Spectral mapping theorem of an abstract non-unitary quantum walk

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Abstract

This paper continues the previous work (Quantum Inf. Process 11 (2019)) by two authors of the present paper about a spectral mapping property of chiral symmetric unitary operators. In physics, they treat non-unitary time-evolution operators to consider quantum walks in open systems. In this paper, we generalize the above result to include a chiral symmetric non-unitary operator whose coin operator only has two eigenvalues. As a result, the spectra of such non-unitary operators are included in the (possibly non-unit) circle and the real axis in the complex plane. We also give some examples of our abstract results, such as non-unitary quantum walks defined by Mochizuki et al. Moreover, we present an application to the Ihara zeta functions and correlated random walks on regular graphs, which are not quantum walks.

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1 Introduction

Discrete-time quantum walks (QWs) are quantum counterparts of random walks, and many authors have studied them from several points of view [1, 6, 18, 27, 28, 31, 34–37, 42, 43] (see [19, 41] for reviews). In closed systems, their time-evolution operators are given as unitary operators on the Hilbert spaces of states describing their positions and inner degrees of freedom. Then the spectral analysis of QWs gives us rich information such as their long-time behavior [10, 24, 36, 43], localization [8, 9, 11, 21], and topological indices [3, 6, 18, 25, 26, 40].

On the other hand, several authors have started the study of quantum walks in open systems [2, 4, 27, 29, 30]. Attal et al. introduced open-quantum random walks to describe such a system using the time-evolution of a density operator [4]. Another way to describe such a system is to use non-unitary operators as the time-evolution operators [27]. Moreover, such operators are not only non-unitary but even non-normal [1]. In general, non-normal operators do not satisfy the spectral theorem, making analysis of their spectra difficult. In this paper, we study the spectral analysis of such non-normal time-evolution operators.

Two of authors of this paper showed any chiral symmetric unitary operator has a spectral mapping property from a self-adjoint operator by the Joukowski transform (divided by two) [34, 35]. More precisely, a bounded operator $u$ is said to have chiral symmetry if there exists a unitary involution $\gamma$, i.e., $\gamma^{-1} = \gamma^* = \gamma$, such that

$$\gamma u \gamma^* = u^*.$$ 

In particular, a unitary operator $u$ on a Hilbert space $\mathcal{H}$ has chiral symmetry if and only if $u$ can be written as a product $u = sc$ of two unitary involutions $s$ and $c$ on $\mathcal{H}$. In this case, there exists a coisometry $d$ from $\mathcal{H}$ to a Hilbert space $K$ such that

$$c = d^*d + (-1)(1 - d^*d),$$

where $d^*d$ and $1 - d^*d$ are the projections onto $\ker(c - 1)$ and $\ker(c + 1)$, respectively [37].

Denoting by $T$ the self-adjoint operator $dsd^*$ on $K$, we have the spectral mapping property

$$\sigma_c(u) \setminus \{\pm 1\} = \varphi^{-1}(\sigma_c(T) \setminus \{\pm 1\}),$$

$$\sigma_p(u) \setminus \{\pm 1\} = \varphi^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \cup \{+1\}^{m_+ + M_+} \cup \{-1\}^{m_- + M_-},$$

where $\varphi(z) = (z + z^{-1})/2$ is the Joukowski transform and $m_\pm = \dim \ker(T \mp 1)$ and $M_\pm = \dim \ker(s \mp 1) \cap \ker d$. Here we use $\{\lambda\}^m$ to denote that the multiplicity of an eigenvalue $\lambda$ is $m$. For classical results and examples of the spectral mapping property, see [35] and the references.
This paper reveals the spectral mapping property for a chiral symmetric non-unitary bounded operator $U$. Let $S$ be a self-adjoint involution and let $d$ be as stated above. Then the chiral symmetric operator $U$ is defined as

$$U = SC,$$

where

$$C = ad^*d + b(1 - d^*d) \quad (a, b \in \mathbb{R}),$$

is a self-adjoint operator with eigenvalues $a$ and $b$. The chiral symmetry $\gamma U \gamma^* = U^*$ of $U$ can be easily check if we take $\gamma = S$. Our goal of the paper is to state the spectrum mapping property of $U$ by using a scaling Joukowski transform $\varphi_{a,b}(z)$ defined in [24].

This abstract non-unitary (possibly non-normal) operator $U$ includes the non-unitary time-evolution operators of quantum walks introduced by Mochizuki, Kim, Obuse [27] and ones of correlated random walks on $k(\geq 2)$-regular graphs. Moreover, $U$ is applicable for the positive supports $U^+$, which are non-unitary, of the time-evolution operators of the (unitary) Grover walks on $k(\geq 2)$-regular graphs. In this case, $U^+$ gives a nice expression of the Ihara zeta function [31]. The organization of the paper is as follows. Sec. 1 briefly introduces the spectral mapping properties and their applications. Sec. 2 gives mathematical preliminaries and main results. Sec. 3 and 4 are the proofs of the spectral mapping properties for eigenvalues and continuous spectra, respectively. In Sec. 5, we give applications of the spectral mapping property. We deal with the Mochizuki-Kim-Obuse model in Sec. 5.1. In Sec. 5.2, we discuss the relation between the positive supports of the Grover walks and the Ihara zeta functions. Sec. 5.3 is devoted to the correlated random walks. The appendix provides a general property of the resolvent set of $U$ and two examples of our model. The two examples presented here are space-homogeneous and space-inhomogeneous quantum walks.

2 Preliminaries and Main Theorem

2.1 Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be (non-trivial) complex Hilbert spaces. We use $d$ to denote a coisometry from $\mathcal{H}$ to $\mathcal{K}$, i.e., $d$ is a bounded linear operator and satisfies

$$dd^* = I_\mathcal{K}, \quad (2.1)$$

where $I_N$ is the identity operator on a Hilbert space $N$. The operator $d$ is the same one as $d_A$ in [35]. For given $a, b \in \mathbb{R}$, we define a bounded self-adjoint operator $C$ on $\mathcal{H}$ as follows

$$C := ad^*d + b(I_{\mathcal{H}} - d^*d) = (a - b)d^*d + bI_{\mathcal{H}}$$

and call it a coin operator.

Let us recall the basic properties of $d$ and $C$. See [35] for detail. We will use the symbol $\sigma(A)$ and $\sigma_p(A)$ to denotes the spectrum and point spectrum of a linear operator $A$ respectively. We observe that $d^*d$ is a non-zero orthogonal projection on $\mathcal{H}$ because $d^*$ is injective and $d$ is surjective. Moreover, the following lemma holds.

**Lemma 2.1.** The spectrum of $C$ consists only of its eigenvalues and that is contained by $\{a, b\}$. Moreover, the following form holds

$$\sigma(C) = \sigma_p(C) = \{a, b\} \quad (2.2)$$

if and only if $d^*d \neq I_{\mathcal{H}}$.

**Proof.** If $d^*d = I_{\mathcal{H}}$, then $C = aI_{\mathcal{H}}$ and so $\sigma(C) = \sigma_p(C) = \{a\}$. If $d^*d \neq I_{\mathcal{H}}$, then (2.2) holds since $d^*d$ and $I_{\mathcal{H}} - d^*d$ are non-zero and orthogonal to each other. \hfill \square

Let $S$ be a unitary involution, which is hence a self-adjoint operator, on $\mathcal{H}$ and we call it a shift operator. We observe that $S^2 = I_{\mathcal{H}}$ holds from $S^* = S$. Moreover, $dS$ is a coisometry from $\mathcal{H}$ to $\mathcal{K}$ because so is $d$ and $S^2 = I_{\mathcal{H}}$. Our model includes that in [35] in the case $a = 1, b = -1$.

**Remark 2.2.** The spectrum of $S$ consists only of its eigenvalues and that is contained by $\{\pm 1\}$. Moreover, $\sigma(S) = \sigma_p(S) = \{\pm 1\}$ if and only if $S \neq \pm I_{\mathcal{H}}$.

**Definition 2.3.** We define a time evolution operator $U$ associated with $S$ and $C$ by

$$U := SC.$$

The discriminant operator $T$ of $U$ is also defined by

$$T := dSd^*.$$

This time-evolution operator is simple but covers a lot of previous unitary and non-unitary walk models. For examples, this time evolution operator can reproduce not only a standard time evolution operator of a discrete-time quantum walk, for example, the split step...
model [18] and the Szegedy walk on a graph [7,39], which are unitary, but also the Mochizuki-Kim-Obuse model [27] and the Bass-Hashimoto expression of the Ihara zeta function [5,14], which are non-unitary (see also examples in Appendix). Moreover this operator $U$ also reproduces a correlated random walk on a graph [20,32].

The operator $U$ is a bounded operator on $\mathcal{H}$ since so is $S$ and $C$. We note that $T$ is a self-adjoint operator equipped with its operator norm $\|T\| \leq 1$ (see [35] for details). Moreover, it has chiral symmetry from $SUS = U^*$.  

**Remark 2.4.** From the definition of $U$,

$$[U,U^*] = (a^2 - b^2)
\begin{pmatrix} S & d^* \end{pmatrix} S$$

holds, where $[A,B] := AB - BA$ denotes the commutator of linear operators $A$ and $B$. Because of the unitarity of $S$, we see that $U$ is normal if and only if $a^2 = b^2$ or $[S,d^*d] = 0$. The equation (2.3) implies that $U$ is not always normal. Note that spectral theory is not applicable to non-normal case, especially to non-unitary case. See Example A.2 and A.3 for non-unitary cases.

A rough estimate of the range of $\sigma(U)$ is immediately obtained from the definition of $U$.

**Corollary 2.5.** The spectrum of $U$ is included in

$$\{ z \in \mathbb{C} \mid \min\{|a|,|b|\} \leq |z| \leq \max\{|a|,|b|\} \}.$$  

This follows from Proposition A.1 which is proven in the appendix. In particular, we emphasize that $\sigma(U)$ does not include the origin if $ab \neq 0$ and the real part of $\sigma(U)$ falls in the range

$$[-\max\{|a|,|b|\}, -\min\{|a|,|b|\}] \cup [\min\{|a|,|b|\}, \max\{|a|,|b|\}].$$

### 2.2 Main Theorem

In what follows, we formulate our assumptions. If $d^*d = I_{\mathcal{H}}$ (respectively, $a = b$, $S = \pm I_{\mathcal{H}}$), then $U = aS$ (resp., $U = bS$, $U = \pm C$), so we get the spectrum of $U$ from that of $S$ (resp., $S$, $C$) immediately. Moreover, if $a = -b$, then $C = a(2d^*d - I_{\mathcal{H}})$, so the spectrum of $U$ is provided by [35] immediately.

**Proposition 2.6.** Assume $d^*d \neq I_{\mathcal{H}}$, $S \neq \pm I_{\mathcal{H}}$, and $a \neq \pm b$. The residual spectrum of $U$ is empty.
This is proved by Remark 3.2 and the chiral symmetry of $U$. Hence, the spectrum of $U$ consists only of the point spectrum and continuous spectrum:

$$\sigma(U) = \sigma_p(U) \cup \sigma_c(U),$$

where $\sigma_c(A)$ denotes the continuous spectrum of a linear operator $A$. Generally speaking, the residual spectrum of a linear operator is empty when it is unitary, or more generally, normal. Because $U$ is not unitary in our paper, the absence of residual spectrum plays a key role in the proof of the continuous spectrum part of our main theorem stated later.

Our goal of the present paper is to state the spectrum mapping property of $U$ by using a scaling Joukowsky transform $\varphi_{a,b}$ defined as follows: Recall that the Joukowsky transform is given by $\varphi(z) := (z + z^{-1})/2$ for $z \in \mathbb{C} \setminus \{0\}$. Then we define the scaled Joukowsky transform $\varphi_{a,b} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ as

$$\varphi_{a,b}(z) := \frac{2\sqrt{-ab}}{a - b} \varphi \left( \frac{z}{\sqrt{-ab}} \right) = \frac{z - abz^{-1}}{a - b}, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.4)$$

Further, if $ab = 0$, $U$ is not invertible, and 0 is included in the spectrum of $U$. In this case, the spectrum of $U$ is not included in the domain of the scaled Joukowsky transform. Hence it is natural to assume that $ab \neq 0$. Therefore, we prove our main theorem under the following conditions:

$$d^*d \neq I_H, \quad S \neq \pm I_H, \quad a \neq \pm b, \quad ab \neq 0. \quad (2.5)$$

Throughout this paper, except for the appendix, we assume that the above conditions hold. We are now in a position to state our main result.

**Theorem 2.7.** The followings hold:

$$\sigma_c(U) \setminus \{\pm a, \pm b\} = \varphi_{a,b}^{-1}(\sigma_c(T) \setminus \{\pm 1\}),$$

$$\sigma_p(U) = \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \cup \{a\}^{m+} \cup \{-a\}^{m-} \cup \{-b\}^{M+} \cup \{b\}^{M-}. \quad (2.7)$$

If $\pm a \in \sigma_c(U)$, then $\pm 1 \in \sigma_c(T)$ and if $\mp b \in \sigma_c(U)$, then $\pm 1 \in \sigma(T)$ respectively. If $\pm 1 \in \sigma_c(T)$, then $\pm a \in \sigma_c(U)$, $\mp b \in \sigma(U)$ respectively. Moreover, for any $\lambda \in \sigma_p(U)$,

$$\dim \ker(U - \lambda) = \begin{cases} \dim \ker(T - \varphi_{a,b}(\lambda)), & \lambda \neq \pm a, \, \pm b, \\ m_+, & \lambda = a, \\ m_-, & \lambda = -a, \\ M_+, & \lambda = -b, \\ M_-, & \lambda = b, \end{cases} \quad (2.8)$$

where

$$m_{\pm} := \dim \ker(T \mp 1), \quad M_{\pm} := \dim [\ker d \cap \ker(S \pm 1)]. \quad (2.9)$$

6
Remark 2.8. Theorem 2.7 shows that the point and continuous spectrum of $U$ except for $\{\pm b\}$ corresponds respectively to that of $T$ via $\varphi_{a,b}$. This is one of the differences between our results and [35]. Also, in the previous study, there are birth eigenvalues $\pm 1$ as an eigenvalue of $U$ that does not originate from $T$. The birth eigenvalue in the present study is $\pm b$. Thus, an eigenvalue of the coin operator for $1 - d^*d$ corresponds to the birth eigenvalues.

For $\lambda \in \sigma(U)$, let $t = \varphi_{a,b}(\lambda)$ then the following equation holds:

$$\lambda = \frac{(a - b)t \pm \sqrt{(a-b)^2t^2 + 4ab}}{2}. \quad (2.10)$$

Recall that $t$ is a real number since $T$ is self-adjoint. This equation indicates that $\sigma(U)$ becomes a subset of real axis if $ab > 0$. Moreover, it may have its imaginary part if $ab < 0$, and $|\lambda| = \sqrt{-ab}$ holds by (2.10) for $\lambda$ satisfying $\text{Im} \lambda \neq 0$. Hence we obtain $\sigma(U) \subset (\mathbb{R} \cup \sqrt{-ab} \mathbb{T})$.

To show that Theorem 2.7 holds, we provide several notations:

$$L := \text{ran}(d^*d) + \text{ran}(Sd^*dS),$$

$$L_1 := d^*(\ker(T^2 - 1)^\bot) + (dS)^*(\ker(T^2 - 1)^\bot),$$

$$L_0^\pm := d^* \ker(T \mp 1),$$

$$L_0 := L_0^+ \oplus L_0^-,$$

$$L_\perp^\pm := L_\perp \cap \ker(S \pm 1),$$

where $E + F = \{e + f \mid e \in E, f \in F\}$, and $M^\perp$ denotes the orthogonal complement of a subspace $M$.

3 Eigenvalues of time evolution

In this section, we will prove the next theorem.
**Theorem 3.1.** The point spectrum of $U$ is given by the following:

$$
\sigma_p(U) = \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \cup \{a\}^{m_+} \cup \{-a\}^{m_-} \cup \{-b\}^{M_+} \cup \{b\}^{M_-}.
$$

(3.1) **eq:sigmapU**

Moreover, for any $\lambda \in \sigma_p(U)$,

$$
\dim \ker(U - \lambda) = \begin{cases} 
\dim \ker(T - \varphi_{a,b}(\lambda)), & \lambda \neq \pm a, \pm b, \\
m_+, & \lambda = a, \\
m_-, & \lambda = -a, \\
M_+, & \lambda = -b, \\
M_-, & \lambda = b.
\end{cases}
$$

**Remark 3.2.** Theorem 3.1 indicates that $\sigma_p(U)$ is reflection symmetry with respect to the real axis. Actually, for all $\lambda \in \sigma_p(U)$, $\varphi_{a,b}(\lambda)$ is a real number and hence

$$
\frac{\lambda - ab\lambda^{-1}}{a-b} = \frac{\bar{\lambda} - ab\bar{\lambda}^{-1}}{a-b}
$$

holds, where $\bar{\eta}$ is the complex conjugate of $\eta \in \mathbb{C}$. This implies $\bar{\lambda} \in \sigma_p(U)$ by (3.1).

The following proposition is important in the sense that it determines the distribution of the spectrum of $U$.

**Proposition 3.3.** For $\lambda \in \sigma_p(U)$ and $\psi \in \ker(U - \lambda) \setminus \{0\}$, then

$$
Tf = \varphi_{a,b}(\lambda)f, \quad f = d\psi.
$$

(3.2) **eq:Tf=phif**

Moreover if $\lambda \neq \pm b$, then $\varphi_{a,b}(\lambda) \in \sigma_p(T)$ and it holds that either

$$
\text{Im } \lambda = 0 \text{ or } |\lambda| = \sqrt{-ab}.
$$

(3.3) **eq:condition_lambda**

**Proof.** We have (3.2) in the same way as [35 Proposition 5.1(1)]. Let $\lambda \neq \pm b$. If $f = 0$, then we obtain $(\lambda - bS)\psi = 0$ from $U\psi = \lambda\psi$. But this contradicts to $\psi \neq 0$ or $\lambda \neq \pm b$. Therefore $f \neq 0$ and $\varphi_{a,b}(\lambda) \in \sigma_p(T)$. Moreover, $\varphi_{a,b}(\lambda)$ becomes a real number since $T$ is self-adjoint, which provides

$$
(\lambda - \bar{\lambda}) (|\lambda|^2 - ab) = 0,
$$

hence we have (3.3). \hfill \Box

**Remark 3.4.** Here we introduce some facts without proof because they can be proved by the similar way in [35].

$$
\mathcal{L}^\perp = \ker d \cap \ker(dS).
$$

(3.4) **eq:Ldecomp**

$$
\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_0^+ \oplus \mathcal{L}_0^-.
$$

(3.5) **eq:Ldecomp**

$$
\mathcal{L}_0^+ \subset \ker(S \mp 1),
$$

(3.6) **L0subS**

$$
\mathcal{H} = \mathcal{L}^\perp \oplus \mathcal{L}_1 \oplus \mathcal{L}_0.
$$

(3.7) **eq:directdecompositionH**
where \( \overline{M} \) denotes the closure of a set \( M \). Moreover, \( U \) is reduced by each closed subspace \( L^\perp, L_1 \) and \( L_0 \). Hence the spectrum of \( U \) is obtained from the each that of the reduced part of \( U \). For a closed subspace \( M \) in a Hilbert space \( \mathcal{N} \) and a linear operator \( A \) on \( \mathcal{N} \), let \( A_M \) denote the restriction of \( A \) to \( M \) if \( A \) is reduced by \( M \).

The restriction of \( d^* \) on \( \ker(T \mp 1) \) is bijective to \( L_0^\pm \) and satisfies

\[
(d^*|_{\ker(T \mp 1)})^{-1} = d|_{L_0^\pm}. \tag{3.8} \tag{eq:surj_d}
\]

The operators \( U_{L_0} \) and \( U_{L^\perp} \) are decomposed as follows:

\[
U_{L_0} = \left(aI_{L_0^+}\right) \oplus \left(-aI_{L_0^-}\right), \quad U_{L^\perp} = \left(-bI_{L^+}\right) \oplus \left(bI_{L^-}\right). \tag{3.9} \tag{eq:decomposeL0andLperp}
\]

**Proposition 3.5.** The following relations hold:

(i) \( \sigma_p(U) \setminus \{\pm a, \pm b\} \subset \sigma_p(U_{L_1}) \cap \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \).

(ii) \( \sigma_p(U) \cap \{\pm a\} \subset \sigma_p(U_{L_0}) \cup \sigma_p(U_{L^\perp}) \).

(iii) \( \sigma_p(U) \cap \{\pm b\} \subset \sigma_p(U_{L_0}) \cup \sigma_p(U_{L^\perp}) \).

**Proof.** The assertion (i)-(iii) is proved by Proposition 3.3 and a similar way in [35, Proposition 5.1] which is based on to give the following relations, for any \( \lambda \in \sigma_p(U) \),

\[
\ker(U - \lambda) \subset \begin{cases} 
L_1, & \text{if } \lambda \neq \pm a, \pm b, \\
L_0 \oplus L^\perp, & \text{if } \lambda = \pm a, \pm b.
\end{cases} \tag{3.10} \tag{eq:kernelU_inclusion}
\]

\[
\begin{array}{ll}
&
\end{array}
\]

**Remark 3.6.** The relation (3.10) implies \( \pm a, \pm b \notin \sigma_p(U_{L_1}) \). Thus Proposition 3.5 gives

\[
\sigma_p(U) \setminus \{\pm a, \pm b\} = \sigma_p(U_{L_1}),
\]

in particular, \( \pm a, \pm b \notin \sigma_p(U_{L_1}) \).

**Proposition 3.7.** The equation

\[
\sigma_p(U_{L_1}) = \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \tag{3.11} \tag{eq:sigmapU}
\]

holds and for any \( \lambda \in \sigma_p(U_{L_1}) \), we obtain

\[
\dim \ker(U - \lambda) = \dim \ker(T - \varphi_{a,b}(\lambda)). \tag{3.12} \tag{eq:sigmapU}
\]
Proof of the equation (3.11). By Proposition 3.5 the left hand side of (3.11) is contained in that of right hand side. Hence it is sufficient to show that the inverse inclusion relation holds. For each \( \lambda \in \mathbb{C} \setminus \{0\} \), a map \( K_\lambda \) from \( \mathcal{K} \) to \( \mathcal{H} \) which is defined by

\[
K_\lambda := (b + \lambda S)d^\ast
\]

is a bounded operator. If \( \lambda \neq \pm b \), then \( b + \lambda S \) is bijection because \( b\lambda^{-1} \) is in the resolvent set of \( S \). In this case,

\[
d(b + \lambda S)^{-1}K_\lambda = I_{\mathcal{K}}
\]

(3.13)

holds. Let \( \lambda \in \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \). Then \( \varphi_{a,b}(\lambda) \in \sigma_p(T) \setminus \{\pm 1\} \) holds and for an eigenvector \( g \in \ker(T - \varphi_{a,b}(\lambda)) \setminus \{0\} \) we define \( \phi := K_\lambda g \). The vector \( g \) is in \( \ker(T^2 - 1)^{\perp} \) since \( \varphi_{a,b}(\lambda) \neq \pm 1 \) and the eigenspaces of a self-adjoint operator associated with different eigenvalues are orthogonal each other, thus \( \phi \in \mathcal{L}_1 \). By the definition of \( \varphi_{a,b} \), we have \( \varphi_{a,b}(\theta) = 1 \) (resp. \( \theta = a, -b \)) since \( \lambda \neq \pm b \). Therefore, we can show that \( \phi \) is a non-zero vector since \( K_\lambda \) is injective from (3.13). Moreover, the equation \( U\phi = \lambda\phi \) holds by \( Tg = \varphi_{a,b}(\lambda)g \). This provides us \( \sigma_p(U_{\mathcal{L}_1}) \supset \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\}) \), hence (3.11) holds.

To get (3.12), we show that the following lemma holds.

**Lemma 3.8.** For \( \lambda \in \sigma_p(U_{\mathcal{L}_1}) \) and \( \psi \in \ker(U - \lambda) \setminus \{0\} \), \( f := d\psi \in \ker(T^2 - 1)^{\perp} \) and \( \psi \) satisfies the following equation:

\[
\psi = \frac{a - b}{\lambda^2 - b^2}K_\lambda f.
\]

(3.14)

In particular, \( \psi \) is in \( \mathcal{L}_1 \).

**Proof.** From Proposition 3.3 we have \( Tf = \varphi_{a,b}(\lambda)f \). Moreover, since \( |\varphi_{a,b}(\lambda)| < 1 \) holds from Remark 3.6 we obtain \( f \in \ker(T^2 - 1)^{\perp} \). It is shown that \( \psi \in \mathcal{L}_1 \) if (3.11) holds.

Next, we prove (3.14). Using the arguments in the proof of [35, Proposition 5.1], the vector \( \psi_0 := \psi - d^\ast f \) is represented by

\[
\psi_0 = (\lambda - bS)^{-1}(aSd^\ast - \lambda d^\ast)f.
\]

By the direct calculation, we see that \( (\lambda - bS)^{-1} = (\lambda + bS)(\lambda^2 - b^2)^{-1} \). Therefore,

\[
\psi = d^\ast f + \psi_0 = \frac{a - b}{\lambda^2 - b^2}K_\lambda \psi.
\]

Hence we get the desired result.

\[
\]
**Proof of (3.12)**

Recall $K_{\lambda} \ker(T - \varphi_{a,b}(\lambda)) \subset \ker(U - \lambda)$ for $\lambda \in \varphi_{a,b}^{-1}(\sigma_p(T) \setminus \{\pm 1\})$. Since $K_{\lambda}$ is injective if $\lambda \in \sigma_p(U_{\lambda})$, the above lemma provide us that $K_{\lambda}$ is injection from $\ker(T - \varphi_{a,b}(\lambda))$ onto $\ker(U - \lambda)$. Hence, we obtain (3.12). □

Summarizing (3.9), Proposition pointspectrum 3.5 and pointUL1 3.7, we get Theorem Maineigenvalue 3.1.

## 4 Continuous spectrum of time evolution

In this section, we will prove the next theorem:

**Theorem 4.1.** The following assertions hold:

$$\sigma_c(U) \setminus \{\pm a, \pm b\} = \varphi_{a,b}^{-1}(\sigma_c(T) \setminus \{\pm 1\}).$$  (4.1) \[eq:MainConti1\]

Moreover, the following hold:

(i) $\pm a \in \sigma_c(U)$ if and only if $\pm 1 \in \sigma_c(T)$.

(ii) If $\mp b \in \sigma_c(U)$, then $\pm 1 \in \sigma(T)$.

(iii) If $\pm 1 \in \sigma_c(T)$, then $\mp b \in \sigma(U)$.

To prove this theorem, we introduce some notations. The symbols

$$\sigma_{ap}(A) := \left\{ z \in \mathbb{C} \mid \text{there exists } \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{N} \text{ such that } \|\psi_n\| = 1 \text{ and } \lim_{n \to \infty} \|(A - z)\psi_n\| = 0 \right\},$$

$\sigma_r(A)$ and $\rho(A)$ are called respectively the approximate point spectrum, residual spectrum and resolvent set of an operator $A$ on a Hilbert space $\mathcal{N}$. In general, the following relation holds (see [16 Theorem 3.1.20]):

$$\sigma_p(A) \cup \sigma_c(A) \subset \sigma_{ap}(A) \subset \sigma(A).$$

**Remark 4.2.** In general, for the directsum operator $A \oplus B$ of linear operators $A$ and $B$ which respectively act on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we have

$$\sigma_r(A \oplus B) = (\sigma_r(A) \cap \sigma_r(B)) \cup (\sigma_r(A) \cap \sigma_c(B)) \cup (\sigma_r(A) \cap \rho(B)) \cup (\sigma_r(A) \cap \rho(B)) \cup (\rho(A) \cap \sigma_r(B)), \quad \sigma_c(A \oplus B) = (\sigma_c(A) \cap \sigma_c(B)) \cup (\sigma_c(A) \cap \rho(B)) \cup (\rho(A) \cap \sigma_c(B))$$ (4.2) \[eq:sigmarDirectsum\] (4.3) \[eq:sigmacDirectsum\]
by the definition of spectra. From $\sigma_c(U\mathcal{L}_1^\perp) = \emptyset = \sigma_c(U\mathcal{L}_1^\perp)$, $\mathcal{L}_1^\perp = \mathcal{L}_0 \oplus \mathcal{L}_1^\perp$ and \[ (\text{3.9}) \] we have
\[
\sigma_c(U) = \sigma_c(U_{\mathcal{L}_1^\perp}) \setminus \{a\}^{m+} \cup \{-a\}^{m-} \cup \{-b\}^{M+} \cup \{b\}^{M-},
\]
(4.4)
\[
\sigma_c(U) = \sigma_c(U_{\mathcal{L}_1^\perp}) \setminus \{a\}^{m+} \cup \{-a\}^{m-} \cup \{-b\}^{M+} \cup \{b\}^{M-}
\]
(4.5)
as $A = U\mathcal{L}_1^\perp$ and $B = U\mathcal{L}_1^\perp$. This and the absence of $\sigma_c(U)$ give $\sigma_c(U_{\mathcal{L}_1^\perp}) \setminus \{\pm a, \pm b\} = \emptyset$ and hence we get
\[
\sigma(U_{\mathcal{L}_1^\perp}) \setminus \{\pm a, \pm b\} \subset \sigma_p(U_{\mathcal{L}_1^\perp}) \cup \sigma_c(U_{\mathcal{L}_1^\perp}).
\]
(4.6)
\[
\sigma(U_{\mathcal{L}_1^\perp}) \setminus \{\pm b\} \subset \sigma_p(U_{\mathcal{L}_1^\perp}) \cup \sigma_c(U_{\mathcal{L}_1^\perp}).
\]
(4.7)

To prove Theorem \[ (\text{4.1}) \] we need the following lemmas.

**Lemma 4.3.** The following holds:
\[
\sigma_c(U_{\mathcal{L}_1^\perp}) \setminus \{\pm a, \pm b\} \subset \varphi_{a,b}^{-1}(\sigma_c(T)).
\]
(4.8)

**Proof.** Let $\lambda \in \sigma_c(U_{\mathcal{L}_1^\perp})$. There exists a sequence $\{\psi_n\}_{n=1}^\infty \subset \mathcal{L}_1$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|(U - \lambda)\psi_n\| = 0$. For $f_n := d\psi_n$ and $\theta_n := \psi_n - d^*f_n$ the fact that $d^*$ is isometric provides us, for all $n \in \mathbb{N}$,
\[
\|\psi_n\|^2 = \|\theta_n\|^2 + \|f_n\|^2.
\]
(4.9)

Assume that $f_n$ tends to zero as $n \to \infty$, then $\lim_{n \to \infty} \|\theta_n\| = 1$. In particular, we can choose $n_0 \in \mathbb{N}$ satisfying $\inf_{n \geq n_0} \|\theta_n\| > 0$. Since $d^*$ is bounded, we obtain
\[
U\psi_n = SC(\theta_n + d^*f_n)
= bS\theta_n + o(1) \quad (n \to \infty).
\]
Thus, it holds that
\[
(U - \lambda)\psi_n = b \left( S - \frac{\lambda}{b} \right) \theta_n + o(1) \quad (n \to \infty).
\]
This implies that $(S - \lambda/b)\theta_n$ tends to zero as $n \to \infty$ because so is $(U - \lambda)\psi_n$. This fact and $\inf_{n \geq n_0} \|\theta_n\| > 0$ give us $\lim_{n \to \infty} \|(S - \lambda/b)\hat{\theta}_n\| = 0$ where $\hat{\theta}_n := \theta_n/\|\theta_n\|$ for $n \geq n_0$. Hence $\lambda/b$ is in $\sigma(S)$. Since the spectrum of $S$ consists of $\pm 1$, we have $\lambda \in \{\pm b\}$.

We assume that $\lambda \neq \pm b$. From the above argument, we see that $f_n$ does not converge to zero. In this case, we can show that $\varphi_{a,b}(\lambda) \in \sigma(T)$ holds from the similar way of the proof in \[ (\text{3.5}) \text{ Lemma 6.2.} \ (1) \]. In the case of $\lambda \neq \pm a, \pm b$, if $\varphi_{a,b}(\lambda) \in \sigma_p(T)$, then it is contradict to \[ (\text{3.11}) \] since $\varphi_{a,b}(\lambda) \neq \pm 1$. Hence, we obtain $\lambda \in \varphi_{a,b}^{-1}(\sigma_c(T))$ for any $\lambda \in \sigma_c(U_{\mathcal{L}_1^\perp}) \setminus \{\pm a, \pm b\}$. □
Lemma 4.4. The following assertion holds:

\[ \varphi_{a,b}^{-1}(\sigma_c(T)) \setminus \{ \pm b \} \subset \sigma_c(U_{1\over 2}). \]

Proof. Let \( \lambda \in \varphi_{a,b}^{-1}(\sigma_c(T)) \setminus \{ \pm b \} \). Since \( \varphi_{a,b}(\lambda) \in \sigma_c(T) \), there exists a sequence \( \{ f_n \} \subset \mathcal{K} \) such that \( \| f_n \| = 1 \) for all \( n \in \mathbb{N} \) and

\[ \lim_{n \to \infty} \| (T - \varphi_{a,b}(\lambda)) f_n \| = 0. \]

Then, we set \( \psi_n := (b + \lambda S)d^* f_n \). It is easy to see that \( \psi_n \in L_1 \) and

\[ \| \psi_n \|^2 = |b|^2 + 2b\text{Re}(\lambda \langle f_n, Tf_n \rangle) + |\lambda|^2 \geq (|b| - |\lambda|)^2 - 2|b||\lambda|\| (T - \varphi_{a,b}(\lambda)) f_n \| \]

holds from \( |\varphi_{a,b}(\lambda)| \leq 1 \). Recall \( |\lambda| = \sqrt{-ab} \) when \( ab < 0 \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), by \( \lambda \neq \pm b \) we have

\[ \lim \inf_{n \to \infty} \| \psi_n \|^2 \geq (|b| - |\lambda|)^2 > 0. \]

Therefore we can choose a subsequence \( \{ \phi_n \} \subset \{ \psi_n \} \) which satisfies \( \inf_n \| \phi_n \| > 0 \). Then,

\[ U\phi_n = SC(b + \lambda S)d^* f_n = \lambda\phi_n + (a - b)\lambda Sd^*(T - \varphi_{a,b}(\lambda)) f_n. \]

Thus, \( \lim_{n \to \infty} \| (U - \lambda)\phi_n \| = 0 \) holds and hence \( \lambda \in \sigma(U_{1\over 2}) \). Furthermore, if \( \lambda \neq \pm a \), then (3.11) and (4.6) imply \( \lambda \in \sigma_c(U_{1\over 2}) \). If \( \lambda = \pm a \), this indicates \( \pm 1 \in \sigma_c(T) \), in particular \( m_{\pm} = 0 \). Recall Remark 3.6 and hence \( \lambda \in \sigma_c(U_{1\over 2}) \) holds by (4.7). Now we obtain our assertion.

\[ \square \]

Proof of Theorem 4.1

Let us first prove (4.1). Lemmas 4.3 and 4.4 provide

\[ \sigma_c(U_{1\over 2}) \setminus \{ \pm a, \pm b \} = \varphi_{a,b}^{-1}(\sigma_c(T)) \setminus \{ \pm a, \pm b \} = \varphi_{a,b}^{-1}(\sigma_c(T) \setminus \{ \pm 1 \}). \]

Thus (4.5) provides

\[ \sigma_c(U) \setminus \{ \pm a, \pm b \} = \varphi_{a,b}^{-1}(\sigma_c(T) \setminus \{ \pm 1 \}). \]

If \( \pm 1 \in \sigma_c(T) \), then \( \pm a \in \sigma_c(U) \) holds from Lemma 4.4 and \( \mp b \in \sigma(U) \) holds by a similar argument of [34, Proof of Lemma 6.2]. Let \( \nu = \pm a, \mp b \). If \( \nu \in \sigma_c(U) \), then (4.5) gives
where

\[ N_\nu = \begin{cases} m_+, & \nu = a, \\ m_-, & \nu = -a, \\ M_+, & \nu = -b, \\ M_-, & \nu = b, \end{cases} \]

and \( \nu \in \sigma_c(U_{\mathbb{Z}^2}) \). Hence discussion of the accumulation point implies \( \pm 1 \in \sigma(T) \). In particular, for \( \nu = \pm a \), \( m_\pm = 0 \) holds and that gives us \( \pm 1 \notin \sigma_p(T) \). Hence \( \pm 1 \) is in \( \sigma_c(T) \) respectively.

\[ \square \]

Theorem 4.7 is a combination of Theorem 3.1 and 4.1.

Remark 4.5. In this subsection, we use the approximate point spectrum to prove our statement. This approach is derived from the absence of the residual spectrum of \( U \) and that is obtained by chiral symmetry and a reflection symmetry of eigenvalues about the real axis.

5 Applications

In this section, we introduce two applications. In the first, we consider Mochizuki-Kim-Obuse (MKO) model which is a non-unitary quantum walk researched by [27]. Next, we consider the application to the Ihara zeta function.

5.1 Mochizuki-Kim-Obuse model

Let \( \gamma > 0, \psi, \theta_j \in [0, 2\pi) \) for \( j = 1, 2 \). We define some matrices \( G, \Phi \) and \( \tilde{C}(\theta_j) \) on \( \mathbb{C}^2 \) as

\[
G := \begin{pmatrix} e^\gamma & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, \quad \Phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad \tilde{C}(\theta_j) = \begin{pmatrix} \cos \theta_j & i \sin \theta_j \\ i \sin \theta_j & \cos \theta_j \end{pmatrix}, \quad j = 1, 2.
\]

We write the multiplication operator on \( \ell^2(\mathbb{Z}; \mathbb{C}^2) \) by \( G, \Phi \) and \( \tilde{C}(\theta_j) \) by the same symbols. For \( \Psi \in \ell^2(\mathbb{Z}; \mathbb{C}^2) \), the operator \( \tilde{S} \) is defined by

\[
(\tilde{S}\Psi)(x) := \begin{pmatrix} \Psi_1(x+1) \\ \Psi_2(x-1) \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}, \quad x \in \mathbb{Z}.
\]

The MKO model is defined \( U_\gamma \) on \( \ell^2(\mathbb{Z}; \mathbb{C}^2) \) by

\[
U_\gamma := \tilde{S}G\Phi\tilde{C}(\theta_2)\tilde{S}G^{-1}\Phi\tilde{C}(\theta_1), \quad \theta_1, \theta_2 \in [0, 2\pi).
\]
In [27], \( \Phi \) is called the Phase operator. For \( \gamma > 0 \), \( e^{-\gamma} \) and \( e^\gamma \) mean the loss and gain of the photon. If the parameter \( \gamma \) is large enough, then the loss and gain amplify. Thus \( G \) is called the loss and gain operator in [27]. In the case of \( \gamma = 0 \) or \( \sin \theta_2 = 0 \), \( U_\gamma \) is unitary. However if \( \gamma > 0 \) and \( \sin \theta_2 \neq 0 \), then \( U_\gamma \) is not unitary. Thus \( U_\gamma \) is a time evolution operator of a non-unitary quantum walk. Moreover, we remark that \( U_\gamma \) is not normal. Thus we can not use some general theories for the spectral analysis. In [27], the parameters \( \theta_1 \) and \( \theta_2 \) may depend on \( x \in \mathbb{Z} \). However, we take \( \theta_1 \) and \( \theta_2 \) such that these do not depend on \( x \in \mathbb{Z} \) in this paper. By the next lemma, the spectrum mapping theorem is applied for the MKO model.

**Lemma 5.1.** [Theorem B] There exists a unitary operator \( \eta \), a self-adjoint unitary operator \( S_{\text{mko}} \) and a self-adjoint operator \( C_{\text{mko}} \) on \( \ell^2(\mathbb{Z}; \mathbb{C}^2) \) such that \( \eta^* U_\gamma \eta = S_{\text{mko}} C_{\text{mko}} \).

*Proof.* We set \( \eta := \sigma_2 \tilde{C}(\theta_1) \tilde{S} \sigma_2 \) where \( \sigma_2 \) is the second Pauli matrix i.e., \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \).

Then \( \eta \) is a unitary operator on \( \mathcal{H} \) since \( \sigma_2 \tilde{C}(\theta_1) \) and \( \tilde{S} \sigma_2 \) are unitary operators. We define \( S_{\text{mko}} \) and \( C_{\text{mko}} \) as \( S_{\text{mko}} := \tilde{S} \tilde{C}(\theta_1) \tilde{S} \sigma_2 \) and \( C_{\text{mko}} := \sigma_2 G \Phi \tilde{C}(\theta_2) G^{-1} \Phi \). Then \( S_{\text{mko}} \) is unitary self-adjoint and \( C_{\text{mko}} \) has only two real eigenvalues since \( \sigma_2 \tilde{C}(\theta_1) \) and \( \tilde{S} \sigma_2 \) are unitary self-adjoint and \( C_{\text{mko}} \) is a \( 2 \times 2 \) Hermitian matrix. Moreover, we obtain \( \eta^* U_\gamma \eta = S_{\text{mko}} C_{\text{mko}} \).

Lemma 5.1 means that the spectra of \( U_\gamma \) and \( S_{\text{mko}} C_{\text{mko}} \) are equal. As a matter of fact, the spectrum of \( U_\gamma \) is included in the unit circle and the real line. In what follows, we consider the case of \( \sin \theta_1 \sin \theta_2 > 0 \). In other cases, we can obtain the results by the same arguments. Let \( \gamma_0, \gamma_1 > 0 \) be

\[
\gamma_i := \frac{1}{2} \log \left( \Gamma_i + \sqrt{\Gamma_i^2 - 1} \right), \quad \Gamma_i = -\frac{1}{ap} \left( 1 + (-1)^{i+1} |qb| \right), \quad i = 0, 1
\]

where \( p := -\sin \theta_1, \quad q := -i \cos(\theta_1), \quad a := \sin \theta_2, \quad b := -ie^{-2t} \cos \theta_2 \). Then, by the direct calculation, we obtain the next proposition.

**Proposition 5.2.** [Theorem C] For any \( \gamma > 0 \), the following equations hold:

\[
\sigma(U_\gamma) = \sigma_c(U_\gamma), \quad \sigma_p(U_\gamma) = \sigma_t(U_\gamma) = \emptyset.
\]

Moreover, the spectrum of \( U_\gamma \) holds that

\[
\sigma(U_\gamma) = \begin{cases}
\{ e^{i\xi} \mid \cos \xi \in [m_\gamma, M_\gamma] \} , & 0 < \gamma \leq \gamma_0, \\
\{ e^{i\xi} \mid \cos \xi \in [-1, M_\gamma] \} \cup [f_-(m_\gamma), f_+(m_\gamma)], & \gamma_0 < \gamma < \gamma_1, \\
[f_-(m_\gamma), f_-(M_\gamma)] \cup [f_+(M_\gamma), f_+(m_\gamma)], & \gamma_1 \leq \gamma
\end{cases}
\]

15
where
\[
f_{\pm}(x) = x \pm \sqrt{x^2 - 1}, \quad m_\gamma := ap \cosh(2\gamma) - |qb|, \quad M_\gamma := ap \cosh(2\gamma) + |qb|.
\]

By Proposition MKOspec 5.2, \(\sigma(U_\gamma)\) is included in the unit circle and the real line. Moreover, the spectrum of \(U_\gamma\) varies greatly with the parameter \(\gamma > 0\). The main theorem is hoped to the applications to the spectral analysis of the MKO model in the spatial dependence case.

### 5.2 Ihara zeta function

Originally, the Ihara zeta function was defined by Y.Ihara in the context of discrete subgroups of the \(p\)-adic special linear group \([17]\). In this paper we introduce the definition of the Ihara zeta function in the graph theoretical setting by \([38]\). Let \(G = (V, E)\) be a connected, finite and \(k\)-regular graph, that is, \(|V| < \infty\) and \(\deg(u) < \infty\). Then the Ihara zeta function can be defined by the following Euler product as an analogue of the Riemann zeta function:
\[
\zeta_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime and reduced cycles of \(G\) and \(|C|\) is the length of the cycle. Let \(A = A(G)\) be the set of symmetric arcs. Here if \(\{u, v\} \in E\), then the induced symmetric arcs are \((u, v)\) and \((v, u)\) which represent the arcs from the origin vertex \(u\) to the terminal vertex \(v\) and vice versa, respectively. For any countable set \(\Omega\), let \(\mathbb{C}^\Omega\) be the linear space whose standard basis are labeled by the elements of \(\Omega\) and inner product is standard. Bass and Hashimoto \([5, 14]\) showed the following determinant expression of the Ihara zeta function.
\[
1/\zeta_G(u) = \det(1_{C^A} - u(B' - J)),
\]

where \(B'\) and \(J\) are operators on \(\mathbb{C}^A\) such that \((B'\psi)(e) = \sum_{e' \in A : o(e) = t(e')} \psi(e')\) and
\[
(J\psi)(e) = \psi(\bar{e}) \quad (5.1) \tag{eq:J}
\]

for any \(\psi \in \mathbb{C}^A\). Remark that \((B')_{e,e'} = 1\) if and only if \(e'\) meets \(e\), and the operation \(J\) corresponds to the back tracking. Thus \((B' - J)_{e,e'} = 1\) if and only if \(e'\) meets \(e\) but \(e' \neq \bar{e}\).

Now let us take further deformation of this expression connecting to quantum walks. Let \(U\) be the time evolution operator of the Grover walk on \(G\) for \(k > 2\) such that
\[
(U\psi)(e) = -\psi(\bar{e}) + \frac{2}{k} \sum_{t(e') = o(e)} \psi(e')
\]
for any $\psi \in \mathbb{C}^A$ and $e \in A$. The positive support of $U$ is defined by

$$(U^+)_{e,e'} = \begin{cases} 
1, & (U)_{e,e'} > 0, \\
0, & (U)_{e,e'} \leq 0.
\end{cases}$$

Then interestingly, we can see that $U^+$ coincides with $B' - J$ because the reflection amplitude of Grover walk corresponding to the back tracking is $2/k - 1 < 0$ and the transmitting amplitude is $2/k > 0$. Thus, $U^+ = B' - J$

An attractive study direction from this fact is an investigation of the extension of this zeta function by $1/\zeta_G(u) := \det(1 - uU^+)$. This theorem is well-known conclusion, but we give another proof. To use the Theorem 2.7 we show the next lemma.

Lemma 5.4. There exist a unitary self-adjoint operator $S$ on $\mathbb{C}^A$ and coisometry $d : \mathbb{C}^A \to \mathbb{C}^V$ satisfying

$$U^+ = S(kd^*d - I_{\mathbb{C}^A}).$$

To this end, we define $K_{\text{in}}$ and $K_{\text{out}}$ as letting $K_{\text{in}} : \mathbb{C}^A \to \mathbb{C}^V$ be an incidence matrix with respect to a terminal vertex of arcs such that $(K_{\text{in}}\psi)(u) = \sum_{t(e)=u} \psi(e)$. The adjoint is expressed by $(K_{\text{in}}^*f)(e) = f(t(e))$. In the same way, let $K_{\text{out}} : \mathbb{C}^A \to \mathbb{C}^V$ be an incidence matrix with respect to origin vertex of arcs such that $(K_{\text{out}}\psi)(u) = \sum_{o(e)=u} \psi(e)$. The adjoint is expressed by $(K_{\text{out}}^*f)(e) = f(o(e))$. Then it is easy to see that

$$B' = K_{\text{out}}^*K_{\text{in}}, \quad M = K_{\text{in}}K_{\text{out}}^*,$$

where $M$ is the adjacency matrix of $G$. Noting $K_{\text{out}} = K_{\text{in}}J$, and setting $d$ by $d = 1/\sqrt{k}K_{\text{in}}$ and $S = J$, we have

$$U^+ = K_{\text{out}}^*K_{\text{in}} - J$$

$$= J(K_{\text{in}}^*K_{\text{in}} - I_{\mathbb{C}^A})$$

$$= S(kd^*d - I_{\mathbb{C}^A})$$

17
Since $\zeta_G(u) = u^{-|A|}/\det(u^{-1} - U^+)$, the problem is switched to spectral analysis on our quantum walk for $a = k - 1$ and $b = -1$ case. By Theorem 3.1 and (3.1), putting $\mu \in \sigma(M)$, we obtain that if $\lambda \in \mathbb{C}$ satisfies $\varphi_{a,b}(\lambda) = \lambda + (k - 1)\lambda^{-1})/k = \mu/k$, then $\lambda \in \sigma(U)$. Such $\lambda$ can be expressed by

$$\lambda = \frac{\mu}{2} \pm \frac{1}{2} \sqrt{\mu^2 - 4(k - 1)}.$$ 

If $|\mu| \leq 2\sqrt{k - 1}$, then putting $x(\mu) = \mu/2$ and $y(\mu) = \sqrt{k - 1 - (\mu/2)^2}$, we have

$$x^2(\mu) + y^2(\mu) = k - 1.$$

Therefore the support of $\lambda \in \sigma(U)$ is included in the circle whose center is 0 and the radius is $\sqrt{k - 1}$ and $\{\mu/2 \pm 1/2 \cdot \sqrt{\mu^2 - 4(k - 1)} \mid |\mu| > 2\sqrt{k - 1}, \mu \in \sigma(M)\}$ on the real line in the complex plain. The former one is called a non-trivial eigenvalues which is an analogue of the pole of the Riemann zeta function.

Finally, let us compute $m_{\pm}$ and $M_{\pm}$ in this case. Let us see $f_+ \in \ker(1 - T)$ implies $d^*f_+ \in \ker(k - 1 - U^+)$ in the following, where $T = (1/k)M$. We can check that if $f_+$ is a constant function, then $f_+ \in \ker(1 - T)$ in this setting. Then by Perron-Frobenius theorem, $\ker(1 - T) = \mathbb{C}\{f_+\}$ because “1” is the largest eigenvalue of the positive matrix $T$. Since $d^*f_+$ is also a constant function by definition of $d^*$ and $f_+$, we have $Sd^*f_+ = d^*f_+$. Then $U^+d^*f_+ = S(kd^*d - I_{C\Lambda})d^*f_+ = (k - 1)Sd^*f_+ = (k - 1)d^*f_+$. On the other hand, if $f_- \in \ker(-1 - T)$ with $f_+ \neq 0$, then $G$ must be a bipartite graph from a general spectral graph theory. Decomposing the vertex set of this bipartite graph into $V = X \sqcup Y$ so that “t(e) \in X, o(e) \in Y” or “o(e) \in X, t(e) \in Y” for any $e \in A$, we obtain that the shape of $f_-$ is

$$f_-(u) = \begin{cases} f_+(u), & u \in X, \\ -f_+(u), & u \in Y. \end{cases}$$

Then $Sd^*f_- = -d^*f$ which implies $U^+d^*f_- = S((k - 1)dd^* - I_{C\Lambda})d^*f_- = Sd^*g = -d^*f_-$. Therefore, by Theorem 3.1, since $m_{\pm} = \dim(\ker(\pm 1 - T))$, we have

$$m_+ = 1, \quad m_- = \begin{cases} 1, & G \text{ is bipartite}, \\ 0, & \text{otherwise}. \end{cases}$$

The value $M_{\pm}$ can be obtained as $M_{\pm} = |E| - |V| + m_{\pm}$ (see e.g., [33] for the proof). Then we have

$$M_{\pm} = \begin{cases} b_1(G), & G \text{ is bipartite}, \\ b_1(G) - 1, & G \text{ is not bipartite}. \end{cases}$$

Here $b_1(G) := |E| - |V| + 1$ is the first Betti number of $G$. 

18
5.3 A correlated random walk on a $k$-regular graph

Let us consider the following random walk on a connected $k$-regular graph $G = (V, E)$ ($k \geq 2$), which is finite. Let $A$ be the set of the symmetric arcs induced by $E$. We propose the time evolution of the correlated random walk with the parameter $p \in [0, 1]$ by

$$(P\psi)(e) = p\psi(\bar{e}) + \frac{1-p}{k-1} \sum_{t(e') = \delta(e), e' \neq \bar{e}} \psi(e')$$

for any $\psi \in \mathbb{C}^A$ and $e \in A$. This means that the probability that a random walker returns back to the same vertex as one at the previous time is $p$, while the probability that it moves to the other neighbor vertices is $1 - p$ and it uniformly chooses a neighbor from the $k - 1$ neighbor vertices. Note that if $p = 1/k$, then the random walk is reduced to the isotropic random walk. Also note that if the graph is the one-dimensional lattice, then this random walk recovers the model in [20, 32]. It is easy to see that $P = SC$ with

$$a = 1, \quad b = \frac{pk - 1}{k - 1},$$

$S = J$, where $J$ is defined in (5.1), and the coisometry $d : \mathbb{C}^A \to \mathbb{C}^V$ is

$$(d\psi)(u) = \frac{1}{\sqrt{k}} \sum_{t(e) = u} \psi(e)$$

for any $\psi \in \mathbb{C}^A$ and $u \in V$. The discriminant operator $T = dS^* = (1/k)K_{in}JK_{in}^* := P_0$ is described by the transition matrix of the isotropic random walk on $G$; that is,

$$(P_0f)(u) = \frac{1}{k} \sum_{t(e) = u} f(\delta(e))$$

for any $f \in \mathbb{C}^V$ and $u \in V$. Then from a standard argument on the spectral graph theory, we have

$$m_+ = 1, \quad m_- = \begin{cases} 1, & G \text{ is bipartite}, \\ 0, & \text{otherwise}, \end{cases}$$

$$M_+ = b_1(G), \quad M_- = \begin{cases} b_1(G), & G \text{ is bipartite}, \\ b_1(G) - 1, & \text{otherwise}. \end{cases}$$

By Theorem 4.1,

$$\sigma(P) \subset \begin{cases} [-1, -r] \cup [r, 1], & 1/k \leq p \leq 1, \\ \{z \in \mathbb{C} \mid |z| = \sqrt{r}\} \cup [-1, -r] \cup [r, 1], & 0 \leq p < 1/k, \end{cases}$$

19
where $r = |pk - 1|/(k - 1)$. Let us see the transition of $\sigma(P)$ by moving the parameter $p$ from $p = 1$ to $p = 0$ in the following. If $p = 1$, then $\sigma(P) = \{\pm 1\}$; the walk is reduced to just a zigzag walk. As $p$ decreases, the intervals $[-1, -r] \cup [r, 1]$ grow up because $r$ is monotone decreasing with respect to $1/k \leq p$. Finally if $p$ reaches to $1/k$, then the two intervals unite. After $p < 1/k$, then the gap on the real axis between $(-r, r)$ appears again, moreover some eigenvalues of $P_0$ are inherited to the circle with the radius $\sqrt{r}$ in the complex plane; the circle grows up while the two intervals shrink because $r$ is monotone increasing with respect to $p < 1/2$. The final radius of the circle and the final gap size on the real axis at $p = 0$ are $1/\sqrt{k-1}$ and $2/(k-1)$, respectively. On the other hand, Theorem 4.1 implies that the geometric information of $b_1(G)$ and the bipartiteness of $G$, which are independent of the parameter $p$, are reflected as the multiplicities $M_\pm$ of the eigenvalues $\pm \sqrt{r}$.

A Appendix

Here we prove the following proposition used for proving Corollary 2.5 and give two examples of our model, which are one-dimensional non-unitary quantum walks. Let $U$ be defined as in Definition 2.3 and allow the case $ab = 0$.

**Proposition A.1.** The following statements hold:

$$\{z \in \mathbb{C} \mid |z| < \min\{|a|, |b|\} \text{ or } \max\{|a|, |b|\} < |z|\} \subset \rho(U), \quad \text{(A.1)}$$

**Proof.** Let $z$ be in the set of the left hand of (A.1). First, we show that $U - z$ is injective. For any $\psi \in \mathcal{H}$ by Schwarz’s inequality we have

$$\|(U - z)\psi\| \geq \|U\psi\| - |z|\|\psi\|.$$  

From the definition of $C$ and what $d^*d$ is a projection, we can show that

$$\min\{|a|, |b|\}\|\psi\| \leq \|C\psi\| \leq \max\{|a|, |b|\}\|\psi\| \quad \text{(A.2)}$$

holds. Moreover, we obtain $\|U\psi\| = \|C\psi\|$ by $S^2 = I_{\mathcal{H}}$. Thus the inequality

$$\|(U - z)\psi\| \geq |c - |z||\|\psi\| \quad \text{(A.3)}$$

holds respectively where $c$ is $\min\{|a|, |b|\}$ or $\max\{|a|, |b|\}$ for each $z$. In both cases $U - z$ is injective.

Next, we will show that $U - z$ is surjective. Recall the direct sum decomposition of $\mathcal{H}$ as follows

$$\mathcal{H} = \ker(U^* - z) \oplus \text{ran}(U - z).$$
Assume that $\phi \in \ker(U^* - \overline{z})$, i.e., $CS\phi = U^*\phi = \overline{z}\phi$. Thus we obtain

$$US\phi = SCS\phi = \overline{z}S\phi.$$ 

This means $S\phi \in \ker(U - \overline{z})$. However, the inequality \((A.3)\) for $\overline{z}$ provides us $\ker(U - \overline{z}) = \{0\}$, since we have $S\phi = 0$. Then, we obtain $\phi = S^2\phi = 0$ and this implies that $\text{ran}(U - z)$ is dense, moreover equals to $\mathcal{H}$ since $U - z$ is bounded from below. Hence, $z \in \rho(U)$ because $U - z$ is bijection. 

We give two examples of our model. First, we give the example of a one-dimensional homogeneous non-unitary quantum walk.

**Example A.2.** For $\phi \in \mathbb{C}^2$ satisfying $\|\phi\| = 1$, let $d: \ell^2(\mathbb{Z}; \mathbb{C}^2) \rightarrow \ell^2(\mathbb{Z})$ as

$$(d\psi)(x) := \langle \phi, \psi(x) \rangle, \quad x \in \mathbb{Z}, \quad \psi \in \ell^2(\mathbb{Z}; \mathbb{C}^2),$$

then $d$ is a coisometry. Furthermore, $d^*d$ has the form as follows and let $S$ be

$$d^*d = \bigoplus_{x \in \mathbb{Z}} \begin{pmatrix} \phi_1^2 & \phi_1 \phi_2 \overline{\phi_2} \\ \phi_1 \phi_2 \overline{\phi_2} & |\phi_2|^2 \end{pmatrix}, \quad S := \begin{pmatrix} p & qL \\ \overline{qL^*} & -p \end{pmatrix},$$

where a pair $(p, q) \in \mathbb{R} \times \mathbb{C}$ satisfies $p^2 + |q|^2 = 1$ and $\phi_j$ denotes the $j$-th component of $\phi$ for $j = 1, 2$, with a direct decomposition $\ell^2(\mathbb{Z}; \mathbb{C}^2) = \bigoplus_{x \in \mathbb{Z}} \mathbb{C}^2$. Then we have

$$Sd^*d - d^*dS = \begin{pmatrix} 2i \text{Im}(q\overline{\phi_1}\phi_2L) & 2p\phi_1\phi_2 + q(|\phi_2|^2 - |\phi_1|^2)L \\ -2p\phi_1\phi_2 + q(|\phi_1|^2 - |\phi_2|^2)L^* & -2i \text{Im}(q\overline{\phi_1}\phi_2L) \end{pmatrix}$$

by the direct calculation, where $\text{Im} A := (A - A^*)/(2i)$ is the imaginary part of a linear operator $A$. From $\sigma(q\overline{\phi_1}\phi_2L) = |q\phi_1\phi_2|\mathbb{T}$, $\text{Im}(q\overline{\phi_1}\phi_2L)$ vanishes only in the case $q\overline{\phi_1}\phi_2 = 0$. Therefore we can show that $[S, d^*d] = 0$ if and only if $q = 0$ and $\overline{\phi_1}\phi_2 = 0$. Also, the space-homogeneous quantum walk $U = SC$ is included in our model.

We next give the example of a one-dimensional inhomogeneous non-unitary quantum walk.

**Example A.3.** Let $(\alpha, \beta) \in (\mathbb{R}, \mathbb{C})$. We set a operator $S: \ell^2(\mathbb{Z}; \mathbb{C}^2) \rightarrow \ell^2(\mathbb{Z}; \mathbb{C}^2)$ as

$$S = \begin{pmatrix} 0 & L \\ L^* & 0 \end{pmatrix},$$

and two matrices

$$C_1 = \begin{pmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{pmatrix}, \quad C_2 = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix},$$
where \( \lambda_{\pm} = \sqrt{\alpha^2 + |\beta|^2} \). For \( j = 1, 2 \), let \( \chi_j \) be the normalized eigenvector of \( C_j \) corresponding to \( \lambda_+ \). We define a coin operator \( C \) on \( \ell^2(\mathbb{Z}; \mathbb{C}^2) \) as a multiplication operator by

\[
C(x) = \begin{cases} 
C_1, & x : \text{odd}, \\
C_2, & x : \text{even}.
\end{cases}
\]

Then \( C \) is represented by \( C = \lambda_+ d^* d + \lambda_- (I - d^* d) \), where \( d : \ell^2(\mathbb{Z}; \mathbb{C}^2) \to \ell^2(\mathbb{Z}) \) define as

\[
(d\Psi)(x) = \langle \chi(x), \Psi(x) \rangle, \quad \Psi \in \ell^2(\mathbb{Z}; \mathbb{C}^2)
\]

and

\[
\chi(x) = \begin{cases} 
\chi_1, & x : \text{odd}, \\
\chi_2, & x : \text{even}.
\end{cases}
\]

Also, we have \([S, d^* d] \neq 0\) by the direct calculation. Thus, the space-inhomogeneous quantum walk \( U = SC \) is included in our model.

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References

[1] K. Asahara, D. Funakawa, M. Seki, Y. Tanaka, An index theorem for one-dimensional gapless non-unitary quantum walks, *Quantum Inf. Process.* 20, 287 (2021).

[2] S. Attal, N. Guillotin-Plantard, C. Sabot, Central Limit Theorems for Open Quantum Random Walks and Quantum Measurement Records, *Ann. Henri Poincaré.* 16, 15-43, (2015).

[3] J. K. Asbóth, H. Obuse, Bulk-boundary correspondence for chiral symmetric quantum walks, *Phys. rev. B.* 88 121406(R) (2013).

[4] S. Attal, F. Petruccione, C. Sabot, I. Sinayskiy, Open Quantum Random Walks. *J. Stat. Phys.* 147, 832-852, (2012).

[5] H. Bass H, The Ihara-Selberg zeta function of a tree lattice, *Internat. J. Math.* 3, 717-797, (1992).
[6] C. Cedzich, T. Geib, F. A. Grunbaum, C. Stahl, L. Velazquez, A. H. Werner, R. F. Werner, The Topological Classification of One-Dimensional Symmetric Quantum Walks, *Ann. Henri Poincaré*. **19**, 325–383, (2018).

[7] D. Emms, E. R. Hancock, S. Severini, R. C. Wilson, A matrix representation of graphs and its spectrum as a graph invariant, *Electron. J. Comb.* **13**, R34, (2006).

[8] T. Fuda, D. Funakawa, A. Suzuki, Localization of a multi-dimensional quantum walk with one defect, *Quantum Inf. Process.* **16**, 203, (2017).

[9] T. Fuda, D. Funakawa, A. Suzuki, Localization for a one-dimensional split-step quantum walk with bound states robust against perturbations, *J. Math. Phys.* **59**(8), (2018).

[10] T. Fuda, D. Funakawa, A. Suzuki, Weak limit theorem for a one-dimensional split-step quantum walk, *Rev. Roum. Math. Pures Appl.* **64**(2-3), 157–165, (2019).

[11] T. Fuda, A. Narimatsu, K. Saito, A. Suzuki, Spectral analysis for a multi-dimensional split-step quantum walk with a defect, *Quantum Stud.*, **9**(1), 93-112, (2022).

[12] Lov K. Grover, A fast quantum mechanical algorithm for database search, Proceedings, STOC 1996, Philadelphia PA USA, 212-219.

[13] C. Godsil, K. Guo, Quantum walks on regular graphs and eigenvalues. *Electron. J. Comb.* **18**, R165, (2011).

[14] K. Hashimoto, Zeta Functions of Finite Graphs and Representations of p-Adic Groups, In:Adv. Stud. Pure Math. **15** 211-280, Academic Press, New York (1989).

[15] Y. Higuchi, N. Konno, I. Sato, E. Segawa, A remark on zeta functions of finite graphs via quantum walks, *Pacific Journal of Math-for-Industry*. **6**, 73-4, (2014).

[16] F. Hiai, K. Yanagi, Hilbert spaces and linear operators, *Makino syoten*. (1995), in Japanese.

[17] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields. *J. Math. Soc. Japan*. **18**,219-235, (1966).

[18] T. Kitagawa, M. S. Rudner, E. Berg, E. Demler, Exploring topological phases with quantum walks, *Phys. Rev. A* **82**, 033429, (2010).

[19] N. Konno, Quantum Walks. Quantum Potential Theory. *Lecture Notes in Mathematics*. vol. 1954, pp. 309–452. Springer, Berlin (2008).
[20] N. Konno, Limit theorems and absorption problems for one-dimensional correlated random walks, *Stoch. Models.* 25, 29–49, (2009).

[21] N. Konno, Localization of an inhomogeneous discrete-time quantum walk on the line. *Quantum Inf. Process.* 9, 405–418, (2010).

[22] N. Konno N, I. Sato, On the relation between quantum walks and zeta functions, *Quantum Inf. Process.* 11, 341-349, (2012).

[23] N. Konno, I. Sato, E. Segawa, Phase measurement of quantum walks: application to structure theorem of the positive support of the Grover walk, *Electron. J. Comb.* 26, Issue 2, (2019).

[24] M. Maeda, A. Suzuki, K. Wada, Absence of singular continuous spectra and embedded eigenvalues for one dimensional quantum walks with general long-range coins, *Rev. Math. Phys.* 34, No. 05, 2250016, (2022).

[25] Y. Matsuzawa, An index theorem for split-step quantum walks. *Quantum Inf. Process.* 19(8), (2020).

[26] Y. Matsuzawa, M. Seki, Y. Tanaka, The bulk-edge correspondence for the split-step quantum walk on the one-dimensional integer lattice. *arXiv:2105.06147*.

[27] K. Mochizuki, D. Kim, H. Obuse, Explicit definition of PT symmetry for non-unitary quantum walks with gain and loss, *Phys. Rev. A.* 93, 062116, (2016).

[28] A. Narimatsu, H. Ohno, K. Wada, Unitary equivalence classes of split-step quantum walks, *Quantum Inf. Process.* 20, 368 (2021).

[29] A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Parity-time synthetic photonic lattices. *Nature.* 488(7410), 167-171, (2012).

[30] A. Regensburger, M. A. Miri, C. Bersch, J. Näger, G. Onishchukov, D. N. Christodoulides, U. Peschel, Observation of defect states in PT -symmetric optical lattices, *Phys. Rev. Lett.* 110, 223902, (2013).

[31] P. Ren, T. Aleksic, D. Emms, R. C. Wilson, E. R. Hancock, Quantum walks, Ihara zeta functions and cospectrality in regular graphs, *Quantum Inf. Process.* 10, 405-417, (2011).

[32] E. Renshaw and R. Henderson, The correlated random walk, *J. Appl. Prob.* 18,403–414, (1981).
[33] E. Segawa, Spectral properties of weighted line digraphs, *RIMS Kokyuroku*. **1956**, 16-28 (2015).

[34] E. Segawa, A. Suzuki, Generator of an abstract quantum walk, *Quantum Stud.: Math. Found.* **3**, 11–30 (2016).

[35] E. Segawa, A. Suzuki, Spectral mapping theorem of an abstract quantum walk, *Quantum Info. Process.* **18**, 333, (2019).

[36] A. Suzuki, Asymptotic velocity of a position-dependent quantum walk. *Quantum Inf. Process.* **15**, 103–119 (2016).

[37] A. Suzuki, Supersymmetry for chiral symmetric quantum walks. *Quantum Inf. Process.* **18**, 363, (2019).

[38] T. Sunada, L-Functions in Geometry and Some Applications, *In: Lecture Notes in Mathematics*. **1201** (1986) pp.266–284, Springer-Verlag, New York.

[39] M. Szegedy, Quantum speed-up of Markov chain based algorithms, *In: Proceedings of the 45th IEEE Symposium on Foundations of Computer Science*, 32-41, (2004)

[40] Y. Tanaka, A constructive approach to topological invariants for one-dimensional strictly local operators. *J. Math. Anal. Appl.* **500**,1 (2021).

[41] S. E. Venegas-Andraca, Quantum walks: a comprehensive review. *Quantum Inf. Process.* **11**, 1015–1106 (2012)

[42] K. Wada, Absence of wave operators for one-dimensional quantum walks, *Lett. in Math. Phys.* **109**, 2571–2583 (2019).

[43] K. Wada, A weak limit theorem for a class of long-range-type quantum walks in 1d, *Quantum Inf. Process.* **19**, 2 (2020).