Sample Complexity of the Robust Linear Observers under Coprime Factors Uncertainty

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Abstract
This paper addresses the end-to-end sample complexity bound for learning the $H_2$ linear observer-based robust Linear Quadratic (LQ) regulators with unknown dynamics for stable Linear Time Invariant (LTI) systems, when given a fixed state feedback gain. The robust synthesis procedure is performed by considering bounded additive model uncertainty on the coprime factors of the plant. The closed-loop identification scheme follows Zhang et al. (2021), where the nominal model of the true plant is identified by constructing a Hankel-like matrix from a single time-series of noisy, finite length input-output data by using the ordinary least squares algorithm from Sarkar et al. (2020). Next, an $H_\infty$ bound on the estimated model error is provided, and the robust observer is designed via convex optimization while considering bounded additive uncertainty on the coprime factors of the model. Our results are consistent with previous methods on learning the linear observer.

Keywords: Linear Observers, Coprime Factorization, LTI Systems, Sample Complexity.

1. Introduction
State estimation is a fundamental problem in control theory and machine learning. The utilization of state observers has been proven to be significant in both detecting and identifying dynamical system faults as well as in monitoring and regulating systems since the work of Luenberger (1966). The existence of disturbances and uncertainties provides significant difficulties in real-world applications, as practically all observer designs are based on the mathematical model of the plant. To this purpose, a number of sophisticated observer designs have been put out as solutions to the high-performance, robust observer-based regulator design challenge, which has lately attracted significant interest. The classical LQ control problem, which involves linear and time-invariant dynamical systems, served as the starting point for the aforementioned research problems, where the goal is to identify the best output feedback law that minimizes the expected value of a quadratic cost. In the past few years, significant research has been put into using cutting-edge statistical and optimization tools from the machine learning framework to approach classical control problems with an eye toward real-world applications, see for instance Dean et al. (2018), Mania et al. (2019), Dean et al. (2020), Zheng et al. (2020). The renowned Kalman Filter (KF) has been the prevalent prediction technique in LQ control problems on the learning framework, see for example Wang et al. (2015), Lee and Lamperski (2020), Tsiamis et al. (2020).

An end-to-end sample-complexity bound of learning observer-based LQ regulator with unknown dynamics for stable LTI systems, that stabilize the true system with high probability is established by incorporating recent advances in finite-time system identification. The resulting sub-optimality
gap is bounded as a function of the level of model uncertainty. The end-to-end sample complexity bound for learning observer-based robust LQ regulators is \( O\left(\sqrt{\frac{\log T}{T}}\right) \), where \( T \) is the time horizon for learning. The observer-based robust control synthesis proposed here achieves the same scaling for the sub-optimality gap as in Tsiamis et al. (2020).

1.1 The Linear Observer Problem

For a discrete-time LTI (Linear and Time Invariant) systems driven by Gaussian process and sensor noise, the state-space model is defined as:

\[
\begin{align*}
    x_{k+1} &= Ax_k + B(u_k + w_k), \\
    y_k &= Cx_k + Du_k + \nu_k,
\end{align*}
\]  

(1)

where \( x_k \in \mathbb{R}^n \) is the state of the system, \( u_k \in \mathbb{R}^m \) is the control input and \( y_k \in \mathbb{R}^p \) is the measurement output with \( w_k \in \mathbb{R}^m, \nu_k \in \mathbb{R}^p \) are Gaussian noise with zero mean, covariance \( \sigma_w^2 I \) and \( \sigma^2 I \) respectively.

Assumption 1 Matrices \( A, B, C, D \) are unknowns. The pair \( (A, B) \) is controllable, and the pair \( (A, C) \) is observable. The matrix \( A \) has spectral radius less than 1, i.e., \( \rho(A) < 1 \).

A state observer for (1) is defined as a system that provides an estimate of the internal state \( x_k \), from measurements of the input \( u_k \) and the output \( y_k \), such that

\[
\lim_{k \to \infty} (x_k - \hat{x}_k) = 0
\]

In this sense the most general form of an observer is described by

\[
\hat{x}(z) = \Psi_1(z)u(z) + \Psi_2(z)y(z)
\]

where \( \Psi_1(z) \) and \( \Psi_2(z) \) are stable transfer function matrices.

It is well known from the separation principle that, a stabilizing controller for system (1) can be viewed as a combination of a state observer and a closed loop feedback control law \( u(z) = F\hat{x}(z) \) for some matrix \( F \). To evaluate the performance of an observer, the state feedback gain matrix \( F \) in this paper is considered given and fixed. In a nutshell, the challenge is to develop the best possible linear observer while taking into account the inherent model uncertainty that arises during the "learning" step. This is accomplished by "learning" the model of an unknown LTI system with high probability in finite time.

1.2 The Main Technical Ingredient

Identifying LTI models from input-output data has been the focus of time-domain identification. Using coprime factors instead of the state space realization of the system has a great advantage as it ensures to work on unstable system identification. The Transfer Function Matrix (TFM) of the plant is written as:

\[
G(z) = \tilde{M}^{-1}(z)\tilde{N}(z) = N(z)M^{-1}(z)
\]  

(2)

where \( \tilde{M}(z), \tilde{N}(z), M(z) \) and \( N(z) \) are stable TFM. Doubly coprime factorization of the TFM of a LTI system plays a key role in many sectors of the factorization approach to filter synthesis and multi-variable control systems analysis. A doubly coprime factorization of a given LTI plant is
closely related to the Parameterization of all stabilizing controllers for this plant. With a realization of the TFM, various formulas to compute doubly coprime factorizations over the ring of stable and proper Rational Matrix Functions (RMF) have been proposed both for standard (proper) and for generalized (improper, singular, or descriptor) systems. The formulas are expressed either in terms of a stabilizable and detectable realization of the underlying TFM or make additional use of a realization of a full or reduced order observer based stabilizing controller.

1.3 Paper Organization

The paper is organized as follows: the general setup and problem formulation is given in Section II. The robust observer synthesis with uncertainty on the coprime factors is included in Section III. A brief discussion on the sub-optimality guarantees with end-to-end sample complexity results are discussed in Section IV. Conclusions and future possible directions are given in Section V. All the proofs are postponed to the Appendices, where literature review and mathematical preliminaries also have been presented briefly.

2. General Setup and Technical Preliminaries

The notation used in this paper is fairly common in control systems. Upper and lower case boldface letters (e.g. \( \mathbf{z} \) and \( \mathbf{G} \)) are used to denote signals and transfer function matrices, and lower and upper case letters (e.g. \( z \) and \( \mathbf{A} \)) are used to denote vectors and matrices. The enclosed results are valid for discrete-time linear systems, therefore \( z \) denotes the complex variable associated with the \( Z \)-transform for discrete-time systems. A LTI system is stable if all the poles of its TFM are situated inside the unit circle for discrete time systems. The TFM of a LTI system is called unimodular if it is square, stable and has a stable inverse. For the sake of brevity the \( z \) argument after a transfer function may be omitted. Some frequently used notation is listed in the next page.

| Nomenclature of Basic Notation |
|-------------------------------|
| TFM | Transfer Function Matrix |
| DCF | Doubly Coprime Factorization |
| RCF, RCF | Right Coprime Factorization, Left Coprime Factorization |
| \( x \overset{def}{=} y \) | \( x \) is by definition equal to \( y \) |
| \( \mathbb{R}(z) \) | Set of all real–rational transfer functions |
| \( \mathbb{R}(z)^{p \times m} \) | Set of \( p \times m \) matrices having all entries in \( \mathbb{R}(z) \) |
| \( T^\varepsilon_{\ell} \) | The TFM of the (closed-loop) map having \( \varepsilon \) as input and \( \ell \) as output |
| \( T^\varepsilon_{Q} \) | The TFM of the (closed-loop) map from the exogenous signal \( \varepsilon \) to the signal \( \ell \) inside the feedback loop, as a function of the Youla parameter \( Q \) |
| \( \|G\|_F \) | Frobenius norm, Schur norm or \( l_2 \) norm of \( G \in \mathbb{R}(z) \), defined as \( \|tr(GG^*)\|^{1/2} \) |
| \( \|G\|_\infty \) | \( \mathcal{H}_\infty \)-norm of \( G \in \mathbb{R}(z) \), defined as \( \sup_{\omega} \sigma_{\max}(G(e^{j\omega})) \) |
| \( \|G\|_{\mathcal{H}_2} \) | \( \mathcal{H}_2 \)-norm of \( G \in \mathbb{R}(z) \), defined as \( \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} \text{tr}\left(G^*(e^{j\omega})G(e^{j\omega})\right)d\omega \right)^{1/2} \) |
| \( pt \) | Notations for true plant (e.g. \( G^{pt} \)) |
| \( md \) | Notations for nominal/estimated model (e.g. \( G^{md} \)) |
| \( G^{pt}, K^{opt} \) | True Plant, Optimal Controller |
| \( G^{md}, K^{nd} \) | Estimated Model, Nominal stabilizing controller for any stable Youla parameter \( Q \) |
2.1 The Youla-Kučera Parameterization

**Proposition 2.1** Given a TFM $K \in \mathbb{R}(z)^{m \times p}$, a fractional representation of the form $K = R^{-1}S$ with $R \in \mathbb{R}(z)^{m \times m}$, $S \in \mathbb{R}(z)^{m \times p}$ is called a left factorization of $K$. If $K = Y^{-1}X$ is a left factorization of $K$ then any other left factorization of $K$ such as $K = R^{-1}S$ is of the form $R = UY$, $S = UX$, for some invertible TFM $U$.

Given a plant $G \in \mathbb{R}(z)^{p \times m}$, a left coprime factorization of $G$ is defined by $G = \tilde{M}^{-1}\tilde{N}$, with $\tilde{N} \in \mathbb{R}(z)^{p \times m}$, $\tilde{M} \in \mathbb{R}(z)^{p \times p}$ both stable and satisfying $\tilde{M}\tilde{Y} + \tilde{N}\tilde{X} = I_p$, for certain stable TFMs $\tilde{X} \in \mathbb{R}(z)^{m \times p}$, $\tilde{Y} \in \mathbb{R}(z)^{p \times m}$. Analogously, a right coprime factorization of $G$ is defined by $G = NM^{-1}$ with both factors $N \in \mathbb{R}(z)^{p \times m}$, $M \in \mathbb{R}(z)^{m \times m}$ being stable and for which there exist $X \in \mathbb{R}(z)^{m \times p}$, $Y \in \mathbb{R}(z)^{m \times m}$ also stable, satisfying $YM + XN = I_m$ (Vidyasagar, 1985, Ch. 4, Corollary 17), with $I_m$ being the identity matrix.

**Definition 2.2** (Vidyasagar, 1985, Ch. 4, Remark pp. 79) A collection of eight stable TFMs $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ is called a doubly coprime factorization of $G$ if $\tilde{M}$ and $M$ are invertible, yield the factorizations $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$, and satisfy the following equality (Bézout’s identity):

$$
\begin{bmatrix}
\tilde{M} & \tilde{N} \\
-X & Y
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} & -N \\
X & M
\end{bmatrix}
= I_{p+m},
\begin{bmatrix}
\tilde{M} & \tilde{N} \\
-X & Y
\end{bmatrix}
\begin{bmatrix}
\tilde{M} & \tilde{N} \\
X & M
\end{bmatrix}
= I_{p+m}.
$$

(3)

**Theorem 2.3** (Ding et al. (1994), Vidyasagar (1985)) The eight TFMs $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ in Definition 2.2 can be formulated as following,

$$
\begin{align*}
M(z) &= I + F(zI - A_F)^{-1}B, \quad N(z) = C_F(zI - A_F)^{-1}B \\
\tilde{M}(z) &= I - C(zI - A_L)^{-1}L, \quad \tilde{N}(z) = C(zI - A_L)^{-1}B_L \\
X(z) &= -F(zI - A_L)^{-1}L, \quad Y(z) = I - F(zI - A_L)^{-1}B_L \\
\tilde{X}(z) &= -F(zI - A_F)^{-1}L, \quad \tilde{Y}(z) = I + C_F(zI - A_F)^{-1}L
\end{align*}
$$

(4)

with $A_F = A + BF$, $A_L = A - LC$, $C_F = C + DF$ and $B_L = B - LD$, where $F$ and $L$ are matrices that ensure $A_F$ and $A_L$ are stable. $F$ is also known as the optimal state feedback controller and $L$ is known as the optimal state estimator. In our closed loop setting, the control law is defined as $u(z) = Fx(z)$ where $F$ is fixed.

**Theorem 2.4** (Youla-Kučera) (Vidyasagar, 1985, Ch. 5, theorem 1) Let $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ be a doubly coprime factorization of $G$. Any controller $K_Q$ stabilizing the plant $G$, can be written as

$$
K_Q = Y_Q^{-1}X_Q = \bar{X}_Q\bar{Y}_Q^{-1},
$$

(5)

where $X_Q$, $\bar{X}_Q$, $Y_Q$ and $\bar{Y}_Q$ are defined as:

$$
X_Q \overset{def}{=} X + Q\tilde{M}, \quad \bar{X}_Q \overset{def}{=} \bar{X} + MQ,
$$

$$
Y_Q \overset{def}{=} Y - Q\tilde{N}, \quad \bar{Y}_Q \overset{def}{=} \bar{Y} - NQ
$$

(6)

for some stable $Q$ in $\mathbb{R}(z)^{m \times p}$. It also holds that $K_Q$ from (5) stabilizes $G$, for any stable $Q$ in $\mathbb{R}(z)^{m \times p}$. 

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Proposition 2.5 Starting from any doubly coprime factorization (3), the following identity

\[
\begin{bmatrix}
U_1 \tilde{M} & U_1 \tilde{N} \\
-2U_2X_Q & U_2Y_Q
\end{bmatrix}
\begin{bmatrix}
\hat{X}_QU_1^{-1} & -NU_2^{-1} \\
X_QU_1^{-1} & MU_2^{-1}
\end{bmatrix}
= I_{p+m}.
\]

provides the class of all doubly coprime factorizations of \( G \), where \( Q \) is stable in \( \mathbb{R}(z)^{m \times p} \) and \( U_1 \in \mathbb{R}(z)^{p \times p} \), \( U_2 \in \mathbb{R}(z)^{m \times m} \) are both unimodular.

2.2 Closed Loop Maps

Given the TFM’s from (4), it’s possible to observe that \( FP(z) = M(z) - I_m \) by denoting the system \((zI - A_F)^{-1}B\) as \( P(z) \in \mathbb{R}(z) \). This relation holds with model uncertainties also when \( M(z) \) is a valid RCF of the plant. By applying the rules of connecting two systems, it follows that \( P(z)M(z)^{-1} = (zI - A)^{-1}B \).

The general form of an observer with stable TFM \( \Psi_1(z), \Psi_2(z) \in \mathbb{R}(z) \) is given by:

\[
\hat{x} = \Psi_1(z)u(z) + \Psi_2(z)y(z)
\]

**Proposition 2.6** For a pseudo-state \( E(z) \) such that \( M(z)E(z) = u(z) \), the LTI system described in (1) can be reformulated as following:

\[
y(z) = N(z)E(z) + N(z)M(z)^{-1}w(z) + \nu(z)
\]

By using (1), it is possible to check:

\[
x(z) = (zI - A)^{-1}BM(z)E(z) + (zI - A)^{-1}Bw(z)
= P(z)E(z) + P(z)M(z)^{-1}w(z)
\]

Furthermore, by using Proposition 2.6, from (9) it is concluded that:

\[
\hat{x}(z) = \Psi_1(z)M(z)E(z) + \Psi_2(z)N(z)E(z) + \Psi_2(z)G(z)w(z) + \Psi_2(z)\nu(z)
\]

**Theorem 2.7** (Ding et al. (1994)) For the LTI system in (1), denote \( P(z) = (zI - A_F)^{-1}B \). When inverse z-transform is applied on \( \hat{x}(z) \) as in (8), we have:

\[
\lim_{k \to \infty} (x_k - \hat{x}_k) = 0
\]

if and only if

\[
\Psi_1(z)M(z) + \Psi_2(z)N(z) = P(z).
\]

**Corollary 2.8** In attempt to solve \( \Psi_1(z)M(z) + \Psi_2(z)N(z) = P(z) \) from Theorem 2.7, we get the following parameterization with \( S(z) \in \mathbb{RH}_\infty \):

\[
\Psi_1(z) = P(z)Y(z) + S(z)\tilde{N}(z)
\]

\[
\Psi_2(z) = P(z)X(z) - S(z)\tilde{M}(z)
\]

Note that the above parameterization requires Bézout’s identity as it is true only if \( YM + XN = I \) and \( \tilde{NM} = \tilde{MN} \). If the closed loop control law \( u(z) = F\tilde{x}(z) \) is allowed, then the relation between \( S(z) \) and the youla parameter \( Q(z) \) is as follows:

\[
FS(z) = Q(z) + \tilde{X}(z)
\]
Proposition 2.9 The closed loop maps from the external noises $w$ and $ν$ to the estimation error $(x - \hat{x})$ is represented by following:

$$T_{(x-\hat{x})w} = \Psi_1(z), \quad T_{(x-\hat{x})ν} = -\Psi_2(z)$$

Note that the stability of the described transfers $T_{(x-\hat{x})w}$ and $T_{(x-\hat{x})ν}$ in Proposition 2.9 can be insured, their Markov parameters decays exponentially.

Remark 1 (Closed-loop maps) The general form of an observer from (8) is deceptive since from this configuration it appears that $Ψ_1$ is designed for the control signal $u$ and $Ψ_2$ is designed for the control signal $y$, but that’s actually not the case. It’s not possible to write $x = Ψ_1(u + w) + Ψ_2Ne$, as the true state $x$ should only be obtained from the state space equation in (1). Also, by combining $Ψ_1 + Ψ_2G = PM^{-1}$ when Bézout’s identity holds, it is evident that the closed-loop transfers from $w$ and $ν$ to the estimation error $(x - \hat{x})$ is just $Ψ_1(z)$ and $-Ψ_2(z)$.

Remark 2 A common way to design an observer in terms of $S(z)$ is to minimize the norm of the transfer functions from the external noises to the estimation error in Proposition 2.9. However, due to the closed-loop setup, the state feedback gain $F$ is given and fixed. Hence, it’s better to consider $F(T_{(x-\hat{x})w}(z))$ and $F(T_{(x-\hat{x})ν}(z))$ to avoid the case that the estimation error $(x - \hat{x})$ having a small norm but amplified by $F$, causing a large gap between $Fx$ and $Fx$, and the later is the best control we can have given this $F$.

Proposition 2.10 The Optimal Linear Observer Problem given a fixed state feedback gain $F$ is defined as:

$$\min_{Q \text{ stable}} \left\| \left[ I_m - Y(z) + Q(z)\Delta N(z) \ X(z) + Q(z)\Delta M(z) \right] \right\|_{H_2}$$

(15)

Proof of Theorem 2.7, Proposition 2.9 and Proposition 2.10 is provided in Appendix B.

Remark 3 The reason of choosing to minimize $\left\| \left[ F\Psi_1 - F\Psi_2 \right] \right\|_{H_2}$ instead of $\left\| \left[ \Psi_1 - \Psi_2 \right] \right\|_{H_2}$ is that the later directly minimizes the estimation error but the state-feedback gain $F$ is designated to be fixed for revealing the cost induced by state estimation. It can be compared to the controller $u = Fx$. The error in $H_2$ cost of the closed loop is bounded by a combination of two parts as following: the cost of picking a suboptimal state feedback gain and the cost induced by inaccurate state estimation. The later is proportional to the norm of transformed error $F(x - \hat{x})$, which is shown in Appendix C.

3. Robust Controller Synthesis: An Observer Approach

Here model uncertainty has been considered in this section. The model in (1) is from a certain learning procedure, and so it differs from the original plant which is presented below.

Given a DCF of the nominal model of the plant $G^\text{nd} = (M^\text{nd})^{-1}N^\text{nd} = N^\text{nd}(M^\text{nd})^{-1}$, we can write the Bézout’s identity that incorporates the coprime factorization of the fixed stabilizing controller for the nominal model $K = (Y)^{-1}X = \bar{X}(\bar{Y})^{-1}$ as:

$$\begin{bmatrix} \bar{M}^\text{nd} & \bar{N}^\text{nd} \\ -X & Y \end{bmatrix} \begin{bmatrix} \bar{Y} & -N^\text{nd} \\ \bar{X} & M^\text{nd} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix}.$$  

(16)

Definition 3.1 (Model Uncertainty Set) The $γ$-radius model uncertainty set (for the nominal plant $G^\text{nd}$ with $\Delta \bar{M}$, $\Delta \bar{N}$ both stable) is defined as:

$$G_γ \overset{\text{def}}{=} \{ G = \bar{M}^{-1}\bar{N} | \bar{M} = (\bar{M}^\text{nd} + \Delta \bar{M}), \bar{N} = (\bar{N}^\text{nd} + \Delta \bar{N}); \quad \| \Delta \bar{M} \Delta \bar{N} \|_\infty < γ \}$$

(17)
Definition 3.2 ($\gamma$-Robustly Stabilizing) A fixed stabilizing controller $K$ of the nominal plant is said to be $\gamma$-robustly stabilizing iff $K$ stabilizes not only $G$ but also all plants $G \in G$. 

Assumption 2 It is assumed that the true plant, denoted by $G^p$, belongs to the model uncertainty set introduced in Definition 3.1, i.e. that there exist stable $\Delta M$, $\Delta N$ with $\| [\Delta M \Delta N] \|_\infty < \gamma$ for which $G^p = (\tilde{M}^{nd} + \Delta M)^{-1}(\tilde{N}^{nd} + \Delta N)$.

In the presence of additive uncertainty on the coprime factors the Bézout’s identity in (16) no longer holds, however, the following holds for certain stable $\Delta M$, $\Delta N$ factors:

$$\begin{bmatrix} (\tilde{M}^{nd} + \Delta M) & (\tilde{N}^{nd} + \Delta N) \\ -X & Y \end{bmatrix} \begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} \Phi & O \\ O & \Phi \end{bmatrix}.$$ (18)

The block diagonal structure of the right hand side term in (18) is due to the fact that $G^p = (\tilde{M}^{nd} + \Delta M)^{-1}(\tilde{N}^{nd} + \Delta N) = (\tilde{N}^{nd} + \Delta N)(\tilde{M}^{nd} + \Delta M)^{-1}$ for the aforementioned certain stable $\Delta M$, $\Delta N$ factors.

Lemma 3.3 A stabilizing controller of the nominal plant $K = (Y)^{-1}X = \tilde{X}(\tilde{Y})^{-1}$ is $\gamma$-robustly stabilizing iff for any stable model perturbations $\Delta M$, $\Delta N$ with $\| [\Delta M \Delta N] \|_\infty < \gamma$ the TFM

$$\Phi_{11} = I_p + [ \Delta M \Delta N ] \begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix},$$ (19)

is unimodular (i.e. it is square, stable and has an inverse $\Phi_{11}^{-1}$ which is also stable) from (18). A similar condition for $\gamma$-robust stabilizability can be formulated in terms of $\Phi_{22}$ TFM, whereas

$$\Phi_{22} = I_m + [ X \ Y ] \begin{bmatrix} \Delta N \\ \Delta M \end{bmatrix}.$$ (20)

Theorem 3.4 The Youla parameterization yields a $\gamma$-robustly stabilizing controller iff its corresponding Youla parameter satisfies $\| \begin{bmatrix} \tilde{Y} \tilde{Q} \\ \tilde{X} \tilde{Q} \end{bmatrix} \|_\infty \leq \frac{1}{\gamma}$, where $Q$ denotes as the Youla parameter.

The proofs for Lemma 3.3 and Theorem 3.4 are given on Appendix B. As an intermediary result, by employing Theorem 3.4 and the standard inequality (??) from Appendix A it is concluded that:

$$\left\| \begin{bmatrix} -\Delta M & -\Delta N \end{bmatrix} \begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} \right\|_\infty \leq \left\| \begin{bmatrix} \Delta M & \Delta N \end{bmatrix} \begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} \right\|_\infty < \gamma \times \frac{1}{\gamma} = 1.$$ 

Although the true plant is defined as $G^p = (\tilde{M}^{nd} + \Delta M)^{-1}(\tilde{N}^{nd} + \Delta N)$, it doesn’t necessary applies that $\tilde{M}^{pt} = (\tilde{M}^{nd} + \Delta M)$. It is considered that

$$\tilde{M}^{pt} = \Phi_{11}^{-1}(\tilde{M}^{nd} + \Delta M), \quad \tilde{N}^{pt} = \Phi_{11}^{-1}(\tilde{N}^{nd} + \Delta N),$$

$$M^{pt} = (M^{nd} + \Delta M)\Phi_{22}^{-1}, \quad N^{pt} = (N^{nd} + \Delta N)\Phi_{22}^{-1},$$

such that the Bézout’s identity holds with $X$, $Y$, $\tilde{X}$ and $\tilde{Y}$. 

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Corollary 3.5 Define $P^{pt}(z) = (zI - (A^{pt} + BF))^{-1}B$ where $A^{pt}$ is a possible state space matrix (up to similarity transformations) for the true plant. The closed loop maps in Proposition 2.9 still holds when replacing $P, \tilde{M}$ and $\tilde{N}$ with $P^{pt}, \tilde{M}^{pt}$ and $\tilde{N}^{pt}$ if (i) the Bézout’s identity (3) holds for $\tilde{M}^{pt}, \tilde{N}^{pt}, M^{pt}, N^{pt}, \tilde{X}, \tilde{Y}, X$ and $Y$ (ii) $FP^{pt}(z) = M^{pt}(z) - I_m$ and (iii) $P^{pt}(z)M^{pt}(z)^{-1} = (zI - A^{pt})^{-1}B$.

Theorem 3.6 The Robust Linear Observer Problem given a fixed state feedback gain $F$ is defined as :

$$\min_{Q \text{ stable}} \max \left\| \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \right\|_{\infty} < \gamma$$

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} \leq \frac{1}{\gamma}.$$  

(21)

Remark 4 (Robustness Condition) When $\Phi_{11}$ is inverted, it needs to be unimodular which means

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} < \frac{1}{\gamma}$$  

necessarily. Equivalently, the initial controller in the closed loop is already robust enough to make it possible for seeking a robust observer. In this case, it is considered that

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} = \alpha < \frac{1}{\gamma}.$$  

Furthermore, the standard closed-loop robust stability constraint,

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} \leq \frac{1}{\gamma},$$

should be brought into consideration since any $Q_*$ solves (21) also corresponds to a robust stabilizing controller.

Canonical min-max formulation in Theorem 3.6 caused by the additive uncertainty on the coprime factors of the robust linear observer synthesis renders the problem non-convex. In order to circumvent this, an upper bound on the cost functional will be derived as below.

Proposition 3.7 For the true plant, $G^{pt} \in G_\gamma$, the robust observer control problem in (21) admits the following upper bound:

$$\min_{Q \text{ stable}} \left\| \begin{bmatrix} (I_m - Y) & X \end{bmatrix} \right\|_{\mathcal{H}_2} + \|Q\|_{\mathcal{H}_2} \frac{1}{1 - \gamma \alpha} \left( \left\| \begin{bmatrix} \tilde{N}^{nd} & \tilde{M}^{nd} \end{bmatrix} \right\|_{\infty} + \gamma \right)$$

s.t.  

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} \leq \frac{1}{\gamma}.$$  

(22)

The objective function in (22) is already affine in $Q$, hence the upper bound from from Proposition 3.7 is convex. Proof for Proposition 3.7 is provided in Appendix B.

Remark 5 In next step, the constraint

$$\left\| \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \right\|_{\infty} \leq \frac{1}{\gamma}$$

is replaced by $\|Q\|_{\infty} < \frac{1 - \gamma \alpha}{\gamma \left\| \begin{bmatrix} N & M \end{bmatrix} \right\|_{\infty}}$. As the new constraint is more restrictive, so replacing it won’t change the feasibility of the upper bound problem in Proposition 3.7. However, it’s possible to miss a potential solution where a trade of optimality has been made for successfully formulating a Semi Definite Programming (SDP) problem to obtain $Q$ numerically.
Proposition 3.8 (SDP formulation for Proposition 3.7)

Let, $Q(z) = \sum_{t=0}^{N} Q_t z^{-t}$, with $Q_t \in \mathbb{R}^{p \times p}$. Here a Finite Time Truncation (FIR) on $Q$ has been performed for numerical purpose. Let $\text{vec}(\cdot)$ be the column vectorization of a matrix. Next we define

$$Q = [Q_0 \quad Q_1 \ldots \quad Q_N]^T \in \mathbb{R}^{(p \times N) \times p}, \quad \eta = [\text{vec}(Q_0) \quad \text{vec}(Q_1) \ldots \quad \text{vec}(Q_N)]^T$$

The optimization problem in Proposition 3.7 after FIR truncation can be expressed as:

$$\min_{\epsilon, \eta, H} \left\| I_m - Y X \right\|_{\mathcal{H}_2} + \frac{1}{1 - \gamma \alpha} \left( \left\| \begin{bmatrix} \tilde{N}^\text{nd} & \tilde{M}^\text{nd} \end{bmatrix} \right\|_{\mathcal{H}_2} + \gamma \right) \epsilon$$

s.t. $H \in \mathcal{S}^{p(N+1)}_+$, $\sum_{i=1}^{n-k} H_{i+k,i} = \frac{1 - \gamma \alpha}{\gamma \left\| \begin{bmatrix} \tilde{N}^\text{nd} & \tilde{M}^\text{nd} \end{bmatrix} \right\|_{\infty}} \delta_k I_p, \quad k = 0 : n,$

$$\left[ \begin{bmatrix} H \\ \eta^T \end{bmatrix} \frac{1 - \gamma \alpha}{\gamma \left\| \begin{bmatrix} \tilde{N}^\text{nd} & \tilde{M}^\text{nd} \end{bmatrix} \right\|_{\infty} I_p \right] \succeq 0,$$

$$\| \eta \| \leq \epsilon. \quad (23)$$

4. Analysis of End-to-End Performance

The performance of the desired observer, together with the fixed state feedback, $\hat{u} = F \hat{x}$ has been considered in this section. Denote the $\mathcal{H}_2$-cost of applying controls, $\hat{u} = F \hat{x}$ and $u = F x$ by $J_{\hat{u}}$ and $J_u$ respectively. Then from Appendix C it is possible to write the following:

$$J_{\hat{u}} - J_u \leq \sum_{k=1}^{m} \sum_{t=0}^{\infty} \left( (Fx_t - F \hat{x}_t)^T (Fx_t - F \hat{x}_t) \right); \quad w_i = e_k \delta_i, \quad (24)$$

where $e_k$ represents the $k^{th}$ standard basis vector in $\mathbb{R}^m$ and $\delta_i$ is the discrete Dirac impulse function. Then, by Proposition 3.7, we have:

$$J_{\hat{u}} - J_u \leq \left\| \hat{u} - u \right\|_2^2$$

$$\leq \left\| \left[ \begin{bmatrix} I_m - Y \\ X \end{bmatrix} \right] \right\|_{\mathcal{H}_2} + \left\| Q \right\|_{\mathcal{H}_2} \frac{1}{1 - \gamma \alpha} \left( \left\| \begin{bmatrix} \tilde{N}^\text{nd} & \tilde{M}^\text{nd} \end{bmatrix} \right\|_{\infty} + \gamma \right) \quad (25)$$

Specially, if the state feedback gain $F$ is given as $F_{\text{opt}}$, the solution of feedback Ricatti equation (30), then by the virtue of separation principle, the cost $J_u$ in (25) becomes the optimal $\mathcal{H}_2$-cost. In this case, (25) immediately gives a bound for the difference in $\mathcal{H}_2$-cost between when applying our designed controller based on $Q$ and the optimal LQG controller. This is further discussed in Appendix C.

From (25), it is evident that $(J_{\hat{u}} - J_u) \sim \mathcal{O}(\frac{\gamma}{1 - \gamma \alpha})$ which indicates that the upper-bound relies heavily on $\alpha$. In other words, the performance of the designed observer depends on the quality of the initial controller in the closed loop.

Remark 6 As $\left\| \begin{bmatrix} \hat{Y} \\ \hat{X} \end{bmatrix} \right\|_{\infty} = \alpha$, the performance of the observer degrades much faster with a bigger $\alpha$. However, as it is required that $\alpha < \frac{1}{\gamma}$, the growth speed of error in $\mathcal{H}_2$ cost is limited - although with the robustness radius $\gamma$ increasing, $\frac{\gamma}{1 - \gamma \alpha}$ diverges eventually.
We integrate the above results with the system identification guarantees of Zhang et al. (2022), to provide end-to-end sample complexity bounds for learning the linear observers of an unknown system. Then following the system identification procedure with probability at least \((1 - \delta)\) where \(\delta\) is the failure probability,

\[
\|[-\Delta_{\hat{M}} - \Delta_{\hat{N}}]\|_\infty \leq \|[X \ Y]\|_{\infty,12c}\beta R \left( \sqrt{\frac{m^2 + pd^2 + d\log(T/\delta)}{T}} \right).
\]

Combining with the robustness radius \(\|[-\Delta_{\hat{M}} - \Delta_{\hat{N}}]\|_\infty < \gamma\), it is necessary to pick a relatively large \(T\) such that the robust observer achieves the error in \(H_2\) cost as in (25).

**Theorem 4.1** Define \(s = 144\|X^{nd} Y^{nd}\|_\infty^2 \beta^2 \mathcal{R}^2\). Then, the error in \(H_2\) cost of applying the control laws \(\hat{u} = F\hat{x}\) and \(u = Fx\) is bounded as in (25) with probability at least \((1 - \delta)\) provided that \(T \geq \max\{T_* \}, \mathcal{D}(T), d_*(T, \delta) \leq 2d_*(\frac{T}{256}, \delta)\), where,

\[
d_*(T, \delta) = \inf\{d|16\beta R \alpha(d) \geq \|\tilde{H}_{0,d,d} - \tilde{H}_{0,\infty}\|_2\}, \mathcal{D}(T) = \{d\in\mathbb{N} | d \leq \frac{T}{cm^2 \log^3(Tm/\delta)}\} \text{ and } f(d) = \sqrt{d} \left( \frac{m + dp + \log(T/\delta)}{T} \right).
\]

Combining Theorem 4.1 with (25), it follows that with high probability the difference \(J_{\hat{u}}\) and \(J_u\) behaves as

\[
J_{\hat{u}} - J_u \sim \mathcal{O}\left( \sqrt{\frac{\log T}{T}} \right) \approx \mathcal{O}\left( \frac{\log T}{T} \right).
\]

Finally, we note here that the resulted sample complexity for learning the linear observers is on par with the existing methods from Tsiamis et al. (2020).

### 4.1 Closed Loop Identification Scheme

Details on the closed-loop identification scheme of a noise contaminated plant \(G^{nd}\) with control input \(u\), noise \(\nu\) (taken \(w = 0\)) and output measurement \(y\) (where \(u\) and \(\nu\) are assumed independent and stationary) is depicted on Figure 2 in (Zhang et al., 2021, Subsection 4.2). The main idea dating back to Anderson (1998) is to identify the stable dual-Youla parameter \(R^{nd}\) rather than \(G^{nd}\) thus recasting the problem in a standard, open-loop identification form. In this way, model uncertainties are additive to the coprime factors of model, not directly on the model. For details on this and identification algorithms, we refer to Appendix G of Zhang et al. (2021).

### 5. Conclusion and Future work

In this paper, we have provided the sample complexity bounds for a observer-based robust LQ regulator synthesis procedure with unknown dynamics, able to cope with uncertainties on coprime factors. We combined finite-time, non-parametric LTI system identification (Sarkar and Rakhlin (2019)) with the Youla parameterization for observer performance evaluation given a fixed state feedback gain under uncertainty on the coprime factors of the plant.

One exciting avenue for future research is the online learning of observer-based LQ control problem under the same type of model uncertainty. One possible direction is to work out the sample complexity of learning: (a) the optimal state feedback (LQR) in tandem with (b) the optimal state-observer (Kalman Filter (Tsiamis et al. (2020))) for a potentially unstable system.
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Appendix

This appendix is divided into following parts. Appendix A presents a brief review of related works on LTI systems with non-asymptotic system identification, observer parameterization, controller parameterization, and robust control in reinforcement learning. Also, a handful of mathematical preliminaries on norm identities and inequalities (Zhou et al. (1996)) are provided here. An overview of the closed loop mapping is stated in Appendix B along with robust synthesis proofs for Lemma 3.3 and Theorem 3.4. This appendix section also completes the suboptimality guarantee proof in Proposition 3.7. Appendix C presents a brief overview on $\mathcal{H}_2$ optimal cost.

Appendix A. Related Works

Recent years have seen a significant amount of research work focused on the finite time (non-asymptotic) learning of the optimal LQ regulator for a "unknown" plant utilizing the modern optimization methods and statistical tools from the learning framework. For related work on the Identification of Dynamical Systems, Controller Design, Robust Control and Optimal Control we refer to the Appendix A of Zhang et al. (2021).

Observer Design: Using the factorization technique for the parameterization of linear observers and associated estimation error dynamics is a classical result. This outcome offers a dual representation of the popular linear controller parameterization and offers fresh information on observer design (Ding et al. (1990)) that may be applied to both robust observer design and observer construction. Ding et al. (1994) outline and address issues with design and parameterization of robust linear observers in the frequency domain. The performance and attributes of advanced state observers are compared in Wang and Gao (2003). These observers were first put out as a solution to the traditional observers’ reliance on a precise mathematical representation of the plant, such as the Kalman filter and the Luenberger observer. Classical approaches to robust Kalman Filtering can be found in Xie and Soh (1994), Einicke and White (1999), Ghaoui and Calafiore (2001), Sayed (2001), Levy and Nikoukhah (2012), where parametric uncertainty is explicitly taken into account during the kalman filter synthesis procedure. The process of designing a Kalman filter for an unknown or partially observed autonomous linear time-invariant system has been discussed in Tsiamis et al. (2020), which
was the first end-to-end sample complexity bounds for an unidentified system’s Kalman filtering. The strategy of Luenberger observer is investigated to suggest a solution to the state and parameter estimation for dynamical systems in Alessandri and Coletta (2001), Kim et al. (2016), Afri et al. (2017), Bernard and Andrieu (2019), Niazi et al. (2022). When the plant dynamics are relatively well understood, the Luenberger observer performs well, but the estimation of the states might not be precise enough in the presence of model perturbations. By utilizing the Lyapunov stability theorem, Gu and Poon (2001) derived a novel resilient observer technique to overcome this problem. More on robust control has been discussed in Zhang et al. (2021).

Norm/Inequality preliminaries: Useful $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm identities and inequalities have been adapted from Zhou et al. (1996) which are essential for the proofs. For more details on this we refer to Appendix B of Zhang et al. (2021).

Appendix B. Details on Proofs

B.1 Closed Loop Details

**Proof for Theorem 2.7**: The proof immediately follow from comparing (10) and (11). Note that the error is guaranteed to converge to the zero vector with respect to k since transfer function matrices $(PM^{-1} - \Psi_2 G)(z)$ and $-\Psi_2(z)$ are stable.

**Proof for Proposition 2.9**: From Theorem 2.7, (10) and (11) we can conclude, 

\[
\begin{align*}
x(z) - \hat{x}(z) &= P(z)M(z)^{-1}w(z) - \Psi_2(z)G(z)w(z) - \Psi_2(z)v(z) \\
&= \left( P(z)M(z)^{-1} - \Psi_2(z)G(z) \right)w(z) - \Psi_2(z)v(z) \\
\end{align*}
\]

(26)

By using values of $\Psi_1(z)$ and $\Psi_2(z)$ from (13), it’s easy to observe that,

\[
\Psi_2(z) + \Psi_2(z)G(z) = P(z)M(z)^{-1}
\]

(27)

Then, the closed loop maps from $w$ and $v$ to $(x - \hat{x})$ denoted by $T_{(x-\hat{x})w}$ and $T_{(x-\hat{x})v}$ respectively gets the following form: error $(x - \hat{x})$ is represented by following:

\[
T_{(x-\hat{x})w} = P(z)M(z)^{-1} - \Psi_2(z)G(z) = \Psi_1(z)
\]

\[
T_{(x-\hat{x})v} = -\Psi_2(z)
\]

**Proof for Proposition 2.10**: The Optimal Linear Observer Problem due to Remark 2 below is:

\[
\min_{Q\text{ stable}} \left\| \begin{bmatrix} FT_{(x-\hat{x})w} & FT_{(x-\hat{x})v} \end{bmatrix} \right\|_{\mathcal{H}_2}
\]
Now, the closed loop maps for the optimal linear control problem above follow from the relationships
\[ \Phi \]
and 
\[ FT_{(x-z)w} = F \Psi_1(z) = F(P(z)Y(z) + S(z)\tilde{N}(z)) \]
\[ = FP(z)Y(z) + FS(z)\tilde{N}(z) \]
\[ = (M(z) - I_m)Y(z) + (Q(z) + \bar{X}(z))\tilde{N}(z) \]
\[ = (M(z)Y(z) + \bar{X}(z)\tilde{N}(z)) - Y(z) + Q(z)\tilde{N}(z) \]
\[ = I_m - Y(z) + Q(z)\tilde{N}(z) \]
\[ FT_{(x-z)u} = -F \Psi_2(z) = -F(P(z)X(z) - S(z)\tilde{M}(z)) \]
\[ = -FP(z)X(z) + FS(z)\tilde{M}(z) \]
\[ = -(M(z) - I_m)X(z) + (Q(z) + \bar{X}(z))\tilde{M}(z) \]
\[ = (-M(z)X(z) + \bar{X}(z)\tilde{M}(z)) + X(z) + Q(z)\tilde{M}(z) \]
\[ = X(z) + Q(z)\tilde{M}(z). \]
Hence, the Optimal Linear Observer Problem have the form:
\[
\begin{align*}
\min_{Q_{stable}} \left\| \begin{bmatrix} FT_{(x-z)w} & FT_{(x-z)u} \end{bmatrix} \right\|_{\mathcal{H}_2} & = \\
\min_{Q_{stable}} \left\| \begin{bmatrix} I_m - Y(z) + Q(z)\tilde{N}(z) & X(z) + Q(z)\tilde{M}(z) \end{bmatrix} \right\|_{\mathcal{H}_2}.
\end{align*}
\]

**B.2 Robust Synthesis Details**

The Bezout identity is retrieved before finding the closed loop maps associated with it by using (18) as below:
\[
\begin{bmatrix}
\Phi_{11}^{-1} & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
\Phi_{11} & 0 \\
0 & \Phi_{22}
\end{bmatrix}
\begin{bmatrix}
I_p & 0 \\
0 & \Phi_{22}^{-1}
\end{bmatrix}
= \\
\begin{bmatrix}
I_p & 0 \\
0 & I_m
\end{bmatrix},
\]
\[
\begin{bmatrix}
\Phi_{11}^{-1} & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
(\bar{M}^{nd} + \Delta_{\bar{M}}) & (\bar{N}^{nd} + \Delta_{\bar{N}}) \\
-\bar{X} & \bar{Y}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} & -(\bar{N}^{nd} + \Delta_{\bar{N}}) \\
\tilde{X} & (\bar{M}^{nd} + \Delta_{\bar{M}})
\end{bmatrix}
\begin{bmatrix}
I_p & 0 \\
0 & \Phi_{22}^{-1}
\end{bmatrix}
= \\
\begin{bmatrix}
I_p & 0 \\
0 & I_m
\end{bmatrix},
\]
\[
\begin{bmatrix}
\Phi_{11}^{-1}(\bar{M}^{nd} + \Delta_{\bar{M}}) & \Phi_{11}^{-1}(\bar{N}^{nd} + \Delta_{\bar{N}}) \\
-\bar{X} & \bar{Y}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} & -(\bar{N}^{nd} + \Delta_{\bar{N}})\Phi_{22}^{-1} \\
\tilde{X} & (\bar{M}^{nd} + \Delta_{\bar{M}})\Phi_{22}^{-1}
\end{bmatrix}
= \\
\begin{bmatrix}
I_p & 0 \\
0 & I_m
\end{bmatrix}. \quad (28)
\]

**Proof for Lemma 3.3:** From DCF matrix \( \Phi \) in (18), \( \Phi_{11} = (\bar{M}^{nd} + \Delta_{\bar{M}})\tilde{Y} + (\bar{N}^{nd} + \Delta_{\bar{N}})\tilde{X} \) and 
\( \Phi_{22} = X(\bar{N}^{nd} + \Delta_{\bar{N}}) + Y(\bar{M}^{nd} + \Delta_{\bar{M}}) \). Next using Bezout identity for nominal model in (16) it follows that
\[
\Phi_{11} = \\
\begin{bmatrix}
(\bar{M}^{nd} + \Delta_{\bar{M}}) & (\bar{N}^{nd} + \Delta_{\bar{N}}) \\
-\bar{X} & \bar{Y}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} \\
\tilde{X}
\end{bmatrix}
+ \\
\begin{bmatrix}
\Delta_{\bar{M}} & \Delta_{\bar{N}}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} \\
\tilde{X}
\end{bmatrix}
= I_p + \\
\begin{bmatrix}
\Delta_{\bar{M}} & \Delta_{\bar{N}}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y} \\
\tilde{X}
\end{bmatrix};
\]

| 15 |
\[ \Phi_{22} = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} (N^{nd} + \Delta_N) \\ (M^{nd} + \Delta_M) \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} N^{nd} \\ M^{nd} \end{bmatrix} + \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = I_m + \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}. \]

Proof of Theorem 3.4 directly follows from Zhang et al. (2021). Before giving the proof for Theorem 3.4, the small gain theorem is stated here.

**Theorem B.1 (Small Gain Theorem)** (Ionescu et al., 1999, Theorem 7.4.1/ page 225) Let \( G_1 \in \mathbb{R}(z)^{p \times m} \) and \( G_2 \in \mathbb{R}(z)^{m \times p} \) be two TFM’s respectively. If \( \|G_1\|_\infty \leq \frac{1}{\gamma} \) and \( \|G_2\|_\infty \leq \gamma \), for some \( \gamma > 0 \), then the closed loop feedback system of \( G_1 \) and \( G_2 \) is internally stable.

**Proof for Theorem 3.4:** For any stable \( Q \) satisfying

\[ \left\| \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \] it follows that \( \Phi_{11} = \left(I_p + \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right) \) is unimodular (square and stable with a stable inverse) due to the fact that:

(a) The term \( \left(I_p + \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right) \) is stable since all factors are stable and (b) We know that \( \left\| \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \right\|_\infty < \gamma \) from the definition of the Model Uncertainty Set. At the same time \( \left( I_p + \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right)^{-1} \) is guaranteed to be stable via the Small Gain Theorem.

Conversely, if a Youla parameter \( Q \) yields a \( \gamma \)-robustly stabilizable controller of the nominal model then necessarily

\[ \left\| \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \].

The proof of this claim is done by contradiction. Assume that

\[ \left\| \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right\|_\infty > \frac{1}{\gamma} \].

Then by the Spectral Mapping Theorem (Douglas, 1972, page 41-42) there must exist \( \left\| \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \right\|_\infty < \gamma \) such that \( \Phi_{11} = \left(I_p + \begin{bmatrix} \Delta_{\tilde{M}} & \Delta_{\tilde{N}} \end{bmatrix} \begin{bmatrix} \tilde{Y}_Q \\ \tilde{X}_Q \end{bmatrix} \right) \) is not unimodular and consequently the Youla parameter \( Q \) does not produce an \( \gamma \)-robustly stabilizable controller, which is a contradiction. The proof ends.

**Proof for Proposition 3.7:** We have the robust observer cost functional as:

\[
\left\| \begin{bmatrix} I_m - Y + Q\Phi_{11}^{-1} (N^{nd} + \Delta_N) & X + Q\Phi_{11}^{-1} (M^{nd} + \Delta_M) \end{bmatrix} \right\|_{\mathcal{H}_2} \\
\leq \left\| \begin{bmatrix} I_m - Y & X \end{bmatrix} \right\|_{\mathcal{H}_2} + \left\| Q \right\|_{\mathcal{H}_2} \left\| \Phi_{11}^{-1} \begin{bmatrix} N^{nd} + \Delta_N \\ M^{nd} + \Delta_M \end{bmatrix} \right\|_{\infty} \\
\leq \left\| \begin{bmatrix} I_m - Y & X \end{bmatrix} \right\|_{\mathcal{H}_2} + \left\| Q \right\|_{\mathcal{H}_2} \left\| \Phi_{11}^{-1} \right\|_{\infty} \left\| \begin{bmatrix} N^{nd} + \Delta_N \\ M^{nd} + \Delta_M \end{bmatrix} \right\|_{\infty}.
\]
where, $\|\Phi_{11}\|_\infty = \left\| \begin{pmatrix} I_p & -\Delta M & -\Delta \tilde{M} & \tilde{Y} \\ \tilde{X} \end{pmatrix} \right\|_\infty$

\[
\leq \left\| I_p + \sum_{j=1}^{\infty} \left( \begin{pmatrix} -\Delta M & -\Delta \tilde{M} & \tilde{Y} \\ \tilde{X} \end{pmatrix} \right)^j \right\|_\infty \\
= 1 + \sum_{j=1}^{\infty} \gamma^j \left\| \begin{pmatrix} \tilde{Y} \\ \tilde{X} \end{pmatrix} \right\|_\infty^j = 1 + \frac{\gamma\alpha}{1 - \gamma\alpha}, \quad \text{with} \quad \alpha = \left\| \begin{pmatrix} \tilde{Y} \\ \tilde{X} \end{pmatrix} \right\|_\infty.
\]

Hence, $\left\| \begin{pmatrix} I_m - Y \\ X \end{pmatrix} \right\|_{\mathcal{H}_2} + \|Q\|_{\mathcal{H}_2} \frac{1}{1 - \gamma\alpha} \left( \left\| \begin{pmatrix} \tilde{N}_{\text{nd}} & \tilde{M}_{\text{nd}} \end{pmatrix} \right\|_{\infty} + \gamma \right)$ is an upper bound for $\left\| \begin{pmatrix} (I_m - Y) + Q\Phi_{11}^{-1}(\tilde{N}_{\text{nd}} + \Delta N) & X + Q\Phi_{11}^{-1}(\tilde{M}_{\text{nd}} + \Delta \tilde{M}) \end{pmatrix} \right\|_{\mathcal{H}_2}$. □

Appendix C. $\mathcal{H}_2$ Optimal Cost

If the pair $(A, B)$ is controllable and the pair $(A, C)$ is observable, then it is appropriate to introduce the time domain-representation of the $\mathcal{H}_2$ cost as:

\[
J = \sum_{k=1}^{m} \left\{ \sum_{t=0}^{\infty} y_t^T y_t, \quad w_t = e_k \delta_t, \quad \nu_k = 0 \right\}
\]

(29)

Here in (29), $e_k$ represents the $k^{th}$ standard basis vector in $\mathbb{R}^m$ and $\delta_t = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$.

The direct feedthrough from $\nu$ to $y$ is assumed to be none in order to obtain a finite $\mathcal{H}_2$ norm for the closed loop system. This characterization of $\mathcal{H}_2$ norm is not usually seen since the motivation of $\mathcal{H}_2$ optimal problem is more naturally stated by average frequency domain characterization, but it would explain the role of difference in control signals here.

The optimal state feedback gain is denoted as $F^{\text{opt}}$. From the well-known Riccati theory, $F^{\text{opt}} = -(D^T D)^{-1}B^T S$ with $S$ being the unique symmetric semidefinite solution to the Algebraic Riccati Equation (ARE):

\[
A^T S + SA - SB(D^T D)^{-1}B^T S + C^T C = 0
\]

(30)

Furthermore, the matrix $A + BF^{\text{opt}}$ is stable.

Denote the optimal $\mathcal{H}_2$ control signal as $u_t^{\text{opt}} = F^{\text{opt}} x_t$, then it’s possible to rewrite the $\mathcal{H}_2$ cost as the following when applying a certain control $u_t$:

\[
J_{u_t} = \sum_{k=1}^{m} \left\{ \sum_{t=0}^{\infty} (u_t - u_t^{\text{opt}})^T D^TD(u_t - u_t^{\text{opt}}) + e_k^T B^T S e_k; \quad w_t = e_k \delta_t \right\}
\]

\[
= \sum_{k=1}^{m} \left\{ \sum_{t=0}^{\infty} (u_t - u_t^{\text{opt}})^T D^TD(u_t - u_t^{\text{opt}}); \quad w_t = e_k \delta_t \right\} + \text{tr}(B^T S B)
\]

In our settings, the control signal used is $\hat{u}_t = F\hat{x}$, where $\hat{x}$ is generated by the designed observer such that it can be written as following:

\[
J_{\hat{u}_t} = \sum_{k=1}^{m} \left\{ \sum_{t=0}^{\infty} (\hat{u}_t - u_t^{\text{opt}})^T D^TD(\hat{u}_t - u_t^{\text{opt}}); \quad w_t = e_k \delta_t \right\} + \text{tr}(B^T S B)
\]
Without loss of generality, it can be assumed that $D^T D = I$ for convenience. It can be observed that the term $((\hat{u}_t - u_t^{\text{opt}}))^T ((\hat{u}_t - u_t^{\text{opt}}))$ is non-negative and follows triangle inequality such that:

$$J_{\hat{u}_t} \leq \sum_{k=1}^{m} \left( \sum_{t=0}^{\infty} [(Fx_t - F\hat{x}_t)^T (Fx_t - F\hat{x}_t)]; \ w_t = e_k \delta_t \right) + \sum_{k=1}^{m} \left( \sum_{t=0}^{\infty} [(u_t^{\text{opt}} - Fx_t)^T (u_t^{\text{opt}} - Fx_t)]; \ w_t = e_k \delta_t \right) + \text{tr}(B^T S B) \quad (31)$$

The $\mathcal{H}_2$ cost of applying the control $u_t = F\hat{x}_t$ is upperbounded as the above. Note that the third term is definitive and the second term is fixed as we are not able to change $F$. Then the only thing we seek to minimize is the first term, equivalently $\|Fx(z) - F\hat{x}(z)\|_2^2$.

A possible future topic for this part is that it is reasonable to reduce $\|u_t^{\text{opt}} - Fx_t\|_2^2$ to compress the upperbound of $J_{\hat{u}_t}$ by learning the optimal feedback gain $F^{\text{opt}}$. To do so, a precise identification of system parameters $A, B, C, D$ is necessary. There already exists such algorithms as in Tsiamis and Pappas (2019) and Sarkar and Rakhlin (2019).

Note that if the optimal state feedback gain $F^{\text{opt}}$ is given, then the second term in the right hand side of inequality (31) is gone. In this case, by expressing the optimal $\mathcal{H}_2$-cost as $\text{tr}(B^T S B)$, our objective function in Theorem 3.6 is directly the error in $\mathcal{H}_2$-cost comparing to the optimal LQG controller for the true plant. Solving it yields a robust LQG controller such that its performance is guaranteed by (25). This provides a new perspective on robust controller design via an observer approach, in parallel with our previous work Zhang et al. (2022).