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Coefficient Estimates for a Subclass of Meromorphic Multivalent $q$-Close-to-Convex Functions

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Abstract: By making use of the concept of basic (or $q$-) calculus, many subclasses of analytic and symmetric $q$-starlike functions have been defined and studied from different viewpoints and perspectives. In this article, we introduce a new class of meromorphic multivalent close-to-convex functions with the help of a $q$-differential operator. Furthermore, we investigate some useful properties such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikeness, and radius of convexity for this new subclass.

Keywords: meromorphic functions; Janowski functions; $q$-calculus; $q$-differential operator

1. Introduction, Definitions and Motivation

Calculus without notion of limits is called $q$-calculus or quantum calculus. This relatively new and advanced field of study became a core of attractions for many well-known mathematicians and physicists due to its various applications in applied mathematics, physics and various engineering areas, see details in [1,2]. The start of this field can be connected with the research of Jackson [3,4], who gave various applications of $q$-calculus and introduced the $q$-analogue of derivative and integral, while the most recent and trending work can be viewed in the study of Aral and Gupta [1,2,5]. Later, Aral and Anastassiou [6–9] gave the $q$-generalization of complex operators, which are known as $q$-Picard and $q$-Gauss–Weierstrass singular integral operators.

In Geometric Function Theory, the role of the $q$-Deference operator is quite significant. Historically speaking, it was Srivastava [10] who used the basic (or $q$-) hypergeometric functions:

$$\nabla \Phi_j \ (\nabla, j \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} = \mathbb{N} \cup \{0\})$$

in Geometric Function Theory (GFT) of Complex Analysis (see, for details, [10]). Very recently, Srivastava’s published review article [11] gave another flavor to this subject. In their published review article [11], Srivastava highlighted the triviality of the so-called $(p, q)$-calculus.

The aforementioned works of Srivastava [10,11] motivated a number of mathematicians to give their findings. In the above-cited work by Srivastava [11], many well-known convolution and fractional $q$-operators were surveyed. For example, in their published
article [12], by use of the concept of convolution, a \( q \)-analogue of Ruscheweyh differential operator was defined by Kanas and Răducanu. More applications of this operator can be seen in the paper [13]. Very recently, by using \( q \)-Differential operator, Srivastava et al. [14] studied a certain subclass of analytic function with symmetric points. Several other authors (see, for example, [15–22]) have studied and generalized the classes of symmetric and other \( q \)-starlike functions from different viewpoints and perspectives. For some more recent investigation about \( q \)-calculus, we may refer the interested reader to [23, 24]. In this article, we introduce a new family of meromorphic multivalent functions associated with Janowski domain using a differential operator and study some of its properties.

By the notation \( A_p \), we denote the family of all meromorphic multivalent functions that are analytic and invariant (or symmetric) under rotations, in the punctured disc \( D = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and with the following normalization conditions

\[
f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n} z^{n+p}, \quad (z \in D).
\]

(1)

Furthermore, a function \( f \) is said to be in the class \( MS_p^s(\alpha) \) \((0 \leq \alpha < 1)\) of meromorphic \( p \)-valent starlike functions, if

\[
f(z) \in MS_p^s(\alpha) \iff \Re \left( \frac{zf'(z)}{pf(z)} \right) < -\alpha.
\]

Next, a function \( f \) is said to be in the class \( MC_p^s(\alpha) \) \((0 \leq \alpha < 1)\) of meromorphic \( p \)-valent convex functions, if

\[
f(z) \in MC_p^s(\alpha) \iff \Re \left( \frac{(zf'(z))'}{pf'(z)} \right) < -\alpha
\]

Clearly, we see that \( MS_p^s(0) = MS_p^s \), the class of meromorphic \( p \)-valent starlike functions. Similarly \( MK_p^s(\alpha) \) denote the class of meromorphic \( p \)-valent close-to-convex functions and defined as

\[
f(z) \in MK_p^s(\alpha) \iff \Re \left( \frac{zf'(z)}{pg(z)} \right) < -\alpha
\]

where \( g(z) \in MS_p^s \).

The \( q \)-derivative (or \( q \)-difference) operator of a function \( f \), where \( 0 < q < 1 \), is defined by

\[
\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad (z \neq 0).
\]

(2)

It can easily be seen that

\[
\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (n \in \mathbb{N}, \ z \in D)
\]

(3)

where

\[
[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n} q^l, \text{ and } [0, q] = 0.
\]

For any non-negative integer \( n \), the \( q \)-number shift factorial is defined by

\[
[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q][2, q][3, q] \cdots [n, q], & n \in \mathbb{N} \end{cases}
\]
Furthermore, the $q$-generalized Pochhammer symbol for $x \in \mathbb{R}$ is given by

$$[x, q]_n = \begin{cases} 1, & n = 0, \\ [x, q][x + 1, q] \cdots [x + n - 1, q], & n \in \mathbb{N}, \end{cases}$$

The differential operator $D_{\mu, q} : \mathcal{A}_p \rightarrow \mathcal{A}_p$ is defined in [25] as

$$D_{\mu, q}f(z) = (1 + [p, q]_\mu) f(z) + \mu q^n z \partial_q f(z).$$

where $\mu \geq 0$.

Now, using Equation (1), one can easily find that

$$D_{\mu, q}f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(1 + [p, q]_\mu + \mu q^n[n, q]\right) a_{p+n} z^{p+n}.$$

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4. If we take 
\[ A = 1 = p, \quad B = -1 \quad \text{and} \quad m = 0 \]
and let \( q \to 1^- \), we have \( \mathcal{MK}^* \), where \( \mathcal{MK}^* \), denotes the class of meromorphic close-to-convex functions.

The following equivalent condition for a function \( f \in \mathcal{MK}_{\mu,q}(p, m, A, B) \) can be verified easily
\[
\left. \begin{array}{c}
\frac{q^p z^q z \partial_q D_{\mu,q}^m f(z)}{|p,q|^2 g(z)} + 1 \\
A + B \frac{q^p z^q z \partial_q D_{\mu,q}^m f(z)}{|p,q|^2 g(z)}
\end{array} \right| < 1.
\] (7)

For our main result, we need the following important result.

**Lemma 1** ([32]). An analytic function \( h(z) \) having series representation
\[
h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n,
\]
and another function \( k(z) \) with the following series representation
\[
k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n.
\]
If \( h(z) \prec k(z) \), then \( |d_n| \leq |k_1|, \) for \( n \in \mathbb{N} \).

2. A Set of Main Results

In this section, we give our main results.

**Theorem 1.** If a function \( f \in A_p \) has a series representation given by Equation (1), then
\( f \in \mathcal{MK}_{\mu,q}(p, m, A, B) \) if it satisfies the following inequality
\[
\sum_{n=1}^{\infty} \left( q^p [p + n, q] (1 + B) [1 + [p, q] \mu + \mu q^p [p + n, q])^m |a_{p+n}| + \frac{2 |p,q| (1 + A)}{p+n} \right) 
\leq \ [p, q] (A - B).
\] (8)

it holds true.

**Proof.** For \( f \) to be in the class \( \mathcal{MK}_{\mu,q}(p, m, A, B) \), we only need to prove the inequality (7).
For this, we consider
\[
\left. \begin{array}{c}
\frac{q^p z^q z \partial_q D_{\mu,q}^m f(z)}{|p,q|^2 g(z)} + 1 \\
A + B \frac{q^p z^q z \partial_q D_{\mu,q}^m f(z)}{|p,q|^2 g(z)}
\end{array} \right| = \frac{q^p z^q z \partial_q D_{\mu,q}^m f(z) + |p,q| g(z)}{|a_{p,q}| g(z) + B q^p z^q z \partial_q D_{\mu,q}^m f(z)}
\]

If the series representation of \( g(z) \) is given by
\[
g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad (z \in \mathbb{D}),
\]
Now, by using Equation (4), and then with the help of Equations (2) and (5), we find, after some simplification, that the above is equal to

\[
\frac{\sum_{n=1}^{\infty} (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] \mu |a_{p+n \mu}| + \sum_{n=1}^{\infty} |p| |a_{p+n \mu}|} {A-B |p| |a_{p+n \mu}| + \sum_{n=1}^{\infty} A |p| |a_{p+n \mu}| + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|} + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|}
\]

\[
\leq \frac{\sum_{n=1}^{\infty} (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] \mu |a_{p+n \mu}| + \sum_{n=1}^{\infty} |p| |a_{p+n \mu}|} {A-B |p| |a_{p+n \mu}| + \sum_{n=1}^{\infty} A |p| |a_{p+n \mu}| + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|} + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|}
\]

Now, for \( g(z) \in M \mathcal{S}_p \), we have

\[ |b_{p+n}| = \frac{2p}{p+n}, \]  

and using the inequality (11) we find that the above is less than 1. The direct part of the proof of our theorem is now completed.

Conversely, let \( f \in \mathcal{MK}_{\mu,q}(p,m,A,B) \) and is given by Equation (1), then from Equation (7), we have for \( z \in \mathbb{D} \),

\[ \frac{q^m D_{p,q}^m f(z)}{D_{p,q}^m f(z)} = \frac{\sum_{n=1}^{\infty} (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] \mu |a_{p+n \mu}| + \sum_{n=1}^{\infty} |p| |a_{p+n \mu}|} {A-B |p| |a_{p+n \mu}| + \sum_{n=1}^{\infty} A |p| |a_{p+n \mu}| + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|} + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|}
\]

Since \(|\Re(z)| \leq |z|\), we have

\[ \Re \left\{ \frac{\sum_{n=1}^{\infty} (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] \mu |a_{p+n \mu}| + \sum_{n=1}^{\infty} |p| |a_{p+n \mu}|} {A-B |p| |a_{p+n \mu}| + \sum_{n=1}^{\infty} A |p| |a_{p+n \mu}| + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|} + B \sum_{n=1}^{\infty} q^n (1 + |p| \mu + \mu^q [p + n q]) q^n [p + n q] |a_{p+n \mu}|} \right\} < 1
\]

Now, choose values of \( z \) on the real axis so that \( \frac{q^m D_{p,q}^m f(z)}{D_{p,q}^m f(z)} \) is real. Upon clearing the denominator in Equation (10) and letting \( z \rightarrow 1^+ \) through real values, we obtain Equation (11). \( \square \)

For an analytic case, the result is as follows:

**Corollary 1.** If a function \( f \in \mathcal{A} \), then \( f \in \mathcal{MK}_{\mu,q}(1,m,A,B) \) if it satisfies the following inequality

\[
\sum_{n=1}^{\infty} \left( q^n [n+1,q](1+B)(1+\mu+\mu^q[n+1,q])^m |a_{n+1}| + \frac{2(1+A)}{n+1} \right) \leq (A-B).
\]

is holds true.

In the next theorem, we obtain the coefficient bounds of the functions of this class.

**Theorem 2.** Let \( f \in \mathcal{MK}_{\mu,q}(p,m,A,B) \) and be of the form (1). Then

\[ |a_{p+n}| \leq \frac{2p |p| q}{I(n) |p| q + (A-B) \sum_{i=0}^{n-1} \frac{1}{p+i}} \]

where

\[ I(n) = (1 + [p,q] \mu + \mu^q [p + n q])^m. \]
Proof. If \( f \in A \) and is in the class \( \mathcal{M}_\mu\alpha(p, m, A, B) \), then it satisfies
\[
-q^p z \partial_z D^m_{\mu, \alpha} f(z) < \frac{1 + Az}{1 + Bz},
\]
with the help of Lemma 1 we obtain the required result. We now put the series expansions of \( h \), which implies that
\[
h(z) = -q^p z \partial_z D^m_{\mu, \alpha} f(z) \frac{1 + Az}{1 + Bz},
\]
then it will have the representation
\[
h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.
\]
which implies that
\[
h(z) < \frac{1 + Az}{1 + Bz}.
\]
However, by simple calculations we get
\[
\frac{1 + Az}{1 + Bz} = 1 + (A - B)z + ....
\]
with the help of Lemma 1 we obtain
\[
|d_n| \leq (A - B)
\]
we now put the series expansions of \( h(z), g(z) \) and \( f(z) \) in Equation (13), simplifying and comparing the coefficients of \( z^n + p \) on both sides
\[
-q^p (1 + [p, q] \mu + \mu q/[p + n, q])^m [p + n, q] a_{p+n} = [p, q] b_{p+n}
\]
\[
+ [p, q] \sum_{i=0}^{n-1} b_{p+i} d_{n-i}
\]
Taking absolute on both sides of this equation and using the triangle inequality with the help of Equation (14), we obtain
\[
(1 + [p, q] \mu + \mu q/[p + n, q])^m q^p [p + n, q] |a_{p+n}| \leq [p, q] |b_{p+n}| + [p, q](A - B) \sum_{i=0}^{n-1} |b_{p+i}|
\]
Now, using Equations (9) and (15), we find that
\[
(1 + [p, q] \mu + \mu q/[p + n, q])^m q^p [p + n, q] |a_{p+n}| \leq [p, q] \frac{2p}{p+n} + [p, q](A - B) \sum_{i=0}^{n-1} \frac{2p}{p+i}
\]
\[
\Rightarrow |a_{p+n}| \leq \frac{2p[p,q]}{l(n)|p+n,q|} \left( \frac{1}{n+1} + (A - B) \sum_{i=0}^{n-1} \frac{1}{p+i} \right)
\]
we obtain the required result. \( \Box \)

Corollary 2. If \( f \in \mathcal{M}_\mu\alpha(1, m, A, B) \) and is of the form (1). Then,
\[
|a_{n+1}| \leq \frac{2}{l(n)|n+1,q|} \left( \frac{1}{n+1} + (A - B) \sum_{i=0}^{n-1} \frac{1}{p+i} \right)
\]
where
\[
l(n) = (1 + \mu + \mu q[n + 1, q])^m.
\]

Next, for our defined functions class, we give results related to growth and distortion.
Theorem 3. If the function $f \in \mathcal{MK}_{p,q}(p,m,A,B)$. Then, for $|z| = r$, we have
\[ \frac{1}{r^p} - \Phi r^p \leq |f(z)| \leq \frac{1}{r^p} + \Phi r^p, \]
where
\[ \Phi = \frac{(p+1)[p,q](A-B)-2p[p,q](1+A)}{q^p(p+1)[p+1,q](1+B)(1+p,q)p+p[p+1,q])}. \]

Proof. Since
\[ |f(z)| = \left| \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \right|, \]
we have
\[ \leq \frac{1}{|z|^p} + \sum_{n=1}^{\infty} |a_{p+n}| |z|^{p+n} \]
\[ = \frac{1}{r^p} + \sum_{n=1}^{\infty} |a_{p+n}| r^{p+n}. \]
Since for $|z| = r < 1$ we have $r^{p+n} < r^p$ and
\[ |f(z)| \leq \frac{1}{r^p} + r^p \sum_{n=1}^{\infty} |a_{p+n}| \]
(16)
Similarly,
\[ |f(z)| \geq \frac{1}{r^p} - r^p \sum_{n=1}^{\infty} |a_{p+n}| \]
(17)
As by Equation (11), we know that
\[ \sum_{n=1}^{\infty} \left( q^p[p+n,q](1+B)(1+[p,q]\mu + \mu q^p[p+n,q])^m|a_{p+n}| + \frac{2p[p,q](1+A)}{p+n} \right) \]
\[ \leq [p,q](A-B). \]
However,
\[ \frac{2p[p,q](1+A)}{p+n} + q^p[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=1}^{\infty} |a_{p+n}| \]
\[ \leq \sum_{n=1}^{\infty} \left( q^p[p+n,q](1+B)(1+[p,q]\mu + \mu q^p[p+n,q])^m|a_{p+n}| + \frac{2p[p,q](1+A)}{p+n} \right). \]
Hence,
\[ \frac{2p[p,q](1+A)}{p+n} + q^p[p+1,q](1+B)(1+[p,q]\mu + \mu q^p[p+1,q])^m \sum_{n=1}^{\infty} |a_{p+n}| \]
\[ \leq [p,q](A-B), \]
which can be written as
\[ \sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{(p+1)[p,q](A-B)-2p[p,q](1+A)}{q^p(p+1)[p+1,q](1+B)(1+p,q)p+p[p+1,q])}. \]
(18)
Now, the required result can easily be obtained if we make use of Equations (16)–(18). \(\square \)

Corollary 3. For a function $f \in \mathcal{MK}_{p,q}(p,1,A,B)$. Then, for $|z| = r$, we have
\[ \frac{1}{r^p} - \Phi r^p \leq |f(z)| \leq \frac{1}{r^p} + \Phi r^p, \]
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where

\[
\Phi = \frac{(p+1)\mu q((A-B)-2p|\mu|q(A+1))}{q^{-m}(1+B)(1+|\mu|q+q^m(1+L_A)^{-1})}.
\]

**Theorem 4.** If \( f \in \mathcal{MK}_{p,q}(p,m,A,B) \) and is of the form (1). Then, for \(|z| = r\)

\[
\|p,q\|_m^m m^m + m^m - \Psi r^p \leq |D^m_q f(z)| \leq \|p,q\|_m^m m^m + m^m + \Psi r^p.
\]

where

\[
\Psi = \frac{(A-B)|\mu|q((p+1)-2p|\mu|q(A+1))}{(p+1)q^m(1+B)(1+|\mu|q+q^m(1+L_A)^{-1})}, \quad \zeta = \sum_{n=1}^m n.
\]

**Proof.** With the help of Equation (2) and using Equation (3), we can easily find that

\[
D^m_q f(z) = \frac{(-1)^m |p,q|_m^m m^m + m^m}{q^{-m}(1+B)|\mu|+q^m(1+L_A)^{-1}} \sum_{n=1}^m [p+n-(m-1),q]a_{p+n}z^{p+n-m}.
\]

Since, for the parameters \( m \leq n, n \geq p+1 \) and also for \(|z| = r < 1\) implies that \( r^{p-n} \leq r^p \) hence, we have

\[
|D^m_q f(z)| \leq \|p,q\|_m^m m^m + m^m + r^p \sum_{n=1}^m [p+n-(m-1),q]a_{p+n}, \quad (19)
\]

Similarly,

\[
|D^m_q f(z)| \geq \|p,q\|_m^m m^m + m^m - r^p \sum_{n=1}^m [p+n-(m-1),q]a_{p+n}. \quad (20)
\]

Now, by using Equation (11), we obtain the following inequality

\[
\left(\frac{2p|\mu|q((A+1)^m)}{p+1} + q^m(1+B)(1+[p,q]u + q^m[p+n,q])\sum_{n=1}^m [p+n,q]a_{p+n}\right)
\]

\[
\leq [p,q]((A-B).
\]

which implies that

\[
q^m(1+B)(1+[p,q]u + q^m[p+n,q])\sum_{n=1}^m [p+n,q]a_{p+n}
\]

\[
\leq \frac{(A-B)|\mu|q((p+1)-2p|\mu|q(A+1))}{(p+1)q^m(1+B)(1+|\mu|q+q^m(1+L_A)^{-1})},
\]

so we have

\[
\sum_{n=1}^m [p+n,q]a_{p+n} \leq \frac{(A-B)|\mu|q((p+1)-2p|\mu|q(A+1))}{(p+1)q^m(1+B)(1+|\mu|q+q^m(1+L_A)^{-1})},
\]

but it can easily be seen that

\[
\sum_{n=1}^m [p+n-(m-1),q]a_{p+n} \leq \sum_{n=1}^m [p+n,q]a_{p+n},
\]

which implies

\[
\sum_{n=1}^m [p+n-(m-1),q]a_{p+n} \leq \frac{(A-B)|\mu|q((p+1)-2p|\mu|q(A+1))}{(p+1)q^m(1+B)(1+|\mu|q+q^m(1+L_A)^{-1})}.
\]

Now, using the last inequality in Equations (19) and (20), we can obtain our result.
Corollary 4. If \( f \in \mathcal{MK}_{p,q}(p,1,A,B) \) and is of the form (1). Then, for \(|z| = r\)
\[
\frac{[p,q]}{q^p+\varepsilon r^{p+1}} - \Psi r^p \leq |\partial_q f(z)| \leq \frac{[p,q]}{q^p+\varepsilon r^{p+1}} + \Psi r^p.
\]
where
\[
\Psi = \frac{(A-B)[p,q][p+1]-2[p,q][1+A]}{(p+1)[A-B](1+A)[p,q][\mu+\mu q^p[p+n,q]]}
\]
and \( \zeta = \frac{u(n+1)}{2} \).

In the next two theorems for the functions class \( \mathcal{M}_{p,q}(p,m,A,B) \), the radii of starlikeness and convexity is given.

Theorem 5. Let \( f \in \mathcal{MK}_{p,q}(p,m,A,B) \). Then, \( f \in \mathcal{MC}_{p}(\alpha) \) for \(|z| < r_1\), where
\[
r_1 = \left( \frac{p(p-a)q^p[p+1][p+n,q][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} \right)^{\frac{1}{p+2}}.
\]

Proof. Let \( f \) be in the class \( \mathcal{MK}_{p,q}(p,m,A,B) \), to prove that \( f \) is in the class \( \mathcal{MC}_{p}(\alpha) \), it will be enough if we show
\[
\left| \frac{zf''(z) + (p+1)f'(z)}{zf''(z) + (1+2\alpha - p)f'(z)} \right| < 1,
\]
Using Equation (1) in conjunction with some elementary calculations, we have
\[
\sum_{n=1}^{\infty} \frac{\sum_{n=1}^{\infty} q^p[p+n,q](1+B)(1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} |a_{n+p}| |z|^{n+2p} < 1,
\]
which gives, after some simplification, that
\[
\sum_{n=1}^{\infty} \frac{q^p[p+1][p+n,q][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} |a_{n+p}| < 1.
\]
Now, inequality (21) will be true, if the following holds
\[
\sum_{n=1}^{\infty} \frac{\sum_{n=1}^{\infty} q^p[p+1][p+n,q][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} |a_{n+p}| < 1.
\]
which implies that
\[
|z|^{n+2p} < \left( \frac{p(p-a)q^p[p+1][p+n,q][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} \right)^{\frac{1}{p+2}},
\]
and so
\[
|z| < \left( \frac{p(p-a)q^p[p+1][p+n,q][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m}{p[p-a]q^p[p+1][1+B](1+[p,q]\mu+\mu q^p[p+n,q])^m} \right)^{\frac{1}{p+2}} = r_1,
\]
we get the required condition. \( \square \)
Corollary 5. Let \( f \in \mathcal{MK}_{p,q}(1,m,A,B) \). Then, \( f \in \mathcal{MC}_p(\alpha) \) for \( |z| < r_1 \), where
\[
r_1 = \left( \frac{1}{(n+1)(n+1+A)(A-B)+2(1+A)} \right)^{\frac{1}{2}} .
\]

Theorem 6. Let \( f \in \mathcal{MK}_{p,q}(p,m,A,B) \). Then, \( f \in \mathcal{MS}_p^\alpha(\alpha) \) for \( |z| < r_2 \), where
\[
r_2 = \left( \frac{1}{(n+1)(n+1+A)(A-B)+2(1+A)} \right)^{\frac{1}{2}} .
\]

Proof. We know that \( f \) is in the class \( \mathcal{MS}_p^\alpha(\alpha) \), if and only if
\[
\left| \frac{zf''(z) + pf(z)}{zf''(z) - (p - 2\alpha)f(z)} \right| < 1.
\]
Now, make use of Equation (1) and, after some simple calculations, we have
\[
\sum_{n=1}^{\infty} \left( \frac{n + p + \alpha}{p - \alpha} \right) |a_{n+p}| |z|^{n+2p} < 1.
\]
(22)

Now, from Equation (11) we can easily obtain
\[
\sum_{n=1}^{\infty} \frac{q^n(p+1)|p+n,q|(1+B)(1+|p,q|\mu + \mu q^p|p+n,q|)^m}{|p,q|(A-B)(p+1)-2p|p,q|(1+A)} |a_n| < 1.
\]
Inequality (22) will be true if
\[
\sum_{n=1}^{\infty} \left( \frac{n + p + \alpha}{p - \alpha} \right) |a_{n+p}| |z|^{n+2p} < \sum_{n=1}^{\infty} \frac{q^n(p+1)|p+n,q|(1+B)(1+|p,q|\mu + \mu q^p|p+n,q|)^m}{|p,q|(A-B)(p+1)-2p|p,q|(1+A)} |a_n| .
\]
This gives
\[
|z|^{n+2p} < \left( \frac{(p-a)q^n(p+1)|p+n,q|(1+B)(1+|p,q|\mu + \mu q^p|p+n,q|)^m}{(n+p+\alpha)(p+1)-2p|p,q|(1+A)} \right)^{\frac{1}{2}} ,
\]
and hence
\[
|z| < \left( \frac{(p-a)(q^n(p+1)|p+n,q|(1+B)(1+|p,q|\mu + \mu q^p|p+n,q|)^m}{(n+p+\alpha)(A-B)} \right)^{\frac{1}{2}} = r_2 .
\]
The proof of our Theorem is now complete. \( \square \)

Corollary 6. Let \( f \in \mathcal{MK}_{p,q}(1,m,A,B) \). Then \( f \in \mathcal{MS}^\alpha(\alpha) \) for \( |z| < r_2 \), where
\[
r_2 = \left( \frac{1}{(n+1)(n+1+A)(A-B)} \right)^{\frac{1}{2}} .
\]

3. Concluding Remarks and Observations

Recently, the basic (or \( q \)-) calculus is a center of attraction for many well-known mathematicians, because of its diverse applications in many areas of Mathematics and Physics see for example [11,33]. In our present investigations, we were essentially motivated by the recent research going on in this field of study, and we have first introduced a new class of meromorphic multivalent \( q \)-close-to-convex function with the help of a \( q \)-differential operator. We next investigate some useful properties such as sufficiency criteria, coefficient estimates, distortion theorem, growth theorem, radius of starlikeness and radius of convexity for this new subclass of meromorphic multivalent \( q \)-close-to-convex functions.
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