Three Favorite Edges Occurs Infinitely Often for One-Dimensional Simple Random Walk

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Abstract
For a one-dimensional simple symmetric random walk $(S_n)$, an edge $x$ (between points $x - 1$ and $x$) is called a favorite edge at time $n$ if its local time at $n$ achieves the maximum among all edges. In this paper, we show that with probability 1 three favorite edges occurs infinitely often. Our work is inspired by Tóth and Werner (Comb Probab Comput 6:359–369, 1997), and Ding and Shen (Ann Probab 46:2545–2561, 2018), disproves a conjecture mentioned in Remark 1 on page 368 of Tóth and Werner (1997).

Keywords Random walk · Favorite edge · Invariance principle for one-side local times · Wiener process

Mathematics Subject Classification 60F15 · 60J55
1 Introduction

Let \((S_n)_{n \in \mathbb{N}}\) be a one-dimensional simple symmetric random walk with \(S_0 = 0\). Following Tóth and Werner [35], we define for any \(i \geq 1\),
\[
\tilde{S}_i := \frac{S_i + S_{i-1} + 1}{2},
\]
which characterizes the edge of \(i\)-th jump (edge \(x\) is between points \(x - 1\) and \(x\)), and also define the “local time on the edge \(x\) at time \(n\)” as follows:
\[
L(x, n) := \# \{1 \leq j \leq n : \tilde{S}_j = x\}.
\] (1.1)

Hereafter, \(#D\) denotes the cardinality of the set \(D\). An edge \(x\) is called a favorite (or most visited) edge of the random walk at time \(n\) if
\[
L(x, n) = \sup_{y \in \mathbb{Z}} L(y, n).
\]

The set of favorite edges of the random walk at time \(n\) is denoted by \(K(n)\). \((K(n))_{n \geq 1}\) is called the favorite edge process of the one-dimensional simple symmetric random walk. We say that three favorite edges occurs at time \(n\) if \(#K(n) = 3\).

**Theorem 1.1** For a one-dimensional simple symmetric random walk, with probability 1, three favorite edges occurs infinitely often.

Theorem 1.1 complements the result in [35] which showed that eventually there are no more than three favorite edges, and disproves a conjecture mentioned in [35, Remark 1 on page 368].

For the related problem of the number of favorite sites of one-dimensional simple symmetric random walks, there are many more references (see Shi and Tóth [33] for an overview). This problem was posed by Erdős and Révész [16–18, 32]. Tóth [34] proved that there are no more than three favorite sites eventually. Ding and Shen [12] proved that with probability 1 three favorite sites occur infinitely often.

Besides the number of favorite sites, a series of papers focus on the asymptotic behavior of favorite sites, see [3, 7, 8, 28]. In addition, there are a number of papers on favorite sites for other processes including Brownian motion, symmetric stable process, Lévy processes, random walks in random environments and so on, see [2, 4, 13–15, 20, 21, 24, 26, 27, 29].

For papers on favorite sites of simple random walks in higher dimensions, we refer to [1, 10, 11, 30].

Our proof of Theorem 1.1 is inspired by [12], which in turn was inspired by [34, 35]. Following [34], we define the number of upcrossings and downcrossings of the site \(x\) by time \(n\) to be
\[
\xi_U(x, n) := \# \{0 < k \leq n : S_k = x, S_{k-1} = x - 1\},
\]
\[
\xi_D(x, n) := \# \{0 < k \leq n : S_k = x, S_{k-1} = x + 1\},
\]
respectively. It is easy to check that

$$\xi_U(x, n) - \xi_D(x - 1, n) = \mathbf{1}_{0 < x \leq S_n} - \mathbf{1}_{[S_n < x \leq 0]}.$$  \hfill (1.2)

Using (1.1) and (1.2), we can easily get (see [34])

$$L(x, n) = \# \{0 < j \leq n : \tilde{S}_j = x\}$$

$$= \# \{0 < j \leq n : S_j = x, S_{j-1} = x - 1\}$$  
$$+ \# \{0 < j \leq n : S_j = x - 1, S_{j-1} = x\}$$

$$= \xi_U(x, n) + \xi_D(x - 1, n)$$

$$= 2\xi_U(x, n) + \mathbf{1}_{[S_n < x \leq 0]} - \mathbf{1}_{[0 < x \leq S_n]}$$

$$= 2\xi_D(x - 1, n) + \mathbf{1}_{[0 < x \leq S_n]} - \mathbf{1}_{[S_n < x \leq 0]}.$$ \hfill (1.3)

For \( r \geq 1 \), let \( f(r) \) be the (possibly infinite) number of times when there are exactly \( r \) favorite edges:

$$f(r) := \# \{n \geq 1 : \#K(n) = r\}.$$

We will show that

$$f(3) = \infty \text{ with probability 1},$$ \hfill (1.4)

which implies Theorem 1.1. The idea for the proof comes from [12]. In the proof, we need a transience result for the favorite edge process of one-dimensional simple symmetric random walks. To prove this transience result, we establish an invariance principle for one-side local times of random walks.

We mention in passing that, similar to [12, 34], we could have defined the following quantity related \( f(r) \):

$$\tilde{f}(r) := \# \{n \geq 1 : \tilde{S}_n \in K(n) \text{ and } \#K(n) = r\},$$

which is the number of times at which a new favorite edge appears, tied with \( r - 1 \) other favorite edges. Using the recurrence of one-dimensional simple symmetric random walk, one can easily see that \( f(3) = \infty \) a.s., is equivalent to \( \tilde{f}(3) = \infty \) a.s. See for instance, the paragraph below [35, Theorem 1.1].

The rest of the paper is organized as follows. In Sect. 2, we establish an invariance principle for one-side local times of random walks, and then use it to prove the transience of the favorite edge process. In Sect. 3, we set up the framework for our proof for Theorem 1.1. In Sect. 4, we give the proof of Theorem 1.1. We emphasize that the idea for the proof of Theorem 1.1 comes from [12], and our main contributions are the invariance principle for one-side local times and the transience of the favorite edge process. In the final section, we give a remark.
2 The Transience of the Favorite Edge Process

In this section, we study the transience of the favorite edge process and prove the following:

**Theorem 2.1**  For any \( \gamma > 11 \), we have

\[
\liminf_{n \to \infty} \frac{\tilde{U}(n)}{\sqrt{n(\log n)^{-\gamma}}} = \infty \quad \text{a.s.,}
\]

(2.1)

where \( \tilde{U}(n) := \min \{|x| : x \in K(n)\} \).

To prove the above theorem, we will establish an invariance principle for one-side local times of random walks in Sect. 2.2. The proof of Theorem 2.1 will be given in Sect. 2.3. In Sect. 2.1, we give a brief introduction for invariance principle for (two-side) local times.

2.1 Invariance Principle for Local Times

Recall that the (site) local time process of the random walk \((S_n, n \geq 0)\) is defined by

\[
\xi(x, n) := \# \{k : 0 \leq k \leq n, S_k = x\}, \quad x \in \mathbb{Z}, \quad n \geq 1.
\]

Define \( \xi^*(n) := \sup_{x \in \mathbb{Z}} \xi(x, n) \).

Let \((W(t))_{t \geq 0}\) be a one-dimensional standard Brownian motion (Wiener process). Recall that the local time process \((\eta(x, t))_{t \geq 0, x \in \mathbb{R}}\) of \((W(t))_{t \geq 0}\) is defined by

\[
\eta(x, t) := \lim_{\varepsilon \downarrow 0} \frac{1}{2 \varepsilon} \int_{0}^{t} \mathbf{1}_{(x-\varepsilon, x+\varepsilon)}(W(s)) \, ds = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0}^{t} \mathbf{1}_{[x,x+\varepsilon]}(W(s)) \, ds.
\]

(2.2)

Révész [31] established the following strong invariance principle with a rate of convergence: On a rich enough probability space, as \( n \to \infty \),

\[
\sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = o(n^{\frac{1}{4} + \varepsilon}) \quad \text{a.s.}
\]

(2.3)

for any \( \varepsilon > 0 \). Csörgő and Horváth [9, Theorems 1 and 2] showed that the Révész’s result is the best possible.

2.2 Invariance Principle for One-Side Local Times

Define the one-side local time of the Wiener process by

\[
\eta_R(x, t) := \lim_{\varepsilon \downarrow 0} \frac{1}{2 \varepsilon} \int_{0}^{t} \mathbf{1}_{[x,x+\varepsilon]}(W(s)) \, ds.
\]
Then by (2.2), we get that for any \( x \in \mathbb{R} \) and \( t \geq 0 \),
\[
\eta_R(x, t) = \frac{1}{2} \eta(x, t). \tag{2.4}
\]

The goal in this subsection is to extend (2.3) to the one-side case, that is, prove the following result.

**Theorem 2.2** On a rich enough probability space \((\Omega, \mathcal{F}, P)\), one can define a Wiener process \((W(t))_{t \geq 0}\) and a one-dimensional simple symmetric random walk \((S_k)_{k \in \mathbb{N}}\) with \( S_0 = 0 \), such that for any \( \varepsilon > 0 \), as \( n \to \infty \), we have
\[
\sup_{x \in \mathbb{Z}} |\xi_D(x, n) - \eta_R(x, n)| = o(n^{\frac{1}{4} + \varepsilon}) \text{ a.s.} \tag{2.5}
\]

Before giving the proof of Theorem 2.2, we prove the following lemma.

**Lemma 2.3** For any \( \varepsilon > 0 \), we have
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{Z}} |\xi_D(x + 1, n) - \xi_D(x, n)| \leq \frac{C}{n^{\frac{1}{4} + \varepsilon}} \quad \text{a.s.} \tag{2.5}
\]

**Proof** The idea of the proof comes from Csáki and Révész [6, Lemma 5], where the corresponding problem for (two-side) local times was considered in a more general setting. First we show that, for all \( m \geq 2 \) and all \( \delta > 0 \), there exists a constant \( C \) (which may depends on \( m \) but not on \( n \)) such that
\[
E(|\xi_D(1, n) - \xi_D(0, n)|^m) \leq C n^{\frac{m}{4} + \delta}. \tag{2.6}
\]

Define
\[
T_k := \sum_{i=1}^{k} (\xi_D(1, \alpha_i) - \xi_D(1, \alpha_{i-1}) - 1), \tag{2.7}
\]
where \( \alpha_0 = 0, \alpha_i = \min\{j > \alpha_{i-1} : S_j = 0, S_{j-1} = 1\}, \forall i \geq 1 \). We claim that
\[
E(\xi_D(1, \alpha_1)) = 1, \quad E(\xi_D^m(1, \alpha_1)) < +\infty, \forall m \geq 2. \tag{2.8}
\]

To prove this claim, we follow Kesten and Spitzer [23, Lemma 2] and define, for \( x, y \in \mathbb{Z}, \tau_0(x) := 0, \tau_i(x) := \min\{n > \tau_{i-1}(x) : S_n = x, S_{n-1} = x + 1\}, \forall i \geq 1 \). Note that \( \alpha_i = \tau_i(0) \). For \( j \geq 0 \),
\[
M_j(x, y) := \sum_{\tau_j(x) < n \leq \tau_{j+1}(x)} \mathbf{1}_{[S_n = y, S_{n-1} = y+1]}
\]
is the number of downcrossings to \( y \) between the \( jth \) and \((j + 1)th\) downcrossings to \( x \). In particular, \( M_0(0, 1) = \xi_D(1, \alpha_1) \). By the strong Markov property, we know
that the distribution of $M_j(0, y)$, $j \geq 0$, is independent of $j$, and the distribution of $M_j(x, y)$, $j \geq 1$, is independent of $j$. Thus we can define for $j \geq 1$,

\[ p(x, y) := P(M_j(x, y) \neq 0) = P(S_n = y, S_{n-1} = y + 1 \text{ for some } n \text{ with } \tau_j(x) < n \leq \tau_{j+1}(x)). \]

We claim that, for any $x, y \in \mathbb{Z}$ with $|x - y| = 1$, it holds that $p(x, y) = p(y, x) = 1/2$. To prove this claim, it suffices to show that $p(0, 1) = p(1, 0) = 1/2$. When $\tau_1(0) < \infty$, there exists $n_0 = \min\{k \geq 0 : S_k = 1 \} < \tau_1(0)$. Then $S_{n_0} = 1, \{S_{n_0+1} = 0\} \subset \{M_0(0, 1) = 0\}$ and $\{S_{n_0+1} = 2\} \subset \{M_0(0, 1) \neq 0\}$. Since $P(\tau_1(0) < \infty) = 1$, we have

\[ p(0, 1) = P(M_0(0, 1) \neq 0, \tau_1(0) < \infty) = P(S_{n_0+1} = 0, M_0(0, 1) \neq 0, \tau_1(0) < \infty) + P(S_{n_0+1} = 2, M_0(0, 1) \neq 0, \tau_1(0) < \infty) = P(S_{n_0+1} = 2) = \frac{1}{2}. \]

Similarly, there exists $n_1 = \min\{k \geq 0 : S_k = 1, S_{k-1} = 2\} < \tau_1(1)$, then

\[ p(1, 0) = P(S_{n_1+1} = 2, M_1(1, 0) \neq 0) + P(S_{n_1+1} = 0, M_1(1, 0) \neq 0) = P(S_{n_1+1} = 0) = \frac{1}{2}. \]

Combining the claim above with the strong Markov property and the fact $S_{\tau_{j+1}(x)} = S_{\tau_{j+1}(x)} = x$, we get

\[ P(M_j(0, 1) = k) = \begin{cases} 1 - p(0, 1) = \frac{1}{2}, & \text{if } k = 0, \\ p(0, 1)[1 - p(1, 0)]^{k-1} p(1, 0) = \frac{1}{2k+1}, & \text{if } k \geq 1. \end{cases} \]

It follows that

\[ E(\xi_D(1, \alpha_1)) = E(M_j(0, 1)) = \sum_{k=1}^{\infty} k \cdot \frac{1}{2k+1} = 1, \]

\[ E(\xi_D^n(1, \alpha_1)) = E(M_j^n(0, 1)) = \sum_{k=1}^{\infty} k^m \cdot \frac{1}{2k+1} < +\infty, \]

which implies that (2.8) holds.

It follows from the strong Markov property that $T_k$ is a sum of i.i.d. r.v.’s with mean 0 and finite moments of all orders. Therefore by Chung [5] and the $L^p$-maximal inequality for martingales, we have

\[ E(|T_k|^m) \leq C_1 k^\frac{m}{2} \]  \hspace{1cm} (2.9)
and for any $\delta_1 > 0$,
\[
E \left( \max_{k \leq n^{1/2+\delta_1}} |T_k|^m \right) \leq C_2 n^{m + m\delta_1/2}. \tag{2.10}
\]

Note that
\[
\xi_D(1, \alpha_{D_n}) - \xi_D(0, n) \leq \xi_D(1, n) - \xi_D(0, n) \leq (\xi_D(0, n) + 1) + 1,
\]

where $D_n = \xi_D(0, n)$. Thus on the event $\{D_n + 1 \leq n^{1/2+\delta_1}\}$, we have
\[
|\xi_D(1, n) - \xi_D(0, n)| \leq \max_{1 \leq k \leq n^{1/2+\delta_1}} |T_k| + 1.
\]

Hence
\[
E(|\xi_D(1, n) - \xi_D(0, n)|^m) 
\leq C E \left( \max_{1 \leq k \leq n^{1/2+\delta_1}} (|T_k| + 1)^m \right) + n^m P \left( D_n + 1 \geq n^{1/2+\delta_1} \right). \tag{2.11}
\]

Let $N_n = N(0, n) = \#\{k : 0 < k \leq n, S_k = 0\}$, then according to Kesten and Spitzer [23], $E(N_n^m) = O(n^{m\varphi})$. Combining this with the fact that $0 \leq D_n \leq N_n$, we get that
$E(D_n^m) = O(n^{m\varphi})$. Then by Markov’s inequality, (2.10) and (2.11), letting $\delta_1$ be small enough, we obtain (2.6).

By repeating the argument above and noticing that $p(-1, 0) = p(0, -1) = 1/2$, we can obtain the inequality (2.6) with $\xi_D(1, n)$ replaced by $\xi_D(-1, n)$.

It is easy to see that for any $x \in \mathbb{Z}$,
\[
E(|\xi_D(x + 1, n) - \xi_D(x, n)|^m) \leq C n^{m/2 + \delta}, \tag{2.12}
\]

where the constant $C$ does not depend on $x$, because $\xi_D(x + 1, n) - \xi_D(x, n)$ is stochastically smaller than $\max(|\xi_D(1, n) - \xi_D(0, n)|, |\xi_D(-1, n) - \xi_D(0, n)|)$.

Choosing $m = \frac{2 + \delta}{\varepsilon}$, we obtain by Markov’s inequality that
\[
P\left( |\xi_D(x + 1, n) - \xi_D(x, n)| > n^{1/2 + \varepsilon} \right) \leq \frac{C}{n^{\varepsilon}}
\]

and hence
\[
P\left( \sup_{|x| < (n \log n)^{1/2}} |\xi_D(x + 1, n) - \xi_D(x, n)| > n^{1/2 + \varepsilon} \right) \leq \frac{2C(\log n)^{1/2}}{n^{\varepsilon/2}}.
\]
Therefore, by the Borel–Cantelli lemma, we get that
\[
\lim_{n \to +\infty} \sup_{|x| \leq (n \log n)^{\frac{1}{2}}} \frac{\left| \xi_D(x + 1, n) - \xi_D(x, n) \right|}{n^{1+\epsilon}} = 0 \quad a.s.
\]

The law of the iterated logarithm for \( S_n \) implies that \( \xi_D(x, n) = 0 \) a.s. for \( |x| \geq (n \log n)^{\frac{1}{2}} \) and \( n \) sufficiently large, hence (2.5) holds.

**Proof of Theorem 2.2** This proof is inspired by Révész [31], and Csörgő and Horváth [9]. Let \((W(t))_{t \geq 0}\) be a Wiener process and define \(\tau_0 := 0, \tau_1 := \inf\{t : t > 0, |W(t)| = 1\}, \tau_n := \inf\{t : t > \tau_{n-1}, |W(t) - W(\tau_{n-1})| = 1\}, \forall n \geq 2\). Then \(X_i = W(\tau_i) - W(\tau_{i-1})(i \geq 1)\) are i.i.d. r.v.'s with \(P\{X_i = 1\} = P\{X_i = -1\} = 1/2\), and \(\tau_i - \tau_{i-1}(i \geq 1)\) are i.i.d. r.v.'s with \(E(\tau_i - \tau_{i-1}) = 1\) and \(E(\tau_i - \tau_{i-1})^2 < \infty\). Put \(\sigma^2 = E(\tau_1 - 1)^2\). Define \(S_k = X_1 + \cdots + X_k = W(\tau_k)\).

Let \(a_i(x) = \eta(x, \tau_{v(i)+1}) - \eta(x, \tau_{v(i)-1}), b_i(x) = \eta_R(x, \tau_{v(i)+1}) - \eta_R(x, \tau_{v(i)-1})(i \in \mathbb{Z}^+)\), where \(v(1) = \min\{k \geq 0, W(\tau_k) = S_k = x\}, v(n) = \min\{k > v(n-1), W(\tau_k)(= S_k) = x\}, \forall n \geq 2\). Then by (2.4), we have
\[
b_i(x) = \frac{a_i(x)}{2}. \tag{2.13}
\]

Kesten [22] showed
\[
\limsup_{n \to \infty} (2n \log \log n)^{-\frac{1}{2}} \cdot \sup_{x \in \mathbb{Z}} \xi(x, n) = 1 \quad a.s. \tag{2.14}
\]

Csörgő and Horváth [9, (2.7)] says that
\[
\max_{-n \leq x \leq n} \left| \sum_{i=1}^{\xi(x,n)} a_i(x) - \eta(x, \tau_n) \right| = O(\log n) \quad a.s.,
\]
which together with (2.4) and (2.13) implies that
\[
\max_{-n \leq x \leq n} \left| \sum_{i=1}^{\xi(x,n)} b_i(x) - \eta_R(x, \tau_n) \right| = O(\log n) \quad a.s. \tag{2.15}
\]

By [9, (2.8)] and (2.13), we have
\[
P \left\{ \max_{-k \leq x \leq k} \left| \sum_{i=1}^{k} \left( b_i(x) - \frac{1}{2} \right) \right| > C_1(k \log k)^{\frac{1}{2}} \right\} \leq C_2k^{-2}. \tag{2.16}
\]
Combining this with the Borel–Cantelli lemma and then using (2.14), we get that
\[
\limsup_{n \to \infty} \max_{-n \leq x \leq n} \left| \sum_{i=1}^{\xi(x,n)} b_i(x) - \frac{\xi(x,n)}{2} \right| \leq C_3 \quad \text{a.s.} \tag{2.17}
\]
Since \(\tau_i - \tau_{i-1}, i \geq 1\), are i.i.d. r.v.’s with \(E \tau_1 = 1\) and \(\sigma^2 = E(\tau_1 - 1)^2 < \infty\), by the law of iterated logarithm, we have
\[
\limsup_{n \to \infty} \frac{|\tau_n - n|}{\sqrt{2\sigma^2 n \log \log n}} = 1 \quad \text{a.s.} \tag{2.18}
\]
By \([9, (2.11)]\) and (2.4), we have
\[
\limsup_{n \to \infty} \frac{\sup x \left| \eta_R(x, n \pm g(n)) - \eta_R(x, n) \right|}{n^{1/3} \cdot (\log n)^{1/2} \cdot (\log \log n)^{1/4}} \leq C_4 \quad \text{a.s.}
\]
where \(g(n) = \left(4\sigma^2 n \log \log n\right)^{1/4}\). (Note that there is a minor typo in the line below \([9, (2.11)]\), the \(-\frac{1}{2}\) there should be \(\frac{1}{2}\).) Thus by (2.18) we have
\[
\limsup_{n \to \infty} \frac{\sup x \left| \eta_R(x, \tau_n) - \eta_R(x, n) \right|}{n^{1/3} \cdot (\log n)^{1/2} \cdot (\log \log n)^{1/4}} \leq C_5 \quad \text{a.s.} \tag{2.19}
\]
Note that \(\xi_D(x, n) = 0\) if \(|x| > n\) and that \(\lim_{n \to \infty} \sup_{|x| > n} \eta_R(x, n) = 0\) a.s. Thus
\[
\limsup_{n \to \infty} \frac{\sup_{|x| > n} \left| \xi_D(x, n) - \eta_R(x, n) \right|}{n^{1/3} + \varepsilon} = 0 \quad \text{a.s.} \tag{2.20}
\]
Now by (2.15), (2.17), (2.19), (2.20) and Lemma 2.3, we obtain
\[
\limsup_{n \to \infty} \frac{\sup x \left| \xi_D(x, n) - \eta_R(x, n) \right|}{n^{1/3} + \varepsilon} \leq \limsup_{n \to \infty} \frac{\sup_{x \leq n} \left| \xi_D(x, n) - \frac{\xi(x,n)}{2} \right| + \sup_{x \leq n} \left| \xi(x,n) - \sum_{i=1}^{\xi(x,n)} b_i(x) \right|}{n^{1/3} + \varepsilon} + \limsup_{n \to \infty} \frac{\sup_{x \leq n} \left| \sum_{i=1}^{\xi(x,n)} b_i(x) - \eta_R(x, \tau_n) \right|}{n^{1/3} + \varepsilon} + \sup_{|x| > n} \frac{\left| \xi_D(x, n) - \eta_R(x, n) \right|}{n^{1/3} + \varepsilon} \leq \limsup_{n \to \infty} \frac{\sup x \left| \xi_D(x, n) - \xi_D(x-1,n) \right| + \sup_{x \leq n} \left| \xi(x,n) - \sum_{i=1}^{\xi(x,n)} b_i(x) \right|}{n^{1/3} + \varepsilon} + \sup_{x \leq n} \frac{\left| \xi(x,n) - \xi(x,n) \right|}{n^{1/3} + \varepsilon}
\]
\[
\begin{align*}
&\limsup_{n \to \infty} \sup_{x \leq n} \left| \sum_{i=1}^{\xi(x,n)} b_i(x) - \eta_R(x, \tau_n) \right| + \sup_{x \leq n} \left| \eta_R(x, \tau_n) - \eta_R(x, n) \right| \\
&+ \limsup_{n \to \infty} \sup_{x \leq n} \left| \xi_D(x, n) - \eta_R(x, n) \right| \\
&+ \limsup_{n \to \infty} \sup_{|x| > n} \left| \xi_D(x, n) - \eta_R(x, n) \right| \\
&= 0 \quad a.s.,
\end{align*}
\]

where we used the following fact
\[
\left| \xi_D(x, n) - \xi_D(x, n) \right| = \left| \xi_D(x, n) - \xi_U(x, n) \right| \leq \left| \xi_D(x, n) - \xi_D(x - 1, n) \right| + 1.
\]

The proof is complete. \(\square\)

### 2.3 Proof of Theorem 2.1

Set
\[
\begin{align*}
\eta^*_R(n) &:= \sup_{x \in \mathbb{R}} \eta_R(x, n), & \eta^*(n) &:= \sup_{x \in \mathbb{R}} \eta(x,n), \\
\xi^*_D(n) &:= \sup_{x \in \mathbb{Z}} \xi_D(x, n), & T_r := \inf \{ n > 0 : \eta_R(0, n) \geq r \}, \\
I_R(h, n) &:= \sup_{|x| \leq h} \eta_R(x, n),
\end{align*}
\]

and
\[
\mathcal{V}(n) := \{ x \in \mathbb{Z} : \eta_R(x, n) = \eta^*_R(n) \}, \quad \mathcal{U}(n) := \{ x \in \mathbb{Z} : \xi_D(x, n) = \xi^*_D(n) \}.
\]

It is easy to see that Theorem 2.1 is a consequence of the following two propositions.

**Proposition 2.4** If \( x \in \mathcal{K}(n) \), then \( x - 1 \in \mathcal{U}(n) \).

**Proposition 2.5** For any \( \gamma > 1 \), we have
\[
\liminf_{n \to \infty} \frac{U(n)}{\sqrt{n}(\log n)^{-\gamma}} = \infty \quad a.s., \quad (2.21)
\]

where \( U(n) := \min \{|x| : x \in \mathcal{U}(n)\} \).

#### 2.3.1 Proof of Proposition 2.4

Assume that \( x \in \mathcal{K}(n) \), that is, \( x \) is a favorite edge at time \( n \). We want to prove that \( x - 1 \in \mathcal{U}(n) \). Let \( \xi_D(x - 1, n) = h \) for some nonnegative integer \( h \). Then by (1.3), we know that \( L(x, n) \in \{2h - 1, 2h, 2h + 1\} \). We will prove \( x - 1 \in \mathcal{U}(n) \) by contradiction. Suppose that \( x - 1 \notin \mathcal{U}(n) \). Then for any \( y \in \mathcal{U}(n) \), we have \( \xi_D(y, n) \geq h + 1 \).
Case 1. $L(x, n) = 2h - 1$ or $2h$. By (1.3), we get that for any $y \in \mathcal{U}(n)$,

$$L(y + 1, n) \geq 2\xi_D(y, n) - 1 \geq 2(h + 1) - 1 = 2h + 1.$$ 

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Case 2. $L(x, n) = 2h + 1$. Now by (1.3), we know that $0 < x \leq S_n$. For any $y \in \mathcal{U}(n)$, we have the following two subcases:

Case 2.1. $y \leq 0$ or $y > S_n$. Now by (1.3), we get that

$$L(y + 1, n) = 2\xi_D(y, n) \geq 2(h + 1) = 2h + 2.$$ 

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Case 2.2. $0 < y \leq S_n$. Now by (1.3), we get that

$$L(y + 1, n) = 2\xi_D(y, n) + 1 \geq 2(h + 1) + 1 = 2h + 3.$$ 

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Thus we must have $x - 1 \in \mathcal{U}(n)$. The proof is complete. \hfill \Box

2.3.2 Proof of Proposition 2.5

To prove Proposition 2.5, we need several lemmas. By [3, (5.1)] and (2.4), we have

Lemma 2.6 For any $\alpha > 5$ and $\varepsilon > 0$, there exists $n_0$ such that, with probability one, we have

$$\eta^*_R(n) > I_R \left( \frac{\sqrt{n}}{(\log n)^{2\alpha+1+\varepsilon}}, n \right) + \frac{1}{2} n^{\frac{1}{2} - \varepsilon}, \quad n \geq n_0.$$ 

By [3, Lemma 5.3] and (2.4), we have

Lemma 2.7 For every $\varepsilon > 0$,

$$\sup_{k \in \mathbb{Z}, t \leq n} \sup_{x \in [k, k+1]} |\eta_R(x, t) - \eta_R(k, t)| = o(n^{\frac{1}{2} + \varepsilon}) \quad a.s.$$ 

Lemma 2.8

$$|\eta^*_R(n) - \xi^*_D(n)| = o(n^{\frac{1}{2} + \varepsilon}) \quad a.s.$$ 

Proof We have

$$\eta^*_R(n) - \xi^*_D(n) = \sup_{x \in \mathbb{R}} \eta_R(x, n) - \sup_{x \in \mathbb{Z}} \xi_D(x, n)$$

$$= \sup_{x \in \mathbb{R}, y \in \mathbb{Z}, |x - y| \leq 1} [\eta_R(x, n) - \xi_D(y, n) + \xi_D(y, n)] - \sup_{x \in \mathbb{Z}} \xi_D(x, n)$$

$\square$ Springer
Similarly, we have
\[
\xi_D^*(n) - \eta_R(n) \leq \sup_{x \in \mathbb{R}, y \in \mathbb{Z}, |x-y| \leq 1} |\eta_R(x, n) - \xi_D(y, n)|.
\]

Hence
\[
|\eta_R(n) - \xi_D^*(n)| \leq \sup_{x \in \mathbb{R}, y \in \mathbb{Z}, |x-y| \leq 1} |\eta_R(x, n) - \xi_D(y, n)|.
\]

Then by Lemma 2.7 and Theorem 2.2, we get
\[
|\eta_R(n) - \xi_D^*(n)| \leq \sup_{x \in \mathbb{R}, y \in \mathbb{Z}, |x-y| \leq 1} |\eta_R(x, n) - \eta_R(y, n) + \eta_R(y, n) - \xi_D(y, n)|
\]
\[
\leq \sup_{x \in \mathbb{R}, y \in \mathbb{Z}, |x-y| \leq 1} |\eta_R(x, n) - \eta_R(y, n)| + \sup_{y \in \mathbb{Z}} |\eta_R(y, n) - \xi_D(y, n)|
\]
\[
= o(n^{1/4 + \varepsilon}).
\]

The proof is complete. \(\square\)

**Proof of Proposition 2.5** By Theorem 2.2 we can define a simple symmetric random walk \(S_n\) and a Wiener process \(W(t)\) on a common probability space such that for each \(\varepsilon > 0\),
\[
\sup_{x \in \mathbb{Z}} |\xi_D(x, n) - \eta_R(x, n)| = o(n^{1/4 + \varepsilon}) \quad a.s. \tag{2.22}
\]

For any \(\alpha > 5\) and \(\varepsilon > 0\), let \(K_n = \max_{x \in \mathbb{Z}, |x| \leq \sqrt{n}(\log n)^{-2\alpha+1+\varepsilon}} \xi_D(x, n)\). By (2.22), Lemmas 2.6 and 2.8,
\[
\xi_D^*(n) \geq \eta_R(n) - cn^{1/4 + \varepsilon}
\]
\[
\geq I_R \left( \frac{\sqrt{n}}{(\log n)^{2\alpha+1+\varepsilon}}, n \right) + \frac{1}{2} n^{1/2 - \varepsilon} - cn^{1/4 + \varepsilon}
\]
\[
\geq K_n + \frac{1}{2} n^{1/2 - \varepsilon} - 2cn^{1/4 + \varepsilon} > K_n
\]

for \(n\) sufficiently large. Thus the most visited edges of \(S_n\) must be larger in absolute value than \(\sqrt{n}(\log n)^{-(2\alpha+1+\varepsilon)}\) for \(n\) large. For any \(\gamma > 11\), choosing \(\alpha\) and \(\varepsilon\) so that \(2\alpha + 1 + \varepsilon < \gamma\), we obtain (2.21). The proof is complete. \(\square\)
Three Consecutive Favorite Edges

We define the inverse edge local times by

\[ T_U(x, k) := \min \{ n \geq 1 : \xi_U(x, n) = k \} \] and \[ T_D(x, k) := \min \{ n \geq 1 : \xi_D(x, n) = k \} \].

For any \( x \in \mathbb{Z} \), define

\[
\begin{align*}
    u(x) & := \sum_{n=1}^{\infty} \mathbf{1}_{\{S_{n-1} = x - 2, S_n = x - 1, x \in K(n), \#K(n) = 3\}} \\
& = \sum_{k=1}^{\infty} \mathbf{1}_{\{x \in K(T_U(x-1, k)), \#K(T_U(x-1, k)) = 3\}} \\
& = \sum_{k=0}^{\infty} \sum_{h=1}^{\infty} \mathbf{1}_{\{x \in K(T_U(x-1, k+1)), \#K(T_U(x-1, k+1)) = 3, L(x, T_U(x-1, k+1)) = h\}}.
\end{align*}
\]

Thus \( f(3) \geq \sum_{x \in \mathbb{Z}} u(x) \).

For \( x \in \mathbb{Z} \), \( h, k \in \mathbb{N} \), we define

\[
A_{x,h}^{(k)} := \{K(T_U(x - 1, k + 1)) = \{x, x + 1, x + 2\}, L(x, T_U(x - 1, k + 1)) = h\}.
\]

(3.1)

Here we use \( T_U(x - 1, k + 1) \) instead of \( T_U(x, k + 1) \), which was used in [12], due to the following two reasons:

(i) If \( x > 0 \), then by the definition of local time on the edge \( x \) at the time \( n = T_U(x, k + 1) \), the set of favorite edges \( K(n) \) is not equal to \( \{x, x + 1, x + 2\} \), since \( L(x + 1, n) \) and \( L(x + 2, n) \) are even numbers and \( L(x, n) \) is an odd number.

(ii) \( T_U(x - 1, k + 1) \) is useful to obtain the lower bound on the first moment in Sect. 3.1.

Note that the definition of \( T_U(x - 1, k + 1) \) implies \( S_{T_U(x - 1, k + 1)} = x - 1 \) and \( \xi_U(x - 1, T_U(k + 1, x - 1)) = k + 1 \). Thus by (1.3), we have for \( x \geq 1 \),

\[
L(x - 1, T_U(x - 1, k + 1)) = 2k + 2 - \mathbf{1}_{\{0 < x - 1\}}.
\]

Hence

\[
L(x - 1, T_U(x - 1, k + 1)) = \begin{cases} 
2k + 1, & \text{if } x > 1, \\
2k + 2, & \text{if } x = 1.
\end{cases}
\]
Again using (1.3), we can easily see that, when $x \geq 1$, the $h$ in $A^{(k)}_{x,h}$ has to be even. In the following, we implicitly assume $x > 1$, which implies that $L(x-1, T_U(x-1, k+1)) = 2k + 1$, unless explicitly mentioned otherwise.

We write the events $A^{(k)}_{x,h}$ in terms of $T_U(x-1, k+1)$ since the events defined this way match the form of the Ray-Knight representation to be discussed later. Let $K_h = \left( \frac{1}{2} (h - 2 \sqrt{h}), \frac{1}{2} (h - \sqrt{h}) \right)$ and define

$$
N_H := \sum_{h=8}^{H} \sum_{k \in K_{2h}} \sum_{x=2}^{+\infty} 1_{A^{(k)}_{x,2h}} \quad \text{and} \quad N := \lim_{H \to \infty} N_H = \sum_{h=8}^{+\infty} \sum_{k \in K_{2h}} \sum_{x=2}^{+\infty} 1_{A^{(k)}_{x,2h}}. \tag{3.2}
$$

We note that for each $h$, the events $A^{(k)}_{x,h}, x \in \mathbb{Z}, k \in K_h$ are mutually disjoint. Since $f(3) \geq \sum_{x \in \mathbb{Z}} u(x)$, we have that $f(3) \geq N$, and thus it is enough to show that $N = \infty$ a.s.

### 3.2 Branching Process and the Ray-Knight Representation

In the remainder of this paper, we denote by $Y_n$ a critical Galton–Watson branching process with geometric offspring distribution, and by $Z_n, R_n$ two related critical Galton–Watson branching processes with immigration. The precise definitions of these processes are as follows: Let $(X_{n,i})_{n,i}$ be i.i.d. geometric variables with mean 1, that is, for all $k \geq 0$, $P(X_{n,i} = k) = \frac{1}{2} \mathbb{P}_{TT}$. We recursively define

$$
Y_{n+1} = \sum_{i=1}^{Y_n} X_{n,i}, \quad Z_{n+1} = \sum_{i=1}^{Z_n+1} X_{n,i}, \quad R_{n+1} = 1 + \sum_{i=1}^{R_n} X_{n,i}. \tag{3.3}
$$

One can check that $Y_n, Z_n$ and $R_n$ are Markov chains with state space $\mathbb{N}$ and transition probabilities:

$$
P(Y_{n+1} = j | Y_n = i) = \pi(i, j) := \begin{cases} 
\delta_0(j), & \text{if } i = 0, \\
2^{-i-j} (i+j-1)! (i-1)! j!, & \text{if } i > 0,
\end{cases} \tag{3.4}
$$

$$
P(Z_{n+1} = j | Z_n = i) = \rho(i, j) := \pi(i + 1, j), \tag{3.5}
$$

$$
P(R_{n+1} = j | R_n = i) = \rho^*(i, j) := \pi(i, j - 1). \tag{3.6}
$$

Let $k \geq 0$ and $x$ be fixed integers. When $x - 1 \geq 1$, we define the following three processes:

1. $(Z^{(k)}_{n})_{n \geq 0}$ is a Markov chain with transition probabilities $\rho(i, j)$ and initial state $Z^{(k)}_0 = k$.
2. $(Y^{(k)}_{n})_{n \geq -1}$ is a Markov chain with transition probabilities $\pi(i, j)$ and initial state $Y^{(k)}_{-1} = k$.
3. $(Y^{(k)}_n)_{n \geq 0}$ is a Markov chain with transition probabilities $\pi(i, j)$ and initial state $Y^{(k)}_0 = Z^{(k)}_{x-1}$.
We assume that the three processes are independent, except that \( Y'_{\langle k \rangle} \) starts from \( Z_{\langle x-1 \rangle}^{\langle k \rangle} \). We patch the above three processes together to a single process as follows:

\[
\Delta_{\langle x \rangle}^{\langle k \rangle} (y) := \begin{cases} 
Z_{\langle x-1 \rangle-y}^{\langle k \rangle}, & \text{if } 0 \leq y \leq x-1, \\
Y_{\langle y \rangle-x}^{\langle k \rangle}, & \text{if } x-1 \leq y < \infty, \\
Y_{\langle y \rangle}^{\langle k \rangle}, & \text{if } -\infty < y \leq 0.
\end{cases}
\]  
(3.7)

By the Ray-Knight theorem on local times of simple random walk on \( \mathbb{Z} \) (c.f. [25, Theorem 1.1]), we know that for any integers \( x \geq 2 \) and \( k \geq 0 \),

\[
(\xi_D(y, T_U(x-1, k+1)), y \in \mathbb{Z}) \overset{\text{law}}{=} (\Delta_{\langle x-1 \rangle}^{\langle k \rangle} (y), y \in \mathbb{Z}).
\]  
(3.8)

Similarly, when \( x-1 \leq 0 \), we define the processes:

1. \( (R_n^{\langle k \rangle})_{n \geq -1} \) is a Markov chain with transition probabilities \( \rho^\ast (i, j) \) and initial state \( R_{\langle -1 \rangle}^{\langle k \rangle} = k \).
2. \( (Y_n^{\langle k \rangle})_{n \geq 0} \) is a Markov chain with transition probabilities \( \pi(i, j) \) and initial state \( Y_{\langle 0 \rangle}^{\langle k \rangle} = k \).
3. \( (Y_n^{\langle \prime \rangle})_{n \geq -1} \) is a Markov chain with transition probabilities \( \pi(i, j) \) and initial state \( Y_{\langle -1 \rangle}^{\langle \prime \rangle} = R_{\langle -1-x \rangle}^{\langle k \rangle} \).

We assume that the three processes are independent, except that \( Y_{\langle \prime \rangle}^{\langle k \rangle} \) starts from \( R_{\langle -1-x \rangle}^{\langle k \rangle} \). In this case, we patch the three processes together as follows:

\[
\Delta_{\langle x \rangle}^{\langle k \rangle} (y) \triangleq \begin{cases} 
Y_{\langle y \rangle}^{\langle \prime \rangle}, & \text{if } -1 \leq y < \infty, \\
R_{\langle y-x \rangle}^{\langle k \rangle}, & \text{if } x-1 \leq y \leq -1, \\
Y_{\langle y \rangle-x-1}^{\langle k \rangle}, & \text{if } -\infty < y \leq x-1.
\end{cases}
\]  
(3.9)

By the Ray-Knight theorem, we get that for the case \( k \geq 0 \), \( x \leq 1 \):

\[
(\xi_D(y, T_U(x-1, k+1)), y \in \mathbb{Z}) \overset{\text{law}}{=} (\Delta_{\langle x-1 \rangle}^{\langle k \rangle} (y), y \in \mathbb{Z}).
\]  
(3.10)

### 3.3 Three Favorite Edges Under Ray-Knight Representation

For \( h \in \mathbb{N} \), define the first hitting times of \( [h, +\infty) \) for \( Y_n^{\langle k \rangle} \) and \( Z_n^{\langle k \rangle} \) to be \( \sigma_h^{\langle k \rangle} \) and \( \tau_h^{\langle k \rangle} \), respectively, and the extinction time of \( Y_n^{\langle k \rangle} \) to be \( \omega^{\langle k \rangle} \). That is,

\[
\sigma_h^{\langle k \rangle} := \min\{n \geq 0 : Y_n^{\langle k \rangle} \geq h\}, \quad \tau_h^{\langle k \rangle} := \min\{n \geq 0 : Z_n^{\langle k \rangle} \geq h\},
\omega^{\langle k \rangle} := \min\{n \geq 0 : Y_n^{\langle k \rangle} = 0\}.
\]  
(3.11)

Using the notation above, we can express \( P(A_{\langle x \rangle}^{\langle k \rangle}) \) in its Ray-Knight representation form. In the remainder of this section, we let \( \tilde{n} := T_U(x-1, k+1) \) for simplicity.
Now, on $A_{x,h}^{(k)}$, we have that $L(x, \tilde{n}) = L(x + 1, \tilde{n}) = L(x + 2, \tilde{n}) = h$, $h > L(y, \tilde{n}) = 2\xi_D(y - 1, \tilde{n}) + 1_{y \geq x}$ for $x \geq 2$, $y \neq x, x + 1, x + 2$. We have the following five cases:

1. $y \leq 0$, $\xi_D(y - 1, \tilde{n}) = \frac{L(y, \tilde{n})}{2} < \frac{h}{2}$;
2. $y \in [1, x - 2]$, $\xi_D(y - 1, \tilde{n}) = \frac{L(y, \tilde{n}) - 1}{2} < \frac{h - 1}{2}$;
3. $y = x - 1$, $\xi_D(y - 1, \tilde{n}) = \frac{L(y, \tilde{n}) - 1}{2} < \frac{h - 1}{2}$;
4. $y = x, x + 1, x + 2$, $\xi_D(y - 1, \tilde{n}) = \frac{L(y, \tilde{n})}{2} = \frac{h}{2}$;
5. $y \geq x + 3$, $\xi_D(y - 1, \tilde{n}) = \frac{L(y, \tilde{n})}{2} < \frac{h}{2}$.

Then by (3.8), we obtain that

$$P(A_{x,h}^{(k)}) = P \left( Y_0^{(k)} = Y_1^{(k)} = Y_2^{(k)} = \frac{h}{2}, \left\{ Y_n^{(k)} < \frac{h}{2}, n \geq 3 \right\}, \left\{ Z_n^{(k)} < \frac{h - 1}{2}, 1 \leq n \leq x - 2 \right\}, \left\{ Y_n^{(k)} < \frac{h}{2}, n \geq 1 \right\} \right). (3.12)$$

For all the notation above, when the initial state of a process is obvious, we omit the superscript “(k)” for simplicity. We will also use the notation $P(\cdot | Y_0 = k)$ to indicate the initial state.

### 3.4 Standard Lemmas

In this subsection, we recall a few lemmas that will be useful later. In what follows, $c_i$ for $i \geq 1$ and $c$ are all constants.

**Lemma 3.1** [12, Lemma 2.2] We have that

1. For $i, j \in \left( \frac{1}{2}(h - 10\sqrt{h}), \frac{1}{2}(h + 10\sqrt{h}) \right)$, there exist positive constants $c$ and $C$ such that $c h^{-\frac{1}{2}} \leq \pi(i, j) \leq C h^{-\frac{1}{2}}$ for all $h \geq 100$.
2. For $i + j = h$, $\pi(i, j) \leq O(1) h^{-\frac{1}{2}}$.
3. For $j < i_1 < i_2$, $\pi(i_1, j) > \pi(i_2, j)$.

**Lemma 3.2** [12, Lemma 2.3] For any $h \in \mathbb{N}$, it holds that $E\tau_h = E Z_{\tau_h} - Z_0$. In particular, for any $0 \leq k \leq h$, we have that $E[\tau_h | Z_0 = k] > h - k$.

**Lemma 3.3** ([34, (6.18)]) There exists a constant $C < \infty$ such that, for any $0 \leq k < h$,

$$E(Z_{\tau_h} | Z_0 = k) \leq h + C h^{\frac{1}{2}}.$$

### 4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by following the idea of [12]. We spell out some details for the reader’s convenience and point out some modifications that need to be made.
4.1 Lower Bound on the First Moment

The following is the counterpart to [12, Lemma 3.1].

**Lemma 4.1** Suppose that \( Z_0 = k \in K_h = \left[ \frac{h - 2 \sqrt{h}}{2}, \frac{h - \sqrt{h}}{2} \right] \). Then there exists a constant \( c > 0 \) such that for any \( h > 4 \),

\[
E \left( \sum_{n=1}^{\tau_{\frac{h}{2}}} \frac{h/2 - Z_n}{h/2} \right) \geq c \sqrt{h}.
\]

**Proof** Let \( M_n = \sum_{s=1}^{n} (Z_s - s) - n(Z_n - n) \), and let \( \mathcal{F}_n = \sigma(Z_0, Z_1, \ldots, Z_n) \). By the proof of [12, Lemma 3.1], we know that \( (M_n) \) is a martingale. By the optional stopping theorem, we get that

\[
E \left( \sum_{n=1}^{\tau_{\frac{h}{2}}} (Z_n - n) \right) = E \tau_{\frac{h}{2}} \left( Z_{\tau_{\frac{h}{2}}} - \tau_{\frac{h}{2}} \right),
\]

and thus

\[
E \left( \sum_{n=1}^{\tau_{\frac{h}{2}}} \frac{h/2 - Z_n}{h/2} \right) = E \tau_{\frac{h}{2}} - \frac{E \left( \sum_{n=1}^{\tau_{\frac{h}{2}}} (Z_n - n) + n \right)}{h/2} = \frac{E \tau_{\frac{h}{2}} - \frac{E \left( \tau_{\frac{h}{2}} Z_{\tau_{\frac{h}{2}}} - \tau_{\frac{h}{2}} \right)}{h/2}}{h/2} - \frac{E \left( 1 + \frac{\tau_{\frac{h}{2}}}{h} + \tau_{\frac{h}{2}} Z_{\tau_{\frac{h}{2}}} - \frac{1}{2} \tau_{\frac{h}{2}} \right)}{h/2}.
\]

Define the process \( M'_n = -\frac{1}{4} Z_n^2 + n Z_n - \frac{1}{2} n^2 + \frac{1}{4} n \). By the proof of [12, Lemma 3.1], we know that \( (M'_n) \) is a martingale. Applying the optional stopping theorem to \( (M'_n) \) at \( \tau_{\frac{h}{2}} \), we get

\[
E \left[ \tau_{\frac{h}{2}} Z_{\tau_{\frac{h}{2}}} - \frac{1}{2} \tau_{\frac{h}{2}}^2 \right] = E \left[ \frac{1}{4} Z_{\tau_{\frac{h}{2}}}^2 - \frac{1}{4} \tau_{\frac{h}{2}}^2 \right] - \frac{Z_0^2}{4} = \frac{1}{4} E \left[ Z_{\tau_{\frac{h}{2}}}^2 - \tau_{\frac{h}{2}}^2 \right] - \frac{Z_0^2}{4}.
\]

Combining (4.1), (4.2) and Lemma 3.2, we get

\[
E \left( \sum_{n=1}^{\tau_{\frac{h}{2}}} \frac{h/2 - Z_n}{h/2} \right) = \left( 1 - \frac{1}{2h} \right) E \tau_{\frac{h}{2}} - \frac{1}{2h} \left( E Z_{\tau_{\frac{h}{2}}}^2 - Z_0^2 \right).
\]
\( \begin{align*}
&= \left(1 - \frac{1}{2h}\right) E \left[ Z_{\tau h^{-1}} - Z_0 \right] - \frac{1}{2h} E \left[ (Z_{\tau h^{-1}} - Z_0)(Z_{\tau h^{-1}} + Z_0) \right] \\
&= \frac{1}{2h} E \left[ \left( 2h - 1 - (Z_{\tau h^{-1}} - Z_0) \right) (Z_{\tau h^{-1}} - Z_0) \right].
\end{align*} \tag{4.3} \)

Obviously \( Z_{\tau h^{-1}} - Z_0 \geq \frac{h}{2} - \frac{h - \sqrt{h}}{2} \geq c_1 \sqrt{h} \). Then by Lemmas 3.2 and 3.3, we obtain

\[ E \left( \sum_{n=1}^{\tau h^{-1}} \frac{h/2 - Z_n}{h/2} \right) \geq \frac{1}{2h} \cdot \left( 2h - 1 - \frac{h - 1}{2} - c_2 \sqrt{h} - \frac{h - \sqrt{h}}{2} \right) \cdot c_1 \sqrt{h} \geq c \sqrt{h}. \tag{4.4} \]

**Proposition 4.2** There exists \( c > 0 \) such that \( EN_H \geq c \log H \) for all \( H \in [50, \infty) \).

**Proof** By the Ray-Knight representation, (3.2) and (3.12), we know that

\[ EN_H = \sum_{h=50}^{H} \sum_{k \in K_{2h}} P \left( Y^{(k)}_0 = Y^{(k)}_1 = Y^{(k)}_2 = h, \left\{ Y^{(k)}_n < h, \forall n \geq 3 \right\} \right) \]

\[ \cdot \sum_{x=2}^{\infty} P \left( \left\{ Z^{(k)}_n < h - \frac{1}{2}, 1 \leq n \leq x - 2 \right\} \cdot \left\{ Y^{(k)}_n < h, \forall n \geq 1 \right\} \right). \]

It follows that

\[ EN_H \geq \sum_{h=50}^{H} \sum_{k \in K_{2h}} P \left( Y^{(k)}_0 = Y^{(k)}_1 \right) \]

\[ = Y^{(k)}_2 = h, \left\{ Y^{(k)}_3 \in \left( h - 5\sqrt{2h}, h - \frac{\sqrt{2h}}{2} \right), Y^{(k)}_n < h, \forall n \geq 4 \right\} \]

\[ \cdot \sum_{x=2}^{\infty} P \left( \left\{ Z^{(k)}_n < h - \frac{1}{2}, 1 \leq n \leq x - 2 \right\} \cdot \left\{ Y^{(k)}_n < h, n \geq 1 \right\} \right) \]

\[ = \sum_{h=50}^{H} \sum_{k \in K_{2h}} \pi(k, h) \cdot \pi(h, h) \cdot \pi(h, h) \]

\[ \cdot \sum_{n \in \left( h - 5\sqrt{2h}, h - \frac{\sqrt{2h}}{2} \right)} \pi(h, m) P \left( Y^{(m)}_n < h, \forall n \geq 1 \right) \]

\[ \cdot \sum_{x=2}^{\infty} P \left( \tau h^{-1} \geq x - 1, \left\{ Y^{(k)}_n < h, \forall n \geq 1 \right\} \right). \tag{4.5} \]
By Lemma 3.1, all the \( \pi(\cdot, \cdot) \)'s in the display above are of the order \( h^{-\frac{1}{2}} \). Since \( Y_n \) is a martingale, applying the optional stopping theorem at \( \sigma_h \wedge \omega \), where \( \sigma_h \) and \( \omega \) are defined in (3.11), we get for \( m \in (h - 5\sqrt{2h}, h - \frac{\sqrt{2h}}{2}) \),

\[
P \left( Y_n^{(m)} < h, \forall n \geq 1 \right) = P \left( Y_n^{(m)} \text{ hits 0 before exceeds } h \right) \geq \frac{h - m}{h} \geq c_3 h^{-\frac{1}{2}}. \tag{4.6}
\]

The last two inequalities hold since

\[
m = E Y_{\sigma_h \wedge \omega} = E \left( Y_{\sigma_h} 1_{\omega \geq \sigma_h} \right) + E \left( Y_{\omega} 1_{\omega < \sigma_h} \right) \\
\geq E \left( Y_{\sigma_h} 1_{\omega \geq \sigma_h} \right) \\
\geq h \left( 1 - P \left( Y_n^{(m)} \text{ hits 0 before exceeds } h \right) \right),
\]

which implies that

\[
P \left( Y_n^{(m)} \text{ hits 0 before exceeds } h \right) \geq \frac{h - m}{h} \geq c_3 h^{-\frac{1}{2}}. \tag{4.7}
\]

By (4.5) and (4.6), we get

\[
EN_H \geq c_4 \sum_{h=50}^H \sum_{k \in K_{2h}} h^{-2} \cdot \sum_{x=2}^{+\infty} P \left( \tau_{h \frac{1}{2}} \geq x - 1, \left\{ Y_n^{(k)} < h, \forall n \geq 1 \right\} \right). \tag{4.8}
\]

By the independence of the processes in the Ray-Knight representation, we have

\[
\sum_{x=2}^{+\infty} P \left( \tau_{h \frac{1}{2}} \geq x - 1, \left\{ Y_n^{(k)} < h, \forall n \geq 1 \right\} \right) \\
\geq \sum_{x=2}^{+\infty} \sum_{l=0}^{h-1} P \left( Z_n^{(k)} < h - \frac{l}{2} \text{ for } 1 \leq n \leq x - 2, Z_{x-1} = l \right) \\
\cdot P \left( Y_n^{(l)} \text{ hits 0 before exceeds } h \right).
\]

By the definitions of \((Z_n)_{n \geq 0}\) and \(\tau_h\), we obtain

\[
Z_0 - Z_{\tau_{h \frac{1}{2}}} \leq h - \frac{\sqrt{2h}}{2} - h + \frac{1}{2} = \frac{1-\sqrt{2h}}{2} \leq 0. \tag{4.9}
\]

Similar to (4.7), we have \( P \left( Y_n^{(l)} \text{ hits 0 before exceeds } h \right) \geq \frac{h-l}{h} \), which together with (4.9) and Lemma 4.1 implies that

\[
\sum_{x=2}^{+\infty} P \left( \tau_{h \frac{1}{2}} \geq x - 1, \left\{ Y_n^{(k)} < h, \forall n \geq 1 \right\} \right)
\]
\[
\begin{align*}
\geq & \sum_{x=2}^{+\infty} \sum_{l=0}^{h} P \left( \tau_{h-\frac{1}{2}} \geq x - 1, Z_{x-2} = l \right) \cdot \frac{h - l}{h} \\
= & \mathbb{E} \left( \sum_{l=0}^{h} \sum_{x=2}^{h-1} \frac{h - l}{h} \cdot 1_{\{ Z_{x-2} = l \}} \right) = \mathbb{E} \left( \sum_{n=0}^{\tau_{h-\frac{1}{2}}-1} \frac{h - Z_n}{h} \right) \\
\geq & \mathbb{E} \left( \sum_{n=1}^{\tau_{h-\frac{1}{2}}-1} \frac{h - Z_n}{h} \right) + \left( \frac{h - Z_{\tau_{h-\frac{1}{2}}}}{h} - \frac{h - Z_0}{h} \right) \\
= & \mathbb{E} \left( \sum_{n=1}^{\tau_{h-\frac{1}{2}}-1} \frac{h - Z_n}{h} \right) \geq c_5 \sqrt{h}.
\end{align*}
\] (4.10)

By (4.8) and (4.10), we obtain

\[
\mathbb{E} N_H \geq c_4 \sum_{h=50}^{H} \sum_{k \in \mathbb{K}_{2h}} h^{-2} \cdot c_5 \sqrt{h} \geq c_6 \sum_{h=100}^{H} h^{-1} \geq c \log H.
\]

The proof is complete. \(\square\)

### 4.2 Upper Bound on the Second Moment

In this subsection, we will give an upper bound on the second moment \(\mathbb{E} N_H^2\) following [12]. If we use the \(N_H\) defined in (3.2)–(3.1), we could not prove the counterparts of [12, Lemmas 3.3, 3.4]. To overcome this, we give a variant \(\tilde{N}_H\) of \(N_H\) with \(N_H \leq \tilde{N}_H\) a.s. In the remainder of this section, we assume \(x \geq 2\) unless explicitly mentioned otherwise.

For any positive integer \(h\), we define

\[
A_{x,2h} := \{ K(T_D(x - 1, h)) = \{x, x + 1, x + 2\}, L(x, T_D(x - 1, h)) = 2h\}
\] (4.11)

and

\[
\tilde{N}_H := \sum_{h=50}^{H} \sum_{x=2}^{\infty} 1_{A_{x,2h}}, \quad \tilde{N} := \lim_{H \to \infty} \tilde{N}_H = \sum_{h=50}^{\infty} \sum_{x=2}^{\infty} 1_{A_{x,2h}}.
\]

We claim that

\[
\bigcup_{k \in \mathbb{K}_{2h}} A_{x,2h}^{(k)} \subset A_{x,2h},
\] (4.12)
where $A^{(k)}_{x, 2h}$ is defined by (3.1). Suppose that $\omega \in A^{(k)}_{x, 2h}$ for some $k \in K_{2h}$. Define

$$\tilde{T} (\omega) := \{ m < T_U (x - 1, k + 1) (\omega) : S_m = x - 1, S_{m-1} = x \},$$

$$K (m) = \{ x, x + 1, x + 2 \}, L (x, m) = 2h \}.$$

By the definition of $A^{(k)}_{x, h}$, we know that $\tilde{T} (\omega)$ consists of one unique element $t (\omega)$. Further by (1.3), we get that $t (\omega) = T_D (x - 1, h) (\omega)$. Hence $\omega \in A_{x, 2h}$ and thus (4.12) holds.

By (4.12), we get that $N_H \leq \tilde{N}_H$ a.s. Now we study the second moment of $\tilde{N}_H$ following [12, Sect. 3.2].

Let $D (n) = (\xi_D (x, n), x \in \mathbb{Z}) \in \mathbb{N}^\mathbb{Z}$ be the random vector that records the number of downcrossings of each site by time $n$. For $l \in \mathbb{N}^\mathbb{Z}$, we use $l (i), i \in \mathbb{Z}$ to denote the $i$-th component of $l$. For $l \in \mathbb{N}^\mathbb{Z}$, define $B_x (l) = \{ \exists n < \infty : D (n) = l, S (n - 1) = x, S (n) = x - 1 \}$. Note that if $B_x (l)$ happens, there exists a unique $n \in \mathbb{N}$ such that $D (n) = l, S (n - 1) = x$ and $S (n) = x - 1$.

Let $P = \{ l \in P (B_x (l)) > 0 \}$. For $Q \in P$, denote $B_x (Q) = \bigcup_{l \in Q} B_x (l)$. Then by virtue of (1.3), we have $A_{x, 2h} = B_x (P_{x, 2h})$, where $P_{x, 2h}$ is the set of $l \in P$ such that

$$l (x - 1) = l (x) = l (x + 1) = h, \ l (i) < h \ for \ all \ i \neq x - 1, x, x + 1.$$

Let $\mathcal{A}$ be the family of all subsets of $P$. For any $x \in \mathbb{Z}$, we define a map $\varphi_x : P \mapsto \mathcal{A}$ by

$$\varphi_x (l) := \{ l^* \in P : l^* (i) < l (i) \ for \ i = x, x + 1, l^* (i) = l (i) \ for \ i \neq x, x + 1 \}.$$

Following the argument of [12, Lemma 3.3], we can prove

**Lemma 4.3** Suppose $x_1, x_2 \in \mathbb{Z}$ and $h$ is a positive integer. If $l^*_i \in \varphi_{x} (l_i)$ with $l_i \in P_{x_i, 2h}, i = 1, 2$, we have that $B_{x_1} (l^*_1) \cap B_{x_2} (l^*_2) = \emptyset$, if $(x_1, l_1) \neq (x_2, l_2)$. Further, we have $B_{x_1} (l^*_1) \cap B_{x_2} (l^*_2) = \emptyset$ if $(x_1, l_1) = (x_2, l_2)$ but $l^*_1 \neq l^*_2$.

The following result is the counterpart of [12, Lemma 3.4].

**Lemma 4.4** There exists a constant $c > 0$ such that for any $x \geq 2, h \geq 50, l \in P_{x, 2h}$,

$$P (B_x (\varphi_x (l))) \geq c P (B_x (l)).$$

**Proof** We consider $l^* \in \varphi_x (l)$ such that $l^* (x) \in [h - \frac{\sqrt{2} h}{2}, h)$ and $l^* (x + 1) \in [h - \frac{\sqrt{2} h}{2}, h)$. According to Lemma 3.1 (1) and (3), there is a constant $c > 0$ such that

$$P (B_x (l^*)) = \frac{P (B_x (l^*) (x)) \cdot \pi (l^* (x), l^* (x + 1)) \cdot \pi (l^* (x + 1), l (x + 2))}{\pi (h, h) \cdot \pi (h, h) \cdot \pi (h, l (x + 2))} \geq 4c.$$

Note that there are about $h/2$ of such $l^* \in \varphi_x (l)$ that satisfy the above inequality. Thus by Lemma 4.3, we get that $P (B_x (\varphi_x (l))) \geq c P (B_x (l))$. \qed
With Lemmas 4.3 and 4.4 in hand, we can follow the proof of [12, Proposition 3.5] to get the following result. We omit the details.

**Proposition 4.5** We have that $\tilde{E} \tilde{N}_H^2 = O(\log H) \cdot E \tilde{N}_H$.

**Corollary 4.6** We have $E N_H^2 = O(\log^2 H)$.

**Proof** By Proposition 4.5 and the Cauchy–Schwarz inequality, we get

$$E \tilde{N}_H^2 \leq O(\log H) \cdot \left( E \tilde{N}_H^2 \right)^{1/2},$$

which implies that

$$E \tilde{N}_H^2 = O(\log^2 H).$$

Since $N_H \leq \tilde{N}_H$ a.s., the desired result follows immediately. $\square$

### 4.3 0–1 Law

Recall that $N = \lim_{H \to \infty} N_H$. First, we show that $N = \infty$ with positive probability.

**Proposition 4.7** There exists a constant $\delta > 0$ such that $P(N = \infty) \geq \delta$.

**Proof** By the Cauchy–Schwarz inequality, we get that

$$E N_H = E N_H 1_{\{N_H > \log \log H\}} + E N_H 1_{\{N_H \leq \log \log H\}} \leq \sqrt{E N_H^2} \cdot P(N_H > \log \log H) + \log \log H.$$

Combining this with Proposition 4.2 and Corollary 4.6, we get that there exist constants $c, \delta > 0$ such that

$$P(N_H > \log \log H) \geq \frac{(E N_H - \log \log H)^2}{E N_H^2} \geq c \frac{\log^2 H}{E N_H^2} \geq \delta,$$

for all sufficiently large $H$. Letting $H \to \infty$, we get that $P(N = \infty) \geq \delta$. $\square$

Recall the definition of $\tilde{U}(n)$ in Theorem 2.1. By Theorem 2.1, we know that

$$\liminf_{n \to \infty} \frac{\tilde{U}(n)}{\sqrt{n(\log n)^{-\gamma}}} = \infty \quad a.s.,$$

where $\gamma > 11$.

Using Proposition 4.7 and (4.14), following the argument of [12, Section 3.3] and applying Kolmogorov’s 0–1 law, we obtain that

$$P(f(3) = \infty) = 1,$$

which completes the proof of Theorem 1.1.
5 Remark

In Sect. 4, we used the transience of the favorite downcrossing site process to show the transience of the favorite edge process. In fact, from Proposition 2.4, we can see that there is a close relation between favorite edges and favorite downcrossing sites. A natural question arises:

How about the number of favorite downcrossing sites of one-dimensional simple symmetric random walk?

In [19], we considered this question and proved the following result.

Theorem 5.1 For a one-dimensional simple symmetric random walk, with probability 1, there are only finitely many times at which there are at least four favorite downcrossing sites and three favorite downcrossing sites occurs infinitely often.

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Declarations

Conflict of interest We have no conflicts of interest.

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