Coherent Collective Excitations in a Superfluid: Spontaneously Broken Symmetries and Fluctuation-Dissipation

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Abstract

This paper presents an elementary theory of the phase-coherent collective excitations in both He II in the gravitational field and atomic Bose-Einstein condensation in a trap. The theory is based on the concept of off-diagonal long-range order by Penrose and Onsager and the quantum theory of Bohm, with emphasis on the broken symmetry in a Bose-Einstein gas with repulsive interactions. It is shown that a spontaneously broken symmetry that accompanies a phonon (Nambu-Goldstone mode) takes place at the surface layer of an inhomogeneous Bose system in the presence of an external field. The spontaneously broken symmetry in a Bose system is described and is shown to manifest itself in both He II and the Bose-Einstein condensation of an imperfect Bose gas - a shell-like structure of the Bose-Einstein condensation in a trap. The broken symmetry gives a coherent explanation for a number of long standing puzzles in He II.

The collective excitations in a liquid helium are essential for a complete understanding of certain properties of the liquid helium at low temperature, particularly its interaction with the external gravitational field, which is similar to that of the Meissner effect in a superconductor in an external magnetic field except for a characteristic difference arising from the statistics. We present a detailed derivation of dispersion relations for the various collective excitations in a surface layer in He II, in which Landau’s two-fluid model breaks down. It is shown in the surface layer that, in addition to the usual surface waves (a gravity and a capillary waves), there is also a transverse collective excitation with entirely different behavior from the phase-coherent longitudinal excitation (phonon). It is further shown that the spontaneously broken symmetry due to fluctuation and dissipation of the system is both a necessary and a sufficient condition for the conservation of energy in an isolated system.

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I. INTRODUCTION

The two-fluid model as a phenomenological theory of superfluidity for He II was a major conceptual breakthrough in quantum hydrodynamics since it gives an excellent account of the data in those cases to which the model is applicable, namely a spatially uniform fluid in a Hilbert space, but it has long been known as an incomplete theory to describe many spatially inhomogeneous problems [1, 2, 3, 4, 53].

Specifically, the remarkable hydrodynamic properties of He II have been explained by the two-fluid model in which He II is considered a mixture of two interpenetrating components: the superfluid with $\nabla \times v_s = 0$ and the normal fluid with viscosity; the two-fluid picture has been confirmed in an experiment with a torsion pendulum [5] and also the theory predicted the correct energy spectra of collective excitations (phonons and rotons) below a $\lambda$ point [1, 6, 7]. However, it cannot explain satisfactorily the peculiarly normal-fluid like behavior of the surface curvature of a rotating He II to this date [8, 9].

Over half of a century the dynamics of quantized vortices have dominated the hydrodynamics of a superfluid. And yet, the mechanism, by which vortices are created and nucleated in a superfluid, is still poorly understood. In particular, the breakdown of superfluidity at the vortex core, which is closely related to the normal fluid-like behavior of the surface of a rotating He II, has remained a mystery in low-temperature physics. In spite of the wealth of experimental data on the apparent breakdown of superfluidity [10, 11, 12, 27], the clue for this long-standing riddle has remained elusive. Before proceeding to the explicit discussion of this problem, we mention some qualitative features of the two-fluid model by Landau [1] that is valid in a Hilbert space. In the absence of an external field, there was no need to address the boundary conditions. Consequently, the model assumes that the fluid cannot actually separated into two parts (superfluid and normal parts). This assumption is not in accord with Mott’s analysis on the surface energy due to velocity discontinuity of a superfluid [13, 14]. The presence of the surface energy implies that the particles in the layer are in excited states with finite energy gap from the ground state [13, 16] and behave like a normal fluid - breakdown of the two-fluid model. We will show this in more detail with explicit calculations on the surface waves in the surface layer.

A degenerate Bose gas exhibits a peculiar condensation phenomenon that is now known as the Bose-Einstein condensation (BEC) which is often described as the momentum conden-
sation; this implies that the condensation is due to the symmetry of the wave function with respect to the interchange of any pair of particles in the degenerate Bose system \[29, 31, 48\]. The BEC shows a remarkable feature that for \( T < T_0 \) the pressure of the condensed state is independent of density, where \( T_0 = 3.31(\hbar^2/m)(N/V) \) [16]; this is the consequence of the degenerate state of the Bose particles with \( \varepsilon = 0 \) (and \( p = 0 \)). Since all real gases are condensed differently by the presence of molecular forces, this mechanism appeared to be devoid of any real physical significance.

The similar phase transition (\( \lambda \)-transition) in liquid helium may be regarded as the condensation phenomenon of a degenerate Bose-Einstein gas modified by the presence of molecular forces. This was first recognized by London who has proposed that the transition between liquid He I and liquid He II - \( \lambda \) transition - is the same process that causes the degenerate Bose gas to condense. In support of London’s view, Feynman [33] argued that, in a liquid-like quantum mechanical system, strongly interacting atoms behave in some respects very much like free particles. In other words, the strong interaction forces do not prevent these particles from behaving as if they are free particles. Hence the inclusion of strong pair-interactions will not alter London’s view on the \( \lambda \) transition as the same process that causes the condensation of an ideal Bose-Einstein gas, so long as there is a pair-interaction however weak the interaction might be [34].

The work of London [29, 31, 43] and the seminal paper by Bogoliubov [34, 35] on a microscopic model of an imperfect Bose gas have laid the basis for much of our theoretical understanding of liquid helium and the Bose-Einstein condensation (BEC). Penrose and Onsager have extended the concept of the Bose-Einstein condensation due to London [29] to a strongly interacting Bose system by introducing the concept of off-diagonal long-range order (ODLRO) [14, 35, 38, 39, 43, 44] and suggested that superfluidity be described by the one particle ground state wave function \( \psi \) which has a macroscopic mean value in the thermodynamic, quasi-equilibrium sense, the simplest example of which is an atomic Bose-Einstein condensation in a trap. Thus the Bose-Einstein condensation in a trap can be defined by ODLRO [39, 43, 45].

In 1942, Daunt and Mendelssohn [49] presented interesting experimental evidence for a striking similarity between the surface flow of He II and the electric surface current in a superconductor, which goes much farther than a superficial similarity. They have noticed that in both cases there are upper limits for the current densities which depend on temperature.
London studied a theoretical model to account for the similarity with the critical surface transfer rate $R_c \approx n\hbar$ where $n$ is the density of superfluid particles per $cm^3$ [53], but was not able to carry through due to the insufficient experimental data; the measurements of Daunt and Mendelssohn with mobile helium films were the only data of which he could make use [48]. He did, however, correctly point out the need to find a quantum mechanical mean field that represents the ground state of a many-particle system, and which can also describe the hydrodynamics of He II. What’s more, he emphasized that the peculiar low temperature transfer processes in both a superconductor and He II should be explained on the basis of that familiar mixture of quantum statistics and classical mechanics [29, 52].

But up to now it has been customary to neglect the gravitational field in the study of quantum fluid for mathematical convenience, [1, 3, 33, 47], which was justified by the extreme weak gravitational force compared to the pair-interaction force between He atoms. However, it is essential to include the external field in the discussion of a surface layer which is defined by the gravitational field and in which the two-fluid model of Landau [1, 33] breaks down. In particular, the BEC in a trap shows this peculiar shell-like structure [13, 24]. This is precisely the reason why the two-fluid model by Landau [1] cannot explain the surface phenomena, since the two-fluid model does not allow a separation of the superfluid and normal parts of the He II. More specifically, one of the problems we would like to address in this paper is to show that there is a striking similarity between the Meissner effect [52] in a superconductor in an external field and the surface phenomena in He II in the gravitational field [49]. The question naturally arises what physical significance has to be given to this analogy. It will be shown here that the surface layer is composed of the excited atoms with the energy gap from the ground state [13] and behaves like a normal fluid with viscosity. The particles in the layer do not interact with the bulk He II. This separation under the gravitational field has been observed by Lamb and Nordsieck [24] and supports Mott’s analysis [13].

In recent years there has been considerable interest in certain basic properties of the atomic Bose condensation in connection with trapped alkali-metal gas at extreme low temperature [62, 63, 64]. In a magnetic trap experiment, Anderson et al., [66] have observed a rapid narrowing of the velocity distribution and density profile of trapped $^{87}\text{Rb}$ gas (and also sodium atoms) at low temperatures. This has been interpreted as evidence for a Bose condensate in the trap. It would be, however, difficult to make any quantitative statement, based on first principles [39, 43], about an actual realization of the Bose-Einstein conden-
sation in a trap. Before making such a dramatic claim, we must be sure that the data which appear to support the BEC in a trap are consistent with the quantum many-body theory, that is, the data on the collective excitations must confirm a phonon spectrum as in Henshaw and Wood’s experiment [6, 7], since the ground state wave function is given by the mean field defined in ODLRO. In a curious circumstance, this pertinent question [68, 69, 70] was never taken seriously in the physics community [62, 63, 64].

Thus the preliminary study, for its final quantitative description, must wait for the correct theoretical analysis of a dynamical proof of the BEC to confirm the realization of a BEC in a trap. Because the weak interaction between the atomic particles in the BEC can be described by the hard sphere approximation as in Bogoliubov’s theory of superfluidity, it is essential to demonstrate the Bogoliubov’s phonon spectrum $\epsilon = \hbar c k$ in the study of collective excitations in the BEC. It is also the purpose of this paper to discuss a variety of additional conceptual details of the broken symmetry [68, 69, 70] in an imperfect Bose gas which completes the theory of the Bose system.

It is extremely challenging to carry out an experimental study of (first) sound wave propagation in a trap to confirm the presence of BEC [71, 72], because 1) the speed of sound wave $c = [4\pi\rho(r)\hbar^2]^{1/2}/M$ depends on both the local density (and the wavelength of a traveling sound wave, which is $\lambda \sim c/\omega = 1/k$); 2) it is impossible to study the quantum effects of the BEC by the de Broglie wave $\lambda = 2\pi\hbar/p$ of a particle in the trap whose wavelength is equal in order of magnitude to the macroscopic size of the trap as $p \rightarrow 0$ in condensation. It should be emphasized, however, that, in liquid helium, the quantum effects become important when the de Broglie wavelength $\lambda = 2\pi\hbar/p$ corresponding to the thermal motion of atoms becomes comparable with the distances between atoms at about the $\lambda$ point [16].

The above discussion applies entirely to the atomic Bose-Einstein condensation in a trap and so remains valid in the liquid helium below the $\lambda$-point as well [33].

It will be shown, therefore, that an experimental confirmation of the Bogoliubov spectrum of a spherical sound wave [34] is both a necessary and a sufficient condition to observe the Bose condensation in a trap. Hence the study of collective excitations occupies a unique place in an interacting Bose system [68, 69, 70, 71, 72]. Just as the phonons that mediate strong attractive electron-pair interactions in a superconductor brought about Cooper pairs which obey Bose statistics and become a superfluid by condensation - the superconducting transition [43, 55, 59], a pair-interaction between Bose particles which brings about $\lambda$ tran-
sition in liquid helium also ensures that, in the absence of the long-range order interaction \[39\], the system possess certain certain collective excitations. More specifically, we study the collective excitations in a BEC to probe the dynamics of interacting Bose particles and thereby to provide a course of experiments to confirm the BEC in a trap \[14, 39, 43, 69, 70\]. Hence this discussion is not merely a question of the mathematical technique to obtain a correct dispersion for a sound wave leading to the Bogoliubov’s phonon spectrum \(\epsilon = \hbar ck\), but is the question of a scientific merit of the Bose-Einstein condensation in a trap \[34\].

The present research was undertaken in the hope that a comprehensive investigation of phase-coherent collective excitations in He II and the atomic BEC in a trap would bring a unified picture to a number of long standing problems in low temperature physics based on first principles. A new perturbation method similar to that of Feynman \[33\] in concept but mathematically more precise technique \[112\], is developed in section III to deal with the region of low energy excitation (phonon) in both He II and BEC in a trap where the conventional perturbation method of quantum field theory fails \[34\]. Along with the concept of ODLRO of Penrose and Onsager for both He II and BEC as a superfluid \[35, 38, 39\], a detailed mathematical analysis is carried out based on Bohm’s theory of quantum mechanics \[40, 108\] and the semi-classical perturbation method of the Lagrangian displacement vectors that overcome a major stumbling block in a finite space problem in the study of a quantum fluid \[112, 114\].

II. BRIEF OUTLINE OF THE RATIONAL

Since the mathematical development is fairly complex and involves the various types of lengthy algebras for each point in the development of the theory, we wish to outline the rational behind the theory, leaving the detailed algebras to later sections.

At low temperature \(T \ll T_\lambda\), Bogoliubov’s theory of superfluidity gives the dispersion relation for the collective excitations of quasi-particles in the form of \(\omega = ck\) in the long wavelength limit (a longitudinal sound wave), where \(c = [4\pi a \rho \hbar^2]^{1/2}/M\) and \(k\) is the wave number \[34\]. However, Bogoliubov assumed in his derivation of the dispersion relation for a phonon that the Bose quantum fluid is homogenous as well as isotropic, and of unlimited extent - of a Hilbert space.

In order to incorporate a well-known theorem (Goldstone) in quantum field theory into
our study of an imperfect Bose gas at low temperature, we review its implications in a many-body system of bosons. The theorem states: the spontaneously broken gauge symmetry in a nuclear many-body system always accompanies a longitudinal phonon, that is, a zero-mass, zero-spin Nambu-Goldstone boson (or the Goldstone mode of symmetry in quantum field theory) \[65, 78, 79, 80, 84, 85, 89, 90\]. The reason why the spontaneously broken symmetry may play a fundamental role is due in part to its practical usefulness as a basic mechanism by which we can explain the apparent breakdown of superfluidity at a nodal surface for a study of surface phenomena in both He II and BEC \[9, 11, 27\], for which a standard perturbation method fails due to the singularity in the mean field.

A proper treatment of the boundary conditions is especially important in a finite space problem; we have developed the method that replaces the physical surface by boundary conditions from which we can separate the surface layer from the bulk fluids by introducing the Lagrangian displacement vectors \[112, 113, 114\]. Moreover, we have established the similarities between the Meissner effect in a superconductor in an external field and the characteristics of the surface layer in He II under the gravitational field. In fact a dominant feature of the similarities is so striking that it is quite reasonable to inquire whether the Goldstone theorem can be equally applied to the imperfect Bose system.

It will be shown, with a specific example, that the broken symmetry is both a necessary and a sufficient condition for the conservation of energy in an isolated Bose system. Thus the spontaneously broken symmetry in a Bose quantum fluid is a fundamental phenomenon that may explain the peculiar normal fluid-like behavior of the surface layer of a rotating He II under the gravitational field; similarly it can also explain the breakdown of superfluidity at a vortex core. This peculiar behavior of a surface layer of a rotating He II has troubled the intuition of many experimental physicists since its first observation by Osborne in 1950 \[8, 9, 28\].

The new BEC in a trap is a small droplet with tens of microns in radius and has a low density, that is to say, the mean particle density \(\bar{\rho}\) satisfies \(\bar{\rho}|a|^3 \ll 1\), where \(a\) is an s wave scattering length. Thus we can adequately describe the pair-interaction by the hard sphere approximation. A simplest procedure we take is to assume the realization of BEC in a trap and inquire how we may prove it. Of course, one may try to confirm a phonon spectrum in experiments as shown in Bogoliubov’s theory of superfluid \[34\]. There are numerous papers on the theories of a Bose quantum fluid \[1, 13, 33, 34, 36\]; they are, however, developed
for a spatially uniform Bose system in a Hilbert space for mathematical convenience. But anyone who is familiar with modern quantum field theories [92, 93] should see at once that the conventional perturbation theories based on the field theories (\textit{i.e.}, the pair interactions through a quantized field) may fail entirely in our problem of atomic BEC in a trap.

In a finite, spatially, inhomogeneous many-body system, it was demonstrated [69] that the most serious difficulty of applying these theories is that the broken symmetry at the nodal surface yields a new set of dispersion relations for the collective excitations, so that the picture of the collective excitations in the previous theories [1, 33, 34] is significantly altered due to the presence of a nodal surface. Similarly, the boundary conditions play a peculiar but decisive role in a finite space problem. Both for this reason, and because the quantum field theoretic (microscopic) approach cannot be applied to a spatially inhomogeneous problem with a boundary [92], it is therefore essential to find a \textit{new mathematical method} by which we may implement the necessary boundary conditions that are consistent with the modified two-fluid model of Landau [1, 13]. This is most simply carried out by introducing the semi-classical perturbation method of Lagrangian displacement vectors [112, 114].

Our discussion of these problems is based on the following physical picture: an incompressible surface excitation involves a breakup of phase coherence over the surface of BEC droplet, whereas longitudinal excitations, however, such as those generated by compression, do not break up the phase coherence in a superfluid and yield the Bogoliubov spectrum for a phonon. Thus the longitudinal and surface excitations in turn identify the type of fluids that support these excitations. The surface waves are essentially transverse in nature. In addition to the usual surface waves, we will show that, in the surface layer, there is also a transverse wave that is driven by entropy change in the layer, which is different from that of temperature waves (second sound) [1, 4].

One of the essential new steps in our study of the broken symmetry is to adopt Mott’s suggestion [13] on the extension of the two-fluid model of Landau [1]. Based on London’s interpretation of the \( \lambda \)-transition [29] and the explanation of energy gap by Bijl, et al., [91], Mott suggested that, to describe the behavior of helium in Rollin films and to derive the critical velocity of a superfluid flow in a capillary, it is necessary to extend the two-fluid model by identifying the superfluid with the atoms in the ground state and the normal fluid with the atoms in the excited states with a finite energy gap (see Figs. 3, 5 and 6 of Ref. [13]). Particularly remarkable, however, is the anticipation of the Onsager-Feynman
quantization of superfluid velocity in a rotating He II as a mechanism to give rise the surface energy by velocity discontinuity [99, 100]. Thus we here define the normal fluid as composed of phonons (and rotons) and the excited atoms with finite energy gap from the ground state as an extended two-fluid model of Landau [11, 16].

It will be shown in this paper that the above characteristic difference between the two waves (the sound wave and the surface wave) provides the reason why the spontaneously broken symmetry must take place at the nodal surface - a breakdown of superfluidity. A proof of the broken symmetry lies at the core of many of unresolved problems in low temperature physics, especially since there is convincing evidence for the break-down of superfluidity at the free surface of rotating He II. Perhaps a more fundamental explanation for the broken symmetry in He II is that one cannot hold the law of conservation of energy in a finite, isolated system by dissipation process unless the symmetry is spontaneously broken at the nodal surface. This is because a phonon cannot interact with a superfluid component, but with the aid of Mott’s extension of the two-fluid model [13], it does, however, interact with a normal fluid, giving rise to a surface energy - a capillary wave by the surface tension.

The above discussion is by no means a complete answer to the surface layer problem. By the well-known arguments of spontaneously broken symmetries [80, 81, 84, 89, 90], this peculiarly universal behavior of a surface layer of a superfluid in a gravitational field cannot be explained by the Goldstone theorem alone, since the degree of broken symmetry depends on the curvature of the free surface and the energy of a superfluid in the system. For example, the breakdown of superfluidity at the vortex core accompanies a roton whose effective mass is $\mu_{ro} = 0.16m_{He}$ (a pseudo-Goldstone boson) [85, 87] but not a massless phonon (a Goldstone boson). Since the non-trivial irreducible linear representation of the group $U(1)$ is a real two-vector $\psi = \psi_1 + i\psi_2$, we write the ground state function in ODLRO as $\psi(r,t) = f(r,t)exp[iS(r,t)]$, where $S(r,t)$ is the action and is a solution of the quantum Hamilton-Jacobi equation [40]. Following Anderson [14] we also interpret the particle field operator $\psi(r,t)$ as our definition of a superfluid which has a macroscopic mean value $\langle \psi(r,t) \rangle = f(r,t)exp[i\phi(r,t)]$ with a definite phase $\phi(r,t)$ [108]. Thus the collective excitations (a gravity wave, a phonon, and a roton) and Bohm’s quantum theory [40] are essential for a complete understanding of properties of a superfluid in a gravitational field, particularly its interaction with the gravitational field and the spontaneously broken symmetry at the nodal surface.
We proceed further in two stages: we first derive collective excitations in a spherically symmetric BEC droplet to show the reason why the hidden symmetry is broken on the nodal surface of the BEC in a trap - a shell-like structure and then extend the calculations to He II. In the second stage of calculation, we discuss the mechanism by which the symmetry is broken by fluctuation-dissipation in terms of the action and the effective quantum mechanical potential [40, 108].

With the spontaneously broken symmetry, I will then discuss the three outstanding problems: a) a breakdown of superfluidity at the vortex core in He II for which there is still no proven mechanism that explains the breakdown in superfluidity at the core [126], and also resolves a long-standing controversy over the form of the Magnus force that has troubled many theoreticians [110] since the first experiment on a vortex quantization by Vinen in 1961 [10]; b) we discuss the Onsager-Feynman quantization of circulation in He II which is an extension of the Bohr-Sommerfeld condition in phase-space for which we show the identifiable direct cause for it to take place in He II with Planck’s quantum condition; c) a longstanding riddle on the curvature of a rotating He II under the gravitational field - the theoretical curvature based on Landau’s two-fluid model is incompatible with that of observed in the experiments [8, 9]. The phenomenon of broken symmetries confined, as it is in both cases, to the geometrical surface, might be explained by the fact that the particles in the surface layer (normal fluid) are free from from the interaction with the bulk superfluid, they can be accelerated only by external forces. With these qualitative remarks as an introduction, we proceed to the development of phenomenological theory of a quantum Bose liquid.

III. PHENOMENOLOGICAL THEORY OF A BOSE QUANTUM FLUID

The broken symmetries in a superconductor have been studied extensively since the remarkable discovery of Josephson junctions [73, 74, 75, 77]. And yet there is no systematic study of the broken symmetry in He II and an imperfect Bose gas in a trap (BEC) to date. Moreover, there is actually a fundamental difference between the ways in which the broken symmetry is realized in a superconductor and in a Bose quantum fluid (He II and BEC). The problem is then to define in a mathematically precise way the broken symmetries in an interacting Bose system in which both the law of conservation of energy and the number density must be preserved.
The basic idea of spontaneously broken symmetry in a superfluid can be explained by the Goldstone theorem alone [80, 86, 89]. The physical principles underlying mathematical analysis in the derivation of the Goldstone theorem are so clear that it is easy to see how the theorem can be applied to a number of problems in low temperature physics. However, we would first study the broken symmetry with the dynamical calculations based on the ground state wave function defined in ODLRO [39] and Bohm’s quantum theory [40]. The fist step is therefore to define the ODLRO and thereby the ground state wave function - mean field. For this purpose we follow Anderson’s analysis [14].

A. Basic Equations

The idea of off-diagonal long-range order (ODLRO) by Penrose and Onsager [39, 43, 45] may be stated precisely as the following: the superfluidity be described as a state in which the reduced density matrix of the condensed Bose system can be factorized,

$$\rho(r, r') = \psi^\dagger(r)\psi(r') + \gamma(|r - r'|).$$

(1)

where $\gamma \to 0$ as $|r - r'| \to \infty$. Here the single particle wave function $\psi(r)$ represents a Bose condensed ground state wave function in ODLRO and describes a superfluid in terms of the single particle wave function. We take $\psi(r)$ as the mean value of the quantum particle field or simply the mean field and consider it as the definition of a superfluid in this paper [38, 39, 45]. More important is the requirement that the wave function be single valued in this measurable part of the density matrix is what leads to the quantization of vorticity in He II.

With the the hard sphere approximation for the repulsive pair interactions for Bose particles [36, 60], one can show that the ground state wave function satisfies the nonlinear Schrödinger equation (Gross-Pitaevskii) in the self-consistent Hartree approximation [116],

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \psi + [V(x)_{\text{ext}} + g_1 |\psi|^2] \psi,$$

(2)

where $g_1 = 4\pi\hbar^2 a/M$ and $a$ is an s-wave scattering length [60, 116]. Eq. (2) is a self-consistent Hartree equation for the Bose condensed wave function. It should be emphasized that the nonlinear term $|\psi|^2$ is invariant under a $U(1)$ group transformation. The Lagrangian from which we can derive Eq. (2) is therefore invariant under the $U(1)$ group provided that
\( V(\vec{x}) \) is invariant, which we assume to be the case [See also Eq. (5a)]. It is also well worth to point out that we can still apply Bohm’s interpretation of quantum theory to Eq. (2) with the understanding that the self-interacting term be a part of the potential defined in his quantum theory \[35, 40\]. Furthermore, the nonlinear term in the nonlinear Schrödinger equation is responsible for an emission of a phonon in a Bose liquid just like the nonlinear Maxwell equations describe a photon emission \[40\].

Since the mathematical analysis of the quantum phenomena with the ground state bounded by a free surface presents much greater difficulties due to a inhomogeneous spatial density, the question naturally arises: what is the critical parameter that can assure us a semi-classical mathematical method as an adequate perturbation method within given limits of accuracy? The answer to this question is obviously Plank’s constant \( \hbar \), since quantum effects become important as the temperature of the system approaches to absolute zero and thus the pair-interaction potential is calculated by the hard-sphere approximation. In the classical limit \( \hbar \to 0 \), the action \( S(\vec{x}) \) is a solution of of the Hamilton-Jacobi equation as is shown below. This implies that if a solution is independent of \( \hbar \), it is then a classical part of the solution. Thus \( \hbar \) gives the fundamental difference between a normal fluid and a superfluid, since the pair-interaction, which is derived from a hard sphere approximation \[60\], brings about the \( \lambda \) transition in liquid helium as in a superconductor (i.e., Cooper pairs).

We also notice that the dispersion relation for the collective excitations in a superfluid is given by \( \omega = ck \), where \( c \) is speed of (first) sound \( c = [4\pi a\rho(r)\hbar^2]^{1/2}/M \) and \( k \) is the wave number in the phonon regime in He II \( (T \ll T_\lambda) \). In order to understand what is involved in such an analysis, we first consider the problem of collective excitations in an atomic BEC in a trap and derive the Bogoliubov spectrum to show that the perturbation method is indeed correct to the first-order.

Thus the whole problem of superfluid dynamics depends on the physical meaning that is ascribed to the particle field operator \( \psi \) in the equation Eq. \[1\] as the macroscopic mean value in a Bose system. Let us now take up the main problem of superfluid dynamics and ask the question of how to apply the mean field \( \psi \) to the solution of the broken symmetry in He II. It is important to keep in mind that \( \psi \) is also a complex order-disorder parameter in the sense of Landau \[1, 56\]. The first step is, therefore, to express the wave function in terms of an action and amplitude that yield the equations of motion which are consistent with the modified two-fluid model of Landau \[1, 13, 16\]. The action satisfies the quantum
Hamilton-Jacobi equation that gives an explanation for a spontaneously broken symmetry in the BEC in a trap. This is precisely the reason why we adopt Bohm’s interpretation of quantum mechanics [40] in our study.

The basic ideas we apply in our discussion is best understood in terms of a more familiar example. We thus begin our discussion by recapitulating the basic concepts of Bohm’s quantum theory. The Bohm’s interpretation of the quantum theory is based on the three special assumptions:

a) the single particle field $\psi$ satisfies Schrödinger’s equation;

b) if we write $\psi(r, t) = f(r, t)\exp[i\frac{\hbar}{\bar{\hbar}}S(r, t)]$, then the particle momentum is restricted to $p = \nabla S(x, t)$ in the classical limit ($\hbar \to 0$);

c) we have a statistical ensemble of particle positions, with the probability, $Pr = |\psi|^2 = f(r, t)^2$.

Plank’s constant $\hbar = 1.054 \times 10^{-27}$ erg sec here plays a critical role in our entire discussion on He II and the BEC; the transition from quantum mechanics to classical mechanics can be formally described as a passage to the limit $\hbar \to 0$ for which the effective quantum mechanical potential becomes $U_{\text{eqmp}} = 0$ (see below). As discussed above, we have modified the first assumption by including the nonlinear term for the pair-interaction of Bose particles, but it is still invariant under the $U(1)$ group transformation and becomes a part of the potential in the wave equation. Thus the wave field $\psi$ satisfies a Schrödinger equation Eq. (2) with the nonlinear interaction term (or simply the nonlinear Schrödinger equation). By deriving the well-known Bogolibov spectrum in quantum liquids [34] from the nonlinear Schrödinger equation, we demonstrate that our modification of the Bohm’s quantum theory does not affect his interpretation. The essential point is to write the wave function in the form $\psi(r, t) = f(r, t)\exp[i\frac{\hbar}{\bar{\hbar}}S(r, t)]$, where $S(r, t)$ is an action and $\rho = |f(r, t)|^2$ may be interpreted as a number density for a system of $N$ Bose particles [40]. In the quantum many-body theory [77], $\psi(r, t)$ has been interpreted as a complex order parameter; it has both an amplitude $f(r, t)$ and a phase $\phi = i\frac{\hbar}{\bar{\hbar}}S(r, t)$ which is coupled to external forces, whereas $f(r, t)$ is merely an internal order parameter in the sense of Landau’s order parameter of the systems [11, 56]. Thus Bohm’s quantum theory does not create any difficulty in the conventional many-body theory and can be viewed as an extension of it [77].

Furthermore, it should be emphasized that Bohm’s quantum theory is essentially equivalent to Feynman’s space-time approach to quantum mechanics [109]. Bohm’s approach
is, however, better suited especially to boundary value problems, because the action $S(r, t)$ satisfies the quantum Hamilton-Jacobi equation (QHJE) exhibiting the broken symmetry by the effective quantum mechanical potential at the nodal surface in the limit $\hbar \to 0$, whereas $\rho = |f(r, t)|^2$ satisfies the equation of continuity from which we can derive boundary conditions at the nodal surface [121]. Both $S(r, t)$ and $\rho = |f(r, t)|^2$ require the boundary conditions for their solutions and are not quite independent from each other [40, 108].

We now derive the following basic equations from the wave equation Eq. (2) with $\psi(r, t) = f(r, t) \exp \left( \frac{i}{\hbar} S(r, t) \right)$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \frac{\nabla S}{M}) = 0 \quad (3a)$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2M} + V(x) - \frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = 0, \quad (3b)$$

where $\rho = f(r, t)^2$, $V(x) \equiv V_{\text{ext}} + (4\pi \hbar^2 a/M) \rho$. Here $(4\pi \hbar^2 a/M) \rho$ is taken as an external potential [33, 35]. Eq. (3b) is the quantum Hamilton-Jacobi equation (QHJE) and reduces to the Hamilton-Jacobi equation in the limit of $\hbar \to 0$.

It may be worth to note the dependence of the potential energy on $(4\pi \hbar^2 a/M) \rho$ that arises from the pair-interaction of an imperfect Bose gas [60]. In the model we study in this paper, the dependence of $\hbar$ in the potential arises from a hard sphere approximation of two-particle scattering through a repulsive potential of the hard-core range $a$; therefore, the speed of the (first) sound $c = [4\pi a \rho \hbar^2]^{1/2}/M$ should depend on $\hbar$ in the analysis of collective excitations in the atomic BEC in a trap and He II.

By the Hamilton-Jacobi theory with Eq. (3b), it follows that the second assumption of Bohm’s quantum theory $p = \nabla S(x)$ is consistent with the usual interpretation of the quantum theory, in the sense that if it holds initially, it will hold for all time. From Eq. (3a), it follows that if $v = \nabla S/M$ and $\psi$ satisfies Schrödinger’s equation, then the third assumption of Bohm’s interpretation implies that the probability $P = |\psi|^2 = f(r, t)^2$ is conserved. Thus if one interprets $\rho = f(r, t)^2$ as a number density, it is also conserved by Eq. (3a). Moreover, $\rho = f(r, t)^2$ manifests itself as a useful function of hidden variables by which we describe the collective excitations in a Bose condensate. The last term in Eq. (3b) is the effective quantum mechanical potential (EQMP) defined by

$$U_{\text{eqmp}} = -\frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = -\frac{\hbar^2}{M} \frac{\nabla^2 f}{M}.$$
In the limit $\nabla \rho = 0$ for a homogeneous system, $U_{eqmp} = 0$ and thus $S(r, t)$ becomes a solution of the Hamilton-Jacobi equation which is consistent with the usual definition of a phase $\phi = S(r, t)/\hbar$.

In general, Eq. (3b) implies, however, that the particle moves under the action of the force that is not entirely derivable from the potential $V(x)$, but which also obtains a contribution from the EQMP. The important aspect of Eq. (4) is that it fluctuates near the nodal surface as $\nabla \rho \to \infty$ and drives the system to undergo the spontaneously broken symmetry at the nodal surface. The concept of EQMP is the main point of Bohm’s quantum theory that shows how Bohr’s correspondence principle breaks down at the nodal surface. Thus we see, in the language of quantum field theory [90], the spontaneously broken local gauge symmetry in He II and the atomic BEC in a trap [65, 78, 79, 80, 84, 85, 89]. As will be shown below, it should be emphasized that there is no unique way of making transition from classical to quantum mechanics other than taking the classical or correspondence-limit $\hbar \to 0$. However a simple quantum system in which the density is uniform $\nabla \rho = 0$ in space yields the correct classical limit.

Next we write the equations for the ensemble average energy in the usual quantum theory [40, 108]:

$$\mathcal{H} = \int \psi^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x)_{ext} + \frac{g}{2} |\psi|^2 \right) \psi d\mathbf{x}, \quad (5a)$$

$$\mathcal{E}_{ave} = \int \left( \frac{\hbar^2}{2M} \nabla |\psi|^2 + V(x)_{ext}|\psi|^2 + \frac{g}{2} |\psi|^4 \right) d\mathbf{x}, \quad (5b)$$

$$\int \psi^\dagger \psi d\mathbf{x} = N. \quad (5c)$$

At this point it is possible to show that the ground state density profile is given in terms of an external potential and the chemical potential upon minimizing the energy functional Eq. (5b) with Eq. (5c) and with the condition $p = 0$ (Penrose-Onsager criterion for BEC):

$$\rho(x) = |\psi(x)|^2 = \frac{M}{4\pi\hbar^2 a} [\mu - V(x)_{ext}], \quad (6)$$

where $\mu$ is a Lagrangian multiplier and is the chemical potential.

Eq. (6) is of the utmost importance in understanding superfluidity of BEC in a trap. It tells us two things: first, that $f(r)$ is a function of thermodynamic variable $\mu$ and the external potential $V(x)_{ext}$, and that the density is inhomogeneous in space and that the standard equation of motion of a superfluid, $d\mathbf{v}(x)/dt = d[\nabla S(x)]/dt/M = -\nabla \mu$ breaks down due
to fluctuations of $U_{eqmp}$ at the free surface, and thus defines the domain of ODLRO; second, that the broken symmetry takes place at the free (nodal) surface to maintain the steady state \[68\]. This is precisely the nature of superfluid that they cannot assume a stationary state under an external field or pressure gradient \[115\]. It may be worth while to point out that Landau \[1\] introduced the chemical potential to describe the a superfluid as a potential flow, i.e., $\omega = \nabla \times \mathbf{v}_s = 0$ and $\frac{d}{dt}\mathbf{v}_s + \nabla [(1/2)\mathbf{v}_s^2 + \mu] = 0$. And hence Landau emphasizes the superfluid velocity and its equation of motion in his two-fluid model \[1, 115\].

In order to obtain the explicit relation between the mean field in ODLRO and many-body ground state wave function, we have to invoke the phase coherence \[68, 69, 70\] which clearly shows the role of the mean field in many-body theory \[14, 39\]. Here the phase-coherence is defined as follows: if we write the ground state wave function in ODLRO as $\psi(r, t) = f(r, t)exp[\frac{i}{\hbar}S(r, t)]$ where $S(r, t)$ is the action (phase), then we may state the phase coherence as

$$\xi \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla_i S_{0, i}(x_0, i, t), \quad (7)$$

where $\xi$ are the Lagrangian displacement vectors \[112, 114\]. The summation of the phase change by an individual atom on the right-hand side is not measurable as it is hidden variable in Bohm’s interpretation, but the left-hand side in the mean-field can be measured in an experiment \[18\]. This is is an excellent example of a hidden variable of a single particle phase in Bohm’s quantum theory \[40\]. One very important point about the phase-coherence is that it is a necessary condition for the existence of superfluidity by ODLRO of Penrose-Onsager theory of Bose-Einstein condensation. The phase-coherence is a striking feature of superfluidity. Its role in a superfluid is remarkable in connection with the spontaneously broken symmetry as discussed below.

Because He II is a Bose system, the ground state wave function is symmetric under particle exchange and has no nodes in the case of a homogeneous system. Based on these properties, Feynman went on to argue that there can be no single particle excitation \[33\], which is essentially equivalent to the phase-coherence in our analysis. Hence the collective excitations we study in the Bose-Einstein condensation in a trap are a phase coherent, longitudinal sound wave (a phonon) \[79\], the dispersion relation which is given in the form of $\omega \simeq ck$ \[68, 70\]. In contrast to the two-fluid model of Feynman in his study of atomic theory of liquid helium \[33\], the ground state of BEC in a trap Eq. (6) is not uniform and has
a nodal surface on which the complex order parameter $\psi(r, t)$ undergoes the second-order phase transition and hence the symmetry is broken spontaneously as will be shown below.

B. Landau’s Two-fluid Model

When liquid He$^4$ is cooled below $T_{\lambda} = 2.19^\circ$K, it enters a new phase known as He II. This thermodynamic transition is marked by a peak in the specific heat, which behaves like $\ln|T - T_{\lambda}|$ on both sides of the transition which represents the onset of Bose condensation - the $\lambda$-transition.

The He II has remarkable hydrodynamic properties, many of which can be explained by Tisza’s phenomenological two-fluid model [4] and by Landau’s two-fluid model [1, 2]. Landau recognized that He II resembles a macroscopic mixture of two noninteracting components: a superfluid phase of zero viscosity and a normal fluid made of collective excitations of phonons (and rotons) at all finite temperature [1, 2, 13]. The flow of the superfluid component is irrotational $\nabla \times \mathbf{v} = 0$; the collective excitations are called the normal fluid with entropy which behaves like a classical fluid with viscosity. The density $\rho_n/\rho$ depends on temperature, where $\rho = \rho_s + \rho_n$. The two-fluid model [1] is essentially correct [33, 132], but a question for its completeness was raised by Osborne in his experiment of a rotating He II [8], because the two-fluid model does not allow a physical separation of the superfluid and normal parts of the fluid [1, 13]. Except for the surface-layer in which the two-fluid model breaks down because of the boundary conditions [121], the model is no more than a convenient description of the phenomena that occur in a fluid where quantum effects are important.

A single particle wave function $\psi(r)$ is introduced as the superfluid order parameter (a mean value) in the definition of an off-diagonal long range order (ODLRO) [39]. Due to a convincing argument of Penrose and Onsager [38, 39] for the theory of Bose-Einstein condensation and liquid helium as an extension of London’s theory of the $\lambda$-transition in a strongly interacting Bose gas [29], ODLRO was generally accepted as the definition of a superfluid [14]. And yet it still lacks an operational (or direct) link to the two-fluid model by ODLRO as the way in which the BCS microscopic theory is shown to be equivalent to the macroscopic Ginzburg-Landau theory by Gor’kov [58].

Although the two-fluid model was a conceptual breakthrough [1, 4] and was quite successful in explaining the dynamics of He II in a homogeneous flow with emphasis on the
superfluid velocity and its equation of motion \( \omega = \nabla \times v_s = 0 \) is known as an incomplete phenomenological theory.

The most important equation of motion that defines a superfluid in both the two-fluid model and the Bose-Einstein condensation follows from a ground state wave function defined in ODLRO \([35, 38, 39, 45, 65]\) and is given by Bohm’s second assumption in his quantum theory \([40]\), \( \psi(r, t) = f(r, t) \exp[\frac{i}{\hbar} S(r, t)] \) in which \( S(r, t) \) is the action, then

\[
\nabla \times v_s = \nabla \times \nabla S(x, t)/M = 0,
\]

(8)

where \( p = \nabla S \). Hence this identity shows its limitation as it is true only in the classical limit \( (\hbar \rightarrow 0) \). If we take the point of view that \( S \) must be a solution of the quantum Hamilton-Jacobi equation Eq. (3b), then \( S \) breaks the gauge symmetry \([14]\); it will be shown below how the fluctuation and dissipation driven by the effective quantum mechanical potential Eq. (4) brings about the spontaneously broken gauge symmetry at the nodal surface.

The most significant achievement of the two-fluid model \([1]\) has been in the analysis of the energy spectra of collective excitations in He II and the experimental data by Henshaw \([6]\) confirmed the prediction of the model in great detail. There are, however, many problems for which the spatial inhomogeneity of the fluid with a boundary poses a major stumbling block, in the study of the two-fluid model and still remains unsolved.

More specifically, Landau \([1, 2]\) studied the collective excitations to explain the macroscopic properties of He II with a particular form of the energy-momentum curve that rises linearly for small momentum \( p = \hbar k \); passes through a maximum, falls to a local minimum, and rises again. The excitations in the linear region are quantized sound waves (phonons); their energy, measured relative to the ground-state energy, is given

\[
\epsilon_k = \hbar ck,
\]

(9)

where at \( T = 1.12^0\text{K} \), data were obtained in the momentum range \( k = p/\hbar = 0.25 \sim 2.5\AA^{-1} \), \( c \approx 238 \text{ m/sec} \) is the speed of (first) sound \([6, 7]\). Near its local minimum of the energy-momentum curve, the energy spectrum can be approximated by a parabola

\[
\epsilon_k = \Delta + \frac{\hbar^2 (k - k_0)^2}{2\mu_r}.
\]

(10)

Landau \([1]\) further proposed that the excitations in this region represent rotons, the quantum analog of smoke rings. In Landau’s two-fluid model, the total free energy arises from
the thermally excited quasi-particles (phonons and rotons), which are treated as an ideal degenerate Bose gas. The theoretical curve that describe the above Eq. (9) and Eq. (10) has been confirmed in great detail by inelastic neutron scattering experiments including the roton parameters $\Delta/k_B = 8.6^\circ K$, $k_0 = 1.91\AA^{-1}$, and $\mu_r = 0.16m_{He}$ in Eq. (10) [6, 95].

However, for over a half century it has been known that the superfluid flow can exist only below Landau’s critical velocity $[\varepsilon/p]_{min}$, where $\varepsilon$ is the energy spectrum of a phonon and roton; the roton limited critical velocity $v_{cr} \approx 60m/sec$ has been observed [1, 97]. Although the Landau’s critical velocity is an important parameter in flow experiments, yet it remained a puzzle why they have observed the apparent breakdown of superfluidity at the core of a vortex line at a velocity well below the Landau’s critical velocity [97].

Another problem that requires a fresh examination is the experiments on a rotating He II by Osborne and Meservey [8, 9]. The measured surface curvature in steady rotation of He II (within a thin surface layer; average depth $5.0 \times 10^{-3} cm$ along the curvature) is very nearly given by $\gamma = \omega^2/g$ which is independent of temperature and is incompatible with the two-fluid model [1], where $\gamma$ is the maximum surface curvature [8, 9]. Both Onsager and Feynman independently have investigated the vorticity in a superfluid and reached the similar conclusion that the peculiarly normal fluid like behavior of the surface of a rotating He II can be explained by the vorticity [99, 100].

A number of theories [103] have been developed to explain the observed data [8, 9], notably Landau and Lifshitz’s vortex sheet model [98], and Hall and Vinen’s vortex line model [101] as suggested by Onsager [99] and Feynman [100]. After a careful observation on a rotating He II, Meservey has concluded none of these theories appeared to be completely adequate to explain his data; He II behaved like a normal fluid in a steady rotation. As emphasized by Mott [13], Meservey speculated that the surface energy might have played a role in a manner analogous to the Meissner effect in a superconductor in which the exclusion of the external magnetic field from a superconductor was attributed to the surface energy as shown in the Ginzburg-Landau phenomenological theory of superconductivity [56, 106].

IV. COLLECTIVE EXCITATIONS IN ATOMIC BEC IN A TRAP

First we review the Bose-Einstein condensation based on a degenerate Bose gas derived by Einstein [29] and then extends the theory to a weakly interacting (imperfect) Bose gas.
A. Condensation in Degenerate Bose gas

In this section, I should like first to review the Bose-Einstein condensation in a degenerate Bose gas as discovered by Einstein [29] to avoid the confusion from the weakly interacting atomic gas in a trap.

At low temperatures, a Bose gas at constant density obeys the following equation [16]:

$$\frac{N}{V} = \frac{g(mT)^{3/2}}{2^{1/2} \pi^{2} \hbar^{3}} \int_{0}^{\infty} \frac{\sqrt{z} dz}{e^{z-\mu/T} - 1},$$

(11)

where $g = 2s + 1$ with $s$ the spin of the Bose particle and $z = \epsilon/T$.

This equation implicitly determines the chemical potential of the gas as a function of its temperature and density ($N/V$). By setting $\mu = 0$ at a temperature determined by the following equation:

$$\frac{N}{V} = \frac{g(mT)^{3/2}}{2^{1/2} \pi^{2} \hbar^{3}} \int_{0}^{\infty} \frac{\sqrt{z} dz}{e^{z-\mu/T} - 1},$$

(12)

which can be expressed in terms of the Riemann zeta function. The critical temperature $T_0$ can be expressed then as

$$T_0 = \frac{3.31}{g^{2/3}} \frac{\hbar^2}{m} \left( \frac{N}{V} \right)^{2/3}. $$

(13)

The total number of particles with $\epsilon > 0$ with $\mu = 0$ is given by

$$N_{\epsilon>0} = \frac{gV(mT)^{3/2}}{2^{1/2} \pi^{2} \hbar^{3}} \int_{0}^{\infty} \frac{\sqrt{z} dz}{e^{z-1}} = N(T/T_c)^{3/2} $$

(14)

Thus the remaining particles with $\epsilon = 0$ is then

$$N_{\epsilon=0} = N[1 - (T/T_0^{3/2})]. $$

(15)

The steady increase of particles in the state with $\epsilon = 0$ is called Bose-Einstein condensation. This peculiar condensation is also known as the momentum-space condensation to emphasize that the cause of the condensation is solely due to the symmetry of wave function for a degenerate ideal Bose gas. This is precisely the reason why the mechanism appeared to have little physical significance, since all real gases are condensed at the critical temperature in the presence of molecular forces.

The energy of the degenerate gas for $T < T_0$ is given by

$$E = \frac{gV(mT)^{3/2}T}{2^{1/2} \pi^{2} \hbar^{3}} \int_{0}^{\infty} \frac{z^{3/2} dz}{e^{z} - 1}, $$

(16)
This integral can be tabulated by the Riemann zeta function $\zeta\left(\frac{5}{2}\right)$ [16], and is given by

$$E = 0.128g \left( m^{3/2}T^{5/2}/\hbar^3 \right) V. \quad (17)$$

The specific heat is then given by

$$C_v = 5E/2T. \quad (18)$$

Next to show the discontinuity of the first derivative of the specific heat $(\partial C_v/\partial T)$ at $T = T_0$, we calculate the energy of the degenerate Bose gas for a small $|T - T_0| \ll \varepsilon$ by expanding the following integrand in terms of small $\epsilon$ near $\mu = 0$:

$$N = N_0(T) + \frac{gV(\mu)^{3/2}}{2^{1/2}\pi^2h^3} \int_0^\infty \sqrt{\epsilon}d\epsilon \left[ \frac{1}{e^{(\epsilon-\mu)/T} - 1} - \frac{1}{e^{\varepsilon T} - 1} \right], \quad (19)$$

from which we obtain after some tedious algebra the chemical potential in terms of $N - N_0$.

$$\mu = -\frac{2\pi^2h^6}{g^2m^3} \left( \frac{N - N_0}{TV} \right)^2. \quad (20)$$

Thus we obtain the energy for $T > T_0$

$$E = E_0 + \frac{3}{2}N_0\mu = E_0 - \frac{3\pi^2h^6}{g^2m^3}N_0 \left( \frac{N - N_0}{TV} \right)^2, \quad (21)$$

where $E_0 = E_0(T)$ denotes the energy for $\mu = 0$.

Using Eq. (18) and Eq. (21), we calculate the difference the discontinuity of the derivative $(\partial C_v/\partial T)_V$ at $T = T_0$

$$\Delta \left( \frac{\partial C_v}{\partial T} \right)_V = -\frac{6\pi^2h^6}{g^2m^3V^2} \left[ N_0 \left( \frac{1}{T} \frac{\partial N_0}{\partial T} \right)^2 \right]_{T=T_0} = -3.66N/T_0, \quad (22)$$

where we have followed the approach of Landau and Lifshitz [16, 30, 116] to calculate the values of the first derivative $\partial C_v/\partial T$ at $T_0 \pm \varepsilon$.

We see therefore that the degenerate Bose gas undergoes the third order phase-transition as shown by London [30] and is quite different from the $\lambda$ transition in a liquid helium which is the second-order phase transition as in a superconductor.

**B. Condensation in Interacting Bose gas**

We now turn to a weakly interacting Bose gas in trap. Within the frame work of quantum field theory, Bogoliubov has developed the theory of superfluidity by quantizing the scalar
fields for Bose particles in a Hilbert space [34]. It is the only accepted approach for the derivation of the dispersion relation $\epsilon_k = \hbar ck$ (or $\omega = ck$) for the phonon in He II. In order to come the Bogoliubov theory of superfluidity [34] to grips with experimental observations of the BEC in a trap, we must derive a phonon dispersion relation for the collective excitations (i.e., a sound wave) in the BEC with proper geometrical corrections [6]. However, the dispersion relation must be consistent with the Bogoliubov’s results [34] or the Bruckner and Sawada’s [36]. In a finite space problem, we cannot quantize the fields [36, 92, 93] and thus we must find an alternative perturbation method which is applicable to a non-uniform system, and yet yields the same dispersion relation consistent with the geometry.

Our first task, then, is to find the condition for the second-order phase transition (a superfluid to a normal fluid) [56] in BEC in a trap. Here we carry out the first stage of the program described in the Introduction for the breakdown of superfluidity, a spontaneously broken symmetry in the BEC [68, 69, 70, 79]. The theory is subtle and complex. The broken symmetry is a quantum interference phenomenon taking place in both a Fermion system [80, 89, 90] and a Bose system [68, 69, 70, 77] at low temperature. Since the spontaneously broken gauge symmetry in a superconductor has been discussed in detail (i.e., the Meissner effect, the flux quantization, and the Josephson effect) in a number of papers employing the quantum field theory [78, 79, 84, 90], we will focus on spontaneously broken symmetries in a Bose system (the Bose-Einstein condensation and He II), which is new [68, 69, 70].

We first discuss the dynamical aspect of the broken symmetry since detailed dynamical studies are necessary to explain why the spontaneously broken symmetry should take place at the nodal surface. The basic algebra of a broken symmetry in a Bose system has been described, in essence, by the semi-classical perturbation of particle orbits given by $x_i = x_{0,i} + \xi(x_{0,i}, t)$, where $\xi$ is a Lagrangian displacement vector [112, 113], and is both tedious and complex [68, 69, 70]. The whole point of employing the Lagrangian displacement vectors in the perturbation analysis is to find an alternative method that permits one to impose appropriate boundary conditions in a finite system for which the conventional field theory approach entirely fails.

It should be emphasized, however, that from now on, we will discuss less of the quantum field theoretical background of the broken symmetry, than a number of its most important consequences for the BEC and He II, many of which can be understood without quantum field theories [34, 36] and are therefore of fundamental interest. Here we will describe the
essential algebras and refer the reader Refs. [68][69][70] for details.

Both He II and the BEC of an imperfect Bose gas are described by Eq. (1) [38][39] as a basic assumption in our study of phase-coherent collective excitations. We have come across the analogy between the \( \lambda \)-transition at \( T_\lambda = 2.176 \text{K} \) in liquid helium (the second-order phase transition) and the peculiar momentum condensation of the ideal Bose gas at \( T_0 = 3.140 \text{K} \) \( \text{i.e., a discontinuity of the derivative of the specific heat (phase transition in the third-order)} \) [29][30], which seems go much farther than the similar condensation, \( \text{i.e., the unique feature of Bose-Einstein condensation of charged bosons provides a clue that exhibits the phenomenon of superconductivity - the phase transition of the second-order at the critical temperature below which the Meissner effect occurs for sufficiently weak magnetic fields} \) [43].

Finally, it has been shown by Penrose and Onsager [38][39] that, in the absence of an external field, the BEC is always present in a spatially uniform system with a periodic boundary conditions whenever a finite fraction of particles have identical momenta [34],

\[
\frac{n_M}{N} = e^{O(1)} \Leftrightarrow \text{BEC},
\]  

(23)

where \( n_M \) is the largest eigenvalue of \( \sigma_1 = N \text{tr}_{2,3\ldots N} (\sigma) \). Here von Neumann’s statistical operator \( \sigma \) known as the density matrix is defined by \( < q'_1 \cdots q'_N | \sigma | q''_1 \cdots q''_N > \) [39][44][45]. In other words, our definition of ODLRO as defined in Eq. (1) is equivalent to Eq. (23), and thus one can see the presence of BEC in a uniform He II in a Hilbert space in which the Bogoliubov theory of superfluidity remains valid [34][35].

The question naturally arises what physical significance must be ascribed to this criteria for the presence of the BEC. The answer to the question is obvious: A single criterion for BEC in either a fluid or a degenerate Bose gas is also applicable to BEC in He II although it is an approximate caculation. It is clear therefore that the theoretical interpretation of superfluidity defined by ODLRO with Bohm’s quantum theory [40] is much more fundamental than that of mere definition of a superfluid by the superfluid velocity and its equation of motion [11]. It should be stressed that the particle field operator \( \langle \psi \rangle \) also has a macroscopic mean value in a superfluid system (the order-disorder parameter). Especially since Bogoliubov’s study of collective excitations in He II [34] establishes the superfluidity of liquid helium below the \( \lambda \) point [39], it is essential in our work to demonstrate that we obtain a correct phonon spectrum for collective excitations in the imperfect Bose gas consistent with
the Bogoliubov spectrum \[34\] and thereby to confirm the realization of the BEC in a trap \[66\].

As Feynman’s picture of a phonon \[33\] is essential in our study of phase-coherent collective excitations in a superfluid, it will be recapitulated here. In a series of papers \[33\], Feynman has laid out an elaborate, complex theory of collective excitations and an atomic theory of the two-fluid model based on the exact partition function with his space-time approach to quantum mechanics \[109\]. The main point of his theory is that the short range interaction of any pair of Bose particles which brings about the superfluidity of He II (the \(\lambda\)-transition) also ensures that the system possesses certain collective excitations (a phonon and a roton) that are the consequence of the off-diagonal long range order (\textit{i.e.}, the absence of a long-range order) \[39, 45\]. Moreover, Feynman has given an argument suggesting that, as emphasized by London \[29\], the \(\lambda\)-transition in liquid helium is the same process as the Bose-Einstein condensation of an ideal Bose gas \[33\]. The essential point of his argument is that the strong pair-interactions in a liquid-like quantum mechanical system do not prevent these particles from moving freely, because the system remains in a state of a superfluid. Also this is, in essence, Bogoliubov’s theory of superfluidity \[34\]. In addition, Feynman has pointed out that a density fluctuation satisfies the conservation of number density: when the density is increased by compression in one part of the system, it is decreased by rarefaction in the other part of the system. This property leads to Feynman’s concept of back-flow in his analysis of a sound wave in the liquid helium \[95\].

Finally, the ground state wave function is assumed to have a positive amplitude similar to that of zero-point fluctuations of the vacuum electromagnetic field in any configuration because the ground state has no \textit{nodes} in a uniform Bose system. With these properties Feynman \[33\] went on to argue that there can be no low-lying single particle excitations and that the only low-energy excitations are long-wavelength sound waves. In Bogoliubov’s theory of superfluidity, a phonon is a quantized sound wave (a Goldstone boson - zero spin, zero mass, scalar particle).

When, and only when an appropriate perturbation technique, which reproduces the Bogoliubov’s spectrum \[34\], is found, does the perturbation method take on a firm ground from which the qualitative description of collective excitations by Feynman \[33\] and the concept of ODLRO by Penrose and Onsager \[39, 45\] can be investigated quantitatively. We will incorporate Feynman’s picture of a phonon in our study of collective excitations by
introducing the Lagrangian displacement vector \[ 112, 114 \] in the particle orbits defined by Bohm’s quantum theory \[ 40 \] in ODLRO, which yields the Bogoliubov spectrum \[ 34 \]. It should be emphasized, however, that the semi-classical method by the Lagrangian displacement vectors is not applicable to a situation for excited states with energies above the energy gap in the roton spectrum, \textit{i.e.}, states in the roton region of excitation \[ 36 \].

As emphasized in the Introduction, the study of collective excitations plays a unique role in interacting many-body systems \[ 68, 69, 70 \]. \textit{In particular, we study the collective excitation in the Bose-Einstein condensation (BEC) to probe the dynamics of interacting Bose particles and thereby to provide a course of experiments to confirm the realization of the BEC in a trap \[ 39, 69, 70 \].} At low temperature \( (T \ll T_\lambda) \), Bogoliubov’s theory of superfluidity predicts the collective excitation energy in the form of \( \omega = ck \) in the long wavelength limit \[ 34 \], where \( c = [4\pi a \rho \hbar^2]^{1/2}/M \) and \( k \) is the wave number. Perhaps, one of the essential points in this paper that requires a study of collective excitations for its explanation is that of broken symmetry in the BEC in a trap - a shell-like structure, since the spontaneously broken gauge symmetry always accompanies a Nambu-Goldstone boson, a mass-less, spin-less particle (phonon) \[ 80 \].

\textbf{C. General Features of the Theory}

The use of the Hamilton-Jacobi equation in solving for the motion of a particle is only matter of convenience. Thus if we write \( v(x_0, t) = \nabla S(x_0, t)/M \) (\textit{i.e.}, a solution of the Hamilton-Jacobi equation), we obtain the following three dynamical equations from Eqs. \( 3 \) and \( 6 \): the equation of motion for a single particle,

\[
M \left( \frac{\partial}{\partial t} v + v \cdot \nabla v \right) = -\nabla \mu, \tag{24}
\]

the equation of continuity,

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0, \tag{25}
\]

where \( \rho \) is henceforth interpreted as the number density, and the equation of state,

\[
\mu(r, t) = \mu_{\text{loc}}[\rho(r, t)] + V_{\text{ext}}, \tag{26}
\]

where \( \mu_{\text{loc}}[\rho(r, t)] \) is the chemical potential in the local density approximation and is introduced to describe a superfluid droplet confined by the external potential with a free
surface. In order to expand the chemical potential in the lowest-order of \( a \), we assume the following conditions: \( aρ^{1/3} \ll 1 \), \( ka \ll 1 \), \( a/λ \ll 1 \), and \( aλ^2ρ \ll 1 \), where \( a \) is an s-wave scattering length in the pair-interaction potential, \( k \) is the wave number in the new BEC \[ \text{[14, 16, 117, 118, 119]} \].

Keeping in mind \( ψ(r,t) = f(r,t)\exp[i\bar{S}(r,t)] \), we expand \( S \), \( v \), and \( ρ \) to the first-order,

\[
\begin{align*}
S(r,t) &= S(r_0,t) + \xi \cdot \nabla_0 S(r_0,t) \quad \text{(27a)} \\
v(x,t) &= \frac{\partial}{\partial t} \xi \quad \text{(27b)} \\
ρ(x,t) &= ρ(x_0) - \nabla_0 \cdot [ρ(x_0)\xi] = ρ(x_0) + δρ, \quad \text{(27c)}
\end{align*}
\]

where \( p(x_0) = 0 \) in the BEC and \( \nabla_0 \) denotes the partial derivative with respect to \( x_0 \) with \( \nabla \rightarrow \nabla_0 - \nabla_0 ξ \cdot \nabla_0 \). Eq. (27c) was derived by substituting Eq. (27b) to Eq. (25) with \( \nabla S(x,t) = p(x,t) \) without taking into account of EQMP in Eq. (3b), then integrating over time.

By virtue of Eq. (27c), the chemical potential may be expanded as \[ \text{[14, 117, 118, 119]} \]

\[
μ(x,t) = μ_{loc}(x_0) + V_{\text{ext}} - \frac{\partial}{\partial ρ}(μ_{loc})[\nabla_0 \cdot (ρ_0\xi)].
\]

In BEC, \( p(x_0) = 0 \), and the equation of motion gives

\[
μ_{loc0}(ρ_0(r_0)) + V_{\text{ext}} = \text{constant} = μ_{loc0}[ρ_0(0)] = μ_0.
\]

If we now set \( V_{\text{ext}} = ω_0^2 r^2 / 2 \), the density profile describes a spherically symmetric, nonuniform Bose-Einstein condensation (SSNBEC). This is the problem from which our investigation started. As a special example of collective excitations in BEC in trap, we limit our analysis to the problem of SSNBEC.

We proceed with the understanding that the conventional microscopic perturbation theory based on the quantum field theory such as employed by Bogoliubov [34] is not applicable to a finite space problem with the boundary conditions [92, 93]. Eq. (6) exhibits peculiar features which show that the density of the ground state is not uniform, but it has a nodal surface. The surface layer of the SSNBEC is described by the distribution of a degenerate Bose gas with \( ε > 0 \) by the Bose statistics with the chemical potential \( μ = 0 \). It is the most convincing experimental proof of the validity of Eq. (6) and thus Eq. (29) - a second-order phase transition [16, 19, 20, 23, 24].
In order to describe the collective excitations in Eq. (6), we linearize Eq. (24) along with Eq. (27b), Eq. (27c), Eq. (28), and Eq. (29). It is straightforward algebra, though somewhat tedious, to arrive at the equation of motion in terms of $\xi$. This algebra is also carried out in Ref. [69]. The result is

$$\frac{\partial^2}{\partial t^2} \xi = \frac{\mu_0}{M} \nabla \sigma - \omega_0^2[(\xi \cdot \nabla)r + (r \cdot \nabla \xi)] - \omega_0^2[\sigma r + \frac{1}{2}r^2 \nabla \sigma].$$

(30)

Here we have taken $V_{ext}(r) = M\omega_0^2r^2/2$, $\partial \mu_{loc}/\partial \rho = 4\pi \hbar^2 a/M$, $\sigma = \nabla \cdot \xi$, and have also dropped the subscript in $r_0$. The perturbation analysis leading to Eq. (30) bears considerable resemblance to Feynman’s atomic theory of the $\lambda$ transition in helium [33], and Eq. (30) leads to the Bogoliubov dispersion relation for a longitudinal sound wave [34] in He II as will be shown below. However, it is not necessary to introduce his concept of a back-flow to derive the dispersion relation for a sound wave, because we insist on the phase-coherence in our analysis which is more narrowly defined concept and is also valid in quantum phenomena.

Although SSNBEC would not be expected to correspond to a realistic shape of condensation in experiments, one would hope that insight gained would be helpful in more realistic geometrical configurations in the prolate spheroidal coordinates. Even the simplest problems in the spherical coordinates, however, appear to have given rise confusion in recent literature [62, 63, 64]. Here we wish to suggest a straightforward approach that automatically includes all the modes of excitations in SSNBEC. The question of collective excitations is put in the form of an initial-boundary value problem in Eq. (30) with the boundary conditions which play a pivotal role in this analysis.

The solutions of Eq. (30) yield the dispersion relations for the collective excitations which fall into two groups: a phonon, a quantized longitudinal sound wave, the spectrum of which is $\omega \simeq ck$ for a typical sound wave; a surface wave under the external force. It should be noted that Eq. (30) is a typical second-order (inhomogeneous) partial differential equation from which we obtain two particular solutions, one corresponding to surface waves and the other to longitudinal sound waves. The general solution to Eq. (30), then, is a combination of the two particular solutions which can be obtained by a different set of boundary conditions.

We now make use of the unique feature of the Lagrangian displacement vectors with the appropriate boundary conditions: $\nabla \times \xi = 0$ which follows from $\nabla \times \nabla S = 0$ with $\nabla S(x_0, t) = M\mathbf{v}(x_0)$, the irrotational motion of a superfluid, and the other boundary condition on the free surface is due to the incompressibility of a fluid $\nabla \cdot \xi = 0$ which follows
from the equation of continuity Eq. (3a) [121].

In order to understand the physical meaning of the above boundary conditions, let us take a quick look at the similarity between the phenomenological theory of the Meissner effect [52] and the surface phenomena in He II, before the introduction of the specific features in Nambu’s gauge invariant calculation of Meissner effect. Just as in a superconductor, the surface layer of He II behaves like a normal fluid. This similarity between the surface phenomena of a superfluid in a gravitational field and the Meissner effect in a superconductor in a static magnetic field led us to study a possible broken symmetry in He II. It is this uniqueness of Eq. (30) that will play the essential role in the explanation of the broken symmetry in He II; it requires the two-sets of boundary conditions for its solutions in a way that is analogous to the London equations \[ \nabla \cdot J \equiv \nabla \cdot v = 0 \text{ and } \nabla \times v = e\mathbf{h}(x)/Mc \] with \( h(x) \) as a magnetic field] which was the first theoretical interpretation of the Meissner effect [52, 78, 79]. We shall bring this similarity further later in this paper along the line of Nambu’s analysis [1, 31, 78]. It should be emphasized that these boundary conditions are also consistent with the extended two-fluid model of Landau [13].

D. Surface Waves

We now solve Eq. (30) with the boundary conditions to show that the symmetry is broken at the free surface. We begin with the boundary conditions \( \nabla \cdot \xi = 0 \) and \( \nabla \times \xi = 0 \) on the free surface [121], which simplifies the algebra considerably. The boundary conditions suggest that we may solve Eq. (30) as a potential flow problem, since \( \nabla^2 \chi = 0 \) and \( \xi_s = -\nabla \chi \). Here the subscript \( s \) stands for the surface waves. There are a number of different surface waves which show different characteristics driven by a different force. We discuss only gravity and capillary waves in SSNBEH.

1. Gravity Waves

We discuss first the gravity wave driven the external trapping force only. The solution of Eq. (30) as a potential flow \( \chi \) is then given by

\[ \chi(r, t) = \sum_{\ell, m} [\chi_+^{\ell}(t)r^\ell + \chi_-^{\ell}(t)r^{-(\ell+1)}]Y_{\ell, m}(\theta, \phi), \] (31)
where we may set $\chi_\ell(t) = 0$ for a quantum liquid droplet. We now expand $\xi_s$ in terms of three orthogonal vector spherical harmonics \[68\],

$$
\xi_s = \sum_{\ell,m} \left[ \xi_{1s}^{\ell,m}(r,t)a_1 + \xi_{2s}^{\ell,m}(r,t)a_2 + \xi_{3s}^{\ell,m}(r,t)a_3 \right].
$$

(32)

Here we have defined the three vector spherical harmonics as

$$
a_1 = e_\ell Y_{\ell,m}(\theta,\phi), \quad a_2 = r \nabla Y_{\ell,m}(\theta,\phi), \quad a_3 = r \times \nabla Y_{\ell,m}(\theta,\phi).
$$

It is now only a matter of elementary algebra to obtain the dispersion relation by substituting Eq. (32) into Eq. (30); the result is

$$
\omega_{\text{surf}}^2 = \ell \omega_0^2.
$$

(33)

This dispersion relation gives the eigenfrequencies of gravity waves under the external trapping force. The frequency for $\ell = 0$ vanishes, since it corresponds a uniform radial oscillation which is not allowed for an incompressible fluid. The $\ell = 1$ mode corresponds to a translational motion of a fluid droplet without any deform of a shape.

Our entire argument critically depends on the interpretation of the dispersion relation Eq. (33). We see at once that the dispersion relation Eq. (33) is independent of the internal dynamics of the imperfect Bose gas described by the interaction term in the nonlinear Schrödinger equation Eq. (2) that yields the quantum ground state Eq. (6) which depends on $\hbar$.

And yet, Eq. (33) implies that the surface wave is a classical wave and thus a fluid that supports the surface wave (a gravity wave) is a normal fluid; these results are consistent with Mott’s analysis of the two-fluid model, and with Lamb and Nordsiek’s observation \[13, 24\]. More importantly, the dispersion relation Eq. (33) shows that the two-fluid model of Landau \[1\] breaks down in the surface layer, since it implies a separation of the superfluid and normal parts of a quantum fluid \[8, 9\].

Before we proceed further, we ask why do we have a classical form of dispersion relation in a quantum mechanical analysis? This question has been studied previously by Anderson, et al., \[115\]. They have shown that it is precisely the nature of a superfluid which cannot assume a stationary state, Eq. (6), under an external field or pressure gradient, but will have a time-dependent order parameter in a superfluid \(\text{i.e., a presence of dissipation}\) \[115\]. Hence the surface layer must be a classical fluid to maintain a stationary state under the
trapping potential and it shows a shell-like structure of BEC in a trap \cite{13, 24}. Similarly we obtain the dispersion relation for a surface wave in a cylindrically symmetric condensate as
\[ \omega_{\text{surf}}^2 = m\omega_0^2 \]
where \( m \) is a mode number \cite{70}.

2. Capillary Waves

So far we have paid no attention on the surface tension on the BEC droplet. In particular, in deriving Eq. (30) we have taken no account of processes of energy dissipation, which occur during the course of initiation and propagation of sound waves in BEC as a result of viscosity of surface layer and heat exchange between different parts of it. Through these processes, however, we are able to conserve the energy of an isolated system. Hence it is this unique characteristic of the surface layer that explains the essence of the broken symmetry - a breakdown of superfluidity.

In order to observe the surface free energy in an experiment, we must study the capillary waves that depend on the surface tension which is expected to be roughly in the order of \( \alpha \propto 0.35 \text{ erg/cm}^2 \) \cite{15, 19, 23} on the surface of BEC. The surface tension in He\( ^4 \) has been studied extensively since the initial analysis of a surface phenomenon by Mott \cite{13, 15}. It will be discussed fully later in the study of surface phenomena in He II.

For the derivation of dispersion relations for the capillary waves, we follow Landau and Lifshitz \cite{120} for the sake of self-contained presentation. However, we shall present it in the framework of our semi-classical approach based on the Lagrangian displacement vectors. And yet we must explain why the presence of the capillary wave is a natural consequence of the surface tension driven by the sound waves and is a necessary consequence of the conservation of energy in an isolated system. It is further suggested that the basis of our justification is provided by Kapitza’s conjecture that the capillary waves are driven by the instability of the fluid \cite{125}.

To describe capillary waves on a spherical droplet driven by the surface-tension, it is necessary to introduce the Laplace formula that gives the equilibrium condition for the surface layer under the action of the external pressure and the surface-tension \cite{120, 122}.

This formula for a spherical droplet is obtained from the thermodynamic equilibrium condition by

\[ \delta W = - \int (p - p_0)\delta \zeta df + \alpha \delta f, \]  

(34)
where $W$ is the work necessary to bring about the volume change, $\alpha$ is the surface-tension. For example, $\alpha$ between liquid helium and its vapor is very small $\alpha = 0.35$ erg/cm$^2$ at 0$^0$K; $p_0$ the constant external pressure to maintain the mechanical equilibrium, and $f$ is the area of a surface.

Next the change of the surface area of separation is given by [122],

$$
\delta f = \int \delta \zeta \left( \frac{1}{R_1} + \frac{1}{R_2} \right) df,
$$

(35)

where $R_1$ and $R_2$ are the principal radii of curvature at a given point of a surface.

Substituting this expression in Eq. (34), we obtain the Laplace formula,

$$
p_1 - p_0 = \alpha \left( \frac{1}{R_1} - \frac{1}{R_2} \right).
$$

(36)

Since the Laplace formula was derived from the Euler equation, it is convenient to apply the boundary conditions derived from the velocity in potential flow of an incompressible fluid $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$ [121], which are equivalent to our boundary conditions $\nabla \cdot \xi = 0$ and $\nabla \times \xi = 0$ provided that the wave amplitude is smaller than the wavelength $a \ll \lambda$, so that we can safely neglect the convective term $(\mathbf{v} \cdot \nabla)\mathbf{v} \ll \partial \mathbf{v}/\partial t$ in the Euler equation. The velocity in potential flow may be expressed as the gradient of scalar function, $\mathbf{v} = \nabla \phi$, the velocity potential. Thus the boundary conditions imply that $\phi$ is a solution of the Laplace equation $\nabla^2 \phi = 0$.

We shall next show how the surface area is calculated in differential geometry. Let us take up the cartesian coordinate problem first. Then a typical surface equation is given by $z = \zeta(x, y)$. And in differential geometry, $d\sigma = |\sec \gamma| dxdy$, with $\cos \gamma = \mathbf{n} \cdot \mathbf{k}$ where $\mathbf{n}$ is the normal vector to the surface, and $\mathbf{k}$ a unit vector in the z-direction, it is now a simple exercise to derive the area of a surface in cartesian coordinates [127] as

$$
f = \int \left[ 1 + \left( \frac{\partial \zeta}{\partial x} \right)^2 + \left( \frac{\partial \zeta}{\partial y} \right)^2 \right]^{1/2} dxdy,
$$

(37)

where $\zeta \ll 1$.

Having thus shown how the surface area in the cartesian coordinates is calculated in differential geometry, let us next calculate the surface area $\delta f$ for a spherical droplet in spherical coordinates with $r = h(\theta, \varphi)$

$$
f = \int_0^{2\pi} \int_0^\pi \left[ 1 + \frac{1}{r^2} \left( \frac{\partial h}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial h}{\partial \varphi} \right)^2 \right]^{1/2} r^2 \sin \theta d\theta d\varphi.
$$

(38)
We now expand the integrand of Eq. (38) in terms of \( \zeta \) which is defined by \( r = r_0 + \zeta \), where \( r_0 = b \).

\[
f \approx \int_0^{2\pi} \int_0^{\pi} \left\{ (r_0 + \zeta)^2 + \frac{1}{2} \left[ \frac{\partial^2 \zeta}{\partial \theta^2} \right] \right\} \sin \theta d\theta d\varphi. \tag{39}
\]

Here Eq. (39) is obtained by expanding the integrand of Eq. (38) with \( \zeta \ll 1 \).

It is a simple algebra to calculate the variation of \( f \) with respect to \( \zeta \),

\[
\delta f = \int_0^{2\pi} \int_0^{\pi} \left\{ 2(r_0 + \zeta) \delta \zeta + \frac{\partial \zeta}{\partial \theta} \frac{\partial \delta \zeta}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \varphi^2} \right\} \sin \theta d\theta d\varphi. \tag{40}
\]

Integrating the second term by parts with respect to \( \theta \), the third term with respect to \( \varphi \), and using Eq. (35), we find

\[
\delta f = \int_0^{2\pi} \int_0^{\pi} \left\{ 2(r_0 + \zeta) - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \varphi^2} \right] \right\} \delta \zeta \sin \theta d\theta d\varphi. \tag{41}
\]

If we divide the integrand by \( (r_0 + \zeta)^2 \approx r_0 (r_0 + 2\zeta) \) and compare the result with Eq. (35), we find the following formula correct to the first order in \( \zeta \)

\[
\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{r_0} - \frac{2\zeta}{r_0^2} - \frac{1}{r_0^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \varphi^2} \right] + p_0 = 0, \tag{42}
\]

Since the condition \((v \cdot \nabla) v \ll \partial v / \partial t\) is assumed in the Euler equation (i.e., for a wave whose amplitude is much smaller than the wavelength), we may then write the equilibrium condition for the surface layer,

\[
\rho_0 \frac{\partial \phi}{\partial t} + \alpha \left\{ \frac{2}{r_0} - \frac{2\zeta}{r_0^2} - \frac{1}{r_0^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \varphi^2} \right] \right\} + p_0 = 0, \tag{43}
\]

where \( \rho_0 \) is the surface density \((gr/cm^2)\) which is finite in the presence of sound waves.

Since \( v_r = \partial \zeta / \partial t = \partial \phi / \partial r \), we obtain the equilibrium condition on \( \phi \) by differentiating with respect to time and omitting the constant terms

\[
\rho_0 \frac{\partial^2 \phi}{\partial t^2} - \alpha \left\{ \frac{2}{r_0} \frac{\partial \phi}{\partial r} + \frac{\partial}{\partial r} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} \right] \right\} = 0, \tag{44}
\]

as an equilibrium condition for the surface layer at \( r = r_0 \).

Next we shall seek a solution to Eq. (44) in the form of a stationary wave: \( \phi = e^{-i\omega t} \eta(r, \theta, \varphi) \), where \( \eta \) satisfies the Laplace equation \( \nabla^2 \eta = 0 \).

Since

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (r) - \frac{L^2}{r^2}, \tag{45a}
\]

\[
L^2 = -\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} \right], \tag{45b}
\]

\[
L^2 Y_{lm} = l(l+1)Y_{lm}, \tag{45c}
\]

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and further assume the spatial variation in the form \( \phi = A e^{i \omega t} r^l Y_{lm}(\theta, \vartheta) \), and thus with Eq. (45), we readily obtain the dispersion relation from Eq. (44) for the capillary waves

\[
\rho_0 \omega^2 + \alpha l [2 - l(l + 1)]/r_0^3 = 0, \quad (46a)
\]

or

\[
\omega^2 = \alpha l (l - 1)(l + 2)/\left(\rho_0 r_0^3\right). \quad (46b)
\]

As in the gravity wave problem Eq. (33), we see from Eq. (46) that \( l = 0 \) is not allowed, since the mode is a uniform radial oscillation, which is impossible for an incompressible fluid; the \( l = 1 \) mode is a translational motion of a spherical drop (i.e., a displacement of the center). Since \( \omega^2 \approx \alpha k_\theta^3/\rho_0 \), where \( k_\theta \approx l/r_0 \), we may neglect the capillary waves in the long wavelength limit \( k_{\text{theta}} \ll a_1 \), where \( a_1 = \left[2 \alpha/(g \rho_0)\right]^{1/2} \) is a capillary constant \[120\]. On the other hand, in the short wavelength limit, the gravity wave can be neglected.

A question then arises naturally: how do we observe the capillary wave alone in BEC in a trap, especially since the gravity waves and the capillary waves can be observed simultaneously by the combined action of capillarity and the external trapping force? This is precisely the reason why it is difficult to observe the capillary waves in a fluid with small viscosity \[26, 125\]. In the course of our work, we became aware of the difficulty of studying sound wave propagation in BEC in a trap, which is essential to the realization of BEC in an atomic trap experiment. Our original motivation for studying the capillary wave was to provide an indirect means of testing the extended two-fluid model, since it may be considered as evidence for the presence of the (first) sound wave dissipation process at the surface layer. We shall next concentrate on (first) sound waves in BEC in a trap.

**E. Compressional Waves - Sound Waves**

We now solve the equation Eq. (30) with the boundary conditions inside the nodal surface \( \nabla \cdot \xi \neq 0 \) but \( \nabla \times \xi = 0 \), i.e., the fluid is now compressible, but it remains irrotational, \( \omega = \nabla \times v_s = \nabla \times \nabla S/M = 0 \) by the second assumption of Bohm’s quantum theory \[40\], so that the superfluid supports the phase coherent sound waves in BEC. One point we would like to emphasize in this paper is that the boundary conditions in terms of the displacement
vector $\xi$ are unique in that they can be used to define the surface layer (normal fluid) and the bulk fluid (superfluid) simultaneously in the discussion of collective excitations for a system of Bose-particles described by the single Schrödinger equation (Gross-Pitaevskii) Eq. (2).

It is also noteworthy that the above boundary conditions are very similar to those of London Equations (i.e., $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{p} = 0$) for the Meissner effect \[52\], which explains why we shall see the similarity between the Meissner effect and the peculiar phenomenon of a superfluid in the gravitational field later in our discussion of He II leading to the theory of spontaneously broken gauge symmetry \[78, 80, 84\].

As the mathematical detail may obscure the essentially simple steps involved, it will be helpful to give a brief outline of the mathematical analysis. First we define the compressibility of the fluid as $\sigma = \nabla \cdot \xi$ from Eq. (27c), and then its spectrum can be obtained from Eq. (30) by taking the divergence on both sides. After straightforward algebra with the help of the vector identity $\nabla \cdot [(\mathbf{r} \cdot \nabla) \xi] = \mathbf{r} \cdot \nabla \sigma + \sigma$ and the condition of superfluidity, $\nabla \times \xi = 0$, we obtain \[69\]

$$\frac{\partial^2}{\partial t^2} \sigma(r, t) = \frac{1}{2} \omega_0^2 (\alpha^2 - r^2) \nabla^2 \sigma - 3 \omega_0^2 r \frac{\partial}{\partial r} \sigma - 5 \omega_0^2 \sigma,$$

where $\alpha^2 = 8\pi \hbar^2 a \rho_0(0)/(M \omega_0)^2$.

The solution of the equation is not entirely trivial. Writing $\sigma(r, t) = S(t)W_n(r)Y_{\ell m}(\theta, \phi)$, we obtain the variable separated equations:

$$\frac{d^2}{dt^2} S(t) + \lambda_n S(t) = 0 \tag{48}$$

and

$$\left(\alpha^2 - r^2\right) \left[ \frac{1}{r} \frac{d}{dr} \left( r W_n(r) \right) - \frac{\ell(\ell + 1)}{r^2} W_n(r) \right] - 6r \frac{d}{dr} W_n(r) + (2\lambda_n/\omega_0^2 - 10) W_n(r) = 0, \tag{49}$$

where $\lambda_n$ is a constant of separation. The eigenvalues are determined by Eq. (49), and by the boundary conditions on $\sigma(r, t)$, i.e., $\sigma(r, t) = 0$ at the free surface and the origin. Here we tacitly assume a small point source at the origin to drive an outgoing spherical sound wave \[120\]. The speed of first sound $c(r) = [4\pi a \rho(r) \hbar^2]^{1/2}/M$ is the maximum at the origin and approaches zero at the free surface. What’s more, we recall that the fluid of the surface
layer, where the two-fluid model of Landau [1, 4, 13] breaks down, is no longer a superfluid but is a normal fluid as shown above. Hence the sound wave propagates to the surface layer and interacts with the normal fluid there giving rise to a surface energy by dissipation - a manifestation of the surface tension.

This is precisely the reason why we investigate the sound wave propagation for the dynamical study of an imperfect Bose gas in the trap. We see a self-consistent picture emerge from the study of a sound wave propagation which conserves the energy of an isolated system. It is indeed remarkable to see that Nature provides such an elegant self-consistent picture through Eq. (30).

The solution of Eq. (48), then, yields the dispersion relation for the collective excitations with the eigenvalues that are a function of the s-wave scattering length, the radial trap frequency, the peak density at the center of the trap, i.e., the speed of first sound \( c_{\text{ctr}} = \sqrt{\frac{4\pi a \rho(0)}{M\omega_0}} \), the trapping frequency \( \omega_0 \), and the wave number with \( k_\theta \approx \ell/r \). To show this explicitly, we transform Eq. (49) to the Sturm-Liouville problem and obtain the eigenvalues in terms of a complete set of functions with the orthogonal properties:

\[
\lambda_n = \int_0^b r^2 (\alpha^2 - r^2)^3 \left( \frac{d}{dr} W_n(r) \right)^2 + \frac{\ell(\ell + 1)}{r^2} W_n^2(r) dr
\]

(50)

and with

\[
\int_0^b r^2 (\alpha^2 - r^2)^2 W_m(r)W_n(r) dr = \delta_{m,n}.
\]

(51)

Here \( \lambda_n = (2\lambda_n/\omega_0^2 - 10) \) and \( b = \alpha = \sqrt{\frac{8\pi a h^2 \rho(0)}{M\omega_0}} = \sqrt{\frac{2}{\omega_0}}c(0)/\omega_0 \) is the radius of the condensate. It is also well worth of pointing out that the upper-limit of the integral in Eq. (50) is a function of speed of first sound \( c_{\text{ctr}} = \sqrt{\frac{4\pi a \rho(0)}{M\omega_0}} \) at the center.

It should be noted that the integral equation Eq. (50) for \( \sigma(r, t) \) remains valid only for the collective excitation energy in the phonon regime. Higher excitation energy spectrum such as a roton excitation cannot studied by the perturbation method described in this paper; there is an energy gap in the roton excitation spectrum, separating the ground state from the excited state for which the Green’s function approach based on the quantum field theory in a spatially homogeneous system is better suited [35, 96]. However, we will discuss later a mechanism by which the rotons are created in the fluctuation-dissipation process that leads to a broken symmetry.

Let us now analyze this integral in some detail since it shows the nature of collective excitations. Equations (50) - (51) exhibit the essence of our results: that the eigenvalues
are, indeed, a function of the speed of (first) sound, \( b = \sqrt{2/\omega_0} c_{ctr} \), where the sound speed \( c_{ctr} = c(0) = [4\pi a\rho(0)\hbar^2/\omega_0]^1/2/M \), and the trapping frequency \( \omega_0 \), for all values of the angular momentum \( \ell \). The functional relation of the speed of (first) sound and the wave number in the excitation spectrum is also clear in this integral representation. Perhaps more important is that the mean field \( \psi(r,t) \) in ODLRO can be perturbed in the particle orbits by the semi-classical method [112, 113, 114] which yields a quantum mechanical result by means of Bohm’s quantum theory [40]. It is also worth while pointing out that the above semi-classical method is actually a well-defined quantum mechanical perturbation method by the phase-coherence since it involves atomic displacements as in Feynman’s qualitative picture of a phonon in his analysis on two-fluid model [33].

Another important point is that the eigenvalue is a function of the speed of first sound not at the free surface but at the center of SSNBEC. This has a simple physical interpretation: an outgoing spherical sound wave can travel with little energy loss toward the free surface where the sound wave (a phonon) interacts with the normal fluid of the surface layer and completely dissipates at the surface, giving rise to a surface energy. It is precisely the nature of the extended two-fluid model of Landau [1] that a phonon cannot interact with a superfluid component as emphasized by Mott [13], but it will, however, interact with a normal fluid. There has been little investigation as yet of sound wave propagation from the deep inside of liquid helium to a surface area under the gravitational field although we can infer the surface wave study by Vinen’s group [26], which would provide a simple demonstration of the spontaneously broken symmetry in He II. The effect due to the interaction between a phonon and the surface layer would be very small since the phonons have a small amplitude; nonetheless we are only interested in the question of basic principle.

The analytical expression Eq. (50) gives in principle a complete solution to the eigenvalue problem. Unfortunately, the eigenvalue equation Eq. (50) for the phonon is not analytically tractable and one must resort to numerical integration. Here we take a different but equivalent approach to the eigenvalue problem; that is, we shall solve Eq. (49) as an eigenvalue equation in a differential equation. This simple but powerful mathematical technique will permit us a detail study of the eigenvalue problem. With the substitution \( W_n(r) = r^{(\ell+1/2)-1/2} Z_n(r) \) together with \( x = r^2/\alpha^2 \), we obtain

\[
x(1-x) \frac{d^2}{dx^2} Z_n + [c - (a + b + 1)x] \frac{d}{dx} Z_n - abZ_n = 0
\] (52)
This is just the Gauss differential equation \[133\] with \(c = \pm(\ell+1/2)+1, a+b = \pm(\ell+1/2)+3,\) and \(ab = (1/4)\lambda_n - 6[\pm(\ell + 1/2) - 1/2].\)

To obtain explicit solutions of Eq. (52), one has to resort to a numerical method. Fortunately a number of simplifications can be made by studying analytic structure of the equation. Eq. (52) has regular singular points at \(x = 0, x = 1\) and \(x = \infty.\) Its solution is the hypergeometric function, which is analytic in the complex plane with a cut from 1 to \(\infty\) along the real axis [133]. A simple, but accurate, numerical method [134] has been employed to evaluate the eigenvalues in the domain \([0, 1]\) in which the solutions are analytic. Since we are interested in the low-lying excited states (a phonon), it is only necessary to find the smallest eigenvalues in the differential equation. This is consistent with Feynman’s picture of a phonon - a sound (longitudinal) wave with a small amplitude [33].

Now returning to Eq. (48) and taking \(S(t) = e^{i\omega t},\) we obtain the dispersion relation as

\[\omega_{ph} = \pm[\lambda_s/2]^{1/2}\omega_0,\]  \hspace{1cm} (53)

where \(\lambda_s\) is the smallest eigenvalues from Eq. (52).

The principal result of this section is Eq. (53); it is given in Fig.1, where the ratio \(\omega_{ph}/\omega_0\) is plotted against the angular momentum \(\ell.\) The pertinent question, ”How do we interpret the Fig. 1” ?, remains. Eq. (53) is a statement of a dispersion law, an energy spectrum \((\varepsilon = \hbar\omega)\) relating to the wave number \(k_\theta\) of a longitudinal spherical wave with the speed of the (first) sound \(c_{ctr} = [4\pi a\rho(0)\hbar^2]^{1/2}/M,\) in the BEC in a trap. A close examination of the dispersion curve shows that in the phonon regime the energy spectrum of a longitudinal spherical wave is nearly linear with respect to \(\ell\) (i.e., the wave number \(k_\theta \approx \ell/r).\) "The existence of the phonon spectrum is a consequence in part of the finite energy gap for particle excitation and hence intimately associated with the statistics of the particles, and that there is no single particle excitation” [33, 34, 36]. A more detailed study of the excitation spectrum Eq. (53) is given later by deriving the Bogoliubov spectrum in the phonon regime \(\omega = ck\) [34] in a homogeneous He II.

Collective excitations in SSNBEC may be defined as solutions for which the dispersion curve Fig. 1 is a part of the complete solution to Eq. (30). In an inhomogeneous medium, the Bogoliubov dispersion relation \(\omega_{ph} = ck\) with \(c(r) = [4\pi a\rho(r)\hbar^2]^{1/2}/M\) must be studied approximately with \(k_\theta \approx \ell/r\) at a given radius in SSNBEC. In particular \(\omega_{ph}/\omega_0 = 1.5972\) for \(\ell = 0\) is a unique value in a finite space problem, which corresponds to a uniform radial
perturbation. Finally, since the eigenvalues are positive for all values of $\ell$, the Landau critical velocity $v_c = \varepsilon_{ph}/p$ is also finite. Hence the imperfect Bose gas in the trap is a superfluid, whereas a degenerate Bose for which $c$ is independent of the scattering length is not, because $c$ and $v_c$ vanish identically, which is consistent with Singh’s analysis [19].

Here the behavior of the dispersion curve as a function of $\ell$ is of particular interest, since the azimuthal wave number $k_\theta$ is given by $k_\theta \approx \ell/r$ in the spherical geometry and the slope of the curve can represent the average speed of a spherical longitudinal sound wave travelling toward the free surface. Moreover, the dispersion relation, Fig.1, also shows how the presence of the repulsive pair-interaction determines the excitation spectrum of a phonon, which is nearly linear with respect to $k_\theta$ as expected from the Bogoliubov spectrum $\omega = ck$, where $c = [4\pi a\rho(r)h^2]^{1/2}/M$, since the slope is the average speed of sound wave [34][116]. For a cylindrically symmetric long condensate [70], we obtain a similar dispersion curve in terms of mode number $m$ with the wave number $k_\theta = m/r$.

To implement the Feynman picture of a phonon [33] in our microscopic study of collective excitations in BEC in a trap, we have made some basic assumptions. One of which, as discussed above, is the semi-classical perturbation theory by the Lagrangian displacement vector $\xi$ in a particle orbit in Bohm’s quantum theory [40] along with ODLRO as an alternative perturbation method, but not a standard quantum field theoretical approach with an interaction Hamiltonian in terms of creation and annihilation operators [34]. It is indeed remarkable to see that the dispersion curve is the anticipated form from the Bogoliubov dispersion relation $\omega = ck$ for a phonon [34].

Next we have defined the compressibility of a fluid as $\sigma = \nabla \cdot \xi$ from Eq. (27c), which was derived from the equation of continuity. Hence there was no need to introduce the concept of the back-flow for a propagation of the sound wave, since the conservation of the number density and the phase-coherence automatically enforced.

It should be stressed, however, that the above results contradict the recent theoretical and experimental work in BEC in a trap [62, 63, 64], but agree well with the Bogoliubov’s result with a proper geometrical correction [33, 34]. The errors made in the previous work [62, 63, 64, 65] are obvious, because the dispersion relation must be a function of the speed of (first) sound $c = [4\pi a\rho(r)h^2]^{1/2}/M$ in the phonon regime at low temperature $T \ll T_\lambda$. Hence much of the previous work on the BEC in an atomic trap, i.e., either experimental or theoretical work, [62, 63, 64] is, to put it mildly, questionable [65]. Moreover, a finite space problem

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in quantum fluids require a new mathematical approach as emphasized in the Introduction [92, 93]. It has been shown that a direct application of the perturbation method to the phase-coherent collective excitations in He II and BEC through the Lagrangian displacement vectors in particle orbits is straightforward and free of ambiguity.

F. Spontaneously Broken Symmetries in Atomic BEC in a Trap

An alternative perturbation method is developed to deal with the regime of low energy excitation in the BEC where the usual quantum field theoretical perturbation theory [34]. The effectiveness of the above elaborate algebra with the Lagrangian displacement vectors can be appreciated by the simplicity of the dispersion relations Eq. (53) and Eq. (33), the first of which predicts a phonon spectrum and the second shows the presence of surface waves. The existence of the surface layer is a consequence in part of the finite energy gap for particle excitation predicted by Mott [13, 16]. The ground state wave function, $\psi(r,t)$, which is equivalent to the order parameter $\eta$ of Ginzburg and Landau [56], undergoes the second-order phase transition, i.e., from a superfluid to a normal fluid at the free surface. A model that can capture the essence of the broken symmetry is a broken $U(1)$ global symmetry that accompanies a phonon as a mass-less and spin-less boson, similar to that of lattice gauge due to a dislocation in the translational symmetry in a lattice. It is therefore natural to identify the underlying basic mechanism for the symmetry breaking as a spontaneously broken symmetry at the free surface which accompanies a phonon as Nambu-Goldstone boson [65, 78, 80, 89, 90]. The difference between the macroscopic quantum phenomena (a quantum fluid) and those of the classical phenomena (a classical fluid) is so clear (i.e., $\hbar$ difference) that it is easy to see how the surface phenomena are intimately associated with the statistics of particles [16].

Finally if we introduce a small disturbance adiabatically at the center of SSNBE to initiate an outgoing longitudinal spherical sound wave, it will travel toward its surface and will be dissipated at the (nodal) surface by the interaction with the normal fluid there, giving rise a surface energy. Thus the conservation of energy of an isolated Bose system is maintained. It is therefore clear that the spontaneously broken symmetry is both a necessary and a sufficient condition to hold the law of conservation of energy. The broken symmetry also explains why it has been exceedingly difficult to confirm the realization of the BEC in
a trap in an experiment, and it also shows a uniqueness of shell-like structure of the BEC in equilibrium in a trap.

The sole purpose of the above discussion of the collective excitations is an attempt to investigate the superfluidity of Bose particles in a trap and the spontaneously broken symmetry from a microscopic point of view. By virtue of superfluidity [34, 39], one expects the dispersion relation Eq. (53) should be satisfied in the trapped gas as evidence for the Bose condensation. On the other hand, the case of broken symmetry can be shown by Eq. (33) which is independent of internal dynamics of the BEC. It should be emphasized that, in a finite, inhomogeneous system, it is the only way one can show explicitly the second-order phase transition, i.e., from a superfluid to a normal fluid - a spontaneously broken symmetry. The discussion here is intentionally somewhat sketchy, but it is sufficient to explain the main points. We also note that, with a few minor changes in the formalism, the method also yields a theory of collective excitations in a long cylindrical condensation [68, 69, 70].

In passing we remark that a similar analysis can be performed for a charge neutral Fermi gas in a trap to study the phase coherent collective excitations [135] with strong repulsive interactions [136, 137]. There is, however, a fundamental difference between a Bose gas and a neutral Fermi gas because of Pauli’s exclusion principle which limits the possible inter-particle interactions in a neutral Fermi system at low temperature. In 1957 Landau [135] predicted that at sufficiently low temperature a new type of sound, which he calls zero sound, propagates freely in He3. Based on Landau’s idea, a more detailed theory of sound propagation has been worked out by Abrikosov and Khalatinikov [137]. At sufficiently low temperature, it is predicted that the attenuation of zero sound is proportional to \( T^2 \) and independent of frequency. But in the phonon regime, the first sound attenuates as \( \omega^2/T^2 \) corresponding to classical viscous attenuation. Both of these temperature and frequency dependence are observed in experiments with He3. The data indeed show that \( (c_0 - c_1)/c_1 \approx 0.040 \) [138, 139].

A detailed study of phase coherent collective excitations and attenuation of sounds would provide the direct experimental confirmation of Landau’s theory for an imperfect Fermi gas in a trap. Theory, in the absence of direct measurements of the sound (first and zero) propagations [135, 136, 137], often could go off down blind alley and it is not meaningful to speak of a Fermi liquid with interactions in a trap as reported in a recent paper [140], since the observation on the sound propagations is the essential requirement for the dynamical
study of the interacting many-body system. Only upon making careful observations of (first
and zero) sound propagations for the collective modes do we understand the dynamics of
the many-body system [138, 139], and perhaps, even provide the qualitative understanding
of an anisotropic BCS-type superfluid state with triplet pairing. The experiments [141, 142,
143, 144, 145] demonstrated that the liquid He³ which undergoes the second-order phase
transition below 2.7 mK and approximately correct specific heat jump suggested by BCS
exhibits the two phases: A-phase as an \( l = 1 \) Anderson-Morel phase [146], and the B-phase
as Balian-Werthamme state [137].

V. COLLECTIVE EXCITATIONS IN HE II

The above analysis may also be applied to He II with the short range interaction with
off-diagonal long range order. It should be emphasized, however, that the basic equations
Eqs. (1)-(2) are far less restrictive than those of the previous microscopic theory [34]. To a
great extent, the present study with the Lagrangian displacement vectors in ODLRO [39],
like the Bogoliubov theory of superfluidity [34], can serve as a model even when it is not
a perfect microscopic theory. Yet the present approach is a satisfactory model for finite
space problems that require the apparatus of a new perturbation method with which one
can formulate a quantitative analysis of Feynman’s theory of two-fluid model [1, 13, 33].

One shortcoming in the present model is that the two-body interaction is sufficiently
small, so that the nonlinear Schrödinger equation (Gross-Pitaevskii) Eq. (2) is assumed to be
applicable to He II. Hence one might raise an objection against the argument of Bogoliubov
[34] on the grounds that his theory is limited to a weakly interacting Bose gas and that the
inclusion of a strong interatomic force is essential to the treatment of collective excitations
in He II. We shall argue, however, following Feynman [33] that London’s view on the BEC
of an ideal Bose gas is essentially correct and that the inclusion of a strong force between
He atoms will not alter the central features of Bose condensation and the superfluidity of
He II [33] as the strong interaction forces in a liquid-like quantum fluid would not prevent
these particles from behaving like a free particles, a degenerate Bose gas [16].

In this section we are concerned with the problems in a uniform superfluid in which we
show the similarities between the Meissner effect in a uniform superconductor and the surface
phenomena in He II in a gravitational field. In the case of He II which is stationary under the
gravitational field, some care is necessary in separating surface waves from the longitudinal
sound waves in the bulk fluid. The entire argument is based on the the boundary conditions
in terms of the Lagrangian displacement vectors by which we can construct the boundary
conditions that are consistent with the basic equations Eqs. (3). The physical boundary
conditions allow a separation of a domain in which a dynamical calculation shows different
properties of fluids; the surface layer is a normal fluid (and solenoidal) and the bulk fluid is
a superfluid (and irrotational) [24].

The physical principles underlying the mathematical technique are so clear that it is easy
to see how the present results can be applied to the explanation of spontaneously broken
symmetry in He II. As a simple model which retains the main feature of the problem,
we consider the free surface of a superfluid in a gravitational field. The simplest correct
procedure, then, is to consider the gravity wave first and then extend the dispersion relation
for both the gravity and capillarity by using the Laplace formula [120].

A. Surface Waves

In the surface layer of He II, there are several different surface waves of which a transverse
wave plays an important role in explaining the broken symmetry just as the transverse
surface current in Nambu’s theory of gauge invariant explanation for the Meissner effect in
a superconductor.

1. Gravity Waves

We shall first consider gravity waves on a free surface of a uniform superfluid in which
the velocity of moving particles is so small that we may neglect the convective term \((v \cdot \nabla)v\)
in comparison with \(\partial v/\partial t\) in the Euler equation. As before we use the boundary conditions
\(\nabla \cdot v = 0\) and \(\nabla \times v = 0\) which can be written \(v = \nabla \phi\) and \(\nabla^2 \phi = 0\) for a potential flow.
Again our boundary conditions are equivalent to the original form \(\nabla \cdot \xi = 0\) and \(\nabla \times \xi = 0\)
in terms of the displacement vector [120] [121].

For an incompressible fluid under the gravitational field, we may write the Euler equation

\[
\left( \frac{\partial v}{\partial t} \right) = -\nabla \left( \frac{p}{\rho_0} \right) + \mathbf{g},
\]

where \(\mathbf{g}\) is the gravitational acceleration.
Eq. (54) can be rewritten in terms of a potential

\[ \nabla \frac{\partial \phi}{\partial t} = -\nabla \left( \frac{p}{\rho_0} + gz \right), \]  

(55)

where we have taken the z-axis vertically upward from the \( x - y \) plane of the equilibrium surface of the fluid under the gravitational field.

Let \( \eta \) be the vertical displacement of the free surface in its oscillation; it is a function of \( x, y, \) and \( t \) on the surface. Since \( p_0 \) is the atmospheric pressure on the free surface, we have

\[ p = p_0 + g\rho_0(\eta - z), \]  

(56)

Substituting \( p \) in Eq. (55), we obtain the equilibrium condition

\[ g\eta + \left( \frac{\partial \phi}{\partial t} \right)_{z=\eta} = f(t). \]  

(57)

Since \( \eta \) is small, we may write \( v_z = \partial \eta / \partial t \) to the same degree of accuracy as in Eq. (54). But \( v_z = \partial \phi / \partial z \), so that

\[ \left( \frac{\partial \phi}{\partial z} \right)_{z=\eta} = \frac{\partial \eta}{\partial t}. \]  

(58)

Without losing a generality, we may take \( f(t) = 0 \) in Eq. (57) and differentiate it with respect to time to obtain the equation of the free surface in oscillations along with Eq. (58).

\[ \left( \frac{\partial \phi}{\partial z} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \right)_{z=\eta} = 0. \]  

(59)

Since \( \eta \ll 1 \), we may take it as zero and solve the equation

\[ \left( \frac{\partial \phi}{\partial z} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \right)_{z=0} = 0, \]  

(60a)

with the boundary condition \( \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \)  

(60b)

We notice that the above derivation of the equilibrium condition Eq. (59) is simpler and it does not lead to any contradiction.

Next we consider surface waves propagating along the x-axis and uniform in the y-direction. We shall then look for a solution in the form

\[ \phi = f(z) \cos(kx - \omega t), \]  

(61)

where \( k = 2\pi/\lambda \) is the wave number. After a brief algebra with the boundary condition \( \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \), we obtain the dispersion relation,

\[ \omega^2 = kg, \]  

(62)
which shows the relation between the wave number and the frequency of a gravity wave \[122\]. The gravity waves are often studied in the context of hydrodynamic instability (the Rayleigh-Taylor instability) in a fluid under the gravitational field, and the implosion hydrodynamics in ICF program \[123\] \[124\]. Just like a spherical droplet, the surface waves are independent of \( h \) and are classical waves in He II.

2. Capillary Waves

In this section we shall consider the free surface of He II that undergoes an infinitesimal displacement under the combined action of a gravitational force and the surface tension. A study of the surface phenomena in He II is important in demonstrating the broken symmetry in quantum fluids, especially since the capillary waves have been observed recently in liquid helium by Vinen’s group \[26\]. In this connection, it should be pointed out that the surface tension of liquid He\(^4\) has been studied extensively by Atkins and others \[15\] \[21\] \[23\] and will be reviewed in the following.

The surface tension of a fluid is a measure of the free energy per unit area in He II which is in stationary state under the gravitational field \[128\]. The calculation of the surface tension based on a degenerate Bose gas model by Singh yields

\[
\sigma = \sigma_0 - \pi mk^2 T^2 \zeta(2)/2\hbar^2 = \sigma_0 - 0.0075T^2,
\]

\[
\sigma_{AN} = 0.3745 - 0.0096 T^2,
\]

where Eq. (63a) is a modified to Eq. (63b) to fit to their data by Atkins and Narahara \[23\], \( \sigma_0 = 0.352 \text{ erg/cm}^2 \) is the surface tension at 0\(^0\) K \[19\], and \( \zeta(2) = \pi^2/6 \). In an effort to obtain further information on \( \sigma_0 \), they have made extensive measurements down to 0.35\(^0\)K.

And yet Atkins and Narahara \[23\] have also provided a different theoretical interpretation with the assumption similar to that of the Debye theory of the specific heat of solids based on the lattice vibrations \[21\] \[22\] \[23\], which yields the surface tension as

\[
\sigma = \sigma_0 - 1.55(\rho/\sigma_0)^{2/3}2\pi\hbar(k_B/2\pi\hbar)^{7/3}T^{7/3},
\]

where \( \rho \) is the density of liquid with the cutoff frequency \( \nu_c = 1.5 \times 10^{11}\text{ sec}^{-1} \) and characteristic temperature \( \theta_c = 2\pi\hbar/k_B \approx 7^0 \text{ K} \).
However, the following empirical curve of $\sigma$ obtained by the least squares fit of the equation $\sigma = \sigma_0 + AT^n$,

$$\sigma = 0.3729 - 0.0081T^{2.5}, \quad (65)$$

shows a better agreement with the data [23].

Although Eq. (63) and Eq. (64) give similar curves (see Fig. 7 and Fig. 5 of Ref. [23]), it is rather remarkable to observe that Singh’s analysis $\sigma_{AN}$ [19] shows the best agreement with the experimental data obtained by Atkins and Narahara [15, 23]. The most compelling argument for this unexpected agreement here is that made above in the case of the BEC in the trap; the surface layer is a normal fluid, and that there is no interaction between the normal fluid in the surface layer and the superfluid beneath the layer. Thus the surface layer is composed of the atoms in the excited states as emphasized by Mott [13], which is consistent with Singh’s calculation of the surface tension [19, 25].

Since we have already discussed the Laplace formula in the study of capillary waves in a spherical droplet of BEC, we begin with Eq. (36) in which $p$ is the pressure inside the fluid in the surface layer, and $p_0$ is a constant atmospheric pressure on the free surface. As before, we assume a potential flow (i.e., $v = \nabla \phi$) of an incompressible fluid for the surface layer and take $p$ from Eq. (56),

$$p = -\rho_0 g \zeta - \rho_0 \frac{\partial \phi}{\partial t}, \quad (66)$$

where $\zeta$ is the displacement of the free surface in the $z$-direction and $\rho_0$ is the surface density of an incompressible fluid in the surface layer.

Taking $p_0 = 0$, we can write down the Laplace equation

$$\rho_0 g \zeta + \rho_0 \frac{\partial \phi}{\partial t} - \alpha(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}) = 0. \quad (67)$$

Differentiating the equation with respect to $t$ and replacing $\partial \zeta/\partial t$ with $\partial \phi/\partial z$ since $v_z = \partial \zeta/\partial t = \partial \phi/\partial z$, we obtain the equilibrium condition on $\phi$ on the free surface $z = 0$,

$$g \frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2} - \frac{\alpha}{\rho_0} \frac{\partial}{\partial z} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0. \quad (68)$$

As in the gravitational wave, we consider a plane wave propagating in the direction of the $x$-axis and look for a solution in the form $\phi = Ae^{kz} \cos(kx - \omega t)$, which yields the dispersion relation as

$$\omega^2 = gk + \alpha k^3/\rho_0. \quad (69)$$
It should be stressed that, in the long wavelength limit $k \ll 1$, the gravity wave dominates over the capillary wave. A peculiar wave motion of the free surface due to surface tension in He II has been observed recently by Vinen’s group [26]. Kapitza has argued that these waves were driven by the instability of the fluid at comparatively small Reynolds numbers [125]. This conjecture can be shown to be true by studying the propagation of sound waves in He II.

We also note that the dispersion relation Eq. (69) clearly suggests that the fluid on the nodal surface behaves like a normal fluid, because Eq. (69) is independent of $\hbar$ as in Eq. (33) for the BEC in a trap. Moreover, the observation of the capillary wave implies that the surface energy is another manifestation of a broken symmetry in the observable domain, where $\alpha \simeq \sigma = 0.3352 - 7.5 \times 10^{-3}T^2 \text{erg/cm}^2$ [19]. Since the two-fluid model does not permit a physical separation of the two-fluids (the superfluid and normal parts of the fluid) [1], the two-fluid model breaks down at the surface layer. It also demonstrates that the law of conservation of energy in the system is maintained, since the sound wave initiated deep inside is dissipated at the surface layer of He II as in the BEC in a trap.

The wavelengths of these capillary waves are actually extremely short, far too short to be observed in any ordinary experiment. It is indeed a challenging experiment [26] in which the presence of capillary waves on the surface of a superfluid layer has been observed with a strong external electric field that mimics the gravitational field. It should be pointed out that this experimental observation supports the theoretical explanation of the spontaneously broken symmetry at the nodal surface of a superfluid as discussed below.

3. Transverse Waves

We have already discussed the surface waves in the surface layer in both BEC and He II. And yet there exists still another type of oscillations inside the surface layer; it is similar to a gravity wave, but is driven by a entirely different mechanism, thermal expansion of an incompressible fluid, since the free surface cannot be a mathematical surface, but it will be a surface layer with finite thickness [9]. This new collective excitation is a transverse wave, defined by $\nabla \cdot \xi = 0$ (solenoidal, that is, free of sources and sinks). And therefore the surface waves are essentially transverse in nature, on the other hand the longitudinal excitation is defined by $\nabla \times \xi = 0$, but not solenoidal, i.e., $\nabla \cdot \xi \neq 0$. It is this unique
characteristic of the displacement vectors that allow separation the surface phenomena from
the bulk phenomena in a fluid.

Since a transverse excitation involves breaking up the phase coherence of collective exci-
tations in the surface layer and plays an important role as shown by Nambu in his study of
a Meissner effect in a superconductor [78], we show here how it comes about in the surface
layer as in a superconductor [52, 120].

Since transverse excitation in the surface layer is driven by thermal expansion, we need
a new set of hydrodynamic equations. First, we write down the equation of conservation of
the entropy:

\[ \frac{\partial s'}{\partial t} + \mathbf{v} \cdot \nabla s_0 = 0, \tag{70} \]

where \( s' \) is defined by \( s = s_0 + s' \), and \( s_0 \) is an equilibrium value and is a function of the
vertical coordinate \( z \).

Next neglecting the convective term \( \mathbf{v} \cdot \nabla \mathbf{v} \) and defining the change in density by
\( \rho = \rho_0 + \rho' \), we write Euler’s equation for a fluid element in the surface layer under a gravitational
field,

\[ \frac{\partial \mathbf{v}}{\partial t} = -\frac{\nabla p}{\rho} + \mathbf{g} = -\frac{\nabla p'}{\rho_0} + \frac{\nabla p_0}{\rho_0^2} \rho', \tag{71} \]

where the expansion of the pressure \( p = p_0 + p' \) and the equilibrium pressure in the gravi-
tational field \( \nabla p_0 = \rho_0 \mathbf{g} \) are used in the derivation.

Since the variation of density is due only to the change in entropy, and not due to the
pressure change, we may write

\[ \rho' = \left( \frac{\partial \rho_0}{\partial s_0} \right)_p s', \tag{72} \]

then we obtain the Euler equation in the form

\[ \frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{g}}{\rho_0} \left( \frac{\partial \rho_0}{\partial s_0} \right)_p s' - \nabla \rho'/\rho_0. \tag{73} \]

In the limit \( (\mathbf{v} \cdot \mathbf{v}) \ll \partial \mathbf{v}/\partial t \), we may neglect the change in the equilibrium density over
distances of the order of a wavelength. Hence we solve the equations [70], [71], and [72] with
the boundary condition \( \nabla \cdot \mathbf{v} = 0 \) which is equivalent to \( \nabla \cdot \xi = 0 \) in Eq. [27c] as discussed
earlier. We shall look for a solution in the form of a plane wave \( \mathbf{v} = A \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \).

Then the boundary conditions give

\[ \mathbf{v} \cdot \mathbf{k} = 0, \tag{74} \]
which implies that the fluid motion is perpendicular to the wave vector, i.e., a transverse wave. Eq. (74) is equivalent to London’s assumption that the super-electrons are assumed to be an incompressible charged fluid $\nabla \cdot j(x, t) = \nabla \cdot v = 0$ [52].

As a detailed derivation is available in the literature [120], we skip the algebra and give the dispersion relation for the transverse wave in the surface layer:

$$\omega^2 = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial s} \right)_p g \frac{ds}{dz} \sin^2 \theta,$$

(75)

where $\theta$ is the angle between the $z$-axis and the wave vector; it is defined by $g \cdot k = gk \cos \theta$. Here we have also dropped the subscript zero to the equilibrium quantities. We now use the thermodynamic identities [16]

$$\frac{ds}{dz} = \left( \frac{\partial s}{\partial p} \right)_T \frac{dp}{dz} = -\rho g \left( \frac{\partial s}{\partial p} \right)_T,$$

(76a)

$$\left( \frac{\partial s}{\partial p} \right)_T = \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial T} \right)_T,$$

(76b)

and

$$\left( \frac{\partial \rho}{\partial s} \right)_p = \frac{T}{c_p} \left( \frac{\partial \rho}{\partial T} \right)_p$$

(76c)

to rewrite the dispersion relation Eq. (75) in an experimentally accessible form as

$$\omega = \left[ \frac{T}{c_p} \right]^{1/2} \frac{g}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p \sin \theta = \left[ \frac{T}{c_p} \right]^{1/2} \frac{g}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p \sin \theta,$$

(77)

where we have taken $z$-axis upward from the $x-y$ plane and thus $\theta = \pi/2$ is the condition for a transverse wave in the $x-y$ plane and $c_p$ is the specific heat per unit mass. For $\theta = 0$, the transverse wave cannot exist in the surface layer. The frequency $\omega$ depends only on the direction of the wave vector.

The presence of the transverse wave demonstrates that the positive surface energy associated with a normal and superfluid interface is indeed a correct remedy for the serious flaw of London’s original theory of the Meissner effect [52]. In fact, this point of view turns out to be more fundamental than that of Ginzburg-Landau theory as far as the Meissner effect is concerned, although it gives the surface energy in terms of the coherence length, but it cannot address the nature of a transverse wave on the surface layer which is essential to the gauge invariant explanation of the Meissner effect in the microscopic theory of BCS theory [56, 78].
Moreover, the surface energy is necessary to maintain the law of conservation of energy associated with the dissipation of sound waves in the surface layer of He II. This description of the surface energy is also consistent with the positive surface energy associated with the normal state joined onto the super-conducting state in the second-order phase transition (the Meissner state) of the Ginzburg-Landau’s theory of a superconductor [56, 78].

The dispersion relation Eq. (77) is different from that of the (second) sound for which the velocity of sound depends markedly on temperature and vanishes at the $\lambda$-point [120]; the second sound is of course a longitudinal wave. The transverse wave in the surface layer is another manifestation of broken symmetry and breaks up the phase-coherence in the collective excitations just as the transverse current on the surface of a superconductor in the Meissner effect [84]. This difference is the basis for the broken symmetry on the free surface of a superfluid - a break-down of the two-fluid model. The dispersion relation Eq. (77) clearly shows that it is driven by thermal expansion of the density in the layer. It should be emphasized that all the surface waves in the surface layer are transverse in nature [122].

B. Compressional waves in He II

Let us now return to the main question on the superfluidity posed in the Introduction, namely the Bogoliubov spectrum for a phonon [34] below the $\lambda$-point in He$^4$. Inside the free surface, the fluid is compressible $\nabla \cdot \xi \neq 0$, but it remains irrotational $\nabla \times \xi = 0$ as it is a superfluid. To obtain the phonon excitation spectrum, first note that the vortices are in an isolated region, so that we may assume a uniform superfluid flow $v(x_0)$, and take the first-order terms $\rho_1$ and $S_1$ varying as $C \exp[i(k \cdot x - \omega t)]$.

It is then straightforward algebra to obtain the first-order linearized equations of motion from Eq. (3b) and Eq. (27),

\begin{align}
- i\omega \rho_1 - \frac{\rho_0}{M} k^2 S_1 &= 0 \quad (78a) \\
- i\omega S_1 + \frac{4\pi \hbar^2 a}{M} \rho_1 + \frac{\hbar^2}{4M \rho_0} k^2 \rho_1 &= 0, \quad (78b)
\end{align}

where we have neglected the external potential $V_{\text{ext}}$ in Eq. (3b), since the gravitational force is too small compared to the short range force between the helium atoms.

We at once obtain the dispersion relation for a longitudinal sound wave. The result is:

$$\omega^2 = \frac{4\pi a \rho_0 \hbar^2 k^2}{M^2} + \frac{\hbar^2}{4M^2}.$$ (79)
We notice that the difference between the phonon spectrum Eq. (79) and the spectrum of the gravity-capillary wave Eq. (69) in the normal fluid is striking [116]. It is also surprising, but pleasant to see that we have obtained the exactly same dispersion relation Eq. (79) as Bogoliubov’s microscopic results, for which, of course, calculations were performed with his well-known interaction Hamiltonian for an imperfect Bose gas in terms of the creation and annihilation operators in quantum field theory [34].

In the phonon regime $k \to 0$, $\omega = ck$, where $c = [4\pi a \rho(r) \hbar^2]^{1/2}/M$. The energy spectrum is characteristic of a sound wave with the speed of first sound $c = [4\pi a \rho(r) \hbar^2]^{1/2}/M$ in an imperfect Bose gas. One may conclude, then, immediately from the above analysis that our approach with the mean field by ODLRO [39] and Bohm’s quantum theory [40] is far more powerful and simpler in the study of superfluid dynamics than that of the quantum field theory.

Both the transverse excitations Eq. (77) and the longitudinal excitations Eq. (79) are essential to the gauge invariant explanation of the Meissner effect [78]; the transverse excitation is viewed as a necessary consequence of the longitudinal excitation in a superconductor.

On the other hand the longitudinal excitations, such as those generated by the density compression Eq. (27c), do not break up phase-coherence in a superfluid He II, that is, the phase coherence $\xi \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla_i S_{0,i}(x_{0,i}, t)$ is preserved, where the ground state wave function is defined in ODLRD as $\psi(r, t) = f(r, t) e^{\frac{i}{\hbar} S(r, t)}$ with the action $S(r, t)$ and $\xi$ the Lagrangian displacement vector. Unlike a superconductor, the transverse excitations do not take place in a superfluid but they do occur only in the surface layer which is a normal fluid. This difference is the basis for a spontaneously broken symmetry in a Bose quantum fluid, He II.

C. Spontaneously Broken Symmetries in He II

The surface layer is the boundary in which the waves show a marked change in the property of the fluid, indicating a second-order phase transition from a superfluid to a normal fluid. The entire argument is based on this characteristic difference between the two dispersion relations: Eqs. (69) and (79). It shows clearly that the fluid in the surface layer behaves like a normal fluid, whereas the compressible fluid inside the free surface behaves like a superfluid supporting the sound wave with its excitation energy $\omega = ck$. 

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where \( c = [4\pi a \rho(r)h^2]^{1/2}/M \) in the phonon regime. Moreover, the free surface cannot be a mathematical surface, but it will be a surface layer with finite thickness e.g., \( \lambda \approx 5.0 \times 10^{-3} \text{ cm} \) [9], in which the transverse wave Eq. (77) is a direct reflection of the break-down of the two-fluid model.

It is therefore natural to identify the underlying basic mechanism for the surface phenomena as a *spontaneously broken symmetry* at the free surface in a Bose system as in Nambu’s argument on the spontaneously broken gauge symmetry in a superconductor [84]. Here we have extended the two-fluid model of Landau [1] in such a manner that, in addition to phonons and rotons, the normal fluid now includes the degenerate, excited Bose particles in the Bose system from the ground state of the condensation [13, 19]. The phonon may be interpreted as the mass-less, spin-less boson, Nambu-Goldstone boson, as \( k \to 0 \) in the broken symmetry [78, 79, 80, 84, 85, 86, 89].

The picture we have presented above also explains how the superfluidity in a vortex core breaks down and why the classical form of the Magnus force in the study of a vortex quantization is indeed correct one [10, 11]. This result may also solve one of the most important and long-standing puzzle in low temperature physics, the parabolic surface of a rotating He II which has troubled the intuition of many experimental physicists over the half century [8, 9, 103].

The surface energy increment due to the interactions with a phonon would be extremely small, however, far too small to be observed in any ordinary experiment except for the indirect observation of a capillary wave. And yet, the above quantitative analysis has been in agreement with its most consistent and systematic analysis of the surface phenomena by Mott [13, 19] and Meservey’s precise observation of the curvature of a rotating He II [9].

In spite of a complicated mathematical analysis, it is indeed remarkable that, from just a simple set of boundary conditions in terms of the Lagrangian displacement vectors, one can establish the similarity between the Meissner effect in a superconductor in an external static magnetic field and the surface phenomena of a superfluid in a gravitational field, which leads to the spontaneously broken gauge symmetry in a Bose system. What’s more, the broken symmetry provides an explanation not only of how the superfluidity is broken at the core of a vortex, but of why a new BEC in a trap has a *shell-like structure* to hold the law of conservation of energy [69].
VI. BROKEN SYMMETRIES IN A BOSE SYSTEM

What has been described in the previous sections is the first step based on the detailed analysis of dynamical equations toward obtaining a more general theory which might offer hopes of understanding the broken symmetries that would be of fundamental importance in a Bose system as the Meissner effect in a superconductor.

Although the broken symmetry has been demonstrated by the dynamical calculations alone as shown above, we pause here, however, to comment on the broken symmetry by Nambu [78, 84], partly because of the inherent elegance of his method, and partly because incidental points of interest which has led Goldstone [80] to discover the general theorem of spontaneously broken gauge symmetry.

It is a mathematical theorem that explains the observations over a wide range of many-body problems. The beauty of the theorem is its simplicity: the symmetry is said to be broken spontaneously when the ground state (the vacuum in the field theory) does not share the same symmetry group with the Lagrangian that describe the dynamics of the system, and that, in the absence of a long-range order [81], accompanies a mass-less boson (a phonon). Hence we gain a new perspective on the broken symmetry in a Bose system.

To see whether the theorem which is the outcome of the study of the Meissner effect in a superconductor (a Fermion system), can be applied to problems in an imperfect Bose system, a careful analysis must be made for the consequences of the theorem. With these preliminary remarks, let us turn to the study of broken symmetry in a Bose system and ask the questions: what is then the spontaneously broken symmetry that accompanies a longitudinal phonon in both He II and BEC? and why?

VII. SURFACE PHENOMENA IN HE II AND MEISSNER EFFECT IN SUPER-CONDUCTORS

So far the analysis has not deviated significantly from that of phase-coherent collective excitations. At this point, however, we shall have to digress to show how the Goldstone theorem [80] came into being in particle physics. In the early 60s, the question of gauge invariance in the study of the Meissner effect has been addressed by a number of authors [59, 78, 79], a proof of gauge invariance in the derivation of the Meissner effect in BCS theory.
lies at the core of the problem of superconductivity, since the Hamiltonian used in the BCS model is not gauge invariant.

A. Nambu’s Gauge Invariant Theory of Meissner Effect and the Goldstone Theorem

In a series of papers, Nambu [78, 84] has laid out an elegantly simple picture of gauge invariant explanation for the Meissner effect. In the BCS-Bogoliubov theory of superconductivity [84], the energy gap parameter $\phi$ is obtained by a generalized Hartree-Fock calculation of the electron-hole interactions $\phi \approx \omega_D \exp[-1/\bar{\omega}]$, where $\omega_D$ is the Debye frequency around the Fermi surface and $\bar{\omega} = N(0)g \approx 0.2 - 0.3$ is the average interaction energy of an electron with unit energy shell of electrons on the Fermi surface, which depends on the interaction strength [50, 51, 61, 126]. Based on the same mechanism for the appearance of energy gap as in the theory of superconductivity in which elementary excitations can be described by a coherent mixture of electrons and holes, real nucleons are regarded as quasi-particle excitations in the mathematical formulation of a dynamical model of binding nucleon-pair into a pion, which is, in essence, a compound model of an idealized pion as a pseudoscalar zero-mass bound state of nucleon-antinucleon pair [84].

In the explanation of gauge invariance of the Meissner effect, Nambu [78] studies a linear relation between the Fourier components of the induced currents $J(q)$ in the ground state of a superconductor and an external vector potential $A(q)$ in the limit $q \to 0$. It has been emphasized that there is a fundamental difference between the transverse and longitudinal current operators in the calculation of matrix elements in the BCS model [55]. Bogoliubov [61] was the first to point out that there exist the collective excited states of quasi-particle pairs, a coherent mixture of electrons and holes driven by the longitudinal current. The pair of quasi-particle is a linear combination of an electron and a hole; it becomes an electron above the Fermi surface, and a hole below the Fermi surface. It was pointed out that the collective excited states such as quasi-particles are essential to the gauge invariance in the Meissner effect. Moreover, the longitudinal current satisfies a sum rule [see Eqs. (1.1), (5.1), and (5.2) of Nambu] from which one concludes that the longitudinal current produces no physical effect - the gauge invariant Meissner effect, whereas any transverse excitation involves breaking up the phase coherence over the Fermi surface of at least one
pair (equivalent to a Cooper bound state) in the super-conducting ground state - keeping the BCS result intact. Eq. (5.2) of Nambu takes the form of the London equations, since the intermediate pair formation (quasi-particles) is suppressed due to the finite energy gap in the limit $q \to 0$. In the previous chapter, we have also derived the dispersion relation for the transverse wave in the surface layer Eq. (77), which breaks phase-coherence in He II. It is a dynamical calculation with the boundary conditions for the surface phenomena in He II and is in remarkably consistent with Nambu’s picture of gauge invariant explanation of the Meissner effect [78, 79, 84].

The models studied by Nambu [78, 84] involve the basic four-fermion interactions. The boson excitations appear as collective modes of the fermion system. Goldstone [80, 86] has, on the other hand, studied the theories in which bosons are taken as elementary fields. These elementary bosons transform by an irreducible representation of a continuous group under which the Lagrangian remains invariant. From these models, Goldstone conjectures that whenever the Lagrangian admits a continuous symmetry group, but the vacuum is not invariant under the same group (whose expectation value of boson fields is not zero), then massless boson states must appear - broken symmetry. The broken symmetry, the energy gap, and the collective excitations are thus all closely related phenomena in a superconductor - the BCS-Bogoliubov theory [126].

On the other hand, the Meissner effect in a superconductor is explained by the gauge invariant theory [77, 78, 79], since the Hamiltonian used in the BCS calculation is not gauge invariant. However, as we have seen in the previous discussion, the phase-coherence is broken by transverse excitations in both He II and the Meissner effect in a superconductor. Hence we can clearly see that the phase-coherence is a more fundamental concept in the study of the broken symmetry.

Although much of what we have discussed on the Goldstone theorem [80, 86] may be regarded as something of a digression from the main object of the paper and is not new either, the discussion of underlying physics is a pedagogical contribution at the very most to our understanding of the broken symmetries which have played an important role in our understanding of superconductivity - the gauge invariant explanation of the Meissner effect in the theory of superconductivity. Hence it is helpful to discuss the broken gauge symmetry in a liquid helium in some detail, both for its intrinsic mathematical interest and as a means of solving long standing problems in low temperature physics.
In order to show why the Goldstone theorem plays an important role in the broken
symmetry, we must cast it in a mathematical form so as to yield a precise mathematical
type. It can be shown by the $\sigma$-model of Gell-Mann and Levy within the framework of
quantum field theory $[85, 89, 94]$: the Lagrangian density for two scalar fields $(\sigma, \pi)$ are
given by
\[ L = \frac{1}{2} \left[ \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi \right] - V(\sigma^2 + \pi^2), \] (80)
where the potential
\[ V(\sigma^2 + \pi^2) = \frac{1}{2} \mu^2 (\sigma^2 + \pi^2) + \frac{1}{4} \lambda (\sigma^2 + \pi^2)^2. \] (81)

One can easily show that the lagrangian density Eq. (80) is invariant under $O(2)$ (or
$U(1)$) group by:
\[ \begin{pmatrix} \tilde{\sigma} \\ \tilde{\pi} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \] (82)

In order to carry out a perturbation analysis, it is necessary to find the minimum values for
the potential. This self-consistent condition yields the minimum conditions if both
\[ \begin{align*}
\partial V/\partial \sigma &= \sigma [\mu^2 + \lambda (\sigma^2 + \pi^2)] \tag{83a} \\
\partial V/\partial \pi &= \pi [\mu^2 + \lambda (\sigma^2 + \pi^2)] \tag{83b}
\end{align*} \]
satisfy simultaneously
\[ \partial V/\partial \sigma = \partial V/\partial \pi = 0. \] (84)

If $\mu^2 < 0$ in Eq. (83), the minimum occurs on the circle $[\sigma^2 + \pi^2]^{1/2} = [-\mu^2/\lambda]^{1/2}$. Thus
in the $\sigma - \pi$ plane, we may assign the vacuum expectation values as
\[ \langle \sigma \rangle_0 = [-\mu^2/\lambda]^{1/2} \quad \text{and} \quad \langle \pi \rangle_0 = 0. \]

Hence the symmetry of ground state (vacuum in quantum field theory) is not invariant under
the same group $O(2)$ and therefore the symmetry is broken spontaneously.

In the $\sigma$ - model of Gell-Mann and Levy $[85]$, they add a small symmetry breaking term
($c\sigma$) to the potential $V$. Then the minimum occurs if
\[ \begin{align*}
\sigma [\mu^2 + \lambda (\sigma^2 + \pi^2)] &= c \quad \text{and} \quad \pi [\mu^2 + \lambda (\sigma^2 + \pi^2)] = 0
\end{align*} \]
The equations show that $c\sigma$ breaks the rotational symmetry in the $\sigma - \pi$ plane around the third axis and hence there is no solution for the equations except for $\pi = 0$ and $\sigma(\mu^2 + \lambda\sigma^2) = c$. In the limit $c \to 0$, either $\sigma = 0$ or $\sigma = [-\mu^2/\lambda]^{1/2}$ as before. Hence the solution takes the minimum value if $\mu^2 < 0$. The nature of symmetry is made much clearer if we perform translation $\tilde{\sigma} = \sigma - <\sigma>$ with $<\sigma>^2 = -\mu^2/\lambda$ and rewrite the Lagrangian density Eq. (80) in terms of $(\tilde{\sigma}, \tilde{\pi})$:

$$
\mathcal{L} = \frac{1}{2}[\partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} + \partial_\mu \pi \partial^\mu \pi] + \mu^2 \tilde{\sigma}^2 - \lambda <\sigma> \tilde{\sigma}(\tilde{\sigma}^2 + \pi^2) - \frac{1}{4}(\pi^2 + \tilde{\sigma}^2)^2, \tag{85}
$$

which shows that $\pi$ field is for a massless Goldstone mode and $\tilde{\sigma}$ field is for a particle with a positive mass $-\mu^2$ [94]. As explained above the presence of a massless particle is necessary in the spontaneously broken symmetry. From this example, a conceptually very simple interpretation of the Goldstone theorem emerges from the mathematics of a simplified $\sigma$-model of Gell-mann and Levy [85].

It will be shown below that there is a unique aspect about the way the spontaneously broken symmetry manifests itself in a Bose system, but it is consistent with the conservation of energy as shown in the previous Gedanken experiment in BEC. Since the above mathematical derivation for the Goldstone theorem in the quantum field theory is valid only in the Hilbert space [94], we now see why it was necessary to show by a semi-classical method that a phonon, whose spectrum is $\omega = ck$, is present in the spontaneously broken symmetry in the Bose system by Eq. (53) and Eq. (79), where $c = [4\pi a\rho(r)\hbar^2]^{1/2}/M$ in the phonon regime. And thus it only remains to justify on physical grounds that the ground state is not invariant under the global symmetry $U(1)$ for the spontaneously broken symmetry to take place in the liquid helium under the gravitational field.

In the following we shall consider a formal similarity between the surface phenomena in He II in a gravitational field and the Meissner effect in a superconductor in weak external magnetic fields. There is, however, a significant difference between the ways in which the symmetry is broken in He II under the gravitational field and a superconductor in an external static magnetics field. In He II, the Hamiltonian Eq. (5a) is invariant under $U(1)$, but the symmetry is broken at the surface layer as the profile of ground state function Eq. (29) is not invariant and looks like a Mexican hat just as a vacuum expectation (ground state) Eq. (81) in quantum field theory [80, 89].
B. London Equations and Surface phenomena in He II

We proceed by first noting that Landau’s two-fluid model which describes beautifully many aspects of He II, in particular, the excitation spectra of phonons and rotons, seems a well-grounded theory despite the fact that underneath all lies the peculiar absence of physical boundaries which brings about the broken symmetries in the liquid helium [120].

Much of the similarities between the surface phenomena in He II and the Meissner effect in a superconductor has been due to the London equations for the Meissner effect in terms of the classical magnetohydrodynamics, \( \nabla \cdot \mathbf{J} = -n_s \nabla \cdot \mathbf{v} = 0 \) and \( \mathbf{Q} = \nabla \times \mathbf{v} - e \mathbf{h}/Mc = 0 \).

If we set \( h = 0 \), the London equations are essentially equivalent to our boundary conditions (\( \nabla \cdot \xi = 0 \) and \( \nabla \times \xi = 0 \)) with the gravitational acceleration \( g = 0 \) [29] along with Bohm’s second assumption \( p = \nabla S \) which implies \( \nabla \times p = 0 \) for a free surface of He II in the absence of gravitational field [121].

It may seem strange that the characteristics of the surface layer of He II under the gravitational field are precisely analogous to those of a superconductor in a static magnetic field (Meissner effect). Yet the similarity is indeed remarkable. It is this critical observation of the surface phenomena in He II that led us to study the broken symmetries. Thus the spontaneously broken symmetry on the surface layer in He II is not quite accidental. However, unlike a superconductor in a uniform magnetic field for which the penetration depth is finite \( \lambda_L = [mc^2/(4\pi n_s e^2)]^{1/2} \) due to the expulsion of the magnetic flux, the penetration depth of the gravitational field in He II is infinite \( \lambda \to \infty \) but the gravitational field strength is negligible compared to the molecular forces; and the thickness of a surface layer is only \( \lambda_d \approx 5.0 \times 10^{-3} \text{cm} \). Thus we see that He II is not ever be free of vorticity under the gravitational field, which allows one to assume the upward diffusion of vortices in a rotating He II as will be shown below.

Since the ground state wave function in ODLRO can be written as \( \psi(r,t) = f(r,t)\exp[\frac{i}{\hbar}S(r,t)] \) [40] [108], we may interpret the broken symmetry of the ground state wave function at the nodal surface, the point at which \( S(r,t) \) must satisfy the prescribed boundary conditions by which we have demonstrated in section IV that a degenerate Bose system in an external field exhibits the second-order phase transition at the nodal surface - a normal fluid with the surface energy onto a superfluid just as the Meissner effect in a superconductor in an external magnetic field, i.e., a spontaneously broken symmetry.
This is analogous to that of the second-order phase transition between the normal state and the super-conducting state based on the free energy function in terms of order parameter in the macroscopic Ginzburg-Landau theory of a superconductor in which the energy gap function $\Delta$ shows the existence of spatial inhomogeneity, i.e., the normal state joined onto the super-conducting state near the second-order phase transition (the Meissner effect) [106]. What’s more, the second-order phase transition in a superconductor with $\kappa = \lambda(T)/\xi(T) \ll 1$, where $\kappa$ is the Ginzburg-Landau parameter, is almost identical to our problems - the superfluid joined on to surface layer (a normal fluid) in He II under the gravitational field or the shell-like structure of BEC in a trap.

It is also well-known that a lattice possesses a translational symmetry, i.e., invariant under $U(1)$ transformation and that the symmetry can be broken by dislocation [167]. In this sense the spontaneously broken symmetry in He II is similar to what a lattice breaks the translational symmetry by dislocation or by impurity - a broken $U(1)$ lattice gauge [77, 81], since the Bose particles in He II are charge neutral which can be described by the global symmetry without a local gauge field like Fermions. Our discussion on the spontaneously broken symmetry in He II should be, therefore, justified by the Goldstone theorem alone [86, 89, 90].

It should be emphasized, however, that Goldstone studied a model that has a single fermion interacting with a single pseudoscalar boson (quantized) field in the long wavelength limit $k = 0$ [86]; the model thus has a boson field at the outset in contrast to Nambu’s dynamical model of a pseudoscalar zero-mass bound states of nucleon-antinucleon pair (soft pions) [84]. Therefore we see that our dynamical study of phonons as collective excitations in He II is a semi-classical approach (a compressional wave) in which we confined our attention to verifying that the phonons (Goldstone modes) are present with a correct dispersion relation for the traveling sound waves in He II and hence our approach is quite different from those of both Nambu’s and Goldstone models. Yet we can still apply the Goldstone theorem for the spontaneously broken symmetry in our problem, because we can construct the Lagrangian from the Hamiltonian Eq. (5a) that possesses a continuous symmetry group $U(1)$ under which the ground state Eq. (6) is not invariant, for it has a nodal surface. What is important here is that the broken symmetry always accompanies a phonon (Nambu-Goldstone boson or Goldstone mode) [65].
In the case of a vortex, the breakdown of superfluidity, however, accompanies a roton whose effective mass is \( \mu_{ro} = 0.16 m_{He} \). In fact there is also a spontaneously broken approximate symmetry for which we add a small symmetry breaking term in the action for the derivation of the Goldstone theorem and from which we see the appearance of low mass, spin-zero bosons, pseudo-Goldstone boson as shown above in the \( \sigma \) model of Gell-Mann and Levy \[85, 87\].

VIII. BROKEN SYMMETRIES BY THE FLUCTUATION-DISSIPATION

Another approach to the study of broken symmetry is to ask a question whether the mean-field approach can explain the spontaneously broken symmetry at the nodal surface in He II based on Bohm’s interpretation of quantum theory \[40\]. The affirmative answer to the question is essential to the basic argument of this paper. Here we discuss the broken symmetry based on the following picture: He II is described by the mean field \( \psi(r, t) \) in ODLRO and has the hidden symmetry which is given in terms of the action \( S(r, t) \). He II breaks this symmetry at the nodal surface by the effective quantum mechanical potential Eq. (4), which drives the particles to fluctuations (i.e., Bohm’s irreducible disturbance) and thereby breaks the phase coherence - a necessary condition for superfluidity. Thus the spontaneously broken symmetry can be readily explained by Bohm’s quantum theory and the mean field defined in ODLRO.

An important step in our argument is based on the phase-coherence of the collective excitations in ODLRO. Since the phase-coherence, \( \xi \cdot \nabla S(x_0, t) = \sum \xi_i \cdot \nabla_i S_0, \psi(x_0, i, t) \), is a necessary condition for a superfluid to exist, if it is broken due to the quantum fluctuation \[40\] at the nodal surface where \( \nabla \rho(x) \) is discontinuous and the effective quantum mechanical potential Eq. (4) drives particles to fluctuate rapidly at the nodal surface of He II, thus the fluid is no longer a superfluid due to breakdown of the phase-coherence at the surface where the boundary conditions play an important role just as the London equations for the Meissner effect in a superconductor. On the other hand, we note that, inside of He II away from the surface layer where the boundary conditions are unimportant, the action \( S(r, t) \) is analytic and the longitudinal sound waves remain phase coherent \[79, 84\].

If, for the purpose of illustration, we assume that a superfluid droplet, which is suspended by an external force and in which the symmetry is not broken, and that a sound wave was
initiated adiabatically at the center of the droplet, then the phonons must bounce back to
the center. Hence we see immediately that we cannot maintain the law of conservation of
energy in an isolated system, because the sound wave initiated at the center of a superfluid
droplet cannot be dissipated at the surface layer. However, if the symmetry is broken, the sound
wave dissipates by the interaction with the normal fluid at the surface layer, which would give rise to a surface energy manifested by a surface
tension - Eq. (63b). This is because acoustic phonons, which possess many attributes of
particle, cannot interact with a superfluid component, as Mott emphasized in his study of
the two-fluid model, it does, however, interact with a normal fluid, giving rise to a
surface energy manifested by the capillary wave Eq. (69). This dissipation process of the
sound wave is the same as in the case of the BEC in a trap. It should be made clear that
this Gedanken experiment of (first) sound wave propagation in a superfluid droplet described
above is different from the experimental study of the propagation of second sound by Hall
and Vinen. In the simple example described above, the energy supplied from an
external source is to drive the phonons to form a degenerate Bose gas with an energy gap out
of Bose particles near the surface layer in a trap. Thus the above picture provides
a simple model to help understand the mathematical treaties of broken symmetry.

It is obvious that the above discussion is not a standard method to describe a dissipation
process. In this we have departed from the Kubo’s approach for the electrical conductivity
in a dissipative medium, which would be a more familiar approach to the
traditional many-body theory. The broken symmetry in the Bose system follows, however,
from the statistics and simplicity of the ground state defined in ODLRO, and explains
a number of unsolved problems in He II. The discussion on Luttinger’s analysis on the
fluctuation and dissipation in a superconductor is given below to show that the general
features of the broken symmetry by the fluctuation and dissipation presented here are similar
to those employed in his analysis, in particular his basic assumption on the chemical potential
in the equation of motion, i.e., Eq.(2.4) of Luttinger.

Since there is no Hamiltonian that describes a thermal gradient and the temperature is
a statistical property of the system, the Kubo’s formula has been derived by using local
variables assuming the local equilibrium. In the absence of an external magnetic field,
the dissipation mechanism must be then temperature dependent.

Indeed, the equations with phenomenological transport coefficients for a superconductor
has been derived in the presence of temperature gradient by Luttinger \cite{172} with additional assumption on the system. From these equations, he derives an expression for the thermal conductivity as a correlation function of the Kubo-type based on the two-fluid model and the basic equation [Eq. 2.4 of Luttinger] which is similar to our Eq. (24) [see also Eq. (26)]. The fundamental assumption on the presence of the chemical potential in the equations of motion implies that a system of steady-state is made up with a free surface and thus the two-fluids in which heat and particle flows are necessary to maintain quasi-equilibrium in the presence of temperature gradient \cite{57,115,172}.

In other words, a superfluid alone cannot maintain a steady state without the broken symmetry at the free surface. In our study of BEC, the chemical potential $\mu$ is introduced in Eq. (6) as a Lagrangian undetermined multiplier, a condition to maintain the number of particles constant in an isolated system. A more elaborate explanation was offered by Luttinger using the mean-occupation number formalism in statistical mechanics \cite{173}, and he gives a full justification for his model of thermal transport coefficients of a superconductor \cite{172}. Precisely, it is also the reason why we have seen the broken symmetry in an isolated system of the BEC at the free surface - a shell-like structure.

The dissipation process of sound waves in the surface layer of He II is essentially similar to that of Luttinger’s model of thermal transport coefficients of a superconductor \cite{171,172}, since his derivation of transport coefficients are appropriate to a normal conductor. Thus there is a discontinuity in the free energy which leads to the second-order phase transition \textit{(i.e., a superconductor transforms to a normal metal)}.

In a simple system of a charged boson gas as a superconductor in equilibrium, Schafroth showed the system exhibits a phase transition of the second kind at a critical temperature $T_e$ \cite{43}. It was further emphasized that, in the presence of the vector potential of an applied weak inhomogeneous magnetic field, the penetration depth of a superconductor is not determined solely by the density of condensed boson $\rho_s$ [see Eq. 4 of Schafroth], but it also depends on the normal fluid $\rho_n$ \cite{43,55}. Thus the charged boson gas $\rho_s$ in the surface layer may be considered as the bound two-electron state due to interactions mediated by phonons in the lattice. Also, it may be interpreted as another manifestation of Bose-Einstein condensation below $T_e$ \cite{16}. Thus the second-order phase-transition at the surface layer in a superconductor in an external magnetic field \cite{78} is similar to that of a shell-like structure of Bose-Einstein condensation in an external potential. The model of a superconductor
by Schafroth gives further evidence of the bound two-electron state in a superconductor \[ \Phi = \frac{n\hbar c}{2e} \], i.e., a correct flux quantization constant realized in experiments by Deaver and Fairbank and also by Doll and Nabauer [157].

In conclusion we note that the broken symmetry at the surface layer is both a necessary and a sufficient condition for an isolated Bose system to conserve the energy at low temperature and to conserve the number of particles in a trap. Although the effect on the surface energy is very small in practice, we are only interested in the question of basic principles, and an arbitrarily small effect on the surface tension is just as good as a large one. Perhaps, I should remind the reader of how a neutrino has been discovered in the \( \beta \) decay in nuclear physics (Fermi theory of \( \beta \) decay) [168]. The discovery of the neutrino was in a similar circumstance as in the second-order phase transition in He II [19, 29, 56, 57]. So far we have outlined only a number of the salient features in theoretical developments for the broken symmetry in a Bose system and we now turn to experimental aspects of the broken symmetry in He II which provide a complete picture of the broken symmetry.

IX. BASIC EXPERIMENTS

Based on the phenomenological theory of broken symmetry presented above and Anderson’s concept of phase slippage we now discuss the four puzzling problems in He II and BEC: 1) the vortex quantization, 2) Magnus force, 3) the curvature of a rotating He II, and 4) a question of possible formation of a super-lattice in He II by compression.

A. Vortices in He II

It is useful to recall that phonons and rotons are not the only collective excitations in He II. There also exist macroscopic excitations, vortices, which involve the flow of large amount of fluid in considerably higher energy. The study of vortex motion in He II is of particular interest, because it helps not only understand the decay of vorticity in a homogeneous turbulence state which is created in a steady state of He II and the subsequent decay of vortices [174], but also the role played by the current-induced vortex motion (i.e., the motion of Abrikosov flux lines [175]) for dissipative processes in type-II superconductors [179, 180, 181].
In a multiply connected region in He II the condition that $\psi(r,t)$ be single-valued leads to the requirement that the phase $S/\hbar$ need not be single-valued but merely need return to its original value $\pm 2n\pi$ on traversing the nodal surface of a vortex. Hence the Onsager-Feynman quantization \cite{99,100} can be stated in a more precise manner,

$$\sum_{\text{cir}} \nabla S \cdot \xi = \oint \nabla S \cdot dl = nh. \tag{86}$$

Here the summation is taken around the nodal surface and $\xi$ is taken to be small since it is an atomic displacement. In He II, if one takes $\nabla S = p = Mv$ by Bohm’s second assumption which states that $S$ is a solution of the Hamilton-Jacobi equation, then $\kappa = \oint v_s \cdot dl$, where $\kappa_0 = 2\pi\hbar/M = 0.997 \times 10^{-3} \text{cm}^2/\text{sec} \ [40, 107]$.

As we have discussed in VIII that Eq. (86) breaks up the phase-coherence $\xi \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla S_{0,i}(x_{0,i}, t)$. Here $p = \nabla S(x, t)$, where $S(x, t)$ is assumed to be analytic and is also independent of time. The phase-coherence is broken spontaneously due to fluctuations in particle orbits driven by the EQMP in the phase-space (i.e., Bohm’s irreducible disturbance) and is also quantized by Planck’s quantum condition. Thus this can be interpreted as the spontaneously broken symmetry manifested by breaking up the phase-coherence. Better yet is the conceptual difference that the spontaneously broken symmetry is more fundamental for the creation of a vortex line \cite{14,40} than that of the Bohr-Sommerfeld quantization of the linear momentum \cite{99,100}, for which there is no underlying mechanism by which a spontaneous quantum jump can be explained other than the statistical law. The absence of an identifiable cause for the spontaneous quantization troubled Einstein whose objection to Bohr’s atomic model is well known \cite{40,41,42}.

This quantized circulation has been confirmed in a superfluid helium at low temperature, He II. In an ingenious experiment by Rayfield and Reif \cite{11} in which they have used ions as a microscopic probe to study the motion of vortices in He II. The motion of ions creates a flow favorable for creating a vortex around the probe which is trapped at the core of the vortex. The probes are interpreted as an ion-structure clustered by van der Waals forces \cite{152,153}. Moreover the motion of the ion-cluster is accurately measured by externally applied electromagnetic forces. An interpretation of the ion mobility in He II involves the nature of collective excitations and their interaction with the ionic probe particles.

Rayfield and Reif also proposed a mechanism for the vortex-ring formation in He II by
ion probes. But there was a difficulty in explaining how the ion probes have been trapped in the vortex-ring \[11, 158, 160\]. Rayfield \[11, 161, 162\] repeated the same experiment with impurity of He\(^3\) (1 part of He\(^3\) per 5.35 × 10\(^3\) parts of He\(^4\)) \[159\] and at the temperature below 0.30\(^0\)K to limit the scattering of rotons with the ion probes which complicates the study of vortex-ring formation in He II \[160\]. Based on the new data, Rayfield has proposed a mechanism \[161, 162\] by which a vortex-ring is being formed in He II with ion probes at the center and which overcomes a logical inconsistency in the previous work \[11\].

More specifically the new experimental data indicate that ”the vortex lines associated with the formation of the vortex-ring is slowly peeled away from the ion complex in the form of a steadily growing loop as the electric field increased”. This picture explains why the ion complex is not required to hop into the ring and there was no discontinuous change in the drift velocity after the ion reaches a critical velocity. Thus the peeling out of vortex line begins when the velocity field of the superfluid in the neighborhood \(\oint \mathbf{v} \cdot d\mathbf{l} = \hbar/M\) (see Fig.2 of Ref.\[161\]). If the process is completed at the ion-critical velocity, the discontinuity results in the curve of velocity vs electric field which shows a spontaneous symmetry breaking in He II \[161\].

Based on the concept of Anderson’s phase slippage \[14\], this process can be described more precisely by 2\(\pi\) phase slip of a vortex line at the inner surface of the macroscopic vortex-ring and the momentum exchange as the ion cluster begins to slow down, since the microscopic vortex line attached to the ion complex can be detached through the interaction with the macroscopic vortex-ring at its inner surface. As emphasized by Rayfield \[161\], it is a difficult task to obtain the dispersion relation for the ion-vortex-ring experimentally, for the velocity field is a superposition of the flow field of ion and vortex-ring. As will be shown below, the vortex-ring formation is very similar to that of a parabolic surface of a rotating He II.

It is noteworthy, however, that we have applied a spontaneously broken symmetry as a mechanism to explain the breakdown of superfluidity at the vortex core instead of the Bogoliubov-de Gennes equations which were applied to the study of the energy gap parameter \(\Delta\) to explain the breakdown of superconductivity in the Abrikosov vortex line \[63, 176\]. The reason for this is that the Bogoliubov equations are valid in a Hilbert space, but not to a problem with a boundary, because they are derived for a uniform system \[93\]. Hence this proves our approach is indeed consistent with the quantum field theory, as it gives us a
correct concept on the breakdown of superfluidity from which we describe the basic notion of broken symmetry.

**B. Circulation Constants \((\kappa)\) and Magnus Force**

As we have discussed above, it is well known that, in addition to phonons, rotons, and a vortex-line, there also exist macroscopic quantum excitations involving the flow of much larger amount of liquid at much higher energy - a vortex-ring in a superfluid. It is also known that the quantized vortices in a neutral superfluid play an important role in the decay of superfluid currents. Likewise in a superconductor in a high magnetic field, the motion of quantized magnetic flux lines is the main mechanism for electrical resistance in superconductors [180].

It is therefore important to understand the dynamics of quantized vortices for a study of dissipation mechanism in both a neutral superfluid and the superconductors. Rayfield and Reif [11] have presented the vortex formation by the exact expressions which are derived from classical hydrodynamics for the energy and translational velocity of a vortex ring, moving in an *incompressible* fluid of density \(\rho\). We follow this procedure since we wish to maintain a close contact with the previous theoretical work, but we wish to offer an entirely different interpretation of the results. The following equations describe the motion of the surface of a vortex-ring in a superfluid [121].

\[
E = \frac{1}{2} \rho \kappa^2 R [\eta - (7/4)] \\
v = \frac{\kappa}{4\pi R} (\eta - 1/4),
\]

where \(\eta = \ln(8R/a)\) and \(a\) is the core radius of a vortex. The experimental data were, then, analyzed by eliminating \(R\).

Furthermore Eq. (87b) and \(\eta = (vE/B)^{1/2} + 1\) with \(\eta \gg 1\) yield

\[
vE = B[\eta - (7/4)](\eta - 1/4) \\
\eta = \ln\{16E[\rho \kappa^2 a(\eta - 7/4)]^{-1}\}
\]

where \(B = \rho \kappa^2 / 8\pi\).

With the condition \(\eta \gg 1\), which is equivalent to taking the hydrodynamic equations to a quantum domain in which the uncertainty principle limits the precise measurements of the
position and momentum of a particle in the system. This is consistent with Bohm’s interpretation of quantum theory \[40\] and Eq. \[86\]. Perhaps more important, Bohm’s irreducible disturbance at the core of a vortex (i.e., fluctuations of particle orbits due to $U_{\text{emp}}$) is precisely the cause of breakdown of superfluidity at the core of a vortex-line. A combination of the above equations gives an approximate equation for $[vE]^{1/2}$,

$$[vE]^{1/2} = B^{1/2}\{\ln E - \ln\left(\frac{vE}{B}\right)^{1/2} - \frac{3}{4}\} + B^{1/2}\{\ln\left(\frac{16}{\rho\kappa^2 a}\right) - 1\}. \tag{88}$$

Since $vE$ is a slowly varying function of $E$, Rayfield and Reif \[11\] were able to use a graphic technique to obtain the values for $\kappa$:

$$\kappa = (1.00 \pm 0.03) \times 10^{-3} cm^2 sec^{-1} \tag{89}$$

$$a = (1.28 \pm 0.13) \AA, \tag{90}$$

where $\rho = 0.1454 g cm^{-3}$ for liquid helium has been used. It is indeed remarkable that the experimental value $\kappa$ with $n = 1$ is, within the limits of estimated error due to a discontinuity, equal to $\kappa_0 = h/M$ where $M$ is the mass of $\text{He}^4$ - the second-order phase transition \[43, 56\]; it is another manifestation of a spontaneously broken symmetry in a superfluid.

The first experiment designed to investigate the question of quantized circulation was Vinen’s closed cylindrical vessel of superfluid in which he measured the transverse vibrations of a fine wire along the axis of rotation from which the Magnus force on the wire was deduced \[10\]. The experiment, however, encountered some difficulties in establishing steady state and in avoiding partial attachment of vortex lines to the wire. The data showed, therefore, a wide range of uncertainty in the observed values of $\kappa$, although a pronounced peak was found near the value of $h/M$.

Although Rayfield and Reif \[11\] did not directly measure the Magnus force to study the vortex quantization, it is essential to understand the dynamics of a charged vortex ring from the point of view of hydrodynamics which provides a greater physical insight into the question of breakdown of superfluidity at the vortex core. In spite of its importance in vortex dynamics, there is still no agreement on a correct form of the Magnus force \[150, 151\]; Thouless, et al., \[110\] pointed out that there are many, conflicting results in the literature, as many as there are theorists active in the field. Perhaps it is not surprising that there should have been such a diverse disagreement, for the Landau’s two-fluid model was confirmed in
an experiment by Andronikashvili [132] and was never before suspected for its breakdown at the nodal surface, although there was clear experimental evidence for the break-down of superfluidity in a rotating He II as early as 1950 in Osborne’s experiment [8].

Out of such seemingly hopeless confusion that has plagued our intuition since the first experiment by Vinen [10], a systematic study of Thouless group over the years has brought the question of a correct form of the Magnus force as a major stumbling block in our understanding of dissipation processes in He II. But a general consensus on the form of Magnus force is still lacking, because there has been no convincing theoretical analysis to date. The failure of strenuous efforts by Thouless group led us to speculate at first that some novel concept is lurking behind this pervasive confusion over the half century [110].

Yet there is one outstanding hint in the way of accepting any of those previous theories: the crux of controversy on the form of the Magnus force is the very simple observation that we have no experimental evidence for the presence of a superfluid component at the vortex core and the data are always analyzed based on the classical form of the Magnus force [10, 11, 121]. The current state on the Magnus force reminds us the $\tau - \theta$ puzzle in nuclear physics that led the discovery of the parity non-conservation in the $\beta$ decay in particle physics by Lee and Yang in the 1956 [148, 149].

Indeed the most significant consequence of the spontaneously broken symmetry is that we adapt the classical form of the Magnus force, since the two-fluid model breaks down at the nodal surface. Thus the core of charged vortex ring may be regarded as a charged thin ring made of a normal fluid (i.e., a smoke ring with an ion probe) that satisfies the equation of motion for a uniform motion:

\[ F + G = 0, \quad \text{with} \quad G = \rho \kappa \times U, \]  

where $F$ is the force by the applied electric field, and $G$ is the hydrodynamic Magnus force per unit length, where $\rho$ is the density of a normal fluid. Here $U$ is the relative velocity of the core element with respect to the fluid. Thus $U = u - u_f$, where $u_f$ denotes the fluid velocity at this position caused by all sources other than the core element. The Magnus force is, therefore, proportional to its velocity of circulation and perpendicular to its direction of motion, and thus causes a precession of the plane of the vibration.
With the empirical dispersion relation of a charged vortex, they obtain the Magnus force:

\[ G_x = -\pi \rho \kappa R^2 \dot{\theta} = -F_x. \] (92)

It should be emphasized at this point that the basic equations Eqs. (87) used in their data analysis are obtained from classical hydrodynamics [121] except for \( \kappa \) value which was determined by the data - a spontaneously broken symmetry by which it is understood that \( \kappa = \oint v_s \cdot dl \), where \( v_s \) is the velocity of a normal fluid (of an inner surface of a vortex - a surface layer) and \( \kappa \) is quantized in units of \( \kappa_0 = h/M = 0.997 \times 10^{-3} \text{cm}^2/\text{sec} \) in He II. Notice that the circulation is not only constant but also quantized on a macroscopic scale in units of \( h/M \), whereas the motion of a vortex-line by an external force should be described by the classical physics.

Moreover, the relation between the Magnus force and the applied external force Eq. (91) can be used to test the self-consistency since the applied field is given by \( F = eE \) which can be measured precisely, although the precise radius of a vortex core is difficult to measure in experiments. The velocity measurements in terms of energy showed that there is a unique relation between the velocity and the energy. Moreover, the energy is proportional to \( \kappa^2 R \), that is \( E \propto \kappa^2 R \). But these points are completely classical in nature except for the \( \kappa \) value by the quantization, Eq. (86).

Once again we remind the reader that the free surface at the vortex core is a nodal surface where the broken symmetry takes place [102] and should therefore behave like a normal fluid. This explains why Rayfield and Reif [11] have observed that a vortex responds like a normal fluid to an ion probe at the core (\( i.e., \) a breakdown of superfluidity). As the superfluid density \( \rho_s \) approaches to zero near the nodal surface of a vortex-line which rotates approximately with the speed of the first sound \( c = \kappa/(\sqrt{8\pi} \zeta) \) with \( \zeta \) the coherence length, the effective quantum mechanical potential (EQMP) fluctuates rapidly to excite the rotons near the vortex line, because on the vortex line \( U_{eqmp} \) is highly singular.

The first experimental observation on the collective excitations of rotons near the vortex line was reported in 1968 [27]. With the picture of a roton as the quantum analog of a smoke ring whose effective mass is \( \mu_{ro} = 0.16m_{He} \) and whose excitation energy is much higher than that of a phonon, we have limited our discussion on the mechanism of how rotons are created near the vortex line to explain the breakdown of superfluidity at the core - a breakdown of Landau’s two-fluid model [1].
The broken symmetry is demonstrated so strikingly in the experiments by Rayfield and Reif \cite{11} that it has finally helped resolve a longstanding controversy over the Magnus force \cite{110, 150, 151}. We have known the erroneous (and confusing) interpretation of the Magnus force, however, for a good many years, since the first experiment on a vortex quantization was analyzed by the classical form of the Magnus force that has given the correct circulation in units of \(\kappa_0 = h/M\) \cite{10, 27, 121}. The reason why this controversy has persisted for so long may be traced to our failure to recognize the broken symmetry at the nodal surface and to take account the fluctuations of particle orbits (i.e., a break down of superfluidity by the breaking up the phase-coherence) driven by the fluctuating EQMP with decreasing radius of a curvature of the free surface at the core \cite{40}. It also explains why Vinen’s experimental data \cite{10} are essentially correct, which was obtained by the classical form of the Magnus force.

We would like to clarify about the confusing remark that one finds in the literature on the Iordanskii’s extension of the Magnus force in the two-fluid model \cite{110}, which was supported by experimental data \cite{111}, but is now questionable, because the extension motivated by the two-fluid model and contradicts Vinen’s experimental data \cite{10}. We find the data are not credible \cite{111} and contradict the analysis by Rayfield and Reif [see Eq. (89) and Eq. (91)].

C. A Rotating He II

From Landau’s two-fluid model \cite{1}, one obtains the equation \(z = (\omega^2 r^2 \rho_n)/(2g\rho)\), where \(\rho = \rho_s + \rho_n\), for the shape of the free surface of a rotating He II, but in experiments one observes instead the temperature independent equation for the curvature \(\gamma = \omega^2/g\) \cite{8, 9}. This observation presents a profound paradox \cite{103}: Meservey \cite{9} has questioned whether the two-fluid model can explain the shape of the free surface without violating hydrodynamics of a superfluid \(\nabla \times \mathbf{v}_s = 0\), which takes nonzero values at discrete points [see the comments following Eq. (86)] in a multiply connected region. Moreover, Mott’s analysis on the surface energy on a rotating cylinder suggests that there might occur the Onsager-Feynman quantization on the surface of a rotating superfluid, which is essentially similar to that of Anderson’s phase slippage, but it cannot still explain how the energy from a rotating superfluid is transferred to the surface \cite{13}. And Reppy and Lane \cite{164} also questioned if a sufficient number of vortex lines were generated and maintained in a rotating
He II during the course of the experiment to satisfy Feynman’s criteria for the areal vortex number density $\Gamma_\Omega \geq \frac{20\Omega}{\kappa}$ with $\kappa = \frac{h}{M} = 0.997 \times 10^{-3} \text{cm}^2/\text{sec} \ [102]$.

We approach this long standing riddle in a rotating He II from an entirely different point of view [8, 9]. Since a superfluid is described by the potential flow $\nabla \times \mathbf{v}_s = 0$, a steady rotating superfluid flow exerts no force on the rotating container (d’Alembert’s paradox). The normal fluid (phonon and roton), on the other hand, exerts drag force on the wall of a rotating container. The fundamental limitation of the two-fluid model brought about by the absence of the surface phenomena defined by an external pressure. In 1950, Osborne was the first to question if the superfluid component rotates when we rotate a container of He II. To answer this question, he observed the contour of the free surface of a rotating He II that was independent of temperature, which is incompatible with the two-fluid model [1].

In order to overcome this difficulty, Landau and Lifshitz proposed a vortex sheet model [98]. On the other hand based on the suggestion by Onsager [99] and Feynman [100], Hall and Vinen [101] extended London’s model [31] and showed that the macroscopic rotation of He II can be achieved despite the condition $\nabla \times \mathbf{v}_s = 0$ if an array of vortices is aligned along the axis of rotation ($\Omega$) and satisfies Feynman’s criteria for the areal vortex number density. We refer the reader Ref. [102] for details of Feynman’s criteria [102, 104, 105].

In contrast to the above models, Lin [189] suggested that the superfluid might have a small viscosity and would thus reach a steady state of uniform rotation in which boundary slip would cause the superfluid component to rotate more slowly than the normal fluid at low velocities. Lin’s insight into the question of whether the superfluid component rotates was quite plausible, but it was difficult to explain the experimental observation by Reppy and Lane [190] that a container of He II can be rotated at a higher speed in the absence of the superfluid motion. Moreover the notion of the boundary slip is incompatible with the two-fluid model of Landau. In particular, the superfluid component in a uniform motion must remain inviscid and irrotational $\nabla \times \mathbf{v} = 0$, but it can be singular only at isolated points of a rotating He II, i.e., $\kappa = \oint d\mathbf{l} \cdot \mathbf{v} = \frac{h}{M} = 0.997 \times 10^{-3} \text{cm}^2/\text{sec}$ at discrete points of the superfluid in motion, at which the broken symmetry takes place to create vortices [11, 13, 100, 132].

To Meservey [9] who has carried out the same experiment on a rotating He II as Osborne’s [8] and whose incisive comments on the previous theoretical models [11, 100, 132] based the his data, and to Mott [13, 14] whose remarkable insights into the surface energy due to
velocity discontinuity with which the present concept of broken symmetry has been deduced with mathematical rigor, we owe them for their valuable insights into the surface phenomena in He II.

In 1964, Meservey [9] concluded that neither the simple vortex line model [102], nor the vortex sheet model [98] is adequate to explain all of his data. In addition, Reppy and Lane [164] also pointed out that there is no reasonable mechanism by which the vortex line model can be realized in actual experiments, since the macroscopic turbulence in the superfluid has been observed in a rotating He II which implies the vortex lines were unstable long before the rotating He II reached a steady state [105, 184]. In fact so random is the upward motion of vortices, diffusion of vortex-lines - that is, short smoke ring like objects, but they are much longer in length than those of rotons - that the vorticity of the system has been evolved into an infinite number of patterns which can be described only by a homogeneous turbulence in He II [174]. Thus the experimental support for Feynman’s vortex line model [102] is doomed to failure; the unusual argument for a stable vortex line model leads to counterintuitive results that cannot still describe the shape of curvature of a rotating He II, since a vortex line is absolutely unstable to the $m = 1$ hose mode [105, 184].

Meservey’s [9] argument for the role of a surface free energy associated with velocity discontinuities suggested by Mott [13] is quite plausible, yet they cannot still explain why the curvature of the surface of rotating He II is temperature independent and the free surface does rotate. This is the principal difficulty of the vortex-nucleation model in a superfluid: at no point can one really exclude the second order phase-transition in order parameter, in view of the break-down of the phase coherence due to fluctuation-dissipation at the free surface.

Also, of course, it is precisely the nature of a superfluid that it cannot have a steady state under a gravitational field whether it is in motion or at rest [14, 115], but it can have a steady state with a free surface on which the two-fluid model breaks down as shown in section III. We have already seen in section IV that, in the dynamical study of the BEC in trap, from which we have discovered a shell-like structure of the BEC in an external field, it was necessary to introduce the chemical potential as an undetermined Lagrangian multiplier to maintain the equilibrium state of an imperfect Bose gas in an external potential [see Eq. (24) and Eq. (26)]. This study of the BEC in a trap has led us to study the spontaneously broken symmetry to explain the shell-like structure.
With the spontaneously broken symmetry, which takes place only locally at the free surface and is purely topological in nature, and Anderson’s $2\pi$ phase slippage of vortices at the free surface [14, 28, 182, 183], it is straightforward to derive the contour of a free surface of a rotating He II [122],

$$z = \frac{\Omega^2}{2g} r^2,$$

(93)

since no superfluid component is present at the surface layer [19, 23].

For the simplest dissipation case, let us consider a small bundle of $n$ vortices which are created at the bottom of rotating cylinder and move upward like micro-bubbles in a beer can. The bundle moves upward in a complicated manner, perhaps undergoing a few twists and reconnections [166] before finally being nucleated at the free surface by the phase slippage, imparting the angular momentum to the surface. This will lead to a $2\pi n$ phase slip and the energy dissipated at the surface will be $n$ times greater than the $2\pi$-event. Hence the surface rotates and reaches a steady state with the angular velocity $\Omega$ after awhile [68, 122]. Here the free surface being a normal fluid plays a role similar to that of a wall in Anderson’s model - that is, nucleation of vortices by a $2\pi n$ phase slippage [14, 28, 182, 183]. Hence the curvature must remain temperature independent, which is consistent with the extended two-fluid model of Landau [1, 13, 19, 58, 115].

It is also worth noting that the above picture of how a parabolic free surface is formed in a rotating He II is similar, in many respects, to that of formation of a vortex-ring by an ion complex in He II [161]. Thus $2\pi$ phase slip is indeed a basic concept that applies to a number of problems including the motion of a vortex in an orifice [14]. In accordance with the broken symmetry, not only should we able to resolve the recent controversy over the Magnus force [110], but also explain how the superfluidity breaks down at the core of a vortex line [106, 126].

With the detailed dynamical calculations in terms of the action of the mean-field defined in ODLRO [39, 68], we have been able to explain a number of problems that have been profoundly puzzling over the last half century [103], such as the curvature of a rotating He II and the breakdown of superfluidity at a vortex core in He II [8, 9, 161].

Before we close this discussion on the rotating He II, one more comment on a recent paper [187] is in order: based on the notion of a regauged space translation, Su and Suzuki have shown a new derivation of the quantization of a vortex-line, and have shown the similarity between the main characteristic of the surface layer of a rotating He II and the Meissner
effect in a superconductor [185, 186]. When a particle displacement is introduced into the Bose system, it invariably perturbs the particle density of the system. Moreover, the gauge field is a dynamical variable which must be analytic unless it is spontaneously broken. We therefore find it difficult to believe the idea of the regauged space translation [186, 187] is tenable. This idea has also led to the self-contradiction in their work as pointed out by Shi [188]. Moreover, there was a fundamental misunderstanding [185, 186] on the gauge invariant explanation of the Meissner effect [78, 79] and the BCS theory [55].

D. A Super-Lattice

Lastly we discuss a possible formation of a close-packed lattice as $a \to 0$ in BEC, which is equivalent to applying high pressure to an atomic gas in a trap. Our ground state wave function describes a degenerate Bose gas modified as little as possible by the presence of the repulsive pair-interaction [see Eq. (2)] [69, 191]. Hence the BEC in a trap provides an excellent test for this controversial problem [193]. Moreover, it is highly unlikely that, from our experience in a superconductor, a long range force could change the essential picture of our discussion in the absence of the electron-pair interactions (Cooper pairs) of electrons in the system [43, 56].

It may be worth emphasizing that Gorkov’s theory of superconductivity [57] demonstrated that both the energy-gap function $\Delta$ and the Ginzburg-Landau order parameter $\Psi$ vary as $e^{-2i\mu t/\hbar}$, where $\mu$ is the Fermi energy. Because of this time dependence, the Fermi level plays an important role in the phenomenology of superconductivity which is different from that of normal metals, and thus the energy gap function due to a virtual electron-electron interaction near the Fermi surface is essential to a superconducting state [54].

Nevertheless, a detailed analysis shows that, in the limit $a \to 0$ [see Eq. (50)] which is equivalent to applying huge compressional pressure to the BEC in a trap, we obtain the dispersion relation for a close-packed lattice with zero-point vibration from Eq. (50) [69]:

$$\omega_{latt} = \left(\frac{5}{2}\right)\omega_0.$$  

(94)

It is again independent of $\hbar$. This implies that if there is actually a transition from BEC to a crystalline ground state, then the ground state is a classical lattice. The above discussion explains to some extent why a simple model based on the imperfect Bose gas that
does not include long range interaction predicts a better logical structure with the classical close-packed lattice than a super-lattice.

The theory of broken symmetry is remarkably successful to give an unequivocal answer to this controversial question of a super-lattice [192, 193]. The physical mechanism by which the close-packed lattice can be realized is again the spontaneously broken symmetry that accompanies a longitudinal phonon as Nambu-Goldstone boson [89, 90]. Since the spontaneously broken symmetry is the consequence of superfluidity described by \( \psi \) in ODLRO, it is highly unlikely that we could observe a super-lattice in an laboratory experiment by compression, since we cannot find a physical mechanism by which we can transform He II into a superconductor [192, 193, 194, 195].

X. FUTURE PROSPECTS AND DISCUSSION

In this paper, we have developed a perturbation method appropriate to the determination of a phonon spectrum of collective excitations in a finite Bose system. The new perturbation method based on ODLRO and the particle orbit perturbations by the Lagrangian displacement vectors in Bohm’s quantum theory shows that a remarkably simple algebra yields not only a correct phonon spectrum of the sound wave but also the dispersion relations for surface waves in a finite space problem. A detailed analysis of the dispersion relations shows, furthermore, that a spontaneously broken symmetry takes place in a boundary layer - breakdown of the two-fluid model. Moreover, we are able to provide the direct cause of the Bohr-Sommerfeld quantization Eq. (86) by the spontaneously broken gauge symmetry. Finally we have shown that the characteristics of surface phenomena in He II in the gravitational field are similar to those of the Meissner effect in a superconductor in a weak external magnetic field.

It seems appropriate to point out in conclusion what was apparent at the outset - namely, that the conventional hydrodynamic perturbation methods to many-body Bose systems which have been employed by many authors in the past, are extremely primitive and are not applicable to the problems with boundary conditions [64]. On the other hand the quantum field theoretical approaches to the problems would face the same criticism; in particular the Bogoliubov-de Gennes equation should not be applied to a finite space problem, since the quantum field theory remains valid only in a Hilbert space [34, 63, 92, 126, 176].
A finite space problem is indeed unique in that a correct approach to the problem requires a new mathematical method. Here we have taken the entirely different approach to incorporate Feynman’s atomic picture of a phonon [33], namely the semi-classical perturbation method with the Lagrangian displacement vector to solve the many-boson problems. The Lagrangian displacement vector is basically the classical perturbation method that has been applied to particle orbits in Bohm’s interpretation of quantum mechanics [40] in ODLRO [38, 39], and yet the results are in the quantum mechanical domain so long as we maintain the phase-coherence and retain the effective quantum potential (EQMP). It is also remarkable to obtain the same dispersion relation Eq. (79) for the collective excitations in He II as Bogoliubov obtained in his theory of superfluidity using quantum field theoretical technique for a uniform He II [34]. In this paper I have described the essential mathematical techniques of calculating the coherent collective excitations in finite space problems and have shown how one can realize the spontaneously broken symmetry in the many-boson system to explain several puzzling experiments.

A few points that would bear a further discussion follow. The first is the accuracy on which Bohm’s interpretation of a quantum-mechanical system depends on a precisely definable phase-coherence by a mean field \( \psi \) that also defines a superfluid in ODLRO. Since the Goldstone theorem [78, 80, 84, 89] implies a zero-mass boson, there is no difficulty of identifying a phonon as a Nambu-Goldstone boson. However, a vortex creation by a broken symmetry accompanies a roton whose effective mass is \( 0.16m_{He} \) that can be treated as a pseudo-Goldstone boson. This large mass can be explained by the effective quantum mechanical potential by which a particle fluctuates to a higher energy (i.e., Bohm’s irreducible disturbance [40]), because the vortex creation involves a bulk of a large superfluid with higher energy and the radius of curvature of a free surface is also extremely small being in the order of many Å. Hence the EQMP [see Eq. (3) and Eq. (4)] in the quantum Hamilton-Jacobi equation plays a crucial role in explaining the spontaneously broken symmetry in He II. Moreover, it is consistent with the \( \sigma \)-model of Gell-Mann and Levy within the framework of quantum field theory [85, 89, 94].

The second, we have introduced, in the spirit of Feynman [33], the Lagrangian displacement vector which provides a new physical concept of phase-coherence in terms of the particle displacements. More importantly, in terms of the Lagrangian displacement vector, the boundary conditions have defined a boundary layer in which the two-fluid model of Landau...
breaks down. One can show that Eq. (3a) and Eq. (3b) with the boundary conditions give the second-order phase transition in He II under the gravitational field - the normal fluid joined a superfluid - just like the Meissner effect in a superconductor in an external magnetic field.

Moreover, the phase-coherent collective excitation we have studied is, in essence, to place Feynman’s picture of a phonon in precise mathematical formulas to obtain the excitation spectrum which turns out to be the same as the Bogoliubov spectrum [33, 34]. Hence we have established the equivalence between Feynman’s atomic theory of liquid helium near absolute zero [33] and Bogoliubov’s theory of superfluidity [34]. More importantly, we have derived the dispersion relation for a sound wave Eq. (53) without Feynman’s concept of a back-flow which has become an unnecessary conjecture because of the equation of continuity Eq. (3a).

If we are to take seriously the usual dynamical proof of BEC [6], we must study the dispersion relation Eq. (53) experimentally, which will be extremely difficult to carry out in the present geometrical configurations for the Bose-Einstein condensation in the trap. It should be emphasized that a typical wavelength of a sound wave $\lambda_{ph} = 2\pi/k_{ph} = (2\pi/\omega_{ph})(4\pi a n \hbar^2)^{1/2}/M$ with a small perturbation in BEC that one can drive is comparable to the size of BEC. Moreover, the speed of sound $c = [4\pi a \rho(r) \hbar^2]^{1/2}/M$ is extremely difficult to measure [129], because it depends on the local density and therefore depends on the wavelength. Its domain of access is so small and difficult (due to a peculiar geometrical configuration of confining magnets) in the device that, with all our experimental efforts, we have been able to probe a small fraction of a surface area with limited success [126, 130].

At the outset, the new BEC in a trap is doomed to failure, because the wavelength $\lambda_{ph} = 2\pi/k_{ph} = (2\pi/\omega_{ph})(4\pi a n \hbar^2)^{1/2}/M$ of a (first) sound is too long for the micron-scale cloud to carry out the dynamical study of collective excitations in BEC. [Recall the wave number $k_{ph} = p/\hbar = 0.25 \sim 2.5 \AA^{-1}$ (or $\lambda_{ph} = 0.06 \sim 2.51 \mu \text{m}$) used in the study of a phonon spectrum at $T = 1.12^9 \text{K}$ in He II]. It is indeed unfortunate that the atomic Bose condensate in a trap was once hailed as a Holy Grail in atomic physics (sic) [131] as an alternative to He II in testing the quantum many-body theory of an interacting Bose gas [64, 116], and yet one must now question for its scientific merit in the light of new developments in the theoretical front [68, 69, 70].

Unfortunately the circumstance in which the new Bose-Einstein condensation in a tap
was promoted and celebrated for its achievement suggests that the scheme is untenable in a healthy scientific community. In spite of Bogoliubov’s well-known theory of superfluidity [34], it is almost incredible that no attention has ever been called on to study the (first) sound wave propagation in BEC as a dynamical proof of the BEC in a trap. Although the two-fluid model of Landau [1] describes remarkably well the hydrodynamic properties of He II, it cannot, however, explain the problems with boundary conditions, since it is developed in a Hilbert space. On the experimental front, Henshaw and Wood have confirmed in great detail the Landau’s prediction on the shape of collective excitation spectrum (phonon and roton) in He II by means of the precise neutron-scattering technique [6]. It may not be worth while, however, to create a larger scale of a droplet of BEC for a study of the sound wave, since a strength of pair-interaction force will not alter the central features of Bose-condensation [33], nor will it modify London’s view on the λ-transition between He I and He II as the same process of the condensation of an ideal Bose-Einstein gas [29].

Experiments, of which there are many, are usually performed on a prolate ellipsoidal condensate in a trap. One of the few exceptions to the universal ellipsoidal condensate was a long cylindrical condensate in which the length of the condensate can be made much longer than the radius i.e., \( L \gg r \gg \lambda \), where \( \lambda \) is the wavelength of a sound wave, and which mimics a uniform condensate for a traveling sound wave. In an experiment on collective excitations, Steinhauer, et al., [129] have created such a cylindrically symmetric, long Bose condensate of \(^{87}\text{Rb}\) atoms that satisfies the above criteria and launched a traveling longitudinal sound wave along the axis of symmetry. But the effort to measure the speed of the sound wave was not successful, for they were not aware of the breakdown of the superfluidity at the end of the condensate. The difficulties in the measurements of the speed of sound waves are enormous. Thus their dispersion relations are qualitatively correct in some respects, but not in detail.

Given the present geometrical configuration of a trapping devise, it is almost impossible to precisely measure the speed of the (first) sound wave in any form of BEC in a trap, because, in addition to the breakdown of superfluidity on the free surface, the speed \( c = \left[ 4\pi \alpha r (\bar{\rho}) \hbar^2 \right]^{1/2} / M \) varies as the sound wave propagates toward to the free surface in a spatially inhomogeneous BEC [69, 129]. Moreover, the sound wave dissipates near the surface layer as discussed above, and thus one must take measurements deep inside of BEC where the boundary conditions are unimportant.
The basic idea described in this paper suggests several other lines of investigation involving the spontaneously broken symmetry in He II: a) there is, first, a natural extension of the present study on the collective excitations to a more general condensation in geometry in a trap [69, 70]; although the present work in a spherical and a cylindrical geometries describes the essential physics, it is too specific in practice and must be extended to a realistic model, a prolate ellipsoidal condensate that has the best possible chance for a future experiment provided the number of trapped particles increase by many times with the much larger dimension of condensation - that is in the order of \( \text{cm} \) [196]; b) second, the stability of a vortex line based on the broken symmetry should be carried out for a detailed study of the upward diffusion of vortices in a rotating He II [184].

As in Anderson’s argument on the singularities of Landau’s order parameter in connection with the energy-dissipation mechanism [77], the whole points of discussion here are essentially the emergent phenomena that follow from the spontaneously broken symmetry. It is also clear that, for this spontaneously broken symmetry to occur on the nodal surface, all that is required of the analyticity of the action \( S(x, t) \) of Bohm’s theory of quantum mechanics (Bohm’s irreducible disturbance). The current controversy [198, 199] over the broken gauge symmetry may be resolved simply by identifying the Goldstone boson in the analysis of collective excitations [65].

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FIG. 1: The ratio $\omega_{ph}/\omega_0 = [2\lambda_s]^{1/2}$ is plotted against the angular momentum $\ell$. It shows how the energy spectrum of phonons varies with the angular momentum valid in the phonon regime ($ck_\theta \ll 1$); and it does not increase without limit as Landau has shown, but must approach a value $\lambda$ transition point.