Non-Linear Effects in a Yamabe-Type Problem with Quasi-Linear Weight

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Abstract

We study the quasi-linear minimization problem on $H^1_0(\Omega) \subset L^q$ with $q = \frac{2n}{n-2}$:

$$\inf_{\|u\|_{L^q(\Omega)} = 1} \int_\Omega (1 + |x|^{\beta} |u|^k) |\nabla u|^2.$$

We show that minimizers exist only in the range $\beta < \frac{kn}{q}$ which corresponds to a dominant non-linear term. On the contrary, the linear influence for $\beta \geq \frac{kn}{q}$ prevents their existence.

1 Introduction

Given a smooth bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, let us consider the minimizing problem

$$S_{\Omega}(\beta, k) = \inf_{u \in H^1_0(\Omega)} I_{\Omega, \beta, k}(u) \quad \text{with} \quad I_{\Omega, \beta, k}(u) = \int_\Omega p(x, u(x)) |\nabla u(x)|^2 \, dx$$

and $p(x, y) = 1 + |x|^{\beta} |y|^k$. Here $q = \frac{2n}{n-2}$ denotes the critical exponent of the Sobolev injection $H^1_0(\Omega) \subset L^q(\Omega)$. We restrict ourselves to the case $\beta \geq 0$ and $0 \leq k \leq q$. The Sobolev injection for $u \in H^{s+1}(\Omega)$ and $\nabla u \in H^s(\Omega)$ gives:

$$I_{\Omega, \beta, k}(u) \leq \|u\|^2_{H^1_0(\Omega)} + C_s \left( \sup_{x \in \Omega} |x|^{\beta} \right) \|u\|^2_{H^{s+1}(\Omega)} \quad \text{for} \quad s \geq \frac{kn}{q(k+2)}$$

so $I_{\Omega, \beta, k}(u) < \infty$ on a dense subset of $H^1_0(\Omega)$. Note in particular that one can have $I_{\Omega, \beta, k}(u) < \infty$ without having $u \in L^\infty_{\text{loc}}(\Omega)$. If $0 \notin \Omega$, the problem is essentially equivalent to the case $\beta = 0$ thus one will also assume from now on that $0 \in \Omega$. The case $0 \in \partial \Omega$ is interesting but will not be addressed in this paper.

As $|\nabla u| = |\nabla u|$ for any $u \in H^1_0(\Omega)$, one has

$$I_{\Omega, \beta, k}(u) = I_{\Omega, \beta, k}(|u|)$$

thus, when dealing with (1), one can assume without loss of generality that $u \geq 0$. 

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The Euler-Lagrange equation formally associated to (1) is
\[
\begin{cases}
  - \text{div} \left( p(x, u(x)) \nabla u \right) + Q(x, u(x)) |\nabla u(x)|^2 = \mu |u(x)|^{q-2} u(x) & \text{in } \Omega \\
  u \geq 0 & \text{on } \partial \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
\]
with \( Q(x, y) = \frac{k}{2} |x|^\beta |y|^{k-2} y \) and \( \mu = S_\Omega(\beta, k) \). However, the logical relation between (1) and (3) is subtle: \( I_{\Omega; \beta, k} \) is not Gateaux differentiable on \( H^1_0(\Omega) \) because one can only expect \( I_{\Omega; \beta, k}(u) = +\infty \) for a general function \( u \in H^1_0(\Omega) \). However, if a minimizer of (1) belongs to \( H^1_0 \cap L^\infty(\Omega) \) then, without restriction, one can assume \( u \geq 0 \) and for any test-function \( \phi \in H^1_0 \cap L^\infty(\Omega) \), one has
\[
\forall t \in \mathbb{R}, \quad I_{\Omega; \beta, k}(\frac{u + t\phi}{\|u + t\phi\|_{L^n}}) < \infty.
\]
A finite expansion around \( t = 0 \) then gives (3) in the weak sense, with the test-function \( \phi \).

The following generalization of (1) will be addressed in a subsequent paper:
\[
S_\Omega(\lambda; \beta, k) = \inf_{u \in H^1_0(\Omega)} \left( \frac{J_{\Omega; \beta, k}(\lambda, u)}{\|u\|_{L^4(\Omega)} = 1} \right) \quad \text{with} \quad J_{\Omega; \beta, k}(\lambda, u) = I_{\Omega; \beta, k}(u) - \lambda \int_\Omega |u|^2. \tag{4}
\]
for \( \lambda > 0 \), which is a compact perturbation of the case \( \lambda = 0 \).

This type of problem is inspired by the study of the Yamabe problem which has been the source of a large literature. The Yamabe invariant of a compact Riemannian manifold \((M, g)\) is:
\[
\mathcal{Y}(M) = \inf_{\phi \in \mathcal{C}^\infty(M; \mathbb{R}_+)} \int_M \left( 4 \frac{n-1}{n-2} |\nabla \phi|^2 + \sigma \phi^2 \right) dV_g
\]
where \( \nabla \) denotes the covariant derivative with respect to \( g \) and \( \sigma \) is the scalar curvature of \( g \); \( \mathcal{Y}(M) \) is an invariant of the conformal class \( \mathcal{C} \) of \((M, g)\). One can check easily that \( \mathcal{Y}(M) \leq \mathcal{Y}(S^n) \). The so called Yamabe problem which is the question of finding a manifold in \( \mathcal{C} \) with constant scalar curvature can be solved if \( \mathcal{Y}(M) < \mathcal{Y}(S^n) \). In dimension \( n \geq 6 \), one can show that unless \( M \) is conformal to the standard sphere, the strict inequality holds using a “local” test function \( \phi \); however, for \( n \leq 5 \), one must use a “global” test function (see [11] for an in-depth review of this historical problem and more precise statements).

Even though problems (1) and (4) seem of much less geometric nature, they should be considered as a toy model of the Yamabe problem that can be played with in \( \mathbb{R}^n \). As it will be shown in this paper, those toy models retain some interesting properties from their geometrical counterpart: the functions \( u_x \) that realise the infimum \( \mathcal{Y}(S^n) \) still play a crucial role in (1) and (4) and the existence of a solution is an exclusively non-linear effect.

Another motivation can be found in the line of [4] for the study of sharp Sobolev and Gagliardo-Nirenberg inequalities. For example, among other striking results it is shown that, for an arbitrary norm \( \|\cdot\| \) on \( \mathbb{R}^n \):
\[
\inf_{\|u\|_{L^2} = 1} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx = \|\nabla h\|^2_{L^2} \quad \text{with} \quad h(x) = \frac{1}{(c + \|x\|^2)^{\frac{n-2}{2}}}
\]
and a constant \( c \) such that \( \|h\|_{L^n} = 1 \). The problem (1) can be seen as a quasi-linear generalisation where the norm \( \|\cdot\| \) measuring \( \nabla u \) is allowed to depend on \( u \) itself.
1.1 Bibliographical notes

The case $\beta = k = 0 \ i.e. \ a$ constant weight $p(x, y) = 1$ has been addressed in the celebrated [2] where it is shown in particular that the equation

$$-\Delta u = u^{q-1} + \lambda u, \quad u > 0$$

has a solution $u \in H^1_0(\Omega)$ if $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{I_{\Omega, 0, 0}(u)}{\int_{\Omega} |u|^2 dx}$.

On the contrary, for $\lambda = 0$, the problem (5) has no solution if $\Omega$ is star-shaped around the origin. In dimension $n = 3$, the situation is more subtle. For example, if $\Omega = \{ x \in \mathbb{R}^3 ; |x| < 1 \}$, then (5) admits solutions for $\lambda \in [\frac{2}{3}, \pi^2]$ but has none if $\lambda \in [0, \frac{\pi^2}{4}]$. See also [6] for the behavior of solutions when $\lambda \to (\pi^2/4)_+$ and for generalizations to general domains.

A first attempt to the case $\beta \neq 0$ but with $k = 0 \ i.e. \ a$ weight that does not depend on $u$, which is the semi-linear case) was achieved in [10]. More precisely, [10] deals with a weight $p \in H^1(\Omega) \cap C(\Omega)$ that admits a global minimum of the form

$$p(x) = p_0 + c|x - a|^\beta + o\left(|x - a|^\beta\right), \quad c > 0.$$  

They show that for $n \geq 3$ and $\beta > 0$, there exists $\lambda_0 \geq 0$ such that (4) admits a solution for any $\lambda \in [\lambda_0, \lambda_1]$ where $\lambda_1$ is the first eigenvalue of the operator $-\Delta - (p(x)\nabla \cdot )$ in $\Omega$, with Dirichlet boundary conditions and for $n \geq 4$ and $\beta > 2$, one can check that $\lambda_0 = 0$. On the contrary, the problem (4) admits no solution if $\lambda \leq \lambda_0^*$ for some $\lambda_0^* \in (0; \lambda_0]$ or for $\lambda \geq \lambda_1$. See [10] for more precise statements.

Similarly, the semi-linear case in which the minimum value of the weight is achieved in more than one point was studied in [9] ; namely in dimension $n \geq 4$ if

$$p^{-1}\left(\inf_{x \in \Omega} p(x)\right) = \{a_0, a_1, \ldots, a_N\}$$

then multiple solutions that concentrate around each of the $a_j$ can be found for $\lambda > 0$ small enough.

For $\lambda = 0$ and a star-shaped domain, it is well known (see [2]) that the linear problem $\beta = k = 0$ has no solution. However, when the topology of the domain is not trivial, the problem (1) has at least one solution (see [5] for $\beta = k = 0$ ; [8] and [10] for $k = 0, \beta \neq 0$).

1.2 Ideas and main results

In this article, the introduction of the fully quasi-linear term $|x|^\beta |u|^k$ in (1) provides a more unified approach and generates a sharp contrast between sub- and super-critical cases. Moreover, the existence of minimizers will be shown to occur exactly in the sub-cases where the nonlinearity is dominant.

The critical exponent for (1) can be found by the following scaling argument. As $0 \in \Omega$, the non-linear term tends to concentrate minimizing sequences around $x = 0$. Let us therefore consider the blow-up of $u \in H^1_0(\Omega)$ around $x = 0$. This means one looks at the function $v_\varepsilon$ defined by :

$$\forall \varepsilon > 0, \quad u(x) = \varepsilon^{-n/q} v_\varepsilon(x/\varepsilon).$$

One has $v_\varepsilon \in H^1_0(\Omega_\varepsilon)$ with $\Omega_\varepsilon = \{ \varepsilon^{-1} y ; y \in \Omega \}$ and $\|v_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)}$. Moreover, the definition of $q$ ensures that $2 - n + \frac{2n}{q} = 0$, thus :

$$I_{\Omega, \varepsilon, k}(u) = \int_{\Omega_\varepsilon} \left(1 + \varepsilon^{\beta - \frac{kn}{q}} |y|^\beta |v_\varepsilon(y)|^k \right) |\nabla v_\varepsilon(y)|^2 dy.$$  

Depending on the ratio $\beta/k$, different situations occur.
• If $\frac{\beta}{k} < \frac{n}{q}$, leading term of the blow-up around $x = 0$ is

$$I_{\Omega;\beta,k}(u) \sim \varepsilon^{-\left(\frac{2k}{n} - \frac{\beta}{k}\right)} \int_{\Omega} |y|^\beta |v_\varepsilon(y)|^k |\nabla v_\varepsilon(y)|^2 dy.$$ 

One can expect the effect of the non-linearity to be dominant and one will show in this paper that (1) admits indeed minimizers in this case.

• If $\frac{\beta}{k} = \frac{n}{q}$, both terms have the same weight and

$$\forall \varepsilon > 0, \quad I_{\Omega;\beta,k}(u) = I_{\Omega;\varepsilon}(v_\varepsilon).$$

One will show that, similarly to the classical case $\beta = k = 0$, the corresponding infimum $S(\beta, k)$ does not depend on $\Omega$ and that (1) admits no smooth minimizer.

• If $\frac{\beta}{k} > \frac{n}{q}$, the blow-up around 0 gives

$$I_{\Omega;\beta,k}(u) \sim \varepsilon \int_{\Omega} |\nabla v_\varepsilon(y)|^2 dy.$$ 

In this case, one can show that the linear behavior is dominant and that (1) admits no minimizer. Moreover, one can find a common minimizing sequences for both the linear and the non-linear problem. A cheap way to justify this is as follows. The problem (1) tends to concentrate $u$ as a radial decreasing function around the origin. Thus, when $\beta/k > n/q$, one can expect $|u(x)|^q \ll 1/|x|^q$ because the right-hand side would not be locally integrable while the left-hand side is required to. In turn, this inequality reads $|x|^\beta |u(x)|^k \ll 1$ which eliminates the non-linear contribution in the minimizing problem (1).

The infimum for the classical problem with $\beta = k = 0$ is (see e.g. [2])

$$S = \inf_{w \in H^1_0(\Omega) \cap H^3/2 \cap L^\infty(\Omega)} \int |\nabla w|^2$$

which does not depend on $\Omega$. Let us now state the main Theorem concerning (1).

**Theorem 1** Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain with $n \geq 3$ and $q = \frac{2n}{n-2}$ the critical exponent for the Sobolev injection $H^1_0(\Omega) \subset L^q(\Omega)$.

1. If $0 \leq \beta < kn/q$ then $S_{\Omega}(\beta, k) > S$ and the infimum for $S_{\Omega}(\beta, k)$ is achieved.

2. If $\beta = kn/q$ then $S_{\Omega}(\beta, k)$ does not depend on $\Omega$ and $S_{\Omega}(\beta, k) \geq S$. Moreover, if $\Omega$ is star-shaped around $x = 0$, then the minimizing problem (1) admits no minimizers in the class $H^1_0 \cap H^{3/2} \cap L^\infty(\Omega)$.

   If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

3. If $\beta > kn/q$ then $S_{\Omega}(\beta, k) = S$ and the infimum for $S_{\Omega}(\beta, k)$ is not achieved in $H^1_0(\Omega)$.

**Remarks.**

1. In the first case, one has $k > 0$, thus results concerning $k = 0$ (such as those of e.g. [9] and [10]) are included either in our second or third case.

2. If the minimizing problem (1) is achieved for $u \in H^1_0(\Omega)$, then $|u|$ is a positive minimizer. In particular, if $\beta < kn/q$, the problem always admits positive minimizers.

3. In the critical case, it is not known whether a non-smooth minimizer could exist in $H^1_0 \setminus (H^{3/2} \cap L^\infty)$.

Such a minimizer could have a non-constant sign.
1.3 Structure of the article

Each of the following sections deals with one sub-case $\beta \leq kn/q$.

2 Subcritical case ($0 \leq \beta < kn/q$) : existence of minimizers

The case $\beta < kn/q$ is especially interesting because it reveals that the non-linear weight $|u|^k$ helps for the existence of a minimizer. Note that $k > 0$ in throughout this section.

Proposition 2 If $0 \leq \beta < \frac{n}{q}$, the minimization problem (1) has at least one solution $u \in H^1_0(\Omega)$. Moreover, one has

$$S(\beta,k) > S$$

where $S$ is defined by (8).

Proof. Let us prove first that the existence of a solution implies the strict inequality in (9). By contradiction, if $S(\beta,k) = S$ and if $u$ is a minimizer for (1) thus $u \neq 0$, one has

$$S = \int_{\Omega} (1 + |x|^\beta |u(x)|^k) |\nabla u(x)|^2 dx > \int_{\Omega} |\nabla u(x)|^2 dx$$

which contradicts the definition of $S$. Thus, if the minimization problem has a solution, the strict inequality (9) must hold.

Let us prove now that (1) has at least one solution $u \in H^1_0(\Omega)$. Let $(u_j)_{j \in \mathbb{N}} \in H^1_0(\Omega)$ be a minimizing sequence for (1), i.e.:

$$I_{\Omega,\beta,k}(u_j) = S_{\Omega}(\beta,k) + o(1), \quad \text{and} \quad \|u_j\|_{L^q} = 1.$$

As noticed in the introduction, one can assume without restriction that $u_j \geq 0$. Up to a subsequence, still denoted by $u_j$, there exists $u \in H^1_0(\Omega)$ such that $u_j(x) \to u(x)$ for almost every $x \in \Omega$ and such that:

$$u_j \to u \quad \text{weakly in} \quad H^1_0 \cap L^q(\Omega),$$

$$u_j \to u \quad \text{strongly in} \quad L^\ell(\Omega) \quad \text{for any} \quad \ell < q.$$

The idea of the proof is to introduce $v_j = u_j^{\frac{k+1}{2}}$ and prove that $v_j$ is a bounded sequence in $W^{1,r}_0 \subset L^p$ for indices $r$ and $p$ such that

$$p \left( \frac{k}{2} + 1 \right) \geq q.$$

The key point is the formula:

$$I_{\Omega,\beta,k}(u_j) = \int_{\Omega} |\nabla u_j|^2 + \left( \frac{k}{2} + 1 \right)^{-2} \int_{\Omega} |x|^\beta |\nabla v_j|^2$$

which gives “almost” an $H^1_0$ bound on $v_j$ (and does indeed if $\beta = 0$). For $r \in [1,2[,$ one has:

$$\int_{\Omega} |\nabla v_j|^r \leq \left( \int_{\Omega} |x|^\beta |\nabla v_j|^2 dx \right)^{r/2} \left( \int_{\Omega} |x|^{-\frac{\beta r}{r-2}} dx \right)^{1-r/2}$$

The integral in the right-hand side is bounded provided $\frac{\beta r}{r-2} < n$. All the previous conditions are satisfied if one can find $r$ such that:

$$1 \leq r < 2, \quad \beta < n \left( \frac{2}{r} - 1 \right), \quad \frac{k}{2} + 1 \geq \frac{q}{p} = q \left( \frac{1}{r} - \frac{1}{n} \right).$$
This system of inequalities boils down to:
\[
1 \leq r < 2, \quad \frac{\beta}{n} < \frac{2}{r} - 1 \leq \frac{2}{q} \left( \frac{k}{2} + 1 + \frac{q}{n} \right) - 1
\]
which is finally equivalent to \(\beta < \frac{kn}{q}\) provided \(k \leq q\). Using the compacity of the inclusion \(W^{1,r}_0 \subset L^p\) and up to a subsequence, one has \(v_j \to v = u^{\frac{k}{2}+1}\) strongly in \(L^p\). Finally, as \(u_j \geq 0\) and \(u \geq 0\), one has:
\[
|u_j - u|^q \leq C \left| u_j^q - u^q \right| = C \left| v^{q/(k/2+1)} - v^{q/(k/2+1)} \right|
\]
and thus \(u_j \to u\) strongly in \(L^q\). One gets \(\|u\|_{L^q} = 1\). The following compacity result then implies that \(u\) is a minimizer for (1).

**Proposition 3** If \(u_j \in H^1_0(\Omega)\) is a minimizing sequence for (1) with \(\|u_j\|_{L^q(\Omega)} = 1\) and such that \(u_j \to u\) in \(L^2(\Omega)\), and \(\nabla u_j \rightharpoonup \nabla u\) weakly in \(L^2(\Omega)\),
the weak limit \(u \in H^1_0(\Omega)\) is a minimizer of the problem (1) if and only if \(\|u\|_{L^q(\Omega)} = 1\).

**Proof.** It is an consequence of the main Theorem of [7, p. 77] (see also [14]) applied to the function:
\[
f(x, z, p) = (1 + |x|^\beta |z|^k |p|^2
\]
which is positive, measurable on \(\Omega \times \mathbb{R} \times \mathbb{R}^n\), continuous with respect to \(z\), convex with respect to \(p\). Then
\[
I(u) = \int_{\Omega} f(x, u, \nabla u) \leq \liminf_{j \to \infty} \int_{\Omega} f(x, u_j, \nabla u_j) = \liminf_{j \to \infty} I(u_j).
\]
If \(u_j\) is a minimizing sequence, then \(I(u) = S_\Omega(\beta, k)\) and \(u\) is a minimizer if and only if \(\|u\|_{L^q} = 1\).

**Remarks**

- The sequence \(u_j\) converges strongly in \(H^1_0(\Omega)\) towards \(u\) because \(\nabla u_j \rightharpoonup \nabla u\) weakly in \(L^2(\Omega)\) and:
  \[
  \int_{\Omega} |\nabla u_j|^2 - \int_{\Omega} |\nabla u|^2 = I(u_j) - I(u) + \int_{\Omega} |x|^\beta u_j^k |\nabla u|^2 - \int_{\Omega} |x|^\beta u^k_j |\nabla u_j|^2.
  \]
  Applying the previous lemma with \(\tilde{f}(x, z, p) = |x|^\beta |z|^k |p|^2\) provides
  \[
  \forall j \in \mathbb{N}, \quad \int_{\Omega} |\nabla u_j|^2 \leq \int_{\Omega} |\nabla u|^2 + o(1)
  \]
  and Fatou’s lemma provides the converse inequality.

- This proof implies also that \(S_\Omega(\beta, k)\) is continuous with respect to \((\beta, k)\) in the range \(0 \leq \beta < \frac{kn}{q}\) and that the corresponding minimizer depends continuously on \((\beta, k)\) in \(L^q(\Omega)\) and \(H^1_0(\Omega)\).
3 Semi-linear case ($\beta > kn/q$) : non-compact minimizing sequence

When $\beta > kn/q$, the problem (1) is under the total influence of the linear problem (8). Let us recall that its minimizer $S$ is independent of the smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) and that this minimizing problem has no solution. According to [2], a minimizing sequence of (8) is given by $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ where:

$$u_\varepsilon(x) = \frac{\varepsilon^{-\frac{n-2}{2}} \zeta(x)}{(\varepsilon + |x|^2)^\frac{n-2}{2}}$$

with $\zeta \in C^\infty(\Omega; [0, 1])$ is a smooth compactly supported cutoff function that satisfy $\zeta(x) = 1$ in a small neighborhood of the origin in $\Omega$. Recall that $\frac{n-2}{2} = n/q$. Recall that $(k+1)(n-2) > kn/q$ for any $k \geq 0$. We know from [2] that

$$\|\nabla u_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{\frac{n-2}{2}}), \quad \|u_\varepsilon\|_{L^q}^2 = K_2 + o(\varepsilon^{\frac{n-2}{2}})$$

and that $S = K_1/K_2$.

The goal of this section is the proof of the following Proposition.

Proposition 4 If $\frac{\beta}{k} > \frac{n}{q}$, one has

$$S_\Omega(\beta, k) = S$$

and the problem (1) admits no minimizer in $H^1_0(\Omega)$. Moreover, the sequence $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ defined by (11) is a minimizing sequence for both (1) and the linear problem (8).

**Proof.** Suppose by contradiction that (1) is achieved by $u \in H^1_0(\Omega)$. Then $u \neq 0$ and therefore the following strict inequality holds:

$$S \leq \int_\Omega |\nabla u|^2 < I_{\Omega, \beta, k}(u) = S_\Omega(\beta, k).$$

Therefore the identity (12) implies that (1) has no minimizer. To prove (12) and the rest of the statement, it is sufficient to show that

$$I_{\Omega, \beta, k} \left(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon\right) = S + o(1)$$

in the limit $\varepsilon \to 0$, because one obviously has $S \leq S_\Omega(\beta, k) \leq I_{\Omega, \beta, k}(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon)$. The limit (13) will follow immediately from the next result.

**Proposition 5** With the previous notations, (13) holds and more precisely, as $\varepsilon \to 0$, one has :

$$\int_\Omega |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx = \begin{cases} C\varepsilon^{2\beta-k-(n-2)} + o\left(\varepsilon^{2\beta-k-(n-2)}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\
O\left(\varepsilon^{(k+2)(n-2)\frac{n-2}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\
O\left(\varepsilon^{(k+2)(n-2)\frac{n-2}{4}}\right) & \text{if } \beta > (k+1)(n-2)\end{cases}$$

with $C = \int_{\mathbb{R}^n} \frac{|x|^\beta+2}{(1 + |x|^2)^\frac{kn-2}{2}+n} dx$ and thus :

$$I_{\Omega, \beta, k} \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{L^q}}\right) = S + \begin{cases} C\varepsilon^{2\beta-k-(n-2)} K_2 + o(\varepsilon^{2\beta-k-(n-2)}) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\
O\left(\varepsilon^{(k+2)(n-2)\frac{n-2}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\
O\left(\varepsilon^{(k+2)(n-2)\frac{n-2}{4}}\right) & \text{if } \beta > (k+1)(n-2)\end{cases}$$
Proof. The only verification is that of (14).

\[
\int_{\Omega} |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 \, dx = (n-2)^2 \varepsilon \int_{\Omega} \frac{\zeta(x)^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx \\
+ \varepsilon \int_{\Omega} \frac{\zeta(x)^{k+2} |\nabla \zeta(x)|^2 |x|^\beta}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx \\
- 2(n-2) \varepsilon \int_{\Omega} \frac{\zeta(x)^{k+1} |x|^{\beta-1} |\nabla \zeta(x)|}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx.
\]

Since \( \zeta \equiv 1 \) on a neighborhood of \( a \) and using the Dominated Convergence Theorem, a direct computation gives

\[
\int_{\Omega} |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 \, dx = (n-2)^2 \varepsilon \int_{\Omega} \frac{\zeta(x)^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx \\
+ \mathcal{O}(\varepsilon^{\frac{k+2(n-2)}{4}}).
\]

Here we will consider the following three subcases.

1. Case \( \beta < (k+1)(n-2) \)

\[
\varepsilon \int_{\Omega} \frac{\zeta(x)^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx + \int_{\mathbb{R}^n \setminus \Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx \\
+ \int_{\Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} (|\zeta(x)|^{k+2} - 1) |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx.
\]

Using the Dominated Convergence Theorem, and the fact that \( \zeta \equiv 1 \) on a neighborhood of 0, one obtains

\[
\varepsilon \int_{\mathbb{R}^n} \frac{\zeta(x)^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx + \mathcal{O}(\varepsilon^{\frac{(k+2)(n-2)}{4}}).
\]

By a simple change of variable, one gets

\[
\varepsilon \int_{\mathbb{R}^n} \frac{\zeta(x)^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dx = \int_{\mathbb{R}^n} \frac{|y|^{\beta+2}}{(1 + |y|^2)^{\frac{\gamma(n-2)}{2} + n}} \, dy + \mathcal{O}(\varepsilon^{\frac{2n-k(n-2)}{4}})
\]

which gives (14) in this case.
2. Case $\beta = (k + 1)(n - 2)$

$$\int_{\Omega} |x|^{\beta} |u_\varepsilon| |\nabla u_\varepsilon|^2 \, dx = (n - 2)^2 \varepsilon \frac{(k + 2)(n - 2)}{4} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx + O\left(\varepsilon \frac{(k + 2)(n - 2)}{4}\right)$$

$$= (n - 2)^2 \varepsilon \frac{(k + 2)(n - 2)}{4} \int_{\Omega} \frac{|\zeta(x)|^{k+2} - 1 |x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx$$

$$+ (n - 2)^2 \varepsilon \frac{(k + 2)(n - 2)}{4} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx + O\left(\varepsilon \frac{(k + 2)(n - 2)}{4}\right)$$

$$= (n - 2)^2 \varepsilon \frac{(k + 2)(n - 2)}{4} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx + O\left(\varepsilon \frac{(k + 2)(n - 2)}{4}\right)$$

One has, for some constants $R_1 < R_2$ :

$$\int_{B(0, R_1)} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx \leq \int_{\Omega} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx \leq \int_{B(0, R_2)} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx$$

with

$$\int_{B(0, R)} \frac{|x|^{k(n-2)+n}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx = \omega_n \int_{0}^{R} r^{k(n-2)+2n-1} \frac{\varepsilon^{k(n-2)+n}}{2} \, dr$$

$$= \frac{1}{2} \omega_n \log \varepsilon + O(1).$$

Consequently, one has :

$$\int_{\Omega} |x|^{\beta} |u_\varepsilon| |\nabla u_\varepsilon|^2 \, dx = O\left(\varepsilon \frac{(k + 2)(n - 2)}{4} \log \varepsilon \right).$$

3. Case $\beta > (k + 1)(n - 2)$

$$\int_{\Omega} |x|^{\beta} |u_\varepsilon| |\nabla u_\varepsilon|^2 \, dx = (n - 2)^2 \varepsilon \frac{(k + 2)(n - 2)}{4} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \, dx + O\left(\varepsilon \frac{(k + 2)(n - 2)}{4}\right).$$

One can apply the Dominated Convergence Theorem :

$$\frac{|\zeta(x)|^{k+2} |x|^\beta+2}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \rightarrow |\zeta(x)|^{k+2} |x|^\beta-(k(n-2)+2n-2) \quad \text{when } \varepsilon \rightarrow 0$$

and

$$\frac{|\zeta(x)|^{k+2} |x|^\beta+2}{\varepsilon + |x|^2} \frac{\varepsilon^{k(n-2)+n}}{2} \leq |\zeta(x)|^{k+2} |x|^\beta-(k(n-2)+2n-2) \in L^1(\Omega).$$

So, it follows that

$$\int_{\Omega} |x|^{\beta} |u_\varepsilon| |\nabla u_\varepsilon|^2 \, dx = O\left(\varepsilon \frac{(k + 2)(n - 2)}{4}\right)$$

which again is (14).
4 The critical case ($\beta = kn/q$): non-existence of smooth minimizers

The critical case is a natural generalization of the well known problem with $\beta = k = 0$. In this section, the following result will be established.

**Proposition 6** If $\beta = kn/q$, one has

$$S_{\Omega}(\beta, k) = S_{\Omega}^{\omega}(\beta, k)$$

(16)

for any two smooth neighborhoods $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ of the origin. Moreover, if $\Omega$ is star-shaped around $x = 0$, the minimization problem (1) admits no solution in the class:

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

**If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.**

The rest of this section is devoted to the proof of this statement. Note that if the minimization problem (1) had a minimizer $u$ with non constant sign in this class of regularity, then $|u|$ would be a positive minimizer in the same class, thus it is sufficient to show that there are no positive minimizers.

4.1 $S_{\Omega}(\beta, k)$ does not depend on the domain

If $\Omega \subset \Omega'$, there is a natural injection $i : H_0^1(\Omega) \rightarrow H_0^1(\Omega')$ that corresponds to the process of extension by zero. Let $u_j \in H_0^1(\Omega)$ be a minimizing sequence for $S_{\Omega}(\beta, k)$. Then $\|i(u_j)\|_{L^\infty(\Omega')} = 1$ thus

$$S_{\Omega'}(\beta, k) \leq I_{\Omega';\beta,k}(i(u_j)) = I_{\Omega;\beta,k}(u_j)$$

and therefore $S_{\Omega'}(\beta, k) \leq S_{\Omega}(\beta, k)$.

Conversely, let us now consider the scaling transformation (6) which, in the case of $\frac{\beta}{k} = \frac{n}{q}$, leaves both $\|u\|_{L^\infty(\Omega)}$ and $I_{\Omega;\beta,k}(u)$ invariant. If $u_j$ is a minimizing sequence on $\Omega$ then $v_j = u_{j,\lambda^{-1}}$ is an admissible sequence on $\Omega$ thus:

$$S_{\Omega}(\beta, k) \leq I_{\Omega;\beta,k}(v_j) = I_{\Omega;\beta,k}(u_j) \rightarrow S_{\Omega}(\beta, k).$$

Conversely, if $v_j$ is a minimizing sequence on $\Omega$ then $u_j = v_j,\lambda^{-1}$ is an admissible sequence on $\Omega$ and :

$$S_{\Omega}(\beta, k) \leq I_{\Omega;\beta,k}(u_j) = I_{\Omega;\beta,k}(v_j) \rightarrow S_{\Omega}(\beta, k).$$

This ensures that $S_{\Omega}(\beta, k) = S_{\Omega}(\beta, k)$ for any $\lambda > 0$.

Finally, given two smooth bounded open subsets $\Omega$ and $\tilde{\Omega}$ of $\mathbb{R}^n$ that both contain 0, one can find $\lambda, \mu > 0$ such that $\Omega, \mu \subset \tilde{\Omega} \subset \Omega, \mu$ and the previous inequalities read

$$S_{\Omega}(\beta, k) \leq S_{\tilde{\Omega}}(\beta, k) \leq S_{\Omega}(\beta, k) \quad \text{and} \quad S_{\Omega}(\beta, k) = S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k)$$

thus ensuring $S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k)$.

4.2 Pohozaev identity and the non-existence of smooth minimizers

Suppose by contradiction that a bounded minimizer $u$ of (1) exists for some star-shaped domain $\Omega$ with $\beta = kn/q$, i.e. $u \in H_0^1 \cap L^\infty(\Omega)$. As mentioned in the introduction $|u|$ is also a minimizer thus, without loss of generality, one can also assume that $u \geq 0$. Moreover, $u$ will satisfy the Euler-Lagrange equation (3) in the weak sense, for any test-function in $H_0^1 \cap L^\infty(\Omega)$.
In the following argument, inspired by [13], one will use \((x \cdot \nabla)u\) and \(u\) as test functions. The later is fine but the former must be checked out carefully. A brutal assumption like \((x \cdot \nabla)u \in H^1_0 \cap L^\infty(\Omega)\) is much too restrictive. Let us assume instead that
\[
u \in H^1_0 \cap H^{3/2} \cap L^\infty \quad \text{and (if } k < 1 \text{) } u^{k-1} \in L^n(\Omega), \quad (17)
\]
Note that if \(v \in H^{3/2}\) then \(|v| \in H^{3/2}\) thus the assumption \(u \geq 0\) still holds without loss of generality. Then one can find a sequence \(\phi_n \in H^1_0 \cap L^\infty(\Omega)\) such that \(\phi_n \to \phi = (x \cdot \nabla)u\) in \(H^{1/2}(\Omega)\) and almost everywhere and such that each sequence of integrals converges to the expected limit:
\[
\begin{align*}
(-\Delta u|\phi_n) & \to (-\Delta u|\phi), \\
(u^k|\phi_n) & \to (u^k|\phi) \\
(u^{k-1}\nabla u|\phi_n) & \to (u^{k-1}\nabla u|\phi) \quad \text{and} \quad (u^{q-1}|\phi_n) \to (u^{q-1}|\phi).
\end{align*}
\]
Indeed, each integral satisfies a domination assumption:
\[
\begin{align*}
|(-\Delta u|\phi_n - \phi)| & \leq \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}}, \\
(u^k|\phi_n - \phi)| & \leq \|u^k\|_{L^{2/(n+1)}} \|\phi_n - \phi\|_{L^{2/n/(n-1)}} \leq C_\Omega \|u^k\|_{L^n} \|\phi_n - \phi\|_{H^{1/2}}, \\
(u^{k-1}\nabla u|\phi_n - \phi)| & \leq \begin{cases}
\|u^{k-1}\|_{L^\infty}\|\nabla u\|_{L^2}\|\phi_n - \phi\|_{L^2} & \text{if } k \geq 1, \\
\|u^{k-1}\|_{L^n}\|\nabla u\|_{L^{2/(n-1)}}\|\phi_n - \phi\|_{L^{2/n/(n-1)}} & \text{if } k < 1, \\
C_\Omega \|u^{k-1}\|_{L^\infty}\|\phi_n - \phi\|_{H^{1/2}} & \text{if } k < 1,
\end{cases}
\end{align*}
\]
Thus, the Euler-Lagrange is also satisfied in the weak sense for the test-function \(\phi = (x \cdot \nabla)u\).

Let us multiply by \((x \cdot \nabla)u\) and integrate by parts:
\[
-\int \nabla (p(x, u)\nabla u) \times (x \cdot \nabla)u + \frac{k}{2} \int |x|^2 |u|^{k-2} |\nabla u|^2 u(x \cdot \nabla)u = S(\beta, k) \int |u|^{q-2} u(x \cdot \nabla)u.
\]
An integration by part in the right-hand side and the condition \(u \in H^1_0(\Omega)\) provide:
\[
S(\beta, k) \int |u|^{q-2} u(x \cdot \nabla)u = -S(\beta, k) \frac{n - 2}{2} \int |u|^q = -\frac{n}{q} S(\beta, k).
\]
The first term of the left-hand side is:
\[
-\int \nabla (p(x, u)\nabla u) \times (x \cdot \nabla)u = B(u) + \int p(x, u)|\nabla u|^2 - \int_{\partial \Omega} p(x, u) (x \cdot \nabla)u \frac{\partial u}{\partial \nu}
\]
with \(B(u)\) define as follows and dealt with by a second integration by part
\[
B(u) = \sum_{i,j} \int_{\Omega} x_j \left(1 + |x|^2 |u|^k \right) \left(\partial_i u \partial_j u \right)
= -B(u) - n \int \nabla u \nabla u - \beta \int |x|^2 |u|^{k-2} |\nabla u|^2 \\
- k \int |x|^2 |u|^{k-2} |\nabla u|^2 u(x \cdot \nabla)u + \int_{\partial \Omega} p(x, u) |\nabla u|^2 (x \cdot \n).n).
\]
On the boundary, \(p(x, u) = 1\) and as \(u \in H^1_0(\Omega)\), one has also \(\nabla u = \frac{\partial u}{\partial \nu} \) where \(\n\) denotes the normal unit vector to \(\partial \Omega\) and in particular \(|\nabla u| = |\frac{\partial u}{\partial \nu}|\); thus
\[
B(u) = -\frac{n}{2} \int \nabla u \nabla u - \frac{\beta}{2} \int |x|^2 |u|^k |\nabla u|^2 - k \int |x|^2 |u|^{k-2} |\nabla u|^2 u(x \cdot \nabla)u + \frac{1}{2} \int_{\partial \Omega} |\frac{\partial u}{\partial \nu}|^2 (x \cdot \n).
\]
The whole energy estimate with \((x \cdot \nabla) u\) boils down to:

\[
\frac{n-2}{2} \int_{\Omega} p(x, u) |\nabla u|^2 + \frac{\beta}{2} \int_{\Omega} |x|^\beta |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial \Omega} |\partial_u|^2 (x \cdot n) = \frac{n}{q} S(\beta, k).
\]

Finally, to deal with the first term, let us multiply (3) by \(u\) and integrate by parts; one gets:

\[
\int_{\Omega} p(x, u) |\nabla u|^2 = \int_{\Omega} (1 + |x|^\beta |u|^k) |\nabla u|^2 = -\frac{k}{2} \int_{\Omega} |x|^\beta |u|^k |\nabla u|^2 + S(\beta, k).
\]

Combining both estimates provides:

\[
\frac{1}{2} \left( \frac{\beta - kn}{q} \right) \int_{\Omega} |x|^\beta |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial \Omega} |\partial_u|^2 (x \cdot n) = 0.
\]

As \(\beta = kn/q\) and \(x \cdot n > 0\) (\(\Omega\) is star-shaped), one gets \(\frac{\partial u}{\partial n} = 0\) on \(\partial \Omega\).

The Euler-Lagrange equation (3) now reads:

\[-p(x, u) \Delta u = \frac{k}{2} |x|^\beta |u|^{k-2} |\nabla u|^2 + \beta |x|^{\beta-2} |u|^k (x \cdot \nabla)u + \mu |u|^{q-2} u\]

which for \(u \geq 0\) boils down to:

\[-p(x, u) \Delta u = \left| x \right|^{\beta-2} u^{k-1} \left( \frac{k}{2} \left| x \right|^2 |\nabla u|^2 + u(x \cdot \nabla)u \right) + \mu u^{q-1} = \left| x \right|^{\beta-2} u^{k-1} \left( \sqrt{\frac{k}{2}} |x| |\nabla u + C u x|^2 \right) - C^2 |x|^\beta u^{k+1} + \mu u^{q-1}\]

with \(2\sqrt{k/2}C = \beta\). For any \(t \in \mathbb{R}\), one has therefore:

\[-\Delta u + tu = \frac{|x|^{\beta-2} u^{k-1}}{p(x, u)} \left( \sqrt{\frac{k}{2}} |x| |\nabla u + C u x|^2 \right) + \frac{\mu u^{q-1}}{p(x, u)} + tu - \frac{C^2 |x|^\beta u^{k+1}}{p(x, u)} = f(t, x).\]

As \(u \in L^\infty\), one can chose \(t > C^2 |x|^\beta \|u\|^2_{L^\infty}\). Then \(f(t, x) \geq 0\) and the maximum principle implies that either \(u = 0\) or \(\frac{\partial u}{\partial n} < 0\) on \(\partial \Omega\). In particular, only the solution \(u = 0\) satisfies simultaneously Dirichlet and Neumann boundary conditions, which leads to a contradiction because \(\|u\|_{L^q} = 1\).

**Remarks**

1. Note that Pohozaev identity (18) prevents the existence of minimizers when \(\beta \geq kn/q\). However, the technique we used in §3 (when \(\beta > kn/q\)) enlightens the leading term of the problem and avoids dealing with artificial regularity assumptions.

2. Similarly, one could check that the computation is also correct if

\[u \in H^1_0 \cap H^2 \cap L^\infty(\Omega) \quad \text{and (if } k < 1 \text{) } u^{k-1} \in L^{n/2}.\]  

Assumption (19) is only preferable over (17) for \(k < 1\). But it requires additional regularity in the interior of \(\Omega\) and would not allow to assume \(u \geq 0\) without loss of generality because in general, \(v \in H^2 \neq |v| \in H^2\).
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