Characteristics of interaction between Gravitons and Photons

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Abstract. The direct detection of gravitational waves from binary mergers has been hailed as the discovery of the century. In the light of recent evidence on the existence of gravitational waves, it is now possible to extract features about matter under extreme conditions and properties of different dynamical spacetimes. In LIGO, gravitational waves were detected using laser interferometry by measuring the spacetime distortion between the hanging and stationary mirrors when the gravitational waves passed by. A number of alternate ways of detecting gravitational waves through electromagnetic counterparts have been suggested. Here, we characterize the interaction between photons and gravitons, quantas of gravitational waves in low-energy theories of gravity, through an effective action of interacting photon degrees of freedom. This could open an alternate possibility to extract information from astrophysical objects indirectly.
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1 Introduction

Existence of gravitation waves is one of the most important features of Einstein’s theory of General relativity (GR) and was first predicted by Einstein. Discovery of gravitational waves can be seen as a test to verify GR and also put constraints on alternate theories of gravity. Its discovery after a hiatus of almost a century from its theoretical prediction is because of the extreme sensitivities of the measurements involved. LIGO confirmed the first detection [1–6] through laser optical interferometry which is very useful for probing signals that are significant at the level of atomic scale.

Among the four fundamental interactions in nature, the electromagnetic and gravitational interactions are long-ranged and mediated by a spin-1 massless photon and a spin-2 massless graviton, respectively. Gravity becomes as strong as the other forces at the Planck scale, hence, understanding the gravitational interactions at the quantum level is believed to be crucial for a consistent description of the birth and evolution of the Universe according to the Big Bang theory.

The success of LIGO in detecting gravitational waves motivated suggestions for utilizing optical measurement techniques which would incur lesser expense [7–10, 10–15]. To make optical measurements a knowledge of interaction of gravitational waves (GW) with light would be required. This question has been first attempted in [16, 17] where different scattering processes between gravitons and photons were studied [18, 19]. However, the problem with measuring such scattering amplitudes or cross-section is that their numerical values are extremely small. Measurement of a single graviton is difficult with current technologies [20], [21]. Nevertheless, it can capture useful physical information of spacetime which could be probed through weak measurement techniques, as will be shown in this article.

Here we show how certain features of graviton-photon interactions can be accessed through optical measurements that indirectly confirm features of spacetime carried by gravitational waves. Throughout our discussion we will also comment on massive gravity theory which is an alternate theory of GR, motivated for solving the problem of Dark Energy and current accelerated expansion of universe, among others. In massive gravity theory [22, 23, 23] (which are also ghost-free [24]), infrared (IR) region of GR is modified by the addition of a mass term leading to gravitons becoming massive and spin-2.

The extremely small numerical values of scattering amplitudes of the graviton-photon interactions would suggest that a tool to amplify the signals involved would be very welcome. Such a scenario is facilitated by recent developments in the field of quantum optics, in particular the weak measurement technique [25–31]. Weak measurements is the name coined to a measurement scenario in quantum mechanics, wherein the empirically measured value (called the weak value) of an observable can yield results beyond the eigenvalue spectrum of the measured observable. This has lead to a number of interesting developments including weak value amplification, of relevance to the present work, which could be useful for enhancing the sensitivity of specific detection schemes.

We briefly discuss the weak field limit of general relativity, followed by the Fierz-Pauli action of massive gravity and Stueckelberg’s technique for restoring gauge symmetry to massive gravity action. This sets the scene for the construction of an effective action for interacting photons in a non-perturbative manner that takes into account the interactions between photons and gravitons. This is followed by some non-trivial features of on-shell equations obtained from the minimization of the effective action. A few scattering amplitudes are next computed between photonic states and it is shown that through weak measurement protocol
these amplitudes can be amplified. Finally, 1-loop quantum corrections are taken into account in order to write quantum effective action for the interacting photons, explicitly at the quadratic level. Further, effective interacting vertices at the quantum level between photons are obtained.

2 Weak field limit of General Relativity

2.1 Free-field theory of massless gravitons

GR can be derived from Einstein-Hilbert action, invariant under the symmetry group of diffeomorphisms:

\[ d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x \]

\[ g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \] (2.1)

\[ \Rightarrow \sqrt{-g'(x')} = \left| \frac{\partial x}{\partial x'} \right| \sqrt{-g(x)}. \]

When we focus on a linearized theory, \( g_{\mu\nu} \) can be thought of as associated with a massless spin-2 particle in a Minkowski background, in accordance with Wigner’s classification of relativistic particle representations of the Poincare group. Therefore, an analysis of the linearized diffeomorphism invariant theory of gravity is equivalent to discuss the properties of a massless spin-2 particle.

The linearized version of symmetry group can be obtained by looking at infinitesimal coordinate transformations leading to,

\[ g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \xi^\rho \partial_\rho g_{\mu\nu}(x) - g_{\rho\nu} \partial_\mu \xi^\rho - g_{\rho\mu} \partial_\nu \xi^\rho + \mathcal{O}(\xi^2). \] (2.2)

If \( h_{\mu\nu} \) is denoted as a first order correction to the metric \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), then \( h_{\mu\nu} \) transforms under coordinate transformations as

\[ h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \mathcal{O}(\xi^2), \] (2.3)

with just the first order expansion term and the higher order terms being neglected. Therefore, under the following transformation,

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \] (2.4)

the Einstein-Hilbert action in its linearized form must be invariant.

The Einstein-Hilbert action is:

\[ S_{EH} = \frac{1}{\kappa^2} \int \mathcal{R}[g] \sqrt{-g} d^4x, \] (2.5)

where \( \kappa \) is a dimension full quantity which is mass\(^{-2} \) in unit of \( \hbar = c = 1. \)

Note that

\[ \sqrt{-g} = 1 + \frac{1}{2} h_\alpha^\alpha + \mathcal{O}(h^2), \] (2.6)

while the first order term in Christoffel symbols are

\[ \Gamma^\rho_{\sigma\mu} = \frac{1}{2} (\partial_\sigma h_{\mu}^\rho + \partial_\mu h_{\sigma}^\rho - \partial^\rho h_{\mu\sigma}). \] (2.7)
The Ricci tensor \( R_{\mu\nu} \) is, thus,
\[
R_{\mu\nu} = \left( \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} \right)
= \frac{1}{2} \left( \partial_\rho \partial_\mu h^\rho_{\nu} + \partial_\rho \partial_\nu h^\rho_{\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h \right). \tag{2.8}
\]

Using the fact \( g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} \), we have following expression
\[
\sqrt{-g} R[g] = \left( 1 + \frac{1}{2} h \right) (\eta^\mu - h^\mu) \frac{1}{2} (\partial_\rho \partial_\mu h^\rho_{\nu} + \partial_\rho \partial_\nu h^\rho_{\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h) + \left( 1 + \frac{1}{2} h \right) (\partial_\rho \partial_\nu h^{\nu \rho} - \Box h)
\approx -\frac{1}{2} h_{\mu\nu} \partial^\sigma \partial^\rho h_{\sigma\rho} + h \partial^\mu \partial^\nu h_{\mu\nu} + \frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} - \frac{1}{2} h \Box h, \tag{2.9}
\]
where in the third line we have neglected the total derivative terms since they contribute at the boundary.

The weak-field action in presence of matter, considering first order correction to metric is
\[
S_{\text{EH}} = \int d^4 x L_{\text{EH}} + S_M,
\]
\[
L_{\text{EH}} = -h^\mu_\sigma \partial^\sigma \partial^\nu h_{\mu\nu} + h \partial^\mu \partial^\nu h_{\mu\nu} + \frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} - \frac{1}{2} h \Box h, \tag{2.10}
\]
where one can explicitly check the invariance under the transformation (2.4).

The Lagrangian can be recast as
\[
L_{\text{EH}} = \frac{1}{2} h_{\mu\nu} O^{\mu\nu\rho\sigma} h_{\rho\sigma}
\]
\[
O^{\mu\nu\rho\sigma} = \left( \frac{1}{2} \eta^{\mu\sigma} \eta^{\nu\rho} + \frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma} \right) \Box + \eta^{\mu\nu} \partial^\rho \partial^\sigma + \eta^{\rho\sigma} \partial^\mu \partial^\nu
- \frac{1}{2} \left( \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\mu\sigma} \partial^\rho \partial^\nu + \eta^{\mu\nu} \partial^\sigma \partial^\rho + \eta^{\rho\sigma} \partial^\nu \partial^\mu \right), \tag{2.11}
\]
where it satisfies the following symmetries
\[
O^{\mu\nu\rho\sigma} = O^{\nu\mu\rho\sigma} = O^{\mu\sigma\rho\nu} = O^{\rho\sigma\mu\nu}. \tag{2.12}
\]

The corresponding equation of motion becomes
\[
O^{\mu\nu\rho\sigma} h_{\rho\sigma} = 8\pi G T^{\mu\nu}, \tag{2.13}
\]
which implies \( \partial_\mu O^{\mu\nu\rho\sigma} = 0 \). Therefore, in momentum space the operator \( O^{\mu\nu\rho\sigma} \) is not invertible and hence, we can’t have a corresponding Green’s function. But that is expected because this is a gauge invariant theory and the gauge has not yet been fixed which will be done next.
2.2 Gauge-fixing

We introduce the following gauge fixing term, de Donder gauge [23],

\[ L_{GF} = -\frac{1}{\alpha}(\partial_\rho h^\rho - \frac{1}{2}\partial_\mu h)(\partial_\sigma h^{\mu\sigma} - \frac{1}{2}\partial^\mu h), \]  

(2.14)

where \( \alpha \) is known as gauge parameter.

With this new term the weak field lagrangian density becomes

\[ \bar{L}_{EH} = L_{EH} + L_{GF}, \]

\[ L_{GF} = \frac{1}{2}h_{\mu\nu}O_{GF}^{\mu\nu\rho\sigma}h^{\rho\sigma}, \]

where

\[ O_{GF}^{\mu\nu\rho\sigma} = \frac{1}{\alpha}\left(2\eta^{\mu\rho}\partial^\sigma - \eta^{\rho\sigma}\partial^\mu - \eta^{\mu\nu}\partial^\rho + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\Box\right), \]

(2.15)

\[ \Rightarrow \bar{L}_{EH} = \frac{1}{2}h_{\mu\nu}(O + O_{GF})^{\mu\nu\rho\sigma}h^{\rho\sigma}. \]

Define

\[ \tilde{\mathcal{O}} = O + O_{GF}, \]

\[ \Rightarrow \tilde{\mathcal{O}}^{\mu\nu\rho\sigma} = \left(\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma} + \frac{1}{2}\eta^{\mu\sigma}\eta^{\nu\rho} - \left(1 - \frac{1}{2\alpha}\right)\eta^{\mu\nu}\eta^{\rho\sigma}\right)\Box + \left(1 - \frac{1}{\alpha}\right)(\eta^{\mu\nu}\partial^\rho - \eta^{\rho\sigma}\partial^\mu - \eta^{\mu\nu}\partial^\rho + \eta^{\rho\sigma}\partial^\mu) + \frac{1}{8}\left(1 - \frac{1}{\alpha}\right)\eta^{\mu\nu}\partial^\rho - \eta^{\rho\sigma}\partial^\mu + \eta^{\nu\rho}\partial^\mu + \eta^{\nu\sigma}\partial^\rho + \eta^{\sigma\rho}\partial^\mu + \eta^{\sigma\nu}\partial^\rho). \]

(2.16)

Therefore, after choosing Feynman gauge \( \alpha = 1 \), in momentum space the Green’s function takes the following form

\[ \Pi_{GR,\mu\nu\rho\sigma} = -\frac{1}{k^2}(\eta_{\mu\rho}\eta^{\nu\sigma} + \eta_{\mu\sigma}\eta^{\nu\rho} - \eta_{\mu\nu}\eta^{\rho\sigma}). \]

(2.17)

From now onwards \( \tilde{\mathcal{O}} \) will be denoted as \( \mathcal{O} \) for sake of convenience. Gauge fixing could also have been achieved using the Fadeev-Popov Ghost [32], [33] method in the path integral formalism.

3 Massive Gravity

3.1 Free Fierz-Pauli Action

The unique action that describes a massive spin-2 particle in flat spacetime in which field is described by a symmetric rank-2 tensor is

\[ S = \int d^Dx \left[ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^{\lambda} h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\rho h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^{\lambda} h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - \delta_{\mu\nu}h^2) \right], \]

(3.1)

known as the Fierz-Pauli action [34–36]. Note that when \( m = 0 \) this becomes the linearized Einstein-Hilbert action, invariant under the following gauge transformation

\[ \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \]

(3.2)

The above action is not gauge invariant, but will be made so by using Stucklberg’s trick [37], [38].
3.2 Equations of motion and degrees of freedom

From the above action, the equation of motion can be seen to be
\[
\Box h_{\mu\nu} - \partial_{\lambda}\partial_{\mu}h_{\lambda\nu} - \partial_{\lambda}\partial_{\nu}h_{\lambda\mu} + \eta_{\mu\nu}\partial_{\sigma}\partial_{\sigma}h - \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\Box h = m^2(h_{\mu\nu} - \eta_{\mu\nu}h). \tag{3.3}
\]

The L.H.S of the above equation consists of the linearized form of the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \) and has zero divergence. Hence by acting \( \partial_{\mu} \) on eq. (3.3), we get
\[
m^2(\partial_{\mu}h_{\mu\nu} - \partial_{\nu}h) = 0. \tag{3.4}
\]
Assuming \( m \neq 0 \) gives
\[
\partial_{\mu}h_{\mu\nu} = \partial_{\nu}h. \tag{3.5}
\]
Plugging eqn. (3.5) into eqn. (3.3) gives us
\[
2h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h = m^2(h_{\mu\nu} - \eta_{\mu\nu}h). \tag{3.6}
\]
Taking trace of the above equation gives us
\[
\Box h - \Box h = -3m^2h = 0 \implies h = 0, \tag{3.7}
\]
which means \( h_{\mu\nu} \) is traceless and transverse. Further, using traceless and transverse property of \( h_{\mu\nu} \) we get following equation of motion
\[
(\Box - m^2)h_{\mu\nu} = 0. \tag{3.8}
\]

The equations of motion give us 10 wave equations and 5 constraints. Therefore, in \( D = 4 \) dimension we have 5-degrees of freedom which implies that these degrees of freedom are nothing but massive, spin-2 gravitons.

3.3 Propagator

To find the propagator of massive gravitons, we need to first write down Fierz-Pauli action in the following form
\[
S = \int d^4x \frac{1}{2}h_{\mu\nu}O^{\mu\nu\alpha\beta}h_{\alpha\beta}. \tag{3.9}
\]
It can be shown that
\[
O^{\mu\nu\alpha\beta}_{\alpha\beta} = (\eta^{\alpha\mu}\eta^{\nu\beta} - \eta^{\mu\nu}\eta_{\alpha\beta})(\Box - m^2) - 2\partial^{[\mu}\partial_{(\alpha}\eta^{\nu]} + \partial^{\mu}\partial^\nu\eta_{\alpha\beta} + \partial_{\alpha}\partial_{\beta}\eta^{\mu\nu}. \tag{3.10}
\]
Therefore the propagator, denoted by \( D_{\alpha\beta,\sigma\lambda} \) is defined in following way
\[
O^{\mu\nu,\alpha\beta}D_{\alpha\beta,\sigma\lambda} = \frac{i}{2}(\delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda} + \delta^{\nu}_{\sigma}\delta^{\mu}_{\lambda}). \tag{3.11}
\]
Then one can check that in momentum space the propagator takes the following form
\[
D_{\alpha\beta,\sigma\lambda} = -\frac{i}{p^2 + m^2}\left[\frac{1}{2}(P_{\alpha\sigma}P_{\beta\lambda} + P_{\alpha\lambda}P_{\beta\sigma}) - \frac{1}{3}P_{\alpha\beta}P_{\sigma\lambda}\right], \tag{3.12}
\]
where
\[
P_{\alpha\beta} = \eta_{\alpha\beta} + \frac{p_{\alpha}p_{\beta}}{m^2}. \tag{3.13}
\]
Notice that in high energy limit i.e., large momenta, we have
\[
D_{\alpha\beta,\sigma\lambda} \simeq -\frac{1}{p^2 + m^2} \frac{p_{\alpha}p_{\beta}p_{\sigma}p_{\lambda}}{m^4} \simeq -\frac{p^2}{m^4}, \tag{3.14}
\]
which implies that standard power counting rules are no longer valid and we can’t talk about renormalizability of this theory. This is not true and can be shown explicitly using Stueckelberg’s trick.
4 Stueckelberg’s trick

Here we will apply the technique introduced by Stueckelberg [23, 37, 39] to sourced massive gravity action in order to restore gauge symmetry. For the sake of simplicity, we rewrite the Lagrangian density with the massless term written separately

\[ S = \int d^4x \left[ \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) + \kappa h_{\mu\nu} T^{\mu\nu} \right]. \] (4.1)

Massless gravitons have gauge symmetry which is broken due to the presence of mass term in the above action. We introduce a new auxiliary field \( V_\mu \), known as Stueckelberg field in the same way as a gauge transformation

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu. \] (4.2)

Note that under this transformation \( \mathcal{L}_{m=0} \) remains invariant since, it is invariant under gauge transformation with the same structure as eq. (3.2). However, the other terms do change and we get

\[ S = \int d^4x \left[ \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 \bar{F}_{\mu\nu} F^{\mu\nu} + 2m^2 (h_{\mu\nu} \partial_\mu V_\nu - h \partial_\mu V_\mu) \right]. \] (4.3)

where we have analogous to the electromagnetic field strength tensor

\[ \bar{F}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu. \] (4.4)

Note that the above action has the following gauge symmetry

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu, \quad V_\mu \rightarrow V_\mu - \xi_\mu. \] (4.5)

We can fix this to \( V_\mu = 0 \) and recover the original action. Therefore, both the actions (4.1) and (4.3) are equivalent. But notice that if we try to take \( m = 0 \) limit it does not go smoothly because one degree of freedom is lost. Hence, we need to make a similar kind of transformation again

\[ V_\mu \rightarrow V_\mu + \partial_\mu \phi. \] (4.6)

With this transformation, the action (4.3) becomes

\[ S = \int d^4x \left[ \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 \bar{F}_{\mu\nu} F^{\mu\nu} - 2m^2 (h_{\mu\nu} \partial_\mu V_\nu - h \partial_\mu V_\mu) \right. \]

\[ \left. - 2m^2 (h_{\mu\nu} \partial_\mu \partial_\nu \phi - h \Box \phi) + \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa V_\mu \partial_\nu T^{\mu\nu} + 2\kappa \phi \partial_\mu \partial_\nu T^{\mu\nu} \right]. \] (4.7)

The resultant action has following two separate gauge symmetries

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad V_\mu \rightarrow V_\mu - \xi_\mu; \]
\[ V_\mu \rightarrow V_\mu + \partial_\mu \Lambda, \quad \phi \rightarrow \phi - \Lambda. \] (4.8)
We can again fix the gauge \( \phi = 0 \) and recover back the action in (4.3), which implies action (4.7) is equivalent to action (4.3). Hence, with new additional fields and gauge symmetries, the new action does the same job as the original one, (4.1).

We now make the following set of scalings

\[
V_\mu \to \frac{1}{m} V_\mu, \quad \phi \to \frac{\phi}{m^2},
\]

(4.9)
such that the action takes the following form

\[
S = \int d^4x \left[ \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 2m(h_{\mu\nu} \partial^\mu \phi^{\nu} - h \partial_\mu \phi^\nu) \\
- 2(h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \Box \phi) + \kappa h_{\mu\nu} T^{\mu\nu} - 2 \frac{\kappa}{m} V_\mu \partial_\nu T^{\mu\nu} + 2 \frac{\kappa}{m^2} \phi \partial_\mu \partial_\nu T^{\mu\nu} \right],
\]

(4.10)
and is equipped with the following gauge symmetries

\[
h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad V_\mu \to V_\mu - m \xi_\mu
\]

\[
V_\mu \to V_\mu + m \partial_\mu \Lambda, \quad \phi \to \phi - m^2 \Lambda.
\]

(4.11)
At this point, assuming that the source is conserved, i.e., \( \partial_\mu T^{\mu\nu} = 0 \) and \( m \to 0 \), we get

\[
S = \int d^4x \left[ \mathcal{L}_{m=0} - \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 2(h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \Box \phi) + \kappa h_{\mu\nu} T^{\mu\nu} \right].
\]

(4.12)
In order to count the total number degrees of freedom we now perform an infinitesimal conformal transformation

\[
(\eta_{\mu\nu} + h_{\mu\nu}) = \Omega(\eta_{\mu\nu} + h'_{\mu\nu})
\]

\[
= (1 + \Pi)(\eta_{\mu\nu} + h'_{\mu\nu})
\]

\[
= \eta_{\mu\nu} + h'_{\mu\nu} + \Pi \eta_{\mu\nu}
\]

(4.13)
where \( \Pi \) is another scalar, therefore, this is nothing but a redefinition of the field \( h_{\mu\nu} \). Under this change, the massless Lagrangian \( \mathcal{L}_{m=0} \) becomes

\[
\mathcal{L}_{m=0}[h] = \mathcal{L}_{m=0}[h'] - \partial_\lambda \Pi \partial^\lambda h' - 2 \partial_\lambda \Pi \partial^\lambda \Pi + 2 \partial_\mu \Pi \partial_\lambda h^{\mu\lambda} + \partial_\mu \Pi \partial^\mu \Pi - \partial_\mu \Pi \partial^\mu h'
\]

\[
- 4 \partial_\mu h^{\mu\nu} \partial_\nu \Pi - 4 \partial_\mu \Pi \partial^\mu \Pi + 4 \partial_\lambda \Pi \partial^\lambda h' + 8 \partial_\lambda \Pi \partial^\lambda \Pi + \kappa \Pi T
\]

(4.14)
and the action in (4.12) takes the following form

\[
S = \int d^4x \left[ \mathcal{L}_{m=0}[h'] + 2 \left[ \partial_\mu \partial^\mu h' - \partial_\nu h^{\mu\nu} \partial_\nu \Pi + \frac{3}{2} \partial_\mu \Pi \partial^\mu \Pi \right] + \frac{3}{2} \partial_\mu \Pi \partial^\mu \Pi \right]
\]

(4.15)
\[
- \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 2(h'_{\mu\nu} \partial^\mu \partial^\nu \phi - h' \Box \phi) + 6 \Pi \Box \phi + \kappa h'_{\mu\nu} T^{\mu\nu} + \kappa \Pi T.
\]

Considering \( \Pi = \phi \) makes all coupled tensor-scalar terms cancel. Performing a couple of integration by parts we arrive at

\[
S = \int d^4x \left[ \mathcal{L}_{m=0}[h'] - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi + \kappa h'_{\mu\nu} T^{\mu\nu} + \frac{1}{\sqrt{6}} \phi' T \right],
\]

(4.16)
with gauge symmetries

\[
h'_{\mu\nu} \rightarrow h'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu
\]
\[
V'_\mu \rightarrow V'_\mu + \partial_\mu \Lambda,
\]

where \( V_\mu \rightarrow V'_\mu = \sqrt{2} V_\mu \), \( \phi \rightarrow \phi' = \sqrt{\frac{2}{3}} \phi \).

Now it is easy to count that in \( D = 4 \) dimensions, there is 1 massless graviton which possesses 2 degrees of freedom, 1 massless vector field which also possesses 2 degrees of freedom and 1 massless scalar, in total making 5 degrees of freedom.

If we now go back to massive action (4.10) and do the conformal transformation (4.13), we will get the following action

\[
S = \int d^4 x \left[ \mathcal{L}_{m=0}[h'] - \frac{1}{2} m^2 (h'_{\mu\nu} h'^{\mu\nu} - h'^2) - \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + 3 \tilde{\phi} (\Box + 2 m^2) \phi \right.
\]
\[
- 2 m (h'_{\mu\nu} \partial^\mu V'^{\nu} - h' \partial_\mu V'^{\mu}) + 3 (2 m \phi \partial_\mu V'^{\mu} + m^2 h' \phi) + \kappa h'_{\mu\nu} T^{\mu\nu}
\]
\[
+ \kappa \phi T - \frac{2}{m} \kappa V_\mu \partial_\nu T^{\mu\nu} + \frac{2}{m^2} \kappa \phi \partial_\mu \partial_\nu T^{\mu\nu} \right].
\]

The gauge symmetries now reads

\[
\delta h'_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + m \Lambda \eta_{\mu\nu}, \quad \delta V_\mu = -m \xi_\mu + \partial_\mu \Lambda
\]
\[
\delta \phi = m \Lambda.
\]

We add two gauge fixing terms to the action

\[
S_{GF1} = - \int d^4 x \left( \partial^\rho h'_{\rho\mu\nu} - \frac{1}{2} \partial_\mu h' + m V_\mu \right)^2,
\]
\[
S_{GF2} = - \int d^4 x \left( \partial_\mu V'^{\mu} + \frac{1}{2} m (h' + 3 \phi) \right)^2.
\]

Introducing these specific gauge fixing terms make the action diagonal as

\[
S + S_{GF1} + S_{GF2} = \int d^4 x \left[ \frac{1}{2} h'_{\mu\nu} (\Box - m^2) h'^{\mu\nu} - \frac{1}{4} h' (\Box - m^2) h' + V_\mu (\Box - m^2) V'^{\mu} + 3 \tilde{\phi} (\Box - m^2) \phi
\]
\[
+ \kappa h'_{\mu\nu} T^{\mu\nu} + \kappa \phi T - \frac{2}{m} \kappa V_\mu \partial_\nu T^{\mu\nu} + \frac{2}{m^2} \kappa \phi \partial_\mu \partial_\nu T^{\mu\nu} \right].
\]

For transverse and traceless energy-momentum tensor, the last three terms of the above action vanishes. This makes vector and scalar degrees of freedom completely decoupled from interaction with matter. The propagators of \( h'_{\mu\nu}, V_\mu, \phi \) in momentum space are now

\[
- \frac{i}{p^2 + m^2} \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}),
\]
\[
- \frac{i}{p^2 + m^2} \frac{1}{2} \eta_{\mu\nu}, \quad - \frac{i}{6 (p^2 + m^2)},
\]

respectively. They all behave as \( \frac{1}{p^2} \) for large momenta, implying that standard power counting arguments are now applicable.
5 Photon-Graviton interaction

5.1 Introduction

Consider a source whose stress-energy tensor $T_{\mu\nu}^{(c)}$ produces gravitational waves (GW) that travel through spacetime to asymptotically flat spacetime and interacts with a medium of photons. Our expectation is that the interaction between photons and gravitons captures the properties of the original source of GW.

The action for such a system would be (follows from eqs. (2.10), (2.15))

$$S = S_{m=0}^{(\text{spin}=2)} + S_{GF} + S_{\text{photon}}$$

$$= \int d^4x \left[ \frac{1}{2} h_{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h_{\alpha\beta} + \kappa h_{\mu\nu} T^{(s=1)}_{\mu\nu} \right]$$

$$= \int d^4x \left[ \frac{1}{2} h_{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h_{\alpha\beta} + \kappa h_{\mu\nu} T^{(c)}_{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

where $\kappa = \sqrt{\frac{8\pi G}{c}}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $T^{(s=1)}_{\mu\nu}$ is the stress-energy tensor of photons.

Therefore, the generating functional can be written as

$$Z[J^{\mu\nu} = 0] = \int \mathcal{D}h_{\mu\nu} \mathcal{D}A_\mu e^{i \int d^4x \left[ \frac{1}{2} h_{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h_{\alpha\beta} + \kappa h_{\mu\nu} T^{(s=1)}_{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \kappa h_{\mu\nu} T^{(s=1)}_{\mu\nu} \right]}$$

$$= \mathcal{N} \int \mathcal{D}A_\mu e^{i \int d^4x \left[ \frac{1}{2} \kappa^2 (T^{(c)}_{\mu\nu} + T^{(s=1)}_{\mu\nu}) D_{\mu\nu,\alpha\beta} T^{(s=1)}_{\alpha\beta} \right] + \frac{1}{2} \kappa^2 (T^{(c)}_{\mu\nu} + T^{(s=1)}_{\mu\nu}) D_{\mu\nu,\alpha\beta} T^{(s=1)}_{\alpha\beta} \right]}$$

Therefore, by integrating out graviton degrees of freedom we get an effective action for photon degrees of medium

$$S^{(s=1)}_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \kappa^2 T^{(c)}_{\mu\nu} D_{\mu\nu,\alpha\beta} T^{(s=1)}_{\alpha\beta} \right]$$

As can be seen from the above equation, the 3rd piece is purely an interacting term, taking into account the effective interaction between photons. For the time being, the interaction term is neglected; which is justified for weak gauge fields, the action involves only quadratic or free part and reduces to

$$S^{(1)}_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \kappa^2 T^{(c)}_{\mu\nu} D_{\mu\nu,\alpha\beta} T^{(s=1)}_{\alpha\beta} \right]$$

where (for massless gravitons)

$$T^{(s=1)}_{\mu\nu} = \eta_{\alpha\beta} F^{\alpha\mu} F^{\beta\nu} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

$$D_{\mu\nu,\alpha\beta} = \frac{1}{2\kappa^2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta})$$
Therefore,

\[ T^{(c)}_{\mu\nu}D_{\mu\alpha\beta}T^{(s=1)\alpha\beta} = T^{(c)}_{\mu\nu} \frac{1}{2} \left( T^{(s=1)}_{\mu\nu} + T^{(s=1)}_{\mu\nu} - \eta_{\mu\nu} \right) \]

\[ = T^{(c)}_{\mu\nu} \frac{1}{2} T^{(s=1)}_{\mu\nu} \]

\[ = T^{(c)}_{\mu\nu} \left( \eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \]

(5.6)

Note that

\[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (A_\mu \Box A_\nu - A_\mu \partial^\mu \partial^\nu A_\nu) \]

\[ = \frac{1}{2} A_\mu \Box \Theta^{\mu\nu} A_\nu \]

\[ \Theta^{\mu\nu} = \eta^{\mu\nu} - \Box. \]

(5.7)

Therefore,

\[ T^{(c)}_{\mu\nu} \left( \eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \]

\[ \equiv \left( \frac{1}{2} T^{(c)}_{\mu\nu} \right) \left[ \eta^{\alpha\beta} (\partial_\alpha A_\mu \partial_\beta A_\nu - \partial_\alpha A_\mu \partial_\nu A_\beta - \partial_\mu A_\alpha \partial_\beta A_\nu + \partial_\mu A_\alpha \partial_\nu A_\beta) \right. \]

\[ \left. - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right]. \]

(5.8)

5.2 Equations of motion

The Lagrangian density in eq. (5.4) can be written as

\[ \mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A_\nu - \partial_\nu A_\mu \partial^\nu A_\mu) + \kappa^2 \left( \frac{1}{2} T^{(c)\mu
u} \right) \left[ \eta^{\alpha\beta} (\partial_\alpha A_\mu \partial_\beta A_\nu - \partial_\alpha A_\mu \partial_\nu A_\beta \right. \]

\[ \left. - \partial_\mu A_\alpha \partial_\beta A_\nu + \partial_\mu A_\alpha \partial_\nu A_\beta) \right] \]

\[ - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \]. \]

(5.9)
Therefore, corresponding equations of motion are

\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\sigma}
\]

\[
\Rightarrow -\partial_\rho F^{\rho\sigma} + \kappa^2 \partial_\rho \left[ \left( \frac{1}{2} T^{(c)\mu\nu} \right) (\eta^{\alpha\beta} \partial_\beta A_\nu \delta_\alpha^\rho \delta_\mu^\sigma + \eta^{\alpha\beta} \delta_\rho^\mu \partial_\alpha A_\mu - \eta^{\alpha\beta} \partial_\nu A_\beta \delta_\alpha^\rho \delta_\mu^\sigma \right.
\]

\[
- \eta^{\alpha\beta} \partial_\alpha A_\mu \delta_\rho^\mu \delta_\beta^\sigma - \eta^{\alpha\beta} \delta_\rho^\mu \partial_\beta A_\mu - \eta^{\alpha\beta} \partial_\nu A_\alpha \delta_\beta^\rho \delta_\mu^\sigma + \eta^{\alpha\beta} \delta_\rho^\mu \partial_\alpha A_\beta + \eta^{\alpha\beta} \partial_\mu A_\alpha \delta_\beta^\rho \delta_\mu^\sigma \left] \right. - \kappa^2 \partial_\rho \left( \frac{1}{2} T^{(c) F^{\rho\sigma}} \right) = 0
\]

\[
\Rightarrow -\partial_\rho F^{\rho\sigma} + \kappa^2 \left[ \partial_\rho \left( \frac{1}{2} T^{(c)\sigma\nu} \partial^\rho A_\nu \right) + \partial_\rho \left( \frac{1}{2} T^{(c)\sigma\nu} \partial^\rho A_\nu \right) - \partial_\rho \left( \frac{1}{2} T^{(c)\mu\nu} \partial_\mu A_\rho \right)
\]

\[
- \partial_\rho \left( \frac{1}{2} T^{(c)\mu\nu} \partial^\rho A_\nu \right) + \partial_\rho \left( \frac{1}{2} T^{(c)\rho\nu} \partial_\nu A_\sigma \right) + \partial_\rho \left( \frac{1}{2} T^{(c)\rho\nu} \partial_\nu A_\sigma \right) \right] - \kappa^2 \partial_\rho \left( \frac{1}{2} T^{(c) F^{\rho\sigma}} \right) = 0
\]

\[
- \left( 1 + \kappa^2 \frac{1}{2} T^{(c)} \right) \partial_\rho F^{\rho\sigma} + 2\kappa^2 \left[ \partial_\rho \left( \frac{1}{2} T^{(c)\sigma\nu} \partial^\rho A_\nu \right) + \frac{1}{2} T^{(c)\sigma\nu} \partial_\nu A_\sigma \right] - \frac{1}{2} T^{(c)\mu\nu} \partial_\mu \partial_\nu A_\sigma \right] - \kappa^2 \frac{1}{2} \partial_\rho T^{(c) F^{\rho\sigma}} = 0.
\]

(5.10)

This can be rewritten as

\[
- \left( 1 + \kappa^2 \frac{1}{2} T^{(c)} \right) \partial_\rho F^{\rho\sigma} - \kappa^2 \frac{1}{2} \partial_\rho T^{(c) F^{\rho\sigma}}
\]

\[
+ 2\kappa^2 \left[ \partial_\rho \left( \frac{1}{2} T^{(c)\sigma\nu} \partial^\rho A_\nu \right) + \frac{1}{2} T^{(c)\sigma\nu} \partial_\nu A_\sigma \right] - \frac{1}{2} \partial_\rho T^{(c)\sigma\nu} \partial_\nu A_\rho\right]
\]

\[
- \frac{1}{2} T^{(c)\sigma\nu} \partial_\nu A_\sigma - \frac{1}{2} T^{(c)\rho\nu} \partial_\nu A_\sigma + \frac{1}{2} T^{(c)\rho\nu} \partial_\nu A_\sigma = 0,
\]

(5.11)

where

\[
\omega_{\mu\nu} = \partial_\mu \partial_\nu - \square, \quad (5.12)
\]

acts as a projection operator along longitudinal polarization of gauge fields.

### 5.3 Features of equations of motion

The terms under-marked in eqn. (5.11) show contribution of longitudinal modes of photons in the on-shell equation. This is one of the characteristic features hidden in the equations of motion.

The Sun of our solar system can be considered as a standard candle which acts as a source of a medium of photons. Measurement of the longitudinal polarization of light from the Sun would be a signature of the GW interacting with the solar medium. It should be emphasized that, the equations of motion of photons carry the information of different properties of the sources that is captured by the stress-energy tensor of the source, for example, for binary mergers one can get the information such as charge, spin, angular momentum and mass of these compact objects.
Another important feature which would help us in putting constraints on the graviton mass and IR domain of GR is that if we consider massive gravitons [23], the equations of motion simply turn into

\[
- \left(1 + \kappa^2 \frac{1}{\Box + m^2} T^{(c)} \right) \partial_\rho F^\rho\sigma - \kappa^2 \frac{1}{\Box - m^2} \partial_\rho T^{(c)} F^\rho\sigma \\
+ 2\kappa^2 \left[ \frac{1}{\Box - m^2} \partial_\rho T^{(c)} \sigma^\rho \partial^\sigma A_\nu + \frac{1}{\Box - m^2} T^{(c)}_{\sigma^\rho \partial^\sigma A_\nu} - \frac{1}{\Box - m^2} \partial_\rho T^{(c)} \sigma^\rho \partial^\sigma A^\rho \\
- \frac{1}{\Box - m^2} T^{(c)} \sigma^\rho \partial^\rho \omega^\sigma A_\nu - \frac{1}{\Box - m^2} T^{(c)}_{\mu^\rho \partial^\rho \omega^\sigma A_\mu} + \frac{1}{\Box - m^2} T^{(c)}_{\mu^\rho \partial^\rho \omega^\sigma A_\mu} \right] = 0,
\]

(5.13)

where \( m \) is the mass of gravitons which follows from the action in eqns. (5.4, 5.5, 4.21). For photons, the stress-energy tensor satisfies the following two important conditions

\[
\partial_\mu T^{(s=1)}_{\mu\nu} = 0, \quad T^{(s=1)} = 0,
\]

(5.14)

which would kill the last three terms in the action, (4.21). This suggests that in the presence of photons, vector and scalar degrees of freedom do not couple with photon degrees of freedom. Hence, they can be essentially treated as free-fields separately. It follows that these degrees of freedom can be integrated out without having any net effect on the effective action of photons, obtained earlier.

Therefore, matching the data with eqn. (5.13), it would be possible to put constraint on the mass of gravitons which in principle should be smaller than the constraint put by LIGO and others [40–45].

### 5.4 Source free gravitons interact with photons and birefringence

If we now consider the source free gravitons interacting with photons, then \( T^{(c)}_{\mu\nu} = 0 \) in eqn. (5.3), leading to

\[
S_{\text{eff}}^{(s=1)} = \int d^4x \left[ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa^2}{2} T^{(s=1)}_{\mu\nu} D_{\mu\alpha\beta} T^{(s=1)}_{\alpha\beta} \right],
\]

(5.15)

where

\[
T^{(s=1)}_{\mu\nu} \frac{1}{2\Box} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) T^{(s=1)}_{\alpha\beta} \\
= T^{(s=1)}_{\mu\nu} \frac{1}{\Box} T^{(s=1)}_{\mu\nu},
\]

(5.16)

Therefore, the effective action for photons becomes

\[
S_{\text{eff}}^{(s=1)} = \int d^4x \left[ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa^2}{2} T^{(s=1)}_{\mu\nu} \frac{1}{\Box} T^{(s=1)}_{\mu\nu} \right].
\]

(5.17)
Now
\[
T^{(s=1)}_{\mu\nu} = \frac{1}{4} T^{(s=1)}_{\mu\nu} = \left( \eta_{\rho\sigma} F^{\rho\mu} F^{\sigma\nu} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \frac{1}{4} \left( \eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)
\]
\[
= \left( F^{\mu}_{\sigma} F^{\nu}_{\sigma} - \frac{1}{4} \eta^{\mu\nu} \eta_{\rho\sigma} F^{\rho\sigma} \right) \frac{1}{4} \left( \eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} \eta_{\mu\nu} \eta_{\rho\sigma} F^{\rho\sigma} \right)
\]
\[
= \left( \eta^{\alpha\beta} \eta_{\mu\nu} \right) F_{\alpha\mu} F_{\beta\nu}
\]
which makes the action in eqn. (5.17)
\[
S^{(s=1)}_{eff} = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa^2}{2} F^{\mu\nu} \right] (F_{\alpha\mu} F_{\alpha\nu}) = 0.
\]

Equations of motion would be
\[
-\partial_{\mu} F^{\mu\nu} + \frac{\kappa^2}{2} \partial_{\mu} \left[ \left( \frac{\partial}{\partial (\partial_{\mu} A_{\nu})} (F^{\kappa\delta}_{\sigma}) \right) \left( F_{\kappa\beta} F_{\delta\beta} \right) + \frac{\kappa^2}{2} \partial_{\mu} \left[ \left( F_{\kappa\beta} F^{\mu\nu} \right) \left( \frac{\partial}{\partial (\partial_{\mu} A_{\nu})} (F_{\kappa\beta} F_{\delta\beta}) \right) \right] \right] = 0.
\]

which leads to
\[
-\partial_{\mu} F^{\mu\nu} + \frac{\kappa^2}{2} \partial_{\mu} \left[ \left( \eta^{\mu\nu} \delta_{\sigma} - \delta_{\sigma} \eta^{\mu\nu} \right) F^{\delta\sigma}_{\kappa} + F^{\kappa}_{\sigma} \left( \eta^{\delta\sigma} \eta^{\mu\nu} - \eta^{\mu\nu} \eta^{\delta\sigma} \right) \right] \frac{1}{4} \left( F_{\kappa\beta} F_{\delta\beta} \right) \]
\[
+ \frac{\kappa^2}{2} \partial_{\mu} F^{\mu\nu} \left( \frac{1}{4} \left( F_{\alpha\beta} F^{\alpha\beta} \right) \right) \frac{1}{4} \left( F_{\mu\nu} F^{\mu\nu} \right) \]
\[
- \frac{\kappa^2}{2} \partial_{\mu} \left( \frac{1}{4} \left( F_{\alpha\beta} F^{\alpha\beta} \right) \right) \frac{1}{4} \left( F_{\mu\nu} F^{\mu\nu} \right) = 0.
\]

We can write this as
\[
\partial_{\mu} F^{\mu\nu} = \left( \partial_{\mu} S^{\mu\nu} \right) \frac{\kappa^2}{2} \equiv J^{\nu}_{_{eff}}, \quad (5.22)
\]
where

\[
S^{\mu\nu} = F^{\delta\nu} \left( F_{\mu\delta} - F^{\delta\mu} \right) - F^{\delta\mu} \left( F_{\nu\delta} - F^{\delta\nu} \right) + F^{\delta\nu} \left( F_{\mu\delta} - F^{\delta\mu} \right) - F^{\delta\mu} \left( F_{\nu\delta} - F^{\delta\nu} \right)
\]

\[+ F^{\delta\nu} \left( F_{\mu\delta} - F^{\delta\mu} \right) - F^{\delta\mu} \left( F_{\nu\delta} - F^{\delta\nu} \right) + F^{\delta\nu} \left( F_{\mu\delta} - F^{\delta\mu} \right) - F^{\delta\mu} \left( F_{\nu\delta} - F^{\delta\nu} \right)
\]

\[= F^{\mu\nu} + F_{\alpha\beta} F^{\alpha\beta} - F_{\alpha\beta} F^{\alpha\beta} F^{\mu\nu}.
\]

(5.23)

Note that

\[S^{\mu\nu} = -S^{\nu\mu} \implies \partial_{\mu\nu} = 0,
\]

(5.24)

which is consistent.

Now let us look at its components

\[
S^{0i} = F^{0i} \left( F^{0\beta} F_{0\beta} + F^{j0} \left( F^{0\beta} F_{j\beta} \right) - F^{j0} \left( F^{0\beta} F_{j\beta} \right) + F^{0i} \left( F^{0\beta} F_{0\beta} \right)
\]

\[+ F^{j0} \left( F^{0\beta} F_{j\beta} \right) - F^{j0} \left( F^{0\beta} F_{j\beta} \right) + F^{j0} \left( F^{0\beta} F_{j\beta} \right) - F^{j0} \left( F^{0\beta} F_{j\beta} \right)
\]

\[= F^{0i} + F^{0i} + F_{0i} F^{00} - F_{0i} F^{00} + F_{0i} F^{00} - F_{0i} F^{00}
\]

\[+ 2F^{0i} \left( E_{\beta} - B_{\beta} \right) + 2(E_{\beta} - B_{\beta}) F^{0i}.
\]

(5.25)

Using \(F^{0i} = E^i\) and \(F^{ij} = \epsilon^{ijk} B_k\) and after some algebra, it can be shown that

\[
S^{0i} = 2B^k \left( E^i B^k \right) - 2B^k \left( E^k B^i \right) - 2E^j \left( E^i E^j \right) - 2E^j \left( B^i B^j \right)
\]

\[+ 2E^j \left( E^i B^j \right) - 2B^i \left( E^i E^j \right) - 2E^j \left( E^j B^i \right) - 3B^i \left( E^i B^j \right)
\]

\[+ 2(E^2 - B^2) \left( E^i B^i \right).
\]

(5.26)

This brings out the first set of modified Maxwell’s equations in presence of gravitons, which follows from eqns. (5.22-5.26)

\[\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{D} = \vec{E} - \vec{S}, \quad D^i = F^{0i} - S^{0i}.
\]

(5.27)

Similarly for spatial indices \((ij)\) we get the following expression

\[
S^{ij} = 2\epsilon^{ijk} B^k \left( E^i + B^i \right) + 2\epsilon^{ijk} E^j \left( E^i + B^i \right) + 2\epsilon^{ijk} B^j \left( E^i + B^i \right)
\]

\[+ 2\epsilon^{ijk} B^j \left( E^i + B^i \right) + 2E^i \left( E^i + B^i \right) - 2E^i \left( E^i + B^i \right) - E^j \left( \frac{1}{E^2} \right) \times \vec{E}^j
\]

\[+ E^j \left( \frac{1}{E^2} \right) \times \vec{E}^i - (E^i + B^i) \left( E^j + (E^j + B^j) \right) E^j.
\]

(5.28)
Now we define the quantity
\[
\tilde{S}^m = \epsilon^{ijm} S^{ij}
\]
\[
= 2 \left[ 2 B^m \frac{1}{\Box} (\vec{E}^2 + \vec{B}^2) + 2 E^2 \frac{1}{\Box} B^m + B^m \frac{1}{\Box} (E^m E^l + B^m B^l) - B^m \frac{1}{\Box} (\vec{E}^2 + \vec{B}^2) - B^l \frac{1}{\Box} (E^m E^l + B^m B^l) + B^m \frac{1}{\Box} (\vec{E}^2 + \vec{B}^2) + 2 (\vec{E} \times \frac{1}{\Box} (\vec{E} \times \vec{B}))^m - (\vec{E} \times \left( \frac{1}{\Box} \vec{B} \right) \times \vec{E})^m \right] ,
\]
(5.29)

which in the vector notation would be
\[
\tilde{S} = 4 \vec{B} \frac{1}{\Box} (\vec{E}^2 + \vec{B}^2) + 4 \vec{E} \vec{B} \vec{B} + 4 (\vec{E} \times \frac{1}{\Box} (\vec{E} \times \vec{B})) - 2 \vec{E} \times \left( \frac{1}{\Box} \vec{B} \right) \times \vec{E}.
\]

Therefore,
\[
\nabla \times \vec{H} = \frac{\partial \vec{B}}{\partial t} , \quad \vec{H} = \vec{B} - \tilde{S}.
\]
(5.30)

The above set of equations bring out that in the presence of the gravitons, the photon medium gets polarized with \( \tilde{S} \) and gains magnetization, denoted by \( \vec{S} \). These are non-linear features which could be useful for detecting GW. Also for the case of massive gravitons, each \( \frac{1}{\Box} \) term gets modified to \( \frac{1}{\Box - m^2} \). These set of non-linear Maxwell equations can put further constraints on the mass of graviton by a comparison of the experimental data with the theoretical prediction.

In birefringent media, uniform plane waves can be decomposed into two orthogonal polarization states (linear or circular) that propagate with two different speeds. The two states develop a phase difference as they propagate, which alters the total polarization of the wave. As we can see the \( \tilde{S} \), \( \vec{S} \) do depend on electric and magnetic fields \( \vec{E}, \vec{B} \) non-linearly.

This shows that permittivities and therefore, refractive indices of this anisotropic medium strongly depends on the \( \vec{E}, \vec{B} \) fields non-linearly, as do the permeabilities. This characterizes the birefringence property of the vacuum. In case of non-vanishing sources these quantities also carry information about physical properties of the compact objects, as discussed above (follows from linearity property of on-shell equation in terms of individual terms in the action).

Therefore, the above set of equations can be used not only in the detection process but also to extract information about compact objects like neutron stars, white-dwarfs, binary mergers.

### 6 Scattering Process between photons in presence of gravitons

#### 6.1 Action in momentum space

Since the detection of single graviton is very challenging from the perspective of present technology, our approach of integrating out the graviton degrees of freedom and writing an effective action for photons which takes into account the interactions between photons and gravitons, would be helpful since there have been impressive advances in the field of photon detection. Now our aim is to write down the interacting part of the action in momentum
space from which the scattering amplitudes can be calculated. Here also we don’t assume any external source producing GW.

Recall eq. (5.18),

\[
\int d^4x \, T^{\mu \nu}(x) \frac{1}{\Delta T_{\mu \nu}(x)} = \int d^4x \left( \eta_{\alpha \beta} F_{\rho \sigma}^\alpha F_{\sigma \nu}^\beta - \frac{1}{4} \eta_{\mu \nu} F_{\rho \sigma}^{\alpha \beta} \right) \frac{1}{\Delta} \left( \eta^\rho_{\alpha \beta} F_{\rho \nu}^\alpha F_{\rho \sigma}^\beta \right) \left( \eta^\sigma_{\mu \nu} F_{\rho \sigma}^{\alpha \beta} \right) \frac{1}{\Delta} \left( F_{\mu \nu} F^{\mu \nu} \right),
\]

(6.1)

which in momentum space takes the following form

\[
(2\pi)^4 \int \frac{4}{d^4k_i \delta^{(4)}(k_1 + k_2 + k_3 + k_4) \frac{1}{(k_3 + k_4)^2}} \times \left[ (k_1.k_2.A(k_1).A(k_2) - k_1.A(k_2)k_2.A(k_1)) \times (k_3.k_4.A(k_3).A(k_4) - k_3.A(k_4)k_4.A(k_3)) \right.
\]

\[
- (k_1.A(k_1)k_2^\mu A^\nu(k_2) - k_1.A(k_2)k_2^\mu A^\nu(k_1)) \times (k_3.A(k_4)k_4.A(k_3) - k_4.A(k_3)k_3.A(k_4))
\]

\[
\left. + (k_1.k_2^\mu A^\nu(k_1)A^\sigma(k_2)A^\rho(k_3)A_\mu(k_4) - k_1.A(k_1)k_2^\mu A^\nu(k_2)A^\sigma(k_3)A_\mu(k_4)) \right].
\]

(6.2)

This can be re-written as

\[
(2\pi)^4 \int \frac{4}{d^4k_i \delta^{(4)}(k_1 + k_2 + k_3 + k_4) \frac{1}{(k_3 + k_4)^2}} \times \left[ A^\mu(k_1)A^\nu(k_2)A^\rho(k_3)A^\sigma(k_4) \right.
\]

\[
- A^\mu(k_1)A^\nu(k_2)A^\rho(k_3)A^\sigma(k_4) - k_1.A(k_1)k_2^\mu A^\nu(k_2)A^\rho(k_3)A_\mu(k_4)
\]

\[
+ k_1.A(k_1)k_2^\mu A^\nu(k_2)A^\rho(k_3)A_\mu(k_4)
\]

\[
+ k_1.A(k_1)k_2^\mu A^\nu(k_2)A^\rho(k_3)A_\mu(k_4)
\]

\[
\left. - k_1.A(k_1)k_2^\mu A^\nu(k_2)A^\rho(k_3)A_\mu(k_4) \right].
\]

(6.3)
6.2 Vertex function and polarization tensors

Now we find out the tree-level scattering amplitudes in terms of the vertex operator denoted by \( V_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) \) which is

\[
V_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = \left[ k_1 \cdot k_2 k_3 \cdot k_4 \eta_{\mu\nu} \eta_{\rho\sigma} - k_1 k_2 \eta_{\mu\rho} k_3 \eta_{\nu\sigma} k_4 - k_1 k_4 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 + k_1 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 k_4
\]

\[
- k_1 k_2 k_3 \eta_{\mu\nu} \eta_{\rho\sigma} + k_1 k_2 k_3 \eta_{\mu\rho} k_4 + k_1 k_2 \eta_{\mu\rho} k_3 \eta_{\nu\sigma} k_4 - k_1 k_2 \eta_{\mu\rho} k_3 k_4
\]

\[
+ k_3 k_4 \eta_{\mu\rho} k_1 \eta_{\nu\sigma} k_2 - k_2 k_3 \eta_{\mu\nu} k_1 \eta_{\rho\sigma} k_4 - k_1 k_2 \eta_{\mu\rho} k_3 k_4 + k_2 k_4 \eta_{\mu\rho} k_1 \eta_{\nu\sigma}
\]

\[
+ k_3 k_4 \eta_{\mu\nu} k_1 \eta_{\rho\sigma} k_2 - k_1 k_4 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 + k_1 k_3 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_4 - \eta_{\mu\nu} \eta_{\rho\sigma} k_1 k_3 k_2 k_4
\]

\[
x \times (2\pi)^4 \frac{\kappa^2}{(k_1 + k_4)^2} \delta^{(4)}(k_1 + k_2 + k_3 + k_4),
\]

(6.4)

and in the case of massive gravitons, the vertex function becomes

\[
V_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = \left[ k_1 \cdot k_2 k_3 \cdot k_4 \eta_{\mu\nu} \eta_{\rho\sigma} - k_1 k_2 \eta_{\mu\rho} k_3 \eta_{\nu\sigma} k_4 - k_1 k_4 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 + k_1 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 k_4
\]

\[
- k_1 k_2 k_3 \eta_{\mu\nu} \eta_{\rho\sigma} + k_1 k_2 k_3 \eta_{\mu\rho} k_4 + k_1 k_2 \eta_{\mu\rho} k_3 \eta_{\nu\sigma} k_4 - k_1 k_2 \eta_{\mu\rho} k_3 k_4
\]

\[
+ k_3 k_4 \eta_{\mu\rho} k_1 \eta_{\nu\sigma} k_2 - k_2 k_3 \eta_{\mu\nu} k_1 \eta_{\rho\sigma} k_4 - k_1 k_2 \eta_{\mu\rho} k_3 k_4 + k_2 k_4 \eta_{\mu\rho} k_1 \eta_{\nu\sigma}
\]

\[
+ k_3 k_4 \eta_{\mu\nu} k_1 \eta_{\rho\sigma} k_2 - k_1 k_4 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_3 + k_1 k_3 \eta_{\mu\nu} k_2 \eta_{\rho\sigma} k_4 - \eta_{\mu\nu} \eta_{\rho\sigma} k_1 k_3 k_2 k_4
\]

\[
x \times (2\pi)^4 \frac{\kappa^2}{(k_1 + k_4)^2 + m^2} \delta^{(4)}(k_1 + k_2 + k_3 + k_4).
\]

(6.5)

Let us denote the polarization tensors of the photons as \( \varepsilon_i \equiv \varepsilon(k_i, \lambda_i) \), where \( \lambda_i \in \{1, -1\} \) and defined by

\[
\varepsilon(\vec{k}, \lambda = 1) = \frac{1}{\sqrt{(k^1)^2 + (k^2)^2}}(k^2, -k^1, 0)
\]

\[
\varepsilon(\vec{k}, \lambda = -1) = \frac{1}{|\vec{k}|}\sqrt{(k^1)^2 + (k^2)^2}(k^1 k^3, k^2 k^3, -(k^1)^2 + (k^2)^2),
\]

(6.6)

which satisfy

\[
\sum_{\lambda=1,2} \varepsilon^{(\lambda)}_i \varepsilon^{(\lambda)}_j = \delta_{ij} - \frac{k^i k^j}{|\vec{k}|^2}.
\]

(6.7)

6.3 Vacuum to 4-photons scattering amplitude

In this section, we compute scattering amplitude of a process in which from vacuum, four photons are produced with the same polarization \( \lambda = 1 \). The scattering amplitude is defined by \( \langle f | S^{(1)} | i \rangle \) with \( S^{(1)} \) interaction term in the action and \( |i\rangle = |0\rangle \) is the initial state and \( |f\rangle = \Pi_{i=1}^4 \hat{a}^\dagger(k_i, \lambda_i = 1) |0\rangle \) is the final state. We can write

\[
\langle f | S^{(1)} | i \rangle = \mathcal{M}(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4) \prod_{i=1}^4 \frac{1}{\sqrt{2\omega_i}}(2\pi)^2,
\]

(6.8)
where $\mathcal{M}$ is the scattering amplitude of the process.

For this case

$$\mathcal{M}_1 = \int \prod_{i=1}^{4} d^4 p_i V_{\mu \nu \rho \sigma}(p_1, p_2, p_3, p_4) \langle 0 | \hat{a}(k_4, \lambda = 1) \hat{a}(k_3, \lambda = 1) \hat{a}(k_2, \lambda = 1) \hat{a}(k_1, \lambda = 1) \times \hat{a}^\dagger(p_1, \lambda = 1) \hat{a}^\dagger(p_2, \lambda = 1) \hat{a}^\dagger(p_3, \lambda = 1) \hat{a}^\dagger(p_4, \lambda = 1) | 0 \rangle \times \varepsilon^\mu(p_1, \lambda = 1) \varepsilon^\nu(p_2, \lambda = 1) \varepsilon^\rho(p_3, \lambda = 1) \varepsilon^\sigma(p_4, \lambda = 1) = \sum_{p \in S_4} V_{\mu \nu \rho \sigma}(k_{p(1)}, k_{p(2)}, k_{p(3)}, k_{p(4)}) \varepsilon^\mu(k_{p(1)}, \lambda = 1) \varepsilon^\nu(k_{p(2)}, \lambda = 1) \varepsilon^\rho(k_{p(3)}, \lambda = 1) \varepsilon^\sigma(k_{p(4)}, \lambda = 1). \quad (6.9)$$

For the case where all the scattered photons have $\lambda = -1$ polarization, we just need to change the polarization tensor in above contraction.

Now we will look at the amplitude where the final state is $\prod_{i=1}^{4} \hat{a}^\dagger(k_i, \lambda_i = 1) | 0 \rangle$ and the polarizations $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ can take any value. The corresponding scattering amplitude would be

$$\mathcal{M}_2 = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \int \prod_{i=1}^{4} d^4 p_i V_{\mu \nu \rho \sigma}(p_1, p_2, p_3, p_4) \langle 0 | \hat{a}(k_4, \lambda_4) \hat{a}(k_3, \lambda_3) \hat{a}(k_2, \lambda_2) \hat{a}(k_1, \lambda_1) \times \hat{a}^\dagger(p_1, \lambda_1) \hat{a}^\dagger(p_2, \lambda_2) \hat{a}^\dagger(p_3, \lambda_3) \hat{a}^\dagger(p_4, \lambda_4) | 0 \rangle \varepsilon^\mu(p_1, \lambda_1) \varepsilon^\nu(p_2, \lambda_2) \varepsilon^\rho(p_3, \lambda_3) \varepsilon^\sigma(p_4, \lambda_4) = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \sum_{p \in S_4} V_{\mu \nu \rho \sigma}(k_{p(1)}, k_{p(2)}, k_{p(3)}, k_{p(4)}) \varepsilon^\mu(k_{p(1)}, \lambda_{p(1)}) \varepsilon^\nu(k_{p(2)}, \lambda_{p(2)}) \varepsilon^\rho(k_{p(3)}, \lambda_{p(3)}) \varepsilon^\sigma(k_{p(4)}, \lambda_{p(4)}) \times \varepsilon^\sigma(k_{p(4)}, \lambda_{p(4)}) \delta_{\lambda_1, \lambda_{p(1)}} \delta_{\lambda_2, \lambda_{p(2)}} \delta_{\lambda_3, \lambda_{p(3)}} \delta_{\lambda_4, \lambda_{p(4)}}. \quad (6.10)$$

### 6.4 Scattering amplitude of decay process

Now we consider $1 \rightarrow 3$ particle decay process where initial state is $|i\rangle = \hat{a}^\dagger(k_1, \lambda_1) | 0 \rangle$ and final state is $|f\rangle = \hat{a}^\dagger(k_2, \lambda_2) \hat{a}^\dagger(k_3, \lambda_3) \hat{a}^\dagger(k_4, \lambda_4) | 0 \rangle$, for which we can write

$$\langle f | S^{(1)} | i \rangle = \mathcal{M}, \quad (6.11)$$

where

$$\mathcal{M} = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \int \prod_{i=1}^{4} d^4 p_i V_{\mu \nu \rho \sigma}(p_1, p_2, p_3, p_4) \varepsilon^\mu(p_1, \lambda_1) \varepsilon^\nu(p_2, \lambda_2) \varepsilon^\rho(p_3, \lambda_3) \varepsilon^\sigma(p_4, \lambda_4) \times \left[ \langle 0 | \hat{a}(k_4, \lambda_4) \hat{a}(k_3, \lambda_3) \hat{a}(k_2, \lambda_2) \hat{a}(k_1, \lambda_1) \hat{a}^\dagger(p_2, \lambda_2) \hat{a}^\dagger(p_3, \lambda_3) \hat{a}^\dagger(p_4, \lambda_4) | 0 \rangle \varepsilon^\mu(p_1, \lambda_1) \varepsilon^\nu(p_2, \lambda_2) \varepsilon^\rho(p_3, \lambda_3) \varepsilon^\sigma(p_4, \lambda_4) \right] \times (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3 - p_4) \prod_{i=1}^{4} \frac{1}{\sqrt{2\omega_i(2\pi)^3}}. \quad (6.12)$$
This can be re-written as

$$\mathcal{M} = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \left[ \prod_{i=1}^{4} d^4 p_i \nu_{\mu p r e} (p_1, p_2, p_3, p_4) \varepsilon^\mu (p_1, \tilde{\lambda}_1) \varepsilon^\nu (p_2, \tilde{\lambda}_2) \varepsilon^\rho (p_3, \tilde{\lambda}_3) \varepsilon^\sigma (p_4, \tilde{\lambda}_4) \times \left[ \right] \times (2\pi)^4 \delta^{(4)} (p_1 - p_2 - p_3 - p_4) \prod_{i=1}^{4} \frac{1}{\sqrt{2\omega_i (2\pi)^2}} \right.$$}

$$= \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \left[ \prod_{i=1}^{4} d^4 p_i \nu_{\mu p r e} (p_1, p_2, p_3, p_4) \varepsilon^\mu (p_1, \tilde{\lambda}_1) \varepsilon^\nu (p_2, \tilde{\lambda}_2) \varepsilon^\rho (p_3, \tilde{\lambda}_3) \varepsilon^\sigma (p_4, \tilde{\lambda}_4) \times \left[ \right] \times (2\pi)^4 \delta^{(4)} (p_1 - p_2 - p_3 - p_4) \prod_{i=1}^{4} \frac{1}{\sqrt{2\omega_i (2\pi)^2}} \right.$$}

$$\times \delta^{(4)} (p_1 - k_1) \delta^{(4)} (p_2 - k_{P(2)}) \delta^{(4)} (p_3 - k_{P(3)}) \delta^{(4)} (p_4 - k_{P(4)}) \delta_{\lambda_1, \lambda_1} \delta_{\lambda_2, \lambda_2} \delta_{\lambda_3, \lambda_3} \delta_{\lambda_4, \lambda_4} + \sum_{m, m' = 2}^{4} \delta^{(4)} (p_1 - k_m) \delta^{(4)} (k_1 - p_n) \delta_{\lambda_1, \lambda_1} \delta_{\lambda_2, \lambda_1} \delta_{\lambda_3, \lambda_1} \delta_{\lambda_4, \lambda_1} \prod_{m' \neq m', n' \neq n \in \{1, 2, 3\}} \delta^{(4)} (k_{m'} - p_{n'}) \delta_{\lambda_{m'}, \lambda_{m'}} \bigg]$$

$$- 3 \delta^{(4)} (p_1 - p_2) \left\{ \sum_{m, m' = 2}^{4} \delta_{\lambda_2, \lambda_1} \delta^{(4)} (k_1 - k_m) \delta_{\lambda_4, \lambda_m} \delta^{(4)} (p_4 - p_{m'}) \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (p_3 - m_{m'}) \right\}$$

$$- 2 \delta^{(4)} (p_1 - p_2) \sum_{m, m' = 2}^{4} \delta_{\lambda_2, \lambda_1} \delta^{(4)} (k_1 - k_m) \delta_{\lambda_4, \lambda_m} \delta^{(4)} (p_4 - p_{m'}) \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (p_3 - m_{m'})$$

$$- \delta^{(4)} (p_1 - p_2) \sum_{m, m' = 2}^{4} \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (k_1 - k_m) \delta_{\lambda_4, \lambda_{m'}} \delta^{(4)} (p_4 - p_{m'}) \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (p_3 - m_{m'})$$

$$- 2 \delta^{(4)} (p_1 - p_2) \sum_{m, m' = 2}^{4} \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (k_1 - k_m) \delta_{\lambda_4, \lambda_{m'}} \delta^{(4)} (p_4 - p_{m'}) \delta_{\lambda_3, \lambda_{m'}} \delta^{(4)} (p_3 - m_{m'})$$

$$\left(6.13\right)$$
where $m = 2$ then $m_\geq = 3, m_\leq = 4$, if $m = 3$ then $m_\leq = 2, m_\geq = 4$ and if $m = 4$ then $m_\leq = 2, m_\geq = 3$ and $m', n'$ can take one value at one time only.

### 6.5 Scattering amplitude of $2 \rightarrow 2$ scattering process

Now we consider $2 \rightarrow 2$ scattering process where initial state is $|i\rangle = \hat{a}^\dagger(k_1, \lambda_1)\hat{a}^\dagger(k_2, \lambda_2)\,|0\rangle$ and final state is $|f\rangle = \hat{a}^\dagger(k_3, \lambda_3)\hat{a}^\dagger(k_4, \lambda_4)\,|0\rangle$, for which we can write

$$
\langle f | S^{(1)} | i \rangle = \mathcal{M},
$$

(6.14)

where

$$
\mathcal{M} = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \int \prod_{i=1}^{4} d^4 p_i \, \mathcal{V}_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) \varepsilon^\mu(p_1, \lambda_1) \varepsilon^\nu(p_2, \lambda_2) \varepsilon^\rho(p_3, \lambda_3) \varepsilon^\sigma(p_4, \lambda_4)
$$

$$
\times \left[ \langle 0 | \hat{a}(k_4, \lambda_4)\hat{a}(k_3, \lambda_3)\hat{a}(p_1, \lambda_1)\hat{a}(p_2, \lambda_2)\hat{a}^\dagger(p_3, \lambda_3)\hat{a}^\dagger(p_4, \lambda_4)\hat{a}^\dagger(k_1, 1)\hat{a}^\dagger(k_2, \lambda_2)\,|0\rangle
\right.
$$

$$
+ \langle 0 | \hat{a}(k_4, \lambda_4)\hat{a}(k_3, \lambda_3)\hat{a}(p_1, \lambda_1)\hat{a}^\dagger(p_3, \lambda_3)\hat{a}(p_2, \lambda_2)\hat{a}^\dagger(p_4, \lambda_4)\hat{a}^\dagger(k_1, 1)\hat{a}(k_2, \lambda_2)\,|0\rangle
$$

$$
+ \langle 0 | \hat{a}(k_4, \lambda_4)\hat{a}(k_3, \lambda_3)\hat{a}^\dagger(p_3, \lambda_3)\hat{a}(p_1, \lambda_1)\hat{a}(p_2, \lambda_2)\hat{a}^\dagger(p_4, \lambda_4)\hat{a}^\dagger(k_1, 1)\hat{a}(k_2, \lambda_2)\,|0\rangle
$$

$$
+ \langle 0 | \hat{a}(k_4, \lambda_4)\hat{a}(k_3, \lambda_3)\hat{a}^\dagger(p_3, \lambda_3)\hat{a}(p_1, \lambda_1)\hat{a}(p_2, \lambda_2)\hat{a}^\dagger(p_4, \lambda_4)\hat{a}^\dagger(k_1, 1)\hat{a}(k_2, \lambda_2)\,|0\rangle
$$

$$
+ \langle 0 | \hat{a}(k_4, \lambda_4)\hat{a}(k_3, \lambda_3)\hat{a}^\dagger(p_3, \lambda_3)\hat{a}(p_1, \lambda_1)\hat{a}(p_2, \lambda_2)\hat{a}^\dagger(p_4, \lambda_4)\hat{a}^\dagger(k_1, 1)\hat{a}(k_2, \lambda_2)\,|0\rangle
$$

$$
+ ((p_3, \lambda_3) \leftrightarrow (p_4, \lambda_4)) \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \prod_{i=1}^{4} \frac{1}{\sqrt{2\omega_i(2\pi)^\frac{3}{2}}},
$$

(6.15)

This can be evaluated explicitly using the commutation algebra between creation and annihilation operators of different modes. 

\[ -21 - \]
6.6 Features of scattering amplitudes

Note that up to one-loop all scattering amplitudes shown above are proportional to $\kappa^2 = \frac{8\pi G\hbar}{c^4}$ which is $O(10^{-77})$. This is very small in magnitude for measurement in any scattering experiment. However, using weak value amplification [46], [47], [48] this magnitude can be amplified through suitable pre-selected and post-selected scattering states, shown below.

Another interesting aspect of these scattering processes is that helicity of photon through these processes is not conserved. This was first shown in [16], and the reason behind this will be discussed below.

The other feature found is that the vertex function $V_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4)$ contains a factor of $\frac{1}{(k_3 + k_4)^2}$ which shows an IR pole at $k_3 + k_4 = 0$ that can be avoided by adding soft photons [49], [50]. This also happens in QED although there vertex function does not have any poles in momentum space. Note that this feature is in-built in photons due to interaction with massless gravitons since asymptotically interaction term in action is finite for soft gravitons but this is not the case for massive gravitons. Therefore, one would expect the absence of the IR pole for massive gravitons. This is indeed the case as there we need to replace $\frac{1}{(k_3 + k_4)^2}$ by $\frac{1}{(k_3 + k_4)^2 + m^2}$.

6.7 Duality symmetry

One of the beautiful features of Maxwell equations in free-space is that it has the symmetry of exchanging electric and magnetic fields. Maxwell equations in free-space are symmetric under the following transformation [51, 52]

$$\vec{E} \rightarrow \vec{B}, \quad \vec{B} \rightarrow -\vec{E}. \quad (6.16)$$

This discrete symmetry is a particular case of the most general continuous transformation, known as dual transformation [53–55], [51, 52]

$$\vec{E} \rightarrow \vec{E} \cos \theta + \vec{B} \sin \theta, \quad \vec{B} \rightarrow \vec{B} \cos \theta - \vec{E} \sin \theta, \quad (6.17)$$

where $\theta$ is an arbitrary angle.

A concrete analysis of conserved quantity (Noether’s charge) corresponding to this continuous symmetry revealed that the pertinent pseudo scalar integrated over spatial hypersurface represents the difference between the number of left- and right-hand circularly polarized photons which is nothing but optical helicity. Hence, this duality symmetry leads to conservation of helicity of light [56–60].

In standard Maxwell’s electromagnetism

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{2} (\vec{E}^2 - \vec{B}^2), \quad (6.18)$$

corresponds to free Maxwell equations

$$\partial_\alpha F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \quad \partial_\alpha * F^{\alpha\beta} = 0$$

$$*F^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}. \quad (6.19)$$

Note that under $*$ transformation

$$F \rightarrow *F \rightarrow **F = -F. \quad (6.20)$$
It is important to note that coupling of matter with photon breaks this symmetry. If Lagrangian acquires a term \( j^\alpha A_\alpha \) which implies equations of motion become

\[
\partial_\alpha F^{\alpha \beta} = -j_\beta = j^\beta \\
\partial_\alpha * F^{\alpha \beta} = -j_\beta^M \text{ (magnetic current).}
\]  

(6.21)

Under dual transformation \( \vec{R} = \vec{E} + i\vec{B} \) transforms as \( \vec{R} \rightarrow \vec{R} e^{-i\theta} \), which shows that

\[
\vec{R} \vec{R} = (\vec{E}^2 - \vec{B}^2) + 2i\vec{E} \cdot \vec{B}.
\]  

(6.22)

Now we define two Lorentz invariant quantities

\[
I_1 = \frac{1}{2} F^{\alpha \beta} F_{\alpha \beta} = \vec{E}^2 - \vec{B}^2,
I_2 = \frac{1}{2} * F^{\alpha \beta} F_{\alpha \beta} = 2 \vec{E} \cdot \vec{B},
\]  

(6.23)

which will be important for the subsequent discussion. Note that

\[
\mathcal{L}_{\text{free}} = \frac{1}{2} \mathfrak{Re}(\vec{R} \cdot \vec{R}) = \frac{1}{2} I_1.
\]  

(6.24)

Under dual transformation

\[
I_1 \rightarrow I_1 \cos 2\theta + I_2 \sin 2\theta, \\
I_2 \rightarrow I_2 \cos 2\theta - I_1 \sin 2\theta,
\]  

(6.25)

the lagrangian density transforms as

\[
\mathcal{L}_{\text{free}} \rightarrow \mathcal{L}_{\text{free}} \cos 2\theta - \frac{1}{4} * F^{\alpha \beta} F_{\alpha \beta} \sin 2\theta.
\]  

(6.26)

Although the above transformation changes Lagrangian density, the Maxwell equations remain unchanged, since

\[
* F^{\alpha \beta} F_{\alpha \beta} = 2 \partial_\alpha (* F^{\alpha \beta} A_\beta),
\]  

(6.27)

where we have used the on-shell relation \( * \partial_\alpha * F^{\alpha \beta} = 0 \) for free-space.

Considering the infinitesimal version of the transformation (6.25)

\[
\mathcal{L}_{\text{free}} \rightarrow \mathcal{L}_{\text{free}} - \theta \partial_\alpha (* F^{\alpha \beta} A_\beta),
\]  

(6.28)

it can be seen that \( \mathcal{L}_{\text{free}} \) changes only by a total derivative which therefore is a symmetry transformation and

\[
J^\alpha = * F^{\alpha \beta} A_\beta,
\]  

(6.29)

is the corresponding conserved charge in the absence of sources. In the presence of sources

\[
\partial_\alpha J^\alpha = \partial_\alpha * F^{\alpha \beta} A_\beta + \frac{1}{2} * F^{\alpha \beta} F_{\alpha \beta} \\
= -j_\beta^M A_\alpha - I_2 = -I_2 \text{ (if we neglect magnetic charges in the universe)}
\]  

(6.30)

\( \propto \vec{E} \cdot \vec{B} \neq 0 \).

And in the presence of gravitons we have a non-zero source which is \( j_{\text{eff}}^\mu = \partial_\mu S^{\mu \nu} \), see eqn. (5.22), which guarantees that \( \vec{E} \cdot \vec{B} \neq 0 \), thereby bringing out the violation of helicity
Hence presence of non-trivial source term in eq. (5.22) leads to violation of helicity conservation in scattering processes. Here

\[ J^0 = \ast F^{0\beta} A_\beta = \frac{1}{2} \varepsilon^{0\beta\gamma\delta} F_{\gamma\delta} A_\beta = F_{23} A_1 - F_{13} A_2 + F_{12} A_3 = \vec{A} \cdot \vec{B}. \]  

(6.31)

Instead duality symmetric modifications of \( J^0 \), \( \vec{J} \) are

\[ J^0 \equiv H = \frac{1}{2} (\vec{A} \cdot \vec{B} - \vec{C} \cdot \vec{E}) \]

\[ \vec{J} \equiv \vec{S} = \frac{1}{2} (\vec{E} \times \vec{A} + \vec{B} \times \vec{C}), \]

(6.32)

where

\[ \vec{\nabla} \times \vec{C} = -\vec{E}, \quad \vec{\nabla} \times \vec{A} = \vec{B}. \]

(6.33)

7 Weak measurements

Here, we will discuss the general weak measurement protocol [63–67]. This will lead naturally to the definition of weak value [68, 69] of an operator.

7.1 The Weak Measurement protocol

In order to perform weak measurement in an experiment we need a large ensemble of particles all prepared in the same initial state denoted by \( |\psi_i\rangle \). Each particle will interact with a separate measuring device which is in the following state

\[ |\Phi\rangle = \int \frac{dq}{\sqrt{\Delta(2\pi)^d}} e^{-\frac{q^2}{4\Delta}} |q\rangle, \]

(7.1)

where \( \Delta \) is the standard deviation of position of the measuring device.

The interaction Hamiltonian is

\[ \hat{H} = \chi(t) \hat{p} \otimes \hat{A}, \]

(7.2)

where \( \hat{p} \) is the momentum operator of the device and \( \hat{A} \) is the operator whose expectation value needs to be measured.

A weak measurement could be an impulsive measurement at time \( t_0 \)

\[ \chi(t) \approx \chi \delta(t - t_0), \]

(7.3)

with the interaction strength \( \chi \) being sufficiently small. The collective state of the particle and device will evolve under the interaction Hamiltonian into the following entangled state

\[ e^{-i\chi \hat{p} \otimes A} |\Phi\rangle \otimes |\Psi_i\rangle = \left( \sum_j |a_j\rangle e^{-i\chi \hat{p} a_j} \langle a_j| \right) |\Phi\rangle \otimes |\Psi_i\rangle \]

\[ = \sum_j \alpha_j e^{-i\chi \hat{p} a_j} |\Phi\rangle \otimes |a_j\rangle, \]

(7.4)

where \( a_j \)'s are the eigenvalues of \( \hat{A} \) corresponding to eigenvector \( |a_j\rangle \) and

\[ \alpha_j = \langle a_j|\Psi_i\rangle. \]

(7.5)
Also,

\[ e^{-i\hat{p}\alpha_j} |\Phi\rangle = \int \frac{dq}{\sqrt{\Delta(2\pi)^\frac{1}{4}}} e^{-\frac{(q-\chi\alpha_j)^2}{4\Delta^2}} |q\rangle. \] (7.6)

The expectation value of \( \hat{A} \) in the initial state can be determined by measuring the shift in the position of the device. This can be seen by considering the probability of position measurement of the device after interaction with the system is

\[ P(q) = \sum_j |\alpha_j|^2 \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(q-\chi\alpha_j)^2}{2\Delta^2}} \]

\[ \approx \frac{1}{\sqrt{2\pi\Delta^2}} \left( 1 - \frac{(q - \sum_j \chi|\alpha_j|^2)}{2\Delta^2} \right) \]

\[ \approx \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(q - \sum_j \chi|\alpha_j|^2)}{2\Delta^2}} \]

\[ = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(q - \chi\langle \hat{A} \rangle)^2}{2\Delta^2}}, \] (7.7)

where in the second line we have considered the uncertainty of device is taken to be much larger than the eigenvalues of operator. Thus, the eqn. (7.7) shows that the probability distribution of the device is Gaussian centered around \( \chi\langle \hat{A} \rangle \). Since for an ensemble of N-particles the uncertainty in measuring the average value of the operator is reduced by \( \frac{1}{\sqrt{N}} \), therefore, for a large collection of identical particles we can measure \( \langle \hat{A} \rangle \) with arbitrary accuracy.

The reason for calling this weak is that after measurement, the state of each particle does not change much. After measurement, the state of the whole system becomes

\[ |\Psi\rangle_f = \frac{1}{\mathcal{N}} \sum_j \alpha_j \frac{1}{\sqrt{\Delta(2\pi)^\frac{1}{4}}} e^{-\frac{(q_0 - \chi\alpha_j)^2}{4\Delta^2}} |q_0\rangle \otimes |a_j\rangle, \] (7.8)

where \( q_0 \) is the measurement outcome of the position of the device and \( \mathcal{N} \) is the normalization constant. In the limit \( \Delta \gg 1 \), \( |\Psi\rangle_f \rightarrow |\Psi\rangle_i \). Hence, the “weakness” behind the measurement is due to lack of information extraction from the individual measurement.

### 7.2 The weak value

A very pertinent feature of weak measurement gets revealed once post-selection is added to it. Here, we will see how post-selection affects the measurement outcome, weak value and the limits under which calculations are valid.

We choose a state \( |b\rangle \) which is an eigenvector of another operator \( \hat{B} \) that need not be the operator \( \hat{A} \); this is the post-selected state. In this case a strong measurement will be performed on the particle (with a separate measuring device), after it has interacted with weak measuring device, to collapse the particle into one of \( \hat{B} \)'s eigenstates. Then, we select only particles that collapse into state \( |b\rangle \) to calculate the average value of \( \hat{A} \). Hence, only the post selected state of the particles are used to calculate the average value of the operator \( \hat{A} \).
After the post selection the state of the whole system (particle+device) turn into following state

\[
|\Omega\rangle = \frac{1}{K} |b\rangle \langle b| e^{-i\chi p B} |\Phi\rangle \otimes |\Psi_1\rangle \\
\approx \frac{1}{K} |b\rangle \langle b| (I - i\chi \hat{\rho} \otimes \hat{A}) |\Phi\rangle \otimes |\Psi_1\rangle \\
= \frac{1}{K} |b\rangle \langle b| \hat{A} |\Psi_1\rangle \left(1 - i\chi \hat{\rho} \frac{\langle b| \hat{A} |\Psi_1\rangle}{\langle b| \Psi_1\rangle} \right) |\Phi\rangle \\
\approx \frac{1}{K} \langle b| \Psi_1\rangle |b\rangle \otimes \left(e^{-i\chi \rho \langle b| \hat{A} |\Psi_1\rangle} \right) |\Phi\rangle ,
\]

where \( K \) is a normalization constant. Thus, the device measures the following quantity

\[
\langle \hat{A} \rangle_w = \frac{\langle b| \hat{A} |\Psi_1\rangle}{\langle b| \Psi_1\rangle} ,
\]

the weak value. The above results are based on following approximations [65]

\[
|\chi^a p^n \langle b| \hat{A} |\psi_1\rangle | \ll |\langle b| \Psi_1\rangle|, \forall p \\
|\chi^a p^n \langle b| \hat{A}^a |\psi_1\rangle | \ll |\langle b| \hat{A} |\psi_1\rangle|, \forall n \geq 2 \\
|\chi p \langle \hat{A} \rangle_w | \ll 1.
\]

Hence, adding a post-selected state to measurement, changes the center of the probability distribution of the device from \( \langle \hat{A} \rangle \) to \( \hat{A}_w \).

### 7.3 Measuring scattering amplitudes using weak measurement protocol

In the scattering amplitudes we are interested in measuring the quantity \( \langle f | i \rangle \) where \( |i\rangle \) is the initial state at past infinity \( t \to -\infty \) and \( |f\rangle \) is the final state at future infinity at \( t \to \infty \) or in the Schrödinger picture \( \langle f | e^{-iHT} | i \rangle |_{T \to \infty} = \langle f | \hat{S} | i \rangle \). Therefore, if we choose operator \( \hat{A} \) in weak measurement protocol to be the scattering matrix \( \hat{S} \), and choose the initial state to be a many-particle state \( |i\rangle = |k_1, k_2, \ldots, k_m\rangle \) and a post-selected state \( |f\rangle = (1 - \epsilon) \langle l_1, \ldots, l_n \rangle + \sqrt{2\epsilon} |k_1, k_2, \ldots, k_m\rangle \) such that \( \epsilon \ll 1 \), we have

\[
\langle \hat{A} \rangle_w = \langle \hat{S} \rangle_w = \frac{(1 - \epsilon) \langle l_1, \ldots, l_n | \hat{S} |k_1, \ldots, k_m\rangle + \sqrt{2\epsilon} \langle k_1, \ldots, k_m | \hat{S} |k_1, \ldots, k_m\rangle}{\sqrt{2\epsilon}} \\
\approx \frac{1}{\sqrt{2\epsilon}} \langle l_1, \ldots, l_n | \hat{S} |k_1, \ldots, k_m\rangle ,
\]

which is a scattering amplitude of \( m \)-particle state with momenta \( k_1, \ldots, k_m \) to \( n \)-particle state with momenta \( l_1, \ldots, l_n \) and a amplifying factor \( \frac{1}{\sqrt{2\epsilon}} \). This facilitates the measurement of scattering amplitude or cross-sections of any process in the theory through the weak measurement protocol by suitably choosing a channel with a collection of particular initial and post-selected states. Using the cascaded weak measurement strategy [70] \( \langle \hat{S} \rangle_w \) can be amplified by \( \mathcal{O}(10^{12}) \).

### 8 Effective action

To take into account quantum corrections, quantum effective action is obtained here by a one-loop computation.
8.1 Mass renormalization

Figure 1 depicts the 1-loop self-energy diagram, computed using Feynman diagrammatic techniques and equal to

\[ -i \int \frac{d^4k}{(2\pi)^4} \Gamma_{\mu\nu\rho\sigma}(p, k, k, p) \frac{\eta^{\mu\rho}}{k^2 + \mu^2} \tag{8.1} \]

where \( \mu \) is mass-regulator of photons and

\[
\begin{align*}
\Gamma_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) &= \left[ k_1.k_2k_3.k_4\eta_{\mu\nu}\eta_{\rho\sigma} - k_1.k_2\eta_{\mu\nu}k_3.k_4.k_4.k_4 - k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 + k_1.k_2.k_3.k_4.k_4.k_4 - k_1.k_2.k_3.k_4.k_4.k_4 - k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 - k_1.k_2.k_3.k_4.k_4.k_4 - k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 - k_1.k_2.k_3.k_4.k_4.k_4 + k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 - k_1.k_2.k_3.k_4.k_4.k_4 + k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 - k_1.k_2.k_3.k_4.k_4.k_4 - k_3.k_4\eta_{\mu\nu}k_1.k_2.k_2.k_2 - k_1.k_2.k_3.k_4.k_4.k_4 \right. \\
& \quad \left. \times \frac{\kappa^2}{(k_3 + k_4)^2} \delta^{(4)}(k_1 + k_2 + k_3 + k_4). \right]
\end{align*}
\]

This leads to

\[
\eta^{\mu\nu}\Gamma_{\mu\nu\rho\sigma}(p, k, k, p) = \frac{\kappa^2}{(p + k)^2} \left[ (p.k)^2 \eta_{\mu\sigma} - p.kk_\sigma p_\mu - p.kp_\sigma k_\mu + p^2 k_\mu k_\sigma \right. \\
& \quad - (p.k)^2 \eta_{\mu\sigma} + p.kp_\mu k_\sigma - p.kk_\mu p_\sigma + p.kp_\mu k_\sigma \right. \\
& \quad - p.kp_\mu k_\sigma - p^2 k_\mu k_\sigma + p.kp_\sigma k_\mu + p.kp_\sigma k_\mu - p^2 k_\mu k_\sigma \\
& \quad - p.kk_\mu p_\sigma + p.kk_\mu p_\sigma - p.kp_\mu k_\sigma + p.kp_\mu k_\sigma + p.kk_\mu p_\sigma - \eta_{\mu\sigma}(p.k)^2 \right] \\
& \quad = \frac{\kappa^2}{(p + k)^2} ((p.k)^2 \eta_{\mu\sigma} - p^2 k_\mu k_\sigma + p.kk_\mu p_\sigma + p.kp_\mu k_\sigma),
\]
which implies that the diagram can be mathematically represented by the following expression

\[
-\kappa^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \mu^2} \frac{1}{(p + k)^2} \left[ (p.k)^2 \eta_{\mu\sigma} - p^2 k_{\mu} k_{\sigma} + p.k k_{\mu} p_{\sigma} + p.k p_{\mu} k_{\sigma} \right]
\]

\[
= -\kappa^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2} \left[ (p.k)^2 \eta_{\mu\sigma} - p^2 k_{\mu} k_{\sigma} + p.k k_{\mu} p_{\sigma} + p.k p_{\mu} k_{\sigma} \right]
\]

\[
= -\kappa^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2} \left[ \frac{p^2 k^2}{4} \eta_{\mu\sigma} + (p^2)^2 x^2 \eta_{\mu\sigma} - p^2 p_{\mu} p_{\sigma} x^2 \right.
\]

\[
- \frac{p^2 k^2 \eta_{\mu\sigma}}{4} + \frac{k^2 p_{\mu} p_{\sigma}}{2} - \frac{p^2 p_{\mu} p_{\sigma} x^2}{2}
\]

\[
= -\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \int_0^1 x^2 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
= -\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \int_0^1 x^2 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
- \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
= -\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \int_0^1 x^2 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
- \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
= -\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \int_0^1 x^2 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
\times \left[ 1 - \frac{\kappa^2}{2} \ln \left( \frac{p^2 x(1 - x) + \mu^2(1 - x)}{4\pi\Lambda^2} \right) \right].
\]

where $\Gamma$, $\psi$ denote Gamma and Euler’s function, respectively. $\Lambda$ is the momentum scale (effective scale) up to which this theory is valid and $\varepsilon = 4 - D$.

We can now safely take $\mu = 0$, since there is no IR divergence when $p = 0$, which gives

\[
-\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(1) \right) \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(1) - \psi(2) \right) \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k + px)^2 + p^2 x(1 - x) + \mu^2(1 - x)]^2}
\]

\[
\times \left[ 1 - \frac{\kappa^2}{2} \ln \left( \frac{p^2 x(1 - x) + \mu^2(1 - x)}{4\pi\Lambda^2} \right) \right]
\]

\[
\frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(1) - \psi(2) \right) \left[ \ln \left( \frac{p^2 x(1 - x) + \mu^2(1 - x)}{4\pi\Lambda^2} \right) - \frac{\kappa^2 p_{\mu} p_{\sigma}}{12} \ln \left( \frac{p^2}{4\pi\Lambda^2} \right) \right].
\]

From the above expression, divergent part can be omitted by adding counterterms and we are left with finite part whose contribution is

\[
-\kappa^2 [\eta_{\mu\sigma} - 3p^2 p_{\mu} p_{\sigma}] \frac{1}{(4\pi)^2} \left( \frac{\psi(1)}{3} - \int_0^1 x^2 \ln(x(1 - x))dx - \frac{1}{3} \ln \left( \frac{p^2}{4\pi\Lambda^2} \right) \right)
\]

\[
+ \frac{\kappa^2 p_{\mu} p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left( \frac{\psi(1) - \psi(2)}{6} p^2 - 2p^2 \int_0^1 x(1 - x) \ln(x(1 - x))dx - \frac{p^2}{3} \ln \left( \frac{p^2}{4\pi\Lambda^2} \right) \right).
\]
If we omit \( \ln \frac{p^2}{4\pi\Lambda^2} \) for time being (to which we will come later), we get

\[
-\frac{i\kappa^2}{(4\pi)^2} \left[ (p^2)^2 \eta_{\mu\sigma} - 3p^2 p_\mu p_\sigma \right] \left[ \frac{1}{(4\pi)^2} \left( \frac{\psi(1)}{3} + \frac{13}{18} \right) + \frac{p^2}{2} \left( \frac{\psi(1) - \psi(2)}{6} + \frac{5}{9} p^2 \right) \right] + \frac{i\kappa^2 p_\mu p_\sigma}{(4\pi)^2} \left( \psi(1) + \frac{13}{6} + \frac{\psi(1) - \psi(2)}{12} + \frac{5}{18} \right) - \frac{i\kappa^2}{(4\pi)^2} \left[ (p^2)^2 \eta_{\mu\sigma} - \frac{7}{6} p_\mu p_\sigma p^2 \right].
\]

This implies that in the effective action following the quadratic term would appear

\[
\kappa^2 A^\mu [\beta \Box \partial_\mu \partial_\nu - \alpha \eta_{\mu\nu} \Box^2] A^\nu.
\]  

Note that if we include the terms containing \( \ln \frac{p^2}{4\pi\Lambda^2} \) then we need to add the following contribution

\[
\frac{i\kappa^2}{(4\pi)^2} \ln \frac{p^2}{4\pi\Lambda^2} \left[ (p^2)^2 \eta_{\mu\sigma} - \frac{7}{6} p_\mu p_\sigma p^2 \right].
\]

This will give rise to term such as

\[
A^\mu \left( \kappa^2 [\alpha' \Box^2 \eta_{\mu\nu} - \beta' \partial_\mu \partial_\nu] \ln \frac{-\Box}{4\pi\Lambda^2} \right) A^\nu.
\]

Therefore, the quadratic part of the effective Lagrangian density of effective photon degrees of freedom (after integrating out the graviton degrees of freedom) up to 1-loop would be

\[
\mathcal{L}_{\text{eff}}^{(2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \kappa^2 A^\mu \left[ \beta - \beta' \ln \frac{-\Box}{4\pi\Lambda^2} \right] \Box \partial_\mu \partial_\nu - \left( \alpha - \alpha' \ln \frac{-\Box}{4\pi\Lambda^2} \right) \eta_{\mu\nu} \Box^2 \right] A^\nu
\]

\[
= \frac{1}{2} A_\mu \left[ (\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) + \kappa^2 \left[ -\beta' \ln \frac{-\Box}{4\pi\Lambda^2} \right] \Box \partial^\mu \partial^\nu - \left( \alpha - \alpha' \ln \frac{-\Box}{4\pi\Lambda^2} \right) \eta^{\mu\nu} \Box^2 \right] A^\nu.
\]

Notice that taking into account the quantum correction of photons up to 1-loop generates the non-local \( \ln \frac{p^2}{4\pi\Lambda^2} \) term in the quadratic part of the effective action. Similar kind of non-local terms were recently found in effective field theory GR in [71], [72].

We will now calculate the dispersion relation (on-shell) due to quantum corrections which takes into account effective interactions with gravitons. But before that we need to choose a gauge and in this case we choose the Lorentz gauge \( \partial_\mu A^\mu = 0 \), due to which the effective Lagrangian density up to quadratic part becomes

\[
\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{2} A_\mu \left[ \Box \eta^{\mu\nu} - \kappa^2 \left( \alpha - \alpha' \ln \frac{-\Box}{4\pi\Lambda^2} \right) \eta^{\mu\nu} \Box^2 \right] A^\nu.
\]

Therefore, the dispersion relation of photons becomes

\[
k^2 \left( 1 + \kappa^2 \left( \alpha - \alpha' \ln \frac{k^2}{4\pi\Lambda^2} \right) k^2 \right) = 0,
\]
which has 2 branches. One of them is usual photon dispersion relation in free-field theory $k^2 = 0$ and other is the non-trivial scale dependent dispersion relation

$$1 + \kappa^2 \left( \alpha - \alpha' \ln \frac{k^2}{4\pi\Lambda^2} \right) k^2 = 0. \quad (8.14)$$

If we approximate $\alpha = \alpha' \approx \frac{1}{(4\pi)^2}$ then the above dispersion relation becomes

$$1 + \frac{\kappa^2}{(4\pi)^2} \left( 1 - \ln \frac{k^2}{4\pi\Lambda^2} \right) k^2 = 0, \quad (8.15)$$

whose solution is

$$k^2 = 4\pi\Lambda^2 e^{1 + \mathcal{W}_{\pm 1} \left( \frac{4\pi\kappa}{\Lambda^2} \right)}, \quad (8.16)$$

where $\mathcal{W}_{\pm 1}(x)$ is W-Lambert function, which has non-zero imaginary part which shows that photon amplitude decays exponentially in time. A similar dispersion relation for gravitons in effective field theory of GR was recently found in [71], [72]. Further, existence of massive photon was observed recently in [73], consistent with the present result.

However, presence of graviton mass gives one additional scale which is completely independent of Planck length scale $\kappa$ and this new scale depends on Cosmological constant [74, 75]. In principle this could also come self-energy contribution to graviton field from inflaton field [76, 77] which leads to inflation of Universe at very early stage. To see how this new scale emerges into effective action at quadratic level we need to replace in (8.3) $(p + k)^2 \to (p + k)^2 + m^2$ which modifies (8.4) to

$$-i\kappa^2 [(p^2)_{\eta\alpha}^2 - 3p^2_{\rho\mu}p_{\sigma}] \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(1) \right) \int_0^1 dx \, x^2 \left( 1 - \varepsilon \ln \left( \frac{p^2 x(1 - x) + m^2 x}{4\pi\Lambda^2} \right) \right)$$
$$\quad + \frac{i\kappa^2 p_{\rho\mu}p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left( \frac{4}{6} + \psi(1) - \psi(2) \right) \int_0^1 dx \left[ p^2 x(1 - x) + m^2 x \right]$$
$$\quad \times \left( 1 - \varepsilon \ln \left( \frac{p^2 x(1 - x) + m^2 x}{4\pi\Lambda^2} \right) \right) \right]. \quad (8.17)$$

from which one can extract out the following finite part

$$-i\kappa^2 [(p^2)_{\eta\alpha}^2 - 3p^2_{\rho\mu}p_{\sigma}] \frac{1}{(4\pi)^2} \left( \frac{\psi(1)}{3} - \int_0^1 dx \, x^2 \ln \left( \frac{p^2 x(1 - x) + m^2 x}{4\pi\Lambda^2} \right) \right)$$
$$\quad + \frac{i\kappa^2 p_{\rho\mu}p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left[ (\psi(1) - \psi(2)) \left( \frac{p^2}{6} + \frac{m^2}{2} \right) - 2 \int_0^1 dx \left( p^2 x(1 - x) + m^2 x \right) \ln \left( \frac{p^2 x(1 - x) + m^2 x}{4\pi\Lambda^2} \right) \right]$$
$$= -i\kappa^2 [(p^2)_{\eta\alpha}^2 - 3p^2_{\rho\mu}p_{\sigma}] \frac{1}{(4\pi)^2} \left[ \frac{\psi(1)}{3} \right.$$
$$\quad + \frac{13}{18} \left( \frac{6}{p^2} - 6 + \frac{18}{p^2} m^2 + 18 \frac{1}{p^4} m^4 + \frac{3}{p^6} m^6 \right) \ln \left( \frac{p^2 + m^2}{4\pi\Lambda^2} \right)$$
$$\quad + \frac{m^2}{p^2} \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + 5 \right) + \frac{m^4}{p^4} \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) + \frac{m^6}{3p^6} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right)$$
$$\quad + \frac{i\kappa^2 p_{\rho\mu}p_{\sigma}}{2} \frac{1}{(4\pi)^2} \left[ (\psi(1) - \psi(2)) \left( \frac{p^2}{6} + \frac{m^2}{2} \right) - \left( \frac{p^2}{3} + m^2 + \frac{m^4}{3p^4} \right) \ln \left( \frac{p^2 + m^2}{4\pi\Lambda^2} \right) \right]$$
$$\quad + \frac{5p^2}{9} + \frac{4m^2}{3} + \frac{m^4}{p^2} \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) + \frac{m^6}{3p^6} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \left]. \quad (8.18)\)
The above expression leads to the following inclusion of non-local operation in the action
\[ \kappa^2 A^\mu \left[ 3 \Box \partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right] \left[ \psi(1) \right] + 13 \frac{\psi(1)}{3} \left( 6 - \frac{18m^2}{\Box} + \frac{18m^4}{\Box^2} - \frac{m^6}{3\Box^3} \right) \ln \left( \frac{-\Box + m^2}{4\pi\Lambda^2} \right) \]
\[ - m^2 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + 5 \right) + m^4 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) - \frac{m^6}{3 \Box^3} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \] \[ A' \]
\[ - 5 \Box + 4m^2 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) + m^6 \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \} \] \[ A'' \].

(8.19)

However, choosing \( \partial_\mu A^\mu = 0 \) gauge leads to the following contribution
\[ -\kappa^2 A^\mu \left[ 3 \Box \partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right] \left[ \psi(1) \right] + 13 \frac{\psi(1)}{3} \left( 6 - \frac{18m^2}{\Box} + \frac{18m^4}{\Box^2} - \frac{m^6}{3\Box^3} \right) \ln \left( \frac{-\Box + m^2}{4\pi\Lambda^2} \right) \]
\[ - m^2 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + 5 \right) + m^4 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) - \frac{m^6}{3 \Box^3} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \] \[ A' \]
\[ \left[ -5 \Box + 4m^2 \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) + m^6 \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \right] \] \[ A'' \].

(8.20)

Taking into account this quantum correction gives the following on-shell condition
\[ p^2 + \kappa^2 p^4 \left[ 3 \Box \partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right] \left[ \psi(1) \right] + 13 \frac{\psi(1)}{3} \left( 6 - \frac{18m^2}{p^2} + \frac{18m^4}{p^4} - \frac{m^6}{3p^6} \right) \ln \left( \frac{p^2 + m^2}{4\pi\Lambda^2} \right) \]
\[ + \frac{m^2}{p^2} \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + 5 \right) + \frac{m^4}{p^4} \left( \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{1}{3} \right) + \frac{m^6}{3 p^6} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) \] \[ = 0. \]

(8.21)

Although, we don’t give an analytic expression of dispersion relation, however, it can be easily checked that tree-level dispersion \( p^2 = 0 \) is no longer valid and is shifted in complex plane to a point which depends on the ratio of \( \kappa, m^2 \) to \( \Lambda^2 \).

Another point that we want to highlight is the following. The above expression suggests that after considering 1-loop correction the propagator of photons would be \( \frac{1}{p^2 + \Sigma(p^2)} \), where \( \Sigma(p^2) \) is the self energy of the photon. A nice property of the theory, apart from IR modification, is that in the UV limit the propagator essentially becomes \( \frac{1}{p^2 + \Sigma(p^2)} \sim -\kappa^2 p^4 \ln \left( \frac{p^2 + m^2}{4\pi\Lambda^2} \right) \).

This shows that the degree of superficial divergence of any diagram is reduced by 2. Hence, taking into quantum corrections at 1-loop level modifies both UV and IR limit of the theory significantly.

### 8.2 Vertex renormalization

Figure 2 depicts the 1-loop vertex diagram, which is computed next. The diagram corresponds to
\[ - \int \frac{d^4k}{(2\pi)^4} \frac{\eta^{\alpha\gamma}}{k^2} \frac{\eta^{\beta\delta}}{(p-k)^2} \mathcal{V}_{\mu\nu\alpha\beta}(p_1, p_2, k, p - k) \mathcal{V}_{\gamma\delta\rho\sigma}(k, p - k, p_3, p_4). \]

(8.22)
Note that the above multiplication generates 400 terms of which many terms cancel after

\[ \eta^{\alpha\gamma} \eta^{\beta\delta} V_{\mu\nu\alpha\beta}(p_1, p_2, k, p - k) \]

Therefore, we need to compute the following expression

\[ \eta^{\alpha\gamma} \eta^{\beta\delta} V_{\mu\nu\alpha\beta}(p_1, p_2, k, p - k) V_{\gamma\delta\rho\sigma}(k, p - k, p_3, p_4) \]

\[ = \left[ p_1.p_2.k.(p - k) \eta_{\mu\nu}(p - k)^\gamma - k.(p - k) \eta^{\gamma\delta} p_{1\nu}p_{2\mu} \right. \\
+ p_{1\nu}p_{2\mu}k^\delta(p - k)^\gamma - p_{1\nu}p_{2}(k - p)\delta_\mu^\gamma \delta_\nu^\delta + p_1.p_2\delta_\mu^\gamma \delta_\nu^\delta (p - k)_\nu + p_1.p_2\delta_\mu^\gamma k_\mu(p - k)^\gamma \\
- p_1.p_2\eta^{\gamma\delta} k_\mu(p - k)_\nu + k.(p - k)\delta_\mu^\gamma p_{1\nu}p_2^\delta - p_2.(p - k)\delta_\mu^\gamma p_{1\nu}k_\mu(p - k)^\gamma \\
+ p_2.(p - k)\eta^{\gamma\delta} p_{1\nu}k_\mu + k.(p - k)\eta^{\gamma\delta} p_{1\nu}k_\mu - p_1.p_2\delta_\mu^\gamma (p - k)_\nu - p_1.p_2\delta_\mu^\gamma p_{2\mu}(p - k)^\gamma \\
+ p_1.p_2\eta^{\gamma\delta} p_{2\mu}(p - k)_\nu - k.(p - k)\eta_{\mu\nu}p_1^\gamma p_2^\delta + p_2.(p - k)\eta_{\mu\nu}p_1^\gamma k_\mu + p_1.p_2\eta_{\mu\nu}p_1^\gamma p_2^\delta(p - k)^\gamma \\
- \eta_{\mu\nu}\eta^{\gamma\delta} p_{1\nu}p_2.(p - k) \right] \frac{\kappa^2}{p^2}, \\
\]

where \( p = p_1 + p_2 = p_3 + p_4 \).

Note that the above multiplication generates 400 terms of which many terms cancel after
some algebraic manipulations and we are left with the following expression which is still large

\[ \frac{\kappa^4}{(p^2)^4} \rho_1^n \rho_2^\alpha \rho_3^\beta \rho_4^\gamma \left[ - \eta_{\alpha \beta} k.(p-k) k_\delta \eta_{\gamma \sigma} \eta_{\mu \nu} (p-k) \rho - \eta_{\alpha \beta} k.(p-k) (p-k)_\gamma k_\sigma \eta_{\delta \rho} \eta_{\mu \nu} \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} k.(p-k) \rho_k \eta_{\mu \nu} + \eta_{\alpha \beta} k.(p-k) (p-k)_\rho k_\delta \eta_{\mu \nu} \eta_{\sigma \rho} \\
- k.(p-k) \eta_{\gamma \sigma} (p-k)_\rho k_\delta \eta_{\mu \nu} \eta_{\beta \mu} + k.(p-k) k_\delta (p-k)_\rho \eta_{\gamma \sigma} \eta_{\mu \nu} \eta_{\beta \mu} \\
+ k.(p-k) (p-k) \rho_k \eta_\delta \eta_{\gamma \sigma} \eta_{\mu \nu} k.(p-k) (p-k)_\rho k_\delta \eta_{\mu \nu} \eta_{\sigma \rho} \\
- \eta_{\alpha \beta} \eta_{\gamma \sigma} (k.(p-k))^2 \eta_{\mu \nu} \eta_{\sigma \rho} + \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} (k.(p-k))^2 \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} k.(p-k) (p-k)_\mu k_\nu \eta_{\sigma \rho} - \eta_{\alpha \beta} k.(p-k) (p-k)_\mu k_\nu \eta_{\sigma \rho} \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} (k.(p-k))^2 \eta_{\mu \nu} \eta_{\sigma \rho} - \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} (k.(p-k))^2 \\
- \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} \eta_{\sigma \rho} (k.(p-k))^2 \\
- \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} \eta_{\sigma \rho} (k.(p-k))^2 \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} \eta_{\sigma \rho} (k.(p-k))^2 \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} \eta_{\sigma \rho} (k.(p-k))^2 \\
+ \eta_{\alpha \beta} \eta_{\gamma \sigma} \eta_{\delta \rho} \eta_{\mu \nu} \eta_{\sigma \rho} (k.(p-k))^2 \right] \]

\[ (8.25) \]
−ηδηανυκ(п−к)γαδρ(п−к)σ + ηανυηασ(п−к)δ(п−к)ρκβδμκ −ηαυηασκβп(п−к)γ(п−к)σ − ηαυηασ(п−к)δκβ(п−к)γκμ +ηγηεηηαυκκαδργσηγσηηαυ(п−к)δ +ηγηεηηαυκκαδργσηγσηηαυ(п−к)δ2 − ηαυηασηγσηηαυ(п−к)δ)2 −ηαυηασκκαδργσηγσηηαυ(п−к)δ +ηγηεηηαυκκαδργσηγσηηαυ(п−к)δ2 − ηαυηασηγσηηαυ(п−к)δ)2 +ηγηεηηαυκκαδργσηγσηηαυ(п−к)δ2 − ηαυηασηγσηηαυ(п−к)δ)2
The computation of each integral coming from such a large expression would be very cumbersome. Therefore, we compute integral of most general term from which, in principle, all the above terms could be computed by contractions.

The most general form of the integral that we need to compute is of the following form

\[
\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(p-k)^2} k^a k^b (p-k)^c (p-k)^d = \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{1}{(k - px)^2 + p^2 x(1-x))^2} k^a k^b (p-k)^c (p-k)^d
\]

\[
= \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{1}{k^2 + p^2 x(1-x))^2} (k + px)^a (k + px)^b (p(x) - k)^c (p(x) - k)^d
\]

\[
= \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{1}{k^2 + p^2 x(1-x))^2} (k^a k^b + (k^a p^b + p^a k^b) x + p^a p^b x^2)
\]

\[
\times (p^c p^d (1-x)^2 - (k^c p^d + p^c k^d)(1-x) + k^c k^d)
\]

\[
= p^c p^d p^e p^f \int \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{x^2(1-x)^2}{k^2 + p^2 x(1-x))^2} dx + p^c p^d \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{(1-x)^2 k^a k^b}{k^2 + p^2 x(1-x))^2}
\]

\[
- \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{x(1-x)(k^a k^b + k^b p^a)(k^c p^d + p^c k^d)}{k^2 + p^2 x(1-x))^2} + \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{k^a k^b k^c k^d}{k^2 + p^2 x(1-x))^2}
\]

\[
+ p^c p^d \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{x^2 k^c k^d}{k^2 + p^2 x(1-x))^2},
\]

\[
(8.26)
\]

where \( D = 4 - \varepsilon \) in dimensional regularization.

Now we compute each of the integrals separately.

\[
\mathcal{I}_1 = \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{x^2(1-x)^2}{k^2 + p^2 x(1-x))^2}
\]

\[
= \frac{1}{(4\pi)^2} \Gamma(\frac{\varepsilon}{2}) \int_0^1 x^2(1-x)^2 [p^2 x(1-x)]^{-\frac{\varepsilon}{2}} dx
\]

\[
= \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(0) \right) \left[ \int_0^1 x^2(1-x)^2 dx - \frac{\varepsilon}{2} \int_0^1 x^2(1-x)^2 \ln(x(1-x)) + \ln \left( \frac{p^2}{4\pi \mu^2} \right) dx \right]
\]

\[
= \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon} + \psi(0) \right) \left[ \frac{1}{30} \left( 1 - \frac{\varepsilon}{2} \ln \left( \frac{p^2}{4\pi \mu^2} \right) \right) + \frac{47\varepsilon}{1800} \right]
\]

\[
\Rightarrow \mathcal{I}_1^{\text{(finite)}} = \frac{1}{(4\pi)^2} \left[ \psi(0) - \frac{1}{30} \ln \left( \frac{p^2}{4\pi \mu^2} \right) + \frac{47}{900} \right]
\]

(8.27)
\[ I_2 = \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{(1-x)^2 k^a k^b}{[k^2 + p^2 x(1-x)]^2} = \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{x^2 k^a k^b}{[k^2 + p^2 x(1-x)]^2} \]
\[ = \frac{\eta^{ab}}{4} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{k^2}{[k^2 + p^2 x(1-x)]^2} \]
\[ = \frac{\eta^{ab}}{4} \left[ \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{x^2}{[k^2 + p^2 x(1-x)]} - \frac{p^2}{2} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{x^3(1-x)}{[k^2 + p^2 x(1-x)]^2} \right] \]
\[ = \frac{\eta^{ab}}{4} \left( \Gamma(-1 + \frac{\varepsilon}{2}) - \Gamma\left(\frac{\varepsilon}{2}\right) \right) \left[ \int_0^1 x^3(1-x) dx - \frac{\varepsilon}{2} \int_0^1 x^3(1-x) \left( \ln \frac{p^2}{4\pi^2} + \ln(x(1-x)) \right) dx \right], \quad (8.28) \]

\[ \Rightarrow I_2 = -\frac{\eta^{ab} p^2}{4(4\pi)^2} \left( \frac{4}{\varepsilon} + \psi(0) + \psi(1) \right) \left[ \frac{1}{20} \left( 1 - \frac{\varepsilon}{2} \ln \frac{p^2}{4\pi^2} \right) + \frac{13}{300} \right] \]
\[ \Rightarrow I_2^{(finite)} = -\frac{\eta^{ab} p^2}{80(4\pi)^2} \left( \psi(0) + \psi(1) - \frac{1}{10} \ln \frac{p^2}{4\pi^2} + \frac{13}{75} \right) \]
\[ I_3 = \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{x(1-x) k^a k^b p^d + k^a k^d p^b p^c + k^b k^c p^a p^d + k^b k^d p^a p^c}{[k^2 + p^2 x(1-x)]^2} \]
\[ = \frac{\eta^{ac} p^d p^c + \eta^{ad} p^d p^c + \eta^{bc} p^a p^d + \eta^{bd} p^a p^c}{(4\pi)^2} \left( \Gamma(-1 + \frac{\varepsilon}{2}) - \Gamma\left(\frac{\varepsilon}{2}\right) \right) \]
\[ \times \left[ \int_0^1 x^2(1-x)^2 dx - \frac{\varepsilon}{2} \int_0^1 x^2(1-x)^2 \left( \ln \frac{p^2}{4\pi^2} + \ln(x(1-x)) \right) \right] \]
\[ = -\frac{\eta^{ac} p^d p^c + \eta^{ad} p^d p^c + \eta^{bc} p^a p^d + \eta^{bd} p^a p^c}{(4\pi)^2} \left( \frac{4}{\varepsilon} + \psi(0) + \psi(1) \right) \]
\[ \times \left[ \frac{1}{30} \left( 1 - \frac{\varepsilon}{2} \ln \frac{p^2}{4\pi^2} \right) + \frac{47}{1800} \right] \]
\[ \Rightarrow I_3^{(finite)} = -\frac{\eta^{ac} p^d p^c + \eta^{ad} p^d p^c + \eta^{bc} p^a p^d + \eta^{bd} p^a p^c}{(4\pi)^2} \left[ \psi(0) + \psi(1) \right] \left( \frac{1}{30} - \frac{1}{15} \ln \frac{p^2}{4\pi^2} + \frac{47}{450} \right) \]
\[ I_4 = \frac{1}{\alpha} \left( \eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} \right) \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{k^4}{[k^2 + p^2 x(1-x)]^2} \]
\[ = \frac{1}{\alpha} \left( \eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} \right) \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{(k^2 + p^2 x(1-x))^2}{[k^2 + p^2 x(1-x)]^2} \]
\[ = \frac{1}{\alpha} \left( \eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} \right) \left( \frac{p^2}{4\pi^2} \right)^2 \left( \frac{\psi(0) + \psi(1)}{30} - \frac{1}{15} \ln \frac{p^2}{4\pi^2} + \frac{47}{900} \right) \]
\[ - \frac{2p^4}{\alpha(4\pi)^2} \left( \eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} \right) \left( \psi(0) + \psi(1) \right) \left( \frac{1}{30} - \frac{1}{15} \ln \frac{p^2}{4\pi^2} - \frac{47}{450} \right), \quad (8.29) \]
where $\alpha$ is a numerical coefficient. Therefore, the net finite contribution would be

$$\Delta I_{\text{finite}} = \left[ p^a p^b p^c p^d I_1^{\text{finite}} + (p^e p^d \eta^{ab} + p^e p^b \eta^{cd}) I_2^{\text{finite}} + I_4^{\text{finite}} - I_3^{\text{finite}} \right].$$

(8.30)

Thus, in the effective action in place of $p^2$ one just need to write $-\Box$ and each $p^a$ term would be replaced by $-i\partial^a$.

Hence, we have just shown the procedure for taking into account the quantum correction at a 4-point vertex in the effective action. We can also see that it contains non-local terms because of the presence of $\ln \frac{\Box}{4\pi^2}$, where $\mu$ is the momentum scale upto which this theory is valid in perturbative manner.

9 Soft gravitons and non-perturbative action of photons

In this section we will discuss the reason why we write down an action describing effective interacting photons which incorporate the effect of interactions with gravitons, eq. (5.15), in a non-perturbative manner.

9.1 Asymptotic dynamics of graviton-photon interaction

Recall that in the original theory we have the the following interaction terms in the photon-graviton action

$$\mathcal{L}_{\text{int}} = \kappa h_{\mu\nu} T^{\mu\nu},$$

$$T_{\mu\nu} = \eta^{\alpha\beta} F_{\mu\alpha} F_{\beta\nu} - \frac{1}{4} \eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

$$= \eta^{\alpha\beta} (\partial_\alpha A_\mu - \partial_\mu A_\alpha)(\partial_\beta A_\nu - \partial_\nu A_\beta) - \frac{1}{2} \eta_{\mu\nu} (\partial_\alpha A_\sigma \partial_\sigma A^\alpha - \partial_\rho A^\rho \partial^\sigma A^\sigma).$$

(9.1)

Now we will go to momentum space description using Fourier transform

$$\hat{h}_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{E_k}} (\hat{h}_{\mu\nu}(k)e^{ik.x} + \hat{h}^\dagger_{\mu\nu}(k)e^{-ik.x})$$

$$\hat{A}_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} (\hat{a}_\mu(p)e^{ip.x} + \hat{a}^\dagger_\mu(p)e^{-ip.x}),$$

(9.2)

where $E_p = |p|$.
This leads to the following expression for stress-energy tensor

\[
\hat{T}_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{E_p E_q}} \left[ \eta^{\alpha\beta}(ip_{\alpha}\hat{\alpha}_{\mu}(p)e^{ip.x} - ip_{\alpha}\hat{\alpha}^{\dagger}_{\mu}(p)e^{-ip.x} - ip_{\beta}\hat{\alpha}_{\nu}(p)e^{ip.x} + ip_{\beta}\hat{\alpha}^{\dagger}_{\nu}(p)e^{-ip.x}) \times (iq_{\beta}\hat{\alpha}_{\nu}(q)e^{iq.x} + iq_{\beta}\hat{\alpha}^{\dagger}_{\nu}(q)e^{-iq.x} - iq_{\alpha}\hat{\alpha}_{\beta}(q)e^{iq.x} + iq_{\alpha}\hat{\alpha}^{\dagger}_{\beta}(q)e^{-iq.x}) \right] \times \left( \frac{1}{2} \eta_{\mu\nu} \left[ (ip_{\rho}\hat{\alpha}_{\sigma}(p)e^{ip.x} - ip_{\rho}\hat{\alpha}^{\dagger}_{\sigma}(p)e^{-ip.x})(iq^{\rho}\hat{\alpha}\sigma e^{iq.x} - iq^{\rho}\hat{\alpha}\sigma^{\dagger}(q)e^{-iq.x}) - (ip_{\rho}\hat{\alpha}_{\sigma}(p)e^{ip.x} - ip_{\rho}\hat{\alpha}^{\dagger}_{\sigma}(p)e^{-ip.x})(iq^{\rho}\hat{\alpha}\sigma^{\dagger}(q)e^{-iq.x} - iq^{\rho}\hat{\alpha}\sigma e^{iq.x}) \right] \right) \right]
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{E_p E_q}} \left\{ \left[ -p.q\hat{\alpha}_{\mu}(p)\hat{\alpha}_{\nu}(q)e^{i(p+q).x} + p.q\hat{\alpha}_{\mu}(p)\hat{\alpha}_{\nu}(q)e^{i(p-q).x} + q_o\hat{\alpha}_{\mu}(p)p.\hat{\alpha}(q)e^{i(p+q).x} - q_o\hat{\alpha}_{\mu}(p)p.\hat{\alpha}(q)e^{i(p-q).x} - q_o\hat{\alpha}_{\mu}(p)p.\hat{\alpha}(q)e^{i(p+q).x} + q_o\hat{\alpha}_{\mu}(p)p.\hat{\alpha}(q)e^{i(p-q).x} \right] \right\}
\]

This allows writing the interacting action corresponding to (9.1) as

\[
S_{int} = \int d^4x \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{E_p E_q} E_k} \left\{ \left[ -p.q\hat{h}^{\mu\nu}(k)\hat{\alpha}_{\mu}(p)\hat{\alpha}_{\nu}(q)e^{i(p+q+k).x} + p.q\hat{h}^{\mu\nu}(k)\hat{\alpha}_{\mu}(p)\hat{\alpha}^{\dagger}_{\nu}(q)e^{i(p+q-k).x} \right] \left[ -q_x\hat{h}_{\mu\nu}(k)q_x\hat{\alpha}_{\mu}(p)\hat{\alpha}(q)e^{i(p+q+k).x} - q_x\hat{h}_{\mu\nu}(k)q_x\hat{\alpha}_{\mu}(p)\hat{\alpha}^{\dagger}(q)e^{i(p+q-k).x} + q_x\hat{h}_{\mu\nu}(k)q_x\hat{\alpha}_{\mu}(p)\hat{\alpha}(q)e^{i(p+q-k).x} + q_x\hat{h}_{\mu\nu}(k)q_x\hat{\alpha}_{\mu}(p)\hat{\alpha}^{\dagger}(q)e^{i(p+q+k).x} \right] \right. \]

(9.3)
which produce corresponding set of Dirac-delta functions after integration over spacetime

\[ \varepsilon_{\mu\nu}(\lambda, k) = \sum_{\lambda} \hat{h}(k, \lambda)e^{\mu\nu}(\lambda, k), \]

where

\[ \hat{h}_{\mu\nu}(k) = \sum_{\lambda} \hat{h}(k, \lambda)e^{\mu\nu}(\lambda, k), \]

\[ \hat{a}_\mu(p) = \sum_{\lambda} \hat{a}(p, \lambda)e_{\mu}(\lambda, p). \]

Here \( e^{\mu\nu}(\lambda, k) \) is the spin-2 polarization tensor, \( e_{\mu}(p, \lambda) \) is spin-1 polarization vector.

In the above expression of \( S_{int} \), we can classify the following terms

\[ I) \ e^{i(p+q+k).x}, \ e^{-i(p+q+k).x} \]

\[ II) \ e^{i(p-q+k).x}, \ e^{i(q-p-k).x} \]

\[ III) \ e^{i(k-p-q).x}, \ e^{i(p+q-k).x} \]

\[ IV) \ e^{i(q-p-k).x}, \ e^{i(p-q-k).x} \]

which produce corresponding set of Dirac-delta functions after integration over spacetime

\[ I) \ \delta^{(4)}(p + q + k) \]

\[ II) \ \delta^{(4)}(p - q + k) \]

\[ III) \ \delta^{(4)}(p + q - k) \]

\[ IV) \ \delta^{(4)}(p - q - k) \]

After the \( \int d^3q \) integration we have the following four kinds of terms

\[ I) \ \delta(E_p + E_k + E_{p+k}) \]

\[ II) \ \delta(E_p + E_k - E_{p+k}) \]

\[ III) \ \delta(E_p - E_k + E_{p-k}) \]

\[ IV) \ \delta(E_p - E_k - E_{p-k}). \]
The first class of terms contribute only if
\[ E_p + E_k + E_{p+k} = 0 \implies |\vec{p}| + |\vec{k}| + |\vec{p} + \vec{k}| = 0. \] (9.10)

This is only possible iff \( \vec{p} = 0 = \vec{k} \) which can only come from soft-photons and soft-gravitons.

The second class of terms would contribute iff
\[ E_p + E_k = E_{p+k} \implies |\vec{p}| + |\vec{k}| = |\vec{p} + \vec{k}| \implies |\vec{p}||\vec{k}| = 2\vec{p} \cdot \vec{k}, \] (9.11)
i.e., when \( \vec{p} = \alpha \vec{k} = \vec{q} = (1 + \alpha)\vec{k} \), which implies that photons and gravitons are co-linear.

The third and fourth class terms can be analyzed in similar fashion. This implies that interaction term can’t be switched off at past and future infinite time limit [78].

9.2 Interaction vertex function

\[
\int T_{\mu\nu}(x) h^{\mu\nu}(x) d^4x = \int d^4x \left[ \eta^{\alpha\beta} (\partial_\alpha A_\mu - \partial_\mu A_\alpha)(\partial_\beta A_\nu - \partial_\nu A_\beta) - \frac{1}{2} \eta_{\mu\nu} (\partial_\rho A_\sigma \partial^\rho A^\sigma - \partial_\rho A_\sigma \partial^\sigma A^\rho) \right] h^{\mu\nu} \\
= \int d^4x \int \frac{d^4p \ d^4q \ d^4k}{(2\pi)^{12}} \left[ \eta^{\alpha\beta} (ip_\alpha A_\mu(p) - ip_\mu A_\alpha(p))(iq_\beta A_\nu(q) - iq_\nu A_\beta(q)) \right. \\
\left. - \frac{1}{2} \eta_{\mu\nu} (-p.A(p).A(q) + p.A(q) + p.A(p)q.A(p)) \right] h^{\mu\nu}(k) e^{i(p+q+k) \cdot x} \\
= \int \frac{d^4p \ d^4q \ d^4k}{(2\pi)^{8}} \delta^{(4)}(p + q + k) \left[ (-p.A_\mu(p)A_\nu(q) + p.A(q)A_\mu(p) + q.A(p)p_\mu A_\nu(q) \\
- p_\mu q_\nu A(p).A(q)) h^{\mu\nu}(k) - \frac{1}{2} \eta_{\mu\nu} h^{\mu\nu}(k)(p.A(q)q.A(p) - p.q.A(p).A(q)) \right] \\
= \int \frac{d^4p \ d^4q \ d^4k}{(2\pi)^{8}} \delta^{(4)}(p + q + k) h^{\mu\nu}(k) A^{\rho}(p)A^\rho(q) \Gamma_{\mu\nu\rho\sigma}(k, p, q). \\
\Gamma_{\mu\nu\rho\sigma}(k, p, q) = \left\{ -p.q\eta_{\mu\nu}\eta_{\rho\sigma} + p_\sigma q_\nu\eta_{\mu\rho} + q_\rho p_\mu\eta_{\nu\sigma} - p_\mu q_\alpha\eta_{\nu\sigma} - \frac{1}{2} \eta_{\mu\nu}(p_\sigma q_\rho - p_\rho q_\sigma) \right\}, \tag{9.12} \]

where \( \Gamma_{\mu\nu\rho\sigma}(k, p, q) \) is the interaction vertex.
9.3 Photon mass renormalization

In fig. 3 we have shown self-energy diagram for photons upto 1-loop is shown. It represents

\[ \int \frac{d^4q}{(2\pi)^4}G_{\mu \nu \rho \sigma}(q)G_{\gamma \delta}(p-q)\Gamma^{\mu \nu \alpha \eta}(-q,p,q,p-q)\Gamma^{\rho \sigma \delta \beta}(q,p-q,-p) \]

\[ = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2((p-q)^2 + \mu^2)}(\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho} - \eta_{\mu \nu} \eta_{\rho \sigma})\eta_{\delta} \]

\[ \times \Gamma^{\mu \nu \alpha \eta}(-q,p,q,p-q)\Gamma^{\rho \sigma \delta \beta}(q,p-q,-p) \]

\[ = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2((p-q)^2 + \mu^2)}(\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho} - \eta_{\mu \nu} \eta_{\rho \sigma})\eta_{\delta} \]

\[ \times \left\{ -p.(q-p)\eta^{\rho \alpha}p^{\eta \eta} + p^\eta(q-p)^\alpha p^\rho(q-p)^\beta \eta^{\rho \delta} - p^\rho(q-p)^\beta p^\eta(q-p)^\alpha \eta^{\rho \delta} + (q-p)^\alpha p^\rho \eta^{\rho \delta} - (q-p)^\beta p^\rho \eta^{\rho \delta} - 1 \right\} \]

\[ \times \left\{ -p.(q-p)\eta^{\rho \alpha}p^{\eta \eta} + p^\eta(q-p)^\alpha p^\rho(q-p)^\beta \eta^{\rho \delta} - p^\rho(q-p)^\beta p^\eta(q-p)^\alpha \eta^{\rho \delta} + (q-p)^\alpha p^\rho \eta^{\rho \delta} - (q-p)^\beta p^\rho \eta^{\rho \delta} - 1 \right\}, \]

where \( \mu \) is photon mass regulator. After some algebraic manipulations the above expression can be seen to be

\[ \int \frac{d^4q}{(2\pi)^4} \int^1_0 \frac{dx}{(q - px)^2 + p^2 x(1 - x) + \mu^2 x^2} \left\{ 3(p.(q-p))^2 \eta^{\alpha \beta} - 2p.(q-p)p^{\alpha}(q-p)^\beta \right. \]

\[ - 2p.(q-p)p^\beta(q-p)^\alpha + 2p^2(q-p)^\alpha(q-p)^\beta + 2p^2(q-p)^2 \eta^{\alpha \beta} - 2(q-p)^2 p^{\alpha} p^{\beta} \}

\[ = \int \frac{d^4q}{(2\pi)^4} \int^1_0 \frac{dx}{(q - px)^2 + p^2 x(1 - x) + \mu^2 x^2} \left\{ 3(p.(q+px-p))^2 - 2p.(p+q+px)p^{\alpha}(q+px-p)^\beta \right. \]

\[ - 2p.(q+px-p)p^\beta(q+px+p)^\alpha + 2p^2(q+px-p)^\alpha(q+px-p)^\beta + 2p^2(q+px-p)^2 \eta^{\alpha \beta} \]

\[ - 2(q+px-p)^2 p^{\alpha} p^{\beta} \}, \]

which can be further simplified into the following form

\[ \int \frac{d^4q}{(2\pi)^4} \int^1_0 \frac{dx}{q^2 + p^2 x(1 - x) + \mu^2 x^2} \]

\[ \times \left( \frac{13}{4} p^2 q^2 \eta^{\alpha \beta} + 5p^4(x-1)^2 \eta^{\alpha \beta} - 2p^2 p^{\alpha} p^{\beta}(x-1)^2 - 2p.q(p^\alpha q^\beta + p^\beta q^\alpha) \right) \]

\[ = \int \frac{d^4q}{(2\pi)^4} \int^1_0 \frac{dx}{q^2 + p^2 x(1 - x) + \mu^2 x^2} \]

\[ \times \left( \frac{13}{4} p^2 q^2 \eta^{\alpha \beta} + 5p^4(x-1)^2 \eta^{\alpha \beta} - 2p^2 p^{\alpha} p^{\beta}(x-1)^2 - 4p^\alpha p^\beta q^2 \right). \]
This expression will get modified for the case of massive gravitons to

$$\int \frac{d^4 q}{(2\pi)^4} \int_0^1 dx \frac{1}{[q^2 + p^2 x(1-x) + \mu^2 x]^2 + m^2 (1-x)} \times \left( \frac{13}{4} p^2 q^2 \eta^{\alpha\beta} + 5 p^4 (x-1)^2 \eta^{\alpha\beta} - 2 p^2 p^\alpha p^\beta (x-1)^2 - 4 p^\alpha p^\beta q^2 \right), \quad (9.17)$$

where the first and last term do not have IR divergence and hence, can be removed. The remaining pieces contribute as

$$\int \frac{d^4 q}{(2\pi)^4} \int_0^1 dx \frac{1}{[q^2 + p^2 x(1-x) + \mu^2 x]^2 + m^2 (1-x)} (5 p^4 (x-1)^2 \eta^{\alpha\beta} - 2 p^2 p^\alpha p^\beta (x-1)^2) = \frac{1}{(4\pi)^2} \left( \frac{2}{\varepsilon + \psi(1)} \right) \left[ \frac{5 p^4}{3} \eta^{\alpha\beta} - \frac{2 p^2 p^\alpha p^\beta}{3} \right] - \frac{\varepsilon}{2} \int_0^1 dx (x-1)^2 \ln [(p^2 x(1-x) + \mu^2 x + m^2 (1-x)) \times (5 p^4 \eta^{\alpha\beta} - 2 p^2 p^\alpha p^\beta)]. \quad (9.18)$$

The coefficient of $\varepsilon$ term, which is an $x$-integral, gives

$$\frac{p^6 - 3 \mu^2 p^4 + (3 m^2 \mu^2 - 3 p^4) p^2 - \mu^6 + 3 m^2 \mu^4 - 3 m^4 \mu^2 + m^6}{p^6} \ln \frac{m^2}{\mu^2} + \frac{1}{3} \ln \frac{m^2}{4\pi \Lambda^2} \text{IR-finite terms.} \quad (9.19)$$

This shows that in the $\mu \to 0$ and $m \to 0$ limit that integral gives IR-divergence which could be attributed to the soft gravitons, where $\Lambda$ is effective momentum scale of the theory. We have seen that during photon-mass renormalization from photons ‘effective action’ eq. (8.11) that such IR term did not arise, because during the process of integrating out the graviton degrees of freedom the contribution of the soft gravitons [79–83] are taken into account leading to a finite answer, apart from a UV-divergent term. Hence, the interacting action for photons, described earlier non-perturbatively, captures all kinds of interactions between gravitons and photons. Therefore, the treatment developed here is non-perturbative in nature.

10 Conclusion

The effective action for photons, developed here, non-perturbatively captures all possible interactions between photons and gravitons at the quantum level. Furthermore, it is shown that through weak measurement protocol one can enhance the strength of the scattering amplitudes or cross-sections of the scattering process between multiple photons which would make it possible to be measured in the laboratory. Polarization measurement of photons will also be able to capture this interaction, an idea also suggested in [84, 85]. Though, S-matrix elements of photon-graviton interaction were calculated before in [86–88], here we have used a different approach in which instead of measuring gravitons directly, inferences can be drawn from measurements on photon states, a task comparatively easier to achieve in current experimental scenarios.

We have also shown how Maxwell’s equations get modified in the presence of gravity. This can capture the properties of the source of GW, such as compact objects, binary mergers, in terms of their stress-energy tensors. Furthermore, the propagation of longitudinal polarization degree of light in the presence of GW is brought out, a feature absent in well-known
Maxwell’s equations in vacuum. Vacuum birefringence property [89, 89–91] is seen to emerge from the modified Maxwell’s equations where polarizability and magnetization non-linearly depend on the electric and magnetic fields.

We also consistently compared results due to gravitons from GR with massive gravity theory which is an IR modified version of GR. This will put a bound on the mass of gravitons, and by studying graviton-photon interaction using photon’s effective action in (5.3) one can suitably modify IR domain of GR. This will have implications to our understanding of Dark energy [92–94] and expansion of the Universe [93, 95, 96].

Finally, modified dispersion of photons has been computed by taking into account one-loop quantum corrections both in case of massive and massless gravitons. This dispersion is shown to be scale dependent (these scales are basically Planck length scale or Planck mass and graviton mass scale), coming from the presence of non-local terms in the quantum effective action at the quadratic level. This dispersion also shows presence of longitudinal degree of freedom for interacting photons, a feature which could also be seen in a different context from the gauge symmetry breaking of on-shell equations (5.11, 5.13) in presence of classical source $T^{(c)}_{\mu\nu}$, although the effective action (5.3) is gauge invariant.

The method of integrating out the graviton degrees of freedom and writing down an effective action for photon degrees of freedom is a very useful technique particularly in 2 + 1-dimensional gravity theories coupled to matter. In 2 + 1-dimensional gravity, it is quite well-known that there are no on-shell graviton degrees of freedom and hence there would not be any external graviton legs in any scattering processes and presence of soft-divergences coming from off-shell graviton (3 degrees of freedom) loops can completely be avoided through this non-perturbative technique shown earlier.

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