Critical temperature of non-interacting Bose gases on disordered lattices

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Abstract. For a non-interacting Bose gas on a lattice we compute the shift of the critical temperature for condensation when random-bond and on-site disorder are present. We evidence that the shift depends on the space dimensionality $D$ and the filling fraction $f$. For $D \to \infty$ (infinite range model), using results from the theory of random matrices, we show that the shift of the critical temperature is negative, depends on $f$, and vanishes only for large $f$. The connections with analogous results obtained for the spherical model are discussed. For $D = 3$ we find that, for large $f$, the critical temperature $T_c$ is enhanced by disorder and that the relative shift does not appreciably depend on $f$; in contrast, for small $f$, $T_c$ decreases, in agreement with the results obtained for a Bose gas in the continuum. We also provide numerical estimates for the shift of the critical temperature due to disorder induced in a non-interacting Bose gas by a bichromatic incommensurate potential.

Keywords: Bose Einstein condensation (theory), disordered systems (theory), optical lattices
1. Introduction

The ideal Bose gas (IBG) is well studied in quantum statistical mechanics [1]–[3] since it provides invaluable qualitative insights into the properties of realistic bosonic systems. Its thermodynamics properties can be explicitly worked out and have been intensively studied both in the thermodynamical limit and in finite systems\(^5\). Nowadays, the study of the IBG has more than just academic interest, since recent progress in manipulating cold atomic systems [5,6] allows for a reliable control of the relative strength of the kinetic and interaction energies: for instance, the possibilities of varying the scattering length through Feshbach resonances [5,6] and superimposing optical lattices (obtaining effective lattice Hamiltonians [7,8]) provide a tool for testing a Bose gas from the strongly interacting regime to the (almost) non-interacting one. In [9] it has been shown that it is possible to tune the s-wave scattering length \(a\) of a \(^{87}\)Rb Bose gas with high precision, and set it almost to zero\(^6\): this allows for long-lived Bloch oscillations in an optical lattice [11] and allows one to perform atom interferometry [12]. When an optical lattice is superimposed

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\(^5\) See a pedagogical treatment of the ideal Bose gas in a finite system in [4].

\(^6\) A small amount of dipolar interaction is still present, but one can further tune \(a\) in order to partially compensate for such residual interaction, as discussed in [10].
on an IBG, for large values of the laser power, the system is described by a tight-binding bosonic model.

The IBG has also been used to understand the effect of disorder [13] and the role played by confining tailorable external potentials, relevant in investigations of trapped ultracold atoms [5, 6]. Studying the effects of controllable or random impurities is again motivated by the huge progress made in the manipulation of ultracold bosonic gases, culminating in the recent experimental observation of Anderson localization for matter waves in a random potential [14, 15]. In a Bose gas, disorder can be induced either by laser speckle [16, 17] (i.e., a laser beam is shone through a diffusive plate and superimposed on the trapping potential) or by an incommensurate bichromatic potential [18] (i.e., an auxiliary lattice is added to the main lattice, with an incommensurate ratio between the two frequencies). Random on-site disorder in the strongly interacting limit induces a Bose glass phase [19, 20], whose characterization has been the object of many theoretical [21–30] and experimental [18] investigations. Finally, ultracold atoms provide a good experimental setup for studying superfluidity in random environments, which is an issue related to the adsorption of $^4$He in porous media [20], [31–36].

The IBG is very interesting also in relation to statistical mechanics models; indeed, the IBG is in the universality class [37] of the spherical model (SM) [38], which displays a phase transition in dimensions larger than or equal to 3. The SM is solvable in any dimension also with an external magnetic field [38–40], and it provides an interesting laboratory in which to investigate a variety of topics, such as long-range interactions [41] and finite size scaling [42]; furthermore, it is the $n \rightarrow \infty$ limit of the $O(n)$ model [43] and, thus, is the leading term of $1/n$ expansions [44]. With disorder, the spin-glass SM with infinite range interactions has been studied intensively [45, 46] and it has been used to analyze the properties of relaxor ferroelectrics [47].

When there is disorder, non-interacting bosons on a lattice are described by the Hamiltonian

$$\hat{H} = -t \sum_{i,j} \hat{A}_{ij} \hat{b}_i^{\dagger} \hat{b}_j + \sum_i \epsilon_i \hat{b}_i^{\dagger} \hat{b}_i, \quad (1)$$

with

$$\hat{A}_{ij} = A_{ij}(1 + c_{ij}). \quad (2)$$

In equations (1) and (2), the lattice sites are denoted by $i, j$, $t$ is the rate of tunneling between neighboring sites, $A_{ij}$ is the adjacency matrix characterizing the lattice (i.e., $A_{ij} = 1$ if $i$ and $j$ are nearest neighbor sites, and 0 if they are not) and $\hat{b}_i, \hat{b}_i^{\dagger}$ are the bosonic operators. The number of sites on a cubic lattice of dimension $D$ is $N_S = L^D$, while the total number of particles is $N_T$. The filling (i.e. the average number of particles per site) is given by

$$f = \frac{N_T}{N_S}. \quad (3)$$

Random on-site disorder is accounted for by $\epsilon_i$, while random-bond disorder is included in the definition of $A_{ij}$ through $c_{ij}$; in the following, $\epsilon_i$ and $c_{ij}$ are regarded as random variables with vanishing average and variance $\sigma^2 \approx 1$, and $c_{ij}$ respectively. We restrict ourself to situations where $t_{ij} \equiv t A_{ij} > 0 (\sigma^2 \lesssim 1)$, since they arise naturally when bosonic lattices...
are disordered by inserting a speckle or by adding an incommensurate optical lattice: in these situations both random-bond disorder $c_{ij}$ and random on-site energies $\epsilon_i$ occur and, typically, $v_o \gtrsim v$. The inclusion of random-bond disorder also allows one to make contact with interesting properties of the spin-glass SM [45, 46].

In this paper we investigate how the critical temperature of a lattice IBG is shifted by the presence of disorder, elucidating the dependence of this shift on the filling. For the continuous Bose gas, the shift in the critical temperature has been derived in [35]. In that paper it is shown that the critical temperature decreases with disorder: as we shall see, the lattice computation reproduces this behavior for small filling, but, on the lattice, different behaviors are possible as a result of the interplay between discreteness and disorder. Indeed, on a lattice, there are two competing effects, since disorder broadens the distribution of the delocalized eigenstates of Hamiltonian (1) and further localizes the ground-state wavefunction: the former (latter) effect tends to lower (increase) the critical temperature. For $D \to \infty$ (i.e., the infinite range model), the analysis can be carried out analytically and one finds that the two effects exactly balance at large filling while, for small filling, the critical temperature decreases. For $D = 3$ the balance is obtained for intermediate filling while, at large (small) filling, the critical temperature increases (decreases); the numerical coefficient that, for small disorder and large filling, determines the critical temperature increase is much smaller for random on-site disorder.

The plan of the paper is the following. In section 2 we determine the critical temperature of non-interacting bosons on a three-dimensional lattice; there, we analyze the limit of large filling and the connection with the SM. Section 3 is devoted to the analysis of the infinite range model without and with disorder. In section 4 we study three-dimensional lattices with random-bond disorder and random on-site energies; in addition, we exhibit numerical results for the shift of the critical temperature due to disorder induced by a bichromatic incommensurate potential. Our concluding remarks are in section 5.

2. Non-interacting bosons on a lattice

In this section we determine the critical temperature for condensation of bosons hopping without disorder on a three-dimensional lattice. We shall show also that, at large filling, the value of this temperature coincides with the one of a mean spherical model.

2.1. Hopping bosons without disorder

In the absence of disorder ($\epsilon_i = 0$ and $c_{ij} = 0$, $\forall i, j$), the Hamiltonian (1) reduces to the boson-hopping Hamiltonian $\hat{H}^{(0)}$:

$$\hat{H}^{(0)} = -t \sum_{i,j} A_{ij} \hat{b}_i^\dagger \hat{b}_j.$$  (4)

By setting $\hat{d}_k = N^{-1/2} \sum_j \hat{b}_j e^{i k \cdot j}$, $\hat{H}^{(0)}$ may be readily diagonalized; for $D = 3$ one gets

$$\hat{H}^{(0)} = \sum_k E_k \hat{d}_k^\dagger \hat{d}_k$$  (5)

with

$$E_k = -2t \left( \cos k_x + \cos k_y + \cos k_z \right).$$  (6)

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We assumed periodic boundary conditions, so that \( k_\ell = 2\pi n_\ell / L \) with \( n_\ell = 0, \ldots, L - 1 \) \((\ell = x, y, z)\).

In computing the grand-canonical partition function, one adds the chemical potential \( \mu \) enforcing the conservation of the total number of particles; as a result, one replaces \( \hat{H}^{(0)} \) with \( \hat{H}^{(0)} - \mu \hat{N} \), where \( \hat{N} = \sum_i \hat{b}_i \hat{b}_i^\dagger = \sum_k \hat{d}_k \hat{d}_k^\dagger \) is the operator counting the total number of particles. Requiring \( \langle \hat{N} \rangle = N_T \) yields

\[
\sum_k \frac{1}{e^{\beta(E_k - \mu)} - 1} = N_T, \tag{7}
\]

which fixes the chemical potential \( \mu \) as a function of \( T \) and \( N_T \). In the thermodynamical limit, \( N_S, N_T \to \infty \) at fixed filling \( f \), equation (7) reads \( \int_{BZ} \left( 1 / (2\pi)^3 \right) (1 / e^{\beta(E_k - \mu)} - 1) = f \), where \( \int_{BZ} \) denotes the integral over the first Brillouin zone: \( \int_{BZ} \left( 1 / (2\pi)^3 \right) d\mathbf{k} \). The critical temperature in the absence of disorder is denoted by \( T_c^{(0)} \) and it is defined by the condition \( \beta_c^{(0)} \equiv \mu_c = E_0 \), where \( E_0 = -6t \) is the ground-state energy:

\[
\int_{BZ} \frac{1}{(2\pi)^3} \frac{1}{e^{\beta(E_k - \mu_c)} - 1} = f \tag{8}
\]

with \( \beta_c^{(0)} = 1/k_B T_c^{(0)} \).

A simple estimate of \( T_c^{(0)} \) may be obtained by keeping only the lowest order of the Taylor expansion of the exponential in equation (8). One gets

\[
\frac{1}{\beta_c^{(0)}} \int_{BZ} \frac{1}{(2\pi)^3} \frac{1}{E_k - \mu_c} \approx f \tag{9}
\]

yielding

\[
k_B T_c^{(0)} \approx \frac{6tf}{W(1)}. \tag{10}
\]

In equation (10) \( W(1) \approx 1.516386 \) and

\[
W(z) \equiv \int_{BZ} \frac{1}{(2\pi)^3} \frac{1}{1 - (1/3)z} \sum_{\ell=1}^3 \cos k_\ell \tag{11}
\]

is the three-dimensional generalized Watson’s integral [40]. Some useful properties of the generalized Watson’s integral are grouped in appendix A.

In order to show that equation (10) provides indeed an accurate estimate of \( T_c^{(0)} \) for large filling, it is most convenient to write the exponential in equation (8) as a geometric series; this yields

\[
f = \sum_{j=1}^{\infty} e^{-6jX_0} [I_0(2jX_0)]^3, \tag{12}
\]

where \( X_0 = \beta_c^{(0)} t \) and \( I_0 \) is the modified Bessel function of the first kind [48]. When \( X_0 \ll 1 \), one can approximate the series in (12) with an integral and—for \( f \gg 1 \)—one obtains \( f \simeq (1/6X_0) \int_0^\infty dx e^{-x}[I_0(x/3)]^3 \). From equation (A.2), one readily gets equation (10).

Equation (12) is very useful for numerically computing \( X_0 \) as a function of \( f \) (or vice versa). Indeed, although the series in (12) is slowly convergent, it is always possible to
choose a large enough integer \( N \) so that equation (12) may be written as

\[
f = \sum_{j=1}^{N} e^{-6jX_0} [I_0 (2jX_0)]^3 + \mathcal{R}(N),
\]

where the remainder of the series \( \mathcal{R} \) may be approximated (with arbitrary precision for a pertinent choice of \( N \)) as

\[
\mathcal{R}(N) \simeq \frac{1}{(4\pi X_0)^{3/2}} \left( \frac{3}{2} \zeta \left( \frac{3}{2} \right) - \sum_{j=1}^{N} \frac{1}{j^{3/2}} \right).
\]

In equation (14) the Riemann zeta function \( \zeta \) and the asymptotic expansion of the modified Bessel function of the first kind \( I_0(x) \simeq e^x / \sqrt{2\pi x} \) valid for large \( x \) [48] have been used.

In figure 1 we plot the ratio \( Q \) between the critical temperature \( T_c^{(0)} \), determined numerically from equations (13) and (14), and the large filling critical temperature \( 6tf/k_B W(1) \) given by equation (10), as a function of the filling \( f \). The solid line corresponds to the \( 1/f \) expansion, which is found to be

\[
Q \equiv \frac{k_B T_c^{(0)}}{6tf/W(1)} \approx 1 + \frac{1}{2f}.
\]

Notice that, at finite filling, \( T_c^{(0)} \) is larger than the critical temperature given for large filling in equation (10).

We conclude this section by observing that, using [49]

\[
\lim_{\tau \to 0} \sum_{n=\pm \infty} \frac{e^{-i\omega_n \tau}}{i\omega_n - x} = \frac{-\beta}{e^{\beta x} - 1}
\]
with \( \beta = \beta_c(0) \) (the sum is over all the Matsubara frequencies \( \omega_n = 2\pi n / \beta_c(0) \) and \( \hbar = 1 \)), equation (8) may be written as
\[
f = k_B T_c(0) \lim_{\tau \to 0^-} \sum_{n=-\infty}^{\infty} e^{-i\omega_n \tau} \int_{\text{BZ}} \frac{dk}{(2\pi)^3} \frac{1}{E_k - \mu_c - i\omega_n},
\]
which is valid for all values of \( f \). From equation (16), equation (9) may be obtained by taking only the contribution from the lowest Matsubara frequency \( \omega_0 \). Contributions from higher Matsubara frequencies \( (n = \pm 1, \pm 2, \ldots) \) lead to a non-linear equation relating \( T_c(0) \) and \( f \).

As we shall see, equation (10) provides also the critical temperature of the SM discussed next.

### 2.2. Correspondence with the mean spherical model

The Hamiltonian of the SM reads
\[
H = -t \sum_{\langle i,j \rangle} S_i S_j,
\]
where the values of the (classical) spins are real and satisfy the constraint \( \sum_i S_i^2 = N_S \).

In the original solution of the SM [38], the constraint is taken into account by writing the partition function as
\[
Z = \int \prod_i dS_i e^{-\beta H \delta(\sum_i S_i^2 - N_S)}.
\]

In contrast, in the so-called mean SM the constraint is satisfied only on average: upon introducing the grand-canonical Hamiltonian \( K = H - \mu(\sum_i S_i^2 - N_S) \), one only requires \( \langle \sum_i S_i^2 \rangle = N_S \), where \( \langle \cdots \rangle \) denotes the thermal average with respect to \( K \).

In order to compare the results from the mean SM with the ones from an IBG with definite filling \( f \), it is most convenient to write the constraint as
\[
\sum_i S_i^2 = f N_S = N_T.
\]

The grand-canonical Hamiltonian is
\[
K = H - \mu \left( \sum_i S_i^2 - f N_S \right),
\]
while the partition function is given by
\[
Z = \int \prod_j dS_j e^{-\beta K}.
\]

For a cubic three-dimensional lattice, performing the Gaussian integration in equation (20) yields
\[
Z = e^{-\beta \mu f N_S} \prod_k \sqrt{\frac{2\pi}{\beta (E_k - 2\mu)}}.
\]

Minimizing the free energy \( F = -k_B T \ln Z \) with respect to \( \mu \) leads to
\[
1 = \frac{k_B T}{f N_S} \sum_k \frac{1}{E_k - 2\mu}.
\]
from which, in the thermodynamical limit, one gets

$$f = k_B T \int_{BZ} \frac{1}{(2\pi)^3 E_k - 2\mu}.$$  \hspace{1cm} (23)

The critical temperature $T_c^{(0)}$ is given by the condition $\mu(T_c^{(0)}) = E_0/2$ yielding

$$k_B T_c^{(0)} = \frac{6tf}{W(1)}. \hspace{1cm} (24)$$

Equation (24) coincides with (10), which has been obtained for the lattice IBG for large filling.

3. Infinite range model

In this section we investigate the $D \to \infty$ limit of the boson-hopping Hamiltonian (1) without and with random-bond disorder. The main advantage of the infinite range model lies both in its solvability and in the insights that it provides for the behavior of a realistic three-dimensional lattice model.

3.1. Without disorder

Let us start by considering the infinite range boson-hopping model ($A_{ij} = 1$ if $i \neq j$). Its Hamiltonian is given by

$$\hat{H}^{(0)} = -\frac{t}{N_S} \sum_{i \neq j} \hat{b}_i^\dagger \hat{b}_j.$$ \hspace{1cm} (25)

The single-particle energies are straightforwardly computed: one finds a non-degenerate ground-state energy $E_0 = -t(1 - 1/N_S)$ and an excited state $E_1 = t/N_S$ with degeneracy $N_S - 1$. Conservation of the total number of particles yields

$$N_T = \frac{1}{e^{\beta E_0 - \mu} - 1} + \frac{N_S - 1}{e^{\beta E_1 - \mu} - 1}. \hspace{1cm} (26)$$

As usual, the critical temperature $T_c^{(0)}$ is obtained from the condition $\mu(T_c^{(0)}) = E_0$ which, in the thermodynamical limit, yields

$$k_B T_c^{(0)} = \frac{t}{\ln (1 + (1/f))}. \hspace{1cm} (27)$$

For $f \gg 1$, equation (27) leads to

$$k_B T_c^{(0)} \approx tf. \hspace{1cm} (28)$$

A plot of the ratio $Q$ between the critical temperature $T_c^{(0)}$ (given by equation (27)) and the large filling critical temperature (given by equation (28)) is provided in figure 2. The corrections to the large filling limit are of order $1/f$; indeed

$$Q = \frac{k_B T_c^{(0)}}{tf} = \frac{1}{f \ln (1 + (1/f))} \approx 1 + \frac{1}{2f}. \hspace{1cm} (29)$$

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Figure 2. Plot of the ratio between the critical temperature $T_c^{(0)}$ and $tf/k_B$ as a function of the filling $f$ (solid line). The dashed line corresponds to the expansion at order $1/f$ given by equation (29).

As for the three-dimensional lattice, the critical temperature at finite filling given by equation (27) is larger than the critical temperature determined for large $f$ in equation (28); it turns out that the $1/f$ correction has the same coefficient as in equation (15). From figure 2 one sees that the expansion (29) is rather good also for intermediate filling, $f \gtrsim 1$.

To conclude this section, we observe that if one adds a diagonal term to the Hamiltonian (25) ($A_{ii} = 1$), one still finds that the critical temperature is determined by equation (27).

3.2. With random-bond disorder

In the presence of random-bond disorder the Hamiltonian is obtained from equation (25) replacing $A_{ij}$ with $\tilde{A}_{i,j}$. Namely,

$$\hat{H} = -\frac{t}{N_S} \sum_{i,j} (1 + c_{ij}) \hat{b}_i^\dagger \hat{b}_j - \sum_{i,j} J_{ij} \hat{b}_i^\dagger \hat{b}_j,$$

(30)

where, as in [45], $J_{ij}$ is a real symmetric matrix with elements normally distributed, having average value $t/N_S$ and variance $J^2/N_S$ (but variance $2J^2/N_S$ on the diagonal). We set

$$v = \frac{J}{t}$$

(31)

and we limit ourselves to $v < 1$. The Hamiltonian (30) provides a mean-field theory description of the bond disordered boson-hopping Hamiltonian discussed in section 4.

In this section we compute the dependence on the filling $f$ of the shift of the critical temperature, $\delta T_c \equiv T_c - T_c^{(0)}$, with $T_c^{(0)}$ defined in (27). We find that while $\delta T_c < 0$ for finite $f$, we have $\delta T_c \to 0$ for $f \to \infty$.

In the infinite range model the density of states can be obtained through an orthogonal transformation mapping the problem to that of a single local impurity in a crystal [45, 50]; one obtains the well-known semi-circular law [51] and a single isolated impurity state which...
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Figure 3. Solid line: density of states $\rho(E)$ (in units of $1/\pi J$) as a function of $E$ (in units of $t$) for the infinite range model with $v = 0.25$—the ground-state energy (33) is represented by a vertical solid line. The eigenvalues $E = 0$ and $-t$ in the absence of disorder ($v = 0$) are indicated by vertical dashed lines.

The same result can be obtained by the replica trick [52]. The density of states $\rho(E)$ is given by

$$\rho(E) = \rho_0(E) + \frac{1}{N_S} \delta(E - E_0),$$

where

$$E_0 = -t(1 + v^2).$$

The density of states $\rho_0(E)$ is given by $\rho_0(E) = 0$ for $|E| > 2J$ and

$$\rho_0(E) = \frac{1}{2\pi J^2} \sqrt{4J^2 - E^2}$$

for $|E| < 2J$. A plot of the spectrum is provided in figure 3: one sees that the ground-state energy decreases (this induces an increase of the critical temperature $T_c$ for the bond disordered model), while there is a spreading in the distribution of the continuous spectrum (instead, this induces a decrease of $T_c$). The shift in the critical temperature is a result of the competition of these two effects.

Using equation (32) one finds that the critical temperature is determined by ($\beta_c = \frac{1}{k_B T_c}$)

$$f = \frac{1}{2\pi J^2} \int_{-2J}^{2J} dE \frac{\sqrt{4J^2 - E^2}}{e^{\beta_c (E - E_0)} - 1},$$

with $E_0$ given by equation (33). Defining $X = \beta_c t$ and $x = E/2J$, equation (35) becomes

$$f = \frac{2}{\pi} \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{e^{X (1 + v^2) + 2X v x} - 1}.$$
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Figure 4. Shift of the critical temperature (in units of $v^2T_c^{(0)}$) as a function of the filling according to equation (40). $T_c^{(0)}$ is the critical temperature for the lattice without bond disorder and it is given by equation (27).

In order to compute $\delta T_c$ for $f \gg 1$, one can, as in section 2, Taylor expand the exponential in the r.h.s. of equation (36) to get

$$f \simeq \frac{1}{\pi \beta_c tv} \int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{x + (1 + v^2)/2v} = \frac{1}{\beta_c t}. \quad (37)$$

From equation (28), one immediately sees that $\delta T_c \simeq 0$.

For any finite value of the filling $f$ and for small $v$, $\delta T_c$ may be computed from the series expansion of the integrand of equation (36); one obtains

$$f \simeq \frac{1}{e^X - 1} + v^2 \frac{Xe^X}{(e^X - 1)^2} \left( \frac{Xe^X}{e^X - 1} - \frac{X}{2} - 1 \right). \quad (38)$$

Upon defining $X_0 = \beta_c^{(0)} t = \ln (1 + 1/f)$ and putting $X = X_0 + \delta X$, one may readily linearize equation (38) around $X_0$. One finds

$$\delta X \simeq v^2 X_0 \left( \frac{X_0e^{X_0}}{e^{X_0} - 1} - \frac{X_0}{2} - 1 \right), \quad (39)$$

from which one gets

$$\frac{\delta T_c}{T_c^{(0)}} \simeq -v^2 \left[ \left( f + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{f} \right) - 1 \right]. \quad (40)$$

Thus $\delta T_c < 0$.

Equation (40) is plotted in figure 4. In figure 5 we compare the result for $\delta T_c$ obtained from equation (40) with the numerical solution of equation (35) for different values of the filling and the disorder. Although equation (40) is valid for small $v$, a rather good agreement is found also for relatively large values of the disorder parameter $v$. 

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Figure 5. Shift of the critical temperature $\delta T_c$ (in units of $T_c^{(0)}$) as a function of the disorder $v$: circles (triangles) are obtained from the numerical solution of equation (35) for $f = 1$ ($f = 0.1$)—solid lines: analytical prediction equation (40). Inset: $\delta T_c/T_c^{(0)}$ from equation (40) versus the filling $f$ for $v = 0.25$ (squares) and $v = 0.5$ (filled circles)—solid lines: equation (40).

In order to compute $f$ to any order in $v$, one observes that equation (36) can be written as

$$f = 2 \sum_{p=1}^{\infty} \frac{I_1(2vXp)}{2vXp} e^{-X(1+v^2)p},$$

where $I_1$ is a modified Bessel function of the first kind [48]. Expanding both $I_1(2vXp)$ and $e^{-X(1+v^2)p}$ in terms of $v$ and using the definition of polylogarithm function, $\text{Li}_n(x) = \sum_{p=1}^{\infty} \frac{x^p}{p^n}$, one gets

$$f = \sum_{k,n=0}^{\infty} \left[ \frac{(-1)^n X^{2k+n}}{n!k!\Gamma(k+2)} \text{Li}_{-(2k+n)}(e^{-X}) \right] v^{2k+2n},$$

where $\Gamma$ is the Gamma function. It is easy to verify that, up to the order $v^2$, equation (42) reduces to equation (38).

We conclude this section by noticing that, for the $D \rightarrow \infty$ SM, the effect of random-bond disorder was worked out long ago [45]: for disorder smaller than a critical value, i.e. for $v < 1$, the authors of [45] found that the critical temperature $T_c$ of the ferromagnetic–paramagnetic transition does not get shifted as a result of bond disorder. Our result remarkably parallels the result obtained in [45].

4. Three-dimensional hopping bosons with bond and on-site disorder

In this section we determine $\delta T_c$ for non-interacting bosons for $D = 3$ with random-bond and on-site disorder. The Hamiltonian is given in equation (1), where $c_{ij}$ ($\epsilon_i$) is a Gaussian random variable with zero average and variance $v^2$ ($\nu_0^2 t^2$). In the following we use the replica trick [46, 53]: we shall determine the Green functions of the disordered system from
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Figure 6. Two-particle vertices for the disorder induced effective interaction.

the replicated action, where disorder enters as an effective attractive interaction between replicas. As usual, one introduces $N$ replicas and, after averaging the partition function on the disorder, performs the limit $N \to 0$. In the following, for the sake of simplicity, we separately consider the effect of random-bond and on-site disorder.

4.1. Random-bond disorder

A standard analysis (reported in appendix B) allows us, after integrating over disorder, to derive an effective replica action as

$$S_{\text{eff}} = \sum_\alpha \int_0^\beta d\tau \left\{ \sum_i \varphi_i^{*\alpha}(\tau) \left( \frac{\partial}{\partial \tau} - \mu \right) \varphi_i^{\alpha}(\tau) - \frac{t}{4} \sum_{ij} A_{ij} \int_0^\beta d\tau \left( \varphi_i^{*\alpha}(\tau) \varphi_j^{\alpha}(\tau) \varphi_i^{*\gamma}(\tau') \varphi_j^{\gamma}(\tau') \right) \right\}$$

$$+ \frac{v^2 t^2}{4} \sum_{ij\alpha\gamma} A_{ij} \left[ \varphi_i^{*\alpha}(\tau) \varphi_j^{\alpha}(\tau) \varphi_i^{*\gamma}(\tau') \varphi_j^{\gamma}(\tau') \right]$$

The fields $\varphi$ represent Bose particles and $\alpha, \gamma = 1, \ldots, N$ are the replica indices. As one can see from (43), bond disorder induces an effective interaction with the two-particle vertex functions depicted in figure 6. Equation (43) is derived under the assumption that $\tilde{A}_{ij} = \tilde{A}_{ji}$ for any realization of disorder.

Defining the Green’s functions

$$G_{ij}(z) = \langle \varphi_i \varphi_j^* \rangle_{v=0} = \sum_k \frac{\delta^{(1)}(i-j)}{E_k - z},$$

with $z = \mu + i\omega_n$ and performing a perturbative evaluation of equation (45) up to the first order in $v^2$, one gets for $N \to 0$,

$$G_{lm} = G_{lm} + \frac{v^2 t^2}{2} \sum_{ij} A_{ij} \left[ G_{il} G_{ij} G_{jm} + G_{ij} G_{jm} G_{il} \right] + O(v^4).$$

Upon introducing the self-energy function $\Sigma(k, z)$, one may write the Dyson equation in Fourier space as follows:

$$G^{-1}(k, z) = G^{-1}(k, z) + [\Sigma(k, z) - \delta\mu],$$

Notice that, before taking the limit, the expression for $G$ is analogous to the standard spinful case where instead of the spin $s$ we have $N$ flavors: $G_{lm} = G_{lm} + (v^2 t^2/2) \sum_{ij} A_{ij} \left[ (2N+1)G_{il} G_{ij} G_{jm} + G_{ij} G_{jm} G_{il} \right] + O(v^4).$
Equation (56) shows that, as a result of the bond disorder, the bandwidth is increased:

\[ G^{-1}(k, z) = E_k - z \]  

and \( \delta \mu \), the shift of the chemical potential, is defined as

\[ \delta \mu \equiv \Sigma(k = 0, z = \mu). \]  

Bose–Einstein condensation occurs when

\[ G(k, z)^{-1}|_{k=0, z=\mu} = -6t - \mu + [\Sigma(0, \mu) - \delta \mu] = 0 \]  

which is solved for

\[ \mu_c = -6t; \]

thus, for bond disorder, the chemical potential at \( T_c \) is given by \( \mu_c + \delta \mu_c = -6t + \Sigma(0, -6t) \).

Upon Fourier transforming equation (46), a straightforward but lengthy computation yields the Fourier component of \( \Sigma \) up to the order \( v^2 \). One gets

\[ \Sigma(k, z) = -v^2 t^2 \left[ 3 \int_{BZ} \frac{d{k'}}{(2\pi)^3} \frac{1}{E_{k'} - z} + \int_{BZ} \frac{d{k'}}{(2\pi)^3} \sum_{i=1}^{3} \cos(k_i' - k_i) \right]. \]  

Elementary computations based on the definition of Watson integral (see equation (11)) show that, introducing the functions

\[ S_1(z) \equiv 3t^2 \int_{BZ} \frac{d{k'}}{(2\pi)^3} \frac{1}{E_{k'} - z} = -\frac{3t^2}{z} W(-z/6t), \]  

\[ S_2(z) \equiv -\frac{t}{2} \int_{BZ} \frac{d{k'}}{(2\pi)^3} \frac{E_{k'}}{E_{k'} - z} = -\frac{t}{2} \left( 1 - W(-z/6t) \right) \]  

and using trigonometric relations, equation (51) may be compactly presented as

\[ \Sigma(k, z) = -v^2 \left( S_1(z) - \frac{1}{6t} S_2(z) E_k \right). \]  

From equation (54) one may readily compute \( \delta \mu_c \), as well as the renormalization of the bandwidth, which, at the critical point, are given by

\[ \delta \mu_c = -v^2 \left( S_1(-6t) + S_2(-6t) \right) = v^2 (2W(1) - 1)t/2 \approx v^2 t, \]  

\[ t_c = t + v^2 \frac{1}{6} S_2(-6t) = t[1 + \frac{1}{12} v^2 (W(1) - 1)] \approx t(1 + \frac{1}{24} v^2). \]  

Equation (56) shows that, as a result of the bond disorder, the bandwidth is increased: this is consistent with the mean-field result obtained in section 3.

The starting point of the computation of \( \delta T_c \) is given by equation (16). One notices that, as a result of the contribution in equation (50) of the self-energy induced by the random-bond disorder, the filling fraction \( f \) is changed to \( f + \delta f \), with \( \delta f \) given by

\[ \delta f = \lim_{\tau \to 0^-} k_B T_c \sum_n e^{-i \omega_n \tau} \int_{BZ} \frac{d{k}}{(2\pi)^3} \frac{1}{E_k - \mu_c - i \omega_n + \Sigma(k, \mu_c + i \omega_n) - \delta \mu_c} \]  

\[ - \frac{1}{E_k - \mu_c - i \omega_n} \]  

\[ \sim - \lim_{\tau \to 0^-} k_B T_c \sum_n e^{-i \omega_n \tau} \int_{BZ} \frac{d{k}}{(2\pi)^3} \frac{[\Sigma(k, \mu_c + i \omega_n) - \delta \mu_c]}{(E_k - \mu_c - i \omega_n)^2}. \]  

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Substituting
\[
\int_{\text{BZ}} \frac{dk}{(2\pi)^3} \frac{1}{(E_k - z)^2} = \frac{1}{3t} \frac{\partial}{\partial z} S_1(z),
\]
\[
\int_{\text{BZ}} \frac{dk E_k}{(2\pi)^3 (E_k - z)^2} = \frac{2}{t} \frac{\partial}{\partial z} S_2(z)
\]
in equation (57) one gets
\[
\delta f = \lim_{\tau \to 0} k_B T_c \sum_n e^{-i\omega_n \tau} \frac{v^2}{3t^2} \left\{ [S_1(z) - S_1(\mu_c)] \frac{\partial}{\partial z} S_1(z) + S_2(z) \frac{\partial}{\partial z} S_2(z)
\right.
\]
\[\left. - S_2(\mu_c) \frac{\partial}{\partial z} S_1(z) \right|_{z = \mu_c + i\omega_n}. \] (60)

For large values of the filling \(f\), the dominant contribution in equation (60) is given by the lowest Matsubara frequency. The explicit computation carried in appendix C yields
\[
\delta f = - \left( \frac{9}{4\pi^2} + \frac{1}{9} W(1) (W(1) - 1) \right) \frac{v^2}{8t} k_B T_c \approx -0.04 \frac{v^2}{t} k_B T_c. \] (61)

As a result of equation (10), one notices that, for large \(f\), there is a linear relation between \(T_c(0)\) and \(f\), which is given by
\[
\frac{W(1)}{6t} k_B T_c(0) = f. \] (62)

At the order \(v^2\), taking into account the shift in the particle density given by equation (61), one also has
\[
\frac{W(1)}{6t} k_B T_c + \delta f = f. \] (63)

From equations (62) and (63) and using equation (61), one obtains
\[
T_c = T_c(0) \left\{ 1 + \frac{3}{4} v^2 \left( \frac{9}{4\pi^2 W(1)} + \frac{1}{9} (W(1) - 1) \right) \right\} \approx T_c(0) \left( 1 + 0.16 v^2 \right), \] (64)

implying that
\[
\frac{\delta T_c}{T_c(0)} \approx 0.16 v^2. \] (65)

\(\delta T_c/T_c(0)\) is then positive and independent on the filling, provided that \(f \gg 1\).

We numerically checked the validity of equation (65). We adopted the following procedure. For several values of \(v\) and \(f\) and for several (100) configurations of hopping disorder and finite sized samples (cubic lattice with \(L\) sites per side with \(L = 5, \ldots, 10\)), we diagonalized exactly the Hamiltonian \(H(L)\), then we averaged the single-particle eigenvalues over all the configurations, computing \(T_c(L)\). Finally, we performed a finite size scaling analysis using the equation \(1/T_c(L) - 1/T_c = c/L [54]\) in order to fit the parameters \(c\) and \(T_c \equiv T_c(\infty)\).

In figure 7 the numerical results obtained for different values of \(v\) and \(f\) are compared with the analytic result (65). One sees that, for small disorder and \(f \gtrsim 1\), there is a

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Figure 7. Shift of the critical temperature $\delta T_c$ (in units of the critical temperature of the ordered lattice) as a function of the disorder strength $v$. The dots are obtained by numerical analysis while the two dash lines are the analytic results: respectively, from the top of the figure, our result (65) valid for large $f$ and the result of [35] for the continuous disordered Bose gas.

good agreement between the two results. For $f$ very small, instead, we recover the result obtained by Lopatin and Vinokur [35]: indeed, for small filling, the lattice details become irrelevant and one may treat the problem as in the continuum limit. Using the results of [35] with $t \sim \hbar^2/2ma_0^2$ ($a_0$ being the lattice constant), one has

$$\frac{\delta T_c}{T_c(0)} \approx -0.05 \frac{v^2}{f^{1/3}}.$$  

The resulting curve for $f = 0.01$ is in very good agreement with the numerical results. The reason for this agreement is that, for a small filling, the occupied single-particle states lie at the bottom of the first energy band rendering the lattice ideal Bose gas basically equivalent to the continuous model.

In appendix D we consider also the effect of an asymmetric random-bond disorder ($c_{ij} \neq c_{ji}$): the final result (D.4) is similar to (65), with a different numerical coefficient.

4.2. Effects of on-site disorder

It is easy to generalize our results to situations where also on-site disorder is present; see equation (1). Introducing the parameter $\gamma$, determined by the delta-correlated on-site random energies $\epsilon_i$ with variance $v_0^2 t^2$, as

$$\gamma \equiv \frac{3v^2 + v_0^2}{3v^2},$$  

(66)
one gets that equation (54) is modified as

\[ \Sigma(k, z) = -v^2 \left( \gamma S_1(z) - \frac{1}{6t} S_2(z) E_k \right). \]  \tag{67}

When only on-site disorder is present \((v^2 \to 0)\), one has that \(\Sigma(k, z) = -(v_0^2/3)S_1(z)\): this signifies that there is no renormalization of the bandwidth induced by a small random on-site disorder. Equation (67) amounts to stating that the first term in equation (60) should be multiplied by \(\gamma\). As a result, when both random hopping and on-site disorder are present, the critical temperature is given by

\[ T_c = T_c^{(0)} \left\{ 1 + \frac{3}{4} v^2 \left( \frac{6 + 3\gamma}{4\pi^2 W(1)} + \frac{1}{9} (W(1) - 1) \right) \right\} \approx T_c^{(0)} \left( 1 + 0.16 v^2 + 0.01 v_0^2 \right). \tag{68} \]

For comparable disorder strengths, \(T_c\) is more sensitive to random-bond disorder, since the numerical coefficient of \(v_0^2\) in equation (68) is smaller than the one associated with \(v^2\). As a result, the dependence of \(T_c\) on a small on-site disorder is extremely weak. We checked numerically that, for \(v = 0\), \(v_0 \lesssim 1\) and large \(f\), \(\delta T_c\) is very small. In contrast, when \(f \ll 1\), \(\delta T_c\) is negative in agreement with the results of [35].

4.3. Incommensurate potentials

An experimental strategy employed to simulate the effect of disorder is based on the use a bichromatic incommensurate lattice [14,18]. The ultracold atomic gas is subjected to a periodic potential of the form \(V(r) = V_{\text{main}}(r) + V_{\text{sec}}(r)\). While the main lattice potential creates the ordered lattice (e.g., an isotropic three-dimensional lattice corresponding to \(V_{\text{main}}(r) = V_0 (\sin^2(k_0x) + \sin^2(k_0y) + \sin^2(k_0z))\)), the addition of a secondary lattice, with a spacing incommensurate with that of the main lattice, leads to the emergence of a quasi-random potential which, for a finite sample, is equivalent, for all practical purposes, to a random potential. If the secondary potential is one-dimensional, \(V_{\text{sec}}(r) = V_1 \sin^2(k_1x)\), then, for large values of the height \(V_0\) of the main lattice, one gets the Bose–Hubbard Hamiltonian (1) with on-site energies \(\epsilon_i\) of the form \(\epsilon_i = -v \cos(2\pi q_i x)\), where \(v\) depends on \(V_1 (v \sim V_1/2)\), \(q = k_1/k_0\) and \(i \equiv (i_x, i_y, i_z)\), with the \(i\)'s integers. In [18], the experimental values of \(\lambda_0 = 2\pi/k_0\) and \(\lambda_1 = 2\pi/k_1\) are 830 and 1076 nm, respectively. Of course, if the secondary lattice is three-dimensional, one would have \(\epsilon_i \propto v(\cos(2\pi q_i x) + \cos(2\pi q_i y) + \cos(2\pi q_i z))\).

In figure 8 we plot, for two values of \(q\), the shift in \(T_c\) for a finite size system \((L = 10)\) as a function of the disorder intensity \(v\) of a one-dimensional secondary lattice for two different values of the filling: we see that, for small \(v\), \(\delta T_c > 0\) while, for intermediate \(v\), \(\delta T_c\) may become negative. For small \(v\), \(\delta T_c \propto v^2\).

5. Conclusions

We investigated how the critical temperature for condensation of an ideal Bose gas on a cubic lattice is shifted as a consequence of random-bond and on-site disorder. Our analysis evidences how the shift in the critical temperature, \(\delta T_c\), depends on the filling fraction \(f\) and on the space dimension \(D\).

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Figure 8. Shift of the critical temperature (in units of the critical temperature $T_c(0)$ of the ordered lattice having a size $L$) as a function of $v$ for a bichromatic incommensurate lattice. Circles (squares): $f = 1$ ($f = 10$) for $q = 0.206$. Diamonds (stars): $f = 1$ ($f = 10$) for $q = 830/1076$. In both cases $L = 10$. Dotted lines (for $q = 0.206$) and dash lines (for $q = 830/1076$) are just a guide for the eye.

For the infinite range model ($D \to \infty$), using results from the theory of random matrices, we found the explicit dependence on $f$ of $\delta T_c$: $\delta T_c$ is negative and vanishes for large $f$, in agreement with known results for the SM [45].

For $D = 3$, we wrote down the corrections to the Green functions due to disorder, finding that, for large $f$, $\delta T_c/T_c(0)$ is positive and does not appreciably depend on $f$. In contrast, for small $f$, $\delta T_c$ is negative and our results are in a very good agreement with the ones obtained in [35] for the continuous disordered Bose gas. The reason for this agreement is that, for small filling (i.e., much smaller than one particle per site), the occupied single-particle states lie at the bottom of the first energy band: this renders the lattice ideal Bose gas basically equivalent to the continuous model investigated in [35].

We presented numerical results for the shift of the critical temperature induced by the disorder generated by a bichromatic incommensurate potential in a realistic finite size system: we found that the critical temperature increases for small values of the height $v$ of the secondary lattice (i.e., small disorder), and decreases for larger values of $v$. The dependence of the shift of the critical temperature upon $v$ is $\delta T_c \sim v^2$.

We observe that, in the replicated action, the disorder enters as a fictitious effective attractive interaction between replicas. From this point of view, some similarity emerges with the problem of determining the shift of the critical temperature in a Bose gas due to the presence of a repulsive interaction [55]–[58] (for a clear presentation of this issue and more references, see [59]): rather counterintuitively, the critical temperature increases for small repulsive interactions, while it decreases for larger interaction. This should be compared with the results obtained for a homogeneous continuous Bose gas with weak disorder [35, 58, 60], where $T_c$ decreases with disorder, as a consequence of the inter-replica attraction. For the ideal Bose gas we showed in this paper that, on a lattice, this result is modified: indeed, for small filling, only the bottom part of the energy spectrum is occupied and the effective attraction among replicas amounts to a decrease of the critical
temperature, in agreement with [35]. In contrast, for large filling, all the single-particle energy band is occupied, including states with effective negative mass, leading to a net increase of \(T_c\).

We have considered non-interacting bosons on lattices with disorder: adding a repulsive interaction among the bosons, and treating it within the same approach as we used, one expects [59] \(T_c\) to be unchanged. Going beyond the mean-field approach of this paper should unveil the interplay between the discreteness induced by the lattice and the interaction in a disordered environment. For this purpose, non-perturbative resummation techniques for computing the self-energy, such as the one discussed in [61], could be relevant for determining \(\delta T_c\). In the same way, it would be very interesting to study the effects of a harmonic confinement on the disordered lattice [62].

To conclude, we mention that if one introduces suitable time-dependent optical lattices [63,64], one should be able to see the effects also of negative effective \(t_{ij}\)s: for non-interacting bosons, this could provide an experimental testing ground for the three-dimensional spin-glass spherical model.

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**Appendix A. Some useful properties of the Watson integrals**

The generalized Watson’s integrals are defined by

\[
W(d, z) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \prod_{i=1}^{d} dk_i \frac{1}{1 - (1/dz) \sum_{j=1}^{d} \cos k_j};
\]

in our paper we are concerned only with \(W(z) \equiv W(3, z)\).

The Watson integral (11) admits an integral representation

\[
W(z) = \int_0^\infty dx \, e^{-x} [I_0(x/3z)]^3
\]

in terms of the modified Bessel function of the first kind \(I_0\). Particularly important is its value at \(z = 1\), which is given by \(W(1) = \frac{\sqrt{\pi}}{32\pi} \Gamma(\frac{1}{24}) \Gamma(\frac{5}{24}) \Gamma(\frac{7}{24}) \Gamma(\frac{11}{24}) \approx 1.516386\) [65].

The Watson function \(W(z)\) can also be written as a probability generating function

\[
W(z) = \sum_{n=0}^\infty \frac{P_n}{z^n}, \quad |z| \geq 1
\]

with \(P_n\) being the probability that a random walker returns after \(n\) steps to the same point of the cubic lattice. In particular,

\[
P_{2n} = \int_{BZ} \frac{dk}{(2\pi)^3} \left( \frac{1}{3} \sum_{\ell=1}^{3} \cos k_{\ell} \right)^{2n}
\]
is the probability for a random walker on a cubic lattice of reaching the starting site after $2n$ steps. Clearly, $P_{2n+1} = 0$ since a closed loop on a square lattice always has an even number of segments. For this reason one can write $W(-z/6t)$, which is needed in the computation of section 4 and appendix C, as

$$W(-z/6t) = \frac{(-6t)^{2n} P_{2n}}{z^{2n}}. \tag{A.5}$$

We notice that $W(z)$ is also the solution of a Riemann p-differential equation, so, when $|z| \geq 6t$, $W(-z/6t)$ and its derivative are well defined.

**Appendix B. Effective replicated action**

Let us consider the Hamiltonian

$$\hat{H} = -t \sum_{i,j} \tilde{A}_{ij} \hat{b}_i^\dagger \hat{b}_j - \mu \sum_i \hat{b}_i^\dagger \hat{b}_i + \sum_i \epsilon_i \hat{b}_i^\dagger \hat{b}_i,$$ \tag{B.1}

where $\tilde{A}_{ij} = A_{ij}(1 + c_{ij})$, with $A_{ij} = 1, 0$ depending on whether $i$ and $j$ are neighbor sites or not, $c_{ij}, \epsilon_i$ being random variables.

The partition function can be written as a coherent-state path integral (in units where $\hbar = k_B = 1$)

$$Z = \int \prod_i D\varphi_i D\varphi_i^\dagger e^{-S}, \tag{B.2}$$

where $S$ is given by

$$S = \int_0^\beta d\tau \left\{ \sum_i \varphi_i^* \left( \frac{\partial}{\partial \tau} - \mu \right) \varphi_i + \sum_i \epsilon_i \varphi_i^* \varphi_i - t \sum_{i,j} \tilde{A}_{ij} \varphi_i^* \varphi_j \right\}. \tag{B.3}$$

The partition function (B.2) depends on the disorder realization of the hopping $\{c\}$ and of the on-site energies $\{\epsilon\}$: $Z = Z(\{c\}, \{\epsilon\})$. Labeling the $N$ replicas by $\alpha = 1, \ldots, N$ and using $\tilde{A}_{ij} = \tilde{A}_{ji}$, one has

$$Z^N(\{c\}, \{\epsilon\}) = \int \prod_{i,\alpha} D\varphi_i^\alpha D\varphi_i^{\alpha*} \exp \left( - \int_0^\beta d\tau \left\{ \sum_{i,\alpha} \varphi_i^{\alpha*} \left( \frac{\partial}{\partial \tau} - \mu + \epsilon_i \right) \varphi_i^\alpha 
- \frac{t}{2} \sum_{i,j,\alpha} \tilde{A}_{ij} \left( \varphi_i^{\alpha*} \varphi_j^\alpha + \varphi_j^{\alpha*} \varphi_i^\alpha \right) \right\} \right). \tag{B.4}$$

If, for simplicity, one assumes that $\epsilon_i = 0$ (only random-bond disorder is present), the averaged effective partition function is

$$\overline{Z^N(\{c\})} = \int \prod_{i,j} dc_{ij} P(c_{ij}) Z^N(\{c\}), \tag{B.5}$$

where $P(c_{ij})$ is the Gaussian probability distribution

$$P(c_{ij}) = \frac{1}{\sqrt{2\pi v^2}} e^{-(c_{ij})^2/2v^2}, \tag{B.6}$$

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with variance \( v^2 \). After integrating over the \( c_{ij} \) s, one gets
\[
Z^N(\{c\}) = \int \prod_i D\varphi_i^a D\varphi_i^{a*} e^{-S_{\text{eff}}},
\]
where \( S_{\text{eff}} \) is given in equation (43). When on-site disorder \( \{\epsilon\} \) is present, one can follow the same procedure: namely, one averages equation (B.4) over disorder with the measure
\[
\int \prod_i d\epsilon_i P_o(\epsilon_i) = \int \prod_i d\epsilon_i (1/\sqrt{2\pi v_o^2 t^2}) e^{-(\epsilon_i)^2/2v_o^2 t^2},
\]
where \( v_o^2 t^2 \) is the variance of on-site random energies (\( v_o \) is a dimensionless quantity).

**Appendix C. Shift of the filling fraction**

Using equations (52)–(54), (58) and (59) in order to write \( \delta f \) only in terms of the Watson function \( W(-z/6t) \), one obtains equation (60). In equation (60) the dominant contribution to \( \delta f \) is lowest the Matsubara frequency \( (\omega_n = 0) \):
\[
\delta f = \frac{k_B T_c v^2}{3t^2} \lim_{z \to \mu_c} \left\{ \left[ S_1(z) - S_1(\mu_c) \right] \frac{\partial}{\partial z} S_1(z) + S_2(z) \frac{\partial}{\partial z} S_2(z) - S_2(\mu_c) \frac{\partial}{\partial z} S_1(z) \right\}. \tag{C.1}
\]
One can compute equation (C.1) taking advantage of the asymptotic behavior of \( W(z) \).
For \( \eta^2 \equiv z^2 - (6t)^2 \to 0 \) one can write
\[
W(-z/6t) = W(1) - \frac{3\sqrt{3}}{2\pi} \sqrt{1 - \frac{(6t)^2}{z^2}} + \mathcal{K} \left( 1 - \frac{(6t)^2}{z^2} \right) + O(\eta^3), \tag{C.2}
\]
where \( \mathcal{K} = \frac{9}{32}(W(1) + 6/((\pi^2 W(1))) \). From equations (52) and (53) one has
\[
S_1(z) = -\frac{3t^2}{z} \left( W(1) - \frac{3\sqrt{3}}{2\pi} \sqrt{1 - \frac{(6t)^2}{z^2}} \right) + O(\eta^2), \tag{C.3}
\]
\[
S_2(z) = -\frac{t}{2} \left( 1 - W(1) + \frac{3\sqrt{3}}{2\pi} \sqrt{1 - \frac{(6t)^2}{z^2}} \right) + O(\eta^2). \tag{C.4}
\]
The derivatives of these functions are
\[
\frac{\partial}{\partial z} S_1(z) = 3t^2 \left( \frac{3\sqrt{3}}{2\pi} \frac{(6t)^2}{z^4} \frac{1}{\sqrt{1 - (6t)^2/z^2}} - \frac{W(1)}{z^2} \frac{2(6t)^2}{z^4} \right) + O(\eta), \tag{C.5}
\]
\[
\frac{\partial}{\partial z} S_2(z) = \frac{t}{2} \left( \frac{3\sqrt{3}}{2\pi} \frac{(6t)^2}{z^3} \frac{1}{\sqrt{1 - (6t)^2/z^2}} + \frac{2(6t)^3}{z^3} \right) + O(\eta). \tag{C.6}
\]
One should notice that equations (C.5) and (C.6) diverges at \( z = -6t \) but that these divergences are canceled in equation (C.1). Inserting equations (C.4)–(C.6) in equation (C.1), one readily finds equation (61).

An alternative way to obtain equation (61) is based on the random-walk probability expansion of the Watson integral presented in appendix A. If one truncates the sum (A.5), the result for \( \delta T_c \) will be zero since only for the infinite sum, one has that
\[
\frac{\partial}{\partial z} W(-z/6t) = -\sum_{n=1}^{\infty} \frac{n(-6t)^{2n} P_{2n}}{z^{2n+1}} \tag{C.7}
\]
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is not convergent at \( z = -6t \). Equation (C.7) implies that only very long random walks are important: thus, expansion in terms of \( n^{-1} \) (the inverse length of the walks) is meaningless since only walks with \( n \to \infty \) are relevant, although \( P_n \) decreases very fast with \( n \). For large \( n \), \( P_{2n} \) has the following asymptotic behavior:

\[
P_{2n} \sim 2 \left( \frac{3}{4\pi n} \right)^{3/2}, \quad n \to \infty.
\]  

Inserting equation (C.8) into equation (A.5) (for \( n \geq 1 \), and \( P_0 = 1 \)) and using equations (52), (53) and (C.1), after taking the limit \( z \to -6t \), one gets equation (61).

Appendix D. Asymmetric random-bond disorder

Here we consider the situation where \( c_{ij} \) and \( c_{ji} \) are two independent random variables, namely

\[
c_{ij} \neq c_{ji}.
\]  

This situation may arise, for instance, when time reversal symmetry is broken.

After integrating over disorder, one gets

\[
G_{lm} = G_{lm} + v_A^2 t^2 \sum_{ij} A_{ij} (N + 1) G_{li} G_{ij} G_{jm} + O(v_A^4)
\]  

(where \( v_A^2 \) is the variance of the asymmetric random-bond disorder), which implies that

\[
\Sigma(k, z) = \frac{v_A^2}{3t} S_2(z) E_k.
\]  

As a result, an asymmetric random-bond disorder yields a value of \( T_c \) given by

\[
T_c = T_c^{(0)} \left\{ 1 + \frac{3}{2} v_A^2 \left( \frac{6}{4\pi^2 W(1)} + \frac{1}{9} (W(1) - 1) \right) \right\} \approx T_c^{(0)} \left( 1 + 0.24 v_A^2 \right).
\]  

In order to convince oneself that time reversal symmetry could play a role, one may consider a system with only two sites and random hopping between them, described by a matrix

\[
\begin{pmatrix}
0 & t_{12} \\
t_{21} & 0
\end{pmatrix}.
\]  

The eigenvalues of (D.5) are

\[
\lambda_{\pm} = \pm \sqrt{t_{12} t_{21}}.
\]  

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If time reversal symmetry holds, one has that
\[ t_{12} = t_{21}. \]  
(D.7)

Averaging on the disorder and assuming \( t_{12} = t_{21} = t \), one gets
\[ \lambda_{\pm} = \pm t. \]  
(D.8)

On the other hand, if time reversal symmetry is broken (i.e., \( t_{12} \neq t_{21} \)), one has
\[ \lambda_{\pm} = \pm t \left( \frac{(1 + \varepsilon)^{3/2} - (1 - \varepsilon)^{3/2}}{3\varepsilon} \right)^{2} \approx \pm t \left( 1 - \frac{\varepsilon^{2}}{12} \right). \]  
(D.9)

In deriving equation (D.9) a uniform distribution for \( t_{12} \) and \( t_{21} \) with width \( 2t\varepsilon \) and mean value \( t \) has been assumed.

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