Effects of Friction Force on the Vortex Motion in Superfluid $^4$He

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Abstract
The motion of a vortex filament in superfluid $^4$He is considered by using the Hall-Vinen (9, 10) phenomenological model for the scattering process between the vortex and thermal excitations in liquid $^4$He. The Hall-Vinen equations are formulated first in the intrinsic geometric parameter space to obtain insights into the physical implications of one aspect of the friction term, associated with the friction coefficient $\alpha$, in the Hall-Vinen notation, as well as a proper rationale for the previous neglect of the other aspect of the friction term, associated with the friction coefficient $\alpha'$. This development also serves to highlight the difficulties arising in making further progress on this route. A reformulation of the Hall-Vinen equation in the extrinsic vortex filament coordinate space is then given. This is shown to provide a useful alternative approach in this regard. Though the friction term associated with $\alpha$, for very small $\alpha$, has little capacity to make significant contribution to the vortex motion in a quantitative way, it is shown to be able to change the vortex motion aspects like the vortex kink characteristics in a qualitative way. This becomes possible because of the ability of the friction term to play the dual roles of driving force and drag force.

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1. Introduction

The Landau [1] model of superfluid $^4$He considers the superfluid below the lambda point as an inviscid, irrotational "background fluid" with thermal excitations, the phonons and rotons moving upon that background. These excitations constitute the normal fluid which would not interact with the superfluid except for the presence of vortices. The vortices scatter the thermal excitations when there is a relative velocity between them, thus generating the so-called "mutual friction" (Feynman [7]). The existence of mutual friction was confirmed by the experiment on the attenuation of second sound in uniformly rotating liquid $^4$He (Hall and Vinen [9], [10]). These experiments, if the plausible assumption is made that a thermal excitation can exchange momentum only in a direction perpendicular to the scattering vortex line, then indicated (Vinen [11]) that the friction force will have a constant and large value for all directions of the relative velocity perpendicular to the rotation axis (with which the vortex lines are aligned) but will be zero for directions parallel to the latter. On the other hand, vortices in liquid $^4$He have a core radius of the order of the quantum coherence length ($10^{-1}$ Angstrom), so the detailed core physics is not very relevant to the dynamical effects of the vortex away from the core. As a result, barring vortex reconnection events (which involve sharp distortions of vortex lines (Paoletti et al. [12]) and the concomitant generation of Kelvin waves associated with helical displacements of the vortex cores (Yepez et al. [13])), vortices in superfluid $^4$He behave like classical vortex filaments, the only difference being that their circulations and core radii exhibit quantum mechanical features. Indeed, the pioneering numerical simulations of Schwarz [14] and [15], which provided considerable insight into superfluid vortex dynamics, are based on the idea that, except on very short length scales, quantized vortices can be regarded as vortex filaments moving according to classical fluid dynamics, with the inclusion of the mutual friction force.

The leading-order behavior of the vortex-induced flow velocity formula in classical fluid dynamics is given by the so-called local induction approximation (LIA) (Da Rios [16], Arms and Hama [17]) which resolves the singularity due to the neglect of the finite vortex core size by an asymptotic calculation. Using the LIA, Da Rios [16] and Betchov [18] derived a set of coupled equations governing the inextensional motion of a vortex filament in an irrotational fluid in terms of time evolution of its intrinsic geometric parameters - curvature and torsion. Hasimoto [19] showed that Da Rios-Betchov equations can be elegantly combined to give a nonlinear Schrodinger equation. The single-soliton solution of this equation (Zakharov and Shabat [20]) provides a description of an isolated loop of helical twisting motion along the vortex line. The LIA is, however, hampered by the fact that the motion of the vortex filament is assumed to be governed solely by the local features on the filament, so distant

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1. Vorticity in superfluid $^4$He is confined to vortices which, as Onsager [2] suggested, are linear topological defects with the superfluid density vanishing at the vortex core. This allows liquid $^4$He to have the angular momentum as in a solid body rotation situation while maintaining irrotationality, as per Landau's [1] theory, over almost the whole volume. This, on the other hand, as Onsager [2] pointed out, leads to the result that the circulation around a vortex line is quantized, which was confirmed in an ingenious experiment due to Vinen [3]. Rayfield and Reif [4] also gave direct evidence for the existence of quantized vortices in superfluid $^4$He via charge carrying vortex rings having one quantum of circulation. Direct observation of quantized vortices has been accomplished in superfluidity manifested in dilute Bose-Einstein condensates produced by laser cooling in laboratory experiments (Pethick and Smith [5]). Direct observation of vortex cores has also been accomplished recently (Bewley et al. [6]) by using small solid hydrogen particles as traces in liquid $^4$He.

2. Our understanding of the microscopic origin of the physical processes underlying the mutual friction is not good (Donnelly [8]) so an adequate theory of the fundamental roton-vortex scattering process in superfluid $^4$He is not at hand yet.
parts of the filament need to remain sufficiently separated during the motion. The large-amplitude solutions in LIA suffer from the violation of this premise as in a self-interaction of the vortex filament.

Interestingly, rotating liquid $^4\text{He}$ appears to be a better system than ordinary fluids for the application of LIA, because,

- the cores of vortices in superfluid $^4\text{He}$ are only few angstroms in diameter (hence validating the asymptotic evaluation implicit in the LIA which holds in the limit of vanishingly small vortex core size);
- interactions between different segments of a vortex filament are important only when the distance between them is of the order of few angstroms ($[14]$, $[15]$).

Upon including the effect of the frictional force exerted by the normal fluid on a vortex line, the self-induced velocity of the vortex line in the reference frame moving with the superfluid according to the LIA is given by the Hall-Vinen equation ($[9]$, $[10]$)

$$v = \gamma \kappa \hat{t} \times \hat{n} + \alpha \hat{t} \times (U - \gamma \kappa \hat{t} \times \hat{n}) - \alpha' \hat{t} \times [(\hat{t} \times (U - \gamma \kappa \hat{t} \times \hat{n}))].$$

Here, $U$ is the normal fluid velocity taken to be constant in space and time and prescribed ($[14]$, $[15]$), $\kappa$ is the average curvature, and $\hat{t}$ and $\hat{n}$ are unit tangent and unit normal vectors respectively, to the vortex filament, and $\gamma = \Gamma ln(c/\kappa a_0)$, where $\Gamma$ is the quantum of circulation, $c$ is a constant of order 1 and $a_0 \approx 1 \cdot 3 \times 10^{-8}$ cm is the effective core radius of the filament. $\alpha$ and $\alpha'$ are the friction coefficients which are usually found to be small (except near the $\lambda$-point) so the short-term vortex motion appears to be only weakly affected by the friction. However, the friction term associated with $\alpha$ (which reflects the fact that a thermal excitation can exchange momentum only in a direction perpendicular to the scattering vortex line) plays the dual roles of driving force and drag force ($[14]$, $[15]$). It can therefore lead to both growth and decay of the vortex line length and hence can produce important qualitative effects. On the other hand, the friction term associated with $\alpha'$ arises partly from the asymmetry in the fundamental roton-vortex scattering and partly from the Magnus effect and is usually dropped under the pretext that $\alpha > \alpha'$ ($\text{Vinen and Nimela}[21]$). Nevertheless, determination of the vortex motion from the Hall-Vinen equation (1) is a highly non-trivial task and numerical simulations ($[14]$, $[15]$) have been so far essentially the only method of investigation. In this paper, we will first do Hasimoto $[19]$ formulation of this problem in the intrinsic geometric parameter space to obtain insights into the underlying physical features as well as highlight difficulties involved in the Hasimoto $[19]$ formulation route. We will then show that a reformulation of the Hall-Vinen equation (1) in the extrinsic vortex filament coordinate space provides a useful alternative approach in this regard - it provides insight into the fundamental importance of the friction term associated with $\alpha$ as well as a proper rationale for the previous neglect of the friction term associated with $\alpha'$.

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3. So this formulation is kinematical in nature - the effect of the vortices on the normal fluid is ignored. Notwithstanding this limitation, this formulation turns out to be useful in shedding light on the physical mechanisms involved. In actuality, $U$ should be determined as part of the solution, once the back reaction of the vortices on the normal fluid has been accounted for.

4. Actual numerical values of $\alpha$ and $\alpha'$ at a couple of typical temperatures taken from table 1 of Schwarz $[14]$ are -

$$T(K) = 1.0 : \alpha = 0.005, \quad \alpha' = 0.003$$

$$T(K) = 1.5 : \alpha = 0.073, \quad \alpha' = 0.018.$$
(as in the numerical simulations ([14], [15])). The friction term associated with \( \alpha \), even for very small \( \alpha \), is shown to be able to change the vortex motion aspects like the vortex kink characteristics in a qualitative way.

2. Formulation of the Hall-Vinen Equation in the Intrinsic Geometric Parameter Space

Let us first note that equation (1) can be rewritten as

\[
\mathbf{v} = (1 - \alpha') \gamma \kappa \hat{b} + \alpha \hat{t} \times \mathbf{U} + \alpha \gamma \kappa \hat{n}
\]

(2)

where \( \hat{b} \) is the unit binormal vector to the vortex filament. Observe that the friction term associated with \( \alpha' \) can be eliminated by renormalizing the vortex strength \( \gamma \).

We take the normal fluid velocity to be

\[
\mathbf{U} = U_2 \hat{n} + U_3 \hat{b}
\]

(3)

and allow the normal fluid velocity to be a function of the arc length \( s \). Equation (2) then becomes

\[
\mathbf{v} = \gamma \kappa \hat{b} + \alpha \left( -U_3 \hat{n} + U_2 \hat{b} \right) + \alpha \gamma \kappa \hat{n}.
\]

(4)

Note the Frenet-Serret formulae for the underlying differential geometry,

\[
\mathbf{x}' = \hat{t}, \quad \hat{t}' = \kappa \hat{n}, \quad \hat{n}' = \tau \hat{b} - \kappa \hat{t}, \quad \hat{b}' = -\tau \hat{n}
\]

(5)

where \( \tau \) is the torsion and primes denote differentiation with respect to the arc length \( s \). (5) leads to

\[
\left( \hat{n} + i \hat{b} \right)' = -i \tau \left( \hat{n} + i \hat{b} \right) - \kappa \hat{t}
\]

which suggests we introduce ([18])

\[
\mathbf{N} \equiv \left( \hat{n} + i \hat{b} \right) e^{i \int \tau(s) ds}, \quad \psi \equiv \kappa(s) e^{i \int \tau(s) ds}.
\]

(6)

Note the following relations,

\[
\mathbf{N} \cdot \hat{t} = 0, \quad \mathbf{N} \cdot \mathbf{N} = 0, \quad \mathbf{N} \cdot \bar{\mathbf{N}} = 2
\]

(7)

where the bar overhead denotes the complex conjugate of the quantity in question. (6) leads to

\[
\mathbf{N}' = -\psi \hat{t}
\]

(8)

and

\[
\hat{t}' = \text{Re} \left( \psi \bar{\mathbf{N}} \right) = \frac{1}{2} \left( \psi \mathbf{N} + \bar{\psi} \bar{\mathbf{N}} \right).
\]

(9)

On the other hand, we have from equations (4), (5) and (6),

\[
\hat{t} = \gamma \left( \kappa' \hat{b} - \kappa \tau \hat{n} \right) + \alpha \gamma \left( \kappa \tau \hat{b} + \kappa' \hat{n} \right) - \alpha \tau \left( U_2 \hat{n} + U_3 \hat{b} \right) + \alpha \kappa U_3 \hat{t} - \alpha \gamma \kappa^2 \hat{t}
\]

\[
= \gamma \text{Re} \left( i \psi' \bar{\mathbf{N}} \right) + \alpha \gamma \text{Re} \left( \psi' \bar{\mathbf{N}} \right) - \alpha \gamma |\psi|^2 \hat{t} + \alpha \text{Re} \left( i \psi \bar{v} \right) \hat{t} + \alpha \text{Re} \left( i v' \bar{\mathbf{N}} \right)
\]

\[
= i \frac{\gamma}{2} \left( \psi' \mathbf{N} - \bar{\psi}' \bar{\mathbf{N}} \right) + \frac{\alpha \gamma}{2} \left( \psi' \mathbf{N} + \bar{\psi}' \bar{\mathbf{N}} \right) - \alpha \gamma |\psi|^2 \hat{t} + i \frac{\alpha}{2} \left( \psi \bar{V} - \bar{\psi} \mathbf{V} \right) \hat{t} + i \frac{\alpha}{2} \left( V' \mathbf{N} - \bar{V}' \bar{\mathbf{N}} \right)
\]

(10)
where the dot overhead denotes differentiation with respect to time $t$, and

$$ V \equiv (U_2 + iU_3) e^{\int \tau(s) ds}. \quad (11) $$

Let

$$ \dot{N} = \sigma N + \beta \hat{t} \quad (12) $$

and on using (7) and (10),

$$ \sigma + \bar{\sigma} = \frac{1}{2} \left( \dot{N} \cdot \dot{N} + \ddot{N} \cdot \dot{N} \right) = \frac{1}{2} \frac{\partial}{\partial t} (N \cdot \ddot{N}) = 0 \quad \text{or} \quad \sigma = iR \quad (13) $$

and

$$ \beta = \dot{N} \cdot \hat{t} = -N \cdot \dot{\hat{t}} = -(i \gamma + \alpha \gamma) \psi' - i \alpha V' \quad (14) $$

where $R$ is some real-valued function.

Substituting (13) and (14), (12) becomes

$$ \dot{N} = iR N + \left[ -(i \gamma + \alpha \gamma) \psi'' - i \alpha V'' \right] \hat{t}. \quad (15) $$

Taking the time derivative of equation (8) and the $s$-derivative of equation (15), and using equations (8) - (10), we obtain

$$ \dot{N}' = -\psi \dot{\hat{t}} - \dot{\psi} \hat{t} = -\psi \left[ i \frac{\gamma}{2} (\psi' \bar{N} - \bar{\psi}' N) + \frac{\alpha \gamma}{2} (\psi' \bar{N} + \bar{\psi}' N) \right. $$

$$ - \alpha \gamma |\psi|^2 \hat{t} + i \alpha \frac{\bar{\psi}}{2} (\psi \bar{V} - \bar{\psi} V) \hat{t} + i \frac{\alpha}{2} (V'\bar{N} - \bar{V}' N) \left] - \psi \hat{t} \quad (16) $$

and

$$ \dot{N}' = i \left( R' N - R \bar{\psi} \hat{t} \right) - [(i \gamma + \alpha \gamma) \psi'' + i \alpha V''] \hat{t} - [(i \gamma + \alpha \gamma) \psi' + i \alpha V'] \frac{1}{2} \left( \psi \bar{N} + \bar{\psi} N \right). \quad (17) $$

Equating (16) and (17), we obtain the following coupled equations,

$$ -\psi + \alpha \gamma |\psi|^2 \psi - i \frac{\alpha}{2} \psi (\psi \bar{V} - \bar{\psi} V) = -iR \psi - [(i \gamma + \alpha \gamma) \psi'' + i \alpha V''] \quad (18) $$

$$ \frac{i \gamma}{2} \psi \bar{\psi}' - \frac{\alpha}{2} \psi \bar{\psi}' + i \frac{\alpha}{2} \bar{\psi} V' = iR' - \frac{1}{2} [(i \gamma + \alpha \gamma) \psi' + i \alpha V''] \bar{\psi}. \quad (19) $$

The almost impossibility of decoupling equations (18) and (19), as they are, highlights the enormous difficulty in fully determining the vortex motion in an analytic way from the Hall-Vinen equation (1). In order to make further progress, appropriate approximations are necessary. Recognizing that $\alpha$ is very small, in a first approximation, equations (18) and (19) may be decoupled to give the nonlinear Schrodinger equation,

$$ \frac{1}{i} \dot{\Phi} \approx \frac{\gamma}{2} |\Phi|^2 \Phi + \gamma \Phi'' \quad (20) $$

where,

$$ \Phi \equiv \psi + \frac{\alpha}{\gamma} V. \quad (21) $$

Equation (20) is the same as that for the ordinary fluid case ([19]) with the superfluid effects now represented by a renormalization of the “wave function” $\psi$, as in (21), in the first approximation.
Noting (6) and (11), (21) may be approximated as follows,

\[
\Phi = \kappa \left[ 1 + \frac{\alpha}{\gamma \kappa} (U_2 + iU_3) \right] e^{i \int \tau ds} \approx \kappa e^{\frac{\alpha}{\gamma} (U_2 + iU_3) + i \int \tau ds}
\]

(22) may be approximated as follows:

\[
\Phi = \kappa \left[ 1 + \frac{\alpha}{\gamma \kappa} (U_2 + iU_3) \right] e^{i \int \tau ds} \approx \kappa e^{\frac{\alpha}{\gamma} (U_2 + iU_3) + i \int \tau ds}
\]

(22) shows that, to first approximation,

- the normal fluid velocity component along \( \hat{n} \) serves to modify the curvature \( \kappa \);
- the normal fluid velocity component along \( \hat{b} \) serves to modify the torsion \( \tau \);

as to be expected.

In recognition of the enormous difficulty involved in making further progress in dealing with equations (18) and (19) in an analytic way, we now consider a reformulation of the Hall-Vinen equation (1) in the extrinsic vortex filament coordinate space.

3. Reformulation of Hall-Vinen Equation in the Extrinsic Vortex Filament Coordinate Space

The extrinsic vortex filament coordinate space formulation considers only small-amplitude vortex motions in the LIA model. So it avoids the problems besetting the large-amplitude solutions due to violation of the basic premise of LIA that distant parts of the vortex filament remain sufficiently separated during the motion.

Consider the vortex filament essentially aligned along the x-axis (Dmitreyev [22], Shivamoggi and Heijst [23]) and take \( \mathbf{U} = U_1 \hat{i}_x + U_2 \hat{i}_y + U_3 \hat{i}_z \); equation (1) then becomes

\[
\mathbf{v} = \left( 1 - \alpha' \right) \gamma \kappa \hat{t} \times \hat{n} + \alpha \hat{t} \times \mathbf{U} + \alpha \gamma \kappa \hat{n} - \alpha' U_1 \hat{t}.
\]

(23)

We assume the deviations from the x-axis to be small,

\[
\mathbf{r} = x \hat{i}_x + y(x, t) \hat{i}_y + z(x, t) \hat{i}_z.
\]

(24)

Then have

\[
\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = y_t \hat{i}_y + z_t \hat{i}_z
\]

(25)

\[
\hat{t} \equiv \frac{d\mathbf{r}}{ds} = \frac{dx}{ds} \hat{i}_x + y \hat{i}_y + z \hat{i}_z.
\]

(26)

We are assuming small-amplitude vortex motions, so we have

\[
\frac{dx}{ds} = \left( 1 + y_x^2 + z_x^2 \right)^{-1/2} \approx 1 - \frac{1}{2} (y_x^2 + z_x^2).
\]

(27)

Next,

\[
\kappa \hat{n} \equiv \frac{d\hat{t}}{ds} = \frac{d\hat{t}}{dx} \frac{dx}{ds} \approx - \left( y_x y_{xx} + z_x z_{xx} \right) \left[ 1 - \frac{1}{2} (y_x^2 + z_x^2) \right] \hat{i}_x
\]

\[
+ \left[ y_{xx} \left( 1 - \frac{1}{2} (y_x^2 + z_x^2) \right) - \left( y_x^2 y_{xx} + y_x z_x z_{xx} \right) \left[ 1 - \frac{1}{2} (y_x^2 + z_x^2) \right] \right] \hat{i}_y
\]

\[
+ \left[ z_{xx} \left( 1 - \frac{1}{2} (y_x^2 + z_x^2) \right) - \left( z_x y_x y_{xx} + z_x z_x z_{xx} \right) \left[ 1 - \frac{1}{2} (y_x^2 + z_x^2) \right] \right] \hat{i}_z
\]

(28)
Substituting (26) - (28), equation (23) gives

\[ y_t = - (1 - \alpha') \gamma z_{xx} + \frac{3\gamma}{2} (y_x^2 + z_x^2) z_{xx} + \alpha U_1 z_x - \alpha U_3 + \alpha \gamma y_{xx} - \alpha' U_1 y_x \] (29)

\[ z_t = - (1 - \alpha') \gamma y_{xx} + \frac{3\gamma}{2} (y_x^2 + z_x^2) y_{xx} + \alpha U_1 y_x + \alpha U_2 + \alpha \gamma z_{xx} - \alpha' U_1 y_x. \] (30)

Putting, \( \Phi \equiv y + iz, \ V \equiv U_2 + iU_3 \) (31)
equations (29) and (30) can be combined to give the nonlinear Schrödinger equation

\[ \frac{1}{i}(\Phi_t - i\alpha V + \alpha' U_1 \Phi_x) + \alpha U_1 \Phi_x = (1 - \alpha') \gamma \Phi_{xx} - i\alpha \gamma \Phi_{xx} - \frac{3\gamma}{2} |\Phi_x|^2 \Phi_{xx}. \] (32)

which describes the propagation of nonlinear Kelvin waves on a vortex filament in a superfluid.

Putting, \( \Phi (x,t) = \chi(x,t) e^{i\alpha V t} \) (33)
equation (32) becomes

\[ \frac{1}{i}(\chi_t + \alpha' U_1 \chi_x) + \alpha U_1 \chi_x = (1 - \alpha') \gamma \chi_{xx} - i\alpha \gamma \chi_{xx} - \frac{3\gamma}{2} |\chi_x|^2 \chi_{xx}. \] (34)

It may be noted that the friction term associated with \( \alpha' \) on the left hand side in equation (34) can be transformed away via the Galilean transformation

\[ q(x,t) \Rightarrow q(x,\tau), \ \tau \equiv t - x/\alpha' U_1 \] (35)

while the friction term associated with \( \alpha' \) on the right hand side can again be eliminated by renormalizing the vortex strength \( \gamma \). This provides a proper rationale for neglecting the friction term associated with \( \alpha' \), as in [14], [15].

4. Nonlinear Localized Structures on a Vortex Filament

Look for a nonlinear localized Kelvin stationary wave solution

\[ \chi(x,t) = \nu \psi(x - u \gamma \tau) e^{i(\sigma x - c \gamma \tau) + \mu \tau} \] (36)

and assume \( \psi \) is slowly-varying; we then obtain from equation (34),

\[ (1 - i\alpha) \psi'' + i [2\sigma (1 - i\alpha) - u] \psi' + \left[ c - (1 - i\alpha) \sigma^2 - i\alpha \sigma U_1 + i\mu \right] \psi + \frac{3\nu \sigma^4}{2} \psi^3 = 0. \] (37)

We now drop \( i\alpha \) from the factors \( (1 - i\alpha) \) (because \( \alpha \) is very small compared with 1), and put

\[ \sigma = \frac{u}{2}, \ \beta \equiv \sigma^2 - c, \ \mu = \frac{\alpha u}{2} \left( U_1 - \frac{u}{2} \right). \] (38)

The first relation implies that the velocity of propagation of the structure is twice the torsion, like the case with Hasimoto’s [19] solution in ordinary fluid. Further, \( \beta \) is a measure of the curvature. Equation (37) then becomes

\[ \psi'' - \beta \psi + \frac{3\nu \sigma^4}{2} \psi^3 = 0 \] (39)
which admits a solitary-wave solution
\[
\psi = \sqrt{\frac{4\beta}{3\nu\sigma^4}} \text{sech} \sqrt{\beta} (x - 2\sigma\gamma\tau)
\] (40)
describing a propagating damped Kelvin kink on a vortex filament in superfluid $^4$He. Observe that the damping parameter $\mu$ vanishes when

(i) either $\alpha = 0$, ordinary fluid case, \hspace{1cm} (41a)

(ii) or $u = 2U_1$, special superfluid case. \hspace{1cm} (41b)

Further, the damping is symmetric with respect to the direction of propagation of the vortex kink (as it should be) - this symmetry is however broken by the normal fluid velocity component $U_1$ along the vortex. The vortex kink growth associated with the latter aspect has interesting qualitative similarities with the Donnelly-Glaberson instability (Cheng et al. [24], Glaberson et al. [25]) of the Kelvin waves on a vortex driven by the normal fluid velocity component along the vortex.

In the second case, namely (41b), the vortex kink is undamped - the nonlinearity in the system, under condition (41b), balances both the dispersion and the mutual friction and the vortex kink amplitude and its propagation speed are determined by the normal fluid velocity $U_1$. Thus, even though the friction term associated with $\alpha$, for very small $\alpha$, has little capacity to make significant contribution to the vortex motion in a quantitative way it appears to be able to change vortex kink dynamics characteristics in a qualitative way. This feature reveals itself in a more striking way on considering the rotating planar vortex filament problem in superfluid $^4$He, as discussed below.

5. Rotating Planar Vortex Filament

Consider a vortex filament with shape $y = y(x)$ lying in a plane which is rotated with angular velocity $\Omega$ (this problem was considered by Hasimoto [26] in the ordinary fluid case).

We now have
\[
\hat{t} = < 1, y_x, 0 > \frac{1}{\sqrt{1 + y_x^2}}
\] (42)

\[
\kappa \hat{n} \equiv \frac{d\hat{t}}{ds} = \frac{d\hat{t}}{dx} ds = \left\langle -\frac{y_x y_{xx}}{(1 + y_x^2)^{3/2}}, \frac{y_{xx}}{(1 + y_x^2)^{3/2}}, 0 \right\rangle \frac{1}{\sqrt{1 + y_x^2}}.
\] (43)

Taking $U = U_1 \hat{i}_x$, and using (42) and (43), equation (23) leads to
\[
(1 - \alpha') \gamma \frac{y_{xx}}{(1 + y_x^2)^{3/2}} - \alpha U_1 \frac{y_x}{(1 + y_x^2)^{1/2}} = -\Omega y.
\] (44)

Note that the vortex filament is rotating in the retrograde sense (i.e., opposite to that of the vortex core). Further, the friction term associated with $\alpha'$ on the left hand side can again be eliminated by renormalizing the vortex strength $\gamma$.

Put,
\[
y_x = \tan \theta
\] (45)
so $\theta$ is the angle between the tangent to the vortex filament and the x-axis. Equation (44) then becomes

$$\gamma \frac{d\theta}{ds} - \alpha U_1 \sin \theta = -\Omega y$$  \hspace{1cm} (46a)

or

$$\gamma \frac{d^2 \theta}{ds^2} - \alpha U_1 \cos \theta \cdot \frac{d\theta}{ds} + \Omega \sin \theta = 0.$$  \hspace{1cm} (46b)

In the ordinary-fluid limit, equation (46b) constitutes Euler's elastica (Love [27]), i.e., the finite deformation of a plane elastic filament of flexural rigidity $B$ under the action of the thrust $F$ applied at its ends,

$$B \frac{d^2 \theta}{ds^2} + F \sin \theta = 0.$$  \hspace{1cm} (47)

On the other hand, equation (46b) indicates a vortex growth via a normal-fluid flow driven instability\(^5\) and confirms that the friction term associated with $\alpha$ plays the dual roles of driving force and drag force ([14], [15]) and hence leads to both growth and decay of the vortex line length. Indeed, assuming $|\Omega/\gamma| \ll 1$ and expanding in powers of $\theta$, equation (46b) can be approximated by

$$\gamma \frac{d^2 \theta}{ds^2} + \alpha U_1 \left(\frac{\theta^2}{2} - 1\right) \frac{d\theta}{ds} + \Omega \theta = 0$$  \hspace{1cm} (48)

which is just the van der Pol equation. If the normal-fluid velocity is in the same direction as that of vorticity in the undisturbed vortex filament, then there is decay/growth of the vortex line length if $|\theta|^2 \gtrsim 4$ and vice versa if the normal-fluid velocity is in the direction opposite to that of the vorticity in the undisturbed vortex filament. The friction term associated with $\alpha$ appears again to be able to change the vortex motion aspects in a qualitative way.

6. Discussion

Beginning with the work of Feynman [7], the phenomenological model of quantized vortices as classical vortex filaments subject to an effective frictional force (simulating interactions with thermal excitations in superfluid $^4$He) has been the standard approach to investigate vortex dynamics in superfluid $^4$He. The theoretical formulations developed in this paper provide insight into the fundamental importance of the friction term associated with the friction coefficient $\alpha$ as well as a proper rationale for the previous neglect of the other friction term associated with the friction coefficient $\alpha'$. Further, though the friction term associated with $\alpha$, for very small $\alpha$, has little capacity to make significant contribution to the vortex motion in a quantitative way it appears to be able to change the vortex motion aspects like the vortex kink characteristics in a qualitative way. This becomes possible because of the ability of the friction term to play the dual roles of driving force and drag force.

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\(^5\)Experiments by Sato et al. [28] on vortex dynamics in liquid $^4$He near $T_\lambda$ indicated vortex growth due to an instability.
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