LOCAL SPECTRA OF OPERATOR WEIGHTED SHIFTS

A. BOURHIM

Abstract. In this note, we study the local spectral properties of unilateral operator weighted shifts.

1. Introduction

Let $L(H)$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $H$, and let $A := (A_n)_{n \geq 0}$ be a sequence of uniformly bounded invertible operators of $L(H)$. Let

$$\widehat{H} = \bigoplus_{n=0}^{+\infty} \oplus H_n,$$

where $H_n = H$ for each $n \geq 0$. It is a Hilbert space when equipped with the inner product

$$\langle (x_n)_n, (y_n)_n \rangle_{\widehat{H}} = \sum_{n=0}^{+\infty} \langle x_n, y_n \rangle_H.$$

Therefore, the corresponding norm is given by

$$\| (x_n)_n \|_{\widehat{H}} = \left( \sum_{n} \| x_n \|_H^2 \right)^{\frac{1}{2}}.$$

The unilateral operator weighted shift, $S_u$, with the weight sequence $A = (A_n)_{n \geq 0}$ is the operator on $\widehat{H}$ defined by

$$S_u(x_0, x_1, x_2, \ldots) = (0, A_0 x_0, A_1 x_1, A_2 x_2, \ldots), \quad ((x_n)_n \in \widehat{H}).$$

Operator weighted shifts were first introduced by A. Lambert [9], and have been studied by many authors (see for example [11], [12], [11], [7], and [8]). In the case when $\dim H = 1$, they are exactly the scalar weighted shifts which have been widely studied. An excellent survey of the investigation of the spectral theory of such operators was given by A. L. Shields [17]. Moreover, several known results for the scalar case have been generalized and extended to the setting of operator weighted shifts. However, the question of determining the local spectral properties for operator weighted shifts is natural and has been initiated in [20]. While, the investigation of these properties for scalar weighted shifts has been studied in [3] and [14]. The main goal of the present note is to study and examine whether or not the results obtained in [3] remain valid for unilateral operator weighted shifts. We

1991 Mathematics Subject Classification. Primary 47B37; Secondary 47A10, 47A11.
give necessary and sufficient conditions for a unilateral operator weighted shift to satisfy Dunford’s condition (C) or Bishop’s property (β). Unlike the scalar weighted shift operators, we show that they are examples of unilateral operator weighted shifts possessing Bishop’s property (β) with large approximate point spectrum and without fat local spectra.

For an operator \( T \in \mathcal{L}(\mathcal{H}) \), let, as usual, \( T^*, \sigma(T), \sigma_{ap}(T), \sigma_p(T), \) and \( r(T) \) denote the adjoint, the spectrum, the approximate point spectrum, the point spectrum, and the spectral radius of \( T \), respectively. Let \( m(T) := \inf\{\|Tx\| : \|x\| = 1\} \) denote the lower bound of \( T \), and note that the sequence \( (m(T^n)^{1/n})_{n \geq 1} \) converges and its limit, denoted \( r_1(T) \), equals sup\( \{m(T^n)^{1/n}\} \) (see [12]). Let \( T \in \mathcal{L}(\mathcal{H}) \); for an element \( x \in \mathcal{H} \), let \( \sigma_T(x), \rho_T(x) := \mathbb{C}\setminus \sigma_T(x) \), and \( r_T(x) := \limsup_{n \to +\infty} \|T^n x\|^{1/n} \) be the local spectrum, the local resolvent, and the local spectral radius of \( T \) at \( x \), respectively (see [3] and [10]). The operator \( T \) is said to have the single–valued extension property at a complex number \( \lambda_0 \in \mathbb{C} \) if for every open disc \( U \) centered at \( \lambda_0 \), the only analytic solution of the equation \( (T - \lambda)f(\lambda) = 0, (\lambda \in U) \) is the zero function \( f \equiv 0 \).

Denote by \( \mathcal{R}(T) \) the set of all complex numbers on which \( T \) fails to have the single–valued extension property and recall that \( T \) is said to have the single–valued extension property provided that \( \mathcal{R}(T) \) is empty. The reader is reminded that in the case \( T \) has the single–valued extension property, the local resolvent of \( x \) is the unique analytic \( \mathcal{H} \)-valued function, \( x(\cdot) \), satisfying \( (T - \lambda)x(\lambda) = x, (\lambda \in \rho_T(x)) \). Also, recall that an operator \( T \in \mathcal{H} \) is said to satisfy Dunford’s condition (C) provided that for every closed subset \( F \) of \( \mathcal{C} \), the linear subspace,

\[
\mathcal{H}_\tau(F) := \{x \in \mathcal{H} : \sigma_T(x) \subset F\}
\]

is closed. Moreover, \( T \) is said to have fat local spectra if \( \sigma_T(x) = \sigma(T) \) for all non-zero \( x \in \mathcal{H} \). It is well known that every operator which satisfies Dunford’s condition (C) has the single–valued extension property and it turns out that Dunford’s condition (C) follows from fat local spectra property.

Throughout this note, let \( S_n \) be a unilateral operator weighted shift with weight sequence \( A := (A_n)_{n \geq 0} \), and let \( (B_n)_{n \geq 0} \) be the sequence given by

\[
B_n = \begin{cases} 
A_{n-1}A_{n-2}...A_1A_0 & \text{if } n > 0 \\
1 & \text{if } n = 0
\end{cases}
\]

Define

\[
r_2(S_n) := \limsup_{n \to +\infty} \left\|B_n^{-1}\right\|^{1/n}, \quad r_3(S_n) := \liminf_{n \to +\infty} \left\|B_n^{-1}\right\|^{1/n},
\]

\[
R_2^+(S_n) := \sup_{x \in \mathcal{H}, x \neq 0} \left\{ \frac{1}{\limsup_{n \to +\infty} \left\|B_n^{-1}x\right\|^{1/n}} \right\} = \sup_{x \in \mathcal{H}, \|x\|=1} \left\{ \frac{1}{\limsup_{n \to +\infty} \left\|B_n^{-1}x\right\|^{1/n}} \right\},
\]
\( R_2^-(S_u) := \inf_{x \in \mathcal{H}, x \neq 0} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n-1}^{-1} x \|^\frac{1}{n}} \right\} = \inf_{x \in \mathcal{H}, \|x\| = 1} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n-1}^{-1} x \|^\frac{1}{n}} \right\}, \)

\( R_3^+(S_u) := \sup_{x \in \mathcal{H}, x \neq 0} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n-1} x \|^\frac{1}{n}} \right\} = \sup_{x \in \mathcal{H}, \|x\| = 1} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n} x \|^\frac{1}{n}} \right\}, \)

and

\( R_3^-(S_u) := \inf_{x \in \mathcal{H}, x \neq 0} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n-1} x \|^\frac{1}{n}} \right\} = \inf_{x \in \mathcal{H}, \|x\| = 1} \left\{ \frac{1}{\limsup_{n \to +\infty} \| B_{n} x \|^\frac{1}{n}} \right\}. \)

Note that

\( r_1(S_u) \leq r_2(S_u) \leq R_2^-(S_u) \leq R_2^+(S_u), \)

and

\( r_3(S_u) \leq R_3^+(S_u) \leq R_3^-(S_u) \leq r(S_u). \)

Note also that for a scalar weighted shift \( S_u \), we have

\( r_1(S_u) \leq r_2(S_u) = R_2^-(S_u) = R_2^+(S_u) \leq r_3(S_u) = R_3^+(S_u) = R_3^-(S_u) \leq r(S_u). \)

Finally, we would like to record and without further mention a notation that we will use repeatedly throughout this note. For every \( x \in \mathcal{H} \), we write

\( x^{(n)} = (0, ..., 0, x, 0, ...) \), \( (n \geq 0) \)

for the element of \( \mathcal{H} \) for which all the coordinates are zero except the \( n \)th coordinate which is equal \( x \), and note that

\[
(1.1) \quad r_{S_u}(x^{(k)}) = \limsup_{n \to +\infty} \| B_{n+k}^{-1} x \|^{\frac{1}{n}}.
\]

2. Preliminaries and elementary background

In this section, we assemble some elementary results that are very much on the straightforward side and therefore the proofs will be omitted.

**Proposition 2.1.** Assume that \( T \in \mathcal{L}(\mathcal{H}) \) is an operator for which \( \bigcap_{n \geq 0} T^n \mathcal{H} = \{0\} \). The following statements hold.

(a) \( \{ \lambda \in \mathbb{C} : |\lambda| \leq r_1(T) \} \subset \sigma_+(x) \) for every non-zero element \( x \in \mathcal{H} \).

(b) \( \sigma_p(T) \subset \{0\} \).

(c) Each \( \sigma_+(x) \) is connected.

(d) \( \sigma(T) \) is a connected set and satisfies \( \{ \lambda \in \mathbb{C} : |\lambda| \leq r_1(T) \} \subset \sigma(T) \).

In particular, if \( \sigma(T) \) is circularly symmetric about the origin, then

\( \sigma(T) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r(T) \}. \)

Evidently, the unilateral operator weighted shift \( S_u \) satisfies the condition that \( \bigcap_{n \geq 0} S_u^n \mathcal{H} = \{0\} \), and its spectrum is rotationally symmetric. Therefore, the next result is an immediate consequence of proposition 2.1.

**Corollary 2.2.** The following statements hold.
Therefore, by the Cauchy-Hadamard formula for the radius of convergence, we get that

Proposition 2.3. For every \( n \geq 1 \), we have

\[
\|S_u^n\| = \sup_{k \geq 0} \|B_{n+k}B_k^{-1}\|, \quad \text{and} \quad m(S_u^n) = \inf_{k \geq 0} \left\{ \frac{1}{\|B_kB_{n+k}\|} \right\}.
\]

Thus,

\[
r(S_u) = \lim_{n \to +\infty} \left[ \sup_{k \geq 0} \|B_{n+k}B_k^{-1}\| \right]^{\frac{1}{n}}, \quad \text{and} \quad r_1(S_u) = \lim_{n \to +\infty} \left[ \inf_{k \geq 0} \left\{ \frac{1}{\|B_kB_{n+k}\|} \right\} \right]^{\frac{1}{n}}.
\]

Proposition 2.4. The adjoint of \( S_u \) is given by

\[
S_u^*x = (A_0^*x_1, A_1^*x_2, A_2^*x_3, \ldots), \quad (x = (x_0, x_1, \ldots) \in \hat{H}).
\]

3. Local spectra of \( S_u \)

We begin this section with the following result that gives a necessary and sufficient condition for \( S_u^* \) to enjoy the single-valued extension property.

Lemma 3.1. The following statements hold.

(a) \( \sigma_p(S_u) = \emptyset \).

(b) \( \{0\} \cup \{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \} \subset \sigma_p(S_u^*) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq R_2^+(S_u) \} \).

(c) \( S_u^* \) has the single-valued extension property if and only if \( R_2^+(S_u) = 0 \). Moreover, we always have

\[
\Re(S_u^*) = \{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \}.
\]

Proof. (a) By proposition 2.1, we have \( \sigma_p(S_u) \subset \{0\} \). As \( S_u \) is injective, we note that \( \sigma_p(S_u) = \emptyset \).

(b) Suppose that \( \lambda \in \mathbb{C} \) is an eigenvalue for \( S_u^* \) and that \((x_n)_n\) is a corresponding eigenvector. We have

\[
(A_0^*x_1, A_1^*x_2, A_2^*x_3, \ldots) = (\lambda x_0, \lambda x_1, \lambda x_2, \ldots).
\]

This shows that

\[
x_n = \lambda^n B_n^{-1}x_0, \quad (n \geq 0).
\]

Therefore,

\[
\|x\|^2 = \sum_{n \geq 0} |\lambda|^{2n} \|B_n^{-1}x_0\|^2.
\]

By the Cauchy-Hadamard formula for the radius of convergence, we get that

\[
|\lambda| \leq \frac{1}{\limsup_{n \to +\infty} \|B_n^{-1}x_0\|^{\frac{1}{n}}}
\]

Thus,

\[
\sigma_p(S_u^*) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq R_2^+(S_u) \}.
\]
Now, let us prove that
\[
\{0\} \cup \{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \} \subset \sigma_p(S_u^*) .
\]
It is clear that for every \( x \in \mathcal{H} \), we have \( S_u^*x^{(0)} = 0 \); hence, \( 0 \in \sigma_p(S_u^*) \).

If \( R_2^+(S_u) = 0 \), then there is nothing to prove; thus, we may assume that \( R_2^+(S_u) > 0 \). Let \( \lambda \in \mathbb{C} \) such that \( |\lambda| < R_2^+(S_u) \). So, there is a non-zero \( x_0 \in \mathcal{H} \) such that \( |\lambda| < \frac{1}{\limsup_{n \to +\infty} \|B_n^{-1}x_0\|^2} \). We have \((S_u^* - \lambda)k_{x_0}(\lambda) = 0\), where
\[
k_{x_0}(\lambda) = \sum_{n \geq 0} \oplus \lambda^n B_n^{-1} x_0 .
\]
This shows that
\[
\{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \} \subset \sigma_p(S_u^*),
\]
and the desired statement holds.

(c) In view of the statement (b) and the fact that \( \Re(S_u^*) \subset \text{int}(\sigma_p(S_u^*)) \), we have \( \Re(S_u^*) \subset \{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \} \).

Conversely, let \( x \) be a non-zero element of \( \mathcal{H} \) and set
\[
U_x := \{ \lambda \in \mathbb{C} : |\lambda| < \frac{1}{\limsup_{n \to +\infty} \|B_n^{-1}x\|^2} \},
\]
and
\[
k_x(\lambda) := \sum_{n \geq 0} \oplus \lambda^n B_n^{-1} x, \ (\lambda \in U_x) .
\]
Since \((S_u^* - \lambda)k_x(\lambda) = 0\), for all \( \lambda \in U_x \), and \( x \) is an arbitrary non-zero element of \( \mathcal{H} \), we have
\[
\{ \lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u) \} = \bigcup_{x \in \mathcal{H}, \ x \neq 0} U_x \subset \Re(S_u^*).
\]
The proof is therefore complete.

The following result refine the local spectral inclusion given in corollary 2.2.

**Proposition 3.2.** For every non-zero \( y = (y_0, y_1, y_2, \ldots) \) \( \in \hat{\mathcal{H}} \), we have
\[
\{ \lambda \in \mathbb{C} : |\lambda| \leq R_2^-(S_u) \} \subset \sigma_{S_u}(y).
\]
In particular, if \( r(S_u) = R_2^-(S_u) \) then \( S_u \) has fat local spectra.

**Proof.** As \( \bigcap_{n \geq 0} S_u^*\hat{\mathcal{H}} = \{0\} \), we have \( 0 \in \sigma_{S_u}(y) \). Thus, we may assume that \( R_2^-(S_u) > 0 \). Let \( O := \{ \lambda \in \mathbb{C} : |\lambda| < R_2^-(S_u) \} \), and let \( x \) be a non-zero element of \( \mathcal{H} \). Consider the following analytic \( \hat{\mathcal{H}} \)-valued function on \( O \),
\[
k_x(\lambda) = \sum_{n \geq 0} \oplus \lambda^n B_n^{-1} x .
\]
We have \((S_u - \lambda)^* k_u(\overline{x}) = 0\) for every \(\lambda \in O\). Now, let \(y = (y_0, y_1, y_2, \ldots) \in \widehat{H}\) such that \(O \cap \rho_{su}(y) \neq \emptyset\). So, for every \(\lambda \in O \cap \rho_{su}(y)\), we have
\[
\sum_{n \geq 0} \langle y_n, B_n^{*-1} x \rangle \lambda^n = \langle y, k_x(\overline{x}) \rangle \widehat{H} = 0.
\]
Hence, for every \(n \geq 0\), we have
\[
\langle y_n, B_n^{*-1} x \rangle \mathcal{H} = 0.
\]
Since \(x\) is an arbitrary element of \(\mathcal{H}\), we have \(y = 0\); and the proof is complete.

In view of proposition 3.2, we note that \(R_2^-(S_u) \leq r_{S_u}(x)\), for all non-zero \(x = (x_0, x_1, x_2, \ldots) \in \widehat{H}\). The following gives more information about local spectral radii of \(S_u\).

**Proposition 3.3.** For every non-zero element \(x = (x_0, x_1, \ldots) \in \widehat{H}\), we have
\[
R_3^-(S_u) \leq r_{S_u}(x) \leq r(S_u).
\]
Moreover, if \(x = (x_0, x_1, \ldots)\) is a non-zero finitely supported element of \(\widehat{H}\), then
\[
R_3^-(S_u) \leq r_{S_u}(x) = \max_{k \geq 0} (r_{S_u}(x_k^{(k)})) \leq R_3^+(S_u).
\]

**Proof.** Let \(x = (x_0, x_1, \ldots)\) be a non-zero element of \(\widehat{H}\); so, there is an integer \(k_0 \geq 0\) such that \(x_{k_0} \neq 0\). Since,
\[
\|S_u^n x\|^2 = \sum_{k=0}^{+\infty} \|B_{n+k} B_k^{-1} x_k\|^2, \forall n \geq 0,
\]
we have
\[
\|B_{n+k_0} B_k^{-1} x_{k_0}\|_{\frac{1}{n+k_0}} \leq \|S_u^n x\|_{\frac{1}{n+k_0}}, \forall n \geq 0.
\]
Now, taking \(\limsup\) as \(n \to +\infty\), we get
\[
R_3^-(S_u) \leq \limsup_{n \to +\infty} \|B_{n+k_0} B_k^{-1} x_{k_0}\|_{\frac{1}{n+k_0}} \leq r_{S_u}(x),
\]
as desired.

(b) Assume that \(x = (x_0, x_1, \ldots)\) is a non-zero finitely supported element of \(\widehat{H}\). As above, we have
\[
\|B_{n+k} B_k^{-1} x_k\|_{\frac{1}{n+k}} \leq \|S_u^n x\|_{\frac{1}{n}}, \forall n, k \geq 0.
\]
By taking \(\limsup\) as \(n \to +\infty\), we get \(r_{S_u}(x_k^{(k)}) \leq r_{S_u}(x), \forall k \geq 0\). Hence,
\[
\max_{k \geq 0} (r_{S_u}(x_k^{(k)})) \leq r_{S_u}(x).
\]
As $\sigma_{s_u}(x) \subset \bigcup_{k \geq 0} \sigma_{s_u}(x_k^{(k)})$, and $r_{s_u}(y) = \max\{|\lambda| : \lambda \in \sigma_{s_u}(y)\}$ for every non-zero $y \in \hat{H}$, we obtain $r_{s_u}(x) \leq \max_{k \geq 0} (r_{s_u}(x_k^{(k)}))$. Hence,

$$r_{s_u}(x) = \max_{k \geq 0} (r_{s_u}(x_k^{(k)})).$$

On the other hand, we have $r_{s_u}(x_k^{(k)}) = r_{s_u}((B_k^{-1}x_k)^{(0)}), \forall k \geq 0$. This shows that

$$r_{s_u}(x) = \max_{k \geq 0} (r_{s_u}(x_k^{(k)})) \leq R_3^+(S_u).$$

Therefore, the desired result holds. \qed

For every $x = (x_0, x_1, \ldots) \in \hat{H}$, we set

$$R_A(x) := \frac{1}{\limsup_{n \to +\infty} \|B_n^{-1}x_n\|^\frac{1}{n}}.$$

Obviously, if $x$ is a non-zero element of $\hat{H}$, then $r_2(S_u) \leq R_A(x) \leq +\infty$.

**Theorem 3.4.** For every non-zero element $x = (x_0, x_1, \ldots) \in \hat{H}$, we have

$$\{ \lambda \in \mathbb{C} : |\lambda| \leq \min_{n \geq 0} (R_A(x), r_3(S_u)) \} \subset \sigma_{s_u}(x).$$

Moreover, if $x = (x_0, x_1, \ldots)$ is a non-zero finitely supported element of $\hat{H}$, then

$$\{ \lambda \in \mathbb{C} : |\lambda| \leq R_3^+(S_u) \} \subset \sigma_{s_u}(x).$$

**Proof.** Let $x = (x_0, x_1, \ldots)$ be a non-zero element of $\hat{H}$. If $\min_{n \geq 0} (R_A(x), r_3(S_u)) = 0$, then there is nothing to prove since $0 \in \sigma_{s_u}(x)$. Thus we may suppose that $\min_{n \geq 0} (R_A(x), r_3(S_u)) > 0$. Now, for each $n \geq 0$, let

$$F_n(\lambda) = -\frac{B_n x_0}{\lambda^{n+1}} - \frac{B_n B_1^{-1} x_1}{\lambda^n} - \frac{B_n B_2^{-1} x_2}{\lambda^{n-1}} - \cdots - \frac{x_n}{\lambda}, \ (\lambda \in \mathbb{C}\{0\}),$$

and

$$G_n(\lambda) = x_0 + \lambda B_1^{-1} x_1 + \lambda^2 B_2^{-1} x_2 + \cdots + \lambda^n B_n^{-1} x_n, \ (\lambda \in \mathbb{C}).$$

We have,

$$F_n(\lambda) = -\frac{1}{\lambda^{n+1}} B_n G_n(\lambda), \ (\lambda \in \mathbb{C}\{0\}). \quad (3.3)$$

By writing $\bar{x}(\lambda) := (f_0(\lambda), f_1(\lambda), f_2(\lambda), \ldots), \ \lambda \in \rho_{s_u}(x)$, we get from the equation,

$$(S_u - \lambda)\bar{x}(\lambda) = x, \ \lambda \in \rho_{s_u}(x),$$

that for every $\lambda \in \rho_{s_u}(x)$, we have

$$\begin{cases} -\lambda f_0(\lambda) = x_0 \\ A_n f_n(\lambda) - \lambda f_{n+1}(\lambda) = x_{n+1} \end{cases} \quad \text{for every } n \geq 0.$$
Therefore, for every \( n \geq 0 \) and for every \( \lambda \in \rho_{S_u}(x) \), we have

\[
f_n(\lambda) = \frac{-B_n x_0}{\lambda^{n+1}} - \frac{B_n B_1^{-1} x_1}{\lambda^n} - \frac{B_n B_2^{-1} x_2}{\lambda^{n-1}} - \cdots - \frac{x_n}{\lambda} = F_n(\lambda).
\]

Since \( \|\tilde{x}(\lambda)\|^2 = \sum_{n \geq 0} \|f_n(\lambda)\|^2 < +\infty \) for every \( \lambda \in \rho_{S_u}(x) \), it then follows that

\[
\lim_{n \to +\infty} F_n(\lambda) = \lim_{n \to +\infty} f_n(\lambda) = 0 \quad \text{for every } \lambda \in \rho_{S_u}(x).
\]

We shall show that (3.4) is not satisfied for most of the points in the open disc \( V(x) := \{ \lambda \in \mathbb{C} : |\lambda| < \min \{R_A(x), r_3(S_u)\} \} \). It is clear that the sequence \( (G_n)_{n \geq 0} \) converges uniformly on compact subsets of \( V(x) \) to the non-zero power series \( G(\lambda) = \sum_{n \geq 0} \lambda^n B_n^{-1} x_n \). Now, let \( \lambda_0 \in V(x) \setminus \{0\} \) such that \( G(\lambda_0) \neq 0 \); there is \( \epsilon > 0 \) and an integer \( n_0 \) such that \( \epsilon < \|G_n(\lambda_0)\| \) for every \( n \geq n_0 \). On the other hand, \( |\lambda_0| < r_3(S_u) \), then there is a subsequence \( (n_k)_{k \geq 0} \) of integers greater than \( n_0 \) such that \( |\lambda_0|^{n_k} \|B_{n_k}^{-1}\| < 1 \). Thus, it follows from (3.3) that for every \( k \geq 0 \), we have

\[
\|F_{n_k}(\lambda_0)\| = \left| \frac{-1}{\lambda_0^{n_k+1}} \|B_{n_k} G_{n_k}(\lambda_0)\| \right| \\
\quad \geq \frac{1}{|\lambda_0^{n_k+1}||B_{n_k}^{-1}|} \|G_{n_k}(\lambda_0)\| \\
\quad \geq \epsilon/|\lambda_0|.
\]

And so, by (3.4), \( \lambda_0 \notin \rho_{S_u}(x) \). Since the set of zeros of \( G \) is at most countable, we have \( \{ \lambda \in \mathbb{C} : |\lambda| \leq \min \{R_A(x), r_3(S_u)\} \} \subset \sigma_{S_u}(x) \).

Now, assume that \( x = (x_0, x_1, \ldots) \) is a non-zero finitely supported element of \( \mathcal{H} \), and \( k_0 \) is the largest integer \( n \geq 0 \) for which \( x_n \neq 0 \). Conserve the same notations as above and note that, for every \( n \geq k_0 \), we have

\[
F_n(\lambda) = \frac{-1}{\lambda^{n+1}}B_n G(\lambda), \quad (\lambda \in \mathbb{C} \setminus \{0\}),
\]

where

\[
G(\lambda) := x_0 + \lambda B_1^{-1} x_1 + \lambda^2 B_2^{-1} x_2 + \cdots + \lambda^{k_0} B_{k_0}^{-1} x_{k_0}, \quad (\lambda \in \mathbb{C}).
\]

Let \( W(x) := \{ \lambda \in \mathbb{C} : |\lambda| < R_3(S_u) \} \), and let \( \lambda_0 \in W(x) \setminus \{0\} \) such that \( G(\lambda_0) \neq 0 \). As \( |\lambda_0| < R_3(S_u) \leq \limsup_{n \to +\infty} \|B_n G(\lambda_0)\|^{1/2} \), we note that the series \( \sum_{n \geq 0} \|F_n(\lambda_0)\|^2 \) diverges. Hence, \( \lambda_0 \in \sigma_{S_u}(x) \), and therefore,

\[
\{ \lambda \in \mathbb{C} : |\lambda| \leq R_3(S_u) \} \subset \sigma_{S_u}(x).
\]

\[ \square \]
For every $x \in \mathcal{H}$, we write 
$$\hat{H}(x) := \bigvee \{(B_n x)^{(n)} : n \geq 0\},$$
where "$\bigvee$" denotes the closed linear span. It is shown in proposition 4.3.5 of [20] that for every non-zero $x \in \mathcal{H}$, we have 
$$\sigma_{S_u}(x^{(n)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(n)})\}, \quad (n \geq 0).$$

We refine this result as follows; our proof is inspired by an argument of [3].

**Proposition 3.5.** Let $x$ be a non-zero element of $\mathcal{H}$, and let $y \in \hat{H}(x)$. The following statements hold.

(a) If $R_A(y) > r_{S_u}(x^{(0)})$, then $\sigma_{S_u}(y) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}$.

(b) If $R_A(y) \leq r_{S_u}(x^{(0)})$, then $\{\lambda \in \mathbb{C} : |\lambda| \leq R_A(y)\} \subset \sigma_{S_u}(y)$.

**Proof.** Let $x$ be a non-zero element of $\mathcal{H}$, and let us first show that 
$$\sigma_{S_u}(x^{(0)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}.$$ 

To do this it suffices to prove that $\{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\} \subset \sigma_{S_u}(x^{(0)})$. Indeed, as in the proof of theorem 3.4 we trivially have 
$$\tilde{x^{(0)}}(\lambda) = (-\frac{x}{\lambda}, -\frac{B_1 x}{\lambda^2}, -\frac{B_2 x}{\lambda^3}, \ldots), \quad (\lambda \in \rho_{S_u}(x^{(0)})).$$

In particular, we have $\|\tilde{x^{(0)}}(\lambda)\|_{\mathcal{H}}^2 = \sum_{k=0}^{+\infty} \frac{\|B_k x\|^2}{|\lambda|^{k+1}}\|, \quad (\lambda \in \rho_{S_u}(x^{(0)}))$. This implies that $\rho_{S_u}(x^{(0)}) \subset \{\lambda \in \mathbb{C} : r_{S_u}(x^{(0)}) \leq |\lambda|\}$. Or, equivalently, 
$$\{\lambda \in \mathbb{C} : |\lambda| < r_{S_u}(x^{(0)})\} \subset \sigma_{S_u}(x^{(0)}).$$

As $\sigma_{S_u}(x^{(0)})$ is a closed set, the desired identity holds.

(a) Assume that $y = \sum_{n=0}^{+\infty} a_n (B_n x)^{(n)}$ is a non-zero element of $\hat{H}(x)$ for which $R_A(y) > r_{S_u}(x^{(0)})$. In this case the function $f(\lambda) := \sum_{n=0}^{+\infty} a_n \lambda^n$ is analytic on the open disc $\{\lambda \in \mathbb{C} : |\lambda| < R_A(y)\}$ which is a neighborhood of $\sigma_{S_u}(x^{(0)})$. Let $r$ be a real number such that $r_{S_u}(x^{(0)}) < r < R_A(y)$, we have 
$$f(S_u, x^{(0)}) := \frac{-1}{2\pi i} \int_{|\lambda|=r} f(\lambda) \tilde{x^{(0)}}(\lambda) d\lambda$$
$$= \frac{-1}{2\pi i} \int_{|\lambda|=r} f(\lambda) \left(- \sum_{n=0}^{+\infty} \frac{S_u^n x^{(0)}}{\lambda^{n+1}}\right) d\lambda$$
$$= y.$$ 

And so, by theorem 2.12 of [19], we have 
$$\sigma_{S_u}(y) = \sigma_{S_u}(f(S_u, x^{(0)})) = \sigma_{S_u}(x^{(0)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}. $$
(b) The proof of the second statement is similar to the one of theorem 3.4. if, for every integer \( n \geq 0 \), we take

\[
F_n(\lambda) := -\left( \frac{a_n}{\lambda^{n+1}} + \frac{a_1}{\lambda^n} + \frac{a_2}{\lambda^{n-1}} + \ldots + \frac{a_n}{\lambda} \right) B_n x, \quad (\lambda \in \mathbb{C}\backslash\{0\}),
\]

and

\[
G_n(\lambda) := a_0 + a_1 \lambda + a_2 \lambda^2 + \ldots + a_n \lambda^n, \quad (\lambda \in \mathbb{C}).
\]

\[\square\]

4. **Dunford’s condition (C) and Bishop’s property (\(\beta\)) for \( S_u \)**

Before outlining the statement of the main results of this section, let us recall a few more notions and properties from the local spectral theory which will be needed in the sequel. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be **hyponormal** if \( \|T^* x\| \leq \|Tx\| \) for all \( x \in \mathcal{H} \). It is said be **subnormal** if it has a normal extension, this means that there is a normal operator \( N \) on a Hilbert space \( \mathcal{K} \), containing \( \mathcal{H} \), such that \( \mathcal{H} \) is a closed invariant subspace of \( N \) and the restriction \( N|_{\mathcal{H}} \) coincides with \( T \). Note that every subnormal operator is hyponormal, but the converse is false (see [5]). For an open subset \( U \) of \( \mathbb{C} \), let \( \mathcal{O}(U, \mathcal{H}) \) denote as usual the Fréchet space of all analytic \( \mathcal{H} \)-valued functions on \( U \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to possess Bishop’s property (\(\beta\)) if the continuous mapping

\[
T_U : \mathcal{O}(U, \mathcal{H}) \rightarrow \mathcal{O}(U, \mathcal{H})
\]

\[
f \mapsto (T - z)f
\]

is injective with closed range for each open subset \( U \) of \( \mathbb{C} \). It is known that hyponormal operators possess Bishop’s property (\(\beta\)) (see [15]) and it turns out that Dunford’s condition (\(C\)) follows from Bishop’s property (\(\beta\)). Let \( \lambda_0 \in \mathbb{C} \); recall that an operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to possess **Bishop’s property (\(\beta\)) at** \( \lambda_0 \) if there is an open neighbourhood \( V \) of \( \lambda_0 \) such that for every open subset \( U \) of \( V \), the mapping \( T_U \) is injective and has a closed range. Note that if \( T \) possesses Bishop’s property (\(\beta\)) at any point \( \lambda \in \mathbb{C} \) then \( T \) possesses Bishop’s classical property (\(\beta\)). Finally, for any operator \( T \in \mathcal{L}(\mathcal{H}) \), we shall denote

\[
\sigma_\beta(T) := \{ \lambda \in \mathbb{C} : T \text{ fails to possess Bishop’s property (\(\beta\)) at } \lambda \}.
\]

It is a closed subset of \( \sigma_{ap}(T) \) (see proposition 2.1 of [3]).

The following result gives necessary conditions for the operator weighted shift, \( S_u \), to enjoy Dunford’s condition (\(C\)).

**Theorem 4.1.** If \( S_u \) satisfies Dunford’s condition (\(C\)), then \( r(S_u) = R_u^+ \). Moreover, for every non-zero \( x \in \mathcal{H} \), we have

\[
\limsup_{n \to +\infty} \|B_n x\|^\frac{1}{n} = \lim_{n \to +\infty} \left[ \sup_{k \geq 0} \frac{\|B_{n+k} x\|}{\|B_k x\|} \right]^\frac{1}{n}.
\]
Proof. To prove $R_3^+(S_u) = r(S_u)$, it suffices to show that $r(S_u) \leq R_3^+(S_u)$. Since each $B_k$ is an invertible operator, we note that

$$R_3^+(S_u) = \sup_{x \in \mathcal{H}, \ x \neq 0} (r_{S_u}(x^{(k)})), \ \forall k \geq 0.$$ 

Now, assume that $S_u$ satisfies Dunford’s condition $(C)$, and let

$$F := \{ \lambda \in \mathbb{C} : |\lambda| \leq R_3^+(S_u) \}.$$ 

It follows from (3.2) that $\hat{\mathcal{H}}_{S_u}(F)$ contains a dense subspace of $\hat{\mathcal{H}}$. As the subspace $\hat{\mathcal{H}}_{S_u}(F)$ is closed, we have $\hat{\mathcal{H}}_{S_u}(F) = \mathcal{H}$; therefore, $\sigma_{S_u}(x) \subset F$ for every $x \in \hat{\mathcal{H}}$. And so, $\sigma(S_u) = \bigcup_{x \in H} \sigma_{S_u}(x) \subset F$ (see proposition 1.3.2 of [10]). Hence, $r(S_u) \leq R_3^+(S_u)$, as desired.

Let $x$ be a non-zero element of $\mathcal{H}$ and let us now establish the identity (4.5). Since $S_u$ satisfies Dunford’s condition $(C)$, we note that $S_u$ restricted to $\mathcal{H}(x)$ satisfies also Dunford’s condition $(C)$ (see proposition 1.2.21 of [10]). Now, note that $(v_n)_{n \geq 0}$ is an orthonormal basis of $\mathcal{H}(x)$, where

$$v_n := \frac{(B_n x)^{(n)}}{\|B_n x\|}, \ (n \geq 0).$$

We have

$$S_u v_n = \frac{\|B_{n+1} x\|}{\|B_n x\|} v_{n+1}, \ (n \geq 0).$$

This shows that $S_u|\mathcal{H}(x)$ is an injective scalar unilateral weighted shift with weight sequence $(\frac{\|B_{n+1} x\|}{\|B_n x\|})_{n \geq 0}$. Therefore, the identity, (4.5), follows from theorem 3.7 of [3].

Unlike the scalar weighted shift operators, generally we do not have $r_1(S_u) = r(S_u)$ if the unilateral operator weighted shift $S_u$ possesses Bishop’s property $(\beta)$ (see example 4.5). But, of course, if $r_1(S_u) = r(S_u)$, then either $S_u$ possesses Bishop’s property $(\beta)$, or $\sigma_\beta(S_u) = \{ \lambda \in \mathbb{C} : |\lambda| = r(S_u) \}$. In [20], H. Zguitti represented a unilateral operator weighted shift as operator multiplication by $z$ on a Hilbert space of formal power series whose coefficients are in $\mathcal{H}$. He therefore adapted T. L. Miller and V. G. Miller’s arguments given in [13] to show that if $S_u$ possesses Bishop’s property $(\beta)$, then $r_2(S_u) = R_1(S_u)$, where $R_1(S_u) = \lim_{n \to +\infty} \inf_{k \geq 0} \left| \inf_{k \geq 0} \|B_{n+k} B_k^{-1}\| \right|^\frac{1}{n}$. Here, we refine this result as follows and provide a direct proof.

**Theorem 4.2.** If $S_u$ possesses Bishop’s property $(\beta)$, then $r_2(S_u) = r_1(S_u)$, and $r(S_u) = R_3^+(S_u)$. Moreover, for every non-zero $x \in \mathcal{H}$, we have

$$(4.6) \quad \lim_{n \to +\infty} \left[ \inf_{k \geq 0} \frac{\|B_{n+k} x\|}{\|B_k x\|} \right]^{\frac{1}{n}} = \lim_{n \to +\infty} \left[ \sup_{k \geq 0} \frac{\|B_{n+k} x\|}{\|B_k x\|} \right]^{\frac{1}{n}}.$$
Proof. Suppose that \(S_u\) possesses Bishop’s property (\(\beta\)) and note that, since \(S_u\) satisfies Dunford’s condition (\(C\)), \(r(S_u) = R^+_3(S_u)\) (see theorem [1]). If \(r_2(S_u) = 0\) then, since \(r_1(S_u) \leq r_2(S_u)\), there is nothing to prove. Thus, we may assume that \(0 < r_2(S_u)\). Now, recall that it is shown in [12] that

\[ r_1(T) = \min\{|\lambda| : \lambda \in \sigma_{ap}(T)\} \]

for any operator \(T \in \mathcal{L}(H)\). And so, in order to show that \(r_2(S_u) = r_1(S_u)\), it suffices to prove that \(U \cap \sigma_{ap}(S_u) = \emptyset\), where \(U := \{\lambda \in \mathbb{C} : |\lambda| < r_2(S_u)\}\). Assume for the sake of contradiction that there is \(\lambda_0 \in U \cap \sigma_{ap}(S_u)\). Since \(\sigma_p(S_u) = \emptyset\), there is \(y = (y_0, y_1, y_2, \ldots) \in \text{cl}(\text{ran}(S_u - \lambda_0))\setminus\text{ran}(S_u - \lambda_0)\). For every \(x \in \mathcal{H}\), set \(k_x(\lambda) := \sum_{i \geq 0} \oplus \lambda B^{-1}_i - 1 y_i\), \((\lambda \in U)\), and note that

\[ (S_u - \lambda)^* k_x(\lambda) = 0, \quad \forall \lambda \in U. \]

In particular, we have

\[ \langle y, k_x(\lambda_0) \rangle_{\mathcal{H}} = 0, \quad \text{for all} \ x \in \mathcal{H}. \tag{4.7} \]

And so, for every \(x \in \mathcal{H}\), we have

\[
\langle \sum_{i \geq 0} \lambda_0^i B^{-1}_i y_i, x \rangle_{\mathcal{H}} = \sum_{i \geq 0} \langle y_i, \lambda_0^i B^{-1}_i - 1 x \rangle_{\mathcal{H}} = \langle y, k_x(\lambda_0) \rangle_{\mathcal{H}} = 0
\]

This implies that

\[ \sum_{i \geq 0} \lambda_0^i B^{-1}_i y_i = 0. \tag{4.8} \]

Now, for every integer \(n \geq 0\), we define on \(U\) the following analytic \(\mathcal{H}\)-valued functions by

\[
f(\lambda) := y - \left( \sum_{i \geq 0} \lambda^i B^{-1}_i y_i \right)^{(0)}, \quad \text{and} \quad f_n(\lambda) := y^n - \left( \sum_{i=0}^{n} \lambda^i B^{-1}_i y_i \right)^{(0)},
\]

where \(y^n := (y_0, \ldots, y_n, 0, 0, \ldots)\). Note that for every integer \(n \geq 0\), we have

\[
f_n(\lambda) = \sum_{i=0}^{n} (S_u^i - \lambda^i) (B^{-1}_i y_i)^{(0)}, \quad (\lambda \in U).
\]

This implies that each \(f_n\) is in \(\text{ran}(S_u^i)_{i\in U}\). But \(f \notin \text{ran}(S_u^i)_{i\in U}\) since, in view of [11], we have \(f(\lambda_0) = y \notin \text{ran}(S_u - \lambda_0)\). On the other hand, for
every compact subset $K$ of $\mathcal{U}$, we have

$$\sup_{\lambda \in K} \|f_n(\lambda) - f(\lambda)\|_{\hat{H}} \leq \|y - y^n\|_{\hat{H}} + \sup_{\lambda \in K} \|\left(\sum_{i>n} \lambda^i B_i y_i\right)^{(0)}\|_{\hat{H}}$$

$$= \|y - y^n\|_{\hat{H}} + \sup_{\lambda \in K} \|\sum_{i>n} \lambda^i B_i y_i\|_{\hat{H}}$$

$$\leq \|y - y^n\|_{\hat{H}} + \sup_{\lambda \in K} \left\{ \sum_{i>n} |\lambda|^i \|B_i^{-1}\| \|y_i\|_H \right\}$$

$$\leq \left(1 + \sup_{\lambda \in K} \left( \sum_{i \geq 0} |\lambda|^{2i} \|B_i^{-1}\|^2 \right)^{\frac{1}{2}} \right) \|y - y^n\|_{\hat{H}}.$$ 

Therefore, $f_n \to f$ in $\mathcal{O}(\mathcal{U}, \hat{H})$. As each $f_n \in \text{ran}(\mathcal{U}_{u}^3)$ and $f \notin \text{ran}(\mathcal{U}_{u}^3)$, we note that $\text{ran}(\mathcal{U}_{u}^3)$ is not closed. We have a contradiction to the fact that $S_u$ possesses Bishop’s property ($\beta$). And so, $\mathcal{U} \cap \sigma_{ap}(S_u) = \emptyset$, as desired.

Now, let $x$ be a non-zero element of $\mathcal{H}$. Since $S_u$ possesses Bishop’s property ($\beta$), the injective scalar unilateral weighted shift $S_u^{\{3\}}$ possesses also Bishop’s property ($\beta$). Thus, applying theorem 3.8 of [3], gives the identity (4.6).

**Remark 4.3.** Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator, and assume that $A_n = T$ for all $n \geq 0$. The corresponding unilateral operator weighted shift, $S_u$, satisfies the following identities.

$$r(S_u) = R^+_3(S_u) = r(T)$$

and

$$r_1(S_u) = r_2(S_u) = R^-_2(S_u) = r_1(T) = \frac{1}{r(T^{-1})}.$$ 

Indeed, we clearly have $r(S_u) = r(T)$ and $r_1(S_u) = r_2(S_u) = r_1(T) = \frac{1}{r(T^{-1})}$.

Since, $R^+_3(S_u) = \sup \left\{ r_T(x) : x \in \mathcal{H}, \ x \neq 0 \right\}$, it follows from proposition 3.3.14 of [10] that $R^+_3(S_u) = r(T)$; therefore, the first identity holds. On the other hand, we have

$$R^-_2(S_u) = \inf \left\{ \frac{1}{r_{T^{-1}}(x)} : x \in \mathcal{H}, \ x \neq 0 \right\}$$

$$= \frac{1}{\sup \left\{ r_{T^{-1}}(x) : x \in \mathcal{H}, \ x \neq 0 \right\}}.$$ 

Again, by proposition 3.3.14 of [10], we have $R^-_2(S_u) = \frac{1}{r(T^{-1})}$; and the second identity follows.

Assume that $T \in \mathcal{L}(\mathcal{H})$ is an invertible operator and that $A_n = T$ for all $n \geq 0$. So, one may think that the corresponding unilateral operator weighted shift, $S_u$, satisfies Dunford’s condition ($C$). It turns out that this is not true in general as the next example shows.

**Example 4.4.** Let $(\epsilon_n)_{n \in \mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, and let $(\omega_n)_{n \in \mathbb{Z}}$ be a positive two-sided sequence for which

(a) $0 < \inf_{n \in \mathbb{Z}} \omega_n \leq \sup_{n \in \mathbb{Z}} \omega_n < +\infty.$
(b) \( \limsup_{n \to +\infty} \left[ \omega_0 \omega_1 \cdots \omega_{n-1} \right]^{\frac{1}{n}} < \lim_{n \to +\infty} \left[ \sup_{k \geq 0} (\omega_k \omega_{k+1} \cdots \omega_{n+k-1}) \right]^{\frac{1}{n}}. \)

Let \( T \) be the scalar invertible bilateral weighted shift on \( \mathcal{H} \), defined by

\[ Te_n = \omega_n e_{n+1}, \quad (n \in \mathbb{Z}). \]

If \( A_n = T \) for all \( n \geq 0 \) then, in view of (b), neither the identity \( (\ref{eq:4.5}) \) nor the identity \( (\ref{eq:4.6}) \) is satisfied for \( e_0 \). Hence, \( S_u \) is without Dunford’s condition (C).

For the construction of a specific example of a positive two-sided sequence satisfying the above conditions, we refer the reader to \( \cite{16} \).

It is shown in theorem 2.5 of \( \cite{18} \) that a nonnormal hyponormal scalar (unilateral or bilateral) weighted shift has fat local spectra (see also theorem 3.7 of \( \cite{2} \)). The next example shows that this result is not valid for hyponormal operator weighted shifts.

**Example 4.5.** Assume that \( (\alpha_n)_{n \geq 0} \) is an orthonormal basis of \( \mathcal{H} \), and let \( (\alpha_n)_{n \geq 0} \) be an increasing positive sequence such that \( \lim_{n \to +\infty} \alpha_n = 1 \).

The diagonal operator, \( T \), with the diagonal sequence \( (\alpha_n)_{n \geq 0} \) (i.e., \( Te_n = \alpha_n e_n, \forall n \geq 0 \)) is invertible and satisfies \( r_1(T) = \alpha_0 < r(T) = 1 \). If \( A_n = T \) for all \( n \geq 0 \), then the unilateral operator weighted shift \( S_u \) is subnormal. Indeed, let \( \mathcal{H}_n = \mathcal{H} \) for all \( n \in \mathbb{Z} \) and let

\[ \tilde{\mathcal{H}} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \]

be the Hilbert space of the two-sided sequences \( (x_n)_{n \in \mathbb{Z}} \) such that

\[ \|(x_n)_{n \in \mathbb{Z}}\|_{\tilde{\mathcal{H}}} := \left( \sum_{n \in \mathbb{Z}} \|x_n\|^2_{\mathcal{H}} \right)^{\frac{1}{2}} < +\infty. \]

Let \( S_0 \) be the bilateral operator weighted shift defined on \( \tilde{\mathcal{H}} \) by

\[ S_0(..., x_{-2}, x_{-1}, [x_0], x_1, x_2, ...) = (... , T x_{-2}, [T x_{-1}], T x_0, T x_1, ...), \]

where for an element \( x = (... , x_{-2}, x_{-1}, [x_0], x_1, x_2, ...) \in \tilde{\mathcal{H}} \), \( [x_0] \) denotes the central (0th) term of \( x \). Note that, since \( T \) is an hermitian operator, \( S_0 \) is a normal extension of \( S_u \). This shows that \( S_u \) is a subnormal operator. Now, we note that for every \( k \geq 0 \), we have

\[ r_{S_u}(e_k^{(0)}) = \limsup_{n \to +\infty} \|T^n e_k\|^{\frac{1}{n}} = \alpha_k < r(S_u) = 1. \]

This shows, on the one hand, that \( S_u \) is without fat local spectra and, on the other hand, that

\[ r_1(S_u) = r_2(S_u) = R_2^+(S_u) = r_3(S_u) = R_3^-(S_u) = \alpha_0 < R_3^+(S_u) = r(S_u) = 1. \]

Therefore, in view of the fact that \( \sigma(S_u) = \sigma_{ap}(S_u) \cup \overline{\sigma_p(S_u)} \), corollary \( \ref{cor:2.2} \) and lemma \( \ref{lem:5.1} \) we have

\[ \sigma_{ap}(S_u) = \{ \lambda \in \mathbb{C} : \alpha_0 \leq |\lambda| \leq 1 \}. \]
Remark 4.6. Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator. If $A_n = T$ for all $n \geq 0$, then $S_n$ is hyponormal if and only if $T$ is also hyponormal. Therefore, to construct a kind of example 4.5, it suffices to take $T$ a hoponormal operator for which there is a non-zero element $x \in \mathcal{H}$ with $r_T(x) < r(T)$.

Finally, we would like to point out that
(a) proposition 3.9 of [3] remain valid for the general setting of operator weighted shift. This is not the case for proposition 3.11 of [3] as it is shown in example 4.4.
(b) after the present note was completed, we began to study the local spectra of bilateral operator weighted shifts; this case is quite difficult. However, we provided some local spectral inclusions and obtained a necessary and sufficient condition for a bilateral operator weighted shift to enjoy the single–valued extension property. Furthermore, we gave necessary and sufficient conditions for such operator to satisfy Dunford’s condition (C) or Bishop’s property ($\beta$). These results are still on a preliminary level, and will appear somewhere else once we get some interesting improvements.

References
[1] A. Ben-Artzi and I. Gohberg, Dichotomy, discrete bohl exponents, and spectrum of block weighted shifts, Integral equations and Operator Theory, 14 (1991) 613-677.
[2] A. Bourhim, C. E. Chidume and E. H. Zerrouali, Bounded point evaluations for cyclic operators and local spectra, Proc. Amer. Math. Soc. 130 (2002), 543-548.
[3] A. Bourhim, On the local spectral properties of weighted shift operators, (submitted).
[4] I. Colojoara and C. Foias, Theory of generalized spectral operators, Gordon and Breach, New York 1968.
[5] J. B. Conway, The Theory of Subnormal Operators, volume 36 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, R.I, 1991.
[6] D. Herrero, Spectral pictures of hyponormal bilateral operator weighted shifts, Proc. Amer. Math. Soc. 109 (1990) 753-763.
[7] C. L. Jiang, J. X. Li, The irreducible decomposition of Cowen-Douglas operators and operator weighted shifts, Acta Sci. Math. (Szeged) 66 (2000), no. 3-4, 679-695.
[8] Y. G. Ji, J. Li, and Y. Yang, A characterization of bilateral operator weighted shifts being Cowen-Douglas operators, Proc. Amer. Math. Soc. 129 (2001), no. 11, 3205-3210.
[9] A. Lambert, Unitary equivalence and reducibility of invertibly weighted shifts, Bull. Austral. Math. Soc., 5 (1971) 157-173.
[10] K. B. Laursen and M. M. Neumann, An introduction to local spectral theory, London Mathematical Society Monograph New Series 20 (2000).
[11] J. X. Li, Y. Q. Ji and S. L. Sun, The essential spectrum and Banach reducibility of operator weighted shifts, Acta Math. Sinica, English Series, Vol 17, 3 (2001) 413-424.
[12] E. Makai and J. Zemánek, The surjectivity radius, packing numbers and boundedness below of linear operators, Integral equations and Operator Theory 6 (1983) 372-384.
[13] T. L. Miller and V. G. Miller, An operator satisfying Dunford’s condition (C) without Bishop’s property ($\beta$), Glasgow Math. J. 40(1998) 427-430.
[14] T. L. Miller, V. G. Miller and M. M. Neumann, Local spectral properties of weighted shifts, J. Operator Theory (to appear).
[15] M. Putinar, Hyponormal operators are subscalar, J. Operator Theory 12 (1984), 385-395.
[16] W. C. Ridge, Approximate point spectrum of a weighted shift, Trans. Amer. Math. Soc. 147(1970) 349-356.
[17] A. L. Shields, *Weighted shift operators and analytic function theory*, in Topics in Operator Theory, Mathematical Surveys, N0 13 (ed. C. Pearcy), pp. 49-128. American Mathematical Society, Providence, Rhode Island 1974.

[18] L. R. Williams, *The local spectra of pure quasinormal operators*, J. Math. Anal. Appl. 187 (1994), 842-850.

[19] L. R. Williams, *Local functional calculus and related results on the single-valued extension property*, Integral Equations and Operator Theory, (to appear).

[20] H. Zguitti, *Théorie spectrale locale appliquée aux différentes classes d’opérateurs*, Thèse de troisième cycle, Univ. Cadi Ayyad, Faculté des Sciences Semlalia, Marrakech-Maroc, N d’ordre 573, (1998).

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

E-mail address: bourhim@ictp.trieste.it