Exclusive semileptonic decays of $B$ mesons into light mesons in the relativistic quark model

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Abstract
The exclusive semileptonic $B \rightarrow \pi(\rho)\ell\nu$ decays are studied in the framework of the relativistic quark model based on the quasipotential approach in quantum field theory. The large recoil momentum of the final $\pi(\rho)$ meson allows for the expansion of the decay form factors at $q^2 = 0$ in the inverse powers of $b$-quark mass. This considerably simplifies the analysis of these decays. The $1/m_b$ expansion is carried out up to the first order. It is found that the $q^2$-dependence of the axial form factor $A_1$ is different from the $q^2$-behaviour of other form factors. We find $\Gamma(B \rightarrow \pi\ell\nu) = (3.1 \pm 0.6) \times |V_{ub}|^2 \times 10^{12} \text{s}^{-1}$ and $\Gamma(B \rightarrow \rho\ell\nu) = (5.7 \pm 1.2) \times |V_{ub}|^2 \times 10^{12} \text{s}^{-1}$. The relation between semileptonic and rare radiative $B$-decays is discussed.

1 Introduction
The investigation of semileptonic decays of $B$ mesons into light mesons is important for the determination of the Cabibbo-Kobayashi-Maskawa matrix element $V_{ub}$, which is the most poorly studied. At present the value of $V_{ub}$ is mainly determined from the endpoint of the lepton spectrum in semileptonic $B$-decays [1]. Unfortunately, the theoretical interpretation of the endpoint region of the lepton spectrum in inclusive $B \rightarrow X_u\ell\bar{\nu}$ decays is very complicated and suffers from large uncertainties [2]. The other way to determine $V_{ub}$ is to consider exclusive semileptonic decays $B \rightarrow \pi(\rho)e\nu$. These are the heavy-to-light transitions with a wide kinematic range. In contrast to the heavy-to-heavy transitions, here we can not expand matrix elements in the inverse powers of the final quark mass. It is also necessary to mention that the final meson has a large recoil momentum almost in the whole kinematical range. Thus the motion of final $\pi(\rho)$ meson should be treated relativistically. If we consider the point of maximum recoil of the final meson, we find that $\pi(\rho)$ bears the large relativistic recoil momentum $|\Delta_{\text{max}}|$ of order of $m_b/2$ and the energy of the same order. Thus at this kinematical point it is possible to expand the matrix element of the weak current both in inverse powers of $b$-quark mass of the initial $B$ meson and in inverse powers of the recoil momentum $|\Delta_{\text{max}}|$ of the final $\pi(\rho)$ meson. As a result the expansion in powers $1/m_b$ arises for the $B \rightarrow \pi(\rho)$ semileptonic form factor at $q^2 = 0$, where $q^2$ is a momentum carried by the lepton pair. The aim of this paper is to realize such expansion in the framework of relativistic quark model. We show that
this expansion considerably simplifies the analysis of exclusive $B \to \pi(\rho)e\nu$ semileptonic decays.

Our relativistic quark model is based on the quasipotential approach in quantum field theory with the specific choice of the $q\bar{q}$ potential. It provides a consistent scheme for calculation of all relativistic corrections at a given order of $v^2/c^2$ and allows for the heavy quark $1/m_Q$ expansion. This model has been applied for the calculations of meson mass spectra [3], radiative decay widths [4], pseudoscalar decay constants [5], heavy-to-heavy semileptonic [6] and nonleptonic [7] decay rates. The heavy quark $1/m_Q$ expansion in our model for the heavy-to-heavy semileptonic transitions has been developed in [8] up to $1/m_b^2$ order. The results are in agreement with the model independent predictions of the heavy quark effective theory (HQET) [9]. The $1/m_b$ expansion of rare radiative decay form factors of $B$ mesons has been carried out in [10] along the same lines as in the present paper.

2 Relativistic quark model

In the quasipotential approach meson is described by the wave function of the bound quark-antiquark state, which satisfies the quasipotential equation [11] of the Schrödinger type [12]:

$$\left(\frac{b^2(M)}{2\mu_R} - \frac{p^2}{2\mu_R}\right)\Psi_M(p) = \int \frac{d^3q}{(2\pi)^3} V(p, q; M)\Psi_M(q),$$

where the relativistic reduced mass is

$$\mu_R = \frac{M^4 - (m_a^2 - m_b^2)^2}{4M^3};$$

$$b^2(M) = \frac{[M^2 - (m_a + m_b)^2][M^2 - (m_a - m_b)^2]}{4M^2},$$

$m_{a,b}$ are the quark masses; $M$ is the meson mass; $p$ is the relative momentum of quarks. While constructing the kernel of this equation $V(p, q; M)$ — the quasipotential of quark-antiquark interaction — we have assumed that effective interaction is the sum of the one-gluon exchange term with the mixture of long-range vector and scalar linear confining potentials. We have also assumed that at large distances quarks acquire universal nonperturbative anomalous chromomagnetic moments and thus the vector long-range potential contains the Pauli interaction. The quasipotential is defined by [3]:

$$V(p, q, M) = \bar{\pi}_a(p)\bar{\pi}_b(-p)\left\{\frac{4}{3}\alpha_S D_{\mu\nu}(k)\gamma^\mu\gamma^\nu + V_{\text{conf}}^{V}(k)\Gamma^a_{\nu\mu} + V_{\text{conf}}^{S}(k)\right\}u_a(q)u_b(-q),$$

where $\alpha_S$ is the QCD coupling constant, $D_{\mu\nu}$ is the gluon propagator; $\gamma_\mu$ and $u(p)$ are the Dirac matrices and spinors; $k = p - q$; the effective long-range vector vertex is

$$\Gamma_\mu(k) = \gamma_\mu + \frac{i\kappa}{2m}\sigma_{\mu\nu}k^\nu.$$
κ is the anomalous chromomagnetic quark moment. Vector and scalar confining potentials in the nonrelativistic limit reduce to

\[ V^V_{\text{conf}}(r) = (1 - \varepsilon)(Ar + B), \quad V^S_{\text{conf}}(r) = \varepsilon(Ar + B), \]  

reproducing \( V_{\text{nonrel}}^{\text{conf}}(r) = V^V_{\text{conf}} + V^V_{\text{conf}} = Ar + B \), where \( \varepsilon \) is the mixing coefficient. The explicit expression for the quasipotential with the account of the relativistic corrections of order \( v^2/c^2 \) can be found in ref. [3]. All the parameters of our model: quark masses, parameters of linear confining potential \( A \) and \( B \), mixing coefficient \( \varepsilon \) and anomalous chromomagnetic quark moment \( \kappa \) were fixed from the analysis of meson masses [3] and radiative decays [4]. Quark masses: \( m_b = 4.88 \text{ GeV} \); \( m_c = 1.55 \text{ GeV} \); \( m_s = 0.50 \text{ GeV} \); \( m_{u,d} = 0.33 \text{ GeV} \) and parameters of linear potential: \( A = 0.18 \text{ GeV}^2 \); \( B = -0.30 \text{ GeV} \) have standard values for quark models. The value of the mixing coefficient of vector and scalar confining potentials \( \varepsilon = -0.9 \) has been primarily chosen from the consideration of meson radiative decays, which are very sensitive to the Lorentz-structure of the confining potential: the resulting leading relativistic corrections coming from vector and scalar potentials have opposite signs for the radiative MI-decays [4]. Universal anomalous chromomagnetic moment of quark \( \kappa = -1 \) has been fixed from the analysis of the fine splitting of heavy quarkonia \( ^3P_J \) states [3].

Recently we have considered the expansion of the matrix elements of weak heavy quark currents between pseudoscalar and vector meson states up to the second order in inverse powers of the heavy quark masses [8]. It has been found that the general structure of leading, subleading and second order \( 1/m_Q \) corrections in our relativistic model is in accord with the predictions of HQET. The heavy quark symmetry and QCD impose rigid constraints on the parameters of the long-range potential of our model. The analysis of the first order corrections [8] allowed to fix the value of effective long-range anomalous chromomagnetic moment of quarks \( \kappa = -1 \), which coincides with the result, obtained from the mass spectra [3]. The mixing parameter of vector and scalar confining potentials has been found from the comparison of the second order corrections to be \( \varepsilon = -1 \). This value is very close to the previous one \( \varepsilon = -0.9 \) determined from radiative decays of mesons [4]. Therefore, we have got QCD and heavy quark symmetry motivation for the choice of the main parameters of our model. The found values of \( \varepsilon \) and \( \kappa \) imply that confining quark-antiquark potential has predominantly Lorentz-vector structure, while the scalar potential is anticonfining and helps to reproduce the initial nonrelativistic potential.

3 \( B \to \pi(\rho)e\nu \) decay form factors

The form factors of the semileptonic decays \( B \to \pi e\nu \) and \( B \to \rho e\nu \) are defined in the standard way as:

\[ \langle \pi(p_{\pi})|q\gamma_\mu b|B(p_B)\rangle = f_+(q^2)(p_B + p_{\pi})_\mu + f_-(q^2)(p_B - p_{\pi})_\mu, \]  

\[ \langle \rho(p_\rho, e)|q\gamma_\mu(1 - \gamma^5)b|B(p_B)\rangle = -(M_B + M_\rho)A_1(q^2)e^*_\mu. \]
\[
+ \frac{A_2(q^2)}{M_B + M_\rho} (e^* p_B)(p_B + p_\rho)_\mu + \frac{A_3(q^2)}{M_B + M_\rho} (e^* p_B)(p_B - p_\rho)_\mu + \frac{2V(q^2)}{M_B + M_\rho} i \epsilon_{\mu \tau \sigma \rho} e^{* \nu} p_B^\tau p_\sigma^\rho, \quad (8)
\]

where \( q = p_B - p_{\pi(\rho)} \), \( e \) is a polarization vector of \( \rho \) meson. In the limit of vanishing lepton mass, the form factors \( f_- \) and \( A_3 \) do not contribute to the decay rates and thus will not be considered.

The matrix element of the local current \( J \) between bound states in the quasipotential method has the form [13]:

\[
\langle \pi(\rho)|J_\mu(0)|B \rangle = \int \frac{d^3 p d^3 q}{(2\pi)^6} \Psi_{\pi(B)}(p) \Gamma_\mu(p, q) \Psi_B(q), \quad (9)
\]

where \( \Gamma_\mu(p, q) \) is the two-particle vertex function and \( \Psi_{\pi, B} \) are the meson wave functions projected onto the positive energy states of quarks.

In the case of semileptonic decays \( J_\mu = \bar{q} \gamma_\mu (1 - \gamma^5) b \) and in order to calculate its matrix element between meson states it is necessary to consider the contributions to \( \Gamma \) from Figs. 1 and 2. Thus the vertex functions look like

\[
\Gamma_\mu^{(1)}(p, q) = \bar{u}_q(p_1) \gamma_\mu (1 - \gamma^5) u_b(q_1) (2\pi)^3 \delta(p_2 - q_2), \quad (10)
\]

and

\[
\Gamma_\mu^{(2)}(p, q) = \bar{u}_q(p_1) \bar{u}_q(p_2) \left\{ \gamma_\mu (1 - \gamma^5) \frac{\Lambda^{(-)}_{k_1}}{\varepsilon_q(k_1) + \varepsilon_q(p_1)} \gamma_1^0 V(p_2 - q_2) + V(p_2 - q_2) \frac{\Lambda^{(-)}_{k_1'}}{\varepsilon_q(k_1') + \varepsilon_q(q_1)} \gamma_1^0 \gamma_\mu (1 - \gamma^5) \right\} u_b(q_1) u_q(q_2), \quad (11)
\]

where \( k_1 = p_1 - \Delta; \quad k_1' = q_1 + \Delta; \quad \Delta = p_B - p_{\pi(\rho)}; \quad \varepsilon(p) = (m^2 + p^2)^{1/2}; \quad \Lambda^{(-)}(p) = \frac{\varepsilon(p) - (m^0 + \gamma^0(\gamma p))}{2\varepsilon(p)}.
\]

Note that the contribution \( \Gamma^{(2)} \) is the consequence of the projection onto the positive-energy states. The form of the relativistic corrections resulting from the vertex function \( \Gamma^{(2)} \) is explicitly dependent on the Lorentz-structure of \( q\bar{q} \)-interaction.

It is convenient to consider the decay \( B \rightarrow \pi(\rho) e \nu \) in the \( B \) meson rest frame. Then the wave function of the final \( \pi(\rho) \) meson moving with the recoil momentum \( \Delta \) is connected with the wave function at rest by the transformation [13]

\[
\Psi_{\pi(\rho)}(p) = D^{1/2}_q(R_{L\Delta}) D^{1/2}_q(R_{L\Delta}) \Psi_{\pi(\rho)}(0, p), \quad (12)
\]

where \( D^{1/2}(R) \) is the well-known rotation matrix and \( R^{W} \) is the Wigner rotation.

The meson wave functions in the rest frame have been calculated by numerical solution of the quasipotential equation (1) [14]. However, it is more convenient to use analytical expressions for meson wave functions. The examination of numerical results for the ground
state wave functions of mesons containing at least one light quark has shown that they can be well approximated by the Gaussian functions

$$\Psi_M(p) \equiv \Psi_{Mo}(p) = \frac{(4\pi)^{3/4} \beta_M^3}{\beta_M^3} \exp\left(-\frac{p^2}{2\beta_M^2}\right),$$  \hspace{1cm} (13)

with the deviation less than 5%.

The parameters are

$$\beta_B = 0.41 \text{ GeV}; \quad \beta_{\pi(\rho)} = 0.31 \text{ GeV}.$$

Substituting the vertex functions (10), (11), with the account of the Lorentz transformation of final meson wave function (12), in the matrix element (9) we can get the expressions for the semileptonic decay form factors defined in eqs. (7) and (8). They are very cumbersome and lengthy especially those which arise from the vertex function $\Gamma^{(2)}$ in Fig. 2, because they explicitly depend on the form of the $q\bar{q}$-interaction potential and it is necessary to integrate both with respect to $d^3p$ and $d^3q$ (see eq. (9)). However, as it was already mentioned in the introduction, at the point of maximum recoil of final $\pi(\rho)$ meson the large value of recoil momentum $|\Delta_{max}| \sim m_b/2$ allows for the expansion in powers of $1/m_b$ for the decay form factors $f_+(0), A_1(0), A_2(0)$ and $V(0)$. This expansion leads to considerable simplification of the formulae. The large value of recoil momentum $|\Delta_{max}|$ permits to neglect $p^2$ in comparison with $\Delta_{max}^2$ in the quark energy $\varepsilon_q(p + \Delta)$ of final meson in the expressions for the form factors originating from $\Gamma^{(2)}$. Thus we can perform one of the integrations in the current matrix element (9) using the quasipotential equation as in the case of the heavy final meson [4,6]. As a result, we get compact formulae suitable for numerical treatment. Neglecting terms of the second order in $1/m_b$ expansion we get for the form factor of $B \to \pi e\nu$ decay:

$$f_+(0) = \sqrt{\frac{E_\pi}{M_B}} \int \frac{d^3p}{(2\pi)^3} \Psi_\pi\left(p + \frac{2\varepsilon_q}{E_\pi + M_\pi} \Delta_{max}\right) \sqrt{\frac{\varepsilon_q(p + \Delta_{max}) + m_q}{2\varepsilon_q(p + \Delta_{max})}}

\times \left\{1 + \frac{M_B - E_\pi}{\varepsilon_q(p + \Delta_{max}) + m_q} + \frac{3}{2} \frac{p^2}{\Delta_{max}^2} \frac{(p\Delta_{max})}{(\varepsilon_q(p) + m_q)(\varepsilon_q(p + \Delta_{max}) + m_q)}

- 2 \left(\frac{p_x^2 + p_y^2}{(\varepsilon_q(p) + m_q)^2 \varepsilon_b(\Delta_{max}) + m_b} \frac{2}{\Delta_{max}^2} \frac{1}{\varepsilon_q(p) + m_q} \left(1 + \frac{M_B - E_\pi}{\varepsilon_b(\Delta_{max}) + m_b}\right)\right)

\times \left(M_B + M_\pi - \varepsilon_b(p) - \varepsilon_q(p) - 2\varepsilon_q\left(p + \frac{2\varepsilon_q}{E_\pi + M_\pi} \Delta_{max}\right)\right)\right\} \Psi_B(p), \hspace{1cm} (14)$$

where we have set [3] the mixing parameter (6) of vector and scalar confining potentials $\varepsilon = -1$ and long-range anomalous chromomagnetic quark moment (5) $\kappa = -1$;

$$|\Delta_{max}| = \frac{M_B^2 - M_\pi^2}{2M_B}; \quad E_\pi = \frac{M_B^2 + M_\pi^2}{2M_B}; \hspace{1cm} (15)$$

\footnote{Such approximation corresponds to the omitting terms of the third order in $1/m_b$ expansion.}
and $z$-axis is chosen in the direction of $\Delta$.

Similar expressions can be written for the form factors of $B \to \rho e \nu$ decay at $q^2 = 0$. They have the same structure as (14) and will be given elsewhere [15].

Let us proceed further and for the sake of consistency carry out the complete expansion of form factor (14) in inverse powers of $b$-quark mass.

The mass of $B$ meson has the following expansion in $1/m_B$ [9]

$$M_B = m_b + \bar{\Lambda} + O\left(\frac{1}{m_b}\right),$$  \hfill (16)

In our model parameter $\bar{\Lambda}$ is equal to the mean value of the light quark energy inside the $B$ meson $\bar{\Lambda} = \langle \epsilon_q \rangle_B \approx 0.54 \text{ GeV}$ [8].

Now we use the Gaussian approximation for the wave functions (13). Then shifting the integration variable $p$ in (14) by $-\frac{\epsilon_q}{E_\pi + M_\pi} \Delta_{\text{max}}$, we can factor out the $\Delta$ dependence of the meson wave function overlap in form factor $f_+$. The result can be written in the form

$$f_+(0) = F_+(\Delta_{\text{max}}^2 \exp(-\zeta \Delta_{\text{max}}^2),$$  \hfill (17)

where $|\Delta_{\text{max}}|$ is given by (15) and

$$\zeta \Delta_{\text{max}}^2 = \frac{2\bar{\Lambda}^2 \Delta_{\text{max}}^2}{(\beta_B^2 + \beta_{\pi}^2)(E_\pi + M_\pi)^2} = \frac{\bar{\Lambda}^2}{\beta_B^2} \eta \left(\frac{M_B - M_\pi}{M_B + M_\pi}\right)^2,$$  \hfill (18)

here $\eta = \frac{2\beta_{\pi}^2}{\beta_B^2 + \beta_{\pi}^2}$ and $\bar{\Lambda}$ is equal to the mean value of light quark energy between $B$ and $\pi$ meson states:

$$\bar{\Lambda} = \langle \epsilon_q \rangle = \frac{m_q^2}{\frac{1}{\pi} \beta_\pi \sqrt{\eta} e^z K_1(z)} \approx 0.53 \text{ GeV},$$  \hfill (19)

where $z = \frac{m_q^2}{2\eta \beta_\pi^2}$.

Substituting the Gaussian wave functions (13) in the expression for the form factor (14) and taking into account (16)–(18), we get up to the first order in $1/m_b$ expansion:

$$f_+(0) = N \exp\left(-\frac{\bar{\Lambda}^2}{\beta_B^2}\right) \left(1 + \frac{1}{m_b} \left(4 \frac{\bar{\Lambda}^2}{\beta_B^2} \eta M_\pi + \frac{2}{3} \frac{p^2}{\bar{\epsilon}_q + m_q}\right) \left(1 + \frac{6}{\sqrt{5} + 2}\right)
- \frac{4}{3(\sqrt{5} + 2)} \left(\frac{p^2}{\bar{\epsilon}_q + m_q}\right) \left(\bar{\Lambda} + M_\pi + 3m_q\right) + \bar{\Lambda} \eta \left(2 \left(1 + \frac{2}{\sqrt{5} + 2}\right)
\times \left(\frac{1}{\bar{\epsilon}_q + m_q}\right) \left(\bar{\Lambda} + M_\pi + 3m_q\right) - 3 + \frac{1}{3} \left(\frac{p^2}{\bar{\epsilon}_q (\bar{\epsilon}_q + m_q)}\right) - \frac{1}{2}\right)\right),$$

where $N = \left(\frac{2\beta_B \beta_{\pi}}{\beta_B^2 + \beta_{\pi}^2}\right)^{3/2} = \left(\frac{\beta_{\pi}}{\beta_B \eta}\right)^{3/2}$ is due to the normalization of Gaussian wave functions in (21); $\bar{\epsilon}_q = \sqrt{p^2 + m_q^2 + \bar{\Lambda}^2 \eta^2}$, i.e. the energy of light quarks in final meson acquires additional contribution from the recoil momentum.
4 Results and discussion

Using the parameters of Gaussian wave functions (13) in the expression (20) for the $B \rightarrow \pi$ transition form factor $f_+(0)$ and in the similar expressions [15] for the $B \rightarrow \rho$ transition form factors $A_1(0)$, $A_2(0)$ and $V(0)$ we get

$$
\begin{align*}
    f_+^{B\rightarrow\pi}(0) & = 0.21 \pm 0.02 & V^{B\rightarrow\rho}(0) & = 0.29 \pm 0.03 \\
    A_1^{B\rightarrow\rho}(0) & = 0.26 \pm 0.03 & A_2^{B\rightarrow\rho}(0) & = 0.30 \pm 0.03.
\end{align*}
$$

(21)

The theoretical uncertainty in (21) result mostly from the approximation of the wave functions by Gaussians (13) and does not exceed 10% of form factor values. We have also calculated the second order terms in $1/m_b$ expansion of form factors [15] and found them to be small (less than 5% of form factor values).

We compare our results (21) for the form factors of $B \rightarrow \pi (\rho) e\nu$ decays with the predictions of quark models [16,17] and QCD sum rules [18,19] in Table 1. There is an agreement between our value of $f_+^{B\rightarrow\pi}(0)$ and QCD sum rule predictions, while our $B \rightarrow \rho e\nu$ form factors are approximately 1.5 times less than QCD sum rule results.

To calculate the $B \rightarrow \pi (\rho)$ semileptonic decay rates it is necessary to determine the $q^2$-dependence of the form factors. Analysing the expressions (14), (17), (18), (20) for the form factor $f_+$ and similar expressions [15] for $A_1$, $A_2$ and $V$, we find that the $q^2$-dependence of these form factors near $q^2 = 0$ is given by

$$
\begin{align*}
    f_+(q^2) & = \frac{M_B + M_\pi}{2\sqrt{M_B M_\pi}} \tilde{\xi}(w) F_+(\Delta_{\text{max}}^2), \\
    A_1(q^2) & = \frac{2\sqrt{M_B M_\rho}}{M_B + M_\rho} \frac{1}{2} (1 + w) \tilde{\xi}(w) A_1(\Delta_{\text{max}}^2), \\
    A_2(q^2) & = \frac{M_B + M_\rho}{2\sqrt{M_B M_\rho}} \tilde{\xi}(w) A_2(\Delta_{\text{max}}^2), \\
    V(q^2) & = \frac{M_B + M_\rho}{2\sqrt{M_B M_\rho}} \tilde{\xi}(w) V(\Delta_{\text{max}}^2),
\end{align*}
$$

(21)–(24)

where $w = \frac{M_B^2 + M_\pi M_\rho - q^2}{2M_B M_\pi M_\rho}$; $F_+(\Delta_{\text{max}}^2)$ is defined by (17) and $A_{1,2}(\Delta_{\text{max}}^2)$, $V(\Delta_{\text{max}}^2)$ are defined similarly. We have introduced the function

$$
\tilde{\xi}(w) = \left( \frac{2}{w + 1} \right)^{1/2} \exp \left( -\eta \frac{\tilde{\Lambda}^2 w - 1}{\beta^2 w + 1} \right),
$$

(25)

which in the limit of infinitely heavy quarks in the initial and final mesons coincides with the Isgur-Wise function of our model [8]. In this limit eqs. (21)–(24) reproduce the leading order prediction of HQET [9].

It is important to note that the form factor $A_1$ in (22) has a different $q^2$-dependence than the other form factors (21), (23), (24). In the quark models it is usually assumed
the pole [16] or exponential [17] \( q^2 \)-behaviour for all form factors. However, the recent QCD sum rule analysis indicate that the form factor \( A_1 \) has \( q^2 \)-dependence different from other form factors [18,19]. In [19] it even decreases with the increasing \( q^2 \) as

\[
A_1(q^2) \simeq \left( 1 - \frac{q^2}{M_b^2} \right) A_1(0) \simeq \frac{2M_B M_\rho}{(M_B + M_\rho)^2} (1 + w) A_1(0). \tag{26}
\]

Such behaviour corresponds to replacing \( \tilde{\xi}(w) \) in (22) by \( \tilde{\xi}(w_{\text{max}}) \).

We have calculated the decay rates of \( B \to \pi(\rho)e\nu \) using our form factor values at \( q^2 = 0 \) and the \( q^2 \)-dependence (21)–(25) in the whole kinematical region (model A). We have also used the usual pole dependence for form factors \( f_+(q^2), A_2(q^2), V(q^2) \) and \( A_1(q^2) = \frac{2M_B M_\rho}{(M_B + M_\rho)^2} (1 + w) \frac{A_1(0)}{\tilde{\xi}(q^2/m_\rho^2)} \) (model B), which corresponds to replacing the function \( \tilde{\xi}(w) \) (25) by the pole form factor. The results are presented in Table 2 in comparison with the quark model [16,17] and QCD sum rule [18,19] predictions. We see that our results for the above mentioned models A and B of form factor \( q^2 \)-dependence coincide within errors. The ratio of the rates \( \Gamma(B \to \rho e\nu)/\Gamma(B \to \pi e\nu) \) is considerably reduced in our model compared to the BSW [16] and ISGW [17] models, with the simple pole and exponential \( q^2 \)-behaviour of all form factors. Meanwhile our prediction for this ratio is in agreement with QCD sum rule results [18,19]. The absolute values of the rates \( \Gamma(B \to \pi e\nu) \) and \( \Gamma(B \to \rho e\nu) \) in our model are close to those from QCD sum rules [19]. The predictions for the rates with longitudinally and transversely polarized \( \rho \) meson differ considerably in these approaches. This is mainly due to different \( q^2 \)-behaviour of \( A_1 \) (see (22), (26) or usual pole dominance model [16]). Thus the measurement of the ratios \( \Gamma(B \to \rho e\nu)/\Gamma(B \to \pi e\nu) \) and \( \Gamma_L/\Gamma_T \) should provide the test of \( q^2 \)-dependence of \( A_1 \) and may discriminate between these approaches.

The differential decay spectra \( \frac{1}{\Gamma} \frac{d\Gamma}{dx} \) for \( B \to \pi(\rho) \) semileptonic transitions, where \( x = \frac{E_\ell}{M_B} \) and \( E_\ell \) is lepton energy, are presented in Fig. 3 (see also [23]).

We can use our results for \( V \) and \( A_1 \) to test the HQET relation [20] between the form factors of the semileptonic and rare radiative decays of \( B \) mesons. Isgur and Wise [20] have shown that in the limit of infinitely heavy \( b \)-quark mass an exact relation connects the form factors \( V \) and \( A_1 \) with the rare radiative decay \( B \to \rho \gamma \) form factor \( F_1 \) defined by:

\[
\langle \rho(p_\rho, e)|\bar{u}\sigma_{\mu\nu}q_v P_R b\rangle B(p_B) = i\epsilon_{\mu\nu\tau\sigma}e^{\ast \nu}p_\rho^{\ast \tau}p_\mu^{\ast \sigma}F_1(q^2) + [(e^{\ast \mu}(M_B^2 - M_\rho^2) - (e^{\ast}q)(p_B + p_\rho)_\mu]G_2(q^2). \tag{27}
\]

This relation is valid for \( q^2 \) values sufficiently close to \( q^2_{\text{max}} = (M_B - M_\rho)^2 \) and reads:

\[
F_1(q^2) = \frac{q^2 + M_B^2 - M_\rho^2}{2M_B} V(q^2) + \frac{M_B + M_\rho}{2M_B} A_1(q^2). \tag{28}
\]

It has been argued in [21,22,19], that in these processes the soft contributions dominate over the hard perturbative ones, and thus the Isgur-Wise relations (28) could be extended to the whole range of \( q^2 \). In [10] we developed 1/\( m_b \) expansion for the rare radiative decay
form factor $F_1(0)$ using the same ideas as in the present discussion of semileptonic decays. It was shown that Isgur-Wise relation (28) is satisfied in our model at leading order of $1/m_b$ expansion. The found value of the form factor of rare radiative decay $B \rightarrow \rho \gamma$ up to the second order in $1/m_b$ expansion is [10]

$$F_1^{B \rightarrow \rho}(0) = 0.26 \pm 0.03.$$  

(29)

Using (28) and the values of form factors (21) we find

$$F_1^{B \rightarrow \rho}(0) = 0.27 \pm 0.03,$$  

(30)

which is in accord with (29). Thus we conclude that $1/m_b$ corrections do not break the Isgur-Wise relation (28) in our model.

5 Conclusions

We have investigated the semileptonic decays of $B$ mesons into light mesons. The recoil momentum of final $\pi(\rho)$ meson is large compared to the $\pi(\rho)$ mass almost in the whole kinematical range. This requires the completely relativistic treatment of these decays. On the other hand, the presence of large recoil momentum, which for $q^2 = 0$ is of order of $m_b/2$, allows for the $1/m_b$ expansion of weak decay matrix element at this point. The contributions to this expansion come both from heavy $b$-quark mass and large recoil momentum of light final meson.

Using the quasipotential approach in quantum field theory and the relativistic quark model, we have performed the $1/m_b$ expansion of the semileptonic decay form factors at $q^2 = 0$ up to the first order. We have determined the $q^2$-dependence of the form factors near $q^2 = 0$. It has been found that the axial form factor $A_1$ has a $q^2$-behaviour different from other form factors (see (21)-(24)). This is in agreement with recent QCD sum rule results [18,19]. The ratios $\Gamma(B \rightarrow \rho \nu)/\Gamma(B \rightarrow \pi \nu)$ and $\Gamma_L/\Gamma_T$ are very senstive to the $q^2$-dependence of $A_1$, and thus their experimental measurement may discriminate between different approaches.

We have considered the relation between semileptonic decay form factors and rare radiative decay form factor [20], obtained in the limit of infinitely heavy $b$-quark. It has been found that in our model $1/m_b$ corrections do not break this relation.

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Table 1 Semileptonic $B \to \pi$ and $B \to \rho$ decay form factors.

| Ref. | $f_+^{B \to \pi}(0)$ | $A_1^{B \to \rho}(0)$ | $A_2^{B \to \rho}(0)$ | $V^{B \to \rho}(0)$ |
|------|----------------------|-----------------------|-----------------------|---------------------|
| our results | 0.21 ± 0.02 | 0.26 ± 0.03 | 0.30 ± 0.03 | 0.29 ± 0.03 |
| 16 | 0.33 | 0.28 | 0.28 | 0.33 |
| 17 | 0.09 | 0.05 | 0.02 | 0.27 |
| 18 | 0.26 ± 0.02 | 0.5 ± 0.1 | 0.4 ± 0.2 | 0.6 ± 0.2 |
| 19 | 0.23 ± 0.02 | 0.38 ± 0.04 | 0.45 ± 0.05 | 0.45 ± 0.05 |

Table 2 Semileptonic decay rates $\Gamma(B \to \pi e \nu)$, $\Gamma(B \to \rho e \nu)$ ($\times |V_{ub}|^2 \times 10^{12} \text{s}^{-1}$) and the ratio of the rates for longitudinally and transversely polarized $\rho$ meson.

| Ref. | $\Gamma(B \to \pi e \nu)$ | $\Gamma(B \to \rho e \nu)$ | $\Gamma_L/\Gamma_T$ |
|------|--------------------------|--------------------------|----------------|
| our results | | | |
| model A | 3.1 ± 0.6 | 5.7 ± 1.2 | 0.6 ± 0.3 |
| model B | 3.0 ± 0.6 | 5.2 ± 1.2 | 0.6 ± 0.3 |
| 16 | 7.4 | 26 | 1.34 |
| 17 | 2.1 | 8.3 | 0.75 |
| 18 | 5.1 ± 1.1 | 12 ± 4 | 0.06 ± 0.02 |
| 19 | 3.6 ± 0.6 | 5.1 ± 1.0 | 0.13 ± 0.08 |

Figure captions

Fig. 1 Lowest order vertex function

Fig. 2 Vertex function with the account of the quark interaction. Dashed line corresponds to the effective potential (4). Bold line denotes the negative-energy part of the quark propagator.

Fig. 3 The differential decay spectra (model A) $\frac{1}{\Gamma} \frac{d\Gamma}{dx}$ for $B \to \pi(\rho)$ semileptonic transitions, where $x = \frac{E_l}{M_B}$ and $E_l$ is lepton energy. Absolute rates $\frac{d\Gamma}{dx}$ can be obtained using $\Gamma(B \to \pi(\rho)e\nu)$ from Table 2.
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This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9505321v2
Fig. 3

\( \frac{1}{G} \frac{dG}{dx} \) vs. \( x \)

- \( \rho \)
- \( \pi \)