Probabilistic Systems with LimSup and LimInf Objectives

Krishnendu Chatterjee\(^1\) and Thomas A. Henzinger\(^1,2\)

\(^1\) EECS, UC Berkeley, USA
\(^2\) EPFL, Switzerland
\{c_krish,tah\}@eecs.berkeley.edu

Abstract. We give polynomial-time algorithms for computing the values of Markov decision processes (MDPs) with limsup and liminf objectives. A real-valued reward is assigned to each state, and the value of an infinite path in the MDP is the limsup (resp. liminf) of all rewards along the path. The value of an MDP is the maximal expected value of an infinite path that can be achieved by resolving the decisions of the MDP. Using our result on MDPs, we show that turn-based stochastic games with limsup and liminf objectives can be solved in NP \(\cap\) coNP.

1 Introduction

A turn-based stochastic game is played on a finite graph with three types of states: in player-1 states, the first player chooses a successor state from a given set of outgoing edges; in player-2 states, the second player chooses a successor state from a given set of outgoing edges; and probabilistic states, the successor state is chosen according to a given probability distribution. The game results in an infinite path through the graph. Every such path is assigned a real value, and the objective of player 1 is to resolve her choices so as to maximize the expected value of the resulting path, while the objective of player 2 is to minimize the expected value. If the function that assigns values to infinite paths is a Borel function (in the Cantor topology on infinite paths), then the game is determined [12]; the maximal expected value achievable by player 1 is equal to the minimal expected value achievable by player 2, and it is called the value of the game.

There are several canonical functions for assigning values to infinite paths. If each state is given a reward, then the \(\text{max}\) (resp. \(\text{min}\)) functions choose the maximum (resp. minimum) of the infinitely many rewards along a path; the \(\text{limsup}\) (resp. \(\text{liminf}\)) functions choose the limsup (resp. liminf) of the infinitely many rewards; and the \(\text{limavg}\) function chooses the long-run average of the rewards. For the Borel level-1 functions \(\text{max}\) and \(\text{min}\), as well as for the Borel level-3 function \(\text{limavg}\), computing the value of a game is known to be in NP \(\cap\) coNP [10]. However, for the Borel level-2 functions \(\text{limsup}\) and \(\text{liminf}\), only special cases have been considered so far. If there are no probabilistic states (in this case, the game is called deterministic), then the game value can be computed in polynomial time using value-iteration algorithms [1]; likewise, if all states are
given reward 0 or 1 (in this case, \textit{limsup} is a Büchi objective, and \textit{liminf} is a coBüchi objective), then the game value can be decided in \(\text{NP} \cap \text{coNP} \) [3]. In this paper, we show that the values of general turn-based stochastic games with \textit{limsup} and \textit{liminf} objectives can be computed in \(\text{NP} \cap \text{coNP}\).

It is known that pure memoryless strategies suffice for achieving the value of turn-based stochastic games with \textit{limsup} and \textit{liminf} objectives [9]. A strategy is \textit{pure} if the player always chooses a unique successor state (rather than a probability distribution of successor states); a pure strategy is \textit{memoryless} if at every state, the player always chooses the same successor state. Hence a pure memoryless strategy for player 1 is a function from player-1 states to outgoing edges (and similarly for player 2). Since pure memoryless strategies offer polynomial witnesses, our result will follow from polynomial-time algorithms for computing the values of Markov decision processes (MDPs) with \textit{limsup} and \textit{liminf} objectives. We provide such algorithms.

An MDP is the special case of a turn-based stochastic game which contains no player-1 (or player-2) states. Using algorithms for solving MDPs with Büchi and coBüchi objectives, we give polynomial-time reductions from MDPs with \textit{limsup} and \textit{liminf} objectives to MDPs with max objectives. The solution of MDPs with max objectives is computable by linear programming, and the linear program for MDPs with max objectives is obtained by generalizing the linear program for MDPs with reachability objectives. This will conclude our argument.

\textbf{Related work.} Games with \textit{limsup} and \textit{liminf} objectives have been widely studied in game theory; for example, Maitra and Sudderth [11] present several results about games with \textit{limsup} and \textit{liminf} objectives. In particular, they show the existence of values in \textit{limsup} and \textit{liminf} games that are more general than turn-based stochastic games, such as concurrent games, where the two players repeatedly choose their moves simultaneously and independently, and games with infinite state spaces. Gimbert and Zielonka have studied the strategy complexity of games with \textit{limsup} and \textit{liminf} objectives: the sufficiency of pure memoryless strategies for deterministic games was shown in [8], and for turn-based stochastic games, in [9]. Polynomial-time algorithms for MDPs with Büchi and coBüchi objectives were presented in [5], and the solution turn-based stochastic games with Büchi and coBüchi objectives was shown to be in \(\text{NP} \cap \text{coNP}\) in [3]. For deterministic games with \textit{limsup} and \textit{liminf} objectives polynomial-time algorithms have been known, for example, the value-iteration algorithm terminates in polynomial time [1].

\section{Definitions}

We consider the class of turn-based probabilistic games and some of its subclasses.

\textbf{Game graphs.} A \textit{turn-based probabilistic game graph} (2\(1/2\)-player game graph) 
\(G = ((S, E), (S_1, S_2, S_P), \delta)\) consists of a directed graph \((S, E)\), a partition \((S_1, S_2, S_P)\) of the finite set \(S\) of states, and a probabilistic transition function \(\delta:\)
$S_P \rightarrow \mathcal{D}(S)$, where $\mathcal{D}(S)$ denotes the set of probability distributions over the state space $S$. The states in $S_1$ are the player-1 states, where player 1 decides the successor state; the states in $S_2$ are the player-2 states, where player 2 decides the successor state; and the states in $S_P$ are the probabilistic states, where the successor state is chosen according to the probabilistic transition function $\delta$. We assume that for $s \in S_P$ and $t \in S$, we have $(s, t) \in E$ if $\delta(s)(t) > 0$, and we often write $\delta(s, t)$ for $\delta(s)(t)$. For technical convenience we assume that every state in the graph $(S, E)$ has at least one outgoing edge. For a state $s \in S$, we write $E(s)$ to denote the set $\{ t \in S \mid (s, t) \in E \}$ of possible successors. The turn-based deterministic game graphs (2-player game graphs) are the special case of the $2^{1/2}$-player game graphs with $S_P = \emptyset$. The Markov decision processes (1$^{1/2}$-player game graphs) are the special case of the $2^{1/2}$-player game graphs with $S_1 = \emptyset$ or $S_2 = \emptyset$. We refer to the MDPs with $S_2 = \emptyset$ as player-1 MDPs, and to the MDPs with $S_1 = \emptyset$ as player-2 MDPs.

**Plays and strategies.** An infinite path, or a *play*, of the game graph $G$ is an infinite sequence $\omega = (s_0, s_1, s_2, \ldots)$ of states such that $(s_k, s_{k+1}) \in E$ for all $k \in \mathbb{N}$. We write $\Omega$ for the set of all plays, and for a state $s \in S$, we write $\Omega_s \subseteq \Omega$ for the set of plays that start from the state $s$. A *strategy* for player 1 is a function $\sigma: S^* \cdot S_1 \rightarrow \mathcal{D}(S)$ that assigns a probability distribution to all finite sequences $\omega \in S^* \cdot S_1$ of states ending in a player-1 state (the sequence represents a prefix of a play). Player 1 follows the strategy $\sigma$ if in each player-1 move, given that the current history of the game is $\omega \in S^* \cdot S_1$, she chooses the next state according to the probability distribution $\sigma(\omega)$. A strategy must prescribe only available moves, i.e., for all $\omega \in S^*$, $s \in S_1$, and $t \in S$, if $\sigma(\omega \cdot s)(t) > 0$, then $(s, t) \in E$. The strategies for player 2 are defined analogously. We denote by $\Sigma$ and $\Pi$ the set of all strategies for player 1 and player 2, respectively.

Once a starting state $s \in S$ and strategies $\sigma, \pi \in \Sigma$ and $\pi \in \Pi$ for the two players are fixed, the outcome of the game is a random walk $\omega^{s, \pi}_s$ for which the probabilities of events are uniquely defined, where an *event* $A \subseteq \Omega$ is a measurable set of plays. For a state $s \in S$ and an event $A \subseteq \Omega$, we write $\Pr^{s, \pi}_s(A)$ for the probability that a play belongs to $A$ if the game starts from the state $s$ and the players follow the strategies $\sigma$ and $\pi$, respectively. For a measurable function $f: \Omega \rightarrow \mathbb{R}$ we denote by $E^{s, \pi}_s[f]$ the expectation of the function $f$ under the probability measure $\Pr^{s, \pi}_s(\cdot)$.

Strategies that do not use randomization are called pure. A player-1 strategy $\sigma$ is pure if for all $\omega \in S^*$ and $s \in S_1$, there is a state $t \in S$ such that $\sigma(\omega \cdot s)(t) = 1$. A memoryless player-1 strategy does not depend on the history of the play but only on the current state; i.e., for all $\omega, \omega' \in S^*$ and for all $s \in S_1$ we have $\sigma(\omega \cdot s) = \sigma(\omega' \cdot s)$. A memoryless strategy can be represented as a function $\sigma: S_1 \rightarrow \mathcal{D}(S)$. A pure memoryless strategy is a strategy that is both pure and memoryless. A pure memoryless strategy for player 1 can be represented as a function $\sigma: S_1 \rightarrow S$. We denote by $\Sigma^{PM}$ the set of pure memoryless strategies for player 1. The pure memoryless player-2 strategies $\Pi^{PM}$ are defined analogously.
Given a pure memoryless strategy $\sigma \in \Sigma^{PM}$, let $G_\sigma$ be the game graph obtained from $G$ under the constraint that player 1 follows the strategy $\sigma$. The corresponding definition $G_\pi$ for a player-2 strategy $\pi \in \Pi^{PM}$ is analogous, and we write $G_{\sigma,\pi}$ for the game graph obtained from $G$ if both players follow the pure memoryless strategies $\sigma$ and $\pi$, respectively. Observe that given a 21/2-player game graph $G$ and a pure memoryless player-1 strategy $\sigma$, the result $G_{\sigma}$ is a player-2 MDP. Similarly, for a player-1 MDP $G$ and a pure memoryless player-1 strategy $\sigma$, the result $G_{\sigma}$ is a Markov chain. Hence, if $G$ is a 21/2-player game graph and the two players follow pure memoryless strategies $\sigma$ and $\pi$, the result $G_{\sigma,\pi}$ is a Markov chain.

**Quantitative objectives.** A quantitative objective is specified as a measurable function $f : \Omega \to \mathbb{R}$. We consider zero-sum games, i.e., games that are strictly competitive. In zero-sum games the objectives of the players are functions $f$ and $-f$, respectively. We consider quantitative objectives specified as lim sup and lim inf objectives. These objectives are complete for the second levels of the Borel hierarchy: lim sup objectives are $\Pi_2$ complete, and lim inf objectives are $\Sigma_2$ complete. The definitions of lim sup and lim inf objectives are as follows.

- **Limsup objectives.** Let $r : S \to \mathbb{R}$ be a real-valued reward function that assigns to every state $s$ the reward $r(s)$. The limsup objective lim sup assigns to every play the maximum reward that appears infinitely often in the play. Formally, for a play $\omega = \langle s_1, s_2, s_3, \ldots \rangle$ we have
  $$\limsup(r)(\omega) = \limsup_i r(s_i)_{i \geq 0}.$$

- **Liminf objectives.** Let $r : S \to \mathbb{R}$ be a real-valued reward function that assigns to every state $s$ the reward $r(s)$. The liminf objective lim inf assigns to every play the maximum reward $v$ such that the rewards that appear eventually always in the play is at least $v$. Formally, for a play $\omega = \langle s_1, s_2, s_3, \ldots \rangle$ we have
  $$\liminf(r)(\omega) = \liminf_i r(s_i)_{i \geq 0}.$$

The objectives lim sup and lim inf are complementary in the sense that for all plays $\omega$ we have $\limsup(r)(\omega) = -\liminf(-r)(\omega)$.

We also define the max objectives, as it will be useful in study of MDPs with lim sup and lim inf objectives. Later we will reduce MDPs with lim sup and lim inf objectives to MDPs with max objectives. For a reward function $r : S \to \mathbb{R}$ the max objective max assigns to every play the maximum reward that appears in the play. Observe that since $S$ is finite, the number of different rewards appearing in a play is finite and hence the maximum is defined. Formally, for a play $\omega = \langle s_1, s_2, s_3, \ldots \rangle$ we have
  $$\max(r)(\omega) = \max(r(s_i))_{i \geq 0}.$$

**Büchi and coBüchi objectives.** We define the qualitative variant of lim sup and lim inf objectives, namely, Büchi and coBüchi objectives. The notion of
qualitative variants of the objectives will be useful in the algorithmic analysis of 2\textsuperscript{1/2}-player games with lim sup and lim inf objectives. For a play \( \omega \), we define \( \operatorname{Inf}(\omega) = \{ s \in S \mid s_k = s \text{ for infinitely many } k \geq 0 \} \) to be the set of states that occur infinitely often in \( \omega \).

- **Büchi objectives.** Given a set \( B \subseteq S \) of Büchi states, the Büchi objective Büchi\((B)\) requires that some state in \( B \) be visited infinitely often. The set of winning plays is Büchi\((B) = \{ \omega \in \Omega \mid \operatorname{Inf}(\omega) \cap B \neq \emptyset \} \).

- **co-Büchi objectives.** Given a set \( C \subseteq S \) of co-Büchi states, the co-Büchi objective coBüchi\((C)\) requires that only states in \( C \) be visited infinitely often. Thus, the set of winning plays is coBüchi\((C) = \{ \omega \in \Omega \mid \operatorname{Inf}(\omega) \subseteq C \} \).

The Büchi and co-Büchi objectives are dual in the sense that Büchi\((B) = \Omega \setminus \operatorname{coBüchi}(S \setminus B)\).

Given a set \( B \subseteq S \), consider a boolean reward function \( r_B \) such that for all \( s \in S \) we have \( r_B(s) = 1 \) if \( s \in B \), and 0 otherwise. Then for all plays \( \omega \) we have \( \omega \in \text{Büchi}(B) \) iff \( \limsup(r_B)(\omega) = 1 \). Similarly, given a set \( C \subseteq S \), consider a boolean reward function \( r_C \) such that for all \( s \in S \) we have \( r_C(s) = 1 \) if \( s \in C \), and 0 otherwise. Then for all plays \( \omega \) we have \( \omega \in \text{coBüchi}(C) \) iff \( \liminf(r_C)(\omega) = 1 \).

**Values and optimal strategies.** Given a game graph \( G \), qualitative objectives \( \Phi \subseteq \Omega \) for player 1 and \( \Omega \setminus \Phi \) for player 2, and measurable functions \( f \) and \( \neg f \) for player 1 and player 2, respectively, we define the value functions \( \langle 1 \rangle_{\text{val}} \) and \( \langle 2 \rangle_{\text{val}} \) for the players 1 and 2, respectively, as the following functions from the state space \( S \) to the set \( \mathbb{R} \) of reals: for all states \( s \in S \), let

\[
\langle 1 \rangle_{\text{val}}^G(\Phi)(s) = \sup_{\sigma \in \Sigma} \inf_{\pi \in H} \mathbb{P}_{s}^{\sigma,\pi}(\Phi);
\]

\[
\langle 1 \rangle_{\text{val}}^G(f)(s) = \sup_{\sigma \in \Sigma} \inf_{\pi \in H} \mathbb{E}_{s}^{\sigma,\pi}[f];
\]

\[
\langle 2 \rangle_{\text{val}}^G(\Omega \setminus \Phi)(s) = \sup_{\pi \in H} \inf_{\sigma \in \Sigma} \mathbb{P}_{s}^{\sigma,\pi}(\Omega \setminus \Phi);
\]

\[
\langle 2 \rangle_{\text{val}}^G(-f)(s) = \sup_{\pi \in H} \inf_{\sigma \in \Sigma} \mathbb{E}_{s}^{\sigma,\pi}[-f].
\]

In other words, the values \( \langle 1 \rangle_{\text{val}}^G(\Phi)(s) \) and \( \langle 1 \rangle_{\text{val}}^G(f)(s) \) give the maximal probability and expectation with which player 1 can achieve her objectives \( \Phi \) and \( f \) from state \( s \), and analogously for player 2. The strategies that achieve the values are called optimal: a strategy \( \sigma \) for player 1 is optimal from the state \( s \) for the objective \( \Phi \) if \( \langle 1 \rangle_{\text{val}}^G(\Phi)(s) = \inf_{\pi \in H} \mathbb{P}_{s}^{\sigma,\pi}(\Phi) \); and \( \sigma \) is optimal from the state \( s \) for \( f \) if \( \langle 1 \rangle_{\text{val}}^G(f)(s) = \inf_{\pi \in H} \mathbb{E}_{s}^{\sigma,\pi}[f] \). The optimal strategies for player 2 are defined analogously. We now state the classical determinacy results for 2\textsuperscript{1/2}-player games with lim sup and lim inf objectives.

**Theorem 1 (Quantitative determinacy).** For all 2\textsuperscript{1/2}-player game graphs \( G = ((S, E), (S_1, S_2, S_P), \delta) \), the following assertions hold.
1. For all reward functions \( r : S \to \mathbb{R} \) and all states \( s \in S \), we have

\[
\langle \langle 1 \rangle \rangle_{val} (\limsup (r))(s) + \langle \langle 2 \rangle \rangle_{val} (\liminf (-r))(s) = 0;
\]

\[
\langle \langle 1 \rangle \rangle_{val} (\liminf (r))(s) + \langle \langle 2 \rangle \rangle_{val} (\limsup (-r))(s) = 0.
\]

2. Pure memoryless optimal strategies exist for both players from all states.

The above results can be derived from the results in [11]; a more direct proof can be obtained as follows: the existence of pure memoryless optimal strategies for MDPs with \( \limsup \) and \( \liminf \) objectives can be proved by extending the results known for Büchi and coBüchi objectives. The results (Theorem 3.19) of [7] proved that if for a quantitative objective \( f \) and its complement \( -f \) pure memoryless optimal strategies exist in MDPs, then pure memoryless optimal strategies also exist in 2\( \frac{1}{2} \)-player games. Hence the pure memoryless determinacy follows for 2\( \frac{1}{2} \)-player games with \( \limsup \) and \( \liminf \) objectives.

### 3 The Complexity of 2\( \frac{1}{2} \)-Player Games with \( \limsup \) and \( \liminf \) Objectives

In this section we study the complexity of MDPs and 2\( \frac{1}{2} \)-player games with \( \limsup \) and \( \liminf \) objectives. We present polynomial time algorithms for MDPs and show that 2\( \frac{1}{2} \)-player games can be decided in \( \text{NP} \cap \text{coNP} \). In the next subsections we present polynomial time algorithms for MDPs with \( \limsup \) and \( \liminf \) objectives by reductions to a simple linear-programming formulation, and then show that 2\( \frac{1}{2} \)-player games can be decided in \( \text{NP} \cap \text{coNP} \). We first present a remark and then present some basic results on MDPs.

**Remark 1.** Given a 2\( \frac{1}{2} \)-player game graph \( G \) with a reward function \( r : S \to \mathbb{R} \) and a real constant \( c \), consider the reward function \( (r + c) : S \to \mathbb{R} \) defined as follows: for \( s \in S \) we have \( (r + c)(s) = r(s) + c \). Then the following assertions hold: for all \( s \in S \)

\[
\langle \langle 1 \rangle \rangle_{val} (\limsup (r + c))(s) = \langle \langle 1 \rangle \rangle_{val} (\limsup (r))(s) + c;
\]

\[
\langle \langle 1 \rangle \rangle_{val} (\liminf (r + c))(s) = \langle \langle 1 \rangle \rangle_{val} (\liminf (r))(s) + c.
\]

Hence we can shift a reward function \( r \) by a real constant \( c \), and from the value function for the reward function \( (r + c) \), we can easily compute the value function for \( r \). Hence without loss of generality for computational purpose we assume that we have reward function with positive rewards, i.e., \( r : S \to \mathbb{R}^+ \), where \( \mathbb{R}^+ \) is the set of positive reals.

### 3.1 Basic results on MDPs

In this section we recall several basic properties on MDPs. We start with the definition of *end components* in MDPs [5, 4] that play a role equivalent to closed recurrent sets in Markov chains.
### End components.

Given an MDP $G = ((S, E), (S_1, S_P), \delta)$, a set $U \subseteq S$ of states is an end component if $U$ is $\delta$-closed (i.e., for all $s \in U \cap S_P$ we have $E(s) \subseteq U$) and the sub-game graph of $G$ restricted to $U$ (denoted $G \upharpoonright U$) is strongly connected. We denote by $E(G)$ the set of end components of an MDP $G$. The following lemma states that, given any strategy (memoryless or not), with probability 1 the set of states visited infinitely often along a play is an end component. This lemma allows us to derive conclusions on the (infinite) set of plays in an MDP by analyzing the (finite) set of end components in the MDP.

**Lemma 1.** [5, 4] Given an MDP $G$, for all states $s \in S$ and all strategies $\sigma \in \Sigma$, we have $\Pr_\sigma^s(\{\omega | \Inf(\omega) \in E(G)\}) = 1$.

For an end component $U \in E(G)$, consider the memoryless strategy $\sigma_U$ that at a state $s$ in $U \cap S_1$ plays all edges in $E(s) \cap U$ uniformly at random. Given the strategy $\sigma_U$, the end component $U$ is a closed connected recurrent set in the Markov chain obtained by fixing $\sigma_U$.

**Lemma 2.** Given an MDP $G$ and an end component $U \in E(G)$, the strategy $\sigma_U$ ensures that for all states $s \in U$, we have $\Pr_\sigma^s(\{\omega | \Inf(\omega) = U\}) = 1$.

### Almost-sure winning states.

Given an MDP $G$ with a Büchi or a coBüchi objective $\Phi$ for player 1, we denote by $W^1_G(\Phi) = \{s \in S | \langle 1 \rangle_{val}(\Phi)(s) = 1 \}$; the sets of states such that the values for player 1 is 1. These sets of states are also referred as the almost-sure winning states for the player and an optimal strategy from the almost-sure winning states is referred as an almost-sure winning strategy. The set $W^1_G(\Phi)$, for Büchi or coBüchi objectives $\Phi$, for an MDP $G$ can be computed in $O(n^3)$ time, where $n$ is the size of the MDP $G$ [2].

### Attractor of probabilistic states.

We define a notion of attractor of probabilistic states: given an MDP $G$ and a set $U \subseteq S$ of states, we denote by $Attr_P(U, G)$ the set of states from where the probabilistic player has a strategy (with proper choice of edges) to force the game to reach $U$. The set $Attr_P(U, G)$ is inductively defined as follows:

$$T_0 = U; \quad T_{i+1} = T_i \cup \{s \in S_P | E(s) \cap T_i \neq \emptyset\} \cup \{s \in S_1 | E(s) \subseteq T_i\}$$

and $Attr_P(U, G) = \bigcup_{i \geq 0} T_i$.

We now present a lemma about MDPs with Büchi and coBüchi objectives and a property of end components and attractors. The first two properties of Lemma 3 follows from Lemma 2. The last property follows from the fact that an end component is $\delta$-closed (i.e., for an end component $U$, for all $s \in U \cap S_P$ we have $E(s) \subseteq U$).

**Lemma 3.** Let $G$ be an MDP. Given $B \subseteq S$ and $C \subseteq S$, the following assertions hold.
1. For all $U \in \mathcal{E}(G)$ such that $U \cap B \neq \emptyset$, we have $U \subseteq W_B^G(\text{B"uchi}(B))$.
2. For all $U \in \mathcal{E}(G)$ such that $U \subseteq C$, we have $U \subseteq W_C^G(\text{coB"uchi}(C))$.
3. For all $Y \subseteq S$ and all end components $U \in \mathcal{E}(G)$, if $X = \text{Attr}_P(Y,G)$, then either (a) $U \cap Y \neq \emptyset$ or (b) $U \cap X = \emptyset$.

3.2 MDPs with limsup objectives

In this subsection we present polynomial time algorithm for MDPs with limsup objectives. For the sake of simplicity we will consider bipartite MDPs.

Bipartite MDPs. An MDP $G = ((S,E),(S_1,S_2),\delta)$ is bipartite if $E \subseteq S_1 \times S_2$. An MDP $G$ can be converted into a bipartite MDP $G'$ by adding dummy states with an unique successor, and $G'$ is linear in the size of $G$. In sequel without loss of generality we will consider bipartite MDPs. The key property of bipartite MDPs that will be useful is as follows: for a bipartite MDP $G = ((S,E),(S_1,S_2),\delta)$, for all $U \in \mathcal{E}(G)$ we have $U \cap S_1 \neq \emptyset$.

Informal description of algorithm. We first present an algorithm that takes an MDP $G$ with a positive reward function $r : S \rightarrow \mathbb{R}^+$, and computes a set $S^*$ and a function $f^* : S^* \rightarrow \mathbb{R}^+$. The output of the algorithm will be useful in reduction of MDPs with limsup objectives to MDPs with max objectives. Let the rewards be $v_0 > v_1 > \cdots > v_k$. The algorithm proceeds in iteration and in iteration $i$ we denote the MDP as $G_i$ and the state space as $S_i$. At iteration $i$ the algorithm considers the set $V_i$ of reward $v_i$ in the MDP $G_i$, and computes the set $U_i = W_i^G(\text{B"uchi}(V_i))$, (i.e., the almost-sure winning set in the MDP $G_i$ for B"uchi objective with the B"uchi set $V_i$). For all $u \in U_i \cap S_1$ we assign $f^*(u) = v_i$ and add the set $U_i \cap S_1$ to $S^*$. Then the set $\text{Attr}_P(U_i,G_i)$ is removed from the MDP $G_i$ and we proceed to iteration $i+1$. In $G_i$ all end components that intersect with reward $v_i$ are contained in $U_i$ (by Lemma 3 part (1)), and all end components in $S^* \setminus U_i$ do not intersect with $\text{Attr}_P(U_i,G_i)$ (by Lemma 3 part(3)). This gives us the following lemma.

Lemma 4. Let $G$ be an MDP with a positive reward function $r : S \rightarrow \mathbb{R}^+$. Let $f^*$ be the output of Algorithm 1. For all end components $U \in \mathcal{E}(G)$ and all states $u \in U \cap S_1$, we have $\max(r(U)) \leq f^*(u)$.

Proof. Let $U^* = \bigcup_{i=0}^k U_i$ (as computed in Algorithm 1). Then it follows from Lemma 3 that for all $A \in \mathcal{E}(G)$ we have $A \cap U^* \neq \emptyset$. Consider $A \in \mathcal{E}(G)$ and let $v_i = \max(r(A))$. Suppose for some $j < i$ we have $A \cap U_j \neq \emptyset$. Then there is a strategy to ensure that $U_j$ is reached with probability 1 from all states in $A$ and then play an almost-sure winning strategy in $U_j$ to ensure B"uchi($r^{-1}(v_j) \cap S^i)$.

Then $A \subseteq U_j$. Hence for all $u \in A \cap S_1$ we have $f^*(u) = v_j \geq v_i$. If for all $j < i$ we have $A \cap U_j = \emptyset$, then we show that $A \subseteq U_i$. The uniform memoryless strategy $\sigma_A$ (as used in Lemma 2) in $G_i$ is a witness to prove that $A \subseteq U_i$. In this case for all $u \in A \cap S_1$ we have $f^*(u) = v_i = \max(r(A))$. The desired result follows.■
Proof. The result is obtained from the following two case analysis.

Transformation to MDPs with max objective. Given an MDP $G = ((S, E), (S_1, S_2), \delta)$ with a positive reward function $r : S \rightarrow \mathbb{R}^+,$ and let $S^*$ and $f^*$ be the output of Algorithm 1. We construct an MDP $\overline{G} = ((\hat{S}, \hat{E}), (\hat{S}_1, \hat{S}_2), \hat{\delta})$ with a reward function $\overline{r}$ as follows:

- $\overline{S} = S \cup \hat{S}^*$; i.e., the set of states consists of the state space $S$ and a copy $\hat{S}^*$ of $S^*$.
- $\overline{E} = E \cup \{(s, \hat{s}) \mid s \in S^*, \hat{s} \in \hat{S}^* \}$ where $\hat{s}$ is the copy of $s$.
- $\overline{S}_1 = S_1 \cup \hat{S}^*$.
- $\overline{\delta} = \delta.$
- $\overline{r}(s) = 0$ for all $s \in S$ and $\overline{r}(s) = f^*(s)$ for $\hat{s} \in \hat{S}^*$, where $\hat{s}$ is the copy of $s$.

We refer to the above construction as limsup conversion. The following lemma proves the relationship between the value function $\langle 1 \rangle_{\text{val}}(\limsup(r))$ and $\langle 1 \rangle_{\text{val}}(\max(\overline{r}))$.

**Lemma 5.** Let $G$ be an MDP with a positive reward function $r : S \rightarrow \mathbb{R}^+.$ Let $\overline{G}$ and $\overline{r}$ be obtained from $G$ and $r$ by the limsup conversion. For all states $s \in S,$ we have
\[ \langle 1 \rangle_{\text{val}}(\limsup(r))(s) = \langle 1 \rangle_{\text{val}}(\max(\overline{r}))(s). \]

**Proof.** The result is obtained from the following two case analysis.

1. Let $\sigma$ be a pure memoryless optimal strategy in $G$ for the objective $\limsup(r).$ Let $C = \{ C_1, C_2, \ldots, C_m \}$ be the set of closed connected recurrent sets in the Markov chain obtained from $G$ after fixing the strategy $\sigma.$ Note that since we consider bipartite MDPs, for all $1 \leq i \leq m,$ we have

\[ \overline{C}_i = C_i \cup \hat{C}_i. \]
Let \( C_i \cap S_1 \neq \emptyset \). Let \( C = \bigcup_{i=1}^{m} C_i \). We define a pure memoryless strategy \( \sigma \) in \( \overline{G} \) as follows

\[
\sigma(s) = \begin{cases} 
\sigma(s) & s \in S_1 \setminus C; \\
\hat{s} & \hat{s} \in \hat{S}^* \text{ and } s \in S_1 \cap C.
\end{cases}
\]

By Lemma 4 it follows that the strategy \( \sigma \) ensures that for all \( C_i \in C \) and all \( s \in C_i \), the maximal reward reached in \( \overline{G} \) is at least \( \max(r(C_i)) \) with probability 1. It follows that for all \( s \in S \) we have

\[
\langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r))(s) \leq \langle \langle 1 \rangle \rangle_{\text{val}}^G (\max(\overline{\sigma}))(s).
\]

2. Let \( \overline{\sigma} \) be a pure memoryless optimal strategy for the objective \( \max(\overline{\sigma}) \) in \( \overline{G} \).

We fix a strategy \( \sigma \in G \) as follows: if at a state \( s \in S^* \) the strategy \( \sigma \) chooses the edge \((s, \hat{s})\), then in \( G \) on reaching \( s \), the strategy \( \sigma \) plays an almost-sure winning strategy for the objective Büchi\((r^{-1}(f^*(s)))\), otherwise \( \sigma \) follows \( \overline{\sigma} \).

It follows that for all \( s \in S \) we have

\[
\langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r))(s) \geq \langle \langle 1 \rangle \rangle_{\text{val}}^G (\max(\overline{\sigma}))(s).
\]

Thus we have the desired result. \( \blacksquare \)

**Linear programming for the max objective in \( \overline{G} \).** The following linear program characterizes the value function \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\max(\overline{\sigma})) \). For all \( s \in S \) we have a variable \( x_s \) and the objective function is \( \min \sum_{s \in S} x_s \). The set of linear constraints are as follows:

\[
\begin{align*}
x_s & \geq 0 & \forall s \in S; \\
x_s & = \overline{\sigma}(s) & \forall s \in \hat{S}^*; \\
x_s & \geq x_t & \forall s \in S_1, (s, t) \in \widetilde{E}; \\
x_s & = \sum_{t \in S} \overline{\sigma}(t) \cdot x_t & \forall s \in \mathcal{F}.
\end{align*}
\]

The correctness proof of the above linear program to characterize the value function \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\max(\overline{\sigma})) \) follows by extending the result for reachability objectives [6]. The key property that can be used to prove the correctness of the above claim is as follows: if a pure memoryless optimal strategy is fixed, then from all states in \( S \), the set \( S^* \) of absorbing states is reached with probability 1. The above property can be proved as follows: since \( r \) is a positive reward function, it follows that for all \( s \in S \) we have \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r))(s) > 0 \). Moreover, for all states \( s \in S \) we have \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\max(\overline{\sigma}))(s) = \langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r))(s) > 0 \). Observe that for all \( s \in S \) we have \( \overline{\sigma}(s) = 0 \). Hence if we fix a pure memoryless optimal strategy \( \sigma \) in \( \overline{G} \), then in the Markov chain \( \overline{G}_\sigma \) there is no closed recurrent set \( C \) such that \( C \subseteq S \). It follows that for all states \( s \in S \), in the Markov chain \( \overline{G}_\sigma \), the set \( S^* \) is reached with probability 1. Using the above fact and the correctness of linear-programming for reachability objectives, the correctness proof of the above linear-program for the objective \( \max(\overline{\sigma}) \) in \( \overline{G} \) can be obtained. This shows that the value function \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r)) \) for MDPs with reward function \( r \) can be computed in polynomial time. This gives us the following result.

**Theorem 2.** Given an MDP \( G \) with a reward function \( r \), the value function \( \langle \langle 1 \rangle \rangle_{\text{val}}^G (\limsup(r)) \) can be computed in polynomial time.
Lemma 3 that for all $v \in \mathbb{R}^+$ then play an almost-sure winning strategy in $U$.

Proof. Let $r(S) = \{v_0, v_1, \ldots, v_k\}$ with $v_0 > v_1 > \cdots > v_k$.

1. Let $G_0 := G; S_s = \emptyset$;
2. for $i := 0$ to $k$ do 
   3.1 $U_i := W_i^{G_i}((\text{coBüchi}((\bigcup_{j \leq i} r^{-1}(v_j)) \cap S'))$;
   3.2 for all $u \in U_i \cap S_i$
      $f_i(u) := v_i$;
   3.3 $S_s := S_s \cup (U_i \cap S_i)$;
   3.4 $B_i := \text{Attr}_P(U_i, G_i)$;
   3.5 $G_{i+1} := G_i \setminus B_i, S^{i+1} := S^i \setminus B_i$;
   3.6 return $S_s$ and $f_s$.

### 3.3 MDPs with liminf objectives

In this subsection we present polynomial time algorithms for MDPs with liminf objectives, and then present the complexity result for $2^{1/2}$-player games with limsup and liminf objectives.

Informal description of algorithm. We first present an algorithm that takes an MDP $G$ with a positive reward function $r : S \to \mathbb{R}^+$, and computes a set $S_s$ and a function $f_s : S_s \to \mathbb{R}^+$. The output of the algorithm will be useful in reduction of MDPs with liminf objectives to MDPs with max objectives. Let the rewards be $v_0 > v_1 > \cdots > v_k$. The algorithm proceeds in iteration and in iteration $i$ we denote the MDP as $G_i$ and the state space as $S^i$. At iteration $i$ the algorithm considers the set $V_i$ of reward at least $v_i$ in the MDP $G_i$, and computes the set $U_i = W_i^{G_i}((\text{coBüchi}(V_i)))$, (i.e., the almost-sure winning set in the MDP $G_i$ for coBüchi objective with the coBüchi set $V_i$). For all $u \in U_i \cap S_i$ we assign $f_s(u) = v_i$ and add the set $U_i \cap S_i$ to $S_s$. Then the set $\text{Attr}_P(U_i, G_i)$ is removed from the MDP $G_i$ and we proceed to iteration $i + 1$. In $G_i$ all end components that contain reward at least $v_i$ are contained in $U_i$ (by Lemma 3 part (2)), and all end components in $S^i \setminus U_i$ do not intersect with $\text{Attr}_P(U_i, G_i)$ (by Lemma 3 part (3)). This gives us the following lemma.

Lemma 6. Let $G$ be an MDP with a positive reward function $r : S \to \mathbb{R}^+$. Let $f_s$ be the output of Algorithm 2. For all end components $U \in \mathcal{E}(G)$ and all states $u \in U \cap S_1$, we have $\min(r(U)) \leq f_s(u)$.

Proof. Let $U^* = \bigcup_{i=0}^k U_i$ (as computed in Algorithm 2). Then it follows from Lemma 3 that for all $A \in \mathcal{E}(G)$ we have $A \cap U^* \neq \emptyset$. Consider $A \in \mathcal{E}(G)$ and let $v_i = \min(r(A))$. Suppose for some $j < i$ we have $A \cap U_j \neq \emptyset$. Then there is a strategy to ensure that $U_j$ is reached with probability 1 from all states in $A$ and then play an almost-sure winning strategy in $U_j$ to ensure coBüchi($\bigcup_{i \leq j} r^{-1}(v_i) \cap$
$S^i$). Then $A \subseteq U_j$. Hence for all $u \in A \cap S_1$ we have $f_\sigma(u) = v_j \geq v_i$. If for all $j < i$ we have $A \cap U_j = \emptyset$, then we show that $A \subseteq U_i$. The uniform memoryless strategy $\sigma_A$ (as used in Lemma 2) in $G_i$ is a witness to prove that $A \subseteq U_i$. In this case for all $u \in A \cap S_1$ we have $f_\sigma(u) = v_i = \min(r(A))$. The desired result follows.

**Transformation to MDPs with max objective.** Given an MDP $G = ((S,E),(S_1,S_P),\delta)$ with a positive reward function $r : S \rightarrow \mathbb{R}^+$, and let $S_*$ and $f_\sigma$ be the output of Algorithm 2. We construct an MDP $\overline{G} = ((\overline{S}, \overline{E}),(\overline{S_1}, \overline{S_P}), \overline{\delta})$ with a reward function $\overline{\tau}$ as follows:

- $\overline{S} = S \cup \hat{S}_*$; i.e., the set of states consists of the state space $S$ and a copy $\hat{S}_*$ of $S$.
- $\overline{E} = E \cup \{(s, \hat{s}) | s \in S_*, \hat{s} \in \hat{S}_* \} \cup \{(\hat{s}, s) | s \in \hat{S}_* \}$; along with edges $E$, for all states $s \in S_*$ there is an edge to its copy $\hat{s}$ in $\hat{S}_*$, and all states in $\hat{S}_*$ are absorbing states.
- $\overline{S}_1 = S_1 \cup \hat{S}_*$.
- $\overline{\delta} = \delta$.
- $\overline{\tau}(s) = 0$ for all $s \in S$ and $\overline{\tau}(s) = f_\sigma(s)$ for $\hat{s} \in \hat{S}_*$, where $\hat{s}$ is the copy of $s$.

We refer to the above construction as liminf conversion. The following lemma proves the relationship between the value function $\langle 1 \rangle_{\text{val}}^G(\liminf(r))$ and $\langle 1 \rangle_{\text{val}}^{\overline{G}}(\max(\overline{\tau}))$.

**Lemma 7.** Let $G$ be an MDP with a positive reward function $r : S \rightarrow \mathbb{R}^+$. Let $\overline{G}$ and $\overline{\tau}$ be obtained from $G$ and $r$ by the liminf conversion. For all states $s \in S$, we have

$$\langle 1 \rangle_{\text{val}}^G(\liminf(r))(s) = \langle 1 \rangle_{\text{val}}^{\overline{G}}(\max(\overline{\tau}))(s).$$

**Proof.** The result is obtained from the following two case analysis.

1. Let $\sigma$ be a pure memoryless optimal strategy in $G$ for the objective $\liminf(r)$. Let $C = \{ C_1, C_2, \ldots, C_m \}$ be the set of closed connected recurrent sets in the Markov chain obtained from $G$ after fixing the strategy $\sigma$. Since $G$ is an bipartite MDP, it follows that for all $1 \leq i \leq m$, we have $C_i \cap S_1 \neq \emptyset$. Let $C = \bigcup_{i=1}^m C_i$. We define a pure memoryless strategy $\overline{\sigma}$ in $\overline{G}$ as follows

$$\overline{\sigma}(s) = \begin{cases} \sigma(s) & s \in S_1 \setminus C; \\ \hat{s} & \hat{s} \in \hat{S}_* \text{ and } s \in S_1 \cap C. \end{cases}$$

By Lemma 6 it follows that the strategy $\overline{\sigma}$ ensures that for all $C_i \in C$ and all $s \in C_i$, the maximal reward reached in $\overline{G}$ is at least $\min(r(C_i))$ with probability 1. It follows that for all $s \in S$ we have

$$\langle 1 \rangle_{\text{val}}^{C_i}(\limsup(r))(s) \leq \langle 1 \rangle_{\text{val}}^{\overline{G}}(\max(\overline{\tau}))(s).$$
2. Let $\sigma$ be a pure memoryless optimal strategy for the objective $\max(\tau)$ in $\overline{G}$.
We fix a strategy $\sigma$ in $G$ as follows: if at a state $s \in S$, the strategy $\sigma$ chooses the edge $(s, \overline{s})$, then in $G$ on reaching $s$, the strategy $\sigma$ plays an almost-sure winning strategy for the objective $\coBuchi(\bigcup_{v_j \geq f_v(s)} r^{-1}(v_j))$, otherwise $\sigma$ follows $\overline{\sigma}$. It follows that for all $s \in S$ we have

$$\langle \langle 1 \rangle \rangle_{val}(\liminf(r))(s) \geq \langle \langle 1 \rangle \rangle_{val}(\max(\tau))(s).$$

Thus we have the desired result. $\blacksquare$

**Linear programming for the max objective in $\overline{G}$.** The linear program of subsection 3.2 characterizes the value function $\langle \langle 1 \rangle \rangle_{val}(\max(\tau))$. This shows that the value function $\langle \langle 1 \rangle \rangle_{val}(\liminf(r))$ for MDPs with reward function $r$ can be computed in polynomial time. This gives us the following result.

**Theorem 3.** Given an MDP $G$ with a reward function $r$, the value function $\langle \langle 1 \rangle \rangle_{val}(\liminf(r))$ can be computed in polynomial time.

### 3.4 2$^{1/2}$-player games with limsup and liminf objectives

We now show that 2$^{1/2}$-player games with limsup and liminf objectives can be decided in NP $\cap$ coNP. The pure memoryless optimal strategies (existence follows from Theorem 1) provide the polynomial witnesses and to obtain the desired result we need to present a polynomial time verification procedure. In other words, we need to present polynomial time algorithms for MDPs with limsup and liminf objectives. Since the value functions in MDPs with limsup and liminf objectives can be computed in polynomial time (Theorem 2 and Theorem 3), we obtain the following result about the complexity 2$^{1/2}$-player games with limsup and liminf objectives.

**Theorem 4.** Given a 2$^{1/2}$-player game graph $G$ with a reward function $r$, a state $s$ and a rational value $q$, the following assertions hold: (a) whether $\langle \langle 1 \rangle \rangle_{val}(\limsup(r))(s) \geq q$ can be decided in NP $\cap$ coNP; and (b) whether $\langle \langle 1 \rangle \rangle_{val}(\liminf(r))(s) \geq q$ can be decided in NP $\cap$ coNP.

**Acknowledgments.** We thank Hugo Gimbert for explaining his results and pointing out relevant literature on games with limsup and liminf objectives. This research was supported in part by the NSF grants CCR-0132780, CNS-0720884, and CCR-0225610, by the Swiss National Science Foundation, and by the COMBEST project of the European Union.

**References**

1. K. Chatterjee and T.A. Henzinger. Value iteration. In 25 Years of Model Checking, LNCS. Springer, 2007.
2. K. Chatterjee, M. Jurdziński, and T.A. Henzinger. Simple stochastic parity games. In CSL’03, volume 2803 of LNCS, pages 100–113. Springer, 2003.
3. K. Chatterjee, M. Jurdziński, and T.A. Henzinger. Quantitative stochastic parity games. In SODA’04, pages 121–130. SIAM, 2004.
4. C. Courcoubetis and M. Yannakakis. Markov decision processes and regular events. In ICALP 90: Automata, Languages, and Programming, volume 443 of Lecture Notes in Computer Science, pages 336–349. Springer-Verlag, 1990.
5. L. de Alfaro. Formal Verification of Probabilistic Systems. PhD thesis, Stanford University, 1997.
6. J. Filar and K. Vrieze. Competitive Markov Decision Processes. Springer-Verlag, 1997.
7. H. Gimbert. Jeux positionnels. PhD thesis, Université Paris 7, 2006.
8. H. Gimbert and W. Zielonka. Games where you can play optimally without any memory. In CONCUR’05, pages 428–442. Springer, 2005.
9. H. Gimbert and W. Zielonka. Perfect information stochastic priority games. In ICALP’07, pages 850–861. Springer, 2007.
10. T. A. Liggett and S. A. Lippman. Stochastic games with perfect information and time average payoff. Siam Review, 11:604–607, 1969.
11. A. Maitra and W. Sudderth, editors. Discrete Gambling and Stochastic Games. Springer, 1996.
12. D.A. Martin. The determinacy of Blackwell games. The Journal of Symbolic Logic, 63(4):1565–1581, 1998.