Centralizers in endomorphism rings

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Abstract. We prove that the centralizer $\text{Cen}(\varphi) \subseteq \text{Hom}_R(M, M)$ of a nilpotent endomorphism $\varphi$ of a finitely generated semisimple left $R$-module $R^M$ (over an arbitrary ring $R$) is the homomorphic image of the opposite of a certain $Z(R)$-subalgebra of the full $m \times m$ matrix algebra $M_m(R[z])$, where $m$ is the dimension (composition length) of $\ker(\varphi)$. If $R$ is a local ring, then we provide an explicit description of the above $\text{Cen}(\varphi)$. If in addition $Z(R)$ is a field and $R/J(R)$ is finite dimensional over $Z(R)$, then we give a formula for the $Z(R)$-dimension of $\text{Cen}(\varphi)$. If $R$ is a local ring, $\varphi$ is as above and $\sigma \in \text{Hom}_R(M, M)$ is arbitrary, then we give a complete description of the containment $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$ in terms of an appropriate $R$-generating set of $R^M$. Using our results about nilpotent endomorphisms, for an arbitrary (not necessarily nilpotent) linear map $\varphi \in \text{Hom}_K(V, V)$ of a finite dimensional vector space $V$ over a field $K$ we determine the PI-degree of $\text{Cen}(\varphi)$ and give other information about the polynomial identities of $\text{Cen}(\varphi)$.

1. INTRODUCTION

Our work was motivated by one of the classical subjects of advanced linear algebra. A detailed study of commuting matrices can be found in many of the text books on linear algebra ([9, 10]). Commuting pairs and $k$-tuples of $n \times n$ matrices have been continuously in the focus of research (see, for example, [5, 6]). If we replace $n \times n$ matrices by endomorphisms of an $n$-generated module, we get a more general situation. The aim of the present paper is to investigate the size and the PI properties of the centralizer $\text{Cen}(\varphi)$ of an element $\varphi$ in the endomorphism ring $\text{Hom}_R(M, M)$.

In general, if $S$ is a ring (or algebra), then the centralizer

$$\text{Cen}(s) = \{ u \in S \mid us = su \}$$

of an element $s \in S$ is a subring (subalgebra) of $S$. Clearly, $\text{Cen}(s)$ satisfies all polynomial identities of $S$. However, $\text{Cen}(s)$ may also satisfy some other polynomial

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identities. Thus the study of the PI properties of Cen(s), particularly in the case of classical rings (e.g. matrix rings), deserves special attention.

Following the observations of Sections 2 and 3 about the nilpotent Jordan normal base, Section 4 contains the following results about the centralizer Cen(ϕ) of a nilpotent endomorphism ϕ of a finitely generated semisimple left R-module \( R^M \) (over an arbitrary ring R).

In Theorem 4.8 we prove that Cen(ϕ) is the homomorphic image of the opposite of a certain \( Z(R) \)-subalgebra of the full \( m \times m \) matrix algebra \( M_m(R[z]) \) over the polynomial ring \( R[z] \), where \( m \) is the dimension (composition length) of \( \ker(\varphi) \). As a consequence, we obtain that Cen(ϕ) satisfies all polynomial identities of \( M^R_m(R[z]) \); in particular, if \( R \) is commutative, then the standard identity \( S_{2m} = 0 \) of degree \( 2m \) holds in Cen(ϕ).

In Theorem 4.10 we obtain a complete characterization of Cen(ϕ) if \( R \) is a local ring.

In Theorem 4.11 we determine the \( Z(R) \)-dimension of Cen(ϕ) in the case when \( R \) is local, \( Z(R) \) is a field and \( R/J(R) \) is finite dimensional over \( Z(R) \).

Section 5 contains further results about centralizers.

In Theorem 5.1 a complete description of Cen(ϕ) is provided in terms of an appropriate \( R \)-generating set of the finitely generated semisimple left \( R \)-module \( R^M \) (over an arbitrary ring \( R \)), with \( \varphi : M \to M \) a so-called indecomposable nilpotent \( R \)-endomorphism. In particular, if \( R \) is commutative, then we prove in Corollary 5.2 that \( \psi \in Cen(\varphi) \) if and only if \( \psi \) is a polynomial expression of \( \varphi \). A nilpotent linear map (or equivalently, a nilpotent \( n \times n \) matrix) is indecomposable if and only if its characteristic and minimal polynomials coincide (see part (3) in Proposition 2.3). The following is a classical result about the centralizer (see [10]). If \( K \) is a field and the characteristic and minimal polynomials of a (not necessarily nilpotent) matrix \( A \in M_n(K) \) coincide, then

\[
Cen(A) = \{ f(A) \mid f(z) \in K[z] \}.
\]

For an indecomposable nilpotent \( \varphi \) the mentioned corollary is a generalization of the above result. We note that the above result on Cen(A) is similar to Bergman’s Theorem ([1]) about the centralizer of a non-constant polynomial in the free associative algebra.

In Theorem 5.3 the containment relation \( Cen(\varphi) \subseteq Cen(\sigma) \) is considered. If \( R \) is a local ring, \( \varphi \) is nilpotent and \( \sigma \in \text{Hom}_R(M,M) \) is arbitrary, then we provide a complete description of the situation \( Cen(\varphi) \subseteq Cen(\sigma) \) in terms of an appropriate \( R \)-generating set of \( R^M \). For two (not necessarily nilpotent) matrices \( A,B \in M_n(K) \), over an algebraically closed field \( K \), the containment \( \text{Cen}(A) \subseteq \text{Cen}(B) \) holds if and only if \( B = f(A) \) for some \( f(z) \in K[z] \) (see [9] part VII, section 39). For a nilpotent \( \varphi \) our description of \( Cen(\varphi) \subseteq Cen(\sigma) \) is a generalization of the above result.

Section 6 is devoted to a study of the polynomial identities of the centralizer. An arbitrary (not necessarily nilpotent) linear map of a finite dimensional vector space, or equivalently, a matrix \( A \in M_n(K) \) over a field \( K \) will be considered. First we show that it suffices to deal with nilpotent matrices.

In Theorem 6.1 the radical and the semisimple component of the centralizer Cen(A) of a nilpotent matrix A is determined. The proof of Theorem 6.1 is based on the explicit presentation of Cen(A) in Theorem 4.10.
Corollary 6.2 is about the polynomial identities of the centralizer of a (not necessarily nilpotent) matrix \( A \in M_n(K) \). The Jordan normal form of \( A \) over the algebraic closure of \( K \) is considered. If \( p \) is the maximum of the number of elementary Jordan matrices of the same size and with the same eigenvalue, and \( T(S) \subseteq K(x_1, x_2, \ldots) \) is the T-ideal of the polynomial identities of the algebra \( S \), then \( T(\text{Cen}(A)) \supseteq T(M_p(K)) \) for a suitable \( q \), which also can be found explicitly. Hence the PI-degree of \( \text{Cen}(A) \) is equal to \( p \).

Since all known results about matrix centralizers are closely connected with the Jordan normal form, it is not surprising that our development depends on the existence of the nilpotent Jordan normal base of a semisimple module with respect to a given nilpotent endomorphism (guaranteed by one of the main theorems of [11]).

2. THE NILPOTENT JORDAN NORMAL BASE

Throughout the paper a ring \( R \) means a (not necessarily commutative) ring with identity, and \( Z(R) \) and \( J(R) \) denote the centre and the Jacobson radical of \( R \), respectively. Let \( \varphi : M \to M \) be an \( R \)-endomorphism of the (unitary) left \( R \)-module \( R^M \). A subset \( \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\} \subseteq M \) is called a nilpotent Jordan normal base of \( R^M \) with respect to \( \varphi \) if each \( R \)-submodule \( Rx_{\gamma,i} \leq M \) is simple, and the set \( \{k_\gamma \mid \gamma \in \Gamma\} \) of integers is bounded. For \( i \geq k_\gamma + 1 \) we assume that \( x_{\gamma,i} = 0 \) holds in \( M \). Now \( \Gamma \) is called the set of (Jordan-) blocks and the size of the block \( \gamma \in \Gamma \) is the integer \( k_\gamma \geq 1 \). Obviously, the existence of a nilpotent Jordan normal base implies that \( R^M \) is semisimple and \( \varphi \) is nilpotent with \( \varphi^n = 0 \neq \varphi^{n-1} \), where

\[
\varphi(M) = \varphi \left( \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} Rx_{\gamma,i} \right) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} R\varphi(x_{\gamma,i}) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma - 1} Rx_{\gamma,i+1}
\]

implies that \( \text{im}(\varphi) = \bigoplus_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma - 1} Rx_{\gamma,i+1} = \bigoplus_{\gamma \in \Gamma', 2 \leq i' \leq k_\gamma} Rx_{\gamma,i'} \), where \( \Gamma' = \{\gamma \in \Gamma \mid k_\gamma \geq 2\} \) and \( \Gamma \setminus \Gamma' = \{\gamma \in \Gamma \mid k_\gamma = 1\} \).
Any element \( u \in M \) can be written as

\[
u = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_{\gamma}} a_{\gamma,i}x_{\gamma,i},
\]

where \( \{(\gamma,i) \mid \gamma \in \Gamma, 1 \leq i \leq k_{\gamma}, \text{ and } a_{\gamma,i} \neq 0\} \) is finite and all summands \( a_{\gamma,i}x_{\gamma,i} \) are uniquely determined by \( u \). Since

\[
\varphi(u) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_{\gamma}} a_{\gamma,i}\varphi(x_{\gamma,i}) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_{\gamma}} a_{\gamma,i}x_{\gamma,i+1} = 0
\]
is equivalent to the condition that \( a_{\gamma,i}x_{\gamma,i+1} = 0 \) for all \( \gamma \in \Gamma, 1 \leq i \leq k_{\gamma} - 1 \), we obtain that

\[
\varphi(u) = 0 \iff u = \sum_{\gamma \in \Gamma} a_{\gamma,k_{\gamma}}x_{\gamma,k_{\gamma}}.
\]

Indeed, \( a_{\gamma,i}x_{\gamma,i} \neq 0 \) \( (1 \leq i \leq k_{\gamma} - 1) \) would imply that \( ba_{\gamma,i}x_{\gamma,i} = x_{\gamma,i} \) for some \( b \in R \) (note that \( Rx_{\gamma,i} \leq M \) is simple), whence

\[
x_{\gamma,i+1} = \varphi(x_{\gamma,i}) = ba_{\gamma,i}\varphi(x_{\gamma,i}) = ba_{\gamma,i}x_{\gamma,i+1} = 0
\]
can be derived, a contradiction. It follows that

\[
\ker(\varphi) = \bigoplus_{\gamma \in \Gamma} Rx_{\gamma,k_{\gamma}}
\]

and \( \dim R(\ker(\varphi)) = |\Gamma| \) in case of a finite \( \Gamma \). The following is one of the main results in \cite{11}.

**2.1. Theorem.** Let \( \varphi : M \to M \) be an \( R \)-endomorphism of the left \( R \)-module \( RM \). Then the following are equivalent.

1. \( RM \) is a semisimple left \( R \)-module and \( \varphi \) is nilpotent.
2. There exists a nilpotent Jordan normal base of \( RM \) with respect to \( \varphi \).

**2.2. Proposition.** Let \( \varphi : M \to M \) be a nilpotent \( R \)-endomorphism of the finitely generated semisimple left \( R \)-module \( RM \). If \( \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_{\gamma}\} \) and \( \{y_{\delta,j} \mid \delta \in \Delta, 1 \leq j \leq l_{\delta}\} \) are nilpotent Jordan normal bases of \( RM \) with respect to \( \varphi \), then there exists a bijection \( \pi : \Gamma \to \Delta \) such that \( k_{\gamma} = l_{\pi(\gamma)} \) for all \( \gamma \in \Gamma \). Thus the sizes of the blocks of a nilpotent Jordan normal base are unique up to a permutation of the blocks.

**Proof.** We apply induction on the index of the nilpotency of \( \varphi \). If \( \varphi = 0 \), then we have \( k_{\gamma} = l_{\delta} = 1 \) for all \( \gamma \in \Gamma, \delta \in \Delta \), and

\[
\bigoplus_{\gamma \in \Gamma} Rx_{\gamma,1} = \bigoplus_{\delta \in \Delta} Ry_{\delta,1} = M
\]

implies the existence of a bijection \( \pi : \Gamma \to \Delta \) (Krull-Schmidt, Kurosh-Ore). Assume that our statement holds for any \( R \)-endomorphism \( \phi : N \to N \) with \( \phi N \) being a finitely generated semisimple left \( R \)-module and \( \phi^{n-2} \neq 0 = \phi^{n-1} \). Consider the situation described in the proposition with \( \varphi^{n-1} \neq 0 = \varphi^n \), then

\[
\text{im}(\varphi) = \bigoplus_{\gamma \in \Gamma', 2 \leq i' \leq k_{\gamma}} Rx_{\gamma,i'}
\]

ensures that

\[
\{x_{\gamma,i'} \mid \gamma \in \Gamma', 2 \leq i' \leq k_{\gamma}\}
\]
is a nilpotent Jordan normal base of the left \( R \)-submodule \( \text{im}(\varphi) \leq M \) of \( R \cdot M \) with respect to the restricted \( R \)-endomorphism \( \varphi : \text{im}(\varphi) \rightarrow \text{im}(\varphi) \). The same holds for

\[
\{ y_{\delta,j'} \mid \delta \in \Delta', 2 \leq j' \leq l_3 \}.
\]

Since we have \( \phi^{n-2} \neq 0 = \phi^{n-1} \) for \( \phi = \varphi \mid \text{im}(\varphi) \), our assumption ensures the existence of a bijection \( \pi : \Gamma' \rightarrow \Delta' \) such that \( k_\gamma - 1 = l_{\pi(\gamma)} - 1 \) for all \( \gamma \in \Gamma' \). In view of

\[
\ker(\varphi) = \bigoplus_{\gamma \in \Gamma'} \text{Rx}_{\gamma,k_\gamma} = \bigoplus_{\delta \in \Delta} \text{R}y_{\delta,l_3}
\]

we obtain that \( |\Gamma| = |\Delta| \) (Krull-Schmidt, Kurosh-Ore), whence \( |\Gamma \setminus \Gamma'| = |\Delta \setminus \Delta'| \) follows. Thus we have a bijection \( \pi^* : \Gamma \setminus \Gamma' \rightarrow \Delta \setminus \Delta' \) and the natural map

\[
\pi \cup \pi^* : \Gamma' \cup (\Gamma \setminus \Gamma') \rightarrow \Delta' \cup (\Delta \setminus \Delta')
\]

is a bijection with the desired property. \( \square \)

We call a nilpotent element \( s \in S \) of the ring \( S \) decomposable if \( es = se \) holds for some idempotent element \( e \in S \) \( (e^2 = e) \) with \( 0 \neq e \neq 1 \). A nilpotent element which is not decomposable is called indecomposable.

\subsection*{2.3-Proposition.} Let \( \varphi : M \rightarrow M \) be a nonzero nilpotent \( R \)-endomorphism of the semisimple left \( R \)-module \( R \cdot M \). Then the following are equivalent.

1. There is a nilpotent Jordan normal base \( \{ x_i \mid 1 \leq i \leq n \} \) of \( R \cdot M \) with respect to \( \varphi \) consisting of one block (thus \( |\Gamma| = 1 \) for any nilpotent Jordan normal base \( \{ x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma \} \) of \( R \cdot M \) with respect to \( \varphi \)).

2. \( \varphi \) is an indecomposable nilpotent element of the ring \( \text{Hom}_R(M,M) \).

3. \( R \cdot M \) is finitely generated and \( \varphi^{d-1} \neq 0 \), where \( d = \text{dim}_R(M) \) is the dimension of \( R \cdot M \).

\textbf{Proof.}

(1)\( \Rightarrow \)\( (3) \): Clearly,

\[
\bigoplus_{1 \leq i \leq n} \text{Rx}_i = M
\]

implies that we have \( d = n \) for the dimension of \( R \cdot M \), whence

\[
\varphi^{d-1}(x_1) = \varphi^{n-1}(x_1) = x_n \neq 0
\]

follows.

(3)\( \Rightarrow \)\( (1) \): Let \( \{ x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma \} \) be a nilpotent Jordan normal base of \( R \cdot M \) with respect to \( \varphi \). Suppose that \( |\Gamma| \geq 2 \), then

\[
n = \max \{ k_\gamma \mid \gamma \in \Gamma \} \leq d - 1,
\]

where \( d = \sum_{\gamma \in \Gamma} k_\gamma = \text{dim}_R(M) \). Thus \( \varphi^{d-1} = \varphi^{(d-1)-n} \circ \varphi^{n} = 0 \), a contradiction.

(1)\( \Rightarrow \)\( (2) \): Suppose that \( \varepsilon \circ \varphi = \varphi \circ \varepsilon \) holds for some idempotent endomorphism \( \varepsilon \in \text{Hom}_R(M,M) \) with \( 0 \neq \varepsilon \neq 1 \). Then

\[
\text{im}(\varepsilon) \circ \text{im}(1-\varepsilon) = M
\]

for the non-zero (semisimple) \( R \)-submodules \( \text{im}(\varepsilon) \) and \( \text{im}(1-\varepsilon) \) of \( R \cdot M \). Now \( \varepsilon \circ \varphi = \varphi \circ \varepsilon \) ensures that \( \varphi : \text{im}(\varepsilon) \rightarrow \text{im}(\varepsilon) \) and \( \varphi : \text{im}(1-\varepsilon) \rightarrow \text{im}(1-\varepsilon) \). Since these restricted \( R \)-endomorphisms are nilpotent, we have a nilpotent Jordan normal base of \( \text{im}(\varepsilon) \) with respect to \( \varphi \mid \text{im}(\varepsilon) \) and a nilpotent Jordan normal base of \( \text{im}(1-\varepsilon) \) with respect to \( \varphi \mid \text{im}(1-\varepsilon) \). The union of these two bases gives a
nilpotent Jordan normal base of $M$ with respect to $\varphi$ consisting of more than one block, a contradiction (the direct sum property of the new base is a consequence of the modularity of the submodule lattice of $R M$).

(2)$\implies$(1): Suppose that $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ is a nilpotent Jordan normal base of $R M$ with respect to $\varphi$ with $|\Gamma| \geq 2$ and fix an element $\delta \in \Gamma$. Consider the non-zero $\varphi$-invariant $R$-submodules

$$N_\delta' = \bigoplus_{1 \leq i \leq k_\delta} R x_\delta,i$$

and

$$N_\delta'' = \bigoplus_{\gamma \in \Gamma \setminus \{\delta\}, 1 \leq i \leq k_\gamma} R x_{\gamma,i},$$

then $M = N_\delta' \oplus N_\delta''$ and define $\varepsilon_\delta : M \to M$ as the natural projection of $M$ onto $N_\delta'$. Then $\varepsilon_\delta(u) = u'$, where $u = u' + u''$ is the unique sum presentation of $u \in M$ with $u' \in N_\delta'$ and $u'' \in N_\delta''$. It is straightforward to see that $\varepsilon_\delta \circ \varepsilon_\delta = \varepsilon_\delta$, $0 \neq \varepsilon_\delta \neq 1$ and $\varepsilon_\delta \circ \varphi = \varphi \circ \varepsilon_\delta$ hold. □

3. THE MODULE STRUCTURE INDUCED BY AN ENDOMORPHISM

Let $R[z]$ denote the ring of polynomials of the commuting indeterminate $z$ with coefficients in $R$. The ideal $(z^k) = R[z]z^k = z^k R[z] \triangleleft R[z]$ generated by $z^k$ will be considered in the sequel. If $\varphi : M \to M$ is an arbitrary $R$-endomorphism of the left $R$-module $R M$, then for $u \in M$ and

$$f(z) = a_1 + a_2 z + \cdots + a_{n+1} z^n \in R[z]$$

(unusual use of indices!) the left multiplication

$$f(z) \ast u = a_1 u + a_2 \varphi(u) + \cdots + a_{n+1} \varphi^n(u)$$

defines a natural left $R[z]$-module structure on $M$. This left action of $R[z]$ on $M$ extends the left action of $R$ on $R M$. Note that

$$z^n \ast u = \varphi^n(u) \text{ and } \varphi(f(z) \ast u) = (zf(z)) \ast u.$$

For any $R$-endomorphism $\psi \in \text{Hom}_R(M, M)$ with $\psi \circ \varphi = \varphi \circ \psi$ we have

$$\psi(f(z) \ast u) = f(z) \ast \psi(u)$$

and hence $\psi : M \to M$ is an $R[z]$-endomorphism of the left $R[z]$-module $R[z] M$. On the other hand, if $\psi : M \to M$ is an $R[z]$-endomorphism of $R[z] M$, then

$$\psi(\varphi(u)) = \psi(z \ast u) = z \ast \psi(u) = \varphi(\psi(u))$$

implies that $\psi \circ \varphi = \varphi \circ \psi$. The centralizer

$$\text{Cen}(\varphi) = \{\psi \mid \psi \in \text{Hom}_R(M, M) \text{ and } \psi \circ \varphi = \varphi \circ \psi\}$$

of $\varphi$ is a $Z(R)$-subalgebra of $\text{Hom}_R(M, M)$ and the argument above gives that

$$\text{Cen}(\varphi) = \text{Hom}_{R[z]}(M, M).$$

For a set $\Gamma \neq \emptyset$, the $\Gamma$-copower $\coprod_{\gamma \in \Gamma} R[z]$ of the ring $R[z]$ is an ideal of the $\Gamma$-direct power ring $\prod_{\gamma \in \Gamma} R[z]$ consisting of all elements $f = (f_\gamma(z))_{\gamma \in \Gamma}$ with a finite set

$$\{\gamma \in \Gamma \mid f_\gamma(z) \neq 0\}$$

of non-zero coordinates. The power (copower) has a natural $(R[z], R[z])$-bimodule structure. If $\Gamma$ is finite, then

$$(R[z])^\Gamma = \prod_{\gamma \in \Gamma} R[z] = \prod_{\gamma \in \Gamma} R[z].$$
If \( \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma \} \) is a nilpotent Jordan normal base of \( RM \) with respect to a nilpotent endomorphism \( \varphi \), then for an element \( f = (f_\gamma(z))_{\gamma \in \Gamma} \) with 
\[
f_\gamma(z) = a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,n_\gamma+1}z^{n_\gamma} \]
the formula 
\[
\Phi(f) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} a_{\gamma,i}x_{\gamma,i} = \sum_{\gamma \in \Gamma} f_\gamma(z) * x_{\gamma,1}
\]
defines a function 
\[
\Phi : \prod_{\gamma \in \Gamma} R[z] \to M.
\]

3.1. Proposition. For a nilpotent endomorphism \( \varphi \in \text{Hom}_R(M,M) \) of the semisimple left \( R \)-module \( RM \), the function \( \Phi \) is a surjective left \( R[z] \)-homomorphism. We have \( \varphi(\Phi(f)) = \Phi(zf) \) for all \( f \in \prod_{\gamma \in \Gamma} R[z] \) and the kernel 
\[
\prod_{\gamma \in \Gamma} J(R)[z] + (z^{k_\gamma}) \subseteq \ker(\Phi) \cap \prod_{\gamma \in \Gamma} R[z]
\]
is a left ideal of the power (and hence of the copower) ring. If \( R \) is a local ring \((R/J(R) \text{ is a division ring})\), then 
\[
\prod_{\gamma \in \Gamma} J(R)[z] + (z^{k_\gamma}) = \ker(\Phi).
\]

Proof. Clearly, 
\[
\sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} Rx_{\gamma,i} = M
\]
implies that \( \Phi \) is surjective. The second part of the defining formula gives that \( \Phi \) is a left \( R[z] \)-homomorphism: 
\[
\Phi(g(z)f) = \sum_{\gamma \in \Gamma} (g(z)f_\gamma(z)) * x_{\gamma,1} = \sum_{\gamma \in \Gamma} g(z) * (f_\gamma(z) * x_{\gamma,1}) = g(z) * \Phi(f),
\]
where \( g(z) \in R[z] \). We also have 
\[
\varphi(\Phi(f)) = \sum_{\gamma \in \Gamma} \varphi(f_\gamma(z) * x_{\gamma,1}) = \sum_{\gamma \in \Gamma} (zf_\gamma(z)) * x_{\gamma,1} = \Phi(zf).
\]
If \( f \in \prod_{\gamma \in \Gamma} J(R)[z] + (z^{k_\gamma}) \), then 
\[
f_\gamma(z) = (a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,n_\gamma+1}z^{n_\gamma})
\]
with \( a_{\gamma,i} \in J(R), 1 \leq i \leq k_\gamma \). Since \( Rx_{\gamma,i} \) is simple, we have \( J(R)x_{\gamma,i} = \{0\} \). Thus \( \varphi^{k_\gamma}(x_{\gamma,1}) = 0 \) implies that \( f_\gamma(z) * x_{\gamma,1} = 0 \), whence \( \Phi(f) = 0 \) follows. Take an element \( g = (g_\gamma(z))_{\gamma \in \Gamma} \) of the direct power and suppose that \( \Phi(f) = 0 \) in \( M \). Then 
\[
\bigoplus_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} Rx_{\gamma,i} = M
\]
implies that \( a_{\gamma,i}x_{\gamma,i} = 0 \) for all \( \gamma \in \Gamma \) and \( 1 \leq i \leq k_\gamma \). Thus \( f_\gamma(z) * x_{\gamma,1} = 0 \) for all \( \gamma \in \Gamma \). It follows that 
\[
\Phi(gf) = \sum_{\gamma \in \Gamma} (g_\gamma(z)f_\gamma(z)) * x_{\gamma,1} = \sum_{\gamma \in \Gamma} g_\gamma(z) * (f_\gamma(z) * x_{\gamma,1}) = 0,
\]
whence \( gf \in \ker(\Phi) \) can be deduced.

If \( R \) is a local ring and \( a_{\gamma,i}x_{\gamma,i} = 0 \) for some \( 1 \leq i \leq k_\gamma \), then \( a_{\gamma,i} \in J(R) \). Thus 
\[
\Phi(f) = 0 \implies f_\gamma(z) = (a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,k_\gamma}z^{k_\gamma-1}) + (a_{\gamma,k_\gamma+1}z^{k_\gamma} + \cdots + a_{\gamma,n_\gamma}z^{n_\gamma}) \in J(R)[z] + (z^{k_\gamma}).
\]
It follows that 
\[
f \in \prod_{\gamma \in \Gamma} J(R)[z] + (z^{k_\gamma}). \quad \square
\]

4. THE CENTRALIZER OF A NILPOTENT ENDOMORPHISM

Let \( X = \{ x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma \} \) be a nilpotent Jordan normal base of \( R^m \) with respect to the nilpotent endomorphism \( \varphi \in \text{Hom}_R(M, M) \). We keep the notations of the previous section and in the rest of the paper we assume that \( R^m \) is finitely generated, i.e., that \( \Gamma \) is finite.

A linear order on \( \Gamma \), say \( \Gamma = \{ 1, 2, \ldots, m \} \), allows us to view an element \( f = (f_\gamma(z))_{\gamma \in \Gamma} \) of \((R[z])^\Gamma\) as a \( 1 \times m \) matrix (a row vector) over \( R[z] \). For an \( m \times m \) matrix \( P = [p_{\delta,\gamma}(z)] \) in \( M_m(R[z]) \) the matrix product
\[
fP = \sum_{\delta \in \Gamma} f_\delta(z)p_\delta
\]
of \( f \) and \( P \) is a \( 1 \times m \) matrix (row vector) in \((R[z])^\Gamma\), where \( p_\delta = (p_{\delta,\gamma}(z))_{\gamma \in \Gamma} \) is the \( \delta \)-th row vector of \( P \) and 
\[
(fP)_\gamma = \sum_{\delta \in \Gamma} f_\delta(z)p_{\delta,\gamma}(z).
\]

We define the subsets
\[
I(X) = \{ P \in M_m(R[z]) \mid P = [p_{\delta,\gamma}(z)] \text{ and } p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma}) \text{ for all } \delta, \gamma \in \Gamma \},
\]
\[
N(X) = \{ P \in M_m(R[z]) \mid P = [p_{\delta,\gamma}(z)] \text{ and } z^{k_\delta}p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma}) \text{ for all } \delta, \gamma \in \Gamma \}
\]
and
\[
M(X) = \{ P \in M_m(R[z]) \mid \ker(f) \subseteq \ker(P) \text{ for all } f \in \ker(\Phi) \}
\]
of \( M_m(R[z]) \). Note that \( I(X) \) and \( N(X) \) are \((R[z], R[z])\)-sub-bimodules of \( M_m(R[z]) \) in a natural way.

For \( \delta, \gamma \in \Gamma \) let \( k_{\delta,\gamma} = k_\gamma - k_\delta \) when \( 1 \leq k_\delta < k_\gamma \leq n \) and \( k_{\delta,\gamma} = 0 \) otherwise.

It can be verified, that the condition \( z^{k_\delta}p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma}) \) is equivalent to \( p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma}) \).

4.1. Remark. If \( E_{\delta,\gamma} \) denotes the \( m \times m \) standard matrix unit over \( R[z] \) with 1 in the \( (\delta, \gamma) \) entry and zeros in the other entries, then \( E_{\delta,\gamma} \in N(X) \) for all \( \delta, \gamma \in \Gamma \) with \( k_\delta \geq k_\gamma \).

4.2. Lemma. \( I(X) \triangleleft M_m(R[z]) \) is a left ideal, \( N(X) \subseteq M_m(R[z]) \) is a subring, \( I(X) \triangleleft N(X) \) is an ideal and \( M(X) \) is a \( Z(R) \)-subalgebra of \( M_m(R[z]) \). If \( R \) is a local ring, then \( N(X) = M(X) \).

Proof. For the elements \( P = [p_{\delta,\gamma}(z)] \) and \( Q = [q_{\delta,\gamma}(z)] \) of \( M_m(R[z]) \) take \( PQ = [r_{\delta,\gamma}(z)] \), where
\[
r_{\delta,\gamma}(z) = \sum_{\tau \in \Gamma} p_{\delta,\tau}(z)q_{\tau,\gamma}(z).
\]
If $Q \in \mathcal{I}(X)$, then $q_{\delta, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ and $J(R)[z] + (z^{k_\gamma}) \subset R[z]$ imply that $r_{\delta, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$. Thus $PQ \in \mathcal{I}(X)$ and $\mathcal{I}(X)$ is a left ideal of $M_m(R[z])$.

If $P, Q \in \mathcal{N}(X)$, then $z^{k_\delta} p_{\delta, \gamma}(z) \in J(R)[z] + (z^{k_\tau})$ and $z^{k_\tau} q_{\tau, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ for all $\delta, \tau, \gamma \in \Gamma$. It follows that

$$z^{k_\delta} p_{\delta, \tau}(z) = g_{\delta, \tau}(z) + z^{k_\tau} h_{\delta, \tau}(z)$$

with $g_{\delta, \tau}(z) \in J(R)[z] \text{ and } h_{\delta, \tau}(z) \in R[z]$. Thus

$$z^{k_\delta} r_{\delta, \gamma}(z) = \sum_{\tau \in \Gamma} z^{k_\delta} p_{\delta, \tau}(z) g_{\tau, \gamma}(z) = \sum_{\tau \in \Gamma} (g_{\delta, \tau}(z) + z^{k_\tau} h_{\delta, \tau}(z)) g_{\tau, \gamma}(z) =$$

$$\sum_{\tau \in \Gamma} (g_{\delta, \tau}(z) g_{\tau, \gamma}(z) + h_{\delta, \tau}(z) z^{k_\tau} g_{\tau, \gamma}(z))$$

and $z^{k_\tau} q_{\tau, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ ensure that $z^{k_\delta} r_{\delta, \gamma}(z)$ is in $J(R)[z] + (z^{k_\gamma})$. Consequently, we obtain that $PQ \in \mathcal{N}(X)$. Hence $\mathcal{N}(X)$ is a subring of $M_m(R[z])$.

If $P \in \mathcal{I}(X)$ and $Q \in \mathcal{N}(X)$, then $p_{\delta, \tau}(z) \in J(R)[z] + (z^{k_\nu})$ and $z^{k_\nu} q_{\tau, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ for all $\delta, \tau, \gamma \in \Gamma$. It follows that

$$p_{\delta, \tau}(z) = u_{\delta, \tau}(z) + z^{k_\nu} v_{\delta, \tau}(z)$$

with $u_{\delta, \tau}(z) \in J(R)[z] \text{ and } v_{\delta, \tau}(z) \in R[z]$. Thus

$$r_{\delta, \gamma}(z) = \sum_{\tau \in \Gamma} p_{\delta, \tau}(z) q_{\tau, \gamma}(z) = \sum_{\tau \in \Gamma} (u_{\delta, \tau}(z) + z^{k_\tau} v_{\delta, \tau}(z)) q_{\tau, \gamma}(z) =$$

$$\sum_{\tau \in \Gamma} (u_{\delta, \tau}(z) q_{\tau, \gamma}(z) + v_{\delta, \tau}(z) z^{k_\tau} q_{\tau, \gamma}(z))$$

and $z^{k_\tau} q_{\tau, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ ensure that $r_{\delta, \gamma}(z)$ is in $J(R)[z] + (z^{k_\gamma})$. Consequently, we obtain that $PQ \in \mathcal{I}(X)$. Hence $\mathcal{I}(X)$ is an ideal of $\mathcal{N}(X)$.

If $P, Q \in \mathcal{M}(X), f \in \ker(\Phi)$ and $c \in Z(R)$, then

$$\Phi(f(cP)) = \Phi(c(fP)) = c\Phi(fP) = 0,$$

$$\Phi(f(P + Q)) = \Phi(fP + fQ) = \Phi(fP) + \Phi(fQ) = 0$$

and $fP \in \ker(\Phi)$ implies that

$$\Phi(f(PQ)) = \Phi((fP)Q) = 0,$$

whence $cP, P + Q, PQ \in \mathcal{M}(X)$, proving that $\mathcal{M}(X)$ is a $Z(R)$-subalgebra of $M_m(R[z])$.

If $R$ is a local ring, then Proposition 3.1 gives that

$$\ker(\Phi) = \prod_{\gamma \in \Gamma} J(R)[z] + (z^{k_\gamma}).$$

Now $e_\delta \in \ker(\Phi)$, where $e_\delta$ denotes the vector with $z^{k_\delta}$ in its $\delta$-coordinate and zeros in all other places.

If $P \in \mathcal{M}(X)$, then $e_\delta P \in \ker(\Phi)$ implies that $z^{k_\delta} p_{\delta, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ for all $\delta, \gamma \in \Gamma$, whence $P \in \mathcal{N}(X)$ follows.

If $P \in \mathcal{N}(X)$ and $f \in \ker(\Phi)$, then $z^{k_\delta} p_{\delta, \gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ for all $\delta, \gamma \in \Gamma$ and $f_{\gamma}(z) = g_{\gamma}(z) + z^{k_\gamma} h_{\gamma}(z)$ with $g_{\gamma}(z) \in J(R)[z]$ and $h_{\gamma}(z) \in R[z]$. Thus

$$(fP)_{\gamma} = \sum_{\delta \in \Gamma} (g_{\delta}(z) + z^{k_\delta} h_{\delta}(z)) p_{\delta, \gamma}(z) = \sum_{\delta \in \Gamma} g_{\delta}(z) p_{\delta, \gamma}(z) + h_{\gamma}(z) z^{k_\delta} p_{\delta, \gamma}(z)$$

is in $J(R)[z] + (z^{k_\gamma})$, whence $fP \in \ker(\Phi)$ and $P \in \mathcal{M}(X)$ follows. □
4.3. Lemma. If the centre \( Z(R) \) of the ring \( R \) is a field such that \( R/J(R) \) is finite dimensional over \( Z(R) \), then we can exhibit a vector space base of the factor \( Z(R) \)-algebra \( \mathcal{N}(X)/\mathcal{I}(X) \) as

\[
\{ b_t z^i E_{\delta,\gamma} + \mathcal{I}(X) \mid \delta, \gamma \in \Gamma, k_{\delta,\gamma} \leq i \leq k_{\gamma} - 1 \text{ and } 1 \leq t \leq [R/J(R) : Z(R)] \},
\]

where the \( b_t \)'s are fixed elements of \( R \) such that

\[
\{ b_t + J(R) \mid 1 \leq t \leq [R/J(R) : Z(R)] \}
\]

is a vector space base of \( R/J(R) \) over \( Z(R) \). Hence

\[
\dim_{Z(R)}(\mathcal{N}(X)/\mathcal{I}(X)) = [R/J(R) : Z(R)] \sum_{\delta, \gamma \in \Gamma} (k_{\gamma} - k_{\delta,\gamma}).
\]

Using \( \Gamma = \{1, 2, \ldots, m\} \) and the assumption that \( k_1 \geq k_2 \geq \ldots \geq k_m \geq 1 \), we obtain that

\[
\dim_{Z(R)}(\mathcal{N}(X)/\mathcal{I}(X)) = [R/J(R) : Z(R)](k_1 + 3k_2 + \cdots + (2m - 1)k_m).
\]

**Proof.** If \( P = [p_{\delta,\gamma}(z)] \) is in \( \mathcal{N}(X) \), then \( p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_{\delta,\gamma}}) \) as observed earlier. Thus we have

\[
p_{\delta,\gamma}(z) = f_{\delta,\gamma}(z) + \left( \sum_{k_{\delta,\gamma} \leq i \leq k_{\gamma} - 1} a_{\delta,\gamma,i} z^i \right) + z^{k_{\gamma}} g_{\delta,\gamma}(z)
\]

for some \( f_{\delta,\gamma}(z) \in J(R)[z] \), \( a_{\delta,\gamma,i} \in R \) and \( g_{\delta,\gamma}(z) \in R[z] \). In view of the definition of \( \mathcal{I}(X) \) we have

\[
P + \mathcal{I}(X) = \sum_{\delta, \gamma \in \Gamma, k_{\delta,\gamma} \leq i \leq k_{\gamma} - 1} (a_{\delta,\gamma,i} z^i E_{\delta,\gamma} + \mathcal{I}(X)) = \sum_{\delta, \gamma \in \Gamma, k_{\delta,\gamma} \leq i \leq k_{\gamma} - 1} \sum_{t = 1}^{[R/J(R) : Z(R)]} c_t(\delta, \gamma, i)(b_t z^i E_{\delta,\gamma} + \mathcal{I}(X)),
\]

where

\[
a_{\delta,\gamma,i} + J(R) = \sum_{t = 1}^{[R/J(R) : Z(R)]} c_t(\delta, \gamma, i)(b_t + J(R))
\]

for some \( c_t(\delta, \gamma, i) \in Z(R) \). Therefore the cosets \( b_t z^i E_{\delta,\gamma} + \mathcal{I}(X) \) generate \( \mathcal{N}(X)/\mathcal{I}(X) \) over \( Z(R) \). It is straightforward to check the \( Z(R) \)-linear independence of these cosets. \( \Box \)

4.4. Lemma. The ideal \( z M_m(R[z]) \unlhd M_m(R[z]) \) is nilpotent modulo \( \mathcal{I}(X) \), more precisely we have \( (z M_m(R[z]))^n \subseteq \mathcal{I}(X) \), where \( n = \max\{k_{\gamma} \mid \gamma \in \Gamma\} \). There is a natural isomorphism between the factor ring \( \mathcal{N}(X)/(\mathcal{N}(X) \cap z M_m(R[z])) + \mathcal{I}(X) \) and the subring

\[
U(X) = \{ U = [\overline{w_{\delta,\gamma}}] \mid \overline{w_{\delta,\gamma}} \in R/J(R) \text{ and } \overline{w_{\delta,\gamma}} = 0 \text{ if } 1 \leq k_{\delta} < k_{\gamma} \leq n \}
\]

of \( M_m(R/J(R)) \):

\[
\mathcal{N}(X)/(\mathcal{N}(X) \cap z M_m(R[z])) + \mathcal{I}(X) \cong U(X)
\]

and this is an \( (R, R) \)-bimodule isomorphism at the same time. The ideal

\[
(\mathcal{N}(X) \cap z M_m(R[z])) + \mathcal{I}(X)/\mathcal{I}(X) \unlhd \mathcal{N}(X)/\mathcal{I}(X)
\]
is nilpotent with \((N(X) \cap zM_m(R[z]) + I(X)/I(X))^n = \{0\}\) and we have the following isomorphism for the iterated factor:

\[
(N(X)/I(X))/(N(X) \cap zM_m(R[z]) + I(X)/I(X)) \cong N(X)/(N(X) \cap zM_m(R[z]) + I(X)).
\]

**Proof.** Since any entry in the product of the matrices \(Q_1, Q_2, \ldots, Q_n \in zM_m(R[z])\) with \(Q_i = [q_{i,k_\gamma}(z)]\) is a sum of terms of the form

\[
z^{(1)}_{k_\gamma}z^{(2)}_{k_\gamma}(z)\cdots z^{(n)}_{k_\gamma}(z) = \sum q_{k_\gamma}(z)q_{(2)}(z)\cdots q_{k_\gamma}(z),
\]

which is in \((z^n)\), and since \((z^n) \subseteq (z^{k_\gamma})\) for each \(\gamma \in \Gamma\), we obtain that \((zM_m(R[z]))^n \subseteq I(X)\). It follows that \((N(X) \cap zM_m(R[z]) + I(X)/I(X))^n = \{0\}\).

If \(P = [p_{k_\gamma}(z)]\) is in \(N(X)\), then

\[
p_{k_\gamma}(z) = u_{k_\gamma} + z^k_f(z) + z^{k_{k_\gamma}}(v_{k_\gamma} + zg_{k_\gamma}(z))
\]

for some \(u_{k_\gamma} \in J(R), f_{k_\gamma}(z) \in J(R[z], v_{k_\gamma} \in R\) and \(g_{k_\gamma}(z) \in R[z]\).

If \(1 \leq k_\delta < k_\gamma \leq n\), then \(1 \leq k_\gamma - k_\delta = k_{k_\gamma}\) and

\[
p_{k_\gamma}(z) - u_{k_\gamma} = z^k_f(z) + z^{k_{k_{k_\gamma}}}(v_{k_\gamma} + zg_{k_\gamma}(z)) \in (J(R[z] + (z^{k_{k_{k_\gamma}}})) \cap (zR[z])\).
\]

If \(1 \leq k_\gamma \leq k_\delta \leq n\), then \(k_{k_\gamma} = 0\), \(z^{k_{k_{k_\gamma}}} = 1\), \((z^{k_{k_{k_\gamma}}}) = R[z]\) and

\[
p_{k_\gamma}(z) - (u_{k_\gamma} + v_{k_\gamma}) = z^k_f(z) + zg_{k_\gamma}(z) \in (J(R[z] + (z^{k_{k_\gamma}})) \cap (zR[z])\).
\]

Thus \([w_{k_\gamma}] \in M_m(R) \cap N(X)\), \(P - [w_{k_\gamma}] \in N(X) \cap zM_m(R[z])\) and

\[
P + ([N(X)] \cap zM_m(R[z]) + I(X)) = [w_{k_\gamma}] + ([N(X) \cap zM_m(R[z])] + I(X))
\]

in \(N(X)/N(X) \cap zM_m(R[z]) + I(X)\), where

\[
u_{k_\gamma} = \begin{cases}
  u_{k_\gamma} + v_{k_\gamma} \\
  u_{k_\gamma}
\end{cases}
\]

if \(1 \leq k_\gamma \leq k_\delta \leq n\) and \(1 \leq k_\delta < k_\gamma \leq n\) respectively.

If \([w'_{k_\gamma}], [w''_{k_\gamma}] \in M_m(R) \cap N(X)\) and

\[
[w'_{k_\gamma}] + ([N(X) \cap zM_m(R[z])] + I(X)) = [w''_{k_\gamma}] + ([N(X) \cap zM_m(R[z])] + I(X),
\]

then \(w'_{k_\gamma} + J(R) = w''_{k_\gamma} + J(R)\) obviously holds in \(R/J(R)\) (for all \(\delta, \gamma \in \Gamma\)). It follows that the assignment

\[
P + ([N(X) \cap zM_m(R[z])] + I(X)) \mapsto [w_{k_\gamma} + J(R)]
\]

is well defined and gives an

\[
N(X)/N(X) \cap zM_m(R[z]) + I(X) \mapsto U(X)
\]

isomorphism. The isomorphism for the iterated factor is obvious. \(\square\)

We note that, if \(\Gamma = \{1, 2, \ldots, m\}\) and \(k_1 > k_2 > \ldots > k_m \geq 1\), then

\[
U(X) = \begin{bmatrix}
R/J(R) & R/J(R) & \cdots & R/J(R) \\
0 & R/J(R) & \cdots & R/J(R) \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & R/J(R) & R/J(R) \\
0 & \cdots & 0 & R/J(R)
\end{bmatrix}
\]

is an upper triangular matrix algebra. In general, if \(k_1 \geq k_2 \geq \ldots \geq k_m \geq 1\), then \(U(X)\) is a blocked upper triangular matrix algebra over \(R/J(R)\) and the T-ideal of the identities of \(U(X)\) is determined by Lewin’s theorem (see [4]).
4.5. Lemma. For $P \in \mathcal{M}(X)$ and $f = (f_{\gamma}(z))_{\gamma \in \Gamma}$ in $(R[z])^\Gamma$ the formula

$$\psi_P(\Phi(f)) = \Phi(f^P)$$

properly defines an $R$-endomorphism $\psi_P : M \longrightarrow M$ of $\mathcal{M}(X)$ such that $\psi_P \circ \phi = \phi \circ \psi_P$. The assignment $\Lambda(P) = \psi_P$ gives an $\mathcal{M}(X)^{op} \longrightarrow \text{Cen}(\phi)$ homomorphism of $Z(R)$-algebras.

**Proof.** Let $g \in (R[z])^\Gamma$. If $\Phi(f) = \Phi(g)$, then $f - g \in \text{ker}(\Phi)$ implies that $(f - g)P \in \text{ker}(\Phi)$, whence $\Phi(f^P) = \Phi(g^P)$ follows. Since $\Phi$ is surjective, it follows that $\psi_P$ is well defined. It is straightforward to check that

$$\psi_P(\Phi(f) + \Phi(g)) = \psi_P(\Phi(f)) + \psi_P(\Phi(g))$$

for all $f, g \in (R[z])^\Gamma$ and $r \in R$. Thus $\psi_P$ is an $R$-endomorphism. In view of

$$\psi_P(\phi(\Phi(f)) = \psi_P(z \Phi(f)) = \psi_P(\Phi(zf)) = \Phi((zf)P) = \Phi(z(\Phi(P))) = z \Phi(f^P) = z \psi_P(\Phi(f)) = \phi(\psi_P(\Phi(f))),$$

the surjectivity of $\Phi$ gives that $\psi_P \circ \phi = \phi \circ \psi_P$. Clearly,

$$\psi_{P+Q} = \psi_P + \psi_Q$$

ensure that $\Lambda$ is a homomorphism of $Z(R)$-algebras. We deal only with the last identity:

$$\psi_{PQ}(\Phi(f)) = \Phi(f^P)Q = \Phi((f^P)Q) = \psi_Q(\Phi(f^P)) = \psi_Q(\psi_P(\Phi(f)))$$

proves our claim. \[ \square \]

4.6. Lemma. $\mathcal{I}(X) \subseteq \text{ker}(\Lambda)$ ($\Lambda$ is defined in Lemma 4.5). If $R$ is a local ring then $\mathcal{I}(X) = \text{ker}(\Lambda)$.

**Proof.** If $P = [p_{\delta,\gamma}(z)]$ is an element of $\mathcal{I}(X)$ and $f = (f_{\gamma}(z))_{\gamma \in \Gamma}$ is an element of $(R[z])^\Gamma$, then $p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma})$ implies that

$$(f^P)_\gamma = \sum_{\delta \in \Gamma} f_{\delta}(z)p_{\delta,\gamma}(z)$$

is in $J(R)[z] + (z^{k_\gamma})$ for all $\gamma \in \Gamma$, whence $f^P \in \text{ker}(\Phi)$ follows by Proposition 3.1. Since $\psi_P(\Phi(f)) = \Phi(f^P) = 0$, we obtain that $\Lambda(P) = \psi_P = 0$, i.e. that $P \in \text{ker}(\Lambda)$. Thus the containment is proved.

If $R$ is a local ring and $P \in \text{ker}(\Lambda)$, then $\Lambda(P) = \psi_P = 0$ implies that $\psi_P(\Phi(f)) = \Phi(f^P) = 0$ for all $f \in (R[z])^\Gamma$. If $1_{\delta}$ denotes the vector in $(R[z])^\Gamma$ with 1 in its $\delta$-coordinate and zeros in all other places, then $1_{\delta}P \in \text{ker}(\Phi)$ and Proposition 3.1 imply that $p_{\delta,\gamma}(z) \in J(R)[z] + (z^{k_\gamma})$. \[ \square \]

4.7. Lemma. If $\phi \circ \phi = \phi \circ \phi$ holds for an $R$-endomorphism $\phi : M \longrightarrow M$ of $\mathcal{M}(X)$, then there exists an $m \times m$ matrix $P \in \mathcal{M}(X)$ such that

$$\phi(\Phi(f)) = \Phi(f^P)$$

for all $f = (f_{\gamma}(z))_{\gamma \in \Gamma}$ in $(R[z])^\Gamma$. 
Theorem. Let \( \varphi : M \rightarrow M \) be a nilpotent \( R \)-endomorphism of the finitely generated semisimple left \( R \)-module \( pM \). Then \( \Lambda : \mathcal{M}(X)^{op} \rightarrow \text{Cen}(\varphi) \) (defined in Lemma 4.5) is a surjective homomorphism of \( Z(R) \)-algebras, where the centralizer \( \text{Cen}(\varphi) \) is a \( Z(R) \)-subalgebra of \( \text{Hom}_R(M, M) \) and \( m = \dim_R(\ker(\varphi)) \).

Proof. Lemma 4.5 ensures that \( \Lambda : \mathcal{M}(X)^{op} \rightarrow \text{Cen}(\varphi) \) is a homomorphism of \( Z(R) \)-algebras. The surjectivity of \( \Lambda \) follows from Lemma 4.7. To conclude the proof it suffices to note that \( m = |\Gamma| = \dim_R(\ker(\varphi)) \). □

Corollary. Let \( \varphi : M \rightarrow M \) be a nilpotent \( R \)-endomorphism of the finitely generated semisimple left \( R \)-module \( R^{\oplus}M \). Then \( \text{Cen}(\varphi) \) satisfies all of the polynomial identities (with coefficients in \( Z(R) \)) of \( M_{m}^{\oplus}(R[z]) \). If \( R \) is commutative, then \( \text{Cen}(\varphi) \) satisfies the standard identity \( S_{2m} = 0 \) of degree \( 2m \) by the Amitsur-Levitski theorem.

Theorem. Let \( R \) be a local ring and \( \varphi : M \rightarrow M \) a nilpotent \( R \)-endomorphism of the finitely generated semisimple left \( R \)-module \( R^{\oplus}M \). Then the centralizer \( \text{Cen}(\varphi) \) of \( \varphi \) is isomorphic to the opposite of the factor \( \mathcal{N}(X)/\mathcal{I}(X) \) as \( Z(R) \)-algebras:

\[
\text{Cen}(\varphi) \cong \left( \mathcal{N}(X)/\mathcal{I}(X) \right)^{op} \cong \mathcal{N}^{op}(X)/\mathcal{I}(X).
\]

If \( f_i = 0 \) are polynomial identities of the \( Z(R) \)-subalgebra \( \mathcal{U}^{op}(X) \) of \( M_{m}^{\oplus}(R/J(R)) \) with \( f_i \in K(x_1, \ldots, x_r), 1 \leq i \leq n \), then \( f_1 f_2 \cdots f_n = 0 \) is an identity of \( \text{Cen}(\varphi) \) (here \( \varphi^n = 0 \neq 
(=^{n-1} \text{and } m = \dim_R(\ker(\varphi))).

Proof. Theorem 4.8 ensures that \( \text{Cen}(\varphi) \cong \mathcal{M}(X)^{op}/\ker(\Lambda) \) as \( Z(R) \)-algebras. In order to prove the desired isomorphism, it suffices to note that for a local ring \( R \) we have \( \mathcal{M}(X) = \mathcal{N}(X) \) and \( \ker(\Lambda) = \mathcal{I}(X) \) by Lemmas 4.2 and 4.6 respectively. Now Lemma 4.4 ensures that

\[
L = (\mathcal{N}(X) \cap zM_m(R[z])) + \mathcal{I}(X)/\mathcal{I}(X) \triangleleft \mathcal{N}(X)/\mathcal{I}(X)
\]

can be viewed as an ideal of \( \text{Cen}(\varphi) \) such that \( L^n = \{0\} \) and

\[
\text{Cen}(\varphi)/L \cong (\mathcal{N}^{op}(X)/\mathcal{I}(X))/L \cong \mathcal{N}^{op}(X)/(\mathcal{N}(X) \cap zM_m(R[z])) + \mathcal{I}(X) \cong \mathcal{U}^{op}(X).
\]

It follows that \( f_i = 0 \) is an identity of \( \text{Cen}(\varphi)/L \). Thus \( f_i(v_1, \ldots, v_r) \in L \) for all \( v_1, \ldots, v_r \in \text{Cen}(\varphi) \), whence we obtain that \( f_1 f_2 \cdots f_n = 0 \) is an identity of \( \text{Cen}(\varphi) \). □
Proposition 2.3 ensures the existence of a nilpotent Jordan normal base $\Phi$.

Proof. By Theorem 4.10 we have

$$\text{Cen}(\varphi) \cong (\mathcal{N}(X)/\mathcal{I}(X))^{\text{op}},$$

and since

$$\dim_{\mathbb{Z}(R)}(\mathcal{N}(X)/\mathcal{I}(X))^{\text{op}} = \dim_{\mathbb{Z}(R)}(\mathcal{N}(X)/\mathcal{I}(X)),$$

the result follows from Lemma 4.3. □

In the nilpotent case, Theorem 4.11 generalizes the formula for the dimension of the centralizer $\text{Cen}(A)$ of a matrix $A \in M_n(K)$ over a field $K$ (see [9,10]).

5. FURTHER PROPERTIES OF THE CENTRALIZERS

5.1. Theorem. If $R$ is a local ring such that $Z(R)$ is a field and $R/J(R)$ is finite dimensional over $Z(R)$. If $\varphi : M \to M$ is a nilpotent $R$-endomorphism of the finitely generated semisimple left $R$-module $M$, then

$$\dim_{\mathbb{Z}(R)}(\text{Cen}(\varphi)) = |R/J(R) : Z(R)|(k_1 + 3k_2 + \cdots + (2m - 1)k_m),$$

where $k_1 \geq k_2 \geq \cdots \geq k_m \geq 1$ are the sizes of the blocks of the nilpotent Jordan normal base $\Phi$ with respect to $\varphi$.

Proof. By Theorem 4.10 we have

$$\text{Cen}(\varphi) \cong (\mathcal{N}(X)/\mathcal{I}(X))^{\text{op}},$$

and since

$$\dim_{\mathbb{Z}(R)}(\mathcal{N}(X)/\mathcal{I}(X))^{\text{op}} = \dim_{\mathbb{Z}(R)}(\mathcal{N}(X)/\mathcal{I}(X)),$$

the result follows from Lemma 4.3. □

In the nilpotent case, Theorem 4.11 generalizes the formula for the dimension of the centralizer $\text{Cen}(A)$ of a matrix $A \in M_n(K)$ over a field $K$ (see [9,10]).
5.2. Corollary. If $R$ is commutative, $R\mathcal{M}$ is semisimple and $\varphi : M \rightarrow M$ is an indecomposable nilpotent element of the ring $\text{Hom}_R(M, M)$, then the following are equivalent.

1. $\psi \in \text{Cen}(\varphi)$.
2. We can find elements $a_1, a_2, \ldots, a_n$ in $R$ such that
   \[ a_1u + a_2\varphi(u) + \cdots + a_n\varphi^{n-1}(u) = \psi(u) \]
   for all $u \in M$. In other words, $\psi$ is a polynomial of $\varphi$.

Proof. It suffices to prove that if $\sum_{1 \leq j \leq d} R y_j = M$ and
   \[ a_1 y_1 + a_2 \varphi(y_1) + \cdots + a_n \varphi^{n-1}(y_1) = \psi(y_1) \]
holds for all $1 \leq j \leq d$, then we have
   \[ a_1 u + a_2 \varphi(u) + \cdots + a_n \varphi^{n-1}(u) = \psi(u) \]
for all $u \in M$. Since $u = b_1 y_1 + b_2 y_2 + \cdots + b_d y_d$ for some $b_1, b_2, \ldots, b_d \in R$ and $b_j a_i = a_i b_j$, we obtain that
   \[ \psi(u) = \sum_{1 \leq j \leq d} b_j \psi(y_j) = \sum_{1 \leq j \leq d} b_j (a_1 y_1 + a_2 \varphi(y_1) + \cdots + a_n \varphi^{n-1}(y_1)) = \]
   \[ = a_1 \left( \sum_{1 \leq j \leq d} b_j y_j \right) + a_2 \varphi \left( \sum_{1 \leq j \leq d} b_j y_j \right) + \cdots + a_n \varphi^{n-1} \left( \sum_{1 \leq j \leq d} b_j y_j \right) = \]
   \[ = a_1 u + a_2 \varphi(u) + \cdots + a_n \varphi^{n-1}(u). \square \]

5.3. Theorem. Let $R$ be a local ring. If $\varphi : M \rightarrow M$ is a nilpotent $R$-endomorphism of the finitely generated semisimple left $R$-module $R\mathcal{M}$ and $\sigma \in \text{Hom}_R(M, M)$ is arbitrary, then the following are equivalent.

1. $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$.
2. We can find an $R$-generating set $\{y_j \in M \mid 1 \leq j \leq d\}$ of $R\mathcal{M}$ and elements $a_1, a_2, \ldots, a_n$ in $R$ such that
   \[ a_1 \psi(y_j) + a_2 \varphi(\psi(y_j)) + \cdots + a_n \varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j)) \]
   for all $1 \leq j \leq d$ and all $\psi \in \text{Cen}(\varphi)$.

Proof.
(1)$\implies$(2): Obviously, if $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$ then
   \[ a_1 y_j + a_2 \varphi(y_j) + \cdots + a_n \varphi^{n-1}(y_j) = \sigma(y_j) \]
implies that
   \[ a_1 \psi(y_j) + a_2 \varphi(\psi(y_j)) + \cdots + a_n \varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j)) \]
for all $\psi \in \text{Cen}(\varphi)$. Theorem 2.1 ensures the existence of a nilpotent Jordan normal basis $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ of $R\mathcal{M}$ with respect to $\varphi$. Consider the natural projection $\varepsilon_\delta : M \rightarrow N_\delta'$ corresponding to the direct sum $M = N_\delta' \oplus N_\delta''$ (see the proof of 2.3), where
   \[ N_\delta' = \bigoplus_{1 \leq i \leq k_\delta} Rx_{\delta,i} \text{ and } N_\delta'' = \bigoplus_{\gamma \in \Gamma \setminus \{\delta\}, 1 \leq i \leq k_\gamma} Rx_{\gamma,i}. \]
Then $\varepsilon_\delta \in \text{Cen}(\varphi)$, whence $\varepsilon_\delta \in \text{Cen}(\sigma)$ follows for all $\delta \in \Gamma$. Thus $\text{im}(\varepsilon_\delta) = N'_\delta$ and

$$\sigma : \text{im}(\varepsilon_\delta) \longrightarrow \text{im}(\varepsilon_\delta)$$

implies that

$$\sigma(x_{\delta,1}) = \sum_{1 \leq i \leq k_\delta} a_{\delta,i} x_{\delta,i} = h_\delta(z) * x_{\delta,1}$$

for some $h_\delta(z) = a_{\delta,1} + a_{\delta,2} z + \cdots + a_{\delta,k_\delta} z^{k_\delta-1}$ in $R[z]$. Since $\varphi \in \text{Cen}(\sigma)$ implies that $\sigma \in \text{Cen}(\varphi)$, it follows that

$$\sigma(\Phi(f)) = \sum_{\gamma \in \Gamma} \sigma(f_{\gamma}(z) * x_{\gamma,1}) = \sum_{\gamma \in \Gamma} f_{\gamma}(z) * \sigma(x_{\gamma,1}) = \sum_{\gamma \in \Gamma} f_{\gamma}(z) * (h_\gamma(z) * x_{\gamma,1}) =$$

$$= \sum_{\gamma \in \Gamma} (f_{\gamma}(z)h_\gamma(z)) * x_{\gamma,1} = \Phi(fH),$$

where $f \in (R[z])^\Gamma$ and $H = \sum_{\gamma \in \Gamma} h_\gamma(z) E_{\gamma,\gamma}$ is a $\Gamma \times \Gamma$ diagonal matrix in $M(X)$ (note that $H \in M(X)$ is a consequence of $\sigma(\Phi(f)) = \Phi(fH)$). In view of Theorem 4.8, the containment $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$ is equivalent to the condition that $\sigma \circ \psi_P = \psi_P \circ \sigma$ for all $P \in M(X)$. Consequently, we obtain that $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$ is equivalent to the following:

$$\Phi(fPH) = \Phi(\sigma(\Phi(f))) = \sigma(\psi_P(\Phi(f))) = \psi_P(\sigma(\Phi(f))) = \psi_P(\Phi(fH)) = \Phi(\psi_P(fH))$$

for all $f \in (R[z])^\Gamma$ and $P \in M(X)$. Thus $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$ implies that $\Phi(f(\Phi(HH - HP))) = 0$, i.e. that $f(\Phi(HH - HP)) \in \ker(\Phi)$ for all $f$ and $P$.

Now we use that $R$ is a local ring. If $\alpha \in \Gamma$ is an index such that

$$k_\alpha = \max\{k_\gamma \mid \gamma \in \Gamma\} = n,$$

then $E_{\alpha,\delta} \in M(X)$ for all $\delta \in \Gamma$ (see Remark 4.1). Take $e = (1)_{\gamma \in \Gamma}$ and $P = E_{\alpha,\delta}$, then the $\delta$-coordinate of

$$e(E_{\alpha,\delta}H - HE_{\alpha,\delta}) = (h_\delta(z) - h_\alpha(z))eE_{\alpha,\delta}$$

is $h_\delta(z) - h_\alpha(z)$. Since

$$(h_\delta(z) - h_\alpha(z))eE_{\alpha,\delta} \in \ker(\Phi) = \prod_{\gamma \in \Gamma} J(R[z] + (t^k_\gamma)),$$

we obtain that $h_\delta(z) - h_\alpha(z) \in J(R[z] + (z^{k_\delta}))$. Thus

$$\sigma(x_{\delta,1}) = h_\delta(z) * x_{\delta,1} = h_\alpha(z) * x_{\delta,1}$$

for all $\delta \in \Gamma$. It follows that

$$\sigma(x_{\gamma,i}) = \sigma(\varphi^{-1}(x_{\gamma,i})) = \varphi^{-1}(\sigma(x_{\gamma,i})) = \varphi^{i-1}(h_\alpha(z) * x_{\gamma,1}) =$$

$$= h_\alpha(z) * \varphi^{i-1}(x_{\gamma,1}) = h_\alpha(z) * x_{\gamma,i} = a_1 x_{\gamma,i} + a_2 \varphi(x_{\gamma,i}) + \cdots + a_n \varphi^{n-1}(x_{\gamma,i}),$$

where $h_\alpha(z) = a_1 + a_2 z + \cdots + a_n z^{n-1}$.

(2)$\implies$(1): $(R$ is an arbitrary ring) Since $1_M \in \text{Cen}(\varphi)$, we obtain that

$$a_1 y_j + a_2 \varphi(y_j) + \cdots + a_n \varphi^{n-1}(y_j) = \sigma(y_j)$$

for all $1 \leq j \leq d$. If $\psi \in \text{Cen}(\varphi)$, then

$$\psi(\sigma(y_j)) = \psi(a_1 y_j + a_2 \varphi(y_j) + \cdots + a_n \varphi^{n-1}(y_j)) =$$

$$= a_1 \psi(y_j) + a_2 \varphi(\psi(y_j)) + \cdots + a_n \varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$$

for all $1 \leq j \leq d$, whence $\psi \circ \sigma = \sigma \circ \psi$ follows. Thus $\text{Cen}(\varphi) \subseteq \text{Cen}(\sigma)$. □
6. THE CENTRALIZER OF AN ARBITRARY LINEAR MAP

If the field $K$ is arbitrary and $E$ is an extension of $K$, we may assume that $M_n(K) \subseteq M_n(E)$. Let us denote the centralizers of $A \in M_n(K)$ in $M_n(K)$ and in $M_n(E)$ by $\text{Cen}_K(A)$ and $\text{Cen}_E(A)$, respectively. Since

$$M_n(E) \cong E \otimes_K M_n(K),$$

we obtain that

$$\text{Cen}_E(A) \cong E \otimes_K \text{Cen}_K(A).$$

If the field $K$ is infinite, then the $K$-algebra $S$ and the $E$-algebra $E \otimes_K S$ have the same polynomial identities. If $T_K(S) \subseteq K(x_1, \ldots, x_n, \ldots)$ and

$$T_E(E \otimes_K S) \subseteq E \otimes_K K(x_1, \ldots, x_n, \ldots) \cong E(x_1, \ldots, x_n, \ldots)$$

are the $T$-ideals of the $K$-algebra $S$ and the $E$-algebra $E \otimes_K S$, respectively, then

$$T(E \otimes_K S) = E \otimes_K T(S).$$

If the field $K$ is finite, this holds for the multilinear identities only. Hence the information on the (at least multilinear) polynomial identities of $\text{Cen}(A)$ for $K$ arbitrary can be derived from the case when $K$ is algebraically closed.

If $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ is the set of all eigenvalues of $A$, then $\text{Cen}(A)$ is isomorphic to the direct product of the centralizers $\text{Cen}(A_i)$, where $A_i$ denotes the block diagonal matrix consisting of all Jordan blocks of $A$ having eigenvalue $\lambda_i$ in the diagonal.

The number of the diagonal blocks in $A_i$ is $\dim(\ker(A_i - \lambda_i I))$, and the size of $A_i$ is $d_i \times d_i$, where $d_i$ is the multiplicity of the root $\lambda_i$ in the characteristic polynomial of $A$. Since

$$\text{Cen}(A_i) = \text{Cen}(A_i - \lambda_i I_i)$$

and $A_i - \lambda_i I_i$ is nilpotent in $M_{d_i}(K)$, we shall consider the case of a nilpotent matrix.

We note that a multiplicative ($K$ vector space) base of $\text{Cen}(A)$ was constructed in [8], where it was also proved that $\text{Cen}(A)$ is uniquely determined by the block structure of the Jordan normal form of $A$.

6.1. Theorem. Let $A \in M_d(K)$ be a nilpotent matrix and let $J(\text{Cen}(A))$ be the Jacobson radical of $\text{Cen}(A)$. The elementary Jordan matrices in the canonical Jordan form of $A$ are indexed by the elements of $\Gamma = \{1, 2, \ldots, m\}$ and we have

$$k_1 \geq k_2 \geq \ldots \geq k_m \geq 1$$

for the sizes of these elementary Jordan blocks. Then

$$\text{Cen}(A)/J(\text{Cen}(A)) \cong M_{p_1}(K) \oplus \cdots \oplus M_{p_v}(K)$$

for some $u$, where $p_e$ is the number of elementary Jordan matrices of size $e \times e$ and $M_{p_e}(K) = \{0\}$ if $p_e = 0$. The index of nilpotency of $J(\text{Cen}(A))$ is bounded from above by $nv$, where $n = \max\{k_i \mid i \in \Gamma\}$ and $v$ is the number of different sizes.

Proof. The matrix $A$ can be considered as a nilpotent $K$-linear map of the vector space $K^d$. The Jordan normal form of $A$ provides a nilpotent Jordan normal base in $K^d$ with block sizes $k_1 \geq k_2 \geq \ldots \geq k_m \geq 1$. Now $k_{i,j} = k_j - k_i$ when $1 \leq k_i < k_j \leq n$ and $k_{i,j} = 0$ otherwise. The application of Theorem 4.10 gives an isomorphism $\text{Cen}(A) \cong (\mathcal{N}(X)/\mathcal{I}(X))^{\text{op}} \cong \mathcal{N}^{\text{op}}(X)/\mathcal{I}(X)$ of $K$-algebras, where

$$\mathcal{N}(X) = \{P \in M_m(K[z]) \mid P = [p_{i,j}(z)] \text{ and } z^k p_{i,j}(z) \in (z^{k_j}) \text{ for all } 1 \leq i, j \leq m\},$$

$$\mathcal{I}(X) = \{P \in M_m(K[z]) \mid P = [p_{i,j}(z)] \text{ and } p_{i,j}(z) \in (z^{k_j}) \text{ for all } 1 \leq i, j \leq m\}$$

with
and \( z^{k_i}p_{i,j}(z) \in (z^{k_i}) \) is equivalent to \( p_{i,j}(z) \in (z^{k_i}) \) (see the three sentences preceding Remark 4.1). Let \( T_i = K[z]/(z^{k_i}) \), and denote by the same symbol \( z \) the element \( z + (z^{k_i}) \) of \( T_i \). Take \( l_{i,j} = k_{j,i} \) for all \( i, j \in \Gamma \) and consider the set

\[
C_A = \begin{bmatrix}
  z^{l_{1,1}}T_1 & z^{l_{1,2}}T_1 & \cdots & z^{l_{1,m}}T_1 \\
  z^{l_{2,1}}T_2 & z^{l_{2,2}}T_2 & \cdots & z^{l_{2,m}}T_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  z^{l_{m,1}}T_m & z^{l_{m,2}}T_m & \cdots & z^{l_{m,m}}T_m
\end{bmatrix} = \sum_{i,j=1}^m z^{l_{i,j}}T_i E_{i,j}
\]

of \( m \times m \) matrices, where the \( E_{i,j} \)'s are the usual matrix units in \( M_m(K) \). It is straightforward to see that the natural matrix addition and multiplication give a \( K \)-algebra structure on \( C_A \). Using a matrix \( P = [p_{i,j}(z)] \) in \( \mathcal{N}(X) \), the map

\[
P + \mathcal{I}(X) \longrightarrow [p_{i,j}(z) + (z^{k_j})]^{\top}
\]

(here \( ^\top \) denotes the transpose) is well defined and provides an \( \mathcal{N}^{\text{op}}(X)/\mathcal{I}(X) \to C_A \) isomorphism of \( K \)-algebras, whence

\[
\text{Cen}(A) \cong C_A
\]

can be derived. Recall that the Jacobson radical of a finite dimensional algebra is equal to the maximal nilpotent ideal of the algebra. Since \( k_1 \geq k_2 \geq \ldots \geq k_m \), the \( K[z] \)-module

\[
T_A = \begin{bmatrix}
  T_1 & T_1 & \cdots & T_1 \\
  T_2 & T_2 & \cdots & T_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  T_m & T_m & \cdots & T_m
\end{bmatrix} = \sum_{i,j=1}^m T_i E_{i,j}
\]

satisfies \( z^{k_i}T_A = 0 \). The intersection \( I = zT_A \cap C_A \) is an ideal of \( C_A \) and \( I^n = I^{k_1} = 0 \). Hence \( I \subseteq J(C_A) \). Since \( l_{i,j} = k_i - k_j \) when \( i < j \) and \( l_{i,j} = 0 \) if \( i > j \) or \( k_i = k_j \), we obtain that

\[
C_A/I = \begin{bmatrix}
  K & \cdots & K & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  K & \cdots & K & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  K & \cdots & K & K & \cdots & K & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  K & \cdots & K & K & \cdots & K & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  K & \cdots & K & K & \cdots & K & K & \cdots & K \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  K & \cdots & K & K & \cdots & K & K & \cdots & K
\end{bmatrix}
\]
Here the diagonal blocks
\[
\begin{bmatrix}
K & \cdots & K \\
\vdots & \ddots & \vdots \\
K & \cdots & K
\end{bmatrix}
\]
are matrix algebras of size \(p_1 \times p_1, p_2 \times p_2, \ldots, p_u \times p_u\). The number of blocks is equal to the number \(v\) of different sizes of elementary Jordan matrices in the canonical Jordan form of \(A\). Hence the lower triangular part of \(C_A/I\)
\[
\begin{bmatrix}
K & \cdots & K \\
\vdots & \ddots & \vdots \\
K & \cdots & K
\end{bmatrix}
\]
consists of \(v \times v\) lower triangular block matrices, is nilpotent of index \(v\) and is equal to the radical of \(C_A/I\). Hence \((J(C_A)^v)^n \subseteq I^n = \{0\}\) and the class of nilpotency of \(J(C_A)\) is bounded by \(nv\). Clearly
\[
C_A/J(C_A) \cong M_{p_1}(K) \oplus \cdots \oplus M_{p_u}(K).
\]

We note that when \(K\) is of characteristic 0, algebras of the type
\[
R_t = \begin{bmatrix}
K[z]/(z^t) & zK[z]/(z^t) \\
zK[z]/(z^t) & K[z]/(z^t)
\end{bmatrix}
\]
appeared in the description of the T-ideals \(T(S)\) of \(K(x_1, \ldots, x_n)\) containing \(T(M_2(K))\) (see [2]). For every unitary algebra \(S\) such that the T-ideal \(T(S)\) strictly contains \(T(M_2(K))\) there exists a nilpotent algebra \(N\) such that \(T(S) = T(R_t \oplus N^2)\) for a suitable \(t\), where the algebra \(N^2\) is obtained from \(N\) by formal adjoint of 1. (Another description of \(T(S) \supset T(M_2(K))\) was given by Kemer [7].) The algebra \(R_3\) appears also in noncommutative invariant theory (see [3]). The algebra of \(G\)-invariants
\[
(K\langle x_1, \ldots, x_n \rangle/(K\langle x_1, \ldots, x_n \rangle \cap T(S)))^G, \quad n \geq 2,
\]
is finitely generated for every finite subgroup $G$ of $GL_n(K)$ if and only if $T(S)$ is not contained in $T(R_3)$.

For a background on algebras with polynomial identity see e.g. the book by Giambruno and Zaicev [4]. Recall that the PI-degree $\text{PIdeg}(S)$ of a PI-algebra $S$ is equal to the maximal $p$ such that the multilinear polynomial identities of $S$ follow from the multilinear polynomial identities of $M_p(K)$.

6.2. Corollary. Let $A$ be an $n \times n$ matrix over an arbitrary field $K$ and let $p$ be the maximal number of equal elementary Jordan matrices in the canonical Jordan form of $A$ over the algebraic closure of $K$. Then

$$\text{PIdeg}(\text{Cent}(A)) = p.$$ 

Proof. Let

$$A = \sum_{i,j=1}^{n} a_{i,j} E_{i,j}$$

and let $E$ be the algebraic closure of $K$. The centralizer $\text{Cent}_K(A) \subseteq M_n(K)$ consists of all matrices

$$B = \sum_{i,j=1}^{n} \xi_{i,j} E_{i,j} \in M_n(K), \quad \xi_{i,j} \in K,$$

such that

$$AB = \sum_{i,k=1}^{n} \left( \sum_{j=1}^{n} a_{i,j} \xi_{j,k} \right) E_{i,k} = \sum_{i,k=1}^{n} \left( \sum_{j=1}^{n} \xi_{i,j} a_{j,k} \right) E_{i,k} = BA.$$ 

Hence the entries $\xi_{i,j}$ of $B$ are the solutions of the system of $n^2$ homogeneous linear equations

$$\sum_{j=1}^{n} a_{i,j} \xi_{j,k} = \sum_{j=1}^{n} \xi_{i,j} a_{j,k}, \quad i, k = 1, \ldots, n.$$ 

The dimension $\text{dim}_K(\text{Cent}_K(A))$ of $\text{Cent}_K(A)$ over $K$ is equal to the dimension $n^2 - r$ of the vector space of the solutions of the system, where $r$ is the rank of the matrix of the system. Since $\text{Cent}_E(A)$ is obtained from the same homogeneous linear system but considered over $E$, we derive that

$$\text{dim}_E(\text{Cent}_E(A)) = \text{dim}_K(\text{Cent}_K(A)),$$

$$\text{Cent}_E(A) \cong E \otimes_K \text{Cent}_K(A).$$

It is well known that the extension of the base field preserves the PI-degree of the algebra and we may assume that $K$ is algebraically closed. Then for a finite dimensional $K$-algebra $S$ with Jacobson radical $J$ the PI-degree of $S$ is equal to the maximal size of the matrix subalgebras of $S/J$. Applying Theorem 6.1 we complete the proof. □

The algebra $\mathcal{C}_A$, as presented in the proof of Theorem 6.1, has a natural $\mathbb{Z}$-grading assuming that the variable $z$ has degree 1. The component of degree 0 is isomorphic to the factor algebra $\mathcal{C}_A/I$. In characteristic 0 this algebra has several remarkable properties obtained by Giambruno and Zaicev (see [4] for detailed exposition).
It plays a key role in their result about the exponent of PI-algebras. If \( c_n(R) \), 
\( n = 0, 1, 2, \ldots \), is the codimension sequence of the PI-algebra \( R \), then

\[
\exp(R) = \lim_{n \to \infty} \sqrt[n]{c_n(R)}
\]

exists and is a nonnegative integer. The algebras \( CA/I \) are also minimal of given exponent. If \( R \) is a finitely generated PI-algebra with the property that \( \exp(R) > \exp(S) \) for any PI-algebra \( S \) such that the polynomial identities of \( S \) strictly contain the polynomial identities of \( R \), then the polynomial identities of \( R \) coincide with the polynomial identities of one of the algebras \( CA/I \).

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