Granular discorectangle in a thermalized bath of hard disks

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By using the Enskog-Boltzmann approach, we study the steady-state dynamics of a granular discorectangle placed in a two-dimensional bath of thermalized hard disks. Hard core collisions are assumed elastic between disks and inelastic between the discorectangle and the disks, with a normal restitution coefficient $\alpha < 1$. Assuming a Gaussian ansatz for the probability distribution functions, we obtain analytical expressions for the granular temperatures. We show the absence of equipartition and investigate both the role of the anisotropy of the discorectangle and of the relative ratio of the bath particles to the linear sizes of the discorectangle. In addition, we investigate a model of a discorectangle with two normal restitution coefficients for collisions along the straight and curved surfaces of the discorectangle. In this case one observes equipartition for a non trivial ratio of normal restitution coefficients.

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I. INTRODUCTION

Granular matter is characterized by the existence of dissipative forces between particles. In order to sustain a collective motion, it is necessary to provide energy continuously. When power supply is sufficiently copious, the assembly of granular particles attains a non equilibrium steady state (NESS)\textsuperscript{1,2}. It is customary to define the granular temperature as the second moment of the velocity distribution. It is a source of both fascination and inconvenience that the well-know properties of temperatures characterizing thermal systems are not necessarily transferable to granular temperatures. In particular, recent work, both theoretical\textsuperscript{2,3,4,5,6,7,8} and experimental\textsuperscript{9,10}, has shown that in a binary granular system the two species have different granular temperatures that are non-trivial functions of the microscopic parameters (mass, size, restitution coefficient,\ldots). Although the absence of equipartition is not surprising for a dissipative system sustained in a NESS, a more complete investigation is necessary since the granular temperatures play an important role in hydrodynamic descriptions of these systems (in particular the absence of equipartition in binary mixtures yields granular temperature gradients which enhance segregation\textsuperscript{11}). In addition, the extension of the fluctuation-dissipation theorem is an important issue in the context of granular gases\textsuperscript{12}. Other consequences of the absence of equipartition include the ability of a binary system to exhibit a segregation phenomena in a “Maxwell demon” experiment\textsuperscript{13}. See also the homogeneous cooling state of a granular mixture\textsuperscript{14} and the impurity problem\textsuperscript{15}.

Most of the above-referenced studies examined assemblies of spherical particles. Yet, in reality, the particles composing granular systems are to some degree anisotropic and, in many cases, strongly so. Even if the particles are smooth, each collision results in some exchange and, possibly loss, of rotational kinetic energy. There are relatively few studies of these systems (but see for example, Refs.\textsuperscript{16,17,18}) and fewer still that focus specifically on equipartition. Huthmann et al.\textsuperscript{19} used kinetic theory to examine the free cooling of a system of granular needles in three dimensions and, more recently, two of the present authors studied a two-dimensional system composed of a single granular needle in a thermalized bath of point particles\textsuperscript{20} in a NESS. For inelastic needle-point collisions, the rotational granular temperature is smaller than the translational one while both are less than the bath temperature. The validity of the theoretical predictions were confirmed by comparison with numerical simulations of the model. While this study provided useful insights, infinitesimal width of the particle is obviously an idealization.

The objective of present article is to consider a more realistic system where both the tracer particle and bath particles are of finite extent. Specifically, we consider a discorectangle in a bath of thermalized hard disks. Fortunately, despite the increased complexity, it is still possible to obtain an analytic solution of the steady state kinetic equations. The principal difference between the discorectangle-disk and needle-point systems is that two kinds of collision are possible in the former compared to one in the latter. Specifically, a disk can collide with either the sides or the caps of the discorectangle. If each type of collision is characterized by different normal restitution coefficients we show that equipartition between the translational and rotational degrees of freedom can be obtained for specific values of these parameters. Consequently, for appropriate ranges of the restitution coefficients the translational granular temperature may be less than or greater than the rotational one.

II. MODEL AND COLLISION RULES

We investigate a two-dimensional system consisting of a discorectangle of total length $L + 2R$, radius $R$ and mass $M$ with a moment of inertia $I$ (The value of which
is given in appendix A). The bath consists of disks of mass \(m\) and of radius \(r\). The vector positions of the center of mass of the discorectangle and a disk particle are denoted by \(\mathbf{r}_1\) and \(\mathbf{r}_2\), respectively. The orientation of the discorectangle is specified by a unit vector \(\mathbf{u}_1\) that points along the long axis. Let \(\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2\) and \(\mathbf{u}^\perp_1\) denote a vector perpendicular to \(\mathbf{u}_1\). A collision between a discorectangle and a disk can take place either on the linear part or on the circular parts of the former,

\[
\mathbf{r}_{12} \cdot \mathbf{u}^\perp_1 = -(R + r),
\]

if \(|\lambda| < L/2\), and

\[
\mathbf{r}_{12} \cdot \mathbf{u}_r = -(R + r),
\]

if \(R > |\lambda| - L/2 > 0\), where \(\mathbf{u}_r\) denotes the unit vector of the collision axis. (see Fig. 1). The relative velocity of the point of contact \(\mathbf{V}\) is given by

\[
\mathbf{V} = \mathbf{v}_{12} + \omega_1 \times \mathbf{OC},
\]

where \(\omega_1\) denotes the angular time derivative and \(\mathbf{OC}\) the vector from the center of the discorectangle to the point of impact.

The pre- and post-collisional quantities (the latter are labeled with a prime) satisfy:

- Total momentum conservation

\[
M\mathbf{v}'_1 + m\mathbf{v}'_2 = M\mathbf{v}_1 + m\mathbf{v}_2.
\]

- Angular momentum conservation with respect to the point of contact

\[
I\omega'\mathbf{k} = I\omega_1\mathbf{k} + m\mathbf{OC} \times (\mathbf{v}_2 - \mathbf{v}_2'),
\]

where \(\mathbf{k}\) is a unit vector perpendicular to the plane.

As a result of the collision, the relative velocity of the contacting points changes instantaneously according to the following relations:

\[
\mathbf{V}'\cdot \mathbf{u}^\perp_1 = -\alpha \mathbf{V} \cdot \mathbf{u}^\perp_1,
\]

\[
\mathbf{V}'\cdot \mathbf{u}_1 = \mathbf{V} \cdot \mathbf{u}_1,
\]

where \(\alpha\) is the normal restitution coefficient and \(\mathbf{u}^\perp_1\) and \(\mathbf{u}_1\) denote the unit vectors of the collision along the rectangular part of the discorectangle. When the collision occurs on the circular parts of the discorectangle, the collision rules are given by

\[
\mathbf{V}'\cdot \mathbf{u}_r = -\alpha \mathbf{V} \cdot \mathbf{u}_r,
\]

\[
\mathbf{V}'\cdot \mathbf{u}_\theta = \mathbf{V} \cdot \mathbf{u}_\theta,
\]

where \(\mathbf{u}_r\) and \(\mathbf{u}_\theta\) denote the unit vectors associated with the circular part of the discorectangle (See Fig 1b).

The tangential restitution coefficient is set to one for the sake of simplicity. This choice is reflected in the form of Eqs. \((7-9)\).

By combining Eqs. \((3)- (6)\) one obtains, after some algebra, the change of the discorectangle momentum \(\Delta \mathbf{p} = M(\mathbf{v}'_1 - \mathbf{v}_1)\) for a collision along the linear part

\[
\Delta \mathbf{p} \cdot \mathbf{u}^\perp_1 = \frac{(1 + \alpha)\mathbf{V} \cdot \mathbf{u}^\perp_1}{\frac{1}{m} + \frac{1}{M} + \frac{\lambda^2}{I}}.
\]

for \(|\lambda| \leq L/2\) and at the two ends of the discorectangle

\[
\Delta \mathbf{p} \cdot \mathbf{u}_r = \frac{(1 + \alpha)\mathbf{V} \cdot \mathbf{u}_r}{\frac{1}{m} + \frac{1}{M} + \frac{L \sin \theta}{I}}.
\]

for \(R > |\lambda| - L/2 > 0\) with \(\cos(\theta) = \frac{\lambda - L/2}{R}\).
III. BOLTZMANN EQUATION

Since we are interested in the homogeneous state, the distribution function \( f(v_1, \omega_1) \) of the discorectangle obeys

\[
\frac{\partial f(v_1, \omega_1)}{\partial t} = N \int \frac{d\theta_1}{2\pi} \int dv_2 \int dr_2 \hat{T}_{12} f(v_1, \omega_1, v_2)
\]

where \( N \) is the total number of disks, \( f(v_1, \omega_1, v_2) \) is the distribution function of the discorectangle and a disk, and \( \hat{T}_{12} \) is the collision operator between a discorectangle and a disk.

Defining the granular temperatures as quadratic average of the appropriate velocity distribution, one has \( T_T = M/2(\bar{v}^2) \) and \( T_R = I(\bar{\omega}^2) \) for the translational and rotational granular temperatures, respectively (the angular brackets denote the average). By taking the second moment with respect of the velocity and of the angular velocity of Eq. (12), one obtains

\[
\frac{2\partial T_T}{M \partial t} = \int dv_1 d\omega_1 \partial_1 (v_1^2 f(v_1, \omega_1))
\]

\[
= N \int dv_1 \int d\omega_1 \int \frac{d\theta_1}{2\pi} \int dv_2 \int dr_2 \hat{T}_{12} f(v_1, \omega_1, v_2)v_1^2,
\]

\[
\frac{\partial T_R}{I \partial t} = \int dv_1 d\omega_1 \partial_1 (\omega_1^2 f(v_1, \omega_1))
\]

\[
= N \int dv_1 \int d\omega_1 \int \frac{d\theta_1}{2\pi} \int dv_2 \int dr_2 \hat{T}_{12} f(v_1, \omega_1, v_2)\omega_1^2.
\]

In the stationary state the time derivatives of the left-hand side of these two equations are equal to zero.

To build the collision operator between the discorectangle and a disk, \( \hat{T}_{12} \), one must include the change in quantities (i.e., velocity and angular momentum) produced during the infinitesimal time interval of the collision. This operator is different from zero only if the two particles are in contact and if the particles were approaching just before the collision [11]. For a collision between a disk and the rectilinear part of the discorectangle, the explicit form of the operator given by

\[
\hat{T}_{12} \propto \Theta(L/2 - |\lambda|) \delta(|r_{12}.u_1^+| - r - R)
\]

\[
\times \left| \frac{d|r_{12}.u_1^+|}{dt} \right| \Theta \left( - \left| \frac{d|r_{12}.u_1^+|}{dt} \right| \right) (b_{12} - 1),
\]

where \( b_{12} \) is an operator that changes post-collisional quantities to pre-collisional quantities and \( \Theta(x) \) is the Heaviside function, and for a collision at two ends of the discorectangle

\[
\hat{T}_{12} \propto \Theta(R + L/2 - |\lambda|) \Theta(|\lambda| - L/2) \delta(|r_{12}.u_r| - (r + R))
\]

\[
\times \left| \frac{d|r_{12}.u_r|}{dt} \right| \Theta \left( - \left| \frac{d|r_{12}.u_r|}{dt} \right| \right) (b_{12} - 1),
\]

where \( b_{12} \) is an operator that converts pre-collisional to post-collisional quantities.

The others terms of the collision operator correspond to the necessary conditions of contact \( \Theta(L/2 - |\lambda|) \delta(|r_{12}.u_1^+| - (r + R)) \), and approach \( \Theta \left( - \left| \frac{d|r_{12}.u_1^+|}{dt} \right| \right) \) in the first case and \( \Theta(R + L/2 - |\lambda|) \Theta(|\lambda| - L/2) \delta(|r_{12}.u_r| - (r + R)) \), and approach \( \Theta \left( - \left| \frac{d|r_{12}.u_r|}{dt} \right| \right) \) in the second case.

By taking the second moments of the distribution function of the discorectangle and after substitution of the collision operator (Eq. (15)), one obtains explicitly for the translational kinetic energy

\[
\int ... \int dv_2 \int dv_1 \int \frac{d\theta_1}{2\pi} \int dv_2 \int dr_2 \hat{T}_{12} f(v_1, \omega_1) \Delta E^T
\]

\[
= \Theta(L/2 - |\lambda|) \delta(|r_{12}.u_1^+| - (r + R))
\]

\[
\times \left| \frac{d|r_{12}.u_1^+|}{dt} \right| \Theta \left( - \left| \frac{d|r_{12}.u_1^+|}{dt} \right| \right) f(v_1, \omega_1) \Delta E^T
\]

\[
+ \Theta(R + L/2 - |\lambda|) \Theta(|\lambda| - L/2) \delta(|r_{12}.u_r| - (r + R))
\]

\[
\times \left| \frac{d|r_{12}.u_r|}{dt} \right| \Theta \left( - \left| \frac{d|r_{12}.u_r|}{dt} \right| \right) f(v_1, \omega_1) \Delta E^T = 0.
\]

A similar equation can be written for rotational kinetic energy. Since the impulse of the collision depends on the location of the impact, it is easy to show that the solution is not a Maxwell distribution function, a property already observed in the model of a needle and points [20]. However, since the deviations from the Maxwell distribution are small, we use it as a trial function. That is we assume that

\[
f(v_1, \omega_1) \propto \exp \left( -\frac{Mv_1^2}{2\gamma_T} - \frac{I\omega_1^2}{2\gamma_R} \right),
\]

where \( \gamma_T \) and \( \gamma_R \) are the ratios of the translational and rotational discorectangle temperatures to the bath temperature, respectively. In summary, in order to obtain the granular temperatures of the discorectangle, it is necessary to: (i) Calculate the change of translational and rotational energy occurring during a collision (ii) Perform an average over all degrees of freedom of the collision integral.

IV. CALCULATION AND RESULTS

A. Energy changes during a collision

When a disk collides with the discorectangle, the change of the translational kinetic energy of the latter
is given by

\[ \Delta E_i^T = \frac{M}{2} ((v_1')^2 - (v_1)^2) \]

\[ = \Delta p \cdot v_1 + \frac{1}{M} \Delta \rho^2 \]

\[ = - (1 + \alpha) \frac{V \cdot u_1^\perp v_1 \cdot u_1^\perp}{\frac{1}{m} + \frac{1}{M} + \frac{\lambda^2}{I}} + \frac{1}{2M} (1 + \alpha)^2 (V \cdot u_1^\perp)^2, \]

(19)

for \( |\lambda| < \frac{L}{2} \), and

\[ \Delta E_i^T = \frac{(1 + \alpha) V \cdot u_r \cdot v_1 \cdot u_r}{\frac{1}{m} + \frac{1}{M} + \frac{\lambda^2 \sin^2 \theta}{I}} \]

\[ + \frac{1}{2M} (1 + \alpha)^2 (V \cdot u_r)^2. \]

(20)

for \( R > |\lambda| - \frac{L}{2} > 0 \).

The collision results in a change of rotational energy, for \( |\lambda| < \frac{L}{2} \),

\[ \Delta E_i^R = \frac{I}{2} ((\omega_1')^2 - (\omega_1)^2) \]

\[ = - \frac{\lambda (1 + \alpha) V \cdot u_1^\perp (\omega_1' + \omega_1)}{\frac{1}{m} + \frac{1}{M} + \frac{\lambda^2}{I}} \]

\[ = - \lambda (1 + \alpha) \frac{V \cdot u_1^\perp \omega_1}{\frac{1}{m} + \frac{1}{M} + \frac{\lambda^2}{I}} \]

\[ + \frac{\lambda^2 (1 + \alpha)^2 (V \cdot u_1^\perp)^2}{2I \left( \frac{1}{m} + \frac{1}{M} + \frac{\lambda^2}{I} \right)^2}. \]

(21)

and for \( R > |\lambda| - L/2 > 0 \),

\[ \Delta E_i^R = - (1 + \alpha) \frac{L \sin \theta V \cdot u_r \cdot \omega_1}{\frac{1}{m} + \frac{1}{M} + \frac{L^2 \sin^2 \theta}{4I}} \]

\[ + \frac{(1 + \alpha)^2 L^2 \sin^2 \theta (V \cdot u_r)^2}{8I \left( \frac{1}{m} + \frac{1}{M} + \frac{L^2 \sin^2 \theta}{4I} \right)^2}. \]

(22)

B. Expressions of the granular temperatures

After inserting Eq. (18) in Eq. (17) it is necessary to perform integrations over each variable in the corresponding equation for the rotational energy. Details of this rather technical task are given in the Appendix B. After some tedious calculation, one obtains the following set of equations

\[ b (cI_{11}^0(a, k) + (1 - c)I_{12}^0(a, k)) = \]

\[ \frac{1 + \alpha}{2} (cI_{03}^0(a, k) + (1 - c)I_{02}^0(a, k)) \]

(23)

\[ a (cI_{11}^1(a, k) + (1 - c)I_{12}^1(a, k)) = \]

\[ \frac{1 + \alpha}{2} (cI_{13}^1(a, k) + (1 - c)I_{12}^3(a, k)) \]

(24)

where

\[ k = \frac{L^2}{4I \left( \frac{1}{m} + \frac{1}{M} \right)}, \]

(25)

\[ a = \gamma_T \frac{M + m}{M + m \gamma_T}. \]

(26)
\[ b = \frac{\frac{M + m}{M + m\gamma_T}}{\gamma_T} \]  
(27)

and

\[ c = \frac{L/2}{L/2 + (r + R)} \]  
(28)

and \( I_m^{np} \) and \( J_m^{np} \) are given by

\[ I_m^{np}(u, v) = \int_0^1 dx \frac{x^{2n}(1 + ux^2)^{p/2}}{(1 + vx^2)^m} \]  
(29)

\[ J_m^{np}(u, v) = \int_0^{\pi} d\theta \frac{\sin^{2n}(\theta)(1 + uv\sin^2 \theta)^{p/2}}{(1 + v\sin^2 \theta)^m} \]  
(30)

Explicit expressions for the integrals appearing in Eqs. 23 and 24 are given in Appendix C. Eq. 24 is an implicit equation for \( a \) that, for a given value of \( \alpha \), can be solved with standard numerical methods. \( b \) is then easily obtained by calculating the ratio of integrals of Eq. 23. Finally, from the values of \( a \) and \( b \), \( \gamma_T \) and \( \gamma_R \) can be obtained from Eqs. 26, 27.

A first check is the elastic case, \( \alpha = 1 \), for which one obtains \( a = b = 1 \) which gives \( \gamma_T = 1 \) and, since \( a/b = \gamma_R/\gamma_T \), \( \gamma_R = 1 \). The temperatures of translational and rotational degrees of freedom are the same and correspond to the bath temperature, a property of an equilibrium system.

In the limit \( R \to 0 \) and \( r \to 0 \), for which \( c \to 1 \) corresponding to a needle in a bath of point particles, one recovers the results of Ref[20]. More interesting is the limit \( R \to 0 \) with \( r \) remaining finite, i.e., a needle in a bath of disks. By using Eq. 25 one obtains that \( c = L/(L + 2r) \). Since \( c < 1 \), unlike the simple needle-point system, there is a contribution resulting from collisions between bath particles and the needle’s tips. This type of collision is more pronounced when the bath particles are larger than the smallest size of the anisotropic particle.

To illustrate the effect of this contribution, Figures 2a and 2b compare the system of a needle in a bath of point particles and the system of a needle in a bath of disks, the radius of disks \( r \) are equal to \( L/4 \) in Fig 2a, and \( r = 9L/2 \) in Fig 2b. It is noticeable that the translational and rotational temperatures of the latter system are smaller than those of the needle in a bath of points, an effect that becomes more pronounced when the radius of the disks becomes larger than the length of the needle. The difference \( \gamma_T - \gamma_R \) are shown in the insets for the needle and the bath of points and for the needle and the bath of disks. Note that the translational temperature is always larger than the rotational temperature and that the difference depends very weakly on the size of the radius of the bath. Additionally, the difference increases when the restitution coefficient decreases from 1 (elastic case), reaches a maximum for a value of \( \alpha \sim 0.3 \) and decreases slightly when the restitution coefficient still decreases.

Figure 3 shows the influence of the anisotropy of the tracer particle. The rotational temperature is still lower than the translational temperature what ever the elongation of the particles, but the effect is greater when the elongation is large. The two upper curves correspond the translational (full curve) and rotational (dashed curve) temperature of the discorectangle when \( L = R \), the two intermediate curve to granular temperatures when \( L = 3R \), and the two lower curves for a discorectangle with \( L = 10R \).

When the anisotropy of the discorectangle approaches zero, i.e., \( L/R \to 0 \), \( c \to 0 \), one can show by using Eqs. 23-24 that

\[ \gamma_T = \frac{1 + \alpha}{2 + \frac{\alpha}{M}(1 - \alpha)}. \]  
(31)

which is the the result of Martin and Piasecki for a spherical tracer particle in a bath of spherical particles. Moreover, the limit \( L/R \to 0 \) leads equipartition between the rotational and translational granular temperatures, a result which is different from a model of pure spherical particles where, in the absence of tangential friction, the particles cannot exchange rotational energy. In physical systems, the particles are never completely spherical and our model shows that if an infinitesimal amount of anisotropy is present, the translational and rotational temperatures are equal in the steady state. Although our analysis provides no quantitative information about the relaxation time to reach this NESS, it is certain to be very long in this limit of small anisotropy.
C. Influence of mass ratio

We consider a homogeneous discorectangle of mass $M$ in a bath of disks each of mass $m$ for which $M \neq m$. When $m/M \to 0$ one obtains, from Eqs. (23)-(24), that

$$\gamma_T = \gamma_R = 1 + \frac{\alpha}{2}$$  \hspace{1cm} (32)

i.e., equipartition between the degrees of freedom of the tracer particle, but not between the bath and the tracer particle. This is, moreover, the same result for a needle in a bath of point particles. We conjecture that this result is general in the sense that we expect equipartition between the different degrees of freedom of the tracer particle in a bath of light particles, whatever the shape of the tracer particle and the dimension of the system. The behavior for finite values of the ratio $m/M$ is shown in Fig. 4. The granular temperatures decrease when the ratio $m/M$ increases, and the translational temperature remains higher than the rotational temperature for each value of $\alpha$.

V. NON UNIFORM RESTITUTION COEFFICIENT

In practice it may be difficult to construct a discorectangle for which the restitution coefficient is constant over the entire perimeter. It is clear, for example, that if the object is composed of a homogeneous viscoelastic material, collisions with the ends will be characterized by a smaller restitution coefficient than collisions with the linear part. This effect becomes more pronounced as the elongation ($L/R$) increases. Other possibilities exist for a non-homogeneous discorectangle composed of two or more materials. For example, a hard material may be used to construct the caps. In addition, the restitution coefficient could depend on the relative velocity of the point of impact \cite{22}, an effect that one neglects here as a first approximation.

As a first approach to describe this possibility, we consider in this section a discorectangle where the restitution coefficient is equal to $\alpha_1$ for a collision along the rectilinear part of the object and equal to $\alpha_2$ for one along the circular part: see Fig. 5. Using the procedure outlined...
above one obtains the following set of closed equations
\[
\begin{align*}
    b \left( (1 + \alpha_1) c I_1^{01}(a, k) + (1 + \alpha_2)(1 - c) J_1^{01}(a, k) \right)
    &= \frac{(1 + \alpha_1)^2}{2} c I_1^{11}(a, k) + \frac{(1 + \alpha_2)^2}{2}(1 - c) J_1^{11}(a, k) \\
    a \left( (1 + \alpha_1) c I_1^{11}(a, k) + (1 + \alpha_2)(1 - c) J_1^{11}(a, k) \right)
    &= \frac{(1 + \alpha_1)^2}{2} c I_2^{11}(a, k) + \frac{(1 + \alpha_2)^2}{2}(1 - c) J_2^{11}(a, k)
\end{align*}
\]

Figure 8 shows the ratio of translational and rotational temperature of a discorectangle with \( R = r \) and \( L = 2R \) to the temperature of the bath as a function of the normal restitution coefficient \( \alpha_2 \) with a fixed \( \alpha_1 = 0.5 \). One notes that the translational temperature becomes smaller than the rotational temperature for \( \alpha_2 > 0.765 \). When the two curves cross, equipartition is recovered, but unlike the limiting cases discussed above of a discorectangle with a uniform restitution coefficient (light bath particles, infinitely small anisotropy), for a non-trivial value of restitution coefficient \( \alpha_2 \) and, moreover, for larger values of \( \alpha_2 \) the ratio of temperatures is inverted.

One can determine in general when equipartition is recovered for a discorectangle with two restitution coefficients. Using the relation \( a = b \) (assumption of equipartition) and Eqs. (33)-(34), one obtains two implicit equations with the three parameters \( \alpha_1, \alpha_2 \) and \( a \). A simple numerical procedure allows us to obtain the \( \alpha_2 \) as a function of \( \alpha_1 \).

Figure 9 shows the equipartition lines in the \((\alpha_1, \alpha_2)\) space. Above each line, corresponding to a given elongation of the rectangle, \( T_R > T_T \) while the reverse inequality applies to the region below the line. It is noticeable that as the elongation increases, the region of the \((\alpha_1, \alpha_2)\) space where \( T_R > T_T \) decreases.

Figure 10 shows the role of the size of the bath particles on the existence of equipartition of a discorectangle of length \( L = SR \). For small bath particles, only a small range of \( \alpha_1 \), (between 0 and \( \sim 0.3 \)) with a smaller range of \( \alpha_2 \) (between \( \sim 0.89 \) and 1) allows equipartition for the tracer particle. For larger bath disks, all values of \( \alpha_1 \) (between 0 and 1) are available with a smaller corresponding range of \( \alpha_2 \).

VI. CONCLUSION

We have investigated the influence of the anisotropy of a tracer particle in a bath of thermalized disks in two dimensions. By using a mean-field approach, we have obtained analytical results for the rotational and translational temperatures. For a homogeneous discorectangle with a uniform normal restitution coefficient, the translational temperature is always higher than the rotational temperature, with the difference depending on the elongation of the tracer particle, the size of the bath particles and the mass ratio.

APPENDIX A: MOMENT OF INERTIA

For a homogeneous discorectangle, the moment of inertia is given by
\[
I_{Oz} = \int \int_S \rho(x^2 + y^2) dxdy
\]

FIG. 7: \( \alpha_1 \) versus \( \alpha_2 \) where equipartition is obtained for \( r = R \) and different values of \( L = R, 2R, \ldots, 5R \) from bottom to top.
and the mass of the system is
\[ M = \int \int_S \rho \, dx \, dy = \rho (\pi R^2 + 2LR). \quad (A2) \]
which gives
\[ I_{Oz} = \rho R \left[ \pi R \left( \frac{R^2}{2} + \left( \frac{L}{2} \right)^2 \right) + \frac{2}{3} L \left( 3R^2 + \left( \frac{L}{2} \right)^2 \right) \right]. \quad (A3) \]
By substituting the density as a function of the total mass of the disc rectangle,
\[ I = M \left[ \frac{2L \left( \sin^2 \left( \frac{\theta}{2} \right) \right)^2}{3} + \pi R \left( \frac{R^2}{2} + \left( \frac{L}{2} \right)^2 \right) \right] \pi R + 2L \]
\[ \frac{2L}{\pi R + 2L} \quad (A4) \]
Two well-known limits are recovered
\[ \lim_{R \to 0} I_{Oz} = \frac{ML^2}{12} \quad (A5) \]
\[ \lim_{L \to 0} I_{Oz} = MR^2 \quad (A6) \]
For an inhomogeneous disc rectangle, the moment of inertia depends on the mass distribution. However, it is possible to determine the lower and upper bounds for allowable values. A trivial lower bound of zero is obtained when the mass is concentrated at the center. Conversely, when the mass is distributed equally at the two extremities of the object (point masses of $M/2$ at a distance of $L + R$ from the center on each side), one obtains
\[ I = M(L/2 + R)^2 \quad (A7) \]
which gives the upper bound for the moment of inertia.

**APPENDIX B: NEEDLE AVERAGE ENERGY LOS**

As for binary mixtures of spheres[3], we use a Gaussian ansatz for the distribution functions and introduce two different temperatures corresponding to the translational and rotational degrees of freedom of the needle. The homogeneous distribution functions of the needle and of the points are then given respectively by
\[ f(v_1, \omega_1) \sim \exp \left( -\frac{Mv_1^2 \gamma_T^{-1}}{2T} - \frac{I \omega_1^2 \gamma_R^{-1}}{2T} \right), \quad (B1) \]
\[ \Phi(v_2) \sim \exp \left( -\frac{mv_2^2}{2T} \right), \quad (B2) \]
where $T$ is the temperature of the bath, $\gamma_T$ and $\gamma_R$ the ratio of the translational (and rotational) temperature of the needle to the bath temperature.

We introduce the vectors $\chi$ and $\nu$ such that
\[ \chi = \frac{1}{\sqrt{2T(M\gamma_T + m)}} (MV_1 + MV_2) \quad (B3) \]
\[ \nu = \frac{mM}{\sqrt{2T(M\gamma_T + m)\gamma_T}} (v_1 - \gamma_T v_2) \quad (B4) \]
The scalar products $V \cdot u_1^+$ and $V \cdot u_r$ can be expressed as
\[ V \cdot u_1^+ = h \left[ (\gamma_T - 1) \chi \cdot u_1^+ + \sqrt{T} \left( \sqrt{\frac{m}{M}} + \sqrt{M \frac{m}{m}} \right) \nu \cdot u_1^+ \right] \]
\[ + \omega_1 \lambda \quad (B5) \]
\[ V \cdot u_r = h \left[ (\gamma_T - 1) \chi \cdot u_r + \sqrt{T} \left( \sqrt{\frac{m}{M}} + \sqrt{M \frac{m}{m}} \right) \nu \cdot u_r \right] \]
\[ + \omega_1 \frac{L}{2} \sin \theta \quad (B6) \]
where $h = \frac{\sqrt{2T}}{\sqrt{M \gamma_T + m}}$.

Let us introduce $\xi = \omega_1 \sqrt{\frac{L}{2 \gamma_T}}$. The translational energy loss is given by the formula
\[
\sum_{p = \pm 1} \left[ \int d\lambda \int \frac{d\theta_1}{2\pi} \int d\chi \int d\nu \int d\xi \exp(-\chi^2 - \nu^2 - \xi^2) |V \cdot u_1^+| \Theta(pV \cdot u_1^+) \Theta(L \frac{L}{2} - |\lambda|) \Delta E_1^T + (R + r) \int d\theta \int \frac{d\theta_1}{2\pi} \int d\chi \int d\nu \int d\xi \exp(-\chi^2 - \nu^2 - \xi^2) |V \cdot u_r| \Theta(pV \cdot u_r) \Delta E_1^T = 0 \quad (B7) \]
Since Eq. \(19\) depends only on \(\chi, u^\perp_1\) and \(\nu, u^\perp_1\), one can freely integrate over the direction of \(u_1\) for the vectors \(\chi\) and \(\nu\), and similarly for Eq. \(18\). The integration over \(\theta_1\) can be easily performed. If we introduce the three dimensional vectors \(G_{u^\perp_1}\) and \(s_{u^\perp_1}\) with components:

\[
G_{u^\perp_1} = (G_1, G_2, G_3) = \left(\sqrt{\frac{2T}{M\gamma_T + m}}(\gamma_T - 1), \sqrt{\frac{2T\gamma_T}{M\gamma_T + m + M}}\lambda \sqrt{\frac{2T\gamma_T}{T}}\right) \tag{B8}
\]

and

\[
s_{u^\perp_1} = (s_1, s_2, s_3) = (\chi, u^\perp_1, \nu, u^\perp_1, \xi) \tag{B9}
\]

Respectively, one has introduces \(G_{u^\perp_r}\) and \(s_{u^\perp_r}\) vectors associated with collisions on the circular parts of the dis-rectangle by changing \(u^\perp_1\) in \(u_r\) and \(\lambda\) by \(\frac{L_s}{2}\sin \theta\). By inserting Eq. \(10\) in Eq. \(7\), the average energy loss can be rewritten as

\[
\sum_{p = \pm 2} \int_{-L/2}^{L/2} d\lambda \int ds_{u^\perp_1} \exp(-s^2) |G_{u^\perp_1} . s_{u^\perp_1}| \Theta(p G_{u^\perp_1} . s_{u^\perp_1})
\]

\[
\left[\frac{1}{2M} \left(\frac{1 + \alpha)^2}{m + 1} + \frac{\lambda^2}{\pi}\right)^2 \right]
\]

\[
\left(\frac{1}{1 + \frac{\lambda}{\pi}} + \frac{\lambda^2}{\pi}\right) \int \frac{2T}{M\gamma_T + m} \left(\gamma_s^2 + \frac{m\gamma_T}{M} s_2\right)
\]

\[
(R + r) \int_0^{2\pi} d\theta \int ds_{u_r} \exp(-s^2) |G_{u_r} . s_{u_r}| \Theta(p G_{u_r} . s_{u_r})
\]

\[
\left[\frac{1}{2M} \left(\frac{1 + \alpha)^2}{m + 1} + \frac{\lambda^2}{\pi}\right)^2 \right]
\]

\[
- \left(\frac{1 + \alpha}{m + 1 + \frac{\lambda^2}{\pi}} \right) \frac{2T}{M\gamma_T + m} \left(\gamma_s^2 + \frac{m\gamma_T}{M} s_2\right) = 0. \tag{B12}
\]

By defining a new coordinate system in which the \(z\)-axis is parallel to \(G\), one find that the integrals of Eq. \(B12\) involve gaussian integrals of the form

\[
\int ds \exp(-s^2) (|G|s_2)^2 \Theta(\pm s_2) G_i s_2 = \frac{\pi}{2} |G|^2 G_i \tag{B13}
\]

and

\[
\int ds \exp(-s^2) (|G|s_2)^3 \Theta(\pm s_2) = \frac{\pi}{2} G^3 \tag{B14}
\]

which finally leads to Eq. \(20\). The equation for rotational energy is derived following exactly the same procedure.

**APPENDIX C: INTEGRALS**

The coupled equations \(39\), \(41\) depend on the eight integrals \(J_0^1(a, k), J_0^3(a, k), J_1^1(a, k), J_1^3(a, k), J_2^0(a, k), J_2^2(a, k), J_2^3(a, k), J_3^1(a, k), J_3^3(a, k)\) which can be expressed in terms of transcendental and special functions. For completeness, we give below their expressions.

For completeness, we give below their expressions.
where $E(x)$ denotes the complete elliptic integral of the second kind.

\[
J_{11}^{11}(a, k) = -\frac{a}{k\sqrt{(1+ak)}}K\left(\sqrt{\frac{ak}{1+ak}}\right) + \frac{\sqrt{1+ak}}{k}E\left(\sqrt{\frac{ak}{1+ak}}\right) - \frac{1-a}{k(k+1)\sqrt{(1+ak)}}\Pi\left(\frac{k}{k+1}, \sqrt{\frac{ak}{1+ak}}\right)
\]

and finally

\[
J_{13}^{13}(a, k) = \frac{1+2ak-3a^2-4a^2k}{2k(k+1)\sqrt{1+ak}}K\left(\sqrt{\frac{ak}{1+ak}}\right) + \frac{\sqrt{1+ak(2ak+3a-1)}}{2k(k+1)}E\left(\sqrt{\frac{ak}{1+ak}}\right) + \frac{k+4a^2k-3a+3a^2-5ak}{2k(k+1)^2\sqrt{(1+ak)}}\Pi\left(\frac{k}{k+1}, \sqrt{\frac{ak}{1+ak}}\right)
\]

\[\text{(C7)}\]

\[\text{(C8)}\]

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