation of \( U(n) \) of highest weight \( k \). Since \( A(\theta) \) commutes with \( U(n-1) \) then \( \pi(A(\theta)) \) is a scalar in each \( U(n-1) \)-submodule \( V_{k(s)} \) of \( V_k \), see (9). Thus a highest weight vector of \( V_{k(s)} \) as \( U(n-1) \)-module is a vector of the \( U(n) \)-module \( V_k \) of weight \( \mu_s \), see (10). Then \( \Phi_k(a(\theta)) = \pi(A(\theta)) \) in each \( V_{k(s)} \) is equal to \( \cos(\theta)^{m+\ell-s} \) times the identity. Also \( \Phi(a(\theta)) \) is a scalar \( \phi_s(a(\theta)) \) in each \( V_{k(s)} \).

The projection of \( a(\theta) \) into \( P_n(\mathbb{C}) \) is \( p(a(\theta)) = (\tan(\theta), 0, \ldots, 0, 1) \). Thus if we make the change of variables \( r = \tan(\theta) \) and \( t = (1 + r^2)^{-1} = \cos^2(\theta) \) we have

\[
(29) \quad \phi_s(a(\theta)) = t^{(m+\ell-s)/2}h_s(t).
\]

In [PT4] it is proved that \( \phi_s(a(\theta)) = (\cos(\theta)^{m+\ell-s}p_s(\sin^2(\theta)) \) where \( p_s \) is a polynomial. Therefore \( h_s(a(\theta)) = p_s(\sin^2(\theta)) \) and \( h_s(t) = p_s(1-t) \). Thus \( H = H(t) \) is polynomial, and it only remains to prove (28).

By taking limit when \( t \to 0 \) in (29) we get

\[
\lim_{t \to 0} t^{(m+\ell-s)/2}h_s(t) = \phi_s(a(\pi/2)).
\]

If \( x \in \mathbb{C} \) with \( |x| = 1 \), then

\[
b(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & x^{-1} \end{pmatrix} \in K
\]

has the following nice property: \( b(x)a(\pi/2) = a(\pi/2)b(x^{-1}) \). Hence

\[
\phi_s(b(x)a(\pi/2)) = x^{m+\ell-s}\phi_s(a(\pi/2)), \quad \phi_s(a(\pi/2)b(x^{-1})) = \phi_s(a(\pi/2))x^{-(m+\ell-s)}.
\]

Therefore if \( s \neq m + \ell \), then \( \phi_s(a(\pi/2)) = 0 \), and we have obtained

(i) if \( s \neq m + \ell \), \( \lim_{t \to 0} t^{(m+\ell-s)/2}h_s(t) = 0 \),

(ii) if \( s = m + \ell \), \( \lim_{t \to 0} t^{(m+\ell-s)/2}h_s(t) = \phi_s(a(\pi/2)) \).
In particular if \( m + \ell + 1 \leq s \leq \ell \) we have
\[
\lim_{t \to 0} t^{(m+\ell-s)/2} h_s(t) = 0.
\]
Hence \( t = 0 \) is a zero of \( h_s \) of order \( k \geq 1 \).

If \( H(t) = \sum_j t^j H_j \) with column vector coefficients \( H_j = (H_{0j}, \ldots, H_{\ell,j})^T \), then from \( DH = \lambda \dot{H} \) we get that the following three term recursion relation holds for all \( j \),
\[
(j - 1)(j - 2) + (j - 1)(A_0 + n) + B_1 - \lambda)H_{j-1} - (2j(j - 1) + j(2A_0 + n) - B_0 - \lambda)H_j + (j + 1)(j + A_0)H_{j+1} = 0.
\]

Therefore if \( m + \ell + 1 \leq s \leq \ell \) we have \( h_s(t) = \sum_{j \geq k} t^j H_{s,j} \), with \( H_{s,k} \neq 0 \). If we put \( j = k - 1 \) from (30) we obtain
\[
(k + m + \ell - s)H_{s,k} = 0.
\]
Hence \( k = s - m - \ell \) which completes the proof of the theorem. \( \square \)

A particular class of matrix-valued differential operators are the hypergeometric ones which are of the form
\[
t(1 - t) \frac{d^2}{dt^2} + (C - tU) \frac{d}{dt} - V
\]
where \( C, U \) and \( V \) are constant square matrices. These were introduced in [T2]. We observe that the differential operator appearing in (25), obtained from the Casimir of \( G \), is closed to be hypergeometric but it is not.

**Definition 3.2.** A differential operator \( D \) is conjugated to a hypergeometric operator if there exists an invertible smooth matrix-valued function \( \Psi = \Psi(t) \) such that the differential operator \( \hat{D} \) defined by \( \hat{D}F = \Psi^{-1}D(\Psi F) \) is of the form
\[
\hat{D} = t(1 - t) \frac{d^2}{dt^2} + \hat{A}_1(t) \frac{d}{dt} + \hat{A}_0,
\]
where \( \hat{A}_1(t) \) is a matrix polynomial of degree 1 and \( \hat{A}_0 \) is a constant matrix.

The differential operator
\[
-D = t(1 - t) \frac{d^2}{dt^2} + (A_0 - tA_1) \frac{d}{dt} + \frac{1}{1 - t}(B_0 + tB_1)H(t)
\]
is conjugated to a hypergeometric operator if and only if there exists an invertible smooth matrix-valued function \( \Psi(t) \) and constant matrices \( C, U, V \), such that
\[
2t(1 - t)\Psi(t)^{-1}\Psi'(t) + \Psi(t)^{-1}(A_0 - t(A_0 + n))\Psi(t) = C - tU
\]
and
\[
t(1 - t)\Psi(t)^{-1}\Psi''(t) + \Psi(t)^{-1}(A_0 - t(A_0 + n))\Psi'(t)
\]
\[
+ \frac{1}{1 - t}\Psi(t)^{-1}(B_0 + tB_1)\Psi(t) = -V.
\]

A problem of this kind was first solved in [R1]. Motivated by the form of the solution found there, in [PT2] a solution of (32) and (33) is looked for among those functions of the form \( \Psi(t) = XT(t) \), where \( X \) is a constant lower triangular matrix with ones in the main diagonal and \( T(t) = \sum_{0 \leq s \leq t} (1 - t)^s E_{ss} \). Then it is proved that \( X \), equal to the Pascal matrix, provides the unique solution to equations (32) and (33), and the following fact is established in Corollary 3.3 in [PT2].
Proposition 3.3. Let \( T(t) = \sum_{i=0}^{\ell} (1-t)^i E_{ii} \), let \( X \) be the Pascal matrix \( X_{ij} = \binom{i}{j} \), and let \( \Psi(t) = XT(t) \). Then \( \tilde{D}F(t) = \Psi^{-1}(t)(-D)(\Psi(t)F(t)) \) is a hypergeometric operator of the form

\[
\tilde{D}F(t) = t(1-t)F''(t) + (C - tU)F'(t) - VF(t)
\]

with

\[
C = \sum_{s=0}^{\ell} (m + \ell - s + 1)E_{ss} - \sum_{s=1}^{\ell} sE_{s,s-1}, \quad U = \sum_{s=0}^{\ell} (n + m + \ell + s + 1)E_{ss},
\]

\[
V = \sum_{s=0}^{\ell} s(n + m + s - k)E_{ss} - \sum_{s=0}^{\ell-1} (\ell - s)(n + s - k)E_{s,s+1}.
\]

Proposition 3.4. Let \( \tilde{E} \) be defined by \( \tilde{E}F(t) = \Psi(t)^{-1}(-E)(\Psi(t)F(t)) \). Then

\[
\tilde{E}F(t) = t(M_0 - tM_1)F''(t) + (P_0 - tP_1)F'(t) - (m - k)VF(t)
\]

with

\[
M_0 = \sum_{s=0}^{\ell} (m + \ell - s)E_{ss} - \sum_{s=1}^{\ell} sE_{s,s-1}, \quad M_1 = \sum_{s=0}^{\ell} (m + \ell - s)E_{ss},
\]

\[
P_0 = \sum_{s=1}^{\ell} ((m + \ell)(m + \ell + 1) + \ell(n - k) - 2s(n + m - k + s))E_{ss}
\]

\[
- \sum_{s=1}^{\ell-1} s(n - k + \ell + 2m + s)E_{s,s-1} + \sum_{s=0}^{\ell-1} (\ell - s)(n - k + s)E_{s,s+1},
\]

\[
P_1 = \sum_{s=0}^{\ell} (m + \ell - s)(m + n + \ell + s + 1)E_{ss} + \sum_{s=0}^{\ell-1} (\ell - s)(n - k + s)E_{s,s+1}.
\]

Remark 3.1. A proof of the above proposition can be reached by a long, direct and careful computation. A shorter path would be as follows: to start by observing that the differential operator \( E^{PR} \) obtained from the differential operator \( E \) which appears in Section 3 of [PR], by making the change of variables \( t = 1 - u \), it is given in terms of our \( \tilde{D} \) and \( \tilde{E} \) by \( E^{PR} = 2(2m + \ell)\tilde{D} - 3\tilde{E} \).

4. THE EIGENVALUES OF \( D \) AND \( E \)

According to Proposition 1.1 and Lemma 2.1 the irreducible spherical functions \( \Phi \) of our pair \( (G, K) \) are eigenfunctions of the differential operators \( \Delta_P \) and \( \Delta_Q \), introduced in [1]. In this situation the eigenvalues \( \Delta_P \Phi(e) \) and \( \Delta_Q \Phi(e) \) are scalars because the irreducible spherical functions are all of height one.

Let \( \Phi = \Phi^{m,k} \) be the irreducible spherical function associated to the representations \( m \) of \( G \) and \( k \) of \( K \),

\[
k = (m + \ell, \ldots, m + \ell, m, \ldots, m)_k,
\]

\[
m = (w + m + \ell, \ldots, w + m + \ell, m + r, m, \ldots, m, -w - r),
\]

where \( 0 \leq w, 0 \leq r \leq \ell \) and \( -m - r \leq w \). See (7) and (8). Let us recall that \( \Phi \) is an spherical function of type \( k \), because \( s_m = s_k \).
Since the differential operators $D$ and $E$ come from $\Delta_P$ and $\Delta_Q$ respectively, (see (5)) the corresponding function $H(t) = H^{m,k}(t)$ satisfies
\[ DH^{m,k} = [\Delta_P \Phi^{m,k}](e)H^{m,k}, \quad EH^{m,k} = [\Delta_Q \Phi^{m,k}](e)H^{m,k}. \]

Now we will concentrate on computing the scalars $[\Delta_P \Phi^{m,k}](e)$ and $[\Delta_Q \Phi^{m,k}](e)$. We start by observing that if $\Delta \in D(G)^K$, then we have that
\[ [\Delta \Phi^{m,k}](e) = \hat{\pi}_m(\Delta). \]

Let us recall that the Casimir operator of $GL(n, \mathbb{C})$ is
\[ \Delta^{(n)} = \sum_{1 \leq i,j \leq n} E_{ij}E_{ji} = \sum_{1 \leq i \leq n} E_{ii}^2 + \sum_{1 \leq i < j \leq n} (E_{ii} - E_{jj}) + 2 \sum_{1 \leq i < j \leq n} E_{ji}E_{ij}. \]
Thus the differential operator $\Delta_P = \sum_{1 \leq j \leq n} E_{n+1,j}E_{j,n+1}$, can be also written as
\[ \Delta_P = \frac{1}{2} \left( (\Delta^{(n+1)} - \Delta^{(n)}) - E_{n+1,n+1}^2 - \sum_{1 \leq j \leq n} (E_{jj} - E_{n+1,n+1}) \right). \]

The highest weight of $V_m$ is
\[ \mu_m = (w + m + \ell)x_1 + (m + \ell)(x_2 + \cdots + x_k) + (m + r)x_{k+1} + m(x_{k+2} + \cdots + x_n) - (w + r)x_{n+1}, \]
then it follows that
\[ \hat{\pi}_m(\Delta^{(n+1)}) = ((w + m + \ell)^2 + (m + \ell)^2(k - 1) + (m + r)^2 + m^2(n - k - 1) \]
\[ + (w + r)^2 + w(k - 1) + w + \ell - r + (w + \ell)(n - k - 1) + 2w + m + \ell + r \]
\[ + (\ell - r)(k - 1) + \ell(k - 1)(n - k - 1) + (m + \ell + w + r)(k - 1) \]
\[ + r(n - k - 1) + m + w + 2r + (m + w + r)(n - k - 1) \] $I_m$.

If $v_0 \in V_k \subset V_m$ is a $gl(n, \mathbb{C})$-highest weight vector, then its weight is
\[ \mu_0 = (m + \ell)(x_1 + \cdots + x_k) + m(x_{k+1} + \cdots + x_n), \]

because
\[ \hat{\pi}_m(E_{n+1,n+1})v_0 = \hat{\pi}_m \left( \sum_{1 \leq j \leq n+1} E_{jj} \right)v_0 - \hat{\pi}_m \left( \sum_{1 \leq j \leq n} E_{jj} \right)v_0 = (s_m - s_k)v_0 = 0. \]

We also have that
\[ \hat{\pi}_m(\Delta^{(n)})v_0 = (k(m + \ell)^2 + (n - k)m^2 + k(n - k)\ell)v_0 \]
and therefore we obtain
\[ \hat{\pi}_m(\Delta_P)v_0 = (w(w + m + \ell + r + n) + r(m + r - k + n))v_0. \]

To compute $\hat{\pi}_m(\Delta_Q)$ we start by writing $\Delta_Q = \sum_{1 \leq i,j \leq n} E_{n+1,i}E_{j,n+1}E_{ij}$ in terms of elements in $D(G)^G$ and $D(K)^K$.

Let us recall that in the proof of Lemma 2.8 we introduced the element
\[ F^{(n)} = \sum_{1 \leq i,j,k \leq n} E_{ki}E_{ij}E_{jk}. \]
which is in the center of the universal enveloping algebra of $\mathfrak{g}(n, \mathbb{C})$. Hence

$$F^{(n+1)} - F^{(n)} = \sum_{1 \leq j,k \leq n} E_{k,n+1} E_{n+1,j} E_{jk} + \sum_{1 \leq i \leq n} E_{ki} E_{i,n+1} E_{n+1,k} + \sum_{1 \leq i,j \leq n} E_{n+1,i} E_{ij} E_{j,n+1}.$$  

We also have

$$\sum_{1 \leq j,k \leq n+1} E_{k,n+1} E_{n+1,j} E_{jk} = \Delta^{(n+1)} + \Delta Q + E_{n+1,n+1}^3 - (n+1)E_{n+1,n+1}^2 + (2E_{n+1,n+1} - 1)\Delta \rho,$$

$$\sum_{1 \leq i \leq n} E_{i,n+1} E_{n+1,k} = \Delta^{(n)} - E_{n+1,n+1} \sum_{1 \leq i \leq n} E_{ii} + \Delta Q + E_{n+1,n+1} \Delta \rho,$$

$$\sum_{1 \leq i,j \leq n} E_{n+1,i} E_{ij} E_{j,n+1} = n \Delta \rho + \Delta Q.$$  

Thus

$$\Delta Q = \frac{1}{3} \left( F^{(n+1)} - F^{(n)} - \Delta^{(n+1)} - \Delta^{(n)} - (3E_{n+1,n+1} + n - 1) \Delta \rho - (n+1)E_{n+1,n+1}^2 + E_{n+1,n+1} \sum_{1 \leq i \leq n} E_{ii} \right).$$

Taking into account that $E_{n+1,n+1} v_0 = 0$ we have

$$\hat{\pi}_m(\Delta Q)v_0 = \frac{1}{3} \left( \hat{\pi}_m(F^{(n+1)}) - \hat{\pi}_m(F^{(n)}) - \hat{\pi}_m(\Delta^{(n+1)}) - \hat{\pi}_m(\Delta^{(n)}) - (n-1)\hat{\pi}_m(\Delta \rho) \right)v_0. \tag{34}$$

Observe that we have already calculated $\hat{\pi}_m(\Delta^{(n+1)})v_0$, $\hat{\pi}_m(\Delta^{(n)})v_0$, $\hat{\pi}_m(\Delta \rho)v_0$ and also $\hat{\pi}_m(F^{(n)})v_0$ in (22).

Since $F^{(n+1)}$ is $\text{GL}(n+1, \mathbb{C})$-invariant it is enough to compute $\hat{\pi}_m(F^{(n+1)})$ on a highest weight vector $v_m$ of $V_m$; hence it is enough to write $F^{(n+1)}$ modulo the left ideal of $U(\mathfrak{g}(n+1, \mathbb{C}))$ generated by $\{E_{ij} : 1 \leq i < j \leq n+1\}$.

To rewrite the summand $E_{ki} E_{ij} E_{jk}$ of $F^{(n+1)}$ we partition the set of indices as we did in (21). This yields to

$$F^{(n+1)} = \sum_{1 \leq k \leq n+1} E_{kk}^3 + \sum_{1 \leq i < j \leq n+1} (E_{ii} - E_{jj})(2E_{ii} + E_{jj} + 2(n + 1 - j) + 1).$$

Thus we obtain

$$\hat{\pi}_m(F^{(n+1)}) = ((w + m + \ell)^3 + (m + \ell)^3(k - 1) + (m + r)^3 + m^3(n - k - 1) - (w + r)^3 + w(k - 1)(2w + 3m + 3\ell + 2n - k + 1) + (w + \ell - r)(2w + 3m + 2\ell + r + 2n - 2k + 1) + (w + \ell)(n - k - 1)(2w + 3m + 2\ell + n - k + 1) + (2w + m + \ell + r)(w + 2m + 2\ell - r + 1) + (\ell - r)(k - 1)(3m + 2\ell + r + 2n - 2k + 1) + \ell(k - 1)(n - k - 1)(3m + 2\ell + n - k + 1).$$
finite-dimensional complex vector space, and let $A, B$ be eigenfunctions of the following differential operators, so we will be dealing with analytic functions around where the coefficient matrices are those given in Propositions 3.3 and 3.4.

From (34), taking into account the previous calculations, we obtain after some careful computation

$$\hat{\pi}_m(\Delta_Q)v_0 = (w(m + \ell - r)(w + m + \ell + r + n) + r(m - k)(m + r - k + n))v_0.$$ 

Therefore we have proved the following proposition.

**Proposition 4.1.** The function $H^{m,k}$ associated to the spherical function $\Phi^{m,k}$ satisfies

$$DH^{m,k} = (w(m + \ell + r + n) + r(m + r - k + n))H^{m,k},$$

$$EH^{m,k} = (w(m + \ell - r)(w + m + \ell + r + n) + r(m - k)(m + r - k + n))H^{m,k}.$$

5. The one-step spherical functions

5.1. The simultaneous eigenfunctions of $D$ and $E$.

In Section 3 we introduced the differential operators $\hat{D} = \Psi^{-1}(-D)\Psi$ and $\hat{E} = \Psi^{-1}(-E)\Psi$. If we put $F^{m,k}(t) = \Psi(t)H^{m,k}(t)$, from Proposition 4.1 we get

$$\hat{D}F^{m,k} = \lambda(w, r)F^{m,k}, \quad \hat{E}F^{m,k} = \mu(w, r)F^{m,k},$$

where

$$\lambda(w, r) = -w(m + \ell + r + n) - r(m + r - k + n),$$

$$\mu(w, r) = -w(m + \ell - r)(w + m + \ell + r + n) - r(m - k)(m + r - k + n).$$

Since spherical functions are analytic functions on $G$ it follows that $F^{m,k}(t)$ are analytic in a neighborhood of $t = 1$. Thus we make the change of variables $u = 1 - t$, so we will be dealing with analytic functions around $u = 0$ which are simultaneous eigenfunctions of the following differential operators,

$$DF = u(1 - u)F'' + (U - C - uU)F' - VF,$$

$$EF = (1 - u)(M_0 - M_1 + uM_1)F'' + (P - P_0 - uP)F' - (m - k)VF,$$

where the coefficient matrices are those given in Propositions 3.3 and 5.4.

The matrix-valued hypergeometric function was introduced in [12]. Let $W$ be a finite-dimensional complex vector space, and let $A, B$ and $C \in \text{End}(W)$.

The hypergeometric equation is

$$u(1 - u)F'' + (C - u(I + A + B))F' - ABF = 0.$$ 

If the eigenvalues of $C$ are not in $-\mathbb{N}_0$, we define the function

$$\mathbf{2F_1}(\frac{A; B}{C}; u) = \sum_{m=0}^{\infty} \frac{u^m}{m!}(C; A; B)_m,$$

where the symbol $(C; A; B)_m$ is defined inductively by $(C; A; B)_0 = 1$ and

$$(C; A; B)_m + 1 = (C + m)^{-1}(A + m)(B + m)(C; A; B)_m,$$
for all \( m \geq 0 \). The function \( z^m F_1(A;B;u) \) is analytic on \( |u| < 1 \) with values in \( \text{End}(W) \). Moreover if \( F_0 \in W \), then \( F(u) = z^m F_1(A;B;u) F_0 \) is a solution of the hypergeometric equation \((38)\) such that \( F(0) = F_0 \). Conversely any solution \( F \), analytic at \( u = 0 \) is of this form.

In the scalar case the differential operator \((31)\) is always of the form given in \((38)\); after solving a quadratic equation we can find \( A, B \) and \( V = AB \). This is not necessarily the case when \( \text{dim}(W) > 1 \). In other words, a differential equation of the form

\[
(39) \quad u(1-u)F'' + (C-uU)F' - VF = 0,
\]

with \( U, V, C \in \text{End}(W) \), cannot always be reduced to the form of \((38)\), because a quadratic equation in a noncommutative setting as \( \text{End}(W) \) may have no solutions. Thus it is also important to give a way to solve \((39)\).

If the eigenvalues of \( C \) are not in \(-N_0\), let us introduce the sequence \([C,U,V]_m \in \text{End}(W)\) by defining inductively \([C;U;V]_0 = I\) and

\[
[C;U;V]_{m+1} = (C + m)^{-1}(U + m(U - 1) + V)[C;U;V]_m,
\]

for all \( m \geq 0 \). Then the function

\[
z^m H_1(U;C;V;u) = \sum_{m=0}^{\infty} \frac{u^m}{m!}[C;U;V]_m,
\]

is analytic on \( |u| < 1 \) and it is the unique solution of \((39)\) analytic at \( u = 0 \), with values in \( \text{End}(W) \), whose value at \( u = 0 \) is \( I \).

Let \( F^{m,k} \) be the function defined by \( F^{m,k}(u) = F^{m,k}(1-t) \). Then the map \( \Phi^{m,k} \mapsto F^{m,k} \) establishes a one-to-one correspondence between the equivalence classes of irreducible spherical functions of \( SU(n + 1) \) of a fixed \( K\)-type as in \((7)\), with certain simultaneous \( \mathbb{C}^{\ell+1}\)-analytic eigenfunctions on the open unit disc of the differential operators \( D \) and \( E \) given in \((37)\).

Therefore

\[
F^{m,k}(u) = z^m H_1(U;C;V;u) F^{m,k}(0),
\]

with \( \lambda = \lambda(w,r) \).

The goal now is to describe all simultaneous \( \mathbb{C}^{\ell+1}\)-valued eigenfunctions of the differential operators \( D \) and \( E \) analytic on the open unit disc \( \Omega \). We let

\[
V_\lambda = \{ F = F(u) : DF = \lambda F, \text{ F analytic on } \Omega \}.
\]

Since the initial value \( F(0) \) determines \( F \in V_\lambda \), we have that the linear map \( \nu : V_\lambda \rightarrow \mathbb{C}^{\ell+1} \) defined by \( \nu(F) = F(0) \) is a surjective isomorphism. Because \( \Delta_P \) and \( \Delta_Q \) commute, both being in the algebra \( D(G)^K \) which is commutative, \( D \) and \( E \) also commute. Moreover, since \( E \) has polynomial coefficients, \( E \) restricts to a linear operator of \( V_\lambda \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
V_\lambda & \xrightarrow{E} & V_\lambda \\
\downarrow \nu & & \downarrow \nu \\
\mathbb{C}^{\ell+1} & \xrightarrow{M(\lambda)} & \mathbb{C}^{\ell+1}
\end{array}
\]

where \( M(\lambda) \) is the \((\ell + 1) \times (\ell + 1)\) matrix given by

\[
M(\lambda) = (M_0 - M_1)(U - C + 1)^{-1}(U + V + \lambda)(U - C)^{-1}(V + \lambda) + (P_1 - P_0)(U - C)^{-1}(V + \lambda) - (m - k) V.
\]
Although the matrix $M(\lambda)$ has a complicated form we are able to know its characteristic polynomial, via an indirect argument (cf. Theorem 10.3 in [GPT1]).

**Proposition 5.1.**

\[
\det(\mu - M(\lambda)) = \prod_{r=0}^{\ell}(\mu - \mu_r(\lambda)),
\]

where $\mu_r(\lambda) = \lambda(m + \ell - r) + r(m + r - k + n)(\ell - r + k)$. Moreover, all eigenvalues $\mu_k(\lambda)$ of $M(\lambda)$ have geometric multiplicity one. In other words all eigenspaces are one-dimensional. If $v = (v_0, \ldots, v_{\ell})^t$ is a nonzero $\mu_r(\lambda)$-eigenvector of $M(\lambda)$, then $v_0 \neq 0$.

**Proof.** Let us consider the polynomial $p \in \mathbb{C}[\lambda, \mu]$ defined by $p(\lambda, \mu) = \det(\mu - M(\lambda))$. For each integer $r$ such that $0 \leq r < \ell$ let $\lambda(w, r) = -w(w + m + \ell + r + n) - r(m + r - k + n)$. Then from (35) and (36) we have

\[
p(\lambda(w, r), \mu_r(\lambda(w, r))) = 0,
\]

for all $w \in \mathbb{Z}$ such that $0 \leq w$ and $0 \leq w + m + r$. Since there are infinitely many such $w$, the polynomial function $w \mapsto p(\lambda(w, r), \mu_r(\lambda(w, r)))$ is identically zero on $\mathbb{C}$. Hence, given $r$ ($0 \leq r < \ell$), we have $p(\lambda, \mu_r(\lambda)) = 0$ for all $\lambda \in \mathbb{C}$.

Now $\Lambda = \{ \lambda \in \mathbb{C} : \mu_r(\lambda) = \mu_r(\lambda) \text{ for some } 0 \leq r < r' \leq \ell \}$ is a finite set, in fact $|\Lambda| \leq \ell(\ell + 1)/2$. Since for any $\lambda$, $p(\lambda, \mu)$ is a monic polynomial in $\mu$ of degree $\ell + 1$, it follows that if $\lambda \in \mathbb{C} - \Lambda$, then

\[
p(\lambda, \mu) = \prod_{r=0}^{\ell}(\mu - \mu_r(\lambda))
\]

for all $\mu \in \mathbb{C}$. Now it is clear that (40) holds for all $\lambda$ and all $\mu$, which completes the proof of the first assertion.

To prove the last two statements of the proposition we point out that the matrix $M(\lambda) = A + B$, where $A$ is a lower triangular matrix and

\[
B = \sum_{s=0}^{\ell-1} \frac{(n + s - 1)(n + s + \ell)(s + k)}{(n + 2s - 1)(n + 2s)} E_{s,s+1}.
\]

Therefore if $v = (v_0, \ldots, v_{\ell})^t$ is a $\mu$-eigenvector of $M(\lambda)$, then $v_0$ determines $v$, because the coefficients of $B$ are not zero, and this implies that the geometric multiplicity of $\mu$ is one and that $v_0 \neq 0$. \hfill \Box

In this way we have proved the following theorem.

**Theorem 5.2.** The simultaneous $\mathbb{C}^{\ell+1}$-valued eigenfunctions of the differential operators $D$ and $E$ analytic on the open unit disc $\Omega$ are the scalar multiples of

\[
F_r(u) = 2H_1(u^{r+\lambda} \nabla_C; \Delta) F_r(0),
\]

where $F_r(0) = (v_0, \ldots, v_r)^t$ is the unique $\mu_r(\lambda)$-eigenvector of $M(\lambda)$ normalized by $v_0 = 1$, for some $0 \leq r < \ell$ and $\lambda \in \mathbb{C}$. Notice that

\[
DF_r = \lambda F_r \quad \text{and} \quad EF_r = \mu_r(\lambda)F_r.
\]
5.2. Spherical functions.

We recall that we were interested to determine the irreducible spherical functions of $G = SU(n+1)$ whose $K$-type $(K = U(n))$ is of the form

$$k = \left( m + \ell, \ldots, m + \ell, m, \ldots, m \right)_{n-k},$$

by giving an explicit expression of their restriction to the one-dimensional abelian subgroup $\mathbb{A}$ introduced in (2). At this point we have established the following characterization.

**Theorem 5.3.** There is a bijective correspondence between the equivalence classes of all irreducible spherical functions of $SU(n+1)$ of a $K$-type $k$ as in (7), and the set of pairs $(\lambda(w, r), \mu(w, r)) \in \mathbb{C} \times \mathbb{C}$ where

$$\lambda(w, r) = -w(m + \ell + r + n) - r(m + r - k + n),$$

$$\mu(w, r) = -w(m + \ell - r)(m + \ell + r + n) - r(m - k)(m + r - k + n),$$

with $w$ a nonnegative integer, $0 \leq r \leq \ell$ and $0 \leq w + m + r$. In particular $\Phi_{m(w, r), k}$ can be obtained explicitly from the following vector-valued function

$$F_{w, r}(u) = 2H_1 \left( \frac{U + \lambda(w, r)}{\ell} : \mu \right) F_{w, r}(0),$$

where $F_{w, r}(0)$ is the unique $\mu(w, r)$-eigenvector of $M(\lambda(w, r))$ such that $F_{w, r}(0) = (1, v_1, \ldots, v_\ell)^T$.

From the above theorem it follows that the pair of eigenvalues $[\Delta_P \Phi](e)$ and $[\Delta_Q \Phi](e)$ characterize the irreducible spherical functions of type $k$ as in (7). This suggests that one should be able to prove that the algebra $D(G)^K$ is generated by $\Delta_P$ and $\Delta_Q$ modulo the two-sided ideal generated by the kernel in $D(K)$ of the representation $\pi$ of highest weight $k$.

Another consequence is that the map $(w, r) \mapsto (\lambda(w, r), \mu(w, r))$ is one-to-one on the set

$$S = \{(w, r) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq w, 0 \leq r \leq \ell, 0 \leq m + w + r\},$$

which is not easy to see from the formulas. We will need the following generalization.

**Proposition 5.4.** Let $(w, r) \in S$ and let $(d, s) \in \mathbb{R} \times \mathbb{R}$ with $d, s > -1$. If $(\lambda(w, r), \mu(w, r)) = (\lambda(d, s), \mu(d, s))$, then $(w, r) = (d, s)$.

**Proof.** Let us consider the algebraic plane curves

$$F(x, y) = x(x + m + \ell + y + n) + y(m + y - k + n) + \lambda,$$

$$G(x, y) = x(m + \ell - y)(x + m + \ell + y + n) + y(m - k)(m + y - k + n) + \mu,$$

where we put $\lambda = \lambda(w, r)$ and $\mu = \mu(w, r)$. Then it is easy to check that $F$ is irreducible and that $F$ and $G$ have no common component.

Now one can see that both curves meet at the following points:

$$(w, r), (w, -r - w - n + k - m), (-w - m - r - \ell - n, r),$$

$$(-w - m - r - \ell - n, k + \ell + w), (-\ell + r - k, k + \ell + w), (-\ell + r - k, -r - w - n + k - m).$$

These are six different points because $(w, r) \in S$ and $1 \leq k \leq n - 1$. Therefore by Bezout’s theorem, see [2], these are the only points of intersection of $F$ and $G$. But the only point of intersection with $x > -1$ and $y > -1$ is $(w, r)$. This completes the proof of the proposition. □
5.3. The orthogonal functions $F_{w,r}$.

The aim of this subsection is to prove that the functions $F = F_{w,r}$ associated to the spherical functions $\Phi_{m(w,r),k}$ are polynomial functions that are orthogonal with respect to a certain inner product.

Let us equip $V_k$ with a $K$-invariant inner product. Then the $L^2$-inner product of two continuous functions $\Phi_1$ and $\Phi_2$ with values in $V_k$ is

$$\langle \Phi_1, \Phi_2 \rangle = \int_G \text{tr}(\Phi_1(g)\Phi_2(g)^*) \, dg,$$

where $dg$ denotes the normalized Haar measure of $G$ and $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$.

In particular if $\Phi_1$ and $\Phi_2$ are two irreducible spherical functions of type $k$, we write as above $\Phi_1 = H_1 \Phi_k$ and $\Phi_2 = H_2 \Phi_k$. Then by using the integral formula given in Corollary 5.12, p. 191 in [He] we obtain

$$\langle \Phi_1, \Phi_2 \rangle = \int_0^1 H_2^*(u) V(u) H_1(u) \, du$$

where $V(u)$ is the weight matrix

$$V(u) = 2n \sum_{\ell=0}^\ell \left( \binom{\ell+k-r-1}{\ell-r} \binom{n-k+r-1}{r} (1-u)^{m+r-u} u^{n-1} E_{rr}. \right)$$

In Theorem 5.3 we introduced the function $\Psi$

$$\Psi(u) = XT(u), \quad T(u) = \sum_{i=0}^\ell u^i E_{ii} \quad \text{and} \quad X = \sum_{i,j} (i_j) E_{ij},$$

to conjugate the differential operator $D$ to a hypergeometric operator $\tilde{D}$.

Therefore in terms of the functions $F_1 = \Psi^{-1} H_1$ and $F_2 = \Psi^{-1} H_2$ we have

$$\langle F_1, F_2 \rangle_W = \int_0^1 F_2(u)^* W(u) F_1(u) \, du,$$

where the weight matrix $W(u) = \Psi^*(u) V(u) \Psi(u)$ is given by

$$W(u) = \sum_{i,j=0}^\ell \sum_{r=0}^\ell \binom{i}{r} \binom{j}{r} \binom{\ell+k-r-1}{\ell-r} \binom{n-k+r-1}{r} (1-u)^{m+r-u} u^{i+j+n-1} E_{ij}.$$

We point out that the weight function $W(u)$ has finite moments of all orders if and only if $m \geq 0$. See [PT2] for some details.

**Proposition 5.5.** The functions $F_{w,r}$ associated to irreducible spherical functions $\Phi_{m(w,r),k}$ of type $k$ are orthogonal with respect to the weight function $W(u)$, that is

$$\langle F_{w,r}, F_{w',r'} \rangle_W = 0, \quad \text{if} \quad (w,r) \neq (w',r').$$

**Proof.** The differential operators $\Delta_P$ and $\Delta_Q$ are symmetric with respect to the $L^2$-inner product, among matrix-valued functions on $G$. Therefore the differential operators $\tilde{D}$ and $\tilde{E}$ are symmetric with respect to the inner product (41), among vector-valued functions on the interval $[0,1]$. This implies that $\tilde{D}$ and $\tilde{E}$ are symmetric with respect to the inner product defined by the weight matrix $W(u)$.

Since the pairs $(\lambda, \mu)$ of eigenvalues of the symmetric operators $\tilde{D}$ and $\tilde{E}$ characterize these $F$’s, see Theorem 5.3 it follows that the $F$’s associated to irreducible spherical functions of type $k$ are orthogonal with respect to this inner product. □
Let $L^2_W$ denote the Hilbert space of all $\mathbb{C}^{\ell+1}$-valued functions on $[0, 1]$, squared integrable with respect to the inner product $\langle \cdot, \cdot \rangle$. Also let $\mathbb{C}^{\ell+1}[u]$ be the $\mathbb{C}^{\ell+1}$-valued polynomial functions.

Let us consider the following linear space,

\[(42) \quad \mathcal{V} = \{ P \in \mathbb{C}^{\ell+1}[u] : (\Psi P)_s = (1 - u)^{s-m-\ell}g_s, \; g_s \in \mathbb{C}[u], \; m + \ell + 1 \leq s \leq \ell \}. \]

Observe that $\mathcal{V} = \mathbb{C}^{\ell+1}[u]$ if and only if $m \geq 0$.

**Lemma 5.6.** $\mathcal{V}$ is a linear subspace of $L^2_W$. Moreover $\mathcal{V}$ is stable under the differential operator $D$.

**Proof.** Let $P \in \mathcal{V}$. Then
\[
\|P\|_W^2 = \int_0^1 P^* \Psi^* V(u) \Psi P \, du = \int_0^1 (\Psi P)^* V(u)(\Psi P) \, du
\]
where $c_s = 2n^{(\ell + k - s - 1)}(\ell - s)$.

\[
\|P\|_W^2 = \sum_{s=0}^{m+\ell} \int_0^1 c_s |(\Psi P)_s|^2 u^{n-1}(1 - u)^{m+\ell-s} \, du + \sum_{s=m+\ell+1}^{\ell} \int_0^1 c_s u^{n-1}(1 - u)^{s-m-\ell} g_s(u)^2 \, du < \infty,
\]

because $\Psi$, $P$ and $g_s$ are polynomial functions. Therefore $P \in L^2_W$.

The proof of the last statement is left to the interested reader. \qed

Let us consider the closure $\overline{\mathcal{V}}$ of the subspace $\mathcal{V}$ in $L^2_W$.

**Proposition 5.7.** Let $F$ be the $\mathbb{C}^{\ell+1}$-valued function associated to an irreducible spherical function $\Phi$ of type $k$. Then $F \in \overline{\mathcal{V}}$.

**Proof.** The function $F = \Psi^{-1}H$ is analytic at $u = 1$ because $H$ and $\Psi^{-1}$ are analytic at $u = 1$. Hence
\[
F = \sum_{j=0}^{\infty} (1 - u)^j F_j, \quad F_j \in \mathbb{C}^{\ell+1}.
\]

It is enough to prove that the partial sums $\sum_{j=0}^N (1 - u)^j F_j$ are in $\mathcal{V}$, because $F$ is analytic on $[0, 1]$. In other words, if $m + \ell + 1 \leq s \leq \ell$ we need to show that
\[
\left( \Psi \sum_{j=0}^N (1 - u)^j F_j \right)_s = (1 - u)^{s-m-\ell}g_s, \text{ for some polynomial } g_s.
\]

We can write $\Psi(u) = \sum_{0 \leq k \leq \ell} (1 - u)^k \Psi_k$, where the coefficients $\Psi_k$ are $(\ell + 1) \times (\ell + 1)$ matrices. Then
\[
(\Psi F)_s = \sum_{j=0}^{\infty} (1 - u)^j \sum_{k=0}^\ell (1 - u)^k (\Psi_k F_j)_s = \sum_{r=0}^\infty (1 - u)^r \sum_{k=0}^{\min(r, \ell)} (\Psi_k F_{r-k})_s.
\]
From Theorem 3.1 we have that $h_s = (\Psi F)_s$ is analytic in $[0, 1]$ and it has a zero of order at least $s - m - \ell$ at $u = 1$. Then we have

$$\min\{r, \ell\} \sum_{k=0}^{r} (\Psi_k F_{r-k})_s = 0, \quad \text{for all } r < s - m - \ell.$$ 

Therefore

$$\left( \Psi \sum_{j=0}^{N} (1 - u)^j F_j \right)_s = \sum_{r=s-m-\ell}^{N+\ell} (1 - u)^r \sum_{k=0}^{\min\{r, \ell\}} (\Psi_k F_{r-k})_s,$$

and thus $\sum_{j=0}^{N}(1 - u)^j F_j \in \mathcal{V}$. This completes the proof. \qed

**Theorem 5.8.** The function $F_{w,r}$ associated to the spherical function $\Phi^{m(w,r),k}$ is a polynomial function.

**Proof.** Set

$$\mathcal{V}_j = \{ P \in \mathcal{V} : \deg P \leq j \}.$$ 

Then $\mathcal{V}_j$ is $D$-stable. Since $D$ is symmetric $\mathcal{V}_j^\perp \cap \mathcal{V}_{j+1}$ is invariant under $D$, and there exists an orthonormal basis of $\mathcal{V}_j^\perp \cap \mathcal{V}_{j+1}$ of eigenvectors of $D$. Then by induction on $j \geq 0$ it follows that there exists an orthonormal basis $\{ P_i \}$ of $\mathcal{V}$ such that $DP_i = \lambda_i P_i$, $\lambda_i \in \mathbb{C}$.

By Proposition 5.7 $F \in \mathcal{V}$. Therefore

$$F = \sum_{j=0}^{\infty} a_j P_j, \quad a_i = \langle F, P_i \rangle_W.$$

Since $F$ is analytic on $[0, 1]$ the series converges not only in $L^2_W$ but also in the topology based on uniform convergence of sequences of functions and their successive derivatives on compact subsets of $(0, 1)$. Therefore

$$\lambda F = DF = D \left( \sum_{j=0}^{\infty} a_j P_j \right) = \sum_{j=0}^{\infty} a_j \lambda_j P_j,$$

and thus $a_i = 0$ unless $\lambda = \lambda_i$. Then

$$F = \sum_{\lambda=\lambda_i} a_i P_i,$$

and since $\dim \mathcal{V}_\lambda < \infty$ we conclude that the function $F$ is a polynomial. \qed

**Proposition 5.9.** The function $F_{w,r}$ is a polynomial function of degree $w$ and its leading coefficient is of the form $F_{d} = (x_0, \ldots, x_r, 0, \ldots, 0)^t$ with $x_r \neq 0$.

**Proof.** From Theorem 5.8 we have that $F = F_{w,r}$ is a polynomial function; let us say that it is of degree $d$ and leading coefficient $F_d$. From Theorem 5.3 we know that $F$ is an eigenfunction of the differential operators $D$ and $E$, namely

$$DF = \lambda(w, r) F, \quad EF = \mu(w, r) F.$$

By looking at the leading coefficient of the polynomials in the above identities we have that

$$(d(d - 1 + U) + V + \lambda(w, r)) F_d = 0, \quad (d(d - 1)M_1 + dP_1 + (m - k)V + \mu(w, r)) F_d = 0.$$
Let us introduce the matrices

\[
L_1 = d(d - 1 + U) + V + \lambda(w, r)
\]

\[
= \sum_{s=0}^{\ell}(\lambda(w, r) - \lambda(d, s))E_{ss} + \sum_{s=0}^{\ell-1}(\ell - s)(n + s - k)E_{s,s+1},
\]

\[
L_2 = d(d - 1)M_1 + dP_1 + (m - k)V + \mu(w, r)
\]

\[
= \sum_{s=0}^{\ell}(\mu(w, r) - \mu(d, s))E_{ss} + \sum_{s=0}^{\ell-1}(\ell - s)(n + s - k)(d - m + k)E_{s,s+1}.
\]

Now we observe that \(L_1\) has a nontrivial kernel if and only if

\[
\lambda(w, r) = \lambda(d, j) \quad \text{for some} \quad 0 \leq j \leq \ell.
\]

In this case, if \(s = \min\{j : \lambda(w, r) = \lambda(d, j)\}\) and \(0 \neq x = (x_0, \ldots, x_\ell) \in \ker(L_1)\), then \(x_s \neq 0\) and \(x_j = 0\) for \(s + 1 \leq j \leq \ell\); this is so because the elements above the main diagonal of \(L_1\) are not zero.

Similarly, \(L_2\) has a nontrivial kernel if and only if \(\mu(w, r) = \mu(d, j)\) for some \(0 \leq j \leq \ell\). Now the elements above the main diagonal of \(L_2\) are all zero or all are not zero. In the first case since \(F_d \in \ker(L_1) \cap \ker(L_2)\) we must have \(\mu(w, r) = \mu(d, s)\). In the second case if \(s' = \min\{j : \mu(w, r) = \mu(d, j)\}\) and \(0 \neq x = (x_0, \ldots, x_\ell) \in \ker(L_2)\), then \(x_{s'} \neq 0\) and \(x_j = 0\) for \(s' + 1 \leq j \leq \ell\). Since \(F_d \in \ker(L_1) \cap \ker(L_2)\) we must have \(s = s'\). In any case we have \(\lambda(w, r) = \lambda(d, s)\) and \(\mu(w, r) = \mu(d, s)\).

From Proposition 5.4 we have that \((w, r) = (d, s)\). In particular \(d = \deg(F_{w,r}) = w\) and the leading coefficient is of the required form. □

6. Matrix orthogonal polynomials

From now on we shall assume that \(m \geq 0\).

In this section we package appropriately the functions \(F_{w,r}\) associated to the spherical functions \(\Phi_{m(w, r), k}\) of one step \(K\)-type, for \(0 \leq w\) and \(0 \leq r \leq \ell\), in order to obtain families of examples of matrix-valued orthogonal polynomials that are eigenfunctions of certain second-order differential operators.

It is important to remark that these examples arise naturally from the group representation theory, in particular from spherical functions.

The theory of matrix-valued orthogonal polynomials, without any consideration of differential equations goes back to M. G. Krein, in \([K1]\) and \([K2]\). Starting with a selfadjoint positive definite weight matrix \(W = W(u)\), with finite moments we can consider the skew-symmetric bilinear form defined for matrix-valued polynomials \(P\) and \(Q\) by

\[
\langle P, Q \rangle_W = \int_{\mathbb{R}} P(u)W(u)Q(u)^* \, du = 0.
\]

By the usual construction this leads to the existence of sequences \((P_w)_{w \geq 0}\) of matrix valued orthogonal polynomials with non singular leading coefficients and \(\deg P_w = w\).

Any sequence of matrix orthogonal polynomials \((P_w)_{w}\) satisfies a three-term recursion relation of the form

\[
u_{w+2}(u) = A_w P_{w+1}(u) + B_w P_w(u) + C_w P_{w-1}(u), \quad w \geq 0,
\]

where we put \(P_{-1}(u) = 0\).
In [D] the study of matrix-valued orthogonal polynomials that are eigenfunctions of certain second-order symmetric differential operator was started. A differential operator with matrix coefficients acting on matrix-valued functions could be made to act either on the left or on the right. One finds a discussion of these two actions in [D]. The conclusion there is that if one wants to have matrix weights \( W(u) \) that are not direct sums of scalar ones and that have matrix polynomials as their eigenfunctions, one should settle for right-hand side differential operators. We agree now that

\[
D = \sum_{i=0}^{s} \partial^i F_i(u), \quad \partial = \frac{d}{du},
\]

acts on \( P(u) \) by means of

\[
PD = \sum_{i=0}^{s} \partial^i (P(u)) F_i(u).
\]

One could make \( D \) act on \( P \) on the right as defined above, and still write down the symbol \( DP \) for the result. The advantage of using the notation \( PD \) is that it respects associativity: if \( D_1 \) and \( D_2 \) are two differential operators, we have

\[
P(D_1 D_2) = (PD_1)D_2.
\]

We define the matrix polynomial \( P_w \) as the \((\ell + 1) \times (\ell + 1)\) matrix whose \( r \)-row is the polynomial \( F_{w,r}(u) \), for \( 0 \leq r \leq \ell \). In other words

\[
(44) \quad P_w = (F_{w,0}, \ldots, F_{w,\ell}).
\]

An explicit expression for the rows of \( P_w \) is given in Theorem 5.3, in terms of the matrix hypergeometric function, namely

\[
F_{w,r}(u) = 2H_{1}\left(\frac{U+\lambda(w,r)}{U-C}; u \right) F_{w,r}(0),
\]

where its value \( F_{w,r}(0) \) at \( u = 0 \) is the \( \mu(w, r) \)-eigenvector of \( M(\lambda(w, r)) \) properly normalized.

We notice that from Proposition 5.5 the rows of \( P_w \) are orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle_W \); then the sequence \( (P_w)_w \) is orthogonal with respect to the weight matrix \( W(u) \),

\[
\langle P_w, P_{w'} \rangle_W = \int_0^1 P_w(u) W(u) P_{w'}(u)^* du = 0, \quad \text{for all } w \neq w'.
\]

From Proposition 5.8 we obtain that each row of \( P_w \) is a polynomial function of degree \( w \). A more careful look at the definition shows that \( P_w \) is a matrix polynomial whose leading coefficient is a lower triangular nonsingular matrix, see Proposition 5.9. In other words this sequence of matrix-valued polynomials fits squarely within Krein’s theory and we obtain the following proposition.

**Proposition 6.1.** The matrix polynomial functions \( P_w \), whose \( r \)-row is the polynomial \( F_{w,r} \), \( 0 \leq r \leq \ell \), form an sequence of orthogonal matrix polynomials with respect to \( W \).

In [D] a sequence of matrix orthogonal polynomials is called classical if there exists a symmetric differential operator of order-two that has these polynomials as eigenfunctions with a matrix eigenvalue.
In the next proposition we show that the \((P_w)_w\) is a family of classical orthogonal polynomials featuring two algebraically independent symmetric differential operators of order two.

**Proposition 6.2.** The polynomial function \(P_w\) is an eigenfunction of the differential operator \(D^t\) and \(E^t\), the transposes of the operators \(D\) and \(E\) appearing in (37). Moreover

\[
P_w D^t = \Lambda_w(D^t)P_w, \quad P_w E^t = \Lambda_w(E^t)P_w,
\]

where \(\Lambda_w(D^t) = \text{diag}(\lambda(w,0), \ldots, \lambda(w,\ell))\) and \(\Lambda_w(E^t) = \text{diag}(\mu(w,0), \ldots, \mu(w,\ell))\).

**Proof.** We have

\[
D^t = \partial^2 u(1-u) + \partial(U^t - C^t - uU^t) - V^t,
\]

\[
E^t = \partial^2 (1-u)(M^t_0 - M^t_1 + uM^t_0) + \partial(P^t_1 - P^t_0 - uP^t_1) - (m-k)V^t,
\]

where the coefficient matrices are given explicitly in Propositions 3.3 and 3.4. Now we observe that

\[
P_w D^t = (1-u)P''_w + P'_w(U^t - C^t - uU^t) - P_w V^t
\]

\[
= u(1-u)(F''_{w,0}, \ldots, F''_{w,\ell})^t + (F'_{w,0}, \ldots, F'_{w,\ell})^t(U - C - uU)^t
\]

\[
- (F_{w,0}, \ldots, F_{w,\ell})^tV^t
\]

\[
= (DF_{w,0}, \ldots, DF_{w,\ell})^t = (\lambda(w,0)F_{w,0}, \ldots, \lambda(w,\ell)F_{w,\ell})^t
\]

\[
= \Lambda_w(D^t)P_w,
\]

where \(\Lambda_w(D^t) = \text{diag}(\lambda(w,0), \ldots, \lambda(w,\ell))\).

Similarly \(P_w E^t = \Lambda_w(E^t)P_w\) where \(\Lambda_w(E^t) = \text{diag}(\mu(w,0), \ldots, \mu(w,\ell))\) and this concludes the proof. \(\square\)

In general given a matrix weight \(W(u)\) and a sequence of matrix orthogonal polynomials \(P_w\) one can dispense of the requirement of symmetry and consider the algebra of all matrix differential operators \(D\) such that

\[
P_w D = \Lambda_w(D)P_w,
\]

where each \(\Lambda_w\) is a matrix. See Section 4 of [GT]. The set of these \(D\) is denoted by \(D(W)\).

Starting with [GPT1], [GPT2], [G1] and [DG] one has a growing collection of weight matrices \(W(u)\) for which the algebra \(D(W)\) is not trivial, i.e., does not consist only of scalar multiples of the identity operator. A first attempt to go beyond the issue of the existence of one nontrivial element in \(D(W)\) was undertaken in [CG], for some examples whose weights are of size two. The study of the full algebra, in one of these cases was considered in [T3].

### 6.1. The three-term recursion relation satisfied by the polynomials \(P_w\).

The aim of this subsection is to indicate how to obtain, from the representation theory, the three term recursion relation satisfied by the sequence of matrix orthogonal polynomials \((P_w)_w\) built up from packages of spherical functions of the pair \((G, K)\). More details can be found in [PT1] for the case \(n = 2\) and in [P2] for the general case.

The standard representation of \(U(n+1)\) on \(\mathbb{C}^{n+1}\) is irreducible and its highest weight is \((1, 0, \ldots, 0)\). Similarly the representation of \(U(n+1)\) on the dual of \(\mathbb{C}^{n+1}\)
The constants $a_i$ are given by

$$a_i(m,k) = \left[ \prod_{j=1}^{n} (k_j - m_i - j + i - 1) / \prod_{j \neq i} (m_j - m_i - j + i) \right]^{1/2},$$

and $b_i(m,k)$ is defined accordingly.

Moreover

$$\sum_{i=1}^{n+1} a_i^2(m,k) = \sum_{i=1}^{n+1} b_i^2(m,k) = 1.$$
In this section we are restricting our attention to spherical functions of type
\[
\mathbf{k} = (m + \ell, \ldots, m + \ell, m, \ldots, m),
\]
with \(m \geq 0\). Thus we only consider \((n + 1)\)-tuples \(\mathbf{m}\) of the form
\[
\mathbf{m}(w, r) = (w + m + \ell, \ldots, m + \ell, m + r, m, \ldots, m, -w - r),
\]
for \(0 \leq w\) and \(0 \leq r \leq \ell\).

Now we introduce the \(rs\)-entries of the matrices \(A_w, B_w\) and \(C_w\):
\[
(A_w)_{rs} = \begin{cases}
  a_{n+1}^2(m(w, r))b_j^2(m(w, r) + e_{n+1}) & \text{if } s = r \\
  a_{k+1}^2(m(w, r))b_j^2(m(w, r) + e_{k+1}) & \text{if } s = r + 1 \\
  0 & \text{otherwise}
\end{cases}
\]
\[
(C_w)_{rs} = \begin{cases}
  a_{n+1}^2(m(w, r))b_j^2(m(w, r) + e_{n+1}) & \text{if } s = r \\
  a_{k+1}^2(m(w, r))b_j^2(m(w, r) + e_{k+1}) & \text{if } s = r + 1 \\
  0 & \text{otherwise}
\end{cases}
\]
\[
(B_w)_{rs} = \begin{cases}
  \sum_{1 \leq j \leq n+1} a_j^2(m(w, r))b_j^2(m(w, r) + e_j) & \text{if } s = r \\
  a_{n+1}^2(m(w, r))b_j^2(m(w, r) + e_{n+1}) & \text{if } s = r + 1 \\
  0 & \text{otherwise}
\end{cases}
\]
where \(a_j^2(m(w, r)) = a_j^2(m(w, r), k)\), \(b_j^2(m(w, r) + e_j) = b_j^2((m(w, r) + e_j, k))\) for \(1 \leq j \leq n + 1\), see \([17]\).

Notice that \(A_w\) is an upper-bidiagonal matrix, \(C_w\) is a lower-bidiagonal matrix and \(B_w\) is a tridiagonal matrix.

For the benefit of the reader we include the following simplified expressions, in the case of one-step \(K\)-type, of the coefficients involved in the definition of the matrices \(A_w, B_w\) and \(C_w\):
\[
a_j^2(m(w, r)) = \frac{(w + k)(w + \ell + n)}{(w + \ell - r + k)(2w + m + n + \ell + r)};
\]
\[
a_{k+1}^2(m(w, r)) = \frac{(\ell - r)(r + n - k)}{(w + \ell - r + k)(w + m + n + 2r - k)};
\]
\[
a_{n+1}^2(m(w, r)) = \frac{(w + m + n + \ell + r - k)(w + m + n + \ell + r)}{(w + m + n + 2r - k)(2w + m + n + \ell + r)};
\]
all the others \(a_j^2(m(w, r))\) are zero. The remaining coefficients are
\[
b_j^2(m(w, r) + e_1) = \frac{(w + 1)(w + \ell + k + 1)}{(w + \ell - r + k + 1)(2w + m + n + \ell + r + 1)};
\]
\[
b_{k+1}^2(m(w, r) + e_{k+1}) = \frac{w(w + \ell + k)}{(w + \ell - r + k - 1)(2w + m + n + \ell + r)};
\]
\[
b_{n+1}^2(m(w, r) + e_{n+1}) = \frac{w(w + \ell + k)}{(w + \ell - r + k)(2w + m + n + \ell + r - 1)};\]
Theorem 6.6. The polynomial functions \( P_w(u) \) satisfy
\[
(1 - u)P_w(u) = A_w P_{w-1}(u) + B_w P_w(u) + C_w P_{w+1}(u),
\]
for all \( w \geq 0 \), where the matrices \( A_w, B_w \) and \( C_w \) were introduced before.

This three term recursion relation can be written in the following way:

\[
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
B_0 & C_0 & 0 \\
A_1 & B_1 & C_1 & 0 \\
0 & A_2 & B_2 & C_2 & 0 \\
0 & 0 & A_3 & B_3 & C_3 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} 
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
\vdots
\end{bmatrix}
\]

Corollary 6.4 and of the definitions of the matrices \( A_w, B_w, C_w \), when we take \( g = a(\theta) \).

We recall that \( \phi(g) \) and \( \psi(g) \) are the one-dimensional spherical functions associated to the \( G \)-modules \( \mathbb{C}^{n+1} \) and \( (\mathbb{C}^{n+1})^* \), respectively. A direct computation gives
\[
\phi(a(\theta)) = \langle a(\theta)e_{n+1}, e_{n+1} \rangle = \cos \theta \quad \text{and} \quad \psi(a(\theta)) = \langle a(\theta)\lambda_{n+1}, \lambda_{n+1} \rangle = \cos \theta.
\]

Then \( \phi(a(\theta))\psi(a(\theta)) = \cos^2(\theta) = t. \) \( \Box \)

Now we take into account that the polynomial functions \( P_w \) introduced in (44), are obtained by right-multiplying the function \( \Phi(w,t) \) by a matrix function on \( t \), independent of \( w \). After the change of variable \( t = 1 - u \) we obtain the following three term recursion relation for the polynomials \( P_w \).

In the open subset \( \{a(\theta) \in A : 0 < \theta < \pi/2\} \) of \( A \), we introduce the coordinate \( t = \cos^2(\theta) \) and define on the open interval \((0,1)\) the matrix-valued function
\[
\Phi(w,t) = (\Phi_s^{m(w,r),k}(a(\theta)))_{0 \leq r, s \leq t}.
\]
From (48) one can prove that
\[
\sum_{s=0}^{\ell} \left( (A_w)_{rs} + (B_w)_{rs} + (C_w)_{rs} \right) = 1,
\]
showing that the square semi-infinite matrix \( M \) appearing in (49) is a stochastic matrix.

In [GPT6] we describe a random mechanism based on Young diagrams that gives rise to a random walk in the set of all Young diagrams of \( 2n+1 \) rows and whose \( 2j \)-th row has \( k_j \) boxes \( 1 \leq j \leq n \), whereby in one unit of time one of the \( m_i \) is increased by one with probability \( a_i^2(m, k) \) see (47).

In the expressions for \( D, E, W, P_w \) and \( M \) the discrete parameters \((m, n)\) enter in a simple analytical fashion. Appealing to some version of analytic continuation, it is clear that this entire edifice remains valid if one allows \((m, n-1)\) to range over a continuous set of real values \((\alpha, \beta)\). The requirement that \( W \) retain the property of having finite moments of all orders translates into the conditions \( \alpha, \beta > -1 \). We will denote the corresponding weight and the orthogonal polynomials by \( W_{\beta, \alpha} \) and \( P_{\beta, \alpha} \). These are some interesting families of new matrix valued Jacobi polynomials.

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