Abstract

The coefficient $c_{sw}$ appearing in the Sheikholeslami-Wohlert improved action is computed to one loop perturbation theory for improved gluon actions including six-link loops. The $O(a)$ improvement coefficients for the dimension three isovector composite operators bilinear in the quark fields are also computed to one loop order of perturbation theory with degenerate non-vanishing quark masses.
1 Introduction

Despite the fact that the quenched approximation is a drastic modification of the theory, it turns out that the low lying hadronic spectrum obtained in this approximation is quantitatively very close to the experimentally observed one. Only recently, with the progress of lattice QCD simulations, have the precise results from CP-PACS [1, 2] shown a systematic deviation not only in the meson sector but also in the baryon masses. It is thus now clear that one has to incorporate the effect of dynamical quarks to have a chance to obtain the correct hadron spectra in the continuum limit. On the other hand, full QCD simulations are extremely computer time consuming compared to those of quenched QCD. Although the fastest computers now available have peak speeds of a few hundred Gflops, it is still difficult to reduce the lattice spacing less than 0.1 fm while keeping the physical volume larger than 2 fm with reasonable statistics. Due to the large scaling violation of the combination of the plaquette gluon action and the Wilson quark action there is no hope to obtain accurate and reliable hadron spectra in the continuum limit by the extrapolation from such lattice spacings.

One of the possible ways to overcome this difficulty is, of course, the use of improved actions to reduce the scaling violation. Among the several proposals, the least ambitious but most practical one known at present is the on-shell $O(a^2)$ improvement program, where only one new term (clover term) has to be added to the quark action (clover action) [3]. In this program the scaling violation is reduced to $O(a^2)$ with the appropriate value of the coefficient of the clover term (clover coefficient). Although no modification for the plaquette gluon action is necessary in the $O(a)$ improvement program, one can still add one or more terms to the plaquette gluon action without significantly increasing CPU time, in order to reduce non-negligible $O(a^2)$ errors present at lattice spacings equal to or larger than 0.1 fm. Recently such a combination (the so-called renormalization group (RG) improved gluon action [4] and the clover action) has been compared to other combinations at a lattice spacing of $\sim 0.2$ fm and it has been shown that this combination has some better properties than others [5]: the rotational symmetry of the static quark potential is well restored and the scaling violation in hadron spectra is reasonably small.

Because of this result, the CP-PACS collaboration decided to use the RG improved gluon action and the clover action for their production run of full QCD simulations [5] whose lattice spacings range from 0.2 fm to 0.1 fm. Unfortunately at the time they started the production runs the value of clover coefficient $c_{sw}$ for the RG improved gluon action was known neither non-perturbatively nor at 1-loop order of perturbation theory. Instead they used a “perturbative mean field” value $c_{sw} = (1 - 0.8412\beta^{-1})^{-3/4}$, where $1 - 0.8412\beta^{-1}$ is the one-loop value for the plaquette. Although this value might be a
dominant contribution at 1-loop order, it is of course necessary to know $c_{sw}$ fully at 1-loop order to be able to estimate how large the errors in this approximation are. To obtain $O(a)$ improvement for correlation functions of composite operators one also has to construct $O(a)$ improved operators \[7\]. For the case of quark bilinear operators of dimension 3 (see subsection 2.4) this involves computing mixing coefficients ($c_X$, $X = A, V, T$) as well as mass dependent correction factors ($1 + am_Y$, $Y = S, P, A, V, T, m$). It would of course also be desirable to push $O(a)$ improvement to the non-perturbative level as it has been done by the Alpha Collaboration \[8\] for the plaquette gluon action.

In this paper we have calculated $c_{sw}$, $c_X$, and $b_Y$ at 1-loop order of perturbation theory for gluon actions including six-link loops and the clover quark action. Our computational method uses the Schrödinger functional \[9\]. Our results for the coefficients $c_X$, and $b_Y$ are in complete agreement with the results of the recent computation of Taniguchi and Ukawa \[10\] who used a completely different method.

2 Definitions

As mentioned in the introduction we are here concerned with constructing $O(a)$ on-shell improved actions for QCD which are to be used in large scale numerical simulations. Starting with the original Wilson action $S_W$ for the quarks, Sheikholeslami and Wohlert \[3\] have shown that $O(a)$ on-shell improvement can be achieved by adding just one extra term $\propto S_{SW}$ to the action. Thus we will be considering total actions of the form

$$S[U, \bar{\psi}, \psi] = S[U] + S_W[U, \bar{\psi}, \psi] + c_{sw}(g_0)S_{SW}[U, \bar{\psi}, \psi],$$

(2.1)

with $N_f \geq 2$ flavors. The part of the action involving the quarks will be specified in more detail below. At this point we only stress that the factor $c_{sw}$ multiplying the SW term is a function of the bare coupling $g_0^2$ which depends on the particular form of the gauge action $S[U]$ chosen. The tree-level value is the same for all gauge actions $c_{sw}(0) = 1$ \[3\] \[1\] and so $c_{sw}$ has a perturbative expansion of the form

$$c_{sw}(g_0) = 1 + c_{sw}^{(1)} g_0^2 + \ldots$$

(2.2)

For the Wilson plaquette action the 1-loop coefficient $c_{sw}^{(1)}$ was computed a long time ago by Wohlert \[11\] (and checked in ref. \[12\]), and recently non-perturbatively for the quenched theory \[13\] and full QCD with $N_f = 2$ flavors \[14\].

\[1\] Here we take Wilson’s $r$-coefficient fixed to be 1, otherwise $c_{sw}(0) = r$
In this paper we will compute $c_{sw}^{(1)}$ for some gauge actions belonging to a general class containing loops up to length 6

$$S[U] = \frac{2}{g_0} \sum_{i=0}^{3} c_i(g_0^2) \sum_{C \in S_i} \mathcal{L}(C)$$  \hspace{1cm} (2.3)

where the $S_i$ denote sets of elementary loops $C$ on the lattice as given in fig. 1. Loops $C$ differ by orientation only are considered equal. Furthermore

$$\mathcal{L}(C) = \text{ReTr}[I - U(C)],$$  \hspace{1cm} (2.4)

Figure 1: Elementary loops in $S_i$, $i = 0, 1, 2, 3$

\footnote{Our notation differs from refs. \cite{16,17} by an interchange of $c_2$ with $c_3$.}
$U(C)$ being the ordered product of the link variables $U_\mu(x)$ along $C$. The coefficients $c_i(g^2_0)$ are regular at $g^2_0 = 0$ and are normalized such that

$$c_0(g^2_0) + 8c_1(g^2_0) + 16c_2(g^2_0) + 8c_3(g^2_0) = 1. \tag{2.5}$$

This class of gauge actions was first considered by Wilson [13] in the framework of block spin renormalization and includes in particular the action proposed by Iwasaki [4]. Among this class are also the on-shell $O(a^2)$ improved pure gauge actions [16]. However we stress that here we do not have the ambition to achieve on-shell $O(a^2)$ improvement for full QCD. The latter ambitious program would involve the addition of various other terms to the action including in particular 4-fermion operators (of dimension 6) which would be very awkward for numerical simulations.

In our explicit computations we have only considered actions with $c_2 = 0$. Apart from the Wilson action ($c_1 = c_3 = 0$), we have studied 4 actions

\begin{align*}
\text{LW} : c_1(0) &= -1/12, \quad c_3(0) = 0 \quad [16], \tag{2.6} \\
\text{RG1} : c_1 &= -0.331, \quad c_3 = 0 \quad [4], \tag{2.7} \\
\text{RG2} : c_1 &= -0.27, \quad c_3 = -0.04 \quad [4], \tag{2.8} \\
\text{RG3} : c_1 &= -0.252, \quad c_3 = -0.17 \quad [15]. \tag{2.9}
\end{align*}

Note for the LW action the $c_i$ are functions of the bare coupling, whereas for the RG actions the $c_i$ are constants.

In ref. [17] it was shown that demanding positivity of the action in the limit $g_0 \to 0$ restricts the tree level coefficients to some convex domain $P$. A sufficient condition for the coefficients to lie in $P$ is

$$c_0(0) + 8\hat{c}_1 + 32\hat{c}_2 + \frac{80}{3}\hat{c}_3 > 0 \tag{2.10}$$

where

$$\hat{c}_i = \frac{1}{2}(c_i(0) - |c_i(0)|). \tag{2.11}$$

Note that all the actions listed above satisfy this condition except RG3.

2.1 The Schrödinger Functional

The work of the Alpha Collaboration has demonstrated that a convenient framework in order to compute improvement coefficients and renormalization constants (and of
course more importantly many quantities of direct physical relevance) is provided by the Schrödinger functional (SF) [9, 18, 7]. Here one considers the theory defined on hypercubic lattices of volume $L^3 \times T$ with cylindrical geometry, i.e. periodic-type boundary conditions in the spatial directions and Dirichlet boundary conditions in the “time” direction, for both gluon and quark fields $U_\mu, \psi$. From now on we set the lattice spacing $a = 1$ in most formulae.

For the case of the Wilson action the SF was described in detail in refs. [9, 18, 7]. The extension of the SF considerations to gauge actions belonging to the class above was first considered by Klassen [19]. We found it conceptually more convenient to set things up slightly differently. In particular in our construction the dynamical variables to be integrated over are chosen independently of the form of the action: they are the spatial link variables and quark fields $U_k(x), \psi(x)$ with times $x_0 = 1, \ldots, T - 1$ and the link variables $U_0(x)$ with times $x_0 = 0, \ldots, T - 1$ (i.e. inside the cylinder). Dirichlet boundary conditions are imposed on the fields $U_k(x), \psi(x)$ at the boundaries $x_0 = 0$ and $x_0 = T$.

What remains to be done is to specify the action in more detail.

### 2.2 The pure gauge action $S[U]$

Taking account of the fact that we are working with the Schrödinger functional geometry, we modify the gauge action above to read

$$S[U] = \frac{2}{g_0^2} \sum_{i=0}^{3} \sum_{C \in S_i} W_i(C, g_0^2) \mathcal{L}(C)$$

(2.12)

with weights $W_i(C, g_0^2)$ which may differ from $c_i(g_0^2)$ for loops $C$ near the boundaries. First we must specify the elements of the classes $S_i$ more precisely. These consist of all loops of the given shape which can be drawn on the cylindrical lattice that involve only the “dynamical links” in the sense specified above, and spatial links on the boundaries at $x_0 = 0$ and $x_0 = T$. In particular rectangles protruding out of the cylinder are not included and hence we do not have to specify further boundary conditions for link variables outside the cylinder.

The most important point now is that to achieve $O(a)$ improvement for all (on-shell) Green functions, a careful choice of the weights $W_i$ is necessary [9]. From general considerations [18] only two independent boundary $O(a)$ counterterms involving the gauge fields $\int d^3 x F_{0k}(x) F_{0k}(x)$ and $\int d^3 x F_{kl}(x) F_{kl}(x)$ at $x_0 = 0$ and $x_0 = T$ are expected. Since we

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3For improvement of some physical quantities arising from local Ward identities a particular choice of $W_i$ may not be necessary.
now have many types of loops at our disposal there are various ways to chose the weights to achieve our goal. A rather natural way, which we call Choice A, is to adjust the \( O(a) \) counterterms just through the plaquettes:

\[
W_0(\mathcal{C}, g_0^2) = \begin{cases} 
  c_s(g_0^2) & \text{if } \mathcal{C} \text{ lies completely on one of the boundaries}, \\
  c_t(g_0^2) & \text{if } \mathcal{C} \text{ just touches one of the boundaries}, \\
  c_0(g_0^2) & \text{otherwise},
\end{cases} \tag{2.13}
\]

and for the other classes \( \mathcal{S}_i, i = 1, 2, 3 \) we simply set:

\[
W_i(\mathcal{C}, g_0^2) = \begin{cases} 
  0 & \text{if } \mathcal{C} \text{ lies completely on one of the boundaries}, \\
  c_i(g_0^2) & \text{otherwise}. 
\end{cases} \tag{2.14}
\]

With this particular choice for complete \( O(a) \) improvement we find to tree level

\[
c_s(0) = \frac{1}{2}, \tag{2.15}
\]

\[
c_t(0) = c_0(0) + 2c_1(0), \tag{2.16}
\]

independently of the (smooth) boundary conditions. The necessity for the extra term in \( c_t(0) \) involving the coefficient of the rectangles is explained in Appendix A.

From the discussion in Appendix A it will also be clear that another admissible special way to set things up, which we call Choice B, is to keep eq.(2.13) but change eq.(2.14) to

\[
W_i(\mathcal{C}, g_0^2) = \begin{cases} 
  0 & \text{if } \mathcal{C} \text{ lies completely on one of the boundaries}, \\
  \frac{3}{2}c_i(g_0^2) & \text{if } i = 1 \text{ and } \mathcal{C} \text{ has exactly 2 links on a boundary}, \\
  c_i(g_0^2) & \text{otherwise}.
\end{cases} \tag{2.17}
\]

Then eq.(2.15) remains valid but instead of eq.(2.16) we must set

\[
c_t(0) = c_0(0). \tag{2.18}
\]

In other words in this case one of the \( O(a) \) boundary counterterms at tree level is adjusted by the coefficient of the rectangular loops of space-length 2 lattice units. The reason for mentioning this choice will become manifest below.

We must still specify the boundary conditions for the gauge field. As in the determination of the running coupling \[3\] we chose

\[
U_k(x)|_{x_0=0} = e^C, \quad U_k(x)|_{x_0=T} = e^{C'}. \tag{2.19}
\]
where $C, C'$ are diagonal traceless antihermitian matrices. As explained in ref. \[9\] the presence of non-trivial boundary conditions induces a constant chromo-electric background field. In the continuum limit the background field is given (up to gauge transformations) by

\[
A_0(x) = 0, \quad A_k(x) = b(x_0), \quad k = 1, 2, 3,
\]

\[
b(x_0) = \frac{1}{T} \left[ (T - x_0)C + x_0 C' \right].
\]

On the lattice one will have a solution to the field equations of the form

\[
U_0(x) = 1, \quad U_k(x) = V(x_0), \quad k = 1, 2, 3,
\]

but for actions having $c_1 \neq 0$ and generic choices of (admissible) boundary weights it is not possible to solve the lattice equations of motion analytically and one must be content with numerical solutions. Although this is of no relevance for numerical simulations, for perturbative computations it is more practical to have analytic solutions at tree level. This is where the special Choice B comes in to play since it is easy to check that in this case

\[
V(x_0) = \exp b(x_0),
\]

with $b$ as in eq.(2.21), is an exact solution (in the limit $g_0 = 0$). On the other hand for Choice A, although the above form of $V$ is a solution to the equations obtained by the variation of links well inside the cylinder, it fails to be a solution for equations obtained by the variation of spatial links at time 1 (or at time $T - 1$). Thus for non-vanishing Abelian background field it is clearly advantageous to use Choice B for the analytical computations.

The Feynman rules are now easily derived by expanding around the background field solution

\[
U_0(x) = \exp[g_0 q_0(x)], \quad U_k(x) = V(x_0) \exp[g_0 q_k(x)].
\]

For the validity of the perturbative expansion it is necessary to know whether the background field is the unique absolute minimum of the action (up to gauge transformations). For the case of the Wilson plaquette action this was proven \[7\] provided the diagonal elements of $C$ lie in a certain so called fundamental domain. The extension of the proof of this theorem to the general class of actions considered here has to our knowledge not been achieved. Klassen \[19\] has however investigated the question numerically and found no evidence to the contrary for a large class of actions. In fact the only cases where Klassen found that the global minimum was not given by eqs.(2.23) and (2.21) were those
with coefficients not in the positivity domain $P$ alluded to above. We hope therefore that perturbation theory remains trustworthy. As a consistency check in our computations we monitored the eigenvalues of the quadratic gauge operator (the inverse free propagator) and checked (for various covariant gauges) that it did not develop a negative eigenvalue. This held even for the action RG3 which is not manifestly positive.

Concerning gauge fixing we used the covariant gauge fixing specified in ref. [9] with

$$S_{gf} = \lambda_0(d^* q, d^* q)$$

(2.25)

with

$$(d^* q)_{\alpha \beta}(x) = \begin{cases} \sum_{\mu} (D_{\mu}^* q_{\mu})_{\alpha \beta}(x) & \text{for } 0 < x_0 < T, \\ L^{-3} \sum_y q_0(0, y)_{\alpha \beta} \delta_{\alpha \beta} & \text{for } x_0 = 0, \\ 0 & \text{for } x_0 = T. \end{cases}$$

(2.26)

The inverse gauge propagator for the case of zero background field is given in Appendix B. Note that the simple choice of gauge fixing given above has the disadvantage that there is for generic $c_i$ no choice of the gauge parameter where the propagator becomes diagonal with respect to the Lorentz indices. It may be possible to introduce other choices of gauge fixing where this is the case but we did not investigate this possibility further. In all cases we obtained the free propagator numerically although analytical expressions are probably available for the case of vanishing background field. The expressions for the vertices are rather lengthy and we do not give them here, they can be obtained from the authors upon request.

### 2.3 The fermion action

As mentioned above for fermion action we consider only the on-shell $O(a)$ improved Wilson action which is obtained by adding the Sheikholeslami-Wohlert interaction term (with coefficient $c_{sw}$) [3]. That is in eq.(2.23) $S_W$ is the standard Wilson fermion action with $r = 1$,

$$S_W[U, \bar{\psi}, \psi] = \sum_x \bar{\psi}(x)(D + m_0)\psi(x) + S_{Wb},$$

(2.27)

where $m_0$ is the bare quark mass and

$$D = \frac{i}{2} \sum_{\mu} \{\gamma_{\mu}(\nabla_{\mu} + \nabla_{\mu}) - \nabla_{\mu}^* \nabla_{\mu} \},$$

(2.28)

\(^4\text{The reader is requested to consult ref. [9] for the specific notations. A more symmetric way to fix the gauge is discussed in the paper by Narayanan and Wolff [20].}\)
is the lattice Wilson Dirac operator. $\nabla_\mu$, $\nabla_\mu^*$ are the forward and backward lattice covariant derivatives

$$\nabla_\mu \psi(x) = \lambda_\mu U_\mu(x) \psi(x + \hat{\mu}) - \psi(x),$$

$$\nabla_\mu^* \psi(x) = \psi(x) - \lambda_\mu^* U_\mu(x - \hat{\mu})^\dagger \psi(x - \hat{\mu}).$$

(2.29)

(2.30)

The additional $U(1)$ constant field

$$\lambda_\mu = \begin{cases} 
1 & \text{if } \mu = 0, \\
e^{i\theta/L} & \text{if } \mu > 0,
\end{cases}$$

(2.31)

corresponds to a free phase in the spatial boundary conditions of the fermions. Finally $S_{\text{WB}}$ in (2.27) is an additional boundary term [7] which is needed to remove $O(a)$ lattice artifacts in the SF framework:

$$S_{\text{WB}} = (c_t - 1) \sum_x \left\{ \delta_{x_0,1} \overline{\psi}(x) \left[ \psi(x) - U_0(x - \hat{0})^{-1} P_+ \psi(x - \hat{0}) \right] \\
+ \delta_{x_0,T-1} \overline{\psi}(x) \left[ \psi(x) - U_0(x) P_- \psi(x + \hat{0}) \right] \right\},$$

(2.32)

where $P_\pm = \frac{1}{2} (1 \pm \gamma_0)$.

$S_{\text{SW}}$ is the Sheikholeslami-Wohlert [8] term:

$$S_{\text{SW}}[U, \bar{\psi}, \psi] = \frac{i}{4} \sum_{x,\mu,\nu} \overline{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x),$$

(2.33)

where the field $F_{\mu\nu}$ is given explicitly by

$$F_{\mu\nu}(x) = \frac{1}{8} \left\{ \begin{array}{c}
U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu(x + \hat{\nu}) U_\nu(x) \dagger \\
+ U_\nu(x) U_\mu(x + \hat{\nu}) U_\nu(x - \hat{\mu}) U_\mu(x - \hat{\nu}) \\
+ U_\mu(x - \hat{\mu}) U_\nu(x - \hat{\nu}) U_\mu(x - \hat{\nu}) U_\nu(x - \hat{\mu}) \\
+ U_\nu(x - \hat{\nu}) U_\mu(x + \hat{\mu}) U_\mu(x + \hat{\nu}) U_\nu(x) \dagger \\
- [\mu \leftrightarrow \nu] \end{array} \right\}.$$

(2.34)

The boundary conditions for the fermion fields have first been specified in ref. [18]; in all our computations we have set the quark field on the boundaries to zero.

10
A general introduction to on-shell $O(a)$ improvement of lattice QCD with Wilson quarks can be found in refs. \cite{21, 22}. In particular there it has been discussed how the renormalization procedure can be carried out in a way consistent with $O(a)$ improvement. In the following we adopt the conventions and notations as introduced in this reference.

### 2.4 Composite operators of dimension 3

On-shell $O(a)$ improvement requires also the introduction of $O(a)$ counterterms for the composite fields. Here we restrict attention to a few gauge invariant composite operators which are bilinear in the quark fields. Assuming $N_f \geq 2$ mass degenerate quark flavors we consider the local isovector fields

\begin{align}
V_a^\mu(x) &= \bar{\psi}(x) \gamma_\mu \frac{1}{2} \tau^a \psi(x), \\
A_a^\mu(x) &= \bar{\psi}(x) \gamma_\mu \gamma_5 \frac{1}{2} \tau^a \psi(x), \\
P_a(x) &= \bar{\psi}(x) \gamma_5 \frac{1}{2} \tau^a \psi(x), \\
T_{\mu\nu}^a(x) &= i\bar{\psi}(x) \sigma_{\mu\nu} \frac{1}{2} \tau^a \psi(x).
\end{align}

Using these definitions and following ref. \cite{7} the corresponding improved vector, axial vector and tensor currents may be parametrized as follows,

\begin{align}
(V_I)_a^\mu &= V_a^\mu + c_V(g_0^2) a \tilde{\partial}_\nu T_{\mu\nu}^a, \\
(A_I)_a^\mu &= A_a^\mu + c_A(g_0^2) a \tilde{\partial}_\mu P_a, \\
(T_I)^a_{\mu\nu} &= T_{\mu\nu}^a + c_T(g_0^2) a \left( \tilde{\partial}_\mu V_{\nu}^a - \tilde{\partial}_\nu V_{\mu}^a \right),
\end{align}

where $\tilde{\partial}$ denotes the symmetric lattice derivative $\tilde{\partial}_\mu f(x) = \frac{1}{2} [f(x + a\hat{\mu}) - f(x - a\hat{\mu})]$.

Throughout the paper we will use a mass-independent renormalization scheme which is compatible with $O(a)$ improvement. In such a scheme the renormalized coupling and quark mass are given by

\begin{align}
g_R^2 &= \tilde{g}_0^2 Z_g(g_0^2, a\mu), \\
m_R &= \tilde{m}_q Z_m(g_0^2, a\mu),
\end{align}

\footnote{The Dirac gamma matrix conventions are as in Appendix A of ref. \cite{7}.}
where \( \mu \) is the renormalization scale. The parameters \( \tilde{g}_0 \) and \( \tilde{m}_q \) are defined as follows

\[
\tilde{g}_0^2 = g_0^2 [1 + b_g(g_0^2)am_q],
\]

\[
\tilde{m}_q = m_q [1 + b_m(g_0^2)am_q], \quad m_q = m_0 - m_c,
\]

where \( m_c \) is the critical bare quark mass. The critical hopping parameter is thus given by \( K_c = 1/(8 + 2am_c) \).

With these definitions the renormalized improved bilinear operators take the form \([7]\)

\[
(X = V, A, P, T)
\]

\[
X_R = Z_X(\tilde{g}_0^2, a\mu)[1 + b_X(g_0^2)am_q]X_I.
\]

For the Wilson plaquette action a non-perturbative determination in the quenched approximation of \( Z_A, b_V \) can be found in \([23]\); of \( c_A \) in \([24]\); and of \( c_V \) in \([25]\).

### 2.5 Computation of the improvement coefficients

One first considers the massless theory. In this case on-shell correlation functions involving only the composite fields are \( O(a) \) improved provided the improvement coefficients \( c_{sw}, c_A, c_V \) and \( c_T \) are properly chosen as functions of the bare coupling \( g_0 \). In the Schrödinger functional framework it is, however, usually easier to perturbatively compute correlation functions also involving the near-boundary quark fields \( \zeta, \bar{\zeta} \) and \( \zeta', \bar{\zeta}' \) \([7, 12]\). \( O(a) \) improved correlation functions are then obtained with the renormalized fields \( \zeta_R, \bar{\zeta}_R \) and \( \zeta'_R, \bar{\zeta}'_R \), which are all related to the bare fields by the same renormalization factor, e.g.

\[
\zeta_R = Z_{\zeta}(\tilde{g}_0^2, a\mu)[1 + b_{\zeta}(g_0^2)am_q]\zeta.
\]

The particular set of correlation functions which we have chosen to compute was precisely as for the case of the Wilson action described in detail in refs.\([12, 26]\). Our calculation determines of course only the 1-loop correction to the tree level values of the improvement coefficients, i.e. the quantities \( c_X^{(1)} \) in the perturbative expansion for a general improvement coefficient \( c_X \):

\[
c_X = c_X^{(0)} + c_X^{(1)} g_0^2 + c_X^{(2)} g_0^4 + \ldots
\]

Only for the computation of \( c_{sw}^{(1)} \) is it necessary to have a non-vanishing background field. Once one knows this coefficient the others can be determined by computing correlation functions in the absence of a background field, for which the programs are considerably less CPU time consuming. In the case of vanishing quark mass \( (m_R L = 0) \) we have
collected data at $T = 2L$, for $\theta = 0$ and $\theta = 1$, with vanishing background field and at $T = L$, only for $\theta = 0$, with non-vanishing background field.

After calculating the improvement coefficients which are necessary in the massless theory one can proceed to the determination of the 1-loop contribution to the various $b$-coefficients that were introduced above and are required for the theory with massive quarks. These coefficients have an expansion in powers of $g_0^2$ of the same form as the expansion (2.48) for the $c$-coefficients. In the massive case one has to study quark correlation functions at fixed values of $m_R L$. The set of correlation functions that we considered is exactly the same as described in detail in ref. [26]. Also in this case the computation can be done at vanishing gauge background field. Data were collected for two values of $m_R L$, namely 0.1 and 0.5 and $\theta = 0$, by taking $T = 2L$.

We remark that we did not directly determine the coefficient $b^{(1)}_S$ for the isovector scalar operator $S^a$ since it can be determined by the relation

$$b^{(1)}_S = -2b^{(1)}_m.$$  \hfill (2.49)

In the quenched approximation one can prove the non-perturbative relation $b_S = -2b_m$ from which the 1-loop relation above follows for the fully interacting theory. We also observe that the improvement coefficient $b_g$ (see Eq.(2.44)) vanishes at tree level and at one loop order it is independent of the pure gauge action: $b^{(1)}_g = 0.012000(2) N_f$ [27].

It should also be noticed that the values of $\tilde{c}_t^{(1)}$ and $b^{(1)}_\zeta$ actually needed for the $O(a)$ improvement of correlation functions involving boundary quark fields depends in general on the precise form of the pure gauge action. This involves not only the different choices of the Wilson loop coefficients $c_i$ ($i = 0, 1, 2, 3$), but also the particular choice made for the SF setup. Our results for $\tilde{c}_t^{(1)}$ and $b^{(1)}_\zeta$ refer specifically to the above mentioned Choice A for all the considered pure gauge actions.

3 Analysis Method

We describe in this section the method that we adopted to extrapolate the asymptotic values of the various improvement coefficients from the 1-loop results obtained on a sequence of finite lattices.

The 1-loop results were produced by Fortran codes in double precision. We have checked the gauge invariance of the results produced with our codes by adopting different

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6 A brief outline of the proof, which is due to M. Lüscher, can be found in the last section of ref. [28]. Please note that in this paper the factor on the rhs of eq.(113) should be $(1 + 2b_m a m_q)^{-1}$.
covariant gauge fixings within the family of eq. (2.25). Checks were also done against pre-existing safe codes in some limiting cases, like the case of the standard Wilson plaquette action or the case of vanishing background field. Further checks about the correctness of the codes and of the whole theoretical setup were obtained \textit{a posteriori} by the observed consistency of independent determinations of the same improvement coefficients.

The effects of rounding errors and their dependence on the lattice size have been investigated by comparing, for a limited set of cases, results obtained by running the same code in single, double and extended precision. From this investigation we have derived a formula which gives a rough estimate of the rounding-error effects as a function of the lattice size. The estimated rounding errors were conservatively propagated through the whole analysis procedure described below. In all the cases, when the analysis output was clearly affected by rounding errors, our estimate was found to be realistic or even conservative.

The starting point of the analysis procedure are the (gauge invariant) correlation functions \( f_X(L; T/L; \theta; m_R) \), evaluated on a sequence of lattices of volume \( L^3 \times T \) at a given fixed value of \( m_R \) and with the spatial boundary conditions for the fermion field specified by \( \theta \), as explained in section 2. Here \( X \) labels the different quark correlation functions within the set considered in refs. [12, 26].

Taking suitable linear combinations of these correlation functions, possibly involving different values of \( \theta \) and \( m_R \), enables us to construct lattice approximants of the improvement coefficients or some simple linear combinations of them, which are the quantities that we will consider from now on. The precise form of these quantities is not given here, because, as already mentioned, they are exactly the same as specified in [12], for the determination of \( K_{(1)}^{(c)}, c_{(A)}^{(1)}, c_{t}^{(1)} \) and \( \tilde{c}_{sw}^{(1)} \), and in [26] for all the remaining improvement coefficients introduced above.

Each sequence \( \{Q\} \) of lattice approximants has, by construction, a well definite limit as \( L \to \infty \) and a large \( L \) expansion of the form:

\[
Q(L; T/L) = Q_0 + \sum_{i=1}^{\infty} [Q_i + q_i \log(L)]L^{-i}, \tag{3.1}
\]

where the lattice artifacts coefficients \( Q_i \) and \( q_i \) depend on the value of \( T/L \). We are of course interested in the asymptotic value \( (Q_0) \) of the sequence, which is obtained by suitable extrapolation to the limit \( L \to \infty \), performed at a fixed the value of \( T/L \).

In order to get the asymptotic value of the various sequences of interest we have used the method of [24], suitably adapted to the case of functions of \( L \) of the form (3.1). The main point of the method is to perform some consecutive blocking steps on the initial data,
represented by the sequence \( \{Q\} \). In the first step, for each value of \( L \), one constructs the linear combination of the data at \( L \) and the two contiguous values of the lattice size \((L-1\) and \(L+1\), in the case \( T = 2L \)) that does not contain terms of order \( L^{-1} \) and \( \log(L)L^{-1} \). In the second step, the terms of order \( L^{-2} \) and \( \log(L)L^{-2} \) are analogously removed and so on. At each step the number of data in the resulting sequence is of course decreased by two units and the rounding-error effects are enhanced, which limits the optimal number of steps to be performed in practice. This method allows for a safe, although often conservative, estimate of the systematic error on the asymptotic value of the sequence under study. To this aim it is important to properly propagate the estimated rounding errors on \( f_X(L; T/L; \theta; m_R L) \) when constructing all the needed linear combinations. Our practical use of the method closely followed the original proposal and detailed description in [29], apart from some details which are discussed below.

The data for the correlation functions \( f_X(L; T/L; \theta; m_R L) \) were collected for values of \( L \) ranging from 4 to \( L_{\text{max}} \), as indicated in table 1. Because our data have been obtained using only double precision and with a value of \( L_{\text{max}} = 32 \), the uncertainties on the values of improvement coefficients that we quote for LW, RG1, RG2 and RG3 actions are systematically larger than the uncertainties obtained in [12, 26] for the standard plaquette action. In addition, we had also to adopt a slightly modified blocking–fit procedure, in some cases where a straightforward application of the above mentioned blocking procedure turned out to lead to a large (order 10%) estimated value of the relative error on the asymptotic quantities we are interested in.

We realized, indeed, that when the lattice artifacts of a given order \( j \) in the asymptotic expansion (3.1), i.e.

\[
[Q_j + q_j \log(L)]L^{-j}
\]

are numerically very small (in comparison with higher order lattice artifacts) for all the considered values of \( L \), removing them through a suitable linear combination strongly enhances rounding-error effects while hardly changing the estimate of the asymptotic value \( (Q_0 \text{ in (3.1)}) \). In such a case we found it convenient for the purpose of improving the precision on \( Q_0 \), to resort to the following hybrid blocking–fit procedure:

- perform standard blocking steps for terms of order from 0 to \( j - 1 \);
- using suitable linear combinations of the reduced data sequence at step \( j - 1 \), determine separately the values of \( Q_j \) and \( q_j \), say \( \bar{Q}_j \) and \( \bar{q}_j \), with some (in practice rather large) relative uncertainty \( \delta Q_j \) and \( \delta q_j \);
- subtract from the reduced data sequence at step \( j - 1 \) the quantity \( [\bar{Q}_j + \bar{q}_j \log(L)]L^{-j} \) and perform standard blocking steps for terms of order \( j + 1, j + 2, \ldots \), stopping as
usual when the number of data in the sequence gets too small or the rounding-error effects too large.

- repeat the procedure of the latter item by replacing $\bar{Q}_j$ and $\bar{q}_j$ with $\bar{Q}_j \pm \delta Q_j$ and $\bar{q}_j \pm \delta q_j$, respectively, taking mean and extreme values in a way that is equivalent to the error estimate adopted for standard blocking steps. In this way one gets a conservative estimate of the uncertainty induced by $\delta Q_j$ and $\delta q_j$.

This blocking–fit procedure is equivalent to the simple blocking procedure of [29] in the ideal case of infinite precision data, but turns out to be more convenient when rounding-error effects are not negligible and lattice artifacts of a given order $j$ are very small.

We remark that requiring $O(\alpha)$ improvement of all the correlation functions introduced in ref.s [12, 26] provides a number of relations between the various improvement coefficients larger than the number of the improvement coefficients themselves, allowing then for several consistency checks. Further consistency checks were done by comparing the values of the b-coefficients at $m_R L = 0.1$ and $m_R L = 0.5$.

After checking the consistency among the asymptotic values obtained from different lattice approximants of the same improvement coefficients, we have sometimes made use of the fact that a particular linear combination of these approximants showed small lattice artifacts, leading to a more precise determination of the asymptotic value than the single approximants themselves.

4 Results and Conclusion

In table 1 we give our final result obtained by the method in the previous section. This table contains a synthesis of all our 1-loop results, including a comparison with the case of the standard plaquette gauge action (label “Plaq.”) [12, 20].

We first notice that tadpole contributions to the $c_{sw}^{(1)}$, denoted $c_{sw}^{\text{tad}}$ in the table, give about 90% of the complete 1-loop contributions for all actions considered here. Therefore the value of $c_{sw}$ taken by the CP-PACS collaboration for their full QCD simulation with RG1 gauge action [3] is very close to the full one-loop value up to order $g_0^2$, and the order $g_0^2$ difference can be calculated from the table: $c_{sw}^{\text{pert.}} = c_{sw}^{CP-PCS} + 0.008 g_0^2 + O(g_0^4)$.

Although only 3 choices of RG improved gauge actions are considered, it seems that RG improved gauge actions generally give a $c_{sw}^{(1)}$ coefficient which is a factor 2 to 2.5 smaller than that of the plaquette action, while for the perturbative improved action (LW) the reduction factor is only about 1.35. This tendency has already been found in the finite part of 1-loop renormalization factors for various quantities [30]. Other improvement
coefficients, the $c_{X}^{(1)}$'s and $b_{X}^{(1)}$'s, as well as $b_{c}^{(1)}$ and $c_{t}^{(1)}$ for the setup denoted as Choice \( A \), are also given in table \[1\]. The same tendency for the size of 1-loop coefficients is also observed.

It is clear, however, that the smallness of 1-loop coefficient for the RG improved action does not imply the smallness of the lattice artifacts for the action. To get some impression of these we have to evaluate lattice artifacts directly from physical observables for the corresponding action. For that purpose, following ref. \[12\], we consider the unrenormalized current quark mass \( m \), defined through the PCAC relation (see eq. (6.13) of ref. \[7\]), which is expected to vanish up to terms of order \( a^{2} \) at \( K = K_{c} \). If we expand \( m \) such that

\[
am = r_{0} + r_{1} C_{F} g_{0}^{2} + O(g_{0}^{4})
\]

at \( K = K_{c} \) and \( x_{0} = T/2 \), the value of \( r_{1} \) represents the magnitude of the remaining cutoff effect for the particular gauge action at 1-loop order, since the tree-level contribution \( r_{0} \) is independent of the pure gauge action. In fig. \[2\] we have plotted \( r_{1} \) for various actions as a function of \( a/L \) for \( T = 2L \) and \( \theta = 0 \) and vanishing boundary gauge field (in which case the tree-level contribution actually vanishes). At large values of \( a/L \) it is observed that this 1-loop lattice artifact is indeed smaller for the RG improved gauge actions than for the LW action, for which it is still smaller than for the plaquette action. We remark however that one cannot draw general conclusions from this simple exercise; the pattern may be different for other correlation functions and, needless to say, at a non-perturbative level we have no statement to make at all.

In the last box of the table the 1-loop value of the critical hopping parameter \( K_{c}^{(1)} \), defined by \( K_{c} = K_{c}^{(0)} + g_{0}^{2} K_{c}^{(1)} + \cdots \), is also given and compared with the results from perturbation theory on an infinite lattice \[30\]. They agree with each other within 5 digits, and this fact also supports the correctness of our calculations.

Recently the coefficients \( c_{X} \) and \( b_{X} \) have been calculated by Taniguchi and Ukawa \[10\] using a completely different method. The two sets of results agree well, although our quoted errors are generally larger than theirs. The good agreement between the central values of the two methods indicates, however, that our error estimates are rather conservative.

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Appendix A

In this appendix we explain the allowable weights of the contributions to the gauge action near the boundary in our setup to avoid order $a$ effects. We remind the reader that in this paper we are not demanding the absence of $O(a^2)$ effects. As explained in ref. [9] there are only two possible boundary counterterms at order $a$ in the pure gauge theory, which are obtained by summing any local lattice expressions for the fields

$$a^4 \text{tr} \left\{ F_{0k} F_{0k} \right\} \quad \text{and} \quad a^4 \text{tr} \left\{ F_{ij} F_{ij} \right\} \quad \text{(A.2)}$$

over the $x_0 = 0$ and $x_0 = T$ hyperplanes.

The first step is to ensure that the approach to the classical continuum limit is $O(a^2)$. Following the discussion of ref. [16] we recall that any given smooth continuum gauge field $A_\mu(y)$, can be arbitrarily well approximated by lattice gauge fields $U_\mu(x)$

$$U_\mu(x) = T \exp \left\{ a \int_0^1 dt A_\mu(x + [1-t]a\hat{\mu}) \right\} \quad \text{(A.3)}$$

One can then derive asymptotic expansions for the lattice operators. For example for $\mathcal{L}(C_i) = \text{ReTr}[I - U(C_i)]$ with $C_i \in \mathcal{S}$ one easily finds for the flat loops in the $\mu - \nu$ plane

$$\mathcal{L}(C_i) = -k_i a^4 \text{tr} F_{\mu\nu}(x_c) F_{\mu\nu}(x_c) + O(a^6), \quad i = 0,1 \quad \text{(A.4)}$$

where the constant $k_i$ depends on the class

$$k_0 = \frac{1}{2}, \quad k_1 = 2 \quad \text{(A.5)}$$

and $x_c$ is the “center of the loop” (otherwise one obtains in general $O(a^5)$ appearing on the rhs of Eq. (A.3) above). Obviously there are similar expressions for the non-planar loops.

The approximation of the integrals by sums in spatial directions can incur only $O(a^2)$ errors because of the periodic boundary conditions. However approximations of the integral by sums in the time direction may result in additional effects of order $a$ through boundary terms. For example for a smooth function $f(z)$ defined on the region $[0,T]$ and dividing the region in $M = T/a$ segments, we have

$$a \sum_{m=1}^{M} f(ma - \frac{1}{2}a) = \int_0^T dz f(z) + O(a^2), \quad \text{(A.6)}$$

$$a \sum_{m=1}^{M-1} f(ma) = \int_0^T dz f(z) - \frac{a}{2} \left( f(0) + f(T) \right) + O(a^2). \quad \text{(A.7)}$$
For planar loops extending in the time direction for which the time coordinate of the center $x_c$ takes half integer values we can invoke Eq. (A.6) to see that their contribution to the action will be as in the continuum up to terms of order $a^2$. However for those loops where $x_c$ takes integer values, which is the case only for the rectangles extending over two time units, we see from (A.7) that the contribution to the action from these rectangles will have an $O(a)$ deficit with respect to the classical continuum expression. For instance at the boundary $T = 0$ we are missing a contribution $-\left(\frac{2c_1}{g_0^2}\right) \int d\mathbf{x} trF_{0k}(0, \mathbf{x})F_{0k}(0, \mathbf{x})$ which can be compensated by adding extra terms involving plaquettes and/or rectangles extending only one unit in the time direction. Two admissible choices are the special setups A, B specified in subsection 2.2.

Eq. (A.7) also explains the fact that $c_s(0) = \frac{1}{2}$, but this is not relevant for our particular boundary conditions for which $F_{jk} = 0$.

**Appendix B**

In this appendix we give the form of the free inverse gauge propagator for non-zero $c_1$ and $c_3$ in the case of vanishing background field. Introducing the Fourier transformed fields through

$$q_0(x) = L^{-3} \sum_p e^{ipx} \tilde{q}_0(p, x), \quad (B.8)$$
$$q_k(x) = L^{-3} \sum_p e^{ipx} e^{ipk/2} \tilde{q}_k(p, x), \quad (B.9)$$

the quadratic part of the gauge action including the gauge-fixing term takes the form

$$S^{(2)}_{\text{gauge}} = -L^{-3} \sum_p \sum_{x_0, y_0=0}^{T-1} \text{tr} \tilde{q}_\mu(-p, x_0)\tilde{q}_\nu(p, y_0)K_{\mu\nu}(p; x_0, y_0). \quad (B.10)$$

Explicitly

$$K_{00}(p; x_0, y_0) = \delta_{x_0, y_0}\left[\left(c_0 + 6c_1 + 8c_3\right)\hat{p}^2 - c_1\hat{p}^4 + c_3(\hat{p}^4 - (\hat{p}^2)^2)\right]$$
$$+ \lambda_0\{2 - \delta_{x_0,0}(1 - \delta_{p,0}) - \delta_{x_0,T-1}\}$$
$$+ (\delta_{x_0-1, y_0} + \delta_{x_0+1, y_0})(c_1\hat{p}^2 - \lambda_0)$$
$$+ c_1\delta_{x_0, y_0} (\delta_{x_0,T-1} + \delta_{x_0,0})(\hat{p}^2 - \hat{p}^4 d_{bc}), \quad (B.11)$$

$$K_{k0}(p; x_0, y_0) = i\tilde{p}_k\left[(\delta_{x_0, y_0} - \delta_{x_0-1, y_0})\left(c_0 + 5c_1 + 8c_3 - \lambda_0 - c_1\hat{p}_k^2 + c_3(\hat{p}_k^2 - \hat{p}^2)\right)\right]$$

19
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K_{0k}(p; x_0, y_0) = -K_{k0}(p; y_0, x_0),

K_{kl}(p; x_0, y_0) = \delta_{x_0,y_0} \delta_{kl} \left[ (c_0 + 8c_1 + 4c_3)p^2 + 2c_0 + 10c_1 + 16c_3 
- c_1(2\hat{p}_k^2 + \hat{p}_k^4 + \hat{p}_k^2\hat{p}_k^2) + c_3(2\hat{p}_k^2 + \hat{p}_k^4 - (\hat{p}_k^2)^2 + \hat{p}_k^2\hat{p}_k^2) 
+ \delta_{x_0,y_0} \hat{p}_k \hat{p}_l \left[ \lambda_0 - c_0 - 8c_1 - 6c_3 + c_1(\hat{p}_k^2 + \hat{p}_l^2) + c_3(\hat{p}_k^2 - \hat{p}_k^2 - \hat{p}_l^2) \right] 
- (\delta_{x_0+1,y_0} + \delta_{x_0-1,y_0}) \left[ \delta_{kl} \{c_0 + 4c_1 + 8c_3 - c_1\hat{p}_k^2 + c_3(\hat{p}_k^2 - 2\hat{p}_k^2) \} \right] 
+ c_3\hat{p}_k \hat{p}_l \right] - (\delta_{x_0+2,y_0} + \delta_{x_0-2,y_0}) \delta_{kl} c_1 
+ c_1 \delta_{kl} \delta_{x_0,y_0} (\delta_{x_0,T-1} + \delta_{x_0,1})(1 - \hat{p}_k^2 d_{bc}), \quad (B.14)

where

\hat{p}_k = 2 \sin p_k/2, \quad \hat{p}_k^2 = \sum_{k=1}^3 \hat{p}_k^2, \quad \hat{p}_k^4 = \sum_{k=1}^3 \hat{p}_k^4. \quad (B.15)

For the boundary coefficient $d_{bc}$ appearing above, we have $d_{bc} = 0$ for Choice A and $d_{bc} = 1$ for Choice B.

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Table
Figure 2: Remaining cutoff effect at 1-loop in the PCAC mass at $x_0 = T/2$ and $K = K_c$ for various actions, for the case $T = 2L$, $\theta = 0$ and vanishing background field.
Table 1: Improvement coefficients at 1-loop for various gluon actions. The results for the standard plaquette action are taken from refs. [12, 26]. All results are obtained from data with $T = 2L$ except $c^{(1)}_{sw}$, which is calculated from $T = L$ data. Tadpole contributions for $c^{(1)}_{sw}$ are also listed. The critical hopping parameter $K^{(1)}_{c}$ (above) is also compared with the result (below) obtained from the usual perturbation theory in infinite volume [30].

| gauge action | Plaq. | LW | RG1 | RG2 | RG3 |
|--------------|-------|----|-----|-----|-----|
| $c_1$        | 0.0   | -1/12 | -0.331 | -0.27 | -0.252 |
| $c_3$        | 0.0   | 0.0 | 0.0 | -0.04 | -0.17 |

| $L_{max}$   | 32 | 24 | 30 | 24 | 24 |
|-------------|----|----|----|----|----|
| $c^{(1)}_{sw}$ | 0.267(1) | 0.196(6) | 0.113(3) | 0.119(5) | 0.109(5) |
| $c^{(1)}_{sw}$ | 0.25 | 0.183 | 0.105 | 0.110 | 0.096 |

| $L_{max}$   | 48 | 32 | 32 | 32 | 32 |
|-------------|----|----|----|----|----|
| $c^{(1)}_{A}$ | -0.005680(2) | -0.004525(25) | -0.002846(11) | -0.003017(12) | -0.002805(20) |
| $c^{(1)}_{V}$ | -0.01225(1) | -0.0103(3) | -0.00730(20) | -0.00757(26) | -0.00709(20) |
| $c^{(1)}_{T}$ | 0.00896(1) | 0.00743(7) | 0.00505(10) | 0.00526(15) | 0.00496(12) |
| $b^{(1)}_{A}$ | -0.07217(2) | -0.0576(11) | -0.0382(8) | -0.0395(15) | -0.0353(12) |
| $b^{(1)}_{V}$ | 0.11414(4) | 0.0881(13) | 0.0550(4) | 0.0572(6) | 0.0510(5) |
| $b^{(1)}_{T}$ | 0.11484(2) | 0.0889(14) | 0.0558(9) | 0.0584(10) | 0.0528(8) |
| $b^{(1)}_{S}$ | 0.14434(4) | 0.1152(22) | 0.0764(16) | 0.0790(30) | 0.0706(24) |
| $b^{(1)}_{A}$ | 0.10434(4) | 0.0790(25) | 0.0477(12) | 0.0502(19) | 0.0444(15) |
| $b^{(1)}_{V}$ | -0.06738(4) | -0.0505(23) | -0.0243(11) | -0.0275(18) | -0.0252(11) |
| $b^{(1)}_{T}$ | -0.01346(1) | -0.0122(4) | -0.00661(21) | -0.00612(23) | -0.00186(20) |

| $L_{max}$   | 32 | 32 | 32 | 32 | 32 |
|-------------|----|----|----|----|----|
| $K^{(1)}_{c}$ | 6.329891(3) | 4.716253(1) | 2.760894(1) | 2.911363(2) | 2.593490(1) |
| $K^{(1)}_{c}$ | 6.3300(1) | 4.7163(1) | 2.7609(1) | 2.9114(1) | 2.5935(1) |