UNLABELED SIGNED GRAPH COLORING

BRIAN DAVIS

ABSTRACT. We extend the work of Hanlon on the chromatic polynomial of an unlabeled graph to define the unlabeled chromatic polynomial of an unlabeled signed graph. Explicit formulas are presented for labeled and unlabeled signed chromatic polynomials as summations over distinguished order-ideals of the signed partition lattice. We also define the quotient of a signed graph by a signed permutation, and show that its signed graphic arrangement is closely related to an induced arrangement on a distinguished subspace.

1. INTRODUCTION

Philip Hanlon computed the unlabeled chromatic polynomial (proper colorings up to automorphism) of an ordinary graph in [4], and showed an analogous result to Stanley’s reciprocity relating the number of acyclic orientations of a graph to an evaluation of its chromatic polynomial [5]. We extend this work to the context of signed graphs, and give a geometric interpretation compatible with the point of view presented by Zaslavsky in [6]. We introduce the quotient graph of a signed graph by a signed-automorphism.

Our principal tool is the definition of an order ideal of the signed partition lattice which partitions proper colorings by containment in flats of the type-B arrangement. In Proposition 3.4 we give an explicit expression for the signed chromatic polynomial of a signed graph. Theorem 3.5 states the unlabeled version. Theorem 4.2 enumerates the unlabeled acyclic orientations of a signed graph. Section 2.3 describes the quotient graph \( \Sigma / \beta \) of a signed graph \( \Sigma \) by a signed automorphism \( \beta \).

2. BACKGROUND

2.1. Signed Graphs & Hyperplane Arrangements. A signed graph \( \Sigma \) is a graph \( G = (V, E) \) with a sign function \( \varepsilon : E \to \{+1, -1\} \) on its edge multiset. We allow multiple edges, half-edges (one endpoint),
loops (both endpoints the same), and free loops (no endpoints). For notational convenience we identify
the vertex set $V$ with the integers $[n]$, and sometimes identify edges with their set of endpoints.

A proper coloring of a signed graph is a map $\sigma : V \to \mathbb{Z}$ respecting the constraints that for edge $e$:

- $\sigma(i) \neq \varepsilon(e) \cdot \sigma(j)$ for $e = \{i, j\}$
- $\sigma(i) \neq 0$ for $e = \{i\}$ a negative loop or a half edge (of either sign)
- There are no proper colorings if $\Sigma$ has edge $e = \emptyset$ a free loop or $e = \{i\}$ a positive loop

We are interested in the signed chromatic polynomial $\chi_\Sigma$, a function whose input is a natural number $k$ and whose output is the number of proper colorings of $\Sigma$ taking values in $[-k, k] \cap \mathbb{Z}$. Note that there are signed graphs that admit no proper coloring, i.e., $\chi_\Sigma \equiv 0$. A nice geometric viewpoint on (signed) graph coloring in terms of lattice points and hyperplane arrangements is described in [2] and [6].

The type-$BC$ Coxeter arrangement (in dimension $n$), denoted $BC_n$, consists of the following hyperplanes:

- $h_{i,j}^+ := \{x \in \mathbb{R}^n : x_i = x_j\}$
- $h_{i,j}^- := \{x \in \mathbb{R}^n : x_i = -x_j\}$
- $h_i := \{x \in \mathbb{R}^n : x_i = 0\}$

where $1 \leq i < j \leq n$. For a signed graph $\Sigma$ with neither free loops nor positive loops, the signed graphic arrangement $\mathcal{B}_\Sigma$ is the sub-arrangement of $BC_n$ that encodes the properness conditions of $\Sigma$-colorings. It is the collection

$$\mathcal{B}_\Sigma := \{h_{e}^{\varepsilon(e)}\}_{e \in E}$$

where we take $h_i^\pm$ to mean $h_i$.

A map $f : [n] \to \mathbb{Z}$ may be associated with the point $(f(1), \ldots, f(n)) \in \mathbb{Z}^n$. A coloring $\sigma$ is proper if and only if, as a point in $\mathbb{Z}^n$, it avoids each hyperplane of $\mathcal{B}_\Sigma$.

Recall that associated to a (central) hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is a partially ordered set $L(\mathcal{A})$ called the intersection lattice, whose elements are the geometric intersections of sub-collections of the arrangement, ordered by reverse containment. These elements are called flats of the arrangement, and

---

1This definition is slightly different from that used in [7]. Translating between them is straightforward.
they partition the space $\mathbb{R}^n$ by associating to each point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the maximal flat (minimal under inclusion) containing $x$. In the context of signed graphic arrangements, the maximal flat is equivalent to the maximal collection of equalities of type $h_e^{\varepsilon(e)}$ satisfied by $x$.

### 2.2. Group Actions.

The (hyperoctahedral) group of signed permutations may be described by its action on $\mathbb{R}^n$ as being the group generated by reflection about the hyperplanes of $BC_n$. It is sufficient to generate by reflections of type $h_{i,j}^+$ and $h_i^-$, thus moving forward we will use a canonical representation of a signed permutation $\beta$ in terms of a $(b, \delta)$ pair, with permutation $b \in S_n$ and switching set $\delta \subseteq [n]$. The action of $\beta$ on the standard basis vector $e_i$ is

$$\beta(e_i) = (-1)^{|\delta \cap \{b(i)\}|} e_{b(i)}.$$

**Example 1.** The point $(1, 2, -3) \in \mathbb{R}^3$ has image $(3, 2, 1)$ under the signed permutation $\beta$ defined by reflection across the hyperplanes $h_{1,3}^+$ and $h_1^-$. We represent $\beta$ by the $(b, \delta)$-pair $((1, 3), \{1\})$.

There is a natural action of $\beta$ on signed graphs with vertex set $[n]$. Given $\Sigma$, we define $\beta(\Sigma)$ to be the signed graph with vertex set $[n]$ and edge multiset resulting from permuting the endpoint set of each edge of $\Sigma$ by permutation $b$. For each edge $e' = b(e)$ of $\beta(\Sigma)$, we set $\varepsilon(e') = (-1)^{|e' \cap \delta|} \varepsilon(e)$.

**Example 2.** Let $\Sigma$ be the signed graph on the left in Figure 1 and $\beta$ be the signed permutation $\{(1)(23)(4), \{234\}\}$. For edge $e = \{1, 2\}$, we have $e' = b(e) = \{1, 3\}$ and $\varepsilon(e') = (-1)^{|e' \cap \delta|} \varepsilon(e) = -1$ since $|e' \cap \delta| = |\{1, 3\} \cap \{234\}| = 1$.

![Figure 1](image.png)

We define a $\Sigma$-automorphism to be a signed permutation under which $\Sigma$ is fixed. Observe that $\mathcal{S}_\Sigma$ is invariant under the action of a $\Sigma$-automorphism.
2.3. **Quotient Signed Graphs.** Let $\Sigma$ be a signed graph with vertex set $[n]$ and $b \in S_n$ have cycle decomposition $C_1 \cdots C_m$, with cycles ordered by minimal element. Then for a signed permutation $\beta = (b, \delta)$, the vertex set $V(\Sigma/\beta)$ of the quotient signed graph $\Sigma/\beta$ is the collection of $s \in [m]$ such that the intersection $C_s \cap \delta$ has even cardinality. The edge multiset of $\Sigma/\beta$ is constructed as follows: for each edge $e$ of $\Sigma$, there is a corresponding edge $e'$ of $\Sigma/\beta$ whose endpoint set is $\{ s \in V(\Sigma/\beta) : C_s \cap e \neq \emptyset \}$.

Observe that edges of $\Sigma$ may become half edges, loops, or even free loops in $\Sigma/\beta$.

Before describing the sign function $\varepsilon$ for the edges of $\Sigma/\beta$, we define some notation. Given a signed permutation $\beta = (b, \delta)$ and an integer $i$ in $[n]$, let $k$ be the minimal element of $C_s$, the $b$-cycle containing $i$. There is a minimal positive integer $l$ such that $b^l(k) = i$. We then define $\beta_{(i)}$ to be the number

$$\beta_{(i)} := |\delta \cap \{ b(k), b^2(k), \ldots, b^l(k) = i \}|.$$

In particular, if $k = i$, then $\beta_{(i)}$ is the cardinality of $\delta \cap C_s$.

The sign of each edge of $\Sigma/\beta$ is determined as follows: for edges $e' \in E(\Sigma/\beta)$ whose endpoint set has cardinality strictly less than that of the associated $e \in E(\Sigma)$, set $\varepsilon(e') = -1$. For each edge $e'$ with distinct endpoints $i$ and $j$, let the sign of $e'$ be given by

$$\varepsilon(e') = \varepsilon(e) \cdot (-1)^{\beta_{(i)} + \beta_{(j)}}.$$

For all other edges, let $\varepsilon(e') = \varepsilon(e)$.

**Example 3.** Given the signed graph $\Sigma$ as on the left in Figure 2 and a signed permutation $\beta = \{(1)(23)(4), \{234\}\}$, we construct the quotient graph $\Sigma/\beta$ (right).

![Figure 2](image-url)
The $b$-cycles $C_1 = (1)$ and $C_2 = (23)$ have even intersection with $\delta$. We compute that $\beta(2) = 2$ since $|\{234\} \cap \{b(2), b^2(2)\}| = 2$. The endpoint set of $e'_1$ is $\{1, 2\}$ since the endpoints of $e_1$ lie in $C_1$ and $C_2$, respectively. We compute $\epsilon(e'_1)$ by

$$\epsilon(e'_1) = \epsilon(e_1) \cdot (-1)^{\beta(1)+\beta(2)} = (-1) \cdot (-1)^{0+2} = -1.$$ 

The endpoint set of edge $e_2$ is $\{2, 3\}$, a subset of $C_2$. Thus the edge $e'_2$ has strictly smaller endpoint set $\{2\}$ and $\epsilon(e'_2) = -1.

3. UNLABELED SIGNED CHROMATIC POLYNOMIALS

For a signed permutation $\beta = (b, \delta)$ we define a sub-collection of hyperplanes of $BC_n$:

$$h^\beta_i := \{x \in \mathbb{R}^n : x_i = (-1)^{\delta \cap \{i\}} x_{b^{-1}(i)}\}.$$ 

We define the flat $\hat{\beta} \in L(BC_n)$ associated to $\beta$ by

$$\hat{\beta} := \bigcap_{i \in [n]} h^\beta_i.$$ 

Note that for $x \in \hat{\beta}$ and $i \in [n]$, we get the following chain of equalities:

$$x_i = (-1)^{|\delta \cap \{i\}|} x_{b^{-1}(i)}$$

$$x_{b^{-1}(i)} = (-1)^{|\delta \cap \{b^{-1}(i)\}|} x_{b^{-2}(i)}$$

$$\vdots$$

$$x_{b(i)} = (-1)^{|\delta \cap \{b(i)\}|} x_i,$$

so that for $i \in C_s$, with $k = \min\{C_s\}$,

(2) \hspace{1cm} x_i = (-1)^{\beta(i)} x_k.$$

Thus $x \in \hat{\beta}$ is specified by the coordinate indexed by the minimal element of each cycle of $b$. Observe that for $k$ the minimal element of $C_s$, we get the equality $x_k = (-1)^{|\delta \cap C_s|} x_k$, so that if $|\delta \cap C_s|$ is odd, then $x_k = 0$.

Lemma 3.1. A signed coloring $\sigma$ is fixed by an automorphism $\beta$ if and only if it is an element of the geometric set $\hat{\beta}$. 

Proof. A point \( x \in \mathbb{R}^n \) is fixed by the action of \( \beta \) if and only if for all \( i \in [n] \), the dot product \( e_i \cdot \beta(x) = x_i \). Observe that

\[
x_i = e_i \cdot \beta(x) = e_i \cdot \left( \sum_{j=1}^{n} x_j \beta(e_j) \right) = e_i \cdot \left( \sum_{j=1}^{n} x_j (-1)^{|\delta \cap \{b(j)\}|} e_{b(j)} \right) = (-1)^{|\delta \cap \{i\}|} x_{b^{-1}(i)},
\]

so that \( x \) is in \( h_i^B \) for all \( i \in [n] \).

Lemma 3.2. The number of proper signed \( k \)-colorings of \( \Sigma \) contained in flat \( \hat{\beta} \) is given by \( \chi_{\Sigma/\beta}(k) \).

Proof. Given a proper \( \Sigma \)-coloring \( \sigma \in \hat{\beta} \), we construct a \( \Sigma/\beta \)-coloring \( \sigma' \) by letting

\[
\sigma'(s) = \sigma(\min\{C_s\}).
\]

We check cases, noting that edges of \( \Sigma \) and \( \Sigma/\beta \) are in bijection. For edge \( e' = b(e) \) of \( \Sigma/\beta \):

(i) If \( e' \) has distinct endpoints, then it is a straightforward application of equations (1) and (2) to show that \( \sigma' \) is proper with respect to \( e' \).

(ii) If \( e' \) is a half edge with endpoint \( s \), then \( \sigma' \) is proper with respect to \( e' \) exactly when \( \sigma'(s) \neq 0 \). If \( e \) is a half edge with endpoint \( i \in C_s \), then properness of \( \sigma \) implies that \( \sigma(i) \neq 0 \), and thus by equation (2) we see that \( \sigma'(s) = \pm \sigma(i) \neq 0 \). If instead \( e \) has distinct endpoints \( i \) and \( j \), where \( j \) is in a \( b \)-cycle whose intersection with \( \delta \) has odd cardinality, then since \( \sigma \) is in \( \hat{\beta} \), we have by equation (2) that \( \sigma(j) = 0 \). Regardless of the sign of edge \( e \), properness of \( \sigma \) implies that \( \sigma(i) \neq 0 \), so that \( \sigma'(s) = \pm \sigma(i) \neq 0 \) and \( \sigma' \) is proper with respect to \( e' \).

(iii) If \( e' \) is a loop at vertex \( s \) and \( e \) is a loop at vertex \( i \in C_s \), then since \( \Sigma \) is properly colored by \( \sigma \), \( \varepsilon(e) = \varepsilon(e') = -1 \). Because \( \sigma \) is in \( \hat{\beta} \), we have that \( \sigma'(s) = \pm \sigma(i) \neq 0 \). It follows that \( \sigma' \) is proper with respect to \( e' \). If instead \( e \) has distinct endpoints \( i \) and \( j \), both in \( C_s \), then \( \varepsilon(e') = -1 \) since \( e' \) has strictly smaller endpoint set. Since \( \sigma \) is in \( \hat{\beta} \), we have that \( \sigma(i) = \pm \sigma(j) \), and by properness for edge \( e \) we see that neither \( \sigma(i) \) nor \( \sigma(j) \) is equal to zero. Thus \( \sigma'(s) = \pm \sigma(i) \neq 0 \) and \( \sigma' \) is proper with respect to \( e' \).

Thus \( \sigma' \) is a proper \( \Sigma/\beta \) coloring.

Given \( \sigma' \) a proper \( \Sigma/\beta \)-coloring, we find its preimage \( \sigma \in \hat{\beta} \) by:

- \( \sigma(\min\{C_s\}) = \sigma'(s) \)
\[ \sigma(b(i)) = (-1)^{\delta \cap (b(i))} \sigma(i), \]

- otherwise, \( \sigma(i) = 0. \)

It remains to show that \( \sigma \) is a proper \( \Sigma \)-coloring.

Again we check cases, noting that there is a bijection between edges of \( \Sigma \) and those of \( \Sigma/\beta \).

(i) If an edge \( e' \) of \( \Sigma/\beta \) has distinct endpoints \( s \) and \( t \), then it is again a straightforward application of equations (1) and (2) to show that \( \sigma \) is proper with respect to \( e \).

(ii) If \( e' \) is a loop at vertex \( s \) and \( e \) is a loop at vertex \( i \in C_s \), then since \( \sigma' \) is proper, \( \epsilon(e) = \epsilon(e') = -1 \) and \( \sigma(i) = \pm \sigma'(s) \neq 0 \). Because \( \epsilon(e) = -1 \) and \( \sigma(i) \neq 0 \), the coloring \( \sigma \) is proper with respect to \( e \). If instead \( e \) has distinct endpoints \( i \) and \( j \), then by equation (2), \( \sigma(i) = (-1)^{\beta(i) + \beta(j)} \sigma(j) \). By equation (1), we have \( \epsilon(e) = \epsilon(e')(-1)^{\beta(i) + \beta(j)} \). Thus since \( \sigma' \) is proper, \( \epsilon(e') = -1 \) and so \( \sigma \) is proper with respect to \( e \).

(iii) If \( e' \) is a half edge at vertex \( s \) and \( e \) a half edge at vertex \( i \), then \( \sigma(i) = \pm \sigma'(s) \neq 0 \) and \( \sigma \) is proper with respect to \( e \). If instead \( e \) has distinct endpoints \( i \) and \( j \), where \( j \) is in a \( b \)-cycle having odd intersection with \( \delta \), then \( \sigma(i) = \pm \sigma'(s) \neq 0 \) and \( \sigma(j) = 0 \), so that the coloring \( \sigma \) is proper with respect to \( e \).

Thus \( \sigma \in \hat{\beta} \) is a proper \( \Sigma \)-coloring. \( \square \)

The unlabeled chromatic polynomial \( \hat{\chi}_\Sigma \) of a signed graph \( \Sigma \) with vertex set \([n]\) is a function whose input is a natural number \( k \) and whose output is the number of proper colorings of \( \Sigma \) up to automorphism taking values in \([-k,k]^n \cap \mathbb{Z}^n \). It is a natural extension of the unlabeled chromatic polynomial of an ordinary graph, as described by Hanlon in [4].

**Theorem 3.3.** For a signed graph \( \Sigma \) with automorphism group \( B \), the unlabeled chromatic polynomial \( \hat{\chi}_\Sigma(k) \) is given by

\[ \hat{\chi}_\Sigma(k) = \frac{1}{|B|} \sum_{\beta \in B} \chi_{\Sigma/\beta}(k). \]

**Proof.** By Burnside’s lemma, the number of orbits in the set of proper \( k \)-colorings of \( \Sigma \) under the action of \( B \) is given by the sum

\[ \frac{1}{|B|} \sum_{\beta \in B} \left| \text{fix}(\beta) \right|, \]
where \( \text{fix}(\beta) \) is the set of proper \( k \)-colorings of \( \Sigma \) which are invariant under the action of \( \beta \). Applying Lemmas 3.1 and 3.2, we see that the summands are given by \( \chi_{\Sigma/\beta}(k) \).

**Definition.** For a signed graph \( \Sigma \), we define \( P(\Sigma) \), an associated sub-poset of \( L(BC_n) \), as follows:

\[
P(\Sigma) := \{ p \in L(BC_n) : p \not\subseteq h^{e(e)}_e, e \in E \}.
\]

**Proposition 3.4.** The chromatic polynomial \( \chi_\Sigma(k) \) of a signed graph \( \Sigma \) with vertex set \([n]\) and no positive or free loops is given by

\[
\chi_\Sigma(k) = \sum_{p \in P(\Sigma)} 2^{n-p(p)}(k)_{n-\rho(p)} = \sum_{i=0}^{n} W_{n-i} 2^i(k)_i,
\]

where for the poset \( P(\Sigma) \), \( \rho \) is the rank function, \( W_i \) is the \( i \)th Whitney number of the second kind, and \( (k)_i \) is the falling factorial.

**Proof.** A coloring is proper if and only if it avoids the signed graphic arrangement, thus integer points on flats not contained by hyperplanes of type \( h^{e(e)}_e \) are precisely the proper colorings. To count signed \( k \)-colorings, notice that on a flat of dimension \( i \) there are \( \binom{k}{i} \) choices for the magnitudes of the coefficients, \( 2^i \) ways to choose the signs, and \( i! \) orderings. Thus on each flat of rank \( i \), there are \( 2^{n-i}(k)_{n-i} \) signed \( k \)-colorings. Summing by co-dimension gives the second equality. \( \square \)

It is possible to compute \( \hat{\chi}_\Sigma(k) \) using Theorem 3.3 and Proposition 3.4, but this requires a rather tedious process of computing \( \Sigma/\beta \) for each \( \beta \in B \), and then forming the poset \( P(\Sigma/\beta) \) in order to compute \( \chi_{\Sigma/\beta}(k) \). Instead, the following simplified calculation can be used.

**Theorem 3.5.** For a signed graph \( \Sigma \) without free or positive loops and with automorphism group \( B \), the unlabeled chromatic polynomial \( \hat{\chi}_\Sigma(k) \) is given by

\[
\hat{\chi}_\Sigma(k) = \frac{1}{|B|} \sum_{\beta \in B} \sum_{p \in P(\Sigma)} 2^{n-\rho(p)}(k)_{n-\rho(p)}.
\]

**Proof.** It is sufficient to recall Lemma 3.2 and observe that the inner sum is over flats contained by \( \hat{\beta} \) and not contained by the signed graphic arrangement. \( \square \)
4. Unlabeled Signed Acyclic Orientations

Similar to the orienting of edges of ordinary graphs, incidences of signed graphs may be oriented, i.e., for a given edge $e$ with endpoint $i$, we orient the incidence by $\tau(e,i) = \pm 1$. An orientation of a signed graph is an orientation of each incidence (free loops have no incidences) subject to the following constraint on each edge $e$ with endpoints $i$ and $j$:

$$\tau(e,i) = -\varepsilon(e)\tau(e,j).$$

Acyclic orientations of a signed graph $\Sigma$ are defined in [8], where it is shown that they are in bijection with maximal connected components (regions) of $\mathbb{R}^n \setminus \mathcal{B}_\Sigma$. This implies that the number of acyclic orientations of a signed graph $\Sigma$ is given by $(-1)^{V(\Sigma)}\chi_\Sigma(-1)$. Let $\Delta(\Sigma)$ denote the set of acyclic orientations of $\Sigma$, and for $o \in \Delta(\Sigma)$, let $r(o)$ be the associated region.

The action of a $\Sigma$ automorphism $\beta$ on $\mathbb{R}^n$ induces an action on the regions of $\mathcal{B}_\Sigma$. This defines an action on each orientation $o$.

**Lemma 4.1.** An acyclic orientation $o \in \Delta(\Sigma)$ is fixed by $\beta$ if and only if $r(o) \cap \hat{\beta} \neq \emptyset$.

**Proof.** Let $\beta$ fix $r(o)$. Then for $y \in r(o)$, the orbit of $y$ under powers of $\beta$ is a subset of $r(o)$, and thus the average $\bar{y}$ of this orbit is in the convex set $r(o)$ and is fixed by $\beta$. By Lemma 3.1 a point in $\mathbb{R}^n$ is fixed by $\beta$ if and only if it is contained by $\hat{\beta}$, thus $r(o) \cap \hat{\beta}$ is non empty since it contains $\bar{y}$.

Conversely, let $y \in r(o) \cap \hat{\beta}$. An automorphism $\beta$ induces a continuous map $\mathbb{R}^n \rightarrow \mathbb{R}^n$, and so preserves path connectedness of $r(o)$. Thus for all $y'$ in $\beta \cdot r(o)$ there exists a path from $y$ to $y'$ in the complement of $\mathcal{B}_\Sigma$. Since $r(o)$ is defined to be a maximal connected subset of the complement of $\mathcal{B}_\Sigma$, $y'$ is in $r(o)$, showing that $r(o)$ is fixed by $\beta$. $\square$

We define an unlabeled acyclic orientation of a signed graph to be an acyclic orientation up to automorphism.

**Theorem 4.2.** The set $\hat{\Delta}$ of unlabeled acyclic orientations of a signed graph $\Sigma$ with signed-automorphism group $B$ has cardinality given by

$$|\hat{\Delta}| = \frac{1}{|B|} \sum_{\beta \in B} (-1)^{V(\Sigma/\beta)}\chi_{\Sigma/\beta}(-1).$$
Proof. The proper colorings of $\Sigma$ contained in $\hat{\beta}$ are precisely the lattice points in $\hat{\beta}\setminus B_{\Sigma}$. It follows from Theorem 2.2 of [1] and Lemma 3.2 that $\chi_{\Sigma/\beta}(k)$ is the characteristic polynomial of the induced arrangement on $\hat{\beta}$, and that the number of regions $r(o)$ of $B_{\Sigma}$ meeting $\hat{\beta}$ is given by $(-1)^{V(\Sigma/\beta)}\chi_{\Sigma/\beta}(-1)$.

Applying Burnside’s Lemma as in the proof of Theorem 3.3 the result follows from Lemma 4.1.

ACKNOWLEDGEMENTS

The author thanks Matthias Beck for suggesting and supervising this work, as well as the anonymous referees for their helpful comments and suggestions. This project was partially supported by the NSF GK-12 program (grant DGE-0841164).

REFERENCES

[1] Christos A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields, Adv. Math. 122 (1996), no. 2, 193–233.
[2] Matthias Beck and Thomas Zaslavsky. Inside-out polytopes, Adv. Math. 205 (2006), no. 1, 134–162.
[3] H. S. M. Coxeter and W. O. J. Moser. Generators and relations for discrete groups, third ed., Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 14.
[4] P. Hanlon. The chromatic polynomial of an unlabeled graph, J. Combin. Theory Ser. B 38 (1985), no. 3, 226–239.
[5] Richard P. Stanley. Acyclic orientations of graphs, Discrete Math. 5 (1973), 171–178.
[6] Thomas Zaslavsky. The geometry of root systems and signed graphs, Amer. Math. Monthly 88 (1981), no. 2, 88–105. MR 606249 (82g:05012)
[7] ______. Signed graph coloring, Discrete Math. 39 (1982), no. 2, 215–228.
[8] ______. Orientation of signed graphs, European J. Combin. 12 (1991), no. 4, 361–375.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40507, U.S.A.

E-mail address: Brian.Davis@uky.edu