MOMENT MAP FLOWS AND THE HECKE CORRESPONDENCE FOR QUIVERS

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ABSTRACT. In a series of papers, Nakajima uses quiver varieties to give a geometric construction of representations of Kac-Moody algebras and other related algebras. The generators of the algebras are constructed using the Hecke correspondence as a correspondence variety. The main result of this paper is a different construction of the Hecke correspondence using gradient flow lines for the norm-square of a moment map. We also prove a similar result for handsaw quivers and give gradient flow constructions of Nakajima’s Lagrangian subvariety and Kashiwara’s operators on crystal bases.

1. Introduction

There is a well known correspondence between Geometric Invariant Theory and symplectic geometry, which relates GIT quotients and symplectic quotients. A famous example is the Donaldson-Uhlenbeck-Yau theorem relating stability of holomorphic bundles to solutions of the Yang-Mills equations. Theorems of this type are often called Kempf-Ness theorems or Hitchin-Kobayashi correspondences.

For many examples of interest, there is also a symplectic geometric interpretation of unstable points in terms of moment map flows. The case of quiver varieties is studied in [6, Theorem 3]. On the symplectic geometry side of the picture, there is a Hamiltonian group action on the vector space of representations of the quiver, an associated moment map and the downwards gradient flow of the norm-square of the moment map converges to a critical point of this function. From the GIT point of view, there is a complex reductive group acting linearly on the space of representations with a notion of stability defined by a choice of parameters. Unstable points have an associated double filtration called the Harder-Narasimhan-Jordan-Hölder (HNJH) filtration and [6, Theorem 3] relates the moment map picture to the GIT picture by showing that the limit of the flow is isomorphic to the graded object of the Harder-Narasimhan-Jordan-Hölder filtration of the initial condition. This theorem is analogous to earlier theorems of [3] and [2] for the Yang-Mills flow and has since been generalised by Hoskins in [9] to reductive group actions on affine spaces.

The goal of this paper is to extend this idea further to flow lines between critical sets. The symplectic geometry determines the flow lines for the norm-square of the moment map and the algebraic side of the picture turns out to be related to Nakajima’s geometric constructions in representation theory from [15], [16], [17] and [18]. Here “flow line” is defined in an approximate sense since we use the exponential image of the negative eigenspace of the Hessian instead of the unstable manifold for the flow; see Definition 3.23.

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Critical points for the flow are direct sums of stable representations and each critical set deformation retracts onto a subset which is identified with a smaller quiver variety. The main object of study is the space of pairs of equivalence classes in these smaller quiver varieties that are connected by flow lines, which we denote by $M\mathcal{F}(Q, v_1, v_2)$ (see Definition 3.27 for precise details).

The first main theorem shows that this space $M\mathcal{F}$ is Nakajima’s Hecke correspondence from $[16]$ Sec. 5, which is denoted $B_k(Q, v)$ in the statement below.

**Theorem 1.1 (Theorem 4.2).** There is a homeomorphism $M\mathcal{F}(Q, v - e_k, v) \cong B_k(Q, v)$.

Theorem 4.13 shows that a similar relationship holds for the spaces $M\mathcal{F}$ and the Hecke correspondence associated to the handsaw quiver varieties of $[18]$.

The second main theorem characterises the Lagrangian subvariety $Z(Q, v_1, v_2)$ from $[16]$ Sec. 7 in terms of broken flow lines.

**Theorem 1.2 (Theorem 4.7).** Choose any $\bar{v} > v_1 + v_2$. Then $([x_1], [x_2]) \in Z(Q, v_1, v_2)$ if and only if there exist representatives $x_1' \in [x_1]$ and $x_2' \in [x_2]$ such that $(x_1', 0)$ and $(x_2', 0)$ are critical points for $\|\mu - \alpha\|^2$ on $\mu_{\bar{v}}^{-1}(0)_\bar{v}$ and there exists $g \in K_\bar{v}$ such that $g \cdot (x_1', 0)$ and $(x_2', 0)$ are connected to the same critical point by a (possibly broken) flow line.

The third main theorem gives a gradient flow interpretation of Nakajima’s geometric description (see $[16]$ Sec. 10.i) of Kashiwara’s operators $\tilde{E}_k$ and $\tilde{F}_k$ from $[10]$.

**Theorem 1.3 (Theorem 4.12).** Let $X$ be an irreducible component of $M(Q, v)_x$ and let $r = \varepsilon_k(X)$ as in $[16]$ (7.3). Then

$$\tilde{E}_k[X] = \left[ M_{k;0}(Q, v - re_k) \times M_{k;r-1}(Q, v - e_k) \cap p_1^{-1}(p_2(M_{k;0}(Q, v - re_k) \times (X \cap M_{k;r}(Q, v))) \right]$$

and

$$\tilde{F}_k[X] = \left[ M_{k;0}(Q, v - re_k) \times M_{k;r+1}(Q, v + e_k) \cap p_3^{-1}(p_2(M_{k;0}(Q, v - re_k) \times (X \cap M_{k;r}(Q, v))) \right].$$

The notational setup for the above theorem requires some background which is given in Section 4.4; however we can give an intuitive description of the theorem here in the introduction: In Nakajima’s geometric construction, Kashiwara’s operators $\tilde{E}_k$ and $\tilde{F}_k$ act on irreducible components of certain subvarieties of quiver varieties, which we can identify with subsets of the critical sets in our construction. Given a dimension vector $v$, fix such a subset of the critical set $C^0_v$ associated to $v$ (see Definition 3.11) and call it $X$. The action of the operators $\tilde{E}_k$ and $\tilde{F}_k$ on $X$ is to flow the subset $X$ up to the critical set $C^0_{v-re_k}$ and then flow the image down to the critical sets $C^0_{v-e_k}$ (for the operator $\tilde{E}_k$) and $C^0_{v+e_k}$ (for the operator $\tilde{F}_k$). The associated irreducible component is the action of the relevant operator on $X$.

1.1. **Some remarks and questions.** Throughout the paper we use unframed quivers, rather than the framed quivers of $[15]$, $[16]$, etc. In $[2]$, Crawley-Boevey constructs an unframed quiver representation associated to a framed quiver representation and identifies the two moduli spaces
(see also Remark 3.6 in this paper) and so the two perspectives are equivalent. In particular, we drop the notation $w$ that Nakajima uses for the dimension vector of the framing as this is now incorporated into the quiver $Q$. The reason for using unframed quivers is to remain in a general situation that can be applied to other types of quivers such as the handsaw quivers of [18].

The gradient flow results and the construction of the spaces of flow lines are valid for any quiver $Q$, however the proof of the relationship with the Hecke correspondence and Kashiwara’s operators in Sections 4.2 and 4.4 uses the condition that $Q$ has no loops.

Given the similarities between the methods of [6] (for quivers), [3] (for the Yang-Mills flow) and [23] (for the Yang-Mills-Higgs flow), it is natural to ask whether there is an analogous interpretation of flow lines for the Yang-Mills flow or the Yang-Mills-Higgs flow on a compact Riemann surface.

As mentioned previously, the moduli space of flow lines is defined in an approximate sense, where the exponential image of the negative eigenspace of the Hessian (which we call the negative slice) plays the role of the unstable manifold of a critical point. Another question is whether the results of Section 4 also apply to flow lines, rather than just approximate flow lines. This is true if one can show that there is a neighbourhood of each critical set on which there is a homeomorphism between the unstable manifold and the negative slice which is given by isomorphisms. If so, then since the isomorphism class of the limit of the downwards flow is an isomorphism invariant by [6, Theorem 3] then one can use this to construct a homeomorphism between spaces of flow lines and approximate flow lines. It is worth noting that the approximate flow lines used here fit the definition of $\varepsilon$-perturbed gradient segments from [12].

1.2. Organisation of the paper. Section 2 contains the background theory for the properties of the norm-square of the moment map on the vector space of complex representations of a quiver. In Section 3 we show how these properties restrict to a singular subset invariant under the group action and define the moduli spaces of flow lines. Section 4 contains the main results of the paper: a gradient flow interpretation of the affine projection, the Hecke correspondence, the Lagrangian subvariety and Kashiwara’s operators. Finally, in Section 4.5 we study the handsaw quiver varieties recently defined in [18] and show that the Hecke correspondence for these quivers also admits a similar gradient flow interpretation.

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2. Results for the smooth space $\text{Rep}(Q, v)$

This section contains the basic results and notational setup used in the rest of the paper. The goal is to study the gradient flow, the structure of the critical sets and the eigenspaces of the Hessian for the function $\|\mu - \alpha\|^2$ on the vector space $\text{Rep}(Q, v)$ (where we can apply theorems for smooth manifolds), before we restrict to the subvariety $\mu_{c}^{-1}(0)$ in Section 3.
2.1. Quiver varieties.

2.1.1. Representations of quivers.

**Definition 2.1.** A *quiver* $Q$ is a directed graph, consisting of vertices $I$, edges $E$, and head/tail maps $h, t : E \rightarrow I$.

A complex representation of a quiver consists of a collection of complex vector spaces $\{V_i\}_{i \in I}$, and $\mathbb{C}$-linear homomorphisms $\{A_a : V_{t(a)} \rightarrow V_{h(a)}\}_{a \in E}$. The dimension vector of a representation is the vector $v := (\dim \mathbb{C} V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$. The vector space of all representations with fixed dimension vector is denoted $\text{Rep}(Q, v) := \bigoplus_{a \in E} \text{Hom}(V_{t(a)}, V_{h(a)})$.

The group $G_v := \prod_{i \in I} \text{GL}(V_i, \mathbb{C})$ acts on the space $\text{Rep}(Q, v)$ via the induced action on each factor $\text{Hom}(V_{t(a)}, V_{h(a)})$

\begin{equation}
(g_i)_{i \in I} \cdot (A_a)_{a \in E} := \left( g_{h(a)} A_a g_{t(a)}^{-1} \right)_{a \in E}.
\end{equation}

The infinitesimal action of the Lie algebra $\mathfrak{g}_v$ at a representation $A \in \text{Rep}(Q, v)$ is denoted $\rho^\mathfrak{c}_A : \mathfrak{g}_v \rightarrow T_A \text{Rep}(Q, v) \cong \text{Rep}(Q, v)$. A calculation shows that

\begin{equation}
\rho^\mathfrak{c}_A(u) := \frac{d}{dt} \bigg|_{t=0} e^{tu} \cdot A = \bigoplus_{a \in E} (u_{h(a)} A_a A_{t(a)} - A_a u_{t(a)}) \in \bigoplus_{a \in E} \text{Hom}(V_{t(a)}, V_{h(a)}).
\end{equation}

The direct sum of all the vector spaces is denoted

$$\text{Vect}(Q, v) := \bigoplus_{i \in I} V_i.$$

Given a representation $A \in \text{Rep}(Q, v)$, we can consider each component $A_a$ as a homomorphism $\text{Vect}(Q, v) \rightarrow \text{Vect}(Q, v)$ via the inclusion $\text{Hom}(V_{t(a)}, V_{h(a)}) \subseteq \text{End}(\text{Vect}(Q, v))$.

2.1.2. Geometric Invariant Theory for quiver varieties. There is a notion of slope-stability for quivers introduced by King in [11], which matches the usual definition of stability from GIT. Recall from [11, Lemma 2.2] that GIT-stability on $\text{Rep}(Q, v)$ is equivalent to defining a lift of the $G_v$-action to a line bundle over $\text{Rep}(Q, v)$. In contrast to the case of GIT on a projective variety (where the line bundle is determined by the projective embedding), in this case the line bundle is the trivial bundle $\text{Rep}(Q, v) \times \mathbb{C}$, and the lift of the action is determined by the choice of a stability parameter.

Given $\alpha = (\alpha_i)_{i \in I} \in \mathbb{Z}^I$, define the lift of the $G_v$-action to $\text{Rep}(Q, v) \times \mathbb{C}$ by

\begin{equation}
g \cdot (A, \xi) := (g \cdot A, \chi_\alpha(g) \xi),
\end{equation}

where the character $\chi : G_v \rightarrow \mathbb{C}$ is defined to be

$$\chi_\alpha(g) = \prod_{i \in I} (\det g_i)^{\alpha_i}.$$
Definition 2.2. An admissible stability parameter for $\text{Rep}(Q, v)$ is a choice of $\alpha = (\alpha_i)_{i \in J} \in \mathbb{Z}^J$ such that
\[ \sum_{i \in J} \alpha_i v_i = 0. \]

Remark 2.3. The subgroup $\{ (\lambda \cdot \text{id})_{i \in J} : \lambda \in \mathbb{C}^* \} \subseteq G_v$ acts trivially on $\text{Rep}(Q, v)$. An equivalent definition of admissibility is that $\alpha$ is an admissible stability parameter if and only if the subgroup of scalar multiples of the identity in $G_v$ also acts trivially on the line bundle $\text{Rep}(Q, v) \times \mathbb{C}$. This is essential for the definition of stability in Definition 2.4, since all points would be unstable if the parameter is not admissible.

The definition of GIT stability and semistability with respect to an admissible stability parameter $\alpha$ is then the usual one (first described for representations of quivers in [11]), which we recall in the following.

Definition 2.4. A representation $A \in \text{Rep}(Q, v)$ is $\alpha$-semistable if, for all nonzero $\xi \in \mathbb{C}$, the closure of the $G_v$-orbit of $(A, \xi)$ in the trivial line bundle $\text{Rep}(Q, v) \times \mathbb{C}$ does not intersect the zero section, i.e.
\[ \overline{G_v \cdot (A, \xi) \cap (\text{Rep}(Q, v) \times \{0\})} = \emptyset. \]

A representation $A \in \text{Rep}(Q, v)$ is $\alpha$-polystable if $A$ is $\alpha$-semistable and the $G_v$-orbit of $(A, \xi)$ in $\text{Rep}(Q, v)$ is closed for all nonzero $\xi \in \mathbb{C}$.

A representation $A \in \text{Rep}(Q, v)$ is $\alpha$-stable if $A$ is $\alpha$-polystable and the isotropy group of $A$ in $G_v$ consists only of the scalar multiples of the identity.

The space of $\alpha$-stable (respectively $\alpha$-semistable and $\alpha$-polystable) representations is denoted $\text{Rep}(Q, v)^{\alpha-st}$ (respectively $\text{Rep}(Q, v)^{\alpha-ss}$ and $\text{Rep}(Q, v)^{\alpha-polyst}$).

Definition 2.5. The GIT quotient of $\text{Rep}(Q, v)$ by $G_v$ with respect to the stability parameter $\alpha$ is
\[ \mathcal{M}_\alpha(Q, v) = \text{Rep}(Q, v) \git\alpha G_v := \text{Rep}(Q, v)^{\alpha-ss} \git G_v = \text{Rep}(Q, v)^{\alpha-polyst} / G_v, \]
where the quotient $\git$ identifies $S$-equivalent orbits (those whose closures intersect) in the usual way.

Remark 2.6. It is sometimes more convenient to divide out by the scalar multiples of the identity (which act trivially) and use the projectivisation $\text{PG}_v$ instead. The quotients $\text{Rep}(Q, v)^{\alpha-ss} \git G_v$ and $\text{Rep}(Q, v)^{\alpha-ss} \git \text{PG}_v$ have the same underlying space, although, in the first case, when computing equivariant cohomology one has to remember the extra factor of $\mathbb{C}^*$ that acts trivially.

When $\alpha = 0$, then the lift of the $G_v$ action to $\text{Rep}(Q, v) \times \mathbb{C}$ is the trivial one, hence all representations $A \in \text{Rep}(Q, v)$ are semistable. Therefore, in this case the GIT quotient $\mathcal{M}_0(Q, v)$ is just the affine quotient $\text{Rep}(Q, v) \git G_v$. Every $G_v$ orbit in $\text{Rep}(Q, v)$ has a unique closed orbit.
in its closure (see [14, Theorem 4, p19] and [13, Sec. 8]), and the points in the affine quotient correspond to these closed orbits. Therefore there is a well-defined projection map

\[(2.4) \quad \pi : M_{\alpha}(Q, v) \to M_0(Q, v)\]

taking an orbit to the unique closed orbit in its closure (where we take the closure in \(\text{Rep}(Q, v)\)).

In analogy with holomorphic bundles, one can also define slope-stability of a representation in terms of the degree and rank (cf. [11]).

**Definition 2.7.** A subrepresentation of a representation \(A \in \text{Rep}(Q, v)\) consists of vector spaces \(\{V_i \subseteq V_i\}_{i \in J}\) such that \(A_a(V'_t(a)) \subseteq V'_h(a)\) for all edges \(a \in E\), and homomorphisms \(\{A'_a : V'_t(a) \to V'_h(a)\}_{a \in E}\) such that \(A'_a\) is the restriction of \(A_a\) to \(V'_t(a)\) for all \(a \in E\).

For a given subrepresentation \(A' \in \text{Rep}(Q, v')\), let \(v' := (\dim_{\mathbb{C}} V'_i)_{i \in J}\) be the associated dimension vector. Then \(A' \in \text{Rep}(Q, v') \subseteq \text{Rep}(Q, v)\). We can now define the degree and rank of a subrepresentation.

**Definition 2.8.** Let \(Q\) be a quiver, \(\alpha = (\alpha_i)_{i \in J}\) an admissible stability parameter, and \(v' = (v_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J\) a dimension vector. The \(\alpha\)-degree of \((Q, v')\) is

\[\deg_{\alpha}(Q, v') := \sum_{i \in J} \alpha_i v_i,\]

and the rank is

\[\text{rank}(Q, v') := \sum_{i \in J} v_i.\]

The \(\alpha\)-slope of \((Q, v')\) is

\[\text{slope}_{\alpha}(Q, v') := \frac{\deg_{\alpha}(Q, v')}{\text{rank}(Q, v')}\]

**Remark 2.9.** The stability parameter \(\alpha\) is admissible for \(\text{Rep}(Q, v)\) if and only if \(\deg_{\alpha}(Q, v) = 0\).

The following theorem of King then shows that, in analogy with holomorphic bundles, \(\alpha\)-stability and \(\alpha\)-semistability have an interpretation in terms of the slopes of subrepresentations.

**Proposition 2.10** (Proposition 3.1 of [11]). Let \(Q\) be a quiver, \(v\) a dimension vector, and \(\alpha\) an admissible stability parameter. A representation \(A \in \text{Rep}(Q, v)\) is \(\alpha\)-stable (resp. \(\alpha\)-semistable) if and only if every proper non-zero subrepresentation satisfies

\[\text{slope}_{\alpha}(Q, v') < 0 \quad (\text{respectively, } \text{slope}_{\alpha}(Q, v') \leq 0).\]

Given a subrepresentation, one would often like to study the question of stability/semistability for that subrepresentation; for example, when classifying the critical sets of \(\|\mu - \alpha\|^2\) in Sections 2.4 and 3.2. This requires a choice of stability parameter for the subrepresentation. In general it is not possible to use the same stability parameter \(\alpha\), since \(\deg_{\alpha}(Q, v')\) may not be zero, and therefore \(\alpha\) may not be admissible for \((Q, v')\). Instead, the correct definition involves subtracting a scalar multiple of the vector \((1)_j\) from \(\mu\) so that \((Q, v')\) has degree zero with respect to the new parameter.
Definition 2.11. Let $Q$ be a quiver, $v$ a dimension vector, and $\alpha = (\alpha_j)_{j \in \mathbb{J}}$ an admissible stability parameter for $(Q, v)$. Given any dimension vector $v' \leq v$, the induced stability parameter on $(Q, v')$ is

$$\alpha' = (\alpha_j - \text{slope}_{\alpha}(Q, v'))_{j \in \mathbb{J}}.$$ 

Note that it is easy to see that the induced stability parameter is admissible, since $\deg_{\alpha'}(Q, v') = \deg_{\alpha}(Q, v') - \deg_{\alpha}(Q, v') = 0$.

The final result of the section is used in Section 4.5.

Lemma 2.12. Let $Q$ be a quiver and let $\bar{Q}$ denote the quiver with the same vertices, but with the direction of all edges reversed. Fix Hermitian structures on the vector spaces $\{V_k\}_{k \in \mathbb{J}}$. Then $A \in \text{Rep}(Q, v)$ is $\alpha$-stable (resp. semistable, polystable) if and only if the adjoint $A^* \in \text{Rep}(\bar{Q}, v)$ is $-\alpha$-stable (resp. semistable, polystable).

Proof. Suppose that there is a subrepresentation with dimension vector $v'$ preserved by $A^*$ and let $\mu = \text{slope}_{-\alpha}(\bar{Q}, v') = -\text{slope}_{\alpha}(Q, v')$. Then the orthogonal complement of the subrepresentation is preserved by $A$ and so $\text{slope}_{\alpha}(Q, v - v') < 0$ (resp. $\leq 0$) since $A$ is $\alpha$-stable (resp. semistable). Therefore, since $\alpha$ is an admissible stability parameter, then $\text{slope}_{\alpha}(Q, v') > 0$ (resp. $\geq 0$) and so $\text{slope}_{-\alpha}(\bar{Q}, v') < 0$ (resp. $\leq 0$). Therefore $A$ is $\alpha$-stable (resp. semistable) if and only if $A^*$ is $-\alpha$-stable (resp. semistable).

Since $A$ is a direct sum of subrepresentations if and only if the adjoint $A^*$ is also a direct sum, then the above argument shows that $A$ is $\alpha$-polystable iff $A^*$ is $-\alpha$-polystable. □

2.2. The algebraic stratification. The Harder-Narasimhan stratification for quivers is defined in analogy with the case of holomorphic bundles (see [1] and [7] for holomorphic bundles, and [19] Section 2 for quivers).

The filtration is denoted by the sequence

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$$

of subrepresentations such that for each $j = 1, \ldots, n$, the quotient $A_j/A_{j-1}$ is the maximal semistable subrepresentation of $A/A_{j-1}$ (where the stability parameter is the one induced on the quotient using Definition 2.11). The associated dimension vectors induce a canonical filtration

$$\{0\} = \text{Vect}(Q, v_0) \subseteq \text{Vect}(Q, v_1) \subseteq \cdots \subseteq \text{Vect}(Q, v_n) = \text{Vect}(Q, v)$$

called the Harder-Narasimhan filtration, and the dimension vectors $v^* = (v_1, v_2 - v_1, \ldots, v_n - v_{n-1})$ form a vector called the Harder-Narasimhan type of the filtration. Note that the inclusion maps in (2.6) are induced from the representation $A$, so that the spaces $\text{Vect}(Q, v_j) \subseteq \text{Vect}(Q, v)$ are all $A$-invariant.

Definition 2.13. The length of the Harder-Narasimhan filtration (2.6) is equal to $n$, the number of non-trivial terms in the filtration.
Definition 2.14. The Harder-Narasimhan stratum with Harder-Narasimhan type $v^*$ is
\begin{equation}
B_{v^*} := \{ A \in \text{Rep}(Q, v) : A \text{ has H-N type } v^* \} \subseteq \text{Rep}(Q, v).
\end{equation}

Since the filtration is canonical then every representation belongs to exactly one Harder-Narasimhan stratum, and so we have a disjoint union
$$
\text{Rep}(Q, v) = \bigcup_{\text{HN types } v^*} B_{v^*}.
$$

There is a partial ordering on the strata given in [19, Definition 3.6] (analogous to that for holomorphic bundles described by Shatz in [20]), and [19, Proposition 3.7] shows that the closure of each stratum $B_{v^*}$ is contained in the union of all $B_{w^*}$ such that $w^* \geq v^*$.

2.2.1. The Jordan-Hölder filtration. Any semistable representation also has a Jordan-Hölder filtration, given by the following

Definition 2.15. Let $A \in \text{Rep}(Q, v)^{\alpha-ss}$ be an $\alpha$-semistable representation. A filtration
$$
\{0\} = \text{Vect}(Q, v_0) \subset \text{Vect}(Q, v_1) \subset \cdots \subset \text{Vect}(Q, v_m) = \text{Vect}(Q, v)
$$
with induced subrepresentations of $A$
$$
0 = A_0 \subset A_1 \subset \cdots \subset A_m = A,
$$
is called a Jordan-Hölder filtration if each quotient representation $A_j/A_{j-1}$ is stable with respect to the stability parameter on $\text{Rep}(Q, v_j - v_{j-1})$ induced by $\alpha$, and each subrepresentation has the same slope.

In contrast to the Harder-Narasimhan filtration, the Jordan-Hölder filtration is not necessarily unique, but the graded object
$$
\text{Gr}^{\text{JH}}(A) = \bigoplus_{j=1}^{m} A_j/A_{j-1}
$$
is unique up to isomorphism.

Combining the Harder-Narasimhan filtration with the Jordan-Hölder filtration, for any representation $A \in \text{Rep}(Q, v)$ we obtain a double filtration called the Harder-Narasimhan-Jordan-Hölder filtration (cf. [6, Sec. 5] for quivers and [4] for holomorphic bundles). Again, this is not necessarily unique, but the graded object $\text{Gr}^{\text{HNJH}}(A)$ is unique up to isomorphism.

The next lemma shows that the graded object of the Jordan-Hölder filtration is related to the projection to the affine quotient from (2.4) (recall that every representation is $\alpha$-semistable when $\alpha = 0$). See also [16, Proposition 3.20].

Lemma 2.16. Let $A \in \text{Rep}(Q, v)$. Then $\pi([A])$ is the isomorphism class of the graded object of the Jordan-Hölder filtration of $A$, where the Jordan-Hölder filtration is taken with respect to the stability parameter zero.
Proof. First note that \( \text{Gr}^{JH}(g \cdot A) \cong g \cdot \text{Gr}^{JH}(A) \) for all \( g \in G_v \), and so the map \([A] \mapsto [\text{Gr}^{JH}(A)]\) is well-defined. Secondly, note that \( \text{Gr}^{JH}(A) \) is in the closure of the \( G_v \)-orbit of \( A \), since one can obtain \( \text{Gr}^{JH}(A) \) from \( A \) by scaling the extension classes. Finally, the \( G_v \)-orbit of \( \text{Gr}^{JH}(A) \) is closed (see [11, Proposition 3.2]), and so this orbit is the unique closed orbit in the closure of \( G_v \cdot A \). \( \square \)

2.3. The symplectic quotient. Another theorem of King ([11, Theorem 6.1]) identifies the GIT quotient of \( \text{Rep}(Q, v) \) with the symplectic quotient. Since this equivalence is central to this paper, then we recall the details here.

Let \( Q \) be a quiver with dimension vector \( v = (v_i)_{i \in J} \), and fix a Hermitian structure on the vector spaces \( V_i \cong \mathbb{C}^{v_i} \). There is an associated symplectic structure on \( \text{Rep}(Q, v) \), defined as follows. Given tangent vectors \( \delta A_1, \delta A_2 \in T_A \text{Rep}(Q, v) \cong \text{Rep}(Q, v) \), define the metric

\[
(2.8) \quad g(\delta A_1, \delta A_2) := \sum_{a \in E} \Re \text{Tr} ( (\delta A_1)_a (\delta A_2)_a^* ),
\]
and symplectic structure

\[
(2.9) \quad \omega(\delta A_1, \delta A_2) := \sum_{a \in E} \Im \text{Tr} ( (\delta A_1)_a (\delta A_2)_a^* ).
\]

Note that \( \omega(\delta A_1, \delta A_2) = g(i\delta A_1, \delta A_2) \), in other words the complex structure \( I = i \cdot \text{id} \) is compatible with the metric. With this complex structure and metric, the space \( \text{Rep}(Q, v) \) has the structure of a Kähler manifold.

With respect to the Hermitian structure on each \( V_i \), one can define the unitary group \( U(V_i) \subset \text{GL}(V_i, \mathbb{C}) \), and therefore the compact subgroup

\[
K_v := \prod_{i \in J} U(V_i) \subset G_v.
\]

The induced action of \( K_v \) on \( \text{Rep}(Q, v) \) is given by

\[
(2.10) \quad (g_j)_{j \in J} \cdot (A_a)_{a \in E} = \left( g_{h(a)} A_a g_{t(a)}^{-1} \right)_{a \in E},
\]
and the infinitesimal action of the Lie algebra \( \mathfrak{k}_v \), denoted \( \rho_A : \mathfrak{k}_v \to T_A \text{Rep}(Q, v) \cong \text{Rep}(Q, v) \), is given by

\[
(2.11) \quad \rho_A(u) := \frac{d}{dt} \bigg|_{t=0} e^{tu} \cdot A = \bigoplus_{a \in E} (u_{h(a)} A_a - A_a u_{t(a)}).
\]

This action is Hamiltonian, i.e. it preserves the symplectic structure and has an associated moment map

\[
\mu : \text{Rep}(Q, v) \to \mathfrak{k}_v^* \quad \text{with} \quad (A_a)_{a \in E} \mapsto \frac{1}{2i} \sum_{a \in E} [A_a, A_a^*]
\]
that satisfies \( d\mu(\delta A) \cdot u = \omega(\rho_A(u), \delta A) \) for all \( \delta A \in \text{Rep}(Q, v) \cong T_A \text{Rep}(Q, v) \), and \( u \in \mathfrak{g} \). Note that when writing the commutator \([A_a, A_a^*]\) we think of each \( A_a \) as an element of \( \text{End}(\text{Vect}(Q, v)) \).
Note also that the above definition implies that $\text{Tr} \mu(A) = 0$, since $\mu(A)$ is constructed from commutators. Therefore, for the symplectic quotient to make sense, we need the following definition.

**Definition 2.17.** Let $Q$ be a quiver, and $v = (v_j)_{j \in J} \in \mathbb{Z}_{\geq 0}^J$ a dimension vector. The central element $\alpha = (i \alpha_j \cdot \text{id}_{V_j})_{j \in J} \in \mathfrak{z}(E^*)$ is an **admissible central element** if $\sum_{j \in J} \alpha_j v_j = 0$.

The **symplectic quotient** with respect to an admissible central element $\alpha$ is $M_\alpha(Q, v) := \mu^{-1}(\alpha) / K_v$.

**Remark 2.18.**
1. The parameter $\alpha$ is admissible if and only if $\alpha$ is a central element of the dual of the Lie algebra of $PK$.
2. Given an admissible stability parameter $(\alpha_j)_{j \in J} \in \mathbb{Z}^J$ one can construct an admissible central element $(i \alpha_j \cdot \text{id}_{V_j})_{j \in J}$ and vice-versa. In the rest of the paper both of these will be denoted $\alpha$, and the meaning will be clear from the context.

**Proposition 2.19.** Let $Q$ be a quiver, $v$ a dimension vector, and $\alpha$ an admissible stability parameter. Then the GIT quotient $\text{Rep}(Q, v)^{\alpha-ss} / G_v$ is homeomorphic to the symplectic quotient $\mu^{-1}(\alpha) / K_v$.

See [9, Theorem 4.2] for a proof.

### 2.4. Critical points of $\|\mu - \alpha\|^2$ and the gradient flow.

Recall from [6, Sec. 3.2] that a representation $A \in \text{Rep}(Q, v)$ is a critical point for $\|\mu - \alpha\|^2$ if and only if the infinitesimal action of $K_v$ satisfies

$$\rho_A(\mu(A) - \alpha) = 0 \quad \text{for all} \ a \in \mathcal{E}. \quad (2.12)$$

More explicitly, this is equivalent to the condition that

$$A_a(\mu(A) - \alpha)_{i(a)} - (\mu(A) - \alpha)_{h(a)} A_a = 0 \quad \text{for all} \ a \in \mathcal{E}. \quad (2.13)$$

This equation implies that the representation $A$ splits into subrepresentations, each of which corresponds to an eigenspace of $i(\mu(A) - \alpha)$ (the factor of $i$ is used so that the eigenvalues are real; see (2.15)). In other words, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $i(\mu(A) - \alpha)$, then for each eigenvalue $\lambda_j$ there exists a dimension vector $v_j$ such that $v_1 + \cdots + v_n = v$, and

$$A = \bigoplus_{j=1}^n A_j, \quad \text{Vect}(Q, v) \cong \bigoplus_{j=1}^n \text{Vect}(Q, v_j) \quad (2.14)$$

where $A_j \in \text{Rep}(Q, v_j)$ for each $j$. Since $\mu(A)$ is constructed from commutators, then $\text{Tr} \mu(A) = 0$ on each subrepresentation, and therefore taking the trace of $i(\mu(A) - \alpha)$ shows that

$$\lambda_j = \text{slope}_\alpha(Q, v_j). \quad (2.15)$$

Moreover, restriction to a subrepresentation with dimension vector $v_j$ induces a new stability parameter $\alpha_j$ on $\text{Rep}(Q, v_j)$ (see Definition 2.11), and a direct sum of representations such as that
described in (2.14) is critical if and only if each $A_j$ is a minimum for $\|\mu - \alpha\|^2$ on $\text{Rep}(Q, v_j)$. See [6, Proposition 1] for more details.

For each critical set there is a corresponding decomposition $v = v_1 + \cdots + v_n$. The critical type of a critical point is the vector $v^* = (v_1, \ldots, v_n)$, where the dimension vectors are ordered by decreasing slope, i.e. $\text{slope}_\alpha(Q, v_i) > \text{slope}_\alpha(Q, v_j)$ if and only if $i < j$. The set of all critical points with critical type $v^*$ is denoted $C_{v^*}$.

**Definition 2.20.** Let $\gamma^{-}(A_0, t) \in \text{Rep}(Q, v)$ denote the downwards gradient flow of $\|\mu - \alpha\|^2$ at time $t$ with initial condition $A_0 \in \text{Rep}(Q, v)$.

It follows from Sjamaar’s compactness result in [22] and the Lojasiewicz inequality technique of Simon in [21] that $\gamma^{-}(A_0, t)$ exists for all time $t \geq 0$ and converges to a unique limit $A_\infty = \lim_{t \to \infty} \gamma(A_0, t)$.

The main theorem of [6] gives an algebraic description of the limit of the downwards gradient flow of $\|\mu - \alpha\|^2$ (see also [9]). This will be used in Section 3.4 to characterise the pairs of critical points connected by a flow line.

**Theorem 2.21.** Let $Q$ be a quiver and $\alpha$ a stability parameter for $Q$. Given a dimension vector $v$ for $Q$, let $A \in \text{Rep}(Q, v)$. Then

1. For any initial condition $A_0 \in \text{Rep}(Q, v)$, the gradient flow $\gamma^{-}(A_0, t)$ converges to a limit $A_\infty$, which is a critical point of $\|\mu - \alpha\|^2$.
2. ([6, Theorem 8, p336]) The limit $A_\infty$ is isomorphic to the graded object of the Harder-Narasimhan-Jordan-Hölder double filtration of $A$.
3. ([6, Proposition 2, p320]) The gradient flow defines a continuous $K_v$-equivariant deformation retraction of each Harder-Narasimhan stratum $B_{v^*}$ onto the associated critical set $C_{v^*}$.

### 2.5. A description of the eigenspaces of the Hessian at a critical point.

Recall from (2.12) that the critical point equation for $\|\mu - \alpha\|^2$ on $\text{Rep}(Q, v)$ is

$$I_{\rho_\Lambda}(\mu(A) - \alpha) = 0 \iff [(\mu(A) - \alpha), A_a] = 0 \text{ for all } a \in \mathcal{E},$$

and recall from (2.14) that any representation satisfying this equation must split into the direct sum of subrepresentations

$$(2.16) \quad A = A_1 \oplus \cdots \oplus A_n,$$

where each $A_j \in \text{Rep}(Q, v_j)$, and $v_1 + \cdots + v_n = v$. In addition, each $A_j$ minimises the function $\|\mu - \alpha_j\|^2$ on $\text{Rep}(Q, v_j)$, where $\alpha_j$ is the stability parameter on $\text{Rep}(Q, v_j)$ induced from $\alpha$. 
Also recall from \((2.15)\) that 
\[ i(\mu(A_j) - \alpha) = \text{slope}_\alpha(Q, v_j) \cdot \text{id} \] 
for each \( j = 1, \ldots, n \), and so \( i(\mu(A) - \alpha) \) has the block-diagonal form
\[
(2.17) \quad i(\mu(A) - \alpha) = \begin{pmatrix}
\lambda_1 \cdot \text{id} & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 \cdot \text{id} & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 \cdot \text{id} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_n \cdot \text{id}
\end{pmatrix},
\]
where \( \lambda_j = \text{slope}_\alpha(Q, v_j) \) for each \( j = 1, \ldots, n \). The eigenvalues in \((2.17)\) are ordered so that \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \) (i.e. the slope increases with \( j \)).

**Definition 2.22.** The derivative of the infinitesimal action is
\[
\delta \rho_A : \mathfrak{t} \times T_A \text{Rep}(Q, v) \to T_A \text{Rep}(Q, v)
\]
\[(2.18) \quad (u, X) \mapsto \left. \frac{d}{dt} \right|_{t=0} \rho_{A+tX}(u). \]

An explicit formula for \( \delta \rho \) is
\[
(2.19) \quad \delta \rho_A(u)(X) = \sum_{a \in \mathcal{E}} [u, X_a].
\]

**Remark 2.23.**
(1) Note that the tangent bundle of \( \text{Rep}(Q, v) \) is trivial, and therefore we can use the trivial connection in the above definition.

(2) From the definition of the complex structure \( I \) in \((2.9)\) we have \( \delta \rho_A(u)(IX) = I \delta \rho_A(u)(X) \).

**Lemma 2.24.** At a critical point \( A \in \text{Rep}(Q, v) \), the Hessian \( H_f : T_A \text{Rep}(Q, v) \to T_A \text{Rep}(Q, v) \) has the form
\[
H_f(\delta A) = -I \rho_A \rho_A^* I \delta A + I \delta \rho_A(\mu(A))(\delta A).
\]

**Proof.** Recall that the gradient of \( f = \frac{1}{2} \| \mu - \alpha \|^2 \) at a representation \( A \in \text{Rep}(Q, v) \) is given by
\[
\text{grad} f(A) = I \rho_A(\mu - \alpha).
\]

Differentiating the gradient of \( f \) at \( A \in \text{Rep}(Q, v) \) in the direction of a tangent vector \( X \in T_A \text{Rep}(Q, v) \) gives us
\[
(2.20) \quad H_f(A)(X) = \nabla_X \text{grad} f(A) = I \delta \rho_A(\mu - \alpha)(X) + I \rho_A d\mu(X)
\]
\[= I \delta \rho_A(\mu - \alpha)(X) - I \rho_A \rho_A^* IX,
\]
where again we use the trivial connection on the tangent bundle of \( \text{Rep}(Q, v) \). \qed

The next two lemmas contain some identities that will be useful in characterising the negative eigenspace of the Hessian. In order to be completely clear about the sign conventions then all the details are included.

**Lemma 2.25.** For any \( A \in \text{Rep}(Q, v) \) we have
\[
(2.21) \quad d\mu(x) = -\rho_A^* I,
\]
where we use the inner product on $\mathfrak{k}_v$ to identify $\mathfrak{k}_v \cong \mathfrak{k}_v^*$. For any $u \in \mathfrak{k}_v$ we also have
\begin{equation}
\rho^*_AI\rho_A(u) = [\mu(A), u].
\end{equation}

**Proof.** Using the moment map equation, we know that
\[ d\mu(X) \cdot v = \omega(\rho_A(v), X) = g(I\rho_A(v), X) = -\langle v, \rho^*_AI X \rangle \]
for all $v \in \mathfrak{k}_v$, where we use $d\mu(X) \cdot v$ to denote the dual pairing $\mathfrak{k}_v^* \times \mathfrak{k}_v \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle$ to denote the inner product on $\mathfrak{k}_v$. This proves (2.21). Setting $X = \rho_A(u)$ gives us
\[ -\langle v, \rho^*_AI \rho_A(u) \rangle \geq d\mu(\rho_A(u)) \cdot v \]
for all $v \in \mathfrak{k}_v$, and so we can identify $\rho^*_A I \rho_A(u) = -d\mu(\rho_A(u))$ (where we use the inner product on $\mathfrak{k}_v$ to identify $\mathfrak{k}_v$ with $\mathfrak{k}_v^*$). Equivariance of the moment map with respect to the action of $K$ gives us
\[ \mu(e^{tu} \cdot A) = e^{tu} \mu(A)e^{-tu} \Rightarrow d\mu(\rho_A(u)) = \left. \frac{d}{dt} \right|_{t=0} e^{tu} \mu(A)e^{-tu} = [u, \mu(A)]. \]
Therefore $\rho^*_A I \rho_A(u) = [\mu(A), u]$, as required.

Differentiating this result at a critical point $A$ gives us
\[ (\delta \rho_A)^*(I \rho_A(u), X) + \rho^*_A I \delta \rho_A(u)(X) = [d\mu(X), u] \]
Using the fact that $I \delta \rho_A(u)(X) = \delta \rho_A(u)(IX)$ then gives us
\[ \rho^*_A (\delta \rho_A(u)(X)) = -\rho^*_A I \delta \rho_A(u)(IX) = [\rho^*_A X, u] - (\delta \rho_A)^*(I \rho_A(u), X). \]
Therefore, we have proven

**Lemma 2.26.**
\begin{align}
(2.23) & \quad \rho^*_A I \delta \rho_A(u)(X) = -[\rho^*_A IX, u] \\
(2.24) & \quad \rho^*_A (\delta \rho_A(u)(X)) = [\rho^*_A X, u] - (\delta \rho_A)^*(I \rho_A(u), X)
\end{align}

The next lemma will be used in the proof of Lemma 2.31.

**Lemma 2.27.** Let $A$ be a critical point of $\|\mu - \alpha\|^2$. Then for any $v \in \mathfrak{k}_v$ we have
\begin{equation}
\rho_A([\mu(A), v]) = \delta \rho_A(\mu(A) - \alpha)(\rho_A(v)).
\end{equation}

**Proof.** Firstly note that $[\mu(A), v] = [\mu(A) - \alpha, v]$ since $\alpha$ is central. Therefore
\[ \rho_A([\mu(A) - \alpha, v]) = \rho_A(\text{ad}_{\mu(A) - \alpha}(v)) \]
\[ = \left. \frac{\partial}{\partial t} \right|_{t=0} \rho_A(\text{Ad}_{\exp(t(\mu(A) - \alpha))}(v)) \]
\[ = \left. \frac{\partial^2}{\partial s\partial t} \right|_{s,t=0} \left( e^{\rho(\mu(A) - \alpha)} e^{sv} e^{-t(\mu(A) - \alpha)} \right) \cdot A \]
\[ = \left. \frac{\partial}{\partial s} \right|_{s=0} \left( \rho_{e^{sv} \cdot A}(\mu(A) - \alpha) - e^{sv} \cdot \rho_A(\mu(A) - \alpha) \right), \]
where $e^{sv} \cdot \rho_A(\mu(A) - \alpha)$ denotes the action of $e^{sv} \in K_v$ on the tangent vector $\rho_A(\mu(A) - \alpha) \in T_A \text{Rep}(Q, v)$, which maps it to an element of $T_{e^{sv}A} \text{Rep}(Q, v)$. This term vanishes since $A$ is a critical point, and so we have

$$\rho_A([\mu(A) - \alpha, v]) = \frac{\partial}{\partial s}\Big|_{s=0} \rho_{e^{sv}A}(\mu(A) - \alpha) = \delta \rho_A(\mu(A) - \alpha)(\rho_A(v))$$

as required. \hfill \square

Since $H_f$ is self-adjoint then the tangent space splits into the orthogonal direct sum of eigenspaces and each eigenvalue is real. The next lemma describes the negative eigenspace of the Hessian.

**Lemma 2.28.** Let $A \in \text{Rep}(Q, v)$ be a critical point of $f = \frac{1}{2}\|\mu - \alpha\|^2$, and let $X \in T_A \text{Rep}(Q, v)$. Suppose that $H_f(X) = \lambda X$ for some $\lambda \neq 0$. Then $X \in \ker \rho_A^*$. Moreover, if $\lambda < 0$ then $X \in \ker \rho_A^* I$ also and so the negative eigenspaces of the Hessian are orthogonal to the $G_v$-orbit through $A$.

**Proof.** Since $f(A) = \|\mu(A) - \alpha\|^2$ is $K_v$-invariant then the non-zero eigenspaces of $H_f(X)$ are orthogonal to the $K_v$-orbit through $A$ and therefore $0 = \frac{1}{\lambda}\rho_A^* H_f(X) = \rho_A^* X$. An explicit proof involves applying $\rho_A^*$ to both sides of the equation $H_f(X) = \lambda X$ and using equations (2.22) and (2.23) gives us

$$\rho_A^* I \delta \rho_A(\mu - \alpha)(X) - \rho_A^* I \rho_A \rho_A^* IX = \lambda \rho_A^* X$$

(2.26)

$$\iff -[\rho_A^* IX, \mu - \alpha] - [\mu - \alpha, \rho_A^* IX] = \lambda \rho_A^* X$$

$$\iff 0 = \lambda \rho_A^* X.$$  

Since $\lambda \neq 0$ then $\rho_A^* X = 0$.

Now suppose that $H_f(X) = \lambda X$ for some $\lambda < 0$. Applying $\rho_A^* I$ to both sides of the eigenvalue equation and using (2.24) and the critical point equation gives us

$$-\rho_A^* \delta \rho_A(\mu - \alpha)(X) + \rho_A^* I \rho_A \rho_A^* IX = \lambda \rho_A^* IX$$

(2.27)

$$\iff -[\rho_A^* X, \mu - \alpha] + \rho_A^* I \rho_A (\rho_A^* IX) = \lambda \rho_A^* IX.$$

Since we have already shown that $\rho_A^* X = 0$, then

$$\rho_A^* I \rho_A (\rho_A^* IX) = \lambda \rho_A^* IX,$$

and so $\rho_A^* IX = 0$, since $\lambda < 0$ and the operator $\rho_A^* I \rho_A$ is non-negative definite. \hfill \square

**Corollary 2.29.** The Hessian $H_f$ preserves $\ker(\rho_A^*)^*$.

**Proof.** Note that $\text{im} \rho_A^* = \text{im} \rho_A + \text{im} I \rho_A$ and so $\ker(\rho_A^*)^* = (\text{im} \rho_A^*)^\perp = (\text{im} \rho_A)^\perp \cap (\text{im} I \rho_A)^\perp = \ker \rho_A^* \cap \ker(\rho_A^* I)$. Let $X \in \ker(\rho_A^*)^*$. Then (2.26) shows that

$$\rho_A^* H_f(X) = \rho_A^* I \delta \rho_A(\mu - \alpha)(X) - \rho_A^* I \rho_A \rho_A^* IX = -[\rho_A^* IX, \mu - \alpha] - [\mu - \alpha, \rho_A^* IX] = 0.$$  

Moreover, (2.27) shows that

$$\rho_A^* H_f(X) = -[\rho_A^* X, \mu - \alpha] + \rho_A^* I \rho_A (\rho_A^* IX) = 0,$$

since $X \in \ker \rho_A^* \cap \ker \rho_A^* I$. Therefore $H_f(X) \in \ker \rho_A^* \cap \ker \rho_A^* I = \ker(\rho_A^*)^*$. \hfill \square
Definition 2.30. Given $\lambda \in \mathbb{R}$, let $V_\lambda = \{ X \in T_A \text{Rep}(Q, v) : H_f(X) = \lambda X \}$ denote the $\lambda$-eigenspace of the Hessian.

Lemma 2.31. $\text{im} \rho_A \subseteq V_0$ and $\text{im} \rho_A^C$ splits into eigenspaces for $H_f$, with $\text{im} \rho_A^C \subseteq \bigoplus_{\lambda \geq 0} V_\lambda$.

Proof. The statement that $\text{im} \rho_A \subseteq V_0$ follows from the fact that the function $\|\mu - \alpha\|_2$ is $K_v$-invariant. One can also explicitly see this from the calculation
\[
H_f(\rho_A(u)) = I\delta\rho_A(\mu(A) - \alpha)(\rho_A(u)) - I\rho_A\rho_A^*I\rho_A(u) \\
= I\delta\rho_A(\mu(A) - \alpha)(\rho_A(u)) - I\rho_A(\mu(A), u) \quad \text{by (2.22)} \\
= I\delta\rho_A(\mu(A) - \alpha)(\rho_A(u)) - I\delta\rho_A(\mu(A) - \alpha)(\rho_A(u)) \quad \text{by (2.25)} \\
= 0.
\]

Since $H_f$ is self-adjoint and preserves $\ker(\rho_A^C)^*$ by Corollary 2.29 then $\text{im} \rho_A^C = (\ker(\rho_A^C)^*)^\perp$ is preserved also, and therefore it splits into eigenspaces for $H_f$. Lemma 2.28 then shows that each eigenvalue must be non-negative.

Given dimension vectors $v_1$ and $v_2$, with corresponding collections of vector spaces $\{V_k^1\}_{k \in \mathcal{I}}$ and $\{V_k^2\}_{k \in \mathcal{I}}$, define the spaces
\[
(2.28) \quad \text{Hom}^0(Q, v_1, v_2) := \bigoplus_{k \in \mathcal{I}} \text{Hom}(V_k^1, V_k^2) \\
(2.29) \quad \text{Hom}^1(Q, v_1, v_2) := \bigoplus_{a \in \mathcal{E}} \text{Hom}(V_{t(a)}^1, V_{h(a)}^2).
\]

The final result of this section is a characterisation of the negative eigenspace of the Hessian in terms of homomorphisms between the subrepresentations that appear in the splitting (2.16).

Proposition 2.32. Let $A$ be a critical point of $\|\mu - \alpha\|^2$ on $\text{Rep}(Q, v)$ corresponding to a decomposition (2.16). For $j, k = 1, \ldots, n$, define $\lambda_{j,k} := \lambda_j - \lambda_k = \text{slope}_\alpha(Q, v_k) - \text{slope}_\alpha(Q, v_j)$, where $\lambda_j$ and $\lambda_k$ are as in (2.17). Then if $\lambda < 0$ we have
\[
V_\lambda = \bigoplus_{j,k : \lambda_{j,k} = \lambda} (\ker \rho_A^C)^* \cap \text{Hom}^1(Q, v_j, v_k).
\]

Proof. Lemma 2.28 shows that when $\lambda < 0$ the negative eigenspace equation reduces to
\[
I\delta\rho_A(\mu - \alpha)(X) = \lambda X.
\]

Applying equations (2.17) and (2.19) completes the proof.

3. Nakajima quiver varieties

In this section we recall the definition of Nakajima quiver varieties and prove the necessary technical results for the main theorems in Section 4.
3.1. Nakajima quiver varieties. The definition of quiver variety given here is equivalent to that given by Nakajima in [15], however we use unframed quivers in order to match the notation used for the gradient flow results of [6]. Crawley-Boevey’s construction in [2] shows that it is sufficient to use unframed quivers to study the framed quiver varieties of [15]. The dimension vector \( w \) for the framing is now part of the quiver \( Q \), since the dimension of \( w \) at the \( k \)th vertex now corresponds to the number of edges from the \( k \)th vertex to the vertex \( \infty \) (see Remark 3.6 for more details) and so we drop the notation for \( w \) from the definition of quiver variety. The quiver varieties are denoted by \( \mathcal{M}(Q, v) \), where \( Q \) is the quiver from Remark 3.6 constructed from the original framed quiver and the dimension vector \( w \).

The motivation for this choice of definition is that we want to apply the same ideas of this section to other types of quiver varieties such as the handsaw quiver varieties of [18] (see Section 4.5). Rather than rederive everything from scratch for handsaw quivers, it is easier to develop the theory in general for unframed quivers and then apply an analog of Crawley-Boevey’s construction from [2] to relate it to the particular quiver variety under consideration.

**Definition 3.1.** Let \( Q \) be a quiver with vertices \( I \) and edges \( E \), and let \( v = (v_i)_{i \in I} \) a dimension vector such that one vertex (call it \( \infty \)) has dimension 1. Define \( J' = J \setminus \{ \infty \} \) be the set of remaining vertices of \( Q \). For such a quiver \( Q \) and dimension vector \( v \), the canonical stability parameter \( \alpha(Q, v) := (\alpha_i)_{i \in J} \) is given by

\[
\alpha_i := \begin{cases} 
- \sum_{j \in J'} v_{ji} & i = \infty \\ \ i \in J'. 
\end{cases}
\]

In this case we define

\[
\text{Vect}_0(Q, v) = \bigoplus_{k \in J'} V_k
\]

to be the direct sum of all the vector spaces except for the one at the vertex \( \infty \).

**Remark 3.2.** Via Crawley-Boevey’s construction in [2] (see also Remark 3.6), for a framed quiver variety this stability parameter is equivalent to the choice of character given by Nakajima in [16, Sec. 3].

**Remark 3.3.** The \( \alpha \)-semistable points are all \( \alpha \)-stable for this choice of stability parameter. To see this, note that there are two cases for a proper subrepresentation: (a) the subrepresentation does not contain the vertex \( \infty \) and so it must have strictly positive slope, or (b) the subrepresentation contains the vertex \( \infty \) and so it must have strictly negative slope. A subrepresentation of an \( \alpha \)-semistable representation must be in case (b).

Next we recall the hyperkähler structure on \( T^* \text{Rep}(Q, v) \). Let \( Q \) be a quiver with vertices \( J \) and edges \( E \). Given an edge \( a \in E \), define \( \bar{a} \) to be a new edge with the opposite orientation, i.e. \( t(\bar{a}) = h(a) \) and \( h(\bar{a}) = t(a) \). Now define a new set of edges \( \bar{E} \) by

\[
\bar{E} := \{ \bar{a} : a \in E \}.
\]
Let $\tilde{Q}$ be a quiver with the same vertices $\mathcal{I}$ as $Q$, but with edges $\mathcal{E} \cup \tilde{\mathcal{E}}$ and head/tail maps defined as above, and let $\bar{Q}$ be the quiver with vertices $\mathcal{I}$ and edges $\tilde{\mathcal{E}}$. Then there is an identification

$$\text{Rep}(\tilde{Q}, \mathbf{v}) \cong T^* \text{Rep}(Q, \mathbf{v}).$$

Given a representation $\tilde{A} \in \text{Rep}(\tilde{Q}, \mathbf{v})$, write $\tilde{A} = (A, B)$, where $A \in \text{Rep}(Q, \mathbf{v})$ and $B \in \text{Rep}(\bar{Q}, \mathbf{v})$. Identifying the tangent space at any point of $\text{Rep}(\tilde{Q}, \mathbf{v})$ with the vector space $\text{Rep}(\tilde{Q}, \mathbf{v})$ we have the following three complex structures

$$(3.3) \quad I : (A, B) \mapsto (iA, iB), \quad J : (A, B) \mapsto (B^*, -A^*), \quad K : (A, B) \mapsto (iB^*, -iA^*),$$

which satisfy the quaternionic relations $I^2 = J^2 = K^2 = -\text{id} = IJK$. Using the metric

$$(3.4) \quad g ((A_1, B_1), (A_2, B_2)) := \Re \text{Tr}(A_1 A_2^* + B_1 B_2^*),$$

we have the following symplectic structures

$$(3.5) \quad \omega_I (\cdot, \cdot) = g(I \cdot, \cdot), \quad \omega_J (\cdot, \cdot) = g(J \cdot, \cdot), \quad \omega_K (\cdot, \cdot) = g(K \cdot, \cdot).$$

Let $\omega_C (\cdot, \cdot) = \omega_J (\cdot, \cdot) + i\omega_K (\cdot, \cdot)$ denote the holomorphic symplectic form. This is given by the following explicit expression

$$(3.6) \quad \omega_C ((A_1, B_1), (A_2, B_2)) = \text{Tr}(A_2 B_1 - B_2 A_1).$$

For each complex structure $I, J, K$, the moment map equation $d\mu(X) \cdot u = \omega(\rho(u), X)$ is then satisfied by the following three moment maps

$$(3.7) \quad \mu_I (A, B) = \frac{1}{2i} \sum_{a \in \mathcal{E}} [A_a, A_a^*] + [B_a, B_a^*] \in \mathfrak{t}^*$$

$$(3.8) \quad \mu_J (A, B) = \frac{1}{2} \sum_{a \in \mathcal{E}} [A_a, B_a] + [A_a^*, B_a^*] \in \mathfrak{t}^*$$

$$(3.9) \quad \mu_K (A, B) = \frac{1}{2i} \sum_{a \in \mathcal{E}} [A_a, B_a] - [A_a^*, B_a^*] \in \mathfrak{t}^*.$$ We also define

$$(3.10) \quad \mu_C (A, B) = \mu_J (A, B) + i\mu_K (A, B) = \sum_{a \in \mathcal{E}} [A_a, B_a] \in \mathfrak{g}^*.$$

**Definition 3.4.** The quiver variety with quiver $Q$, stability parameter $\alpha$, and dimension vector $\mathbf{v}$ is the hyperkähler quotient

$$\mathcal{M}_\alpha^{HK} (Q, \mathbf{v}) := \mu_I^{-1}(\alpha) \cap \mu_C^{-1}(0)/K.$$ Since the space $\mu_C^{-1}(0)$ is closed and $G_\mathbf{v}$-invariant, then the definitions of $\alpha$-stability and $\alpha$-semistability restrict from $T^* \text{Rep}(Q, \mathbf{v}) \cong \text{Rep}(\tilde{Q}, \mathbf{v})$ to $\mu_C^{-1}(0)$. In particular, Proposition 2.19 also applies to the space $\mu_C^{-1}(0)$, and we have an identification of the GIT quotient with the hyperkähler quotient

$$(3.11) \quad \mathcal{M}_\alpha^{HK} (Q, \mathbf{v}) \cong \mu_C^{-1}(0)^{\alpha-ss} / G_\mathbf{v}.$$
It is also well known from \[8\] that the hyperkähler structure on \(\text{Rep}(\tilde{Q}, v)\) descends to the quotient \(M^{HK}_{\alpha}(Q,v)\) (see also \[5\] for a different proof).

When the quiver is an affine Dynkin diagram then the quiver varieties associated to two generic stability parameters are diffeomorphic (see [15, Corollary 4.2]). In the sequel we need that two quiver varieties are homeomorphic when one parameter is a positive scalar multiple of the other. Below we give an elementary proof of this result.

**Lemma 3.5.** If \(\beta = k\alpha\) for some real scalar \(k > 0\) then \(M^{HK}_{\beta}(Q,v) \cong M^{HK}_{\alpha}(Q,v)\).

**Proof.** Let \((A, B)\) be a solution to \(\mu_I(A,B) = \alpha\). Then \(\mu_I(\sqrt{k}A, \sqrt{k}B) = \beta\). Since \(\mu_C\) is a homogeneous quadratic polynomial in \((A, B)\) then solutions to \(\mu_C(A,B) = 0\) are preserved by scaling and so we have a continuous map \(M^{HK}_{\alpha}(Q,v) \to M^{HK}_{\beta}(Q,v)\). Similarly, the inverse \((A, B) \mapsto \frac{1}{\sqrt{k}}(A,B)\) is continuous and so \(M^{HK}_{\beta}(Q,v) \cong M^{HK}_{\alpha}(Q,v)\).

Equivalently, one can also note that the stability condition from Definition 2.4 is preserved if we multiply the stability parameter by a positive non-zero scalar. The same is true for the slope-stability condition from Proposition 2.10. \(\square\)

**Remark 3.6.** (1) Definition 3.4 differs slightly from that given by Nakajima in [15], which also involves a framing of the quiver. It was first pointed out by Crawley-Boevey in [2] that these framed quiver varieties can be interpreted as a quiver variety of the form described above (see also [6] Remark 2). We briefly recall this construction in the notation of this paper since it is relevant to the current section. Given a quiver \(Q'\) with vertices \(J'\) and edges \(E'\), dimension vector \(v' = (v_i)_{i \in J'}\), and framed dimension vector \(w' = (w_i)_{i \in J'}\) in the notation of [15], let \(Q\) be a new quiver with vertices \(J = J' \cup \{\infty\}\) and edges \(E = E' \cup T\), where \(T\) consists of \(w_i\) edges from \(\infty\) to each edge \(i \in J'\). Also let \(v = (v', 1)\) be the dimension vector obtained from \(v'\) by adjoining a 1 for the new vertex \(\infty\). Since the construction of \((Q,v)\) described above has a vertex with dimension one, then it has a stability parameter \(\alpha(Q,v)\) as defined in Definition 3.1. Crawley-Boevey then shows in [2] that the quotient \(M^{HK}_{\alpha}(Q,v)\) is the same as Nakajima’s definition of quiver variety \(M(Q,v', w')\).

(2) The stability parameter \(\alpha(Q,v)\) induces the same stability condition on \(\mu^{-1}_C(0)\) as Nakajima’s stability condition for the framed quiver \((Q', v', w')\) from [16] Sec. 3.ii. To see this, note that the stability condition induced by \(\alpha(Q,v)\) is that \((A, B) \in \mu^{-1}_C(0)\) is \(\alpha\)-stable if and only if every subrepresentation has negative slope, which occurs if and only if every subrepresentation contains the vertex \(\infty\). This is equivalent to condition (2) of [16] Lemma 3.8.

We also have the affine quotient \(M^{HK}_0(Q,v) := \mu^{-1}_C(0)\big// G_v\). The Kempf-Ness theorem shows that \(M^{HK}_0(Q,v) \cong \mu^{-1}_I(0) \cap \mu^{-1}_C(0) / K_v\). Note that this space is contractible, since \(\mu_I\) and \(\mu_C\) are homogeneous polynomials and so the equations \(\mu_I = 0\) and \(\mu_C = 0\) are preserved under the \(K_v\)-equivariant transformation \((A,B) \mapsto (tA,tB)\). Restricting (2.4) to the closed \(G_v\)-invariant
subspace $\mu_C^{-1}(0)$ gives us the projection map

$$\pi : M_{HK}^0(Q, v) \to M_{HK}^0(Q, v)$$

that takes an equivalence class $[(A, B)]$ to the unique closed $G_v$-orbit in the closure of the orbit $G_v \cdot (A, B)$. (See [18] (3.18).)

3.2. Structure of critical points. In this section we describe the structure of representations that are critical points of $\|\mu - \alpha\|^2$ on $\mu_C^{-1}(0)$. First, we define what it means for a representation to be critical for $\|\mu - \alpha\|^2$ on the singular space $\mu_C^{-1}(0)$.

**Definition 3.7.** A point $x \in \mu_C^{-1}(0) \subset T^*\Rep(Q, v)$ is critical for $\|\mu - \alpha\|^2$ if and only if $x$ is critical for $\|\mu - \alpha\|^2$ on the smooth space $T^*\Rep(Q, v)$.

Let $\text{Crit}(Q, v, \alpha)$ denote the set of all critical points of $\|\mu - \alpha\|^2$ on $\mu_C^{-1}(0)$.

Since $\mu_C^{-1}(0)$ is singular then this definition needs some justification. Returning to the smooth space $\Rep(Q, v)$ for the moment, recall that the gradient flow of $\|\mu - \alpha\|^2$ on $\Rep(Q, v)$ is generated by the action of $G_v$. As noted in [6 Prop. 15], since the gradient flow of $\|\mu - \alpha\|^2$ on $\Rep(Q, v)$ preserves any $G_v$-invariant closed subset, then the flow is contained in $\mu_C^{-1}(0)$ if the initial condition is contained in $\mu_C^{-1}(0)$. Therefore we can define the gradient flow on the subset to be the restriction of the gradient flow on the smooth space $\Rep(Q, v)$ and Theorem 2.21 will apply. In particular, if we define the critical points of $\|\mu - \alpha\|^2$ on $\mu_C^{-1}(0)$ as in Definition 3.7 then we have a Morse stratification of the space $\mu_C^{-1}(0)$. (See [6] or [9].)

We also have the following property of critical points on the smooth space $\Rep(Q, v)$.

**Lemma 3.8.**

1. Let $A \in \Rep(Q, v)$ be a critical point of $\|\mu - \alpha\|^2$. Then $A$ minimises the value of $\|\mu - \alpha\|^2$ on the orbit $G_v \cdot A$.

2. Given any $A \in \Rep(Q, v)$, consider the orbit closure $\overline{G_v \cdot A}$. The representations minimising $\|\mu - \alpha\|^2$ on $G_v \cdot A$ are precisely those in the $K_v$-orbit of critical points in $\overline{G_v \cdot A}$ that contains the limit of the downwards gradient flow of $\|\mu - \alpha\|^2$ with initial condition $A$.

**Proof.** Recall that the Harder-Narasimhan type is $G_v$-invariant and so $G_v \cdot A$ is contained in the Harder-Narasimhan stratum of $A$. The result of [6 Corollary 2, p334] shows that the critical point $A$ minimises the value of $\|\mu - \alpha\|^2$ on the Harder-Narasimhan stratum and therefore it must do so on the $G_v$-orbit also.

Recall Reineke’s result [19 Prop. 3.7] that says the closure of a Harder-Narasimhan stratum $\Rep(Q, v)_{v^*}$ is contained in the union

$$\Rep(Q, v)_{v^*} \subset \bigcup_{w^* \geq v^*} \Rep(Q, v)_{w^*}.$$ 

Therefore the closure $\overline{G_v \cdot A}$ is also contained in this union.

To see that this is minimised by a unique $K_v$-orbit, first note that the minimum of $\|\mu - \alpha\|^2$ on $\overline{G_v \cdot A}$ is not attained by any point in $\Rep(Q, v)_{w^*}$ for $w^* > v^*$, since (a) the minimum of
Corollary 3.9. Let \( x \in \mu^{-1}(0) \subseteq \text{Rep}(\tilde{Q},\nu) \) be a critical point of \( \|\mu - \alpha\|^2 \). Then \( x \) minimises the value of \( \|\mu - \alpha\|^2 \) on the orbit \( \mathcal{G}_\nu \cdot x \subseteq \mu^{-1}(0) \).

(2) Given any \( x \in \mu^{-1}(0) \), consider the orbit closure \( \overline{\mathcal{G}_\nu \cdot x} \). The minimum of \( \|\mu - \alpha\|^2 \) on \( \overline{\mathcal{G}_\nu \cdot x} \) is precisely the \( K_\nu \)-orbit of critical points in \( \mathcal{G}_\nu \cdot x \) that contains the limit of the downwards gradient flow of \( \|\mu - \alpha\|^2 \) with initial condition \( x \).

The rest of this section contains more details about the structure of critical points in \( \mu^{-1}(0) \) with respect to the stability parameter \( \alpha(Q,\nu) \) from Definition 3.1. Let \( x \in \mu^{-1}(0) \) be a critical point. Recall from (2.16) that \( x \) must split into subrepresentations and from (2.17) that the value of the moment map on each subrepresentation is determined by the slope. Each of the subrepresentations is semistable with respect to the induced stability parameter.

Since the vertex \( \infty \) has dimension 1 then only one of the subrepresentations (call it \( x_1 \)) in the decomposition (2.16) can have non-zero dimension vector at this vertex. Let \( \nu_1 = (v'_i)_{i \in J} \) be the dimension vector for this subrepresentation. A calculation shows that the induced stability parameter is

\[
\alpha'_i = \begin{cases} 
\frac{1}{1+\sum_{j \in J'} v'_{j}} & i \in J' \\
\frac{1+\sum_{j \in \infty} v'_{j}}{1+\sum_{j \in J'} v'_{j}} & i = \infty
\end{cases}
\]

which is a positive scalar multiple of the stability parameter \( \alpha(Q,\nu') \). Remark 3.3 then shows that \( x_1 \) is stable with respect to the induced stability parameter and Lemma 3.5 shows that the induced stability parameter is equivalent to the parameter from Definition 3.1.

From (3.11) we see that all of the other subrepresentations must then have the same slope. Let \( x_2 \) denote the sum of all the subrepresentations in (2.16) that do not contain the vertex \( \infty \). Then (2.17) shows that \( \mu_I(x_2) = 0 \).

The above argument is summarised in the following proposition.

Proposition 3.10. Let \( x \in \mu^{-1}(0) \) be a critical point of \( \|\mu - \alpha\|^2 \). Then \( x \) splits into two subrepresentations \( x_1 \) and \( x_2 \) with respective dimension vectors \( \nu_1 \) and \( \nu_2 \). The induced values of...
the moment map are \( \mu_1(x_1) = k \alpha(Q, v_1) \) and \( \mu_1(x_2) = 0 \), where \( k > 0 \) is the scalar from \( \text{(3.13)} \). The subrepresentation \( v_1 \) is stable with respect to the induced stability parameter.

Moreover, any representation \( x \in \mu_{C}^{-1}(0) \) of the form \( x = x_1 \oplus x_2 \) where \( \mu_1(x_1) = k \alpha(Q, v_1) \) and \( \mu_1(x_2) = 0 \) is a critical point.

Note that the equation \( \mu_1(x_2) = 0 \) always has a solution \( x_2 = 0 \). Given a dimension vector \( v_1 \leq v \) with \( M_{H,K}^\alpha(Q, v) \) nonempty, fix a collection of subspaces \( V_i^{(1)} \subset V_k \). The inner product on \( \text{Vect}(Q, v) \) then induces a direct sum decomposition \( \text{Vect}(Q, v) \cong \text{Vect}(Q, v_1) \oplus \text{Vect}(Q, v - v_1) \).

Using this, we define a subset \( C_{v_1}^0 \) of the critical points which take the form \((x_1, 0)\) with respect to this decomposition. This will appear in the constructions of Section [4].

**Definition 3.11.** Let \( C_{v_1} \) denote all of the critical points of \( ||\mu - \alpha||^2 \) on \( \mu_{C}^{-1}(0) \) for which the stable subrepresentation containing the vertex \( \infty \) from the decomposition in Proposition 3.10 has dimension vector \( v_1 \).

Given the dimension vector \( v_1 = (v'_i)_{i \in J} \) and associated vector spaces \( \{V'_i\}_{i \in J} \) such that \( \text{dim} C V'_i = v'_i \), fix an inclusion \( V'_i \to V_i \) for each \( i \in J \). Let \( C_{v_1}^0 \subset C_{v_1} \) be the subset consisting of representations of the form \((x_1, 0)\) that also preserve \( \bigoplus_{i \in J} V'_i \).

**Lemma 3.12.** Given the fixed inclusion \( V'_i \hookrightarrow V_i \) for each \( i \in J \) from Definition 3.11 let \( K_{v_1} \) denote the associated subgroup of \( K_v \). Then

\[
C_{v_1}/K_v \cong M_{\alpha}(Q, v_1) \times M_0(Q, v - v_1), \quad \text{and} \quad C_{v_1}^0/K_{v_1} \cong M_{\alpha}(Q, v_1).
\]

Moreover, there is a \( K_{v_1} \)-equivariant deformation retract of \( C_{v_1} \) onto \( C_{v_1}^0 \).

**Proof.** Recall Proposition 3.10 above. Restricting [6] Prop. 12 to \( \mu_{C}^{-1}(0) \) shows that \( C_{v_1}/K_v \cong M_{\alpha}(Q, v_1) \times M_0^{H,K}(Q, v - v_1) \). Similarly, we obtain \( C_{v_1}^0/K_{v_1} \cong M_{\alpha}(Q, v_1) \times \{0\} \). Since solutions to \( \mu_1(x_2) = 0 \) are invariant under scaling by a real parameter, then the \( K_{v_1} \)-equivariant deformation retract is given by \((x_1, x_2) \mapsto (x_1, tx_2) \) for \( 0 \leq t \leq 1 \). \( \square \)

**Lemma 3.13.** The connected components of \( \text{Crit}(Q, v, \alpha) \) are the sets \( C_{v_1} \) for each dimension vector \( 0 \leq v_1 \leq v \) such that \( v_1 \) has dimension 1 at the vertex \( \infty \) and the set \( C_{v_1} \) is nonempty.

**Proof.** Recall Crawley-Boevey’s result from [2] which says that the quiver varieties \( M_{H,K}^{\alpha}(Q, v_1) \) are connected. Lemma 3.12 shows that \( C_{v_1}^0 \) fibres over this space with connected fibres and so it must also be connected. Therefore \( C_{v_1} \) is connected, since it deformation retracts onto \( C_{v_1}^0 \).

Using the estimate in [6] Lemma 14], we can construct a neighbourhood around each point in \( C_{v_1} \) that does not intersect \( \text{Crit}(Q, v, \alpha) \setminus C_{v_1} \). Taking the union of these neighbourhoods gives an open neighbourhood of \( C_{v_1} \) in \( \mu_{C}^{-1}(0) \) that does not intersect \( \text{Crit}(Q, v, \alpha) \setminus C_{v_1} \). Therefore \( C_{v_1} \) is open in \( \text{Crit}(Q, v, \alpha) \) (which has the subspace topology induced from \( \mu_{C}^{-1}(0) \)).

We can construct similar neighbourhoods for all other critical points of the form \( C_{v'} \) with \( v' \neq v_1 \). Therefore \( C_{v_1} \) is contained in a closed subset of \( \mu_{C}^{-1}(0) \) that does not intersect \( \text{Crit}(Q, v, \alpha) \setminus C_{v_1} \), and so \( C_{v_1} \) must be closed in \( \text{Crit}(Q, v, \alpha) \).
Therefore we have shown that $C_{v_1}$ cannot be a proper subset of a connected component of $\text{Crit}(Q, v, \alpha)$. Since $C_{v_1}$ is connected then it must be a connected component. \hfill \Box

### 3.3. Local slices around the critical points

Returning to the smooth space $\text{Rep}(Q, v)$ for the moment, recall the following local slice theorem from [6, Lemma 18].

**Lemma 3.14.** Let $A \in \text{Rep}(Q, v)$. The function
\[
\psi : (\ker \rho_A^C)^\bot \times \ker(\rho_A^C)^* \to \text{Rep}(Q, v)
\]
\[
(u, \delta A) \mapsto \exp(u) \cdot (A + \delta A)
\]
is a diffeomorphism from a neighbourhood of $(0, 0)$ in $(\ker \rho_A^C)^\bot \times \ker(\rho_A^C)^*$ to a neighbourhood of $A$ in $\text{Rep}(Q, v)$.

The next result is a restriction of Lemma 3.14 from $T^*\text{Rep}(Q, v) = \text{Rep}(\tilde{Q}, v)$ to $\mu_{-1}^{-1}(0)$. The local slices in Lemma 3.14 are sufficiently small neighbourhoods of zero in $\ker(\rho_A^C)^*$. On the space $\mu_{-1}^{-1}(0)$, we replace $\ker(\rho_A^C)^*$ with the slice $S_x$ defined below.

**Definition 3.15.** Let $x \in \mu_{-1}^{-1}(0) \subset T^*\text{Rep}(Q, v) = \text{Rep}(\tilde{Q}, v)$ and let $\rho_x^C$ denote the infinitesimal action of $G_v$ on $T_x(T^*\text{Rep}(Q, v))$. The slice through $x$ is defined to be
\[
S_x = \{ \delta x \in T^*\text{Rep}(Q, v) : \delta x \in \ker(\rho_x^C)^* \text{ and } x + \delta x \in \mu_{-1}^{-1}(0) \}.
\]

We then have the following result.

**Corollary 3.16.** Let $x = (A, B) \in \mu_{-1}^{-1}(0)$. The function
\[
\psi : (\ker \rho_x^C)^\bot \times S_x \to \mu_{-1}^{-1}(0)
\]
\[
(u, \delta x) \mapsto \exp(u) \cdot (x + \delta x)
\]
is a homeomorphism from a neighbourhood of $(0, 0)$ in $(\ker \rho_x^C)^\bot \times S_x$ to a neighbourhood of $x$ in $\mu_{-1}^{-1}(0)$.

**Proof.** Let $y \in \mu_{-1}^{-1}(0)$ be sufficiently close to $x$ such that Lemma 3.14 applies in $\text{Rep}(\tilde{Q}, v)$. Therefore we can write
\[
y = \exp(u) \cdot (x + \delta x)
\]
for unique $u \in (\ker \rho_A^C)^\bot$ and $\delta x \in \ker(\rho_A^C)^*$. Since $x + \delta x = \exp(-u) \cdot y \in \mu_{-1}^{-1}(0)$, then $\delta x \in S_x$. Therefore $\psi$ surjects onto a neighbourhood of $x \in \mu_{-1}^{-1}(0)$. Since it is the restriction of a local diffeomorphism then it is injective, continuous and has a continuous inverse. Therefore $\psi$ is a local homeomorphism. \hfill \Box

Next we study the subset of the slice corresponding to the negative eigenspace of the Hessian. Recall from Section 2.5 that we have the following description of the tangent space at a critical point on the ambient smooth space $\text{Rep}(\tilde{Q}, v) \cong T^*\text{Rep}(Q, v)$.

- Since the Hessian is self-adjoint, then the tangent space splits into eigenspaces for the Hessian at $x$. 

• The tangent space also decomposes according to the splitting of the representation into subrepresentations from (2.14). This has the form

\[ T_x \text{Rep}(\tilde{Q}, v) \cong \bigoplus_{j,k=1}^n \text{Hom}^1(\tilde{Q}, v_j, v_k). \]

• The negative eigenspaces of the Hessian are characterised by maps from the subrepresentations of large slope into subrepresentations of small slope. If we order the subrepresentations by increasing slope as in (2.17), then Proposition 2.32 shows that the negative eigenspaces of the Hessian are

\[ (3.14) \quad V(x)^- = \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \cap \ker(\rho^C_x)^*. \]

**Definition 3.17.** Let \( x \in \mu_{\mathbb{C}}^{-1}(0) \) be a critical point for \( \|\mu_I - \alpha\|^2 \) and let \( V(x)^- = \bigoplus_{\lambda < 0} V_\lambda \) denote the negative eigenspace of the Hessian at \( x \) on the smooth space \( T^* \text{Rep}(\tilde{Q}, v) \). The negative slice through \( x \in \mu_{\mathbb{C}}^{-1}(0) \) is

\[ S_x^- := V(x)^- \cap S_x. \]

There is also a slice theorem for the restriction to the negative slice, which we use in Section 4.4. Here we order the subrepresentations for the critical point by the condition that \( j > k \) if and only if \( \text{slope}_\alpha(\tilde{Q}, v_j) > \text{slope}_\alpha(\tilde{Q}, v_k) \) (see (2.17)).

**Lemma 3.18.** Let \( x \in \mu_{\mathbb{C}}^{-1}(0) \) be critical for \( \|\mu_I - \alpha\|^2 \) and let \( \delta x \in \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \) such that \( x + \delta x \in \mu_{\mathbb{C}}^{-1}(0) \) is in the neighbourhood from Corollary 3.16. Then there exists \( g \in G_v \) such that

\[ g \cdot (x + \delta x) - x \in S_x^- . \]

**Proof.** Let \( \rho_x^- \) denote the restriction of \( \rho_x^C \) to the subspace \( \mathfrak{g}_x^- := \bigoplus_{j>k} \text{Hom}^0(\tilde{Q}, v_j, v_k) \). Note that

\[ \rho_x^- : \mathfrak{g}_x^- \to \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \]

We have the orthogonal decomposition \( \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \cong \text{im} \rho_x^- \oplus \ker(\rho_x^-)^* \) as well as the \( a \ priori \) results

\[ (3.15) \quad \text{im} \rho_x^- = \text{im} \rho_x^C \cap \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \]

\[ (3.16) \quad \ker(\rho_x^-)^* \supset \ker(\rho_x^C)^* \cap \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k). \]

Since

\[ \text{im} \rho_x^- \oplus \ker(\rho_x^-)^* \cong \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) = \left( \text{im} \rho_x^C \oplus \ker(\rho_x^C)^* \right) \cap \bigoplus_{j>k} \text{Hom}^1(\tilde{Q}, v_j, v_k) \]

Therefore, there exists \( g \in G_v \) such that

\[ g \cdot (x + \delta x) - x \in S_x^- . \]
Proposition 3.20. will always be associated to a dimension vector. from the context since, when we talk about stable representations in this section, the representation always chosen to be the stability parameter from Definition 3.1. The meaning will always be clear.

Gradient flow of points in the negative slice.

Let \( x \in \mathbb{R}^n \) and let \( \delta x, \delta y \) and so

Lemma 3.19. parameter.

Proof. The definition of \( S_x^- \) together with (3.14) shows that

\[
S_x^- = \{ \delta x \in \text{Hom}(\tilde{Q}, \mathbf{v}_2, \mathbf{v}_1) : \delta x \in \mu_{\mathbb{C}}^{-1}(0) \}.
\]

Let \( x = (A, B) \) and \( \delta x = (\delta A, \delta B) \) be the decomposition with respect to \( T^* \text{Rep}(Q, \mathbf{v}) \cong \text{Rep}(Q, \mathbf{v}) \oplus \text{Rep}(\tilde{Q}, \mathbf{v}) \) from Section 3.1. Since \( x \in \mu_{\mathbb{C}}^{-1}(0) \), then we have

\[
0 = \mu_{\mathbb{C}}((A, B) + (\delta A, \delta B)) = \mu_{\mathbb{C}}(A, B) + \sum_{a \in \mathcal{E}} [A_a, \delta B_a] + [A_a, B_a] + [\delta A_a, B_a]
\]

\[
= d\mu_{\mathbb{C}}(\delta A, \delta B) + \sum_{a \in \mathcal{E}} [\delta A_a, \delta B_a],
\]

Since \( (\delta A, \delta B) \in \text{Hom}(\tilde{Q}, \mathbf{v}_2, \mathbf{v}_1) \) then \( [\delta A_a, \delta B_a] = 0 \) for each \( a \in \mathcal{E} \), and so the condition \( x + \delta x \in \mu_{\mathbb{C}}^{-1}(0) \) simplifies to \( \delta x \in \ker d\mu_{\mathbb{C}}. \)

3.4. Gradient flow of points in the negative slice. In this section the stability parameter \( \alpha \) is always chosen to be the stability parameter from Definition 3.1. The meaning will always be clear from the context since, when we talk about stable representations in this section, the representation will always be associated to a dimension vector.

Proposition 3.20. Let \( \mathbf{v}_1 < \mathbf{v} \) be a dimension vector such that the vertex \( \infty \) has dimension 1, and let \( x = (x_1, 0) \in C_{\mathbf{v}_1}^0 \). Then for any \( \delta x \in S_x^- \), there exists a dimension vector \( \mathbf{v}_2 \) and a representation \( x' \in \mu_{\mathbb{C}}^{-1}(0)^{a_{\text{st}}} \subset T^* \text{Rep}(Q, \mathbf{v}_2) \) such that the representation \( (x_1, 0) + \delta x \) has the form \( (x', 0) \).
Proof. Recall the description of $S_x^-$ from Lemma 3.19. Proposition 3.10 shows that the representation $(x_1, 0)$ induces a direct sum decomposition $V_k = V_k^{(1)} \oplus V_k'$ for each vertex $k \in \mathcal{I}$.

Let $V_k'' \subset V_k'$ be the subspace given by ker $\delta x$, and let $(V_k'')^⊥$ be the orthogonal complement of $V_k''$ in $V_k'$. Note that for each non-zero $v \in (V_k'')^⊥$ there exists an edge $a \in \mathcal{E} \cup \mathcal{E}$ such that $t(a) = k$ and $(\delta x)_a v$ is a non-zero vector in $V_k^{(1)}$ (where $(\delta x)_a v$ denotes either $\delta A_a v$ or $\delta B_a v$ depending on whether $a \in \mathcal{E}$ or $a \in \mathcal{E}$. Define $V_k^{(2)} = V_k^{(1)} \oplus (V_k'')^⊥$ for each $k \in \mathcal{I}$, let $v_2$ be the associated dimension vector and define $\text{Vect}(Q, v_2) = \bigoplus_{k \in \mathcal{I}} V_k^{(2)}$.

Therefore $(x_1, 0) + \delta x$ preserves $\text{Vect}(Q, v_2)$. Since $x_1 \in \mu_c^{-1}(0)^{α-st} \subset T^* \text{Rep}(Q, v_1)$ then every subrepresentation contains the vertex $∞$. For every $k \in \mathcal{I}$, $\delta x$ maps every non-zero vector in $(V_k'')^⊥$ to a vector in $\text{Vect}(Q, v_1)$ and so every subrepresentation of $(x_1, 0) + \delta x$ contains the vertex $∞$. Therefore there exists $x' \in \mu_c^{-1}(0)^{α-st} \subset T^* \text{Rep}(Q, v_2)$ such that $(x', 0) = (x_1, 0) + \delta x$. \hfill \Box

Remark 3.21. It follows from the construction in the proof that $v_1 < v_2$.

Combining this with the algebraic description of the limit of the gradient flow from Theorem 2.21 gives us the following result, which shows that the limit of the gradient flow with initial condition $(x_1, 0) + \delta x$ is contained in the $G_v$-orbit. In other words, the initial condition is isomorphic to the limit of the flow.

Proposition 3.22. Let $v_1 < v$ be a dimension vector, and let $x = (x_1, 0) \in C_{v_1}^0 \subset \mu_c^{-1}(0) \subset T^* \text{Rep}(Q, v)$. Then for any $\delta x \in S_x^-$ there exists $g \in G_v$ such that

$$g \cdot ((x_1, 0) + \delta x) = \lim_{t \to \infty} \gamma^−((x_1, 0) + \delta x, t).$$

Proof. Proposition 3.20 shows that there exists a dimension vector $v_2$ and $x' \in \mu_c^{-1}(0)^{α-st} \subset \text{Rep}(Q, v_2)$ such that $(x_1, 0) + \delta x = (x', 0)$. Since $x'$ is stable, then

$$(x', 0) = G^\text{HNJH} \cdot (x', 0),$$

and therefore Theorem 2.21 implies that $(x_1, 0) + \delta x = (x', 0)$ is isomorphic to the limit of the flow with initial conditions $(x', 0)$. \hfill \Box

Definition 3.23. Given dimension vectors $0 \leq v_1 < v_2 \leq v$, let $\mathcal{C}(Q, v_1, v_2, v)$ be the space of pairs $((x_1, 0), (x_2, 0)) \in C_{v_1}^0 \times C_{v_2}^0$ such that there exists $\delta x \in S_{x_1}^−$ satisfying $(x_1, 0) + \delta x \cong (x_2, 0)$.

Remark 3.24. (1) Since we can always scale $\delta x$ by a 1-PS of $G_v$ then this definition is equivalent to requiring that $\delta x$ be in some neighbourhood of zero in $S_{x_1}^−$.

(2) One can think of the space $\mathcal{C}(Q, v_1, v_2, v)$ as the space of pairs of critical points connected by an approximate flow line. Here “approximate” means that the negative slice plays the role of the unstable manifold in the usual definition of flow line.

(3) Approximate flow lines will correspond to flow lines if one can show that each point in the negative slice is isomorphic to a point in the unstable manifold, and vice versa. In general,
such an isomorphism would be non-trivial; it is not true (except for some degenerate cases) that the negative slice is equal to the unstable manifold.

The next lemma shows that the definitions are independent of the choice of dimension vector for the ambient space $\mu_{C}(0)_v \subset \text{Rep}(Q,v)$.

**Lemma 3.25.** Let $v_1 < v_2 < v$. Then $((x_1,0),(x_2,0)) \in \mathcal{C}(Q,v_1,v_2,v)$ if and only if $((x_1,0),x_2) \in \mathcal{C}(Q,v_1,v_2,v_2)$.

**Proof.** For each $k \in J$, let $V_k$ and $V_k^{(2)}$ denote the respective vector spaces corresponding to the dimension vectors $v$ and $v_2$. Fix inclusions $V_k^{(2)} \hookrightarrow V_k$. This induces an inclusion $\mu_{C}^{-1}(0)_{v_2} \hookrightarrow \mu_{C}^{-1}(0)_{v}$. Since the moment map $\mu_I$ satisfies $\mu_I(x_1 \oplus x_2) = \mu_I(x_1) \oplus \mu_I(x_2)$ for a direct sum of representations, then a direct sum of representations will remain a direct sum of representations under the gradient flow of $\|\mu_I - \alpha\|^2$. In particular, the subspace $\mu_{C}^{-1}(0)_{v_2} \subset \mu_{C}^{-1}(0)_{v}$ is preserved by the flow equations. \hfill $\square$

**Remark 3.26.** In view of the above lemma, it is natural to ask why we don’t just restrict attention to the case $v = v_2$ in order to simplify the notation. The reason is that some of the constructions of Section 4 involve more than one pair of critical sets and so we often need to choose an ambient dimension vector which is larger than the dimension of all the critical sets.

Since the points in $\mathcal{C}$ are independent of the dimension vector for the ambient space, then the space of pairs of equivalence classes connected by a flow line can be defined independently of the dimension vector for the ambient space.

**Definition 3.27.** Let $v_1 < v_2$ and define $\mathcal{M}(Q,v_1,v_2) \subset \mathcal{M}(Q,v_1) \times \mathcal{M}(Q,v_2)$ to be the space of pairs of equivalence classes $([x_1],[x_2])$ such that there exist representatives $x_1' \in [x_1]$ and $x_2' \in [x_2]$ such that $((x_1',0),x_2') \in \mathcal{C}(Q,v_1,v_2,v)$. Note that Lemma 3.25 gives a homeomorphism $\mathcal{C}(Q,v_1,v_2,v) \cong \mathcal{C}(Q,v_1,v_2,v)$ for all $v > v_2$.

The projection maps $\mathcal{M}(Q,v_1,v_2) \to \mathcal{M}(Q,v_1)$ and $\mathcal{M}(Q,v_1,v_2) \to \mathcal{M}(Q,v_2)$ are denoted by $p_{flow}^1$ and $p_{flow}^2$ respectively.

4. **Gradient flow interpretation of Nakajima’s varieties**

This section contains the main results of the paper: a gradient flow interpretation of the varieties used in Nakajima’s constructions from [16] and [18]. Section 4.1 studies the projection $\pi: \mathcal{M}(Q,v) \to \mathcal{M}_0(Q,v)$, Section 4.2 studies the Hecke correspondence, Section 4.3 studies the Lagrangian subvariety from [16 Sec. 7], Section 4.4 gives a gradient flow interpretation of Kashiwara’s operators from [16 Sec. 10.i] and Section 4.5 gives a gradient flow interpretation of the Hecke correspondence for the handsaw quiver varieties from [18].

In this section, $Q$ denotes the quiver with doubled edges associated to a framed representation via Remark 3.6. $v$ denotes the dimension vector of $Q$ (which contains all the information about the dimension vectors $v'$ and $w'$ for the framed representation) and $\alpha = \alpha(Q,v)$ denotes the canonical
stability parameter associated to the quiver $Q$ and dimension vector $v$ from Definition 3.1. The notation for $Q$ and $v$ is omitted if the meaning is clear from the context. $\mathcal{M}(Q,v)$ denotes the moduli space associated to the quiver $Q$, dimension vector $v$ and stability parameter $\alpha(Q,v)$, and $\mathcal{M}_0(Q,v)$ denotes the affine quotient.

The dimension vector $\bar{v}$ always denotes the dimension vector for the ambient space $\mu_{C^{-1}}(0)\bar{v}$. We consider the gradient flow on this space, however in view of Lemma 3.25 the choice of dimension vector $\bar{v}$ is irrelevant, as long as it is large enough.

4.1. Projection to the affine quotient. The result of this section is that the affine projection $\pi: \mathcal{M}(Q,v) \to \mathcal{M}_0(Q,v)$ is given by the gradient flow of $\|\mu_I\|^2$ (note that we take the flow with respect to the zero stability parameter).

Proposition 4.1. Let $x \in \mu_{C^{-1}}(0)\alpha_{st}$ and let $\gamma^-(x,t)$ denote the downwards gradient flow of $\|\mu_I\|^2$ at time $t$ with initial condition $x$. Then

$$\pi([x]) = \left[\lim_{t \to \infty} \gamma^-(x,t)\right].$$

Proof. All points are semistable with respect to the zero stability parameter and so the Harder-Narasimhan-Jordan-Hölder double filtration is just the Jordan-Hölder filtration. Therefore Theorem 2.21 shows that $\lim_{t \to \infty} \gamma^-(x,t)$ is isomorphic to $\text{Gr}^{\text{JH}}(x)$, the graded object of the Jordan-Hölder filtration of $x$.

The graded object of the Jordan-Hölder filtration $\text{Gr}^{\text{JH}}(x)$ is contained in the unique closed orbit in $G_v \cdot x$ (cf. [11, Proposition 3.2]). Therefore

$$\pi([x]) = \left[\text{Gr}^{\text{JH}}(x)\right] = \left[\lim_{t \to \infty} \gamma^-(x,t)\right].$$

4.2. The Hecke correspondence. In this section we use $Q$ to denote the quiver with doubled edges associated to a framed representation via Remark 3.6. For technical reasons (see the proof of Lemma 4.1) we assume there are no edges with head and tail at the same vertex.

Let $k \in \mathcal{I}$ be a vertex of the quiver $Q$, and let $e_k$ be the dimension vector that is equal to 1 on the $k^{th}$ vertex and zero at all other vertices. Throughout this section we use $v = (v_k)_{k \in \mathcal{I}}$ to denote a dimension vector with $v_k > 0$ and we denote the vector spaces at each vertex by

$$\text{Vect}(Q,v - e_k) \cong \bigoplus_{\ell \in \mathcal{I}} V^{(1)}_{\ell} \quad \text{and} \quad \text{Vect}(Q,v) \cong \bigoplus_{\ell \in \mathcal{I}} V^{(2)}_{\ell}.$$

The goal is to consider the gradient flow of $\|\mu_I - \alpha\|^2$ on the space $\mu_{C^{-1}}(0)\bar{v}$ and give a gradient flow interpretation of the Hecke correspondence in terms of approximate flow lines from the critical set $C^0_{v,e_k}$ to the critical set $C^0_{v}$.

For notation, we denote the vector spaces at each vertex by

$$\text{Vect}(Q,v - e_k) \cong \bigoplus_{\ell \in \mathcal{I}} V^{(1)}_{\ell} \quad \text{and} \quad \text{Vect}(Q,v) \cong \bigoplus_{\ell \in \mathcal{I}} V^{(2)}_{\ell}.$$
By construction, there is always a vertex \( x \in \mathcal{I} \) such that \( \dim V_x = 1 \). Define the subset

\[
\text{Hom}_0^0(Q, v - e_k, v) := \{ \xi \in \text{Hom}_0^0(Q, v - e_k, v) : \xi_\infty = \text{id} \}.
\]

The **Hecke correspondence** is

\[
\mathcal{B}_k(Q, v) := \{ ([x_1], [x_2]) \in \mathcal{M}(Q, v - e_k) \times \mathcal{M}(Q, v) : \exists \xi \in \text{Hom}_0^0(Q, v - e_k, v) \setminus \{0\} \text{ s.t. } \xi x_1 = x_2 \xi \}.
\]

(cf. [16] Section 5). In other words, \( \mathcal{B}_k(Q, v) \) consists of pairs of equivalence classes that are “intertwined” by a nonzero homomorphism \( \xi \in \text{Hom}_0^0(Q, v - e_k, v) \).

[16] Theorem 5.7] shows that the Hecke correspondence is smooth and that the homomorphism \( \xi \) is injective.

**Theorem 4.2.** There is a homeomorphism \( \mathcal{M}^\mathcal{F}(Q, v - e_k, v) \cong \mathcal{B}_k(Q, v) \).

The proof reduces to Lemmas 4.3 and 4.4 below. Lemma 4.3 shows that there is a map from \( \mathcal{M}_k(Q, v - e_k, v) \) to \( \mathcal{B}_k(Q, v) \).

**Lemma 4.3.** Choose \( k \in \mathcal{I} \) and let \( v = (v_k)_{k \in \mathcal{I}} \) be a dimension vector satisfying \( v_k > 0 \). If \( ([x_1], [x_2]) \in \mathcal{C}(Q, v - e_k, v, \nu) \) then \( ([x_1], [x_2]) \in \mathcal{B}_k(Q, v) \). This gives a continuous map \( \mathcal{M}^\mathcal{F}(Q, v - e_k, v) \to \mathcal{B}_k(Q, v) \).

**Proof.** By definition, there exists \( \delta x \in S_x^\mathcal{I}_1 \) and \( g' \in G_\nu \) such that

\[
g' \cdot ((x_1, 0) + \delta x) = (x_2, 0).
\]

The construction in the proof of Proposition 3.20 shows that \( x_1 + \delta x \) is a representation in \( \mu^{-1}_\nu(0) \). Restricting (4.3) to \( \text{Vect}(Q, v) \) gives us

\[
g(x_1 + \delta x) = x_2 g \quad \text{for } g \in G_\nu.
\]

Here juxtaposition of homomorphisms denotes composition. The group action (2.1) is denoted with a “·" as in (1.3). Both sides of the above equation are elements of \( \mu^{-1}_\nu(0) \). Given a fixed splitting \( V^{(2)}_k = V^{(1)}_k \otimes \mathbb{C} \), we have an induced splitting

\[
\text{Rep}(Q, v) = \text{Hom}^1(Q, v - e_k, v) \oplus \text{Hom}^1(Q, e_k, v).
\]

With respect to this decomposition, the left-hand side has components

\[
gx_1 \in \text{Hom}^1(Q, v - e_k, v), \quad g\delta x \in \text{Hom}^1(Q, e_k, v),
\]

and the right-hand side decomposes as

\[
x_2 g|_{v-e_k} \in \text{Hom}^1(Q, v - e_k, v), \quad x_2 g|_{e_k} \in \text{Hom}^1(Q, e_k, v),
\]

where \( g|_{v-e_k} \) and \( g|_{e_k} \) denote the restriction of \( g : \text{Vect}(Q, v) \to \text{Vect}(Q, v) \) to the subspaces \( \text{Vect}(Q, v - e_k) \) and \( \text{Vect}(Q, e_k) \) respectively. Note also that \( g x_1 = g|_{v-e_k} x_1 \), since \( x_1 \in \text{Rep}(Q, v - e_k) \).

Therefore we have shown that \( g|_{v-e_k} x_1 = x_2 g|_{v-e_k} \), where \( g|_{v-e_k} \in \text{Hom}^0(Q, v - e_k, v) \) is non-zero. Scaling \( g|_{v-e_k} \) by a scalar multiple of the identity we can arrange it so that the component of
g that maps the vertex \( \infty \) to itself is the identity. Setting \( \xi = g|_{\mathcal{V} - \mathbf{e}_k} \) in the definition of the Hecke correspondence \([1,2]\) shows that the pair of representations satisfies \((|x_1|, |x_2|) \in \mathcal{B}_k(Q, \mathcal{V}).\) \(\square\)

The next lemma shows that elements of the Hecke correspondence define pairs of critical points connected by a flow line.

**Lemma 4.4.** Let \((|x_1|, |x_2|) \in \mathcal{B}_k(Q, \mathcal{V}).\) Then there exist representatives \(x'_1 \in [x_1]\) and \(x'_2 \in [x_2]\) such that \(((x'_1, 0), (x'_2, 0)) \in \mathcal{C}(Q, \mathcal{V} - \mathbf{e}_k, \mathcal{V}).\) This gives a continuous map \(\mathcal{B}_k(Q, \mathcal{V}) \to \mathcal{M}(Q, \mathcal{V} - \mathbf{e}_k, \mathcal{V}).\)

**Proof.** Since \(\xi\) is injective, then there exists an orthogonal decomposition \(V^2_k \cong \text{im} \xi_k \oplus \mathbb{C}\) and isomorphisms \(\xi_\ell : V^{(1)}_\ell \to V^{(2)}_\ell\) for each \(\ell \neq k.\) This induces inclusions \(\text{Vect}(Q, \mathcal{V} - \mathbf{e}_k) \hookrightarrow \text{Vect}(Q, \mathcal{V})\) and \(\mu^{-1}_\mathbb{C}(0)_{\mathcal{V} - \mathbf{e}_k} \hookrightarrow \mu^{-1}_\mathbb{C}(0)_{\mathcal{V}}.\) Via Lemma \([3,25]\) and the inclusion \(\mu^{-1}_\mathbb{C}(0)_{\mathcal{V} - \mathbf{e}_k} \hookrightarrow \mu^{-1}_\mathbb{C}(0)_{\mathcal{V}},\) we can consider \(x'_1, x'_2 \in \mu^{-1}_\mathbb{C}(0)_{\mathcal{V}}.\) By assumption, there exists \(\xi \in \text{Hom}^0_\mathbb{C}(Q, \mathcal{V} - \mathbf{e}_k, \mathcal{V}) \setminus \{0\}\) and representatives \(x'_1 \in [x_1]\) and \(x'_2 \in [x_2]\) such that

\[
\xi x'_1 = x'_2 \xi.
\]

We want to show that there exist \(g \in G_\mathcal{V}\) and \(\delta x \in \mathcal{S}_{x'_1}^-\) such that

\[
g \cdot (x'_1 + \delta x) = x'_2.
\]

We can choose \(u \in \text{Hom}^0(Q, \mathcal{V}, \mathcal{V})\) such that \(\ker u = \text{Vect}(Q, \mathcal{V} - \mathbf{e}_k)\) and \(\xi + u \in \text{Hom}^0_\mathbb{C}(Q, \mathcal{V}, \mathcal{V})\) is also injective. Note that \(ux_1 = 0\) and that \(im u\) and \(im \xi\) are linearly independent.

Note also that \(x'_2 u \in \text{Hom}^1(Q, \mathbf{e}_k, \mathcal{V}).\) Since there are no loops in \(Q,\) then the image of \(x'_2 u\) is contained in \(\bigoplus_{\ell \neq k} V^{(2)}_\ell.\) Since \(\xi_\ell\) is an isomorphism for each \(\ell \neq k\) then we can define \(\tilde{\delta x} = \xi^{-1}x'_2 u \in \text{Hom}^1(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k).\) This homomorphism has the property that

\[
\xi \tilde{\delta x} = x'_2 u.
\]

The observation that \(u \tilde{\delta x} = 0\) shows that \(\xi + u \in \text{Hom}^0_\mathbb{C}(Q, \mathcal{V}, \mathcal{V})\) satisfies

(1) \(\xi + u\) is injective, and

(2) \((\xi + u)(x'_1 + \delta x) = x'_2(\xi + u)\).

The final step is to construct \(\delta x \in \ker (\rho_{x'_2}^\mathcal{C})^*\) with the same properties. Using the decomposition \(V^{(2)}_k \cong V^{(1)}_k \oplus \mathbb{C},\) the infinitesimal group action at \(x'_1\) restricts to

\[
\rho_{x'_2}^\mathcal{C} : \text{Hom}^0(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k) \to \text{Hom}^1(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k)
\]

\[
u \mapsto -x'_1 u
\]

The inner product on \(\text{Hom}^1(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k)\) then induces a decomposition

\[
\text{Hom}^1(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k) \cong \text{im} \rho_{x'_2}^\mathcal{C} \oplus \ker (\rho_{x'_2}^\mathcal{C})^*.
\]

Let \(\delta x'\) be the component of \(\tilde{\delta x}\) in \(\text{im} \rho_{x'_2}^\mathcal{C}\). Then \(\delta x' = x'_1 v\) for some \(v \in \text{Hom}^0(Q, \mathbf{e}_k, \mathcal{V} - \mathbf{e}_k),\) and we define

\[
\delta x = \tilde{\delta x} - \delta x' \in \ker (\rho_{x'_2}^\mathcal{C})^*.
\]
Then \((4.5)\) shows that
\[
\xi \delta x = \xi (\delta x - x_1 v) = x_2 u - \xi x_1 v = x_2 (u - \xi v).
\]
Note that since \(x'_1 \in \mu^{-1}_C(0)_V - e_k\), \(\text{im} \delta x \subset \text{Vect}(Q, v - e_k)\) and \(\ker u \subset \text{Vect}(Q, v - e_k)\), \(\ker v \subset \text{Vect}(Q, v - e_k)\), then \(u(x'_1 + \delta x) = 0\) and \(v(x'_1 + \delta x) = 0\). Since \(\text{im} \xi\) and \(\text{im} u\) are linearly independent, then \(g = \xi + u - \xi v \in \text{Hom}^0(Q, v)\) is an isomorphism and therefore \(g \in G_v\). We then have
\[
g(x'_1 + \delta x) = (\xi + u - \xi v)(x'_1 + \delta x) = \xi x_1 + \xi \delta x = x'_2 (u - \xi v).
\]
Therefore \(g \cdot (x'_1 + \delta x) = x'_2\), as required. \(\square\)

4.3. The Lagrangian subvariety. Let \(v_1\) and \(v_2\) be two dimension vectors for a quiver \(Q\). There are inclusion maps
\[
i_1 : M_0(Q, v_1) \rightarrow M_0(Q, v_1 + v_2) \\
i_2 : M_0(Q, v_2) \rightarrow M_0(Q, v_1 + v_2),
\]
which, when coupled with the projection maps \(\pi_j : M(Q, v_j) \rightarrow M_0(Q, v_j)\) for \(j = 1, 2\), give maps
\[
i_1 \circ \pi_1 : M(Q, v_1) \rightarrow M_0(Q, v_1 + v_2) \\
i_2 \circ \pi_2 : M(Q, v_2) \rightarrow M_0(Q, v_1 + v_2).
\]
Define
\[
Z(Q, v_1, v_2) := \{([x_1], [x_2]) \in M(Q, v_1) \times M(Q, v_2) : i_1 \circ \pi_1([x_1]) = i_2 \circ \pi_2([x_2])\}.
\]
This is the Lagrangian subvariety from [16] Section 7. The main result of this section is an interpretation of \(Z(Q, v_1, v_2)\) in terms of broken flow lines.

The first lemma is the key technical result of the section.

**Lemma 4.5.** Let \([x] \in M(Q, v)\) and let \([x_0] = \pi[x] \in M_0(Q, v)\). Then there exist representatives \(x' \in [x]\) and \(x'_0 \in [x'_0]\) such that
\[
\begin{align*}
(1) \quad & (x', 0) \text{ and } (x'_0, 0) \text{ are critical for } ||\mu - \alpha||^2 \text{ on } \mu^{-1}_C(0)_v, \text{ and} \\
(2) \quad & x' \text{ and } x'_0 \text{ are connected by a broken flow line.}
\end{align*}
\]

**Proof.** Since \(x\) is \(\alpha\)-stable then it is isomorphic to a point \(x'\) in the minimum of \(||\mu - \alpha||^2\) on \(\mu^{-1}_C(0)_v\). Proposition \([3, 10]\) gives a description of the critical sets for \(||\mu - \alpha||^2\) on the ambient space \(\mu^{-1}_C(0)_v\), which shows that \((x', 0)\) is critical.

Since \((x_0, 0)\) is the direct sum of stable representations with respect to the zero stability parameter then the \(G_v\)-orbit is closed. Let \((x'_0, 0)\) be the limit of the flow of \(||\mu - \alpha||^2\) with initial condition \((x_0, 0)\). Since the flow is contained in the \(G_v\)-orbit (which is closed) then the limit must be isomorphic to the initial condition. Therefore \((x'_0, 0)\) is critical and isomorphic to \((x_0, 0)\).

The second part of the proof requires inductively constructing flow lines using the Jordan-Hölder filtration of \(x'\). Consider the \(\alpha\)-stable representation \(x'\) on \(\mu^{-1}_C(0)_v\). Since \((x', 0)\) is critical on
\(\mu_C^{-1}(0)\varphi\) and \(x'\) is \(\alpha\)-stable then it minimises \(\|\mu - \alpha\|^2\) on \(\mu_C^{-1}(0)\varphi\). Similarly, \(x'_0\) is critical for \(\|\mu - \alpha\|^2\) on \(\mu_C^{-1}(0)\varphi\).

Using the Jordan-Hölder filtration, one can repeatedly choose points in the negative slice to construct a broken flow line from \(x'_0\) to \(x'\). Since \(x'\) is \(\alpha\)-stable then the Jordan-Hölder filtration must be by subrepresentations of increasing \(\alpha\)-slope (otherwise \(x'\) would have a destabilising subrepresentation). In \(S_{\alpha}^-\), choose the term \(\delta x_0\) corresponding to the first non-trivial extension in the filtration (if necessary, scale using the \(G\) representatives corresponding to the extension class of the Jordan-Hölder filtration). In \(S_{\alpha}^-\), choose the term \(\delta x_0\) corresponding to the first non-trivial extension in the filtration (if necessary, scale using the \(G\) representatives corresponding to the extension class of the Jordan-Hölder filtration). In particular, if \(x_0 + \delta x_0\) is the direct sum of \(\alpha\)-stable representations and so the limit \(\lim_{t \to \infty} \gamma^{-}(x_0 + \delta x_0, t)\) is isomorphic to \(x_0 + \delta x_0\). Now repeat with the second non-trivial extension in the Jordan-Hölder filtration, and so on until the representation is \(\alpha\)-stable. Each step corresponds to a flow line and so the whole process corresponds to a broken flow line. The limit is isomorphic to \(x_0 + \delta x\), where \(\delta x\) corresponds to the extension class of the Jordan-Hölder filtration.

By construction, the limit is isomorphic to \(x'\). Therefore there exists \(g \in K\varphi\) (which preserves the minimum) such that there is a broken flow line between \(g \cdot x'\) and \(x'_0\). Since \(g \cdot x' \in [x]\) and \(g \cdot x'\) minimises \(\|\mu_I - \alpha\|^2\) on \(\mu_C^{-1}(0)\varphi\) then we can replace \(x'\) with \(g \cdot x'\) in the statement of the lemma.

The result follows by noting that the gradient flow of \(\|\mu_I - \alpha\|^2\) on \(\mu_C^{-1}(0)\varphi\) preserves the inclusion \(\mu_C^{-1}(0)\varphi \hookrightarrow \mu_C^{-1}(0)\varphi\) given by \(x \mapsto (x, 0)\). \(\square\)

The next lemma gives the converse of the above result.

**Lemma 4.6.** If \(x_1 \in C_{v_1}\) and \(x_2 \in C_{v_2}\) are connected to the same critical point \(x_0\) by a broken flow line, then \(Gr^{JH}(x_1) \cong Gr^{JH}(x_2)\). In particular, if \((x_1, 0) \in C_{v_1}^0\) and \((x_2, 0) \in C_{v_2}^0\) are connected to the same critical point by a broken flow line then \(((x_1, 0), (x_2, 0)) \in Z(Q, v_1, v_2)\).

**Proof.** First we prove the result for the special case where \(x_1\) and \(x_2\) are connected by a flow line.

If \(x_1\) and \(x_2\) are connected by a flow line then they are in the closure of the same group orbit \(G \cdot y\) (for example, set \(y = x_1 + \delta x\) for \(\delta x \in S_{\alpha}^\times\)). By [13, p377], the unique closed orbit in \(\overline{G \cdot y}\) is contained in both \(\overline{G_{v_1} \cdot x_1 \cap G_{v_2} \cdot y}\) and \(\overline{G_{v_2} \cdot x_2 \cap G_{v_1} \cdot y}\) and therefore it is contained in \(\overline{G_{v_1} \cdot x_1 \cap G_{v_2} \cdot x_2}\).

Since this closed orbit corresponds to the orbit of the graded object of the Jordan-Hölder filtration (cf. [11, Prop. 3.2]) then \(Gr^{JH}(x_1) \cong Gr^{JH}(x_2)\).

Inductively applying this result shows that the same is true for any two critical points connected by a broken flow line. Therefore \(Gr^{JH}(x_1) \cong Gr^{JH}(x_0) \cong Gr^{JH}(x_2)\). If \(x_1 \in C_{v_1}^0\) and \(x_2 \in C_{v_2}^0\) then by Lemma 3.12 we have \([x_1] \in M(Q, v_1)\) and \([x_2] \in M(Q, v_2)\). The above result together with Lemma 2.16 shows that \(((x_1), (x_2)) \in Z(Q, v_1, v_2)\). \(\square\)

**Theorem 4.7.** Choose any \(\bar{\varphi} > v_1 + v_2\). Then \(((x_1), (x_2)) \in Z(Q, v_1, v_2)\) if and only if there exist representatives \(x'_1 \in [x_1]\) and \(x'_2 \in [x_2]\) such that \((x'_1, 0)\) and \((x'_2, 0)\) are critical points for \(\|\mu_I - \alpha\|^2\) on \(\mu_C^{-1}(0)\varphi\) and there exists \(g \in K\varphi\) such that \(g \cdot (x'_1, 0)\) and \((x'_2, 0)\) are connected to the same critical point by a broken flow line.

**Proof.** First suppose that \(((x_1), (x_2)) \in Z(Q, v_1, v_2)\). Let \([x_0] := i_1 \circ \pi_1([x_1])\). By definition, \([x_0] = i_2 \circ \pi_2([x_2])\) also. Since the orbit of \(x_0\) is closed then it is the direct sum of stable representations,
where stability is defined with respect to the zero stability parameter (cf. [11, Proposition 3.2]). This is a closed orbit in the closure of $G_V \cdot x_1$ and therefore $x_0$ is isomorphic to $Gr^{JH}(x_1)$. Similarly, $x_0$ is also isomorphic to $Gr^{JH}(x_2)$.

Lemma 4.5 then shows that there exist representatives $y_0 \cong z_0 \cong x_0$, $x'_1 \cong x_1$ and $x'_2 \cong x_2$ such that $(x'_1,0)$ and $y_0$ are critical points connected by a broken flow line and that the same is true for $(x'_2,0)$ and $z_0$. Since $y_0$ and $z_0$ are isomorphic and critical then there exists $g \in K_V$ such that $g \cdot y_0 = z_0$. Since the flow is $K_V$-equivariant, then $g \cdot (x'_1,0)$ is connected to $z_0$ by a broken flow line.

For the converse, first note that $Gr^{JH}(g \cdot (x'_1,0)) \cong Gr^{JH}(x'_1,0)$. Lemma 4.6 then shows that $([x_1],[x_2]) = ([x'_1],[x'_2]) \in Z(Q,v_1,v_2)$. □

4.4 The varieties $\mathcal{M}_{k,r}(Q,v)$ and the geometric analogue of Kashiwara’s operators. In this section we give a gradient flow interpretation of Nakajima’s description from [10] Sec. 10.i of Kashiwara’s operators $\hat{E}_k$ and $\hat{F}_k$ (see [10]). The operators are defined for quivers without loops and the definition of the projection map $p : \mathcal{M}_{k,r}(Q,v) \to \mathcal{M}_{k,0}(Q,v-\alpha e_k)$ requires that the quiver does not have loops, so we also impose this assumption throughout this section.

First, recall the definition of $\mathcal{M}_{k,r}(Q,v)$ from [10] (4.3). Fix a vertex $k$ and consider the subset of $\mathcal{M}(Q,v)$ given by equivalence classes of representations $[x]$ such that $\sum_{h(a) = k} \text{im} \, x_a$ has codimension equal to $r$.

Note that if we fix a subspace $V'_k \subset V_k$ with $\text{codim} \, V'_k = r$ and fix a direct sum decomposition $V_k = V'_k \oplus V''_k$ then each equivalence class $[x] \in \mathcal{M}_{k,r}(Q,v)$ has a representative $x$ such that $\sum_{h(a) = k} \text{im} \, x_a \subset V'_k$.

Restricting $x$ to $\mu^{-1}_C(0)_{v-\alpha e_k}$ gives a stable subrepresentation $x_0$ such that $[x_0] \in \mathcal{M}_{k,0}(Q,v-\alpha e_k)$. The map $p : \mathcal{M}_{k,r}(Q,v) \to \mathcal{M}_{k,0}(Q,v-\alpha e_k)$ given by $[x] \mapsto [x_0]$ is the projection map defined in [10] (4.4).

The first result of the section is that $[x]$ and $[x_0]$ are related by gradient flow lines.

**Lemma 4.8.** Let $[x] \in \mathcal{M}_{k,r}(Q,v)$ and let $[x_0] := p([x])$. Then there exist representatives $x' \in [x]$ and $x'_0 \in [x_0] := p([x])$ such that

1. $(x',0)$ and $(x'_0,0)$ are critical for $\|\mu_I - \alpha\|^2$ on $\mu^{-1}_C(0)_v$, and
2. $((x',0),(x'_0,0)) \in \mathcal{C}(Q,v-\alpha e_k,v,v)$.

In particular, we see that $p([x]),[x]) \in M_{\mathcal{F}}(Q,v-\alpha e_k,v)$.

**Proof.** Let $[x] \in \mathcal{M}_{k,r}(Q,v)$. Since $x$ is stable by definition, then Proposition 3.10 shows that there is a representative $x' \in [x]$ such that $(x',0)$ is critical. Let $[x_0] = p([x]) \in \mathcal{M}_{k,0}(Q,v-\alpha e_k)$ be the projection of $[x]$ and let $x'_0 \in [x_0]$ be a representative such that $(x'_0,0)$ is critical. Since $[x'_0] = p([x'])$, then using the action of $K_V$ (which preserves the critical sets) we can arrange it so that $x'_0$ is a subrepresentation of $x'$.

Let $\delta x = (x',0) - (x'_0,0) \in \mu^{-1}_C(0)_v$. Then $(\delta x)_a$ is zero for all edges $a$ such that $t(a) \neq k$. Therefore $\delta x$ consists of homomorphisms from $V''_k$ to $\bigoplus_{a : t(a) = k} V_{h(a)}$. We want to show that $x' = x'_0 + \delta x$ is isomorphic to a point in the negative slice $S^{-}_{x'_0}$.
Consider a one-parameter subgroup $\mathbb{C}^* \subset G_{\mathbb{V}}$ which acts on $V''_k$ with negative weight and $V'_k$ and $V_\ell$ with the same positive weight for all $\ell \neq k$. Then this subgroup fixes $(x'_0, 0)$ and acts by scalar multiplication on $\delta x$. Therefore there exists $g \in G_{\mathbb{V}}$ such that $x'_0 + \delta x' := g \cdot (x'_0 + \delta x)$ satisfies the condition that $\delta x'$ is in the neighbourhood where the slice theorem (Corollary 3.16) applies. The negative slice theorem (Lemma 3.18) then shows that there exists $g' \in G_{\mathbb{V}}$ such that $g' \cdot (x'_0 + \delta x) - x'_0 \in S_{x'_0}^-$. Since $x'_0 + \delta x$ is isomorphic to $x'_0 + \delta x'$ for some $\delta x' \in S_{x'_0}^-$ then (after possibly acting on $(x', 0)$ by $K_{\mathbb{V}}$) we have $((x', 0), (x'_0, 0)) \in \mathcal{C}(Q, v - re_k, v, \bar{v})$. \hfill \Box

The next lemma gives a converse result.

**Lemma 4.9.** Let $[x] \in M_{k,r}(Q, v)$ and $[x_0] \in M_{k,0}(Q, v - re_k)$. Suppose that there exist representatives $x' \in [x]$ and $x'_0 \in [x_0]$ such that $(x', 0)$ and $(x'_0, 0)$ are critical points for $\|\mu - \alpha\|^2$ on $\mu_{\mathbb{C}^*}^{-1}(0)_{\mathbb{V}}$ and that $((x', 0), (x'_0, 0)) \in \mathcal{C}(Q, v - re_k, v, \bar{v})$. Then $[x_0] = p([x])$.

**Proof.** Since $((x', 0), (x'_0, 0)) \in \mathcal{C}(Q, v - re_k, v, \bar{v})$ then there exists $\delta x \in S_{x'_0}^-$ such that $x'_0 + \delta x$ flows down to $x'$. Proposition 3.22 shows that $x' \cong x'_0 + \delta x$. Therefore there exists $g \in G_{\mathbb{V}}$ such that $g \cdot x' = x'_0 + \delta x$.

Since $[x'_0] = [x_0] \in M_{k,0}(Q, v - re_k)$ and $[g \cdot x'] = [x] \in M_{k,r}(Q, v)$ then there exists a direct sum decomposition $V_k = V'_k \oplus V''_k$, where $\dim V''_k = r$ and

$$\sum_{h(a) = k} \text{im}(x'_0)_a = V'_k.$$

Since $\delta x \in S_{x'_0, 0}^-$ then $\delta x$ consists solely of homomorphisms from $V''_k$ to $\bigoplus_{a : t(a) = k} V(h(a))$. Therefore $x'_0$ is the restriction of $g \cdot x'$ to $\mu_{\mathbb{C}^*}^{-1}(0)_v - re_k$ and so $p([x]) = p([g \cdot x']) = [x'_0] = [x_0]$. \hfill \Box

The previous two lemmas combine to give the following proposition

**Proposition 4.10.** Let $[x] \in M_{k,r}(Q, v)$. Then there exists a unique $[x_0] \in M_{k,0}(Q, v - re_k)$ such that $([x], [x]) \in \mathcal{M}(Q, v - re_k, v)$. Moreover, $[x_0] = p([x])$.

**Proof.** Lemma 4.8 shows that $(p([x]), [x]) \in \mathcal{M}(Q, v - re_k, v)$, which takes care of the existence part of the proof. Uniqueness follows from Lemma 4.9. \hfill \Box

**Remark 4.11.** As a consequence, we can interpret the projection map $p : M_{k,r}(Q, v) \to M_{k,0}(Q, v - re_k)$ in terms of the projection map $p^1_{\text{flow}} : \mathcal{M}(Q, v - re_k, v) \to M(Q, v - re_k)$. More precisely, for every $[x] \in M_{k,r}(Q, v)$ we have

$$\{(p([x]), [x])\} = (M_{k,0}(Q, v - re_k) \times \{[x]\}) \cap \mathcal{M}(Q, v - re_k, v)$$

and so

$$p([x]) = p^1_{\text{flow}}((M_{k,0}(Q, v - re_k) \times \{[x]\}) \cap \mathcal{M}(Q, v - re_k, v)).$$

Therefore, restricting the projection $p^1_{\text{flow}}$ to $M_{k,0}(Q, v - re_k) \times M_{k,r}(Q, v) \cap \mathcal{M}(Q, v - re_k, v)$ gives the Grassmann bundle of $M_{k,0}(Q, v - re_k) \times M_{k,r}(Q, v) \cap \mathcal{M}(Q, v - re_k, v)$. More precisely, for every $[x] \in M_{k,r}(Q, v)$ we have

$$\{(p([x]), [x])\} = (M_{k,0}(Q, v - re_k) \times \{[x]\}) \cap \mathcal{M}(Q, v - re_k, v)$$

and so

$$p([x]) = p^1_{\text{flow}}((M_{k,0}(Q, v - re_k) \times \{[x]\}) \cap \mathcal{M}(Q, v - re_k, v)).$$
We can use this to give a gradient flow interpretation of Nakajima’s geometric construction of Kashiwara’s operators \( \tilde{E}_k \) and \( \tilde{F}_k \) from [16 Sec. 10.i]. Recall the projections from Definition 3.27

\[
\begin{align*}
\mathcal{MF}(Q, v - r e_k, v - e_k) & \xrightarrow{p_1} \mathcal{MF}(Q, v - r e_k, v) \\
\mathcal{M}(Q, v - r e_k) & \xrightarrow{p_2} \mathcal{MF}(Q, v - e_k, v) \\
\mathcal{M}(Q, v - r e_k) & \xrightarrow{p_3} \mathcal{MF}(Q, v - e_k, v + e_k)
\end{align*}
\]

(The change in notation from \( p_{\text{flow}} \) to avoid confusion between the different cases.)

**Theorem 4.12.** Let \( X \) be an irreducible component of \( \mathcal{M}(Q, v)_x \) and let \( r = \varepsilon_k(X) \) as in [16 (7.3)]. Then

\[
\tilde{E}_k[X] = \left[ (\mathcal{M}_{k,0}(Q, v - r e_k) \times \mathcal{M}_{k,r-1}(Q, v - e_k)) \cap p_1^{-1}(p_2(\mathcal{M}_{k,0}(Q, v - r e_k) \times (X \cap \mathcal{M}_{k,r}(Q, v))) \right]
\]

and

\[
\tilde{F}_k[X] = \left[ (\mathcal{M}_{k,0}(Q, v - r e_k) \times \mathcal{M}_{k,r+1}(Q, v + e_k)) \cap p_3^{-1}(p_2(\mathcal{M}_{k,0}(Q, v - r e_k) \times (X \cap \mathcal{M}_{k,r}(Q, v))) \right].
\]

**Proof.** The proof follows by using Remark 4.11 together with Nakajima’s construction from [16 Sec. 10.i].

### 4.5. Hecke correspondences for handsaw quiver varieties

In this section we show that the Hecke correspondence for the handsaw quiver varieties studied in [18] admits a similar gradient flow interpretation to that in Section 4.2.

First, recall the definition of handsaw quiver varieties from [18 Sec. 2]. Let \( n \) be a positive integer and consider the quiver

The analog of the vector space \( \operatorname{Rep}(Q, v) \) is

\[
\mathcal{M}_n(V, W) = \bigoplus_{k=1}^{n-2} \operatorname{Hom}(V_k, V_{k+1}) \oplus \bigoplus_{k=1}^{n-1} \operatorname{Hom}(V_k, V_k) \oplus \bigoplus_{k=1}^{n-1} \operatorname{Hom}(W_k, V_k) \oplus \bigoplus_{k=2}^{n} \operatorname{Hom}(V_{k-1}, W_k).
\]
The analog of the complex moment map equation is

\begin{equation}
\mu(B_1, B_2, a, b) = [B_1, B_2] + \sum_{k=2}^{n-1} a_k b_k = 0
\end{equation}

A representation \((B_1, B_2, a, b)\) is stable if for any choice of subspaces \(S_k \subset V_k\) such that \(a_k(W_k) \subset S_k\) and \(\oplus S_k\) is preserved by \(B_1\) and \(B_2\) we must have \(S_k = V_k\) for all \(k\).

In order to use the constructions of Sections 2 and 3 we need to show that the handsaw quivers can be related to unframed quivers via an analog of Crawley-Boevey’s construction from [2]. Given a handsaw quiver as above, consider the new quiver \(Q\) constructed by adding an extra vertex of dimension 1 as in Remark 3.6. There are \(\dim W_k\) edges from the extra vertex to \(V_k\) (the homomorphisms correspond to the columns of \(a_k\) and are labelled \(a_k^j\) for \(j = 1, \ldots, \dim W_k\)) and \(\dim W_{k+1}\) edges from \(V_k\) to the extra vertex (the homomorphisms correspond to the rows of \(b_{k+1}\) and are labelled \(b_{k+1}^j\) for \(j = 1, \ldots, \dim W_k\)). The example below shows what happens for the case \(\dim W_1 = 1, \dim W_2 = 2, \dim W_{n-1} = 1\) and \(\dim W_n = 1\).

Equation (4.7) is then equivalent to

\begin{equation}
[B_1, B_2] + \sum_{k=2}^{n-1} \sum_{j=1}^{\dim W_k} a_k^j b_k^j = 0.
\end{equation}

This defines a singular subset \(\mu^{-1}(0) \subset \text{Rep}(Q, v)\). The stability condition then corresponds to slope stability for the parameter \(-\alpha(Q, v)\) from Definition 3.1 (the parameter \(\alpha(Q, v)\) corresponds to costability; see [16 Def. 2.2]).

The definition of the Hecke correspondence for handsaw quiver varieties in [18 Sec. 5] is analogous to the definition for framed quiver varieties given in [16], except now \(\xi \in \text{Hom}^0_k(Q, v - e_k)\) (i.e. \(\xi\) now maps in the opposite direction and is surjective instead of injective). The condition is

\begin{equation}
\xi B_1 = B_1^1 \xi, \quad \xi B_2 = B_2^1 \xi, \quad \xi a^2 = a^1, \quad b^2 = b^1 \xi
\end{equation}

Let \(P_k(Q, v)\) denote the space of pairs of equivalence classes \(([B_1^1, B_2^1, a^1, b^1]), ([B_1^2, B_2^2, a^2, b^2])\) with representatives satisfying (4.9). \(P_k(Q, v)\) is the Hecke correspondence for handsaw quiver varieties.

Rather than rewrite the proofs from Section 4.2 for this definition where \(\xi\) is surjective and maps in the opposite direction, it is easier to reverse all the arrows in the quiver and take the adjoint of all the homomorphisms (interchanging the roles of \(a\) and \(b\)). Then the methods of Section 4.2...
apply without change. Lemma 2.12 then shows that \((B_1, B_2, a, b)\) is \(-\alpha(Q, v)\) stable if and only if \((B_1^*, B_2^*, b^*, a^*)\) is \(\alpha(\bar{Q}, v)\) stable. Replacing \(b^*\) with \(-b^*\) does not change the stability condition and so taking the adjoint in this way allows us to apply all the results of Section 4.2. The handsaw quiver then becomes

\[
\begin{array}{cccccc}
\circ & \circ & \cdots & \cdots & \circ & \circ \\
B_2 & B_1^* & B_2^* & \cdots & B_1 & B_2 \\
V_{n-1} & V_{n-2} & \cdots & \cdots & V_1 & V_0 \\
W_n & W_{n-1} & \cdots & \cdots & W_2 & W_1 \\
\vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\
\end{array}
\]

The moment map equation (4.7) for the original handsaw quiver becomes

\[
\mu(B_1^*, B_2^*, b^*, a^*) = -[B_1^*, B_2^*] - \sum_{k=2}^{n-1} b_k^* a_k^* = 0
\]

which is exactly the moment map equation for this new handsaw quiver.

Let \(\bar{Q}\) denote the quiver obtained by applying the analog of Crawley-Boevey’s construction to this new handsaw quiver. Since the singular subset is \(G_v\)-invariant then the results for the structure of the critical sets and negative eigenspace of the Hessian from Section 3 carry over to this case as well. Moreover, one can construct the negative slice in exactly the same way. The slice theorems and Propositions 3.20 and 3.22 apply in the same way as before. Therefore we can construct analogous spaces \(\mathcal{C}(\bar{Q}, v_1, v_2, \bar{v})\) and \(\mathcal{M}(\bar{Q}, v_1, v_2)\) to those from Definitions 3.23 and 3.27 where now we take the flow of \(\|\mu - \alpha\|^2\) on the space of solutions to (4.10) inside \(\text{Rep}(\bar{Q}, v)\).

Taking the adjoint of equation (4.9), we see that \((B_1, B_2, a, b)\) satisfies (4.9) if and only if

\[
(B_1^*)^* \xi^* = \xi^* (B_1^*)^*, \quad (B_2^*)^* \xi^* = \xi^* (B_2^*)^*, \quad (a^2)^* \xi^* = (a^1)^*, \quad (b^2)^* = \xi^* (b^1)^*
\]

Now the map \(\xi^*\) is injective and an analogous method to the proof of Lemmas 4.3 and 4.4 gives us

**Theorem 4.13.** There is a homeomorphism \(\mathcal{M}(\bar{Q}, v - e_k, v) \cong \mathcal{P}_k(Q, v)\).

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