Dynamical mass generation in 2+1-dimensional electrodynamics in an external magnetic field.

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Abstract.
The influence of a magnetic field on the mass generation in 2+1 dimensional QED is considered. It is shown that the magnetic field is a catalyst of the generation of a fermion dynamical mass. The mass arises in the system with arbitrary number of fermions, not only with \( N \leq 4 \), as it is in the system without the magnetic field. The polarization tensor is calculated for a constant magnetic field.
Introduction.

It was shown in [1-4] that a constant magnetic field is a strong catalyst of dynamical mass generation and flavor symmetry breaking because of the interaction between fermions. This effect is due to the dimensional reduction \((D \rightarrow D-2)\) in dynamics of pairing of fermions in a magnetic field. It is connected with restriction of the motion of charged particles in directions that are perpendicular to the magnetic field. The phenomenon of the catalysis was illustrated by the Nambu–Jona–Lasinio model (NJL) in 2+1 and 3+1 dimensions, quantum electrodynamics (QED) in 3+1 dimensions. The crucial role of the lowest Landau level (LLL) in catalyzing the spontaneous symmetry breaking was emphasized. The role of the LLL in the effect of dynamical symmetry breaking is similar to the role of the Fermi surface in the BCS theory.

Here we investigate effect of the dynamical mass generation in 2+1 electrodynamics (QED\(_3\)) in an external magnetic field. There are \(N\) flavors in the system, initial lagrangian is massless. This model is interesting from two points of view. At first, QED\(_3\) can serve as an effective theory for the description of longwave excitations in planar systems in the condensed matter theories [7]. QED\(_3\) also has properties reminiscent of QCD and other four-dimensional gauge theories [5, 6]. The investigation of the Schwinger-Dyson equation for the fermion self energy in \(1/N\) expansion without magnetic field indicates that there exists critical number of fermions \((N_c \simeq 3 \div 4)\) [5], the chiral symmetry is broken for \(N < N_c\). In such a way QED\(_3\) has two phases (massive and massless) depending on parameters of the theory, as it was in NJL and QED\(_4\) (in ladder approximation). But in contrast to QED\(_4\) polarization effects must be taken into account to obtain the chiral phase transition. With turning on a strong magnetic field in QED\(_3\) the dimensional reduction occurs and dynamical mass of fermions is generated for arbitrary number of flavors.

This article contains three sections and appendix: in the first section the dimensional reduction of the space is considered with help of the Schwinger-Dyson equation, in the second section the effects that are connected with vacuum polarization are under consideration, in the third section the expression for a mass of fermions is obtained. The polarization tensor for the QED\(_3\) in the constant external magnetic field is obtained in appendix.

1. The Schwinger-Dyson equation for the system with a magnetic field.

In the presence of a magnetic field the infrared region is responsible for the mass generation. This justifies using the ladder approximation because the full vertex \(\Gamma^\mu\) is replaced by its infrared asymptotics \(\gamma^\mu\) (Dirac matrices). Let us consider the Schwinger-Dyson (SD) equation in the improved ladder approximation (the vacuum polarization is taken into account).

\[
G(x, y) = S(x, y) - ie^2 \int d^3 z d^3 t S(x, z) \gamma^\mu G(z, t) \gamma^\nu G(t, y) D_{\mu\nu}(t - z),
\]

where \(G(x, y) = -i \left\langle 0 \left| T \psi(x) \bar{\psi}(y) \right| 0 \right\rangle\) is the full electron propagator, \(D_{\mu\nu}(x - y) = i \left\langle 0 \left| T A_\mu(x) A_\nu(y) \right| 0 \right\rangle\) is the full photon propagator, \(S\) is the free fermion propagator.

\[
S(x, y) = \exp \left( \frac{ie}{2} (x - y)^\mu A^{ext}_\mu(x + y) \right) \tilde{S}(x - y), \quad A^{ext} = (0, -\frac{B}{2} x_2, \frac{B}{2} x_1).
\]
The expression for $\tilde{S}$ will be written down further. The full electron propagator can be represented as follows:

$$G(x, y) = \exp \left( \frac{i e}{2} (x - y)^\mu A^\text{ext}_\mu (x + y) \right) \tilde{G}(x - y).$$ \hspace{1cm} (3)

In terms of $r = x - y, R = \frac{x + y}{2}$ the SD equation takes the form:

$$\tilde{G}(r) = \tilde{S}(r) - i e^2 \int d^3 R_1 d^3 r_1 \tilde{S} \left( \frac{r - r_1}{2} - R_1 \right) \gamma^\mu \tilde{G}(r_1) \gamma^\nu \tilde{G} \left( \frac{r - r_1}{2} + R_1 \right) \cdot D_{\mu\nu}(-r_1) \exp \left( i e (r + r_1) A^\text{ext}(R_1) \right).$$ \hspace{1cm} (4)

Transforming this into the momentum space, we obtain:

$$\tilde{G}(p) = \tilde{S}(p) - i e^2 \int \frac{d^3 k_\perp d^2 R_\perp d^2 q_\perp d k_0}{(2\pi)^5} e^{-iq_\perp R} \tilde{S} \left( p_0, p_\perp + eA(R) + \frac{q_\perp}{2} \right) \cdot \gamma^\mu \tilde{G}(k) \gamma^\nu \tilde{G} \left( p_0, p_\perp + eA(R) - \frac{q_\perp}{2} \right) D_{\mu\nu}(k_0 - p_0, k_\perp - p_\perp - 2eA(R)), \hspace{1cm} (5)

where $q_\perp = (q_1, q_2)$. Schwinger found the exact expression for the free electron propagator in an external magnetic field $[11]$:

$$S = \exp \left( ie (x - y)^\mu A^\text{ext}_\mu \left( \frac{x + y}{2} \right) \right) \tilde{S}(x - y),$$

$$\tilde{S}(k) = -i \int \, ds \exp \left( -ism^2 + isk^2_0 - isk^2 tg(eBk) \right) \cdot (k^\mu \gamma_\mu + (k^2 \gamma^1 - k^1 \gamma^2)tg(eBk))(1 + \gamma^1 \gamma^2 tg(eBk)).$$ \hspace{1cm} (6)

According to $[10]$ $\tilde{S}$ can be decomposed over the Landau level poles:

$$\tilde{S}(k) = -\exp \left( -\frac{k_\perp}{|eB|} \right) \sum_{n=0}^{\infty} \frac{(-1)^n D_n(eB, k)}{m^2 - k^2_0 - 2|eB|n},$$ \hspace{1cm} (7)

where

$$D_n(eB, k) = (m + k^0 \gamma^0) \left( 1 - i\gamma^1 \gamma^2 \text{sign}(eB) \right) L_n \left( \frac{2k^2_\perp}{|eB|} \right) - (1 + i\gamma^1 \gamma^2 \text{sign}(eB)) L^1_{n-1} \left( \frac{2k^2_\perp}{|eB|} \right) + 4(k^1 \gamma^1 + k^2 \gamma^2) L^1_{n-1} \left( \frac{2k^2_\perp}{|eB|} \right).$$

Therefore, in strong enough field, the sum can be reduced to only one term ($k, m_{dyn} \ll \sqrt{e|B|}$).

$$\tilde{S}(p) = e^{-\frac{e^2p^2}{2}} \frac{1}{\gamma_0 p_0 - m} (1 - i\gamma^1 \gamma^2 \cdot \text{sign}(B)); \hspace{1cm} (8)$$
\( \ell = 1/\sqrt{e|B|} \) is a magnetic length (let us take \( B > 0 \)). The main contribution comes from the LLL. And this leads to the dimensional reduction. It is natural to search the full electron propagator in the form

\[
\tilde{G}(p) = e^{-\ell^2 p_\perp^2} g(p_0).
\]  

(9)

If we substitute this into (5), we obtain the equation for \( g \):

\[
g(p_0) \gamma^0 p_0 = 1 - i \gamma^1 \gamma^2 - \frac{ie^2}{4(2\pi)^3} (1 - i \gamma^1 \gamma^2) \int dk_0 \gamma^\mu \gamma^\nu g(k_0) \gamma^\nu g(p_0) D^0_{\mu\nu}(k_0 - p_0),
\]  

(10)

\[
D^0_{\mu\nu}(p_0) = \int d^2 t_\perp e^{-\ell^2 t_\perp^2 / 2} D_{\mu\nu}(p_0, t_\perp).
\]

Let us consider the matrix structure of the full electron propagator. We can write the SD equation in two equivalent forms:

\[
G(x, y) = S(x, y) - ie^2 \int d^3 z d^3 t S(x, z) \gamma^\mu G(z, t) \gamma^\nu G(t, y) D_{\mu\nu}(t - z),
\]

or

\[
G(x, y) = S(x, y) - ie^2 \int d^3 z d^3 t G(x, z) \gamma^\mu G(z, t) \gamma^\nu S(t, y) D_{\mu\nu}(t - z).
\]

Therefore the full propagator takes the form (see(8))

\[
G = \frac{1 - i \gamma^1 \gamma^2}{2} \xi \frac{1 - i \gamma^1 \gamma^2}{2} = P \cdot \xi, \quad P^2 = P,
\]  

(11)

where \( \xi \) is a matrix \( 4 \times 4 \). In the standard representation

\[
P \begin{pmatrix} a & ? & b & ? \\ ? & ? & ? & ? \\ c & ? & d & ? \\ ? & ? & ? & ? \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]  

(12)

In such a way a strong magnetic field restricts matrix structure of the full electron propagator to \( G \in GL(2, \mathbb{C}) \). Hence the equation for the full electron propagator reduces to the form:

\[
\Lambda (2 \tilde{g} \gamma^0 p_0 - 2) \Lambda = -\frac{ie^2}{4(2\pi)^3} 2 \Lambda \int dk_0 \gamma^\mu \Lambda 2 \tilde{g}(k_0) \Lambda \gamma^\nu \Lambda 2 \tilde{g}(p_0) \Lambda D^0_{\mu\nu}(k_0 - p_0),
\]

\[
\Lambda = \frac{1 - i \gamma^1 \gamma^2}{2}, \quad g = \Lambda 2 \tilde{g} \Lambda,
\]  

(13)
where $\tilde{g}$ is in general the linear combination of four matrices $I, \gamma^0, \gamma^3, \gamma^5$. Because of identities $\Lambda \gamma^2 \Lambda = \Lambda \gamma^1 \Lambda = 0$, $[\Lambda, \tilde{g}] = 0$, we get

$$\frac{1}{\tilde{g}(p_0)} = p_0 \gamma^0 + \frac{ie^2}{(2\pi)^3} \int dk_0 \gamma^0 \tilde{g}(k_0) \gamma^0 D^0_{00}(k_0 - p_0).$$

(14)

This equation has the one-dimensional form what justifies the term 'dimensional reduction'. The system of equations for scalar functions can be obtained if we put

$$G = e^{-\rho^2/\ell^2} \frac{1}{A\gamma^0 p_0 - B(1 - i\gamma^1 \gamma^2)}, \quad \tilde{g} = \frac{1}{A\gamma^0 p_0 - B},$$

(15)

$$p_0 (1 - A(p_0)) = -\frac{ie^2}{(2\pi)^3} \int dk_0 \frac{A(k_0)k_0}{A^2 k_0^2 - B^2} D^0_{00}(k_0 - p_0),$$

(16)

$$B(p_0) = -\frac{ie^2}{(2\pi)^3} \int dk_0 \frac{B(k_0)}{A^2 k_0^2 - B^2} D^0_{00}(k_0 - p_0),$$

(17)

Where $A, B$ are functions of one variable $p_0$. Transforming this into the Euclidean region ($p_0 \rightarrow i\rho$), we obtain:

$$B(p) = \frac{e^2}{(2\pi)^3} \int dk \frac{B(k)}{A^2 k^2 + B^2} D(k - p),$$

(18)

$$p(1 - A) = \frac{e^2}{(2\pi)^3} \int dk \frac{Ak}{A^2 k^2 + B^2} D(k - p),$$

(19)

$$D(k) = -\int d^2 p_\perp e^{-i^2 \rho^2/2} D_{00}(ik, p_\perp).$$

(20)

2. The vacuum polarization in QED$_3$ in an external magnetic field.

Let us investigate the influence of the strong magnetic field on the vacuum polarization. For this purpose let us write the SD equation for the polarization tensor:

$$\Pi^{\mu\nu}(x, x') = -ie^2 Tr \int d^3 y d^3 z \ G(z, x) \gamma^\mu G(x, y) \Gamma^\nu(y, z|x'),$$

(21)

where $\Gamma^\nu$ is the full vertex, $G$ is the full electron propagator. Since in the strong magnetic field the full propagator has the structure (11) and

$$1 - i\gamma^1 \gamma^2 \gamma^\mu 1 - i\gamma^1 \gamma^2 = \gamma^0 \delta^{\mu, 0} 1 - i\gamma^1 \gamma^2 /2, \quad \Pi^{\mu\nu} = \Pi^{\nu\mu},$$

(22)
then only $\Pi^{00}$ does not vanish. Because of $k_\mu \Pi^{\mu\nu} = k_0 \Pi^{00} = 0$, which is a consequence of the gauge invariance, $\Pi^{\mu\nu}(k) = 0$. Thus the LLL does not contribute to the polarization tensor. It means that photons become almost free in very strong magnetic field. The contribution of the next levels into the $\Pi^{00}$ is $\sim 1/\sqrt{eB}$ (see appendix).

The suppression of the vacuum polarization is connected with absence of the longitudinal (with respect to the magnetic field) components of the polarization tensor in QED$_3$. In general it takes the form (restrictions are put on by the gauge invariance and by the presence of only one singled out direction):

$$
\Pi^{\mu\nu} = \Pi_0 \left( g_{\mu\nu} k^2 - k_\mu k_\nu \right) + \Pi_\perp \left( -g_\perp^{\mu\nu} k^2_\perp - k_\mu k_\nu \right) + \Pi_\parallel \left( g_\parallel^{\mu\nu} k^2_\parallel - k_\mu k_\nu \right) \tag{23}
$$

$$
k^2_\perp = k_1^2 + k_2^2, \ k_\mu = \delta^{\mu0} k^0, \ g_\parallel = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ g_\perp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{24}
$$

The last term in (23), which grows with the increase of the field in four-dimensional case [1], vanishes in QED$_3$. Thus, the contribution of the vacuum polarization to the effect of a mass generation is small when $e^2 \ll \sqrt{eB}$. Further it will be taken into account in the following way:

$$
D_{00} \approx \left( 1 - \frac{k^2_0}{k^2} \right) \frac{1}{k^2(1 + \nu_0 Ne^2 \ell)} + \frac{\lambda k^2_0}{k^4}, \ \nu_0 = 0.14037 \tag{25}
$$

3. The dynamical mass of fermions in the strong magnetic field.

Let us analyse equations (18,19). It is convinient to rewrite them in the dimensionless form:

$$
t(1 - A(t)) = \alpha_0 \int ds \frac{s A(s)}{A^2 s^2 + M^2} U(s - t), \tag{26}
$$

$$
M(t) = \alpha_0 \int ds \frac{M(s)}{A^2 s^2 + M^2} U(s - t), \tag{27}
$$

$$
M(s) \equiv \ell B(\ell k), \ A(s) \equiv A(\ell k), \ \alpha_0 = \frac{e^2 \ell}{4\pi^2 (1 + \nu_0 Ne^2 \ell)},
$$

$$
U(s) = e^{s^2/2} \int_{|s|}^\infty dt \frac{e^{-t^2/2}}{t} - e^{s^2/2} s^2 \int_{|s|}^\infty dt \frac{1 - \lambda(1 + \Pi(0))}{t^3} e^{-t^2/2}. \tag{28}
$$

Further it will be shown that only infrared region is responsible for the mass generation, therefore we need to know only infrared behavior of the $U$. The first term in (28) has the logarithmic singularity near $s = 0$. The second term is connected with a gauge and
is nonsingular, therefore its contribution is small. This fact reflects approximate gauge invariance of the ladder approximation. Further only singular term will be considered: $U(s) \approx -\ln(s)$. Let us suppose that $B(p^2)$ decreases as function of $p^2$, $A(p^2)$ tends to the finite value $A_\infty$ when $p \gg B(0)$. Let us find $A_\infty$. If we differentiate (26) and omit derivative of $A$ at $s \gg M(0)$, we obtain:

$$1 - A(t) = \alpha_0 \int ds \frac{sA(s)}{A^2 s^2 + M^2} \frac{dU(t - s)}{dt}$$

$$\rightarrow \alpha_0 \int ds \frac{sA(s)}{A^2 s^2 + M^2} U(t) = 0. \quad (29)$$

And so $A_\infty = 1$. To estimate $A(0)$ let us make the next approximation:

$$\frac{1}{A^2 p^2 + B^2} \approx \frac{1}{p^2 + B^2(0)}, \quad (30)$$

which works well both for $p \gg B(0)$ and for $p < B(0)$. Then after differentiating the expression in (26) at $t = 0$ we get:

$$1 - A(0) \approx \alpha_0 \int \frac{ds A(s)}{s^2 + M^2(0)} A(0) \frac{\alpha_0 \pi}{M(0)} = A(0) \left(1 + \frac{\alpha_0 \pi}{M(0)}\right)^{-1}. \quad (31)$$

We used infrared asymptotics of $U$ and the fact that (30) behaves as $\delta$-function. Let us find $M(0)$, it defines a fermion dynamical mass. At first let us prove that a dynamical mass is generated for arbitrary $e^2$, $N$, if $\ell m_{\text{dyn}} \ll 1$. Approximate equation

$$M(t) = \alpha_0 \int ds \frac{M(s)}{s^2 + M^2(0)} U(s - t). \quad (32)$$

can be reduced to the Shrödinger equation as a result of transformation:

$$V(x) = -\alpha_0 \int ds e^{isx} U(s), \quad \Psi(x) = \frac{1}{2 \pi} \int ds \frac{M(s)}{s^2 + M^2(0)} e^{isx}, \quad (33)$$

$$\frac{d^2}{dx^2} \Psi + (E - V) \Psi = 0, \quad E = -M^2(0). \quad (34)$$

Using the first (singular) term in (26), we obtain

$$V(x) = -\frac{\alpha_0 \pi^{3/2}}{\sqrt{2}} e^{x^2/2} Erfc\left(\frac{x}{\sqrt{2}}\right), \quad Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt, \quad (35)$$

$$V(x) = -\frac{\alpha_0 \pi}{x}, \quad x \gg 1.$$
Therefore the potential is longrange. With the help of variational method, it is easy to show that equation (34) has eigenvalues in discrete spectrum if \( U(s) > 0 \) when \( s \to 0 \). In our case it means that mass is generated for arbitrary \( e^2 \), \( N \) if \( \ell m_{\text{dyn}} \ll 1 \). Let us find \( M(0) \). If we put \( s = 0 \) in (32), we obtain

\[
M(0) = -\alpha_0 \int ds \frac{M(s)}{s^2 + M^2(0)} \ln|s|.
\]  

(36)

Because of \( M(0) \ll 1 \) and the expression to be integrated is singular near \( s=0 \) then the infrared region gives the main contribution to the integral. Therefore we can replace \( M(s) \) in the right-hand side to the \( M(0) \). Then

\[
M(0) \approx -\pi \alpha_0 \ln M(0).
\]

(37)

Then we obtain the expression for the dynamical mass of fermions:

\[
m_{\text{dyn}} \approx \sqrt{|eB|} \frac{M(0)}{A(0)} \approx \frac{e^2}{4\pi(1 + \nu_0 Ne^2\ell)} \left(1 - \frac{1}{\ln \alpha_0 \pi}\right) \left(\ln(\alpha_0 \pi) + \ln(-\ln(\alpha_0 \pi))\right).
\]

(38)

\[
\alpha_0 = \frac{e^2 \ell}{4\pi^2(1 + \nu_0 Ne^2\ell)}.
\]

It is interesting to consider the case of the weak field in comparison with charge: \( e^2 \gg \sqrt{eB} \). In this case

\[
m_{\text{dyn}} = -\frac{1}{4\pi \nu_0 N} \sqrt{|eB|} \left(1 - \frac{1}{\ln \alpha_0 \pi}\right) \left(\ln(\alpha_0 \pi) + \ln(-\ln(\alpha_0 \pi))\right), \quad \alpha_0 = \frac{1}{4\pi^2 \nu_0 N}.
\]

(39)

When \( N \gg 1 \) the condition \( \ell m_{\text{dyn}} \ll 1 \) is true, i.e. magnetic field can be turned off without breaking of the approximation \( e^2 \ell \ll 1 \). It is in agreement with the fact that without a magnetic field fermions are massless when \( N \gg 1 \).

This solution for \( m_{\text{dyn}} \) is non–analytic by the coupling constant and it can’t be evaluated in the perturbation theory. It is in the full agreement with statements in [12],[13] about analytic dependence of the ground state energy in the one–dimensional Shrödinger equation with the long–range potential (\( V(x) \sim 1/x \), \( x \gg 1 \)). Non–analytical dependence of \( m_{\text{dyn}}(\alpha_0) \) essentially differs QED\(_3\) and 2+1 dimensional NJL where dependence of the \( m_{\text{dy}n} \) on the coupling constant is analytical:

\[
m_{\text{dyn}} = |eB| \frac{NG}{2\pi}, \quad G \to 0,
\]

(40)

where \( N \) is number of flavors, \( G \) is a coupling constant in the NJL model. The analytical dependence of \( m_{\text{dyn}}(G) \) is due to the short–range character of the potential \( V \sim \delta(x) \)

\[1\] We can choose \( \exp(-\varepsilon x) \) as a trial function.

\[2\] by using iterations.
in this model. In ref. [3] it is shown that the dependence of the ground state energy on the coupling constant in the one–dimensional Shrödinger equation

\[ \frac{d^2\Psi}{dx^2} + (E - \alpha V)\Psi = 0 \]

is analytical if the next condition takes place:

\[ \left| \int_{-\infty}^{\infty} (1 + |x|)V(x)dx \right| < \infty , V(x) \leq 0. \] (41)

This condition is true for the NJL model and broken for QED3.

**Conclusion.**

With the help of the SD equation the problem of the dynamical mass generation in QED3 in presence of the strong magnetic field was considered. The main effects that are due to the magnetic field are:

1. Fermions become massive for arbitrary number of flavors in the system although without an external field the mass is generated only for \( N < N_c \approx 4.32 \). This is connected in its turn with the dimensional reduction \( (D \to D - 2) \), which takes place in this problem.

2. In the case of a very strong magnetic field the vacuum polarization almost does not contribute to the result. Photons become almost free in strong magnetic field. It is connected with absence of the longitudinal (with respect to the magnetic field) directions. This effect essentially differs QED3 from QED4.

3. Strong magnetic field shifts the mass generation to the infrared region \( (p \leq m_{dyn}) \). In the QED3 without external magnetic field the middle region \( m_{dyn} < p < N e^2/8 \) is responsible for the mass generation.

4. The dependence of the \( m_{dyn} \) on the coupling constant is non–analytical in contrast to the NJL model.

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Appendix.

The polarization tensor in presence of a constant magnetic field.

The polarization tensor can be evaluated in the second order of the perturbation theory using the method of [9].

\[ \Pi_{\mu\nu}(k) = -ie^2 tr \langle \gamma_\mu S(p) \gamma_\nu S(p-k) \rangle , \langle ... \rangle = \int \frac{d^3 p}{(2\pi)^3} \ldots \] (42)

\( S \) is the free fermion propagator. Let us represent it in the next form:

\[ S(k) = -i \int_0^\infty ds \exp \left(-is \left( m^2 - k_0^2 + k_\perp^2 \frac{tg z}{z} \right) \right) \cdot \left( k^\mu \gamma_\mu + m + (k^2 \gamma^1 - k_1 \gamma^2)tg z \right) \left( 1 + \gamma^1 \gamma^2 tg z \right) = \]

\[ = -i \int_0^\infty ds \exp \left(-is \left( m^2 - k_0^2 + k_\perp^2 \frac{tg z}{z} \right) \right) \left( k^0 \gamma^0 + m \frac{e^{i\sigma_3 z}}{cos z} + \frac{\gamma k_\perp}{cos^2 z} \right), \] (43)

After some computations we have obtained the expression for the polarization tensor:

\[ \Pi_{\mu\nu}(k) = \Pi(k) \left( g_{\mu\nu} k^2 - k_\mu k_\nu \right) + \Pi_{\perp}(k) \left( -g^\perp_{\mu\nu} k_\perp^2 - k^\perp_\mu k^\perp_\nu \right), \] (44)

\[ \Pi = e^{i\pi/4} \frac{e^2}{4\pi \sqrt{\pi}} \int_0^\infty \frac{ds}{\sqrt{s}} \int_{-1}^1 \frac{dv}{2} e^{-is\varphi_0} \left( z \cos z v - v \cos z \sin z v \right), \] (45)

\[ \Pi_{\perp} = e^{i\pi/4} \frac{e^2}{4\pi \sqrt{\pi}} \int_0^\infty \frac{ds}{\sqrt{s}} \int_{-1}^1 \frac{dv}{2} e^{-is\varphi_0} \cos z v - \cos z \sin^3 z v 2z - \Pi. \] (46)

The sign ‘−’ before the \( g_{\perp} \) is due to the \( k_\perp^2 = k_1^2 + k_2^2 \). The full photon propagator is equal to:

\[ D_{\mu\nu} = \frac{1}{k^2 (1 + \Pi)} \left( g_{\mu\nu} - k^\perp_\mu k^\perp_\nu - k_\mu k_\nu \right) \]

\[ + \frac{1}{k^2 + k^2 \Pi - k_\perp^2 \Pi_{\perp}} \left( g^\perp_{\mu\nu} + k^\perp_\mu k^\perp_\nu \right) + \lambda k_\mu k_\nu k^4. \] (47)
In Euclidean notation the quantities $\Pi, \Pi_\perp$ take the form:

$$\Pi = \frac{e^2}{4\pi \sqrt{\pi}} \int_0^\infty ds \int_{-1}^1 dv \frac{1}{2} e^{-s\varphi} \left( \frac{z \, ch \, zv}{sh \, z} - \frac{zv \, cth \, z \, sh \, zv}{sh \, z} \right),$$

$$\Pi_\perp = \frac{e^2}{4\pi \sqrt{\pi}} \int_0^\infty ds \int_{-1}^1 dv \frac{1}{2} e^{-s\varphi} \frac{ch \, z - ch \, zv}{sh^3 z} 2z - \Pi,$$

$$\varphi = m^2 + \frac{1 - v^2}{4} k_3^2 + \frac{ch \, z - ch \, zv}{2z \, sh \, z} k_\perp^2, \quad k_3 = ik_0. \tag{50}$$

Both integrals are finite for $|B| > 0$. Because we need only (48), let us investigate it in detail. For zero momentum

$$\Pi(m) = \frac{e^2 \ell}{4\pi \sqrt{\pi}} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-zm^2 z} \left( \frac{cth \, z}{z} - \frac{1}{sh^2 z} \right) \tag{51}$$

$$\Pi(0) \approx 0.14037 e^2 \ell = \nu_0 \, e^2 \ell$$

While $p_\perp^2$ increases, $p_\perp^2$ (48) slowly decreases. Because the leading contribution is given from the infrared region, we can replace $\Pi(p)$ to the $\Pi(0)$. If there are $N$ flavors in the system then all $\Pi$ must be multiplied on $N$ because $N$ equalent diagramms contribute to (12).
Literature.

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