Subgroups of $SF(\omega)$ and the relation of almost containedness

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Abstract. The relations of almost containedness and orthogonality in the lattice of groups of finitary permutations are studied in the paper. We define six cardinal numbers naturally corresponding to these relations by the standard scheme of $P(\omega)$. We obtain some consistency results concerning these numbers and some versions of the Ramsey theorem.

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1 Preliminaries

1.1 Introduction

The paper is motivated by investigations of various versions of van Douwen’s diagram, i.e. the set of relations between six cardinals related to simple properties of almost disjointness and almost containedness, for example see [2], [3], [4], [6], [7], [14], [17]. The following theorem proved by P. Matet in [13] became one of the motivating results in this direction:

Let $(\omega)^\omega$ be the set of all partitions of $\omega$ having infinitely many classes. Let $\leq$ be the order on $(\omega)^\omega$ defined by: $E_1 \leq E_2$ if $E_2$ is finer than $E_1$. Then assuming the continuum hypothesis there is a filter $F \subset (\omega)^\omega$ such that for every $(\Sigma^1_1 \cup \Pi^1_1)$-coloring $\delta: (\omega)^\omega \to 2$ there is a partition $E \in F$ such that $\delta$ is constant on the set of all infinite partitions coarser than $E$. 

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The statement is a variant (and a consequence) of the dualized version of Ramsey’s theorem proved by T.Carlson and S.Simpson in [5]. The argument of P.Matet uses the observation that the tower cardinal (we denote it by \(t_d\)) for the ordering of infinite partitions is uncountable. Here \(t_d\) is defined by the same scheme as \(t\) for the lattice \(P(\omega)\) of all subsets of \(\omega\) (see [7]). Moreover it is proved in [13] that the tower cardinal for partitions is \(\omega_1\) in ZFC and it is proved in [6] that the size of a maximal almost orthogonal family of partitions must be \(2^{\omega}\).

The lattice of partitions under the order reversing \(\leq\) was studied in [12] and [3]. It is shown there that the corresponding cardinal invariants look differently. For example \(p\) and \(t\) defined in this case coincide with classical \(p\).

Note that this lattice can be also defined to be the lattice of 1-closed subgroups of \(\text{Sym}(\omega)\) (i.e. the automorphism groups of structures with only unary predicates). Indeed, the corresponding isomorphism of these lattices maps a partition \(E\) to the group \(G_E\) of all permutations preserving the \(E\)-classes. On the other hand for any \(E\) the group \(G_E\) is uniquely determined by the subgroup \(G_E \cap SF(\omega)\) of the group \(SF(\omega)\) of all finitary permutations of \(\omega\). The embedding obtained \(E \rightarrow G_E \cap SF(\omega)\) maps almost trivial equivalence relations into the ideal \(IF\) of all finite subgroups of \(SF(\omega)\).

This motivates further questions. For example it is interesting to find a variant of the result of P.Matet in the lattice of all subgroup of the group \(SF(\omega)\). Since the corresponding tower cardinal is involved in this question, a general problem of description the corresponding van Douwen’s diagram for this lattice seems relevant.

The paper is devoted to these questions.

We use standard set theoretic conventions and notation. \([\omega]^\omega\) and \([\omega]^{<\omega}\) stand for all infinite and all finite subsets of \(\omega\) respectively. For \(k \in \omega \setminus \{0\}\), let \([\omega]^k\) be the set of all \(k\)-element subsets of \(\omega\). By \((\omega)\) we denote the set of all partitions of \(\omega\). A partition is finite if it has finitely many pieces (classes). The set of all finite partitions will be denoted by \((\omega)^{<\omega}\) and the set of all infinite partitions will be denoted by \((\omega)^\omega\).

In our paper we often identify \(n \in \omega\) with \(\{0, \ldots, n - 1\}\).

## 1.2 Almost containedness

Van Douwen’s diagrams are due to [7] and [17], where the case of \(P(\omega)\) was considered. The term was used in [3] where the case of partitions was studied. The general idea can be described as follows.

Let \(L\) be a lattice with 0 and 1, and let \(I\) be an ideal of \(L\). We say that \(a, b \in L \setminus I\) are orthogonal if \(a \wedge b \in I\). The element \(a\) is almost contained in \(b\) (we denote it by \(a \leq_{a} b\)) if \(a \leq b \lor c\) for some \(c \in I\). We write \(a =_{a} b\) if \(a \leq_{a} b\) and \(b \leq_{a} a\). For any \(a \in L\) we put \(a_I = \{b : b =_{a} a\}\). It is clear that the relation \(\leq_{a}\) becomes the usual almost containedness if we consider the lattice \((P(\omega), \subseteq)\) with respect to the ideal of finite subsets of \(\omega\).

In general, to characterize a lattice \(L\) under these relations we need some further notions. We say that \(a\) splits \(b\) if there are \(c, d \leq b\) not in \(I\) such that \(c \leq a\) and \(d, a\) are orthogonal. A family \(\Gamma \subset L \setminus I\) is a splitting family if for every \(b \in L \setminus I\) there
exists $a \in \Gamma$ that splits $b$. We say that $\Gamma$ is a reaping family if for each $a \in L \setminus I$ there is some $b \in \Gamma$ such that $b \leq_a a$ or $a, b$ are orthogonal. We also define a family $\Gamma \subset L \setminus I$ to be $\leq$-centered if any finite intersection of its elements is not in $I$.

We can now associate to $L$ the following cardinals. Define $a_I$ to be the least cardinality of a maximal family of pairwise orthogonal elements from $L \setminus 1_I$. Let $p_I$ be the least cardinality of a $\leq$-centered family $\Gamma$ such that there is no $b \in L \setminus I$ such that $b$ is a lower bound of $\Gamma$ under $\leq_a$ and the family $\Gamma \cup \{b\}$ is still $\leq$-centered. Similarly, define $t_I$ (the tower cardinal) as the least cardinality of a $\leq_a$-decreasing $\leq$-centered chain without lower $\leq_a$-bound consistent (in the sense of $\leq$-centeredness) with the family. The cardinals $s_I, r_I$ are the corresponding (least) cardinals for splitting families and reaping families respectively. It is worth noting that $p_I$ and $t_I$ can be undefined (for example if for any $a \in L$ the set $\{b : b \leq a\}$ is finite). Also, $(L, I)$ does not necessarily have a splitting family (for example if $L$ is an atomic boolean algebra and $I$ is trivial). So $s_I$ can be undefined too. On the other hand, it is clear that $p_I \leq t_I$ if they are defined.

The last cardinal $h_I$ is defined as follows. A family $\Sigma$ of maximal families of pairwise orthogonal elements in $L \setminus 1_I$ is shattering if for every $a \in L \setminus I$ there are $\Gamma \in \Sigma$ and distinct $b, c \in \Gamma$ which are not orthogonal to $a$. Let $h_I$ be the least cardinality of a shattering family in $L$.

The following lemma seems to be folklore.

**Lemma 1.1** If $s_I$ is defined then $h_I \leq s_I$.

**Proof.** Take a splitting family $\Gamma = \{c_\nu : \nu < s\}$. For each $\nu < s$ choose $\Psi_\nu$ to be a maximal family of pairwise orthogonal elements such that $c_\nu \in \Psi_\nu$. Let us check that the set of these families is shattering. Let $c \in L \setminus I$. Since $\Gamma$ is a splitting family there is $\nu$ and $a, b \leq c$ such that $a \leq c_\nu$ and $b$ is orthogonal to $c_\nu$. By our construction there is $d \in \Psi_\nu$ not orthogonal to $b$. So $\Psi_\nu$ shatters $c$ by $c_\nu$ and $d$. $\square$

**Remark.** In the case of the lattice $(P(\omega), \cup, \cap)$ and the ideal $[\omega]^{<\omega}$ of all finite subsets of $\omega$ the introduced numbers are exactly the classical cardinals $a, h, p, r, s, t$ (all of them occur in [17]). Indeed, our definitions of $a_I, r_I, s_I, h_I$ are formulated as the corresponding classical ones in [7] and in [17]. The classical $t$ is the least cardinality of a $\leq_a$-decreasing chain in $P(\omega)$ without $\leq_a$-bound. The classical $p$ is defined as follows. We say that a family $\Gamma \subseteq [\omega]^{\omega}$ is $\leq_a$-centered if every its finite subset $\Gamma'$ has an infinite pseudointersection: a set $X \subseteq [\omega]^{\omega}$ almost contained in each element of $\Gamma'$. Then the classical $p$ is the least cardinality of a $\leq_a$-centered family from $P(\omega)$ without lower $\leq_a$-bound. So, there is no assumption on $\subseteq$-centeredness as in the definitions of $p_I$ and $t_I$. On the other hand we do not need such assumptions because any $\leq_a$-centered family from $P(\omega)$ is centered. So $p = p_{[\omega]^{<\omega}}$ and $t = t_{[\omega]^{<\omega}}$. $\square$

Note that the definitions of the above cardinals make sense if we consider $L/_{=a}$ under the reverse order $\geq_a$ replacing the ideal $I$ by $1_I$. In the case of $P(\omega)$ the converse cardinals are equal to the corresponding cardinals for $\subseteq$ because $P(\omega)$ is a Boolean algebra. The fact that this is not true in general is quite important for the lattice of subgroups of $SF(\omega)$ and for partitions.
In the latter case we can consider partitions as subsets of \( \omega^2 \) under the inclusion (denoted by \( \subset_{\text{pairs}} \)). The lattice that we get (with operations \( \lor_{\text{pairs}} \) and \( \land_{\text{pairs}} \)) is converse to the lattice \((\omega, \leq)\). Let \( IF \) be the ideal of partitions (in \( (\omega, <_{\text{pairs}}) \)) obtained from \( id_\omega \) by adding a finite set of pairs. Then the class \( 1_{IF} \) is exactly \( (\omega)^{<\omega} \).

Note that papers \([12]\) and \([3]\) study cardinal invariants of this lattice. On the other hand the relation of almost containedness of partitions studied in \([13]\) and \([6]\) can be defined as follows:

\[
Y \leq^* X \iff (\exists Z \in IF)(X \subset_{\text{pairs}} Y \lor_{\text{pairs}} Z).
\]

As a result we have that the cardinal \( a_d, p_d, t_d, h_d, s_d, r_d \) studied in \([13]\) and \([6]\) are the converse cardinals for the pair \(((\omega), <_{\text{pairs}}, IF)\).

### 1.3 The lattice of subgroups of \( SF(\omega) \)

Let \( SF(\omega) \) be the group of all finitary permutations of \( \omega \). This means that the elements of \( SF(\omega) \) are exactly the permutations \( g \) with finite support, where \( \text{supp}(g) = \{ x : g(x) \neq x \} \). The algebraic structure of subgroups of \( SF(\omega) \) is described in \([15]\), \([16]\). The aim of our paper is to study the van Douwen’s invariants of the lattice of subgroups of \( SF(\omega) \).

Throughout the paper \( LF \) is the lattice of all subgroups of \( SF(\omega) \) and \( IF \) is the ideal of all finite subgroups. We say that \( G_1 \) and \( G_2 \) from \( LF \setminus IF \) are orthogonal if their intersection is in \( IF \). The group \( G_1 \) is almost contained in \( G_2 \) \( (G_1 \leq_a G_2) \) if \( G_1 \) is a subgroup of a group finitely generated over \( G_2 \) by elements of \( SF(\omega) \). Let \( SF(\omega)_{IF} = \{ G \leq SF(\omega) : SF(\omega) \text{ is finitely generated over } G \} \). As in Section 1.2 we define the cardinal numbers \( a_{SF}, p_{SF}, t_{SF}, r_{SF}, h_{SF}, s_{SF} \). For example, \( a_{SF} \) is the least cardinality of a maximal family of pairwise orthogonal elements from \( LF \setminus SF(\omega)_{IF} \) and \( p_{SF} \) is the least cardinality of a \( \leq_a \)-centered family of elements in \( LF \setminus IF \) with no lower \( \leq_a \)-bound \( \leq \)-consistent (in the sense of \( \leq \)-centeredness) with the family.

We put a topology on \( LF \) in the following way. Let \( H \leq SF(\omega) \) be finite and \( A \subset \omega \) is a finite set containing the union of the supports of the elements of \( H \). Let \([H,A]\) be the set of all subgroups of \( SF(\omega) \) such that the groups that they induce on \( A \) are equal to \( H \) (we think of \( H \) as a permutation group on \( A \)). The topology that we consider is defined by the base consisting of all sets \([H,A]\). This topology is metrizable: fix an enumeration \( A_0, A_1, \ldots \) of all finite subsets of \( \omega \) and define

\[
d(G_1, G_2) = \sum\{2^{-n} : \text{the groups induced by } G_1 \text{ and } G_2 \text{ on } A_n \text{ are not the same} \}.
\]

Note that the space \( LF \) is complete. A function \( \delta : LF \to n, n \in \omega, \) is then called a Borel (respectively \( \Sigma^1_1 \cup \Pi^1_1 \)) coloring if \( \delta^{-1}(i) \) is Borel (respectively analytic or coanalytic) for every \( i < n \) (where \( n \in \omega \) is viewed as \( \{0, \ldots, n - 1\} \)).

Consider the set \( LF_1 \) of all groups of the form \( SF(\omega) \cap G \) where \( G \) is 1-closed. We identify elements of \( LF_1 \) with elements of \( 2^{\omega \times \omega} \) (the corresponding partitions). Then
it is easily seen that the topology on $LF_1$ induced by the topology above becomes the restriction of the product topology on $2^{\omega \times \omega}$ where $2$ considered discrete. A theorem of T.Carlson and S.Simpson from [5] can be restated as follows: for every $(\Sigma_1^1 \cup \Pi_1^1)$-coloring $\delta : LF_1 \to 2$ there exists $G \in LF_1 \setminus SF(\omega)_{IF}$ such that $\delta$ is constant on the elements of $LF_1 \setminus SF(\omega)_{IF}$ containing $G$. The corresponding theorem for $P(\omega)$ proved by F.Galvin and K.Prikry in [8] is stated as follows: for every $(\Sigma_1^1 \cup \Pi_1^1)$-coloring (originally: Borel coloring; see Remark 2.6 in [5]) $\delta : P(\omega) \to 2$ there exists an infinite $A \in P(\omega)$ such that $\delta$ is constant on the set of all infinite subsets of $A$. It is shown in [5] that this theorem is a consequence of the Carlson-Simpson theorem.

2 A variant of Matet’s theorem

The proof of the theorem of Matet stated in Section 1.1 (this is Proposition 8.1 from [13]) uses the Carlson-Simpson’s theorem and Proposition 4.2 from [13] asserting that $\mathfrak{t}_n$ is uncountable. We will use the same strategy. Our version of this theorem will be given in Section 2.2.

2.1 Comparing $(LF, IF)$ with $(P(\omega), Fin)$

We begin with the following useful lemma.

**Lemma 2.1** Let $G \in LF \setminus IF$ and $m \in \omega$. Then:

(i) there exists a non-trivial $\rho \in G$ such that $\text{supp}(\rho) \cap m = \emptyset$, 

(ii) moreover, for any $H \subset \text{Sym}(m)$ and any sequence $G_0, G_1, \ldots, G_n \in LF \setminus IF$ of groups orthogonal to $G$ the above $\rho$ can be chosen such that additionally $\langle H, \rho \rangle \cap G_i = \langle H \rangle \cap G_i$, for $i \leq n$.

**Proof.** (i). Suppose that the lemma is not true. Choose a minimal $A = \{a_0, \ldots, a_k\} \subset m$ such that there are infinitely many $g \in G$ such that $m \cap \text{supp}(g) \subset A$. Then $A \neq \emptyset$. We fix some non-trivial $g_0$ with that property and consider all tuples $g(\bar{a}) = (g(a_0), \ldots, g(a_k))$ for the above $g$’s. If everyone of these tuples has non-empty intersection with $\text{supp}(g_0)$ then there is $i \leq k$ such that $g(a_i)$ is the same for infinitely many $g$’s. Clearly, for such $g$ and $g'$ the set $m \cap \text{supp}(g^{-1} \cdot g')$ is a subset of $A \setminus \{a_i\}$. This contradicts the minimality of $A$.

Choose $g$ as above with $g(\bar{a}) \cap \text{supp}(g_0) = \emptyset$ additionally. It is easily seen that $g^{-1} \cdot g_0 \cdot g$ fixes $m$ pointwise. This contradicts our assumption.

(ii). Suppose the contrary. By (i) we can find $i \leq n$ such that for infinitely many $\rho \in G$ with $\text{supp}(\rho) \cap m = \emptyset$ there is $g \in \langle H \rangle$ satisfying $g \cdot \rho \in G_i$. Since $\langle H \rangle$ is finite, there is $g_0 \in \langle H \rangle$ such that $g_0 \cdot \rho \in G_i$ for infinitely many $\rho \in G$. Hence, for infinitely many $\rho, \rho' \in G$, $\rho^{-1} \cdot \rho \in G_i$, which contradicts orthogonality. □

This lemma alows us to imitate some arguments from [8] and [12].
**Lemma 2.2** For any $G \in LF \setminus IF$ there is $H \in LF \setminus IF$ such that $H \leq G$ and the sublattice of $(LF/ =_a, \leq_a)$ of all elements below the $=_a$-class of $H$ is isomorphic to $(\mathcal{P}(\omega)/\text{Fin}, \subseteq_a)$.  
In particular $h \leq h_{SF}$, $s \leq s_{SF}$, $r_{SF} \leq r$, $t_{SF} \leq t$ and $p_{SF} \leq p$.

**Proof.** Given $G$ we construct an infinite sequence $\sigma_i \in G$, $i \in \omega$, so that the supports of these $\sigma_i$ are pairwise disjoint. It is easy to arrange that all $\sigma_i$ are of prime orders, say $p_m$. Let $H$ the group generated by all $\sigma_i$, $i \in \omega$. Then $H$ is isomorphic to the direct sum of all $\mathbb{Z}/p_m\mathbb{Z}$.

To each $A \subseteq \omega$ we associate the subgroup of $H$ generated by all these $\sigma_i$ with $i \in A$. This maps $(\mathcal{P}(\omega)/\text{Fin}, \subseteq_a)$ to the sublattice of $(LF/ =_a, \leq_a)$ of all elements below the $=_a$-class of $H$. It is clear that this map is an isomorphism. The rest is easy. $\Box$

The following easy statement is based on an argument which will be typical below.

**Lemma 2.3** For any countable sequence $G_0 > G_1 > ...$ of elements of $LF \setminus IF$ there is a group $G \in LF \setminus IF$ such that $G \leq_a G_i$, $i \in \omega$ and the family $\{G : i \in \omega\} \cup \{G\}$ is $\leq$-centered.

**Proof.** Assume we have a decreasing sequence $G_0 > G_1 > ...$ in $LF \setminus IF$. For every $i \in \omega$ choose non-trivial $g_i \in G_i$ such that $\text{supp}(g_i)$ is disjoint from the supports of the previous elements. We can do this by Lemma 2.1(i). Let $G$ be the group generated by all these $g_i$. Then $G \in LF \setminus IF$, the family $\{G\} \cup \{G_i : i \in \omega\}$ is centered and $G \leq_a G_i$ for every $i \in \omega$. $\Box$

The following lemma uses another type of arguments.

**Lemma 2.4** Let $P_1, ..., P_i, ...$ be a sequence of pairwise disjoint infinite subsets of $\omega$ defining a partition $E_0$ of $\omega$. Let $G_0 = \text{Aut}(\omega, P_1, ..., P_i, ...) \cap SF(\omega)$ be the subgroup of $SF(\omega)$ corresponding to a 1-closed subgroup of $\text{Sym}(\omega)$ defined by $P_i$, $i \in \omega \setminus \{0\}$. Then any proper supergroup of $G_0$ has this form for a partition coarser than $E_0$.

**Proof.** Let $g$ be a finitary permutation such that $g(a) = b \in P_j$ for $a \in P_i$, $i \neq j$. Let $a' \in P_i \setminus \text{supp}(g)$ and $b' \in P_j \setminus \text{supp}(g)$. Below we denote the transposition of $x$ and $y$ by $(x, y)$. It is clear that the element $(a, a') \cdot g^{-1} \cdot (b, b') \cdot g \cdot (a, a')$ (which belongs to $\langle G_0, g \rangle$) is the transposition $(a', b')$. This yields that the group inducing $SF(P_i \cup P_j)$ and acting trivially on $\omega \setminus (P_i \cup P_j)$, is a subgroup of $\langle G_0, g \rangle$. The rest is clear. $\Box$

**Theorem 2.5**

$$p_{SF} = p = t_{SF}.$$ 

**Proof.** $p \leq p_{SF}$. Take an $\leq$-centered family $\Gamma \subseteq LF \setminus IF$ of cardinality $p_{SF}$ without bounds in $LF$, as in the definition of $p_{SF}$. Consider the set $P_p$ of all pairs $(H, F)$ where $H \subseteq SF(\omega)$ is a finite set of permutations with pairwise disjoint supports and $F$ is a finite subfamily of $\Gamma$. We define a forcing relation on $P_p$ as follows:

$$(H, F) \leq (H', F') \text{ iff } H' \subset H, F' \subset F \text{ and each } \alpha \in H \setminus H',$$
is contained in all \( G \in F' \).

Clearly \( P_p \) is \( \sigma \)-centered and satisfies the ccc. For any \( k \in \omega \) and \( G \in \Gamma \) the family

\[
\{(H, F) \in P_p : k < |H|, G \in F\}
\]

is dense in \( P_p \) by centeredness of \( \Gamma \) and Lemma 2.1(i). For a generic \( \Phi \) define

\[
G_0 = \left< \bigcup \{H : (H, F) \in \Phi\} \right>.
\]

It is easily seen that for any \( G \in \Gamma \), the group \( G_0 \) is almost contained in \( G \). Since \( G_0 \) is generated by elements with pairwise disjoint supports, the family \( \Gamma \cup \{G_0\} \) is \( \leq \)-centered. This is a contradiction.

By a theorem of M.Bell [1] \( \text{MA}_\kappa(\sigma\text{-centered}) \) is equivalent to \( \kappa < p \). Therefore, \( p \leq p_{SF} \).

Let us prove \( t_{SF} \leq p \). Since \( p_{SF} \leq t_{SF} \), we will have the equality of these three invariants.

We use the method of Theorem 3.2 of [3]. By Lemma 3.1 of this paper:

\( p \) is the minimal cardinal of form \(|A| + |\mathcal{T}|\) where \( A \cup \mathcal{T} \) is a family of infinite subsets of \( \omega \) with the finite intersection property so that \( \mathcal{T} \) is well-ordered by \( \supseteq^* \) and there is no infinite \( T \) so that \( T \) is almost contained in each member of \( \mathcal{T} \) and \( A \cup \{T\} \) has the finite intersection property.

Assume that \( A \cup \mathcal{T} \) is as above and

\[
A = \{A_\alpha : \alpha < \lambda\}, \quad \mathcal{T} = \{T_\beta : \beta < \mu\}.
\]

In particular \( p = \lambda + \mu \). Let \( \{S_\alpha : \alpha < \lambda\} \) be a properly \( \subset^* \)-descending family of subsets of \( \omega \). We define a sequence \( G_\alpha \), \( \alpha < \lambda + \mu \) as follows.

If \( \alpha < \lambda \) let \( G_\alpha \) be generated by the following subgroups of \( SF(\omega) \):
- the pointwise stabiliser of \( \omega \setminus \{4i, 4i + 1 : i \in A_\alpha\} \),
- the pointwise stabiliser of \( \omega \setminus \{4i : i \notin A_\alpha\} \),
- the pointwise stabiliser of \( \omega \setminus \{4i + 1 : i \notin A_\alpha\} \),
- the group generated by all transpositions \((4i, 4i + 1), i \in T_\beta\).

If \( \alpha = \lambda + \beta \) where \( \beta < \mu \) let \( G_\alpha \) be generated all transpositions \((4i, 4i + 1), i \in T_\beta\).

Using the argument of Lemma 2.4 we obtain that

\[
G_{\lambda + \beta} \leq_a G_\gamma \leq_a G_\alpha, \quad \text{where } \alpha < \gamma < \lambda \text{ and } \beta < \mu.
\]

Now it is easy to see that the sequence \( G_\alpha \), \( \alpha < \lambda + \mu \), is \( \leq_a \)-decreasing.

If \( F \subset \lambda + \mu \) is finite let

\[
B = \bigcap \{A_\alpha : \alpha \in \lambda \cap F\} \cap \bigcap \{T_\beta : \lambda + \beta \in F\}.
\]

The group generated by all transpositions \((4i, 4i + 1), i \in B\), is an infinite subgroup of any \( G_\gamma \) with \( \gamma \in F \). This shows the family \( G_\alpha \), \( \alpha < \lambda + \mu \), is \( \leq \)-centered.
If $H \leq a G_\alpha$ for all $\alpha < \lambda + \mu$, then $H$ is finitely generated over some family of transpositions $(4i, 4i + 1), i \in T$. Since $T$ is almost contained in all $T_\beta, \beta < \mu$, the assumptions above imply that $\mathcal{A} \cup \{T\}$ does not have the finite intersection property. The latter easily implies that $\{G_\alpha : \alpha < \lambda + \mu\} \cup \{H\}$ is not $\leq$-centered. □

Corollary 2.6  
(i) The following inequalities are true in $(LF, IF)$:

$$\omega_1 \leq p_{SF} = t_{SF} \leq 2^{\omega};$$

(ii) $p_{SF}$ and $t_{SF}$ are equal to continuum under Martin’s Axiom.

2.2 A version of Matet’s theorem

We now prove the main result of the section.

Theorem 2.7 Assuming MA there is a filter $F \subset LF \setminus IF$ such that for every $(\Sigma^1_1 \cup \Pi^1_1)$-coloring $\delta : LF \to 2$ there is $G \in F$ such that $\delta$ is constant on the set of all infinite subgroups of $G$.

Proof. Let $G_0 < SF(\omega)$ be generated by an infinite sequence $\{g_1, \ldots, g_i, \ldots\}$ with pairwise disjoint supports where each $g_i$ is a cycle of the prime length $p_i$. Then $G_0$ is isomorphic to $\Sigma_{i<\omega} \mathbb{Z}/p_i \mathbb{Z}$. It is easy to see that if $G \leq G_0$ then $G$ is generated by a subset of the set $\{g_i : i \in \omega\}$. Then the lattice of all subgroups of $G_0$ is isomorphic to $(P(\omega), \subseteq)$. We identify $G \leq G_0$ with the corresponding subset of $\omega$. Notice that then the topology defined in Section 1.3, on $\{G : G \leq G_0\}$ becomes the product topology on $2^{\omega}$. Also, $\{G : G \leq G_0\}$ is a closed subset of $LF$.

We now use the strategy of Proposition 8.1 from [13]. Let $\delta_\alpha, \alpha < 2^{\omega}$, be an enumeration of all $(\Sigma^1_1 \cup \Pi^1_1)$-colorings $\delta : LF \to 2$. We construct a descending tower of subgroups of $G_0$. Supposing that $G_\beta, \beta < \alpha$, have already been selected, use Corollary 2.6(ii) to find $G_\alpha \leq G_0$ such that the family $\{G_\gamma : \gamma \leq \alpha\}$ is $\leq$-centered and $G_\alpha \leq a G_\gamma$ for all $\gamma < \alpha$. By the Galvin-Prikry theorem (8) there is an infinite subset of the set of generators of $G_\alpha$ such that all its infinite subsets have the same color with respect to the coloring induced by $\delta_\alpha$. This shows that $G_\alpha$ can be chosen such that all its infinite subgroups have the same color with respect to $\delta_\alpha$.

Let $F$ be the filter generated by the tower obtained. It follows from the construction that $F$ satisfies the conditions of the theorem. □

3 The diagram in $(LF, IF)$

We begin this subsection with the following algebraic lemma.

Lemma 3.1 Let $G_0, \ldots, G_{n-1}$ be a sequence of infinite groups from $LF$ not $a$-equivalent to $SF(\omega)$. Then for any $k, m \in \omega$, $k > 0$, and $H \subset Sym(m)$ there is a non-trivial
finitary permutation $\rho$ consisting of $(k+1)$-cycles such that $\text{supp}(\rho) \subseteq \omega \setminus m$ and for every $i < n$,

$$\langle H, \rho \rangle \cap G_i = \langle H \rangle \cap G_i.$$  

Proof. For each $i < n$ set

$$S_i = \{ g \in G_i : \exists g_0, g_1 (g_0 \in \langle H \rangle \land (m \cap \text{supp}(g_1) = \emptyset) \land (g = g_0 \cdot g_1)) \}$$

It is easily seen that each $S_i$ is a group. Choose a family $\{D_{j,0} : 0 \leq j < n\}$ of pairwise disjoint finite sets such that for every $j$, $D_{j,0} \subset \omega \setminus m$ and $S_j$ does not induce $\text{Sym}(D_{j,0})$. Let $D_{j,1}, ..., D_{j,k}$ be sets from $\omega \setminus m$ of the same size as $D_{j,0}$. We may assume that every pair from $\{D_{j,i} : 0 \leq i \leq k; j < n\}$ has empty intersection. For every $0 < i \leq k$ and $j < n$ we choose a bijection $f_{j,i}$ from $D_{j,0}$ onto $D_{j,i}$ such that it is not induced by any element of $S_j$. The existence of such $f_{j,i}$ is a consequence of the fact that for any bijections $f, g : D_{j,0} \to D_{j,i}$ induced by $S_j$, the bijection $g^{-1} \cdot f$ defines a permutation on $D_{j,0}$ induced by $S_j$.

We now define a permutation $\rho$ with the support $\bigcup \{D_{j,i} : 0 \leq i \leq k; 0 \leq j < n\}$ as follows. If $x \in D_{j,i}, 0 < i < k$, then $\rho(x) = f_{j,i+1}(f_{j,i}^{-1}(x))$. If $x \in D_{j,0}$, then $\rho(x) = f_{j,1}(x)$. For $x \in D_{j,k}$ we put $\rho(x) = f_{j,k}^{-1}(x)$. Let us check that $\rho$ satisfies the conclusion of the lemma. It is clear that $\rho$ consists of cycles of length $k + 1$. Suppose, that for some $g_0 \in \langle H \rangle$ the element $g = g_0 \cdot g'$, $0 < l \leq k$, is contained in some $G_j$. Thus $g \in S_j$ and by our construction $g$ maps $D_{j,0}$ onto $D_{j,l}$ by $f_{j,l}$. Since $S_j$ does not induce $f_{j,l}$, we have a contradiction. \qed

The van Douwen cardinals for $(LF, IF)$ are described in the following theorem.

**Theorem 3.2**  
(i) The following inequalities are true in $(LF, IF)$:

$$\omega_1 \leq p_{SF} = t_{SF} \leq h_{SF} \leq s_{SF} \leq 2^{\omega},$$

$$\omega_1 \leq a_{SF}, r_{SF} \leq 2^{\omega};$$

(ii) All the coefficients are equal to continuum under Martin’s Axiom;

(iii) Each of the following equalities is consistent with $\{\text{ZFC} + \omega_1 < 2^{\omega}\}$:

$$a_{SF} = \omega_1, s_{SF} = \omega_1, r_{SF} = \omega_1.$$  

Proof.  
(i). The inequality $h_{SF} \leq s_{SF}$ is shown in Lemma [1.1]. The inequality $\omega_1 \leq p_{SF}$ follows from Theorem [2.3].

To show that $t_{SF} \leq h_{SF}$, suppose $\mu < t_{SF}$ and $\Psi = \{\Psi_\nu : \nu < \mu\}$ is a shattering family of maximal families of pairwise orthogonal elements in $L \setminus SF(\omega)_{IF}$. Choose an infinite $G_0 < SF(\omega)$ generated by an infinite sequence of elements with pairwise disjoint supports and of prime orders. Taking intersections with appropriate elements of $\Psi_\nu$'s we construct a $\leq_a$-decreasing $\leq$-centered sequence of infinite groups $\{G_\nu : \nu < \mu\}$ such that $G_\nu$ is not shattered by $\Psi_\gamma$ for all $\gamma < \nu$. We can arrange that the groups are $=_a$-distinct. Since $\mu < t_{SF}$, there is $G \in LF \setminus IF$ such that $G <_a G_\nu$ for all
\(\nu < \mu\). We may assume that \(G < G_0\). If \(G\) is shattered by a pair \(H_1, H_2\) from some family from \(\Psi\) then \(H_1\) and \(H_2\) provide two disjoint infinite sets \(S_1 \subseteq G \cap H_1\) and \(S_2 \subseteq G \cap H_2\) such that each \(S_i\) consists of elements with pairwise disjoint supports. Since \(G_{<a} G_{\nu}\) every \(S_i\) has an infinite intersection with \(G_{\nu}\). This shows that every \(G_{\nu}\) is shattered by \(H_1\) and \(H_2\). This is a contradiction.

To prove \(\omega_1 \leq r_{SF}\) it suffices to show that if a family \(\Psi \subseteq LF \setminus IF\) is countable then there exists \(G \in LF \setminus IF\) such that for every \(G' \in \Psi\) the groups \(G, G'\) are not orthogonal and \(G' \not\leq a G\). Let \(\{G_0, G_1, \ldots\}\) be an enumeration of \(\Psi\). Assume that each member of \(\Psi\) occurs infinitely often. We construct two sequences \(g_0, g_1, \ldots\) and \(h_0, h_1, \ldots\) of finitary permutations with pairwise disjoint supports such that for all \(i, j \in \omega\) we have \(\text{supp}(g_i) \cap \text{supp}(h_j) = \emptyset\) and \(g_i, h_i \in G_i\). It is easily seen that Lemma 2.1(i) implies the existence of such sequences. Let \(\hat{G}_1 = \langle\{g_i : i \in \omega\}\rangle\) and \(\hat{G}_2 = \langle\{h_i : i \in \omega\}\rangle\). Clearly, \(\hat{G}_1\) and \(\hat{G}_2\) are orthogonal but they are not orthogonal to any \(G_i\) (since each member of \(\Psi\) is enumerated infinitely often). Now it is easy to see that \(G = \hat{G}_1\) satisfies the conditions that we need.

To prove the inequality \(\omega_1 \leq a_{SF}\) take a countable \(\Psi \subseteq LF \setminus SF(\omega)_{IF}\). We construct a group \(G\) by induction. Fix an enumeration of \(\Psi : G_0, G_1, \ldots\). Let \(H\) be the set of the elements which have been constructed at the first \(n - 1\) steps. At the \(n\)-th step we choose a permutation \(\rho\) as in Lemma 3.1 with respect to \(G_0, \ldots, G_n\) and \(m\) large enough. It is easily seen that that the group generated by this sequence is orthogonal to any group from \(\Psi\).

(iii). Assume MA. By Theorem 2.5 we have \(p_{SF} = 2^\omega\).

To prove \(r_{SF} = 2^\omega\) we introduce a ccc forcing notion \(P_\tau\) as follows. Consider the family of all pairs \((H, H')\) where \(H, H' \subseteq SF(\omega)\) are finite and the supports of any two elements of \(H \cup H'\) have empty intersection. The order is defined as follows \((H, H') \leq (F, F')\) iff \(F \subseteq H\) and \(F' \subseteq H'\). Let \(\Psi \subseteq LF \setminus IF\) have cardinality < \(2^\omega\). For any \(k \in \omega\) and \(G \in \Psi\) the family

\[\{(H, H') \in P_\tau : k < |H' \cap G|, k < |H \cap G|\}\]

is dense in \(P_\tau\) by Lemma 2.1(i) (see also the previous part of the proof). For a generic \(\Phi\) define \(G_0 = \langle\bigcup\{H : (H, H') \in \Phi\}\rangle\). It is easy to see that for any \(G \in \Psi\), the groups \(G\) and \(G_0\) are not orthogonal and \(G\) is not contained in \(G_0\) under \(\leq a\). Thus \(\Psi\) is not reaping.

To show \(a_{SF} = 2^\omega\), given an infinite family \(\Gamma \subseteq LF\) of infinite groups define a forcing notion \(P_a\) as follows. Let \(P_a\) be the set of all pairs \((H, F)\) where \(F\) is a finite subset of \(\Gamma\) and \(H\) is a finite set of permutations such that their supports are pairwise disjoint. We define \((H, F) \leq (H', F')\) iff \(H' \subseteq H, F' \subseteq F\) and each \(h \in \langle H \rangle \setminus \langle H'\rangle\) is not contained in any \(G \in F'\). It is easily verified that \(P_a\) is a ccc forcing notion.

Consider \(P_a\) with respect to \(\Psi \subseteq LF \setminus SF(\omega)_{IF}\) of cardinality < \(2^\omega\). Clearly, the following sets are dense in \(P_a\) (apply Lemma 3.1 in the second case):

\[\Sigma_G = \{(H, F) : G \in F\}; G \in \Psi\]

\[\Sigma_l = \{(H, F) : \text{the number of the elements of } H \text{ is greater than } l\}; l \in \omega\]
By MA we have a filter $\Phi \subset P_a$ meeting all these $\Sigma$’s. It is easy to see that the group $G_0 = \langle \bigcup \{H : (H, F) \in \Phi\} \rangle$ is orthogonal to any group from $\Psi$.

(iii). $\text{Con}(\text{ZFC} + a_{SF} = \omega_1 < 2^\omega)$. We start with an arbitrary countable family $\Psi_0 \subset \text{LF} \setminus \text{SF}(\omega)_{IF}$ of parwise orthogonal groups. Take a sequence

$$\Psi_0 \subset \Psi_1 \subset ... \subset \Psi_\gamma \subset ..., \gamma < \omega_1,$$

by a finite support iteration

$$(P_\gamma, Q_\gamma : \gamma < \omega_1)$$

of the forcing $P_a$ (from the previous part of the proof) applied to potential $\Psi_\gamma$’s. The canonical name for $P_\gamma$ of $\Psi_{\gamma+1}$ is obtained from the canonical name of $\Psi_\gamma$ by adding the canonical name of the group $G_\gamma$ defined by $Q_\gamma$ as $G_0$ by $P_a$ above. Let $\Phi$ be generic for $P_\omega$ and $\Phi_\gamma$ be the corresponding restriction to $P_\gamma$. It is easily seen that $P_\omega$ fulfils the ccc. Since $\omega_1$ is regular and each group $G$ in $\text{LF}[\Phi]$ is defined by a countable set of finitary permutations, it is contained in some $\text{LF}[\Phi_\gamma]$. Suppose that some $G$ is orthogonal to each group from $\Psi_\gamma$. Thus by Lemma 2.1(ii) every set $D_n = \{(H, F) \in Q_\gamma[\Phi_\gamma] : n < |H \cap G|\}$ is dense in $Q_\gamma[\Phi_\gamma]$. So the group

$$G_\gamma = \langle \bigcup \{H : (H, F) \in \Phi_{\gamma+1}/\Phi_\gamma\} \rangle$$

is not orthogonal to $G$. This shows that the set $\bigcup \{\Psi_\gamma : \gamma < \omega_1\}$ is a maximal family of pairwise orthogonal groups in $\text{LF}[\Phi]$.

The cases $\text{Con}(\text{ZFC} + s_{SF} = \omega_1 < 2^\omega)$ and $\text{Con}(\text{ZFC} + r_{SF} = \omega_1 < 2^\omega)$ can be handled in a similar way. In the first case constructing $G_\gamma$ we apply the forcing $P_r$. In the second one at every step we should apply $P_a$ to the whole family $\text{LF} \setminus (IF \cup \text{SF}(\omega)_{IF})$. So every $G \in \text{LF}[\Phi] \setminus (IF \cup \text{SF}(\omega)_{IF})$ is orthogonal to some $G_\gamma$. □

We conjecture that $h_{SF} = h$, $s_{SF} = s$ and $r_{SF} = r$. Note that the corresponding equalities hold for the lattice of partitions under $\leq_{pairs}$ [3]. At the moment we cannot adapt the arguments of [3] to our case. The case of $a$ is also open.

4 Other versions. Remarks

4.1 Another version of Matet’s theorem

As we noted in Introduction the lattice of partitions under the reverse order is a sublattice of $\text{LF}$. This suggests that in the lattice $\text{LF}$ the most natural variant of the theorem of P.Matet cited there (which is Proposition 8.1 from [13]) is the following one.

**Theorem 4.1** Assuming the continuum hypothesis there is an ideal $I \subset \text{LF} \setminus \text{SF}(\omega)_{IF}$ such that for every $(\Sigma^1_1 \cup \Pi^1_1)$-coloring $\delta : \text{LF} \to 2$ there is $G \in I$ such that $\delta$ is constant on the set of all supergroups of $G$ which do not belong to $\text{SF}(\omega)_{IF}$. 
Proof. Let $P_1, ..., P_i, ...$ be a sequence of pairwise disjoint infinite subsets of $\omega$ defining a partition $E_0$ of $\omega$. Then $G_0 = Aut(\omega, P_1, ..., P_i, ...) \cap SF(\omega)$ is the subgroup of $SF(\omega)$ obtained from the corresponding 1-closed subgroup of $Sym(\omega)$. By Lemma 2.4 any proper supergroup of $G_0$ has this form for a partition coarser than $E_0$.

We may now consider the set $L_0 = \{ G : G_0 \leq G \leq SF(\omega) \}$ as a sublattice of partitions coarser than $E_0$. Notice that then the topology defined in Section 1.3, on $L_0$ becomes the product topology on $2^{\omega \times \omega}$. This follows from the fact that any finite permutation group (on a finite subset of $\omega$) induced by a group $G$ from $L_0$ is a finite 1-closed permutation group and can be identified with a partition induced by the partition corresponding to $G$. Moreover, it is easy to see that $L_0$ is closed in $LF$.

We now use the Matet’s theorem. Take an ideal $I_0$ of $L_0$ provided by this theorem. Then $I_0$ generates an ideal of $LF$. This works as $I$ in the statement. □

In fact Lemma 2.4 is a dual version of Lemma 2.2. In particular we have the following corollary.

Proposition 4.2 Let $h_d, p_d, r_d, s_d, t_d$ be dual cardinal invariants of the lattice of partitions defined by the scheme of Section 1.2 (defined as in [6]). Let us consider $LF/\approx_a$ with respect to the converse ordering $\geq_a$ and let $h_{SF}^d, p_{SF}^d, r_{SF}^d, s_{SF}^d, t_{SF}^d$ be the corresponding sequence of cardinal invariants defined with respect to $SF(\omega)_{\approx_F}$ as an ideal of this converse lattice.

Then $h_d \leq h_{SF}^d, r_{SF}^d \leq r_d, s_d \leq s_{SF}^d$ and $t_{SF}^d = p_{SF}^d = \omega_1$.

Proof. By Lemma 2.4 we have

$$h_d \leq h_{SF}^d, r_{SF}^d \leq r_d, s_d \leq s_{SF}^d, t_{SF}^d \leq t_d, p_{SF}^d \leq p_d.$$ 

Since $p_d = t_{SF}^d = \omega_1$ (see [13]), we have the statement of the lemma. □

Using this proposition and the material of papers [13], [6], [2], [14] we obtain the following relations:

$r_{SF}^d \leq r_d \leq \min(r, \partial, non(\mathcal{M}), non(\mathcal{N}))$ and $\max(cov(\mathcal{N}), cov(\mathcal{M}), s, b) \leq s_d \leq s_{SF}^d$.

Moreover the following relations are consistent with ZFC:

$r_d \leq add(\mathcal{M}), r_d > b, s_d > cof(\mathcal{M}), s_d \leq r, s_d < \partial$.

We mention the following questions:

1. Is $a_{SF}^d = 2^\omega$?
2. What is relation between $h_{SF}^d$ and $h$?
3. Are the following relations consistent with ZFC:

$r_{SF}^d > b, s_{SF}^d \leq r, s_{SF}^d < \partial$?
4.2 Remarks

Our results suggest the investigation of van Douwen’s cardinals for the lattice of all closed subgroups of $\text{Sym}(\omega)$. The definition of the $a$-order in this case must be as follows: $G \leq_a G'$ iff there exists a finite set $X$ of finitary permutations such that $G$ is a subgroup of the closed group generated by $G'$ and $X$. It is worth noting that some results of [10] can be interpreted in this vein for some converse coefficients (for example, see Observation 3.3 in [10]).

However, one can notice that the lattice of all closed subgroups admits several constructions which make some of the van Douwen’s cardinals trivial. For example, the group $\mathbb{Z}$ with the natural action on itself can be considered as a closed subgroup of $\text{Sym}(\omega)$. It is clear that for every $n \in \omega$ no closed subgroups split any $n\mathbb{Z}$. So, $s_I$ is undefined. On the other hand it is worth noting that for every $n \in \omega$, $n\mathbb{Z} = a \text{Sym}(\omega)$. Indeed, fixing some representatives $a_i$ of all the orbits, add the transpositions of the pairs $a_i, a_i + n$. This induces all permutations on every orbit. Adding transpositions of some pairs from distinct orbits we get $\text{Sym}(\omega)$.

Another easy observation is that $r_I = 1$ in this case. Indeed, the Prüfer group $\mathbb{Z}(p^{\infty})$ with the natural action on itself forms a reaping family.

It is interesting to compare the lattices that we consider here with the lattice $P(\omega)$ of all subsets of $\omega$ and the ideal of finite subsets. Since $=_a$ is a congruence of $P(\omega)$, the orthogonality of infinite $a$ and $b$ means the absence of $c$ such that $c \leq_a a$ and $c \leq_a b$. So the van Douwen’s cardinals can be defined only in terms of $\leq_a$ (and originally it was so). On the other hand, this does not hold in lattices of subgroups of $\text{Sym}(\omega)$. Indeed, let $\sigma$ be a transposition of some pair in $\omega$. Then $\mathbb{Z}\sigma$ induces a closed subgroup of $\text{Sym}(\omega)$ which is $a$-equivalent to $\mathbb{Z}$ with the above action. Clearly, the intersection of these groups is trivial.

In the case of $(LF, IF)$ the corresponding example is as follows. Let infinite $A, B, C \subset \omega$ define a partition of $\omega$ and $R$ be a bijection between $A$ and $B$. Let $E_0 = A^2 \cup B^2 \cup \text{id}_{C \times C}$ and $E_1 = R \cup \text{id}_{C \times C}$. It is easily seen that $E_0$ and $E_1$ are orthogonal equivalence relations, but $E_1 \leq_a E_0$. The groups $G_{E_0} \cap SF(\omega)$ and $G_{E_1} \cap SF(\omega)$ have the same properties.

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