We derive simple formulae for the transmittance $T$ and reflectance $R$ of Gaussian-Schell beams incident upon any stratified dielectric structure by using second-order classical coherence theory in the space-frequency picture. The formalism is applied to a particular structure consisting of a double-layer, with balanced gain and loss, satisfying the parity-time symmetry conditions. It is shown that sources with a low degree of spatial coherence, on the order of the wavelength, can induce large resonant peaks in the transmitted and reflected amplitudes. The resonance peaks vanish as the spatial coherence increases.

Thirty years ago, W. Wang and E. Wolf addressed the problem of transmission and reflection properties of partially coherent beams through a general stratified medium [1]. The examples given by the authors relied on the space-frequency formalism of classical coherence theory and on the quasi-homogeneous approximation for the source field. Also, their definitions for "reflectance" and "transmittance" relied on the far-zone approximation for the transmitted and reflected fields. Experimental results confirming some aspects of the theory were published shortly after the formulation of the theoretical approach [2]. Unfortunately, their work did not receive much attention through the years despite the interesting results they obtained, such as the strong dependence of the reflection and transmission coefficients on the spatial coherence properties of the incident field. Spectral changes in the transmitted and reflected beams were also considered later [3, 4].

Since the seminal work of C. M. Bender and S. Boettcher on non-Hermitian Hamiltonians having parity-time symmetry [5], optics has been passing through a profound transformation regarding the fundamental properties of the interaction between radiation and matter [6, 7]. This was initially achieved due to an analogy existent between the paraxial wave equation of optics and the time-dependent Schrödinger equation. The complex, quantum, non-Hermitian Hamiltonian is mapped into a complex, classical, refractive index whose imaginary part is related to gain and loss properties of the material medium. Despite the large amount of new effects discovered in non-Hermitian optical settings, the role of classical optical coherence, and how it influences the interaction between radiation and non-Hermitian matter, has just begun to be explored in scattering systems [8–12]. It was found that deterministic non-Hermitian materials can drastically alter the directions [9] and the frequency spectrum of the scattered radiation, giving rise to a non-Hermitian Wolf effect [11]. A recent generalization of light scattering from random, parity-time-symmetric materials, has demonstrated that a new class of localized, statistically stationary materials, can be constructed from parity-time-symmetry concepts [12]. It now seems evident that the combination of classical coherence theory with non-Hermitian concepts give rise to many intriguing phenomena yet unexplored. With this in mind, the objective of this paper is to extend and put in a more modern language the problem of transmission and reflection properties of (spatially) partially coherent beams through dielectric layers containing gain and loss, satisfying the conditions of parity-time symmetry. Furthermore, we intend to observe how the spatial coherence properties of the incident beam influences the transmittance and reflectance of the structure.

Let us begin by considering a homogeneous, isotropic, linear and planar material, described by a piecewise constant complex refractive index $n(z) = n_G(z) + i n_I(z)$, positioned in the interval $0 \leq z \leq D$, embedded in vacuum. A stochastic incident scalar monochromatic light beam $\psi_l(x,z,\omega)$, arriving from vacuum ($z < 0$), strikes the material at $z = 0$. We assume that all field quantities are independent of the $y$ direction. As a consequence, two beams are originated outside the material: a reflected beam, $\psi_r(x,z,\omega)$, and a transmitted one, $\psi_t(x,z,\omega)$. We write each beam as a sum of plane waves (angular spectrum) and omit the frequency dependence on $\omega$ from now on,

$$
\psi_l(x,z) = \int_{-\infty}^{\infty} \psi_l(k_x) e^{ik_x x} e^{iz\sqrt{k^2 - k_0^2}} dk_x, \quad (z \leq 0),
$$

$$
\psi_r(x,z) = \int_{-\infty}^{\infty} \psi_r(k_x) e^{ik_x x} e^{-iz\sqrt{k^2 - k_0^2}} dk_x, \quad (z \leq 0),
$$

$$
\psi_t(x,z) = \int_{-\infty}^{\infty} \psi_t(k_x) e^{ik_x x} e^{iz\sqrt{k^2 - k_0^2}} dk_x, \quad (z \geq D),
$$

where $k = \omega/c$ is the wavenumber with $c$ being the speed of light in vacuum and $\psi_l(k_x)$ is the Fourier transform of $\psi_l(x,0)$:

$$
\psi_l(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_l(x,0) e^{-ik_x x} dx \quad (l = i, r, t).
$$

The continuity of the fields and their normal derivatives imposed at the interfaces $z = 0$ and $z = D$ connect all spectral
amplitudes. Here, our interest is in the transmitted spectral amplitude \( t(k_x) \) at wavevector \( k_x \), defined by
\[
t(k_x) = \frac{\psi_i(k_x)}{\psi_s(k_x)}. \tag{3}
\]

The transmitted beam can then be written as
\[
\psi_t(x, z) = \int_{-\infty}^{\infty} \psi_i(k_x) e^{i k_x^2 \sqrt{k^2 - k_x^2}} dk_x,
\]
and its statistical properties can be calculated from the cross-spectral density
\[
W_t(x_1, x_2, z_0) = \langle \psi_t^*(x_1, z_0) \psi_t(x_2, z_0) \rangle_\omega,
\tag{5}
\]
where \( W_t(x_1, x_2, z_0) = W_t(x, x, z_0) \) is the spectral density of the transmitted radiation and we define the transmittance \( T(z_0) \) at the plane \( z = z_0 \) as
\[
T(z_0) = \sqrt{2\pi} \int_{-\infty}^{\infty} S_t(x, z_0) dx, \tag{6}
\]
where \( S_t(x, 0) = W_t(x, x, 0) \) is the spectral density of the incident field at \( z = 0 \) and the \( 2\sqrt{\pi} \) factor is included to guarantee that \( T = 1 \) when \( |t(k_x)| = 1 \). Then, we have the following expression for the transmittance:
\[
T(z_0) = \sqrt{2\pi} \int_{-\infty}^{\infty} S_t(x, z_0) dx = \sqrt{2\pi} \int_{-\infty}^{\infty} S(t(k_x)) dx = \sqrt{2\pi} \int_{-\infty}^{\infty} S_t(x, z_0) dx.
\]

The transmitted beam can then be written as
\[
\psi_t(x, z) = \int_{-\infty}^{\infty} \psi_i(k_x) e^{i k_x^2 \sqrt{k^2 - k_x^2}} dk_x,
\]
and its statistical properties can be calculated from the cross-spectral density
\[
W_t(x_1, x_2, z_0) = \langle \psi_t^*(x_1, z_0) \psi_t(x_2, z_0) \rangle_\omega,
\tag{5}
\]
where \( W_t(x_1, x_2, z_0) = W_t(x, x, z_0) \) is the spectral density of the transmitted radiation and we define the transmittance \( T(z_0) \) at the plane \( z = z_0 \) as
\[
T(z_0) = \sqrt{2\pi} \int_{-\infty}^{\infty} S_t(x, z_0) dx, \tag{6}
\]
where \( S_t(x, 0) = W_t(x, x, 0) \) is the spectral density of the incident field at \( z = 0 \) and the \( 2\sqrt{\pi} \) factor is included to guarantee that \( T = 1 \) when \( |t(k_x)| = 1 \). Then, \( S_t(x_1, x_2, z_0) \) is well described by a Gaussian-Schell model, which is independent of \( z_0 \). To obtain \( T(z_0) \), \( S_t(x_1, x_2, z) \) was used and \( S_t(k_x) = W_t(k_x, k_x) \). We assume that the coherence properties of the incident beam, is well described by a Gaussian-Schell model,
\[
W_t(x_1, x_2, 0) = \sqrt{S_t(x_1, 0) S_t(x_2, 0) \mu_t(x_1 - x_2, 0), \tag{11}
\]

with
\[
S_t(x, 0) = S_0 \exp \left( -\frac{x^2}{\Delta^2} \right), \tag{12}
\]

The function \( \mu_t(x_1, x_2, 0) \) is the spectral degree of coherence of the incident beam, \( S_0 \) is a constant amplitude, \( \delta \) is related to the beam’s width at \( z = 0 \) and \( \Delta \) is the coherence length at \( z = 0 \). The usual monochromatic, spatially coherent, Gaussian beam is obtained directly from the above formulas in the limit \( \Delta \to \infty \), implying that all points located in the transverse plane are fully correlated. Direct substitution of Eq. (12) into Eq. (11) and then into Eq. (8) yields
\[
W_t(k_x, k_x) = \frac{S_0 \delta^2 \Delta}{(2\pi)^{3/2} \sqrt{\Delta^2 + 2\delta^2}} \exp \left( -\frac{\Delta^2 k_x^2}{2 + \Delta^2/\delta^2} \right)
\tag{13}
\]

In the limit \( \Delta \to \infty \), Eq. (13) becomes separable, as expected. The function \( S_t(k_x) \) is readily obtained from Eq. (13):
\[
S_t(k_x) = \frac{S_0 \delta^2 \Delta}{(2\pi)^{3/2} \sqrt{\Delta^2 + 2\delta^2}} \exp \left( -\frac{\Delta^2 k_x^2}{2 + \Delta^2/\delta^2} \right).
\tag{14}
\]

Then, Eq. (6) can be rewritten by using Eq. (10) and Eq. (14) as
\[
T = \frac{2\pi}{S_0^2} \int_{-\infty}^{\infty} S_t(k_x) |t(k_x)|^2 dk_x = \frac{\delta \Delta}{\sqrt{\pi} \sqrt{\Delta^2 + 2\delta^2}} \int_{-\infty}^{\infty} \exp \left( -\frac{\Delta^2 k_x^2}{2 + \Delta^2/\delta^2} \right) |t(k_x)|^2 dk_x.
\tag{15}
\]

The transmittance \( T \) expressed by Eq. (15) is the main result of our analysis. It is important to emphasize that it is generally valid, independent of the type of dielectric interfaces existent in the interval \([0, D]\). The only restriction imposed on Eq. (15) is that the incident field be a Gaussian-Schell beam described by Eq. (12). The reader can then verify that for \( |t(k_x)|^2 = 1 \), Eq. (15) reduces to \( T = 1 \), which justifies the inclusion of the factor \( \sqrt{2\pi} \) in the definition of \( T \) [Eq. (6)]. Thus, once the spectral amplitude \( t(k_x) \) for a specific system is known, a direct integration can be performed to obtain the transmittance properties of the dielectric.

The reflectance \( R \) can be defined in the same way as the transmittance through the definition of the reflected spectral amplitude
\[
r(k_x) = \frac{\psi_r(k_x)}{\psi_s(k_x)}.
\tag{16}
\]
By performing the same analysis as before, we obtain the formula
\[
R = \frac{\delta \Delta}{\sqrt{\pi} \Delta^2 + 2\delta^2} \int_{-\infty}^{\infty} \exp \left( -\frac{\Delta^2 k_x^2}{2 + \frac{\delta^2}{\Delta^2}} \right) |r(k_x)|^2 dk_x. \tag{17}
\]

For conservative (Hermitian) systems where \(|t(k_x)|^2 + |r(k_x)|^2 = 1\) is satisfied for every transverse wavevector \(k_x\), it is readily found that \(T + R = 1\), as expected.

We are now in a position to consider specific results of numerical simulations involving the transmission of a Gaussian-Schell beam through non-Hermitian structures. Our main task is to understand how the transmittance \([\text{Eq. (15)}]\), and the reflectance \([\text{Eq. (17)}]\), depend on the non-Hermitian parameter \(n_1\) as well as on the coherence length \(\Delta\) of the Gaussian-Schell beam. To this end, it only remains to obtain the transmitted and reflected spectral amplitudes \(t(k_x)\) and \(r(k_x)\). This is easily achieved by using the transfer matrix \(M\) approach for the system under consideration \([13]\). Suppose we find that the \(M\)-matrix has the form
\[
M(k_x) = \begin{bmatrix} M_{11}(k_x) & M_{12}(k_x) \\ M_{21}(k_x) & M_{22}(k_x) \end{bmatrix}, \tag{18}
\]

with \(M_{ij}\) being complex-valued entries. Then, the spectral amplitudes are given by \([13]\)
\[
t(k_x) = \frac{\det[M(k_x)]}{M_{22}(k_x)},
\]
\[
r(k_x) = -\frac{M_{21}(k_x)}{M_{22}(k_x)}, \tag{19}
\]
where \(\det[M(k_x)] = M_{11}(k_x)M_{22}(k_x) - M_{12}(k_x)M_{21}(k_x)\) is the determinant of the transfer matrix.

**Fig. 1.** A Gaussian-Schell beam (width \(\delta\) and coherence length \(\Delta\) [Eq. (12)]) incident upon a non-Hermitian double-layer structure of length \(2L\) having parity-time symmetry. Layer 1 \((2)\) has refractive index \(n_1 = n_R + in_l (n_2 = n_l^* )\). The spectral amplitudes \(\psi_i(k_x)\) \((i = i, r, l)\) of each plane wave with wavevector \(k_x\), outside the material, are also illustrated.

In order to apply the above formalism, let us consider a double-layer system having parity-time symmetry. By this we mean a dielectric layer with refractive index \(n_1 = n_R + in_l\), with \(n_R\) and \(n_l\) real-valued, occupying the region \([0, L]\) and another layer with refractive index \(n_2 = n_l^*\) occupying the region \([L, 2L]\). The total length of the system being \(2L\) and we assume both layers to be embedded in vacuum \((n = 1)\). See Figure 1 for an illustration of the geometry. These parity-time-symmetric double-layer structures have attracted a lot of attention due to its unusual response to optical fields. Effects such as unconventional lasing modes \([14]\), invisibility \([15, 16]\), complete transmission through epsilon-near-zero layers \([17]\) and nonlinear scattering \([18]\) have been considered recently.

For double-layer systems, the elements of the transfer matrix \(M\) are given explicitly by
\[
M_{11}(k_x) = \cos(k_x L) \cos(k_x L) - \frac{1}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \sin(k_x L) \sin(k_x L)
\]
\[
\quad + \frac{i}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \sin(k_x L) \cos(k_x L), \tag{20}
\]
\[
M_{12}(k_x) = -\frac{1}{2} \left( \frac{k_1}{k_2} - \frac{k_2}{k_1} \right) \sin(k_x L) \sin(k_x L)
\]
\[
\quad + \frac{i}{2} \left( \frac{k_1}{k_2} - \frac{k_2}{k_1} \right) \sin(k_x L) \cos(k_x L), \tag{21}
\]
\[
M_{21}(k_x) = -\frac{1}{2} \left( \frac{k_1}{k_2} - \frac{k_2}{k_1} \right) \sin(k_x L) \sin(k_x L)
\]
\[
\quad + \frac{i}{2} \left( \frac{k_1}{k_2} - \frac{k_2}{k_1} \right) \sin(k_x L) \cos(k_x L), \tag{22}
\]
\[
M_{22}(k_x) = \cos(k_x L) \cos(k_x L) - \frac{1}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \sin(k_x L) \sin(k_x L)
\]
\[
\quad + \frac{i}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \sin(k_x L) \cos(k_x L), \tag{23}
\]
where \(k_x = \sqrt{1 - k_2^2}, k_1 = \sqrt{n_1^2 k_2^2 - k_2^2}\) and \(k_2 = \sqrt{n_2^2 k_2^2 - k_2^2}\). We chose to include the elements of \(M\) here since they are not easy to find (at least in this form) but can be derived by using appropriate boundary conditions at all interfaces. Without losing generality, we keep \(n_R = 1.5, kL = 10\) and \(k\delta = 10\) fixed in what follows and investigate how \(R\) and \(T\) depend on the parameters \(\Delta\) and \(n_l\) since they are the most relevant ones to our analysis. Figure 2(A) shows a density plot of \(T(n_l, k\Delta)\) for \(n_l \in [-0.2, 0.2]\) and \(k\Delta \in [1, 6]\). It is readily seen from this plot that there are two strong resonant transmission peaks (with \(T \sim 10\)) located around \(n_l \approx \pm 0.151\) that vanish as \(k\Delta\) increases (large spatial coherence). The black lines in Figure 3 show the cross-section of \(T\) for \(k\Delta = 1\) (continuous) and \(k\Delta = 6\) (dashed). One can verify from these plots that the transmitted peaks are only evident in the low coherence regime of the source, more specifically, for sources whose coherence length is of the order of the wavelength.
Curiously, the transmittance is symmetric with respect to the inversion round \( n_I = 0 \).

![Transmittance T and reflectance R obtained from Eq. (15) and Eq. (17)](image)

The transmittance displays a more asymmetric character, as can be visualized in part (B) of Fig. 2. The scale is logarithm for better visualization since the peak located at \( n_I = 0.151 \) is much higher than the one located at 0.151. This asymmetric effect in the reflectance is characteristic of non-Hermitian systems. Notice that exchanging the imaginary parts of both, \( n_1 \) and \( n_2 \), is equivalent to the assumption that the incident field is striking the first interface from the other side of the structure. Therefore, the formalism clearly contains non-Hermitian features. The red lines shown in Figure 3 are cross-sections of Figure 2(B) at \( k\Delta = 1 \) (continuous) and \( k\Delta = 6 \) (dashed) as a function of \( n_I \). The intense peaks at \( n_I = \pm 0.151 \) are obtained in the low coherence regime and they disappear as soon as the coherence length \( \Delta \) increases as it is clearly depicted in Figure 3. Therefore we reach the counter-intuitive conclusion that poorly coherent sources induce coherent phenomena such as resonance.

In closing, we remark that the formalism developed here can be easily generalized to \((2+1)\) dimensions, meaning that one can be used directly for non-Gaussian-Schell beams once \( S_0(k_z) \) is known. All these final considerations are currently under investigation and will be published elsewhere.

**Fig. 3.** Transmittance \( T \) and reflectance \( R \) obtained from Eq. (15) and Eq. (17), respectively, as a function of the imaginary part of the complex refraction index, \( n_I \), for two distinct values of the coherence length \( k\Delta \). Parameters used: \( k\delta = 10, kL = 10 \) and \( n_R = 1.5 \). For better visualization, the reflectance axis is written in logarithm scale.

**DISCLOSURES**

The authors declare no conflicts of interest.

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