STRING STRUCTURES, 2-GROUP BUNDLES, AND A
CATEGORIFICATION OF THE FREED–QUINN LINE BUNDLE

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Abstract. For a 2-group constructed from a finite group and 3-cocycle, we provide an
explicit description of the bicategory of flat 2-group bundles on an oriented surface in
terms of weak representations of the fundamental group. We show that this bicategory
encodes (flat) string structures. Furthermore, we identify the space of isomorphism
classes of objects with Freed and Quinn’s line bundle appearing in Chern–Simons theory
of a finite group.

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1. INTRODUCTION AND STATEMENT OF RESULTS

A 2-group is a monoidal groupoid $(\mathcal{G}, \otimes)$ whose objects and morphisms admit $\otimes$-inverses. In this paper we focus on 2-groups $\mathcal{G}$ that are categorical central extensions of a finite group $G$
\begin{equation}
1 \rightarrow \text{pt} \rightarrow U(1) \rightarrow \mathcal{G} \rightarrow G \rightarrow 1
\end{equation}
classified by a 3-cocycle $\alpha \in Z^3(G; U(1))$. Regarding $\mathcal{G}$ as smooth 2-group in the sense of [SP11] allows one to consider the bicategory of principal $\mathcal{G}$-bundles over a manifold $X$. Endowing $\mathcal{G}$ with the discrete topology specifies a sub-bicategory of flat $\mathcal{G}$-bundles. A brief overview of our results is as follows.

Theorem 1.1. Let $\Sigma$ be a connected, oriented surface.

1. A flat $\mathcal{G}$-bundle on $\Sigma$ is determined by a weak homomorphism $\pi_1(\Sigma) \rightarrow \mathcal{G}$.
2. Fibers of the forgetful functor $\pi: \text{Bun}_G(\Sigma) \rightarrow \text{Bun}_G(\Sigma)$ from (flat) $\mathcal{G}$-bundles to (flat) $G$-bundles are equivalent to bicategories of (flat) string structures on principal $G$-bundles.
3. Taking isomorphism classes along the fibers of $\pi$ determines a $U(1)$-bundle on $\text{Bun}_G(\Sigma)$. The associated line bundle is the Freed–Quinn line bundle whose global sections are the value of Chern–Simons theory on $\Sigma$ for the group $G$ with 3-cocycle $\alpha$.

The main technical work is to assemble explicit algebraic descriptions of the bicategory of (flat) $\mathcal{G}$-bundles on a manifold $X$. The results above are then proved completely explicitly via computations in group cohomology. This leads to connections with some expected structures in geometry and physics. The remainder of this introduction gives precise formulations of the above three statements and surveys some of these connections.

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1.1. (Weak) 2-groups, string geometry, and topological modular forms. The majority of the previous literature on 2-group principal bundles concerns strict Lie 2-groups, meaning that the associator in the monoidal groupoid \((G, \otimes)\) is trivial e.g., see [BS07, BS09, Woc11, SW11, NW13a]. Such 2-group principal bundles are closely related to bundles of categories, as we now explain. Every category \(C\) has a strict 2-group \(\text{Aut}(C)\) of auto-equivalences: multiplication in \(\text{Aut}(C)\) is composition of functors, and therefore is strictly associative. Applying an associated bundle construction to an \(\text{Aut}(C)\)-principal bundle yields a bundle of categories with fiber \(C\). Conversely, given a bundle of categories, the fiberwise autoequivalences yield a bundle of strict 2-groups. When \(\text{Aut}(C)\) has the structure of a smooth 2-group, one can construct a bicategory of smooth \(\text{Aut}(C)\)-principal bundles.

The 2-groups that arise from extensions as in (1.1) are strict if and only if \(\alpha\) is the trivial cocycle. This makes the symmetries captured by a general (weak) 2-group somewhat mysterious: they do not arise as the symmetries of any category. This further obscures the differential geometry of their ensuing principal bundles. Indeed, very little has been written about this flavor of 2-group principal bundle apart from the definition [ST11, Definition 70].

Perhaps the most important example of a nontrivial extension (1.1) is the string group. This is a smooth 2-group arising as a categorical central extension [SP11, Theorem 2]

\[
1 \to \text{pt} \xrightarrow{\text{/}} U(1) \to \text{String}(n) \to \text{Spin}(n) \to 1, \quad n \in \mathbb{N}
\]

classified by \([\lambda] \in \text{H}^3(U(1); U(1)) \simeq \text{H}^4(B\text{Spin}(n); \mathbb{Z})\), the universal fractional Pontryagin class. As outlined in §1.4, just as the spin group controls spin structures, the string group controls string structures.

Such string structures are orientation data for the cohomology theory of topological modular forms (TMF), meaning that vector bundles with string structure beget Thom isomorphisms in TMF. Lurie constructs the string orientation through an equivariant refinement of TMF with respect to any compact smooth 2-group, a feature called 2-equivariance [Lur09, §5.1-5.3].

A differential geometric description of TMF has evaded discovery for over 30 years. This is partly because there are so many seemingly distinct approaches (e.g., [Seg88, Seg04, HK04, ST04, ST11, BDR04, DH11, Cos11a, Cos11b]), and partly because it has been difficult to construct a map from geometric objects to TMF. Lurie’s more robust packaging using 2-equivariance rigidifies this problem considerably. In short, an understanding of the geometry behind 2-equivariance ought to make it harder to guess the “wrong” geometric model for TMF. Furthermore, a putative model for TMF possessing the correct form of 2-equivariance automatically has a map to TMF, as sketched in [Lur09, §5.5].

The geometric origins of 2-equivariance are undoubtedly wrapped up in the string group and string structures. One additional clue comes from physics, where string structures originated as choices of anomaly cancellation for 2-dimensional quantum field theories [Kil87, Wit88]. A central assertion of the Stolz–Teichner program links these structures in physics with the string orientation of TMF. Specifically, a theory of (2-dimensional, chiral) free fermions is expected to construct a cocycle representative of the TMF Thom class with the string group encoding symmetries of the theory [ST04, §5] [DH11, §5]. A universal Thom class would then come from realizing the free fermions as a representation of a category whose morphisms are 2-dimensional cobordisms \(\Sigma\) with a String\((n)\)-principal bundle \(\mathcal{P} \to \Sigma\). More formally, this is a 2-dimensional bordism category over the 2-stack \(\smallint \text{String}(n)\); we refer to [ST11, §1.7] for a discussion of bordism categories over stacks. At a conceptual level, this fits nicely with Lurie’s point of view: the moduli of 2-group principal bundles over elliptic curves is an expected home for 2-equivariant elliptic cohomology; see also [Rez16].

Alas, realizing this picture in its entirety remains out of reach at present. To make progress, below we give a detailed and explicit description of the moduli of 2-group principal bundles over Riemann surfaces for smooth 2-groups of the form (1.1) where \(G\) is finite. Our methods are chosen with an eye towards the case of an arbitrary compact smooth 2-group of the form (1.1) (in particular String\((n)\)), as well as generalizations to 2-group bundles.
over cobordisms. Assuming that $\mathcal{G}$ is finite simplifies some of the geometry we encounter, but the nontrivial associator $\alpha$ has a rather dramatic effect. We expect similar behavior for arbitrary smooth 2-groups.

**Remark 1.2.** We comment on an important (but subtle) point: MacLane’s coherence theorem shows that any monoidal category is equivalent to a strict monoidal category [Mac71, Ch. 7]. In particular, any 2-group is equivalent to a strict 2-group. However, this result fails in the category of smooth 2-groups: String($n$) is a counter-example. Indeed, [BL04] show that there is no (finite-dimensional) strict smooth 2-group sitting in the extension (1.2), whereas Schommer-Pries constructs precisely such an extension in (weak) smooth 2-groups.

In short, there are ways to view String($n$) as a strict 2-group, but such descriptions are not compatible with its smooth geometry from [SP11]. Hence the study of smooth principal String($n$)-bundles requires working in the setting of weak 2-groups.

We are similarly obliged to work with the weak 2-groups $\mathcal{G}$ that are the focus of this paper, rather than being able to choose strictifications.

### 1.2. 2-group principal bundles.

For a smooth 2-group $\mathcal{G}$, the moduli of principal $\mathcal{G}$-bundles $\text{Bun}_{\mathcal{G}}(X)$ is naturally a bicategory. When $\mathcal{G}$ is of the form (1.1) there is a forgetful functor $\pi : \text{Bun}_{\mathcal{G}}(X) \to \text{Bun}_G(X)$ to the well-studied category of ordinary $G$-bundles on $X$.

Studying the fibers of $\pi$ gives a toehold for understanding $\text{Bun}_{\mathcal{G}}(X)$.

Recall that when $X$ is connected there is an equivalence of categories

$$\text{Bun}_G(X) \simeq \text{Hom}(\pi_1(X), G)/G$$

with the action groupoid whose objects are homomorphisms $\pi_1(X) \to G$ and whose morphisms come from postcomposing with the conjugation action of $G$ on itself. After one chooses a presentation of $\pi_1(X)$, an object in (1.3) can be specified by a tuple of elements in $G$ satisfying relations. For example, the standard presentation for the genus $g$-surface

$$\pi_1(\Sigma_g) = \{a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1\}$$

leads to a description of $G$-bundles as $2g$-tuples of elements in $G$ with a property

$$\text{Bun}_G(\Sigma_g) \simeq \{g_1, h_1, \ldots, g_g, h_g \in G \mid \prod_{i=1}^g (g_i, h_i) = 1\}/G.$$ 

Our first result gives similarly explicit descriptions for flat $\mathcal{G}$-bundles; the following is a precise statement of Theorem 1.1 part (1).

**Theorem 1.3.** For $\mathcal{G}$ as in (1.1) and $X$ a manifold with contractible universal cover, there is an equivalence of bicategories

$$\text{Bun}_{\mathcal{G}}^\flat(X) \simeq \text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/\mathcal{G}),$$

where objects in the bicategory on the right are weak homomorphisms $\pi_1(X) \to \mathcal{G}$. When $X = \Sigma_g$ is a connected oriented surface of genus $g$ with fundamental group presented as (1.4), the isomorphism class of a flat $\mathcal{G}$-bundle on $\Sigma_g$ is determined by a $2g$-tuple as in (1.5) together with an isomorphism $\sigma$ between objects of $\mathcal{G}$

$$\prod_{i=1}^g [g_i, h_i] \xrightarrow{\sigma} 1, \quad \sigma \in A.$$  

We view (1.6) as a categorification of the relation $\prod [g_i, h_i] = 1$: for ordinary $G$-bundles, this relation is a condition on the tuple of elements in $G$, whereas for $\mathcal{G}$-bundles it is additional data. In the genus 1 and genus 2 cases, one can visualize this categorified relation in terms of the pictures.
where for the torus the usual condition $hg = gh$ to define a $G$-bundle is replaced by the data $h \otimes g \sim g \otimes h$ of an isomorphism in the monoidal groupoid $\mathcal{G}$. Similarly in genus 2, the condition of the equality $h_1^{-1}g_1^{-1}h_1g_1 = g_2^{-1}h_2^{-1}g_2h_2$ is promoted to the data of an isomorphism in $\mathcal{G}$.

1.3. Chern–Simons theory and moduli spaces of flat 2-group bundles. A compact Lie group $G$ and a cohomology class $[\alpha] \in H^3(BG; U(1))$ are the input data to Chern–Simons theory. When $G$ is finite, Freed and Quinn construct a 3-dimensional topological field theory $\text{CS}(G, \alpha)$ whose value on oriented surfaces is given by [FQ93]

$$\text{CS}(G, \alpha)(\Sigma) = \Gamma(\text{Bun}_G(\Sigma); L)$$

(1.8)

for a certain complex line bundle $L \to \text{Bun}_G(\Sigma)$ called the Freed–Quinn line bundle. Recall that in the case of a finite group, Chern–Simons theory is the same as 3-dimensional Dijkgraaf–Witten theory [DW90].

The geometry of 2-group bundles can also be used to construct a line bundle over $\text{Bun}_G(\Sigma)$; together with Proposition 1.8 and Theorem 1.10 below, the following is the precise statement of Theorem 1.1 part (3).

**Theorem 1.4.** For $\mathcal{G}$ as in (1.1) and $X$ a manifold with contractible universal cover, the forgetful functor

$$\pi: \text{Bun}_\mathcal{G}^\flat(X) \to \text{Bun}_G(X)$$

(1.9)

is a cloven 2-fibration of bicategories. For $X = \Sigma$ an oriented surface, taking isomorphism classes of the fibers of (1.9) produces a $U(1)$-bundle on $\text{Bun}_G(\Sigma)$. The line bundle associated to this $U(1)$-bundle is the Freed–Quinn line bundle on $\text{Bun}_G(\Sigma)$ associated to $\alpha \in Z^3(G; U(1))$.

Showing that (1.9) is a 2-fibration is basically formal; however, our particular choice of cleavage reveals a concise algebraic description of the fibers. This leads to explicit formulas for the $U(1)$-bundle on $\text{Bun}_G(\Sigma)$ that we can then compare to Freed and Quinn’s construction.

**Remark 1.5.** Ganter has explained how the Freed–Quinn line bundle over the moduli stack of elliptic curves supplies a twist for complex analytic equivariant elliptic cohomology [Gan09, §2]. In light of this, the above provides a glimpse at the differential geometry underlying Lurie’s 2-equivariant elliptic cohomology [Lur09, §5].

**Remark 1.6.** Chern–Simons theory is closely related to Stolz and Teichner’s proposed cocycle representative of the TMF Thom class. Roughly, this cocycle comes from a trivialization of a family of Chern–Simons theories [ST04, §5.3].

**Remark 1.7.** In the physics literature, 2-group symmetries have appeared in a variety of places but only recently are being recognized as such. For example, Benini–Córdova–Hsin explain how “obstruction to symmetry fractionalization” in the condensed matter literature is an instance of a 2-group symmetry [BCH19]. These authors also give a short survey of 2-groups in the physics literature [BCH19, page 4].
1.4. Bicategories of string structures and bicategories of 2-group bundles. Finally, we explain how the bicategory $\text{Bun}_G(X)$ is related to string structures. To warn up, recall that a metrized vector bundle $V \to X$ is oriented if it admits a reduction of structure group to $\text{SO}(n) < O(n)$, and is spin if it admits a lift of structure group along $\text{Spin}(n) \to \text{SO}(n)$. Such a lift determines a principal $\text{Spin}(n)$-bundle $P \to X$ with a map $P \to \text{Fr}(V)$ to the $\text{SO}(n)$-frame bundle of $V$ that is equivariant for the homomorphism $\text{Spin}(n) \to \text{SO}(n)$. In this way, spin structures form a category whose morphisms are isomorphisms of $\text{Spin}(n)$-bundles with compatible maps to $\text{Fr}(V)$. When a spin structure on $V$ exists, isomorphism classes of objects are in bijection with $H^1(X; \mathbb{Z}/2)$. This approach also has an evident notion of flat spin structure: a flat oriented vector bundle has as structure group $\text{SO}(n)$ with the discrete topology, and a spin structure is a reduction of structure group to $\text{Spin}(n)$ with the discrete topology.

In parallel to the above, one may define a string structure on a metrized spin vector bundle $V \to X$ as a lift of structure group along $\text{String}(n) \to \text{Spin}(n)$, where $\text{String}(n)$ is defined by (1.2). Lifts of structure groups involving smooth 2-groups involve principal bundles whose total space is a smooth stack (see §3), yielding a bicategory of string structures on the vector bundle $V$. This definition of string structure has seen precious few applications in differential geometry, in part because the category of smooth $\text{String}(n)$-bundles (using Schommer-Pries’s finite-dimensional string 2-group) remains largely unexplored. Waldorf gives a different definition of string structure [Wal13] in terms of trivializations of 2-gerbes. This definition also yields a bicategory of string structures on $V$, and has seen several geometric applications (e.g., [Bun11, Wal15, Bec16]), but its connection to the string 2-group is less transparent.

In both Schommer-Pries’s and Waldorf’s definitions, a vector bundle $V \to X$ admits a string structure if and only if $V$ is spin and has vanishing fractional Pontryagin class, $\frac{p}{2}(V) = 0$. Nikolaus–Waldorf [NW13b, §6] show that Waldorf’s definition agrees with a definition of string structure using principal string bundles for an infinite-dimensional model for the string 2-group. However, to date there is no comparison with Schommer-Pries’s definition.

Motivated by these different perspectives, we formulate two definitions of string structure on $G$-bundles for a finite group $G$ and 2-group of the form (1.1) classified by a 3-cocycle $\alpha$: Definition 3.22 is modelled on the Schommer-Pries approach, and Definition 3.24 is modelled on Waldorf’s approach. Our definitions are compatible with the original ones in the case that the 3-cocycle $\alpha$ is pulled back from a representative of $[\lambda] \in H^3(B\text{Spin}(n); U(1))$ under a homomorphism $G \to \text{Spin}(n)$. Our results on principal $G$-bundles show that the two definitions are equivalent.

**Proposition 1.8.** For a principal $G$-bundle $P \to X$ and a 3-cocycle $\alpha \in Z^3(G; U(1))$, the bicategories of string structures (respectively, flat string structures) on $P$ in Definitions 3.22 and 3.24 are equivalent.

**Remark 1.9.** We anticipate that Proposition 1.8 will generalize to categorical extensions of arbitrary compact Lie groups (in particular $G = \text{Spin}(n)$ and the 2-group $\text{String}(n)$), but the techniques below rely on $G$ being finite.

The proof of Proposition 1.8 follows from the following, which is a precise phrasing of Theorem 1.1 part (2).

**Theorem 1.10.** For a smooth manifold $X$, the fiber of $P \in \text{Bun}_G(X)$ of the 2-fibration $\pi: \text{Bun}_G(X) \to \text{Bun}_G(X)$ (respectively, $\pi: \text{Bun}^0_G(X) \to \text{Bun}^0_G(X)$) from $G$-bundles (respectively, flat $G$-bundles) to $G$-bundles is equivalent to the bicategory of trivializations of the 2-gerbe (respectively, flat 2-gerbe) on $X$, denoted $\lambda_{P,\alpha}$ and determined by $P \to X$ and $\alpha \in Z^3(G; U(1))$.

1.5. Outline. In §2, we review basic definitions and constructions of smooth 2-groups as group objects in geometric stacks. Our presentation mostly follows [SP11]; we additionally
provide explicit formulas for various structures in smooth 2-groups of the form (1.1) that are required for later computations.

Next we recall Schommer-Pries’s definition of (smooth) 2-group principal bundles in §3. Proposition 3.5 unpacks this definition for smooth 2-groups of the form (1.1), giving an explicit Čech-style description of $G$-bundles. This illuminates a connection with the geometry of 2-gerbes and Waldorf’s definition of string structure, as we explain in §§3.3-3.4. This leads to the proof of Proposition 1.8. Then Theorem 1.10 follows from Proposition 3.17, Corollary 3.18, and Corollary 3.21.

In §4, we use Proposition 3.5 to give an explicit and purely algebraic description of $\text{Bun}_G(\Sigma)$ in terms of group cohomological information. This description leads to the equivalence of bicategories in Theorem 1.3, proved as Proposition 4.2, as well as the 2-fibration statement in Theorem 1.4, proved as Lemma 4.4.

The choice of cleavage for this 2-fibration is specified in §4.4, which also gives a completely explicit description of the trivializations in Theorem 1.10.

Finally, in §5 we explain the connection with the Freed–Quinn line bundle. This follows from explicit computation, using the algebraic description of $\text{Bun}_G(\Sigma)$ from the previous section. We finish the proof of Theorem 1.3 via Proposition 5.4, which characterizes isomorphism classes of $G$-bundles over a connected oriented surface. Finally, we identify the Freed Quinn line bundle with isomorphism classes of the fibers of $\pi$ in Proposition 5.5, completing the proof of Theorem 1.4.

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2. Review of smooth 2-groups

We give a brief introduction to smooth 2-groups following [SP11]. The 2-groups of interest for our purposes are comparatively simple objects built from explicit group-theoretic data; see §2.3. With this in mind, the purpose of this section is to show that our definitions and constructions are compatible with pre-existing ones. We refer to the references [Ler10, SP11] for details on bibundles and smooth 2-groups.

2.1. Lie groupoids and bibundles.

Definition 2.1. A Lie groupoid $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$ is a groupoid object in the category of smooth manifolds for which the source and target maps $s, t: \Gamma_1 \to \Gamma_0$ are surjective submersions. Define functors and natural transformations between Lie groupoids as internal to the category of smooth manifolds.

The collection of Lie groupoids, functors, and natural transformations form the strict 2-category $\text{LieGpd}$.

Example 2.2. Any manifold $X$ determines a Lie groupoid $\{X \rightrightarrows X\}$ with only identity morphisms, and a smooth functor between Lie groupoids of this sort is the same data as a smooth map. Hence, we obtain a fully-faithful functor $\text{Man} \to \text{LieGpd}$. Throughout, we use the same notation for a manifold $X$ and the Lie groupoid determined by $X$. In particular, the 1-point manifold pt determines the (terminal) Lie groupoid, pt.

Example 2.3. Given a Lie group $G$ acting on a smooth manifold $M$, the action groupoid $M//G = \{G \times M \rightrightarrows M\}$ has $M$ as its objects, $G \times M$ as its morphisms, source map the projection, and target map the action map. Composition is determined by the group multiplication in $G$, and identity morphisms are given by the identity element $1 \in G$. For $G$-manifolds $M$ and $N$, a $G$-equivariant map $M \to N$ determines a functor $M//G \to N//G$. 

Example 2.4. Given a surjective submersion $Y \to X$, let $Y^{[k]}$ denote the $k$th fibered product of $Y$ over $X$, e.g., $Y^{[2]} = Y \times_X Y$. The Čech groupoid associated to $Y$ is $\{ Y^{[2]} \rightrightarrows Y \}$, with source and target maps given by the projections out of the fibered product. Composition is given by the map $Y^{[3]} = Y \times_X Y \times_X Y \to Y \times_X Y = Y^{[2]}$ determined by projection to the first and third factors of $Y^{[3]}$, and the identity is the diagonal map $Y \to Y \times_X Y = Y^{[2]}$.

Definition 2.5. A left action of a Lie groupoid $\Gamma$ on a manifold $P$ consists of the data of maps $a: P \to \Gamma_0$ and $\mu: \Gamma_1 \times_{\Gamma_0} P \to P$. The fibered product uses the map $a$ (the anchor map) and the target map $t: \Gamma_1 \to \Gamma_0$ in $\Gamma$. The map $\mu$ is further required to be compatible with composition in $\Gamma$. A right action is defined similarly as a map $\mu: P \times_{\Gamma_0} \Gamma_1 \to P$, where the fibered product uses the source map $s: \Gamma_1 \to \Gamma_0$.

Definition 2.6. For Lie groupoids $\Gamma$ and $\Gamma'$, a bibundle from $\Gamma$ to $\Gamma'$ is a manifold $P$ with commuting actions by $\Gamma$ and $\Gamma'$, where $\Gamma$ acts on the left and $\Gamma'$ acts on the right. Part of the data of these actions are anchor maps $P \to \Gamma_0$ and $P \to \Gamma'_0$. Hence, a bibundle is often written as a diagram,

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{P} & \Gamma'_1 \\
\downarrow & & \downarrow \\
\Gamma_0 & \xrightarrow{P} & \Gamma'_0.
\end{array}
\]

A bibundle $P$ from $\Gamma$ to $\Gamma'$ is principal if $P \to \Gamma_0$ is a principal $\Gamma'$-bundle, meaning $P \to \Gamma_0$ is a submersion and the map $P \times_{\Gamma_0} \Gamma_1 \to P \times_{\Gamma_0} P$ is a diffeomorphism. We refer to [Ler10, §3.2] for details.

Hereafter, all bibundles will be assumed to be principal. We will denote a bibundle from $\Gamma$ to $\Gamma'$ by an arrow $\Gamma \to \Gamma'$.

Example 2.7. Given a manifold $X$ (regarded as a Lie groupoid) and an action groupoid $M/\!/G$, a bibundle $X \to M/\!/G$ is a principal $G$-bundle $P \to X$ with a $G$-equivariant map $P \to M$. In this case, the principal condition for the bibundle $X \to M/\!/G$ is precisely that the right action of $G$ on $P$ makes $P \to X$ into a principal $G$-bundle.

Definition 2.8. The bicategory Bibun has as objects Lie groupoids, 1-morphisms (principal) bibundles, and 2-morphisms equivariant maps of bibundles. We use the same notation for a Lie groupoid (a priori an object in LieGpd) when considered as an object in Bibun.

Remark 2.9. The bicategory Bibun is sometimes called the Morita category of Lie groupoids. It can also be identified with the category of differentiable stacks, e.g., see [BX11, §2] for an overview.

As proved in [Pro96], there is an equivalence $\text{Bibun} \simeq \text{LieGpd}[W^{-1}]$. Here, $W$ is the class of functors between Lie groupoids that are fully faithful and essentially surjective. The bicategory $\text{LieGpd}[W^{-1}]$ is the localization of $\text{LieGpd}$ at this class of 1-morphisms.

Localization comes equipped with a 2-functor

\[
\text{LieGpd} \to \text{LieGpd}[W^{-1}] \simeq \text{Bibun}.
\]

In particular (2.2) sends functors between Lie groupoids to bibundles between Lie groupoids. This construction can be made explicit, and is called bundlization. In brief, the manifold underlying the bundlization of a functor $F: \Gamma \to \Gamma'$ is given by the pullback $P := F_0'^* \Gamma_1'$ of the target map $t: \Gamma_1 \to \Gamma_0'$; one shows that $P$ admits commuting actions by $\Gamma$ and $\Gamma'$, see [Ler10, Remark 3.27] for details. More generally, 2-commuting diagrams in LieGpd are sent under (2.2) to 2-commuting diagrams in Bibun. Hence, many constructions in LieGpd can be imported to Bibun under (2.2).

Example 2.10. This example contrasts the bibundle maps in Example 2.7 with the image of bundlization, (2.2). A functor $F: \to M/\!/G$ between Lie groupoids is the same data
as a map $F_0: X \to M$ ($G$ plays no role). The bundlization of this functor is the trivial $G$-bundle $G \times X \to X$ with the equivariant map $G \times X \to M$ determined by the composition $G \times X \xrightarrow{id_G \times F_0} G \times M \xrightarrow{act} M$

where the last map is the $G$-action on $M$.

The following are examples of non-invertible maps in $\text{LieGpd}$ whose images in $\text{Bibun}$ under bundlization are invertible.

**Example 2.11.** Given a surjective submersion $Y \to X$, consider the Čech groupoid $\{Y^{[2]} \rightrightarrows Y\}$ from Example 2.4. There is a canonical functor of Lie groupoids $\{Y^{[2]} \rightrightarrows Y\} \to X$, which is easily seen to be fully faithful and essentially surjective. Hence, the image of the objects $\{Y^{[2]} \rightrightarrows Y\}$ and $X$ under (2.2) determine isomorphic objects in $\text{Bibun}$. On the other hand, $\{Y^{[2]} \rightrightarrows Y\} \to X$ determines an isomorphism in $\text{LieGpd}$ if and only if the map $Y \to X$ admits a global section.

**Example 2.12.** Let $P \to M$ be a principal $G$-bundle. Then there is a canonical functor between Lie groupoids $P/G \to M$ which is fully faithful and essentially surjective. Hence, the images of $P/G$ and $M$ under (2.2) are isomorphic objects in $\text{Bibun}$. As objects of $\text{LieGpd}$, $P/G$ and $M$ are isomorphic if and only if $P \to M$ is a trivializable $G$-bundle.

2.2. Smooth 2-groups.

**Definition 2.13.** A smooth 2-group $G$ is a group object in $\text{Bibun}$.

More explicitly, a 2-group is the data of a Lie groupoid $G$, 1-morphisms $m: G \times G \to G$, $1: pt \to G$, and 2-morphisms $a, l, r$,

$G \times G \times G \xrightarrow{id_G \times m} G \times G \xrightarrow{m} G \xrightarrow{1 \times id_G} G \times G \xrightarrow{l} G \times pt \xrightarrow{id_G \times 1} G \times G \xrightarrow{r} G \times G$

These data are required to satisfy the property that the map $(p_1, m): G \times G \to G \times G$

is an equivalence, as well as pentagon and triangle identities for $m, l,$ and $r$ [SP11, Definition 41 and 63]. Here, $a$ is an *associator*, $l$ is a *left-unitor* and $r$ is a *right-unitor*. We note that the equivalence (2.3) implies that multiplicative inverses exist in $G$.

**Definition 2.14 ([SP11, Definitions 42 and 43]).** Let $2\text{Grp}$ denote the bicategory whose objects are smooth 2-groups, 1-morphisms are 1-homomorphisms between smooth 2-groups, and 2-morphisms are 2-homomorphisms between smooth 2-groups.

**Example 2.15.** An ordinary Lie group $G$ with its multiplication $m$ and identity element $1$ determines a smooth 2-group where $a, l, r$ are all identity 2-morphisms.

**Example 2.16.** Let $A$ be an abelian Lie group. The bundlization of the functor $pt/A \times pt/A \to pt/A$ induced by multiplication in $A$ determines a multiplication bibundle $m$ for $pt/A$. Similarly, define $1: pt \to pt/A$ from the inclusion of the identity element of $A$. Taking $a, l,$ and $r$ to be identities then defines a 2-group that we also denote by $pt/A$.

**Example 2.17.** The *string group* is a smooth 2-group that sits in the exact sequence (1.2).

**Remark 2.18.** A smooth 2-group has an underlying monoidal category (forgetting the topology on $G$) with monoidal product determined by multiplication in $G$. 


2.3. Finite smooth 2-groups. The main examples of smooth 2-groups in this paper have finitely many isomorphism classes of objects. Following the work of Sinh [Sin75], these are determined by an ordinary group $G$ and a 3-cocycle $Z^3(G; A)$ where $A$ is a $G$-module, as described in the following example; see also [SP11, Example 68].

**Example 2.19** (2-groups from group theoretic data). Let $G$ be a finite group and $A$ an abelian Lie group endowed with a right $G$-action. Given a 3-cocycle $\alpha \in Z^3(G, A)$, define a smooth 2-group with objects $G$ and morphisms $G \ltimes A$, where both the source and target maps are defined by the projection to $G$. The multiplication bibundle $m$ is the bundlization of the functor determined by multiplication in $G$ and $G \ltimes A$. The 2-morphism $a$ is determined by the 3-cocycle $\alpha$, where the pentagon condition for $a$ follows from the cocycle condition for $\alpha$. The 2-morphisms $l$ and $r$ are determined up to natural isomorphism. Hence, $(G, A, \alpha)$ determines a 2-group with a finite set of isomorphism classes of objects.

**Example 2.20** (1-homomorphisms between 2-groups). Given a pair of 2-groups $(G, A, \alpha)$ and $(G', A', \alpha')$ as in the previous example, a 1-homomorphism $F$ between them is given by the data of a triple $(\rho, f, \gamma)$. Here $\rho: G \to G'$ is a group homomorphism, determining the value of $F$ on objects, while $f: A \to A'$ is a homomorphism of abelian groups, determining the value of $F$ on morphisms. Finally, a map $\gamma: G \times G \to A'$ provides a collection of morphisms in the target 2-group that witnessing compatibility of $F$ with the multiplication i.e., monoidal structure. The 1-homomorphism property on this data requires that $f$ is a homomorphism of $G$-modules (using $\rho$), and compatibility with associativity

$$f(\alpha(g, h, k))\gamma(gh, k)(\gamma(g, h) \cdot \rho(k)) = \gamma(g, hk)\gamma(h, k)\alpha'(\rho(g), \rho(h), \rho(k)).$$

as an equality of elements of $A'$ (written multiplicatively).

**Example 2.21** (2-homomorphisms). Given a pair of 1-homomorphisms $(\rho, f, \gamma)$ and $(\rho', f', \gamma')$ as above, a 2-homomorphism exists when $\rho = \rho'$ and $f = f'$. In this case, the data of the 2-homomorphism is that of a 1-cocycle $\eta: G \to A'$ (which provides a family of isomorphisms $\eta(g)$ between $\rho(g)$ and $\rho'(g)$) satisfying

$$\eta(gh)\gamma(g, h) = \gamma'(g, h)(\eta(g) \cdot \rho(h))\eta(h)$$

for all $g, h \in G$.

Motivated by the string group in Example 2.17, we will generally restrict attention to smooth 2-groups given by data $(G, A, \alpha)$ that determine central extensions

$$1 \to \text{pt}/A \to G \to G \to 1$$

of the ordinary finite group $G$ (regarded as a 2-group) by the 2-group $\text{pt}/A$ from Example 2.16. The sequence (2.6) defines a central extension of $G$ by $\text{pt}/A$ if and only if the $G$-action on $A$ is trivial [SP11, Lemma 84], and so $\alpha \in Z^3(G; A)$ is a 3-cocycle for the trivial $G$-module $A$.

**Remark 2.22.** A 3-cocycle $\alpha \in Z^3(G; A)$ is normalized if

$$\alpha(1, g_1, g_2) = \alpha(g_1, 1, g_2) = \alpha(g_1, g_2, 1) = 1,$$

where $1 \in G$ is the identity element. Every 3-cocycle is cohomologous to a normalized one. Hence, the description of 1-homomorphisms above implies that one may assume $\alpha$ is normalized when specifying a 2-group in terms of data $(G, A, \alpha)$. In this case, one can choose $l$ and $r$ to be identity 2-morphisms. Hereafter we shall assume that all 3-cocycles are normalized. Up to equivalence, one can fix unit and counit isomorphisms

$$i_g = \text{id}: 1 \to g \otimes g^{-1}, \quad e_g = \alpha(g^{-1}, g, g^{-1}): g^{-1} \otimes g \to 1$$

for all objects $g \in G$, i.e., $i$ is the identity and $e$ is determined by $\alpha$. These choices provide an adjoint equivalence for every object of $G$ and so imply that the resulting 2-group is
coherent [BL04, Definition 7]. As a consequence we get canonical isomorphisms

$$\phi: h^{-1}g^{-1} \to (gh)^{-1}, \quad \phi = \alpha(g,h,h^{-1}g^{-1}) \in A,$$

(2.7) $$\tilde{\phi}: g \to (g^{-1})^{-1} \quad \tilde{\phi} = \alpha(g^{-1},g,g^{-1}) \in A.$$

These explicit morphisms play a role in the coherence structures when we formulate the conjugation action of $G$ on the bicategory of principal $G$-bundles in section 4.4.

We will use the following lemma to give a simple description of automorphisms of the trivial $G$-bundle in the next section.

**Lemma 2.23.** Let $G$ be a smooth 2-group as in (2.6) and $X$ be a smooth manifold. A bibundle $X \to G$ is equivalent to the data of a smooth (and hence locally constant) map $X \to G$ and the data of a principal $A$-bundle $P \to X$.

**Proof.** We first note that the Lie groupoid underlying $G$ is given by

$$G \simeq G \sslash A \simeq \coprod_{g \in G} pt \sslash A \quad (2.9)$$

for the trivial $A$-action on $G$. By Example 2.7, a bibundle $X \to G$ is therefore the data of a principal $A$-bundle $P \to X$ and an $A$-equivariant map $P \to G$. But since the $A$-action on $G$ is trivial, this latter data is the same as an $A$-invariant map, which is equivalent to a map $X \to G$. Hence a bibundle $X \to G$ is a map $X \to G$ and a principal $A$-bundle $P \to X$. □

### 3. Principal bundles for 2-groups

Given a smooth 2-group $G$, we shall consider principal $G$-bundles. There is a very general theory for principal bundles with structure group an $\infty$-group [NSS15a, NSS15b], but this level of generality is a bit excessive for our intended applications. Instead, we take Schommer-Pries’s definition of principal bundles for smooth 2-groups [SP11, Definition 70]. This ends up being very similar to an earlier incarnation of 2-group principal bundles due to Bartels [Bar04].

The main goal of the section is to prove a technical result describing 2-group bundles in terms of Čech cocycle data (Proposition 3.5). This description also provides a connection between the moduli of 2-group bundles and the geometry of 2-gerbes (Proposition 3.17). The main upshot is that our definition of $G$-bundle refines the following topological definition. First, let $BG$ denote the classifying space of the group $G$ and $\alpha: BG \to B^3A$ be a map classifying a class $[\alpha] \in H^3(G;A)$. In an abuse of notation, define $B\mathcal{G}$ as the homotopy pullback,

$$\begin{array}{ccc}
B\mathcal{G} & \longrightarrow & * \\
\downarrow & & \downarrow \\
X & \longrightarrow & BG \\
\alpha & \longrightarrow & B^3A
\end{array} \quad (3.1)$$

One possible (topological) definition of a $\mathcal{G}$-bundle is the indicated lift $X \to B\mathcal{G}$. Such a lift is equivalent to a homotopy between the composition $X \to BG \to B^3A$ and the constant map $X \to * \to B^3A$. A 2-gerbe is topologically classified by a map $X \to B^3A$, and hence the required homotopy is a topological trivialization of this 2-gerbe. We will show that Schommer-Pries’s definition of smooth principal $G$-bundle leads to precisely this data, but in the world of smooth stacks; see Proposition 3.17.

#### 3.1. Principal $\mathcal{G}$-bundles and Čech cocycles.
Definition 3.1. A (right) action of a smooth 2-group \( \mathcal{G} \) on a Lie groupoid \( \Gamma \) is the data of a 1-morphism (i.e., bibundle) \( \mu: \Gamma \times \mathcal{G} \to \Gamma \) and 2-isomorphisms

\[
\begin{align*}
\Gamma \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\text{id}_\Gamma \times \mu} \Gamma \times \mathcal{G} \\
\Gamma \times \text{pt} & \xrightarrow{1 \times \text{id}_\Gamma} \Gamma \times \mathcal{G}
\end{align*}
\]

(3.2)

in \( \text{Bibun} \), which are required to satisfy compatibility properties with the 2-morphisms defining \( \mathcal{G} \); see [SP11, Definition 44].

For a smooth 2-group \( \mathcal{G} \), there is a bicategory denoted \( \mathcal{G}\text{-Bibun} \) whose objects are Lie groupoids with a right \( \mathcal{G} \)-action (also called \( \mathcal{G} \)-objects), 1-morphisms are \( \mathcal{G} \)-equivariant bibundles, and 2-morphisms are \( \mathcal{G} \)-equivariant maps between bibundles; see [SP11, §3.2 and Definition 63] for details. For \( X \) a smooth manifold, let \( \text{Bibun}/X \) denote the slice bicategory whose objects are Lie groupoids with a bibundle to \( X \), and let \( \mathcal{G}\text{-Bibun}/X \) denote the \( \mathcal{G} \)-equivariant slice bicategory.

Definition 3.2 ([SP11, Definition 70]). A (right) principal \( \mathcal{G} \)-bundle over \( X \) is an object \( \mathcal{P} \in \mathcal{G}\text{-Bibun}/X \) for which there exists a surjective submersion \( u: Y \to X \) and a \( \mathcal{G} \)-equivariant isomorphism \( Y \times \mathcal{G} \xrightarrow{\sim} u^* \mathcal{P} \) over \( Y \). Here, \( Y \) is a manifold regarded as a Lie groupoid, and we identify \( u \) with a 1-morphism in \( \text{Bibun} \). Let \( \text{Bun}_G(X) \) denote the bicategory whose objects are principal \( \mathcal{G} \)-bundles over \( X \), 1-morphisms are \( \mathcal{G} \)-equivariant bibundles \( \mathcal{P} \to \mathcal{P}' \) of \( \mathcal{G} \)-bundles over \( X \), and 2-morphisms are \( \mathcal{G} \)-equivariant isomorphisms between bibundles.

Example 3.3. For \( G \) an ordinary Lie group regarded as smooth 2-group, \( \text{Bun}_G(X) \) is equivalent to the usual category of principal \( G \)-bundles over \( X \).

Example 3.4. For an abelian Lie group \( A \) and the corresponding one-object 2-group \( \text{pt}/\!/A \), \( \text{Bun}_{\text{pt}/\!/A}(X) \) is equivalent to the usual bicategory of \( A \)-gerbes over \( X \), e.g., see [SP11, Example 73].

Our next goal is to describe the category \( \text{Bun}_G(X) \) in terms of Čech-style data. Before doing so, we review the Čech-data description of the category \( \text{Bun}_G(X) \) where \( G \) is an ordinary Lie group. We require some notation: for a submersion \( u: Y \to X \) and \( I \subset \{ 1, \ldots, n \} \) let \( p_I: Y^{[n]} \to Y^{[|I|]} \) denote the projection onto the factors in the fibered product indexed by \( I \). For example, \( p_{12}: Y^{[3]} \to Y^{[2]} \) denotes the projection to the first two factors, \( Y^{[3]} = Y \times_X Y \times_X Y \to Y \times_X Y = Y^{[2]} \), \((y_1,y_2,y_3) \mapsto (y_1,y_2)\).

Now let \( P \to X \) be a principal \( G \)-bundle for an ordinary Lie group \( G \). Fix a surjective submersion \( u: Y \to X \) and a trivialization \( \text{triv}: Y \times G \to u^* P \). The data of \( P \) to \( X \) is equivalent to the data of the trivial bundle \( Y \times G \to Y \) together with gluing data on the intersections \( Y^{[2]} \) that satisfy compatibility conditions. Specifically, over \( Y^{[2]} \) consider the following composition

\[
\tilde{\rho}: Y^{[2]} \times G \xrightarrow{p_2^{\text{triv}}} p_2^* u^* P \xrightarrow{p_1^* (\rho^{\text{triv}})^{-1}} Y^{[2]} \times G.
\]

This automorphism of the trivial bundle is equivalent to the data of a map \( \rho: Y^{[2]} \to G \). Pulling back further to \( Y^{[3]} \), we see that \( p_{12} \rho \circ p_{23} \rho = p_{13} \rho \), i.e., the cocycle condition

\[
\rho(y_1,y_2)\rho(y_2,y_3) = \rho(y_1,y_3) \in G, \quad \text{for all } (y_1,y_2,y_3) \in Y^{[3]}.
\]

There is an analogous Čech-style description for 2-group bundles.

Proposition 3.5. Fix a smooth manifold \( X \) and a 2-group \( \mathcal{G} \) determined by the data of a trivial \( G \)-module \( A \) and a 3-cocycle \( \alpha \in Z^3(G; A) \). The bicategory \( \text{Bun}_G(X) \) of principal \( G \)-bundles on \( X \) is equivalent to the bicategory defined as follows:
• Objects are triples \((u, \rho, \gamma)\), where \(u : Y \to X\) is a surjective submersion, \(\rho : Y^{[2]} \to G\), and \(\gamma : Y^{[3]} \to A\), satisfying

\[
\rho(y_1, y_2)\rho(y_2, y_3) = \rho(y_1, y_3) \iff d\rho = 1,
\]


\[
\frac{\gamma(y_1, y_2, y_3)\gamma(y_1, y_3, y_4)}{\gamma(y_1, y_2, y_4)\gamma(y_2, y_3, y_4)} = \frac{\alpha(\rho(y_1, y_2), \rho(y_2, y_3), \rho(y_3, y_4))}{\gamma(\rho(y_1, y_2), y_3, y_4)} \iff \frac{1}{d\gamma} = \rho^*\alpha.
\]

• 1-morphisms from \((u_1, \rho_1, \gamma_1)\) to \((u_2, \rho_2, \gamma_2)\) are triples \((u, t, \eta)\). Here \(u : Y \to X\) is a common refinement of \(u_1\) and \(u_2\)

\[
\begin{array}{ccc}
  Y & \xleftarrow{t} & X; \\
  f_1 & \xrightarrow{u} & u_1 \\
  f_2 & \xrightarrow{u_2} & Y_2
\end{array}
\]

\(t : Y \to G\), and \(\eta : Y^{[2]} \to A\). We require these data to satisfy two compatibility conditions. First, for all \((y_1, y_2) \in Y^{[2]}\)

\[
t(y_1)(\rho_1(y_1, y_2)) = (\rho_2(y_1, y_2))t(y_2) \in G
\]

and second, for all \((y_1, y_2, y_3) \in Y^{[3]}\) we have the equality of elements in \(A\),

\[
\eta(y_1, y_2)(\rho(y_2, y_3)) = \gamma(\eta(y_1, y_2), \eta(y_2, y_3))
\]

\[
\frac{\eta(y_1, y_2)\eta(y_2, y_3)}{\eta(y_1, y_3)} = \frac{\alpha(\rho(y_1, y_2), \rho(y_2, y_3), t(y_3))}{\alpha(\rho_2(y_1, y_2), t(y_2), \rho_1(y_2, y_3))}.
\]

• 2-morphisms from \((u_1', t_1, \eta_1)\) to \((u_2', t_2, \eta_2)\) (where these are 1-morphisms from \((u_1, \rho_1, \gamma_1)\) to \((u_2, \rho_2, \gamma_2)\)) are determined by pairs \((u, \omega)\). Here \(u : Y \to X\) is a common refinement of \(u_1'\) and \(u_2'\) via maps \(f_i' : Y \to \tilde{Y}_i'\), and \(\omega\) is a map \(Y \to A\). We require the compatibility condition \(t_1 \circ f_1' = t_2 \circ f_2'\), and that for all \((y_1, y_2) \in Y^{[2]}\),

\[
\eta_2(y_1, y_2)\omega(y_2) = \omega(y_1)\eta_1(y_1, y_2) \in A.
\]

Two such pairs \((u_1, \omega_1), (u_2, \omega_2)\) define the same 2-morphism if there is a common refinement \(u\) of \(u_1\) and \(u_2\) over which the pullbacks of \(\omega_1\) and \(\omega_2\) are equal.

The above follows essentially from the same reasoning as in the case of ordinary principal \(G\)-bundles. There are no real surprises, but the argument is a bit tedious so we defer the proof to §A.

Remark 3.6. By the previous theorem, any fixed open cover \(u : Y \to X\) determines a sub-bicategory of \(G\)-bundles on \(X\) given by Čech data subordinate to \(Y\). When \(u : Y \to X\) is a good open cover, the inclusion of this sub-bicategory induces an equivalence. Indeed, for an arbitrary \(G\)-bundle \(P\) and a good cover \(Y \to X\), there is guaranteed to exist a trivialization \(u^*P \simeq Y \times G\) and the higher categorical data trivializes as well. The covers in examples below will always be good.

3.2 Flat \(G\)-bundles.

Definition 3.7. Let \(G\) be a smooth 2-group. The discrete 2-group underlying \(G\) is the Lie groupoid with the same objects and morphisms as \(G\) but endowed with the discrete topology, and the bibundles \(m\) and \(1\) endowed with the discrete topology. We denote the discrete 2-group underlying \(G\) by \(G_d\).

\[\text{We are abusing notation here (and in all the equations below for 1- and 2-morphisms), suppressing the maps involving the common refinement \(f_1, f_2\). By } \rho_1(y_1, y_2) \text{ we mean } \rho_1(f_1(y_1), f_1(y_2)); \text{ by } \gamma_2(y_1, y_2, y_3) \text{ we mean } \gamma_2(f_2(y_1), f_2(y_2), f_3(y_2)); \text{ etc.}\]
We observe that there is a canonical 1-homomorphism
\begin{equation}
\mathcal{G}_\delta \to \mathcal{G}
\end{equation}
of smooth 2-groups; concretely this comes from endowing the identity bibundle on \( \mathcal{G} \) with the discrete topology and taking 2-morphisms associated with identity maps on underlying sets for the remaining data of a 1-homomorphism.

**Definition 3.8.** Let \( \mathcal{G} \) be a smooth 2-group. A flat \( \mathcal{G} \)-bundle on a manifold \( X \) is a \( \mathcal{G}_\delta \)-bundle. The 1-homomorphism (3.6) determines a functor
\[
\text{Bun}_{\mathcal{G}_\delta}(X) := \text{Bun}_{\mathcal{G}_\delta}(X) \to \text{Bun}_{\mathcal{G}}(X)
\]
from flat \( \mathcal{G} \)-bundles to \( \mathcal{G} \)-bundles on \( X \).

The following is an immediate consequence of the definition.

**Corollary 3.9.** Let \( \mathcal{G} \) be as in (2.6), so that in particular its underlying group is finite and hence discrete. A flat \( \mathcal{G} \)-bundle on a manifold \( X \) is specified by the same data as in Proposition 3.5, but where \( \gamma: Y^{[3]} \to A \) is a locally constant function. A 1-isomorphism between flat \( \mathcal{G} \)-bundles is given by the same data, but where \( \eta: Y^{[2]} \to A \) is locally constant. A 2-isomorphism between 1-isomorphisms is given by a locally constant function \( \omega: Y \to A \).

**Example 3.10.** Starting with \( \mathcal{G} \) as in (2.6), consider a flat \( \mathcal{G} \)-bundle over \( X = \mathbb{R}^2/\mathbb{Z}^2 \). By Proposition 3.5 and Remark 3.6, we may describe an object of this bicategory as the data of \((u, \rho, \gamma)\), where \( u: \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \) is the universal cover, \( \rho: (\mathbb{R}^2)^{[2]} \cong \mathbb{R}^2 \times \mathbb{Z}^2 \to G \) and \( \gamma: (\mathbb{R}^2)^{[3]} \cong \mathbb{R}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2 \to A \), which satisfy the following equations for all \( y_i \in \mathbb{R}^2 \):
\[
\rho(y_1, y_2) \rho(y_2, y_3) = \rho(y_1, y_3) \quad \frac{1}{d\gamma}(y_1, y_2, y_3, y_4) = \alpha(\rho(y_1, y_2), \rho(y_2, y_3), \rho(y_3, y_4))
\]
Because \( G \) is discrete, \( \rho \) is equivalent to the data of a homomorphism \( \bar{\rho}: \mathbb{Z}^2 \to G \). Since the \( \mathcal{G} \)-bundle is flat, \( \gamma \) is locally constant and specified by a map \( \tilde{\gamma}: \mathbb{Z}^2 \times \mathbb{Z}^2 \to A \). Finally, we have the condition \( 1/d\tilde{\gamma} = \bar{\rho}^* \alpha \) as \( A \)-valued functions on \( (\mathbb{Z}^2)^3 \).

The above example is generalized in §4.1, where we show that \((\bar{\rho}, \tilde{\gamma})\) as above is equivalent to the data of a functor \( \text{pt}/\pi_1(X) \to \text{pt}/\mathcal{G} \) between bicategories; see Proposition 4.2.

### 3.3. From \( \mathcal{G} \)-bundles to trivializations of 2-gerbes.

For a 2-group \( \mathcal{G} \) as in (2.6), there is a natural forgetful functor from principal \( \mathcal{G} \)-bundles to principal bundles for the underlying ordinary group \( G \). The goal of this section is to describe the fibers of this forgetful functor; this will use the language of gerbes and 2-gerbes.

**Lemma 3.11.** Let \( \mathcal{G} \) be a smooth 2-group arising as an extension of an ordinary group \( G \) as in (2.6) and \( X \) be a smooth manifold. There is a forgetful functor
\begin{equation}
\pi: \text{Bun}_{\mathcal{G}}(X) \to \text{Bun}_G(X)
\end{equation}
from the bicategory of \( \mathcal{G} \)-bundles on \( X \) to the category of \( G \)-bundles on \( X \).

**Proof.** Proposition 3.5 shows that part of the data of a cocycle for a \( \mathcal{G} \)-bundle is a cocycle for a \( G \)-bundle, and that part of the data of a 1-isomorphism of \( \mathcal{G} \)-bundles is an isomorphism of \( G \)-bundles. Furthermore, 2-isomorphic 1-morphisms determine the same Čech data, implying that this assignment gives a well-defined functor. \(\square\)

**Remark 3.12.** The functor (3.7) can also be described as the associated bundle for the 1-homomorphism \( \mathcal{G} \to G \), where then \( P := \mathcal{P} \times_G G \) is an ordinary \( G \)-bundle. Equivalently, the 1-homomorphism \( \mathcal{G} \to G \) takes isomorphism classes of objects in \( \mathcal{G} \), and so \( P \) comes from taking isomorphism classes along fibers of \( \mathcal{P} \to X \).
Definition 3.14. A \( \omega \)-gerbe on a manifold \( X \) is a surjective submersion and \( \gamma: Y^{[3]} \to A \) is a Čech 2-cocycle. A 1-isomorphism \( (u, \gamma) \to (u', \gamma') \) between gerbes is a mutual refinement \( u'' \): \( Y \to X \) of \( u, u' \) and a 1-cochain \( \eta: Y^{[2]} \to A \) such that \( d\eta = \gamma/\gamma' \) where in this formula \( \gamma \) and \( \gamma' \) are pulled back to \( u'' \). A 2-isomorphism \( (u, \eta) \to (u', \eta') \) is a mutual refinement \( u'' \): \( Y \to A \) of \( u, u' \) and a 0-cochain \( \omega: Y \to A \) with \( d\omega = \eta/\eta' \), where in this formula \( \eta \) and \( \eta' \) are pulled back to \( u'' \).

Because 3-stacks are beyond the scope of this paper, we simply define 2-gerbes in terms of analogous Čech data.

Definition 3.14. A 2-gerbe on a manifold \( X \) is a pair \( (u, \lambda) \) where \( u: Y \to X \) is a surjective submersion and \( \lambda: Y^{[4]} \to A \) is a Čech 3-cocycle. A 1-isomorphism \( (u, \lambda) \to (u', \lambda') \) between 2-gerbes is a mutual refinement \( u'' \): \( Y \to X \) of \( u, u' \) and a 2-cochain \( \gamma: Y^{[3]} \to A \) such that \( d\gamma = \lambda/\lambda' \) where in this formula \( \lambda \) and \( \lambda' \) are pulled back to \( u'' \). A 2-isomorphism \( (u, \gamma) \to (u', \gamma') \) is a mutual refinement \( u'' \): \( Y \to X \) of \( u, u' \) and a 1-cochain \( \eta: Y^{[2]} \to A \) with \( d\eta = \gamma/\gamma' \), where in this formula \( \gamma \) and \( \gamma' \) are pulled back to \( u'' \). Proceeding as in the previous definition, one obtains a 3-category of 2-gerbes (though we will not use this structure below).

The trivial 2-gerbe \( 1_A \) is defined by the submersion \( u = \text{id}: X \to X \) and the 3-cocycle \( \lambda: X \to A \) given by the constant map to \( 1_A \in A \). A trivialization of a 2-gerbe is a 1-isomorphism \( \gamma: 1_A \to (u, \lambda) \), i.e., a 2-cochain \( \gamma: Y^{[3]} \to A \) such that \( d\gamma = \frac{\lambda}{\lambda} \) \( \text{ mod } 1 \). Proceeding as in the previous definition, trivializations of a 2-gerbe are the objects of a bicategory (possibly empty) [Ste04].

Lemma 3.15. Given a group \( G \) and 3-cocycle \( \alpha \in Z^3(G; A) \), any \( G \)-bundle \( P \to X \) determines a 2-gerbe on \( X \) that is unique up to 1-isomorphism.

Proof. Cocycle data \( \rho: Y^{[2]} \to G \) for the \( G \)-bundle \( P \) is uniquely determined up to refinement of the cover and conjugation of \( \rho \) by a \( G \)-valued function. Then the function

\[
\lambda_{P, \alpha}: Y^{[4]} \to A, \quad \lambda_{P, \alpha}(y_1, y_2, y_3, y_4) = \alpha(\rho(y_1, y_2), \rho(y_2, y_3), \rho(y_3, y_4))
\]

determines a 3-cocycle on \( X \) with values in \( A \), i.e., a 2-gerbe on \( X \). A different choice of cocycle data \( \rho': Y^{[2]} \to G \) for \( P \) determines a 1-isomorphic 2-gerbe, with 1-isomorphism determined by the \( G \)-valued function on a mutual refinement of the submersions for \( \rho \) and \( \rho' \). \( \square \)

Remark 3.16. Using the appropriate notion of submersion (e.g., see [SP11, Definition 37]) one can generalize Definition 3.14 to 2-gerbes over general Lie groupoids rather than just manifolds. In this definition, the submersion \( Y = pt \to pt//G \) has \( Y^{[4]} = G^{\times 3} \), and a 3-cocycle \( \alpha \in Z^3(G; A) \) determines a 2-gerbe on \( pt//G \). The 2-gerbe in Lemma 3.15 is the pullback along the bibundle \( X \to pt//G \) classifying \( P \).

Proposition 3.17. There is an equivalence of bicategories of \( G \)-bundles with underlying \( G \)-bundle \( P \) and trivializations of the 2-gerbe \( \lambda_{P, \alpha} \).

Proof. In terms of the cocycle data from Proposition 3.5, we fix a map \( \rho \) determining the \( G \)-bundle \( P \to X \). In view of (3.4), the data \( \gamma \) trivializes the 2-gerbe (3.8). Different choices of Čech cocycle data lead to the claimed isomorphism of \( G \)-bundles and compatible 1-isomorphism of trivializations of 2-gerbes. \( \square \)

Applying Proposition 3.17 to a fixed \( G \)-bundle, we obtain the following.
Corollary 3.18. Let \( G \) be as in the statement of the previous proposition. The fibers of the forgetful functor \( \pi: \text{Bun}_G(X) \to \text{Bun}_G(X) \) are non-canonically equivalent to the bicategory of gerbes on \( X \), where a choice of equivalence \( \text{fib}(P) \cong \text{Gerbe}_A(X) \) for the fiber at \( P \in \text{Bun}_G(X) \) is determined by a choice of \( G \)-bundle \( P \) whose underlying \( G \)-bundle is \( P \).

Remark 3.19. The bicategory \( \text{Gerbe}_A(X) \) has a monoidal structure coming from multiplication on \( A \). Furthermore, there is a (weak) action \( \text{Bun}_G(X) \times \text{Gerbe}_A(X) \to \text{Bun}_G(X) \): using the description of \( \text{Bun}_G(X) \) from Proposition 3.17, a \( G \)-bundle and gerbe determines a new \( G \)-bundle by modifying the trivialization of a 2-gerbe by a 1-gerbe. This action does not change the underlying \( G \)-bundle. Informally, this allows one to enhance the above corollary: the fibers of \( \pi: \text{Bun}_G(X) \to \text{Bun}_G(X) \) are a torsor for the symmetric monoidal bicategory \( \text{Gerbe}_A(X) \).

We finish this subsection by specializing the results to the case of discrete groups and 2-groups, or equivalently of flat bundles. In particular, specializing (3.7), we obtain a functor,

\[
(3.9) \quad \pi: \text{Bun}_G^\ast(X) \to \text{Bun}_G(X)
\]

from the bicategory of flat \( G \)-bundles to the category of flat \( G \)-bundles.

Definition 3.20. In complete analogy to Definition 3.14, a flat 2-gerbe is a 2-gerbe where the abelian Lie group \( A \) is endowed with the discrete topology. A trivialization of such a 2-gerbe is an \( A \)-valued 2-cochain where \( A \) is endowed with the discrete topology

Corollary 3.21. Let \( G \) be as in (2.6). A lift of a flat \( G \)-bundle to a flat \( G \)-bundle along (3.9) is equivalent data to a flat trivialization of the 2-gerbe \( \lambda_{P,\alpha} \) in (3.8). The fibers of (3.9) are non-canonically isomorphic to the bicategory of flat 1-gerbes on \( X \).

Proof. The result follows immediately from Proposition 3.17, after noting that for finite \( G \), the 2-gerbe determined by (3.8) is always flat. \( \square \)

3.4. Bicategories of string structures. We are now in a position to make the comparison between different approaches to the notion of a “string structure,” following the overview in §1.4. The first of these uses Schommer-Pries’s definition of String\((n)\) as a 2-group extension of Spin\((n)\) by \( \text{pt}/U(1) \) classified by \( [\lambda] \in H^3(B\text{Spin}(n);U(1)) \); then a string structure on a Spin\((n)\)-principal bundle \( P \to X \) is a lift to a String\((n)\)-principal bundle \( P \to X \). Motivated by this we make the following definition.

Definition 3.22. Given a finite group \( G \) and a 3-cocycle \( \alpha \in Z^3(G;U(1)) \), an \( \alpha \)-string structure (respectively, flat \( \alpha \)-string structure) on a \( G \)-bundle \( P \to X \) is a \( G \)-bundle (respectively, flat \( G \)-bundle) whose underlying \( G \)-bundle is \( P \). Together with 1- and 2-isomorphisms between \( G \)-bundles, \( \alpha \)-string structures on \( P \) form a bicategory.

Remark 3.23. A homomorphism \( f: G \to \text{Spin}(n) \) determines a degree 3 class \( f^*\lambda =: [\alpha] \in H^3(BG;U(1)) \). Choose a representative \( \alpha \), and suppose that \( V \simeq P \times_G \mathbb{R}^n \) is a vector bundle associated to a \( G \)-bundle \( P \) via \( G \overset{f}{\to} \text{Spin}(n) \to \text{SO}(n) \). Then a string structure in Schommer-Pries’s sense determines an \( \alpha \)-string structure on \( P \) in the sense of Definition 3.22.

A second approach to string structures is due to Waldorf [Wal13]. Every principal spin bundle \( P \to X \) determines a 2-gerbe on \( X \) whose Dixmier–Douady class is the pullback of \( [\lambda] \) along the map \( X \to B\text{Spin}(n) \) classifying the spin structure on \( V \). Waldorf defines a string structure on \( V \) to be a trivialization of this 2-gerbe. We modify this definition as follows.

Definition 3.24. Given a finite group \( G \) and a 3-cocycle \( \alpha \in Z^3(G;U(1)) \), an \( \alpha \)-string structure (respectively, flat \( \alpha \)-string structure) on a \( G \)-bundle \( P \to X \) is a trivialization (respectively, flat trivialization) of the 2-gerbe \( \lambda_{P,\alpha} \) defined in (3.8). Together with 1- and 2-isomorphisms between (flat) trivializations of a 2-gerbe, \( \alpha \)-string structures on \( P \) form a bicategory.
Remark 3.25. Let \( f : G \to \text{Spin}(n) \), \( \alpha \), and \( V \) be as in Remark 3.23. Then a trivialization of the 2-gerbe determined by \( P \) determines a trivialization of the induced 2-gerbe over \( G \); hence a string structure in the sense of Waldorf gives rise to an \( \alpha \)-string structure.

Now it is straightforward to show that the above definitions of \( \alpha \)-string structure are equivalent.

Proof of Proposition 1.8. This follows from Definitions 3.22 and 3.24 together with Proposition 3.17 for \( \alpha \)-structures and Corollary 3.21 for flat \( \alpha \)-string structures. \( \square \)

4. Holonomy of flat \( G \)-bundles

Ordinary \( G \)-bundles over a connected space \( X \) have the algebraic description (1.3). The main goal of this section is to construct a 2-categorical generalization for flat 2-group bundles over a manifold \( X \) with contractible universal cover, i.e., when \( X \) is a \( K(\pi_1(X), 1) \).

Let \( \text{pt} \#/ G \) denote the one-object bicategory determined by the monoidal category underlying an essentially finite 2-group \( G \). We construct an equivalence of bicategories

\[
\text{Bun}_G^b(X) \cong \text{Bicat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G),
\]

which implies Theorem 1.1 part (1) when \( X = \Sigma \) is an oriented surface of genus \( g > 0 \). Composing with the canonical homomorphism \( G \to G \) then gives a forgetful functor

\[
\text{Bicat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G) \to \text{Cat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G) \simeq \text{Hom}(\pi_1(X), G)/G
\]

extracting the \( G \)-bundle underlying a flat \( G \)-bundle. This gives a completely algebraic description of the functor \( \pi : \text{Bun}_G^b(X) \to \text{Bun}_G(X) \). We use this to prove the first statement in Theorem 1.4, namely that the functor \( \pi \) is a 2-fibration. We go on to show that a choice of cleavage for this 2-fibration identifies \( \pi \) with the Grothendieck construction applied to a (tri) functor (see [Buc14, Construction 3.3.5])

\[
F_\pi : \text{Bun}_G^b(X)^\text{op} \to \text{Bicat}.
\]

One can make choices such that the values of \( F_\pi \) encode the geometry of \( G \)-bundles studied in the previous section: \( F_\pi \) assigns to a principal bundle \( P \in \text{Bun}_G^b(X) \) the bicategory of trivializations of the gerbe \( \lambda_{P, \alpha} \) from Corollary 3.21. This description permits the comparison with the Freed–Quinn line bundle in the next section.

4.1. The functor bicategory \( \text{Bicat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G) \). In what follows, \( \pi_1(X) \) will alternately denote the one-object category or the one-object bicategory determined by \( \pi_1(X) \); the intended meaning should be clear in context. Define \( \text{Bicat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G) \) as the bicategory of functors, invertible natural transformations and invertible modifications. In preparation for our construction of the functor (4.1), we establish notation and collect some formulas encoding the structure of this bicategory.

The following description of objects, 1-morphisms and 2-morphisms in \( \text{Bicat}(\text{pt} \#/ \pi_1(X), \text{pt} \#/ G) \) is a specialization of [GU16, §3.1.1-3.1.3]. Objects are homomorphisms \( \pi_1(X) \to G \) of 2-groups, and are specified as in (2.4) by data

\[
\text{Obj} = \{ \rho : \pi_1(X) \to G, \gamma : \pi_1(X)^\times^2 \to A \mid d\gamma = (\rho^*\alpha)^{-1} \}
\]

where \( \rho \) is a homomorphism and \( \gamma \) is a 2-cochain. A 1-morphism \( (\rho', \gamma') \to (\rho, \gamma) \) is determined by a pair \( t \in G, \eta : \pi_1(X) \to A \) satisfying the properties

\[
\frac{\eta(a)\eta(b)}{\eta(ab)} = \frac{\gamma'(a,b)}{\gamma(a,b)} \frac{\alpha(t, \rho'(a), \rho'(b))}{\alpha(\rho(a), t, \rho'(b))}, \quad t \rho't^{-1} = \rho.
\]

The condition (4.5) guarantees that the isomorphisms \( \eta(a) \sim \rho'(a)t \) between objects in \( G \) for each \( a \in \pi_1(X) \) are compatible. Finally, a 2-morphism \( \omega : (t, \eta) \Rightarrow (t', \eta') \) is specified by an element \( \omega \in A \). Such a 2-morphism exists if and only if \( \eta = \eta' \) and \( t = t' \), and the compatibility conditions are trivial.

Remark 4.1. We note our convention \( d\gamma(a, b, c) = \frac{\gamma(b, c)\gamma(a, bc)}{\gamma(ab, c)\gamma(ab)} \) is the inverse to the convention in [GU16]; our convention is consistent with the coboundary map for Čech cochains.
Composition of 1-morphisms in $\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G})$ is given by the assignment

$$(\rho_1, \gamma_1) \xrightarrow{(t,\eta)} (\rho_2, \gamma_2) \xrightarrow{(s,\xi)} (\rho_3, \gamma_3) \Rightarrow (\rho_1, \gamma_1) \xrightarrow{(s,t,\theta)} (\rho_3, \gamma_3)$$

where

$$\theta(a) = \xi(a)\eta(a)\frac{\alpha(s,t,\rho_1(a))\alpha(\rho_3(a),s,t)}{\alpha(s,\rho_2(a),t)}.$$ 

Composition of 1-morphisms is associative up to specified 2-morphism. Given

$$(\rho_1, \gamma_1) \xrightarrow{(t,\eta)} (\rho_2, \gamma_2) \xrightarrow{(s,\xi)} (\rho_3, \gamma_3) \xrightarrow{(r,\theta)} (\rho_4, \gamma_4)$$

this 2-morphism is determined by $\alpha(r,s,t) \in A$. To spell this out, denote the two possible compositions by $(rst,Y) := (r,\theta) \circ (s,\xi) \circ (t,\eta)$ and $(rst,Z) := (r,\theta) \circ ((s,\xi) \circ (t,\eta))$. Note that $Y(a), Z(a) : (rst)\rho_1(a) \to \rho_4(a)(rst)$ are morphisms in $\mathcal{G}$, and in fact $Y(a) = Z(a)$ in $A$. Hence we can view $\alpha(r,s,t)$ as a 2-morphism $Y(a) \Rightarrow Z(a)$.

The identity 1-morphism in $\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G})$ is $(1_G, 1_A)$. Given a 1-morphism $(t,\eta)$, let $(t,\eta)^{-1}$ be the 1-morphism with the property $(t,\eta) \circ (t,\eta)^{-1} = (1_G, 1_A)$. Then $(t,\eta)^{-1} = (t^{-1}, \xi)$ where

$$\xi(a) = \frac{1}{\eta(a)}\frac{\alpha(t,\rho_1(a),t^{-1})}{\alpha(t,\rho_2(a),t, t^{-1})}\alpha(t, t^{-1}, \rho_2(a)).$$

4.2. **Weak representations and flat $\mathcal{G}$-bundles.** Using the results and notation from above, we may now compare $\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G})$ with $\text{Bun}_G^\flat(X)$ and $\text{Bun}_G(X)$. To start, identify $\rho$ in (4.4) with a flat $G$-bundle on $X$ with holonomy determined by $\rho : \pi_1(X) \to G$. The condition $tp^t^{-1} = \rho$ for 1-morphisms in (4.5) shows that the underlying $G$-bundles associated to $\rho$ and $\rho'$ are isomorphic. This observation gives an evident functor $\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G}) \to \text{Bun}_G(X)$, which we lift to the functor $\text{Descend}$ below.

**Proposition 4.2.** Let $\mathcal{G}$ be as in (2.6) and $X$ be a connected manifold. There is a fully faithful functor $\text{Descend}$ sitting in the diagram

$$\begin{array}{ccc}
\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G}) & \xrightarrow{\text{Descend}} & \text{Bun}_G^\flat(X) \\
\pi \downarrow & \cong & \downarrow \pi \\
\text{Cat}(pt/\pi_1(X), pt/\mathcal{G}) & \xrightarrow{\cong} & \text{Bun}_G(X)
\end{array}$$

where $\pi$ is determined by post-composing with the functor $pt/\mathcal{G} \to pt/G$ associated with the 1-homomorphism $\mathcal{G} \to G$, and $\pi$ is the functor (3.9). When $X$ has a contractible universal cover, $\text{Descend}$ is an equivalence of bicategories.

**Proof.** We will compare the bicategory of functors $\text{Bicat}(pt/\pi_1(X), pt/\mathcal{G})$ as described above with the Čech data for $\mathcal{G}$-bundles relative to the universal cover $\tilde{X} \to X$. We will show that these lead to *isomorphic* bicategories (i.e., bijections on objects, 1-morphism and 2-morphisms). Since the Čech data determines a sub-bicategory of $\text{Bun}_G^\flat(X)$, the result follows.

For $Y = \tilde{X} \to X$, we have $Y^{[k]} = \tilde{X} \times \pi_1(X)^{\times (k-1)}$. Under this identification, the maps $p_1 : Y^{[k]} \to Y^{[l]}$ can be reformulated in terms of multiplication in $\pi_1(X)$: for example, $p_{12} : Y^{[3]} \to Y^{[2]}$ corresponds to

$$\tilde{X} \times \pi_1(X)^{\times 2} \to \tilde{X} \times \pi_1(X)$$

$$(x, a, b) \mapsto (x, a),$$

while $p_{13} : Y^{[3]} \to Y^{[2]}$ corresponds to

$$\tilde{X} \times \pi_1(X)^{\times 2} \to \tilde{X} \times \pi_1(X)$$

$$(x, a, b) \mapsto (x, ab).$$
Since the universal cover is connected, the (locally constant) cocycle data from Corollary 3.9 associated to a flat $\mathcal{G}$-bundle on $X$ satisfying the hypotheses is equivalent to maps $\rho : \pi_1(X) \to G$, $\gamma : \pi_1(X) \times \pi_1(X) \to A$.

The cocycle condition requires that $\rho$ be a homomorphism and that $\rho^* \alpha = 1/d\gamma$ (from (3.4)). From (4.4), this is the same data as a functor $\text{pt}/\pi_1(X) \to \text{pt}/\mathcal{G}$. Similarly, a 1-isomorphism between $\mathcal{G}$-bundles satisfying the hypotheses is equivalent to an element $t \in G$ and a map $\eta : \pi_1(X) \to A$ such that the compositions $X \times \pi_1(X) \to \pi_1(X) \to A$ satisfy (3.5). Comparing with (4.5), this is equivalent to a natural isomorphism between functors $\text{pt}/\pi_1(X) \to \text{pt}/\mathcal{G}$. Finally, a 2-isomorphism between $\mathcal{G}$-bundles is an element $\omega \in A$, where a 2-isomorphism between 1-isomorphisms exists if and only if $\eta = \eta'$ and $t = t'$. This recovers invertible modifications between natural isomorphisms of functors. This verifies the existence of the fully faithful functor $\text{Descend}$ with the claimed properties. When $X$ has a contractible universal cover, the functor $\text{Descend}$ is an equivalence by Remark 3.6.

The compatibility of $\text{Descend}$ with the map to the category $\text{Bun}_G(X)$ follows from the description of this category in (1.3) together with the translation between Čech cocycle data and homomorphisms $\pi_1(X) \to G$ in describing flat $G$-bundles. □

4.3. The functor $\pi$ is a 2-fibration. A 2-fibration between bicategories $C$ and $D$ is a functor $\pi : C \to D$ where certain 1- and 2-morphisms in $D$ can be lifted to $C$; we refer to [Buc14, Definition 3.1.5] for the precise definition. In our case of interest the definition simplifies considerably.

**Lemma 4.33.** Let $C$ be a 2-groupoid and $D$ be a 1-groupoid viewed as a 2-groupoid with only identity 2-morphisms. Then a functor $\pi : C \to D$ is a 2-fibration if for any $y \in C$, $x \in D$ and $t : x \to \pi(y)$, there is a 1-isomorphism $t : \hat{x} \to y$ with $\pi(t) = t$.

**Proof.** The definition of 2-fibration has 3 parts [Buc14, Definition 3.1.5]. Part (1) follows from the assumption that the lift $\hat{t}$ exists and the fact that $C$ is a 2-groupoid. Part (2) follows from (1) together with the fact that $D$ is a 1-category. Part (3) follows from the fact that $C$ is a 2-groupoid, and hence all 1-isomorphisms and 2-isomorphisms are cartesian [Buc14, Proposition 3.1.11]; see [Buc14, Definitions 3.1.1 and 3.1.4] for the definition of cartesian. □

The following proves the first part of Theorem 1.4.

**Lemma 4.34.** When $X$ has a contractible universal cover, the functor $\pi : \text{Bun}_G(\mathcal{G})(X) \to \text{Bun}_G(X)$ is a 2-fibration.

**Proof.** Since 2-fibrations are stable under equivalences of bicategories, Proposition 4.2 shows that the statement of the lemma is equivalent to showing that $\tilde{\pi} : \text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/\mathcal{G}) \to \text{Cat}(\text{pt}/\pi_1(X), \text{pt}/\mathcal{G})$ is a 2-fibration. To this end, we apply Lemma 4.33 and the presentation (1.3) of the target. We observe that it suffices to construct a 1-morphism $(t, \eta) : (\rho', \gamma') \to (\rho, \gamma)$ given the input data of a morphism $t : \rho' \to \rho$ in $\text{Hom}(\pi_1(X), G)/\mathcal{G}$ and an object $(\rho, \gamma) \in \text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/\mathcal{G})$.

To this end, define $\gamma' : \pi_1(X) \times \pi_1(X) \to G$ and $\eta : \pi_1(X) \to G$ by

$$\gamma'(a, b) := \gamma(a, b) \alpha(t^{-1}, \rho(a)t, \rho'(b)) \alpha(\rho(a), t, \rho(b)) \alpha(t, t^{-1}, \rho(b)t) \alpha(\rho(a), \rho(b), t), \quad \eta(a) := \alpha(t, t^{-1}, \rho(a)t)^{-1}.$$  

A short computation using the cocycle condition for $\alpha$ shows that these data supply an object $(\rho', \gamma') \in \text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/\mathcal{G})$ and a 1-morphism $(t, \eta) : (\rho', \gamma') \to (\rho, \gamma)$. By inspection, the image of this 1-morphism under $\pi$ is the morphism $t : \rho' \to \rho$ in $\text{Hom}(\pi_1(X), G)/\mathcal{G}$, completing the proof. □

4.4. A choice of cleavage for $\pi$. Given the 2-fibration $\pi$, the functor $F_x : \text{Bun}_G(X)^{op} \to \text{Bicat}$ as in (4.3) exists and is unique up to isomorphism [Buc14, Construction 3.3.5]. To compare with other (classical) constructions, it will be useful to fix a choice of functor and have explicit formulas for its values.
Remark 4.5. Because the target of (4.2) is the action groupoid \( \text{Hom}(\pi_1(X), G)/G \), the functor \( F_\pi \) is the same data as a (right) \( G \)-action on \( \text{Bun}_G^X(X) \cong \text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/G) \).

Our choice of \( F_\pi \) is induced from the natural right action of \( G \) on itself by conjugation, which gives a monoidal functor \( G \to \text{Aut}(\text{pt}/G) \), and hence \( G \to \text{Aut}(\text{Bicat}(\text{pt}/\pi_1(X), \text{pt}/G)) \). Both of these functors factor through the quotient \( G \to G \).

Define the value of (4.3) on objects as
\[
F_\pi(P) = \{\text{trivializations of } \lambda_{P, \alpha}\},
\]
i.e., the bicategory of trivializations of the 2-gerbe \( \lambda_{P, \alpha} \) over \( X \). After identifying \( P \) with a homomorphism \( \rho \), \( F_\pi(P) = F_\pi(\rho) \) is the bicategory whose objects are \( \gamma \) (with \( \rho^*\alpha = 1/d\gamma \) for a fixed \( \rho \)), 1-morphisms are \( \eta \) (which we identify with \((1, \eta) = (t, \eta)\)), and 2-morphisms \( \omega \) using the notation from §4.1. For \( \varphi : P' \to P \) an isomorphism of \( G \)-bundles over \( X \), we require an equivalence of bicategories,
\[
F_\pi(\varphi) : \{\text{trivializations of } \lambda_{P, \alpha}\} \xrightarrow{\sim} \{\text{trivializations of } \lambda_{P', \alpha}\}.
\]
Such an equivalence exists because there is an isomorphism of 2-gerbes, \( \varphi^*\lambda_{P, \alpha} \cong \lambda_{P', \alpha} \), and hence their bicategories of trivializations are equivalent; however, the equivalence is only unique up to 1-isomorphism, which is itself only unique up to unique 2-isomorphism. To fix choices, first we identify \( P \) and \( P' \) with homomorphisms \( \rho \) and \( \rho' \) and the isomorphism \( \varphi \) of \( G \)-bundles with an element \( g \in G \) such that \( \rho \) is conjugate to \( \rho' \) via \( g \). Then define the functor \( F_\pi(\varphi) = F_\pi(g) \) as follows. The value of \( F_\pi(g) \) on objects is \( F_\pi(g)(\rho, \gamma) := (\rho^g, \gamma^g) \) where
\[
\rho^g(a) := g^{-1}(\rho(a)g), \quad \gamma^g(a, b) := \gamma(a, b) \frac{\alpha(g^{-1}, \rho(a)g, \rho^g(b))\alpha(\rho(a), g, \rho^g(b))}{\alpha(g, g^{-1}, \rho(b)g)\alpha(\rho(a), \rho(b), g)}
\]
which is the same as the chosen lift in the proof of Lemma 4.4. Define the value of \( F_\pi(g) \) on morphisms as \( F_\pi(g)((t, \eta) : (\rho_1, \gamma_1) \to (\rho_2, \gamma_2)) := (t^g, \eta^g) \) where
\[
(t^g, \eta^g) := m(g, \rho_2)^{-1} \circ ((t, \eta) \circ m(g, \rho_1)), \quad m(g, \rho)(-) := (g, \alpha(g, g^{-1}, \rho(-)g)^{-1}).
\]
The value of \( F_\pi(g) \) on a 2-morphism \( u : (t, \eta) \to (t, \xi) \) is \( F_\pi(g)(u) := (u : (t^g, \eta^g) \to (t^g, \xi^g)) \). In addition to this data on objects, 1-morphisms, and 2-morphisms, the functor \( F_\pi(g) : F_\pi(\rho) \to F_\pi(\rho') \) requires the data of 2-morphisms that witness the compatibility between compositions of 1-morphisms: given \( (\rho_1, \gamma_1) \xrightarrow{(t, \eta)} (\rho_2, \gamma_2) \xrightarrow{(s, \xi)} (\rho_3, \gamma_3) \) we require a 2-morphism
\[
F_\pi(g)(s, t) : (F_\pi(g)(s, \xi)) \circ (F_\pi(g)(t, \eta)) \to F_\pi(g)((s, \xi) \circ (t, \eta)).
\]
We define
\[
F_\pi(g)(s, t) := \frac{\alpha(g^{-1}, sg, g^{-1}tg)\alpha(s, g, g^{-1}tg)}{\alpha(g, g^{-1}, tg)\alpha(s, t, g)}.
\]
It is routine (but tedious) to verify that the above data are compatible, completing the construction of the functor \( F_\pi(g) : F_\pi(\rho) \to F_\pi(\rho') \).

The functor (4.3) also requires the data of natural isomorphisms
\[
(4.7) \quad F_\pi(g, h) : F_\pi(g) \circ F_\pi(h) \Rightarrow F_\pi(hg),
\]
which is the data of a 1-morphism \( F_\pi(g, h)_{(\rho, \gamma)} : (F_\pi(g) \circ F_\pi(h))(\rho, \gamma) \to F_\pi(hg)(\rho, \gamma) \) for each object \( (\rho, \gamma) \in \text{Bun}_G^X(X) \) together with a 2-morphism witnessing naturality. We define the 1-morphism by
\[
F_\pi(g, h)_{(\rho, \gamma)} = m(hg, \rho)^{-1} \circ (m(h, \rho) \circ m(g, \rho^h)).
\]
The 2-morphism data for (4.7) sits in the diagram

\[
((\rho^h_1)g, (\gamma^h_1)g) \xrightarrow{F_\pi(g,h)(\rho_1, \gamma_1)} (\rho^h_1, \gamma^h_1) \\
((t^h)^g, (\eta^h)^g) \xrightarrow{\alpha} (t^h, \eta^g) \\
((\rho^h_2)g, (\gamma^h_2)g) \xrightarrow{F_\pi(g,h)(\rho_2, \gamma_2)} (\rho^h_2, \gamma^h_2)
\]

and we define it as

\[F_\pi(g,h)(t, \eta) = \alpha(g^{-1}, h^{-1}, thg) \alpha(hg, g^{-1}, h^{-1}) \alpha(t, h, g).\]

One must then check that the above data satisfy the following compatibility conditions to define the natural isomorphism (4.7).

**Lemma 4.6.** Given

\[((\rho_1, \gamma_1) \xrightarrow{(t, \eta)} (\rho_2, \gamma_2) \xrightarrow{(s, \xi)} (\rho_3, \gamma_3))\]

the following diagram commutes:

\[
\begin{array}{ccc}
\alpha(s, h, t, g, 1) = 1 & \xrightarrow{F_\pi(h, g)(s, t)} & F_\pi(h(g, h)(s, t)) \\
(s, h, \xi, \eta) \circ (t^h)^g \circ (\eta^h)^g & \xrightarrow{F_\pi(g, h)(s, t)} & F_\pi(g(h, h)(s, t)) \\
 F_\pi(g, h)(t, \eta) & \xrightarrow{F_\pi(h, g)(t, \eta)} & F_\pi(h(g, h)(t, \eta)) \\
((s, h, \xi, t, g)^h) & \xrightarrow{F_\pi(g, h)(t, \eta)} & F_\pi(h(g, h)(t, \eta)) \\
\end{array}
\]

\[
\begin{array}{ccc}
\alpha(1, s, h, t, g) = 1 & \xrightarrow{F_\pi(g, h)(s, t)} & F_\pi(g(h, h)(s, t)) \\
((s, h, \xi, t, g)^h) \circ (t^h)^g \circ (\eta^h)^g & \xrightarrow{F_\pi(h, g)(t, \eta)} & F_\pi(h(g, h)(t, \eta)) \\
 F_\pi(g, h)(t, \eta) & \xrightarrow{F_\pi(h, g)(t, \eta)} & F_\pi(h(g, h)(t, \eta)) \\
((s, h, \xi, t, g)^h) \circ (t^h)^g \circ (\eta^h)^g & \xrightarrow{F_\pi(g, h)(t, \eta)} & F_\pi(h(g, h)(t, \eta)) \\
\end{array}
\]

**Proof.** The statement amounts to the equality

\[
\frac{F_\pi(h, g)(t, \eta)F_\pi(g, h)(s, \xi)F_\pi(h)(s, t)}{F_\pi(g, h)(s, \xi)F_\pi(h(g, h)(s, t))} = 1 \in A
\]

After unraveling the definitions, this follows from judicious use of the cocycle condition for \(\alpha\). \hfill \qed

Finally, there is associativity data for the \(F_\pi(g, h)\) to assemble to a functor (4.3): given \(g, h, k \in G\) the following diagram must commute up to specified isomorphism:

\[
\begin{array}{ccc}
\xrightarrow{1} & \xrightarrow{\alpha} \\
((F_\pi(g)) \circ (F_\pi(h)) \circ (F_\pi(k)) & \rightarrow & ((F_\pi(g)) \circ (F_\pi(h(k))) \\
F_\pi(hg) \circ F_\pi(k) & \xrightarrow{u} & F_\pi(g) \circ F_\pi(kh)
\end{array}
\]

The arrows are defined as follows. The composition of functors is strictly associative, hence the top arrow is 1. The vertical arrows are determined by horizontal composition; explicitly
their values on objects and 1-morphisms are given by
\[
(F_\pi(g, h) \cdot \text{id}_{F_\pi(k)})_{(\rho, \gamma)} = F_\pi(g, h)_{(\rho^k, \gamma^k)}
\]
\[
(F_\pi(g, h) \cdot \text{id}_{F_\pi(k)})_{(t, \eta)} = F_\pi(g, h)_{(tk^\eta, \eta^k)}
\]
\[
(\text{id}_{F_\pi(g)} \cdot F_\pi(h, k))_{(\rho, \gamma)} = F_\pi(g, F_\pi(h, k)_{(\rho, \gamma)})
\]
\[
(\text{id}_{F_\pi(g)} \cdot F_\pi(h, k))_{(t, \eta)} = F_\pi(g)(F_\pi(h, k)_{(t, \eta)}) = F_\pi(h, k)_{(t, \eta)}
\]

To define the modification \(u\) witnessing associativity, for each \((\rho, \gamma) \in \text{Bun}_G^b(X)\) we require a 2-morphism (i.e., an element of \(A\))
\[
u_{(\rho, \gamma)} : [F_\pi(hg, k) \circ [F_\pi(g, h) \cdot \text{id}_{F_\pi(k)}]]_{(\rho, \gamma)} \to [F_\pi(g, kh) \circ [\text{id}_{F_\pi(g)} \cdot F_\pi(h, k)]]_{(\rho, \gamma)}
\]
which we take to be
\[
\nu_{(\rho, \gamma)} = \frac{\alpha(g, g^{-1}, g)\alpha(kh, (kh)^{-1}, kh)}{\alpha(k, h, g)\alpha((hg)^{-1}, hg, (hg)^{-1})}
\]
This satisfies the required condition, namely an equality of elements of \(A\)
\[
u_{(\rho_1, \gamma_1)} \cdot [F_\pi(g, kh) \circ [\text{id}_{F_\pi(g)} \cdot F_\pi(h, k)]]_{(t, \eta)} = \nu_{(\rho_2, \gamma_2)} \cdot [F_\pi(hg, k) \circ [F_\pi(g, h) \cdot \text{id}_{F_\pi(k)}]]_{(t, \eta)}.
\]
To verify this equality, we first note \(\nu_{(\rho, \gamma)}\) is independent of \((\rho, \gamma)\), and only depends on \(k\), \(h\), and \(g\). Thus when we check the equality in \(A\), it is sufficient to verify that
\[
[F_\pi(g, kh) \circ [\text{id}_{F_\pi(g)} \cdot F_\pi(h, k)]]_{(t, \eta)} = [F_\pi(hg, k) \circ [F_\pi(g, h) \cdot \text{id}_{F_\pi(k)}]]_{(t, \eta)},
\]
which follows from unravelling the definitions and using that \(\alpha\) is a normalized 3-cocycle.

This completes the description of our choice of cleavage for the 2-fibration \(\pi\).

5. A categorification of the Freed–Quinn line bundle

For this section let \(\Sigma\) be a connected oriented surface and let \(A = U(1)_\delta\), the unitary group with discrete topology. We will drop the subscript “\(\delta\)” in this section: hereafter \(U(1)\) will carry the discrete topology. Let \(\mathcal{G}\) be the 2-group of the form \((1.1)\). Our goal is to compare \(\text{Bun}_G^b(\Sigma)\) with the Freed–Quinn line bundle described in \((1.8)\).

5.1. Review of the Freed–Quinn line bundle. For a finite group \(G\), a 3-cocycle \(\alpha \in \frac{Z^3(G; U(1))}{\text{Aut}(G)}\), and an oriented surface \(\Sigma\), Freed and Quinn construct a line bundle \([\text{FQ93}, \text{Theorem 1.7}]\) \(L_\alpha \to \text{Bun}_G(\Sigma)\). Following the terminology from \([\text{Gau99}, \S 2.3-2.4]\), we refer to \(L_\alpha\) as the Freed–Quinn line bundle.

For \(\Sigma\) connected, \(L_\alpha\) admits an easy description on the equivalent groupoid,
\[
\prod_{[\rho]} \text{pt}/\text{Aut}(\rho) \xrightarrow{\sim} \text{Hom}(\pi_1(\Sigma), G)/G \simeq \text{Bun}_G(\Sigma).
\]
Here we have chosen a representative \(\rho : \pi_1(\Sigma) \to G\) of each conjugacy class \([\rho]\), and \(\text{Aut}(\rho) < G\) is the subgroup centralizing \(\rho\).

By inspection, this inclusion is fully faithful and essentially surjective, and hence an equivalence. A line bundle over the source of \((5.1)\) is then specified by a 1-dimensional representation of \(\text{Aut}(\rho)\) for each conjugacy class \([\rho]\). We note that an element \(t \in \text{Aut}(\rho)\) determines a \(G\)-bundle \(P_\rho \times \Sigma \times S^1\); consider the principal bundle \(P_\rho \times I \to \Sigma \times I\), where \(P_\rho\) is the bundle determined by \(\rho\), and then glue the boundaries together using the principal bundle automorphism determined by \(t \in \text{Aut}(\rho)\). Note that this bundle is classified by the map \(\rho_t : \pi_1(\Sigma) = \pi_1(\Sigma) \times \mathbb{Z} \to G\) given by \(\rho_t(a, n) = \rho(a)t^n\).

**Definition 5.1.** The Freed–Quinn line bundle \(L_\alpha \to \prod_{[\rho]} \text{pt}/\text{Aut}(\rho)\) is determined by the following data. For each conjugacy class \([\rho]\), we use our representative \(\rho : \pi_1(\Sigma) \to G\) to define a homomorphism
\[
(5.2) \quad \text{Aut}(\rho) \to U(1), \quad t \mapsto [\rho(t)\alpha, [\Sigma \times S^1]] \in U(1),
\]
where \([\Sigma \times S^1] \in H_3(\Sigma \times S^1; \mathbb{Z})\) is the fundamental class.

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5.2. Recovering the Freed–Quinn line bundle. Recall the forgetful 2-fibration \( \pi : \text{Bun}_G(\Sigma) \to \text{Bun}_G(\Sigma) \) that sends a \( G \)-bundle to its underlying \( G \)-bundle.

**Definition 5.2.** For the functor \( F_\pi \) defined in \S 4.4, define the presheaf of sets

\[
(F_\pi)_*: \text{Bun}_G(\Sigma)^{\text{op}} \to \text{Bicat} \to \text{Set}
\]

where the second arrow sends a bicategory to its set of equivalence classes of objects.

Explicitly, the presheaf \([F_\pi]\) implements an equivalence relation \( \sim \) on objects of \( \text{Bun}_G(\Sigma) \). We have \( (\rho, \gamma) \sim (\rho', \gamma') \) if and only if \( \rho = \rho' \) and there exists an \( \eta \) such that \( \gamma/\gamma' = d\eta \), or (equivalently) \( \gamma \) and \( \gamma' \) are isomorphic trivializations of the 2-gerbe \( \rho^* \alpha \).

**Lemma 5.3.** The presheaf of sets \((5.3)\) canonically has the structure of a principal \( U(1) \)-bundle on \( \text{Bun}_G(\Sigma) \).

**Proof.** This follows from Corollary 3.21 and the fact that flat 2-gerbes are classified by \( H^2(\Sigma; U(1)) \cong U(1) \) when \( \Sigma \) is an oriented surface (we emphasize that \( U(1) \) carries the discrete topology).

For each fixed \( \rho : \pi_1(\Sigma) \to G \), Lemma 5.3 gives a non-canonical bijection between the value \([F_\pi]\)(\(\rho\)) and \( U(1) \). We fix such a bijection depending on a choice of representative of the fundamental class

\[ [\Sigma] \in H_2(\pi_1(\Sigma); \mathbb{Z}) \cong H_2(\Sigma; \mathbb{Z}) \]

of the connected, oriented surface \( \Sigma \), where the isomorphism above uses that \( \Sigma \) has contractible universal cover. Let \( \Sigma \in \mathbb{Z}_2(\pi_1(\Sigma); \mathbb{Z}) \) be such a choice of representative of \( [\Sigma] \) and consider the map

\[
(F_\pi)(\rho) \to U(1), \quad (\rho, \gamma) \mapsto \langle \gamma, \Sigma \rangle \in U(1).
\]

The following completes the proof of Theorem 1.3, where \( \sigma \) in the statement of the theorem is the value of the pairing \((5.4)\) for a \( G \)-bundle determined by \((\rho, \gamma)\), and the choice of representative of the fundamental class is specified using the presentation of the fundamental group, e.g., see [Vaf86]. We work out the case of a torus explicitly in (5.5) below.

**Proposition 5.4.** The map \((5.4)\) is a bijection: a pair of objects \((\rho, \gamma), (\rho, \gamma') \in \text{Bun}_G(\Sigma)\) is isomorphic if and only if the values \((5.4)\) are equal.

**Proof.** Using that \( \rho^* \alpha = \frac{1}{\Sigma} \), for any pair \((\rho, \gamma)\) and \((\rho, \gamma')\) the ratio \( \gamma/\gamma' \) is closed, i.e., is a 2-cocycle. By Poincaré duality, the pairing between the underlying class \( [\gamma/\gamma'] \) and \([\Sigma]\) is zero if and only if \( \gamma/\gamma' \) is exact. We conclude that \( [\gamma, \Sigma] = [\gamma', \Sigma] \) if and only if \( \gamma/\gamma' = d\eta \) is exact. This is precisely the equivalence relation implemented by \([F_\pi]\), and hence the assignment \((5.4)\) is injective. Since the pairing is \( U(1) \)-equivariant, surjectivity is also clear.

The following completes the proof of Theorem 1.4.

**Proposition 5.5.** The line bundle associated to the \( U(1) \)-principal bundle \((5.3)\) is isomorphic to the Freed–Quinn line bundle \( \mathcal{L}_\alpha \to \text{Bun}_G(\Sigma) \). The isomorphism is specified by a choice of chain-level representative of the fundamental class of \( \Sigma \).

**Proof.** It suffices to prove the claimed isomorphism over the equivalent subgroupoid \((5.1)\). Then for \( t \in \text{Aut}(\rho) \) determining a 1-isomorphism \( (t, \eta) : (\rho, \gamma) \to (\rho, \gamma') \) in \( \text{Bun}_G(\Sigma) \), we need to show that the \( G \)-equivariance structure from Lemma 5.3 agrees with \((5.2)\). We require a chain-level description of the Künneth isomorphism

\[ H_2(\Sigma; \mathbb{Z}) \otimes H_1(\Sigma; \mathbb{Z}) \cong H_3(\Sigma \times S^1; \mathbb{Z}), \quad [\Sigma] \otimes [S^1] \mapsto [\Sigma \times S^1] \]

that permits a chain-level description of the fundamental class of \( \Sigma \times S^1 \) in terms of choices

\[ e \in Z_1(\pi_1(S^1); \mathbb{Z}) \subset \mathbb{Z}[\pi_1(S^1)], \quad [\Sigma] = \sum_i a_i \otimes b_i \in Z_2(\pi_1(\Sigma); \mathbb{Z}) \subset \mathbb{Z}[\pi_1(\Sigma)] \otimes \mathbb{Z}[\pi_1(\Sigma)] \]
for the representatives of the fundamental classes of $\Sigma$ and $S^1$, respectively. As an element of the bar complex of $\pi_1(\Sigma \times S^1)$, this product of cycles is given by the shuffle product [Wei94, Proposition 8.6.13 and Exercise 8.6.5], and so

$$[\Sigma \times S^1] = \sum_{i \in I} (a_i \otimes b_i \otimes e - a_i \otimes e \otimes b_i + e \otimes a_i \otimes b_i) \in H^3(\pi_1(\Sigma) \times \mathbb{Z}; \mathbb{Z}).$$

Let $\rho_i : \pi_1(\Sigma) \times \mathbb{Z} \to G$ be the homomorphism determined by $\rho$ and $t \in \text{Aut}(\rho) < G$. We compute

$$\langle \rho^\ast \alpha, [\Sigma \times S^1] \rangle = \langle \alpha, (\rho_1)_\ast [\Sigma \times S^1] \rangle = \prod_{i \in I} \alpha(\rho(a_i), \rho(b_i)) \alpha(t, \rho(a_i), \rho(b_i))$$

$$= \prod_{i \in I} dq(a_i, b_i) : \frac{\gamma_2(a_i, b_i)}{\gamma_1(a_i, b_i)}$$

$$= \langle dq, \Sigma \rangle \cdot \frac{\langle \gamma_2, \Sigma \rangle}{\langle \gamma_1, \Sigma \rangle} = \langle \gamma_2, \Sigma \rangle \langle \gamma_1, \Sigma \rangle$$

where the third line follows from (4.5), and in the last line we use that $\langle dq, \Sigma \rangle = 0$. In the above chain of equalities, we start with the isomorphism between fibers in the definition of the Freed–Quinn line bundle, and end with the isomorphism between fibers dictated by [Wei94, §3.4] as a function of the pairs of commuting elements with an equivariance property for Willerton’s twisted elliptic characters in $G$ [Wil08, §3.4] as a function of the pairs of commuting elements with an equivariance property for the action of conjugation.

5.3. **Explicit computations for the torus and elliptic characters.** In the case when $\Sigma = S^1 \times S^1$ is genus 1, we can compute explicit formulas for the equivariant line bundle. For $a$ and $b$ the standard generators of $\pi_1(\Sigma) \cong \mathbb{Z} \times \mathbb{Z}$, a representative of the fundamental class is $a \otimes b - b \otimes a \in Z_2(\pi_1(\Sigma); \mathbb{Z})$. We observe that

$$\sigma = \langle \gamma, \Sigma \rangle = \frac{\gamma(a, b)}{\gamma(b, a)}$$

is an isomorphism $a \otimes b \sim b \otimes a$ in $\mathcal{G}$; compare the picture (1.7). As in the proof of Proposition 5.5, we compute

$$\frac{\langle \gamma^2, \Sigma \rangle}{\langle \gamma_1, \Sigma \rangle} = \frac{\alpha(t, \rho_1(b), \rho_1(a))\alpha(\rho_2(a), t, \rho_1(b))\alpha(\rho_2(b), t, \rho_1(a))\alpha(\rho_2(a), \rho_2(b), t)}{\alpha(t, \rho_1(a), \rho_1(b))\alpha(\rho_2(b), t, \rho_1(a))\alpha(\rho_2(a), \rho_2(b), t)}.$$

This is precisely the transformation property for Willerton’s twisted elliptic characters in $G$ [Wil08, §3.4] as a function of the pairs of commuting elements with an equivariance property for the action of conjugation.

**Appendix A. The proof of Proposition 3.5**

We define a bicategory $\mathcal{C}$ of Čech-style gluing data for principal $\mathcal{G}$-bundles on $X$. The first result is the following.

**Theorem A.1.** The functor

$$\text{Glue} : \mathcal{C} \to \text{Bun}_G(X)$$

induces an equivalence of 2-categories.

**Proof of Theorem A.1.** We begin with a principal $\mathcal{G}$-bundle $\mathcal{P}$ on $X$, and choose a submersion $u : Y \to X$ over which we can fix an isomorphism $d_0 : Y \times \mathcal{G} \to u^* \mathcal{P}$. We pull back along the two projections $p_i : Y^{[2]} \to Y$ and obtain a canonical automorphism $p_{ij}^\ast u^* \mathcal{P} \Rightarrow p_{ij}^\ast u^* \mathcal{P}$, which yields an automorphism $\Phi$ of $Y^{[2]} \times \mathcal{G}$.

Next consider the pullback of $\Phi$ along the three projections $p_{ij} : Y^{[3]} \to Y^{[2]}$, yielding a pair of canonical 1-isomorphisms $p_{ij}^\ast u^* \mathcal{P} \to p_{ij}^\ast u^* \mathcal{P}$. This correspond under $d_0$ to the 1-automorphisms of $Y^{[3]} \times \mathcal{G}$ given by $p_{ij}^\ast \Phi \circ p_{ij}^\ast \Phi$ and $p_{ij}^\ast \Phi$. In the classical setting, these
isomorphisms are equal; in the bicategory of bibundles, we instead obtain a 2-isomorphism

\[(A.1) \quad \tilde{\Gamma} : p_{12}^*\Phi \circ p_{23}^*\Phi \to p_{13}^*\Phi.\]

This 2-isomorphism in turn satisfies a compatibility condition on \(Y[4]\), where we obtain a pair of 2-isomorphisms between the 1-automorphisms \((p_{12}^*\Phi \circ p_{23}^*\Phi) \circ p_{34}^*\Phi\) and \(p_{14}^*\Phi\).

To summarize, from a bundle \(\mathcal{P}\), the cover \(u : Y \to X\), and the trivialization \(d_0\) we obtain the following data:

- A 1-automorphism \(\tilde{\Phi} : Y[2] \times \mathcal{G} \to Y[2] \times \mathcal{G}\)
- \(\tilde{\Gamma} : p_{12}^*\Phi \circ p_{23}^*\Phi \to p_{13}^*\Phi\)

\[(A.2) \quad p_{134}^*\tilde{\Gamma} \circ p_{123}^*\tilde{\Gamma} = p_{124}^*\tilde{\Gamma} \circ p_{234}^*\tilde{\Gamma} \circ \alpha,\]

where here \(\alpha\) denotes the relevant associativity isomorphism.

Define the objects of \(\tilde{\mathcal{C}}\) to be triples \((u, \tilde{\Phi}, \tilde{\Gamma})\) satisfying condition \((A.2)\).

To define the 1-morphisms in \(\tilde{\mathcal{C}}\), consider a 1-isomorphism \(\mathcal{T} : \mathcal{P}_1 \to \mathcal{P}_2\) of principal \(\mathcal{G}\)-bundles on \(X\) and a surjective submersion \(u : Y \to X\) with fixed trivializations \(d_i^0 : Y \times \mathcal{G} \to u^*\mathcal{P}_i\) for \(i = 1, 2\). With respect to these trivializations, \(u^*\mathcal{T}\) gives rise to a 1-automorphism \(\tilde{\mathcal{T}}\) of the trivial \(\mathcal{G}\)-bundle \(Y \times \mathcal{G}\). This 1-automorphism \(\tilde{\mathcal{T}}\) is compatible with the 1-automorphisms \(\tilde{\Phi}_i\) on \(Y[2]\) via a specified 2-morphism \(\tilde{\eta}\). On \(Y[3]\), \(\tilde{\eta}\) is compatible with the 2-morphisms \(\Gamma_i\), in the sense that the pair of natural 2-morphisms between \((p_{12}^*\mathcal{T} \circ p_{12}^*\Phi_1) \circ p_{23}^*\tilde{\Phi}_1\) and \(p_{13}^*\tilde{\Phi}_2 \circ p_{13}^*\mathcal{T} \circ p_{13}^*\tilde{\Phi}_2\) are equal. Indeed, this follows from the identity

\[(A.3) \quad p_{13}^*\tilde{\eta} \circ \tilde{\Gamma}_1 \circ \alpha = \tilde{\Gamma}_2 \circ \alpha^{-1} \circ p_{23}^*\tilde{\eta} \circ \alpha \circ p_{12}^*\tilde{\eta},\]

where the \(\alpha\)'s are associativity isomorphisms. Hence, from \((\mathcal{T}, u, d_0^1, d_0^2)\), we have extracted the following data:

- \(\tilde{T} : Y \times \mathcal{G} \to Y \times \mathcal{G}\) a 1-automorphism of the trivial \(\mathcal{G}\)-bundle;
- \(\tilde{\eta} : p_{12}^*\tilde{T} \circ \tilde{\Phi}_1 \Rightarrow \tilde{\Phi}_2 \circ p_{23}^*\tilde{T}\), a 2-isomorphism satisfying the condition \((A.3)\).

Define the 1-morphisms of \(\tilde{\mathcal{C}}\) between two objects \((u, \tilde{\Phi}_1, \tilde{\Gamma}_1)\) and \((u, \tilde{\Phi}_2, \tilde{\Gamma}_2)\) to be a triple \((u, \tilde{T}, \tilde{\eta})\) as above. For objects \((u_1, \tilde{\Phi}_1, \tilde{\Gamma}_1)\) and \((u_2, \tilde{\Phi}_2, \tilde{\Gamma}_2)\) with \(u_1\) not necessarily equal to \(u_2\), a 1-morphism is a common refinement \(u\) of the submersions \(u_1\) and \(u_2\) together with a 1-morphism between the pullbacks to \(u\) of the original objects.

This construction also leads to the definition of composition of 1-morphisms in \(\tilde{\mathcal{C}}\). Without loss of generality we may assume all of the objects and 1-morphisms are defined over the same surjective submersion \(u : Y \to X\). The following diagram in \(\tilde{\mathcal{C}}\)

\[(u, \tilde{\Phi}_1, \tilde{\Gamma}_1) \xrightarrow{(u, \tilde{T}_1, \tilde{\eta}_1)} (u, \tilde{\Phi}_2, \tilde{\Gamma}_2) \xrightarrow{(u, \tilde{T}_2, \tilde{\eta}_2)} (u, \tilde{\Phi}_3, \tilde{\Gamma}_3).\]

determines a diagram of (trivial) \(\mathcal{G}\)-bundles over \(Y\). So define the composition \((u, \tilde{T}_2, \tilde{\eta}_2) \circ (u, \tilde{T}_1, \tilde{\eta}_1)\) as the triple \((u, \tilde{S}, \tilde{\theta})\) where \(\tilde{S} : Y \times \mathcal{G} \to Y \times \mathcal{G}\) is the composition \(\tilde{T}_2 \circ \tilde{T}_1\) of 1-isomorphisms of (trivial) principal \(\mathcal{G}\)-bundles, and \(\tilde{\theta}\) is the following composition of 2-isomorphisms of principal \(\mathcal{G}\)-bundles:

\[(A.4) \quad p_1^*(\tilde{T}_2 \circ \tilde{T}_1) \circ \tilde{\Phi}_1 \longrightarrow (p_1^*\tilde{T}_2 \circ p_1^*\tilde{T}_1) \circ \tilde{\Phi}_1 \overset{\alpha}{\longrightarrow} p_1^*\tilde{T}_2 \circ (p_1^*\tilde{T}_1 \circ \tilde{\Phi}_1)
\]

One can check that this pair \((\tilde{S}, \tilde{\theta})\) satisfies the compatibility condition \((A.2)\) and hence defines a 1-morphism in \(\tilde{\mathcal{C}}\).
A similar argument motivates us to define a 2-morphism in $\mathcal{C}$ between triples $(u_1, T_1, \tilde{\eta}_1)$ and $(u_2, T_2, \tilde{\eta}_2)$ to be represented by an 2-isomorphism $\tilde{\omega} : T_1 \to T_2$ of (trivial) $G$-bundles (on a suitable common refinement $u$ of $u_1$ and $u_2$), compatible with the 2-morphisms $\tilde{\eta}_1$,

$$\tilde{\eta}_1 \circ p_1^* \tilde{\omega} = p_2^* \tilde{\omega} \circ \tilde{\eta}_2.$$ 

Two pairs $(u, \tilde{\omega})$ and $(u', \tilde{\omega}')$ define the same 2-morphisms if they agree over a common refinement. One can check that there are natural 2-morphisms in $\mathcal{C}$ of 1-morphisms (A.4) weakly associative.

In summary, we have defined a two-category $\mathcal{C}$ with

- objects $(u, \tilde{T}, \tilde{\eta})$ satisfying the condition (A.2);
- 1-morphisms $(u, \tilde{T}, \tilde{\eta})$ satisfying the condition (A.3);
- 2-morphisms equivalence classes $\left[ (u, \tilde{\omega}) \right]$.

By construction, each object, 1-morphism, or 2-morphism in $\mathcal{C}$ is exactly the data that we need to construct an object, 1-morphism, or 2-morphism in $\text{Bun}_G$ making the composition of 1-morphisms (A.4) weakly associative.

Furthermore, by construction this gives a quasi-inverse to $\text{Glue}$, and hence the theorem is proved.

The next goal is to identify a nice subcategory $\mathcal{C}_A \subset \mathcal{C}$ in which the data of the 1-automorphisms $\Phi$ and $\tilde{T}$ are particularly tractable. We begin with some results and definitions about 1-automorphisms of the trivial $G$-bundle.

**Proposition A.2.** Let $G$ be a smooth 2-group specified by the data of a finite group $G$, abelian Lie group $A$, and 3-cocycle $\alpha \in Z^3(G; A)$ for the trivial $G$ action on $A$ (see (2.6)).

The data of a $G$-equivariant bibundle $\Phi$ giving an automorphism of the trivial $G$-bundle $X \times G \to X$ is equivalent to the data of a bibundle $\Phi : X \to G$.

Under this equivalence, the composition of two automorphisms $\Phi_1 \circ \Phi_2$ corresponds to the composition

$$X \xrightarrow{\Phi_1 \times \Phi_2} G \times G \xrightarrow{m} G.$$

Furthermore, the associativity 2-isomorphism $(\Phi_1 \circ \Phi_2) \circ \Phi_3 \sim \tilde{\Phi}_1 \circ (\tilde{\Phi}_2 \circ \tilde{\Phi}_3)$ is determined by the associator for $G$.

**Proof.** Recall that “equivariance” for a bibundle $\Phi : X \times G \to X \times G$ over $X$ is an additional structure, namely an isomorphism of bibundles between the following compositions:

$$X \times G \times G \xrightarrow{\Phi \times \text{id}_G} X \times G \times G \xrightarrow{\text{id}_X \times m} X \times G; \text{ and}$$

$$X \times G \times G \xrightarrow{\text{id}_X \times m} X \times G \xrightarrow{\Phi} X \times G.$$

Given an equivariant bibundle $\Phi : X \times G \to X \times G$ over $X$, define a bibundle $S(\Phi) : X \to G$ to be the composition $\text{proj} \circ (\Phi \circ (\text{id}_X, 1_G))$:

$$X \xrightarrow{(\text{id}_X, 1_G)} X \times G \xrightarrow{\Phi} X \times G \xrightarrow{\text{proj}} G.$$

Conversely, given a bibundle $\Phi : X \to G$, define an equivariant bibundle $T(\Phi) : X \times G \to X \times G$ to be the composition

$$X \times G \xrightarrow{(\text{id}_X, \Phi) \times \text{id}_G} X \times G \times G \xrightarrow{\text{id}_X \times m} X \times G.$$

Here the equivariant structure comes from the associativity isomorphism for the 2-group $G$. 

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The equivalence of the data mentioned in the proposition follows from the definitions of the maps $S$ and $T$. \hfill \Box

Combining Proposition A.2 with Lemma 2.23 we obtain the following elementary description of the data of an automorphism:

**Corollary A.3.** The data of a $G$-equivariant bibundle $\Phi: X \times G \to X \times G$ is equivalent to the data of a principal $A$-bundle $P_\Phi \to X$ together with a map $\phi: X \to G$. Composition of bibundles $\Phi$ corresponds to taking the tensor product of the corresponding principal $A$-bundles (which is possible because $A$ is abelian) and the product in $G$ of the maps $X \to G$.

**Definition A.4.** We will say that the automorphism $\Phi$ is $A$-trivial if the underlying $A$-bundle is the trivial bundle $X \times A \to X$. We say that it is $A$-trivialized if it is equipped with an isomorphism to an $A$-trivial automorphism, and $A$-trivializable if such an isomorphism exists but is not specified.

**Remark A.5.** There are many $A$-trivial automorphisms over a given $X$, each corresponding to a choice of map $X \to G$. An isomorphism of $A$-trivial automorphisms $\Phi_1, \Phi_2$ corresponding to the data $(X \times A, \phi_1), (X \times A, \phi_2)$ can exist only if $\phi_1 = \phi_2$, and in that case is given by the data of an isomorphism of $A$-bundles $X \times A \to X \times A$, or equivalently a map $X \to A$.

**Remark A.6.** Since every principal $A$-bundle is locally trivial, every automorphism $\Phi$ of $X \times G$ can be pulled back along some surjective submersion $u: Y \to X$ such that $u^* \Phi$ is $A$-trivializable.

**Definition A.7.** We define a subcategory $\tilde{C}^A$ of $\tilde{C}$ as follows:

- objects are triples $(u, \tilde{\Phi}, \tilde{\Gamma})$ with $\tilde{\Phi}$ an $A$-trivial automorphism of $Y^{[2]} \times G$;
- 1-morphisms are triples $(u, \tilde{T}, \tilde{\eta})$ with $\tilde{T}$ an $A$-trivial automorphism of $Y \times G$;
- 2-morphisms are as in $\tilde{C}$.

It follows from Remark A.6 that $\tilde{C}^A$ is equivalent to $\tilde{C}$. We conclude the following.

**Proposition A.8.** The 2-functor $\text{Glue}$ restricts to an equivalence of 2-categories

\begin{equation}
\text{Glue}: \tilde{C}^A \to \text{Bun}_G X.
\end{equation}

Finally, we re-interpret the data of objects, 1-morphisms, and 2-morphisms of $\tilde{C}^A$ in terms of group cohomology data, through the lens of Corollary A.3 and Remark A.5.

Let us begin with an object of $\tilde{C}^A$, given by a triple $(u, \tilde{\Phi}, \tilde{\Gamma})$. In particular, $\tilde{\Phi}$ is an $A$-trivial automorphism of $Y^{[2]} \times G$, so under Corollary A.3, it is determined by the data of the trivial $A$-bundle $Y^{[2]} \times A \to Y^{[2]}$ together with a map $Y^{[2]} \to G$, which we will denote by $\rho$. Next, $\tilde{\Gamma}$ is an isomorphism of $A$-trivial automorphisms of $Y^{[3]} \times G$,

$$
\tilde{\Gamma}: p_{12}^* \tilde{\Phi} \circ p_{23}^* \tilde{\Phi} \to p_{13}^* \tilde{\Phi}.
$$

Under the equivalence of Corollary A.3, the two automorphisms correspond to the maps $p_{12}^* \rho \cdot p_{23}^* \rho$ and $p_{13}^* \rho$ from $Y^{[3]}$ to $G$. As in Remark A.5, the existence of the isomorphism $\tilde{\Gamma}$ implies that these two maps must be equal; furthermore the data of $\tilde{\Gamma}$ is equivalent to a choice of a map $\gamma: Y^{[3]} \to A$. Finally, the compatibility condition

$$
p_{134}^* \gamma \circ p_{123}^* \Gamma = p_{124}^* \Gamma \circ p_{234}^* \Gamma \circ \alpha
$$

corresponds to the following equality of maps $Y^{[4]} \to G$:

\begin{equation}
\gamma \circ p_{123}^* \gamma = \gamma \circ p_{234}^* \gamma \cdot \alpha(p_{12}^* \rho, p_{23}^* \rho, p_{34}^* \rho).
\end{equation}

Similarly, let us consider a 1-morphism of $\tilde{C}^A$

$$(u, \tilde{T}, \tilde{\eta}): (u, \tilde{\Phi}_1, \tilde{\Gamma}_1) \to (u, \tilde{\Phi}_2, \tilde{\Gamma}_2).$$

Using Corollary A.3, Remark A.5 and an analogous argument to the above, $\tilde{T}$ is equivalent to a map $t: Y \to G$ such that $p_1^* t \cdot \rho_1 = p_2^* t \cdot \rho_2$, and $\tilde{\eta}$ corresponds to a map
\[ \eta : Y^{[2]} \to A. \] 

The compatibility condition (A.3) can be translated as the following equality in \( A \): for each triple \((y_1, y_2, y_3) \in Y^{[3]}\), we have

\[ \tag{A.7} \]

\[
\frac{\eta(y_1, y_2)\eta(y_2, y_3)}{\eta(y_1, y_3)} = \frac{\gamma_1(y_1, y_2, y_3)}{\gamma_2(y_1, y_2, y_3)} \cdot \frac{\alpha(t(y_1), \rho_1(y_1, y_2, \rho_1(y_2, y_3)))}{\alpha(t(y_1), \rho_2(y_1, y_2, \rho_2(y_2, y_3)))}. \]

Finally, consider a pair of 1-morphisms \((u, \tilde{T}_1, \tilde{\eta}_1) : (u, \Phi_1, \tilde{\Gamma}_1) \to (u, \Phi_2, \tilde{\Gamma}_2)\) in \( \mathcal{C}^A \), and let \((u, \tilde{\omega})\) be a 2-morphism \((u, \tilde{T}_1, \tilde{\eta}_1) \to (u, \tilde{T}_2, \tilde{\eta}_2)\).

Again using Corollary A.3, Remark A.5, one can see that this means that the maps \( t_1, t_2 : Y \to G \) corresponding to \( \tilde{T}_1, \tilde{T}_2 \) are equal. The data of \( \tilde{\omega} \) is equivalent to a map \( \omega : Y \to A \); this map is subject to the compatibility condition that for all pairs \((y_1, y_2) \in Y^{[2]}\),

\[ \tag{A.8} \]

\[
\eta_2(y_1, y_2) \cdot \omega(y_2) = \omega(y_1) \cdot \eta_1(y_1, y_2). \]

Both vertical and horizontal composition of 2-morphisms is given by multiplication in \( A \). Combining the above discussion with Proposition A.8 gives Proposition 3.5.

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