Strong rigidity of constant curvature Finsler manifolds

Asanjarani A. Bidabad B.
Amirkabir University of Technology (Tehran Polytechnic),
Faculty of Mathematics and Computer Sciences,
424 Hafez Ave., 15914 Tehran, Iran.
bidadad@aut.ac.ir

Abstract

Here, an extension of the Obata-Tanno’s theorem to Finsler geometry is established and the following rigidity result is obtained: Every complete connected Finsler manifold of positive constant flag curvature is isometrically homeomorphic to an n-sphere equipped with a certain Finsler metric, and vice versa.

1 Introduction.

Rigidity describes quite different concepts in geometry. Historically, the first rigidity theorem, proved by Cauchy in 1813, states that if the faces of a convex polyhedron were made of metal plates and the edges were replaced by hinges, the polyhedron would be rigid [6]. Although rigidity problems were of immense interest to engineers, intensive mathematical study of these types of problems has occurred only relatively recently [7]. Geometrically, sometimes an object is said to be rigid if it has flexibility and not elasticity, that is to say, invariant by each isometry. In Riemannian geometry the sectional curvature is invariant under isometries. Hence a space of positive constant curvature is transformed into the same space by each isometry. This fact is sometimes described as the ”strong rigidity” of a space of constant curvature. So far, in Finsler geometry, the encountered rigidity results are rather slightly weaker and they say that under such and such assumptions about the flag curvature -analogous to the sectional curvature in Riemannian geometry- the underlying Finsler structure must be either Riemannian or locally Minkowskian. A famous treatise of this kind due to Akbar-Zadeh [2] in 1988 who has established in compact case the following rigidity theorem; Let \((M, g)\) be a compact without boundary Finsler manifold of constant flag curvature \(K\). If \(K < 0\), then \((M, g)\) is Riemannian. If \(K = 0\), then \((M, g)\) is locally Minkowskian.

In 1997 Foulon [8] addresses the case of strictly negative flag curvature by imposing the additional hypothesis that the curvature be covariantly constant along a
distinguished vector field on the homogeneous bundle of tangent half lines. He shows that, under these conditions, as in the Akbar-Zadeh’s theorem, the Finsler structure is Riemannian. Next in 2002, Foulon came back to this problem and presented a strong rigidity theorem for symmetric compact Finsler manifolds with negative curvature and proves that it is isometric to a locally symmetric negatively curved Riemannian space [9]. This extends the Akbar-Zadeh’s rigidity theorem, to a so called, strong rigidity one.

Shen in 2005 [15] observes the case of negative but not necessarily constant flag curvature by imposing the additional hypothesis that the S-curvature be constant and proved that the Akbar-Zadeh’s rigidity theorem still holds.

In 2007 following the several rigidity theorems in the two joint papers [11] and [12], Kim proved that [13]: any compact locally symmetric Finsler manifold with positive constant flag curvature is Riemannian.

One of the present authors has also established in a joint paper in 2007, some rigidity theorems as an application of connection theory in Finsler geometry [5].

Here, we study the strong rigidity of Finsler manifolds of positive constant curvature and show that; Let \((M, g)\) be an \(n\)-dimensional complete connected Finsler manifold. Then it is of positive constant flag curvature \(K > 0\) if and only if \((M, g)\) is isometrically homeomorphic to an \(n\)-sphere equipped with a certain Finsler metric. Particularly, this result completes the Akbar-Zadeh’s rigidity theorem for positive constant flag curvature. Meanwhile, we have extended the Obata-Tanno’s theorem to Finsler geometry as follows. Let \((M, g)\) be a complete connected Finsler manifold of dimension \(n \geq 2\). In order that there is a non-trivial solution of \(\nabla^H \nabla^H \rho + C^2 \rho g = 0\) on \(M\), it is necessary and sufficient that \((M, g)\) be isometric to an \(n\)-sphere of radius \(1/C\).

This result leads us to illustrate a definition for an \(n\)-sphere in Finsler geometry.

2 Preliminaries.

Let \((M, g)\) be an \(n\)-dimensional Finsler manifold and \(TM \rightarrow M\) the bundle of its tangent vectors. Using the local coordinates \((x^i, y^i)\) on \(TM\), called the line elements, we have the local field of frames \(\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}\) on \(TTM\).

Let \(\rho : M \rightarrow \mathbb{R}\) be a scalar function on \(M\) and consider the following second order differential equation

\[
\nabla^H \nabla^H \rho = \phi g,
\]

(2.1)

where \(\nabla^H\) is the Cartan horizontal covariant derivative and \(\phi\) is a function of \(x\) alone, then we say that Eq. (2.1) has a solution \(\rho\). The connected component of a regular hypersurface defined by \(\rho = \text{constant}\), is called a level set of \(\rho\). We denote by \(\text{grad}\rho\) the gradient vector field of \(\rho\) which is locally written in the form \(\text{grad}\rho = \rho^i \frac{\partial}{\partial x^i}\), where \(\rho^i = g^{ij} \rho_j\) and \(i, j, ...\) run over the range 1, ..., \(n\).

The partial derivatives \(\rho_j\) are defined on the manifold \(M\) while \(\rho^i\) the components of \(\text{grad}\rho\) may be defined on its slit tangent bundle \(TM_0\). Hence, \(\text{grad}\rho\) can be considered as a section of \(\pi^*TM \rightarrow TM_0\), the pulled-back tangent bundle over \(TM_0\), and its trajectories lie on \(TM_0\).
One can easily verify that the canonical projection of the trajectories of the vector field $\text{grad} \rho$ are geodesic arcs on $M$. Therefore, we can choose a local coordinates $(u^1 = t, u^2, \ldots, u^n)$ on $M$ such that $t$ is the parameter of the geodesic containing the projection of a trajectory of the vector field $\text{grad} \rho$ and the level sets of $\rho$ are given by $t = \text{constant}$. These geodesics are called $t$-geodesics. Since in this local coordinates, the level sets of $\rho$ are given by $t = \text{constant}$, then $\rho$ may be considered as a function of $t$ only. In the sequel we will refer to these level sets and this local coordinates as $t$-levels and adapted coordinates, respectively.

Let $(M, g)$ be a Finsler manifold and $\rho$ a non-trivial solution of Eq. (2.1) on $M$. Then in an adapted coordinates, components of the Finsler metric tensor $g$ are given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n2} & \cdots & g_{nn} \end{pmatrix},$$

and $t$ may be regarded as the arc-length parameter of $t$-geodesics. It can be easily verified that the Finsler metric form of $M$ is given by

$$ds^2 = (dt)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta,$$

where $f_{\gamma\beta}$ given by $g_{\gamma\beta} = \rho'^2 f_{\gamma\beta}$ are components of a Finsler metric tensor on a $t$-level of $\rho$ and $g_{\gamma\beta}$ is the induced metric tensor of this $t$-level. Here and every where in this paper, the Greek indices $\alpha, \beta, \gamma, \ldots$ run over the range 2, 3, ..., $n$.

If $g(\text{grad} \rho, \text{grad} \rho) = 0$ in some points of $M$, then $M$ possesses some interesting properties. A point $o$ of $(M, g)$ is called a critical point of $\rho$ if the vector field $\text{grad} \rho$ vanishes at $o$, or equivalently if $\rho'(o) = 0$.

**Lemma A.** Let $(M, g)$ be an $n$-dimensional Finsler manifold which admits a non-trivial solution $\rho$ of Eq. (2.1). If $\rho$ admits a critical point, then any $t$-level $M$ with Finsler metric form $ds^2 = f_{\gamma\beta} du^\gamma du^\beta$ has the positive constant flag curvature $\rho''(0)$.

### 3 A special solution.

Let $(M, g)$ be an $n$-dimensional Finsler manifold and $\rho : M \to \mathbb{R}$ a solution of Eq. (2.1). If $\phi$ is a linear function of $\rho$ with constant coefficients, then we say that $\rho$ is a special solution of Eq. (2.1). Hence, any special solution of Eq. (2.1) can be written in the form

$$\nabla^H \nabla^H \rho = (-K \rho + B)g,$$

where $K$ and $B$ are constants. Along any geodesic with arc-length $t$, Eq. (3.3) reduces to the ordinary differential equation

$$\frac{d^2 \rho}{dt^2} = -K \rho + B.$$
Now for the special case $K = C^2 > 0$ and $B = 0$, we have
\[ \frac{d^2 \rho}{dt^2} + C^2 \rho = 0. \]  
(3.5)

By a suitable choice of the arc-length $t$, a solution of Eq. (3.5) is given by
\[ \rho(t) = A \cos Ct, \]  
(3.6)
and its first derivative is
\[ \rho'(t) = -AC \sin Ct. \]  
(3.7)

We can see at a glance, that it might appear two critical points corresponding to $t = 0$ and $t = \frac{\pi}{C}$ on $M$, where these points are periodically repeated. Hence, if $\rho$ is a non-trivial solution of Eq. (3.5), then it can be written in the following form
\[ \rho(t) = \frac{-1}{C} \cos Ct \quad (A = \frac{-1}{C}). \]  
(3.8)

Taking into account Eq. (2.2), the metric form of $M$ becomes
\[ ds^2 = dt^2 + (\sin Ct)^2 ds^2, \]  
(3.9)
where $ds^2$ is the metric form of a $t$-level of $\rho$ given by $ds^2 = f_{\gamma\beta}du^\gamma du^\beta$.

### 4 Finsler manifolds of positive constant flag curvature.

Let $(x, y)$ be the line element of $TM$ and $P(y, X) \subset T_x(M)$ a 2-plane generated by the vectors $y$ and $X$ in $T_x(M)$. Then the flag curvature $K(x, y, X)$ with respect to the plane $P(y, X)$ at a point $x \in M$ is defined by
\[ K(x, y, X) := \frac{g(R(X, y)y, X)}{g(X, X)g(y, y) - g(X, y)^2}, \]
where $R(X, y)y$ is the $h$-curvature tensor of Cartan connection. If $K$ is independent of $X$, then $(M, g)$ is called space of scalar curvature. If $K$ has no dependence on $x$ or $y$, then the Finsler manifold is said to be of constant curvature, see for example [1]. It can be easily verified that the components of the $h$-curvature tensor of Cartan connection in adapted coordinates are given by (see page 7 in [3])
\[ R^\alpha_{\gamma\delta\gamma\beta} = -R^\alpha_{\gamma\delta\gamma\beta} = (\frac{\partial^m}{\partial^r}) \delta^\alpha_{\gamma}, \]
\[ R^1_{\gamma\delta\gamma\beta} = -R^1_{\gamma\delta\gamma\beta} = -\rho' \rho'' f_{\gamma\beta}, \]  
(4.10)
\[ R^\alpha_{\delta\gamma\beta} = \overline{R}^\alpha_{\delta\gamma\beta} - (\rho'')^2 (f_{\gamma\beta} \delta^\alpha_{\delta} - f_{\delta\beta} \delta^\alpha_{\gamma}), \]
where $\overline{R}^\alpha_{\delta\gamma\beta}$ are components of $h$-curvature tensor related to the metric form $\overline{ds}^2$ on a $t$-level of $\rho$. 

4
**Proposition 1.** Let $(M, g)$ be an $n$-dimensional Finsler manifold. Then it is of constant flag curvature $K = C^2 > 0$, if and only if, there is a non-trivial solution of $\nabla^H \nabla^H p = (-C^2 \rho + B)g$ on $M$.

**Proof.** Let $(M, g)$ be a Finsler manifold, then it is of constant flag curvature $K$, if and only if the components of the $h$-curvature tensor are given by

$$R^i_{hjk} = K(\delta^i_hg_{jk} - \delta^i_jg_{hk}).$$

Using Eq. (4.11), one can easily drive the differential equation

$$\ddot{A} + KAg = 0,$$

where $A$ is the Cartan torsion tensor, $\dot{A}_{ijk} := (\nabla^H A_{ijk})y^s$ and $\ddot{A}_{ijk} := (\nabla^H \nabla^H A_{ijk})y^sy^t$ (see for example [1] or [4]).

Fix any $X, Y, Z \in \pi^*TM$ at $v \in I_xM = \{w \in T_xM, F(w) = 1\}$. Let $c : \mathbb{R} \rightarrow M$ be the unit speed geodesic in $(M, F)$ with $\frac{dc}{dt}(0) = v$ and $\dot{c} := \frac{dc}{dt}$ be the canonical lift of $c$ to $TM_0$. Let $X(t), Y(t)$ and $Z(t)$ denote the parallel sections along $\dot{c}$ with $X(0) = X$, $Y(0) = Y$ and $Z(0) = Z$. Put $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$ and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$. Indeed along geodesics, we have $\frac{dA}{dt} = \dot{A}$, $\frac{dA}{dt} = \ddot{A}$ and Eq. (4.12) becomes

$$\frac{d^2A(t)}{dt^2} + K A(t) = 0.$$  

(4.13)

The general solution of this differential equation is $A(t) = A_0 \cos \sqrt{K}t + B_0 \sin \sqrt{K}t$. Therefore, Eq. (3.5) which represents a special case of Eq. (3.3) along geodesics, has a non-trivial solution on $M$.

Conversely, let $\rho$ given by Eq. (3.8) be a solution of Eq. (3.3) on $M$. Then there is an adapted coordinate on $M$ for which the components of $h$-curvature are given by three equations (4.10). Hence, first and second equations of Eqs. (4.10) satisfy

$$R^i_{hjk} = \frac{-\rho''}{\rho'}(\delta^i_hg_{jk} - \delta^i_jg_{hk}).$$

(4.14)

Differentiate Eq. (3.8) and replace the first and third derivatives of $\rho$, we obtain $\frac{\rho'''}{\rho'} = C^2$. Therefore, Eq. (4.11) is satisfied by first two equations of Eqs. (4.10).

For checking the third equation of Eqs. (4.10), we recall that as we saw in section 3, $\rho$ has critical points on $M$. Thus, subject to Lemma A, the $t$-levels of $\rho$ are spaces of the positive constant curvature $\rho'^2(0) = C^2$. Therefore, the third equation of Eqs. (4.10) becomes

$$R^i_{\delta\gamma\beta} = (C^2 - \rho'^2)(f_{\gamma\beta}\delta^i_\delta - f_{\delta\beta}\delta^i_\gamma).$$

Substituting $g_{\alpha\beta} = \rho^2 f_{\alpha\beta}$ and first and second derivatives of $\rho$, we obtain

$$R^i_{\delta\gamma\beta} = C^2(g_{\gamma\beta}\delta^i_\delta - g_{\delta\beta}\delta^i_\gamma).$$

Consequently, Eq. (4.11) is satisfied by all three components of Cartan $h$-curvature tensor in adapted coordinates, and the Finsler manifold $(M, g)$ is of constant flag curvature $K = C^2$. \qed
Now, we are in a position to prove an extension of the Obata-Tanno’s theorem for Finsler manifolds.

**Theorem 1.** Let \((M, g)\) be a complete connected Finsler manifold of dimension \(n \geq 2\). In order that there is a non-trivial solution of

\[
\nabla^H \nabla^H \rho + C^2 \rho = 0
\]

on \(M\), it is necessary and sufficient that \((M, g)\) be isometric to an \(n\)-sphere of radius \(1/C\).

*Proof.* If there is a non-trivial solution of Eq. (3.5) on \((M, g)\), then according to Proposition 1, it is of positive constant flag curvature \(C^2\). Preceding argument in the section 3 shows that the metric form of \((M, g)\) is given by Eq. (3.9), which is the polar form of a Finsler metric on a standard sphere of radius \(1/C\) [10]. Therefore, we have the sufficiency.

Conversely, if \((M, g)\) is isometric to an \(n\)-sphere of radius \(1/C\), then the metric form of \(M\) is given by \(ds^2 = (dt)^2 + \sin^2(Ct) d\sigma^2\), where \(d\sigma^2\) is the metric form of a \(t\)-level of \(M\). This is the polar form of a Finsler metric on an \(n\)-sphere in \(\mathbb{R}^{n+1}\) with the positive constant curvature \(C^2\). Now if we substitute the derivative of \(\rho(t) = -1/C \cos Ct\), in the metric form of \(M\), we obtain \(ds^2 = (dt)^2 + \rho'(t) d\sigma^2\), where \(\rho(t)\) is a non-trivial solution of the second order differential equation (3.5), or equivalently (4.15) along geodesics.

Now taking into account the number of critical points of \(\rho\), we have

**Corollary 1.** Let \((M, g)\) be a complete connected Finsler manifold. In order that there is a non-trivial solution of \(\nabla^H \nabla^H \rho + C^2 \rho = 0\), it is necessary and sufficient that \((M, g)\) be isometrically homeomorphic to an \(n\)-sphere.

*Proof.* Let \((M, g)\) admit a non-trivial solution of \(\nabla^H \nabla^H \rho + C^2 \rho = 0\), then from Theorem 1 we know that it is isometric to an \(n\)-sphere of radius \(1/C\). On the other hand, \(M\) is complete and Proposition 1 says that, the Finsler manifold \((M, g)\) is of positive constant curvature. Therefore, by means of extension of the Meyers’s theorem for Finsler manifolds [2], \(M\) is compact. Thus, the function \(\rho\) admits its absolute maximum and minimum values on \(M\). Consequently, \(\rho\) has two critical points on \(M\) and from extension of the Milnor’s theorem for Finsler manifolds [14], \((M, g)\) is homeomorphic to an \(n\)-sphere.

Conversely, let \((M, g)\) be isometrically homeomorphic to an \(n\)-sphere of radius \(1/C > 0\), then corresponding to Theorem 1, there is a non-trivial solution of \(\nabla^H \nabla^H \rho + C^2 \rho = 0\), on \(M\).

Following the Obata-Tanno’s theorem in Riemannian geometry a unit sphere is characterized by existence of solution of the differential equation \(\nabla^H \nabla f + fg = 0\), where \(f\) is a certain function on \(M\) and \(\nabla\) is the Levi-Civita connection associated to the Riemannian metric \(g\) [10]. Similarly, Theorem 1 shows that in Finsler geometry a unit sphere is characterized by existence of solution of \(\nabla^H \nabla^H \rho + C^2 \rho = 0\), where
$\rho$ is a certain function on $M$. In analogy with Riemannian geometry, this leads to a definition for an $n$-sphere in Finsler geometry as follows: Let $(M, g)$ be a complete connected Finsler manifold of positive constant flag curvature, then it is said to be a Finslerian $n$-sphere, if it is isometrically homeomorphic to an $n$-sphere.

**Theorem 2.** Let $(M, g)$ be an $n$-dimensional complete connected Finsler manifold. Then it is of positive constant flag curvature $K = C^2$, if and only if, $(M, g)$ is isometrically homeomorphic to an $n$-sphere of radius $1/C$ endowed with a certain Finsler metric.

**Proof.** Let $(M, g)$ be of positive constant curvature $C^2$. As a consequence of Proposition 1, there is a non-trivial solution of Eq. (3.5) on $M$ and by means of Corollary 1 it is isometrically homeomorphic to an $n$-sphere of radius $1/C$ equipped with a certain Finsler metric form $ds^2 = (dt)^2 + \sin^2(Ct)ds^2$.

Conversely, let $(M, g)$ be a Finsler manifold which is isometrically homeomorphic to an $n$-sphere of radius $1/C > 0$. Then, by means of Corollary 1 $M$ admits a non-trivial solution of Eq. (3.5). It follows from Proposition 1 that $(M, g)$ is of positive constant flag curvature $C^2$. $\square$

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That is, $M$ is homeomorphic to an $n$-sphere and $g$ is isometric to a certain Finsler metric on an $n$-sphere.
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