A simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the $3x + 1$ problem.

Daudin, Jean-Jacques  
jeanjacques.daudin@gmail.com

Pierre, Laurent  
laurent.pierre@u-paris10.fr

June 1, 2018

Summary. This paper gives a simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the $3x+1$ problem (see [5] and [2]). This representation permits to compute all the ascending Collatz sequences $(f^{(i)}(n), i = 1, b-1)$ with a last value $f^{(b)}(n) = 1$. Other periodic sequences connected to 1 are also identified.

1 Basic elements

In the presentation of the book "The Ultimate Challenge: The 3x+1 Problem", [3], J.C. Lagarias write The $3x + 1$ problem, or Collatz problem, concerns the following seemingly innocent arithmetic procedure applied to integers: If an integer $x$ is odd then "multiply by three and add one", while if it is even then "divide by two". The $3x + 1$ problem asks whether, starting from any positive integer, repeating this procedure over and over will eventually reach the number 1. Despite its simple appearance, this problem is unsolved. We refer to this book and other papers from the same author for a good review of the context and the references.

1.1 Definitions

Let $n \in \mathbb{N}$.

Direct algorithm

\[ T(n) = \begin{cases} 3n + 1 & \text{if } n \equiv 1 \pmod{2} \\ n/2 & \text{if } n \equiv 0 \pmod{2} \end{cases} \]

Inverse algorithm

\[ U(n) = \begin{cases} 2n \quad \text{and} \quad \frac{n-1}{3} & \text{if } n \equiv 4 \pmod{6} \end{cases} \]
Graph $G(n)$

Let $(n_1, n_2) ∈ \mathbb{N}^2$. $n_1$ and $n_2$ are connected by an edge if $n_1 = T(n_2)$ or $n_2 = T(n_1)$. $G(n)$ is the subset of the integers connected to $n$.

Conjecture "3x + 1"

$∀ n ∈ \mathbb{N}, ∃ k ∈ \mathbb{N}: T^k(n) = 1$. An equivalent assertion is $G(1) = \mathbb{N}^*$.

2 Restriction to odd integers

2.1 $f$ and $h$

If the "3x + 1" conjecture is true for the odd integers it is also true for the even ones by definition of $T$. The expressions of $T$ and $U$ restricted to odd terms are the following with $n$ odd:

- $T$ becomes $f$: $f(n) = (3n + 1)2^{-j(3n+1)}$ with $j(3n + 1)$ the power of 2 in the prime factors decomposition of $3n + 1$. $f$ is often called the "Syracuse function".

- $U$ becomes $h$, see [1]:

$$h(n) = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{3} \\ \frac{2^{k-1}}{3}, k = 2, 4, 6... & \text{if } n \equiv 1 \pmod{3} \\ \frac{2^{k-1}}{3}, k = 1, 3, 5... & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

The expression of $h$ comes from the following:

$$f(n) = (3n + 1)2^{-j(3n+1)} ⇒ n = \frac{f(n)2^{j(3n+1)} - 1}{3} ∈ \mathbb{N} \quad (1)$$

There are 3 cases

- $f(n) ≡ 0 \pmod{3}$ ([1]) is impossible,

- $f(n) ≡ 1 \pmod{3} ⇒ f(n) = 3x + 1$ with $x ∈ \mathbb{N}$

$$\frac{f(n)2^{j(3n+1)} - 1}{3} = \frac{(3x + 1)2^{j(3n+1)} - 1}{3} = \frac{x.2^{j(3n+1)} + 2^{j(3n+1)} - 1}{3} ∈ \mathbb{N} \text{ if } j(3n + 1) \text{ even (see Lemma 5 on page II)}$$

- $f(n) ≡ 2 \pmod{3} ⇒ f(n) = 3x + 2$ with $x ∈ \mathbb{N}$

$$\frac{f(n)2^{j(3n+1)} - 1}{3} = \frac{(3x + 2)2^{j(3n+1)} - 1}{3} = \frac{x.2^{j(3n+1)} + 2^{j(3n+1)+1} - 1}{3} ∈ \mathbb{N} \text{ if } j(3n + 1) \text{ odd (see Lemma 5 on page II)}$$
2.2 Graph $g(n)$

Let $(n_1, n_2)$ be odd integers. $n_1$ and $n_2$ are connected by an edge if $n_1 = f(n_2)$ or $n_1 = f(n_2)$. $g(n)$ is the subset of the odd integers connected to $n$.

3 Properties of $g(1)$

Many authors have tried to give a representation of $g(1)$. Goodwin [2] gives a short history of their work and provides some more results. This paper follows the same line of research and the theorems [1][2] and [3] are not new. However we tried to give a simple and clear presentation of the results and the proofs. The theorems [4] and [5] are new as long as we know.

3.1 Expression of $n \in g(1)$ as a sum of fractions

Theorem 1. Let $n \in g(1)$. $\exists (b, a > u_1 > u_2, \ldots > u_b = 0) \in \mathbb{N}^{b+2}$:

$$n = \frac{2^a}{3^b} - \sum_{i=1, b}^{u_i} \frac{2^a}{3^{b-i+1}}.$$ 

Note that $\frac{2^a}{3^b} \geq 1 \Rightarrow a \geq b \log_2 3 - \log_3 2$.

Proof. $n \in g(1) \iff \exists b : n \in h^{(b)}(1)$. The proof uses induction with $b$. Theorem 1 is true for $b = 1$ because $h(1) = \{ \frac{2^k-1}{3}, k = 2, 4, 6, \ldots \}$, and for $b = 2$ because

$$h^{(2)}(1) \subset \left\{ \frac{1}{3} \left( \frac{2^{k_1} - 1}{3} \cdot 2^{k_2} - 1 \right), k_1 = 2, 4, 6; k_2 \in \mathbb{N} \right\}$$

$$\subset \left\{ \frac{2^{k_1+k_2}}{3^2} - \frac{2^{k_2}}{3^2}, k_1 = 2, 4, 6; k_2 \in \mathbb{N} \right\}$$

Assume that theorem 1 is true for $l \leq b - 1$.

$$h^{(b)}(1) \subset \left\{ \frac{1}{3} \left[ \left( \frac{2^a}{3^{b-1}} - \sum_{i=1, b-1}^{u_i} \frac{2^a}{3^{b-i+1}} \right) 2^k - 1 \right], (a > u_1 > \ldots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\}$$

$$\subset \left\{ \frac{2^{a+k}}{3^b} - \sum_{i=1, b-1}^{u_i+k} \frac{2^{a+k}}{3^{b-i+1}} - \frac{2^{a+k}}{3}, (a > u_1 > \ldots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\}$$

The last expression has the form claimed in theorem 1. □

Note that $k_1$ is even but $k_l$ may be odd or even for $l > 1$. Thus $a - u_1$ is even. If $a - u_1 = 2$, $h(1) = 1$, so the first "interesting" value is $a - u_1 = 4$. 

3
Proof. An alternative proof of theorem I on the previous page using \( f : n \in g(1) \iff \exists b \in \mathbb{N} : f^b(n) = 1 \). \( b \) is the number of odd integers (excluding 1) in the sequence from \( n \) to 1.

Induction with \( b \):
Let \( b = 1 \) and \( 3n + 1 = 2^{j(3n+1)}x \) be a partial prime factors decomposition of \( 3n + 1 \).

\[
\begin{align*}
f(n) &= (3n + 1)2^{-j(3n+1)} \\
&= 2^{j(3n+1)}x2^{-j(3n+1)} \\
&= x
\end{align*}
\]

\( b = 1 \Rightarrow f(n) = 1 \Rightarrow x = 1 \Rightarrow 3n + 1 = 2^{j(3n+1)} \Rightarrow n = \frac{2^{j(3n+1)}}{3} - \frac{1}{3} \).

Let \( b = 2 \). \( b = 2 \Rightarrow f(f(n)) = 1 \Rightarrow f(n) = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3} \).

\[
\begin{align*}
f(n) &= (3n + 1)2^{-j(3n+1)} \Rightarrow (3n + 1)2^{-j(3n+1)} = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}.
\end{align*}
\]

Therefore

\[
n = \frac{2^{j(3n+1)+j(3f(n)+1)}}{3^2} - \frac{2^{j(3n+1)}}{3^2} - \frac{1}{3}.
\]

Assuming that the theorem is true till \( b - 1 \) we have to prove that it is true for \( b \).

\[
\begin{align*}
f(n) &= (3n + 1)2^{-j(3n+1)} = \frac{2^a}{3^{b-1}} - \sum_{i=1}^{b-1} \frac{2^{u_i}}{3^{b-i}}.
\end{align*}
\]

Therefore

\[
n = \frac{2^a+j(3n+1)}{3^b} - \sum_{i=1}^{b-1} \frac{2^{u_i+j(3n+1)}}{3^{b-i+1}} - \frac{1}{3}.
\]

Note that the general form of \( u_i \) is thus \( u_{b-i} = \sum_{l=1}^{i} j[3f^{(l-1)}(n)+1] \), with \( f(0) = \text{Id} \), and \( a = \sum_{l=1,b} j[3f^{(l-1)}(n)+1] \).

3.2 Admissible tuple \((b, a > u_1 > u_2, ..., u_b = 0)\)

Only some values of \((b, a > u_1 > u_2, ..., u_b = 0)\) give an integer \( n \) in theorem 1, most of them do not.

Definition 1. A tuple \((b, a \geq \frac{\log_3 a}{\log_2}, a > u_1 > u_2, ..., u_b = 0)\) of \( b + 1 \) integers is admissible if \( \frac{a^b}{3} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N} \).

The admissible parity of \( u_i - u_{i+1} \) is determined by the remainder modulo 3 of the integer obtained at step \( i \) (see the definition of \( h \) in section 2.1 on page 2).

Lemma 1. Let \( \frac{2^k - 1}{3} \in \mathbb{N} \) and \( \frac{2^k - 1}{3} \equiv v \pmod{3} \). Then \( \frac{2^{k+2} - 1}{3} \equiv v + 1 \pmod{3} \).
Proposition 1. Let \( b \leq \frac{n^{2k-1}}{3} \equiv \frac{n^{2k+6}-1}{3} \) (mod 3).

Proof. \( \frac{n^{2k-1}}{3} \equiv v \) (mod 3) \( \Rightarrow \) \( \frac{n^{2k-1}}{3} = 3x + v \),

\[
\frac{n^{2k+2} - 1}{3} = 4 \frac{n^{2k} - 1}{3} + 1 \\
= 4(3x + v) + 1 \\
= v + 1 \quad \text{(mod 3)}
\]

Lemma 2. Let \( n \in \mathbb{N} \) and \( n_1 = \frac{n^{2k-1}}{3} \notin \mathbb{N} \). Then \( \forall l \in \mathbb{N} \), \( \frac{n_l^{2k-1}}{3} \notin \mathbb{N} \).

The lemma indicates that if \( (b, a > u_1 > u_2, \ldots > u_b = 0) \) is admissible and \( k \) has not the correct parity, the tuple \( (b + 1, a + k > u_1 + k > u_2 + k, \ldots > u_b + k, u_{b+1} = 0) \) is not admissible and all tuples based on it are also not admissible. Conversely, if \( (b, a > u_1 > u_2, \ldots > u_b = 0) \) is admissible, \( (b-1, a - u_{b-1} > u_1 - u_{b-1} > u_2 - u_{b-1}, \ldots > u_b - u_{b-1} = 0) \) is also admissible and all such successive reduced tuples till \( (1, a - u_1 > u_1 - u_1 = 0) \)

Proof. \( n_1 = \frac{n^{2k-1}}{3} = \frac{2}{3} \), with \( p \) and 3 relatively prime. \( \frac{n_l^{2k-1}}{3} = \frac{2^l - 1}{3} = \frac{p^{2 - 3}}{2} \). Suppose that \( p^{2 - 3} = x \in \mathbb{N} \). Then \( p^2 = 9x + 3 \) that is impossible because \( p \) and 3 are relatively prime.

3.3 Structure of \( g(1) \)

Lemma 3. \( g(1) \) is a tree with an additional loop in its root 1.

Proof. Let \( h^* \) be a modified version of \( h \): \( h^*(1) = \frac{2^{k-1}}{3}, k = 4, 6, 8, \ldots \),

\[
g^*(1) = \{1 \cup h^*(1) \cup h[h^*(1)] \cup h^*(1) \} \quad \text{The case } n \in h(n_1) \cap h(n_2) \quad \text{with } n_1 \neq n_2, \text{ is impossible because there is only one } \frac{f(n)}{3} \text{. Thus } g^*(1) \text{ is a tree because any } n \in g^*(1) \text{ cannot have two different parents. } g(1) \text{ is equal to } g^*(1) \text{ with a supplementary loop at node 1}.
\]

The following definition 2 and proposition 3 are not nessassary for the proof of theorem 4 on page 7 and may be skipped.

Definition 2. \( g^*(1)[t, s] \subset g^*(1) \) is the graph generated by the admissible tuples with \( b \leq t, 4 \leq a - u_1 \leq 2 + 6s \text{ and } u_i - u_{i+1} \leq 6s \).

Proposition 1. \( |g^*(1)[t, s]| = 1 + \frac{3s(2s)^t - 1}{2s-1} \), with \( |A| \) the cardinal of the set \( A \).

Proof. Lemma 1 on the preceding page implies that for each node of the tree there are \( 3s \) admissible children of which \( 2s \) have children.

Note that with \( s = 1 \) one obtains that the ratio of integers pertaining to \( g^*(1)[t, 1] \) and less than \( \max(g^*(1)[t, 1]) \simeq \frac{2^{t+4}}{3} \) is greater than \( \frac{2}{3} \left( \frac{2}{3} \right)^t \).
Lemma 4. Let \((b, u_0 > u_1 > u_2, \ldots > u_b = 0)\) be an admissible tuple. Let \(j < b\) and \(u'_i = u_i + 2.3^{b-j-1}\) if \(i \leq j\) and \(u'_i = u_i\) if \(i > j\). Then the tuple \((b, u'_0 > u'_1 > u'_2, \ldots > u'_b = 0)\) is admissible.

Proof. Let \(n = \frac{2u_0}{3^b} - \sum_{i=1,b} \frac{2u_i}{3^{b-i}}\), and \(x = \frac{2u'_0}{3^b} - \sum_{i=1,b} \frac{2u'_i}{3^{b-i}}\).

\[
x - n = \left(\frac{2u_0}{3^b} - \sum_{i=1,j} \frac{2u_i}{3^{b-i+1}}\right) \left(2.3^{b-j-1} - 1\right)
= \left(\frac{2u_0}{3^b} - \sum_{i=1,j} \frac{2u_i}{3^{b-i+1}}\right) \left(q3^{b-j}\right)
= q \left(\frac{2u_0}{3^j} - \sum_{i=1,j} \frac{2u_i}{3^{j-i+1}}\right)
\]

Lemma 7 on page 11 implies that \(q \in \mathbb{N}\) and lemma 2 on the preceding page implies that the second term is integer, therefore \(x\) is integer. \(\square\)

We introduce an alternative notation for the tuple \((b, u_0 > u_1 > u_2, \ldots > u_b = 0)\).

Let \(v_i = u_{i-1} - u_i, i = 1, \ldots, b\). The tuple \((b, \sum_{i=1, b} v_i, \sum_{i=2, b} v_i, \ldots, v_b)\) is equal to the tuple \((b, u_0 > u_1 > u_2, \ldots > u_b = 0)\). The alternative notation for this tuple is \((b, v_1, v_2, \ldots, v_b)\).

Note that \(v_i = j(3f(b-i)(n) + 1)\), with \(n\) given by theorem 1 on page 3, see the second proof of theorem 1 on page 3.

Theorem 2. Let \(v_i \in \mathbb{N}, i = j, \ldots, b\) with \(1 \leq v_i \leq 2.3^{b-i}\) and \(b > 1\). For each tuple \((v_2, v_3, \ldots, v_b) \ni v_1\) even with \(4 \leq v_1 \leq 2.3^{b-1}\) such that \((b, \sum_{i=1,b} v_i, \sum_{i=2,b} v_i, \ldots, v_b)\) is admissible.

Proof. The cardinal number of \(F = \{v_2, \ldots, v_b\}\) is

\[
|\{v_2, \ldots, v_b\}| = \prod_{i=2,b} 2.3^{b-i} = 2^{b-1}3^{\sum_{i=2,b}(b-i)} = 2^{b-1}3^{\sum_{k=0,b-2} k} = 2^{b-1}3^{\frac{(b-2)(b-1)}{2}}
\]

Let \(E\) be the set of the admissible \(\{v_1, v_2, \ldots, v_b\}\). \(\#E\) is equal to the product of the number of admissible nodes with children at each step excepted the last one with sterile nodes taken into account. At the first step \(v_1\), this number is \(\frac{2}{3}3^{b-1}\). Then for each \(v_1\) there are \(3^{b-2}\) possible admissible values for \(v_2\). From these values only \(\frac{2}{3}3^{b-2}\) have children, and so on till the last step with one admissible node (with or without child for this last step). The product is equal to \(2^{b-1}3^{\frac{(b-2)(b-1)}{2}}\).
Let \( t : E \mapsto F \) with \( t(v_1, v_2, \ldots, v_b) = (v_2, \ldots, v_b) \). \( t \) is injective because \( t(v'_1, v'_2, \ldots, v'_b) = t(v_1, v_2, \ldots, v_b) \Rightarrow (v'_2, \ldots, v'_b) = (v_2, \ldots, v_b). \) \( n = \frac{a^n}{3^n} - \sum_{i=1}^{b} \frac{2^{u_i}}{3^{v_i}} \in \mathbb{N} \) and \( n' = \frac{a^{n'}}{3^{n'}} - \sum_{i=1}^{b} \frac{2^{u_i}}{3^{v_i}} \in \mathbb{N} \). Therefore \( n' - n = \frac{a^{n'}}{3^{n'}} - \frac{a^n}{3^n} = 2^n \frac{a^{n'-n-1}}{3^{n'-n}} = 2^n \frac{2^{u'_1-v_1-1}}{3^{v_1}} \in \mathbb{N} \) and thus \( v'_1 - v_1 = p.2.3^{b-1} \) with \( p \geq 1 \). Therefore \( v'_1 = v_1 \) and \( t \) is injective. \#E = \#F and \( t \) injective imply that \( t \) is bijective and that only one \( v_1 \leq 2.3^{b-1} \) corresponds to a \( t \)-uple \((v_2, \ldots, v_b)\). 

Note that the number of admissible \( \{v_2, \ldots, v_b\} \) corresponding to one \( v_1 \) is

\[
\frac{2^{b-1}3^{\frac{(b-2)(b-1)}{2}}}{3^{b-1}} = 2^{b-2}3^{\frac{(b-2)(b-3)}{2}}.
\]

\[\square\]

| \( v_1 \) | 4 | 4 | 8 | 8 | 10 | 10 | 14 | 14 | 16 | 16 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( v_2 \) | 3 | 5 | 2 | 6 | 1 | 5 | 4 | 6 | 1 | 3 | 2 | 4 |
| \( v_3 \) | 2 | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 1 |
| \( b \) | 17 | 35 | 75 | 2417 | 151 | 4849 | ... | 1242755 |

Table 1: The 12 admissible tuples with \( b = 3 \)

Among the possible sequences \((v_2, \ldots, v_b)\) allowed by the theorem 2 on the previous page some are specially interesting such as the strictly ascending sequence \((f^{(i)}(n), i = 1, b-1)\) (see \( n = 151 \) in the above table as an example), given in the following corollary.

**Corollary.** \( \forall b \in \mathbb{N}, \exists n \in g(1) : \forall i \in (1, b-1), f^{(i)}(n) > f^{(i-1)}(n) \).

**Proof.** \( n \) is obtained with \( v_1 = 1, i = 2 : b \), and \( v_1 \) given by theorem 2 on the preceding page and lemma 4 on the previous page. \[\square\]

The Wirsching-Goodwin representation of the nodes of \( g(1) \) obtained with \( b \) steps (see [2]) may be now stated in the following theorem. Let \( g^*(1, b) = \{ n \in g(1) : f^{(b)}(n) = 1 \text{ and } f^{(b-1)}(n) \neq 1 \} \) and \( v'_1 \) the value of \( v_1 \) whose existence is proven in theorem 2 on the preceding page.

**Theorem 3.** There is a one to one relation between \( g^*(1, b) \) with \( b > 1 \) and the set of the tuples \((b, v'_1, v_2', \ldots, v'_b)\) with \( v'_1 = v_i + 2.3^{b-i}c_i, c_i \in \mathbb{N}^*, v_i \in \mathbb{N}, i = 2, \ldots, b \) with \( 1 \leq v_i \leq 2.3^{b-i} \) and \( 4 \leq v_1 = v'_1 \leq 2.3^{b-1} \).

**Proof.** Direct from theorem 2 on the previous page and lemma 4 on the preceding page. \[\square\]

For each \( b \) and \((v_2, \ldots, v_b)\) there is a unique \( v_1 \in (4, 2.3^{b-1} + 2) \). The theorem 4 gives its value.
Theorem 4. \( v_i^* = a - \sum_{i=2}^b v_i \) with
\[
a \equiv \log_2 \left( \sum_{i=1}^b 2^{u_i} 3^{i-1} \mod 3^b \right) \pmod{2.3^{b-1}}.
\]

Proof. \( n = \frac{2^a}{3^b} - \sum_{i=1}^b \frac{2^{u_i}}{3^{b-i+1}} \Rightarrow 2^a = n3^b + \sum_{i=1}^b 2^{u_i} 3^{i-1} \).

The group \((\mathbb{Z}/3^b \mathbb{Z})^*\) is cyclic of order \(2.3^{b-1}\) and generated by \(2 \mod 3^b\) (see [4]). This means that \(F: \mathbb{Z}/2.3^{b-1} \mathbb{Z} \to (\mathbb{Z}/3^b \mathbb{Z})^*\)
\[
i \mod 2.3^{b-1} \mapsto 2^i \mod 3^b.
\]
is defined and bijective. So we can use its reciprocal \(F^{-1}\) and call it \(\log_2\).

E.g. \(\log_2(7 \mod 9) = 4 \mod 6\) since \(2^4 = 16 \equiv 2^{10} = 1024 \equiv 7 \pmod{9}\).

\(2^a \equiv \sum_{i=1}^b 2^{u_i} 3^{i-1} \pmod{3^b}\) implies
\[
a \mod 2.3^{b-1} = \log_2 \left( \sum_{i=1}^b 2^{u_i} 3^{i-1} \mod 3^b \right)
\]

\(\square\)

3.4 Ascending Collatz sequences excepted the last term

It is possible to give explicitly \(a\) and \(v_1\) in some particular cases such as \((v_2 = v_3 = \ldots = v_b = 1)\) and any \(b\). The following theorem defines all the strictly ascending sequence \((f^{(i)}(n), i = 1, b - 1)\) with a last value \(f^{(b)}(n) = 1\).

Theorem 5. Let \((v_2 = v_3 = \ldots = v_b = 1)\) then \(v_1^* = 3^{b-1} + 1\), with the corresponding
\[
n = \frac{2^{b-1} 3^{b-1} + 1}{3^b} - \sum_{i=1}^b \frac{2^{b-i}}{3^{b-i+1}}.
\]

Proof. With induction with \(b\). The theorem is true for \(b = 2\) and \(b = 3\) (see the above table). Assume that it is true till \(b - 1\). Let \(n_j = \frac{2^{j-1} 3^{b-1} + 1}{3^j} - \sum_{i=1}^j \frac{2^{j-i}}{3^{j-i+1}}, j = 1, b,\) the values obtained at step \(j\) with \(b\) total steps, and \(m_j = \frac{2^{j-1} 3^{b-2} + 1}{3^j} - \sum_{i=1}^j \frac{2^{j-i}}{3^{j-i+1}}, j = 1, b - 1,\) the values obtained at step \(j\) with \(b - 1\) total steps. The \(m_j\) are integers by the induction hypothesis. Note that
\[
3^{b-1} + 1 = 3^{b-2} + 1 + 2.3^{b-2} = v_1^*(b - 1) + 2.3^{b-2}
\]
Proposition 2. Let \( v_i = k \in \mathbb{N}^* \), \( i = 2, b \). \( v_1 \) is defined by the relation \( 2^{v_{i-1} - k}(2^k - 3) \equiv 1 \) (mod \( 3^b \)).

\[ v_1 = k(b - i) \text{, therefore } \sum_{i=1}^{i=b} 2^{v_{i-1} - 1} = 2^{k(b-1)}. \sum_{i=1}^{i=b} (2^k)^{-i+1}3^{-i-1} = \frac{2^{k(b-1)}}{3^k-3} \]

\[ a = v1 + (b-1)k \Rightarrow 2^{v_{i-1} + (b-1)k} = \frac{2^{k(b-1)}}{3^k-3} \text{ (mod } 3^b) \]

\[ 2^{v_{i-1} + (b-1)k}(2^k - 3) \equiv 2^{k(b-1)} \text{ (mod } 3^b) \Rightarrow 2^{v_{i-1} - k}(2^k - 3) \equiv 1 \text{ (mod } 3^b). \]

Table 2: \( v_i^* \) for \( b = 2, \ldots, 6 \) and \( v_i = 1, i = 2 \ldots b \)

| \( v_i^* \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( n \) |
|-----------|-------|-------|-------|-------|-------|-----|
| 4         | 1     |       |       |       |       | 3   |
| 10        | 1     | 1     |       |       |       | 151 |
| 28        | 1     | 1     | 1     |       |       | 265 |
| 82        | 1     | 1     | 1     | 1     |       | 318 |
| 244       | 1     | 1     | 1     | 1     | 1     | ... |

Therefore the lemma 4 on page 6 implies that \( n_1, \ldots, n_{b-1} \) are integers. We have to prove that \( n_b \in \mathbb{N} \). This is true if \( n_{b-1} \equiv 2 \) (mod 3).

Using Lemma 8 on page 12 one obtains

\[ n_{b-1} \equiv m_{b-1} + (-1)^b \equiv \begin{cases} 1 + 1 = 2 \text{ (mod } 3) & \text{if } b \equiv 0 \text{ (mod } 2) \\ 0 - 1 = 2 \text{ (mod } 3) & \text{if } b \equiv 1 \text{ (mod } 2) \end{cases} \]

A simpler (but not directly related to theorem 3) proof of theorem 3 is the following: a Collatz sequence \( (n_0, n_1, \ldots, n_b) \) is growing if \( \forall i, n_{i+1} = (3n_i + 1)/2 \). Therefore \( \exists k \in \mathbb{N}^* \), \( n_0 = k2^b - 1 \) and \( n_b = k3^b - 1 \) if \( n_i = k3^b2^{i-1} - 1 \). If the following term of the sequence is 1, then \( \exists l \in \mathbb{N}^*, 1 = (k3^{b+1} - 2)/2^l \). Thus \( k3^{b+1} = 2 + 2^l \) and \( 2^l \equiv -2 \) (mod \( 3^{b+1} \)), which is equivalent to \( l - 1 \equiv 3^b \) (mod \( 2 \times 3^b \)). The first term of this Collatz sequence is \( n_0 = (1 + 2(2p+1)3^b)(2/3)^{b+1} - 1 \) with \( p \in \mathbb{N} \).

Theorem 3 is generalized by the following proposition.

**Proposition 2.** Let \( v_i = k \in \mathbb{N}^*, i = 2, b \). \( v_1 \) is defined by the relation \( 2^{v_i - k}(2^k - 3) \equiv 1 \) (mod \( 3^b \)).

**Proof.** \( u_i = k(b - i) \), therefore \( \sum_{i=1}^{i=b} 2^{v_{i-1} - 1} = 2^{k(b-1)} \sum_{i=1}^{i=b} (2^k)^{-i+1}3^{-i-1} = \frac{2^{k(b-1)}}{3^k-3} \).

Theorem 4 implies that \( 2^a \equiv \frac{2^{kb-3}}{3^k-3} \) (mod \( 3^b \))

\[ a = v1 + (b-1)k \Rightarrow 2^{v_{i-1} + (b-1)k} = \frac{2^{k(b-1)}}{3^k-3} \text{ (mod } 3^b) \]

\[ 2^{v_{i-1} + (b-1)k}(2^k - 3) \equiv 2^{k(b-1)} \text{ (mod } 3^b) \Rightarrow 2^{v_{i-1} - k}(2^k - 3) \equiv 1 \text{ (mod } 3^b). \]
If \( k = 1, 2 \) one obtains explicitly all the values of \( v_1 \), but this is not true for \( k \geq 3 \).
If \( k = 1, 2^{n-1} \equiv -1 \pmod{3^b} \Rightarrow v_1 - 1 \equiv 3^{b-1} \pmod{2 \cdot 3^{b-1}} \).
If \( k = 2, v_1 - 2 \equiv 0 \pmod{2 \cdot 3^{b-1}} \).
If \( k = 3, 5^b v_{i-3} \equiv 1 \pmod{3^b} \Rightarrow v_1 \equiv 3 - \log_2(5 \pmod{3^b}) \pmod{2 \cdot 3^{b-1}} \).

This kind of result can be extended to any periodic sequence \((v_2, v_3, \ldots)\). For example the sequence \( v_{2i} = 1, v_{2i+1} = 2, i = 1, \ldots (b-1)/2 \) implies that \( 2^{v_1} \equiv -20 \pmod{3^b} \). This result is obtained by dividing \( \sum_{i=1,b} 2^u \cdot 3^{b-1} \) in two separate geometric series that gives \( 3^{b+1} - 2^{3(b+1)} - 12 \left(3^{b-1} - 2^{3(b+1)}\right) \). Note that this particular sequence is associated to a globally increasing Collatz sequence till the penultimate term.

### 3.5 Structure of \( g(n) \)

The structure of \( g(n) \) for \( n \in \mathbb{N} \), is similar to the structure of \( g(1) \) (for \( n \in g(1) \) or \( n \notin g(1) \)). Proofs are very similar to the case of \( g(1) \) and are not given here. Theorem 6 is slightly modified:

**Theorem 6.** Let \( m \in g(n), n \equiv (1, 2) \pmod{3} \). \( \exists (b, a > u_1 > u_2, \ldots > u_b = 0) \in \mathbb{N}^{b+2} : \)

\[
m = n \cdot \frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}}.
\]

Theorem 3 remains true for \( g(n) \):

**Theorem 7.** If \( n \equiv (1, 2) \pmod{3} \), there is a one to one relation between \( g(n, b) \) with \( b > 1 \) and the set of the tuples \((b, v'_1, v'_2, \ldots, v'_b)\) with \( v'_i = v_i + 3^{b-i}c_i, c_i \in \mathbb{N}^*, v_i \in \mathbb{N}, i = 2, \ldots b \) with \( 1 \leq v_i \leq 2 \cdot 3^{b-i} \) and \( 1 \leq v_1 = v'_1 \leq 2 \cdot 3^{b-1} \).

If \( n \equiv 2 \pmod{3} \), \( v_1 \) is odd. If \( n \equiv 1 \pmod{3} \), \( v_1 \) is even. The Theorem 4 becomes

**Theorem 8.** Let \( n \equiv (1, 2) \pmod{3} \).

\[
v'_1 = a - \sum_{i=2,b} v_i\quad \text{with} \quad a \equiv \log_2 \left(\sum_{i=1,b} 2^{u_i} \cdot 3^{i-1} \pmod{3^b}\right) - \log_2(n \pmod{3^b}) \pmod{2 \cdot 3^{b-1}}.
\]

Finally the proposition 2 may be extented to any starting number:

**Proposition 3.** Let \( n \equiv (1, 2) \pmod{3} \) and \( v_i = k \in \mathbb{N}^*, i = 2, b, v_1 \) is defined by the relation \( n \cdot 2^{u_b-k} (2^k - 3) \equiv 1 \pmod{3^b} \).

This proposition with \( k = 1 \) allows to define all the strictly ascending sequences from \( m \) to \( n \), \((f^{(i)}(m), i = 1, b)\) with \( f^{(b)}(m) = n \). Contrarily to the case of \( g(1) \), the sequence \( v_1 = \ldots v_b = 1 \) exists and the proposition 3 imply that \( n \equiv -1 \pmod{3^b} \). \( n \) is odd therefore \( n = 2p \cdot 3^b - 1 \), with the associate value \( m = 2^{b+1}p - 1 \). Moreover the proposition 3 shows that \( v_1 = \ldots v_b = 2 \) imply that \( n \equiv 1 \pmod{3^b} \). \( n \) is odd therefore \( n = 2p \cdot 3^b + 1 \), with the associate value \( m = 2^{b+1}p + 1 \). Note that this includes the case \( p = 0 \) and \( n = m = 1 \).
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A Proofs of Lemmas

Lemma 5.

\[
2^k \equiv \begin{cases} 
2 & \text{if } k \text{ odd} \\
1 & \text{if } k \text{ even} 
\end{cases} \pmod{3}
\]

Proof. 2 \equiv −1 (mod 3). So if k is even then \(2^k \equiv (−1)^k = 1 \pmod{3}\).

If k is odd then \(2^k \equiv (−1)^k = −1 \equiv 2 \pmod{3}\). □

Lemma 6. \(\frac{2^k+1}{3^{k+1}} \in \mathbb{N} \text{ and } \equiv 1 \pmod{3}\)

Proof. By induction. The lemma is true for \(k = 0\).

Assuming the lemma true for \(k−1\) implies that \(2^{3^{k−1}}+1 = 3^k x\) with \(x \equiv 1 \pmod{3}\).

\[
2^{3^k} + 1 = \left(\frac{2^{3^k+1}}{3^{k+1}}\right)^3 + 1
\]

\[
= \left(\frac{3^k x - 1}{3^{k+1}}\right)^3 + 1
\]

\[
= 3^{3^k} x^3 - 3^{2k+1} x^2 + 3^{k+1} x
\]

\[
\frac{2^{3^k} + 1}{3^{k+1}} = 3^{3^k−1} x^3 - 3^k x^2 + x \equiv x \equiv 1 \pmod{3}
\]

□

Lemma 7. \(\frac{2^{3^k} - 1}{3^{k+1}} \equiv 1 \pmod{3}\)

Proof.

\[
\frac{2^{3^k} - 1}{3^{k+1}} = \frac{2^{3^k} + 1}{3^{k+1}}(2^{3^k} - 1)
\]

\[
\equiv 2^{3^k} - 1 \pmod{3}, \text{ see Lemma [5]}\]

\[
\equiv 1 \pmod{3}, \text{ see Lemma [5]}
\]

□
Lemma 8. Let \( n_b \) given by theorem 5 on page 8.

\[
n_b \equiv \begin{cases} 
0 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\
1 \pmod{3} & \text{if } b \equiv 1 \pmod{2} 
\end{cases}
\]

Proof. By induction. The lemma is true for \( b = 2 \) and \( b = 3 \) because \( n_2 = 3 \) and \( n_3 = 151 \). Assume that the lemma is true till \( b - 1 \).

\[
\begin{align*}
n_b - m_{b-1} & = \frac{2^{b-1}2^{3^{b-1}+1}}{3^b} - \frac{\sum_{i=1}^b \frac{2^{b-i}}{3^{b-i+1}} - \frac{2^{b-2}2^{3^{b-2}+1}}{3^{b-1}}}{3^b} + \sum_{i=1}^{b-1} \frac{2^{b-1-i}}{3^{b-1}} \\
& = \frac{2^{b-1}2^{3^{b-1}+1}}{3^b} - \frac{2^{b-1}}{3^b} - \frac{2^{b-2}2^{3^{b-2}+1}}{3^{b-1}} \\
& = 2^{b-1} \left( \frac{2^{3^{b-1}} + 1}{3^b} - \frac{2^{3^{b-2}} + 1}{3^{b-1}} \right) \\
& \equiv 2^{b-1}(2 - 1) \pmod{3}, \quad \text{see lemma 6 on the previous page}
\end{align*}
\]

\[
n_b \equiv m_{b-1} + 2^{b-1} \equiv \begin{cases} 
1 + 2 \equiv 0 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\
0 + 1 = 1 \pmod{3} & \text{if } b \equiv 1 \pmod{2} 
\end{cases}
\]