Dynamical Collapse of Boson Stars

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Abstract

We study the time evolution in a system of $N$ bosons with a relativistic dispersion law interacting through a Newtonian gravitational potential with coupling constant $G$. We consider the mean field scaling where $N$ tends to infinity, $G$ tends to zero and $\lambda = GN$ remains fixed. We investigate the relation between the many body quantum dynamics governed by the Schrödinger equation and the effective evolution described by a (semi-relativistic) Hartree equation. In particular, we are interested in the super-critical regime of large $\lambda$ (the sub-critical case has been studied in \cite{2, 19}), where the nonlinear Hartree equation is known to have solutions which blow up in finite time. To inspect this regime, we need to regularize the interaction in the many body Hamiltonian with an $N$ dependent cutoff that vanishes in the limit $N \to \infty$. We show, first, that if the solution of the nonlinear equation does not blow up in the time interval $[-T, T]$, then the many body Schrödinger dynamics (on the level of the reduced density matrices) can be approximated by the nonlinear Hartree dynamics, just as in the sub-critical regime. Moreover, we prove that if the solution of the nonlinear Hartree equation blows up at time $T$ (in the sense that the $H^{1/2}$ norm of the solution diverges as time approaches $T$), then also the solution of the linear Schrödinger equation collapses (in the sense that the kinetic energy per particle diverges) if $t \to T$ and, simultaneously, $N \to \infty$ sufficiently fast. This gives the first dynamical description of the phenomenon of gravitational collapse as observed directly on the many body level.

1 Introduction and main results

We consider systems of gravitating bosons known as boson stars. Assuming the particles to have a relativistic dispersion, but the interaction to be treated classically (Newtonian gravity), we arrive at the $N$-particle Hamiltonian

$$H_{\text{grav}} = \sum_{j=1}^{N} \sqrt{1 - \Delta x_j} - G \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

acting on the Hilbert space $L_2^2(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ containing all functions symmetric with respect to arbitrary permutations (here we use units with $\hbar = 1$, $c = 1$, and $m = 1$, where $m$ denotes the mass of the bosons).
We are interested in the mean field limit where $N \to \infty$, $G \to 0$ so that $NG =: \lambda$ remains fixed. In other words, we are going to study a family of systems, parametrized by the number of bosons $N$, described by the $N$ particle Hamiltonian

$$H_N = \sum_{j=1}^{N} \sqrt{1 - \Delta x_j} - \frac{\lambda}{N} \sum_{i < j}^{N} \frac{1}{|x_i - x_j|}.$$  \tag{1.1}$$

The system is critical, and it behaves very differently depending on the value of the coupling constant $\lambda > 0$. The criticality of the system is a consequence of the fact that the kinetic energy scales, for large momenta, like the potential energy (both scales as an inverse length). The potential energy can be made arbitrarily large (and negative) by moving the particles closer and closer together ($N$ particle in a box of volume $\ell^3$ have a potential energy of the order $N\ell^{-1}$, taking also into account the $1/N$ factor in front of the interaction energy). However, in order to localize particles in a small volume we have to pay a price in terms of kinetic energy (to localize $N$ particles within a box of volume $\ell^3$, we need an energy proportional to $N\ell^{-1}$). This simple observation implies that, for small values of the coupling constant $\lambda$, the kinetic energy dominates the potential energy, and that, for sufficiently large $\lambda$, the kinetic energy needed to bring particles together is not sufficient to compensate for the gain in the potential energy.

For every $N \in \mathbb{N}$, there exists therefore a critical coupling constant $\lambda_{\text{crit}}(N)$ such that $H_N$ is bounded below for all $\lambda < \lambda_{\text{crit}}(N)$ and such that

$$
\inf_{\psi \in L^2(\mathbb{R}^3^N)} \frac{\langle \psi, H_N \psi \rangle}{\|\psi\|^2} = -\infty
$$

for all $\lambda > \lambda_{\text{crit}}(N)$. It was proven in [17] that the critical constant is given, as $N \to \infty$, by the critical coupling constant for the Hartree energy functional

$$E_{\text{Hartree}}(\varphi) = \int dx \left| (1 - \Delta)^{1/4} \varphi(x) \right|^2 - \frac{\lambda}{2} \int dx dy \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x - y|}. \tag{1.2}$$

More precisely, it was proven in [17] that, as $N \to \infty$, $\lambda_{\text{crit}}(N) \to \lambda_{\text{crit}}^H$, where

$$\frac{1}{\lambda_{\text{crit}}^H} = \sup_{\varphi \in L^2(\mathbb{R}^3), \|\varphi\|=1} \frac{1}{2} \frac{\int dx dy |\varphi(x)|^2 |\varphi(y)|^2 |x - y|^{-1}}{\int dx \|\nabla|^{1/2} \varphi(x)\|^2}. $$

Note that, with this definition, $E_{\text{Hartree}}(\varphi) \geq 0$ for all $\varphi \in H^{1/2}(\mathbb{R}^3)$ if $\lambda \leq \lambda_{\text{crit}}^H$, while

$$\inf_{\varphi \in H^{1/2}(\mathbb{R}^3), \|\varphi\|=1} E_{\text{Hartree}}(\varphi) = -\infty$$

if $\lambda > \lambda_{\text{crit}}^H$. It is also possible (see [17]) to give bounds on the fluctuations of $\lambda_{\text{crit}}(N)$ around $\lambda_{\text{crit}}^H$:

$$\lambda_{\text{crit}}^H (1 - c_1 N^{-1/3}) \leq \lambda_{\text{crit}}(N) \leq \lambda_{\text{crit}}^H (1 + c_2 N^{-1})$$

for appropriate constants $c_1, c_2 > 0$. The value of $\lambda_{\text{crit}}^H$ is not explicitly known. By Kato's inequality, $|x_i - x_j|^{-1} \leq (\pi/2) |\nabla x_j|$, it is easy to see that $\lambda_{\text{crit}}^H \geq (4/\pi) \approx 1.3$. In [18, 17], it is also shown that $\lambda_{\text{crit}}^H \leq 2.7$.

Let us discuss first the subcritical case $\lambda < \lambda_{\text{crit}}^H$. In this case, the Hamiltonian (1.1) has, at least for sufficiently large $N$ (so that $\lambda < \lambda_{\text{crit}}(N)$), a unique realization as a self-adjoint operator on
$L^2_\psi(\mathbb{R}^{3N})$ and therefore generates the one-parameter group of unitary transformation $U_N(t) = e^{-iH_Nt}$, with $t \in \mathbb{R}$. The unique global solution of the $N$-particle Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \quad \psi_{N,t=0} = \psi_N \in L^2_\psi(\mathbb{R}^{3N})$$  \hspace{1cm} (1.3)$$

which governs the time evolution of an arbitrary initial $N$-particle wave function $\psi_N$ is then given by $\psi_{N,t} = U_N(t) \psi_N$.

Consider now the time evolution (1.3) of a factorized initial data $\psi_N = \varphi^{\otimes N}$ for some $\varphi \in L^2(\mathbb{R}^3)$ (here we use the notation $\varphi^{\otimes N}(x) = \prod_{j=1}^N \varphi(x_j)$, where $x = (x_1, \ldots, x_N)$). Of course, factorization is not preserved by the evolution. Nevertheless, because of the mean field character of the interaction, one may expect factorization of the evolved wave-function $\psi_{N,t}$ to be restored, in an appropriate sense, in the limit of large $N$. If we assume, formally, that

$$\psi_{N,t} \simeq \varphi^{\otimes N},$$  \hspace{1cm} (1.4)$$

then it is easy to derive a self-consistent equation for the evolution of the one-particle orbital $\varphi_t$. In fact, if (1.4) is correct, the potential experienced by the particles can be approximated by an averaged, mean field, potential given by the convolution $-\lambda |.|^{-1} * |\varphi_t|^2$. Therefore, (1.4) implies that $\varphi_t$ must evolve according to the semirelativistic nonlinear Hartree equation

$$i\partial_t \varphi_t = \sqrt{1 - \Delta} \varphi_t - \lambda \left( \frac{1}{|.|} * |\varphi_t|^2 \right) \varphi_t$$  \hspace{1cm} (1.5)$$

with initial data $\varphi_{t=0} = \varphi$. The question now is in which sense can the factorization (1.4) hold true. It turns out that (1.4) should be understood on the level of the marginal densities associated with $\psi_{N,t}$.

Let $\gamma_{N,t} = |\psi_{N,t}\rangle \langle \psi_{N,t}|$ be the orthogonal projection onto the solution of the $N$-particle Schrödinger equation $\psi_{N,t} = e^{-iH_Nt} \psi_N$ with factorized initial data $\psi_N = \varphi^{\otimes N}$. Then, for $k = 1, \ldots, N$, we define the $k$-particle marginal (or reduced) density $\gamma^{(k)}_{N,t}$ associated with $\psi_{N,t}$ by taking the partial trace of $\gamma_{N,t}$ over the last $N - k$ particles; that is

$$\gamma^{(k)}_{N,t} = \text{Tr}_{k+1,k+2,\ldots,N} \gamma_{N,t}.$$  

In other words, $\gamma^{(k)}_{N,t}$ is defined as a non-negative trace class operator on $L^2(\mathbb{R}^{3k})$ with kernel

$$\gamma^{(k)}_{N,t}(x_k, y_k) = \int dx_{N-k} \gamma_{N,t}(x_k, x_{N-k}; y_k, x_{N-k}) = \int dx_{N-k} \psi_{N,t}(x_k, x_{N-k}) \overline{\psi_{N,t}(y_k, x_{N-k})},$$

where we introduced the notation $x_k = (x_1, \ldots, x_k)$, $y_k = (y_1, \ldots, y_k)$, $x_{N-k} = (x_{k+1}, \ldots, x_N)$.

The first rigorous proof of the validity of (1.4) on the level of the marginal densities (and hence the first derivation of (1.5)) has been obtained, for the subcritical regime, in [2]. More precisely, under the condition that $\lambda < 4/\pi$ (which is smaller than $\lambda_{\text{crit}}^H$), it is proven in [2] that, for every fixed $k \geq 1$, and for every $t \in \mathbb{R}$,

$$\text{Tr} \left| \gamma^{(k)}_{N,t} - |\varphi_t| \langle \varphi_t |^{\otimes k} \right| \rightarrow 0$$  \hspace{1cm} (1.6)$$


as $N \to \infty$. Here $\varphi_t$ is the solution of the semirelativistic nonlinear Hartree equation \((1.5)\) with initial data $\varphi_{t=0} = \varphi$. Note that the convergence of the marginal densities implies that, for an arbitrary $k$-particle observable $J^{(k)}$, we have

$$\langle \psi_{N,t}, (J^{(k)} \otimes 1^{(N-k)})\psi_{N,t} \rangle \to \langle \varphi_t \otimes k, J^{(k)} \varphi_t \otimes k \rangle$$

as $N \to \infty$. In this sense, the solution of the $N$-particle Schrödinger equation $\psi_{N,t}$ can be approximated, for large $N$, by products of the solution of the one-particle semirelativistic Hartree equation \((1.5)\). Observe here that the semirelativistic Hartree equation \((1.5)\) is locally well-posed in the energy space $H^{1/2}(\mathbb{R}^3)$, for arbitrary $\lambda \in \mathbb{R}$. In fact, it is proven in [16] that, for every $\varphi \in H^{1/2}(\mathbb{R}^3)$, there exists a maximal $0 < T \leq \infty$ and a unique solution $\varphi_t \in C((-T, T), H^{1/2}(\mathbb{R}^3))$ of \((1.5)\) in the time interval $t \in (-T, T)$. Here, either $T = \infty$ (and then the solution is global), or $T < \infty$ and $\|\varphi_t\|_{H^{1/2}} \to \infty$ as $t \to T$ or as $t \to -T$ (in this case, $\varphi_t$ blows up in finite time). For $\lambda < \lambda_{\text{crit}}^{H}$, the unique solution $\varphi_t$ is also shown to be global (hence $T = \infty$ in this case); for this reason, \((1.6)\) makes sense for all $t \in \mathbb{R}$.

Note that the result \((1.6)\) is just one of the several results concerning the derivation of effective evolution equations from first principle quantum dynamics which have been obtained in the last years. Other results of this type concern the derivation of the non-relativistic Hartree equation in the mean-field limit (see for example [22, 8, 1, 21, 10, 19, 11, 12]) and the derivation of the (non-relativistic) Gross-Pitaevskii equation for the description of the dynamics of initially trapped Bose-Einstein condensates (see [4, 5, 6, 7] and, very recently, [20]). Although most of these papers deal with non-relativistic particles, the authors of [19] also consider semirelativistic bosons interacting through a Newtonian potential, in the subcritical regime $\lambda < \lambda_{\text{crit}}^{H}$. They improve \((1.6)\), by giving an explicit bound on the rate of the convergence; more precisely, they show that

$$\text{Tr} \left| \gamma^{(k)}_{N,t} - \|\varphi_t\|_{\otimes k} \right| \leq \frac{C_{k,t}}{\sqrt{N}}$$

\[(1.7)\]

for a constant $C_{k,t} = C_t \sqrt{k}$, where $C_t$ grows at most exponentially in $t \in \mathbb{R}$ (under additional assumptions on the dispersion of $\varphi_t$, $C_t$ is bounded uniformly in $t$).

So far, we considered the subcritical regime $\lambda < \lambda_{\text{crit}}^{H}$. Let us discuss now the supercritical regime $\lambda > \lambda_{\text{crit}}^{H}$. The criticality of the Hartree energy functional remarked earlier can also be observed on the level of the time-dependent semirelativistic nonlinear Hartree equation \((1.5)\). On the one hand, \((1.5)\) is globally well-posed for $\lambda < \lambda_{\text{crit}}^{H}$. On the other hand, it turns out that, for $\lambda > \lambda_{\text{crit}}^{H}$, \((1.5)\) has solutions that blowup in finite time. More precisely, it was proven in [9] that, for every spherically symmetric $\varphi \in H^1(\mathbb{R}^3)$ with $E_{\text{Hartree}}(\varphi) < 0$ such that $\|x|\varphi| < \infty$, the unique maximal solution $\varphi_t \in C((-T, T), H^{1/2}(\mathbb{R}^3))$ of \((1.5)\) with initial data $\varphi_{t=0} = \varphi$ blows up in finite time, in the sense that $T < \infty$ and $\|\varphi_t\|_{H^{1/2}} \to \infty$, as $t \to T$ or as $t \to -T$.

Solutions of \((1.5)\) exhibiting blowup in finite time are supposed to describe, within the framework of Chandrasekhar’s theory, the gravitational collapse of bosons stars. The expectation that blowup solutions of the semirelativistic Hartree equation describe the collapse of boson stars is based on the unproven assumption that the many-body dynamics can be approximated by the Hartree dynamics also in the supercritical regime $\lambda > \lambda_{\text{crit}}^{H}$ and all the way up to the time of the (nonlinear) blowup. In this paper, we give a rigorous proof of this physical assumption. We show, first of all, that the convergence \((1.7)\) also holds in the supercritical case $\lambda \geq \lambda_{\text{crit}}^{H}$, if the norm $\|\varphi_t\|_{H^{1/2}}$ stays bounded in the interval $[0, t]$. Moreover, we prove that the convergence of the $N$-particle Schrödinger evolution...
towards the Hartree dynamics does not only hold in the sense of (1.7); instead, it also holds (again assuming that the norm \( \| \varphi_s \|_{H^{1/2}} \) remains bounded in \([0, t]\)) with respect to the stronger energy norm, at least for the one-particle marginal density (this means that \((1 - \Delta)^{1/4} \langle \varphi_t \rangle(1 - \Delta)^{1/4} \) converges to \((1 - \Delta)^{1/4} |\varphi_t| (1 - \Delta)^{1/4} \) in the trace norm, as \(N \to \infty\)). Note that this is the first proof of the convergence of the many body Schrödinger evolution towards the Hartree dynamics with respect to the energy norm, not only for supercritical semirelativistic boson stars, but for any mean field system. As a consequence of the convergence in energy, we show that the solution of the Hartree equation (1.5) really describes the collapse of the many body system.

Let us now describe our results in more details. We are interested in the dynamics generated by the \(N\)-particle Hamiltonian (1.1) in the supercritical regime \(\lambda > \lambda_{\text{crit}}^N\) (although at the end our results will also hold in the sub-critical regime). The first issue that we have to face is that, since the form defined by \(H_N\) is not bounded below, the Hamiltonian \(H_N\) does not necessarily have a unique realization as a self-adjoint operator on the Hilbert space \(L^2_s(\mathbb{R}^{3N})\) (and thus the time-evolution is not necessarily well-defined). For this reason we choose a sequence \(\alpha = (\alpha_N)_{N \geq 1}\) with \(\alpha_N > 0\) for all \(N \in \mathbb{N}\) and \(\alpha_N \to 0\) as \(N \to \infty\), and we define the regularized \(N\)-particle Hamiltonian

\[
H_N^\alpha = \sum_{j=1}^N \sqrt{1 - \Delta_{x_j}} - \frac{\lambda}{N} \sum_{i<j} \frac{1}{|x_i - x_j| + \alpha_N}.
\]

The regularized Hamiltonian \(H_N^\alpha\) defines now a quadratic form on \(L^2_s(\mathbb{R}^{3N})\) which is clearly bounded below, for every \(N \in \mathbb{N}\), since

\[
\langle \psi_N, H_N^\alpha \psi_N \rangle \geq -\frac{\lambda N}{2\alpha_N} \| \psi_N \|^2.
\]

By Friedrichs theorem, \(H_N^\alpha\) has a unique extension as a self-adjoint operator on \(L^2_s(\mathbb{R}^{3N})\), with domain \(H^{1/2}(\mathbb{R}^{3N})\). Hence \(H_N^\alpha\) generates the one-parameter group of unitary transformations \(U_N^\alpha(t) = e^{-itH_N^\alpha}, t \in \mathbb{R}\), and therefore the \(N\)-particles Schrödinger equation

\[
i \partial_t \psi_{N,t} = H_N^\alpha \psi_{N,t} \quad \text{with initial condition } \psi_{N,t=0} = \psi_N
\]

is globally well-posed (it has the unique solution \(\psi_{N,t} = e^{-iH_N^\alpha t} \psi_N\), for all \(t \in \mathbb{R}\)).

From the physical point of view, the introduction of the cutoff \(\alpha\) is justified by the observation that on very short length scales, the Newtonian potential is effectively regularized by the presence of other forces (such as electromagnetic or nuclear forces) or because of general relativity effects. The results that we will state and prove below concern the limit of large \(N\) and small \(\alpha_N\). How fast \(\alpha_N\) tends to zero is irrelevant to establish the convergence of the Schrödinger evolution to the Hartree dynamics in the trace norm, analogously to (1.6) (although, of course, the rate of the convergence depends on \(\alpha_N\)). On the other hand, to show convergence in the energy norm, we will need to assume that \(\alpha_N\) does not converge to zero too fast; more precisely, we will suppose that there exists \(\beta > 0\) such that \(N^\beta \alpha_N \to \infty\). This condition, which allow for any power law decay, still leaves a lot of freedom in the choice of \(\alpha_N\) (physically, conditions on the decay of \(\alpha\) translate into restrictions of the range of systems for which the approximation of the many body evolution by the Hartree dynamics is applicable).

We study the time evolution generated by the regularized Hamiltonian (1.8) on factorized initial data \(\psi_N = \varphi^\otimes N\) for \(\varphi \in H^2(\mathbb{R}^3)\). We compare the marginal densities associated with the solution of the \(N\) particle Schrödinger equation \(\psi_{N,t} = e^{-iH_N^\alpha t} \psi_N\) with products of the solution to the Hartree equation (1.5). Note that the cutoff disappears in the limiting Hartree equation, because of the assumption that \(\alpha_N \to 0\) as \(N \to \infty\) (part of the proof of the convergence will consists in estimating
the distance between the solution $\varphi_t$ of (1.5) and the solution $\varphi_t^{(a)}$ of a regularized Hartree equation with interaction $-\lambda/|x|$ replaced by the regularized interaction $-\lambda/(|x| + \alpha)$ in the limit of small $\alpha$.

The first main result of this paper is the following theorem. Under the assumption that the solution $\varphi_t$ of (1.5) has a bounded $H^{1/2}$-norm in the interval $[-T, T]$ (which means that there is no blowup, up to time $T$), we prove the convergence of the marginal densities associated with the solution $\psi_{N,t}$ of the $N$-particle Schrödinger equation (1.9) to the orthogonal projections onto products of $\varphi_t$.

Theorem 1.1. Fix $\lambda \in \mathbb{R}$, $\varphi \in H^2(\mathbb{R}^3)$ with $\|\varphi\| = 1$ and set $\psi_N = \varphi \otimes N$. Let $\psi_{N,t} = e^{-iH_N t} \psi_N$ be the evolution of the the initial wave function $\psi_N$ with respect to the Hamiltonian (1.8), and let $\gamma_{N,t}^{(1)}$ be the one-particle reduced density associated with $\psi_{N,t}$.

Denote by $\varphi_t$ the solution of the nonlinear Hartree equation (1.3) with initial data $\varphi_{t=0} = \varphi$. Fix $T > 0$ such that

$$\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \quad (1.10)$$

Then there exists a constant $C = C(\kappa, T, \|\varphi\|_{H^2})$ such that

$$\text{Tr} |\gamma_{N,t}^{(1)} - \langle \varphi_t \rangle \langle \varphi_t \rangle| \leq C \left( \frac{1}{\sqrt{N}} + \alpha_N \right). \quad (1.11)$$

for all $t \in \mathbb{R}$ with $|t| \leq T$, and for all $N$ sufficiently large. In particular, if $\alpha_N \to 0$ as $N \to \infty$, it follows that $\gamma_{N,t}^{(1)} \to \langle \varphi_t \rangle \langle \varphi_t \rangle$ in trace-norm, as $N \to \infty$.

Remarks. The existence of $T > 0$ such that (1.10) is satisfied is a consequence of the local well-posedness of the nonlinear Hartree equation (1.5), see [16]. Similar methods to the ones used to prove (1.11) can be employed to show the convergence of higher order marginals $\gamma_{N,t}^{(k)}$ with the same rate $(N^{-1/2} + \alpha_N)$ for any fixed $k \in \mathbb{N}$. If we are satisfied with a slower rate for higher marginals, a simple argument, outlined in Section 2 of [19], shows that (1.11) immediately implies that for any $k \in \mathbb{N},$

$$\text{Tr} |\gamma_{N,t}^{(k)} - \langle \varphi_t \rangle \langle \varphi_t \rangle^{\otimes k}| \leq C \sqrt{k \left( \frac{1}{\sqrt{N}} + \alpha_N \right)}. \quad (1.11)$$

The first ingredient of the proof of Theorem 1.1 is the observation that the bound (1.10) on the $H^{1/2}$-norm of the solution of the Hartree equation (1.5), together with the assumption $\varphi \in H^2(\mathbb{R}^3)$ on the initial data, implies an upper bound on the $H^{1/2}$-norm of the solution of the regularized Hartree equation

$$i\partial_t \varphi_t^{(a)} = \sqrt{1 - \Delta} \varphi_t^{(a)} - \lambda \left( \frac{1}{|\cdot| + \alpha} \ast |\varphi_t^{(a)}|^2 \right) \varphi_t^{(a)}, \quad (1.12)$$

uniform in the cutoff $\alpha > 0$ (actually, we prove that $\|\varphi_t - \varphi_t^{(a)}\|_{H^{1/2}}$ is small, of the order $\alpha^{1/2}$). By the propagation of regularity for the solution of the Hartree equation (both the original equation (1.5) and the regularized equation (1.12)), we also obtain a bound for the norm $\|\varphi_t^{(a)}\|_{H^2}$ uniform in $\alpha > 0$ and in $t \in [-T, T]$.

The second ingredient in the proof of Theorem 1.1 is the method developed in [21] to establish the convergence to the Hartree dynamics for a system of non-relativistic bosons. This method is based on the use of a Fock space representation of the many boson system, and on the study of the dynamics of coherent states. When analyzing the time-evolution of an initial coherent state, it
is possible to isolate the Hartree component of the evolution. Moreover, as first observed in [15] (and later in [13]), the evolution of the fluctuations around the Hartree dynamics can be expressed through a two-parameter group of unitary evolutions with an explicit time-dependent generator. The problem then reduces to deriving a bound for the growth of the number of particle operator (which measures the “number” of fluctuations, after second quantization) with respect to this evolution. The crucial observation is that the bound derived in [21] for non-relativistic particles can be easily extended to the relativistic setting, once a uniform bound on \( \| \varphi_t^{(\alpha)} \|_{H^1} \) is available.

In Section 2 we show the necessary bounds on the solution \( \varphi_t^{(\alpha)} \) of the regularized equation (1.12). In Section 3 we introduce the Fock space representation, we define the coherent states, and we discuss some of their main properties. Then, in Section 4 we show that the evolution of initial coherent states can be approximated by the Hartree dynamics, and we use this fact to conclude the proof of Theorem 1.1.

Theorem 1.2. Fix \( \varphi \in H^2(\mathbb{R}^3) \) with \( \| \varphi \| = 1 \) and set \( \psi_N = \varphi^{\otimes N} \). Consider an arbitrary sequence \( \alpha_N > 0 \) with \( \alpha_N \to 0 \) and such that \( N^\beta \alpha_N \to \infty \) as \( N \to \infty \), for some \( \beta > 0 \). Let \( \psi_{N,t} = e^{-iH_N t} \psi_N \) be the evolution of the initial wave function \( \psi_N \) generated by the Hamiltonian (1.8) and let \( \gamma_{N,t}^{(1)} \) be the one-particle reduced density associated with \( \psi_{N,t} \).

Denote by \( \varphi_t \) the solution of the nonlinear Hartree equation (1.3) with initial data \( \varphi_{t=0} = \varphi \). Fix \( T > 0 \) such that

\[
\kappa := \sup_{|t| \leq T} \| \varphi_t \|_{H^{1/2}} < \infty. \tag{1.13}
\]

Then there exists a constant \( C = C(\kappa, T, \| \varphi \|_{H^2}, \beta) \) such that

\[
\text{Tr} \left| (1 - \Delta)^{1/4} \left( \gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right) (1 - \Delta)^{1/4} \right| \leq C \left( \frac{1}{N^{1/4}} + \alpha_N^{1/2} \right) \tag{1.14}
\]

for all \( t \in \mathbb{R} \) with \( |t| \leq T \), and for all \( N \) sufficiently large. In particular, it follows that \( \gamma_{N,t}^{(1)} \to |\varphi_t\rangle \langle \varphi_t| \) in energy-norm, as \( N \to \infty \).

Remark. We believe that the same arguments used to show (1.14) can be extended to prove the convergence (in energy norm) of the higher marginal \( \gamma_{N,t}^{(k)} \). To keep the paper readable, we do not follow this direction here. Note that a simple argument, similar to the one presented in Section 2 of [19] (and mentioned in the remark after Theorem 1.1), shows that (1.14) implies

\[
\text{Tr} \left| (1 - \Delta x_t)^{1/4} \left( \gamma_{N,t}^{(k)} - |\varphi_t\rangle \langle \varphi_t|^{\otimes k} \right) (1 - \Delta x_t)^{1/4} \right| \leq C \sqrt{k} \left( \frac{1}{N^{1/4}} + \alpha_N^{1/2} \right)
\]

for all \( k \in \mathbb{N} \).

The proof of Theorem 1.2 is based again on a Fock space representation of the many body system, and on the use of coherent states as initial data. As in the proof of Theorem 1.1 the
Hartree component of the evolution of the initial coherent states can be isolated, and the dynamics of the fluctuations can be written through a two-parameter group of unitary transformations with an explicit generator. To obtain convergence in the energy norm, however, instead of controlling the growth of the number of particle operator, we need to control the growth of the kinetic energy operator with respect to the fluctuation dynamics. Technically, this step (contained in Proposition 5.2) is the most challenging part of our paper. In a sense, the fact that we can control the growth of the kinetic energy of the fluctuations implies that, although on the $N$-particle level we are considering a supercritical regime, after subtracting the (supercritical) Hartree dynamics, the system on the level of the fluctuations is subcritical. In Section 5 we prove Proposition 5.2) is the most challenging part of our paper. In a sense, the fact that we can control the growth of the number of particle operator, we need to control the growth of the kinetic energy of the fluctuations is subcritical. In Section 5 we prove the convergence to the Hartree dynamics for initial coherent states and we complete the proof of Theorem 1.2 assuming Proposition 5.2 to hold true. In Section 6 we prove Proposition 5.2.

Theorems 1.1 and 1.2 show that, as long as a bound on the $H^{1/2}$-norm of the solution $\varphi_t$ of the Hartree equation (1.5) is available, the evolution of the marginal densities can still be approximated by $\varphi_t$. Next, we ask what happens if the solution $\varphi_t$ of (1.5) exhibits blowup. Under the assumption that $\varphi_t$ blows up as $t \to T$, for some $0 < T < \infty$, we show that also the solution of the regularized $N$-particle Schrödinger equation (1.9) collapses, if $t \to T$, and, simultaneously, $N \to \infty$. The $N$-particle wave function $\psi_{N,t}$ collapses in the sense that the kinetic energy per particle, which remains finite, uniformly in $N$, up to time $T$, diverges to infinity as $t \to T$ if simultaneously, $N \to \infty$. In order to make sure that the solution of the $N$-particle Schrödinger equation remains close to the solution of the Hartree equation as $t$ approaches the nonlinear blowup time, we have to assume that $N$ diverges to infinity sufficiently fast. Physically, this condition imposes restrictions to the range of many body systems for which the Hartree approximation is valid close to the blowup time. From a different point of view (if we think of the number of particles $N$ as fixed), it tells us how close to the nonlinear blowup time we can expect the Hartree dynamics to be a good approximation for the real many body quantum evolution.

**Corollary 1.3.** Fix $\varphi \in H^2(\mathbb{R}^3)$ with $\| \varphi \| = 1$ and set $\psi_N = \varphi^{\otimes N}$. Consider an arbitrary sequence $\alpha_N > 0$ with $\alpha_N \to 0$ and $N^\beta \alpha_N \to \infty$ as $N \to \infty$, for some $\beta > 0$. Let $\psi_{N,t} = e^{-iH_N^0 t} \psi_N$ be the evolution of the initial wave function $\psi_N$ generated by the Hamiltonian (1.8) and let $\gamma_{N,t}^{(1)}$ be the one-particle reduced density associated with $\psi_{N,t}$.

Denote by $\varphi_t$ the solution of the nonlinear Hartree equation (1.5) with initial data $\varphi_{t=0} = \varphi$. Suppose that $T_c > 0$ is the first time of blow-up for $\varphi_t$. In other words, assume that

$$\kappa_t := \sup_{0 < s < t} \| \varphi_s \|_{H^{1/2}} < \infty$$

for all $t < T_c$, and

$$\| \varphi_t \|_{H^{1/2}} \to \infty \quad \text{as} \quad t \to T_c^-.$$

Then, for any fixed $t \in [0, T_c)$ there exists a constant $C_t > 0$ such that

$$\|(1 - \Delta_{x_1})^{1/4} \psi_{N,t} \|^2 \text{Tr} (1 - \Delta)^{1/2} \gamma_{N,t}^{(1)} < C_t$$

uniformly in $N \in \mathbb{N}$. Moreover, for $t \in [0, T_c)$, there exists $N(t) \in \mathbb{N}$ with $N(t) \to \infty$ as $t \to T_c^-$, and such that

$$\|(1 - \Delta_{x_1})^{1/4} \psi_{N(t),t} \|^2 = \text{Tr} (1 - \Delta)^{1/2} \gamma_{N(t),t}^{(1)} \to \infty \quad \text{as} \quad t \to T_c^-.$$  \hspace{1cm} (1.15)

In other words, the kinetic energy per particle is uniformly bounded in $N$, if $0 \leq t < T_c$ but it diverges in the limit $t \to T_c^-$, if at the same time, the number of particles tends to infinity sufficiently fast.
Remark 1. The existence of blow-up for solutions of the nonlinear Hartree equation (1.5) has been proven in [9] under the assumption that the initial data \( \varphi \) is spherically symmetric and that it has negative energy \( E_{\text{Hartree}}(\varphi) < 0 \); see (1.2) (this is possible if \( \lambda > \lambda_{\text{crit}}^H \)).

Remark 2. The fact that \( \psi_{N,t} \) collapses at some point in the interval \([0,T_c]\) follows already from the blow-up of \( \varphi_t \) at time \( T_c \) and from the Theorem 1.1. This fact, which was pointed out to us by R. Seiringer, follows from the general observation that the kinetic energy of an \( L^2 \)-limit is always smaller than the limit of the kinetic energy. This argument, however, does not prove that the collapse takes place at time \( T_c \) nor that the blow-up of the Hartree equation accurately describes it.

Proof of Corollary 1.3. Set \( \beta_N = N^{-1/4} + \alpha_N \); then \( \beta_N \to 0 \) as \( N \to \infty \). For every \( 0 < t < T_c \) there exists, by Theorem 1.2, a constant \( C_t \), depending on \( \beta,\kappa,t,\|\varphi\|_{H^2} \), such that

\[
\text{Tr} \left| (1 - \Delta)^{1/4} \left( \gamma_{N,s}^{(1)} - |\varphi_s\rangle\langle\varphi_s| \right) (1 - \Delta)^{1/4} \right| \leq C_t \beta_N
\]

for all \( 0 < s < t \). In particular this implies that

\[
\left| \text{Tr} (1 - \Delta)^{1/2} \gamma_{N,t}^{(1)} - \|\varphi_t\|_{H^{1/2}}^2 \right| \leq C_t \beta_N .
\]

For \( 0 < t < T_c \), choose now \( N(t) \) sufficiently large, so that \( \gamma_{N(t)} \leq (T_c - t)/C_t \) (this is certainly possible because \( \beta_N \to 0 \) as \( N \to \infty \)). Then

\[
\left| \text{Tr} (1 - \Delta)^{1/2} \gamma_{N(t),t}^{(1)} - \|\varphi_t\|_{H^{1/2}}^2 \right| \to 0
\]

as \( t \to T_c \). Since \( \|\varphi_t\|_{H^{1/2}} \to \infty \) as \( t \to T_c \), this implies that

\[
\text{Tr} (1 - \Delta)^{1/2} \gamma_{N(t),t}^{(1)} \to \infty .
\]

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2 Bounds on solutions of nonlinear Hartree equations

In this section, we study properties of the solution of the nonlinear Hartree equation (1.5). In particular, we need to compare the solution of (1.5) with the solution of regularized Hartree equations (like (2.1), with \( \alpha > 0 \)). To this end, we first need to establish the property of propagation of initial regularity, under the assumption of a bound on the \( H^{1/2} \)-norm.

Proposition 2.1 (Propagation of regularity). Fix \( s > 1/2 \) and \( \alpha \geq 0 \). Let \( \varphi \in H^s(\mathbb{R}^3) \) with \( \|\varphi\| = 1 \). Let \( \varphi_t \) denote the solution to the nonlinear Hartree equation

\[
\begin{equation}
\tag{2.1}
i\hbar \partial_t \varphi_t = \sqrt{1 - \Delta} \varphi_t - \lambda \left( \frac{1}{|\cdot| + \alpha} \ast |\varphi_t|^2 \right) \varphi_t
\end{equation}
\]
with the initial condition $\varphi_{t=0} = \varphi$. Fix $T > 0$ such that

$$\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty.$$ \hfill (2.2)

Then there exists a constant $\nu = \nu(\kappa, T, s, \|\varphi\|_{H^s}) < \infty$ (but independent of $\alpha$) such that

$$\sup_{|t| \leq T} \|\varphi_t\|_{H^s} \leq \nu.$$ \hfill (2.3)

**Proof.** We follow here the proof of \cite[Lemma 3]{16} with some modifications. Let $J(\varphi) := ((|\cdot| + \alpha)^{-1} * |\varphi|^2)\varphi$. Then we claim that

$$\|J(\varphi)\|_{H^s} \lesssim \|\varphi\|_{H_1^{1/2}}^2 \|\varphi\|_{H^s}, \quad \text{for all } \varphi \in H^s(\mathbb{R}^3).$$ \hfill (2.4)

In fact, $\|J(\varphi)\|_{H^s} \lesssim \|J(\varphi)\|_2 + \|(-\Delta)^{s/2}J(\varphi)\|_2$ and

$$\|J(\varphi)\|_2 \lesssim \left\|\left(\frac{1}{\cdot} + \alpha \ast |\varphi|^2\right)\varphi\right\|_2 \lesssim \left\|\frac{1}{\cdot} \ast |\varphi|^2\right\|_{\infty} \|\varphi\|_2 \lesssim \|\varphi\|_{H_1^{1/2}}^2 \|\varphi\|_2.$$ \hfill (2.5)

Moreover,

$$\|(-\Delta)^{s/2}J(\psi)\|_2 = \left\|(-\Delta)^{s/2} \left(\frac{1}{|\cdot| + \alpha} \ast |\varphi|^2\right)\varphi\right\|_2 \lesssim \left\|\frac{1}{|\cdot|} \ast (-\Delta)^{s/2}|\varphi|^2\right\|_6 \|\varphi\|_3 + \left\|\frac{1}{|\cdot| + \alpha} \ast |\varphi|^2\right\|_{\infty} \|(-\Delta)^{s/2}\varphi\|_2 \lesssim \|(-\Delta)^{s/2}|\varphi|^2\|_6 \|\varphi\|_{H^{1/2}} + \|\varphi\|_{H_1^{1/2}}^2 \|\varphi\|_{H^s} \lesssim \|\varphi\|_{H_1^{1/2}}^2 \|\varphi\|_{H^s}.$$ \hfill (2.6)

Here we used the generalized Leibniz rule (see Lemma \cite{2.4}) in the first inequality. In the second inequality, we used the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality $\|\varphi\|_3 \lesssim \|\varphi\|_{H^{1/2}}$ to bound the first term, and Kato's inequality

$$\sup_{x \in \mathbb{R}^3} \int dy \frac{|\varphi(y)|^2}{|x - y|} \leq \frac{\pi}{2} \int dy \left| \nabla \varphi(y) \right|^2 \leq \frac{\pi}{2} \|\varphi\|_{H^{1/2}}^2$$ \hfill (2.7)

to bound the second term. Finally, in the third inequality, we used again the generalized Leibniz rule. This shows (2.4).

Next, we write $\varphi_t$ as

$$\varphi_t = e^{-i \sqrt{-\Delta} t} \varphi + i \lambda \int_0^t ds e^{-i \sqrt{-\Delta} (t-s)} \left(\frac{1}{|\cdot| + \alpha_N} \ast |\varphi_s|^2\right) \varphi_s$$ \hfill (2.8)

and we obtain, by (2.4) and (2.2), that

$$\|\varphi_t\|_{H^s} \lesssim \|\varphi\|_{H^s} + \int_0^t \|J(\varphi_\tau)\|_{H^s} \, d\tau \lesssim \|\varphi\|_{H^s} + \kappa^2 \int_0^t \|\varphi_\tau\|_{H^s} \, d\tau.$$ \hfill (2.9)

The proposition now follows applying Gronwall's inequality to (2.9).
Next, under the assumption that the solution $\varphi_t$ of the Hartree equation (1.5) with initial data $\varphi \in H^2(\mathbb{R}^3)$ stays bounded in $H^{1/2}$ in the interval $[-T,T]$, we show the vicinity (in the $H^{1/2}$-norm) of the solution $\varphi_t^{(\alpha)}$ of the regularized equation (2.1), with $\alpha > 0$ and small, to $\varphi_t$.

**Proposition 2.2.** Fix $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\| = 1$ and let $\varphi_t$ denote the solution of the nonlinear Hartree equation (1.5) with initial condition $\varphi_t=0 = \varphi$. Let $T > 0$ be such that

$$\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty.$$  \hspace{1cm} (2.10)

For $\alpha > 0$, let $\varphi_t^{(\alpha)}$ be the solution to the regularised Hartree equation

$$i\partial_t \varphi_t^{(\alpha)} = \sqrt{1 - \Delta} \varphi_t^{(\alpha)} - \lambda \left( \frac{1}{|x| + \alpha} \ast \|\varphi_t^{(\alpha)}\|^2 \right) \varphi_t^{(\alpha)}$$ \hspace{1cm} (2.11)

with initial condition $\varphi_t=0 = \varphi$.

Then there exists a constant $C = C(T, \kappa, \|\varphi\|_{H^1}) < \infty$, such that

$$\|\varphi_t - \varphi_t^{(\alpha)}\|_2 \leq C \alpha \quad \text{for all } |t| \leq T \text{ and all } \alpha > 0.$$ \hspace{1cm} (2.12)

Moreover, if we assume additionally that $\varphi \in H^2(\mathbb{R}^3)$, then we can also find a constant $D = D(T, \kappa, \|\varphi\|_{H^2}) < \infty$ such that

$$\|\varphi_t - \varphi_t^{(\alpha)}\|_{H^{1/2}} \leq D \alpha^{1/2},$$ \hspace{1cm} (2.13)

for all $|t| \leq T$ and $0 < \alpha < 1$.

**Proof.** We start by proving (2.12). Since $\varphi \in H^1(\mathbb{R}^3)$, we can find, by (2.10) and Proposition 2.1 $\nu = \nu(T, \kappa, \|\varphi\|_{H^1}) < \infty$ such that

$$\sup_{|t| \leq T} \|\varphi_t\|_{H^1} \leq \nu.$$ \hspace{1cm} (2.14)

Let $t \in [-T,T]$. We have

$$\frac{d}{dt} \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 = -2 \frac{d}{dt} \text{Re} \langle \varphi_t, \varphi_t^{(\alpha)} \rangle$$

$$= 2\lambda \text{Im} \left\langle \varphi_t, \left( \frac{1}{|x|} \ast |\varphi_t|^2 - \frac{1}{|x| + \alpha} \ast |\varphi_t^{(\alpha)}|^2 \right) \varphi_t^{(\alpha)} \right\rangle$$

$$= 2\lambda \text{Im} \left\{ \left\langle \varphi_t, \left( \frac{\alpha}{|x|(|x| + \alpha)} \ast |\varphi_t|^2 \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle + \left\langle \varphi_t, \left( \frac{1}{|x| + \alpha} \ast (|\varphi_t|^2 - |\varphi_t^{(\alpha)}|^2) \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle \right\}.$$}

Therefore

$$\frac{d}{dt} \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \leq 2|\lambda| \left\{ \left| \left\langle \varphi_t, \left( \frac{\alpha}{|x|(|x| + \alpha)} \ast |\varphi_t|^2 \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle \right|$$

$$+ \left| \left\langle \varphi_t, \left( \frac{1}{|x| + \alpha} \ast (|\varphi_t|^2 - |\varphi_t^{(\alpha)}|^2) \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle \right| \right\}.$$ \hspace{1cm} (2.15)
The first summand in the r.h.s. of (2.13) can be estimated as
\[
\left| \left\langle \varphi_t, \left( \frac{\alpha}{|x|(|x|+\alpha)} * |\varphi_t|^2 \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle \right| \leq \left\| \frac{\alpha}{|x|(|x|+\alpha)} * |\varphi_t|^2 \right\|_\infty \|\varphi_t - \varphi_t^{(\alpha)}\|_2 \\
\leq \alpha \left\| \frac{1}{|x|^2} * |\varphi_t|^2 \right\|_\infty \|\varphi_t - \varphi_t^{(\alpha)}\|_2 \\
\leq \alpha \|\varphi_t - \varphi_t^{(\alpha)}\|_2 \|\varphi_t\|_{H^1} \lesssim \alpha \nu^2 \|\varphi_t - \varphi_t^{(\alpha)}\|_2.
\]

The second summand can be estimated by
\[
\left| \left\langle \varphi_t, \left( \frac{1}{|x|+\alpha} * (|\varphi_t|^2 - |\varphi_t^{(\alpha)}|^2) \right)(\varphi_t^{(\alpha)} - \varphi_t) \right\rangle \right| \\
\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx \, dy \, |\varphi_t(x)| \frac{1}{|x-y|+\alpha} |\varphi_t(y) - \varphi_t^{(\alpha)}(y)| \left( |\varphi_t(y)| + |\varphi_t^{(\alpha)}(y)| \right) |\varphi_t^{(\alpha)}(x) - \varphi_t(x)| \\
\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx \, dy \, |\varphi_t(x)|^2 \frac{1}{|x-y|^2} |\varphi_t(y) - \varphi_t^{(\alpha)}(y)|^2 \\
+ 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx \, dy \left( |\varphi_t(y)|^2 + |\varphi_t^{(\alpha)}(y)|^2 \right) |\varphi_t^{(\alpha)}(x) - \varphi_t(x)|^2 \\
\leq \left\| \frac{1}{|x|^2} * |\varphi_t|^2 \right\|_\infty \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 + 2 (\|\varphi_t||_2^2 + \|\varphi_t^{(\alpha)}||_2^2) \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \\
\lesssim (\|\varphi_t||_{H^1}^2 + 1) \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \lesssim (1 + \nu^2) \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2,
\]

where on the last line we used Hardy’s inequality
\[
\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} dy \, \frac{|\varphi(y)|^2}{|x-y|^2} \leq 4 \int_{\mathbb{R}^3} dy \, |\nabla \varphi(y)|^2 \leq 4 \|\varphi\|_{H^1}^2.
\]

Thus, (2.15) gives
\[
\frac{d}{dt} \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \lesssim (1 + \nu^2) \left( \alpha \|\varphi_t - \varphi_t^{(\alpha)}\|_2 + \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \right) \\
\lesssim (1 + \nu^2) \left( \alpha^2 + \|\varphi_t - \varphi_t^{(\alpha)}\|_2^2 \right).
\]

By Gronwall’s inequality, we find $C = C(\nu, T)$ with
\[
\|\varphi_t - \varphi_t^{(\alpha)}\|_2 \leq C \alpha
\]
for all $\alpha > 0$.

Next, we prove (2.13). To this end, it is enough to show that there exists $D = D(T, \kappa, \|\varphi\|_{H^2})$ such that
\[
\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^{(\alpha)})\|_2 \lesssim D \alpha^{1/2}.
\]

Note that, since $\varphi \in H^2(\mathbb{R}^3)$, we can find, by (2.10) and Proposition 2.1, $\nu = \nu(T, \kappa, \|\varphi\|_{H^2}) < \infty$ such that
\[
\sup_{|t| \leq T} \|\varphi_t\|_{H^2} \leq \nu.
\]

We write $\varphi_t$ and $\varphi_t^{(\alpha)}$ using their Duhamel expansions
\[
\varphi_t = e^{-i\sqrt{1-\Delta} t} \varphi + i \lambda \int_0^t ds \, e^{-i\sqrt{1-\Delta}(t-s)} \frac{1}{|x|} * |\varphi_s|^2 \varphi_s
\]
and
\[ \varphi^{(\alpha)}_t = e^{-i\sqrt{-1-\alpha}t} \varphi + i\lambda \int_0^t ds \, e^{-i\sqrt{-1-\alpha}(t-s)} \left( \frac{1}{|x| + \alpha} \ast |\varphi^{(\alpha)}_s|^2 \right) \varphi^{(\alpha)}_s \]
respectively. Thus
\[
\|(\Delta)^{1/4}(\varphi_t - \varphi^{(\alpha)}_t)\|_2 \leq |\lambda| \int_0^t ds \left\{ \|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \\
+ \|(\Delta)^{1/4} \left( \frac{\alpha}{|x|(|x| + \alpha)} \ast |\varphi_s|^2 \right) \varphi^{(\alpha)}_s \|_2 \\
+ \|(\Delta)^{1/4} \left( \frac{1}{|x| + \alpha} \ast (|\varphi_s|^2 - |\varphi^{(\alpha)}_s|^2) \right) \varphi^{(\alpha)}_s \|_2 \right\} 
\] (2.21)
Further decomposing the second and third term in the parenthesis we find
\[
\|(\Delta)^{1/4}(\varphi_t - \varphi^{(\alpha)}_t)\|_2 \leq |\lambda| \int_0^t ds \left\{ \|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \\
+ \|(\Delta)^{1/4} \left( \frac{\alpha}{|x|(|x| + \alpha)} \ast |\varphi_s|^2 \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \\
+ \|(\Delta)^{1/4} \left( \frac{1}{|x| + \alpha} \ast (|\varphi_s|^2 - |\varphi^{(\alpha)}_s|^2) \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \right\} 
\] (2.22)
The first term is bounded by
\[
\|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \leq \|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) \|_6 \|\varphi_s - \varphi^{(\alpha)}_s\|_3 \\
+ \left\| \frac{1}{|x|} \ast |\varphi_s|^2 \right\|_\infty \|(\Delta)^{1/4} \left( \varphi_s - \varphi^{(\alpha)}_s \right) \|_2
\] (2.23)
where we used the generalized Leibniz rule (see Lemma 2.4). Next we observe that, by Kato’s inequality (2.7) and by (2.20), we have \|.|^{-1} \ast |\varphi_s|^2\|_\infty \lesssim \|\varphi_s\|_{H^{1/2}} \lesssim \nu^2. This, combined with the bound
\[
\|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) \|_6 \lesssim \|\varphi_s\|_3^3 \lesssim \|\varphi_s\|^2_{H^{1/2}} \lesssim \nu^2 
\] (2.24)
implies (using also (2.12)) that
\[
\|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) (\varphi_s - \varphi^{(\alpha)}_s) \|_2 \lesssim \nu^2 \|\varphi_s - \varphi^{(\alpha)}_s\|_{H^{1/2}} \lesssim \nu^2 \alpha + \nu^2 \|(\Delta)^{1/4} \left( \varphi_s - \varphi^{(\alpha)}_s \right) \|_2 
\] (2.25)
To prove (2.21), we rewrite \|.|^{-1} \ast |\varphi_s|^2 = -4\pi (\Delta)^{-1} |\varphi_s|^2. Then
\[
\|(\Delta)^{1/4} \left( \frac{1}{|x|} \ast |\varphi_s|^2 \right) \|_6 \lesssim \|(\Delta)^{-3/4} |\varphi_s|^2 \|_6 = \| G_{3/2} \ast |\varphi_s|^2 \|_6 .
\] (2.26)
Here \( G_s, s \in (0,3) \), is the kernel of the operator \((\Delta)^{-s/2}\) which is explicitly given by
\[
G_{3/2}(x) = c_{3/2} |x|^{-3/2} 
\] (2.27)
with \( c_{3/2} = \pi^2 \sqrt{2}/\Gamma(\frac{3}{2}) \). From (2.28), we conclude by the Littlewood-Hardy-Sobolev inequality that
\[
\|(-\Delta)^{1/4}\left(\frac{1}{|x|} * |\varphi_s|^2\right)\|_6 \leq \|G_{3/2} * |\varphi_s|^2\|_6 \lesssim \|\varphi_s\|_3^4
\]
and thus (2.23) follows.

The second term on the r.h.s. of (2.22) is estimated again by the generalized Leibniz rule as
\[
\|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_2 \lesssim \|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_\infty \|\varphi_s - \varphi_s^{(\alpha)}\|_2 + \|\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\|_\infty \|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(\alpha)})\|_2. 
\]
(2.28)

Since
\[
\|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_\infty \leq \|\frac{\alpha}{|x|(|x| + \alpha)}\|_2 \|(-\Delta)^{1/4}|\varphi_s|^2\|_2 \lesssim \alpha^{1/2} \|(-\Delta)^{1/4}|\varphi_s|_3\| \|\varphi_s\|_6 \lesssim \alpha^{1/2} \|\varphi_s\|_{H^1}^2 \lesssim \alpha^{1/2} \nu^2 
\]
(2.29)

and, by (2.16),
\[
\|\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\|_\infty \leq \alpha \|\frac{1}{|x|^2} * |\varphi_s|^2\|_\infty \lesssim \alpha \|\varphi_s\|_{H^1}^4 \lesssim \alpha \nu^2 
\]
(2.30)

we find, using (2.12), that
\[
\|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_2 \lesssim \alpha^{3/2} \nu^2 + \alpha \nu^2 \|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(\alpha)})\|_2. 
\]
(2.31)

The third summand in (2.22) is estimated (again using (2.16)) as
\[
\|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\varphi_s\|_2 \lesssim \|\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\|_\infty \|(-\Delta)^{1/4}\varphi_s\|_2 + \|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_3 \|\varphi_s\|_6 \lesssim \alpha \|\varphi_s\|_{H^1}^4 \|\varphi_s\|_{H^{1/2}} \lesssim \alpha \nu^3
\]
(2.32)

where we used (2.30), the Sobolev inequality \( \|\varphi_s\|_6 \lesssim \|\varphi_s\|_{H^1} \) and the bound
\[
\|(-\Delta)^{1/4}\left(\frac{\alpha}{|x|(|x| + \alpha)} * |\varphi_s|^2\right)\|_3 \lesssim \alpha \|\frac{1}{|x|^2} * |(-\Delta)^{1/4}|\varphi_s|^2\|_3 \lesssim \alpha \|(-\Delta)^{1/4}(\varphi_s \varphi_s)\|_{3/2} \lesssim \alpha \|\varphi_s\|_{H^{1/2}} \|\varphi_s\|_{H^{1/2}}^3
\]
where we used the Littlewood-Hardy-Sobolev inequality, and, in the last inequality, the generalized Leibniz rule (see Lemma 2.4 below).
The fourth summand on the r.h.s. of (2.22) is bounded by
\[
\|(-\Delta)^{1/4} \left( \frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(a)}|^2) \right) \varphi_s \|_2 \\
\lesssim \left\| \left( (-\Delta)^{1/4} \frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(a)}|^2) \right) \|_{2+\varepsilon} \frac{\|\varphi_s\|_{2(2+\varepsilon)}}{2} \right) \\
+ \frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(a)}|^2) \right\|_6 \|(-\Delta)^{1/4}\varphi_s\|_3
\] (2.33)
for arbitrary \( \varepsilon > 0 \). The second term on the r.h.s. of (2.33) is estimated by
\[
\left\| \frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(a)}|^2) \right\|_6 \|(-\Delta)^{1/4}\varphi_s\|_3 \lesssim \|\varphi_s\|_{H^1} \left\| \frac{1}{|x|} \right\|_1 \|\varphi_s|^2 - |\varphi_s^{(a)}|^2 \right\|_6 \\
\lesssim \|\varphi_s\|_{H^1} \left\| \varphi_s - \varphi_s^{(a)} \right\|_1 \|\varphi_s\|_{H^2} \\
\lesssim \|\varphi_s\|_{H^1} \left( \|\varphi_s\|_2 + \|\varphi_s^{(a)}\|_2 \right) \|\varphi_s - \varphi_s^{(a)}\|_3 \\
\lesssim \nu \|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(a)})\|_2.
\] (2.34)
where we used the Sobolev inequality on the first, the Hardy-Littlewood-Sobolev inequality on the second, the Hölder inequality on the third, and, finally, again the Sobolev inequality in the fourth line.

As for the first term on the r.h.s. of (2.33), we notice that
\[
\left\| \left((-\Delta)^{1/4} \frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(a)}|^2) \right) \|_{2+\varepsilon} \frac{\|\varphi_s\|_{2(2+\varepsilon)}}{2} \right) \\
\lesssim \left\| (-\Delta)^{1/4} \frac{1}{|x| + \alpha} \right\|_{2+\varepsilon} \left\| \varphi_s - \varphi_s^{(a)} \right\|_1 \|\varphi_s\|_{H^2} \\
\lesssim \left\| (-\Delta)^{1/4} \frac{1}{|x| + \alpha} \right\|_{2+\varepsilon} \left\| \varphi_s - \varphi_s^{(a)} \right\|_2 \|\varphi_s\|_{H^2} \\
\lesssim \nu \alpha^{-\varepsilon}
\] (2.35)
where in the last step we used the bound
\[
\left\| (-\Delta)^{1/4} \frac{1}{|x| + \alpha} \right\|_{2+\varepsilon} \lesssim \alpha^{-\varepsilon}
\] (2.36)
for all \( \varepsilon > 0 \). The bound (2.36) follows from the pointwise estimate
\[
\left| \left((-\Delta)^{1/4} \frac{1}{|x| + \alpha} \right) \right| \lesssim \frac{1}{(|x| + \alpha)^{3/2}}
\] (2.37)
valid for all \( x \in \mathbb{R}^3 \). To show (2.37), we observe that
\[
\left( (-\Delta) \frac{1}{(|x| + \alpha)} \right)(x) = -\frac{2\alpha}{|x(|x| + \alpha)^3}
\]
and therefore
\[
\left| \left((-\Delta)^{1/4} \frac{1}{|x| + \alpha} \right) \right| \lesssim (-\Delta)^{-3/4} \frac{\alpha}{|x(|x| + \alpha)^3} \lesssim \int dy \frac{1}{|x - y|^{3/2}} \frac{\alpha}{|y|(|\alpha + |y|)^3}.
\] (2.38)
We assume first that $|x| \geq \alpha$. From (2.38) we find
\[
\left|\left(\frac{-\Delta}{|x| + \alpha}\right)^{1/4}\left(x\right)\right| \lesssim \int_{|x-y| \geq |x|/2} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|(|\alpha + |y|)^3} + \int_{|x-y| \leq |x|/2} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|(|\alpha + |y|)^3}
\]
\[
\lesssim \frac{1}{|x|^{3/2}} \int dy \frac{\alpha}{|y|(|\alpha + |y|)^3} + \frac{1}{|x|^3} \int_{|x-y| \leq |x|/2} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|}
\]
where we used the fact that $|x-y| \leq |x|/2$ implies $|y| \geq |x|/2$. Explicit computation implies that
\[
\left|\left(\frac{-\Delta}{|x| + \alpha}\right)^{1/4}\left(x\right)\right| \lesssim \frac{1}{|x|^{3/2}} \lesssim \frac{1}{(|x| + \alpha)^{3/2}} \quad \text{for all } |x| \geq \alpha.
\] (2.39)
For $|x| \leq \alpha$ we notice that, by (2.38),
\[
\left|\left(\frac{-\Delta}{|x| + \alpha}\right)^{1/4}\left(x\right)\right| \lesssim \int_{|x-y| \geq \alpha} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|(|\alpha + |y|)^3} + \int_{|x-y| \leq \alpha} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|(|\alpha + |y|)^3}
\]
\[
\lesssim \frac{1}{\alpha^{3/2}} \int dy \frac{\alpha}{|y|(|\alpha + |y|)^3} + \frac{1}{\alpha^2} \int_{|x-y| \leq \alpha} \frac{dy}{|x-y|^{3/2}} \frac{\alpha}{|y|}
\] (2.40)
Since $|x| \leq \alpha$ and $|x-y| \leq \alpha$ imply that $|y| \leq 2\alpha$, the last term is bounded, for $|x| \leq \alpha$, by
\[
\int_{|x-y| \leq \alpha} \frac{dy}{|x-y|^{3/2}|y|} \lesssim \int_{|x-y| \leq \alpha} \frac{dy}{|x-y|^{5/2}} + \int_{|y| \leq 2\alpha} \frac{1}{|y|^{5/2}} \lesssim \alpha^{1/2}
\]
Inserting back in (2.30), it follows that
\[
\left|\left(\frac{-\Delta}{|x| + \alpha}\right)^{1/4}\left(x\right)\right| \lesssim \frac{1}{\alpha^{3/2}} \lesssim \frac{1}{(|x| + \alpha)^{3/2}} \quad \text{for all } |x| \leq \alpha.
\] Together with (2.39), this implies (2.37) and therefore (2.35). Combining (2.34) with (2.35), we obtain the bound
\[
\left\|(-\Delta)^{1/4}\left(\frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(\alpha)}|^2)\right)\varphi_s\right\|_2 \leq \nu \left\|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(\alpha)})\right\|_2 + \nu \alpha^{1-\varepsilon}
\] (2.41)
for all $\varepsilon > 0$.

The fifth summand in (2.22) is estimated as
\[
\left\|(-\Delta)^{1/4}\left(\frac{1}{|x| + \alpha} * (|\varphi_s|^2 - |\varphi_s^{(\alpha)}|^2)\right)\varphi_s\right\|_2 \leq \nu \left\|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(\alpha)})\right\|_2 + \nu \alpha^{1-\varepsilon}
\] (2.42)
where we used the generalized Leibniz rule in the first inequality and the bounds (2.12) and (2.37) in the last inequality.
Inserting the estimates (2.25), (2.31), (2.32), (2.41), and (2.42) into (2.22) yields (using that \( \nu \geq 1 \) and the assumption \( \alpha \leq 1 \))

\[
\left\|(-\Delta)^{1/4}(\varphi_t - \varphi_t^{(\alpha)})\right\|_2 \lesssim \nu^3 \int_0^t \left\{\left\|(-\Delta)^{1/4}(\varphi_s - \varphi_s^{(\alpha)})\right\|_2 + \alpha^{1/2}\right\} ds.
\] (2.43)

Eq. (2.43) follows by Gronwall’s lemma.

In the next corollary, we summarize the consequences of the bound on the \( H^{1/2} \)-norm of \( \varphi_t \) (and of the assumption \( \varphi \in H^2(\mathbb{R}^3) \) on the initial data), that will play a crucial role in the many body analysis.

**Corollary 2.3.** Fix \( s \geq 2 \) and \( \varphi \in H^s(\mathbb{R}^3) \). Let \( \varphi_t \) and, for any \( \alpha > 0 \), \( \varphi_t^{(\alpha)} \) be the solutions of the nonlinear Hartree equations (1.5) and, respectively, (2.11) with initial data \( \varphi_{t=0} = \varphi_t^{(\alpha)} = \varphi \) (\( \varphi_t \) is the maximal local solution of (1.5) in \( H^{1/2}(\mathbb{R}^3) \); \( \varphi_t^{(\alpha)} \), on the other hand, is known to exist globally in \( H^{1/2}(\mathbb{R}^3) \)). Fix \( T > 0 \) such that

\[
\kappa := \sup_{|t| \leq T} \left\|\varphi_t\right\|_{H^{1/2}} < \infty.
\]

Then there exists \( \nu = \nu(s, T, \kappa, \|\varphi\|_{H^s}) < \infty \) independent of \( \alpha \) such that

\[
\sup_{|t| \leq T} \left\|\varphi_t^{(\alpha)}\right\|_{H^s} \leq \nu
\]

for all \( \alpha > 0 \) small enough.

**Proof.** Since \( s \geq 2 \), Proposition 2.2 implies that

\[
\sup_{|t| \leq T} \left\|\varphi_t^{(\alpha)}\right\|_{H^{1/2}} \leq 2\kappa
\]

for sufficiently small \( \alpha > 0 \). The claim follows then by Proposition 2.1.

To conclude this section, we state the generalized Leibniz rule for fractional derivatives. For a proof of this lemma, see [14].

**Lemma 2.4 (Generalized Leibniz Rule).** Suppose that \( 1 < p < \infty \), \( s \geq 0 \), \( \alpha \geq 0 \), \( \beta \geq 0 \), and \( 1/p_i + 1/q_i = 1/p \) with \( i = 1, 2 \), \( 1 < q_1 \leq \infty \), \( 1 < p_i \leq \infty \). Then there exists a constant \( c = c(p, p_1, p_2, s, \alpha, \beta) < \infty \) such that

\[
\left\|(-\Delta)^{s/2}(fg)\right\|_p \leq c \left( \left\|(-\Delta)^{(s+\alpha)/2}f\right\|_{p_1} \left\|(-\Delta)^{-\alpha/2}g\right\|_{q_1} + \left\|(-\Delta)^{-\beta/2}f\right\|_{p_2} \left\|(-\Delta)^{(s+\beta)/2}g\right\|_{q_2} \right)
\]

for all measurable functions \( f, g \) for which the r.h.s. is finite.
3 Fock space representation

In this section, we introduce a Fock-space representation of our system, and we define coherent states. The bosonic Fock space over $L^2(\mathbb{R}^3, dx)$ is defined by

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2(\mathbb{R}^3, dx)^{\otimes_n} = \mathbb{C} \oplus \bigoplus_{n \geq 1} L^2_0(\mathbb{R}^{3n}, dx_1 \ldots dx_n),$$

with the convention $L^2(\mathbb{R}^3)^{\otimes_0} = \mathbb{C}$. Vectors in $\mathcal{F}$ are sequences $\psi = \{\psi^{(n)}\}_{n \geq 0}$ of $n$-particle wave functions $\psi^{(n)} \in L^2_0(\mathbb{R}^{3n})$. On $\mathcal{F}$, we introduce the scalar product

$$\langle \psi_1, \psi_2 \rangle = \sum_{n \geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^{3n})} = \overline{\psi_1^{(0)}} \psi_2^{(0)} + \sum_{n \geq 1} \int dx_1 \ldots dx_n \overline{\psi_1^{(n)}}(x_1, \ldots, x_n) \psi_2^{(n)}(x_1, \ldots, x_n).$$

It is simple to check that, with this inner product, $\mathcal{F}$ is a Hilbert space. States with $N$ particles and with wave function $\psi_N \in L^2_0(\mathbb{R}^{3N})$ are described on $\mathcal{F}$ by the sequence $\{\psi^{(n)}\}_{n \geq 0}$ where $\psi^{(n)} = 0$ for all $n \neq N$ and $\psi^{(N)} = \psi_N$. The vector $\{1, 0, 0, \ldots\} \in \mathcal{F}$ is called the vacuum, and will be denoted by $\Omega$.

The number of particles operator $\mathcal{N}$ acts on $\mathcal{F}$ according to $(\mathcal{N} \psi)^{(n)} = n \psi^{(n)}$ for all $n \in \mathbb{N}$. Eigenvectors of $\mathcal{N}$ are vectors of the form $\{0, \ldots, 0, \psi^{(m)}, 0, \ldots\}$ with a fixed number of particles.

For arbitrary $f \in L^2(\mathbb{R}^3)$ we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathcal{F}$ by

$$(a^*(f)\psi^{(n)})(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

and

$$(a(f)\psi^{(n)})(x_1, \ldots, x_n) = \sqrt{n+1} \int dx f(x) \psi^{(n+1)}(x, x_1, \ldots, x_n).$$

The operators $a^*(f)$ and $a(f)$ are unbounded, densely defined, closed operators. The creation operator $a^*(f)$ is the adjoint of the annihilation operator $a(f)$ (note that by definition $a(f)$ is anti-linear in $f$), and they satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0. \quad (3.2)$$

We will also make use of operator valued distributions $a_x^*$ and $a_x$ ($x \in \mathbb{R}^3$), defined so that

$$a^*(f) = \int dx f(x) a_x^*$$

and

$$a(f) = \int dx f(x) a_x$$

for every $f \in L^2(\mathbb{R}^3)$. The canonical commutation relations assume the form

$$[a_x, a_y^*] = \delta(x - y), \quad [a_x, a_y] = [a_x^*, a_y^*] = 0.$$

The number of particle operator, expressed through the distributions $a_x, a_x^*$, is given, formally, by

$$\mathcal{N} = \int dx a_x^* a_x.$$

The following standard lemma provides some useful bounds to control creation and annihilation operators in terms of the number of particle operator $\mathcal{N}$. 

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Lemma 3.1. Let \( f \in L^2(\mathbb{R}^3) \). Then we have

\[
\|a(f)\psi\| \leq \|f\| \|N^{1/2}\psi\| \quad \text{and} \quad \|a^*(f)\psi\| \leq \|f\| (N + 1)^{1/2} \|\psi\| \tag{3.4}
\]

For an arbitrary \( \psi \in \mathcal{F} \), we define the one-particle density \( \gamma^{(1)}_\psi \) associated with \( \psi \) as the positive trace class operator on \( L^2(\mathbb{R}^3) \) with kernel given by

\[
\gamma^{(1)}_\psi(x; y) = \frac{1}{\langle \psi, N\psi \rangle} \langle \psi, a^*_y a_x \psi \rangle. \tag{3.5}
\]

By definition, \( \gamma^{(1)}_\psi \) is a positive trace class operator on \( L^2(\mathbb{R}^3) \) with \( \text{Tr} \gamma^{(1)}_\psi = 1 \). For every \( N \)-particle state with wave function \( \psi_N \in L^2(\mathbb{R}^{3N}) \) (described on \( \mathcal{F} \) by the sequence \( \{0, 0, \ldots, \psi_N, 0, 0, \ldots\} \)) it is simple to see that this definition is equivalent to the standard definition.

For any sequence \( \alpha = (\alpha_N) \), with \( \alpha_N \to 0 \) as \( N \to \infty \), we define the Hamiltonian \( \mathcal{H}_N^\alpha \) on \( \mathcal{F} \) by

\[
(\mathcal{H}_N^\alpha)^{(n)} = \sum_{j=1}^{n} (1 - \Delta x_j)^{1/2} - \frac{\lambda}{N} \sum_{i<j} \frac{1}{|x_i - x_j| + \alpha_N}.
\]

Using the distributions \( a_x, a^*_x, \mathcal{H}_N^\alpha \) can be rewritten, formally, as

\[
\mathcal{H}_N^\alpha = \int dx \, a^*_x (1 - \Delta x)^{1/2} a_x - \frac{\lambda}{2N} \int dx \, xy a^*_x a_y a_x.
\tag{3.6}
\]

By definition, the Hamiltonian \( \mathcal{H}_N^\alpha \) leaves sectors of \( \mathcal{F} \) with a fixed number of particles invariant. Moreover, it is clear that on the \( N \)-particle sector, \( \mathcal{H}_N^\alpha \) agrees with the Hamiltonian \( \mathcal{H}_N^\alpha \). We will study the dynamics generated by the operator \( \mathcal{H}_N^\alpha \). In particular we will consider the time evolution of coherent states, which we introduce next.

For \( f \in L^2(\mathbb{R}^3) \), we define the Weyl-operator

\[
W(f) = \exp(a^*(f) - a(f)) \tag{3.7}
\]

and the coherent state \( \psi(f) \in \mathcal{F} \) with one-particle wave function \( f \) by \( \psi(f) = W(f)\Omega \). Notice that

\[
\psi(f) = W(f)\Omega = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{(a^*(f))^n \Omega}{n!} = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^{\otimes n}, \tag{3.8}
\]

where \( f^{\otimes n} \) indicates the Fock-vector \( \{0, \ldots, 0, f^{\otimes n}, 0, \ldots\} \). This follows from

\[
\exp(a^*(f) - a(f)) = e^{-\|f\|^2/2} \exp(a^*(f)) \exp(-a(f))
\]

which is a consequence of the fact that the commutator \([a(f), a^*(f)] = \|f\|^2\) commutes with \( a(f) \) and \( a^*(f) \). From Eq. (3.8) we see that coherent states are superpositions of states with different number of particles (the probability of having \( n \) particles in \( \psi(f) \) is given by \( e^{-\|f\|^2/2n} n! \)).

In the following standard lemma we collect some important and well known properties of Weyl operators and coherent states.

Lemma 3.2. Let \( f, g \in L^2(\mathbb{R}^3) \).
i) The Weyl operators satisfy the relations
\[ W(f)W(g) = W(g)W(f)e^{-2i\text{Im}(f,g)} = W(f+g)e^{-i\text{Im}(f,g)}. \]

ii) \( W(f) \) is a unitary operator and
\[ W(f)^* = W(f)^{-1} = W(-f). \]

iii) We have
\[ W^*(f)a_xW(f) = a_x + f(x), \quad \text{and} \quad W^*(f)a_x^*W(f) = a_x^* + \overline{f}(x). \]

iv) From iii) we see that coherent states are eigenvectors of annihilation operators
\[ a_x\psi(f) = f(x)\psi(f) \quad \Rightarrow \quad a(g)\psi(f) = \langle g, f \rangle_{L^2}\psi(f). \]

v) The expectation of the number of particles in the coherent state \( \psi(f) \) is given by \( \|f\|^2 \), that is
\[ \langle \psi(f), N\psi(f) \rangle = \|f\|^2. \]

Also the variance of the number of particles in \( \psi(f) \) is given by \( \|f\|^2 \) (the distribution of \( N \) is Poisson), that is
\[ \langle \psi(f), N^2\psi(f) \rangle - \langle \psi(f), N\psi(f) \rangle^2 = \|f\|^2. \]

vi) Coherent states are normalized but not orthogonal to each other. In fact
\[ \langle \psi(f), \psi(g) \rangle = e^{-\frac{1}{2}(\|f\|^2+\|g\|^2-2\langle f, g \rangle)} \quad \Rightarrow \quad |\langle \psi(f), \psi(g) \rangle| = e^{-\frac{1}{2}\|f-g\|^2}. \]

4 Time evolution of coherent states and proof of Theorem 1.1

In this section we study the time evolution of an initial coherent state \( \psi(\sqrt{N}\varphi) = W(\sqrt{N}\varphi)\Omega \), for \( \varphi \in H^2(\mathbb{R}^3) \) with \( \|\varphi\| = 1 \). The expected number of particles in the coherent state \( \psi(\sqrt{N}\varphi) \) is \( N \). Therefore, we may expect the evolution generated by \( H_N^{\varphi} \) on \( \psi(\sqrt{N}\varphi) \) to have a mean-field character. In particular we may expect that \( e^{-itH_N^{\varphi}}\psi(\sqrt{N}\varphi) \approx \psi(\sqrt{N}\varphi_t) \) where \( \varphi_t \) solves the nonlinear Hartree equation \[1.3\]. We will prove that this is indeed the case, under the assumption that \( \varphi_t \) remains bounded in \( H^{1/2}(\mathbb{R}^3) \) in the time interval \([-T, T]\).

**Theorem 4.1.** Fix \( \varphi \in H^2(\mathbb{R}^3) \) with \( \|\varphi\| = 1 \) and an arbitrary sequence \( \alpha_N > 0 \) such that \( \alpha_N \to 0 \) as \( N \to \infty \). Let \( \psi(N,t) = e^{-itH_N^{\varphi}}W(\sqrt{N}\varphi)\Omega \) be the evolution of the initial coherent state \( W(\sqrt{N}\varphi)\Omega \) generated by the Hamiltonian \[3.6\]. Denote by \( \Gamma_{N,t}^{(1)} \) the one-particle reduced density associated with \( \psi(N,t) \).

Let \( \varphi_t \) be the solution of the nonlinear Hartree equation \[1.3\], with initial data \( \varphi_{t=0} = \varphi \). Fix \( T > 0 \) so that
\[ \kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty. \] (4.1)

Then there exists \( C = C(T, \kappa, \|\varphi\|_{H^2}) < \infty \) such that
\[ \text{Tr} \left| \Gamma_{N,t}^{(1)} - \langle \varphi_t \rangle \langle \varphi_t \rangle \right| \leq C \left( \frac{1}{N} + \alpha_N \right) \]
for all \( t \in \mathbb{R} \) with \( |t| \leq T \).
Proof. The proof of Theorem 4.1 is analogous to the proof of Theorem 3.1 in [21]. For completeness (and because some of these arguments will be used later on), we explain here the main steps.

Since $|\varphi_t\rangle\langle\varphi_t|$ is a rank one projection, it is enough to show that

$$\|\Gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\|_{\text{HS}} \leq C \left( \frac{1}{N} + \alpha_N \right)$$

where $\|A\|_{\text{HS}} = \text{Tr} A^* A$ is the Hilbert-Schmidt norm of $A$. This follows from the remark$^1$ that the operator $\Gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|$ can only have one negative eigenvalue. Since the trace vanishes, the absolute value of the negative eigenvalue must be the same as the sum of all positive eigenvalues. For this reason, the trace norm is twice the operator norm, which is of course bounded by the Hilbert-Schmidt norm.

Suppose now that $\varphi_t^{(\alpha_N)}$ denote the solution of the regularized Hartree equation

$$i\partial_t \varphi_t^{(\alpha_N)} = \sqrt{1 - \Delta} \varphi_t^{(\alpha_N)} - \lambda \left( \frac{1}{1 + \alpha_N} * |\varphi_t^{(\alpha_N)}|^2 \right) \varphi_t^{(\alpha_N)}$$

with initial data $\varphi_{t=0}^{(\alpha_N)} = \varphi$. By Proposition 2.2 (see, in particular, (2.12)), and since

$$\big\| |\varphi_t^{(\alpha_N)}\rangle\langle\varphi_t^{(\alpha_N)}| - |\varphi_t\rangle\langle\varphi_t| \big\|_{\text{HS}} \leq 2 \|\varphi_t - \varphi_t^{(\alpha_N)}\|,$$

it is enough to prove that

$$\|\Gamma_{N,t}^{(1)} - |\varphi_t^{(\alpha_N)}\rangle\langle\varphi_t^{(\alpha_N)}|\|_{\text{HS}} \leq \frac{C}{N}$$

for a constant $C$ depending on $T, \kappa, \|\varphi\|_{H^2}$. In order to prove (4.3) we write the difference of the kernels of $\Gamma_{N,t}^{(1)}$ and $|\varphi_t^{(\alpha_N)}\rangle\langle\varphi_t^{(\alpha_N)}|$ as (compare with (3.4) in [21])

$$\Gamma_{N,t}^{(1)}(x; y) - \varphi_t^{(\alpha_N)}(x) \varphi_t^{(\alpha_N)}(y)$$

$$= \frac{\varphi_t^{(\alpha_N)}(y)}{\sqrt{N}} \left\langle \Omega, W^*(\sqrt{N} \varphi) e^{iH_{N,t}}(a_x - \sqrt{N} \varphi_t^{(\alpha_N)}(x)) e^{-iH_{N,t}} W(\sqrt{N} \varphi) \Omega \right\rangle$$

$$+ \frac{\varphi_t^{(\alpha_N)}(x)}{\sqrt{N}} \left\langle \Omega, W^*(\sqrt{N} \varphi) e^{iH_{N,t}}(a_y - \sqrt{N} \varphi_t^{(\alpha_N)}(y)) e^{-iH_{N,t}} W(\sqrt{N} \varphi) \Omega \right\rangle$$

$$+ \frac{1}{N} \left\langle \Omega, W^*(\sqrt{N} \varphi) e^{iH_{N,t}}(a_y - \sqrt{N} \varphi_t^{(\alpha_N)}(y))(a_x - \sqrt{N} \varphi_t^{(\alpha_N)}(x)) e^{-iH_{N,t}} W(\sqrt{N} \varphi) \Omega \right\rangle.$$  

(4.4)

Then, following (3.5)-(3.8) in [21], we can define the two-parameter group of unitary transformations $U_N(t; s)$ by the Schrödinger equation

$$i\partial_t U_N(t; s) = L_N(t) U_N(t; s) \quad \text{and} \quad U_N(s; s) = 1$$

(4.5)

$^1$We learned this argument from R. Seiringer
It was observed by Hepp in [15] and then by Ginibre-Velo in [13] that

Therefore it follows from (4.4) that

with the generator

\[ \tilde{L}_N(t) = \int dx \, a_x^* \left( 1 - \Delta_x \right)^{1/2} a_x - \lambda \int dx \left( \frac{1}{|x| + \alpha_N} \cdot |\varphi_i^{(\alpha_N)}|^2 \right) (x) \, a_x^* a_x \]

\[ - \lambda \int dx dy \frac{1}{|x - y| + \alpha_N} \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_y a_x \]

\[ - \frac{\lambda}{2} \int dx dy \frac{1}{|x - y| + \alpha_N} \left( \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_x a_y + \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_y a_x \right) \]

\[ - \frac{\lambda}{2N} \int dx dy \frac{1}{|x - y| + \alpha_N} a_x^* a_y^* a_x a_y. \]

It was observed by Hepp in [15] and then by Ginibre-Velo in [13] that

\[ \mathcal{U}_N^*(t; 0) a_x \mathcal{U}_N(t; 0) = W^* (\sqrt{N} \varphi) e^{i\mathcal{H}_N t} (a_x - e^{i\mathcal{H}_N t} W (\sqrt{N} \varphi)). \]

Therefore it follows from (4.4) that

\[ \Gamma_{N,t}^{(1)}(x, y) - \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) = \frac{1}{N} \langle \Omega, \mathcal{U}_N(t; 0)^* a_y^* a_x \mathcal{U}_N(t; 0) \Omega \rangle \]

\[ + \frac{\varphi_i^{(\alpha_N)}(x)}{\sqrt{N}} \langle \Omega, \mathcal{U}_N(t; 0)^* a_y^* \mathcal{U}_N(t; 0) \Omega \rangle \]

\[ + \frac{\varphi_i^{(\alpha_N)}(y)}{\sqrt{N}} \langle \Omega, \mathcal{U}_N(t; 0)^* a_x \mathcal{U}_N(t; 0) \Omega \rangle. \]

To get an optimal bound on the error, we also introduce, similarly to (3.9) and (3.10) in [21], the modified evolution \( \tilde{\mathcal{U}}_N(t; s) \) defined by the equation

\[ i\partial_t \tilde{\mathcal{U}}_N(t; s) = \tilde{L}_N(t) \tilde{\mathcal{U}}_N(t; s) \]

with \( \tilde{\mathcal{U}}_N(s; s) = 1 \)

with the time-dependent generator

\[ \tilde{L}_N(t) = \int dx \, a_x^* \left( 1 - \Delta_x \right)^{1/2} a_x - \lambda \int dx \left( \frac{1}{|x| + \alpha_N} \cdot |\varphi_i^{(\alpha_N)}|^2 \right) (x) \, a_x^* a_x \]

\[ - \lambda \int dx dy \frac{1}{|x - y| + \alpha_N} \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_y a_x \]

\[ - \frac{\lambda}{2} \int dx dy \frac{1}{|x - y| + \alpha_N} \left( \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_x a_y + \varphi_i^{(\alpha_N)}(x) \varphi_i^{(\alpha_N)}(y) a_y a_x \right) \]

\[ - \frac{\lambda}{2N} \int dx dy \frac{1}{|x - y| + \alpha_N} a_x^* a_y^* a_x a_y. \]

Since \( \mathcal{U}_N \) commutes with the parity operator \((-1)^N\), we have

\[ \langle \Omega, \tilde{\mathcal{U}}_N(t; 0)^* a_y \tilde{\mathcal{U}}_N(t; 0) \Omega \rangle = \langle \Omega, \tilde{\mathcal{U}}_N(t; 0)^* a_x^* \tilde{\mathcal{U}}_N(t; 0) \Omega \rangle = 0. \]
Therefore, we can write

\[
\begin{aligned}
\Gamma^{(1)}_{N,t}(x; y) - \varphi_i^{(\alpha_N)}(x)\varphi_i^{(\alpha_N)}(y) \\
= \frac{1}{N} \langle \Omega, U_N(t; 0) a_y^\dagger a_x U_N(t; 0) \Omega \rangle \\
+ \frac{\varphi_i^{(\alpha_N)}(x)}{\sqrt{N}} \left( \langle \Omega, U_N^*(t; 0) a_y^\dagger \left( U_N(t; 0) - \tilde{U}_N(t; 0) \right) \Omega \rangle + \langle \Omega, \left( U_N^*(t; 0) - \tilde{U}_N^*(t; 0) \right) a_y^\dagger \tilde{U}_N(t; 0) \Omega \rangle \right) \\
+ \frac{\varphi_i^{(\alpha_N)}(y)}{\sqrt{N}} \left( \langle \Omega, U_N^*(t; 0) a_x \left( U_N(t; 0) - \tilde{U}_N(t; 0) \right) \Omega \rangle + \langle \Omega, \left( U_N^*(t; 0) - \tilde{U}_N^*(t; 0) \right) a_x \tilde{U}_N(t; 0) \Omega \rangle \right)
\end{aligned}
\]

which leads, after multiplying with a Hilbert-Schmidt observable \( J \) and taking the trace, to the bound

\[
\left| \text{Tr} J \left( \Gamma^{(1)}_{N,t} - |\varphi_i^{(\alpha_N)}\rangle \langle \varphi_i^{(\alpha_N)}| \right) \right| \leq \frac{\| J \|_{\text{HS}}}{N} \langle \Omega, U_N(t; 0) \Omega \rangle_{\Omega, U_N(t; 0)} \\
+ \frac{2\| J \|_{\text{HS}}}{\sqrt{N}} \| (U_N(t; 0) - \tilde{U}_N(t; 0)) \Omega \| \| (N + 1)^{1/2} U_N(t; 0) \Omega \| \\
+ \frac{2\| J \|_{\text{HS}}}{\sqrt{N}} \| (U_N(t; 0) - \tilde{U}_N(t; 0)) \Omega \| \| (N + 1)^{1/2} \tilde{U}_N(t; 0) \Omega \|.
\]

(4.11)

To conclude the proof of the theorem, we combine the last bound with Proposition 4.2, Proposition 4.3 and Proposition 4.3 below.

The next proposition shows that expectations of powers of the number of particle operator, evolved with respect to the fluctuation dynamics \( U_N \), stay bounded up to time \( T \). Note that to prove Theorem 4.1 it would be enough to have (4.12) for \( k = 1 \) and for \( \psi = \Omega \); for later use, however, it is useful to consider arbitrary \( k \in \mathbb{N} \) and \( \psi \in \mathcal{F} \).

**Proposition 4.2.** Suppose that the assumptions of Theorem 4.1 are satisfied. Suppose moreover that the unitary evolution \( U_N(t; s) \) is defined as in (4.5). Then, for every \( k \in \mathbb{N} \), there exists \( C = C(k, T, \kappa, \| \varphi \|_{H^2}) \) such that

\[
\langle U_N(t; 0) \psi, N^k U_N(t; 0) \psi \rangle \leq C \langle \psi, (N + 1)^{2k+2} \psi \rangle
\]

(4.12)

for every \( \psi \in \mathcal{F} \) and every \( t \in \mathbb{R} \) with \( |t| \leq T \).

A similar estimate is also needed to control the growth of the expectation of the number of particle operator with respect to the modified dynamics \( \tilde{U}_N \) introduced in (4.9).

**Proposition 4.3.** Suppose that the assumption of Theorem 4.1 are satisfied. Suppose moreover that the unitary evolution \( U_N(t; s) \) is defined as in (4.5). Then there exists \( C = C(T, \kappa, \| \varphi \|_{H^2}) \) such that

\[
\langle \tilde{U}_N(t; 0) \Omega, N^3 \tilde{U}_N(t; 0) \Omega \rangle \leq C
\]

for every \( \psi \in \mathcal{F} \) and every \( t \in \mathbb{R} \) with \( |t| \leq T \).

Finally, we need to show that, in the second and in the third term on the r.h.s. of (4.11), it is possible to extract one more factor \( N^{-1/2} \) from the difference between the two evolutions.
Proposition 4.4. Suppose that the assumption of Theorem 4.1 are satisfied. Suppose moreover that the unitary evolutions $\tilde{U}_N(t; s)$ and $U_N(t; s)$ are defined as in (4.3) and in (4.9). Then there exists $C = C(T, \kappa, \|\varphi\|_{H^2})$ such that

$$
\left\| \left( U_N(t; 0) - \tilde{U}_N(t; 0) \right) \right\| \Omega \leq \frac{C}{\sqrt{N}}.
$$

The proof of these three propositions can be obtained in the exact same way as the proof of Proposition 3.3, Lemma 3.8 and Lemma 3.9 in [21]. This follows by the observation that, on the one hand, the kinetic energy (given by the second quantization of the dispersion $(1 - \Delta)^{1/2}$), which is the only term in the generators $L_N(t)$ and $\tilde{L}_N(t)$ which differs from the generators in [21], commutes with the number of particle operator (and with all its powers). The other important remark is that by the assumptions in Theorem 4.1 (in particular, by (4.1)), and by Corollary 2.3, there exists the only term in the generators $L_N$ of Proposition 3.3, Lemma 3.8 and Lemma 3.9 in [21]. This follows by the observation that, on the one hand, the kinetic energy (given by the second quantization of the dispersion $(1 - \Delta)^{1/2}$), which is the only term in the generators $L_N(t)$ and $\tilde{L}_N(t)$ which differs from the generators in [21], commutes with the number of particle operator (and with all its powers). The other important remark is that by the assumptions in Theorem 4.1 (in particular, by (4.1)), and by Corollary 2.3, there exists $\nu = \nu(\kappa, T, \|\varphi\|_{H^2}) < \infty$ independent of $\alpha_N$ such that

$$
\sup_{|t| \leq T} \|\varphi_t^{(\alpha_N)}\|_{H^1} \leq \nu.
$$

The uniform bound on the $H^1$-norm of $\varphi_t$ is the only property of $\varphi_t$ that is used in the proof of Proposition 3.3, Lemma 3.8 and Lemma 3.9 of [21].

Note that the main idea in the proof of Proposition 4.2 is the introduction of yet another modified dynamics $W_N(t; s)$ defined by

$$
i \partial_t W_N(t; s) = \mathcal{M}_N(t) W_N(t; s) \quad \text{with } W_N(s; s) = 1 \quad \text{for all } s \in \mathbb{R},
$$

with the time-dependent generator

$$
\mathcal{M}_N(t) = \int dx \ a_x^* \left( 1 - \Delta_x \right)^{1/2} a_x - \lambda \int dx \ \left( \frac{1}{|x| + \alpha_N} \right) |\varphi_t^{(\alpha_N)}|^2 (x) a_x^* a_x
$$

$$
- \lambda \int dx dy \ \frac{1}{|x-y| + \alpha_N} \varphi_t^{(\alpha_N)}(x) \varphi_t^{(\alpha_N)}(y) a_x^* a_x
$$

$$
- \frac{\lambda}{2} \int dx dy \ \frac{1}{|x-y| + \alpha_N} \left( \varphi_t^{(\alpha_N)}(x) \varphi_t^{(\alpha_N)}(y) a_x^* a_y + \varphi_t^{(\alpha_N)}(x) \varphi_t^{(\alpha_N)}(y) a_y a_x \right) \tag{4.14}
$$

$$
- \frac{\lambda}{\sqrt{N}} \int dx dy \ \frac{1}{|x-y| + \alpha_N} a_x^* \left( \varphi_t^{(\alpha_N)}(y) a_y + \varphi_t^{(\alpha_N)}(y) a_y \right) \left( \mathbf{1}_M(N) a_y + \mathbf{1}_M(N) a_y \right) a_x
$$

$$
- \frac{\lambda}{2N} \int dx dy \ \frac{1}{|x-y| + \alpha_N} a_x^* a_y a_y a_x.
$$

where, for every $M > 0$, $\mathbf{1}_M(s) = 1$ for $s \leq M$, and $\mathbf{1}_M(s) = 0$ if $s > M$ ($\mathbf{1}_M$ is the characteristic function of $(-\infty, M]$). At the end $M$ is chosen as $M = \text{const} \cdot N$. One of the main steps in the proof of Proposition 4.2 is a bound for the growth of the expectation of the number of particles w.r.t. the cutoffed dynamics $W_N(t; s)$. We state this result explicitly, because similar ideas are used also in the next section for the proof of Theorem 5.1. The proof of the next lemma is analogous to the proof of Lemma 3.5 in [21].

Lemma 4.5. Suppose that the assumptions of Proposition 4.2 are satisfied. Let $W_N$ be defined as the unitary evolution (4.13) with generator (4.14) and with $M \leq \text{const} \cdot N$. Then, for every $k \in \mathbb{N}$ there exists $C = C(\text{const}, k, T, \kappa, \|\varphi\|_{H^2})$ such that

$$
\langle W_N(t; 0) \Omega, N^k W_N(t; 0) \Omega \rangle \leq C
$$

for all $t \in \mathbb{R}$ with $|t| \leq T$. 
The kernel of the one-particle reduced density $\gamma^{(1)}_{N,t}$ associated with $\psi_{N,t} = e^{-iH_N t}\psi_N$ is therefore given by

$$\gamma^{(1)}_{N,t}(x; y) = \frac{d^2_{\psi}}{N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{-i\theta_1 N} e^{i\theta_2 N} \langle \mathcal{U}^{\theta_1}_N(t; 0)\Omega, a_x^* a_x \mathcal{U}^{\theta_2}_N(t; 0)\Omega \rangle$$

(4.17)

where we introduced the notation $a_x(t) = e^{iH_N t}a_x e^{-iH_N t}$. As in (4.5)-(4.7) of [21], we find

$$\gamma^{(1)}_{N,t}(x; y) - \varphi_t^{(\alpha_N)}(y) \varphi_t^{(\alpha_N)}(x) = \frac{d^2_{\psi}}{N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{-i\theta_1 N} e^{i\theta_2 N} \langle \mathcal{U}^{\theta_1}_N(t; 0)\Omega, a_y^* a_x \mathcal{U}^{\theta_2}_N(t; 0)\Omega \rangle$$

$$+ \frac{d_{\psi}}{\sqrt{N}} \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \mathcal{U}^{\theta}_N(t; 0)\Omega, a_y^* \varphi^{(N-1)} \rangle$$

$$+ \frac{d_{\psi}}{\sqrt{N}} \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \mathcal{U}^{\theta}_N(t; 0)\Omega, a_x \varphi^{(N-1)} \rangle$$

(4.18)

where $\varphi^{(N-1)}$ actually denotes the vector $\{0, \ldots, 0, \varphi^{(N-1)}, 0, \ldots\} \in \mathcal{F}$ and where $\mathcal{U}^{\theta}_N(t; 0)$ is defined as the unitary evolution in (4.5), with $\varphi_t^{(\alpha_N)}$ replaced by $e^{i\theta} \varphi_t^{(\alpha_N)}$ (it is important to observe that if $\varphi_t^{(\alpha_N)}$ solves the nonlinear Hartree equation, also $e^{i\theta} \varphi_t^{(\alpha_N)}$ is a solution). Therefore, we conclude that

$$\int dx dy \left| \gamma^{(1)}_{N,t}(x; y) - \varphi_t^{(\alpha_N)}(x) \varphi_t^{(\alpha_N)}(y) \right|^2$$

$$\leq 2 \frac{d^4_{\psi}}{N^2} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} \| \mathcal{U}^{1/2}_N(t; 0)\Omega \|^2 \| \mathcal{U}^{1/2}_N(t; 0)\Omega \|^2 + \frac{4}{N} \int dx |f_N(x)|^2$$

(4.19)

with

$$f_N(x) = \frac{d^2_{\psi}}{N} \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \mathcal{U}^{\theta}_N(t; 0)\Omega, a_x \varphi^{(N-1)} \rangle.$$ 

(4.20)

Proceeding exactly as in Lemma 4.2 of [21], we find a constant $C = C(T, \kappa, \|\varphi\|_{H^2})$ such that

$$\int dx |f_N(x)|^2 \leq C$$

uniformly in $N$. Proposition 4.2 implies therefore that there exists $C = C(T, \kappa, \|\varphi\|_{H^2})$ with

$$\left\| \gamma^{(1)}_{N,t} - |\varphi_t^{(\alpha)}\rangle \langle \varphi_t^{(\alpha)}| \right\|_{HS} \leq \frac{C}{\sqrt{N}}$$

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Therefore, it follows from Proposition 2.2 (in particular, (2.12)) that
\[
\left\| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right\|_{HS} \leq C \left( \frac{1}{\sqrt{N}} + \alpha_N \right).
\]
Since \( |\varphi_t\rangle\langle\varphi_t| \) is a rank one projection, and since \( \text{Tr} \gamma_{N,t}^{(1)} = \text{Tr} |\varphi_t\rangle\langle\varphi_t| = 1 \), the trace norm of the difference \( \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \) is at most two times its Hilbert-Schmidt norm. This completes the proof of the theorem.

\[\square\]

5 Convergence in energy

We study again the evolution of initial coherent states in the Fock space. This time, we establish the convergence of the one-particle reduced density towards the solution of the Hartree equation in the energy norm. As a consequence, we obtain a proof of Theorem 1.2.

**Theorem 5.1.** Fix \( \varphi \in H^2(\mathbb{R}^3) \) with \( \|\varphi\| = 1 \) and a sequence \( \alpha_N > 0 \) such that \( \alpha_N \to 0 \) and \( N^{\beta} \alpha_N \to \infty \) as \( N \to \infty \), for an appropriate \( \beta > 0 \). Let \( \psi(N,t) = e^{-itH_S} W(\sqrt{N}\varphi)\Omega \) be the evolution of the initial coherent state \( W(\sqrt{N}\varphi)\Omega \) generated by the Hamiltonian (3.6). Denote by \( \Gamma_{N,t}^{(1)} \) the one-particle reduced density associated with \( \psi(N,t) \).

Let \( \varphi_t \) be the solution of the nonlinear Hartree equation (1.5) with initial data \( \varphi_{t=0} = \varphi \). Fix \( T > 0 \) so that
\[
\kappa := \sup_{|t| \leq T} \|\varphi_t\|_{H^{1/2}} < \infty.
\]
Then there exists \( C = C(\beta, T, \kappa, \|\varphi\|_{H^2}) < \infty \) such that
\[
\left\| (1-\Delta)^{1/4} \left( \Gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right) (1-\Delta)^{1/4} \right\|_{HS} \leq C \left( \frac{1}{\sqrt{N}} + \alpha_N^{1/2} \right)
\]
for all \( t \in \mathbb{R} \) with \( |t| \leq T \).

**Remark.** From (5.2) we can conclude, using arguments similar to the ones used below in the proof of Theorem 1.2 (starting from Eq. (5.10)), that
\[
\text{Tr} \left| (1-\Delta)^{1/4} \left( \Gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right) (1-\Delta)^{1/4} \right| \leq C \left( \frac{1}{\sqrt{N}} + \alpha_N^{1/2} \right).
\]

**Proof.** Denote now by \( \varphi_t^{(\alpha_N)} \) the solution of the regularized Hartree equation (4.2) with initial data \( \varphi_{t=0}^{(\alpha_N)} = \varphi \). Since
\[
\left\| (1-\Delta)^{1/4} \left( |\varphi_t\rangle\langle\varphi_t| - |\varphi_t^{(\alpha_N)}\rangle\langle\varphi_t^{(\alpha_N)}| \right) (1-\Delta)^{1/4} \right\|_{HS} \lesssim \|\varphi_t - \varphi_t^{(\alpha_N)}\|_{H^{1/2}}
\]
and using Proposition 2.2 (see, in particular, (2.13)), it is enough to prove that there exists a constant \( C = C(\kappa, T, \|\varphi\|_{H^2}) \) such that
\[
\left\| (1-\Delta)^{1/4} \left( \Gamma_{N,t}^{(1)} - |\varphi_t^{(\alpha_N)}\rangle\langle\varphi_t^{(\alpha_N)}| \right) (1-\Delta)^{1/4} \right\|_{HS} \leq \frac{C}{\sqrt{N}}.
\]

(5.3)
To show (5.3), we use again the representation (4.8) for the kernel of $\Gamma^{(1)}_{N,t}$, which implies that

$$\left( (1 - \Delta)^{1/4} \left( \Gamma^{(1)}_{N,t} - |\varphi_{t}^{(\alpha N)}\rangle \langle \varphi_{t}^{(\alpha N)} | \right) (1 - \Delta)^{1/4} \right)(x,y) = \frac{1}{N} \left\langle \Omega, \mathcal{U}_{N}(t;0)^{\ast} (1 - \Delta_{y})^{1/4} a_{y}^{\ast} (1 - \Delta_{x})^{1/4} a_{x} \mathcal{U}_{N}(t;0)\Omega \right\rangle$$

$$+ \frac{(1 - \Delta)^{1/4} \varphi_{t}^{(\alpha N)}(x)}{\sqrt{N}} \left\langle \Omega, \mathcal{U}_{N}(t;0)^{\ast} (1 - \Delta_{y})^{1/4} a_{y}^{\ast} \mathcal{U}_{N}(t;0)\Omega \right\rangle$$

$$+ \frac{(1 - \Delta)^{1/4} \varphi_{t}^{(\alpha N)}(y)}{\sqrt{N}} \left\langle \Omega, \mathcal{U}_{N}(t;0)^{\ast} (1 - \Delta_{x})^{1/4} a_{x} \mathcal{U}_{N}(t;0)\Omega \right\rangle$$

(5.4)

With a Schwarz inequality, we find

$$\|(1 - \Delta)^{1/4} (\Gamma^{(1)}_{N,t} - |\varphi_{t}\rangle \langle \varphi_{t}|)(1 - \Delta)^{1/4}\|_{HS}^{2} \lesssim \frac{1}{N^{2}} \left\langle \mathcal{U}_{N}(t;0)\Omega, \mathcal{K} \mathcal{U}_{N}(t;0)\Omega \right\rangle^{2} + \frac{\|\varphi_{t}^{(\alpha N)}\|^{2}_{H^{2}}}{N} \left\langle \mathcal{U}_{N}(t;0)\Omega, \mathcal{K} \mathcal{U}_{N}(t;0)\Omega \right\rangle,$$

(5.5)

where we defined

$$\mathcal{K} = \int dx (1 - \Delta_{x})^{1/4} a_{x}^{\ast} (1 - \Delta_{x})^{1/4} a_{x}$$

(5.6)

to be the kinetic energy operator. The theorem follows now from Proposition 5.2 below.

The key point, in the proof of Theorem 5.1, and also in the proof of Theorem 1.2, is the following proposition, which controls the growth of the expectation of the kinetic energy with respect to the fluctuation dynamics $\mathcal{U}_{N}$.

**Proposition 5.2.** Suppose that the assumptions of Theorem 5.1 are satisfied. Suppose moreover that the unitary evolution $\mathcal{U}_{N}(t;s)$ is defined as in (4.5). Then there exists $C = C(T, \kappa, \|\varphi\|_{H^{2}})$ such that

$$\left\langle \mathcal{U}_{N}(t;0)\Omega, \mathcal{K} \mathcal{U}_{N}(t;0)\Omega \right\rangle \leq C$$

(5.7)

for $t \in \mathbb{R}$ with $|t| \leq T$.

The proof of Proposition 5.2 is given in Section 9. The bound on the growth of the expectation of $\mathcal{K}$ can also be used to conclude the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Using the representation (4.18), we obtain

$$\left\| (1 - \Delta)^{1/4} (\gamma^{(1)}_{N,t} - |\varphi_{t}^{(\alpha N)}\rangle \langle \varphi_{t}^{(\alpha N)} |) (1 - \Delta)^{1/4} \right\| (x,y) \right\| (1 - \Delta_{y})^{1/4} a_{y}^{\ast} \mathcal{U}_{N}^{\theta_{1}}(t;0)\Omega \right\|$$

$$\lesssim \frac{d_{N}^{2}}{N} \int_{0}^{2\pi} \int_{0}^{2\pi} d\theta_{1} d\theta_{2} \left\| (1 - \Delta_{x})^{1/4} a_{x} \mathcal{U}_{N}^{\theta_{1}}(t;0)\Omega \right\|$$

$$+ \frac{d_{N}}{\sqrt{N}} \left\| (1 - \Delta_{x})^{1/4} \varphi_{t}^{(\alpha N)}(x) \right\| \int_{0}^{2\pi} d\theta_{1} \left\| (1 - \Delta_{y})^{1/4} a_{y} \mathcal{U}_{N}^{\theta_{1}}(t;0)\Omega \right\|$$

(5.8)

where $\mathcal{U}_{N}^{\theta}(t;0)$ is defined as the unitary evolution in (4.5), with $\varphi_{t}^{(\alpha N)}$ replaced by $e^{i\theta} \varphi_{t}^{(\alpha N)}$. Taking the square and integrating over $x, y$, we find

$$\int dx dy \left\| (1 - \Delta)^{1/4} (\gamma^{(1)}_{N,t} - |\varphi_{t}^{(\alpha N)}\rangle \langle \varphi_{t}^{(\alpha N)} |)(1 - \Delta)^{1/4} \right\| (x,y) \right\|^{2}$$

$$\lesssim \frac{1}{N} \left( \int_{0}^{2\pi} d\theta \left\langle \mathcal{U}_{N}^{\theta}(t;0)\Omega, \mathcal{K} \mathcal{U}_{N}^{\theta}(t;0)\Omega \right\rangle \right)^{2} + \frac{1}{\sqrt{N}} \int_{0}^{2\pi} d\theta \left\langle \mathcal{U}_{N}^{\theta}(t;0)\Omega, \mathcal{K} \mathcal{U}_{N}^{\theta}(t;0)\Omega \right\rangle.$$
Proposition 5.2 implies that there exists $C = C(T, \kappa, \|\varphi\|_{H^2})$ such that
\[
\left\| (1 - \Delta)^{1/4} \left( \gamma_{N,t}^{(1)} - |\varphi_t^{(\alpha_N)}\rangle \langle \varphi_t^{(\alpha_N)} | \right) (1 - \Delta)^{1/4} \right\|_{HS} \leq \frac{C}{N^{1/4}}. \quad (5.10)
\]
By Proposition 2.2 we obtain therefore
\[
\left\| (1 - \Delta)^{1/4} \left( \gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t | \right) (1 - \Delta)^{1/4} \right\|_{HS} \leq C \left( \frac{1}{N^{1/4}} + \alpha_N^{1/2} \right). \quad (5.11)
\]
Note also that (5.8) implies that
\[
\left| \int dx \left( (1 - \Delta)^{1/4} \left( \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4} - \| (1 - \Delta)^{1/4} \varphi_t \|_2 \right) \right) \right| \nonumber
\]
\[
\leq \frac{1}{N^{1/4}} \left( \| \varphi_t^{(\alpha_N)} \|_{H^{1/2}}^2 + \int_0^{2\pi} d\theta \left| \langle \mathcal{U}_N^0(t; 0) \Omega, \mathcal{K} \mathcal{U}_N^0(t; 0) \Omega \rangle \right| \right) \quad (5.12)
\]
and therefore, by Proposition 5.2
\[
\left| \text{Tr} (1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4} - \| (1 - \Delta)^{1/4} \varphi_t \|_2 \right| \leq \frac{C}{N^{1/4}}. \quad (5.13)
\]
Again, Proposition 2.2 implies that
\[
\left| \text{Tr} (1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4} - \| (1 - \Delta)^{1/4} \varphi_t \|_2 \right| \leq \frac{C}{N^{1/4}}. \quad (5.14)
\]
Last equation, together with (5.11), implies that
\[
\left\| \frac{(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}}{\text{Tr}[(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}]} - \frac{|(1 - \Delta)^{1/4} \varphi_t \rangle \langle (1 - \Delta)^{1/4} \varphi_t |}{\| (1 - \Delta)^{1/4} \varphi_t \|_2^2} \right\|_{HS} \nonumber
\]
\[
\leq \frac{\| (1 - \Delta)^{1/4} \left( \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4} - |\varphi_t\rangle \langle \varphi_t | \right) \|_{HS}}{\text{Tr}[(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}]} + \frac{\| \text{Tr}[(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4} - \| (1 - \Delta)^{1/4} \varphi_t \|_2^2]}{\text{Tr}[(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}]} \quad (5.15)
\]
\[
\leq C \left( \frac{1}{N^{1/4}} + \alpha_N^{1/2} \right). \nonumber
\]
On the l.h.s. of (5.15) we are now comparing a density matrix with a rank-one projection. The trace norm of their difference is at most twice the corresponding Hilbert-Schmidt norm and thus we find
\[
\text{Tr} \left| \frac{(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}}{\text{Tr}[(1 - \Delta)^{1/4} \gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}]} - \frac{(1 - \Delta)^{1/4} \varphi_t \rangle \langle (1 - \Delta)^{1/4} \varphi_t |}{\| (1 - \Delta)^{1/4} \varphi_t \|_2^2} \right| \leq C \left( \frac{1}{N^{1/4}} + \alpha_N^{1/2} \right). \quad (5.16)
\]
Combining last equation with (5.14), we finally obtain

\[
\text{Tr} \left| (1 - \Delta)^{1/4}(\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|)(1 - \Delta)^{1/4} \right|
\]

\[
\lesssim \| (1 - \Delta)^{1/4}\varphi_t \|^2 \text{Tr} \frac{(1 - \Delta)^{1/4}\gamma_{N,t}^{(1)}(1 - \Delta)^{1/4}}{\| (1 - \Delta)^{1/4}\varphi_t \|^2} - \frac{(1 - \Delta)^{1/4}|\varphi_t\rangle\langle\varphi_t|(1 - \Delta)^{1/4}}{\| (1 - \Delta)^{1/4}\varphi_t \|^2}
\]

\[
\leq C \left( \frac{1}{N^{1/4}} + \alpha_t^{1/2} \right)\]

which concludes the proof of the theorem.

6 Control of the growth of the kinetic energy

The goal of this section is to prove Proposition 5.2 which gives control on the growth of the expectation of the kinetic energy operator \( K \) with respect to the fluctuation dynamics \( U_N(t;0) \).

**Proof of Proposition 5.2.** Recall the definition (4.5) of the unitary maps \( U_N(t;s) \) describing the evolution of the fluctuations; note that the generator \( L_N(t) \) of \( U_N(t;s) \) is defined in terms of the solution \( \varphi_t^{(\alpha_N)} \) of the regularized Hartree equation

\[
i\partial_t \varphi_t^{(\alpha_N)} = \sqrt{1 - \Delta} \varphi_t^{(\alpha_N)} - \lambda \left( \frac{1}{|\cdot| + \alpha_t} \ast |\varphi_t^{(\alpha_N)}|^2 \right) \varphi_t^{(\alpha_N)}.
\]

(6.1)

In the rest of this section, we will use the shorthand notation \( \phi_t \equiv \varphi_t^{(\alpha_N)} \). By (5.1), by the assumption \( \varphi \in H^2(\mathbb{R}^3) \) on the initial data, and by Corollary 2.3, there exists \( \nu = \nu(T, \kappa, \|\varphi\|_{H^2}) \) such that

\[
\sup_{|t| \leq T} \| \phi_t \|_{H^2} \leq \nu
\]

(6.2)

uniformly in \( N \) (\( \phi_t = \varphi_t^{(\alpha_N)} \) depends on \( N \) through the cutoff \( \alpha_N \)).

We compare the growth of \( K \) along the fluctuation dynamics \( U_N \) and along a new dynamics \( \tilde{W}_N \) defined through the equation

\[
i\partial_t \tilde{W}_N(t;s) = \tilde{M}_N(t) \tilde{W}_N(t;s) \quad \text{with} \quad \tilde{W}_N(s;s) = 1 \quad \text{for all} \ s \in \mathbb{R},
\]

(6.3)
with the time-dependent generator

\[
\tilde{\mathcal{M}}_N(t) := \int dx \, a_x^*(1 - \Delta_x)^{1/2}a_x - \lambda \int dx \left( \frac{1}{|x| + \alpha_N} * |\phi_t|^2 \right) a_x^*a_x \\
- \lambda \int dx \, dy \frac{1}{|x - y| + \alpha_N} \phi_t(x) \phi_t(y) a_x^*a_y \\
- \frac{\lambda}{2} \int dx \, dy \frac{1}{|x - y| + \alpha_N} \left\{ \phi_t(x) \phi_t(y) a_x^*a_y + \phi_t(x) \phi_t(y) a_xa_y \right\} \\
- \frac{\lambda}{\sqrt{N}} \int dx \, dy \frac{1}{|x - y| + \alpha_N} \left\{ \phi_t(y) a_x^*a_y^* 1_{\vartheta N}(\mathcal{N}) a_x + \phi_t(y) a_x^* 1_{\vartheta N}(\mathcal{N}) a_y a_y \right\} \\
- \frac{\lambda}{2N} \int dx \, dy \frac{1}{|x - y| + \alpha_N} a_x^*a_y^* 1_{\vartheta N}(\mathcal{N}) a_y a_x
\]

where \(1_{\vartheta N}(s)\) is the characteristic function of the interval \((-\infty, \vartheta N]\), that is \(1_{\vartheta N}(s) = 1\) if \(s \leq \vartheta N\) and \(1_{\vartheta N}(s) = 0\) otherwise. Here \(\vartheta \leq 1\) will be fixed later to be sufficiently small.

We split

\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K} \mathcal{U}_N(t; 0)\Omega \rangle = \langle \tilde{\mathcal{W}}_N(t; 0)\Omega, \mathcal{K} \tilde{\mathcal{W}}_N(t; 0)\Omega \rangle + \langle \mathcal{U}_N(t; 0) - \tilde{\mathcal{W}}_N(t; 0)\Omega, \mathcal{K} \tilde{\mathcal{W}}_N(t; 0)\Omega \rangle \\
+ \langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K} \mathcal{U}_N(t; 0) - \tilde{\mathcal{W}}_N(t; 0)\Omega \rangle \\
\leq \langle \tilde{\mathcal{W}}_N(t; 0)\Omega, \mathcal{K} \tilde{\mathcal{W}}_N(t; 0)\Omega \rangle + \|\mathcal{W}_N(t; 0)\Omega\| \|\mathcal{U}_N(t; 0)\rangle \| \mathcal{W}_N(t; 0)\Omega\| \\
+ \|\mathcal{K} \mathcal{U}_N(t; 0)\Omega\| \|\mathcal{U}_N(t; 0) - \tilde{\mathcal{W}}_N(t; 0)\Omega\|.
\]

Proposition 5.2 now follows from Proposition 6.1, Proposition 6.2 and Proposition 6.3 and from the assumption that \(N^{\beta} \alpha_N \to \infty\) for some \(\beta > 0\).

The first ingredient in the proof of Proposition 5.2 is a bound for the growth of the kinetic energy \(\mathcal{K}\) and of its square w.r.t. the cutoffed evolution \(\tilde{\mathcal{W}}_N(t; s)\). This is the content of the next Proposition, which will be proven in Section 6.1.

**Proposition 6.1.** Suppose that the assumptions of Proposition 5.2 are satisfied (but here the assumption \(N^{\beta} \alpha_N \to \infty\) for some \(\beta > 0\) will not be used). Let the evolution \(\tilde{\mathcal{W}}_N(t; s)\) be defined according to (6.3), with generator (6.4), and suppose that \(\vartheta > 0\) is small enough. Then there exists \(C = C(\vartheta, T, \kappa, \|\varphi\|_{H^2})\) such that

\[
\langle \tilde{\mathcal{W}}_N(t; 0)\Omega, \mathcal{K}^2 \tilde{\mathcal{W}}_N(t; 0)\Omega \rangle \leq C
\]

for all \(t \in \mathbb{R}\) with \(|t| \leq T\).

The second ingredient to prove Proposition 5.2 is a weak bound on the growth of the expectation of \(\mathcal{K}^2\) with respect to the dynamics \(\mathcal{U}_N(t; s)\); the proof of the following proposition is given in Section 6.2.

**Proposition 6.2.** Suppose that the assumptions of Proposition 5.2 are satisfied. Then there exists \(C = C(T, \kappa, \|\varphi\|_{H^2})\) such that

\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2 \mathcal{U}_N(t; 0)\Omega \rangle \leq C \left( N^2 + \frac{N^2}{\alpha_N^2} \right)
\]

for all \(t \in \mathbb{R}\) with \(|t| \leq T\).
Finally, we need to compare the two dynamics $\mathcal{U}_N(t;s)$ and $\tilde{\mathcal{W}}_N(t;s)$. The next proposition is shown in Section 6.3.

**Proposition 6.3.** Suppose that the assumptions of Proposition 6.2 are satisfied. Let the evolution $\tilde{\mathcal{W}}_N(t;s)$ be defined according to (6.3), with generator (6.4), and with $\vartheta > 0$. Then, for any $k \in \mathbb{N}$, there exists $C = C(k, \vartheta, T, \kappa, \|\varphi\|_{H^2})$ such that

$$\left\| \left(\mathcal{U}_N(t;0) - \tilde{\mathcal{W}}_N(t;0)\right) \Omega \right\| \leq \frac{C}{N^k} \left(1 + \frac{1}{\alpha_N}\right)$$

(6.8)

for all $t \in \mathbb{R}$ with $|t| \leq T$.

### 6.1 Growth of $K^2$ with respect to regularized dynamics

The goal of this section is to prove Proposition 6.1. We will make systematic use of the bound (6.2) (recall that in this section we use the shorthand notation $\phi_t \equiv \varphi_t^{(\alpha)}$).

Observe that, by the definition (6.4) of $\tilde{\mathcal{M}}_N(t)$, we have

$$K^2 \lesssim \tilde{\mathcal{M}}_N^2(t) + \left( \int dx \left( \frac{1}{|.| + \alpha_N} \ast |\phi_t|^2 \right) a_x^* a_x \right)^2$$

$$+ \left( \int dx \, dy \frac{1}{|x-y| + \alpha_N} \tilde{\phi}_t(x) \phi_t(y) a_y^* a_x \right)^2$$

$$+ \left( \frac{1}{2} \int dx \, dy \frac{1}{|x-y| + \alpha_N} \left\{ \phi_t(x) \phi_t(y) a_x^* a_y^* + h.c. \right\} \right)^2$$

$$+ \left( \frac{1}{\sqrt{N}} \int dx \, dy \frac{1}{|x-y| + \alpha_N} \left\{ \phi_t(y) a_x^* a_y^* \mathbf{1}_{\partial N}(N) a_x + h.c. \right\} \right)^2$$

$$+ \left( \frac{1}{2N} \int dx \, dy \frac{1}{|x-y| + \alpha_N} a_x^* a_y^* \mathbf{1}_{\partial N}(N) a_y a_x \right)^2$$

(6.9)

where h.c. denotes the hermitian conjugate. First of all, we note that, by Lemma 6.4 below, the last term is bounded by

$$\left( \frac{1}{2N} \int dx \, dy \frac{1}{|x-y| + \alpha_N} a_x^* a_y^* \mathbf{1}_{\partial N}(N) a_y a_x \right)^2 \lesssim \vartheta^2 K^2$$

Therefore, choosing $\vartheta > 0$ sufficiently small, we find

$$K^2 \lesssim \tilde{\mathcal{M}}_N^2(t) + \left( \int dx \left( \frac{1}{|.| + \alpha_N} \ast |\phi_t|^2 \right) a_x^* a_x \right)^2$$

$$+ \left( \int dx \, dy \frac{1}{|x-y| + \alpha_N} \tilde{\phi}_t(x) \phi_t(y) a_y^* a_x \right)^2$$

$$+ \left( \frac{1}{2} \int dx \, dy \frac{1}{|x-y| + \alpha_N} \left\{ \phi_t(x) \phi_t(y) a_x^* a_y^* + h.c. \right\} \right)^2$$

$$+ \left( \frac{1}{\sqrt{N}} \int dx \, dy \frac{1}{|x-y| + \alpha_N} \left\{ \phi_t(y) a_x^* a_y^* \mathbf{1}_{\partial N}(N) a_x + h.c. \right\} \right)^2.$$  

(6.10)

To bound the second term on the r.h.s. of the last equation, we note that

$$\int dx \left( \frac{1}{|.| + \alpha_N} \ast |\phi_t|^2 \right) a_x^* a_x \leq \sup_x \left( \frac{1}{|.|} \ast |\phi_t|^2 \right) N \lesssim \|\phi_t\|^2_{H^{1/2}N}.$$
Since moreover \( \mathcal{N} \) commutes with the operator on the l.h.s., we conclude that

\[
\left( \int dx \left( \frac{1}{|x| + \alpha_N} \ast |\phi_t|^2 \right) a_x^* a_x \right)^2 \lesssim \mathcal{N}^2. \tag{6.11}
\]

Analogously, the third term on the r.h.s. of (6.10) is bounded by

\[
\left( \int dx \, dy \frac{1}{|x - y| + \alpha_N} \overline{\phi_t(x)} \phi_t(y) a_y^* a_x \right)^2 \lesssim \mathcal{N}^2. \tag{6.12}
\]

Next, the terms on the third line of (6.10) can be controlled as follows. Let

\[
A = \int dx \, dy \frac{1}{|x - y| + \alpha_N} \phi_t(x) \phi_t(y) a_x^* a_y^*.
\]

Since \((A + A^*)^2 \lesssim 2(AA^* + A^*A)\), we find

\[
\langle \psi, (A + A^*)^2 \psi \rangle \lesssim \int dx \, dy \, dx' \, dy' \frac{\phi_t(x) \phi_t(y)}{|x - y| + \alpha_N} \frac{\overline{\phi_t(x')} \overline{\phi_t(y')}}{|x' - y'| + \alpha_N} \langle \psi, a_x^* a_y^* a_x a_y \psi \rangle
\]

\[
+ \int dx \, dy \, dx' \, dy' \frac{\phi_t(x) \phi_t(y)}{|x - y| + \alpha_N} \frac{\overline{\phi_t(x')} \overline{\phi_t(y')}}{|x' - y'| + \alpha_N} \langle \psi, [a_x^* a_y^*, a_x a_y] \psi \rangle
\]

\[
\lesssim \int dx \, dy \, dx' \, dy' \frac{\phi_t(x) |\phi_t(y)|}{|x - y| + \alpha_N} \frac{\overline{\phi_t(x')} |\phi_t(y')|}{|x' - y'| + \alpha_N} \|a_x a_y \psi\| \|a_x a_y \psi\|
\]

\[
+ \int dx \, dy \, dx' \, dy' \frac{\phi_t(x) |\phi_t(y)|}{|x - y| + \alpha_N} \frac{\overline{\phi_t(x')} |\phi_t(y')|}{|x' - y'| + \alpha_N} \|a_y \psi\| \|a_x \psi\|
\]

\[
+ \int dx \, dy \, dx' \, dy' \frac{\phi_t(x) |\phi_t(y)|}{|x - y| + \alpha_N} \frac{\overline{\phi_t(x')} |\phi_t(y')|}{|x' - y'| + \alpha_N} \|a_x \psi\| \|a_y \psi\|
\]

for arbitrary \( \psi \in \mathcal{F} \). Here we used that

\[
[a_x a_y, a_x^* a_y^*] = a_y^* a_x \delta(y - x') + a_x^* a_x \delta(y - y') + a_y^* a_y \delta(x - x') + a_x^* a_y \delta(x - y') + \delta(x - x') \delta(y - y') + \delta(y' - x) \delta(y - x'). \tag{6.13}
\]

With Schwarz inequality, we obtain

\[
\langle \psi, (A + A^*)^2 \psi \rangle \lesssim \int dx \, dy \, dx' \, dy' \frac{|\phi_t(x)|^2 |\phi_t(y)|^2}{(|x - y| + \alpha_N)^2} \|a_x^* a_y \psi\|^2
\]

\[
+ \int dx \, dy \, dx' \, dy' \frac{|\phi_t(x)|^2 |\phi_t(y)|^2}{(|x - y| + \alpha_N)^2} \|a_x^* \psi\|^2 + \int dx \, dy \frac{|\phi_t(x)|^2 |\phi_t(y)|^2}{(|x - y| + \alpha_N)^2} \|\psi\|^2
\]

\[
\lesssim \|\phi_t\|_{H^1}^2 \|\phi_t\|^2 \langle \psi, (\mathcal{N} + 1)^2 \psi \rangle.
\]

Thus

\[
\left( \frac{1}{2} \int dx \, dy \frac{1}{|x - y| + \alpha_N} \left\{ \phi_t(x) \phi_t(y) a_x^* a_y^* + \text{h.c.} \right\} \right)^2 \lesssim (\mathcal{N} + 1)^2. \tag{6.14}
\]

Now, we estimate the terms on the fourth line of (6.10). Let

\[
B = \frac{1}{\sqrt{\mathcal{N}}} \int dx \, dy \frac{1}{|x - y| + \alpha_N} a_x^* a_y^* \phi_t(y) \mathbf{1}_{\mathcal{N}}(\mathcal{N}) a_x
\]
Then \((B + B^*)^2 \lesssim BB^* + B^*B\). The term \(BB^*\) can be bounded by
\[
\langle \psi, BB^* \psi \rangle = \frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \phi_t(y) \overline{\phi_t}(y') \times \langle 1_{\theta N}(N - 2) \psi, a_x^* a_y^* a_x a_y 1_{\theta N}(N - 2) \psi \rangle
\]
\[
= \frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \phi_t(y) \overline{\phi_t}(y') \times \langle 1_{\theta N}(N - 2) \psi, a_x^* a_y^* a_x a_y 1_{\theta N}(N - 2) \psi \rangle + \frac{1}{N} \int dx \, dy \, dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x - y'| + \alpha_N} \phi_t(y) \overline{\phi_t}(y') \times \langle 1_{\theta N}(N - 2) \psi, a_x^* a_y^* a_x a_y 1_{\theta N}(N - 2) \psi \rangle
\]
for every \(\psi \in \mathcal{F}\). From Schwarz inequality, we find
\[
\langle \psi, BB^* \psi \rangle \lesssim \frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{(|x - y| + \alpha_N)^2} \| a_x^* a_y 1_{\theta N}(N - 2) \psi \|^2 + \frac{1}{N} \int dx \, dy \, dy' \frac{1}{(|x - y'| + \alpha_N)^2} \| a_x a_y 1_{\theta N}(N - 2) \psi \|^2 \lesssim \frac{1}{N} \| \phi_t \|^2_{H^1} (N + 1)^{3/2} 1_{\theta N}(N - 2) \psi \|^2.
\]

The term \(B^*B\), on the other hand, is given by
\[
\langle \psi, B^*B \psi \rangle = \frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \overline{\phi_t}(y) \phi_t(y') \times \langle 1_{\theta N}(N - 1) \psi, a_x^* a_y a_x^* a_y 1_{\theta N}(N - 1) \psi \rangle
\]
\[
= \frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \overline{\phi_t}(y) \phi_t(y') \times \langle 1_{\theta N}(N - 1) \psi, a_x^* a_x^* a_y a_x a_y 1_{\theta N}(N - 1) \psi \rangle + \frac{1}{N} \int dx \, dx' \, dy \, dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \overline{\phi_t}(y) \phi_t(y') \times \langle 1_{\theta N}(N - 1) \psi, a_x^* a_x a_y^* 1_{\theta N}(N - 1) \psi \rangle
\]
The first term on the r.h.s. is bounded in absolute value by
\[
\frac{1}{N} \int dx \, dy \, dx' dy' \frac{1}{|x - y|} \frac{1}{|x' - y'|} \| a_x a_y 1_{\theta N}(N - 1) \psi \| \| a_y a_x 1_{\theta N}(N - 1) \psi \| \lesssim \frac{1}{N} \left( \sup_x \int dx \frac{1}{(|x - y|)^2} \int dx \, dx' \psi \| a_x a_y 1_{\theta N}(N - 1) \psi \|^2 \right) \lesssim \frac{1}{N} \| \phi_t \|^2_{H^1} (N + 1)^{3/2} 1_{\theta N}(N - 1) \psi \|^2.
\]

When we insert (6.13) in the second term on the r.h.s. of (6.17), we obtain contributions quartic in the creation and annihilation operators of the form
\[
\frac{1}{N} \int dx \, dy \, dy' \frac{1}{|x - y| + \alpha_N} \frac{1}{|y - y'| + \alpha_N} \overline{\phi_t}(y) \phi_t(y') \langle 1_{\theta N}(N - 1) \psi, a_x^* a_y^* a_x a_y 1_{\theta N}(N - 1) \psi \rangle
\]

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whose absolute value can be bounded by

\[
|\frac{1}{N} \int dx \, dy \, dz \, \frac{1}{|x-y| + \alpha_N |y-z| + \alpha_N} \phi_t(y) \phi_t(z) \left( 1_{\partial N}(N-1) \psi, a^*_{y} a^*_{z} a_x \mathbf{1}_{\partial N}(N-1) \psi \right) |
\leq \frac{1}{N} \int dx \, dy \, dz \, \frac{1}{|x-y| + \alpha_N |y-z| + \alpha_N} \left| \phi_t(y) \right| \left| \phi_t(z) \right| \left| a^*_{y} a^*_{z} a_x \mathbf{1}_{\partial N}(N-1) \psi \right| \left| a_x a^*_{y} a^*_{z} \mathbf{1}_{\partial N}(N-1) \psi \right|
\leq \frac{1}{N} \left( \sup_y \int dy \, \frac{1}{|x-y|^2} \left| \phi_t(y) \right|^2 \right) \int dx \, dy \, \left| a_x a^*_{y} \mathbf{1}_{\partial N}(N-1) \psi \right|^2 + \frac{1}{N} \left( \sup_x \int dx \, \frac{1}{|x-y|^2} \left| \phi_t(y) \right|^2 \right) \int dx \, dy \, \left| a^*_{y} a_x \mathbf{1}_{\partial N}(N-1) \psi \right|^2
\leq \frac{1}{N} \left\| \phi_t \right\|^2_{H^1} \left\| (N + 1) \mathbf{1}_{\partial N}(N-1) \psi \right\|^2.
\]

The other terms arising when we insert (6.13) in the second summand on the r.h.s. of (6.17) (both the quartic and the quadratic terms) can be bounded analogously. Together with (6.18), we conclude that

\[
\langle \psi, B^* B \psi \rangle \lesssim \frac{1}{N} \left\| \phi_t \right\|^2_{H^1} \left\| (N + 1)^{3/2} \mathbf{1}_{\partial N}(N-1) \psi \right\|^2.
\]  

(6.19)

From (6.15) and (6.19), we find

\[
\left( \frac{1}{\sqrt{N}} \int dx \, dy \, \frac{1}{|x-y| + \alpha_N} \left\{ \phi_t(y) a^*_{y} a^*_{x} \mathbf{1}_{\partial N}(N) a_x + \text{h.c.} \right\} \right)^2 \lesssim \frac{1}{N} (N + 1)^3 \mathbf{1}_{\partial N}(N-2).
\]  

(6.20)

Combining (6.11), (6.12), (6.14), and (6.20) we conclude (since \( \mathbf{1}_{\partial N} \leq 1 \)) that

\[
\mathcal{K}^2 \lesssim \hat{\mathcal{M}}^2_N(t) + (N + 1)^3.
\]  

(6.21)

Next, we observe that there exists a constant \( C = C(T, \kappa, \left\| \varphi \right\|_{H^2}) \) such that

\[
\langle \hat{W}_N(t; 0) \Omega, (N + 1)^3 \hat{W}_N(t; 0) \Omega \rangle \leq C
\]  

(6.22)

for all \( |t| \leq T \). The proof of this bound is analogous to the proof of Proposition 4.3 (see Lemma 3.5, with \( M = \theta N \), and its proof in [21]). The only difference is that the generator \( \hat{\mathcal{M}}_N(t) \) contains a cutoff also in the quartic term (while in Proposition 4.3 the cutoff appeared only in the cubic term of the generator \( \mathcal{M}_N(t) \)); this difference does not play any role in the proof of (6.22) because the quartic term (with or without cutoff) commutes with the number of particles operator \( \mathcal{N} \) (and thus with its powers).

Finally, we control the growth of the expectation of \( \hat{\mathcal{M}}^2_N(t) \). To this end we compute, using (6.3),

\[
\frac{d}{dt} \langle \hat{W}_N(t; 0) \Omega, \hat{\mathcal{M}}^2_N(t) \hat{W}_N(t; 0) \Omega \rangle = \langle \hat{W}_N(t; 0) \Omega, \left( \hat{\mathcal{M}}_N(t) \hat{\mathcal{M}}_N(t) + \hat{\mathcal{M}}_N(t) \hat{\mathcal{M}}_N(t) \right) \hat{W}_N(t; 0) \Omega \rangle
\]

and thus

\[
\frac{d}{dt} \langle \hat{W}_N(t; 0) \Omega, \hat{\mathcal{M}}^2_N(t) \hat{W}_N(t; 0) \Omega \rangle^{1/2} \lesssim \langle \hat{W}_N(t; 0) \Omega, \hat{\mathcal{M}}^2_N(t) \hat{W}_N(t; 0) \Omega \rangle^{1/2}.
\]  

(6.23)
We have
\[
\hat{\mathcal{M}}_N(t) = -\lambda \int dx \left( \frac{1}{|.| + \alpha_N} * (\bar{\phi}_t \phi_t + \bar{\phi}_t \phi_t) \right)(x) a_x^* a_x \\
- \lambda \int dx dy \frac{1}{|x-y| + \alpha_N} \left( \bar{\phi}_t(x) \phi_t(y) + \bar{\phi}_t(x) \phi_t(y) \right) a_y^* a_x \\
- \lambda \int dx dy \frac{1}{|x-y| + \alpha_N} \left( \phi_t(x) \phi_t(y) a_y^* a_x^* + \text{h.c.} \right) \\
- \frac{\lambda}{\sqrt{N}} \int dx dy \frac{1}{|x-y| + \alpha_N} \left( \phi_t(y) a_y^* a_y 1_{\partial N}(N) a_x + \text{h.c.} \right).
\] (6.24)

Observe that from the (regularized) Hartree equation (6.1) we easily find that \( \| \dot{\phi}_t \| \lesssim \| \phi_t \|_{H^1} \) and
\[
\| \nabla \phi_t \| \lesssim \left\| (1 - \Delta) \phi_t \right\|_2 + \| \nabla \left( \frac{1}{|.| + \alpha} * |\phi_t|^2 \right) \phi_t \|_2 \\
\lesssim \| \phi_t \|_{H^2} + \left\| \frac{1}{|.| + \alpha} * |\phi_t|^2 \right\|_3 \| \phi_t \|_6 + \left\| \frac{1}{|.| + \alpha} * |\phi_t|^2 \right\|_\infty \| \nabla \phi_t \|_2 \\
\lesssim \| \phi_t \|_{H^2} + \| \phi_t \|_{H^{2/1}}^2 \| \phi_t \|_{H^{1/1}}.
\] (6.25)

This implies, by (6.2), that there exists a constant \( C = C(T, \kappa, \| \varphi \|_{H^2}) \) such that
\[
\| \dot{\phi}_t \|_{H^1} \leq C \quad \text{for all } t \in \mathbb{R} \text{ with } |t| \leq T.
\] (6.26)

Next, we bound the square of the terms on the r.h.s. of (6.24). Similarly to (6.11) and (6.12), we find
\[
\left( \int dx \left( \frac{1}{|.| + \alpha_N} * (\bar{\phi}_t \phi_t + \bar{\phi}_t \phi_t) \right)(x) a_x^* a_x \right)^2 \lesssim \| \phi_t \|_{H^1} \| \dot{\phi}_t \|_{H^1} N^2
\] (6.27)
and
\[
\left( \int dx dy \frac{1}{|x-y| + \alpha_N} \left( \bar{\phi}_t(x) \phi_t(y) + \bar{\phi}_t(x) \phi_t(y) \right) a_x^* a_x \right)^2 \leq \| \phi_t \|_{H^1} \| \dot{\phi}_t \|_{H^1} N^2.
\] (6.28)
Moreover, similarly to (6.14), we obtain
\[
\left( \int dx dy \frac{1}{|x-y| + \alpha_N} \left( \phi_t(x) \phi_t(y) + \phi_t(x) \phi_t(y) \right) a_y^* a_x^* + \text{h.c.} \right)^2 \lesssim \| \phi_t \|_{H^1} \| \phi_t \| (N + 1)^2.
\] (6.29)
Finally, analogously to (6.19) (replacing \( \phi \) with \( \dot{\phi} \)) we have
\[
\left( \frac{1}{\sqrt{N}} \int dx dy \frac{1}{|x-y| + \alpha_N} \phi_t(y)a_x^* a_y^* 1_{\partial N}(N) a_x + \text{h.c.} \right)^2 \lesssim \frac{1}{N} \| \phi_t \|_{H^1}^2 (N + 1)^3.
\] (6.30)
From (6.24), (6.27), (6.28), (6.29), (6.30), we find, using (6.22), that
\[
\langle \hat{\mathcal{W}}_N(t; 0) \Omega, \hat{\mathcal{M}}_N(t) \hat{\mathcal{W}}_N(t; 0) \Omega \rangle \lesssim 1.
\]
Eq. (6.23) then implies that there exists \( C = C(\kappa, T, \| \varphi \|_{H^2}) \) such that
\[
\langle \hat{\mathcal{W}}_N(t; 0) \Omega, \hat{\mathcal{M}}_N^2(t) \hat{\mathcal{W}}_N(t; 0) \Omega \rangle \leq C
\]
for all \( t \in \mathbb{R} \) with \( |t| \leq T \). Proposition 6.1 now follows from (6.21) and (6.22). \( \square \)
Lemma 6.4. There exists a universal constant $C > 0$ such that

$$\left( \int dx dy \frac{1}{|x - y| + \alpha} a_x^* a_y^* 1_{\vartheta N} (\mathcal{N}) a_y a_x \right)^2 \leq C \vartheta^2 K^2 \quad (6.31)$$

for all $\alpha, \vartheta > 0$.

Proof. Denote

$$\tilde{V} = \int dx dy \frac{1}{|x - y| + \alpha} a_x^* a_y^* 1_{\vartheta N} (\mathcal{N}) a_y a_x .$$

Then $\tilde{V}$ (and thus $\tilde{V}^2$) leaves the number of particles invariant and, on the $n$-particle sector, we have

$$(\tilde{V}^2)^{(n)} = \left( \frac{1}{N} \sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j| + \alpha} \right)^2$$

if $n \leq \vartheta N$ (6.32)

and $(\tilde{V}^2)^{(n)} = 0$ when $n > \vartheta N$. Using the operator inequality (see, for example, Lemma 9.1 in [2])

$$\frac{1}{|x - y|^2} \lesssim (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{1/2}$$

we find

$$(\tilde{V}^2)^{(n)} \lesssim \frac{n^2}{N^2} \sum_{1 \leq i < j \leq n} \frac{1}{(|x_i - x_j| + \alpha)^2} \lesssim \frac{n^2}{N^2} \sum_{1 \leq i < j \leq n} (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{1/2}$$

$$\lesssim \frac{n^2}{N^2} \left( \sum_{j=1}^n (1 - \Delta_x_j)^{1/2} \right)^2 \lesssim \vartheta^2 (K^2)^{(n)} \quad (6.33)$$

and the lemma is proven. \qed

### 6.2 Weak bounds on growth of $K^2$ with respect to fluctuation dynamics

In this subsection, we show Proposition 6.2. Again, we will need the estimate (6.2); recall also that in this section we use the shorthand notation $\phi_t \equiv \varphi_{\alpha N}^{(\alpha_N)}$ for the solution of (6.1).

We write

$$\langle \mathcal{U}_N(t;0) \Omega, K^2 \mathcal{U}_N(t;0) \Omega \rangle$$

$$= \int dx dy \left\langle \mathcal{U}_N(t;0) \Omega, (1 - \Delta_x)^{1/4} a_x^* (1 - \Delta_x)^{1/4} a_x \right.$$ 

$$\left. \times (1 - \Delta_y)^{1/4} a_y^* (1 - \Delta_y)^{1/4} a_y \mathcal{U}_N(t;0) \Omega \right\rangle$$

$$= \int dx dy \left\langle \Omega, (1 - \Delta_x)^{1/4} \mathcal{U}_N(t;0) a_x^* \mathcal{U}_N(t;0) (1 - \Delta_x)^{1/4} \mathcal{U}_N^* (t;0) a_x \mathcal{U}_N(t;0) \right.$$ 

$$\left. \times (1 - \Delta_y)^{1/4} \mathcal{U}_N^* (t;0) a_y^* \mathcal{U}_N(t;0) (1 - \Delta_y)^{1/4} \mathcal{U}_N^* (t;0) a_y \mathcal{U}_N(t;0) \Omega \right\rangle .$$

Next we use that (see (4.7))

$$\mathcal{U}_N^* (t;0) a_x \mathcal{U}_N(t;0) = W^* (\sqrt{N} \varphi) e^{i \mathcal{H}_N t} (a_x - \sqrt{N} \phi_t(x)) e^{-i \mathcal{H}_N t} W(\sqrt{N} \varphi)$$

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to conclude that
\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2\mathcal{U}_N(t; 0)\Omega \rangle
= \int dx dy \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, (1 - \Delta_x)^{1/4} (a_x^* - \sqrt{N}\phi_t(x)) (1 - \Delta_y)^{1/4} (a_y - \sqrt{N}\phi_t(x)) \right>
\times (1 - \Delta_y)^{1/4} (a_y^* - \sqrt{N}\phi_t(y)) (1 - \Delta_y)^{1/4} (a_y - \sqrt{N}\phi_t(y)) e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>.
\] (6.34)

For \( f \in L^2(\mathbb{R}^3) \), let
\[
\pi(f) = a^*(f) + a(f) = \int dx \left( f(x) a_x^* + \overline{f}(x) a_x \right).
\]
Then, from (6.34), we obtain
\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2\mathcal{U}_N(t; 0)\Omega \rangle
= \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \mathcal{K}^2 e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ 2\sqrt{N} Re \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \mathcal{K} \pi((1 - \Delta)^{1/4} \phi_t) e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ 2N \|\phi_t\|^2_{H^{1/2}} \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \mathcal{K} e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ N \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \pi^2((1 - \Delta)^{1/4} \phi_t) e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ 2N^{3/2} \|\phi_t\|^2_{H^{1/2}} \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \pi((1 - \Delta)^{1/4} \phi_t) e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ N^2 \|\phi_t\|^4_{H^{1/2}}.
\]
Using Schwarz inequality, we find
\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2\mathcal{U}_N(t; 0)\Omega \rangle \leq \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \mathcal{K}^2 e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ N \left< e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega, \pi^2((1 - \Delta)^{1/4} \phi_t) e^{-i\mathcal{H}_N^t} W(\sqrt{N}\varphi)\Omega \right>
+ N^2 \|\phi_t\|^4_{H^{1/2}}.
\] (6.35)

Next, we observe that, for arbitrary \( f \in L^2(\mathbb{R}^3) \) and \( \psi \in \mathcal{F} \),
\[
\langle \psi, \pi^2(f)\psi \rangle = \|\pi(f)\psi\|^2 \lesssim \|a(f)\psi\|^2 + \|a^*(f)\psi\|^2 \lesssim \|f\|^2 \langle \psi, (\mathcal{N} + 1)\psi \rangle.
\] (6.36)

Moreover, we have
\[
\mathcal{H}_N^\alpha = \mathcal{K} - \mathcal{V}, \quad \text{where} \quad \mathcal{V} = \frac{\lambda}{2N} \int dx dy \frac{1}{|x - y| + \alpha} a_x^* a_y a_x.
\]
Since \( \mathcal{V} \lesssim \frac{1}{\sqrt{N} \alpha} N^2 \)
and since \( \mathcal{V}, \mathcal{N} = 0 \), we conclude that
\[
\mathcal{K}^2 \lesssim (\mathcal{H}_N^\alpha)^2 + \mathcal{V}^2 \lesssim (\mathcal{H}_N^\alpha)^2 + \frac{1}{N^2 \alpha^2} N^4.
\] (6.37)

Inserting (6.36) and (6.37) in (6.35), we find
\[
\langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2\mathcal{U}_N(t; 0)\Omega \rangle
\lesssim \left< W(\sqrt{N}\varphi)\Omega, (\mathcal{H}_N^\alpha)^2 W(\sqrt{N}\varphi)\Omega \right>
+ \frac{1}{N^2 \alpha^2} \left< W(\sqrt{N}\varphi)\Omega, N^4 W(\sqrt{N}\varphi)\Omega \right>
+ N \|\phi_t\|^2_{H^{1/2}} \left< W(\sqrt{N}\varphi)\Omega, (\mathcal{N} + 1) W(\sqrt{N}\varphi)\Omega \right>
+ N \|\phi_t\|^4_{H^{1/2}} \left< W(\sqrt{N}\varphi)\Omega, \mathcal{K}^2 W(\sqrt{N}\varphi)\Omega \right>
+ \frac{1}{N^2 \alpha^2} \left< W(\sqrt{N}\varphi)\Omega, N^4 W(\sqrt{N}\varphi)\Omega \right>
+ N \|\phi_t\|^2_{H^{1/2}} \left< W(\sqrt{N}\varphi)\Omega, (\mathcal{N} + 1) W(\sqrt{N}\varphi)\Omega \right>
+ N \|\phi_t\|^4_{H^{1/2}}.
\] (6.38)
Using the properties of Weyl operators listed in Lemma 3.2, it is simple to check that
\[ \langle W(\sqrt{N} \varphi)\Omega, (\mathcal{N} + 1) W(\sqrt{N} \varphi)\Omega \rangle \lesssim N \quad \text{and} \quad \langle W(\sqrt{N} \varphi)\Omega, N^4 W(\sqrt{N} \varphi)\Omega \rangle \lesssim N^4. \] (6.39)
Moreover, we have
\[ \langle W(\sqrt{N} \varphi)\Omega, \mathcal{K}^2 W(\sqrt{N} \varphi)\Omega \rangle \]
\[ = \int \, dx \, dy \, \langle W(\sqrt{N} \varphi)\Omega, (1 - \Delta_x)^{1/4} a_x^* (1 - \Delta_x)^{1/4} a_x \]
\[ \times (1 - \Delta_y)^{1/4} a_y^* (1 - \Delta_y)^{1/4} a_y W(\sqrt{N} \varphi)\Omega \rangle \]
\[ = \int \, dx \, dy \, \langle \Omega, (1 - \Delta_x)^{1/4} (a_x^* - \sqrt{N} \varphi(x)) (1 - \Delta_x)^{1/4} (a_x - \sqrt{N} \varphi(x)) \]
\[ \times (1 - \Delta_y)^{1/4} (a_y^* - \sqrt{N} \varphi(y)) (1 - \Delta_y)^{1/4} (a_y - \sqrt{N} \varphi(y)) W(\sqrt{N} \varphi)\Omega \rangle \]
\[ = N^2 \| \varphi \|^4_{H^{1/2}} + N \| \varphi \|^2_{H^1}. \] (6.40)
Inserting (6.39) and (6.40) into (6.38), and using the bound (6.2), we conclude that
\[ \langle \mathcal{U}_N(t; 0)\Omega, \mathcal{K}^2 \mathcal{U}_N(t; 0)\Omega \rangle \lesssim N^2 + \frac{N^2}{\alpha^2}. \]
This completes the proof of Proposition 6.2.

\[ \square \]
6.3 Comparison of fluctuation dynamics with regularized dynamics

In this section, we prove Proposition 6.3.

We rewrite
\[ \| (\mathcal{U}_N(t; 0) - \mathcal{W}_N(t; 0))\Omega \| = \| (1 - \mathcal{U}_N^* (t; 0) \mathcal{W}_N(t; 0))\Omega \| \]
\[ \leq \int_0^t \| \mathcal{U}_N^*(s; 0) \mathcal{L}_N(s) - \mathcal{M}_N(s) \mathcal{W}_N(s; 0)\Omega \| \, ds \] (6.41)
\[ \leq \int_0^t \| (\mathcal{L}_N(s) - \mathcal{M}_N(s)) \mathcal{W}_N(s; 0)\Omega \| \, ds. \]
We recall that
\[ \mathcal{M}_N(t) - \mathcal{L}_N(t) = \frac{\lambda}{\sqrt{N}} \int \, dx \, dy \, \frac{1}{|x - y| + \alpha_N} \phi_t(y) a_x^* a_y^* (1 - \mathbf{1}_{\vartheta_N(N)}) a_x + \text{h.c.} \]
\[ + \frac{\lambda}{2N} \int \, dx \, dy \, \frac{1}{|x - y| + \alpha_N} a_x^* a_y^* (1 - \mathbf{1}_{\vartheta_N(N)}) a_y a_x. \] (6.42)
Analogously to (6.20), but with \( \mathbf{1}_{\vartheta_N} \) replaced by \( 1 - \mathbf{1}_{\vartheta_N} \), the square of the terms on the first line of the last equation can be bounded by
\[ \left( \frac{1}{\sqrt{N}} \int \, dx \, dy \, \frac{1}{|x - y| + \alpha_N} \phi_t(y) a_x^* a_y^* (1 - \mathbf{1}_{\vartheta_N(N)}) a_x + \text{h.c.} \right)^2 \]
\[ \lesssim \frac{1}{N} (N + 1)^3 (1 - \mathbf{1}_{\vartheta_N(N - 2)}). \] (6.43)
As for the second term on the r.h.s. of (6.42), its square can be estimated as follows.

\[
\left( \frac{1}{N} \int \frac{1}{|x - y| + \alpha_N} a_x a_y^*(1 - \Theta_N(N)) a_y a_x \right)^2
\]

\[
= \frac{1}{N^2} \int \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \times (1 - \Theta_N(N - 2)) a_x^* a_y^* a_y a_x a_x^* a_y^* a_y (1 - \Theta_N(N - 2))
\]

From \(a_y a_x a_x^* a_y^* = a_x a_y a_y a_x + [a_y a_x, a_x^* a_y^*]\), and from (6.13), we conclude that

\[
\left( \frac{1}{N} \int \frac{1}{|x - y| + \alpha_N} a_x a_y^*(1 - \Theta_N(N)) a_y a_x \right)^2 
\]

\[
\lesssim \frac{1}{N^2} \int \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \times (1 - \Theta_N(N - 2)) a_x^* a_y^* a_y a_x^* a_y a_x (1 - \Theta_N(N - 2))
\]

\[
+ \frac{1}{N^2} \int \frac{1}{|x - y| + \alpha_N} \frac{1}{|x' - y'| + \alpha_N} \times (1 - \Theta_N(N - 2)) a_x a_y^* a_y^* a_y a_x (1 - \Theta_N(N - 2))
\]

\[
+ \frac{1}{N^2} \int \frac{1}{|x - y| + \alpha_N} (1 - \Theta_N(N - 2)) a_x^* a_y a_x (1 - \Theta_N(N - 2))
\]

\[
\lesssim \frac{1}{N^2} (N + 1)^4 (1 - \Theta_N(N - 2))
\]

From (6.33) and (6.35), we find that

\[
(\hat{\mathcal{M}}_N(t) - \mathcal{L}_N(t))^2 \lesssim \left( \frac{1}{N} + \frac{1}{N^2} \right) (N + 1)^4 (1 - \Theta_N(N - 2))
\]

(6.46)

For every \(k \in \mathbb{N}\), we have \((1 - \Theta_N(N - 2)) \leq (N - 2)^k/(\Theta N)^k\). Therefore

\[
(\hat{\mathcal{M}}_N(t) - \mathcal{L}_N(t))^2 \lesssim \left( \frac{1}{N} + \frac{1}{N^2} \right) \frac{(N + 1)^{k+4}}{(\Theta N)^k}
\]

(6.47)

Analogously to Proposition 4.3 (see Lemma 3.8 and its proof in [21]), there is \(C = C(k, \kappa, T, \|\varphi\|_{H^2})\) such that

\[
\|\hat{\mathcal{W}}_N(t; 0)\Omega, (N + 1)^{k+4}\| \leq C
\]

for all \(|t| \leq T\). From (6.46), we find that, for every \(k \in \mathbb{N}\), there exists \(C = C(\Theta, k, T, \kappa, \|\varphi\|_{H^2})\) such that

\[
\left\| (\hat{\mathcal{M}}_N(t) - \mathcal{L}_N(t))\hat{\mathcal{W}}_N(t; 0)\Omega \right\| \leq \frac{C}{N^k} \left( \frac{1}{N} + \frac{1}{N\alpha_N} \right).
\]

The proposition now follows from (6.41).

\[\square\]

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