Minimal Surfaces with Planar Boundary Curves

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1 Introduction

In 1956, Shiffman [Sh] proved that any compact minimal annulus with two convex boundary curves (resp. circles) in parallel planes is foliated by convex planar curves (resp. circles) in the intermediate planes. In 1978, Meeks conjectured that the assumption the minimal surface is an annulus is unnecessary [M]; that is, he conjectured that any compact connected minimal surface with two planar convex boundary curves in parallel planes must be an annulus.

Partial results have been proven in the direction of this conjecture. Schoen [Sc1] proved the Meeks conjecture in the case where the two boundary curves share a pair of reflectional symmetries in planes perpendicular to the planes containing the boundary curves. Another interesting result related to the Meeks conjecture has been proven by Meeks and White (Theorem 1.2, [MW2]). Recall that a minimal surface $M$ is called stable if, with respect to any normal variation that vanishes on $\partial M$, the second derivative of the area functional is positive. The minimal surface is unstable if there exists such a variation with negative second derivative for the area functional, and it is almost-stable if the second derivative is nonnegative for all such variations and is zero for some nontrivial variation. Recall also that a subset of $\mathbb{R}^3$ is called extremal if it is contained in the boundary of its convex hull. The result of Meeks and White is that if $\Gamma$ is an extremal pair of smooth disjoint convex curves in distinct planes, then exactly one of the following holds:

1) $\Gamma$ is not the boundary of any connected compact minimal surface, with or without branch points.

2) $\Gamma$ is the boundary of exactly one minimal annulus and this annulus is almost-stable. In this case, $\Gamma$ bounds no other connected branched minimal surface.
3) $\Gamma$ is the boundary of exactly two minimal annuli, one stable and one unstable.

Other partial results toward the Meeks conjecture have been proven by Meeks and White for stable surfaces [MW1], [MW2]. They have proven the conjecture for stable and almost-stable minimal surfaces that have two convex boundary curves lying in parallel planes such that

1) the two boundary curves have a common plane of reflective symmetry perpendicular to the planes containing them, or

2) the two boundary curves are reflected into each other by a plane parallel to the planes containing them.

The first of these two conditions has been extended by Meeks and White to boundary curves lying in nonparallel planes, but still forming an extremal set. The second of these two conditions is generalized to nonparallel planes by Theorem 2.1 in the next section.

In section 3 we consider a more general setting: compact connected minimal surfaces, with a pair of boundary curves (not necessarily convex) in distinct planes, that have least-area amongst all orientable surfaces with the same boundary. When the planes containing these two boundary curves are either parallel or “sufficiently close” to parallel, and when the boundary curves themselves are “sufficiently close” to each other, one can draw specific conclusions about the geometry and topology of the surfaces, as in Theorem 3.1 and Theorem 3.2. (These theorems are formal statements about surfaces that can be physically realized by experimentation with wire frames and soap films.)

In the final section 4 we state two well known results on the existence of compact minimal surfaces with planar boundary curves in parallel planes. These results are easily proven using the maximum principle for minimal surfaces, but do not appear elsewhere in the literature. A corollary of these results is a generalization of a result of Nitsche [N1]: Let $M$ be any compact minimal annulus with two planar boundary curves of diameters $d_1$ and $d_2$ in parallel planes $P_1$ and $P_2$; if the distance between $P_1$ and $P_2$ is $h$, then the inequality $h \leq \frac{3}{2} \max\{d_1, d_2\}$ is satisfied. The corollary we prove here does not assume the minimal surface $M$ is an annulus and has the strengthened conclusion $h \leq \max\{d_1, d_2\}$. We also include a similar result for nonminimal constant mean curvature surfaces.

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2 Topological Uniqueness in the Stable Case

Meeks and White [MW1] proved the following, which we generalize in Theorem 2.1: Let $\alpha$ be a smooth convex plane curve in $\mathbb{R}^3$, let $P_0$ be a plane parallel to the plane containing $\alpha$, and let $\text{Ref} : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection in the plane $P_0$. If $\Sigma$ is a connected stable or almost-stable compact minimal surface with boundary $\alpha \cup \text{Ref}(\alpha)$, then $\Sigma$ is an embedded annulus.

Theorem 2.1 is similar to the above result, but we assume $\alpha$ does not lie in a plane parallel to $P_0$. We assume only that $\alpha$ lies on one side of $P_0$, thus $\alpha$ and $\text{Ref}(\alpha)$ do not lie in parallel planes. In order to state this precisely, we introduce some notation.

Consider all closed half-planes in $\mathbb{R}^3$ with boundary the $x_1$-axis. Let $\omega(0)$ be the half-plane containing the positive $x_2$-axis, and let $\omega(\theta)$ be the half-plane making an angle of $\theta$ with $\omega(0)$. (Thus the half-plane containing the positive $x_3$-axis is $\omega(\pi/2)$.) Let $R_\theta$ be a rotation of angle $\theta$ about the $x_1$-axis.

**Definition 2.1** The wedge between $\omega(\gamma)$ and $\omega(\beta)$ of angle $\beta - \gamma$ in $\mathbb{R}^3$ is $W_{\omega,\beta} = \cup_{\gamma \leq \alpha \leq \beta} \omega(\alpha)$.

**Theorem 2.1** If $M \subseteq W_{\gamma,\beta}$ is a stable or almost-stable compact connected minimal surface with boundary $\partial M = C \cup R_{\beta-\gamma}(C)$, where $C$ is a strictly convex curve contained in $\omega(\gamma)$, then $M$ is an embedded annulus.

We now describe the natural free boundary problem for a wedge. Suppose $C$ is a strictly convex Jordan curve in $\omega(\beta)$ and $M$ is a compact branched minimal surface such that the boundary $\partial M$ consists of $C$ and a nonempty collection of immersed curves in the $x_1x_2$-plane. We may assume that $0 \leq \beta \leq \pi/2$, for if not, by reflections through the $x_1x_3$-plane and $x_1x_2$-plane, we have a congruent problem where this is so. If $M$ is orthogonal to the $x_1x_2$-plane along $\partial M \cap \omega(0)$, then $M$ is called a solution of the free boundary value problem for $C$ and $W_{0,\beta}$. By the maximum principle (see, for example, [Sc1]), if $M$ is such a solution, then the portion of $\partial M$ lying in the $x_1x_2$-plane actually lies in $\omega(0)$. If, with respect to any normal variation of $M$ that vanishes on $C$, the second derivative of the area functional is positive, then $M$ is a stable solution to the free boundary value problem. Similarly, we can define when $M$ is unstable or almost-stable [MW1], [MW2].

**Corollary 2.1** Suppose $M$ is a stable or almost-stable solution to the free boundary value problem for $C$ and $W_{0,\beta}$, then $M$ is an embedded annulus.
The proof of Theorem 2.1 requires the use of the Jacobi operator and Jacobi fields. Following the notation of Choe [C], three types of Killing vector fields in $\mathbb{R}^3$ are $\phi_n$, $\phi_l$, and $\phi_p$, where these three vector fields are the variation vector fields produced by translating in the direction of the unit vector $n$, rotating around a straight line $l$, and homothetically expanding from a point $p$, respectively. Meeks and White [MW1] were interested in $\phi_n$, and here we are interested in $\phi_p$.

Let $S$ be a smooth immersed surface in $\mathbb{R}^3$. The horizon of $S$ with respect to $\phi_p$, denoted by $H(S; \phi_p)$, is the set of all points of $S$ at which $\phi_p$ is a tangent vector of $S$. A connected subset $D$ of $S$ is called a visible set with respect to $\phi_p$ if $D$ is disjoint from $H(S; \phi_p)$. If $M$ is a minimal surface in $\mathbb{R}^3$, then $\phi_n(M^\perp)$, $\phi_l(M^\perp)$, and $\phi_p(M^\perp)$ (the projection of $\phi_n$, $\phi_l$, and $\phi_p$ onto the normal bundle of $M$, respectively) are Jacobi fields [C]. If $M$ has a strictly proper open connected subset $D$ such that $\partial D \subseteq H(M; \phi_p)$, then it follows from the Smale index theorem (see, for example, [C], p199) that $D$ is almost-stable or unstable, and hence $M$ is unstable. We use this fact in the following proof.

**Proof of Theorem 2.1.** By a rotation of $\mathbb{R}^3$ if necessary, we may assume $M \subseteq W_{-\beta, \beta}$, with $\partial M = C \cup R_{-2\beta}(C)$, $C \subseteq \omega(\beta)$, $R_{-2\beta}(C) \subseteq \omega(-\beta)$, and $0 < \beta \leq \pi/2$. In fact $\beta < \pi/2$, since otherwise (by the maximum principle) $M$ would be a pair of disks, which is not connected. The orthogonal projection $\Lambda$ from $\mathbb{R}^3$ to the $x_1x_2$-plane maps $C$ and $R_{2\beta}(C)$ to a strictly convex smooth Jordan curve $\Lambda(C)$ in $\omega(0)$. By Theorem 2 of [Sc1], $M$ is embedded and $M = \{(x_1, x_2, \pm u(x_1, x_2)) \mid (x_1, x_2) \in \Omega\}$, where $\Omega$ is a compact region in $\omega(0)$, and $u(x_1, x_2)$ is a nonnegative function defined on $\Omega$. Furthermore, since $M_0^+ = M \cap \{x_3 \geq 0\}$ and $M_0^- = M \cap \{x_3 \leq 0\}$ have locally bounded slope in their interiors [Sc1], and since the maximum principle immediately implies that the tangent planes of $M$ along $C$ and $R_{-2\beta}(C)$ can never by vertical, the normal vectors of $M$ are horizontal only on $M \cap \omega(0)$.

Assume $M$ is not an annulus. Then $\partial \Omega = \Lambda(C) \cup C_1 \cup \cdots \cup C_k$, where $k > 1$, and $C_i$ is a curve in the compact region of $\omega(0)$ bounded by $\Lambda(C)$, for all $i$. Note that the curves $C_i$ are also planar geodesics in $M$. We now check that each $C_i$ must be strictly convex. If not, then there exists an $i$ and a $q \in C_i$ such that the Gaussian curvature of $M$ at $q$ is zero. Thus in a small neighborhood of $q$, $M \cap T_q(M)$ is a set of at least three curves crossing at $q$. This implies $M_0^+$ cannot be a graph, contradicting Theorem 2 of [Sc1].

There must be at least one point in the interior of $M_0^+$ at which the Gaussian curvature $K$ vanishes. This follows from an argument identical to the argument given in the third paragraph of the proof of Theorem 3.1 in [MW1], so we do not
repeat it here. Thus there exists a point \( q \in \text{Int}(\Omega) \) such that the curvature \( K \) at \((q, u(q))\) vanishes. Let \( p \) be a point on the \( x_1 \)-axis which is also contained in \( T_{(q, u(q))}(M) \). By translating if necessary we may assume that \( p \) is the origin \( \vec{0} \). Thus \((q, u(q)) \in H(M; \phi_{\vec{0}})\). (If \( T_{(q, u(q))}(M) \) does not intersect the \( x_1 \)-axis, then we can replace \( \phi_{\vec{0}} \) with \( \phi_{\vec{e}_1, n} = \vec{e}_1 \), and the remaining arguments of this proof follow through.)

Since \( G \) is a conformal map with branch points, \( H(M; \phi_{\vec{0}}) \) consists of smooth curve segments, whose endpoints meet in even numbers at isolated points in the interior of \( M \). In particular, at least four such curves meet at the point \((q, u(q))\), because it is a branch point of \( G \). Note also that \( H(M; \phi_{\vec{0}}) \) intersects each of \( C, R_{-2\beta}(C) \), and each \( C_i \) at exactly two points, as these are strictly convex planar curves. Let \( Z = \Lambda(H(M; \phi_{\vec{0}})) \). Since \( Z \) is homeomorphic to \( H(M_0^+; \phi_{\vec{0}}) \), it has the same structure. Hence \( Z \) meets each \( C_i \) and also \( \Lambda(C) \) exactly twice.

Now form a topological space \( \hat{\Omega} \) from \( \Omega \) by identifying each \( C_i \) to a point, and note that \( \hat{\Omega} \) is topologically a disk. The corresponding set \( \hat{Z} \) (of \( Z \)) in \( \hat{\Omega} \) is a graph in which each vertex (except for the two vertices on \( \Lambda(C) = \partial \hat{\Omega} \)) has an even number of edges. Furthermore, the vertex \( q \) has at least four edges. It follows that \( \hat{\Omega} - \hat{Z} \) contains at least one connected component \( \hat{U} \) whose closure does not intersect \( \Lambda(C) = \partial \hat{\Omega} \). Let \( U \) be the corresponding region in \( \Omega \), and let \( \tilde{U} = \{ (x_1, x_2, x_3) \in M \mid (x_1, x_2) \in U \} \). Then \( \partial \tilde{U} \subseteq H(M, \phi_{\vec{0}}) \), and therefore \( \tilde{U} \) is an almost-stable or unstable proper subset of \( M \), so \( M \) must be unstable (by the Smale index theorem). This contradiction implies that \( k = 1 \) and \( M \) is an annulus.

**Proof of Corollary 2.1.** Let \( M \) satisfy the conditions of Corollary 2.1, and let \( \text{Ref}(M) \) be the reflection of \( M \) across \( \omega(0) \). It follows from the Schwarz reflection principle (see, for example, [O1], Lemma 7.3) that \( M \cup \text{Ref}(M) \) is a smooth minimal surface which satisfies the conditions of Theorem 2.1 and therefore is an embedded annulus. By the arguments above, \( \partial M \) intersects \( \omega(0) \) along a single curve (since \( k = 1 \)), and \( M \) is a graph over \( \omega(0) \), so \( M \) itself must be an embedded annulus.

**Remark.** It is not possible to generalize the Meeks and White result or Theorem 2.1 to the case where a boundary curve has a 1-to-1 perpendicular projection to a strictly convex planar curve, but is not itself a planar curve. A counterexample can be constructed as follows: Let the \( x_1 \),\( x_2 \)-plane be the plane in which the strictly convex projection lies. Consider the two line segments connecting the points \((0,0,1)\) and \((N,0,1)\), \((0,\delta,1)\) and \((N,\delta,1)\), respectively, for some large \( N \). Connect them at the ends with two horizontal semicircles of radius \( \frac{\delta}{2} \). (The semicircles are \( \{ x_1 \leq 0 \} \cap \{ x_3 = 1 \} \cap \{ x_1^2 + (x_2 - \frac{\delta}{2})^2 = (\frac{\delta}{2})^2 \} \) and \( \{ x_1 \geq N \} \cap \{ x_3 = 1 \} \cap \{ (x_1 - N)^2 + (x_2 - \frac{\delta}{2})^2 = (\frac{\delta}{2})^2 \} \).
A small perturbation gives us a smooth strictly convex curve in the plane \( \{ x_3 = 1 \} \), which is symmetric with respect to reflection in the plane \( \{ x_2 = \frac{d}{2} \} \). Along each of the two almost straight segments deform the third coordinate of the curve by making \( n \) smooth vertical dips almost to the \( x_1x_2 \)-plane, so that the curve remains symmetric with respect to the plane \( \{ x_2 = \frac{d}{2} \} \). Calling this resulting loop \( \alpha \), it is easy to imagine a stable embedded minimal surface of genus \( n - 1 \) with boundary \( \alpha \cup \text{Ref}(\alpha) \), where Ref is reflection through the \( x_1x_2 \)-plane (see Figure 1).

\[ \square \]

3 Topological Uniqueness for Least-Area Surfaces

In this section, we extend our considerations to include non-convex boundary curves.

Let \( \alpha \) and \( \beta \) be two \( C^2 \) planar Jordan curves (not necessarily convex) in parallel planes \( P_0 \) and \( P_1 \), respectively. Without loss of generality, we may assume \( P_0 = \{ x_3 = 0 \} \) and \( P_1 = \{ x_3 = 1 \} \). Let \( \alpha(t) \) be a one-parameter family of planar curves (lying in planes \( P_t = \{ x_3 = t \} \) parallel to \( P_0 \)) that are vertical translations of the curve \( \alpha = \alpha(0) \) in the direction of the plane \( P_1 \) at constant unit speed (making \( \frac{\partial}{\partial t} \alpha(t) \) a constant vector) so that \( \alpha(1) \) and \( \beta \) both lie in \( P_1 \). Consider the points of \( \beta \) where \( \alpha(1) \) and \( \beta \) intersect. Assume these intersections are transverse, and that there are a finite number \( n \) of them. We will call these points \( \{ p_1, ..., p_n \} \) on \( \beta \) the crossing points of \( \alpha \) and \( \beta \). Since \( \alpha(1) \) intersects \( \beta \) transversely, every vertex in the graph \( \alpha(1) \cup \beta \) has exactly four edges emanating from it. Furthermore, to each of the connected components of \( P_1 \setminus \{ \alpha(1) \cup \beta \} \) we can uniquely assign either a plus sign or a minus sign so that the following statement is true: No two adjacent components have the same sign, and the single component which is not compact has a minus sign (see Figure 2). Define \( s \) to be the number of components with a plus sign. Define \( \mathcal{W} \) to
be the union of the open components of $P_1 \setminus \{\alpha(1) \cup \beta\}$ which are assigned a plus sign.

For any set $B \subseteq P_1$, let $C(B) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2, 1) \in B\}$ be the cylindrical domain in $\mathbb{R}^3$ over the set $B$. Let $B_{\varepsilon,P_1}(p)$ be an open $\varepsilon$-ball in $P_1$ about a point $p \in P_1$.

**Theorem 3.1** Suppose that $W$ is the unique least-area set in $P_1$ with boundary equal to $\alpha(1) \cup \beta$. Let $\{p_1, \ldots, p_n\} = \alpha(1) \cap \beta$ be the crossing points. Then there exists $\varepsilon > 0$ and $t_0 \in (0, 1)$ so that for all $t \in [t_0, 1)$, there is an orientable minimal surface $M(t) \subset \mathbb{R}^3$ with boundary $\alpha(t) \cup \beta$ having the following properties:

1) $M(t)$ has least-area amongst all orientable surfaces with the same boundary.

2) $M(t)$ is embedded.

3) $M(t) \setminus C(\cup_{i=1}^n B_{\varepsilon,P_1}(p_i))$ is a graph over the set $W \subset P_1$.

4) For each $i$, $M(t) \cap C(B_{\varepsilon,P_1}(p_i))$ is homeomorphic to a disk with total absolute curvature less than $2\pi$.

5) $M(t)$ has genus $\frac{n-s}{2}$.

**Remark.** The assumption that $M(t)$ is orientable is natural, since $\alpha(t) \cup \beta$ is extremal and therefore any compact non-orientable minimal surface with this boundary cannot be embedded. And any surface which is not embedded cannot be least-area, since one can easily decrease area by adding topology and desingularizing at intersection points.
Remark. The assumption that $W$ is the least-area set in $P_1$ with boundary equal to $\alpha(1) \cup \beta$ is really necessary. For $\alpha$ and $\beta$ for which this assumption does not hold, it follows from the results of [HS] and [D] that there exists an embedded minimal surface with the same boundary and with area strictly less than that of the $M(t)$ described above. Again, embeddedness of this new surface with lesser area implies that it is also orientable. \hfill $\square$

Remark. With further arguments, one can show that in a small vertical neighborhood of each crossing point, $M(t)$ is ”approximately helicoidal” in the following sense: For each crossing point $p_j \in P_1$, there exists a portion $S \subset \mathbb{R}^3$ of a helicoid bounded by a pair of infinite lines and with total absolute curvature less than $2\pi$, and there exist homotheties $\phi_t$ centered at $p_j$ for each $t \in (0, 1)$, such that for any sequence $t_i \to 1$ with $t_i \to 1$, $\{\phi_{t_i}(M(t_i))\}_{i=1}^{\infty}$ converges to $S$. (We say that a sequence of surfaces $\{S_i\}_{i=1}^{\infty}$ converges as $i \to \infty$ to a surface $S$ in $\mathbb{R}^3$ if, for any compact region $B$ of $\mathbb{R}^3$, there exists an integer $N_B$ such that for $i > N_B$, $S_i \cap B$ is a normal graph over $S$, and $\{S_i \cap B\}_{i=1}^{\infty}$ converges to $S$ in the $C^1$-norm.) \hfill $\square$

Before giving the proof of Theorem 3.1, we give an useful preliminary lemma. For an oriented surface $M \subset \mathbb{R}^3$, let $\text{dist}_M(A, B)$ be the distance in $M$ between two sets $A \subset M$ and $B \subset M$. Let $\partial M$ be the boundary of $M$. For each point $q \in M$, let $K_q$ be the Gaussian curvature of $M$ at $q$, and let $\vec{N}_q$ be the oriented unit normal vector of $M$ at $q$. Let $\vec{e}_3 = (0, 0, 1)$, and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^3$.

Lemma 3.1 Let $M$ be an oriented minimal surface lying in the closed region $\{t \leq x_3 \leq 1\}$ between the two distinct planes $P_1$ and $P_1$, and consider a point $p \in M$. Suppose there exists a positive constant $A$ such that $\text{dist}_M(p, \partial M) \geq 1/\sqrt{A}$ and $|K_q| < A$ for all $q \in M$. Then, for $T := 1 - 2\sqrt{A}(1-t)$,

$$|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > T.$$ 

Proof. Note that $T < 1$, and that the result is obvious if $T < 0$, so we assume $T \in [0, 1)$. Suppose that $|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 \leq T$. Then there exists a unit vector $\vec{T} \in T_pM$ so that $\langle \vec{T}, \vec{e}_3 \rangle \geq \sqrt{T - T}$, and there exists a unit speed geodesic $\gamma(s) \subset M, s \in [0, 1/\sqrt{A}]$ so that $\gamma(0) = p$ and $\gamma'(0) = \vec{T}$, where $t = \frac{\partial}{\partial s}$. Let $k_9(s)$ be the geodesic curvature of $\gamma(s)$. Since $|K_q| < A$ and $M$ is minimal, $|k_9(s)| < \sqrt{A}$ for all $s \in [0, 1/\sqrt{A}]$. Thus $|\gamma'''(s)| < \sqrt{A}$. Writing $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ in terms of coordinates in $\mathbb{R}^3$, we have $|\gamma_3''(s)| < \sqrt{A}$. Then for $s \in [0, 1/\sqrt{A}]$,

$$|\gamma_3'(s) - \gamma_3'(0)| = \left| \int_0^s \gamma_3'' \right| \leq \int_0^s |\gamma_3''| < \sqrt{A} \cdot s.$$
and thus $\gamma_3'(s) > \gamma_3'(0) - \sqrt{A} \cdot s$. Therefore

$$\int_0^{\gamma_3'(0)} \gamma_3'(s) ds > \int_0^{\gamma_3'(0)} (\gamma_3'(0) - \sqrt{A} \cdot s) ds = \gamma_3'(0) \cdot \frac{\gamma_3'(0)}{\sqrt{A}} - \frac{1}{2} \sqrt{A} \left( \frac{\gamma_3'(0)}{\sqrt{A}} \right)^2,$$

and so

$$\gamma_3 \left( \frac{\gamma_3'(0)}{\sqrt{A}} \right) - \gamma_3(0) > \frac{(\gamma_3'(0))^2}{2\sqrt{A}} \geq \frac{1 - T}{2\sqrt{A}} = 1 - t,$$

since $\gamma_3'(0) = \langle \vec{T}, \vec{e}_t \rangle \geq \sqrt{1 - T}$. Hence the vertical change $\gamma_3(\gamma_3'(0)/\sqrt{A}) - \gamma_3(0)$ of the geodesic $\gamma(s) \in \mathcal{M}$ is greater than $1 - t$, and since $1 - t$ is the distance between the planes $P_t$ and $P_1$, $\mathcal{M}$ cannot lie between $P_t$ and $P_1$. This contradiction proves the lemma. \hfill $\square$

We now give the proof of Theorem 3.1:

**Proof.** It was shown in [HS] that there exists an orientable minimal surface $M(t)$ with boundary $\alpha(t) \cup \beta$ that has least-area amongst all orientable surfaces with the same boundary, and that $M(t)$ is embedded, with finite genus, and with bounded curvature. $M(t)$ cannot have any interior branch points ([O2]), and since its boundary is extremal, it cannot have any boundary branch points ([N2], section 366).

We divide the proof into steps.

We fix $\epsilon$ to be a small positive number such that $\frac{1}{\epsilon}$ is much greater than the maximum planar curvature of $\alpha \cup \beta$. (In steps 3 and 6 we will add further constraints on $\epsilon$.) Let $N_{\epsilon,P_1}(\alpha(1) \cup \beta) := \{ p \in P_1 \mid \text{dist}(p, \alpha(1) \cup \beta) < \epsilon \}$ be the $\epsilon$ neighborhood of $\alpha(1) \cup \beta$ in $P_1$.

**Step 1:** $M(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cup \beta))$ is a collection of graphs over $P_1$ for $t$ sufficiently close to 1. In fact, for any $y \in (0,1)$, there exists a $t_y \in (0,1)$ such that for all $t \in [t_y,1)$, $|\langle \vec{N}_p, \vec{e}_\delta \rangle| > y$ for all $p \in M(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cup \beta))$.

By Corollary 4 of [Sc2] there exists a universal constant $c \geq 1$ such that $|K_q| < \frac{c}{r}$, where $K_q$ is the Gaussian curvature at some point $q$ of a stable minimal surface and $r$ is the distance from $q$ to the boundary of the surface. Thus for all $q \in M(t) \setminus C(N_{\epsilon/2,P_1}(\alpha(1) \cup \beta))$, $|K_q| < \frac{4c}{\epsilon^2}$. We now apply Lemma 3.1 with $\mathcal{M} = M(t) \setminus C(N_{\epsilon/2,P_1}(\alpha(1) \cup \beta))$ and $A = \frac{4c}{\epsilon^2}$. Let $p$ be any point in $M(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cup \beta))$. Note that $\text{dist}_\mathcal{M}(p, \partial \mathcal{M}) \geq \epsilon/2 = \sqrt{c}/\sqrt{A} \geq 1/\sqrt{A}$. Hence by Lemma 3.1,

$$|\langle \vec{N}_p, \vec{e}_\delta \rangle|^2 > 1 - 2\frac{2\sqrt{c}}{\epsilon}(1 - t).$$

So for any $t \in [1 - \frac{\epsilon}{4\sqrt{c}}, 1)$, we have $|\langle \vec{N}_p, \vec{e}_\delta \rangle|^2 > 0$. Thus for $t \in [1 - \frac{\epsilon}{4\sqrt{c}}, 1)$ the normal vector is never horizontal on $M(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cup \beta))$, implying that
For any \( \delta > 0 \) is sufficiently close to 1. This contradiction implies step 2.

\[ P \] just consider a surface that is a graph away from crossing points over each open small disks twisted by approximately \( \pi \) radians at each crossing point. Choosing \( \delta = (1/2)(\text{Area}(\tilde{W}) - \text{Area}(W)) \), and choosing \( t \geq t_\delta \), we have

\[ \text{Area}(M(t)) > \frac{1}{2}(\text{Area}(\tilde{W}) + \text{Area}(W)) > \text{Area}(\tilde{M}(t)) \]

But \( M(t) \) is least-area, so this contradiction implies \( \tilde{W} = W \). This shows step 3.
To prepare for step 4, consider $N_{\epsilon,P_1}(\alpha(1) \cap \beta) := \{p \in P_1 \mid \text{dist}(p, \alpha(1) \cap \beta) < \epsilon\}$, the $\epsilon$ neighborhood in $P_1$ of the crossing points.

**Step 4:** Along the curves $(\alpha(t) \cap \beta) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$ the normal vector to $M(t)$ must become arbitrarily close to vertical, uniformly as $t \to 1$.

To show this, for $t$ sufficiently close to 1 we consider a compact connected piece of a catenoid $C(r_1, r_2, t)$ with the following properties:

- $C = C(r_1, r_2, t)$ has vertical axis and is bounded by circles $P_t \cap C$ and $P_1 \cap C$, of radii $r_2$ and $r_1$ respectively, in the planes $P_t$ and $P_1$.

- $r_2$ is much smaller than $r_1$, and $r_1$ is much smaller than $\epsilon$.

- The normal vector to $C$ is close to vertical everywhere on $C$.

Choosing $r_2$ and $r_1$ small enough and choosing $t$ close enough to 1, $C$ can be placed in the vertical cylinder over any open component of $P_1 \setminus (\alpha(1) \cup \beta)$ assigned a minus sign, so that $C \cap M(t) = \emptyset$, by step 3. We can then translate $C$ horizontally until it makes first contact with $M(t)$ at any given point $p \in \alpha(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$. (The maximum principle implies that first contact cannot be at an interior point of $M(t)$. See Figure 3.) By the maximum principle, the normal vector of $M(t)$ at $p$ must be even more vertical than it is at the same point on $C$. Hence the normal vector of $M(t)$ at $p$ is close to vertical. Reflecting $C$ through the plane $P_{1+\frac{1}{2}t}$, the same argument shows that the normal vector of $M(t)$ any point of $\beta \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$ must also be close to vertical.

Additionally, we have just shown that the projection of $M(t) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$ to $P_1$ lies entirely inside $W$, for otherwise $C$ (or its reflection through $P_{1+\frac{1}{2}t}$) could be translated in such a way as to make first contact at an interior point.

At that first boundary point $p \in \partial M(t)$ of contact, the catenoid $C$ (or its reflection through $P_{1+\frac{1}{2}t}$) makes a small angle with the horizontal plane containing the boundary curve. Among all catenoid pieces of the type $C(r_1, r_2, t)$ contacting $M(t)$ only at $p$ there is a greatest lower bound $\theta(p, t)$ for this angle, and for all boundary points $p \in (\alpha(t) \cup \beta) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$, the lower bound $\theta(p, t)$ approaches zero as $t$ approaches 1 (see Figure 3). Let

$$\theta_0(t) = \max\{\theta(p, t) : p \in (\alpha(t) \cup \beta) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))\}.$$  

Since $(\alpha(t) \cup \beta) \setminus C(N_{\epsilon,P_1}(\alpha(1) \cap \beta))$ is compact, $\lim_{t \to 1} \theta_0(t) = 0$.

**Step 5:** $M(t) \setminus C(N_{2\epsilon,P_1}(\alpha(1) \cap \beta))$ is a graph over $W$, for $t$ sufficiently close to 1. In fact, for any $y \in (0, 1)$, there exists a $t_y \in (0, 1)$ such that if $t \in [t_y, 1)$, then $|\langle \tilde{N}_p, \tilde{e}_3 \rangle| > y \forall p \in M(t) \setminus C(N_{2\epsilon,P_1}(\alpha(1) \cap \beta))$. 

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Figure 3: A Pictoral View of \( \theta(p,t) \)

Since we have already proved this for \( M(t) \setminus C(N_{e,P_1}(\alpha(1) \cup \beta)) \) in steps 1 and 2, we may restrict our considerations here to any points \( p \in M(t) \cap C(N_{e,P_1}(\alpha(1) \cup \beta)) \setminus N_{2e,P_1}(\alpha(1) \cap \beta) \). Let \( \hat{p} \) be the vertical projection of \( p \) into \( P_1 \), and let \( r \) be the distance of \( \hat{p} \) to \( \alpha(1) \cup \beta \). (Note that \( r \leq \epsilon \).) Assume for the moment that the closest point in \( \alpha(t) \cup \beta \) to \( p \) is a point in \( \beta \).

Consider the bounded cylinder \( Cyl = C(B_{r/2},P_1(\hat{p})) \cap \{1 - 2r \tan \theta_0(t) \leq x_3 \leq 1\} \). \( Cyl \) is a finite solid cylinder of radius \( r/2 \) and height \( 2r \tan \theta_0(t) \) with two planar horizontal circular boundary disks. Let \( \partial Cyl \) be its boundary. It follows from the construction of the catenoid barriers in the previous step (see Figure 3) that \( M(t) \cap \partial Cyl \) is contained entirely in the cylindrical part of \( \partial Cyl \), and is disjoint from the upper and lower horizontal boundary disks. To avoid unnecessary complications near the crossing points, we have increased \( \epsilon \) to \( 2\epsilon \).

Let \( \mathcal{M} = Cyl \cap M(t) \). Then \( \mathcal{M} \) lies between the planes \( P_1 - 2r \tan \theta_0(t) \) and \( P_1 \). Since for any point \( q \in \mathcal{M} \), \( \text{dist}_{\mathcal{M}(t)}(q, \alpha(t) \cup \beta) \geq r/2 \), there exists a universal constant \( c \geq 1 \) such that \( |K_q| < \frac{4c}{r^2} \) [Sc2], like in step 1. Let \( A = 4c/(r^2) \), and thus \( \text{dist}_{\mathcal{M}}(p, \partial \mathcal{M}) \geq r/2 = \sqrt{c}/\sqrt{A} \geq 1/\sqrt{A} \). Hence we may apply Lemma 3.1 to conclude that

\[
|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > 1 - 2\frac{\sqrt{c}}{r}2r \tan \theta_0(t)
\]

For any \( y \in (0,1) \), choose \( t_y \in (0,1) \) so that for all \( t \in [t_y,1) \),

\[
\tan \theta_0(t) \leq \frac{1 - y^2}{8\sqrt{c}}
\]

Then, for \( t \in [t_y,1) \), \( |\langle \vec{N}_p, \vec{e}_3 \rangle| > y \forall p \in M(t) \setminus C(N_{2e,P_1}(\alpha(1) \cap \beta)) \).

When the closest point in \( \alpha(t) \cup \beta \) to \( p \) is a point in \( \alpha(t) \), we can reflect \( M(t) \) across the plane \( P_{(1+t)/2} \) and we have the same situation, but the roles of \( \alpha(t) \) and \( \beta \) are reversed. So the above argument applies to this case as well. Step 5 is shown.
Step 6: For each crossing point $p_i$, $C(B_{2\epsilon,P_i}(p_i)) \cap M(t)$ is a disk with total curvature less than $2\pi$, for $t$ sufficiently close to 1.

We could have originally chosen $\epsilon$ small enough so that the two curve segments $\alpha(t) \cap C(B_{2\epsilon,P_i}(p_i))$ and $\beta \cap C(B_{2\epsilon,P_i}(p_i))$ have total curvature as close to zero as we wish. We may then choose $t$ sufficiently close to 1 so that $\partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i))) \setminus \{\alpha(t) \cup \beta\}$ approximates arbitrarily closely a pair of circular arcs, each of less than $\pi$ radians, and so that the exterior angles at the four singular points of $\partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$ are each arbitrarily close to $\frac{\pi}{2}$. (This follows from the fact, as shown in steps 1 and 5, that for any given $y \in (0,1)$ we have $|\langle \vec{N}_q, \vec{e}_3 \rangle| > y$ on $M(t) \setminus C(N_{2\epsilon,P_i}(\alpha(1) \cap \beta))$ for $t$ sufficiently close to 1.) So for $\epsilon$ small enough and $t$ sufficiently close to 1, $\partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$ has total curvature less than $4\pi$.

If $M(t) \cap C(B_{2\epsilon,P_i}(p_i))$ is not a disk, then there exists a smooth Jordan curve $\sigma \subset \text{Int}(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$ with the following properties:

- $\sigma$ is a smooth approximation to $\partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$ that lies on the boundary of a convex region in $\mathbb{R}^3$ and has total curvature less than $4\pi$. (This property is possible because $\partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$ is extremal.)

- $\sigma$ separates $M(t) \cap C(B_{2\epsilon,P_i}(p_i))$ into two components, one component $A_1$ is bounded by $\sigma$, and the other component $A_2$ is an annulus bounded by $\sigma \cup \partial(M(t) \cap C(B_{2\epsilon,P_i}(p_i)))$.

- The component $A_1$ is not a disk.

However, section 3 of [MY] then implies that $A_1$ must be a disk. This contradiction implies that $M(t) \cap C(B_{2\epsilon,P_i}(p_i))$ is actually a disk. The Gauss-Bonnet theorem then implies that the total absolute curvature satisfies

$$\int_{M(t) \cap C(B_{2\epsilon,P_i}(p_i))} |K|dA < 2\pi.$$

This shows step 6. Dividing our original choice for $\epsilon$ by 2, step 6 implies the fourth item of the theorem.

Thus $M(t)$ with $t$ close to 1 is a graph away from crossing points over components of $P_i \setminus \{\alpha(1) \cup \beta\}$ assigned a plus sign. And $M(t)$ is a disk in small vertical neighborhoods of each crossing point. One can then easily check that the Euler characteristic of $M(t)$ is $s - n$, and therefore the genus of $M(t)$ is $\frac{n - s}{2}$. $\square$

Just as the Meeks and White result was extended to the case of a wedge by Theorem 2.1 in the previous section, likewise Theorem 3.1 can be extended to the
case of a wedge. This is done in Theorem 3.2, and the proof is essentially the same as for Theorem 3.1. Let $\beta$ be a $C^2$ Jordan curve in the interior of the half-plane $\omega(0)$, and let $\alpha$ be a $C^2$ Jordan curve in the interior of the half-plane $\omega(\frac{\pi}{2})$. Let $\alpha(t) = R_{\frac{\pi}{2}}(\alpha)$ be the rotation about the $x_1$-axis of $\alpha$ into the half-plane $\omega(\frac{\pi}{2}(1-t))$. Suppose $\alpha(1)$ and $\beta$ intersect transversely at a finite number of points. Call these $n$ points the *crossing points* of $\alpha$ and $\beta$. Define $s$ and $W$ just as before.

**Theorem 3.2** Suppose that $W$ is the unique least-area set in $\omega(0)$ with boundary equal to $\alpha(1) \cup \beta$. Let $\{p_1, \ldots, p_n\} = \alpha(1) \cap \beta$ be the crossing points. Then there exists $\epsilon > 0$ and $t_0 \in (0, 1)$ so that for all $t \in [t_0, 1)$, there is an orientable minimal surface $M(t) \subset \mathbb{R}^3$ with boundary $\alpha(t) \cup \beta$ having the following properties:

1) $M(t)$ has least area amongst all orientable surfaces with the same boundary.

2) $M(t)$ is embedded.

3) $M(t) \setminus C(\cup_{i=1}^n B_{\epsilon,\omega(0)}(p_i))$ is a graph over the set $W \subset \omega(0)$.

4) For each $i$, $M(t) \cap C(B_{\epsilon,\omega(0)}(p_i))$ is homeomorphic to a disk with total absolute curvature less than $2\pi$.

5) $M(t)$ has genus $\frac{n-s}{2}$.

### 4 Non-Existence Results

The results we prove in this section are about existence of compact minimal and constant mean curvature surfaces with a given pair of boundary curves $C_1, C_2$ in parallel horizontal planes. When $C_1$ and $C_2$ are convex and the vertical projection of $C_1$ into the plane containing $C_2$ does not intersect $C_2$, it is well known that such a minimal surface does not exist. For completeness we give the proof here.

**Lemma 4.1** If $C_1$ and $C_2$ are convex curves in parallel planes and if the perpendicular projection of $C_1$ into the plane containing $C_2$ is disjoint from $C_2$, then there does not exist a compact connected minimal surface with boundary $C_1 \cup C_2$.

**Proof.** We may assume $C_1$ and $C_2$ are contained in planes parallel to the $x_1x_2$-plane, and by hypothesis we may assume $C_1$ and $C_2$ lie on opposite sides of and are disjoint from the plane $\{x_1 = 0\}$. Suppose there exists a compact connected minimal
surface $M$ with boundary $C_1 \cup C_2$, then let $\text{Ref}(M)$ be the reflection of $M$ through the plane $\{x_1 = 0\}$. Translate $\text{Ref}(M)$ in the direction of the $x_2$-axis until it is disjoint from $M$, then translate it back until the first moment when $M$ and $\text{Ref}(M)$ intersect. This point of intersection must be in the interiors of both $M$ and $\text{Ref}(M)$, and $M \neq \text{Ref}(M)$, contradicting the maximum principle. 

We use the same kind of technique to prove the next lemma. A slab in $\mathbb{R}^3$ is a region lying between two parallel planes. A slab is called vertical if its two boundary planes are perpendicular to the $x_1x_2$-plane.

**Lemma 4.2** Suppose $C_1$ and $C_2$ are two planar curves in horizontal planes, and these two planes are of distance $h_1$ apart. If $C_1 \cup C_2$ is contained in a vertical slab of width $h_2 < h_1$, then there does not exist a compact connected minimal surface with boundary $C_1 \cup C_2$.

**Proof.** We may assume that the boundary planes of the vertical slab are parallel to the $x_1$-axis, and we may assume the $x_1$-axis is equidistant from the two planes containing $C_1$ and $C_2$ and also from the boundary planes of the vertical slab. Suppose there exists a compact connected minimal surface $M$ with boundary $C_1 \cup C_2$. Let $R_{\frac{\pi}{2}}(M)$ be a rotation of $M$ by $\frac{\pi}{2}$ radians about the $x_1$-axis. Translate $R_{\frac{\pi}{2}}(M)$ in the direction of the $x_1$-axis until it is disjoint from $M$, then translate it back until the first moment when $M$ and $R_{\frac{\pi}{2}}(M)$ intersect. Since $h_2 < h_1$, this point of intersection must be in the interiors of both $M$ and $R_{\frac{\pi}{2}}(M)$, and $M \neq R_{\frac{\pi}{2}}(M)$, contradicting the maximum principle. 

**Corollary 4.1** If $C_1$ and $C_2$ are planar curves with diameters $d_1$ and $d_2$ in parallel planes, and these two planes are of distance $h$ apart, and these two curves bound a compact connected minimal surface $M$, then $h \leq \max\{d_1, d_2\}$.

**Proof.** We may assume $C_1$ and $C_2$ lie in horizontal planes of distance $h$ apart. Let $P_1$ be a vertical plane that is tangent to both $C_1$ and $C_2$, so that $C_1 \cup C_2$ lies entirely to one side of $P_1$. There exists another plane $P_2$ parallel to $P_1$ such that $C_1 \cup C_2$ lies entirely within the vertical slab bounded by $P_1$ and $P_2$, and the distance between $P_1$ and $P_2$ is at most $\max\{d_1, d_2\}$. The result follows from Lemma 4.2.

As stated in the introduction, the above corollary is a generalization of a result by Nitsche [N1], where it is assumed that $M$ is an annulus and gives the weaker conclusion $h \leq \frac{3}{2} \max\{d_1, d_2\}$. Even our result here perhaps does not give the strongest possible
result for an upper bound on $h$. For example, if $M$ is an annular subregion of a catenoid with a vertical axis in $\mathbb{R}^3$, and the boundary $\partial M$ is two circles (with radii $r_1$ and $r_2$, respectively) in horizontal planes, then the two horizontal planes containing $\partial M$ are of distance less than $\frac{2}{3} (r_1 + r_2)$ apart. One can easily see this by elementary considerations of the generating curve $y = \cosh(x)$ of the catenoid. Since $\frac{2}{3} (r_1 + r_2) \leq \frac{2}{3} \max\{d_1 = 2r_1, d_2 = 2r_2\}$, we know that Corollary 4.1 is not the strongest possible result in the case of this catenoid $M$.

Lemma 4.2 cannot be extended to surfaces of constant non-zero mean curvature, as the round cylinder shows, but something can be said about the possible values of the mean curvature in the case of nonminimal constant mean curvature surfaces:

**Proposition 4.1** Let $C = \{x_1^2 + x_2^2 \leq r^2, 0 \leq x_3 \leq d\} \subset \mathbb{R}^3$, with $r < \frac{d}{2}$. If $\Sigma \subset C$ is an embedded constant mean curvature surface with mean curvature $H$ (with respect to the inward pointing normal) and boundary in the planes $P_0 = \{x_3 = 0\}$ and $P_d = \{x_3 = d\}$, then

$$H > \frac{d^2 - 12r^2}{2r(d^2 - 4r^2)} - \frac{32r^2d^2}{(d^2 - 4r^2)^3}.$$  

Thus when $d$ is large relative to $r$, the mean curvature $H$ of $\Sigma$ is bounded away from zero. Furthermore, the limiting value for this lower bound as $d \to \infty$ is equal to the mean curvature of a cylinder of radius $r$.

**Proof.** Consider the embedded annular surfaces in $C$ which are surfaces of rotation about the $x_3$-axis with boundary curves $\{x_1^2 + x_2^2 = r^2, x_3 = 0\}$ and $\{x_1^2 + x_2^2 = r^2, x_3 = d\}$ and with generating curves that are arcs of a circle. Among this 1-parameter family of surfaces the mean curvature (with respect to the inward pointing normal) is always larger than $\frac{d^2 - 12r^2}{2r(d^2 - 4r^2)} - \frac{32r^2d^2}{(d^2 - 4r^2)^3}$. Shrinking this family of surfaces from a cylinder of radius $r$ to the surface which makes first contact with $\Sigma$, we may apply the maximum principle [Sc1] to conclude the result. \qed

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