On a Lower Bound for $\| (4/3)^k \|$  

Yury PUPYREV  
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Abstract

We prove, that

$$\left\| \left( \frac{4}{3} \right)^k \right\| > \left( \frac{4}{9} \right)^k \quad \text{for} \quad k \geq 6,$$

where $\| \cdot \|$ is a distance to the nearest prime.

1 Introduction

In 1994 Bennett \[\text{[1]}\] considered a generalization of Waring’s problem, namely, a problem on the order $g_N(k)$ of the additive basis

$$S_N^{(k)} = \{1^k, N^k, (N+1)^k, \ldots \}, \quad N \geq 2,$$

of the set of positive integers. He established the following estimates for $\| (1 + 1/N)^k \|$: 

$$\left\| \left( 1 + \frac{1}{N} \right)^k \right\| > 3^{-k} \quad \text{for} \quad 4 \leq N \leq k \cdot 3^k$$

and with their help obtained the representation

$$g_N(k) = N^k + \left\lceil \left( 1 + \frac{1}{N} \right)^k \right\rceil - 2$$

for $4 \leq N \leq (k + 1)^{(k-1)/k} - 1.$

He concluded that he needed the inequality

$$\left\| \left( \frac{4}{3} \right)^k \right\| > \left( \frac{4}{9} \right)^k \quad \text{for} \quad k \geq 6,$$  \quad (1)

for the representation

$$g_3(k) = 3^k + \left\lceil \left( \frac{4}{3} \right)^k \right\rceil - 2.$$
In 2007 Zudilin [6], by modifying Baker’s construction, namely, by considering Padé approximations to the remainder of the series
\[
\frac{1}{(1 - z)^{m+1}} = \sum_{n=0}^{\infty} \binom{m+n}{m} z^n
\]
and by receiving sharp estimates for the \(p\)-adic orders of the arising binomial coefficients, arrived at the bound
\[
\left\| \left( \frac{4}{3} \right)^k \right\| > 0.4914^k \quad \text{for} \quad k \geq K,
\]
where \(K\) is an effective constant.

In 2009 this author [4] received an exact value of \(K\), but it was too big for checking (1) for \(6 \leq k < K\).

In this paper using the same method as Zudilin, but with another set of parameters, we receive the bound (1) for \(k \geq 17545718\), and check it using a lemma similar to [2, Proposition 1], with software for remaining \(k\).

Thus, we prove the following result.

**Theorem 1.** We have
\[
\left\| \left( \frac{4}{3} \right)^k \right\| > \left( \frac{4}{9} \right)^k \quad \text{holds for} \quad k \geq 6.
\]

2 **Padé Approximations**

Following [6], we fix two positive integers \(a\) and \(b\), with \(3a \leq b\), and write
\[
\left( \frac{4}{3} \right) 2^{(b+1)} = \left( \frac{16}{9} \right)^{(b+1)} = 2^{b+1} \left( 1 + \frac{1}{8} \right)^{-(b+1)}
\]
\[
= 2^{b+1} \sum_{l=0}^{\infty} \binom{b+l}{b} \left( \frac{1}{8} \right)^l (-1)^l
\]
\[
= 2^{b+1} 2^{-3a} \sum_{l=0}^{\infty} \binom{b+l}{b} 2^{3(a-l)} (-1)^{-l}
\]
\[
= 2^{b-3a+1} (-1)^a \sum_{l=0}^{\infty} \binom{b+l}{b} 2^{3(a-l)} (-1)^{(a-l)}
\]
\[
\equiv 2^{b-3a+1} (-1)^a \sum_{l=a}^{\infty} \binom{b+l}{b} 2^{3(a-l)} (-1)^{(a-l)} \pmod{\mathbb{Z}}
\]
\[
\equiv 2^{b-3a+1} (-1)^a \sum_{\nu=0}^{\infty} \binom{a+b+\nu}{b} \left( -\frac{1}{8} \right)^\nu \pmod{\mathbb{Z}}. \quad (2)
\]
So, we are going to consider Padé approximations to the function

\[ F(z) = F(a, b; z) = \sum_{\nu=0}^{\infty} \left( \frac{a + b + \nu}{b} \right) z^\nu. \]  

(3)

For any positive integer \( n \leq b \) we find \( ^{[6]} \)

\[ Q_n(z^{-1}) = \frac{(a + b + n)!}{(a + n - 1)! n!(b - n)!} \int_0^1 t^{a + n - 1} (1 - t)^{b - n} (1 - z^{-1}t)^n \, dt \]  

(4)

and

\[ R_n(z) = \frac{(a + b + n)!}{(a + n - 1)! n!(b - n)!} \cdot z^n \times \int_0^1 t^n (1 - t)^{a + n - 1} (1 - zt)^{-(a + b + n) - 1} \, dt \]

such that

\[ Q_n(z^{-1}) F(z) = P_n(z^{-1}) + R_n(z), \]

(6)

is performed with polynomial \( P_n(x) \in \mathbb{Z}[x], \deg P_n \leq n - 1. \)

3 Arithmetic argument

For every prime \( p > \sqrt{a + b + n} \) we set

\[ e_{p,n} = \min_{\mu \in \mathbb{Z}} \left( -\left\{ \frac{a + n}{p} \right\} + \left\{ \frac{a + n + \mu}{p} \right\} + \left\{ \frac{\mu}{p} \right\} \right) + \left\{ \frac{a + b + n}{p} \right\} \]

\[ -\left\{ \frac{a + b + n}{p} \right\} + \left\{ \frac{a + b + \mu}{p} \right\} + \left\{ \frac{n - \mu}{p} \right\} \), \]

\[ e'_{p,n} = \min_{\mu \in \mathbb{Z}} \left( -\left\{ \frac{a + n + \mu}{p} \right\} + \left\{ \frac{a + n}{p} \right\} + \left\{ \frac{\mu}{p} \right\} \right) + \left\{ \frac{a + b + n}{p} \right\} \]

\[ -\left\{ \frac{a + b + n}{p} \right\} + \left\{ \frac{a + b + \mu}{p} \right\} + \left\{ \frac{n - \mu}{p} \right\} \), \]

and for

\[ \Phi = \Phi(a, b, n) = \prod_{p > \sqrt{a + b + n}} p^{e_{p,n}}, \quad \Phi' = \Phi'(a, b, n) = \prod_{p > \sqrt{a + b + n}} p^{e'_{p,n}}, \]

by lemmas 3 and 4 in \( ^{[6]} \), we have

\[ \Phi^{-1} Q_n(x), \Phi^{-1} P_n(x) \in \mathbb{Z}[x], \]

\[ \Phi'^{-1} (n + 1) Q_{n+1}(x), \Phi'^{-1} (n + 1) P_{n+1}(x) \in \mathbb{Z}[x]. \]
4  A Bound for \(\|(4/3)^k\|\)

For \(a, b,\) and \(n\) we write

\[
a = \alpha m, \quad b = \beta m, \quad n = \gamma m, \quad m \in \mathbb{N}.
\]

Our aim is to find a lower bound for the absolute value of \(\varepsilon_k\), where

\[
\left(\frac{4}{3}\right)^k = M_k + \varepsilon_k, \quad M_k \in \mathbb{Z}, \quad 0 < |\varepsilon_k| < \frac{1}{2}.
\]

For \(k \geq 3\) we write \(k = 2(\beta m + 1) + j\) with positive integers \(m\) and \(j < 2\beta\). We multiply both sides of (6) by \(\tilde{\Phi}^{-1}2^{b-3a+1}(-1)^a\) (where \(\tilde{\Phi}\) is equal to \(\Phi\) or to \(\Phi'/(n + 1)\); we discuss this choice in what follows) and put \(z = -1/8\):

\[
Q_n(-8)\tilde{\Phi}^{-1}3^j \cdot \left(\frac{4}{3}\right)^j 2^{b-3a+1}(-1)^a F(a, b, -\frac{1}{8}) = P_n(-8)\tilde{\Phi}^{-1}2^{b-3a+1+2j}(-1)^a + R_n\left(-\frac{1}{8}\right)\tilde{\Phi}^{-1}2^{b-3a+1+2j}(-1)^a.
\]

(7)

From (2) and (3) one can find that

\[
\left(\frac{4}{3}\right)^j 2^{b-3a+1}(-1)^a F(a, b, -\frac{1}{8}) \equiv \left(\frac{4}{3}\right)^{2(b+1)+j} \pmod{\mathbb{Z}} = \left(\frac{4}{3}\right)^k,
\]

so the left-hand side can be written as \(M''_k + \varepsilon_k\) and one can rewrite (7) as

\[
Q_n(-8)\tilde{\Phi}^{-1}3^j \cdot \varepsilon_k = M''_k + R_n\left(-\frac{1}{8}\right)\tilde{\Phi}^{-1}2^{b-3a+1+2j}(-1)^a.
\]

(8)

At this point we should check if the number \(M''_k\) is distinct from zero. Lemma 2 in [6] guarantees that for \(n\) or for \(n + 1\) we have \(M''_k \neq 0\). So \(\tilde{\Phi} = \Phi\) if (8) holds for \(n\) with \(M''_k \neq 0\), and \(\tilde{\Phi} = \Phi'/(n + 1)\) if (8) holds for \(n + 1\) with \(M''_k \neq 0\). (This odd way of working with \(\tilde{\Phi}\) becomes more understandable once we have determined \(a, b,\) and \(n\).)

So, assuming that

\[
|R_n\left(-\frac{1}{8}\right)\tilde{\Phi}^{-1}2^{b-3a+1+2j}(-1)^a| < \frac{2}{3},
\]

(9)

from (8) we have

\[
|Q_n(-8)\tilde{\Phi}^{-1}3^j| \cdot |\varepsilon_k| \geq |M''_k| - |R_n\left(-\frac{1}{8}\right)\tilde{\Phi}^{-1}2^{b-3a+1+2j}(-1)^a| > \frac{1}{3},
\]

and so

\[
|\varepsilon_k| > \frac{\tilde{\Phi}}{3^j+1|Q_n(-8)|} \geq \frac{\tilde{\Phi}}{3^{2\beta}|Q_n(-8)|},
\]

4
which means that
\[ |\varepsilon_k| > \frac{\Phi}{3^{2\beta}|Q_n(-8)|}. \tag{10} \]
or
\[ |\varepsilon_k| > \frac{\Phi'}{(n+1)3^{2\beta}|Q_{n+1}(-8)|}, \tag{11} \]
depending on the choice made in (8).

5 A Bound for $\Phi$

For evaluating $\Phi$ and $\Phi'$ we consider the functions
\[
\varphi(x) = \min_{0 \leq y < 1} (-\{(\alpha + \gamma)x\} + \{(\alpha + \gamma)x - y\} + \{y\} - \{(\alpha + \beta + \gamma)x\} + \{(\alpha + \beta)x + y\} + \{\gamma x - y\}),
\]
\[
\varphi'(x) = \min_{0 \leq y < 1} (-\{(\alpha + \gamma)x + y\} + \{(\alpha + \gamma)x\} + \{y\} - \{(\alpha + \beta + \gamma)x\} + \{(\alpha + \beta)x + y\} + \{\gamma x - y\}),
\]
which take the values $e_{n,p}$ and $e'_{n,p}$, respectively at the point $m/p$.

All the solutions $x$ of the equation $\varphi(x) = 1$ form the set of intervals in $[0, 1)$, which should contain $\{x\}$. If we denote $A_i$ and $B_i$ the left and right points of this intervals, respectively, then the condition $A_i \leq \{m/p\} < B_i$ (i.e. $e_{n,p} = \varphi(m/p) = 1$) is equivalent to
\[ A_i + N \leq \frac{m}{p} < B_i + N, \quad N \in \mathbb{N}, \]
($\mathbb{N}$ is the set of non-negative integers), or the same
\[ p \in \left(\frac{m}{B_i + N}, \frac{m}{A_i + N}\right), \quad N \in \mathbb{N}. \]

This means that all the prime numbers $p$ such that
\[
p \in \left[\frac{m}{\sqrt{a + b + n}}\right]^{-1} \bigcup_{N=0}^{\infty} \bigcup_i \left(\frac{m}{B_i + N}, \frac{m}{A_i + N}\right)
\]
(the inequality $p > \sqrt{a + b + n}$ entails $m/(B_i + N) \geq \sqrt{a + b + n}$, and one can find the bound for $N$) go to $\Phi$. So we have
\[
\log \Phi \geq \sum_{N=0}^{\left[\frac{m}{\sqrt{a + b + n}}\right]^{-1}} \sum_i \left(\theta\left(\frac{m}{A_i + N}\right) - \theta\left(\frac{m}{B_i + N}\right)\right), \tag{12}
\]
where $\theta(x) = \sum_{p \leq x, \ p \text{ is prime}} \log p$.

The same works for $\varphi'(x)$. And it is proved in [6], that the sets for $\varphi(x)$ and $\varphi'(x)$ differs only on a set of zero measure.
6 Analytic and Arithmetical Bounds

Let us take $\alpha = 3$, $\beta = 9$, $\gamma = 4$.

For $\Phi$ we have the set of intervals
\[
\left[\frac{1}{8}, \frac{1}{7}\right] \cup \left[\frac{3}{16}, \frac{1}{5}\right] \cup \left[\frac{3}{8}, \frac{2}{5}\right] \cup \left[\frac{9}{16}, \frac{4}{7}\right] \cup \left[\frac{11}{16}, \frac{5}{7}\right] \cup \left[\frac{15}{16}, 1\right)
\]

For $\Phi'$ the difference will only be in the right-end points of the intervals.

We will use the following bounds for $\theta(x)$ [5]: the upper bound
\[
\theta(x) < 1.001102 \cdot x \quad \text{if} \quad x > 0,
\]
and the lower bound
\[
\theta(x) > 0.998 \cdot x \quad \text{if} \quad x > 487,381.
\]

Substituting them in (12) and taking the sum for $N = 0, 1$ we obtain
\[
\log \Phi > 1.639533 \cdot m \quad \text{for} \quad m \geq 974,762. \tag{13}
\]
The same bound holds for $\Phi'$.

We need to estimate the values of $|Q_n(z^{-1})|$, $|R_n(z)|$, and $|Q_{n+1}(z^{-1})|$, $|R_{n+1}(z)|$ at the point $z = -1/8$, and to estimate $\Phi'$. We begin with
\[
\log \left(\frac{(16m)!}{(7m-1)!(4m)!(5m)!}\right) \quad \text{and} \quad \log \left(\frac{(16m+1)!}{(7m)!(4m+1)!(5m-1)!}\right).
\]

We use Stirling’s formulae [3]
\[
\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n)},
\]
and we find
\[
\log \left(\frac{(16m)!}{(7m-1)!(4m)!(5m)!}\right) = \log \left(\frac{(16m)!}{(7m)!(4m)!(5m)!}\right) + \log(7m)
\]
\[
< \frac{1}{2} \log(2\pi) + \frac{1}{2} \log(16) + \frac{1}{2} \log(m) + 16m \log(16) + 16m \log(m) + 16m
\]
\[
+ \frac{1}{12 \cdot 16m}
\]
\[
- \left(\frac{1}{2} \log(2\pi) + \frac{1}{2} \log(7) + \frac{1}{2} \log(m) + 7m \log(7) + 7m \log(m) - 7m\right)
\]
\[
- \left(\frac{1}{2} \log(2\pi) + \frac{1}{2} \log(4) + \frac{1}{2} \log(m) + 4m \log(4) + 4m \log(m) - 4m\right)
\]
\[
- \left(\frac{1}{2} \log(2\pi) + \frac{1}{2} \log(5) + \frac{1}{2} \log(m) + 5m \log(5) + 5m \log(m) - 5m\right)
\]
\[
+ \log(7) + \log(m).
\]
Since
\[-\log(2\pi) + \frac{1}{2} \left( \log(16) - \log(7) - \log(4) - \log(5) \right) - \log(m) + \frac{1}{12 \cdot 16m} + \log(7) + \log(m) \leq 0 \quad \text{for} \quad m \geq 974762.
\]

one can have
\[
\log \left( \frac{(16m)!}{(7m-1)!(4m)!(5m)!} \right) < (16 \log(16) - 7 \log(7) - 4 \log(4) - 5 \log(5)) \cdot m
\]
for \( m \geq 974762 \). (14)

Now,
\[
\log \left( \frac{(16m+1)!}{(7m)!(4m+1)!(5m-1)!} \right)
= \log \left( \frac{(16m)!}{(7m)!(4m)!(5m)!} \right) + \log \left( \frac{(16m+1)(5m)}{4m+1} \right)
< \log \left( \frac{(16m)!}{(7m)!(4m)!(5m)!} \right) + \log \left( \frac{(16m+4)(5m)}{4m+1} \right)
= \log \left( \frac{(16m)!}{(7m)!(4m)!(5m)!} \right) + \log(20m),
\]
and in a similar way we conclude that
\[
\log \left( \frac{(16m+1)!}{(7m)!(4m+1)!(5m-1)!} \right) < (16 \log(16) - 7 \log(7) - 4 \log(4) - 5 \log(5)) \cdot m + 1
\]
for \( m \geq 974762 \). (15)

For the integral in (5) we write the estimates
\[
\int_0^1 t^{4m} (1-t)^{7m-1} (1-zt)^{-16m-1} dt
\leq \left( \max_{t \in [0,1]} t^4 (1-t)^7 \left( 1 + \frac{t}{8} \right)^{-16} \right)^{m-1} \int_0^1 t^4 (1-t)^6 \left( 1 + \frac{t}{8} \right)^{-17} dt,
\]
\[
\int_0^1 t^{4m+1} (1-t)^{7m} (1-zt)^{-16m-2} dt
\leq \left( \max_{t \in [0,1]} t^4 (1-t)^7 \left( 1 + \frac{t}{8} \right)^{-16} \right)^{m-1} \int_0^1 t^5 (1-t)^7 \left( 1 + \frac{t}{8} \right)^{-18} dt,
\]
and so
\[
\log \left( \int_{0}^{1} t^{4m} (1 - t)^{7m-1} (1 - zt)^{-16m-1} \, dt \right) < -7.884160 \cdot (m - 1) - 8.568400,
\]
\[
\log \left( \int_{0}^{1} t^{4m+1} (1 - t)^{7m} (1 - zt)^{-16m-2} \, dt \right) < -7.884160 \cdot (m - 1) - 10.140038.
\]

Let us check inequality (9) for \( n \) and \( n + 1 \):
\[
\log \left| R_n \left( \frac{-1}{8} \right) \tilde{\Phi}^{-1} 2^{b-3a+1+2j} (-1)^a \right| < 17.147682 \cdot m + \log \left( \frac{1}{8} \right)^4 \cdot m - 7.884160 \cdot (m - 1) - 8.568400
\]
\[
- 1.639533 \cdot m + 35 \log(2)
\]
\[
< -0.693777 \cdot m + 23.575912 < \log \left( \frac{2}{3} \right) \quad \text{for} \quad m \geq 974762,
\]
\[
\log \left| R_{n+1} \left( \frac{-1}{8} \right) \tilde{\Phi}^{-1} 2^{b-3a+1+2j} (-1)^a \right| < 17.147682 \cdot m + 1 + \log \left( \frac{1}{8} \right)^4 \cdot m + \log \left( \frac{1}{8} \right) - 7.884160 \cdot (m - 1)
\]
\[
- 10.140038 - 1.639533 \cdot m + \log(4) \log(m) + 1 + 35 \log(2)
\]
\[
< -0.693777 \cdot m + \log(4) \log(m) + 21.924832 < \log \left( \frac{2}{3} \right) \quad \text{for} \quad m \geq 974762.
\]

So, inequality (9) holds, and we can move on.

For integral in (11), in the same way as for the one in (5), one has
\[
\log \left( \int_{0}^{1} t^{7m-1} (1 - t)^{5m} (1 + 8t)^{4m} \, dt \right) < -0.945755 \cdot (m - 1) - 1.725707,
\]
\[
\log \left( \int_{0}^{1} t^{7m} (1 - t)^{5m-1} (1 + 8t)^{4m+1} \, dt \right) < -0.945755 \cdot (m - 1) + 0.878883.
\]
Now we can calculate the bounds in (10) and (11). We begin with (10):

\[
\log |\varepsilon_k| > 1.639533 \cdot m - 18 \log(3) - 17.147682 \cdot m \\
+ 0.945755 \cdot (m-1) + 1.725707 \\
> -14.562394 \cdot m - 18.995070 \\
> -0.81 \cdot k > \log \left(\frac{4}{9}\right) \cdot k \quad \text{for} \quad k \geq 17545718.
\]

For (11) we have

\[
\log |\varepsilon_k| > 1.639533 \cdot m - \log(4m+1) - 18 \log(3) - 17.147682 \cdot m \\
+ 0.945755 \cdot (m-1) - 0.878883 \\
> -14.562414 \cdot m - \log(4) \log(m) - 22.599660 \\
> -14.562414 \cdot m - 22.599660 \\
> -0.81 \cdot k > \log \left(\frac{4}{9}\right) \cdot k \quad \text{for} \quad k \geq 17545718.
\]

So, we have

\[
\left\| \left(\frac{4}{3}\right)^k \right\| > \left(\frac{4}{9}\right)^k \quad \text{for} \quad k \geq 17545718.
\]

7 The Final check

We need to check inequality (11) for 6 \leq k \leq 17545717. Following [2], we prove next lemma.

**Lemma 1.** Let \( m \) be a positive integer, and assume that the number \( 4^m \) contain no block of \( h \) consecutive 0, or 2, in its ternary expansion. Then the inequality

\[
\left\| \left(\frac{4}{3}\right)^k \right\| \geq \left(\frac{4}{9}\right)^k 
\]

holds for all

\[
m \left(\frac{\log 4}{\log 9}\right) + \frac{h}{2} \leq k \leq m.
\]

**Proof.** We give a proof by contradiction. Assume that \( k \) is in the specified interval, but (16) is not true. Then for some integer \( M_1 \) we have one of the next two equalities:

\[
M_1 = \left(\frac{4}{3}\right)^k \pm \epsilon_1, \quad \text{where} \quad 0 < \epsilon_1 < \left(\frac{4}{9}\right)^k,
\]
so, with some integer $M_2$

$$4^m = 3^k M_2 + \epsilon_2, \quad \text{where} \quad 0 < \epsilon_2 < 4^m 3^{-k}.$$  

Since $m (\log 4/\log 9) \leq k - h/2$, we have

$$0 < \epsilon_2 < 9^{k-h/2} \cdot 3^{-k} = 3^{k-h},$$

but this means, that a block of $h$ digits of the number $4^m$, which are responsible for powers $3^{k-h}, \ldots, 3^{k-1}$, consists of 2, or 0. \(\square\)

For specified $m$ the software calculates $h(m)$ defined in lemma \(\ref{lemma1}\) descends to the new value of $m$ prescribed by (17), and so on. We started calculations with $m = 17545718$ and stopped at $m = 5$. Results of all the steps are given in Table 1.

| $m$  | $h$  | $m$  | $h$  | $m$  | $h$  |
|------|------|------|------|------|------|
| 1    | 17545718 | 14    | 53046 | 11    | 27    |
| 2    | 11229269 | 15    | 33955 | 10    | 28    |
| 3    | 7186741  | 16    | 21737 | 11    | 29    |
| 4    | 4599523  | 17    | 13917 | 8     | 30    |
| 5    | 2943702  | 18    | 8911  | 9     | 31    |
| 6    | 1883977  | 19    | 5708  | 8     | 32    |
| 7    | 1205753  | 20    | 3658  | 10    | 33    |
| 8    | 771689   | 21    | 2347  | 8     | 34    |
| 9    | 493886   | 22    | 1507  | 7     | 35    |
| 10   | 316094   | 23    | 968   | 5     | 36    |
| 11   | 202307   | 24    | 622   | 7     | 37    |
| 12   | 129482   | 25    | 402   | 10    |      |
| 13   | 82874    | 26    | 263   | 5     |      |

So Theorem \(\ref{lemma1}\) is proved.

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