Positive definite functions on semilattices

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Abstract
We introduce a notion of positive definiteness for functions \(f: P \to \mathbb{R}\) defined on meet semilattices \((P, \preceq, \land)\) and prove several properties for these functions. In addition, we utilize the LDL\textsuperscript{T} decomposition of meet matrices in order to explore the properties of multivariate positive definite arithmetic functions \(f: \mathbb{Z}_d^+ \to \mathbb{R}\). Finally, we give a series of examples and counterexamples of positive definite functions.

Keywords: arithmetic function, positive definite function, meet matrix, GCD matrix, semilattice

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1. Introduction

Recently, a notion of positive definite arithmetic functions \(f: \mathbb{Z}_+ \to \mathbb{R}\) was introduced in [6]. The definition given in [6] is closely connected to the structure theory [8] and positive definiteness [1] of GCD matrices defined on the divisor lattice \((\mathbb{Z}_+, |)\).

In this paper, we explore an extension of the aforementioned notion of positive definiteness for a more general class of functions \(f: P \to \mathbb{R}\) defined on meet semilattices \((P, \preceq, \land)\). We discuss several properties that positive definiteness imposes on three classes of functions:

(A) functions \(f: P \to \mathbb{R}\) defined on meet semilattices \((P, \preceq, \land)\),

(B) multivariate functions \(f: P_1 \times \cdots \times P_d \to \mathbb{R}\) defined on the product of meet semilattices \((P_i, \preceq_{P_i}, \land_{P_i})\), \(i \in \{1, \ldots, d\}\),

(C) multivariate arithmetic functions \(f: \mathbb{Z}_d^+ \to \mathbb{R}\).

We remark that class (C) is a special case of class (B), whereas class (B) is a special case of the most general case (A). On the other hand, the results of [6] can thus be seen as a special case of all of the above classes.

For recent theoretical work on multivariate arithmetic functions, we refer to the excellent treatise by Tóth [10].

We give the basic notations and definitions related to meet matrices in Subsections 1.1 and 1.2. In Section 2 we give the general definition of positive definiteness for functions belonging to class (A) and establish the basic properties induced by positive definiteness on these functions. In Section 3 we first prove a series of results relating to the structure of meet matrices with Cartesian product form to support our analysis. In Section 4 we discuss how the results of the previous sections can be specialized to class (C) of multivariate arithmetic functions. Finally, we present a series of examples of positive definite functions in several variables as well as accompanying counterexamples in Section 5. We end this paper with conclusions on the results.

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1.1. Meet semilattices

Let \((P, \preceq)\) be a nonempty poset equipped with a partial order relation \(\preceq\). We denote the greatest lower bound of \(x, y \in P\) by
\[
x \land y = \sup\{z \in P \mid z \preceq x \text{ and } z \preceq y\},
\]
which is called the meet of \(x\) and \(y\) provided that it exists. The triplet \((P, \preceq, \land)\) is called a meet semilattice if \(x \land y\) exists for all \(x, y \in P\). The poset \((P, \preceq)\) is called locally finite if the interval
\[
[x, y]_P = \{z \in P \mid x \preceq z \preceq y\}
\]
is finite for all \(x, y \in P\), i.e., any two elements \(x, y \in P\) are separated by at most a finite number of elements subject to the partial ordering \(\preceq\).

A finite nonempty set \(S \subset P\) is called meet closed if \(x \land y \in S\) for all \(x, y \in S\). On the other hand, the set \(S\) is called lower closed if having any \(x \in P\) with \(x \preceq y\) for some \(y \in S\) implies that \(x \in S\). While a lower closed set is naturally meet closed, the converse is generally not true.

1.2. Table of notations

The special notations used throughout this paper are listed in the following table.

| Symbol | Description |
|--------|-------------|
| \(\hat{0}_P\) | The least element \(\hat{0}_P\) of the poset \((P, \preceq_P)\) such that \(\hat{0}_P \preceq_P x\) for all \(x \in P\). |
| \(*_P\) | The \(P\)-convolution of functions \(f, g: P \times P \to \mathbb{R}\) defined by setting \((f *_P g)(x, y) = \sum_{x \preceq_P z \preceq_P y} f(x, z)g(z, y), \ x, y \in P\). |
| \(\zeta_P\) | The incidence function \(\zeta_P(x, y) = 1\) if \(x \preceq_P y, x, y \in P\), and 0 otherwise. |
| \(\delta_P\) | The incidence function \(\delta_P(x, y) = 1\) if \(x = y, x, y \in P\), and 0 otherwise. |
| \(\mu_P\) | The Möbius function of the poset \((P, \preceq_P)\) is the inverse of \(\zeta_P\) under \(*_P\). |
| \(|\cdot|\) | The divisibility relation of positive integers: \(x | y \iff (y/x) \in \mathbb{Z}_+, x, y \in \mathbb{Z}_+\). |

For an introduction to lattices and incidence functions, see for example [7, 9].

Remark 1.1. The convolution of incidence functions as well as the Dirichlet convolution of one and several variables are all usually denoted as \(f \ast g\) in the literature. In this paper, we adopt a different convention in order to distinguish these binary operations.

2. Positive definite functions defined on semilattices

The properties of functions \(f: P \to \mathbb{R}\) with poset domains can be neatly characterized by introducing the notion of meet matrices.

Definition 2.1. Let \((P, \preceq, \land)\) be a meet semilattice, \(S = \{x_1, \ldots, x_n\}\) a finite nonempty subset of \(P\), and \(f: P \to \mathbb{R}\) a function. The matrix \(A = (S)_f\) defined by setting
\[
A_{i,j} = f(x_i \land x_j), \quad i, j \in \{1, \ldots, n\},
\]
is called the meet matrix of \(S\) with respect to \(f\).

We can now define a general notion of positive definiteness for functions defined on meet semilattices.

Definition 2.2. Let \((P, \preceq, \land)\) be a meet semilattice. A function \(f: P \to \mathbb{R}\) is called positive definite if the meet matrix \((S)_f\) is positive semidefinite for all finite sets \(S \subset P, S \neq \emptyset\).

We can employ the properties of meet matrices to obtain a characterization for Definition 2.2.
Theorem 2.3. Let \((P, \preceq, \wedge)\) be a locally finite meet semilattice. Let the finite nonempty sets \(S_i \subset P, i \in \mathbb{Z}^+\), be a covering of \(P\) such that

\[
P = \bigcup_{i=1}^{\infty} S_i.
\]

Then \(f: P \to \mathbb{R}\) is positive definite if and only if \((S_m)_f\) is positive semidefinite for all \(m \in \mathbb{Z}^+\).

Proof. Without loss of generality, the covering \((S_i)_i^\infty\) can be assumed to be nested in the sense that \(S_1 \subset S_2 \subset \cdots \subset P\) since it is always possible to construct a nested covering \((S'_i)_i^\infty\) of \(P\) by setting \(S'_1 = S_1\) and \(S'_i = S'_{i-1} \cup S_i, i \geq 2\).

The “only if” direction follows immediately from the definition. To show the converse, let us assume that the matrix \((S_m)_f\) is positive semidefinite for all \(m \in \mathbb{Z}^+\). Let \(S\) be an arbitrary finite and nonempty subset of \(P\). Then there is a positive integer \(m\) such that \(S \subset S_m\). The claim that the matrix \((S)_f\) is positive semidefinite now follows from the fact that it is a principal submatrix of the positive semidefinite matrix \((S_m)_f\), and every principal submatrix of a positive semidefinite matrix is always positive semidefinite (see, e.g., [11, Observation 7.1.2]). \(\Box\)

Example 2.4. If the locally finite meet semilattice \((P, \preceq, \wedge)\) consists of elements satisfying \(x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots\), then it is called a chain. In this case, a covering for \(P\) is given by the sets \(S_m = \{x_1, \ldots, x_m\}\), and the positive definiteness of a function \(f: P \to \mathbb{R}\) can be determined by proving the positive definiteness of the meet matrices \((\{x_1, \ldots, x_m\})_f\) for all \(m \in \mathbb{Z}^+\).

The \(LDL^T\) decompositions of meet matrices provide an excellent way to characterize positive definite functions \(f: P \to \mathbb{R}\). Since the \(LDL^T\) decomposition may be interpreted as an inertia preserving transformation of a matrix, we can deduce a criterion for the positive definiteness of functions \(f: P \to \mathbb{R}\) using the decomposition theory of meet matrices.

Theorem 2.5. Let \((P, \preceq, \wedge, \hat{0}_P)\) be a locally finite meet semilattice with the least element \(\hat{0}_P\). Let us assume that there exists a sequence of finite sets \(S_i \subset P, i \in \mathbb{Z}^+\), that cover \(P\) such that

\[
P = \bigcup_{i=1}^{\infty} S_i.
\]

Then \(f: P \to \mathbb{R}\) is positive definite if and only if

\[
(f_r *_P \mu_P)(\hat{0}_P, x) \geq 0 \quad \text{for all } x \in P,
\]

where the restricted incidence function \(f_r(\hat{0}_P, x) = f(x), x \in P\).

Proof. The covering \((S_i)_i^\infty\) can be assumed to be nested in the sense that \(S_1 \subset S_2 \subset \cdots \subset P\) (see the remark at the beginning of the proof of Theorem 2.3).

We first remark that under the assumptions of this theorem, it is possible to construct another covering for \(P\) consisting of only lower closed sets. We proceed by describing this construction.

Let us define the sets

\[
T_i = \{y \in P \mid y \preceq x, x \in S_i\}, \quad i \in \mathbb{Z}^+.
\]

Let \(i \in \mathbb{Z}^+\) be arbitrary. By the reflexivity of the partial order relation, it holds that \(x \preceq x\) for all \(x \in P\). Hence \(S_i \subset T_i\). In consequence, the sets \(T_i, i \in \mathbb{Z}^+\), form a covering for \(P\). Moreover, the set \(T_i\) must be finite since – due to the assumption that the ambient meet semilattice is locally finite – each \(y \in \hat{T}_i\) lies in the finite interval \(y \in [\hat{0}_P, x]^P\) for some \(x \in S_i\).

Next, let us show that the sets \(T_i\) are lower closed. To this end, let \(y \in T_i\) and \(z \in P\) be such that \(z \preceq y\). By construction, \(y \preceq x\) for some \(x \in S_i\). The transitive property of the partial order relation means that \(z \preceq y \preceq x \Rightarrow z \preceq x \Rightarrow z \in T_i\). Hence \(T_i\) is lower closed.

Due to the previous discussion, we may assume that the covering \((T_i)_i^\infty\) of \(P\) consists of finite and nonempty lower closed sets \(T_i, i \in \mathbb{Z}^+\). By Theorem 2.3, it suffices to show that \((T_i)_f\) is positive semidefinite.
for all $i \in \mathbb{Z}_+$. Without loss of generality, we may assume that the elements of $T_i = \{x_1, \ldots, x_n\}$ are ordered $x_i \preceq x_j \Rightarrow i \leq j$. It follows from the decomposition theory of meet matrices [2] Theorem 12] and from the formula of the Möbius function of a lower closed set [3 Example 1] that $(T_i)_f = EDE^T$, where $E$ is an $n \times n$ matrix defined as $E_{ij} = 1$ if $x_j \preceq x_i$, $E_{ij} = 0$ otherwise, and $D = \text{diag}(d_1, \ldots, d_n)$, where

$$d_i = (f_r \ast_P \mu_P)(\hat{0}_P, x_i), \quad i \in \{1, \ldots, n\},$$

where $f_r(\hat{0}_P, x) = f(x)$ for all $x \in P$.

The diagonal matrix $D$ is clearly positive semidefinite precisely when $d_i \geq 0$ for all $i \in \{1, \ldots, n\}$. Since $(T_i)_f$ is a congruence transformation of $D$, Sylvester’s law of inertia implies that $(T_i)_f$ is positive semidefinite if and only if $d_i \geq 0$ for all $i \in \{1, \ldots, n\}$.

**Corollary 2.6.** Let $(P, \preceq, \wedge, \hat{0}_P)$ be a locally finite semilattice and let $S_1 \subset S_2 \subset \cdots \subset P$ be covering for $P$, where each $S_i$ is lower closed. Let $f: P \to \mathbb{R}$ be of the form

$$f(x) = (g_r \ast_P \zeta_P)(\hat{0}_P, x), \quad x \in P,$$

where $g_r(\hat{0}_P, x) = g(x) \geq 0$ for all $x \in P$. Then $f$ is positive definite.

**Proof.** The Möbius inversion formula [3] Proposition 3.7.1 implies that

$$(f_r \ast_P \mu_P)(\hat{0}_P, x) = g(x) \geq 0,$$

where $f_r(\hat{0}_P, x) = f(x)$ for all $x \in P$. \qed

2.1. Properties of positive definite functions of the form $f: P \to \mathbb{R}$

Positive definiteness in the sense of Definition 2.2 is preserved under the following fundamental arithmetical operations.

**Theorem 2.7.** Let $f, g: P \to \mathbb{R}$ be positive definite functions. Then

(i) $af$ is positive definite for any scalar $a \geq 0$.

(ii) $f + g$ is positive definite.

(iii) $fg$ is positive definite.

**Proof.** The proofs are carried out analogously to [3 Theorem 4.4]. However, for completeness, we give the proofs below.

Let $f: P \to \mathbb{R}$ and $g: P \to \mathbb{R}$ be positive definite. Let $S = \{x_1, \ldots, x_n\}$ be a finite, nonempty subset of $P$ ordered such that $x_i \preceq x_j \Rightarrow i \leq j$.

(i) Multiplication of $f$ by a constant $a \geq 0$ preserves the positive semidefiniteness of the respective meet matrices since $x^T(S)_a f x = a x^T(S) f x \geq 0$ for all $x \in \mathbb{R}^n$.

(ii) Positive definiteness is preserved under addition of functions $f$ and $g$ since $x^T(S)_f + x^T(S)_g x = x^T(S)_{f + g} x \geq 0$ for all $x \in \mathbb{R}^n$.

(iii) In the case $fg$, the corresponding meet matrix can be written as a Hadamard product $(S)f_{fg} = (S)_f \odot (S)_g$ of two positive semidefinite matrices. By the Schur product theorem [3] Theorem 7.5.3], it follows that the resulting matrix is also positive semidefinite. \qed

Positive definite functions have the following monotonicity property.

**Theorem 2.8.** Let $f: P \to \mathbb{R}$ be positive definite. Then

(i) $f(x) \geq 0$ for all $x \in P$.

(ii) $f(x) \leq f(y)$ for $x \preceq y, x, y \in P$. 

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Proof. (i) Let $x \in P$ be arbitrary. Then by taking the singleton $S = \{x\}$, we obtain from the definition of positive definiteness that $(S)_f = f(x) \geq 0$.

(ii) Let $x, y \in P$ be such that $x \preceq y$. Then by considering the set $S = \{x, y\}$, we obtain

$$0 \leq \begin{vmatrix} f(x \wedge x) & f(x \wedge y) \\ f(x \wedge y) & f(y \wedge y) \end{vmatrix} = f(x)f(y) - (f(x))^2$$

and the assertion follows immediately. 

3. Multivariate functions

As a special case of the general definition given in Section 2, we study the positive definiteness of multivariate functions $f: P_1 \times \cdots \times P_d \to \mathbb{R}$, where $(P_i, \preceq_{P_i}, \wedge_{P_i})$ is a meet semilattice for all $i \in \{1, \ldots, d\}$. We begin by inspecting the poset $(P_1 \times \cdots \times P_d, \preceq_{P_1 \times \cdots \times P_d})$, where the product order $\preceq_{P_1 \times \cdots \times P_d}$ is defined by setting

$$x \preceq_{P_1 \times \cdots \times P_d} y \iff x_i \preceq_{P_i} y_i \text{ for all } i \in \{1, \ldots, d\},$$

where we denote $x = (x_1, \ldots, x_d) \in P_1 \times \cdots \times P_d$ and $y = (y_1, \ldots, y_d) \in P_1 \times \cdots \times P_d$ as ordered tuplets.

Let us define the pairing $(\cdot, \cdot)_{P_1 \times \cdots \times P_d}$ on the set $P_1 \times \cdots \times P_d$ by setting

$$(x, y)_{P_1 \times \cdots \times P_d} := (x_1 \wedge_{P_1} y_1, x_2 \wedge_{P_2} y_2, \ldots, x_d \wedge_{P_d} y_d) \quad \text{for all } x, y \in P_1 \times \cdots \times P_d.$$

It is straightforward to verify that this pairing defines the meet of the poset $(P_1 \times \cdots \times P_d, \preceq_{P_1 \times \cdots \times P_d})$. Let $x = (x_1, \ldots, x_d) \in P_1 \times \cdots \times P_d$ and $y = (y_1, \ldots, y_d) \in P_1 \times \cdots \times P_d$. Then

$$
\sup\{z \in P_1 \times \cdots \times P_d \mid z \preceq_{P_1 \times \cdots \times P_d} x \text{ and } z \preceq_{P_1 \times \cdots \times P_d} y\}
= \sup\{(z_1, \ldots, z_d) \in P_1 \times \cdots \times P_d \mid z_i \preceq_{P_i} x_i \text{ and } z_i \preceq_{P_i} y_i \text{ for all } i \in \{1, \ldots, d\}\}
= (x_1 \wedge_{P_1} y_1, \ldots, x_d \wedge_{P_d} y_d) = (x, y)_{P_1 \times \cdots \times P_d}.
$$

This justifies identifying $(x, y)_{P_1 \times \cdots \times P_d} = x \wedge_{P_1 \times \cdots \times P_d} y$ for $x, y \in P_1 \times \cdots \times P_d$.

We begin by inspecting the special case $d = 2$ in Subsection 3.1 due to its superior notational simplicity. In Subsection 3.2 we will consider the $d$-variational setting.

3.1. Decompositions of meet matrices of the form $(S \times T)_f$

By Definition 2.2, the positive definiteness of functions $f: P \times Q \to \mathbb{R}$ can be established by considering meet matrices $A = (S \times T)_f$ with

$$A_{i,j} = f(x_i \wedge_{P \times Q} x_j), \quad i, j \in \{1, \ldots, n\},$$

where $S \times T = \{x_1, \ldots, x_n\}$ is a subset of the poset $(P \times Q, \preceq_{P \times Q})$ ordered such that $x_i \preceq_{P \times Q} x_j \Rightarrow i \leq j$.

Remark 3.1. If $S = \{x_1, \ldots, x_n\}$ and $T = \{y_1, \ldots, y_m\}$, then

$$x_i = (x_1 + [i-1]/m), y_{i} + \text{mod}(i-1,m))$$

for $i \in \{1, \ldots, nm\}$. This is also the lexicographic ordering of the elements in $S \times T$. A connection between the Kronecker product and the lexicographic ordering will be used in Proposition 3.3 to derive an $LDL^T$ decomposition for matrices of the form (1).

We review some basic properties of meet matrices of the form $(S \times T)_f$. The following well known result applies to the Möbius function of a Cartesian product.

Lemma 3.2 (cf. [3] Proposition 3.8.2). Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be locally finite posets. The Möbius function of the poset $(P \times Q, \preceq_{P \times Q})$ is given by

$$\mu_{P \times Q}(x, y) = \mu_P(x_1, y_1) \mu_Q(x_2, y_2), \quad x, y \in P \times Q.$$

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In the following proposition we present the well-known factorization \cite{2} Theorem 12 in the case that the underlying poset is a Cartesian product of posets. This form supports our study of multivariate functions. For completeness, we give a short proof below.

**Proposition 3.3.** Let \((P, \leq_P, \land_P)\) and \((Q, \leq_Q, \land_Q)\) be locally finite meet semilattices and suppose that \(S = \{x_1, \ldots, x_n\} \subset P\) and \(T = \{y_1, \ldots, y_m\} \subset Q\) are meet closed sets ordered such that \(x_i \leq_P x_j \Rightarrow i \leq j \) and \(y_i \leq_Q y_j \Rightarrow i \leq j \), respectively. Then

\[
(S \times T)_f = (E \otimes F)\Lambda (E \otimes F)^T,
\]

where \(\otimes\) denotes the Kronecker product, \(E\) is the \(n \times n\) matrix and \(F\) is the \(m \times m\) matrix defined by setting

\[
E_{i,j} = \begin{cases} 
1, & \text{if } x_j \leq_P x_i \\
0, & \text{otherwise}
\end{cases}, \quad \text{and} \quad F_{i,j} = \begin{cases} 
1, & \text{if } y_j \leq_Q y_i \\
0, & \text{otherwise}
\end{cases},
\]

and \(\Lambda = \text{diag}(c(1 + \lfloor \frac{i-1}{m} \rfloor), 1 + \text{mod}(i-1,m)))_{i=1}^{nm}, \) where

\[
c(i,j) = \sum_{x \preceq_P x_1 \text{ and } y \preceq_Q y_1} f(x_1, y_1)\mu_S(x_1, x_i)\mu_T(y_1, y_j).
\]

**Proof.** Let us enumerate the rows and columns of the \(nm \times nm\) Kronecker product \((E \otimes F)\) by the multi-indices \(i, j \in \{1, \ldots, n\} \times \{1, \ldots, m\}\) in lexicographic order, see Table 1 for an illustration. Using this convention, the elements of the Kronecker product can be expressed concisely as

\[
(E \otimes F)_{i,j} = E_{i_1,j_1}F_{i_2,j_2}, \quad i, j \in \{1, \ldots, n\} \times \{1, \ldots, m\},
\]

where \(i = (i_1, i_2)\) and \(j = (j_1, j_2)\) are ordered pairs. By Remark \ref{r3.1} we can number the corresponding elements of \(S \times T\) by \(x_k = (x_{k_1}, y_{k_2})\) for \(k = (k_1, k_2) \in \{1, \ldots, n\} \times \{1, \ldots, m\}\). We also denote \(\Lambda_i = c(i_1, i_2)\) for \(i = (i_1, i_2) \in \{1, \ldots, n\} \times \{1, \ldots, m\}\).

Let us recall that by the Möbius inversion \cite{9} Proposition 3.7.1 we now have

\[
\Lambda_i = \sum_{k: x_k \preceq_P x_i} f(x_k)\mu_S(x_k, x_i) \iff f(x_i) = \sum_{k: x_k \preceq_P x_i} \Lambda_k. \tag{2}
\]

Hence

\[
((E \otimes F)\Lambda (E \otimes F)^T)_{i,j} = \sum_{k \in \{1, \ldots, n\} \times \{1, \ldots, m\}} \Lambda_k E_{i_1,k_1}F_{i_2,k_2}E_{j_1,k_1}F_{j_2,k_2} = \sum_{k \in \{1, \ldots, n\} \times \{1, \ldots, m\}} \Lambda_k \zeta_P(x_{k_1}, x_{i_1})\zeta_Q(y_{k_2}, y_{j_1}) \zeta_P(x_{k_1}, x_{j_1})\zeta_Q(y_{k_2}, y_{j_2}) = \sum_{k \in \{1, \ldots, n\} \times \{1, \ldots, m\}} \Lambda_k \zeta_P(x_{k_1}, x_{i_1} \land_P x_{j_1})\zeta_Q(y_{k_2}, y_{j_2} \land_Q y_{j_2}),
\]

where the last equality follows from the universal property \(x \preceq y, z \Leftrightarrow x \preceq y \land z\). Now

\[
((E \otimes F)\Lambda (E \otimes F)^T)_{i,j} = \sum_{k \in \{1, \ldots, n\} \times \{1, \ldots, m\}} \Lambda_k \zeta_P(x_{k_1}, y_{k_2}) = \sum_{k \in \{1, \ldots, n\} \times \{1, \ldots, m\}} \Lambda_k \zeta_P(x_{k_1}, x_{i_1} \land_P x_{j_1}) = \sum_{k: x_k \preceq_P x_i \land_P x_j} \Lambda_k = f(x_i \land_P x_j),
\]

where the final equality follows from (2). \qed
3.2. Decompositions of meet matrices of the form \((S_1 \otimes \cdots \otimes S_d)_f\)

Recursive application of Proposition 3.3 can be used to yield the following generalized matrix decomposition.

**Proposition 3.8.** Let \((P_i, \leq_{P_i}, \wedge_{P_i})\) be locally finite meet semilattices and suppose that \(S_i = \{x^{(i)}_1, \ldots, x^{(i)}_{n_i}\} \subset P_i\) are finite meet closed sets ordered such that \(x^{(i)}_j \leq_{P_i} x^{(i)}_k \Rightarrow j \leq k\) for all \(i \in \{1, \ldots, d\}\). Then
\[
(S_1 \otimes \cdots \otimes S_d)_f = (E^{(1)} \otimes \cdots \otimes E^{(d)}) \Lambda (E^{(1)} \otimes \cdots \otimes E^{(d)})^T,
\]
where ⊗ denotes the Kronecker product. Here, the \( n_i \times n_i \) matrices \( E^{(i)} \) are defined by setting

\[
E^{(i)}_{j,k} = \begin{cases} 
1, & \text{if } x_k^{(i)} \preceq_{P_i} x_j^{(i)} \\
0, & \text{otherwise},
\end{cases}
\]

for all \( i \in \{1, \ldots, d\} \). In addition, \( \Lambda \) is a diagonal matrix with the diagonal elements

\[
\sum_{k: x_k \preceq \sum_{i} x_i} f(x_k) \mu_{S_1 \times \cdots \times S_d}(x_k, x_i),
\]

where we set \( x_k = (x_{k_1}^{(1)}, \ldots, x_{k_d}^{(d)}) \) and the multi-indices \( i \in \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_d\} \) are enumerated according to the lexicographic order.

Proposition 3.8 is another special instance of [2, Theorem 12].

4. Multivariate arithmetic functions \( f: \mathbb{Z}_+^d \to \mathbb{R} \)

Let us begin by defining an extended GCD operator \((\cdot, \cdot)_d: \mathbb{Z}_+^d \times \mathbb{Z}_+^d \to \mathbb{Z}_+^d\) by setting

\[
(x, y)_d := (\gcd(x_1, y_1), \ldots, \gcd(x_d, y_d))
\]

for all \( x = (x_1, \ldots, x_d) \in \mathbb{Z}_+^d \) and \( y = (y_1, \ldots, y_d) \in \mathbb{Z}_+^d \).

The extended GCD operator can be used to give a definition for the positive definiteness of multivariate arithmetic functions as follows.

**Definition 4.1.** A function \( f: \mathbb{Z}_+^d \to \mathbb{R} \) is positive definite if the matrix

\[
(f((x, y)_d))_{x, y \in S}
\]

is positive semidefinite for all finite \( S \subset \mathbb{Z}_+^d, S \neq \emptyset \).

This definition can be expressed in terms of generalized Smith matrices.

**Theorem 4.2.** A function \( f: \mathbb{Z}_+^d \to \mathbb{R} \) is positive definite if and only if the matrix \((\{1, \ldots, m\}_d)^d\) is positive semidefinite for all \( m \in \mathbb{Z}_+ \).

**Proof.** The assertion follows immediately from Theorem 2.3 since the sets \( S_m = \{1, \ldots, m\}_d, m \in \mathbb{Z}_+, \) constitute a covering for \( \mathbb{Z}_+^d \).

In the following, the Dirichlet convolution of \( f: \mathbb{Z}_+^d \to \mathbb{R} \) and \( g: \mathbb{Z}_+^d \to \mathbb{R} \) is defined as

\[
(f *_d g)(i_1, \ldots, i_d) = \sum_{k_1 | i_1, \ldots, k_d | i_d} f(k_1, \ldots, k_d) g\left(\frac{i_1}{k_1}, \ldots, \frac{i_d}{k_d}\right).
\]

The identity under \(*_d\) is

\[
\delta_d(i_1, \ldots, i_d) = \delta(i_1) \cdots \delta(i_d),
\]

where \( \delta(1) = 1 \) and \( \delta(k) = 0 \) otherwise.

Let \( \mu \) denote the arithmetic Möbius function defined by setting

\[
\mu(n) = \begin{cases} 
1, & \text{if } n = 1, \\
(-1)^m, & \text{if } n \text{ is the product of } m \text{ distinct primes}, \\
0, & \text{otherwise}.
\end{cases}
\]

Now by defining \( \mu_d \) as

\[
\mu_d(i_1, \ldots, i_d) = \mu(i_1) \cdots \mu(i_d)
\]

and letting \( \zeta_d \) be defined as \( \zeta_d(i_1, \ldots, i_d) = 1 \) for all \( i_1, \ldots, i_d \in \mathbb{Z}_+ \), we have

\[
\mu_d *_d \zeta_d = \delta_d.
\]
Theorem 4.3. Let $d \geq 1$.

(i) A function $f: \mathbb{Z}_+^d \to \mathbb{R}$ is positive definite if and only if

$$\langle f * d \mu_d \rangle(i_1, \ldots, i_d) \geq 0 \quad \text{for all } i_1, \ldots, i_d \in \mathbb{Z}_+. $$

(ii) Let $f(x_1, x_2, \ldots, x_d) = g_1(x_1)g_2(x_2) \cdots g_d(x_d)$. Then $f$ is positive definite if and only if there exists $I \subset \{1, \ldots, d\}$, where $\# I$ is even, such that

$$g_i \ast_1 \mu(j) \leq 0 \quad \text{for all } i \in I \text{ and } j \in \mathbb{Z}_+$$

and

$$g_i \ast_1 \mu(j) \geq 0 \quad \text{for all } i \notin I, \ j \in \mathbb{Z}_+. $$

Proof. (i) Due to Theorem 4.2, it suffices to consider the positive semidefiniteness of matrices $([1, \ldots, m]^d)$ for all $m \in \mathbb{Z}_+$. On the other hand, Proposition 3.8 implies that the matrix $([1, \ldots, m]^d)_f$ is a congruence transformation of $\Lambda = \text{diag}(c(i_1, \ldots, i_d))_{(i_1, \ldots, i_d) \in \{1, \ldots, m\}^d}$ (multi-indices enumerated in lexicographic order), where

$$c(i_1, \ldots, i_d) = \sum_{k_{j|i_d}} f(k_1, \ldots, k_d) \mu\left(\frac{i_1}{k_1}\right) \cdots \mu\left(\frac{i_d}{k_d}\right) = \langle f * d \mu_d \rangle(i_1, \ldots, i_d).$$

Since the congruence transformation preserves the inertia of any matrix, this concludes the proof of part (i).

(ii) For $f(x_1, \ldots, x_d) = g_1(x_1) \cdots g_d(x_d)$, it holds that

$$\langle f * d \mu_d \rangle(i_1, \ldots, i_d) = \sum_{k_{j|i_d}} f(k_1, \ldots, k_d) \mu\left(\frac{i_1}{k_1}\right) \cdots \mu\left(\frac{i_d}{k_d}\right) = \sum_{k_{i_1}} g_1(k) \mu\left(\frac{i_1}{k}\right) \cdots \sum_{k_{i_d}} g_d(k) \mu\left(\frac{i_d}{k}\right) = (g_1 \ast_1 \mu)(i_1) \cdots (g_d \ast_1 \mu)(i_d),$$

which yields the assertion. \hfill \Box

5. Examples of positive definite functions of several variables

We begin this section by illustrating the monotonicity property of Theorem 2.8 for two lattices.

Example 5.1 (Monotonicity).

(a) Let $(P, \wedge, \hat{0}_P) = (\mathbb{Z}_+^2, (\cdot, \cdot), 1)$ be the two-dimensional divisor lattice. In this case, $\zeta_P(x, y) = 1$ whenever $x_1 | y_1$ and $x_2 | y_2$ for $x = (x_1, x_2) \in \mathbb{Z}_+^2$ and $y = (y_1, y_2) \in \mathbb{Z}_+^2$ and $0$ otherwise.

(b) Let $(P, \wedge, \hat{0}_P) = (\mathbb{Z}_+^2, \text{min}_{2}, 1)$ be the two-dimensional MIN lattice, where

$$\min_2((x_1, y_1), (x_2, y_2)) = (\min(x_1, x_2), \min(y_1, y_2))$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{Z}_+^2$. In this case, $\zeta_P(x, y) = 1$ whenever $x_1 \leq y_1$ and $x_2 \leq y_2$ for $x = (x_1, x_2) \in \mathbb{Z}_+^2$ and $y = (y_1, y_2) \in \mathbb{Z}_+^2$ and $0$ otherwise.

It is not difficult to see that here one can obtain an absolutely increasing positive definite function $f: P \to \mathbb{R}$ with respect to the partial order relation by defining $f(x) = g_r \ast_P \zeta_P)(\hat{0}_P, x)$ with $g(x) = g_r(\hat{0}_P, x) = 1$ for all $x \in P$ (see Corollary 2.4). Illustrations of the relative magnitude of values that these positive definite functions take with respect to each lattice (a) and (b) are given in Figure 1.
Example 5.2.

(a) Let $f(x, y) = \gcd(x, y)$. Now the generalized GCD matrix $A = (\{1, \ldots, m\}^2)f$ has the form

$$A(i_1, i_2, j_1, j_2) = \gcd(\gcd(i_1, j_1), \gcd(i_2, j_2)) = \gcd(i_1, i_2, j_1, j_2),$$

where the rows and columns are enumerated by the multi-indices $(i_1, i_2, j_1, j_2) \in \{1, \ldots, m\}^2$. Since $\{1, \ldots, m\}$ is meet closed, the matrix $A$ has rank $m$ and its full-rank submatrix can be found by considering the rows and columns with indices $(i_1, i_2, j_1, j_2) = (1, 1, 2, 2), \ldots, (m, m)$. In particular, $A$ admits to a congruence transformation $P A P^T = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$, where $B_{i,j} = \gcd(i, j)$, $i, j \in \{1, \ldots, m\}$, and $P$ is an elimination matrix (including pivots) targeting all linearly dependent rows of $A$. Since $B$ is well known to be positive definite for all $m \in \mathbb{Z}_+$, we conclude that the matrix $A$ is positive semidefinite and the function $f$ is positive definite in consequence.

(b) Let $f(x, y) = \text{lcm}(x, y)$. The characteristic polynomial of $((1, 2)^2)f$ is

$$p(\lambda) = (\lambda - 1)(\lambda^3 - 6\lambda^2 + 1),$$

which has one negative root. Therefore $f$ is not positive definite.

The trick we used in part (a) of Example 5.2 can be generalized.

Example 5.3. Let $(P, \preceq, \wedge, \hat{0}_P)$ be a locally finite meet semilattice and suppose that there exists a covering $(S_i)_{i=1}^\infty$ consisting of finite and nonempty subsets of $P$. Let us assume that $g: P \to \mathbb{R}$ is a positive definite function.

Define the function $f: P \times \cdots \times P \to \mathbb{R}$ by setting $f(x_1, \ldots, x_d) = g(x_1 \wedge \cdots \wedge x_d)$ and let $S = \{x_1, \ldots, x_m\}$ be a finite lower closed subset of $P$ ordered such that $x_i \preceq x_j \Rightarrow i \leq j$. It suffices to show the positive semidefiniteness of the meet matrix $(S)f$ for all finite and nonempty lower closed sets $S \subset P$.

\footnote{Notice that by the remarks in the beginning of the proof of Theorem 2.5 we can now find a lower closed covering of $P$. Thus it is sufficient to prove the positive semidefiniteness of the meet matrix $(S)f$ for all finite and nonempty lower closed sets $S \subset P$.}
semidefiniteness of the meet matrix

\[ A_{(i_1, \ldots, i_d), (j_1, \ldots, j_d)} = g(x_{i_1} \land x_{j_1} \land \ldots \land x_{i_d} \land x_{j_d}), \]

where \((i_1, \ldots, i_d), (j_1, \ldots, j_d) \in \{1, \ldots, m\}^d\) enumerate the rows and columns, respectively. Define the \(m \times m\) matrix \(B\) by setting \(B_{ij} = g(x_i \land_p x_j)\) for all \(i, j \in \{1, \ldots, m\}\). Since \(S\) is factor closed, it is especially meet closed, and thus \(A\) contains only \(m\) linearly independent rows and columns. Hence there exists an elimination matrix \(P\) (with pivots) such that

\[ PAP^T = \begin{pmatrix} B & O \\ O & O \end{pmatrix}. \]

Now the multivariate function \(f\) is positive definite precisely when the univariate function \(g\) is positive definite. Some examples of positive definite multivariate arithmetic functions \(f: \mathbb{Z}_+^2 \rightarrow \mathbb{R}\) are \(f(x, y) = \text{gcd}(x, y)^\alpha\) for \(\alpha > 0\) and \(f(x, y) = \text{lcm}(x, y)^\alpha\) for \(\alpha < 0\).

**Example 5.4.** Assume that \(f: \mathbb{Z}_+^2 \rightarrow \mathbb{R}\) is of the form

\[ f = g \ast_2 \zeta_2, \]

where \(g\) is always nonnegative. Then by Corollary 2.6, \(f\) is positive definite.

Typical examples of nonnegative functions covered by Example 5.4 are combinatorial number-theoretic functions counting the number of integers satisfying certain conditions. For example, number of solutions of certain congruences, see [7].

**Example 5.5.** Let us consider Ramanujan’s sum \(C(m, n)\), see [7]. It is well known that

\[ C(m, n) = \sum_{d | \gcd(m, n)} d \mu\left(\frac{n}{d}\right). \]

Let

\[ P(m, n) = \begin{cases} n, & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases} \]

Now

\[ C(m, n) = \sum_{d | m \text{ and } e | n} P(d, e) \zeta\left(\frac{m}{d}\right) \mu\left(\frac{n}{e}\right) = (P \ast_2 (\zeta \mu))(m, n) \]

and thus

\[ (C \ast_2 \mu_2)(m, n) = (P \ast_2 (\zeta \mu) \ast_2 \mu_2)(m, n) \]
\[ = (P \ast_2 \delta(\mu \ast_1 \mu))(m, n) \]
\[ = \sum_{d | \gcd(m, n)} d \delta\left(\frac{m}{d}\right) \mu\ast_1 \mu\left(\frac{n}{d}\right) \]
\[ = \begin{cases} 0, & \text{if } m \nmid n, \\ m(\mu \ast_1 \mu)\left(\frac{n}{m}\right), & \text{if } m | n. \end{cases} \]

For suitable \(m\) and \(n\), the term \((\mu \ast_1 \mu)(n/m)\) can be positive, negative, or zero since

\( (\mu \ast_1 \mu)(1) = 1, \)
\( (\mu \ast_1 \mu)(p) = -2, \)
\( (\mu \ast_1 \mu)(p^2) = 1, \)
\( (\mu \ast_1 \mu)(p^k) = 0 \text{ for } k \geq 3, \)

and \(\mu \ast_1 \mu\) is multiplicative. Therefore, \(C(m, n)\) is not positive definite.
Conclusions

In this work, we have generalized the recently introduced notion of positive definiteness of arithmetic functions \( f : \mathbb{Z}_+ \rightarrow \mathbb{R} \) to functions defined on posets. In particular, we have shown that the major results proved for positive definite arithmetic functions in [6] generalize in a natural way to functions \( f : P \rightarrow \mathbb{R} \) with poset domains.

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