Some High-Order Iterative Methods for Nonlinear Models Originating from Real Life Problems

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Received: 13 July 2020; Accepted: 28 July 2020; Published: 31 July 2020

Abstract: We develop a sixth order Steffensen-type method with one parameter in order to solve systems of equations. Our study’s novelty lies in the fact that two types of local convergence are established under weak conditions, including computable error bounds and uniqueness of the results. The performance of our methods is discussed and compared to other schemes using similar information. Finally, very large systems of equations (100 $\times$ 100 and 200 $\times$ 200) are solved in order to test the theoretical results and compare them favorably to earlier works.

Keywords: local convergence; Steffensen’s method; Banach space; system of equations

MSC: 65H05; 65G99

1. Introduction

A plenty of problems from Biology, Chemistry, Economics, Engineering, Mathematics, and Physics are converted to a mathematical expression of the following form

$$F(u) = 0.$$  \hspace{1cm} (1)

Here, $F : \Omega \subset \mathbb{B} \to \mathbb{B}$ is differentiable, $\mathbb{B}$ is a Banach space and $\Omega$ is nonempty and open. Closed form solutions are rarely found, so iterative methods [1–16] are used converging to the solution $u_\ast$.

In particular, we propose the following new scheme

$$y_p = u_p - \left[ u_p + F(u_p), u_p; F \right]^{-1} F(u_p)$$

$$z_p = u_p - \lambda \left[ u_p + F(u_p), u_p; F \right]^{-1} \left( F(u_p) + F(y_p) \right) - (1 - \lambda) \left[ u_p, y_p; F \right]^{-1} F(u_p)$$

$$u_{p+1} = z_p - \left[ z_p + F(z_p), z_p; F \right]^{-1} F(z_p),$$ \hspace{1cm} (2)

$u_0 \in \Omega$ is an initial point and $\lambda \in \mathbb{R}$ is a free parameter. In addition to this, $\left[ \cdot, \cdot; F \right] : \Omega \times \Omega \to \ell(\mathbb{B}, \mathbb{B})$ is a divided difference of order one.

We shall present two convergence analyses. Later, we present the advantages over other methods using similar information.

2. Local Convergence Analysis I

We assume that $\mathbb{B} = \mathbb{R}$. We use method (2) with standard Taylor expansions [9] for studying local convergence.
Theorem 1. Suppose that mapping \( F \) is \( s \) sufficient differentiable on \( \Omega \), with \( u_0 \in \Omega \), a simple zero of \( F \). We also consider that the inverse of \( F \), \( F^{-1} \), is \( s \)-smooth on \( \ell(B,\mathcal{B}) \). Then, \( \lim_{p \to \infty} u_p = u_0 \) provided that \( u_0 \) is close enough to \( u_* \). Moreover, the convergence order is six.

Proof. Set \( \epsilon_p = u_p - u_* \) and \( Q_p = \frac{F'(u_*)^{-1}}{p!} \), where \( (\epsilon_p)^\gamma = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^\gamma \), \( \epsilon_p \in \mathbb{R}^p \). We shall use some Taylor series expansions, first for \( F(u_p) \) and \( F\left(u_p + F(u_p)\right) \):

\[
F(u_p) = Q_1 \epsilon_p + Q_2 \epsilon_p^2 + O(\epsilon_p^3) \tag{3}
\]

and

\[
F\left(u_p + F(u_p)\right) = (Q_1 + Q_1^2) \epsilon_p + (3Q_1Q_2 + Q_2 + Q_1^2Q_2) \epsilon_p^2 + O(\epsilon_p^3),
\]

respectively.

By using the expressions (3) and (4) in the first substep of scheme (2), we have

\[
\tilde{\epsilon}_p = y_p - u_* = b_1 \epsilon_p^2 + b_2 \epsilon_p^3 + b_3 \epsilon_p^4 + O(\epsilon_p^5),
\]

where

\[
b_1 = \frac{Q_2}{Q_1} + Q_2,
b_2 = \frac{2Q_3}{Q_1} - \frac{2Q_2^2}{Q_1^2} + \frac{2Q_3^2}{Q_1^2} + Q_1Q_3 + 3Q_3 - Q_2^2,
\]

and

\[
b_3 = \frac{3Q_2^3}{Q_1} - 2Q_1Q_2Q_3 + Q_3^3 + Q_1Q_1^2Q_4 + 4Q_1Q_4 + 6Q_4 - 7Q_2Q_3 + \frac{3Q_4}{Q_1} + \frac{4Q_3^2}{Q_1^2} + \frac{5Q_3}{Q_1} - \frac{10Q_2Q_3}{Q_1}.
\]

Secondly, we expand \( F(y_p) \)

\[
F(y_p) = Q_1 \tilde{\epsilon}_p^2 + Q_2 \tilde{\epsilon}_p^3 + O(\tilde{\epsilon}_p^4).
\]

In view of (3)–(6), we get in the second substep of scheme (2)

\[
\epsilon_p = u_{p+1} - u_* = z_p - u_* = b_4 \epsilon_p^4 + O(\epsilon_p^5),
\]

where

\[
b_4 = \frac{3Q_2^2}{Q_1} + \frac{2Q_1^2}{Q_1} + Q_2^2 - \lambda \left( \frac{4Q_1^2}{Q_1} + \frac{Q_2^2}{Q_1} + \frac{3Q_2^2}{Q_1^2} \right).
\]

Thirdly, we need the expansions for \( F(z_p) \) and \( F\left(z_p + F(z_p)\right) \)

\[
F(z_p) = Q_1 \epsilon_p + Q_2 \epsilon_p^2 + O(\epsilon_p^3),
\]

Hence, by (5) and (8), we get

\[
F(z_p + F(z_p)) = b_5 \epsilon_p + b_6 \epsilon_p^2 + O(\epsilon_p^3),
\]

leading together with the third substep of method (2) to

\[
\epsilon_{p+1} = u_{p+1} - u_* = b_{7p} \epsilon_p^6 + O(\epsilon_p^7),
\]

where
\[ b_5 = Q_1 + Q_1^2, \]
\[ b_6 = 3Q_1Q_2 + Q_2 + Q_1^2Q_2 \]
and
\[ b_7 = 3Q_1Q_2 + 2Q_2^2 - \lambda \left( \frac{4Q_2^2}{Q_2} + \frac{3Q_2^2}{Q_2} + Q_2^2 \right). \]

According to Theorem 1, the applicability of method (2) is limited to mappings \( F \) with derivatives up to the seventh order.

Now, we choose \( B = \mathbb{R}, \Omega = [-\frac{3}{2}, \frac{1}{2}] \) and define a function \( f \), as follows:

\[ f(\xi) = \begin{cases} 
\xi^3 \ln \xi^2 + \xi^5 - 4\xi^3, & \xi \neq 0 \\
0, & \xi = 0
\end{cases} \quad (11) \]

We have the following derivatives of function \( f \):

\[
\begin{align*}
    f'(\xi) &= 3\xi^2 \ln \xi^2 + 2\xi^4 - 4\xi^3 + 2\xi^2, \\
    f''(\xi) &= 12\xi \ln \xi^2 + 20\xi^3 - 12\xi^2 + 10\xi, \\
    f'''(\xi) &= 12 \ln \xi^2 + 60\xi^2 - 12\xi + 22.
\end{align*}
\]

However, \( f'''(\xi) \) is not bounded on \( \Omega \), so Section 2, cannot be used. In this case, we have a more general alternative given in the upcoming section.

### 3. Local Convergence Analysis II

Consider \( a \geq 0 \) and \( b > 0 \). Let \( w_0 : [0, \infty) \times [0, \infty) \to [0, \infty) \) be a increasingly continuous map with \( w_0(0,0) = 0 \).

Suppose equation

\[ w_0(at,t) = 1 \quad (12) \]

has \( \rho_1 \) as the smallest positive zero. In addition, we assume that \( w : [0, \rho_1) \times [0, \rho_1) \to [0, \infty) \) is a increasingly continuous map with \( w(0,0) = 0 \).

Consider functions \( g_1 \) and \( h_1 \) defined on semi open interval \( [0, \rho_1) \) as follow:

\[ g_1(t) = \frac{w(bt,t)}{1 - w_0(at,t)}, \]

and

\[ h_1(t) = g_1(t) - 1. \]

By these definitions, we have \( h_1(0) = -1 \) and \( h_1(t) \to \infty \) as \( t \to \rho_1^- \). Subsequently, the intermediate value theorem assures that function \( h_1 \) has minimum one solution in \( (0, \rho_1) \). Let \( r_1 \) be the minimal such zero.

The expression

\[ w_0(t,g_1(t)t) = 1 \quad (13) \]

has the smallest positive zero \( \rho_2 \). Set \( \rho_3 = \min\{\rho_1, \rho_2\} \).

We construe the functions \( g_2 \) and \( h_2 \) on interval \( [0, \rho_3) \) in the following way.
\[ g_2(t) = g_1(t) + \frac{b|1 - \lambda|w\left(bt, \left(1 + g_1(t)\right) t\right)g_1(t)}{\left(1 - w_0(at, t)\right)\left(1 - w_0\left(t, g_1(t)^t\right)\right)}, \]

and

\[ h_2(t) = g_2(t) - 1. \]

We yield \( h_2(0) = -1 \) and \( h_2(t) \to \infty \) since \( t \to \rho^3_3 \). The \( r_2 \) stand for the minimal such zero of function \( h_2 \) on \((0, \rho_3)\).

The equation

\[ w_0\left( ag_2(t)t, g_2(t)t\right) = 1 \]  \( \tag{14} \)

has \( \rho_4 \) as the smallest positive solution. Set \( \rho = \min\{\rho_3, \rho_4\} \). Define functions \( g_3 \) and \( h_3 \) on \([0, \rho)\) as

\[ g_3(t) = \frac{w\left(bg_2(t)t, g_2(t)t\right)g_2(t)}{1 - w_0\left( ag_2(t)t, g_2(t)t\right)}, \]

and

\[ h_3(t) = g_3(t) - 1. \]

We obtain \( h_3(0) = -1 \) and \( h_3(t) \to \infty \) as \( t \to \rho^- \). The \( r_3 \) imply the minimal zero of \( h_3 \) on \((0, \rho)\).

Moreover, define

\[ r = \min\{r_i\}, \text{ for } i = 1, 2, 3. \]  \( \tag{15} \)

Accordingly, we have

\[ 0 \leq w_0(at, t) < 1, \]

\[ 0 \leq w_0\left(t, g_1(t)t\right) < 1, \]  \( \tag{16} \)

\[ 0 \leq w_0\left( ag_2(t)t, g_2(t)t\right) < 1, \]  \( \tag{17} \)

and

\[ 0 \leq g_i(t) < 1, \]  \( \tag{18} \)

for all \( t \in [0, r) \).

\( S(v, c) \) denotes the open ball centered at \( v \in \mathbb{B} \) and of radius \( c > 0 \). By \( S(v, c) \), we denote the closure of \( S(v, c) \).

We use the following conditions \((A)\) in order to study the local convergence:

\((a_1)\) \( F : \Omega \to \mathbb{B} \) is a differentiable operator in the Fréchet sense, \( [\cdot, \cdot; F] : \Omega \times \Omega \to \ell(\mathbb{B}, \mathbb{B}) \) is a divided difference of order one. In addition to this, we assume that \( u_\ast \in \Omega \) is a simple zero of \( F \). At last, the inverse of operator \( F, F'(u_\ast)^{-1} \in \ell(\mathbb{B}, \mathbb{B}) \).

\((a_2)\) Let \( w_0 : [0, \infty] \times [0, \infty) \to [0, \infty) \) be a increasingly continuous function with \( w_0(0, 0) = 0 \), parameters \( a \geq 0 \) and \( b > 0 \), such that for each \( u, y \in \Omega \)

\[ \left\| F'(u_\ast)^{-1}\left([u, y; F] - F'(u_\ast)\right)\right\| \leq w_0(\|u - u_\ast\|, \|y - u_\ast\|), \]

\[ \|I + [u, u_\ast; F]\| \leq a, \]

and

\[ \|[u, u_\ast; F]\| \leq b. \]

Set \( \Omega_0 = \Omega \cap S(u_\ast, \rho_1) \), where \( \rho_1 \) exists and is given by \((12)\).
Theorem 2. Under the hypotheses (A) further consider that \( u_0 \in S(u_*, r) - \{u_*\} \). Accordingly, the proceeding assertions hold

\[
\{u_p\} \subseteq S(u_*, r),
\]

\[
\lim_{p \to \infty} u_p = u_*,
\]

\[
\|y_p - u_*\| \leq g_1(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\| < r,
\]

\[
\|z_p - u_*\| \leq g_2(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\|,
\]

and

\[
\|u_{p+1} - u_*\| \leq g_3(\|u_p - u_*\|) \|u_p - u_*\| \leq \|u_p - u_*\|.
\]

In addition, the \( u_* \) is the unique solution of \( F(u) = 0 \) in the set \( \Omega_1 \) mentioned in hypothesis (a5).

Proof. We first show items (20)–(24) by adopting mathematical induction. Because \( p \in S(u_*, r) - \{u_*\} \) hold and by condition (a2), we have

\[
\|p + F(p) - u_*\| = \|(I + [p, u_*; F])(p - u_*)\|
\leq \|I + [p, u_*; F]\| \|p - u_*\|
\leq a \|p - u_*\|
\]

and

\[
\|F(p)\| = \|F(p) - F(u_*)\|
\leq \|[p, u_*; F](p - u_*)\|
\leq \|[p, u_*; F]\| \|p - u_*\|
\leq b \|p - u_*\|
\]

so \( p + F(p) - u_* \) and \( F(p) \) belong in \( S(u_*, R) \). Afterwards, for \( u, y, z \in S(u_*, r) - \{u_*\} \), and

\[
\|F'(u_*)^{-1}(u, y; F - F'(u_*))\| \leq w_0(\|u - u_*\|, \|y - u_*\|)
\leq w_0(r, r) < 1,
\]

so the Banach lemma on invertible operators \([3–5, 12]\) gives \([u, y; F]^{-1} \subseteq \ell(\mathbb{B}, \mathbb{B})\), and

\[
\|u, y; F]^{-1}F'(u_*)\| \leq \frac{1}{1 - w_0(\|u_0 - u_*\|, \|y - u_*\|)}.
\]

It also follows that \( y_0 \) is defined.

Adopting (15), (16), (19) (for \( i = 1 \), (a2), (a3), (25) and \( y_0 \), we get
where $y_0 - u_*$

$$
\|y_0 - u_*\| = \left\| u_0 - u_* - [u_0 + F(u_0), u_0; F]^{-1}F(u_0) \right\|
$$

$$
= \left\| u_0 + F(u_0), u_0; F \right\|^2 \left\| (u_0 + F(u_0), u_0; F) - [u_0, u_*; F](u_0 - u_*) \right\|
$$

$$
\leq \left\| u_0 + F(u_0), u_0; F \right\|^2 \left\| F(u_0) - [u_0, u_*; F](u_0 - u_*) \right\| \cdot \|u_0 - u_*\|
$$

$$
\leq \frac{w\left(\|F(u_0)\|, \|u_0 - u_*\|\right) \cdot \|u_0 - u_*\|}{1 - w_0(\|u_0 - u_*\|, \|u_0 - u_*\|)}
$$

(26)

It also follows that

$$
[0 - y_0\| \leq \|y_0 - u_*\| + \|u_0 - u_*\| \leq \|u_0 - u_*\| + \|u_0 + F(u_0), u_0; F\|^{-1}F(u_0) \leq \|y_0 - u_*\| + \|y_0 - u_*\| + \gamma\|u_0 - u_*\|)
$$

$$
\leq \gamma\|u_0 - u_*\|
$$

(27)

Next, by (15), (19) (for $i = 2$) and (25)–(28), we get, in turn, that

$$
\|z_0 - u_0\| \leq \|y_0 - u_*\| + 1 - \lambda \left\| u_0 + F(u_0), u_0; F \right\|^2 \|F(u_0) - [u_0 + F(u_0), u_0; F]^{-1}F(u_0) \|
$$

$$
\leq \left\| u_0 + F(u_0), u_0; F \right\|^2 \|F(u_0) - [u_0, u_*; F](u_0 - u_*) \right\| \cdot \|u_0 - u_*\|
$$

$$
\leq \gamma\|u_0 - u_*\|
$$

(29)

so $z_0 \in S(u_*, r)$ (for $z_0 \neq u_*$ ) and (23) holds for $p = 0$. 


We have by (15), (18) and (29)
\[
\left\| F'(u_*)^{-1}(z_0 + F(u_0), z_0; F) - F'(u_*) \right\| \leq \varpi_0 \left( b \| z_0 - u_* \|, \| z_0 - u_* \| \right) 
\leq \varpi_0 \left( bg_2(\| u_0 - u_* \|), g_2(\| u_0 - u_* \|) \| u_0 - u_* \| \right) 
\leq \varpi_0 \left( bg_2(r), g_2(r) \right) < 1
\]
Accordingly, \([z_0 + F(z_0), z_0; F]^{-1} \in \ell(\mathbb{B}, \mathbb{B})\) and
\[
\left\| [z_0 + F(z_0), z_0; F]^{-1}F'(u_*) \right\| \leq \frac{1}{1 - \varpi_0 \left( bg_2(\| u_0 - u_* \|), g_2(\| u_0 - u_* \|) \| u_0 - u_* \| \right)}.
\]  

(30)

It also follows that \(u_1\) is well defined by (30) and the last substep of method (2) for \(n = 0\). Then, as in (25) and (26) (for \(z = 3\)) and (30), we obtain in turn
\[
\|u_1 - u_*\| \leq \frac{\varpi(b \| z_0 - u_* \|, \| z_0 - u_* \|) \| z_0 - u_* \|}{1 - \varpi_0 \left( a \| z_0 - u_* \|, \| z_0 - u_* \| \right)} 
\leq \frac{\varpi \left( bg_2(\| u_0 - u_* \|), g_2(\| u_0 - u_* \|) \| u_0 - u_* \| \right)}{1 - \varpi_0 \left( a g_2(\| u_0 - u_* \|), g_2(\| u_0 - u_* \|) \| u_0 - u_* \| \right)} 
\leq g_3(\| u_0 - u_* \|) \| u_0 - u_* \| \leq \| u_0 - u_* \|,
\]
so, \(u_1 \in S(u_*, r)\) (for \(u_1 \neq u_*\)) and (24) holds for \(n = 0\). Subsequently, substituting \(u_0, y_0, z_0, u_1\) by \(u_m, y_m, z_m, u_{m+1}\), respectively. Hence, the induction for (30) and (22)–(24) is complete. Using the estimation
\[
\|u_{m+1} - u_*\| < \alpha \| u_m - u_* \| < r,
\]
where \(\alpha = g_3(\| u_0 - u_* \|) \in [0, 1]\), we deduce that \(\lim_{m \to \infty} u_m = u_*\) and \(u_{m+1} \in S(u_*, r)\).

Finally, we want to illustrate that the required solution is unique. Therefore, let \(T = [u_*, y_*, F]\) for \(y_* \in \Omega_1\), so that \(F(y_*) = 0\). Then, by (a2) and (a5), we get
\[
\| F'(u_*)^{-1}(T - F'(u_*)) \| \leq \varpi_0(0, \| u_* - y_* \|) 
\leq \varpi_0(0, r) < 1,
\]
so \(T^{-1} \in \ell(\mathbb{B}, \mathbb{B})\). Finally, \(u_* = y_*\) is deduced from 0 = \(F(u_*) - F(y_*) = T(u_* - y_*)\). □

**Remark 1.** Another way of defining functions \(g_i, h_i\) and radii \(r_i\) is as follows:

Let \(\alpha = \max \{ b, a \}, i = 1, 2, 3\). Subsequently, as in (12)–(18), we shall have instead:

Suppose that equation
\[
\varpi_0(\alpha t, t) = 1
\]
has a smallest positive solution \(\bar{\rho}_1\). Let \(\bar{\nu} : [0, \bar{\rho}_1] \times [0, \bar{\rho}_1] \to \mathbb{R}_+\) be a increasing continuous function with \(\bar{\nu}(0, 0) = 0\).

Let functions \(\bar{g}_1\) and \(\bar{h}_1\) be defined in the interval \([0, \bar{\rho}_1]\) by
\[
\bar{g}_1(t) = \frac{\bar{\nu}(bt, t)}{1 - \varpi_0(\alpha t, t)} \quad \text{and} \quad \bar{h}_1(t) = \bar{g}_1(t) - 1.
\]

The \(\bar{r}_1\) stands for the smallest positive root of \(\bar{h}_1(t) = 0\) in \((0, \bar{\rho}_1)\). Moreover, define functions \(\bar{g}_2, \bar{g}_3, \bar{h}_2\) and \(\bar{h}_3\) on the closed interval \([0, \bar{\rho}_1]\), as follows:
\[ \begin{align*}
\bar{g}_2(t) &= \bar{g}_1(t) + \frac{b|1 - \lambda|w\left(bt, \left(1 + \bar{g}_1(t)\right)\bar{g}_1(t)\right)}{\left(1 - \tilde{w}_0(at, t)\right)^2}, \\
\bar{g}_3(t) &= \frac{w\left(b\bar{g}_2(t), \bar{g}_2(t)\right)}{1 - \tilde{w}_0(at, t)}, \\
\tilde{h}_2(t) &= \bar{g}_2(t) - 1 \\
\text{and} \\
\tilde{h}_3(t) &= \bar{g}_3(t) - 1.
\end{align*} \]

The \( \tilde{r}_2 \) and \( \tilde{r}_3 \) serve as the minimal positive roots of \( \tilde{h}_2(t) = 0 \) and \( \tilde{h}_3(t) = 0 \) on closed interval \([0, \tilde{p}_1]\), respectively. Subsequently, Theorem 2 can be written by using the “bar” conditions and functions, with \( \tilde{r} = \min\{\tilde{r}_i\} \).

**Remark 2.** The convergence of method (2) to \( u_* \) is established under the conditions of Theorem 1. However, the order convergence under the conditions of Theorem 2 can be established by using (COC) and (ACOC) (for the details, please see Section 5).

4. Numerical Examples

Here, we monitor the convergence conditions on three problems (1)–(3). We choose \([u, y; F] = \int_0^1 F(y + \theta(u - y))\,d\theta\) in the examples. We can confirm the verification the hypotheses of Theorem 2 for the given choices of the “\( w \)” functions and parameters \( a \) and \( b \).

**Example 1.** Here, we investigate the application of our results on Hammerstein integral equations (see [9], pp. 19–20) for \( \mathbb{B} = C[0, 1] \) as follows:

\[ F(u(s_1)) = u(s_1) - \frac{1}{5} \int_0^1 S(s_1, s_2)u(s_2)^3\,ds_2 = 0, \quad u \in C[0, 1], \quad s_1, s_2 \in [0, 1], \tag{34} \]

where

\[ S(s_1, s_2) = \begin{cases} s(1 - s_2), & s \leq s_2, \\ (1 - s)s_2, & s_2 \leq s. \end{cases} \]

We use \( \int_0^1 \phi(t)\,dt \approx \sum_{k=1}^{8} w_k\phi(t_k) \) in (34), where \( t_k \) and \( w_k \) are the absicssas and weights, respectively. Using \( u(t_j) \) for \( u_j \) (\( j = 1, 2, 3, \ldots, 8 \)), leads to

\[ 5u_i - 5 - \sum_{k=1}^{8} a_{jk}u_k^3 = 0, \quad j = 1, 2, 3, \ldots, 8, \]

\[ a_{jk} = \begin{cases} w_kt_k(1 - t_j), & k \leq j, \\ w_kt_j(1 - t_k), & j < k. \end{cases} \]

The values of \( t_k \) and \( w_k \) when \( k = 8 \), are illustrated in Table 1. Subsequently, we have

\[ u_* = (1.002096 \ldots, 1.009900 \ldots, 1.019727 \ldots, 1.026436 \ldots, 1.026436 \ldots, 1.019727 \ldots, 1.009900 \ldots, 1.002096 \ldots)^T. \]

Accordingly, we set \( \tilde{w}_0(s_1, s_2) = w(s_1, s_2) = \frac{3}{\tilde{a}}(s_1 + s_2), \quad a = \frac{163}{80} \) and \( b = \frac{83}{80} \). The radii for Example 1 are listed in Tables 2 and 3:
Table 1. Abscissas and weights for $k = 8$.

| $j$ | $t_j$ | $w_j$ |
|-----|-------|-------|
| 1   | 0.01985507175123188415821957... | 0.05061426814518812957626567... |
| 2   | 0.10166676129318663020422303... | 0.111190517226687235227217800... |
| 3   | 0.23723379504183550709113047... | 0.15685322939434636898110... |
| 4   | 0.40828267875217509753026193... | 0.1813418916891809148257522... |
| 5   | 0.5917173214782490246973807... | 0.1813418916891809148257522... |
| 6   | 0.76276620495816449290886952... | 0.15685332293894364366898110... |
| 7   | 0.89833323870681336979577696... | 0.111190517226687235227217800... |
| 8   | 0.98014492824876811584178043... | 0.05061426814518812957626567... |

Table 2. Convergence radii for Example 1.

| $\lambda$ | $r_1$ | $r_2$ | $r_3$ | $r$ |
|-----------|-------|-------|-------|-----|
| 0         | 5.25452 | 3.87208 | 4.09301 | 3.87208 |
| 0.5       | 5.25452 | 4.26006 | 4.42602 | 4.26006 |
| 1         | 5.25452 | 5.25452 | 5.25452 | 5.25452 |

Table 3. Convergence radii for Example 1 with bar functions.

| $\lambda$ | $r_1$ | $r_2$ | $r_3$ | $r$ |
|-----------|-------|-------|-------|-----|
| 0         | 5.25452 | 3.67748 | 3.87626 | 3.67748 |
| 0.5       | 5.25452 | 4.07351 | 4.17413 | 4.07351 |
| 1         | 5.25452 | 5.25452 | 4.89162 | 4.89162 |

Example 2. Here, we choose as integral equation [17,18], for $\mathbb{B} = [0, 1]$ as

$$\left[F(\mu)\right](\gamma_1) = \mu(\gamma_1) - \int_0^{\gamma_1} G(\gamma_1, \gamma_2) \left(\mu(\gamma_2) - \frac{\mu(\gamma_2)^2}{2}\right) d\gamma_2 = 0,$$

where

$$G(\gamma_1, \gamma_2) = \begin{cases} 
(1 - \gamma_2)\gamma_2, & \gamma_2 \leq \gamma_1, \\
\gamma_1(1 - \gamma_2), & \gamma_1 \leq \gamma_2.
\end{cases}$$

Because $\mathbb{B} = [0, 1]$ so, $F : C[0, 1] \to C[0, 1]$ is given as

$$\left[F(\mu)\right](\gamma_1) = \mu(\gamma_1) - \int_0^{\gamma_1} G(\gamma_1, \gamma_2) \left(\mu(\gamma_2) - \frac{\mu(\gamma_2)^2}{2}\right) d\gamma_2.$$  

We get

$$\left\| \int_0^{\gamma_1} G(\gamma_1, \gamma_2) d\gamma_2 \right\| \leq \frac{1}{8};$$

Moreover,

$$\left[F'(\mu)\eta\right](\gamma_1) = \eta(\gamma_1) - \int_0^{\gamma_1} G(\gamma_1, \gamma_2) \left(\frac{3}{2} \mu(\gamma_2)^2 + \mu(\gamma_2)\right) \eta(\gamma_2) d\gamma_2,$$

so $\mu_*(\gamma_1) = 0$, since $F'(\mu_*(\gamma_1)) = I$,

$$\left\| F'(\mu_*)^{-1}(F'(\mu) - F'(\eta)) \right\| \leq \frac{1}{8} \left(\frac{3}{2} \|\mu - \eta\| + \|\mu - \eta\|\right).$$

Hence, we have

$$w_0(s, t) = w(s, t) = \frac{1}{16} \left[\frac{3}{2} (\sqrt{s} + \sqrt{t}) + s + t\right], a = \frac{53}{16} \text{ and } b = \frac{37}{16}.$$
Therefore, our results can be utilized even though \( F' \) is not bounded on \( \Omega \). The radii for Example 2 are given in Table 4.

**Table 4.** Convergence radii for Example 2 with bar functions.

| \( \lambda \) | \( r_1 \)   | \( r_2 \)   | \( r_3 \)   | \( r \)   |
|-------------|---------|---------|---------|---------|
| 0           | 1.03137 | 0.50240 | 0.61211 | 0.50240 |
| 0.5         | 1.03137 | 0.61199 | 0.70738 | 0.61199 |
| 1           | 1.03137 | 1.03137 | 1.03137 | 1.03137 |

**Example 3.** We assume the following differential equations

\[
\begin{align*}
q'_1(\mu) - q_1(\mu) - 1 &= 0 \\
q'_2(\eta) - (e - 1)\eta - 1 &= 0 \\
q'_3(\theta) - 1 &= 0
\end{align*}
\tag{40}
\]

characterizes the progress/movement of a molecule in 3D with \( (\mu, \eta, \theta) \in \Omega \) for \( q_1(0) = q_2(0) = q_3(0) = 0 \). The required solution \( v = (\mu, \eta, \theta)^T \) describes to \( K := (q_1, q_2, q_3) : \Omega \to \mathbb{R}^3 \) given as

\[
K(v) = \left( e^{\mu} - 1, \frac{e^{\mu} - 1}{2} \eta^2 + \eta, \theta \right)^T = 0.
\tag{41}
\]

It follows from (41) that

\[
K'(v) = \begin{bmatrix}
e^{\mu} & 0 & 0 \\
0 & (e - 1)\eta + 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

which yields

\[
w_0(s, t) = \frac{1}{2}(e - 1)(s + t), \ w(s, t) = \frac{1}{2}e(s + t), \ a = \frac{1}{2}(e + 3), \text{ and } b = \frac{1}{2}(e + 1).
\]

We depicted the radii of Example 3 in Tables 5 and 6.

**Table 5.** Convergence radii for Example 3.

| \( \lambda \) | \( r_1 \)   | \( r_2 \)   | \( r_3 \)   | \( r \)   |
|-------------|---------|---------|---------|---------|
| 0           | 0.1388596 | 0.921375 | 0.083356 | 0.083356 |
| 0.5         | 0.1388596 | 0.921375 | 0.086297 | 0.086297 |
| 1           | 0.1388596 | 0.1388596 | 0.1388596 | 0.1388596 |

**Table 6.** Convergence radii for Example 3 with bar functions.

| \( \lambda \) | \( r_1 \)   | \( r_2 \)   | \( r_3 \)   | \( r \)   |
|-------------|---------|---------|---------|---------|
| 0           | 0.1388596 | 0.0487471 | 0.1229551 | 0.0487471 |
| 0.5         | 0.1388596 | 0.0487471 | 0.1377815 | 0.0487471 |
| 1           | 0.1388596 | 0.1388596 | 0.1380780 | 0.1380780 |

**Example 4.** By the example of Section 2, for \( \Omega = \mathbb{B} = \mathbb{R}, \ f(\xi) = 0 \), we get

\[
w_0(s, t) = w(s, t) = \frac{96.66297}{2}(s + t), \ a = \frac{5}{2}, \text{ and } b = \frac{3}{2}.
\]

The radii of method (2) for Example 4 are listed in Tables 7 and 8.
Table 7. Convergence radii for Example 4.

| λ  | r_1    | r_2    | r_3    | r     |
|----|--------|--------|--------|-------|
| 0  | 0.00344841 | 0.00239612 | 0.00256623 | 0.00239612 |
| 0.5| 0.00344841  | 0.00267769  | 0.00280807  | 0.00267769  |
| 1  | 0.00344841  | 0.00344841  | 0.00344841  | 0.00344841  |

Table 8. Convergence radii for Example 4 with bar functions.

| λ  | r_1    | r_2    | r_3    | r     |
|----|--------|--------|--------|-------|
| 0  | 0.00344841 | 0.00225955 | 0.00246765 | 0.00225955 |
| 0.5| 0.00344841  | 0.00267769  | 0.00280807  | 0.00267769  |
| 1  | 0.00344841  | 0.00344841  | 0.00344841  | 0.00344841  |

5. Applications with Large Systems

We choose \( \lambda = 0, \lambda = 0.5 \) and \( \lambda = 1 \) in our scheme (2), called by \((PS1), (PS2)\) and \((PS3)\), respectively. Now, we compare our schemes with a 6th-order iterative methods suggested by Abbasbandy et al. [19] and Hueso et al. [20], among them we picked the methods (8) and (14–15) (for \( t_1 = -\frac{9}{4} \) and \( s_2 = \frac{9}{8} \)), respectively, known as \((AS)\) and \((HS)\). Moreover, a comparison of them has been done with the 6th-order iterative methods given by Wang and Li [21], among their method we chose expression (6), denoted by and \((WS)\). At the last, we contrast (2) with sixth-order scheme given by Sharma and Arora [22], we pick expression (13), known as \((SM)\). The details of all the iterative expressions are given, as follows:

**Method AS:**

\[
y_j = u_j - \frac{2}{3} F'(u_j)^{-1} F(u_j),
\]

\[
z_j = u_j - \left[ 1 + \frac{21}{8} F'(u_j)^{-1} F'(y_j) - \frac{9}{2} \left( F'(u_j)^{-1} F'(y_j) \right)^2 + \frac{15}{8} \left( F'(u_j)^{-1} F'(y_j) \right)^3 \right] F'(u_j)^{-1} F(u_j),
\]

\[
u_{j+1} = z_j - \left[ 3l + \frac{5}{2} F'(u_j)^{-1} F'(y_j) + \frac{1}{2} \left( F'(u_j)^{-1} F'(y_j) \right)^2 \right] F'(u_j)^{-1} F(z_j).
\]

**Scheme HS:**

\[
y_j = u_j - F'(u_j)^{-1} F(u_j),
\]

\[
H(u_j, y_j) = F'(u_j)^{-1} F(y_j), \quad H(y_j, u_j) = F'(y_j)^{-1} F(u_j),
\]

\[
G_3(u_j, y_j) = s_1 I + s_2 H(y_j, u_j) + s_3 H(u_j, y_j) + s_4 H(y_j, u_j)^2,
\]

\[
z_j = u_j - G_3(u_j, y_j) F'(u_j)^{-1} F(u_j),
\]

\[
u_{j+1} = z_j.
\]

where \( s_1, s_2, s_3, \) and \( s_4 \) are real numbers.

**Iterative method WS:**

\[
y_j = u_j - F'(u_j)^{-1} F(u_j),
\]

\[
z_j = y_j - \left[ 2I - F'(u_j)^{-1} F'(y_j) \right] F'(u_j)^{-1} F(y_j),
\]

\[
u_{j+1} = z_j - \left[ 2I - F'(u_j)^{-1} F'(y_j) \right] F'(u_j)^{-1} F(z_j).
\]

**scheme SM:**

\[
y_j = u_j - \frac{2}{3} F'(u_j)^{-1} F(u_j),
\]

\[
z_j = u_j - \left[ p I + F'(u_j)^{-1} F'(y_j) \left( q I + r F'(u_j)^{-1} F'(y_j) \right) \right] F'(u_j)^{-1} F(u_j),
\]

\[
u_{j+1} = z_j - \left[ \frac{5}{2} I - \frac{3}{2} F'(u_j)^{-1} F'(y_j) \right] F'(u_j)^{-1} F(z_j),
\]

At the last, we contrast (2) with sixth-order scheme given by Sharma and Arora [22], we pick expression (13), known as \((SM)\). The details of all the iterative expressions are given, as follows:
where \( p = \frac{23}{18}, q = -3 \) and \( r = \frac{9}{8} \).

The \((j)\), \( \|F(u_j)\|, \|u_{j+1} - u_j\|, \) and \( \rho^* \approx \frac{\log \left[ \|u_{j+1} - u_j\|/\|u_j - u_{j-1}\| \right]}{\log \left[ \|u_j - u_{j-1}\|/\|u_{j-1} - u_{j-2}\| \right] } \) stands for index of iteration, absolute residual errors in the function \( F \), error between two successive iterations and computational convergence order, respectively. There values are listed in Tables 9–11. Moreover, the quantity \( \eta \) is the final obtained value of \( \|u_{j+1} - u_j\| \).

The estimation of all the above parameters have been calculated by Mathematica-9. For minimizing the round-off errors, we have chosen multiple precision arithmetic with 1000 digits of mantissa. The term \( b_1 \) (\( \pm b_2 \)) symbolizes the \( b_1 \times 10^{(\pm b_2)} \) in all mentioned tables. We adopted the command “AbsoluteTiming[]” in order to calculate the CPU time. We run our programs three times and depicted the average CPU time in Table 12, also one can observe the times used for each iterative command.

The configuration of the used computer is given below:
- Processor: Intel(R) Core(TM) i7-4790 CPU @ 3.60 GHz
- Made: HP
- RAM: 8:00 GB
- System type: 64-bit-Operating System, x64-based processor

**Example 5.** Here, we deal with a boundary value problem from Ortega and Rheinboldt [9], given by

\[
y'' = \frac{y^3 + 6y' + 1}{2} - \frac{3}{2 - x}, \quad y(0) = 0, \quad y(1) = 1. \tag{46}
\]

We assume

\[u_0 = 0 < u_1 < u_2 < u_3 < \cdots < u_p, \text{ where } u_{p+1} = u_p + h, \quad h = \frac{1}{p}, \tag{47}\]

partition of the interval \([0, 1]\) and \( y_0 = y(u_0) = 0, \quad y_1 = y(u_1), \ldots, \quad y_{n-1} = y(u_{n-1}), \quad y_n = y(u_p) = 1. \)

Now, we discretize expression (46) by adopting following numerical formula for derivatives

\[
y_j' = \frac{y_{j+1} - y_{j-1}}{2h}, \quad y_j'' = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2}, \quad j = 1, 2, \ldots, p - 1,
\]

which leads to

\[
y_{j+1} - 2y_j + y_{j-1} - \frac{h^2}{2}y_j^3 - \frac{3}{2}h(y_{k+1} - y_{k-1}) - \frac{3}{2 - u_j}h^2 - \frac{1}{h^2} = 0, \quad j = 1, 2, \ldots, p - 1,
\]

\((p - 1) \times (p - 1)\) system of nonlinear equations.

For specific value of \( p = 7 \), we have a \( 6 \times 6 \) system and the required solution is

\[u_* = (0.07654393\ldots, 0.1658739\ldots, 0.2715210\ldots, 0.3984540\ldots, 0.553864\ldots, 0.7486878\ldots)^T.
\]

The computational estimations are listed in Table 9 on the basis of initial approximation

\[y_j^{(0)} = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)^T.
\]
For this purpose, we denote $\Delta$ and $u$ results are listed in Table 10 based on the initial guess $u$ as the respective step sizes in the directions of Example 6.

The classical 2D Bratu problem \cite{23,24} is given by

\[ u_{\mu\mu} + u_{\theta\theta} + Ce^u = 0, \]
\[ \Omega = \{ (\mu, \theta) \in 0 \leq \mu \leq 1, 0 \leq \theta \leq 1 \}, \] with boundary hypotheses $u = 0$ on $\Omega$. \hspace{1cm} (48)

By adopting finite difference discretization, we can deduced the above PDE (48) to a nonlinear system. For this purpose, we denote $\Delta_{i,j} = u(\mu_i, \theta_j)$ as numerical solution at the grid points of the mesh. In addition to this, $M_1$ and $M_2$ stand for the number of steps in the directions of $\mu$ and $\theta$, respectively. The $h$ and $k$ called as the respective step sizes in the directions of $\mu$ and $\theta$. Adopt the following central difference formula to $u_{\mu\mu}$ and $u_{\theta\theta}$

\[ u_{\mu\mu}(u_i, \theta_j) = \Delta_{i+1,j} - 2\Delta_{i,j} + \Delta_{i-1,j} \frac{h^2}{h^2}, \hspace{0.5cm} C = 0.1, \hspace{0.2cm} \theta \in [0, 1], \] \hspace{1cm} (49)

leads to us

\[ \Delta_{i,j+1} + \Delta_{i,j-1} - \Delta_{i,j} + \Delta_{i+1,j} + \Delta_{i-1,j} + h^2C \exp \left( \Delta_{i,j} \right) \] \hspace{1cm} \hspace{1cm} (50)

For obtaining a large system of $100 \times 100$, we choose $M_1 = M_2 = 11$, $C = 0.1$ and $h = \frac{1}{11}$. The numerical results are listed in Table 10 based on the initial guess $u_0 = \frac{0.1 \left( \sin(\pi hi) \sin(\pi hj) \right)}{7}$, $i = j = 10$.  

---

**Table 9.** Comparisons of different methods on a Boundary value problem in Example 5.

| Methods | $j$ | $\|F(u_j)\|$ | $\|u_{j+1} - u_j\|$ | $\rho^*$ | $\|u_{j+1} - u_j\|/\|u_j - u_{j-1}\|$ |
|---------|-----|-------------|-----------------|---------|----------------------------------|
| AS      | 1   | 1.9(-4)     | 6.1(-4)         |         | 5.133234733(-7)                  |
|         | 2   | 8.7(-27)    | 2.8(-26)        |         | 5.920693970(-7)                  |
|         | 3   | 8.1(-161)   | 2.6(-160)       |         | 5.9985                           |
| HS      | 1   | 1.3(-4)     | 5.7(-4)         |         | 8.252588019(-3)                  |
|         | 2   | 8.1(-23)    | 2.8(-22)        |         | 2.013636332(-16)                 |
|         | 3   | 3.5(-114)   | 1.0(-113)       |         | 4.9954                           |
| WS      | 1   | 2.6(-4)     | 1.1(-3)         |         | 1.977528884(-7)                  |
|         | 2   | 1.1(-25)    | 3.1(-25)        |         | 2.448277731(-7)                  |
|         | 3   | 8.6(-155)   | 2.4(-154)       |         | 5.9973                           |
| SM      | 1   | 7.3(-5)     | 2.7(-3)         |         | 7.804847473(-7)                  |
|         | 2   | 8.2(-29)    | 2.7(-28)        |         | 9.053257416(-7)                  |
|         | 3   | 1.1(-172)   | 3.6(-172)       |         | 2.302562088(-9)                  |
| PS1     | 1   | 4.9(-6)     | 1.6(-5)         |         | 2.474537279(-9)                  |
|         | 2   | 1.6(-38)    | 4.8(-38)        |         | 6.0010                           |
|         | 3   | 9.3(-234)   | 2.7(-233)       |         | 2.302562088(-9)                  |
| PS2     | 1   | 1.1(-5)     | 3.7(-5)         |         | 4.513404180(-9)                  |
|         | 2   | 3.7(-36)    | 1.1(-35)        |         | 4.108378955(-9)                  |
|         | 3   | 2.4(-219)   | 7.2(-219)       |         | 6.0013                           |
| PS3     | 1   | 1.9(-5)     | 6.5(-5)         |         | 7.168046437(-9)                  |
|         | 2   | 1.9(-34)    | 5.6(-34)        |         | 6.434316571(-9)                  |
|         | 3   | 6.6(-209)   | 1.9(-208)       |         | 6.434316717(-9)                  |
Table 10. Comparisons of different methods on two-dimensional (2D) Bratu problem in Example 6.

| Methods | $j$ | $\|F(u_j)\|$ | $\|u_{j+1} - u_j\|$ | $\rho^*$ | $\frac{\|u_{j+1} - u_j\|}{\|u_{j} - u_{j-1}\|}$ |
|---------|-----|---------------|-----------------|-------|------------------|
| AS      | 1   | 4.4(−15)     | 2.4(−14)        |       |                  |
|         | 2   | 6.9(−95)     | 3.5(−94)        | 5.9994| 1.428095547(−12) |
|         | 3   | 7.9(−574)    | 3.9(−573)       | 5.9994| 1.973434769(−12) |
| HS      | 1   | 2.1(−13)     | 1.2(−12)        |       |                  |
|         | 2   | 2.1(−71)     | 1.2(−70)        | 7.368055345(−11)|
|         | 3   | 1.7(−361)    | 9.3(−361)       | 4.9997| 3.495510769(+1)  |
| WS      | 1   | 5.0(−19)     | 2.9(−18)        |       |                  |
|         | 2   | 1.7(−122)    | 1.0(−121)       | 1.754949400(−16)|
|         | 3   | 3.1(−743)    | 1.8(−742)       | 5.9999| 1.666475363(+1)  |
| SM      | 1   | 4.4(−15)     | 2.4(−14)        |       |                  |
|         | 2   | 7.1(−95)     | 3.6(−94)        | 5.9994| 1.433541371(−12) |
|         | 3   | 9.2(−574)    | 4.5(−573)       | 5.9994| 1.433541371(−12) |
| PS1     | 1   | 9.1(−21)     | 5.3(−20)        |       |                  |
|         | 2   | 1.2(−134)    | 7.1(−134)       | 3.060974255(−18)|
|         | 3   | 6.9(−818)    | 4.0(−817)       | 6.0000| 3.068006721(−18) |
| PS2     | 1   | 1.9(−20)     | 1.1(−19)        |       |                  |
|         | 2   | 1.7(−132)    | 1.0(−131)       | 6.095821945(−18)|
|         | 3   | 1.1(−804)    | 6.7(−804)       | 6.0000| 6.105210728(−18) |
| PS3     | 1   | 3.1(−20)     | 1.8(−19)        |       |                  |
|         | 2   | 6.7(−131)    | 3.9(−130)       | 1.016575545(−17)|
|         | 3   | 6.3(−795)    | 3.7(−794)       | 6.0000| 1.017799424(−17) |

Example 7. Let us consider the following nonlinear system

$$F(x) = \begin{cases}
u^2_j u_{j+1} - 1 = 0, & 1 \leq j \leq p - 1, \\
x_p^2 u_1 - 1 = 0.
\end{cases} \quad (51)$$

For specific value $p = 200$, we have $200 \times 200$ system, and chose the following starting point

$$x^{(0)} = (1.25, 1.25, 1.25, \cdots, 1.25)^T.$$

The $u_* = (1, 1, 1, \cdots, 1)^T$ is the required solution of system 7. Table 11 provides the numerical results.

Remark 3. On the basis of Tables 9–11, we conclude that our methods namely PS1, PS2 and PS3 perform better in the contrast of existing schemes AS, HS, SM and SM on the basis of residual errors, errors between two consecutive iterations, and asymptotic error constant. In addition, our methods also demonstrate the stable computational order of convergence. Finally, we concluded that our methods not only perform better than existing methods in numerical results, but also take half of the CPU time in contrast to other existing methods (results can be easily found in Table 12).
To extend the suitability of these iterative methods, we only use hypotheses on the first derivative in the published version of the manuscript. Examples of equations, favorable comparisons to other methods can be found in Section 4.

| Methods | $j$ | $\|F(u_j)\|$ | $\|u_{j+1} - u_j\|$ | $\rho$ | $\frac{\|u_{j+1} - u_j\|}{\|u_j - u_{j-1}\|^2}$ |
|---------|-----|----------------|-----------------|--------|----------------------------------|
| $AS$    | 1   | 5.2(−3)       | 1.7(−3)         |        |                                  |
|         | 2   | 6.2(−21)      | 2.1(−21)        |        |                                  |
|         | 3   | 1.7(−128)     | 5.8(−129)       | 6.000  | 7.695242316(−5)                  |
| $HS$    |     |                |                 |        |                                  |
|         | 1   | 2.3(−3)       | 7.7(−4)         |        |                                  |
|         | 2   | 5.4(−20)      | 1.8(−20)        |        | 8.659247536(−2)                  |
|         | 3   | 3.9(−103)     | 1.3(−103)       | 5.000  | 3.689254113(−15)                 |
| $WS$    |     |                |                 |        |                                  |
|         | 1   | 3.5(−3)       | 1.2(−3)         |        |                                  |
|         | 2   | 4.0(−22)      | 1.3(−22)        |        | 5.299207889(−5)                  |
|         | 3   | 8.4(−136)     | 2.8(−136)       | 6.000  | 5.303300859(−5)                  |
| $SM$    |     |                |                 |        |                                  |
|         | 1   | 3.0(−3)       | 1.0(−3)         |        |                                  |
|         | 2   | 1.4(−22)      | 4.6(−23)        |        | 4.671758076(−5)                  |
|         | 3   | 1.3(−138)     | 4.3(−139)       | 6.000  | 4.674761498(−5)                  |
| $PS1$   |     |                |                 |        |                                  |
|         | 1   | 5.1(−3)       | 1.7(−3)         |        |                                  |
|         | 2   | 8.4(−21)      | 2.8(−21)        |        | 1.130483172(−4)                  |
|         | 3   | 1.6(−127)     | 5.5(−128)       | 6.000  | 1.131370850(−4)                  |
| $PS2$   |     |                |                 |        |                                  |
|         | 1   | 1.0(−1)       | 3.3(−2)         |        |                                  |
|         | 2   | 4.0(−12)      | 1.3(−12)        |        | 9.906447117(−4)                  |
|         | 3   | 1.7(−74)      | 5.8(−75)        | 5.9989 | 1.018233765(−4)                  |
| $PS3$   |     |                |                 |        |                                  |
|         | 1   | 3.3(−1)       | 1.1(−1)         |        |                                  |
|         | 2   | 1.3(−8)       | 4.3(−9)         |        | 2.565472254(−3)                  |
|         | 3   | 5.6(−53)      | 1.9(−53)        | 5.9943 | 2.828427114(−3)                  |

Table 11. Comparisons of different methods on Example 7.

| Methods | Example 5 | Example 6 | Example 7 | Total Time | Average Time |
|---------|-----------|-----------|-----------|------------|--------------|
| $AS$    | 0.465330  | 210.079553| 356.906591| 567.451474 | 189.1504913  |
| $HS$    | 0.583412  | 189.541919| 366.511753| 556.637084 | 185.5456947  |
| $WS$    | 0.274193  | 128.377322| 182.956711| 311.608226 | 103.8694087  |
| $SM$    | 1.130812  | 126.641140| 401.627979| 529.399931 | 176.4666437  |
| $PS1$   | 0.101071  | 120.094370| 52.204957 | 172.40398  | 57.46769933  |
| $PS2$   | 0.100071  | 117.901198| 52.146903 | 170.148172 | 56.71605733  |
| $PS3$   | 0.100083  | 117.923227| 51.972773 | 169.96083  | 56.665361    |

According to the CPU time, method $PS3$ is taking the lowest time for executing the results. All of the other schemes $AS$, $HS$, $SM$, and, $SM$ consuming at least double CPU timing as compare to our methods namely $PS1$, $PS2$ and $PS3$. So, we conclude that our methods provide results faster than the other existing methods.

### 6. Conclusions

We presented a new family of Steffensen-type methods with one parameter. The local convergence is studied in Section 2 while using Taylor expansion and derivative up to the order seven, when $B = \mathbb{R}^l$. To extend the suitability of these iterative methods, we only use hypotheses on the first derivative in Section 3 and Banach space valued operators. This way, we also find computable error bounds on $\|u_\rho - u_*\|$ as well as uniqueness results based on generalized Lipschitz-type real functions. Numerical examples of equations, favorable comparisons to other methods can be found in Section 4.

**Author Contributions:** M.Z.U.: Validation; Review & Editing, R.B. and I.K.A.: Conceptualization; Methodology; Validation; Writing—Original Draft Preparation; Writing—Review & Editing. All authors have read and agreed to the published version of the manuscript.
**Funding:** Research and development office (RDO) at the ministry of Education, Kingdom of Saudi Arabia. Grant no (HIQI-22-2019).

**Acknowledgments:** This project was funded by the research and development office (RDO) at the ministry of Education, Kingdom of Saudi Arabia. Grant No. (HIQI-22-2019). The authors also, acknowledge with thanks research and development office (RDO-KAU) at King Abdulaziz University for technical support.

**Conflicts of Interest:** The authors declare no conflict of interest.

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