On the Relation between the Hybrid Minimum Principle and Hybrid Dynamic Programming: a Linear Quadratic Example

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Abstract: Hybrid optimal control problems are studied for systems where autonomous and controlled state jumps are allowed at the switching instants and in addition to running costs, switching between discrete states incurs costs. Key aspects of the analysis are the relationship between the Hamiltonian and the adjoint process in the Hybrid Minimum Principle before and after the switching instants, the boundary conditions on the value function in Hybrid Dynamic Programming at these switching times, as well as the relationship between the adjoint process in the Hybrid Minimum Principle and the gradient process of the value function in Hybrid Dynamic Programming. The results are illustrated through an analytic example with linear dynamics and quadratic costs.

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1. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems. On one hand, the generalization of the Pontryagin Maximum Principle (PMP) [Pontryagin et al. (1962)] results in the Hybrid Minimum Principle (HMP) [Clarke and Vinter (1989a,b); Garavello and Piccoli (2005); Pasquariello et al. (2011); Shaikh and Caines (2007b); Sussmann (1999a,b); Taringoo and Caines (2011, 2013); Xu and Antsaklis (2004)] that gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the hybrid trajectory of the system corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest in the Hybrid Minimum Principle is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [Shaikh and Caines (2007a)].

On the other hand, the generalization of Dynamic Programming [Bellman (1966)] for hybrid systems results in the theory of Hybrid Dynamic Programming (HDP) which employs the notion of the optimal cost to go for the hybrid optimal control problem as its fundamental notion. Under the assumption of smoothness of the value function, the Principle of Optimality results in the celebrated Hamilton-Jacobi-Bellman (HJB) equation of HDP [Barles et al. (2010); Bensoussan and Menaldi (1997); Dhar-matti and Ramaswamy (2005)].

In past work of the authors (see [Pakniyat and Caines (2013, 2014a,c)]) the results of the Hybrid Minimum Principle are given for the general class of hybrid optimal control problems with autonomous and controlled state jumps and in the presence of a large range of running, terminal and switching costs. It is further proved in [Pakniyat and Caines (2014c)] that the adjoint process in the HMP and the gradient of the value function in HDP are governed by the same dynamic equation and have the same boundary conditions and hence are identical to each other. An illustrative example was provided in [Pakniyat and Caines (2014a)] with scalar nonlinear dynamics equations and nonlinear costs. This paper elaborates the HMP - HDP relationship once more by demonstrating the analytical construction of the value function from its gradient process in the case of hybrid optimal control problems with linear dynamics and quadratic costs.

2. HYBRID OPTIMAL CONTROL PROBLEMS

For the hybrid optimal control problems (HOCP) defined in [Pakniyat and Caines (2013, 2014a,b,c)], consider the infimization of the following hybrid cost

$$J(t_0, t_f, h_0, L; L) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_q (x_q (s), u(s)) \, ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1}q_j}} (x_{q_{j-1}} (t_j^-) + g (x_{q_j} (t_j)))$$

subject to the dynamics

$$\dot{x}_q (t) = f_q (x_q (t), u(t)) \quad \text{a.e. } t \in [t_i, t_{i+1}],$$

where $0 \leq i \leq L$ and $t_{L+1} = t_f < \infty$, with the initial condition

$$h_0 = (q_0, x_0) = (g_0, x_0),$$

$$x_0 \in \mathbb{R}^n, \quad g_0 \in \mathbb{R}^n,$$

$$u(t) \in \mathbb{R}^m,$$

$$f_q \in \mathbb{R}^n \to \mathbb{R}^n,$$

$$l_q \in \mathbb{R}^n \to \mathbb{R},$$

$$c_{\sigma} \in \mathbb{R},$$

$$\sigma \in \{1, \ldots, \ell \},$$

$$\ell \in \mathbb{N}.$$

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and the switching jumps maps
\[ x(q_j(t)) = \xi(x_{q_j-1}(t_-)) = \xi(\lim_{t \to t_+} x_{q_j}(t^+)) \quad (4) \]
where \( 1 \leq j \leq L \).

3. HYBRID MINIMUM PRINCIPLE

Theorem 1 [Pakniyat and Caines (2014a,b,c, 2015)] Define the family of system Hamiltonians by
\[ H_q(x, \lambda, u) = \lambda^T f_q(x, u) + l_q(x, u) \quad (5) \]
x, \( \lambda \in \mathbb{R}^n, u \in U, q_j \in Q \). Then along the optimal trajectory \( q^*, x^* \), there exists an adjoint process \( \lambda^o \) such that
\[ \dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}(x^o, \lambda^o, u^o), \quad (6) \]
\[ \lambda^o(t_j^-) = \lambda^o(t_j) = \nabla \xi^T \lambda^o(t_j^+) + p \nabla m + \nabla c_o, \quad (7) \]
amost everywhere \( t \in [t_0, t_f] \) with
\[ x^o(t_0) = x_0, \quad (8) \]
\[ x^o(t_j) = \xi(x^o(t_-)), \quad (9) \]
\[ \lambda^o(t_f) = \nabla \xi^T \lambda^o(t_j^-) + p \nabla m + \nabla c_o, \quad (10) \]
where \( p \in \mathbb{R} \) when \( t_j \) indicates the time of an autonomous switching, and \( p = 0 \) when \( t_j \) indicates the time of a controlled switching.

Moreover, the Hamiltonian is minimized with respect to the control input
\[ H_{q^o}(x^o, \lambda^o, u^o) \leq H_{q^o}(x^o, \lambda^o, u) \quad (12) \]
for all \( u \in U \); and at a switching time \( t_j \) the Hamiltonian satisfies
\[ H_{q_{j-1}}(t_j^-) = H_{q_j}(t_j) \equiv H_{q_j}(t_j^+) \quad (13) \]

4. HYBRID DYNAMIC PROGRAMMING

Consider the hybrid system (2) and the HOCP for the hybrid cost (1). For simplicity of notation, in the rest of the paper and unless otherwise stated, we use \( x \) instead of \( x^o \) in order to indicate that \( x \) refers to the general solution of the corresponding HOCP passing through it. We adapt the same notation for \( q^o, \ t_f^o \), etc.

At a time \( t \in [t_0, t_f] \) that corresponds to some \( 1 \leq j \leq L + 1 \) such that \( t \in (t_{j-1}, t_j) \) and for the state \( h = (q, x) \) in the hybrid state space, the cost to go function is formed as
\[ J(t_f, q, x, L - j + 1; L - j + 1) = \int_{t_j}^{t_f} l_q(x, u) ds + \sum_{i=j}^{L-j} c_{q_{i-1}q_i} (t_i, x_{q_{i-1}}(t_i^-)) \]
\[ + \sum_{j=1}^{L-j} l_q(x_{q_j}(s), u(s)) ds + g(x_{q_j}(t_j^-)) \quad (14) \]
and the value function \( V \) is defined as the optimal cost to go over the family of hybrid control inputs, i.e.
\[ V(t, q, x, L - j + 1) := \inf_{L-j+1} J(t, t_f, q, x, L - j + 1; L - j + 1) \quad (15) \]

Theorem 2 [Pakniyat and Caines (2014b, 2015)] If at the instant \( t \) and the hybrid state \( (q, x) \) the value function \( V \) is differentiable then it necessarily satisfies the following Hamilton-Jacobi-Bellman (HJB) equation
\[ -\frac{\partial V}{\partial t}(t, q, x, L - j + 1) = \inf_u H_q(x, \frac{\partial V}{\partial x}(t, q, x, L - j + 1), u) \quad (16) \]
where
\[ H_q(x, \frac{\partial V}{\partial x}, u) := l_q(x, u) + \frac{\partial V}{\partial x}^T f_q(x, u) \quad (17) \]
In addition, the value function satisfies the following terminal time condition
\[ V(t_f, q_f, x, 0) = g(x) \quad (18) \]
and the boundary conditions
\[ V(t, q, x, L - j + 1) = \min_{\sigma \in \Sigma} \{ V(t, q_j, \xi_o(x), L - j) + c_o(x) \} \quad (19) \]

where \( \Sigma_j = \Sigma \) if \( t_j \) is a time of a controlled switching. In the case of an autonomous switching, the set \( \Sigma_j \) reduces to a subset of admissible discrete inputs which are consistent with the switching manifold condition \( m_{q, q_j}(x) = 0 \).}

5. THE RELATIONSHIP BETWEEN THE MINIMUM PRINCIPLE AND DYNAMIC PROGRAMMING

Theorem 3 [Pakniyat and Caines (2014c)] If the functions \( f_q \) and \( l_q \) are continuously differentiable for all \( q \in Q \), and the value function \( V \) is twice continuously differentiable almost everywhere in Lebesgue sense on \( \mathbb{R} \times \mathbb{R}^n \) then at all Lebesgue points and times the gradient of the value function and the adjoint process for the corresponding (optimal) switching sequence are governed by the same differential equations, i.e.
\[ \frac{d}{dt} \nabla V = -\frac{\partial}{\partial x} f_q(x^o, u^o)^T \nabla V - \frac{\partial}{\partial x} l_q(x^o, u^o) \quad (21) \]
and
\[ \frac{d}{dt} \lambda^o = -\frac{\partial}{\partial x} f_q(x^o, u^o)^T \lambda^o - \frac{\partial}{\partial x} l_q(x^o, u^o) \quad (22) \]
and have the same boundary conditions, i.e.
\[ \nabla V(t_f, q_f, x(t_f), 0) = \nabla g(x^o(t_f)) \quad (23) \]
\[ \nabla V(t_j, q_{j-1}, x(t_j^-), L - j + 1) = \nabla \xi_j^T \lambda_j(t_j^-) + \nabla V(t_j^+, q_j, x(t_j^+), L - j) \]
\[ + p \nabla m_{q, q_j}(t_j^-) + p \nabla c_{q, q_j}(t_j^-) \quad (24) \]
for the gradient of the value function, and
\[ \lambda^o(t_j^-) = \nabla g(x^o(t_j^-)) \quad (25) \]
\[ \lambda^o(t_j) = \nabla \xi_j^T \lambda_{q, q_j}(t_j^+) + p \nabla m_{q, q_j}(t_j^-) + p \nabla c_{q, q_j}(t_j^-) \quad (26) \]
for the adjoint process. Hence, from the uniqueness of the solutions of (21) and (22) that are identical almost everywhere on \( t \in [t_0, t_f] \), it is concluded that the adjoint process locally describes the gradient of the value function, i.e.
\[ \lambda^o = \nabla V \quad (27) \]
almost everywhere in Lebesgue sense on \( \mathbb{R} \times \mathbb{R}^n \).
6. ILLUSTRATIVE EXAMPLE

Consider the hybrid system with the indexed vector fields
\[ \dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f_1(x,u) = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix} \] (28)
and
\[ \dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f_2(x,u) = \begin{bmatrix} x_2 \\ u \end{bmatrix} \] (29)
where autonomous switchings occur on the switching manifold described by
\[ m(x_1(t_s),x_2(t_s)) \equiv x_2(t_s) = 0 \] (30)
Assume that the hybrid optimal control problem is defined as the minimization of the total cost functional
\[ J = \int_{t_0}^{t_f} \frac{1}{2} u^2 dt + \frac{1}{2} (x_1(t_s))^2 + \frac{1}{2} (x_2(t_f) - v_{ref})^2 \] (31)

### 6.1 The HMP Results

Employing the HMP, the corresponding Hamiltonians are defined by
\[ H_1 = \lambda_1 x_2 + \lambda_2 (-x_1 + u) + \frac{1}{2} u^2 \] (32)
and
\[ H_2 = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2} u^2 \] (33)
The Hamiltonian minimization with respect to \( u \) (Eq. (12)) gives
\[ u^* = -\lambda_2 \] (34)
for both \( q = 1 \) and \( q = 2 \).
The adjoint process dynamics (7) is then
\[ \dot{\lambda}_1 = \frac{\partial H_1}{\partial x_1} = \lambda_2 \] (35)
\[ \dot{\lambda}_2 = \frac{\partial H_1}{\partial x_2} = -\lambda_1 \] (36)

for \( q = 1 \) and
\[ \dot{\lambda}_1 = \frac{\partial H_2}{\partial x_1} = 0 \] (37)
\[ \dot{\lambda}_2 = \frac{\partial H_2}{\partial x_2} = -\lambda_1 \] (38)

for \( q = 2 \).
The terminal condition for the adjoint process (10) gives
\[ \lambda_1(t_s) = \frac{\partial g}{\partial x_1} = 0 \] (39)
\[ \lambda_2(t_s) = \frac{\partial g}{\partial x_2} = x_2(t_f) - v_{ref} \] (40)
Eq. (37) together with (39) give
\[ \lambda_1 = 0 \quad t \in [t_s,t_f] \] (41)
Substituting (41) into (38) gives
\[ \lambda_2 = 0 \] (42)
and from (40) we get
\[ \lambda_2 = x_2(t_f) - v_{ref} \quad t \in [t_s,t_f] \] (43)
The boundary conditions (11) on adjoint processes at the switchings instant give
\[ \lambda_1(t_s) = \lambda_1(t_s+) + \frac{\partial c}{\partial x_1} + p \frac{\partial m}{\partial x_1} = x_1(t_s) \] (44)
\[ \lambda_2(t_s) = \lambda_2(t_s+) + \frac{\partial c}{\partial x_2} + p \frac{\partial m}{\partial x_2} = x_2(t_f) - v_{ref} + p \] (45)
Eq. (44) and (45) serve as a terminal condition for the adjoint processes dynamics (35) and (36) which has a general solution of the form
\[ \lambda_1 = A \sin (t + \alpha) \quad t \in [t_0,t_s] \] (46)
\[ \lambda_2 = A \cos (t + \alpha) \quad t \in [t_0,t_s] \] (47)
Then the system state dynamics (6) becomes
\[ \dot{x}_1 = \frac{\partial H_1}{\partial \lambda_1} = x_2 \] (48)
\[ \dot{x}_2 = \frac{\partial H_1}{\partial \lambda_2} = -x_1 + u^* = -x_1 - \lambda_2 = -x_1 - A \cos (t + \alpha) \] (49)
which has a general solution of the form
\[ x_1 = -\frac{1}{2} A \sin (t + \alpha) + B \sin (t + \beta) \] (50)
\[ x_2 = -\frac{1}{2} A \cos (t + \alpha) - \frac{1}{2} A \sin (t + \alpha) + B \cos (t + \beta) \] (51)
that need to satisfy the initial condition \( x_1(t_0) = x_{10} \) and \( x_2(t_0) = x_{20} \). At the switching time \( t_s \) the continuity condition on \( x_1 \) and \( x_2 \) is deduced from (9)
\[ x_1(t_s+) = x_1(t_s) = x_1(t_s-) \] (52)
\[ x_2(t_s+) = x_2(t_s) = x_2(t_s-) = 0 \] (53)
which form the initial conditions required for
\[ \dot{x}_1 = \frac{\partial H_2}{\partial \lambda_1} = x_2 \] (54)
\[ \dot{x}_2 = \frac{\partial H_2}{\partial \lambda_2} = u^* = -\lambda_2 = v_{ref} - x_2(t_f) \] (55)

The solution of these equations is given as
\[ x_1(t) = x_1(t_s) + \frac{1}{2} \left( v_{ref} - x_2(t_f) \right)(t - t_s)^2 \] (56)
\[ x_2(t) = (v_{ref} - x_2(t_f))(t - t_s) \] (57)
The equation (57) is expressed in terms of \( x_2(t_f) \). In order to write an explicit form for \( x_2(t) \) we evaluate (57) at \( t_f \) to write
\[ x_2(t_f) = (v_{ref} - x_2(t_f))(t_f - t_s) \] (58)
or
\[ x_2(t_f) (1 + t_f - t_s) = v_{ref} (t_f - t_s) \] (59)
Thus
\[ x_2(t_f) = \frac{v_{ref} (t_f - t_s)}{1 + t_f - t_s} \] (60)
Substituting (60) into (56) and (57) we get
\[ x_1(t) = x_1(t_s) + \frac{v_{ref}}{2(1 + t_f - t_s)} (t - t_s)^2 \] (61)
\[ x_2 = \frac{v_{ref}}{1 + t_f - t_s} (t - t_s) \] (62)
for \( t \in [t_s,t_f] \). The Hamiltonian continuity condition at \( t_s \) gives
\[ \lambda_1(t_s-) x_2(t_s-) - \lambda_2(t_s-) (x_1(t_s-) + \lambda_2(t_s-)) + \frac{1}{2} \lambda_2(t_s-) \] (63)
With a change of signs on both sides of the equation and using the equations (52) and (53) this equality becomes
\[ \dot{\lambda}_2(t_s^-)x_{1s} + \frac{1}{2}\dot{\lambda}_2(t_s^-)^2 = \frac{1}{2}\dot{\lambda}_2(t_s)^2 \] (64)

Hence, by solving simultaneously the following 6 equations for the given \( t_0 = 0, x(t_0) = [x_{10}, x_{20}]^T, t_f \) and \( v_{ref} \) the values of the 6 unknown parameters \( A, \alpha, B, \beta, t_s \) and \( p \) are determined.

\[ B = x_{10} \] (65)
\[ -\frac{1}{2}A\sin(\alpha) + B\cos(\beta) = x_{20} \] (66)
\[ A\cos(t_s + \alpha) = \frac{v_{ref}}{1+t_f-t_s} + p \] (67)
\[ A\cos(t_s + \alpha) - \frac{1}{2}A\sin(t_s + \alpha) + B\cos(t_s + \beta) = 0 \] (69)
\[ A\cos(t_s + \alpha) \left( -\frac{1}{2}A\sin(t_s + \alpha) + B\sin(t_s + \beta) \right) + \frac{1}{2}A^2\cos^2(t_s + \alpha) = \frac{1}{2} \left( \frac{v_{ref}}{1+t_f-t_s} \right)^2 \] (70)

where (65) and (66) are derived from the system initial condition in (50) and (51); equations (67) and (68) are derived from the adjoint boundary conditions (44) and (45); the relation (69) is the result of the switching manifold condition (53), and (70) represents the Hamiltonian continuity condition (64). For the values of \( t_0 = 0, x_{10} = 1, x_{20} = -0.5, t_f = 5 \) and \( v_{ref} = 1 \) the results are demonstrated in Fig. 1.

### 6.2 HDP results from their relationship to the HMP results

Employing Theorem 3 we want to construct the value function satisfying the results of Theorem 2. To this end we rewrite the system dynamics equations (28) and (29) and the \( \nabla V \) dynamics equation (21) in the matrix form, i.e. for \( q_1 \) the first dynamics equations become
\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \nabla_{x_1} V \\ \nabla_{x_2} V \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \nabla_{x_1} V \\ \nabla_{x_2} V \end{bmatrix} \] (71)

and for \( q_2 \) the second dynamics equations are presented as
\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \nabla_{x_1} V \\ \nabla_{x_2} V \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \nabla_{x_1} V \\ \nabla_{x_2} V \end{bmatrix} \] (72)

The state transition matrix for these equations are denoted by \( \phi_1 \), where
\[ \phi_1(t,t_0) = \frac{1}{2} \begin{pmatrix} 1 & (t-t_0) & \frac{(t-t_0)^3}{6} & -\frac{(t-t_0)^2}{2} \\ 0 & 1 & \frac{(t-t_0)^2}{2} & 0 \\ 0 & 0 & 1 & -\frac{(t-t_0)}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (73)

Partitioning \( \phi_2 \), the solution of (72) for \( t \in [t_s, t_f] \) can be written as
\[ x(t_f) = \phi_{211}(t_f, t) x(t) + \phi_{212}(t_f, t) \nabla V(t) \] (75)
\[ \nabla V(t_f) = \phi_{221}(t_f, t) x(t) + \phi_{222}(t_f, t) \lambda(t) \] (76)

which gives
\[ \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{(t_f-t)^3}{6} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla V_x(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} \] (77)

and
\[ \begin{bmatrix} \nabla V_x(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \nabla V_x(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} \] (78)

Substituting the above expressions for \( x(t_f) \) and \( \nabla V(t_f) \) into (23) results in
\[ \begin{bmatrix} \nabla V(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \left[ \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} - \frac{v_{ref}}{v_{ref}} \right] + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \nabla V_x(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} \] (79)

which from (78) gives
\[ \begin{bmatrix} \nabla V_x(t_f) \\ \nabla V_{x_2}(t_f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -v_{ref} \end{bmatrix} \left[ x_2(t_f) - v_{ref} \right] \] (80)

Substitution of \( x_2(t_f) \) from (77) gives
\[ \nabla V(t) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{t_f-t+1} \end{bmatrix} \left[ x(t) - \frac{v_{ref}}{v_{ref}} \right] \] (81)

Thus the value function in the second dynamics with no switching ahead (i.e. after the switching time \( t_s \)) should be of the form

![Graph](image-url)
\[ V(t, q_2, x, 0) = \frac{1}{2} x^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{t_f - t + 1} \end{bmatrix} x + \frac{v_{ref}}{t_f - t + 1} x + \alpha_2(t) \]  

(82)

Since \( V \) in (82) satisfies the HJB equation (16), the time dependent constant term \( \alpha_2(t) \) is determined to be

\[ \alpha_2(t) = \frac{v_{ref}^2}{2(t_f - t + 1)} \]  

(83)

giving the value function in the second dynamics with no remaining switching as

\[ V(t, q_2, x, 0) = \frac{1}{2} x^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{t_f - t + 1} \end{bmatrix} x + \frac{v_{ref}}{t_f - t + 1} x + \frac{v_{ref}^2}{2(t_f - t + 1)^2} \]  

(84)

Similarly, partitioning \( \phi_1 \) in (73) which is the solution of (71) in \( t \in [t_0, t_f] \) gives

\[ x(t_f - t) = \phi_{11}(t, t) x(t) + \phi_{12}(t, t_f) V(t) \]  

(85)

\[ V(t, t_f) = \phi_{21}(t, t_f) x(t) + \phi_{22}(t, t_f) V(t) \]  

(86)

which is equivalent to

\[ \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} \cos(t_f - t) & \sin(t_f - t) \\ -\sin(t_f - t) & \cos(t_f - t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -v_{ref} \\ 0 \end{bmatrix} \]  

(87)

\[ \begin{bmatrix} \nabla V(t_f) \\ \nabla V(t_f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \begin{bmatrix} \cos(t_f - t) & \sin(t_f - t) \\ -\sin(t_f - t) & \cos(t_f - t) \end{bmatrix} \begin{bmatrix} \nabla V(t) \\ \nabla V(t) \end{bmatrix} \]  

(88)

From (24) \( \nabla V(t_f) \) is given as

\[ \begin{bmatrix} \nabla V(t_f) \\ \nabla V(t_f) \end{bmatrix} = \begin{bmatrix} \nabla V(t_f) \\ \nabla V(t_f) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]  

(89)

where \( \nabla V(t_f) \) is determined from (81). Hence,

\[ \begin{bmatrix} \nabla V(t_f) \\ \nabla V(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t_f - t + 1} \end{bmatrix} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \]  

(90)

Substituting \( x^t(t) \) and \( \lambda^a(t) \) from (87) and (88) respectively results in

\[ \begin{bmatrix} \cos(t_f - t) & \sin(t_f - t) \\ -\sin(t_f - t) & \cos(t_f - t) \end{bmatrix} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \begin{bmatrix} -v_{ref} \\ 0 \end{bmatrix} \]  

(91)

which can be expressed as

\[ \begin{bmatrix} \nabla V(t_f) \\ \nabla V(t_f) \end{bmatrix} = \begin{bmatrix} \cos \delta_s + \frac{2}{\delta_s} \sin \delta_s \\ \sin \delta_s + \frac{2}{\delta_s} \cos \delta_s \end{bmatrix} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \begin{bmatrix} 0 & -v_{ref} \end{bmatrix} \]  

(92)

with

\[ \delta_s := t_f - t, \quad \delta_{fs} := t_f - t_f \]  

(93)

Hence,

\[ V(t, q_1, x, 1) = \begin{bmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ s_{1,2}(t) \end{bmatrix} \]  

(94)

where

\[ K_{11}(t) = \frac{2 + (\delta_{fs} + \frac{1}{2})(1 + \cos 2\delta_s)}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

(95)

\[ K_{12}(t) = \frac{(\delta_{fs} + \frac{1}{2}) \sin 2\delta_s - \delta_s}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

(96)

\[ K_{21}(t) = \frac{(\delta_{fs} - \frac{1}{2}) \sin 2\delta_s - \delta_s}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

(97)

\[ K_{22}(t) = \frac{2 + (\delta_{fs} - \frac{1}{2})(1 - \cos 2\delta_s)}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

and

\[ s_{1,1}(t) = \frac{\delta_{fs} + 1}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

(98)

\[ s_{1,2}(t) = \frac{\delta_{fs} + 1}{2(\delta_{fs} + 1) - 2\delta_s^2 + 2\cos \delta_s + \delta_f \delta_s \sin 2\delta_s} \]  

(99)

\[ p - \frac{v_{ref}}{\delta_{fs} + 1} = \pm \frac{\sqrt{2v_{ref}}}{2} \sqrt{\left(\frac{2v_{ref}}{t_f - t + 2}\right)^2 + (x_1^t(t))^2} \]  

(100)

i.e. at any instant \( t \) and for any given continuous state \( x(t) \) the value function for the first dynamics with one switching left is locally described by

\[ V(t, q_1, x, 1) = \frac{1}{2} x^T K(t) x + s_1(t)^T T x + \alpha_1(t) \]  

(101)

and \( \alpha_1(t) \) determined from the substitution of (99) in the HJB equation (16) which gives its dynamics as
\[ \dot{\alpha}_1 = \frac{1}{2} (s_{1,2} (t))^2 \] (102)
and its value at \( t_s \) determined from the value function boundary condition (20) as
\[ \alpha_1 (t_s) = \alpha_2 (t_s) = \frac{v_{\text{ref}}^2}{2 (t_f - t_s + 1)} \] (103)

In summary, the solution to the HJB partial differential equation of Hybrid Dynamic Programming (16) is determined indirectly from its gradient process \( V \) which is uniquely identified by the HMP - HDP relationship equations presented in Theorem 3. The HJB equation is employed only for the determination of the time-varying constant terms \( \alpha_i (t) \) which do not appear in \( V \) the gradient of the value function.

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