A pure radiation spacetime with Cosmological constant and Closed Timelike Curves

Faizuddin Ahmed

Hindustani Kendriya Vidyalaya, Dinesh Ojah Road, Guwahati-781005, India

Abstract

Here we present an axially symmetric pure radiation solution of the field equations with a Cosmological constant. The spacetime is free-from curvature singularity and admits a non-twisting, shear-free null geodesic congruence. The physical interpretation of this solution, based on the study of the equation of the geodesics deviation, will be presented. It is demonstrated that this spacetime can be understood as exact transverse gravitational waves with amplitude $A_+$ of “+” polarization modes, which propagate on anti-de Sitter backgrounds. The spacetime admits closed timelike curves (CTCs) which develop at some particular moment from an initial spacelike hypersurface in a causally well behaved manner in constant z-plane. The presented spacetime is a four dimensional generalizations of the Misner space in curved spacetime.

Keywords: non-vacuum spacetime, pure radiation, Cosmological constant, closed timelike curves

PACS numbers: 04.20.Jb, 04.20.-q, 04.30.Nk, 04.20.Gz

1 Introduction

The Weyl tensor of a Lorentzian conformally non-flat four-dimensional manifold distinguishes at each point at most four null directions. These directions,
known already to Cartan [1], are nowadays called principal null directions (PNDs). In the case of vacuum spacetimes (with or without Cosmological constant), a principal null direction is always geodesic and shear-free [2]. If the number of principal null directions at each point of the spacetime is equal to one, then the Riemann tensor of such a spacetime is, by definition, of type N. The Petrov algebraic classification of the Weyl tensor and the asymptotic forms of radiative fields of spatially bounded sources (by peeling theorem [3, 4]) demonstrate that solutions of type N play a fundamental role in the theory of gravitational radiation. The various algebraically special Petrov types have some interesting physical interpretations in the context of gravitational radiation. Particularly for our interest, type N spacetime, it has only one repeated principal null direction of multiplicity 4, which means that all four PNDs coincide. The non-vanishing components of the Weyl scalar is $\Psi_4$ and this corresponds to transverse waves propagate along the geodesic null congruences. A comprehensive discussion of the Petrov classification would be in [5, 6, 7, 8, 9, 10]. To determine the Petrov type of a metric, one only needs to calculate the Weyl scalars in the null tetrad ($k, l, m, \bar{m}$, where $k, l$ are real and $m, \bar{m}$ are complex conjugate of one another).

The physical meaning of the different Weyl scalar functions constructed from the independent components of the Weyl tensor are:

- $\Psi_0$: transverse wave component propagating in the $l$ direction,
- $\Psi_1$: longitudinal wave component propagating in the $l$ direction,
- $\Psi_2$: Coulomb-like component,
- $\Psi_3$: longitudinal wave component propagating in the $k$ direction,
- $\Psi_4$: transverse wave component propagating in the $k$ direction.

All solutions of the vacuum Einstein equations with in general non-zero Cosmological constant, which are of type N with non-twisting null congruences, are known. A complete class of such non-twisting type N vacuum solutions with Cosmological constant $\Lambda$ was found by Garcia Diaz and Plebaski.
and analysed further by Ozsvath, Robinson and Rozga \cite{12} and by Bicak and Podolsky metric \cite{13} (see also \cite{5, 14, 15, 16, 17, 18}). These Einstein spaces represent exact pure gravitational waves which propagate in constant curvature backgrounds, \textit{i.e.}, in Minkowski, de Sitter or anti-de Sitter Universe (for $\Lambda = 0$, $\Lambda > 0$ or $\Lambda < 0$, respectively). The special class type N vacuum solutions with $\Lambda \neq 0$ being considered are those null geodesic congruences whose null tangent vector field, is non-diverging and shear-free. These are the generalization of Kundt metrics \cite{19, 20}, a general class of solutions to the Einstein field equations which admit a non-expanding, shear-free, and twist-free null geodesic congruence. Kundt also identified new solutions of various algebraically special types, among them specifically type N spacetimes representing plane-waves which exhibit geometrically different properties than the famous $pp$-waves.

The Cosmological constant Einstein field equations with pure radiation field are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad \text{or} \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

(1)

where $\mu, \nu = 1, 2, 3, 4$. Here $R_{\mu\nu}$ is the Ricci tensor, $R$ is the scalar curvature, $g_{\mu\nu}$ is the metric tensor, $\Lambda$ is the cosmological constant and $T_{\mu\nu}$ is the energy-momentum tensor. The energy-momentum tensor of pure radiation field \cite{5} is

$$T^{\mu\nu} = 2 \Phi_{22} k^\mu k^\nu = \rho k^\mu k^\nu, \quad k^\mu k_\mu = 0,$$

(2)

where $\rho$ is the energy density of pure radiation field and $k^\mu$ is tangent vector field of the null congruence the radiation propagates along. The local conservation law $T^\mu_{;\nu} = 0$ implies that the radiation propagates along geodesics

$$k_{\mu;\nu} k^\mu = 0.$$  

(3)

Taking trace of the field equations we get

$$R = 4 \Lambda.$$

(4)
Substituting this into the field equations (1) we have

\[ R_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu} = \Lambda g_{\mu\nu} + \rho k_\mu k_\nu. \] (5)

Some known solutions of the Einstein field equations admits closed time-like curves (CTCs) which violates the causality condition. A small samples of CTC spacetime are in [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. One way of classifying such causality violating spacetimes would be to categorize as either eternal time machine spacetime in which CTCs either form everywhere or preexist (e.g. [33, 34, 35] etc.) or as time machine spacetime in which CTCs develop at some particular in a causally well behaved manner (e.g. [36, 37, 38] etc.). However, many of the CTC spacetimes constructed so far, suffer from some severe physical problems for time machine spacetimes. The CTC spacetimes discussed in [39, 40, 41, 42, 43] violate the Weak energy condition (WEC) indicating unrealistic matter-energy sources, and hence are unphysical. The WEC states that for any physical (timelike) observer the energy density is non-negative, which is the case for all known types of (classical) matter fields. Godel’s Cosmological model [33] and Mallet’s solution [44] does not admit a partial Cauchy surface (an initial spacelike hypersurface). The time machine models in [45, 46, 47] violate the Strong energy condition and/or there is a curvature singularity [48, 49, 50]. Some other CTC spacetimes admit a naked singularity and may represent Cosmic Time Machines. The author and collaborators constructed CTC spacetimes with a naked singularity which may represent such Cosmic Time Machines [51, 52, 53], quite recently. Hawking proposed a Chronology Protection Conjecture [54] which states that the laws of physics will always prevent a spacetime to form CTCs. However, the general proof of Chronology protection conjecture has not yet existed. A four dimensional curved spacetime, satisfying all the energy conditions, but with causality violating, primarily that CTCs evolve smoothly from an initial spacelike hypersurface in a causally well-behaved manner, would be acceptable as time machine spacetime than an eternal one.
2 Analysis of the Pure Radiation Spacetime

Consider the following axisymmetric spacetime in \((r, \phi, z, t)\) coordinates

\[
ds^2 = g_{rr} dr^2 + g_{zz} dz^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + 2 g_{z\phi} dz d\phi,
\]

where the different metric functions are

\[
\begin{align*}
g_{rr} &= \coth^2(\alpha r), \\
g_{zz} &= \sinh^2(\alpha r), \\
g_{\phi\phi} &= -\sinh t \sinh^2(\alpha r), \\
g_{t\phi} &= \frac{1}{2} g'_{\phi\phi}, \\
g_{z\phi} &= \beta z \sinh(\alpha r),
\end{align*}
\]

where prime denotes derivative with respect to time. Here \(\phi\) coordinate is periodic \(\phi \in [0, 2\pi)\), where \(\alpha > 0\) and \(\beta > 0\) are real number. We have labelled the coordinates as \(x^1 = r, x^2 = \phi, x^3 = z\) and \(x^4 = t\). The ranges of the other coordinates are \(0 \leq r < \infty, -\infty < z < \infty\) and \(-\infty < t < \infty\). The metric has signature \((+, +, +, -)\) and the determinant of the corresponding metric tensor \(g_{\mu\nu}\) is

\[
det g = -\frac{1}{4} \cosh^2(\alpha r) \sinh^4(\alpha r) \cosh^2 t,
\]

which vanishes at \(r = 0\).

The spacetime (6) is a non-vacuum solution of the field equations and the non-components of the Einstein tensor are

\[
G_{\mu}^\nu = 3 \alpha^2, \quad G_{\phi}^t = -2 \beta \csch^2(\alpha r) \sech t,
\]
where \( \text{csch} = \text{cosech} \). The non-zero components of the Ricci tensor \( R_{\mu \nu} \) are

\[
\begin{align*}
R_{rr} &= -3 \alpha^2 \coth^2(\alpha r), \\
R_{\phi \phi} &= \beta + 3 \alpha^2 \sinh t \sinh^2(\alpha r), \\
R_{\phi z} &= R_{z \phi} = -3 \alpha^2 \beta z \sinh^2(\alpha r), \\
R_{\phi t} &= R_{t \phi} = \frac{3}{2} \alpha^2 \cosh t \sinh^2(\alpha r), \\
R_{zz} &= -3 \alpha^2 \sinh^2(\alpha r).
\end{align*}
\]

(10)

The spacetime (6) is free-from the curvature singularity since the curvature invariants

\[
\begin{align*}
R_{\mu \mu} &= R = -12 \alpha^2, \\
R_{\mu \nu} R_{\mu \nu} &= 36 \alpha^4, \\
R_{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} &= 24 \alpha^4, \\
R_{\mu \nu \rho \sigma} R^{\rho \sigma \tau \kappa} R_{\tau \kappa} &= -48 \alpha^6, \\
R_{\mu \nu \rho \sigma \tau} R_{\mu \nu \rho \sigma \tau} &= 36 \alpha^4,
\end{align*}
\]

(11)

are non-vanishing constants.

Consider a null vector \( k_{\mu} = \delta_{\phi}^\mu \) for metric (6). The non-zero component of the energy-momentum tensor (1) is

\[
T_{\phi \phi} = \rho.
\]

(12)

From the field equations (5) using (10) and (12), we get

\[
\Lambda = -3 \alpha^2, \quad \rho = \beta > 0.
\]

(13)

The energy density of pure radiation field which is constant satisfy the null energy condition (NEC) for any null vector.

Consider the Killing vector \( \eta = \partial_\phi \) which has the normal form

\[
\eta^\mu = (0, 1, 0, 0).
\]

(14)
Its co-vector form is

\[ \eta_\mu = \left( 0, -\sinh t \sinh^2(\alpha r), \beta z \sinh^2(\alpha r), -\frac{1}{2} \cosh t \sinh^2(\alpha r) \right). \quad (15) \]

Eqn. (14) satisfies the Killing equation \( \eta_{\mu;\nu} + \eta_{\nu;\mu} = 0 \). For cyclically (or circularly) symmetric metric, the norm of the Killing vector is spacelike, closed orbits \([55, 56, 57, 58, 59, 60, 61]\). For axially symmetry metric, the norm of the spacelike Killing vector must vanish at \( r = 0 \) \([5, 57, 62, 63, 64, 65, 66, 67, 68, 69, 70]\). In our case, the norm of the Killing vector \( \partial_\phi \) is

\[ X = \eta_\mu \eta^\mu = g_{\mu\nu} \eta^\mu \eta^\nu = g_{\phi\phi} = -\sinh t \sinh^2(\alpha r), \quad (16) \]

which is spacelike in the region \( t < 0 \) and vanish as \( r \to 0^+ \). The norm of the Killing vector \( \eta_\mu \) changes sign for \( t > 0 \) implying the formation of timelike curves which we discuss in the last section. However, the spacetime (6) fails to satisfy the elementary flatness condition. In our metric, we find that

\[ \frac{(\nabla_\mu X)(\nabla^\mu X)}{4X} \to -1 - \beta^2 z^2 \operatorname{csch} t, \quad (17) \]

as \( r \to 0^+ \) for \( t < 0 \). That means, the spacetime defined by (6) is not locally flat near the symmetry axis. According to MacCallum et al \([71]\), the symmetry axis is not regular, it is singular. According to Stephani et al \([5]\), there are conical singularities (rods or struts) on the axis.

### 3 Classification and kinematical properties of the spacetime

For the Petrov classification of metric (6), we can construct the following set of null tetrad vectors \( (k, l, m, \bar{m}) \). They are

\[ k_\mu = (0, 1, 0, 0), \quad (18) \]
$l_\mu = \left( 0, \frac{1}{2} \sinh t \sinh^2(\alpha r), -\beta z \sinh^2(\alpha r), \frac{1}{2} \cosh t \sinh^2(\alpha r) \right)$, \hspace{1cm} (19)

$m_\mu = \frac{1}{\sqrt{2}} \left( \coth(\alpha r), 0, i \sinh(\alpha r), 0 \right)$, \hspace{1cm} (20)

$\bar{m}_\mu = \frac{1}{\sqrt{2}} \left( \coth(\alpha r), 0, -i \sinh(\alpha r), 0 \right)$, \hspace{1cm} (21)

where $i = \sqrt{-1}$. The set of tetrad vector are such that the metric tensor for the line element (6) can be expressed as

$$g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu.$$ \hspace{1cm} (22)

We set up an orthonormal frame $e^{(a)} = \{e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}\}$, $e^{(a)} \cdot e^{(b)} = \delta_{ab}$ which consists of three spacelike unit vectors $e^{(i)}$, $i = 1, 2, 3$ and one timelike vector $e^{(4)} = u$. These frame components can conveniently be expressed using the corresponding natural null tetrad vectors (18)—(21) as

$$k = \frac{1}{\sqrt{2}}(e^{(4)} + e^{(2)}), \quad l = \frac{1}{\sqrt{2}}(e^{(4)} - e^{(2)}),$$

$$m = \frac{1}{\sqrt{2}}(e^{(1)} + i e^{(3)}), \quad \bar{m} = \frac{1}{\sqrt{2}}(e^{(1)} - i e^{(3)}).$$ \hspace{1cm} (23)

Using the set of null tetrad vectors (18)—(21) we calculate the five Weyl scalars of which, only

$$\Psi_4 = C_{\mu\nu\rho\sigma} l^\mu \bar{m}^\nu l^\rho \bar{m}^\sigma = -\frac{\beta}{2}$$ \hspace{1cm} (24)

is non-vanishing, while rest are $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$. The Weyl tensor satisfies the Bel criteria $C_{\mu\nu\rho\sigma} k^\sigma = 0$. Physically, the non-zero Weyl scalar $\Psi_4$ denotes a transverse wave component propagating along the principal null direction $k^\mu$ of multiplicity 4. Thus the metric (6) is of type N in the Petrov classification scheme.

In General Relativity, optical scalars refer to a set of three scalar functions $\Theta$ (expansion), $\sigma$ (shear) and $\omega$ (twist/rotation) describing the propagation
of a geodesic null congruences. Using the null tetrad (18) we calculate the Optical scalars the expansion, the twist and the shear [5] and they are

\[ \Theta = \frac{1}{2} k_{;\mu} = 0, \quad \omega^2 = \frac{1}{2} k_{[\mu;\nu]} k^{\mu;\nu} = 0, \]
\[ \sigma \bar{\sigma} = \frac{1}{2} k_{(\mu;\nu)} k^{\mu;\nu} - \Theta^2 = 0. \]  

Hence the spacetime (6) thus provides an example of non-twisting, shear-free type N pure radiation spacetimes with Cosmological constant. We can define CSI spacetimes as those Lorentzian spacetimes for which all the curvature invariants constructed from the metric, the Riemann tensor and its derivative of arbitrary order are constant. Accordingly, CSI stands for Constant Scalar Invariants and CSI spacetimes generally have a non-zero cosmological constant. CSI spacetimes physically are interesting, as it is guaranteed that the curvature invariants do not blow up. The presented spacetime belongs to CSI spacetimes since all the curvature invariants are constants and are polynomials in terms of Cosmological constant \( \Lambda \).

4 The Relative Motions of Test Particles

In order to analyze the effects of the gravitational and matter fields of the above solution, it is natural to investigate the specific influence of various components of these fields on the relative motion of a free test particle. Such a local characterization of spacetimes, based on the equation of geodesic deviation frame, was described by Pirani [72, 73] and Szekeres [74, 75] (see also [76, 77])

\[ \frac{D^2 Z^\mu}{d\tau^2} = -R^\mu_{\nu\rho\sigma} u^\nu Z^\rho u^\sigma, \]  

where \( u \cdot u = -1 \), is the four-velocity of a free test particle (observer), and \( Z(\tau) \) is the displacement vector. By projecting (26) onto the orthonormal frame \( e_{(a)} \) given by equations (23) we get

\[ \ddot{Z}^{(i)} = -R^{(i)}_{(4)(j)(4)} Z^{(j)}, \]  

where

\[ D^2 \]
where \( i, j = 1, 2, 3, \ e_{(4)} = u \) and
\[
R^{(i)}_{(4)(j)(4)} = e^{(i)}_{\mu} u^\nu e^{(j)}_{\rho} u^\sigma R_{\mu\nu\rho\sigma}.  \tag{28}
\]
The frame components of the displacement vector \( Z^{(j)} \equiv e^{(j)}_{\mu} Z^\mu \) determine directly the distance between close test particles, and
\[
\ddot{Z}^{(i)} = e^{(i)}_{\mu} \left( \frac{D^2 Z^\mu}{d\tau^2} \right),  \tag{29}
\]
are their physical relative accelerations. Eq. (26) also implies
\[
\ddot{Z}^{(4)} = -u^\mu (\frac{D^2 Z^\mu}{d\tau^2}) = R_{\mu\nu\rho\sigma} u^\mu u^\nu Z^\rho u^\sigma = 0,  \tag{30}
\]
so that \( Z^{(4)} = a_0 \tau + b_0, \ a_0, b_0 \) are constants. Setting \( Z^{(4)} = 0 \) all test particles are synchronized by the proper time. From the standard definition of the Weyl tensor for metric (6) we get
\[
R^{(i)(j)(4)(4)} = C^{(i)(j)(4)(4)} + \frac{1}{2} (\delta_{ij} S^{(4)(4)} - S^{(i)(j)}) - \frac{\Lambda}{3} \delta_{ij},  \tag{31}
\]
where \( C^{(i)(j)(4)(4)} \) are the components of the Weyl tensor, \( R = 4 \Lambda \) and \( S^{(a)(b)} \) respectively denote the trace and the traceless part of the Ricci tensor.

In the natural tetrad (18)—(21), the only non-vanishing Weyl scalars is given by (24) so that
\[
C^{(1)(4)(1)(4)} = \frac{1}{2} \Re \Psi_4, \quad C^{(2)(4)(2)(4)} = -\frac{1}{2} \Re \Psi_4,
\]
\[
C^{(1)(4)(2)(4)} = -\frac{1}{2} \Im m \Psi_4 = 0,  \tag{32}
\]
where \( \Re, \Im m \) stands for real and complex. From the field equations (5) we get
\[
S_{\mu\nu} = R_{\mu\nu} - \frac{R}{4} g_{\mu\nu} = 2 \Phi_{22} k_\mu k_\nu.
\]
Hence
\[
S^{(i)(j)} = 2 \Phi_{22} e^{(i)}_{\mu} e^{(j)}_{\nu} k_\mu k_\nu,
\]
\[
= \Phi_{22} e^{(i)}_{\mu} (e^{(4)}_{\mu} + e^{(2)}_{\mu}) (e^{(4)}_{\nu} + e^{(2)}_{\nu}),
\]
\[
S^{(4)(4)} = \Phi_{22} e^{(4)}_{\mu} (e^{(4)}_{\mu} + e^{(2)}_{\mu}) (e^{(4)}_{\nu} + e^{(2)}_{\nu}),
\]
\[
= \Phi_{22}.  \tag{33}
\]
Therefore, the equation of geodesic deviation (27) takes the form

\[
\ddot{Z}^{(1)} = -\frac{1}{2} \frac{1}{2} \phi_{22} Z^{(1)} - \frac{1}{2} \phi_{22} Z^{(1)},
\]

\[
\ddot{Z}^{(2)} = -\frac{1}{2} \frac{1}{2} \phi_{22} Z^{(2)} + \frac{1}{2} \phi_{22} Z^{(2)},
\]

\[
\ddot{Z}^{(3)} = -\frac{1}{2} \frac{1}{2} \phi_{22} Z^{(3)} + \frac{1}{2} \phi_{22} Z^{(3)},
\]

where \( A_+ = \frac{1}{2} \frac{1}{2} \phi_{22} \).

Equations (34)—(36) are well suited for physical interpretation. All test particles move isotropically one with respect to the other \( (\ddot{Z}^{(i)} = \frac{1}{2} \frac{1}{2} Z^{(i)}), \)

\( i = 1, 2, 3 \) if \( \phi_{22} = 0 \) and the amplitude \( A_+ = 0 \), \( i.e., \) the Weyl scalars \( \phi_4 \) vanish. No gravitational wave is present in this case. This agrees with the fact that the presented vacuum solution is conformally flat. The only conformally flat vacuum solution with \( \Lambda < 0 \) is the anti-de Sitter spacetime, maximally symmetric solution of constant negative curvature. This explains the resulting isotropic motions. Thus, the terms proportional to \( \Lambda \) in (34)—(36) represent the influence of the anti-de Sitter background. Clearly, the relative motions of nearby test particles depend on:
1. the Ricci scalar $R$ (or the cosmological constant $\Lambda$) responsible for overall background isotropic motions;

2. the terms depending on the local free gravitational field, and consisting of only transverse (ingoing) component with amplitudes given by $\Re \Psi_4$ representing the effect of gravitational waves on the particles in type N space-time;

3. the energy-momentum tensor $T_{(a)(b)}$ (pure radiation field $\Phi_{22}$) terms describing interaction of matter-content which affects the motion.

The structure of the equations (34)–(36) thus interpretate that the solution (6) represents an exact gravitational waves which propagate in the background Universe (an anti-de Sitter space). Note that, the amplitudes $\Phi_{22}$ and $\Psi_4$ which represent radiation depend on the real number $\beta > 0$.

Since $A_+ = \frac{1}{2} \Re \Psi_4 = -\frac{\beta}{4}$, and $\Phi_{22} = \frac{1}{2} S_{\mu \nu} l^\mu l^\nu = \frac{\beta}{2}$, Eq. (34)–(36) reduces to

$$\ddot{Z}^{(1)} = \frac{\Lambda}{3} Z^{(1)}, \quad \Lambda < 0,$$

(37)

$$\ddot{Z}^{(2)} = \frac{\Lambda}{3} Z^{(2)} + A_+ Z^{(2)} = -\left(\frac{\beta}{4} - \frac{\Lambda}{3}\right) Z^{(2)},$$

(38)

$$\ddot{Z}^{(3)} = \frac{\Lambda}{3} Z^{(3)} + A_+ Z^{(3)} = -\left(\frac{\beta}{4} - \frac{\Lambda}{3}\right) Z^{(3)}.$$  

(39)

The solution of the above equation (37)–(39) are

$$Z^{(1)}(\tau) = A_1 \cos\left(\sqrt{-\frac{\Lambda}{3}} \tau\right) + B_1 \sin\left(\sqrt{-\frac{\Lambda}{3}} \tau\right),$$

(40)

$$Z^{(2)}(\tau) = A_2 \cos\left(\sqrt{\frac{\beta}{4} - \frac{\Lambda}{3}} \tau\right) + B_2 \sin\left(\sqrt{\frac{\beta}{4} - \frac{\Lambda}{3}} \tau\right),$$

(41)

$$Z^{(3)}(\tau) = A_3 \cos\left(\sqrt{\frac{\beta}{4} - \frac{\Lambda}{3}} \tau\right) + B_3 \sin\left(\sqrt{\frac{\beta}{4} - \frac{\Lambda}{3}} \tau\right),$$

(42)

where $A_i, B_i, i = 1, \ldots, 3$ are arbitrary constants.
5 Closed Timelike Curves of the Spacetime

The presented spacetime (6) admits closed timelike curves which appear after a certain instant of time, and thus the metric violates the causality condition.

Consider an azimuthal closed curve $\gamma$ defined by $r = r_0, z = z_0$ and $t = t_0$, where $r_0 > 0$, $z_0$, $t_0$ are constants. From metric (6) we have

$$ds^2 = - \sinh t \sinh^2(\alpha r) d\phi^2. \quad (43)$$

These curves are null for $t = 0$, spacelike throughout for $t = t_0 < 0$, but become timelike for $t = t_0 > 0$, which indicates the presence of CTCs. Hence CTCs form at a definite instant of time satisfying $t = t_0 > 0$. These curves evolve from an initial spacelike $t = \text{const}$ hypersurface. This can be ascertained by determining the sign of the component $g^{tt}$ in the metric tensor $g^{\mu\nu}$. From metric (6) we find that

$$g^{tt} = \frac{4 \text{sech}^2 t}{\sinh^2(\alpha r)} \left[ \beta^2 z^2 + \sinh t \right]. \quad (44)$$

A hypersurface $t = \text{const} = t_0$ is spacelike provided $g^{tt} < 0$ for $t = t_0 < 0$, timelike provided $g^{tt} > 0$ for $t = t_0 > 0$ and null $g^{tt} = 0$ for $t = t_0 = 0$. In our case, the hypersurface $t = \text{const} = t_0 < 0$ is spacelike ($r \neq 0$) and can be chosen as initial conditions over which the initial data may be specified. Here we have chosen constant $z = \text{planes}$ defined by $z = z_0$, where $z_0$, a constant equal to zero such that the hypersurface $t = \text{const} = t_0 < 0$ is spacelike ($g^{tt} = \frac{4 \sinh t}{\cosh^2 t \sinh^2(\alpha r)}$) and null hypersurface $t = \text{const} = t_0 = 0$. There is a Cauchy horizon at $t = t_0$ for any such spacelike hypersurface $t = \text{const} = t_0 < 0$. The null curve $t = t_0$ serves as the Chronology horizon (since $g^{tt} = 0$) which separates the spacetime a chronal region without CTCs to a non-chronal region with CTCs. Hence the spacetime evolve from an initial spacelike hypersurface in a causally well-behaved and the formation of CTCs takes place from causally well-behaved initial conditions in $z = \text{const}$-plane.
The formation of CTCs here is thus identical to the metric of Misner space \([78]\). The Misner space is a locally flat metric and regular everywhere.

The Misner space metric in 2D \([78, 79]\) is

\[
\text{ds}_{\text{Misn}}^2 = -2 dT dX - T dX^2,
\]

(45)

where \(-\infty < T < \infty\) and \(X\) coordinate is periodic. The Misner space metric is a flat space and regular everywhere. The curves \(T = T_0\), where \(T_0\) is a constant, are closed since \(X\) is periodic. The curves \(T = T_0 < 0\) are spacelike, but become timelike for \(T = T_0 > 0\) and null curve \(T = T_0 = 0\) serves as the Chronology horizon. The second type of curves, namely, \(T = T_0 > 0\) are closed timelike curves (CTCs).

An important note is that for constant \(r\) and \(z\), the line element (6) reduces to

\[
\text{ds}^2 = -\sinh^2(\alpha r) (\cosh t dt d\phi + \sinh t d\phi^2),
\]

(46)
a Misner space metric form. Thus, the presented metric is a four-dimensional generalization of flat Misner space in curved spacetime.

6 Conclusions

In this paper, we presented an axially symmetric solution of the field equations with pure radiation as that for matter-energy content which satisfies the energy condition and a Cosmological constant. The spacetime admits a non-expanding, non-twisting, and shear-free null geodesic congruence and belongs to type N in the Petrov classification scheme. The physical interpretation of the presented solution, based on the study of the equation of the geodesic deviation, was presented. It was demonstrated that this spacetime can be understood as exact gravitational waves consisting of transverse wave components with amplitude \(A_+\) of “+” polarization modes, which propagate on anti-de Sitter backgrounds. The spacetime admits closed timelike curves which develop at some particular moment from an initial spacelike hypersurface in a causally well behaved manner in constant \(z - \text{planes}\). Most of
the CTC spacetimes violate one or more energy condition or need unrealistic matter source and/or does not admit a partial Cauchy surface and hence are unphysical. The presented spacetime is free-from all these problems and a four-dimensional generalization of flat Misner space in curved spacetime.

References

[1] E. Cartan, C. R. Acad. Sci. Paris 174, 857 (1922).

[2] J. N. Goldberg and R. K. Sachs, Acta Phys. Polon. Suppl. 22, 13 (1962).

[3] R. K. Sachs, Proc. R. Soc. London Ser. A 264, 309 (1961).

[4] R. K. Sachs, Recent Developments in General Relativity, pp. 521-562, Pergamon Press, Oxford (1962).

[5] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact Solutions to Einstein's Field Equations, Cambridge University Press, Cambridge (2003).

[6] A. Z. Petrov, Uch. Zapiski Kazan. Gos. Univ. 114(8), 55 (1954). English translation A. Z. Petrov, Gen. Rel. Grav. 32(8), 1665 (2000).

[7] J. B. Griffiths and J. Podolsky, Exact Space-Time in Einstein’s General Relativity, Cambridge University Press, Cambridge (2009).

[8] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press, Oxford (1998).

[9] R. Penrose, Ann. Phys. 10, 171 (1960).

[10] Eric Poisson, A Relativist’s Toolkit : The Mathematics of Black-hole Mechanics, Cambridge University Press, Cambridge (2004).

[11] A. Garcia Diaz and J. F. Plebanski, J. Math. Phys. 22, 2655 (1981).
[12] I. Ozsvath, I. Robinson, and K. Rozga, J. Math. Phys. 26, 1755 (1985).

[13] J. Bicak and J. Podolsky, J. Math. Phys. 40, 4495 (1999); J. Math. Phys. 40, 4506 (1999).

[14] J. Podolsky, Ph.D. dissertation, Department of Theoretical Physics, Charles University, Prague (1993).

[15] N. Van den Bergh, E. Gunzig, and P. Nardone, Class. Quantum Grav. 7, L175 (1990).

[16] H. Salazar, A. Garcia Diaz, and J. F. Plebanski, J. Math. Phys. 24, 2191 (1983).

[17] S. T. C. Siklos, Galaxies, Axisymmetric Systems and Relativity, edited by M A H MacCallum, Cambridge University Press, Cambridge (1985).

[18] J. Podolsky and M. Ortaggio, Class. Quantum Grav. 20, 1685 (2003).

[19] W. Kundt, Z. Phys. 163, 77 (1961).

[20] W. Kundt, Proc. R. Soc. A 270, 328 (1962).

[21] W. J. van Stockum, Proc. R. Soc. Edin. 57, 135 (1937).

[22] C. W. Misner and A. H. Taub, Sov. Phys. JETP 28, 122 (1969).

[23] B. Carter, Phys. Rev. 174, 1559 (1968).

[24] W. B. Bonnor, Class. Quantum Grav. 18, 1381 (2001).

[25] W. B. Bonnor, Class. Quantum Grav. 19, 5951 (2002).

[26] W. B. Bonnor, Int. J. Mod. Phys. D 12, 1705 (2003).

[27] W. B. Bonnor and B. R. Steadman, Gen. Rel. Grav. 37, 1833 (2005).

[28] F. Lobo and P. Crawford, NATO Science Series 95, 289 (2003).
[29] S. V. Krasnikov, Class. Quantum. Grav. 15, 997 (1998).
[30] D. Sarma, M. Patgiri and F. Ahmed, Gen. Rel. Grav. 46, 1633 (2014).
[31] F. Ahmed, In Press Ann. Phys. (2017).
[32] F. Ahmed, Prog. Theor. Exp. Phys. 2017, 043E02 (2017).
[33] K. Gödel, Rev. Mod. Phy. 21, 447 (1949).
[34] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
[35] J. R. Gott, Phys. Rev. Lett. 66, 1126 (1991).
[36] A. Ori, Phys. Rev. Lett. 95, 021101 (2005).
[37] F. Ahmed, B. B. Hazarika and D. Sarma, Euro. Phys. J. Plus 131, 230 (2016).
[38] F. Ahmed, Commun. Theor. Phys. 67, 189 (2017).
[39] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).
[40] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).
[41] M. Alcubierre, Class. Quantum Grav. 11, L73 (1994).
[42] A. E. Everett, Phys. Rev. D 53, 7365 (1996).
[43] A. E. Everett and T. A. Roman, Phys. Rev. D 56, 2100 (1997).
[44] R. L. Mallett, Found. Phys. 33, 1307 (2003).
[45] Y. Soen and A. Ori, Phys. Rev. D 54, 4858 (1996).
[46] K. D. Olum, Phys. Rev. D 61, 124022 (2000).
[47] A. Ori and Y. Soen, Phys. Rev. D 49, 3990 (1994).
[48] F. J. Tipler, Phys. Rev. D 9, 2203 (1974).

[49] A. Ori, Phys. Rev. D 76, 044002 (2007).

[50] K. D. Olum and A. E. Everett, Found. Phys. Lett. 18, 379 (2005).

[51] D. Sarma, F. Ahmed and M. Patgiri, Adv. HEP 2016, 2546186 (2016).

[52] F. Ahmed, Adv. HEP 2017, 7943649 (2017).

[53] F. Ahmed, In Press Adv. HEP (2017).

[54] S. W. Hawking, Phys. Rev. D 46, 603 (1992).

[55] B. Carter, Commun. Math. Phys. 17, 233 (1970) ; J. Math. Phys. 10, 70 (1969).

[56] J. Bičák and B. D. Schmidt, J. Math. Phys. 25, 600, (1984).

[57] M. Mars and J. M. M. Senovilla, Class. Quantum Grav. 10, 1633 (1993).

[58] A. Barnes, Class. Quantum Grav. 17, 2605 (2000).

[59] A. Barnes, Class. Quantum Grav. 18, 5511 (2001).

[60] M. Heusler, Black Hole Uniqueness Theorems, Cambridge University Press, Cambridge (1996).

[61] A. A. Garcia, C. Campuzano, arXiv pre-Print : gr-qc/0310054.

[62] M. Mars and J. M. M. Senovilla, Class. Quantum Grav. 12, 2071 (1995).

[63] M. A. H. MacCallum, Gen. Rel. Grav. 30, 131 (1998).

[64] M. Mars and J. M. M. Senovilla, Lect. Notes in Phys. Vol. 423, F. J. Chinea, L. M. Gonzales-Romero (editors), Springer, Berlin, pp. 141-146 (1993).
[65] M. Mars, Ph. D. Thesis, Universitat de Barcelona (1995).

[66] J. Carot, J. M. M. Senovilla and R. Vera, Class. Quantum Grav. 16, 3025 (1999).

[67] J. Carot, Class. Quantum Grav. 14, 2675 (2000).

[68] J. C. N. de Araujo and Anzhong Wang, Gen. Rel. Grav. 32, 1971 (2000).

[69] A. Wang, Phys. Rev. D 68, 064006 (2003).

[70] A. Y. Miguelote, M. F. A. da Silva, Anzhong Wang and N. O. Santos, Class. Quantum Grav. 18, 4569 (2001).

[71] M. A. H. MacCallum and N. O. Santos, Class. Quantum Grav. 15, 1627 (1998).

[72] F. A. E. Pirani, Acta Phys. Polon. 15, 389 (1956).

[73] F. A. E. Pirani, Phys. Rev. 105, 1089 (1957).

[74] P. Szekeres, J. Math. Phys. 6, 1387 (1965).

[75] P. Szekeres, J. Math. Phys. 7, 751 (1966).

[76] J. Bičák and J. Podolský, J. Math. Phys. 40, 4506 (1999).

[77] J. Podolský and M. Ortaggio, Class. Quantum Grav. 20, 1685 (2003).

[78] C. W. Misner, Relativity Theory and Astrophysics I. Relativity and Cosmology, Lectures in Applied Mathematics vol. 8, edited by J. Ehlers, American Mathematical Society, Providence (1967).

[79] D. Levanony and A. Ori, Phys. Rev. D 83, 044043 (2011).

[80] S. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge (1973).