Well-posedness for some third-order evolution differential equations: a semigroup approach

FLANK D. M. BEZERRA, ALEXANDRE N. CARVALHO AND LUCAS A. SANTOS

Abstract. In this paper, we discuss the well-posedness of the Cauchy problem associated with the third-order evolution equation in time

\[ u_{ttt} + Au + \eta A^{1/3} u_{tt} + \eta A^{2/3} u_t = f(u) \]

where \( \eta > 0 \), \( X \) is a separable Hilbert space, \( A : D(A) \subset X \to X \) is an unbounded sectorial operator with compact resolvent, and for some \( \lambda_0 > 0 \) we have \( \text{Res}(A) > \lambda_0 \) and \( f : D(A^{7/3}) \subset X \to X \) is a nonlinear function with suitable conditions of growth and regularity.

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1. Introduction

In this paper, we discuss the well-posedness of the Cauchy problem associated with the following third-order evolution equation in time

\[ u_{ttt} + Au + \eta A^{1/3} u_{tt} + \eta A^{2/3} u_t = f(u) \] (1.1)

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where \( \eta > 0 \), \( X \) is a separable Hilbert space and \( A : D(A) \subseteq X \to X \) is an unbounded sectorial operator with compact resolvent, and for some \( \lambda_0 > 0 \) we have \( \text{Re} \sigma(A) > \lambda_0 \), that is, \( \text{Re} \lambda > \lambda_0 \) for all \( \lambda \in \sigma(A) \), where \( \sigma(A) \) is the spectrum of \( A \). This allows us to define the fractional power \( A^{-\alpha} \) of order \( \alpha \in (0, 1) \) according to [2, Formula 4.6.9] and [16, Theorem 1.4.2], as a closed linear operator on its domain with inverse \( A^{\alpha} \).

Denote by \( X^\alpha = D(A^\alpha) \) for \( \alpha \in [0, 1) \), taking \( A^0 := I \) on \( X^0 := X \) when \( \alpha = 0 \). Recall that \( X^\alpha \) is dense in \( X \) for all \( \alpha \in (0, 1] \), for details see [2, Theorem 4.6.5]. The fractional power space \( X^\alpha \) endowed with the norm

\[
\| \cdot \|_{X^\alpha} := \| A^\alpha \cdot \|_X
\]

is a Banach space. It is not difficult to show that \( A^\alpha \) is the generator of a strongly continuous analytic semigroup on \( X \), that we will denote by \( \{ e^{-tA^\alpha} : t \geq 0 \} \), see [16] for any \( \alpha \in [0, 1) \). With this notation, we have \( X^{-\alpha} = (X^\alpha)' = \text{the completion of } X \) with respect to the norm \( \| A^{-\alpha} \cdot \|_X \), for all \( \alpha > 0 \), see [2] for the characterization of the negative scale.

Our interest in the model (1.1) is due to the fact that the geometric properties of the equation can change sharply depending on the size of the parameter \( \eta \).

Let \( X_{-1} \) denote the extrapolation space of \( X \) generated by \( A \), and \( \{ X^\alpha_{-1} : \alpha \geq 0 \} \) the fractional power scale generated by operator \( A \) in \( X_{-1} \), see [2,3] for more details.

Here, \( f : D(A^{\frac{1}{3}}) \subseteq X \to X \) is a nonlinear function with suitable growth conditions and regularity in (1.1) for different cases of \( \eta > 0 \); namely, we consider:

- If \( 0 < \eta < 1 \), then we prove that the Cauchy problem defined by the linear equation associated with (1.1) is ill-posed; consequently, the Cauchy problem defined by (1.1) is ill-posed for any nonlinear function \( f \), under the point of view of the theory of strongly continuous semigroups of bounded linear operators;
- If \( \eta > 1 \), then we assume that \( f \) is an \( \epsilon \)-regular map relative to the pair \( (X^{\frac{1}{3}}, X) \) for \( \epsilon \geq 0 \); that is, there exist constants \( c > 0 \), \( \rho > 1 \), \( \gamma(\epsilon) \) with \( \rho \epsilon \leq \gamma(\epsilon) < \frac{1}{3} \) such that \( f : X^{\frac{1}{3}+\epsilon} \to X^{\gamma(\epsilon)} \) and

\[
\| f(\phi_1) - f(\phi_2) \|_{X^{\gamma(\epsilon)}} \leq c \| \phi_1 - \phi_2 \|_{X^{\frac{1}{3}+\epsilon}} \left( 1 + \| \phi_1 \|^{\rho-1}_{X^{\frac{1}{3}+\epsilon}} + \| \phi_2 \|^{\rho-1}_{X^{\frac{1}{3}+\epsilon}} \right), \quad (1.2)
\]

for any \( \phi_1, \phi_2 \in X^{\frac{1}{3}+\epsilon} \), see [3, Definition 2], [7,8] for more details.

For a better understanding of the \( \epsilon \)-regular map relative to the pair \( (X^{\frac{1}{3}}, X) \) for \( \epsilon \geq 0 \), we construct the following diagram (Fig. 1).

The evolution equations of third order in time have been studied extensively in the Hilbert setting and much progress has been achieved. For instance, in [15] the abstract linear equations of third order in time in the form

\[
u_{ttt} + Au = 0 \quad (1.3)
\]

are analyzed and results on (non)existence of solution are obtained in the sense of generalized solutions. In [4], the abstract linear equations of third order in time in the form
(1.3) are analyzed and results on (non)existence, stability and regularity of solution are obtained via theory of fractional powers of closed and densely defined operators; namely, the authors explicitly calculate the fractional powers of linear operator associated with this type of system \[
\begin{bmatrix}
0 & -I & 0 \\
0 & 0 & -I \\
A & 0 & 0
\end{bmatrix}
\] and they discuss solvability of the fractional equation governed by \[
\begin{bmatrix}
0 & -I & 0 \\
0 & 0 & -I \\
A & 0 & 0
\end{bmatrix}^\alpha
\] for \(\alpha \in (0, 1]\) via Balakrishnan formula. In particular, we can also quote [1, 6, 12, 13, 17–20, 22, 23], where MGT equation is studied in different contexts and results of existence, stability and regularity of solutions are obtained.

What we have obtained cannot be obtained by previous works, the presence of the term \(\eta_1 A^\frac{1}{3} u_{tt} + \eta_2 A^\frac{2}{3} u_t\) in (1.1) with \(\eta_1 = \eta_2\) allows us to obtain well posedness results, see Sects. 4.1 and 4.2, as well as reduce the order of the equation, see Sect. 4.2.

The article is organized in the following way. In Sect. 2, we present general facts on spectral behavior of our problem. In Sect. 3, we consider the case \(0 \leq \eta < 1\) and we obtain the result that shows that the problem (1.1) is ill-posed under the point of view of the theory of strongly continuous semigroups of bounded linear operators. In Sect. 4, we consider the case \(\eta > 1\) and we obtain a result of existence, stability and regularity of solutions for (1.1). In this section, we also discuss the case \(\eta = 1\). Finally, in Sect. 5 we explore our results to present an application associated with Dirichlet Laplacian operator.

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2. Functional framework

We first introduce some notations, we consider \(Z = X^\frac{2}{3} \times X^\frac{1}{3} \times X\) endowed with the norm given by

\[
\left\| \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right\|_Z = \|u\|^2_{X^\frac{2}{3}} + \|v\|^2_{X^\frac{1}{3}} + \|w\|^2_X, \quad \forall \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in Z.
\]

Let (1.1) be the original equation of third order. Note that using the change variable

\[
v = u_t + A^\frac{1}{3} u,
\]
and the function

\[ w = v_t, \]

we can rewrite the (1.1) as follows, a first-order evolution equation in time for \( w \)

\[ w_t + (\eta - 1)A^{\frac{1}{3}}w + A^{\frac{2}{3}}v = f(u). \quad (2.1) \]

The initial value problem associated with Eq. (2.1) as Cauchy problem in \( Z \)

\[ \frac{d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \mathbb{B}(\eta) \begin{bmatrix} u \\ v \\ w \end{bmatrix} = F \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right), \quad t > 0, \quad (2.2) \]

and

\[ \begin{bmatrix} u \\ v \\ w \end{bmatrix}(0) = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \quad (2.3) \]

where \( v = u_t \) and \( w = v_t \) and the unbounded linear operator \( \mathbb{B}(\eta) : D(\mathbb{B}(\eta)) \subset Z \to Z \) is defined by

\[ D(\mathbb{B}(\eta)) = X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}}, \quad (2.4) \]

and

\[ \mathbb{B}(\eta) \begin{bmatrix} u \\ v \\ w \end{bmatrix} := x \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A^{\frac{1}{3}}u - v \\ -w \\ A^{\frac{2}{3}}v + (\eta - 1)A^{\frac{1}{3}}w \end{bmatrix}, \quad \forall \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}}. \quad (2.5) \]

The nonlinearity \( F \) given by

\[ F \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ f(u) \end{bmatrix}, \quad (2.6) \]

where \( f : D(A^{\frac{1}{3}}) \subset X \to X \) is a Lipschitz continuous function on bounded sets.

**Lemma 2.1.** The following conditions hold.

(i) The linear operator \( \mathbb{B}(\eta) \) is closed and densely defined;

(ii) Zero belongs to the resolvent set \( \rho(\mathbb{B}(\eta)) \); namely, the resolvent operator of \( \mathbb{B}(\eta) \) is the bounded linear operator \( \mathbb{B}^{-1}(\eta) : Z \to Z \) given by

\[ \mathbb{B}^{-1}(\eta) = \begin{bmatrix} A^{\frac{1}{3}} & (\eta - 1)A^{-\frac{2}{3}} & A^{-1} \\ 0 & (\eta - 1)A^{-\frac{4}{3}} & A^{-\frac{2}{3}} \\ 0 & -I & 0 \end{bmatrix}. \]

Moreover, \( \mathbb{B}(\eta) \) has compact resolvent;

(iii) The spectrum set of \( -\mathbb{B}(\eta) \), \( \sigma(-\mathbb{B}(\eta)) \), is given by

\[ \sigma(-\mathbb{B}(\eta)) = \left\{ \lambda \in \mathbb{C} : \lambda \in \sigma(-A^{\frac{1}{3}}) \right\} \cup \left\{ c_{\eta} \lambda \in \mathbb{C} : \lambda \in \sigma(-A^{\frac{1}{3}}) \right\} \cup \left\{ d_{\eta} \lambda \in \mathbb{C} : \lambda \in \sigma(-A^{\frac{1}{3}}) \right\}, \]
where \( \sigma(-A^{1\over 3}) \) denote the spectrum set of \(-A^{1\over 3}\) and

\[
c_\eta = \frac{1}{2} \left[ (\eta - 1) + \sqrt{\eta^2 - 2\eta - 3} \right]
\]

and

\[
d_\eta = \frac{1}{2} \left[ (\eta - 1) - \sqrt{\eta^2 - 2\eta - 3} \right],
\]

**Proof.** To prove part i) we take a sequence \((\begin{bmatrix} u_n & v_n & w_n \end{bmatrix}, \mathbb{B}(\eta) \begin{bmatrix} u_\alpha & v_\alpha & w_\alpha \end{bmatrix})\) is the graph of \(\mathbb{B}(\eta)\), which converges in \(Z\) to \((\begin{bmatrix} \phi & \psi \end{bmatrix}, \begin{bmatrix} \nu & \zeta \end{bmatrix})\). Then, we have

\[
u_n \to \nu \quad \mbox{in} \quad X^{2\over 3}
\]

\[
u_n \to \nu \quad \mbox{in} \quad X^{1\over 3}
\]

\[
u_n \to \nu \quad \mbox{in} \quad X
\]

and

\[
A^{1\over 3} u_n - v_n \to \nu \quad \mbox{in} \quad X^{2\over 3}
\]

\[
-w_n \to \mu \quad \mbox{in} \quad X^{1\over 3}
\]

\[
A^{2\over 3} v_n + (\eta - 1)A^{1\over 3} w_n \to \zeta \quad \mbox{in} \quad X.
\]

From this, we easily conclude that

\[
\psi = -\mu \in X^{1\over 3}.
\]

From the closedness of \(A^{1\over 3}\), we obtain

\[
A^{1\over 3} \phi + (\eta - 1) \psi \in X^{1\over 3} \quad \mbox{and} \quad A^{1\over 3} \left( A^{1\over 3} \phi + (\eta - 1) \psi \right) = \zeta,
\]

that is,

\[
\phi \in X^{2\over 3} \quad \mbox{and} \quad A^{2\over 3} \phi + (\eta - 1)A^{1\over 3} \psi = \zeta.
\]

From the closedness of \(A^{1\over 3}\), we also obtain

\[
A^{2\over 3} \phi - A^{1\over 3} \phi \in X^{1\over 3} \quad \mbox{and} \quad A^{1\over 3} \left( A^{2\over 3} \phi - A^{1\over 3} \phi \right) = A^{2\over 3} \nu
\]

that is,

\[
\phi \in X^{1\over 3} \quad \mbox{and} \quad A^{1\over 3} \phi - \phi = \nu.
\]

From this, we conclude that \(\begin{bmatrix} \phi & \psi \end{bmatrix} \in D(\mathbb{B}(\eta))\) and \(\mathbb{B}(\eta) \begin{bmatrix} \phi & \psi \end{bmatrix} = \begin{bmatrix} \nu & \zeta \end{bmatrix} \).
For the proof of $ii$), the result easily follows from $B^{-1}_{(\eta)} : Z \to Z$ to be given by

$$B^{-1}_{(\eta)} = \begin{bmatrix} (1-\eta)^{\frac{1}{2}} A^{-\frac{1}{2}} & A^{-1} \\ 0 & (1-\eta)^{\frac{1}{2}} A^{-\frac{1}{2}} \end{bmatrix}.$$

Finally, after considering the eigenvalue problem for the operator $-B_{(\eta)}$,

$$-B_{(\eta)} u = \lambda u,$$

and after straightforward calculations, part $iii$) follows immediately from the fact that $B_{(\eta)}$ has compact resolvent.

$$\square$$

3. Ill-posed problems

In this section, we consider the case $0 \leq \eta < 1$. We prove that the Cauchy problem defined by the linear equation associated with (1.1) is ill-posed in $Z$; consequently, the Cauchy problem defined by (1.1) is ill-posed for any nonlinear function $f$ in $Z$, under the point of view of the theory of strongly continuous semigroups of bounded linear operators.

**Theorem 3.1.** Let $0 \leq \eta < 1$ and let $B_{(\eta)}$ be the unbounded linear operator defined in (2.4)–(2.5). Then, the problem (2.2)–(2.3) is ill-posed in the sense that it does not generate a strongly continuous semigroup of bounded linear operators on the state space $Z$.

**Proof.** If $-B_{(\eta)}$ generates a strongly continuous semigroup $\{e^{-B_{(\eta)}t} : t \geq 0\}$ on $Z$, it follows from Pazy [21, Theorem 1.2.2] that there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|e^{-B_{(\eta)}t}\|_{L(Y)} \leq Me^{\omega t} \quad \text{for } 0 \leq t < \infty. \quad (3.1)$$

Moreover, from Pazy [21, Remark 1.5.4] we have

$$\{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\} \subset \rho(-B_{(\eta)}). \quad (3.2)$$

Where $\rho(-B_{(\eta)})$ denotes the resolvent set of the operator $-B_{(\eta)}$.

From Lemma 2.1, we can consider a sequence $(\lambda_k z_{\eta})_k \in \sigma(-B_{(\eta)})$, for $k = 1, 2, 3, \ldots$, with $\lambda_k \in \sigma(-A^{\frac{1}{2}})$ and $|\lambda_k| \to \infty$ as $k \to \infty$. Note that

$$\arg(\lambda_k z_{\eta}) = \arctan\left(\frac{2\sqrt{-\eta^2 + 2\eta + 3}}{1 - \eta}\right)$$

and since $0 \leq \eta < 1$, we have

$$0 < \arg(\lambda_k z_{\eta}) < \pi/2.$$

Figure 2. Semi-lines contained the eigenvalues of $-\mathcal{B}(\eta)$, $0 \leq \eta < 1$

for every $k \geq 1$ and $|\lambda_k z_\eta| \to \infty$ as $k \to \infty$. Then, we conclude that

$$\sigma(-\mathcal{B}(\eta)) \cap \{\lambda \in \mathbb{C} : \Re\lambda > \omega\} \neq \emptyset.$$ 

This contradicts (3.2) and therefore $-\mathcal{B}(\eta)$ cannot be the infinitesimal generator of a strongly continuous semigroup on $Z$. \[\square\]

Thanks to Lemma 2.1, we have the following illustration of the eigenvalues of $-\mathcal{B}(\eta)$ (Fig. 2).

4. Parabolic differential equations

In this section, we consider the case $\eta > 1$. This section is divided into three subsections, entitled ‘Sectoriality,’ ‘Analysis by reducing the order’ and ‘Final remark,’ where a sectoriality of $\mathcal{B}(\eta)$ defined in (2.4)–(2.5) is proved, a result of well posedness is proved using a different point of view for (1.1) via reducing the order, and the limit case $\eta = 1$ is observed, respectively.

4.1. Sectoriality

Namely, thanks to Lemma 2.1 we have the following illustration of the eigenvalues of $-\mathcal{B}(\eta)$ (Fig. 3).

Initially, we prove the following theorem on the sectoriality of the operator $\mathcal{B}(\eta)$ for $\eta > 1$. 
Theorem 4.1. Let $\eta > 1$. The unbounded linear operator $\mathbb{B}(\eta)$ defined in (2.4)–(2.5) is a sectorial operator.

Proof. In this proof, $M$ will denote a positive constant, not necessarily the same one. Let $\lambda \in \mathbb{C}$, then

$$\lambda I - \mathbb{B}(\eta) = \begin{bmatrix} \lambda I - A^{\frac{1}{3}} & I & 0 \\ 0 & \lambda I & I \\ 0 & -A^{\frac{1}{3}} & \lambda I - (\eta - 1)A^{\frac{1}{3}} \end{bmatrix}.$$  

From Lemma 2.1, it follows that

$$\rho(\mathbb{B}(\eta)) = \rho(A^{\frac{1}{3}}) \cap \rho(c_\eta A^{\frac{1}{3}}) \cap \rho(d_\eta A^{\frac{1}{3}}).$$  

Note that for $\lambda \in \rho(\mathbb{B}(\eta))$ we have

$$(\lambda I - \mathbb{B}(\eta))^{-1} = D_\eta(\lambda)^{-1} \begin{bmatrix} \left(\lambda I - c_\eta A^{\frac{1}{3}}\right) & \left(\lambda I - (\eta - 1)A^{\frac{1}{3}}\right) & I \\ 0 & \left(\lambda I - A^{\frac{1}{3}}\right) & \left(\lambda I - (\eta - 1)A^{\frac{1}{3}}\right) - \left(\lambda I - A^{\frac{1}{3}}\right) \\ 0 & \left(\lambda I - A^{\frac{1}{3}}\right) & \lambda \left(\lambda I - A^{\frac{1}{3}}\right) \end{bmatrix}$$

where

$$D_\eta(\lambda) = (\lambda I - A^{\frac{1}{3}})(\lambda I - c_\eta A^{\frac{1}{3}})(\lambda I - d_\eta A^{\frac{1}{3}})$$

Figure 3. Semi-lines contained the eigenvalues of $-\mathbb{B}(\eta)$ and $0 < \theta_\eta < \pi / 2$, $\eta > 1$.
with
\[ c_\eta = \frac{1}{2} \left( \eta - 1 + \sqrt{\eta^2 - 2\eta - 3} \right) \tag{4.3} \]
\[ d_\eta = \frac{1}{2} \left( \eta - 1 - \sqrt{\eta^2 - 2\eta - 3} \right). \tag{4.4} \]

Since Re\(c_\eta\) > 0 and Re\(d_\eta\) > 0, for \(\eta > 1\), \(c_\eta A^{\frac{1}{3}}\) and \(d_\eta A^{\frac{1}{3}}\) are sectorial operators. Let \(S_{A^{\frac{1}{3}}}, S_{c_\eta A^{\frac{1}{3}}}, S_{d_\eta A^{\frac{1}{3}}}\) be sectors such that \(S_{A^{\frac{1}{3}}} \subset \rho(A^{\frac{1}{3}}), S_{c_\eta A^{\frac{1}{3}}} \subset \rho(c_\eta A^{\frac{1}{3}}), S_{d_\eta A^{\frac{1}{3}}} \subset \rho(d_\eta A^{\frac{1}{3}})\) and

\[
\|\lambda I - A^{\frac{1}{3}}\|_{\mathcal{L}(X)} < \frac{M}{|\lambda|}, \quad \text{for each } \lambda \in S_{A^{\frac{1}{3}}},
\]

\[
\|\lambda I - c_\eta A^{\frac{1}{3}}\|_{\mathcal{L}(X)} < \frac{M}{|\lambda|}, \quad \text{for each } \lambda \in S_{c_\eta A^{\frac{1}{3}}},
\]

\[
\|\lambda I - d_\eta A^{\frac{1}{3}}\|_{\mathcal{L}(X)} < \frac{M}{|\lambda|}, \quad \text{for each } \lambda \in S_{d_\eta A^{\frac{1}{3}}}
\]

for some \(M > 0\). We will prove that \(B_\eta\) is a sectorial operator using the sector

\[ S_{B(\eta)} := S_{A^{\frac{1}{3}}} \cap S_{c_\eta A^{\frac{1}{3}}} \cap S_{d_\eta A^{\frac{1}{3}}}. \]

It is immediate that \(S_{B(\eta)} \subset \rho(B(\eta))\). If \(\lambda \in S_{B(\eta)}\) and \(\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in Z\) with \(\|\mathbf{u}\|_Z \leq 1\), then writing

\[ (\lambda I - B(\eta))^{-1} \mathbf{u} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \]

where

\[
\varphi_1 = \left( \lambda^2 I - (\eta - 1)\lambda A^{\frac{1}{3}} - A^{\frac{2}{3}} \right) D_\eta(\lambda)^{-1} u \\
+ \left( -\lambda I + (\eta - 1)A^{\frac{1}{3}} \right) D_\eta(\lambda)^{-1} v + D_\eta(\lambda)^{-1} w,
\]

\[
\varphi_2 = \left( \lambda^2 I - \eta A^{\frac{1}{3}} + (\eta - 1)A^{\frac{2}{3}} \right) D_\eta(\lambda)^{-1} v + \left( -\lambda I + A^{\frac{1}{3}} \right) D_\eta(\lambda)^{-1} w,
\]

\[
\varphi_3 = \left( \lambda A^{\frac{2}{3}} - A \right) D_\eta(\lambda)^{-1} v + \left( \lambda^2 I - \lambda A^{\frac{1}{3}} \right) D_\eta(\lambda)^{-1} w.
\]

In order to conclude that

\[ \|\varphi_1\|_{X^{\frac{1}{2}}} + \|\varphi_2\|_{X^{\frac{1}{3}}} + \|\varphi_3\|_X < \frac{M}{|\lambda|}, \]

we only need to show that \(\lambda A^{\frac{1}{3}} D_\eta(\lambda)^{-1}, A^{\frac{2}{3}} D_\eta(\lambda)^{-1}, \lambda^2 D_\eta(\lambda)^{-1} \in \mathcal{L}(X)\) and are bounded by \(M/|\lambda|\), which is clear because \(D_\eta(\lambda)^{-1} \in \mathcal{L}(X)\) and

\[ \|D_\eta(\lambda)^{-1}\|_{\mathcal{L}(X)} < \frac{M}{|\lambda|^3}. \]

□
Lemma 4.2. Let $Z$ denote the extrapolation space of $Z$ generated by $B_{(\eta)}$. Then

$$Z = X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}.$$ 

Proof. The extrapolation space of $Z$ is the completion of the normed space $(Z, \|B_{(\eta)}^{-1}\|Z)$. Since for $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in Z$, we have

$$B_{(\eta)}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \eta A^{-\frac{1}{2}} u + (1-\eta) A^{-\frac{3}{2}} v + A^{-1} w \\
(1-\eta) A^{-\frac{1}{2}} u + A^{-\frac{2}{2}} w \end{bmatrix},$$

and consequently

$$\|B_{(\eta)}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}} = \|\eta A^{-\frac{1}{2}} u + (1-\eta) A^{-\frac{3}{2}} v + A^{-1} w\|_{X^{\frac{1}{2}}} + \|v\|_X + \|w\|_X \leq \eta \|A^{-\frac{1}{2}} u\|_X + \max\{1, \eta - 1\}\|v\|_X + 2 \|A^{-\frac{3}{2}} w\|_X \leq C_{1\eta} \|\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X \times X^{-\frac{1}{2}} \times X^{-\frac{3}{2}}},$$

where $C_{1\eta} = \max\{2, \eta\} > 0$.

On the other hand

$$\|\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}} = \|u\|_{X^{\frac{1}{2}}} + \|v\|_X + \|w\|_{X^{-\frac{1}{2}}} = \frac{1}{\eta} \eta A^{-\frac{1}{2}} u\|_{X^{\frac{1}{2}}} + \|v\|_X + \|w\|_{X^{-\frac{1}{2}}} \leq \frac{1}{\eta} \eta A^{-\frac{1}{2}} u + (1-\eta) A^{-\frac{3}{2}} v + A^{-1} w\|_{X^{\frac{1}{2}}} + \left(2 - \frac{1}{\eta}\right) \|v\|_X + \left(\frac{1}{\eta} + 1\right) \|w\|_{X^{-\frac{1}{2}}} \leq \frac{1}{\eta} \eta A^{-\frac{1}{2}} u + (1-\eta) A^{-\frac{3}{2}} v + A^{-1} w\|_{X^{\frac{1}{2}}} + \left(2 + \eta - \frac{2}{\eta}\right) \|v\|_X + \left(\frac{1}{\eta} + 1\right) \|(1-\eta) A^{-\frac{1}{2}} v + A^{-\frac{3}{2}} w\|_{X^{\frac{1}{2}}} \leq C_{2\eta} \|B_{(\eta)}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}},$$

for some $C_{2\eta} = 2 + \eta - \frac{2}{\eta} > 0$.

Hence, there exist $C_{1\eta} > 0$ and $C_{2\eta} > 0$ such that

$$\|B_{(\eta)}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}} \leq C_{1\eta} \|\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X \times X^{-\frac{1}{2}} \times X^{-\frac{3}{2}}} \leq C_{1\eta} C_{2\eta} \|B_{(\eta)}^{-1}\begin{bmatrix} u \\ v \\ w \end{bmatrix}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}},$$

completions of $(X^{\frac{1}{2}} \times X^{\frac{1}{2}} \times X, \|B_{(\eta)}^{-1}\|_{X^{\frac{1}{2}} \times X^{\frac{1}{2}} \times X})$ and $(X^{\frac{1}{2}} \times X^{\frac{1}{2}} \times X, \|B_{(\eta)}^{-1}\|_{X^{\frac{1}{2}} \times X \times X^{-\frac{1}{2}}})$ coincide. \hfill \Box
Consider the closed extension of $B(\eta)$ to $Z_{-1}$ (see [2, p. 262]) and still denote it by $B(\eta)$. Then, $B(\eta)$ is a sectorial and positive operator in $Z_{-1}$ with the domain $Z_{-1} = X^{\frac{3}{2}} \times X^{\frac{3}{2}} \times X$; the imaginary powers of $B(\eta)$ are bounded. Our next concern will be to obtain embeddings of the spaces from the fractional power scale $Z_{-1}^\alpha, \alpha \geq 0$, generated by $(B(\eta), Z_{-1})$.

Before we can proceed, we need the following general interpolation result.

**Proposition 4.3.** Let $X_i, Y_i, M_i, i = 1, 2, 3$ be the Banach spaces such that $X_1 \subset X_0$, $Y_1 \subset Y_0$, $M_1 \subset M_0$ topologically and algebraically. Then

\[
[X_0 \times Y_0 \times M_0, X_1 \times Y_1 \times M_1]_\theta = [X_0, X_1]_\theta \times [Y_0, Y_1]_\theta \times [M_0, M_1]_\theta,
\]

for any $\theta \in (0, 1)$.

**Proof.** The proof is an immediate consequence of the definition of complex interpolation spaces in [24, Sect. 1.9.2]. □

The following result also may be established by a straightforward extension of [7, Theorem 2] so we omit its proof.

**Lemma 4.4.** Denote by $\{Z_{-1}^\alpha : \alpha \in [0, 1]\}$ the partial fractional power scale generated by operator $B(\eta)$ in $Z_{-1}$. Then

\[
Z_{-1}^{k+\alpha} = X^{k+1+\alpha} \times X^{k+\alpha} \times X^{k-1+\alpha}, \quad \alpha \in [0, 1], \quad k \in \mathbb{N}.
\]

For better understanding the relation of the fractional power scale-spaces of the operator $B(\eta)$, we construct the following diagram (Fig. 4).

4.2. Analysis by reducing the order

A different point of view can be considered, we can be considered the reduction of order

\[
v = u_t + A^\frac{1}{3} u \quad (4.6)
\]
for positive time, where \( u \) is an unknown function to be determined, and we obtain the following equation of second order

\[
v_{tt} + A^\frac{3}{2}v + (\eta - 1)A^\frac{1}{2}v_t = g(v) \tag{4.7}
\]

for positive time, where \( \eta > 0 \) and

\[
g(v) = f(u). \tag{4.8}
\]

Note that we can see (4.7) as follows

\[
\begin{bmatrix} v \\ v_t \\ \end{bmatrix}_t + \begin{bmatrix} 0 & -I \\ A^\frac{3}{2} (\eta - 1)A^\frac{1}{2} & \end{bmatrix} \begin{bmatrix} v \\ v_t \\ \end{bmatrix} = \begin{bmatrix} 0 \\ g(v) \\ \end{bmatrix}, \quad \begin{bmatrix} v(0) \\ v_t(0) \\ \end{bmatrix} = \begin{bmatrix} v_0 \\ w_0 \\ \end{bmatrix} \in X^\frac{1}{2} \times X. \tag{4.9}
\]

Solving (4.9) we find

\[
e^{-\Lambda t} \begin{bmatrix} v_0 \\ w_0 \\ \end{bmatrix} = \begin{bmatrix} v(t, v_0, w_0) \\ v_t(t, v_0, w_0) \\ \end{bmatrix} \in X^\frac{1}{2} \times X \text{ where } \Lambda : D(\Lambda) \subset X^\frac{1}{2} \times X \rightarrow X^\frac{1}{2} \times X \text{ is defined by, } D(\Lambda) = X^\frac{3}{2} \times X^\frac{1}{2}
\]

\[
\Lambda \begin{bmatrix} v_0 \\ w_0 \\ \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A^\frac{3}{2} (\eta - 1)A^\frac{1}{2} & \end{bmatrix} \begin{bmatrix} v_0 \\ w_0 \\ \end{bmatrix} = \begin{bmatrix} -w_0 \\ A^\frac{3}{2} v_0 \\ \end{bmatrix}.
\]

It follows that \( \mathbb{R}^+ \ni t \mapsto v(t, v_0, w_0) \in X^\frac{1}{2} \) is a continuous function. We can then try to solve

\[
u_t + A^\frac{1}{2}u = v(t, v_0, w_0), \quad u(0) = u_0 \tag{4.10}
\]

through the variation of constants formula

\[
u(t) = e^{-A^\frac{1}{2}t}u(0) + \int_0^t e^{-A^\frac{1}{2}(t-s)}v(s)ds \tag{4.11}
\]

for positive time.

We next solve (4.7) following the same ideas in [8–11]. Next, we use (4.11) to obtain a local solution to our original differential equation.

4.3. Final remark

We also aim to consider the case \( \eta = 1 \). We note that the initial value problem associated with Eq. (2.1) as the Cauchy problem in \( Z \)

\[
\frac{d}{dt} \begin{bmatrix} u \\ v \\ \end{bmatrix} + B_1 \begin{bmatrix} u \\ v \\ \end{bmatrix} = F_1 \left( \begin{bmatrix} u \\ v \\ \end{bmatrix} \right), \quad t > 0, \tag{4.12}
\]
and

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}(0) = \begin{bmatrix} u_0 \\
v_0 \\
w_0
\end{bmatrix},
\] (4.13)

where \(v = u_t\) and \(w = v_t\) and the unbounded linear operator \(\mathbb{B}(1) : D(\mathbb{B}(1)) \subset Z \rightarrow Z\) is defined by

\[
D(\mathbb{B}(1)) = X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}},
\] (4.14)

and

\[
\mathbb{B}(1) \begin{bmatrix} u \\
v \\
w
\end{bmatrix} := \begin{bmatrix} A_{\frac{1}{3}} & -I & 0 \\
0 & 0 & -I \\
0 & A_{\frac{2}{3}} & 0
\end{bmatrix} \begin{bmatrix} u \\
v \\
w
\end{bmatrix} = \begin{bmatrix} A_{\frac{1}{3}}u - v \\
-\frac{w}{A_{\frac{2}{3}}v}
\end{bmatrix},
\]

\[\forall \begin{bmatrix} u \\
v \\
w
\end{bmatrix} \in X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}}.
\] (4.15)

The nonlinearity \(F_1\) is given by (2.6), where \(f : D(A_{\frac{2}{3}}) \subset X \rightarrow X\) is a Lipschitz continuous function on bounded sets.

In this case, we already know some properties; namely

**Lemma 4.5.** Let \(\mathbb{B}(1)\) be the unbounded linear operator defined in (4.14)–(4.15). The following conditions hold.

(i) The unbounded linear operator \(\mathbb{B}(1)\) is closed and densely defined;
(ii) Zero belongs to the resolvent set \(\rho(\mathbb{B}(1))\); namely, the resolvent operator of \(\mathbb{B}(1)\) is the bounded linear operator \(\mathbb{B}^{-1}(1) : Z \rightarrow Z\) given by

\[
\mathbb{B}^{-1}(1) = \begin{bmatrix} A_{\frac{1}{3}} & 0 & A_{\frac{1}{3}}^{-1} \\
0 & 0 & A_{\frac{2}{3}}^{-1} \\
0 & -I & 0
\end{bmatrix},
\] (4.16)

Moreover, \(\mathbb{B}(1)\) has compact resolvent;
(iii) The spectrum set of \(-\mathbb{B}(1)\), \(\sigma(-\mathbb{B}(1))\), is given by

\[
\sigma(-\mathbb{B}(1)) = \{\lambda \in \mathbb{C} : \lambda \in \sigma\left(-A_{\frac{1}{3}}\right)\} \cup \{\lambda i \in \mathbb{C} : \lambda \in \sigma\left(-A_{\frac{1}{3}}\right)\}
\]
\[
\cup \{-\lambda i \in \mathbb{C} : \lambda \in \sigma\left(-A_{\frac{1}{3}}\right)\},
\]

where \(\sigma(-A_{\frac{1}{3}})\) denote the spectrum set of \(-A_{\frac{1}{3}}\) (Fig. 5).

**Proof.** The proof of \(i\) easily follows from (4.15). For the proof of \(ii\), the result easily follows from (4.16). Finally, for the proof of \(iii\) the result easily follows from (4.15). \(\square\)

**Theorem 4.6.** Let \(\eta > 0\) and let \(\mathbb{B}(\eta)\) be the unbounded linear operator defined as in (2.4)–(2.5). The unbounded linear operator \(-\mathbb{B}(\eta)\) is not a dissipative operator on the state space \(Z\).
Proof. Indeed, let \( u \in X^1 \) be such that \( u \neq 0 \) and let \( \begin{bmatrix} \frac{u}{2A^\frac{1}{3} u} \\ 0 \end{bmatrix} \in Z^1 \). Note that
\[
\langle \mathbb{B}(\eta) \begin{bmatrix} \frac{u}{2A^\frac{1}{3} u} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{u}{2A^\frac{1}{3} u} \\ 0 \end{bmatrix} \rangle = -\langle A^\frac{1}{3} u, u \rangle_{X^\frac{3}{2}}
\]
and consequently
\[
\text{Re} \left( \langle -\mathbb{B}(\eta) \begin{bmatrix} \frac{u}{2A^\frac{1}{3} u} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{u}{2A^\frac{1}{3} u} \\ 0 \end{bmatrix} \rangle \right) > 0
\]
and the prove is complete. \( \square \)

Theorem 4.6 ensures that the linear operator \(-\mathbb{B}(\eta)\) is not an infinitesimal generator of a strongly continuous semigroup of contractions in \( Z \). A satisfactory understanding of the model in (1.1) for \( \eta = 1 \) will be the subject of a future work.

5. Local solvability

In this subsection, we present a result of local solvability for a boundary-initial value problem of the type (1.1). Namely, let \( \Omega \subset \mathbb{R}^N, N \geq 3 \), be a bounded smooth domain, and the initial-boundary value problems
\[
\begin{cases}
  u_{ttt} - \Delta u + \eta(\Delta)^\frac{1}{2} u_{tt} + \eta(\Delta)^\frac{2}{3} u_t = f(u), & t > 0, \quad x \in \Omega, \\
  u(0, x) = \varphi(x), \quad u_t(0, x) = \xi(x), \quad u_{ttt}(0, x) = \psi(x), & x \in \Omega, \\
  u(t, x) = 0, \quad t \geq 0, \quad x \in \partial \Omega,
\end{cases}
\]
where \( \eta > 1 \).
The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ in (5.1) is a continuously differentiable function satisfying for some $1 < \rho < \frac{3N+4}{3N-8}$ the growth condition
\begin{equation}
|f'(s)| \leq C (1 + |s|^\rho - 1).
\end{equation}
(5.2)

Here, we consider $X = L^2(\Omega)$ and the negative Laplacian operator
\[ Au = -\Delta u, \]
with domain
\[ D(A) = H^2(\Omega) \cap H^1_0(\Omega) \]
which is a sectorial operator and it bounded imaginary powers, and consequently the spaces $X^\alpha$, $\alpha \in [0, 1]$, are characterized with the aid of complex interpolation as
\[ X^\alpha = \left[ L^2(\Omega), H^2(\Omega) \cap H^1_0(\Omega) \right]_\alpha \]
and
\[ X^{-\alpha} = \left( H^2_0(\Omega) \right)' \]
where $[\cdot, \cdot]_\alpha$ denotes the complex interpolation function (see [2,14]). In particular
\[ X = X^0 = L^2(\Omega), X^{\frac{1}{2}} = H^1_0(\Omega), X^{-\frac{1}{2}} = (H^1_0(\Omega))' \]
and
\[ X^{1} = H^2(\Omega) \cap H^1_0(\Omega). \]

With this setup, we will consider problem (5.1) in the form (2.2)–(2.3) with $u_0 = \varphi, v_0 = A_1^\frac{1}{4} \varphi + \xi$, and $w_0 = A_1^\frac{1}{4} \xi + \psi$.

Let $F : Z^1_{-1} \to Z^{\alpha}_{-1}, \alpha \geq 0$, be a locally Lipschitz continuous map, as well as in (2.6). Recall that a mild solution of (2.2)–(2.3) on $[0, \tau]$ is a function $\left[ \begin{array}{c} u \\ v \\ w \end{array} \right] (t, \left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right]) \in C([0, \tau], Z^1_{-1})$ which satisfies
\[ \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] (t, \left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right]) = e^{-B_\eta t} \left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right] + \int_0^t e^{-B_\eta (t-s)} F \left( \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] (s, \left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right]) \right) ds, \]
for $t \in [0, \tau]$. We say that (2.2)–(2.3) is locally well posed in $Z^1_{-1}$ is for any $\left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right] \in Z^1_{-1}$ there is a unique mild solution
\[ t \mapsto \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] (t, \left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right]) \]
of (2.2)–(2.3) defined on a maximal interval of existence $[0, t_u, v_0, w_0)$ and depending continuously on the initial data $\left[ \begin{array}{c} u_0 \\ v_0 \\ w_0 \end{array} \right]$.

As a consequence of the Sobolev embeddings, we obtain the following result cf. [11, Proposition 1.3.8].
Proposition 5.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^m$ and $(A, D(A))$ be a sectorial operator in $L^p(\Omega)$, $1 < p < \infty$, with $D(A) \subset W^{2m,p}(\Omega)$ for some $m \geq 1$. Let $X_p = L^p(\Omega)$ and let $X^\alpha_p$ be the domain of the fractional power $A^\alpha$, endowed with the graphic norm, $\alpha \geq 0$. Then for $\alpha \in [0, 1]$, the following inclusion holds:

$$X^\alpha_p \subset W^{s,q}(\Omega)$$

if $2m\alpha - \frac{N}{p} \geq s - \frac{N}{q}$, $1 < p \leq q < \infty$, $s \geq 0$.

We can now formulate the following result.

Theorem 5.2. The problem (2.2)–(2.3) with $\eta > 1$ is locally well posed in $Z_{-1}$ whenever $f$ satisfies (5.2) for some $1 < \rho < \frac{3N+4}{3N-8}$.

Proof. The map $F$ defined as in (2.6) is Lipschitz continuous on bounded sets from $Z_{-1}$ into $Z^{-\frac{1+\sigma}{3}} \times X^{\frac{\sigma}{2}} \times X^{-\frac{1+\sigma}{3}}$ whenever $0 < \sigma \leq \tilde{\sigma}$, and $\tilde{\sigma} = \min\{1, (\rho - 1)(2 - \frac{3}{N}) + 1\}$. Indeed, if $B$ is a bounded subset of $Z_{-1}$ and $\left[\begin{array}{c} u_1 \\ v_1 \\ w_1 \\
_1 \end{array}\right], \left[\begin{array}{c} u_2 \\ v_2 \\ w_2 \\
_2 \end{array}\right] \in B$, we have

$$\|F\left(\left[\begin{array}{c} u_1 \\ v_1 \\ w_1 \\
_1 \end{array}\right]\right) - F\left(\left[\begin{array}{c} u_2 \\ v_2 \\ w_2 \\
_2 \end{array}\right]\right)\|_{Z_{-1}} \leq c_1 \|f(u_1) - f(u_2)\|_{X^{-\frac{1+\sigma}{3}}}.$$ 

Since $L^{\frac{6N}{3N+4(1-\sigma)}}(\Omega) \hookrightarrow X^{-\frac{1+\sigma}{3}}$ we obtain

$$\|F\left(\left[\begin{array}{c} u_1 \\ v_1 \\ w_1 \\
_1 \end{array}\right]\right) - F\left(\left[\begin{array}{c} u_2 \\ v_2 \\ w_2 \\
_2 \end{array}\right]\right)\|_{Z_{-1}} \leq c_2 \|f(u_1) - f(u_2)\|_{L^{\frac{6N}{3N+4(1-\sigma)}}(\Omega)}$$

and thanks to (5.2) there exists $C > 0$ such that

$$\forall s_1, s_2 \in \mathbb{R}, \ |f(s_1) - f(s_2)| \leq C|s_1 - s_2|(1 + |s_1|^\rho - 1 + |s_2|^\rho - 1)$$

and consequently

$$\|F\left(\left[\begin{array}{c} u_1 \\ v_1 \\ w_1 \\
_1 \end{array}\right]\right) - F\left(\left[\begin{array}{c} u_2 \\ v_2 \\ w_2 \\
_2 \end{array}\right]\right)\|_{Z_{-1}} \leq c_3 \|u_1 - u_2\|_{X^{\frac{\sigma}{2}}} \left(1 + \|u_1\|_{L^{\frac{6N(\rho-1)}{3N+4(1-\sigma)}}(\Omega)}^{\rho-1} + \|u_2\|_{L^{\frac{6N(\rho-1)}{3N+4(1-\sigma)}}(\Omega)}^{\rho-1}\right) \leq c_4 \left\|\begin{array}{c} u_1 \\ v_1 \\
_1 \end{array} - \begin{array}{c} u_2 \\ v_2 \\
_2 \end{array}\right\|_{Z_{-1}}.$$ 

The proof now follows from [16].

Data availability All data generated or analyzed during this study are included in this published article [and its supplementary information files].

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Flank D. M. Bezerra  
Departamento de Matemática  
Universidade Federal da Paraíba  
João Pessoa PB 58051-900  
Brazil  
E-mail: flank@mat.ufpb.br

Alexandre N. Carvalho  
Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação  
Universidade de São Paulo, Campus de São Carlos  
Caixa Postal 668, São Carlos SP 13560-970  
Brazil  
E-mail: andcarva@icmc.usp.br

Lucas A. Santos  
Instituto Federal da Paraíba  
Itaporanga PB 58780-000  
Brazil  
E-mail: lucas92mat@gmail.com

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