Multiuser Detection for Random Access Bandwidth Request in WiMAX

Md Mashud Hyder
School of Electrical Engineering and Computer Science
The University of Newcastle, NSW 2308, Australia

Abstract—Random access is a multiple access communication protocol where the users simultaneously communicate with a base station (BS) in an uncoordinated fashion. In this work, we consider the problem of multiuser detection in a random access bandwidth request context. We propose an enhanced random access scheme where the fixed/low-mobility M2M devices pre-equalize their random access codes using the estimated frequency response of the slowly-varying wireless channel. Consequently, we have developed two different multiuser detection algorithms. The first algorithm works in a greedy way where it performs cross-correlation of the received signal with a set of decoder sequences and detects active users based on the correlation output. We derive the condition under which the algorithm can detect a given number of active users with high probability. Subsequently, we demonstrate an efficient decoder design procedure which enhances the user detection performance. A basis mismatched sparse recovery technique has been applied in the second algorithm which exploit an inherent structure of the random access protocol. The performance of the proposed schemes is demonstrated in a WiMAX network environment.

Index Terms—WiMAX, M2M, Sparse representation, Random Access.

I. INTRODUCTION

The orthogonal frequency-division multiple access (OFDMA) scheme has been adopted by the IEEE 802.16 WiMAX standards. The future wireless communication network has to support a large number of fixed/low-mobility machine-to-machine (M2M) devices that will transmit bursty, small data packets (e.g. meter readings, sensor reports etc.) under a valid security association with the network [1]. This corresponds to a heavily uplink-based traffic model following a Poisson distribution. Random access based bandwidth request (BR) is preferable for such traffic that exploits the benefits of statistical multiplexing to support a large number of devices with a fixed overhead. In the IEEE WiMAX standards, the BR procedure starts with the allocation of a predefined set of subcarriers by the BS in a specific time slots. The subscriber stations (SSs) which wish to request for bandwidth to the BS can take the opportunity by modulating a randomly selected code onto the allocated subcarriers. At the receiver end, the BS is required to detect the multiple active subscribers. Once the BR subscribers are resolved, the BS will allocate bandwidth for the corresponding subscribers. However, the process will be efficient when the base station can separate the BR subscribers successfully.

The idea of random access channel is related to the concept of multiple access channel (MAC) in network information theory [2]. The concept of MAC has been applied successfully in CDMA systems [3]. However, the application of classic MAC channel analysis in the present scenario is not straightforward (see [4] and references therein). The problem of multiuser detection (MUD) in a random access on-off channel has been studied in [4]–[6]. The work in [4] solves the MUD problem using a sparse signal recovery framework and derives necessary conditions under which the orthogonal matching pursuit and Lasso algorithms can detect active users successfully. However, the proposed method assumes perfect synchronization among all users, which is difficult to guarantee in practice. A modified sparse representation framework has been proposed in [5] where the above assumption has been relaxed. A reduced dimension MUD method has been proposed in [6]. It has been shown that the computational complexity of the reduced dimension MUD is lower than conventional correlation based MUD. However, all the methods assume that the transmitted signal is subject to flat fading channel. In a metropolitan wireless network the communication channel generally subject to frequency selective fading and hence the above assumption may not hold in practice. Furthermore, the sparse representation of the received signal is not straightforward for selective fading channel.

Multiuser detection in IEEE 802.16 based BR has been studied in [7]–[10]. A simple solution to enhance MUD is to increase the number of BR channels to accommodate more users/devices. However, under the existing schemes, only a handful of codes can be detected reliably per channel per frame in presence of multiple access interference (MAI) from different codes, and random noise and frequency-selective fading in the multipath wireless channel [7]. Therefore, a lot of random access channels are required, which would substantially reduce the payload capacity of the overall system. Hence, random access has been considered as one of the key bottlenecks by the IEEE 802.16p working group on M2M communications. To overcome this limitation, the recent IEEE 802.16p amendment has proposed several solutions, mostly based on access control over the MAC layer [8]. On the other hand, a number of works have already appeared in the literature concerning this problem, e.g. [9] and [10]. However, these schemes require significant modification to the existing standards and are not suitable for a network supporting both M2M and non-M2M traffic.

In this work we propose an enhanced random access MUD in M2M communication environment. The MUD problem has been resolved by two different algorithms. The contributions
of the work can be summarized as follows:

1) We propose a pre-equalized random access scheme. We assume that the BR subscribers only have an approximate knowledge about their channel frequency response. The BR subscribers pre-equalize the random access codes by using the estimated channel frequency response and transmit to the BS. The imperfect knowledge of channel frequency response results in an additional noise term in the received signal at the BS. We analyze the statistical property of the noise term and develop a suitable data model of the received signal for MUD.

2) We apply a very simple correlation based approach called CMUD for resolving the MUD problem where the received signal is correlated with a set of decoder sequences. We develop the necessary condition under which the CMUD can resolve a given number of active users from the received signal.

3) The necessary condition will show that one can enhance the performance of CMUD by controlling some properties of the decoder sequences. We then propose an algorithm to design efficient decoder sequences.

4) The computational complexity of CMUD is very low, but it cannot detect a large number of active users. To enhance the user detection performance, we resolve the MUD problem by using a basis mismatched sparse signal recovery algorithm. We exploit some inherent properties of MUD and formulate an optimization problem which can enhance the performance of the underlying basis mismatched sparse recovery algorithm. However, the optimization problem is non-convex in general. We apply a Lagrangian Dual Relaxation method to solve the optimization problem.

Notations: Superscript $\top$ denotes matrix transpose. $E(x)$ denotes expected value of $x$. For a complex number $x$, its real and imaginary parts will be denoted by $|x|_r$ and $|x|_i$ respectively. A component of the matrix $C$ at its $i$-th row and $j$-th column will be indicated by $C_{i,j}$ and the $j$-th column of $C$ will be represented by $C_j$. The cardinality of a set $T$ will be denoted by $|T|$. $1_L$ denotes a vector of length $L$ whose all components are one. The $p$-norm of a vector is defined as $\|x\|_p = (\sum |x_i|^p)^{1/p}$. diag($x$) refers to a diagonal matrix with vector $x$ on its diagonal.

II. DATA MODEL AND PROBLEM STATEMENT

A. General System Model

Consider a single-cell WiMAX network with time division duplex (TDD) OFDMA physical layer. A BR channel is comprised of $L$ randomly chosen subcarriers over one OFDM uplink symbol. Suppose the indices of BR subcarriers are $\{j_m : m = 1, 2, \ldots, L\}$. When a subscriber station (SS) intends to send a BR to the base station then it selects an available uplink BR slot and sends a BR packet to the BS. The packet consists of a BR code, which is a $L$-bit pseudo random binary sequence (PRBS) chosen with equal probability from a bank of $K$ codes $[11]$, where $L < K$. For rest of the sequel, we denote the BR code-matrix as $C \in \mathbb{R}^{L \times K}$ where every column of $C$ represents an independent code. Each $C_{i,j}$ is modulated by binary phase shift keying (BPSK) i.e. $C_{i,j} \in \{-1, +1\}$. The code-matrix is known to BS and every subscriber.

B. Pre-Equalization

In this proposed random access model, the SS does not send the random access code directly to the BS, instead it transmits a pre-equalized version of the code. According to the IEEE 802.16 standards, the first OFDM symbol of each WiMAX frame is a preamble transmitted by the BS, where the subcarriers are BPSK modulated with a boosted pilot sequence $[11]$. Typically, the SSs use this information to estimate the channel frequency response (CFR) for the OFDM demodulation process. A SS can pre-equalize its BR code using this estimated CFR exploiting the channel reciprocity of the TDD system [12]. A number of pre-equalization techniques are available. For more details, please refer to [13].

Consider a time instant when $M$ number of SSs are simultaneously contending on the same BR channel. Let, the $m$-th SS selects the $k_m$-th column of the code matrix $C \in \mathbb{R}^{L \times K}$, where $m = 1, 2, \ldots, M$. Considering zero-forcing (ZF) pre-equalization [13], the transmitted code over the $j_l$-th subcarrier from the $m$-th SS be

$$x_{l,m} = \frac{C_{l,k_m}}{\hat{h}_{l,m}}; \text{ for } l \in \{1, 2, \ldots, L\},$$

where $\hat{h}_{l,m}$ is the pilot-aided CFR of $j_l$-th subcarrier estimated by the $m$-th SS. To be more precise, $\hat{h}_{l,m}$ can be expressed as

$$\hat{h}_{l,m} = h_{l,m} + e_{l,m},$$

where $h_{l,m}$ is the actual frequency response of the $j_l$-th subcarrier and $e_{l,m} \sim \mathcal{CN}(0, \sigma^2_{e,m})$ is a zero-mean complex Gaussian noise with variance $\sigma^2_{e,m}$.

In the BS, after down-conversion to baseband and OFDM demodulation, the received signal from the $m$-th SS over the $j_l$-th subcarrier be

$$y_{l,m} = x_{l,m}\hat{h}_{l,m} = C_{l,k_m}\frac{\hat{h}_{l,m}}{h_{l,m}}; \text{ for } l \in \{1, 2, \ldots, L\},$$

where $\hat{h}_{l,m}$ is the effective channel experienced by the BS due to transmission form $m$-th SS over the $j_l$-th subcarrier. Note that, although the position of the BS and the SS will remain almost fixed for stationary/slow-moving M2M devices, the channel is continually affected by the movement of the external scatterers in the surrounding environment [14]. Consequently, $\hat{h}_{l,m}$ will differ from $h_{l,m}$. To generalize, the effective channel can be modelled as

$$\hat{h}_{l,m} = \alpha h_{l,m} + \eta_{l,m},$$

where $\eta_{l,m} \sim \mathcal{CN}(0, \sigma^2_{\eta,m})$ is a zero-mean complex Gaussian noise and $\alpha$ is some deterministic complex valued constant.

Considering the above phenomena, the combined received signal at the BS from all $M$ stations over the $j_l$-th subcarrier will be

$$y_l = \sum_{m=1}^{M} \left\{ C_{l,k_m}\frac{\alpha h_{l,m} + \eta_{l,m}}{h_{l,m} + e_{l,m}} \right\} + \vartheta_l,$$
where \( \vartheta_t \sim \mathcal{CN}(0, \sigma_t^2) \) is a zero-mean complex Gaussian noise.

Equation (5) can be represented as

\[
y_t = \sum_{m=1}^{M} \{ C_{l,k,m} - \lambda_{l,k,m} C_{l,k,m} \} + \vartheta_t,
\]

where \( \lambda_{l,k,m} \) is a ratio of complex variables, i.e.

\[
\lambda_{l,k,m} = \frac{(1 - \alpha) h_{l,m} + \epsilon_{l,m} - \eta_{l,m}}{h_{l,m} + \epsilon_{l,m}}.
\]

Let us construct

\[
y = [y_1, y_2, ..., y_L]^\top.
\]

C. Formal Problem Statement

Let us define the active users set as

\[
S = \{ k_m : m = 1, 2, \cdots M \}
\]

Our goal is to detect the set of active codes, i.e., \( S \) from the received noisy data \( y \).

D. Sparse Representation

Typically, at a particular BR opportunity under a BS, the total number of BR terminals \( M \ll K \). This condition should be met by any random access based communication network. Otherwise, by using probability theory, it can be shown that two or more BR terminals will collide by selecting the same BR code with high probability. It is well known that if multiple users transmit same code then the BS cannot separate the corresponding users. To maintain \( M \ll K \), different approaches have been considered by the Third Generation Partnership Project [16]. For example, the collision resolution algorithms such as random backoff can limit the value of \( M \)

In the following, we shall assume that any BR code will be used by at most one BR terminal at a particular BR opportunity [4]. Let \( \hat{x} \in \mathbb{R}^K \) be a vector such that its \( i \)-th component will be 1 only if \( i \in S \), and zero otherwise. Using the vector we can represent the data model in (3) as

\[
y = C \hat{x} - u + \vartheta
\]

where \( u = Q \hat{x} \), and

\[
Q = \begin{bmatrix}
C_{1,1} \lambda_{1,1} & C_{1,2} \lambda_{1,2} & \cdots & C_{1,K} \lambda_{1,K} \\
C_{2,1} \lambda_{2,1} & C_{2,2} \lambda_{2,2} & \cdots & C_{2,K} \lambda_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
C_{L,1} \lambda_{L,1} & C_{L,2} \lambda_{L,2} & \cdots & C_{L,K} \lambda_{L,K}
\end{bmatrix},
\]

with \( Q_{i} = 0 \) if \( i \notin S \). Since \( M \ll K \) by assumption, the vector \( \hat{x} \) is sparse. Thus, MUD can be seen as a problem of estimating the sparse vector \( \hat{x} \) from the received data \( y \). Now suppose that we have some prior knowledge about the number of active BR users i.e., \( M \). Then we can postulate that \( \hat{x} \) has a probability density function

\[
p_x(\hat{x}) = (1 - \epsilon) \delta_0(\hat{x}) + \epsilon \delta_1(\hat{x})
\]

where \( \epsilon = \frac{M}{K} \), \( \delta_1(x) = \delta_0(x - 1) \) and \( \delta_0(x) \) is a Dirac delta function.

III. Correlation Based Code Detector

In this section, we apply a simple correlation based algorithm to detect active BR codes. In detecting active codes, we take real part of \( y \) in (10) because imaginary part contains only noise and interference. In the following, we shall study some statistical properties of \( \lambda \) which will be helpful for developing the code detection algorithm.

A. Statistical Properties of \( \lambda \)

In general, the components of CFR are modelled as zero mean complex Gaussian random variable [15], i.e., \( h_{l,m} \sim \mathcal{CN}(0, \sigma_{h,m}^2) \). Hence, the channel power can be approximated by its variance. The variance of CFR, i.e. \( \sigma_{h,m}^2 \) may not same for every \( m = 1, 2, \cdots M \). However, the BS always equalizes the channel power of the active users through the ranging procedure. In effect, the variances of CFR of all users must remain in a known interval due to the ranging process. Consequently, we can use the value of average channel power as an estimate of \( \sigma_{h,m}^2 \) for all \( m \). Similarly, BS can estimate \( \sigma_{e,m}^2 \) and \( \sigma_{\eta,m}^2 \) by using pilot-aided synchronization procedure. Thereby, we shall use the following assumption.

Assumption 1: \( \sigma_{h,i}^2 = \sigma_{h}^2, \sigma_{e,i}^2 = \sigma_{e}^2, \sigma_{\eta,i}^2 = \sigma_{\eta}^2; \forall i \in \{1, 2, \cdots M\} \)

According to (7), \( \lambda_{l,m} \) is a ratio of two complex quantities. By applying the concept of [15] and using Assumption[1] every \( \lambda_{l,m} \) can be modelled using a complex random variable \( \lambda = [\lambda]_r + i[\lambda]_i \) with probability density function:

\[
f([\lambda]_r, [\lambda]_i) = \frac{(1 - |\rho|^2)\sigma_{e}^2\sigma_{\eta}^2}{\pi (\sigma_{e}^2|\lambda|_r^2 + \sigma_{\eta}^2 - 2\rho_\eta[\lambda]_r[\sigma_{e}][\sigma_{\eta}])^2} \tag{13}
\]

where \( \sigma_{e}^2 = (1 - \alpha)^2\sigma_{h}^2 + \sigma_{\eta}^2, \sigma_{\eta}^2 = \sigma_{e}^2 + \sigma_{\eta}^2, \sigma_{h}^2 = \sigma_{h}^2 + \sigma_{\eta}^2, \) and \( \rho = (1 - \alpha)\sigma_{e}^2 + \sigma_{\eta}^2 \). Furthermore, \( \{\lambda_{l,m}\}_{l,m} \) are independent from each other. In Appendix-A, we derive the mathematical expectation and variance of \( [\lambda]_r \). In the following we denote them by \( \mu_r \) and \( \sigma^2_r \) respectively.

Since \( C_{\ell,j} \) are real valued, the expected value of \( [u_{j}]_r \) in (10) with respect to \( \hat{x} \) and \( \lambda \) is

\[
E([u_{j}]_r) = E\left( \sum_{\ell=1}^{K} C_{j,\ell}[\lambda]_r [\lambda]_r \hat{x}_{\ell} \right)
\]

\[
= \epsilon \mu_r \sum_{\ell=1}^{K} C_{j,\ell}
\]

\[
\tag{14}
\]
By using a similar procedure, it can be verified that the covariance matrix of $[u]_r$ is

$$\text{Cov}([u]_r) = \epsilon \sigma_r^2 \begin{bmatrix} \|C(1,:)\|_2^2 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \|C(K,:)\|_2^2 \end{bmatrix}$$

where $C(\ell,:) \in \mathbb{C}$ denotes $\ell$-th row of $C$. In the present scenario $C_{j,\ell} \in \{+1,-1\}$, hence

$$\text{Cov}([u]_r) = M \sigma_r^2 I. \quad (15)$$

**B. Correlation based multiuser detection (CMUD) algorithm**

The CMUD algorithm is summarized in Table I. It starts with an empty set $\mathbb{T}$ as an initial estimate of indices of active codes and residual vector $z(0) = y$. To detect the presence of an active BR code, the code detector correlates the residual with a decoder matrix $D \in \mathbb{R}^{L \times K}$. In Step 2 of $j$-th iteration, the algorithm takes every column of $D$ and computes its cross-product with the residual $z^{(j-1)}$. It then construct a set $I$ such that

$$I = \{ \ell \in \{1,2,\cdots,K\} : |D^\top z^{(j-1)}|_r > \kappa \}$$

where $\kappa$ is a predefined threshold. In Step 3, the set $I$ is added to the active code set i.e., $T = T \cup I$. However, if it is found that $I$ is empty then we assume all active codes in $y$ has been detected by the algorithm and hence terminate to Step 6. In Step 4, the residual $z^{(j)}$ is updated by subtracting the selected active codes from $z^{(j-1)}$. The residual $z^{(j)}$ represents the part of active codes $\{C_{\ell}\}_{\ell \in S}$ that has not been detected yet along with noise. In Step 5, we update $\kappa$ to a new value and repeat Steps 1-5. The algorithm needs a decoder matrix $D$ and threshold $\kappa$ as its input. In the following section, we shall demonstrate some procedures for choosing those parameters. Furthermore, the algorithm needs an estimate of $M$. We shall describe a simple procedure in Section III-E to obtain a rough estimate of the parameter. When $D = C$, the CMUD will be closely related to the reduced dimension decision-feedback (RDDFB) algorithm [6]. However, we shall show that an appropriate choice of $D$ can increase the performance of CMUD significantly. The computational complexity of CMUD is very small. At every iteration, the major complexity involves in computing $[D^\top r^{(j-1)}]_r$, which requires $LK$ flops.

**C. Performance Analysis**

In this section, we develop the conditions under which the CMUD will successfully detect all active codes. Our performance measure is based on the probability of code detection error by CMUD. In particular, we define the probability of code detection error as

$$P_e = Pr\{\hat{T} \neq S\}. \quad (16)$$

| TABLE I |
| --- |
| **CORRELATION BASED MULTIUSER DETECTION (CMUD)** |

| Input: Code matrix $C$, a real valued decoder matrix $D \in \mathbb{R}^{L \times K}$, date vector $y$, threshold $\kappa$ and an approximation of $M$, i.e., $M_0$. |
| Initialization: Set $T = \emptyset$, $z^{(0)} = y$, $j = 0$. |
| Loop: for $j = 1, 2, \cdots, M_0$ |
| 1. Set $I = \emptyset$. |
| 2. Construct $I = \{ \ell \in \{1,2,\cdots,K\} : |D_{\ell}^\top z^{(j-1)}|_r > \kappa \}$. |
| 3. If $I = \emptyset$: Set $T = T \cup I$. |
| Else: Go to Step 6. |
| 4. Update $z^{(j)} = z^{(j-1)} - \sum_{\ell \in I} C_{\ell}$. |
| 5. Update $\kappa$. |
| End Loop |
| 6. Output: Indices of active codes $\hat{T} = T$. |

Let us define

$$\alpha = \max_{\ell} \left\{ \sum_{j=1}^{K} D_{\ell}^\top C_j \right\} \quad (17)$$

$$\beta = \max_{\ell} \left\{ \max_{\ell \neq \ell'} |D_{\ell}^\top C_{\ell'}| \right\} \quad (18)$$

$$\gamma = \max_{\ell} |D_{\ell}|_2. \quad (19)$$

**Lemma 1:** Let $y = \sum_{\ell \in S} C_{\ell} - u + \vartheta$ where $\#S = M$ and the random vector $\vartheta$ has zero mean complex Gaussian distribution with covariance $\sigma_\vartheta^2 I$ and the random vector $[u]_r$ is defined as in (10). Let $D \in \mathbb{R}^{L \times K}$ be a decoder matrix such that $D_{\ell}^\top C_{\ell} = 1; \forall \ell$. Set

$$\tau = \epsilon \mu_{\alpha} + \gamma \sqrt{2(1 + \nu) \log K} \sqrt{M_0 \sigma_\vartheta^2 + \sigma_\vartheta^2 / 2} \quad (20)$$

for some given $\nu > 0$ and $M_0 \geq M$. Assume that the decoder matrix $D$ satisfies the following condition:

$$\tau + M_0 \beta < 0.5. \quad (21)$$

If we choose a threshold $\kappa$ such that

$$\tau + M_0 \beta < \kappa < 1 - M_0 \beta - \tau \quad (22)$$

then the probability of code detection error (16) by CMUD will be upper bounded by

$$P_e \leq (\pi(1 + \nu) \log K)^{-1/2} K^{-\nu}. \quad (23)$$

**Proof:** See Appendix-B.

**D. Decoder Design**

Lemma 1 states that for given values of $M$ and noise variances, the CMUD will detect all active codes efficiently if the matrix $D$ satisfies the following two conditions:

$$\epsilon \mu_{\alpha} + \gamma \tau + M_0 \beta < 0.5. \quad (24)$$

$$D_{\ell}^\top C_{\ell} = 1; \forall \ell. \quad (25)$$

where we denote $\tau = \sqrt{2(1 + \nu) \log K} \sqrt{M_0 \sigma_\vartheta^2 + \sigma_\vartheta^2 / 2}$. Hence, to obtain an optimum decoder, we have to design $D$ that minimizes the left side of (24). In Section III-E, we shall demonstrate a procedure of decoder design based on
the objective. As will be seen later, the procedure requires to solve a high dimensional optimization problem. Note that the base station needs not to redesign \( D \) at every BR request interval, instead it will be redesigned only when the values of \( M \) and noise variances change significantly. Thereby, the proposed scheme will be compliant with the IEEE 802.16 standards. Nevertheless, the base station always seeks low complexity algorithm. A low complex decoder design procedure will be proposed in Section III-D2 which adopts a popular approach called minimum mean square error (MMSE) decoder \[13\]. The MMSE decoder will not minimize the left side of (24) directly, however, exhibits moderate number of BR code detection performance.

1) Decoder-I: Let \( \hat{C}(\ell) \) be a matrix constructed from \( C \) by retaining all its columns except the \( \ell \)-th column. To obtain an optimum decoder based on Lemma 1, we need to solve the following optimization problem:

\[
\{D_\ast, \hat{\alpha}_\ast, \hat{\beta}_\ast, \hat{\gamma}_\ast\} = \arg \min_{D, \hat{\alpha}, \hat{\beta}, \hat{\gamma}} \epsilon \mu_r \hat{\alpha} + M_0 \hat{\beta} + \hat{\gamma}\]

subject to, \( D_j^\top C_\ell \leq 1, \) for \( \ell = 1, 2, \ldots, K \)

\[
- \hat{\alpha} \leq \sum_{j=1}^{K} C_j^\top D_\ell \leq \hat{\alpha}, \text{ for } \ell = 1, 2, \ldots, K
\]

\[
- \hat{\beta} I_{K-1} \leq \{\hat{C}(\ell)^\top D_\ell \} \leq \hat{\beta} I_{K-1}, \text{ for } \ell = 1, 2, \ldots, K
\]

\[
\|D_\ell\|_2 \leq \hat{\gamma}, \text{ for } \ell = 1, 2, \ldots, K.
\]

The optimization is convex and can be solved efficiently by using a Primal-Dual algorithm \[19\]. However, the optimization needs to solve for \( L.K + 3 \) number of variables. In the following, we propose a low complex decoder design procedure.

2) Decoder-II: The decoder design strategy is based on the MMSE criterion \[13\]. In particular, for every \( \ell \in \{1, 2, \ldots, K\} \), we design \( D_\ell \in \mathbb{R}^L \) that minimizes \( E(\|\hat{x}_\ell - D_\ell^\top y\|^2) \) along with the constraint \( D_j^\top C_\ell = 1 \). Here, the expectation is with respect to \( \hat{x}, \lambda \) and noise vector \( \vartheta \). Let us define the index set \( \mathbb{U} := \{1, 2, \ldots, K\} \). By using (11) and the fact that \( D_j^\top C_\ell = 1 \), we see that

\[
D_j^\top y = \hat{x}_\ell + \sum_{j \in \mathbb{U} \setminus \ell} D_j^\top C_j \hat{x}_j - \sum_{j \in \mathbb{U}} D_j^\top Q_j \hat{x}_j + D_j^\top \vartheta
\]

Note that the value of \( \epsilon \) in (12) is small. Hence, neglecting the values of \( E(\|\hat{x}_\ell - D_\ell^\top C_j Q_j^\top D_j \hat{x}_k\|) |\neq j \) and \( E(\|\hat{x}_m - D_m^\top Q_m \hat{x}_m\|) |\neq m \), and using a similar procedure of (13), we obtain that

\[
E(\|\hat{x}_\ell - D_\ell^\top y\|^2) = \epsilon \sum_{j \in \mathbb{U}} \|D_j^\top C_j \|^2 + \epsilon \sum_{j \in \mathbb{U}} E(\|D_j^\top Q_j \|^2) + 2 \epsilon \sum_{j \in \mathbb{U}} E(\|D_j^\top C_j Q_j^\top D_j \|) + \|D_\ell\|^2 \sigma_\vartheta^2 / 2
\]

\[
\leq \epsilon \sum_{j \in \mathbb{U}} \|D_j^\top C_j \|^2 + \epsilon \sum_{j \in \mathbb{U}} \|D_j^\top C_j \|^2 E(\lambda^2) + 2 \epsilon \sum_{j \in \mathbb{U}} \|D_j^\top C_j \|^2 \sigma_\vartheta^2 / 2
\]

\[
\approx \delta D_\ell^\top RD_\ell + \|D_\ell\|^2 \sigma_\vartheta^2 / 2
\]

where \( \delta = \epsilon (1 + E(\|\lambda_j\|^2) - 2 \mu_r) \) and \( R = \sum_{j \in \mathbb{U}} C_j C_j^\top \). Now we have to minimize (29) with respect to \( D_\ell \) along with the constraint \( D_j^\top C_\ell = 1 \). Using the procedure of Lagrangian multiplier, it can be verified that the optimization has a closed form of solution:

\[
D_\ell = \frac{(\delta R + I R_\sigma^2 / 2)^{-1} C_\ell}{C_\ell^\top (\delta R + I R_\sigma^2 / 2)^{-1} C_\ell}. \tag{30}
\]

Note that whenever the number of active users or channel noise variance changes significantly, we have to compute the inverse term \( (\delta R + I R_\sigma^2 / 2)^{-1} \) to obtain an updated \( D \), which is computationally expensive. We can reduce the computational complexity by using the following procedure. Since \( L < K \), the matrix \( R \) can be shown to be positive definite (see \[20\]). Hence, the eigenvalue decomposition of \( R \) has the form: \( R = U \Lambda U^\top \) where \( U^\top U = I \) and \( \Lambda \) is a diagonal matrix whose \( k \)-th entry is the \( k \)-th eigenvalue of \( R \). Denote \( v = U^\top C_\ell \). Then it can be verified that

\[
D_\ell = \frac{U(\delta \Lambda + I R_\sigma^2 / 2)^{-1} v}{\sum_{\ell=1}^{L} \|v|_{\ell}\|^2} \tag{31}
\]

Since \( (\delta \Lambda + I R_\sigma^2 / 2) \) is a diagonal matrix, its inverse can be computed easily. Furthermore, for a given \( C \) the matrix \( R \) is fixed, hence we can compute the eigenvalue decomposition in priori. Note that the minimization of (29) will minimize the values of \( \delta \) and \( \gamma \) in (24). However, it does not consider minimizing \( \beta \).

E. Estimate of total number of active users

The CMUD algorithm in Table I requires an approximation of the total number of active users \( M \) which is unknown in practice. In the following, we demonstrate an approximate procedure to estimate \( M \). The energy received at BS due to real parts of \( y \) to be (using \[6\])

\[
G = \|y\|^2 = \sum_{\ell=1}^{L} \sum_{m=1}^{M} \{C_{\ell,k,m} \lambda_{\ell,m} - [\lambda_{\ell,m}] C_{\ell,k,m} \} + [v_{\ell}]^2 \tag{32}
\]

Taking expectation of \( G \) with respect to \( \vartheta \) and \( \lambda \), we see

\[
E(G) = E \left( \sum_{\ell=1}^{L} \left\{ \sum_{m=1}^{M} \{C_{\ell,k,m} \lambda_{\ell,m} - [\lambda_{\ell,m}] C_{\ell,k,m} \} + [v_{\ell}]^2 \right\} \right).
\]

The second term of right side represents the cross product of different parameters which is small compared to other terms. By neglecting this term, it can be verified that

\[
E(G) = ML - 2 mL \mu_r + M L E(\|\lambda\|^2) + L \sigma_{\vartheta}^2 / 2 \tag{34}
\]

Hence, by setting \( E(G) = \|y\|^2 \) in (34) one can obtain a rough estimate of \( M \).
IV. SPARSE SIGNAL RECOVER BASED APPROACH

A. Basis mismatched sparse recover representation

The CMUD algorithm is computationally very efficient but cannot detect large number of active users. Furthermore, CMUD requires the information of variances of channel mismatch parameters i.e., $\sigma_h^2, \sigma_e^2, \sigma_n^2$. On the other hand, MUD being a sparse recovery problem, can be solved by using sparse recovery algorithms. Although the sparse recovery algorithms are relatively computationally demanding, they have been demonstrated the capability of resolving larger number of active users provided that the information of channel mismatch parameters are available. However, if the information of $\sigma_h^2, \sigma_e^2, \sigma_n^2$ are unavailable then we cannot efficiently apply the standard sparse recovery algorithms to the data model in (10). To overcome the channel mismatch limitations, we pose the user detection as a basis mismatched sparse signal recovery problem and develop an efficient algorithm to resolve the MUD problem. To understand the concept of basis mismatch, rewrite the data model in (10) as:

$$y - \vartheta = [C - Q] \hat{x}$$

where we need to estimate $\hat{x}$. Since $\hat{x}$ is sparse, $y - \vartheta$ is a linear combination of few selected columns of $[C - Q]$. Hence, we say that $y - \vartheta$ has a sparse representation on the basis $[C - Q]$. The basis mismatch problem arises when we do not have accurate knowledge about a part of the basis. In the present scenario we do not have actual information about Q.

B. Solution strategy

In this work, we adopt a total least-squares (TLS) optimization with sparsity constraint to handle the basis mismatch problem and obtain an estimate of $\hat{x}$. The sparsity constraint TLS algorithm aims to solve the following optimization problems:

$$\{x, Q\} = \arg \min_{x,Q} G_p(x, Q)$$

(35)

where,

$$G_p(x, Q) = \|y_i - (C - Q)x\|_2^2 + \|Q\|_F^2 + 2\frac{\xi}{\bar{\alpha}}\|x\|_p$$

where $0 \leq p \leq 1$ and the value of $\xi$ depends on noise level. The sparsity constraint $G_p(x, Q)$ has been imposed by the $\ell_p$ norm ($0 < p < 1$). The optimization problem is nonconvex and no effective optimization solvers is available that can guarantee the convergence to the global minimum of (35). To handle the problem, the work in [22] splits the optimization into two parts and develop an iterative algorithm to resolve $Q$ and $x$ in alternative fashion. In particular, given the iterate $Q^{(i)}$ at iteration $i \geq 0$, the iterate $x^{(i)}$ is obtained by solving the following optimization

$$x^{(i)} = \arg \min_{x} G_p(x, Q^{(i)})$$

(36)

With $x^{(i)}$ available, $Q^{(i+1)}$ is found as

$$Q^{(i+1)} = \arg \min_{Q} G_p(x^{(i)}, Q)$$

(37)

The procedure is repeated at every iteration until a convergence criterion is satisfied. Note that the second optimization has a closed form of solution [22]:

$$Q^{(i+1)} = \left(1 + \|x^{(i)}\|_2^2\right)^{-1} \left(Cx^{(i)} - |y_i|\right)^\top$$

(38)

The work in [22] sets $p = 1$, which results in a convex optimization problem in (39). However, it has been shown in [23], [24] that some nonconvex relaxations of $\ell_1$ norm i.e., $(0 < p < 1)$ are more efficient in finding sparse solutions. Furthermore, they take lower number of iterations compared to the $\ell_1$ based optimization. Again, in the present case $x_i \in \{1, 0\}$ which has not been considered in the optimization [25]. In this section, we first adopt a nonconvex relaxation of $\ell_1$ norm, i.e., $\ell_p$ norm $(0 < p < 1)$ for the optimization in (36). Second, we incorporate the constraint $\hat{x_i} \in \{1, 0\}$ with the optimization. Finally, we propose an algorithm that applied an iterative reweighted least-square technique to minimize a sequence like (35). We shall demonstrate by simulations that the optimization yields better user detection.

The optimization problem in (36) is nonconvex for $0 < p < 1$. The work in [21] suggests an iterative algorithm called FOCUSS to solve the optimization. At $j$-th iteration, the algorithm solves a reweighted $\ell_2$ minimization problem:

$$\hat{x}^{(j)} = \arg \min_{x} S_p^{(j)}(x, Q^{(i)})$$

(39)

with,

$$S_p^{(j)}(x, Q^{(i)}) = G_p(x, Q^{(i)}) = \frac{\xi}{2} \|y_i - (C - Q^{(i)})x\|_2^2 + \frac{1}{2}x^\top W_p^{(j-1)}x$$

where $W_p^{(j-1)} = \text{diag}\{x_1^{(j-1)}p-2, \ldots, x_K^{(j-1)}p-2\}$. The optimization is convex and has closed form solution. Starting from an initial estimate $\hat{x}^{(0)}$, the algorithm updates $\hat{x}^{(j)}$ in every iteration. The algorithm terminates when $\|\hat{x}^{(j)} - \hat{x}^{(j-1)}\|$ becomes small. It has been shown in [24] that the cost function $S_p^{(j)}(x, Q^{(i)})$ is monotonically non-increasing on the sequence $\{\hat{x}^{(j)}\}_{j=1}^\infty$. Furthermore, the limit of any convergent subsequence of $\{\hat{x}^{(j)}\}_{j=1}^\infty$ is a stationary point of the problem in (36).

To incorporate the constraint $\hat{x_i} \in \{0, 1\}$ with every iteration of the FOCUSS algorithm, we consider the following optimization problem:

$$\hat{x}^{(j)} = \arg \min_{x} S_p^{(j)}(x, Q^{(i)})$$

(40)

Sub. to. $x_i(\hat{x}_i - 1) = 0; \quad t = 1, 2, \cdots, K$. (41)

Due to the binary constraint in (41), the primal problem above is not convex. Hence, we propose a Lagrangian Dual Relaxation method to optimize (41). The Lagrangian associate to (40)-(41) is

$$\mathcal{L}(x, \mu) = x^\top P_p^{(j-1)}x - \langle \xi_{\hat{y}_i}^p (C - Q^{(i)}) + \mu \rangle x$$

(42)

where, $P_p^{(j-1)} =$

$$\left( W_p^{(j-1)} + \frac{\xi}{2} (C - Q^{(i)})^\top (C - Q^{(i)}) + \text{diag}(\mu) \right)$$

(43)

and $\mu \in \mathbb{R}^K$ is the dual variable. The dual function of (42) is $g(\mu) = \inf_{x} \mathcal{L}(x, \mu)$. Since $\mathcal{L}(x, \mu)$ is a convex function of
Putting the value of $x$ in (42), the dual function is

$$g(\mu) = \frac{1}{4} \left( \frac{1}{(C - Q^{(i)})^T[y]_r + \mu} \right) \left( \xi(C - Q^{(i)})^T[y]_r + \mu \right).$$

(45)

Now we need to find $\mu$ that maximize (45). Once we find $\mu$, we can compute $x$ using (44). We apply a gradient ascent method to maximize (45). The method requires to compute the gradient of (45) with respect to $\mu$. At first note that for any vectors $r$ and $s$ independent from $\mu$, we have

$$\frac{\partial [r^\top (P^{(j-1)})^{-1} s]}{\partial \mu} = \text{Tr} \left\{ \left( \frac{\partial [r^\top (P^{(j-1)})^{-1} s]}{\partial (P^{(j-1)})^{-1}} \right) \frac{\partial (P^{(j-1)})^{-1}}{\partial \mu} \right\}$$

(46)

where

$$\frac{\partial [r^\top (P^{(j-1)})^{-1} s]}{\partial (P^{(j-1)})^{-1}} = - \left( (P^{(j-1)})^{-1} s r^\top (P^{(j-1)})^{-1} \right)^\top$$

(47)

By applying chain rule and using (46), it can be verified that the first order derivative of (45) with respect to $\mu$ is

$$\frac{\partial g(\mu)}{\partial \mu} = \frac{1}{4} \left[ - \text{Diag} \left( (P^{(i-1)})^{-1} z v^\top (P^{(i-1)})^{-1} \right) - (P^{(i-1)})^{-1} v + (P^{(i-1)})^{-1} z \right]$$

(48)

where,

$$z = (\xi(C - Q^{(i)})^T[y]_r - \mu); \quad v = (\xi(C - Q^{(i)})^T[y]_r + \mu)$$

and $\text{Diag}(A)$ denotes a vector constructed from the diagonal components of the matrix $A$.

By using the results, we are now ready to develop an iterative algorithm to solve the basis mismatched sparse recovery problem. Motivated by the objectives of (35) and (39), the algorithm aims to find the sequence $\{\hat{x}^{(i)}, Q^{(i)}\} \subseteq \mathbb{R}$ that will satisfy:

$$\tilde{G}_p(\hat{x}^{(i)}, Q^{(i)}) \leq G_p(\hat{x}^{(i-1)}, Q^{(i-1)})$$

(49)

where, $\tilde{G}_p(\hat{x}^{(i)}, Q^{(i)}) = \left\| \left[ y_r - (C - Q^{(i)}) \hat{x}^{(i)} \right] \frac{1}{2} + \left\| Q^{(i)} \right\|_2^2 + \frac{1}{\xi} \left[ \hat{x}^{(i)} \right]^\top W^{(i-1)} \hat{x}^{(i)} \right\|_2^2$

(50)

with the constraint in (41). Note that the main difference of the objective in (49) with (35) is that instead of $\ell_1$ norm, we incorporate a reweighted quadratic term in (49). Similar to (22), we split (49) into two parts. Given the iterate $Q^{(i-1)}$ the algorithm first minimizes $G_p(x, Q^{(i-1)})$ for $\hat{x}^{(i)}$ with constraint (41). Once $\hat{x}^{(i)}$ is obtained, the algorithm solves $Q^{(i)}$ by using (38). The final $\ell_p$-constrained TLS algorithm is given in Table II. In step-2, we compute $\hat{x}^{(i)}$ by applying a standard gradient ascent algorithm (19). In step-3, we compute $\tilde{G}_p(\hat{x}^{(i)}, Q^{(i-1)})$ using (44). We apply a gradient ascent using (44). Note that Step-2 ensures that $S_p^{(i)}(\hat{x}^{(i)}, Q^{(i-1)}) \leq S_p^{(i)}(\hat{x}^{(i-1)}, Q^{(i-1)})$. By using the fact, it can be verified that the algorithm satisfies $G_p(\hat{x}^{(i)}, Q^{(i-1)}) \leq G_p(\hat{x}^{(i-1)}, Q^{(i-1)})$ at every iteration.

C. Performance analysis of sparse recovery algorithms

The optimization in (35) is nonconvex, hence it is difficult to analyse the user detection performance of the sparse TLS algorithm. However, if the information of variances of channel mismatch parameters i.e., $\sigma_n^2, \sigma_n^2, \sigma_n^2$ are available then we can represent the data model in (10) as

$$y = Cx + w$$

(51)

where $w = -u + \theta$. By using the proof of Lemma 3, it can be shown that $\left[ w \right]$ has Gaussian distribution with covariance matrix $\sigma_n^2 I$ where $\sigma_n^2 = M \sigma_n^2 + \sigma_n^2$. Then we can apply Lasso to recover $\hat{x}$:

$$\hat{x}_s = \arg\min_x \frac{1}{2} ||y_r - Cx||_2^2 + \frac{1}{\xi} ||x||_1.$$  

(52)

Define the set $\mathcal{L} = \{i \in \{1, 2, \ldots, K\} : |x_{s_i}| > 0\}$. The following lemma obtained from (21) describes the user detection performance of Lasso.

Lemma 2: Suppose that $M < L$ and the regularization parameter $\xi$ satisfies $\xi < \sqrt{2L\log(K)}$. Then the Lasso will produce a unique solution $\hat{x}_s$ satisfying $\mathcal{L} \subseteq \mathcal{S}$ with probability greater than $1 - 4 \exp(-c_2 L^2/\xi^2)$ where $c_1, c_2$ are some constants whose values depend on the matrix $C$.

V. Simulation & Results

To evaluate the performance of the proposed algorithms, we develop a Monte Carlo simulation model using MATLAB. The simulation scenarios are on a WiMAX network based on the OFDMA/TDD profile. The key simulation parameters are outlined in Table I. Total $L = 144$ subcarriers are reserved for the BR purpose. The modulation pulse is a root-raised-cosine function with a roll-off 0.22 and duration $10T_s$. The BR terminals follow a mixed channel model specified by...
ITU IMT-2000 standards: Ped-A, Ped-B, and Veh-A. The BR terminals select the channel models with equal probability. The mobile speed varies in the interval $[0, 5]$ m/s for Ped-A, Ped-B channels, and $[5, 20]$ m/s for Veh-A. We assume that the variances of channel mismatch noise $e$ and $\eta$ are equal i.e., $\sigma^2_e = \sigma^2_\eta$. We follow the direction of [13] and set $\alpha = e^{i\theta}$ where $\theta = 5^\circ$. Four different methods are applied for code detection: i) CMUD with Decoder-I (Section IV-D1) will be denoted by D-I. ii) CMUD with Decoder-II (Section IV-D2) will be denoted by D-II. iii) Lasso as recommended in [21], and iv) $\ell_p$ (with $p = 0.5$) constrained TLS (Section IV-B) will be denoted by TLS.

| Parameters | Notation | Values |
|------------|----------|--------|
| Base Frequency | $f_{sys}$ | 2.3 GHz |
| Channel Bandwidth | $f$ | 5 MHz |
| Sampling Frequency | $f_n$ | 5.6 MHz |
| FFT Size | $N$ | 512 |
| Subcarrier Spacing | $\Delta f$ | 10.94 KHz |
| OFDM symbol time | $T_s$ | 91.4 $\mu$s |
| Cyclic Prefix (CP) Time | $T_{cp}$ | $T_s/8$ |
| No. of CP Samples | $N_{cp}$ | 64 |
| Frame Duration | $T_f$ | 5 ms |

| TABLE III  
SIMULATION PARAMETERS |
|---------------------------|
| Parameters | Notation | Values |
|---------------|---------|--------|
| Base Frequency | $f_{sys}$ | 2.3 GHz |
| Channel Bandwidth | $f$ | 5 MHz |
| Sampling Frequency | $f_n$ | 5.6 MHz |
| FFT Size | $N$ | 512 |
| Subcarrier Spacing | $\Delta f$ | 10.94 KHz |
| OFDM symbol time | $T_s$ | 91.4 $\mu$s |
| Cyclic Prefix (CP) Time | $T_{cp}$ | $T_s/8$ |
| No. of CP Samples | $N_{cp}$ | 64 |
| Frame Duration | $T_f$ | 5 ms |

![Figure 1](image1.png)

Fig. 1. Performance of different algorithms with low noise variances. We set $\sigma_e = 0.01$, $\sigma_\eta = 0.01$ and $K = 256$.

At first we evaluate the code detection performance by different algorithms at low noisy environment in Figure-1. This simulation will illustrate the capability of detecting largest number of active users by different algorithms. As can be seen, the D-II shows worst user detection performance whereas D-I performs much better than D-II. For example, with $P_e = 0.1$, the D-II can detect only 3 users whereas D-I is capable of detecting upto 10 users. As being claimed, Lasso and TLS performs much better than correlation based approach. The TLS can detect 52 users with $P_e = 0.1$ which is almost 5 times larger than D-I. Finally the figure also reveal the fact that, at low noisy environment, both Lasso and TLS performs similarly.

We now investigate performances of different algorithms at high noisy environment. Figure-2 illustrates performances of different algorithms with measurement noise variance $\sigma^2_\eta$. Figure-2(a) compares performance of CMUD decoders. The performance of a particular algorithm changes slowly with $\sigma_\eta$. For example, with $M = 4$ users the $P_e$ of D-I are 0.1 and 0.18 for $\sigma_\eta = 0.2$ and 0.5 respectively. Again $P_e$ are 0.57 and 0.61 for $\sigma_\eta = 0.2$ and 0.5 respectively with $M = 6$ users. As before, the performance of D-II is worse compared to other algorithms. With $P_e \leq 0.12$, the D-II can detect only 2 users. Furthermore, comparing Figure-2(a) with Figure-1 we see that the user detection performances of all algorithms drop with increasing noise variances. Figure-2(b) compares performances between Lasso and TLS. As can be seen the code detection performance of both algorithms decreases with increasing $\sigma_\eta$. For TLS with $M = 12$, the $P_e$ are 0.12 and 0.24 for $\sigma_\eta = 0.2$ and 0.5 respectively. The code detection performance gap increases with increasing the number of total active users. For example, with $M = 20$ the $P_e$ are 0.66 and 0.92 for $\sigma_\eta = 0.2$ and 0.5 respectively. The performance of Lasso also changes with $\sigma_\eta$. With $M = 8$ the $P_e$ are 0.1 and 0.12 for $\sigma_\eta = 0.2$ and 0.5 respectively. Note that while Figure-1 shows that both Lasso and TLS performs similarly at low noisy environment, their performance do not remain similar at high noisy environment in Figure-2(b).

Figure-3 shows the effect of channel mismatch on the performance of code detection by different algorithms. Figure-3(a) shows that the performance of CMUD decreases rapidly with increasing $\sigma_\eta$. For D-II with $M = 3$, the value of $P_e$ are 0.2 0.45 and 0.79 for $\sigma_\eta = 0.1, 0.15$ and 0.2 respectively. The performance of D-I is relatively better than D-II. For example with $M = 3$ the value of $P_e$ are 0.02 0.025 and 0.23 for $\sigma_\eta = 0.1, 0.15$ and 0.2 respectively. The performance of D-I degrades rapidly at $\sigma_\eta = 0.2$. In contrast, the TLS performs much better than CMUD decoders. In Figure-3(c) we see that with $M = 12$ the value of $P_e$ are 0.07, 0.12 and 0.54 receptively for $\sigma_\eta = 0.1, 0.15$ and 0.2. In a similar setting (i.e., with $M = 12$), the value of $P_e = 1$ for D-I (see Figure-3(a)). Figure-3(b) also exhibits that TLS performs better than Lasso. With $M = 12$ the value of $P_e$ for Lasso are 0.09, 0.24 and 0.74 receptively for $\sigma_\eta = 0.1, 0.15$ and 0.2.

In Section IV-D we have assumed that any BR code will be used by at most one BR subscriber and hence $x_i \in \{0, 1\}$. This assumption will be valid with high probability when the number of active BR users is much smaller than the number of available BR codes i.e., $M \ll K$. According to the simulation setting in Figure-3(b) the TLS can detect almost $M = 20$ users with high probability when $\sigma_e = 0.1, \sigma_\eta = 0.1, \sigma_\theta = 0.2$ and $K = 256$. As the value of $M$ becomes larger we should increase the value of $K$ to remain consistent with the above assumption. To this aim, we investigate the user detection performance of D-I and TLS as a function of $K$ in Figure-4. As can be seen, the TLS can detect 18 users with $P_e = 0.11, 0.12, 0.2$ and 0.23 for $K = 200, 250, 300$ and 400 respectively.

VI. CONCLUSION

Multiuser detection in random access channels plays an important role in OFDMA wireless network to support services with bursty traffic. However, the existing algorithms
Fig. 2. Code detection performance at different measurement noise levels $\sigma_\vartheta$. We set $\sigma_e = 0.15$, $\sigma_\eta = 0.15$ and $K = 256$.

Fig. 3. Code detection performance at different channel mismatch noise levels $\sigma_e$ and $\sigma_\eta$ where $\sigma_e = \sigma_\eta$. We set $\sigma_\vartheta = 0.2$ and $K = 256$.

Fig. 4. Code detection performance for different values of $K$. (a) CMUD with D-I (b) TLS. We set $\sigma_e = 0.1$, $\sigma_\eta = 0.1$, $\sigma_\vartheta = 0.2$. 
suffer from low efficiency when the base station does not have accurate knowledge about the channel frequency response of active users. In this work, two different solutions have been proposed to alleviate the difficulty. The first solution applied a very simple correlation based algorithm. The theoretical analysis of the algorithm provides a direction to achieve optimum performance from the algorithm. The second solution applied a variant of sparse recovery algorithm which exhibits better user detection performance, however, with additional computational complexity.

**APPENDIX A**  
**MATHEMATICAL VARIANCE OF $\lambda$**

Denote the covariance matrix of $[\lambda], [\lambda]_i$ as

$$V([\lambda], [\lambda]_i) = \begin{bmatrix} \sigma^2_e & \sigma_{ri} \\ \sigma_{ri} & \sigma^2_i \end{bmatrix}$$

where $\sigma^2_e = E(\lambda|^2_e) - (E(\lambda_e))^2$, $\sigma_{ri} = E(\lambda_r|\lambda_i) - E(\lambda_r)E(\lambda_i)$. Then $E(\lambda_r)$ can be calculated as

$$E(\lambda_r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda_r f(\lambda_r, \lambda_i) d\lambda_r d\lambda_i$$

We evaluate the integral in polar coordinate. Let us define

$$\gamma = \sqrt{\frac{1 - \rho^2}{\sigma^2_e}},$$

$$[\lambda] = t \sin \theta - \frac{\rho \sigma_u}{\sigma_v}, \quad [\lambda]_i = t \cos \theta + \frac{\rho \sigma_u}{\sigma_v},$$

$$\hat{\alpha} = \frac{\rho \sigma_u}{\sigma_v}, \quad \beta = -\frac{\rho \sigma_u}{\sigma_v}, \quad \Gamma = \frac{(1 - |\rho|^2)\sigma^2_e}{\pi \sigma^2_v}.$$  

Then by combining (54)-(56) along with the pdf in (13) we get

$$E(\lambda_r) = \int_0^{2\pi} \int_0^{\infty} \left( \frac{t^2 \cos \theta}{(t^2 + \gamma^2)^2} + \frac{t \hat{\alpha}}{(t^2 + \gamma^2)^2} \right) dtd\theta$$

$$= \Gamma \int_0^{2\pi} \left[ -\frac{\pi}{4\gamma} \cos \theta + \frac{\hat{\alpha}}{2\gamma} \right] d\theta$$

$$= \frac{\Gamma \pi \hat{\alpha}}{4\gamma} = \hat{\alpha}.$$  

Next we compute $E(\lambda^2_r)$,

$$E(\lambda^2_r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda_r^2 f(\lambda_r, \lambda_i) d\lambda_r d\lambda_i.$$  

Using the similar conversion in (55)-(56), we have

$$E(\lambda^2_r) = \Gamma \int_0^{2\pi} \int_0^{\infty} \left( \frac{t^3 \cos \theta}{(t^2 + \gamma^2)^2} + \frac{2t \hat{\alpha} t \cos \theta}{(t^2 + \gamma^2)^2} + \frac{t \hat{\alpha}^2}{(t^2 + \gamma^2)^2} \right) dtd\theta$$

$$= \Gamma \int_0^{2\pi} \int_0^{\infty} \left( \frac{t^3 \cos \theta}{(t^2 + \gamma^2)^2} + \frac{\Gamma \pi \hat{\alpha}^2}{\gamma^2} \right) dtd\theta.$$  

There is no closed form solution of the first integral term in (59). In fact, the solution become unbounded when the upper limiting value of $t \to \infty$. However, the solution can be bounded by some limiting value of $t < \infty$. In practice, $\sigma_h$ is bounded and $\sigma_h > \sigma_e, \sigma_n$, hence there is very low probability of $(|\lambda|, |\lambda|_i)$ taking very large magnitude. To obtain a reliable bound for $t$, we first find a threshold $T$ for $(|\lambda|, |\lambda|_i)$ such that both $|\lambda_r|$ and $|\lambda_i|$ will remain below $T$ with high probability. Once we obtain a reliable $T$, we can compute a bound on $t$ by using (55). To approximate the value of $T$, we utilize a concept called percentile level [26]. The $\phi$ percentile level denoted by $T_\phi$ is defined as:

$$\int_{-T_\phi}^{T_\phi} \int_{-T_\phi}^{T_\phi} f(|\lambda_r|, |\lambda_i|) d|\lambda_r| d|\lambda_i| \geq \phi$$

The choice of $\phi$ is dictated by two conflicting requirements: If $\phi$ is close to 1, the estimate of threshold $T_\phi$ is reliable but the sample space of $(|\lambda|, |\lambda|_i)$ is large; if $\phi$ is small, the sample space is reduced but the estimate is less reliable. In our experiments, we set $\phi = 0.95$.

**APPENDIX B**  
**PROOF OF LEMMA 1**

Our proof is closely based on the works [27, Thm. 6] and [28, Thm. 4]. Let us define the event

$$\Sigma = \{ \max_{1 \leq \ell \leq K} |D^\top_{\ell} (-u + \theta)|_r < \tau \}.$$  

We shall demonstrate that $\Sigma$ occurs with high probability and that under (21) whenever $\Sigma$ occurs, the active codes will be detected correctly. At first we provide few lemmas which will be used to prove the result.

**Lemma 3:** Suppose that $\theta$ is a zero mean complex Gaussian random vector with covariance matrix $\sigma^2_{\theta} I$ and $u$ is a random vector as defined in (10). If $(\pi(1 + \nu) \log K)^{-1/2} K^{(1 + \nu) - 1} < 1$ for some $\nu > 0$, then the event $\Sigma$ in (61) occurs with probability at least $1 - (\pi(1 + \nu) \log K)^{-1/2} K^{-\nu}$.

**Proof:** Note that $D^\top_{\ell} u$ is a linear combination of random variables $\{\lambda_{j,k}\}$ and by using the properties of $\{\lambda_{j,k}\}$ discussed in Section III-A, each variable $\lambda_{j,k}$ is independent from other with mean $\mu_\ell$ and variance $\sigma^2_{\ell}$. By central limit theorem and using (13)-(15) we see that $D^\top_{\ell} u$ has a Gaussian distribution with mean $\epsilon_\ell \mu_\ell \sum_{j=1}^{K} D^\top_{\ell} C_j$, and variance $M \sigma^2_{\ell} \|D^\top_{\ell} C_j\|_2^2$. Furthermore, since $\theta$ has zero mean i.i.d. Gaussian distribution and entries of $\theta$ are independent from $u$, we get that $|D^\top_{\ell} (-u + \theta)|_r$ has Gaussian distribution with mean $-\epsilon_\ell \mu_\ell \sum_{j=1}^{K} D^\top_{\ell} C_j$ and variance $M \sigma^2_{\ell} \|D^\top_{\ell} C_j\|_2^2 + \|D^\top_{\ell} C_j\|_2^2 \sigma^2_{\ell}/2$.

The random variables $\{D^\top_{\ell} (-u + \theta)|_r\}_{\ell=1}^{K}$ are jointly Gaussian. Hence

$$Pr(\Sigma) = Pr(\max_{1 \leq \ell \leq K} |D^\top_{\ell} (-u + \theta)|_r < \tau) \geq \prod_{\ell=1}^{K} Pr(|D^\top_{\ell} (-u + \theta)|_r < \tau)$$

Denote $\mu_\ell = -\epsilon_\ell \mu_\ell \sum_{j=1}^{K} D^\top_{\ell} C_j$ and $\sigma^2_{\ell} = M \sigma^2_{\ell} \|D^\top_{\ell} C_j\|_2^2 + \|D^\top_{\ell} C_j\|_2^2 \sigma^2_{\ell}/2$. By using Gaussian tail bound, we have

$$Pr(|D^\top_{\ell} (-u + \theta)|_r < \tau) \geq 1 - \sqrt{\frac{2}{\pi \tau \epsilon_\ell}},$$

By using (17)-(19) we have $\frac{\sigma_{\ell}}{\sqrt{\epsilon_\ell}} \leq \sqrt{\frac{M \sigma^2_{\ell} \|D^\top_{\ell} C_j\|_2^2 + \|D^\top_{\ell} C_j\|_2^2 \sigma^2_{\ell}}{\epsilon_\ell}}$.
Similarly, under the event \( \Sigma \):

\[
\sqrt{\frac{2}{\pi \tau - \mu \ell}} e^{\frac{-(\tau - \mu \ell)^2}{2\pi \ell}} \leq (\pi(1+\nu) \log K)^{-1/2} K^{-(1+\nu)}.
\]

(64)

Now combining (62), (63), (64) we obtain that

\[
Pr(\Sigma) \geq \left( 1 - (\pi(1+\nu) \log K)^{-1/2} K^{-(1+\nu)} \right)^K.
\]

(65)

Applying the inequality \((1-x)^K \geq 1 - Kx\), valid for \(K \geq 1\) and \(x \leq 1\), we have

\[
Pr(\Sigma) \geq 1 - (\pi(1+\nu) \log K)^{-1/2} K^{-\nu}.
\]

(66)

Lemma 4. Let \( y = \sum_{\ell \in S} C_\ell - u + \vartheta \) where the random vector \( \vartheta \) has zero mean Gaussian distribution with covariance \( \sigma_\vartheta^2 I \) and mean and covariance of the random vector \( u \) is defined as in (14) and (15) respectively. Let \( D \in \mathbb{R}^{K \times K} \) be a decoder matrix such that \( D_j^\top C_\ell = 1; \forall \ell \). Then under event \( \Sigma \), we have

\[
\max_{j \in S} \{|D_j^\top y_r|\} \leq M + \beta + \tau
\]

(67)

\[
\max_{j \in S} \{|D_j^\top y_r|\} \geq 1 - (M - 1)\beta - \tau.
\]

(68)

Proof: At first we see that under the event \( \Sigma \),

\[
\max_{j \in S} \{|D_j^\top y_r|\} = \max_{j \in S}\left\{ \left| \sum_{\ell \in S} D_j^\top C_\ell + D_j^\top \left( [-u + \vartheta]_r \right) \right| \right\}
\]

\[
\leq \max_{j \in S} \sum_{\ell \in S} |D_j^\top C_\ell| + \max_{j \in S} |D_j^\top [ -u + \vartheta]_r| \leq M + \beta + \tau.
\]

(69)

Similarly, under the event \( \Sigma \)

\[
\max_{j \in S} \{|D_j^\top y_r|\} = \max_{j \in S}\left\{ \left| \sum_{\ell \in S\setminus j} D_j^\top C_\ell + D_j^\top \left( [-u + \vartheta]_r \right) \right| \right\}
\]

\[
\geq 1 - \max_{j \in S} \sum_{\ell \in S\setminus j} |D_j^\top C_\ell| - \max_{j \in S} |D_j^\top [-u + \vartheta]_r| \geq 1 - (M - 1)\beta - \tau.
\]

(70)

Now we are ready to proof the Lemma 1. The proof follows by the principle of mathematical induction. Suppose that the event \( \Sigma \) occurs and \( D \) satisfies (21). As (21) holds, we always find a threshold \( \kappa \) satisfying (22). In the first iteration of CMUD \( z^{(0)} = y \). The Lemma 4 implies that by setting a threshold \( \kappa \) as in (22) we can select at least one active code. Now suppose at \((\ell > 1)\)-th iteration \( \mathbb{T} \) consists of the unique indices of active codes only and \( \# \mathbb{T} = m \) with \( m < M \). We have

\[
z^{(\ell-1)} = \sum_{j \in S\setminus \mathbb{T}} C_j - u + \vartheta
\]

(71)

By using the similar procedure in (69) and (70) it can be verified that

\[
\max_{j \in S\setminus \mathbb{T}} \{|D_j^\top z_r^{(\ell-1)}|\} \leq (M - m)\beta + \tau
\]

\[
\max_{j \in S\setminus \mathbb{T}} \{|D_j^\top z_r^{(\ell-1)}|\} \geq 1 - (M - m - 1)\beta - \tau.
\]

Clearly the threshold \( \kappa \) satisfies \((M - m)\beta + \tau < \kappa - 1 - (M - m - 1)\beta - \tau \). Hence, CMUD will detect active code indices from \( \{S \setminus \mathbb{T} \} \). Now consider the iterate \( p \), when all active codes are detected i.e., \( \mathbb{T} = S \). Then \( z^{(p-1)} = -u + \vartheta \). Hence, as (61) occurs, \( |D_j^\top z_r^{(p-1)}| \tau < \kappa < \text{CMUD} \) will not detect any further code.

References

[1] “Machine-to-Machine (M2M) communications study report,” IEEE 802.16 Broadband Wireless Access Working Group, 2010.

[2] T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: John Wiley & Sons, 1991.

[3] J. Andrews, “Interference cancellation for cellular systems: a contemporary overview,” Wireless Communications, IEEE, vol. 12, no. 2, pp. 19–29, April 2005.

[4] A. K. Fletcher, S. Rangan, and V. K. Goyal, “On-off random access channels: A compressed sensing framework,” 2009.

[5] L. Applebaum, W. Bajwa, M. Duarte, and R. Calderbank, “Multiuser detection in asynchronous on-off random access channels using lasso,” in Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on, Sept 2010, pp. 130–137.

[6] Y. Xie, Y. Eldar, and A. Goldsmith, “Reduced-dimension multiuser detection,” Information Theory, IEEE Transactions on, vol. 59, no. 6, pp. 3858–3874, June 2013.

[7] J. Krinock, M. Singh, M. Paff, A. Lonkar, L. Fung, and C. Lee, “Comments on OFDMA ranging scheme described in IEEE 802.16-01/01H1,” document IEEE, vol. 802, 2001.

[8] “IEEE Standard for Air Interface for Broadband Wireless Access Systems—Amendment 1: Enhancements to Support Machine-to-Machine Applications,” IEEE Std 802.16p-2012, pp. 1–82, 2012.

[9] H. Wu, C. Zhu, R. J. La, X. Liu, and Y. Zhang, “FASA: Accelerated S-ALOHA using access history for event-driven M2M communications,” Networking, IEEE/ACM Transactions on, vol. PP, no. 99, pp. 1–13, 2013.

[10] Y. Liu, C. Yuen, J. Chen, and C. Cao, “A scalable Hybrid MAC protocol for massive M2M networks,” in Wireless Communications and Networking Conference (WCNC), 2013 IEEE, 2013, pp. 250–255.

[11] “IEEE Standard for Local and metropolitan area networks,” IEEE802.16-2009, pp. 1–2080, 2009.

[12] D. G. Jeong and M.-J. Kim, “Effects of channel estimation error in MC-CDMA/TDD systems,” in Vehicular Technology Conference (VTC), 2000 IEEE 51st, vol. 3, 2000, pp. 1773–1777.

[13] K. Fazel and S. Kaiser, Multicarrier and spread spectrum systems: from OFDM and MC-CDMA to LTE and WiMAX. Wiley, com. 2008.

[14] J. B. Andersen, J. O. Nielsen, G. F. Pedersen, G. Bauch, and G. Dietl, “Doppler spectrum from moving scatterers in a random environment,” Wireless Communications, IEEE Transactions on, vol. 8, no. 6, pp. 3270–3277, 2009.

[15] R. Baxley, B. Walkenhorst, and G. Acosta-Marum, “Complex Gaussian ratio distribution with applications for error rate calculation in fading channels with imperfect CSI” in Global Telecommunications Conference (GLOBECOM 2010), 2010 IEEE, Dec 2010, pp. 1–5.

[16] “GPPP TR 37.868 V11.0.0: Third generation partnership project; technical specification group radio access network; study on RAN improvements for machine-type communications.”

[17] H. S. Jiang, S. M. Kim, K. S. Ko, J. Cha, and D. K. Sung, “Spatial group based random access for M2M communications,” Communications Letters, IEEE, vol. 18, no. 6, pp. 961–964, June 2014.

[18] S. Verdu, Multiuser Detection, 1st ed. New York, NY, USA: Cambridge University Press, 1998.

[19] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[20] P. Stoica, J. Li, and M. Xue, “Transmit codes and receive filters for pulse compression radar systems,” Dept. Elect. Comput. Engi., Tech. Report, 2007.
[21] M. Wainwright, “Sharp thresholds for high-dimensional and noisy sparsity recovery using $\ell_1$-constrained quadratic programming (lasso),” Information Theory, IEEE Transactions on, vol. 55, no. 5, pp. 2183–2202, May 2009.

[22] H. Zhu, G. Leus, and G. Giannakis, “Sparsity-cognizant total least-squares for perturbed compressive sampling,” Signal Processing, IEEE Transactions on, vol. 59, no. 5, pp. 2002–2016, May 2011.

[23] E. Candes, M. Wakin, and S. Boyd, “Enhancing sparsity by reweighted $\ell_1$ minimization,” Journal of Fourier Analysis and Applications, pp. 877–905, 2008.

[24] I. Gorodnitsky and B. Rao, “Sparse signal reconstruction from limited data using focuss: a re-weighted minimum norm algorithm,” Signal Processing, IEEE Transactions on, vol. 45, no. 3, pp. 600–616, Mar 1997.

[25] M. Hyder and K. Mahata, “Focuss is a convex-concave procedure,” in Acoustics, Speech and Signal Processing (ICASSP), 2011 IEEE International Conference on, May 2011, pp. 4216–4219.

[26] A. Papoulis, Probability, Random Variables, and Stochastic Processes. New York: McGraw-Hill, 1991.

[27] J. Tropp and A. Gilbert, “Signal recovery from random measurements via orthogonal matching pursuit,” Information Theory, IEEE Transactions on, vol. 53, no. 12, pp. 4655–4666, Dec 2007.

[28] Z. Ben-Haim, Y. Eldar, and M. Elad, “Coherence-based performance guarantees for estimating a sparse vector under random noise,” Signal Processing, IEEE Transactions on, vol. 58, no. 10, pp. 5030–5043, Oct 2010.