1. Introduction

A spanning tree of a graph is a collection of edges which connects all the vertices and has no cycles. Spanning trees were first investigated by Kirchhoff in his study of electrical resistor networks [Kir47]; in particular he showed that the determinant of the combinatorial Laplacian counts spanning trees.

The uniform random spanning tree (UST) is a well studied probability model, related to several other probability models. For example, the loop-erased random walk of Lawler (see [Law91, Law99, LL10]) was shown by Pemantle [Pem91] to have the same distribution as the paths connecting vertices in the uniform spanning tree. The abelian sandpile model of self-organized criticality was shown by Majumdar and Dhar [MD92] to be closely related to spanning trees (recurrent states in the sandpile model are in bijection with spanning trees). Lawler, Schramm, and Werner [LSW04] showed that the branches of the spanning tree on $\mathbb{Z}^2$ converge in the scaling limit to SLE$_2$ and the “peano curve” winding between the spanning tree and its dual converges to SLE$_8$.

We show here how to compute the probabilities of various connection topologies for uniform random spanning trees on graphs embedded in surfaces. As an application, we show how to compute the “intensity” of the loop-erased random walk in $\mathbb{Z}^2$, that is, the probability that the walk from $(0, 0)$ to $\infty$ passes through a given vertex or edge. For example, the probability that it passes through $(1, 0)$ is $5/16$; this confirms a conjecture from 1994 about the stationary sandpile density on $\mathbb{Z}^2$. We do the analogous computation for the triangular lattice, honeycomb lattice and $\mathbb{Z} \times \mathbb{R}$, for which the probabilities are $5/18$, $13/36$, and $1/4 - 1/\pi^2$ respectively.

Our techniques involve applying the vector bundle Laplacian [Ken11] and asymptotics of the “Green’s function derivative” for planar graphs, together with a generalization of the grove counting techniques of [KW11a] to graphs on annuli.

1.1. Response matrices and groves. Let $\mathcal{G}$ be a graph (undirected, with multiple edges and self-loops allowed), and let $c: \mathcal{G} \rightarrow \mathbb{R}_{>0}$ be a positive conductance on each edge. Our graphs will be finite, except in sections 7 and 8 where we take limits to infinite lattices. Let $\mathcal{N}$ be a subset of $\mathcal{G}$’s vertices such that every vertex of $\mathcal{G}$ is connected to some vertex of $\mathcal{N}$. The triple $(\mathcal{G}, c, \mathcal{N})$ is a resistor network. Associated to this data is the Dirichlet-to-Neumann matrix (also called
the response matrix) \( L \), defined as follows. Given a function \( f : \mathcal{N} \to \mathbb{R} \), find its harmonic extension \( h \) on \( \mathcal{G} \), that is a function on the vertices of \( \mathcal{G} \) that is harmonic on \( \mathcal{G} \setminus \mathcal{N} \) and has values \( f \) on \( \mathcal{N} \). Then \( L(f) = -\Delta(h)|_{\mathcal{N}} \) is a linear function of \( f \), where \( \Delta \) is the (positive semidefinite) graph Laplacian. In electrical terms, \( L(f) \) gives the current flow into the nodes \( \mathcal{N} \) when they are held at \( f \) volts. While it is not obvious from this definition, \( L \) is a symmetric negative semidefinite matrix.

**Circular planar networks** are planar resistor networks where \( \mathcal{N} \) is a subset of the vertices on the outer face listed in cyclic order. These networks were studied in [CdV94, CdVGV96, CIM98], where the set of matrices which occur as response matrices were classified: they are precisely the matrices whose “non-interlaced” minors are nonnegative. (A non-interlaced minor is one in which there are no 4 indices \( a < b < c < d \) for which \( a \) and \( c \) are rows and \( b \) and \( d \) are columns or vice versa.) Furthermore, these authors showed how to construct a circular planar network having a given such response matrix \( L \).

In a resistor network a **grove** is a spanning forest (set of edges with no cycles) in which every component contains at least one vertex in \( \mathcal{N} \). (Our term grove is a generalization, to arbitrary graphs and arbitrary connections, of the groves defined by Carroll and Speyer in [CS04].) A spanning tree on a large graph (such as \( \mathbb{Z}^2 \)) can be studied by cutting the large graph into two subgraphs which are joined at nodes along their boundaries. The spanning tree restricted to either subgraph is a grove. In [KW11a] we studied the natural probability measure on groves (where each grove occurs with probability proportional to the product of its edge weights), showing for circular planar graphs how to compute the probability that a random grove has a given connection topology in terms of the entries in \( L \).

### 1.2. Graphs on surfaces.

We study here the same problem for a graph \( \mathcal{G} \) embedded on a surface \( \Sigma \). Here the usual notion of response matrix is not rich enough to extract information about the underlying topological structure of a grove. Given a resistor network on a surface \( \Sigma \), a natural generalization of the response matrix is a matrix-valued function \( \mathcal{L} \) on the representation variety \( \text{Hom}(\pi_1(\Sigma), H) \) of flat \( H \)-connections on \( \mathcal{G} \); here \( H = \mathbb{C}^* \) or \( \text{SL}_2(\mathbb{C}) \). We show here how \( \mathcal{L} \) can be used to compute connection probabilities of (certain types of) groves on \( \mathcal{G} \).

The question of characterizing which matrices \( \mathcal{L} \) occur as a function of the topology of \( \Sigma \) remains open. See Lam and Pylyavskyy [LP12] for related work in the case when the surface \( \Sigma \) is an annulus.

We give special attention to the case where the surface \( \Sigma \) is an annulus; this is the easiest case beyond the planar one (but already quite involved) and also has applications to the study of spanning trees on planar graphs. Connectivity questions within a spanning tree involving the path to \( \infty \) can be studied by viewing \( \infty \) as one of the nodes of the surface graph, on a boundary by itself. When the other vertices are on the same face, the relevant surface is the annulus.

### 1.3. Applications to planar graphs.

Using these techniques one can in principle compute the probability that the path of the uniform spanning tree from \( a \) to \( b \) in a planar graph passes through a given set of edges or vertices (as in Figure 1), assuming the response matrix \( \mathcal{L} \) can be evaluated. When one of the nodes is \( \infty \), it is more convenient to work with the Green’s function \( G \). For \( \mathbb{Z}^2 \) and the honeycomb and triangular lattices, the usual Green’s function \( G \) is known, and we modify the \( H \)-connection and use the translation and 180° rotational symmetries of these graphs.
Figure 1. A portion of the uniform spanning tree on $\mathbb{Z}^2$, with the path from $(0,0)$ to $\infty$ shown in bold. The uniform spanning tree on $\mathbb{Z}^2$ can be constructed as a weak limit of uniform spanning trees on large boxes. The limiting measure exists, is unique, and is supported on trees of $\mathbb{Z}^2$ [Pem91]. Almost surely, within the uniform spanning tree of $\mathbb{Z}^2$, each vertex has a unique infinite path starting from it [BLPS01] (see also [Lyo14]). The path to infinity is a loop-erased random walk (LERW) [Pem91] (see also [Wil96]).

to extract the additional information in $\mathcal{G}$ in closed form. For $\mathbb{Z}^2$, our method shows that the probability that the path from the origin to $\infty$ passes through a particular edge or vertex (see Figure 2) is in $\mathbb{Q}(1/\pi)$. For the honeycomb and triangular lattices these probabilities are in $\mathbb{Q}(\sqrt{3}/\pi)$ (see Figure 10 and Figure 11).

For example, we show that the probability that the loop-erased random walk in $\mathbb{Z}^2$ from $(0,0)$ to $\infty$ contains the point $(1,0)$ is $5/16$. (See Figure 1.) This value was predicted by Levine and Peres [LP13] and Poghosyan and Priezzhev [PP11], by relating this probability to the average density of the stationary abelian sandpile model.

The connection between the spanning trees and the abelian sandpile model was discovered by Majumdar and Dhar [MD92], and Priezzhev [Pri94] used this connection to compute the height distribution of the abelian sandpile model, in terms of two integrals that could not be evaluated in closed form. Grassberger evaluated these integrals numerically, and conjectured that the stationary density of the sandpile on $\mathbb{Z}^2$ is $17/8$. Later Jeng, Piroux, and Ruelle [JPR06] showed how to express one of these two integrals in terms of the other, and determined the sandpile height distribution in closed form, under the assumption that the remaining integral, which numerically is $0.5 \pm 10^{-12}$, is exactly $1/2$. Our derivation of this probability that LERW passes through $(1,0)$ confirms these conjectures (although our methods are different), and shows that this aforementioned integral is exactly $1/2$.

While we were writing up our results, Poghosyan, Priezzhev, and Ruelle independently found another proof that the probability of visiting $(1,0)$ is $5/16$ [PPR11].
Figure 2. Intensity of loop-erased random walk on $\mathbb{Z}^2$. The origin is at the lower-left, and directed edge-intensities as well as vertex-intensities of the LERW are shown. (See also Figure 9.)
(They also asked about the probability about visiting other points, and remarked that the probability that the LERW visits \((1,1)\) is numerically close to \(2/9\). This differs from the true value of \(1/4 - 1/(4\pi) + 1/(2\pi^2)\) by about \(10^{-3}\).

There are some interesting coincidences in the (undirected) edge intensities of loop-erased random walk. For each of the square, triangular, and honeycomb lattices, there are several groups of edges which are unrelated by any symmetry of the lattice for which the undirected edge intensities are identical (see Figures 9, 10, and 11). We do not have an explanation for this phenomenon.

2. Bundles and connections

Let \(\mathcal{G}\) be a graph. Given a fixed vector space \(V\), a \textit{V-bundle}, or simply a \textit{vector bundle} \(B\) on \(\mathcal{G}\) is the choice of a vector space \(V_v\) isomorphic to \(V\) for every vertex \(v\) of \(\mathcal{G}\). We identify the vector bundle with the vector space \(V^{\mathcal{G}} = \oplus_v V_v \cong V^{|\mathcal{G}|}\). A \textit{section} of a vector bundle \(B\) is an element of \(V^{\mathcal{G}}\).

If \(H\) is a subgroup of \(\text{Aut}(V)\), an \textit{H-connection} \(\Phi\) is the choice for each directed edge \(e = (v,w)\) of \(\mathcal{G}\) of an isomorphism \(\phi_{v,w} \in H\) between the corresponding vector spaces \(\phi_{v,w} : V_v \to V_w\), with the property that \(\phi_{v,w} = \phi_{w,v}^{-1}\). This isomorphism is called the \textit{parallel transport} of vectors in \(V_v\) to vectors in \(V_w\). Given an oriented cycle \(\gamma\) in \(\mathcal{G}\) starting at \(v\), the \textit{monodromy} of the connection is the element of \(\text{Aut}(V_v)\) which is the product of these isomorphisms around \(\gamma\). Monodromies starting at different vertices on \(\gamma\) are conjugate.

Two connections \(\Phi = \{\phi_e\}\) and \(\Phi' = \{\phi'_e\}\) are \textit{gauge equivalent} if there are maps \(\psi_v : V_v \to V_v\) such that \(\phi_{v,w} \circ \psi_v = \psi_w \circ \phi'_{v,w}\) for all vertices \(v\) and \(w\) of \(\mathcal{G}\).

It is useful to extend the notion of bundle and connection to the edges as well: define for each edge \(e\) of \(\mathcal{G}\) a copy \(V_e\) of \(V\), and define maps \(\phi_{e,v} : V_v \to V_e\) whenever \(v\) is an endpoint of \(e\), with the property that \(\phi_{e,w} \circ \phi_{e,v} = \phi_{e,w}\) whenever edge \(e\) joins vertices \(v\) and \(w\).

A \textit{line bundle} is a \(V\)-bundle where \(V \cong \mathbb{C}\), the 1-dimensional complex vector space. In this case if we choose a basis for each \(\mathbb{C}\) then the parallel transport is just multiplication by an element of \(\mathbb{C}^* = \mathbb{C} \setminus \{0\}\). Furthermore, the monodromy of a cycle is in \(\mathbb{C}^*\) and does not depend on the start vertex of the cycle.

In this paper we will take \(V = \mathbb{C}^1\) or \(\mathbb{C}^2\), and use \(H = \mathbb{C}^*\) or \(H = \text{SL}_2(\mathbb{C})\)-connections.

2.1. Laplacian \(\Delta\). Let \(\mathcal{G}\) be a graph with an \(H\)-connection and let \(c : E \to \mathbb{R}_{>0}\) be a conductance associated to each edge. We then define \(\Delta : V_\mathcal{G} \to V_\mathcal{G}\) acting on sections by the formula

\[
\Delta f(v) = \sum_{w : (v,w) \in E} c_{v,w}(f(v) - \phi_{w,v}f(w)).
\]

A section is said to be \textit{harmonic} if it is in the kernel of \(\Delta\).

We define an operator \(d\) from sections of the bundle over vertices to sections over the edges, for an oriented edge \(e = (v,w)\), by

\[
\text{df}(e) = c_{v,w}(\phi_{v,e}f(v) - \phi_{w,e}f(w)),
\]

and its "adjoint"

\[
d^*\omega(v) = \sum_{e \sim v} \phi_{v,e}\omega(e).
\]

Then the Laplacian can be written \(\Delta = d^*d\) [Ken11].
A cycle-rooted spanning forest (CRSF) in a graph is a set of edges each of whose components contains a unique cycle, that is, has as many vertices as edges. A component of a CRSF is called a cycle-rooted tree (CRT).

**Theorem 2.1** ([For93] [Ken11]). For a $\mathbb{C}^*$-connection,

$$\det \Delta = \sum_{\text{CRSFs}} \prod_{\text{edges } e} c(e) \prod_{\text{cycles } C} \left( 2 - w(C) - \frac{1}{w(C)} \right),$$

where the sum is over cycle-rooted spanning forests, where the first product is over all edges of the CRSF and the second is over cycles of the CRSF, and $w(C)$ is the monodromy of the cycle $C$.

For a $U_1$-connection, $\Delta$ is Hermitian and positive semidefinite [Ken11]. The monodromy of a cycle is in $U_1$ and so $2 - w(C) - 1/w(C) \geq 0$. We can define a probability measure on CRSFs where each CRSF has probability proportional to

$$\prod_{\text{edges } e} c(e) \prod_{\text{cycles } C} \left( 2 - w(C) - \frac{1}{w(C)} \right),$$

provided there is a CRSF with nonzero weight.

A similar result holds for an $SL_2(\mathbb{C})$-connection. Now $\Delta$ is a quaternion-Hermitian matrix, that is, a matrix with entries in $GL_2(\mathbb{C})$ which satisfies $\Delta_{i,j}^* = \Delta_{j,i}^*$, where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ 

Its $q$-determinant counts CRSFs:

**Theorem 2.2** ([Ken11]). For an $SL_2(\mathbb{C})$-connection,

$$q\det \Delta = \sum_{\text{CRSFs}} \prod_{\text{edges } e} c(e) \prod_{\text{cycles } C} \left( 2 - \text{Tr } w(C) \right),$$

where the sum is over cycle-rooted spanning forests, where the first product is over all edges of the CRSF and the second is over cycles of the CRSF, and $w(C)$ is the monodromy of the cycle $C$.

In the case of an $SU_2$ connection, any cycle with a nontrivial monodromy has a positive weight, so these weights define a natural probability measure, provided there is a CRSF with nonzero weight.

For information on $q$-determinants, see [Dys70]; for the purposes of this paper one can define $q\det M = \sqrt{\det M}$, where if $M$ is an $N \times N$ matrix with entries in $GL_2(\mathbb{C})$ then $M$ is the $2N \times 2N$ matrix with $\mathbb{C}$ entries obtained by replacing each entry of $M$ by its $2 \times 2$ block of complex numbers. In the cases of primary interest $M$ is a quaternion-Hermitian, or “self-dual”, matrix; for self-dual matrices $q\det$ is a polynomial in the matrix entries. Matrices with $GL_2(\mathbb{C})$ entries enjoy many of the properties of usual matrices: for example, multiplication and addition work the same way. The inverse of a self-dual matrix is well-defined and is both a left- and right-inverse, see e.g., [Dys70].

### 2.2. Dirichlet boundary conditions.

If $B \subset \mathcal{G}$ is a set of vertices, the Laplacian with Dirichlet boundary conditions at $B$ is defined on sections over $\mathcal{G} \setminus B$ by the same formula (2.1) above with $v \in \mathcal{G} \setminus B$ and the sum over all of $\mathcal{G}$. In other words $\Delta$ is just a submatrix of the usual Laplacian on $\mathcal{G}$. Its determinant also has an interpretation. A CRSF on a graph with boundary $B$ is a set of edges such that each component is either a CRT not containing any vertex of $B$ or else a tree containing a single vertex of $B$. (When $B = \emptyset$ this specializes to the previous definition.) In this setting Theorems 2.1 and 2.2 have the same statements (where tree components do not have any monodromy term). See [Ken11].
2.3. **Green’s function** $G$. The usual Green’s function $G$ for the standard Laplacian (with Dirichlet boundary conditions) is the inverse of the Laplacian. It has the probabilistic interpretation that $G_{p,q}$ is $(\sum r c_{q,r})^{-1}$ times the expected number of visits to $q$ of a simple random walk started at $p$ (and stopped at the boundary); equivalently, it is $(\sum r c_{q,r})^{-1}$ times the sum over all paths from $p$ to $q$ which do not hit the boundary, of the probability of the path.

In the case of a graph with connection, the Green’s function $G$ is again the inverse of the Laplacian, and has a similar probabilistic interpretation:

**Proposition 2.3.** $G_{p,q}$ is $(\sum r c_{q,r})^{-1}$ times the sum over all paths from $p$ to $q$ of the product of the parallel transports along the path (from $q$ to $p$) times the path probability (from $p$ to $q$), when the sum converges absolutely. This sum converges absolutely for finite connected graphs with boundary and $U_1$ or $SU_2$ connections, and will be matrix-valued in the case of an $SU_2$ connection.

**Proof.** Using the above definition of $G$ as a sum over paths,

$$\sum_r G_{p,r} \Delta_{r,q} = G_{p,q} \sum_{r \sim q} c_{q,r} - \sum_{r \sim q} G_{p,r} c_{r,q} \phi_{q,r},$$

and since any nontrivial path to $q$ must have last step from a neighbor of $q$, this equals zero unless $p = q$ and the path has length 0, in which case the second sum is zero and the first term is $(\sum_r c_{q,r})^{-1} \sum_r c_{r,q} = 1$. \hfill $\square$

2.4. **Response matrix** $L^\prime$. Let $\mathcal{N}$ be a nonempty set of nodes of $G$, and $n = |\mathcal{N}|$. For each node $v$ pick a preferred basis for $V_v$, the vector space over $v$.

We define an $n \times n$ matrix $L^\prime = L^\prime G$ (with entries in $H$), the response matrix, or Dirichlet-to-Neumann matrix, from this data: $L^\prime : V^\mathcal{N} \to V^\mathcal{N}$ is (minus) the Schur complement of the Laplacian $\Delta$ to $\mathcal{N}$. That is, $L^\prime$ is defined as follows. Order the vertices so that $\mathcal{N}$ comes first. In this ordering the Laplacian is

$$\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$  

Then $L^\prime = -A + BC^{-1}B^*$. Note that $C$ is the Laplacian with Dirichlet boundary conditions at $\mathcal{N}$. Since $\det C$ is a weighted sum of cycle-rooted groves (defined below) and $G$ is connected to $\mathcal{N}$, $\det C$ is positive for connections in $U_1$ or $SU_2$, and $\det C$ is generically nonzero for other connections.

From the viewpoint of harmonic functions, $L^\prime$ is the Dirichlet-to-Neumann matrix: given $f \in V^\mathcal{N}$, find the unique section $h$ with boundary values $f$ at the nodes and harmonic at the interior (non-node) vertices. Then $L^\prime f = -\Delta h$ evaluated at the nodes. To see this, let $h_1$ be $h$ at the interior vertices, that is, $h = \begin{bmatrix} f \\ h_1 \end{bmatrix}$. If

$$\Delta h = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} f \\ h_1 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix},$$

then $B^* f + Ch_1 = 0$, i.e., $h_1 = -C^{-1}B^* f$, and then $c = Af - BC^{-1}B^* f = -L^\prime f$.

The response matrix $L^\prime$ has entries which are functions of the parallel transports. See [Theorem 4.2](#) below for an explicit probabilistic interpretation of the entries of $L^\prime$. In order to define $L^\prime$ as a matrix one must choose a basis in $V_v$ for each node $v$. Base changes in the $V_v$ then act on $L^\prime$ by conjugation by diagonal matrices with entries in $H$.  

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Lemma 2.4. When the Laplacian \( \Delta \) is nonsingular, the response matrix \( L \) is given by
\[
L = -(\Delta^{-1}|_N)^{-1} = -(\mathcal{G}|_N)^{-1}.
\]

Proof. Write \( \Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \) where \( A \) is the submatrix indexed by the nodes. Submatrix \( C \) is invertible since it is the Laplacian with Dirichlet boundary conditions. We have
\[
\Delta = \begin{bmatrix} A - BC^{-1}B^* & BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B^* & C \end{bmatrix}
\]
and using \( L = -A + BC^{-1}B^* \),
\[
\Delta^{-1} = \begin{bmatrix} I & 0 \\ -C^{-1}B^* & C^{-1} \end{bmatrix} \begin{bmatrix} -L^{-1} & \mathcal{L}^{-1}BC^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} -L^{-1} & \ast \\ \ast & \ast \end{bmatrix}.
\]

If \( \mathcal{G} \) is the Green’s function with boundary at node \( n \), then \( \mathcal{G} = \tilde{\Delta}^{-1} \), where the Dirichlet Laplacian \( \tilde{\Delta} \) is obtained from \( \Delta \) simply by removing row and column \( n \). Since \( L \) has the same response as \( \Delta \) on the set \( N \), the response matrix of \( \tilde{\Delta} \) is just \( L \) with row and column \( n \) removed. Thus \( [L_{i,j}]_{i=1,...,n-1} = -(\tilde{\Delta}^{-1}|_N \setminus \{n\})^{-1} \), or equivalently,
\[
(2.2) \quad [L_{i,j}]_{i=1,...,n-1} = -([\mathcal{G}_{i,j}]_{i=1,...,n-1})^{-1}.
\]
By perturbing \( \Delta \), we see that (2.2) holds even if the Laplacian \( \Delta \) is singular, so long as the Dirichlet Laplacian is nonsingular.

3. Graphs on surfaces

Let \( \Sigma \) be an oriented surface, possibly with boundary, and \( \mathcal{G} \) a graph embedded on \( \Sigma \) in such a way that complementary components (the connected components of the surface after it is cut along the edges of \( \mathcal{G} \)) are contractible or peripheral annuli (that is, an annular neighborhood of a boundary component). We call the pair \( (\mathcal{G}, \Sigma) \) a surface graph (see Figure 3).

**Figure 3.** On the left is a graph \( \mathcal{G} \) with wired boundary conditions: the outer boundary is one vertex (and the bottom edge is a self-loop) embedded in an annulus whose inner boundary is one of the squares of the grid. There is a “zipper” (edges crossing a dual path) connecting the inner boundary to the outer boundary of the annulus, and edges crossing the zipper have parallel transport \( z \) from their left endpoint to their right endpoint. We have labeled four of the vertices on the boundary of \( \Sigma \), which we call nodes. On the right is a schematic diagram of the surface graph.
3.1. **Nodes and interior vertices.** For each boundary component $C$ of the surface $\Sigma$ there is a “peripheral” cycle on $G$, bounding the annular complementary component whose other boundary is $C$. We select from this cycle a (possibly empty) set of vertices. The union of these special vertices over all boundary components will be the nodes $N$; the non-node vertices are **interior vertices** (even though these may be on the boundary of $\Sigma$).

Planar maps, in which $\Sigma$ is a topological disk, are examples of surface graphs: these are called **circular planar graphs** in [CIM98]. In this case the nodes are a subset of the vertices on the outer face.

3.2. **Flat bundles.** Given a surface graph $(G, \Sigma)$, a vector bundle on $G$ with connection $\Phi$ is **flat** if it has trivial monodromy around any loop which is contractible on $\Sigma$. In this case, the monodromy around a loop only depends on the homotopy class of the (pointed) loop in $\pi_1(\Sigma)$, and so the monodromy determines a representation of $\pi_1(\Sigma)$ into $\Aut(V_p)$, where $p$ is the base point. This representation depends on the base point $p$ for $\pi_1$; choosing a different base point will conjugate the representation.

Conversely, let $\rho \in \Hom(\pi_1(\Sigma), \Aut(V))$ be a representation of $\pi_1(\Sigma)$ into $\Aut(V)$; there is a unique (up to gauge equivalence) flat bundle with monodromy $\rho$. It is easy to construct: for example start with a trivial bundle on a spanning tree of $G$; for each additional edge the parallel transport along it is determined by the topological type of the resulting cycle created.

In the case of a line bundle, $\Aut(\mathbb{C}) = \mathbb{C}^*$ is abelian and the monodromy of a loop is well defined without regard to base point. Moreover in this case $\rho(\gamma)$ only depends on the homology class of the loop $\gamma$, since any map from $\pi_1(\Sigma)$ to an abelian group factors through $H_1(\Sigma)$.

4. **The response matrix and probabilities**

4.1. **Circular planar graphs.** In the case of a planar graph, there is no monodromy and $L = L$ is a matrix of real numbers. This case was analyzed by [CdV94], see also [CdVGV96, CIM98]. Colin de Verdière showed that response matrices $L$ of planar graphs are characterized by having nonnegative “non-interlaced” minors. Given two disjoint subsets of nodes $R$ and $S$, we say that $R$ and $S$ are non-interlaced if $R$ and $S$ are contained in disjoint intervals in the circular order on the nodes. When $|R| = |S|$, the corresponding minor is $\det(L_{R,S}) \geq 0$ (the determinant of the submatrix whose rows are indexed by $R$ and columns by $S$).

In [KW11a, Proposition 2.8] (see also [CIM98, Lemma 4.1]), there is an interpretation of the entries of $L$ in terms of groves. A **grove** is a spanning forest with the property that every component contains at least one node. The weight of a grove is the product of the conductances of its edges.

**Theorem 4.1** ([CIM98, KW11a, Pom01]). For disjoint non-interlaced subsets $R, S \subset \mathcal{N}$ with $|R| = |S|$, $\det(L_{R,S})$ is a ratio of two terms: the denominator is the weighted sum of groves in which every node is in its own component, and the numerator is the weighted sum of groves in which the nodes in $R$ are connected pairwise with nodes in $S$, and other nodes are in their own component.

In particular this proves that the non-interlaced minors of $L$ are nonnegative.

Groves can be grouped into subsets according to the way they partition the nodes (that is, the way the nodes are connected in a grove). For example, a grove
of type 1, 2 | 3, 4, 5 | 6 is one in which nodes 1 and 2 are in a tree, nodes 3, 4, 5 are in a second tree, and node 6 is in its own tree. For a partition $\sigma$ of the nodes, we let $Z[\sigma]$ denote the weighted sum of groves of type $\sigma$. For circular planar graphs with $n$ nodes on the boundary, we previously showed [KW11a] how to compute the ratio $Z[\sigma]/Z[1|2|\cdots|n]$ for any planar partition $\sigma$ of $\{1, 2, \ldots, n\}$. This ratio is an integer-coefficient polynomial in the $L_{i,j}$ [KW11a].

It is useful to allow the partition $\sigma$ to have missing indices, such as $1, 2|4, 5|6$. The nodes with the missing labeled are treated as internal vertices which can occur in any part, so that, e.g., $Z[1, 2|4, 5|6] = Z[1, 2, 3|4, 5|6] + Z[1, 2|3, 4, 5|6] + Z[1, 2|4, 5|3, 6]$. 

4.2. $L$ matrix entries. Like in Theorem 4.1 in the case of a flat bundle on a surface graph there is a combinatorial interpretation of the entries of $L$. A collection of edges of a surface graph $(\mathcal{G}, \Sigma)$ is a cycle-rooted grove (CRG) if each component is either a CRT (a component containing one cycle) not containing a node, or a tree containing at least one node. Moreover for each CRT component, the cycle must be topologically nontrivial. A CRG is distinguished from a CRSF the tree components may contain several nodes, while in a CRSF the tree components contain a unique node.

A CRG has a weight which is the product of its edge conductances times the product over its cycles of $2 - 1/w$ (for a line bundle) or $2 - \text{Tr}(w)$ (for an $SL_2(\mathbb{C})$-bundle), where $w$ is the monodromy around the cycle. For a partition $\sigma$ of the nodes, we define

$$Z[\sigma] := \text{weighted sum of cycle-rooted groves of type } \sigma$$

$$2^\mathcal{G} := \text{weighted sum of cycle-rooted groves in which all nodes are connected}$$

For example, the weighted sum of CRSFs is $2^\mathcal{G}[1|2|\cdots|n]$. Suppose the partition $\sigma$ is a partial pairing, i.e., $\sigma$ consists of doubleton and singleton parts, say $\sigma = r_1, s_1|\cdots|r_k, s_k|t_1|\cdots|t_l$. We can define

$$2^\mathcal{G}[r_1|s_1|t_1|\cdots|t_l] := \sum_{\text{CRGs of type } \sigma} (\text{weight of CRG}) \times \prod_{i=1}^{k} \text{parallel transport to } r_i \text{ from } s_i$$

for line bundles (so that the structure group is commutative and the above product makes sense), and for vector bundles when $\sigma$ has only one doubleton part.

**Theorem 4.2.** If $i \neq j$, then

$$L_{i,j} = \frac{2^\mathcal{G}[\{i\}(\text{nodes other than } i \text{ and } j \text{ in singleton parts})]}{2^\mathcal{G}[1|2|\cdots|n]}.$$ 

**Proof.** Let us first do the line bundle case. Let $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ be the Laplacian of $\mathcal{G}$, with $A$ indexed by the nodes $\mathcal{N}$.

We make a new graph $\mathcal{G}'$ by adding an edge $e_{i,j}$ (with unit conductance) to $\mathcal{G}$ which connects $i$ and $j$ and has parallel transport $z$ when directed from $i$ to $j$. Let $\tilde{\Delta}$ be the line bundle Laplacian on the new graph $\mathcal{G}'$, with Dirichlet boundary conditions at the nodes except nodes $i$ and $j$, that is, $\tilde{\Delta} = \begin{bmatrix} a & b \\ b^* & C \end{bmatrix}$ where $a = \begin{bmatrix} A_{i,i} + 1 & A_{i,j} - z^{-1} \\ A_{j,i} - z & A_{j,j} + 1 \end{bmatrix}$ and $b$ is the $i$th and $j$th column of $B$. 


By Theorem 2.1 (and its extension discussed in section 2.2), \(-[z](\det \tilde{\Delta})\) is a sum of CRSFs with each node except \(i,j\) in its own tree component, and nodes \(i,j\) in a cycle containing edge \(e_{i,j}\), and the weight includes the parallel transport of the path in \(\mathcal{G}\) from \(j\) to \(i\). (Here \([z^\alpha]f(z)\) refers to the coefficient of \(z^\alpha\) in \(f(z)\).) We can write

\[
\tilde{\Delta} = \begin{bmatrix} a - bC^{-1}b^* & bC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ b^* & C \end{bmatrix},
\]

\[
\det \tilde{\Delta} = \det[a - bC^{-1}b^*]\det C.
\]

However

\[
[z] \det \tilde{\Delta} = -[z^0][a - bC^{-1}b^*]_{i,j} = -L_{i,j}.
\]

Finally, \(\det C\) is the sum of CRSFs.

The proof in the \(\text{SL}_2(\mathbb{C})\)-bundle case is similar. Let \(\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\) be the \(\text{SL}_2(\mathbb{C})\)-bundle Laplacian of \(\mathcal{G}\). Add an edge \(e_{i,j}\) to \(\mathcal{G}\) from node \(i\) to node \(j\) with parallel transport \(M \in \text{SL}_2(\mathbb{C})\). As above let \(\tilde{\Delta} = \begin{bmatrix} a & b \\ b^* & C \end{bmatrix}\) where \(a = \begin{bmatrix} A_{i,i} + I & A_{i,j} - M^{-1} \\ A_{j,i} - M & A_{j,j} + I \end{bmatrix}\) and \(b\) is the \(i\)th and \(j\)th column of \(B\). Now \(\det \tilde{\Delta}\) gives a weighted sum of CRSFs with each node except \(i,j\) in its own tree component, where the weight is the product of the monodromies along the cycles. If the CRSF contains a cycle that uses edge \(e_{i,j}\), then the monodromy of this cycle will depend on \(M\), and otherwise, the weight of the CRSF does not depend on \(M\). We write

\[
\qdet \Delta' = C_0 + \sum_\omega C_\omega (2 - \text{Tr}(K_\gamma M)),
\]

where the sum is over configurations \(\omega\) with a cycle \(\gamma\) containing edge \(e_{i,j}\), \(K_\gamma\) is the parallel transport to \(i\) from \(j\) in the cycle \(\gamma\), and \(C_0, C_\gamma\) and \(K_\gamma\) do not depend on \(M\).

We have (4.1) in this case as well, where \(C\) does not depend on \(M\). Letting \(D = bC^{-1}b^*\), which does not depend on \(M\), we can write

\[
a - bC^{-1}b^* = \begin{bmatrix} A_{i,i} + I - D_{i,i} & A_{i,j} - D_{i,j} - M^{-1} \\ A_{j,i} - D_{j,i} - M & A_{j,j} + I - D_{j,j} \end{bmatrix}.
\]

This is a \(2 \times 2\) matrix with entries in \(\text{GL}_2(\mathbb{C})\). The reader may check that for a \(2 \times 2\) matrix with entries in \(\text{GL}_2(\mathbb{C})\),

\[
\qdet \begin{bmatrix} zI & Y \\ Y^* & zI \end{bmatrix} = xz - \det Y = xz - \frac{1}{2} \text{Tr} YY^*.
\]

Consequently

\[
\qdet(a - bC^{-1}b^*) = \frac{1}{2} \text{Tr}[(A_{i,j} - D_{i,j} - M)(A_{i,j}^* - D_{i,j}^* - M)] + C_1 = \text{Tr}[(A_{i,j} - D_{i,j})M] + C_2
\]

where \(C_1\) and \(C_2\) do not depend on \(M\). Comparing with (4.2) we see that, since \(M\) was arbitrary,

\[
- \sum_\omega C_\omega K_\omega = A_{i,j} - D_{i,j} = [A - BC^{-1}B^*]_{i,j} \det C = -L_{i,j}.
\]

Principal minors of \(L\) also have probabilistic interpretations:
**Theorem 4.3.** Suppose $T \subset N$ and $Q = \{q_1, \ldots, q_\ell\} = N \setminus T$. Then

$$\det \mathcal{L}_T^T = (-1)^{|T|} \frac{\mathcal{Z}[q_1, q_2, \ldots, q_\ell]}{\mathcal{Z}[1, 2, \ldots, n]}.$$  

**Proof.** Order the vertices of $G$ by first $N \setminus T$, then $T$, then the internal nodes. In this order we have

$$\Delta = \begin{bmatrix} A_1 & A_2 & B_1 \\ A_2^* & A_3 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix}. $$

Then

$$\mathcal{L} = - \begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} C^{-1} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}$$

and $\det \mathcal{L}_T^T = \det(-A_3 + B_2C^{-1}B_2^*)$. The proof follows from the identity

$$\begin{bmatrix} A_3 & B_2 \\ B_2^* & C \end{bmatrix} = \begin{bmatrix} A_3 - B_2C^{-1}B_2^* & B_2C^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ B_2^* & C \end{bmatrix}$$

upon taking determinants: the left-hand side determinant is the weighted sum of CRGs of $G_T$, the right-hand side determinant is $(-1)^{|T|}$ times the product of $\det \mathcal{L}_T^T$ and $\det C$ which counts CRSFs. In the SL$_2(\mathbb{C})$-case we used the fact that

$$q\det \begin{bmatrix} X & Y \\ 0 & I \end{bmatrix} = q\det \begin{bmatrix} X & 0 \\ Y & I \end{bmatrix} = q\det X.$$  

For line bundles there is an interpretation of more general minors:

**Theorem 4.4.** Suppose $Q = \{q_1, \ldots, q_\ell\}$, $R = \{r_1, \ldots, r_k\}$, $S = \{s_1, \ldots, s_k\}$, and $T$ are disjoint sequences of nodes for which $|R| = |S|$ and $N = Q \cup R \cup S \cup T$. Then

$$(-1)^{|T|} \det \mathcal{L}_{R,T}^{S,T} = \sum_{\text{permutations } \rho} (-1)^{\rho} \frac{\mathcal{Z}[r_1^{(1)}, \ldots, r_k^{(1)}, s_1, \ldots, s_k]}{\mathcal{Z}[1, 2, \ldots, n]}.$$  

**Proof.** We use a block LU-factorization of $\Delta$, as in the previous proof, to find

$$(-1)^{|R|+|T|} \det \mathcal{L}_{R,T}^{S,T} \det \Delta_T^I = \det \Delta_{R,T,I}^S,$$

where $I$ is the set of internal nodes. $\det \Delta_T^I = \mathcal{Z}[1, \ldots, n]$. So we need to evaluate $\det \Delta_{R,T,I}^S$. The proof now follows the proof of Theorem 2.1 which is found in [Ken11, proof of Theorem 1]. Write $\Delta = d^*d$ where $d$ is the operator from sections over the vertices to sections over the edges. Then $\Delta_{R,T,I}^{S,T} = d_{S,T,I}^*d_{R,T,I}$ where $d_X$ is the restriction of $d$ to sections over $X$. By the Cauchy-Binet theorem,

$$\det d_{S,T,I}^*d_{R,T,I} = \sum_Y \det(d_{S,T,I}^Y)^* \det d_{R,T,I}^Y,$$

where the sum is over collections of edges $Y$ of cardinality $|S \cup T \cup I|$. The nonzero terms in the sum are collections of edges in which each component is a CRT if we glue $r_i$ to $s_i$ for each $i$. Equivalently, each component is either a CRT or a tree containing a unique $r \in R$ and $s \in S$. The weight of a component with a cycle is $2 - w - 1/w$ where $w$ is the monodromy of the cycle; the weight of a path is the parallel transport to the $r$ from the $s$. It remains to compute the signature of each configuration.

This signature is the same as the signature in the case of a trivial bundle, which is determined by [CIM98] to be the signature of the permutation from $R$ to $S$ determined by the pairing.  

$\square$
4.3. Cycle-rooted grove probabilities.

**Theorem 4.5.** The probability of any topological type of CRG involving only two-node connections and loops is a function of $\mathcal{L}$ and the weighted sum of CRSFs.

**Proof.** By a result of Kenyon [Ken11] (based on a theorem of Fock and Goncharov [FG06]), on a graph embedded on a surface with no nodes one can compute the probability of any topological type of CRSF (that is, the probability that a CRSF has a given set of homotopically nontrivial cycles up to isotopy) from the determinant of the Laplacian considered as a function on the space of flat $\text{SL}_2(\mathbb{C})$-connections. Indeed, as $X$ runs over all possible “finite laminations”, that is, isotopy classes of collections of finite, pairwise disjoint, topologically nontrivial simple loops on the surface, the products $\prod_{\text{cycles in } X} (2 - \text{Tr } w)$ form a basis for a vector space (the vector space of regular functions on the representation variety) and the Laplacian determinant is an element of this vector space. In other words, Theorem 2.2 above shows that $\det \Delta = \sum_X C_X \prod_{\text{cycles in } X} (2 - \text{Tr } w)$, where $X$ runs over finite laminations; such an expression determines each coefficient $C_X$ uniquely.

To compute the probability that a random CRG $Y$ on $\Sigma$ has a fixed topology of node connections, add edges $e_{i,j}$ to $G$ connecting endpoints of all two-node connections $i \rightarrow j$ of $Y$; the resulting graph $G'$ can be embedded on a surface $\Sigma'$ containing $\Sigma$, and the union of $Y$ and the new edges is a CRSF on $G'$. (We obtain $\Sigma'$ from $\Sigma$ by gluing a single strip running from $i$ to $j$ for each $e_{i,j}$; in this way cycles containing different $e_{i,j}$s are in different homotopy classes.)

Any CRG on $G$ with the same node connection type as $Y$ can be completed to a CRSF on $\Sigma'$ by adding the edges $e_{i,j}$. Conversely (since each added edge $e_{i,j}$ is in a different homotopy class on $\Sigma'$) each CRSF of this topological type comes from a CRG on $G$ with the same connection type as $Y$.

The flat connection on $G$ can be extended to a flat connection on $G'$ by taking generic parallel transports $\phi_{i,j}$ along the $e_{i,j}$.

It remains to show that the Laplacian determinant of $G'$ is a function of $\mathcal{L}$, the new parallel transports $\Phi = \{\phi_{i,j}\}$, and det $C$. However $\Delta^{G'} = \Delta + S_\Phi$ where $S_\Phi$ is supported on the nodes; using $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ we have

$$\det \Delta^{G'} = \det \begin{bmatrix} A + S_\Phi & B \\ B^* & C \end{bmatrix} = \det \begin{bmatrix} A + S_\Phi - B C^{-1} B^* & B C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B^* & C \end{bmatrix} = \det(-L + S_\Phi) \det C. \quad \square$$

5. Basic surface graphs

The simplest non-circular-planar case is the annulus. Since $\pi_1(\Sigma)$ is abelian in this case it usually suffices to consider a line bundle rather than a two-dimensional bundle. The $\mathcal{L}$ matrix then depends on a single variable $z \in \mathbb{C}^*$ which is the monodromy of a flat connection. For simplicity we choose a connection which is the identity on all edges except for the edges crossing a “zipper”, that is, a dual path connecting the boundaries; these edges have parallel transport $z$.

Suppose $(G, \Sigma)$ is a surface graph on an annulus with $n_1$ nodes on one boundary component and $n_2$ on the other. Then $\mathcal{L}$ is an $(n_1 + n_2)$-dimensional matrix with entries which are rational in $z$. Let $\mathcal{L}_0 = \mathcal{L}[1][2] \cdots [n]$ be the weighted sum of
CRSFs of $G$ (CRGs in which each node is in a separate component). We have

$$\mathcal{Z}(1|2|\cdots|n) = \sum_k \alpha_k (2 - z - z^{-1})^k,$$

where $\alpha_k$ is the weighted sum of CRSFs with $k$ cycles winding around the annulus.

While we have not attempted to show that every connection probability can be computed via the $\mathcal{L}$-entries, we present here some cases of small $n_1, n_2$.

5.1. Annulus with $(2, 0)$ boundary nodes. Suppose there are two nodes on one boundary and none on the other. Then $\mathcal{L}_{1,2}(z)$ counts connections to 1 from 2. There are only two topologically different configurations, which are illustrated in Figure 4. We have the following theorem.

**Theorem 5.1.** $\frac{\partial}{\partial z} \log \mathcal{L}_{1,2}(z)|_{z=1}$ is the probability that the LERW to 1 from 2 crosses the zipper.

**Proof.** Let $A_k, B_k,$ and $\alpha_k$ (respectively) be the weighted sum of cycle-rooted groves which contain $k$ cycles winding around the annulus, and in which nodes 1 and 2 are (respectively) connected by a path not crossing the zipper, connected by a path crossing the zipper, or are not connected. Then

$$\mathcal{Z}[1|2] = \sum_{k \geq 0} [A_k (2 - z - 1/z)^k + B_k z (2 - z - 1/z)^k] = A_0 + B_0 z + O((z - 1)^2)$$

$$\mathcal{Z}[2|1] = \sum_{k \geq 0} \alpha_k (2 - z - 1/z)^k = \alpha_0 + O((z - 1)^2)$$

By Theorem 4.2 $\mathcal{L}_{1,2} = \mathcal{Z}[2|1]/\mathcal{Z}[1|2]$, so

$$\frac{\partial}{\partial z} \log \mathcal{L}_{1,2}(z) = \frac{\partial_z \mathcal{Z}[2|1]}{\mathcal{Z}[2|1]} - \frac{\partial_z \mathcal{Z}[1|2]}{\mathcal{Z}[1|2]} = \frac{B_0}{A_0 + B_0} + O(z - 1). \quad \square$$

![Figure 4](image)

**Figure 4.** The two topologically distinct ways to connect the nodes on an annulus with $(2, 0)$ boundary nodes.

5.2. Annulus with $(1, 1)$ boundary nodes. We let $c_j$ denote the weighted sum of CRGs connecting 1 and 2 and such that the path from 1 to 2 crosses the zipper $j$ times algebraically (see Figure 5). Then $\mathcal{Z}[1|2] = \sum_{j \in \mathbb{Z}} c_j z^j$, from which one can extract $c_j$ for each $j$. Theorem 4.2 shows that $\mathcal{L}_{1,2} = \mathcal{Z}[2|1]/\mathcal{Z}[1|2]$, so $\mathcal{L}_{1,2}$ and $\mathcal{Z}[1|2]$ together determine the winding distribution (which we knew already from Theorem 4.5). But $\mathcal{Z}[1|2]$ is also a function of $z$, so $\mathcal{L}_{1,2}$ does not by itself determine the winding distribution. But from $\mathcal{L}_{1,2}$ we can extract the expected number of algebraic crossings of the zipper via

$$\mathbb{E}[\# \text{ algebraic crossing of zipper}] = \left. \frac{\partial}{\partial z} \log \mathcal{L}_{1,2} \right|_{z=1}.$$
5.3. **Annulus with \((3, 0)\) boundary nodes.** This is a case which can be derived from the \((2, 0)\) case using [Theorem 4.4](#). Suppose nodes 1, 2, 3 are in counterclockwise order on the inner boundary, with a counterclockwise zipper between nodes 1 and 3. Consider the case when all nodes are connected; there are three possible configurations \(A_1, A_2, A_3\) correspond to which complementary component of the triple connection the outer boundary component lies (see Figure 6). The numerator of \(L_{1,3}^{2,3}\) is \(A_1 + A_2 + zA_3\), the numerator of \(L_{1,2}^{2,3}\) is \(zA_1 + A_2 + zA_3\), and the numerator of \(L_{2,1}^{2,3}\) is \(zA_1 + A_2 + A_3\). These three quantities suffice to determine \(A_1, A_2, A_3\).

![Figure 5](image-url). Different numbers of windings of the path to 1 from 2. These configurations contribute to \(c_1, c_0, c_{-1}\), and \(c_{-2}\) respectively.

![Figure 6](image-url). The three topologically distinct subcases when the three nodes are connected (for the annulus with \((3, 0)\) nodes).

5.4. **Annulus with \((3, 1)\) boundary nodes.** This is a case we will need when we do the LERW computations. Suppose there are 4 nodes in all, with nodes 1, 2, 3 on the inner boundary in counterclockwise order and 4 on the outer boundary. Suppose the zipper starts between 1 and 3 and is oriented counterclockwise, as in the figure. We wish to compute the ratios \(Z[1, 2|3, 4]/Z[1, 2|3, 4]\) and \(Z[1, 2|3, 4]/Z[1, 2, 3, 4]\) (as before, \(Z[\sigma]\) denotes the weighted sum of groves of type \(\sigma\), for the trivial bundle).

Recall that \(\mathcal{Z}_{1/3}^{2/4}\) denotes the weighted sum of cycle-rooted groves of type \(1/3\), times the parallel transport of the path to node 1 from 2 and the path to node 3 from node 4, so that for a trivial bundle, \(\mathcal{Z}_{1/3}^{2/4} = Z[1, 2|3, 4]\). Because of the path connecting nodes 3 and 4, in fact there will be no cycles in the CRG. Similarly, \(\mathcal{Z}_{1/2}^{3/4}\) and \(\mathcal{Z}_{2/1}^{3/4}\) denote the weighted sum of CRGs of type \(1/2\) and \(2/1\), times their respective parallel transports.

By [Theorem 4.4](#), \(\mathcal{Z}_{1/2}^{3,4}\) has numerator counting connections \(1/2\) and \(1/2\) (with a minus sign). Similarly for \(\mathcal{Z}_{1/3}^{3,4}\) and \(\mathcal{Z}_{1,4}^{2,3}\). With \(Z_0\) denoting the sum of CRSFs,
we have
\[
Z_0 \frac{\det \mathcal{G}_{i,j}^{2,4}}{\mathcal{G}_{i,j}^{1,3}} = \mathcal{G}_{i,j}^{2,4} - \mathcal{G}_{i,j}^{1,3} = \mathcal{G}_{i,j}^{2,4} - \mathcal{G}_{i,j}^{1,3}
\]
(5.1)
\[
Z_0 \frac{\det \mathcal{G}_{i,j}^{3,4}}{\mathcal{G}_{i,j}^{3,2}} = \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{3,2} = \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{3,2}
\]
(5.1)
\[
Z_0 \frac{\det \mathcal{G}_{i,j}^{3,4}}{\mathcal{G}_{i,j}^{2,1}} = \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{3,2} = \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{3,2}
\]
As a consequence
\[
\mathcal{G}_{i,j}^{1,3} - \mathcal{G}_{i,j}^{1,2} \mathcal{G}_{i,j}^{3,4} + \mathcal{G}_{i,j}^{1,3} \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{1,2} \mathcal{G}_{i,j}^{3,4} - \mathcal{G}_{i,j}^{1,3} \mathcal{G}_{i,j}^{3,4} + \mathcal{G}_{i,j}^{1,2} \mathcal{G}_{i,j}^{3,4}
\]
(5.2)
When \( z \to 1 \) both the numerator and denominator of (5.2) converge to 0, so we evaluate the limiting ratio using l’Hôpital’s rule. We can expand \( \mathcal{G}_{u,v} = \mathcal{L}_{u,v} + (z-1)L_{u,v} + O((z-1)^2) \), where \( L_{u,v} \) is symmetric and \( L_{u,v} \) is antisymmetric. Then in the limit \( z \to 1 \) this gives
\[
\frac{Z[1,2,3,4]}{Z[1,2,3,4]} = \lim_{z \to 1} \frac{\mathcal{G}_{i,j}^{1,3} + h(z-1)^2}{1 - z^2} = -L_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,2}L_{3,4} - L_{1,3}L_{2,4}
\]
(5.3)
It is useful to express this above formula in terms of the Green’s function where the \( n \)th node is the boundary. We can express \( \mathcal{L}_{i,n} = -\sum_{j=1}^{n-1} L_{i,j} \) and \( \mathcal{G}_{i,j}^{3,4}/\mathcal{G}_{i,j}^{1,3} = \det[\mathcal{G}_{i,j}]_{i=1,...,n-1} \) using Theorem 4.3. Let \( \mathcal{G}_{i,n} = G_{i,n} + (z-1)G'_{i,n} + O((z-1)^2) \). Then using (2.2) to express \( \mathcal{G} \) in terms of \( \mathcal{G} \) and taking the limit \( z \to 1 \), some algebraic manipulation yields
\[
\frac{Z[1,2,3,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{1,2} - G_{1,3}
\]
(5.4)
Corollary 6.6 gives a much easier way to convert an \( L \)-formula into a \( G \)-formula.
While the left-hand sides of these formulas (5.3) and (5.4) are symmetric in nodes 1 and 2, the right-hand sides are not. This asymmetry is due to the location of the zipper, and moving the zipper would change the values of the \( L_{i,j} \) and the \( G'_{i,j} \) (albeit in a predictable way). If we keep the zipper between nodes 1 and 3, then we should expect a different formula for \( Z[1,3,2,4] \) than what we would get by permuting the indices 1, 2, 3 in the formula for \( Z[1,2,3,4] \). Indeed, if we carry out the computations as above, we obtain
\[
\frac{Z[1,2,3,4]}{Z[1,2,3,4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,2}L_{3,4} - L_{1,3}L_{2,4}
\]
(5.5a)
\[
\frac{Z[1,2,3,4]}{Z[1,3,2,4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4}
\]
(5.5b)
\[
\frac{Z[2,3,1,4]}{Z[1,2,3,4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,4}L_{2,3} - L_{1,3}L_{2,4}
\]
(5.5c)
and
\[
\frac{Z[1,2,3,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{1,2} - G_{1,3}
\]
(5.6a)
\[
\frac{Z[1,3,2,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1}
\]
(5.6b)
\[
\frac{Z[2,3,1,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{2,3} - G_{1,3}
\]
(5.6c)
5.5. **Pair of pants with** $(2, 0, 0)$ **boundary nodes.** The following two annulus cases are most easily viewed as special cases of the case when the surface $\Sigma$ is a pair of pants with 2 nodes on one boundary and no other nodes, see Figure 7. Put an $SL_2(\mathbb{C})$ bundle with monodromies $A$ and $B$ around the two central holes $C_A$ and $C_B$, and supported on zippers from the holes to the boundary between nodes 1 and 2.

The parallel transport of a path to node 1 from node 2 is of the form

1. $I$, if the path has both holes on its right
2. $AB$, if the path has both holes on its left
3. $(AB)^{-k}A(AB)^k$ for some $k \in \mathbb{Z}$, if the path has the lower hole on its left and the upper hole on its right, and $k$ is the algebraic number of crossings that a dual path from the lower hole to the left boundary makes across the upper zipper
4. $(AB)^{-k}B(AB)^k$ for some $k \in \mathbb{Z}$, if the path has the lower hole on its right and the upper hole on its left, and $k$ is the algebraic number of crossings that a dual path from the upper hole to the left boundary makes across the lower zipper

We let $c^{(RR)}$, $c^{(LL)}$, $c_k^{(LR)}$, and $c_k^{(RL)}$ (for $k \in \mathbb{Z}$) denote the weighted sum of cycle-rooted groves of the above types. We further let $c_{\ell}^{(RR)}$ and $c_{\ell}^{(LL)}$ denote the number of cycle-rooted groves of type $c^{(RR)}$ and $c^{(LL)}$ in which there are $\ell \in \mathbb{N}$ loops that surround both holes.

We need to choose matrices $A$ and $B$ for which $\det A = 1$ and $\det B = 1$, and it is convenient to choose them so that $\operatorname{Tr}(A) = 2$ and $\operatorname{Tr}(B) = 2$ (so that loops which surround one hole but not the other have weight 0), and so that $AB$ is diagonal. We can take

\[
A = \begin{bmatrix}
\frac{2x}{x-1} & \frac{y}{y(x+1)} \\
\frac{x-1}{y(x+1)} & \frac{x}{x+1}
\end{bmatrix} \quad B = \begin{bmatrix}
\frac{2x}{x+1} & -\frac{y}{x+1} \\
\frac{y(x+1)}{x-1} & \frac{x-1}{x+1}
\end{bmatrix} \quad AB = \begin{bmatrix}
x & 0 \\
0 & 1/x
\end{bmatrix}
\]

![Figure 7](image-url) **Figure 7.** Some of the possible topological types for the path between nodes 1 and 2 when the surface is a pair of pants with both nodes on one boundary. The lower zipper (in red) has parallel transport $A$, and the upper zipper (in blue) has parallel transport $B$. For each diagram, the parallel transport of the path to 1 from 2 is shown.
for variables $x$ and $y$. Then since $AB$ is diagonal, it is straightforward to evaluate

$$ (AB)^{-k}A(AB)^k = \begin{bmatrix} \frac{2x}{x+1} & \frac{yx-2k}{2} \\ \frac{(x-1)x^{2k}}{y(x+1)^x} & \frac{x+1}{x+1} \end{bmatrix} $$

$$ (AB)^{-k}B(AB)^k = \begin{bmatrix} \frac{2x}{x+1} & -\frac{xy-2k-1}{2} \\ \frac{(x-1)x^{1+2k}}{y(x+1)^x} & \frac{x+1}{x+1} \end{bmatrix}. $$

$$ Z[2^1] = I \sum_{\ell \in \mathbb{N}} c^{(RR)}_{\ell} (2 - x - 1/x)^\ell + AB \sum_{\ell \in \mathbb{N}} c^{(LL)}_{\ell} (2 - x - 1/x)^\ell $$

$$ + \sum_{k \in \mathbb{Z}} c^{(LR)}_k (AB)^{-k}A(AB)^k + \sum_{k \in \mathbb{Z}} c^{(RL)}_k (AB)^{-k}B(AB)^k. $$

Only the last two sums contribute to the 1,2 entry of $Z[1]$: $Z[1]_{1,2} = y \sum_{k \in \mathbb{Z}} [c^{(LR)}_k x^{-2k} - c^{(RL)}_k x^{-2k-1}].$

This is a Laurent series in $x$ from which one can extract the coefficients $c^{(LR)}_k$ and $c^{(RL)}_k$. Once these are known, the coefficients $c^{(RR)}_\ell$ and $c^{(LL)}_\ell$ can be extracted from $Z[1]_{1,1}$ and $Z[1]_{2,2}$.

5.6. **Annulus with (2, 2) boundary nodes.** On the annulus with 4 nodes, put nodes 1, 2 on the outer boundary and 3, 4 on the inner boundary. Suppose we wish to compute the probability of the connections 13|24 and 14|23. This computation can be used to compute the probability that an edge $e$ is on the LERW from node 1 to node 2 (an equivalent calculation was done in [Ken00a]).

Insert an extra edge $e_{34}$ from 3 to 4; this “splits” the inner boundary into two (see Figure 8). This is then a special case of the construction of section 5.5. In the notation of that section, it suffices to use the limit $x \to -1$ and the value $x = 1$ to distinguish crossings 13|24 and 14|23: $Z_{1,2}^1$ in the limit $x \to -1$ gives the sum and in the case $x = 1$ the difference of the two desired quantities.

![Figure 8. Computing crossings 13|24 and 14|23.](image)

5.7. **Annulus with (4, 0) boundary nodes.** When 4 nodes are on the outer boundary and none on the inner (and nodes 1, 2, 3, 4 are in counterclockwise order), the case we have not yet discussed is the 14|23 case: there are three subcases depending on whether the paths from 1 to 4 and 2 to 3 go left or right of the inner boundary.
Again this is a special case of the \((2,0,0)\) case if we put an extra edge between nodes 3 and 4.

In this case the only possible parallel transports to 1 from 2 are (using the connection from that section)

\[
AB, I, B, A, B^{-1}AB,
\]

and only in the first two cases is there a possible extra loop surrounding both \(C_A\) and \(C_B\). Thus

\[
\mathcal{Z}_{1,2} = I(\alpha_0^{(RR)} + \alpha_1^{(RR)}(2 - x - 1/x)) + AB(\alpha_0^{(LL)} + \alpha_1^{(LL)}(2 - x - 1/x))
+ Bc_0^{(RL)} + A\alpha_0^{(LR)} + B^{-1}AB\alpha_1^{(LR)}.
\]

The 1,2 entry in \(\mathcal{Z} \) is

\[
[\mathcal{Z}_{1,2}]_{1,2} = -\frac{y}{x}\alpha_0^{(RL)} + y\alpha_0^{(LR)} + \frac{y}{x^2}\alpha_1^{(LR)}.
\]

From this we can extract the three cases of interest.

6. Annular-one surface graphs

Suppose that the graph has \(n\) nodes and is embedded on an annulus such that nodes 1, \ldots, \(n - 1\) are on the inner boundary of the annulus arranged in counterclockwise order, and node \(n\) is by itself on the outer boundary, and that the zipper is between nodes \(n - 1\) and 1 and directed in the counterclockwise direction (from \(n - 1\) to 1), as in section 5.4. We call these annular-one surface graphs; they are the next case after circular planar graphs, and they play an important role in our loop-erased random walk calculations in section 8. Annular-one surface graphs of course include the \((1,1)\) and \((3,1)\) cases that we did in the last section, but for expository purposes we treated those special cases separately. We are interested in computing, for any partition \(\sigma\) in which \(n\) is not in a component by itself, the weighted sum of groves of type \(\sigma\), which we denote \(Z[\sigma]\). We show how to compute \(Z[\sigma]/Z[1, 2, \ldots, n]\) in terms of the response matrix \(\mathcal{Z}\), and \(Z[\sigma]/Z[1, 2, \ldots, n]\) in terms of the Green’s function \(\mathcal{G}\).

6.1. Reduction to partial pairings.

**Theorem 6.1.** For a circular planar graph with \(n\) nodes, for any partition \(\sigma\) of the nodes, we can write \(Z[\sigma] = \sum_{m} \alpha_m Z[\tau_m]\), where the \(\tau_m\)’s are partial pairings.

**Proof.** Let \(i\) be the smallest node label that is in a part of \(\sigma\) of size more than 2, and let \(s\) be the size of this part. We measure the “complexity” of partition \(\sigma\) by \(n(n-i)+s\). Let \(j\) be the next-smallest item in \(i\)’s part of \(\sigma\). Let \(\sigma^*\) denote the partition obtained from \(\sigma\) by “de-listing” \(j\), i.e., by regarding \(j\) as an internal vertex which can occur in any of \(\sigma^*\)’s parts. If \(\sigma\) has \(k\) parts, then we can write \(Z[\sigma^*] = \sum_{\ell=1}^{k} Z[\sigma^*\text{ with } j\text{ added to } \ell\text{th part}]\). One of these terms is \(Z[\sigma]\), so

\[
Z[\sigma] = Z[\sigma^*] - \sum_{\ell} Z[\sigma^*\text{ with } j\text{ added to } \ell\text{th part}],
\]

where the sum runs over all parts of \(\sigma^*\) except the one containing \(i\). Because the graph is circular planar, unless \(j\) is added to a part of \(\sigma^*\) that is “covered” by the part containing \(i\), there will be no groves of that partition type. Each nonzero term on the right has smaller complexity than \(\sigma\), so we can iterate this process...
to eventually express $Z[\sigma]$ as a linear combination of $Z[\tau]$’s where $\tau$ is a partial pairing. □

For example,

$$Z[1, 5, 8|2, 3, 4|6, 7] = Z[1, 8|2, 3, 4|6, 7] - Z[1, 8|2, 3, 4, 5|6, 7] - Z[1, 8|2, 3, 4|5, 6, 7]$$

$$= Z[1, 8|2, 4|6, 7] - Z[1, 8|2, 5|6, 7] - Z[1, 8|2, 4|5, 7].$$

There can be multiple such linear combinations for a given partition.

**Theorem 6.2.** For an annular-one surface graph with $n$ nodes, for any partition $\sigma$ of the nodes, we can write $Z[\sigma] = \sum_m a_m Z[\tau_m]$, where the $\tau_m$’s are partial pairings.

**Proof.** If the part containing $n$ has size two, say that it is $\{h, n\}$, then we list the remaining nodes in the order $h + 1, h + 2, \ldots, n - 1, 1, 2, \ldots, h - 1$, and do the reductions described above for circular planar graphs. These will not increase the size of $n$’s part. If the part containing $n$ has more than two nodes, then we first reduce its size by internalizing nodes in the part other than $n$ until it has size two. If $n$ started out in a singleton part, we start out as in the circular planar case (with the order $1, \ldots, n - 1$) until a node gets adjoined to $n$’s part. □

A similar reduction can be done for the annulus with 2 nodes on each boundary, but not for the annulus with 2 nodes on one boundary and 3 on the other.

6.2. **Partial pairings in terms of the response matrix.** Recall the computation $Z[1, 2|3, 4]/Z[1|2|3|4]$ in § 5.4 [Theorem 4.3] provided a family of equations for subdeterminants of the response matrix in terms of grove partition functions. We solved these equations for the grove partition functions in terms of the subdeterminants, and took the limit $z \to 1$ to express $Z[1, 2|3, 4]/Z[1|2|3|4]$ in terms of $L_{u,v}$ and $L'_{u,v}$. We follow the same approach here. For ease of exposition we focus on complete pairings. Partial pairings are handled in the same way, except that the subdeterminants have extra rows and columns and minus signs corresponding to the internalized nodes (recall [Theorem 4.3]). The singleton nodes have no effect on the determinant formulas, except insofar as they affect the values of the $L_{u,v}$ and $L'_{u,v}$ variables.

For complete pairings, there are $n - 1$ ways to connect the two boundaries (ignoring windings). When the annulus is cut along this connection, the domain becomes planar, so there are $C_{n/2} - 1$ ways to pair up the remaining nodes, where $C_k$ is the $k$th Catalan number. We have

$$(n - 1)C_{n/2 - 1} = (n - 1)\frac{(n - 2)!}{(n/2 - 1)!(n/2)!} = \frac{1}{2} \frac{n!}{(n/2)!(n/2)!} = \frac{1}{2} \binom{n}{n/2}.$$

So the number of annular pairings equals the number of equations arising from the determinant formula. In fact, there is a natural bijection between the $L'$-determinants and the annular pairings which is based on the cycle lemma of Dvoretzky and Motzkin [DM47], which we use in Appendix A.2 to show that these equations are linearly independent for any even $n$.

In any directed pairing, the connection between node $n$ and the other boundary determines whether or not and in what direction that any other directed pair crosses the zipper. Reversing the direction of any directed pair (other than the pair containing $n$) that crosses the zipper introduces a factor of $z^2$. Since only even powers of $z$ appear, it is convenient to change variables to $\zeta = z^2$. For example, $\mathcal{P}^{5|2|6}_{3|1|4} = \zeta \mathcal{P}^{3|2|6}_{3|1|4}$. 

\[\]
For more compact notation let $\tilde{Z}^i_\sigma := \tilde{Z}_\sigma / \tilde{Z}[1|2| \ldots |n]$. When expanding an $L$-determinant into a signed sum of $Z^i_\sigma$'s, where $\sigma$ is a directed pairing, $Z^i_\sigma$ can be put into a canonical form $\zeta^\text{power} Z^i_\sigma'$, where $\sigma'$ is a directed pairing in which the pairs are directed counterclockwise around the annulus. Our goal is to solve for $\tilde{Z}^i_\tau = \tilde{Z}^i_\tau(\zeta)$ in terms of the $L$-determinants and $\zeta$, and take the limit $\zeta \to 1$.

The system of linear equations can be represented by a matrix $A_n$. When recording the linear equation corresponding to $\det_\tau L$, we can re-order $R$ and $S$ in any manner, and this would just scale row $\det_\tau L$ of $A_n$ by $\pm 1$, which has no effect on our ability to solve for the $Z^i_\sigma$'s. But the signs in $A_n^{-1}$ are surprisingly nice when we order $R$ and $S$ in a manner that corresponds to $\det_\tau L$'s associated pairing in the aforementioned bijection. This canonical ordering is described in Appendix A.1.

$A_2$ is just the $1 \times 1$ matrix whose entry is 1 (since $Z^2[1]/Z[1|2] = L_{1,2} = \det L_1$):

$$A_2 = \det L_1^2 \begin{bmatrix} 1 \end{bmatrix}$$

The matrix $A_4$ encodes the system of equations (5.1) we saw for the $(3, 1)$ case:

$$A_4 = \begin{bmatrix} \det L^2_{1,3} & \det L^4_{3,2} & \det L^3_{2,1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -\zeta & 1 \end{bmatrix}$$

The first two rows of the next matrix $A_6$ are

$$\begin{bmatrix} \det L^2_{1,3,5} & \det L^4_{1,2,5} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta & -1 & 0 \end{bmatrix}$$

Each $\det L^S_\tau$ is a signed-sum of $(n/2)!$ of the $Z^i_\tau(\zeta)$'s, but not all of these $\sigma$'s can be embedded in the annulus, so the rows generally have fewer than $(n/2)!$ nonzero entries. For each pairing $\sigma$ that embeds in the annulus, the column $Z^i_\sigma$ contains $2^{n/2-1}$ nonzero entries: for each pair $\{i, j\}$ in $\sigma$, except the pair containing $n$, either $i \in R$ and $j \in S$ or else $j \in R$ and $i \in S$.

The inverses $A_n^{-1}$ of these matrices are

$$A_2^{-1} = Z^2_1 = \det L_1^2 \begin{bmatrix} 1 \end{bmatrix}$$
\[ A^{-1}_4 = \begin{vmatrix} 1 & \zeta & 1 \\ \zeta & 1 & 1 \\ \zeta & \zeta & 1 \end{vmatrix} \times \frac{1}{(1 - \zeta)^2} \]

and the first two rows of \( A^{-1}_6 \) are

\[
\begin{vmatrix} \zeta + 1 & \zeta + 1 & \zeta^2 + \zeta & 2\zeta & \zeta + 1 & 2 \\
\zeta & 1 & \zeta & \zeta & \zeta & \zeta^2 & \zeta & \zeta & 1 & 1 \end{vmatrix} \times \frac{1}{(1 - \zeta)^2}
\]

Notice that in the above examples, each entry of the inverse matrix \( A^{-1}_n \) is a polynomial in \( \zeta \) with non-negative integer coefficients and degree at most \( n/2 - 1 \), divided by \( (1 - \zeta)^{n/2 - 1} \). Notice also that these polynomials, when evaluated at \( \zeta = 1 \), depend only on the row, not upon the column. In fact, these observations hold for general \( n \), and are a consequence of an explicit combinatorial expression for the inverse annular matrix \( A^{-1}_n \) that we provide in Appendix A.2.

For example, row \( \mathcal{Z}[4][5][6] \) of \( A^{-1}_6 \) (the second row given above) tells us

\[
\frac{\mathcal{Z}[4][5][6]}{\mathcal{Z}[1][3][4][5][6]} = \frac{1}{(1 - \zeta)^2} \left\{ \zeta \det \mathcal{L}[2, 4, 6] + \det \mathcal{L}[1, 3, 5] + \zeta \det \mathcal{L}[5, 2, 6] + \zeta \det \mathcal{L}[3, 1, 4] + \zeta^2 \det \mathcal{L}[2, 4, 3, 5] + \zeta \det \mathcal{L}[4, 1, 6] + \zeta^2 \det \mathcal{L}[5, 2, 4, 3] + \zeta \det \mathcal{L}[3, 5, 6] + \zeta \det \mathcal{L}[4, 3, 4, 2] + \zeta^2 \det \mathcal{L}[5, 6, 2, 4, 1] + \zeta \det \mathcal{L}[3, 5, 2, 4, 1] + \zeta^2 \det \mathcal{L}[4, 5, 6, 2, 3, 1] \right\}.
\]

We can expand out the \( \mathcal{L} \)-determinants into sums of products of the \( \mathcal{L}_{i,j} \), each of which depends on \( \zeta \), where \( \mathcal{L}_{j,i}(\zeta) = \mathcal{L}_{i,j}(1/\zeta) \). Since the denominator is \( (1 - \zeta)^{n/2 - 1} \), to evaluate the limit \( \zeta \to 1 \), we can differentiate the numerator and denominator \( n/2 - 1 \) times with respect to \( \zeta \) and then set \( \zeta \) to 1. The denominator of course becomes \( (-1)^{n/2 - 1}(n/2 - 1)! \). The numerator will consist of monomials of degree \( n/2 \) in the quantities

\[
\frac{\partial}{\partial \zeta} \mathcal{L}_{i,j} \bigg|_{\zeta = 1}, \quad \frac{\partial^2}{\partial \zeta^2} \mathcal{L}_{i,j} \bigg|_{\zeta = 1}, \quad \ldots, \quad \frac{\partial^{n/2 - 1}}{\partial \zeta^{n/2 - 1}} \mathcal{L}_{i,j} \bigg|_{\zeta = 1}.
\]

Surprisingly, in each case all the terms involving higher order derivatives of \( \mathcal{L}_{i,j} \) cancel upon setting \( \zeta \) to 1. The terms involving the first derivative of \( \mathcal{L}_{i,n} \) also cancel at \( \zeta = 1 \). We prove this cancellation in Appendix A.3. This cancellation is convenient, since there are fewer quantities that we need to evaluate. Recall that

\[
L_{i,j} = \mathcal{L}_{i,j} \bigg|_{z = 1} = \mathcal{L}_{i,j} \bigg|_{\zeta = 1}.
\]
Let us define
\[ L_{i,j} := \frac{\partial}{\partial z} L_{i,j} \bigg|_{z=1} = 2 \frac{\partial}{\partial \zeta} L_{i,j} \bigg|_{\zeta=1}. \]

**Theorem 6.3.** For all positive even \( n \) and pairings \( \sigma \) of \( \{1, \ldots, n\} \), \( Z_{\sigma}/Z_{1[2] \cdots n} \) is a polynomial of degree \( n/2 \) in the quantities
\[ \{ L_{i,j} : 1 \leq i < j \leq n \} \quad \text{and} \quad \{ L'_{i,j} : 1 \leq i < j \leq n - 1 \}. \]

See [105] for an example. We prove this theorem in the appendix.

**Conjecture 6.4.** The coefficients in the polynomials in Theorem 6.3 are all integers. (We have verified this for all \( \sigma \) for all \( n \leq 10 \).)

### 6.3. Formulas using the Green’s function.

**Theorem 6.5.** Let \( \sigma \) be a partial pairing, in which the nodes \( Q \) are in singleton parts, the nodes \( T \) are internalized, and \( n \notin Q \cup T \). In the above formulas expressing \( Z[\sigma]/Z[1[2] \cdots n] \) in terms of \( L \)-determinants, we can replace each \((-1)^{|T|}\) \( \det L_{R,T} \) by \( \det \mathcal{G}_{R,Q} \), where \( \mathcal{G}_{i,j} = \mathcal{G}_{i,j} \) for \( j < n \) and \( \mathcal{G}_{i,n} = 1 \), and the result will be a formula for \( Z[\sigma]/Z[1,2,\ldots,n] \).

**Proof.** Since the higher derivatives of \( L_{i,j} \) and the first derivative of \( L_{i,n} \) always cancel out, we may compute \( Z[\sigma] \) using any convenient choice of \( L_{i,n} \), and for present purposes it is convenient to make the choice for which \( \sum_j L_{i,j} = 0 \) for each \( i \). Then writing the sequence \( S \) as \( S = S^*,n \), where \( n \notin S^* \), we may express the determinant \( \det L_{R,T} \) as
\[ \det L_{R,T} = - \sum_{i \notin S \cup T} \det L_{S,i,T}. \]

Suppose for now that both \( R,Q \) and \( S^*,Q,T \) are in sorted order. We use Jacobi’s formula on each summand and the fact that \( L_{1,\ldots,n-1} \) and \( \mathcal{G}_{1,\ldots,n-1} \) are negative inverses to obtain
\[ \frac{\det L_{R,T}}{\det L_{1,\ldots,n-1}} = - \sum_{i \notin S \cup T} (-1)^{S^*_i + i + |\{ s \in S^*: s > i \}|} \det \left( - \mathcal{G}_{1,\ldots,n-1} \setminus \{ S^*,i,T \} \right). \]

Since \( \{1, \ldots, n-1\} \setminus (R,T) = S^*,Q \) and \( \{1, \ldots, n-1\} \setminus \{ S^*,i,T \} = R,Q \setminus \{ i \} \),
\[ \frac{\det L_{R,T}}{\det L_{1,\ldots,n-1}} = (-1)^{R + \sum S^* + |R| + |Q|} \sum_{i \in R \cup Q} (-1)^{S^* + |\{ s \in S^*: s > i \}|} \det \mathcal{G}_{R,Q}(\{ i \}). \]

When we expand \( \det \mathcal{G}_{R,Q} \) along column \( n \), we obtain
\[ \det \mathcal{G}_{R,Q} = \sum_{j=1}^{|R|+|Q|} (-1)^{j+|R|} \det \mathcal{G}_{R,Q} \text{ with jth item removed} \]

If the \( j \)th item of \( R,Q \) is \( i \), then \( i-j = |\{ s \in S^*: s < i \}| \), so
\[ \det \mathcal{G}_{R,Q} = (-1)^{|R|+|Q|} \sum_{i \in R \cup Q} (-1)^{S^* + |\{ s \in S^*: s > i \}|} \det \mathcal{G}_{R,Q}(\{ i \}) \]
and so
\[ \frac{\det L_{R,T}}{\det L_{1,\ldots,n-1}} = (-1)^{R + \sum S^* + |R| + |Q| + |R| + |S^*|} \det \mathcal{G}_{R,Q}. \]
Since \( R \cup S^* = \{1, 2, \ldots, 2|R| - 1\} \), which adds up to \(|R|\) modulo 2, we have
\[
(6.2) \quad \frac{(-1)^{|T|} \det \mathcal{L}^{S,T}_{R,T}}{\det -\mathcal{L}^{1,\ldots,n-1}_{1,\ldots,n-1}} = \det \mathcal{G}^{S,Q}_{R,Q}.
\]
Observe that if we relax the assumption that \( R, Q, T, \) and \( S^*, Q, T \) are in sorted order, the left- and right-hand sides of the above equation change the same number of times. Hence this equation holds regardless of the relative order of the indices in \( R, S, Q, \) and \( T \).

Finally, recall that \( Z[1, 2, \ldots, n] = \det -\mathcal{L}^{1,\ldots,n-1}_{1,\ldots,n-1} \).

**Corollary 6.6.** For a complete pairing \( \sigma \), the Green’s function formula for \( Z[\sigma]/Z[1, 2, \ldots, n] \) can be obtained from the response-matrix formula for \( Z[\sigma]/Z[1|2| \cdots |n] \) simply by replacing each \( L_{i,j} \) with \( G_{i,j} \) and each \( L'_{i,j} \) with \( G'_{i,j} \), and then setting \( G_{i,n} = 1 \).

**Corollary 6.7.** For a partial pairing \( \sigma \) in which node \( n \) is paired, the Green’s function formulas for \( Z[\sigma]/Z[1, 2, \ldots, n] \) are invariant under the addition of global constant to the Green’s function.

**Proof.** The column indexed by \( n \) in \( \det \mathcal{G}^{S,Q}_{R,Q} \) is all-ones. \( \Box \)

6.4. **Windings.** We can also extract information about the windings of the paths within a grove pairing in a manner similar to that described in the (1, 1) case. For a given directed pairing \( \sigma \), we have
\[
\mathcal{Z}[\sigma] = \sum_k z^k Z[\sigma, (k)],
\]
where \( Z[\sigma, (k)] \) is the weighted sum of groves of type \( \sigma \) in which the algebraic number of zipper crossings (involving all pairs in \( \sigma \)) is \( k \). Then
\[
\mathbb{E}_{\text{[algebraic number of zipper crossings]}} = \lim_{z \to 1} \frac{\partial}{\partial z} \log \frac{\mathcal{Z}[\sigma]}{\mathcal{Z}[1|2| \cdots |n]}.
\]

7. **The Green’s function and its monodromy-derivative**

To carry out our loop-erased random walk computations for various lattices, we will use our formulas for the connection probabilities in annular-one graphs developed in section 6, and for this we need the Green’s function \( G \) together with its derivative \( G' \) with respect to a zipper monodromy. We will need \( G \) and \( G' \) for both the full lattice, and the lattice after some of its edges have been cut.

7.1. **Green’s function and potential kernel.** The Green’s function \( G_{u,v} \) is infinite for recurrent lattices such as \( \mathbb{Z}^2 \), but there is a quantity known as the potential kernel \( A_{u,v} \) which behaves like a Green’s function, except that \( A_{u,u} = 0 \), and \( G_{u,v} \) and \( A_{u,v} \) have the opposite sign convention (see [Spi76]). Suppose that a graph \( \mathcal{G} \) is the intersection of \( \mathbb{Z}^2 \) or another lattice \( \mathbb{L} \) with a region surrounding the origin, with “wired boundary conditions”, i.e., all the lattice vertices in \( \mathbb{L} \) that are not in the region are merged into a single vertex in \( \mathcal{G} \) that plays the role of boundary. If \( R \) denotes the electrical resistance within \( \mathcal{G} \) from the origin to the boundary, then
\[
G_{u,v} = R - A_{u,v} + o(1),
\]
where the error term tends to 0 for fixed \( u \) and \( v \) as \( R \to \infty \). For translation invariant lattices, \( A_{u,v} \) depends only on \( u - v \), and is written as \( a_{u-v} \).
Since all of our formulas for crossings of the annulus are invariant when a global constant is added to the Green’s function (involving terms such as $G_{1,2} - G_{1,3}$), it is straightforward to take the limit $\lim_{L \to \infty}$ of these formulas by replacing each $G_{u,v}^L$ in the formula with $-A_{u,v}^L$, which we shall also denote by $\bar{G}_{u,v}^L$.

For convenience let us work with a modified finite graph $\bar{G}$ approximating the lattice $L$, obtained by adjoining an edge with conductance $-1/R$ to node $n$ of $G$, defining node $n$ of $\bar{G}$ to be the other endpoint of this edge. This has the effect of making the resistance in $\bar{G}$ from the origin to node $n$ exactly zero.

For any partition $\sigma$ for which $n$ is not in a singleton part, we have $Z[\sigma] = -Z[\sigma]/R$, and so in particular we can compute $Z[\sigma]/Z = Z[\sigma]_G/Z[\bar{G}]$ by working with the Green’s function $G$ of this modified graph. We define $\bar{G}_{u,v}$ and $\bar{G}'_{u,v}$ by the expansion

$$G_{u,v}^L = \bar{G}_{u,v} + (z - 1)\bar{G}'_{u,v} + O((z - 1)^2).$$

Then in the limit $G \to L$, we have $\bar{G}_{u,v} \to -A_{u,v}^L$.

It is well-known how to compute the potential kernel on periodic lattices by taking the Fourier coefficients of the characteristic polynomial of the lattice $[\text{Spi76}]$. The potential kernel can also be computed for any “isoradial” graph by doing local computations $[\text{Ken02}]$. The square, triangular, and honeycomb lattices are both periodic and isoradial, so for these lattices either method can be employed. For the square lattice the potential kernel takes values in $\mathbb{Q} + i\mathbb{Q}$, while for the triangular and honeycomb lattices it takes values in $\mathbb{Q} + i\mathbb{Q}$.

We shall make use of the following smoothness result:

**Lemma 7.1 ([Stö50]).** For points $z = (z_1, z_2)$ far from $(0, 0)$, the potential kernel on $\mathbb{Z}^2$ behaves like

$$A_{0,z}^{\mathbb{Z}^2} = \frac{1}{2\pi} \log |z| + \frac{3}{2} \log \frac{2 + \gamma}{2\pi} + O(1/|z|^2),$$

where $\gamma$ is Euler’s constant.

The asymptotics of the Green’s function has also been studied on other vertex-transitive $2$-dimensional periodic lattices $[\text{FU96}, \text{KS04}]$, and also on other isoradial graphs $[\text{Ken02}, \text{Thm. 7.3}] [\text{Büc08}, \text{Thm. A.2}]$. In particular, for the triangular lattice the potential kernel is asymptotically $O(|z| + \log \sqrt{12} + \gamma)/(2\pi\sqrt{3}) + O(1/|z|^2)$, and for the honeycomb lattice it is $O(\log|z| + 2 + \gamma)\sqrt{3}/(2\pi) + O(1/|z|^2)$.

7.2. Derivative of the Green’s function.

7.2.1. Infinite sum formula. Let $S$ be the adjacency matrix of the zipper, i.e.,

$$S_{k,\ell} = \begin{cases} 1 & \text{there is a zipper edge directed from } k \text{ to } \ell \\ 0 & \text{otherwise}. \end{cases}$$

Then $\Delta(z) = \Delta_0 + (1 - z^{-1})S + (1 - z)S^*$, so

$$\Delta(z)^{-1} = \left(\Delta_0 + (1 - z^{-1})\Delta_0^{-1}S + (1 - z)\Delta_0^{-1}S^*\right)^{-1}
= \Delta_0^{-1} - (1 - z^{-1})\Delta_0^{-1}S\Delta_0^{-1} + \Delta_0^{-1}S^*\Delta_0^{-1} + O((z - 1)^2)
\left(G_{u,v} - (z - 1) \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}\bar{G}_{\ell,v} - G_{u,\ell}\bar{G}_{k,v}) + O((z - 1)^2)\right)$$
The sum is over zipper edges \((k, \ell)\) in which the zipper direction is from \(k\) to \(\ell\), and \(c_{k,\ell}\) is the conductance of edge \((k, \ell)\). The linear term in \(z - 1\) gives us the desired derivative:

\[
G'_{u,v} = \partial_z G_{u,v} |_{z=1} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}G_{\ell,v} - G_{u,\ell}G_{k,v}).
\]

For the modified graph \(\bar{G}\), we of course have

\[
\bar{G}'_{u,v} = \partial_z \bar{G}_{u,v} |_{z=1} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(\bar{G}_{u,k}\bar{G}_{\ell,v} - \bar{G}_{u,\ell}\bar{G}_{k,v}).
\]

For a vertical zipper in \(\mathbb{Z}^2\) (or the triangular lattice or honeycomb lattice) started in the face whose lower-left corner is the origin, directed downwards towards infinity, we define

\[
\bar{G}^{rL}_{u,v} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(\bar{G}^L_{u,k}\bar{G}^L_{\ell,v} - \bar{G}^L_{u,\ell}\bar{G}^L_{k,v}).
\]

For fixed \(u\) and \(v\), for zipper edges \((k, \ell)\) at a distance \(r\) from the origin, it is straightforward to use the smoothness result in Lemma 7.1 to show that edge \((k, \ell)\) contributes \(O(r^{-2}\log r)\) to the sum so this sum is absolutely convergent.

We would like to know that \(G'^L_{u,v}\) converges to \(\bar{G}^{rL}_{u,v}\) as defined in (7.3) for a sequence of \(G\)'s converging to \(L\). For our purposes in section 8 when we analyze loop-erased random on the lattice, we do not need this convergence of \(G'\) for every sequence of \(G\)'s converging to \(L\), it will suffice to have convergence for some sequence of \(G\)'s tending to \(L\). Perhaps the easiest way to show this is to exploit the reflection symmetry that each of the square, triangular, and honeycomb lattices possess.

**Lemma 7.2.** If \(L\) is the square, triangular, or honeycomb lattice, and \(L \in \mathbb{N}\), let \(G_L = [-L^3, L^3] \cap \mathbb{Z}^3\) be the off-center box surrounding the origin and zipper (as in Figure 3 except with the lower boundary much closer to the origin than the other boundaries), where the lower boundary of the box is aligned with an axis of reflection symmetry of the lattice \(L\). Let \(u, v\) be fixed points in \(L\). Then

\[
\lim_{L \to \infty} G'^L_{u,v} = G^{rL}_{u,v}.
\]

**Proof.** We can approximate \(G^L_{u,w}\) and \(G^L_{v,w}\) (for \(w\) within distance \(L\) of the origin) using the Green’s function of the lattice intersected with the upper-half plane. More precisely, we approximate \(G^L_{p,q}\) by

\[
G^L_{p,q} := -A^L_{p,q} + A^L_{p^*,q},
\]

where \(p^* = p - (0, 2L)\) is the reflection of \(p\) through the lower boundary of the box, and \(A^L\) is the potential kernel of the lattice. By construction \(G^L_{p,q}\) is zero for \(q\) along the lower side of the box, and by the smoothness result from Lemma 7.1 \(G^L_{p,q} = O(1/L^2)\) along the other three sides of the box. Both \(G^L_{p,q}\) and \(G^L_{p,q} = O(1/L^2)\) along the other three sides of the box. Both \(G^L_{p,q}\) and \(G^L_{p,q}\) are harmonic in both \(p\) and \(q\) within the box (except on the boundary), and \(G^L_{p,q}\) is zero along all four sides of the box. By the maximal principle for harmonic functions, for \(p\) and \(q\) within \(G_L\), we have

\[
|G^L_{p,q} - G^L_{p,q}| = O(1/L^2),
\]

i.e.,

\[
G^L_{p,q} = -A^L_{p,q} + A^L_{p^*,q} - A^L_{0^*,0} + O(1/L^2).
\]
Next we compare the contribution of a zipper edge \((k, \ell)\) to \(\tilde{G}_{u,v}^G\) and \(G_{u,v}^L\):

\[
-(\tilde{G}_{u,k}^G, \tilde{G}_{v,\ell}^G - \tilde{G}_{u,\ell}^G, G_{k,v}^G) = - \left( -A_{u,k}^L + A_{u,k}^L, -A_{v,\ell}^L + A_{v,\ell}^L, -A_{\ell,\ell}^L + A_{\ell,\ell}^L \right)
+ O(L^{-2} \log L)
\]

Recall that \(u\) and \(v\) are fixed, so they are within distance \(O(1)\) of the origin. The second term is \(O(L^{-2} \log L)\). If the zipper edge \((k, \ell)\) is at distance \(r\) from the origin, then the next four terms largely cancel one another and add up to \(O(1/\log L)\). Upon summing over all zipper edges, we find \(\tilde{G}_{u,v}^G = G_{u,v}^L + O(L^{-1} \log L)\). 

7.2.2. Zipper deformations. The next task we have is to evaluate in closed form the infinite sum in (7.3). This we can do for many lattices \(L\), including the square lattice, triangular lattice, and honeycomb lattice, although it is not clear how to do this for arbitrary lattices.

We shall need to deform the path that the zipper takes. In general deforming the zipper while keeping its endpoints fixed has no effect on \(G_{u,v}^G\), unless the zipper is deformed across either \(u\) or \(v\). If the zipper is moved across \(u\) in the direction of the arrow on the zipper, then \(G_{u,v}^G\) decreases by \(G_{u,v}^G\), and similarly, moving the zipper across \(v\) (in the direction of the arrow) increases \(G_{u,v}^G\) by \(G_{u,v}^G\). We can also move the endpoint of the zipper by adding a new zipper edge \((k, \ell)\) (or removing an old one) near the endpoint of the zipper, which of course just adds (or removes) one term to the summations (7.2) and (7.3).

7.2.3. Closed-form evaluation of \(G^G\) on \(\mathbb{Z}^2\). Next we evaluate \(G^G\) for \(\mathbb{Z}^2\). The first step is to rotate the entire lattice \(180^\circ\) about the terminal square of the zipper. The rotation of course preserves the lattice \(\mathbb{Z}^2\), and maps \(u\) and \(v\) to \((1,1)-u\) and \((1,1)-v\) respectively, but now the zipper goes up to infinity rather than down to infinity. Let \(G^G_{u,v}\) denote \(G^G\) with the repositioned zipper. We have

\[
G^G_{u,v} = G^G_{(1,1)-u,(1,1)-v}.
\]

The next step is to deform the zipper so that it once again goes downwards. We can deform the initial segment of the zipper so that it goes downwards, then circles around back up along a large-radius circle, and then continues back up as before.
By Lemma 7.2, the summands along the zipper starting with the large-radius circle and the subsequent path to infinity are negligible. So we have
\[ \tilde{G}_{u,v} = \tilde{G}_{(1,1),u,(1,1)-v} = \tilde{G}_{(1,1)-u,(1,1)-v} + \text{another term if zipper was deformed across } u \text{ or } v. \]

Next we move the location of the start of the zipper, translating it by \( v + u - (1,1) \), by adding a finite number of new zipper edges. Then we deform the zipper again, making it go straight down; we have to add another term if the zipper gets deformed across either \( (1,1) - u \) or \( (1,1) - v \). Because the lattice \( \mathbb{Z}^2 \) is invariant under such translations, translating the starting face of the zipper is equivalent to translating the vertices in the opposite direction. Thus we have
\[ \tilde{G}_{u,v} = \tilde{G}_{v,u} + \text{finite number of terms.} \]

Finally we use the antisymmetry of \( \tilde{G}_{u,v} \):
\[ \tilde{G}_{u,v} = \frac{\text{finite number of terms}}{2}. \]

This procedure is perhaps better explained by way of an example. We can write
\[
\tilde{G}_{(0,0),(2,1)} = \tilde{G}_{(1,1),(-1,0)} = \tilde{G}_{(1,1),(-1,0)} + \tilde{G}_{(1,1),(-1,0)} = \frac{1}{2\pi^2} + \frac{1}{\pi} - \frac{32}{5}.
\]

In like manner we can compute \( \tilde{G}_{u,v} \) for any pair of vertices \( u \) and \( v \) in \( \mathbb{Z}^2 \). The answer will always be in \( \mathbb{Q} + \frac{1}{\pi} \mathbb{Q} + \frac{1}{2\pi^2} \mathbb{Q} \).

7.2.4. Closed-form evaluation of \( \tilde{G}' \) on the triangular lattice. We can compute \( \tilde{G}' \) on the triangular lattice in essentially the same manner as for \( \mathbb{Z}^2 \). The key properties of the lattice that we used is that it is invariant under 180° rotations, and that for any pair of vertices there is a lattice-invariant translation that maps the first vertex to the second vertex.

7.2.5. Closed-form evaluation of \( \tilde{G}' \) on the honeycomb lattice. The honeycomb lattice is invariant under 180° rotations and is vertex-transitive. However, there are not lattice-invariant translations between any pair of vertices: we can partition the vertices into two color classes, black and white, such that any lattice-preserving translation will map the black vertices to the black vertices and the white vertices to the white vertices.

Suppose \( u \) is a black vertex and \( v \) is a white vertex. After a 180° rotation about a hexagon, \( u \) is mapped to a white vertex \( u' \) and \( v \) is mapped to a black vertex \( v' \). We can then translate \( u' \) to \( v \) and \( v' \) to \( u \) while preserving the lattice. This allows us to compute \( \tilde{G}'_{u,v} \) when \( u \) is black and \( v \) is white (or vice versa).

Since \( \tilde{G}'_{u,v} \) is harmonic in both \( u \) and \( v \) (except along the zipper), when \( u \) and \( v \) have the same color, \( \tilde{G}'_{u,v} \) can be expressed as
\[ \tilde{G}'_{u,v} = \frac{1}{3} (\tilde{G}'_{u,w_1} + \tilde{G}'_{u,w_2} + \tilde{G}'_{u,w_3}) \]

(plus another term if one of the edges \( (v, w_i) \) crosses the zipper), where the \( w_i \)'s are the neighbors of \( v \), and the right-hand side we can compute by the above method.
7.3. Cutting edges. Suppose that in the vector bundle setting, we know the Green’s function $G = G^G$ for a graph $G$, and we wish to know the Green’s function for the graph $G \setminus \{s, t\}$ obtained by deleting an edge $\{s, t\}$ of $G$. Recall that $c_{s,t}$ denotes the conductance of edge $(s, t)$, and let us denote by $\tau$ the parallel transport to $s$ from $t$, so that $\Delta_{s,t}^G = -c_{s,t} \tau$ and $\Delta_{t,s}^G = -c_{s,t} \tau^*$. Then it is readily checked that

$$\Delta_{s,t}^G \{s, t\} = \delta_{u,w} - \frac{(G^G_{u,s} - G^G_{u,t} \tau^*)(G^G_{s,v} - \tau G^G_{t,v})}{\alpha_{s,t}}$$

where

$$(\tau)_{s,t} = G_{s,s} + G_{t,t} - G_{s,t} \tau^* - \tau G_{t,s} - 1/c_{s,t}$$

(which is a scalar). Indeed, if we let $f(u, v)$ denote the purported Green’s function on the right-hand side of (7.4), then $f(u, v) = 0$ when either $u$ or $v$ is the boundary, and we have

$$\sum_v f(u, v) \Delta_{v,w}^G = \delta_{u,w} - \frac{(G^G_{u,s} - G^G_{u,t} \tau^*)(\delta_{s,w} - \tau \delta_{t,w})}{\alpha_{s,t}}.$$ 

If $w \neq s$ and $w \neq t$ then $\Delta_{v,w}^G = \Delta_{v,w}^G$, so

$$\sum_v f(u, v) \Delta_{v,w}^G = \delta_{u,w} \quad \text{(if $w \neq s$ and $w \neq t$).}$$

Suppose now $w = s$. Then

$$\sum_v f(u, v) \Delta_{v,s}^G \{s, t\} = \sum_v f(u, v) \Delta_{v,s}^G + f(u, t) \alpha_{t,s} \tau^* - f(u, s) \alpha_{t,s}$$

$$= \delta_{u,s} - \frac{G^G_{u,s} - G^G_{u,t} \tau^*}{\alpha_{t,s}} + \left[ G^G_{u,t} - \frac{(G^G_{u,s} - G^G_{u,t} \tau^*)(G^G_{s,v} - \tau G^G_{t,v})}{\alpha_{t,s}} \right] \alpha_{t,s} \tau^*$$

$$- \left[ G^G_{u,s} - \frac{(G^G_{u,s} - G^G_{u,t} \tau^*)(G^G_{s,v} - \tau G^G_{t,v})}{\alpha_{t,s}} \right] \alpha_{t,s}$$

$$= \delta_{u,s} - \frac{G^G_{u,s} - G^G_{u,t} \tau^*}{\alpha_{t,s}} \alpha_{t,s} \left[ 1 - \alpha_{t,s} + \left( G^G_{t,s} \tau^* - G^G_{t,t} - G^G_{s,s} + \tau G^G_{t,s} \right) \right]$$

$$= \delta_{u,s}$$

by the choice of $\alpha_{t,s}$. The case $w = t$ is similar.

Let us return to the line bundle setting, with a zipper monodromy of $z$, that we are interested in near $z = 1$. If $(s, t)$ is a zipper edge then $\tau = z$ or $\tau = 1/z$ (depending on the zipper direction), and otherwise $\tau = 1$. Let $\tau' = \partial_z \tau|_{z=1}$. Recall that $G = G|_{z=1}$ and $G' = \partial_z G|_{z=1}$. From (7.4) it is evident that

$$G^G_{u,v} \{s, t\} = G_{u,v} - \frac{(G^G_{u,s} - G^G_{u,t})(G^G_{s,v} - G^G_{t,v})}{\alpha_{s,t}}$$

where

$$\alpha_{s,t} = G_{s,s} + G_{t,t} - 2G_{s,t} - 1/c_{s,t}.$$ 

We have

$$\partial_z \alpha_{s,t} = \partial_z G_{s,s} + \partial_z G_{t,t} - (\partial_z G_{s,t}) \tau^* - G_{s,t} \partial_z \tau^* - G_{s,s} \partial_z \tau - (\partial_z \tau) G_{s,t} - \tau \partial_z G_{t,s}$$

$$\partial_z \alpha_{s,t}|_{z=1} = 0.$$
Using this, we can differentiate (7.4) with respect to the zipper monodromy $z$ and set $z = 1$ to obtain

\begin{equation}
(G^G_{u,v})' = G'_{u,v} - \frac{G'_{u,s} - G'_{u,t} + G_{u,t}'}{a_{s,t}}(G_{s,v} - G_{t,v})/a_{s,t}
\end{equation}

\begin{equation}
+ \frac{(G_{u,s} - G_{u,t})(G'_{s,v} - G'_{t,v} - \tau'G_{t,v})}{a_{s,t}}.
\end{equation}

One final edge-cutting formula is

\begin{equation}
Z^G_{\{s,t\}} = 1 - c_{s,t}(G_{s,s} + G_{t,t} - 2G_{s,t}).
\end{equation}

This holds because directed edge $(s,t)$ occurs in a uniform spanning tree of $G$ with probability $c_{s,t}(G_{s,s} - G_{s,t})$, and likewise directed edge $(t,s)$ may occur in the spanning tree.

These formulas (7.6), (7.7), (7.8) and (7.9) of course apply to the modified graph $\bar{G}$ simply by replacing $G$ and $G'$ with $\bar{G}$ and $\bar{G}'$.

8. Loop-erased random walk

In this section we show how to compute the probability that the LERW from $(0,0)$ to $\infty$ in $\mathbb{Z}^2$ (or the triangular or honeycomb lattices) passes through any given vertex or edge.

8.1. Preliminary remarks. We let $P_{v,w}$ denote the probability that the LERW started from $(0,0)$ to $\infty$ passes through edge $(v,w)$, in the direction from $v$ to $w$. Likewise we let $P_w$ denote the probability that the LERW passes through vertex $w$. It is straightforward that

\begin{equation}
P_w = \sum_{v: v \sim w} P_{w,v} = \sum_{v: v \sim w} P_{v,w} + \delta_{w,0}.
\end{equation}

Our strategy is to compute these edge probabilities.

We find it conceptually convenient to work with finite graphs $G$, set up our equations for spanning tree and grove event probabilities in terms of the finite-graph Green’s function $G^G$ and its derivative $(G^G)'$ using the formulas from section 6, and then afterwards take the limit $G \to L$ using the formulas from section 7. We are at liberty to use any convenient sequence $G_k$ of finite graphs that converge to the lattice $L$, since the limiting measure on spanning trees of $L$ is independent of the choice of $G_k$, and the event that the LERW from $(0,0)$ to $\infty$ uses a given edge is a measurable event in the limiting measure. So we choose a sequence $G_k$ for which it is convenient to compute $(G^G_k)'$, as described in section 7.2. These graphs $G_k$ have a wired boundary vertex (see Figure 3) that we will label $\infty$, even though the graphs are finite. The other vertices of $G_k$ we will label by their coordinates in $\mathbb{Z}$. We define $P^{G_k}_{v,w}$ and $P^{G_k}_w$ in the same way as we defined $P_{v,w}$ and $P_w$; for fixed $v$ and $w$, $\lim_{G \to L} P^{G}_{v,w} = P_{v,w}$.

Once an edge traversal probability $P^{L}_{v,w}$ has been computed, finding the corresponding probability $P^{L}_{w,v}$ for the reversed edge is straightforward:

**Lemma 8.1** \cite{Ken00b}. $P^{L}_{v,w} - P^{L}_{w,v} = c_{v,w}(\bar{G}_{0,v} - \bar{G}_{0,w})$

**Proof.** Let the origin 0 be at the center of a large box with wired boundary whose size we will send to infinity. For simple random walk started at 0, the expected number of traversals of $(v,w)$ minus the expected number of traversals of $(w,v)$ is
the edge conductance $c_{v,w}$ times the difference in Green’s functions. The same also holds for loop-erased random walk started at 0, since cycles are reversible. □

The intensity of the undirected edge $\{v,w\}$ is $P^L_{v,w} = P^D_{v,w} + P^L_{w,v}$, and the undirected edge intensities turn out to be nicer numbers than the directed edge intensities. The vertex intensities are easily calculated from the undirected edge intensities, and given the potential kernel of $L$, the directed edge intensities are easily recovered from the undirected intensities.

8.2. **Computation of directed edge intensities.** Suppose $G$ is a finite connected graph, $u$, $v$, $w$, and $r$ are vertices of $G$, and $\{v,w\}$ is an edge of $G$. If the path connecting $u$ to $r$ within a spanning tree of $G$ passes through an edge $(v,w)$ in the direction from $v$ to $w$, then deleting this edge results in a grove of type $u,v|w,r$, and conversely, adjoining edge $(v,w)$ to a grove of type $u,v|w,r$ yields a spanning tree of $G$ in which the path from $u$ to $r$ passes through the edge $(v,w)$ in the direction from $v$ to $w$. If $u$ is the vertex considered to be the origin in $G$ and $r$ is the wired boundary, then

$$P^G_{v,w} = \frac{Z[u,v|w,r]}{Z[u,v,w,r]}.$$

For example, consider the probability that the LERW from $(0,0)$ to $\infty$ in $\mathbb{Z}^2$ uses the directed edge from $(1,1)$ to $(1,0)$. As described in section 8.1, we approximate $\mathbb{Z}^2$ by a large finite grid $\mathcal{G}$ with wired boundary. There are four vertices of interest, so we declare them to be nodes. Their coordinates are $(1,0)$, $(1,1)$, $(0,0)$, and $\infty$, and we also refer to them as nodes 1, 2, 3, and 4. Since $\mathcal{G}$ is planar and nodes 1, 2, and 3 bound the same face, we can view $\mathcal{G}$ as a surface graph embedded on the annulus, and place a zipper from the central face to the outer boundary. This annular surface graph with four nodes is the example we showed earlier in Figure 3, and which we also show schematically on the right. Recall equation (5.6c) for annular-one graphs:

$$\frac{Z[3,2|1,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G''_{2,3} - G''_{1,3}.$$

As discussed in section 7.1, we can use either the Green’s function $G'$ for the original graph $\mathcal{G}$, or for the graph $\bar{\mathcal{G}}$ which has an auxiliary vertex connected to the boundary. Using the method described in section 7 we compute

$$\bar{G}^Z_{\mathbb{Z}^2}^{(1,0)} (1,1) (0,0) \quad \bar{G}^Z_{\mathbb{Z}^2}^{(1,0)} (1,1) (0,0)$$

$$(1,0) \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{bmatrix} (1,0) \begin{bmatrix} 0 & -\frac{3}{32} & -\frac{5}{32} \\ -\frac{3}{32} & 0 & -\frac{1}{16} \\ \frac{5}{32} & \frac{1}{2\pi} & 0 \end{bmatrix}$$

We evaluate $\lim_{\mathcal{G} \to \mathbb{Z}^2} P^G_{(1,1),(1,0)}$ by substituting $\bar{G}^Z_{\mathbb{Z}^2}$ for $G' = G^\mathcal{G}$ and $\bar{G}^{Z^2}$ for $G'' = (G^\mathcal{G})'$:

$$(8.1) \quad P^Z_{(1,0),(1,0)} = \frac{3}{32} + \frac{1}{2\pi} - \frac{5}{32} - \frac{1}{\pi} + \frac{1}{4} = 3 - \frac{1}{2\pi}. $$
The edge \((1,1)(1,0)\) was close enough to the origin for all three node \((0,0)\), \((1,1)\), and \((1,0)\) to be incident to the same face, which made it straightforward to apply our formulas for annular networks to compute the edge intensity. For edges further away, more work is required.

For an arbitrary directed edge \((v,w)\), we identify a set of edges to cut so as to place the starting point \(u\) of the LERW and the endpoints of the edge on the same face, and call the cut graph \(\tilde{G}\). The vertices \(u, v,\) and \(w\), together with the endpoints of all of the cut edges, comprise the nodes on the inner boundary of the annulus. We number these nodes in counterclockwise order so that the zipper starts between nodes 1 and \(n-1\). The vertex labeled \(\infty\) becomes node \(n\), which is on the other boundary of the annulus. The cut graph is an annular-one surface graph.

A grove of type \(u, v \mid w, r\) in \(G\) may contain some of the cut edges, and upon removing these edges, it becomes a grove of some other type in \(\tilde{G}\). We can enumerate all possible subsets of the cut edges and all possible grove types \(\sigma\) in \(\tilde{G}\) which combine to form a grove of type \(u, v \mid w, r\) in \(G\), and thereby express \(Z^G[u, v \mid w, r]\) as a linear combination of \(Z^{\tilde{G}}[\sigma]\)'s. For example, to compute the intensity of the directed edge \((4,2)(4,1)\) in \(Z^2\), we can cut four edges, as shown at right. In this case there are 10 nodes, and the grove \(Z[1,2,5][3,4,10][6][7,8,9]\) together with cut-edges \((2,7)\) and \((4,6)\) are among those which combine to form a grove of type \([9,5][4,10]\) in the original graph. Using Theorem 6.2, for each such grove type \(\sigma\) we can express \(Z^{\tilde{G}}[\sigma]\) as a linear combination of \(Z^{\tilde{G}}[\tau]\)'s, where the \(\tau\)'s are partial pairings of the nodes of \(\tilde{G}\) in which node \(n\) is paired. For each such partial pairing \(\tau\), we can use Theorem 6.5 to express \(Z^{\tilde{G}}[\tau]/Z^{\tilde{G}}\) as the \(\zeta \to 1\) limit of a power of \((1 - \zeta)\) times a linear combination (with polynomial in \(\zeta\) coefficients) of determinants of matrices whose entries are of the form

\[
G_{i,j}^{\tilde{G}} + \zeta \frac{1}{2} G_{i,j}^{\tilde{G}}.
\]

We can then replace each Green’s function entry \(G_{i,j}^{\tilde{G}}\) with the potential kernel \(\tilde{G}_{i,j}\) and each \(G_{i,j}^{\tilde{G}}\) with \(G_{i,j}^{\tilde{L}}\). We then take the limit where \(G\) tends to the infinite lattice, which replaces each \(G_{i,j}^{\tilde{G}}\) with \(G_{i,j}^{\tilde{L}}\) and each \(G_{i,j}^{\tilde{G}}\) with \(G_{i,j}^{\tilde{L}}\), where \(\tilde{L}\) is the infinite lattice with some edges cut. Each of these can then be evaluated in closed form using the formulas in section 8. We then multiply by \(Z^{\tilde{L}}/Z^L\) using (7.9) to obtain \(Z^L[u, v \mid w, \infty]/Z^L\), which is the directed edge intensity.

There are many steps in these computations, but the whole process can be handled by computer, and it was not previously known that these LERW intensities were computable or had a closed form expression. We record the results of our LERW intensity computations for the square lattice in Figure 9 for the honeycomb lattice in Figure 10 and for the triangular lattice in Figure 11.

### 8.3. Loop-erased random walk on \(Z \times \mathbb{R}\)

We consider next a weighted version of \(Z^2\), where each horizontal edge has weight \(c\), and each vertical edge has weight \(1/c\). This graph is isoradial, so we may compute the Green’s function using [Ken02]. Because the lattice is symmetric under a \(180^\circ\) rotation and invariant under translations, we can also compute \(G'\). This gives us all the necessary information we need to compute the probability that LERW from \((0,0)\) passes through \((1,0)\). It is...
convenient to let $c = \tan \theta$. After a computation similar to the ones above, we find that the LERW passes through vertex $(1, 0)$ with probability

$$\frac{1}{4} + \frac{\theta}{2\pi} - \frac{\theta^2}{\pi^2 \sin^2 \theta} \left( 1 - \frac{2\theta}{\pi} \right).$$

When $\theta = \pi/4$, we have $c = 1$, and this above probability reduces to $5/16$, in agreement with our earlier calculation for $\mathbb{Z}^2$.

In the isoradial embedding of the lattice into the plane, if the horizontal edges have length 1, then the vertical edges have length $c$. An interesting special case is the limit $c \to 0$. Then random walk on this weighted graph converges to a standard Brownian motion in the vertical direction, except at a Poisson set of times with

**Figure 9.** Undirected edge intensities of loop-erased random walk on $\mathbb{Z}^2$. For $x \geq 1$, the edges $(x, x)(x, x-1)$ and $(x, x-1)(x+1, x-1)$ appear to have identical intensities, despite there being no lattice symmetry that would imply this. The cases $x = 1$ and $x = 2$ are shown here, we have also checked the cases $x = 3, 4, 5$. 
A Brownian motion on a random walk on this graph is then a continuous-time random walk on intensity 1, where the walk jumps left or right with equal probability. The random walk on this graph is then a continuous-time random walk on \( \mathbb{Z} \) in the horizontal direction and a Brownian motion on \( \mathbb{R} \) in the vertical direction. From the above formula, we see that in this limit, the probability that the LERW passes through \((1,0)\) converges to \(1/4 - 1/\pi^2\).
The determinant of the annular matrix is surprisingly simple, it is a power
of 1 − ζ, so that while this fact is not specifically used in the computation of grove
partition functions, we give a derivation in Appendix A.4.

Appendix A. The annular matrix

Recall the matrix that we introduced in section § 6.2 for computing grove partition functions for pairings in which n − 1 nodes are on one boundary of an annulus and the last node is on the other boundary. The rows are indexed by subsets of {1, ..., n} of size n/2 and the columns are indexed by annular pairings. Since n is even and positive, we let k = n/2 − 1. In this appendix we derive the key properties of these matrices that we use. We review the “cycle lemma” in Appendix A.1 which gives a canonical association between the rows and columns of the matrix and simplifies the subsequent analysis. In Appendix A.2 we show that the matrix is nonsingular and give a combinatorial description of the inverse. In Appendix A.3 we derive a formula about the inverse annular matrix which shows that the higher order derivatives of \( \mathcal{L}_{ij} \) do not appear in the formulas for the grove partition functions. The determinant of the annular matrix is surprisingly simple, it is a power
of 1 − ζ, so that while this fact is not specifically used in the computation of grove
partition functions, we give a derivation in Appendix A.4.

Figure 11. Undirected edge intensity of loop-erased random walk
on the triangular lattice. The edge-intensities of \((x, x)(x, x−1)\) and
\((x, x−1)(x+1, x−1)\) are identical for \(x = 1, 2, 3\), and perhaps all \(x\), despite there being no lattice symmetry that would imply this.
A.1. Annular pairings, subsets, and the cycle lemma. Recall that a standard Dyck path of order \( k \) has \( 2k + 1 \) points, numbered 0, 1, \ldots, 2\( k \), and 2\( k \) steps, numbered 1, \ldots, 2\( k \), where each step is either +1 or −1, and the partial sums are non-negative. Dyck paths are among the structures enumerated by the \( k \)th Catalan number \( \frac{(2k)!}{k(k+1)} \), and are in bijective correspondence with non-crossing pairings of \{1, \ldots, 2\( k \)\} (see e.g., [Sta99 exercise 6.19(rn)]).

We define a “cyclic Dyck path” of order \( k \) to have 2\( k + 1 \) points and 2\( k + 1 \) steps, with the points 0 and 2\( k + 1 \) identified, and the steps numbered 1, \ldots, 2\( k + 1 \), where one of the 2\( k + 1 \) steps is 0 (the “flat step”) and the other steps are ±1 and define a standard Dyck path of order \( k \) when read in cyclic order starting after the flat step. Cyclic Dyck paths are in bijective correspondence with annular perfect matchings that have 2\( k + 1 \) nodes on one boundary (corresponding to the 2\( k + 1 \) steps) and one node (2\( k + 2 \) steps) on the other boundary. The annular perfect matching pairs node 2\( k + 2 \) with the flat step of the cyclic Dyck path, and every +1 step is paired with its associated −1 step in the usual way for standard Dyck paths.

Given a cyclic Dyck path of order \( k \), the set of down steps is a subset of \{1, \ldots, 2\( k + 1 \)\} of size \( k \). The “cycle lemma” bijection of Dvoretzky and Motzkin [DM47] (see also [DZ90]) states that for each subset of \{1, \ldots, 2\( k + 1 \)\} of size \( k \), there is a unique cyclic Dyck path giving rise to it in this way, i.e., the set of down steps uniquely determines which of the remaining steps is the flat step. In our setting, \( k = n/2 - 1 \), in \( \det \mathcal{L}^S_R \), the indices in \( R \) give the locations of the up steps and the flat step, and the indices in \( S \) give the locations of the down steps together with 2\( k + 2 \). The flat step (in \( R \)) gets paired up with 2\( k + 2 \) (in \( S \)), and the chords under the Dyck path determine the rest of the pairing. The following example illustrates the bijection adapted to \((n - 1, 1)\)-annular pairings:

\[
\begin{align*}
\{1,2,5,9,10,12\} & \Rightarrow \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \\
\{3,4,6,7,8,11\} & \Rightarrow \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \\
\Rightarrow \begin{array}{c}
5|2|10|9|1|12 \\
4|6|7|8|11|3 \\
\end{array}
\end{align*}
\]

In the reverse direction, we have

\[
1,11|2,6|3,12|4,5|7,10|8,9 \Rightarrow \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \\
\Rightarrow 5|2|10|9|1|12 \\
\Rightarrow 4,6,7,8,11,3
\]

Notice that we obtain the same \( 2 \times (k + 1) \) array of numbers, where the columns represent the annular pairing, and the rows represent the sets \( R \) and \( S \).

Suppose \( S \subset \{1, \ldots, 2k + 2\} \) is a set for which 2\( k + 2 \in S \) and \( |S| = k + 1 \). Suppose \( \tau \) is a cyclic Dyck path of order \( k \) (which we may interpret as an annular pairing). We let \( S \cdot \tau \) denote the number of up-steps of \( \tau \) in \( S \). We say that an up-step of \( \tau \) is wrapped if its corresponding down step has a smaller index. We let \( S : \tau \) denote the number of wrapped up-steps of \( \tau \) in \( S \).

Define

\[
(A.1) \quad A_{S,\tau} = \begin{cases} 
(-1)^{|S\cdot\tau|} & \text{if } S \setminus \{2k+2\} \text{ is obtained by taking one endpoint from each chord in } \tau, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( A_{S,\tau} \) is the entry of the annular matrix \( A_{2k+2} \) corresponding to row \( \det \mathcal{L}^S_R \) and column \( \mathcal{F}[\tau] \) (where \( R = \{1, \ldots, 2k + 2\} \setminus S \) and the indices of \( R \) and \( S \) ordered
as described above). When $A_{S,\tau}$ is nonzero, it can be rewritten as

$$A_{S,\tau} = (-1)^{S \cdot \tau} \cdot \# \text{ indices in } S \setminus (2k + 2) \text{ after } \tau\text{'s flat step} - \# \text{ down steps of } \tau \text{ after } \tau\text{'s flat step.}$$

### A.2. Inverse annular matrix

Next we show that the inverse of the annular matrix can be expressed in terms of objects known as “cover-inclusive Dyck tilings.”

Dyck tilings were independently introduced by Kenyon and Wilson [KW11b] and Shigechi and Zinn-Justin [SZJ12], and were studied further in [Kim12, KMPW14, FN12]. For any pair of Dyck paths $\lambda$ and $\mu$ of order $k$, if the path $\mu$ dominates $\lambda$ in the sense that at each position $\mu$ is higher than $\lambda$, then the region between $\lambda$ and $\mu$, denoted by $\lambda/\mu$, is a (rotated) skew Young diagram, which can be tiled by $\sqrt{2} \times \sqrt{2}$ squares rotated by $45^\circ$. A Dyck tile is obtained from a Dyck path by replacing each vertex of the Dyck path with such a rotated $\sqrt{2} \times \sqrt{2}$ square, and then gluing the squares together. A cover-inclusive (c.i.) Dyck tiling of $\lambda/\mu$ is a tiling of $\lambda/\mu$ by Dyck tiles such that the Dyck paths associated to any two Dyck tiles either cover disjoint portions of the horizontal axis, or the region covered by one tile is a subset of the region covered by the other tile, with the larger tile underneath the smaller tile. The diagram at right shows an example c.i. Dyck tiling of the region between two Dyck paths. This definition extends naturally to cyclic Dyck paths $\lambda$ and $\mu$ with flat steps at the same location.

**Theorem A.1.** Let $\lambda$ be a cyclic Dyck path of order $k$, and let $S \subset \{1, \ldots, 2k + 2\}$ have size $k + 1$ and contain $2k + 2$. Define

$$B_{\lambda, S} = \sum_{\mu} \left[ \# \text{ of c.i. Dyck tilings of } \lambda/\mu \right] \times \zeta^{-\# \text{ indices in } S \setminus (2k + 2) \text{ after } \lambda\text{'s flat step} \text{ } \times \right.$$  

$$\zeta^{\# \text{ down steps of } \lambda \text{ after } \lambda\text{'s flat step}}.$$

Then the matrix $(B_{\lambda, S}) / (1 - \zeta)^k$ is the inverse of the annular matrix $A_{2k + 2}$.

Furthermore, $B_{\lambda, S}$ is a polynomial in $\zeta$ (i.e., no negative powers) of degree at most $k$.

To prove this theorem we start with a lemma:

**Lemma A.2.** Let $\tau$ and $\mu$ be cyclic Dyck paths of order $k$ on a $(2k + 1)$-cycle, with the flat step of $\mu$ at position $2k + 1$.

$$\sum_{\text{subsets } S \text{ obtained by taking one endpoint from each chord of } \tau} (-1)^{S \cdot \tau} \cdot \zeta^{S \cdot \mu} = \begin{cases} (-1)^{\mu/\tau} (1 - \zeta)^k & \text{if it is possible to push down some chords of } \tau \text{ to obtain } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

(If the flat steps of $\tau$ and $\mu$ are different, then it is not possible to push down chords of $\tau$ to obtain $\mu$, so the second case applies.)

**Proof.** Suppose that $\tau$’s flat step is also located at $2k + 1$. Then $S : \tau = 0$ for any $S$. There are two subcases:

1. Suppose each chord $\{i, j\}$ of $\tau$ connects an up-step of $\mu$ to a down-step of $\mu$.
   Then it is possible “push down” some of the chords of $\tau$ to obtain $\mu$.
   Let $S_0$ be the set of the down steps of $\mu$, so $S_0 \cdot \mu = 0$. For any other set $S$ obtained from taking one endpoint from each chord of $\tau$, $S \cdot \mu$ is precisely
the number of chords of $\tau$ on which $S$ and $S_0$ disagree. Thus the sum is $(-1)^{S_0\cdot\tau}(1 - \zeta)^k$. Now $S_0\cdot\tau$ is the number chords of $\tau$ that we push down to obtain $\mu$. Each time a chord of $\tau$ is pushed down, the area between the modified Dyck path and $\mu$ changes by an odd amount, so the parity of the area of $\mu/\tau$ is the parity of $S_0\cdot\tau$.

(2) Otherwise, there is some chord $\{i, j\}$ of $\tau$ for which both $i$ and $j$ are up steps in $\mu$. For each set $S$ in the sum, let $S'$ be the symmetric difference of $S$ with $\{i, j\}$. Then $\zeta^{S'\cdot\mu} = \zeta^{S\cdot\mu}$ and $\zeta^{S\cdot\tau} = \zeta^{S'\cdot\tau} = 1$, but $(-1)^{S\cdot\tau} = -(1)^{S'\cdot\tau}$, so the sum is 0 in this case.

Next suppose that the flat step of $\tau$ differs from the flat step of $\mu$. There are several subcases:

(1) Suppose there is an unwrapped chord of $\tau$ at two up-steps or two down-steps of $\mu$. Then by the argument of the previous paragraph the sum is 0.

(2) Suppose $\tau$ has a wrapped chord $(2k + 1, j)$, and $\mu$ has an up-step at $j$. Let $S'$ be the symmetric difference of $S$ with $\{2k + 1, j\}$. Then the terms corresponding to $S$ and $S'$ add up to 0.

(3) Suppose $\tau$ has an unwrapped chord $(i, 2k + 1)$, and $\mu$ has a down-step at $i$. Let $S'$ be the symmetric difference of $S$ with $\{i, 2k + 1\}$. Then the terms corresponding to $S$ and $S'$ add up to 0.

(4) Suppose $\tau$ has a wrapped chord $(2k + 1, j)$, and $\mu$ has a down-step at $j$. Then the subpath of $\tau$ consisting of steps $1, \ldots, j - 1$ is a Dyck path, whereas the subpath of $\mu$ on the same interval has more up-steps than down-steps. This implies subcase 1 occurs within this interval.

(5) Suppose $\tau$ has an unwrapped chord $(i, 2k + 1)$, and $\mu$ has an up-step at $i$. Then the subpath of $\tau$ consisting of steps $i + 1, \ldots, 2k$ is a Dyck path, whereas the subpath of $\mu$ on the same interval has more down-steps than up-steps, which again implies subcase 1 occurs.

Proof of Theorem A.4. In an earlier article we proved [KW11b, Theorem 1.5] that if $\lambda$ and $\tau$ are standard Dyck paths of order $k$, then

\begin{equation}
\sum_{\mu \text{ above } \lambda} (-1)^{|\lambda/\mu|}[\# \text{ of c.i. Dyck tilings of } \lambda/\mu] \times \begin{cases}
1 & \text{if it is possible to push down some chords of } \tau \text{ to obtain } \mu, \\
0 & \text{otherwise}
\end{cases} = \begin{cases}
1 & \text{if } \lambda = \tau, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

When $\lambda$ and $\tau$ are cyclic Dyck paths of order $k$, this formula is still true: If $\lambda$ and $\tau$ have their flat steps at the same place, then it is a straightforward consequence of the formula for standard Dyck paths, and if $\lambda$ and $\tau$ have their flat steps in different places, the formula trivially holds since there are no $\mu$’s between $\lambda$ and $\tau$.

In each nonzero summand of (A.4), $\lambda$ is dominated by $\mu$ which is dominated by $\tau$. Since $|\lambda/\mu| + |\mu/\tau| = |\lambda/\tau|$, we can multiply both sides of (A.4) by $(-1)^{|\lambda/\tau|}$ to effectively replace $(-1)^{|\lambda/\mu|}$ with $(-1)^{|\mu/\tau|}$.

If $\lambda$ has its flat step at position $2k + 1$, then so does $\mu$, so we can multiply both sides of (A.4) by $(1 - \zeta)^k$ and then use Lemma A.2 to replace $(-1)^{|\mu/\tau|} \times (1 - \zeta)^k$ with the conditional expression on the left-hand side with a summation over $S$:
\[
\sum_{\mu \text{ above } \lambda} \left[ \text{# of c.i. Dyck tilings of } \lambda / \mu \right] \times \\
\sum_{S: \text{ each chord of } \tau \text{ intersects } S \text{ once}} (-1)^{S: \tau} \zeta^{S: \tau} \zeta^{S: \mu} = \begin{cases} 
(1 - \zeta)^k & \text{if } \lambda = \tau, \\
0 & \text{otherwise}. 
\end{cases}
\]

Changing the order of summation, we obtain

\[
(A.5) \quad \sum_{S: \text{ each chord of } \tau \text{ intersects } S \text{ once}} (-1)^{S: \tau} \zeta^{S: \tau} \sum_{\mu \text{ above } \lambda} \left[ \text{# of c.i. Dyck tilings of } \lambda / \mu \right] \zeta^{S: \mu} = \begin{cases} 
(1 - \zeta)^k & \text{if } \lambda = \tau, \\
0 & \text{otherwise}. 
\end{cases}
\]

Next we use formula (A.1) for \(A_{S, \tau}\) and definition (A.3) for \(B_{\lambda, S}\) (using that \(\lambda\)'s flat step is at position \(2k + 1\)) to rewrite the summand of (A.5) as \(A_{S, \tau} B_{\lambda, S}\). Since \(A_{S, \tau}\) is zero unless each chord of \(\tau\) intersects \(S\) once, we can extend the summation to include all \(S\), and obtain

\[
(A.6) \quad \sum_{S} B_{\lambda, S} A_{S, \tau} = \begin{cases} 
(1 - \zeta)^k & \text{if } \lambda = \tau, \\
0 & \text{otherwise}. 
\end{cases}
\]

for cyclic Dyck paths \(\lambda\) and \(\tau\) of order \(k\) when \(\lambda\) has its flat step at location \(2k + 1\).

Next we argue that (A.6) also holds when \(\lambda\)'s flat step is in other locations. For a given \(\lambda\), \(S\), and \(\tau\), suppose that we cyclically decrease all the indices by the same amount modulo \(2k + 1\) (except \(2k + 2\), which indexes the node on the other boundary), to obtain \(\lambda'\), \(S'\), and \(\tau'\). When \(B_{\lambda, S} A_{S, \tau}\) is nonzero, we see from (A.3) and (A.2) that \(B_{\lambda', S'} A_{S', \tau'}\) differs from it by a power of \(\zeta\). The indices in \(S'\) after \(\tau'\)'s flat step correspond to the indices in \(S\) after \(\tau\)'s flat step together with the indices in \(S'\) after \(\lambda'\)'s flat step, unless \(\tau'\)'s flat step gets wrapped around, at which point the number of indices of \(S'\) after \(\tau'\) flat step drops by \(k\). The downsteps of \(\tau'\) after \(\tau'\)'s flat step similarly correspond to the downsteps of \(\tau\) after \(\tau\)'s flat step together with the downsteps of \(\lambda'\) after \(\lambda'\)'s flat step, until \(\tau'\)'s flat step gets wrapped around, at which point there is a similar jump by \(k\). Thus we have

\[
B_{\lambda', S'} A_{S', \tau'} = B_{\lambda, S} A_{S, \tau} \times \zeta^{\# \text{down steps of } \lambda' \text{ after } \lambda'\text{'s flat step}} / \zeta^{\# \text{down steps of } \tau' \text{ after } \lambda'\text{'s flat step}}.
\]

Upon summing over \(S'\), we see that we obtain (A.6) scaled by a power of \(\zeta\), which is still zero when \(\lambda \neq \tau\), and the power of \(\zeta\) is 1 when \(\lambda = \tau\). Thus the identity (A.6) holds for general cyclic Dyck paths \(\lambda\) and \(\tau\).

Next we check that \(B_{\lambda, S}\) is a polynomial in \(\zeta\). Referring to (A.3), \(\mu\) and \(\lambda\) have their flat step in the same place. Consider the indices in \(S\) after \(\lambda\)'s flat step. Each such index contributes a factor \(\zeta^{-1}\) in (A.3), but also a factor of \(\zeta\) if \(\mu\) has an up step at that index. But because \(\mu\) dominates \(\lambda\), for each such down step of \(\mu\) after the flat step, there is also a down step of \(\lambda\) after the flat step, which also contributes a factor of \(\zeta\). Thus \(B_{\lambda, S}\) has no negative powers of \(\zeta\).
Next we bound the degree of $B_{\lambda,S}$. For each down step of $\lambda$ after $\lambda$’s flat step, there must also be an up step of $\lambda$, and hence also of $\mu$. Each such up step of $\mu$ makes no net contribution to the power of $\zeta$, so the degree of $B_{\lambda,S}$ is at most $k$. □

A.3. Cancellation of higher order derivatives. We start with an identity:

**Theorem A.3.** Suppose $\sigma$ and $\tau$ are cyclic Dyck paths of order $k$, and $U \subset \{1, \ldots, 2k+2\}$, and $|U| \leq \ell < k$. Then

$$
\sum_{S} \left[ \frac{\zeta d}{d\zeta} \right]^{\ell-|U|} B_{\sigma,S} \right|_{\zeta=1} \times A_{S,\tau} \times (-1)^{|S \cap U|} = 0.
$$

**Proof.** For a general finite graph with $n = 2k+2$ nodes and with general parallel transports, define $\mathcal{Z}[\sigma]$ as in the case of a planar graph or annular-one graph. Let $R = \{1, \ldots, n\} \setminus S$, and define

$$
D_{R}^{S} = \sum_{\sigma} A_{S,\sigma} \mathcal{Z}[\sigma].
$$

For annular graphs $D_{R}^{S} = \det \mathcal{L}_{R}^{S}$, but for general graphs these two quantities will be different. We can “recover” the $\mathcal{Z}[\sigma]$’s from these $D_{R}^{S}$’s by multiplying by $A_{n}^{-1}$:

$$
\mathcal{Z}[\sigma] = \frac{1}{(1-\zeta)^{k}} \sum_{S} B_{\sigma,S}(\zeta) D_{R}^{S},
$$

where $k = n/2 - 1$ and $B_{\sigma,S}$ was defined in (A.3).

Let us consider now the complete graph on $n$ nodes, so that $\mathcal{L} = -\Delta$ and $\mathcal{L}_{i,j}$ is the edge weight between nodes $i$ and $j$, times the parallel transport to $i$ from $j$. Note that in this case $\mathcal{L}$ is (except for the diagonal entries) a general Hermitian matrix. Each $D_{R}^{S}$ is a polynomial in the entries of $\mathcal{L}$, with coefficients that involve powers of $\zeta$. (For the complete graph, any matrix times the vector of $\mathcal{Z}[\sigma]$’s will yield polynomials in the $\mathcal{L}_{i,j}$’s for $i \neq j$.)

We change variables by setting $\zeta = e^{t}$. Then $\frac{d}{dt} \mathcal{L}_{i,j} = -\frac{d}{dt} \mathcal{L}_{i,j}$. The $\zeta \to 1$ limit is of course equivalent to $t \to 0$, and $(1-e^{t})^{k}$ has a zero of order $k$ at $t = 0$, with $\frac{d}{dt}(1-e^{t})^{k} = (-1)^{k}k!$.

For general nonzero edge weights and smooth (in $t$) parallel transports on the complete graph, $\mathcal{Z}[\sigma]$ is finite and $\mathcal{Z}[1|2] \cdots [n] = 1$, so for any $\ell < k$ it must be that we get zero when we differentiate the numerator from (A.8), i.e., $\sum_{S} B_{\sigma,S}(e^{t}) D_{R}^{S}$, $\ell$ times with respect to $t$ and then set $t$ to 0:

$$
0 = \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{S} \frac{d^{\ell-m}}{dt^{\ell-m}} B_{\sigma,S}(e^{t}) \bigg|_{t=0} \times \frac{d^{m}}{dt^{m}} D_{R}^{S} \bigg|_{t=0}.
$$

Since $\mathcal{L}$ is generic, we can rescale the $m$th derivative of each $\mathcal{L}_{i,j}$ by a factor of $\beta^{m}$, and deduce that for each $m, \ell$ with $m \leq \ell < k$

$$
0 = \sum_{S} \frac{d^{\ell-m}}{dt^{\ell-m}} B_{\sigma,S}(e^{t}) \bigg|_{t=0} \times \frac{d^{m}}{dt^{m}} D_{R}^{S} \bigg|_{t=0}.
$$

Next we write $D_{R}^{S}(\lambda)$ for the polynomial function of a Hermitian matrix, which, when evaluated on the response matrix $\mathcal{L}$ of the complete graph, gives $D_{R}^{S} = D_{R}^{S}(\mathcal{L})$ (see (A.12)). Let $d_{i}$ denote the differential operator for which $d_{i} \mathcal{L}_{i,j} = \frac{1}{2} \frac{d}{dt} \mathcal{L}_{i,j}$ and $d_{i} \mathcal{L}_{i,j} = \frac{1}{2} \frac{d}{dt} \mathcal{L}_{j,i}$ but $d_{i} \mathcal{L}_{h,j} = 0$ for $h, j \neq i$. For the complete graph, each monomial of the polynomial $D_{R}^{S}$ includes each index $i$ exactly once. (This also
holds for annular graphs, though of course the polynomials are different.) Using this property of these polynomials, we can write the mth derivative of \( \mathcal{L} = \mathcal{L}(t) \) as follows:

\[
(A.10) \quad \frac{d^m}{dt^m} D_R^S(\mathcal{L}) = \sum_{i_1, i_2, \ldots \in \{1, \ldots, n\}} d_{i_1} \cdots d_{i_m} D_R^S(\mathcal{L}).
\]

Given a set \( U \) of \( m \) nodes, for each \( i \in U \) and each \( j \) we rescale \( \frac{d}{dt} \mathcal{L}_{i,j} \bigg|_{t=0} \) by a factor of \( \beta \), without changing any of the other derivatives at \( t = 0 \). If \( i, j \in U \), then we rescale \( \frac{d^2}{dt^2} \mathcal{L}_{i,j} \bigg|_{t=0} \) by a factor of \( \beta^2 \). (Recall that \( \mathcal{L} \) is generic Hermitian, so we can do this.) The coefficient of \( \beta^m \) within the mth derivative is obtained from (A.10) by including only those terms for which \( i_1, \ldots, i_m \) is a permutation of \( U \). Then substituting (A.10) into (A.9) and taking the coefficient of \( \beta^m \), we find

\[
(A.11) \quad 0 = \sum_{S} \frac{d^{m-t}}{dt^{m-t}} B_{\sigma,S}(e^t) \bigg|_{t=0} \times D_R^S(\mathcal{L}^{(U)}) \bigg|_{t=0},
\]

where \( \mathcal{L}^{(U)} \) where is the Hermitian matrix obtained from \( \mathcal{L} \) by replacing \( \mathcal{L}_{i,j} \) with \( \frac{d}{dt} \mathcal{L}_{i,j} \) for each \( i \in U \) and \( j \notin U \) or \( i \notin U \) and \( j \in U \), and replacing \( \mathcal{L}_{i,j} \) with \( \frac{d^2}{dt^2} \mathcal{L}_{i,j} \) for each \( i, j \in U \).

From our definition of \( D_R^S \), for the complete graph we have

\[
(A.12) \quad D_R^S(\mathcal{L}^{(U)}) = \sum_{\rho} A_{S,\rho} \prod_{\{i,j\} \in \rho} \mathcal{L}_{i,j}^{(U)},
\]

where the sum is over annular directed pairings \( \rho \). Next we take the pairing \( \tau \), and for each pair \( \{i, j\} \) of \( \tau \), we rescale \( \mathcal{L}_{i,j}^{(U)} \) by a factor \( \gamma \). Then the coefficient of \( \gamma^{n/2} \) in (A.11) only arises when \( \rho = \tau \) in the above sum, so

\[
(A.13) \quad 0 = \sum_{S} \frac{d^{m-t}}{dt^{m-t}} B_{\sigma,S}(e^t) \bigg|_{t=0} \times A_{S,\tau} \bigg|_{t=0} \times \prod_{\{i,j\} \in \tau} \mathcal{L}_{i,j}^{(U)} \bigg|_{t=0}.
\]

For each \( S \), the product term in the formula takes the same (generically nonzero) value, except for a sign, which is given by the parity of \( S \cap U \). So we cancel this factor (keeping the sign), and obtain (A.7). □

We now restate and prove Theorem 6.3

**Theorem A.4.** Suppose that an annular-one graph has \( n \) nodes, and that \( \sigma \) is a partial pairing of the \( \{1, \ldots, n\} \) which has \( k+1 \) pairs, one of which contains \( n \). Then \( Z_{\sigma}/Z_{[1, \ldots, n]} \) is a polynomial of degree \( k+1 \) in the quantities

\[
\{ L_{i,j} : 1 \leq i < j \leq n \} \quad \text{and} \quad \{ L'_{i,j} : 1 \leq i < j \leq n-1 \}.
\]

**Proof.** Let \( Q \) denote the singleton nodes of \( \sigma \), and \( T \) the unlisted / internal nodes. Let \( h(i) \) denote the ith element of \( \{1, \ldots, n\} \setminus (Q \cup T) \). For \( S \subset \{1, \ldots, 2k+2\} \) with \( 2k+2 \in S \) and \( |S| = k+1 \), and \( R = \{1, \ldots, 2k+2\} \setminus S \), the relevant determinants are of the form \( D_R^S = \det \mathcal{L}_{h(R),T}^{(S)} \), and (A.8) holds for these \( D_R^S \)’s. Thus

\[
\frac{Z[\sigma]}{Z[1, \ldots, n]} = \frac{(-1)^k}{k!} \sum_{m=0}^{k} \binom{k}{m} \sum_{S} \frac{d^{k-m}}{dt^{k-m}} B_{\sigma,S}(e^t) \bigg|_{t=0} \times \frac{d^m}{dt^m} \det \mathcal{L}_{h(R),T}^{(S)} \bigg|_{t=0}.
\]
We rewrite the $n$th derivative of $\det L^{h(S),T}_{\nu}^{h(R),T}$ using the differential operators $d_i$, as in (A.10). Let $U$ be the set of nodes for which we applied $d_i$ an odd number of times. If for some $i$ we applied $d_i$ more than once, then the size of the set $U$ will be less than $m$, and then by (A.7) (with $\tau = \sigma$), the coefficient of such terms is 0.

Next, suppose the variable $L_{i,n}$ is differentiated. Since $n$ is in each set $h(S)$, we may as well replace $U$ with $U \setminus \{n\}$ and introduce a global sign, but then since $|U|$ is smaller, we see from (A.7) that the coefficient of such terms is 0.

A.4. Determinant.

**Theorem A.5.** The determinant of the annular matrix is

$$\det A_n = (1 - \zeta)^{2^{n-2} - \frac{n}{2}}.$$  

**Proof.** Since $\det A_n$ is a polynomial in $\zeta$, and the formula for $A_n^{-1}$ is well defined whenever $\zeta \neq 1$, it follows that $\det A_n$ can only have a root at $\zeta = 1$.

We split the original zipper into $n - 1$ zippers each with parallel transport $z^{1/(n-1)}$, and then deform these zippers so that their endpoints lie in each of the $n - 1$ intervals between the nodes. When we deform a zipper across node $i \neq n$ in the counterclockwise direction, the parallel transport from $i$ to any other node $j$ is multiplied by $z^{1/(n-1)}$. For each column of the annular matrix, say indexed by directed pairing $\sigma$, the column is scaled by $z^{1/(n-1)}$ according to whether node $i$ is a source or destination in $\sigma$. Likewise, each row of the annular matrix, say indexed by $\det L^{S}_{R}$, is scaled by $z^{1/(n-1)}$ according to whether $i \in R$ or $i \in S$. The effect of deforming these zippers is to conjugate the annular matrix $A_n$ by a diagonal matrix, yielding a new more symmetric matrix $A_n^*$ for which $\det A_n^* = \det A_n$.

We change variables to

$$w = z^{2/(n-1)} = \zeta^{1/(n-1)}$$

so that the nonzero entries of $A_n^*$ are integral powers of $w$. For example, the first two rows of $A_n^*$ are

$$\begin{bmatrix}
\frac{w^3}{w} & \frac{w^2}{w} & \frac{w}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w} \\
\frac{1}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w} & \frac{1}{w}
\end{bmatrix}$$

and the other rows are determined by cyclic rotations.

From the determinant formula for $\det L^{S}_{R}$, we see that the diagonal entries of $A_n^*$ are all 1. Consider the column indexed by directed pairing $\sigma$. Since $A_n^*$ is symmetric under cyclic rotations of the indices $1, \ldots, n-1$, let us assume for convenience that $\sigma$ pairs $n-1$ to $n$, so that we can write

$$\sigma = a_{1,1} \cdots a_{n-2,1} | a_{n,1} | a_{n-2,1} | \cdots | a_{1,1} .$$

Referring to the above bijection, since $n-1$ pairs to $n$, for each $j$ we have $a_{j,0} < a_{j,1}$. Column $\sigma$ contains $2^{n/2-1}$ nonzero entries, one for each sequence $f_1, \ldots, f_{n/2-1}$ of $n/2 - 1$ 0’s and 1’s, where the row is indexed by

$$\begin{bmatrix}
\det L_{a_{1,1}-f_1, \ldots, a_{n,1} \cdots a_{n/2-1,1}-f_{n/2-1}, n}^{a_{1,1}, \ldots, a_{n/2-1,1}, f_{n/2-1}, n-1}
\end{bmatrix}.$$
Since the pair \((a_j,0,a_j,1)\) crosses \(a_j,1-a_j,0\) zippers, this pair contributes \(w^{f_j}\) to the matrix entry. In particular, all nonzero nondiagonal entries of \(A_n^*\) have positive powers of \(w\). This implies
\[
\det A_n|_{\zeta=0} = \det A_n^*|_{w=0} = 1.
\]

For a given column, the row that maximizes the power of \(w\) is the one for which \(f_0,\ldots,f_{n/2-1} = 1,\ldots,1\), and the power is the area under the Dyck path. The mapping from a column \(\sigma\) to the row \(\det L_{\sigma}\) which has the highest power of \(w\) is also a bijection, in fact it is a simple variant of the cycle lemma bijection. Hence the leading coefficient of the polynomial \(\det A_n^*\) is \(\pm 1\), and the degree is
\[
\deg \det A_n^* = (n-1) \times \sum_{\text{Dyck paths of length } n-2} \text{area under Dyck path}.
\]

For Dyck paths of length \(2k = n-2\), the above sum is (see Sloane’s A008549)
\[
4^k - \binom{2k+1}{k} = 2^{n-2} - \binom{n-1}{n/2-1} = 2^{n-2} - \frac{1}{2} \binom{n}{n/2}.
\]

Because \(\det A_n\) has a root only at \(\zeta = 1\), the constant term is 1, and the degree is \(2^{n-2} - \frac{1}{2} \binom{n}{n/2}\), the determinant formula follows. \(\square\)

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