Harnack’s estimate for a mixed local–nonlocal doubly nonlinear parabolic equation

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Abstract
We establish Harnack’s estimates for positive weak solutions to a mixed local and nonlocal doubly nonlinear parabolic equation, of the type
\[ \partial_t (|u|^{p-2}u) - \text{div} \ A(x, t, u, Du) + Lu(x, t) = 0, \]
where the vector field \( A \) satisfies the \( p \)-ellipticity and growth conditions and the integro-differential operator \( L \) whose model is the fractional \( p \)-Laplacian. All results presented in this paper are provided by using sharp tools in the doubly nonlinear theory together with quantitative estimates.

Mathematics Subject Classification
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1 Introduction and main results

In this paper, we consider the following mixed local–nonlocal doubly nonlinear equation of the form
\[ \partial_t (|u|^{p-2}u) - \text{div} \ A(x, t, u, Du) + Lu(x, t) = 0 \quad \text{in} \quad \Omega_T := \Omega \times (0, T), \quad (1.1) \]
where \( p > 1 \) denotes the summability and the operator \( A : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is a Carathéodory function fulfill the \( p \)-ellipticity and growth conditions:
\[
\begin{align*}
|A(x, t, u, \xi) &\cdot \xi| \geq c_0 |\xi|^p, \\
|A(x, t, u, \xi)| &\leq c_1 |\xi|^{p-1}
\end{align*}
\]
for almost every \((x, t) \in \Omega_T\) and every \((u, \xi) \in \mathbb{R} \times \mathbb{R}^n\) with constants \(0 < c_0 \leq c_1 < \infty\), where the integro-differential operator \( L \) is defined by

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\[ \mathcal{L}u(x,t) := 2 \text{ P.V.} \int_{\mathbb{R}^n} K(x,y,t)|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t)) \, dy \]

\[ = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} K(x,y,t)|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t)) \, dy, \quad (1.2) \]

where the symbol P.V. means “in the principal value sense”. Further, the kernel \( K(x,y,t) \) is assumed to be a nonnegative measurable function on \( \mathbb{R}^n \times \mathbb{R}^n \times (0, T) \) fulfilling

\[ K(x,y,t) = K(y,x,t) \]

and

\[ \frac{\Lambda_{-1}}{|x-y|^{n+sp}} \leq K(x,y,t) \leq \frac{\Lambda}{|x-y|^{n+sp}} \]

for almost every \((x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, T)\), with \( \Lambda \geq 1 \) being a constant and \( s \in (0, 1) \) being fractional differentiability.

Our Eq. (1.1) contains the prototype equation

\[ \partial_t \left( |u|^{p-2}u \right) - \Delta_p u + (-\Delta)^s_p u = 0, \quad (1.3) \]

where \( \Delta_p u := \text{div} \left( |Du|^{p-2}Du \right) \) denotes the classical \( p \)-Laplacian with \( Du = (\partial_i u)_{1 \leq i \leq n} \) being the gradient of \( u \) with respect to the space-variable \( x \). We know that \( \Delta_p u \) is the gradient vector field of the \( p \)-energy \( W_0^{1,p}(\Omega) \ni u \mapsto \int_\Omega |Du|^p \, dx \). Analogously, the fractional \( p \)-Laplace operator \(-\Delta)^s_p u\), defined by (1.2) with \( \Lambda = 1 \), can be understood as the gradient vector field \( \nabla E(u) \) on \( W_0^{s,p}(\Omega) \), where \( E(u) \) denotes an energy functional on \( W_0^{s,p}(\Omega) \), defined by

\[ W_0^{s,p}(\Omega) \ni u \mapsto E(u) := \frac{1}{p} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

Here the fractional Sobolev space \( W_0^{s,p}(\Omega) \) (see e.g. [37] for details) is given by

\[ W_0^{s,p}(\Omega) := \left\{ \| u \|_{W^{s,p}(\mathbb{R}^n)} < \infty : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\}, \]

where the norm \( \| u \|_{W^{s,p}(\mathbb{R}^n)} \) is defined by

\[ \| u \|_{W^{s,p}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |u(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

It is worth remarking that, for a Sobolev function \( u \in W_0^{1,p}(\Omega) \), there holds that

\[ \lim_{s \nearrow 1} (1 - s) \int_{B_\varepsilon \cap B_{\varepsilon}^c} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy = C(n,p) \int_{B_\varepsilon} |Du(x)|^p \, dx \quad \forall B_\varepsilon \subseteq \Omega, \]

which can be seen in [4]. Furthermore, as expected, the solution of \(-\Delta)^s_p u = 0\) converges strongly to \(-\Delta_p u = 0\) in \( L^p(\Omega) \cap W_0^{1,p}(\Omega) \) as \( s \nearrow 1 \). We refer to [11, Sect.1.4, Comments] for a precise description of the limit case \( s \nearrow 1 \). The fundamental tools and regularity theory of the fractional Sobolev space and interesting nonlocal problems during the last decades are addressed in the comprehensive literatures [34, 37] and references therein.

The prototype Eq. (1.3) is considered as the mixed type of the so-called Trudinger equation

\[ \partial_t \left( |u|^{p-2}u \right) - \Delta_p u = 0 \quad (1.4) \]
and the nonlocal Trudinger equation
\[ \partial_t \left( |u|^{p-2} u \right) + (-\Delta)^s_p u = 0. \] (1.5)

One of the features of Eqs. (1.4) and (1.5) is the homogeneity, that is, the solution to (1.4) or (1.5) can be scaled by any scale factor. As seen in the literatures [7–9, 32, 33], this homogeneity is advantage in De Giorgi’s measure theoretic lemmata. In 1968, by the use of Moser’s iteration, Trudinger first proved a Harnack inequality for nonnegative solutions to (1.4) in his pioneer work [43]. Later, in the case \( p > 2 \), Gianazza and Vespri [24] extended the Trudinger’s result to more general equation, replaced \( \Delta p u \) by \( A = A(x, t, u, Du) \), where the vector field \( A \) satisfies the \( p \)-ellipticity and growth conditions. In a doubling Borel measure framework, Kinnunen and Kuusi [29] succeeded proving a Harnack inequality for positive weak solutions to (1.4). Their proofs are based on the Moser scheme combined with the Sobolev and Caccioppoli type inequalities.

For the nonlocal parabolic problem, there are many literatures [1, 12, 19, 31, 34, 35, 38, 42] and references therein about a nonlocal parabolic equation of the form
\[ \partial_t u + L u = 0, \] (1.6)

although we will give the brief overview of literatures related to the regularity results for (1.6). Kassmann and Schwab [31] showed a weak Harnack inequality and the Hölder regularity for solutions of the nonlocal heat Eq. (1.6) with \( p = 2 \) and extra force term \( f \in L^\infty \). Alternatively, Kim [28] showed a Harnack inequality with nonlocal tail for (1.6) with \( p = 2 \). Later, Strömqvist [40, 41] obtained the local boundedness and a Harnack inequality with nonlocal tail for weak solutions to (1.6) in the case \( p > 2 \). As far as we know, this is a first contribution of Di Castro, Kuusi and Palatucci [16, 17] in the parabolic setting. Alternatively, in the case \( p > 2 \), Ding, Zhang and Zhou [19] proved the local boundedness and the Hölder regularity for weak solutions to (1.6) with a source term \( f = f(x, t, u) \) satisfying some structural conditions. The common underlying feature of all such papers is the approach by the nonlocal De Giorgi-Nash-Moser method. Very recently, Brasco, Lindgren and Strömqvist [12] showed the Hölder regularity for (local) weak solutions in the case \( p \geq 2 \), whose proof is completely different to previous approaches in view of using the iterated discrete differentiation method together with a Morrey type embedding. On the contrast, in the doubly nonlinear setting, Banerjee, Garain and Kinnunen [3] first proved the local boundedness of positive solutions to (1.5). The regular form of (1.1), that is,
\[ \partial_t u - \Delta_p u + (-\Delta)^s_p u = 0, \] (1.7)

where the fractional order \( s \in (0, 1) \) and summability \( p > 1 \), is motivated by not a just purely mathematical interest, but also biological modeling. Indeed, there is an application of the mixed local and nonlocal operator \( u \mapsto -\Delta_p u + (-\Delta)^s_p u \) to a logistic equation from the viewpoint of biological problems, see [20] and references therein. Very recently, Garain and Kinnunen [22, 23] showed a Harnack estimate and the local Hölder regularity for weak solutions to (1.7) in the case \( p > 1 \). Alternatively, Fang, Shang and Zhang [21] showed the local boundedness and the Hölder regularity for weak solutions to (1.7).

To the best of our knowledge, this paper contributes to new results for positive weak solutions to our mixed local–nonlocal doubly nonlinear parabolic Eq. (1.1) and the technical novelties of this paper are the local and nonlocal Moser’s iteration scheme and mollification arguments with a detailed description. Although, as long as we employ the approach in this paper, it is worth remarking that, the positivity condition of solutions cannot be removable readily, because the power nonlinearity with respect to the possibly sign-changing solution
itself makes the situation more difficult. Therefore we need to employ another approach like the so-called “expansion of positivity”. This phenomenon often occurs in the usual doubly nonlinear parabolic equation of the form
\[
\partial_t (|u|^q - 1) - \triangle_p u = 0, \quad q > 0, \quad p > 1.
\]
In the doubly nonlinear framework, we make the full use of the De Giorgi’s measure theoretic approach because it is very flexible. We refer to [7–9, 32, 33] for a detailed description.

Before formulating the main results, we need to introduce the notion of weak solutions to (1.1) as follows. Let \( T \) be the class of test functions defined by
\[
\mathcal{T} := \left\{ \varphi \in L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)) \mid \varphi(x, 0) = \varphi(x, T) = 0 \text{ a.e. } x \in \Omega \right\},
\]
where the Sobolev space \( W^{1,p}_0(\Omega) \) with zero boundary value is defined by
\[
W^{1,p}_0(\Omega) := \left\{ u \in W^{1,p}(\Omega) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\}.
\]

**Definition 1** (Weak solution) Suppose that the vector field \( A \) and the nonlocal operator \( \mathcal{L} \) satisfy the conditions (2.1)–(2.3). A measurable function \( u = u(x, t) \) defined on \( \mathbb{R}^n \times (0, T) \) in the class
\[
u \in L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(0, T; L^{sp-1}_p(\mathbb{R}^n))
\]
is a weak sub(super)-solution to (1.1) iff
\[
\int_\Omega (|u|^{p-2}u : \partial_t \phi + A(x, t, u, Du) \cdot D\phi) \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} U(x, y, t)K(x, y, t)(\phi(x, t) - \phi(y, t)) \, dx \, dy \, dt
\]
\[
\leq 0
\]
for every nonnegative testing function \( \phi \in \mathcal{T} \) with the shorthand notation
\[
U(x, y, t) := |u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t)).
\]
We say that \( u \) is a weak solution to (1.1) if and only if \( u \) is simultaneously a weak super and subsolution to (1.1).

We remark that, following the argument as in [6, 39], the time continuity in \( L^p \) for weak solutions \( u \) to (1.1) in the sense of Definition 1 can be derived, that is,
\[
u \in C([0, T]; L^p(\Omega)).
\]
The precise proof is seen in [36, Proposition 3.4].

We are ready to take on the main theorem as follows.

**Theorem 1.1** (Weak Harnack estimate) Let \( p > 1 \), \( s \in (0, 1) \) and fix the power \( q \) with \( 0 < q < \frac{n+p}{n} (p - 1) \). Suppose that the vector field \( A \) and the integro-differential operator \( \mathcal{L} \) satisfy the conditions (2.1)–(2.3). Suppose further that a weak supersolution \( u \) to (1.1) in the sense of Definition 1 satisfies \( u \geq m > 0 \) in \( \mathbb{R}^n \times (0, T) \). Then, for any \( \delta \in (0, 1) \) there exists a constant \( C \equiv C(n, s, p, c_0, c_1, \Lambda, \delta, q) \) such that
\[
\left( \int_\Omega u^q \, dx \, dt \right)^{\frac{1}{q}} \leq C \inf_{Q^\pm_\delta(z_0)} u
\]
holds whenever concentric space-time cylinders \( Q^\pm_\delta(z_0) \subset Q^\pm(\delta_0) \subset \Omega_T \) with \( 0 < \delta \leq 1 \).
As a by-product of Theorem 1.1, we have the following theorem.

**Theorem 1.2** (Harnack estimate) Suppose that the vector field $\mathbf{A}$ and the integro-differential operator $\mathcal{L}$ satisfy the conditions (2.1)–(2.3). With $p > 1$, $s \in (0, 1)$ let $u$ be a weak solution to (1.1) in the sense of Definition 1 fulfilling $u \geq m > 0$ in $\mathbb{R}^n \times (0, T)$. Then, for any $\sigma \in (0, 1)$ there exists a constant $C \equiv C(n, s, p, c_0, c_1, \Lambda, \sigma)$ such that

$$
\sup_{Q_{\sigma\rho}(z_0)} u \leq C \inf_{Q_{\rho}(z_0)} u
$$

holds whenever concentric space-time cylinders $Q^{\pm}_{\sigma\rho}(z_0) \subset Q_{\rho}(z_0) \subset \Omega_T$ with $0 < \rho \leq 1$.

**Structure of the paper**

In Sect. 2, we list the notation that will be used throughout the paper, then state the structural assumption and finally collect some auxiliary tools. In Sect. 3, we derive quantitative estimates for supersolutions; in particular, we derive the Reversed Hölder inequality (Proposition 3.3) in Sect. 3.1, then prove the log-type estimates (Lemma 3.4 and Proposition 3.6) and a useful lemma (Lemma 3.8) in Sect. 3.2. Section 4 is devoted to the local boundedness (Proposition 4.3) for subsolutions. Sections 5 and 6 give the proofs of Theorems 1.1 and 1.2, respectively. In Appendix A we give the full proof of a Caccioppoli type estimate for supersolutions (Lemma 3.1). Finally, in Appendix B, we prove Lemmata 3.5 and 3.9.

**2 Notation and preliminary materials**

In this section, we will record our notation, describe the structural assumptions on (1.1) and collect some auxiliary materials that will be helpful at various stages of the paper.

**2.1 Notation**

In this brief section we introduce the notation that will be used. In the present paper, we shall fix exponents $p > 1$ and $0 < s < 1$, while a bounded open subset $\Omega$ of $\mathbb{R}^n$, with $n \geq 2$. For $T \in (0, \infty)$, let $\Omega_T := \Omega \times (0, T)$ be a space-time cylinder. More generally, for $0 \leq t_1 < t_2 \leq T$, we denote by $\Omega_{t_1, t_2} := \Omega \times (t_1, t_2) \subset \Omega_T$ the subcylinder. We shall denote in a standard way $B_{\rho}(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < \rho \}$, the open ball with center $x_0 \in \mathbb{R}^n$ and radius $\rho > 0$. We omit denoting $B_{\rho}$ instead of $B_{\rho}(x_0)$ when being clear from the context or all the balls considered will share the same center. For $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, let us define space-time cylinders as follows

$$
Q_{-\rho}(z_0) := B_{\rho}(x_0) \times (t_0 - \rho^p, t_0),
Q_{+\rho}(z_0) := B_{\rho}(x_0) \times (t_0, t_0 + \rho^p),
Q_{\rho}(z_0) := B_{\rho}(x_0) \times (t_0 - \rho^p, t_0 + \rho^p),
$$

where $\rho > 0$ is a radius and $\rho^p$ is a time length. The $\lambda$-dilate of $Q_{\pm \rho}(z_0)$ and $Q_{\rho}(z_0)$ with $\lambda > 0$ are denoted by $\lambda Q_{\pm \rho}(z_0) := Q_{\pm \rho}(z_0)$ and $\lambda Q_{\rho}(z_0) := Q_{\lambda \rho}(z_0)$, respectively.

As customary, we write $w(t) := w(\cdot, t)$, which means the time-slice value at time $t \in (0, T)$ for functions $w$, defined on a space-time region.
With $B \subset \mathbb{R}^k$, $k \geq 1$ being a measurable subset with finite and positive measure $|B| > 0$, and with $g : B \to \mathbb{R}$, being an integral function, we denote by

$$(g)_B \equiv \int_B g(x) \, dx := \frac{1}{|B|} \int_B g(x) \, dx$$

its integral average. In this paper, all the measures addressed will be the Lebesgue measure on Euclidean space $\mathbb{R}^k$, with $k \geq 1$.

For open sets $E$ and $F$ in $\mathbb{R}^k$, $k \geq 1$, the symbol $E \subseteq F$ denotes the closure of $E$ is compactly contained in $F$.

For a set $S \subset \mathbb{R}^n \times \mathbb{R}$ we write

$$S \cap \{ u > k \} := \{(x, t) \in S : u(x, t) > k \},$$
$$S \cap \{ u < k \} := \{(x, t) \in S : u(x, t) < k \}.$$

Next, we briefly recall the nonlocal tail that naturally appears when dealing with nonlocal operators like $\mathcal{L}$. This nonlocal quantity is originally introduced in [16].

Let $p \in [1, \infty)$ and $s \in (0, 1)$. We quantify the nonlocal tail in $B_\varrho(x_0)$ as follows:

$$\text{Tail}(u, B_\varrho(x_0)) := \left( Q^{sp} \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{|v(x)|^{p-1}}{|x - x_0|^{n+sp}} \, dx \right)^{\frac{1}{p-1}}.$$

the tail space $L_{sp}^{p-1}(\mathbb{R}^n)$ is defined by requiring that $v \in L_{sp}^{p-1}(\mathbb{R}^n)$ if and only if $v \in L_{loc}^{p-1}(\mathbb{R}^n)$ and

$$\text{Tail}(u, B_\varrho(x_0)) < \infty \quad \forall x_0 \in \mathbb{R}^n, \forall \varrho > 0.$$

By definition, we see that

$$L_{sp}^{p-1}(\mathbb{R}^n) := \left\{ v \in L_{loc}^{p-1}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|v(x)|^{p-1}}{(1 + |x|)^{n+sp}} \, dx < \infty \right\}.$$

The parabolic nonlocal tail of a function $v \in L^\infty(t_1, t_2 ; L_{loc}^{p-1}(\mathbb{R}^n))$ is defined by

$$\text{Tail}_\infty(u, Q_{\rho; t_1, t_2}) := \left( \sup_{t_1 < t < t_2} Q^{sp} \int_{\mathbb{R}^n \setminus B_{\rho}(x_0)} \frac{|v(x, t)|^{p-1}}{|x - x_0|^{n+sp}} \, dx \right)^{\frac{1}{p-1}},$$

with $Q_{\rho; t_1, t_2} := B_{\rho}(x_0) \times (t_1, t_2)$, which is presented in [41]. This quantity naturally appears in Theorem 4.4, as seen below.

Finally, we will list the general notation. Throughout the paper, $c, C, \cdots$ denote different positive constants in a given context. Relevant dependencies on parameters will be emphasized using parentheses, e.g., $c \equiv c(n, s, p)$ means that $c$ depends on $n, s$ and $p$. For the sake of readability, the dependencies of the constants will be often omitted within the chains of estimates. Furthermore, the equation number (\cdot)$_\ell$ denotes the \ell-th line of the Eq. (\cdot).

### 2.2 Setting

In order to consider weak solutions to the mixed local and nonlocal doubly nonlinear parabolic Eq. (1.1), we impose the assumption that will be used in the course of paper. The vector field $A = A(x, t, u, \xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ appearing on (1.1) is assumed to be measurable
with respect to \((x, t) \in \Omega_T\) for every \((u, \xi) \in \mathbb{R} \times \mathbb{R}^n\) and continuous with respect to \((u, \xi)\) for almost everywhere \((x, t) \in \Omega_T\). We further suppose that \(A\) fulfills the structure condition

\[
\begin{cases}
A(x, t, u, \xi) \cdot \xi \geq c_0 |\xi|^p, \\
|A(x, t, u, \xi)| \leq c_1 |\xi|^{p-1}
\end{cases}
\]  

(2.1)

with \(p > 1\) and the structure constants \(c_0\) and \(c_1\). The integro-differential operator \(L\) appearing on (1.1) is taken in the Cauchy principal value:

\[Lu(x, t) := 2 \text{P.V.} \int_{\mathbb{R}^n} K(x, y, t)|u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t)) \, dy,\]

where the kernel \(K : \mathbb{R}^n \times \mathbb{R}^n \times (0, T) \rightarrow [0, \infty)\) is a measurable function fulfilling the symmetric property

\[K(x, y, t) = K(y, x, t)\]

(2.2)

and

\[
\frac{\Lambda^{-1}}{|x - y|^{n+s}} \leq K(x, y, t) \leq \frac{\Lambda}{|x - y|^{n+s}}
\]

(2.3)

with \(\Lambda \geq 1\) and a fractional order \(s \in (0, 1)\) for every \((x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, T)\). Notice that, when \(\Lambda = 1\) the operator \(L\) coincides the fractional \(p\)-Laplacian \((-\Delta)_p^\Lambda\).

### 2.3 Auxiliary materials

In this subsection, we collect the auxiliary material used throughout the paper.

Firstly, we recall a very useful inequality that controls the fractional integral term, which is a variant version of [16, Lemma 3.1].

**Lemma 2.1** Let \(p \geq 1\) and \(\varepsilon \in (0, 1]\). Then,

\[|a|^p \leq |b|^p + c\varepsilon|b|^p + (1 + c\varepsilon)|a - b|^p.
\]

holds true whenever \(a, b \in \mathbb{R}^k, k \geq 1\), where \(c = c(p) := 2p(p - 1)^p\).

**Proof** For the reader’s convenience we give the short proof that is slightly different to [16, Lemma 3.1], but similar to [15, Lemma 4.3].

Due to the fundamental theorem of calculus and Young’s inequality, we infer that, for any \(\varepsilon \in (0, 1]\)

\[
(|b| + |a - b|)^p - |b|^p = p \int_0^{|a-b|} (|b| + t)^{p-1} \, dt \\
\leq p(|b| + |a - b|)^{p-1}|a - b| \\
\leq (p - 1)\varepsilon(|b| + |a - b|)^p + \varepsilon^{1-p}|a - b|^p.
\]

(2.4)

Now, taking \(\varepsilon = \frac{1}{2(p-1)}\) yields in particular that

\[|a - b| \leq 2|b|^p + 2^p(p - 1)^{p-1}|a - b|^p.
\]

Inserting this back to (2.4), we gain

\[|b| + |a - b|)^p - |b|^p \leq 2(p - 1)\varepsilon|b|^p + [2^p(p - 1)^p\varepsilon + \varepsilon^{1-p}]|a - b|^p \\
\leq c(p)\varepsilon|b|^p + (1 + c(p)\varepsilon)\varepsilon^{1-p}|a - b|^p.
\]
with \( c(p) = 2^p (p - 1)^p \). Thus, this in turn implies that
\[
|a|^p - |b|^p \leq (|b| + |a - b|)^p - |b|^p \leq c(p)\varepsilon |b|^p + (1 + c(p)\varepsilon)1 - p|a - b|^p,
\]
as desired. \( \square \)

We further retrieve the algebraic estimate; the proof is in [2, Lemma 2.2] in the case \( 0 < \beta < 1 \) and in [26, inequality (2.4)] in the case \( \beta > 1 \).

**Lemma 2.2** (Algebraic inequality I) For every \( \beta > 0 \) there exists a constant \( c(\beta) \) such that
\[
c^{-1}\left| \xi^{\beta - 1}\xi - |\eta|^{\beta - 1}\eta \right| \leq ((|\xi| + |\eta|)^{\beta - 1}|\xi - \eta| \leq c\left| |\xi|^{\beta - 1}\xi - |\eta|^{\beta - 1}\eta \right|
\]
holds true whenever \( \xi, \eta \in \mathbb{R} \).

The above algebraic inequality allows us to derive the following useful inequality:

**Lemma 2.3** (Algebraic inequality II) For all \( \alpha \in (1, \infty) \) there is a constant \( c \equiv c(\alpha) \) such that
\[
c^{-1} \leq \frac{(|\xi|^{\alpha - 2}\xi - |\eta|^{\alpha - 2}\eta)(\xi - \eta)}{(|\xi| + |\eta|)^{\alpha - 2}(\xi - \eta)^2} \leq c
\]
holds whenever \( \xi, \eta \in \mathbb{R} \) with \( \xi \neq \eta \); in particular,
\[
(|\xi|^{\alpha - 2}\xi - |\eta|^{\alpha - 2}\eta)(\xi - \eta) \geq 0 \quad \forall \xi, \eta \in \mathbb{R}.
\]

The next inequality is another necessary tool to control the fractional integral term, whose proof is in [3, Lemma 2.9].

**Lemma 2.4** Let \( a, b > 0, \tau_1, \tau_2 \geq 0 \). Then for all \( p > 1 \) there exists a constant \( c \equiv c(p) \) such that
\[
|b - a|^{p-2}(b - a) (\tau_1^p a^{-\varepsilon} - \tau_2^p b^{-\varepsilon}) \geq c \xi(\varepsilon) \left| \tau_2 b^\varepsilon - \tau_1 a^\varepsilon \right|^p - \left( \xi(\varepsilon) + 1 + \varepsilon^{(p-1)} \right) |\tau_2 - \tau_1|^{p-2} (b^\varepsilon + a^\varepsilon),
\]
where \( \varepsilon \in (0, p - 1) \), \( \alpha := p - 1 - \varepsilon \) and the function \( \varepsilon \mapsto \xi(\varepsilon) \) is explicitly given by
\[
\xi(\varepsilon) := \begin{cases} \varepsilon p^p / \alpha & \text{if } 0 < \alpha < 1 \\ \varepsilon (p / \alpha)^p & \text{otherwise.} \end{cases}
\]

We next deduce the Gagliardo–Nirenberg inequality of parabolic type, which is retrieved from [18, Chapter I.4]. The proof is also seen in [36, Lemma 2.3].

**Lemma 2.5** (Gagliardo-Nirenberg inequality) Let \( 1 \leq p, r < \infty \) and \( 0 \leq t_1 < t_2 \leq T \).

Assume that
\[
v \in L^\infty(t_1, t_2; L^r(B_\rho(x_0))) \cap L^p(t_1, t_2; W^{1,p}(B_\rho(x_0))).
\]

Then there exists a constant \( c \equiv c(n, p, r) \) such that
\[
\iint_{Q_{\rho, t_1, t_2}} |v|^p \frac{dx}{a} \, dt \leq c \rho^p \left( \sup_{t \in (t_1, t_2)} \iint_{B_\rho(x_0)} |v(t)|^r \, dx \, dt \right)^{\frac{p}{r}} \iint_{Q_{\rho, t_1, t_2}} \left( |Dv|^p + \frac{|v|^p}{\rho} \right) \, dx \, dt,
\]
where \( Q_{\rho, t_1, t_2} := B_\rho(x_0) \times (t_1, t_2) \).

\( \square \) Springer
The following inequality holds on the Sobolev space $W^{1,p}_0(\Omega)$, whose proof can be seen in [14, Lemma 2.1].

**Lemma 2.6** Let $1 < p < \infty$, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary. Then there exists a constant $c \equiv c(n, s, p, \Omega)$ such that

$$
\int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq c \int_{\Omega} |Dv|^p \, dx
$$

holds wherever $v \in W^{1,p}_0(\Omega)$.

We finally present the so-called “simple but fundamental lemma” due to Giaquinta and Giusti [25, Lemma 1.1] or [27, Lemma 6.1, Page 191].

**Lemma 2.7** Let $A, B \geq 0$, $\beta > 0$ and $0 < \sigma < \tau$. Suppose that any nonnegative bounded function $Z = Z(s)$ defined on $[\sigma, \tau]$ satisfies

$$
Z(s) \leq \frac{3}{4} Z(t) + A(t - s)^{-\beta} + B
$$

for every $\sigma \leq s < t \leq \tau$. Then there exists a constant $c$, depending only on $\beta$, such that

$$
Z(\sigma) \leq c \left( A(\tau - \sigma)^{-\beta} + B \right).
$$

**2.4 Mollification in time**

In this subsection we introduce the exponential mollification in time, originally devised in [30]. This mollification is a breakthrough tool that can overcome the lack of weak differentiable in time for solutions. For this reason, this technique is applied to various doubly nonlinear equations, as seen in the literatures [6–9, 36]. With $E \subset \mathbb{R}^k$ being a bounded domain, let us define, for $v \in L^1(ET)$ and $h \in (0, T)$,

$$
[v]_h(x, t) := \frac{1}{h} \int_0^t e^{\frac{t-s}{h}} v(x, s) \, ds, \quad (x, t) \in E \times [0, T]. \tag{2.7}
$$

The backward version of $[v]_h$ is given by

$$
[v]_\bar{h}(x, t) := \frac{1}{h} \int_t^T e^{\frac{t-s}{h}} v(x, s) \, ds, \quad (x, t) \in E \times [0, T].
$$

In this setting, we summarize the properties of $[v]_h$ and $[v]_\bar{h}$ displayed below, whose detailed proof can be seen in the literatures [30, Lemma 2.2] and [5, Appendix B].

**Lemma 2.8** Assume that $v \in L^1(ET)$ and $p \in [1, \infty)$, where $E_T := E \times (0, T)$. Then the mollifications $[v]_h$ and $[v]_\bar{h}$ have the following properties:

(i) If $v \in L^p(ET)$, then $[v]_h \in L^p(ET)$ and the inequality holds true:

$$
\| [v]_h \|_{L^p(ET)} \leq \| v \|_{L^p(ET)}.
$$

Furthermore,

$$
[v]_h \rightarrow v \text{ strongly in } L^p(ET) \text{ as } h \searrow 0.
$$

A same statement for $[v]_\bar{h}$ holds true.
(ii) If \( v \in L^p(E_T) \), then \([v]_h\) and \([v]_h^\dagger\) have weak time derivatives being in \( L^p(E_T) \) and solve the ODE:

\[
\partial_t[v]_h = -\frac{[v]_h - v}{h}; \quad \partial_t[v]_h^\dagger = \frac{[v]_h^\dagger - v}{h}.
\]

(iii) If \( v \in L^p(0, T; W^{1,p}_0(E)) \), then \([v]_h \in L^p(0, T; W^{1,p}_0(E)) \). Furthermore, there holds that \([v]_h \to v\) strongly in \( L^p(0, T; W^{1,p}_0(E)) \) as \( h \searrow 0 \). The same implication for \([v]_h^\dagger\) holds true.

(iv) If \( v \in L^p(0, T; L^p(E)) \), then \([v]_h\) and \([v]_h^\dagger\) belong to \( C([0, T]; L^p(E)) \).

(v) If \( v \in L^\infty(0, T; L^2(E)) \), then \([v]_h\) \( C([0, T]; L^2(E)) \). Moreover we have

\[
[v]_h \to v \quad \text{in} \quad C([0, T]; L^2(E)) \quad \text{as} \quad h \searrow 0.
\]

Hereafter, we shall apply Lemma 2.8 with \( E = \Omega \) or \( E = \Omega \times \Omega \) on many times.

Using Lemma 2.8, we deduce the following lemma.

**Lemma 2.9** With \( p \geq 1 \), suppose that \( v \in L^\infty(0, T; L^p(\Omega)) \) and abbreviate \( w := |v|^{\frac{p-2}{2}}v \).

Set

\[
\langle v \rangle_h := \begin{cases} 
[w]_h^\frac{p}{p-1} \text{ if } v \neq 0, \\
0 & \text{ if } v = 0.
\end{cases}
\]

Then, \( \langle v \rangle_h \in C([0, T]; L^p(\Omega)) \) and moreover \( \langle v \rangle_h \to v \) in \( C([0, T]; L^p(\Omega)) \) as \( h \searrow 0 \).

**Proof** First of all, by the assumption that \( v \in L^\infty(0, T; L^p(\Omega)) \), \( w := |v|^{\frac{p-2}{2}}v \) belongs to \( L^\infty(0, T; L^2(\Omega)) \). Thus, Lemma 2.8-(v) yields that

\[
[w]_h \in C([0, T]; L^2(\Omega)) \tag{2.8}
\]

and

\[
[w]_h \to w \quad \text{in} \quad C([0, T]; L^2(\Omega)) \quad \text{as} \quad h \downarrow 0. \tag{2.9}
\]

We shall show the first implication. For this, we now distinguish two cases between \( p \geq 2 \) and \( 1 \leq p < 2 \). As we are considering the case \( p \geq 2 \), applying Lemma 2.2 with \( \beta = \frac{p}{2} \), we infer that for every \( 0 \leq t_1 < t_2 \leq T \)

\[
\left| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \right|^{\frac{p}{2}} \leq c \left| \langle v \rangle_h \right|^{\frac{p-2}{2}} \left| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \right| \leq c \left| w \right|_h(t_1) - \left| w \right|_h(t_2)
\]

and therefore, this together with (2.9) implies that

\[
\left\| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \right\|^{p}_{L^p(\Omega)} \leq c \left\| \langle w \rangle_h(t_1) - \langle w \rangle_h(t_2) \right\|^{2}_{L^2(\Omega)} \to 0
\]
as \( t_2 - t_1 \to 0 \). In the remaining case \( 1 \leq p < 2 \), we again use Lemma 2.2 with \( \beta = \frac{p}{2} \) to get

\[
\left| \langle v \rangle_h(t_1) + \langle v \rangle_h(t_2) \right|^{\frac{p-2}{2}} \left| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \right| \leq c \left| \langle v \rangle_h \right|^{\frac{p-2}{2}} \left| \langle v \rangle_h(t_1) - \langle v \rangle_h \right|^{\frac{p-2}{2}} \left| \langle v \rangle_h(t_2) \right| = c \left| w \right|_h(t_1) - \left| w \right|_h(t_2)
\]

and thus,

\[
\int_{\Omega} \left( |\langle v \rangle_h(t_1) + \langle v \rangle_h(t_2)| \right)^{\frac{p-2}{2}} \left| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \right|^2 \, dx \leq c \left\| \langle w \rangle_h(t_1) - \langle w \rangle_h(t_2) \right\|^{2}_{L^2(\Omega)} \to 0
\]
as \( t_2 - t_1 \to 0 \). Using this, (2.8) and Hölder’s inequality with the exponent \( \left( \frac{2}{p}, \frac{2}{2-p} \right) \), we infer that

\( \Box \) Springer
Therefore we conclude that, for every $p \geq 1$,
\[
\| \langle v \rangle_h(t_1) - \langle v \rangle_h(t_2) \|_{L^p(\Omega)} \to 0 \quad \text{as} \quad t_2 - t_1 \to 0,
\]
as desired. Similarly as above, by the use of (2.9) we deduce that,
\[
\| (v)_h(t) - v(t) \|_{L^p(\Omega)}^p \\
\leq \begin{cases} 
2c \| [w]_h(t) - w(t) \|_{L^2(\Omega)}^2 & \text{for } p \geq 2 \\
2c \| [w]_h(t) - w(t) \|_{L^2(\Omega)}^2 \left( \| [w]_h(t) \|_{L^2(\Omega)}^2 + \| w(t) \|_{L^2(\Omega)}^2 \right)^{2-p} & \text{for } 1 \leq p < 2 
\end{cases} \\
\to 0
\]
as $h \searrow 0$, showing the second implication and therefore, the proof is complete. \qed

### 2.5 Mollified weak formulation of (1.8)

We next list the mollified version of (1.8) in Definition 1, whose proof is based on the Fubini theorem for the double integral, see [36, Lemma 2.10] for the details. This enables us to choose various testing functions in rigorous way.

**Lemma 2.10** Let $u$ be a weak sub(super)-solution to (1.1) in the sense of Definition 1. Then for every nonnegative $\varphi \in T$, there holds that
\[
\iint_{\Omega_T} (\partial_t [u|^{p-2}u]_h \varphi + [A(x,t,u,Du)]_h \cdot D\varphi) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} [U(x,y,t)K(x,y,t)]_h (\varphi(x,t) - \varphi(y,t)) \, dx \, dy \, dt \\
\leq \begin{cases} 
(\geq) \int_{\Omega} |u|^{p-2}u(0) \left( \frac{1}{h} \int_0^r e^{\frac{r}{h}} \varphi(x,s) \, ds \right) \, dx. & \text{for } p \geq 2 \\
(=) \int_{\Omega} |u|^{p-2}u(0) \left( \frac{1}{h} \int_0^r e^{\frac{r}{h}} \varphi(x,s) \, ds \right) \, dx. & \text{for } 1 \leq p < 2 
\end{cases}
\] (2.10)

### 3 Quantitative estimates for supersolutions

In this section we give quantitative estimates for supersolutions. This section is twofold. First we prove the Reverse Hölder inequality for supersolutions, whose proof is based on
the Moser’s iteration scheme. Besides, we shall derive the log-type estimate, that will be key ingredient to prove Theorem 1.1.

### 3.1 Reverse Hölder’s inequality

In order to prove the Reverse Hölder inequality (Proposition 3.3), we need some preliminary results.

The first step is to establish the following Caccioppoli type inequality.

**Lemma 3.1** (Caccioppoli type estimate for supersolutions) Let \( u \) be a weak supersolution to (1.1) fulfilling \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0 - \varrho^p, t_0) \). With \( p > 1 \) let \( \varepsilon \in (0, p - 1) \) and set \( \alpha := p - 1 - \varepsilon \). Then

\[
\sup_{t \in (t_0 - \varrho^p, t_0)} \int_{B_\varrho(x_0) \times \{t\}} u^\alpha \varphi^p \, dx + \int_{Q_\varepsilon^-(z_0)} |Du|^p u^{-\varepsilon-1} \varphi \, dx \, dt + \int_{t_0 - \varrho^p}^{t_0} \int_{B_\varrho(x_0) \times B_\varrho(x_0)} \frac{|u(x, t)^\alpha \varphi(x, t) - u(y, t)^\alpha \varphi(y, t)|^p}{|x - y|^{n + sp}} \, dx \, dy \, dt \\
\leq c \int_{Q_\varepsilon^-(z_0)} u^\alpha \varphi^{p-1} \varphi \, dx \, dt + c \int_{Q_\varepsilon^-(z_0)} |D\varphi|^p \, dx \, dt + c \int_{t_0 - \varrho^p}^{t_0} \int_{B_\varrho(x_0) \times B_\varrho(x_0)} \frac{(u(x, t)^\alpha + u(y, t)^\alpha)|\varphi(x, t) - \varphi(y, t)|}{|x - y|^{n + sp}} \, dx \, dy \, dt \\
+ c \left( \sup_{x \in \text{supp}\varphi(\cdot, t)} \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{dy}{|x - y|^{n + sp}} \right) \int_{Q_\varepsilon^-(z_0)} u^\alpha \varphi \, dx \, dt
\]

holds whenever nonnegative \( \varphi \in C_0^\infty(Q_\varepsilon^-(z_0)) \), where the constant \( c \equiv c(p, \varrho_0, c_1, \Lambda, \varepsilon) \) blows up as \( \varepsilon \searrow 0 \) and \( \varepsilon \not\nearrow p - 1 \).

For the readability, we shall postpone the proof to Appendix A.

The subsequent lemma is the starting point of Moser’s iteration scheme that will be argued later in the proof of Proposition 3.3.

**Lemma 3.2** Let \( u \) be a weak supersolution to (1.1) fulfilling \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0 - \varrho^p, t_0) \). Let \( \varepsilon \in (0, p - 1) \) and set \( \alpha := p - 1 - \varepsilon \) and \( \kappa := \frac{n + sp}{p} \). Then for any concentric \( Q_r^{-}(z_0) \subset Q_\varepsilon^-(z_0) \subset \Omega_T \) the quantitative estimate

\[
\left( \int_{Q_r^{-}(z_0)} u^\alpha \varphi \, dx \, dt \right)^{\frac{1}{\alpha}} \leq c \frac{1}{\alpha} \left[ \left( \frac{\varrho}{r} \right)^n \left( \frac{\varrho}{Q - r} \right)^{n + sp} (1 + \varepsilon^{(1-s)p}) \right]^{\frac{1}{\alpha}} \left( \int_{Q_\varepsilon^-(z_0)} u^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}}
\]

holds true, where \( c = c(n, s, p, \varepsilon) \) blows up as \( \varepsilon \searrow 0 \) or \( \varepsilon \not\nearrow p - 1 \).

**Proof** Take a cut-off function \( \varphi \in C_0^\infty(Q_\varepsilon^-(z_0)) \) satisfying

- \( \text{supp} \varphi \subset Q_{\varrho \varepsilon r}^{-}(z_0) \), \( 0 \leq \varphi \leq 1 \) in \( Q_\varepsilon^-(z_0) \), \( \varphi \equiv 1 \) on \( Q_r^{-}(z_0) \);
- \( |D\varphi| \leq \frac{c}{Q - r} \), \( |\varphi| \leq \frac{c}{(Q - r)^p} \).

We now observe that

\[
\int_{Q_r^{-}(z_0)} (u^\varrho \varphi)^p \, dx \, dt = \int_{Q_r^{-}(z_0)} (u^\varrho \varphi)^p \, dx \, dt \leq \left( \frac{\varrho}{r} \right)^{n + p} \int_{Q_\varepsilon^-(z_0)} (u^\varrho \varphi)^p \, dx \, dt.
\]
Applying the Gagliardo–Nirenberg inequality (2.6) in Lemma 2.5 to \( v = u^\frac{\alpha}{p} \phi \) and the Caccioppoli type estimate (Lemma 3.1) yield that

\[
\iint_{Q_{\varepsilon}(z_0)} (u^\frac{\alpha}{p} \phi)^p \, dx \, dt \\
\leq c Q^p \left( \sup_{t \in (t_0 - \varepsilon^p, t_0)} \int_{B_{\varepsilon}(x_0)} (u^\frac{\alpha}{p} \phi)^p \, dx \right)^\frac{p}{n} \iint_{Q_{\varepsilon}(z_0)} \left( |D(u^\frac{\alpha}{p} \phi)|^p + \left| \frac{u^\frac{\alpha}{p} \phi}{Q} \right|^p \right) \, dx \, dt \\
\leq \frac{c}{Q^{n+p}} \left[ \iint_{Q_{\varepsilon}(z_0)} (u^\alpha \phi^{p-1} |\phi_t| + u^\alpha |D\phi|^p + \frac{u^\alpha \phi}{Q^p}) \, dx \, dt \\
+ \int_{t_0 - \varepsilon^p}^{t_0} \iint_{B_{\varepsilon}(x_0) \times B_{\varepsilon}(x_0)} (u(x, t)^\alpha + u(y, t)^\alpha) |\phi(x, t) - \phi(y, t)|^p \frac{dy}{|x - y|^{n + sp}} \, dx \, dy \, dt \\
+ \left( \sup_{x \in supp \phi \setminus t} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x_0)} \frac{dy}{|x - y|^{n + sp}} \right) \iint_{Q_{\varepsilon}(z_0)} u^\alpha \phi^p \, dx \, dt \right] ^\kappa. 
\]

Thus, combining this with the preceding estimate gives

\[
\left( \iint_{Q_{\varepsilon}(z_0)} u^{\alpha \kappa} \, dx \, dt \right)^\frac{1}{\alpha \kappa} \leq \frac{c}{r^{n+p}} \left[ \frac{1}{(Q - r)^p} \iint_{Q_{\varepsilon}(z_0)} u^\alpha \, dx \, dt \\
+ \frac{1}{(Q - r)^p} \int_{t_0 - \varepsilon^p}^{t_0} \iint_{B_{\varepsilon}(x_0) \times B_{\varepsilon}(x_0)} (u(x, t)^\alpha + u(y, t)^\alpha) |D\phi|^p \frac{dy}{|x - y|^{n + (1-s)p}} \, dx \, dy \, dt \\
+ \left( \sup_{x \in B_{\frac{r+\varepsilon}{2}}(x_0)} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x_0)} \frac{dy}{|x - y|^{n + sp}} \right) \iint_{Q_{\varepsilon}(z_0)} u^\alpha \, dx \, dt \right] ^\kappa
\]

\[
=: \frac{c}{r^{n+p}} [I + II + III]^\kappa, \tag{3.1}
\]

where the definition of I–III are clear from the context and note that, the constant \( c \) appearing on the final line has the singularity at \( \varepsilon = 0 \) and \( \varepsilon = p - 1 \).

We will estimate I–III separately. Rearranging the term I yields

\[
I = \frac{1}{(Q - r)^p} \iint_{Q_{\varepsilon}(z_0)} u^\alpha \, dx \, dt. \tag{3.2}
\]
By exchanging the role of \(u(x, t)\) and \(u(y, t)\), we have
\[
\mathbb{II} = \frac{2}{(Q - r)^p} \int_{t_0 - Q_0}^{t_0} \int_{B_y(x_0) \times B_y(x_0)} \frac{u^\alpha(x, t)}{|x - y|^{n - (1 - s)p}} \, dx \, dy \, dt
\]
\[
= \frac{2}{(Q - r)^p} \int_{Q_0(t_0)} \int_{Q_0(t_0)} u^\alpha(x, t) \left( \int_{B_y(x_0)} \frac{dy}{|x - y|^{n - (1 - s)p}} \right) \, dx \, dt
\]
\[
= \frac{c(n, s, p)}{(Q - r)^p} Q^{(1-s)p+n+s} \int_{Q_0(t_0)} u^\alpha \, dx \, dt, \tag{3.3}
\]
where, in the second line, we computed that
\[
\int_{B_y(x_0)} \frac{dy}{|x - y|^{n - (1 - s)p}} = |B_y| \left( \frac{Q}{1 - s} \right)^{p - n + s}.
\]
We shall handle the term \(\mathbb{III}\). Since
\[
|y - x| \leq |y - x_0| + |x - x_0| = |y - x_0| + |x - x_0|
\]
and
\[
|y - x| \geq |y - x_0| - |x_0 - x| \geq Q - \frac{Q + r}{2} = \frac{Q - r}{2}
\]
holds whenever \(x \in B_{2C}(x_0)\) and \(y \in \mathbb{R}^n \setminus B_y(x_0)\), we have
\[
\frac{|y - x_0|}{|y - x|} \leq \frac{2\rho}{Q - r} \quad \forall x \in B_{2C}(x_0), \quad \forall y \in \mathbb{R}^n \setminus B_y(x_0).
\]
Thus,
\[
\mathbb{III} \leq c \left( \frac{Q}{Q - r} \right)^{n + s} \left( \sup_{x \in B_{2C}(x_0)} \int_{\mathbb{R}^n \setminus B_y(x_0)} \frac{dy}{|y - x_0|^{n + s}} \right) \int_{Q_0(t_0)} u^\alpha \, dx \, dt
\]
\[
= c \left( \frac{Q}{Q - r} \right)^{n + s} \int_{Q_0(t_0)} u^\alpha \, dx \, dt
\]
\[
= c \left( \frac{Q}{Q - r} \right)^{n + s} \int_{Q_0(t_0)} u^\alpha \, dx \, dt, \tag{3.4}
\]
where, in the second line we used that \(\int_{\mathbb{R}^n \setminus B_y(x_0)} \frac{dy}{|y - x_0|^{n + s}} = c(n, s, p) Q^{(1-s)p}\), noticing that \(r^{n + s} = r^{n_k}\) and inserting the preceding estimates (3.2)–(3.4) back to (3.1) concludes that
\[
\int_{Q_0(t_0)} (u^\alpha v^\alpha)^{n_k} \, dx \, dt
\]
\[
\leq c \left( \frac{Q}{Q - r} \right)^{n + s} \left( 1 + q^{(1-s)p} + \left( \frac{Q}{Q - r} \right)^{n + s} q^{(1-s)p} \right) \int_{Q_0(t_0)} u^\alpha \, dx \, dt
\]
\[
\leq c \left( \frac{Q}{Q - r} \right)^{n_k} \left( \left( \frac{Q}{Q - r} \right)^{n + s} \left( 1 + q^{(1-s)p} \right) \int_{Q_0(t_0)} u^\alpha \, dx \, dt \right) ^{n_k},
\]
which proves the claim.
\(\square\)
We are ready to prove the Reverse Hölder’s inequality about supersolutions.

**Proposition 3.3** (Reverse Hölder’s inequality) Assume that \( u \) is a weak supersolution to (1.1) fulfilling \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0 - \varrho^p, t_0) \). Fix a parameter \( \sigma_0 \in (0, 1) \). Then, for any \( Q_\varrho(z_0) \Subset \Omega_T \), there exists a positive constant \( C = C(n, s, p, c_0, c_1, \Lambda, \sigma_0) \) such that

\[
\left( \frac{\Omega}{\int\int_{Q_{\varrho}(z_0)} u^q \, dx \, dt} \right)^{\frac{1}{q}} \leq C \left( \frac{1 + \varrho^{(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)^{\frac{\alpha \kappa}{np}} \left( \frac{\Omega}{\int\int_{Q_{\varrho}(z_0)} u^r \, dx \, dt} \right)^{\frac{1}{r}}
\]

holds whenever \( \sigma_0 \leq \sigma < \tau \) and \( 0 < \gamma < q \leq \frac{\alpha \kappa}{n} (p - 1) \).

**Proof** First of all, for every \( q, \gamma \) satisfying \( q > \gamma \), we choose \( k \in \mathbb{N} \) so that

\[
\frac{\log \frac{q}{\gamma}}{\log \kappa} \leq k \leq \frac{\log \frac{q}{\gamma}}{\log \kappa} + 1
\]

and for \( q_0 < \gamma \), we set \( q = q_k = q_0 k^k \). Next, for \( i = 0, 1, \ldots, k \) let us define

\[
\varrho_i := \left( \tau - (\tau - \sigma) \frac{1 - 2^{-i}}{1 - 2^{-k}} \right) \varrho, \quad Q_i := Q_{\varrho_i}(z_0) = B_{\varrho_i}(x_0) \times (t_0 - \varrho_i^p, t_0),
\]

and therefore the following inclusions trivially hold:

\[
\varrho_0 = \tau \varrho \supseteq \cdots \supseteq \varrho_i \supseteq \varrho_k = \sigma \varrho, \quad Q_0 = Q_{\tau \varrho} \supset Q_i \supset \cdots \supset Q_k = Q_{\sigma \varrho}.
\]

Applying Lemma 3.2 with \( r = \varrho_{i+1}, q = \varrho_i \) and

\[
q_{i+1} = \alpha \kappa, \quad q_i = \alpha \quad (\Rightarrow q_i = q_0 k^i),
\]

we infer that, for \( i = 0, 1, \ldots, k - 1 \),

\[
\left( \frac{\Omega}{\int\int_{Q_i} u^{q_{i+1}} \, dx \, dt} \right)^{\frac{1}{q_{i+1}}} \leq c^{\frac{1}{\alpha \kappa}} \left[ \left( \frac{\varrho_i}{\varrho_{i+1}} \right)^n \left( \frac{Q_i}{Q_{\varrho_{i+1}}} \right)^{n+sp} \left( 1 + \varrho_i^{(1-s)p} \right)^{\frac{\alpha \kappa}{np}} \right]^{\frac{1}{q_i}} \left( \frac{\Omega}{\int\int_{Q_i} u^{q_i} \, dx \, dt} \right)^{\frac{1}{q_i}}.
\]

We observe that

\[
\frac{\varrho_i}{Q_{\varrho_i} - \varrho_{i+1}} = \frac{\tau - (\tau - \sigma) \frac{1 - 2^{-i}}{1 - 2^{-k}}}{(\tau - \sigma) \frac{2^{i+1} \tau}{1 - 2^{-k}}} \leq \frac{2^{i+1}}{\alpha \kappa} \frac{\varrho_i}{\varrho_{i+1}},
\]

and \( \varrho_i^{(1-s)p} \leq \varrho_0^{(1-s)p} = \varrho^{(1-s)p} \), which junction with (3.5) yields

\[
\left( \frac{\Omega}{\int\int_{Q_i} u^{q_{i+1}} \, dx \, dt} \right)^{\frac{1}{q_{i+1}}} \leq c^{\frac{1}{\alpha \kappa}} \left[ \left( \frac{\varrho_i}{\varrho_{i+1}} \right)^n \left( \frac{2^{i+1} \tau}{\tau - \sigma} \right)^{n+sp} \left( 1 + \varrho_i^{(1-s)p} \right) \right]^{\frac{1}{q_i}} \left( \frac{\Omega}{\int\int_{Q_i} u^{q_i} \, dx \, dt} \right)^{\frac{1}{q_i}}.
\]
Note that, this estimate (3.6) is only valid for $0 < q_i < p - 1$ since the constant $c$ possesses the singularity at $q_i = 0$ or $q_i = p - 1$. For the rest of the argument, let us denote in short, for $i = 0, 1, \ldots, k$

$$Y_i := \left( \iint_{Q_i} u^{q_i} \, dx \, dt \right)^{\frac{1}{q_i}}.$$

Iterating (3.6), for $i = 0, 1, \ldots, k$, gives that

$$\left( \iint_{Q_{\sigma \rho}} u^{q_i} \, dx \, dt \right)^{\frac{1}{q_i}} = Y_k \leq c^{\frac{k}{q_k}} \left( \frac{q_k-1}{q_k} \right)^{\frac{1}{q_k}} 2^{(n+sp) \frac{k-1}{q_k-1}} \left( \frac{1 + \varrho^{(1-s)p}}{(\sigma - \sigma)^{n+sp}} \right)^{\frac{1}{q_k-1}} Y_{k-1}$$

$$\vdots$$

$$\leq c^{S(k)} 2^{(n+sp)T(k)} M(k) \left( \frac{1 + \varrho^{(1-s)p}}{(\sigma - \sigma)^{n+sp}} \right)^{S(k)} Y_0,$$  \hspace{1cm} (3.7)

where

$$S(k) := \sum_{i=0}^{k} \frac{1}{q_i}, \quad T(k) := \sum_{i=0}^{k} \frac{i}{q_i}$$

$$M(k) := \prod_{i=1}^{k} \left( \frac{Q_{i-1}}{Q_i} \right)^{\frac{1}{q_i}}.$$  

We estimate the above quantities $S(k)$, $T(k)$ and $M(k)$. Since

$$\gamma \kappa^{k-1} \leq q_0 \kappa^k = q_k \iff q_0 \geq \gamma / \kappa,$$

a straightforward computation implies that

$$S(k) = \sum_{i=0}^{k} \frac{1}{q_0 \kappa^i} = \frac{1}{q_0 (\kappa - 1)} \left[ 1 - \left( \frac{1}{\kappa} \right)^k \right] \leq \frac{k^2}{\gamma (\kappa - 1)}$$

and

$$T(k) = \sum_{i=0}^{k} \frac{i}{q_0 \kappa^i} = -\frac{k}{q_0 (\kappa - 1) \kappa^{k-1}} + \frac{k}{q_0 (\kappa - 1)^2} \left[ 1 - \left( \frac{1}{\kappa} \right)^k \right] \leq \frac{k^2}{\gamma (\kappa - 1)^2}.$$  

Also, we estimate that

$$M(k) = \exp \left( \sum_{i=1}^{k} \frac{1}{q_0 \kappa^i} \log \left( \frac{Q_{i-1}}{Q_i} \right) \right) \leq \sum_{i=1}^{k} \frac{1}{q_0 \kappa} (\log Q_{i-1} - \log Q_i)$$

$$\leq \exp \left( \frac{1}{\gamma} \log \left( \frac{1}{\sigma} \right) \right) \leq \left( \frac{1}{\sigma_0} \right)^{\frac{1}{\gamma}}.$$  

Therefore,

$$C_{\text{prod}}(k) := c^{S(k)} 2^{(n+sp)T(k)} M(k).$$

\footnote{Since $\{q_i\}_{i=1}^{k}$ is increasing sequence, it is enough to consider that $0 < q = q_k = q_{k-1} < (p-1)\kappa$.}
Choosing the testing function in (2.10) as desired.

\[ C_{\text{prod}}(k) \leq c \gamma^{\frac{2}{s}} \frac{2(n+sp)}{\gamma(n+1)} \left( \frac{1}{\sigma_0} \right)^{\frac{1}{p}} =: C, \]

where the constant \( C \) depends on \( n, s, p, c_0, c_1, \Lambda, \gamma \) and \( \sigma_0 \). Combining this with (3.7) and applying Hölder’s inequality to \( Y_0 \), we finally deduce that, for every \( \sigma_0 \leq \sigma < 1 \) and \( \gamma < q \leq \kappa(p-1) = \frac{n+p}{n}(p-1) \),

\[ \left( \int_{Q_{t_0}} u^q \right)^{\frac{1}{q}} \leq C \left( \int_{Q_{t_0}} \left( 1 + \varphi (1-s)^p \right) \gamma(\kappa - 1) \right)^{\frac{1}{p}} \],

as desired.

\[ \blacksquare \]

### 3.2 Log-type estimates

To begin, we prove the log-type Caccioppoli estimate.

**Lemma 3.4** (Log-type Caccioppoli estimate) Let \( 0 \leq t_1 < t_2 \leq T \) and suppose that \( u \) is a weak supersolution to (1.1) such that \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_1, t_2) \). Then, there exists a positive constant \( c \equiv c(p, c_0, c_1) \) such that, for every nonnegative \( \varphi \in C_0^\infty(\Omega_{t_1, t_2}) \), the following quantitative estimate holds true:

\[ \int_{\Omega_{t_1, t_2}} \varphi^p |D(\log u)|^d \text{dxdt} - \left[ \int_{\Omega} \varphi^p \log u \text{dxdy} \right]_{t = t_1}^{t_2} \]

\[ \leq c \int_{\Omega_{t_1, t_2}} |D\varphi|^p \text{dxdy} + c \int_{\Omega_{t_1, t_2}} \varphi^{p-1} |\varphi_t| |\log u| \text{dxdy} + c \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t) K(x, y, t) \left( u(x, t)^{1-p} \varphi(x, t)^p - u(y, t)^{1-p} \varphi(y, t)^p \right) \text{dxdydt}, \]

where \( \Omega_{t_1, t_2} := \Omega \times (t_1, t_2) \).

**Proof** First of all, for every \( 0 \leq t_1 < t_2 \leq T \) and \( \varepsilon > 0 \) small enough, we define the following Lipschitz cut-off function:

\[ \chi = \chi_{\varepsilon}(t) := \begin{cases} 0, & t \in [0, t_1), \\ \frac{1}{\varepsilon} (t - t_1), & t \in [t_1, t_1 + \varepsilon), \\ 1, & t \in [t_1 + \varepsilon, t_2 - \varepsilon], \\ \frac{1}{\varepsilon} (t - t_2), & t \in (t_2 - \varepsilon, t_2], \\ 0, & t \in (t_2, T]. \end{cases} \]

Choosing the testing function in (2.10) as

\[ \varphi(x, t) = \chi_{\varepsilon}(t) u(x, t)^{1-p} \varphi(x, t)^p \quad \text{with} \quad 0 \leq \varphi \in C_0^\infty(\Omega_T) \]

gives the estimates below respectively: The parabolic term is split into two terms

\[ \int_{\Omega_T} \partial_t \left[ |u|^{p-2} u \right] \chi_{\varepsilon} u^{1-p} \varphi^p \text{dxdt} = \int_{\Omega_T} \chi_{\varepsilon} \varphi^p \partial_t \left[ u^{p-1} \right] \left( u^{1-p} - \left[ u^{p-1} \right] \right) \text{dxdt} \]
+ \int_{\Omega_T} \chi_\varepsilon \varphi^p \partial_t [u^{p-1}]_h [u^{p-1}]_h^{-1} \, dx \, dt
\quad =: I_1 + I_2.

Apply Lemma 2.8-(ii), we have

\[ I_1 = \int_{\Omega_T} \chi_\varepsilon \varphi^p \frac{u^{p-1} - [u^{p-1}]_h}{[u^{p-1}]_h u^{p-1}} \, dx \, dt \]
\[ = - \frac{1}{h} \int_{\Omega_T} \chi_\varepsilon \varphi^p \left( \frac{u^{p-1} - [u^{p-1}]_h}{[u^{p-1}]_h u^{p-1}} \right)^2 \, dx \, dt \leq 0. \]

By integration by parts, we infer that

\[ I_2 = \int_{\Omega_T} \chi_\varepsilon \varphi^p \partial_t \left( \log [u^{p-1}]_h \right) \, dx \, dt \]
\[ = - \int_{\Omega_T} \left( \chi_\varepsilon' \varphi^p + p \varphi^{p-1} \varphi' \chi_\varepsilon \right) \log [u^{p-1}]_h \, dx \, dt \]
\[ = - \int_{t_1}^{t_1+\varepsilon} \int_{\Omega} \varphi^p \log [u^{p-1}]_h \, dx \, dt + \int_{t_2-\varepsilon}^{t_2} \int_{\Omega} \varphi^p \log [u^{p-1}]_h \, dx \, dt \]
\[ - p \int_{\Omega_{t_1-t_2}} \varphi^{p-1} \varphi' \chi_\varepsilon \log [u^{p-1}]_h \, dx \, dt. \]

At this stage, we claim the following result. The proof is postponed and will be presented in Appendix B. □

**Lemma 3.5** As \( h \downarrow 0 \)

\[ \log [u^{p-1}]_h \longrightarrow \log u^{p-1} \quad \text{in} \quad L^{\frac{p}{p-1}}(\Omega_{t_1}, t_2). \]

Thanks to this lemma and the preceding estimates, we infer that

\[ \lim_{h \downarrow 0} \sup \int_{\Omega_T} \partial_t \left[ |u|^{p-2} u \right] \chi_\varepsilon u^{1-p} \varphi^p \, dx \, dt \]
\[ \leq \lim_{h \downarrow 0} \sup (I_1 + I_2) \]
\[ \leq - \int_{t_1}^{t_1+\varepsilon} \int_{\Omega} \varphi^p \log u^{p-1} \, dx \, dt + \int_{t_2-\varepsilon}^{t_2} \int_{\Omega} \varphi^p \log u^{p-1} \, dx \, dt \]
\[ - p \int_{\Omega_{t_1-t_2}} \varphi^{p-1} \varphi' \chi_\varepsilon \log u^{p-1} \, dx \, dt \]
\[ = -(p-1) \int_{t_1}^{t_1+\varepsilon} \int_{\Omega} \varphi^p \log u \, dx \, dt + (p-1) \int_{t_2-\varepsilon}^{t_2} \int_{\Omega} \varphi^p \log u \, dx \, dt \]
\[ - p(p-1) \int_{\Omega_{t_1-t_2}} \varphi^{p-1} \varphi' \chi_\varepsilon \log u \, dx \, dt. \] (3.8)

On the other hand, since by Lemma 2.8-(ii) \([A(x, t, u, Du)]_h \longrightarrow A(x, t, u, Du)\) in \( L^{\frac{p}{p-1}}(\Omega_T) \) as \( h \downarrow 0 \), we observe that

\[ \lim_{h \downarrow 0} \int_{\Omega_T} [A(x, t, u, Du)]_h \cdot D \left( \chi_\varepsilon u^{1-p} \varphi^p \right) \, dx \, dt \]
\[= \int \int_{\Omega_T} A(x, t, u, Du) \cdot D \left( \chi_\varepsilon u^{1-p} \varphi^p \right) \, dx \, dt \]
\[= \int \int_{\Omega_T} \chi_\varepsilon A(x, t, u, Du) \cdot \left( -(p - 1)u^{-p} Du \varphi^p + p \varphi^{-p-1} D \varphi u^{1-p} \right) \, dx \, dt \]
\[\leq p \int \int_{\Omega_T} \left( \chi_\varepsilon \varphi |D(\log u)| \right)^{p-1} \left( c_1 \chi_\varepsilon \varphi \right) \, dx \, dt - c_0(p - 1) \int \int_{\Omega_T} \chi_\varepsilon \varphi |D(\log u)|^p \, dx \, dt, \]

which together with Young’s inequality with conjugate exponents \(\left( \frac{p}{p-1}, p \right)\) now yields,

\[\lim_{h \to 0} \int_{\Omega_T} [A(s, t, u, Du)]_h \cdot D \left( \chi_\varepsilon u^{1-p} \varphi^p \right) \, dx \, dt \]
\[\leq -\frac{p-1}{2} c_0 \int_{\Omega_{t_2}} \chi_\varepsilon |D(\log u)|^p \, dx \, dt + c(p, c_0, c_1) \int_{\Omega_{t_1,t_2}} \chi_\varepsilon |D\varphi|^p \, dx \, dt. \quad (3.9)\]

As argued in Step 2 in Appendix A, we obtain

\[\lim_{h \to 0} \int_{0}^{T} \int_{\mathbb{R}^n \times \mathbb{R}^n} [U(x, y, t)K(x, y, t)]_h \chi_\varepsilon(t) \left( u(x, t)^{1-p} \varphi(x, t)^p \right) \, dx \, dy \, dt \]
\[= \int_{0}^{t_2} \int_{0}^{t_2} U(x, y, t)K(x, y, t) \chi_\varepsilon(t) \left( u(x, t)^{1-p} \varphi(x, t)^p \right) \, dx \, dy \, dt \]
\[- u(x, t)^{1-p} \varphi(x, t)^p \right) \, dx \, dy \, dt \quad (3.10)\]

and

\[\lim_{h \to 0} \int_{\Omega} |u|^{p-2} u(0) \left( \frac{1}{h} \int_{0}^{T} e^{-\frac{\varepsilon}{p}} \chi_\varepsilon(s)u(x, s)^{1-p} \varphi(x, s)^p \, ds \right) \, dx = 0. \quad (3.11)\]

Therefore, collecting the preceding estimates \((3.8)–(3.11)\) and subsequently, passing to the limit \(\varepsilon \searrow 0\), we finally gain

\[\begin{align*}
(p - 1) \left[ \int_{\Omega} \varphi^p \log u \, dx \right]_{t=t_1}^{t_2} - p(p - 1) \int \int_{\Omega_{t_1,t_2}} \varphi^{p-1} \varphi' \log u \, dx \, dt \\
- \frac{p-1}{2} c_0 \int_{\Omega_{t_1,t_2}} |D(\log u)|^p \, dx \, dt + c(p, c_0, c_1) \int \int_{\Omega_{t_1,t_2}} |D\varphi|^p \, dx \, dt \\
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t) \left( u(x, t)^{1-p} \varphi(x, t)^p \right) \, dx \, dy \, dt \\
- u(x, t)^{1-p} \varphi(x, t)^p \right) \, dx \, dy \, dt \geq 0,
\end{align*}\]

which in turn implies the desired inequality. \(\square\)

Next we deduce the log-type estimate for supersolutions.

**Proposition 3.6** Assume that \(u\) is a weak supersolution to (1.1) satisfying \(u \geq m > 0\) in \(\mathbb{R}^n \times (t_0 - \varrho^p, t_0 + \varrho^p)\). Given \(\sigma \in (0, 1)\), let \(\chi = \chi(x) \in C^\infty(\mathbb{R}^n)\) be a smooth radial symmetric function such that

\[
\begin{cases}
0 \leq \chi \leq 1 & \text{in } \mathbb{R}^n, \quad \text{supp } \chi \subset B_{\frac{1+\sigma}{c} \varepsilon}(x_0); \\
\chi \equiv 1 & \text{on } B_{\sigma \varepsilon}(x_0), \quad |D \chi| \leq \frac{1}{(1+\sigma)\varrho^p}.
\end{cases}
\]
Set
\[ \beta := \int_{B_{\varrho}(x_0)} \chi^p(x) \log u(x, t_0) \, dx. \]

Then, there exist positive constant \( \overline{C} = \overline{C}(n, s, p, \sigma) \) such that
\[
\left| Q_{\sigma\varrho}(z_0) \cap \{ \log u > \lambda + \beta \} \right| \leq \frac{\overline{C}}{\lambda^{p-1}} \left| Q_{\sigma\varrho}(z_0) \right|
\]
and
\[
\left| Q_{\sigma\varrho}(z_0) \cap \{ \log u < -\lambda + \beta \} \right| \leq \frac{\overline{C}}{\lambda^{p-1}} \left| Q_{\sigma\varrho}(z_0) \right|
\]

Before proving Proposition 3.6, we need the following technical lemma.

**Lemma 3.7** With \( p > 1 \) let \( g(\tau) := \frac{1 - \tau^{1-p}}{1 - \tau} \) for \( \tau \in (0, 1) \) and \( A > 0 \). Then,
\[
(1 - \tau)^p \left( g(\tau) + A \right) \leq (A - (p - 1)) \left[ \log \left( \frac{1}{\tau} \right) \right]^p \quad \forall \tau \in (0, 1)
\]
holds true.

**Proof** Rearranging \( g(\tau) \) as
\[
g(\tau) = -\frac{p - 1}{1 - \tau} \int_{\tau}^{1} q^{-p} \, dq = -(p - 1) \int_{\tau}^{1} q^{-p} \, dq.
\]

By Hölder’s inequality, we infer that
\[
\frac{1}{1 - \tau} \log \left( \frac{1}{\tau} \right) = \int_{\tau}^{1} q^{-1} \, dq \leq \left( \int_{\tau}^{1} q^{-p} \, dq \right)^{\frac{1}{p}}
\]
and therefore,
\[
(1 - \tau)^p g(\tau) \leq -(p - 1) \left[ \log \left( \frac{1}{\tau} \right) \right]^p.
\]

Since \( 1 \leq 1/\varrho \) for any \( \varrho \in (\tau, 1) \),
\[
(1 - \tau)^p \leq \left( \int_{\tau}^{1} \frac{1}{\varrho} \, d\varrho \right)^{p} = \left[ \log \left( \frac{1}{\tau} \right) \right]^p.
\]

Combining these displays, the result in turn follows. \( \square \)

**Proof of Proposition 3.6** Following the argument in the literatures [29, Lemma 6.1] and [17, Lemma 1.3], we prove the claim.

**Step 1.** In the first step, we derive the preliminarily result. For this, let us define
\[
V(t) := \frac{1}{N} \int_{B_{\varrho}(x_0)} (\log u(x, t) - \beta) \chi(x)^p \, dx, \quad N := \int_{B_{\varrho}(x_0)} \chi(x)^p \, dx
\]
with \( \beta = \int_{B_{\varrho}(x_0)} \chi(x)^p \log u(x, t_0) \, dx \). By definition,
\[
V(t_0) = 0, \quad |B_{\sigma\varrho}| \leq N \leq |B_{\varrho}|.
\]
Applying Lemma 3.4 to \(\varphi(x, t) = \chi(x)\) yields that
\[
\int_{t_1}^{t_2} \int_{B_{e}(x_0)} \chi^p |D(\log u)|^p \, dx \, dt - \left[ \int_{B_{e}(x_0)} \chi^p \log u \, dx \right]_{t=t_1}^{t_2} \leq c \int_{t_1}^{t_2} \int_{B_{e}(x_0)} |D\chi|^p \, dx \, dt
\]
\[+ c \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t)(u(x, t)^{1-p} \chi(x, t)^p - u(y, t)^{1-p} \chi(y, t)^p) \, dx \, dy \, dt \]
whenever \(t_0 - (\sigma \varrho)^p \leq t_1 < t_2 \leq t_0 + (\sigma \varrho)^p\). In view of the weighted Poincaré inequality, it holds that
\[
\int_{B_{e}(x_0)} |D(\log u - \beta)|^p \chi^p(x) \, dx \geq \frac{1}{cQ^p} \int_{B_{e}(x_0)} |\log u - \beta - V(t)|^p \chi^p(x) \, dx
\]
\[\geq \frac{1}{cQ^p} \int_{B_{\sigma e}(x_0)} |\log u - \beta - V(t)|^p \, dx \]
and therefore we get, for every \(t_0 - (\sigma \varrho)^p \leq t_1 < t_2 \leq t_0 + (\sigma \varrho)^p\),
\[
\frac{1}{cQ^p N} \int_{t_1}^{t_2} \int_{B_{e}(x_0)} |\log u - \beta - V(t)|^p \, dx + \left[ V(t) \right]_{t_1}^{t_2} \leq c \int_{t_1}^{t_2} \int_{B_{e}(x_0)} |D\chi|^p \, dx \, dt
\]
\[+ c \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t)(u(x, t)^{1-p} \chi(x, t)^p - u(y, t)^{1-p} \chi(y, t)^p) \, dx \, dy \, dt \]
\[= I + II , \quad (3.12)\]
where the definition of \(I\) and \(II\) are obvious from the context.

By definition, \(I\) is estimated as
\[
I \leq \frac{c|B_{e}(x_0)|}{(1 + \sigma)^p Q^p} \frac{t_2 - t_1}{N}.
\]

The fractional integral term \(II\) is split into two terms:
\[
II = \frac{c}{N} \left( \int_{t_1}^{t_2} \int_{B_{e} \times B_{e}} (\cdots) \, dx \, dy \, dt + 2 \int_{t_1}^{t_2} \int_{B_{e} \times (\mathbb{R}^n \setminus B_{e})} (\cdots) \, dx \, dy \, dt \right)
\]
\[= \frac{c}{N} (II_1 + 2II_2) . \quad (3.13)\]

In order to estimate \(II_1\), we distinguish between the two cases \(u(x, t) > u(y, t)\) and \(u(x, t) \leq u(y, t)\). In the first case, applying Lemma 2.1 with \(a = \chi(x), b = \chi(y)\) and
\[\varepsilon = \delta \frac{u(x, t) - u(y, t)}{u(x, t)} \in (0, 1) \quad \text{with} \quad \delta \in (0, 1),\]
the integrand of \(II_1\) is estimated as
\[
U(x, y, t)K(x, y, t) \left( \frac{\chi(x)^p}{u(x, t)^p} - \frac{\chi(y)^p}{u(y, t)^p} \right)
\]
\[ K(x, y, t) (u(x, t) - u(y, t))^ {p-1} \left( \frac{\chi(x)^p}{u(x, t)^p} - \frac{\chi(y)^p}{u(y, t)^p} \right) \]

\[ \leq K(x, y, t) \left( \frac{u(x, t) - u(y, t)}{u(x, t)} \right)^ {p-1} \chi(y)^p \left[ 1 + c\delta \frac{u(x, t) - u(y, t)}{u(x, t)} - \left( \frac{u(x, t)}{u(y, t)} \right)^ {p-1} \right] \]

\[ + c\delta^1 - p K(x, y, t) |\chi(x) - \chi(y)|^p \]

\[ = K(x, y, t) \left( \frac{u(x, t) - u(y, t)}{u(x, t)} \right)^ p \left[ g \left( \frac{u(y, t)}{u(x, t)} \right) + c\delta \right] + c\delta^1 - p K(x, y, t) |\chi(x) - \chi(y)|^p, \]

where in the last line we let \( g(\tau) := \frac{1 - \tau^{1-p}}{1 - \tau} \) for \( \tau \in (0, 1) \). Therefore, Lemma 3.7 applied with \( \tau \equiv \frac{u(y, t)}{u(x, t)} \) and \( A \equiv c\delta \) and choosing \( \delta \equiv \frac{p - 1}{2^p c} \) implies that

\[ U(x, y, t) K(x, y, t) \left( \frac{\chi(x)^p}{u(x, t)^p} - \frac{\chi(y)^p}{u(y, t)^p} \right) \]

\[ \leq - \frac{(p - 1)(2^p - 1)}{2^p} K(x, y, t) \left[ \log \left( \frac{u(x, t)}{u(y, t)} \right) \right]^ p \chi(y)^p \]

\[ + c \left( \frac{p - 1}{2^p c} \right)^ {1 - p} K(x, y, t) |\chi(x) - \chi(y)|^p. \]

In the case \( u(x, t) = u(y, t) \), this estimate trivially holds and, by the interchanging the role of \( u(x, t) \) and \( u(y, t) \), we also recover this estimate in the case \( u(x, t) < u(y, t) \). Therefore, we finally deduce that

\[ \mathbf{II}_1 \leq -c \int_{t_1}^{t_2} \int_{B_\rho \times B_\rho} K(x, y, t) \left| \log \left( \frac{u(x, t)}{u(y, t)} \right) \right|^ p \chi(y)^p \, dx \, dy \, dt \]

\[ + c \int_{t_1}^{t_2} \int_{B_\rho \times B_\rho} K(x, y, t) |\chi(x) - \chi(y)|^p \, dx \, dy \, dt \]

\[ \leq -c \int_{t_1}^{t_2} \int_{B_\rho \times B_\rho} K(x, y, t) \left| \log \left( \frac{u(x, t)}{u(y, t)} \right) \right|^ p \chi(y)^p \, dx \, dy \, dt \]

\[ + \frac{c\Lambda}{(1 + \sigma)^p \rho^p} (t_2 - t_1) |B_\rho| \sup_{x \in B_\rho} \left( \int_{B_\rho} \frac{dy}{|x - y|^{n-(1-s)p}} \right) \]

\[ \leq \frac{c}{(1 - s) (1 + \sigma)^p} \frac{\rho^{n-sp}}{(t_2 - t_1)} \]

(3.14)

for a constant \( c \equiv c(n, s, p, \Lambda) \), where we again used that

\[ \int_{B_\rho} \frac{dy}{|x - y|^{n-(1-s)p}} = |B_\rho| \frac{\rho^{(1-s)p}}{(1 - s) p}. \]

Next, recalling the fact that \( \text{supp} \chi \in B_{\frac{1+\rho}{2}} (x_0) \) and

\[ \left( \frac{u(x, t) - u(y, t)}{u(x, t)^{p-1}} \right) \leq 1 \quad \forall (x, y, t) \in B_{\frac{1+\rho}{2}} (x_0) \times (\mathbb{R}^n \setminus B_\rho (x_0)) \times (t_1, t_2), \]

we infer that

\[ \mathbf{II}_2 = 2 \int \int_{B_{\frac{1+\rho}{2}} (x_0) \times (\mathbb{R}^n \setminus B_\rho (x_0))} U(x, y, t) K(x, y, t) u(x, t)^{1-p} \chi(x)^p \, dx \, dy \, dr \]

\( \square \) Springer
\[
\leq 2 \int_{B_{\frac{1+\sigma}{2}}(x_0) \times (\mathbb{R}^n \setminus B_{\sigma}(x_0))} (u(x, t) - u(y, t))^2 K(x, y, t)u(x, t)^{1-p} \, dx \, dy \, dt
\]
\[
\leq 2\Lambda |B_{\frac{1+\sigma}{2}}(x_0)| (t_2 - t_1) \sup_{x \in B_{\frac{1+\sigma}{2}}(x_0)} \int_{\mathbb{R}^n \setminus B_{\sigma}(x_0)} \frac{dy}{|x - y|^{n+sp}}.
\]

Similarly as before, since
\[
\frac{|y - x_0|}{|y - x|} \leq \frac{|y - x| + |x - x_0|}{|y - x|} = 1 + \frac{|x - x_0|}{|y - x|}
\]
and
\[
|y - x| \geq |y - x_0| - |x_0 - x| \geq \varrho - \frac{1 + \sigma}{2}\varrho = \frac{1 - \sigma}{2}\varrho
\]
holds again for any \(x \in B_{\frac{1+\sigma}{2}}(x_0)\) and \(y \in \mathbb{R}^n \setminus B_{\sigma}(x_0)\), we get
\[
\frac{|y - x_0|}{|y - x|} \leq \frac{2}{1 - \sigma} \quad \forall x \in B_{\frac{1+\sigma}{2}}(x_0), \quad \forall y \in \mathbb{R}^n \setminus B_{\sigma}(x_0)
\]
and therefore,
\[
\mathbf{II}_2 \leq c \frac{|B_{\frac{1+\sigma}{2}}(x_0)|}{(1 - \sigma)^n}\frac{(t_2 - t_1)}{n+sp} \sup_{x \in B_{\frac{1+\sigma}{2}}(x_0)} \int_{\mathbb{R}^n \setminus B_{\sigma}(x_0)} \frac{dy}{|x - x_0|^{n+sp}}
\]
\[
\leq c \frac{(1 + \sigma)^n}{(1 - \sigma)^n}\varrho^{n-sp}(t_2 - t_1)
\]
(3.15)

with a constant \(c \equiv c(n, s, p, \Lambda)\), where \(\int_{\mathbb{R}^n \setminus B_{\sigma}(x_0)} \frac{dy}{|y - x_0|^{n+sp}} = \frac{c(n)}{sp}\varrho^{-sp}\) is used again.

Merging estimates (3.14)–(3.15) with (3.13), we obtain
\[
\mathbf{II} \leq \frac{c}{N} \left(\frac{1}{(1 + \sigma)^p} + \frac{(1 + \sigma)^n}{(1 - \sigma)^n}\varrho^{n-sp}(t_2 - t_1)\right).
\]

Combining the preceding estimates with (3.12) implies that
\[
\frac{1}{c\varrho^p |B_{\sigma}|} \int_{t_1}^{t_2} \int_{B_{\sigma}(x_0)} |\log u - \beta - V(t)|^p \, dx \, dt + \frac{V(t_2) - V(t_1)}{t_2 - t_1}
\]
\[
\leq \frac{c|B_{\sigma}|}{N(1 + \sigma)^p\varrho^p} + \frac{c}{N} \left(\frac{1}{(1 + \sigma)^p} + \frac{(1 + \sigma)^n}{(1 - \sigma)^n}\varrho^{n-sp}\right),
\]

which together with the fact that \(|B_{\sigma}| \geq N \geq \sigma^n|B_{\sigma}|\) yields
\[
\frac{1}{c\varrho^p |B_{\sigma}|} \int_{t_1}^{t_2} \int_{B_{\sigma}(x_0)} |\log u - \beta - V(t)|^p \, dx \, dt + \frac{V(t_2) - V(t_1)}{t_2 - t_1} \leq C(\varrho^{-p} + \varrho^{-sp})
\]
with a constant \(c \equiv c(n, s, p, \Lambda, \sigma)\). Denoting
\[
w(x, t) := \log u(x, t) - \beta - C(\varrho^{-p} + \varrho^{-sp})(t - t_0),
\]
\[
W(t) := V(t) - C(\varrho^{-p} + \varrho^{-sp})(t - t_0)
\]
with noticing that \(W(t_0) = 0\), the above display becomes
\[
\frac{1}{c\varrho^p |B_{\sigma}|} \int_{t_1}^{t_2} \int_{B_{\sigma}(x_0)} |w - W(t)|^p \, dx \, dt + \frac{W(t_2) - W(t_1)}{t_2 - t_1} \leq 0.
\]
Since by the monotone increasing of the function $t \mapsto W(t)$, $W(t)$ is differentiable for almost everywhere $t \in (t_0 - \varphi^p, t_0 + \varphi^p)$, letting $t_2 \searrow t_1 = t$ in the above formula gives that

$$\frac{1}{c \varphi^p |B_{\varphi}|} \int_{B_{\varphi}(x_0)} |w - W(t)|^p \, dx + W'(t) \leq 0$$  \hspace{1cm} (3.16)

for almost everywhere $t \in (t_0 - \varphi^p, t_0 + \varphi^p)$.

**Step 2.** In this step we shall estimate the Lebesgue measure of

$$\Sigma^{-}_\lambda(t) := \left\{ x \in B_{\varphi}(x_0) : w(x, t) > \lambda \right\} \text{ for } t \in (t_0 - (\varphi \varphi), \varphi^p)$$

and

$$\Sigma^{+}_\lambda(t) := \left\{ x \in B_{\varphi}(x_0) : w(x, t) < -\lambda \right\} \text{ for } t \in (t_0, t_0 + (\varphi \varphi))^p,$$

where $\lambda > 0$ is arbitrarily given. Since $W(t) < W(t_0) = 0$ for $t \in (t_0 - (\varphi \varphi), t_0)$ it holds that

$$\int_{B_{\varphi}(x_0)} |w - W(t)|^p \, dx \geq \int_{\Sigma^{-}_\lambda(t)} |\lambda - W(t)|^p \, dx = (\lambda - W(t))^p |\Sigma^{-}_\lambda(t)|,$$

and therefore, by (3.16), for almost everywhere $t \in (t_0 - (\varphi \varphi), t_0)$, we have

$$\frac{|\Sigma^{-}_\lambda(t)|}{c \varphi^p |B_{\varphi}|} + \frac{W'(t)}{(\lambda - W(t))^p} \leq 0.$$

Integrating with respect to $t \in (t_0 - (\varphi \varphi), t_0)$ yields

$$\int_{t_0 - (\varphi \varphi)}^{t_0} |\Sigma^{-}_\lambda(t)| \, dt \leq \frac{\bar{C}}{\lambda^{p-1}} |Q^{-}_{\varphi}(z_0)|$$

for a constant $\bar{C} = \bar{C}(n, s, p, \sigma)$. Therefore, since $t - t_0 < 0$ for $t \in (t_0 - (\varphi \varphi), t_0)$ we conclude that

$$|Q^{-}_{\varphi}(z_0) \cap \{ \log u > \lambda + \beta \}| \leq \int_{t_0 - (\varphi \varphi)}^{t_0} |\Sigma^{-}_\lambda(t)| \, dt \leq \frac{\bar{C}}{\lambda^{p-1}} |Q^{-}_{\varphi}(z_0)|,$$

as desired. By the same reasoning, we can deduce that

$$\int_{t_0}^{t_0 + (\varphi \varphi)} |\Sigma^{+}_\lambda(t)| \, dt \leq \frac{\bar{C}}{\lambda^{p-1}} |Q^{+}_{\varphi}(z_0)|$$

and hence

$$|Q^{+}_{\varphi}(z_0) \cap \{ \log u < -\lambda + \beta \}| \leq \int_{t_0}^{t_0 + (\varphi \varphi)} |\Sigma^{+}_\lambda(t)| \, dt \leq \frac{\bar{C}}{\lambda^{p-1}} |Q^{+}_{\varphi}(z_0)|.$$

This completes the proof of the lemma. \qed

We conclude this section by deducing the following helpful lemma that will be used later in the proof of Theorem 1.1.

**Lemma 3.8** Let $p > 1$ and $s \in (0, 1)$. If $u$ is a weak supersolution to (1.1) satisfying $u \geq m > 0$ in $\mathbb{R}^n \times (0, T)$, then $v = u^{-1}$ is a positive subsolution to (1.1) replaced $A$ by $\tilde{A}$, where the vector field $\tilde{A} : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$\tilde{A}(x, t, y, \xi) := -\tilde{y}^{2(p-1)} A (x, \tilde{y}^{-1}, -\tilde{y}^{-2} \xi) \text{ with } \tilde{y} := \min \left\{ y, \frac{1}{m} \right\}.$$
and possesses the same structure constants as $A$.

**Proof** First of all, by definition we see that $\tilde{A}$ is measurable with respect to $(x, t) \in \Omega_T$ for every $(y, \zeta) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to $(y, \zeta)$ for almost everywhere $(x, t) \in \Omega_T$. A straightforward computation shows that $\tilde{A}$ fulfills the structure condition

$$
\begin{cases}
\tilde{A}(x, t, y, \zeta) \cdot \zeta \geq c_0 |\zeta|^p \\
|\tilde{A}(x, t, y, \zeta)| \leq c_1 |\zeta|^{p-1}
\end{cases}
$$

and therefore, the vector field $\tilde{A}$ is Carathéodory map fulfilling the same structural condition as $A$.

Let $t_1 \in (0, T)$ be an arbitrary time. For any $\delta > 0$ small, let us define the cutoff function $\psi_\delta(t)$ with respect to time as

$$
\psi_\delta(t) := \begin{cases}
0, & t \in (0, t_1), \\
\frac{1}{\delta}(t - t_1), & t \in [t_1, t_1 + \delta), \\
1, & t \in [t_1 + \delta, T).
\end{cases}
$$

Now, in the weak formulation (2.10), testing $\varphi = \psi_\delta u^{2(1-p)} \phi$ for any nonnegative $\phi \in \mathcal{T}$, we have

$$
\begin{align*}
\int_0^T \int_{\Omega_T} \left( \partial_t [u^{p-1}]_h \psi_\delta u^{2(1-p)} \phi + [A(x, t, u, Du)]_h \cdot D \left( \psi_\delta u^{2(1-p)} \phi \right) \right) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} [U(x, y, t)K(x, y, t)]_h \psi_\delta(t) \\
\left( u(x, t)^{2(1-p)} \phi(x, t) - u(y, t)^{2(1-p)} \phi(y, t) \right) \, dx \, dy \, dt \\
\geq \int_{\Omega_T} u(0)^{p-1} \left( \frac{1}{h} \int_0^T e^{\frac{t}{h}} \psi_\delta(s) u^{2(1-p)} \phi(x, s) \, ds \right) \, dx.
\end{align*}
$$

For the evolutionary term,

$$
\begin{align*}
\int_0^T \int_{\Omega_T} \partial_t [u^{p-1}]_h u^{2(1-p)} \psi_\delta \phi \, dx \, dt &= \int_0^T \int_{\Omega_T} \partial_t [u^{p-1}]_h \left( u^{2(1-p)} - [u^{p-1}]_h^{-2} \right) \psi_\delta \phi \, dx \, dt \\
&\quad + \int_0^T \int_{\Omega_T} \partial_t [u^{p-1}]_h [u^{p-1}]_h^{-2} \psi_\delta \phi \, dx \, dt \\
&=: I_h + II_h.
\end{align*}
$$

Lemma 2.8-(ii) and inequality (2.5) with $\alpha = 3$ in Lemma 2.3 imply that

$$
I_h = \int_0^T \int_{\Omega_{t_1,T}} \psi_\delta \phi \frac{u^{p-1} - [u^{p-1}]_h}{h} \cdot \frac{[u^{p-1}]_h^2 - u^{2(p-1)}}{[u^{p-1}]_h^2 u^{2(p-1)}} \, dx \, dr \leq 0.
$$

For the observation of $II_h$, we require the following lemma, whose proof will be given in Appendix B.

**Lemma 3.9** Suppose that $u \geq m > 0$ in $\mathbb{R}^n \times (0, T)$ and fix $t_1 \in (0, T)$. Then

$$
[u^{p-1}]_h^{-1} \rightarrow u^{1-p} = v^{p-1} \text{ in } L^{\frac{p}{p-1}}(\Omega_{t_1,T}) \text{ as } h \downarrow 0.
$$
By Lemma 3.9, we have

\[
\lim_{h \searrow 0} \mathbf{I}_h = \lim_{h \searrow 0} \int_{\Omega_{1,t}} \psi_\beta \phi \partial_t \left( -[u^{p-1}]_h \right) \, dx \, dt
\]

\[
= \lim_{h \searrow 0} \int_{\Omega_{1,t}} [u^{p-1}]_h \partial_t (\psi_\beta \phi) \, dx \, dt
\]

\[
= \int_{\Omega_{1,t}} v^{p-1} \partial_t (\psi_\beta \phi) \, dx \, dt
\]

\[
= \int_{t_1}^{t_1+\delta} \int_{\Omega} v^{p-1} \phi \, dx \, dt + \int_{\Omega_{1,t}} v^{p-1} \psi_\beta \partial_t \phi \, dx \, dt.
\]

Combining the preceding estimates above, we infer that

\[
\limsup_{h \searrow 0} \int_{\Omega_T} \partial_t [u^{p-1}]_h u^{2(1-p)} \psi_\beta \phi \, dx \, dt
\]

\[
\leq \limsup_{h \searrow 0} (\mathbf{I}_h + \mathbf{II}_h) \leq \int_{t_1}^{t_1+\delta} \int_{\Omega} v^{p-1} \phi \, dx \, dt + \int_{\Omega_{1,t}} v^{p-1} \psi_\beta \partial_t \phi \, dx \, dt
\] (3.18)

We shall estimate the spatial term involving \( A \). As mentioned before, since

\[
[A(x, t, u, Du)]_h \longrightarrow A(x, t, u, Du) \quad \text{in} \quad L^{\frac{p}{p-1}}(\Omega_T),
\]

one can easily check that

\[
\lim_{h \searrow 0} \int_{\Omega_T} [A(x, t, u, Du)]_h \cdot D \left( u^{2(1-p)} \psi_\beta \phi \right) \, dx \, dt
\]

\[
= \int_{\Omega_T} A(x, t, u, Du) \cdot D \left( u^{2(1-p)} \psi_\beta \phi \right) \, dx \, dt
\]

\[
= \int_{\Omega_{1,t}} \psi_\beta v^{2(p-1)} A(x, t, v^{-1}, -v^{-2} Dv) \cdot D\phi \, dx \, dt
\]

\[
+ 2(p - 1) \int_{\Omega_{1,t}} \psi_\beta \psi_\phi v^{2p-1} A(x, t, v^{-1}, -v^{-2} Dv) \cdot D\phi \, dx \, dt
\]

\[
=: \mathbf{I} + \mathbf{II},
\]

where the definition of \( \mathbf{I} \) and \( \mathbf{II} \) are clear form the context.

By definition, \( \mathbf{I} \) is rewritten as

\[
\mathbf{I} = -\int_{\Omega_{1,t}} \psi_\beta \tilde{A}(x, t, v, Du) \cdot D\phi \, dx \, dt.
\]

For term \( \mathbf{II} \) we have

\[
\mathbf{II} = -2(p - 1) \int_{\Omega_{1,t}} \psi_\beta \phi v^{-1} \tilde{A}(x, t, v, Du) \cdot Dv \, dx \, dt
\]

\[
\leq -2(p - 1) \int_{\Omega_T} \psi_\beta \phi v^{-1} c_0 |Dv|^p \, dx \, dt \leq 0.
\]

Combining the preceding estimates gives

\[
\lim_{h \searrow 0} \int_{\Omega_T} [A(x, t, u, Du)]_h \cdot D \left( u^{2(1-p)} \psi_\beta \phi \right) \, dx \, dt
\]
with the shorthand notation

\[ H \text{arnack's estimate for a mixed local–nonlocal doubly... Page 27 of 45} \]

\[ 40 \]

we get

\[ \text{we have the bound} \]

\[ \lim_{h \to 0} \int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} [U(x, y, t)K(x, y, t)]_h \psi_h(t) \]

\[ \frac{u(x, t)^{2(1-p)} \phi(x, t) - u(y, t)^{2(1-p)} \phi(y, t)}{dx \, dy \, dt} \]

\[ = \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t) \psi_h(t) \]

\[ \frac{u(x, t)^{2(1-p)} \phi(x, t) - u(y, t)^{2(1-p)} \phi(y, t)}{dx \, dy \, dt} \]

\[ =: \text{III} \]

In order to estimate III, we now follow the argument considered in [3]. Since

\[ U(x, y, t) = u(x, t)^{p-1}u(y, t)^{p-1} |v(x, t) - v(y, t)|^{p-2} (v(x, t) - v(y, t)), \]

the integrand of III is written as

\[ U(x, y, t)K(x, y, t) \left( u(x, t)^{2(1-p)} \phi(x, t) - u(y, t)^{2(1-p)} \phi(y, t) \right) \]

\[ = -V(x, y, t)K(x, y, t) \left[ \left( \frac{v(x, t)}{v(y, t)} \right)^{p-1} \phi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{p-1} \phi(y, t) \right] \]

with the shorthand notation

\[ V(x, y, t) := |v(x, t) - v(y, t)|^{p-2} (v(x, t) - v(y, t)). \]

We now distinguish between two cases \( v(x, t) \geq v(y, t) \) and \( v(x, t) < v(y, t) \). As we are considering the case \( v(x, t) \geq v(y, t), \) since

\[ \frac{v(x, t)}{v(y, t)} \geq 1, \quad \frac{v(y, t)}{v(x, t)} \leq 1 \]

we have the bound

\[ V(x, y, t)K(x, y, t) \left[ \left( \frac{v(x, t)}{v(y, t)} \right)^{p-1} \phi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{p-1} \phi(y, t) \right] \]

\[ \geq V(x, y, t)K(x, y, t) (\phi(x, t) - \phi(y, t)). \]

When \( v(x, t) < v(y, t), \) since

\[ \frac{v(x, t)}{v(y, t)} < 1, \quad \frac{v(y, t)}{v(x, t)} > 1 \]

we get

\[ V(x, y, t) \left[ \left( \frac{v(x, t)}{v(y, t)} \right)^{p-1} \phi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{p-1} \phi(y, t) \right] \]

\[ \geq |v(x, t) - v(y, t)|^{p-2} (v(y, t) - v(x, t)) (\phi(y, t) - \phi(x, t)) \]

\[ = V(x, y, t) (\phi(x, t) - \phi(y, t)) \]
as well. Consequently, we conclude that

$$\text{III} \leq -\int_{t_1}^{T} \int_{\mathbb{R}^n \times \mathbb{R}^n} V(x, y, t) K(x, y, t) \psi_{\delta}(t) (\phi(y, t) - \phi(x, t)) \, dx \, dy \, dt. \quad (3.20)$$

Since by $u \geq m > 0$ in $\mathbb{R}^n \times (0, T)$ and $\psi_{\delta}(s) \leq 1$ it holds that

$$\lim_{h \searrow 0} \int_{\Omega} u(0)^{p-1} \left( \frac{1}{h} \int_{0}^{T} e^{\frac{t}{h}} u^{2(1-p)} \psi_{\delta}(s) \phi(x, s) \, ds \right) \, dx = 0, \quad (3.21)$$

collecting these estimations (3.18)–(3.21) and, in (3.17), passing to the limit $h \searrow 0$ and subsequently, sending $\delta \searrow 0$, we conclude that

$$\int_{\Omega_{1,T}} (v^{p-1} \partial_t \phi + \tilde{\kappa}(x, t, v, Dv) \cdot D\phi) \, dx \, dt - \int_{\Omega} v(x, t_1)^{p-1} \phi(x, t_1) \, dx$$

$$+ \int_{t_1}^{T} \int_{\mathbb{R}^n \times \mathbb{R}^n} V(x, y, t) K(x, y, t) (\phi(x, t) - \phi(y, t)) \, dx \, dy \, dt \leq 0. \quad (3.22)$$

for every nonnegative $\phi \in \mathcal{T}$. Now, we claim that

$$\lim_{t_1 \searrow 0} \int_{\Omega} \phi(x, t_1) \, dx = 0. \quad (3.23)$$

Indeed, for any nonnegative $\phi \in \mathcal{T}$, set

$$\langle \phi \rangle_h := \left( [\phi_h^\frac{p}{2}]_h \right) \frac{2}{p}.$$

By Hölder’s inequality, we estimate

$$0 \leq \int_{\Omega} \phi(x, t_1) \, dx = \int_{\Omega} (\phi(x, t_1) - \langle \phi \rangle_h(x, t_1)) \, dx + \int_{\Omega} \langle \phi \rangle_h(x, t_1) \, dx$$

$$\leq \left( \| \phi(t_1) - \langle \phi \rangle_h(t_1) \|_{L^p(\Omega)} + \| \langle \phi \rangle_h(t_1) \|_{L^p(\Omega)} \right) |\Omega| \frac{p-1}{p}.$$

Since by Lemma 2.9, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\| \phi(t_1) - \langle \phi \rangle_h(t_1) \|_{L^p(\Omega)} < \varepsilon$$

whenever $h \in (0, \delta)$ and $t_1 \in [0, T]$. This implies that

$$0 \leq \int_{\Omega} \phi(x, t_1) \, dx \leq \left( \varepsilon + \| \langle \phi \rangle_h(t_1) \|_{L^p(\Omega)} \right) |\Omega| \frac{p-1}{p} \quad (3.24)$$

and moreover, in view of Lemma 2.9 and $\phi(x, 0) = 0$,

$$\| \langle \phi \rangle_h(t_1) \|_{L^p(\Omega)} \rightarrow \| \langle \phi \rangle_h(0) \|_{L^p(\Omega)} = 0 \text{ as } t_1 \searrow 0$$

holds true. Passing to the limit $t_1 \searrow 0$ in (3.24) gives

$$0 \leq \lim_{t_1 \searrow 0} \int_{\Omega} \phi(x, t_1) \, dx \leq \varepsilon |\Omega| \frac{p-1}{p}.$$

and subsequently, letting $\varepsilon \searrow 0$ proves (3.23).

Thus, using (3.23) and the assumption that $v^{-1} \equiv u \geq m > 0$ in $\mathbb{R}^n \times (0, T)$, we have

$$0 \leq \lim_{t_1 \searrow 0} \int_{\Omega} v(x, t_1)^{p-1} \phi(x, t_1) \, dx \leq m^{1-p} \lim_{t_1 \searrow 0} \int_{\Omega} \phi(x, t_1) \, dx = 0.$$
and therefore, using this and letting \( t_1 \searrow 0 \) in (3.22) conclude that
\[
\int_{\Omega_T} (v^{p-1} \partial_t \phi + \tilde{A}(x,t,v,Dv) \cdot D\phi) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} V(x,y,t) K(x,y,t) (\phi(x,t) - \phi(y,t)) \, dx \, dy \, dt \leq 0,
\]
proving the claim. \( \square \)

## 4 Quantitative estimates for subsolutions

In this section, we give quantitative estimates for subsolutions. We begin by deriving the reverse H"{o}lder inequality for supersolutions, then we prove the local boundedness of subsolutions under the assumption that \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0, t_0 + q^p) \). By invoking [13], Inequality (A.4)) and the same reasoning as Lemma 3.1, the following lemma holds true:

**Lemma 4.1** (Caccioppoli type estimate for subsolutions) Let \( u \) be a weak subsolution to (1.1) fulfilling \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0, t_0 + q^p) \). With \( p > 1 \) and \( s \in (0,1) \) let \( \epsilon \geq 1 \) and set \( \alpha := p - 1 + \epsilon \). Then

\[
\sup_{t \in (t_0, t_0 + q^p)} \int_{B_\rho(x_0) \times [t]} u^\alpha \phi^p \, dx + \int_{Q^+_\rho(x_0)} |Du|^p u^{\epsilon-1} \phi^p \, dx \, dt \\
+ \int_{t_0}^{t_0 + q^p} \int_{B_{\rho}(x_0) \times B_{\rho}(x_0)} |u(x,t)^{\frac{\alpha}{p}} \phi(x,t) - u(y,t)^{\frac{\alpha}{p}} \phi(y,t)|^p \, dx \, dy \, dt \\
\leq c \int_{Q^+_\rho(x_0)} u^\alpha \phi^{p-1} |\phi| \, dx \, dt + c \int_{Q^+_\rho(x_0)} u^\alpha |D\phi|^p \, dx \, dt \\
+ c \int_{t_0}^{t_0 + q^p} \int_{B_{\rho}(x_0) \times B_{\rho}(x_0)} \frac{(u(x,t)^\alpha + u(y,t)^\alpha)|\phi(x,t) - \phi(y,t)|^p}{|x - y|^{n+sp}} \, dx \, dy \, dt \\
+ c \left( \sup_{x \in \text{supp}\ \phi(-,t)} \int_{\mathbb{R}^n \setminus B_{\rho}(x_0)} \frac{dy}{|x - y|^{n+sp}} \right) \int_{Q^+_\rho(x_0)} u^\alpha \phi^p \, dx \, dt
\]
holds whenever nonnegative \( \phi \in C_0^\infty(Q^+_{\rho}(x_0)) \), where the constant \( c \equiv c(p,c_0,c_1,\Lambda, \epsilon) \).

Following the same reasoning of Lemma 3.2, we obtain the Reverse H"{o}lder type lemma for subsolutions.

**Lemma 4.2** Suppose that a weak subsolution \( u \) to (1.1) satisfies \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0, t_0 + q^p) \). For any \( \epsilon \geq 1 \) let \( \alpha := p - 1 + \epsilon \) and \( \kappa := \frac{n+sp}{n} \). Then for any concentric cylinders \( Q^+_{\rho}(z_0) \subset Q^+_\rho(x_0) \subset \Omega_T \) the quantitative estimate

\[
\left( \int_{Q^+_\rho(x_0)} u^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}} \leq c \frac{1}{\alpha} \left[ \left( \frac{\rho}{r} \right)^n \left( \frac{\rho}{r - \rho} \right)^{n+sp} (1 + \epsilon^{1-s})^p \right] \left( \int_{Q^+_\rho(x_0)} u^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}}
\]
holds true, where \( c \equiv c(n,s,p,c_0,c_1,\Lambda, \epsilon) \) blows up as \( \epsilon \searrow 0 \).

**Proposition 4.3** (Local boundedness for subsolutions) Suppose that a weak subsolution \( u \) to (1.1) satisfies that \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0, t_0 + q^p) \). Let \( \sigma \) and \( \tau \) fulfill \( 0 < \sigma < \tau \) and therefore using this and letting \( t_1 \searrow 0 \) conclude that
Let $\tau \leq 1$ and let $\gamma > 0$ be an arbitrary number. Then, there exists a positive constant $c = c(n, s, p, c_0, c_1, \Lambda, \gamma)$ such that

$$
\sup_{Q^+_{\tau\rho}(z_0)} u \leq c \left( \sigma^{-(1-s)p} \frac{1 + \varrho^{(1-s)p}}{(\tau - \rho)^{n+sp}} \right)^{\frac{n+p}{p \varrho}} \left( \iint_{Q^+_{\tau\rho}} u^\gamma \, dx \, dt \right)^{\frac{1}{\varrho}},
$$

holds whenever concentric cylinders $Q^+_{\tau\rho}(z_0) \subset Q^+_{\tau_0}(z_0) \subset \Omega_T$.

**Proof** Let $q_0 > p - 1$. Fix $0 < \sigma \leq \theta_1 < \theta_2 \leq \tau \leq 1$ and define, for $i = 0, 1, 2 \ldots$

$$
\rho_i := \theta_1 \rho + \frac{(\theta_2 - \theta_1) \rho}{2^i}; \quad Q_i := Q^+_{\rho_i}(z_0) = B_{\rho_i}(x_0) \times (t_0, t_0 + \rho_i^p)
$$

and therefore the following inclusions hold inductively:

$$
q_0 = \theta_2 q \geq \cdots \geq q_i \geq \rho \infty = \theta_1 q, \quad Q_0 = Q^+_{\theta_2 q} \supset \cdots \supset Q_i \supset \cdots \supset Q_\infty = Q^+_{\theta_1 q}.
$$

Applying Lemma 4.2 with $r = \rho_{i+1}$, $\rho = \rho_i$ and

$$
q_{i+1} = \alpha \kappa, \quad q_i = \alpha
$$

which means that $q_i = q_0 \kappa^i \not\to \infty$ as $i \to \infty$, we get for $i = 0, 1, \ldots$,

$$
\left( \iint_{Q_{i+1}} u^{q_{i+1}} \, dx \, dt \right)^{\frac{1}{q_{i+1}}} \leq c \left( \frac{\rho_i}{\rho_{i+1}} \right)^n \left( \frac{\rho_i}{\rho_i - \rho_{i+1}} \right)^{n+sp} (1 + \rho_i^{(1-s)p}) \left( \iint_{Q_i} u^{q_i} \, dx \, dt \right)^{\frac{1}{q_i}}. \quad (4.1)
$$

One readily checks the three bounds:

$$
\frac{\rho_i}{\rho_{i+1}} \leq 2, \quad \frac{\rho_i}{\rho_i - \rho_{i+1}} \leq \frac{2^{i+1}\theta_2}{\theta_2 - \theta_1} \quad \text{and} \quad \rho_i^{(1-s)p} \leq \rho^{(1-s)p}.
$$

These bounds and (4.1) give that

$$
Y_{i+1} \leq c \left[ 2^n \left( \frac{2^{i+1}\theta_2}{\theta_2 - \theta_1} \right)^{n+sp} (1 + \rho_i^{(1-s)p}) \right]^{\frac{1}{q_i}} Y_i
$$

$$
\leq c \left[ 4^{(n+sp)\frac{i}{q_i}} \left( \frac{\theta_2}{\theta_2 - \theta_1} \right)^{n+sp} (1 + \rho_i^{(1-s)p}) \right]^{\frac{1}{q_i}} Y_i, \quad (4.2)
$$

where we again used the shorthand notation

$$
Y_i := \left( \iint_{Q_i} u^{q_i} \, dx \, dt \right)^{\frac{1}{q_i}} \quad \forall i = 0, 1, 2, \ldots
$$

Iterating (4.2), we obtain that, for all $k \in \mathbb{N}$,

$$
Y_k \leq c^{S(k)} 4^{(n+sp)T(k)} \left[ \frac{\theta_2}{\theta_2 - \theta_1} \right]^{n+sp} (1 + \rho_i^{(1-s)p})^{S(k)} Y_0, \quad (4.3)
$$
where
\[ S(k) := \sum_{i=0}^{k} \frac{1}{q_i}, \quad T(k) := \sum_{i=0}^{k-1} \frac{i}{q_i}. \]

As argued before, since
\[ \lim_{k \to \infty} S(k) = \frac{\kappa}{q_0(\kappa - 1)} = \frac{n + p}{q_0 p} \]
and
\[ \lim_{k \to \infty} T(k) = \frac{\kappa}{q_0(\kappa - 1)^2}, \]
passing to the limit \( k \to \infty \) in (4.3), we arrive at
\[ \sup_{Q_{p}(\nu)} u = \lim_{k \to \infty} Y_k \leq \left[ C \left( \frac{\theta_2}{\theta_2 - \theta_1} \right)^{n+sp} \left( 1 + \varrho^{(1-s)p} \right) \right] \frac{n+p}{\rho p} \left( \iint_{Q_{\tau p}^0} u^{q_0} \, dx \, dt \right)^{\frac{1}{q_0}} \]

with \( C = C(n, s, p, c_0, c_1, 1, \Lambda) \). We now choose \( q_0 = p \) and \( \beta \in (0, p) \). Young’s inequality gives
\[ \sup_{Q_{p}(\nu)} u \leq \left[ C \left( \frac{\theta_2}{\theta_2 - \theta_1} \right)^{n+sp} \left( 1 + \varrho^{(1-s)p} \right) \right] \frac{n+p}{\rho p} \left( \iint_{Q_{\tau p}^0} u^{p-\beta} \, dx \, dt \right)^{\frac{1}{p-\beta}}, \tag{4.4} \]

where in the last line we carefully estimated that
\[ \frac{\theta_2^{n+sp}}{\theta_2} \leq \frac{\tau}{\theta_2} \leq \frac{\tau^p}{\theta_2^{n+sp}} \leq \frac{\sigma^{-(1-s)p}}{\theta_2^{n+sp}}. \]

Set \( \gamma := p - \beta > 0 \) and define, for \( s \in [\sigma, \tau] \), \( \mathcal{Z}(s) := \sup_{Q_{p}(\nu)} u \). Since \( \mathcal{Z}(s) \) is nonnegative and bounded by (4.4), Lemma 2.7 with
\[ A \equiv \left( C \sigma^{-(1-s)p} \left( 1 + \varrho^{(1-s)p} \right) \right) \frac{n+p}{\rho p} \left( \iint_{Q_{\tau p}^0} u^\gamma \, dx \, dt \right)^{\frac{1}{\gamma}} \]
yields that
\[ \sup_{Q_{p}(\nu)} u \leq C(n, s, p, c_0, c_1, 1, \gamma) \left( \sigma^{-(1-s)p} \left( 1 + \varrho^{(1-s)p} \right) \right) \frac{n+p}{\rho p} \left( \iint_{Q_{\tau p}^0} u^\gamma \, dx \, dt \right)^{\frac{1}{\gamma}}, \]
finishes the proof. \( \square \)

We note that, removing the positivity condition \( u \geq m > 0 \) and adding the restriction \( p \geq 2 \), we can deduce the local boundedness with tail as follows. The proof can be seen in [36, Theorem 1.1].
Theorem 4.4 (Local boundedness with tail for subsolutions) Let $2 \leq p < \infty$, $s \in (0, 1)$ and let $u$ be a possibly sign-changing weak subsolution to (1.1) in the sense of Definition 1. Then, for every forward space-time cylinder $Q_\tau^+(z_0) \subseteq \Omega_T$ and every $\delta \in (0, 1)$ there exists a constant $c \equiv c(n, s, p, c_0, c_1, \Lambda)$ such that

$$
\sup_{Q_\tau^+(z_0)} u \leq \delta \text{Tail}_\infty \left( u_+, Q_\tau^+(z_0) \right) + c(1 + \delta^{(1-s)p} + \delta^{1-p}) \frac{p+n}{p+1} \left( \iint_{Q_\tau^+(z_0)} u_+^p \, dx \, dt \right)^{\frac{1}{p}},
$$

where $u_+ := \max\{u, 0\}$ denotes the positive part of $u$.

5 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1.

Proof We shall prove Theorem 1.1 step by step; the proof goes now in three different steps.

Step 1: Setting. For a supersolution $u$ to (1.1) satisfying $u > m \geq 0$ in $\mathbb{R}^n \times (0, T)$, we set

$$
\beta := \int_{B_\rho(x_0)} \chi^p(x) \log u(x, t_0) \, dx
$$

with the cut-off function appearing on Lemma 3.6, and define

$$
v^- := u e^{-\beta} \quad \text{and} \quad v^+ := u e^\beta.
$$

By Lemma 3.8, $v^-$ and $v^+$ are weak supersolution to (1.1) and subsolution to (1.1) replaced $A$ by $\tilde{A}$, respectively. Thus, applying Lemma 3.6 with $\sigma = \tau$, we infer that

$$
\left| Q_{\tau \rho}^-(z_0) \cap \{ \log v^- > \lambda \} \right| \leq \frac{C}{\lambda^{p-1}} \left| Q_{\tau \rho}^-(z_0) \right|
$$

and

$$
\left| Q_{\tau \rho}^+(z_0) \cap \{ \log v^+ > \lambda \} \right| \leq \frac{C}{\lambda^{p-1}} \left| Q_{\tau \rho}^+(z_0) \right|,
$$

(5.1)

where $\lambda > 0$ and $C = C(n, s, p, \tau)$.

Step 2: Bombieri-Giusti type estimate. Following the idea from [10, Theorem 1.4], we derive the Bombieri-Giusti type estimate as follows: given $\delta \in (0, 1)$, let us set

$$
\Phi(\tau) := \log \left( \iint_{Q_{\tau \rho}^-(z_0)} (v^+)^{q-} \, dx \, dt \right)^{\frac{1}{q}}, \quad \Psi(\tau) := \log \left( \iint_{Q_{\tau \rho}^+(z_0)} (v^-)^q \, dx \, dt \right)^{\frac{1}{q}}
$$

for every $\delta \leq \tau \leq \frac{1+\delta}{2}$, $q^+ > 0$ and $0 < \gamma < q^- < \frac{n+p}{n}(p-1)$. By Hölder’s inequality and (5.1)$_2$ with $\lambda = \Phi(\tau)/2 > 0$, we have, for $0 < \gamma < q$,

$$
\iint_{Q_{\tau \rho}^+(z_0)} (v^+)^{q^+} \, dx \, dt = \frac{1}{|Q_{\tau \rho}^+|} \left[ \iint_{Q_{\tau \rho}^+ \cap \{ v^+ \leq \exp(\Phi(\tau)/2) \}} (v^+)^\gamma \, dx \, dt + \iint_{Q_{\tau \rho}^+ \cap \{ v^+ > \exp(\Phi(\tau)/2) \}} (v^+)^\gamma \, dx \, dt \right]
$$
\[\leq \exp (\gamma \Phi(\tau)/2) + \left( \iiint_{Q_{\tau e}}^+(v^+)q^+ \, dx \, dt \right)^{\frac{1}{\sigma^+}} \left( \frac{|Q_{\tau e}^+ \cap \{v^+ > \exp(\Phi(\tau)/2)\}|}{|Q_{\tau e}^+|} \right)^{1 - \frac{1}{\sigma^+}}\]

\((5.1)\)

\[\leq \exp (\gamma \Phi(\tau)/2) + \exp (\gamma \Phi(\tau)) \left( \frac{C}{(\Phi(\tau)/2)^{p-1}} \right)^{1 - \frac{\gamma}{\sigma^+}}.\]

Choosing \(\Phi(\tau)\) enough large so that

\[0 < \log \left( \frac{(\Phi(\tau)/2)^{p-1}}{C} \right) \leq q^+ \Phi(\tau) \iff \exp(-q^+ \Phi) \leq \frac{C}{(\Phi(\tau)/2)^{p-1}} < 1,\]

which in turn implies

\[\iiint_{Q_{\tau e}}^+(v^+)q^+ \, dx \, dt \leq \exp (\gamma \Phi(\tau)/2) \left[ 1 + \exp(\gamma \Phi(\tau)/2) \left( \frac{C}{(\Phi(\tau)/2)^{p-1}} \right)^{1 - \frac{\gamma}{\sigma^+}} \right].\]

Since by selecting the exponent \(\gamma\) such that

\[\frac{2}{3} q^+ \geq \gamma = \frac{2}{3} \Phi(\tau)^{-1} \log \left( \frac{(\Phi(\tau)/2)^{p-1}}{C} \right) \iff \exp(\gamma \Phi(\tau)/2) = \left( \frac{C}{(\Phi(\tau)/2)^{p-1}} \right)^{\frac{1}{2} - \frac{\gamma}{\sigma^+}}, \quad \frac{2}{3} - \frac{\gamma}{q^+} \geq 0 \quad (5.2)\]

it follows that

\[\exp(\gamma \Phi(\tau)/2) \left( \frac{C}{(\Phi(\tau)/2)^{p-1}} \right)^{1 - \frac{\gamma}{\sigma^+}} = \left( \frac{C}{(\Phi(\tau)/2)^{p-1}} \right)^{\frac{2}{3} - \frac{\gamma}{\sigma^+}} \leq 1,\]

we therefore gain

\[\iiint_{Q_{\tau e}}^+(v^+)q^+ \, dx \, dt \leq 2 \exp(\gamma \Phi(\tau)/2)\]

with \(\gamma\) being as in (5.2).

**Step 3: Conclusion.** A careful re-reading of the proof of Lemma 4.3 shows that the lower and upper bounds for \(\sigma\) and \(\tau\) can be replaced by \(\delta\) and \(1 + \delta\), respectively. Now we apply Lemma 4.3 thereby obtaining, whenever \(\delta \leq \sigma < \tau \leq \frac{1+\delta}{2}\):

\[\exp \Phi(\sigma) = \left( \iiint_{Q_{\sigma e}}^+(v^+)q^+ \, dx \, dt \right)^{\frac{1}{q^+}} \leq \sup_{Q_{\sigma e}^+} v^+ + c \left( \sigma^{-(1-s)p} \frac{1 + \sigma^{(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)^{\frac{n+p}{p}} \left( \iiint_{Q_{\tau e}^+}^+(v^+)q^+ \, dx \, dt \right)^{\frac{1}{q^+}} \]

\[\leq c \delta^{-(1-s)p} \frac{1 + \delta^{(1-s)p}}{(\tau - \delta)^{n+sp}} \left( \iiint_{Q_{\tau e}^+}^+(v^+)q^+ \, dx \, dt \right)^{\frac{n+p}{p}} (2 \exp(\gamma \Phi(\tau)/2))^{\frac{1}{q}},\]

which together with the fact that \(q \leq 1\) and (5.2) yields that

\[\Phi(\sigma) \leq \frac{1}{\gamma} \left( c \delta^{-(1-s)p} \frac{1 + \delta^{(1-s)p}}{(\tau - \delta)^{n+sp}} \right)^{\frac{n+p}{p}} + \frac{\Phi(\tau)}{2}.\]
\[
\Phi(\tau) = \frac{\Phi(\tau)}{2} \left[ 1 + \frac{3(n + p)}{p} \frac{\log \left( \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)}{\log \left( \frac{(\Phi(\tau)/2)^{p-1}}{C} \right)} \right].
\]

We further select a suitably large \(\Phi(\tau)\) such that

\[
\frac{3(n + p)}{p} \frac{\log \left( \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)}{\log \left( \frac{(\Phi(\tau)/2)^{p-1}}{C} \right)} \leq \frac{1}{2} \iff \Phi(\tau) \geq 2 \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \left( \frac{6(n+p)}{p(p-1)} \right)^{\frac{1}{C}}
\]

and therefore, we gain, whenever \(\delta \leq \sigma < \tau \leq 1 + \frac{2}{\delta}\)

\[
\Phi(\sigma) \leq \frac{3}{4} \Phi(\tau)
\]

provided \(\Phi(\tau) \geq \overline{C}_0 \left( \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)^{\frac{6(n+p)}{p(p-1)}} \).

In the opposite case \(\Phi(\tau) < \overline{C}_0 \left( \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)^{\frac{6(n+p)}{p(p-1)}} \), we infer that

\[
\psi(\sigma) = (\Phi(\sigma) - \Phi(\tau)) + \Phi(\tau) = \frac{1}{q^+} \log \left( \frac{\int \int_{Q_{\tau_0}^+} (v^+)\,q^+\,dxdt}{\int \int_{Q_{\tau_0}^+} (v^+)\,q^+\,dxdt} \right) + \Phi(\tau)
\]

\[
\leq \frac{1}{q^+} \log \left( \frac{1}{|Q_{\tau_0}^+|} \int \int_{Q_{\tau_0}^+} (v^+)\,q^+\,dxdt \right) + \Phi(\tau)
\]

\[
= \frac{n + p}{q^+} \log \left( \frac{T}{\sigma} \right) + \Phi(\tau)
\]

\[
\leq \frac{n+p}{q^+} \log \left( \frac{1 + \delta}{2\delta} \right) + \overline{C}_0 \left( \frac{c\delta^{-(1-s)p}}{(\tau - \sigma)^{n+sp}} \right)^{\frac{6(n+p)}{p(p-1)}}
\]

holds whenever \(\delta \leq \sigma < \tau \leq 1 + \frac{2}{\delta}\). Bearing in mind these observations, we conclude that

\[
\Phi(\sigma) \leq \frac{3}{4} \Phi(\tau) + \frac{A}{(\tau - \sigma)^{n+sp}} \left( \frac{6(n+p)}{p(p-1)} \right)^{\frac{1}{C}} + \frac{n+p}{q^+} \log \left( \frac{1 + \delta}{2\delta} \right)
\]

with \(A = A(n, s, p, c_0, c_1, \Lambda, \delta)\). As a consequence, applying Lemma 2.7 with \(Z(\sigma) = \Phi(\sigma)\), we have

\[
\left( \int \int_{Q_{kq}^+} (v^+)\,q^+\,dxdt \right)^{\frac{1}{q^+}}
\]
\[ = \Phi(\delta) \leq C(n, s, p, c_0, c_1, \Lambda, \delta) \left[ \frac{1}{(1 - \delta)^{(n+sp)/p(p-1)}} + \frac{1}{q^+} \log \left( \frac{1 + \delta}{2\delta} \right) \right] \]

Since the exponent \( q^+ \) is arbitrary, we are allowed to pass to the limit \( q^+ \to \infty \) in the above display, and this finally yields that

\[
\sup_{Q_x^\tau} v^+ \leq C(n, s, p, c_0, c_1, \Lambda, \delta). \tag{5.3}
\]

Similarly, for term \( \Psi(\tau) \) we estimate that

\[
\iint_{Q_{\tau_0}^-} (v^-)^\gamma \, dx \, dt \leq 2 \exp(\gamma \Psi(\tau)/2)
\]

with suitable choice \( \gamma \) and large enough \( \Psi \) in the following:

\[
\frac{2}{3} q^- \geq \gamma = \frac{2}{3} \Psi(\tau)^{-1} \log \left( \frac{\Psi(\tau)/2^{p-1}}{C} \right) \quad \text{and} \quad 0 < \log \left( \frac{\Psi(\tau)/2^{p-1}}{C} \right) \leq q^- \Psi(\tau).
\]

Noticing that Lemma 3.2 is valid for every \( 0 < \gamma < q^- < \frac{n+p}{n}(p-1) \) and repeating the above proof, we gain, for \( 0 < q^- < \frac{n+p}{n}(p-1) \),

\[
\Psi(\sigma) \leq \frac{3}{4} \Psi(\tau) + \frac{B}{(\tau - \sigma)^{(n+sp)/p(p-1)}} + \frac{n + p}{q^-} \log \left( \frac{1 + \delta}{2\delta} \right)
\]

with \( B = B(n, s, p, c_0, c_1, \Lambda, \delta) \). Thus, Lemma 2.7 implies that

\[
\left( \iint_{Q_{\tau_0}^-} (v^-)^q \, dx \, dt \right)^{\frac{1}{q^-}} = \Psi(\delta) \leq B(n, s, p, c_0, c_1, \Lambda, \delta) \left[ \frac{1}{(1 - \delta)^{(n+sp)/p(p-1)}} + \frac{1}{q^-} \log \left( \frac{1 + \delta}{2\delta} \right) \right]
\]

\[
=: D(n, s, p, c_0, c_1, \Lambda, \delta, q^-) \tag{5.4}
\]

From definition of \( v^\pm \), (5.3) and (5.4), the assertion finally follows. \( \square \)

### 6 Proof of Theorem 1.2

In this final section we report the proof of Theorem 1.2.

**Proof** Let \( 0 < \rho \leq 1 \). Given \( \sigma \in (0, 1) \), from Theorem 1.1 with \( \delta = \frac{1 + \sigma}{2} \) it readily follows that

\[
\left( \iint_{Q_{1 - \rho\sigma}^\tau(z_0)} u^q \, dx \, dt \right)^{\frac{1}{q}} \leq C \inf_{Q_{1 - \rho\sigma}^\tau(z_0)} u.
\]

Since \( u \) is a weak subsolution to (1.1), Proposition 4.3 yields that

\[
\sup_{Q_{\sigma\rho}(z_0)} u \leq \frac{c}{(1 - \sigma)^{(n+sp)/pq}} \left( \iint_{Q_{1 - \rho\sigma}^\tau(z_0)} u^q \, dx \, dt \right)^{\frac{1}{q}}.
\]
which together with the fact that \( \inf_{Q_{\alpha+\rho}^+ (z_0)} u \leq \inf_{Q_{\rho}^+ (z_0)} u \) concludes the proof. \( \square \)

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**Appendix A: Proof of Lemma 3.1**

**Proof of Lemma 3.1** Although the argument used here is almost same as Proposition 3.4 and Lemma 3.8 (also [36, Proposition 3.1]), for the sake of completeness we shall nevertheless give the full proof. For this, we divide it into four different steps.

**Step 1: Beginning.** Let \( t_1 \in (t_0 - q^h_0, t_0) \) be an arbitrary number. Given \( \delta > 0 \) small, take the cutoff function \( \psi_\delta (t) \) with respect to time, defined by

\[
\psi_\delta (t) := \begin{cases} 
0, & t \in (t_0 - q^h_0, t_1), \\
\frac{1}{\delta} (t - t_1), & t \in [t_1, t_1 + \delta), \\
1, & t \in [t_1 + \delta, t_0). 
\end{cases}
\]

In the weak formulation (2.10), we switch the test function \( \varphi \) to \( \varphi^p \psi_\delta (t) u^{-\varepsilon} \) with nonnegative \( \varphi \in C_0^\infty (Q_{\rho}^+ (z_0)) \). Then the evolutional term is divided into two terms as follows:

\[
\int \int \Omega_T \partial_t [u^{p-2} u] h \varphi^p \psi_\delta (t) u^{-\varepsilon} \, dx \, dt = \int \int_{Q_{\rho}^+ (z_0)} \varphi^p \psi_\delta (t) \partial_t [u^{p-1}] h \left( u^{-\varepsilon} - [u^{p-1}]_{h}^{-\frac{p-1}{p-1-\varepsilon}} \right) \, dx \, dt \\
+ \int \int_{Q_{\rho}^+ (z_0)} \varphi^p \psi_\delta (t) \partial_t [u^{p-1}] h [u^{p-1}]_{h}^{-\frac{p-1}{p-1-\varepsilon}} \, dx \, dt 
\]

:= I_1 + I_2,

where the definition of \( I_1 \) and \( I_2 \) are obvious form the context. Abbreviating

\[
f_h := [u^{p-1}]_{h}^{-\frac{1}{p-1-\varepsilon}} \iff [u^{p-1}]_{h} = f_h^{p-1},
\]

and Lemma 2.8-(ii) and (2.5) with \( \alpha = 1 + \frac{p-1}{\varepsilon} \) in Lemma 2.3 imply that

\[
I_1 = \int \int_{Q_{\rho}^+ (z_0)} \varphi^p \psi_\delta (t) \frac{u^{p-1} - [u^{p-1}]_{h}}{h} \cdot \frac{[u^{p-1}]_{h}^{\frac{1}{p-1}} - u^{\varepsilon}}{[u^{p-1}]_{h}^{\frac{1}{p-1-\varepsilon}} u^{\varepsilon}} \, dx \, dt \\
= \int \int_{Q_{\rho}^+ (z_0)} \varphi^p \psi_\delta (t) \frac{u^{p-1} - f_h^{p-1}}{h} \cdot \frac{f_h^{\varepsilon} - u^{\varepsilon}}{f_h^{\varepsilon} u^{\varepsilon}} \, dx \, dt \leq 0.
\]

From integration by parts, it follows that

\[
I_2 = \frac{p - 1}{p - 1 - \varepsilon} \int \int_{Q_{\rho}^+ (z_0)} \varphi^p \psi_\delta (t) \partial_t [u^{p-1}]_{h}^{\frac{p-1-\varepsilon}{p-1}} \, dx \, dt \\
= - \frac{p - 1}{p - 1 - \varepsilon} \int \int_{Q_{\rho}^+ (z_0)} (p \varphi^{p-1} \psi_\delta + \varphi^p \psi_\delta') [u^{p-1}]_{h}^{\frac{p-1-\varepsilon}{p-1}} \, dx \, dt.
\]
The preceding estimates above and the fact that \( [u^{p-1}]_h^{1/p^T} \rightarrow u \) in \( L^p(\Omega_T) \) by Lemma 2.8-(iii) give that

\[
\limsup_{h \searrow 0} \int_{\Omega_T} \partial_t [u|^{p-2}u]_h \psi_p \psi_\delta(t) u^{-\epsilon} \ dx \ dt \\
\leq \limsup_{h \searrow 0} (I_1 + I_2) \\
\leq - \frac{p-1}{p-1-\epsilon} \int_{Q_\delta(t_0)} (p \psi_p \partial_t \psi_\delta + \psi_p \psi_\delta') u^{p-1-\epsilon} \ dx \ dt \\
\leq - \frac{p-1}{p-1-\epsilon} \int_{t_1}^{t_1+\delta} \int_{B_\delta(x_0)} u^{p-1-\epsilon} \varphi_p \ dx \ dt \\
+ \frac{p(p-1)}{p-1-\epsilon} \int_{Q_{\delta}(t_0)} \psi_p \partial_t \varphi_p |u|^{p-1-\epsilon} \psi_\delta \ dx \ dt. \quad (A.1)
\]

The spatial term can be dealt with comparatively easily. Indeed, by the use of convergence \([Du]^{p-2}Du_h \rightarrow [Du]^{p-2}Du\) in \( L^{p-1}(\Omega_T) \) as \( h \searrow 0 \) via Lemma 2.8-(ii) and Young’s inequality with the exponents \( \left( \frac{p}{p-1}, p \right) \) yields that, for any \( \theta > 0 \),

\[
\lim_{h \searrow 0} \int_{\Omega_T} [Du]^{p-2}Du_h \cdot D \left( \varphi_p \psi_\delta(t) u^{-\epsilon} \right) \ dx \ dt \\
= \int_{\Omega_T} [Du]^{p-2}Du \cdot D \left( \varphi_p \psi_\delta(t) u^{-\epsilon} \right) \ dx \ dt \\
= \int_{\Omega_T} [Du]^{p-2}Du \cdot (p \varphi_p \partial_t \varphi_p u^{-\epsilon} + \varphi_p Du^{-\epsilon}) \ dx \ dt \\
\leq p \int_{Q_{\delta}(t_0)} [Du]^{p-1} u^{-\epsilon} \partial_t \psi_\delta \ dx \ dt \\
\leq \left( (p-1)\theta - \epsilon \right) \int_{\Omega_T} |Du|^{p-1} u^{-\epsilon} \varphi_p \psi_\delta \ dx \ dt \\
+ c(\theta) \int_{Q_{\delta}(t_0)} |D\varphi|^{p} u^{-\epsilon} p + (\epsilon + 1)(p-1) \varphi_p \psi_\delta \ dx \ dt,
\]

which, by taking \( \theta = \frac{\epsilon}{2(p-1)} \), yields in particular that

\[
\lim_{h \searrow 0} \int_{\Omega_T} [Du]^{p-2}Du_h \cdot D \left( \varphi_p \psi_\delta(t) u^{-\epsilon} \right) \ dx \ dt \\
\leq - \frac{\epsilon}{2} \int_{\Omega_T} |Du|^{p-1} u^{-\epsilon} \varphi_p \psi_\delta \ dx + c(\epsilon) \int_{Q_{\delta}(t_0)} |D\varphi|^{p} u^{p-1-\epsilon} \psi_\delta \ dx \ dt, \quad (A.2)
\]

where we have computed that \( -\epsilon p + (\epsilon + 1)(p-1) = p - 1 - \epsilon \).

**Step 2: Fractional term.** Firstly, we will show that

\[
\int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} [U(x, y, t)K(x, y, t)]_h (\phi(x, t) - \phi(y, t)) \ dx \ dy \ dt \\
\rightarrow \int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t) (\phi(x, t) - \phi(y, t)) \ dx \ dy \ dt \quad (A.3)
\]
in the limit \( h \searrow 0 \), where we abbreviated \( \phi \equiv \varphi^{p} \psi_{\delta}(t)u^{-\varepsilon} \) for short. In order to prove (A.3) we consider the quantity

\[
(\Pi)_{h} := \left| \int_{0}^{T} \int_{\Omega} \left[ (U(x, y, t)K(x, y, t) - U(x, y, t)K(x, y, t)) \phi(x, t) - \phi(y, t) \right] dx dy \right|
\]

\[
\leq \int_{0}^{T} \int_{\Omega} \left| \phi(x, t) - \phi(y, t) \right| dx dy + 2 \int_{0}^{T} \int_{\Omega} \left| U(x, y, t) \left| x - y \right|^{{\frac{1}{p}}(n+sp)} \right| dx dy \]

\[
=: (\Pi_{1})_{h} + 2(\Pi_{2})_{h},
\]

with denoting \( D_{\Omega} := (\mathbb{R}^{n} \times \mathbb{R}^{n}) \setminus (\Omega^{c} \times \Omega^{c}) \), where the definition of \( (\Pi_{1})_{h} \) and \( (\Pi_{2})_{h} \) are clear from the context. We now claim the following: As \( h \searrow 0 \),

\[
\left[ U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right]_{h} \longrightarrow U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \text{ in } L^{{\frac{p}{p-1}}} (\Omega_{T}^{2}),
\]

where \( \Omega_{T}^{2} := \Omega \times \Omega \times (0, T) \). Indeed, since

\[
\left\| U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right\|^{\frac{p}{p-1}}_{L^{{\frac{p}{p-1}}} (\Omega_{T}^{2})} < \infty
\]

Lemma 2.8-(i) with \( E = \Omega^{2} \) implies (A.4). Therefore, using (A.4), Hölder’s inequality and Lemma 2.6, we have

\[
(\Pi_{1})_{h} = \int_{0}^{T} \int_{\Omega} \left[ U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right]_{h}
\]

\[
- U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \times \frac{\left| \phi(x, t) - \phi(y, t) \right|}{|x - y|^{-{\frac{1}{p}}(n+sp)}} dx dy
\]

\[
\leq c \left( \int_{0}^{T} \int_{\Omega} \left[ UK(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right]_{h}
\]

\[
- U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right|^{\frac{p}{p-1}} dx dy \right)^{\frac{p-1}{p}}
\]

\[
\times \left( \int_{\Omega} |D\phi(x)|dx \right)^{\frac{1}{p}}
\]

\[
\rightarrow 0
\]

as \( h \searrow 0 \), where we used the shorthand notation \( UK(x, y, t) = U(x, y, t)K(x, y, t) \).

We next prove that \( (\Pi_{1})_{h} \longrightarrow 0 \). Take a ball \( B_{R} = B_{R}(0) \) satisfying \( B_{R} \supset \Omega \). By \( \phi(y, t) = 0 \) for any \( (y, t) \in (B_{R} \setminus \Omega) \times (0, T) \) and Hölder’s inequality, we obtain

\[
(\Pi_{2})_{h} (R) := \left| \int_{0}^{T} \int_{\Omega \setminus (B_{R \setminus \Omega})} \left[ U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \right]_{h}
\]

\[
- U(x, y, t)K(x, y, t)|x - y|^{-{\frac{1}{p}}(n+sp)} \times \frac{\left| \phi(x, t) \right|}{|x - y|^{-{\frac{1}{p}}(n+sp)}} dx dy
\]

\[
\leq \left( \int_{0}^{T} \int_{\Omega \setminus (B_{R \setminus \Omega})} \left| \phi(x, t) \right|^{p} \frac{|x - y|^{n+sp}}{dx dy} \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_0^T \int_{\Omega \times (B_R \setminus \Omega)} \left[ U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \right]_h \right.
\]
\[
- U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \left[ \frac{\rho}{p-1} \right]_h dxdydt \right)^{\frac{p-1}{p}} \leq (\text{III})^{\frac{1}{p}} (\text{IV})^{\frac{p-1}{p}},
\]

where

\[
\text{III} := \int_0^T \int_{\Omega \times (B_R \setminus \Omega)} \frac{\rho}{p-1} |x - y|^{n+sp} dxdydt
\]

and

\[
\text{IV} := \int_0^T \int_{\Omega \times (B_R \setminus \Omega)} \left[ U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \right]_h \]
\[
- U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \left[ \frac{\rho}{p-1} \right]_h dxdydt.
\]

Since

\[|x - y| \geq d_R := \text{dist} (\text{supp } \phi(\cdot, t), B_R \setminus \Omega) \quad \forall (x, y) \in \text{supp } \phi(\cdot, t) \times (B_R \setminus \Omega)\]

and \(\text{supp } \phi = Q^{-}_0(z_0)\), we estimate

\[
\text{III} = \int_{Q^{-}_0(z_0)} \left( \int_{B_R \setminus \Omega} \frac{dy}{|x - y|^{n+sp}} \right) |\phi(x, t)| dxdt
\]
\[
\leq m^{-\varepsilon} \|\phi\|_{L^\infty(Q^{-}_0(z_0))} |Q^{-}_0(z_0)| \left( \int_{\mathbb{R}^n \setminus B_d(x)} \frac{dy}{|x - y|^{n+sp}} \right)
\]
\[
= m^{-\varepsilon} \|\phi\|_{L^\infty(Q^{-}_0(z_0))} |Q^{-}_0(z_0)| \cdot \frac{c(n)}{d^{-sp}_R}.
\]

Next, we estimate the integrand of IV:

\[
\left[ U_K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right]_h - U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \left[ \frac{\rho}{p-1} \right]_h dxdydt
\]
\[
\leq c(p) \left( \left[ U_K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right]_h \right)^{\frac{p}{p-1}} + U_K(x, y, t) |x - y|^{\frac{1}{p}(n+sp)} \left[ \frac{\rho}{p-1} \right]_h.
\]

By the use of Hölder’s inequality and the assumption (2.3), we infer that

\[
\left[ U_K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right]_h = \left[ \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} U_K(x, y, s)|x - y|^{\frac{1}{p}(n+sp)} ds \right]
\]
\[
\leq \Lambda \left( \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} ds \right)^{\frac{1}{p}} \left( \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} \frac{|U_K(x, y, s)|}{|x - y|^{\frac{p-1}{p}(n+sp)}} ds \right)^{\frac{p-1}{p}}
\]

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\[ \begin{align*}
&= \Lambda(1 - e^{-\frac{t}{\delta}})^{\frac{1}{p}} \left[ \frac{|U(x, y, t)|^{\frac{p}{p-1}}}{|x - y|^{n+sp}} \right]_h^{p-1}
\end{align*} \]
that is,
\[ \left[ U K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right]_h^{p-1} \leq \Lambda^{\frac{p}{p-1}} (1 - e^{-\frac{t}{\delta}})^{\frac{1}{p}} \left[ \frac{|U(x, y, t)|^{\frac{p}{p-1}}}{|x - y|^{n+sp}} \right]_h. \]
Similarly, we have
\[ \left| U K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right|^{p-1} \leq \Lambda^{\frac{p}{p-1}} \frac{|U(x, y, t)|^{p}}{|x - y|^{n+sp}}. \]
Collecting the preceding estimates above, the integrand of IV is estimated as
\[ \left[ U K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \right]_h - U K(x, y, t)|x - y|^{\frac{1}{p}(n+sp)} \]
\[ \leq c \Lambda^{\frac{p}{p-1}} \left( \left[ \frac{|U(x, y, t)|^{\frac{p}{p-1}}}{|x - y|^{n+sp}} \right]_h + \left[ \frac{|U(x, y, t)|^{\frac{p}{p-1}}}{|x - y|^{n+sp}} \right]_h \right) \]
\[ = c \Lambda^{\frac{p}{p-1}} \left( \left[ \frac{|u(x, t) - u(y, t)|^{p}}{|x - y|^{n+sp}} \right]_h + \left[ \frac{|u(x, t) - u(y, t)|^{p}}{|x - y|^{n+sp}} \right]_h \right) \]
\[ =: c \Lambda^{\frac{p}{p-1}} W_h(x, y, t). \]
Since \( W_h \in L^1(\Omega \times (B_R \setminus \Omega) \times (0, T)) \) holds true, Lemma 2.8-(i) applied with \( E = \Omega \times (B_R \setminus \Omega) \) yields that
\[ \lim_{h \downarrow 0} W_h \rightarrow W := 2 \left[ \frac{|u(x, t) - u(y, t)|^{p}}{|x - y|^{n+sp}} \right] \in L^1(\Omega \times (B_R \setminus \Omega) \times (0, T)) \]
as \( h \downarrow 0 \). Therefore using the dominated convergence theorem, we conclude that
\[ \lim_{h \downarrow 0} \text{IV} = 0. \quad (A.7) \]
Merging (A.6) with (A.7) in (A.5) and, subsequently, sending \( R \uparrow \infty \), we finally arrive at the resulting convergence (A.3).

Step 3: Taking the limit as \( h \downarrow 0 \) and \( \delta \downarrow 0 \). Since by \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_0 - \varphi^p, t_0) \), we have
\[ \lim_{h \downarrow 0} \int_{\Omega} |u|^{p-2}u(0) \left( \frac{1}{R} \int_{t_0}^{T} e^{-\frac{t}{\delta}} \phi(x, s) \, ds \right) \, dx = 0 \]
with \( \phi = \varphi^p \psi_\delta(t)u^{-\varepsilon} \). Therefore, combining this with the observations (A.1)–(A.3) and passing to the limit as \( h \downarrow 0 \) in the weak formulation (2.10) with the testing function \( \varphi^p \psi_\delta(t)u^{-\varepsilon} \), we gain
\[ \frac{p-1}{p-1-\varepsilon} \int_{t_0}^{t_1+\delta} \int_{B} u^{p-1-\varepsilon} \varphi \, dx \, dt + \frac{\varepsilon}{2} \int_{Q} |D^u|^p u^{-\varepsilon-1} \varphi^p \psi_\delta \, dx \, dt \]
\[ \leq \frac{p(p-1)}{p-1-\varepsilon} \int_{Q} \varphi_{|t|} u^{p-1-\varepsilon} \psi_\delta \, dx \, dt + c \int_{Q} |D\varphi|^p u^{p-1-\varepsilon} \psi_\delta \, dx \, dt \]
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\[ + \int_{t_1}^{t_0} \int_{\mathbb{R}^n \times \mathbb{R}^n} U K(x, y, t) \left( \varphi^p(x, t) u(x, t)^{1-\varepsilon} - \varphi^p(y, t) u(y, t)^{1-\varepsilon} \right) \psi_\delta(t) \, dx \, dy \, dt. \]

Passing to the limit \( \delta \searrow 0 \) combined with Lebegue’s differential theorem and the dominated convergence theorem implies that

\[
\int_{B_\varepsilon(x_0) \times \{t_1\}} u^{p-1-\varepsilon} \varphi^p \, dx + \int_{B_\varepsilon(x_0)} |Du|^p u^{-\varepsilon-1} \varphi^p \, dx \, dt \leq \left( p + \frac{2p(p-1)}{\varepsilon(p-1-\varepsilon)} \right) \int_{t_1}^{t_0} \int_{B_\varepsilon(x_0)} \varphi^{p-1} |\varphi| u^{p-1-\varepsilon} \, dx \, dt + c \left( \frac{p-1-\varepsilon}{p-1} + \frac{2}{\varepsilon} \right) \int_{t_1}^{t_0} \int_{B_\varepsilon(x_0)} |D\varphi|^p u^{p-1-\varepsilon} \, dx \, dt + c \left( \frac{p-1-\varepsilon}{p-1} + \frac{2}{\varepsilon} \right) \int_{t_1}^{t_0} \int_{\mathbb{R}^n \times \mathbb{R}^n} U K(x, y, t) \left( \varphi^p(x, t) u(x, t)^{-\varepsilon} - \varphi^p(y, t) u(y, t)^{-\varepsilon} \right) \, dx \, dy \, dt.
\]

Since \( \frac{p-1-\varepsilon}{p-1} < 1 \) and \( \frac{p-1-\varepsilon}{p-1-\varepsilon} > 1 \) the constant

\[ C(\varepsilon, p) := c \left( p + \frac{2p(p-1)}{\varepsilon(p-1-\varepsilon)} \right) \]

bounds the all constants appearing on the right-hand side of (A.8), and we know that \( C(\varepsilon, p) \) blows up as \( \varepsilon \searrow 0 \) or \( \varepsilon \nearrow p - 1 \). In the first term on the left-hand side of (A.8) we take the supremum over \( t_1 \in (t_0 - \varrho^p, t_0) \), while in the others we let \( t_1 \searrow t_0 - \varrho^p \). This finally leads to

\[
\sup_{t_1 \in (t_0 - \varrho^p, t_0)} \int_{B_\varepsilon(x_0) \times \{t_1\}} u^{p-1-\varepsilon} \varphi^p \, dx + \int_{B_\varepsilon(x_0)} |Du|^p u^{-\varepsilon-1} \varphi^p \, dx \, dt \leq C \int_{Q_0^+(\varepsilon, x_0)} \varphi^{p-1} |\varphi| u^{p-1-\varepsilon} \, dx \, dt + C \int_{Q_0^+(\varepsilon, x_0)} |D\varphi|^p u^{p-1-\varepsilon} \, dx \, dt + C \int_{t_0 - \varrho^p}^{t_0} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t) \left( \varphi^p(x, t) u(x, t)^{-\varepsilon} - \varphi^p(y, t) u(y, t)^{-\varepsilon} \right) \, dx \, dy \, dt.
\]

**Step 4: Conclusion.** In this final step, we are ready to conclude the whole proof, again by estimating the fractional term appearing on the right-hand side of (A.9). For this, we estimate separately as follows:

\[
V := \int_{t_0 - \varrho^p}^{t_0} \int_{\mathbb{R}^n \times \mathbb{R}^n} U(x, y, t)K(x, y, t) \left( \varphi^p(x, t) u(x, t)^{-\varepsilon} - \varphi^p(y, t) u(y, t)^{-\varepsilon} \right) \, dx \, dy \, dt = \int_{t_0 - \varrho^p}^{t_0} \int_{B_\varepsilon(x_0) \times B_\varepsilon(x_0)} (\cdots) \, dx \, dy \, dt + 2 \int_{t_0 - \varrho^p}^{t_0} \int_{B_\varepsilon(x_0) \times (\mathbb{R}^n \setminus B_\varepsilon(x_0))} (\cdots) \, dx \, dy \, dt
\]

with the obvious meaning of \( V_1 \) and \( V_2 \). Applying Lemma 2.4 with \( a = u(y, t) \), \( \tau_1 = \varphi(y, t) \) and \( b = u(y, t) \), \( \tau_2 = \varphi(x, t) \), the integrand of \( V_1 \) is estimated as

\[ U(x, y, t)K(x, y, t) \left( \varphi^p(x, t) u(x, t)^{-\varepsilon} - \varphi^p(y, t) u(y, t)^{-\varepsilon} \right) \]
In this final appendix, we report the proof of Lemmata 3.5 and 3.9.

**Appendix B: Proof of Lemmata 3.5 and 3.9**

\[
\frac{-\Lambda c(p)\zeta(\varepsilon)}{|x-y|^{n+sp}} \left| \varphi(x,t)u(x,t)^{a} - \varphi(y,t)u(y,t)^{a} \right|^p \\
+ \lambda \left( \zeta(\varepsilon) + 1 + \varepsilon^{-(p-1)} \right) \left| \varphi(x,t) - \varphi(y,t) \right|^p \cdot \frac{u(x,t)^{a} + u(y,t)^{a}}{|x-y|^{n+sp}} 
\]

and therefore

\[
V_1 \leq -\Lambda c(p)\zeta(\varepsilon) \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times B_{\varrho}(x_0)} \left| \varphi(x,t)u(x,t)^{a} - \varphi(y,t)u(y,t)^{a} \right|^p \frac{dx dy dt}{|x-y|^{n+sp}} \\
+ \lambda \left( \zeta(\varepsilon) + 1 + \varepsilon^{-(p-1)} \right) \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times B_{\varrho}(x_0)} \left| \varphi(x,t) - \varphi(y,t) \right|^p \left( u(x,t)^{a} + u(y,t)^{a} \right) \frac{dx dy dt}{|x-y|^{n+sp}} 
\]

On the other hand, using the positivity of \( u \), the term \( V_2 \) is estimated as

\[
V_2 \leq \Lambda \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times (\mathbb{R}^n \setminus B_{\varrho}(x_0))} \frac{U(x,y,t)}{|x-y|^{n+sp}} |\varphi^p(x,t)u(x,t)^{-\varepsilon}| \, dx \, dy \, dt \\
= \Lambda \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times (\mathbb{R}^n \setminus B_{\varrho}(x_0)) \cap \{ u(x,t) \geq u(y,t) \}} (\cdots) \, dx \, dy \, dt \\
+ \Lambda \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times (\mathbb{R}^n \setminus B_{\varrho}(x_0)) \cap \{ u(x,t) < u(y,t) \}} (\cdots) \, dx \, dy \, dt \\
\leq \Lambda \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times (\mathbb{R}^n \setminus B_{\varrho}(x_0)) \cap \{ u(x,t) \geq u(y,t) \}} \frac{(u(x,t) - u(y,t))^{p-1}}{|x-y|^{n+sp}} \varphi^p(x,t)u(x,t)^{-\varepsilon} \, dx \, dy \, dt \\
\leq \Lambda \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times (\mathbb{R}^n \setminus B_{\varrho}(x_0)) \cap \{ u(x,t) \geq u(y,t) \}} \frac{u(x,t)^{p-1-\varepsilon}}{|x-y|^{n+sp}} \varphi^p(x,t) \, dx \, dy \, dt \\
\leq \Lambda \left( \sup_{x \in \text{supp} \varphi(\cdot,t)} \int_{\mathbb{R}^n \setminus B_{\varrho}(x_0)} \frac{dy}{|x-y|^{n+sp}} \right) \int \int_{Q_{\varrho}^c(x_0)} u(x,t)^{a} \varphi^p(x,t) \, dx \, dt. 
\]

Joining the preceding estimates for \( V_1 \) and \( V_2 \) we get

\[
V \leq -\Lambda c(p)\zeta(\varepsilon) \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times B_{\varrho}(x_0)} \left| \varphi(x,t)u(x,t)^{a} - \varphi(y,t)u(y,t)^{a} \right|^p \frac{dx dy dt}{|x-y|^{n+sp}} \\
+ \lambda \left( \zeta(\varepsilon) + 1 + \varepsilon^{-(p-1)} \right) \int_{t_0 - \varrho^p}^{t_0} \int \int_{B_{\varrho}(x_0) \times B_{\varrho}(x_0)} \left| \varphi(x,t) - \varphi(y,t) \right|^p \left( u(x,t)^{a} + u(y,t)^{a} \right) \frac{dx dy dt}{|x-y|^{n+sp}} \\
+ 2\lambda \left( \sup_{x \in \text{supp} \varphi(\cdot,t)} \int_{\mathbb{R}^n \setminus B_{\varrho}(x_0)} \frac{dy}{|x-y|^{n+sp}} \right) \int \int_{Q_{\varrho}^c(x_0)} u(x,t)^{a} \varphi^p(x,t) \, dx \, dt. 
\]

Inserting this estimate back to (A.9), we finally arrive at the desired estimate and therefore, the lemma is completely proved. \( \square \)

**Appendix B: Proof of Lemmata 3.5 and 3.9**

In this final appendix, we report the proof of Lemmata 3.5 and 3.9.
Proof of Lemma 3.5  
Since by \( u \geq m > 0 \) in \( \mathbb{R}^n \times (t_1, t_2) \) there holds that, for all \( t \in (t_1, t_2) \)
\[
[u^{p-1}]_h = \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} u(s)^{p-1} \, ds \geq m^{p-1} \left( \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} \, ds \right)
= m^{p-1}(1 - e^{-\frac{t}{h}}).
\]
(B.1)

This together with an elementary estimate \(| \log s | \leq \frac{|s-1|}{\min\{s, 1\}} \) for \( s > 0 \) implies that
\[
\left| \log[u^{p-1}]_h - \log u^{p-1} \right| = \left| \log \left( \frac{[u^{p-1}]_h}{u^{p-1}} \right) \right| \leq \frac{1}{\min\{[u^{p-1}]_h, u^{p-1}\}} \left[ [u^{p-1}]_h - u^{p-1} \right]
\leq \frac{1}{m^2(p-1)(1 - e^{-\frac{t}{h}})} \left[ [u^{p-1}]_h - u^{p-1} \right]
\]
for every \((x, t) \in \Omega_{t_1, t_2}\). Therefore, Lemma 2.8-(ii) concludes that
\[
\left\| \log[u^{p-1}]_h - \log u^{p-1} \right\|_{L^{\frac{p}{p-1}}(\Omega_{t_1, t_2})} \leq \frac{1}{m^2(p-1)(1 - e^{-\frac{t}{h}})} \left[ [u^{p-1}]_h - u^{p-1} \right] \rightarrow 0,
\]
as desired. \( \square \)

Proof of Lemma 3.9  
Again, by (B.1)
\[
[u^{p-1}]_h^{1} - u^{1-p} \leq \frac{1}{m^2(p-1)(1 - e^{-\frac{t}{h}})} \left[ [u^{p-1}]_h - u^{p-1} \right]
\]
holds whenever \((x, t) \in \Omega_{t_1, T}\). Hence
\[
\left[ [u^{p-1}]_h^{1} - u^{1-p} \right]_{L^{\frac{p}{p-1}}(\Omega_{t_1, T})} \leq \frac{1}{m^2(p-1)(1 - e^{-\frac{t}{h}})} \left[ [u^{p-1}]_h - u^{p-1} \right]_{L^{\frac{p}{p-1}}(\Omega_{t_1, T})} \rightarrow 0,
\]
finishing the proof. \( \square \)

References
1. Abdellaoui, B., Attar, A., Bentifour, R., Peral, I.: On fractional \( p \)-Laplacian parabolic problem with general data. Ann. Mat. Pura Appl. (4) 197(2), 329–356 (2018)
2. Acerbi, E., Fusco, N.: Regularity for minimizers of nonquadratic functionals: the case \( 1 < p < 2 \). J. Math. Anal. Appl. 140(1), 115–135 (1989)
3. Banerjee, A., Garain, P., Kinnunen, J.: Some local properties of subsolutions and supersolutions for a doubly nonlinear nonlocal parabolic \( p \)-Laplace equation. Ann. Mat. Pura Appl. 210(4), 1717–1751 (2022)
4. Bourgain, J., Brezis, H., Mironescu, P.: Limiting embedding theorems for \( W^{s,p} \) when \( s \uparrow 1 \) and applications. J. Anal. Math. 87, 77–101 (2002)
5. Bögelein, V., Duzaar, F., Marcellini, P.: Parabolic systems with \( p,q \)-growth: a variational approach. Arch. Ration. Mech. Anal. 210(1), 219–267 (2013)
6. Bögelein, V., Dietrich, N., Vestberg, M.: Existence of solutions to a diffusive shallow medium equation. J. Evol. Equ. 21(1), 845–889 (2021)
7. Bögelein, V., Duzaar, F., Korte, R., Scheven, C.: The higher integrability of weak solutions of porous medium systems. Adv. Nonlinear Anal. 8(1), 1004–1034 (2019)
8. Bögelein, V., Duzaar, F., Korte, R., Scheven, C.: Higher integrability for doubly nonlinear parabolic systems. J. Math. Pures Appl. 143, 31–72 (2020)
9. Bögelein, V., Duzaar, F., Liao, N.: On the Hölder regularity of signed solutions to a doubly nonlinear equation. J. Funct. Anal. 281(9), 109–173 (2021)
10. Bombieri, E., Giusti, E.: Harnack’s inequality for elliptic differential equations on minimal surfaces. Invent. Math. 15, 24–46 (1972)
11. Brasco, L., Lindgren, E.: Higher Sobolev regularity for the fractional $p$-Laplace equation in the superquadratic case. Adv. Math. 304, 300–354 (2017)
12. Brasco, L., Lindgren, E., Strömqvist, M.: Continuity of solutions to a nonlinear fractional diffusion equation. J. Evol. Equ. 21(4), 4319–4381 (2021)
13. Brasco, L., Parini, E.: The second eigenvalue of the fractional $p$ Laplacian. Adv. Calc. Var. 9(4), 323–355 (2016)
14. Bucher, S., da Silva, J.V., de Miranda, L.H.: A System of Local / Nonlocal $p$-Laplacians: The Eigenvalue Problem and Its Asymptotic Limit as $p \to \infty$, arXiv:2001.05985, (2020)
15. Cozzi, M.: Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes. J. Funct. Anal. 272(11), 4762–4837 (2017)
16. Di Castro, A., Kuusi, T., Palatucci, G.: Nonlocal Harnack inequalities. J. Evol. Equ. 21(4), 4319–4381 (2021)
17. Di Castro, A., Kuusi, T., Palatucci, G.: Local behavior of fractional $p$-minimizers. Ann. Inst. H. Poincaré Anal. Non Linéaire 33(5), 1279–1299 (2016)
18. DiBenedetto, E.: Degenerate Parabolic Equations. Universitext, Springer-Verlag, New York (1993)
19. Ding, M., Zhang, C., Zhou, S.: Local boundedness and Hölder continuity for the parabolic fractional $p$-Laplace equations. Calc. Var. Partial Differ. Equ. 60, 38 (2021)
20. Dipierro, S., Lippi, E.P., Valdinoci, E.: (Non)local logistic equations with Neumann conditions, arXiv:2101.02315, (2021)
21. Fang, Y., Shang, B., Zhang, C.: Regularity theory for mixed local and nonlocal parabolic $p$-Laplace equations. J. Geom. Anal. 32(1), 1–33 (2022)
22. Garain, P., Kinnunen, J.: On the regularity theory for mixed local and nonlocal quasilinear parabolic $p$-Laplace equations, arXiv:2108.02986, (2021)
23. Garain, P., Kinnunen, J.: Weak Harnack inequality for a mixed local and nonlocal parabolic equation, arXiv:2105.15016, (2021)
24. Gianazza, U., Vespri, V.: A Harnack inequality for solutions of doubly nonlinear parabolic equations. J. Appl. Funct. Anal. 1(3), 271–284 (2006)
25. Giaquinta, M., Giusti, E.: On the regularity of the minima of variational integrals. Acta Math. 148, 31–46 (1982)
26. Giaquinta, M., Modica, G.: Remarks on the regularity of the minimizers of certain degenerate functionals. Manuscr. Math. 57(1), 55–99 (1986)
27. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific Publishing Company, Tuck Link, Singapore (2003)
28. Kim, Y.C.: Nonlocal Harnack inequalities for nonlocal heat equations. J. Differ. Equ. 267, 6691–6757 (2019)
29. Kinnunen, J., Kuusi, T.: Local behavior of solutions to doubly nonlinear parabolic equations. Math. Ann. 337(3), 705–728 (2007)
30. Kinnunen, J., Lindqvist, P.: Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. Ann. Mat. Pura Appl. (4) 185(3), 411–435 (2006)
31. Kassmann, M., Schwab, R.W.: Regularity results for nonlocal parabolic equations. Riv. Math. Univ. Parma (N.S.) 5(1), 183–212 (2014)
32. Kuusi, T., Misawa, M., Nakamura, K.: Regularity estimates for the $p$-Sobolev flow. J. Geom. Anal. 30, 1918–1964 (2020)
33. Kuusi, T., Misawa, M., Nakamura, K.: Global existence for the $p$-Sobolev flow. J. Differ. Equ. 279, 245–281 (2021)
34. Kuusi, T., Palatucci, G. (eds.): Recent Developments in Nonlocal Theory. De Gruyter, Berlin/Boston (2018)
35. Mazón, J.M., Rossi, J.D., Toledo, J.: Fractional $p$-Laplacian evolution equations. J. Math. Pures Appl. (9) 105(6), 810–844 (2016)
36. Nakamura, K.: Local Boundedness of a mixed local-nonlocal doubly nonlinear equation. J. Evol. Equ. 22(3), 75 (2022)
37. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker’s guide to the fractional Sobolev space. Bull. Sci. Math. 136(5), 521–573 (2012)
38. Puhst, D.: On the evolutionary fractional $p$-Laplacian. Appl. Math. Res. Express AMRX 2, 253–273 (2015)
39. Sturm, S.: Existence of weak solutions of doubly nonlinear parabolic equations. J. Math. Anal. Appl. 455(1), 842–863 (2017)
40. Strömqvist, M.: Local boundedness of solutions to non-local parabolic equations modeled on the fractional $p$-Laplacian. J. Differ. Equ. 266(12), 7948–7979 (2019)
41. Strömqvist, M.: Harnack’s inequality for parabolic nonlocal equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 36(6), 1709–1745 (2019)
42. Vázquez, J.L.: The Dirichlet problem for the fractional $p$-Laplacian evolution equation. J. Differ. Equ. 260(7), 6038–6056 (2016)
43. Trudinger, N.S.: Pointwise estimates and quasilinear parabolic equations. Commun. Pure Appl. Math. 21, 205–226 (1968)

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