An application of the exponential spline for the approximation of a function and its derivatives in the presence of a boundary layer

I A Blatov¹, A I Zadorin², E V Kitaeva³

¹ Volga Region State University of Telecommunications and Informatics, ul. L’va Tolstogo,23, 443010, Samara, Russia
² Sobolev Institute of Mathematics, pr. Koptyuga,4, 630090, Novosibirsk, Russia
³ Korolev Samara State University, Moskovskoe sh., 34, 443086, Samara, Russia

E-mail: zadorin@ofim.oscsbras.ru, blatow@mail.ru

Abstract. The problem of exponential spline interpolation of functions having large gradients in the exponential boundary layer is considered. The spline of the class $C^2[0, 1]$ is constructed as the sum of a polynomial of the second degree and a boundary-layer function on each grid interval. Estimates of the error in the approximation of a function and its derivatives are obtained. These estimates are uniform in small parameter. The limiting behavior of the exponential spline is investigated, when the perturbing parameter tends to infinity or to zero. In the first case, the spline becomes cubic, and in the second case it becomes parabolic. The results of numerical experiments are presented.

1. Introduction

Various convection-diffusion processes with predominant convection are modeled on the basis of singularly perturbed problems. The solution of singular perturbed problem has large gradients in the boundary layer. The problem of the interpolation of functions with large gradients in the boundary layer is of interest. It is shown [1] that the interpolation of functions with large gradients in the boundary layer with using of cubic spline on a uniform grid is inefficient. It is proved that the interpolation error can grow without limit when the small parameter $\varepsilon$ tends to zero. In [1] it is proved that in the case of Shishkin mesh [2] the interpolation error can grow if $\varepsilon$ decreases. In [1] the modification of the cubic spline on Shishkin mesh [2] is made, as result the error of the cubic spline becomes uniform with respect to $\varepsilon$.

In [3] for interpolated functions with large gradients in the exponential boundary layer the exponential spline is applied. In the case of a uniform grid error estimates are obtained, which are uniform with respect to the parameter $\varepsilon$. These results are summarized in [4]. Note that generalized splines have been studied in a number of works, in particular in [5], [6], [7]. However, important questions of the convergence uniform in the parameter $\varepsilon$ of interpolation processes for such splines are very poorly investigated.

An interesting question is the approximation of not only the function, but also its derivatives. In this paper we estimate the error in the approximation of the derivatives of functions with large gradients in the boundary layer on the basis of the exponential spline. We assume that
the function is defined at the nodes of the uniform grid. We also study the limiting properties of exponential splines.

Introduce the notations. Let $\varepsilon \in (0, 1]$, $\Omega = \{x_n : x_n = nh, n = 0, 1, \ldots, N, \ x_0 = 0, x_N = 1\}$ be a uniform grid of the interval $[0, 1]$ with the step $h = 1/N$. By $C$ and $C_j$ we mean positive constants that do not depend on the parameter $\varepsilon$ and the step $h$. Suppose that $C[a, b]$ is the space of functions continuous on $[a, b]$ with the norm $\| \cdot \|_{C[a, b]}$. We denote by $C_h[0, 1]$ the linear normed space of functions from $C[0, 1]$ having one-sided second-order derivatives at the points $x = 0$ and $x = 1$, with the norm $\| u \|_{C, h} = \| u \|_{C[0, 1]} + h^2 (|u''(0)| + |u''(1)|)$.

2. Formulation of the problem and main results

Let us a function $u(x)$ be decomposed in the form of the sum of regular and singular components:

$$u(x) = q(x) + \gamma \Phi(x),$$

where

$$|q^{(j)}(x)| \leq C_1, \ 0 \leq j \leq 4, \ \Phi(x) = e^{-\alpha x/\varepsilon}, \ \alpha > 0, \ x \in [0, 1].$$

Assume that the regular component $q(x)$ and the constant $\gamma$ are not given in the representation (1), the function $\Phi(x)$ is known, its derivatives grow unboundedly at the boundary $x = 0$ if $\varepsilon \to 0$. The component $\gamma \Phi(x)$ is responsible for the large gradients of the function $u(x)$ in the boundary layer. Such a representation is valid for the solution of a singularly perturbed problem [2].

We assume that the function $u(x)$ is given at the nodes of the grid $\Omega, \ u_n = u(x_n), n = 0, 1, \ldots, N$.

Define the space of $L$-splines taking into account that the interpolated function $u(x)$ has the representation (1). Let

$$SL(\Omega, 3, 1) = \{S(x) \in C^2[0, 1] : S(x) = a_n + b_n x + c_n x^2 + d_n e^{-\alpha x/\varepsilon}, \ x \in [x_n, x_{n+1}], 0 \leq n \leq N-1\}.$$  

We define $L$-spline $S(x; u) \in SL(\Omega, 3, 1)$ for the function $u(x)$ from the interpolation conditions

$$S(x_n; u) = u(x_n), \ 0 \leq n \leq N, \ S''(0; u) = u''(0), \ S''(1; u) = u''(1).$$

**Theorem 1** For a function $u(x)$ having decomposition (1), there exists a unique interpolating spline $S(x; u) \in SL(\Omega, 3, 1)$ satisfying the conditions (2), and for some constant $C$ and $j = 0, 1, 2$ the following error estimates are valid

$$\| S^{(j)}(x; u) - u^{(j)}(x) \|_{C[0, 1]} \leq C \min \left\{ h^{3-j}, \frac{h^{4-j}}{\varepsilon} \right\}, \ \varepsilon \in (0, 1].$$

From (3) there follows the uniform in $\varepsilon \in (0, 1]$ convergence of the third order accuracy of the interpolation process (2) for functions of the form (1), as well as uniform convergence of the second and first orders for the first and second derivatives, respectively.

Let $S_3(x; u) \in C^2[0, 1]$ be a cubic spline with interpolation conditions (2). Similarly $S_2(x; u) \in C^1[0, 1]$ is a parabolic spline with interpolation conditions:

$$S_2(x_n; u) = u(x_n), \ 0 \leq n \leq N, \ S''_2(1 - 0; u) = u''(1).$$
**Theorem 2** For any function \( u(x) \in C_b[0,1] \) the interpolation splines \( S_2(x;u) \) and \( S_3(x;u) \) exist, they are unique, and for any fixed \( h \) the following formulas hold:

\[
\lim_{\varepsilon \to +\infty} \| S(x;u) - S_3(x;u) \|_{C[0,1]} = 0, \tag{4}
\]
\[
\lim_{\varepsilon \to 0+0} \| S(x;u) - S_2(x;u) \|_{C[0,1]} = 0. \tag{5}
\]

As a corollary of Theorem 2 we obtain the following result. If \( u(x) \in C^4[0,1] \), then for some constant \( M \) the following estimates hold:

\[
\| S_3(x;u) - u(x) \|_{C[0,1]} \leq M h^4, \tag{6}
\]
\[
\| S_2(x;u) - u(x) \|_{C[0,1]} \leq M h^3. \tag{7}
\]

Note that in (4)–(7) the form of the function \( u(x) \) is not related to the representation (1). The estimate (6) is well known, but for the parabolic spline the estimate (7) is an interesting, because in this case the nodes of the spline with interpolation nodes coincide.

For the convergence of the spline, a more accurate result is also true. Let us \( u(x) \in C_b[0,1] \). Then

\[
\lim_{\varepsilon/h \to +\infty} \| S(x;u) - S_3(x;u) \|_{C[0,1]} = 0. \tag{8}
\]

### 3. Proof of Theorems

The estimate (3) for \( j = 0 \) is given in [4] and is proved in [3]. It remains to prove the estimates of the error in the approximation of the derivatives.

We represent the spline \( S(x,u) \) in terms of basic splines:

\[
S(x;u) = \sum_{n=-3}^{N-1} \alpha_n N_{n,3}(x), \tag{8}
\]

where the exponential basis splines \( N_{n,3}(x) \) are constructed recurrently.

First we set \( N_{n,1}(x) \) :

\[
N_{n,1}(x) = \begin{cases} 
\frac{1-e^{\alpha(x_n-x)/\varepsilon}}{1-e^{-\alpha h/\varepsilon}}, & x \in [x_n, x_{n+1}), \\
\frac{e^{\alpha(x_{n+1}-x)/\varepsilon}-e^{-\alpha h/\varepsilon}}{1-e^{-\alpha h/\varepsilon}}, & x \in [x_{n+1}, x_{n+2}), \\
0, & x \notin [x_n, x_{n+2})
\end{cases} \tag{9}
\]

Exponential \( B \)-splines of higher orders are defined recurrently:

\[
N_{n,k+1}(x) = \frac{1}{h} \int_{x_n}^{x} (N_{n,k}(s) - N_{n+1,k}(s)) ds, \quad k = 1, 2. \tag{10}
\]

In accordance with [3], the following lemma holds.

**Lemma 1** Let the function \( u(x) \) be representable in the form (1). Then there exists a function \( \tilde{S}(x;u) \in SL(\Omega, 3, 1) \) and the constant \( C > 0 \) such that the following estimates are valid

\[
\| \tilde{S}^{(j)}(x;u) - u^{(j)}(x) \|_{C[0,1]} \leq C \min\{h^3-j, \frac{h^{4-j}}{\varepsilon}\}, \varepsilon \in (0, 1), \ 0 \leq j \leq 2. \tag{11}
\]
We introduce the function $\text{err}(x) = S(x; u) - \tilde{S}(x; u)$ and represent it in the form

$$\text{err}(x) = \sum_{n=-3}^{N-1} \beta_n N_{n,3}(x),$$

(12)

where $N_{n,k}(x)$ are normalized exponential $B$-splines corresponding to (9), (10), generators for each $k = 1, 2, 3$ partition of the unit [3]. We estimate the coefficients $\beta_n$, taking into account that $u(x)$ corresponds to (1). Let us consider two cases: 1) $h/\epsilon \leq C$ and 2) $h/\epsilon > C$, where $C$ is some constant.

In the first case the following estimate

$$\max_{-3 \leq n \leq N-1} |\beta_n| \leq C \max\{h^4/\epsilon, h^4\}$$

(13)
is proved [3].

In the second case according to [3]

$$\max_{-3 \leq n \leq N-1} |\beta_n| \leq C h^3.$$  

(14)

By virtue of (10), $N'_{i,k}(x) = \frac{1}{h}(N_{i,k-1}(x) - N_{i+1,k-1}(x))$. Therefore, the following relations hold:

$$\text{err}'(x) = \frac{1}{h} \sum_{n=-3}^{N-1} \beta_n (N_{n,2}(x) - N_{n+1,2}(x)),$$

(15)

$$\text{err}''(x) = \frac{1}{h^2} \sum_{n=-3}^{N-1} \beta_n (N_{n+2,1}(x) - 2N_{n+1,1}(x) + N_{n,1}(x)).$$

(16)

Since by the non-negativity of exponential $B$-splines and the fact that they form a partition of unity, for any $i$ will be $0 \leq N_{i,k}(x) \leq 1$. Then from (11), (13), (15), (16) we obtain the estimate (3) for $j = 1, 2$ in the first case. The estimate (3) for $j = 1, 2$ in the second case is obtained similarly with account of (14). This proves Theorem 1.

Now we dwell on the proof of Theorem 2. We represent the cubic spline $S_3(x; u)$ in terms of basic splines:

$$S_3(x; u) = \sum_{n=-3}^{N-1} g_n \tilde{N}_{n,3}(x).$$

(17)

First we set $\tilde{N}_{n,1}(x)$:

$$\tilde{N}_{n,1}(x) = \begin{cases} \frac{x-x_n}{x_{n+1}-x_n}, & x \in [x_n, x_{n+1}] \\ \frac{x_{n+2}-x}{x_{n+2}-x_{n+1}}, & x \in [x_{n+1}, x_{n+2}], -1 \leq n \leq N - 1 \\ 0, & x /\in [x_n, x_{n+2}] \end{cases}.$$  

(18)

Polynomial $B$-splines of higher orders are defined recurrently:

$$\tilde{N}_{n,k+1}(x) = \frac{1}{h} \int_{x_n}^{x} (\tilde{N}_{n,k}(s) - \tilde{N}_{n+1,k}(s)) ds, \quad k = 1, 2.$$  

(19)

From (9), (18) it follows that for any $n$

$$\lim_{\epsilon \to 0+0} \int_{0}^{1} |N_{n,1}(x) - \tilde{N}_{n,0}(x)| dx = 0, \quad \lim_{\epsilon \to +\infty} \| N_{n,1}(x) - \tilde{N}_{n,1}(x) \|_{C([0,1])} = 0.$$
Then, taking into account (10), (19) we obtain
\[
\lim_{\varepsilon \to 0+0} \| N_{i,k}(x) - \tilde{N}_{i,k-1}(x) \|_{C[0,1]} = 0, \quad \lim_{\varepsilon \to +\infty} \| N_{i,k}(x) - \tilde{N}_{i,k}(x) \|_{C[0,1]} = 0.
\]

So,
\[
\lim_{\varepsilon \to 0+0} N_{i,k}(x) = \tilde{N}_{i,k-1}(x), \quad \lim_{\varepsilon \to +\infty} N_{i,k}(x) = \tilde{N}_{i,k}(x), \quad x \in [0,1],
\]

Taking into account (8), (17), (20) we obtain that in the limit the coefficients \(\alpha_n\) and \(g_n\) satisfy the same system of equations if \(\varepsilon \to +\infty\). This proves that \(\lim_{\varepsilon \to +\infty} \alpha_n = g_n\). This proves (4).

The case \(\varepsilon \to 0\) is treated similarly. This proves Theorem 2.

4. Results of numerical experiments
We define the function of the form (1)
\[
u(x) = \cos \frac{\pi x}{2} + e^{-x}, \quad x \in [0,1].
\]

Table 1. Errors of the cubic spline on the uniform mesh

| \(\varepsilon\) | \(2^4\) | \(2^5\) | \(2^6\) | \(2^7\) | \(2^8\) | \(2^9\) |
|----------------|--------|--------|--------|--------|--------|--------|
| 1              | 2.82 \times 10^{-7} | 1.76 \times 10^{-8} | 1.16 \times 10^{-9} | 1.02 \times 10^{-10} | 4.30 \times 10^{-12} | 2.68 \times 10^{-13} |
| 10^{-1}        | 3.43 \times 10^{-4} | 2.33 \times 10^{-5} | 1.51 \times 10^{-6} | 9.58 \times 10^{-8} | 6.03 \times 10^{-9} | 4.11 \times 10^{-10} |
| 10^{-2}        | 0.43   | 8.38 \times 10^{-2} | 9.72 \times 10^{-3} | 8.00 \times 10^{-4} | 5.59 \times 10^{-5} | 3.65 \times 10^{-6} |
| 10^{-3}        | 9.88   | 4.58   | 1.93   | 0.66   | 0.15   | 2.03 \times 10^{-2} |
| 10^{-4}        | 1.05 \times 10^{2} | 5.23 \times 10^{1} | 2.58 \times 10^{1} | 1.25 \times 10^{1} | 5.90   | 2.59   |
| 10^{-5}        | 1.06 \times 10^{3} | 5.30 \times 10^{2} | 2.64 \times 10^{2} | 1.32 \times 10^{2} | 6.56 \times 10^{1} | 3.24 \times 10^{1} |
| 10^{-6}        | 1.06 \times 10^{4} | 5.30 \times 10^{3} | 2.65 \times 10^{3} | 1.33 \times 10^{3} | 6.62 \times 10^{2} | 3.30 \times 10^{2} |
| 10^{-7}        | 1.06 \times 10^{5} | 5.30 \times 10^{4} | 2.65 \times 10^{4} | 1.33 \times 10^{4} | 6.63 \times 10^{3} | 3.30 \times 10^{3} |
| 10^{-8}        | 1.06 \times 10^{6} | 5.30 \times 10^{5} | 2.65 \times 10^{5} | 1.33 \times 10^{5} | 6.63 \times 10^{4} | 3.31 \times 10^{4} |

The results of the calculations are summarized in four tables. The tables show the maximum errors in computing the function and its derivatives on the basis of splines, calculated at the nodes of the thickened grid obtained from the given computational grid by dividing each of its grid intervals into 10 equal parts.

In Table 1 are given the errors of cubic spline in the case of the function (21). The error increases without limit as the parameter \(\varepsilon\) decreases. This shows that the use of a cubic spline on a uniform grid is not effective in the presence of a boundary layer.

Next, we investigate the accuracy of the approximation of a function and its derivatives on the basis of the exponential spline. Errors and calculated orders of accuracy of the exponential spline for the function (1) are given in the Table 2. In Table 3 are given the errors and calculated accuracy orders for the first derivative of the function (21), and in Table 4 the corresponding results for second derivative. The numerical results from Tables 2, 3 and 4 are in agreement with the estimates of Theorem 1.
Table 2. Errors and calculated accuracy orders for the exponential spline

| $\varepsilon$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ |
|----------------|-------|-------|-------|-------|-------|-------|
| 1              | 1.56·10^{-4} | 9.75·10^{-6} | 6.05·10^{-7} | 3.77·10^{-8} | 2.36·10^{-9} | 1.47·10^{-10} |
| 10^{-1}       | 1.25·10^{-3} | 7.18·10^{-5} | 4.17·10^{-6} | 2.50·10^{-7} | 1.53·10^{-8} | 9.46·10^{-10} |
| 10^{-2}       | 3.73·10^{-3} | 4.33·10^{-4} | 2.22·10^{-5} | 1.30·10^{-6} | 7.81·10^{-7} | 4.08·10^{-8} |
| 10^{-3}       | 3.83·10^{-3} | 4.83·10^{-4} | 6.02·10^{-5} | 7.43·10^{-6} | 8.62·10^{-7} | 9.21·10^{-8} |
| 10^{-4}       | 3.95·10^{-3} | 4.83·10^{-4} | 6.05·10^{-5} | 7.57·10^{-6} | 9.46·10^{-7} | 1.18·10^{-7} |
| 10^{-5}       | 3.96·10^{-3} | 4.86·10^{-4} | 6.06·10^{-5} | 7.57·10^{-6} | 9.46·10^{-7} | 1.18·10^{-7} |
| 10^{-6}       | 3.97·10^{-3} | 4.86·10^{-4} | 6.06·10^{-5} | 7.57·10^{-6} | 9.46·10^{-7} | 1.18·10^{-7} |
| 10^{-7}       | 3.97·10^{-3} | 4.86·10^{-4} | 6.06·10^{-5} | 7.57·10^{-6} | 9.46·10^{-7} | 1.18·10^{-7} |
| 10^{-8}       | 3.96·10^{-3} | 4.86·10^{-4} | 6.06·10^{-5} | 7.57·10^{-6} | 9.46·10^{-7} | 1.18·10^{-7} |

Table 3. Errors and calculated accuracy orders for the first derivative of the function (21)

| $\varepsilon$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ |
|----------------|-------|-------|-------|-------|
| 1              | 2.36·10^{-4} | 2.91·10^{-4} | 3.61·10^{-5} | 4.49·10^{-6} |
| 10^{-1}       | 1.53·10^{-2} | 1.86·10^{-3} | 2.23·10^{-4} | 2.71·10^{-5} |
| 10^{-2}       | 6.29·10^{-2} | 1.27·10^{-2} | 2.21·10^{-3} | 2.89·10^{-4} |
| 10^{-3}       | 7.76·10^{-2} | 1.9183·10^{-2} | 4.58·10^{-3} | 1.04·10^{-3} |
| 10^{-4}       | 7.32·10^{-2} | 2.02     | 2.07     | 2.14     |
| 10^{-5}       | 7.95·10^{-2} | 2.01·10^{-2} | 5.00·10^{-3} | 1.24·10^{-3} |
| 10^{-6}       | 7.95·10^{-2} | 2.01·10^{-2} | 5.04·10^{-3} | 1.26·10^{-3} |
| 10^{-7}       | 7.95·10^{-2} | 2.01·10^{-2} | 5.04·10^{-3} | 1.26·10^{-3} |
| 10^{-8}       | 7.95·10^{-2} | 2.01·10^{-2} | 5.04·10^{-3} | 1.26·10^{-3} |
Table 4. Errors and calculated accuracy orders for the second derivative of the function (21).

| ε       | 2^2   | 2^3   | 2^4   | 2^5   | 2^6   | 2^7   |
|---------|-------|-------|-------|-------|-------|-------|
| 1       | 4.55 · 10^{-2} | 1.08 · 10^{-2} | 2.62 · 10^{-3} | 6.42 · 10^{-4} | 1.58 · 10^{-4} | 3.95 · 10^{-5} |
| 10^{-1} | 2.60 · 10^{-1} | 6.72 · 10^{-2} | 1.66 · 10^{-2} | 4.08 · 10^{-3} | 1.01 · 10^{-3} | 2.51 · 10^{-4} |
| 10^{-2} | 7.87 · 10^{-1} | 3.35 · 10^{-1} | 1.33 · 10^{-1} | 4.03 · 10^{-2} | 1.04 · 10^{-2} | 2.57 · 10^{-3} |
| 10^{-3} | 9.05 · 10^{-1} | 4.63 · 10^{-1} | 2.26 · 10^{-1} | 1.07 · 10^{-1} | 4.74 · 10^{-2} | 1.87 · 10^{-2} |
| 10^{-4} | 9.18 · 10^{-1} | 4.76 · 10^{-1} | 2.40 · 10^{-1} | 1.19 · 10^{-1} | 5.90 · 10^{-2} | 2.88 · 10^{-2} |
| 10^{-5} | 9.20 · 10^{-1} | 4.78 · 10^{-1} | 2.41 · 10^{-1} | 1.21 · 10^{-1} | 6.03 · 10^{-2} | 3.01 · 10^{-2} |
| 10^{-6} | 9.20 · 10^{-1} | 4.78 · 10^{-1} | 2.41 · 10^{-1} | 1.21 · 10^{-1} | 6.05 · 10^{-2} | 3.02 · 10^{-2} |
| 10^{-7} | 9.20 · 10^{-1} | 4.78 · 10^{-1} | 2.41 · 10^{-1} | 1.21 · 10^{-1} | 6.05 · 10^{-2} | 3.02 · 10^{-2} |

5. Conclusion

The problem of the interpolation of functions with large gradients in an exponential boundary layer is investigated. On the uniform grid the exponential spline which is exact on the boundary layer component of the interpolated function is constructed. For the constructed spline error estimates are obtained that are uniform with respect to a small parameter. This advantageously distinguishes the constructed spline from the cubic spline, whose error increases with decreasing of perturbation parameter. It is proved that the exponential spline turns into cubic if the perturbing parameter tends to infinity, and parabolic if this parameter tends to zero.

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