Scalarons mimicking Dark Matter in the Hu-Sawicki model of $f(R)$ gravity

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In this paper, we conduct a study on the scalar field obtained from $f(R)$ gravity via Weyl transformation of the spacetime metric $g_{\mu\nu}$ from the Jordan frame to the Einstein frame. The scalar field is obtained as a result of the modification in the geometrical part of Einstein’s field equation of General Relativity. For the Hu-Sawicki model of $f(R)$ gravity, we find the effective potential of the scalar field and calculate its mass. Our study shows that the scalar field (also named as scalaron) obtained from this model has the chameleon property, i.e. the scalaron becomes light in the low-density region while it becomes heavy in the high-density region of matter. Then it is found that the scalaron can be regarded as a dark matter (DM) candidate since the scalaron mass is found to be quite close to the mass of ultralight axions, a prime DM candidate. Thus the scalaron in the Hu-Sawicki model of $f(R)$ gravity behaves as DM. Further, a study on the evolution of the scalaron mass with the redshift is also carried out, which depicts that scalaron becomes light with expansion of the Universe and with different rates at different stages of the Universe.

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I. INTRODUCTION

One of the major challenges of present day physics and cosmology is uncovering the nature of mysterious dark matter (DM) [1–5]. The first prediction of the existence of DM dates back to the early 1930s when J. H. Oort first postulated that more mass must be contained within the Milky Way galaxy over and above the visible amount to hold the stars in their respective orbits [6]. Then came Swiss astronomer Fritz Zwicky, who studied the Coma Cluster and found similar evidence of missing mass within the cluster [7, 8]. Two most prominent pieces of evidences that argue about the existence of DM are namely, galactic rotation curve [9–11] and gravitational lensing [10, 12]. Various other evidences that support for non-luminous matter existence include the large mass-to-light ratio in galaxy clusters [13], very high x-ray luminosity of the Bullet Cluster (1E0657–558) [14], density parameters obtained from the observations of distant type Ia supernovae [15, 16], etc. The Planck satellite observations of the Cosmic Microwave Background (CMB) radiation [17] have also provided fascinating evidence for the existence of DM through the determination of the cosmological parameters. These observations have also confirmed the standard $\Lambda$CDM cosmological paradigm by showing that baryonic matter alone cannot explain the mass content of the Universe. Observations of the Bullet Cluster also provide strong evidence for the existence of DM. In this cluster, the baryonic matter and the DM components are separated due to a long-ago collision of its two components [14, 18]. Using the Planck data [17] on the CMB radiation, measurements of the cosmological parameters imply that the Universe is composed of $\sim 4 - 5\%$ baryons, $\sim 25\%$ non-baryonic DM, and $\sim 70\%$ dark energy. A few particles have been claimed as DM candidates [19–21]. Some of them are, namely, weakly interacting massive particles (WIMPs), standard model (SM) neutrinos, sterile neutrinos, axions, supersymmetric candidates (neutralinos, sneutrinos, gravitinos, axinos), etc. WIMPs are non-baryonic, which includes lightest SUSY particles, specially the neutralino, and it is considered as the most probable candidate of DM. Axions, which are also prime candidates of DM, are bosons that were first proposed to solve the strong CP problem. However, after almost eight decades since the DM concept was introduced, the DM particle is still missing from the table of elementary particles of nature, i.e. the fundamental nature of DM remains a mystery.

In the last three decades various issues and consequent limitations (specially, the limitation related to the late time cosmic acceleration [15, 16]) of Einstein’s General Theory of Relativity (GTR) have come to light, leading to the conclusion that GTR is not the ultimate theory of gravitational interaction. Theories of modified gravity [22] were proposed so that the gravitational interaction other than the one described by GTR, could be justified. In modified gravity models, we have $f(R)$ gravity models, Braneworld models, Gauss-Bonnet dark energy models, etc. [23–26]. Out of these, the simplest class of modified gravity theories is the $f(R)$ gravity [27, 28]. Here, modification occurs in the part describing the geometry of Einstein’s field equation. It is modified by replacing the Ricci scalar $R$ of the Einstein-Hilbert action with a function $f(R)$ of $R$. There are two variational approaches to derive the field equations in $f(R)$ models, (i) metric formalism and (ii) palatini formalism. In metric formalism, matter is minimally coupled with the metric, and the energy-momentum tensor is independently conserved. In Palatini formal-
The formation of large-scale structure in the Universe dominated by DM and driven to accelerated expansion by $f(R)$ gravity is studied in Ref. [36]. These studies have motivated us to take into consideration of modified theories of gravity to understand DM. In our work, we discuss the DM problem in $f(R)$ gravity and modify field equation. We apply the metric formalism in $f(R)$ gravity to explain DM. The scalar field plays an important role in $f(R)$ gravity. This scalar field also called scalaron is derived from the modification of gravitational theory. We consider the modification of the Einstein-Hilbert action [39] as it is one of the few known viable models within the framework of modified gravity which is able to satisfy solar system tests.

Our paper is organized in six sections. In the section II, we discuss the simplest type of modified gravity, i.e. $f(R)$ gravity using metric formalism. Here, we obtain the modified field equations. The Weyl transformation from the Jordan frame to the Einstein frame adds an extra degree of freedom. Then we obtain the potential of the scalaron which helps us to calculate the effective potential of the scalaron. In the section III, we discuss the simplest type of modified gravity with matter field $\Phi$. Here, we apply the metric formalism. So, variation of the action (1) with respect to the metric $g_{\mu\nu}$ leads to the equation of motion,

$$\left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + f_R(R) R_{\mu\nu} - g_{\mu\nu} \left[ \frac{f(R)}{2} - \Box f_R(R) \right] - \nabla_{\mu} \nabla_{\nu} f_R(R) = \kappa^2 T_{\mu\nu},$$

(2)

where $f_R(R)$ is the derivative of $f(R)$ with respect to $R$. And, the energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} [g^{\mu\nu}, \Phi] = \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m [g^{\mu\nu}, \Phi])}{\delta g_{\mu\nu}}.$$  

(3)

The term within the parentheses in Eq. (2) is the Einstein tensor and hence remaining terms together on the left side give the modified part of the Einstein’s field equation. Now, the trace of Eq. (2) is given as

$$3 \Box f_R(R) - R + R f_R(R) - 2 f(R) = \kappa^2 T^{\mu}_{\mu},$$

(4)

which shows that the Ricci scalar $R$ becomes dynamical to be present as a new scalar degree of freedom if $f(R) \neq R$. In GTR, where $f(R) = R$, Eq. (4) simply leads to the trivial solution $R = -\frac{4}{3} \kappa^2 T^{\mu}_{\mu}$. In a useful form this equation can be written as

$$\Box f_R(R) = \frac{1}{3} \left[ R + 2 f(R) - R f_R(R) + \kappa^2 T^{\mu}_{\mu} \right].$$

(5)

This equation is analogous to the scalar field equation of Klein-Gordon type in a potential if we consider that $f_R(R)$ is the extra scalar degree of freedom of $f(R)$ gravity.
B. Weyl Transformation

To study the dynamics of the new scalar field, we require the Weyl transformation. It is the frame transformation from the Jordan frame $g_{\mu\nu}$ to Einstein frame $\tilde{g}_{\mu\nu}$ [41], which we consider in the form:

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{\sqrt{2} \kappa \phi} g_{\mu\nu} \equiv (1 + f_R(R)) g_{\mu\nu},$$

(6)

where $\phi$ is the new scalar field named as scalaron. It is to be noted that via this transformation the Ricci scalar $R$ can be expressed as a function of the scalaron field $\phi$. Our intention is to obtain the field equations in the Einstein frame to study the effect of the scalaron as dark matter. For this purpose we have to write the action (1) in the Einstein frame using this transformation Eq. (6). However, before doing so, let us rewrite the action (1) as given by

$$S = \int d^4x \sqrt{-g} \left[ R + R f_R(R) \frac{1}{2\kappa^2} - U(R) \right] + \int d^4x \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m[g^{\mu\nu}, \Phi],$$

(7)

where

$$U(R) = \frac{1}{2\kappa^2} \left[ R f_R(R) - f(R) \right],$$

(8)

which can be treated as the potential of a new supplementary scalar field in the scalar-tensor equivalence of $f(R)$ gravity [42, 43] in Jordan frame itself [42]. This means that there is an extra scalar degree of freedom of $f(R)$ gravity in Jordan frame as already specified by the trace Eq. (4). Later we will see that $U(R)$ is related with the scalar field potential in the Einstein frame.

Now, under the Weyl transformation (6), the Ricci scalar $\tilde{R}$ in Jordan frame and $\hat{R}$ in Einstein frame is related by the equation:

$$R = \Omega^2 \left[ \hat{R} + 6 \Box \ln \Omega - 6 \tilde{g}^{\mu\nu} \nabla_\mu (\ln \Omega) \nabla_\nu (\ln \Omega) \right],$$

(9)

where $\Omega^2 \equiv 1 + f_R(R)$ is the conformal factor. Thus, using the transformation (6) and the consequent relation (9), the original action (1) can be written in Einstein frame as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \hat{R} + \int d^4x \sqrt{-\tilde{g}} \left[ -\frac{1}{2} \tilde{g}^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - V(\phi) \right] + \int d^4x \sqrt{-\tilde{g}} e^{-2\sqrt{2} \kappa \phi} \tilde{\mathcal{L}}_m[g^{\mu\nu}, \Phi],$$

(10)

where

$$V(\phi) = \frac{U(\phi)}{\Omega^2} = \frac{1}{2\kappa^2} \frac{R f_R(R) - f(R)}{(1 + f_R(R))^2}$$

(11)

is the potential of the scalaron field $\phi$. Since, here $R = R(\phi)$, so in view of this we may consider that $U(R) \equiv U(\phi)$. This also justifies our conjecture mentioned above related with it in Eq. (8). Hence, as usual Eq. (11) also shows the relationship of the scalar field potentials in Jordan frame and Einstein frame.

The equation of motion for the scalaron field is obtained by the variation of Eq. (10) with respect to the scalaron field $\phi$, which is given as

$$\sqrt{-\tilde{g}} \left[ \Box \phi - V(\phi) \right] + \frac{\delta}{\delta \phi} \left( \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m[g^{\mu\nu}, \Phi] \right) = 0,$$

(12)

where

$$\Box = \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu (\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu).$$

(13)

Since,

$$\delta \tilde{g}^{\mu\nu} = \frac{2\kappa}{\sqrt{6}} e^{\sqrt{2} \kappa \phi} \delta \phi \tilde{g}^{\mu\nu} = \frac{2\kappa}{\sqrt{6}} g^{\mu\nu} \delta \phi,$$

(14)

So, we can have

$$\frac{\delta}{\delta \phi} = \frac{2\kappa}{\sqrt{6}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}}.$$

(15)
Substituting Eq. (15) in Eq. (12), we get

\[ \Box \phi = V(\phi)_{,\phi} + \frac{\kappa}{\sqrt{6}} e^{-2\sqrt{2} \phi} T^\mu_\mu. \]  

(16)

This equation is similar to the Klein-Gordon equation for a scalar field with an effective potential and it corresponds to Eq. (5) in Jordan frame as mentioned above. Thus, we may rewrite this equation as

\[ \Box \phi = \frac{dV_{eff}(\phi)}{d\phi}, \]  

(17)

where

\[ \frac{dV_{eff}(\phi)}{d\phi} = V(\phi)_{,\phi} + \frac{\kappa}{\sqrt{6}} e^{-2\sqrt{2} \phi} T^\mu_\mu. \]  

(18)

Integrating this equation, we obtain the effective potential of the scalaron as

\[ V_{eff}(\phi) = V(\phi) - \frac{1}{4} e^{-2\sqrt{2} \phi} T^\mu_\mu. \]  

(19)

It is to be noted that the effective potential of scalaron includes the trace of the energy-momentum tensor \( T^\mu_\mu \), which indicates that the matter distribution has an important role to play on the potential of the scalaron. It turns out that the effective mass of scalaron depends on the type of matter distribution and this is a necessary requirement for scalaron to exhibit the chameleon mechanism, which will be clear from the next sections.

### III. SCALARON MASS

Here, we will derive the mass of scalaron taking into account the effect of the matter distribution \( T^\mu_\mu \) from the extremum condition of the effective potential. For this purpose we first calculate the derivative of the effective potential as

\[ V_{eff}(\phi)_{,\phi} = V(\phi)_{,\phi} + \frac{\kappa}{\sqrt{6}} e^{-2\sqrt{2} \phi} T^\mu_\mu \]

\[ = \frac{1}{\sqrt{6\kappa}} \left[ \frac{R(1 - f_R(R)) + 2f(R) + \kappa^2 T^\mu_\mu}{(1 + f_R(R))^2} \right]. \]  

(20)

It would be worthwhile to mention that the comparison of this equation with Eq. (5) justifies our earlier conclusion about the later equation in relation with Eq. (16). This, in fact, is the result of the correspondence of physical situations between the Jordan frame and the Einstein frame.

The effective potential has an extremum at \( \phi = \phi_{min} \). So at this situation,

\[ V_{eff}(\phi)_{,\phi} \big|_{\phi = \phi_{min}} = 0. \]  

(21)

Applying this condition to Eq. (20) through the Weyl transformation, we get,

\[ R_0 - R_0 f_R(R_0) + 2f(R_0) + \kappa^2 T^\mu_\mu = 0, \]  

(22)

where \( R_0 \) is the value of \( R \) corresponding to \( \phi = \phi_{min} \). This equation can be used to find the value of \( R_0 \) for a given matter distribution. Now, the second derivative of the effective potential is obtained as

\[ V_{eff}(\phi)_{,\phi\phi} = V(\phi)_{,\phi\phi} - \frac{2\kappa^2}{3} e^{-2\sqrt{2} \phi} T^\mu_\mu \]  

\[ = \frac{R f_R(R) - 3R - 4f(R)}{3(1 + f_R(R))^2} + \frac{1}{3f_R(R)} - \frac{2}{3} \frac{\kappa^2 T^\mu_\mu}{(1 + f_R(R))^2}. \]  

(23)

At extremum, the second derivative of the effective potential gives us the square of the scalaron mass. Thus using Eq. (22) in this Eq. (23) the scalaron mass square can be obtained as follows:

\[ m_{\phi}^2 = V_{eff}(\phi)_{,\phi\phi} \big|_{\phi = \phi_{min}} \]

\[ = \frac{1}{3(1 + f_R(R_0))} \left[ 1 + f_R(R_0) \right] f_R(R_0) - R_0 \]  

(24)
In the high curvature regime, where $|f_R(R)| \ll 1$ and $|f / R| \ll 1$, this mass square term can be written as

$$m_\phi^2 \approx \frac{1}{3} \left[ \frac{1 + f_R(R_0)}{f_{RR}(R_0)} - R_0 \right].$$

(25)

Although this equation looks independent of matter distribution, i.e. of $T^\mu_\mu$, but as we have already seen that $R_0$ is determined by Eq. (22) as a function of the trace of the energy-momentum tensor $T^\mu_\mu$. Hence, the scalaron mass changes according to the trace of the energy-momentum tensor $T^\mu_\mu$, i.e. with the matter distributions.

IV. Hu-Sawicki $f(R)$ Gravity and Chameleon Mechanism

To study the chameleon mechanism in the behaviour of scalaron field, we will use the Hu-Sawicki $f(R)$ gravity model. The Hu-Sawicki model of $f(R)$ gravity was proposed by Wayne Hu and Ignacy Sawicki in 2007 [38]. It represents one of the few known viable functional forms of $f(R)$ gravity with the interesting feature of being able to satisfy solar system tests. This model is introduced to explain the current situation of the accelerating Universe without taking into account a cosmological constant term. In this model,

$$f(R) = -m^2 \frac{c_1(R/m^2)^n}{c_2(R/m^2)^n + 1},$$

(26)

where $n > 0$, $c_1$ and $c_2$ are two dimensionless model parameters. The parameter $m^2$ represents the mass (energy) scale and for our case we will consider it equivalent to the present energy scale of the Universe, which is the current observed value of the cosmological constant $\Lambda$ [44]. In the high curvature regime, where $R \gg m^2$, Eq. (26) can be simplified as

$$\lim_{m^2/R \to 0} f(R) \approx -\frac{c_1}{c_2} m^2 + \frac{c_1}{c_2} m^2 \left(\frac{m^2}{R}\right)^n.$$

(27)

Therefore, we can have

$$f_R(R) = -\frac{c_1}{c_2} m^2 \left(\frac{m^2}{R}\right)^n+1,$$

(28)

and

$$f_{RR}(R) = \frac{n(n + 1)}{R} \frac{c_1}{c_2} \left(\frac{m^2}{R}\right)^n+1.$$  

(29)

Substituting Eqs. (28) and (29) into Eq. (25) for $R = R_0$ we may obtain the expression for the mass square of scalaron in the Hu-Sawicki model of $f(R)$ gravity, which is given by

$$m_\phi^2 = \frac{1}{3} \frac{R_0}{n(n+1)} \left[ 1 - n(n+2) \frac{c_1}{c_2} \left(\frac{m^2}{R_0}\right)^n+1 \right].$$

(30)

Similarly, substituting these equations into Eq. (22), the equation to determine $R_0$ from the condition of minimum of the potential for this model is found as

$$R_0 + (n - 2) \frac{c_1}{c_2} m^2 \left(\frac{m^2}{R_0}\right)^n - 2 \frac{c_1}{c_2} m^2 + \kappa^2 T^\mu_\mu = 0.$$  

(31)

In high-curvature regime $R \gg m^2$, the above equation gives,

$$R_0 \approx -\kappa^2 T^\mu_\mu + 2 \frac{c_1}{c_2} m^2.$$  

(32)

If we assume the pressureless ($\rho = 0$) dust model of the Universe, then the matter contribution is approximately expressed as $T^\mu_\mu = -\rho$, where $\rho$ is the matter-energy density of the Universe. Hence, for this model of the Universe Eq. (32) takes the form:

$$R_0 \approx \kappa^2 \rho + 2 \frac{c_1}{c_2} m^2.$$  

(33)
FIG. 1: Variations of scalaron mass $m_{\phi}$ with energy density $\rho$ for the model parameters $c_1 = 2.23$ and $c_2 = 0.08$ [45], but for three different values of the parameter $n$. This set of values of $c_1$ and $c_2$ are used in all other figures wherever necessary.

Again Eq. (30) can be rewritten as

$$m_{\phi}^2 = \frac{m^2 c_2}{3n(n+1)c_1} \left( \frac{R_0}{m^2} \right)^{n+2} - \frac{(n+2)R_0}{3(n+1)}.$$  \hspace{1cm} (34)

As we already know that in high-curvature regime, $R/m^2 \gg 1$, so in this case the second term on the R.H.S. of above equation becomes negligible in comparison of the first, and hence for this case we may write,

$$m_{\phi}^2 \approx \frac{m^2 c_2}{3n(n+1)c_1} \left( \frac{R_0}{m^2} \right)^{n+2}.$$ \hspace{1cm} (35)

Finally, substituting the value of $R_0$ obtained from Eq. (33) in Eq. (35), we get the expression of the scalaron mass square in terms of matter density of the Universe and the Hu-Sawicki model parameters as given by

$$m_{\phi}^2 = \frac{\kappa^2 \rho + 2 \frac{c_1}{c_2} m^2}{3n(n+1)\frac{1}{c_2} m^{2(n+1)}}.$$ \hspace{1cm} (36)

It should be noted at this point that the best-fit values of the model parameters $c_1$ and $c_2$ are obtained from various observational tests as discussed in the Ref. [45]. However, from our study we found that the values of $c_1$ and $c_2$ have to be $> 0$, otherwise the results become undefined. Hence, for our rest of the study we will use only one particular set of best-fit values of $c_1$ and $c_2$, i.e. $c_1 = 2.23$ and $c_2 = 0.08$ obtained from the observational $H(z)$ test [45].

From Eq. (36), we can see that the mass of the scalaron behaves as a monotonically increasing function of the matter density $\rho$ as depicted in Fig. 1. This means that the scalaron becomes heavy in high-density region, whereas it becomes light in the low-density region of matter. This particular behaviour exhibited by the scalaron field is one of the screening mechanisms, which is known as the chameleon mechanism. In Fig. 1, we have chosen three values of $n$ keeping the value of $c_1 = 2.23$ and $c_2 = 0.08$ according to our consideration as mentioned above to get a comparative assessment of the way the scalaron mass changes with respect to the matter density $\rho$ for different values of the parameter $n$. It is seen that for each ascending value of $n$, the scalaron mass increases rapidly.

A. Singularity Problem

There is a singularity problem in the Hu-Sawicki model of $f(R)$ gravity. To understand this problem that arises in the model we first consider the scalaron potential without taking into account the matter contribution. The scalaron potential is already obtained as an explicit function of the Ricci scalar $R$, which is given by Eq. (11). Using the Weyl transformation (6), we may obtain the scalaron potential and effective potential as an explicit function of the scalaron field $\phi$. Thus, the scalaron potential
given by Eq. (11) can be expressed in terms of $\phi$ for the Hu-Sawicki model as

\[
\frac{V(\phi)}{V_0} = \frac{-n \frac{c_1}{c_2} - n \frac{c_1}{c_2} \left(1 - e^{\sqrt{2/3} \kappa \phi} \right) \frac{n}{m^2} - \frac{n}{c_2} + c_1 \left(1 - e^{\sqrt{2/3} \kappa \phi} \right) \frac{n}{c_2} - \frac{n}{c_2}}{e^{2\sqrt{2/3} \kappa \phi} \left(1 - e^{-\sqrt{2/3} \kappa \phi} \right) + 1},
\]

(37)

where $V_0 = m^2/2\kappa^2$ is the normalization factor and the Weyl conformal transformation (6) gives the relation between $R$ and $\phi$ as

\[
e^{\sqrt{2/3} \kappa \phi} = 1 - \frac{c_1}{c_2} n \left(\frac{m^2}{R}\right)^{n+1}.
\]

(38)

In this relation we have used Eq. (28).

![Variations of scalaron potential $V(\phi)$ for the Hu-Sawicki model as a function of scalaron field $\phi$. The left panel is for $n = 1.3$, while the right panel is for $n = 1.8$ and 2.3.]

FIG. 2: Variations of scalaron potential $V(\phi)$ for the Hu-Sawicki model as a function of scalaron field $\phi$. The left panel is for $n = 1.3$, while the right panel is for $n = 1.8$ and 2.3.

The behaviour of scalaron potential as a function of the scalaron field for the Hu-Sawicki model given in the above Eq. (37) is depicted in Fig. 2. In the first panel of this figure, the variation of the potential with the field is shown exclusively for the parameters chosen as $n = 1.3$, $c_1 = 2.23$ and $c_2 = 0.08$. As seen from this plot, the potential decreases moderately with the increasing values of $\kappa \phi$ from the negative side and becomes minimum at $\kappa \phi_{\text{min}} \sim \kappa \phi_{\text{max}} \sim 0.05$. After that the potential rises rapidly to terminate itself at $\kappa \phi = 0$ with a particular higher value. Again, we have studied the effect of variation of values of $n$ on the potential $V(\phi)$ for a fixed value of $c_1$ and $c_2$ as shown in the right panel of this figure. This panel is for $n = 1.3, 1.8, 2.3$ with values of $c_1$ and $c_2$ as in the case of the first panel. It is seen from this panel that as the parameter $n$ is varied to higher values, the scalaron potential increases rapidly from that for smaller values of $n$, but it descends to a particular value at $\kappa \phi = 0$ for all values of $n$. It is to be noted that in both of these two cases no potential exists for the positive values of the field.

Next, we add the matter contribution term to study the effective potential of the scalaron field. From Eq. (19) the normalized effective potential of the scalaron field is obtained as

\[
\frac{V_{\text{eff}}(\phi)}{V_0} = \frac{V(\phi)}{V_0} - \frac{\kappa^2 T^\mu_\mu}{2m^2 e^{2\sqrt{2/3} \kappa \phi}}.
\]

(39)

This effective potential is calculated for the Hu-Sawicki model using the pressureless dust model of the Universe as mentioned earlier. The result of this calculation is depicted in Fig. 3, which is independent of different model parameter values except $m^2$. Here, it is seen that by considering the matter contribution as a positive value of matter-energy density, the potential could be slightly shifted from its original minimum and also made to behave as an almost smoothly falling function of the field. Hence, the minimum of the effective potential occurs at closer to zero of the field value. We find that the minimum of the effective potential occurs so close to zero of the scalaron field that it can easily go to zero at the minimum. This indicates that the curvature singularity can be readily attained because $R \to \infty$ for $\phi \to 0$ as can be seen from Eq. (38). This is the so-called curvature singularity problem \[46\] and the infrared modified viable $f(R)$ gravity models usually experience this problem.
FIG. 3: Variation of effective scalaron potential as a function of scalaron field for \(-T_\mu^\mu \sim \rho_{\text{crit}}\). This potential is found to be independent of Hu-Sawicki model parameters \(n, c_1\) and \(c_2\).

B. Higher Order Correction

As can be seen from the previous subsection, the singularity problem arises in the Hu-Sawicki model in the large curvature regime. Hence, it is assumed that this problem can be mended by improving the potential structure of the scalaron field in the high curvature region. A well-known way to deviate the singularity problem is to add the higher curvature term \([47]\). In \(f(R)\) gravity, the most suitable higher-curvature correction term is \(\alpha R^2\), where \(\alpha\) is a dimensionful constant parameter. With this correction term the Hu-Sawicki model (26) can be modified as

\[
f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} + \alpha R^2.
\] (40)

In the high curvature limit \(R \gg m^2\), this Eq. (40) can be rewritten as

\[
f(R) \approx -\frac{c_1}{c_2} m^2 + \frac{c_1}{c_2} m^2 \left(\frac{m^2}{R}\right)^n + \alpha R^2.
\] (41)

FIG. 4: Variation of scalaron potential \(V(\phi)\) as a function of scalaron field for the Hu-Sawicki model with the \(R^2\) correction. The parameter \(\alpha\) for this plot is taken as \(\alpha = 10^{54}\) GeV\(^{-2}\). This value of \(\alpha\) is used for the subsequent figures if not mentioned otherwise.

Similarly, with this correction term in the large curvature regime, Eq. (38) which gives the relation between \(R\) and \(\phi\) can be
modified as
\[ e^{\sqrt{2/3}k\phi} = 1 - \frac{c_1}{c_2} \frac{1}{\alpha} (\frac{m^2}{R})^{n+1} + 2\alpha R. \] (42)

Therefore, for the \( R^2 \) corrected Hu-Sawicki model (40), the scalaron potential is obtained as
\[ \frac{V(\phi)}{V_0} = \frac{c_1}{c_2} + \left( e^{\sqrt{2/3}k\phi} - 1 \right)^2 / 4\alpha m^2. \] (43)

The behaviour of this corrected potential \( V(\phi) \) with respect to the scalaron field is shown in Fig. 4 for the parameters \( c_1 = 2.23, c_2 = 0.08 \) and \( \alpha = 10^{14} \text{ GeV}^{-2} \). From this figure it is seen that for the positive value of \( \phi \), the potential is modified to become finite in the large curvature region. The reason for choosing this particular large value of \( \alpha \) is mentioned below.

![Variations of effective scalaron potential as a function of scalaron field for the Hu-Sawicki model with the \( R^2 \) correction are drawn for contributions of two matter distributions: \( -T_\mu^\mu \sim 10^{18} \rho_{\text{crit}} \) and \( 10^{27} \rho_{\text{crit}} \).](image)

FIG. 5: Variations of effective scalaron potential as a function of scalaron field for the Hu-Sawicki model with the \( R^2 \) correction are drawn for contributions of two matter distributions: \( -T_\mu^\mu \sim 10^{18} \rho_{\text{crit}} \) and \( 10^{27} \rho_{\text{crit}} \).

Subsequently, we deduce the effective potential of the scalaron field for the \( R^2 \) corrected model (40). As earlier, in this case also the effective potential is obtained in the form:
\[ \frac{V_{\text{eff}}(\phi)}{V_0} = \frac{V(\phi)}{V_0} - \frac{\kappa^2 T_\mu^\mu}{2m^2 e^{2\sqrt{2/3}k\phi}}. \] (44)

The variation of this effective potential as a function of \( k\phi \) is shown in Fig. 5 for two matter distributions, viz., \( -T_\mu^\mu \sim 10^{18} \rho_{\text{crit}} \) and \( 10^{27} \rho_{\text{crit}} \). It is seen that with the increasing matter distribution to \( 10^{27} \rho_{\text{crit}} \) the minimum of the effective potential rises substantially from its approximately zero level for the matter distribution \( 10^{18} \rho_{\text{crit}} \). It is also noticed that the matter distribution \( 10^{18} \rho_{\text{crit}} \) has no significant effect on the potential as the variation of the effective potential with this matter distribution is almost similar to the plot of Fig. 4. Fig. 6 is drawn to show explicitly the matter contribution to the effective potential (44) with the matter distribution \( 10^{27} \rho_{\text{crit}} \). Moreover, we see from Fig. 5 and 6 that \( k\phi_{\text{min}} \sim 1 \) is achieved when \( -T_\mu^\mu \sim 10^{27} \rho_{\text{crit}} \). However, the value of \( T_\mu^\mu \) for obtaining \( k\phi_{\text{min}} \sim 1 \) as well as the level of raising the minimum of the effective potential depends on the parameter values chosen. Although the value \( \alpha = 10^{14} \text{ GeV}^{-2} \) is very large, this value of \( \alpha \) is chosen so that the effect of matter contribution can be realised. Again, as we go on decreasing the value of \( \alpha \), the value of \( k\phi_{\text{min}} \) also becomes smaller. As discussed in Ref. [41], the background effect can be ignored if \( k\phi_{\text{min}} \) is small enough. Therefore, to obtain a smaller \( k\phi_{\text{min}} \) as well as to account for the matter contribution we have to choose a reasonable value of \( \alpha \). From the relation between \( R \) and \( \phi \) given by Eq. (42) with \( R^2 \) correction it is seen that, here \( R \to \infty \) corresponds to \( k\phi \to \infty \) (or \( \phi \to \infty \)) in contrast to the relation given by Eq. (38). This implies that here a small value of \( k\phi \) corresponds to small curvature \( R \). For example, when \( k\phi_{\text{min}} = 0 \) (as in the case of \( -T_\mu^\mu \sim 10^{18} \rho_{\text{crit}} \)), the corresponding value of \( R \) is \( R_0 = (c_1 m^4/2\alpha c_2^3)^{1/3} \) for \( n = 1 \), which is a finite value. Thus we see that for large matter-energy density in the high curvature regime, the potential has a minimum not at \( R \to \infty \), but at finite value of \( R \) and also this minimum is not so minute as to let it easily go to zero unlike the original effective potential shown in Fig. 4. Instead, here the singularity is moved away to infinity so that it is not easily attainable. Hence, the singularity problem is resolved.
At this stage, we want to calculate the mass of scalaron from the $R^2$ corrected Hu-Sawicki model (40). In Fig.1 we have already seen how the scalaron mass increases with matter energy density. This extremely large mass of scalaron is a consequence of the singularity problem as mentioned above, because to subdue the singularity problem we need larger energy density in the effective potential, which appears as some order higher in magnitude in the scalaron mass, as seen from Eq. (36). Thus we should get a reasonably light scalaron in the high density region, once we solve the singularity problem. In this context, it needs to be mentioned that according to the analysis in Ref. [29], the upper bound for the scalaron mass should be such that

$$m_\phi < O(1) \text{ GeV}. \quad (45)$$

Now, from Eq. (41), we get

$$f_R(R) = -\frac{c_1}{c_2} \left( \frac{m^2}{R} \right)^{n+1} + 2\alpha R. \quad (46)$$

Similarly, from this equation, we obtain

$$f_{RR}(R) = n(n+1) \frac{c_1}{c_2} \left( \frac{m^2}{R} \right)^{n+1} \frac{1}{R} + 2\alpha. \quad (47)$$

Substituting these values of $f_R(R)$ and $f_{RR}(R)$ for the higher order correction in the Hu-Sawicki model in the scalaron mass Eq. (25), we found the scalaron mass as

$$m_\phi^2 = \frac{1}{3} \left[ \frac{1 - n(n+2) \frac{c_1}{c_2} \left( \frac{m^2}{R} \right)^{n+1}}{n(n+1) \frac{c_1}{c_2} m^{-2} \left( \frac{m^2}{R} \right)^{n+2} + 2\alpha} \right]. \quad (48)$$

Simplifying the above equation with the higher curvature condition $R \gg m^2$, the scalaron mass is deduced as

$$m_\phi^2 \approx \frac{1}{6\alpha}. \quad (49)$$

Hence, the mass of the scalaron becomes constant and depends only on the value of $\alpha$ in the high curvature region.

**V. PROPERTIES OF SCALARON MASS**

In this section, we calculate the scalaron mass range relevant to the present Universe and discuss its implications. Also we calculate and discuss the evolution of scalaron mass with the cosmological redshift.
A. Scalaron Mass Scale

As already mentioned, if we consider our Universe under the dust approximation, then the typical energy density of galaxy can be found as $\rho_0 \sim 3 - 5 \times 10^{-25}$ g/cm$^3 \sim 2 - 3 \times 10^{-42}$ GeV$^4$ [48]. The scalaron mass range corresponding to this range of energy density of galaxy can be computed from Eq. (48) with a reasonable value of $a = 10^{44}$ GeV$^{-2}$. Under these considerations the computed mass range of scalaron is found to be

$$m_\phi \sim 0.556 - 1.383 \times 10^{-23} \text{ eV.} \quad (50)$$

This result naturally satisfies the upper bound $m_\phi < \mathcal{O}(1)$ GeV, which means that the mass of the scalaron should be less than or of the order of 1 GeV. The mass range of the scalaron calculated in Eq. (50) is comparable to the mass of ultralight axions, which has an approximate mass of $\sim 10^{-22}$ eV [49]. This DM candidate is referred to as the fuzzy cold dark matter (FCDM). In fact, the mass range given in Eq. (50) is closer to the scalaron mass value $m_\phi \sim 3 - 5 \times 10^{-24}$ eV obtained for the Starobinsky model with $R^2$ correction [41]. As mentioned in [50], this lower mass of the scalaron does not create any complication. In fact, it can be said that this scalaron makes small contributions to the amount of DM in the present Universe. However, DM candidates in the mass range $10^4$ GeV $\gtrsim m_\phi \gtrsim 10^{-22}$ eV could potentially create huge amount of DM. From a theoretical point of view although such a low mass seems strange, yet observations as well as experimental evidences permit considerably lighter scalar fields ($m \lesssim 10^{-33}$ eV) [51] so that observations of accelerated expansion of the Universe could be understood. Furthermore, in Ref.[52] the authors investigated the possibility of a condensate of a very light vector boson as the dark matter and shows that the mass of such massive vector boson should satisfy the condition: $m_\phi \gtrsim \Omega_{DM}^2 \mathcal{H}_0 b = 6.6 \times 10^{-35}$ eV when the mass of the boson $\phi$ is comparable to the Hubble parameter $H$ for its upper bound is equal to the Planck’s mass. But for such a low mass the Compton wavelength is too large. In order to allow structure formation, the Compton wavelength has to be on the kpc scale. This requirement gives a sharper bound on the lowest mass as

$$1 \text{kpc} < \frac{\hbar}{\Delta p} = \frac{\hbar}{m_\phi v_{esc}} \Rightarrow m_\phi \geq 1.67 \times 10^{-24} \text{ eV} \quad (51)$$

It is seen that our calculated scalaron mass range satisfies this lowest vector boson mass condition. Usually, it is speculated that the mass of various DM candidates ranges from $10^{-22}$ eV (FCDM) [49, 53] to thousands of solar masses including the primordial black holes [54, 55]. It would be appropriate to mention at this point that the relic abundance of scalaron as dark matter has been discussed in detail in Ref.[41].

Moreover, it can be seen from the literature [56–58] that when the scalaron mass $m_\phi$ approaches to Hubble parameter $H$, the scalaron field starts to oscillate, and the scalaron field oscillates in a slow-varying manner when $m_\phi \gg H$. Our work interprets this scalaron as an axion cold DM particle, which satisfies the chameleon mechanism. And for axions it has also been mentioned in the Ref. [59] that during late times, at low curvature of the Universe, the slow-varying oscillation of the axion field leads to the fact that the axion energy density behaves as $\rho_\phi \sim a^{-3}$, where $a$ is the scale factor, and hence it satisfies the perfect fluid continuity equation. Whereas, in our work, the perfect fluid continuity equation is not satisfied. This is due to the fact that we have taken into account the pressureless dust model of the Universe. In this theory, the energy is not conserved and the energy density of the chameleon does not satisfy the continuity equation. Hence, with expansion of the Universe, the chameleon energy density does not behave as $\rho_\phi \sim a^{-3}$.

Again, curiosity arises as to how the scalaron would behave near the surface as well as at the center of very dense objects, such as neutron stars. The density of neutron stars is around $10^{15} - 10^{16}$ g/cm$^3$ [60]. Using this data in Eq. (48), we computed the scalaron mass inside a neutron star as well as near its surface, and found the scalaron mass $\sim 10^{-19}$ eV deep inside the star and $\sim 10^{-31}$ eV near the surface. Thus, the scalaron satisfies the chameleon mechanism, with lighter mass near the surface and heavier mass at the center of neutron stars. However, detailed study is required in this direction, especially to see the characteristics of neutron stars in the light of chameleon mechanism. Furthermore, the potential in our work is chameleonic and as such the inflationary constraints on the potential discussed in [61, 62] do not apply. As our chameleonic potential is unaffected by inflationary constraints, the behaviour of the scalar field in our theory can be studied for neutron stars. Neutron stars have been studied in $f(R)$ theories that use the chameleon mechanism in [63, 64]. In these works authors have demonstrated how the scalar field changes with the minimum of its effective potential at the core, near the boundary and outside the neutron stars.

B. Evolution of Scalaron Mass

Finally, in this section we study how the mass of the scalaron evolves as a function of cosmological redshift for specific parameter values of the model. This is to clearly see the behaviour of scalaron at different stages of the expanding Universe under the $\Lambda$CDM scenario. For this purpose, we used Eq. (48) to obtain the required expression for the mass of the scalaron as a function of the cosmological redshift $z$. 
In the Hu-Sawicki model, for the flat $\Lambda$CDM model of cosmic expansion the Ricci scalar can be expressed as [38]

$$R_0 \approx \kappa^2 \rho_0 (1 + z)^3 + 2\frac{c_1}{c_2} m^2,$$

(52)

where $\rho_0$ is the present average matter density of the Universe, since the matter density in the $\Lambda$CDM model is given by $\rho = \rho_0 (1 + z)^3$. Now, Eq. (52) can be used in Eq. (48) to obtain the scalaron mass as a function of the redshift $z$.

\[\text{FIG. 7: Variation of scalaron mass in the Hu-Sawicki model as a function of redshift $z$ for the parameter $n = 2.5$. First two plots are for $\alpha = 10^{52}$ and the last plot is for two values of $\alpha$ as shown. All these plots are drawn for the present average matter density of the Universe $\rho_0 = 4 \times 10^{-23} \text{g/cm}^3$.}\]

The variation of scalaron mass as a function of redshift for different parameter values is shown in Fig. 7. The first two panels are for $\alpha = 10^{54} \text{GeV}^{-2}$ and the last panel is drawn for two values of $\alpha$, $\alpha = 10^{54}$ and $10^{56} \text{GeV}^{-2}$. The parameter $n = 2.5$ and the present average matter density of the Universe $\rho_0 = 4 \times 10^{-23} \text{g/cm}^3$ are used for all these plots. From this figure, it can be seen that the scalaron mass gradually increases with redshift, but this rate of increment is different at different stages of the Universe’s evolution. That is, at high redshifts, the mass of the scalaron is also high. This means that with expansion of the Universe, the scalaron mass becomes light. Thus the scalaron was very heavy in the very early Universe in comparison to its present range of values as mentioned already. This is in accordance with the chameleon mechanism. Again it can be seen that the scalaron mass increases with redshift up to a particular $z$ value and then the curve becomes flat. This gives us an interesting result in analogy with the matter-dominated phase of the Universe. Moreover, for both the values of $\alpha$ the scalaron mass increases with redshift in a much similar fashion, however the lower value of $\alpha$ gives heavier scalaron than that given by its higher value.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have discussed about the scalaron, which is a new scalar field introduced from the modification of gravity via Weyl transformation of the spacetime metric $g_{\mu\nu}$ from the Jordan frame to the Einstein frame. We have assumed that the fluctuation of the scalaron around the potential minimum can be regarded as dark matter.

In the first part, we discuss the $f(R)$ gravity which is one of the simplest modified gravity theories, using metric variational approach. We found that the Weyl transformation from the Jordan frame to the Einstein frame adds a new degree of freedom. Then we obtained the potential of the scalaron which helped us to calculate the effective potential of the scalaron. We calculated the mass of the scalaron with a matter effect $T^\mu_\mu$. We found that the mass of the scalaron is dependent on the matter contribution.

In the second part, the chameleon mechanism is studied in the framework of the Hu-Sawicki model of $f(R)$ gravity. The mass of the scalaron field for the Hu-Sawicki model shows its relation with the matter-energy density via the property known as chameleon mechanism, which is a screening mechanism for solar system tests of $f(R)$ gravity. For increasing values of the model parameter $n$, the scalaron mass increases rapidly. Here, we found the mass of the scalaron to be very large. This large mass is a consequence of the singularity problem that appears in Hu-Sawicki model of $f(R)$ gravity. So, to obtain a smaller mass of the scalaron field and to deviate from the singularity problem, we implemented the higher curvature correction to the original Hu-Sawicki model. As a result of the $R^2$ correction, we found the scalaron mass to be small in the large curvature region. Also, the singularity is pushed to infinity so that it is not easily attainable.

Furthermore, we discussed the properties of the scalaron field in the present Universe. The scalaron mass was calculated to be very light $\sim 10^{-23} \text{eV}$, and satisfies the upper bound for the scalaron mass $m_\phi < \mathcal{O}(1) \text{ GeV}$. This mass of the scalaron field is quite closer to the mass of ultralight axion, which has an approximate mass scale of $\sim 10^{-22} \text{eV}$. This ultralight axion is referred to as fuzzy cold DM. However, studies have shown that this low mass does not create any issue. Instead, it can be said that this scalaron makes small contributions to the amount of DM in the current Universe. Moreover, to explain observations of

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accelerated expansion much lighter scalar fields \((m \lesssim 10^{-33}\text{eV})\) are permitted by experimental evidences. Thus, we can come to the conclusion that the Hu-Sawicki model of \(f(R)\) gravity can explain dark matter. In this work, we have implemented the pressureless dust model and hence, the scalaron (or axion) energy density does not scale as \(\rho_\phi \sim a^{-3}\) and consequently it does not satisfy the perfect fluid continuity equation.

Finally, the evolution of the scalaron mass with the cosmological redshift is examined. It is seen that the mass of the scalaron increases with redshift up to a particular \(z\) value and then becomes constant. It is known that the most distant objects manifest larger redshifts. Thus, our study shows that in the early Universe the scalaron was heavy and with expansion of the Universe, the scalaron becomes light. This is in accordance with the chameleon mechanism. Moreover, the rate of increase of scalaron mass with the redshift within the range of effective redshift values is different at different stages of evolution of the Universe.

Lastly, it should be mentioned that it would be interesting to extend further our work to study the interaction of the scalaron with SM particles. Another interesting aspect would be to apply this scalaron DM in other fields of particle physics to resolve different issues. Also, in near future we plan to work on other viable models of \(f(R)\) gravity to study DM. Further possibility is considering a scalar field which can be a perfect fluid along with the chameleon used in our theory. Such \(f(R, \phi)\) theories have been studied in [65, 66]. Since dark matter can have multiple components, this would be an interesting future work.

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