FREE SUBGROUPS OF 3-MANIFOLD GROUPS

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Abstract. We show that any closed hyperbolic 3-manifold \( M \) has a co-final tower of covers \( M_i \to M \) of degrees \( n_i \) such that any subgroup of \( \pi_1(M_i) \) generated by \( k_i \) elements is free, where \( k_i \geq n_i^C \) and \( C = C(M) > 0 \). Together with this result we prove that \( \log k_i \geq C\text{sys}_1(M_i) \), where \( \text{sys}_1(M_i) \) denotes the systole of \( M_i \), thus providing a large set of new examples for a conjecture of Gromov. In the second theorem \( C_1 > 0 \) is an absolute constant. We also consider a generalization of these results to non-compact finite volume hyperbolic 3-manifolds.

1. Introduction

Let \( \Gamma < \text{PSL}_2(\mathbb{C}) \) be a cocompact Kleinian group and \( M = \mathbb{H}^3/\Gamma \) be the associated quotient space. It is a closed orientable hyperbolic 3-orbifold, it is a manifold if \( \Gamma \) is torsion-free. We will call a group \( \Gamma \) \( k \)-free if any subgroup of \( \Gamma \) generated by \( k \) elements is free. We denote the maximal \( k \) for which \( \Gamma \) is \( k \)-free by \( N_{fr}(\Gamma) \) and we call it the free rank of \( \Gamma \). For example, if \( S_g \) is a closed Riemann surface of genus \( g \), then its fundamental group satisfies \( N_{fr}(\pi_1(S_g)) = 2g - 1 \). In this note we prove that for any Kleinian group as above there exists an exhaustive filtration of normal subgroups \( \Gamma_i \) of \( \Gamma \) such that 
\[ N_{fr}(\pi_1(M_i)) \geq \text{vol}(M_i)^C, \]
where \( C = C(\Gamma) > 0 \) is a constant.

In geometric terms the result can be stated as follows.

**Theorem 1.** Let \( M \) be a closed hyperbolic 3-orbifold. Then there exists a co-final tower of regular finite-sheeted covers \( M_i \to M \) such that
\[ N_{fr}(\pi_1(M_i)) \geq \text{vol}(M_i)^C, \]
where \( C = C(M) \) is a positive constant which depends only on \( M \).

The proof of the theorem is based on the previous results of Baumslag, Shalen and Wagreich [BS89, SW92], Belolipetsky [B13], and Calegari–Emerton [CE11]. Let us emphasize that although some of the results use arithmetic techniques, our theorem applies to all closed hyperbolic 3-orbifolds. A result of similar flavor but for another property of 3-manifold groups was obtained by Long, Lubotzky and Reid in [LLR08]. Indeed, in some parts our construction comes close to their argument.

Together with Theorem 1 we obtain the following theorem of independent interest:

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Theorem 2. Any closed hyperbolic 3-orbifold admits a sequence of regular manifold covers $M_i \to M$ such that
\[ N_{fr}^\prime(\pi_1(M_i)) \geq (1 + \varepsilon)^{\text{sys}_1(M_i)}, \]
where $\varepsilon > 0$ is an absolute constant and $\text{sys}_1(M_i)$ is the length of a shortest closed geodesic in $M_i$.

This type of bound was stated by Gromov [G87, Section 5.3.A] for hyperbolic groups in general, but later turned into a conjecture (see [G09, Section 2.4]). We refer to the introduction of [B13] for a related discussion and some other references. In [G09], Gromov particularly mentioned that the conjecture is open even for hyperbolic 3-manifold groups. The first set of examples of hyperbolic 3-manifolds for which the conjecture is true was presented in [B13]. These examples were all arithmetic. Our theorem significantly enlarges this set.

We review the construction of covers $M_i \to M$ and prove a lower bound for their systoles in Section 2. Theorems 1 and 2 are proved in Section 3. In Section 4 we consider a generalization of the results to non-compact finite volume 3-manifolds. Their groups always contain a copy of $\mathbb{Z} \times \mathbb{Z}$, so have $N_{fr}^\prime = 1$, however, we can modify the definition of the free rank so that it becomes non-trivial for the non-compact manifolds: we define $N_{fr}^\prime(\Gamma)$ to be the maximal $k$ for which the group $\Gamma$ is $k$-semifree, where $\Gamma$ is called $k$-semifree if any subgroup generated by $k$ elements is a free product of free abelian groups. With this definition at hand we can extend Gromov’s conjecture to the groups of finite volume non-compact manifolds. In Section 4 we prove:

Theorem 3. Any finite volume hyperbolic 3-orbifold admits a sequence of regular manifold covers $M_i \to M$ such that
\[ N_{fr}^\prime(\pi_1(M_i)) \geq (1 + \varepsilon)^{\text{sys}_1(M_i)}, \]
where $\varepsilon > 0$ is an absolute constant.

To conclude the introduction we would like to point out one important detail. While in Theorems 2 and 3 we have an absolute constant $\varepsilon > 0$, the constant in Theorem 1 depends on the base manifold. In [B13] it was shown that in arithmetic case $C(M)$ is also bounded below by a universal positive constant. Existence of a bound of this type in general remains an open problem.

Question 1. Do there exist an absolute constant $C_0 > 0$ such that for any $M$ in Theorem 1 we have $C(M) \geq C_0$.

2. Preliminaries

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a lattice, i.e. a finite covolume discrete subgroup. By Mostow–Prasad rigidity, $\Gamma$ admits a discrete faithful representation into $\text{SL}_2(\mathbb{C})$ with the entries in some (minimal) number field $E$. Since $\Gamma$ is finitely generated, there is a finite set
of primes \( S \) in \( E \) such that \( \Gamma < \text{SL}_2(\mathcal{O}_{E,S}) \), where \( \mathcal{O}_{E,S} \) denotes the ring of \( S \)-integers in \( E \).

Following Calegari–Emerton [CE11], we can consider an exhaustive filtration of normal subgroups \( \Gamma_i \) of \( \Gamma \) which gives rise to a co-final tower of hyperbolic 3-manifolds covering \( \mathbb{H}^3/\Gamma \). The subgroups \( \Gamma_i \) are defined as follows. From the description of \( \Gamma \) given above it follows that it is residually finite and for all but finitely many primes \( p \in \mathcal{O}_E \) there is an injective map \( \phi_p : \Gamma \to \text{SL}_2(\hat{\mathcal{O}}_{E,p}) \) (where \( \hat{\mathcal{O}}_{E,p} \) denotes the \( p \)-adic completion of the ring of integers of \( E \)). Let \( p \) be a rational prime such that for any prime \( \mathfrak{p} \in \mathcal{O}_E \) which divides \( p \), the correspondent map \( \phi_{\mathfrak{p}} \) is injective (this holds for almost all primes \( p \)). We can write \( p\mathcal{O}_E = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m} \).

For any \( j = 1, \ldots, m \), the ring \( \hat{\mathcal{O}}_{E,\mathfrak{p}_j} \) contains \( \mathbb{Z}_p \) as a subring and is a \( \mathbb{Z}_p \)-module of dimension \( d_j = e_j f_j \), where \( f_j \) is the degree of the extension of residual fields \( [\mathcal{O}_E/\mathfrak{p}_j : \mathbb{Z}/p\mathbb{Z}] \). If we fix \( j \) and a basis \( b_1^j, \ldots, b_{d_j}^j \in \hat{\mathcal{O}}_{E,\mathfrak{p}_j} \) as \( \mathbb{Z}_p \)-module, we have a natural ring homomorphism \( \psi_j : \hat{\mathcal{O}}_{E,\mathfrak{p}_j} \to M_{d_j \times d_j}(\mathbb{Z}_p) \) given by \( \psi_j(x) = (x_{rs}) \) if \( x b_k^j = \sum_{r=1}^{d_j} x_{rs} b_k^j \).

Let \( \psi : \prod_{j=1}^m \text{SL}_2(\hat{\mathcal{O}}_{E,\mathfrak{p}_j}) \to \text{GL}_N(\mathbb{Z}_p) \) be given diagonally by the blocks \( \psi_1, \ldots, \psi_m \), where \( N = 2 \sum_{j=1}^m d_j \). Let \( \phi = \psi \circ \prod_{j=1}^m \phi_{\mathfrak{p}_j} : \text{SL}_2(\mathcal{O}_{E,S}) \to \text{GL}_N(\mathbb{Z}_p) \). The Zariski closure of the image of \( \phi \) is a group \( \hat{G} < \text{GL}_N(\mathbb{Z}_p) \) of dimension \( d \geq 6 \) (cf. [CE11, Example 5.7]). It is a \( p \)-adic analytic group which admits a normal exhaustive filtration

\[
G_i = \hat{G} \cap \ker\left( \text{GL}_N(\mathbb{Z}_p) \to \text{GL}_N(\mathbb{Z}_p/p^i\mathbb{Z}_p) \right).
\]

This filtration gives rise to a filtration of \( \Gamma \) via the normal subgroups \( \Gamma_i = \phi^{-1}(G_i) \). The filtration \( (\Gamma_i) \) is exhaustive because \( \phi \) is injective.

Associated to each of the subgroups \( \Gamma_i \) of \( \Gamma \) is a finite-sheeted cover \( M_i \) of \( M = \mathbb{H}^3/\Gamma \), and by the construction the sequence \( (M_i) \) is a co-final tower of covers of \( M \). By Minkowski’s lemma, almost all groups \( G_i \) are torsion-free, hence associated \( M_i \) are smooth hyperbolic 3-manifolds. Therefore, when it is needed we can assume that \( M \) is a manifold itself.

We will require a lower bound for the systole of \( M_i \). Such a bound is essentially provided by Proposition 10 of [GL14], which can be seen as a generalization of a result of Margulis [M82] (see also [LLR08]). The main difference is that we do not restrict to arithmetic manifolds. The main technical difference is that while in [op. cit.] the authors consider matrices with real entries we do it for \( p \)-adic numbers, which requires replacing norm of a matrix by the height of a matrix. This technical part is more intricate, however, as it is shown below, it does not affect the main argument.

**Lemma 2.1.** Suppose \( M \) is a compact manifold. Then there is a constant \( c_1 = c_1(M) > 0 \) such that \( \text{sys}_1(M_i) \geq c_1 \log n_i \), where \( n_i = [\Gamma : \Gamma_i] \).

**Proof.** Since \( M \) is compact, we can apply the Milnor–Schwarz lemma. Therefore, if we fix a point \( o \in \mathbb{H}^3 \), then \( \Gamma \) has a finite symmetric set of generators \( X \) such that
Let Claim 1: the minimal length of a word in $\gamma \in \Gamma$ we have

$$C_1 d_X(\gamma_1, \gamma_2) - C_2 \leq d(\gamma_1(o), \gamma_2(o)) \leq \frac{1}{C_1} d_X(\gamma_1, \gamma_2) + C_2,$$

where $d(\cdot, \cdot)$ denotes the distance function in $\mathbb{H}^3$, $d_X(\gamma_1, \gamma_2) = |\gamma_1^{-1} \gamma_2|_X$ and $|\gamma|_X$ is the minimal length of a word in $X$ which represents $\gamma$. For any $i \geq 1$, we define

$$\text{sys}(\Gamma_i, X) = \min\{d_X(1, \gamma) \mid \gamma \in \Gamma_i \setminus \{1\}\}.$$

Claim 1: Let $\delta_M > 0$ be the diameter of $M$. For any $i \geq 1$, we have

$$\text{sys}_1(M_i) \geq C_1 \text{sys}(\Gamma_i, X) - C_2 - 2\delta_M.$$

To prove the claim, consider the Dirichlet fundamental domain $D(o)$ of $\Gamma$ in $\mathbb{H}^3$ centered in $o$. It is easy to see that any point $x \in D(o)$ satisfies $d(x, o) \leq \delta_M$. Now let $\alpha_i \subset M_i$ be a closed geodesic realizing the systole of $M_i$. As $M_i \rightarrow M$ is a local isometry, the image of $\alpha_i$ in $M$ has the same length (counted with multiplicity). Denote the image by $\alpha_i$ again. We can suppose that $x_i \in D(o)$ is a lift of $\alpha_i(0)$. Thus, there exists a unique nontrivial $\gamma_i \in \Gamma_i$ such that $\text{sys}_1(M_i) = d(x_i, \gamma_i(x_i))$. Note that $d(x_i, o) = d(\gamma_i(x_i), \gamma_i(o)) \leq \delta_M$, therefore, by the triangle inequality we have

$$\text{sys}_1(M_i) \geq d(o, \gamma_i(o)) - 2\delta_M \geq C_1 d_X(1, \gamma_i) - C_2 - 2\delta_M \geq C_1 \text{sys}(\Gamma_i, X) - C_2 - 2\delta_M.$$

Now our problem is reduced to proving that $\text{sys}(\Gamma_i, X)$ grows logarithmically as a function of $[\Gamma : \Gamma_i]$. In order to do so we use arithmetic of the field $E$ in an essential way.

Let $S(E)$ be the set of all places of $E$, $S_\infty$ be the set of archimedean places, and $S_p$ be the set of places corresponding to the prime ideals $p_1, \ldots, p_m$, which appear in the definition of $M_i$. For any $x \in E$, we define the height of $x$ by $H(x) = \prod_{v \in S(E)} \max\{1, |x|_v\}$. Recall that for any $x, y \in E$ and an archimedean place $v$, we have $|x + y|_v \leq 4 \max\{|x|_v, |y|_v\}$, and for any non-archimedean place $u$, we have $|x + y|_u \leq \max\{|x|_v, |y|_u\}$. Therefore, the height function satisfies $H(x + y) \leq 4^{\#S_\infty} H(x) H(y)$.

We can generalize the definition of height for matrices with entries in $E$. Thus, for any $M = (m_{ij}) \in \text{SL}_2(E)$, we define $H(M) = \prod_{v \in S(E)} \max\{1, |m_{ij}|_v\}$. We note that $H(M) \geq \max\{H(m_{ij})\}$.

Claim 2: For any $M, N \in \text{SL}_2(E)$, we have $H(MN) \leq 4^{\#S_\infty} H(M) H(N)$.

Indeed, any entry $x$ of $MN$ can be written as $x = au + bt$ with $a, b$ entries of $M$ and $u, t$ entries of $N$. Therefore, for any $v \in S_\infty$,

$$\max\{1, |x|_v\} \leq 4 \max\{1, |a|_v, |b|_v\} \max\{1, |u|_v, |t|_v\} \leq 4 \max\{1, |m_{ij}|_v\} \max\{1, |n_{ij}|_v\}.$$
For non-archimedean places we have the same inequality without the factor 4. Now if $MN = (x_{ij})$, then these inequalities show that

$$H(MN) = \prod_{v \in S(E)} \max\{1, |x_{ij}|_v\} \leq 4^{#S_{\infty}} H(M)H(N).$$

Next we want to estimate from below the height of $\gamma$ for any nontrivial $\gamma \in \Gamma_i$.

**Claim 3:** There exists a constant $C_3 > 0$ such that for any $\gamma \in \Gamma_i \setminus \{1\}$ we have $H(\gamma) \geq C_3 p^{ni}$, where $n = [E : \mathbb{Q}]$.

Indeed, let $\gamma = \gamma_{r_1} \cdots \gamma_{r_w(\gamma)} \in \Gamma_i$ be a nontrivial element with $\gamma_{r_j} \in X$ and $w(\gamma) = d_X(1, \gamma)$. We now recall the definition of the group $\Gamma_i$. If we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for any $l = 1, \ldots, m$ we have

$$\left( \begin{array}{cc} \psi_l(a) & \psi_l(b) \\ \psi_l(c) & \psi_l(d) \end{array} \right) \equiv \left( \begin{array}{cc} I_{d_l} & 0 \\ 0 & I_{d_l} \end{array} \right) \mod (p^i \mathbb{Z}_p).$$

By the definition of $\psi_l$ we have that $(a - 1)b_j^l, bbb_j^l, cbl_j^l, (d - 1)b_j^l \in p^i \hat{O}_{E,p_l}$ for any $1 \leq j \leq d_l$. Taking $C^* = \min_{l,j} \{|b_j^l|_{p_l}\} > 0$, we obtain

$$C^* \max\{|a - 1|_{p_l}, |b|_{p_l}, |c|_{p_l}, |d - 1|_{p_l}\} \leq \text{Norm}(p_l)^{-\epsilon_{i_l}},$$

for any $l = 1, \ldots, m$. This is because $|p|_{p_l} = \text{Norm}(p_l)^{-\epsilon_l}$ by definition, where for an ideal $I \subset \mathcal{O}_E$ the norm of $I$ is equal to $\text{Norm}(I) = \#(\mathcal{O}_E/I)$.

Recall that the Product Formula says that for any nonzero $x \in E$ we have $\prod_v |x|_v = 1$. Since $\gamma$ is nontrivial, at least one of the numbers $\{a - 1, b, c, d - 1\}$ is not zero. Therefore, if we apply the Product Formula for any nonzero element in this set, we obtain

$$\max\{H(a - 1), H(b), H(c), H(d - 1)\} \geq \prod_{l=1}^{m} C^* \text{Norm}(p_l)^{\epsilon_{i_l}} = (C^*)^{m} p^{ni}.$$

Moreover, by the estimate of the height of a sum we have

$$\max\{H(a - 1), H(b), H(c), H(d - 1)\} \leq 4^{#S_{\infty}} \max\{H(a), H(b), H(c), H(d)\},$$

therefore,

$$H(\gamma) \geq \max\{H(a), H(b), H(c), H(d)\} \geq \frac{(C^*)^{m} p^{ni}}{4^{#S_{\infty}}} = C_3 p^{ni}.$$ 

This proves Claim 3.

We can now finish the proof of the lemma. If we take $C_4 = 4^{#S_{\infty}} \max\{H(M) \mid M \in X\}$, we have

$$C_3 p^{ni} \leq H(\gamma) \leq (4^{#S_{\infty}})^{d_X(1,\gamma)-1} \max\{H(M) \mid M \in X\}^{d_X(1,\gamma)} \leq C_4^{d_X(1,\gamma)}.$$
This estimate holds for any nontrivial $\gamma \in \Gamma_i$, hence $C_3 p^{n_i} \leq C_4^{\text{sys}(\Gamma_i, X)}$ for any $i$. On the other hand, there exists a constant $C_5 > 0$ such that $[\Gamma : \Gamma_i] \leq C_5 p^{d(G)}$. These inequalities together imply that

$$\text{sys}(\Gamma_i, X) \geq \frac{n}{\dim(G) \log(C_4)} \log([\Gamma : \Gamma_i]) + \frac{\log(C_3 C_5^{\dim(G)})}{\log(C_4)}.$$  

Since $[\Gamma : \Gamma_i] \to \infty$ and $\text{sys}_1(M_i)$ is bounded below by a positive constant, we conclude that there exists a constant $c_1 = c_1(o, \delta_M, p, \psi_1, \ldots, \psi_m) = c_1(M) > 0$ such that $\text{sys}_1(M_i) \geq c_1 \log([\Gamma : \Gamma_i])$ for any $i \geq 1$. □

Note that the constant $c_1$ depends on $M$ (cf. Question 1). If $M$ is arithmetic, then by [KSV07] we can take $c_1 = \frac{2}{3} - \epsilon$ for a small $\epsilon > 0$ assuming $n_i$ is sufficiently large. In general case the argument of [KSV07] does not apply, while the proof of Lemma 2.1 does not provide a sufficient level of control over the constants.

3. Proofs of Theorems 1 and 2

Following [B13], we define the systolic genus of a manifold $M$ by

$$\text{syst}(M) = \min \{ g \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g) \},$$

where $S_g$ denotes a closed Riemann surface of genus $g > 0$.

Let $M$ be a closed hyperbolic 3-manifold with sufficiently large systole $\text{sys}_1(M)$. By [B13, Theorem 2.1], we have

$$\log \text{syst}(M) \geq c_2 \cdot \text{sys}_1(M),$$

where $c_2 > 0$ is an absolute constant (for any $\delta > 0$, assuming $\text{sys}_1(M)$ is sufficiently large, we can take $c_2 = \frac{1}{2} - \delta$).

The second ingredient of the proof is a theorem of Calegary–Emerton [CE11], which implies that for the sequences of covers defined in Section 2 we have

$$\dim H_1(M_i, \mathbb{F}_p) \geq \lambda \cdot p^{(d-1)i} + O(p^{(d-2)i})$$

for some rational constant $\lambda \neq 0$. Recall that we have dimension $d = \dim(G) \geq 6$ and the degree of the covers $M_i \to M$ grows like $p^d$. Hence we can rewrite (2) in the form

$$\dim H_1(M_i, \mathbb{F}_p) \geq c_3 \text{vol}(M_i)^{5/6},$$

where $c_3 > 0$ is a constant depending on $M$ and we assume that $\text{vol}(M_i)$ is sufficiently large.

We note that in contrast with the previous related work, the theorem of [CE11] applies to non-arithmetic manifolds as well as to the arithmetic ones.

Now recall a result of Baumslag–Shalen [BS89, Appendix]. They show that if $\text{syst}(M) \geq k$ and $\dim H_1(M, \mathbb{Q}) \geq k + 1$, then $\pi_1(M)$ is $k$-free. In a subsequent paper [SW92], Shalen and Wagreich proved that the same conclusion holds if $\text{syst}(M) \geq k$ and $\dim H_1(M, \mathbb{F}_p) \geq k + 2$ [loc. cit., Proposition 1.8].
We now bring all the ingredients together. Given a closed hyperbolic 3-orbifold $M$, for the sequence $(M_i)$ of its manifold covers defined in Section 2 we have:

$$\text{sysg}(M_i) \geq e^{c_2 \cdot \text{sys}_1(M_i)} \quad \text{(by (1))}$$

and

$$\dim H_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \text{vol}(M_i)^{5/6} \quad \text{(by (3))}.$$ 

Hence by the theorem from [SW92] cited above we obtain

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq \text{vol}(M_i)^C,$$

where $C = C(M) > 0$ and we assume that $\text{vol}(M_i)$ is sufficiently large. This proves Theorem 1.

For the second theorem recall that the systole of a hyperbolic 3-manifold is bounded above by the logarithm of its volume. Indeed, a manifold $M$ with a systole $\text{sys}_1(M)$ contains a ball of radius $r = \text{sys}_1(M)/2$. The volume of a ball in $\mathbb{H}^3$ is given by $\text{vol}(B(r)) = \pi(\sinh(2r) - 2r)$, hence we get

$$\text{vol}(M) \geq \pi(\sinh(\text{sys}_1(M)) - \text{sys}_1(M)) \sim \frac{\pi}{2} e^{\text{sys}_1(M)};$$

$$\text{vol}(M) \geq e^{c \cdot \text{sys}_1(M)}, \quad \text{as } \text{sys}_1(M) \to \infty.$$ 

By Lemma 2.1, the systole of the covers $M_i \to M$ grows as $i \to \infty$. Therefore, we can bound both $\text{sysg}(M_i)$ and $\dim H_1(M_i, \mathbb{F}_p)$ below by an exponential function of $\text{sys}_1(M_i)$ with an absolute constant in exponent. Theorem 2 now follows immediately from the theorem of [SW92].

Remark 3.1. It follows from the proof that for any $\delta > 0$, assuming $\text{sys}_1(M_i)$ is large enough, we can take $\varepsilon$ in Theorem 2 equal to $e^{\mathcal{N}_{fr}(\Gamma)} - 1$. The same bound applies for the constant in Theorem 3, which we prove in the next section.

4. Generalization to finite volume hyperbolic 3-manifolds

Let $\Gamma \subset \text{PSL}_2(\mathbb{C})$ be a finite covolume Kleinian group. The quotient $M = \mathbb{H}^3/\Gamma$ is a finite volume orientable hyperbolic 3-orbifold, which can be either closed or non-compact with a finite number of cusps. The group $\Gamma$ is a relatively hyperbolic group with respect to the cusp subgroups. In this section we discuss a generalization of Gromov’s conjecture and our results to this class of groups.

We call $\Gamma$ a $k$-semifree group if any subgroup of $\Gamma$ generated by $k$ elements is a free product of free abelian groups. The maximal $k$ for which $\Gamma$ is $k$–semifree is denoted by $\mathcal{N}_{fr}(\Gamma)$. With this definition, we can generalize Gromov’s conjecture to relatively hyperbolic groups. Although the injectivity radius of manifolds with cusps vanish, their systole is still bounded away from zero. Therefore, a natural generalization of Gromov’s conjecture would be that $\mathcal{N}_{fr}(\Gamma)$ is bounded below by an exponential
function of the systole of the associated quotient space $M$. Theorem 3, which we prove in this section, can be considered as an evidence for this conjecture.

We need to modify the definition of the systolic genus of a manifold $M$ in the following way:

$$\text{sysg}(M) = \min\{g > 1 \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g)\},$$

where $S_g$ denotes a closed Riemann surface of genus $g$. We excluded the genus $g = 1$ in order to adapt the definition to the non-compact finite volume 3-manifolds which otherwise would all have $\text{sysg} = 1$.

Let $M$ be a finite volume hyperbolic 3-manifold with sufficiently large systole $\text{sys}_1(M)$. By [B13, Theorem 2.1], if $M$ is closed, we have

$$\log \text{sysg}(M) \geq c_2 \cdot \text{sys}_1(M),$$

(4)

where $c_2 > 0$ is an absolute constant. We now discuss a generalization of this result to non-compact finite volume 3-manifolds. The first step in the proof of the theorem in [B13] is an application of the theorem of Schoen–Yau and Sacks–Uhlenbeck, which allows to homotop a $\pi_1$-injective map of a surface of genus $g > 1$ into $M$ to a minimal immersion. This result was recently generalized to the finite volume hyperbolic 3-manifolds in the work of Collin–Hauswirth–Mazet–Rosenberg [CHMR17] and Huang–Wang [HW17] (see in particular [HW17, Theorem 1.1]). So let $S_g$ be a closed immersed least area minimal surface in $M$. In order to establish (4) for $M$ we can suppose that $S_g$ is embedded. Indeed, since $\pi_1(M)$ is LERF [A13, Corollary 9.4] there exists a finite covering $\tilde{M}$ of $M$ such that $S_g$ is embedded and $\pi_1$-injective in $\tilde{M}$. Moreover, $g \geq \text{sysg}(\tilde{M})$ and $\text{sys}_1(\tilde{M}) \geq \text{sys}_1(M)$. If $S_g$ has no accidental parabolic curves, then the systole of $S_g$ with respect to the induced metric satisfies $\text{sys}_1(S_g) \geq \text{sys}_1(M)$ and the rest of the proof in [B13] applies without any changes.

In the presence of accidental parabolics, we can apply the following lemma.

**Lemma 4.1 (Compression Lemma).** Let $M$ be a non-compact hyperbolic 3-manifold of finite volume. Suppose that there exists a $\pi_1$-injective embedded closed surface $S_g \subset M$, for some genus $g \geq 2$, such that $S_g$ has an accidental parabolic simple curve $\alpha$. Then there exist disjoint tori $T_1, \ldots, T_n \subset M$, one for each cusp $C(T_i)$ of $M$, such that the compact 3-manifold $M' = M \setminus \bigcup_{i=1}^n C(T_i)$ has a properly incompressible and boundary-incompressible surface $S_{g', p}$ with $g' \geq \frac{g}{2}$ and $1 \leq p \leq 2$.

**Proof.** Suppose that $\alpha$ is associated to a parabolic isometry corresponding to a cusp $C = T_0 \times [0, \infty)$ of $M$, where $T_0$ is a maximal torus. Since $S_g$ is compact we can consider a torus $T = T_0 \times \{t_0\} \subset C$ for some $t_0 > 0$ sufficiently large such that $S_g \subset M \setminus T_0 \times [0, t_0)$. We denote by $\beta \subset T$ the corresponding simple curve homotopic to $\alpha$.

We first show that there exists an embedding $f : S_g \to M$ homotopic to the embedding $\iota : S_g \to M$ such that $f$ is transversal to some torus $T_1 \subset C$ and $f(S_g) \cap$
$T_1 \times [0, \infty) \subset \mathcal{C}$ is an annulus with boundary curves $f(\alpha_0), f(\alpha_1)$, where $\alpha_0, \alpha_1$ are the boundary curves of a collar neighborhood of $\alpha$ in $S_g$.

As an application of the Jaco–Shalen Annulus Theorem [Jaco80, Theorem VIII.13], there exists an embedding $H_0 : S^1 \times [0, 1] \to M$ with $H(\theta, 0) = \alpha(\theta)$ and $H(\theta, 1) = \beta(\theta)$ (see [OT03, Lemma 2.1]). We can suppose that $H_0$ is transversal to $S_g$ and $T$ and is such that if we denote by $\mathcal{A}$ the image $H_0(S^1 \times [0, 1])$, then $\mathcal{A} \cap S_g = \alpha$ and $\mathcal{A} \cap M \setminus T \times [0, \infty) = \beta$.

Let $\mathcal{D}$ be a collar neighborhood of $\alpha$ in $S_g$ contained in a tubular neighborhood $\pi : E \subset M \to \mathcal{A}$ such that $\mathcal{D} \cap \mathcal{A} = \alpha$. Since $\pi : E \to \mathcal{A}$ is trivial, we can deform $\mathcal{D}$ into $E$ preserving the boundary and moving $\alpha$ along $\mathcal{A}$. We get a new annulus $\mathcal{D}' \subset M$ with $\partial \mathcal{D}' = \alpha_0 \cup \alpha_1$ and $\mathcal{D}' \cap T = \beta$.

Let $\psi$ be the diffeomorphism between $\mathcal{D}$ and $\mathcal{D}'$ given by the deformation. We can suppose that $\psi$ is the identity in a small neighborhood of the boundary. We now define the map $f : S_g \to M$ by $f(x) = x$ if $x \notin \mathcal{D}$ and $f(y) = \psi(y)$ if $y \in \mathcal{D}$. It is a smooth embedding homotopic to the inclusion.

By transversality, for some $0 < t_1 < t_0$ we have a torus $T_1 = T_0 \times \{t_1\}$ and a subannulus $\hat{D} \subset \mathcal{D}$ such that $f$ is transversal to $T_1$ and

$$f(S_g \cap M \setminus T_1 \times [0, \infty)) = f(S_g \setminus \text{int}(\hat{D})) \quad \text{and} \quad f(\partial(S_g \setminus \text{int}(\hat{D}))) = f(\partial \hat{D}) \subset T_1.$$  

This shows that embedding $f$ has the desired properties.

Now, for the torus $T_1$ constructed above, there exist disjoint tori $T_2, \ldots, T_n$ in the cusps of $M$ such that the corresponding cusps $\mathcal{C}(T_j) \cap \mathcal{C}(T_1) = \emptyset$ for all $j = 2, \ldots, n$ and $f(S_g \setminus \text{int}(\hat{D})) \subset M' = M \setminus \bigcup_{i=1}^n \mathcal{C}(T_i)$, and we have that $f(S_g \setminus \text{int}(\hat{D})) \subset M'$ is a proper submanifold of $M'$.

Note that $f(S_g \setminus \text{int}(\hat{D}))$ is connected with two boundary curves if $\alpha$ does not separate and has two components with a boundary curve if $\alpha$ separates it. In the latter case we consider the component with the maximal genus. Hence in both cases we have a surface $S_{g',p}$ with $g' \geq \frac{2}{3}$ and $1 \leq p \leq 2$ and a proper embedding $f : S_{g',p} \to M'$.

Recall that a properly embedded surface $F$ in a compact 3-manifold $N$ with boundary is called boundary-compressible if either $F$ is a disk and $F$ is parallel to a disk in $\partial N$, or $F$ is not a disk and there exists a disk $D \subset N$ such that $D \cap F = c$ is an arc in $\partial D$, $D \cap \partial N = c'$ is an arc in $\partial D$, with $c \cap c' = \partial c = \partial c'$ and $c \cup c' = \partial D$, and either $c$ does not separate $F$ or $c$ separates $F$ into two components and the closure of neither is a disk. Otherwise, $F$ is boundary-incompressible (see [Jaco80, Chapter III]).

Since $S_g \subset M$ is $\pi_1$-injective, it follows from the definition and our construction that $S_{g',p} \subset M'$ is incompressible and boundary-incompressible.

We now apply to $S_{g',p}$ a result of Adams and Reid [AR00, Theorem 5.2]. Since $\text{sys}_1(M) = \text{sys}_1(M')$, it immediately implies inequality (4).

The theorem of Calegary–Emerton applies to non-cocompact groups as well as to the cocompact ones.
We finally recall a result of Anderson–Canary–Culler–Shalen [ACCS96]. They show that if \( \text{sysg}(M) \geq k \) and \( \dim H_1(M, \mathbb{F}_p) \geq k + 2 \) for some prime \( p \), then \( \pi_1(M) \) is \( k \)-semifree [loc. cit., Corollary 7.4]. This theorem generalizes the previous results in [BS89, SW92] to non-compact hyperbolic 3–manifolds. Its proof also makes an essential use of topology of 3-manifolds.

Similar to Section 3, we bring together all the ingredients considered above.

Given a finite volume hyperbolic 3-orbifold \( M \), for the sequence \( (M_i) \) of its manifold covers defined in Section 2 we have:

\[
\text{sysg}(M_i) \geq e^{c_2 \text{sys}_1(M_i)} \quad \text{(by (4))},
\]

and

\[
\dim H_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \text{vol}(M_i)^{5/6} \quad \text{(by Calegari–Emerton)}.
\]

The fact that a manifold \( M \) with systole \( \text{sys}_1(M) \) contains a ball of radius \( r = \text{sys}_1(M)/2 \) is not necessarily true for non-compact finite volume hyperbolic 3-manifolds but it is still possible to bound the volume by an exponential function of the systole. By Lakeland–Leininger [LL14, Theorem 1.3], we have

\[
\text{vol}(M) \geq e^{c \text{sys}_1(M)}, \quad \text{as } \text{sys}_1(M) \rightarrow \infty
\]

(with \( c = \frac{3}{4} - \delta \) for any \( \delta > 0 \), assuming \( \text{sys}_1(M) \) is sufficiently large).

Although we do not have a generalization of Lemma 2.1, we do know that \( \text{sys}_1(M_i) \rightarrow \infty \) with \( i \) because the sequence of covers \( M_i \rightarrow M \) is co-final. Therefore, we can bound both \( \text{sysg}(M_i) \) and \( \dim H_1(M_i, \mathbb{F}_p) \) below by an exponential function of \( \text{sys}_1(M_i) \) with an absolute constant in exponent and Theorem 3 now follows from the theorem of [ACCS96].

\[\Box\]

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