Consistent Riccati Expansion and Solvability

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Abstract: A consistent Riccati expansion (CRE) is proposed for solving nonlinear systems with the help of a Riccati equation. A system is defined to be CRE solvable if it has a CRE. Various integrable systems are CRE solvable. Furthermore, it is also revealed that many CRE solvable systems, including the Korteweg-de-Vries, Kadomtsev-Petviashvili, nonlinear Schrödinger and sine-Gordon equations, possess a common determining equation which describes interactions between a soliton and a cnoidal wave.

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1 Introduction

A trouble and tedious but very important problem is to find exact solutions of nonlinear systems. To solve this problem, some elegant approaches such as the inverse scattering transformation\textsuperscript{[1]}, Darboux-Bäcklund transformations\textsuperscript{[2]}, and nonlinearizations of Lax pairs\textsuperscript{[3]} have been established. However, these effective methods are inscrutable or mystical for people who are neither mathematicians nor theoretical physicists. Thus it is very necessary to establish some types of quite simple and understandable methods to construct exact solutions. For instance, the hyperbolic tangent (tanh) function expansion method\textsuperscript{[4]}, the auxiliary equation (say, Riccati equation) method\textsuperscript{[5]} and the homogeneous balance method\textsuperscript{[6]} are usually used to find (quasi-) traveling solitary wave solutions. Nevertheless, it is unfortunate that these feasible methods might lose some essential information of the original nonlinear systems, and consequently, only some quite special solutions can be obtained. In fact, the simple methods mentioned above can be further developed to find much more general solutions retrieving the missing essential properties. In addition, one can even clarify the integrability of some types of nonlinear systems. In Refs.\textsuperscript{[7–9]}, we have generalized the tanh function expansion method to find not only various interaction solutions between different types of excitations but also possible new integrable systems.

In the next section, we propose a consistent Riccati expansion (CRE) method, which can be considered as an extension version of the usual Riccati equation method and the tanh function expansion method, such that we could find new solutions of nonlinear systems and strong signals to clarify possible
integrable models. Based on the new method, we define a model is CRE solvable if it has a CRE. The method is systematically illustrated by some important nonlinear systems. Section 3 is devoted to clarifying the possible CRE solvable cases of the fifth order KdV system. In Section 4, it is exhibited that many CRE solvable systems possess a same determining equation which describes interesting interactions between a soliton and a cnoidal wave. The last section is a short summary and discussion.

2 Consistent Riccati expansion for some nonlinear systems

For a given derivative nonlinear polynomial system,

\[ P(x, t, u) = 0, \quad x = \{x_1, x_2, \ldots, x_n\}, \]

we aim to look for the following possible truncated expansion solution

\[ u = \sum_{i=0}^{n} u_i R^i(w), \]

where \( R(w) \) is a solution of the Riccati equation

\[ R_w = a_0 + a_1 R + a_2 R^2, \]

\( n \) should be determined from the leading order analysis of (1), and all the expansion coefficient functions \( u_i \) should be determined by vanishing the coefficients of the like powers of \( R(w) \) after substituting (2) with (3) into (1).

**Definition.** If the system for \( u_i (i = 0, \ldots, n) \) and \( w \) obtained by vanishing all the coefficients of the powers of \( R(w) \) after substituting (2) with (3) into (1) is consistent, or not over-determined, we call the expansion (2) is a consistent Riccati expansion (CRE) and the nonlinear system (1) is defined CRE solvable.

In the following, we apply the proposed new method to nine concrete examples, as a result, we can find that a variety of integrable systems are actually CRE solvable.

**Example 1. The Korteweg de-Vries (KdV) equation.** For the KdV equation\(^{[10]}\)

\[ u_t + 6uu_x + u_{xxx} = 0, \]

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the leading order analysis leads to $n = 2$. Substituting (2) with (3) and $n = 2$ into (1) yields

\[ 0 = 12a_2 w_x u_2 (u_x + 2a_2^2w_x^2)R^5 + 6\left[u_2 w_x (2u_x + 9a_2^2w_x^2) a_1 + a_2^2u_1 w_x^3 + 3w_x a_2^2(w_x u_2)_x + 3a_2 u_1 u_2 w_x + u_2 w_x^2\right]R^4 + \left[4u_2 w_x (3u_x + 10a_2^2w_x^2) a_0 + 38a_2^2 w_x^4 + 6a_2^2 w_x (u_x w_x)_x + 6a_1 w_x \left[3u_1 u_2 + 2a_2^2 u_1 w_x^3 + 5a_2 (w_x u_2)_x\right] + 6(w_1 u_2)_x + 2a_2 \left[u_2 (w_1 + w_x x) + 3w (2u_0 u_2 + u_2^3) + 3 (u_x w_x)_x\right]\right]\] \[+ \left[2a_0 w_x [2u_x a_2 (2u_1 a_2 + 13 u_2 a_1) + 12 a_2 (w_x u_2)_x + 9 u_1 u_2] + 8u_2 w_x^3 a_1^2 + a_1^2 w_x \left[7 u_1 w_x^2 a_2 + 12 (w_x u_2)_x + a_1^2 g_2 w_x (w_x u_1)_x 12 a_0 u_2 w_x + 6 (u_x w_x)_x + 6 \left[u_2 w_x^2 + u_2 + u_2 w_x + 6 u_1 u_x + 6 (u_x w_0)\right] R^2 + \left\{16 a_2 a_0 w_x u_2 w_x^3 + 16 a_2^2 a_0 w_x^2 \left(2 a_1 w_x^2 + 9 (w_x u_2)_x\right) + 6u_2 w_x (u_x w_x)_x + 12 a_0 u_2 w_x + 6 (u_x w_x)_x + 2 u_x (w_x x + w_1) + 6 a_1^2 w_x + 3 a_1^2 w_x (u_x w_x)_x + a_1^2 (u_x w_x)_x + u_1 (w_xx + w_x + 6 u_0 w_x) + u_1 + u_1 w_x + 6 (u_0 u_1)\right\} R + a_0^2 [6 a_1 u_2 w_x^3 + 2 a_1 w_x^3 + 6 w_x (w_x u_2) u_2] + a_0 (a_2^2 w_x + 3 a_1 w_x (u_x w_x)_x + 3 (u_x w_x)_x + u_1 (w_x x + w_1 + 6 u_0 w_x)] + u_0 + 6 u_0 u_0 x + u_0 w_x. \right]\] (5)

Setting zero the coefficients of all the same powers of $R$ in (5), we have six over-determined equations for only four undetermined functions $u_0$, $u_1$, $u_2$ and $w$. It is fortunate that for many integrable models, these kinds of over-determined systems may be consistent. In this case, from the coefficients of $R^5$, $R^4$ and $R^3$, we can simply find

\[ u_2 = -2a_2^2 w_x^2, \] \[ u_1 = -2a_2 (a_1 w_x^2 + w_xx), \] \[ u_0 = \frac{1}{6} \left( w_x - \frac{1}{2} \left(a_x^2 + 8a_0 a_2\right) w_x^2 - a_1 w_xx - \frac{2}{3} w_x - \frac{1}{2} w_x^2\right). \] (6a,b,c)

The coefficient of $R^2$ in (5) becomes identically zero by using (6). Then from the coefficient of $R$ in (5), we find the $w$ equation

\[ w_x = -w_xx + \frac{3}{2} \frac{w_x^2}{w_x} + \frac{1}{2} \frac{w_x^2}{w_x} + \lambda w_x, \quad \delta \equiv (a_1^2 - 4 a_0 a_2). \] (7)

Finally, one can verify that the coefficient of $R^0$ in (5) is identically zero. Evidently, the condition in the Definition is satisfied, and thus the KdV equation (1) is CRE solvable. It is noted that the $w$ equation (7) is a generalization of the Schwarzian KdV equation (without $\delta$ term) and the potential modified KdV equation (cancelling the rational term).

In summary, we have the following theorem:

**Theorem 1.** If $w$ is a solution of Eq. (7), then

\[ w = -\frac{\lambda}{6} - a_1 w_xx - \frac{1}{4} (a_1^2 + 4a_0 a_2) w_x^2 + \frac{1}{2} \frac{w_xx}{w_x} + \frac{1}{4} \frac{w_x^2}{w_x} - 2a_2 (a_1 w_x^2 + w_xx) R - 2a_2^2 w_x^2 R^2 \] (8)

is a solution of the KdV equation (1) with $R \equiv R(w)$ being a solution of the Riccati equation (3).
Example 2. The Kadomtsev-Petviashvili (KP) equation. Applying the same procedure to the KP equation \[11\]
\[
(u_t + 6uu_x + u_{xxx})_x + \gamma u_{yy} = 0,
\]
we can write down the following theorem. It is noted that the detailed derivation or proof is omitted since the procedure is exactly same to the KdV case, which is also valid for the following theorems for other different models.

**Theorem 2.** If \( w \) is a solution of

\[
\left( C + S - \frac{\delta}{2} w_x^2 + \frac{\gamma}{2} K^2 \right)_x + \gamma K_y = 0,
\]

with

\[
C \equiv \frac{w_t}{w_x}, \quad S \equiv \frac{w_{xxx}^2}{3 w_x^2} - \frac{1}{2} w_x^2 - \frac{\gamma w_y}{6 w_x^2} - 2a_2(a_1 w_x^2 + w_{xx})R - 2a_2^2 w_x^2 R^2
\]

then

\[
u = -\frac{1}{6} w_t - a_1 w_{xx} - \frac{1}{6} (a_1^2 + 8a_0a_2) w_x^2 - \frac{2 w_{xxx}}{3 w_x} + \frac{2 w_{xx}}{w_x} - \frac{\gamma w_y}{6 w_x^2} - 2a_2(a_1 w_x^2 + w_{xx})R - 2a_2^2 w_x^2 R^2
\]

is a CRE of the KP equation.

**Example 3. The Boussinesq equation.** The theorem for the Boussinesq equation \[12\]

\[
u_{tt} + (6uu_x + u_{xxx})_x = 0,
\]

is stated as follows:

**Theorem 3.** If \( w \) is a solution of

\[
\left( S - \frac{\delta}{2} w_x^2 + \frac{1}{2} C^2 \right)_x + C_t = 0,
\]

then

\[
u = -a_1 w_{xx} - \frac{1}{6} (a_1^2 + 8a_0a_2) w_x^2 - \frac{2 w_{xxx}}{3 w_x} + \frac{1 w_{xx}^2}{2 w_x^2} - \frac{1 w_t^2}{6 w_x^2} - 2a_2(a_1 w_x^2 + w_{xx})R - 2a_2^2 w_x^2 R^2
\]

is a CRE of the Boussinesq equation.

**Example 4. The AKNS (Ablowitz-Kaup-Newell-Segur) system.** For the AKNS system \[13\]

\[
p_t + \frac{1}{2} ibp_{xx} - ip^2 q = 0, \quad b^2 = 1, \quad \lambda = 1,
\]

\[
q_t - \frac{1}{2} ibq_{xx} + iq^2 p = 0,
\]

the theorem reads

**Theorem 4.** If \( w \) is a solution of

\[
C_t + \frac{1}{8} (2S - 12C^2 - 16b\lambda C + \delta w_x^2)_x = 0,
\]

then

\[
p = \sqrt{b} \left[ a_2 w_x R + ib \frac{w_t}{w_x} + i\lambda + \frac{w_{xx}}{2 w_x} + \frac{1}{2} w_x \right] e^{iu},
\]

\[14\]

\[15\]
\[ q = \sqrt{b} \left[ a_2 w_x R - i b w_t w_x - i \lambda + \frac{1}{2} w_{xx} + \frac{1}{2} a_1 w_x \right] e^{-i u}, \]  

(18b)
is a solution of the AKNS system \(^{16}\) with the consistent conditions for the ‘phase’ \( u \)

\[
\begin{align*}
    u_x &= 2bC + \lambda, \\
    u_t &= 3bC^2 + 4\lambda C - \frac{1}{4} b w_x^2 + \frac{3}{2} b \lambda^2 - \frac{b}{2} S.
\end{align*}
\]

(19b)

It is remarkable that the consistent condition \( u_{xt} = u_{tx} \) of (19) is nothing but (17).

**Example 5. Sine-Gordon (sG) equation.**

For the sine-Gordon equation \(^{14}\)

\[
    v_{xt} - 2m \sin(v) = 0, \tag{20}
\]

in order to use the CRE method, we have to transform it to a derivative polynomial form

\[
    u_t u_x - uu_{xt} + mu^3 - mu = 0, \tag{21}
\]

through

\[
    u = \exp(iv). \tag{22}
\]

For the variant form \(^{21}\) of the sG equation, we can establish the following theorem:

**Theorem 5.** If \( w \) is a solution of the consistent system

\[
\begin{align*}
    (CC_{xt} - C_x C_t)\lambda + mC(C^2 - \lambda^2) &= 0, \tag{23a} \\
    \lambda(\delta C^2 w_x^2 + 2CC_{xx} + 2C^2 S - C_x^2) - 2C^2 m &= 0, \tag{23b}
\end{align*}
\]

then Eq. (21) has a CRE solution

\[
    u = \frac{2}{m} w_t w_x a_2^2 R^2 + \frac{2a_2}{m} (w_{xt} + a_1 w_x w_t) R + \frac{(w_{xt} + a_1 w_x w_t)^2}{2mw_x w_t}. \tag{24}
\]

It should be emphasized that for the sG equation, one \( w \) function would satisfy two equations \(^{23}\), however we can still call it CRE solvable because those two equations are consistent, i.e., \( C_{xx} = C_{xxx} \) is identically satisfied.

**Example 6. Modified asymmetric Veselov-Novikov (VN) equation.**

For the modified asymmetric VN equation \(^{15}\)

\[
\begin{align*}
    u_t - u_{xxx} - 3u_x v_x - \frac{3}{2} w_{xx} &= 0, \\
    v_y &= u^2, \tag{25a}
\end{align*}
\]

it is straightforward to find the transformation theorem as follows:

**Theorem 6.** If \( w \) is a solution of

\[
    C_y = \frac{\delta}{4} w_x (w_x K_x + 4K w_{xx}) + \frac{3K^3}{4K^2} - \frac{3}{2} K K_{xx} + \frac{1}{2} SK_x + KS_x + K_{xxx}, \tag{26}
\]

then the VN system \(^{25}\) possesses the CRE form

\[
\begin{align*}
    u &= -\frac{b}{2} \sqrt{-w_x w_y} \left( a_1 + \frac{w_{xy}}{w_x w_y} + 2a_2 R \right), \\
    v &= v_0 - a_2 w_x R. \tag{27}
\end{align*}
\]
where \( v_0 \) is related to \( w \) by a consistent system

\[
\begin{align*}
    v_{0x} &= \frac{\delta}{12} w_x^2 - \frac{1}{2} a_1 w_{xx} + \frac{C - S}{3} - \frac{w_x^2}{w_x^2}, \\
    v_{0y} &= \frac{K_x^2}{4K} - \frac{K_x}{2w_x} (a_1 w_x^2 + w_{xx}) + \frac{\delta}{4} K w_x^2 - \frac{a_1}{2} K a_1 w_{xx} - \frac{K w_{xx}^2}{4 w_x^2}.
\end{align*}
\]

(28)

It is noted that the compatibility condition \( v_{0xy} = v_{0yx} \) is nothing but the \( w \) equation (26).

**Example 7. Dispersive water wave (DWW) equation.** For the DWW equation [16]

\[
\begin{align*}
    u_t - (u_{xx} - 3vu_x + 3v^2 - 3u^2)_x &= 0, \quad (29a) \\
    v_t - (v_{xx} + 3vv_x + v^3 - 6uv)_x &= 0, \quad (29b)
\end{align*}
\]

we have the following CRE theorem:

**Theorem 7.** The DWW equation (29) has a CRE

\[
\begin{align*}
    u &= a_2 w_x R^2 + a_2 (w_{xx} + a_1 w_x^2) R + \frac{1}{2} (a_1 + b\sqrt{-\delta}) w_x + v_{0x} + a_0 a_2 w_x^2, \\
    v &= a_2 w_x R + \frac{1}{2} (a_1 + b\sqrt{-\delta}) w_x + v_0, \quad b^2 = 1,
\end{align*}
\]

(30)

with \( \{w, v_0\} \) being a solution of the coupled system

\[
\begin{align*}
    w_t &= (w_{xx} - 3v_0 w_x)_x - w_x [\delta w_x^2 + 3b\sqrt{-\delta}(w_{xx} - v_0 w_x) - 3v_0^2], \quad (31a) \\
    v_{0t} &= (v_{0xx} - 3v_0 v_{0x} + v_0^3)_x. \quad (31b)
\end{align*}
\]

It is clear that to prove the CRE solvability we can simply take \( v_0 = 0 \). However, nonzero \( v_0 \) will lead to more exact solutions of DWW system.

**Example 8. Burgers equation.** Here, we write down the CRE theorem for a simple C-integrable model, the Burgers equation

\[
\begin{align*}
    u_t &= u_{xx} + 2uu_x. \quad (32)
\end{align*}
\]

**Theorem 8.** If \( w \) is a solution of

\[
\begin{align*}
    C_t &= \left( \frac{1}{2} C^2 - S + 2C_x - \frac{\delta}{2} w_x^2 \right)_x, \quad (33)
\end{align*}
\]

then

\[
\begin{align*}
    u &= \frac{w_t - w_{xx}}{2w_x} - \frac{a_1}{2} w_x - a_2 w_x R.
\end{align*}
\]

(34)

is a CRE solution of the Burgers equation (32).

The above examples reveal that for many kinds of integrable models, the truncated expansions based on the Riccati equation will lead to some \( w \) equations, which are the generalization of the Schwarzian equations of the original nonlinear systems because they will reduce back to their Schwarzian forms when \( \delta = 0 \).

It is naturally expected that this elegant property will be lost for nonintegrable systems. Here, we just present one non-CRE solvable example.
Example 9. **Non-integrable KdV-Burgers (KdV-B) equation.** For the KdV-B equation

\[ u_t + 6uu_x + \nu u_{xx} + u_{xxx} = 0, \]  

(35)

with the same procedure as previous cases, the substitution of the Riccati expansion

\[ u = u_0 + u_1 R + u_2 R^2 \]

(36)

into the model will result in

\[ u = \frac{\nu^2}{150} - \frac{1}{5}(a_1 w + \ln w)_x - \frac{1}{6} w_t - \frac{1}{6}(a_1^2 - 8a_0a_2)w_x^2 - a_1 w_{xx} - \frac{2}{3} \frac{w_{xxx}}{w_x} + \frac{1}{2} \frac{w_{xx}^2}{w_x^2} \]

\[ - \frac{2a_2}{5}(5w_{xx} + \nu w_x + 5a_1 w_x^2)R - 2a_2^2 w_x^2 R^2. \]

(37)

However, owing to the nonintegrability of the model, two non-completely consistent equations have to be introduced as

\[ C_x = -S_x - \delta w_x w_{xx} - \frac{\nu^3}{125} \frac{\nu}{5}(2S + \delta w_x^2), \]

(38a)

\[ C_t = S_{xxx} + \frac{15}{2} \delta w_x^2 w_x^2 - \frac{5}{2} \delta w_x^2 - \frac{\nu}{3125}(50S + \nu^2)(50S - \nu^2 + 25C) - \nu S_{xx} \]

\[ + \frac{\delta}{25}(25S - 25C - 11\nu^2)w_x w_{xx} - \frac{1}{5} \delta (C + 9S)w_x^2 + \delta w_x^2 w_{xx} \]

\[ - \frac{1}{5} \nu \delta^2 w_x^4 + \left(2S + 2\delta w_x^2 - C - \frac{11}{25}\nu^2\right)S_x. \]

(38b)

It is noted that if \( \nu = 0 \), then two equations in (38) will be degenerated to one equation (37). While for \( \nu \neq 0 \), the consistent condition \( C_{xx} - C_{xt} \) is not identically zero except for

\[ S_{xxx} + \left(2\delta w_x^2 + 2S + \frac{\nu^2}{225}\right)S_x + \frac{\nu}{45} \delta^2 w_x^4 + \delta w_x^2 w_{xx} + \frac{19}{45} \delta \nu S w_x^2 + \frac{\delta}{225} (\nu^2 + 2025S)w_x w_{xx} \]

\[ + \frac{1}{3} \nu S_x + \frac{5}{6} \nu \delta w_x^2 + \frac{\nu}{28125} (50S^2 - \nu^2) + \frac{15}{2} \nu \delta w_x^2 = 0. \]

(39)

Consequently, the non-integrable KdV-Burgers equation is non-CRE solvable.

### 3 Searching for integrable systems via CRE

The results concerning the examples in the last section provide us an approach to find CRE solvable systems which may be strongly integrable systems. Now, as an illustration, we try to clarify some possible CRE solvable systems from the general fifth order KdV type equation

\[ u_t = u_{xxxxx} + au^2 u_x + bu_x u_{xx} + cu u_{xxx} \]

(40)

with three arbitrary constants \( a, b \) and \( c \). It is known from the Painlevé analysis or the existence of higher order general symmetries, there exist three and only three integrable models from Eq. (40), namely, the Sawada-Kortera (SK), Kaup-Kupershmidt (KK) and fifth order KdV equations. In the following, we are interested in picking up these integrable systems again by finding CRE solvable systems from (40).
Substituting (2) with (3) and \( n = 2 \) (which is determined by the leading order analysis) into (40), we have

\[-2u_2w_xa_2[360a_4^4w_x^4 + 6u_2a_2^2w_x^2(b + 2c) + au_2^2]R^7 + \sum_{i=0}^{6} K_iR^i = 0, \tag{41}\]

where \( K_i \) \((i = 1, \ldots, 6)\) are complicated \( w \)-dependent but \( R \)-independent functions.

Vanishing the coefficient of \( R^7 \) in (41) requires that \( u_2 \) must be proportional to \( w_x^2 \). Because the scaling of \( u \) will not change the solvability of the model, we can simply take

\[u_2 = -2a_2^2w_x^2, \tag{42}\]

without loss of generality and thus

\[a = 3b + 6c - 90. \tag{43}\]

Using the relations (42) and (43), (41) is simplified to

\[-20w_5^5a_2^5(2b + 3c - 84)(2a_1a_2w_x^2 + 2a_2w_{xx} + u_1)R^6 + \sum_{i=0}^{5} K_iR^i = 0. \tag{44}\]

To eliminate the \( R^6 \) term in (44), two cases should be classified: (i) \( 2b + 3c - 84 = 0 \), and (ii) \( 2b + 3c - 84 \neq 0 \).

After tedious analysis, we find that six over-determined equations obtained from vanishing coefficients of \( R^i \) for \( i = 0, 1, 2, \ldots, 5 \) are not consistent for the first case. So we only need to consider the second case. In this latter case, we have

\[u_1 = -2a_2a_1w_x^2 - 2a_2w_{xx}. \tag{45}\]

Substituting (45) into (44) yields

\[-8w_5^3a_2^3(b + c - 30)[(a_2^2 + 8a_0a_2)w_x^4 + 6w_x^2(a_1w_{xx} + u_0) + 4w_xw_{xxx} - 3w_{xx}^2]R^5 + \sum_{i=0}^{4} K_iR^i = 0. \tag{46}\]

To discussion further from (46), we find two nontrivial situations: (1) \( b \neq 30 - c \), and (2) \( b = 30 - c \).

Case (1): \( b \neq 30 - c \). In this case, we have

\[u_0 = \frac{1}{6}(a_1^2 + 8a_2a_0)w_x^2 - a_1w_{xx} - \frac{2}{3}S - \frac{1}{2}w_x^2. \tag{47}\]

Substituting (47) into (46) leads to

\[4w_5^4a_2^4(2b + c - 45)\left(\frac{1}{2}w_x^2 - S\right)R^4 + \sum_{i=0}^{3} K_iR^i = 0. \tag{48}\]

Because the coefficient of \( R^4 \) in (48) is independence of \( t \)-derivative, this term can be canceled only for

\[c = 45 - 2b. \tag{49}\]

After using the relation (49), (48) is simplified to

\[\frac{1}{3}w_3^3a_2^3\left[(18 - b)(40S^3w_x^2 - 16S_{xx} + 20\delta S_{wx}^2 - \delta^2w_x^4 - 4S^2) - 12C\right]R^3 + \sum_{i=0}^{2} K_iR^i = 0. \tag{50}\]
Vanishing the coefficient of $R^3$ in (50), we have

$$C = \frac{18 - b}{12} (40\delta w^2_{xx} - 16S_{xx} + 20\delta S w^2_x - \delta^2 w^4_x - 4S^2).$$

(51)

With the help of (51), we can find that the remaining terms of (50) become

$$(75 - 4b) \left\{ \frac{w_x a_4^2}{3} \left[ \delta^2 w^4_x - \delta (3S w^3_x + 16S w_x w_x^2 + 15w^3_{xx}) + w_x (S^2 + 2S_{xx} x) \right] R^2 + \sum_{i=0}^{1} K_i R^i \right\} = 0.$$

(52)

It is clear that except for

$$b = \frac{75}{4},$$

the $w$ equation (51) is inconsistent with those of $K_2 = K_1 = K_0 = 0$. In conclusion, we obtain a CRE solvable system, which is just the known KK equation (18)

$$u_t = u_{xxxxx} + \frac{15}{2} u_{xxx} + \frac{75}{4} u_{xx} + \frac{45}{4} u^2 u_x.$$

(54)

While the related CRE solvable theorem reads:

**Theorem 9.** If $w$ is a solution of

$$C = \frac{\delta^2}{16} w^2_x + \frac{5}{4} \delta w^2_x + S_{xx} + \frac{5}{2} \delta w^2_x + \frac{1}{4} S^2,$$

then

$$u = -\frac{2}{3} w_{xxx} - a_1 w_{xx} - \frac{1}{6} (8a_0 a_2 + a_1^2) w^2_x - 2a_2 (w_{xx} + a_1 w^2_x) R - 2a_2^2 w^2_x R^2$$

(55)

is a CRE of the KK equation (18).

**Case 2:** $b = 30 - c$. In this case, (18) becomes

$$12w_x a_4^2 (15 - c) \left\{ w^3_x (S + 2u_0) + [(4a_0 a_2 + a_1^2) w^2_x + 2a_1 S] w^4_x + w^4_x w_x (2S + 3a_1 w_{xx}) + w^3_{xx} \right\} R^4$$

$$+ \sum_{i=0}^{3} K_i R^i = 0. \quad (56)$$

The coefficient of $R^4$ in (57) shows that two further subcases should be considered: $c \neq 15$ and $c = 15$.

**Case 2.1:** $c \neq 15$. Vanishing the coefficient of $R^4$ of (57) results in

$$u_0 = \frac{1}{4} w^2_x - \frac{1}{2} w_{xxx} - a_1 w_{xx} - \frac{1}{4} (a_1^2 + 4a_0 a_2) w^2_x + \lambda(t),$$

(58)

where $\lambda(t) \equiv \lambda$ is an integration function of $t$ and should be determined later by consistent conditions.

Due to the result (58), (57) becomes

$$-\frac{1}{4} w_x^2 a_4^2 \left\{ (c - 16) \delta^2 w^4_x - 4\delta (3cS - 2c\lambda - 40S) w^2_x + 4(c - 12)(2S_{xx} - 5\delta w^2_{xx}) + 4S[S(c - 16) + 4c\lambda] + 16(C - 3c\lambda^2) \right\} R^3 + \sum_{i=0}^{2} K_i R^i = 0.$$

(59)

Eliminating the $R^3$ term of (59) yields

$$C = \frac{\delta^2}{16} (16 - c) w^4_x + \frac{\delta}{4} (3cS - 2c\lambda - 40S) w^2_x - \frac{c - 12}{4} (2S_{xx} - 5\delta w^2_{xx}) + 3c\lambda^2 + cS\lambda + 4S^2 - \frac{c}{4} S^2.$$  

(60)
After substituting (60) into (59) and vanishing the remaining terms $R_i$, $i = 2, 1, 0$, we find that the expansion (2) with $n = 2$ is CRE only for

$$c = 10, \lambda_\ell = 0.$$  \hspace{0.5cm} (61)

Otherwise two additional $w$ constraints have to be inserted. Therefore, we have the CRE theorem for the known fifth order KdV equation

$$u_t = u_{xxxxx} + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx}.$$  \hspace{0.5cm} (62)

**Theorem 10.** If $w$ is a solution of

$$C = \frac{3}{8}\delta^2w_x^4 + \frac{5}{2}\delta(S + 2\lambda)w_x^2 + S_x^2\frac{5}{2}\delta w_{xx}^2 + \frac{3}{2}S^2 + 30\lambda^2 + 10\lambda S,$$  \hspace{0.5cm} (63)

then the CRE

$$u = -\frac{1}{4}(a_1^2 + a_0a_2)w_x^2 - a_1w_{xx} - \frac{1}{2}w_{xxx}w_x^2 + \frac{1}{4}w_x^2 + \lambda - 2a_2(a_1w_x^2 + w_{xx})R - 2a_2^2w_x^2R^2$$  \hspace{0.5cm} (64)

is a solution of the fifth order KdV equation (62).

**Case 2.2:** $c = 15$. Applying the similar analysis to this second subcase, we conclude that the only possible CRE model is just the so-called SK equation \[19\]

$$u_t = u_{xxxxx} + 45u^2u_x + 15(uu_{xx})_x.$$  \hspace{0.5cm} (65)

The corresponding CRE theorem is summarized as below.

**Theorem 11.** If $\{w, g\}$ is a solution of the consistent equations

$$C = S_{xx} + 4S^2 + \delta^2w_x^4 - 5\delta(S + 3g)w_x^2 - \frac{5}{2}\delta w_{xx}^2 + 45g^2 + 30gS + 15g_{xx},$$  \hspace{0.5cm} (66a)

$$(g_x^2 + \delta g^2 w_x^2 - 4g^3 - 2gg_{xx} - 2g^2S)_x = 0,$$  \hspace{0.5cm} (66b)

$$g_t = S_x(5g\delta w_x^2 - 9g^2 - 8gS) + g_x\left[\delta^2w_x^4 - \delta(5S - 3g)w_x^2 - 9g^2 - \frac{5}{2}\delta w_{xx}^2 - 6gS + S_{xx} - 3g_{xx} + 4S^2\right]$$

$$-4\delta^2gw_{xx}w_x^2 + 3\delta g(3g + 5S)w_xw_{xx} - gS_{xxx} + \frac{15}{2w_x}\delta gw_{xx}^3,$$  \hspace{0.5cm} (66c)

then the CRE

$$u = g - \frac{1}{3}w_{xxx} - a_1w_{xx} - \frac{1}{3}(2a_0a_2 + a_1^2)w_x^2 - 2a_2(w_{xx} + a_1w_x^2)R - 2a_2^2w_x^2R^2$$  \hspace{0.5cm} (67)

is a solution of the SK equation (65).

To guarantee the SK is CRE solvable, it is sufficient to take $g = 0$ in (66). However, the entrance of $g$ is still consistent because the consistent condition $g_{xxxxx} = g_{xxxxx}$ of (66b) and (66c) is nothing but (66d).

All in all, from the above analysis, we have proved that there are only three CRE solvable systems of the form \[10\] which are just three known integrable systems. Therefore, the conclusion is coincided with that from the Painlevé analysis and/or the existent conditions of high order symmetries.
4 Common interaction behavior between soliton and cnoidal waves for CRE solvable systems

Using the DT or BT related symmetry reduction approach, the interactions between a soliton and a cnoidal wave for the KdV equation, super-symmetric KdV, KP equations and AKNS systems have been discovered in [7–9, 20]. It is interesting that many different CRE solvable systems share similar interactions between a soliton and a cnoidal wave. Moreover, the solutions characterizing the interactions between a soliton and a cnoidal wave for many different CRE solvable systems possess a common form

\[ w = k_1 x + l_1 y + \omega_1 t + W(k_2 x + l_2 y + \omega_2 t), \]

where \( W(k_2 x + l_2 y + \omega_2 t) = W(\xi) \equiv W \) satisfies

\[ W_\xi^2 = C_0 + C_1 W_1 + C_2 W_1^2 + C_3 W_1^3 + C_4 W_1^4, \quad W_1 \equiv W_\xi. \]  

(69)

It is clear that (69) has explicit solutions expressed in terms of Jacobi elliptic functions. If the interaction solution between a soliton and a cnoidal wave is allowed for a given CRE solvable model, the only difference lies in the relations among the constants \( C_0, C_1, C_2, C_3, C_4, k_1, k_2, \omega_1, \omega_2, l_1 \) and \( l_2 \) (\( l_2 = l_1 = 0 \) for 1+1 dimensional systems).

For simplicity, blow we just list the required relations among the constants for CRE solvable systems without details.

The KdV equation (4) asks for

\[
\begin{align*}
C_4 &= \delta, \\
C_1 &= \frac{3k_2 C_0}{k_1} + \frac{2k_1 \lambda + k_3 \delta - 2\omega_1}{k_2^3}, \\
C_2 &= \frac{3k_2^2 C_0}{k_1^2} + \frac{4\lambda + 3k_2^2 \delta}{k_1^4} - \frac{\omega_2 k_1 + 3k_2 \omega_1}{k_1 k_2^3}, \\
C_3 &= \frac{k_2^3 C_0}{k_1^3} + \frac{2\lambda}{k_2 k_1} + \frac{3k_1 \delta}{k_2} - \frac{\omega_2 k_1 + \omega_1 k_2}{k_2^2 k_1^3}, \\
\end{align*}
\]

(70)

while all the other constants remain free.

For the KP equation (9), there exist only three constraints among eleven constants,

\[
\begin{align*}
C_4 &= \delta, \\
C_2 &= \frac{3k_2^2 C_0 k_2}{k_1^4} + \frac{2k_2 C_1 k_2}{k_1} - \frac{(k_1^2 l_2^2 - k_2^2 l_1^2) \gamma}{k_1 k_2^4} - \frac{\omega_2 k_1 - k_2 \omega_1}{k_1 k_2^3}, \\
C_3 &= \frac{-2k_2^3 C_0 k_2}{k_1^4} + \frac{k_2 C_2}{k_1^2} + \frac{2\delta k_1}{k_2} - \frac{\omega_2 k_1 - \omega_1 k_2}{k_2^2 k_1^3} - \frac{2(k_1^2 l_2^2 - k_2^2 l_1^2) \gamma}{k_1^3 k_2^3}. \\
\end{align*}
\]

(71)

The Boussinesq system (13) has the constraints as

\[
\begin{align*}
C_4 &= \delta, \\
C_2 &= \frac{-3k_2^2 C_0 k_2}{k_1^4} + \frac{2k_2 C_1 k_2}{k_1} + \frac{k_2^2 \delta}{k_2^2} - \frac{\omega_2^2 k_1 - k_2^2 \omega_1^2}{k_1 k_2^3}, \\
C_3 &= \frac{-2k_2^3 C_0 k_2}{k_1^4} + \frac{k_2 C_2}{k_1^2} + \frac{2\delta k_1}{k_2} - \frac{2(2k_1^2 \omega_2^2 - k_2^2 \omega_1^2 - k_1 k_2 \omega_1 \omega_2)}{3k_1^3 k_2^3}. \\
\end{align*}
\]

(72)
For the SK model \(65\), the constant constraints can be written as
\[
C_4 = -\delta, \\
C_1 = \frac{k_1}{k_2} (2C_2 k_2^2 - 3C_3 k_1 k_2 - 4\delta k_1^3), \\
\omega_1 = -9\delta^2 k_1^5 - \frac{3}{2} k_1 k_2 \delta (5C_3 k_1^3 - 2C_2 k_2 k_1^2 + 10C_0 k_1^2) - \frac{k_3^3}{2} (9C_0 C_3 k_2^2 - 2k_1 C_2^2 k_2 + 3C_2 C_3 k_1^2), \\
\omega_2 = 45\delta^2 k_2 k_1^4 + 3k_2^3 (C_0 k_1^3 - 5C_2 k_2 k_1^2 + 14C_3 k_1^3) \delta + k_2^3 (C_2 k_2 - 3k_1 C_3)^2. 
\] (73)

For the KK system \(54\), we have the following constant constraints
\[
C_4 = \delta, \\
C_1 = \frac{4k_1^3 \delta}{k_2^2} + \frac{k_1}{k_2^2} (2C_2 k_2 - 3C_3 k_1), \\
C_0 = \frac{3k_1^2 \delta}{k_2^2} + \frac{k_1}{k_2^2} (C_2 k_2 - 2C_3 k_1), \\
\omega_1 = \frac{9}{4} \delta^2 k_1^5 + \frac{3}{4} k_1^3 k_2 \delta (C_2 k_2 - 3C_3 k_1) + \frac{k_1 k_2^2}{16} (C_2 k_2 - 3C_3 k_1)^2, \\
\omega_2 = \frac{9}{4} \delta^2 k_2 k_1^4 + \frac{3}{4} k_1^3 k_2^2 \delta (C_2 k_2 - 3C_3 k_1) + \frac{k_3^3}{16} (C_2 k_2 - 3k_1 C_3)^2. 
\] (74)

For the usual fifth order KdV equation \(62\), there are two sets of constant constraints. The first set reads
\[
C_4 = \delta, \\
C_0 = \frac{k_1}{k_2} (C_1 k_1^3 - C_2 k_1 k_2 + C_3 k_1^2 k_2 - \delta k_1^3), \\
\omega_1 = 30\lambda^2 k_1 + \lambda (15C_2 k_1 k_2^2 + 10k_1^3 k_2 - 10C_1 k_1^3 - 15C_3 k_2 k_1^2) - 2C_1 C_3 k_1 k_2 + 5C_3 \delta k_2^3 k_1^3 + \frac{13}{4} C_2 C_3 k_2 k_1^2 - \frac{15}{4} C_2 \delta k_2^3 k_1^3 + \frac{19}{2} iC_3 k_1^2 k_2 + \frac{1}{4} C_2 C_1 k_2^2 - \frac{13}{2} \delta^2 k_1^5 - \frac{21}{8} C_3^2 k_1^3 k_2^2 - \frac{5}{8} C_2^2 k_1 k_2^4, \\
\omega_2 = \frac{k_2}{8} [k_1^4 (4C_1 C_3 - 3C_2^2) - 2k_1 k_2^2 (8\delta C_1 - 5C_2 C_3) - k_2^3 (15C_2^2 + 4C_2 \delta) + 44k_1^4 (k_2 C_3 - \delta k_1)] + 30k_2 \lambda^2 - 5k_2 \lambda (C_2 k_2^2 - 3k_1 C_2 k_3 + 6 \delta k_1^3), 
\] (75)
and the second is given by
\[
C_4 = \frac{\delta}{9}, \quad C_3 = \frac{4\delta k_1}{9k_2}, \quad C_2 = \frac{2(\delta k_1^2 - 6\lambda)}{3k_2^2}, \\
C_0 = \frac{k_1}{3k_2^2} (12k_1 \lambda + 3C_1 k_3^3 - \delta k_1^3), \\
\omega_1 = \frac{16}{3} \lambda \delta k_1^3 - 12C_1 \lambda k_3^3 - 40k_1 \lambda^2, \quad \omega_2 = 56\lambda^2 k_2. 
\] (76)

For the AKNS system \(16\), nine arbitrary constants \(C_i\), \(i = 0, 1, \ldots, 4, k_1, k_2, \omega_1, \omega_2\) should satisfy three constraints
\[
C_4 = \delta, \\
C_3 = \frac{2k_1 \delta}{k_2} + C_1 k_2^2 k_1^2 - 2C_0 k_1^3 k_2^2 + \frac{8(\delta \lambda k_1 + \omega_1) (k_1 \omega_2 - k_2 \omega_1)}{k_2^2 k_1^4}, \\
C_2 = \frac{k_1^2 \delta}{k_1} + \frac{2C_1 k_2}{k_1} - 3C_0 k_2^2 k_1^2 + \frac{4(\delta \lambda k_1 k_2^2 + 3k_2 \omega_1 - k_1 \omega_2) (k_1 \omega_2 - k_2 \omega_1)}{k_2^2 k_1^4}. 
\] (77)
For the sG equation \( (21) \), there are five constraints in the form of

\[
\begin{align*}
C_0 &= \frac{k_1^2 \omega_1^2}{k_2^2 \omega_2^2} - \frac{2mk_1\omega_1(k_1k_2\lambda^2 - \omega_1\omega_2)}{\omega_2^2k_2^2\lambda(k_1\omega_2 - k_2\omega_1)}, \\
C_1 &= \frac{2\delta k_1\omega_1}{k_2^2 \omega_2^2}(k_1\omega_2 + k_2\omega_1) - \frac{2m\lambda k_1(k_1\omega_2 + 2k_2\omega_1)}{k_2^2 \omega_2^2(k_1\omega_2 - k_2\omega_1)} + \frac{2m\omega_1(2k_1\omega_2 + k_2\omega_1)}{k_2^2 \omega_2^2\lambda(k_1\omega_2 - k_2\omega_1)}, \\
C_2 &= \frac{\delta}{k_2^2 \omega_2^2}(k_1^2\omega_2^2 + k_2^2\omega_2^2 + 4k_1k_2\omega_1\omega_2) - \frac{2m(2k_2\omega_1 + k_1\omega_2)}{k_2^2 \omega_2^2(k_1\omega_2 - k_2\omega_1)} + \frac{2m(2k_2\omega_1 + k_1\omega_2)}{k_2^2 \omega_2^2\lambda(k_1\omega_2 - k_2\omega_1)}, \\
C_3 &= \frac{2\delta}{k_2^2 \omega_2^2}(k_1\omega_2 + k_2\omega_1) - \frac{2m(k_2^2\lambda^2 - \omega_2^2)}{k_2^2 \omega_2^2\lambda(k_1\omega_2 - k_2\omega_1)}, \\
C_4 &= \delta, \\
\end{align*}
\]

(78)

while all the other constants \( \omega_1, \omega_2, k_1, k_2, \lambda, \) and \( \delta \) are free.

The VN system \( (25) \) requires four constraints

\[
\begin{align*}
C_4 &= \delta, \\
C_3 &= \frac{k_2l_2C_1}{k_1l_1} - \frac{(k_2l_1 + k_1l_2)(C_0l_2^2k_2^2 - \delta l_1^2 l_2^2)}{k_1^2 l_1^2 k_2 l_2}, \\
C_2 &= \frac{k_1l_1 \delta}{k_2l_2} + \left(\frac{k_2}{k_1} + \frac{l_2}{l_1}\right) C_1 - \left(\frac{k_2}{k_1} + \frac{l_2}{l_1}\right) C_0, \\
\omega_2 &= \frac{k_2\omega_1}{k_1} - \frac{k_2(k_1l_2 - k_2l_1)[C_0l_2^2k_2^2(k_1l_2 + 2k_2l_1) - l_1k_1(C_1l_2^2k_2^2 - \delta l_1^2 l_2^2)]}{4l_1^2 l_2^2 k_1^2 k_2^2}. \\
\end{align*}
\]

(79)

It should be pointed out that not all the CRE solvable systems possess interaction solutions between a soliton and a cnoidal wave. For instance, two C-integrable systems, the Burgers equation \( (32) \) and the DWW system \( (29) \), do not have the \( w \) solution in the form of \( (68) \). In fact, it is well known that for the Burgers equation, even the single cnoidal wave solution is not allowed.

## 5 Summary and discussions

In summary, with the help of the Riccati equation, we have proposed a simple CRE method by which many kinds of nonlinear systems can be solved. It is found that various integrable models such as the KdV, fifth KdV, KP, Boussinesq, AKNS, NLS, sG, SK, KK, DWW, VN and Burgers models are CRE solvable. Though not all the integrable systems are CRE solvable, it is strongly indicated that the CRE solvable systems are integrable. Therefore, it is feasible to use the CRE method to detect CRE solvable systems first and then one can go further to check other integrable properties. This idea is applied to a general fifth order KdV type system, and it is demonstrated that the only possible CRE solvable systems are the SK, KK and the usual fifth order KdV equations, which coincides with the integrability classifications via the Painlevé analysis and the existence of higher order symmetries.

In fact, only to pick out CRE solvable models from general systems, one may use some special solutions of the Riccati equation. For instance, taking \( a_1 = 0, \ a_0 = 1 \) and \( a_2 = -1 \), the Riccati equation \( (8) \) possesses the solution \( R = \tanh(w) \), and accordingly, the simplified CRE can be termed as consistent tanh expansion (CTE). Obviously, the CRE solvable systems are CTE solvable, and vice versa.

For many CRE solvable systems, there is a common interesting exact interaction solution between a soliton and a cnoidal periodic wave, determined by Eqs. \( (68) \) and \( (69) \). The only differences are the
constant constraints related to the possible dispersion relations and average backgrounds. The detailed interactions between solitons and cnoidal waves for the KdV equation, the KP equation and the AKNS system have been discussed in several references [7–9, 20]. The more about the method and the associated $w$ equations needs further study.

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