Fractional diffusion limit for a kinetic Fokker-Planck equation with diffusive boundary conditions in the half-line

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Abstract

We consider a particle with position \((X_t)_{t \geq 0}\) living in \(\mathbb{R}_+\), whose velocity \((V_t)_{t \geq 0}\) is a positive recurrent diffusion with heavy-tailed invariant distribution when the particle lives in \((0, \infty)\). When it hits the boundary \(x = 0\), the particle restarts with a random strictly positive velocity. We show that the properly rescaled position process converges weakly to a stable process reflected on its infimum. From a P.D.E. point of view, the time-marginals of \((X_t, V_t)_{t \geq 0}\) solve a kinetic Fokker-Planck equation on \((0, \infty) \times \mathbb{R}_+ \times \mathbb{R}\) with diffusive boundary conditions. Properly rescaled, the space-marginal converges to the solution of some fractional heat equation on \((0, \infty) \times \mathbb{R}_+\).

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1 Introduction

In the last two decades, many mathematical works showed how to derive anomalous diffusion limit results, also called fractional diffusion limits, from different kinetic equations with heavy-tailed equilibria. In short, these types of results state that the properly rescaled density of the position of a particle subject to some kinetic equation, is asymptotically non-gaussian. A case of particular interest is when the scaling limit of the position of the particle is a stable process, of which the time-marginals satisfies the fractional heat equation.

Mellet, Mischler and Mouhot [39] showed a fractional diffusion limit result for a linearized Boltzmann equation with heavy-tailed invariant distribution using Fourier transform arguments, a result which was improved by Mellet [38] using a moments method. The proofs rely entirely on analytic tools. A similar result was shown in Jara, Komorowski and Olla [30], although it is derived from an \(\alpha\)-stable central limit theorem for additive functional of Markov chains, i.e. from a probabilistic result.

Anomalous diffusion limits also occur for kinetic equations with degenerate collision frequency, see Ben Abdallah, Mellet and Puel [2], and in transport of particles in plasma, see Cesbron, Mellet and Trivisa [20]. In [14], Bouin and Mouhot propose a unified approach to derive fractional diffusion limits from several linear collisional kinetic equations.

In [34], Lebeau and Puel showed a fractional diffusion limit result for a one-dimensional Fokker-Planck equation with heavy-tailed equilibria. From a probabilistic point of view, the corresponding toy model is a one-dimensional particle whose velocity is subject to a restoring
force $F$ and random shocks modeled by a Brownian motion $(B_t)_{t \geq 0}$, which leads to the following stochastic differential equation:

$$V_t = v_0 + \int_0^t F(V_s)ds + B_t, \quad X_t = x_0 + \int_0^t V_s ds,$$

where $(x_0, v_0) \in \mathbb{R}^2$, $(V_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ are the velocity and position processes of the particle. The force at stake is $F(v) = -\frac{\beta v}{1 + v^2}$, where $\beta \in (1, 5) \setminus \{2, 3, 4\}$ leading to an invariant measure which behaves as $(1 + |v|)^{-\beta}$ as $|v| \to \infty$. Their result states that in this case, the properly rescaled position process resembles a stable process in large time. When $\beta > 5$, Nasreddine and Puel [40] established that $(e^{1/2} X_{t/\epsilon})_{t \geq 0}$ resembles a Brownian motion as $\epsilon \to 0$, corresponding to a classical diffusion limit type theorem. Then Cattiaux, Nasreddine and Puel [16] later showed that in the critical case $\beta = 5$, the same result holds up to a logarithmic correction term.

This phenomenon was actually observed by physicists who discovered experimentally that atoms cooled by a laser diffuse anomalously, see for instance Castin, Dalibard and Cohen-Tannoudji [15], Sagi, Brook, Almog and Davidson [41] and Marksteiner, Ellinger and Zoller [37]. A theoretical study (see Barkai, Aghion and Kessler [1]) modeling the motion of atoms precisely by (1) proved with quite a high level of rigor the observed phenomenons.

Then, using probabilistic techniques, Fournier and Tardif [25] treated all cases of (1) (i.e. $\beta > 0$) for a slightly larger class of symmetric forces. When $\beta \geq 5$, the limiting distribution is Gaussian whereas when $\beta \in (1, 5)$, they show that the following convergence in finite dimensional distributions holds, for any initial condition $v_0 \in \mathbb{R}$:

$$\left( \epsilon^{1/\alpha} X_{t/\epsilon} \right)_{t \geq 0} \overset{f.d.}{\longrightarrow} \left( \sigma_{\alpha} Z_t^{\alpha} \right)_{t \geq 0} \quad \text{as } \epsilon \to 0,$$

where $(Z_t^\alpha)_{t \geq 0}$ is a symmetric $\alpha$-stable process with $\alpha = (\beta + 1)/3$, and $\sigma_{\alpha}$ is some positive diffusive constant. Naturally, they recover the result of [16, 34, 40] and even go beyond, treating the case $\beta \in (0, 1)$ which was new. In this regime, the velocity is null recurrent, and the rescaled process was shown to converge to a symmetric Bessel process of dimension $\delta \in (0, 2)$. Then the position process naturally converges to an integrated symmetric Bessel process, which is no longer Markov. Their proof heavily relies on Feller’s representation of diffusion processes through their scale functions and speed measures, enabling them to treat all cases at once, even the critical cases $\beta = 1, 2, 5$. This method was generalized in [9] and (2) was shown to be a special case of an $\alpha$-stable central limit theorem for additive functional of one-dimensional Markov processes. In a companion paper, Fournier and Tardif [24] also showed that these results hold in any dimension and the proofs are much more involved than in dimension 1.

As it is very natural in kinetic theory to consider gas particles interacting with a surface in thermodynamical equilibrium, corresponding to the case of diffusive boundary conditions, we propose in this article to study a version of the process $(X_t)_{t \geq 0}$ living in $\mathbb{R}_+$ and reflected diffusively through its velocity when the particles hits 0. In other words, we consider the case of a particle governed by (1), which interacts with a wall located at $x = 0$. When the particle hits the boundary, it reemerges from the wall with a random velocity distributed according to some probability measure.

The aim of this paper is to study the scaling limit of such a particle. More precisely, we will show that in the Lévy regime ($\beta \in (1, 5)$), the rescaled position process converges in law to a stable process reflected on its infimum. This result should be related to the recent articles of Cesbron, Mellet and Puel [18, 19] and Cesbron [17], which extend the results obtained in [2, 38, 39]. They deal with the kinetic scattering equation describing particles living in the half-space or in bounded domains, with specular and / or diffusive boundary conditions. They obtain as a scaling limit a fractional heat equation with some boundary conditions depending on the original boundary conditions. This differs from the normal diffusive case where one would always obtain the classical heat equation with Neumann boundary conditions. Here, the kinetic equation at
stake is different, but we obtain a similar limiting equation as in [17–19] with however some different behaviour at the boundary. We refer to the PDE section below for more details. In [17, 18], the limiting PDE is clearly identified, but they have a uniqueness issue due to the weakness of their solutions. This problem was solved in dimension 1 in [19]. We emphasize that we have no such issue and that the limiting process is quite explicit. We also point out that their result only holds for $\alpha \in (1, 2)$.

We should also mention the works of Komorowski, Olla, Ryzhik [33] and Bogdan, Komorowski, Marino [13], which are, to our knowledge, the only probabilistic results of fractional diffusion limits with boundary interactions. They both study a scattering equation (linear Boltzmann), as in [17–19], but with a mixed reflective / transmissive / absorbing boundary conditions. We refer to the comments section below, where these papers are further discussed.

Let us now introduce more formally the model studied. We will denote by $\mathbb{N} = \{1, 2, 3, \cdots \}$ the set of positive integers. Let $v_0 > 0$, $(B_{t})_{t \geq 0}$ be a Brownian motion and $(M_{n})_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables whose law $\mu$ is supported in $(0, \infty)$. Everything is assumed to be independent. The object at stake in this article is the strong Markov process $(X_{t}, V_{t})_{t \geq 0}$ valued in $E = ((0, \infty) \times \mathbb{R}) \cup \{0\} \times (0, \infty)$ and defined by the following stochastic differential equation

$$
\begin{align*}
X_{t} &= x_0 + \int_{0}^{t} V_{s} ds, \\
V_{t} &= v_0 + \int_{0}^{t} F(V_{s}) ds + B_{t} + \sum_{n \in \mathbb{N}} (M_{n} - V_{\tau_{n}}) 1_{\{\tau_{n} \leq t\}}, \\
\tau_{1} &= \inf\{t > 0, X_{t} = 0\} \text{ and } \tau_{n+1} = \inf\{t > \tau_{n}, X_{t} = 0\},
\end{align*}
$$

where $F$ fulfills Assumption 1 below and $(x_0, v_0) \in E$. This equation is well-posed and we refer to Subsection 2.1 for more details. It describes the motion of a particle evolving in $[0, \infty)$ and being reflected when it hits 0. More precisely, the velocity $(V_{t})_{t \geq 0}$ and position $(X_{t})_{t \geq 0}$ processes are governed by (1) when $X_{t} > 0$, and the particle is reflected through the velocity when it hits the boundary, i.e. when $X_{t} = 0$. Note that $t \mapsto V_{t}$ is a.s. càdlàg and that the jumps only occur when the particle hits the boundary, i.e. when $t = \tau_{n}$ for some $n \in \mathbb{N}$. In this case, the value of the velocity after the jump is

$$
V_{\tau_{n}} = V_{\tau_{n}^-} + \Delta V_{\tau_{n}} = M_{n}
$$

As for every $n \in \mathbb{N}$, $M_{n} > 0$ a.s., the particle reaches $(0, \infty)$ instantaneously after hitting the boundary and thus spend a strictly positive amount of time in $(0, \infty)$. Hence the zeros of $(X_{t})_{t \geq 0}$ are countable and the successive hitting times $(\tau_{n})_{n \in \mathbb{N}}$ are well defined. Finally, we point out that since a solution $(X_{t}, V_{t})_{t \geq 0}$ of (1) reaches $\{0\} \times \mathbb{R}$ with a necessarily non-positive velocity, the process $(X_{t-}, V_{t-})_{t \geq 0}$ is valued in $((0, \infty) \times \mathbb{R}) \cup \{0\} \times (-\infty, 0]$.

Let us mention some related works, dealing with Langevin-type models reflected in the half-space with diffusive-reflective boundary conditions. In [27, 28], Jacob studies the classical Langevin process, namely the process $(B_{t}, \int_{0}^{t} B_{s} ds)_{t \geq 0}$, reflected at a partially elastic boundary. In [26], Jabir and Profeta study a stable Langevin process with diffusive-reflective boundary conditions. Roughly, their work treats the well-posedness of the equations and wether or not, the obtained process is conservative, observing some phase transitions. The question of the existence of a scaling limit does not make sense since the processes are intrinsically self-similar (at least in the purely reflective case).

In the rest of the article, the process $(X_{t}, V_{t})_{t \geq 0}$ will always be defined by (1), and will be referred to as the "free process". On the other hand, $(X_{t}, V_{t})_{t \geq 0}$ will always refer to the process defined by (3) and will be called the "reflected process" or "reflected process with diffusive boundary conditions". Let us now be a little more precise on the assumptions we will suppose. Regarding the force field $F$, we assume the following.

**Assumption 1.** The Lipschitz and bounded force $F : \mathbb{R} \rightarrow \mathbb{R}$ is such that $F = \frac{\beta}{2} \Theta'$, where $\beta \in (1, 5)$ and $\Theta : \mathbb{R} \rightarrow (0, \infty)$ is a $C^1$ even function satisfying $\lim_{v \rightarrow \pm \infty} |v| \Theta(v) = 1$. 


This assumption is a little bit stronger than the one in [25], and thus (2) holds for the free process. The typical force we have in mind is \( F(v) = -\frac{\beta}{2} \frac{v}{1+v^2} \) studied in [16,34,40], corresponding to the function \( \Theta(v) = (1 + v^2)^{-1/2} \). The value of the diffusive constant \( \sigma_\alpha \) from (2) is given by
\[
\sigma_\alpha = 3^{1-2\alpha} \frac{2^{1-\alpha} \pi^{\eta}}{\Gamma^2(\alpha) \sin(\pi\alpha/2)}, \quad \text{where} \quad c_\beta = \left( \int_\mathbb{R} \Theta^\beta(v) dv \right)^{-1} \tag{4}
\]
Regarding the probability measure \( \mu \) governing the reflection, we will have two different assumptions.

**Assumption 2.**

(i) There exists \( \eta > 0 \) such that \( \mu \) has a moment of order \( (\beta + 1)/2 + \eta \).

(ii) There exists \( \eta > 0 \) such that \( \mu \) has a moment of order \( (\beta + 1)(\beta + 2)/6 + \eta \).

Since \( \beta > 1 \), Assumption 2-(ii) is obviously stronger than Assumption 2-(i). These assumptions are satisfied by many probability measures of interest such as the Gaussian density, every sub-exponential distributions, and many heavy-tailed distributions.

During the whole paper, Assumption 1 is always in force and Assumption 2 will be mentioned when necessary.

For a family of processes \( \{(Y^\epsilon_t)_{t \geq 0}\}_{\epsilon > 0} \), we say that \( (Y^\epsilon_t)_{t \geq 0} \xrightarrow{f.d.} (Y^0_t)_{t \geq 0} \) as \( \epsilon \to 0 \) if for all \( n \geq 1 \), for all \( t_1, \ldots, t_n \geq 0 \), the vector \( (Y^\epsilon_{t_i})_{1 \leq i \leq n} \) converges in law to \( (Y^0_{t_i})_{1 \leq i \leq n} \) in \( \mathbb{R}^n \). Most of the convergence results obtained are actually stronger than convergence in finite dimensional distributions, i.e. we obtain convergence in law of processes in the space of càdlàg functions. As the usual Skorokhod topology is not suited for convergence of continuous processes to a discontinuous process, we will instead use a weaker topology, namely the \( M_1 \)-topology, and we refer to Section 2.2 for more details. The following theorem is the main result of this paper.

**Theorem 3.** Grant Assumptions 1 and 2-(i), and let \( (X_t, V_t)_{t \geq 0} \) be a solution of (3) starting at \( (0, v_0) \) with \( v_0 > 0 \). Let \( (Z^\alpha_t)_{t \geq 0} \) be a symmetric stable process with \( \alpha = (\beta + 1)/3 \) and such that
\[
\mathbb{E}[e^{\xi Z^\alpha_t}] = \exp(-t\sigma_\alpha|\xi|^\alpha) \quad \text{where} \quad \sigma_\alpha \text{ is defined in (4)}.
\]
Let \( (R^\alpha_t)_{t \geq 0} \) be the stable process reflected on its infimum, i.e. \( R^\alpha_t = Z^\alpha_t - \inf_{s \in [0,t]} Z^\alpha_s \). Then we have
\[
(1/\alpha X_{t/\epsilon})_{t \geq 0} \xrightarrow{f.d.} (R^\alpha_t)_{t \geq 0} \quad \text{as} \quad \epsilon \to 0.
\]
Moreover, if we grant Assumption 2-(ii), this convergence in law holds in the space of càdlàg functions endowed with the \( M_1 \)-topology.

It is likely that this theorem could be extended for any initial condition \( (x_0, v_0) \in E \) with some small adjustments but for the sake of simplicity we will only consider the case \( x_0 = 0 \) and \( v_0 > 0 \).

Let us try to explain informally why the limiting process should be the stable process reflected on its infimum. Remember that when \( X_t > 0 \), the process is governed by (1), and therefore, we should expect the limit process to behave like a stable process in \((0, \infty)\). Only the behavior at the boundary is to be identified. When \( X_t \) reaches the boundary with a very high speed, corresponding to a jump in the limit, it is suddenly reflected and slowed down, as the new velocity is distributed according to \( \mu \); and the process somehow restarts afresh. As a consequence, we should expect the following behavior for the limiting process: when it tries to jump across the boundary, the jump is "cut" and the process restarts from 0. This is exactly the behavior of \( (R^\alpha_t)_{t \geq 0} \): when \( R^\alpha_t > 0 \), it behaves as \( Z^\alpha_t \) and when \( Z^\alpha_t \) jumps below its past infimum, the latter one is immediately "updated", corresponding to \( R^\alpha_t \) trying to cross the boundary and being set to 0.

We emphasize that \( (R^\alpha_t)_{t \geq 0} \) is a Markov process, see [5, Chapter 6, Proposition 1]. Let us now point out an interesting phenomena: the limiting process really depends on the way we
reflect the initial process. For instance, let us consider (3) with a specular boundary condition, i.e. when the process hits the boundary, it is reflected with the same incoming velocity, which is flipped. Then it is easy to see that, since the force field is symmetric, \((X_t, \sgn(V_t) V_t)_{t \geq 0}\) is a solution of the corresponding reflected equation, where \((X_t, V_t)_{t \geq 0}\) is a solution of (1). Then by (2), the limiting process is \((|Z^a_t|)_{t \geq 0}\) whose behavior at the boundary is different from \((R^a_t)_{t \geq 0}\): when it tries to cross the boundary, the process is moved back in \(\mathbb{R}_+\) by a mirror reflection. Note that \((|Z^a_t|)_{t \geq 0}\) is a Markov process only in the symmetric case, so it is not clear what happens in the disymmetric case.

Unlike the Brownian motion, there is no unique way to reflect a stable process. Indeed, by a famous result of Paul Lévy, it is well known that \((|B_t|)_{t \geq 0} \overset{d}{=} (B_t - \inf_{s \in [0,t]} B_s)_{t \geq 0}\). This is to be related with the fact that unlike the Laplacian, the fractional Laplacian is a non-local operator. We believe that, with a little bit of work, we could extend Theorem 3 to the diffusive regime, i.e. when \(\beta \geq 5\), and the rescaled process would converge to a reflected Brownian motion.

On our way to establish Theorem 3, we will encounter a singular equation which describes the motion of a particle reflected at a completely inelastic boundary, which is very close to the equation studied by Bertoin in [8]. We will study a solution of the following stochastic differential equation.

\[
\begin{cases}
X_t = \int_0^t V_s ds, \\
V_t = v_0 + \int_0^t F(V_s) ds + B_t - \sum_{0 < s \leq t} V_s - 1_{\{X_s = 0\}},
\end{cases}
\tag{5}
\]

where \(v_0 > 0\). Let \(a > 0\) and consider a solution \((X^a_t, V^a_t)_{t \geq 0}\) of (3) starting at \((0, v_0)\) for the particular choice \(\mu = \delta_a\). Then, informally, \((X^a_t, V^a_t)_{t \geq 0}\) should tend to \((X_t, V_t)_{t \geq 0}\) as we let \(a \to 0\). Since for every \(a > 0\), the rescaled process \((e^{1/\alpha} X^a_{t/\epsilon})_{t \geq 0}\) converges in law to \((R^a_t)_{t \geq 0}\), it should be expected that \((X_t)_{t \geq 0}\) has the same scaling limit. We will see that it is indeed the case, see Theorem 10 below. While it is clear that we can construct a solution to (3), it is non-trivial that (5) possesses a solution and let us quickly explain why.

Consider a solution \((X_t, V_t)_{t \geq 0}\) of (1) starting at \((0, 0)\), then one can easily see that 0 is an accumulation point of the instants at which \(X_t = 0\). Now if we consider a solution \((X_t, V_t)_{t \geq 0}\) of (5), its energy is fully absorbed when the particle hits the boundary i.e. when \(X_t = 0\), the value of the velocity is \(V_t = V_{t^-} - V_{t^-} = 0\). Hence it is not clear at all whether the particle will ever reemerge in \((0, \infty)\) or will remain stuck at 0. As we will see, it turns out that (5) admits a conservative solution, which never gets stuck at 0.

In [7,8], Bertoin studies equation (5) in the special case \(F = 0\) and he establishes the existence and uniqueness (in some sense). We will not be interested in studying the uniqueness of (5) in our case but we will use the construction developed in [7] and [8].

Comments and comparison with the litterature

At this point, we should discuss the works of Komorowski, Olla, Ryzhik [33] and Bogdan, Komorowski, Marino [13]. In [33], they study a scattering equation with a reflective / transmissive / absorbing boundary conditions. When the particle hits the boundary \(x = 0\), it is either reflected (by flipping the incoming velocity), either unchanged (transmitted), or killed. Their model is such that the rescaled particle always satisfies \(X^\alpha_0 = x\) for some \(x > 0\), and they find the following limiting process. Consider a symmetric \(\alpha\)-stable process \((Z^\alpha_t)_{t \geq 0}\) started at \(x\) with \(\alpha \in (1, 2)\), as well as its successive crossing times \((\sigma_n)_{n \in \mathbb{N}}\) at the level 0. Consider also i.i.d. random variables \((\xi_n)_{n \in \mathbb{N}}\) such that \(\mathbb{P}(\xi_1 = 1) = p_+, \mathbb{P}(\xi_1 = -1) = p_-\) and \(\mathbb{P}(\xi_1 = 0) = p_0\), with \(p_+, p_-, p_0 > 0\) such that \(p_+ + p_- + p_0 = 1\). The reflected stable process \((R^\alpha_t)_{t \geq 0}\) is then defined as follows: \(R^\alpha_t = Z^\alpha_t\) on \([0, \sigma_1)\), and for any \(n \in \mathbb{N}\) and any \(t \in [\sigma_n, \sigma_{n+1})\),

\[
R^\alpha_t = \left( \prod_{k=1}^{n} \xi_k \right) Z^\alpha_t.
\]

5
In other words, each time \( (Z^n_t)_{t \geq 0} \) crosses the boundary, the trajectory of \( (R^n_t)_{t \geq 0} \) is transmitted with probability \( p_+ \), reflected with probability \( p_- \) or absorbed with probability \( p_0 \). As it is well-known, a symmetric stable process with index \( \alpha > 1 \) eventually touches 0, but crosses the boundary infinitely many times before doing so, see for instance [5, Chapter VIII, Proposition 8]. Therefore \( \sigma_\infty = \lim_{n \to \infty} \sigma_n \) is a.s. finite but since \( p_0 > 0 \), the reflected process is a.s. absorbed before \( \sigma_\infty \) and \( (R^n_t)_{t \geq 0} \) is naturally set to 0 after \( \sigma_\infty \).

In [13], they study the very same model, but the probability \( p_0^\prime \) for the kinetic process to be absorbed when it hits the boundary vanishes as \( \epsilon \to 0 \) and behaves as \( 1/|\log \epsilon| \). The limiting process obtained is the same as above with \( p_0 = 0 \) for \( t < \sigma_\infty \), and the process is absorbed at 0 at \( t = \sigma_\infty \).

Observe that in both cases, the limiting process is killed before (or precisely when) hitting the boundary. In the present paper, we thus have a substantial additional difficulty which is to characterize the limiting process when starting from 0. Indeed, a symmetric stable process started from 0 touches 0 infinitely many times immediately after (when \( \alpha > 1 \)), so that the behavior of the limiting process started from 0 is not trivially defined. We cannot avoid this difficulty as the kinetic process is restarted with some small velocity.

We emphasize that these kinds of results are very recent and were mostly treated from an analytic point of view, see [17–19]. With the papers [13, 33], our work seems to be the only probabilistic study of fractional diffusion limit with boundary conditions. Our proof borrows different tools from stochastic analysis such as excursion theory, Wiener-Hopf factorization and a bit of enlargement of filtrations.

Regarding Assumption 2-(i), we believe that it is near optimality. As we will see, it appears naturally in the proofs at several places. Moreover, we believe the limiting process should differ at criticality, i.e. when \( v \mapsto \mu((v, \infty)) \) is regularly varying with index \( (\beta + 1)/2 \) as \( v \to \infty \). Roughly, the restarting velocities are no longer negligible and, the limiting process directly enters the domain after hitting \( 0 \) by a jump with "law" \( x^{-\alpha}dx \). Observe that this is the reason why they [18, 19] have to assume \( \alpha > 1 \). We think that, at criticality, we might obtain the same limiting distribution as in [18, 19].

Assumption 2-(ii) is technical and we believe the convergence in the \( M_1 \)-topology should also hold under Assumption 2-(i).

The present results might be extended, through the same line of proof, to integrated powers of the velocity, i.e. to \( X_t = \int_0^t \operatorname{sgn}(V_s)|V_s|\gamma ds \) and some \( \gamma > 0 \). At least, we already know from [9, Theorem 5] that an \( \alpha \)-stable central limit theorems holds for the free process.

Finally, it would be interesting to study what happens in higher dimensions (using the results of [24]), as well as a more complete model, with both diffusive and specular boundary conditions: when the particle hits the boundary, it is reflected diffusively with probability \( p \in (0, 1) \) and specularly with probability \( 1 - p \).

**Informal PDE description of the result**

Let us express our result with a kinetic theory point of view, making a bridge with the P.D.E. papers [16–19, 34, 40]. Let us denote by \( f_t \) the law of the process \( (X_t, V_t)_{t \geq 0} \) solution to (3) starting at \((0, v_0)\) with \( v_0 > 0 \), i.e. \( f_t(dx, dv) = \mathbb{P}(X_t \in dx, V_t \in dv) \) which is a probability measure on \( \mathbb{R}_+ \times \mathbb{R} \). Then, see Proposition 33 and Remark 34, \((f_t)_{t \geq 0}\) is a weak solution of the kinetic Fokker-Planck equation with diffusive boundary conditions

\[
\begin{align*}
\partial_t f_t + v \partial_x f_t &= \frac{1}{2} \partial_v^2 f_t - \partial_v[F(v)f_t] & \text{for } (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\
 v f_t(0, v) &= -\mu(v) \int_{(-\infty, 0)} w f_t(0, w)dw & \text{for } (t, v) \in (0, \infty)^2 \\
f_0 &= \delta_{(0,v_0)}
\end{align*}
\]
where we assume for simplicity that \( \mu(dv) = \mu(v)dv \).

We now set \( \rho_t(dx) = P(R_t^x \in dx) \) where \( (R_t^x)_{t \geq 0} \) is the limiting process defined in Theorem 3. Then, see Proposition 35 and Remark 36, \( (\rho_t)_{t \geq 0} \) is a weak solution of

\[
\begin{align*}
\partial_t \rho_t(x) &= \frac{\sigma_x}{2} \int_{\mathbb{R}} \rho_t(x - z) \mathbb{1}_{|x > z|} - \rho_t(x) + z\partial_x \rho_t(x) \mathbb{1}_{|x| < z} \, dz & \text{for } (t, x) \in (0, \infty)^2, \\
\int_0^\infty \rho_t(x) \, dx &= 1 & \text{for } t \in (0, \infty),
\end{align*}
\]

We believe that the above equation might be well-posed, just as for the heat equation \( \partial_t \rho_t(x) = \partial_x \rho_t(x) \) on \((0, \infty)^2\) where the Neumann boundary condition \( \partial_x \rho_t(0) = 0 \) can classically be replaced by the constraint \( \int_0^\infty \rho_t(x) \, dx = 1 \). The following statement immediately follows from Theorem 3.

**Corollary 4.** Grant Assumptions 1 and 2-(i). Let \( g_t(dx) = \int_{v \in \mathbb{R}} f_t(dx, dv) = P(X_t \in dx) \) and set with an abuse of notation \( g_t^\alpha(x) = \epsilon^{-1/\alpha} g_{t/\epsilon}(\epsilon^{-1/\alpha} x) \), that is \( g_t^\alpha \) is the pushforward measure of \( g_t \) by the map \( x \mapsto \epsilon^{1/\alpha} x \). It holds that for each \( t \geq 0 \), \( g_t^\alpha \) converges weakly (in the sense of measures) to \( \rho_t \) as \( \epsilon \to 0 \).

It is classical, see e.g. Bertoin [5, Chapter VI, Proposition 3], that for any fixed \( t \geq 0 \), \( R_t^\alpha \) has the same law as \( \sup_{s \in [0,t]} Z_s^\alpha \). Hence, the results of Doney-Savov [22, Theorem 1] tell us that \( \rho_t \) has a continuous density \( \rho_t(x) \) satisfying the following asymptotics:

\[
\rho_t(x) \sim At^{-1-\alpha} \quad \text{as } x \to \infty, \quad \text{and} \quad \rho_t(x) \sim Bt^{-1/2} x^{\alpha/2-1} \quad \text{as } x \to 0,
\]

for some constants \( A, B > 0 \) and for all \( t > 0 \).

Regarding the limiting fractional diffusion equation, we stress that it is not the same as the one from [17–19]. Both corresponding Markov processes possess the same infinitesimal generator, with however a different domain: they both behave like a stable process when strictly positive but are not reflected in the same manner when hitting 0. We do not find the same limiting P.D.E. inside the domain: they have an additional term expressing that particles can jump from the boundary to the interior of the domain, whereas in our case the process \( (R_t^\alpha)_{t \geq 0} \) leaves 0 continuously. More precisely, their limiting P.D.E. can be written as

\[
\partial_t \rho_t(x) = \int_{\mathbb{R}} \rho_t((x - z) +) - \rho_t(x) + z\partial_x \rho_t(x) \mathbb{1}_{|x| < z} \, dz \\
= \int_{\mathbb{R}} \rho_t(x - z) \mathbb{1}_{x > z} - \rho_t(x) + z\partial_x \rho_t(x) \mathbb{1}_{|x| < z} \, dz + \rho_t(0) \frac{x^\alpha}{\alpha},
\]

together with some boundary condition ensuring the mass conservation.

**Plan of the paper and sketch of the proof**

Once the limiting process is identified, it is very natural to try to establish the scaling limit of \( (X_t - \inf_{s \in [0,t]} X_s)_{t \geq 0} \), where \( (X_t)_{t \geq 0} \) is defined by (1). It should be clear that we need more than the convergence in finite dimensional distributions, and we need at least the convergence of past supremum and infimum. This is why we will use the convergence in the \( M_1 \)-topology.

In Section 2, we first explain why (3) is well-posed. Then we recall and define rigorously the notion of convergence in the \( M_1 \)-topology. This section ends with Theorem 7, which states that the convergence (2) actually holds in the space of càdlàg functions endowed with the \( M_1 \)-topology, strengthening the result of [25]. The proof is postponed to Section 5.

In Section 3, we study the particle reflected at a completely inelastic boundary, i.e. the solution of (5). We will see that \( (X_t - \inf_{s \in [0,t]} X_s)_{t \geq 0} \) plays a central role in the construction
of a solution to (5). This construction is essentially the same as in [7, 8]. Then we prove, see Theorem 10, that \((X_t - \inf_{s \in [0,t]} X_s)_{t \geq 0}\) and \((X_t)_{t \geq 0}\) have the same scaling limit, which is \((R^a_t)_{t \geq 0}\). The proof relies on Theorem 7, the continuous mapping theorem and Skorokhod’s representation theorem.

In Section 4, we finally establish the scaling limit of \((X_t)_{t \geq 0}\). The proof consists in using the scaling limit of \((X_t - \inf_{s \in [0,t]} X_s)_{t \geq 0}\) and to compare this process with \((X_t)_{t \geq 0}\). More precisely, we will show two comparison results. First we will see that \(X_t \geq X_t - \inf_{s \in [0,t]} X_s\). Then, inspired by the work of Bertoin [7, 8] and its construction of a solution to (5), we will show how we can construct a solution to (3) from the free process \((X_t, V_t)_{t \geq 0}\). From this construction, we will remark that, up to a time-change \((A'_t)_{t \geq 0}\), we have \(X_{A'_t} \leq X_t - \inf_{s \in [0,t]} X_s\). Then the proof is almost complete if we can show that \(A'_t \sim t\) as \(t \to \infty\) and we will see that it is indeed the case. Subsections 4.3 and 4.6 are dedicated to the proof of this result. We believe that these are the most technical parts of the paper. We stress that Assumption 2 is only used in Subsection 4.6.

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2 Preliminaries

2.1 Well-posedness

In this subsection, we explain quickly why (3) possesses a unique solution. To do so, we will regard (1) as an ordinary differential equation with a continuous random source. The force \(F\) being Lipschitz continuous, for any \((x_0, v_0) \in \mathbb{R}^2\), for every \(\omega \in \Omega\), there exists a unique global solution \((X_t(\omega), V_t(\omega))_{t \geq 0}\) to

\[
V_t(\omega) = v_0 + \int_0^t F(V_s(\omega))ds + B_t(\omega), \quad X_t(\omega) = x_0 + \int_0^t V_s(\omega)ds.
\]

One can easily construct a solution to (3) by "gluing" together solutions of (1) until their first hitting times of \([0,t] \times \mathbb{R}\). First consider the solution \((X^1_t, V^1_t)_{t \geq 0}\) of (1) starting at \((x_0, v_0) \in E\), define \(\tau_1 = \inf\{t > 0, X^1_t = 0\}\) and set \((X_t, V_t)_{t \in [0,\tau_1]} = (X^1_t, V^1_t)_{t \in [0,\tau_1]}\). Then consider the solution \((X^2_t, V^2_t)_{t \geq 0}\) of (1) starting at \((0, M_1)\) with \(B_t\) replaced by \(B_{t - \tau_1} - B_{\tau_1}\), define \(\tau_2 - \tau_1 = \inf\{t > 0, X^2_t = 0\}\) and set \((X_t, V_t)_{t \in [\tau_1, \tau_2]} = (X^2_t, V^2_t)_{t \in [\tau_1, \tau_2]}\). Iterating this operation indefinitely, we obtain a process \((X_t, V_t)_{t \in [0, \tau_\infty)}\) defined on \([0, \tau_\infty)\) where \(\tau_\infty = \lim_{n \to \infty} \tau_n\), which solves (3) on \([0, \tau_\infty)\).

Consider now two solutions \((X^1_t, V^1_t)_{t \in [0, \tau^1_\infty)}\) and \((X^2_t, V^2_t)_{t \in [0, \tau^2_\infty)}\) of (3) with the same initial condition, together with their respective sequence of hitting times \((\tau^1_n)_{n \in \mathbb{N}}\) and \((\tau^2_n)_{n \in \mathbb{N}}\). For every \(\omega \in \Omega\), \((X^1_t(\omega), V^1_t(\omega))_{t \geq 0}\) and \((X^2_t(\omega), V^2_t(\omega))_{t \geq 0}\) are two solutions of (6) on the time interval \([0, \tau^1_\infty(\omega) \wedge \tau^2_\infty(\omega))\). Hence they are equal on this interval and \(\tau^1_\infty(\omega) = \tau^2_\infty(\omega)\), and we can extend this reasoning to deduce that \(\tau^1_\infty = \tau^2_\infty = \tau_\infty\) and that \((X^1_t, V^1_t)_{t \in [0, \tau_\infty)}\) and \((X^2_t, V^2_t)_{t \in [0, \tau_\infty)}\) are equal. Therefore, uniqueness holds for (3) for each \(\omega \in \Omega\). Note that so far, we did not need the use of filtrations.

For a solution \((X_t, V_t)_{t \geq 0}\) of (3), we set \((\mathcal{F}_t)_{t \geq 0}\) for the usual completion of the filtration generated by the process. Then \((X_t, V_t)_{t \geq 0}\) is a strong Markov process in the filtration \((\mathcal{F}_t)_{t \geq 0}\). Since \((\tau_n)_{n \in \mathbb{N}}\) is the sequence of successive hitting times of \((X_t, V_t)_{t \geq 0}\) in \([0, \tau_\infty) \times \mathbb{R}\), it is a sequence of \((\mathcal{F}_t)_{t \geq 0}\)-stopping times and we deduce from the strong Markov property that the sequence
$(\tau_{n+1} - \tau_n)_{n \in \mathbb{N}}$ is i.i.d. and therefore $\tau_n \to \infty$ as $n \to \infty$ almost surely. In other words, any solution of (3) is global.

Finally, we stress that uniqueness in law holds for (3) as pathwise uniqueness for S.D.E. implies uniqueness in law. To summarize, (3) is well-posed, i.e. there exists a unique and global solution to (3) for any initial condition $(x_0, v_0) \in E$.

2.2 M₁-topology and the scaling limit of the free process

The main result of [25], see (2), is a convergence in the finite dimensional distributions sense and we cannot hope to obtain a convergence in law as a process in the usual Skorokhod distance, namely the $J_1$-topology. This is due to the fact that the space of continuous functions is closed in the space of càdlàg functions endowed with the $J_1$-topology. But the process may converge in a weaker topology and we will show that the process actually converges in the $M_1$-topology, first introduced in the seminal work of Skorokhod [43]. In this subsection, we recall the definition and a few properties of this topology. All of the results stated here may be found in Skorokhod [43] or in the book of Whitt [44, Chapter 12].

For any $T > 0$, we denote by $\mathcal{D}_T = \mathcal{D}([0,T], \mathbb{R})$ the usual sets of càdlàg functions on $[0,T]$ valued in $\mathbb{R}$. For a function $x \in \mathcal{D}_T$, we define the completed graph $\Gamma_{T,x}$ of $x$ as follows:

$$\Gamma_{T,x} = \{(t, z) \in [0,T] \times \mathbb{R}, z \in [x(t-), x(t)]\}.$$  

The $M_1$-topology on $\mathcal{D}_T$ is metrizable through parametric representations of the complete graphs. A parametric representation of $x$ is a continuous non-decreasing function $(u, r)$ mapping $[0, 1]$ onto $\Gamma_{T,x}$. Let us denote by $\Pi_{T,x}$ the set of parametric representations of $x$. Then the $M_1$-distance on $\mathcal{D}_T$ is defined for $x_1, x_2 \in \mathcal{D}_T$ as

$$d_{M_1}(x_1, x_2) = \inf_{(u, r) \in \Pi_{T,x}} (\|u_1 - u_2\| \vee \|r_1 - r_2\|),$$

where $\|\cdot\|$ is the uniform distance. The metric space $(\mathcal{D}_T, d_{M_1})$ is separable and topologically complete.

Let us now denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ the set of càdlàg functions on $\mathbb{R}_+$. We introduce for any $t \geq 0$, the usual restriction map $r_t$ from $\mathcal{D}$ to $\mathcal{D}_t$. Then the $M_1$-distance on $\mathcal{D}$ is defined for $x, y \in \mathcal{D}$ as

$$d_{M_1}(x, y) = \int_0^\infty e^{-t} (d_{M_1}(r_t(x), r_t(y)) \wedge 1) \, dt.$$  

Again, the metric space $(\mathcal{D}, d_{M_1})$ is separable and topologically complete. We now briefly recall some characterization of converging sequences in $(\mathcal{D}, d_{M_1})$. To this end, we first introduce for $x \in \mathcal{D}$ and $\delta, T > 0$, the following oscillation function

$$w(x, T, \delta) = \sup_{t \in [0,T]} \sup_{t_\delta- \leq t_1 < t_2 < t_3 \leq t_\delta+} d(x(t_2), [x(t_1), x(t_3)]),$$

where $t_\delta- = 0 \vee (t - \delta)$, $t_\delta+ = T \wedge (t + \delta)$ and $d(x(t_2), [x(t_1), x(t_3)])$ is the distance of $x(t_2)$ to the segment $[x(t_1), x(t_3)]$, i.e.

$$d(x(t_2), [x(t_1), x(t_3)]) = \begin{cases} 0 & \text{if } x(t_2) \in [x(t_1), x(t_3)], \\ \|x(t_2) - x(t_1)\| \wedge \|x(t_2) - x(t_3)\| & \text{otherwise}. \end{cases}$$

We finally introduce, for $x \in \mathcal{D}$, the set $\text{Disc}(x) = \{ t \geq 0, \Delta x(t) \neq 0 \}$ of the discontinuities of $x$. We have the following characterization, which can be found in Whitt [44, Theorem 12.5.1 and Theorem 12.9.3].

**Theorem 5.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}$ and let $x$ in $\mathcal{D}$. Then the following assertions are equivalent.
In any case, we will write variable valued in Theorem 6.

We are now ready to characterize convergence in law for sequences of random variables valued in $D$. We have the following result, see for instance [43, Theorem 3.2.1 and Theorem 3.2.2].

**Theorem 6.** Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables valued in $D$ and let $X$ be a random variable valued in $D$ such that for any $t \geq 0$, $\mathbb{P}(\Delta X(t) \neq 0) = 0$. Then the following assertions are equivalent.

1. $(X_n)_{n \in \mathbb{N}}$ converges in law to $X$ as $n \to \infty$ in $D$ endowed with the $M_1$-topology.

2. The finite dimensional distributions of $X_n$ converge to those of $X$ in some dense subset of $\mathbb{R}_+$ containing 0, and for any $T > 0$ and any $\eta > 0$ we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}(w(X_n, T, \delta) > \eta) = 0.$$ 

The following theorem shows that the convergence proved in [25] can be enhanced, in a stronger convergence.

**Theorem 7.** Grant Assumption 1 and let $(X_t, V_t)_{t \geq 0}$ be a solution of (1) starting at $(0, v_0)$ with $v_0 \in \mathbb{R}$. Let $(Z^n_t)_{t \geq 0}$ be a symmetric stable process with $\alpha = (\beta + 1)/3$ and such that $\mathbb{E}[e^{t\xi^n_t}] = \exp(-t\sigma_\alpha|\xi|^\alpha)$, where $\sigma_\alpha$ is defined by (4). Then we have

$$\left(\frac{e^{1/\alpha}X_t}{\epsilon}\right)_{t \geq 0} \to (Z^n_t)_{t \geq 0} \quad \text{as } \epsilon \to 0$$

in law for the $M_1$-topology.

The proof is postponed to Section 5.

### 3 The particle reflected at an inelastic boundary and its scaling limit

In this section, we construct a weak solution of (5) following Bertoin [7,8]. Although the author uses a Brownian motion instead of the process $(V_t)_{t \geq 0}$, some of the trajectorial properties shown in [7,8] still hold in our case.

Consider a weak solution $(X_t, V_t)_{t \geq 0}$ of (1) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting some Brownian motion $(B_t)_{t \geq 0}$, starting at $(0, v_0)$ where $v_0 \geq 0$. We introduce

$$\mathcal{X}_t = X_t - \inf_{s \in [0, t]} X_s. \quad (7)$$

Before introducing the solution of (5), we will first study a little bit the process $(\mathcal{X}_t)_{t \geq 0}$. It is a continuous process, which has the same motion as $X_t$ as long as $X_t > \inf_{s \in [0, t]} X_s$. When $X_t$ reaches its past infimum, it is necessarily with a non-positive velocity. When this happens

w-the interval $[0, d_0]$, where $d_0 = \inf\{s \geq t_0, V_s = 0\}$. On this time interval,

$X_t = \inf_{s \in [0, t]} X_s$ and thus $\mathcal{X}_t = 0$. It is interesting and useful to study

$$\mathcal{I}_X = \{t \geq 0, \mathcal{X}_t = 0\}.$$
As explained in [7, page 2025], the following random set appears naturally in the study of \( I_X \):

\[
H = \{ t \geq 0, V_t < 0, X_t = 0 \text{ and } \exists \epsilon > 0, \forall s \in [t - \epsilon, t), X_s > 0 \}.
\]

Each element of \( H \) is the first time at which \( X_t \) reaches its infimum, after an "excursion" above its infimum. Each point of \( H \) is necessarily isolated and thus \( H \) is countable. We are now able to state the following lemma, which is identical to [7, Lemma 2], and which characterizes the set \( I_X \).

**Lemma 8.** The decomposition of the interior of \( I_X \) as a union of disjoint intervals is given by

\[
\tilde{I}_X = \bigcup_{s \in H} [s, d_s],
\]

where \( d_s = \inf \{ u > s, V_u = 0 \} \). Moreover, the boundary \( \partial I_X = I_X \setminus \tilde{I}_X \) has zero Lebesgue measure.

**Proof.** The idea is to use the result from [7] and the Girsanov theorem. Indeed, the same result is proved in [7] when \( F = 0 \) and \( v_0 = 0 \).

**Step 1:** We first show that the results holds when \( v_0 = 0 \). We first define the following local martingales

\[
L_t = -\int_0^t F(s) dB_s \quad \text{and} \quad \mathcal{E}(L)_t = \exp \left( L_t - \frac{1}{2} \langle L \rangle_t \right)
\]

For any \( T > 0 \), \( \mathbb{E} [e^{\frac{1}{2}(\mathcal{L}_T)^2}] \leq e^{\frac{1}{2}T\|F\|_\infty^2} < \infty \), by Assumption 1, and therefore, by the Novikov criterion and the Girsanov theorem, the measure \( \mathcal{Q}_T = \mathcal{E}(L)_T \cdot \mathbb{P} \) is a probability measure on \((\Omega, \mathcal{F}_T)\) and the process

\[
V_t = \int_0^t F(s) ds + B_t = B_t - \langle B, L \rangle_t
\]

is an \((\mathcal{F}_t)_{t \in [0,T]}\)-Brownian motion starting at 0 under \( \mathcal{Q}_T \). We deduce from Lemma 2 in [7] that \( \mathcal{Q}_T \)-a.s.,

\[
\tilde{I}_X \cap [0,T] = \left( \bigcup_{s \in H} [s, d_s] \right) \cap [0,T]
\]

and \( \partial I_X \cap [0,T] \) has zero Lebesgue measure. Since this holds for every \( T > 0 \), and since \( \mathcal{Q}_T \sim \mathbb{P} \), this establishes our result for the initial condition \((0,0)\).

**Step 2:** We show the results when \( v_0 > 0 \). Let us define \( \tau = \inf \{ t > 0, X_t = 0 \} \). Since \( v_0 > 0 \), it is clear that \( \tau > 0 \) a.s. and that \( \tau \in H \) and

\[
I_X \cap [0,d_\tau] = \{ 0 \} \cup [\tau,d_\tau].
\]

Now the process \((\tilde{X}_t, \tilde{V}_t)_{t \geq 0} = (X_{t+d_\tau} - X_{d_\tau}, V_{t+d_\tau})_{t \geq 0}\) is a solution of (1) starting at \((0,0)\). Moreover, since \( X_{d_\tau} = \inf_{s \in [0,d_\tau]} X_s \), we clearly have on the event \( \{ d_\tau < t \} \),

\[
\inf_{s \in [0,t]} X_s = X_{d_\tau} + \inf_{s \in [0,t-d_\tau]} X_s,
\]

which yields

\[
I_X = \{ 0 \} \cup [\tau,d_\tau] \cup \tilde{I}, \text{ where } \tilde{I} = \left\{ t \geq d_\tau, \tilde{X}_{t-d_\tau} = \inf_{s \in [0,t-d_\tau]} \tilde{X}_s \right\} = I_X \cap [d_\tau, \infty).
\]

By the first step,

\[
\tilde{I} = \bigcup_{s \in H \cap [d_\tau, \infty)} [s, d_s],
\]

whence the result. \(\square\)
We are now ready to complete the construction of the solution of (5). We introduce the following time-change, as well as its right-continuous inverse

$$A_t = \int_0^t 1_{\{X_s > 0\}} ds \quad \text{and} \quad T_t = \inf \{ s > 0, A_s > t \}.$$  

We claim that by using the same arguments as in [7, page 2024-2025], or by using the Girsanov theorem as in the proof of Lemma 8, it holds that

$$\mathcal{X}_t = \int_0^t V_s 1_{\{X_s > 0\}} ds = \int_0^t V_s dA_s. \quad \text{(8)}$$

Therefore, by the change of variables theorem for Stieltjes integrals, we get

$$\mathcal{X}_t := \mathcal{X}_{T_t} = \int_0^{T_t} V_s dA_s = \int_0^{T_t} V_s ds, \quad \text{where} \quad V_t := V_{T_t}. \quad \text{(9)}$$

We finally set $G_t = \mathcal{F}_{T_t}$ and state the main theorem of this section.

**Theorem 9.** There exists a $(G_t)_{t \geq 0}$-Brownian motion such that $(\mathcal{X}_t, V_t)_{t \geq 0}$ is a solution of (5) on $(\Omega, \mathcal{F}, (G_t)_{t \geq 0}, \mathbb{P})$.

**Proof.** We have already seen that $\mathcal{X}_t = \int_0^t V_s ds$. Concerning the velocity, we paraphrase the proof of Proposition 1 in [8] and start by decomposing the process $(V_t)_{t \geq 0}$ as follows

$$V_t = v_0 + \int_0^{T_t} 1_{\{X_s = 0\}} dV_s + \int_0^{T_t} 1_{\{X_s > 0\}} dV_s =: v_0 + C_t + D_t.$$

Let us first deal with $D_t$. Using (1), we get

$$D_t = \int_0^{T_t} 1_{\{X_s > 0\}} F(V_s) ds + \int_0^{T_t} 1_{\{X_s > 0\}} dB_s.$$  

The last term is a $(G_t)_{t \geq 0}$-local martingale whose quadratic variation at time $t$ equals $A_{T_t} = t$ by the very definition of $(T_t)_{t \geq 0}$ and thus, it is a Brownian motion that we will denote $(B_t)_{t \geq 0}$. Regarding the second term, we use again the change of variables for Stieltjes integrals, to deduce that $\int_0^{T_t} 1_{\{X_s > 0\}} F(V_s) ds = \int_0^{T_t} F(V_s) dA_s = \int_0^t F(V_s) ds$. We have proved that $D_t = \int_0^t F(V_s) ds + B_t$. We now deal with $C_t$. First we note that the semimartingale

$$\int_0^t 1_{\{s \in \partial \mathcal{X}_t\}} dV_s = \int_0^t 1_{\{s \in \partial \mathcal{X}_t\}} F(V_s) ds + \int_0^t 1_{\{s \in \partial \mathcal{X}_t\}} dB_s$$

is a.s. null. Indeed since $\partial \mathcal{X}$ has zero Lebesgue measure by Lemma 8, the first term is obviously equal to zero. By the same argument, the second term is a local martingale whose quadratic variation is equal to zero, and therefore it is null. Then we can write as in [7]

$$C_t = \int_0^{T_t} 1_{\{s \in \mathcal{X}_t\}} dV_s = \int_0^{T_t} 1_{\{s \in \mathcal{X}_t\}} F(V_s) ds = \sum_{u \in \mathcal{H}, u \leq T_t} (V_u - V_u) = - \sum_{u \in \mathcal{H}, u \leq T_t} V_u.$$

In the third equality, we used Lemma 8 and the fact that, by definition, $T_t \notin \mathcal{L}_X$ and in the fourth that $V_u = 0$ for every $u \in \mathcal{H}$. To every point $u \in \mathcal{H}$ corresponds a unique jumping time $s$ of $(T_t)_{t \geq 0}$ at which $\mathcal{X}_t$ hits the boundary, i.e. $u = T_s$ and $\mathcal{X}_{T_s} = X_s = 0$. Indeed, the flat sections of $(A_t)_{t \geq 0}$ are precisely $\mathcal{L}_X$, and therefore, for every $u \in \mathcal{H}$, we have $A_u = A_{du}$ and thus if we set $s = A_u$, then $T_s = u$ and $T_s = d_u$. Hence we have

$$C_t = - \sum_{0 < s \leq t} V_{T_s} 1_{\{X_{T_s} = 0\}} = - \sum_{0 < s \leq t} V_s 1_{\{X_s = 0\}},$$

which completes the proof. \qed
We are now ready to study the scaling limits of \((X_t)_{t \geq 0}\) and \((X_t)_{t \geq 0}\).

**Theorem 10.** Let \((X_t)_{t \geq 0}\) and \((X_t)_{t \geq 0}\) be defined as in (7) and (9). Let also \((Z_t^\alpha)_{t \geq 0}\) be a symmetric stable process with \(\alpha = (\beta + 1)/3\) and such that \(\mathbb{E}[e^{t\xi Z_t^\alpha}] = e^{-t|\xi|^{\alpha}}\) where \(\sigma_\alpha\) is defined by (4). Let \(R_t^\alpha = Z_t^\alpha - \inf_{s \in [0,t]} Z_s^\alpha\). Then we have

\[
\left( e^{1/\alpha X_{t/\epsilon}} \right)_{t \geq 0} \xrightarrow{\epsilon \to 0} (R_t^\alpha)_{t \geq 0} \quad \text{and} \quad \left( e^{1/\alpha X_{t/\epsilon}} \right)_{t \geq 0} \xrightarrow{\epsilon \to 0} (R_t^\alpha)_{t \geq 0}
\]

in law for the \(M_1\)-topology.

**Proof.** The convergence of \(e^{1/\alpha X_{t/\epsilon}}\) is straightforward by the continuous mapping theorem and Theorem 7. Indeed the reflection map from the probability space to the \(\mathbb{R}\), we get that

\[
\text{Theorem 10.}
\]

Let \(X_t\) be a symmetric stable process with \(\lambda\), we introduce the time-change process \(Z_t^\alpha\) defined for every \(t \geq 0\) by \(y(t) = x(t) - 0 \wedge \inf_{s \in [0,t]} x(s)\), is continuous with respect to the \(M_1\)-topology, see Whitt [44, Chapter 13, Theorem 13.5.1].

We now study \((X_t)_{t \geq 0} = (X_{t/\epsilon})_{t \geq 0}\). By Skorokhod’s representation theorem, there exist a family of processes \((X_t^\alpha)_{t \geq 0}\) indexed by \(\epsilon > 0\), and a reflected symmetric stable process \((R_t^\alpha)_{t \geq 0}\), both defined on the probability space \([0,1], B([0,1]), \lambda\), where \(\lambda\) denotes the Lebesgue measure on \([0,1]\), such that for every \(\epsilon > 0\), \((X_t^\alpha)_{t \geq 0} = (X_{t/\epsilon})_{t \geq 0}\) and such that,

\[
\lambda - \text{a.s., } d_{M_1}(e^{1/\alpha X_t^\alpha}, (R_t^\alpha)_{t \geq 0}) \to 0 \text{ as } \epsilon \to 0.
\]

Let us denote by \(J\) be the set of discontinuities of \((R_t^\alpha)_{t \geq 0}\). Then, by [44, Chapter 12, Lemma 12.5.1], we get that

\[
\lambda - \text{a.s., for every } t \notin J, \quad e^{1/\alpha X_t^\alpha} \to R_t^\alpha.
\]

We introduce the time-change process \(A_t^\alpha = \int_0^1 1_{e^{1/\alpha X_s^\alpha} > 0} ds\) for every \(\epsilon > 0\) and every \(t \geq 0\). Then we get by the Fatou lemma that \(\lambda - \text{a.s.}, \text{for every } t \geq 0, \)

\[
\int_0^t \liminf_{\epsilon \to 0} 1_{e^{1/\alpha X_s^\alpha} > 0} ds \leq \liminf_{\epsilon \to 0} A_t^\alpha \leq \limsup_{\epsilon \to 0} A_t^\alpha \leq t.
\]

Since \(J\) is countable, we have \(\lambda - \text{a.s.}, \text{for a.e. } s \in [0,t], 1_{R_s^\alpha > 0} \leq \liminf_{\epsilon \to 0} 1_{e^{1/\alpha X_s^\alpha} > 0}\). Since the zero set of the reflected stable process is a.s. Lebesgue-null, we conclude that \(\lambda - \text{a.s.}, \text{for every } t \geq 0, A_t^\alpha \to t\) as \(\epsilon \to 0\).

Let us denote by \((T_t^\alpha)_{t \geq 0}\) the right-continuous inverse of \((A_t^\alpha)_{t \geq 0}\). As an immediate consequence, we have \(\lambda - \text{a.s.}, \text{for every } t \geq 0, T_t^\alpha \to t\) as \(\epsilon \to 0\). Since the \(M_1\)-topology on the space of non-increasing functions reduces to pointwise convergence on a dense subset including 0, see [44, Corollary 12.5.1], we have \((T_t^\alpha)_{t \geq 0} \to (\text{id}_t)_{t \geq 0}\) as \(\epsilon \to 0\) in law in the \(M_1\)-topology, where \(\text{id}_t = t\) is the identity process.

By a simple substitution, we see that \((\epsilon A_{t/\epsilon})_{t \geq 0} = (A_t^\alpha)_{t \geq 0}\), from which we deduce that \((\epsilon T_{t/\epsilon})_{t \geq 0} \xrightarrow{\epsilon \to 0} (A_t^\alpha)_{t \geq 0}\) and that \((\epsilon T_{t/\epsilon})_{t \geq 0} \to (\text{id}_t)_{t \geq 0}\) as \(\epsilon \to 0\) in law in the \(M_1\)-topology. Hence, by a generalization of Slutsky theorem, see for instance [11, Section 3, Theorem 3.9]), we get that \((\epsilon^{1/\alpha} X_{t/\epsilon}, \epsilon T_{t/\epsilon})_{t \geq 0}\) converges in law to \((R_t^\alpha, \text{id}_t)_{t \geq 0}\) in \(\mathcal{D} \times \mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)\) endowed with the \(M_1\)-topology.

We are now able to conclude. Let us denote by \(\mathcal{D}_\uparrow\) the set of càdlàg and non-decreasing functions from \(\mathbb{R}_+\) to \(\mathbb{R}_+\), and \(C_\uparrow\) the set of continuous and strictly increasing functions from \(\mathbb{R}_+\) to \(\mathbb{R}_+\). Then the composition map, from \(\mathcal{D} \times \mathcal{D}_\uparrow\) to \(\mathcal{D}\), which maps \((x,y)\) to \(x \circ y\), is continuous on \(\mathcal{D} \times C_\uparrow\), see [44, Chapter 13, Theorem 13.2.3]. Hence \((\epsilon^{1/\alpha} X_{t/\epsilon})_{t \geq 0} = (\epsilon^{1/\alpha} X_{T_{t/\epsilon}})_{t \geq 0}\) converges to \((R_t^\alpha)_{t \geq 0}\) by the continuous mapping theorem. □
4 The reflected process with diffusive boundary condition

In this section, we finally study the process $(X_t, V_t)_{t \geq 0}$, solution of (3), and its scaling limit. To establish our result, we will rely on the convergence of $(X_t)_{t \geq 0}$ from Theorem 10. First, we will show that $X_t \geq \lambda_t = X_t - \inf_{s \in [0, t]} X_s$ where $(X_t)_{t \geq 0}$ and $(\lambda_t)_{t \geq 0}$ are the solutions to (3) and (1) with the same Brownian motion. Then, inspired by the work of Bertoin [7], we will give another construction of $(X_t, V_t)_{t \geq 0}$ and we will prove that, up to a time-change, $X_t \leq \lambda_t$. Finally, we will conclude since the time-change at stake is asymptotically equivalent to $t$. We believe that the proof of the limit of the time-change is the most technical part of the paper and this will the subject of Subsections 4.3 and 4.6.

4.1 A comparison result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space supporting a Brownian motion $(B_t)_{t \geq 0}$ and a sequence of i.i.d. $\mu$-distributed random variables $(M_n)_{n \in \mathbb{N}}$, independent of each other. Consider the solution $(X_t, V_t)_{t \geq 0}$ of (3), starting at $(0, v_0)$ where $v_0 > 0$, as well as its sequence of hitting times $(\tau_n)_{n \in \mathbb{N}}$. We also consider on the same probability space the solution $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ of (1) starting at $(0, v_0)$, with the same driving Brownian motion $(B_t)_{t \geq 0}$. Let also $(\lambda_t)_{t \geq 0}$ be defined as in (7), i.e. $\lambda_t = X_t - \inf_{s \in [0, t]} X_s$. We have the following proposition.

Proposition 11. Almost surely, for any $t \geq 0$, $X_t \geq \lambda_t$.

Proof. Step 1: We first prove that a.s. for any $t \geq 0$, $V_t \leq V_t$ and to do so, we use the classical comparison theorem for O.D.E.’s. We prove recursively that for any $n \in \mathbb{N}$, a.s. for any $t \in [0, \tau_n)$, $V_t \leq V_t$. This is true for $n = 1$, since the processes are both solutions of the well-posed O.D.E. on $[0, \tau_1)$, with the same starting point. Hence they are equal on this interval.

Now let us assume that for some $n \in \mathbb{N}$, a.s. for any $t \in [0, \tau_n)$, $V_t \leq V_t$. Then a.s. $V_{\tau_{n+1}} = V_{\tau_{n+1}} - V_{\tau_{n+1}} \leq 0$. We also see that $(\tilde{V}_t)_{t \geq 0}$ and $(\tilde{V}_t)_{t \geq 0}$ are two solutions of the same O.D.E. on the interval $[\tau_n, \tau_{n+1}]$. Indeed, a.s. for any $t \in [\tau_n, \tau_{n+1})$, we have

$$V_t = M_n + \int_{\tau_n}^t F(V_s)ds + B_t - B_{\tau_n} \quad \text{and} \quad \tilde{V}_t = V_{\tau_n} + \int_{\tau_n}^t F(V_s)ds + B_t - B_{\tau_n}.$$ 

Since $M_n > 0$ with probability one, we have a.s. $M_n \geq V_{\tau_n}$ and thus, by the comparison theorem, we deduce that a.s., for any $t \in [\tau_n, \tau_{n+1})$, $V_t \leq \tilde{V}_t$. This achieves the first step.

Step 2: We conclude. Almost surely, for any $0 \leq s \leq t$, we have

$$X_t - X_s = \int_s^t V_s ds \leq \int_s^t \tilde{V}_s ds = \tilde{X}_t - X_s \leq \lambda_t,$$

the last inequality holding since $\lambda_t \geq 0$ a.s. This implies that a.s., $\inf_{s \in [0, t]} X_s \geq X_t - \lambda_t$, i.e. $X_t \geq \lambda_t$, for any $t \geq 0$. $\square$

4.2 A second construction

In this subsection, we give another construction of $(X_t, V_t)_{t \geq 0}$, which is inspired by the construction given by Bertoin [7] of the reflected Langevin process at an inelastic boundary.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $(B_t)_{t \geq 0}$ and a sequence of i.i.d. $\mu$-distributed random variables $(M_n)_{n \in \mathbb{N}}$, independent from the Brownian motion. We set $(\mathcal{F}_t)_{t \geq 0}$ to be the filtration generated by $(B_t)_{t \geq 0}$ after the usual completions, and we introduce the filtration $(\mathcal{G}_t)_{t \geq 0}$ defined for every $t \geq 0$ as

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma \{(M_n)_{n \in \mathbb{N}}\}.$$ 

It is clear that $(B_t)_{t \geq 0}$ remains a Brownian motion in the filtration $(\mathcal{G}_t)_{t \geq 0}$. Next, we consider the strong solution $(X_t, V_t)_{t \geq 0}$ of (1) starting at $(0, v_0)$, which remains a strong Markov process
in this filtration. We set $\sigma_0 = 0$, then we set $\tau_1 = \inf\{t > 0, X_t = 0\}$ and we define recursively the sequence of random times
\[
\sigma_n = \inf \{ t > \tau_n, V_t = M_n \} \quad \text{and} \quad \tau_{n+1} = \inf \{ t > \sigma_n, X_t = X_{\sigma_n} \},
\]
where $n \in \mathbb{N}$. We have the following lemma.

**Lemma 12.** The random times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are $(\mathcal{G}_t)_{t \geq 0}$-stopping times which are almost surely finite. Moreover, the sequence $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})_{n \geq 2}$ forms a sequence of identically distributed random variables and the subsequences $(\tau_2n - \sigma_{2n-1}, \sigma_{2n} - \sigma_{2n-1})_{n \geq 1}$ and $(\tau_{2n+1} - \sigma_{2n}, \sigma_{2n+1} - \sigma_{2n})_{n \geq 1}$ form sequences of i.i.d. random variables.

**Proof.** The process $(V_t)_{t \geq 0}$ being a recurrent diffusion, see for instance [25], we have almost surely, $\lim inf_{t \to \infty} V_t = -\infty$ and $\lim sup_{t \to \infty} V_t = \infty$. Moreover, by Theorem 7, we also have $\lim inf_{t \to \infty} X_t = -\infty$ and $\lim sup_{t \to \infty} X_t = \infty$ a.s. Hence, the random times previously defined are almost surely finite. Moreover, it is clear by a simple induction that the random times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are $(\mathcal{G}_t)_{t \geq 0}$-stopping times. Indeed, if $\sigma_n$ is a $(\mathcal{G}_t)_{t \geq 0}$-stopping time for some $n \geq 0$, then $\tau_{n+1} = \inf \{ t > \sigma_n, X_t = X_{\sigma_n} \}$ is obviously a $(\mathcal{G}_t)_{t \geq 0}$-stopping time and since $\sigma_{n+1}$ is the first hitting of zero after $\tau_{n+1}$ of the $(\mathcal{G}_t)_{t \geq 0}$-adapted process $(V_t - M_{n+1})_{t \geq 0}$, it is also a stopping time for $(\mathcal{G}_t)_{t \geq 0}$.

Then, for any $n \geq 2$, applying the strong Markov property of $(X_t, V_t)_{t \geq 0}$ at time $\sigma_{n-1}$, we see that for any $n \geq 2$, $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})$ has the same law as $(\tau_2 - \sigma_1, \sigma_2 - \sigma_1)$. The fact that the subsequences form sequences of i.i.d. random variables follows again from the strong Markov property and the fact that for any $n \geq 2$, $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})$ only depends on $(B_t - B_{\sigma_{n-1}})_{t \in [\sigma_{n-1}, \sigma_n]}$ and $(M_{n-1}, M_n)$.

We are finally ready to start the construction of the reflected process. We define the $(\mathcal{G}_t)_{t \geq 0}$-adapted processes
\[
\mathcal{X}_t = \sum_{n \in \mathbb{N}} (X_t - X_{\sigma_{n-1}}) 1_{\{\sigma_{n-1} \leq t < \tau_n\}} \quad \text{and} \quad \mathcal{V}_t = 1_{\{X_t > 0\}} V_t.
\]  
(10)

We refer to Figure 1 for a visual representation. For every $n \in \mathbb{N}$, the process $(\mathcal{X}_t)_{t \geq 0}$ has the same trajectory as $(X_t)_{t \geq 0}$ on $[\sigma_{n-1}, \tau_n]$ shifted by $X_{\sigma_{n-1}}$ and is null on $(\tau_n, \sigma_n]$. Now by the very definition of $\sigma_n$ and $\tau_n$, $X_t$ is above $X_{\sigma_{n-1}}$ on $[\sigma_{n-1}, \tau_n]$, and it should be clear that a.s., for every $t \geq 0$, $\mathcal{X}_t \geq 0$ and that
\[
\mathcal{I}_\mathcal{X} = \{ t \geq 0, \mathcal{X}_t = 0 \} = \{ 0 \} \cup \left( \bigcup_{n \in \mathbb{N}} [\tau_n, \sigma_n] \right).
\]  
(11)

As in (9), we also have
\[
\mathcal{X}_t = \int_0^t \mathcal{V}_s d\mathcal{Z}_s
\]  
(12)

Indeed, this can be easily checked using that for every $s \geq 0$, $1_{\{\mathcal{X}_s > 0\}} = \sum_{n \in \mathbb{N}} 1_{\{\sigma_{n-1} \leq s < \tau_n\}}$ and that $X_{\tau_n} = X_{\sigma_{n-1}}$. As in the previous subsection, we can compare $\mathcal{X}_t$ to $\mathcal{X}_t = X_t - \inf_{s \in [0,t]} X_s$.

**Proposition 13.** Almost surely, for any $t \geq 0$, $\mathcal{X}_t \leq \mathcal{X}_t$.

**Proof.** The result follows almost immediately from the definition of $(\mathcal{X}_t)_{t \geq 0}$. Indeed, almost surely, for any $n \in \mathbb{N}$ and any $t \geq 0$,
\[
X_{\sigma_{n-1}} 1_{\{\sigma_{n-1} \leq t < \tau_n\}} \geq \inf_{s \in [0,t]} X_s \times 1_{\{\sigma_{n-1} \leq t < \tau_n\}}
\]
and therefore we have almost surely, for any $t \geq 0$,
\[
\mathcal{X}_t = \sum_{n \in \mathbb{N}} (X_t - X_{\sigma_{n-1}}) 1_{\{\sigma_{n-1} \leq t < \tau_n\}} \leq \mathcal{X}_t \times \sum_{n \in \mathbb{N}} 1_{\{\sigma_{n-1} \leq t < \tau_n\}} \leq \mathcal{X}_t,
\]
which achieves the proof.
We introduce the \( G_t \) adapted time-change \( A'_t = \int_0^t 1_{(\chi_s > 0)} ds \) as well as its right-continuous inverse \( T'_t \), and we define the processes

\[
X_t = \mathcal{X}_{T'_t} = \int_0^{T'_t} V_s dA'_s = \int_0^t V_s ds \quad \text{where} \quad V_t = V_{T'_t}.
\]

We finally define the filtration \( (\mathcal{F}_t)_{t \geq 0} = (\mathcal{G}_{T'_t})_{t \geq 0} \). We have the following theorem.

**Theorem 14.** There exists an \( (\mathcal{F}_t)_{t \geq 0} \)-Brownian motion such that the process \( (X_t, V_t)_{t \geq 0} \) is a solution of \( (3) \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \).

**Proof.** The proof is very similar to the proof of Theorem 9. We start by decomposing the process \( (V_t)_{t \geq 0} \) as follows

\[
V_t = v_0 + \int_0^{T'_t} 1_{(\chi_s = 0)} dV_s + \int_0^{T'_t} 1_{(\chi_s > 0)} dV_s =: v_0 + C_t + D_t.
\]

As in the proof of Theorem 9, we easily see, using (1), the definition of \( (T'_t)_{t \geq 0} \) and the change of variables theorem for Stieltjes integrals, that

\[
D_t = \int_0^t F(V_s) ds + B_t,
\]

where \( (B_t)_{t \geq 0} = (\int_0^{T'_t} 1_{(\chi_s > 0)} dB_s)_{t \geq 0} \) is an \( (\mathcal{F}_t)_{t \geq 0} \)-Brownian motion. We now deal with \( C_t \) and we write

\[
C_t = \int_0^{T'_t} 1_{(s \in I_x)} dV_s = \sum_{n \in \mathbb{N}, \tau_n \leq T'_t} (V_{\tau_n} - V_{\tau_n^-}) = \sum_{n \in \mathbb{N}, \tau_n \leq T'_t} (M_n - V_{\tau_n^-}).
\]

In the second inequality, we used (11) and that, by definition, \( T'_t \notin I_x \). Let us now define for all \( n \in \mathbb{N} \), \( \tau_n = \sum_{k=1}^n (\tau_k - \sigma_{k-1}) \). Then by definition of \( A'_t \), we have for all \( n \in \mathbb{N} \), \( A'_{\tau_n} = \tau_n \), which leads to \( \tau_n = T'_{\tau_n^-} \). Indeed the flat section of \( (A'_t)_{t \geq 0} \) consists in \( \cup_{n \in \mathbb{N}} [\tau_n, \sigma_n] \) and thus the jumping times of \( (T'_t)_{t \geq 0} \) are precisely the times \( \tau_n \). Then we get

\[
C_t = \sum_{n \in \mathbb{N}} (M_n - V_{\tau_n^-}) 1_{\{\tau_n \leq t\}}.
\]

Finally it is clear that \( \tau_1 = \inf \{t > 0, X_t = 0\} \) and that \( \tau_{n+1} = \inf \{t > \tau_n, X_t = 0\} \). \( \square \)
Convergence of the time-change

The goal of this subsection is to see that the time change \((A'_t)_{t \geq 0}\) is asymptotically equivalent to \(t\), i.e. the size of \(\sigma_n - \tau_n\) is small compared to the size of \(\tau_n - \sigma_{n-1}\). Recall we are concerned with process \((X_t)_{t \geq 0}\) defined in (10) and \(A'_t = \int_0^t \mathbf{1}_{\{X_s > 0\}} \, ds\). The main result of this subsection is the following result.

**Proposition 15.** Under Assumption 2-(i), we have \(t^{-1} A'_t \xrightarrow{P} 1\) as \(t \to \infty\).

We recall from from Lemma 12 that \((\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})_{n \geq 2}\) is a sequence of identically distributed random variables and each element is equal in law to the random variable \((\tau, \sigma)\) defined as follows: let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space supporting an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion \((B_t)_{t \geq 0}\) and two independent \(\mu\)-distributed random variables \(V_0\) and \(M\), also independent of the Brownian motion. Consider the process \((X_t, V_t)_{t \geq 0}\) solution of (1) starting at \((0, V_0)\). Then \(\tau\) and \(\sigma\) are defined as

\[
\tau = \inf\{t > 0, X_t = 0\} \quad \text{and} \quad \sigma = \inf\{t \geq \tau, V_t = M\}.
\]

For \(t \geq 0\), we define \(N_t = \sup\{n \geq 0, \sigma_n \leq t\}\), for which \(\sigma_{N_t} \leq t < \sigma_{N_t+1}\). Then the time-change \((A'_t)_{t \geq 0}\) satisfies, see (10),

\[
A'_t = \sum_{k=1}^{N_t} (\tau_k - \sigma_{k-1}) + \tau_{N_t+1} \wedge t - \sigma_{N_t}.
\]

Roughly, the reason why Proposition 15 is true is that \(\sigma - \tau\) is actually small compared to \(\tau\). More precisely, we will show that \(\tau\) and \(\sigma\) have exactly the same probability tail.

First, since \((X_t)_{t \geq 0}\) resembles a symmetric stable process as \(t\) is large, we should expect \(\mathbb{P}(\tau > t)\) to behave like the probability for a symmetric stable process started at \(\eta > 0\), to stay positive up to time \(t\), which is well-known to behave like \(t^{-1/2}\) as \(t \to \infty\).

Then, since \((V_t)_{t \geq 0}\) is positive recurrent, we should expect \(\mathbb{P}(\sigma - \tau > t)\) to have a lighter tail than \(\tau\). Indeed, \(X_t\) reaches 0 at time \(\tau\) with some random negative velocity \(V_\tau\), and \(\sigma - \tau\) is the amount of time it takes for \((V_t)_{t \geq 0}\) to reach \(M\), which should not be too big, thanks to the positive recurrence of the velocity and the assumption on \(\mu\).

However, we did not manage to employ this strategy, as the law of \(V_\tau\) is unknown and it is not clear at all how to get an exact asymptotic of \(\mathbb{P}(\tau > t)\) by approaching \((X_t)_{t \geq 0}\) by its scaling limit. We will rather use tools introduced in Berger, Béthencourt and Tardif [3].

We state the following crucial lemma which describes the tails of \(\tau\) and \(\sigma\). Its proof is rather technical and a bit independent of the rest, so it is postponed to Subsection 4.6.

**Lemma 16.** Grant Assumptions 2-(i). Then there exists a constant \(C > 0\) such that

\[
\mathbb{P}(\tau > t) \sim \mathbb{P}(\sigma > t) \sim Ct^{-1/2} \quad \text{as} \quad t \to \infty.
\]

With these results at hand, we are able to prove Proposition 15.

**Proof of Proposition 15.** We seek to show that \((1 - t^{-1} A'_t)\) converges to 0 in probability as \(t \to \infty\). From (13), we have

\[
t - A'_t = \sum_{k=1}^{N_t} (\sigma_{k-1} - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}} + \sigma_{N_t}
\]

\[
= \sum_{k=1}^{N_t} (\sigma_{k-1} - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}} + \sum_{k=1}^{N_t} (\sigma_k - \sigma_{k-1})
\]

\[
= \sum_{k=1}^{N_t} (\sigma_k - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}}.
\]


Since by definition, $\sigma_{N_t+1} \geq t$, we have the following bound:

$$0 \leq 1 - \frac{A^t}{t} \leq \frac{1}{t} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k) = \frac{\sigma_1 - \tau_1}{t} + \frac{1}{t} \sum_{k=2}^{N_t+1} (\sigma_k - \tau_k).$$

Obviously, the first term on the right-hand side almost surely vanishes as $t \to \infty$. We now use Lemma 16 to study the asymptotic behavior of $\sum_{k=2}^{N_t+1} (\sigma_k - \tau_k)$ as $t \to \infty$. We divide the rest of the proof in two steps.

**Step 1:** We first show that

$$\lim_{A \to \infty} \limsup_{t \to \infty} P(N_t \geq A^{1/2}) = 0. \quad (14)$$

To show this, we introduce $\tilde{N}_t = \min\{n \geq 1, \sum_{k=1}^{n} (\sigma_{2k} - \sigma_{2k-1}) \leq t\}$. Since for any $n \geq 1$, we have $\sigma_{2n} \geq \sum_{k=1}^{n} (\sigma_{2k} - \sigma_{2k-1})$, it should be clear that for any $t > 0$, $N_t \leq \tilde{N}_t/2$. Next, by Lemma 12, $(\sigma_{2n} - \sigma_{2n-1})_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables whose common law is that of $\sigma$. Thanks to Lemma 16, we can apply the classical $\alpha$-stable central limit theorem, and it is clear that

$$\frac{1}{n^2} \sum_{k=1}^{n} (\sigma_{2k} - \sigma_{2k-1}) \xrightarrow{L} \mathcal{S}^{1/2} \quad \text{as } n \to \infty,$$

where $\mathcal{S}^{1/2}$ is a positive $1/2$-stable random variable, see [23, Chapter XII.6 Theorem 2]. Then it is immediate that $\tilde{N}_t/t^{1/2}$ converges in law to some random variable $N_\infty$. Finally, we get that

$$\limsup_{t \to \infty} P(N_t \geq A^{1/2}) \leq \limsup_{t \to \infty} P(\tilde{N}_t \geq 2A^{1/2}) = P(N_\infty \geq 2A).$$

Letting $A \to \infty$ shows that (14) holds.

**Step 2:** We show that $t^{-1} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k)$ converges to 0 as $t \to \infty$ in probability. First, for any $n \geq 1$, we can write

$$\sum_{k=1}^{n} (\sigma_k - \tau_k) = \sum_{k \leq n, k \text{ even}} (\sigma_k - \tau_k) + \sum_{k \leq n, k \text{ odd}} (\sigma_k - \tau_k), \quad (15)$$

which is then by Lemma 12 the sum of two sums of i.i.d. random variables distributed as $\sigma - \tau$. Moreover, by Lemma 16 and Lemma 31 in Appendix A with $X = \tau$ and $Y = \sigma - \tau$, the tail of $\sigma - \tau$ is lighter than the tail of $\sigma$:

$$\lim_{t \to \infty} t^{1/2} P(\sigma - \tau > t) = 0.$$

This entails, see Proposition 32 with $\alpha = 1/2$, that the two terms in the right-hand-side of (15) divided by $n^2$, converges to 0 as $n \to \infty$ in probability. At this point, we conclude that

$$\frac{1}{n^2} \sum_{k=2}^{n} (\sigma_k - \tau_k) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \quad (16)$$

Finally, for any $\eta > 0$ and any $A > 0$, we have

$$P\left(t^{-1} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k) > \eta\right) \leq P\left(t^{-1} \sum_{k=1}^{[At^{1/2}]} (\sigma_k - \tau_k) > \eta\right) + P(N_t + 1 \geq At^{1/2}).$$

Making $t \to \infty$ using (16), then $A \to \infty$ using (14), we complete the step. \qed
4.4 Scaling limit of \((X_t)_{t \geq 0}\)

In this subsection, we show that under Assumption 2-(ii), the scaling limit of \((X_t)_{t \geq 0}\) is the stable process reflected on its infimum. The following proposition will help us showing the second part of Theorem 3.

**Proposition 17.** Grant Assumptions 1 and 2-(ii), and let \((X_t, \mathcal{G}_t)_{t \geq 0}\) be defined by (10) with \(v_0 > 0\). Then we have, in law for the \(M_1\)-topology,

\[
(\epsilon^{1/\alpha} X_{t/\epsilon})_{t \geq 0} \to (R_t^\alpha)_{t \geq 0}\quad \text{as } \epsilon \to 0,
\]

where \(R_t^\alpha = Z_t^\alpha - \inf_{s \in [0,t]} Z_s^\alpha\) and \((Z_t^\alpha)_{t \geq 0}\) is the stable process from Theorem 3.

**Proof.** We consider, for \(v_0 > 0\), the processes \((X_t, \mathcal{G}_t)_{t \geq 0}\) and \((X_t, \mathcal{G}_t)_{t \geq 0}\) defined in Sections 3 and 4, starting at \((0, v_0)\) and both constructed from the process \((X_t, V_t)_{t \geq 0}\) solution of (1) also starting at \((0, v_0)\). We recall that almost surely, for all \(t \geq 0\),

\[
X_t = \int_0^t V_s 1_{\{X_s > 0\}} ds \quad \text{and} \quad \bar{X}_t = \int_0^t V_s 1_{\{\bar{X}_s > 0\}} ds,
\]

see (8), (10) and (12). We show that, under Assumption 2-(ii), we have for any \(T > 0\),

\[
\Delta_{T, \epsilon} := \sup_{t \in [0,T]} \epsilon^{1/\alpha} (X_{t/\epsilon} - \bar{X}_{t/\epsilon}) \overset{P}{\to} 0 \quad \text{as } \epsilon \to 0,
\]

which will prove the result thanks to Theorem 10 and [11, Section 3, Theorem 3.1]. Since a.s. for any \(t \geq 0\), we have \(X_t \geq \bar{X}_t\) by Proposition 13, we can write

\[
0 \leq X_t - \bar{X}_t = \int_0^t V_s 1_{\{X_s > 0\} \cap \{\bar{X}_s = 0\}} ds \leq \int_0^t (0 \lor V_s) 1_{\{\bar{X}_s = 0\}} ds \leq \sum_{k=1}^{N_t + 1} M_k(\sigma_k - \rho_k),
\]

where for any \(k \geq 1\), \(\rho_k = \inf\{t \geq \tau_k, V_t = 0\} \leq \sigma_k\) and \(N_t = \sup\{n \geq 0, \sigma_n \leq t\}\), as in the previous subsection. Indeed, for any \(t \geq 0\), we have

\[
\mathcal{I}_{\bar{X}} \cap [0,t] = \bigcup_{k=1}^{N_t + 1} [\tau_k, \sigma_k],
\]

and for any \(k \geq 1\), the velocity \(V_s\) is non-positive on \([\tau_k, \rho_k]\) and is smaller than \(M_k\) on \([\rho_k, \sigma_k]\). The sequence \((M_n(\sigma_n - \rho_n))_{n \in \mathbb{N}}\) is a sequence of i.i.d random variables and we claim that there exists \(\delta' \in (0, 1 - \alpha/2)\) such that \(\mathbb{E}[(M_1(\sigma_1 - \rho_1))^\alpha/2 + \delta'] < \infty\), see Lemma 18-(ii) below. This implies by the Markov inequality that \(t^{\alpha/2} \mathbb{P}(M_1(\sigma_1 - \rho_1) > t) \to 0\) as \(t \to \infty\). Therefore, by Proposition 32, we have

\[
\frac{1}{n^{2/\alpha}} \sum_{k=1}^{n} M_k(\sigma_k - \rho_k) \overset{P}{\to} 0 \quad \text{as } n \to \infty.
\]  

(17)

But since we have

\[
\Delta_{T, \epsilon} \leq \epsilon^{1/\alpha} \sum_{k=1}^{N_{T/\epsilon} + 1} M_k(\sigma_k - \rho_k),
\]

it comes that for any \(\eta > 0\) and any \(A > 0\),

\[
\mathbb{P}(\Delta_{T, \epsilon} > \eta) \leq \mathbb{P}(\epsilon^{1/\alpha} \sum_{k=1}^{\lfloor A\epsilon^{-1/2} \rfloor} M_k(\sigma_k - \rho_k) > \eta) + \mathbb{P}(N_{T/\epsilon} + 1 \geq A\epsilon^{-1/2}).
\]

Letting \(\epsilon \to 0\) using (17), and letting \(A \to \infty\) using (14), we conclude that \(\mathbb{P}(\Delta_{T, \epsilon} > \eta) \to 0\) as \(\epsilon \to 0\).
4.5 Proof of the main result

Proof of Theorem 3. Step 1: We start by showing the convergence in the finite dimensional sense. Let \((X_t, V_t)_{t \geq 0}\) and \((X'_t, V'_t)_{t \geq 0}\) be solutions of (3) and (1) starting at \((0, v_0)\). Let also \((X_t')_{t \geq 0}\) be defined as in (7). Let \(n \geq 1, t_1, \ldots, t_n > 0\) and \(x_1, \ldots, x_n \geq 0\). By Proposition 11, we have

\[
\mathbb{P}\left( e^{1/\alpha} X_{t_1/\epsilon} \geq x_1, \ldots, e^{1/\alpha} X_{t_n/\epsilon} \geq x_n \right) \geq \mathbb{P}\left( e^{1/\alpha} \mathcal{X}_{t_1/\epsilon} \geq x_1, \ldots, e^{1/\alpha} \mathcal{X}_{t_n/\epsilon} \geq x_n \right),
\]

from which we deduce, by Theorem 10, that

\[
\liminf_{\epsilon \to 0} \mathbb{P}\left( e^{1/\alpha} X_{t_1/\epsilon} \geq x_1, \ldots, e^{1/\alpha} X_{t_n/\epsilon} \geq x_n \right) \geq \mathbb{P}\left( R^\alpha_{t_1} \geq x_1, \ldots, R^\alpha_{t_n} \geq x_n \right).
\]

Let us now consider the process \((X_t, 2\mathcal{M}_t)_{t \geq 0}\) starting at \((0, v_0)\), recall (10), built from the process \((X_t, V_t)_{t \geq 0}\), as well as the time change \((A'_t)_{t \geq 0} = (\int_0^t 1_{\{X_s > 0\}} ds)_{t \geq 0}\) and its right-continuous inverse \((T'_t)_{t \geq 0}\). Then by Theorem 14, the process \((X_t, V_t)_{t \geq 0} = (X_{T'_t}, V_{T'_t})_{t \geq 0}\) is a solution of (3). By Proposition 15, we have

\[
(eT'_{t_1/\epsilon}, \ldots, eT'_{t_n/\epsilon}) \overset{P}{\to} (t_1, \ldots, t_n) \quad \text{as} \ \epsilon \to 0. \tag{18}
\]

Let \(\delta > 0\) such that for every \(k \in \{1, \ldots, n\}, \delta < t_k\). We introduce the events

\[
A_{\epsilon} = \bigcap_{k=1}^n \left\{ e^{1/\alpha} X_{t_k/\epsilon} \geq x_k \right\}, \quad B_{\epsilon, \delta} = \bigcap_{k=1}^n \left\{ |eT'_{t_k/\epsilon} - t_k| \leq \delta \right\},
\]

and

\[
C_{\epsilon, \delta} = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} e^{1/\alpha} X_{s/\epsilon} \geq x_k \right\}, \quad D_{\epsilon, \delta} = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} e^{1/\alpha} \mathcal{X}_s_{/\epsilon} \geq x_k \right\}.
\]

By (18), it is clear that \(\mathbb{P}(B_{\epsilon, \delta}) \to 0\) as \(\epsilon \to 0\). We easily see that \(A_{\epsilon} \cap B_{\epsilon, \delta} \subset C_{\epsilon, \delta}\), and \(C_{\epsilon, \delta} \subset D_{\epsilon, \delta}\) by Proposition 13. Therefore \(\mathbb{P}(A_{\epsilon} \cap B_{\epsilon, \delta}) \leq \mathbb{P}(C_{\epsilon, \delta}) \leq \mathbb{P}(D_{\epsilon, \delta})\). As a consequence, we have \(\mathbb{P}(A_{\epsilon}) \leq \mathbb{P}(D_{\epsilon, \delta}) + \mathbb{P}(B_{\epsilon, \delta})\), whence

\[
\limsup_{\epsilon \to 0} \mathbb{P}(A_{\epsilon}) \leq \limsup_{\epsilon \to 0} \mathbb{P}(D_{\epsilon, \delta}).
\]

We claim that the following convergence holds

\[
\lim_{\epsilon \to 0} \mathbb{P}(D_{\epsilon, \delta}) = \mathbb{P}(D_{\delta}) \quad \text{where} \quad D_{\delta} = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} R^\alpha_s \geq x_k \right\}. \tag{19}
\]

Since a.s., \(t_1, \ldots, t_n\) are not jumping times of \((R^\alpha_t)_{t \geq 0}\), \(\mathbb{P}(D_{\delta}) \to \mathbb{P}(R^\alpha_{t_1} \geq x_1, \ldots, R^\alpha_{t_n} \geq x_n)\) as \(\delta \to 0\). Hence (19) would imply

\[
\limsup_{\epsilon \to 0} \mathbb{P}\left( e^{1/\alpha} X_{t_1/\epsilon} \geq x_1, \ldots, e^{1/\alpha} X_{t_n/\epsilon} \geq x_n \right) \leq \mathbb{P}\left( R^\alpha_{t_1} \geq x_1, \ldots, R^\alpha_{t_n} \geq x_n \right),
\]

which would achieve the first step. We now show that (19) holds.

By Theorem 10 and Skorokhod’s representation theorem, there exist a family of processes \((X'_t)_{t \geq 0}\) indexed by \(\epsilon > 0\), and a reflected symmetric stable process \((R^\alpha_t)_{t \geq 0}\), both defined on the probability space \(([0, 1], \mathcal{B}([0, 1]), \lambda)\), where \(\lambda\) denotes the Lebesgue measure on \([0, 1]\), such that for every \(\epsilon > 0\), \((X'_t)_{t \geq 0} \overset{d}{=} (X_{t/\epsilon})_{t \geq 0}\) and such that,

\[
\lambda - \text{a.s.,} \quad d_{M_1}(e^{1/\alpha} X'_{t/\epsilon})_{t \geq 0}, (R^\alpha_t)_{t \geq 0} \overset{\epsilon \to 0}{\longrightarrow} 0.
\]

We now use the fact the \(M_2\)-topology, originally introduced by Skorokhod in his seminal paper [43], is weaker than the \(M_1\)-topology. The convergence in \(D\) endowed with \(M_2\) can be
characterized, see [43, page 267], as follows: a sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\) in \(D\) endowed with the \(\mathbf{M}_2\) if and only if
\[
\inf_{u \in [s,t]} x_n(u) \to \inf_{u \in [s,t]} x(u) \quad \text{and} \quad \sup_{u \in [s,t]} x_n(u) \to \sup_{u \in [s,t]} x(u)
\]
for any \(0 \leq s < t\) points of continuity of \(x\). Therefore, since \(\lambda\text{-a.s., for any } k \in \{1, \ldots, n\}, t_k - \delta\)
and \(t_k + \delta\) are points of continuity of \((R^n_t)_{t \geq 0}\), we deduce that \(\lambda\text{-a.s., for any } k \in \{1, \ldots, n\}\),
the following convergence holds
\[
\sup_{s \in [t_k - \delta, t_k + \delta]} \epsilon^{1/\alpha} X_s^\epsilon \to \sup_{s \in [t_k - \delta, t_k + \delta]} R^n_s, \quad \text{as } \epsilon \to 0,
\]
which implies (19).

**Step 2:** We now grant Assumption 2-(ii) and we seek to show that, under this assumption,
\((\epsilon^{1/\alpha} X_{t/\epsilon})_{t \geq 0}\) is tight for the \(\mathbf{M}_1\)-topology. Here we use the representation \((X_t, V_t)_{t \geq 0} = (X^T_t, V^T_t)_{t \geq 0}\), see Step 1. Our goal is to show that the conditions of Theorem 6-(ii) are satisfied.

By the first step, it suffices to show that for any \(\eta > 0\), for any \(T > 0\),
\[
\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \mathbb{P}\left( w(\epsilon^{1/\alpha} X_{t/\epsilon}, T, \delta) > \eta \right) = 0,
\]
(20)

By Proposition 17, the process \((\epsilon^{1/\alpha} X_{t/\epsilon})_{t \geq 0}\) is tight and by Proposition 15 and Dini’s theorem,
we have the following convergence in probability:
\[
\sup_{t \in [0,T]} \left| \epsilon^{T_{t/\epsilon}} - t \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \epsilon \to 0.
\]

Let \(T > 0\) and \(\delta \in (0, 1)\), we will first place ourselves on the event \(A^\delta_{T, \delta} = \{ \sup_{t \in [0,T]} |\epsilon^{T_{t/\epsilon}} - t| < \delta \}\). On this event, \(\epsilon^{T_{t/\epsilon}} \in ((t - \delta)/\epsilon, (t + \delta)/\epsilon)\) for any \(t \in [0,T]\). Let \(t \in [0,T]\) and recall that \(t_{\delta}^- = 0 \vee (t - \delta)\) and \(t_{\delta}^+ = T \wedge (t + \delta)\). We set \(t_{\delta}^- = 0 \vee (t - 2\delta)\) and (abusively) \(t_{2\delta}^+ = (T + 1) \wedge (t + 2\delta)\). Then on the event \(A^\delta_{T, \delta}\), we have
\[
\sup_{t_{\delta}^- \leq t_2 < t_4 \leq t_{\delta}^+} \frac{d(X_{t_4'}/t/\epsilon, X_{t_4'/t/\epsilon})}{d(X_{t_2'}/t/\epsilon, X_{t_2'/t/\epsilon})} \leq \sup_{t_{2\delta}^- \leq t_2 < t_4 \leq t_{2\delta}^+} \frac{d(X_{t_4'}/t/\epsilon, X_{t_4'/t/\epsilon})}{d(X_{t_2'}/t/\epsilon, X_{t_2'/t/\epsilon})},
\]
from which we deduce that
\[
w(\epsilon^{1/\alpha} X_{t/\epsilon}, T, \delta) \leq w(\epsilon^{1/\alpha} X_{t/\epsilon}, T + 1, \delta)
\]
on the event \(A^\delta_{T, \delta}\). As a consequence, we have for any \(\eta, \delta > 0\) and for any \(T > 0\)
\[
\mathbb{P}\left( w(\epsilon^{1/\alpha} X_{t/\epsilon}, T, \delta) > \eta \right) \leq \mathbb{P}\left( w(\epsilon^{1/\alpha} X_{t/\epsilon}, T + 1, \delta) > \eta \right) + \mathbb{P}\left( \sup_{t \in [0,T]} |\epsilon^{T_{t/\epsilon}} - t| \geq \delta \right).
\]
Therefore, since \((\epsilon^{1/\alpha} X_{t/\epsilon})_{t \geq 0}\) is tight, we have for any \(\eta > 0\), for any \(T > 0\)
\[
\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \mathbb{P}\left( w(\epsilon^{1/\alpha} X_{t/\epsilon}, T, \delta) > \eta \right) = 0,
\]
which completes the proof.
4.6 Some persistence problems

The aim of this subsection is to prove Lemma 16. We recall that the random variable \((\tau, \sigma)\) are defined as follows: let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space supporting an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion \((B_t)_{t \geq 0}\) and two independent \(\mu\)-distributed random variables \(V_0\) and \(M\), also independent of the Brownian motion. Consider the process \((X_t, V_t)_{t \geq 0}\) solution of (1) starting at \((0, V_0)\). During the whole subsection, \(\alpha = (\beta + 1)/3\). Then \(\tau\) and \(\sigma\) are defined as

\[
\tau = \inf\{t > 0, X_t = 0\} \quad \text{and} \quad \sigma = \inf\{t \geq \tau, V_t = M\}.
\]

We first introduce the random times

\[
T_0 = \inf\{t \geq 0, V_t = 0\} \quad \text{and} \quad \rho = \inf\{t \geq \tau, V_t = 0\}.
\]

It should be clear that a.s.,

\[
T_0 \leq \tau \leq \rho \leq \sigma,
\]

see Figure 2. Indeed since \(V_t\) is strictly positive on \((0, T_0)\), so is \(X_t\) and \(T_0 \leq \tau\). Moreover, since \(V_\tau\) is non-positive, \(M > 0\) and \((V_t)_{t \geq 0}\) is continuous, \(\rho \leq \sigma\). Lemma 16 heavily relies on the two following lemmas.

**Lemma 18.**

(i) Under Assumption 2-(i), we can find some \(\delta \in (0, 1/2)\) such that \(\mathbb{E}[T_0^{1/2+\delta}] < \infty\) and \(\mathbb{E}[(\sigma - \rho)^{1/2 + \delta}] < \infty\). As a consequence, we have

\[
\lim_{t \to \infty} t^{1/2} \mathbb{P}(T_0 > t) = \lim_{t \to \infty} t^{1/2} \mathbb{P}(\sigma - \rho > t) = 0.
\]

(ii) Under Assumption 2-(ii), there exists \(\delta' \in (0, 1/\alpha - 2)\) such that \(\mathbb{E}[(M(\sigma - \rho))^{\alpha/2 + \delta'}] < \infty\).

**Lemma 19.** Grant Assumptions 2-(i). Then there exists a constant \(C > 0\) such that

\[
\mathbb{P}(\tau - T_0 > t) \sim \mathbb{P}(\rho - T_0 > t) \sim Ct^{-1/2} \quad \text{as} \quad t \to \infty.
\]

**Proof of Lemma 16.** Using Lemma 31 in Appendix A with \(X = \tau - T_0\) and \(Y = T_0\) and Lemmas 18-(i) and 19, it comes

\[
\mathbb{P}(\tau > t) \sim Ct^{-1/2} \quad \text{as} \quad t \to \infty.
\]

By Lemma 31 again with \(X = \rho - T_0\) and \(Y = \sigma - \rho + T_0\), we get

\[
\mathbb{P}(\sigma > t) \sim Ct^{-1/2} \quad \text{as} \quad t \to \infty,
\]

which completes the proof.
We now seek to show Lemma 18. To do so, we will use Feller’s representation of regular diffusions i.e. we represent the velocity process through its scale function $s$ and its speed measure $m$:

$$s(v) = \int_0^v \Theta^{-\beta}(u)du \quad \text{and} \quad m(v) = \Theta^{\beta}(v).$$

Remember from Assumption 1 that $\Theta : \mathbb{R} \rightarrow (0, \infty)$ is a $C^1$ even function such that $F = \frac{\beta}{2} \Theta'$ and satisfying $\lim_{v \rightarrow \pm \infty} |v| \Theta(v) = 1$. The function $s$ is an increasing bijection from $\mathbb{R}$ to $\mathbb{R}$ and we denote by $s^{-1}$ its inverse. We also define the function $\psi = s' \circ s^{-1}$. Now consider another Brownian motion $(W_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ (or an enlargement of the space) and for $v \in \mathbb{R}$, we set

$$W^v_t = s(v) + W_t, \quad A^v_t = \int_0^t [\psi(W^v_s)]^{-2}ds \quad \text{as well as} \quad \rho^v_t = \inf \{s > 0, A^v_s > t\}.$$

Then the process defined by $V^v_t = s^{-1}(W^v_t)$ and $X^v_t = \int_0^t V^v_s ds$ is a solution of (1) starting at $(v, 0)$. This result is standard and we refer to Kallenberg [32, Chapter 23, Theorem 23.1 and its proof] for more details. As a consequence, using the substitution $u = \rho^v_s$, we can write that almost surely, for any $t \geq 0$,

$$X^v_t = \int_0^t s^{-1}(W^v_s)ds = \int_0^{\rho^v_t} \phi(W^v_s)ds,$$

where for any $v \in \mathbb{R}$, $\phi(v) = s^{-1}(v)/\psi^2(v)$. We set, for any $v \in \mathbb{R}$, $T^v_0 = \inf \{t \geq 0, V^v_t = 0\}$, the first hitting time of $(V^v_t)_{t \geq 0}$ at the level 0. We also note that since $s(v) \rightarrow \pm \infty$ as $v \rightarrow \pm \infty$ and $\int_{\mathbb{R}} m(v)dv < \infty$ because $\beta > 1$, the process $(V^v_t)_{t \geq 0}$ is a positive recurrent diffusion. Note that Assumption 1 yields the following asymptotics:

$$m(v) \sim v^{-\beta}, \quad s(v) \sim (\beta + 1)^{-1}v^{\beta+1}, \quad \phi(v) \sim (\beta + 1)^{\frac{1-2\beta}{\beta+\gamma}}v^{\frac{1-2\beta}{\beta+\gamma}} \quad \text{as} \quad v \rightarrow \infty. \quad (22)$$

Finally, we stress that by the strong Markov property, since $V_0$ is $\mu$-distributed, for any non-negative functional $G$ of continuous functions, we have

$$\mathbb{E}[G((V_t)_{t \geq 0})] = \int_0^\infty \mathbb{E}[G((V^v_t)_{t \geq 0})] \mu(dv). \quad (23)$$

Proof of Lemma 18. Step 1: We first deal with $T_0$. Let $\delta \in (0, 1/2)$, then by (23) and the H"{o}lder inequality, we have

$$\mathbb{E}\left[T_0^{1/2+\delta}\right] = \int_0^\infty \mathbb{E}\left[\left(T^v_0\right)^{1/2+\delta}\right] \mu(dv) \leq \int_0^\infty \mathbb{E}\left[T^v_0\right]^{1/2+\delta} \mu(dv).$$

We will show that the quantity on the right-hand-side is finite if $\delta$ is small enough and to do so, we need to understand the behavior of $\mathbb{E}[T^v_0]$ as $v$ tends to infinity. We use Kac’s moment formula (see for instance L"{o}cherbach [36, Corollary 3.5]) which, applied to our case, tells us that

$$\mathbb{E}[T^v_0] = s(v) \int_v^\infty m(u)du + \int_0^v s(u)m(u)du \leq s(v) \int_\mathbb{R} m(u)du + \int_0^v s(u)m(u)du. \quad (24)$$

By (22), we deduce that $\int_0^v s(u)m(u)du \sim (2(\beta + 1))^{-1}v^2$ as $v \rightarrow \infty$. Since $s(v) \sim (\beta + 1)^{-1}v^{\beta+1}$ and $\beta + 1 > 2$, the dominant term on the right-hand-side in (24) is the first one and we deduce that there exists some positive constant $K$ such that for any $v \geq 0$, $\mathbb{E}[T^v_0] \leq K(1 + v^{\beta+1})$. Hence we have

$$\mathbb{E}\left[T_0^{1/2+\delta}\right] \leq \int_0^\infty \mathbb{E}\left[T^v_0\right]^{1/2+\delta} \mu(dv) \leq K^{1/2+\delta} \int_0^\infty (1 + v^{\beta+1})^{1/2+\delta} \mu(dv),$$

which is finite by Assumption 2-(i) if $\delta = \eta/\beta + 1$. 

23
Step 2: Note that the random time \( \sigma - \rho \) only depends on the speed process. More precisely, since \( \rho \) is a stopping time and \( V_\rho = 0 \), the process \( (V_t + \rho)_t \geq 0 \) is a solution of \( dY_t = F(Y_t)dt + dB_t \) starting at 0, and thus, \( \sigma - \rho \) is equal in law to \( T_M^0 = \inf \{ t \geq 0, V_t^0 = M \} \). Since \( M \) is independent of \( (V_t)_t \geq 0 \), we can write, as in the first step, that
\[
\mathbb{E} \left[ (\sigma - \rho)^{1/2+\delta} \right] = \int_0^\infty \mathbb{E} \left[ (T_v^0)^{1/2+\delta} \right] \mu(dv) \leq \int_0^\infty \mathbb{E} \left[ T_v^0 \right]^{1/2+\delta} \mu(dv),
\]
where \( T_v^0 = \inf \{ t \geq 0, V_t^0 = v \} \). We use again Kac’s moment formula, see [36, Corollary 3.5]:
\[
\mathbb{E} \left[ T_v^0 \right] = s(v) \int_\mathbb{R} m(u)du - s(v) \int_v^\infty m(u)du - \int_0^v s(u)m(u)du \leq s(v) \int_\mathbb{R} m(u)du,
\]
and we can conclude as in the previous step that \( \mathbb{E} \left[ T_v^0 \right] \leq K(1 + \nu^{\beta+1}) \), whence \( \mathbb{E}[(\sigma - \rho)^{1/2+\delta}] \) is finite with the same value of \( \delta \).

Step 3: We deal with the second item and we grant Assumption 2-(ii). Let \( \delta' \in (0, 1 - \alpha/2) \).
Since \( M \) is independent of \( (V_t)_t \geq 0 \) and \( \mu \) is the law of \( M \), we get
\[
\mathbb{E} \left[ (M(\sigma - \rho)\nu^{\beta+\delta})^{1/2+\delta'} \right] = \int_0^\infty \nu^{\alpha/2+\delta'} \mathbb{E} \left[ (T_v^0)^{1/2+\delta'} \right] \mu(dv) \leq \int_0^\infty \nu^{\alpha/2+\delta'} \mathbb{E} \left[ T_v^0 \right]^{1/2+\delta'} \mu(dv).
\]
Then, since \( \mathbb{E} \left[ T_v^0 \right] \leq K(1 + \nu^{\beta+1}) \) as in Step 2,
\[
\mathbb{E} \left[ (M(\sigma - \rho)\nu^{\beta+\delta})^{1/2+\delta'} \right] \leq K^{\alpha/2+\delta'} \int_0^\infty \nu^{\alpha/2+\delta'} (1 + \nu^{\beta+1})^{\alpha/2+\delta'} \mu(dv)
\]
which is finite by Assumption 2-(ii) if \( \delta' = \eta/(\beta + 2) \), recall that \( \alpha = (\beta + 1)/3 \).

We now seek to show Lemma 19, and we will need to estimate the moments of \( X_{T_0^u} \).

Lemma 20. Grant Assumption 2-(i), then we have \( \mathbb{E} \left[ X_{T_0^u}^{\alpha/2} \right] < \infty \).

Proof. For any \( v > 0 \), we set \( \theta_v^0 = \inf \{ t \geq 0, W_t^v = 0 \} \). Since \( s(v) \geq 0 \) if and only if \( v \geq 0 \), and \( V_t^v = s^{-1}(W_{t/\theta_v^0}) \) with \( (\rho_t^v)_{t \geq 0} \) the inverse of \( (A_t^v)_{t \geq 0} \), it should be clear that \( \mathbb{P}(T_v^0 = A_v^0) = 1 \).
Hence, following (21), we have for any \( v > 0 \), almost surely
\[
X_{T_v^0}^v = \int_0^{\theta_v^0} \phi(W_{\tau}) d\tau.
\]

Step 1: We first introduce the non-negative random variable
\[
Z_{v,\alpha} = \int_0^{\theta_v^0} (W_s^v)^{1/\alpha-2} ds.
\]

By the scaling of the Brownian motion, the law of \( (W_t^v, 0 \leq t \leq \theta_v^0) \) is the same as the law of \( (s(v)W_t^{v^{-1}(1)}, 0 \leq t \leq |s(v)|^2\theta_v^{v^{-1}(1)}(1)) \), and as a consequence,
\[
\text{the law of } Z_{v,\alpha} \text{ is the same as the law of } |s(v)|^{1/\alpha} Z_{v^{-1}(1),\alpha}.
\]

In fact, the law of \( Z_{v^{-1}(1),\alpha} \) is explicit and it holds that \( Z_{v^{-1}(1),\alpha} \) has the same law as \( \alpha^2/G_{\alpha} \) where \( G_{\alpha} \) is a random variable whose law is the Gamma distribution of parameter \( (\alpha, 1) \), see for instance Letemplier-Simon [35, page 93]. In particular, we have \( \mathbb{P}(Z_{v^{-1}(1),\alpha} \geq x) \sim cx^{-\alpha} \) as \( x \to \infty \) for some \( c > 0 \). Therefore, it holds that \( \mathbb{E}[Z_{v^{-1}(1),\alpha}] < \infty \).

Step 2: We conclude. Since \( \alpha = (\beta + 1)/3 \), we have \( 1/\alpha - 2 = (1 - 2\beta)/(\beta + 1) \) and by (22), there exists a constant \( c > 0 \) such that \( \phi(v) \sim cv^{1/\alpha-2} \) as \( v \to \infty \). Remark that \( 1/\alpha - 2 < 0 \) as \( \beta > 1 \), and therefore there exists a constant \( C > 0 \) such that for any \( v \geq 0 \), \( \phi(v) \leq Cv^{1/\alpha-2} \).
As a consequence, we have for any $v > 0$, almost surely $0 < X_{T_0}^v < CZ_{v, \alpha}$. Putting the pieces together, by (23) and the previous step, we can write

$$
\mathbb{E}\left[X_{T_0}^{\alpha/2}\right] = \int_0^\infty \mathbb{E}\left[(X_{T_0}^v)^{\alpha/2}\right] \mu( dv) \leq C^{\alpha/2} \int_0^\infty \mathbb{E}\left[(Z_{v, \alpha})^{\alpha/2}\right] \mu( dv)
$$

whence

$$
\mathbb{E}\left[X_{T_0}^{\alpha/2}\right] \leq C^{\alpha/2}\mathbb{E}[Z_{s-1(1, \alpha)}^{\alpha/2}] \int_0^\infty [s(v)]^{1/2} \mu( dv).
$$

We can conclude since there exists a constant $K > 0$ such that for any $v > 0$, $s(v) \leq K(1+v)^{\beta+1}$, see the proof of Lemma 18, and $\int_0^\infty (1+v)^{(\beta+1)/2} \mu( dv) < \infty$ by Assumption 2-(i).

We now introduce the process $(\bar{V}_t, \bar{X}_t)_{t \geq 0} = (V_{t+T_0}, X_{t+T_0} - X_{T_0})_{t \geq 0}$ which is a solution of (1) starting at $(0,0)$ and is independent of $X_{T_0}$. We emphasize that, since the restoring force $F$ is odd by assumption, the processes $(\bar{V}_t)_{t \geq 0}$ and $(\bar{X}_t)_{t \geq 0}$ are symmetric. We also stress that the stopping times $\tau - T_0 = \inf\{ t > 0, \bar{X}_t = -X_{T_0}\}$ and $\rho - T_0 = \inf\{ t \geq \tau - T_0, \bar{V}_t = 0\}$ only depend on $(\bar{V}_t)_{t \geq 0}$, $(\bar{X}_t)_{t \geq 0}$ and $X_{T_0}$. Let us introduce the supremum and infimum of $(\bar{X}_t)_{t \geq 0}$: $\xi_t = \sup_{s \in [0,t]} \bar{X}_s$ and $\Lambda_t = \inf_{s \in [0,t]} \bar{X}_s$. Let us also define for $t \geq 0$

$$
g_t = \sup\{ s \leq t, \bar{V}_s = 0\} \quad \text{and} \quad d_t = \inf\{ s \geq t, \bar{V}_s = 0\}.
$$

Then we have the following inclusions of events:

$$
\{ \Lambda_t > -X_{T_0}\} \subset \{ \tau - T_0 > t\} \subset \{ \rho - T_0 > t\} \subset \{ \Lambda_t > -X_{T_0}\}.
$$

The first two inclusions are straightforward since $\{ \tau - T_0 > t\} = \{ \Lambda_t > -X_{T_0}\}$, $d_t \geq t$ and $\tau \leq \rho$. Remember that $\rho - T_0$ is the first zero of $\bar{V}_t$ after $\tau - T_0$ and thus it should be clear that $\{ \Lambda_{g_t} \leq -X_{T_0}\} = \{ \tau - T_0 \leq d_t\} \subset \{ \rho - T_0 \leq g_t\}$, which establishes the third inclusion since $g_t \leq t$. Since $(\bar{X}_t)_{t \geq 0}$ is symmetric, we then have

$$
\mathbb{P}(\xi_{d_t} < X_{T_0}) \leq \mathbb{P}(\tau - T_0 > t) \leq \mathbb{P}(\rho - T_0 > t) \leq \mathbb{P}(\xi_{g_t} < X_{T_0}).
$$

We will show the following lemma which, combined with (26), immediately implies Lemma 19.

**Lemma 21.** Grant Assumptions 2-(i). Then there exists a constant $C > 0$ such that

$$
\mathbb{P}(\xi_{g_t} < X_{T_0}) \sim \mathbb{P}(\xi_{d_t} < X_{T_0}) \sim t^{-1/2}.
$$

To prove this result, we will heavily use some results developed in [3], which rely on works about Itô’s excursion theory and the links with Lévy processes, in particular by Bertoin [5] and Vallois, Salminen and Yor [42]. The proof is rather long and we will segment it (again) into smaller pieces, see Lemmas 25 and 26 below.

We use the following standard trick: let $e = e(q)$ be an exponential random variable of parameter $q > 0$, independent of everything else, then we will look at the quantities $\mathbb{P}(\xi_{g_{e}} < X_{T_0})$ and $\mathbb{P}(\xi_{d_t} < X_{T_0})$ instead of looking at $\mathbb{P}(\xi_{g_t} < X_{T_0})$ and $\mathbb{P}(\xi_{d_t} < X_{T_0})$. We have nothing to lose doing this since a combination of the Tauberian theorem and the monotone density theorem (see Theorem 24 below) tells us that having an asymptotic of $\mathbb{P}(\xi_{g_t} < X_{T_0})$ (respectively $\mathbb{P}(\xi_{d_t} < X_{T_0})$) as $t \to \infty$ is equivalent to having an asymptotic of $\mathbb{P}(\xi_{g_{e}} < X_{T_0})$ (respectively $\mathbb{P}(\xi_{d_{e}} < X_{T_0})$) as $q \to 0$, and we will first study $\mathbb{P}(\xi_{g_{e}} < x)$ and $\mathbb{P}(\xi_{d_{e}} < x)$ for a fixed $x > 0$.

The first reason we do this is that it brings independence between the quantities we are interested in. The second reason is that by doing this, we actually have explicit formulas for some quantities of interest, for instance for $\mathbb{P}(\xi_{g_{e}} < x)$ and, as we will see, there is a strong link with some Lévy process associated to $(\bar{X}_t, \bar{V}_t)_{t \geq 0}$ and fluctuation’s theory for Lévy processes, see for instance [5, Chapter VI].
The velocity process \((\tilde{V}_t)_{t \geq 0}\) possesses a local time at 0 and we will denote by \((\gamma_t)_{t \geq 0}\) its right-continuous inverse. The latter is a subordinator and we will denote by \(\Phi\) its Laplace exponent, i.e. \(E[e^{-\gamma s}] = \exp(-t\Phi(q))\). The process \((\tilde{V}_t)_{t \geq 0}\) being positive recurrent, we have \(E[\gamma_1] < \infty\), see [6, Chapter 2, page 22], and we choose to normalize the local time so that \(E[\gamma_1] = 1\), whence \(\Phi(q) \sim q\) as \(q \to 0\). The strong law of large number for subordinators entails that a.s. \(t^{-1}\gamma_t \to 1\) as \(t \to \infty\). We will first prove the following which tells us that we only need to study \(P(\xi_{ge} < x)\).

**Lemma 22.** There exists a function \(f: (0, \infty) \to [0, 1]\) such that \(f(q) \to 1\) as \(q \to 0\) and such that for any \(q, x > 0\),

\[
P(\xi_{ge} < x) f(q) \leq P(\xi_{de} < x) \leq P(\xi_{ge} < x).
\]  

(27)

**Proof.** Let us define for any \(t > 0\) the processes

\[
I_t = \int_{g_t}^{d_t} V_s ds = \bar{X}_{d_t} - \bar{X}_{g_t} \quad \text{and} \quad \Delta_t = I_t + \bar{X}_{g_t} - \xi_{ge} = \bar{X}_{d_t} - \xi_{ge}.
\]

Then we can express \(\xi_{de}\) in terms of \(\xi_{ge}\) and \(\Delta_t\): remarking that \((\bar{X}_t)_{t \geq 0}\) is monotonic on every excursion of \((\tilde{V}_t)_{t \geq 0}\), we get that \(\sup_{s \in [g_t, d_t]} X_s = \bar{X}_{g_t} \vee \bar{X}_{d_t}\). Therefore, if \(\Delta_t \leq 0\), then \(\bar{X}_{d_t} \leq \xi_{ge}\) and \(\xi_{de} = \xi_{ge}\). On the other hand, if \(\Delta_t > 0\), then \(\xi_{de} = \bar{X}_{d_t} = \xi_{ge} + \Delta_t\). All in all, we have

\[
\xi_{de} = \xi_{ge} \mathbb{1}_{\{\Delta_t \leq 0\}} + (\xi_{ge} + \Delta_t) \mathbb{1}_{\{\Delta_t > 0\}}.
\]  

(28)

We can factorize functionals of the trajectories of \((\tilde{V}_t)_{t \geq 0}\) before time \(g_t\) and functionals of the trajectories between \(g_t\) and \(d_t\). More precisely, it is shown in [42, Theorem 9] that the processes \((\tilde{V}_u)_{0 \leq u \leq g_t}\) and \((\tilde{V}_{u+g_t})_{0 \leq u \leq d_t-g_t}\) are independent. Therefore, \(I_t\) is independent of \((\xi_{ge}, \bar{X}_{ge} - \xi_{ge})\). Moreover, it is shown in [3, Corollary 4.6] that \(\xi_{ge}\) and \(\bar{X}_{ge} - \xi_{ge}\) are i.i.d., see also Lemma 23 below. Hence the random variables \(I_t, \xi_{ge}\) and \(\bar{X}_{ge} - \xi_{ge}\) are mutually independent and thus \(\xi_{ge}\) is independent of \(\Delta_t = I_t + \bar{X}_{ge} - \xi_{ge}\). We set \(f(q) = P(\Delta_t \leq 0)\) and we deduce from (28) that (27) holds for any \(x > 0\).

Let us now show that \(f(q) \to 1\) as \(q \to 0\) and let us denote by \(n\) the excursion measure of \((\tilde{V}_t)_{t \geq 0}\) away from zero. Let \(E\) the set of excursions, i.e. the set of continuous functions \(\varepsilon = (\varepsilon_t)_{t \geq 0}\) such that \(\varepsilon_0 = 0\) and such that there exists \(\ell(\varepsilon) > 0\) for which \(\varepsilon_s \neq 0\) for every \(s \in (0, \ell(\varepsilon))\) and \(\varepsilon_s = 0\) for every \(s \geq \ell(\varepsilon)\). Then by Theorem 9 in [42], we have for any measurable bounded function \(G: \mathbb{R} \to \mathbb{R}\),

\[
E[G(I_e)] = \frac{1}{\Phi(q)} \int_E G(\int_0^{\ell(\varepsilon)} \varepsilon_s ds) (1 - e^{-q\ell(\varepsilon)}) n(d\varepsilon).
\]

But \(\Phi(q) \sim q\) as \(q \to 0\) and \(\int_E \ell(\varepsilon)n(d\varepsilon) = 1 < \infty\) since \(E[\gamma_1] = 1\) (by a direct application of the Master formula in the context of excursion theory), we get by dominated convergence that

\[
E[G(I_e)] \xrightarrow[q \to 0]{} \int_E G(\int_0^{\ell(\varepsilon)} \varepsilon_s ds) \ell(\varepsilon)n(d\varepsilon),
\]

and thus \(I_e\) converges in law as \(q \to 0\).

Remember that \(\xi_{ge} - \bar{X}_{ge}\) is equal in law to \(\xi_{ge}\). Moreover \(\xi_{ge} \to \infty\) in probability as \(q \to 0\) because \(g_t\) tends to infinity in probability as \(q \to 0\) and because \(\xi_t = \sup_{s \in [0, t]} X_s\) tends to infinity in probability (Theorem 7 clearly implies that \(t^{-1/\alpha}I_t\) converges in law as \(t \to \infty\)). Since \(I_e\) converges in law and \(\xi_{ge} - \bar{X}_{ge}\) converges in probability to \(\infty\) as \(q \to 0\), we have that \(\Delta_t = I_e - (\xi_{ge} - \bar{X}_{ge})\) converges to \(-\infty\) in probability, so that \(P(\Delta_e \leq 0) = f(q)\) converges to 1 as \(q \to 0\). \(\square\)

To handle the quantity \(P(\xi_{ge} < x)\), we will rely on a Wiener-Hopf factorization of the bivariate Lévy process \((\gamma_t, Z_t)_{t \geq 0} = (\gamma_t, X_{g_t})_{t \geq 0}\), which is developed in [3, Appendix A]. Let us denote by \((S_t)_{t \geq 0}\) the supremum process of \((Z_t)_{t \geq 0}\), i.e. \(S_t = \sup_{s \in [0, t]} Z_s\) and let \((R_t)_{t \geq 0} = (S_t - Z_t)_{t \geq 0}\)
be the reflected process, which is a strong Markov process that also possesses a local time at 0, see [5, Chapter VI] and we denote by \((\sigma_t)_{t \geq 0}\) its right-continuous inverse. The process \((\sigma_t, \theta_t, H_t)_{t \geq 0} = (\sigma_t, \gamma_t, S_{\sigma_t})_{t \geq 0}\) is a trivariate subordinator, see [3, Lemma A.1 in Appendix A]. Its Laplace exponent is denoted by \(\kappa\), i.e. for any \(\alpha, \beta, \delta \geq 0\),

\[
\mathbb{E}
\left[
\exp
\left(-\alpha \sigma_t - \beta \theta_t - \delta H_t\right)
\right]
= \exp\left(-t \kappa(\alpha, \beta, \delta)\right).
\]

Finally we introduce the renewal function \(\mathcal{V}\) defined on \([0, \infty)\) by

\[
\mathcal{V}(x) = \int_{0}^{\infty} \mathbb{P}(H_t \leq x) dt,
\]

which is non-decreasing and right-continuous. We state the following lemma, which is borrowed from [3, Proposition 4.4 and Corollary 4.6], which we will use several times.

**Lemma 23.** There exists a constant \(k > 0\) such that for every \(\alpha, \beta, \delta \geq 0\), we have the following Fristedt formula

\[
\kappa(\alpha, \beta, \delta) = k \exp\left(\int_{0}^{\infty} \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-at - \beta r - \delta x}}{t} \mathbbm{1}_{\{x \geq 0\}} \mathbb{P}(\gamma_t \in dr, Z_t \in dx) dt\right).
\]

Recall that \(e = e(q)\) is an exponential random variable of parameter \(q > 0\) independent of everything else. Then for every \(\lambda, \mu \geq 0\), we have

\[
\mathbb{E}\left[e^{-\lambda \xi_{g_e} - \mu (\xi_{g_e} - \bar{X}_{g_e})}\right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)} \frac{\kappa(0, q, 0)}{\kappa(0, q, \mu)}.
\]

In particular, \(\xi_{g_e}\) and \(\xi_{g_e} - \bar{X}_{g_e}\) are independent and have the same law.

Note that, when applying [3, Corollary 4.6], we used the fact that \((Z_t)_{t \geq 0}\) is symmetric so that if \((\bar{Z}_t)_{t \geq 0} = (-Z_t)_{t \geq 0}\) is the dual process, then the corresponding Laplace exponent \(\hat{\kappa} = \kappa\). Let us finally introduce two last tools that are key to the proof of our result and that we will use several times. Let us first remind Frullani’s identity which holds for every \(b \in (0, 1)\), see for instance [5, page 73]:

\[
\log b = \int_{0}^{\infty} \frac{e^{-x} - e^{-bx}}{x} dx.
\]

Let us also remind the following classical theorem, which can be obtained by a careful application of the classical Karamata’s Tauberian theorem and the monotone density theorem, see for instance [12, Theorem 1.7.1 page 37 and Theorem 1.7.2 page 39].

**Theorem 24.** Let \(u : \mathbb{R}_+ \to \mathbb{R}_+\) be a monotone function. Let us denote by \(\mathcal{L}u\) its Laplace transform i.e. for any \(\lambda > 0\), \(\mathcal{L}u(\lambda) = \int_{0}^{\infty} e^{-\lambda x} u(x) dx\). Let \(\rho > -1\) and \(\Gamma\) be the usual Gamma function, then the two following assertions are equivalent.

(i) \(u(x) \sim (\rho + 1) x^\rho\) as \(x \to \infty\).

(ii) \(\lambda \mathcal{L}u(\lambda) \sim \Gamma(\rho + 2) \lambda^{-\rho}\) as \(\lambda \to 0\).

We have the following lemma.

**Lemma 25.** Recall that \(e = e(q)\) is an exponential random variable of parameter \(q > 0\) independent of everything else. There exists \(k > 0\) such that for any \(x > 0\), we have

\[
\mathbb{P}(\xi_{g_e} < x) \sim q \mathbb{P}(\xi_{d_e} < x) \sim k q^{1/2} \mathcal{V}(x).
\]

Moreover, there exists a constant \(M > 0\) such that for any \(x > 0\), for any \(q \in (0, 1)\),

\[
\mathbb{P}(\xi_{d_e} < x) \leq \mathbb{P}(\xi_{g_e} < x) \leq M q^{1/2} \mathcal{V}(x).
\]
Proof. By Lemma 23, for any $q > 0$, we have

$$\lambda \int_{0}^{\infty} e^{-\lambda x} P(\xi_{q} < x) dx = E \left[ e^{-\lambda \xi_{q}} \right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)}.$$ 

We introduce for any $q > 0$ the function $V_{q}$ defined on $[0, \infty)$ by

$$V_{q}(x) = E \left[ \int_{0}^{\infty} e^{-q t H_{t}} 1_{\{H_{t} \leq x\}} dt \right],$$

which is such that

$$\lambda \int_{0}^{\infty} e^{-\lambda x} V_{q}(x) dx = E \left[ \int_{0}^{\infty} e^{-q t H_{t} - \lambda H_{t}} dt \right] = \int_{0}^{\infty} e^{-t \kappa(0, q, \lambda)} dt = \frac{1}{\kappa(0, q, \lambda)}.$$

Hence, by injectivity of the Laplace transform, we get $P(\xi_{q} < x) = \kappa(0, q, 0) V_{q}(x)$ for any $x, q > 0$. For any $x > 0$, $V_{q}(x)$ increases to $V(x)$ as $q \to 0$. Hence, by Lemma 22, to show (30) and (31), it is enough to show that $\kappa(0, q, 0) \sim kq^{1/2}$ as $q \to 0$ for some $k > 0$.

By the Fristedt formula from Lemma 23 and by symmetry of $(V_{t})_{t \geq 0}$, and thus of $(Z_{t})_{t \geq 0}$ (observe $(Z_{t})_{t \geq 0}$ and $(-Z_{t})_{t \geq 0}$ share the same $(\gamma_{t})_{t \geq 0}$), we also have

$$\kappa(0, q, 0) = k \exp \left( \int_{0}^{\infty} \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} \mathbb{1}_{\{x \leq t\}} P(\gamma_{t} \in dr, Z_{t} \in dx) dt \right).$$

Then we can write

$$\log[k(0, q, 0)]^{2} = 2 \log k + \int_{0}^{\infty} \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} P(\gamma_{t} \in dr, Z_{t} \in dx) dt$$

$$= 2 \log k + \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-t} - e^{-qr}}{t} P(\gamma_{t} \in dr) dt$$

$$= 2 \log k + \int_{0}^{\infty} e^{-t} - e^{-\Phi(q)t} dt$$

$$= 2 \log k + \log \Phi(q)$$

by Frullani’s formula (29). Therefore $\kappa(0, q, 0) = k(\Phi(q))^{1/2} \sim kq^{1/2}$ as $q \to 0$. 

We need to show one last result, after which we will able to prove Lemma 21, which will close this subsection.

Lemma 26. There exists a constant $v_{\alpha} > 0$ such that $V(x) \sim v_{\alpha} x^{\alpha/2}$ as $x \to \infty$.

Proof. Let us first remark that we have for any $\lambda > 0$,

$$\lambda \int_{0}^{\infty} e^{-\lambda x} V(x) dy = E \left[ \int_{0}^{\infty} e^{-\lambda H_{t}} dt \right] = \int_{0}^{\infty} e^{-t \kappa(0, 0, \lambda)} dt = \frac{1}{\kappa(0, 0, \lambda)}.$$

Then, by Theorem 24, since $V$ is non-decreasing and $\alpha/2 > -1$, it is enough to show that there exists a constant $c_{\alpha} > 0$ such that $\kappa(0, 0, \lambda) \sim c_{\alpha} \lambda^{\alpha/2}$ as $\lambda \to 0$.

To do so, we use the convergence in law of the rescaled Lévy process $(e^{1/\alpha} Z_{t/\epsilon})_{t \geq 0}$ to the symmetric stable process $(Z_{t})_{t \geq 0}$, see Proposition 30 below. Moreover, since $t^{-1} \gamma_{t} \to 1$ as $t \to \infty$, we get

$$(e^{1/\alpha} Z_{t/\epsilon})_{t \geq 0} \to (t, Z_{t})_{t \geq 0} \quad \text{as} \quad \epsilon \to 0,$$

in law for the usual Skorokhod topology. Indeed, since $(\gamma_{t}, Z_{t})_{t \geq 0}$ is Lévy, only the convergence in law of $(t^{-1} \gamma_{t}, t^{-1/\alpha} Z_{t})$ to $(1, Z_{1}^{\alpha})$ as $t \to \infty$ is required, see Jacod-Shiryaev [29, Chapter VII, Corollary 3.6], and the convergence follows from Slutsky’s lemma.
Let us set, for every \( \alpha, \beta, \lambda \geq 0 \),
\[
\kappa'(\alpha, \beta, \lambda) = k \exp \left( \int_0^\infty \int_{(0,\infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r - \lambda x}}{t} 1_{\{x \geq 0\}} \mathbb{P}(\epsilon \gamma t/\epsilon \in dr, \epsilon^{1/\alpha} Z t/\epsilon \in dx) dt \right),
\]
where \( k > 0 \) is the constant from Lemma 23. We have
\[
\log \kappa'(0, 0, \lambda) = \log k + \int_0^\infty t^{-1} \mathbb{E} \left[ (e^{-t} - e^{-\lambda^{1/\alpha} Z t/\epsilon}) 1_{\{Z t/\epsilon > 0\}} \right] dt
\]
\[
= \log k + \int_0^\infty t^{-1} \mathbb{E} \left[ (e^{-t} - e^{-\lambda^{1/\alpha} Z t/\epsilon}) 1_{\{Z t/\epsilon > 0\}} \right] dt + \int_0^\infty t^{-1} (e^{-\epsilon t} - e^{-t}) \mathbb{P}(Z t \geq 0) dt
\]
\[
= \log \kappa(0, 0, \lambda^{1/\alpha}) - \frac{1}{2} \log \epsilon.
\]
In the third equality, we used Frullani’s identity and the fact that \( \mathbb{P}(Z t \geq 0) = 1/2 \). Therefore, we have \( \kappa'(0, 0, \lambda) = \epsilon^{-1/2} \kappa(0, 0, \lambda^{1/\alpha}) \). Then it is shown in [3, Proposition B.2 in Appendix B] that the convergence (32) entails that for any \( \alpha, \beta, \lambda \geq 0 \),
\[
\kappa'(\alpha, \beta, \lambda) \xrightarrow{\epsilon \to 0} k \exp \left( \int_0^\infty \int_{(0,\infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha \kappa^{1/\alpha} t - \lambda x}}{t} \mathbb{P}(Z t^\alpha \in dx) dt \right) =: \bar{\kappa}(\alpha, \beta, \lambda).
\]
Now for \( \delta > 0 \), choosing \( \lambda = 1 \) and \( \epsilon = \delta^\alpha \), we see that \( \kappa(0, 0, \delta) = \delta^{\alpha/2} \kappa^\alpha(0, 0, 1) \), which is equivalent to \( \delta^{\alpha/2} \bar{\kappa}(0, 0, 1) \), as desired. □

**Proof of Lemma 21.** Let \( \epsilon = \epsilon(q) \) be an independent exponential random variable of parameter \( q > 0 \). Since \( t \mapsto \mathbb{P}(\xi_{\gamma t} < X T_0) \) and \( t \mapsto \mathbb{P}(\xi_{\gamma t} < X T_0) \) are non-decreasing, since \( -1/2 > -1 \), and since
\[
\mathbb{P}(\xi_{\gamma t} < X T_0) = q \int_0^\infty e^{-qt} \mathbb{P}(\xi_{\gamma t} < X T_0) dt \quad \text{and} \quad \mathbb{P}(\xi_{\gamma t} < X T_0) = q \int_0^\infty e^{-qt} \mathbb{P}(\xi_{\gamma t} < X T_0) dt,
\]
by Theorem 24, the result is equivalent to \( \mathbb{P}(\xi_{\gamma t} < X T_0) \sim \mathbb{P}(\xi_{\gamma t} < X T_0) \sim \bar{C} q^{1/2} \) as \( q \to 0 \) for some \( \bar{C} > 0 \). Since \( (X t)_{t \geq 0} \) is independent of \( X T_0 \), we have
\[
\mathbb{P}(\xi_{\gamma t} < X T_0) = \int_0^\infty \mathbb{P}(\xi_{\gamma t} < x) \mathbb{P}(X T_0 \in dx).
\]
By (30), we know that for any \( x > 0 \), \( q^{-1/2} \mathbb{P}(\xi_{\gamma t} < x) \to k \mathcal{V}(x) \) as \( q \to 0 \), and by (31), we have \( q^{-1/2} \mathbb{P}(\xi_{\gamma t} < x) \leq M \mathcal{V}(x) \). Finally, by Lemmas 20 and 26, \( \mathbb{E}[\mathcal{V}(X T_0)] < \infty \) and thus we can apply the dominated convergence theorem, which tells us that
\[
\lim_{q \to 0} q^{-1/2} \mathbb{P}(\xi_{\gamma t} < X T_0) = k \mathbb{E}[\mathcal{V}(X T_0)].
\]
The same proof holds for \( \mathbb{P}(\xi_{\gamma t} < X T_0) \). □

## 5 \textit{M}_1\text{-convergence of the free process}

In this section, we give the proof of Theorem 7. Let \( (X t, V t)_{t \geq 0} \) be the solution to (1) starting at \( (0, v_0) \) where \( v_0 \in \mathbb{R} \). Let \( (Z t^\alpha)_{t \geq 0} \) be the symmetric stable process with \( \alpha = (\beta + 1)/3 \) as in the statement. As the convergence in the finite dimensional distribution sense was already proved in [25], we only need to show the tightness for the \textit{M}_1-topology. By Theorem 6, we need that for any \( T > 0 \), for any \( \eta > 0 \),
\[
\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \mathbb{P}(w(\epsilon^{1/\alpha} X T/\epsilon, T, \delta) > \eta) = 0,
\]
(33)
The idea of the proof is as follows: we first show that the convergence holds for \((X_t, V_t)_{t \geq 0}\) starting at \((0,0)\). To do so, we will show that some Lévy process \((Z_t)_{t \geq 0}\) associated to \((X_t)_{t \geq 0}\) converges in the \(J_1\)-topology to a symmetric stable process (which is immediately implied by the finite dimensional distribution convergence). Therefore, \((Z_t)_{t \geq 0}\) converges also in the \(M_1\)-topology and (33) is then satisfied by \((Z_t)_{t \geq 0}\), which will imply that (33) is also satisfied by \((X_t)_{t \geq 0}\). We will then extend the convergence to processes starting at \((0, v_0)\).

5.1 Some preliminary results

Before jumping in to the proof, we we recall some results that were used / proved in [25]. The first result can be found in Jeulin-Yor [31] and Biane-Yor [10], and represents symmetric stable processes using one Brownian motion.

**Theorem 27 (Biane-Yor).** Let \((W_t)_{t \geq 0}\) be a Brownian motion and \((\tau_t)_{t\geq0}\) the inverse of its local time at 0. Let \(\alpha \in (0, 2)\) and consider, for \(\eta > 0\), the process

\[ K^\eta_t = \int_0^t \text{sgn}(W_s)|W_s|^{1/\alpha - 2}1_{\{|W_s| > \eta\}}ds. \]

Then the process \((K^\eta_t)_{t \geq 0}\) converges a.s. toward a process \((K_t)_{t \geq 0}\) as \(\eta \to 0\), uniformly on compact time intervals. Moreover the process \((S^\eta_t)_{t \geq 0}\) is a symmetric stable process and for all \(t \geq 0\) and all \(\xi \in \mathbb{R}\), \(\mathbb{E}[\exp(i\xi S^\eta_t)] = \exp(-\kappa_\alpha t|\xi|^{\alpha})\), where \(\kappa_\alpha = \frac{2^{\alpha-1} \pi^{\alpha/2} \Gamma(\frac{\alpha}{2})}{\alpha\Gamma(1-\frac{\alpha}{2})}\).

We now summarize the intermediate results that can be found in [25, Lemmas 6 and 9], enabling the authors to prove their main result, which is stated in (2). We mention that the main tool used in the proofs is the theory of scale function and speed measure.

**Theorem 28 (Fournier-Tardif).** Let \((X_t, V_t)_{t \geq 0}\) be a solution of (1), with \(\beta \in (1,5)\) and starting at \((0,0)\). There exists a Brownian motion \((W_t)_{t \geq 0}\) such that for any \(\varepsilon > 0\), there exist a continuous process \((H^\varepsilon_t)_{t \geq 0}\) and a continuous, increasing and bijective time-change \((\rho^\varepsilon_t)_{t \geq 0}\), adapted to the filtration generated by \((W_t)_{t \geq 0}\), and having the following properties:

(i) For any \(\varepsilon > 0\), \((X_t/\varepsilon)_{t \geq 0}\) \(\xrightarrow{d}\) \((H^\varepsilon_t)_{t \geq 0}\).

(ii) For every \(t \geq 0\), a.s., \(\rho^\varepsilon_t \xrightarrow{\varepsilon \to 0} \tau_t\), where \((\tau_t)_{t\geq0}\) is the inverse of the local time at 0 of \((W_t)_{t \geq 0}\).

(iii) Almost surely, for every \(t \geq 0\), \(\sup_{[0,t]} \left| e^{1/\alpha H^\varepsilon_t} - \theta K^\varepsilon_t \right| \xrightarrow{\varepsilon \to 0} 0\), where \(\theta = (\beta + 1)/3\), where

\[ \theta = (\beta + 1)^{1/\alpha - 2}c_3^{1/\alpha} \] and \(\theta (K_t)_{t \geq 0}\) is the process from Theorem 27.

Recall that \(c_3\) is given in (4). We now slightly improve their result, showing the convergence of past infimum and supremum.

**Proposition 29.** Grant Assumption 1 and let \((X_t, V_t)_{t \geq 0}\) be a solution of (1) with \(\beta \in (1,5)\) and starting at \((0,0)\). Let \((Z^\alpha_t)_{t \geq 0}\) be a symmetric stable process with \(\alpha = (\beta + 1)/3\) and such that \(\mathbb{E}[e^{\xi Z^\alpha_t}] = \exp(-\sigma_\alpha \xi^ \alpha)\). Then we have for every \(0 \leq s < t\),

\[ \inf_{u \in [s,t]} e^{1/\alpha} X_{u/\varepsilon} \xrightarrow{\varepsilon \to 0} \inf_{u \in [s,t]} Z^\alpha_u \]
and

\[ \sup_{u \in [s,t]} e^{1/\alpha} X_{u/\varepsilon} \xrightarrow{\varepsilon \to 0} \sup_{u \in [s,t]} Z^\alpha_u \] as \(\varepsilon \to 0\).

**Proof.** By item (i) of Theorem 28, it is enough to show that for any \(0 \leq s < t\), a.s., we have

\[ \inf_{u \in [s,t]} e^{1/\alpha} H^\varepsilon_{\rho^\varepsilon_u} \xrightarrow{\varepsilon \to 0} \inf_{u \in [s,t]} \theta K_{\tau_u} \]
and

\[ \sup_{u \in [s,t]} e^{1/\alpha} H^\varepsilon_{\rho^\varepsilon_u} \xrightarrow{\varepsilon \to 0} \sup_{u \in [s,t]} \theta K_{\tau_u}. \]

We will only show the result for the infimum as the proof for the supremum is identical.
Step 1: We first show that a.s., for any $s < t$, $\inf_{u \in [s,t]} K_{\tau_u} = \inf_{u \in [\tau_u, \tau_t]} K_u$, which is not straightforward since $t \mapsto \tau_t$ is discontinuous. We will first treat the case $\alpha \in (0, 1)$. Observe that in this case, \( \int_0^t |W_s|^{1/\alpha - 2} ds < \infty \) since $1/\alpha - 2 > -1$ and thus we have

\[
K_{\tau_u} = \int_0^{\tau_u} \text{sgn}(W_s)|W_s|^{1/\alpha - 2} ds = \sum_{r \leq s \leq \tau_u} \int_r^{\tau_u} \text{sgn}(W_s)|W_s|^{1/\alpha - 2} ds,
\]

which has finite variations and no drift part. For every $u \geq 0$, $(W_t)_{t \geq 0}$ is of constant sign on the time-interval $[\tau_u, \tau_t]$ and consequently $t \mapsto K_t$ is monotone on every such interval. Hence, the infimum is necessarily reached at the extremities i.e. $\inf_{r \in [\tau_u, \tau_t]} K_r = \min\{K_{\tau_u}, K_{\tau_t}\}$.

If $\alpha \in [1, 2)$, we approximate $(K_t)_{t \geq 0}$ by the processes $(K^\rho_t)_{t \geq 0}$, from Theorem 27. Similarly, we have for every $t \geq 0$, $K^\rho_u = \sum_{\epsilon \in (0, \rho]} \int_{\tau_u}^{\tau_u + \epsilon} \text{sgn}(W_u)|W_u|^{1/\alpha - 2} 1_{(|W_u| > \epsilon)} du$, and thus, by the previous reasoning, we have $\inf_{u \in [s,t]} K^\rho_{\tau_u} = \inf_{u \in [\tau_u, \tau_t]} K^\rho_u$. We can write

\[
\left| \inf_{u \in [s,t]} K_{\tau_u} - \inf_{u \in [\tau_u, \tau_t]} K_u \right| \leq \left| \inf_{u \in [s,t]} K_{\tau_u} - \inf_{u \in [\tau_u, \tau_t]} K^\rho_{\tau_u} \right| + \left| \inf_{u \in [\tau_u, \tau_t]} K^\rho_{\tau_u} - \inf_{u \in [\tau_u, \tau_t]} K_u \right| \\
\leq \sup_{u \in [s,t]} |K^\rho_{\tau_u} - K_{\tau_u}| + \sup_{u \in [\tau_u, \tau_t]} |K^\rho_u - K_u| \\
\leq 2 \sup_{u \in [0,\tau_t]} |K^\rho_u - K_u|.
\]

By Theorem 27, $(K^\rho_t)_{t \geq 0}$ converges a.s. to $(K_t)_{t \geq 0}$ uniformly on compact time intervals, and thus the last term vanishes as $\eta \to 0$.

Step 2: For every $\epsilon > 0$, $t \mapsto \rho_t^\epsilon$ and $t \mapsto H_t^\epsilon$ are almost surely continuous. Therefore, a.s. for every $s < t$, $\inf_{u \in [s,t]} \epsilon^{1/\alpha} H_u^{\rho_t^\epsilon} = \inf_{u \in [\rho_t^\epsilon, \rho_s^\epsilon]} \epsilon^{1/\alpha} H_u^\epsilon$. Hence we can write, by Step 1,

\[
\left| \inf_{u \in [s,t]} \epsilon^{1/\alpha} H_u^\epsilon - \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_{\tau_u} \right| \leq \left| \inf_{u \in [s,t]} \epsilon^{1/\alpha} H_u^\epsilon - \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_u \right| + \left| \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_u - \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_u \right| \\
\leq \sup_{u \in [0,T]} \left| \epsilon^{1/\alpha} H_u^\epsilon - \theta K_u \right| + \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_u - \inf_{u \in [\rho_s^\epsilon, \rho_t^\epsilon]} \theta K_u
\]

where $T = \sup_{\epsilon \in (0,1)} \rho_t^\epsilon$ is a.s. finite by item (ii) of Theorem 28. Almost surely, the first term on the right-hand-side goes to 0 thanks to Theorem 28-(iii). By item (iii), we have for any $0 \leq s < t$, almost surely, $\rho_s^\epsilon \to \tau_s$ and $\rho_t^\epsilon \to \tau_t$ as $\epsilon \to 0$. Since $(K_t)_{t \geq 0}$ is continuous, the second term vanishes as $\epsilon \to 0$, which completes the proof.

5.2 Convergence of the associated Lévy process

In this subsection, we consider a solution $(X_t, V_t)_{t \geq 0}$ of (1) starting at $(0, 0)$. The velocity process possesses a local time at 0 that we will denote by $(L_t)_{t \geq 0}$ and we will also denote by $(\tau_t)_{t \geq 0}$ its right-continuous inverse. The latter is a subordinator. The process $(V_t)_{t \geq 0}$ is positive recurrent which implies that $\mathbb{E}\{\gamma_1\} < \infty$ and we choose to normalize the local time so that $\mathbb{E}\{\gamma_1\} = 1$. The strong law of large number for subordinators entails that a.s. $t^{-1}\gamma_t \to 1$ as $t \to \infty$. This also implies the same result for $(L_t)_{t \geq 0}$, and by Dini theorem, we get that a.s., for any $t \geq 0$,

$$
\sup_{s \in [0,t]} |\epsilon L_{s/\epsilon} - s| \xrightarrow{\epsilon \to 0} 0
$$

Next we define $(Z_t)_{t \geq 0} = (X_{\gamma_t})_{t \geq 0}$ which is a pure jump Levy process with finite variations and should be seen the following way:

$$
Z_t = \int_0^{\gamma_t} V_s ds = \sum_{s \leq t} \int_{\gamma_s}^{\gamma_t} V_u du.
$$

We establish an $\alpha$-stable central limit theorem for the Levy process $(Z_t)_{t \geq 0}$, which seems more or less clear in the light of (2) and the strong law of large number for $(\gamma_t)_{t \geq 0}$.
Proposition 30. Let \((Z_t^\alpha)_{t \geq 0}\) be the stable process of Proposition 29. Then we have
\[
(\epsilon^{1/\alpha} Z_{t/\epsilon})_{t \geq 0} \longrightarrow (Z_t^\alpha)_{t \geq 0}
\]
in law in the \(J_1\)-topology

Proof. As \((Z_t)_{t \geq 0}\) is a Lévy process, it is enough to show that \(t^{-1/\alpha}Z_t\) converges in law to \(Z_t^\alpha\), see for instance Jacod-Shiryaev [29, Chapter VII, Corollary 3.6]. Let \(z \in \mathbb{R}\) and \(\delta > 0\). On the one hand, we have
\[
\mathbb{P}\left(t^{-1/\alpha} Z_t \geq z, |\gamma_t - t| \leq \delta t\right) + \mathbb{P}\left(|\gamma_t - t| > \delta t\right),
\]
and on the other hand, we have
\[
\mathbb{P}\left(t^{-1/\alpha} Z_t \geq z\right) \geq \mathbb{P}\left(|\gamma_t - t| \leq \delta t\right) - \mathbb{P}\left(t^{-1/\alpha} Z_t < z, |\gamma_t - t| \leq \delta t\right).
\]
It follows from the strong law of large number that \(\mathbb{P}(|\gamma_t - t| > \delta t)\) converges to 0 as \(t \to \infty\). Now on the event \(\{|\gamma_t - t| \leq \delta t\}\), we have \(\gamma_t \in [(1-\delta)t, (1+\delta)t]\), and thus, reminding that \(Z_t = X_{\gamma_t}\), we have
\[
\mathbb{P}\left(t^{-1/\alpha} Z_t \geq z, |\gamma_t - t| \leq \delta t\right) \leq \mathbb{P}\left(\sup_{s \in [1-\delta, 1+\delta]} t^{-1/\alpha}X_{st} \geq z\right),
\]
and
\[
\mathbb{P}\left(t^{-1/\alpha} Z_t < z, |\gamma_t - t| \leq \delta t\right) \leq \mathbb{P}\left(\inf_{s \in [1-\delta, 1+\delta]} t^{-1/\alpha}X_{st} < z\right).
\]
The two quantities on the right-hand-side of the above equations converge by Proposition 29 to \(\mathbb{P}(\sup_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z)\) and \(\mathbb{P}(\inf_{s \in [1-\delta, 1+\delta]} Z_s^\alpha < z)\) as \(t\) tends to infinity. Putting the pieces together, we have
\[
\lim_{t \to \infty} \sup_{\delta \to 0} \mathbb{P}\left(t^{-1/\alpha} Z_t \geq z\right) \leq \mathbb{P}\left(\sup_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z\right),
\]
and
\[
\lim_{t \to \infty} \inf_{\delta \to 0} \mathbb{P}\left(t^{-1/\alpha} Z_t \geq z\right) \geq \mathbb{P}\left(\inf_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z\right),
\]
Since almost surely, 1 is not a jumping time of \((Z_t^\alpha)_{t \geq 0}\), it should be clear that by letting \(\delta \to 0\), we can conclude that \(\lim_{t \to \infty} \mathbb{P}(t^{-1/\alpha} Z_t \geq z) = \mathbb{P}(Z_t^\alpha \geq z)\). \(\square\)

5.3 Proof of Theorem 7

Proof of Theorem 7. Step 1: As explained above, we start by showing that Theorem 7 holds for a solution \((X_t, V_t)_{t \geq 0}\) starting at \((0, 0)\). We need to show that (33) holds. Let \((Z_t)_{t \geq 0} = (X_{\gamma_t})_{t \geq 0}\) as in the previous subsection, where \((\gamma_t)_{t \geq 0}\) is the inverse of the local time \((L_t)_{t \geq 0}\) at 0 of \((V_t)_{t \geq 0}\). Since the \(M_1\)-topology is weaker than the \(J_1\)-topology, by Proposition 30 and Theorem 6, we get, for any \(T > 0\), for any \(\eta > 0\),
\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \mathbb{P}(w(\epsilon^{1/\alpha} Z_{t/\epsilon}, T, \delta) > \eta) = 0.
\]
Now we show that for any \(T > 0\), for any \(\eta > 0\) and any \(\delta \in (0, 1)\),
\[
\mathbb{P}\left(w(\epsilon^{1/\alpha} X_{t/\epsilon}, T, \delta) > \eta\right) \leq \mathbb{P}\left(w(\epsilon^{1/\alpha} Z_{t/\epsilon}, T + 1, 2\delta) > \eta\right) + \mathbb{P}\left(\sup_{t \in [0, T]} |\epsilon L_{t/\epsilon} - t| \geq \delta\right) \tag{35}
\]
This will achieve the first step by (34). We first introduce for \(t \geq 0\),
\[
g_t = \sup\{s \leq t, V_s = 0\} \quad \text{and} \quad d_t = \inf\{s \geq t, V_s = 0\}.
\]
Note that \( g_t \) and \( d_t \) can be expressed in terms of the local time and its inverse, i.e. \( g_t = \gamma_{L_t^-} \) and \( d_t = \gamma_{L_t^+} \). We also introduce the random function \( \nu(t) \) such that \( \nu(t) = 1 \) if the excursion straddling the time \( t \) is positive and \( \nu(t) = -1 \) if it is negative, i.e. \( \nu(t) = \mathbf{1}_{\{V_t > 0\}} - \mathbf{1}_{\{V_t < 0\}} \).

Let \( T > 0 \) and \( \delta \in (0, 1) \), we first place ourselves on the event \( A_{T, \delta} = \{ \sup_{t \in [0, T]} |tL_t^- - t| < \delta \} \). Let \( t \in [0, T] \) and \( t_{\delta^-} \leq t_1 < t_2 < t_3 \leq t_{\delta^+} \), where \( t_{\delta^-} = 0 \lor (t - \delta) \) and \( t_{\delta^+} = T \land (t + \delta) \). We also introduce \( t_{2\delta^-} = 0 \lor (t - 2\delta) \) and \( t_{2\delta^+} = (T + 1) \land (t + 2\delta) \). We emphasize that, since we are on the event \( A_{T, \delta} \), \( d_{t_{\delta^-}} = \gamma_{L_t^-} \leq \gamma_{(t+\delta)/\epsilon} \) for every \( t \in [0, T] \). We first bound the distance \( d(e^{1/\alpha}X_{t_{2\delta^-}}, [e^{1/\alpha}X_{t_{2\delta^-}}, e^{1/\alpha}X_{t_{3\delta^-}}]) = e^{1/\alpha}d(X_{t_{2\delta^-}}, [X_{t_{2\delta^-}}, X_{t_{3\delta^-}}]) \). Without loss of generality, we will assume that \( X_{t_{1\epsilon}} \leq X_{t_{3\epsilon}} \).

- First case: \( X_{t_{1\epsilon}} \leq X_{t_{2\epsilon}} \leq X_{t_{3\epsilon}} \). Then we have
  \[
  d(X_{t_{2\epsilon}}, [X_{t_{1\epsilon}}, X_{t_{3\epsilon}}]) = 0 \leq \sup_{t_{2\delta^-} \leq t_1 < t_2 < t_3 \leq t_{2\delta^+}} d(Z_{t_{2\epsilon}}, [Z_{t_{1\epsilon}}, Z_{t_{3\epsilon}}]).
  \]

- Second case: \( X_{t_{2\epsilon}} < X_{t_{1\epsilon}} \leq X_{t_{3\epsilon}} \). In this case, \( d(X_{t_{2\epsilon}}, [X_{t_{1\epsilon}}, X_{t_{3\epsilon}}]) = X_{t_{1\epsilon}} - X_{t_{2\epsilon}} \). Let us note that, since \( \gamma_t \geq 0 \) on every excursion of \( V_t \geq 0 \), \( t_{2\epsilon} \leq t_3 \) can not belong to the same excursion, i.e. \( d_{t_{1\epsilon}} \leq g_{t_{2\epsilon}} \). We define, for \( i \in \{1, 2, 3\} \) and \( \epsilon > 0 \), the positive real numbers \( u_{i\epsilon} \) defined as follows
  1. \( u_{2\epsilon} = g_{t_{2\epsilon}} \) if \( \nu(t_{2\epsilon}) = 1 \), \( u_{2\epsilon} = d_{t_{2\epsilon}} \) if \( \nu(t_{2\epsilon}) = -1 \) and \( u_{2\epsilon} = t_{2\epsilon} \) if \( \nu(t_{2\epsilon}) = 0 \).
  2. For \( i \in \{1, 3\} \), \( u_{i\epsilon} = d_{t_{i\epsilon}} \) if \( \nu(t_{i\epsilon}) = 1 \), \( u_{i\epsilon} = g_{t_{i\epsilon}} \) if \( \nu(t_{i\epsilon}) = -1 \) and \( u_{i\epsilon} = t_{i\epsilon} \) if \( \nu(t_{i\epsilon}) = 0 \).

Therefore, we have \( X_{u_{2\epsilon}} < X_{u_{1\epsilon}} \) and \( X_{u_{2\epsilon}} < X_{u_{3\epsilon}} \), so that necessarily, \( u_{1\epsilon} < u_{2\epsilon} < u_{3\epsilon} \). Now we stress that, if \( r \geq 0 \) is such that \( V_r = 0 \) (i.e. such that \( \nu(r) = 0 \)), then the zero set of \( V_t \geq 0 \) has no isolated points, either \( r = d_t \) or \( r = g_t \). In any case, we always have for every \( i \in \{1, 2, 3\} \), \( u_{i\epsilon} = \gamma_{t_{i\epsilon}^-} \) or \( u_{i\epsilon} = \gamma_{t_{i\epsilon}^+} \). Let \( \theta > 0 \) and remember that on the event \( A_{T, \delta} \), we have \( L_{t_{i\epsilon}} \in ((t_i - \delta)/\epsilon, (t_i + \delta)/\epsilon) \) for every \( i \in \{1, 2, 3\} \). Using the fact that \( \gamma_t \) is increasing and that \( X_t \geq 0 \) is continuous, we can always find \( s_1 < s_2 < s_3 \in [t_{2\delta^-}, t_{2\delta^+}] \) such that
  \[
  X_{t_{2\epsilon}} \geq X_{s_{2\epsilon}} - \frac{\theta}{2}, \quad X_{t_{1\epsilon}} \leq X_{s_{1\epsilon}} + \frac{\theta}{2} \quad \text{and} \quad X_{t_{3\epsilon}} \leq X_{s_{3\epsilon}} + \frac{\theta}{2},
  \]
  which leads to
  \[
  d(X_{t_{2\epsilon}}, [X_{t_{1\epsilon}}, X_{t_{2\epsilon}}]) \leq (Z_{s_{2\epsilon}} - Z_{s_{1\epsilon}}) \wedge (Z_{s_{3\epsilon}} - Z_{s_{2\epsilon}}) + \theta = d(Z_{s_{2\epsilon}}, [Z_{s_{1\epsilon}}, Z_{s_{3\epsilon}}]) + \theta.
  \]
  Since this holds for every \( \theta > 0 \), we deduce the following bound
  \[
  d(X_{t_{2\epsilon}}, [X_{t_{1\epsilon}}, X_{t_{2\epsilon}}]) \leq \sup_{t_{2\delta^-} \leq t_1 < t_2 < t_3 \leq t_{2\delta^+}} d(Z_{t_{2\epsilon}}, [Z_{t_{1\epsilon}}, Z_{t_{2\epsilon}}]).
  \]

- Third case: \( X_{t_{1\epsilon}} \leq X_{t_{3\epsilon}} < X_{t_{2\epsilon}} \). We can adapt the previous case to deduce that
  \[
  d(X_{t_{2\epsilon}}, [X_{t_{1\epsilon}}, X_{t_{2\epsilon}}]) \leq \sup_{t_{2\delta^-} \leq t_1 < t_2 < t_3 \leq t_{2\delta^+}} d(Z_{t_{2\epsilon}}, [Z_{t_{1\epsilon}}, Z_{t_{2\epsilon}}]).
  \]

To summarize, we proved that, on the event \( A_{T, \delta} \), the following bound holds
  \[
  w(\epsilon^{1/\alpha}X_{t_{1\epsilon}}, T, \delta) \leq w(\epsilon^{1/\alpha}Z_{t_{1\epsilon}}, T + 1, 2\delta).
  \]
  This implies (35). We proved Theorem 7 in the case \( v_0 = 0 \).

**Step 2:** We consider the solution \( (X_t, V_t)_{t \geq 0} \) of (1) starting at \((0, v_0)\) where \( v_0 \in \mathbb{R} \). We show that there exists a constant \( C > 0 \) such that for any \( T > 0 \),

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |V_t| \right] \leq C(1 + T^{1/2}). \tag{36}
\]
To this aim, we start by studying \( \sup_{t \in [0,T]} |V_t|^{\beta+1} \) and we introduce, recall Assumption 1, the even function \( \ell \) defined as
\[
\ell(v) = 2 \int_{0}^{v} \Theta^{-\beta}(u) \int_{0}^{u} \Theta^{\beta}(w)dwdu,
\]
which solves the Poisson equation \( \ell' F + \frac{1}{2} \ell'' = 1 \). Then by the Itô formula, we have
\[
\ell(V_t) = \ell(v_0) + t + \int_{0}^{t} \ell'(V_s)dB_s.
\]
Moreover, remember that \( |v| \Theta(v) \to 1 \) as \( v \to \pm \infty \) and that \( \beta > 1 \). As a consequence, there exist positive constants \( c, c' > 0 \) such that
\[
\ell(v) \sim c|v|^{\beta+1} \quad \text{and} \quad \ell'(v) \sim c'\text{sgn}(v)|v|^{\beta} \quad \text{as} \quad v \to \pm \infty.
\]
Therefore, there exist positive constants \( M, M' > 0 \) such that for all \( v \in \mathbb{R} \),
\[
|v|^{\beta+1} \leq M(1 + \ell(v)) \quad \text{and} \quad |\ell'(v)|^2 \leq M'(1 + v^{2\beta}). \tag{37}
\]
Using (37) and Doob’s inequality, we get
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\beta+1} \right] \leq M \left[ 1 + \ell(v_0) + T + 4\mathbb{E}\left[ \left( \int_{0}^{T} |\ell'(V_s)|^2 ds \right)^{1/2} \right] \right]
\leq M \left[ 1 + \ell(v_0) + T + 4(M'T)^{1/2} \right] + 4M(M'T)^{1/2}\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\beta} \right]. \tag{38}
\]
Finally, using Young’s inequality with \( p = \beta + 1 \) and \( q = (\beta + 1)/\beta \), we find some \( C > 0 \) such that
\[
4M(M'T)^{1/2}\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\beta} \right] \leq CT^{(\beta+1)/2} + \frac{1}{2}\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^\beta \right]^{(\beta+1)/\beta}
\leq CT^{(\beta+1)/2} + \frac{1}{2}\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\beta+1} \right],
\]
which, inserted in (38), implies that there exists a constant \( K > 0 \) such that
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\beta+1} \right] \leq K(1 + T^{(\beta+1)/2}).
\]
We deduce (36) by Hölder’s inequality again.

Step 3: We finally show the result for any solution \( (X_t, V_t)_{t \geq 0} \) starting at \( (0, v_0) \), where \( v_0 \in \mathbb{R} \), and we extend the technique used in [25, page 21]. If we set \( T_0 = \inf\{t \geq 0, V_t = 0\} \), then the process
\[
(\bar{X}_t, \bar{V}_t)_{t \geq 0} = (X_{t+T_0} - X_{T_0}, V_{t+T_0})_{t \geq 0}
\]
is a solution of (1) starting at \( (0, 0) \). Therefore, by the first step, the process \( (\epsilon^{1/\alpha} \bar{X}_{t/\epsilon})_{t \geq 0} \) converges to \( (Z_t^{(\alpha)})_{t \geq 0} \) in the space \( D \) endowed with the \( \mathcal{M}_1 \)-topology. Then, by a version of the Slutsky lemma, see for instance [11, Section 3, Theorem 3.1], it is enough to show that for any \( T > 0 \),
\[
\sup_{t \in [0,T]} \epsilon^{1/\alpha} |X_{t/\epsilon} - \bar{X}_{t/\epsilon}| \xrightarrow{P} 0 \quad \text{as} \quad \epsilon \to 0. \tag{39}
\]
Indeed, the \( \mathcal{M}_1 \)-topology is weaker than the topology induced by the uniform convergence on compact time-intervals. We distinguish two cases. First,
\[
1_{\{T_0 \geq t/\epsilon\}} |X_{t/\epsilon} - \bar{X}_{t/\epsilon}| \leq 1_{\{T_0 \geq t/\epsilon\}} |X_{t/\epsilon}| + 1_{\{T_0 \geq t/\epsilon\}} |X_{t/\epsilon} + T_0 - X_{T_0}| \leq \int_{0}^{2T_0} |V_s|ds = D_1.
\]
Second,
\[
1_{\{T_0 < t/\varepsilon\}} |X_{t/\varepsilon} - \bar{X}_{t/\varepsilon}| \leq |X_T| + 1_{\{T_0 < t/\varepsilon\}} |X_{t/\varepsilon} + T_0 - X_{t/\varepsilon}| \leq D_1 + 1_{\{T_0 < t/\varepsilon\}} \int_{t/\varepsilon}^{t/\varepsilon + T_0} |V_s| ds.
\]
Hence, if we set \(D_{t,\varepsilon}^2 = 1_{\{T_0 < t/\varepsilon\}} \int_{t/\varepsilon}^{t/\varepsilon + T_0} |V_s| ds\), we get
\[
\sup_{t \in [0, T]} \varepsilon^{1/\alpha} |X_{t/\varepsilon} - \bar{X}_{t/\varepsilon}| \leq \varepsilon^{1/\alpha} D_1 + \sup_{t \in [0, T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2.
\]
The first term converges almost surely to 0 as \(\varepsilon \to 0\). Regarding the second one, we have
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2 \Big| \mathcal{F}_T \right] \leq \varepsilon^{1/\alpha} T_0 \mathbb{E} \left[ \sup_{t \in [0, T]} 1_{\{T_0 < t/\varepsilon\}} \sup_{s \in [t/\varepsilon, t/\varepsilon + T_0]} |V_s| \right] \leq \varepsilon^{1/\alpha} T_0 1_{\{T_0 < t/\varepsilon\}} \mathbb{E} \left[ \sup_{t \in [0,2T/\varepsilon]} |V_s| \right] \leq \varepsilon^{1/\alpha} T_0 C(1 + (2T)^{1/2} e^{-1/2})
\]
by Step 2. This last quantity almost surely goes to 0 as \(\alpha \in (0, 2)\). This implies the convergence in probability of \(\sup_{t \in [0, T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2\) to 0, so that (39) holds.

**Appendix**

**A Some useful results**

**Lemma 31.** Let \(X\) and \(Y\) be two positive random variables and let \(C > 0\). Assume that we have \(\mathbb{P}(X > t) \sim Ct^{-1/2}\) as \(t \to \infty\). Then the following assertions are equivalent.

(i) \(\lim_{t \to \infty} t^{1/2} \mathbb{P}(Y > t) = 0\).

(ii) \(\mathbb{P}(X + Y > t) \sim Ct^{-1/2}\) as \(t \to \infty\).

**Proof.** Let us first show that (i) implies (ii). Let \(\delta \in (0, 1)\) and write
\[
\mathbb{P}(X + Y > t) = \mathbb{P}(X + Y > t, X \geq \delta t) + \mathbb{P}(X + Y > t, X \leq \delta t)
\]
\[
\leq \mathbb{P}(X > \delta t) + \mathbb{P}(Y > (1 - \delta)t).
\]
We deduce that
\[
C = \lim \inf_{t \to \infty} t^{1/2} \mathbb{P}(X > t) \leq \lim \inf_{t \to \infty} t^{1/2} \mathbb{P}(X + Y > t) \leq \lim \sup_{t \to \infty} t^{1/2} \mathbb{P}(X + Y > t) \leq C \delta^{-1/2}.
\]

Letting \(\delta \to 1\) completes the first step.

We now show that (ii) implies (i). We first remark that
\[
\{Y > 3t\} \subset \bigcup_{n \in \mathbb{N}} \{X + Y > (n + 1)t\} \cap \{X \leq nt\}.
\]
Indeed, if \(Y > 3t\), we set \(n + 2 = \left[ \frac{X + Y}{t} \right] \geq 3\), and get \(\frac{X + Y}{t} \leq \frac{X + Y}{t} - 3 \leq n\) and \(\frac{X + Y}{t} > n + 1\).

Now let \(N \in \mathbb{N}\), then we have \(\bigcup_{n > N} \{X + Y > (n + 1)t\} \cap \{X \leq nt\} \subset \{X + Y > (N + 2)t\}\) and therefore
\[
\mathbb{P}(Y > 3t) \leq \mathbb{P}(X + Y > (N + 2)t) + \sum_{n=1}^{N} \mathbb{P}(X + Y > (n + 1)t, X \leq nt).
\]
For any \(n \geq 1\), we have
\[
\mathbb{P}(X + Y > (n + 1)t, X \leq nt) = \mathbb{P}(X + Y > (n + 1)t) - \mathbb{P}(X + Y > (n + 1)t, X > nt)
\]
\[
\leq \mathbb{P}(X + Y > (n + 1)t) - \mathbb{P}(X > (n + 1)t),
\]
and thus \( \lim_{t \to \infty} t^{1/2} \mathbb{P}(X + Y > (n + 1)t, X \leq nt) = 0 \), from which we deduce that for any \( N \in \mathbb{N} \),

\[
\limsup_{t \to \infty} t^{1/2} \mathbb{P}(Y > 3t) \leq C(N + 2)^{-1/2}.
\]

Letting \( N \to \infty \) completes the proof. \( \square \)

**Proposition 32.** Let \( (X_n)_{n \in \mathbb{N}} \) be i.i.d positive random variables and let \( \alpha \in (0, 1) \) such that \( \lim_{n \to \infty} t^n \mathbb{P}(X_1 > t) = 0 \). If \( S_n = \sum_{k=1}^{n} X_k \), then \( n^{-1/\alpha} S_n \) converges to 0 in probability as \( n \to \infty \).

**Proof.** We will show that the Laplace transform of \( n^{-1/\alpha} S_n \) converges to 1 as \( n \to \infty \). We have for any \( \lambda > 0 \),

\[
\mathbb{E} \left[ e^{-\lambda n^{-1/\alpha} S_n} \right] = \mathbb{E} \left[ e^{-\lambda n^{-1/\alpha} X_1} \right]^{n},
\]

and thus

\[
\log \mathbb{E} \left[ e^{-\lambda n^{-1/\alpha} S_n} \right] \sim n \log \mathbb{E} \left[ e^{-\lambda n^{-1/\alpha} X_1} - 1 \right] = -\frac{\lambda}{n^{1/\alpha-1}} \int_0^\infty e^{-\lambda u/n^{1/\alpha}} \mathbb{P}(X_1 > u) du.
\]

Then we use that for \( u \geq n^{1/\alpha} \), \( \mathbb{P}(X_1 > u) \leq \mathbb{P}(X_1 > n^{1/\alpha}) \) and we get

\[
\frac{\lambda}{n^{1/\alpha-1}} \int_0^\infty e^{-\lambda u/n^{1/\alpha}} \mathbb{P}(X_1 > u) du \leq \frac{\lambda}{n^{1/\alpha-1}} \int_0^{n^{1/\alpha}} \mathbb{P}(X_1 > u) du + e^{-\lambda n} \mathbb{P}(X_1 > n^{1/\alpha}).
\]

The second term on the right-hand-side converges to 0 by assumption. Regarding the first term, since \( \mathbb{P}(X_1 > u) = o(u^{-\alpha}) \) as \( u \to \infty \) and since \( \lim_{x \to \infty} \int_1^x u^{-\alpha} du = \infty \), we have \( \int_0^{n^{1/\alpha}} \mathbb{P}(X_1 > u) du = o(\int_1^{n^{1/\alpha}} u^{-\alpha} du) = o(n^{1/\alpha-1}) \) as \( n \to \infty \), which completes the proof. \( \square \)

**B On the P.D.E result**

In this subsection, we formalize the P.D.E. result briefly exposed in the introduction. We first explain how the law of \( (X_t, V_t)_{t \geq 0} \) is linked with a kinetic Fokker-Planck equation with diffusive boundary conditions. For every \( \varphi \in C_c^2(\mathbb{R}) \), we define \( \mathcal{L} \varphi = \mathbb{P} \varphi + \frac{1}{2} \varphi'' \). Then \( \mathcal{L} \) is the infinitesimal generator of the (free) speed process \( (V_t)_{t \geq 0} \). We also denote by \( \mathcal{L}^* \) its adjoint operator which is such that \( \mathcal{L}^* \varphi = \frac{1}{2} \varphi'' - [F \varphi]' \).

**Proposition 33.** Let \( (X_t, V_t)_{t \geq 0} \) be a solution to (3) starting at \( (x_0, v_0) \in ((0, \infty) \times \mathbb{R}) \cup \{(0) \times (0, \infty)\} \). Let us denote by \( f(dt, dx, dv) = \mathbb{P}(X_t \in dx, V_t \in dv) dt \) which is a measure on \( \mathbb{R}_+^2 \times \mathbb{R}_- \). There exist two measures \( \nu_- \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}_-) \) and \( \nu_+ \in \mathcal{M}(\mathbb{R}_+^2 \times \mathbb{R}_-) \) such that for every \( \varphi \in C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R}) \), we have

\[
\varphi(0, x_0, v_0) + \int_{\mathbb{R}_+^2 \times \mathbb{R}} \left[ \partial_t \varphi + v \partial_x \varphi + \mathcal{L} \varphi \right] f(ds, dx, dv) + \int_{\mathbb{R}_+^2} \varphi(s, 0, v) \nu_+(ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) \nu_-(ds, dv) = 0. \tag{40}
\]

Moreover the measures \( \nu_- \) and \( \nu_+ \) satisfy \( \nu_+(dt, dv) = \mu(dv) \int_{w \in \mathbb{R}_-} \nu_-(dt, dw) \).

**Proof.** Let \( \varphi \in C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R}) \), then by Itô’s formula, and passing to the expectation, we get for every \( T > 0 \) that

\[
\mathbb{E} [\varphi(T, X_T, V_T)] = \varphi(0, x_0, v_0) + \int_0^T \mathbb{E} \left[ (\partial_t \varphi + v \partial_x \varphi + \mathcal{L} \varphi)(s, X_s, V_s) \right] ds + \sum_{n \in \mathbb{N}} \mathbb{E} \left[ (\varphi(\tau_n, 0, M_n) - \varphi(\tau_n, 0, V_{\tau_n-})) \mathbbm{1}_{\{\tau_n \leq t\}} \right].
\]
The local martingale part \( f_0^T \partial_t \varphi(s, X_t, V_t) dB_t \) is indeed a true martingale since \( \partial_t \varphi \) is bounded. Let us now define the measures \( \nu_- \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}_-) \) and \( \nu_+ \in \mathcal{M}(\mathbb{R}_+^2) \) by

\[
\nu_- (dt, dv) = \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt, V_{\tau_n -} \in dv) \quad \text{and} \quad \nu_+ (dt, dv) = \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt, M_n \in dv).
\]

Notice that for any \( T > 0 \), we have \( \nu_-([0, T] \times \mathbb{R}_-) = \nu_+([0, T] \times \mathbb{R}_+) = \mathbb{E}[\sum_{n \in \mathbb{N}} 1_{\{\tau_n \leq T\}}] \), which is finite since \( (\tau_{n+1} - \tau_n)_{n \geq 1} \) is an i.i.d. sequence of positive random variables. This also justifies the above exchange between \( \mathbb{E} \) and \( \sum \). Therefore we have for any \( T > 0 \)

\[
\mathbb{E}[\varphi(T, X_T, V_T)] = \varphi(0, x_0, v_0) + \int_{\mathbb{R}_+^2 \times \mathbb{R}} [\partial_t \varphi + v \partial_x \varphi + \mathcal{L} \varphi] 1_{\{s \leq T\}} f(ds, dx, dv) + \int_{\mathbb{R}_+^2} \varphi(s, 0, v) 1_{\{s \leq T\}} \nu_+ (ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) 1_{\{s \leq T\}} \nu_- (ds, dv).
\]

Choosing \( T \) large enough so that the support of \( \varphi \) is included in \([0, T] \times \mathbb{R}_+ \times \mathbb{R} \), we get the desired identity.

The relation between the two measures comes from the fact that for every \( n \in \mathbb{N} \), \( \tau_n \) and \( M_n \) are independent and that \( M_n \) is \( \mu \)-distributed. Indeed it is clear that, since \( (M_n)_{n \in \mathbb{N}} \) is i.i.d. and also independent from the driving Brownian motion, \( M_n \) is independent from \((X_t, V_t)_{0 \leq t < \tau_n}\) for every \( n \in \mathbb{N} \). Hence we have

\[
\nu_+ (dt, dv) = \mu (dv) \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt) = \mu (dv) \int \nu_- (dt, dw),
\]

which achieves the proof. \( \square \)

**Remark 34.** Assume for simplicity that \( \mu (dv) = \mu(v) dv \). The preceding proposition shows that \( f \) is a weak solution of

\[
\begin{cases}

\partial_t f + v \partial_x f = \mathcal{L}^* f & \text{for } (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\
v f(t, 0, v) = -\mu(v) \int_{(-\infty, 0]} w f(t, 0, w) dw & \text{for } (t, v) \in (0, \infty)^2 \\
f(0, \cdot, \cdot) = \delta(0,v_0)
\end{cases}
\]

Informally, it automatically holds that \( \nu_+ (ds, dv) = v f(s, 0, v) 1_{\{v > 0\}} ds dv \) and \( \nu_- (ds, dv) = -v f(s, 0, v) 1_{\{v < 0\}} ds dv \).

For similar notions of weak solutions associated to closely related equations, we refer to Jabir-Profeta [26, Theorem 4.2.1] and Bernou-Fournier [4, Definition 4].

**Proof.** We assume that \( f(dt, dx, dv) = f(t, x, v) dx dv dt \) with \( f \) smooth enough. Let \( \varphi \) be a function belonging to \( C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R}) \). We perform some integrations by parts. We have

\[
\int_{\mathbb{R}_+^2 \times \mathbb{R}} f \partial_t \varphi = -\varphi(0, x_0, v_0) - \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi \partial_t f \quad \text{and} \quad \int_{\mathbb{R}_+^2 \times \mathbb{R}} f \mathcal{L} \varphi = \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi \mathcal{L}^* f.
\]

Regarding the integration by part in \( x \), we have

\[
\int_{\mathbb{R}_+^2} v f \partial_x \varphi = - \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi v \partial_x f - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} v \varphi(s, 0, v) f(s, 0, v) dv ds.
\]

Inserting the previous identities in (40), it comes that

\[
\int_{\mathbb{R}_+^2 \times \mathbb{R}} [\partial_t f + v \partial_x f - \mathcal{L}^* f] \varphi + \int_{\mathbb{R}_+} v \varphi(s, 0, v) f(s, 0, v) dv ds = \int_{\mathbb{R}_+^2} \varphi(s, 0, v) \nu_+ (ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) \nu_- (ds, dv).
\]

37
Since this holds for every $\varphi \in C_c^\infty(\mathbb{R}_+ \times (0, \infty) \times \mathbb{R})$, we first conclude that $\partial_t f + v \partial_x f = \mathcal{L}^\alpha f$ for $(t, x, v) \in (0, \infty)^2 \times \mathbb{R}$. Then in a second time, we see that

$$v f(s, 0, v) d s d v = \nu_+(d s, d v) - \nu_-(d s, d v),$$

i.e. $\nu_+(d s, d v) = v f(s, 0, v) 1_{\{v > 0\}} d s d v$ and $\nu_-(d s, d v) = -v f(s, 0, v) 1_{\{v < 0\}} d s d v$. Then, since $\nu_+(d s, d v) = \mu(d v) \int_{w \in \mathbb{R}_-} \nu_-(d s, d w)$, we conclude that $v f(t, 0, v) = -\mu(v) \int_{(-\infty, 0)} w f(t, 0, w) d w$ for $(t, v) \in (0, \infty)^2$.

We finally study the limiting fractional diffusion equation. We have the following result.

**Proposition 35.** Let $(R_t^\alpha)_{t \geq 0} = (Z^\alpha_t - \inf_{s \in [0, t]} Z^\alpha_s)_{t \geq 0}$ where $(Z^\alpha_t)_{t \geq 0}$ is the stable process from Theorem 3. Let us denote by $\rho_t(d x) = \mathbb{P}(R_t^\alpha \in d x)$. The following assertions hold.

(i) If $\alpha \in (0, 1)$, then for every $\varphi \in C_c^\infty(\mathbb{R}_+)$, we have

$$\int_{\mathbb{R}_+} \varphi(x) \rho_t(d x) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(d x) d s,$$

where

$$\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \frac{\varphi((x + z)_+) - \varphi(x)}{|z|^{1+\alpha}} d z = \frac{\sigma_\alpha}{2 \alpha} \int_0^\infty \frac{\varphi'(y)(y - x)}{|y - x|^{1+\alpha}} d y,$$

(ii) If $\alpha \in [1, 2)$, then for every $\varphi \in C_c^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$, we have

$$\int_{\mathbb{R}_+} \varphi(x) \rho_t(d x) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(d x) d s,$$

where

$$\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \text{P.V.} \int_{\mathbb{R}} \frac{\varphi((x + z)_+) - \varphi(x)}{|z|^{1+\alpha}} d z = \frac{\sigma_\alpha}{2 \alpha} \text{P.V.} \int_0^\infty \frac{\varphi'(y)(y - x)}{|y - x|^{1+\alpha}} d y,$$

where P.V. stands for principal values.

**Proof.** We will denote by $\nu(d z) = \frac{\sigma_\alpha}{2 \alpha} |z|^{-1-\alpha} d z$ the Lévy measure of the symmetric stable process $(Z^\alpha_t)_{t \geq 0}$ and we set $I_t^\alpha = \inf_{s \in [0, t]} Z^\alpha_s$. 

**Item (i):** Let us denote by $\Pi(dt, d z)$ the random Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $d t \otimes \nu(d z)$, associated with $(Z^\alpha_t)_{t \geq 0}$. Since $\alpha \in (0, 1)$, $(Z^\alpha_t)_{t \geq 0}$ has finite variations and we have $Z^\alpha_t = \int_0^t \int_{\mathbb{R}_+} \Pi(d s, d z)$. The process $(R^\alpha_t)_{t \geq 0}$ is a pure jump Markov process and we check that $\Delta R^\alpha_t = (R^\alpha_{t_+} + \Delta Z^\alpha_{t_+}) - R^\alpha_t - R^\alpha_{t_-}$. If first $\Delta Z^\alpha_t > -R^\alpha_{t_-} = -Z^\alpha_t + I^\alpha_t$, then $Z^\alpha_t > I^\alpha_t$ so that $I_t^\alpha = I_{t_-}^\alpha$ and $\Delta R^\alpha_t = \Delta Z^\alpha_t$. If next $\Delta Z^\alpha_t \leq -R^\alpha_{t_-}$, then $Z^\alpha_t \leq I^\alpha_t$ so that $I_t^\alpha = I_{t_-}^\alpha$ and thus $R^\alpha_t = 0$ i.e. $\Delta R^\alpha_t = -R^\alpha_{t_-}$. Therefore, we have $R^\alpha_t = \int_0^t \int_{\mathbb{R}_+} [(R^\alpha_{t_+} + z) + R^\alpha_{t_-}] \Pi(d s, d z)$. By Itô’s formula, we get that for any $\varphi \in C_c^\infty(\mathbb{R}_+)$,

$$\mathbb{E} [\varphi(R^\alpha_t)] = \varphi(0) + \int_0^t \int_{\mathbb{R}} \mathbb{E} [\varphi((R^\alpha_s + z)_+) - \varphi(R^\alpha_s)] \nu(d z) d s,$$

which exactly means $\int_{\mathbb{R}_+} \varphi(x) \rho_t(d x) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(d x) d s$, where the operator $\mathcal{L}^\alpha \varphi(x)$ is defined as $\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}_+} \varphi((x + z)_+) - \varphi(x) |z|^{-1-\alpha} d z$. It only remains to prove the second identity for $\mathcal{L}^\alpha$. We assume that $x > 0$, the proof for $x = 0$ being similar. We have

$$\frac{2}{\sigma_\alpha} \mathcal{L}^\alpha \varphi(x) = \frac{\varphi(0) - \varphi(x)}{x^\alpha} x^{-\alpha} + \int_{-x}^0 \frac{\varphi(x + z) - \varphi(x)}{|z|^{1+\alpha}} d z + \int_0^\infty \frac{\varphi(x + z) - \varphi(x)}{|z|^{1+\alpha}} d z$$

Then, performing carefully two integration by parts, we see that

$$\int_{-x}^0 \frac{\varphi(x + z) - \varphi(x)}{|z|^{1+\alpha}} d z = -\frac{\varphi(0) - \varphi(x)}{x^\alpha} x^{-\alpha} + \frac{1}{\alpha} \int_{-x}^0 \frac{\varphi'(x + z) z}{|z|^{1+\alpha}} d z.$$
and
\[ \int_0^\infty \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz = \frac{1}{\alpha} \int_0^\infty \frac{\varphi'(x+z)z}{|z|^{1+\alpha}} dz. \]

Putting the pieces together, we get
\[ \frac{2}{\sigma_x} \mathcal{L}^\alpha \varphi(x) = \frac{1}{\alpha} \int_{-x}^\infty \frac{\varphi'(x+z)z}{|z|^{1+\alpha}} dz = \frac{1}{\alpha} \int_0^\infty \frac{\varphi'(y)(y-x)}{|y-x|^{1+\alpha}} dy. \]

Item (ii): When \( \alpha \in [1, 2) \), the second expression of \( \mathcal{L}^\alpha \) is obtained as in the previous case. We need to remove the small jumps and work with the pure jump Lévy process \( (Z^\alpha_t)_{t \geq 0} \) with Lévy measure \( \nu_\delta(dz) = 1_{\{|z|>\delta\}}\nu(dz) \). Since \( \nu \) is symmetric, \( Z^\alpha_t \to Z^\alpha_t \) in law as \( \delta \to 0 \) for every \( t \geq 0 \). This implies, see [29, Chapter VII, Corollary 3.6], that \( (Z^\alpha_t)_{t \geq 0} \) converges in law to \( (Z^\alpha_t)_{t \geq 0} \) in the space of càdlàg functions endowed with the \( \mathbf{J}_1 \)-topology as \( \delta \to 0 \). But since the reflection map is continuous with respect to this topology, see [44, Chapter 13, Theorem 13.5.1], the continuous mapping theorem implies that \( (R^\alpha_t)_{t \geq 0} = (Z^\alpha_t - \inf_{s \in [0,t]} Z^\alpha_s)_{t \geq 0} \) converges weakly to \( (R^\alpha_t)_{t \geq 0} \) as \( \delta \to 0 \). In particular, if we set \( \rho^\delta_t(dx) = \mathbb{P}(R^\alpha_t \subset dx) \), then the probability measure \( \rho^\delta_t \) converges weakly to \( \rho_t \) as \( \delta \to 0 \).

But since \( (R^\alpha_t)_{t \geq 0} \) has finite variations, we can use the very same argument as in the first step to see that for every function \( \varphi \in C^\infty_c(\mathbb{R}_+) \), we have
\[ \int_{\mathbb{R}_+} \varphi(x) \rho^\delta_t(dx) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho^\delta_s(dx) ds, \]
where \( \mathcal{L}^\alpha \) is such that for every \( x \geq 0 \)
\[ \mathcal{L}^\alpha \varphi(x) = \frac{\sigma_x}{2} \int_{|z|>\delta} \frac{\varphi((x+z)_+) - \varphi(x)}{|z|^{1+\alpha}} dz. \]

First, it is clear that \( \int_{\mathbb{R}_+} \varphi(x) \rho^\delta_t(dx) \) converges to \( \int_{\mathbb{R}_+} \varphi(x) \rho_t(dx) \) as \( \delta \to 0 \). We will now conclude by showing that, if \( \varphi'(0) = 0 \),
\[ \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho^\delta_s(dx) ds \to \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds \quad \text{as} \quad \delta \to 0. \]  
(41)

**Step 1:** To do so, we show in Step 2 that there exists a positive constant \( C_\varphi \) such that
\[ \|\mathcal{L}^\alpha \varphi - \mathcal{L}^\alpha \varphi\|_{\infty} \leq C_\varphi \delta^{2-\alpha}. \]  
(42)

This shows that \( \mathcal{L}^\alpha \varphi \) is a continuous and bounded function. Moreover, since
\[
\left| \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds - \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho^\delta_s(dx) ds \right|
\leq \int_0^t \int_{\mathbb{R}_+} |\mathcal{L}^\alpha \varphi(x) - \mathcal{L}^\alpha \varphi(x)| \rho^\delta_s(dx) ds
\leq \int_0^t \int_{\mathbb{R}_+} |\mathcal{L}^\alpha \varphi(x) - \mathcal{L}^\alpha \varphi(x)| \rho_s(dx) ds
+ \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho^\delta_s(dx) ds - \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds,
\]
it is clear that (42) implies (41). Indeed the first term clearly goes to 0 as \( \delta \to 0 \) and the second term converges since for every \( s \in [0,t], \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho^\delta_s(dx) \to \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) \) as \( \mathcal{L}^\alpha \varphi \) is a continuous and bounded function and we can thus apply the dominated convergence theorem.

**Step 2:** We show that (42) holds. Let us first explicit a bit more \( \mathcal{L}^\alpha \varphi(x) \). Let us remark that since \( \varphi'(0) = 0 \), there is no need for principal values for \( \mathcal{L}^\alpha \varphi(0) = \int_0^\infty (\varphi(z) - \varphi(0))|z|^{-1-\alpha} dz \). If \( x > 0 \), we have for any \( \epsilon < x \),
\[
\int_{-\epsilon}^\infty \varphi((x+z)_+) - \varphi(x) - z\varphi'(x) 1_{\{|z|<\epsilon\}} dz = \int_{\mathbb{R}} \varphi((x+z)_+) - \varphi(x) - z\varphi'(x) 1_{\{|z|<\epsilon\}} dz.
\]  
(43)
For the very same reason, the same identity holds for $\mathcal{L}_x^{\alpha, \delta} \varphi(x)$ and therefore, for any $x > 0$

$$\mathcal{L}_x^{\alpha, \delta} \varphi(x) - \mathcal{L}_x^{\alpha, \delta} \varphi(x) = \int_{|z| < \delta} \frac{\varphi((x + z)_{+}) - \varphi(x) - z \varphi'(x) \mathbb{1}_{\{z < x\}}}{|z|^{1+\alpha}} \, dz$$

$$= \frac{\varphi(0) - \varphi(x)}{\alpha} (x^{-\alpha} - \delta^{-\alpha}) \mathbb{1}_{\{\delta > x\}}$$

$$+ \int_{(\delta < x)} \frac{\varphi(x + z) - \varphi(x) - z \varphi'(x) \mathbb{1}_{\{z < x\}}}{|z|^{1+\alpha}} \, dz.$$ 

Since $\varphi \in C_c^\infty(\mathbb{R}^+)$ and $\varphi'(0) = 0$, it is clear that if we set $D_\varphi = ||\varphi''||_\infty$ we have for any $x \geq 0$ and any $z \geq -x$, $|\varphi(x + z) - \varphi(x) - z \varphi'(x)| \leq D_\varphi z^2/2$. Since $\varphi'(0) = 0$, we also have for any $x \geq 0$, $|\varphi'(x)| \leq D_\varphi|x|$, from which we get that $|\varphi(x + z) - \varphi(x)| \leq D_\varphi|x| + D_\varphi z^2$. In any case, we see that for any $x \geq 0$ and any $z \geq -x$

$$|\varphi(x + z) - \varphi(x) - z \varphi'(x) \mathbb{1}_{\{z < x\}}| \leq 2 D_\varphi z^2,$$

It comes that

$$\left| \int_{-\delta \wedge x}^\delta \frac{\varphi(x + z) - \varphi(x) - z \varphi'(x) \mathbb{1}_{\{z < x\}}}{|z|^{1+\alpha}} \, dz \right| \leq \frac{4 D_\varphi}{2 - \alpha} \delta^{2-\alpha}.$$ 

We also get

$$\left| \frac{\varphi(0) - \varphi(x)}{\alpha} (x^{-\alpha} - \delta^{-\alpha}) \mathbb{1}_{\{\delta > x\}} \right| \leq \frac{D_\varphi}{\alpha} x^{-\alpha} \delta^{-2\alpha} \mathbb{1}_{\{\delta > x\}} \leq \frac{D_\varphi}{\alpha} \delta^{-2\alpha}.$$ 

All in all, we showed that for any $x > 0$, $|\mathcal{L}_x^{\alpha} \varphi(x) - \mathcal{L}_x^{\alpha, \delta} \varphi(x)| \leq C_\varphi \delta^{2-\alpha}$. When $x = 0$, we have

$$\left| \mathcal{L}_x^{\alpha} \varphi(0) - \mathcal{L}_x^{\alpha, \delta} \varphi(0) \right| \leq \int_0^\delta \frac{\varphi(z) - \varphi(0)}{|z|^{1+\alpha}} \, dz \leq \frac{D_\varphi}{2 - \alpha} \delta^{2-\alpha} \leq C_\varphi \delta^{2-\alpha}.$$ 

This shows that (42) holds.

\begin{remark}
The above proposition shows that $\rho$ is a weak solution of

\begin{align*}
\partial_t \rho_t(x) &= \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \rho_t(x - z) \mathbb{1}_{\{x > z\}} - \rho_t(x) + z \partial_x \rho_t(x) \mathbb{1}_{\{|z| < x\}} \, dz \quad \text{for } (t, x) \in (0, \infty)^2, \\
\int_0^\infty \rho_t(x) \, dx &= 1 \quad \text{for } t \in (0, \infty), \\
\rho_0(\cdot) &= \delta_0.
\end{align*}

The term $z \partial_x \rho_t(x) \mathbb{1}_{\{|z| < x\}}$ is useless when $\alpha \in (0, 1)$.

\end{remark}

\begin{proof}
As usual, we do as if $\rho_t$ was sufficiently regular so that all the computations below hold true. By the way, it might actually be the case, see for instance Chaumont-Malecki [21], but this is not our purpose. Consider a function $\varphi \in C_c^\infty((0, \infty))$. Then, by Proposition 35, we have that

$$\frac{d}{dt} \int_{\mathbb{R}_+} \varphi(x) \rho_t(x) \, dx = \int_{\mathbb{R}_+} \rho_t(x) \mathcal{L}_x^{\alpha} \varphi(x) \, dx = \frac{\sigma_\alpha}{2} [I_t(\varphi) - J_t(\varphi)],$$

where, recalling (43),

$$I_t(\varphi) = \int_{\mathbb{R}_+} \int_{-\infty}^{\infty} \rho_t(x) \frac{\varphi(x + z) - \varphi(x) - z \varphi'(x) \mathbb{1}_{\{|z| < x\}}}{|z|^{1+\alpha}} \, dz \, dx$$

and, since $\varphi(0) = 0$,

$$J_t(\varphi) = \int_{\mathbb{R}_+} \frac{\varphi(x) \rho_t(x)}{\alpha x^\alpha}.$$ 

40
We first focus on $I_t(\varphi)$. Exchanging integrals, we get
\[
I_t(\varphi) = \int_{\mathbb{R}} |z|^{-1-\alpha} \int_{(-z)\vee 0}^\infty \rho_t(x)[\varphi(x+z) - \varphi(x) - z\varphi'(x)1_{\{|z|<x\}}]dxdz.
\]
We next write, for any $z \in \mathbb{R}$,
\[
\int_{(-z)\vee 0}^\infty \rho_t(x)[\varphi(x+z) - \varphi(x) - z\varphi'(x)1_{\{|z|<x\}}]dx = \int_{z\vee 0}^\infty \rho_t(x)\varphi(x)dx - z \int_{(-z)\vee 0}^\infty \rho_t(x)\varphi(x)dx - z \int_{(-z)\vee 0}^\infty \rho_t(x)\varphi'(x)1_{\{|z|<x\}}dx.
\]
Regarding the third term, we have \( \int_{(-z)\vee 0}^\infty \rho_t(x)\varphi'(x)1_{\{|z|<x\}}dx = \int_{|z|}^\infty \rho_t(x)\varphi'(x)dx \), so that, performing an integration by part, we have
\[
z \int_{(-z)\vee 0}^\infty \rho_t(x)\varphi'(x)1_{\{|z|<x\}}dx = z\rho_t(|z|)\varphi(|z|) - z \int_{(-z)\vee 0}^\infty \varphi(x)\partial_x \rho_t(x)1_{\{|z|<x\}}dx.
\]
All in all, it holds that
\[
I_t(\varphi) = \int_{\mathbb{R}^+} \varphi(x) \int_{\mathbb{R}} \rho_t(x-z)1_{\{x>z\}} - \rho_t(x)1_{\{x>z\}} + z\partial_x \rho_t(x)1_{\{|z|<x\}} dzdx + \int_{\mathbb{R}} \frac{z}{|z|^{1+\alpha}} \rho_t(|z|)\varphi(|z|)dz.
\]
The last term is equal to zero since the integrand is an odd function. Finally, we write
\[
J_t(\varphi) = \int_{\mathbb{R}^+} \varphi(x) \int_{-\infty}^{-x} \frac{\rho_t(x)}{|z|^{1+\alpha}} dz.
\]
Recombining all the terms, we get that
\[
\int_{\mathbb{R}^+} \rho_t(x)L^\alpha \varphi(x)dx = \int_{\mathbb{R}^+} \varphi(x)A \rho_t(x)dx,
\]
where $A \rho_t$ is defined for every $x > 0$ as
\[
A \rho_t(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \rho_t(x-z)1_{\{x>z\}} - \rho_t(x) + z\partial_x \rho_t(x)1_{\{|z|<x\}} dz.
\]
Therefore, we conclude that for any $\varphi \in C^\infty_c((0,\infty))$, we have
\[
\int_{\mathbb{R}^+} \varphi(x)[\partial_t \rho_t(x) - A \rho_t(x)]dz = 0,
\]
which is enough to deduce that for any $t > 0$ and any $x > 0$, $\partial_t \rho_t(x) = A \rho_t(x)$. ☐

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