Modeling of Political Systems using Wasserstein Gradient Flows

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Abstract—The study of complex political phenomena such as parties’ polarization calls for mathematical models of political systems. In this paper, we aim at modeling the time evolution of a political system whereby various parties selfishly interact to maximize their political success (e.g., number of votes). More specifically, we identify the ideology of a party as a probability distribution over a one-dimensional real-valued ideology space, and we formulate a gradient flow in the probability space (also called a Wasserstein gradient flow) to study its temporal evolution. We characterize the equilibria of the arising dynamic system, and establish local convergence under mild assumptions. We calibrate and validate our model with real-world time-series data of the time evolution of the ideologies of the Republican and Democratic parties in the US Congress. Our framework allows to rigorously reason about various political effects such as parties’ polarization and homogeneity. Among others, our mechanistic model can explain why political parties become more polarized and less inclusive with time (their distributions get “tighter”), until all candidates in a party converge asymptotically to the same ideological position.

I. INTRODUCTION

In American politics, it is puzzling that while most of the American people have moderate opinions about main issues tackled by politicians [1], [2], [3], the Republican and Democratic parties are taking positions that are far from the public’s moderate ideology and that are becoming increasingly polarized [4], [5], [6]. For instance, [1] shows that (i) middle of the road positions are predominant in the public’s ideology and that (ii) there is little to no increase in mass polarization. However, at the same time, the ideological overlap between parties is decreasing, and parties are getting more and more polarized. In a society designed for political representation, why are politicians taking more extreme positions when the majority of the public opts for centrist positions?

This apparent contradiction calls for mechanistic mathematical models to help politicians and voters better understand their socio-political positions, and possibly optimize for their acts and decisions. The goal of this paper is to develop a mathematical model for the ideologies of political parties. Specifically, we leverage the theory of gradient flows in probability spaces to formulate a dynamic model for the ideology evolution of political parties. In contrast with existing literature, where a party is usually lumped in its average ideological position, our approach allows us to consider the parties’ full ideological distributions. This way, we can study various political effects such as polarization, homogeneity, and inclusiveness.

a) Related Literature: Political systems are usually modeled with a utility maximizing approach [7], [8], [9], which finds its root in the Downsian model. In the Downsian model, competition among utility-maximizing parties is modeled in a one-dimensional space, representing the ideological position of each party (i.e., negative values are left positions and positive values are right positions). Each candidate of a political party rationally opts for the policy that maximizes their utility, and, given the policy announcement of candidates, voters maximize their expected utility. The Downsian model predicts that the positions of two competing parties reach consensus, converging to the median of voters’ positions.

The Downsian model does not capture empirically observed phenomena such as polarization, and as such it has undergone many extensions and revisions. For instance, if the ideology space is multi-dimensional, then parties can focus on “orthogonal” political issues, which prevents them from converging to the same ideological position. In [8] instead, parties and citizens maximize their quadratic preferences and follow Markov strategies. The model predicts convergence to an alternance of policies. Finally, the utility-maximizing dynamic model in [9] explains why and how parties adopt non-moderate policies.

More recently, [10] proposed a satisficing dynamical model to study polarization of political parties. This model, based on studies showing that people tend to be non-maximizers [11], does not assume that voters maximize their utility, but rather that they opt for candidates who are “good enough”. Then, parties opportunistically adjust the average of their ideology to maximize their number of votes. Among others, this model explains political polarization well.

Our work considers modeling of political systems, specifically parties’ positioning over an ideology. We probabilistically formalize the parties’ utility maximization problem as a gradient flow in the probability space, also called Wasserstein gradient flow. Wasserstein gradient flows were pioneered by [12], who showed that the Fokker-Plank equation can be considered as a gradient flow in the space of probability distributions endowed with the Wasserstein distance, a distance between probability distributions based on the theory of optimal transport [13], [14]. The intuition was then extended and formalized in [15], with a whole theory of gradient flows in metric spaces and its specialization to the probability space. For an introduction to Wasserstein gradient flows, we refer to [16]. Initially, Wasserstein gradient flows
mainly found application in the theory of partial differential equations: Many partial differential equations can be seen, and therefore studied, as Wasserstein gradient flows; e.g., see [17], [18]. More recently, Wasserstein gradient flows also found application in machine learning [19], [20], [21], [22], reinforcement learning [23], [24], and, more generally, optimization theory [25], [26], [27].

b) Contributions: Motivated by the satisficing dynamical model of [10], we provide a model of political systems which accounts for the parties’ full ideological distribution, and not only for the average position of its candidates. More specifically, our contributions is threefold. First, we formulate a Wasserstein gradient flow to model the dynamics of the ideological distributions of political parties aiming at maximizing their political success. Second, we study the arising equilibria and their convergence properties. Finally, we validate our model with data from the US Congress [28].

c) Organization: The remainder of this paper is organized as follows. In Section II, we review the model of [10]. In Section III, we extend it to account for the parties’ full ideological distributions, and study its theoretical properties. In Section IV, we perform numerical simulations and validate our model. Finally, Section V draws the conclusions of this paper. All proofs are relegated to the appendix.

II. A SATISFICING DYNAMICAL MODEL

In this section, we review the satisficing dynamical model from [10]. We present the model in Section II-A, and study its equilibria in Section II-B.

A. The Satisficing Dynamical Model from [10]

Empirical research has demonstrated that the US political spectrum is well captured by a one-dimensional real-valued ideology space: Left positions (i.e., negative values on the real line) represent liberals and right positions (i.e., positive values on the real line) represent conservatives [10]. Each party is modeled by the average ideological position of its candidates in this one-dimensional space, denoted here by $y_i \in \mathbb{R}$. Surveys show that the public’s ideology has a unimodal distribution, with a peak at centrist positions, which can be well approximated by a Gaussian distribution $\rho(x)$ [10]. Without loss of generality, we assume that $\rho$ is zero-mean and has the standard deviation $\sigma_0 \in \mathbb{R}_{>0}$.

In contrast to the Downsian model, voters do not maximize their utility, but rather opt for the party with which they are satisfied. Should they be satisfied with more than one party, they vote randomly for one of them. Satisfaction is measured via a so-called satisficing function $s_i(d_i)$, where $d_i = |x - y_i|$ is the ideological distance between the voter and party $i$. The semantics is as follows: $s_i(d_i)$ is the probability that a voter with ideological position $x$ is satisfied with party $i$ (which has the average ideological position $y_i$). When the distance between the voter’s position $x$ and the party’s position $y_i$ increases, the probability of being satisfied with the party decreases according to

$$s_i(d_i) := \exp \left( -\frac{d_i^2}{2\sigma_i^2} \right),$$

where $\sigma_i \in \mathbb{R}_{>0}$ represents the tolerance that voters have to parties with different ideologies than theirs. Henceforth, we will assume that all $\sigma_i$ are identical; i.e., $\sigma_i = \sigma \in \mathbb{R}_{>0}$. If $\sigma$ is large, more voters with positions far from party $i$’s average ideological position are likely satisfied with the party and might vote for it. Accordingly, in the simplified case of two parties, a voter opts for party 1 if

- it is satisfied with party 1 only, happening with probability $s_1(d_1)(1 - s_2(d_2))$;
- it is satisfied with both parties, happening with probability $s_1(d_1)s_2(d_2)$, and randomly decides to vote for party 1, happening therefore with probability $\frac{1}{2}s_1(d_1)s_2(d_2)$.

Thus, the probability that a voter at position $x$ votes for party 1 is

$$p_1(x|y_1, y_2) = s_1(d_1)(1 - s_2(d_2)) + \frac{1}{2}s_1(d_1)s_2(d_2),$$

and the expected total number of voters for party 1 is

$$V_1(y_1, y_2) := \mathbb{E}_\rho[p_1(x|y_1, y_2)] = \int_\mathbb{R} p_1(x|y_1, y_2)d\rho(x),$$

where $d\rho(x) = \rho(x)dx$. Of course, all expressions for party 2 are symmetric.

The model in [10] assumes that each party opportunistically aims at maximizing its number of votes $V_1$. Thus, the continuous-time evolution of each party’s average ideological position is captured by a gradient flow. Namely, each party moves in the direction that increases its number of votes, at a speed proportional to the potential gain:

$$\dot{y}_1(t) = k\nabla y_1 V_1(y_1(t), y_2(t)) \quad y_1(0) = y_{1,0},$$
$$\dot{y}_2(t) = k\nabla y_2 V_2(y_1(t), y_2(t)) \quad y_2(0) = y_{2,0},$$

where $y_{1,0}, y_{2,0} \in \mathbb{R}$ are given initial conditions and $k \in \mathbb{R}_{>0}$ is a positive constant (determined from empirical data).

B. Theoretic Analysis

Interestingly, in some cases, (4) predicts that parties do not converge to the same ideological position, but rather polarize and converge to asymmetric positions. To identify in which configurations polarization is an equilibrium, we formulate the following assumption on the parameters of the system:

Assumption 1 (Adapted from [10]). We have $\sigma/\sigma_0 < \sigma_c$, where $\sigma_c \approx 0.807$ is the unique real-valued root of $3\sigma_0^2 + 5\sigma_c^4 - 3\sigma_c^2 - 1 = 0$.

Assumption 1 is satisfied whenever voters are not too tolerant; i.e., $\sigma$ is sufficiently small compared to $\sigma_0$. Then, parties’ polarization is an equilibrium if and only if Assumption 1 holds true. Otherwise, a consensus is reached.

Proposition 1 (Adapted from [10]). Let Assumption 1 hold. Then, the dynamic system (4) admits three equilibria:

- the unstable symmetric equilibrium $y_1^* = y_2^* = 0$;
- two locally asymptotically stable asymmetric equilibria $y_1^* = -y_2^* = y^*$ with

$$y^* = \pm \sigma \sqrt{\frac{\sigma_0^2 + \sigma_0^2}{\sigma^2 + 2\sigma_0^2} \ln \left( \frac{(\sigma^2 + \sigma_0^2)^3}{4\sigma^4(\sigma^2 + 2\sigma_0^2)} \right)}.$$
Moreover, if Assumption 1 does not hold, then \( y_1^* = y_2^* = 0 \) is the only equilibrium, and it is asymptotically stable.

The proof is deferred to the appendix. In words, Proposition 1 asserts that if voters are not too tolerant (i.e., Assumption 1 holds true), then parties’ polarization is a locally asymptotically stable equilibrium, while the outcome of the Downsian model (i.e., both parties sharing the public’s ideology, namely \( y_1^* = y_2^* = 0 \)) is an unstable equilibrium. If the tolerance \( \sigma \) is increased so that Assumption 1 is violated, then the system undergoes a pitchfork bifurcation, and \( y_1^* = y_2^* = 0 \) is the unique (asymptotically stable) equilibrium. In this case, both parties asymptotically converge to the average of the public’s ideology, as predicted by the Downsian model. The model was validated with data from the US Congress [28]; see Fig. 1.

III. Model

We now present our model. In Section III-A, we extend the model from [10], reviewed in Section II, to capture the full ideological distributions of political parties. In Section III-B, we study the convergence properties of the arising equilibria. We conclude with some discussion in Section III-C.

A. A Distributional Model

The model in [10] represents the position of party \( i \) as a single point \( y_i \in \mathbb{R} \) on the ideology space. Since parties are usually heterogeneous (indeed, not all candidates share the same ideological position), we extend the model to account for the full ideological distribution, and therefore represent the position of party \( i \) as a probability distribution over the real line. We denote it by \( \mu_i \in \mathcal{P}_2(\mathbb{R}) \), where \( \mathcal{P}_2(\mathbb{R}) \) is the space of probability distributions over the real line with finite second moment. For instance, a “tight” distribution (e.g., Gaussian with low variance) suggests that a party is quite homogeneous around its average ideological position, and culminates in a delta distribution \( \delta_{\bar{y}} \), indicating that the party is homogeneous and all candidates share the ideological position \( \bar{y} \in \mathbb{R} \). Conversely, a “diffused” distribution (e.g., uniform with high variance) models a heterogeneous party, with very different ideological positions. In this setting, the share of candidates of party \( i \) with an ideological position between \( a \) and \( b \) is \( \mu_i((a, b)) = \int_a^b d\mu_i(x) \), and the party’s average ideological position is \( E_{\mu_i}[y_i] = \int_\mathbb{R} y_i d\mu_i(y_i) \).

Accordingly, similarly to (3), the expected total number of votes for party 1 is

\[
V_1(\mu_1, \mu_2) = \int_\mathbb{R} \int_\mathbb{R} \bar{V}_1(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2)
= \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} p_1(x|y_1, y_2) d\rho(x) d\mu_1(y_1) d\mu_2(y_2).
\]

In plain words, \( \bar{V}_1(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) \) is the number of votes that party 1 receives considering the candidates of party 1 with ideology \( y_1 \) and the candidates of party 2 with ideology \( y_2 \) (cf. Eq. (3)). Thus, the total number of votes results from integration over the ideology space of the two parties. Since \( \bar{V}_1 \) is non-negative, by Tonelli’s theorem [29], the order of integration in (6) does not matter. Again, the expression for \( V_2 \) is symmetric.

As in Section II, we suppose that parties aim at maximizing the expected total number of votes, and we adopt a gradient flow approach to model the continuous-time evolution of the parties’ full ideological distributions. We resort to the theory of Wasserstein gradient flows, also known as gradient flows in the Wasserstein space. The Wasserstein space is the space of probability distributions with finite second moment endowed with the Wasserstein distance, defined by

\[
W(\mu, \nu) = \left( \min_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 d\gamma(x, y) \right)^{1/2},
\]

where \( \Gamma(\mu, \nu) \subset \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \) is the set of joint probability distributions (referred to as transport plans) with marginals \( \mu_1 \) and \( \mu_2 \). We refer to the set of minimizers \( \Gamma_o(\mu, \nu) \subset \Gamma(\mu, \nu) \) as the set of optimal transport plans [13], [14]. Intuitively, the Wasserstein distance represents the minimum transportation cost to transport one distribution \( \mu \) into another distribution \( \nu \), when moving one unit of mass from \( x \) to \( y \) costs \( |x - y|^2 \).

In this setting, the gradient flow equations read

\[
\dot{\mu}_1(t) = k \nabla_{\mu_1} V_1(\mu_1(t), \mu_2(t)) \quad ; \quad \mu_1(0) = \mu_{1,0} \quad (7)
\]

\[
\dot{\mu}_2(t) = k \nabla_{\mu_2} V_2(\mu_1(t), \mu_2(t)) \quad ; \quad \mu_2(0) = \mu_{2,0},
\]

where \( \mu_{1,0}, \mu_{2,0} \in \mathcal{P}_2(\mathbb{R}) \) are given initial conditions and \( k \in \mathbb{R}_{>0} \) is a positive constant (determined from empirical data). The expressions \( \nabla_{\mu} V \) are to be intended in the sense of Wasserstein [15]. Namely, the “time derivative” \( \dot{\mu}(t) \) is the tangent vector of an absolutely continuous (w.r.t. the Wasserstein distance) trajectory of probability measures \( t \mapsto \mu(t) \); it can be identified with the velocity vector field \( v : \mathbb{R} \rightarrow \mathbb{R} \) solving (in the sense of distributions) the continuity equation \( \dot{\mu}(t) + \nabla \cdot (v(t) \mu(t)) = 0 \). We refer to [15, Chapter 8] for details. Instead, the “Wasserstein gradient” of a function \( V : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R} \) at \( \mu \in \mathcal{P}_2(\mathbb{R}) \) is denoted by \( \nabla_{\mu} V(\mu) : \mathbb{R} \rightarrow \mathbb{R} \) and is the square integrable function (w.r.t. the measure \( \mu \)) which approximates \( V \) “linearly”: i.e.,

\[
V(\nu) = \int_{\mathbb{R} \times \mathbb{R}} (\nabla_{\mu} V(\mu)(x))(y - x)d\gamma(x, y) + o(W(\mu, \nu)).
\]
for any optimal transport plan $\gamma \in \Gamma_o(\mu, \nu)$ between $\mu$ and $\nu$. Here, $o(W(\mu, \nu))$ denotes terms which are at least quadratic in the Wasserstein distance. For details, we refer to [15, Chapter 10] and [30]. We will study the convergence properties of this model in the next section.

B. Theoretic Analysis

For our theoretic analysis, we assume that a sufficiently regular solution to (7) exists:

**Assumption 2** (Well-posed). The dynamic system (7) admits a locally absolutely continuous solution $\mu_i : [0, +\infty) \to \mathcal{P}(\mathbb{R})$ such that $\mu_i(0) = \mu_{i,0}$ for $i \in \{1, 2\}$.

Note that we are not assuming that $\mu_i(t)$ is absolutely continuous with respect to the Lebesgue measure, but that the curve $t \mapsto \mu_i(t)$ is absolutely continuous (seen as a curve between two metric spaces). Assumption 2 holds for gradient flows of the form $\dot{\mu} = -\nabla V$ [15, Chapter 11], provided that $V$ is sufficiently well behaved. Its study for systems of the form of (7) is left to future research.

We now give an explicit expression for the Wasserstein gradients. This will help us to study equilibria, but also to implement (7) numerically:

**Lemma 2** (Wasserstein gradient). The Wasserstein gradient of $V_1 : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ (with respect to $\mu$) at $(\mu_1, \mu_2)$ is the function $\nabla_{\mu_1} V_1(\mu_1, \mu_2) : \mathbb{R} \to \mathbb{R}$ defined by

$$\nabla_{\mu_1} V_1(\mu_1, \mu_2)(y_1) = \int_{\mathbb{R}} \nabla y_1 \tilde{V}_1(y_1, y_2) d\mu_2(y_2),$$

where $\nabla y_1 \tilde{V}_1(y_1, y_2)$ is the usual gradient of the real-valued function $\tilde{V}_1(y_1, y_2)$ defined in (3). The expression for $\nabla_{\mu_2} V_2$ is analogous.

We refer to the appendix for a proof. Armed with an explicit expression for Wasserstein gradients, we can now study the equilibria of (7). As usual, $(\mu_1^*, \mu_2^*)$ is defined to be an equilibrium if the right hand side of (7) evaluates to 0. Therefore, we look for $(\mu_1^*, \mu_2^*)$ so that the Wasserstein gradients $\nabla_{\mu_1} V_1(\mu_1, \mu_2)$ evaluate to the zero function in $L^2(\mathbb{R}, \mathbb{R}; \mu_i)$ (i.e., $\mu_i$-a.e.). The resulting “distributional” equilibria are compatible with the point-wise equilibria of the satisficing model (cf. Proposition 1):

**Lemma 3** (Equilibria). Let Assumption 1 hold. Then, the dynamic system (4) admits the following equilibria:

- a symmetric equilibrium $\mu_1^* = \mu_2^* = \delta_0$;
- two asymmetric equilibria $\mu_1^* = \delta_{y^*}, \mu_2^* = \delta_{-y^*}$, where $y^*$ results from (5).

Moreover, if Assumption 1 does not hold, then (4) admits the equilibrium $\mu_1^* = \mu_2^* = \delta_0$.

The proof is reported in the appendix. Lemma 3 does not characterize all equilibria, but it suggests that some of the equilibria of (4) are delta distributions, namely ideological distributions where all candidates share the same ideological position. In the next theorem, we show that some of these equilibria are attractive:

**Theorem 4** (Convergence). Let Assumption 1 hold and let $y^*$ as in (5). Then, there exists $\varepsilon > 0$ such that if $\mu_{1,0}, \mu_{2,0} \in \mathcal{P}(\mathbb{R})$ are supported on $y^* + [-\varepsilon, \varepsilon]$ and $-y^* + [-\varepsilon, \varepsilon]$, then $\mu_1(t)$ and $\mu_2(t)$ converge weakly to $\delta_{y^*}$ and $\delta_{-y^*}$ in $\mathcal{P}(\mathbb{R})$, respectively. Similarly, if $\mu_{1,0}, \mu_{2,0} \in \mathcal{P}(\mathbb{R})$ are supported on $-y^* + [-\varepsilon, \varepsilon]$ and $y^* + [-\varepsilon, \varepsilon]$, then $\mu_1(t)$ and $\mu_2(t)$ converge weakly to $\delta_{-y^*}$ and $\delta_{y^*}$ in $\mathcal{P}(\mathbb{R})$, respectively. Moreover, if Assumption 1 does not hold, then there exists $\varepsilon > 0$ such that if $\mu_{1,0}, \mu_{2,0} \in \mathcal{P}(\mathbb{R})$ are supported on $[-\varepsilon, \varepsilon]$ and $[-\varepsilon, \varepsilon]$, then $\mu_1(t)$ and $\mu_2(t)$ both converge weakly in $\mathcal{P}(\mathbb{R})$ to $\delta_0$ in $\mathcal{P}(\mathbb{R})$.

We refer to the appendix for a proof. In plain words, Theorem 4 implies that whenever the support of the initial ideological distributions is sufficiently close to the equilibrium, then the ideological distributions of both parties converge (weakly) to two delta distributions, supported at the equilibrium of the satisficing model from [10]. This allows for the following interpretation: Parties eventually become entirely homogeneous, with all candidates converging to the same ideological position. We will provide empirical evidence of the conclusions of Theorem 4 in the next section.

**Remark.** Theorem 4 does not provide a notion of local asymptotic stability. For instance, it does not allow us to conclude that $(\delta_{y^*}, -\delta_{y^*})$ is locally asymptotically stable (with stability defined with respect to the Wasserstein distance). Indeed, for all $\eta > 0$, there exists $\tilde{\mu}_i$ $\eta$-close to $\delta_{y^*}$ (i.e., $W(\delta_{y^*}, \tilde{\mu}_i) \leq \eta$) not supported on $y^* + [-\varepsilon, \varepsilon]$ and thus for which Theorem 4 does not apply); e.g., for all $\eta \in \mathbb{N}$ with $\eta$ sufficiently large $\tilde{\mu}_1 := (1 - \frac{n^2}{\pi^2}) \delta_{y^*} + \frac{n^2}{\pi^2} \delta_{y^* - \eta}$ is $\eta$-close to $\delta_{y^*}$, since $W(\tilde{\mu}_1, \delta_{y^*}) = \sqrt{1 - \frac{n^2}{\pi^2}} \cdot 0 + \frac{n^2}{\pi^2} n^2 = \eta$, but it is clearly not supported on $y^* + [-\varepsilon, \varepsilon]$. We leave the study of local asymptotic stability region to future work.

C. Discussion

Few comments are in order. First, we do not restrict ourselves to a specific class of probability distributions (e.g., Gaussian, continuous, or discrete), but we work in the probability space $\mathcal{P}(\mathbb{R})$, which includes all probability distributions over the real line, provided that their second moment is finite. Second, since probability distributions are normalized, $\mu_i((a, b))$ is not the total number of candidates with an ideological position between $a$ and $b$, but the share of candidates. This way we can directly deploy the rich theory of optimal transport, formalized for probability distributions, without introducing normalization terms. Third, we do not prove that $\mu_i(t)(A)$ converges to $\mu_i^*(A)$ for all Borel sets $A$ (i.e., strong convergence), but that the integral of each continuous function with quadratic growth converges (i.e., weak convergence). The interpretation is as follows: We do not perform a “microscopic” analysis on each portion of the ideology space, but rather a “macroscopic” analysis for all aggregated quantities resulting from an integral (such as

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1We say that $\mu(t)$ converges weakly in $\mathcal{P}(\mathbb{R})$ to $\mu^*$ if for all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $|f(x)| \leq A + Bx^2$, $A, B \in \mathbb{R}$, we have $\lim_{t \to \infty} \int_{\mathbb{R}} f(y) d\mu(t)(y) = \int_{\mathbb{R}} f(y) d\mu^*(y)$.
mean, number of votes, second moment, etc.). Fourth, our model predicts convergence to delta distributions, representing homogeneous parties. Yet, it can be regularized (e.g., via an entropy term), so that equilibria yield more heterogeneous ideologies. We leave this analysis to future work.

IV. RESULTS

In this section, we present our numerical results. We present a simulation in Section IV-A and compare our model with data in Section IV-B. In the appendix, we study the setting with three parties.

A. Simulations

For simulation purposes, we approximate all probability distributions (i.e., ideological distributions) with discrete measures of 300 particles (representing 300 candidates) and approximate the dynamics (7) by

\[
\begin{align*}
\mu_1(t + 1) &= (\text{Id} + \tau k \nabla \mu_1 V_1(\mu_1(t), \mu_2(t))) \# \mu_1(t), \\
\mu_2(t + 1) &= (\text{Id} + \tau k \nabla \mu_2 V_2(\mu_1(t), \mu_2(t))) \# \mu_2(t),
\end{align*}
\]

where \(\text{Id}\) is the identity map on \(\mathbb{R}\), \(\tau \in \mathbb{R}_{\geq 0}\) is the step-size, and \((\cdot)\#\) denotes the pushforward operator for probability measures [15]. In turn, (8) stipulates that a particle of \(\mu_1\) at position \(x\) is displaced to \(x + \tau k \nabla \mu_1 V_1(\mu_1(t), \mu_2(t))(x)\). The public’s distribution \(\rho(x)\) is a zero-mean Gaussian distribution with standard deviation \(\sigma_0 = 0.93\), determined from data of the US Congress [10]. We use the nominal parameters \(k = 0.5\) and \(\sigma = 0.6\). The initial distributions \(\mu_{1,0}\) and \(\mu_{2,0}\) are samples from truncated Gaussian distributions (truncated at 0 and 0.8, and \(-0.8\) and 0, respectively), originally with mean \(-0.25\) (for party 1), 0.25 (for party 2), and standard deviation 0.15 (for both parties); see Fig. 2.

We show the results of our simulations in Figs. 3 and 4. As can be seen in Fig. 3, our approach is indeed capable of modeling the time evolution of the parties’ ideological distributions. Thus, we can infer features such as parties’ inclusiveness and homogeneity, and not only their average ideological positions. Our simulation confirms that parties become more polarized with time and less inclusive (Fig. 3), until they both converge to two distinct delta distributions (Fig. 4, top), as predicted by Theorem 4. Again, this result allows for the following interpretation: Parties eventually become homogeneous, with all candidates sharing the same ideological position. At equilibrium, parties get the same number of votes, and 27% of the public does not vote. Finally, our model predicts that political polarization increases monotonically with time and eventually converges: The Wasserstein distance between the parties’ ideological distributions, showing that polarization increases monotonically, and converges to 0.67.

B. Validation and Parameters Fitting

We validate our model with data from the US Congress, as in [10]. Specifically, we use a combined dataset of representatives and senators of the Democratic and Republican parties in the US Congress [28]. The dataset comprises the ideology score of every candidate in each party, during the period 1861–2015. Fig. 1 shows the time evolution of the parties’ average ideological position, together with their standard deviations. We fit the parameters \(k\) and \(\sigma\) to minimize the mean squared error, quantified via the Wasserstein distance between the true and the predicted ideological distribution of every party. Formally, given the true trajectory \(\{ (\hat{\mu}_1(0), \hat{\mu}_2(0)), \ldots, (\hat{\mu}_1(T), \hat{\mu}_2(T)) \}\) for a horizon of length \(T\), we solve

\[
k, \sigma \in \arg \min \frac{1}{T} \sum_{t=1}^{T} W(\hat{\mu}_1(t), \mu_1(t))^2 + W(\hat{\mu}_2(t), \mu_2(t))^2
\]

s.t. (8), \(\mu_1(0) = \hat{\mu}_1(0), \mu_2(0) = \hat{\mu}_2(0)\).
The identification yields $k = 0.0264$ and $\sigma = 0.389$. The comparison of the model’s performance with data is shown in Fig. 5. Our model captures the overall behavior of the data. However, it disregards oscillations that are probably due to exogenous impact factors, such as the historical context, election rounds, and political campaigns.

V. CONCLUSION

We presented a satisficing dynamical model for political competition between two parties. Rather than lumping parties in their average ideological position as in [10], our model predicts the dynamic behavior of the full ideological distribution. Under the assumption that parties aim at maximizing the expected total number of votes, we formulated a Wasserstein gradient flow for the time evolution of their ideological distributions. Our model predicts that parties become more homogeneous and polarized with time, until their ideological distributions converge to asymmetric delta distributions. We provided theoretic and numerical support for our findings, and we validated our model with data from the US Congress.

Our model captures the trend in the data, but it disregards impact factors such as the historical context, election rounds, and political campaigns. These aspects, together with further theoretic analysis (e.g., regularization), connections with dynamic game theory [31] and uncertainty propagation [32], and case studies (e.g., asymmetric initial ideological distributions), are possible avenues for future research.

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