ON POSITIVE MAPS, ENTANGLEMENT AND QUANTIZATION

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Abstract. We outline the scheme for quantization of classical Banach space results associated with some prototypes of dynamical maps and describe the quantization of correlations as well. A relation between these two areas is discussed.

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1. INTRODUCTION

The aim of this paper is to bring together two areas, theory of positive maps on $\mathcal{C}^*$-algebras and theory of entanglement considered as a peculiar feature of non-commutative Radon measures. Both topics are at the heart of quantum theory, thus in particular, in the foundations of quantum information theory. It will be shown that such pure quantum features as: peculiar behaviour of positive maps, quantum correlations, entanglement, and quantum stochastic dynamics, can be easily obtained within the framework of the $\mathcal{C}^*$-algebraic approach to Quantum Mechanics. This approach sheds new light on entanglement and quantum features of correlations of non-commutative systems. In particular, some emphasis will be put on evolution of entanglement.

The paper is organized as follows. Section 2 provides sufficient preparation for the concept of “quantization” of classical results related to prototypes of dynamical maps. Section 3 is concerned with entanglement and the coefficient of quantum correlations. The latter is again an example of “quantization” of a classical concept. The last section contains a brief discussion of applications of the presented results to the description of quantum dynamical systems. We will discuss the evolution of entanglement for some selected models as well as relations between classification of positive maps and measures of entanglement.

2. POSITIVE MAPS

In this section we compile some basic facts on the theory of positive maps on $\mathcal{C}^*$-algebras. To begin with, let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{C}^*$-algebras (with unit), $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$, $\mathcal{A}^+ = \{a \in \mathcal{A}_h : a \geq 0\}$ - the set of all positive elements in $\mathcal{A}$, and $S(\mathcal{A})$
the set of all states on $\mathcal{A}$. In particular

$$(\mathcal{A}_h, \mathcal{A}^+) \text{ is an ordered Banach space.}$$

We say that a linear map $\alpha : \mathcal{A} \to \mathcal{B}$ is positive if $\alpha(\mathcal{A}^+) \subset \mathcal{B}^+$.

The theory of positive maps on non-commutative algebras can be viewed as a jig-saw-puzzle with pieces whose exact form is not well known. Therefore, as we address this paper to a readership interested in quantum mechanics and quantum information theory, we will focus our attention on “quantization” procedure of some classical (Banach space) results in order to facilitate access to some main problems of that theory.

We begin with the classical Banach-Stone result ([1], [3]): if a unital linear map $T : C(X) \to C(Y)$, where $X, Y$ are compact Hausdorff spaces, is either isometric or an order-isomorphism then it is also an algebraic isomorphism. Thus, even in the Banach space setting, the order and algebraic structures are strongly related. The Banach-Stone theorem has the following non-commutative generalization (Kadison, [14]): a unital isometric or order isomorphic linear map $\alpha : \mathcal{A}_h \to \mathcal{B}_h$ must preserve the Jordan product $(a, b) \mapsto 1/2(ab + ba))$. In other words, this result indicates the role of a specific algebraic structure - the Jordan structure - in operator algebras and that remark will be frequently used throughout the paper. Moreover, such a result makes it legitimate to study and to classify $C^*$-algebras $\mathcal{A}$ by a detailed analysis of ordered Banach spaces $\mathcal{A}_h$. However, this is a very difficult task. In particular, it was soon realized that one of the basic problems is the answer to the following question: which compact convex sets can arise as the state spaces of unital $C^*$-algebras (again a very difficult task!).
To describe the next result we need some preliminaries. Let \((\Omega, \mu)\) be a measure space. Here and subsequently, \(\mu\) stands for a probability measure. The triple \((\text{semigroup } \{S_t\}, \Omega, \mu)\) will denote the classical dynamical system where \(S_t : \Omega \to \Omega\) is a one parameter family of measure preserving maps. The phase functions \(f : \Omega \to \mathbb{C}\) evolve according to the Koopman operators 

\[ V_tf(\omega) = f(S_t\omega), \quad \omega \in \Omega. \]

It is known that the Koopman operators \(V_t\) are isometries on the Banach space \(L^p = L^p(\Omega, \mu), p \geq 1\) of \(p\)-integrable functions and unitary operators when restricted to the Hilbert space \(L^2\) and the transformations \(S_t\) are automorphisms. The relation of the point dynamics with the Koopman operators is clarified by asking the question: what types of isometries on \(L^p\) spaces are implementable by point transformations? For \(L^p\) spaces \(p \neq 2\), all isometries induce underlying point transformations, i.e. if \(\|Vf\| = \|f\|\) for all \(f \in L^p\), then \(V\) is given by an underlying measurable point transformation \(S\) and a certain function \(h\) according to \((Vf)(x) = h(x)f(Sx)\). Such theorems on the implementability of isometries on \(L^p\) spaces, \(p \neq 2\), are known as Banach-Lamperti theorems \([4], [19]\). They are of great importance for the Misra-Prigogine-Courbage theory \([31]\) which is trying to reconcile irreversible phenomena with the basic dynamical laws.

Again, one may “quantize” Banach-Lamperti theorems \([40]\), see also \([2]\). To this end one should use the so called non-commutative (quantum) \(L_p\)-spaces. Namely, using the “quantized” measure theory, let \(\{\mathcal{A}, \varphi\}\) be a von Neumann algebra with faithful normal trace and let \(L_p(\mathcal{A}, \varphi), p \geq 1\), be the corresponding quantum \(L_p\)-space, i.e. a Banach space of operators which is closed under an appropriate norm. Assume that \(T : L_p(\mathcal{A}, \varphi) \to L_p(\mathcal{A}, \varphi)\) is a linear map. Then \(T\) is an \(L_p\)-isometry
if and only if

$$T(x) = WBJ(x), \quad x \in L_p(A, \varphi) \cap A$$

where $W \in A$ is a partial isometry, $B$ a selfadjoint operator affiliated with $A$, $J$ a normal Jordan isomorphism mapping $A$ into a weakly closed $^*$-subalgebra of $A$ such that $W^*W = J(I) = supp(B)$ and $B$ commutes strongly with $J(A)$. Again, we can see the importance of the Jordan structure.

The third example we wish to recall is associated with a very strong notion of positivity: the so called complete positivity (CP). Namely, a linear map $\tau : A \to B$ is CP iff

$$\tau_n : M_n(A) \to M_n(B); \ [a_{ij}] \mapsto [\tau(a_{ij})]$$

is positive for all $n$.

To explain the basic motivation for that concept we need the following notion:

an operator state of $A$ on a Hilbert space $K$ is a CP map $\tau : A \to B(K)$. Having that concept we can recall the Stinespring result, [37], which is the generalization of GNS construction and which was the starting point for a general interest in the concept of complete positivity.

For operator state $\tau$ there is a Hilbert space $H$, a $^*$-representation $\pi : A \to B(H)$ and a partial isometry $V : K \to H$ for which

$$\tau(a) = V^*\pi(a)V.$$

Following the quantization “route”, it was shown

- (Choi, [8]) if $\tau : A \to B$ is a CP order isomorphism then it is a $^*$-isomorphism. This can be considered as a final “quantization” of the Banach-Stone theorem.
• (Arverson, 3) the Hahn-Banach theorem and its order-theoretical version (due to Krein) has a nice generalization for non-commutative structures in terms of CP maps: Let \( \mathcal{N} \) be a closed self-adjoint subspace of \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) containing the identity and let \( \tau : \mathcal{N} \to \mathcal{B}(\mathcal{H}) \) be a CP map. Then \( \tau \) possesses an extension to a CP map \( \tilde{\tau} : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \).

It is worth pointing out that plain positivity is not enough for these generalizations. Moreover, Arverson’s extension theorem is the basis of the CP ideology in open system theory.

Up to now we considered linear positive maps on an algebra without entering into the (possible) complexity of the underlying algebra. The situation changes when one is dealing with composed systems (for example in the framework of open system theory). Namely, there is a need to use the tensor product structure. In particular, again, we wish to consider positive maps but now defined on the tensor product of two \( \mathbb{C}^* \)-algebras, \( \tau : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \). But now the question of order is much more complicated. Namely, there are various cones determining the order structure in the tensor product of algebras (cf. 38)

\[ C_{inj} = (\mathcal{A} \otimes \mathcal{B})^+ \supseteq \ldots \supseteq C_\beta \supseteq \ldots \supseteq C_{pro} = \text{conv}(\mathcal{A}^+ \otimes \mathcal{B}^+) \]

and correspondingly in terms of states (cf. 24)

\[ S(\mathcal{A} \otimes \mathcal{B}) \supseteq \ldots \supseteq S_\beta \supseteq \ldots \supseteq \text{conv}(S(\mathcal{A}) \otimes S(\mathcal{B})). \]

Here, \( C_{inj} \) stands for the injective cone, \( C_\beta \) for a tensor cone, while \( C_{pro} \) for the projective cone. The tensor cone \( C_\beta \) is defined by the property: the canonical bilinear mappings \( \omega : \mathcal{A}_h \times \mathcal{B}_h \to (\mathcal{A}_h \otimes \mathcal{B}_h, C_\beta) \) and \( \omega^* : \mathcal{A}_h^* \times \mathcal{B}_h^* \to (\mathcal{A}_h^* \otimes \mathcal{B}_h^*, C_\beta) \) are positive. The cones \( C_{inj}, C_\beta, C_{pro} \) are different unless either \( \mathcal{A} \), or \( \mathcal{B} \), or both \( \mathcal{A} \) and \( \mathcal{B} \) are abelian. This feature is the origin of various positivity concepts for
non-commutative composed systems and it was Stinespring who used the partial
transposition (transposition tensored with identity map) for showing the difference
among $C_\beta$ and $C_{inj}$ and $C_{pro}$. Clearly, in dual terms, the mentioned property
corresponds to the fact that separable states are different from the set of all states
and that there are various special subsets of states if both subsystems are truly
quantum.

To summarize one can say that contrary to the plain positivity, CP property
plays a dominant role in the programme of quantization of classical results for
composed systems. However, as it will be discussed in the final section, other types
of positivity are helpful for better understanding the relations between various
subsets of states, the algebraic and the order structure.

3. QUANTUM CORRELATIONS

Now we wish to discuss problems associated with the partial order structure of
tensor product of $C^*$-algebras which are related to quantum information theory.
Our first remark is the observation that quantum information theory relies on the
fact that the restriction of a pure state of a composed system to a subsystem is,
in general, not pure. Moreover, if the restriction of a pure state is pure then the
state of the composed system is of the product form. This leads to the observation
that the coupling of observables of a composed system by an entangled state offers
additional possibilities for information exchange as well as a chance to reproduce
states. All that follows from the fact that entangled states exhibit non-classical
correlations.
To be more precise, let $A_1 \subseteq B(\mathcal{H}_1)$ and $A_2 \subseteq B(\mathcal{H}_2)$ be two concrete C*-algebras and define, for a state $\omega$ on $A_1 \otimes A_2$, the following map:

$$(r_1 \omega)(A) \equiv \omega(A \otimes 1) \quad ((r_2 \omega(B) = \omega(1 \otimes B))$$

where $A \in A_1$ ($B \in A_2$). $r_1(2)\omega$ is a state on $A_1(2)$. Moreover: Let $(r_1 \omega)$ be a pure state on $A_1$. Then $\omega$ can be written as a product state on $A_1 \otimes A_2$.

Let $\omega$ be a state on $A_1 \otimes A_2$. The entanglement of formation, EoF, of $\omega$ can be defined as ([23], see also [5])

$$E(\omega) = \inf_{\mu \in M_\omega(S)} \int_S d\mu(\varphi) S(\varphi)$$

where $S(\cdot)$ stands for the von Neumann entropy, i.e. $S(\varphi) = -Tr_\varphi \log \varphi$ where $\varphi$ is the density matrix determining the state $\varphi$. We want to stress that other entropy-functions can be used! The given definition of EoF is based on the decomposition theory and in particular $M_\omega(S) \equiv \{\mu : \omega = \int_S \nu d\mu(\nu)\}$. We recall that the separable states are those which are in the closure of the convex hull of simple tensors (so tensor products of subsystem states) while an entangled state stands for a non-separable one. One can prove [23]

**Theorem 3.1.** A state $\omega \in S$ is separable if and only if $E(\omega)$ is equal to 0.

Let us denote the set of all states on $A \equiv A_1 \otimes A_2$ ($A_1$, $A_2$) by $S(A)$ ($S(A_1)$, $S(A_2)$ respectively). Obviously, $r_1 \omega$ is in $S(A_1)$, $i = 1,2$. Next, take a measure $\mu$ on $S(A)$. Then, using the restriction maps $r_i$ one can define measures $\mu_i$ on $S(A_i)$ in the following way: for a Borel subset $F_i \subset S(A_i)$ we put

$$\mu_i(F_i) = \mu(r_i^{-1}(F_i)), \quad i = 1,2.$$  

Having measures $\mu_1$ and $\mu_2$, both coming from the given measure $\mu$ on $S(A)$, one can define new measure $\boxplus \mu$ on $S(A_1) \times S(A_2)$ which encodes classical correlations.
between the two subsystems described by $A_1$ and $A_2$ respectively (see [21] for details). The measure $\mathbb{E}_\mu$ leads to the concept of coefficient of local (quantum) correlations for $\phi \in S(A), a_1 \in A_1, a_2 \in A_2$, which is defined as
\[
d(\phi, a_1, a_2) = \inf_{\mu \in M_\phi(S(A))} |\phi(a_1 \otimes a_2) - (\int_\xi d(\mathbb{E}_\mu)(\xi))(a_1 \otimes a_2)|
\]

The crucial property of the coefficient of quantum correlations is that $d(\phi, \cdot, \cdot)$ is equal to 0 if and only if the state $\phi$ is separable ([21], [22]). The advantage of using $d(\cdot)$ lies in the fact that that concept looks more operational and that it does not use an entropy function. Moreover, $d(\cdot)$ is nothing else but the “quantization” of the classical concept of coefficient of independence. Hence, we got a strong indication that entangled states contain new type of correlations which are called quantum.

4. SOME APPLICATIONS

4.1. QUANTUM STOCHASTIC DYNAMICS ([25]-[29], [30]). It is well known that in the theory of classical particle systems one of the basic objectives is to produce, describe and analyze dynamical systems with an evolution originated from stochastic processes in such a way that their equilibrium states are given Gibbs states (cf. [20]). A well known illustration of such an approach are systems with the so called Glauber dynamics [11]. To carry out the analysis of dynamical systems with evolution originated from stochastic processes, it is convenient to use the theory of Markov processes in the framework of $L_p$-spaces. In particular, for the Markov-Feller processes, using the unique correspondence between the process and the corresponding dynamical semigroup, one can give a recipe for the construction
of Markov generators (see [20]). The correspondence uses the concept of conditional expectation which can be nicely characterized within the (classical) $L_p$-space framework (cf. [32]). Furthermore, (classical) $L_p$ spaces are extremely useful in a detailed analysis of the ergodic properties of the evolution.

However, as contemporary science is based on quantum mechanics, it is again legitimate to look for a quantization of the above approach. That task was carried out in the setting of quantum mechanics and the main ingredient of the quantization was the concept of generalized conditional expectation and Dirichlet forms defined in terms of non-commutative (quantum) $L_p$-spaces. We already met these spaces in the description of quantized Banach-Lamperti theorems. The advantage of using quantum $L_p$-spaces for the quantization of stochastic dynamics lies in the fact that we can follow the traditional “route” of analysis of dynamical systems and that it is possible to have a single scheme for the quantum counterparts of stochastic dynamics of both jump and diffusive type.

Turning to concrete dynamical systems, for example to jump type evolutions, we recall that one of the essential ingredients of the $L_p$-space approach to the analysis of such evolutions, is the usage of local knowledge. To illustrate that idea let us consider a region $\Lambda_I$ (usually finite) and its environment $\Lambda_{II}$. Then, performing an operation over $\Lambda_I$ (e.g. a block-spin flip or a symmetry transformation) one is changing locally the reference state. Such a change can be expressed in terms of generalized conditional expectations. Guided by the classical theory, one can define, now in terms of generalized conditional expectations, the infinitesimal generator of quantum dynamics. It is important to note that such a dynamics is the result of local operations (associated with the mentioned local knowledge about the system).
Then, having defined the dynamics, we should pose the natural question of its nontriviality. By this we understand, first of all, that the infinitesimal generator of the dynamics is \textit{not} a function of the hamiltonian defining the reference Gibbs state. This requirement arises in a natural way from the methodology of constructing stochastic dynamics as sketched in the preceding paragraph. In fact, it has been shown \cite{30} that generators defined within the $L_p$-space setting satisfy the above requirement. On the other hand, in order to confirm that the constructed dynamics are interesting, and the genuine quantum counterparts of classical dynamical maps it is necessary to study the evolution of entanglement and correlations as measures of coupling between two subsystems caused by local (e.g. block-spin flip) operations. Going in that direction, an analysis of stochastic quantum models based on reference systems determined by Ising type and XXZ hamiltonians (\cite{17}, see also \cite{16}) was done. It has shown the tendency of enhancement of quantum correlations. In the first example, based on one dimensional Ising model with nearest neighbor interactions, the lack of production of quantum correlations was shown. This is to be expected because the Ising model illustrates a behaviour typical of classical interactions (cf \cite{6}). The second example, based on the quantum XXZ model with more interesting and complicated features of propagation, provides clear signatures of production of quantum correlations.

4.2. \textbf{POSITIVE MAPS VERSUS ENTANGLEMENT.} The analysis of evolution of entanglement which was described in the previous subsection indicates that there is a need for an operational measure of entanglement. This demand is strengthened by the observation that the amount of states that can be used for quantum information is measured by the entanglement. On the other hand, the
programme of classification of entanglement seems to be a very difficult task. In particular, it was realized that the first step must presumably take the full classification of all positive maps. To see this let us take a positive map $\alpha_{1,t}: A_1 \to A_1$, $t$ being the time, and consider the evolution of a density matrix $\rho$ ($\rho$ determines the state $\phi \in S(A \otimes B)$), i.e. we wish to study $(\alpha_{1,t} \otimes id_2)^d \rho$. Here $(\alpha_{1,t} \otimes id_2)^d$ stands for the dual map, i.e. for the dynamical map in the Schrödinger picture. Then, if $\rho$ is an entangled state, $(\alpha_{1,t} \otimes id_2)^d \rho$ may develop negative eigenvalues and thus lose consistency as a physical state. That observation was the origin of rediscovery, now in the physical context, of Stinespring’s argument saying that the tensor product of transposition with the identity map can distinguish various cones in the tensor product structure (see Section 2). This led to the criterion of separability ([33], [13]) saying that only separable states are globally invariant with respect to the family of all positive maps. However, criterions of that type are not operational. Even worse, they are strongly related to a classification of positive maps. In particular, the old open problem concerning the description of non-decomposable maps was revived. To describe that problem we need (cf [35]):

Let $\tau: A \to B(H)$ be a linear, positive map. $\tau$ is called decomposable if there exists a Hilbert space $K$, a bounded linear map $V: H \to K$ and a C*-homomorphism $\pi: A \to B(K)$ such that $\tau = V^* \pi V$. C*-homomorphism means that $\pi(\{a, b\}) = \{\pi(a), \pi(b)\}$ where $\{\cdot, \cdot\}$ stands for anticommutator, i.e. $\pi$ preserves the Jordan structure! A more subtle notion is the following: $\tau$ is locally decomposable if for $0 \neq x \in H$, there exists a Hilbert space $K_x$, $V_x: K_x \to H$ and a C*-homomorphism $\pi_x$ of $A$ to $B(K_x)$ such that

$$V_x \pi_x(a) V_x^* x = \tau(a) x$$
for all $a \in A$.

It is known (39, 7) that for the case $M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$ with $k = 2 = l$ and $k = 2$, $l = 3$ all positive maps are decomposable. Then, the criterion for separability simplifies significantly. Namely, to verify separability it is enough to analyse $(\tau \otimes id)^d$, with $\tau$ being the transposition, as other positive maps are just convex combinations of CP maps (they always map states into states) and the composition of CP map with $\tau \otimes id$.

The situation changes dramatically when both $k$ and $l$ are larger than 2. In that case there are plenty of non-decomposable maps (see 15 and the references given there) and to analyse entanglement one cannot restrict oneself to study $\tau \otimes id$. Thus, a full description of positive maps is needed. Furthermore, one can construct examples of entangled states using concrete non-decomposable maps (see 12). However, the classification of non-decomposable maps is a difficult task which is still not completed (36, 18).

We want to close the section with an important remark. Namely, if $d(\phi, A) = 0$ for any $A \in A_1 \otimes A_2$ then, using the description of locally decomposable maps, one can show that the state $\phi$ is separable [22]. This result shows how strong the interplay between separability and certain subtle features of positive maps is. However, this is not unexpected as the full correspondence between Schrödinger and Heisenberg picture relies on the underlying algebraic structure and geometry of the state space, see [11, 9], and [10].

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