Deep Inelastic Scattering in the Color Glass Formalism

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I discuss some phenomenological consequences of the Color Glass formalism for deep inelastic scattering.

1 Dipole scattering at small x

The Color Glass Condensate (CGC) is the effective theory for small x gluons derived from QCD [1, 2]. Since the theoretical framework of CGC has been established, a considerable progress has been made in understanding its properties. In this talk, I will discuss applications of the CGC formalism to the calculation of deep inelastic scattering (DIS). The relevant quantities to be discussed are the \( \sigma^{\gamma^*p}_{\text{total}} \) and the \( F_2 \) structure function. At small x, one can compute them as follows:

\[
\sigma^{\gamma^*p}_{\text{total}}(x, Q^2) = \sum_{T,L} \int_0^1 dz \int d^2 r_\perp |\Psi_{T,L}(z, r_\perp; Q^2)|^2 \sigma_{\text{dipole}}(x, r_\perp),
\]

\[
F_2(x, Q^2) = \left( \frac{Q^2}{4\pi^2\alpha_{\text{EM}}} \right) \sigma^{\gamma^*p}_{\text{total}}(x, Q^2),
\]

where \( \Psi_{T/L}(z, r_\perp; Q^2) \) are the LC wavefunctions of transverse/longitudinal virtual photons splitting into a \( q\bar{q} \) dipole \( (r_\perp: \text{transverse size}, z: \text{a fraction of the photon's longitudinal momentum carried by the quark}) \), and \( \sigma_{\text{dipole}}(x, r_\perp) \) is the dipole-proton cross section given by \( r_\perp = x_\perp - y_\perp, b_\perp = (x_\perp + y_\perp)/2 \)

\[
\sigma_{\text{dipole}}(x, r_\perp) = 2 \int d^2 b_\perp (1 - S_x(x_\perp, y_\perp)), \quad S_x(x_\perp, y_\perp) \equiv \frac{1}{N_c} \langle \text{tr}(V_{x_\perp}^\dagger V_{y_\perp}) \rangle_x.
\]

The average of the product of Wilson lines \( V_{x_\perp}^\dagger V_{y_\perp} \) is computable in CGC (see [1] for details). Therefore, once we obtain \( \sigma_{\text{dipole}} \) based on CGC, we can immediately calculate \( \sigma^{\gamma^*p}_{\text{total}} \) and \( F_2 \) from the formulae (1) and (2). Below we explain three important features of \( \sigma_{\text{dipole}} \) as computed within CGC.

(I) Geometric scaling persists beyond \( Q_s^2 \) up to \( Q^2 \sim Q_s^4/\Lambda_{\text{QCD}}^2 \) [3].

Geometric scaling is a new scaling phenomenon at small x [4] meaning that \( \sigma^{\gamma^*p}_{\text{total}} \) depends upon \( Q^2 \) and x only via their specific combination \( \xi \equiv Q^2 R_0^2(x) \),

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2 Present address
with $R_0^2(x) \propto x^\lambda$. This can be naturally explained by CGC at low $Q^2$, below the saturation scale $Q_s^2$ (\sim a few GeV$^2$). For example, the energy dependence of the saturation scale is found to be $Q_s^2(x) \propto x^{-\beta}$, which is consistent with that of $1/R_0^2(x)$. The geometric scaling up to much higher values of $Q^2$ (\sim 100 GeV$^2$) can also be understood within the framework of CGC through the solution to the BFKL equation subjected to a saturation boundary condition at $Q^2 \sim Q_s^2(x)$. Indeed, the solution exhibits approximate scaling

$$1 - S_x(r_\perp) \simeq (r_\perp^2 Q_s^2(x))^\gamma,$$

(4)

within a window $1 \lesssim \ln(Q^2/Q^2_s) \ll \ln(Q_s^2/\Lambda_{QCD}^2)$, which is roughly consistent with phenomenology. The power $\gamma$ (more precisely, the difference $1 - \gamma$) is called anomalous dimension since it is different from the naive DGLAP value $\gamma = 1$ (i.e., $1 - S_x(r) \propto r_\perp^2$). Namely, the LO BFKL equation yields $\gamma \simeq 0.64$ \cite{3}, which does not change even with the higher order effects included \cite{5}.

(II) $\sigma_{\text{dipole}}$ can be analytically computed within CGC \cite{6}.

The evolution in CGC is described by a renormalization group equation for the weight function which governs the correlation of color sources representing large $x$ partons. One can solve it approximately in two different kinematical regimes \cite{7}, namely, in the weak field regime (low density, no saturation) and in the strong field regime (deeply at saturation). The first regime is relevant for the scattering of a small dipole ($r_\perp \ll 1/Q_s(x)$), while the second one applies for a relatively large dipole ($r_\perp \gg 1/Q_s(x)$). In both regimes, the weight function can be represented as a Gaussian. In between them, one can use a simple interpolation to get a Gaussian effective theory which may be used globally \cite{6}. In particular, one can use this theory to deduce an explicit analytical form of the dipole $S$-matrix valid in the scaling region, $Q^2 = 1/r_\perp^2 \lesssim Q_s^2(x)/\Lambda_{QCD}^2$:

$$S_x(r_\perp) = \exp \left\{ -\kappa \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1 - e^{ik_\perp \cdot r_\perp}}{k_\perp^2} \ln \left[ 1 + \left( \frac{Q_s^2(x)}{k_\perp^2} \right)^\gamma \right] \right\},$$

(5)

where $Q_s^2(x) \propto x^{-c_\alpha^2}$, and $\kappa$ is a known numerical constant \cite{6}. This formula reproduces (4) in the extended scaling regime and has the correct behavior in the saturation regime $S_x(r_\perp) \propto \exp\left\{ -\frac{1}{2} \ln^2 r_\perp^2 Q_s^2(x) \right\}$ \cite{7}.

(III) $\sigma_{\text{dipole}}(x, r_\perp)$ saturates Froissart bound at high energy \cite{8}.

Consider the high energy scattering of a small $q\bar{q}$ dipole off a proton. With increasing energy, a black disk (BD) in which the dipole is strongly absorbed
\( (S_x(r_\perp, b_\perp) \simeq 0) \) appears near the center of the proton, where the gluon density is larger, and then expands over the target. At high energy, \( \sigma_{\text{dipole}} \simeq 2\pi R^2(x, 1/r_\perp^2) \), so we need to compute the growth of the BD radius \( R(x, 1/r_\perp^2) \) with decreasing \( x \). This was done in Ref. [8] under the following assumptions: (i) At impact parameters \( b_\perp \) outside of (but not too far away from) the BD, the gluon density is relatively low (but still perturbative), and grows rapidly with \( 1/x \) due to BFKL evolution. The dipole scattering is then dominated by relatively nearby color sources, within a disk of size \( 1/Q_s(b_\perp) \) around \( b_\perp \). (Sources which lie further away contribute less because of color screening at saturation [6, 8].) (ii) With increasing energy at fixed \( b_\perp \), the local \( S \)-matrix decreases, and eventually becomes negligible \( (S_x(r_\perp, b_\perp) \simeq 0) \), because of saturation. This corresponds to the unitarity limit \( T \simeq 1 \) for the scattering amplitude \( T_x(r_\perp, b_\perp) \equiv 1 - S \). When this happens, the point \( b_\perp \) enters the BD. (iii) For sufficiently large energies, the BD enters the tail of the hadron wavefunction, where the gluon density decreases exponentially with \( b_\perp \). Then the expansion of the BD is controlled by the competition between the BFKL growth of the gluon density with \( 1/x \) and its exponential decrease with \( b_\perp \): \( T_x(r_\perp, b_\perp) \propto e^{\bar{\omega} x \ln 1/x} e^{-\mu b} \), with \( \mu \) being the lightest mass gap in the correct channel for dipole-hadron scattering. Then the BD radius is found to be proportional to \( \ln(1/x) \) which yields the result saturating Froissart bound:

\[
\sigma_{\text{dipole}}(x, r_\perp) \simeq \frac{\pi}{2} \left( \frac{\omega_s}{m_\pi} \right)^2 \ln^2 \frac{1}{x}.
\]

Here we took \( \mu = 2m_\pi \) with \( m_\pi \) the pion mass.

Whereas points (i) and (ii) above refer to the perturbative (yet non-linear) physics which is proper to the CGC formalism (and, in particular, also to the Balitsky-Kovchegov (BK) equation [9]), on the other hand, point (iii) should be seen as a consequence of confinement, which is not encoded in the present formalism. A purely perturbative evolution based on the BK equation leads to the violation of Froissart bound [10, 11] because of long-range interactions which, in real world, are removed by confinement [8]. In fact, in Ref. [8], the result (6) has been obtained by following the perturbative evolution with non-perturbative initial conditions over a limited energy interval, and by arguing that, in the presence of confinement, the same result would hold up to arbitrary high energies.
2 Phenomenological consequences [12]

Now we use all the features (I) (II) and (III) to compute the physical quantities (1) and (2). First of all, let us give a rough estimate of \( \sigma_{\gamma^* p, \text{total}} \) with some approximations. We assume transverse homogeneity of the proton, ignore quark mass and the longitudinal cross section (which is much smaller than the transverse one), and use scaling form (4) for \( S_x(r_{\perp}) \) in the extended scaling window, while simply \( S_x(r_{\perp}) \approx 0 \) in the saturation regime. Then we obtain the following leading behaviors (numerical coefficients are suppressed)

\[
\sigma_{\gamma^* p, \text{total}}(x, Q^2) \sim \begin{cases} 
\ln\left(\frac{Q^2}{s(x)/Q^2}\right) & Q^2 < Q^2_s \\
\left(\frac{Q^2_s}{Q^2}\right)^\gamma & Q^2_s < Q^2 < Q^4_s/\Lambda_{\text{QCD}}^2 \end{cases} .
\]  

(7)

Note first that the scaling in \( S_x(r_{\perp}) \) (w.r.t. \( r_{\perp}^2 Q^2_s \)) has been converted into the experimentally observed scaling w.r.t. \( Q^2/Q^2_s(x) \). Next, interestingly, this result is consistent with the experimental data while this is admittedly a very rough estimate. In particular, the slope of the data plotted against \( Q^2/Q^2_s(x) \) (in log-log scale) is in rather good agreement with the BFKL value \( \gamma \approx 0.64 \) in the extended scaling window.

In (II), we have shown a formula (5) for the \( S \)-matrix. This immediately gives the dipole cross section \( \sigma_{\text{dipole}}(x, r_{\perp}) = 2\pi R^2[1 - S_x(r_{\perp})] \) with \( R \) being the hadron radius. In fact there are other models for the dipole cross section which incorporate the effects of saturation (for references, see [1]). The first one is by Golec-Biernat and Wüsthoff \( \sigma_{\text{dipole}}^{\text{GBW}}(x, r_{\perp}) = \sigma_0[1 - \exp(-r_{\perp}^2 Q^2_s(x)/4)] \) with \( Q^2_s = \text{GeV}^2(x/x_0)^\lambda \), and has only three parameters \((\sigma_0, x_0, \lambda)\). This gives the naive value for the exponent \( \gamma = 1 \). The second one is by Bartels, Golec-Biernat and Kowalski \( \sigma_{\text{dipole}}^{\text{BGK}}(x, r_{\perp}) = \sigma_0[1 - \exp\{-\pi^4\alpha_s/\sigma_0\} r_{\perp}^2 xG(x, \mu^2)\} \) having five parameters \((\sigma_0, \text{two for } \mu^2 \text{ and two for the initial condition of } xG)\). This includes the DGLAP evolution and thus improves significantly over the GBW model at high \( Q^2 \). These simple parametrizations were quite successful in fitting HERA data. As compared with such previous parametrizations, our formula reproduces the correct QCD behaviors in both the saturation and extended scaling regimes. Nevertheless, our purpose with eq. (5) is not to provide another fit — this equation is rather crude, and is expected to apply only in a limited kinematical regime which is only marginally explored at HERA —, but rather to use this estimate for \( \sigma_{\text{dipole}} \) to draw some qualitative conclusions, rooted in QCD, about the expected behavior of the DIS structure functions at sufficiently small \( x \). One of these conclusions is summarized in eq. (7). Other results of this type will be presented somewhere else [12]. Here, we shall briefly comment...
on one of these conclusions, related to the high energy limit of $F_2$. Note that the result (6) comes from the leading order behavior of the BD at small $x$ for fixed dipole size $r_\perp$. Since $F_2$ is given by the integration over various size of the dipoles, we need to consider also the dipole size dependence of the BD radius $R(x, 1/r_\perp^2)$, which is given in [8]. Then, using the dipole cross section $2\pi R^2(x, 1/r_\perp^2)$, one can compute $F_2$ in the high energy limit. The results is $F_2(x, Q^2) \propto \ln^3(1/x)$, and thus $F_2$ and $\sigma_{\gamma^*p}^{\text{total}}$ appear to violate Froissart bound. This is not a problem because the virtual photon is not a (properly normalized) hadronic state, but rather a superposition of such states (one for each dipole size). The additional factor of $\ln(1/x)$ comes from integrating over all dipole states with the photon wavefunction [12].

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