Strictly working in the framework of the nonrelativistic quantum mechanics of a spin $\frac{1}{2}$ particle coupled to an external electromagnetic field, we show, by explicit construction, the existence of a charge conjugation operator matrix which defines the corresponding antiparticle wave function and leads to the galilean and gauge invariant Schroedinger-Pauli equation satisfied by it.

Key words: charge conjugation; galilean relativity; gauge invariance.

1. Introduction

In a recent paper \(^1\), Cabo et al showed the existence of the nonrelativistic limit $C_{nr}$ of the charge conjugation operation $C$ for the Dirac equation of a 4-spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ coupled to an external electromagnetic potential $(\phi, \vec{A})$. At low velocities of the Dirac particle with respect to the velocity of light in vacuum $c$, the “large components” $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of $\Psi$ satisfy the Schroedinger-Pauli equation

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2m} (-\nabla^2 + \frac{q^2}{\hbar^2 c^2} \vec{A}^2 + \frac{iq}{\hbar c} \nabla \cdot \vec{A} + 2\frac{iq}{\hbar c} \vec{A} \cdot \nabla - \frac{q}{\hbar c} \vec{\sigma} \cdot \vec{B} - 2mq\phi) \begin{pmatrix} u \\ v \end{pmatrix}$$

(1)

where $q$ and $m$ are the electric charge and mass respectively, $\vec{\sigma}$ are the Pauli matrices, $\vec{B} = \nabla \times \vec{A}$ is the magnetic field and, at each space time point $\begin{pmatrix} \vec{x} \\ t \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2$. The charge conjugate Pauli spinor $\psi_c$ representing spin $\frac{1}{2}$ antiparticles (e.g. positrons) if $\psi$ represents spin $\frac{1}{2}$ particles (e.g. electrons) is given by

$$\psi_c = \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = C_{nr} \begin{pmatrix} u \\ v \end{pmatrix}$$

(2)

where

$$C_{nr} = KM, \quad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(3)

is the nonrelativistic limit of the charge conjugation matrix of the Dirac equation, which up to a sign is given by \(^3\)

$$C = i\gamma^2 \gamma_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

(4)

$K$ is the complex conjugation antilinear and hemitian ($K^\dagger = K$) operation. $\psi_c$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = \frac{1}{2m} (-\nabla^2 - \frac{q^2}{\hbar^2 c^2} \vec{A}^2 + \frac{iq}{\hbar c} \nabla \cdot \vec{A} + 2\frac{iq}{\hbar c} \vec{A} \cdot \nabla - \frac{q}{\hbar c} \vec{\sigma} \cdot \vec{B} - 2mq\phi) \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix}.$$ 

(5)

As it was proved in reference 1, both (1) and (5) are transformed into each other by the operator $C_{nr}$, thus reaffirming the galilean character of the approximation $C_{nr}$ to $C$. This is a non trivial result specially because of the general belief that charge conjugation is a symmetry that exists only in the relativistic regime. \(^4\)
In this note we discuss the previous result without appealing to the limiting process, namely, strictly working in the context of the galilean group, for simplicity of its connected component \( G_0 \), and of its universal covering group \( \hat{G}_0 \) (section 2). From the lagrangian density \( \mathcal{L} \) for the equation (1), and using \( C_{nr} \), we construct the lagrangian density \( \mathcal{L}_c \) for equation (5), and prove the galilean invariance of these equations by proving this invariance for \( \mathcal{L} \) and \( \mathcal{L}_c \). We also verify the gauge invariance of \( \mathcal{L}_c \) (section 3).

2. Galilean group, its universal covering group, and spinors

The connected component of the galilean group \( G_0 \) consists of the set of \( 4 \times 4 \) matrices

\[
g = \begin{pmatrix} R & \bar{V} \\ 0 & 1 \end{pmatrix}
\]

with \( R \) in the 3-dimensional rotation group \( SO(3) \), boost velocity \( \bar{V} \) in \( \mathbb{R}^3 \), composition law

\[
g_2 g_1 = \begin{pmatrix} R_2 & \bar{V}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & \bar{V}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & \bar{V}_2 + R_2 \bar{V}_1 \\ 0 & 1 \end{pmatrix},
\]

identity

\[
\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and inverse

\[
\begin{pmatrix} R & \bar{V} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^{-1} & -R^{-1} \bar{V} \\ 0 & 1 \end{pmatrix}.
\]

\( G_0 \) is a non abelian, non compact, connected but non simply connected six dimensional Lie group; like the connected component of the Lorentz group, its topology is that of the cartesian product of the real projective space with ordinary 3-space \( i.e. \mathbb{R}P^3 \times \mathbb{R}^3 \). The action of \( G_0 \) on spacetime is given by

\[
G_0 \times \mathbb{R}^4 \to \mathbb{R}^4, \quad (g, \begin{pmatrix} \bar{x} \\ t \end{pmatrix}) \mapsto \begin{pmatrix} \bar{x} \\ t \end{pmatrix} = g \begin{pmatrix} \bar{x}' \\ t' \end{pmatrix} = \begin{pmatrix} R \bar{x}' + \bar{V} t' \\ t' \end{pmatrix}.
\]

Since one has the action

\[
\mu : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (R, \bar{x}) \mapsto R \bar{x},
\]

then \( G_0 \) is isomorphic to the semidirect sum \( \mathbb{R}^3 \rtimes \mu SO(3) : \begin{pmatrix} R & \bar{V} \\ 0 & 1 \end{pmatrix} \mapsto (\bar{V}, R) \) with composition law

\[
(\bar{V}', R')(\bar{V}, R) = (\bar{V}' + R' \bar{V}, R'R).
\]

The universal covering group of \( G_0 \) is given by the \( \mathbb{Z}_2 \)-bundle

\[
\mathbb{Z}_2 \to \hat{G}_0 \xrightarrow{\Pi} G_0
\]

where

\[
\hat{G}_0 = \{ \hat{g} = \begin{pmatrix} T & \bar{V} \\ 0 & 1 \end{pmatrix}, \quad T \in SU(2), \quad \bar{V} \in \mathbb{R}^3 \},
\]

and \( \Pi \) is the \( 2 \to 1 \) group homomorphism

\[
\Pi(\hat{g}) = \begin{pmatrix} \pi(T) & \bar{V} \\ 0 & 1 \end{pmatrix}
\]
with \( \pi : SU(2) \to SO(3) \) the well known projection

\[
\pi \left( \begin{array}{cc} z & w \\ -\bar{w} & \bar{z} \end{array} \right) = \left( \begin{array}{cc} Rez^2 - Rew^2 & Imz^2 + Imw^2 & -2Rezw \\ -1Imz^2 + Imw^2 & Rez^2 + Rew^2 & 2Imzw \\ 2Rezw & 2Imzw & |z|^2 - |w|^2 \end{array} \right).
\]  (9c)

\( \hat{G}_0 \) is simply connected and has the topology of \( S^3 \times \mathbb{R}^3 \). Since \( SU(2) \) acts on \( \mathbb{R}^3 \):

\[
\hat{\mu} : SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3, \ (T, \bar{V}) \mapsto \pi(T)\bar{V},
\]  (10)

one has the group isomorphism

\[
\hat{G}_0 \ni \left( \begin{array}{c} T \\ \bar{V} \end{array} \right) \mapsto (\bar{V}, T) \in \mathbb{R}^3 \times \hat{\mu} SU(2);
\]  (11)

the composition law in \( \hat{G}_0 \) is given by

\[
\left( \begin{array}{c} T' \\ \bar{V}' \end{array} \right) \left( \begin{array}{c} T \\ \bar{V} \end{array} \right) = \left( \begin{array}{c} T'T \\ \bar{V}' + \pi(T')\bar{V} \end{array} \right),
\]  (12)

while the identity and inverse are respectively given by

\[
\left( \begin{array}{c} I \\ \bar{V} \end{array} \right), \quad I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]  (12a)

and

\[
\left( \begin{array}{c} T \\ \bar{V} \end{array} \right)^{-1} = \left( \begin{array}{cc} T^{-1} & -\pi(T^{-1})\bar{V} \\ 0 & 1 \end{array} \right).
\]  (12b)

Turning back to physics, for each mass value \( m > 0 \), \( \hat{G}_0 \) acts on the infinite dimensional Hilbert space \( \mathcal{L}^2_1 \) of continuously differentiable and square integrable \( \mathbb{C}^2 \)-valued functions \( \left( \begin{array}{c} u \\ v \end{array} \right) \) on \( \mathbb{R}^4 \), the Schroedinger-Pauli spinors. This action is defined as follows: \(^5\)

\[
\hat{\mu}_m : \hat{G}_0 \times \mathcal{L}^2_1 \to \mathcal{L}^2_1, \ (\left( \begin{array}{c} T \\ \bar{V} \end{array} \right), \left( \begin{array}{c} u \\ v \end{array} \right)) \mapsto \left( \begin{array}{c} T \\ 1 \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right),
\]  (13)

\( \hat{\mu}_m \) is equivalent to the representation

\[
\tilde{\hat{\mu}}_m : \hat{G}_0 \to \text{End}(\mathcal{L}^2_1), \quad \tilde{\hat{\mu}}_m(\hat{g})(\left( \begin{array}{c} u \\ v \end{array} \right)) = \hat{g} \cdot \left( \begin{array}{c} u \\ v \end{array} \right).
\]  (13.a)

At each \( t \) one has the inner product

\[
\left( \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right), \left( \begin{array}{c} u_1 \\ v_1 \end{array} \right) \right)(t) = \int d^3\bar{x}(\bar{u}_2(\bar{x}, t)u_1(\bar{x}, t) + \bar{v}_2(\bar{x}, t)v_1(\bar{x}, t))
\]  (14a)

and the norm

\[
\| \left( \begin{array}{c} u \\ v \end{array} \right) \|^2(t) = \left( \left( \begin{array}{c} u \\ v \end{array} \right), \left( \begin{array}{c} u \\ v \end{array} \right) \right)(t) = \int d^3\bar{x}(|u(\bar{x}, t)|^2 + |v(\bar{x}, t)|^2).
\]  (14b)

The galilean transformation of the charge conjugate spinor \( \psi_c \) is given by

\[
\psi_c \mapsto \tilde{\hat{g}} \cdot \psi_c, \quad \left( \begin{array}{c} \bar{T} \\ 0 \end{array} \right) \left( \begin{array}{c} -\bar{v} \\ \bar{u} \end{array} \right) \left( \begin{array}{c} \bar{x} \\ t \end{array} \right) = e^{\frac{im}{\hbar}(\bar{V} \cdot \bar{x} + \frac{1}{2}|\bar{V}|^2)t} \left( \begin{array}{c} -\bar{v}(\pi(\bar{T})\bar{x} + \bar{V}t) \\ \bar{u}(\pi(\bar{T})\bar{x} + \bar{V}t) \end{array} \right).
\]  (15)
Finally, the galilean transformations of the electromagnetic potential \((\phi, \vec{A})\) and the magnetic field \(\vec{B}\) are

\[
\phi(x, t) = \phi'(\vec{x}', t'), \quad \vec{A}(x, t) = R\vec{A}'(\vec{x}', t'), \quad \vec{B}(x, t) = R\vec{B}'(\vec{x}', t')
\]

with \(\vec{x} = R\vec{x}' + \vec{v}t\) and \(t = t'\).

Remark: Representations associated with different values of the mass are inequivalent. 6

3. Lagrangian formulation and galilean and gauge invariances

The Pauli equations (1) and (5) can be formulated within the lagrangian framework. The lagrangian for equation (1) is

\[
L = \frac{i\hbar}{2}\left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi\right)\psi - \psi^\dagger\left(\frac{\partial}{\partial t} - \frac{iq}{\hbar}\phi\right)\psi + \frac{\hbar^2}{2m}(\nabla + \frac{iq}{\hbar c}\vec{A})\psi^\dagger \cdot \left(\nabla - \frac{iq}{\hbar c}\vec{A}\right)\psi - \frac{q\hbar}{2mc}\psi^\dagger \vec{\sigma} \cdot \vec{B}\psi
\]

= \frac{i\hbar}{2}(\psi^\dagger \psi - \psi^\dagger \psi) + \frac{\hbar^2}{2m}(\nabla \psi^\dagger \cdot \nabla \psi - \psi^\dagger \cdot \nabla \psi) - \frac{q\hbar}{2mc}\psi^\dagger \vec{A} \cdot \nabla \psi - \frac{q\hbar}{2mc}\psi^\dagger \vec{B} \cdot \psi + q\hbar \psi^\dagger \phi \psi,
\]

and equation (1) amounts to the variational equation

\[
\frac{\delta}{\delta \psi^\dagger(x, t)} S = 0
\]

where \(S\) is the action

\[
S = \int dt \int d^3x L(x, t).
\]

Under the charge conjugation operation

\[
L \rightarrow L_c = KL = -\frac{i\hbar}{2}\left(\frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi\right)\psi_c - \psi^\dagger_c\left(\frac{\partial}{\partial t} - \frac{iq}{\hbar}\phi\right)\psi_c + \frac{\hbar^2}{2m}(\nabla - \frac{iq}{\hbar c}\vec{A})\psi^\dagger_c \cdot \left(\nabla + \frac{iq}{\hbar c}\vec{A}\right)\psi_c + \frac{q\hbar}{2mc}\psi^\dagger_c \vec{\sigma} \cdot \vec{B}\psi_c
\]

= -\frac{i\hbar}{2}(\psi^\dagger_c \psi_c - \psi^\dagger \psi^\dagger \psi_c) + \frac{\hbar^2}{2m}(\nabla \psi_c^\dagger \cdot \nabla \psi_c + \psi^\dagger \cdot \nabla \psi^\dagger \cdot \nabla \psi) - \frac{q\hbar}{2mc}(\psi^\dagger_c \vec{A} \cdot \nabla \psi^\dagger_c - \psi^\dagger_c \vec{A} \cdot \nabla \psi^\dagger \cdot \vec{A} \psi^\dagger_c) + \frac{q\hbar}{2mc}(\psi^\dagger_c \vec{A} \cdot \vec{B} \psi_c + q\hbar \psi^\dagger \phi \psi_c.
\]

To pass from (17) to (20), the identity \(M^\dagger M = 1\) is inserted at each term of (17), and the fact that \(M \vec{A}^\dagger M = M(\vec{A}_1, \vec{A}_2, \vec{A}_3)M^\dagger = (-\vec{A}_1, \vec{A}_2, -\vec{A}_3)\) is used; then the complex conjugation operation \(K\) completes the transformation.

The total action for the particle-antiparticle system is

\[
S_{tot} = S + S_c = \int dt \int d^3x (L(x, t) + L_c(x, t))
\]

and equation (5) is obtained from \(S_{tot}\) or \(S_c\) as

\[
\frac{\delta}{\delta \psi^\dagger_c(x, t)} S_{tot} = \frac{\delta}{\delta \psi^\dagger_c(x, t)} S_c = 0.
\]

The lagrangian \(L\) and therefore the equation (1), are invariant under the galilean transformations (13), (15) and (16) for \(\psi, \psi_c, (\phi, \vec{A})\) and \(\vec{B}\), respectively. To prove it, we use the facts that \(\nabla = R^{-1}\nabla'\) where \(\nabla = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla\) and \(\nabla' = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla\). If \(L'\) and \(L_c'\) are the transformed lagrangian densities for particles and antiparticles, then, from equation (20),

\[
L_c(x, t) = K L(x, t) = KL'(x', t') = L_c'(x', t')
\]
and therefore the galilean invariance of equation (5) is also proved.

Finally, both $\mathcal{L}$ and $\mathcal{L}_c$, and therefore the equations (1) and (5), are gauge invariant under the transformations $\psi \rightarrow e^{i\Lambda} \psi$, $\psi_c \rightarrow e^{-i\Lambda} \psi_c$, $\phi \rightarrow \phi - \frac{h}{q} \frac{\partial}{\partial t} \Lambda$ and $\vec{A} \rightarrow \vec{A} + \frac{h c}{q} \nabla \Lambda$, where $\Lambda$ is an arbitrary differentiable function of $(\vec{x}, t)$.

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