THE BRJUNO FUNCTIONS OF THE BY-EXCESS, ODD, EVEN AND ODD-ODD CONTINUED FRACTIONS AND THEIR REGULARITY PROPERTIES

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Abstract. The Brjuno function was introduced by Yoccoz to study the linearizability of holomorphic germs and other one-dimensional small divisor problems. The Brjuno functions associated with various continued fractions including the by-excess continued fraction were subsequently investigated: it was conjectured that the difference between the classical Brjuno function and the even part of the Brjuno function associated with the by-excess continued fraction extends to a Hölder continuous function of the whole real line. In this paper, we prove this conjecture and we extend its validity to the more general case of Brjuno functions with positive exponents.

Moreover, we study the Brjuno functions associated to the odd and even continued fractions introduced by Schweiger. We show that they belong to all $L^p$ spaces, $p \geq 1$. We prove that the Brjuno function associated to the odd continued fraction differs from the classical Brjuno function by a Hölder continuous function. On the other hand, the Brjuno function associated to the even continued fraction differs from the classical Brjuno function by a sum of a Hölder continuous function and a Brjuno-type function associated to the odd-odd continued fraction, introduced in the study of the best approximations of the form odd/odd.

1. Introduction

Let $x \in \mathbb{R} \setminus \mathbb{Q}$: the regular continued fraction (RCF) expansion of $x$ is defined by

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}} =: [a_0; a_1, a_2, \ldots, a_n, \ldots],$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. The principal convergents of $x$ are the finite truncations $P_n(x)/Q_n(x) := [a_0; a_1, \ldots, a_n]$ for $n \geq 0$. We call $x \in \mathbb{R}$ a Brjuno number if $x$ satisfies the Brjuno condition: $\sum_{n=0}^{\infty} \frac{\log(Q_{n+1}(x))}{Q_n(x)} < \infty$. The Brjuno condition was introduced by A.D. Brjuno [Brj71, Brj72]: in the case of germs of holomorphic diffeomorphisms of one complex variable with an indifferent fixed point, extending the fundamental work of C.L. Siegel [Sie42], Brjuno proved that all germs with linear part $\lambda = e^{2\pi xi}$ are linearizable if $x$ is a Brjuno number. In 1988, J.-C. Yoccoz [Yoc88, Yoc95] proved that this condition is also necessary.

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Figure 1. Numerical computation of Yoccoz’s Brjuno function associated to the Gauss map at $10^4$ uniformly distributed random points in the interval $(0,1)$.

Similar results hold for the local conjugacy problem of analytic diffeomorphisms of the circle [Yoc02], for some complex area–preserving maps [Mar90, Dav94, CM22]. In a somewhat different but closely related context, it is conjectured that the Brjuno condition is optimal for the existence of real analytic invariant circles in the standard family [Mac88, Mac89, MS92]. See also [BG01, Gen15] for related results.

Yoccoz’s work used a rigorous renormalization technique and the $\text{PGL}_2(\mathbb{Z})$ action on $\mathbb{R} \setminus \mathbb{Q}$ plays an important role: he thus found convenient to reformulate the Brjuno condition in terms of an arithemtical function, the Brjuno function, which satisfies a cocycle equation [MMY01] under this action, see [MMY01, Appendix 5]. The Gauss map $G : (0,1] \to [0,1]$ is defined by $G(x) = \{1/x\}$, where $\{t\}$ is the fractional part of $t$. Let $x_n = G^n(\{x\})$, $\beta_n(x) = x_0 x_1 \cdots x_n$, $n \geq 0$, and $\beta_{-1} \equiv 1$. J.-C. Yoccoz [Yoc88] defined the Brjuno function $B : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ defined by

$$B(x) = - \sum_{n=0}^{\infty} \beta_{n-1}(x) \log x_n,$$

see Figure 1 for the graph. The set of Brjuno numbers can be characterized as the set where $B(x) < \infty$. Indeed, there is a positive constant $C$ such that

$$(1.2) \quad \left| B(x) - \sum_{n=0}^{\infty} \log(Q_{n+1}(x)) \right| < C.$$

Yoccoz proved that $B(x) = - \log x + xB(1/x)$ on $(0,1) \setminus \mathbb{Q}$ and $B(x) = B(x + 1)$ thus the set of Brjuno numbers is invariant under the modular group $\text{PSL}_2(\mathbb{Z})$. In his work on the linearizability of quadratic polynomials, Yoccoz showed that $\Upsilon(x) := B(x) + \log r(x)$ defined on the set of Brjuno numbers is uniformly bounded, where $r(x)$ is the convergence radius of the linearization of the quadratic polynomial $P_\lambda(z) = \lambda z + z^2$ with $\lambda = e^{2\pi x i}$ [Yoc95, BC04]. With the second author, Moussa and Yoccoz conjectured that $\Upsilon$ is 1/2-Hölder continuous [MMY97], a fact prompted also by the numerical study in [Mar90]. It was later proved that

$^1$In his work Yoccoz actually uses the nearest-integer continued fraction map, which will be denoted $A_{1/2}$ in our work and is introduced below, instead of the Gauss map to define the Brjuno function. However the difference between the two functions is a positive Hölder continuous function which extends to the whole real line, as proved in [MMY97], thus they can both be used equivalently for investigating small divisor problems.
the interpolation function $\Upsilon$ extends continuously to $\mathbb{R}$ [BC06] and that if $\eta > 1/2$, $\Upsilon$ cannot be $\eta$-Hölder continuous [Ché08]. The restriction $\Upsilon$ to the set of the high type numbers is $1/2$-Hölder continuous [CC15], where the high type numbers are the real numbers whose nearest-integer continued fraction partial quotients are at least $N_0$, for some large fixed $N_0 \geq 2$.

The interplay between the Hölder regularity properties and various actions of (subgroups of) the modular group, already at the heart of the works [MMY97, MMY01, MMY06, LMNN10] will be our object of the investigation. We will discuss the $L^p$ and the Hölder regularity properties of the difference between $B$ and the Brjuno functions associated with the by-excess, odd and even continued fractions regarded as classical continued fractions by Vallée [Val03, Val06]. These three Brjuno functions satisfy cocycle equations under actions of $\text{PSL}_2(\mathbb{Z})$, the index 2-, and an index 3-congruence subgroups of level 2, respectively, see Appendix C for a proof of this fact. We will give further motivations for our research and state the main theorems in the following subsections.

1.1. Brjuno functions associated with the by-excess continued fractions. Let $0 \leq \alpha \leq 1$. The $\alpha$-continued fraction map is given by

$$A_\alpha(x) = |1/x - [1/x + 1 - \alpha]| \text{ for } x \in (0, \bar{\alpha}],$$

where $\bar{\alpha} = \max\{\alpha, 1-\alpha\}$. This one-parameter family of maps contains the the regular, the nearest-integer and the by-excess continued fractions as three particular cases, corresponding respectively to $\alpha = 1, 1/2, 0$. Nakada [Nak81] investigated metrical properties of the $\alpha$-continued fractions and explicitly computed the absolutely continuous invariant measure for $\frac{1}{2} \leq \alpha \leq 1$. The dependence on the parameter $\alpha$ of the metric entropy of the map $A_\alpha$ has been object of several studies in [LM08, CMPT10, CT13].

The $(\alpha, \nu)$-Brjuno function is defined by

$$B_{\alpha,\nu}(x) = -\sum_{n=0}^{\infty} (\beta_{\alpha,n-1}(x))^{\nu} \log A_\alpha^n(x) \text{ for } x \in \mathbb{R} \setminus \mathbb{Q},$$

where $x_{\alpha,0} = |x - [x + 1 - \bar{\alpha}]|$ 3 and $\beta_{\alpha,n}(x) = x_0 A_{\alpha,1}(x_0) \cdots A_{\alpha,n}(x_0)$, $n \geq 0$ and $\beta_{\alpha,-1} \equiv 1$. From the definition it follows easily that $B_{\alpha,\nu}$ satisfies

$$B_{\alpha,\nu}(x) = -\log x + x^{\nu} B_{\alpha,\nu}(1/x) \text{ for } x \in (0, \bar{\alpha}) \setminus \mathbb{Q},$$

$$B_{\alpha,\nu}(x) = B_{\alpha,\nu}(-x) \text{ for } x \in (0, 1 - \bar{\alpha}) \setminus \mathbb{Q} \text{ and } B_{\alpha,\nu}(x) = B_{\alpha,\nu}(x + 1) \text{ for } x \in \mathbb{R} \setminus \mathbb{Q}. \text{ The difference } B - B_{\alpha,1} \text{ is uniformly bounded for all } \alpha \in [1/2, 1] \text{ and, as we have recalled several times, for } \alpha = 1/2 \text{ (the nearest-integer continued fraction), the difference } B - B_{1/2,1} \text{ is } 1/2\text{-Hölder continuous.}$$

These (generalized) Brjuno functions with a positive real exponent $\nu$, which will be called $\nu$-Brjuno functions. In our recent study with Petrykiewicz and Schindler [LMPS], we investigated the BMO property of the $\nu$-Brjuno functions 4 associated with the $\alpha$-continued fractions motivated by the study of (quasi) modular forms in [Pet14, Pet17].

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2In [Nak81], the $\alpha$-continued fraction map is defined by $f_{\alpha}(x) = |1/x - [1/x + 1 - \alpha]|$ for $x \neq 0, x \in [\alpha - 1, \alpha).

3We correct the initial setup of $x_{\alpha,0} = |x - [x + 1 - \alpha]|$ in [LMNN10] to $x_{\alpha,0} = |x - [x + 1 - \bar{\alpha}]|$ since $x_{\alpha,0}$ should be $x$ for $x \in (0, \pi]$.

4We called the $\nu$-Brjuno function the $k$-Brjuno function in [LMPS].
The graph of \( B(x) - [B_{0,1}(x) + B_{0,1}(-x)] \).

The \( \alpha \)-Brjuno functions for \( \alpha \in [0, 1/2) \) are explored in [LMNN10], where it’s proved that \( B_{\alpha, \nu} - B_{\alpha, \nu} \) is uniformly bounded for \( \alpha \in (0, 1/2) \). Moreover, the authors studied the Brjuno function \( B_{0,1} \) associated with the by-excess continued fraction (\( \alpha = 0 \)). The \((0, 1)\)-Brjuno function \( B_{0,1} \), which is called the semi-Brjuno function, has a different property: indeed one can prove that

\[
B(x) - [B_{0,1}(x) + B_{0,1}(-x)]
\]

is uniformly bounded. The corresponding continued fraction to \( A_0 \) is called the by-excess continued fraction which has the expansion given by

\[
x = \frac{1}{a_1 - \frac{1}{\ddots - \frac{1}{a_n - \ddots}}}
\]

for \( x \in (0,1] \),

where \( a_n = \lceil 1/A_{0}^{n-1}(x) \rceil \). In [LMNN10], from a numerical computation of \( B(x) - [B_{0,1}(x) + B_{0,1}(-x)] \), see also Figure 2a, and by analogy with what had been proved for the difference between \( B \) and \( B_{1/2,1} \), it was conjectured that \( B(x) - [B_{0,1}(x) + B_{0,1}(-x)] \) is 1/2-Hölder continuous.

In this paper (see Section 2) we will prove this conjecture. Moreover we will also prove an extension of this conjecture to all the \((0, \nu)\)-Brjuno functions for \( 0 < \nu \leq 2 \):

**Theorem 1.** For \( 0 < \nu \leq 2 \), the function \( B_{1,\nu}(x) - [B_{0,\nu}(x) + B_{0,\nu}(-x)] \) can be extended from \( \mathbb{R} \setminus \mathbb{Q} \) to \( \mathbb{R} \) a \( \nu/2 \)-Hölder continuous function.

### 1.2. Brjuno functions of the odd continued fraction and the even continued fraction.

Schweiger [Sch82,Sch84] first investigated continued fractions with the odd partial quotients and the even partial quotients, which are called the odd continued fraction (OCF) and the even continued fraction (ECF), respectively. We partition \( [0,1] \) into \( I_n := \left[ \frac{1}{n+1}, \frac{1}{n} \right] \), for \( n \in \mathbb{N} \). The continued fraction maps of the ECF and the OCF are defined by

\[
A_{\text{odd}}(x) = \begin{cases} 
\frac{1}{x} - (2k - 1), & \text{if } x \in I_{2k-1}, \\
(2k + 1) - \frac{1}{x}, & \text{if } x \in I_{2k},
\end{cases}
\quad \text{and} \quad
A_{\text{even}}(x) = \begin{cases} 
2k - \frac{1}{x}, & \text{if } x \in I_{2k-1}, \\
\frac{1}{x} - 2k, & \text{if } x \in I_{2k},
\end{cases}
\]

for \( k \in \mathbb{N} \), see Figure 5 and 6a for their graphs. Analogously to what we did for defining the \( \nu \)-Brjuno function associated with the \( \alpha \)-continued fraction maps, we denote respectively
The graphs of $B_{\text{even},1}$ (upper-left), $B_{\text{oo},1}$ (upper-right), $B_{\text{even},1} + \frac{1}{2} B_{\text{oo},1}$ (lower-left) and $B - [B_{\text{even},1} + \frac{1}{2} B_{\text{oo},1}]$ (lower-right).

with $B_{\text{odd},\nu}$ and $B_{\text{even},\nu}$ the $\nu$-Brjuno functions associated with the OCF and the ECF, defined by

\[(1.7)\quad B_{\text{odd},\nu}(x) = -\sum_{n=0}^{\infty} \beta_{\nu,n-1}(x_{o,0}) \log x_{o,n} \quad \text{and} \quad B_{\text{even},\nu}(x) = -\sum_{n=0}^{\infty} \beta_{\nu,n-1}(x_{e,0}) \log x_{e,n},\]

respectively, by setting $x_{o,0} = x_{e,0} = \|x\|_{2Z} := \min_{n \in \mathbb{Z}} \{x, 2n\}$,

\[(1.8)\quad x_{o,n} := A_{\text{odd}}^n(x_{o,0}), \quad x_{e,n} := A_{\text{even}}^n(x_{e,0}),\]

$\beta_{o,n}(x) := \prod_{i=0}^{n} x_{o,i}$, and $\beta_{e,n}(x) := \prod_{i=0}^{n} x_{e,i}$, $n \geq 0$ with $\beta_{o,-1}(x) = \beta_{e,-1}(x) = 1$, see Figure 2b and 3 for their graphs. By definition, both $B_{\text{odd},\nu}$ and $B_{\text{even},\nu}$ are even $2\mathbb{Z}$-periodic functions, i.e. they both satisfy

\[(1.9)\quad f(x) = f(-x) \text{ for all } x \in (0, 1) \setminus \mathbb{Q} \text{ and } f(x) = f(x + 2) \text{ for all } x \in \mathbb{R} \setminus \mathbb{Q}.\]

It is well known that there is a very strict connection between the RCF and the geodesic flow on the modular surface $\mathbb{H}/\text{SL}_2(\mathbb{Z})$ [Ser85]. Boca and Merriman established connections of the OCF with the geodesic flow on $\mathbb{H}/\Gamma$, and of the ECF with that on $\mathbb{H}/\Theta$ [BM18], where
For function and the semi-Brjuno function: the classical Brjuno function and given by the lacunary Fourier series $R_0$ for details.

With these functional equations, the OCF- and ECF-Brjuno function can be interpreted as $B$ is determined by the finiteness of the even Brjuno function with exponent $B(1.12)$ $\Gamma \Theta$ study of regularity properties of automorphic forms under the action of congruence subgroups $\Gamma$ and $\Theta$ of level 2, respectively. The OCF- and ECF Brjuno functions satisfy (1.9) and

$$B_{\text{odd},\nu}(x) = -\log x + x^\nu B_{\text{odd},\nu}(1 - 1/x) \text{ for all } x \in (0, 1) \setminus \mathbb{Q},$$

$$B_{\text{even},\nu}(x) = -\log x + x^\nu B_{\text{even},\nu}(-1/x) \text{ for all } x \in (0, 1) \setminus \mathbb{Q}.$$ With these functional equations, the OCF-, ECF-Brjuno function can be interpreted as cocycles under the actions of $\Gamma \sqcup (-1/0 \ 1) \Gamma$, $\Theta \sqcup (-1/0 \ 1) \Theta$, respectively, see Appendix C for the details.

We prove a Hölder regularity property of $B_{\text{odd},\nu}$ which shows how the difference between the classical Brjuno function and $B_{\text{odd},\nu}$ is as regular as that between the classical Brjuno function and the semi-Brjuno function:

**Theorem 2.** For $\nu > 0$, the function $B_{1,\nu}(x) - B_{\text{odd},\nu}(x)$ is uniformly bounded. Moreover, for $0 < \nu \leq 2$, $B_{1,\nu}(x) - B_{\text{odd},\nu}(x)$ can be extended to a $\nu/2$-Hölder continuous function on $\mathbb{R}$.

A further motivation for the study of $B_{\text{even},\nu}$ comes from the surprising role that this function plays in a very classical problem. In the 1850s Riemann suggested that the function given by the lacunary Fourier series $S(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{k^2}$ was an example of a continuous but nowhere differentiable function. After the classical investigations of Hardy and Littlewood and especially Gerver’s [Ger71] and Jaffard’s [Jaf96] works, show that $S$ is only differentiable at rational numbers of the reduced form with odd numerator and denominator. Itatsu [Ita81] reproved the non-differentiability and differentiability on the rationals by using a relation between $S$ and the theta function $\theta(x) = \sum_{n \in \mathbb{Z}} \exp(i\pi n^2 x)$, which is an automorphic form of weight $1/2$ under the action of $\Theta$.

More recently, Revoal and Seuret [RS15] dealt with Dirichlet series

$$F_s(x, t) = \sum_{k=1}^{\infty} \frac{\exp(i\pi k^2 x + 2\pi ikt)}{k^s}$$

for $s \in (1/2, 1]$ as a generalization of $S$. They proved that the convergence of $F_s$ depends on Brjuno-type conditions given by two formulas which relate $F_s(x, t)$ to the Gauss map $G$ and to the even continued fraction map $A_{\text{even}}$, see [RS15, Theorem 3] for the formulas. The formula relating $F_s$ to $A_{\text{even}}$ is much simpler and one can see how the convergence of $F_s(x, t)$ is determined by the finiteness of the even Brjuno function with exponent $1/2$, that is, the function $F_1(x)$ converges if $B_{\text{even},1/2}(x) < \infty$ and $\sum_{n=0}^{\infty} (x A_{\text{even}}(x) \cdots A_{n-1}(x))^{1/2} < \infty$.

In [RS15], they stated that $F_1(x)$ converges if $\sum_{n=0}^{\infty} |xT(x) \cdots T^{n-1}(x)|^{1/2} \log(1/|T^n(x)|) < \infty$ and $\sum_{n=0}^{\infty} |xT(x) \cdots T^{n-1}(x)|^{1/2} < \infty$, where $T : [-1, 1] \setminus \{0\} \to [-1, 1]$ defined by $x \mapsto -1/x \mod 2$. The statements are equivalent since $|T(|x|)| = A_{\text{even}}(x)$. We note, moreover, that even the second condition

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We will show that the $B_{1,\nu}(x) < \infty$ can be described by a combination of $B_{\text{even},\nu}$ and a Brjuno-type function $B_{\alpha,\nu}$ associated with the odd-odd continued fraction (OOCF), see (4.3) for the definition. Kim, Lingmin and the first author [KLL22] defined the OOCF algorithm to study Diophantine approximation properties related to the parities of principal convergents.

**Theorem 3.** For any $\nu > 0$, the function $B_{1,\nu}(x) - [B_{\text{even},\nu}(x) + \frac{1}{2}B_{\text{oo},\nu}(x)]$ is uniformly bounded.

A numerical computation of $B - [B_{\text{even},1} + \frac{1}{2}B_{\text{oo},1}]$ leads to the graph in Figure 3: we conjecture that the function $x \mapsto B_{1,\nu}(x) - [B_{\text{even},\nu}(x) + \frac{1}{2}B_{\text{oo},\nu}(x)]$ extends to the whole real line as a $Z$-periodic $1/2$-Hölder continuous function.

The paper is organized as follows. In Section 2, we generalize the Hölder continuity property of the Brjuno functions associated with the nearest-integer continued fraction to the $(1/2,\nu)$-Brjuno functions and prove Theorem 1. In Section 3, we introduce the Brjuno function associated with the OCF and prove Theorem 2. In Section 4, we define the Brjuno function associated with the ECF and to the OOCF and prove Theorem 3.

### 2. The semi-Brjuno function

In this section, we study the semi-Brjuno function $B_{\alpha,\nu}$ defined in [LMNN10] as in (1.3) with $\alpha = 0$ associated to the by-excess continued fraction and we give a proof of Theorem 1.

Let $\alpha \in [0,1]$. Let us consider the linear space of real valued measurable functions $f$ such that

$$f(x) = f(-x) \text{ for } x \in (\alpha - 1, 0) \setminus \mathbb{Q} \quad \text{and} \quad f(x) = f(x + 1) \text{ for } x \in \mathbb{R} \setminus \mathbb{Q}.$$  

On the functions satisfying (2.1), for $\alpha \in [0,1]$ and $\nu \geq 0$, we consider the linear operator $T_{\alpha,\nu}$ is defined by

$$T_{\alpha,\nu}f(x) = x^\nu f(1/x) \quad \text{for } x \in (0, \bar{\alpha}),$$

where $T_{\alpha,\nu}f$ is extended to the whole real line by requiring that it satisfies (2.1), i.e. $T_{\alpha,\nu}f(x) = x^\nu f(A_\alpha(x))$. Then we have the functional equation

$$[(1 - T_{\alpha,\nu})B_{\alpha,\nu}](x) = -\log(|x - |x - \alpha + 1||),$$

where the function on the right hand side satisfies (2.1) and the value is $-\log x$ for $x \in (0, \bar{\alpha})$.

Let us consider a function $g$ defined on an interval $I \subset \mathbb{R}$. For $0 < \eta \leq 1$, we define the Hölder’s $\eta$-seminorm by

$$|g|_\eta := \sup_{x < x' \in I} \frac{|g(x) - g(x')|}{|x - x'|^\eta}.$$  

We define the norm $\|g\|_\eta := |g|_\eta + \|g\|_\infty$. We say that a uniformly bounded function $g$ is $\eta$-Hölder continuous if $\|g\|_\eta < \infty$ and denote by $\mathcal{C}^\eta_1$ the set of the $\eta$-Hölder continuous functions. Note that $\mathcal{C}^1_1$ is the space of the Lipschitz continuous functions and it is different from the space of continuously differentiable functions, usually denoted by $C^1$.

From now on, we recall some properties of the operator $T_{1/2,\nu}$ in [MMY97, Thm 4.4 and 4.6]. The spectral radius of $T_{1/2,\nu}$ is strictly less than 1 on the space of functions in $L^p$ could be reformulated in terms of the formalism of Brjuno function slightly generalizing what shown in [MMY06].
satisfying (2.1) for $\alpha = 1/2$ and $p \geq 1$. Then the operator $B_{1/2, \nu} := (1 - T_{1/2, \nu})^{-1}$ is well-defined.

**Proposition 2.1.** If $f \in C^\eta_{[0,1/2]}$ for $\nu < 2\eta$, then we have $B_{1/2, \nu} f \in C^{\nu/2}_{[0,1/2]}$.

From the above proposition, it is shown that the difference $B_{1,1} - B_{1/2, 1}$ is $1/2$-Hölder continuous. It can be generalized to the Brjuno functions with the exponents $0 < \nu \leq 2$.

**Theorem 2.2.** For $0 < \nu \leq 2$, the difference $B_{1, \nu} - B_{1/2, \nu}$ can be extended from $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{R}$ as a $\nu/2$-Hölder continuous $\mathbb{Z}$-periodic function.

**Proof.** Let us denote the even part and the odd part of $B_{1, \nu}$ by $B_{1, \nu}^+$ and $B_{1, \nu}^-$, respectively. For $x \in [0, 1/2]$, we have

$$B_{1, \nu}^+(x) = -\log x - B_{1, \nu}^-(x) + x^\nu B_{1, \nu}^-(1/x) + x^\nu B_{1, \nu}^+(1/x).$$

Let $\Delta(x) = B_{1, \nu}^+(x) - B_{1/2, \nu}^-(x)$. Then we have $\Delta(x) = -B_{1, \nu}^-(x) + x^\nu B_{1, \nu}^-(1/x) + x^\nu \Delta(1/x)$. Then, $\Delta(x) = -B_{1/2, \nu} B_{1, \nu}^-(x) + \sum_{n=0}^{\infty} T_{1/2, \nu}^n(x^\nu B_{1, \nu}^-(1/x))$. By letting $\varepsilon_n(x) = \text{sgn}(\frac{1}{A_{1/2}^n(x)} - \lfloor \frac{1}{A_{1/2}^n(x)} + \frac{1}{2} \rfloor)$, each term of the second summand is

$$T_{1/2, \nu}^n(x^\nu B_{1, \nu}^-(1/x)) = \beta_{1/2, n-1}^\nu(x) (A_{1/2}^n(x))^{\nu} B_{1, \nu}^-(A_{1/2}^n(x)) = \beta_{1/2, n}^\nu(x) B_{1, \nu}^-(A_{1/2}^n(x)).$$

Therefore, $\Delta = -B_{1/2, \nu} B_{1, \nu}^- + \sum_{n=1}^{\infty} \varepsilon_n T_{1/2, \nu}^n$. It is shown that $\sum_{n=1}^{\infty} \varepsilon_n T_{1/2, \nu}^n f \in C^{\nu/2}_{[0,1/2]}$ if $f \in C^\eta_{[0,1/2]}$ for $\nu < 2\eta$ in [MMY97]. Combining with Proposition 2.1, we have $\Delta \in C^{\nu/2}_{[0,1/2]}$. □

We recall the functional equation in (1.4) for $\alpha = 0$ and $\alpha = 1/2$ as

\begin{align*}
(2.3) & \quad B_{0, \nu}(x) = -\log x + x^\nu B_{0, \nu}(-x^{-1}) \text{ for } x \in (0, 1) \setminus \mathbb{Q}, \\
(2.4) & \quad B_{1/2, \nu}(x) = -\log x + x^\nu B_{1/2, \nu}(x^{-1}) \text{ for } x \in (0, 1/2) \setminus \mathbb{Q}.
\end{align*}

**Proof of Theorem 1.** Let $\nu \in (0, 2]$. By Theorem 2.2 and $B_{0, \nu}(x) = B_{0, \nu}(x + 1)$, it is sufficient to show that $\Delta(x) := B_{1/2, \nu}(x) - [B_{0, \nu}(x) + B_{0, \nu}(1 - x)]$ is $\nu/2$-Hölder continuous. Since $\Delta$ is an even $\mathbb{Z}$-periodic function, it is enough to show that $\Delta \in C^{\nu/2}_{[0,1/2]}$.

For $x \in [0, 1/2]$, $A_0(x) = A_{1/2}(x)$ if $A_0(x) < 1/2$, otherwise, $A_0(x) = 1 - A_{1/2}(x)$. From the functional equations in (2.3) and (2.4), we have

$$\Phi(x) := \Delta(x) - x^\nu \Delta(A_{1/2}(x))$$

$$= -x^\nu B_{0, \nu}(A_0(x)) - B_{0, \nu}(1 - x) + x^\nu [B_{0, \nu}(A_{1/2}(x)) + B_{0, \nu}(1 - A_{1/2}(x))]$$

$$= x^\nu B_{0, \nu}(1 - A_0(x)) - B_{0, \nu}(1 - x),$$

which means that $\Delta = B_{1/2, \nu} \Phi$. See Figure 4 for the graphs of $\Phi$ with $\nu = 1$ and $\nu = 1/2$. By Proposition 2.1, it is enough to show that $\Phi \in C^\eta_{[0,1/2]}$ for $\eta > \nu/2$. We will prove it by the following two steps:
Let us assume that \( x \). Kraaikamp and Nakada [KN01] observed that \( x \) and \( m \)

\[ \nu \]

\[ \frac{1}{x} \]

\[ B \]

\[ \sum_{k=1}^{n-1} ((1 - (k - 1)x)^\nu - (1 - kx)^\nu) \log(1 - kx), \quad \text{where } n = \left\lfloor \frac{1}{x} \right\rfloor. \]

(2) Then, we will show that \( \Phi(x) - x^\nu \log x \) is of \( C_{[0,1/2]}^\eta \) for \( \eta > \nu/2 \). Since \( x^\nu \log x \) is \( \eta \)-Hölder continuous for \( \eta \in (0, \nu) \), it completes the proof.

To show (1), we introduce the jump transformation \( F_{jump} \) of \( A_0 \) with respect to the interval \([0, 1/2]\) which is defined by

\[ F_{jump}(x) = A_0^{m(x)+1}(x), \]

where \( m(x) = \min\{n \geq 0 : A_0^n \in [0, 1/2]\} \). Let us denote by

\[ \psi(x) := -\sum_{k=0}^{m(x)} (kx - (k - 1))^\nu \log \left( \frac{(k + 1)x - k}{kx - (k - 1)} \right), \]

which is equal to \( B_{0,\nu}(x) - (xA_0(x) \cdots A_0^{m(x)}(x))^\nu B_{0,\nu}(F_{jump}(x)) \). Since \( A_0 + G \equiv 1 \), we have

\[ x^\nu B_{0,\nu}(1 - A_0(x)) = (xG(x))^\nu B_{0,\nu}(1 - G^2(x)) - x^\nu \log(G(x)). \]

Let us assume that \( x \in \left[ \frac{1}{n+1}, \frac{1}{n} \right] \) for some \( n \geq 2 \). Then, \( xG(x) = 1 - nx \). Since \( 1 - x \in \left[ \frac{n-1}{n}, \frac{n}{n+1} \right] \), we have \( m(1 - x) = n - 1 \). Thus \( (1 - x)A_0(1 - x) \cdots A_0^{m(1-x)}(1 - x) = 1 - nx \).

Kraaikamp and Nakada [KN01] observed that \( 1 - G^2(x) = F_{jump}(1 - x) \). Thus, we have

\[ x^\nu B_{0,\nu}(1 - A_0(x)) - B_{0,\nu}(1 - x) = -x^\nu \log(G(x)) - \psi(1 - x). \]

Since \( x^\nu \log(G(x)) = x^\nu \log(1/x - n) \) and

\[ -\psi(1 - x) = \sum_{k=0}^{n-1} (1 - kx)^\nu \log \left( \frac{1 - (k + 1)x}{1 - kx} \right) \]

\[ = \sum_{k=1}^{n-1} ((1 - (k - 1)x)^\nu - (1 - kx)^\nu) \log(1 - kx) + (1 - (n - 1)x)^\nu \log(1 - nx), \]
we obtain (2.6).

We will show (2) as follows. Let \( h(x) := ((1 - (n - 1)x) - x^\nu) \log(1 - nx) \) and
\[
g(x) := \sum_{k=1}^{n-1} ((1 - (k - 1)x) - (1 - kx)^\nu) \log(1 - kx).
\]

Then, \( h(x) + g(x) = \Phi(x) - x^\nu \log x \).

We have
\[
\lim_{x \to \frac{1}{n}} h(x) = \lim_{x \to \frac{1}{n}} \frac{(1 - nx + x)^\nu - x^\nu}{1 - nx} (1 - nx) \log(1 - nx) = 0.
\]

Thus, \( h(x) + g(x) \) is continuous on \((0, 1/2]\). Given \( x \in [0, 1/2] \), let \( n = \lfloor 1/x \rfloor \). Note that \( x \to 0 \) is equivalent to \( n \to \infty \). Since \( 0 \leq 1 - nx \leq \frac{1}{n+1} \), \( \lim_{x \to 0} (1 - nx) = 0 \). Thus we have \( \lim_{x \to 0} h(x) = 0 \). Since the maximal length of the subintervals in the partition \( \{1 - kx : k = 0, \ldots, n\} \) approaches 0 when \( x \to 0 \), by using the Riemann–Stieltjes integral, we have
\[
\lim_{x \to 0} g(x) = \int_0^1 \nu y^{\nu - 1} \log y \, dy = \left[ y^\nu \log y - \frac{1}{\nu} y^\nu \right]_0^1 = -\frac{1}{\nu}.
\]

Also we have
\[
(2.7) \quad g(x) \sim \int_{1-nx}^1 \nu y^{\nu - 1} \log y \, dy = -(1 - nx)^\nu \log(1 - nx) + (1 - nx)^\nu/\nu - 1/\nu
\]
as \( x \to 0 \).

Let us define \( h(0) = 0 \) and \( g(0) = -1/\nu \). Then \( h + g \) is continuous and uniformly bounded on \([0, 1/2] \). We will complete the proof by showing that for some \( \eta > \nu/2 \), for all \( x' \in [0, 1/2] \), there exists
\[
\lim_{x \to x'} \frac{|(h + g)(x) - (h + g)(x')|}{|x - x'|^\eta}.
\]

The functions \( h, g \) are differentiable on \((0, 1/2] \setminus \bigcup_{n=2}^\infty \{1/n\} \). The function \( g \) is right and left differentiable on \( \frac{1}{n+1} \). The function \( h \) is right differentiable on \( \frac{1}{n+1} \). If \( \eta \in (\nu/2, \min\{\nu, 1\}) \), then
\[
\lim_{x \to \frac{1}{n+1}} \frac{|h(x)|}{|x - 1/n|^\eta} = \lim_{x \to \frac{1}{n+1}} \frac{|(1 - nx + x)^\nu - x^\nu|}{n^{-\eta}|1 - nx|} |1 - nx|^{1-\eta} \log \frac{1}{1 - nx} = 0.
\]

Thus, for \( x \in (0, 1/2] \), we have \( \lim_{x' \to x} \frac{|(h + g)(x) - (h + g)(x')|}{|x - x'|^\eta} = 0 \).

For \( x \in \left[\frac{1}{n+1}, \frac{1}{n}\right] \), we have \( 0 \leq 1 - nx \leq x \). Thus we have
\[
\lim_{x \to 0} \frac{|h(x)|}{x^\eta} \leq \lim_{x \to 0} \frac{|h(x)|}{|1 - nx|^\eta} = 0.
\]

By (2.7), we have
\[
\lim_{x \to 0} \frac{|g(x) + 1/\nu|}{x^\eta} \lesssim \lim_{x \to 0} (1 - nx)^{\nu-\eta} \log(1 - nx) - 1/\nu = 0.
\]

Thus \( \lim_{x \to 0} \frac{|(h + g)(x) + 1/\nu|}{x^\eta} = 0 \) and then \( h + g \in C^\eta_{[0, 1/2]} \) for \( \eta \in (\nu/2, \min\{\nu, 1\}) \). \( \square \)
3. THE OCF-BRJUNO FUNCTION

In this section, we recall some known arithmetic and measure-theoretical properties of the OCF. Then we prove Theorem 2 giving the Hölder regularity of the ν-Brjuno functions associated to the OCF. From now on, let us denote the reciprocal of the golden mean by \( g = \frac{\sqrt{5} - 1}{2} \). The OCF map \( A_{\text{odd}} \) defined as (1.6) is ergodic with respect to a probability measure

\[
 dm_{\text{odd}}(x) := \frac{1}{3\log(g^{-1})} \left( \frac{1}{g^{-1} - 1 + x} + \frac{1}{g^{-1} + 1 - x} \right) dx,
\]

see [Sch82, Thm 1]. We denote by \([t]_{\text{odd}} := 2[t/2] + 1\) a nearest odd integer of \( t \). For \( x \in \mathbb{R} \), the OCF partial quotients \((a_{o,n}, \varepsilon_{o,n}), n \geq 0\) are defined by

\[
 a_{o,0} := \lfloor x \rfloor_{\text{odd}}, \quad \varepsilon_{o,0} := \text{sign}(x - a_{o,0}), \quad a_{o,n} := \left\lfloor \frac{1}{x_{o,n-1}} \right\rfloor_{\text{odd}} \quad \text{and} \quad \varepsilon_{o,n} := \text{sign}\left( \frac{1}{x_{o,n-1}} - a_{o,n} \right),
\]

where \( x_{o,n} \) is an \( A_{\text{odd}} \)-orbit of \( x \) defined as in (1.8). Every \( x \in \mathbb{R} \) has the following OCF-expansion:

\[
 x = a_{o,0} + \frac{\varepsilon_{o,0}}{a_{o,1} + \frac{\varepsilon_{o,1}}{a_{o,2} + \ddots + \frac{\varepsilon_{o,n-1}}{a_{o,n} + \cdots}}},
\]

The \( n \)th principal convergent of the OCF is the truncated continued fraction expansion denoted by

\[
 \frac{p_{o,n}}{q_{o,n}} := a_{o,0} + \frac{\varepsilon_{o,0}}{a_{o,1} + \frac{\varepsilon_{o,1}}{a_{o,2} + \cdots + \frac{\varepsilon_{o,n-1}}{a_{o,n}}}}.
\]

Let \( p_{o,-1} = 1, q_{o, -1} = 0, p_{o,0} = a_{o,0} \) and \( q_{o,0} = 1 \). We recall the following basic properties of \( p_{o,n}/q_{o,n} \).
Remark 3.1. Since we have
\[
\begin{pmatrix}
p_{o,n} & p_{o,n-1} \\
q_{o,n} & q_{o,n-1}
\end{pmatrix} = \begin{pmatrix} a_{o,0} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{o,1} & 1 \\ \varepsilon_{o,0} & 1 \end{pmatrix} \begin{pmatrix} a_{o,2} & 1 \\ \varepsilon_{o,1} & 1 \end{pmatrix} \ldots \begin{pmatrix} a_{o,n} & 1 \\ \varepsilon_{o,n-1} & 1 \end{pmatrix},
\]
the following recursive formulas holds:
\[
p_{o,n} = a_{o,n}p_{o,n-1} + \varepsilon_{o,n-1}p_{o,n-2} \quad \text{and} \quad q_{o,n} = a_{o,n}q_{o,n-1} + \varepsilon_{o,n-1}q_{o,n-2}
\]
for \( n \geq 1 \), and
\[
p_{o,n}q_{o,n-1} - p_{o,n-1}q_{o,n} = (-1)^{n+1}\varepsilon_{o,n} + \varepsilon_{o,n-1} - \varepsilon_{o,n} - \varepsilon_{o,n-1}.
\]
By using \( a_{o,n} + \varepsilon_{o,n}x_{o,n} = 1/x_{o,n-1} \), we have
\[
x = \frac{p_{o,n} + p_{o,n-1}\varepsilon_{o,n}x_{o,n}}{q_{o,n} + q_{o,n-1}\varepsilon_{o,n}x_{o,n}} \quad \text{and} \quad x_{o,n} = -\varepsilon_{o,n}\frac{q_{o,n}x - p_{o,n}}{q_{o,n-1}x - p_{o,n-1}}.
\]
Thus we have
\[
\beta_{o,n} = x_{o,0}x_{o,1}x_{o,2} \cdots x_{o,n} = (q_{o,n}x - p_{o,n}) \prod_{i=0}^{n}(-\varepsilon_{o,i}) = \frac{1}{q_{o,n+1} + \varepsilon_{o,n+1}q_{o,n}x_{o,n+1}}.
\]
Note that \( q_{o,n} \) can be less than \( q_{o,n-1} \) if \( a_{o,n} = 1 \) and \( \varepsilon_{o,n-1} = -1 \). We note however that for all \( n \) we have
\[
\varepsilon_{o,n}q_{o,n-1}q_{o,n}^{-1} > g^{-1} - 2,
\]
\[
q_{o,n} > gg_{o,n-1}.
\]
Thus we have
\[
g^{-1} - 2 < \varepsilon_{o,n} \frac{q_{o,n-1}}{q_{o,n}} < g^{-1}.
\]
Combining (3.4) with (3.5), we have
\[
1 - g \leq \beta_{o,n}(x)q_{o,n+1} \leq g^{-1}.
\]
If \( x_{o,k} > g \) for some \( 0 \leq k \leq n \), then \( x_{o,k+1} = (x_{o,k})^{-1} - 1 \). Since \( x_{o,k}x_{o,k+1} = 1 - x_{o,k} < 1 - g = g^2 \), we have
\[
\beta_{o,n} \leq g^n \quad \text{for} \quad n \geq 1.
\]
Remark 3.2. Let \( \nu > 0 \). From (3.8) and (3.9), we have \( q_{o,n} \geq \frac{1 - g}{g^{n-1}} \), thus
\[
\sum_{n=0}^{\infty} \frac{1}{q_{o,n}^\nu} \leq \sum_{n=0}^{\infty} \frac{g^{\nu(n-1)}}{(1 - g)^\nu} =: c_{1,\nu}.
\]
Since \( \log q_{o,n} \lesssim \nu q_{o,n}^{\nu/2} \). Thus, there exists a positive constant \( c_\nu \) such that
\[
\sum_{n=0}^{\infty} \log q_{o,n} \lesssim c_\nu \sum_{n=0}^{\infty} \frac{1}{q_{o,n}^{\nu/2}} \lesssim c_\nu \sum_{n=0}^{\infty} \frac{g^{\nu(n-1)/2}}{(1 - g)^{\nu/2}} =: c_{2,\nu}.
\]
Similarly, since \( Q_n \geq (n-1)^{\nu/2} \), there are constants \( C_1, C_2 > 0 \) such that
\[
\sum_{n=0}^{\infty} Q_{n}^{-\nu} \leq C_{1,\nu} \quad \text{and} \quad \sum_{n=0}^{\infty} Q_{n}^{-\nu} \log Q_n \leq C_{2,\nu}.
\]
3.1. **Uniform boundedness.** In this subsection, we show the first assertion of Theorem 2. From (1.2), the following two propositions imply \( B_{1,\nu} \sim B_{\text{odd},\nu} \in L^\infty \).

**Proposition 3.3.** There exists \( C_\nu > 0 \) such that for all \( x \in \mathbb{R} \setminus \mathbb{Q} \),

\[
\left| \sum_{n=0}^{\infty} \frac{\log Q_{n+1}}{Q_n^\nu} - \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^\nu} \right| \leq C_\nu,
\]

where \( P_n/Q_n \) is the principal convergent of the RCF.

**Proposition 3.4.** There exists \( C_\nu > 0 \) such that for all \( x \in \mathbb{R} \setminus \mathbb{Q} \),

\[
\left| B_{\text{odd},\nu}(x) - \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^\nu} \right| \leq C_\nu.
\]

Harton and Kraaikamp [HK02] described a relation between the RCF and the OCF and gave an algorithm for obtaining the OCF-expansion from the RCF-expansion. By their result, we have the following lemma immediately, see also [Har03, Section 2.1 and Theorem 2.1] and [Kra91].

**Lemma 3.5.**

1. If \( p_{o,k}/q_{o,k} = P_n/Q_n \) for some \( n \), then \( p_{o,k+1}/q_{o,k+1} = P_{n+1}/Q_{n+1} \) or \( (P_{n+1} + P_n)/(Q_{n+1} + Q_n) \).
2. If \( p_{o,k}/q_{o,k} \) is not a RCF convergent, then \( p_{o,k}/q_{o,k} = (P_{n+1} + P_n)/(Q_{n+1} + Q_n) \) and \( p_{o,k+1}/q_{o,k+1} = P_{n+1}/Q_{n+1} \) for some \( n \).

From the above lemma, \( \{p_k/q_k\} \) is a subsequence of

\[
\frac{P_{-1}}{Q_{-1}}, \frac{P_{-1} + P_0}{Q_{-1} + Q_0}, \frac{P_0}{Q_0}, \frac{P_0 + P_1}{Q_0 + Q_1}, \frac{P_1}{Q_1}, \ldots, \frac{P_n}{Q_n}, \frac{P_n + P_{n+1}}{Q_n + Q_{n+1}}, \frac{P_{n+1}}{Q_{n+1}}, \ldots
\]

containing all \( P_n/Q_n \) and containing some \( (P_n + P_{n+1})/(Q_n + Q_{n+1}) \). Note that the sequence as in (3.11) is not in order of denominators. By using the above lemma, we can show that the \( \nu \)-Brjuno condition of the RCF is equivalent to \( \sum_{n=0}^{\infty} (\log q_{o,n+1})/q_{o,n}^\nu < \infty \).

**Proof of Proposition 3.3.** By Lemma 3.5, if \( p_k/q_k \) is a RCF convergent \( P_n/Q_n \), then we have

\[
\left| \frac{\log q_{o,k+1}}{(q_{o,k})^\nu} - \frac{\log Q_{n+1}}{Q_n^\nu} \right| \leq \frac{\log 2}{Q_n^\nu}.
\]

If \( p_{o,k}/q_{o,k} \) is not a RCF convergent, then we have

\[
\frac{\log q_{o,k+1}}{(q_{o,k})^\nu} \leq \frac{\log Q_{n+1}}{Q_{n+1}^\nu}.
\]

Combining with Remark 3.2, we have

\[
\left| \sum_{k=0}^{\infty} \log q_{o,k+1}/q_{o,k}^\nu - \sum_{n=0}^{\infty} \log Q_{n+1}/Q_n^\nu \right| \leq \sum_{n=0}^{\infty} \log Q_n + \log 2/Q_n^\nu \leq C_{1,\nu} \log 2 + C_{2,\nu}.
\]
Proof of Proposition 3.4. We have

\[-B_{\text{odd},\nu}(x) + \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}^{-\nu}}{q_{o,n}^{\nu}} = \sum_{n=0}^{\infty} \beta_{o,n}^{-\nu} \log \beta_{o,n}^{-\nu} + \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^{\nu}} \]

\[= \sum_{n=0}^{\infty} \beta_{o,n-1}^{-\nu} \log \beta_{o,n} q_{o,n+1} - \sum_{n=0}^{\infty} \beta_{o,n-1}^{-\nu} \log \beta_{o,n-1}^{-\nu} + \sum_{n=0}^{\infty} \left( \frac{1}{q_{o,n}} - \beta_{o,n-1}^{-\nu} \right) \log q_{o,n+1}. \]

By (3.8) and Remark 3.2, we have

\[
\left| \sum_{n=0}^{\infty} \beta_{o,n-1}^{-\nu} \log \beta_{o,n} q_{o,n+1} \right| \leq \sum_{n=0}^{\infty} \frac{\log (1/g)}{g^{\nu} q_{o,n}^{\nu}} \leq \frac{\log (1/g)}{g^{\nu}} c_{1,\nu},
\]

\[
\left| \sum_{n=0}^{\infty} \beta_{o,n-1}^{-\nu} \log \beta_{o,n-1}^{-\nu} \right| \leq \sum_{n=0}^{\infty} \frac{\log (1/g) + \log q_{o,n}}{g^{\nu} q_{o,n}^{\nu}} \leq \frac{c_{1,\nu} \log (1/g) + c_{2,\nu}}{g^{\nu}}.
\]

From (3.4), we have

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(q_{o,n}^{\nu})^{-\nu}} - \beta_{o,n-1}^{-\nu} \right) \log q_{o,n+1} \leq \frac{1}{g^{\nu}} \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^{\nu}} \leq \frac{c_{2,\nu}}{g}. 
\]

If \(\nu \geq 1\), then (3.12) is bounded above by

\[
\sum_{n=0}^{\infty} \beta_{o,n}^{-\nu} \log q_{o,n+1} \leq \frac{1}{g} \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^{\nu}} \leq \frac{c_{2,\nu}}{g}.
\]

If \(0 < \nu < 1\), then (3.12) bounded above by

\[
\sum_{n=0}^{\infty} \beta_{o,n}^{-\nu} \log q_{o,n+1} \leq \sum_{n=0}^{\infty} \frac{\log q_{o,n+1}}{q_{o,n}^{\nu}} \leq \frac{c_{1,\nu}}{g}.
\]

\[
\sum_{n=0}^{\infty} \max \left\{ \frac{1}{\beta_{o,n-1}^{-\nu}}, q_{o,n}^{-1} \beta_{o,n}^{-\nu} \log q_{o,n+1} \right\} \leq \sum_{n=0}^{\infty} \max \left\{ \frac{1}{\beta_{o,n}^{-\nu}}, q_{o,n}^{-1} \frac{q_{o,n-1}}{q_{o,n}} \log q_{o,n+1} \right\} \]

\[
\leq \sum_{n=0}^{\infty} \max \left\{ \beta_{o,n}^{-\nu}, \frac{1}{g^{\nu} q_{o,n+1}} \right\} \log q_{o,n+1} \leq \sum_{n=0}^{\infty} \frac{1}{g^{\nu} q_{o,n+1}} \log q_{o,n+1} \leq \frac{c_{1,\nu}}{g}.
\]

\[
3.2. \textbf{Hölder continuity.} \text{ In this subsection, we prove that } B_{1,\nu} - B_{\text{odd},\nu} \text{ is } \nu/2\text{-Hölder continuous for } 0 < \nu \leq 2. \text{ In order to prove it, we investigate an operator which is derived from the functional equation of } B_{\text{odd},\nu} \text{ as in (1.12). Let us denote by } X \text{ the space of measurable functions } f \text{ satisfies (1.9). Let } T_{\text{odd},\nu} f \text{ be a function in } X \text{ whose values on } (0, 1) \text{ are defined by}
\]

\[
T_{\text{odd},\nu} f(x) = x\nu f(1 - x^{-1}) \text{ for } x \in (0, 1).
\]

For \(p \in [1, \infty]\), we consider the space

\[
X_{\text{odd},p} := \{ f \text{ satisfies (1.9) and } f|_{[0,1]} \in L^p([0,1], m_{\text{odd}}) \}.
\]
equipped with the $p$-norm $\|f\|_p = \left( \int_0^1 |f|^p dm_{\text{odd}} \right)^{1/p}$. The OCF-Brjuno function $B_{\text{odd},\nu}$ is a function in an even $2\mathbb{Z}$-periodic function satisfying 
\[(1 - T_{\text{odd},\nu})B_{\text{odd},\nu}(x) = -\log(\|x\|_{2\mathbb{Z}}).
\]
Since $m_{\text{odd}}$ is an absolutely continuous measure and its density function is bounded above and below, as vector spaces $L^p([0, 1], dx)$ and $L^p([0, 1], dm_{\text{odd}})$ coincide, and we ensure that $-\log(\|x\|_{2\mathbb{Z}}) \in X_{\text{odd},p}$ for all $p \geq 1$. The following theorem tells us that $1 - T_{\text{odd},\nu}$ is invertible and $B_{\text{odd},\nu}$ is a unique solution of the above functional equation on the space $X_{\text{odd},p}$ for every $p \geq 1$.

**Proposition 3.6.** For all $\nu \geq 0$ and all $p \geq 1$, the operator $T_{\text{odd},\nu}$ is a bounded linear operator on $X_{\text{odd},p}$, and $\rho(T_{\text{odd},\nu}) \leq g^\nu$, where $\rho(T)$ is the spectral radius of an operator $T$.

**Proof.** For given $p \geq 1$ and $f \in X_{\text{odd},p}$, we have
\[(3.14) \quad T_{\text{odd},\nu}^m f(x) = \beta_{o,n-1}^\nu(x) f(A_{\text{odd}}^n(x)).
\]
By (3.9), we have
\[
\int_0^1 |T_{\text{odd},\nu}^n f(x)|^p dm_{\text{odd}}(x) = \int_0^1 |\beta_{o,n-1}^\nu(x)|^p f(A_{\text{odd}}^n(x))|^p dm_{\text{odd}}(x) \leq g^{(n-1)p} \int_0^1 |f|^p dm_{\text{odd}}.
\]
Then, the operator norm $\|T_{\text{odd},\nu}^n\|_p \leq g^{(n-1)p}$, which implies $\rho(T_{\text{odd},\nu}) \leq g^\nu$. \qed

We call $I_n = \left[ \frac{1}{n+1}, \frac{1}{n} \right]$ for $n \in \mathbb{N}$ the branches of $A_{\text{odd}}$ on which the restriction of $A_{\text{odd}}$ is invertible. The branches of $A_{\text{odd}}^n$ are defined by
\[I_{m_1} \cap A_{\text{odd}}^{-1} I_{m_2} \cap A_{\text{odd}}^{-2} I_{m_3} \cdots \cap A_{\text{odd}}^{-(n-1)} I_{m_n},
\]
where $\{m_i\}_{1}^{n} \in \mathbb{N}^n$.

The splitting number $n(x, x')$ for $x, x' \in [0, 1]$ is the greatest integer $n$ such that $x$ and $x'$ belong to the same branch or to two adjacent branches of $A_{\text{odd}}^n$. We denote by $\delta(x, x') := n(x, x') - m$, where $m$ is the greatest integer such that $x, x'$ belong to the same branch of $A_{\text{odd}}^n$. From now on, for $x, x' \in [0, 1]$, let us denote by
\[
\begin{cases}
  x_n = A_{\text{odd}}^n(x), & a_n = a_{o,n}(x), \quad \varepsilon_n = \varepsilon_{o,n}(x), \quad p_n = p_{o,n}(x), \quad q_n = q_{o,n}(x), \\
  x'_n = A_{\text{odd}}^n(x'), & a'_n = a_{o,n}(x'), \quad \varepsilon'_n = \varepsilon_{o,n}(x'), \quad p'_n = p_{o,n}(x'), \quad q'_n = q_{o,n}(x').
\end{cases}
\]

**Remark 3.7.** We write $n = n(x, x')$ and $\delta = \delta(x, x')$. For all $\ell \leq n - \delta$, we have
\[(3.15) \quad a_{\ell} = a'_{\ell}, \quad \varepsilon_{\ell} = \varepsilon'_{\ell}, \quad p_{\ell} = p'_{\ell}, \quad q_{\ell} = q'_{\ell},
\]
\[(3.16) \quad |\beta_{o,\ell}(x) - \beta_{o,\ell}(x')| = |(q_{\ell} x - p_{\ell}) - (q'_{\ell} x' - p'_{\ell})| = q_{\ell} |x - x'| \quad \text{(by (3.4))},
\]
\[(3.17) \quad |x - x'| = |\beta_{o,\ell}(x)\beta_{o,\ell-1}(x') - \beta_{o,\ell-1}(x)\beta_{o,\ell}(x')| = |x_{\ell} - x'_{\ell}| \beta_{o,\ell-1}(x)\beta_{o,\ell-1}(x').
\]

Dividing the following four cases, we will see that $\delta = 0, 1, 2$ and a relation between $\{a_n, \varepsilon_n, p_n, q_n\}$ and $\{a'_n, \varepsilon'_n, p'_n, q'_n\}$.

(A) If the points $x, x'$ belong to the same branch of $A_{\text{odd}}^n$, then $\delta = 0$.

(B) If the points $x, x'$ belong to the same branch of $A_{\text{odd}}^{n-1}$, and there is $k \geq 1$ such that
\[(3.18) \quad \frac{1}{2k+2} \leq x_{n-1} \leq \frac{1}{2k+1} \leq x'_{n-1} \leq \frac{1}{2k},
\]
Let $x'' := p_n/q_n$ and $x'' := A_{odd}^n(x'')$. Then $x''$ is between $x$ and $x'$. We have $x'' = (2k+1)^{-1} \beta_{o,n-1}(x'') = q_n^{-1}$, $x'' = 0$. By (3.2), we have $|x-x''| = |(q_n x - p_n)(q_n x'' - p_n - 1)|$. Combining with (3.4), we have

$$|x-x''| = q_n^{-1} \beta_{o,n}(x) \quad \text{and} \quad |x' - x''| = q_n^{-1} \beta_{o,n}(x').$$

(C) If the points $x, x'$ belong to the same branch of $A_{odd}^{n-2}$ and there is $k \geq 1$ such that

$$2 \frac{4k+1}{4k+1} \leq x_{n-2} \leq \frac{1}{2k} \leq x'_{n-2} \leq \frac{2}{4k-1},$$

then $\delta = 2$, and $x_{n-1}, x'_{n-1} \in \left[1/2, 1\right]$. We have

$$a_{n-1} = a_{n-1}' + 2 = 2k+1, \ \epsilon_{n-1} = -\epsilon_{n-1}' = -1, \ a_n = a_n' = 1, \ \epsilon_n = \epsilon_n' = 1,$$

$$\begin{align*}
q_n &= q_n' + 2q_{n-2}, \\
p_n &= p_n' + 2p_{n-2}.
\end{align*}$$

Let $x' := p_n/q_n$. Then $x'$ is between $x$ and $x'$. We have $x_{n-2}' = (2k)^{-1}, x_{n-1}' = 1$ and $x_n' = 0$. Then $\beta_{o,n-2}'(x') = \beta_{o,n-1}(x') = q_n^{-1}$. By the same argument in (B), we have

$$|x-x'| = q_n^{-1} \beta_{o,n}(x) \quad \text{and} \quad |x' - x'| = q_n^{-1} \beta_{o,n}(x').$$

(D) If the points $x, x'$ belong to the same branch of $A_{odd}^{n-1}$ and there is $k \geq 1$ such that

$$\frac{1}{2k+1} \leq x_{n-1} \leq \frac{1}{2k} \leq x'_{n-1} \leq \frac{1}{2k-1},$$

but at least one of $x_{n}, x'_{n}$ is not in $[1/2, 1]$, i.e.

$$\max \{1 - x_n, 1 - x'_n\} \geq 1/2,$$

then $\delta = 1$. We have

$$\begin{align*}
a_n &= 2k+1, \ \epsilon_n = -1, \\
a_n' &= 2k-1, \ \epsilon_n = 1, \\
q_n &= q_n' + 2q_{n-1}, \\
p_n &= p_n' + 2p_{n-1}.
\end{align*}$$

Let $x' := \frac{p_n + p_n'}{q_n + q_n'}, a_i := a_{o,i}(x'')$ and $\varepsilon_i := \varepsilon_{o,i}(x'')$. Then $x''$ is between $x$ and $x'$. We have $a_i'' = a_i$ and $\varepsilon_i'' = \varepsilon_i$ for $i \leq n - 1$. Since $x_{n-1}' = (2k)^{-1}, x_n' = 1$, from the same argument in (B), we have

$$\begin{align*}
|x-x''| &= \frac{2\beta_{o,n-1}(x)}{q_n + q_n'}(1 - x_n) \quad \text{and} \quad |x' - x''| = \frac{2\beta_{o,n-1}(x')}{q_n + q_n'}(1 - x_n).
\end{align*}$$

The following theorems extend the results showed in [MMY97] for the NICF map $A_{1/2}$ to the OCF map $A_{odd}$. The theorems tell us how the operator $T_{odd,v}$ preserves the Hölder continuity.

**Theorem 3.8.** Let $f \in C_{(0,1]}$ for $0 < \eta \leq 1$. Let $\eta = \min\{\eta, \nu/2\}$.

1. For any $m \geq 0$ and $\nu > 0$, $T_{odd,v}^m f$ is $\overline{\eta}$-Hölder continuous.
(2) If \( \nu > 2\eta \) (thus \( \eta' = \eta \)), then \( T_{\text{odd},\nu} \) is a bounded linear operator on \( C^\eta_{[0,1]} \) with 
\[
\rho(T_{\text{odd},\nu}C^\eta_{[0,1]}) \leq g^{\nu - 2\eta} < 1.
\]
Therefore, the operator \( B_{\text{odd},\nu} := (1 - T_{\text{odd},\nu})^{-1} \) is well-defined in \( C^\eta_{[0,1]} \) and fulfills \( B_{\text{odd},\nu} = \sum_{m=0}^{\infty} T_{\text{odd},\nu}^m \).

(3) If \( \nu = 2\eta \) (thus \( \eta = \eta = \nu/2 \)), then there exists \( c_{3,\nu} > 0 \) such that
\[
\|T_{\text{odd},\nu}^m\|_{C^{\nu/2}} \leq c_{3,\nu} \text{ for all } m \geq 0.
\]

**Theorem 3.9.** Let \( f \in C^\eta_{[0,1]} \) for \( 0 < \eta \leq 1 \).

(1) If \( \nu < 2\eta \), then the function \( B_{\text{odd},\nu}f = \sum_{m=0}^{\infty} T_{\text{odd},\nu}^m f \) is in \( C^{\nu/2}_{[0,1]} \).

(2) If \( \nu = 2\eta \), then the function \( B_{\text{odd},\nu}f \) is in \( C^\eta_{[0,1]} \) for \( 0 < \eta < \nu/2 \).

To prove the above theorem, we need to estimate \( |T_{\text{odd},\nu}^m f(x) - T_{\text{odd},\nu}^m f(x')| \) in terms of \( |x - x'|^{\nu/2} \). For this aim, we state the following auxiliary properties.

**Lemma 3.10.** Let \( x, x' \in [0,1] \).

(1) There exists \( \kappa_1 > 1 \) such that for all \( \ell < n(x, x') \),
\[
(3.26) \quad \kappa_1^{-1} \beta_{\ell,\ell}(x') < \beta_{\ell,\ell}(x) < \kappa_1 \beta_{\ell,\ell}(x').
\]

(2) There exists \( \kappa_2 > 1 \) such that for all \( \ell \geq n(x, x') \),
\[
(3.27) \quad \max\{\beta_{\ell,\ell}(x), \beta_{\ell,\ell}(x')\} \leq \kappa_2 |x - x'|^{1/2}.
\]

(3) Let \( J \) be a branch of \( A^m_{\text{odd}} \) which has an end point on \( p_{o,m}/q_{o,m} \). Then we have
\[
\frac{1 - g}{q_{o,m}^2} \leq |J| \leq \frac{g^{-1}}{q_{o,m}^2}, \quad \text{and} \quad g^2 q_{o,m}^2 \leq \left| \frac{dA^m_{\text{odd}}(x)}{dx} \right| \leq \frac{q_{o,m}^2}{(1 - g)^2} \text{ for } x \in J.
\]

(4) Let \( J, J' \) be adjacent branches of \( A^m_{\text{odd}} \). Then there exists \( \kappa_3 > 1 \) such that
\[
(3.28) \quad \kappa_3^{-1}|J'| \leq |J| \leq \kappa_3 |J'|,
\]
where \( |J| \) is the length of \( J \).

Since the proof are analogous to \[\text{MMY97, Lemma 1.10-13, Theorem 4.2, 4.4}, \] we give a full proof of the above theorems and lemma in Appendix A.

**Proposition 3.11.** Let
\[
B_{\text{odd},\nu}^+(x) = \frac{B_{\text{odd},\nu}(x) + B_{\text{odd},\nu}(1-x)}{2} \quad \text{and} \quad B_{\text{odd},\nu}^-(x) = \frac{B_{\text{odd},\nu}(x) - B_{\text{odd},\nu}(1-x)}{2}.
\]

Then, \( B_{\text{odd},\nu}^+ \) is an even \( \mathbb{Z} \)-periodic function such that
\[
B_{\text{odd},\nu}^+(x) = \log \frac{1}{x} - \frac{1 - (1-x)^\nu}{2} \log \frac{1-x}{x} + x^\nu B_{\text{odd},\nu}^+(\frac{1}{x}) \text{ for } x \in (0,1/2),
\]
and \( B_{\text{odd},\nu}^- \) is an even \( 2\mathbb{Z} \)-periodic function such that \( B_{\text{odd},\nu}^+(1+x) = -B_{\text{odd},\nu}^+(x) \) and
\[
(3.28) \quad B_{\text{odd},\nu}^-(x) = \frac{1 - (1-x)^\nu}{2} \log \frac{1-x}{x} + x^\nu B_{\text{odd},\nu}^-(1 - \frac{1}{x}) \text{ for } x \in (0,1/2).
\]
Proof. By using (1.12), for \( x \in (0, 1/2) \), we have

\[
B_{\text{odd},\nu}(1-x) = \log \frac{1}{1-x} + (1-x)^\nu B_{\text{odd},\nu}\left(\frac{1}{1-x} - 1\right)
= \log \frac{1}{1-x} + (1-x)^\nu \left[ \log \frac{1-x}{x} + \frac{x^\nu}{(1-x)^\nu} B_{\text{odd},\nu}\left(\frac{1}{x}\right) \right]
= \log \frac{1}{1-x} + (1-x)^\nu \log \frac{1-x}{x} + x^\nu B_{\text{odd},\nu}\left(\frac{1}{x}\right).
\]

Combining (1.12) with the above equation, we have the conclusion. □

From Theorem 2.2, to show the Hölder continuity of \( B_{1,\nu} - B_{\text{odd},\nu} \), we will show that \( B_{\text{odd},\nu} - \frac{B_{1,\nu}}{2} \) is Hölder continuous. We split \( B_{\text{odd},\nu} - \frac{B_{1,\nu}}{2} \) by an even \( \mathbb{Z} \)-periodic function \( B_{\text{odd},\nu}^+ \) and an even \( 2\mathbb{Z} \)-periodic function \( B_{\text{odd},\nu}^- \). Thus it is enough to show that \( B_{\text{odd},\nu}^+ - \frac{B_{1,\nu}}{2} \) is Hölder continuous and \( B_{\text{odd},\nu}^- \) is Hölder continuous.

Proof of Theorem 2. Let \( \Delta := B_{\text{odd},\nu}^+ - B_{1,\nu}/2 \). For \( x \in (0, 1/2] \), we have

\[
\Delta(x) = -\frac{1 - (1-x)^\nu}{2} \log \frac{1-x}{x} + x^\nu \left[ B_{\text{odd},\nu}^+ \left(1 - \frac{1}{x}\right) - B_{1,\nu}/2 \left(1 - \frac{1}{x}\right) \right].
\]

Thus, \( (1 - T_{1,\nu})\Delta(x) = -\frac{1 - (1-x)^\nu}{2} \log \frac{1-x}{x} \), which is in \( C^\eta_{[0,1/2]} \) for \( \nu/2 < \eta < \nu \). By Proposition 2.1, \( \Delta \) is in \( C^{\nu/2}_{[0,1/2]} \). Equation (3.28) is equivalent to

\[
(1 - T_{\text{odd},\nu}) B_{\text{odd},\nu}^-(x) = \frac{1 - (1-x)^\nu}{2} \log \frac{1-x}{x} \in C^\eta_{[0,1]} \text{ for } \nu/2 < \eta < \nu.
\]

By Theorem 3.9-(1), we have \( B_{\text{odd},\nu}^- \in C^{\nu/2}_{[0,1]} \). □

4. The ECF- and OOCF-Brjuno functions

In this section, we recall some basic properties of the ECF and the OOCF and define a Brjuno-like function associated with the OOCF. We will prove Theorem 3 which tells us that the corresponding Brjuno numbers can be characterized by a combination of the Brjuno functions associated with the ECF and with the OOCF.

4.1. The even continued fraction. The ECF map \( A_{\text{even}} \) as in (1.6) is ergodic with respect to \( dm_{\text{even}}(x) := \frac{dx}{1-x^2} \), which is a \( \sigma \)-finite measure with infinite total mass [Sch82, Thm 2]. We denote by \( [t]_{\text{even}} := \lfloor (t + 1)/2 \rfloor \) a nearest even integer of \( t \). For \( x \in \mathbb{R} \), we denote the ECF partial quotients \((a_{e,n}, \varepsilon_{e,n})\) by

\[
a_{e,0} := [x]_{\text{even}}, \; \varepsilon_{e,0} := \text{sign}(x - a_{e,0}), \; a_{e,n} := \frac{1}{|x_{e,n-1}|_{\text{even}}} \quad \text{and} \quad \varepsilon_{e,n} := \text{sign}\left(\frac{1}{x_{e,n-1}} - a_{e,n}\right),
\]

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where $x_{e,n}$ is an $A_{even}$-orbit of $x$ defined as in (1.8). The ECF expansion of $x$ is

$$
\frac{a_{e,0} + \frac{\varepsilon_{e,0}}{a_{e,1} + \frac{\varepsilon_{e,1}}{\ddots + \frac{\varepsilon_{e,n}}{a_{e,n+1} + \ddots}}}}{q_{e,n}}
$$

The $n$th principal convergent of the ECF is the truncated continued fraction expansion denoted by

$$
\frac{p_{e,n}}{q_{e,n}} := \frac{a_{e,0} + \frac{\varepsilon_{e,0}}{a_{e,1} + \frac{\varepsilon_{e,1}}{\ddots + \frac{\varepsilon_{e,n-1}}{a_{e,n}}}}}{q_{e,n}}
$$

Then, $p_{e,n}/q_{e,n}$ for all $n \geq 0$ hold the same properties in Remark 3.1. Note that $\{q_{e,n}\}$ is an increasing sequence.

4.2. The odd-odd continued fraction. Kim, Liao and the first author defined the following interval map

$$
A_{oo}(x) = \begin{cases} 
\frac{kx-(k-1)}{k-(k+1)x}, & x \in \left[\frac{k-1}{k}, \frac{2k-1}{2k+1}\right], \\
\frac{k-(k+1)x}{kx-(k-1)}, & x \in \left[\frac{2k-1}{2k+1}, \frac{k}{k+1}\right],
\end{cases}
$$

for $k \geq 1$,

and investigated the corresponding continued fraction, which is called the odd-odd continued fraction (OOCF) [KLL22]. See Figure 6b for the graph of $A_{oo}$. The OOCF expansion has
the form

\[
1 - \frac{1}{a_{oo,1} + \varepsilon_{oo,1}},
\]

\[
2 - \frac{1}{a_{oo,2} + \varepsilon_{oo,2}},
\]

\[
\vdots
\]

\[
2 - \frac{1}{a_{oo,n} + \varepsilon_{oo,n}} \cdot \cdot \cdot
\]

where \(a_{oo,n} \in \mathbb{N}\) and \(\varepsilon_{oo,n} \in \{1, -1\}\) for \(a_{oo,n} \geq 2\) and \(\varepsilon_{oo,n} = 1\) for \(a_{oo,n} = 1\). For \(n \geq 1\), the \(n\)th principal convergent of the OOCF is defined by

\[
\frac{p_{oo,n}}{q_{oo,n}} := 1 - \frac{1}{a_{oo,1} + \varepsilon_{oo,1}},
\]

\[
2 - \frac{1}{a_{oo,2} + \varepsilon_{oo,2}},
\]

\[
\vdots
\]

\[
2 - \frac{1}{a_{oo,n} + \varepsilon_{oo,n}} \cdot \cdot \cdot
\]

Let \(p_{oo,-1} = -1\), \(q_{oo,-1} = 1\), \(p_{oo,0} = 1\) and \(q_{oo,0} = 1\). We have the following recursive relations:

\[
\begin{align*}
    p_{oo,n} &= (2a_{oo,n} + \varepsilon_{oo,n} - 1)p_{oo,n-1} + \varepsilon_{oo,n-1}p_{oo,n-2}, \\
    q_{oo,n} &= (2a_{oo,n} + \varepsilon_{oo,n} - 1)q_{oo,n-1} + \varepsilon_{oo,n-1}q_{oo,n-2},
\end{align*}
\]

for \(n \geq 1\).

We denote by \(x_{oo,n} = A_{oo}^n(x)\). We define a function \(\iota : [0, 1] \rightarrow [0, 1]\) by \(\iota(x) = \frac{1-x}{1+x}\). Note that \(\iota^2 = \text{id}\) and the dynamical systems \((I, A_{even}, m_{even})\) and \((I, A_{oo}, m_{oo})\) are conjugate to each other via \(\iota\). We have

\[
x = \frac{p_{oo,n} + q_{oo,n-1}x}{q_{oo,n} + q_{oo,n-1}x}, \quad \text{and} \quad \iota(x_{oo,n}) = -\varepsilon_{oo,n} \frac{q_{oo,n}x - p_{oo,n}}{q_{oo,n-1}x - p_{oo,n-1}}.
\]

By letting

\[
\beta_{oo,n}(x) := (x_{oo,0} + 1) \prod_{i=0}^{n} \iota(x_{oo,i}), \quad \text{and} \quad \beta_{oo,-1}(x) := 1,
\]

we define the odd-odd-Brjuno function with a positive exponent \(\nu\) by

\[
B_{oo,\nu}(x) = -\sum_{n=0}^{\infty} (\beta_{oo,n-1}(x_{oo,n}))^\nu \log \iota(x_{oo,n}).
\]

We note that \(B_{oo,\nu}(x) = (\{x\} + 1)^\nu B_{even,\nu}(\iota(\{x\}))\). See Figure 3 for the graph of \(B_{oo,1}\).

From (4.2) and the fact that \(|p_{oo,n+1}q_{oo,n} - p_{oo,n}q_{oo,n+1}| = 2\), we have

\[
\beta_{oo,n}(x) = |q_{oo,n}x - p_{oo,n}| = \frac{2}{q_{oo,n+1} + \varepsilon_{oo,n+1}q_{oo,n}x(x_{oo,n+1}).
\]
4.3. **Proof of Theorem 3.** We call a rational \( p/q \) in lowest terms an \( \infty \)-rational if \( p \) and \( q \) have different parities, and a 1-rational if \( p \) and \( q \) are both odd. The ECF principal convergents \( p_{e,n}/q_{e,n} \) are the best-approximating rationals among the \( \infty \)-rationals [SW14]. On the other hand, \( p_{oo,n}/q_{oo,n} \) are the best-approximating rationals among the 1-rationals [KLL22]. To show Theorem 3, we separate the series \( \sum_{n=0}^{\infty} (\log Q_{n+1})/Q_n^\nu \) by the sum over \( n \) such that \( P_n/Q_n \) is an \( \infty \)-rational, and the sum over \( n \) such that \( P_n/Q_n \) is a 1-rational.

**Proposition 4.1.** There exists \( C_\nu > 0 \) such that, for all \( x \in \mathbb{R} \setminus \mathbb{Q} \),

\[
\left| \sum_{k=1}^{\infty} \frac{\log q_{e,k+1}}{(q_{e,k})^\nu} - \sum_{n=0}^{\nu} \frac{\log Q_{n+1}}{Q_n^\nu} \right| < C_\nu.
\]

**Proof.** Kraaikamp and Lopes investigated a relation between the ECF and the RCF convergents. We recall the relation as follows. If \( p_{e,k}/q_{e,k} = P_n/Q_n \) for some \( n \), then either \( q_{e,k+1} = Q_{n+1} \) or \( q_{e,k+1} = Q_{n+1} + Q_n \). If \( p_{e,k}/q_{e,k} \) is not an RCF convergent, then \( q_{e,k} = (i-1)Q_n + Q_{n-1} \) and \( q_{e,k+1} = iQ_n + Q_{n-1} \) for some \( 2 \leq i \leq a_{n+1} \), see [KL96, Remark 2 and Proof of Lemma 1] for more details.

Thus, if \( p_{e,k}/q_{e,k} \) is a RCF convergent, then

\[
0 \leq \frac{\log q_{e,k+1}}{(q_{e,k})^\nu} - \frac{\log Q_{n+1}}{Q_n^\nu} \leq \frac{\log 2}{Q_n^\nu}.
\]

If \( p_{e,k}/q_{e,k} \) is not a RCF convergent, then

\[
\frac{\log q_{e,k+1}}{(q_{e,k})^\nu} = \frac{\log (iQ_n + Q_{n-1})}{(i-1)Q_n + Q_{n-1})^\nu} \leq \frac{\log Q_n + \log s_\nu}{Q_n^\nu}.
\]

where \( s_\nu := \sup_{i \geq 2} \frac{\log (i+1)}{(i-1)^\nu} \) < \( \infty \). Since every \( \infty \)-rational RCF convergent is an ECF convergent, we have

\[
\left| \sum_{k=1}^{\infty} \frac{\log q_{e,k+1}}{(q_{e,k})^\nu} - \sum_{n=0}^{\nu} \frac{\log Q_{n+1}}{Q_n^\nu} \right| \leq \sum_{n=1}^{\infty} \frac{\log Q_n + \log s_\nu}{Q_n^\nu} \leq C_{1,\nu} \log s_\nu + C_{2,\nu},
\]

here \( C_{1,\nu}, C_{2,\nu} \) are constants as in (3.10). \( \square \)

**Remark 4.2.** Let \( x = [a_0; a_1, a_2, \ldots, a_n, \ldots] \). The intermediate convergent \( P_{n,i}/Q_{n,i} \) is defined by

\[
P_{n,i} = iP_{n-1} + P_{n-2} \quad \text{and} \quad Q_{n,i} = iQ_{n-1} + Q_{n-2}, \quad \text{for } 0 \leq i \leq a_n.
\]

Every \( p_{k}/q_{k}^{\infty} \) is a RCF principal convergent or an intermediate convergent. If \( P_n/Q_n \) is a 1-rational, then \( P_{n+1,i}/Q_{n+1,i} \) is not an OOCF principal convergent for any \( 1 \leq i < a_n \). If \( P_n/Q_n \) is an \( \infty \)-rational, then every 1-rational \( P_{n+1,i}/Q_{n+1,i} \) is an OOCF principal convergent, see [KLL22, Section 5].

**Proposition 4.3.** There exists \( C_\nu > 0 \) such that, for all \( x \in \mathbb{R} \setminus \mathbb{Q} \), one has

\[
\left| \sum_{k=1}^{\infty} \frac{\log q_{oo,k+1}}{q_{oo,k}} - \sum_{n=0}^{\nu} \frac{\log Q_{n+1}}{Q_n^\nu} \right| < C_\nu.
\]
Proof. Since if $P_n/Q_n$ is a 1-rational, then $P_{n+1}/Q_{n+1}$ is an $\infty$-rational, from the facts in Remark 4.2, if $p_{oo,k}/q_{oo,k}$ is a RCF principal convergent $P_n/Q_n$, then $q_{oo,k+1} = 2Q_{n+1} + Q_n$. Then, \[
abla \frac{\log q_{oo,k+1}}{(q_{oo,k})^\nu} - \frac{\log Q_{n+1}}{Q_n^\nu} \leq \frac{\log 3}{Q_n^\nu}.
\] Moreover, if $p_{oo,k}/q_{oo,k}$ is not a RCF principal convergent, then we have $q_{oo,k} = iQ_n + Q_{n-1}$ and $q_{oo,k+1} = (i+2)Q_n + Q_{n-1}$ for some $n$ and $1 \leq i \leq a_n - 2$. Then, \[
abla \frac{\log q_{oo,k+1}}{(q_{oo,k})^\nu} = \frac{\log((i+2)Q_n + Q_{n-1})^\nu}{(iQ_n + Q_{n-1})^\nu} \leq \frac{\log Q_n}{Q_n^\nu} + \frac{\log(i+3)}{(iQ_n)^\nu} \leq \frac{\log Q_n + \log s_\nu}{Q_n^\nu},
\] where $s_\nu := \sup_{i \geq 1} \frac{\log(i+3)}{i^\nu} < \infty$. In summary, we have
\[
\left| \sum_{k=1}^n \frac{\log q_{oo,k+1}}{(q_{oo,k})^\nu} - \sum_{n=1}^{P_n/Q_n \text{ rational}} \frac{\log Q_n}{Q_n^\nu} \right| < \sum_{n=1}^\infty \frac{\log 3s_\nu + \log Q_n}{Q_n^\nu} \leq C_{1,\nu} \log (3s_\nu) + C_{2,\nu},
\] where $C_{1,\nu}$ and $C_{2,\nu}$ are constants in (3.10). \hfill \Box

**Proposition 4.4.** There exists $C_\nu > 0$ such that, for all $x \in \mathbb{R} \setminus \mathbb{Q}$, one has \[
|B_{even,\nu}(x) - \sum_{n=0}^\infty \frac{\log q_{e,n+1}}{(q_{e,n})^\nu}| \leq C_\nu.
\]

**Proposition 4.5.** There exists $C_\nu > 0$ such that, for all $x \in \mathbb{R} \setminus \mathbb{Q}$, one has \[
\left| \frac{1}{2} B_{oo,\nu}(x) - \sum_{n=0}^\infty \frac{\log q_{o,n+1}}{(q_{o,n})^\nu} \right| \leq C_\nu.
\]

We can show the above propositions in a similar way to [LMNN10, Theorem 13]. For completeness, we give their proofs in Appendix B. Combining the above propositions with (1.2), we have Theorem 3.

**Remark 4.6.** The operator $T_{1,\nu}$ in (2.2) associated to the classical Brjuno function has the same spectral property to Proposition 3.6, see [MMY01, Theorem 2.6], which implies $B_{1,\nu} \in L^p(dx)$. Since $B_{even,\nu}$ and $B_{oo,\nu}$ are positive functions, $\|B_{even,\nu}\|_p \leq \|B_{even,\nu} + \frac{1}{2} B_{oo,\nu}\|_p = \|B_{1,\nu} + \psi\|_p < \infty$ with an $L^\infty$ function $\varphi$.

However, $L^p(dx)$ is not equivalent to $L^p(dm_{even})$. The space $L^p(dm_{even})$ is a proper subspace of $L^p(dx)$. It is caused from the infinite total mass of $dm_{even} = \frac{dx}{1-x^2}$. The operator associated to the ECF-Brjuno function on $L^p(dm_{even})$ has spectral radius exactly 1 which does not guarantee the existence of the solution of the functional equation (1.13) in $L^p(dm_{even})$.

If $\nu \leq 1$, then $B_{even,\nu}$ is not $L^p(dm_{even})$ for all $p \geq 1$. If $\nu = 1$, then the value $B_{even,\nu}$ in the neighborhood of 1 has positive constant lower bound, and if $\nu < 1$, then $B_{even,\nu}(x)$ goes to infinity when $x$ goes to 1. If $\varepsilon \ll \frac{1}{2}$, then $A^k_{even}(x) \in [\varepsilon, 1]$ for $i \leq \lceil \frac{1}{\varepsilon} - 2 \rceil$. Then, \[
B_{even,\nu}(1 - \varepsilon) = \sum_{k=0}^n (1 - k\varepsilon)^\nu \log \frac{1 - k\varepsilon}{1 - (k+1)\varepsilon} + (1 - (n+1)\varepsilon)^\nu B_{even,\nu} \left( \frac{1 - n\varepsilon}{1 - (n+1)\varepsilon} \right),
\]
where \( n = \lceil \frac{1}{2} - 2 \rceil \). Since \(-\log x \gtrsim 1 - x \) near 1, each term of the second summand \((1 - k\varepsilon)^\nu \log \frac{1 - k\varepsilon}{1 - (k+1)\varepsilon} \gtrsim \varepsilon (1 - k\varepsilon)^{\nu - 1}\). Combining with \((1 - k\varepsilon) < 2\varepsilon\) and \( \nu \leq 1 \), we have
\[
\sum_{k=0}^{n} (1 - k\varepsilon)^{\nu - 1} \gtrsim \frac{1}{(2\varepsilon)^{\nu - 1}}.
\]
Thus we have
\[
\int_{0}^{1} \frac{B_{\text{even},\nu}(x)\,dx}{1 - x^2} \geq \int_{0}^{1/2} \frac{(B_{\text{even},\nu}(1 - \varepsilon))\,d\varepsilon}{\varepsilon (2 - \varepsilon)} \geq \int_{0}^{1/2} \left( \frac{1 - \varepsilon}{\varepsilon^{1 - \nu}} \right)^\nu \frac{d\varepsilon}{\varepsilon} \gtrsim \int_{0}^{1/2} \frac{d\varepsilon}{\varepsilon (1 - \nu)p + 1} = \infty.
\]

**Appendix A. Proof of Lemma 3.10, Theorem 3.8 and 3.9**

**Proof of Lemma 3.10.** Recall that we denote by \( n = n(x, x') \), \( \delta = \delta(x, x') \) and \( g = \frac{\sqrt{x - 1}}{2} \). We follow the notations in (3.2).

1. For \( \ell < n - \delta \), by (3.8) and (3.15), we have \( g(1 - g) \leq \frac{\beta_{o,\ell}(x)}{\beta_{o,\ell}(x')} \leq g^{-1}(1 - g)^{-1} \). For \( n - \delta \leq i \leq n - 1 \), by (3.18), (3.20) and (3.22), we have \( \frac{1}{3} \leq x_i/x_i' \leq 3 \). Thus, we have the conclusion by taking \( \kappa_1 = 9g^{-1}(1 - g)^{-1} \).

2. Since \( \beta_{o,\ell} \leq \beta_{o,n} \) for \( \ell \geq n \), it is enough to show that (3.27) holds for \( \ell = n \). For each case (B), (C) and (D), we have \( |x - x'| = |x - x''| + |x'' - x| \).

In the case (D), without loss of generality, let us assume that \( q_n > q'_n \). From (3.25), (3.23) and (3.8), we have
\[
|x - x'| \geq \frac{\beta_{o,n-1}(x) + \beta_{o,n-1}(x')}{q_n + q'_n} \geq \frac{1 - g}{2q_n^2} \geq \frac{(1 - g)g^2}{2} \beta_{o,n-1}^2(x).
\]
By Lemma 3.10-(1), we have \( |x - x'| \geq (1 - g)g^2 \kappa_1^{-2} \beta_{o,n-1}^2(x') \), then
\[
\max\{\beta_{o,n}(x), \beta_{o,n}(x')\} \leq \sqrt{2}\kappa_1 g^{-1}(1 - g)^{-1/2} |x - x'|^{1/2}. \tag{A.1}
\]
In the case (B) and (C), from (3.19) and (3.21),
\[
|x - x'| = q_n^{-1} (\beta_{o,n}(x) + \beta_{o,n}(x')) \geq q_n^{-1} \max\{\beta_{o,n}(x), \beta_{o,n}(x')\}.
\]
From (3.8) and Lemma 3.10-(1), we have \( q_n^{-1} \geq g\beta_{o,n-1}(x) \geq \kappa_1^{-1} g\beta_{o,n-1}(x') \). Since \( \beta_{o,n-1}(x) \geq \beta_{o,n}(x) \), we have \( |x - x'| \geq g \max\{\kappa_1^{-2} \beta_{o,n}^2(x), \kappa_1^{-1} \beta_{o,n}^2(x')\} \), which implies that
\[
\max\{\beta_{o,n}(x), \beta_{o,n}(x')\} \leq g^{-1/2} \kappa_1 |x - x'|^{1/2}. \tag{A.2}
\]
In the case (A), we have \( |x_n - x_{n+1}'| \geq 1 \) since \( x \) and \( x' \) do not belong to two adjacent branches of \( A_{\text{odd}}^{n+1} \). Thus we have \( |x_n - x_n'| \geq x_n x_n' \). Suppose that \( x_n > x_n' \). If \( x_n' \geq x_n/2 \), then we have \( |x_n - x_n'| \geq x_n/x_n' \). If \( x_n' < x_n/2 \), then we also have \( |x_n - x_n'| = x_n |1 - x_n'| \geq x_n/2 > x_n'/2 \). Symmetrically, we can show that if \( x_n \leq x_n' \), then we have \( |x_n - x_n'| \geq x_n/2 \). From (3.17) and Lemma 3.10-(1), we have
\[
|x - x'| \geq \frac{1}{2} \beta_{o,n-1}(x) \beta_{o,n-1}(x') \max\{x_n^2, x_n'^2\}
\]
\[
\geq \frac{\max\{\beta_{o,n-1}(x), \beta_{o,n-1}(x')\}^2}{2\kappa_1^2} \max\{x_n^2, x_n'^2\} \geq \frac{1}{2\kappa_1^2} \max\{\beta_{o,n}(x), \beta_{o,n}(x')\}^2,
\]
which implies that
\[
\max\{\beta_{o,n}(x), \beta_{o,n}(x')\} \leq \sqrt{2}\kappa_1 |x - x'|^{1/2}. \tag{A.3}
\]
Combining (A.1), (A.2) and (A.3), we have \(\max\{\beta_{o,n}(x), \beta_{o,n}(x)\} \leq \kappa_2|x - x'|^{1/2}\) by letting \(\kappa_2 = \sqrt{2\kappa_1}g^{-1}(1 - g)^{-1/2}\).

(3) By (3.3), the end-points of \(J\) are attained by putting \(x_m = 0\) or 1 in \(\frac{p_m + p_m - \varepsilon_m x_m}{q_m + q_m - \varepsilon_m x_m}\), which are \(\frac{p_m}{q_m}\) and \(\frac{p_m + \varepsilon_m p_m - \varepsilon_m}{q_m + \varepsilon_m q_m - 1}\). Thus \(|J|^{-1} = \frac{p_m - p_m - \varepsilon_m p_m - \varepsilon_m}{q_m + q_m - \varepsilon_m q_m - 1} = q_m(q_m + \varepsilon_m q_m - 1)\). By (3.3), (3.2) and (3.4), we have \(|\frac{dA_m}{dx}(x)| = |q_m + q_m - \varepsilon_m x_m|^2\) for \(x \in J\). From (3.7), we have

\[q_m^2(g^{-1} - 1) < |J|^{-1} < q_m^2(g^{-1} + 1)\] and \(q_m^2(g^{-1} - 1)^2 < |\frac{dA_m}{dx}(x)| < q_m^2(g^{-1} + 1)^2\).

(4) Let \(x\) be the common end-point of \(J\) and \(J'\).

If \(A^m_{odd}(x) = 0\), then we have \(x = \frac{p_m}{q_m}\) and the other end-points are \(\frac{p_m + p_m - \varepsilon_m}{q_m + \varepsilon_m q_m - 1}\) by (3.3). By Lemma 3.10-(3), \(\frac{q_m^2 - 1}{q_m^2 + 1} \leq |J|/|J'| \leq \frac{q_m^2 + 1}{q_m^2 + 1} = 2g^{-1} + 1\).

If \(A^m_{odd}(x) = 1\), then we have \(x = \frac{p_m + \varepsilon_m p_m - \varepsilon_m}{q_m + \varepsilon_m q_m - 1}\) and one of other end-points is \(\frac{p_m}{q_m}\). Let \(\frac{p_m}{q_m}\) be the other end-point. By (3.24), we have \(p_m = \frac{p_m}{q_m} + 2q_m - 1\) and \(q_m = \frac{q_m}{q_m} + 2q_m - 1\). Since \(\frac{q_m + q_m - \varepsilon_m}{q_m + q_m - 1}\) or \(\frac{2 + \varepsilon_m q_m}{q_m}\). By (3.15), we have \(p_m - 1\) and \(q_m - 1\). By (3.6), \(q_m/q_m = 1 + 2q_m - 1/2\) \(\leq 1 + 2g^{-1}\). Thus \(1 \leq \frac{(2 + \varepsilon_m)q_m}{q_m} \leq 3(1 + 2g^{-1})\).

We have a conclusion by letting \(\kappa_3 = 3(1 + 2g^{-1})\).

Proof of Theorem 3.8. (1) Let \(x, x' \in [0, 1]\), \(m \geq 0, 0 < \eta \leq 1\) and \(\nu > 0\). We need to estimate \(|T^m odd, \nu f(x) - T^m odd, \nu f(x')|\) for \(f \in C^\eta\). Let \(n = n(x, x')\). We consider the following three cases:

(a) \(m > n\); From (3.14) and Lemma 3.10-(2), we have

\[|T^m odd, \nu f(x) - T^m odd, \nu f(x')| = |\beta_{o,m-1}^\nu(x) f(x_m) - \beta_{o,m-1}^\nu(x') f(x_m')|\]
\[\leq ||f||_G \beta_{o,n}^\nu(x) \beta_{o,m-1}^\nu(x_{n+1}) + \beta_{o,n}^\nu(x') \beta_{o,m-1}^\nu(x_{n+1})|\]
\[\leq 2\kappa_2^n ||f||_G \beta_{e,m-n-1}^\nu ||f||_G |x - x'|^{\nu/2}.

(A.4)

(b) \(m \leq n - \delta\); We have

\[|T^m odd, \nu f(x) - T^m odd, \nu f(x')| \leq \beta_{o,m-1}^\nu(x) |f(x_m) - f(x_m')| + |f(x_m')||f(x_m)||x - x'|^{\nu/2}.
\]

Since \(x\) and \(x'\) are contained in the same branch of \(A^m\), say \(J\), from Lemma 3.10-(3) and (3.8), we have

\[|x_m - x_m'| \leq \sup_{x \in J} |\frac{dA_m}{dx}(x)| |x - x'| \leq \frac{q_m^2}{1 - g^2} |x - x'| \leq \frac{(\beta_{o,m-1}^\nu(x))^{-1}}{g^2(1 - g)^2} |x - x'|.
\]

From Lemma 3.10-(1), (3.16) and (3.8), we have

\[|\beta_{o,m-1}^\nu(x) - \beta_{o,m-1}^\nu(x')| \leq \nu \max\{\beta_{o,m-1}(x), \beta_{o,m-1}(x')\} |x - x'|^\nu \leq \nu \kappa_2^{\nu-1} |x - x'|^\nu \leq \nu \kappa_2^{\nu-1} \beta_{o,m-1}^\nu(x) |x - x'|.
\]
Combining the above three equations, we have
\[
|T_{odd,\nu}^m f(x) - T_{odd,\nu}^m f(x')| \\
\leq \left\{ \beta_{\nu,m-1}(x) \left( \frac{(\beta_{o,m-1}(x))^{-2}}{g^2(1-g)^2} |x - x'| \right)^{\eta} |f|_\eta + \|f\|_{c_0} \nu k_1^{\nu-1} g^{-1}(\beta_{o,m-1}(x))^{-2} |x - x'| \right\}^{\eta} \\
= |x - x'|^{\eta} \beta_{\nu,m-1}(x) \left\{ \left( \frac{1}{g(1-g)^2} \right)^{2\eta} |f|_\eta + \|f\|_{c_0} \nu k_1^{\nu-1} g \left( \frac{|x - x'|^{1-\eta}}{\beta_{o,m-1}(x)} \right) \right\}.
\]

From (3.8) and Lemma 3.10-(3), we have \(|x - x'| (\beta_{o,m-1}(x))^{-2} \leq (1-g)^{-2} q_m^2 |J| \leq (1-g)^{-2} g^{-1}.

By letting \(K = (g^{-1}(1-g)^{-1})^{2\eta} + \nu k_1^{\nu-1} g^{-1}(1-g)^{-2})^{1-\eta},\) we have
\[
(A.5) \quad |x - x'|^{-\eta} |T_{odd,\nu}^m f(x) - T_{odd,\nu}^m f(x')| \leq K \|f\|_{c_0} \beta_{o,m-1}(x).
\]

(c) \(n - \delta < m \leq n;\) It is enough to consider the case of (B), (C) and (D). Then, \(x\) and \(x'\) belong to the adjacent branches of \(A^m_{\nu}\). In each case, we defined \(x''\) as the common end-point of the adjacent branches. From (A.5) and Lemma 3.10-(1), we have
\[
|T_{odd,\nu}^m f(x) - T_{odd,\nu}^m f(x')| \leq |T_{odd,\nu}^m f(x) - T_{odd,\nu}^m f(x'')| + |T_{odd,\nu}^m f(x'') - T_{odd,\nu}^m f(x')| \\
\leq 2K \|f\|_{c_0} \max \left\{ \beta_{\nu,m-1}(x) |x - x'|^{\eta}, \beta_{\nu,m-1}(x') |x' - x''|^{\eta} \right\} \\
(A.6) \quad \leq 2K \|f\|_{c_0} \nu k_1^{\nu-2\eta} \beta_{\nu,m-1}(x') |x - x'|^{\eta}.
\]

Thus we have the first assertion.

(2) and (3): Let \(\nu \geq 2\eta, f \in C^m_{[0,1]}\) and \(x, x' \in [0,1].\) By (3.9) and (3.14), we have
\[
\|T_{odd,\nu}^m f\|_{c_0} \leq g^{(m-1)\nu} \|f\|_{c_0}.
\]

If \(m \leq n,\) by (A.5) and (A.6), we have
\[
|T_{odd,\nu}^m f(x)|_\eta \leq 2K \|f\|_{c_0} \nu k_1^{\nu-2\eta} g^{(m-1)(\nu-2\eta)}.
\]

By the above arguments in (a) and (3.9), for all \(m > n,\) we have
\[
|T_{odd,\nu}^m f(x) - T_{odd,\nu}^m f(x')| \leq 2k_2^\nu \|f\|_{c_0} |x - x'|^{\eta} |x - x'|^{\nu/2 - \eta} g^{(m-n-2)\nu}.
\]

From (3.17) and Lemma 3.10-(1), we have \(|x - x'| = |x_{n-2} - x'_{n-2}| \beta_{\nu,n-2}(x) \beta_{\nu,n-2}(x') \leq g^{2(n-2)}.

Since \((n-2)(\nu - 2\eta) + (m - n - 2)\nu > (m - 2)(\nu - 2\eta) - 2\nu,\) we have
\[
|T_{odd,\nu}^m f(x)|_\eta \leq 2k_2^\nu \|f\|_{c_0} g^{(m-2)(\nu-2\eta)-2\nu}.
\]

Combining the above equations, for all \(m \geq 0,\) we have
\[
\|T_{odd,\nu}^m f\|_{c_0} \leq \max\{2k_2^\nu g^{-2\nu}, 2K \nu k_1^{\nu-2\eta}\} g^{(m-2)(\nu-2\eta)} \|f\|_{c_0},
\]
which implies that
\[
\rho(T_{odd,\nu}) \leq g^{\nu-2\eta}.
\]

If \(\nu > 2\eta,\) then \(g^{\nu-2\eta}\) is less than 1. Thus the operator \(B_{odd,\nu}\) is well-defined on \(C^m_{[0,1]}\) and it can be represented by the Neumann series as in the statement. If \(\nu = 2\eta,\) then we have the statement by taking \(c_{3,\nu} = \max\{2k_2^\nu g^{-2\nu}, 2K\}.\)

\(\square\)
Proof of Theorem 3.9. Assume that $\nu \leq 2\eta$. Let $x, x' \in [0, 1]$ and $n = n(x, x')$. From (A.4), (A.5) and (A.6), we have
\[
|B_{\text{odd}, \nu}f(x) - B_{\text{odd}, \nu}f(x')| \leq \sum_{m=0}^{n} |T_{\text{odd}, \nu}^m f(x) - T_{\text{odd}, \nu}^m f(x')| + \sum_{m=n+1}^{\infty} |T_{\text{odd}, \nu}^m f(x) - T_{\text{odd}, \nu}^m f(x')| \\
\leq c\|f\|_{\eta} \left( |x - x'|^{\eta - \nu/2} \sum_{m=0}^{n} \beta_{o,m-1}(x)^{\nu - 2\eta} + \sum_{m=n+1}^{\infty} g^{(m-n-2)\nu} \right) |x - x'|^{\nu/2},
\]
where $c = \max\{2\kappa_2^\nu, 2K\kappa_1^{2-2\eta}g^{-\nu}\}$. By (3.8) and (3.9), $|x - x'|^{\eta - \nu/2} \sum_{m=0}^{n} (\beta_{o,m-1}(x))^{\nu - 2\eta}$ is bounded above by
\[
(A.7) \quad \frac{|x - x'|^{\eta - \nu/2}}{\beta_{o,n-1}(x)^{2\eta - \nu}} \sum_{m=0}^{n} (x_m x_{m+1} \cdots x_{n-1})^{2\eta - \nu} \leq \frac{(|J|q_n^{2\eta - \nu/2})}{(1 - g)^{2\eta - \nu}}.
\]
If $x, x'$ are in the same branch, then let $J$ be the branch of $A^n$, otherwise, i.e. $x, x'$ are in the adjacent branches of $A^n$, then let $J$ be the union of the branches. From Lemma 3.10-(3) and (4), $|x - x'|^2 q_n^2 \leq |J| q_n^2 < 2\kappa_3 g$. If $2\eta > \nu$, then (A.7) is uniformly bounded and
\[
|B_{\text{odd}, \nu}f(x) - B_{\text{odd}, \nu}f(x')| \leq c'\|f\|_{\eta} |x - x'|^{\nu/2},
\]
where $c' = c \left( \frac{(2\kappa_3 g)^{\eta - \nu/2}}{(1 - g)^{\eta - \nu} + g^{\nu}} \right)$, which proves the first assertion.
If $2\eta = \nu$, then (A.7) is bounded above by $n(x, x')$. Thus we have
\[
|B_{\text{odd}, \nu}f(x) - B_{\text{odd}, \nu}f(x')| \leq c\|f\|_{\eta} (n(x, x') + 1 + g^{\nu} - (1 - g^{-1})) |x - x'|^{\nu/2}.
\]
Since $x, x'$ are in the same branch of $A_{odd}^{-2}$, say $J'$, we have $|x - x'| \leq |J'| \leq g^{-1} q_{n-2}^2$ by Lemma 3.10-(3). By (3.8), we have $q_{n-2}^2 \leq (1 - g)^{-1} \beta_{o,n-3}(x) \leq (1 - g)^{-1} g^{n-3}$, thus, we have $|x - x'| \leq g^{2n-7}(1 - g)^{-2}$, which means that $n \leq \frac{\log|x - x'|^{-1} + \log(g^{-1})}{2 \log(1/g)}$. There are constants $c_1' > 0$ and $\kappa_2'' > 0$ such that
\[
|B_{\text{odd}, \nu}f(x) - B_{\text{odd}, \nu}f(x')| \leq \|f\|_{\eta} (\kappa_1'' \log |x - x'|^{-1} + \kappa_2'' |x - x'|^{\nu/2}).
\]
If $\eta' < \nu/2$, then $|B_{\text{odd}, \nu}f(x) - B_{\text{odd}, \nu}f(x')| \leq \|f\|_{\eta} (\kappa_1'' \log |x - x'|^{-1} + \kappa_2'' |x - x'|^{\nu/2 - \eta'}) |x - x'|^{\nu/2 - \eta'}$ and $(\kappa_1'' \log |x - x'|^{-1} + \kappa_2'' |x - x'|^{\nu/2 - \eta'})$ is uniformly bounded above. 

\[\Box\]

**Appendix B. Proof of Proposition 4.4 and 4.5**

**Proof of Proposition 4.4.** For $x \in \mathbb{R} \setminus \mathbb{Q}$, let
\[
I_*(x) := \{n \in \mathbb{N} : x_{e,n} \in (1/3, 1)\} = \{n : a_{e,n+1} = 2\}, \\
I_{**}(x) := \{n \in \mathbb{N} : x_{e,n} \in (0, 1/3)\} = \{n : a_{e,n+1} \geq 4\}.
\]
Let $x_{e,m+1}, \ldots, x_{e,m+\ell} \in (1/2, 1]$ and $x_{e,m}, x_{e,m+\ell+1} \not\in (1/2, 1]$. By the same argument in [LMNN10, Proof of Thm 13], for $0 < t \leq \ell - 1$, we have
\[
(B.1) \quad (x_{e,-1}x_{e,0} \cdots x_{e,m}x_{e,m+1} \cdots x_{e,m+\ell})^\nu \log \frac{1}{x_{e,m+\ell+1}} < (x_{e,1}x_{e,0} \cdots x_{e})^\nu \frac{3}{\ell + 2},
\]
and for \( t = 0 \), we have \((x_{e-1} x_{e,0} \cdots x_{e,m})^\nu \log (1/x_{e,m+1}) < (x_{e-1} x_{e,0} \cdots x_{e,m})^\nu t^{-1}\). Then, we have
\[
\sum_{t=0}^{\ell-1} (x_{e-1} x_{e,0} \cdots x_{e,m+t})^\nu \log \frac{1}{x_{e,m+t+1}} < 3(x_{e-1} x_{e,0} \cdots x_{e,m})^\nu.
\]
If \( x_{e,m+1}, \ldots, x_{e,m+\ell} \in (1/3, 1/2) \) and \( x_{e,m}, x_{e,m+\ell+1} \not\in (1/3, 1/2) \), then
\[
(B.2)
\sum_{t=0}^{\ell-1} (x_{e-1} x_{e,0} \cdots x_{e,m+t})^\nu \log \frac{1}{x_{e,m+t+1}} \leq \sum_{t=0}^{\ell-1} (x_{e,0} \cdots x_{e,m} x_{e}^\nu)^\nu \log 3 < (3) (x_{e,0} x_{e,1} x_{e,n})^\nu.
\]
With a similar argument of [LMNN10], we have
\[
\sum_{n \in I(x)} (x_{e-1} x_{e,0} x_{e,1} \cdots x_{e,n-1})^\nu \log \frac{1}{x_{e,n}} < 3.
\]
Then, it is sufficient to show that there exists \( C_\nu > 0 \) such that
\[
(B.3)
\left| \sum_{n \in I_*} \beta_{e,n-1}^\nu \log \frac{1}{x_{e,n}} - \sum_{n \in I_*} \frac{\log q_{e,n+1}^\nu}{(q_{e,n})^\nu} \right| < C_\nu.
\]
The left hand side of (B.3) is bounded above by
\[
(B.4)
\sum_{n \in I_*} \left| \beta_{e,n-1}^\nu \log \beta_{e,n} q_{e,n+1} \right| + \sum_{n \in I_*} \beta_{e,n-1}^\nu \log \beta_{e,n-1}^\nu + \sum_{n \in I_*} \left\{ \frac{1}{(q_{e,n})^\nu} - (\beta_{e,n-1})^\nu \right\} \log q_{e,n+1}^\nu.
\]
From now on, let \( n \in I_* \). Since \( x_{e,n} \leq 1/3 \) and \( a_{n+1}^e \geq 4 \), we have \( q_{e,n-1}/q_{e,n} \leq 1/3 \) and \( 1 + \varepsilon_e n q_{e,n-1}(q_{e,n})^{-1} x_{e,n} \geq 2/3 \). By the fact that (3.4) holds for the ECF, we have
\[
(B.5)
\beta_{e,n} \leq \frac{1}{q_{e,n} + \varepsilon_e n q_{e,n-1} x_{e,n}} \leq \frac{1}{q_{e,n+1} - q_{e,n}} \leq 3 + \frac{1}{2q_{e,n+1}}.
\]
Thus the first summand of (B.4) is bounded above by
\[
(B.6)
\sum_{n \in I_*} \beta_{e,n-1}^\nu \log \beta_{e,n} q_{e,n+1} \leq \sum_{n \in I_*} \frac{3^\nu \log (3/2)}{2^\nu q_{e,n}^\nu} < \frac{3^\nu \log (3/2)}{4^\nu}.
\]
Since \( \{q_{e,n}\} \) is increasing, \( 1 + \varepsilon_e n q_{e,n-1}(q_{e,n})^{-1} x_{e,n} \leq 2 \). Thus we have
\[
(\beta_{e,n-1})^{-1} = q_{e,n}(1 + \varepsilon_e n q_{e,n-1}(q_{e,n})^{-1} x_{e,n}) \leq 2q_{e,n}.
\]
Note that if \( n \) is the \( i \)th number in \( I_* \), then we have \( q_n > 3^{i-1} \), since \( q_n \geq 3q_{n-1} \) if \( n \in I_* \), and \( \{q_n\}_{n \in \mathbb{N}} \) is increasing. By (B.5) and the fact that \( \log N \leq N^{\nu/2} \) for large enough \( N \), the second summand of (B.4) is bounded above by
\[
(B.7)
\sum_{n \in I_*} \frac{3^\nu \log 2(q_{e,n})^\nu}{(2q_{e,n})^\nu} \leq S_\nu + \sum_{n \in I_*} \frac{3^\nu}{(2q_{e,n})^{\nu/2}} \leq S_\nu + \frac{3^{3/2}}{6^{\nu/2} - 2^{\nu/2}}.
\]
where \( S_\nu = \sum_{n \leq N} \frac{3^\nu \log(2q_{e,n})}{(2q_{e,n})^\nu} \). Since (3.4) holds for the ECF, we have \( \varepsilon_{e,n} \beta_{e,n} q_{e,n-1} + \beta_{e,n-1} q_{e,n} = 1 \). By (B.6), the last summand of (B.4) is bounded above by

\[
\tag{B.8} c_\nu \sum_{n \in I_*(x)} \max \left\{ \frac{1}{q_{e,n}} \nu - 1, \frac{\beta_{e,n-1}}{q_{e,n}} \right\} \left| \varepsilon_{e,n} \frac{q_{e,n-1}}{q_{e,n}} \beta_{e,n} \right| \log q_{e,n+1},
\]

for some \( c_\nu > 0 \). If \( \nu \geq 1 \), then (B.8) is bounded above by

\[
\sum_{n \in I_*(x)} \beta_{e,n} \log q_{e,n+1} \leq \frac{3}{2} \sum_{n \in I_*(x)} \frac{\log q_{e,n+1}}{q_{e,n+1}} \leq \sum_{i=1}^\infty \frac{1}{i^{3/2}}.
\]

If \( 0 < \nu < 1 \), then (B.8) bounded above by

\[
\sum_{n \in I_*(x)} \max \left\{ \frac{1}{q_{e,n}} \nu - 1, \beta_{e,n-1} \right\} \frac{q_{e,n-1}}{q_{e,n}} \beta_{e,n} \log q_{e,n+1}
\]

\[
\leq \sum_{n \in I_*(x)} \max \left\{ \frac{\beta_{e,n}}{q_{e,n}} \nu - 1 - \frac{3q_{e,n-1}}{2q_{e,n}}, \frac{q_{e,n}}{q_{e,n+1}} \right\} \log q_{e,n+1} \leq \sum_{n \in I_*(x)} \max \left\{ \beta_{e,n} \nu - 1 - \frac{3q_{e,n-1}}{2q_{e,n}} \right\} \log q_{e,n+1}
\]

\[
\leq \sum_{n \in I_*(x)} \frac{3}{2q_{e,n+1}} \log q_{e,n+1} \lesssim \nu \sum_{i=1}^\infty \frac{1}{(3^{\nu/2})^i}.
\]

\[
\text{Proof of Proposition 4.5. For } x \in \mathbb{R} \setminus \mathbb{Q}, \text{ let}
\]

\[
J_*(x) = \{ n : x_{o,n} \in [0, 1/2) \} \quad \text{and} \quad J_{**}(x) = \{ n : x_{o,n} \in [1/2, 1] \}.
\]

Since \( \iota A_{oo} = A_{even} \iota \), if \( y = \iota(x) \), then \( \iota(x_{oo,n}) = y_{e,n} \). By the arguments in the proof of Proposition 4.4, we have

\[
\sum_{n \in J_*(x)} [(x+1)\iota(x_{oo,-1})\iota(x_{oo,0}) \cdots \iota(x_{oo,n-1})]^{\nu} \log \frac{1}{\iota(x_{oo,n})} < 6.
\]

Thus, similarly, It is sufficient to show that there exists \( C_\nu > 0 \) such that

\[
\tag{B.9} \frac{1}{2} \sum_{n \in I_*(x)} \beta_{oo,n-1} \log \frac{1}{\iota(x_{oo,n})} - \sum_{n \in I_*(x)} \frac{\log q_{oo,n+1}}{q_{oo,n}} < C_\nu.
\]

By (4.4), we have \( \beta_{oo,n-1} q_{oo,n} + \varepsilon_{oo,n} \beta_{oo,n} q_{oo,n-1} = 2 \). Also, we have \( \iota(x_{oo,n}) \leq 1/3 \), \( q_{oo,n-1}/q_{oo,n} \leq 1 \) and \( q_{oo,n+1} \geq 3 q_{oo,n} \), with the same argument in the proof of Proposition 4.4, we have the conclusion.

\[
\text{APPENDIX C. THE BRJUNO FUNCTIONS AS COCYCLES}
\]

In this section, we show how to interpret \( B_{0,\nu}, B_{odd,\nu} \) and \( B_{even,\nu} \) as cocycles under the actions on the real line respectively of the modular group \( \text{PSL}_2(\mathbb{Z}) \), \( \Gamma \) and \( \Theta \). This extends to these functions the results obtained for the Brjuno function in Appendix 5 of [MMY01]. Let \( G \) be a group and \( G \) act on the left on a set \( X \). Let \( A \) be an abelian ring, \( A^* \) be the
The automorphic factor should satisfy
\[ \chi(g_1 g_0, x) = \chi(g_1, g_0 x)\chi(g_0, x). \]

Then \( M^X \) has a \( \mathbb{Z}^{[G]} \)-module structure.

Let \( C^n := C^n(G, M^X) = \text{Map}(G, M^X) \). We consider the cochain complex
\[ \cdots \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} \cdots \]
defined by
\[
(d^n \varphi)(g_0, \ldots, g_n) = g_0 \varphi(g_1, \ldots, g_n) + \sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}).
\]

An \( n \)-coboundary is an element of the image of \( d^{n-1} \). An \( n \)-cocycle is an element of the kernel of \( d^n \). We identify \( C^0 \) with \( M^X \). If \( f \) is a 1-coboundary of \( \varphi \in M^X \), then
\[ f(g)(x) = (d^0 \varphi)(g)(x) = g_0 \varphi(x) - \varphi(x) = \chi(g^{-1}, x)\varphi(g^{-1}x) - \varphi(x). \]

A 1-cocycle \( c \) satisfies
\[
(d^1 c)(g_0, g_1) = g_0 c(g_1) - c(g_0 g_1) + c(g_0) = 0.
\]

Thus, \( c(g_0 g_1) = c(g_0) + g_0 c(g_1) \). Let \( \tilde{c}(g) = c(g^{-1}) \). Then, we have
\[ \tilde{c}(g_0 g_1; x) = \tilde{c}(g_1; x) + \chi(g_1, x)\tilde{c}(g_0; g_1 x). \]

### C.1. Semi-Brjuno function and the modular group action.

The modular group \( \text{PSL}_2(\mathbb{Z}) \) is generated by \( T(x) = x + 1 \) and \( \sigma(x) = -1/x \).

**Proposition C.1.** For \( g \in \text{PSL}_2(\mathbb{Z}) \setminus \{I\} \) and \( x_0 \in \mathbb{R} \setminus \mathbb{Q} \), there exist unique integer \( r \geq 1 \) and \( g_1, \ldots, g_r \in \{T, T^{-1}, \sigma\} \) such that \( g = g_r \cdots g_1 \) with \( g_i g_{i-1} \neq I \), and
\[ x_{i-1} \in (-\infty, -1) \cup (0, 1) \text{ if } g_i = \sigma, \]
where \( x_i = g_i g_{i-1} \cdots g_1 x_0 \).

**Proof.** We have \( \sigma = T \sigma T \sigma T \). For \( g = \sigma, r \) and \( g_i \)'s are determined as follows.

1. If \( x_0 \in (-1, -\frac{1}{2}) \), then \( r = 5 \) and \( g_5 \cdots g_1 = T \sigma T \sigma T \) since \( T(x_0) \in (0, \frac{1}{2}) \) and \( T \sigma T(x_0) \in (-\infty, -1) \).
2. If \( x_0 \in (-\frac{1}{2}, -\frac{1}{3}) \), then \( r = 9 \) and \( g_9(g_8 \cdots g_4)g_3 g_2 g_1 = T(T \sigma T \sigma T)T \sigma T \) since \( T \sigma T(x_0) \in (-1, -\frac{1}{2}) \).
3. Inductively, if \( x_0 \in (-\frac{1}{n}, -\frac{1}{n+1}) \), then \( r = 4n+1 \) and \( g_r \cdots g_1 = T(h_r \cdots h_1)T \sigma T \), where \( h_r \cdots h_1 \) is the representation of \( \sigma \) for \( x \in (-\frac{1}{n+1}, -\frac{1}{n}) \).
4. For \( x_0 \in (n, n+1) \), by using \( \sigma = \sigma^{-1} \), we have \( r = 4n + 1 \) and \( g_r \cdots g_1 = \sigma^{-1} T^{-1}(h_1^{-1} \cdots h_r^{-1})T^{-1}. \)
To show the uniqueness, assume that \( g = g_r \cdots g_1 = I \). Let \( i_1 < i_2 < \cdots < i_k \) be indices such that \( g_{i_\ell} = \sigma \). Since \( 1 = g'(x_0) = \prod_{\ell=1}^r g'_i(x_{i-1}) = \prod_{\ell=1}^k \frac{1}{x_{i_\ell}^{x_{i_\ell-1}}} \), we have

\[
|x_{i_1-1} x_{i_2-1} \cdots x_{i_k-1}| = 1.
\]

If \( |x_{i_{\ell-1}}| > 1 \), then \( x_{i_{\ell-1}} < -1 \) by Condition (C.4). Then, \( x_{i_{\ell}} \in (0, 1) \) and \( x_{i_{\ell+1}-1} = x_{i_{\ell}} \pm (i_{\ell+1}-i_{\ell}-1) \in \mathbb{R} \setminus (0, 1) \). By Condition (C.4), \( x_{i_{\ell+1}-1} < -1 \). Since \( g_{i_1} \cdots g_{i_*} g_r \cdots g_i = I \) for all \( i \), if \( |x_{i_{\ell}-1}| > 1 \) for some \( \ell \), then \( |x_{i_{\ell}-1}| > 1 \) for all \( \ell \). By (C.5), it cannot happen, thus we have \( |x_{i_{\ell-1}}| \leq 1 \) for all \( \ell \). If \( |x_{i_{\ell}-1}| < 1 \) for some \( \ell \), then it contradicts to (C.5). Therefore \( x_{i_{\ell-1}} = 1 \) for all \( \ell \). It contradicts to \( x_0 \) is irrational. Thus, there are no \( g_i \)'s such that \( g_r \cdots g_1 = I \).

\[ \square \]

**Corollary C.2.** Let \( A \) be an abelian ring. For two maps \( t : \mathbb{R} \setminus \mathbb{Q} \to A^* \) and \( s : (0, 1) \setminus \mathbb{Q} \to A^* \), there exists a unique automorphic factor \( \chi \) such that

\[
\begin{align*}
\chi(T, x) &= t(x) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\sigma, x) &= s(x) \quad \text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{align*}
\]

**Proof.** We define by

\[
\begin{align*}
\chi(T^{-1}, x) &= (t(x-1))^{-1} \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \text{ and} \\
\chi(\sigma, x) &= (s(-1/x))^{-1} \quad \text{for } x \in (-\infty, -1) \setminus \mathbb{Q}.
\end{align*}
\]

Then it follows (C.1). From Proposition C.1, for any \( g \in \text{PSL}_2(\mathbb{Z}) \) and \( x_0 \in \mathbb{R} \setminus \mathbb{Q} \), there exists \( g_i \)'s such that \( g = g_r \cdots g_1 \) following (C.4), by (C.1), if there exists such an automorphic factor \( \chi \), then

\[
\chi(g, x_0) = \prod_{i=1}^r \chi(g_i, g_{i-1} \cdots g_1 x_0),
\]

which means that \( \chi \) is uniquely determined.

To show the existence, it is enough to show that \( \chi(g h, x_0) = \chi(g, h x_0) \chi(h, x_0) \). Let

\[
g = g_M \cdots g_1 \quad \text{and} \quad h = h_N \cdots h_1
\]

be the representations of \( g \) for \( h x_0 \) and \( h \) for \( x_0 \) following (C.4), respectively. From the uniqueness of the representations, the representation of \( g h \) for \( x_0 \) is

\[
g h = g_M \cdots g_{i+1} h_{N-i} \cdots h_1,
\]

where \( g_\ell \) is \( h_{N-\ell}^{-1} \) for \( 1 \leq \ell \leq i \). From (C.6), we have \( \chi(g_\ell, h_{N-\ell} y) \chi(h_{N-\ell}, y) = 1 \) for \( 1 \leq \ell \leq i \), where \( y = h_{N-\ell} \cdots h_1 x_0 \).

\[ \square \]

**Corollary C.3.** Let \( A \) be an abelian ring, \( \chi \) be an automorphic factor, \( M \) be an \( A \)-module such that having the \( \mathbb{Z}^{[\text{PSL}_2(\mathbb{Z})]} \)-module structure associated with \( \chi \). For two maps \( \tilde{c}_T : \mathbb{R} \setminus \mathbb{Q} \to M \) and \( \tilde{c}_\sigma : (0, 1) \setminus \mathbb{Q} \to M \), there exists a unique cocycle \( \tilde{c} : \text{PSL}_2(\mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Q}) \to M \) such that

\[
\begin{align*}
\tilde{c}(T; x) &= \tilde{c}_T(x) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \text{ and} \\
\tilde{c}(\sigma; x) &= \tilde{c}_\sigma(x) \quad \text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{align*}
\]

**Proof.** We define by

\[
\begin{align*}
\tilde{c}(T^{-1}; x) &= -\chi(T^{-1}, x) \tilde{c}(T; x - 1) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \text{ and} \\
\tilde{c}(\sigma; x) &= -\chi(\sigma, x) \tilde{c}(\sigma; -1/x) \quad \text{for } x \in (-\infty, -1) \setminus \mathbb{Q}.
\end{align*}
\]
It follows (C.3). For any \( g \in \text{PSL}_2(\mathbb{Z}) \) and \( x_0 \in \mathbb{R} \), there exist \( g_i \)'s in \( \{ T, T^{-1}, \sigma \} \) such that \( g = g_r \cdots g_1 \) following (C.4), we have
\[
\tilde{c}(g; x_0) = \sum_{i=1}^{r} \chi(g_{i-1} \cdots g_1, x_0)\tilde{c}(g_i; g_{i-1} \cdots g_1 x_0),
\]
which means that \( \tilde{c} \) is uniquely determined.

To show the existence, it is enough to show that \( \tilde{c} \) satisfies \( \tilde{c}(gh; x_0) = \tilde{c}(h; x_0) + \chi(h, x_0)\tilde{c}(g; h x_0) \). With the same set up as (C.7), from (C.2) and (C.3), we have
\[
\tilde{c}(g_i; h_{N-\ell} y) + \tilde{c}(h_{N-\ell}; y) = 0, \text{ where } y = h_{N-\ell-1} \cdots h_1 x_0
\]
then we are done. \( \square \)

Let \( A = \mathbb{R} \), \( t(x) = 1 \), \( s(x) = x^\nu \) on \((0, 1)\) for \( \nu \in \mathbb{R} \), \( \tilde{c}_T(x) = 0 \) and \( \tilde{c}_\sigma(x) = -\log x \) on \((0, 1)\). There exist an automorphic factor \( \chi \) and a cocycle \( \tilde{c} \) such that
\[
\begin{cases}
\chi(T^n, x) = 1 & \text{for } n \in \mathbb{Z}, \ x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\sigma, x) = x^\nu & \text{for } x \in (0, 1) \setminus \mathbb{Q},
\end{cases}
\quad \text{and } \begin{cases}
\tilde{c}(T; x) = 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\tilde{c}(\sigma; x) = -\log x & \text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{cases}
\]

From (2.3), \( B_{0, \nu}(x) = B_{0, \nu}(x + 1) \) for \( \mathbb{R} \setminus \mathbb{Q} \) and (C.2), we have
\[
\begin{cases}
(d^0 B_{0, \nu})(\sigma; x) = \chi(\sigma, x)B_{0, \nu}(\sigma x) - B_{0, \nu}(x) = \log x & \text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
(d^0 B_{0, \nu})(T; x) = \chi(T^{-1}, x)B_{0, \nu}(T^{-1}(x)) - B_{0, \nu}(x) = 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

By the uniqueness of the cocycle \( \tilde{c} \), we have \( \tilde{c} = d^0(-B_{0, \nu}) \), i.e. the 1-coboundary of \( -B_{0, \nu} \) is a 1-cocycle under the \( \text{PSL}_2(\mathbb{Z}) \)-action.

### C.2. OCF-Brjuno function and the \( \Gamma \)-action.

From now on, let us denote by \( U(x) = -x \). Let \( \tilde{\Gamma} = \Gamma \sqcup UT \), i.e.
\[
\tilde{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \pmod{2} \right\}.
\]

The group \( \tilde{\Gamma} \) is generated by \( \tau(x) = x + 2 \), \( \Delta(x) = 1 - 1/x \) and \( U(x) = -x \).

**Proposition C.4.** For all \( g \in \tilde{\Gamma} \setminus \{ I \} \) and \( x_0 \in \mathbb{R} \setminus \mathbb{Q} \), there exist unique integer \( r \geq 1 \) and \( g_1, \ldots, g_r \in \{ \tau, \tau^{-1}, \Delta, \Delta^{-1}, U \} \) such that \( g = g_r \cdots g_1 \) with \( g_i g_{i-1} \neq I \)
\[
\begin{align*}
x_{i-1} & \in (-1, 1) =: J_U \quad \text{if } g_i = U, \\
x_{i-1} & \in (1, \infty) =: J_\Delta \quad \text{if } g_i = \Delta \text{ and} \\
x_{i-1} & \in (0, 1) =: J_{\Delta^{-1}} \quad \text{if } g_i = \Delta^{-1},
\end{align*}
\]
where \( x_i := g_i g_{i-1} \cdots g_1 x_0 \).

**Proof.** Since \( \Delta^3 = I \), \( U = \tau^\pm 1 U \tau^\pm 1 \), \( \tau^{-1} \Delta(x) = -1 - \frac{1}{x} = U \Delta U(x) \), and \( (\tau^{-1} \Delta)^3 = I \), we have the following relations:
\[
\begin{align*}
\Delta &= \Delta^{-2} \text{ and } \Delta^{-1} = \Delta^2, \\
\Delta &= \tau U \Delta U \text{ and } \Delta^{-1} = U \Delta^{-1} U \tau^{-1}, \\
\Delta &= \tau \Delta^{-1} \tau \Delta^{-1} \tau \text{ and } \Delta^{-1} = \tau^{-1} \Delta \tau^{-1} \Delta \tau^{-1}.
\end{align*}
\]

(1) If \( g = U \) and \( x_0 \in (2n-1, 2n+1) \) for \( n \in \mathbb{Z} \), then \( r = 2n + 1 \) and \( g_{2n+1} \cdots g_1 = \tau^{-n} U \tau^{-n} \).

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(2) For \(g = \Delta\) and \(x_0 \in (-\infty, -1)\), we will use (C.11). Since \(Ux_0 \in (1, \infty) = J_\Delta\) and \(\Delta Ux_0 \in (\frac{1}{2}, 1) \subset J_U\), by (1), for \(x_0 \in (-2n - 1, -2n + 1), n \in \mathbb{N}\), we have \(r = 2n + 4\) and \(g_r \cdots g_1 = \tau U \Delta^n U \tau^{-1}\).

(3) For \(g = \Delta^{-1}\) and \(x_0 \in (1, 2)\), we will use (C.11). We have \(\tau^{-1} x_0 \in (-1, 0) = J_U\), \(U \tau^{-1} x_0 \in (0, 1) = J_{\Delta^{-1}}\) and \(\Delta^{-1} U \tau^{-1} x_0 \in (1, \infty)\). Since \(\Delta^{-1} U \tau^{-1} x_0 \in (2n - 1, 2n + 1)\) for some \(n \in \mathbb{N}\) if and only if \(x_0 \in \left(\frac{2n+2}{2n+1}, \frac{2n}{2n-1}\right)\), by (1), for \(x_0 \in \left(\frac{2n+2}{2n+1}, \frac{2n}{2n-1}\right)\), we have \(r = 2n + 4\) and \(g_r \cdots g_1 = \tau^{-n} U \tau^{-n} \Delta^{-1} U \tau^{-1}\).

(4) For \(g = \Delta^{-1}\) and \(x_0 \in (-\infty, -1)\), we will use (C.10) and (2). By (2), we have a decomposition of \(\Delta\) on \((-2n - 1, -2n + 1)\) as \(\tau U \Delta^n U \tau^{-1}\). Since \(\Delta x_0 \in (1, 2) \subset J_\Delta\), for \(x \in (-2n - 1, -2n + 1), n \in \mathbb{N}\), we have \(r = 2n + 5\) and \(g_r \cdots g_1 = \Delta U \Delta^n U \tau^{-1}\).

(5) For \(g = \Delta^{-1}\) and \(x_0 \in (3, \infty)\), we will use (C.11). By (C.10) and (C.11), we have \(\Delta^{-1} U = \Delta(U) = \Delta U \Delta^{-1}\). Thus,

\[
\text{(C.13)}
\Delta^{-1} = U(\Delta^{-1} U)\tau^{-1} = U \Delta U \Delta^{-1}. 
\]

Since \(\tau^{-1} x_0 \in (1, \infty) = J_\Delta\), \(\Delta^{-1} x_0 \in (0, 1) \subset J_U\), \(\tau U \Delta^{-1} x_0 \in (1, 2) \subset J_\Delta\), \(\Delta U \tau^{-1} x_0 \in (0, \frac{1}{2}) \subset J_U\), we have \(r = 6\) and \(g_r \cdots g_1 = U \Delta U \Delta^{-1}\).

(6) For \(g = \Delta\) and \(x_0 \in (0, \frac{1}{2})\), we will use (C.10) and (3). Since \(0, \frac{1}{2}) \subset J_{\Delta^{-1}}\) and \(\Delta^{-1}(0, \frac{1}{2}) = (1, 2)\), by using \(\Delta = \Delta^{-1} \Delta^{-1} = (\tau^{-n} U \tau^{-n} \Delta^{-1} U^{-1}) \Delta^{-1}\), for \(x \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)\), we have \(r = 2n + 5\) and \(g_r \cdots g_1 = \tau^{-n} U \tau^{-n} \Delta^{-1} U \tau^{-1}\).

(7) For \(g = \Delta\) and \(x_0 \in (-\frac{1}{2}, 0)\), we will use (C.13). By (C.13), we have \(\Delta = \tau \Delta^{-1} U \tau^{-1} \Delta^{-1} U\). Since \((-\frac{1}{2}, 0) \subset J_U\), \(U x_0 \in (0, \frac{1}{2}) \subset J_{\Delta^{-1}}\), \(\tau^{-1} \Delta^{-1} U x_0 \in (1, 0) \subset J_U\) and \(\tau^{-1} \Delta^{-1} U x_0 \in (0, 1) \subset J_{\Delta^{-1}}\), we have \(r = 6\) and \(g_r \cdots g_1 = \tau \Delta^{-1} U \tau^{-1} \Lambda^{-1} U\).

In summary, we have a representation of \(\Delta\) on \(D_1 := (-\infty, -1) \cup (-\frac{1}{2}, \frac{1}{2}) \cup (1, \infty)\) and we have a representation of \(\Delta^{-1}\) on \(D_2 := (-\infty, -1) \cup (0, 2) \cup (3, \infty)\) following (C.9).

We will show that \(\Delta\) has a representation on \((-\frac{1}{2}, \frac{1}{2}) \cup (1, \frac{1}{2})\) following (C.9) inductively. If \(\Delta\) has a representation on an interval \((a, b) \subset (-\frac{1}{2}, \frac{1}{2})\), then \(\Delta^{-1}\) has a representation on \((a, b)\) with \(\Delta^{-1} = \Delta^2\) since \(\Delta(a, b) \subset (2, 3) \subset D_1\). Then, \(\Delta\) has a representation on \(J = \tau^{-1} \Delta^{-1} (a, b)\) since \(\Delta = \tau \Delta^{-1} \Delta^{-1} \tau\) and \(\tau J = \Delta \tau^{-1} (a, b) \subset (\frac{3}{5}, \frac{5}{3}) \subset D_2\). Let us define a sequence \(a_i\) and \(b_i\) by

\[
\begin{cases}
    a_{i+1} = \tau^{-1} \Delta^{-1} a_i = -\frac{a_i-1}{2-a_i}, \text{ where } a_0 = -1, \\
    b_{i+1} = -\frac{b_i}{2-b_i}, \text{ where } b_0 = -\frac{1}{2}.
\end{cases}
\]

Then, if \(\Delta\) has a representation on \((a_i, a_{i+1})\) and \((b_{i+1}, b_i)\), then \(\Delta\) has a representation on \((a_{i+1}, a_{i+2})\) and \((b_{i+2}, b_{i+1})\).

For \(g = \Delta\) and \(x_0 \in (-1, -\frac{2}{3})\), we have \(\tau x_0 \in (1, \frac{3}{4}) \subset D_1\) and \(\tau \Delta^{-1} \tau x_0 \in (-\infty, -1) \subset D_1\). Thus, \(\Delta\) on \((a_0, a_1) = (-1, -\frac{2}{3})\) has a representation by using (C.12). For \(g = \Delta^{-1}\) and \(x_0 \in (2, \frac{5}{2})\), we have \(\tau^{-1} x_0 \in (0, \frac{1}{2}) \subset D_1\), \(\Delta^{-1} x_0 \in (-\infty, -1)\), \(\tau^{-1} \Delta^{-1} x_0 \in (0, -3) \subset D_1\). Thus, \(\Delta^{-1}\) has a representation on \((2, \frac{5}{2})\) by (C.12). By (C.10), \(\Delta\) has a representation on \(\Delta(2, \frac{5}{2}) = (\frac{1}{2}, \frac{3}{5})\). By (C.11), \(\Delta\) has a representation on \((b_1, b_0) = (\frac{3}{5}, -\frac{1}{2})\). Since \(a_i \to -\frac{\sqrt{5}-1}{2}\) and \(b_i \to -\frac{\sqrt{5}-1}{2}\) as \(i \to \infty\), \(\Delta\) has a representation on \((-1, -\frac{1}{2})\). By using (C.11), we have a representation of \(\Delta\) on \((\frac{1}{2}, 1)\).

Finally, by using (C.10), we can show that \(\Delta^{-1}\) has a representation on \((-1, 0) \cup (2, 3)\).

To show the uniqueness, we assume that there exist \(g_i \in \{\tau, \tau^{-1}, \Delta, \Delta^{-1}, U\} \) for \(1 \leq i \leq r\) following (C.9) such that \(g_r \cdots g_1 = I\). Let \(i_1 < i_2 < \cdots < i_k\) be the indices such that
\(g_{i\ell} = \Delta\) or \(\Delta^{-1}\). Then we have

\[
(C.14) \quad \prod_{i=1}^{r} g_i(x_{i-1}) = 1.
\]

If \(x_{i\ell-1} > 1\), then \(x_{i\ell} = 1 - \frac{1}{x_{i\ell-1}} \in (0, 1)\). Then, \(g_{i\ell+1} = U, \tau\) or \(\tau^{-1}\). If \(g_{i\ell+1} = \tau^{-1}\), then \(g_j = \tau^{-1}\) for all \(j > i\ell\). Then \(\ell = k\). If \(g_{i\ell+1} = \tau\), then \(g_{i\ell+j} = \tau\) for all \(j < i\ell+1 - i\ell\). Then \(x_{i\ell+1} > 1\) and \(g_{i\ell+1} = \Delta\). If \(g_{i\ell+1} = U\), then \(x_{i\ell+1} \in (-1, 0)\). Then \(g_j = \tau^{-1}\) for all \(j > i\ell + 1\). Since \(g_{i\ell} g_{i\ell+1} \cdots g_{i\ell} g_{i\ell+r} \cdots g_i = I\) for all \(i\) is also following (C.9), we conclude that if \(x_{i\ell-1} > 1\) for some \(\ell\), then \(x_{i\ell-1} > 1\) and \(g_{i\ell} = \Delta\) for all \(\ell\). Then \(\prod_{i=1}^{r} g_i(x_{i-1}) = \prod_{\ell=1}^{k} x_{i\ell-1}^2\). It contradicts to (C.14). Therefore, \(x_{i\ell-1} \in (0, 1)\) and \(g_{i\ell} = \Delta^{-1}\) for all \(\ell\). Thus, \(\prod_{i=1}^{r} g_i(x_{i-1}) = \prod_{\ell=1}^{k} (1 - x_{i\ell-1})^{-2} = 1\). It contradicts to \(|1 - x_{i\ell-1}| < 1\).

**Corollary C.5.** Let \(A\) be an abelian ring. For three maps \(t : \mathbb{R} \setminus \mathbb{Q} \to A^*, s : (0, 1) \setminus \mathbb{Q} \to A^*\) and \(u : (0, 1) \setminus \mathbb{Q} \to A^*\), there exists a unique automorphic factor \(\chi\) such that

\[
\begin{align*}
\chi(\tau, x) &= t(x) \quad &\text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\Delta^{-1}, x) &= s(x) \quad &\text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
\chi(U, x) &= u(x) \quad &\text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{align*}
\]

**Proof.** We define by

\[
\begin{align*}
\chi(\tau^{-1}, x) &= (t(x - 2))^{-1} \quad &\text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(U, x) &= (u(-x))^{-1} \quad &\text{for } x \in (-1, 0) \setminus \mathbb{Q}, \text{ and} \\
\chi(\Delta, x) &= (s(1 - 1/x))^{-1} \quad &\text{for } x \in (1, \infty) \setminus \mathbb{Q}.
\end{align*}
\]

From Proposition C.4, with the same arguments as in the proof of Corollary C.2, we have the existence and uniqueness of such an automorphic factor. \(\square\)

**Corollary C.6.** Let \(A\) be an abelian ring, \(\chi\) be an automorphic factor and \(M\) be an \(A\)-module such that having the \(\mathbb{Z}^{[\mathbb{R}]}\)-module structure associated with \(\chi\). For three maps \(\tilde{c}_\tau : \mathbb{R} \setminus \mathbb{Q} \to M\), \(\tilde{c}_{\Delta^{-1}} : (0, 1) \setminus \mathbb{Q} \to M\) and \(\tilde{c}_U : (0, 1) \setminus \mathbb{Q} \to M\), there exists a unique cocycle \(\tilde{c} : \Gamma \times (\mathbb{R} \setminus \mathbb{Q}) \to M\) such that

\[
\begin{align*}
\tilde{c}(\tau; x) &= \tilde{c}_\tau(x) \quad &\text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\tilde{c}(\Delta^{-1}; x) &= \tilde{c}_{\Delta^{-1}}(x) \quad &\text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
\tilde{c}(U; x) &= \tilde{c}_U(x) \quad &\text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{align*}
\]

**Proof.** We define by

\[
\begin{align*}
\tilde{c}(\tau^{-1}; x) &= -(t(x - 2))^{-1}\tilde{c}_\tau(x - 2) \quad &\text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\tilde{c}(U; x) &= -(u(-x))^{-1}\tilde{c}_U(-x) \quad &\text{for } x \in (-1, 0) \setminus \mathbb{Q}, \text{ and} \\
\tilde{c}(\Delta; x) &= -(s(1 - 1/x))^{-1}\tilde{c}_{\Delta^{-1}}(1 - 1/x) \quad &\text{for } x \in (1, \infty) \setminus \mathbb{Q}.
\end{align*}
\]

From Proposition C.4, with the same arguments as in the proof of Corollary C.3, we have the existence and the uniqueness of such a cocycle. \(\square\)

Let \(A = \mathbb{R}\), \(t(x) = 1, s(x) = x^\nu\) on \((0, 1)\) for \(\nu \in \mathbb{R}\), \(u(x) = 1\), \(\tilde{c}_\tau(x) = 0, \tilde{c}_{\Delta^{-1}}(x) = -\log x\) on \((0, 1)\) and \(\tilde{c}_U(x) = 0\) on \((0, 1)\). There exist a unique automorphic factor \(\chi\) and a unique
cyclic $\hat{c}$ such that
\[
\begin{align*}
\chi(\tau^n, x) &= 1 & \text{for } n \in \mathbb{Z}, x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\Delta^{-1}, x) &= x' & \text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
\chi(U, x) &= 1 & \text{for } x \in (0, 1) \setminus \mathbb{Q},
\end{align*}
\]
and
\[
\begin{align*}
\hat{c}(\tau; x) &= 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\hat{c}(\Delta^{-1}; x) &= -\log x & \text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
\hat{c}(U; x) &= 0 & \text{for } x \in (0, 1) \setminus \mathbb{Q}.
\end{align*}
\]
By (1.9), (1.12) and (C.2), the 1-coboundary of $B_{\text{odd,} \nu}$ satisfies
\[
\begin{align*}
(d^0B_{\text{odd,} \nu})(\Delta^{-1}; x) &= \chi(\Delta, x)B_{\text{odd,} \nu}(\Delta x) - B_{\text{odd,} \nu}(x) = \log x & \text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
(d^0B_{\text{odd,} \nu})(U; x) &= \chi(U, x)B_{\text{odd,} \nu}(U x) - B_{\text{odd,} \nu}(x) = 0 & \text{for } x \in (0, 1) \setminus \mathbb{Q}, \\
(d^0B_{\text{odd,} \nu})(\tau; x) &= \chi(\tau^{-1}, x)B_{\text{odd,} \nu}(\tau^{-1} x) - B_{\text{odd,} \nu}(x) = 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{align*}
\]
Thus $\hat{c} = d^0(-B_{\text{odd,} \nu})$, i.e. the 1-coboundary of $-B_{\text{odd,} \nu}$ is a 1-cocycle under the $\tilde{\Gamma}$-action.

### C.3. ECF-Bruhat function and the $\Theta$-action.

Let $\tilde{\Theta} = \Theta \sqcup U \Theta$ and $\Theta$, i.e.
\[
\tilde{\Theta} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \right\}.
\]

The group $\tilde{\Theta}$ is generated by $\tau(x) = x + 2$, $\sigma(x) = -1/x$ and $U(x) = -x$.

**Proposition C.7.** For $g \in \Theta \setminus \{I\}$ and $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, there exist unique integer $r \geq 1$ and $g_1, \ldots, g_r \in \{\tau, \tau^{-1}, \sigma, U\}$ such that $g = g_r \cdots g_1$ with $g_r g_{i-1} \neq 1$, and
\[
\begin{align*}
& x_{i-1} \in \left( -\infty, -1 \right) \cup (0, 1) \quad \text{if } g_i = \sigma, \text{ and} \\
& x_{i-1} \in (-1, 1) \quad \text{if } g_i = U,
\end{align*}
\]
where $x_i = g_r g_{r-1} \cdots g_1 x_0$.

**Proof.** We have $U = \tau^{\pm 1} U \tau^{\pm 1}$ and $\sigma U = U \sigma$. Thus,
\begin{enumerate}
\item If $g = U$ and $x_0 \in (2n - 1, 2n + 1)$ for $n \in \mathbb{Z}$, then we have $r = 2n + 1$ and $g_r \cdots g_1 = \tau^{-n} U \tau^{-n}$.
\item For $g = \sigma$ and $x_0 \in (1, \infty)$, if $x_0 \in (2n - 1, 2n + 1)$ for $n \in \mathbb{N}$, we have $r = 2n + 3$ and $g_r \cdots g_1 = U \sigma \tau^{-n} U \tau^{-n}$.
\item For $g = \sigma$ and $x_0 \in (-1, 0)$, if $x_0 \in \left( -\frac{1}{2n-1}, -\frac{1}{2n+1} \right)$ for $n \in \mathbb{N}$, we have $r = 2n + 3$ and $g_r \cdots g_1 = \tau^n U \tau^n \sigma U$.
\end{enumerate}

To show the uniqueness, let us assume that there are $g_i, i = 1, \ldots, r$, such that $g_r \cdots g_1 = I$ and $r$ is minimal. Let $i_1 < i_2 < \cdots < i_k$ be indices $i$ such that $g_i = \sigma$. For given $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, let us denote by $x_i = g_i x_{i-1}$. Note that $\prod_{i=1}^k g_i'(x_{i-1}) = g'(x_0) = 1$. Since $g_i'(x) = 1$ if $g = \pm \tau$ and $g_i'(x) = x^{-2}$, we have $\prod_{i=1}^k |x_{i-1}| = 1$.

If $x_{i-1} < -1$, then $x_i = \frac{1}{x_{i-1}} \in (0, 1)$. Then, $g_i = \tau, \tau^{-1}$ or $U$. If $g_i = \tau$, then $g_{i+j} = \tau$ for all $j \geq 1$, thus $\ell = k$. If $g_i = U$, then $x_{i-1} = x_{i+1} - 1 \in (-1, 0)$. Then $g_{i+1} = \tau^\pm$ for $j \leq i_{i+1} - i_j - 1$ and $|x_{i+1-1}| > 1$. If $g_i = \tau^{-1}$, then $g_{i+j} = \tau^{-1}$ for $j \leq i_{i+1} - i_j - 1$ and $x_{i+1-1} < -1$. Since $g_1 \cdots g_k g_r \cdots g_1 = I$ for all $i$, if $|x_{i-1}| > 1$ for some $\ell$, then $|x_{i-1}| > 1$ for all $\ell$. It contradicts to $\prod_{i=1}^k |x_{i-1}| = 1$.

We have $|x_{i-1}| \leq 1$ for all $\ell$. If $|x_{i-1}| < 1$ for some $\ell$, then it contradicts to $\prod_{i=1}^k |x_{i-1}| = 1$. Therefore $x_{i-1} = 1$ for all $\ell$. It contradicts to $x_0$ is irrational. Thus, there are no $g_i$'s such that $g_r \cdots g_1 = I$. \hfill \Box
Corollary C.8. Let \( A \) be an abelian ring. For three maps \( t : \mathbb{R} \setminus \mathbb{Q} \to A^* \), \( s : (0,1) \setminus \mathbb{Q} \to A^* \) and \( u : (0,1) \setminus \mathbb{Q} \to A^* \), there exists a unique automorphic factor \( \chi \) such that

\[
\begin{align*}
\chi(\tau,x) &= t(x) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\sigma,x) &= s(x) \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
\chi(U,x) &= u(x) \quad \text{for } x \in (0,1) \setminus \mathbb{Q}.
\end{align*}
\]

Proof. We define by

\[
\begin{align*}
\chi(\tau^{-1},x) &= (t(x-1))^{-1} \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(\sigma,x) &= (s(-1/x))^{-1} \quad \text{for } x \in (-\infty,-1) \setminus \mathbb{Q}, \text{ and} \\
\chi(U,x) &= (u(-x))^{-1} \quad \text{for } x \in (-1,0) \setminus \mathbb{Q}.
\end{align*}
\]

By the same arguments as in the proof of Corollary C.2, we have the existence and the uniqueness of such an automorphic factor. \( \square \)

Corollary C.9. Let \( A \) be an abelian ring, \( \chi \) be an automorphic factor, \( M \) be an \( A \)-module such that \( M^X \) having the \( \mathbb{Z}[\Theta] \)-module structure associated with \( \chi \). For three maps \( \hat{c}_\tau : \mathbb{R} \setminus \mathbb{Q} \to M \), \( \hat{c}_\sigma : (0,1) \setminus \mathbb{Q} \to M \) and \( \hat{c}_U : (0,1) \setminus \mathbb{Q} \to M \), there exists a unique cocycle \( c : \tilde{\Theta} \times (\mathbb{R} \setminus \mathbb{Q}) \to M \) such that

\[
\begin{align*}
\hat{c}(\tau;x) &= \hat{c}_\tau(x) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\hat{c}(\sigma;x) &= \hat{c}_\sigma(x) \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
\hat{c}(U;x) &= \hat{c}_U(x) \quad \text{for } x \in (0,1) \setminus \mathbb{Q}.
\end{align*}
\]

Proof. We define by

\[
\begin{align*}
\hat{c}(\tau^{-1};x) &= -\chi(\tau^{-1},x)\hat{c}_\tau(x-1) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\hat{c}(\sigma;x) &= -\chi(\sigma,x)\hat{c}_\sigma(-1/x) \quad \text{for } x \in (-\infty,-1) \setminus \mathbb{Q}, \\
\hat{c}(U;x) &= -\chi(U,x)\hat{c}_U(-x) \quad \text{for } x \in (-1,0) \setminus \mathbb{Q}.
\end{align*}
\]

With the same arguments of the proof of Corollary C.3, we have the existence and the uniqueness of such a cocycle. \( \square \)

Let \( A = \mathbb{R} \), \( t(x) = 1 \), \( s(x) = x^\nu \) on \( (0,1) \) for \( \nu \in \mathbb{R} \), \( u(x) = 1 \) on \( (0,1) \), \( \hat{c}_\tau(x) = 0 \), \( \hat{c}_\sigma(x) = -\log x \) on \( (0,1) \) and \( \hat{c}_U(x) = 0 \) on \( (0,1) \). There exist a unique automorphic factor \( \chi \) and a unique cocycle \( \hat{c} \) such that

\[
\begin{align*}
\chi(n,x) &= 1 \quad \text{for } n \in \mathbb{Z}, x \in \mathbb{R} \setminus \mathbb{Q}, \\
\chi(x) &= x^\nu \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
\chi(U,x) &= 1 \quad \text{for } x \in (0,1) \setminus \mathbb{Q},
\end{align*}
\]

and

\[
\begin{align*}
\hat{c}(\tau;x) &= 0 \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\
\hat{c}(\sigma;x) &= -\log x \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
\hat{c}(U;x) &= 0 \quad \text{for } x \in (0,1) \setminus \mathbb{Q}.
\end{align*}
\]

By (1.9), (1.13) and (C.2), the 1-coboundary of \( B_{\text{even},\nu} \) satisfies

\[
\begin{align*}
(d^0B_{\text{even},\nu})(\sigma;x) &= \chi(\sigma,x)B_{\text{even},\nu}(\sigma x) - B_{\text{even},\nu}(x) = \log x \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
(d^0B_{\text{even},\nu})(U;x) &= \chi(U,x)B_{\text{even},\nu}(U x) - B_{\text{even},\nu}(x) = 0 \quad \text{for } x \in (0,1) \setminus \mathbb{Q}, \\
(d^0B_{\text{even},\nu})(\tau;x) &= \chi(\tau^{-1},x)B_{\text{even},\nu}(\tau^{-1} x) - B_{\text{even},\nu}(x) = 0 \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{align*}
\]

Thus \( \hat{c} = d^0(-B_{\text{even},\nu}) \), i.e. the 1-coboundary of \( -B_{\text{even},\nu} \) is a 1-cocycle under the \( \tilde{\Theta} \)-action.
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