Robust Estimation of High-dimensional non-Gaussian Autoregressive Models

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Abstract

High dimensional non-Gaussian time series data are increasingly encountered in a wide range of applications. Conventional estimation methods and technical tools are inadequate when it comes to ultra high dimensional and heavy-tailed data. We investigate robust estimation of high dimensional autoregressive models with fat-tailed innovation vectors by solving a regularized regression problem using convex robust loss function. As a significant improvement, the dimension can be allowed to increase exponentially with the sample size to ensure consistency under very mild moment conditions. To develop the consistency theory, we establish a new Bernstein type inequality for the sum of autoregressive models. Numerical results indicate a good performance of robust estimates.

1 Introduction

High-dimensional data analysis has been increasingly important in the information era with the rapid explosion of massive data sets. Traditional statistical techniques become unavailable when the number of features greatly exceeds the number of observations. To overcome the curse of high-dimensionality, regularization methods are increasingly popular among a wide body of literatures, which also leads to developing the analysis of such estimators, see, for example, Tibshirani (1996), Zou and Hastie (2005), Bickel et al. (2009), Negahban et al. (2012), Loh and Wainwright (2012). However, the majority of the recent investigations impose the assumptions of Gaussianity or sub-Gaussianity on both the explanatory variables and the noises, which turn out to be far too restrictive. For instance, Cont (2001) studied various statistical properties of asset return with the presence of heavy tails in different types of financial market. Also, non-Gaussianity has been
observed in the application of portfolio allocation (Kim et al. (2012)) and many others. To address the issue of fat-tailness, Huber introduced robust regression (Huber (1973)) as a pioneer work and more robust estimation methods have been unveiled (Huber (1992) and Catoni (2012)). In a more recent work, Fan et al. (2017) studied the robust estimation of high-dimensional mean regression without light tail assumption in linear regression setting. Also, robust estimation can be applied to covariance and precision matrices for the i.i.d. case. See Catoni (2016), Fan et al. (2017), Minsker (2018) among others.

Besides the high-dimensionality and the heavy tail property, modern data also exhibit temporal dependence, making the aforementioned technique invalid. Thus, the analysis of high-dimensional time series from non-Gaussian or even heavy-tailed distribution becomes crucial in many fields including risk management (Koopman and Lucas (2008)), brain network (Friston (2011)) and geophysical dynamic studies (Kondrashov et al. (2005)). However, despite its wide applications, literature remains quiet on robust estimation of time series, while only regularization methods alone were generalized to such setting (see, for example, Chen et al. (2013) and Basu and Michailidis (2015)). In a very recent work, Zhang (2019) established the near optimal convergence rate of the huber mean estimator for non-parametric stationary process, adopting the framework of functional dependence measures introduced by Wu (2005). In this paper, we will build a sharper result on the estimation of the transition matrix in a VAR model, while ultra-high dimensionality and heavy tail can be allowed.

To be specific, we consider a stationary vector autoregressive model VAR(d) generated by

\[
X_i = A_1 X_{i-1} + A_2 X_{i-2} + \cdots + A_d X_{i-d} + \epsilon_i, \quad i = 1, \ldots, n, \tag{1.1}
\]

where \(X_i = (X_{i1}, \ldots, X_{ip}) \in \mathbb{R}^p\), \(A_i \in \mathbb{R}^{p \times p}\) are the transition matrices and \(\epsilon_i \in \mathbb{R}^p\) are i.i.d. innovation (or noise) vectors. We aim to estimate the transition coefficients \(A_i\) from a realization of the random vectors \(\{x_i\}\) without light tail assumption on \(\epsilon_i\). This type of stochastic process has a wide variety of applications, ranging from finance development (Shan (2005)) to economics and econometrics (Juselius (2006)). Moreover, recovering the sparsity pattern of the transition matrices is equivalent to learning the Granger causality of two stochastic sequences (Granger (1969)). Hence, an extensive body of work has been completed for this purpose, including various regularized estimators. See Hamilton (1994) for ridge penalty, Hsu et al. (2008), Nardi and Rinaldo (2011) and Basu and Michailidis (2015) for \(\ell_1\) penalty and Han et al. (2015) for Dantzig-type estimator.
In this paper, we propose to achieve the consistent estimation of the transition matrices via two different approaches of mathematical programming: Lasso-type estimator and Dantzig-type estimator. Since bad behaviors of $X_i$ at the tails will be inherited from the wild behaviors of the noise vectors, our work is different from the transition estimates in the aforementioned literatures in that the consistency can be shown under very mild moment conditions on the noise vectors $\varepsilon_i$, while Gaussian distribution of $\varepsilon_i$ is assumed in both Basu and Michailidis (2015) and Han et al. (2015). Compared with Zhang (2019) which also incorporates the absence of light tail assumption, we improve the convergence rate by a factor of $(\log n)^2$, due to the sharp Bernstein inequality that we established for the VAR model. Another improvement is that unlike other literatures concerning stationary VAR model, we only require the spectral radius of the transition to lie in the unit ball, which is the necessary and sufficient condition of stationarity. However, most of the existing work imposes a slightly stronger condition that the operator norm of the transition matrix is strictly less than 1 to ensure desirable performance.

The paper is organized as follows. In Section 2, we propose a new concept, spectral decay index, to exploit the condition on the spectral radius and establish a sharp Bernstein inequality for VAR models. In Section 3, we consider robust estimation of the transition matrix in high dimensional VAR models. In particular, we employ a clipping technique combined with Lasso regression to derive an estimator of the transition coefficients and prove the convergence guarantee under some sparsity assumptions. An alternative approach to completing consistent estimation via Dantzig estimator and compares the performance of the two methodologies in terms of the rate of convergence. In Section 4, we conduct a simulation study to assess the empirical performance of robust estimators. All of the proofs are relegated to Section 5.

Now we introduce some notations. For a vector $\beta = (\beta_1, \ldots, \beta_p)^\top$, let $|\beta|_1 = \sum_i |\beta_i|$, $|\beta|_2 = (\sum_i \beta_i^2)^{1/2}$ and $|\beta|_\infty = \max_i |\beta_i|$. For a matrix $A = (a_{ij})_{i,j \leq p} \in \mathbb{R}^{p \times p}$, let $\lambda_i$, $i = 1, \ldots, p$, be its eigenvalues and $\rho(A) = \max_i |\lambda_i|$ be the spectral radius. Denote $\|A\|_1 = \max_j \sum_i |a_{ij}|$, $\|A\|_\infty = \max_i \sum_j |a_{ij}|$, spectral norm $\|A\|_2 = \sup_{|x|_2 \neq 0} |Ax|_2/|x|_2$ and Frobenius norm $\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2}$. Moreover, let $\|A\|_{\max} = \max_{i,j} |a_{ij}|$ be the entrywise maximum norm and $|A|$ be a matrix after taking absolute value of $A$, i.e. $|A| = (|a_{ij}|)_{i,j}$. For a random variable $X$ and $q > 0$, define $\|X\|_q = (\mathbb{E}[X^q])^{1/q}$. For two real numbers $x, y$, set $x \vee y = \max(x, y)$. For two sequences of positive numbers $\{a_n\}$ and $\{b_n\}$, we write $a_n \lesssim b_n$ if there exists some constant $C > 0$, such that $a_n/b_n \leq C$ as $n \to \infty$, and also write $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. We use $c_0, c_1, \ldots$ and $C_0, C_1, \ldots$ to denote some universal positive constants whose values may vary.
in different context. Throughout the paper, we consider the high dimensional regime, allowing the dimension $p$ to grow with the sample size $n$, that is, we assume $p = p_n \to \infty$ as $n \to \infty$.

2 Bernstein-type Inequality for VAR Models

Without loss of generality, we shall work with VAR(1) models

$$X_i = AX_{i-1} + \varepsilon_i, \quad i = 1, \ldots, n,$$

(2.1)

where $X_i = (X_{i1}, \ldots, X_{ip})^\top \in \mathbb{R}^p$, $A \in \mathbb{R}^{p \times p}$ is the transition matrix, and $\varepsilon_i, i \in \mathbb{Z}$, are i.i.d. innovation vectors with mean zero. We remark that a general VAR model of order $d$ can be reformulated as a VAR(1) model by appropriately redefining the random vectors, i.e., it is equivalent to representing (1.1) by $Z_i = \tilde{A}Z_{i-1} + \tilde{\varepsilon}_i$, where

$$Z_i = \begin{pmatrix} X_i \\ X_{i-1} \\ \vdots \\ X_{i-d+1} \end{pmatrix} \in \mathbb{R}^{dp}, \quad \tilde{A} = \begin{pmatrix} A_1 & A_2 & \ldots & A_{d-1} & A_d \\ I_p & 0 & \ldots & 0 & 0 \\ 0 & I_p & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_p & 0 \end{pmatrix} \in \mathbb{R}^{dp \times dp}, \quad \tilde{\varepsilon}_i = \begin{pmatrix} \varepsilon_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{dp}.$$

It is well known that the process (2.1) is stationary if and only if the spectral radius $\rho(A) < 1$ (Lütkepohl (2005)). However, in earlier work of estimating large VAR(1) models, it is often assumed that $\|A\| < 1$ (Han et al. (2015)), which turns out to be a stronger condition than $\rho(A) < 1$ in view of the fact $\rho(A) \leq \|A\|$. As a simple example, for a $2 \times 2$ matrix $A$ with diagonals zero and off-diagonal entries $a$ and $b$, elementary calculation suggests $\|A\| = \max\{|a|, |b|\}$ while the necessary and sufficient condition for $\rho(A) < 1$ is $|ab| < 1$. For many cases, it could happen that $\|A\| \geq 1$ while $\rho(A) < 1$. Basu and Michailidis (2015) proposed stability measures for high dimensional time series to capture temporal and cross-section dependence; see the definitions and more examples therein.

From a different viewpoint, we try to fill in the gap between the spectral radius of a matrix and its operator norm. Intuition can be gained from the proposition below. It provides a sufficient and necessary condition for $\rho(A) < 1$ by relating to the spectral norm.

Proposition 2.1. There exists some constant $0 < c_1 < 1$ such that $\rho(A) \leq c_1$ if and only if there
exists some finite integer \( t \) and some constant \( 0 < c_2 < 1 \) such that \( \|A^t\| \leq c_2 \). Specifically, for a symmetric matrix \( A \), the above holds for \( t = 1 \).

Proposition 2.1 inspires us to introduce a new concept: spectral decay index.

**Definition 2.2.** For any matrix \( A \in \mathbb{R}^{p \times p} \) such that \( \rho(A) < 1 \), define the spectral decay index as
\[
\tau = \min\{t \in \mathbb{Z}^+ : \|A^t\| \leq \rho\}
\]
for some \( 0 < \rho < 1 \).

Proposition 2.1 shows the existence of a finite spectral decay index for any matrix \( A \) satisfying \( \rho(A) < 1 \). If \( \|A\| < 1 \) holds, we can simply get \( \tau = 1 \) by letting \( \rho = \|A\| \). In high dimensions, an interesting feature is that the spectral decay index \( \tau \) may increase with the dimension \( p \); see Example 2.3. Thus, in general, \( \tau \) may not be of a constant order when \( p \) increases and we should explicitly write \( \tau = \tau_p \) to capture its dependence on \( p \). In the rest of the paper, we simply use \( \tau \) for ease of notation and we fix \( \rho \) in Definition 2.2 as a constant strictly smaller than 1.

**Example 2.3.** Consider a non-symmetric matrix \( A \in \mathbb{R}^{p \times p} \), where all diagonal entries are set to be \( \lambda \) and all superdiagonals of \( A \) are \( \lambda^2 \) for some \( 0 < \lambda < 1 \). For the induced VAR(1) model, the \( j \)-th entry of \( X_i \) only depends on its value at time \( i - 1 \) and the \( (j+1) \)-th entry at time \( i - 1 \). In this case, spatial dependence also results in temporal dependence. We set \( \rho = 0.9 \) and take \( \lambda = 0.65 \) or \( \lambda = 0.75 \). As shown in Figure 1, in both cases, \( \tau \) increases with \( p \) and in fact it can be observed that \( \rho(A) < 1 \) while \( \|A\| > 1 \). With this design, when \( p \) diverges, \( \tau \) also goes to infinity.

![Figure 1: The graph of the spectral decay index \( \tau \) versus the dimension \( p \).](image-url)
In this paper, our main goal is to robustly estimate the transition matrix $A$ under the framework (2.1) when $\varepsilon_i$ can be non-sub-Gaussian or fat-tailed. Before presenting the methodology and result, we shall first introduce a useful probability tool: Bernstein’s inequality. The celebrated Bernstein’s inequality (Bernstein (1946)) provides an exponential concentration inequality for sums of independent random variables which are uniformly bounded. To fix the idea, let $X_1, \ldots, X_n$ be independent random variables such that $E X_i = 0$, $\text{Var}(X_i) = \sigma_i^2 < \infty$, and $|X_i| \leq M$ for all $i$. Denote $S_n = \sum_{i=1}^n X_i$. Then for any $x > 0$, one has

$$
P(S_n \geq x) \leq \exp \left\{ - \frac{x^2}{2 \sum_{i=1}^n \sigma_i^2 + 2Mx/3} \right\} \quad (2.2)$$

The uniform boundedness condition can be relaxed and some extensions have been made to generalize its validity to the random variables with a finite exponential moment; see Massart (2007) and Wainwright (2019) for some variants. Inequality (2.2) suggests two types of bound for tail probability: sub-Gaussian-type tail $\exp\{-x^2/(C \sum_{i=1}^n \sigma_i^2)\}$ in terms of the variance of $S_n$ and sub-exponential-type tail $\exp\{-x/(CM)\}$ in terms of the uniform bound $M$. This characteristic makes Bernstein’s inequality a powerful tool when analyzing the concentration behavior of independent random variables.

Despite the extensive literature on the concentration results for independent cases, it is still a challenging problem to establish such inequalities with an exponential bound for dependent processes in the presence of possible temporal and cross-sectional dependence. Some progress has been made in the relevant work. For instance, an exponential-type inequality was derived for sums of Markov chains in Douc et al. (2008) under some drift condition. Merlevède et al. (2009) produced Bernstein-type bounds for sums of strong mixing processes. Adopting the framework of functional dependence measures, Zhang (2019) established a Bernstein-type inequality involving the dependence adjusted moment of the underlying process. We shall provide a new Bernstein-type inequality for the sum of transformed VAR models in (2.1) with the mild assumption $\rho(A) < 1$.

**Theorem 2.4.** Let $X_i$ be generated from a stationary VAR(1) model given by (2.1). Assume that $\rho(A) < 1$, $E \varepsilon_{ij} = 0$ and $E \varepsilon_{ij}^2 = \sigma_j^2 < \infty$ for $j = 1, \ldots, p$. Denote $\sigma^2 = \max_j \sigma_j^2$. Let $G : \mathbb{R}^p \to \mathbb{R}$ be a function with $|G(u)| \leq M$ for all $u \in \mathbb{R}^p$. Suppose there exists a vector $g = (g_1, \ldots, g_p)^\top$ with $g_i \geq 0$ for $1 \leq i \leq p$ and $|g|_1 \leq 1$ such that the following Lipschitz condition holds:

$$
|G(u) - G(v)| \leq g^\top |u - v|, \quad \text{for all } u, v \in \mathbb{R}^p. \quad (2.3)
$$
Then for any \( x > 0 \), we have

\[
P\left( \sum_{i=1}^{n} G(X_i) - \mathbb{E}G(X_i) \geq x \right) \leq 2 \exp \left\{ - \frac{x^2}{C_1 \sigma^2 \gamma^2 \tau^2 (\tau \vee n) + C_2 \tau M x} \right\},
\]

(2.4)

where \( \tau \) is the spectral decay index defined in Definition 2.2, \( \gamma = \max_{t=0,1,...,\tau-1} \|A^t\| \) and the constants \( C_1 \) and \( C_2 \) are given by

\[
C_1 = \frac{32e^2}{\sqrt{2\pi}} \left( \frac{1}{\rho^2 \log(1/\rho)} \right)^3, \quad C_2 = \frac{8e}{\log(1/\rho)}.
\]

In the special case of univariate autoregression models, namely \( p = 1 \) in (2.1), \( A \) is reduced to a scalar and \( |A| < 1 \) should be satisfied to ensure the stationarity. Further assume that the random variable \( X_i \) is uniformly bounded by \( M \). Since \( \tau = 1, \gamma = 1 \) and \( \rho = |A| \), with the identity function \( G \), Theorem 2.4 suggests a concentration inequality for sums of VAR(1) processes

\[
P(S_n \geq x) \leq 2 \exp \left\{ - \frac{x^2}{C_1 \|X\|_2^2 + C_2 M x} \right\}.
\]

In the multivariate case, it still follows that the sum of each component process satisfies the above concentration inequality by setting \( G(x) = x_j \) for any \( 1 \leq j \leq p \), provided that the uniform bound \( M \) applies to each coordinate \( X_{ij} \).

The Lipschitz condition (2.3) is an essential assumption for Theorem 2.4. We remark that the assumption that \( |g|_1 \leq 1 \) is not too restrictive, as one can always apply the theorem to the function \( G/|g|_1 \) to make it satisfied.

**Remark 1.** If \( \tau \) and \( \gamma \) are both bounded by some constant, our tail probability inequality in Theorem 2.4 is as sharp as the classical Bernstein’s inequality (2.2). Merlevède et al. (2009) established an exponential-type concentration with an additional \((\log n)^2\) in the denominator of the exponential inequality:

\[
P(S_n \geq x) \leq \exp \left\{ - \frac{x^2}{n \nu^2 + M^2 + M(\log n)^2 x} \right\},
\]

(2.5)

where \((X_i)\) is a strong mixing process of mean 0 and upper bound \( M \) in magnitude, and \( \nu^2 \) is the asymptotic variance of \( \sum_{i=1}^{n} X_i/\sqrt{n} \). Zhang (2019) also derived a tail probability bound in the presence of the \((\log n)^2\) term:

\[
P(S_n \geq x) \leq \exp \left\{ - \frac{x^2}{C_1(n\|X\|_2^2 + M^2) + C_2 M(\log n)^2 x} \right\}
\]

(2.6)
where $C_1, C_2$ are some universal constants and $\|X\|_2$ is the dependence adjust measure of the process $(X_i)$. Compared with (2.5) and (2.6), our result is strictly sharper by removing the additional factor $(\log n)^2$, even if the mild orders $\nu^2 = O(1)$ and $\|X\|_2 = O(1)$ are assumed in the last two displays respectively.

The favorable VAR structure enables us to obtain a sharp bound on the tail probability. To the best of our knowledge, Theorem 2.4 is the first probabilistic result on dependent sequences that recovers the optimal rate in the classical Bernstein’s inequality. By a careful check of the proof of Theorem 2.4, we see that the result can also be applied to functions $G : \mathbb{R}^{2p} \rightarrow \mathbb{R}$ that intakes two random vectors at consecutive times, i.e. $G(X_{i+1}, X_i)$. We summarize this result in the corollary below, which is useful in robust estimation of VAR models presented in the next section.

**Corollary 2.5.** Let $X_i$ be generated from a stationary VAR(1) model given by (2.1). Assume that $\rho(A) < 1$, $\mathbb{E} \varepsilon_{ij} = 0$ and $\mathbb{E} \varepsilon_{ij}^2 = \sigma_j^2 < \infty$ for $j = 1, \ldots, p$. Denote $\sigma^2 = \max_j \sigma_j^2$. Let $G : \mathbb{R}^{2p} \rightarrow \mathbb{R}$ be a function with $|G(u)| \leq M$ for all $u \in \mathbb{R}^{2p}$. Suppose there exists a vector $g = (g_1, \ldots, g_{2p})^T$ with $g_i \geq 0$ for $1 \leq i \leq 2p$ and $|g|_1 \leq 1$ such that

$$|G(u) - G(v)| \leq g^T |u - v|, \text{ for all } u, v \in \mathbb{R}^{2p}. $$

Then for any $x > 0$, we have

$$\mathbb{P}\left(\sum_{i=1}^n G(X_{i+1}, X_i) - \mathbb{E}G(X_{i+1}, X_i) \geq x\right) \leq 2 \exp\left\{-\frac{x^2}{C_1 \sigma^2 \gamma^2 \tau^2 (\tau \vee n) + C_2 \tau Mx}\right\} \tag{2.7}$$

where $\tau, \gamma, C_1$ and $C_2$ are the same as defined in Theorem 2.4.

3 Robust Estimation of High Dimensional VAR Models

3.1 Lasso-type Estimation

In this section, we shall estimate the transition matrix robustly in the ultra high-dimensional regime, allowing $\log p = o(n^b)$ for some constant $0 < b < 1$. To fix the idea, let $a_j^\top$ be the $j$-th row of $A$. Let $s_j$ be the cardinality of the support set of $a_j$, i.e., $s_j = |\text{supp}(a_j)| = |\{i : a_{ij} \neq 0\}|$. Denote $s = \max_{1 \leq j \leq p} s_j$ and $S = \sum_{i=j}^p s_j$. Suppose that the scaling condition $s \sqrt{(\log p)/n} \rightarrow 0$ is satisfied. To ensure the well behavior of the true coefficients, it is assumed that $a_j$ is interior point of an $\ell_1$ ball with sufficiently large radius.
To account for robustness, we first clip the data vector to obtain \( \tilde{X}_i \) by

\[
\tilde{X}_{ij} = X_{ij} \min\{1, T/|X_{ij}|\}, \quad j = 1, \ldots, p,
\]

for some threshold \( T \), which depends on \( n \) and \( p \) and will be determined later. Similar clipping technique was also employed by Zhu and Zhou (2017). Based on the clipped data \( \tilde{X}_i \), we propose to estimate \( A \) by solving the following Lasso problem:

\[
\hat{A} = \arg \min_{B \in \mathbb{R}^{p \times p}} \frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i - B\tilde{X}_{i-1}|^2 + \lambda|B|_1,
\]

where \( \lambda > 0 \) is a tuning parameter. Problem (3.2) is equivalent to \( p \) sub-problems:

\[
\hat{a}_j = \arg \min_{b \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} (\tilde{X}_{ij} - b^\top \tilde{X}_{i-1})^2 + \lambda|b|_1,
\]

Notice that the true transition matrix \( A \) is the minimizer of the population squared error loss with the original data, i.e.

\[
A = \arg \min_{B \in \mathbb{R}^{p \times p}} \mathbb{E} \left[ \sum_{i=1}^{n} |X_i - BX_{i-1}|^2 \right].
\]

Before proceeding, we state the main assumptions on the process (2.1) required in studying the properties of the robust estimator \( \hat{A} \).

**Assumptions.**

(A1) \( \mathbb{E}\varepsilon_i = 0; \max_{1 \leq j \leq p} \|\varepsilon_{ij}\|_2 = \sigma < \infty; \max_{1 \leq j \leq p} \|X_{ij}\|_q = \mu_q < \infty \) for some \( q > 4 \).

(A2) \( \lambda_{\min}(\Sigma_0) \geq \kappa \) for some constant \( \kappa > 0 \), where \( \Sigma_0 = \mathbb{E}(X_iX_i^\top) \).

(A3) \( \mu_q \sigma \tau s \sqrt{1 \vee (\tau/n)}[(\log p)/n]^{1/q} \rightarrow 0 \).

(A3') \( \mu_q \sigma \tau \sqrt{S(1 \vee (\tau/n))}[(\log p)/n]^{1/q} \rightarrow 0 \).

Assumption (A1) imposes polynomial moment conditions for the underlying VAR process. Assumption (A2) requires that the covariance of \( X_i \) is well-conditioned. Assumption (A3) assumes a vanishing scaling property. If \( \tau, \gamma \) and \( \sigma \) are of a constant order, (A3) is reduced to the scaling condition that involves \( s, n, p \) only and requires the sparsity \( s \) of a smaller order compared with \( (n/\log p)^{1/2-1/q} \).

**Theorem 3.1.** Let Assumptions (A1), (A2) and (A3) be satisfied. Choose \( T \asymp \mu_q(n/\log p)^{1/q} \) in (3.1). Let \( \hat{A} \) be the solution of (3.2) with \( \lambda \asymp \sigma \gamma \mu_q(\|A\|_\infty + 1) \sqrt{1 \vee (\tau/n)}[(\log p)/n]^{1/2-1/q} \). It
holds that
\[
\| \hat{A} - A \|_{\infty} \lesssim \sigma \tau \gamma \mu_q s(\| A \|_{\infty} + 1) \sqrt{1 \vee \left( \frac{\tau}{n} \right)} \left( \frac{\log p}{n} \right)^{1/2-1/q}
\] (3.4)

with probability at least \( 1 - 8p^{-c} \) for some constant \( c > 0 \). If Assumption (A3') is further satisfied, it also holds that
\[
\| \hat{A} - A \|_F \lesssim \sigma \gamma \tau \mu_q (\| A \|_{\infty} + 1) \sqrt{S(1 \vee (\tau/n))} \left( \frac{\log p}{n} \right)^{1/2-1/q}
\] (3.5)

with probability at least \( 1 - 8p^{-c} \) for some constant \( c > 0 \).

Now we compare our result to some existing literature concerning robust estimation. In the extensively studied regression setting where each feature is independent and identically distributed, Fan et al. (2017) obtained a convergence rate of \( \sqrt{s \log p/n} \) for the regression parameter, provided that the true signal \( \beta^* \) is exactly sparse and the \( q \)-th moment of the error term exists. However, the features \( X_i \) are still assumed to have sub-Gaussian tail. They further shows that their result is as optimal as regularized LAD estimator (Wang (2013)) and achieves the minimax rate (Raskutti et al. (2011)) for weakly sparse model under the light tails. In our result, the presence of \( \tau, \gamma \) accounts for the temporal dependence among the random sequence \( (X_i) \). But Theorem 3.1 can easily be adopted to the independent case. Consider the degenerate transition matrix \( A = 0 \), then no dependence among \( X_i \) will be observed and \( \tau = \gamma = 1 \). To perform further comparison, suppose \( \tau, \gamma = O(1) \). As mentioned earlier, fat tail of \( X_i \) will be inherited from the fat tail of \( \varepsilon \)'s, we only assume the existence of the \( q \)-th moment of each component process, which is much weaker than the sub-Gaussian assumption required in Fan et al. (2017). This yields us a slightly slower convergence rate \( [s \log p/n]^{1/2-1/q} \). With the existence of moments at higher orders, the rate will be closer to the optimal one \( \sqrt{s \log p/n} \).

In the scenario of transition estimation for high dimensional VAR models, proposition 4.1 of Basu and Michailidis (2015) delivers a similar convergence rate by regularized estimator under the Gaussian assumption. They proposed the stability measure to capture temporal and cross-section dependence and used Hanson-Wright inequality to verify the restricted eigenvalue condition, which is only applicable for Gaussian noises. Different from that, we establish a Bernstein-type inequality with spectral decay index describing the dependence structure, which can be used to deal with fat-tailed processes.
3.2 Dantzig-type estimation

In this section, we shall discuss another methodology of achieving robust estimation. Different from Lasso-based estimation, this Dantzig-type estimator is built upon a linear programming problem, subject to a certain constraint. To fix the idea, denote by $\Sigma_k$ the autocovariance matrix of the process $(X_i)$ at lag $k$, namely, $\Sigma_k = \mathbb{E}(X_iX_{i+k}^\top)$. We will still assume $X_i$ is generated by the VAR(1) model (2.1). Then the celebrated Yule-Walker equation can be written as

$$A = \Sigma_0^{-1}\Sigma_1,$$  \hspace{1cm} (3.6)

which suggests a good estimate $\hat{A}$ should have a small error in terms of $\|\Sigma_0\hat{A} - \Sigma_1\|_{\text{max}}$. Without direct access to the autocovariance matrices $\Sigma_0$ and $\Sigma_1$, a natural approach is to find nice estimators for them. Due to the heavy tails of $X_i$, the natural sample auto-covariance fails to deliver a sharp convergence rate, hence robustness need to be taken into consideration. We will still consider the clipping transformation of the original data as proposed in section 4.

Let $\tilde{X}_i$ be defined as in (3.1). Consider the robust estimators of the auto-covariance with lag 0 and lag 1:

$$\hat{\Sigma}_k = \frac{1}{n} \sum_{i=0}^{n-1} \tilde{X}_i\tilde{X}_{i+k}^\top, \quad \text{for } k = 0, 1.$$

By applying the Bernstein inequality in Theorem 2.4, we have the following lemma regarding the statistical error of $\hat{\Sigma}_0$ and $\hat{\Sigma}_1$.

**Lemma 3.2.** Let Assumption (A1) be satisfied. Choose $T \asymp \mu_q (n/\log p)^{1/q}$ in (3.1). Let $\lambda_0 \asymp \sigma\mu_q\gamma\tau\sqrt{\Gamma\vee (\tau/n)(\log p)/n}^{1/2-1/q}$. Then with probability at least $1 - 8p^{-c'}$ for some constant $c' > 0$, it holds that

$$\|\hat{\Sigma}_0 - \Sigma_0\|_{\text{max}} \leq \lambda_0 \quad \text{and} \quad \|\hat{\Sigma}_1 - \Sigma_1\|_{\text{max}} \leq \lambda_0.$$

Motivated by the Yule-Walker equation (3.6), we propose to estimate $A$ by solving the following convex programming:

$$\hat{A} = \arg\min_{B \in \mathbb{R}^{p \times p}} |B|_1 \quad \text{s.t.} \quad \|\hat{\Sigma}_0B - \hat{\Sigma}_1\|_{\text{max}} \leq \lambda,$$  \hspace{1cm} (3.7)

where $\lambda > 0$ is a tuning parameter. The intuition behind the problem (3.7) is that we aim to select a sparse matrix that approximately satisfies the Yule-Walker equation. Observe that problem (3.7) can be solved in parallel. To this end, let $u_j$ be the unit vector in which the $j$-th entry is 1 and
otherwise 0. Then we see that (3.7) is equivalent to \( p \) subproblems:

\[
\hat{a}_j = \arg \min_{b \in \mathbb{R}^p} |b|_1 \text{ s.t. } |\hat{\Sigma}_0 b - \hat{\Sigma}_1 u_j|_\infty \leq \lambda, \quad j = 1, \ldots, p
\]  

(3.8)

Thus, we can obtain \( \hat{A} \) by simply concatenating all the columns \( \hat{a}_j \), i.e. \( \hat{A} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p) \). This optimization format has the form of Dantzig selector (Candes et al. (2007)), which is also closely related to the estimation method in Han et al. (2015).

The next theorem delivers an upper bound on the statistical accuracy. Let \( a_1, a_2, \ldots, a_p \) be columns of \( A \) and denote \( s^* = \max_{1 \leq j \leq p} |\text{supp}(a_j)| \).

**Theorem 3.3.** Let Assumption (A1) be satisfied. Choose \( T \approx \mu_q(n/\log p)^{1/q} \) in (3.1). Let \( \hat{A} \) be the solution of (3.7) with \( \lambda \approx (\|A\|_1 + 1)\mu_q\sigma \gamma \tau \sqrt{1 \vee (\tau/n)}(|\log p|/n)^{1/2 - 1/q} \) and \( T \approx \mu_q(n/\log p)^{1/q} \). Then with probability at least \( 1 - 8\rho^{-c} \) for some constant \( c > 0 \), the following estimation bounds hold:

\[
\|\hat{A} - A\|_{\max} \lesssim \sigma \mu_q \gamma \tau \sqrt{1 \vee (\tau/n)}\|\Sigma_0^{-1}\|_1 (\|A\|_1 + 1) \left( \frac{\log p}{n} \right)^{1/2 - 1/q},
\]

(3.9)

\[
\|\hat{A} - A\|_1 \lesssim \sigma \mu_q s^* \gamma \tau \sqrt{1 \vee (\tau/n)}\|\Sigma_0^{-1}\|_1 (\|A\|_1 + 1) \left( \frac{\log p}{n} \right)^{1/2 - 1/q}.
\]

(3.10)

**Remark 2.** We see that the error of the Dantzig estimator is almost identical to that of Lasso estimator, provided that both \( \|\Sigma_0^{-1}\| \) and \( \|A\|_1 \) are of constant order. The upper bound in (3.10) is only greater than that in (3.4) by a factor of \( \sqrt{s} \). In most model with sparsity structure, \( s \) is always treated nearly as a constant, so the factor \( \sqrt{s} \) is almost negligible for theoretical analysis. However, the sparsity assumption here is much weaker: the number of non-zero entries of each column of the transition matrix is at most \( s \). In contrast, it is required in section 4 that the number of non-zero elements of the entire matrix does not exceed \( s \). This weaker sparsity assumption gives us more flexibility to implement estimation with statistical guarantees.

Next, we compared our result to another Dantzig-based estimation in Han et al. (2015). It is assumed that \( \|A\| < 1 \) in Han et al. (2015), as opposed to \( \rho(A) < 1 \). If \( \|A\| < 1 \), the spectral decay index \( \tau \) and its relative \( \gamma \) are both equal to 1. Moreover, Han et al. (2015) assumes the innovation vectors have multivariate normal distribution and their result can’t be generalized to heavy-tailed distribution. Hence, our result only loses a factor of \( (|\log p|/n)^{1/q} \) by absorbing robustness. Since the convergence rate in (3.10) is almost identical to that in (3.4), we refer readers to section 3.1 for more detailed comparisons.
4 Simulation Study

In this section, we evaluate the finite sample performance of both robust Lasso and Dantzig estimators that are proposed in Section 3 and compare with the traditional Lasso and Dantzig methods. We consider the model (2.1), where $\varepsilon_{ij}$’s are i.i.d. Student’s $t$-distributions with $df = 5$ or 10. Let $s = \lfloor \log p \rfloor$. For the true transition matrix $A = (a_{ij})$, we consider the following designs.

1. Banded: $A = (\lambda|i-j| \mathbf{1}_{\{|i-j| \leq s\}})$ and $\lambda = 0.5$.

2. Block diagonal: $A = \text{diag}\{A_i\}$, where each $A_i \in \mathbb{R}^{s \times s}$ follows the structure in Example 2.3 with $\lambda_i \sim \text{Unif}(−0.8, 0.8)$.

3. Toeplitz: $A = (\rho|i-j|)$ and $\rho = 0.5$.

4. Random Sparse: $a_{ii} \sim \text{Unif}(−0.8, 0.8)$ and $a_{ij} \sim N(0, 1)$ for $(i, j) \in C \subset \{(i, j): i \neq j\}$ where $C$ is randomly selected and $|C| = s^2$.

The designs in (1), (3) and (4) are further scaled by $2\rho|A|$ to ensure that the spectral radius of the transition matrix is smaller than 1. Thus we have a symmetric sparse and weakly sparse matrix in (1) and (3) respectively, while (2) and (4) generate asymmetric matrices. We take the numerical setup: (i) $n = 50, p = 10$; (ii) $n = 50, p = 30$; (iii) $n = 100, p = 30$.

In each repetition, we generate a process of length $2n$. We run the estimation procedure in (3.2) or (3.7) based on $\{X_1, \ldots, X_n\}$ over a finite grid of tuning parameters. For each $\lambda$ in the grid, denote the estimator by $\hat{A}(\lambda)$. Then $\lambda$ is chosen such that $n^{-1} \sum_{i=n+1}^{2n} |X_i - \hat{A}(\lambda)X_{i-1}|^2$, the average prediction error on $\{X_{n+1}, \ldots, X_{2n}\}$, is minimized. The following tables reports the average $\|\hat{A} - A\|_F$ (estimation error in Frobenius norm) based on 500 repetitions. As comparisons, we also obtained by robust methods and the traditional versions based on 500 repetitions (Lasso programming in Tibshirani (1996) and Dantzig-based estimator in Han et al. (2015)) in different scenarios of designs and innovations.
| $n = 50, p = 10$ | Method       | Banded  | Block  | Toeplitz | Random |
|------------------|--------------|---------|--------|----------|--------|
| $\varepsilon_{ij} \sim t_5$ | Lasso        | 0.764   | 0.782  | 0.805    | 0.627  |
|                  | Robust-Lasso | 0.726   | 0.739  | 0.711    | 0.497  |
|                  | Dantzig      | 0.769   | 1.108  | 0.725    | 0.835  |
|                  | Robust-Dantzig| 0.762   | 0.961  | 0.685    | 0.834  |
| $\varepsilon_{ij} \sim t_{10}$ | Lasso        | 0.748   | 0.765  | 0.701    | 0.752  |
|                  | Robust-Lasso | 0.724   | 0.758  | 0.691    | 0.671  |
|                  | Dantzig      | 0.764   | 1.093  | 0.704    | 1.540  |
|                  | Robust-Dantzig| 0.750   | 1.061  | 0.688    | 1.423  |

| $n = 50, p = 30$ | Method       | Banded  | Block  | Toeplitz | Random |
|------------------|--------------|---------|--------|----------|--------|
| $\varepsilon_{ij} \sim t_5$ | Lasso        | 1.340   | 1.804  | 1.182    | 1.617  |
|                  | Robust-Lasso | 1.252   | 1.634  | 1.174    | 1.482  |
|                  | Dantzig      | 1.276   | 2.337  | 1.186    | 2.175  |
|                  | Robust-Dantzig| 1.265   | 2.109  | 1.170    | 2.134  |
| $\varepsilon_{ij} \sim t_{10}$ | Lasso        | 1.262   | 1.705  | 1.176    | 1.564  |
|                  | Robust-Lasso | 1.257   | 1.635  | 1.172    | 1.533  |
|                  | Dantzig      | 2.279   | 2.100  | 1.178    | 2.150  |
|                  | Robust-Dantzig| 2.264   | 2.049  | 1.172    | 2.019  |

| $n = 100, p = 30$ | Method       | Banded  | Block  | Toeplitz | Random |
|------------------|--------------|---------|--------|----------|--------|
| $\varepsilon_{ij} \sim t_5$ | Lasso        | 1.212   | 1.220  | 1.145    | 1.188  |
|                  | Robust-Lasso | 1.170   | 1.187  | 1.114    | 1.162  |
|                  | Dantzig      | 1.255   | 2.216  | 1.156    | 2.421  |
|                  | Robust-Dantzig| 1.173   | 1.812  | 1.121    | 2.293  |
| $\varepsilon_{ij} \sim t_{10}$ | Lasso        | 1.189   | 0.938  | 1.128    | 1.173  |
|                  | Robust-Lasso | 1.178   | 0.935  | 1.122    | 1.140  |
|                  | Dantzig      | 1.211   | 1.447  | 1.140    | 2.075  |
|                  | Robust-Dantzig| 1.194   | 1.379  | 1.122    | 2.000  |
From statistical perspective, the tables indicate that both of our robust estimation methods noticeably outperformed the regular Lasso and Dantzig, when the innovation vectors have fat tail and the transition matrix enjoys a sparsity pattern. The differences became less significant if the tail of the innovation distribution becomes lighter. In a nutshell, our robust methods is more advantageous in tackling non-Gaussian time series.

5 Proofs

5.1 Proofs of Results in Section 2

In this subsection, we provide the proofs of Proposition 2.1 and Theorem 2.4.

**Proof of Proposition 2.1.** On the one hand, if $\rho(A) \leq c_1 < 1$ and $\varepsilon > 0$ is given, the matrix $B = A / (\rho(A) + \varepsilon)$ has spectral radius strictly less than 1. By Theorem 5.6.12 of Golub and Van Loan (2013), $B$ is convergent in the sense that $\lim_{t \to \infty} B^t = 0$. Thus, $\|B^t\| \to 0$ as $k \to \infty$ and there exists some $N = N(\varepsilon, A)$ such that $\|B^t\| < 1$ for all $k \geq N$. This is just the statement that $\|A^t\| \leq [\rho(A) + \varepsilon]^t$ for all $t \geq N$, implying there exists some $t < \infty$ and some constant $c_2$ such that $\|A^t\| \leq c_2 < 1$ since $\varepsilon > 0$ is arbitrary. The proof for the other side is easy by the fact that $\|\rho(A)^t\| = \rho(A^t) \leq \|A^t\|$ for any $t$.

**Proof of Theorem 2.4.** Define the filtration $\{\mathcal{F}_i\}$ with $\mathcal{F}_i = \sigma(\varepsilon_i, \varepsilon_{i-1}, \ldots)$, and the projection $P_j(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_j) - \mathbb{E}(\cdot | \mathcal{F}_{j-1})$. Conventionally it follows that $P_j(G(X_i)) = 0$ for $j \geq i + 1$. We can write

$$\sum_{i=1}^{n} G(X_i) - \mathbb{E}G(X_i) = \sum_{i=1}^{n} \left( \sum_{j=-\infty}^{n} P_j(G(X_i)) \right) =: \sum_{j=-\infty}^{n} L_j,$$

where $L_j = \sum_{i=1}^{n} P_j(G(X_i))$. By the Markov inequality, for any $\lambda > 0$, we have

$$\mathbb{P}\left( \sum_{i=1}^{n} G(X_i) - \mathbb{E}G(X_i) \geq 2x \right) \leq \mathbb{P}\left( \sum_{j=-\infty}^{0} L_j \geq x \right) + \mathbb{P}\left( \sum_{j=1}^{n} L_j \geq x \right) \leq e^{-\lambda x} \mathbb{E}\left[ \exp\left\{ \lambda \sum_{j=-\infty}^{0} L_j \right\} \right] + e^{-\lambda x} \mathbb{E}\left[ \exp\left\{ \lambda \sum_{j=1}^{n} L_j \right\} \right], \quad (5.1)$$

We shall bound the right-hand side of (5.1) with a suitable choice of $\lambda > 0$. Observing that $\{L_j\}_{j \leq n}$ is a sequence of martingale differences with respect to $\{\mathcal{F}_j\}$, we firstly seek an upper
bound on \(\mathbb{E}[e^{\lambda L_j}|\mathcal{F}_{j-1}]\). By the Lipschitz condition (2.3) and the boundedness of \(G\), it follows that

\[
|L_j| \leq \sum_{i=1}^{n} \min \left\{ \| \mathbb{E}[G(X_i)|\mathcal{F}_j] - \mathbb{E}[G(X_i)|\mathcal{F}_{j-1}] \|, 2M \right\}
\]

\[
\leq \sum_{i=1}^{n} \min \left\{ g^T|A^i-j|\mathbb{E}[|\varepsilon_j - \varepsilon_j'||\mathcal{F}_j|, 2M \right\},
\]

(5.2)

where \(\varepsilon'_j\) is an i.i.d. copy of \(\varepsilon_j\). For notational convenience, we denote \(b_i^T = g^T|A^i|\) and \(\eta_j = \mathbb{E}(|\varepsilon_j - \varepsilon'_j||\mathcal{F}_j)\). Then we have

\[
|L_j| \leq 2M \sum_{i=1}^{n} \mathbb{I}(b^T\eta_j \geq 2M) + \sum_{i=1}^{n} b^T\eta_j(b^T\eta_j \leq 2M) =: I_j + II_j.
\]

For \(j \leq 0\) and \(k \geq 2\), by the triangle inequality, it holds that

\[
\mathbb{E}[|L_j|^k|\mathcal{F}_{j-1}] \leq \left[ \left( \mathbb{E}[|I_j|^k|\mathcal{F}_{j-1}] \right)^{1/k} + \left( \mathbb{E}[|II_j|^k|\mathcal{F}_{j-1}] \right)^{1/k} \right]^k \leq (\|I_j\|_k + \|II_j\|_k)^k.
\]

(5.3)

Moreover,

\[
\|I_j\|_k \leq 2M \sum_{i=-j}^{\infty} \mathbb{I}(b^T\eta_j \geq 2M)\|_k \leq 2M \sum_{i=-j}^{\infty} \mathbb{P}(\{b^T\eta_j\}^2 \geq (2M)^2) \|_k^{1/k}.
\]

(5.4)

Recall the definitions of \(\gamma\) and \(\tau\). We have \(|b|_1 \leq |g|_1\|A^i\|_{\max} \leq \|A^i\| \leq \rho^{-1}\gamma\rho^{i/\tau}\), which implies

\[
\mathbb{E}[(b^T\eta)^2] \leq 2\sigma^2|b|_1^2 \leq 2\gamma^2\sigma^2\rho^{2i/\tau-2}.
\]

By the Markov inequality, we obtain from (5.4) that for \(k \geq 2\),

\[
\|I_j\|_k \leq 2M \left( \frac{\gamma\mu_2}{\rho M} \right)^{2/k} \frac{\rho^{-2j/k\tau}}{1-\rho^{2/k\tau}},
\]

(5.5)

where \(\mu_2 = \sqrt{2}\sigma\). In view of the fact \(1-x \geq -x\log x\) for \(x \in (0, 1)\), we can further relax the bound in (5.5). Applying the Stirling formula, for \(k \geq 2\), we can obtain

\[
\|I_j\|^k \leq k\tau^k \rho^{-2/\tau} \left( \frac{M}{\log(1/\rho)} \right)^k \left( \frac{\gamma\mu_2}{\rho M} \right)^2 \rho^{-2j/\tau} \leq \frac{1}{\sqrt{2\pi}} \left( \frac{\gamma\mu_2}{\rho^2 M} \right)^2 k! \tau^k \left( \frac{eM}{\log(1/\rho)} \right)^k \rho^{-2j/\tau}.
\]
Define the constants
\[ C_1 = \frac{1}{\sqrt{2\pi}} \left( \frac{\gamma\mu_2}{\rho^2 M} \right)^2, \quad \text{and} \quad C_2 = \frac{eM}{\log(1/\rho)}. \]

Then we can simply write
\[ \|I_j\|_k^k \leq C_1 k! \tau^k C_2^k \rho^{-2j/\tau}. \quad (5.6) \]

Analogously, for \( k \geq 2 \), we can also get
\[ \|II_j\|_k^k \leq \sum_{i=-j}^{\infty} \left\{ \mathbb{E}[(g^\top \eta_{ij})^2 (2M)^{k-2}] \right\}^{1/k} \leq C_1 k! \tau^k C_2^k \rho^{-2j/\tau}. \quad (5.7) \]

By (5.3), (5.6) and (5.7), we have
\[ \mathbb{E}[|L_j|^{k} | \mathcal{F}_{j-1}] \leq C_1 k! \tau^k (C_2')^k \rho^{-2j/\tau}, \quad (5.8) \]

where \( C_2' = 2C_2 = 2eM/\log(1/\rho) \). Now we are ready to derive an upper bound for \( \mathbb{E}[e^{\lambda L_j} | \mathcal{F}_{j-1}] \).

By the Taylor expansion, we have
\[ \mathbb{E}[e^{\lambda L_j} | \mathcal{F}_{j-1}] = 1 + \mathbb{E}[\lambda L_j | \mathcal{F}_{j-1}] + \sum_{k=2}^{\infty} \frac{1}{k!} \mathbb{E}[\lambda^k L_j^k | \mathcal{F}_{j-1}]. \]

Notice that \( \mathbb{E}[L_j | \mathcal{F}_{j-1}] = 0 \). For \( 0 < \lambda < (C_2')^{-1} \), we have
\[ \mathbb{E}[e^{\lambda L_j} | \mathcal{F}_{j-1}] \leq 1 + C_1 \rho^{-2j/\tau} \sum_{k=2}^{\infty} \left( C_2' \tau \lambda \right)^{k \leq} \exp \left\{ \frac{C_1' \tau^2 \rho^{-2j/\tau} \lambda^2}{1 - C_2' \tau \lambda} \right\}, \]

where the constant
\[ C_1' = C_1 (C_2')^2 = \frac{1}{\sqrt{2\pi}} \left( \frac{2\gamma\mu_2 e}{\rho^2 \log(1/\rho)} \right)^2, \]

Thus, recursively conditioning on \( \mathcal{F}_0, \mathcal{F}_{-1}, \ldots \), we have for \( 0 < \lambda < (C_2')^{-1} \),
\[ \mathbb{P} \left( \sum_{j=-\infty}^{0} L_j \geq x \right) \leq e^{-\lambda x} \mathbb{E} \left[ \exp \left\{ \lambda \sum_{j=-\infty}^{0} L_j \right\} \right] \leq e^{-\lambda x} \exp \left\{ \frac{C_1' \tau^2 (1 - \rho^{2/\tau})^{-1} \lambda^2}{1 - C_2' \tau \lambda} \right\}. \]
Specifically, choosing \( \lambda = x[C_2'\tau x + 2C_1'\tau^2(1 - \rho^{2/\tau})^{-1}]^{-1} \) yields
\[
\mathbb{P}\left( \sum_{j=-\infty}^{0} L_j \geq x \right) \leq \exp\left\{ -\frac{x^2}{4C_1'\tau^2(1 - \rho^{2/\tau})^{-1} + 2C_2'\tau x} \right\} \\
\leq \exp\left\{ -\frac{x^2}{C_1'\rho^{-2}(\log(1/\rho))^{-1}\tau^3 + 2C_2'\tau x} \right\} \\
= \exp\left\{ -\frac{x^2}{C''_1\tau^3 + 2C_2'\tau x} \right\} ,
\]
(5.9)
where \( C''_1 = C_1'\rho^{-2}(\log(1/\rho))^{-1} \). We can deal with \( L_j \) for \( j \geq 1 \) by similar arguments and obtain
\[
\mathbb{E}[e^{\lambda L_j} | \mathcal{F}_{j-1}] \leq \exp\left\{ \frac{C'_1\tau^2\lambda^2}{1 - C''_1\tau\lambda} \right\} \text{ for } j \geq 1.
\]
In a similar way as deriving (5.9), it follows that
\[
\mathbb{P}\left( \sum_{j=1}^{n} L_j \geq x \right) \leq \exp\left\{ -\frac{x^2}{4C_1'\tau^2n + 2C_2'\tau x} \right\} \leq \exp\left\{ -\frac{x^2}{C''_1\tau^3n + 2C_2'\tau x} \right\} .
\]
(5.10)
Combining (5.1), (5.9) and (5.10), we have
\[
\mathbb{P}\left( \sum_{i=1}^{n} G(X_i) - \mathbb{E}[G(X_i)] \geq x \right) \leq \exp\left\{ -\frac{x^2}{4C_1'\tau^2n + 4C_2'\tau x} \right\} + \exp\left\{ -\frac{x^2}{4C''_1\tau^3 + 4C_2'\tau x} \right\} \\
\leq 2 \exp\left\{ -\frac{x^2}{4C_1'\tau^2(\tau \vee n) + 4C_2'\tau x} \right\} ,
\]
which implies (2.4).

Proof of Corollary 2.5. By a slight modification of the Lipschitz condition (5.2), the proof follows directly from that of Theorem 2.4 without any extra technical difficulty.

5.2 Proofs of Results in Section 3

To prove Theorem 3.1, we firstly introduce some preparatory lemmas.

**Lemma 5.1.** Let Assumption (A1) be satisfied. Choose \( T \asymp \mu_q(n/\log p)^{1/q} \rightarrow \infty \) in (3.1). Define \( \tilde{L}_j(b) = n^{-1} \sum_{i=1}^{n} (\tilde{X}_{ij} - b^\top \tilde{X}_{i-1})^2 \). For \( \lambda \asymp \sigma\tau\gamma\mu_q(||A||_\infty + 1)\sqrt{1 \vee (\tau/n)}[(\log p)/n]^{1/2-1/q} \), it holds that
\[
\mathbb{P}\left( 2\|\nabla \tilde{L}_j(a_j)\|_\infty \leq \lambda, \text{ for all } 1 \leq j \leq p \right) \geq 1 - 4p^{-c_1}
\]
(5.11)
for some constant $c_1 > 0$.

**Proof of Lemma 5.1.** We consider the first component of $\nabla \tilde{L}_j(a_j)$, denoted by $\nabla \tilde{L}_{j1}(a_j)$. Other components can be manipulated analogously. Let $G(X_i, X_{i-1}) = 2(\tilde{X}_{i1} - \tilde{X}_{i-1}a_j)\tilde{X}_{(i-1)1}$. Then we can write

$$
\nabla \tilde{L}_{j1}(a_j) = \frac{1}{n} \sum_{i=1}^{n} G(X_i, X_{i-1}).
$$

Notice that $|G| \leq 2(\|A\|_{\infty} + 1)T^2$ and $G(u) - G(v) \leq g^T|u - v|$, where $|g|_1 \leq 4(\|A\|_{\infty} + 1)T$. For notational convenience, denote $d(\tau, n) = 1 \lor (\tau/n)$. By Corollary 2.5, for $x = c'\sigma\gamma\tau \sqrt{d(\tau, n)} \sqrt{\log p}/n$, we have

$$
\mathbb{P}\left( \left| \nabla \tilde{L}_{j1}(a_j) - \mathbb{E}[\nabla \tilde{L}_{j1}(a_j)] \right| \geq 4T(\|A\|_{\infty} + 1) x \right) \leq 4 \exp\left\{ - \frac{(c')^2 \log p}{2C_1} \right\}. \quad (5.12)
$$

In view of $\mathbb{E}[\nabla L_n(a_j)] = 0$, the triangle inequality and $|\tilde{X}_{ij}| \leq |X_{ij}|$, we have

$$
\begin{align*}
|\mathbb{E}[\nabla \tilde{L}_{j1}(a_j)]| &= |\mathbb{E}[\nabla \tilde{L}_{j1}(a_j)] - \mathbb{E}[\nabla L_{j1}(a_j)]| \\
&= 2\mathbb{E}\left[ \left| (\tilde{X}_{ij} - a_j^T\tilde{X}_{i-1})\tilde{X}_{(i-1)1} - (X_{ij} - a_j^TX_{i-1})X_{(i-1)1} \right| \right] \\
&\leq \mathbb{E}\left[ \left| X_{(i-1)1}(\tilde{X}_{ij} - X_{ij}) \right| \right] + |a_j|_1^T\mathbb{E}\left[ \left| X_{(i-1)1}(\tilde{X}_{i-1} - X_{i-1}) \right| \right] \\
&\quad + \mathbb{E}\left[ \left| X_{ij}(X_{(i-1)1} - \tilde{X}_{(i-1)1}) \right| \right] + |a_j|_1^T\mathbb{E}\left[ \left| X_{i-1}(\tilde{X}_{(i-1)1} - X_{(i-1)1}) \right| \right]. \quad (5.13)
\end{align*}
$$

Since $|\tilde{X}_{ij} - X_{ij}| \leq |X_{ij}|1\{|X_{ij}| \geq T\}$, by Hölder’s inequality and Markov’s inequality, we have

$$
\mathbb{E}\left[ \left| X_{(i-1)1}(X_{ij} - \tilde{X}_{ij}) \right| \right] \lesssim \|X_{(i-1)1}X_{ij}\|_21\{|X_{ij}| \geq T\} \|_2 \lesssim \mu_q^2 \left( \frac{Hq}{T} \right)^{q/2}.
$$

Other terms in (5.13) can be dealt with similarly. With the choice of $T$, we can get

$$
|\mathbb{E}[\nabla \tilde{L}_{j1}(a_j)]| \lesssim (\|A\|_{\infty} + 1)\mu_q^2 \left( \frac{Hq}{T} \right)^{q/2} \lesssim (\|A\|_{\infty} + 1)\mu_q^2 \sqrt{\frac{\log p}{n}} = o(T(\|A\|_{\infty} + 1) x).
$$

Letting $\lambda = 4T(\|A\|_{\infty} + 1) x$ and $c' > 2\sqrt{C_1}$, it follows from (5.12) that

$$
\mathbb{P}\left( 2|\nabla \tilde{L}_{j1}(a_j)| \geq \lambda \right) \leq 4 \exp\left\{ - \frac{(c')^2 \log p}{2C_1} \right\}.
$$

By the Bonferroni inequality, we have

$$
\mathbb{P}\left( 2|\nabla \tilde{L}_j(a_j)| \geq \lambda, \text{ for all } 1 \leq j \leq p \right) \leq 4p^{-c_1}
$$

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where \( c_1 = 2^{-1}C_1^{-1}c'^2 - 2 > 0 \).

Define a cone \( C(S) = \{ \Delta \in \mathbb{R}^p : |\Delta S^c|_1 \leq 3|\Delta_S|_1 \} \) for a subset \( S \subseteq \{1, 2, \ldots, p\} \). We shall verify a restricted eigenvalue (RE) condition on the set \( C(S) \) in the lemma below.

**Lemma 5.2.** Let Assumptions (A1), (A2) and (A3) be satisfied. Choose \( T \asymp \mu_q(n/\log p)^{1/q} \). Then for all \( \Delta \in C(S) \),

\[
\Delta^\top \nabla^2 \tilde{L}_j(a_j) \Delta \geq \frac{\kappa_2}{2} |\Delta|_2^2
\]  

holds with probability at least \( 1 - 4p^{-c_2} \) for some constant \( c_2 > 0 \).

**Proof of Lemma 5.2.** Denote \( \tilde{X} = (\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1})^\top \). Then \( \nabla^2 \tilde{L}_j(a_j) = 2\tilde{X}^\top \tilde{X}/n =: \Gamma \). We shall first show that with probability at least \( 1 - 4p^{-c_2} \) for some positive constant \( c_2 \), it holds that

\[
|v^\top (\Gamma - \mathbb{E}\Gamma) v| \leq t, \quad \forall v \in \mathbb{R}^p, \quad |v|_0 \leq 2s, \quad |v|_2 \leq 1,
\]  

(5.15)

where \( t = c_1 \mu_q \gamma \tau s \sqrt{d(\tau,n)}(\log p/n)^{1/4-1/q} \). By Theorem 2.4, for any fixed \( u \) such that \( |u|_2 \leq 1 \),

\[
P \left( |u^\top (\Gamma - \mathbb{E}\Gamma) u| \geq t \right) \leq 4 \exp \left\{ - c_3 \sqrt{n \log p} \right\}.
\]

Following the same spirit of the \( \varepsilon \)-net argument in lemma 15 of Loh and Wainwright (2012), we can obtain that (5.15) holds with probability at least

\[
1 - 4 \exp \left\{ 2s \log 9 + 2s \log p - c_3 \sqrt{n \log p} \right\} \geq 1 - 4p^{-c_2},
\]

provided that \( p \to \infty \) and \( s \sqrt{(\log p)/n} \to 0 \). By Lemma 12 in Loh and Wainwright (2012) and (5.15), it further implies that with probability greater than \( 1 - 4p^{-c_2} \),

\[
|v^\top (\Gamma - \mathbb{E}\Gamma) v| \leq 27t \left( |v|_2^2 + \frac{|v|_1^2}{s} \right), \quad \forall v \in \mathbb{R}^p.
\]  

(5.16)

Note that when \( \Delta \in C(S) \),

\[
|\Delta|_1 = |\Delta_S|_1 + |\Delta S^c|_1 \leq 4|\Delta_S|_1 \leq 4\sqrt{s}|\Delta_S|_2 \leq 4\sqrt{s}|\Delta|_2.
\]
Combining (5.16) and (5.19), we can establish the following RE condition for all \( \Delta \in \hat{C} \):

\[
\Delta^\top \mathbb{E}[\Gamma] \Delta = 2\mathbb{E}[(\tilde{X}_1^\top \Delta)^2] \geq 2 \left( \Delta^\top \mathbb{E}[X_1 X_1^\top] \Delta - \Delta^\top \mathbb{E}[X_1 X_1^\top - \tilde{X}_1 \tilde{X}_1^\top] \Delta \right)
\geq 2 \left( \Delta^\top \mathbb{E}[X_1 X_1^\top] \Delta - \left\| \mathbb{E}[X_1 X_1^\top - \tilde{X}_1 \tilde{X}_1^\top] \right\|_\infty |\Delta|_2^2 \right)
\geq 2\kappa |\Delta|_2^2 - 2|\Delta|_2^2 \left\| \mathbb{E}[X_1 X_1^\top - \tilde{X}_1 \tilde{X}_1^\top] \right\|_\infty ,
\]

(5.17)

where for any \( 1 \leq j, k \leq p \),

\[
|\mathbb{E}[X_{ij} X_{ik} - \tilde{X}_{ij} \tilde{X}_{ik}]| \leq \sqrt{\mathbb{E}[(X_{ij}^2 X_{ik}^2)]} \left( \mathbb{P}(|X_{ij}| \geq T) + \mathbb{P}(|X_{ik}| \geq T) \right) \lesssim \mu_q^2 \sqrt{\frac{\log p}{n}} .
\]

(5.18)

Then it follows that

\[
\Delta^\top \mathbb{E}[\Gamma] \Delta \geq 2\kappa |\Delta|_2^2 - 32s\mu_q^2 \sqrt{\frac{\log p}{n}} |\Delta|_2^2 \geq \kappa |\Delta|_2^2 .
\]

(5.19)

Combining (5.16) and (5.19), we can establish the following RE condition

\[
\nabla^2 L_j(\tilde{a}_j) \geq \kappa |\Delta|_2^2 - 27t(|\Delta|_2^2 + |\Delta|_1^2/s) \geq \kappa |\Delta|_2^2 - 459t|\Delta|_2^2 \geq \frac{\kappa}{2} |\Delta|_2^2 ,
\]

for all \( \Delta \in C(S) \) with probability no less than \( 1 - 4p^{-c_2} \).

**Proof of Theorem 3.1.** Let \( \tilde{\Delta}_j = \tilde{a}_j - a_j \) for \( j = 1, \ldots, p \). As the solution of (3.3), \( \tilde{a}_j \) satisfies

\[
\tilde{L}_j(\tilde{a}_j) + \lambda |\tilde{a}_j|_1 \leq \tilde{L}_j(a_j) + \lambda |a_j|_1 ,
\]

which together with convexity implies,

\[
0 \leq \tilde{L}_j(\tilde{a}_j) - \tilde{L}_j(a_j) - (\nabla \tilde{L}_j(a_j), \tilde{\Delta}_j) \leq \lambda (|a_j|_1 - |\tilde{a}_j|_1) + |\nabla \tilde{L}_j(a_j)|_\infty \tilde{\Delta}_j_1 .
\]

(5.20)

Denote by \( A \) and \( B \) the events in Lemma 5.1 and Lemma 5.2 respectively. Then \( \mathbb{P}(A \cap B) = 1 - \mathbb{P}(A^c \cup B^c) \geq 1 - 8p^{-c} \) for \( c = \min\{c_1, c_2\} \). Conditioned on the event \( A \), (5.20) implies

\[
0 \leq |a_j, s|_1 - |\tilde{a}_j, s|_1 - |\tilde{a}_j, s^c|_1 + \frac{1}{2} |\tilde{\Delta}_j|_1 \leq |\tilde{\Delta}_j, s|_1 - |\tilde{\Delta}_j, s^c|_1 + \frac{1}{2} |\tilde{\Delta}_j|_1 = \frac{3}{2} |\tilde{\Delta}_j, s|_1 - \frac{1}{2} |\tilde{\Delta}_j, s^c|_1 ,
\]

which further implies \( \tilde{\Delta}_j \in C(S) \) for all \( 1 \leq j \leq p \). Conditioned on the event \( B \), by (5.11) and the
second inequality in (5.20), we have
\[ \frac{\kappa}{2} |\hat{\Delta}_j|^2 \leq (\lambda + |\nabla L_n(a_j.)|_\infty) |\hat{\Delta}_j|_1 \leq 6\sqrt{s}\lambda|\hat{\Delta}_j|_2. \quad (5.21) \]

This immediately shows
\[ |\hat{\Delta}_j|_2 \leq \frac{12\sqrt{s}\lambda}{\kappa} \approx \sigma \mu q \gamma \tau (\|A\|_\infty + 1) \sqrt{s \cdot d(\tau, n)} \left( \frac{\log p}{n} \right)^{\frac{1}{2} - \frac{1}{q}}, \quad \text{for all } 1 \leq j \leq p \]  

(5.22)
as well as
\[ |\hat{\Delta}_j|_1 \lesssim \sigma \mu q \gamma \tau s (\|A\|_\infty + 1) \sqrt{d(\tau, n)} \left( \frac{\log p}{n} \right)^{\frac{1}{2} - \frac{1}{q}}, \quad \text{for all } 1 \leq j \leq p. \]

Hence, (3.4) follows in view of \( \|\hat{A} - A\|_\infty = \max_j |\hat{\Delta}_j|_1 \). Moreover, if we consider the estimation of \( \text{Vec}(A) = (a_1^T, a_2^T, \ldots, a_p^T)^T \in \mathbb{R}^p \) with the sparsity parameter \( S = \sum_{i=j}^p s_j \), by similar arguments of verifying the RE condition in Lemma 5.2, (5.14) becomes
\[ 2\Delta^T \left[ I_p \otimes \left( \frac{\hat{X}^T \hat{X}}{n} \right) \right] \Delta \geq \frac{\kappa}{2} |\Delta|^2, \quad \text{for all } \Delta \in \mathbb{R}^p. \]

Thus, similarly as (5.22), (3.5) follows. \( \square \)

Next we shall provide the proof on the consistency of the robust Dantzig-type estimator.

**Proof of Lemma 3.2.** Let \( \lambda_0 = c \sigma \mu q \gamma \sqrt{1 \vee (\tau/n)} [(\log p)/n]^{1/2 - 1/q} \) for some positive constant \( c \).

Applying Theorem 2.4 to the \((m, l)\)-th entry of \( \hat{\Sigma}_0 \), we have
\[ P \left( \frac{1}{n} \sum_{i=0}^{n-1} \bar{X}_{im} \bar{X}_{il} - \mathbb{E}[\bar{X}_{im} \bar{X}_{il}] \geq \lambda_0 \right) \leq 4 \exp \left\{ - \frac{c^2 \log p}{2C_1} \right\} = 4p^{-c^2/(2C_1)}. \]

By (5.18) in the proof of Lemma 5.2, we see that
\[ \left| \mathbb{E}[\bar{X}_{im} \bar{X}_{il}] - \mathbb{E}[X_{im} X_{il}] \right| \lesssim \mu^2_q \sqrt{\frac{\log p}{n}} = o(\lambda_0). \]

Therefore,
\[ P \left( \frac{1}{n} \sum_{i=0}^{n-1} \bar{X}_{im} \bar{X}_{il} - \mathbb{E}[X_{im} X_{il}] \geq \lambda_0 \right) \leq P \left( \frac{1}{n} \sum_{i=0}^{n-1} \bar{X}_{im} \bar{X}_{il} - \mathbb{E}[\bar{X}_{im} \bar{X}_{il}] \geq \lambda_0/2 \right) \]
\[ \leq 4p^{-c^2/(8C_1)}. \]

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Choosing $c > 4C_1$ and taking a union bound yields

$$
P(\|\hat{\Sigma}_0 - \Sigma_0\|_{\text{max}} \geq \lambda_0) \leq 4p^{-c'},$$

for some constant $c' > 0$. By Corollary 2.5, similar arguments apply to $\Sigma_1$, which delivers $\|\hat{\Sigma}_1 - \Sigma_1\|_{\text{max}} \leq \lambda_0$ with probability at least $1 - 4p^{-c'}$. In conclusion, it holds simultaneously that $\|\hat{\Sigma}_0 - \Sigma_0\|_{\text{max}} \leq \lambda_0$ and $\|\hat{\Sigma}_1 - \Sigma_1\|_{\text{max}} \leq \lambda_0$ with probability at least $1 - 8p^{-c'}$.

**Proof of Theorem 3.3.** We first show that $A$ is feasible to the convex programming (3.7) for $\lambda = (\|A\|_1 + 1)\lambda_0$ with high probability. By the Yule-Walker equation (3.6) and Lemma 3.2, we have

$$
\|\hat{\Sigma}_0 A - \hat{\Sigma}_1\|_{\text{max}} \leq \|\hat{\Sigma}_0 A - \Sigma_1\|_{\text{max}} + \|\Sigma_1 - \hat{\Sigma}_1\|_{\text{max}}
$$

$$
\leq \|\hat{\Sigma}_0 - \Sigma_0\|_{\text{max}} \|A\|_1 + \|\Sigma_1 - \hat{\Sigma}_1\|_{\text{max}} \leq \lambda,
$$

with probability no less than $1 - 8p^{-c'}$. Therefore, conditioned on the event in Lemma 3.2, we conclude that $|\hat{a}_j| \leq |a_j|$ for all $j = 1, \ldots, p$ and hence $\|\hat{A}\|_1 \leq \|A\|_1$. Then we have

$$
\|\hat{A} - A\|_{\text{max}} = \|\Sigma_0^{-1}(\Sigma_0\hat{A} - \hat{\Sigma}_1 + \hat{\Sigma}_1 - \Sigma_1)\|_{\text{max}}
$$

$$
\leq \|\Sigma_0^{-1}(\Sigma_0\hat{A} - \hat{\Sigma}_0 \hat{A} + \hat{\Sigma}_0 \hat{A} - \hat{\Sigma}_1 + \hat{\Sigma}_1 - \Sigma_1)\|_{\text{max}} + \|\Sigma_0^{-1}(\hat{\Sigma}_1 - \Sigma_1)\|_{\text{max}}
$$

$$
\leq \|\Sigma_0^{-1}\|_1 \|\Sigma_0 - \hat{\Sigma}_0\|_{\text{max}} \|\hat{A}\|_1 + \|\Sigma_0^{-1}\|_1 \|\hat{\Sigma}_0 \hat{A} - \hat{\Sigma}_1\|_{\text{max}} + \|\Sigma_0^{-1}\|_1 \|\hat{\Sigma}_1 - \Sigma_1\|_{\text{max}}.
$$

By Lemma 3.2 and the feasibility of $\hat{A}$, we have

$$
\|\hat{A} - A\|_{\text{max}} \leq \|\Sigma_0^{-1}\|_1 (\lambda_0 \|A\|_1 + \lambda + \lambda_0) = 2\|\Sigma_0^{-1}\|_1 \lambda.
$$

Now we shall bound $\|\hat{A} - A\|_1$ from above. Denote by $S_j$ the support of $a_j$ for $j = 1, \ldots, p$. Then for any $1 \leq j \leq p$, we have

$$
|\hat{a}_j - a_j|_1 = |\hat{a}_{j,S_j} - a_{j,S_j}|_1 + |\hat{a}_j|_1 - |\hat{a}_{j,S_j}|_1
$$

$$
\leq |\hat{a}_{j,S_j} - a_{j,S_j}|_1 + |a_j|_1 - |\hat{a}_{j,S_j}|_1
$$

$$
\leq 2|\hat{a}_{j,S_j} - a_{j,S_j}|_1 \leq 4s\|\Sigma_0^{-1}\|_1 \lambda.
$$

(5.23)
Since (5.23) holds for all $1 \leq j \leq p$, we conclude that

$$\|\hat{A} - A\|_1 \leq 4s\|\Sigma^{-1}_0\|_1 \lambda \lesssim \mu_q s \sigma \gamma \tau \sqrt{d(\tau, n)} \|\Sigma^{-1}_0\|_1 (\|A\|_1 + 1) \left(\frac{\log p}{n}\right)^{1/2 - 1/q}.$$ 

6 Concluding Remarks

High-dimensional time series models arise in a wide range of disciplines. In this paper, we have made contributions towards the robust estimation theory of high dimensional VAR models in the presence of fat tails. Equipped with the new concept spectral decay index, the main tools we developed are Beinstein type inequalities for sums of transforms of high-dimensional VAR processes, based on which, we were able to robustly estimate the transition matrix with the existence of finite polynomial moments. The convergence rate depends on the spectral decay index, the moment condition, the dimension and the sample size. To perform statistical inference of the estimates such as hypothesis testing and construction of simultaneous confidence bands, one needs to develop the more refined result in terms of asymptotic distributional theory. The latter is more challenging and we leave it as future work.

References

Basu, S. and Michailidis, G. (2015). Regularized estimation in sparse high-dimensional time series models. *The Annals of Statistics*, 43(4):1535–1567.

Bernstein, S. (1946). The theory of probabilities.

Bickel, P. J., Ritov, Y., Tsybakov, A. B., et al. (2009). Simultaneous analysis of lasso and dantzig selector. *The Annals of Statistics*, 37(4):1705–1732.

Candes, E., Tao, T., et al. (2007). The dantzig selector: Statistical estimation when p is much larger than n. *The annals of Statistics*, 35(6):2313–2351.

Catoni, O. (2012). Challenging the empirical mean and empirical variance: a deviation study. *Annales de l’IHP Probabilités et statistiques*, 48(4):1148–1185.
Catoni, O. (2016). Pac-bayesian bounds for the gram matrix and least squares regression with a random design. arXiv preprint arXiv:1603.05229.

Chen, X., Xu, M., Wu, W. B., et al. (2013). Covariance and precision matrix estimation for high-dimensional time series. The Annals of Statistics, 41(6):2994–3021.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance, 1(2):223–236.

Douc, R., Guillin, A., and Moulines, E. (2008). Bounds on regeneration times and limit theorems for subgeometric markov chains. Annales de l'IHP Probabilités et statistiques, 44(2):239–257.

Fan, J., Li, Q., and Wang, Y. (2017). Estimation of high dimensional mean regression in the absence of symmetry and light tail assumptions. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 79(1):247–265.

Friston, K. J. (2011). Functional and effective connectivity: a review. Brain connectivity, 1(1):13–36.

Golub, G. H. and Van Loan, C. F. (2013). Matrix computations, 4th edition. Johns Hopkins University Press.

Granger, C. W. (1969). Investigating causal relations by econometric models and cross-spectral methods. Econometrica: journal of the Econometric Society, pages 424–438.

Hamilton, J. D. (1994). Time series analysis. Princeton university press.

Han, F., Lu, H., and Liu, H. (2015). A direct estimation of high dimensional stationary vector autoregressions. Journal of Machine Learning Research.

Hsu, N.-J., Hung, H.-L., and Chang, Y.-M. (2008). Subset selection for vector autoregressive processes using lasso. Computational Statistics & Data Analysis, 52(7):3645–3657.

Huber, P. J. (1973). Robust regression: asymptotics, conjectures and monte carlo. The annals of statistics, 1(5):799–821.

Huber, P. J. (1992). Robust estimation of a location parameter. In Breakthroughs in statistics, pages 492–518. Springer.
Juselius, K. (2006). *The cointegrated VAR model: methodology and applications*. Oxford university press.

Kim, Y. S., Giacometti, R., Rachev, S. T., Fabozzi, F. J., and Mignacca, D. (2012). Measuring financial risk and portfolio optimization with a non-gaussian multivariate model. *Annals of operations research*, 201(1):325–343.

Kondrashov, D., Kravtsov, S., Robertson, A. W., and Ghil, M. (2005). A hierarchy of data-based enso models. *Journal of climate*, 18(21):4425–4444.

Koopman, S. J. and Lucas, A. (2008). A non-gaussian panel time series model for estimating and decomposing default risk. *Journal of Business & Economic Statistics*, 26(4):510–525.

Loh, P.-L. and Wainwright, M. J. (2012). High-dimensional regression with noisy and missing data: Provable guarantees with nonconvexity. *The Annals of Statistics*, pages 1637–1664.

Lütkepohl, H. (2005). *New introduction to multiple time series analysis*. Springer Science & Business Media.

Massart, P. (2007). *Concentration inequalities and model selection*, volume 6. Springer.

Merlevède, F., Peligrad, M., Rio, E., et al. (2009). Bernstein inequality and moderate deviations under strong mixing conditions. In *High dimensional probability V: the Luminy volume*, pages 273–292. Institute of Mathematical Statistics.

Minsker, S. (2018). Sub-gaussian estimators of the mean of a random matrix with heavy-tailed entries. *The Annals of Statistics*, 46(6A):2871–2903.

Nardi, Y. and Rinaldo, A. (2011). Autoregressive process modeling via the lasso procedure. *Journal of Multivariate Analysis*, 102(3):528–549.

Negahban, S. N., Ravikumar, P., Wainwright, M. J., and Yu, B. (2012). A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. *Statistical Science*, 27(4):538–557.

Raskutti, G., Wainwright, M. J., and Yu, B. (2011). Minimax rates of estimation for high-dimensional linear regression over $\ell_q$-balls. *IEEE transactions on information theory*, 57(10):6976–6994.
Shan, J. (2005). Does financial development ‘lead’ economic growth? A vector auto-regression appraisal. *Applied Economics*, 37(12):1353–1367.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(1):267–288.

Wainwright, M. J. (2019). *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press.

Wang, L. (2013). The l1 penalized lad estimator for high dimensional linear regression. *Journal of Multivariate Analysis*, 120:135–151.

Wu, W. B. (2005). Nonlinear system theory: Another look at dependence. *Proceedings of the National Academy of Sciences*, 102(40):14150–14154.

Zhang, D. (2019). Robust estimation of the mean and covariance matrix for high dimensional time series. *Statistica Sinica*, to appear.

Zhu, Z. and Zhou, W. (2017). Taming heavy-tailed features by shrinkage. *arXiv preprint arXiv:1710.09020*.

Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net. *Journal of the royal statistical society: series B (statistical methodology)*, 67(2):301–320.