Supersymmetric modified Korteweg–de Vries equation: bilinear approach

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Received 30 July 2004, in final form 4 March 2005
Published 29 April 2005
Online at stacks.iop.org/Non/18/1597

Abstract

A proper bilinear form is proposed for the \(N = 1\) supersymmetric modified Korteweg–de Vries equation. The bilinear Bäcklund transformation for this system is constructed. As applications, some solutions are presented for it.

Mathematics Subject Classification: 35Q51, 35Q58

1. Introduction

The celebrated Korteweg–de Vries (KdV) equation is one of the most important systems in mathematical physics. It has wide applications and numerous interesting properties, such as soliton solutions, an infinite number of conservation laws, Bäcklund and Darboux transformations and solvability in terms of inverse scattering transformation. The KdV equation has many extensions, and one of them is the supersymmetric KdV (sKdV) equation constructed by Manin and Radul [1] (see also [2]). Since then, this \(N = 1\) sKdV system has been studied extensively, and many interesting properties have been established. For example, it is shown that the sKdV equation has a bi-Hamiltonian structure [3], the Painlevé property [4], infinitely many symmetries, the Darboux transformation [5] and the Bäcklund transformation (BT) [6] and bilinear forms [7–9].

Closely related to the sKdV equation, the \(N = 1\) supersymmetric modified Korteweg–de Vries (sMKdV) equation is introduced by Mathieu [2] and Yamanaka and Sasaki [10], respectively. It reads as

\[ \Psi_t + \Psi_{xxx} - 3\Psi (\partial_x \Psi) (\partial_x \Psi) - 3(\partial_x \Psi)^2 \Psi_x = 0, \]

or

\[ \Psi_t + \Psi_{xxx} - 3\Psi (\partial_x \Psi) (\partial_x \Psi) = 0, \]
where the field $\Psi = \Psi(x, \theta, t)$ is a Grassmann odd variable depending on the superspace variables $(x, \theta)$ and time, $t$, and $D = \partial/\partial \theta + \theta(\partial/\partial x)$ is the usual super derivative. It is shown that equation (1) is related to the $N = 1$ sKdV through a Miura type of transformation [2,10]. The sMKdV equation shares the common conserved quantities with the supersymmetric sine–Gordon equation.

It is known that Hirota’s bilinear approach is a very effective method for constructing particular solutions for soliton systems [11]. This method has been extended to the supersymmetric case in [7,9]. In particular, Carstea, Grammaticos and Ramani constructed the soliton type of solutions for the $N = 1$ sKdV equation. In a recent paper [12], the bilinearization of the sMKdV equation has been considered; that is, the system (1) is transformed into

\[
(SD_t + SD_x^3)(f \cdot f) = 0,
\]
\[
(SD_t + SD_x^3)(g \cdot g) = 0,
\]
\[
SD_x(g \cdot f) = 0,
\]
\[
D^2_x(g \cdot f) = 0,
\]

via $\Psi = D \ln(g/f)$, where the Hirota derivative is defined as

\[
SD_m^m D^n x f \cdot g = \left( D_{\theta_1} - D_{\theta_2} \right) \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right)^m \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n f(x_1, t_1, \theta_1)g(x_2, t_2, \theta_2) \bigg|_{x_1 = x_2, t_1 = t_2, \theta_1 = \theta_2}.
\]

Since there are only two tau functions $f$ and $g$, four bilinear equations result in more restrictions than necessary.

The purpose of this paper is to present our results on the bilinear approach to the sMKdV equation. We will show that there is indeed a proper bilinearization for this equation.

This paper is organized as follows: in the next section, we will transform the sMKdV equation into bilinear form. In section 3, we construct a bilinear BT for the sMKdV system. We will show that this BT in turn provides us a new Lax operator for the sMKdV equation. Then section 4 will be devoted to the construction of some solutions. The final section contains our discussion and conclusion.

2. Bilinearization

It is noted that the sMKdV equation (1) can be rewritten as

\[
\Psi_t + D[D\Psi_{xx} + 3\Psi\Psi_x(D\Psi) - 2(D\Psi)^3] = 0,
\]

which suggests the following substitution:

\[
\Psi = D\Phi.
\]

Thus the system (6) is transformed into its potential form,

\[
\Phi_t + \Phi_{xxx} - 2\Phi_x + 3(D\Phi)(D\Phi_x)\Phi_x = 0.
\]

To get its bilinearization, we take

\[
\Phi = \ln \frac{g}{f},
\]

where $f$ and $g$ are two Grassmann even functions. Then straightforward calculation yields

\[
(D\Phi)(D\Phi_x) = \frac{D_x^2(g \cdot f)}{fg} - D \left( \frac{\partial}{\partial f} \right).
\]
where
\[ \mathbb{B} = SD_x(g \cdot f), \]
while
\[
\Phi_t + \Phi_{xxx} - 2\Phi^3_x = \frac{(D_t + D^3_x)(g \cdot f)}{fg} - 3 \left( \frac{D^2_x g \cdot f}{f g} D_x(g \cdot f) \right). \tag{10}
\]
\[
\Phi_x = \frac{D_x(g \cdot f)}{fg}. \tag{11}
\]
Substituting the above expressions (9)–(11) into equation (7), we obtain
\[
\Phi_t + \Phi_{xxx} - 2\Phi^3_x + 3(D\Phi)(D\Phi_x)\Phi_x = \frac{(D_t + D^3_x)(g \cdot f)}{fg} - 3 \frac{D_x(g \cdot f)}{fg} D\left( \frac{\mathbb{B}}{g f} \right). \tag{12}
\]
Therefore, we have the following bilinearization for the sMKdV equation:
\[
(D_t + D^3_x)(g \cdot f) = 0, \tag{13}
\]
\[
SD_x(g \cdot f) = 0. \tag{14}
\]

Remark.
- Unlike the previous attempt in [12], our bilinearization (13) and (14) constitutes two equations for two tau functions. This bilinearization is a natural generalization of those for the classical MKdV equation [13]. To exhibit the relation between our bilinearization and the one proposed in [12] we need the following identity,
\[
[S(D_t + D^3_x)f \cdot g]g^2 - f^2[S(D_t + D^3_x)g \cdot g] = 2S((D_t + D^3_x)f \cdot g) \cdot df + 6D_x(SD_x f \cdot g) \cdot (D_x g \cdot f),
\]
which can be checked easily. Thus, our equations (13) and (14) show that one of the two equations (2) and (3) implies the other. However, equation (5) presents a certain redundant constraint. We will see in the following that this equation makes finding interesting soliton solutions impossible.
- We observe that equation (14) may be replaced by a more general one,
\[
SD_x(g \cdot f) = \kappa fg,
\]
where \(\kappa\) is a Grassmann odd constant. However equations (13) and (14) are adequate for constructing soliton solutions.

3. Bäcklund transformation

It is well known that BT is a useful concept and an effective tool for soliton systems as well as a characteristic of integrability. In this section, we will derive a bilinear BT for our sMKdV system. Our results are summarized in the following.

Proposition. Suppose that \((f, g)\) is a solution of equations (13) and (14); then \((\tilde{f}, \tilde{g})\) satisfying the following relations,
\[
D_t f \cdot \tilde{g} - \lambda D_x g \cdot \tilde{f} = \mu f \tilde{g} - \lambda \mu f \tilde{g}, \tag{15}
\]
\[
Sf \cdot \tilde{g} + \lambda Sg \cdot \tilde{f} = v f \tilde{g} + \lambda v f \tilde{g}, \tag{16}
\]
\[
(D_t + D^3_x - 3\mu D^2_x + 3\mu^2 D_x) f \cdot \tilde{f} = 0, \tag{17}
\]
\[
(D_t + D^3_x - 3\mu D^2_x + 3\mu^2 D_x) g \cdot \tilde{g} = 0. \tag{18}
\]
is another solution of (13) and (14), where \( \lambda, \mu \) are ordinary (even) constants and \( \nu \) is an odd constant.

**Proof.** We consider the following:

\[
P_1 = 2[[SD_x f \cdot g] \tilde{f} \tilde{g} - fg[SD_x \tilde{f} \cdot \tilde{g}]],
\]

\[
P_2 = [(D_x + D^2_\lambda) f \cdot g] \tilde{f} \tilde{g} + fg[(D_x + D^2_\lambda) \tilde{f} \cdot \tilde{g}].
\]

We will show that the above equations (15)–(18) imply \( P_1 = 0 \) and \( P_2 = 0 \). We first work on the case of \( P_1 \). We will use various bilinear identities which, for convenience, are presented in the appendix.

\[
P_1 \overset{(11)}{=} S[(D_x f \cdot \tilde{g}) \cdot \tilde{f} \tilde{g} + f \tilde{g} \cdot (D_x \tilde{f} \cdot g)] + D_x[(Sf \cdot \tilde{g}) \cdot \tilde{f} \tilde{g} + f \tilde{g} \cdot (S \tilde{f} \cdot g)]
\]

\[
\overset{(15, 16)}{=} S \left[ \left( \lambda D_x g \cdot \tilde{f} + \mu f \tilde{g} \right) \cdot \tilde{f} \tilde{g} + f \tilde{g} \cdot \left( -\frac{1}{\lambda} D_x f \cdot \tilde{g} - \mu \tilde{f} g \right) \right]
\]

\[
+ D_x \left[ \left( -\lambda S g \cdot \tilde{f} + \nu f \tilde{g} \right) \cdot \tilde{f} \tilde{g} + f \tilde{g} \cdot \left( \frac{1}{\lambda} Sf \cdot \tilde{g} - \nu \tilde{f} g \right) \right]
\]

\[
= \frac{\lambda}{\lambda} S[(D_x g \cdot \tilde{f}) \cdot \tilde{f} \tilde{g}] - \lambda D_x[(Sg \cdot \tilde{f}) \cdot \tilde{f} \tilde{g}]
\]

\[
- \frac{1}{\lambda} S[fg \cdot (D_x f \cdot \tilde{g})] + \frac{1}{\lambda} D_x[fg \cdot (S f \cdot \tilde{g})]
\]

\[
= 0.
\]

We now come to the second part of the proof.

\[
P_2 \overset{(43, 44)}{=} (D_x f \cdot \tilde{f}) \tilde{g} \tilde{g} g + f \tilde{f} (D_x \tilde{g} \cdot g) + (D^2_x f \cdot \tilde{f}) \tilde{g} \tilde{g} g + f \tilde{f} (D^2_\lambda \tilde{g} \cdot g)
\]

\[
- 3D_x[(D_x f \cdot \tilde{g}) \cdot (D_x g \cdot \tilde{f})],
\]

but

\[
D_x[(D_x f \cdot \tilde{g}) \cdot (D_x g \cdot \tilde{f})] \overset{(15)}{=} D_x[(\mu f \tilde{g} - \lambda \mu \tilde{f} g) \cdot (D_x g \cdot \tilde{f})]
\]

\[
= \mu D_x[f \tilde{g} \cdot (D_x g \cdot \tilde{f})] - \lambda \mu D_x[f \tilde{g} \cdot (D_x g \cdot \tilde{f})]
\]

\[
\overset{(15)}{=} \mu D_x[f \tilde{g} \cdot (D_x g \cdot \tilde{f})] + \mu D_x[f \tilde{g} \cdot (-D_x f \cdot \tilde{g} + \mu f \tilde{g})]
\]

\[
= \mu D_x[f \tilde{g} \cdot (D_x g \cdot \tilde{f})] + \mu^2 D_x[\tilde{g} \cdot f \tilde{g}]
\]

\[
\overset{(45)}{=} \mu(D^2_x f \cdot \tilde{f}) \tilde{g} \tilde{g} g - \mu f \tilde{f} (D^2_\lambda g \cdot g) + \mu^2 D_x[\tilde{g} \cdot f \tilde{g}]
\]

\[
= \mu(D^2_x f \cdot \tilde{f}) \tilde{g} \tilde{g} g - \mu f \tilde{f} (D^2_\lambda g \cdot g) + \mu^2(D_x f \cdot \tilde{f}) \tilde{g} \tilde{g} g - \mu^2(D_x \tilde{g} \cdot g) f \tilde{f}
\]

\[
= \left( \mu D^2_x - \mu^2 D_\lambda \right) f \cdot \tilde{f} \tilde{g} \tilde{g} g - f \tilde{f} (\mu D^2_\lambda + \mu^2 D_\lambda) g \cdot \tilde{g} g,
\]

and thus

\[
P_2 \overset{(17, 18)}{=} [(D_x + D^2_\lambda - 3\mu D^2_\lambda + 3\mu^2 D_\lambda) f \cdot \tilde{f} \tilde{g} \tilde{g} g - f \tilde{f} \tilde{g} \tilde{g} g (D_x + D^2_\lambda - 3\mu D^2_\lambda + 3\mu^2 D_\lambda) g \cdot \tilde{g} g]
\]

\[
= 0.
\]

This completes our proof. \( \square \)

**Remark.** There are three constants in our BT (15)–(18). In principle, these constants may take arbitrary values, but to construct interesting solutions, we are not allowed to set \( \lambda \) to zero. Indeed, \( \lambda \) can be an arbitrary constant but zero and further may be renormalized to unity. The true Bäcklund parameter is \( \mu \), as will be clear at the end of the section.
Next we will demonstrate that a spectral problem can be derived from the above BT. To this end, we assume
\[ u = \bar{f}, \quad v = \bar{g}, \]  
then by simple manipulation, we have
\[ (v - \lambda u)_x = -\Phi_1(v + \lambda u) - \mu(v - \lambda u), \]  
\[ Dv + \lambda Du = -(D\Phi)(v - \lambda u) - v(v + \lambda u), \]  
which constitute the spatial part of the spectral problem for our system (6). To obtain a more compact form, we introduce
\[ U = v - \lambda u, \quad V = v + \lambda u \]  
in these variables, and the spectral problem (20) can be rewritten simply as
\[ U_x = -\Phi_1V - \mu U, \]  
\[ DV = -(D\Phi)U - \nu V. \]  
It is interesting to note that we may have a scalar Lax operator for the sMKdV equation. Indeed, letting \( \nu = 0 \) and eliminating one of the wave functions, \( V \), we arrive at the following Lax operator,
\[ L = \partial_x - \Phi_1D^{-1}(D\Phi), \quad \text{or} \quad L = \partial_x - (D\Psi)D^{-1}\Psi, \]  
and now our system (6) has the Lax representation as follows:
\[ \frac{d}{dt} L = [L, (L^3)_{\geq 0}]. \]  
To our knowledge, this Lax operator is new, and we remark here that the Lax operator (24) is in a form of the constrained soliton systems. Indeed, it can be recovered from a reduction of the so-called supersymmetric AKNS Lax operator studied in [14].

4. Solutions

For a given system, Hirota’s bilinear form is ideal for constructing particular solutions, and so is BT. For the sMKdV equation, we may use either Hirota’s perturbation method or BT to calculate its soliton type of solutions. Since the calculation involved here is straightforward, although it is cumbersome, we just list the results:

1-soliton:
\[ f = 1 + \exp \eta, \quad g = 1 - \exp \eta, \]  
where \( \eta = k_1 x - k_2 t + 2 \theta \xi \) and \( \xi \) is an arbitrary Grassmann odd constant. Thus, our solution in the original variable is
\[ \Psi = \frac{2(\xi + \theta k)}{\exp(\eta) - \exp(-\eta)}. \]  
2-soliton:
\[ f = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12} \exp(\eta_1 + \eta_2), \]  
\[ g = 1 - \exp(\eta_1) - \exp(\eta_2) + A_{12} \exp(\eta_1 + \eta_2), \]  
where
\[ A_{12} = \left( \frac{k_1 - k_2}{k_1 + k_2} \right) \left[ \frac{k_1 - k_2 - 2 \xi_1 \xi_2}{k_1 + k_2} + 2 \theta \frac{(k_2 \xi_1 - k_1 \xi_2)}{k_1 + k_2} \right] \]  
(28)
and
\[ \eta_i = k_i x - k_i^3 t + \theta \xi_i, \quad i = 1, 2 \]
and \( \xi_1, \xi_2 \) are arbitrary Grassmann odd constants. The form of the last term can be reformed into the same one as presented by Carstea et al. [9] for the sKdV equation. It is interesting to note that the above \( f \) and \( g \) do not solve equation (5); thus one is not able to construct the above 2-solitons using the bilinearization (2)–(5) of Ghosh and Sarma [12].

5. Conclusion and discussion

In this paper, we studied the sMKdV equation from the viewpoint of Hirota’s bilinear method. In contrast to the results of Ghosh and Sarma [12], we demonstrated that there exists a simpler bilinearization for this system. We further obtained a bilinear BT, which leads us to a new Lax operator for the sMKdV equation.

We may convert the spatial part of our bilinear BT (15) and (16) into the following form:
\[ \Phi_x + \bar{\Phi}_x + \frac{\mu}{2} [\exp (\bar{\Phi} - \Phi) - \exp (\Phi - \bar{\Phi}) + (D \Phi)(D \bar{\Phi})] \frac{\exp \Phi - \exp \bar{\Phi}}{\exp \Phi + \exp \bar{\Phi}} = 0, \]
where we assume \( \lambda = 1 \) and \( \nu = 0 \). To compare with the BT of MKdV equation, we let \( \Phi = i \bar{Q} \), \( \bar{\Phi} = i \bar{Q} \), and then our BT (29) takes the following form,
\[ Q_x + \bar{Q}_x + \mu \sin (\bar{Q} - Q) - (D \bar{Q})(D \bar{Q}) \tan \left( \frac{\bar{Q} - Q}{2} \right) = 0, \]
which is a generalization of the BT for the MKdV equation. It is interesting to derive the corresponding nonlinear superposition formula.

Acknowledgments

This work was done when the authors visited the Abdus Salam International Centre for Theoretical Physics. We would like to thank the ICTP for support and hospitality. We also would like to thank the anonymous referees for the interesting comments. QPL is supported in part by National Natural Science Foundation of China under grant number 10231050 and the Ministry of Education of China, and XBH is supported by National Natural Science Foundation of China under grant number 10171100.

Appendix: some bilinear identities

In this appendix, we list the relevant bilinear identities, which can be proved directly. Here \( a, b, c \) and \( d \) are arbitrary even functions of the independent variables \( x, t \) and \( \theta \).

\[(SDa \cdot b)cd - ab(SDc \cdot d) = \frac{1}{2}S[(D_x a \cdot d) \cdot cb + ad \cdot (D_x c \cdot b)]\]
\[+ \frac{1}{2}D_x[(S_x a \cdot d) \cdot cb + ad \cdot (S_x c \cdot b)], \quad (A1)\]
\[S[(D_x a \cdot b) \cdot ab] = D_x[(S_x a \cdot b) \cdot ab], \quad (A2)\]
\[(Da \cdot b)cd + ab(D_x c \cdot d) = (D_x a \cdot d)cb + ad(D_x c \cdot b), \quad (A3)\]
\[(D^2_x a \cdot b)cd + ab(D^2_x c \cdot d) = (D^2_x a \cdot d) \cdot cb + ad \cdot (D^2_x c \cdot b)\]
\[ - 3D_x(D_x a \cdot c) \cdot (D_x b \cdot d), \quad (A4)\]
\[D_x[(D_x a \cdot b) \cdot cb + ab \cdot (D_x c \cdot d)] = (D^2_x a \cdot d)cb - ad(D^2_x c \cdot b), \quad (A5)\]
\[D_x ab \cdot cd = (D_x a \cdot d)cb - ad(D_x c \cdot b). \quad (A6)\]
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