Minisuperspaces: Symmetries and Quantization

Abhay Ashtekar * Ranjeet Tate † Claes Uggla ‡

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Abstract

In several of the class A Bianchi models, minisuperspaces admit symmetries. It is pointed out that they can be used effectively to complete the Dirac quantization program. The resulting quantum theory provides a useful platform to investigate a number of conceptual and technical problems of quantum gravity.

1. Introduction

Minisuperspaces are useful toy models for canonical quantum gravity because they capture many of the essential features of general relativity and are at the same time free of the technical difficulties associated with the presence of an infinite number of degrees of freedom. This fact was recognized by Charlie Misner quite early and, under his leadership, a number of insightful contributions were made by the Maryland group in the sixties and the seventies. Charlie’s own papers are so thorough and deep that they have become classics; one can trace back to them so many of the significant ideas in this area. Indeed, it is a frequent occurrence that a beginner in the field gets excited by a new idea that seems beautiful and subtle only to find out later that Charlie was well aware of it. It is therefore a pleasure and a privilege to contribute an article on minisuperspaces to this Festschrift –of course, Charlie himself may already know all our results!

*Physics Department, Syracuse University, Syracuse, NY 13244-1130, USA and Inter-University Center for Astronomy and Astrophysics, Pune, 411017, India. This author was supported in part by NSF grant PHY90-16733, by a travel grant from the United Nations Development Office and by research funds provided by Syracuse University.

†Physics Department, University of California, Santa Barbara, CA 93106-9530, USA. This author was supported in part by NSF grant PHY90-16733 and by research funds provided by Syracuse University.

‡Physics Department, Syracuse University, Syracuse, NY 13244-1130, USA and Department of Physics, University of Stockholm, Vanadisvägen 9, S-113 46 Stockholm, Sweden. This author was supported by a grant from the Swedish National Science Research Council and by research funds provided by Syracuse University.
In this paper we shall use the minisuperspaces associated with Bianchi models to illustrate some techniques that can be used in the quantization of constrained systems—including general relativity—and to point out some of the pitfalls involved. We will carry out canonical quantization and impose quantum constraints to select the physical states *a la* Dirac. However, as is by now well-known, the Dirac program is incomplete; it provides no guidelines to introduce the inner product on the space of these physical states. For this, additional input is needed. A possible strategy \[1\] is to use appropriate symmetries of the classical theory.

Consider for example a “free” particle of mass $\mu$ moving in a stationary space-time (where the Killing field is everywhere timelike). The constraint $P^a P_a + \mu^2 = 0$ implies that the physical states in the quantum theory must satisfy the Klein-Gordon equation $\nabla^a \nabla_a \phi - \mu^2 \phi = 0$. To select the inner product on the space of physical states, one can use the fact \[2, 3\] that the space of solutions admits a unique Kähler structure which is left invariant by the action of the timelike Killing field. The isometry group of the underlying spacetime is unitarily implemented on the resulting Hilbert space; the classical symmetry is promoted to a symmetry of the quantum theory. In fact there is a precise sense \[4\] in which this last condition selects the Hilbert space structure of the quantum theory uniquely. We will see that a similar strategy is available for all class A Bianchi models except types VIII and IX. In these models, therefore, the Dirac program can be completed and the resulting mathematical framework can be used to test many of the ideas that have been proposed in quantum cosmology. For example, one can investigate if the resulting Hilbert space admits a preferred wave function which can be taken to be the ground state and hence a candidate for the “wave function of the universe”. We shall find that, in general, the answer is in the negative.

The paper is organized as follows. In section 2, we recall some of the key features of the mathematical description of Bianchi models. In section 3, we point out that in type I, II, VI$_0$ and VII$_0$ models, the supermetric on each minisuperspace admits a null Killing vector whose action leaves the potential term in the scalar (or, Hamiltonian) constraint invariant. Such Killing fields are called *conditional symmetries* \[5\]. We then quantize these models by requiring that the symmetries be promoted to the quantum theory. We conclude in section 4 with a discussion of the ramifications of these results for quantum gravity in general and quantum cosmology in particular.

### 2. Hamiltonian cosmology

In this paper, we shall consider spatially homogeneous models which admit a three-dimensional isometry group which acts simply and transitively on the preferred spatial slices. (Thus, we exclude, e.g., the Kantowski-Sachs models.) In this case, one can choose, on the homogeneous slices, a basis of (group invariant) 1-forms $\omega^a (a = 1 - 3)$
such that
\[ d\omega^a = -\frac{1}{2}C^a_{bc}\omega^b \wedge \omega^c, \tag{1} \]
where \( C^a_{bc} \) are the components of the structure constant tensor of the Lie algebra of the associated Bianchi group. The vanishing or nonvanishing of the trace \( C^b_{ab} \) divides these models into two classes called class A and class B respectively. This classification is important for quantization because while a satisfactory Hamiltonian formulation is available for all class A models, the standard procedure for obtaining this formulation fails in the case of general class B models (although the possibility that a modified procedure may eventually work is not ruled out. See, e.g., the suggestion made in the last section of [6].) Since the Hamiltonian formulation is the point of departure for canonical quantization, from now on we will consider only the class A models.

Let us simplify matters further by restricting ourselves to those spatially homogeneous metrics which can be diagonalized. (This can be achieved in the type VIII and IX models by exploiting the gauge freedom made available by the vector or the diffeomorphism constraint. In the remaining models there is a loss of generality, which, however, is mild in the sense that the restriction corresponds only to fixing certain constants of motion. For details, see [6].) Thus, we consider space-time metrics of the form:
\[ (4)ds^2 = -N^2(t)dt^2 + \sum_{a=1}^{3} g_{aa}(t)(\omega^a)^2, \tag{2} \]
where \( N(t) \) is the lapse function (see, e.g., [7]). Since the trace of the structure constants vanishes, they can be expressed entirely in terms of a symmetric, second rank matrix \( n_{ab} \):
\[ C^a_{bc} = \epsilon_{mbc}n^{ma}, \tag{3} \]
where \( \epsilon_{mbc} \) is a 3-form on the 3-dimensional Lie algebra. The signature of \( n^{am} \) can then be used to divide the class A models into various types. In the literature, one generally uses the basis which diagonalizes \( n^{am} \) and then expresses \( C^a_{bc} \) as [8]:
\[ C^a_{bc} = n^{(a)}\epsilon_{abc} \quad \text{(no sum over } a) \tag{4} \]
The constants \( n^{(a)} \) are then used to characterize the different class A Bianchi types. A convenient set of choices is given in the following table [8]:

|     | I | II | VI_0 | VII_0 | VIII | IX |
|-----|---|----|------|-------|------|----|
| \( n^{(1)} \) | 0 | 1 | 1 | 1 | 1 | 1 |
| \( n^{(2)} \) | 0 | 0 | -1 | 1 | 1 | 1 |
| \( n^{(3)} \) | 0 | 0 | 0 | 0 | -1 | 1 |
To simplify the notation and calculations, let us use the Misner parametrization of the diagonal spatial metric in (2)

\[
\begin{align*}
\mathbf{g} & \equiv g_{ab} = \text{diag}(g_{11}, g_{22}, g_{33}) \equiv e^{2\beta} . \\
\beta & = \beta^0 \text{diag}(1, 1, 1) + \beta^+ \text{diag}(1, 1, -2) + \beta^- \text{diag}(\sqrt{3}, -\sqrt{3}, 0) .
\end{align*}
\]

This parametrization leads to conformally inertial coordinates for the Lorentz (super) metric defined by the “kinetic term” in the scalar constraint. For Bianchi types I, II, VI\textsubscript{0} and VII\textsubscript{0}, the scalar constraint (which, incidentally, can be obtained directly by applying the ADM procedure to all class A models) now takes the form

\[
C(\beta^A, p_A) := \frac{1}{24N} e^{-3\beta^0} \eta^{AB} p_A p_B + \frac{1}{2} N e^{\beta^0 + 4\beta^+} \left( n^{(1)} e^{2\sqrt{3}\beta^-} - n^{(2)} e^{-2\sqrt{3}\beta^-} \right)^2 = 0 .
\]

Here the upper case latin indices \(A, B\ldots\) range over 0, +, −; \(p_A\) are the momenta canonically conjugate to \(\beta^A\) and the matrix \(\eta^{AB}\) is given by \(\text{diag}(-1, 1, 1)\). To further simplify matters, one usually chooses the “Taub time gauge”, i.e. one chooses the lapse function to be \(N_T = 12 \exp 3\beta^0\). (This choice of gauge is also known as Misner’s supertime gauge.) With this lapse, the scalar constraint takes the form:

\[
C_T = \frac{1}{2} \eta^{AB} p_A p_B + U_T = 0 ,
\]

\[
U_T = 6 e^{4(\beta^0 + \beta^+)} \left( n^{(1)} e^{2\sqrt{3}\beta^-} - n^{(2)} e^{-2\sqrt{3}\beta^-} \right)^2 .
\]

Consequently, the dynamics of all these Bianchi models is identical to that of a particle moving in a 3-dimensional Minkowski space under the influence of a potential \(U_T\). Finally, because we have restricted ourselves to metrics which are diagonal, the vector constraint is identically satisfied. Thus, the configuration space is 3-dimensional and there is one nontrivial constraint. The system therefore has two true degrees of freedom.

3. Quantization

To quantize these systems we must allow the two true degrees of freedom to undergo quantum fluctuations. Following the Dirac theory of constrained systems, let us consider, to begin with, wave functions \(\phi(\vec{\beta})\) where \(\vec{\beta} \equiv (\beta^0, \beta^+, \beta^-)\). The physical states can then be singled out by requiring that they should satisfy the quantum version of the constraint equation (7):

\[
\Box \phi - \mu^2(\vec{\beta}) \phi = 0 , \text{ where}
\]

\[
\Box = - \left( \frac{\partial}{\partial \beta^0} \right)^2 + \left( \frac{\partial}{\partial \beta^+} \right)^2 + \left( \frac{\partial}{\partial \beta^-} \right)^2 , \text{ and}
\]

\[
\mu^2(\vec{\beta}) = 12 e^{4(\beta^0 + \beta^+)} \left( n^{(1)} e^{2\sqrt{3}\beta^-} - n^{(2)} e^{-2\sqrt{3}\beta^-} \right)^2 .
\]
Thus, the physical states are solutions of a “massive” Klein-Gordon equation where
the mass term, however, is “position dependent”: it is a potential in minisuperspace.
Our first task is to endow the space of these states with the structure of a complex
Hilbert space. It is here that we need to extend the Dirac theory of quantization of
constrained systems.

Consider, for a moment, type I models where the potential vanishes. In this case, we
are left with just the free, massless Klein-Gordon field in a 3-dimensional Minkowski
space. Therefore, to endow the space of solutions with the structure of a Hilbert
space, we can use the standard text-book procedure. First, decompose the fields
into positive and negative frequency parts (with respect to a timelike Killing vector
field) and restrict attention to the space $V^+ \subseteq \phi^+$ of positive frequency fields.
Then, introduce on $V^+$ the inner-product:

$$\langle \phi^+_1, \phi^+_2 \rangle := -2i\Omega(\phi^+_1, \phi^+_2),$$  \hspace{1cm} (9)

where “overbar” denotes complex conjugation and where $\Omega$ is the natural symplectic
structure on the space of solutions to the Klein-Gordon equation:

$$\Omega(\phi_1, \phi_2) := \int_\Sigma d^2 S^A (\phi_2 \partial_A \phi_1 - \phi_1 \partial_A \phi_2),$$  \hspace{1cm} (10)

$\Sigma$ being any (2-dimensional) Cauchy surface on $(M, \eta^{AB})$. Alternatively, we can
restrict ourselves to the vector space $V$ of real solutions to the Klein-Gordon equation
and introduce on this space a complex structure $J$ as follows: $J \circ \phi = i(\phi^+ - \phi^-)$. This $J$
is a real-linear operator on $V$ with $J^2 = -1$. It enables us to “multiply” real
solutions $\phi \in V$ by complex numbers: $(a + ib) \circ \phi := a\phi + bJ \circ \phi$, which is again in $V$. Thus $(V, J)$ can be regarded as a complex vector space. Furthermore, $J$ is compatible
with the symplectic structure in the sense that $\Omega(J\phi_1, \phi_2)$ is a symmetric, positive
definite metric on $V$. Therefore, $\langle \cdot, \cdot \rangle := \Omega(J\cdot, \cdot) - i\Omega(\cdot, \cdot)$ is a Hermitian inner product
on the complex vector space $(V, J)$ and thus $(V, J, \langle \cdot, \cdot \rangle)$ is a Kähler space $[2, 3, 4]$. There is a natural isomorphism between the complex vector spaces $V^+$ and $(V, J)$:

$$\phi^+ = \frac{1}{2}(\phi - iJ\phi).$$

Furthermore, this map preserves the Hermitian inner products on the two spaces. Thus, the two descriptions are equivalent. While we introduced $J$ in
terms of positive frequency fields, one can also proceed in the opposite direction and
treat $J$ as the basic object. One would then use the above isomorphism to define the
positive frequency fields. In fact, it turns out that the description in terms of $(V, J)$
can be extended more directly to the Bianchi models with non-vanishing potentials
(as well as to other contexts such as quantum field theory in curved space-times).
This is the strategy we will adopt.

Let us make a small digression to discuss the problem of finding the required operator
$J$ in a general context and then return to the Bianchi models. Let us suppose we
are given a real vector space $V$ equipped with a symplectic structure $\Omega$. Thus,

\[^{1}\]Throughout this paper, our aim is to convey only the main ideas. Therefore, we will not make
Ω : \( V \otimes V \mapsto \mathbb{R} \) is a second rank, anti-symmetric, non-degenerate tensor over \( V \).

A 1-parameter family of canonical transformations on \( V \) consists of linear mappings \( U(\lambda) \) from \( V \) to itself \((\lambda \in \mathbb{R})\) such that \( \Omega(U(\lambda) \circ v, U(\lambda) \circ w) = \Omega(v, w) \) for all \( v, w \) in \( V \). The generator \( T \) of this family, \( T := dU(\lambda)/d\lambda \), is therefore a linear mapping from \( V \) to itself satisfying \( \Omega(T \circ v, w) = -\Omega(T \circ v, w) \). It is generally referred to as an infinitesimal canonical transformation. These operators can be regarded as symmetries on \( \{V, \Omega\} \). The mathematical question of interest to us is: can one endow \( V \) with the structure of a complex Hilbert space on which \( U(\lambda) \) are unitary operators?

Since \( V \) is equipped with a symplectic structure \( \Omega \), to “Hilbertize” \( V \), it is natural to seek a complex structure \( J \) on it which is compatible with \( \Omega \) in the sense discussed above; the resulting Kähler structure can then provide the Hermitian inner-product on the complex vector space \( \{V, J\} \). The issue of existence and uniqueness of such complex structures was discussed in detail in [4] and we will only report the final result here. The generating function \( F_T(v) \) of the canonical transformation under consideration is simply \( F_T(v) := \frac{1}{2} \Omega(v, Tv) \). If this is positive definite, then there exists a unique complex structure \( J \) which is compatible with \( \Omega \) such that \( U(\lambda) \) are unitary operators on the resulting Kähler space: \( \langle U(\lambda) \circ v, U(\lambda) \circ w \rangle = \langle v, w \rangle \), where, as before, the inner-product is given by

\[
\langle v, w \rangle = \Omega(Jv, w) - i\Omega(v, w). \tag{11}
\]

Thus, if one has available a preferred canonical transformation on the real symplectic space \( \{V, \Omega\} \) whose generating function is positive definite, one can endow it unambiguously with the structure of a complex Hilbert space.

The preferred complex structure is defined as follows. Using \( T \), let us first introduce (only as an intermediate step) a fiducial, real inner-product \( (.,.) \) on \( V \) as follows:

\[
(v, w) := \frac{1}{2} \Omega(v, T w). \tag{12}
\]

This is indeed an inner product: it is symmetric because \( T \) is an infinitesimal canonical transformation and the resulting norm is positive definite because it equals the generating functional \( F_T(v) \). Let \( \bar{V} \) denote the Hilbert space obtained by Cauchy completing \( V \) w.r.t. \( (.,.) \). It is easy to check that \( T^2 \) is a symmetric, negative operator on \( \bar{V} \), whence it admits a self-adjoint extension which we also denote by \( T^2 \).

Then,

\[
J := -(-T^2)^{-\frac{1}{2}} \cdot T \tag{13}
\]

is a well-defined operator with \( J^2 = -1 \). Using the expression [12] of the inner-product on \( \bar{V} \), it is easy to check that \( J \) is indeed compatible with \( \Omega \). We can digressions to discuss the subtle but often important issues from functional analysis. In particular, we will not specify the precise domains of various operators nor shall we discuss topologies on the infinite dimensional spaces with respect to which, for example, the symplectic structures are to be continuous.
therefore introduce a new *Hermitian* inner product \( \langle \cdot, \cdot \rangle \) on the *complex* vector space \( (\bar{V}, J) \) via (11). The Cauchy completion \( H \) of this complex pre-Hilbert space is then the required Hilbert space of physical states. (Thus, the inner-product (12) and the resulting real Hilbert space \( \bar{V} \) were introduced only as mathematical tools to enable us to construct the physical Hermitian structure.) Finally, it is easy to show that \( J \) commutes with \( U(\lambda) \). It then follows, from the fact that \( U(\lambda) \) are canonical transformations and from the definition (11) of the Hermitian inner-product, that \( U(\lambda) \) are unitary. The proof of uniqueness of \( J \) is straightforward but more involved [2, 3, 4].

In the case when \( V \) is the space of solutions to the Klein-Gordon equation in Minkowski space-time (as is the case in the Bianchi type I models) the standard complex structure (corresponding to the positive/negative frequency decomposition) can be obtained by choosing for \( T \) the operator \( L_t \) where \( t^A \) is any time translation Killing field in Minkowski space. More generally, one can think of \( J \) as arising from “a positive and negative frequency decomposition defined by \( T \)”. More precisely, \( J \) is the unitary operator in the *polar decomposition* [12] of the operator \( T \) on the Hilbert space \( \bar{V} \).

With this general machinery at hand, we can now return to the Bianchi types II, VI\(_0\), VII\(_0\). Denote by \( V \) the space of real solutions \( \phi \) to the operator constraint equation (8). Elements of \( V \) represent the physical quantum states of the system. As observed earlier, \( V \) is naturally equipped with a symplectic structure \( \Omega \) (see (10)) and our task is to find a compatible complex structure \( J \). For this, we need a preferred canonical transformation on \( V \) with a positive generating function. The situation in type I models suggests that we attempt to construct this transformation using an appropriate Killing field \( t^A \) of the flat supermetric \( \eta^{AB} \) on the 3-dimensional mini-superspace. However, now, the constraint equation (8) involves a nontrivial potential term \( \mu^2(\vec{\beta}) \). Hence, for \( \mathcal{L}_t \) to be a well-defined operator on \( V \) – i.e, for \( \mathcal{L}_t \phi \) to be again a solution to (8) – it is essential that \( \mathcal{L}_t \mu^2 = 0 \). As mentioned earlier, such a Killing field is called a *conditional symmetry* [5]. Fortunately, the models under consideration do admit such a conditional symmetry. An inspection of the potential term \( \mu^2(\vec{\beta}) \) shows that it is invariant under the action of the diffeomorphism generated by the null Killing field

\[
t^A := \left( \frac{\partial}{\partial \beta^0} \right)^A - \left( \frac{\partial}{\partial \beta^+} \right)^A
\]

(14)

of the supermetric \( \eta^{AB} \). The question therefore reduces to that of positivity of the generating functional of the canonical transformation \( T \circ \phi := \mathcal{L}_t \phi \). A straightforward calculation shows that the generating functional \( F_T(\phi) \equiv \frac{1}{2} \Omega(\phi, T\phi) \) is simply the

\footnote{The origin of this symmetry can be traced back to the fact that each of the type I–VII\(_0\) models admits a diagonal automorphism [8, 13] and –like all vacuum models– enjoys a scale invariance.}
“conserved energy” associated with the null Killing field:

\[
F_T(\phi) = \frac{1}{2} \int_{\Sigma} \left( \partial_A \phi \partial_B \phi - \eta_{AB} (\partial^C \phi \partial_C \phi + \mu^2 \phi^2) \right) t^A dS^B ,
\]  

(15)

where \( \Sigma \) is a spacelike Cauchy surface. Since the potential is nonnegative and \( t^A \) is everywhere future-directed, it follows that \( F_T(\phi) \) is indeed positive. Hence, by carrying out a polar decomposition of the operator \( T \equiv L_t \), we obtain a Kähler structure on \( V \) which provides us with the required complex Hilbert space of physical states of the system. On this Hilbert space, the 1-parameter family of diffeomorphisms generated by the conditional symmetry \( t^A \) is unitarily implemented.

To summarize, because Bianchi types I, II, VI\(_0\) and VII\(_0\) admit conditional symmetries whose generating functions are positive definite, in these cases, one can satisfactorily complete the Dirac quantization program. In the more familiar language of positive and negative frequency decomposition, the Hilbert space of physical states consists of positive frequency solutions \( \phi^+ \) of the quantum constraint equation (8):

\[
\phi^+ = \frac{1}{2} (\phi - iJ \circ \phi) \text{ where } \phi \text{ is a real solution and } J = -(-T^2)^{-\frac{1}{2}} \cdot T .
\]

The inner product \( \langle \phi^+_1, \phi^+_2 \rangle \) is constructed from the familiar probability current, i.e., the symplectic structure \( \langle \Omega(\phi^+_1, \phi^+_2) \rangle = -2i \Omega(\phi^+_1, \phi^+_2) \). Finally, the 1-parameter family of diffeomorphisms generated by \( t^A \) is unitarily implemented in this Hilbert space. Motions along the integral curves of \( t^A \) can be interpreted as “time evolution” in the classical theory both in the minisuperspace and in space-times: \( t^A \) is future directed in the minisuperspace and, in any space-time defined by the classical field equations, \( \beta^0 \) — an affine parameter of \( t^A \) — increases monotonically with physical time. In the quantum theory, this time evolution is unitary.

This mathematical structure can now be used to probe various issues in quantum cosmology. We conclude with an example.

Let ask whether there is a “preferred” state in the Hilbert space which can be taken to be the ground state. The question is well-posed because on the physical Hilbert space there is a well-defined Hamiltonian which generates time-evolution in the sense described above. On real solutions to the quantum constraint, the Hamiltonian is given by \( \hat{H} \circ \phi = Jh \mathcal{L}_t \phi \); while in the positive frequency representation of physical states it is given by \( \hat{H} \circ \phi^+ = ih \mathcal{L}_t \phi^+ \). We will show that \( \hat{H} \) is a non-negative operator with zero only in the continuous part of its spectrum. This will establish that \( \hat{H} \) does not admit a (normalizable) ground state. Thus, the most direct procedure to select a “preferred” state fails in all these models. Furthermore, a detailed examination of the mathematical structures involved suggests that as long as the configuration space – spanned by \( \beta^\pm \) – is noncompact, there is in fact no preferred state in the physical Hilbert space. It would appear that to obtain such a state, which could be taken, e.g., as the wavefunction of the universe in these models, one must modify in an essential way the broad quantization program proposed by Dirac.
Finally, we outline the proof of the technical assertion made above. Note first, that the expectation value of $\hat{H}$ in any physical state is given by

$$\frac{\langle \phi, \hat{H}\phi \rangle}{\langle \phi, \phi \rangle} = \frac{\hbar}{\Omega(J\phi, J\phi)} \frac{\Omega(J\phi, J\phi)}{\Omega(J\phi, J\phi)} = \hbar \frac{(\phi, \phi)}{(\phi, (-T^2) \frac{1}{2} \phi)} \geq 0, \quad (16)$$

where we have used the expression (11) of the physical, Hermitian inner product $\langle ., . \rangle$, the expression (12) of the fiducial inner product $(. , .)$ and the fact that $(-T^2)$ is a positive definite operator with respect to $(. , .)$. Since the expectation values of $\hat{H}$ are non-negative, it follows that its spectrum is also non-negative. To show that zero is in the continuous part of the spectrum, it is convenient to work with the real Hilbert space $\tilde{V}$ defined by the inner product $(. , .)$. Using the fact that the potential $\mu^2(\vec{\beta})$ in (8) is smooth, non-negative and takes values that are arbitrarily close to zero, one can show that the expectation values of $(-T^2)$ on $\tilde{V}$ can also approach arbitrarily close to zero. Hence zero is in the spectrum of $(-T^2)$, which implies that the spectrum of $(-T^2)^{-\frac{1}{2}}$ is unbounded above. It therefore follows from (14) that one can find physical states $\phi$ in which the expectation values of $\hat{H}$ are arbitrarily close to zero. Hence zero is in the spectrum of $\hat{H}$. Finally, if it were in the discrete part of the spectrum, there would exist a physical state $\phi_0$ which is annihilated by $L_t$. In this state, in particular, $\langle \phi_0, \hat{H}\phi_0 \rangle = 2\hbar(\phi_0, \phi_0)$ must vanish. However, it is straightforward to show that

$$2(\phi_0, \phi_0) = \int_\Sigma \left( (L_t\phi_0)^2 + \left( \frac{\partial \phi_0}{\partial \beta^0} \right)^2 + \mu^2(\vec{\beta})\phi_0^2 \right) d^2x \quad (17)$$

where $\Sigma$ is any $\beta^0 = \text{const.}$ 2-plane. Hence $(\phi_0, \phi_0)$ can vanish iff $\phi_0$ and $L_t\phi_0$ vanish on $\Sigma$, i.e., iff $\phi_0$ is the zero solution to (8). Hence zero must belong only to the continuous part of the spectrum of $\hat{H}$.

4. Discussion

In the literature on spatially homogeneous quantum cosmology, the emphasis has been on type I and type IX models. Type I is the simplest and its quantization has been well-understood for sometime now. In fact, the Hamiltonian description of this model is the same as that of the strong coupling limit of general relativity ($G_{\text{Newton}} \rightarrow \infty$), whence its quantization has been discussed also in the context of this limit. Type IX is the most complicated of the spatially compact models and exhibits, in a certain well-defined sense, chaotic behavior in the classical theory. It is also the most “realistic” of the Bianchi models as far as dynamics of general relativity in

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\footnote{The type VIII model has many of the interesting features of the type IX. However, it seems not to have drawn as much attention in quantum cosmology because it is spatially open. Many of the remarks that follow on the type IX model are applicable to the type VIII model as well.}
strong field regions near singularities is concerned. In the early work on quantum cosmology therefore, the focus of attention was type IX models; type I was studied mainly as a preliminary step towards the quantum theory of type IX. Unfortunately, the “potential” in the scalar constraint of type IX models is so complicated that quantization attempts met with only a limited success. In particular, until recently, not a single exact solution to the quantum constraints was known, whence one could not even begin to address the issues we have discussed in this paper.

The reason we could make further progress in this paper is that we analysed the intermediate models in some detail. Although they are not as “realistic” as type IX, we have seen that these models do have an interesting structure. In particular, unlike in the type I models, the potential in the scalar constraint is non-zero, whence their dynamics is quite non-trivial already in the classical theory. For example, we still do not know the explicit form of a complete set of constants of motion in type VI$_0$ and type VII$_0$ space-times. In spite of this, we could complete the Dirac quantization program because these models admit an appropriate (i.e., future-directed) conditional symmetry. In physical terms, quantization is possible because one can define an internal time variable on these minisuperspaces in a consistent fashion.

The resulting mathematical framework is well-suited for addressing a number of conceptual issues in quantum gravity in general and quantum cosmology in particular.

First is the issue of time. Using the conditional symmetry $t^A$, we were able to construct a complex structure $J$ such that the 1-parameter family of transformations on the physical quantum states $\phi$ induced by $t^A$ are implemented by a 1-parameter family of unitary operators on the Hilbert space. Hence, using the affine parameter along the vector field $t^A$ as a “time” variable, one can deparametrize the theory and express time evolution through a Schrödinger equation: it is the introduction of the complex structure – or, the use of only positive frequency fields – that provides a “square-root” of the quantum scalar constraint which can be re-interpreted as the Schrödinger equation. In the type I model, for example, the “square-root” takes the following form:

$$i\hbar L_t \phi^+ (\vec{\beta}) = \hbar \left( (-\Delta)^{1/2} - i\frac{\partial}{\partial \beta^+} \right) \phi^+ (\vec{\beta})$$

where $\Delta = (\partial/\partial \beta^+)^2 + (\partial/\partial \beta^-)^2$ is the Laplacian in the $\beta^\pm$ plane. Thus the argument $\beta^0$ of $\phi^+$ can be regarded as time and $\hat{H} \equiv \hbar ((-\Delta)^{1/2} - i(\partial/\partial \beta^+))$ can be regarded as the Hamiltonian. (The second term in the expression for $\hat{H}$ arises simply because our evolution is along a null Killing field $t^A$ rather than a timelike one.) Now, there exist in the literature several distinct approaches to the issue of time in full

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4Over the past two years, a handful of solutions have been obtained [14, 15] using new canonical variables [16] in terms of which the potential term disappears. While this development does represent progress, the solutions obtained thus far are rather special and there are too few to construct a useful Hilbert space.
quantum gravity, including those that suggest that one should generalize quantum mechanics in a way in which there is in fact no preferred time variable. It would be fruitful indeed to apply these ideas to the minisuperspaces considered in this paper and compare the resulting quantum descriptions with the one obtained here using the conditional symmetry.

A second issue is the strategy to select inner products. Since we do not have access to any symmetry group in general relativity, it has been suggested \cite{17, 18} that we should let the “reality conditions” determine the inner product. More precisely, the strategy is the following: Find a sufficient number of real classical observables a la Dirac—i.e. functions on the phase space which weakly commute with the constraints—and demand that the inner product between quantum states be so chosen that the corresponding quantum operators are Hermitian. In many examples, this condition suffices to pick out the inner product uniquely. In the minisuperspaces considered in this paper, on the other hand, we have selected the inner-product using a completely different strategy: we exploited the fact that these models admit a conditional symmetry of an appropriate type, thereby side-stepping the issue of isolating a complete set of Dirac observables. It would be interesting to identify the Dirac observables, use the “reality conditions” strategy and compare the resulting inner product with the one we have introduced. (Equivalently, the question is whether the real Dirac observables can be promoted to operators which are Hermitian with respect to the inner product we have introduced.) This program has been completed in type I and II models \cite{19}. While the two inner products do agree in a certain sense, subtleties arise already in type II models. However, the question remains open in types VI\(_0\) and VII\(_0\).

A third issue is the fate of classical singularities in the quantum descriptions. Every (non-flat) classical solution belonging to the models considered here has a singularity and one sometimes appeals to “the rule of unanimity” \cite{20} to argue that in such cases, singularities must persist also in quantum theory. On the other hand, we have found that the quantum evolution is given by a 1-parameter family of unitary operators. In all cases considered in this paper, the quantum Hilbert spaces are well-defined and quantum evolution is unitarily implemented. This appears, at least at first sight, to be a violation of the rule of unanimity and it is important to understand the situation in detail. Is it the case that the classical singularities simply disappear in the quantum theory? Or, do they persist but in a “tamer” fashion? In the space-time picture provided by classical general relativity, evolution is implemented by hyperbolic equations which simply breakdown at curvature singularities. Is this loss of predictability recovered in the quantum theory? That is, in quantum theory, can

\footnote{The situation is similar in full 2+1 gravity, where, every classical cosmological solution begins with a “little bang” where (there is no curvature singularity —hence the prefix “little”— but where) the spatial volume goes to zero. Inspite of this, the mathematical framework of quantum theory is complete and well-defined.}
we simply “evolve through the singularity”? It would be useful to apply the semi-classical methods available in the literature to these models both to gain physical insight into this issue and to probe the limitations of the semi-classical methods themselves. (Some results pertaining to these issues are discussed in [18, 19].)

Finally, further work is needed to achieve a more complete understanding of the quantum physics of these models. What we have constructed is the Hilbert space of physical states. Any self-adjoint operator on this space may be regarded as a quantum Dirac observable. However, unless the operator has a well-defined classical analog it is in general not possible to interpret it physically. In all the models considered here, the generator of the “time translation” defined by the conditional symmetry is one such observable: its classical analog is simply \( p_0 - p_+ \), the momentum corresponding to the conditional symmetry. However, since these models have two (configuration) degrees of freedom, the reduced phase space is four dimensional whence one expects there to exist four independent Dirac observables. The open question therefore is that of finding the three remaining observables. In the type I model, there is no potential, whence every generator of the Lorentz group defined by the flat supermetric \( \eta^{AB} \) is a Dirac observable. Thus, it is easy to find the complete set both classically and quantum mechanically. In the type II model, the task is already made difficult by the presence of a non-zero potential. However, now the potential is rather simple and can actually be eliminated by a suitable canonical transformation [19]. (In full general relativity, canonical transformations with the same property exist [16, 22]. However, the resulting supermetric is curved. In the type II model, the new supermetric is again flat; only the global structure of the constraint surface is different from that in the type I model.) In terms of these new canonical variables it is easy to find the full set of Dirac observables both classically and quantum mechanically. Thus, in the type I and II models, the issue of the physical interpretation is under control. In the type VI\(_0\) and VII\(_0\) models, on the other hand, the issue is wide open. Resolution of this issue will also shed light on the question of uniqueness of the inner product and clarify the issue of singularities discussed above.

What is the situation with respect to the more complicated Bianchi models, types VIII and IX? Now, the conditional symmetry which played a key role in quantum theory no longer exists: the origin of this symmetry can be traced to the presence of a diagonal automorphism group which happens to be zero dimensional in type VIII and IX models (see references in footnote 2). However, it is possible that these models admit some “hidden” symmetries –i.e. symmetries which are not induced by space-time diffeomorphisms. There is indeed a striking feature along these lines, first pointed out by Charlie himself [9]: in the type IX models, there exists an asymptotic constant

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\(^6\)In addition, to obtain physical interpretations in the quantum theory of constrained dynamical systems such as the ones we are considering, one has to deparametrize the theory and express phase space variables (such as the 3-metric and extrinsic curvature) in terms of the Dirac operators and the “time”variable. (see e.g. [21, 18, 19]).
of motion precisely at the early chaotic stage. Can one use it for quantization?
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