Thurston boundary of the Teichmüller space is the space of big bang singularities of 2+1 gravity

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Abstract. ADM formalism of vacuum general relativity in constant mean extrinsic curvature spatial harmonic (CMCSH) gauge is used to study the asymptotic behaviour of the solution curves of the dynamics on spacetimes of the type \( \Sigma_g \times R \), \( g > 1 \), where \( \Sigma_g \) is a closed Riemann surface of genus \( g \). Configuration space of the gauge fixed dynamics is identified with the Teichmüller space (\( T\Sigma_g \approx \mathbb{R}^{6\text{genus}-6} \)) of \( \Sigma_g \). Utilizing the properties of the Dirichlet energy of certain harmonic map, estimates derived from the associated elliptic equations in conjunction with few standard results of surface theory, we show that every solution curve runs off the edge of the Teichmüller space at the limit of big bang singularity and attaches to the space of projective measured laminations/foliations (\( \mathcal{PML} \) or \( \mathcal{PMF} \)) or namely the Thurston boundary of the Teichmüller space. This result while identifies the complete solution space of the Einstein equations on flat spacetimes of the type \( \Sigma_g \times R \), also yields yet another way to compactify the Teichmüller space.

1. Introduction

2+1 gravity formulated on the spacetimes of the type \( \Sigma_g \times R \), where \( \Sigma_g \) is the closed (compact without boundary) Riemann surface of genus \( g > 1 \), is of considerable interests in mathematical relativity despite the fact that it does not allow for gravitational waves degrees of freedom and as such is devoid of straightforward physical significance. However, it becomes extremely important while studying ‘3 + 1’ gravity on spacetimes of certain topological type. [9] studied the Einstein equations for vacuum spacetimes with spatial topology being the circle bundle over \( S^2 \). Later [7, 6, 8] studied the vacuum Einstein equations for spacetimes with spatial topology being circle bundles over higher genus Riemann surfaces (\( g > 1 \)), where 3+1 gravity is reduced to 2+1 gravity coupled to a wave map which has the hyperbolic plane as its target space. In addition to these classical analysis, considerable attention has been paid quantum mechanically [11, 10, 4], where 2+1 gravity is essentially treated as a toy model for 3+1 quantum gravity.

Despite such physical motivations to study 2+1 gravity as a tool for studying physically interesting 3+1 gravity, 2+1 gravity is itself a mathematically rich topic with several open issues even in the purely classical level. Considerable amount of work has been done on purely classical 2+1 gravity. Prof. Moncrief [1] reduced the Einstein equations in 2+1 dimensions to a Hamiltonian system over the Teichmüller space,
where the phase space of the dynamics was identified with the co-tangent bundle of the Teichmüller space \((\approx R^{12g-12})\). Later [12] proved the global existence of the Einstein equations on spacetimes of the type \(\Sigma_g \times R, g > 1\) by controlling the Dirichlet energy (a proper function on the Teichmüller space) of an associated harmonic map. Prof. Moncrief’s extensive analysis of 2+1 gravity (Constant mean curvature spatial harmonic gauge) in [13] led to the several classical results of Teichmüller theory, which were obtained by means of purely relativistic/Riemannian geometric analysis. This included, e.g., the homeomorphism between the Teichmüller space and the space of holomorphic quadratic differentials (transverse-traceless tensors in the context of relativity) etc. In the same paper, the term ‘Relativistic Teichmüller theory’ was coined. Through studying a Hamilton Jacobi equation whose complete solution determines all the solution curves of the reduced Einstein equations and a Monge-Ampere type equation which allows for a more explicit characterization of these solution curves, he defined a family of ray structures on the Teichmüller space of \(\Sigma_g\). Studying the behaviour of the associated Dirichlet energy, Prof. Moncrief [13] has conjectured that each of these solution curves runs off the edge of the Teichmüller space at the limit of big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations \((PML, PMF)\). This, in principle, if holds true, then classifies the big bang singularities of ‘2 + 1’ gravity as the points on the Thurston boundary and serves as another means to compactify the Teichmüller space.

[14] studied the spacetimes of simplicial types (a dense subset in the space of all spacetimes) in cosmological time gauge and obtained a similar result that is the past singularity corresponds to the isometric action of fundamental group of \(\Sigma_g\) on certain Real tree, that is, in other words, a point on the Thurston boundary is associated to the initial singularity. However, their study does not involve direct quantitative analysis of the Einstein equations in a sense that it is mostly of qualitative nature, and also works in a gauge different from CMCSH gauge. Later, based on the work of [14], [15] used a barrier arguments to control the constant mean curvature slices relative to the cosmic time ones near the big bang singularities and thereby to show that Thurston boundary points are attained, at the limit, by the former as well as the latter. Despite the fact that these results conform to the conjecture of Prof. Moncrief to a large extent, they lack direct arguments and also differ in the choice of gauge and whether this result is gauge invariant is currently unknown. Therefore, it is worth proving the conjecture by a direct analysis of Einstein evolution and constraint equations in CMCSH gauge and the definition of the Thurston boundary of the Teichmüller space.

In addition to the general relativistic perspective, M. Wolf [31] established the homeomorphism between the space of holomorphic quadratic differential and Teichmüller space of \(\Sigma_g\) by utilizing the complex analytic properties such as Beltrami differential (stretching) of the associated harmonic map. One may naively assert that the wolf’s result might be directly applicable to the relativistic case, since, the transverse-traceless tensor of GR may be associated to a holomorphic quadratic differential. However, in M. wolf’s case, the domain is kept fixed while the dynamics of the target
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surface is considered and therefore the available machinery became useful, contrary to the relativistic case, when the domain (conformal structure) is varying while the target is fixed (an interior point of the Teichmüller space). Therefore, traditional machinery becomes useless and we are left with only tools which are accessible through GR.

In this paper, we aim to study the ‘2 + 1’ gravity on vacuum spacetimes of the type $\Sigma_g \times \mathbb{R}$ in constant mean extrinsic spatial harmonic gauge (CMCSH). Utilizing the direct estimates from the Einstein evolution and constraint equation in conjunction with a few established results from [13] and the theory of Riemann surface, we show via a direct argument that indeed Moncrief’s conjecture hold true, that is, at the limit of big-bang singularity, the conformal geometry degenerates and every corresponding solution curve attaches to the Thurston boundary. The structure of the paper is as follows. We begin with a rough sketch of the classical proof of the Thurston compactification and introduce necessary backgrounds from the theory of Riemann surfaces such as holomorphic quadratic differentials, the associated measured foliations and their transverse measure etc. Then we study the reduced Einstein equations through a conformal technique and obtain the estimates necessary from the associated elliptic PDEs. Finally, we state the relativistic interpretation of the concepts introduced from surface theory and show using the estimates obtained that the conjecture hold true, that is, at the limit of big-bang singularity, every solution curve runs off the edge of the Teichmüller space and attaches to the space of Porjective measured foliations/laminations. We conclude by discussing the validity of the conjecture with the inclusion of the cosmological constant and suitable matter sources.

2. Notations and facts

We denote the ‘2 + 1’ spacetimes by $\tilde{M}$ with its topology being $\Sigma_g \times \mathbb{R}$. Here, $\Sigma_g$ is a closed (compact without boundary) Riemann surface with genus $g > 1$. Space of Riemannian metrics on $\Sigma_g$ is denoted by $\mathcal{M}$ and its closed submanifold $\mathcal{M}_{-1}$ is defined as follows

$$\mathcal{M}_{-1} = \{ \gamma \in \mathcal{M} | R(\gamma) = -1 \}. \quad (1)$$

Space of symmetric 2-tensor fields are denoted by $S^2(\Sigma_g)$. The $L^2$ inner product with respect to the metric $\gamma \in \mathcal{M}$ between any two elements $A$ and $B$ of $S^2(\Sigma_g)$ is defined as

$$< A, B >_{L^2} := \int_{\Sigma_g} A_{ij} B_{kl} \gamma^{ik} \gamma^{jl} \mu_\gamma, \quad (2)$$

where $\mu_\gamma = \sqrt{\det(\gamma_{ij})} dx^1 \wedge dx^2$ is the volume form on $\Sigma_g$. Abusing notation we will use $\mu_\gamma$ for both $\det(\gamma_{ij})$ and the volume form. Unless otherwise stated, we will consider every element of $\mathcal{M}$ in isothermal coordinate that is $\mathcal{M} \ni \gamma := e^{\xi(z)} |dz|^2, \xi : \Sigma_g \to \mathbb{R}$. The rough Laplacian $\Delta_g$ is defined to have non-negative spectrum on $\Sigma_g$, that is, $\Delta_g := -g^{ij} \nabla_i \nabla_j$. 
3. Background on Teichmüller space and Thurston compactification

Teichmüller space is studied from algebraic topologic perspective [17, 16], complex analytic perspective [18, 16], and Riemannian geometric perspective[19]. Here, we will focus mainly on the later as the Teichmüller space while viewed from Riemannian geometric perspective, naturally appears as the configuration space of the vacuum Einstein gravity (with or without a positive cosmological constant) on $\Sigma_g \times \mathbb{R}$. Nevertheless, we will state the algebraic topologic definition of the Teichmüller space and show how this is connected to the Einstein gravity. The Teichmüller space of $\Sigma_g$ is defined as the space of homomorphisms of the fundamental group of $\Sigma_g$ into the isometry group of its universal cover that is the hyperbolic plane modulo the action of the isometry group by conjugation. If the Poincare disk model of the hyperbolic plane is assumed, then the Teichmüller space turns out to be

$$T \Sigma_g := \frac{\text{Hom}(\pi_1(\Sigma_g), \text{PSL}_2 \mathbb{R})}{\text{PSL}_2 \mathbb{R} \text{conj}},$$

dimension of which may be calculated as follows. $\text{Hom}(\pi_1(\Sigma_g), \text{PSL}_2 \mathbb{R})$ is mod out by the $\text{PSL}_2 \mathbb{R}$ conjugation as to remove the base point of the homotopy (at the level of loops). This definition precisely identifies the the ways to equip $\Sigma_g$ with distinct conformal structures (or hyperbolic structures). $\pi_1(\Sigma_g)$ is to be viewed as discrete and faithful subgroup of $\text{PSL}_2 \mathbb{R}$ and as such is finitely generated. It is sufficient to consider its $2g$ generators. The dimension of $\text{PSL}_2 \mathbb{R}$ is 3 and action by conjugation by an element of $\text{PSL}_2 \mathbb{R}$ produces equivalence class (gauge transformation is physics language). In addition, the generators $(A_i, B_i)_{i=1}^{g}$ satisfy the commutation relation $\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = \text{id}$ implying the representation $\rho \in \text{Hom}(\pi_1(\Sigma_g), \text{PSL}_2 \mathbb{R})/\text{PSL}_2 \mathbb{R} \text{conj}$ would satisfy $\prod_{i=1}^{g} \rho(A_i) \rho(B_i) \rho(A_i)^{-1} \rho(B_i)^{-1} = \text{id}$ as well. Therefore we lose $3 + 3 = 6$ degrees of freedom out of $2g \times 3 = 6g$ and the dimension of the Teichmuller space turns out to be $6g - 6$. Let us now show how this is related to vacuum Einstein dynamics. Vacuum Einstein equations in $2+1$ dimension reads

$$R_{\mu\nu} = 0,$$

where $(\mu, \nu)$ correspond to the spacetime indices. Now, in $2+1$ dimension, vanishing of Ricci tensor ($R_{\mu\nu}$) implies vanishing of the full Riemann tensor (or the sectional curvature) and therefore, the solutions of the Einstein equations are necessarily the flat spacetimes and consequently isometric to the Minkowski spacetimes. Now we are interested in flat spacetimes foliated by $\Sigma_g$. In order to obtain the solution space, we therefore need to identify the space of homomorphisms of $\pi_1(\Sigma_g \times \mathbb{R})$ into the isometry group of the flat spacetimes, which in this case is the full Poincare group ISO(2,1). Now $\pi_1(\Sigma_g \times \mathbb{R}) \approx \pi_1(\Sigma_g)$ and therefore the solution space is described as

$$\text{Eins} = \frac{\text{Hom}(\pi_1(\Sigma_g), \text{ISO}(2,1))/\text{ISO}(2,1)\text{conj}},$$

where $\text{Eins}$ is the space of solutions of the equation (4). In the similar way, we may compute the dimension of $\text{Eins}$. Note that now the isometry group ISO(2,1) has...
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dimension 6 and therefore following the exact same procedure, we obtain the dimension of
$E_{12}$ to be $12g - 12$. Therefore, the full solution space is twice the dimension of the
Teichmüller space. One immediate guess would be that the co-tangent bundle $T^* T \Sigma_g$
of the Teichmüller space acts as the full solution space, which is precisely the case as
shown in [1, 13]. $T^* T \Sigma_g$ is indeed the phase space of the reduced dynamics. We will get
back to this point in detail later. Let us get back to the concepts of geodesic currents,
measured laminations and foliations, which will be required to prove the conjecture.

Let us now introduce few elementary concepts from the theory of Riemann surface.
From elementary hyperbolic geometry, we know that there exists a unique geodesic
between any two distinct points lying on the boundary of the Poincare disc (in this
model of the hyperbolic 2-space). Therefore, we define the set of all un-oriented
geodesics on $\tilde{\Sigma}_g$ (lift of $\Sigma_g$ to its universal cover) as the $Z_2$ graded double boundary
of $\tilde{\Sigma}$ i.e., $G(\tilde{\Sigma}_g) = \{ \text{The set of all un-oriented geodesics on } \tilde{\Sigma} \} \approx (S^1_\infty \times S^1_\infty - \Delta)/Z_2$. A
geodesic current is a radon measure on $G(\tilde{\Sigma})$ which is invariant under
$\pi_1(\Sigma_g)$ action (see [25, 24] for more details and See [22] for details about radon measure). The property
of radon measure which would be of particular interest to us is that it is locally finite.
In a sense, a geodesic current is essentially an assignment of radon measure to the
open sets of $G(\tilde{\Sigma})$, which remain invariant under the action of the fundamental group
$\pi_1(\Sigma_g)$. This $\pi_1(\Sigma_g)$ invariance property of the geodesic current allows one to define
it on the space of geodesics on $\Sigma_g$ i.e., $G(\Sigma_g) = G(\tilde{\Sigma}_g)/\pi_1(\Sigma_g)$ (note that action of
$\pi_1(\Sigma_g)$ extends continuously to $\partial \Sigma_g$). Now, for a closed hyperbolic surface of genus
greater than 1, $\pi_1(\Sigma_g)$ while viewed as a proper discrete subgroup of the isometry group
of the hyperbolic plane that is $PSL_2 \mathbb{R}$, consists of hyperbolic (also called loxodromic)
elements only (see [17, 23] for detailed classification of the types of isometries of $H^2$).
Each element of $\pi_1(\Sigma_g)$ has an axis geodesic along which it acts by translation and in
general it has two fixed points: one attracting, one repelling. Therefore each element
of $\pi_1(\Sigma_g)$, a homotopy class of nontrivial loops, has a unique geodesic representative.
Whenever, we will consider the length of a non-trivial closed curve on $\Sigma_g$, we will always
mean the length of the geodesic in its homotopy class. Once we have defined the geodesic
current, the space of geodesic lamination will follow. A geodesic lamination is a closed
subset on $\Sigma_g$ which is the union of disjoint simple geodesics. A measured lamination is
defined as a geodesic lamination equipped with a transverse measure (invariant under
translations along the leaves of the lamination). Clearly, space of measured lamination
is a subset of the space of geodesic current. A geodesic foliation may be thought of as
the union of the the geodesics which are also integral curves of a vector field. Zeros
of the vector field correspond to the singularity of the foliation. One may similarly
assign a transverse measure to the foliation promoting it to a measured foliation. There
is a natural homeomorphism between the space of measured lamination and measured
foliation (via a straightening map; see Fig [1]). For now, we will provide a rough sketch
of the proof of the Thurston compactification of the Teichmüller space. This will be
crucial in understanding the proof of Prof. Moncrief’s conjecture that is the relativistic
interpretation.
3.1. Space of projective lamination as the Thurston boundary of the Teichmüller space

In this section, we provide a rough sketch of the proof of Thurston compactification of the Teichmüller space by the space of projective measured laminations ($\mathcal{PML}$). The details may be found in [26, 27]. Here we show that a sequence diverging in Teichmüller space converges in the space of projective measured laminations ($\mathcal{PML}$), which is a compact subset of the space of geodesic currents. A $\pi_1(\Sigma_g)$--invariant measure on $G(\tilde{\Sigma}_g)$ may be defined as

$$L = \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2},$$  \hspace{1cm} (6)

where $(e^{i\alpha}, e^{i\beta}) \in (S^1 \times S^1) \setminus \Delta$ and $\Delta$ represents diagonal. This measure is called the Liouville measure. Liouville measure corresponding to $X$ is denoted by $L_X$, which satisfies the following for any $\gamma \in G(\Sigma_g)$

$$i(\gamma, L_X) = l_X(\gamma).$$  \hspace{1cm} (7)

The intersection property may be interpreted as follows. Let us consider a closed non-trivial geodesic $\gamma \in G(\Sigma_g)$. Lift $\gamma$ to the universal cover and consider its intersection with the set of geodesics transverse to its lift $\tilde{\gamma}$ that is $i(\gamma, L)$ is defined as $\int_E L(E \cap \tilde{\gamma})$, where $E \subset G(\tilde{\Sigma}_g)$ is the set of geodesics transverse to $\tilde{\gamma}$. Few lines of calculations show that this integral is indeed the length of $\gamma$ with respect to the hyperbolic metric (scalar curvature $= -1$). Note that Liouville measure may be used to define a geodesic current on $G(\Sigma_g)$ due to its $\pi_1(\Sigma_g)$--invariance property. Now, let $(X, f)$ be a marked hyperbolic surface (and thus $\in T\Sigma$) such that $f : \Sigma_g \to X = H^2/\pi_1(\Sigma_g)$ is a homeomorphism. Liouville measure provides a well defined map from the Teichmüller space $T\Sigma$ to the space of currents. Here we just provide a brief description of the Thurston compactification of the Teichmüller space, necessary for the current purpose. For details, the readers are referred to the excellent book [26], where the proof of the stated theorems may be found.

**Lemma 0** [26, 27] The map $(X, f) \to L_X$ is a proper embedding of $T\Sigma_g$ into the space of currents $\text{Curr}(\Sigma_g)$ given by the intersection number that is, for all closed curves $\alpha$ in $\Sigma_g$,

$$i(\alpha, L_X) = l_X(\alpha)$$  \hspace{1cm} (8)

defines a proper embedding of $T\Sigma_g$ into $\text{Curr}(\Sigma_g)$.

**Proof:** See [26, 27].

We are now ready to establish the Thurston compactification. Let us first state a lemma first.

**Lemma 1** [26, 27] For any marked hyperbolic surface $\Sigma_g$ with the marking $(X, f)$, we have the following result

$$i(L_X, L_X) = \pi^2|\chi(\Sigma_g)|,$$  \hspace{1cm} (9)

where $\chi(\Sigma_g) = 2(1 - g)$ is the Euler characteristics of $\Sigma_g$. Remarkably, this is a topological invariant. Let’s denote the map $(X, f) \to L_X$ by $L$. We have the following...
Lemma 2:

\[ L : T \Sigma \rightarrow \mathbb{CP} \text{Curr}(\Sigma_\mathbb{g}) = (\text{Curr}(\Sigma_\mathbb{g}) - 0)/(\mu \sim t\mu, \mu \in \text{Curr}(\Sigma_\mathbb{g}), t \in \mathbb{R}_{>0}) \]

is injective.

Proof: Let \([f : \Sigma \rightarrow X]\) and \([h : \Sigma \rightarrow Y]\) be two elements of \( T \Sigma \). Then

\[ [L_X] = [L_Y] \Rightarrow L_X = tL_Y. \] (10)

Now we use the previous lemma and obtain

\[ \pi^2|\chi(\Sigma_\mathbb{g})| = i(L_X, L_X) = i(tL_Y, tL_Y) \]

(11)

\[ = t^2i(L_Y, L_Y) = t^2\pi^2|\chi(\Sigma_\mathbb{g})|, \]

i.e.,

\[ t = 1, \] (12)

as \( t \in \mathbb{R}_{>0} \) and therefore \( L_X = L_Y \).

As we have defined earlier, a lamination \( L \) on \( \Sigma_\mathbb{g} \) is a closed subset which is the union of disjoint simple geodesics and the geodesics in \( L \) are called the leaves of the lamination. An important property of these leaves is that they do not intersect each other that is if \( \lambda, \alpha \in L \), then the following is satisfied

\[ i(\lambda, \alpha) = 0. \] (13)

If we associate a transverse measure to the leaves of \( L \), then we obtain a measured lamination denoted by \( \mathcal{ML} \). We may of course construct the projective measured laminations \( \mathcal{PML} \) through the following identification

\[ \mathcal{PML} = (\mathcal{ML} - \{0\})/(\lambda \sim t\lambda, \lambda \in \mathcal{ML}, t > 0). \] (14)

Clearly the leaves of a measured lamination define a subset in the space of all geodesics and therefore, the projective measured lamination \( \mathcal{PML} \) may be identified as a subset of the space of geodesic currents. It is in fact a compact subset, which may be proven utilizing an elementary result from topology namely the Tychonof’s theorem [28, 29].

Another important observation is to note that the image of the Teichmüller space under the map \( L \) i.e., \( L(T\Sigma_\mathbb{g}) \) and \( \mathcal{PML} \) are disjoint. This follows from the definition of the geodesic lamination that is \( i(\lambda, \lambda) = 0 \ \forall \lambda \in \mathcal{PML} \), while \( i(L_X, L_X) = \pi^2|\chi(\Sigma_\mathbb{g})| \neq 0, \ \forall X \in T\Sigma_\mathbb{g} \). Now we finish the Thurston compactification

Lemma 3: The closure of \( T\Sigma_\mathbb{g} \subset \mathbb{CP} \text{Curr}(\Sigma) \) is precisely \( T\Sigma_\mathbb{g} \cup \mathcal{PML} \).

Proof: Let say \([f_n : \Sigma \rightarrow X_n]\) is a sequence that diverges in \( T\Sigma_\mathbb{g} \). Then obviously, \([[L_X]]) \subset \mathcal{PML} \) converges to some element of \( \mathcal{PML} \) due to the fact that \( \mathcal{PML} \) is a compact subset of \( \text{Curr}(\Sigma_\mathbb{g}) \) (passing up to subsequence). Then \( \exists t_n \) such that \( \lim_{n \rightarrow \infty} t_n L_{X_n} = \mu \in \mathcal{PML} \). Now from the divergence criteria, there exists a simply closed curve \( \alpha \in \Sigma_\mathbb{g} \), such that

\[ \lim_{n \rightarrow \infty} l_{X_n}(\alpha) = \infty. \] (15)
But, $\infty > i(\alpha, \mu) = i(\alpha, t_n L_{X_n}) = t_n l_{X_n}(\alpha)$ and thus we must have

$$\lim_{n \to \infty} t_n = 0. \quad (16)$$

Now we see the following

$$i(\mu, \mu) = i(\lim_{n \to \infty} t_n L_{X_n}, \lim_{n \to \infty} t_n L_{X_n}), \quad (17)$$
$$\quad = \lim_{n \to \infty} t_n^2 i(L_{X_n}, L_{X_n}), \quad (18)$$
$$\quad = \lim_{n \to \infty} t_n^2 \pi^2 |\chi(\Sigma)|, \quad (19)$$
$$\quad = 0, \quad (20)$$

and therefore, $\mu \in \mathcal{PML}$. 

### 3.2. Homeomorphism between $\mathcal{ML}$, $\mathcal{MF}$, and $\mathcal{QD}$

Let us first define a holomorphic quadratic differential on a Riemann surface $\Sigma_g$. A holomorphic quadratic differential is a holomorphic section of the symmetric square of the holomorphic cotangent bundle of $\Sigma_g$. It may be defined locally as follows. Let \{ $z_a : U_a \to \mathbb{C}$ \} be an atlas for $\Sigma_g$. A holomorphic quadratic differential $\Phi$ on $\Sigma_g$ is locally expressible on the chart $z_a$ as $\Phi_a(z_a) dz_a^2$ with the following properties: [1] $\Phi_a : z_a(U_a) \to \mathbb{C}$ is holomorphic, i.e., $\frac{\partial \Phi}{\partial \bar{z}_a} = 0$, and [2] $\Phi_a(z_a)(\frac{dz_a}{dz_b})^2 = \Phi_b(z_b)$ for two different charts $z_a : U_a \to \mathbb{C}$ and $z_b : U_b \to \mathbb{C}$. The second condition precisely states the invariance of $\Phi dz^2$ under coordinate transformations. Let us denote the space of holomorphic quadratic differentials on $\Sigma_g$ by $\mathcal{QD}$. By the famous theorem of Hubbard and Masur [32], there is a homeomorphism between the space of holomorphic quadratic differentials on $\Sigma_g$ by $\mathcal{QD}$. By the famous theorem of Hubbard and Masur [32], there is a homeomorphism between the space of holomorphic quadratic differentials on $\Sigma_g$ by $\mathcal{QD}$. One may simply associate a vertical or horizontal foliation with $\Phi \in \mathcal{QD}$ (upto isotopy and whitehead moves; see [31] for details about whitehead moves). For details, the reader is referred to [33]. For now we will only need this homeomorphism property. Given a holomorphic quadratic differential $\Phi(z) dz^2$ in some chart, the transverse measure (except at the zeros of $\Phi$, which correspond to the singularities of the foliation) to the vertical foliation and horizontal foliation associated with $\Phi$ are defined as follows

$$\mu_{vert}(A) := \int_A |\mathcal{R} \left( \sqrt{\Phi(z)} dz \right) |, \quad (21)$$
$$\mu_{hor}(A) := \int_A |\mathcal{I} \left( \sqrt{\Phi(z)} dz \right) |, \quad (22)$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, respectively. We will use this definitions later while considering the Einstein flow on $\Sigma_g$ exclusively. Given a measured foliation, one may obtain a measured lamination via a suitable straightening map [30, 34] (or collapsing a lamination yields a foliation). Therefore, there is a homeomorphism between $\mathcal{MF}$ and $\mathcal{ML}$. Figure (1) depicts the mechanism of yielding a lamination from a foliation. For our purposes, we will only use the homeomorphism between $\mathcal{QD}$ and $\mathcal{MF}$. All of these spaces remain homeomorphic to each other at the level of projective spaces.
3.3. Harmonic Maps

Let us now describe the harmonic maps. This will be crucial later in studying the Einsteinian dynamics. Let us consider the map $\mathcal{E} : (M, g) \rightarrow (N, \rho)$ (M and N are considered to be two closed Riemann surfaces) and define the Dirichlet energy

$$
E[\mathcal{E}; g, \rho] = \frac{1}{2} \int_M \rho_{\alpha\beta} \partial \mathcal{E}^\alpha \partial \mathcal{E}^\beta g^{ij} \mu_g.
$$

From the expression of the Dirichlet energy, it is obvious that it only depends on the conformal structure of the domain, that is, a conformal transformation $g_{ij} \mapsto e^{2\delta} g_{ij}, \delta : M \rightarrow \mathbb{R}$ leaves $E$ invariant. Harmonic maps are defined to be the critical points of this the Dirichlet energy functional in the space of $\mathcal{E}$. The critical points of $E$ may be computed as follows. On the bundle $T^* M \otimes \mathcal{E}^{-1}TN$ (while restricted to the image), one has the following connection

$$
\nabla_i A_j^\alpha := \partial_i A_j^\alpha + N \Gamma^\alpha_{\beta\gamma} A^\beta_j \frac{\partial \psi^\gamma}{\partial x^i} - M \Gamma^k_{ij} A_k^\alpha,
$$

for $A \in \text{section}(T^* M \otimes \mathcal{E}^{-1}TN)$. Using this definition of the connection, few lines of calculation yield the harmonicity condition

$$
g^{ij} \partial_i \partial_j \psi^\alpha - g^{ij} M \Gamma^k_{ij} \partial_k \psi^\alpha + N \Gamma^\alpha_{\beta\gamma} \partial_i \psi^\beta \partial_j \psi^\gamma g^{ij} = 0.
$$

From [35, 36], we know that there is a harmonic map homotopic to identity i.e., $\mathcal{E} \in \mathcal{D}_0$ and in fact such a map is an orientation preserving diffeomorphism. If we take $\mathcal{E}$ to be the identity map i.e., $M = N = \Sigma_g$, then the harmonicity condition reduces to the following

$$
-g^{ij} \left( \Gamma[g]_{ij}^\alpha - \Gamma[\rho]_{ij}^\alpha \right) = 0.
$$
This condition will be of extreme importance when we fix the spatial gauge of the Einstein equations and also in the later part of the analysis. The Dirichlet energy of this identity map is computed to be

\[
E[id; g, \rho] = \frac{1}{2} \int_{\Sigma_g} \rho_{ij} g^{ij} \mu_g.
\]

(27)

Note that the conformal and diffeomorphism invariance of \( E[id; g, \rho] \) allow it to be a function on the Teichmüller space of \( \Sigma_g \) and in particular a proper function (that is the inverse of the compact sets are compact)\[19, 31, 36, 37\].

4. Einstein flow on \( \Sigma_g \times \mathbb{R} \)

We will use the ADM formalism of general relativity in order to obtain a Cauchy problem for ‘2 + 1’ gravity. The ADM formalism of ‘2+1’ gravity splits the spacetime described by a ‘2+1’ dimensional Lorentzian manifold \( \tilde{M} \) into \( \mathbb{R} \times \Sigma_g \) with each level set \( \{t\} \times \Sigma_g \) of the time function \( t \) being an orientable 2-manifold diffeomorphic to a Cauchy hypersurface (assuming the spacetime admits a Cauchy hypersurface) and equipped with a Riemannian metric. Such a split may be executed by introducing a lapse function \( N \) and shift vector field \( X \) belonging to suitable function spaces and defined such that

\[
\partial_t = N \tilde{n} + X
\]

(28)

with \( t \) and \( \tilde{n} \) being time and a hypersurface orthogonal future directed timelike unit vector i.e., \( \tilde{g}(\tilde{n}, \tilde{n}) = -1 \), respectively. The above splitting writes the spacetime metric \( \tilde{g} \) in local coordinates \( \{x^\alpha\}_{\alpha=0}^2 = \{t, x^1, x^2\} \) as

\[
\tilde{g} = -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)
\]

(29)

where \( g_{ij} dx^i \otimes dx^j \) is the induced Riemannian metric on \( \Sigma_g \). In order to describe the embedding of the Cauchy hypersurface \( \Sigma_g \) into the spacetime \( \tilde{M} \), one needs the information about how the hypersurface is curved in the ambient spacetime. Thus, one needs the second fundamental form \( k \) defined as

\[
K_{ij} = -\frac{1}{2N}(\partial_t g_{ij} - (L_X g)_{ij}),
\]

(30)

trace of which \( (tr_g K = \tau = g^{ij}K_{ij}, g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := g^{-1}) \) is the mean extrinsic curvature of \( \Sigma_g \) in \( \tilde{M} \) and \( L \) denotes the Lie derivative operator. The vacuum Einstein equations

\[
R_{\mu\nu}(\tilde{g}) - \frac{1}{2} R(\tilde{g}) \tilde{g}_{\mu\nu} = 0
\]

(31)

may now be expressed as the evolution and constraint equations (Gauss and Codazzi equations) of \( g \) and \( k \)

\[
\begin{align*}
\partial_t g_{ij} &= -2NK_{ij} + (L_X g)_{ij}, \\
\partial_t K_{ij} &= -\nabla_i \nabla_j N + N(R_{ij} + \tau K_{ij} - 2k^k_{ij}k_{kj}) + (L_X k)_{ij}, \\
0 &= R(g) - |K|^2 + (tr_g K)^2, \\
0 &= \nabla^i K_{ij} - \nabla_j tr_g K.
\end{align*}
\]

(33)
Note that there is no canonical way way to split the spacetimes, that is, the choice of a spacelike hypersurface is not unique. In order to choose a slice and study its evolution under Einstein flow, we must fix the gauge. In our case, the most convenient choice is the constant mean extrinsic curvature spatial harmonic gauge used by [38]. In this gauge, \( \tau = tr_g K \) is constant throughout the hypersurface (\( \partial_i \tau = 0 \)) and therefore is chosen to play the role of time

\[
t = \text{monotonic function of } \tau. \tag{34}
\]

Spatial harmonic gauge is precisely the vanishing of the tension vector field \(-\bar{g}^{ij} \left( \Gamma[g]_{ij}^k - \hat{\Gamma}[\hat{g}]_{ij}^k \right)\), where \( \hat{g} \) is an arbitrary background metric or in other words, the harmonicity of the identity map defined in the previous section. This choice of gauge yields the following two elliptic equations for the lapse function and the shift vector field, respectively

\[
\Delta_g N + N(H^T)^2 + \frac{\tau^2}{2} = \partial_t \tau, \tag{35}
\]

\[
\Delta_g X^i - R^i_j X^j = (\nabla^i N) \tau - 2 \nabla^j N K_{ij} + (2 N K^i k - 2 \nabla_j X^k) \tag{36}
\]

This Cauchy problem (with initial data \((g_0, k_0, N, X)\)) with constant mean extrinsic curvature and spatially harmonic gauge is referred to as CMCSH Cauchy problem.

### 4.1. well-posedness:

[38] proved a local well posedness theorem for the Cauchy problem for a family of elliptic-hyperbolic systems that included the ‘\( n+1 \)’ dimensional vacuum Einstein equations in CMCSH gauge, \( n \geq 2 \). They also proved that the conservation of gauges and constraint. In addition to the local well-posedness, [12] proved a global existence theorem for the expanding solutions in the same gauge through controlling the Dirichlet energy of an associated harmonic map for any \( \tau \in (-\infty, 0) \). Therefore, the well-posedness of the Cauchy problem is established and we do not wish to repeat the same here. Interested readers are referred to these articles.

### 4.2. Reduced Dynamics

Given a scalar function \( \psi : \Sigma_g \to \mathbb{R} \), we define a set of conformal variables \((\gamma, \kappa^{TT})\) in terms of the physical variables \((g, \kappa^{TT})\) by setting

\[
(g_{ij}, K^{TT}_{ij}) = (e^{2\psi} g_{ij}, e^{-4\psi} K^{TT}_{ij}), \tag{37}
\]

where \( R(\gamma) = -1 \) (Uniformization theorem guarantees that such \( \gamma \) exists if \( \text{genus} (\Sigma_g) > 1 \)) and the second fundamental form is written as follows

\[
K = K^{TT} + \frac{\tau}{2} g, \tag{38}
\]
by using the momentum constraint. Here $\kappa^{TT}$ is transverse-traceless with respect to $\gamma$, that is,

$$\nabla[\gamma]_{j}^{TT}k^{TT}_{ij} = 0,$$

$$\gamma_{ij}^{TT} = 0,$$

if and only if $\kappa^{TT}$ is transverse-traceless with respect to $g$. Naturally

$$\kappa^{TT}_{ij} = K^{TT}_{ij}.\quad (41)$$

$\psi$ can be found by solving the Hamiltonian constraint which now takes the form of the following semilinear elliptic PDE namely the Lichnerowicz equation

$$-2\Delta_{\gamma}\psi + 1 + e^{-2\psi}|\kappa^{TT}|^{2}_{\gamma} - \frac{e^{2\psi}\tau^{2}}{2} = 0. \quad (42)$$

Using sub and super solution technique $[39, 40]$, it is established that there is a unique solution $\psi[\gamma, \kappa^{TT}, \tau]$ of the Lichnerowicz equation. Indeed, this equation will be crucial to our analysis towards proving the main theorem. The phase space of the reduced dynamics now may be defined as

$$\{(\gamma_{ij}, \kappa^{TT}_{ij}) | \gamma \in M_{-1}, tr_{\gamma}\kappa^{TT} = 0 = \nabla[\gamma]_{j}^{TT}k^{TT}_{ij}\}.$$ 

In reality, the true dynamics assumes a metric lying in the orbit space $M_{-1}/D_{0}$, $D_{0}$ being the group of diffeomorphisms (of $\Sigma_{g}$) isotopic to identity. This is a consequence of the fact that if $\gamma_{ij} \in M_{-1}, k^{TT}_{ij}, N$, and $X^{i}$ solve the Einstein equations, so do $((\phi^{-1})^{*}\gamma)_{ij}, (\phi_{*}\kappa^{TT})_{ij}, (\phi^{-1})^{*}N = N \circ \phi^{-1}$, and $(\phi_{*}X)^{i}$, where $\phi \in D_{0}$ and $\ast$, and $\ast$ denote the pullback and push-forward operations on the cotangent and tangent bundles of $M$, respectively. Now, $M_{-1}/D_{0}$ is precisely the Teichm"{u}ller space of $\Sigma_{g}$ and following $[19]$, the transverse-traceless tensor $\kappa^{TT}$ models the tangent space at $\gamma$. Therefore, we obtain the Teichm"{u}ller space ($6g - 6$ dimensional) $T\Sigma_{g}$ as the configuration space, while the cotangent bundle ($12g - 12$ dimensional) of $T\Sigma_{g}$ serves as the phase space of the reduced dynamics. This is precisely what was stated previously in section 2 while relating the full solution space of the vacuum Einstein equations and the Teichm"{u}ller space through its algebraic topologic definition.

Now we will obtain a series of estimates which will be useful for the later analysis. Note that an standard maximum principle while applied to the Lichnerowicz equation (42) yields the following

$$\tau^{2}e^{4\psi} - 2e^{2\psi} - 2\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma} \leq 0. \quad (43)$$

Noting the discriminant of the quadratic form $\tau^{2}e^{4\psi} - 2e^{2\psi} - 2\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma}$ to be $4 + 8\tau^{2}\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma}$ is strictly positive, the inequality is satisfied only for a specific range of $e^{2\psi}$ i.e.,

$$\left(e^{2\psi} - \frac{1 + \sqrt{1 + 2\tau^{2}\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma}}}{\tau^{2}}\right)\left(e^{2\psi} - \frac{1 - \sqrt{1 + 2\tau^{2}\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma}}}{\tau^{2}}\right) \leq 0. \quad (44)$$

But, $e^{2\psi} > 0$ and therefore, we must have

$$e^{2\psi} \leq \frac{1 + \sqrt{1 + 2\tau^{2}\sup_{\gamma}|\kappa^{TT}|^{2}_{\gamma}}}{\tau^{2}}. \quad (45)$$
Similarly, at a minimum, the following holds
\[ \tau^2 e^{4\psi} - 2e^{2\psi} \geq 0, \tag{46} \]
that is,
\[ e^{2\psi} \geq \frac{2}{\tau^2}, \tag{47} \]
where the equality holds if and only if
\[ \kappa^{TT} \equiv 0. \tag{48} \]

In summary, we have the following estimate of the conformal factor from the Lichnerowicz equation
\[ \frac{2}{\tau^2} \leq e^{2\psi} \leq 1 + \sqrt{1 + 2\tau^2 \sup |\kappa^{TT}|^2}, \tag{49} \]
which will be useful later. Now we will obtain an estimate of \( |K^{TT}|^2_g = e^{-4\psi}|K^{TT}|^2_g \). Note that in 2 dimensions, the momentum constraint
\[ \nabla [g] K^j_i - \nabla_i tr_g K = 0 \tag{50} \]
implies that \( K \) is a codazzi tensor \([13, 12]\) i.e.,
\[ \nabla [g] K^j_k - \nabla [g] K^j_k = 0. \tag{51} \]

Taking covariant divergence of this equation, commuting covariant derivatives, and utilizing the relation \( R[\bar{g}]_{ijkl} = \frac{R[\bar{g}]}{2}(g_{ik}g_{jl} - g_{il}g_{jk}) \), one obtains the following elliptic equation for \( |K^{TT}|^2_g \)
\[ -\Delta_g(|K^{TT}|^2_g) - 2|K^{TT}|^2_g(|K^{TT}|^2_g - \frac{1}{2}\tau^2) = 2\nabla_k(\kappa^{TT}_ij)\nabla^k(K^{TT}_{ij}). \tag{52} \]
Noting \( \nabla_k(\kappa^{TT}_ij)\nabla^k(K^{TT}_{ij}) \geq 0 \) and applying a maximum principle yields
\[ |K^{TT}|^2_g \leq \frac{\tau^2}{2}. \tag{53} \]

Lastly, we will obtain the estimate for the lapse function after choosing the following time coordinate
\[ t := -\frac{1}{\tau}. \tag{54} \]
The allowed time range in this coordinate is \((0, \infty)\). The lapse equation (35) now reads
\[ \Delta_g N + N(|K^{TT}|^2_g + \frac{\tau^2}{2}) = \tau^2. \tag{55} \]
Once again, a standard maximum principle applied to the lapse equation together with the estimate (53) yields the following estimate of \( N \)
\[ 1 \leq N \leq 2. \tag{56} \]

Now we will describe Prof. Moncrief’s ray structure \([13]\) of the Teichmüller space, which will be of crucial in obtaining the main result. The ray structure defined by Prof.
Moncrief is the following equation

\[ \rho_{ij} = |K|^2 g_{ij} + 2\tau (K_{ij} - \frac{1}{2}\tau g_{ij}) \]

\[ = (|K^{TT}|^2 g + \frac{\tau^2}{2}) g_{ij} + 2\tau K^{TT}_{ij} \]

\[ = (e^{-4\psi}|\kappa^{TT}|^2 g + \frac{\tau^2}{2}) e^{2\psi} g_{ij} + 2\tau \kappa^{TT}_{ij} \]

where \( \rho \) is a fixed metric satisfying \( R(\rho) = -1 \) (and therefore lies inside the Teichmüller space) and \( g_{ij} \) is solved in terms of \( \rho_{ij} \). For the detailed derivation of this expression, one may consult the relevant section of [13]. This is entitled in [13] as the 'Gauss' map equation. For our purpose, the derivation of this map is tangential and hence, we do not wish to repeat the same here. The vital question is whether such \((g_{ij}, K^{TT}_{ij}, N, X)\) actually solves the Einstein equations for all \( \tau \) given an initial \((g_{ij}, K^{TT}_{ij}, N_0, X_0)\) satisfying the constraint equations. This is equivalent to solving for conformal variables \((\gamma_{ij}, \kappa^{TT}_{ij}, \psi)\) and associated lapse function \(N\) and shift vector field \(X\). This is exactly shown in [13] through studying the associated Hamilton Jacobi equation for the reduced dynamics. When this lagrangian formulation is cast into a more natural Hamiltonian one, one clearly sees that the original Einstein-Hilbert action may be written as follows

\[ S = \int_{I \subset \mathbb{R}} \int_{\Sigma_g} \left( \mu_g (-\kappa^{TT}_{ij} + \tau \delta_{ij} \frac{\partial g_{ij}}{\partial t} - N H - X^i P_i) \right) d^2x dt, \]

where \( H := \mu_g \kappa^{TT}_{ij} K^{TT}_{ij} - \frac{1}{2} \tau^2 \mu_g - \mu_g R(g), \) and \( P_i := \nabla [g]_{ij} (2\mu_g K^{TT}_{ij} - \tau \mu_g \delta_{ij}) \). Note that vanishing of \( H \) and \( P_i \) is precisely equivalent to \((g_{ij}, K^{TT}_{ij})\) satisfying the Hamiltonian and momentum constraints. When both of these constraints are satisfied we obtain the reduced action

\[ S_{\text{reduced}} = \int_{I \subset \mathbb{R}} \int_{\Sigma_g} \mu_g (-\kappa^{TT}_{ij} + \tau \delta_{ij} \frac{\partial g_{ij}}{\partial t}) d^2x dt, \]

which through the conformal transformation (37) becomes

\[ S_{\text{reduced}} = \int_{I \subset \mathbb{R}} \left( \int_{\Sigma_g} (-\mu_g \kappa^{TT}_{ij} \frac{\partial g_{ij}}{\partial t} - \frac{\partial \tau}{\partial t} \mu_g g^2 x) \right) dt, \]

where the boundary in time terms are ignored, because, they do not contribute to the equations of motions in classical level. The Hamiltonian of this reduced dynamics can be read off as follows from the expression of the previous action

\[ H_{\text{reduced}} = \int_{\Sigma_g} \frac{\partial \tau}{\partial t} \mu_g. \]

Substituting the time coordinate from equation (54) into the expression of the reduced Hamiltonian together with the Hamiltonian constraint yields

\[ H_{\text{reduced}} = 2 \int_{\Sigma_g} |K^{TT}|^2 g - 8\pi\chi, \]

where \( \chi = 2(1 - g) < 0 \) is the Euler characteristics of \( \Sigma_g \). This reduced Hamiltonian can be related to the Dirichlet energy of the Gauss map. The Dirichlet energy (conformally
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invariant on the domain) associated to the Gauss map (57) is given as

\[ E[\text{id}; g, \rho] = \frac{1}{2} \int_{\Sigma} \mu_g g^{ij} \rho_{ij} = \frac{1}{2} \int_{\Sigma} \mu_\gamma g^{ij} \rho_{ij} = E[\text{id}; \gamma, \rho] \]

(63)

Therefore, we have the following relation between the Dirichlet energy of the Gauss map and the reduced Hamiltonian of the dynamics

\[ H_{\text{reduced}} = E[\text{id}; \gamma, \rho] - 4\pi \chi. \]

(64)

Let us consider that the Teichmüller space \( T\Sigma_g \) is parametrized by \( \{q_\alpha\}_{1}^{6g-6} \), which may be of the Fenchel-Neilsen type (see [17] for details about Fenchel-Neilsen parametrization). For \( \gamma \in T\Sigma_g \), we may write

\[ \gamma := \gamma(q_\alpha), \]

(65)

\[ \frac{\partial \gamma_{ij}}{\partial q_\alpha} := l^{TT}_{\alpha ij}. \]

(66)

This is precisely because the tangent space of \( T\Sigma_g \) at \( \gamma \) can be modelled by the transverse-traceless (with respect to \( \gamma \) or any metric conformal to \( \gamma \)) 2-tensor \( l^{TT}_{\alpha ij} \). Now we observe the following

\[ \frac{\partial E[\text{id}; \gamma(q), \rho]}{\partial q_\alpha} = \frac{1}{4} \int_{\Sigma} \mu_\gamma \left( \gamma^{mn} \gamma^{ij} \rho_{ij} - 2 \gamma^i \gamma^j \rho_{ij} \right) \frac{\partial \gamma_{mn}}{\partial q_\alpha}, \]

(67)

which after substituting \( \rho_{ij} = (|K^{TT}|_g^{2} + \tau^2)g_{ij} + 2\tau K^{TT}_{ij} = (e^{-4\psi}|\kappa^{TT}|_\gamma^{2} + \tau^2)e^{2\psi}g_{ij} + 2\tau \kappa^{TT}_{ij} \) yields

\[ \frac{\partial E[\text{id}; \gamma(q), \rho]}{\partial q_\alpha} = -\tau \int_{\Sigma} \mu_\gamma \kappa^{TT}_{ij} \frac{\partial \gamma_{mn}}{\partial q_\alpha}. \]

(68)

Now let us go back to equation (60) and substitute \( \gamma = \gamma(q) \). we immediately obtain

\[ S_{\text{reduced}} = \int_{I \subset \mathbb{R}} \left( ( \int_{\Sigma} -\mu_\gamma \kappa^{TT}_{ij} \frac{\partial \gamma_{ij}}{\partial q_\alpha} ) \dot{q}_\alpha - H_{\text{reduced}}(\gamma(q), p, \rho) \right) dt \]

(69)

\[ = \int_{I \subset \mathbb{R}} (p^\alpha \dot{q}_\alpha - H_{\text{reduced}}(\gamma(q), p, \rho)) dt, \]

which upon utilizing equations (64) and 68 leads to

\[ \frac{\partial H_{\text{reduced}}(\gamma(q), p, \rho)}{\partial q_\alpha} = \tau p^\alpha. \]

(70)

Here \( \{(q_\alpha, p^\alpha)\}_{\alpha=1}^{6g-6} \) parametrizes the phase space i.e., the co-tangent bundle of \( T\Sigma_g \). Now using the time defined in (54), we may construct a principle functional

\[ S(g, \gamma(q), \rho) = -T(E[\text{id}; \gamma(q), \rho] - 4\pi \chi) \]

(71)

which then clearly satisfies

\[ p^\alpha = \frac{\partial S}{\partial q_\alpha}, \]

(72)

\[ -\frac{\partial S}{\partial T} = E[\text{id}; \gamma(q), \rho] - 4\pi \chi = H_{\text{reduced}}(q, p, \gamma(q)), \]

(73)
that is, $\mathcal{S}$ satisfies the Hamilton-Jacobi equation
\[
-\frac{\partial S}{\partial T} = H_{\text{reduced}}(q, p, \gamma(q))
\]  
for all $T \in (0, \infty)$. In other words $\mathcal{S}$ is dynamically complete. For detailed analysis (arguments behind dynamical completeness of $\mathcal{S}$), the reader is referred to the relevant sections of [13]. Here we only require the fact that through the solution of this Hamilton-Jacobi equation, the Gauss map equation defined in (57) solves the Einstein equation for all $T \in (0, \infty)$ or equivalently for all $\tau \in (-\infty, 0)$ and defines a ray-structure based at $\rho$ of the Teichmüller space parametrized by the transverse-traceless conformally invariant 2-tensor $\kappa_{ij}^{TT}$. The conformal metric $\gamma_{ij} = e^{-2\psi} g_{ij} \in T \Sigma_g$ indeed approaches to $\rho_{ij}$ at the limit $\tau \to 0^-$. Therefore, if we run the Einstein flow in the reverse direction, then expression of $\gamma_{ij}$ in terms of $\rho_{ij}$ and $\kappa_{ij}^{TT}$ obtained from the Gauss map equation defines a ray-structure of the Teichmüller space parametrized by $\kappa_{ij}^{TT}$ i.e., for a fixed $\rho$, two different $\kappa_{ij}^{TT}$ corresponding to two different rays. An implicit solution [13] of the Gauss map equation (57) gives
\[
\gamma_{ij} = e^{2\psi} g_{ij} = e^{2\psi} \left( \frac{2\tau^3}{\mu_p} \frac{\rho^{ik} \mu \gamma^{jl} \kappa_{kl}^{TT}}{1 + \sqrt{1 + 2\tau^2 \mu_p^2 |\kappa^{TT}|_g^2}} \right) + \tau^2 \left( \frac{1 + 2\tau^2 \mu_p^2 |\kappa^{TT}|_g^2}{\mu_p^2} \right) \rho_{ij}.
\]  
Using this equation (which is effectively same as the Gauss map equation), [13] constructed a fully non-linear elliptic equation namely the Monge-Ampere equation and showed that a unique solution of the such equation exists. Recently [41] showed using direct analytic technique that such a unique solution exists for all $\tau \in (-\infty, 0)$. Essentially, these analysis are in a sense complementary to the Hamilton-Jacobi theory and provides a more explicit description of the ray structure of the Teichmüller space. Analyzing the associated Monge-Ampere equation, [13] explicitly showed that every solution curve of the reduced dynamics in the configuration space ($T \Sigma_g$) approaches a point (\rho) lying in the interior of the Teichmüller space, that is,
\[
\lim_{\tau \to 0^-} \gamma_{ij} = \rho_{ij}.
\]  
Note that the choice of $\rho$ is arbitrary as long as it does not leave the compact sets of $T \Sigma_g$, and therefore, one may vary $\rho$ over $T \Sigma_g$ to obtain the full ray-structure of the Teichmüller space. We do not provide the complete calculations regarding the $\tau \to 0^-$ behavior of the solution curve as it is derived and described in details by Prof. Moncrief in [13]. Readers are referred to the relevant sections of the same. We only need the information that the Gauss map equation indeed provides a solution ray in the Teichmüller space for the Einstein equations. Forward time asymptotics of each such ray corresponds to an interior point which also realizes the infimum of the Dirichlet energy (and the reduced Hamiltonian). Each member of a family of rays which asymptotically approach the point $\rho \in T \Sigma_g$ corresponds to a unique choice of $\kappa^{TT}$ and none of the two rays of a same family intersect each other (except at $\rho$, where they approach at $\tau \to 0^-$).
Forward in time limit of the solution curves is well studied in [13]. Therefore, without repeating the same here, we will proceed to study the other limit, that is, \( \tau \to -\infty \) limit which corresponds to the big bang singularity. In this limit the solution curve leaves every compact set of the Teichmüller space, which may be obtained through studying the time evolution of the Dirichlet energy (a proper function on \( T\Sigma_g \)) of the Gauss map. The time is chosen to be \( t = \frac{-1}{\tau} \) (54). From equation (63), the time derivative of the \( |K^{TT}|^2_g \) reads

\[
\frac{d}{dt} \int_{\Sigma} |K^{TT}|^2_g \mu_g = \frac{d}{d\tau} \int_{\Sigma} (\frac{\tau^2}{2} + R(g)) \mu_g, \\
= \tau^2 \frac{d}{d\tau} \int_{\Sigma} (\frac{\tau^2}{2} + R(g)) \mu_g, \\
= \tau^3 \int_{\Sigma} \mu_g + \tau^2 \int_{\Sigma} \mu_g (-2N\tau), \\
= \tau \int_{\Sigma} N|K^{TT}|^2_g \mu_g, \\
= -\frac{1}{t} \int_{\Sigma} N|K^{TT}|^2_g \mu_g,
\]

where, we have used the lapse equation \( \Delta_g N + N(|K^{TT}|^2_g + \frac{\tau^2}{2}) = \tau^2 \), the Hamiltonian constraint \( |K^{TT}|^2_g = \frac{\tau^2}{2} + R(g) \), and the evolution equation \( \frac{\partial \mu_g}{\partial t} = -2NK_{ij} + (L_X g)_{ij} \).

Utilizing the estimate of the lapse function (56), we immediately obtain

\[
-\frac{2}{t} \int_{\Sigma} |K^{TT}|^2_g \mu_g \leq \frac{d}{dt} \int_{\Sigma} |K^{TT}|^2_g \mu_g \leq -\frac{1}{t} \int_{\Sigma} |K^{TT}|^2_g \mu_g,
\]

integration of which yields at \( t \to 0 \) limit

\[
\frac{\text{const.}}{t} \leq \int_{\Sigma} |K^{TT}|^2_g \mu_g \leq \frac{\text{const.}}{t^2}.
\]

Using the expression of the Dirichlet energy \( E[id; \gamma, \rho] \) from equation (63), the following estimate is obtained at the limit \( \tau \to -\infty \) i.e., \( t \to 0 \)

\[
\frac{2\text{const.}}{t} - 4\pi \chi \leq E_\gamma \leq \frac{2\text{const.}}{t^2} - 4\pi \chi,
\]

which clearly implies that the Dirichlet energy blows up at limit of big bang singularity i.e., at the limit \( t \to 0 \) or equivalently \( \tau \to -\infty \). An immediate interpretation of such limiting behavior would be that the corresponding Einstein solution curve defined by the Gauss map leaves every compact set in the Teichmüller space (configuration space). This is precisely the consequence of the fact that the Dirichlet energy is a proper function on the Teichmüller space (see [19] for the detailed proof of the properness of the Dirichlet energy). Therefore, every solution curve leaves the Teichmüller space at the limit of big-bang. However, we do not know where they attach, that is, realizing \( T\Sigma_g \) as its image in the space of projective current by the map \( L \) (as defined in (8)), the solution curve diverging in the Teichmüller space must converge somewhere (follows from the compactness of the space of projective current). But, we do not yet know the identity of that space. In fact we would like to show in the following sections that every solution
curve indeed attaches to the Thurston boundary of the Teichmüller space. In addition to the backward in time asymptotic behavior of the Dirichlet energy, we also observe the monotonic decay of the same in the time forward direction

$$\frac{d}{dt} E[id; \gamma, \rho] = 2 \frac{d}{dt} \int_{\Sigma} |K^{TT}|^2 g_{\mu \nu},$$

$$= 2 \tau \int_{\Sigma} N|K^{TT}|^2 g_{\mu \nu} < 0.$$ 

d\[E[id; \gamma, \rho]\] = 0 if and only if \(K^{TT} \equiv 0\) (or \(\kappa^{TT} \equiv 0\)) and \(d\[E[id; \gamma, \rho]\] \rightarrow 0\) at the limit \(\tau \rightarrow 0\). The solution corresponding to \(\kappa^{TT} \equiv 0\) is nothing but the fixed points of the reduced Einstein evolution equations. Substituting \(\kappa^{TT} = 0\) is the Lichnerowicz equation (42) yields

$$-2\Delta \gamma \psi + 1 - e^{2\psi} \frac{\tau^2}{2} = 0,$$

which has a unique solution

$$e^{2\psi} = \frac{2}{\tau^2}.\tag{84}$$

The reduced evolution equation reads

$$\frac{\partial \gamma_{ij}}{\partial t} = e^{-2\psi} \left( -(\partial_t e^{2\psi} \gamma_{ij} - 2N\kappa^{TT}_{ij} - e^{2\psi} N\tau \gamma_{ij} + (L_X e^{2\psi} \gamma)_{ij} \right), \tag{85}$$

which, upon substituting \(e^{2\psi} = \frac{2}{\tau^2}, \kappa^{TT} = 0\) and utilizing the lapse equation, shift equation, and Hamiltonian constraint yields

$$\frac{\partial \gamma_{ij}}{\partial t} = 0.\tag{86}$$

Few lines of simple calculation yields \(\partial_t \kappa^{TT}_{ij} = 0\) as well. These fixed points characterized by \((\gamma_{ij}, \kappa^{TT}_{ij} = 0, N = 2, X^i = 0), R(\gamma) = -1,\) are indeed stable fixed points for arbitrary large data (even though the Dirichlet energy controls the \(H^1 \times L^2\) norm of the data \((\gamma, \kappa^{TT})\), finite dimensionality of the phase space implies that control on this norm is sufficient). This is precisely a consequence of the monotonic decay of the Dirichlet energy and \(d\[E[id; \gamma, \rho]\] \equiv 0\) precisely at these fixed points (Dirichlet energy acts as a Lyapunov function). Therefore, these fixed point solutions are asymptotically stable and every solution curve approaches one of this fixed points in forward infinite time \(t\).

This point even though is described in details in \([13, 12]\), is extremely important will be of use in obtaining the main result. Summarizing this section together with the results of \([13]\), we have the following theorem

**Theorem 1:** Let \(\Sigma_g\) be a closed (compact without boundary) Riemann surface of genus \(g > 1\). The data \((\gamma, \kappa^{TT}, \tau, N, X)\) defined through the Gauss map equation (57) and elliptic equations (35-36) solve the Einstein dynamical equations iff they also solve the constraints. Such solution asymptotically approaches the fixed point solution \((R(\gamma) = -1, \kappa^{TT} = 0, N = 2, X^i = 0)\) of the dynamical equations at the limit \(\tau \rightarrow 0\) and every such solution runs off the edge of the configuration space (Teichmüller space) at the limit of the big-bang singularity \((\tau \rightarrow -\infty)\).

Now we enter into the final phase where we utilize available results stated in the previous sections and obtain the main result.
5. Asymptotic behavior of the solution curve at big-bang and Thurston boundary

In the previous section, we have established that every solution curve runs off the edge of the Teichmüller space. However, we do not apriori know whether they actually converge to the Thurston boundary. However, when realizing the Teichmüller space as a subset of the space of Projective current (which is compact), these solution curves must converge somewhere. Therefore, let us name this boundary as the Einstein boundary of the Teichmüller space and denote it by \( \text{Ein}_g \). Our goal in this section is to show that this boundary is indeed equivalent to the Thurston boundary that is \( \mathcal{T}\Sigma_g^{\text{Th}} = \mathcal{T}\Sigma_g \cup \partial \mathcal{T}\Sigma_g^{\text{Th}} \approx \mathcal{T}\Sigma_g \cup \text{Ein}_g \). Note that Michael Wolf [31] obtained a compactification of the Teichmüller space through the use of holomorphic quadratic differential and he proved that his compactification is indeed equivalent to the Thurston compactification. In our case, we are automatically equipped with a holomorphic quadratic differential \( \kappa_{TT} \) (the transverse-traceless tensor). However, importantly, Wolf’s analysis is quite different from ours (and complementary in nature) in a sense that the Einsteinian dynamics occurs in the domain of the associated harmonic map while Wolf’s dynamics materializes in the target space. However, before constructing the quadratic differential and associated entities, we will need a few more estimates.

Now we will show the Boundedness of \( |\kappa_{TT}|_g^2 \) at the limit \( \tau \to -\infty \). Note that the following entity is conformally invariant

\[
P = \int_{\Sigma_g} \sqrt{|K_{TT}|_g^2} \mu_g = \int_{\Sigma_g} \sqrt{|\kappa_{TT}|_g^2} \mu_\gamma.
\]  

(87)

Applying Cauchy-Swartz inequality, Hamiltonian constraint \( |K_{TT}|_g^2 = \frac{\tau^2}{2} + R(g) \), and time defined in (54), we immediately obtain

\[
\left( \int_{\Sigma_g} \sqrt{|\kappa_{TT}|_g^2} \mu_\gamma \right)^2 = \left( \int_{\Sigma_g} \sqrt{|K_{TT}|_g^2} \mu_g \right)^2 \leq \left( \int_{\Sigma_g} |K_{TT}|_g^2 \mu_g \right) \left( \int_{\Sigma_g} \mu_g \right) = \left( \int_{\Sigma_g} \left( \frac{\tau^2}{2} + R(g) \right) \mu_g \right) \left( \int_{\Sigma_g} \mu_g \right) = \frac{\tau^2}{2} \left( \int_{\Sigma_g} \mu_g \right)^2 + 4\pi \chi \int_{\Sigma_g} \mu_g \leq \frac{1}{2\tau^2} V(g)^2,
\]

where we have used Gauss-Bonet theorem \( \int_{\Sigma_g} R(g) \mu_g = 4\pi \chi, \chi = 2(1 - g) < 0 \) is the Euler characteristics. On the other hand, we know that the volume \( V(g) \) of (\( \Sigma_g, g \)) approaches zero at the big-bang. However, we will study the evolution of \( V(g) \) and obtain a more precise estimate in terms of \( |\tau| \). Time differentiating \( V(g) = \int_{\Sigma_g} \mu_g \) yields

\[
\frac{dA(g)}{dt} = \frac{1}{2} \int_{\Sigma_g} g^{ij} \partial_t g_{ij} \mu_g,
\]  

(88)

which together with the evolution equation \( \partial_t g_{ij} = -2N(K_{ij}^{TT} + \frac{\tau}{2} g_{ij}) + (L_X g)_{ij} \) yields

\[
\frac{dA(g)}{dt} = \int_{\Sigma_g} (\tau N + \nabla[g]_i X^i) \mu_g = -\tau \int_{\Sigma_g} N \mu_g,
\]  

(89)
where the total covariant divergence term is dropped following Stokes theorem. Utilizing the estimate of the lapse function $1 \leq N \leq 2$ \textit{(56)} and $t = -\frac{1}{\tau}$ \textit{(54)}, we immediately achieve the following bound for the time derivative of the volume $V(g)$

$$\frac{1}{t} \leq \frac{dV(g)}{dt} \leq \frac{2}{t},$$

\textit{(90)}

integration of which yields the following at the limit $\tau \to -\infty$ or $t \to 0$

$$\text{constant} \cdot t^2 \leq V(g(t)) \leq \text{constant} \cdot t$$

\textit{(91)}

Therefore, we using the inequality $0 \leq \int_{\Sigma_g} \sqrt{|\kappa^{TT}|^2 \gamma_\mu} \leq \frac{1}{2\tau^2}(V(g))^2$, we obtain

$$0 \leq \lim_{t \to 0/\tau \to -\infty} \left( \int_{\Sigma} \sqrt{|\kappa^{TT}|^2 \gamma} \right) \leq C < \infty,$$

\textit{(92)}

for some constant $C$. Here note an important fact that $|\kappa^{TT}|^2_\gamma$ may only approach zero asymptotically without ever becoming exactly zero. This follows from the fact that $\kappa^T \equiv 0$ throughout $\Sigma_g$ is only attained at the fixed point which corresponds to an interior point of the Teichmüller space. But we have shown in the previous section that every solution curve runs off the edge of the Teichmüller space at big-bang ($\tau \to -\infty/\ t \to 0$). Therefore, $|\kappa^{TT}|^2_\gamma$ may only limit to zero without ever achieving it and we will obtain a suitable power-law decay of $|\kappa^{TT}|^2_\gamma$ at the limit $\tau \to -\infty$. Now we also have the following

$$\lim_{t \to 0/\tau \to -\infty} \int_{\Sigma_g} \mu_g = \lim_{t \to 0} \int_{\Sigma_g} e^{2\psi} \mu_\gamma = 0,$$

\textit{(93)}

and thus

$$\lim_{t \to 0/\tau \to -\infty} e^{2\psi} = 0.$$

\textit{(94)}

Note that we have the following inequality \textit{(49)}

$$\frac{2}{\tau^2} \leq e^{2\psi} \leq 1 + \sqrt{1 + 2\tau^2 \sup |\kappa^{TT}|^2_\gamma}.$$

\textit{(95)}

In a sense, we must have

$$e^{2\psi} \sim \frac{h^2(x)}{\tau^\alpha},$$

\textit{(96)}

for some $2 \geq \alpha > 0$ and suitable function $h$ satisfying $h^2(x) > 0 \ \forall x \in \Sigma_g$ (and pointwise norm ($L^\infty$) of $h^2(x)$ must be finite in order for the limit $\lim_{\tau \to -\infty} e^{2\psi}$ to be defined), at the limit $t \to 0$ or $\tau \to -\infty$ and $|\kappa^{TT}|^2_\gamma$ will be constrained by this. Let us say $|\kappa^{TT}|^2_\gamma$ behaves as follows at the limit $\tau \to -\infty$

$$|\kappa^{TT}|^2_\gamma \sim \frac{f^2(x)}{\tau^\beta},$$

\textit{(97)}

for some $\beta \geq 0$ and integrable function $f$, $x \in \Sigma_g$ \textit{(from (92)}, one has $|\kappa^{TT}|^2_\gamma \leq \frac{x^{2\alpha \psi}}{2} \sim \frac{h^4(x)}{2|\tau^{2\alpha - 2}}$ pointwise and therefore, $f^2(x) \leq \frac{h^4(x)}{2|\tau^{2\alpha - 2}}$; in fact it is not hard to see from \textit{(94, 95)}, that for suitable $\alpha$ and $\beta$, pointwise norm of $f^2$ is bounded; actually follows from
the fact that $\kappa^{TT}$ is an element of a finite dimensional vector space and therefore all norms are equivalent). Also note that $\beta$ cannot be negative because then

$$\int_{\Sigma_g} \sqrt{|\kappa^{TT}|^2_{\gamma}} \mu_\gamma \sim \frac{C}{|\tau|^\beta/2} \quad (98)$$

would not be finite at $\tau \to -\infty$, where $C$ is a constant. Following the previous estimate of $e^{2\psi}$ $(49,95)$ obtained from the Lichnerowicz equation, we may essentially infer that $|\kappa^{TT}|^2_{\gamma}$ can not grow or decay too fast, otherwise $e^{2\psi}$ might not decay to zero or decay faster than $\frac{2}{\tau}$ (a contradiction to the estimate $(49,95)$) at $\tau \to -\infty$. Following $(49,95)$, we must have

$$2 \geq \alpha \geq \frac{\beta}{2} + 1. \quad (99)$$

This yields

$$1 \leq \alpha \leq 2, \quad (100)$$

$$0 \leq \beta \leq 2. \quad (101)$$

But $\beta$ should in fact be strictly less than $\alpha$, which should follow from the expression of the Dirichlet energy

$$E[id; \gamma, \rho] = 2 \int_{\Sigma_g} |K^{TT}|^2_g \mu_g - 4\pi \chi = 2 \int_{\Sigma_g} |\kappa^{TT}|^2_{\gamma} e^{-2\psi} \mu_\gamma - 4\pi \chi \quad (102)$$

$$\sim |\tau|^{\alpha-\beta} \int_{\Sigma_g} \frac{f^2(x)}{h^2(x)} \mu_\gamma \sim C|\tau|^{\alpha-\beta} \quad (103)$$

for some suitable constant $C > 0$. Now Dirichlet energy must blow up at the limit $\tau \to -\infty$ (we have shown in the last section), that is,

$$\alpha > \beta, \quad (104)$$

and more precisely following the estimate $(81)$ and $t = -\frac{1}{\tau}$,

$$1 \leq \alpha - \beta \leq 2. \quad (105)$$

Therefore, we must have

$$1 \leq \alpha \leq 2, \quad (106)$$

$$0 \leq \beta < 1, \quad (107)$$

and the following power law behavior for $|\kappa^{TT}|^2_{\gamma}$

$$\sup_{\Sigma_g} |\kappa^{TT}|^2_{\gamma} \sim \frac{C^2}{|\tau|^{\beta}} \quad (108)$$

for a suitable constant $C^2 < \infty$ at the limit $\tau \to -\infty/ t \to 0$.

Let us now define the quadratic differential and study the property associated with its measured foliation. As mentioned previously, we have a natural holomorphic quadratic differential associated to the Einstein flow due to the fact that corresponding
to each transverse-traceless tensor $\kappa^{TT}$, we may associate a holomorphic quadratic differential. Here, we define the following quadratic differential

$$
\phi_\tau = |\tau|^{\frac{\beta}{2}} (\kappa_{11}^{TT} - \iota \kappa_{12}^{TT}) dz^2
= \phi_\tau(z) dz^2,
= \phi_{11}^{TT} (dx^2 - dy^2) + 2\phi_{12}^{TT} dxdy + \iota(\phi_{12}^{TT} (dx^2 - dy^2) - 2\phi_{11}^{TT} dxdy),
= \kappa + \iota \xi,
$$

where $\phi_{ij}^{TT} = |\tau|^{\frac{\beta}{2}} \kappa_{ij}^{TT}$, $0 \leq \beta < 1$. Transverse-traceless property of $\phi_{ij}^{TT}$ precisely implies $\partial \phi_\tau / \partial \bar{z} = 0$ i.e., $\phi_\tau$ is holomorphic (note that we are in isothermal coordinates). Define the $L^2$ norm of this quadratic differential as follows

$$
|\phi_\tau|^2_{L^2} = \int_{\Sigma_g} |\phi^{TT}|^2_{\gamma} \mu_\gamma,
$$

which approaches to a finite constant at $\tau \to -\infty$ i.e.,

$$
\lim_{\tau \to -\infty} |\phi_\tau|^2_{L^2} = \lim_{\tau \to -\infty} \int_{\Sigma_g} |\tau|^\beta f^2(x) |\tau|^\beta \mu_\gamma = C
$$

for some suitably chosen $0 < C < \infty$. This result will be important later. Now, once we have a quadratic differential, immediately, we obtain horizontal and vertical measured foliations associated with this holomorphic quadratic differential. The transverse measures of the vertical measured foliation and horizontal measured foliation are (follows from (21) and 22)

$$
\mu_{vert}(C) = \oint_{\Sigma_g} \sqrt{\kappa + \sqrt{\kappa^2 + \xi^2}/2},
$$

$$
\mu_{hor}(C) = \oint_{\Sigma_g} \sqrt{\kappa^2 - \xi^2} \kappa/2
$$

respectively. Let us consider that the tangent vector field to the curve $C$ be $u^1 \partial / \partial x_1 + u^2 \partial / \partial x_2$ and denote this by $(u^1, u^2)^T$. The term $\kappa^2$ may be written as the bi-linear form $\phi_{ij}^{TT} u^i u^j (d\lambda)^2$, where $\lambda$ is the parameter along $C$. Similarly, the term $\xi$ may be written as $\kappa_{im}^{TT} J^m u^i u^j = \kappa_{im}^{TT} u^i u^m$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $v^m = J^m u^j$ that is $v = (-u^2, u^1)^T$. More importantly, we see that the following holds in isothermal coordinate ($\gamma = \gamma(z) |dz|^2$, $\gamma(z) = e^{\delta(z)}$, $\delta(z) : \Sigma_g \to \mathbb{R}$)

$$
\gamma(u, v) = \gamma(z)(-u^1 u^2 + u^2 u^1) = 0,
$$

that is, $u$ and $v$ are orthogonal to each other. This is precisely a consequence of the existence of a global isothermal coordinate on $\Sigma_g$ (as mentioned in the beginning, we use isothermal coordinate throughout). The transverse measure to the vertical foliation may be written as follows

$$
\mu_{vert}(C) = \oint_{\Sigma_g} \sqrt{\phi_{ij}^{TT} u^i u^j + \sqrt{(\phi_{ij}^{TT} u^i u^j)^2 + (\phi_{ij}^{TT} u^i v^j)^2}} d\lambda.
$$
Let us now compute the $\gamma$—length of a geodesic in the homotopy class $[\mathcal{C}]$ and relate it to its transverse measure associated to the measured foliation of the holomorphic quadratic differential $\phi_x$. Through the unique solution of the Monge-Ampere equation, we define a ray structure of the Einstein equations. Therefore analyzing the asymptotic behaviour of the Monge-Ampere equation is in principle same as analysing the Gauss map equation. In addition, analysis of the Gauss map equation seems more tractable, because, we have a handful of estimates from the elliptic equations associated with Einstein dynamics. Using the Gauss map equation, we obtain

\[
\rho_{ij} u^i u^j = |K|^2 g_{ij} u^i u^j + 2\tau K_{ij} u^i u^j - \tau^2 g_{ij} u^i u^j, \tag{117}
\]

\[
= (|K^{TT}|^2 g + \frac{\tau^2}{2}) g_{ij} u^i u^j + 2\tau K_{ij}^{TT} u^i u^j
= (e^{-4\psi}|K^{TT}|^2 g + \frac{\tau^2}{2}) e^{2\psi} \gamma_{ij} u^i u^j + 2\tau \kappa_{ij}^{TT} u^i u^j.
\]

We do know the fact that $\rho \in T\Sigma_g$ is fixed and the $\rho$—length of $\mathcal{C}$ is bounded (due to the properness of the Dirichlet energy which remains finite at the interior of the Teichmüller space). Following the Gauss map equation, we have the following

\[
|\rho_{ij} u^i u^j| = \left| \left\{ |K^{TT}|^2 g + \frac{\tau^2}{2} \right\} g_{ij} u^i u^j + 2\tau \kappa_{ij}^{TT} u^i u^j \right|,
\]

\[
\geq \left| \left\{ |K^{TT}|^2 g + \frac{\tau^2}{2} \right\} g_{ij} u^i u^j \right| - 2|\tau \kappa_{ij}^{TT} u^i u^j|,
\]

\[
= ||\kappa^{TT}|^2 e^{-2\psi} \gamma_{ij} u^i u^j + \frac{\tau^2}{2} e^{2\psi} \gamma_{ij} u^i u^j| - 2|\tau \kappa_{ij}^{TT} u^i u^j|,
\]

\[
\geq 2\sqrt{\frac{|\kappa^{TT}|^2 \tau^2}{2} (\gamma_{ij} u^i u^j)^2} - 2|\tau \kappa_{ij}^{TT} u^i u^j|,
\]

that is,

\[
\frac{1}{\sqrt{2}} \sqrt{\frac{|\kappa^{TT}|^2 \gamma_{ij} u^i u^j}{\tau^2}} \leq |\kappa_{ij}^{TT} u^i u^j| + \frac{1}{2|\tau|} \rho_{ij} \alpha^i \alpha^j. \tag{119}
\]

Now from (97), let’s say that the infimum of $f^2$ be $C_f^2$ which is strictly positive provided that we stay away from the zeros (finite number) of the quadratic differential $\phi_x$ (corresponds to the singularities of the associated measured foliation). This is indeed the case. Let us consider that the quadratic differential has zeros at $(z_1, z_2, ..., z_n)$, $n < \infty$. Consider $\epsilon$ disks $D_\epsilon(z_i)$ around each of the zeros. As these zeros correspond to the the singularity of the associated measured foliation, we will consider the transverse measure on $\Sigma' = \Sigma - \{ \cup_{i=1}^n D_\epsilon(z_i) \}$. On $\Sigma'$, the previous inequality becomes

\[
\frac{|C_f|}{\sqrt{2}} \gamma_{ij} u^i u^j \leq ||\tau \frac{\kappa^{TT}}{\tau} u^i u^j| + \frac{1}{2|\tau|} \rho_{ij} \alpha^i \alpha^j, \tag{120}
\]

\[
\lim_{\tau \to -\infty} \frac{|C_f|}{\sqrt{2}} \gamma_{ij} u^i u^j \leq \lim_{\tau \to -\infty} |\phi^{TT}_{ij} u^i u^j| + \lim_{\tau \to -\infty} \frac{1}{2|\tau|} \rho_{ij} \alpha^i \alpha^j. \tag{121}
\]
Now notice the fact that $0 \leq \beta < 1$ (106) and thus the last term drops out of the inequality ($\rho-$length of $C$ is finite and independent of $\tau$). Therefore, we obtain the following inequality

$$\lim_{\tau \to -\infty} \left| \frac{C_f}{\sqrt{2}} \gamma_{ij} u^i u^j \right| \leq \lim_{\tau \to -\infty} |\phi_{ij}^{TT} u^i u^j|. \quad (122)$$

Let us analyze the Gauss-map equation in a different way

$$|\rho_{ij} u^i u^j - 2\tau \kappa_{ij}^{TT} u^i u^j| = \left| \left\{ |K^{TT}|^2 g + \frac{\tau^2}{2} \right\} g_{ij} u^i u^j. \right| \quad (123)$$

Now utilizing the estimate of $|K^{TT}|^2$ from (49), we obtain

$$|2\tau \kappa_{ij}^{TT} u^i u^j - |\rho_{ij} u^i u^j| \leq \tau_0 e^{2\psi} \gamma_{ij} u^i u^j, \quad (124)$$

which utilizing the estimate (53) yields

$$|2\tau \kappa_{ij}^{TT} u^i u^j - |\rho_{ij} u^i u^j| \leq \left( 1 + \sqrt{1 + \frac{2\tau_0 e^{2\psi}}{\kappa_{ij}^{TT}}} \right) \gamma_{ij} u^i u^j. \quad (125)$$

Substituting the estimate (108) into the previous inequality leads to

$$||\tau|^\beta \kappa_{ij}^{TT} u^i u^j| \leq \left( \frac{1}{2\tau^{1-\frac{\beta}{2}}} + \frac{\sqrt{\tau |^\beta + 2C_2^2\tau^2}}{2\tau} \right) \gamma_{ij} u^i u^j + \frac{1}{2\tau^{1-\frac{\beta}{2}}} |\rho_{ij} u^i u^j|,$$

that is,

$$|\phi_{ij}^{TT} u^i u^j| \leq \left( \frac{1}{2\tau^{1-\frac{\beta}{2}}} + \frac{\sqrt{\tau |^\beta + 2C_2^2\tau^2}}{2\tau} \right) \gamma_{ij} u^i u^j + \frac{1}{2\tau^{1-\frac{\beta}{2}}} |\rho_{ij} u^i u^j| \quad (126)$$

and therefore, using $0 \leq \beta < 2$ from (106), at the limit $\tau \to -\infty$

$$\lim_{\tau \to -\infty} \frac{|\phi_{ij}^{TT} u^i u^j|}{\gamma_{ij} u^i u^j} \leq \lim_{\tau \to -\infty} \left( \frac{1}{2\tau^{1-\frac{\beta}{2}}} + \frac{\sqrt{\tau |^\beta + 2C_2^2\tau^2}}{2\tau} \right) \quad (127)$$

In a sense, we have at $\tau \to -\infty$

$$\frac{|C_f|}{\sqrt{2}} \gamma_{ij} u^i u^j \leq |\phi_{ij}^{TT} u^i u^j| \leq \frac{|C|}{\sqrt{2}} \gamma_{ij} u^i u^j, \quad (128)$$

with $0 < C_f^2 < C^2 < \infty$. Therefore, without loss of generality, we may write

$$C' \gamma_{ij} u^i u^j = |\phi_{ij}^{TT} u^i u^j|, \quad (129)$$

for a suitably chosen constant satisfying $C'$ satisfying $|C_f| < C' < |C|$. This is an important expression obtained at the limit of big-bang ($\tau \to -\infty$). On the other hand, the expression for the transverse measure of the vertical foliation reads (116)

$$\mu_{vert}(C) = \oint_C \left| \frac{\phi_{ij}^{TT} u^i u^j + \sqrt{(\phi_{ij}^{TT} u^i u^j)^2 + (\phi_{ij}^{TT} u^i u^j)^2}}}{2} \right| d\lambda. \quad (130)$$
We still need to obtain an estimate for the term $\kappa_{TT} u^i v^j$. In addition to the transverse measure to the vertical foliation, we also have the following transverse measure to the horizontal foliation of the holomorphic quadratic differential $\Phi$

$$\mu_{\text{hor}}(C) = \oint_C \sqrt{\frac{(\phi_{ij}^{TT} u^i u^j)^2 + \phi_{ij}^{TT} u^i v^j - \phi_{ij}^{TT} u^i u^j}{2}} |d\lambda|. \quad (131)$$

In the analysis of Wolf [31], it is shown that this transverse measure associated to the horizontal foliation collapses asymptotically. In a sense, the high energy (Dirichlet) maps collapses the vertical direction of a measured foliation [30, 31]. However, note that such property is different from our case. This is, because, in Wolf’s [31] construction, the domain is fixed while the target is varied, that is, the dynamics occurs in the target space. In our case, the dynamics takes place in the domain. Therefore, we can not utilize available machinery such as Beltrami differential $\nu := \frac{|Wz|}{|W\bar{z}|}$ ($W : \Sigma_g(\gamma) \to \Sigma_g(\rho)$ and harmonic) or the associated Bochner equation controlling the behaviour of $\nu$ to show that $\mu_{\text{hor}}$ vanishes and therefore, $\kappa_{ij}^{TT} u^i v^j$ approaches zero asymptotically. Once again the Gauss map equation (57) comes to the rescue and notably it is of purely relativistic origin. The Gauss-map equation reads

$$\rho_{ij} = (e^{-4\psi} |\kappa_{TT}|^2 \gamma + \frac{\tau^2}{2})e^{2\psi} \gamma_{ij} + 2\tau \kappa_{ij}^{TT}, \quad (132)$$

which upon contrating with $\zeta$ and $\eta$ yields

$$\rho_{ij} \zeta^i \eta^j = (e^{-4\psi} |\kappa_{TT}|^2 \gamma + \frac{\tau^2}{2})e^{2\psi} \gamma_{ij} \zeta^i \eta^j + 2\tau \kappa_{ij}^{TT} \zeta^i \eta^j. \quad (133)$$

Performing exactly similar analysis as before, we may arrive without much difficulty to the following relation at the limit of big-bang ($\tau \to -\infty$)

$$C' |\gamma_{ij} \zeta^i \eta^j| = |\phi_{ij}^{TT} \zeta^i \eta^j| \quad (134)$$

Now using (116), we immediately obtain the following at the limit when $\tau$ approaches $-\infty$

$$\mu_{\text{vert}}(C) = \oint_C \sqrt{\frac{(\phi_{ij}^{TT} u^i u^j)^2 + (\phi_{ij}^{TT} u^i v^j)^2 + C''^2 (\gamma_{ij} u^i v^j)^2}{2}} |d\lambda|. \quad (135)$$

But, from the orthogonality of $u$ and $v$ (115), we immediately observe that $|\phi_{ij}^{TT} u^i v^j| = C' |\gamma_{ij} u^i v^j| = 0$ which leads to the following expression for the vertical measure of curve $C$ with respect to the foliation defined by the holomorphic quadratic differential $\phi_{T}$

$$\mu_{\text{vert}}(C) = \oint_C \sqrt{|\phi_{ij}^{TT} u^i u^j|} |d\lambda|. \quad (136)$$

The asymptotic vanishing of the term $\xi = \phi_{TT} u^i v^j$ precisely implies that the transverse measure of the associated horizontal foliation vanishes i.e.,

$$\mu_{\text{hor}}(C) = \oint_C \sqrt{\frac{(\phi_{ij}^{TT} u^i u^j)^2 + C''^2 (\gamma_{ij} u^i v^j)^2 - \phi_{ij}^{TT} u^i u^j}{2}} |d\lambda| \quad (137)$$

\[= 0. \]
Figure 2. At the limit $\tau \to -\infty$, a non-trivial element $\gamma$ of $\pi_1(\Sigma_g)$ aligns itself with a leave of Horizontal foliation and thus its transverse measure with respect to the horizontal foliation collapses.

Thus, the high Dirichlet energy limit (while viewed as a proper function on the Teichmüller space of the domain) precisely indicates that the transverse measure to the horizontal foliation associated to the quadratic differential $\phi_\tau$ defined in terms of $\kappa^{TT}$ (or equivalently $\phi^{TT}$) vanishes. Note that, the metric $\gamma$, the quadratic differential $\phi_\tau$, and the dynamics of the associated measured foliations are related to each other via Einstein flow. In a sense, Einstein flow drives the solution curve in such a way that the measured foliation behaves in this way at the limit of big-bang singularity.

Therefore, we obtain the following crucial relation in the big-bang limit ($\tau \to -\infty$)

$$\mu_{\text{vert}}(C) \sim \sqrt{C'} \oint_C \sqrt{\gamma_{ij} \dot{\alpha}^i \alpha^j} d\lambda = \sqrt{C'} l_\gamma(C).$$

(138)

Summarizing this section, we state the following theorem.

**Theorem 2:** Let $\Sigma_g$ be a closed (compact without boundary) Riemann surface of genus $g > 1$ and the data $(\gamma, \kappa^{TT}, \tau, e^{\psi}, N, X)$ defined through the Gauss map equation (57), Lichnerowicz equation (42), and the elliptic equations (35-36) solve the reduced Einstein equations. The ratio of the transverse measure of any non-trivial element of $\pi_1(\Sigma_g)$ with respect to the vertical measured foliation of the natural holomorphic quadratic differential $\phi_\tau := |\tau|^\frac{1}{2} (\kappa^{TT}_{11} - \iota \kappa^{TT}_{12}) dz^2 = (\phi^{TT}_{11} - \iota \phi^{TT}_{12}) dz^2, 0 \leq \beta < 1$ and its hyperbolic length that is length with respect to the metric $\gamma$ approaches to a finite constant at the limit of big-bang singularity i.e., $\tau \to -\infty$. The transverse measure associated with the horizontal foliation collapses to zero at the same limit.
6. Compactification theorem

In this section we claim that the Thurston compactification of the Teichmüller space is equivalent to our relativistic compactification. Let us denote the Einstein compactification of $\mathcal{T} \Sigma^g$ by $\mathcal{T} \Sigma^g_{Ein}$. In this section, we claim that the following theorem holds

**Theorem 3:** $\mathcal{T} \Sigma^g_{Th} \approx \mathcal{T} \Sigma^g_{Ein}$.

Before proving this theorem, we need a few additional concepts and two lemmas. Let us consider the function space $\Omega = \mathbb{R}^G(\Sigma g)$, where the space of geodesics on $\Sigma g$ is denoted as $G(\Sigma g)$, which may be obtained by $\pi_1(\Sigma g)$ action on the space of geodesics on $H^2$, that is $S^1_\infty \times S^1_\infty - \Delta$. Essentially $\Omega$ consists of functions which takes an element of $G(\Sigma g)$ and associate a positive number to it (in this case, length to be precise). It can essentially be viewed as the space of geodesic current given that the association of a positive number to each element of $G(\Sigma g)$ is $\pi_1(\Sigma g)$ invariant. We may construct the following map

$$l : \mathcal{T} \Sigma_g \rightarrow \Omega$$

$$\gamma \mapsto l_\gamma : G(\Sigma g) \rightarrow R_{>0}.$$ (139)

We may projectivize the space $\Omega$ as follows

$$P\Omega = \Omega / (\beta \sim t\beta, t > 0, \beta \in \Omega)$$ (140)

and subsequently obtain the following injetive map

$$\pi \circ l : \mathcal{T} \Sigma_g \rightarrow P\Omega.$$ (141)

Clearly, the map $l$ can be identified with the Louville current defined in section (2). The injectivity of $\pi \circ l$ follows from the injectivity of the map $L$ of section (2). Similarly, we may construct the following map from the space of measured lamination to $\Omega$

$$\nu : \mathcal{MF} \rightarrow \Omega$$

$$\mathcal{F} \mapsto (i(\mathcal{F}, \gamma_1) = \int_{\gamma_1 \in G(\Sigma g)} \mu_F, i(\mathcal{F}, \gamma_2) = \int_{\gamma_2 \in G(\Sigma g)} \mu_F, ....)$$, where $(\gamma_1, \gamma_2, ....) \in G(\Sigma g)$. Here, $\mu_F$ corresponds to the transverse measure associated with $\mathcal{F} \in \mathcal{MF}$. Clearly the space of measured geodesic foliation is a subset of the space of all geodesics and therefore, we have the following

$$\mathcal{PMF} := (\mathcal{MF} - \{0\}) / (\mathcal{F} \sim t\mathcal{F}, t > 0, \mathcal{F} \in \mathcal{MF})$$ (143)

$$= \pi \circ \nu(\mathcal{MF}) \subset \mathcal{P}\Omega.$$

Note that $\pi \circ \nu$ is injective. Following the one-one correspondence between $\mathcal{PML}$ and $\mathcal{PMF}$ as described in section (2), we may conclude that $\mathcal{PMF}$ and $\mathcal{T} \Sigma_g$ are disjoint in the space of projective currents i.e., $\mathcal{P}\Omega$. Thurston compactification, essentially, is given as $\mathcal{T} \Sigma_g = \mathcal{T} \Sigma_g \cup \mathcal{PMF}$. Now we state the following crucial lemma.

**Lemma*:** Let the sequence $\{\gamma_\tau\}$ leave all the compact sets in $\mathcal{T} \Sigma_g$ at the limit of big-bang i.e., $\tau \rightarrow -\infty$. The $\pi \circ l(\gamma_\tau)$ converges if and only if $\pi \circ \nu(\mathcal{F}_\tau)$ converges and subsequently
both have the same limit in $\mathcal{P}\Omega$. Here $\mathcal{F}_\tau$ is the measured foliation corresponding to the holomorphic quadratic differential $\phi_\tau = (\phi_{TT}^{11} - i\phi_{TT}^{12})dz^2$.

**Proof:** $\{\gamma_\tau\}$ diverges in $T\Sigma_g$ (maybe identified with its image in $\mathcal{P}\Omega$ under the map $\pi \circ l$) and therefore $\lim_{\tau \to -\infty} l_\tau(C) = \infty$ for some $C \in G(\Sigma_g)$. Now, it must converge to $\mathcal{P}\Omega$ due to the fact that the later is compact (passing to the level of subsequence). Therefore, $\exists \{\lambda_\tau\}$ such that $\lim_{\tau \to -\infty} \lambda_\tau l_\tau(C) = \mathcal{L} < \infty$. Now, utilizing the following estimate derived in the previous section, we have at the limit $\tau \to -\infty$

$$l_\tau(C) \sim C^{-\frac{1}{2}} i(\mathcal{F}, C) = i(\frac{1}{\sqrt{C}}\mathcal{F}, C), \quad (144)$$

we may immediately obtain that $\lim_{\tau \to -\infty} \lambda_\tau i(\frac{1}{\sqrt{C}}\mathcal{F}_\tau, \alpha) = \mathcal{L}$ i.e., equal to the limit of $\lambda_\tau l_\tau(C)$. Moreover, $\lambda_\tau \frac{1}{\sqrt{C}}\mathcal{F}_\tau$ converges in $\mathcal{P}\Omega$ or $\frac{1}{\sqrt{C}}\mathcal{F}_\tau$ converges in $\mathcal{P}\mathcal{M}\mathcal{F}$ (image of $\mathcal{P}\mathcal{M}\mathcal{F}$ in $\mathcal{P}\Omega$ under $\pi \circ \nu$ is identified with $\mathcal{P}\mathcal{M}\mathcal{F}$ and multiplication of a measured foliation by an overall constant yields the same foliation at the level of projective space). The reverse may be obtained in a similar way.

\[
\begin{array}{ccc}
\partial T\Sigma_g^{Ein} & \leftrightarrow & \partial T\Sigma_g^{Th} \\
\mathcal{T}T & \downarrow & \mathcal{W}_Q \\
S\mathcal{Q}\mathcal{D} & \leftrightarrow & S\mathcal{Q}\mathcal{D} \\
\mathcal{F} & \downarrow & \mathcal{W}_F \\
\mathcal{P}\mathcal{M}\mathcal{F} & \leftrightarrow & \mathcal{P}\mathcal{M}\mathcal{F}
\end{array}
\]

Notice the diagram above. This together with the results established so far will help us to finish the proof of the theorem 3. Prof. Moncrief has shown in [13] that no two solution ray (defined by the Gauss-map, satisfying constraint and gauges, and uniquely satisfying reduced Einstein equations through Hamilton-Jacobi equation) intersect each other (except at the limit $\tau \to 0$, where they may asymptotically approach each other). This gives a homeomorphism between the Teichmüller space $T\Sigma_g$ and the space of transverse-traceless tensors. However, each such transverse-traceless tensor $\kappa^{TT}$ has a holomorphic quadratic differential $\phi_\tau$ associated to it. Moreover, each such holomorphic quadratic differential represents measured foliation (with zeros of the quadratic differential being the singularities of the foliation), which follows from the classical result of Hubbard and Masur [32]. Now let us consider $\{\gamma_\tau\}$ leaves every compact set in the Teichmüller space and converges to the $\partial T\Sigma_g^{Ein}$. Associated with the sequence $\{\gamma_\tau\}$, there is a sequence of quadratic differential $\{\phi_\tau dz^2\}$ (defined in 109) from relativistic dynamics and such a unique sequence satisfies (as is shown in (112))

$$\lim_{\tau \to -\infty} |\phi_\tau|_2^2 = C, \quad (145)$$
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for some suitable constant $0 < C < \infty$. This is precisely equivalent to saying that as

the sequence $\{\gamma_\tau\}$ converges in $\partial T\Sigma_g^{Ein}$, $\{\phi_\tau\}$ approaches the $6g-7$ dimensional sphere $SQD$ in the space of holomorphic quadratic differential $QD$ and is defined by

$$|\phi_{\tau}|^2_{L^2} = C. \quad (146)$$

Now associated to the sequence $\{\phi_\tau\}$, there exists a unique sequence of measured foliations $\{F_\tau\}$. Now, the lemma* enters into the picture. Lemma* precisely states that the limits of the sequence $\{\gamma_\tau\}$ and the sequence $\{F_\tau\}$ are the same in the space of projective current ($\mathbb{P}Curr$) and lies on the space of Projective measured foliations ($\mathcal{PMF}$). Therefore, the space $\partial T\Sigma_g^{Ein}$ is precisely the space of projective measured foliations $\mathcal{PMF}$. But, $\mathcal{PMF}$ is nothing but the Thurston boundary of the Teichmüller space in $\mathbb{P}Curr$. Therefore

$$\partial T\Sigma_g^{Ein} \approx \partial T\Sigma_g^{Th}, \quad (147)$$

which completes the proof of

$$T\Sigma_g^{Th} \approx T\Sigma_g^{Ein}. \quad (148)$$

In addition note that each of $\partial T\Sigma_g^{Ein}$ and $\mathcal{PMF}$ are homeomorphic to $SQD$. In a sense the maps $TT, F, F \circ TT$ are all homeomorphisms (their injectivity and surjectivity is trivial to show explicitly using the uniqueness of the solutions of the reduced Einstein equations via Gauss map and Lemma*; for $\gamma_\tau$ inside $T\Sigma_g$, define $id : T\Sigma_g \rightarrow T\Sigma_g$, and for $\tau \rightarrow -\infty$, define a map between $\partial T\Sigma_g^{Ein}$ and $\partial T\Sigma_g^{Th}$ using $TT$ which assigns a $\phi_\tau$ for $\gamma_\tau$ through the solution of the reduced Einstein equations, and $F$ which assigns a $F_\tau$ to each $\phi_\tau$, then using lemma*, it is straightforward to establish the required homeomorphism). In a sense, we sketch a proof of Prof. Moncrief’s conjecture each of the solution curves of reduced Einsteinian dynamics runs off the edge of the Teichmüller space at the limit of big-bang singularity and attaches to the Thurston boundary of the Teichmüller space, that is, the space of projective measured laminations or foliations ($\mathcal{PML}, \mathcal{PMF}$). As a bonus, we also have in this relativistic settings that the space $SQD \subset QD$ is homeomorphic to $\partial T\Sigma_g^{Ein}$ and therefore $\mathcal{PMF}$. In a sense, we also recover Wolf’s result. Now we will describe the possible two mechanisms of approaching the boundary of the Teichmüller space in the next section.

7. Approaching $\partial T\Sigma_g$

Let us consider the Fenchel Neilsen coordinates of the Teichmüller space. Figure (5) shows the pants decomposition of the Teichmüller space and the associated Fenchel-Neilen co-ordinates (see [33] for the details of Fenchel-Neilen parametrization and pants decomposition). Such parametrization is given by length of $3g-3$ nontrivial (nontrivial in $\pi_1(\Sigma_g)$) geodesics $\{l_i\}_{i=1}^{3g-3}$ along with $3g-3$ associated twist parameters $\{\theta_{i=1}^{3g-3}\}$ (twist is performed about the same geodesic). The two possible mechanisms of attaining the boundary of the Teichmüller space are descried below.
7.1. Pinching of \( \Sigma \)

Now \( \gamma(l^n_i) \) denote a series of hyperbolic metrics and let \( \theta_i = 0 \ \forall i = 1, 2, 3, \ldots, 3g - 3 \). Letting any one of the \( l_i \) tend to infinity i.e., \( \lim_{n \to \infty} l^n_i = \infty \) implies approaching the boundary \( \partial T \Sigma \). Using collar lemma (see [19] for the detailed proof of the collar lemma), we immediately obtain there is a non-trivial geodesics transverse to \( l_i \) with length \( \approx \lim_{n \to \infty} e^{-l^n_i} \). This is the pinching mechanism described in figure (4). Note that the nontrivial (in \( \pi_1(\Sigma_g) \)) geodesic \( \gamma_2 \) collapses while the hyperbolic length \( l_1 \) of \( \gamma_1 \) approaches infinity. Now, the Dirichlet energy of the harmonic map between the varying metric \( \gamma(l^n_i) \) defined is a continuous proper function on the Teichmüller space. Therefore, the sequence of Dirichlet energy associated with the diverging sequence of metrics (or degenerating to be precise) \( \gamma(l^n_i) \) can not stay in a compact set that is the sequence blows up. Therefore we have the following correspondence

\[
\lim_{n \to \infty} l^n_i \to \infty \Rightarrow \lim_{n \to \infty} E_{\gamma(l^n_i)} \to \infty.
\]  

Notice that multiple non-trivial geodesics \( \gamma_i \) (and the corresponding transverse ones) may show the pinching behavior at once and each such limit corresponds to distinct points on \( \partial T \Sigma_g \).

7.2. Wringing of \( \Sigma_g \) by its neck

In order to explain the approach to \( \partial T \Sigma_g \) through wringing of \( \Sigma_g \), we need to introduce the symplectic geometry of the Teichmüller space [42, 43]. Using the parametrization...
Figure 4. Pinching mechanism collapsing the hyperbolic length of $\gamma_2$, while hyperbolic length of $\gamma_1$ approaches infinity.

$$(l_i, \theta_i)_{i=1}^{3g-3}$$ of Teichmuller space, define the symplectic form

$$\omega = \sum_{i=1}^{3g-3} dl_i \wedge d\theta_i,$$

which is preserved under the flow of the vector field $v = -\frac{\partial}{\partial \theta_i}$ and satisfies

$$\omega(-\frac{\partial}{\partial \theta_i}, \cdot) = dl_i(X).$$

The conserved Hamiltonian is nothing but the length $l_i(X)$. Here, $\theta_i$ is the twist parameter about $i$th geodesic. Therefore, flow of the vector field $-\frac{\partial}{\partial \theta_i}$ preserves the length $l_i$ of the geodesics about which $\Sigma_g$ is twisted. After $n$ such twists, the length of the geodesics transverse to the $i$th geodesic increases by $nl_i$. The wringing of $\Sigma_g$ about $i$th geodesic corresponds to the limit $n \to \infty$. Let the length of the transverse geodesic before the twist be $L^T$. After performing $n$ twists, the length becomes $\sim L^T + nl_i$ and therefore, the wringing corresponds to the fact that $\lim_{n \to \infty} \frac{l_i}{L^T + nl_i} = 0$. This is the other mechanism to approach the boundary of the Teichmüller space. Note that every points on the boundary $\partial T\Sigma_g$ can be obtained through a combination of these two basic operations and in every situation, the Dirichlet energy approaches infinity.

8. Conclusion

Despite the fact that ‘2+1’ gravity is devoid of a straightforward physical significance due to lack of gravitational waves degrees of freedom, it is of extreme importance while studying ‘3+1’ gravity on spacetimes of certain topological type ($S^2 \times S^1 \times \mathbb{R}, T^2 \times S^1 \times \mathbb{R},$...
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Figure 5. Pants decomposition of the hyperbolic surface $\Sigma_g$: hyperbolic length of $\gamma_i$ together with the twist about the same geodesic $\gamma_i$ parametrizes the Teichmüller space.

and $\Sigma_g \times S^1 \times \mathbb{R}$). As mentioned in the introduction, several studies have been done on this topic while the ‘3 + 1’ gravity has been realized as the ‘2 + 1’ gravity coupled to a wave map, where Teichmüller space of $\Sigma_g$ plays a crucial role. In 2 + 1 case, the configuration space is the Teichmüller space and we’ve shown here that the space of big-bang singularity is realized as the Thurston boundary of the Teichmüller space. At big-bang, the conformal geometry degenerates via pinching and wringing of $(\Sigma_g, \gamma)$. This result essentially characterizes the the complete solution space as well as identifies that the reduced Einstein flow can naturally be used to compactify the Teichmüller space. While such result is obtained by studying the purely vacuum gravity, a natural question arises whether inclusion of a positive cosmological constant might yield the same result. [12, 44] studied the vacuum GR in $2 + 1$, $n \geq 2$ case, where the future in time (i.e., $\tau \to 0$) behavior seems to persist. Therefore, it would be interesting to include a positive cosmological constant and check whether the Thurston boundary is approached in the big-bang limit. In addition, if one includes matter source and focus on the evolution of the gravitational degrees of freedom, can the big-bang limit be realized as the Thurston boundary? Could the Teichmüller degrees of freedom of ‘3+1’ gravity on $S^1$ bundle over $\Sigma_g \times \mathbb{R}$ realize the Thurston boundary in the same limit? what is the implication of such limiting behavior in classical level in quantizing ‘2+1’ gravity or ‘3+1’ gravity on these special topologies? can this characterization of the space of singularities be extended to higher dimensional gravity?
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