A NOTE ON THE CONVEX HULL OF FINITELY MANY PROJECTIONS OF SPECTRAHEDRA

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Abstract. A spectrahedron is a set defined by a linear matrix inequality. A projection of a spectrahedron is often called a semidefinitely representable set. We show that the convex hull of a finite union of such projections is again a projection of a spectrahedron. This improves upon the result of Helton and Nie [3], who prove the same result in the case of bounded sets.

1. Introduction

Let $A_0, A_1, \ldots, A_n$ be real symmetric $k \times k$ matrices. The set
\[
\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0 \},
\]
where $\succeq 0$ means positive semidefiniteness, is called a spectrahedron. Spectrahedra are generalizations of polyhedra and occur as feasible sets for semidefinite optimization.

A projection of a spectrahedron to a subspace of $\mathbb{R}^n$ is often called a semidefinitely representable set. Helton and Nie [3] conjecture that every convex semialgebraic set is such a projection. See for example [1, 2, 3, 4, 5, 6, 7] for more detailed information on spectrahedra and their projections.

We prove that the convex hull of finitely many projections of spectrahedra is again a projection of a spectrahedron. This generalizes Theorem 2.2 from Helton and Nie [3], which is the same result in the case that all sets are bounded or that the convex hull is closed.

2. Result

Proposition 2.1. If $S \subseteq \mathbb{R}^n$ is a projection of a spectrahedron, then so is $cc(S)$, the conic hull of $S$.

Proof. Since $S$ is a projection of a spectrahedron we can write
\[
S = \left\{ x \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^m : A + \sum_{i=1}^n x_i B_i + \sum_{j=1}^m z_j C_j \succeq 0 \right\},
\]
with suitable real symmetric $k \times k$-matrices $A, B_i, C_j$. Then with
\[
C := \{ x \in \mathbb{R}^n \mid \exists \lambda, r \in \mathbb{R}, z \in \mathbb{R}^m : \lambda A + \sum_{i=1}^n x_i B_i + \sum_{j=1}^m z_j C_j \succeq 0 \land \prod_{i=1}^n \left( \frac{\lambda x_i}{r} \right) \succeq 0 \}
\]
we have $C = cc(S)$ (note that $C$ is a projection of a spectrahedron, since the conjunction can be eliminated, using block matrices).
To see "\(C\)" let some \(x\) fulfill all the conditions from \(C\), first with some \(\lambda > 0\). Then \(a := \frac{1}{\lambda} x\) belongs to \(S\), using the first condition only. Since \(x = \lambda a, x \in \text{cc}(S)\). If \(x\) fulfills the conditions with \(\lambda = 0\), then \(x = 0\), by the last \(n\) conditions in the definition of \(C\). So clearly also \(x \in \text{cc}(S)\).

For "\(\exists\)" take \(x \in \text{cc}(S)\). If \(x \neq 0\) then there is some \(\lambda > 0\) and \(a \in S\) with \(x = \lambda a\). Now there is some \(z \in \mathbb{R}^m\) with \(A + \sum a_i B_i + \sum z_j C_j \geq 0\). Multiplying this equation with \(\lambda\) shows that \(x\) fulfills the first condition in the definition of \(C\). But since \(\lambda > 0\), the other conditions can clearly also be satisfied with some big enough \(r\). So \(x\) belongs to \(C\). Finally, \(x = 0\) belongs to \(C\), too.

\[\Box\]

**Remark 2.2.** The additional \(n\) conditions in the definition of \(C\) avoid problems that could occur in the case \(\lambda = 0\). This is the main difference to the approach of Helton and Nie in [3].

**Corollary 2.3.** If \(S_1, \ldots, S_t \subseteq \mathbb{R}^n\) are projections of spectrahedra, then also the convex hull \(\text{conv}(S_1 \cup \cdots \cup S_t)\) is a projection of a spectrahedron.

**Proof.** Consider \(\tilde{S}_i := S_i \times \{1\} \subseteq \mathbb{R}^{n+1}\), and let \(K_i\) denote the conic hull of \(\tilde{S}_i\) in \(\mathbb{R}^{n+1}\). All \(\tilde{S}_i\) and therefore all \(K_i\) are projections of spectrahedra, and thus the Minkowski sum \(\tilde{K} := K_1 + \cdots + K_t\) is also such a projection. Now one easily checks
\[
\text{conv}(S_1 \cup \cdots \cup S_t) = \{x \in \mathbb{R}^n \mid (x, 1) \in \tilde{K}\},
\]
which proves the result. \[\Box\]

**Example 2.4.** Let \(S_1 := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy \geq 1\}\) and \(S_2 = \{(0, 0)\}\). Both subsets of \(\mathbb{R}^2\) are spectrahedra, so the convex hull of their union,
\[
\text{conv}(S_1 \cup S_2) = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\},
\]
is a projection of a spectrahedron.

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