A model for predicting J-integral of surface cracks in round bars under combined mode I loading

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Abstract. This study presents a method to develop an analytical aspect for J-integral prediction of surface crack in round bars under combined mode I loading. This technique is based on the local limit load approach considering a plastic deformation across the crack ligament. Then, the developed limit load is then combined with the reference stress method to predict J-integral along the crack front of surface crack in cylindrical bars.

1. Introduction

In fracture mechanics analysis, the crack shapes are idealized as a three-dimensional semi-elliptical crack shapes [1, 2]. There are several other crack shapes for examples straight-fronted crack [3], circular crack front [4] and sickle-shaped crack front [5, 6]. In analyzing the crack behaviors, finite element analysis (FEA) is frequently used to investigate the fracture characteristic along the crack front. While, the solution of SIFs under bending [7-10] and tension [11-15] loadings are well documented and established for the cylindrical-shape bars. The SIFs under combined loadings are also explicitly obtained for similar mode of failure using a superposition method [16, 17] for wide ranges of crack geometries. However, J-integral behavior of such conditions is significantly limited [11, 12, 15, 18-21].

In addition, the calculation of J-integral using FEA required special skills and it is also an exhausted work when modelling a three-dimensional finite element model containing a crack. Therefore, a simplified J-integral estimation scheme is highly required. However, the use of FEA is still important for the crack geometries which are not fully explored and not widely available elsewhere. The crack geometries in pipes [21, 22] and plates [1, 23-26] are tremendously available in the literature but it is not for solid round bars [27]. There are two popular J-integral estimation schemes are currently available for examples the general electric/electric power research institute (GE/EPRI) approach [2] and reference stress approach [23-25]. Several solutions of GE/EPRI are also available in handbook [27-28]. However, such solutions are limited for several practical purposes of components containing cracks. In reference stress approach, limit load solution for particular crack configurations are required to estimate J-integral. A good limit load solution able to be used to predict J-integral successfully. According to Lei [22] limit load can be divided into two categories: a global limit load which is the limit load is defined as the overall plastic deformation of the cracked...
component. While a local limit load analysis represented the localized plastic deformation across the crack ligament. The advantages and disadvantages of these approaches are still investigated [26, 29]. However, the reference stress approach method is the priority because it is not required a complete stress-strain material data compared with the GE/EPRI method [1, 23-25]. Recent publications on the similar discussed issues can be found in [26-31]. However, no literature found on the limit load of solid cylindrical bar under combined mode I loading. Therefore, this paper presents the method developed to predict the J-integral along the crack front for surface crack in round bar under combined forces. The results are then compared with the results obtained numerically as presented in [38].

2. Reference stress method
In the reference stress method, the total J-integral, \( J = J_e + J_p \) is estimated using the elastic J-integral, mechanical properties and the limit load of the cracked component. The prediction of J is expressed as:

\[
J = \frac{E \varepsilon_{ref}}{L_r \varepsilon_{ref}} + \frac{L_r \sigma_0}{2E \varepsilon_{ref}}
\]

where \( \varepsilon_{ref} \) is the reference strain determined from true stress-strain data of the material with \( E \) denotes Young’s modulus and \( L_r \) is a proximity of plastic collapse defined by:

\[
L_r = \frac{\sigma_{ref} \sigma_{ref}}{\sigma_a \sigma_b} = \frac{F}{F_e} = \frac{M}{M_e}
\]

where \( F \) and \( M \) are the applied loads on the cracked component corresponding to the normalizing limit loads \( F_e \) and \( M_e \), respectively. In this work, the normalizing limit loads are actually the combined tension and bending limit loads. It is developed according to the elastic-perfectly-plastic material behaviour and it is appeared in Appendix A. The plastic collapse behaviour, \( L_r \) in Eq. (14) may be described by tension and bending parameters. However, it will produce similar values of \( L_r \). Generally, an optimized normalizing load is used in the calculation to predict J-integral and the detail about the procedure can be found in [22]. In Eq. (13), \( J_e \) can be calculated either using FEA or Eq. (11). The elastic solutions for various types of crack geometries are available for example pipes [1, 22, 23], plates [23-25] and solid round bar [33], which is emphasized the advantage of reference stress approach. Similar methodology is implemented this work as mentioned in [23-25].

3. Determination of combined limit loads
Under combined bending and tension loadings, a simplified equivalent von Mises stress, \( \sigma \) is expressed which is simplified when \( \sigma_c = r_a = r_b = 0.0 \):

\[
\sigma = \sigma_a + \sigma_b
\]

where \( \sigma_a \) and \( \sigma_b \) are tension and bending stresses, respectively and expressed as the following expression:

\[
\sigma_a = \frac{4F}{\pi D^2}
\]

\[
\sigma_b = \frac{32M}{\pi D^3}
\]

where \( D \) is a diameter of the bar. Substituting Eqs. (4) and (5) into Eq. (3) and given that \( M = \eta F D \), yield the following expression:

\[
\sigma = (1+8\eta) \left( \frac{4F}{\pi D^2} \right)
\]

where, \( \eta \) is a ratio between bending moment and tension force. Eq. (6) can also be arranged in terms of combined tension force or bending moment, respectively as the following expressions:
The tensile limit load for a surface crack in a round bar can be expressed by Lei [23]:

\[ F = \frac{\pi D^2 \sigma_0}{4(1+8\eta)} \]  

(7)

\[ M = \frac{\eta \pi D^3 \sigma_0}{4(1+8\eta)} \]  

(8)

where \( \xi_{a_{+b}} \) is a normalised tension limit load under combined loadings. In addition it is a function of crack geometries and loading. While, \( \sigma_0 \) is a yield strength of the material. Substituting Eqs. (7) and (8) into Eq. (9), yield the following expression:

\[ \xi_{a_{+b}} = \frac{1}{1+8\eta} \left( \frac{\sigma_0/\sigma_0}{L_r} \right) \]  

(10)

where \( L_r \) in Eq. (10) is solved using Eq. (1) for the given \( \sigma_0 \) and \( \eta \). \( J \) and \( J_e \) can be obtained directly from non-linear and linear finite element analysis along the crack front except the point on the intersection between the crack front and an outer surface. This is due to the singularity problems occurred at that point [5].

4. Conclusion

Limit load for surface crack in cylindrical bar is developed based on the local ligament approach and the reference stress method. According to the literature survey, there is no limit load for surface cracks in cylindrical bars. It is important to develop since the development of finite element model requires high skills and knowledge. The formulated analytical model is then compared with the results obtained numerically in the second part of this paper [38].

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Appendix A – Determination of Function \( h^* \)

By assuming that the material used behaved as an elastic-perfectly plastic material model. From Fig. A1, force and moment equilibrium can be expressed as follow

\[ \sum F = F \]  

(A1)

\[ A_1\sigma_0 - A_2\sigma_0 - A_3\sigma_\psi = F \]  

(A2)

Substituting the parameters listed in Table A1 into Eq. A2 yield the following:

\[ F = R^2\sigma_\psi = 2\alpha + \sin \alpha \]  

(A3)

Referring to Fig. A2, \( \sin \alpha = h/R \) and assumed that \( \delta = h/R \), then, \( \sin \alpha = \delta \). Therefore, Eq. (A3) can be as follows:

\[ F = R^2\sigma_\psi = 2\alpha + \delta \]  

(A4)

Rearranging Eq. (A4) into the following expression:

\[ \delta = \frac{F}{R^2\sigma_\psi} - 2\alpha \]  

(A5)

Moment equilibrium shown in Fig. A1 (b) is as followed:
\[
\sum M = M = \left[ \sigma_0 A_1 (y_1 - h) + \sigma_0 A_2 (h - y_2) + \sigma_0 A_3 (h + y_3) \right] = M
\] (A6)

\[
\left( \frac{\pi}{3} \right) \left( \frac{\cos \alpha}{\pi - \psi} \right) \left( \frac{1 - \cos^3 \alpha}{\psi} \right)
\]

Table A1. Geometrical properties of the circular bar.

| Segment 1 | Segment 2 | Segment 3 |
|-----------|-----------|-----------|
| Area, \(A\) | \(A_1 = \frac{R^2}{2} (\pi - \psi)\) | \(A_2 = \frac{R^2}{2} \psi\) | \(A_3 = \frac{R^2}{2} \pi\) |
| Centroid area, \(y\) | \(y_1 = \left( \frac{4R}{3} \left[ \frac{\cos \alpha}{\pi - \psi} \right] \right)\) | \(y_2 = \left( \frac{4R}{3} \left[ \frac{1 - \cos^3 \alpha}{\psi} \right] \right)\) | \(y_3 = \frac{4R}{3\pi}\) |

\[\alpha = \cos^{-1} \left( \frac{1}{2} \sin \theta \right)\]

\[\psi = 2\alpha + \sin \alpha\]

Fig. A1. The illustration of circular bar geometry, (a) subjected to combined bending and tension loadings, and (b) stress distribution in the bar.

Fig. A2. Cross-sectional area of the circular bar.

Replacing the parameters in Eq. (A7) from Table A1, yields the following expression:
\[
\frac{M}{R^2\sigma_o} = \frac{4R}{3}\cos^3\alpha + h\psi \tag{A8}
\]

Substituting \(\psi\) into Eq. (A8) to form the Eq. (A9):
\[
\frac{M}{R^2\sigma_o} = \frac{4R}{3}\cos^3\alpha + 2\alpha\delta + \delta^2 \tag{A9}
\]

Replacing Eq. (A4) into Eq. (A9), therefore:
\[
\frac{M}{R^2\sigma_o} = \left[\frac{F}{R^2\sigma_o}\right]^2 - 2\alpha \left[\frac{F}{R^2\sigma_o}\right] + \left(\frac{4}{3}\cos^3\alpha\right) \tag{A10}
\]

Given that the loading ratio, \(\vartheta\) is as in Eq. (A11):
\[
\vartheta = \frac{M}{FR} \tag{A11}
\]

Then, substituting in Eq. (A11) into Eq. (A1) to yield the following expression:
\[
\frac{F\vartheta}{R^2\sigma_o} = \left[\frac{F}{R^2\sigma_o}\right]^2 - 2\alpha \left[\frac{F}{R^2\sigma_o}\right] + \left(\frac{4}{3}\cos^3\alpha\right) \tag{A12}
\]

Rearranging the Eq. (A12) into locus form as follows:
\[
\left[\frac{F}{R^2\sigma_o}\right]^2 - \left(2\alpha + \vartheta\right)\left[\frac{F}{R^2\sigma_o}\right] + \left(\frac{4}{3}\cos^3\alpha\right) = 0 \tag{A13}
\]

Solve this quadratic Eq. (A13) and take the positive root as the following expression:
\[
\frac{F}{R^2\sigma_o} = \frac{\alpha + \frac{\vartheta}{2}}{\sqrt{\alpha^2 + \alpha\vartheta + \frac{\vartheta^2}{4} - \frac{4}{3}\cos^3\alpha}} = \delta_1 \tag{A14}
\]

Using similar procedure described earlier for the moment loading as in Eq. (A15):
\[
\frac{M}{R^2\sigma_o} = \frac{\alpha\vartheta + \frac{\vartheta^2}{2}}{\sqrt{\alpha^2\vartheta^2 + \alpha\vartheta^3 + \frac{\vartheta^4}{4} - \frac{4\vartheta^2}{3}\cos^3\alpha}} = \delta_2 \tag{A15}
\]
where $F$ in Eq. (A14) can be simplified as in Eq. (A16). It is then called $F_L^*$ to represent the combined tension limit load as follows:

$$F_L^* = \delta_1 R^3 \sigma_v$$  \hspace{1cm} (A16)

where:

$$\delta_1 = \left[ \frac{\alpha + \frac{\vartheta}{2}}{\sqrt{\alpha^2 + \alpha \vartheta + \frac{\vartheta^2}{4} - \frac{4}{3} \cos^3 \alpha}} \right]$$  \hspace{1cm} (A17)

while $M$ in Eq. (A15) can be simplified as in Eq. (A18). It is then called $M_L^*$ to represent the combined moment limit load as follows:

$$M_L^* = \delta_2 R^3 \sigma_v$$  \hspace{1cm} (A18)

where:

$$\delta_2 = \left[ \frac{\alpha \vartheta + \frac{\vartheta^2}{2}}{\sqrt{\alpha^2 \vartheta^2 + \alpha \vartheta^3 + \frac{\vartheta^4}{4} - \frac{4 \vartheta^2}{3} \cos^3 \alpha}} \right]$$  \hspace{1cm} (A19)

In calculating the $h_I^*$ function subjected to combined loadings, plastic J-integral, $J_{p-FE} = J - J_e$ is used where $J$ is a total J-integral can be obtained directly from FEA and $J_e$ can be predicted through the Eq. (11). From Eq. (12), $F_L$ is a normalizing limit load for F can be replaced by Eq. (A16). On the other hand, $F$ is obtained through Eq. (27). Rearrange Eq. (12) as follow:

$$h_I^* \left( \frac{a}{b}, \frac{x}{h}, \frac{n}{h}, \lambda \right) = \frac{J_{p-FE}}{\alpha \sigma_v D F_L^*(F/F_L^*)^{1/4}}$$  

$$= \frac{J_{p-FE}}{J_{p-normal}}$$  \hspace{1cm} (A20)

$$h_I^* \left( \frac{a}{b}, \frac{x}{h}, \frac{n}{h}, \lambda \right) = \frac{J_{p-FE}}{\alpha \sigma_v D M_L^*(M/M_L^*)^{1/4}}$$  

$$= \frac{J_{p-FE}}{J_{p-normal}}$$  \hspace{1cm} (A21)

where Eqs. (A20) and (A21) are both can be used to determine the $h_I^*$-function under combined mode I loadings, respectively.

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