STRAIN GRADIENT THEORY OF POROUS SOLIDS WITH INITIAL STRESSES AND INITIAL HEAT FLUX

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ABSTRACT. In this paper we present a strain gradient theory of thermoelastic porous solids with initial stresses and initial heat flux. First, we establish the equations governing the infinitesimal deformations superposed on large deformations. Then, we derive a linear theory of prestressed porous bodies with initial heat flux. The theory is capable to describe the deformation of chiral materials. A reciprocity relation and a uniqueness result with no definiteness assumption on the elastic constitutive coefficients are presented.

1. Introduction. In recent years considerable attention has been given to the mechanical behaviour of porous elastic solids (see, e.g., [9, 16, 33], and references therein). In [10, 32] Nunziato and Cowin established a theory of elastic materials with voids for the treatment of porous solids. This theory introduces an additional degree of kinematical freedom. There has been very much written in the last years on the theories of elastic solids with inner structure in which the deformation is described not only by the usual vector displacement field, but by other vector or tensor fields as well. The origin of the theories of continua with microstructure goes back to the papers of Mindlin [27], Eringen and Suhubi [12] and Green and Rivlin [18]. Mindlin [27] formulated a theory of an elastic solid which has some properties of a crystal lattice as a result of the inclusion of the idea of the unit cell. Mindlin begins with the general concept of an elastic continuum each material point of which is itself a deformable medium. In the theory developed by Mindlin each material point is constrained to deform homogeneously. In this theory the degrees of freedom for each material point are twelve: three translations, $u_i$, and nine microdeformations, $\chi_{ij}$. A special class of bodies with microstructure [13] is characterized by a microdeformation tensor of the form $\psi\delta_{ij}$, where $\psi$ is the microstretch function (or microdilatation function) and $\delta_{ij}$ is the Kronecker delta. In this case the material points undergo a uniform microdilatation (a breathing motion). The linear equations which describe the behavior of an elastic body with this kind of microstructure coincide with the equations of the linear theory of elastic materials with voids established by Cowin and Nunziato [10] (cf. Eringen [13]). In what follows we shall refer to this model as a porous elastic continuum.

The linear theory of elastic materials with voids is the simplest theory of elastic bodies that takes into account the microstructure of the material. The theory of

2010 Mathematics Subject Classification. 74B10, 74B15, 74E20, 74F05, 74H25.

Key words and phrases. Porous materials, prestressed bodies, thermoelastic solids, chiral materials, bodies with initial heat flux, uniqueness.
elastic solids with microstructure is characterized by constitutive functions which depend on the deformation gradient, microdeformation and microdeformation gradient. Rymarz [35], Brulin and Hjalmars [5], and Hjalmars [19] have shown that in order to obtain a consistent grade level for the microstructure theory it is necessary to add the second-order displacement gradient to the above independent constitutive variables. The equations of motion, constitutive equations, and the boundary conditions of the theory of nonsimple elastic bodies of grade 2 were first established by Toupin ([37, 38]). The linear theory has been developed by Mindlin [27] and Mindlin and Eshel [28]. In the first part of this paper we use the results established by Toupin [38] and Nunziato and Cowin [32] to derive a non-linear theory of porous thermoelastic solids of grade 2. This work is motivated by the recent interest in the study of non-simple solids (see, e.g., [1, 2, 3, 4, 15, 36]) and the use of the continuum with microstructure as model for engineering materials (see, e.g., [11, 30]).

We establish a strain gradient theory of thermoelastic porous solids with initial stresses and initial heat flux. The theory of initially stressed elastic bodies is of considerable interest both from the mathematical and the technical point of view. It provides a natural extension of the classical theory of elasticity, as the initial configuration needs no longer be unstressed. The theory also establishes a basis for discussing stability of finite deformations (see Knops and Wilkes [23]). The theory of initially stressed bodies has given impetus to theoretical research into the equations of elastic bodies for which little or no information is known concerning the elasticities (see Knops and Payne [24], Sections 2.1, 4.4, 8.3 and Knops [25]). We note the recent interest in the study of prestressed bodies with inner structure (see, e.g., [39] and references therein). By using the non-linear theory of classical thermoelasticity , Ieşan [20] established a linear theory of thermoelasticity with initial stresses and initial heat flux. Some theorems in this theory have been established by Chirita [8] and Navarro and Quintanilla [29]. Martinez and Quintanilla [26] have established the theory of small thermoelastic deformations superposed on a large deformation at non-uniform temperature in the context of the nonsimple materials of grade 2. An existence result and continuous dependence of solutions upon the initial state and supply terms have been also presented.

This paper is structured as follows. In Section 2 we establish a non-linear theory of thermoelastic solids in which the independent constitutive variables are the deformation gradient, the second-order displacement gradient, volume fraction field, the gradient of volume fraction field, temperature and temperature gradient. Section 3 presents the equations governing the infinitesimal thermoelastic deformations superposed on large deformations at non-uniform temperature. In Section 4 we establish a linear theory of prestressed porous bodies with initial heat flux. Section 5 is devoted to reciprocity and uniqueness results in the strain gradient theory of prestressed thermoelastic porous solids.

2. Basic equations. In this section we present a non-linear theory of thermoelastic porous solids of grade 2. We consider a body that at time \( t_0 \) occupies the region \( B \) of the Euclidean three-dimensional space and is bounded by the smooth surface \( \partial B \). Let \( (t_0, t_1) \) be a given interval of time The motion of the body is referred to a fixed system of rectangular Cartesian axes and to the reference configuration. The coordinates of a typical point in the reference configuration are \( X_K \). The coordinates of this particle at time \( t \) are denoted by \( x_i \), where

\[
x_i = x_i(X_1, X_2, X_3, t), \quad (X_K, t) \in B \times (t_0, t_1).
\]
If the deformation is possible in a real material, then we have

$$J = \det \left( \frac{\partial x_i}{\partial X_K} \right) > 0.$$  \hspace{1cm} (2)

We assume the continuous differentiability of $x_i$ with respect to each of the variables $X_K$ and $t$ as many times as required. The concept of a distributed body asserts that the mass density $\rho$ at time $t$ has the decomposition

$$\rho = \nu \gamma,$$  \hspace{1cm} (3)

where $\gamma$ is the density of the matrix material and $\nu$ is the volume fraction field. Throughout this paper, a superposed dot denotes the material time derivative. Partial derivatives are denoted by commas preceding a subscript: differentiation with respect to $x_i$ is indicated by small Latin indices and differentiation with respect to $X_K$ by Latin capitals, thus $\partial f/\partial X_K = f_{,K}$ and $\partial f/\partial x_i = f_{,i}$. The usual Cartesian summation convention is used. Let $B$ be at rest relative to the considered system of reference. We consider an arbitrary region $P$ of the continuum at time $t$ bounded by the surface $\partial P$, and we suppose that $P$ is the corresponding region at time $t_0$, bounded by the surface $\partial P$. Following [18, 32, 38], the conservation of energy, for every regular region $P$ of $B$ and every time $t$, can be expressed as

$$\int_P \rho_0 (\dot{v}_i \dot{v}_i + \kappa \dot{\nu} \ddot{\nu} + \dot{\epsilon}) dv = \int_P \rho_0 (f_i v_i + l \dot{\nu} + S) dv +$$

$$+ \int_{\partial P} (T_i v_i + M_{ji} v_{i,j} + H \dot{\nu} + Q) da,$$  \hspace{1cm} (4)

where $dv$ and $da$ are elements of volume and area in the reference configuration, $\rho_0$ is the mass density in the reference configuration, $\epsilon$ is the internal energy per unit mass, $f_i$ is the body force per unit mass, $l$ is the extrinsic equilibrated body force per unit mass, $S$ is the heat supply per unit mass, $T_i$ is the stress vector associated with the surface $\partial P$ and measured per unit area of $\partial P$, $M_{ji}$ is the hypertraction associated with the surface surface $\partial P$ and measured per unit area of $\partial P$, $H$ is the equilibrated stress associated with the surface $\partial P$ and measured per unit undeformed area, $Q$ is the heat flux across the surface $\partial P$ and measured per unit area of $\partial P$, $\kappa$ is the equilibrated coefficient of inertia, and $v_i = \dot{x}_i$. We use the method given by Green and Rivlin [18] to obtain the equations of motion from the balance of energy and the invariance requirements under superposed rigid body motions. First, we consider motions of the body which differ from those given by (1) only by superposed uniform rigid body translational velocities, the continuum occupying the same position at time $t$. We assume that $\rho_0, \dot{\epsilon}, f_i, l, S, T_i, M_{ji}, H, Q$ and $\kappa$ are unaltered by such rigid body velocities. Then, from (4) we obtain the balance of momentum. From this law we get

$$T_i = T_{Ki} N_K,$$  \hspace{1cm} (5)

and the local form of the balance of momentum,

$$T_{Ki,K} + \rho_0 f_i = \rho_0 \ddot{x}_i.$$  \hspace{1cm} (6)

Here, $T_{Ki}$ is the first Piola-Kirchhoff stress tensor and $N_K$ are the components of the unit outward normal vector to the surface $\partial P$. If we use (5) and (6) then the relation (4) reduces to

$$\int_P \rho_0 (\dot{\epsilon} + \kappa \ddot{\nu}) dv =$$  \hspace{1cm} (7)
= \int_{\mathcal{P}} (T_{ki}v_{i,K} + \rho_0 \dot{v} + \rho_0 S)dv + \int_{\partial\mathcal{P}} (M_{ji}v_{i,j} + H\dot{v} + Q)da.

If we apply this equation to a region which in the reference state was a tetrahedron bounded by coordinate planes through the point \( X \) and by a plane whose unit normal is \( N_K \), we obtain

\[ (M_{ji} - M_{Kji}N_K)v_{i,j} + (H - H_KN_K)\dot{v} + Q - Q_KN_K = 0, \tag{8} \]

where \( M_{Kji} \) is the hyperstress tensor, \( H_K \) is the equilibrated stress and \( Q_K \) is the heat flux, associated with surfaces in the deformed body which were originally coordinate planes perpendicular to the \( X_K \)-axes through the point \( X \), measured per unit area of these planes. If we use (8) in (7) and apply the resulting equation to an arbitrary region, then we find the local form of the conservation of energy

\[ \rho_0 \dot{\varepsilon} = (T_{ki} + M_{Lji,L}X_{K,j})v_{i,K} + M_{Kji}X_{L,j}v_{i,LK} + H_K\dot{v}_K - g\dot{v} + Q_{K,K} + \rho_0 S, \tag{9} \]

where \( g \) is defined by

\[ H_{K,K} + g + \rho_0 l = \rho_0 \dot{\varepsilon}. \tag{10} \]

The function \( g \) is called the intrinsic equilibrated body force (cf. [4], [5]). This function is specified by a constitutive equation. We introduce the notations

\[ M_{KLi} = M_{Kji}X_{L,j}, \quad S_{Ki} = T_{Ki} + M_{LKi,L}. \tag{11} \]

The equation (9) becomes

\[ \rho_0 \dot{\varepsilon} = S_{Ki}v_{i,K} + M_{KLi}v_{i,KL} + H_K\dot{v}_K - g\dot{v} + Q_{K,K} + \rho_0 S. \tag{12} \]

Let us denote

\[ p_{ij} = S_{Ki}x_{j,K}. \tag{13} \]

We consider a motion of the body which differs from the given motion only by a superposed uniform rigid body angular velocity, and we assume that \( \rho_0, \dot{\varepsilon}, T_{Ki}, M_{KLi}, H_K, g, Q_K \) and \( S \) are not affected by such motion. Then, the equation (12) implies that

\[ p_{ij} + M_{KLi}x_{j,KL} = p_{ji} + M_{KLi}x_{i,KL}. \tag{14} \]

We postulate the entropy production inequality in the form

\[ \int_{\mathcal{P}} \rho_0 \dot{\eta}dv - \int_{\mathcal{P}} \frac{1}{T} \rho_0 Sdv - \int_{\partial\mathcal{P}} \frac{1}{T} Qda \geq 0, \tag{15} \]

for every part \( P \) of \( B \) and every time. Here \( \eta \) is the entropy per unit mass, and \( T \) is the absolute temperature, which is assumed to be positive. If we introduce the Helmholtz free-energy \( \psi \) by

\[ \psi = e - T\eta, \tag{16} \]

then the equation (12) can be presented in the form

\[ \rho_0 (\dot{\psi} + \dot{T}\eta + T\dot{\eta}) = (T_{Ki} + M_{KLi,L})\dot{x}_{i,K} + M_{KLi}\dot{x}_{i,KL} + H_K\dot{v}_K - g\dot{v} + Q_{K,K} + \rho_0 S. \tag{17} \]

A thermoelastic material is defined as one for which the following constitutive equations hold

\[ \psi = \psi(\zeta), \quad T_{Ki} = T_{Ki}(\zeta), \quad M_{KLi} = M_{KLi}(\zeta), \]

\[ H_K = H_K(\zeta), \quad g = g(\zeta), \quad \eta = \eta(\zeta), \quad Q_K = Q_K(\zeta), \]

\[ M_{ij} = M_{ij}(\zeta, N_L), \quad H = H(\zeta, N_L), \quad Q = Q(\zeta, N_L), \tag{18} \]
where \( \zeta = (x_{i,K}, x_{i,KL}, \nu, \nu_K, T, T_L, X_K) \). The response functions are assumed to be sufficiently smooth. For a given deformation, \( v_{ij} \) and \( \dot{\nu} \) in (8) may be chosen arbitrarily so that, on the basis of the constitutive equations (18) we find

\[
M_{ij} = M_{Kij}N_K, \quad H = H_KN_K, \quad Q = Q_KN_K.
\]  

(19)

If we use (19) in (15) then we obtain the following local form of the second law of thermodynamics

\[
\rho_0T\dot{\eta} - \rho_0S - Q_{K,K} + \frac{1}{T}Q_KT_K \geq 0.
\]  

(20)

From (17), (18) and (20) we get

\[
\left( T_{Ki} + M_{KLi,L} - \frac{\partial \sigma}{\partial x_{i,K}} \right) \dot{x}_{i,K} + \left( M_{KLi} - \frac{\partial \sigma}{\partial x_{i,KL}} \right) \dot{x}_{i,KL} + \left( H_K - \frac{\partial \sigma}{\partial \nu_K} \right) \dot{\nu}_K - \left( g + \frac{\partial \sigma}{\partial \nu} \right) \dot{\nu} - \left( \rho_0\eta + \frac{\partial \sigma}{\partial T} \right) \dot{T} -
\]

\[
- \frac{\partial \sigma}{\partial \nu} \dot{T}_K + \frac{1}{T}Q_KT_K \geq 0,
\]  

(21)

where \( \sigma = \rho_0\psi \). Following Toupin [38] and Mindlin [27], the skew symmetric part of \( M_{KLi} \), with respect to \( K \) and \( L \), makes no condition to the rate of work over any closed surface in the body, or over the boundary. We shall assume that \( M_{KLi} = M_{LKi} \) and that there is no kinematical constraint. From the inequality (21) we get [14]

\[
\sigma = \tilde{\sigma}(x_{i,K}, x_{i,KL}, \nu, \nu_K, T, X_M),
\]

\[
T_{Ki} = \frac{\partial \sigma}{\partial x_{i,K}} - \left( \frac{\partial \sigma}{\partial x_{i,KL}} \right)_{,L}, \quad M_{KLi} = \frac{\partial \sigma}{\partial x_{i,KL}},
\]

\[
H_K = \frac{\partial \sigma}{\partial \nu_K}, \quad g = -\frac{\partial \sigma}{\partial \nu}, \quad \rho_0\eta = -\frac{\partial \sigma}{\partial T},
\]

and

\[
Q_KT_K \geq 0.
\]  

(23)

From (11) and (22) we find that

\[
S_{Ki} = -\frac{\partial \sigma}{\partial x_{i,K}}.
\]  

(24)

In view of (22) the energy equation (17) reduces to

\[
\rho_0T\dot{\eta} = Q_{K,K} + \rho_0S.
\]  

(25)

The functions \( \sigma \) and \( Q_K \) must be invariant under Euclidean displacements. It can be shown (see [32], [38]) that the functions \( \sigma \) and \( Q_K \) are expressible in the form

\[
\sigma = \tilde{\sigma}(E_{KL}, G_{KLM}, \nu, \nu_K, T, X_L),
\]

\[
Q_A = \tilde{Q}_A(E_{KL}, G_{KLM}, \nu, \nu_K, T, T_M, X_L),
\]

(26)

where

\[
2E_{KL} = x_{i,K}x_{i,L} - \delta_{KL}, \quad G_{KLM} = x_{i,M}x_{i,KL}.
\]  

(27)

Here \( \delta_{KL} \) is the Kronecker delta. From (22), (24), (26) and (27) we obtain

\[
S_{Ki} = \frac{\partial \tilde{\sigma}}{\partial E_{KL}} x_{i,L} + \frac{\partial \tilde{\sigma}}{\partial G_{MLK}} x_{i,LM}, \quad M_{KLi} = \frac{\partial \tilde{\sigma}}{\partial G_{KLM}} x_{i,M}.
\]  

(28)
We note that if $S_K$, and $M_{KL}$ have the form (28) then the equation (14) is identically satisfied. It follows from (11) and (28) that
\[ T_{ki} = \frac{\partial \tilde{s}}{\partial E_{KL} x_{i,L}} + \frac{\partial \sigma}{\partial G_{MLK} x_{i,LM}} - \left( \frac{\partial \tilde{\sigma}}{\partial G_{KLM}} x_{i,M} \right)_L. \]  
(29)

By (22) and (26),
\[ \rho \alpha = -\frac{\partial \tilde{s}}{\partial T}, \quad g = -\frac{\partial \tilde{s}}{\partial \nu}, \quad H_K = \frac{\partial \tilde{\sigma}}{\partial \nu_K}. \]  
(30)

As in classical thermoelasticity (see, e.g., Carlson [7]) the inequality (23) implies that
\[ \tilde{Q}_{ij}(E_{KL}, G_{KLM}, \nu, \nu_K, T, 0, X_L) = 0. \]  
(31)

The basic equations consists of the equations of motion (6) and (10), the equation of energy (25), the constitutive equations (26), (28)-(30), and the geometrical equations (27). To the field equations we must adjoin boundary conditions and initial conditions. We suppose that $B$ is a bounded region with Lipschitz boundary $\partial B$. The boundary $\partial B$ consists in the union of a finite number of smooth surfaces, smooth curves (edges) and points. Let $C$ be the union of the edges. We introduce the surface gradient $D_K$ defined by $D_K = (\delta_{KL} - N_K N_L)\partial/\partial X_L$, and the notation $Df = f_K N_K$. Following Toupin [38] and Mindlin [32] we can write
\[ \int_{\partial B} (T_i v_i + M_{j,i}v_{i,j}) da = \int_{\partial B} (P_i v_i + R_i Dv_i) da + \int_C \Gamma_i v_i dl, \]  
(32)

where
\[ P_i = T_{Ki} N_K - D_L(M_{KL} N_K) + (D_P N_P) M_{LK} N_K N_L, \]  
(33)
\[ R_i = M_{KL} N_K N_L, \quad \Gamma_i = < M_{KL} N_K \zeta_L >, \quad \zeta_L = \xi_{KLM} s_K N_M. \]

Here, $< f >$ denotes the difference in values of $f$ as a given point on an edge is approached from either side, $s_K$ are the components of the unit vector tangent to $C$, and $\varepsilon_{KLM}$ is the alternating symbol.

Let $S_i (r = 1, 2, \ldots, 8)$ be subsets of $\partial B$ so that $S_1 \cup S_2 = \overline{S}_3 = S_4 = \overline{S}_5 \cup S_6 = \overline{S}_7 \cup S_8 = \partial B$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset$. We consider the boundary conditions
\[ x_i = \tilde{x}_i, v_i = \tilde{v}_i \text{ on } S_1 \times (t_0, t_1), \quad P_i = \tilde{P}_i \text{ on } S_2 \times (t_0, t_1), \]
\[ Dx_i = \tilde{d}_i \text{ on } S_3 \times (t_0, t_1), \quad R_i = \tilde{R}_i \text{ on } S_4 \times (t_0, t_1), \]
\[ \nu = \tilde{\nu} \text{ on } S_5 \times (t_0, t_1), \quad H_K N_K = \tilde{H} \text{ on } S_6 \times (t_0, t_1), \]
\[ T = \tilde{T} \text{ on } S_7 \times (t_0, t_1), \quad Q_K N_K = \tilde{Q} \text{ on } S_8 \times (t_0, t_1), \quad T_i = \tilde{T}_i \text{ on } C \times (t_0, t_1), \]  
(34)

where $\tilde{x}_i, \tilde{d}_i, \tilde{\nu}, \tilde{T}, \tilde{P}_i, \tilde{R}_i, \tilde{H}, \tilde{Q}$ and $\tilde{T}_i$ are prescribed functions. The initial conditions are
\[ x_i(X_K, 0) = x_{i0}(X_K), \quad \dot{x}_i(X_K, 0) = \xi_{i0}(X_K), \]  
(35)
\[ \nu(X_K, 0) = \nu_{0}(X_K), \quad \dot{\nu}(X_K, 0) = \zeta_{0}(X_K), \quad \eta(X_K, 0) = \eta_{0}(X_K), (X_K) \in \overline{B}, \]

where $x_{i0}, \xi_{i0}, \nu_{0}, \zeta_{0}$ and $\eta_{0}$ are given functions. We assume that: (i) $\rho_0$ and $\kappa$ are continuous and strictly positive on $\overline{B}$; (ii) $f_{ij}, l$ and $S$ are continuous on $\overline{B} \times [t_0, t_1]$; (iii) $x_{i0}, \xi_{i0}, \nu_{0}, \zeta_{0}$ and $\eta_{0}$ are continuous on $\overline{B}$; (iv) $\tilde{x}_i$ are continuous on $S_1 \times (t_0, t_1), \tilde{d}_i$ is continuous on $S_5 \times (t_0, t_1)$, $\tilde{\nu}$ is continuous on $S_5 \times (t_0, t_1)$, $\tilde{T}$ is continuous on $S_7 \times (t_0, t_1)$ and $\tilde{d}_i$ are continuous in time and piecewise regular on $S_3 \times (t_0, t_1)$; (v) $\tilde{P}_i, \tilde{R}_i, \tilde{H}$ and $\tilde{Q}$ are continuous in
time and piecewise regular on \(S_2 \times (t_0, t_1), S_4 \times (t_0, t_1), S_6 \times (t_0, t_1)\) and \(S_8 \times (t_0, t_1)\), respectively; (vi) \(\tilde{\Gamma}_i\) are continuous in time and piecewise regular on \(C \times (t_0, t_1)\).

3. **Thermoelastic deformations superposed on a non-linear deformation.**

In this section we derive the equations governing the infinitesimal thermoelastic deformations superposed on nonlinear deformations of non-simple porous bodies. In the case of simple materials, the theory of a thermoelastic body, deformed from a state of zero stress and which is subsequently subjected to small perturbations, was investigated in various works (see, e.g., [17, 21, 22, 23] and references therein). We consider three states of the body: the reference configuration \(B\) investigated in various works (see, e.g., [17, 21, 22, 23] and references therein). We refer all quantities to the configuration \(B\) in the primary state and to \(y\) in the secondary state. If the point \(X\) in the reference configuration \(B\) moves to \(x\) in the primary state and to \(y\) in the secondary, then \(u_i = y_i - x_i\) is the incremental displacement. We have

\[
y_i = y_i(X_K, t), \quad (X_K, t) \in B \times (t_0, t_1),
\]

with \(\det(\partial y_i/\partial X_K) > 0\). The thermomechanical quantities associated with the secondary state will be denoted with an asterisk. Now let

\[
u = \nu^* - \nu, \quad \theta = T^* - T.
\]

We assume that \(u_i = \varepsilon u_i^*, \varphi = \varepsilon \varphi^*, \theta = \varepsilon \theta^*\), where \(\varepsilon\) is a constant small enough for squares and higher powers to be neglected, and \(u_i^*, \varphi^*, \theta^*\) are independent of \(\varepsilon\). The problem consists in establishing the equations, boundary conditions and initial conditions for \(u_i, \varphi \) and \(\theta\), when the functions \(x_i(X_K, t), \nu(X_K, t)\) and \(T(X_K, t)\) are known. We refer all quantities to the configuration \(B\). We have

\[
u = u_i(X_K, t), \quad \varphi = \varphi(X_K, t), \quad \theta = \theta(X_K, t), \quad (X_K, t) \in B \times (t_0, t_1).
\]

The equations of motion of the secondary deformation are

\[
T_{Ki,K}^* + \rho_0 f_{i,K}^* = \rho_0 \ddot{y}_i.
\]

In the secondary motion the balance of equilibrated forces is

\[
H_{K,L}^* + g^* + \rho_0 v^* = \rho_0 k^* v^*.
\]

The energy equation can be written in the form

\[
\rho_0 T^* \dot{\theta}^* = Q_{K,L}^* + \rho_0 S^*.
\]

In view of the constitutive equations, we have

\[
s^* = \tilde{s}(E_{KL}^*, G_{KL}^*, \nu^*, \nu_{K,L}^*, T^*, X_L),
\]

\[
S_{K,L}^* = \frac{\partial s^*}{\partial E_{KL}^*} y_{i,L} + \frac{\partial s^*}{\partial G_{ML}^*} y_{i,M}, \quad M_{K,L}^* = \frac{\partial s^*}{\partial G_{KL}^*} y_{i,M},
\]

\[
T_{Ki,K}^* = S_{K,L}^* - M_{K,L}^*, \quad H_{K}^* = \frac{\partial s^*}{\partial v_{K}^*}, \quad g^* = -\frac{\partial s^*}{\partial v_{L}^*}, \quad \rho_0 \dot{\theta}^* = -\frac{\partial s^*}{\partial T^*},
\]

\[
Q_{K}^* = \tilde{Q}_{K}(E_{MN}^*, G_{MNP}^*, \nu^*, \nu_{L}^*, T^*, T_{L}^*, X_M),
\]
where
\[ 2E_{KL}^* = y_iKy_iL - \delta_{KL}, \quad G_{KLM}^* = y_iMy_iKL. \]

We consider the following boundary conditions for the secondary deformation
\[ y_i = \bar{y}_i \text{ on } S_1(t_0, t_1), \quad P_i^* = \bar{P}_i^* \text{ on } S_2 \times (t_0, t_1), \]
\[ D_i = \bar{d}_i^* \text{ on } S_3 \times (t_0, t_1), \quad R_i^* = \bar{R}_i^* \text{ on } S_4 \times (t_0, t_1), \]
\[ \nu^* = \bar{\nu}^* \text{ on } S_5 \times (t_0, t_1), \quad H_K^*N_K = \bar{H}_K^* \text{ on } S_6 \times (t_0, t_1), \]
\[ T^* = \bar{T}_K^* \text{ on } S_7 \times (t_0, t_1), \quad Q_K^*N_K = \bar{Q}_K^* \text{ on } S_8 \times (t_0, t_1), \]
\[ \Gamma_i^* = \bar{\Gamma}_i^* \text{ on } C \times (t_0, t_1), \]
where \( \bar{y}_i, \bar{d}_i^*, \bar{\nu}^*, \bar{T}_K^*, \bar{P}_i^*, \bar{R}_i^*, \bar{H}_K^*, \bar{Q}_K^* \) and \( \bar{\Gamma}_i^* \) are given. In the secondary motion we have the following initial conditions
\[ y_i(X_K, 0) = y_i^0(X_K), \quad \bar{y}_i(X_K, 0) = \zeta_0^0(X_K), \]
\[ \nu^*(X_K, 0) = \zeta_0^*(X_K), \quad \eta^*(X_K, 0) = \eta_0^0(X_K), \quad (X_K) \in \bar{B}, \]
where \( y_i^0, \zeta_0^0, \bar{\nu}^0, \zeta_0^* \) and \( \eta_0^0 \) are prescribed functions. The entropy production inequality implies that
\[ Q_K^*T_K^* \geq 0. \]

Let us derive the equations, boundary conditions and initial conditions for the functions \( u_i, \varphi \) and \( \theta \). From (27) and (43) we find that
\[ E_{KL}^* = E_{KL} + \varepsilon_{KL}, \quad G_{KLM}^* = G_{KLM} + \gamma_{KLM}, \]
where
\[ \varepsilon_{KL} = \frac{1}{2}(x_iKu_iL + x_iLx_iK), \quad \gamma_{KLM} = x_iMu_iKL + x_iKLx_iM. \]

If we take into account (42) and (48), then to a second-order approximation we obtain
\[ \frac{\partial \sigma^*}{\partial E_{KL}^*} = \frac{\partial \sigma}{\partial E_{KL}} + A_{KLMN}\varepsilon_{MN} + B_{KLRS}\gamma_{RST} + b_K\varphi + \]
\[ + D_{KLM}\varphi, + \beta_{KL}\theta, \]
\[ \frac{\partial \sigma^*}{\partial G_{KLM}^*} = \frac{\partial \sigma}{\partial G_{KLM}} + B_{SKLM}\varepsilon_{RS} + C_{KLMRS}\gamma_{RST} + N_{KLM}\varphi + \]
\[ + F_{SKLM}\varphi, - d_{KLM}\theta, \]
\[ \frac{\partial \sigma^*}{\partial \nu^*} = \frac{\partial \sigma}{\partial \nu} + b_{KL}\varepsilon_{KL} + N_{KLM}\gamma_{KLM} + \xi\varphi + B_K\varphi, - b\theta, \]
\[ \frac{\partial \sigma^*}{\partial \nu^*_K} = \frac{\partial \sigma}{\partial \nu_K} + D_{MK}\varepsilon_{MN} + F_{KMNP}\gamma_{MNP} + A_{KL}\varphi, + B_K\varphi - a_K\theta, \]
\[ \frac{\partial \sigma^*}{\partial T^*} = \frac{\partial \sigma}{\partial T} - \beta_{KL}\varepsilon_{KL} - d_{KLM}\gamma_{KLM} - b\varphi - a_K\varphi, - a\theta, \]
\[ Q_M^* = Q_M + R_{MLN}\varepsilon_{LN} + J_{MKLN}\gamma_{KLN} + C_M\varphi \]
\[ + L_{MK}\varphi, + E_M\theta + K_{ML}\theta, \]
where

\[ A_{KLMN} = \frac{\partial^2 \sigma}{\partial E_{KL} \partial E_{MN}}, \]
\[ B_{KLrst} = \frac{\partial^2 \sigma}{\partial E_{KL} \partial G_{RST}}, \]
\[ b_{KL} = \frac{\partial^2 \sigma}{\partial E_{KL} \partial \nu}, \]
\[ \zeta_{KLM} = \frac{\partial^2 \sigma}{\partial G_{KLM} \partial E_{MN}}, \]
\[ \beta_{KL} = \frac{\partial^2 \sigma}{\partial G_{KLM} \partial \nu}, \]
\[ \nu_{KLM} = \frac{\partial^2 \sigma}{\partial E_{KL} \partial G_{RST}}, \]
\[ \xi = \frac{\partial^2 \sigma}{\partial \nu^2}, \]
\[ a_K = \frac{\partial^2 \sigma}{\partial T \partial \nu}, \]
\[ J_{MKLN} = \frac{\partial Q_M}{\partial G_{KLN}}, \]
\[ C_M = \frac{\partial Q_M}{\partial \nu}, \]
\[ L_{MK} = \frac{\partial Q_M}{\partial \nu}, \]

Clearly,

\[ A_{KLMN} = A_{MNKL}, \]
\[ B_{KLrst} = B_{LKRST}, \]
\[ b_{KL} = b_{LK}, \]
\[ \zeta_{KLM} = \zeta_{LMK}, \]
\[ \beta_{KL} = \beta_{LK}, \]
\[ \nu_{KLM} = \nu_{LKM}, \]
\[ R_{MLN} = R_{MNL}, \]
\[ J_{MKLN} = J_{MLKN}. \]

Let us note that (31) implies

\[ R_{KLM} = 0, J_{KLMN} = 0, C_K = 0, L_{MK} = 0, E_K = 0 \text{ if } T_R = 0. \]  

(52)

We introduce the notations

\[ t_{Ki} = T_{Ki} - T_{Ki}, \]
\[ s_{Ki} = S^*_{Ki} - S_{Ki}, \]
\[ m_{KL} = M^*_{KL} - M_{KL}, \]
\[ \sigma_K = H^*_{K} - H_{K}, \]
\[ \chi = g^* - g, \gamma = \rho_0 (\eta^* - \eta), \]
\[ \Phi_K = Q^*_K - Q_K, \]
\[ F_i = f^*_i - f_i, \]

(53)

From (11), (42) and (53) we get

\[ t_{Ki} = s_{Ki} - m_{LKi,L}. \]  

(54)

From (6), (39), (53) and (54) we find the equations

\[ s_{Ki,K} - m_{LKi,LL} + \rho_0 F_i = \rho_0 \ddot{u}_i. \]  

(55)

Similarly, from (10), (37), (40) and (53) we obtain

\[ \sigma_{K,K} + \chi + \rho_0 L = \rho_0 \kappa \ddot{\phi}. \]  

(56)

From (25), (37), (41) and (53) we get

\[ T \gamma + \rho_0 \theta \ddot{\eta} = \Phi_{K,K} + \rho_0 s. \]  

(57)

We introduce the notations

\[ P_{KL} = \frac{\partial \sigma}{\partial E_{KL}}, \]
\[ \Pi_{KLM} = \frac{\partial \sigma}{\partial G_{KLM}}. \]  

(58)
It follows from (28), (42), (49), (53) and (58) that

\[ s_{Ki} = p_{KLi} + \nu_{MLK} x_{LM} + P_{KL} u_{i,LM} + \Pi_{LMK} u_{i,LM}, \]

\[ m_{KLi} = v_{KLM} x_{i,M} + \Pi_{KLM} u_{i,M}, \]

\[ \sigma_K = \zeta_{MNP} E_{MN} + f_{KMN} \gamma_{NMP} + \alpha_{KL} \varphi_{,L} + b_K \varphi - a_K \theta, \]

\[ \chi = -b_K \varphi - v_{KLM} \gamma_{KL} - \xi \varphi - b_K \varphi_{,K} + b \theta, \]

\[ \gamma = \beta_{KL} \varphi_{,KL} + d_{KLM} \varphi_{,KL} + b \varphi + a_K \varphi_{,K} + a \theta, \]

\[ \Phi_M = R_{MLN} \epsilon_{LN} + J_{MKLN} \gamma_{KL} + C_M \varphi + L_{MK} \varphi_{,K} + E_M \theta + k_{ML} \theta_{,L}, \]

where

\[ p_{KLi} = A_{KLMN} \epsilon_{MN} + B_{KLRST} \gamma_{RST} + b_K \varphi + \zeta_{KLM} \varphi_{,M} - \beta_{KL} \theta, \]

\[ \nu_{KLM} = B_{RSLK} \epsilon_{RS} + C_{MRST} \gamma_{RST} + \nu_{KLM} \varphi + f_{SKLM} \varphi_{,S} - d_{KLM} \theta. \]

The functions \( s_{Ki} \) and \( m_{KLi} \) can be written in the form

\[ s_{Ki} = (A_{KLRST} x_{i,L} + B_{RSLK} x_{i,LM}) \epsilon_{RS} + \]

\[ + (B_{KLRST} x_{i,L} + C_{MRST} x_{i,LM}) \gamma_{RST} + \]

\[ + (b_K x_{i,L} + \nu_{MLK} x_{i,LM}) \varphi + (\zeta_{KLS} x_{i,L} + f_{SKLM} x_{i,LM}) \varphi_{,S} - \]

\[ - (\beta_{KL} x_{i,L} + d_{KLM} x_{i,LM}) \theta + P_{KL} u_{i,LM} + \Pi_{MLK} u_{i,LM}, \]

\[ m_{KLi} = B_{RSLK} x_{i,LM} \epsilon_{RS} + C_{MRST} x_{i,LM} \gamma_{RST} + \nu_{KLM} \varphi + f_{SKLM} \varphi_{,S} - d_{KLM} x_{i,LM} \theta + \Pi_{KLM} u_{i,M}. \]

The basic equations for the functions \( u, \varphi \) and \( \theta \) consist of the equations of motion (55) and (56), the equation of energy (57), the constitutive equations (59) and (60), and the geometrical equations (48). From (34) and (44) we obtain the following boundary conditions

\[ u_i = \bar{u}_i \text{ on } S_1 \times (t_0, t_1), \]

\[ P_i = \bar{P}_i \text{ on } S_2 \times (t_0, t_1), \]

\[ D u_i = \bar{w}_i \text{ on } S_3 \times (t_0, t_1), \]

\[ \mathcal{R}_i = \bar{R}_i \text{ on } S_4 \times (t_0, t_1), \]

\[ \varphi = \bar{\varphi} \text{ on } S_5 \times (t_0, t_1), \]

\[ \sigma_K N_K = \bar{h} \text{ on } S_6 \times (t_0, t_1), \]

\[ \theta = \bar{\theta} \text{ on } S_7 \times (t_0, t_1), \]

\[ \Phi_K N_K = \bar{\Phi} \text{ on } S_8 \times (t_0, t_1), \]

where \( \bar{u}_i = \tilde{u}_i - \bar{u}, \bar{w}_i = \tilde{w}_i - \bar{w}, \bar{\varphi} = \tilde{\varphi} - \bar{\varphi}, \bar{\theta} = \tilde{\theta} - \bar{\theta}, \bar{P}_i = \tilde{P}_i - \bar{P}_i, \bar{R}_i = \tilde{R}_i - \bar{R}_i, \bar{h} = \tilde{H} - \bar{H}, \bar{\Phi} = \tilde{Q} - \bar{Q}, \bar{\mathcal{L}}_i = \tilde{\Gamma}_i - \bar{\Gamma}_i, \text{ and we have used the notations} \]

\[ \mathcal{P}_i = t_{Ki} N_K - D_L (m_{KLi} N_K) + (D_{\rho N_p}) m_{KLi} N_K N_L, \]

\[ \mathcal{R}_i = m_{KLi} N_K N_L, \]

\[ \mathcal{L}_i = < m_{KLi} N_K \zeta_L >. \]

The relations (35) and (45) lead to the following initial conditions

\[ u_i(X, 0) = u^0_i \text{, } \dot{u}_i(X, 0) = \dot{v}^0_i \text{,} \]

\[ \varphi(X, 0) = \varphi^0 \text{, } \dot{\varphi}(X, 0) = \varphi^0 \text{, } \gamma(X, 0) = \gamma^0 \text{,} \]

where \( u^0_i = y^0_i - x^0_i, v^0_i = \xi^0_i - \xi^0, \varphi^0 = \hat{\varphi}^0 - \hat{\varphi}^0, \psi^0 = \hat{\psi}^0 - \hat{\psi}^0, \) and \( \gamma^0 = \rho_0 (\hat{\gamma}^0 - \hat{\gamma}^0). \)

From (36) we obtain

\[ R_{MLN} \epsilon_{LN} \theta_{,M} + J_{MKLN} \gamma_{KL} \theta_{,M} + C_M \varphi_{,M} + L_{MK} \varphi_{,K} \theta_{,M} + E_M \varphi_{,M} + k_{ML} \theta_{,L} \theta_{,L} + \Phi_M T_M + Q_M T_M \geq 0, \]

for any \( \epsilon_{KL}, \gamma_{KL}, \varphi, \varphi_{,K}, \theta \text{ and } \theta_{,K}. \)
4. Linear theory of materials with initial stresses and initial heat flux.
In this section we present the strain gradient theory of elastic materials with initial stresses and initial heat. Following Green [17], we now assume that the primary configuration $B'$ is identical with the reference configuration $B$. We suppose that $B$ is subjected to an initial stress, and is at non-uniform temperature $T$. We consider that the distribution of voids in $B'$ is non-uniform so that we assume that $ν = ν$, where $ν$ is a prescribed function. We then assume that the configuration $B''$ is obtained from $B$ (or $B'$) by an infinitesimal deformation. We introduce the notations: $u_i$ is the displacement vector, $φ$ is the change in the volume fraction field, $θ$ is the temperature variation. We have

$$x_i = δ_{iK}X_K, J = 1, ν = ν, E_{KL} = 0, G_{KLM} = 0, T = T,$$
$$P_{KL} = \frac{∂σ}{∂E_{KL}}, Π_{KLM} = \frac{∂σ}{∂G_{KLM}}, H_K = \frac{∂σ}{∂ν_K}, g = -\frac{∂σ}{∂ν},$$
(66)
$$ρ_0 η = -\frac{∂σ}{∂T}, Q_K = \bar{Q}_K(0, 0, ν, ν, T, T, M, X_N),$$
the derivatives being evaluated at $E_{KL} = 0$, $G_{KLM} = 0$, $ν = ν$ and $T = T$. The functions $P_{KL}$, $Π_{KLM}$ and $H_K$ define the initial stresses in the body $B$. The functions $Q_K$ are the components of the initial heat flux. The work of preceding section can be applied to this special case and yields a strain gradient theory of thermoelastic porous solids which are under initial stresses and initial heat flux. We assume that the body is in equilibrium in the state $B$. We introduce the notations

$$u_K = δ_{iK}u_i, ε_{KL} = \frac{1}{2}(u_{KL} + u_{L,K}), κ_{KLM} = u_{M,KL}.$$  
(67)
In this case, from (48) we find that

$$ε_{KL} = ε_{KL}, γ_{KLM} = κ_{KLM}.$$  
(68)
If we denote

$$τ_{KL} = δ_{iLSK_i}, μ_{KLM} = δ_{iMmKN_i},$$  
(69)
then from (59), (60) and (67) we obtain the following constitutive equations

$$τ_{KL} = A_{KLMN}ε_{MN} + B_{KLRST}ε_{RST} + b_{KLV}φ + ζ_{KLM}φ_{,M} -$$
$$- β_{KL}θ + P_{KM}u_{L,M} + Π_{MKN}u_{M,N},$$
$$μ_{KLM} = B_{RSKLM}ε_{RS} + C_{KLMRST}ε_{RST} + ν_{KLM}φ +$$
$$+ f_{SKLM}φ, S - d_{KLM}θ + Π_{KLM}u_{M,N},$$
$$σ_K = γ_{MN}ε_{MN} + f_{KMNP}φ_{,MN} + α_{KL}φ_{,L} + b_{K}φ - a_{K}θ,$$
$$χ = -b_{KL}ε_{KL} - ν_{KLM}κ_{KLM} - ξφ - b_{K}φ_{,K} + bθ,$$
$$γ = β_{KL}ε_{KL} + d_{KLM}κ_{KLM} + bφ + a_{K}φ_{,K} + aθ,$$
$$Φ_M = R_{MLN}ε_{LN} + J_{MKL}κ_{KLN} + C_{M}φ + L_{MK}φ_{,K} + E_{M}θ + k_{ML}θ_{,L}.$$  
(70)
In view of (67) and (69) the equations (55) can be written in the form

$$τ_{KL,K} - m_{MN}ε_{MN} + ρ_0 F_M = ρ_0 u_M,$$  
(71)
where $F_M = δ_{iM}F_i$. We can write the field equations by using small indices. Thus, the basic equations of the strain gradient theory of prestressed thermoelastic porous solids consist of the equations of motion

$$τ_{ji,j} - μ_{kji,kj} + ρ_0 F_i = ρ_0 u_i, σ_{j,j} + χ + ρ_0 L = ρ_0 κφ,$$  
(72)
the equations of energy
\[ T \gamma = \Phi_{j,j} + \rho_0 s, \] (73)

the constitutive equations
\[
\begin{align*}
\tau_{ij} &= A_{ijrs} e_{rs} + B_{ijpq} k_{pqr} + b_{ij} \varphi + \zeta_{ij} \varphi, \kappa - \beta_{ij} \theta + P_{m} u_{j,m} + \Pi_{mn} u_{j,mn}, \\
\mu_{ijk} &= B_{rsijk} e_{rs} + C_{ijkpq} k_{pqr} + \nu_{ijk} \varphi + f_{mijk} \varphi, m - d_{ijk} \theta + \Pi_{ijm} u_{k,m}, \\
\sigma_i &= \zeta_{rsi} e_{rs} + f_{ipqr} k_{pqr} + \alpha_{ij} \varphi, j + b_{i} \varphi - a_i \theta, \\
\chi &= -b_{ij} e_{ij} - \nu_{ijk} \kappa_{ijk} - \xi \varphi - b_{i} \varphi, j + b \theta, \\
\gamma &= \beta_{ij} e_{ij} + d_{ijk} \kappa_{ijk} + b_{i} \varphi + a_i \varphi, j + a \theta, \\
\Phi_i &= R_{irs} e_{rs} + J_{ipqr} k_{pqr} + C_i \varphi + L_{ij} \varphi, j + E_i \theta + k_{ij} \theta, j,
\end{align*}
\]

and the geometrical equations
\[ 2e_{ij} = u_{i,j} + u_{j,i}, \quad \kappa_{ijk} = u_{k,ij}, \] (75)
on \( B \times (t_0, t_1) \). From \( 51 \) we get
\[
A_{ijrs} = A_{rsij}, \quad B_{ijpq} = B_{jipq} = B_{ijpq}, \quad b_{ij} = b_{ij},
\]
\[
\zeta_{ij} = \zeta_{ij}, \quad \beta_{ij} = \beta_{ij}, \quad C_{ijkpq} = C_{pqrij} = C_{ijkpq},
\]
\[
\nu_{ijk} = \nu_{ijk}, \quad f_{ij} = f_{ij}, \quad d_{ijk} = d_{ijk}, \quad \alpha_{ij} = \alpha_{ij}, \quad R_{ij} = R_{ij}, \quad J_{pq} = J_{pq}.
\] (76)
The coefficients from \( 74 \) are defined by \( 50 \), and are evaluated at \( E_{KL} = 0 \), \( G_{KLM} = 0 \), \( v = \nu \) and \( T = \bar{T} \). From \( 52 \) we get
\[ R_{ij} = 0, \quad J_{ij} = 0, \quad C_i = 0, \quad L_{ij} = 0, \quad E_i = 0 \quad \text{if} \quad T_{,j} = 0. \] (77)
To the system of equations \( 72 \)-\( 75 \) we adjoin the initial conditions \( 64 \) and the boundary conditions
\[
\begin{align*}
u_i &= \bar{\nu}_i \text{ on } S_1 \times (t_0, t_1), \quad P_i = \bar{P}_i \text{ on } S_2 \times (t_0, t_1), \\
D u_i &= \bar{D} u_i \text{ on } S_3 \times (t_0, t_1), \quad R_i = \bar{R}_i \text{ on } S_4 \times (t_0, t_1), \\
\varphi &= \bar{\varphi} \text{ on } S_5 \times (t_0, t_1), \quad \sigma_j n_j = \bar{n} \text{ on } S_6 \times (t_0, t_1), \\
\theta &= \bar{\theta} \text{ on } S_7 \times (t_0, t_1), \quad \Phi_j n_j = \bar{\Phi} \text{ on } S_8 \times (t_0, t_1), \\
\mathcal{L}_i &= \bar{\mathcal{L}}_i \text{ on } C \times (t_0, t_1),
\end{align*}
\]
where \( n_i = \delta_{ik} N_{ik} \), and \( P_i, R_i \) and \( \mathcal{L}_i \) are given by
\[
\begin{align*}
P_i &= (\tau_{ij} - \mu_{kij} n_j) n_j - D_j (\mu_{kij} n_k) + (D_j n_j) \mu_{rsi} n_r n_s, \\
R_i &= \mu_{rsi} n_r n_s, \quad \mathcal{L}_i = < \mu_{rsi} n_r \zeta_s > .
\end{align*}
\] (79)
In view of \( 75 \) and \( 76 \) we can express the equations \( 74 \) in the form
\[
\begin{align*}
\tau_{ij} &= a_{ijrs} u_{r,s} + b_{ijpq} u_{r,pq} + b_{ij} \varphi + \zeta_{ij} \varphi, \kappa - \beta_{ij} \theta, \\
\mu_{ijk} &= b_{rsijk} u_{r,s} + c_{ijkpq} u_{r,pq} + \nu_{ijk} \varphi + f_{mijk} \varphi, m - d_{ijk} \theta, \\
\sigma_i &= \zeta_{rsi} u_{r,s} + f_{ipqr} u_{r,pq} + b_{i} \varphi + \alpha_{ij} \varphi, j - a_i \theta, \\
\chi &= -b_{ij} u_{i,j} - \nu_{ijk} u_{k,ij} - \xi \varphi - b_{i} \varphi, i + b \theta, \\
\gamma &= \beta_{ij} u_{i,j} + d_{ijk} u_{k,ij} + b_{i} \varphi + a_i \varphi, j + a \theta, \\
\Phi_i &= R_{irs} u_{s,r} + J_{ipqr} u_{r,pq} + C_i \varphi + L_{ij} \varphi, j + E_i \theta + k_{ij} \theta, j,
\end{align*}
\]
where
\[
\begin{align*}
a_{ijrs} &= A_{ijrs} + P_{sij} \delta_{rs}, \quad b_{ijpq} = B_{ijpq} + \Pi_{pqj} \delta_{ir}, \quad c_{ijkpq} = C_{ijkpq}.
\end{align*}
\] (81)
It follows from (58), (76) and (81) that
\[ a_{ijs} = a_{rsij}, \quad b_{ijpq} = b_{ijpq}, \quad c_{ijpq} = c_{pqrij} = c_{ijkpq}. \] (82)
The coefficients from (80) depend on the initial stresses and the temperature \( T \).
If the body is isotropic then we have \( P_{ij} = p\delta_{ij}, \Pi_{ij} = 0 \), where \( p \) is a specified function of \( x_j \).
When the body has a constant reference temperature then \( Q_i = 0, R_{ijpq} = 0, J_{ijpq} = 0, C_i = 0, L_{ij} = 0 \) and \( E_i = 0 \).

The strain gradient theory of elasticity is an adequate tool to describe the deformation of chiral solids (see Papanicolpoulos [34] and references therein). By using the results of [28] and [34] we find that in the case of isotropic chiral elastic solids with a constant reference temperature, the constitutive equations (74) become
\[ \tau_{ij} = \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + b\varphi \delta_{ij} + f(\varepsilon_{ikm}\kappa_{kjm} + \varepsilon_{jkm}\kappa_{ikm}) - \beta\theta \delta_{ij} + pu_{i,j}, \]
\[ \mu_{ijk} = \frac{1}{2} \alpha_1 (\kappa_{rr} \delta_{jk} + 2\kappa_{krk} \delta_{ij} + \kappa_{rr} \delta_{ik}) + \alpha_2 (\kappa_{rr} \delta_{jk} + \kappa_{rr} \delta_{ik}) + 2\alpha_3 \kappa_{rrk} \delta_{ij} + 2\alpha_4 \kappa_{iijk} + \alpha_5 (\kappa_{kj} + \kappa_{ki}) + b_1 \delta_{ij} \varphi_{,k}, \]
\[ + b_2 (\delta_{ik} \varphi_{,j} + \delta_{jk} \varphi_{,i}) + f(\varepsilon_{iks}\varepsilon_{js} + \varepsilon_{jks}\varepsilon_{is}), \]
\[ \sigma_i = b_1 \kappa_{rr} + 2b_2 \kappa_{irr} + \alpha \varphi_{,i}, \quad \chi = -be_{rr} - \xi \varphi + \beta_1 \theta, \]
\[ \gamma = \beta e_{rr} + \beta_1 \varphi + a \theta, \quad \Phi_i = k \theta_{,i}, \]
where the coefficients are prescribed functions of \( x_k \).

The behaviour of chiral materials has received in recent years a widespread attention. Examples of chiral materials include auxetic materials, bones, carbon nanotubes, as well as some porous composites.

5. **Uniqueness.** In this section we use the method of [6] to establish reciprocity and uniqueness results in the strain gradient theory of prestressed thermoelastic porous solids with a constant reference temperature \( T_0 \). The field equations consist of the equations (72), (73) and the following constitutive equations
\[ \tau_{ij} = a_{ijs} u_{r,s} + b_{ijpq} u_{r,pq} + b_{ij} \varphi + \zeta_{ijk} \varphi_{,k} - \beta_{ij} \theta, \]
\[ \mu_{ijk} = b_{i} + c_{ijk} u_{r,s} + c_{ijpq} u_{r,pq} + \nu_{ijk} \varphi + f_{mi} \varphi_{,m} - d_{ijk} \theta, \]
\[ \sigma_i = \zeta_{rjs} u_{r,s} + f_{ijpq} u_{r,pq} + b_i \varphi + \alpha_{ij} \varphi_{,j} - a_i \theta, \]
\[ \chi = -b_3 u_{i,j} - \nu_{ijk} u_{k,i} - \xi \varphi - b_i \varphi_{,i} + b \theta, \]
\[ \gamma = \beta_{ij} u_{i,j} + d_{ijk} u_{k,i,j} + b \varphi + a \varphi_{,i} + a \varphi, \quad \Phi_i = k \theta_{,j}. \] (83)
We assume that the coefficients in (83) are prescribed functions which are continuously differentiable on \( B \) and satisfy the relations (76) and (82).

We first establish a reciprocity relation which involves two processes at different times. In what follows we suppose that the underlying time interval is \( I = (0, \infty) \).

If \( f \) is a continuous function on \( B \times I \), then we denote by \( \hat{f} \) the function defined by
\[ \hat{f}(x, t) = \int_0^t f(x, s) ds, \quad (x, t) \in B \times I. \] (84)

We introduce the notation
\[ G = \rho_0 \delta + T_0 \gamma^0. \] (85)
An immediate consequence of the equation (73) and the initial conditions (64) is the following proposition: if the functions \( \gamma \) and \( \Phi_i \) satisfy the equations (73) and the conditions (64) then we have
\[ T_0 \gamma = \Phi_{j,i} + G, \] (86)
on \( B \times I \). Let us consider two external data systems \( \mathcal{D}^{(\alpha)} = (F^{(\alpha)}, L^{(\alpha)}, s^{(\alpha)}, \tilde{u}^{(\alpha)},\tilde{w}^{(\alpha)}, \tilde{v}^{(\alpha)}, \tilde{\rho}^{(\alpha)}, \tilde{\phi}^{(\alpha)}, \tilde{\psi}^{(\alpha)}, \tilde{\Lambda}^{(\alpha)}, u^{(\alpha)}_i, \varphi^{(\alpha)}_i, \gamma^{(\alpha)}_i, \gamma^{(\alpha)}_i, \chi^{(\alpha)}_i, \gamma^{(\alpha)}_i, \phi_i^{(\alpha)} ) \) for all \( r,s \), \( i = 1,2 \) and the constitutive equations (83) we obtain
\( x^{(\alpha)}_i = \Phi^{(\alpha)}_i n_j - D_j(\mu^{(\alpha)}_{kji,k}) n_k + (D_j n_j) \mu^{(\alpha)}_{rsi} n_r n_s, \)
\( R_i^{(\alpha)} = \mu^{(\alpha)}_{rsi} n_r n_s, L_i^{(\alpha)} = < \mu^{(\alpha)}_{rsi} n_r \zeta_s >, \)
\( \Phi^{(\alpha)}_i = \Phi^{(\alpha)}_j n_j, \pi^{(\alpha)} = \sigma^{(\alpha)}_i n_j, G^{(\alpha)} = \rho_0 \Theta^{(\alpha)} + T_0 \gamma^{(\alpha)} \). \]

Let us denote
\( Q_{\alpha\beta}(r,s) = \int_B [\rho_0 F_i^{(\alpha)}(x,r) u_i^{(\beta)}(x,s) + \rho_0 L_i^{(\alpha)}(x,r) \varphi^{(\beta)}_i(x,s) - \frac{1}{T_0} \varphi^{(\beta)}_i(x,s) dv + \int_{B} [P_i^{(\alpha)}(x,r) u_i^{(\beta)}(x,s) + R_i^{(\alpha)}(x,r)Du_i^{(\beta)}(x,s) + \pi^{(\alpha)}(x,r) \varphi^{(\beta)}_i(x,s) - \frac{1}{T_0} \varphi^{(\beta)}_i(x,s) dv] \]
\( K_{\alpha\beta}(r,s) = \int_B [\rho_0 u_i^{(\alpha)}(x,r) u_i^{(\beta)}(x,s) + \rho_0 N_i^{(\alpha)}(x,r) \varphi^{(\beta)}_i(x,s) - \frac{1}{T_0} \varphi^{(\beta)}_i(x,s) dv], \)

for all \( r,s \in I \).

A general reciprocity relation is expressed by the following theorem.

**Theorem 5.1.** Assume that the conductivity tensor \( k_{ij} \) is symmetric and the relations (76) and (82) hold. Let
\( D_{\alpha\beta}(r,s) = Q_{\alpha\beta}(r,s) - K_{\alpha\beta}(r,s), \)
for all \( r,s \in I \). Then
\( D_{\alpha\beta}(r,s) = D_{\beta\alpha}(s,r), (\alpha, \beta = 1,2). \)

**Proof.** We introduce the notations
\( N_{\alpha\beta}(r,s) = \tau^{(\alpha)}_{ij} u^{(\beta)}_{i,j}(s) + \mu^{(\alpha)}_{ij} u^{(\beta)}_{i,j}(s) + \sigma^{(\alpha)}_{ij} \varphi^{(\beta)}_{i,j}(s) - \chi^{(\alpha)}_i \varphi^{(\beta)}_i(s) - \gamma^{(\alpha)}_i \varphi^{(\beta)}_i(s), r,s \in I. \)

In (91), for convenience, we have suppressed the argument \( x \). With the help of constitutive equations (83) we obtain
\( N_{\alpha\beta}(r,s) = \alpha_{ijm} u^{(\alpha)}_{m,i}(r) u^{(\beta)}_{i,j}(s) + c_{ijpq} u^{(\alpha)}_{k,pq}(r) u^{(\beta)}_{i,j}(s) + \xi^{(\alpha)}_i \varphi^{(\beta)}_i(s) + \alpha^{(\alpha)}_i \varphi^{(\beta)}_i(s) - \alpha^{(\alpha)}_i \varphi^{(\beta)}_i(s) + b_{ijpq} [u^{(\alpha)}_{i,j}(r) u^{(\beta)}_{m,pq}(s) + u^{(\beta)}_{i,j}(s) u^{(\alpha)}_{m,pq}(r)] + b_{ij} [u^{(\alpha)}_{i,j}(r) \varphi^{(\beta)}_i(s) + u^{(\beta)}_{i,j}(s) \varphi^{(\alpha)}_i]\)
\[ u_{i,j}^{(\beta)}(s) = \beta_{ij}[u_{i,j}^{(\alpha)}(r)\theta^{(\beta)}(s) + u_{i,j}^{(\beta)}(s)\theta^{(\alpha)}(r)] + \nu_{ij,k}^{(\beta)}[u_{k,ij}^{(\alpha)}(r)\phi^{(\beta)}(s) + u_{k,ij}^{(\beta)}(s)\phi^{(\alpha)}(r)] + \tau_{ip}^{(\beta)}[u_{m,pq}^{(\alpha)}(r)\varphi^{(\beta)}_{i,s} + u_{m,pq}^{(\beta)}(s)\varphi^{(\alpha)}_{i,s}] + u_{i,j}^{(\alpha)}(s)\phi^{(\beta)}_{i,t} - \nu_{ij,k}^{(\alpha)}[u_{k,ij}^{(\beta)}(r)\theta^{(\alpha)}(s) + u_{k,ij}^{(\alpha)}(s)\theta^{(\beta)}(r)] + b_{i}[\varphi^{(\beta)}_{i}(r)\phi^{(\beta)}(s) + \phi^{(\beta)}_{i}(s)\phi^{(\alpha)}(r)] - a_{i}[\phi^{(\alpha)}_{i}(r)\theta^{(\beta)}(s) + \theta^{(\beta)}(s)\phi^{(\alpha)}(r)] + \phi^{(\beta)}_{i}(s)\theta^{(\alpha)}(r) - b[\theta^{(\alpha)}(r)\phi^{(\beta)}(s) + \theta^{(\beta)}(s)\phi^{(\alpha)}(r)] \]

On the basis of (82) and (92) we conclude that

\[ N_{\alpha\beta}(r,s) = N_{\beta\alpha}(s,r), \quad (\alpha, \beta = 1, 2), \]  

for all \( r, s \in I \). On the other hand, in view of (72), (86) and (91) we obtain

\[ N_{\alpha\beta}(r,s) = \left[ (r^{(\alpha)}(r) - \mu_{kj,k}^{(\alpha)}(r))u_{k,j}^{(\beta)}(s) + \mu_{kj}^{(\alpha)}(r)u_{k,j}^{(\beta)}(s) + \sigma_{ij}^{(\alpha)}(r)\phi^{(\beta)}(s) - \frac{1}{T_{0}}\tilde{\theta}_{ij}^{(\alpha)}(r)\phi^{(\beta)}(s)]_{,j} + \rho_{0}F_{i}^{(\alpha)}(r)u_{i}^{(\beta)}(s) + \rho_{0}L_{i}^{(\alpha)}(r)\varphi^{(\beta)}(s) - \frac{1}{T_{0}}G^{(\alpha)}(r)\theta^{(\beta)}(s) - \rho_{0}\tilde{\varphi}_{i}^{(\alpha)}(r)u_{i}^{(\beta)}(s) - \rho_{0}\kappa_{ij}^{(\alpha)}(r)\varphi^{(\beta)}(s) + \frac{1}{T_{0}}\Phi_{ij}^{(\alpha)}(r)\theta^{(\beta)}(s) \right] 

If we integrate the above relation on \( B \) and use the divergence theorem, (87) and (88) we find

\[ \int_{B} N_{\alpha\beta}(r,s) dv = \int_{\partial B} \left[ (r^{(\alpha)}(r)u_{k,j}^{(\beta)}(s) + \mu_{kj,k}^{(\alpha)}(r)u_{k,j}^{(\beta)}(s) + \sigma_{ij}^{(\alpha)}(r)\phi^{(\beta)}(s) - \frac{1}{T_{0}}\tilde{\theta}_{ij}^{(\alpha)}(r)\phi^{(\beta)}(s)]_{,j} + \rho_{0}F_{i}^{(\alpha)}(r)u_{i}^{(\beta)}(s) + \rho_{0}L_{i}^{(\alpha)}(r)\varphi^{(\beta)}(s) - \frac{1}{T_{0}}G^{(\alpha)}(r)\theta^{(\beta)}(s) - \rho_{0}\tilde{\varphi}_{i}^{(\alpha)}(r)u_{i}^{(\beta)}(s) - \rho_{0}\kappa_{ij}^{(\alpha)}(r)\varphi^{(\beta)}(s) + \frac{1}{T_{0}}\Phi_{ij}^{(\alpha)}(r)\theta^{(\beta)}(s) \right] da + \int_{B} [\rho_{0}F_{i}^{(\alpha)}(r)u_{i}^{(\beta)}(s) + \rho_{0}L_{i}^{(\alpha)}(r)\varphi^{(\beta)}(s) - \frac{1}{T_{0}}G^{(\alpha)}(r)\theta^{(\beta)}(s)] dv - K_{\alpha\beta}(r,s). \]

In view of (32), (87) and (88) we can write the relation (94) in the form

\[ \int_{B} N_{\alpha\beta}(r,s) dv = Q_{\alpha\beta}(r,s) - K_{\alpha\beta}(r,s). \]

From (89), (93) and (95) we obtain the desired result. \( \square \)

Now we use the reciprocity relation (90) to derive the following result.

**Theorem 5.2.** Assume that \( A = (u_{i}, \varphi, \theta, \tau_{ij}, \mu_{ij}, \sigma_{i}, \chi, \gamma, \Phi_{i}) \) is a solution corresponding to the external data system \( (F_{i}, L, s, \bar{u}_{i}, \bar{\varphi}, \bar{\theta}, \bar{\tau}_{i}, \bar{\mu}_{i}, \bar{\sigma}_{i}, \bar{\chi}, \bar{\gamma}, \bar{\Phi}_{i}, i_{0}, \varphi^{0}, \psi^{0}, \gamma^{0}) \). Let the constitutive coefficients be as in Theorem 1 and let

\[ F(r,s) = \int_{B} [\rho_{0}F_{i}(r)u_{i}(s) + \rho_{0}L(r)\varphi(s) - \frac{1}{T_{0}}G(r)\theta(s)] dv + \int_{\partial B} [P_{i}(r)u_{i}(s) + R_{i}(r)Du_{i}(s) + \pi(r)\varphi(s) - \frac{1}{T_{0}}\tilde{\Phi}(r)\theta(s)] da + \int_{C} L_{i}(r)u_{i}(s) dl, \]
for all \( r, s \in I \). Then
\[
\frac{d}{dt} \left\{ \int_B \left( \rho_0 u_i u_i + \rho_0 \kappa \varphi^2 \right) dv + \frac{1}{T_0} \int_0^t \int_B k_{ij} \theta_{ij} \theta_{ij} dt \right\} =
\]
\[
= \int_0^t [F(t-s,t+s) - F(t+s,t-s)] ds + \int_B \left[ \rho_0 [\dot{u}_i(2t)u_i(0) +
\]
\[
+ \dot{u}_i(0)u_i(2t)] + \rho_0 \kappa [\dot{\varphi}(2t)\varphi(0) + \dot{\varphi}(0)\varphi(2t)] \right] dv. \tag{97}
\]

Proof. In view of (90) we get
\[
\int_0^t D_{11}(t+s,t-s) ds = \int_0^t D_{11}(t-s,t+s) ds. \tag{98}
\]
We apply the relation (98) to the process \( A^{(1)} = A \). From (88), (89) and (97) we obtain
\[
\int_0^t D_{11}(t+s,t-s) ds = \int_0^t F(t+s,t-s) ds - \int_0^t \int_B \rho_0 \dot{u}_i(t+s)u_i(t-s) +
\]
\[
+ \rho_0 \kappa \dot{\varphi}(t+s)\varphi(t-s) - \frac{1}{T_0} k_{ij} \theta_{ij}(t+s)\theta_{ij}(t-s) ds dv. \tag{99}
\]
In a similar way we get
\[
\int_0^t D_{11}(t-s,t+s) ds = \int_0^t F(t-s,t+s) ds - \int_0^t \int_B \rho_0 \dot{u}_i(t-s)u_i(t+s) +
\]
\[
+ \rho_0 \kappa \dot{\varphi}(t-s)\varphi(t+s) - \frac{1}{T_0} k_{ij} \theta_{ij}(t-s)\theta_{ij}(t+s) ds dv. \tag{100}
\]
If we use the relations
\[
\int_0^t \ddot{u}(t+s)v(t-s) ds = \ddot{u}(2t)v(0) - \ddot{u}(t)v(t) + \int_0^t \dot{u}(t+s)\dot{v}(t-s) ds,
\]
\[
\int_0^t \ddot{v}(t)u(t+s) ds = \ddot{v}(t)u(t) - \ddot{v}(0)u(2t) + \int_0^t \dot{v}(t-s)\dot{u}(t+s) ds,
\]
\[
\int_0^t u(t+s)\dot{v}(t-s) ds = u(t)v(t) - u(0)v(2t) + \int_0^t \dot{u}(t+s)v(t-s) ds,
\]
then the relation (99) can be expressed in the form
\[
\int_0^t D_{11}(t+s,t-s) ds = \int_0^t F(t+s,t-s) ds - \int_B \left\{ \rho_0 [\ddot{u}_i(2t)u_i(0) -
\]
\[
- \ddot{u}_i(t)u_i(t) + \int_0^t \ddot{u}_i(t-s)u_i(t+s) ds] + \rho_0 \kappa [\ddot{\varphi}(2t)\varphi(0) -
\]
\[
- \ddot{\varphi}(t)\varphi(t) + \int_0^t \ddot{\varphi}(t-s)\ddot{\varphi}(t+s) ds] - \frac{1}{T_0} k_{ij} \ddot{\theta}_{ij}(t)\ddot{\theta}_{ij}(0) +
\]
\[
+ \int_0^t \theta_{ij}(t+s)\theta_{ij}(t-s) ds \right\} dv.
\]
The relation (100) can be transformed in a similar way. If we take into account (98) then we conclude that the relation (97) holds. \( \square \)

Theorem 2 forms the basis of the following uniqueness result.
Theorem 5.3. Assume that
(i) \( \rho_0 \) and \( \kappa \) are strictly positive;
(ii) \( k_{ij} \) is symmetric and positive definite;
(iii) \( a \neq 0 \) on \( B \);
(iv) the symmetry relations (76) and (82) hold.
Then the boundary-initial-value problem has at most one solution.

Proof. If there are two solutions then their difference \((u_i, \varphi, \theta, \tau_{ij}, \mu_{ijk}, \sigma_i, \chi, \gamma, \Phi_i)\) corresponds to null data. Then, from (97) we conclude that
\[
\frac{d}{dt} \left( \int_B (\rho_0 u_i u_i + \rho_0 \kappa \varphi^2) dv + \frac{1}{T_0} \int_0^t \int_B k_{ij} \hat{\theta}_i \hat{\theta}_j \, dv \, dt \right) = 0.
\]
Since \( u_i, \varphi \) and \( \theta \) vanish initially we get
\[
\int_B (\rho_0 u_i u_i + \rho_0 \kappa \varphi^2) dv + \frac{1}{T_0} \int_0^t \int_B k_{ij} \hat{\theta}_i \hat{\theta}_j \, dv \, dt = 0. \tag{101}
\]
In view of hypothesis (i) and (ii), from (101) we obtain
\[
u_i = 0, \, \varphi = 0, \, \hat{\theta}_i = 0 \text{ on } B \times [0, \infty).
\tag{102}
\]
By (102) we find that \( \theta, \tau_{ij} = 0 \) on \( B \times [0, \infty) \), so that \( \theta(x, t) = \vartheta(t), (x, t) \in B \times [0, \infty) \), and from the constitutive equations we obtain \( \Phi_i = 0 \) on \( B \times [0, \infty) \). In view of (83) we get \( \gamma = a \vartheta \) on \( B \times [0, \infty) \). The energy equation implies that \( T_0 a \gamma \neq 0 \). Since \( a \) and \( T_0 \) are different from zero we find that \( \gamma = \text{constant} \) on \( B \times [0, \infty) \). From the initial conditions and the hypothesis (iii) we conclude that \( \theta = 0 \) on \( B \times [0, \infty) \) and the proof is complete. \qed

Let \( u \) and \( v \) be functions on \( B \times [0, \infty) \) that are continuous in time. We denote by \( u \ast v \) the convolution of \( u \) and \( v \),
\[
u \ast v(x, t) = \int_0^t u(x, t - \tau) v(x, \tau) \, d\tau, \, (x, t) \in B \times [0, \infty).
\]
We introduce the notations
\[
j(t) = t, \, \mathcal{F}_i^{(\alpha)} = \rho_0 (j \ast F_i^{(\alpha)} + tv_i^{0(\alpha)} + u_i^{0(\alpha)}),
\]
\[
\mathcal{H}^{(\alpha)} = \rho_0 (j \ast L^{(\alpha)} + \kappa \tau \rho^{0(\alpha)} + \kappa \varphi^{0(\alpha)}), \, t \in [0, \infty). \tag{103}
\]
Theorem 1 implies the following reciprocal theorem.

Theorem 5.4. Assume that the conductivity tensor \( k_{ij} \) is symmetric and the relations (76) and (82) hold. Let \( A^{(\alpha)} \) be a solution corresponding to the external data system \( D^{(\alpha)} \), \( (\alpha = 1, 2) \). Then
\[
\int_B [\mathcal{F}_i^{(1)} \ast u_i^{(2)} + \mathcal{H}^{(1)} \ast \varphi^{(2)} - \frac{1}{T_0} \mathcal{F}_i^{(2)} \ast \theta^{(1)}] \, dv + \int_{\partial B} j \ast [\mathcal{F}_i^{(1)} \ast u_i^{(2)} + R_i^{(1)} \ast Du_i^{(2)} + \pi^{(1)} \ast \varphi^{(2)} - \frac{1}{T_0} \mathcal{F}_i^{(2)} \ast \theta^{(1)}] \, da + \int_C j \ast [\mathcal{F}_i^{(1)} \ast u_i^{(2)}] \, dl = \int_B [\mathcal{F}_i^{(2)} \ast u_i^{(1)} + \mathcal{H}^{(2)} \ast \varphi^{(1)} - \frac{1}{T_0} \mathcal{F}_i^{(1)} \ast \theta^{(2)}] \, dv + \int_{\partial B} j \ast [\mathcal{F}_i^{(2)} \ast u_i^{(1)} + R_i^{(2)} \ast Du_i^{(1)} + \pi^{(2)} \ast \varphi^{(1)} - \frac{1}{T_0} \mathcal{F}_i^{(1)} \ast \theta^{(2)}] \, da + \int_C j \ast [\mathcal{F}_i^{(2)} \ast u_i^{(1)}] \, dl. \tag{104}
\]
Proof. We take in (90), \( r = \tau \) and \( s = t - \tau \) and integrate from 0 to \( t \). With the aid of (88) we find that

\[
\int_B \left[ \rho_0 \bar{F}_i^{(1)} \ast \bar{u}_i^{(2)} + \rho_0 L_i^{(1)} \ast \varphi_i^{(2)} - \frac{1}{T_0} G_i^{(1)} \ast \theta_i^{(2)} \right] dv + \\
+ \int_{\partial B} \left[ P_i^{(1)} \ast \bar{u}_i^{(2)} + P_i^{(1)} \ast D u_i^{(2)} + \chi^{(1)} \ast \varphi_i^{(2)} - \frac{1}{T_0} \bar{\Phi}_i^{(1)} \ast \theta_i^{(2)} \right] da + \\
+ \int_C L_i^{(1)} \ast \bar{u}_i^{(2)} - \int_B \left[ \rho_0 \dddot{u}_i^{(1)} \ast \bar{u}_i^{(2)} + \rho_0 \kappa_i^{(1)} \ast \varphi_i^{(2)} - \frac{1}{T_0} \dddot{\Phi}_i^{(1)} \ast \theta_i^{(2)} \right] dv = \\
= \int_B \left[ \rho_0 F_i^{(2)} \ast \bar{u}_i^{(1)} + \rho_0 L_i^{(2)} \ast \varphi_i^{(1)} - \frac{1}{T_0} \dddot{G}_i^{(2)} \ast \theta_i^{(1)} \right] dv + \int_{\partial B} \left[ P_i^{(2)} \ast \bar{u}_i^{(1)} \right] + (105) \\
+ R_i^{(2)} \ast D u_i^{(1)} + \chi^{(2)} \ast \varphi_i^{(1)} - \frac{1}{T_0} \dddot{\Phi}_i^{(2)} \ast \theta_i^{(1)} \right] da + \int_C L_i^{(2)} \ast \bar{u}_i^{(1)} \right] dl - \\
- \int_B \left[ \rho_0 \dddot{u}_i^{(2)} \ast \bar{u}_i^{(1)} + \rho_0 \kappa_i^{(2)} \ast \varphi_i^{(1)} - \frac{1}{T_0} \dddot{\Phi}_i^{(2)} \ast \theta_i^{(1)} \right] dv.
\]

If we take the convolution of the relation (105) with \( j(t) = t, t \in [0, \infty) \), and use the relations

\[
 j \ast \dddot{u}_i^{(\alpha)} = \dddot{u}_i^{(\alpha)} - \dddot{u}_i^{0(\alpha)} - \dddot{u}_i^{0(\alpha)}, \ j \ast f = 1 \ast \hat{f},
\]

(103), and the symmetry of conductivity tensor, then we obtain the desired result.

\( \square \)

Following Nowacki [31] we can derive various applications of the reciprocal theorem. The existence and uniqueness results established by Navaro and Quintanilla [29] can be extended to the theory of non-simple prestressed solids with voids.

Acknowledgments. I express my gratitude to the referees for their helpful suggestions.

REFERENCES

[1] E. C. Aifantis, Exploring the applicability of gradient elasticity to certain micro/ nano reliability problems, Microsystem Technology, 15 (2009), 109–115.
[2] G. Amendola, M. Fabrizio and J. M. Golden, Thermodynamics of a non-simple heat conductor with memory, Quart. Appl. Math., 69 (2011), 787–806.
[3] G. Amendola, M. Fabrizio and J. M. Golden, Second gradient viscoelastic fluids: Dissipation principle and free energies, Meccanica, 47 (2012), 1859–1868.
[4] H. Askes and E. C. Aifantis, Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results, Int. J. Solids Struct., 48 (2011), 1962–1990.
[5] O. Brulin and S. Hjalmars, Linear grade consistent micropolar theory, Int. J. Eng. Sci., 19 (1981), 1731–1738.
[6] L. Brun, Methodes energetiques dans les systemes evolutifs lineaires, J. Mecanique, 8 (1969), 125–166.
[7] D. E. Carlson, Linear Thermoelasticity, in Handbuch der Physik, vol. VIa/2, (ed. C. Truesdell), Springer-Verlag, Berlin-Heidelberg-New York, 1972.
[8] S. Chirita, Uniqueness and continuous dependence results for the incremental thermoelasticity, J. Thermal Stresses, 5 (1982), 331–346.
[9] O. Coussy, Mechanics and Physics of Porous Solids, John Wiley and Sons, Chichester, 2010.
[10] S. C. Cowin and J. W. Nunziato, Linear elastic materials with voids, J. Elasticity, 13 (1983), 125–147.
[11] T. Dillard, S. Forest and P. Ienny, Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams, Eur. J. Mech.-A/Solids, 25 (2006), 526–549.
[12] A. C. Eringen and E. S. Suhubi, Nonlinear theory of simple microelastic solids, Int. J. Eng. Sci., 2 (1964), 189–203.
A. C. Eringen, *Microcontinuum Field Theories. I: Foundations and Solid*, Springer-Verlag, New York, Berlin, Heidelberg, 1999.

M. Fabrizio and A. Morro, *Mathematical Problems in Linear Viscoelasticity*, SIAM Studies in Applied Mathematics 12, Philadelphia, PA, USA, 1992.

S. Forest, J. M. Cardona and R. Sievert, Thermoelectricity of second-grade media, in *Continuum Thermomechanics, The Art and Science of Modelling Material Behaviour*, Paul Germain’s Anniversary Volume, (eds. G.A. Maugin, R. Drouot and F. Sidoroff.), Kluwer Academic Publishers, (2000), 163–176.

P. Giovine, Linear wave motions in continua with nano-pores, in *Wave Processes in Classical and New Solids*, (ed. P. Giovine), Publisher: InTech, (2012), 62–86.

A. E. Green, *Thermoelastic stresses in initially stressed bodies*, Proc. Roy. Soc. London, Ser. A, 266 (1962), 1–19.

A. E. Green and R. S. Rivlin, Multipolar continuum mechanics, *Arch. Rational Mech.Anal.*, 17 (1964), 113–147.

S. Hjalmars, Non-linear micropolar theory, in *Mechanics of Micropolar Media*, (eds. O. Brulin and R.K.T. Hsieh), *World Scientific, Singapore*, (1982), 147–189.

D. Iesan, Incremental equations in thermoelasticity, *J. Thermal Stresses*, 3 (1980), 41–56.

D. Iesan, *Prestressed Bodies*, Pitman Research Notes in Mathematics Series 195, Longman Scientific and Technical, Longman House, Harlow, Essex, UK and John Wiley & Sons, Inc., New York, 1989.

D. Iesan, *Thermoelastic Models of Continua*, Kluwer Academic, Dordrecht, 2004.

R. J. Knops and E. W. Wilkes, Theory of elastic stability, in *Handbuch der Physik*, (ed. C. Truesdell), Springer-Verlag, Berlin Heidelberg-New York, 1973.

R. J. Knops and L. E. Payne, *Uniqueness Theorems in Linear Elasticity*, Springer Tracts in Natural Philosophy, vol. 19, Berlin-Heidelberg-New York, 1971.

R. J. Knops, Uniqueness and continuous data dependence in the elastic cylinders, *Int. J. Non-Linear Mech.*, 36 (2001), 489–499.

F. Martinez, F. and R. Quintanilla, On the incremental problem in thermoelasticity of non-simple materials, *Zeit. Angew.Math. Mech.*, 78 (1998), 703–710.

R. D. Mindlin, Microstructure in linear elasticity, *Arch. Rational Mech.Anal.*, 16 (1964), 51–78.

R. D. Mindlin and N. N. Eshel, On first strain gradient theories in linear elasticity, *Int. J. Solids Struct.*, 4 (1968), 109–124.

C. B. Navarro and R. Quintanilla, On existence and uniqueness in incremental thermoelasticity, *Zest. Angew. Math. Mech.*, 35 (1984), 206–215.

P. Neff and S. Forest, A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling existence and minimizers, identification of moduli and computational results, *J. Elasticity*, 87 (2007), 239–276.

W. Nowacki, *Theory of Asymmetric Elasticity*, Polish Scientific Publishers, Warszawa and Pergamon Press, Oxford, New York, Paris, Frankfurt, 1986.

J. W. Nunziato and S. C. Cowin, A nonlinear theory of elastic materials with voids, *Arch. Rational Mech. Anal.*, 72 (1979/80), 175–201.

A. Ochsner, G. E. Murch and M. J. S. Lemos, *Cellular and Porous Materials*, Wiley-VCH, Weinheim, 2008.

S. A. Papanicolopulos, Chirality in isotropic linear gradient elasticity, *Int. J. Solids Struct.*, 48 (2011), 745–752.

R. A. Toupin, Elastic materials with couple stresses, *Arch. Rational Mech. Anal.*, 11 (1962), 385–414.

R. A. Toupin, Theories of elasticity with couple-stress, *Arch. Rational Mech. Anal.*, 17 (1964), 85–112.

J. R. Vinson and R. L. Sierakowski, *The Behaviour of Structures Composed of Composite Materials*, Second edition, Kluwer Acad. Publ., Dordrecht, 2002.

Received March 2013; revised May 2013.

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