Abstract: We propose an efficient algorithm for the optimal control problems (OCPs) of nonlinear switched systems that optimizes the control input and switching instants simultaneously for a given switching sequence. We consider the switching instants as the optimization variables and formulate the OCP based on the direct multiple shooting method. We derive a linear equation to be solved in Newton’s method and propose a Riccati recursion algorithm to solve the linear equation efficiently. The computational time of the proposed method scales linearly with respect to the number of time stages of the horizon as the standard Riccati recursion. Numerical experiments show that the proposed method converges with a significantly shorter computational time than the conventional methods.

Keywords: Hybrid Model Predictive Control, Optimization and Model Predictive Control, Optimal Control, Switched Systems, Hybrid Systems

1. INTRODUCTION

Switched systems are a class of hybrid systems made up of several dynamical subsystems and the switching laws of active subsystems. Many practical control systems are modeled as switched systems, such as automobiles with different gears (Ngo et al. (2012)), electrical circuit systems (Kouro et al. (2009)), and mechanical systems with contacts (Grizzle et al. (2001); Li and Wensing (2020)).

Optimal control plays a significant role in the planning (e.g., trajectory optimization in robotics) and control, that is, model predictive control (MPC), of dynamical systems, including switched systems. There is sufficient literature on the optimal control of linear switched systems, for example, on dynamic programming for linear hybrid systems (Borrelli et al. (2005)), mixed integer linear programming (Bemporad et al. (2000)), multiparametric programming (Herceg et al. (2013)), or embedding transformation (Wu et al. (2019)). However, it is generally difficult to solve the optimal control problems (OCPs) for nonlinear switched systems because of a need to solve nonlinear and combinatorial optimization problems. A practical research topic of the OCPs of nonlinear switched systems is to optimize the continuous control input and switching instants for a given switching sequence, which reduces the OCP to a continuous optimization problem. However, it is still difficult to solve such problems within a short computational time, which is crucial for MPC. Xu and Antsaklis (2004) proposed a two-stage framework in which the upper stage solves the optimization problem to find the optimal switching times under the fixed control input (the solution to the lower stage problem), and the lower stage solves the OCP (both continuous-time OCPs and discrete-time OCPs can be incorporated) under fixed switching times (the solution to the upper stage problem) to find the optimal control input. However, it was reported that their method required several minutes to converge because they solved a particular form of continuous-time OCP in the lower stage. Farshidian et al. (2017) improved the approach proposed by Xu and Antsaklis (2004), by solving the discrete-time OCP in the lower stage using a Newton-type method. However, their method required several seconds because the problem was still decomposed into two stages as Xu and Antsaklis (2004), and therefore the real-time application of their algorithm is difficult.

By contrast, our previous study (Katayama et al. (2020)) applied a Newton-type method to simultaneously optimize the control input and switching time. It used the Newton-Krylov method and succeeded in the MPC of a compass-like walking robot modeled as a nonlinear switched system with state jumps in real time. However, the Newton-Krylov method generally requires a careful tuning of settings such as the number of Krylov iterations and preconditioning; otherwise, it can lack numerical stability compared with the direct methods that compute the inverse matrix of the Hessian explicitly.

In this paper, we propose an efficient algorithm for the OCP of nonlinear switched systems that optimizes the control input and switching instants simultaneously for a given switching sequence. We formulate the OCP based on the direct multiple shooting method (Bock and Plitt (1984); Diehl et al. (2005)) with regard to the switching instance as the optimization variable, as well as the state and control input. We derive the linear equation to be solved in Newton’s method and propose a Riccati recursion algorithm to solve the linear equation efficiently. The computational time of the proposed method scales linearly with respect to the number of time stages of the horizon as the standard Riccati recursion (Frison (2016); Nielsen...
Next, we assume there is only one switch from $L$ where problems with multiple switches on the horizon (Katayama following formulation and the proposed algorithm to the direct multiple-shooting method. In Section 3, we derive the linear equation for Newton’s method and the Riccati recursion algorithm for the linear equation. In Section 4, we discuss the numerical experiments and demonstrate the effectiveness of the proposed method. Finally, in Section 5, we conclude our study with a brief summary and a discussion on future work.

**Notation:** We describe the Jacobians and the Hessians of a differentiable function by certain vectors as follows: $\nabla_x f(x)$ denotes $\left( \frac{df}{dx} \right)(x)$, and $\nabla_{xy} g(x,y)$ denotes $\frac{\partial^2 g}{\partial x \partial y}(x,y)$.

### 2. OPTIMAL CONTROL PROBLEMS OF SWITCHED SYSTEMS

#### 2.1 Continuous-Time Optimal Control Problem

We consider the OCP for the following nonlinear switched system consisting of $M$ subsystems

$$
\dot{x}(t) = f_q(t)(x(t), u(t)), \quad f_q : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}, \quad (1)
$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the continuous state, $u(t) \in \mathbb{R}^{n_u}$ denotes the piecewise continuous control input, and $q \in \{1, \ldots, M\}$ denotes the index of the active subsystem. We also define the switching sequence $\sigma = (q_1, q_2, \ldots, q_m)$ and the switching time sequence $t_{s\sigma} = (t_{s1}, t_{s2}, \ldots, t_{sm-1})$, where $t_{si}$ denotes an instance of the switch of the active subsystem from subsystem $i$ to subsystem $i+1$ over the time horizon $[t_0, t_f]$. Note that we do not consider the state jumps at the switch or the state-dependent condition of the switch in this study, the former of which indicates that the state trajectory of (1) is not smooth but continuous over the horizon. However, in principle, it is possible to extend the proposed method to such cases; the former can be achieved by adding the state jump equation in the proposed Riccati recursion, and the latter can be attained by adding the pure-state constraint representing the switching condition just before the switching instance. The OCP for the switched systems for a given switching sequence $\sigma$ is defined as follows: find $u$ and the switching time sequence $t_{s\sigma}$ that minimize the cost function

$$
J = \varphi(t_f)(x(t_f)) + \sum_{i=1}^{m} \int_{t_{si-1}}^{t_{si}} L_{q_i}(x(\tau), u(\tau))d\tau, \quad (2)
$$

where $t_{mf} = t_f$, $\varphi(\cdot, \cdot) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ denotes the terminal cost, and $L_{q_i}(\cdot, \cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}$ denotes the stage cost, subject to (1).

Next, we assume there is only one switch from $q_1$ to $q_2$ over the horizon $[t_0, t_f]$; that is, we assume $\sigma = (q_1, q_2)$ and $t_{s2} = t_{sig}$ for simplicity. Note that it is trivial to extend the following formulation and the proposed algorithm to the problems with multiple switches on the horizon (Katayama et al. (2020)), which we will further discuss in Section 3.

#### 2.2 Direct Multiple Shooting

For numerical computation, we discretize the OCP based on the direct multiple shooting method (Bock and Plitt (1984); Diehl et al. (2005)) and the forward Euler method. We divide the horizon $[t_0, t_f]$ into $N$ steps, define the time step $\Delta t = (t_f - t_0)/N$, and introduce $i_0$ as an integer that satisfies $i_0\Delta t \leq t_0 < (i_0 + 1)\Delta t$. Here, $i_0$ denotes the time stage at which the switch occurs, that is, the switch occurs between stages $i_0$ and $i_0 + 1$. We introduce the state on the horizon as $x_0, \ldots, x_N$ and control input $u_0, \ldots, u_{N-1}$, where $x_i$ and $u_i$ correspond to $x(t_0 + i\Delta t)$ and $u(t_0 + i\Delta t)$, respectively. Note that we include $x_0$ in the optimization variables so that the proposed method can be combined with the real-time algorithm of MPC (Diehl et al. (2005)). We also introduce the variables just after the switch as $x_i := x(t_{si})$ and $u_i := u(t_{si})$. The cost function (2) is then discretized as

$$
J = \varphi(x_N) + \sum_{i=i_0}^{i_0+N-1} L_{q_i}(x_i, u_i)\Delta t + L_{q_0}(x_{i_0}, u_{i_0})\Delta t_{i_0} + \sum_{i=0}^{i_0-1} L_{q_i}(x_i, u_i)\Delta t, \quad (3)
$$

where $\Delta t_{i_0} := t_{si} - t_0 - i_0\Delta t$. The state equation is discretized as

$$
x_i + f_{q_i}(x_i, u_i)\Delta t - x_{i-1} = 0, \quad i \in \{1, \ldots, i_0 - 1\}, \quad (4)
$$

$$
x_i + f_{q_i}(x_i, u_i)\Delta t = x_0, \quad i \in \{i_0, \ldots, N-1\}, \quad (5)
$$

and

$$
x_i + f_{q_i}(x_i, u_i)\Delta t - x_{i+1} = 0, \quad i \in \{i_0 + 1, \ldots, N\}. \quad (6)
$$

The constraint on the initial state,

$$
x_0 - x(t_0) = 0, \quad (7)
$$

is also imposed because we regard $x_0$ as the optimization variable. The discretized OCP is now defined as follows: find $x_0, \ldots, x_N, x_{i_0}, \ldots, x_N, u_0, \ldots, u_{N-1}$, and $t_{s2}$ that minimize the cost function (3) subject to (4)–(8).

#### 2.3 Optimality Conditions

To derive the optimality conditions, the necessary conditions for optimal control, we define the Hamiltonian

$$
H_q(x, u, \lambda) := L_q(x, u) + \lambda^T f_q(x, u). \quad (9)
$$

The optimality conditions are then derived by the calculus of variations (Bryson and Ho (1975)) as follows:

$$
\nabla_x H_{q_1}(x_i, u_i, \lambda_{i+1})\Delta t + \lambda_{i+1} - \lambda_i = 0, \quad i \in \{1, \ldots, i_0 - 1\}, \quad (10)
$$

$$
\nabla_x H_{q_1}(x_i, u_i, \lambda_{i+1})\Delta t = \lambda_i - \lambda_{i-1} = 0, \quad (11)
$$

$$
\nabla_x H_{q_2}(x_i, u_i, \lambda_{i+1})\Delta t + \lambda_{i+1} - \lambda_i = 0, \quad (12)
$$

$$
\nabla_x H_{q_2}(x_i, u_i, \lambda_{i+1})\Delta t = \lambda_{i+1} - \lambda_i = 0, \quad i \in \{i_0 + 1, \ldots, N - 1\}, \quad (13)
$$

$$
\nabla_u H_{q_1}(x_i, u_i, \lambda_{i+1})\Delta t = 0, \quad i \in \{1, \ldots, i_0 - 1\}, \quad (14)
$$

$$
\nabla_u H_{q_1}(x_i, u_i, \lambda_{i+1})\Delta t = 0, \quad i \in \{i_0 + 1, \ldots, N - 1\}, \quad (15)
$$

$$
\nabla_u H_{q_2}(x_i, u_i, \lambda_{i+1})\Delta t = 0, \quad (16)
$$

$$
\nabla_u H_{q_2}(x_i, u_i, \lambda_{i+1})\Delta t = 0, \quad (17)
$$

(2017)). Through numerical experiments, we show that the proposed method converges within a significantly short computational time (10 – 100 ms), whereas the methods proposed by Xu and Antsaklis (2004) and Farshidian et al. (2017) consumed more than several seconds per upper stage iteration.
\[ \nabla_u H_{q_i}(x_i, u_i, \lambda_{i+1}) \Delta u = 0, \quad i \in \{i_0 + 1, \ldots, N - 1\}, \] 

and
\[ H_{q_i}(x_i, u_i, \lambda_i) - H_{q_i}(x_{i+1}, u_{i+1}, \lambda_{i+1}) = 0, \]

where \( \lambda_0, \ldots, \lambda_N, \lambda_i \) are the Lagrange multipliers with respect to the constraints (4)–(8).

3. RICCATI RECURSION FOR OPTIMAL CONTROL PROBLEMS OF SWITCHED SYSTEMS

3.1 Linearization for Newton's Method

The optimality conditions (4)–(19) are linearized with respect to the constraints (4)–(8).

**Intermediate stages without a switch:** At the intermediate

**Terminal stage:** At the terminal stage \((i = N)\), we have
\[ Q_{x,N}\Delta x_N - \Delta x_N + \bar{l}_x,N = 0, \]

where we define \( Q_{x,N} := \nabla_{xx} \varphi_{q_i}(x_N) \) and define \( \bar{l}_{x,N} \) using the left-hand side of (14).

**Intermediate stage:** At stage \( s \) (switching instant \( t_s \)), we have
\[ Q_{x,s}\Delta x_s + Q_{u,s}\Delta u_s + A^T_s \Delta \lambda_{i+1} - \Delta x_s = 0, \]

and
\[ A_s \Delta x_s + B_s \Delta u_s - \Delta x_{i+1} - f_s \Delta t_s + \bar{x}_s = 0, \]

where we define \( Q_{x,s} := \nabla_{xx} H_{q_s}(x_s, u_s, \lambda_{i+1})(\Delta t - \Delta t_s), Q_{u,s} := \nabla_{uu} H_{q_s}(x_s, u_s, \lambda_{i+1})(\Delta t - \Delta t_s), Q_{u,u,s} := \nabla_{uu} H_{q_s}(x_s, u_s, \lambda_{i+1})(\Delta t - \Delta t_s), A_s := \nabla_u H_{q_s}(x_s, u_s)(\Delta t - \Delta t_s), \) and \( B_s := \nabla_u H_{q_s}(x_s, u_s)(\Delta t - \Delta t_s) \). We also define \( \bar{x}_s, l_{x,s}, \) and \( \bar{l}_{x,s} \) using the left-hand sides of (6), (12), and (17), and \( h_{x,s} := \nabla_x H(x_s, u_s, \lambda_{i+1}), h_{u,s} := \nabla_u H(x_s, u_s, \lambda_{i+1}), \) and \( f_s := f_{q_i}(x_s, u_s) \).

**Terminal stage:** At the terminal stage \((i = N)\),
\[ P_N = Q_{x,N}, \quad z_N = -\bar{l}_N \]

is given as the standard Riccati recursion. In the forward recursion, we have \( \Delta x_N \) and compute \( \Delta \lambda_N = P_N \Delta x_N - z_N \).

**Intermediate stages without a switch:** At the intermediate

We derive the Riccati recursion to solve the linear equation for Newton’s method (20)–(31). As the Riccati recursion for the standard OCP (Frison (2016); Nielsen (2017)), our goal is the series of matrices \( P_i \) and vectors \( \tau_i \) such that
\[ \Delta \lambda_i = P_i \Delta x_i - \tau_i \] holds.

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\[ \Delta \lambda_i = P_i \Delta x_i - \tau_i \] holds.
where we define

\[ H_i := Q_{xu,i} + A_i^T P_{i+1} B_i, \]
\[ G_i := Q_{uu,i} + B_i^T P_{i+1} B_i, \]
\[ K_i := -G_i^{-1} H_i, \quad k_i := -G_i^{-1} (B_i^T P_{i+1} x_i - B_i^T z_{i+1} + u_i), \]

and

\[ P_i := F_i - K_i^T G_i K_i, \quad z_i := A_i^T (z_{i+1} - P_{i+1} x_i) - l_{x,i} - H_i k_i. \]

In the forward recursion, we have \( \Delta x_i \) and compute \( \Delta u_i \) and \( \Delta \lambda_i \) from \( \Delta x_i \) as

\[ \Delta u_i = K_i \Delta x_i + k_i, \]
\[ \Delta \lambda_i = P_i \Delta x_i - z_i, \]

and compute \( \Delta x_{i+1} \) from (23).

Intermediate stages with a switch: Next, we derive the Riccati recursion for intermediate stages with a switch, that is, at stages \( s \) and \( i_s \). Suppose that we have \( P_{i_s+1} \) and \( z_{i_s+1} \) satisfying (40). To obtain the Riccati recursion, we derive the relations \( \Delta \lambda_i, \Delta u_i, \Delta \lambda_i, \Delta u_{i+1}, \Delta \lambda_{i+1}, \) and \( \Delta x_s \) with respect to \( \Delta x_i \). This problem is equivalent to factorizing the linear equation (41). First, by using (26), (40) with \( i = i_s + 1 \), and

\[ \Delta u_s = K_s \Delta x_s + k_s + T_s \Delta t_s, \]

we can reduce (41) into

\[
\begin{bmatrix}
-I & Q_{xu,i} & Q_{uu,i} & A_i^T & h_{x,i} & s
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_i
\Delta x_i
\Delta u_i
\Delta \lambda_{i+1}
\Delta u_{i+1}
\end{bmatrix}
= -\begin{bmatrix}
\bar{I}_{x,i} & s
\end{bmatrix}
\begin{bmatrix}
\xi
\end{bmatrix}
\]

where we define \( F_i, H_s, G_s, K_s, k_s, P_s, \) and \( z_s \) according to (34)–(38)

\[
\psi_{x,i} := h_{x,i} + A_i^T P_{i+1} x_i, \quad \psi_{u,i} := h_{u,i} + B_i^T P_{i+1} f_s, \quad \xi := -\psi_{x,s} + K_s^T \psi_{u,s}, \]
\[ T_s := -G_s^{-1} \psi_{u,s}, \quad T_{s+1} := \psi_{x,s} + K_s^T \psi_{u,s}, \]
\[ \eta_s := h_s - f_s^T (P_{i+1} x_i - z_{i+1}) - \psi_{u,s} k_s. \]

We further factorize (43) using (29),

\[ \Delta \lambda_s = P_s \Delta x_s - z_s - \Gamma_s \Delta t_s, \]

and

\[ \Delta u_i = K_i \Delta x_i + k_i + T_i \Delta t_s, \]

and then obtain

\[
\begin{bmatrix}
-I & F_i - K_i^T G_i K_i & \Gamma_i & \Gamma_i & 1
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_{s,i}
\Delta x_{s,i}
\Delta u_{s,i}
\Delta \lambda_{s,i+1}
\Delta u_{s,i+1}
\end{bmatrix}
= -\begin{bmatrix}
\bar{I}_{x,i} + A_i^T (P_{i+1} x_i - s_i) + H_i k_i
\end{bmatrix}
\begin{bmatrix}
\xi
\end{bmatrix}
\]

where we define \( F_{s,i}, H_{s,i}, G_{s,i}, K_{s,i}, \) and \( k_{s,i} \) according to (34)–(37) and

\[
\psi_{x,i} := h_{x,i} + A_i^T P_{i+1} f_s, \quad \psi_{u,i} := h_{u,i} + B_i^T P_{i+1} f_s, \quad \xi := -\psi_{x,i} + K_i^T \psi_{u,i}, \]
\[ T_{s+1} := -G_i^{-1} \psi_{u,i}, \quad \xi := \psi_{x,i} + K_i^T \psi_{u,i}, \]
\[ \bar{\xi}_s := \xi - 2 \Gamma_i f_i + f_i^T P_{s+1} f_s - \psi_{u,s} G_i^{-1} \psi_{u,i}, \]
\[ \bar{\eta}_s := \eta - \Gamma_i^T x_i + f_i^T (P_{s+1} x_i - s_i) + \psi_{x,i} k_i. \]

Finally, we have recursions

\[ P_i := F_i - K_i^T G_i K_i - \xi \Gamma_i \Gamma_i^T, \]
\[ z_i := A_i^T (z_{i+1} - P_{i+1} x_i) - l_{x,i} - H_i k_i + \xi \Gamma_i \bar{\eta}_s. \]

We can then compute \( \Delta t_s \) by

\[ \Delta t_s = -\xi \Gamma_i \Gamma_i^T \Delta x_i - \xi \Gamma_i \bar{\eta}_s \]

and \( \Delta \lambda_i \) from (40) with \( i = i_s \). In the forward recursion, we have \( \Delta \lambda_i \), and compute \( \Delta t_s, \Delta u_s, \Delta \lambda_{i+1}, \) and \( \Delta x_s \) from (58), (49), (40), and (46). Subsequently, we compute \( \Delta u_s, \Delta \lambda_s, \) and \( \Delta x_{i+1} \) from (42), (48), and (26).

Note that \( \xi_s > 0 \) holds under Assumption 3.1. By the contrapositive of this property, if \( \xi_s \leq 0 \), Assumption 3.1 does not hold, that is, the stationary point is not a local minimum (a maximum or a saddle point). This property is verified as follows: First, recall that the Newton iteration corresponds to solving the QP (32) and the Riccati recursion corresponds to applying dynamic programming (DP) to the QP (Frison (2016); Nielsen (2017)). By Lemma 16.1 of Noceald and Wright (2006) and Assumption 3.1 (positive definiteness of the reduced Hessian), the QP (32) has a unique solution. By solving this QP backward in time, we obtain the following QP subproblem: find \( \Delta u_i, \Delta t_s, \Delta \lambda_{i+1}, \) and \( \Delta x_{i+1} \) that minimizes a cost function whose quadratic term is given by

\[ \frac{1}{2} \begin{bmatrix}
\Delta x_{s,i}
\Delta u_{s,i}
\Delta t_s
\Delta \lambda_{i+1}
\Delta x_{s,i+1}
\Delta u_{s,i+1}
\Delta t_{s+1}
\Delta \lambda_{i+2}
\end{bmatrix}
\begin{bmatrix}
\bar{I}_{x,i} + A_i^T (P_{i+1} x_i - s_i) + H_i k_i
\end{bmatrix}
\begin{bmatrix}
\xi
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\Delta x_s
\Delta u_s
\Delta t_s
\Delta \lambda_i
\end{bmatrix}
\begin{bmatrix}
\bar{I}_{x,i} + A_i^T (P_{i+1} x_i - s_i) + H_i k_i
\end{bmatrix}
\begin{bmatrix}
\xi
\end{bmatrix}
\]

where the last term originates from the cost-to-go function of stage \( i_s + 1 \), subject to (29) and (26). This QP subproblem has a unique solution as well as (32). By eliminating \( \Delta u_i, \Delta t_s, \Delta u_s, \) and \( \Delta x_{i+1} \) from the above QP (59), we obtain another QP: find \( \Delta \lambda_i \) that minimizes a cost function whose quadratic term is given by

\[ \frac{1}{2} \begin{bmatrix}
\Delta x_{s,i}
\Delta u_{s,i}
\Delta t_s
\Delta \lambda_{i+1}
\Delta x_{s,i+1}
\Delta u_{s,i+1}
\Delta t_{s+1}
\Delta \lambda_{i+2}
\end{bmatrix}
\begin{bmatrix}
\bar{I}_{x,i} + A_i^T (P_{i+1} x_i - s_i) + H_i k_i
\end{bmatrix}
\begin{bmatrix}
\xi
\end{bmatrix}
\]

Since this QP must have a solution, \( \xi_s > 0 \) holds.

Initial stage: At the beginning of the forward recursion, we compute \( \Delta x_0 \) from (31).

3.3 Step Size Selection

The full-step Newton’s method can be very aggressive at the beginning of Newton’s iterations. In such cases, the magnitude of the switching time direction, \( \Delta t_s \), is excessively large, and the solution diverges or converges to the saddle points. To avoid such situations, we selected the step size based on the switching time. We assumed that each switching time, \( t_s \), must lie on \( [t_s, \min, t_s, \max] \). Subsequently, we chose the step size \( \alpha \) by applying the fraction-to-boundary rule (Wächter and Biegler (2006)) for the inequality constraints \( t_s - t_s, \min > 0 \) and \( t_s, \max - t_s > 0 \).
Algorithm 1 Computation of Newton direction by the proposed Riccati recursion

Input: Initial state $x(t_0)$ and the current solution $x_0$, $x_1$, $x_N$, $x_s$, $u_0$, ..., $u_{N-1}$, $u_s$, $A_0$, $A_s$, $\lambda_0$, $\lambda_s$, and $t_s$.

Output: Newton directions $\Delta x_0$, $\Delta x_N$, $\Delta x_s$, $\Delta u_0$, $\Delta u_s$, $\Delta u_{N-1}$, $\Delta u_N$, $\Delta \lambda_0$, $\Delta \lambda_s$, and $\Delta t_s$.

1. Form the linear equations, i.e., compute the coefficient matrices and residuals of (20)–(31).
2. Compute $P_N$ and $z_N$ from (33).
3. for $i = N, N-1, \ldots, s + 1$ do
4. Compute $P_i$ and $z_i$ from (23)–(38).
5. end for
6. Compute $P_s$, $z_s$, and $P_{s-1}$ from (23)–(38), (44)–(47), and (51)–(57), respectively.
7. for $i = s + 1, \ldots, 1$ do
8. Compute $P_i$ and $z_i$ from (23)–(38).
9. end for
10. Compute $x_0$ from (31).
11. for $i = 0, 1, \ldots, s - 1$ do
12. Compute $\Delta u_i$, $\Delta \lambda_i$, and $\Delta x_{i+1}$ from (39), (40), and (23), respectively.
13. end for
14. Compute $\Delta t_s$, $\Delta u_s$, $\Delta \lambda_s$, $\Delta x_s$, $\Delta u_N$, $\Delta \lambda_N$, and $\Delta x_{i+1}$ from (58), (49), (40), (29), (42), (48), and (26), respectively.
15. for $i = s + 1, \ldots, N$ do
16. Compute $\Delta u_i$, $\Delta \lambda_i$, and $\Delta x_{i+1}$ from (39), (40), and (23).
17. end for
18. Compute $\Delta \lambda_N$ from (40).

3.4 Algorithm and Convergence

We summarize the single Newton iteration using the proposed Riccati recursion algorithm in Algorithm 1. As shown in Algorithm 1, the proposed Riccati recursion computes the Newton direction for a given solution. In the first step, we form the linear equations of Newton’s method, that is, compute the coefficient matrices and residuals of (20)–(31) (line 1). Second, we perform the backward Riccati recursion and compute $P_i$ and $z_i$ for $i \in \{1, \ldots, N-s\}$ (lines 3–9). Finally, we perform the forward Riccati recursion and compute the Newton directions for all the variables (lines 10–18).

We also summarize Newton’s method for the OCP of the switched system by the proposed Riccati recursion (line 3), we determine the $\ell_2$-norm of the residual of the optimality conditions (4)–(19) become smaller than a prespecified threshold ($\epsilon$ in Algorithm 2). We refer to the $\ell_2$-norm of the residual of the optimality conditions as “Opt. error” in the following sections. After each iteration of the proposed Riccati recursion (line 3), we determine the

Algorithm 2 Newton’s method for the OCP of the switched system by the proposed Riccati recursion

Input: Initial guess of the solution $x_0$, $x_N$, $x_s$, $u_0$, $u_{N-1}$, $u_s$, $A_0$, $A_s$, $\lambda_0$, $\lambda_s$, and $t_s$, and the termination criteria $\epsilon \geq 0$.

Output: Optimal solution $x_0$, $x_N$, $x_s$, $u_0$, $u_{N-1}$, $u_s$, $A_0$, $A_s$, $\lambda_0$, $\lambda_s$, and $t_s$.

1: Set $i_s$ such that it satisfies $i_s \Delta \tau \leq t_s - t_0 < (i_s + 1) \Delta \tau$.
2: while $\text{Opt. error} > \epsilon$ do (at $k$-th iteration)
3: Compute the Newton directions $\Delta x_N^k$, $\Delta x_s^k$, $\Delta u_N^k$, $\Delta u_s^k$, $\Delta \lambda_N^k$, $\Delta \lambda_s^k$, and $\Delta t_s^k$ using Algorithm 1 based on the current solution.
4: Choose step size $\alpha \ (0 < \alpha \leq 1)$.
5: Update the solution by $x_{N}^{k+1} \leftarrow x_{N}^{k+1} + \alpha \Delta x_N^k$, $u_{0}^{k+1} \leftarrow u_{0}^{k+1} + \alpha \Delta u_0^k$, $\lambda_{0}^{k+1} \leftarrow \lambda_{0}^{k+1} + \alpha \Delta \lambda_0^k$, and $t_s^{k+1} \leftarrow t_s^{k+1} + \alpha \Delta t_s^k$.
6: Update $i_s$ such that it satisfies $i_s \Delta \tau \leq t_s^{k+1} - t_0 < (i_s + 1) \Delta \tau$.
7: end while

Note that the proposed method can be directly applied to the OCP with multiple switches on the horizon. When there are multiple switches on the horizon, we compute the coefficient matrices and residuals in (24)–(30) for each switch in line 1 of Algorithm 1, apply line 6 of Algorithm 1 for each switch in the backward Riccati recursion, and apply line 14 of Algorithm 1 for each switch in the forward Riccati recursion.

The proposed method is in substance a Newton’s method for the optimization problem with equality constraints as the standard direct multiple shooting method, and we can similarly discuss the convergence. The difference is that the switching stage, $i_s$, can change depending on $t_s$ at each iteration, which implies that the problem structure, that is, the cost function (2) and equality constraints (4)–(7), can change between the iterations. If $i_s$ does not change, that is, the optimal switching instant $t_s$ satisfies $i_s \Delta \tau < t_s - t_0 < (i_s + 1) \Delta \tau$, and the sequence of the switching instant at each iteration $\{t_s^k\}$ satisfies $i_s \Delta \tau < t_s^k - t_0 < (i_s + 1) \Delta \tau$ for the same $i_s$ with $\alpha = 1$, then the proposed method achieves quadratic convergence under Assumption 3.1, for example, by Theorem 18.4 of Nocedal and Wright (2006).

4. NUMERICAL EXPERIMENTS

We conducted two numerical experiments to show the effectiveness of the proposed method. The proposed al-
The second example is a switched system consisting of three nonlinear subsystems that were treated in Xu and Antsaklis (2004) and Farshidian et al. (2017). The dynamics of the subsystems are given by

\[
\begin{align*}
    f_1(x, u) &= \begin{bmatrix} x_1 + u_1 \sin(x_1) \\ -x_2 - u_1 \cos(x_2) \end{bmatrix}, \\
    f_2(x, u) &= \begin{bmatrix} x_2 + u_1 \sin(x_2) \\ -x_1 - u_1 \cos(x_1) \end{bmatrix}, \\
    f_3(x, u) &= \begin{bmatrix} -x_1 - u_1 \sin(x_1) \\ x_2 + u_1 \cos(x_2) \end{bmatrix},
\end{align*}
\]

and the stage cost is given by

\[
L_q(x, u) = \frac{1}{2} ||x - x_{ref,2}||^2 + \frac{1}{2} ||u||^2, \quad q \in \{1, 2\},
\]

where \(t_0 = 0, t_f = 3\), and \(x_{ref} = [1, -1]^T\). We discretize the continuous-time OCP into \(N = 220\) steps. The switching sequence is given by \(\sigma = (1, 2, 3)\). The initial state is given by \(x(t_0) = [2, 3]^T\), and the initial guess of the solution is given by \(x_i = x(t_0)\), \(t_1, t_2, t_3 = [0.5, 1.0]\). As in the previous example, we use the fraction-to-boundary rule in subsection 3.3 in the step-size selection such that \(t_1, t_2 \in [t_0, t_f]\).

Figure 2 shows the \(\log_{10}\) scaled Opt. error and the switching instants with respect to the number of Newton iterations. As shown in Fig. 2, the proposed method converges after 70 Newton iterations. The proposed method takes only 1.153 ms per Newton iteration and 80 ms for convergence, which is significantly faster than the existing methods (Xu and Antsaklis (2004) and Farshidian et al. (2017)). Farshidian et al. (2017) reported that the method proposed by Xu and Antsaklis (2004) took 26 s per upper stage gradient iteration and Farshidian et al. (2017) took 26 s per upper stage gradient iteration on an Intel Core-i7 CPU @2.7 GHz. As in the previous example, the solution provided by the proposed method is almost identical to the continuous-time counterpart reported in Xu and Antsaklis (2004); for example, our results show that \(t_1 = 0.2335\) [s] and \(t_2 = 1.0179\) [s], and the solution of the continuous time was \(t_1 = 0.2262\) [s] and \(t_2 = 1.0176\) [s].

A future improvement of the proposed method is the Hessian convexification and globalization. When the initial guess of the switching time is significantly different from the optimal one, the proposed method may converge or converge to saddle points because the Hessian may contain a negative curvature, that is, Assumption 3.1 may not hold. For example, the proposed method with only the fraction-to-boundary rule for step-size selection converges to a saddle point when we use the initial guess \(t_1 = 1.0\) [s] and \(t_2 = 2.0\) [s], for which Xu and Antsaklis (2004) and Farshidian et al. (2017) succeed in convergence to the optimal solution with line searches for the upper-stage problem (optimization of the switching instants for a fixed control input). Furthermore, it is not straightforward to use the existing Hessian convexification method, such as the Gauss-Newton Hessian approximation, because the
Fig. 2. log_{10} scaled Opt. error and the switching instants \( t_1 \) and \( t_2 \) with respect to the iterations of Example 2.

structure of the Hessian (59) is different from that of the standard OCPs.

5. CONCLUSION

We proposed an efficient algorithm for the OCP of nonlinear switched systems that optimizes the control input and switching instants simultaneously for a given switching sequence. We formulated the OCP based on the direct multiple shooting method with regard to the switching instance as the optimization variable, as well as the state and control input. We derived a linear equation for Newton’s method and a Riccati recursion algorithm to solve the linear equation. The computational time of the proposed method scales linearly with respect to the length of the horizon as the standard Riccati recursion (Frison (2016); Nielsen (2017)). We conducted numerical experiments and demonstrated that the proposed method converges with a significantly shorter computational time compared with the previous two-stage methods.

As part of our future work, we plan to extend the proposed method to systems with state-dependent switching conditions and state jumps to model mechanical systems that have contact with the environment (Grizzle et al. (2001); Katayama et al. (2020); Li and Wensing (2020)). Subsequently, we need to extend the proposed algorithm to problems with the pure-state constraints that represent the switching conditions, for example, by using the approach proposed by Sideris and Rodriguez (2011). Future work also includes the Hessian convexification and globalization of the proposed method to avoid divergence or convergence to saddle points, even when the initial guess of the solution (particularly the initial guess of the switching instant) is significantly different from the optimal solution.

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