THE PERTURBED PHOTOGRAVITATIONAL RESTRICTED THREE–BODY PROBLEM: 
ANALYSIS OF RESONANT PERIODIC ORBITS

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ABSTRACT. In the framework of the perturbed photo–gravitational restricted three–body problem, the first order exterior resonant orbits and the first, third and fifth order interior resonant periodic orbits are analyzed. The location, eccentricity and period of the first order exterior and interior resonant orbits are investigated in the unperturbed and perturbed cases for a specified value of Jacobi constant \( C \).

It is observed that as the number of loops increases successively from one loop to five loops, the period of infinitesimal body increases in such a way that the successive difference of periods is either 6 or 7 units. It is further observed that for the exterior resonance, as the number of loops increases, the location of the periodic orbit moves towards the Sun whereas for the interior resonance as the number of loops increases, location of the periodic orbit moves away from the Sun. Thereby we demonstrate that the location of resonant orbits of the given order moves away from the Sun when perturbation is included.

The evolution of interior first order resonant orbit with three loops is studied for different values of Jacobi constant \( C \). It is observed that when the value of \( C \) increases, the size of the loop decreases and degenerates finally into a circle, the eccentricity of periodic orbit decreases and location of the periodic orbit moves towards the second primary body.

1. Introduction. The model of the restricted three-body problem plays a significant role in constructing the motion of artificial satellites. This model can be also used to describe the motion of the planets, the minor planets and the comets. The

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restricted problem gives a precise photography not only regarding the motion of the Moon but also the motion of other natural satellites. In addition, the applications of the restricted three–body problem is not limited to celestial mechanics, but also in theoretical physics, quantum mechanics and mathematics. It can be used to describe the motion of an infinitesimal body not only under the mutual gravitational field but also by radiation pressure from one (or both) of the primaries, in this case the problem is called the photogravitational restricted problem, which has a special significance in the investigations of dynamics of solar systems.

From a physical point of view, it is unbelievable to take into account the celestial objects as point masses without physical dimensions. This is a clash for the realistic cases of the celestial objects. Further the objects will suffer from deformation in its shapes at poles regard to the effect of rotation. For this reason, the oblate spheroid bodies are a good approximation for the most celestial objects. In this situation, there are some contributions constructed by [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [25] and [37]. In addition the existence of the equilibrium points and their stability in the restricted three-body problem by considering the effect of the zonal harmonics parameters with respect to both primaries are studied by [12].

The aforementioned reasons push us to study the resonance phenomena, in the framework of photogravitational restricted three–body problem, by taking the first primary is radiating and the second primary is oblate spheroid. The importance of the phenomena of resonance in the space dynamics was studied by [36, 28]. They established that among the solar system, the occurrence of commensurability between the pairs of mean motions is more frequent than in a chance of distribution. The existence of a mean motion resonance between a pair of objects can lead to a repeating geometrical configuration of the orbits which guarantees stability even if the resonance is not exact, since there is still the possibility of stable liberal motion around an equilibrium point.

It is important to have an understanding of the dynamics of resonance and to develop analytical models that precisely reflect the true nature of resonant interactions. Since the late twentieth century until today, enormous number of researches have enriched the study of restricted three–body problem.

But the effects of many perturbing forces has not been studied in many of such interesting problems, such as the non–sphericity of one or both primaries, radiation pressure, relativistic effect, Poynting–Robertson drag, etc. But the more effective forces are due to radiation pressure and oblateness. This is considered one of our motivation to study the effect of these perturbing forces on the exterior and interior resonant periodic orbits.

The study of resonance in the solar system has received wide attention in the recent past. A resonance can arise due to a simple numerical relationship between frequencies or periods. The periods involved can be the rotational period and orbital period of a single body or orbital periods of more than two bodies. The former is termed as spin-orbit coupling and the latter as orbit-orbit coupling. For instance, the Earth-Moon spin orbit is 1:1, while the Neptune-Pluto has a 3 : 2 orbit-orbit resonance [28]. A number of articles has emerged related to the study of resonance in the solar system [39, 14, 17] and resonance of orbits of planets outside solar system [19, 20, 21, 23].

[26] have studied the consequences higher order resonances 7 : 2 and 10 : 3 with Jupiter and resonances 4 : 7, 5 : 9, 7 : 2 with Mars. [27] have shown that some asteroid families are at higher order mean resonances 7:3 and 9:4. Outer asteroid
belt also shows higher resonances of order 8:5, 7:4, and 9:5 [22]. [15, 18] have shown that many of the Trans–Neptune objects (TNO) show higher order resonances of order 5:2, 7:3, 7:4, and 9:5. It was pointed out by [40], that higher order resonances 5:2, 7:4, 9:5 have some role in the formation of planetary systems. [13] have studied the stability of higher order resonances in the restricted three body problem in which the primaries are the Sun-Jupiter and the Sun-Neptune systems.

[16] investigated numerically by using the Poincaré surface of section the stability of evolution of family $f$ for the Earth–Moon system. They demonstrated that the resonances of third order are the main cause of the reduction of the stability region of retrograde satellites. They also computed many branches of family $f$ through the conguration of family features to determine the stability region.

In the frame Sitnikov problem, [33] studied the families of three–dimensional periodic orbits, which bifurcating from self-resonant orbits of the Sitnikov family at double, triple and quadruple period of the bifurcation orbit. They have shown that the branch families close upon themselves and remain 3D up to their terminations having two common members with the Sitnikov family. They also studied the evolution of some calculated families by varying the parameter of mass ratio. They investigated that these families are isolated and disappear gradually in 3D decreasing to point size.

Recently [29] studied two different families of periodic orbits using PSS for Sun–Saturn system, when Saturn is oblate spheroids, One is Sun centered and the other is Saturn centered. In addition $f$ family orbits for the same system and their stability are also studied under the perturbation of the solar radiation pressure by [30]. In this context, [31] and [32] analyzed periodic orbits around both primaries which can be used as a transfer trajectory for Sun–Mars and Sun–Earth system under the perturbation of oblateness. They also gave the prediction of periodic orbits using regression and the stability analysis. Furthermore, the interior first order resonance is analyzed in details by [35].

This paper aims to analyse periodic orbits of different orders of resonance, both interior and exterior, using Poincaré surface of section (PSS) in the photo gravitational restricted three–body problem for two systems, namely, the Sun–Earth and the Sun–Mars systems. The effect of resonance on the dynamical structure, location and period of orbits will be studied extensively.

This paper is organized as follows: In Section 2, the description of mathematical system and the equations of motion for the perturbed restricted three–body problem are stated, while the mathematical tools for solving these equations and the strategy of estimate resonant ratio from a surface section are discussed 3.

The periodic orbits corresponding to exterior first order resonance with number of loops varying from 1 to 5 have been studied in Section 4. In Section 5, we have investigated periodic orbits corresponding to interior first order resonance with number of loops varying from 2 to 8. Sections (6, 7) includes analysis of interior third order and fifth order resonant periodic orbits respectively and conclusions are drawn in Section 8.

2. Model description. By following [38], we shall recall the equations of motion of the infinitesimal body in the dimensionless synodic coordinates as

$$\ddot{x} - 2n\dot{y} = \Omega_x$$

$$\ddot{y} + 2n\dot{x} = \Omega_y$$

(1)
where $\Omega$ denotes the potential function, which is governed by

$$\Omega = \frac{n^2}{2} [(1 - \mu) r_1^2 + \mu r_2^2] + \frac{q(1 - \mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3}$$  \hspace{1cm} (2)$$

and

$$r_1^2 = (x + \mu)^2 + y^2$$
$$r_2^2 = (x - 1 + \mu)^2 + y^2$$

here $q$ is the radiation pressure (the reduction of mass parameter) given by

$$q = 1 - \frac{F_p}{F_g}$$

where $F_p$ and $F_g$ denote the solar radiation and the gravitational forces respectively.

The mean motion $n$ is given by,

$$n = \sqrt{1 + \frac{3}{2} A_2}$$  \hspace{1cm} (3)$$

where $A_2$ is oblateness coefficient and is ruled by

$$A_2 = \frac{R_e^2 - R_p^2}{5R^2}$$

Here, $R_e$ and $R_p$ are the equatorial and polar radii of the second primary and $R$ is the separation distance between primaries bodies. Further Eqs. 1 and 2 admit Jacobi integral in the following form

$$C = n^2 (x^2 + y^2) + \frac{2q(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + \frac{\mu A_2}{r_2^3} - \dot{x}^2 - \dot{y}^2$$ \hspace{1cm} (4)$$

Poincaré surface of section (PSS) is a qualitative approach of analysing dynamical system first introduced by [34]. This technique is very effective, so widely used for analysing periodic, quasi periodic orbits and chaotic orbits. [29] explained PSS in detailed for Sun-Saturn system. PSS containing large number of points known as *sea*. Region made by collection of points in the PSS are known as *island*. Isolated point in the PSS gives chaotic orbit. The point located at the center of the *island* gives periodic orbit whereas other points of the *island* gives quasi-periodic orbit. Size of the *island* in the PSS gives stability of the periodic orbit corresponding to point located at the center of the *island*. [28] explained analysis of periodic orbits using PSS. [24] employed multiple PSS method to find quasi-periodic orbits around the equilibrium points $L_1$ and $L_2$ in the Sun-Earth system. Hence the next section will be devoted to the the strategy of estimating resonant ratio from a surface section.

3. **Estimation of resonant ratio.** In order to obtain orbital elements of the test particle at any time it is necessary to know the position $(x, y)$ and velocity $(\dot{x}, \dot{y})$ of the particle, which corresponds to a point in the four-dimensional phase space. We have constructed surfaces of sections on the $X\dot{X} - \dot{X}$ plane by taking initial conditions along $OX$- axis by using intervals of length 0.001.

In this setting, the position vector $\underline{r}$ and the velocity $\underline{V}$ of the infinitesimal body is ruled by

$$\underline{r} = x\hat{i} + y\hat{j}$$
$$\underline{V} = \dot{x}\hat{i} + [\dot{y} + n(x + \mu)]\hat{j}$$ \hspace{1cm} (5)$$
where $\hat{i}$ and $\hat{j}$ are the unit vectors in the directions of the orthogonal axes of the rotating frame and $n$ is given by Eq. (2). Using Eq. 5 the magnitudes of the velocity $V$ and the angular momentum $h$ are

$$V = \sqrt{\dot{x}^2 + [\dot{y} + n(x + \mu)]^2}$$

$$h = (x + \mu) [\dot{y} + n(x + \mu)]$$

with the help of Eq. 6, one can calculate the semi-major axis $a$ and the eccentricity $e$ by the following formulæ

$$a = \left[ \frac{2}{r_1} - \frac{\dot{x}^2 + [\dot{y} + n(x + \mu)]^2}{1 - \mu} \right]^{-1}$$

$$e = \sqrt{1 - \frac{(x + \mu)^2 [\dot{y} + n(x + \mu)]^2}{a(1 - \mu)}}$$

and the period of planet’s orbit $T_P$ is obtained by the relation

$$(1 + \frac{3}{2} A_2) T_P = 2\pi$$

thereby we can use Kepler’s third law to get

$$\frac{T_1}{T_2} = \left( \frac{a_1}{a_2} \right)^{3/2}$$

So using semi–major axis of orbit of second primary body and semi-major axis of orbit of infinitesimal mass, we can obtain order of resonance, which can also be determined from the number of islands in PSS. This is considered as one of the characteristics of the resonance [28].

At the end of the current section, the coefficient of oblateness is taken as $A_2 = 0.0001$ for both systems. Therefore, the period of Earth’s orbit is $T_E \approx 6.282714$ and the period of Mars’s orbit is $T_M \approx 6.282714$. While the semi–major axis of Earth’s orbit is $a_E = 1.00000011$ and the semi–major axis of Mars’s orbit $a_M \approx 1.0003$.

Eqs. (4 - 9) are constructed in the framework of the perturbed photo–gravitational restricted three–body problem. These equations will agree with the corresponding equations of the classical case ($(q = 1$ and $A_0 = 0)$, see [28].

The locations of stable periodic orbits can be easily identified from the surface of section. Furthermore, these orbits can be evaluated by many techniques, such as diffraction corrected method, K-S transformation etc. But identification of unstable periodic orbits is more difficult. In this case we can calculate the trajectory of the infinitesimal body in the restricted three–body problem, even if the period is unknown. The technique of the determination of $p : q$ ratio for a potential resonant orbit from its location is investigated. This technique includes the use of Poincaré sections and two-body approximations as initial conditions.

Thereby from Eq. (9) we can obtain the ratio between periods by

$$\frac{p}{q} = \frac{T_q}{T_p}$$

The difference between $p$ and $q$ is equal to order of the resonance which can be determined using PSS of that orbits, as the number of islands visible in PSS indicate the difference value of $p$ and $q$. 

The values of \(a\), \(e\) and \(T_p\) are used to calculate the approximate resonant orbit. In addition the period \(p\) in the \(p : q\) ratio can be determined from the number of loops, thereby \(q\) can be evaluated from the value of \(p\) with the help of Eq. (10). Hence the values of location and velocity, which are obtained from Poincaré section and the ratio of \(p : q\) from the two-body approximation are used as the initial condition in the correction scheme to evaluate the desired resonant orbit in the perturbed restricted three–body problem related to Sun–Earth and the Sun–Mars systems. Generally the above investigations summarize the strategy on estimating resonant ratio from a surface section.

4. Exterior first order resonance. We have analyzed four families of periodic orbits for exterior first order resonance in the Sun-Earth and the Sun-Mars systems. Periodic orbit having exterior resonance are inner loops orbits. Family-I is unperturbed by radiation pressure and oblateness coefficient (i.e. \(q = 1\) and \(A_2 = 0\)). Perturbation due to oblateness alone is considered in Family-II (i.e. \(q = 1\) and \(A_2 = 0.0001\)). Family-III is characterized by perturbation due to radiation pressure only (i.e. \(q = 0.9845\) and \(A_2 = 0\)) and in Family-IV both radiation pressure and oblateness are included (i.e. \(q = 0.9845\) and \(A_2 = 0.0001\)). For simplicity in writing the head rows of Tables, numerical estimates for relevant quantities of family, solar radiation pressure, oblateness parameter, number of loops, location of the periodic orbit, number of islands, resonance order, eccentricity, time period of the orbit, ratio of the orbital periods and Jacobi constant will be denoted by \(FA, SR, OB, NL, LO, NI, RO, EC, TP, RP\) and \(JC\).

| Family | \(SR\) | \(OB\) | \(NL\) | \(LO\) | \(NI\) | \(RO\) | \(EC\) | \(TP\) | \(RP\) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| I      | 0.93904| 0      | 1      | 1:2    | 0.40895| 13     | 0.49936|        |
| II     | 0.88740| 1      | 2:3    | 0.32301| 14     | 0.66633|        |
| III    | 0.85623| 1      | 3:4    | 0.29337| 26     | 0.74943|        |
| IV     | 0.83547| 1      | 4:5    | 0.28015| 32     | 0.79977|        |

Since the period of Earth’s orbit \(T_E \approx 6.282714\) units, it can be noticed that as the number of loops increases successively from one-loop to five loops, the period of
the orbit of infinitesimal mass increases in such a way that the successive difference of periods differ either by 6 or 7 units as shown in Table 1. The period of one-loop orbit of Family I is 13, while that of two-loops orbit is 19 with a difference of 6 units.

Now the period of three loops orbit is 26 so that the difference in the period of two loops and three loops orbits is 7 units. Resonance of order 1 : 2 indicates that the time taken by Earth to orbit twice around the Sun is equal to the time taken by the infinitesimal mass to orbit once around the Sun. Also period of Mars's orbit $T_M \approx 6.282714$ units. So, in a similar manner it can be noticed that as the number of loops increases successively from one loop to five loops, the period of the orbit of infinitesimal mass increases in such a way that the successive difference of periods differ either by 6 or 7 units as shown in Table 2.

Table 2. Analysis of exterior first order resonance in the perturbed Sun-Mars system

| FA | SR | OB | NL | LO | NI | RO | EC | TP | RP |
|----|----|----|----|----|----|----|----|----|----|
| I  | 1  | 0  | 1  | 1  | 2  | 3  | 4  | 5  | 6  |
| 1  | 0.939000 | 1.2 | 0.40852 | 13 | 0.50015 |
| 2  | 0.887370 | 1.3 | 0.32284 | 19 | 0.66692 |
| 3  | 0.856190 | 1.4 | 0.29324 | 26 | 0.75031 |
| 4  | 0.837433 | 4.5 | 0.28006 | 32 | 0.80033 |
| 5  | 0.820715 | 5.6 | 0.27323 | 38 | 0.83368 |
| II | 1  | 0.0001 | 1  | 1  | 2  | 3  | 4  | 5  | 6  |
| 1  | 0.93875 | 1.2 | 0.40866 | 13 | 0.50017 |
| 2  | 0.88708 | 1.3 | 0.32302 | 19 | 0.66697 |
| 3  | 0.85588 | 1.4 | 0.29346 | 26 | 0.75037 |
| 4  | 0.83512 | 4.5 | 0.28029 | 32 | 0.80039 |
| 5  | 0.82040 | 5.6 | 0.27347 | 38 | 0.83374 |
| III | 0.9845 | 0 | 1  | 1  | 2  | 3  | 4  | 5  | 6  |
| 1  | 0.97891 | 1.2 | 0.35887 | 13 | 0.53027 |
| 2  | 0.93795 | 1.3 | 0.26069 | 19 | 0.70019 |
| 3  | 0.91204 | 1.4 | 0.22397 | 26 | 0.78521 |
| 4  | 0.89379 | 4.5 | 0.20662 | 32 | 0.83643 |
| 5  | 0.88069 | 5.6 | 0.19722 | 38 | 0.87067 |
| IV | 0.9845 | 0.0001 | 1  | 1  | 2  | 3  | 4  | 5  | 6  |
| 1  | 0.97861 | 1.2 | 0.35895 | 13 | 0.53041 |
| 2  | 0.93761 | 1.3 | 0.26090 | 19 | 0.70018 |
| 3  | 0.91167 | 1.4 | 0.22423 | 26 | 0.78529 |
| 4  | 0.89358 | 4.5 | 0.20691 | 32 | 0.83652 |
| 5  | 0.88029 | 5.6 | 0.19752 | 38 | 0.87076 |

The period of one loop orbit of Family I is 13, while that of two loops orbit is 19 with a difference of 6 units. Now the period of three loops orbit is 26 so that the difference in the period of two loops and three loops orbits is 7 units. Resonance of order 1 : 2 indicates that the time taken by Mars to orbit twice around the Sun is equal to the time taken by the infinitesimal mass to orbit once around the Sun. Also period of Mars's orbit $T_M \approx 6.282714$ units. So, in a similar manner it can be noticed that as the number of loops increases successively from one loop to five loops, the period of the orbit of infinitesimal mass increases in such a way that the successive difference of periods differ either by 6 or 7 units as shown in Table 2.

The PSS of periodic orbit with number of loops one and two are shown in Fig.1 and Fig.2.

Fig.1a depicts single loop orbit corresponding to $x_0 = 0.93904$ for the classical case with $q = 1$ and $A_2 = 0$, which corresponds to unperturbed motion. The inner single loop orbit has a period 13, and PSS at $x_0 = 0.93904$ gives single island as shown in Fig.1b. Using the characteristics of resonance for PSS, number of islands indicates order of the resonance. So the resulting orbit is of first order exterior
resonance, because the ratio of periods $T_1/T_2 \approx 0.49936$ which indicates that the resonance is 1 : 2 type.

In Fig.1c, we have shown a single loop periodic orbit with perturbation $q = 0.9845$ and $A_2 = 0.0001$ with period 13 corresponding to $x_0 = 0.97870$. The PSS at $x_0 = 0.97870$ gives single island as shown in Fig.1d. In this case, the value of $T_1/T_2 \approx 0.52779$ which indicates that the resonance is of the order 1 : 2. It can be noticed that the size of the single-loop orbit has reduced due to the effect of perturbations due to radiation and oblateness.

In Fig.2, we have displayed two loops periodic orbits with period 19 corresponding to $x_0 = 0.88740$ in the unperturbed case. Then the (The) eccentricity of the orbit is 0.32301. The PSS at $x_0 = 0.88740$ as shown in Fig.2b consists of a single island and hence the order of the resonance is one. Furthermore, $T_1/T_2 = 0.66633$ indicates that the ratio of resonance is 2 : 3. Fig.2c and Fig.2d respectively show two loops periodic orbits and the PSS corresponding to $x_0 = 0.93764$ in the perturbed case with $q = 0.9845$ and $A_2 = 0.0001$. The period is same as in the unperturbed case. The eccentricity of the orbit is 0.26136 and $T_1/T_2 = 0.69919$.

As in the single loop case, the size of the loops in the orbit and the eccentricity have reduced due to perturbations. The order of the resonance is one and its ratio is 2 : 3. Similar analysis have been conducted for periodic orbits with number of loops ranging from 3 to 5 for Family I and Family IV in the unperturbed as well as perturbed case and arrived at similar conclusions as in single loop and two loops cases.
5. Interior first order resonance. In Table 3 and Table 4, we have displayed relevant quantities for first order interior resonance for number of loops varying 2 to 8 for the Sun-Earth and the Sun-Mars systems. It should be noted that for internal first order resonance single loop orbit does not exist. For $C = 2.93$ we have divided the table into four families in which Family I is without perturbation and other families incorporating perturbation.

Fig. 3a depicts two loops orbit at $x_0 = 0.29385$ without perturbation and corresponding PSS is given in Fig. 3b. The two loops orbit at $x_0 = 0.31222$ for the Sun-Earth with perturbation and its PSS are given in Fig. 3c and Fig. 3d. In all the cases, it can be seen that no loop is formed around each of the primaries and the orbit is around the larger primary, namely the Sun. Further, the size of the loop reduces due to perturbation.

In the case of interior resonance for Sun-Earth system, Fig. 4 shows three loops orbits for different Jacobi constant values, when the mass reduction value $q = 0.9845$ and the parameter of oblateness $A_2 = 0.0001$. It is clear that the size of loop decreases with increasing the value of Jacobi constant $C$. This observation is investigated in Sub-Fig. [4a - 4f], when $C = 2.95, 2.97, 2.99, 3.01, 3.02$ and $3.03$. While Fig. 5 presents the variation in PSS of these three loops periodic orbits, for the different Jacobi values and the parameters of oblateness and reduction of mass. It is observed that the shape of the PSS changes and finally degenerates in to circle, see Sub-Fig. [5a - 5f]. In addition Fig. 4 and Fig. 5 show the evolution of three loops orbits, with perturbation for Family-IV of the Sun-Earth system, together with its PSS for different values of $C$. During the evolution, the loops becomes smaller and smaller and finally the orbit degenerates into a circle.
Fig. 6 shows variation in the location of the periodic orbits of first order interior and exterior resonance in ideal case (i.e. $q = 1$ and $A_2 = 0$) and in perturbed case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) in the Sun-Earth system. It is clearly seen that for the external resonance as the number of loops increases location of the periodic orbit moves towards the Sun whereas for the internal resonance as the number of loops increases, location of the periodic orbit moves away from the Sun.

In this context the location of exterior or interior first order resonant orbits moves away from the Sun whenever perturbation is included. From locations of orbits it can be seen that exterior resonant orbits with and without perturbation are nearer to the Earth whereas interior resonant orbits are nearer to the Sun. So, for the orbit having same number of loops, location of interior resonant orbit is nearer to the Sun in comparison to exterior resonant orbit.

Fig. 7 shows variation in the eccentricity of periodic orbits of first order interior and exterior resonance in classical case (i.e. $q = 1$ and $A_2 = 0$) and in perturbed case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) in the Sun-Earth system. It is clearly seen that for all four cases eccentricity of the periodic orbit decreases as number of loops increases. Eccentricity of interior resonant periodic orbit in ideal case is highest among all the four cases.

As perturbation increases eccentricity of the periodic orbit decreases. Eccentricity of exterior resonant periodic orbit in perturbed case is lowest among all the cases.

Table 3. Analysis of interior first order resonance in the perturbed Sun-Earth system

| FA | SR | OB | NL | LO | NI | RO | EC | TP | RP |
|----|----|----|----|----|----|----|----|----|----|
| I  | 1  | 0  | 2  | 0.29385 | 2:1 | 0.5353 | 07 | 2.00000 |
|    | 3  | 0.47692 | 3:2 | 0.37506 | 13 | 1.50000 |
|    | 4  | 0.55735 | 1 | 4:3 | 0.32483 | 19 | 1.35330 |
|    | 5  | 0.60105 | 5:4 | 0.30258 | 26 | 1.24991 |
|    | 6  | 0.62515 | 6:5 | 0.29074 | 32 | 1.19981 |
|    | 7  | 0.64650 | 7:6 | 0.28366 | 38 | 1.16633 |
|    | 8  | 0.66000 | 8:7 | 0.27897 | 44 | 1.14185 |
| II | 1  | 0.0001 | 2  | 0.29375 | 2:1 | 0.53567 | 07 | 2.00015 |
|    | 3  | 0.47675 | 3:2 | 0.37525 | 13 | 1.50009 |
|    | 4  | 0.55713 | 4:3 | 0.32506 | 19 | 1.33339 |
|    | 5  | 0.60080 | 5:4 | 0.30283 | 26 | 1.25001 |
|    | 6  | 0.62788 | 6:5 | 0.29100 | 32 | 1.19992 |
|    | 7  | 0.64627 | 7:6 | 0.28391 | 38 | 1.16635 |
|    | 8  | 0.65980 | 8:7 | 0.27921 | 44 | 1.14180 |
| III | 0.9845 | 0 | 2  | 0.31234 | 2:1 | 0.47832 | 07 | 2.15851 |
|    | 3  | 0.50990 | 3:2 | 0.30776 | 13 | 1.58182 |
|    | 4  | 0.59888 | 4:3 | 0.25104 | 19 | 1.39854 |
|    | 5  | 0.64770 | 5:4 | 0.22523 | 26 | 1.30827 |
|    | 6  | 0.67801 | 6:5 | 0.21135 | 32 | 1.25452 |
|    | 7  | 0.69851 | 7:6 | 0.20301 | 38 | 1.21879 |
|    | 8  | 0.71327 | 8:7 | 0.19758 | 44 | 1.19323 |
| IV | 0.9845 | 0.0001 | 2  | 0.31222 | 2:1 | 0.47848 | 07 | 2.15875 |
|    | 3  | 0.50970 | 3:2 | 0.30799 | 13 | 1.58195 |
|    | 4  | 0.59861 | 4:3 | 0.25137 | 19 | 1.39860 |
|    | 5  | 0.64740 | 5:4 | 0.22554 | 26 | 1.30839 |
|    | 6  | 0.67768 | 6:5 | 0.21168 | 32 | 1.25465 |
|    | 7  | 0.69819 | 7:6 | 0.20333 | 38 | 1.21887 |
|    | 8  | 0.71290 | 8:7 | 0.19793 | 44 | 1.19338 |
four cases. So, for the orbit having same number of loops, eccentricity of interior resonant orbit is more in comparison to exterior resonant orbit.

Fig. 8 shows variation in location and eccentricity of three loops interior resonant periodic orbit due to variation in $C$ in perturbed case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) in the Sun-Earth system. As $C$ increases, location of periodic orbit moves

Table 4. Analysis of interior first order resonance in the perturbed Sun-Mars system.

| $FA$ | $SR$ | $OB$ | $NL$ | $LO$ | $NI$ | $RO$ | $EC$ | $TP$ | $RP$ |
|------|------|------|------|------|------|------|------|------|------|
| I 1 0 | 2 0.29386 2.1 0.53352 0.7 2.00090 | 3 0.47693 3.2 0.37504 13 1.50067 |
| II 1 0.0001 | 4 0.55734 4.3 0.32482 19 1.33393 | 5 0.60102 5.4 0.30258 26 1.25055 |
| III 0.9845 0 | 6 0.62808 6.5 0.29075 32 1.20005 | 7 0.64637 7.6 0.28369 38 1.16792 |
| IV 0.9845 0.0001 | 8 0.65954 8.7 0.27911 44 1.14323 | 9 0.67797 9.8 0.27444 50 1.11980 |

Table 5. Variation in three loops orbit due to variation in $C$ for $q = 0.9845$ and $A_2 = 0.0001$ in the Sun-Earth system

| JC | LO | NI | RO | EC | TP | RP |
|----|----|----|----|----|----|----|
| 2.93 | 0.50970 | 0.30799 | 1.58195 | |
| 2.95 | 0.53653 | 0.27320 | 1.57665 | |
| 2.97 | 0.59860 | 0.25130 | 1.59936 | |
| 2.99 | 0.64738 | 0.22530 | 1.39905 | |
| 3.01 | 0.67764 | 0.21167 | 1.25532 | |
| 3.02 | 0.69509 | 0.20330 | 1.21963 | |
| 3.03 | 0.71273 | 0.19796 | 1.19426 | |
away from the Sun and eccentricity decreases. Figs. (9 – 11) show similar results for the Sun-Mars system.

6. Interior resonance of third order. In this section we analyze the third order resonant orbits with different number of loops for different parameters of the orbit. The variations of position and eccentricity for different values of the Jacobi constant $C$ for seven loops orbit is given in Table 7, for the Sun-Earth system and in Table 8 for the Sun-Mars system.

Figs. (14a – 12c) are seven loops orbits for $C = 2.93, 2.96$ and 2.98 respectively for $q = 0.9845$ and $A_2 = 0.0001$. It can be noticed that the size of the periodic orbit as well as size of the loops decrease with increase in the value of $C$ when the parameter of reduction of mass $q$ and oblateness coefficient $A_2$ are fixed. However,

\[
\begin{array}{cccccccc}
JC & LO & NI & RO & EC & TP & RP \\
2.93 & 0.50971 & 0.30798 & 1.58266 \\
2.97 & 0.56750 & 0.23303 & 1.57184 \\
3.01 & 0.65608 & 0.11815 & 1.55901 \\
3.02 & 0.69590 & 0.06651 & 1.55431 \\
3.03 & 0.72200 & 0.01541 & 1.55908 \\
\end{array}
\]
Figure 4. Variation in three loops orbit due to interior first order resonant when \( q = 0.9845 \) and \( A_2 = 0.0001 \) in the Sun-Earth system

the period of the orbits remain same as 26 indicating that an increase in \( C \) decreases the orbital velocity of the particle.

Similar observation have been made in the Sun-Mars system too. The PSS at \( x = 0.39923 \) for \( C = 2.93 \), \( q = 0.9845 \) and \( A_2 = 0.0001 \) in the Sun-Earth system given in Fig.13a showing three islands indicating third order of resonance. Fig.13b is the magnified version of one of the islands of Fig. 13a.

Numerical estimates of position, eccentricity, period and other relevant quantities for third order interior resonance, with number of loops varying from 7 to 14 for the Sun-Earth and the Sun-Mars systems are shown in Table 9 and Table 10. They are divided in to two families; Family I and Family II. Seven loops orbit from Family I having period 26 while Family II having period 32. These orbits are 7 : 4 resonant orbits. Thereby for the given number of loops, and the given resonance, period of Family II orbit is more than period of Family I orbit.
Orbits of Family I are around the first primary only, whereas, orbits of Family II are around the both primaries in which one of the loop of the orbit is around

Table 7. Variation in third order interior resonant seven loops orbit due to variation in $C$ in the Sun-Earth system

| JC | LO | NI | RS | EC | TP | RP |
|----|----|----|----|----|----|----|
| 2.93 | 0.39923 | 3 | 7:4 | 0.39522 | 26 | 1.86449 |
| 2.96 | 0.42824 | 3 | 7:4 | 0.35353 | 26 | 1.85476 |
| 2.98 | 0.44991 | 3 | 7:4 | 0.32298 | 26 | 1.84836 |
Table 1. Variation in location of the periodic orbit of the first order interior and exterior resonant periodic orbit for \( C = 2.93 \) in perturbed case (\( q = 0.9845, A_2 = 0.0001 \)) and ideal case (\( q = 1, A_2 = 0 \)) for the Sun-Earth system.

| No. of Loops | Location | Ideal Ext | Ideal Int | Perturbed Ext | Perturbed Int |
|--------------|----------|-----------|-----------|---------------|---------------|
| 0            | Location |           |           |               |               |
| 1            | Location |           |           |               |               |
| 2            | Location |           |           |               |               |
| 3            | Location |           |           |               |               |
| 4            | Location |           |           |               |               |
| 5            | Location |           |           |               |               |
| 6            | Location |           |           |               |               |
| 7            | Location |           |           |               |               |
| 8            | Location |           |           |               |               |

Figure 6. Variation in location of the periodic orbit of the first order interior and exterior resonant periodic orbit for \( C = 2.93 \) in perturbed case (\( q = 0.9845, A_2 = 0.0001 \)) and ideal case (\( q = 1, A_2 = 0 \)) for the Sun-Earth system.

Table 2. Variation in eccentricity of the first order interior and exterior resonant periodic orbit for \( C = 2.93 \) in perturbed case (\( q = 0.9845, A_2 = 0.0001 \)) and ideal case (\( q = 1, A_2 = 0 \)) cases in the Sun-Earth system.

| No. of Loops | Eccentricity | Ideal Ext | Ideal Int | Perturbed Ext | Perturbed Int |
|--------------|--------------|-----------|-----------|---------------|---------------|
| 0            | Eccentricity |           |           |               |               |
| 1            | Eccentricity |           |           |               |               |
| 2            | Eccentricity |           |           |               |               |
| 3            | Eccentricity |           |           |               |               |
| 4            | Eccentricity |           |           |               |               |
| 5            | Eccentricity |           |           |               |               |
| 6            | Eccentricity |           |           |               |               |
| 7            | Eccentricity |           |           |               |               |
| 8            | Eccentricity |           |           |               |               |

Figure 7. Variation in eccentricity of the first order interior and exterior resonant periodic orbit for \( C = 2.93 \) in perturbed case (\( q = 0.9845, A_2 = 0.0001 \)) and ideal case (\( q = 1, A_2 = 0 \)) cases in the Sun-Earth system.

Figure 8. Variation in location and eccentricity of first order interior three loops orbit when \( q = 0.9845 \) and \( A_2 = 0.0001 \) in Sun-Earth system.

the second primary body, namely, Earth and Mars. Periodic orbits with loops 7, 8, 10, 11, 13 and 14 of Family I with third order interior resonance for the Sun-Earth system for \( q = 0.9845, A_2 = 0.0001 \) and \( C = 2.93 \) are shown in Fig. 14a, Figs. b, c, d, e and f respectively.

The PSS at \( x_0 = 0.5616 \) for \( C = 2.93, q = 0.9845 \) and \( A_2 = 0.0001 \) in the Sun-Earth system given in Fig. 15a contains three islands indicating third order of
7. Interior resonance fifth order. Numerical estimates of different orbital elements of fifth order interior resonance with number of loops varying from 11 to 23 for the Sun-Earth and the Sun-Mars systems are shown in Table 11 and Table 12.
Figure 12. Variation in interior third order resonant seven loops orbit for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun - Earth system.

They are divided into two families; Family I and Family II. Seventeen loops orbit

Table 8. Variation in third order interior resonant seven loops orbit due to variation in $C$ in the Sun-Mars system

| JC  | LO  | NI  | RS  | EC  | TP  | RP   |
|-----|-----|-----|-----|-----|-----|------|
| 2.93| 0.39923 | 3   | 7:4 | 0.39521 | 26 | 1.86534 |
| 2.96| 0.42823 | 0.35353 | 1.85564 |
| 2.98| 0.44991 | 0.32232 | 1.84921 |

Table 9. Third order interior resonance in the Sun-Mars system

| FA  | NL  | LO  | NI  | RO  | EC  | TP  | RP   |
|-----|-----|-----|-----|-----|-----|-----|------|
| I   |     |     |     |     |     |     |      |
| 7   | 0.39923 | 7:4 | 0.39522 | 26 | 1.86449 |
| 8   | 0.46231 | 8:5 | 0.34307 | 32 | 1.69384 |
| 10  | 0.54635 | 3   | 10:7 | 0.28318 | 44 | 1.50281 |
| 11  | 0.57915 | 11:8 | 0.26511 | 51 | 1.44432 |
| 13  | 0.61796 | 13:10 | 0.24063 | 63 | 1.36221 |
| 14  | 0.63585 | 14:11 | 0.22924 | 70 | 1.33343 |
| II  |     |     |     |     |     |     |      |
| 7   | 0.56160 | 7:4 | 0.27346 | 32 | 1.47147 |
| 9   | 0.62620 | 3   | 9:6 | 0.23625 | 44 | 1.34696 |
| 11  | 0.66415 | 11:8 | 0.21766 | 57 | 1.27850 |
| 13  | 0.68886 | 13:10 | 0.20702 | 70 | 1.23510 |
(a) PSS at $x_0 = 0.39923$ and $C = 2.93$  
(b) Enlarged PSS at $x_0 = 0.39923$ for $C = 2.93$

Figure 13. PSS of interior third order resonant seven loops orbit of family I for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun - Earth system.

(a) Seven loops orbit at $x_0 = 0.39923$  
(b) Eight loops periodic orbit at $x_0 = 0.46231$

(c) Ten loops periodic orbit at $x_0 = 0.54635$  
(d) Eleven loops periodic orbit at $x_0 = 0.57515$

(e) Thirteen loops periodic orbit at $x_0 = 0.61796$  
(f) Fourteen loops periodic orbit at $x_0 = 0.63358$

Figure 14. Family I interior third order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun-Earth system.
The perturbed photogravitational restricted problem

Figure 15. PSS of interior third order resonant seven loops orbits from family II for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun-Earth system.

Table 10. Third order interior resonance in the Sun-Mars system

| FA | NL  | LO  | NI  | RO  | EC   | TP   | RP   |
|----|-----|-----|-----|-----|------|------|------|
| I  |     |     |     |     |      |      |      |
| 7  | 0.39923 | 7:4 | 0.39521 | 26 | 1.86534 |
| 8  | 0.46235 | 8:5 | 0.34303 | 32 | 1.69452 |
| 10 | 0.54630 | 3   | 0.28320 | 44 | 1.50361 |
| 11 | 0.57521 | 11:8| 0.26506 | 51 | 1.44487 |
| 13 | 0.61783 | 13:10| 0.24068 | 63 | 1.36310 |
| 14 | 0.63380 | 14:11| 0.29291 | 70 | 1.33366 |

| II |     |     |     |     |      |      |      |
| 7  | 0.56158 | 7:4 | 0.27346 | 32 | 1.47220 |
| 9  | 0.62616 | 9:6 | 0.23626 | 44 | 1.34767 |
| 11 | 0.66413 | 11:8| 0.21764 | 57 | 1.2794 |
| 13 | 0.68878 | 13:10| 0.20702 | 70 | 1.23584 |

Table 11. Fifth order interior resonance in the Sun-Earth system

| FA | NL  | LO  | NI  | RO  | EC   | TP   | RP   |
|----|-----|-----|-----|-----|------|------|------|
| I  |     |     |     |     |      |      |      |
| 11 | 0.36792 | 11:6| 0.42357 | 38 | 1.96103 |
| 12 | 0.41350 | 12:7| 0.38286 | 44 | 1.82329 |
| 13 | 0.45107 | 13:8| 0.35190 | 51 | 1.72225 |
| 14 | 0.48277 | 14:9| 0.32751 | 57 | 1.64406 |
| 16 | 0.53281 | 16:11| 0.29211 | 70 | 1.53139 |
| 17 | 0.55295 | 17:12| 0.27893 | 76 | 1.48915 |

| II |     |     |     |     |      |      |      |
| 15 | 0.58150 | 15:10| 0.26122 | 70 | 1.43180 |
| 17 | 0.61344 | 17:12| 0.24307 | 82 | 1.37065 |
| 19 | 0.63733 | 5   | 0.23053 | 95 | 1.29660 |
| 21 | 0.65640 | 21:16| 0.22124 | 107| 1.29228 |
| 23 | 0.67095 | 23:18| 0.21460 | 120| 1.26649 |
The PSS at $x_0 = 0.36792$ for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun-Earth system given in Fig. 17a contains five islands indicating fifth order of resonance. Fig. 17b is magnified version of one of the islands of Fig. 17a. Orbits with loops varying from 11, 12, 13, 14, 16 and 17 of Family I with fifth order interior resonance for the Sun-Earth system for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ are shown in Fig. 18a, b, c, d, e and f respectively.

Periodic orbits with loops varying from 15, 17, 19, 21 and 23 of Family II for the Sun-Earth system are shown in Fig. 19a, b, c, d and e respectively.

| Table 12. Fifth order interior resonance in the Sun-Mars system |
|-----------------|---|---|---|---|---|---|---|
| $FA$ | $NL$ | $LO$ | $NI$ | $RO$ | $EC$ | $TP$ | $RP$ |
|---|---|---|---|---|---|---|---|
| I | 11 | 0.36790 | 11:6 | 0.42359 | 38 | 1.96199 |
| | 12 | 0.41345 | 12:7 | 0.38280 | 44 | 1.82430 |
| | 13 | 0.45118 | 13:8 | 0.35180 | 51 | 1.72276 |
| | 14 | 0.48285 | 14:9 | 0.32744 | 57 | 1.64463 |
| | 16 | 0.53273 | 16:11 | 0.29216 | 70 | 1.53227 |
| | 17 | 0.55264 | 17:12 | 0.27912 | 76 | 1.49047 |
| II | 15 | 0.58152 | 15:10 | 0.26127 | 70 | 1.43243 |
| | 17 | 0.61335 | 17:12 | 0.24310 | 82 | 1.37146 |
| | 19 | 0.63740 | 19:14 | 0.23048 | 95 | 1.32710 |
| | 21 | 0.65623 | 21:16 | 0.22130 | 107 | 1.29319 |
| | 23 | 0.67119 | 23:18 | 0.21447 | 120 | 1.26667 |
Figure 17. PSS of interior fifth order resonant eleven loops orbits from Family II for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun-Earth system

(a) PSS at $x_0 = 0.36792$

(b) Enlarged PSS at $x_0 = 0.36792$

Figure 18. Family I interior fifth order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ in the Sun-Earth system

(a) Eleven loops orbit at $x_0 = 0.36792$

(b) Twelve loops periodic orbit at $x_0 = 0.41350$

(c) Thirteen loops periodic orbit at $x_0 = 0.45107$

(d) Fourteen loops periodic orbit at $x_0 = 0.48277$

(e) Sixteen loops periodic orbit at $x_0 = 0.53281$

(f) Seventeen loops periodic orbit at $x_0 = 0.55295$
The variation in the location of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun-Earth system for Family I and Family II is shown in Fig. 20 against the number of loops of the periodic orbits. It can be noticed that the location of the orbit shifts towards the second primary body as the number of loops increases. Family I and II of third order resonance and Family I of fifth order resonance contains periodic orbit having 11 and 13 loops. From location of these orbits as shown in Fig. 20, it is clear that for the given number of loops, as order of resonance increases location of periodic orbits moves towards the Sun.

The variation in the eccentricity of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun-Earth system for Family I and Family II are shown in Fig. 21 against the number of loops of the periodic orbits. It can be noticed that the eccentricity of the orbit decreases as the number of loops increases. Also, eccentricity of the Family I orbit is higher than the Family II orbit for the
given order of resonance. From eccentricity of 11 and 13 loops orbits as shown in Fig. 21, it is clear that for the given number of loops, as order of resonance increases eccentricity increases.

The variation in the period of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun-Earth system for Family I and Family II is shown in Fig. 22 against the number of loops of the periodic orbits. It can be noticed that the period of the orbit increases as the number of loops increases. Also, period of the Family II orbit is higher than the Family I orbit for the given order of resonance. From period of 11 and 13 loops orbits shown in Fig. 22, it is clear that for the given number of loops, as order of resonance increases, the period decreases, which is obvious. In addition, Figs. (23 – 25) show similar results for the Sun-Mars system.

8. Conclusion. We have studied exterior and interior first, third and fifth order resonances in the photogravitational restricted three-body problem, by numerical methods for the Sun-Earth and the Sun-Mars systems considering the Sun as a radiating body and Earth or Mars as an oblate spheroid. In this context, the first order exterior and interior resonant orbits, location, eccentricity and period of the periodic orbits are analyzed with and without perturbation for $C = 2.93$. It is observed that for the given order of resonance, period of the orbit is increased by exactly 6 or 7 units as number of loops is increased by 1 because period of the Earth’s orbit is 6.282714 and period of Mars’s orbit is 6.282714 units.
Figure 22. Variation in period of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ Sun-Earth system

Figure 23. Variation in location of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ Sun-Mars system

Figure 24. Variation in eccentricity of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ Sun-Mars system

It is concluded that for the external resonance as the number of loops increases location of the periodic orbit moves towards the Sun where as for the internal resonance, as the number of loops increases, location of the periodic orbit moves away from the Sun. Also, location of exterior or interior first order resonant orbits moves away from the Sun as perturbation included. While from location of orbits, we can conclude that exterior resonance orbits with and without perturbation are nearer to the Earth where as interior resonance orbits are nearer to the Sun. So, for
Eccentricity of the periodic orbit decreases as number of loops increases for both interior and exterior resonance in both perturbed and unperturbed cases. Also, for the orbit having same number of loops, eccentricity of interior resonant orbit is more in comparison to exterior resonant orbit. We also observe that for the given order of resonance as perturbation increases eccentricity of the periodic orbit decreases.

Furthermore, we study the evolution of three loops orbit for interior first order resonance by changing value of Jacobi constant \( C \). As value of \( C \) increases, size of the loop reduces, and hence the shape of the orbit changes and finally it becomes circle. Thus, as \( C \) increases, eccentricity of the periodic orbit decreases and location of the periodic orbit moves towards the second primary body, namely, Earth or Mars.

Regard to the location of third and fifth order resonant orbits for \( C = 2.93 \), \( q = 0.9845 \) and \( A_2 = 0.0001 \) shifts towards the second primary as the number of loops increases. Third and fifth order resonant orbits are divided into two families. Orbits of Family I are around the first primary only, whereas, orbits of Family II are around both the primaries in which one of the loops of the orbit is around the second primary body, namely, Earth or Mars. It is concluded that for the given number of loops, as order of resonance increases location of periodic orbits moves towards the Sun. Also, eccentricity of the orbit decreases as the number of loops increases, and eccentricity of Family I orbit is higher than Family II orbit for the given order of resonance.

It can be observed that for the given number of loops, as order of resonance increases eccentricity increases. Period of the first, third and fifth order resonance orbit increases as the number of loops increases. Also, period of Family II orbit is higher than the Family I orbit for the given order of resonance. Further we notice that for the given number of loops, as order of resonance increases period decreases, which is obvious.

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