The n+n+alpha System in a Continuum Faddeev Formulation

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The continuum Faddeev equations for the neutron-neutron-alpha ($n+n+\alpha$) system are formulated for a general interaction as well as for finite rank forces. In addition, the capture process $n+n+\alpha \rightarrow ^{6}\text{He}+\gamma$ is derived.

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I. INTRODUCTION

In recent years the study of quantum halo systems experienced increased interest in the nuclear as well as the atomic few-body community. For a recent review see Ref. [1]. The nucleus $^6\text{He}$ is of particular interest, since it constitutes the lightest two-neutron halo nucleus with a $^4\text{He}$ core. Being an effective three-body system, the properties of the ground state have been explored using either Faddeev [2–4] or Hyper-spherical Harmonics (HH) [5–8]. More recently the ground state has also been calculated with multi-cluster methods [9–17] as well as with GFMC [18]. Those multi-cluster methods include various techniques like the microscopic dynamical multi-configuration three-cluster model [9], the stochastic variational method [10], the multi-cluster dynamic model (MDMP and AMDMP), the hybrid-TV model, a combination of the cluster orbital shell model (COSM) [15] and the extended cluster model (ECM) [16], the refined resonating group method [11, 14], and the coupled-rearrangement-channel variational method with Gaussian basis functions [17]. In addition, the beta decay to the alpha+d continuum has been studied [19, 20]. Of interest is also the two-neutron capture process $^4\text{He}(2n,\gamma)^{6}\text{He}$ as a possible route bridging the instability gap at nuclear mass $A=5$ [21].

The situation is quite different in the continuum of two neutrons and an $\alpha$-particle. There is a well established $2^+$ resonance [22], but further resonant structures are still under debate [7, 23–30]. Up to now more indirect approaches in understanding the resonance structure have been carried out, e.g. the four-body distorted wave approach, leading to three-body continuum excitations of two-neutron Borromean halo nuclei [23, 28]. Furthermore, complex scaling in Coulomb break-up reactions has been employed [31]. In addition, an extension of the HH method on a Lagrange mesh [32] has been used to study three-body continuum states. This is at least a four-body problem with great uncertainties about the reaction mechanisms and the interactions entering these much more complicated systems.

Thus, the currently predominant approach to continuum calculations for the pure $n+n+\alpha$ system is the HH method [7, 8, 26–28, 32]. A Faddeev approach is to the best of our knowledge still missing. Only for the $^6\text{Li}$ nucleus, a Faddeev treatment of the deuteron-alpha ($d-\alpha$) system has been employed [33], which however, did not have to face the challenge of three-to-three scattering. This also refers to the pioneering work by Koike [34, 35] on $d-\alpha$ scattering.

The aim of this investigation is to fill that gap. For the $n+n+\alpha$ system one faces the situation of three free particles being in the initial channel and leading again to three free particles in the exit channel. In other words, one has to deal with three-to-three scattering. Scattering of three free incoming particles to three free outgoing ones in a Faddeev approach has been initiated in [36, 37] in the context of the three-body photo disintegration of $^3\text{He}$. This path has also been followed in the same context and in a Faddeev approach by Meijgaard and Tjon [38]. Into the matrix element for the photo disintegration enters the three-nucleon to three-nucleon scattering wave function, which has been evaluated in Ref. [38] and then inserted into the photodisintegration matrix element. However, evaluating the wave function is a completely unnecessary complication, since this process is initiated by the three-nucleon bound state. One can directly derive a Faddeev equation for the three-body break up amplitude, in which the driving term contains the action of the current operator on the $^3\text{He}$ ground state. Then, the complete final state interaction is generated by a Faddeev integral kernel for the amplitude. This considerably simplifies the technical part of a calculation, since no disconnected processes occur. This very procedure has been pioneered in Refs. [36, 37] and is being applied in state-of-the-art calculations, see e.g. [39, 40].

The same procedure can trivially be adapted to the capture process $n+n+\alpha \rightarrow ^{6}\text{He}+\gamma$, as will be displayed in the
present investigation. This capture process is relevant for the production rate of $^6$He in astrophysical environments characterized by high neutron and alpha densities e.g. those related to supernova shock fronts. In e.g. Ref. [21] this three-body process is approximated by sequential two-body processes, whereas in principle a genuine three-body reactions needs to be calculated. Furthermore, the $nn\alpha \rightarrow nn\alpha$ amplitude is relevant for determining the the next order coefficient in the virial equation of state in low-density matter.

From a technical point of view the Faddeev approach to the $n + n + \alpha$ continuum is strongly needed, since the currently predominant approach, namely the Hyper-spherical Harmonics (HH) approach, still faces open challenges. It is already known that in the break up process $n + d \rightarrow n + n + p$ a strong FSI peak appears for the $n - n$ subsystem. In the Faddeev approach using Jacobi momenta this can be mapped out correctly, whereas when changing to the hyper-spherical angle, the convergences is quite poor for this particular configuration. In the HH method, the control of the coupling potentials can be a painful exercise, whereas in the Faddeev approach using Jacobi variables the dynamics is perfectly well under control in all details. This same situation must be expected in $n + n + \alpha$ scattering, where the three-body S-matrix is characterized by continuous quantum numbers describing how the energy is distributed over the relative motion. There are strong initial and final state interaction peaks, which in a discrete representation through hyper-spherical K-quantum numbers are difficult to map out correctly. As stated above only a technically reliable approach as the Faddeev one will guarantee the validity of the results when searching for $^6$He resonances.

The paper is organized as follows. In Section II we derive the coupled Faddeev equations for the three-to-three scattering amplitude, followed by a partial wave decomposition in Section III. The Faddeev equations will be solved by iteration yielding a multiple scattering series. This will be outlined in Section IV. Since most of the Faddeev type investigations of the $n + n + \alpha$ system are based on finite rank forces, we also present in Section V a continuum formulation based on separable forces. Furthermore, we discuss the the unitarity relation for the three-to-three amplitude in Section VI. Finally the capture process $n + n + \alpha \rightarrow ^6$He$+\gamma$ will be discussed for the Faddeev scheme in Section VII. Then we summarize in section VII. Technical details about the partial wave decomposition and an efficient way of treating the three-body singularities are given in the Appendices.

II. THE FADDEEV EQUATIONS FOR THE NN$\alpha$ SYSTEM

In developing the formal expression for the transition amplitude between three free particles interacting with short-range, strong interactions, we start from the triad of Lippmann-Schwinger (LS) equations acting on a three-particle initial state given by

$$\Phi_\alpha^{(+)} = |p_\alpha|^{(+)}|q_\alpha\rangle,$$

(2.1)

where $|p_\alpha\rangle^{(+)}$ is a two-body scattering state, and the index $\alpha = 1, 2, 3$ indicates the three choices of pairs characterized by the third particle, the spectator. Furthermore, $V^\alpha = \sum_{\beta \neq \alpha} V_\beta$, where $V_\beta (\beta = 1, 2, 3)$ are the pair forces. Three-body forces can in principle be incorporated in a straightforward fashion. However, we will only concentrate on two-body forces here. The triad of LS equations,

$$\Psi_0^{(+)} = \Phi_\alpha^{(+)} + G_\alpha V^\alpha \Phi_0^{(+)}$$

(2.2)

define the scattering wave uniquely. The channel Green’s function is given by $G_\alpha^{-1} = (E + i\varepsilon - H_0 - V_\alpha)^{-1}$. We use standard Jacobi momenta $p_\alpha$ and $q_\alpha$ and their quantum numbers as basis states.

By suitable multiplication of the three equations in the triad from the left by $V_\beta$ one obtains the transition operators $U_{\alpha\beta} \equiv (V_\beta + V_\gamma)\Phi_0^{(+)}$, with $\beta \neq \alpha, \gamma \neq \alpha$, which fulfill the set of equations

$$U_{\alpha\beta} = \sum_{\beta \neq \alpha} t_\beta \Phi_0 + \sum_{\beta \neq \alpha} t_\gamma G_\alpha U_{\beta\gamma},$$

(2.3)

where $\Phi_0 = |p\rangle|q\rangle$ is the free three-particle state.

The three-body break up operator is given by

$$U_{00} = \sum_{\gamma} V_\gamma \Phi_0^{(+)}$$

(2.4)

Again, from the triad follows

$$U_{00} = \sum_{\gamma} t_\gamma \Phi_0 + \sum_{\gamma} t_\gamma G_0 U_{\gamma0}.$$

(2.5)
Iterating Eq. (2.3) one obtains the multiple scattering series

$$U_{00} = \sum_{\gamma} t_{\gamma} \Phi_0 + \sum_{\gamma} t_{\gamma} G_0 \sum_{\beta \neq \gamma} t_{\beta} \Phi_0 + \sum_{\gamma} t_{\gamma} G_0 \sum_{\beta \neq \gamma} t_{\beta} G_0 \sum_{\alpha \neq \beta} t_{\alpha} \Phi_0 + \cdots .$$  \hspace{1cm} (2.6)

Instead of working with the coupled set of Eq. (2.3) and the relation of Eq. (2.5) for the three-body break-up operator, one can generate the multiple scattering series directly by decomposing $U_{00}$ as

$$U_{00} \equiv \sum_{\gamma} U_{\gamma},$$  \hspace{1cm} (2.7)

and choosing $U_{\gamma}$ to obey the coupled set of Faddeev equations

$$U_{\gamma} = t_{\gamma} + t_{\gamma} G_0 \sum_{\alpha \neq \gamma} U_{\alpha} .$$  \hspace{1cm} (2.8)

Indeed, iterating Eq. (2.8) and inserting the result into Eq. (2.7) leads exactly to the multiple scattering series from above. Explicitly, we have a set of three coupled equations

$$U_1 = t_1 + t_1 G_0 (U_2 + U_3)$$

$$U_2 = t_2 + t_2 G_0 (U_3 + U_1)$$

$$U_3 = t_3 + t_3 G_0 (U_1 + U_2) .$$  \hspace{1cm} (2.9)

We also observe, that comparing Eqs. (2.7) and (2.4) leads to

$$U_{\gamma} \equiv V_{\gamma} \Psi_0^{(+)} .$$  \hspace{1cm} (2.10)

When considering the $n+n+\alpha$ system, we need to incorporate the identity of the two neutrons. Fixing arbitrarily the alpha particle as spectator and label it as “1” and the two neutrons as particles “2” and “3”, the scattering wave function $\Psi_0^{(+)}$ must be antisymmetric under the exchange of particles “2” and “3”. Thus, defining the transposition operator $P_{23}$, the scattering wave function must fulfill $P_{23} \Psi_0^{(+)} = - \Psi_0^{(+)}$. Using this in Eq. (2.10) leads to

$$U_3 = - P_{23} U_2 .$$  \hspace{1cm} (2.11)

Thus, for the $n+n+\alpha$ system we only have 2 coupled equations,

$$U_1 = t_1 + t_1 G_0 (1 - P_{23}) U_2$$

$$U_2 = t_2 + t_2 G_0 (- P_{23} U_2 + U_1) .$$  \hspace{1cm} (2.12)

More precisely, one has to apply the driving terms to the free state $\Phi_{0,a}$, which is antisymmetric under exchange of the two neutrons:

$$\Phi_{0,a} \equiv (1 - P_{23}) |p_1 q_1 \rangle |0 m_2 m_3 \rangle \left| \frac{1}{2} \frac{1}{2} \right> .$$  \hspace{1cm} (2.13)

leading to

$$U_1 \Phi_{0,a} = t_1 \Phi_{0,a} + t_1 G_0 (1 - P_{23}) U_2 \Phi_{0,a}$$

$$U_2 \Phi_{0,a} = t_2 \Phi_{0,a} + t_2 G_0 (- P_{23} U_2 \Phi_{0,a} + U_1 \Phi_{0,a}) .$$  \hspace{1cm} (2.14)

The full break-up operator is given by

$$U_{00} \Phi_{0,a} = U_1 \Phi_{0,a} + (1 - P_{23}) U_2 \Phi_{0,a} .$$  \hspace{1cm} (2.15)

For the on shell break-up amplitude one has to evaluate the matrix element $\langle \Phi_{0,a}' | U_{00,a} | \Phi_{0,a} \rangle$, where in the final state momenta as well as spin magnetic quantum numbers are changed,

$$\Phi_{0,a} = (1 - P_{23}) |p_1 q_1 \rangle |0 m_2' m_3' \rangle \left| \frac{1}{2} \frac{1}{2} \right> .$$  \hspace{1cm} (2.16)
III. PARTIAL WAVE DECOMPOSITION

In order to solve the coupled equations, Eqs. (2.14), two sets of partial wave basis states are needed:

$$|p_1 q_1 \alpha_1 \rangle = \sum_{\mu_1} C(j_1 \lambda_1 J, \mu_1 M_1 - \mu_1) |p_1 (l_1 s_1) j_1 \mu_1 \rangle |q_1 \lambda_1 M_1 - \mu_1 \rangle \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) 1$$

$$|p_2 q_2 \alpha_2 \rangle = \sum_{\mu_2} C(j_2 I_2 J, \mu_2 M_2 - \mu_2) |p_2 (l_2 s_2) j_2 \mu_2 \rangle |q_2 (\lambda_2 \frac{1}{2} \frac{1}{2} \frac{1}{2}) I_2 M_2 - \mu_2 \rangle \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) 1.$$  (3.1)

The details of a partial wave decomposition of Eqs. (2.14) is well known (see e.g. [39]), and we refer to [46] for details. Employing the states of Eq. (3.1), the coupled equations, Eq. (2.14) read

$$\langle p_1' q_1' \alpha_1'| U_{1,a} = \frac{\delta(q_1'-q_1)}{q_1'} t_{\alpha_1'}(p_1' p_1, E_{q_1}) C_{\alpha_1}^{m_2+m_3}(\theta_1)$$

$$+ \left(1 + (-1)^{l_1+s_1}\right) \int dx \int dq_2' q_2'^2 t_{\alpha_1'}(p_1' \pi_1(q_1' q_2' x), E_{q_1'}) G_0(\pi_1(q_1' q_2' x), q_1')$$

$$\sum_{\alpha_2} G_{\alpha_1' \alpha_2}(q_1' q_2') \langle \pi_2(1_q' q_2' x) q_2' \alpha_2'| U_{2,a} \rangle,$$

$$\langle p_2' q_2' \alpha_2'| U_{2,a} = \frac{\delta(q_2'-q_2)}{q_2'} t_{\alpha_2'}(p_2' p_2, E_{q_2}) D_{\alpha_2}^{m_2,m_3}(\theta_1)$$

$$- \delta(q_2'-q_2) t_{\alpha_2'}(p_2' p_2, E_{q_2}) \tilde{D}_{\alpha_2}^{m_2,m_3}(\theta_1)$$

$$+ \int dx \int dq_1' q_1'^2 t_{\alpha_1'}(p_1' \pi_3(q_1' q_1' x), E_{q_1'}) G_0(\pi_3(q_1' q_1' x), q_1')$$

$$\sum_{\alpha_1} \hat{H}_{\alpha_1' \alpha_1}(q_1' q_1') \langle \pi_4(q_1' q_1' x) q_1' \alpha_1'| U_{1,a} \rangle,$$

$$- \int dx \int dq_2'' q_2''^2 t_{\alpha_2'}(p_2' \pi_5(q_2' q_2'' x), E_{q_2'}) G_0(\pi_5(q_2' q_2'' x), q_2')$$

$$\sum_{\alpha_2} \hat{I}_{\alpha_2' \alpha_2}(q_2' q_2'' x) \langle \pi_6(q_2' q_2'' x) q_2'' \alpha_2'| U_{2,a} \rangle,$$  (3.2)

where

$$C_{\alpha_1}^{m_2+m_3}(\theta_1) = (1 + (-1)^{l_1+s_1}) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (s_1, m_2 m_3) \sum_{m_1} (l_1 s_1 j_1, m_1, m_2 + m_3)$$

$$\left((j_1 \lambda_1 J, m_1 + m_2 + m_3, 0, M) Y_{l_1 m_1}(\theta_1, 0) \sqrt{\frac{\lambda_1}{4\pi}}\right),$$  (3.3)

with $\lambda_1 = 2\lambda_1 + 1$.

$$D_{\alpha_2}^{m_2,m_3}(\theta_1) \equiv D_{\alpha_2}^{m_2,m_3}(\theta_2 \theta_3 q_2)$$

$$= \delta_{\alpha_2}^{\lambda_2} \sum_{\mu} (j_2 I_2 J', \mu, M' - \mu) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (j_2' j_2, \mu - m_3, m_3) Y_{j_2' m_2 - m_3}^* (\hat{q}_2)$$

$$\left((\lambda_2 \frac{1}{2} I_2, M' - \mu - m_2, m_2) Y_{\lambda_2 M' - m_2}^* (\hat{q}_2)\right)$$  (3.4)

and

$$\tilde{D}_{\alpha_2}^{m_2,m_3}(\theta_1) \equiv \tilde{D}_{\alpha_2}^{m_2,m_3}(\theta_2 \theta_3 q_2)$$

$$= \delta_{\alpha_2}^{\lambda_2} \sum_{\mu} (j_2' I_2 J', \mu M' - \mu) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (j_2' j_2, \mu - m_2, m_2) Y_{j_2' m_2 - m_3}^* (\hat{q}_2)$$
\[ \left( \lambda'_2 \frac{1}{2} I'_2, M' - \mu - m_3, m_3 \right) Y_{\lambda'_2 M' - \mu - m_3}^*(\hat{q}_2). \] (3.5)

The ‘shifted’ momenta \( \pi_i \) are given as

\[
\begin{align*}
\pi_1 &= \sqrt{\alpha^2 q_1' + q_2'^2 + 2 \alpha q_1' q_2' x} \\
\pi_2 &= \sqrt{q_2'^2 + \beta q_1'^2 + 2 \beta q_1' q_2' x} \\
\pi_3 &= \sqrt{q_2'^2 + \beta^2 q_1'^2 + 2 \beta q_1' q_2' x} \\
\pi_4 &= \sqrt{\alpha^2 q_2'^2 + q_1'^2 + 2 \alpha q_1' q_2' x} \\
\pi_5 &= \sqrt{\beta^2 q_2'^2 + q_2''^2 + 2 \beta q_2' q_2'' x} \\
\pi_6 &= \sqrt{q_2'^2 + \beta^2 q_2''^2 + 2 \beta q_2' q_2'' x},
\end{align*}
\] (3.6)

where

\[
\begin{align*}
\alpha &= \frac{1}{2} \\
\beta &= \frac{m_\alpha}{m + m_\alpha} \\
\gamma &= \frac{2m + m_\alpha}{2(m + m_\alpha)} \\
\beta' &= \frac{m}{m + m_\alpha}.
\end{align*}
\] (3.7)

Here \( m \) is the neutron mass, and \( m_\alpha \) the mass of the \(^4\)He nucleus.

We refer to the Appendix \[\text{A}\] for some details of the derivation and the expressions of the purely geometric quantities \( G_{\alpha_1' \alpha_2'}(q_1' q_2' x), H_{\alpha_1' \alpha_1'}(q_2' q_1' x), \) and \( I_{\alpha_1' \alpha_2''}(q_2'^2 x) \). Furthermore, \( t_{\alpha_1}(p_1 p_1, E_{q_1}) \) is the two-neutron \( t \)-matrix and \( t_{\alpha_2}(p_2 p_2, E_{q_2}) \) the one for the neutron-\( \alpha \) pair.

Due to the free Green’s functions \( G_0 \) and the \( x \)-integration over it, one encounters the well known logarithmic singularities of any three-body problem. These singularities can be reliably treated \[39, 47\]. However, the method suggested in Refs. \[48, 49\], appears to be beneficial for here, since not only kernels contain logarithmic singularities but also the driving terms. We illustrate this new method with an example in Appendix \[B\].

### IV. THE MULTIPLE SCATTERING SERIES

A well established way to solve a coupled set of Faddeev equations is to generate the multiple scattering series. For the 3N system is is laid out e.g. in Ref. \[39\]. Schematically the Eqs. \( 3.2 \) have the form

\[ U = U^{(0)} + KU \] (4.1)

which, when iterated yield

\[ U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots, \] (4.2)

with

\[ U^{(n)} = KU^{(n-1)}, n = 1, 2, \cdots. \] (4.3)

The first few terms of this series are depicted in Fig. \[1\] The driving terms of Eqs. \( 3.2 \), sketched in the upper row of Fig. \[1\] are necessarily disconnected, since a two-body \( t \)-matrix can not act on three particles.

Let us consider the terms of second order in the two-body \( t \)-matrix (indicated in the second row of Fig. \[1\]):

\[
\langle p_1' q_1' \alpha_1' | U^{(1)}_{1,1,1} \rangle = \\
(1 + (-1)^{\tilde{s}_1}) \int dx \int dq_2' \int dq_2 dq_1 \ t_{\alpha_1'}(p_1' \pi_1(q_1' q_2' x), E_{q_1'}) \ G_0(\pi_1(q_1' q_2' x), q_1') \\
\sum_{\alpha_2} G_{\alpha_1' \alpha_2'}(q_1' q_2' x) \left[ \frac{\delta(q_2' - q_2)}{q_2'} t_{\alpha_2'}(\pi_2(q_1' q_2' x) p_2, E_{q_2}) \ D_{\alpha_2}^{m_2 m_3}(\theta_1) \right]
\]
\[ - \frac{\delta(q_1' - \tilde{q}_1)}{q_2^2} t_{\alpha_2'}(\pi_2(q_1'\tilde{q}_2x)\tilde{p}_2, E_{q_2}) \hat{D}_{\alpha_2'}^{m_2,m_3}(\theta_1) \]
\[ = \left( 1 + (-1)^{l_1 + l_2} i \right) \int dx \sum_{\alpha_2'} \left[ t_{\alpha_1'}(p_1'\pi_1(q_1'q_2x), E_{q_1'}) G_0(\pi_1(q_1'q_2x), q_1') ight. \]
\[ G_{\alpha_1',\alpha_2'}(q_1'q_2x)p_2, E_{q_2}) \hat{D}_{\alpha_2'}^{m_2,m_3}(\theta_1) \]
\[- t_{\alpha_1'}(p_1'\pi_1(q_1'\tilde{q}_2x), E_{q_1'}) G_0(\pi_1(q_1'\tilde{q}_2x), q_1') G_{\alpha_1',\alpha_2'}(q_1'\tilde{q}_2x) \]
\[ t_{\alpha_1'}(\pi_2(q_1'\tilde{q}_2x)\tilde{p}_2, E_{q_2}) \hat{D}_{\alpha_2'}^{m_2,m_3}(\theta_1). \] (4.4)

The only singular function under the x-integral is the free Green’s function, which leads in the \( q_1' - q_2 \) and \( q_1' - \tilde{q}_2 \) planes of external momenta to the well known logarithmic singularities. The same is true for

\[ \langle p_2'q_2\alpha_2'|U_{2,1}^{(1)} \rangle = \int dx \int dq_1'q_2' t_{\alpha_2'}(p_2'\pi_3(q_2'q_1x), E_{q_2'}) G_0(\pi_3(q_2'q_1x), q_2') \]
\[ \sum_{\alpha_1'} H_{\alpha_1',\alpha_2'}(q_1'q_1x) t_{\alpha_1'}(\pi_4(q_1'q_1x)p_1, E_{q_1}) C_{\alpha_1'}^{m_2+m_3}(\theta_1) \]
\[- \int dx \int dq_2'q_2''' t_{\alpha_2''}(p_2''\pi_5(q_2''q_2'x), E_{q_2''}) G_0(\pi_5(q_2''q_2'x), q_2') \]
\[ \sum_{\alpha_1''} I_{\alpha_2''}(q_2''q_2'x) t_{\alpha_2''}(\pi_6(q_2''q_2'x)p_2, E_{q_2''}) \hat{D}_{\alpha_2''}^{m_2,m_3}(\theta_1) \]
\[- \frac{\delta(q_2'' - \tilde{q}_2)}{q_2^2} t_{\alpha_2''}(\pi_6(q_2''q_2'x)\tilde{p}_2, E_{q_2''}) \hat{D}_{\alpha_2''}^{m_2,m_3}(\theta_1) \]
\[ = \int dx t_{\alpha_2'}(p_2'\pi_3(q_2'q_1x), E_{q_2'}) G_0(\pi_3(q_2'q_1x), q_2') \]
\[ \sum_{\alpha_1'} H_{\alpha_1',\alpha_2'}(q_2'q_1x) t_{\alpha_1'}(\pi_4(q_2'q_1x)p_1, E_{q_1}) C_{\alpha_1'}^{m_2+m_3}(\theta_1) \]
\[- \int dx \left[ t_{\alpha_1'}(p_2'\pi_5(q_2'q_2x), E_{q_2'}) G_0(\pi_5(q_2'q_2x), q_2') \right. \]
\[ \sum_{\alpha_1''} I_{\alpha_2''}(q_2'q_2x) t_{\alpha_2''}(\pi_6(q_2'q_2x)p_2, E_{q_2''}) \hat{D}_{\alpha_2''}^{m_2,m_3}(\theta_1) \]
\[- t_{\alpha_2'}(p_2'\pi_5(q_2'q_2x), E_{q_2'}) G_0(\pi_5(q_2'q_2x), q_2') \]
\[ \sum_{\alpha_2''} I_{\alpha_2''}(q_2'q_2x) t_{\alpha_2''}(\pi_6(q_2'q_2x)\tilde{p}_2, E_{q_2''}) \hat{D}_{\alpha_2''}^{m_2,m_3}(\theta_1). \] (4.5)

Indeed, the free Green’s functions \( G_0 \) lead in the only remaining x-integral to logarithmic singularities.

In order to safely apply the kernel to the previous amplitude, that amplitude has to be a smooth function. This is only the case for the next higher order, being of third order in \( t \), sketched in the third row of Fig. 1. The third order term reads

\[ \langle p_1'q_1\alpha_1'|U_{1,1}^{(2)} \rangle = \left( 1 + (-1)^{l_1 + l_2} i \right) \int dx \int dq_2'q_2'^{2} t_{\alpha_2'}(p_1'\pi_1(q_1'q_2x), E_{q_1'}) G_0(\pi_1(q_1'q_2x), q_1') \]
\[ \sum_{\alpha_2'} G_{\alpha_1',\alpha_2'}(q_1'q_2x) \left[ \int dy t_{\alpha_2''}(\pi_2(q_1'q_2x)\pi_3(q_2'q_1y), E_{q_2''}) G_0(\pi_3(q_2'q_1y), q_2') \right. \]
\[ \sum_{\alpha_1''} H_{\alpha_1'',\alpha_2''}(q_1'q_1y) t_{\alpha_1''}(\pi_4(q_2'q_1y)p_1, E_{q_1}) C_{\alpha_1''}^{m_2+m_3}(\theta_1) \]
\[- \int dy \left[ t_{\alpha_2''}(\pi_2(q_1'q_2x)\pi_5(q_2'q_2y), E_{q_2''}) G_0(\pi_5(q_2'q_2y), q_2') \right. \]
\[
\sum_{\alpha_1'' \alpha_2} I_{\alpha_1'' \alpha_2''}(q_2' q_2 y) t_{\alpha_2''}(\pi_6(q_2' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- t_{\alpha_2''}(\pi_5(q_2' q_2 y), E_{q_2}) G_0(\pi_5(q_2' q_2 y), q_2') \\
\sum_{\alpha_1'' \alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2' q_2 y) t_{\alpha_2''}(\pi_6(q_2' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1)
\]

Correspondingly one obtains
\[
\langle p_2' q_2' \alpha_1'' | U_{2,a}^{(2)} \rangle = \\
\sum_{\alpha_1''} \int dx \int dq_2' q_2' x t_{\alpha_2''}(p_2' \pi_3(q_2' q_1 x), E_{q_2'}) G_0(\pi_3(q_2' q_1 x), q_2') \\
H_{\alpha_2'' \alpha_1''}(q_2' q_1 x) \left(1 + (-1)^{l_1 + s_1}\right) \\
\int dy \left[t_{\alpha_2''}(\pi_4(q_2' q_1 x) \pi_1(q_1 q_2 y), E_{q_1'}) G_0(\pi_3(q_2' q_1 y), q_1') \right. \\
\sum_{\alpha_2''} G_{\alpha_1'' \alpha_2''}(q_1 q_2 y) t_{\alpha_2''}(\pi_2(q_2' q_1 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- t_{\alpha_2''}(\pi_4(q_2' q_1 x) \pi_1(q_1 q_2 y), E_{q_1'}) G_0(\pi_3(q_2' q_1 y), q_1') \\
\sum_{\alpha_2''} G_{\alpha_1'' \alpha_2''}(q_1 q_2 y) t_{\alpha_2''}(\pi_2(q_2' q_1 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- \int dx \int dq_2'' (q_2'' x^2 t_{\alpha_2''}(p_2'' \pi_5(q_2'' q_2, x), E_{q_2'}) G_0(\pi_5(q_2'' q_2, x), q_2') \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2) \left[\int dy t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_1 y), E_{q_2'}) G_0(\pi_3(q_2'' q_1 y), q_2'' \right. \\
\sum_{\alpha_2''} H_{\alpha_2'' \alpha_1''}(q_2'' q_1 y) t_{\alpha_2''}(\pi_4(q_2'' q_1 y)p_1, E_{q_1}) C_{\alpha_2''}^m(\theta_1) \\
- \int dy \left[t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_1 y), E_{q_2'}) G_0(\pi_3(q_2'' q_1 y), q_2'' \right. \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2 y) t_{\alpha_2''}(\pi_6(q_2'' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_2 y), E_{q_2'}) G_0(\pi_5(q_2'' q_2 y), q_2'') \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2 y) t_{\alpha_2''}(\pi_6(q_2'' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \right].
\]

All three-fold integrals in Eqs. 4.10 and 4.44 are of the same type: two angular integrations, where each one leads to logarithmic singularities in the corresponding momenta, one of which is external and the other the intermediate integration variable. It is not difficult to see that the intermediate momentum integration over products of logarithms leads to smooth functions in the external momenta. Therefore the third order amplitudes in \(t\) can serve as driving terms for the application of the kernels, and thus leading to all higher order amplitudes:

\[
\langle p_1' q_1' \alpha_1' | U_{1,a}^{(m)} \rangle = \\
\left(1 + (-1)^{l_1 + s_1}\right) \int dx \int dq_2' q_2' x t_{\alpha_2'}(p_1' \pi_1(q_1' q_2 x), E_{q_1'}) G_0(\pi_1(q_1' q_2 x), q_1') \\
\sum_{\alpha_2'} G_{\alpha_1' \alpha_2'}(q_1' q_2 x) \langle p_2' \pi_3(q_2' q_1 x), E_{q_2'} \rangle G_0(\pi_3(q_2' q_1 x), q_2') \\
\langle p_2' q_2' \alpha_1'' | U_{2,a}^{(m)} \rangle = \\
\int dx \int dq_2' q_2' x t_{\alpha_2'}(p_2' \pi_3(q_2' q_1 x), E_{q_2'}) G_0(\pi_3(q_2' q_1 x), q_2') \\
\sum_{\alpha_2'} H_{\alpha_2' \alpha_1''}(q_2' q_1 x) \langle p_3(q_2' q_1 x)q_1' \alpha_1'' | U_{1,a}^{(m-1)} \rangle \\
\int dy \left[t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_1 y), E_{q_2'}) G_0(\pi_3(q_2'' q_1 y), q_2'') \right. \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2 y) t_{\alpha_2''}(\pi_6(q_2'' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- \int dy \left[t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_1 y), E_{q_2'}) G_0(\pi_3(q_2'' q_1 y), q_2'') \right. \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2 y) t_{\alpha_2''}(\pi_6(q_2'' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \\
- t_{\alpha_2''}(\pi_6(q_2'' q_2 y) \pi_3(q_2'' q_2 y), E_{q_2'}) G_0(\pi_5(q_2'' q_2 y), q_2'') \\
\sum_{\alpha_2''} I_{\alpha_1'' \alpha_2''}(q_2'' q_2 y) t_{\alpha_2''}(\pi_6(q_2'' q_2 y)p_2, E_{q_2}) D_{\alpha_2''}^m(\theta_1) \right].
\]
Similarly, the second equation, Eq. (3.2) becomes

\[ - \int dx \int dq \frac{d}{dq^2} dq^2 \, t_{\alpha_1}(p_1 \pi_5(q_1^2 q_2^2 \, x), E_{q_1}) G_{0}(\pi_5(q_1^2 q_2^2 \, x), q_2) \]

\[ \sum_{\alpha_2} I_{\alpha_2}(\pi_6(q_1^2 q_2^2 \, x) q_2 \alpha_2 | U_{2,a}^{(n-1)} , \]

(4.8)

with \( n = 3, 4, \cdots \).

The resulting series \( \sum_{n=3}^{\infty} \langle p' | q' | U_{1,a}^{(n)} \rangle \) and \( \sum_{n=3}^{\infty} \langle p' | q' | U_{2,a}^{(n)} \rangle \) can safely be summed e.g. via Padé summation. For the corresponding three-nucleon amplitudes the above considerations were made in Ref. [38].

V. FINITE RANK FORCES

So far, Faddeev type studies of light nuclei treating the discrete structures have been based on finite rank forces [33]. Therefore, it appears useful to also formulate the \( n \alpha \) system in the continuum in this fashion. For the sake of a simple notation we choose a rank-1 separable t-matrix,

\[ t_{\alpha}(p p', E_q) = h_{\alpha}(p) \tau_{\alpha}(q) h_{\alpha}(p') . \]  

(5.1)

Then Eqs. (3.2) take the form

\[ \langle p'_1 q'_1 \alpha'_1 | U_{1,a} \rangle = \frac{\delta(q'_1 - q_1)}{q_1^2} \tau_{\alpha'_1}(E_{q_1}) h_{\alpha'_1}(p_1) C_{\alpha'_1\alpha_1}^{m_2+m_3}(\theta_1) \]

\[ + \left( 1 + (-1)^{\ell'_1 + s'_1} \right) \int dx \int dq'_2 q'_2 \, \tau_{\alpha'_2}(E_{q_1}) h_{\alpha'_2}(\pi_1(q'_1 q'_2 x)) \]

\[ G_{0}(\pi_1(q'_1 q'_2 x), q'_1) G_{\alpha'_1 \alpha'_2}(q'_1 q'_2 x) \langle \pi_2(q'_1 q'_2 x) q'_2 \alpha'_2 | U_{2,a} \rangle \]

\[ \equiv h_{\alpha'_1}(p'_1) Z_{\alpha'_1}(q'_1), \]  

(5.2)

where the new unknown single variable amplitude is

\[ Z_{\alpha'_1}(q'_1) = \frac{\delta(q'_1 - q_1)}{q_1^2} \tau_{\alpha'_1}(E_{q_1}) h_{\alpha'_1}(p_1) C_{\alpha'_1\alpha_1}^{m_2+m_3}(\theta_1) \]

\[ + \left( 1 + (-1)^{\ell'_1 + s'_1} \right) \int dx \int dq'_2 q'_2 \, \tau_{\alpha'_2}(E_{q_1}) h_{\alpha'_2}(\pi_1(q'_1 q'_2 x)) \]

\[ G_{0}(\pi_1(q'_1 q'_2 x), q'_1) G_{\alpha'_1 \alpha'_2}(q'_1 q'_2 x) \langle \pi_2(q'_1 q'_2 x) q'_2 \alpha'_2 | U_{2,a} \rangle . \]  

(5.3)

Similarly, the second equation, Eq. (3.2) becomes

\[ \langle p'_2 q'_2 \alpha'_2 | U_{2,a} \rangle = \frac{\delta(q'_2 - q_2)}{q_2^2} h_{\alpha'_2}(p'_2) \tau_{\alpha'_2}(E_{q_2}) h_{\alpha'_2}(p_2) D_{\alpha'_2}^{m_2,m_3}(\theta_1) \]

\[ + \sum_{\alpha'_2} \int dx \int dq'_2 q'_2 \, h_{\alpha'_2}(p'_2) \tau_{\alpha'_2}(E_{q_2}) h_{\alpha'_2}(\pi_3(q'_2 q'_1 x)) \]

\[ G_{0}(\pi_3(q'_2 q'_1 x), q'_2) H_{\alpha'_2 \alpha'_1}(q'_2 q'_1 x) \langle \pi_4(q'_2 q'_1 x) q'_1 \alpha'_1 | U_{1,a} \rangle \]

\[ - \sum_{\alpha'_2} \int dx \int dq'_2 q'_2 \, h_{\alpha'_2}(p'_2) \tau_{\alpha'_2}(E_{q_2}) h_{\alpha'_2}(\pi_5(q'_2 q'_2 x)) \]

\[ G_{0}(\pi_5(q'_2 q'_2 x), q'_2) I_{\alpha'_2 \alpha'_2'}(q'_2 q'_2 x) \langle \pi_6(q'_2 q'_2 x) q'_2 \alpha'_2 | U_{2,a} \rangle \]

\[ \equiv h_{\alpha'_2}(p'_2) V_{\alpha'_2}(q'_2) , \]  

(5.4)

with

\[ V_{\alpha'_2}(q'_2) = \frac{\delta(q'_2 - q_2)}{q_2^2} \tau_{\alpha'_2}(E_{q_2}) h_{\alpha'_2}(p_2) D_{\alpha'_2}^{m_2,m_3}(\theta_1) \]
and iterations exhibits the same features as discussed in detail in the previous section. Thus, we write

These two equations, Eqs. (5.6) and (5.7), form as set of coupled one-dimensional integral equations. The low order

Then we insert the functions $Z$ and $V$ under the integrals

and

These two equations, Eqs. (5.6) and (5.7), form as set of coupled one-dimensional integral equations. The low order

iterations exhibits the same features as discussed in detail in the previous section. Thus, we write

and

From Eqs. (5.6) and (5.7) we can read off the different orders. For the lowest order we obtain,

and

The second order is given by

\[ Z^{(1)}_{\alpha_1}(q'_1) = \left( 1 + (-1)^{i_1+i'_1} \right) \int dx \int dq'_1 dq'_2 \sum_{\alpha_2} \tau_{\alpha_2}(E_{q'_1}) h_{\alpha_2}(\pi_1(q'_1 q'_2 x)) \]
\begin{equation}
\alpha = \delta - \delta = \rho - \rho = \sum_{\alpha_1^i} \left( E_{\alpha_1^i}(q_1^i, q_2^i, x) \right) \tau_1 q_1^i \left( q_1^i, q_2^i, x \right) \ G_0(q_1^i, q_2^i, x, q_1^i) \\
\sum_{\alpha_2^j} G_{\alpha_1^i, \alpha_2^j}(q_1^i, q_2^i, x) \ h_{\alpha_1^i}(\tau_2 q_1^i, q_2^i, x) V_{\alpha_2^j}(q_1^i) \\
= \left( 1 + (-1)^{i_1 + i_2} \right) \int dx \int dq_1^i dq_2^i \sum_{\alpha_2^j} \tau_1 \left( E_{\alpha_1^i}(q_1^i, q_2^i, x) \right) \ G_0(q_1^i, q_2^i, x, q_1^i) \\
\sum_{\alpha_2^j} G_{\alpha_1^i, \alpha_2^j}(q_1^i, q_2^i, x) \ h_{\alpha_1^i}(\tau_2 q_1^i, q_2^i, x) \left[ \delta(q_1^i - q_2^i) \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \right] \\
- \frac{\delta(q_1^i - q_2^i)}{q_2^i} \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \right]
\end{equation}

and

\begin{equation}
V_{\alpha_2^j}(q_1^i) = \int dx \int dq_1^i dq_2^i \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \\
\sum_{\alpha_1^i} H_{\alpha_2^j, \alpha_1^i}(q_2^i, q_1^i) \ h_{\alpha_1^i}(\tau_3 q_2^i q_1^i, x) V_{\alpha_2^j}(q_1^i) \\
= \int dx \int dq_1^i dq_2^i \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \\
\sum_{\alpha_1^i} H_{\alpha_2^j, \alpha_1^i}(q_2^i, q_1^i) \ h_{\alpha_1^i}(\tau_3 q_2^i q_1^i, x) \frac{\delta(q_1^i - q_2^i)}{q_2^i} \tau_2 q_1^i(p_1) C_{\alpha_1^i}^{m_2, m_3}(\theta_1) \\
- \int dx \int dq_1^i dq_2^i \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \\
\sum_{\alpha_1^i} H_{\alpha_2^j, \alpha_1^i}(q_2^i, q_1^i) \ h_{\alpha_1^i}(\tau_3 q_2^i q_1^i, x) \left[ \delta(q_1^i - q_2^i) \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \right] \\
- \frac{\delta(q_1^i - q_2^i)}{q_2^i} \tau_2 q_1^i(p_2) D_{\alpha_2^j}^{m_2, m_3}(\theta_1) \right]
\end{equation}
\[- \int dx \left[ \tau_{a_2} (q_2') h_{a_2'} (\pi_5 (q_2' q_5 x), q_2) + \sum_{\alpha_2'} I_{\alpha_2'a_2'} (q_2' q_5 x) h_{\alpha_2'} (\pi_6 (q_2' q_5 x)) \tau_{\alpha_2'a_2'} (q_2') h_{a_2'} (p_2) D^{m_2, m_3}_{\alpha_2'} (\theta_1) \right. \\
\left. - \tau_{a_2'} (q_2') h_{a_2'} (\pi_5 (q_2' q_5 x)) G_0 (\pi_5 (q_2' q_5 x), q_2) + \sum_{\alpha_2'} I_{\alpha_2'a_2'} (q_2' \pi_5 x) h_{\alpha_2'} (\pi_6 (q_2' q_5 x)) \tau_{\alpha_2'a_2'} (q_2') h_{a_2'} (p_2) \tilde{D}^{m_2, m_3}_{\alpha_2'} (\theta_1) \right]\]

\[= \tau_{a_2'} (q_2') \sum_{\alpha_1} \tau_{\alpha_1} (q_1) h_{\alpha_1} (p_1) C_{\alpha_1'}^{m_2 + m_3} (\theta_1) + \int dx h_{a_2'} (\pi_3 (q_2' q_3)) G_0 (\pi_3 (q_2' q_3), q_2) H_{\alpha_2'a_2'} (q_2' q_3) h_{\alpha_1} (\pi_4 (q_2' x)) \]

\[- \tau_{a_2'} (q_2') \sum_{\alpha_2'} \left[ \tau_{\alpha_2'a_2'} (q_2) h_{\alpha_2'} (p_2) D^{m_2, m_3}_{\alpha_2'} (\theta_1) \right. \\
\left. - \int dx h_{a_2'} (\pi_5 (q_2' q_5 x)) G_0 (\pi_5 (q_2' q_5 x), q_2) I_{\alpha_2'a_2'} (q_2' q_5 x) h_{\alpha_2'} (\pi_6 (q_2' q_5 x)) \right. \\
\left. - \tau_{\alpha_2'a_2'} (q_2') h_{\alpha_2'} (\pi_6 (q_2' q_5 x)) G_0 (\pi_6 (q_2' q_5 x), q_2) I_{\alpha_2'a_2'} (q_2' q_5 x) \right] \quad \text{(5.13)} \]

As for general forces the x-integration leads to logarithmic singularities in the external momenta. The next order, however, gives smooth functions

\[Z_{a_1'}^{(2)} (q_1') = \left( 1 + (-1)^{l_1 + s_1} \right) \tau_{a_1'} (q_1') \int \left[ dq_2 q_2'^2 \sum_{a_2'} G_{a_1'a_2'} (q_1' q_2 x) h_{a_1'} (\pi_1 (q_1' q_2 x)) G_0 (\pi_1 (q_1' q_2 x), q_1') \right. \\
\left. \sum_{\alpha_1'} \tau_{\alpha_1'} (q_1) h_{\alpha_1'} (p_1) C_{\alpha_1'}^{m_2 + m_3} (\theta_1) \right. \\
\left. \int dy h_{a_2'} (\pi_3 (q_2' q_3 y)) G_0 (\pi_3 (q_2' q_3 y), q_2') H_{\alpha_2'a_2'} (q_2' q_3 y) h_{\alpha_1'} (\pi_4 (q_2' y)) \right. \\
\left. - \sum_{\alpha_2'} \left[ \tau_{\alpha_2'a_2'} (q_2) h_{\alpha_2'} (p_2) D^{m_2, m_3}_{\alpha_2'} (\theta_1) \right. \\
\left. \int dy h_{a_2'} (\pi_5 (q_2' q_5 y)) G_0 (\pi_5 (q_2' q_5 y), q_2') I_{\alpha_2'a_2'} (q_2' q_5 y) h_{\alpha_2'} (\pi_6 (q_2' y)) \right. \\
\left. - \tau_{\alpha_2'a_2'} (q_2') h_{\alpha_2'} (\pi_6 (q_2' y)) G_0 (\pi_6 (q_2' y), q_2') I_{\alpha_2'a_2'} (q_2' q_5 y) \right] \right] \quad \text{(5.14)} \]

and

\[V_{a_2'}^{(2)} (q_2') = \int \left[ dq_2 q_2'^2 \tau_{a_2'} (q_2') h_{a_2'} (\pi_3 (q_2' q_1' x)) G_0 (\pi_3 (q_2' q_1' x), q_2') \right. \\
\left. \sum_{\alpha_1'} \tau_{\alpha_1'} (q_1) h_{\alpha_1'} (p_1) C_{\alpha_1'}^{m_2 + m_3} (\theta_1) \right. \\
\left. \int dy h_{a_2'} (\pi_3 (q_2' q_3 y)) G_0 (\pi_3 (q_2' q_3 y), q_2') H_{\alpha_2'a_2'} (q_2' q_3 y) h_{\alpha_1'} (\pi_4 (q_2' y)) \right. \\
\left. - \sum_{\alpha_2'} \left[ \tau_{\alpha_2'a_2'} (q_2) h_{\alpha_2'} (p_2) D^{m_2, m_3}_{\alpha_2'} (\theta_1) \right. \\
\left. \int dy h_{a_2'} (\pi_5 (q_2' q_5 y)) G_0 (\pi_5 (q_2' q_5 y), q_2') I_{\alpha_2'a_2'} (q_2' q_5 y) h_{\alpha_2'} (\pi_6 (q_2' y)) \right. \\
\left. - \tau_{\alpha_2'a_2'} (q_2') h_{\alpha_2'} (\pi_6 (q_2' y)) G_0 (\pi_6 (q_2' y), q_2') I_{\alpha_2'a_2'} (q_2' q_5 y) \right] \right] \]
\[
\sum_{\alpha_1} H_{\alpha_2,\alpha_1}(q'_2 q'_1 x) \ h_{\alpha_1}(\pi_4(q'_2 q'_1 x)) \ Z_{\alpha_1}^{(1)}(q'_1)
\]
\[- \int dx \int dq'_2 \ G_0(\pi_5(q'_2 q'_2 x), q'_2) \ G_0(\pi_5(q'_2 q'_2 x), q'_2) \sum_{\alpha_2} I_{\alpha_2}^{\alpha_2}(q'_2 q'_2 x) \ h_{\alpha_2}(\pi_5(q'_2 q'_2 x)) \ V_{\alpha_2}^{(1)}(q'_2)
\]
\[= \tau_{\alpha_1}(q'_2) \int dq'_1 q'_1^2
\]
\[\int dx \ h_{\alpha_1}(\pi_3(q'_2 q'_1 x)) \ G_0(\pi_3(q'_2 q'_1 x), q'_2) \sum_{\alpha_1} H_{\alpha_2,\alpha_1}(q'_2 q'_1 x) \ h_{\alpha_1}(\pi_4(q'_2 q'_1 x))
\]
\[(1 + (-1)^{l_1 + l_2}) \ \tau_{\alpha_1}(q'_1) \sum_{\alpha_1} \tau_{\alpha_2}(q'_2) \ h_{\alpha_2}(p_2) \ D_{\alpha_2}^{m_2, m_3}(\theta_1)
\]
\[- \tau_{\alpha_2}(q'_2) \ h_{\alpha_2}(p_2) \ D_{\alpha_2}^{m_2, m_3}(\theta_1)
\]
\[- \tau_{\alpha_1}(q'_2) \int dq''_2 q''_2^2
\]
\[\int dx \ h_{\alpha_2}(\pi_5(q''_2 q''_2 x)) \ G_0(\pi_5(q''_2 q''_2 x), q''_2) \sum_{\alpha_2} I_{\alpha_2}^{\alpha_2}(q''_2 q''_2 x) \ h_{\alpha_2}(\pi_6(q''_2 q''_2 x))
\]
\[\left[ \begin{array}{c}
\tau_{\alpha_1}(q'_1) \ h_{\alpha_1}(p_1) \ C_{\alpha_1}^{m_2, m_3}(\theta_1) \\
\int dy \ h_{\alpha_2}(\pi_3(q'_2 q_1 y), q'_2) \ h_{\alpha_1}(\pi_4(q_2 y))
\end{array} \right]
\]
\[\left[ \begin{array}{c}
\int dy \ h_{\alpha_2}(\pi_3(q'_2 q_1 y), q'_2) \ h_{\alpha_1}(\pi_4(q_2 y))
\end{array} \right]
\]
\[- \tau_{\alpha_2}(q'_2) \int dq''_2 q''_2^2
\]
\[\int dx \ h_{\alpha_2}(\pi_5(q''_2 q''_2 y), q''_2) \sum_{\alpha_2} I_{\alpha_2}^{\alpha_2}(q''_2 q''_2 y) \ h_{\alpha_2}(\pi_6(q''_2 q''_2 y))
\]
\[- \tau_{\alpha_2}(q'_2) \int dq''_2 q''_2^2
\]
\[\int dx \ h_{\alpha_2}(\pi_5(q''_2 q''_2 y), q''_2) \sum_{\alpha_2} I_{\alpha_2}^{\alpha_2}(q''_2 q''_2 y) \ h_{\alpha_2}(\pi_6(q''_2 q''_2 y))
\]
\[h_{\alpha_2}(\pi_6(q''_2 q''_2 y))
\]
\[. \ (5.15)
\]

As shown in the previous section, there appear three-fold integrals. Two of them are over angles, leading to logarithmic singularities, and which are then eliminated by the third intermediate momentum integral. Thus we end up starting with \(n = 3\):

\[Z_{\alpha_1}^{(n)}(q'_1) =
\]
\[\left(1 + (-1)^{l_1 + l_1'}\right) \int dx \int dq'_2 q'_2^2 \ \tau_{\alpha_1}(q'_1) \ h_{\alpha_1}(\pi_1(q'_1 q'_2 x)) \ G_0(\pi_1(q'_1 q'_2 x), q'_1)
\]
\[\sum_{\alpha_2} G_{\alpha_2,\alpha_1}(q'_1 q'_2 x) \ h_{\alpha_2}(\pi_2(q'_1 q'_2 x)) \ V_{\alpha_2}^{(n-1)}(q'_2)
\]
\[= \left(1 + (-1)^{l_1 + l_1'}\right) \ \tau_{\alpha_1}(q'_1) \int dq'_2 q'_2^2 \ \int dx \ h_{\alpha_1}(\pi_1(q'_1 q'_2 x)) \ G_0(\pi_1(q'_1 q'_2 x), q'_1)
\]
\[\sum_{\alpha_2} G_{\alpha_2,\alpha_1}(q'_1 q'_2 x) \ h_{\alpha_2}(\pi_2(q'_1 q'_2 x)) \ V_{\alpha_2}^{(n-1)}(q'_2) . \ (5.16)
\]

Correspondingly,

\[V_{\alpha_2}^{(n)}(q'_2) =
\]
\[
\begin{align*}
\int dx \int dq_1'q_1'^2 \tau_{\alpha_2'}(q_2) h_{\alpha_2'}(\pi_3(q_2'q_1'^3 x)) G_0(\pi_3(q_2'q_1'^3 x),q_2) \\
\sum_{\alpha_1'} H_{\alpha_1',\alpha_1}(q_2'q_1'^3 x) h_{\alpha_1'}(\pi_4(q_2'q_1'^3 x)) Z_{\alpha_1'}^{(n-1)}(q_1') \\
- \int dx \int dq_2' q_2'^2 \tau_{\alpha_2''}(q_2) h_{\alpha_2''}(\pi_5(q_2'q_2'' x)) G_0(\pi_5(q_2'q_2'' x),q_2) \\
\sum_{\alpha_1''} I_{\alpha_1'',\alpha_2''}(q_2'q_2'' x) h_{\alpha_2''}(\pi_6(q_2''q_2'' x)) V_{\alpha_2''}^{(n-1)}(q_2''). 
\end{align*}
\]

Again, the singular integrals can be rewritten according to the method given in Appendix B.

VI. THE UNITARITY RELATIONS

The scattering states \(\Psi_0^{(+)}\) depend on the initial state quantum numbers

\[
\Psi_0^{(+)} \equiv \Psi_{p_1q_1,m_2m_3}^{(0)} \tag{6.1}
\]

like the initial state

\[
\Phi_{0,a} \equiv \Phi_{p_1q_1,m_2m_3}^{0} \tag{6.2}
\]

Using the full Green's operator, \(G \equiv (E + i\epsilon - H)^{-1}\), to the full Hamiltonian, \(\Psi_{p_1q_1,m_2m_3}^{(0)}\) obeys the equation

\[
\Psi_{p_1q_1,m_2m_3}^{(+) = \Phi_{p_1q_1,m_2m_3}^{0} + GV\Phi_{p_1q_1,m_2m_3}^{0} \cdot \tag{6.3}
\]

A second scattering state is defined by

\[
\Psi_{p_1q_1,m_2m_3}^{(-)} = \Phi_{p_1q_1,m_2m_3}^{0} + G^*V\Phi_{p_1q_1,m_2m_3}^{0} \cdot \tag{6.4}
\]

Both are related to each other as

\[
\Psi_{p_1q_1,m_2m_3}^{(-)} = \Psi_{p_1q_1,m_2m_3}^{(+)} + (G^* - G)V\Phi_{p_1q_1,m_2m_3}^{0} \tag{6.5}
\]

The S-matrix is defined as

\[
S_{p_1'q_1',p_1q_1}^{m_1',m_2m_2} = \langle \Psi_{p_1'q_1',m_1'm_3}|\Psi_{p_1q_1,m_2m_3}^{(+)} \rangle. \tag{6.6}
\]

Inserting Eq. (6.5) leads to

\[
S_{p_1'q_1',p_1q_1}^{m_1',m_2m_2} = \langle \Psi_{p_1'q_1',m_1'm_3}|\Psi_{p_1q_1,m_2m_3}^{(+)} \rangle \\
-2\pi i\delta(E' - E)\langle \Phi_{p_1q_1,m_2m_3}^{0}|V|\Psi_{p_1q_1,m_2m_3}^{(+)} \rangle. \tag{6.7}
\]

Now, we have due to general considerations

\[
\langle \Psi_{p_1'q_1',m_1'm_3}|\Psi_{p_1q_1,m_2m_3}^{(+)} \rangle = \langle \Phi_{p_1q_1,m_2m_3}^{0}|\Phi_{p_1q_1,m_2m_3}^{0} \rangle \tag{6.8}
\]

Consequently,

\[
S_{p_1'q_1',p_1q_1}^{m_1',m_2m_2} = \langle \Phi_{p_1'q_1',m_1'm_3}|\Phi_{p_1q_1,m_2m_3}^{0} \rangle.
The dependence on $S$

For the last equation we used the definition of the transition operator $U^{00}$.

Since the scattering states in the definition of $S$ belong to the same Hamiltonian one has to have

$$\delta_{\sigma'\sigma}^{m_1' m_2 m_3'} \equiv \delta_{\sigma'\sigma}^{m_1' m_2 m_3'} \delta(E' - E),$$

where $E = \frac{E_p^2}{m} + \frac{q^2}{2m}$ (setting the $\alpha$-particle as spectator), and correspondingly as similar expression for $E'$. The unitarity relation simply follows from the completeness relation spanned by the scattering states:

$$\sum_{m_2' m_3'} |\psi_{\sigma'}^{(+)}(q_1', m_2', m_3')\rangle = \sum_{m_2' m_3'} \int d^3 p_1 d^3 q_1' |\psi_{\sigma}^{(+)}(p_1', m_2', m_3')\rangle |\psi_{\sigma'}^{(-)}(p_1', m_2', m_3')\rangle |\psi_{\sigma'}^{(+)}(q_1', m_2', m_3')\rangle,$$

or in terms of the $S$-matrix elements

$$\langle \Phi^0_{p_1, q_1, m_1} | \Phi^0_{p_1, q_1, m_2, m_3} \rangle = \sum_{m_2,m_3} \int d^3 p_1 d^3 q_1 |S_{p_1, q_1, m_1'}| S_{p_1, q_1, m_1''} \delta_{m_1', m_2} \delta_{m_1'', m_3}.$$

This can be rewritten in terms of the matrix elements of $U^{00}$. Using the completeness relation,

$$\sum_{m_2', m_3'} \int d^3 p_1 d^3 q_1 |\Phi^0_{p_1, q_1, m_2', m_3'}\rangle \langle \Phi^0_{p_1, q_1, m_2, m_3'}| = 1,$$

and

$$\delta(E'' - E') \langle \Phi^0_{p_1', q_1', m_2, m_3', m_2' m_3} | U^{00} | \Phi^0_{p_1, q_1, m_2, m_3} \rangle =$$

$$\langle \Phi^0_{p_1', q_1', m_2, m_3'} | \delta(H_0 - E') U^{00} | \Phi^0_{p_1, q_1, m_2, m_3} \rangle,$$

leads to

$$\delta(E' - E') \left[ i \langle \Phi^0_{p_1, q_1, m_2, m_3} | U^{00} | \Phi^0_{p_1', q_1', m_2, m_3'} \rangle^* - i \langle \Phi^0_{p_1', q_1', m_2, m_3'} | U^{00} | \Phi^0_{p_1, q_1, m_2, m_3} \rangle \right]$$

$$+ \frac{2\pi}{\sqrt{2}} \sum_{m_2, m_3} \int d^3 p_1 d^3 q_1 \delta(E'' - E) =$$

$$\langle \Phi^0_{p_1', q_1', m_2, m_3'} | U^{00} | \Phi^0_{p_1, q_1, m_2, m_3} \rangle^* \langle \Phi^0_{p_1, q_1, m_2, m_3} | U^{00} | \Phi^0_{p_1', q_1', m_2, m_3} \rangle = 0.$$ (6.15)

More interesting is the partial wave decomposed version for the on shell matrix element

$$\langle p_1', q_1, \alpha_1 | p_1 q_1, \alpha_1 \rangle = U^{00}_{\alpha_1', \alpha_1} (p_1' p_1) \delta_{J', J} \delta_{M_1', M_1}.$$ (6.16)

The dependence on $p_1', p_1$ is sufficient since on shell $q_1 = \sqrt{2M_1(E - \frac{E_p^2}{m})}, q_1' = \sqrt{2M_1(E' - \frac{E_p^2}{m})}$.

As shown in the Appendix the matrix element $U^{00}_{\alpha_1', \alpha_1} (p_1', p_1)$ obeys the unitarity relation

$$i U^{00}_{\alpha_1' \alpha_1'} (p_1, p_1') - i U^{00}_{\alpha_1' \alpha_1} (p_1', p_1)$$

$$+ 2\pi \sum_{\alpha_1} \int \delta(E'' - E) U^{00}_{\alpha_1' \alpha_1} (p_1', p_1') U^{00}_{\alpha_1' \alpha_1} (p_1', p_1) = 0.$$ (6.17)

The corresponding relation for the partial wave projected $S$-matrix element is

$$\sum_{\alpha_1', \alpha_1} \int \delta(E'' - E) \delta(E'' - E') S_{\alpha_1' \alpha_1'} (p_1, p_1') S_{\alpha_1' \alpha_1} (p_1, p_1) = \delta_{\alpha_1' \alpha_1} \frac{\delta(p_1 - p_1')\delta(q_1 - q_1')}{p_1^2 q_1^2}.$$ (6.18)

Note that not only discrete quantum numbers span the columns and rows of the $S$-matrix but also the continuous quantum numbers $p_1', p_1$ which describe how the energy is continuously distributed among the two relative motions.
VII. THE CAPTURE PROCESS N+N+α → ^6HE

The matrix element for the capture process is simply related to the time reversed photo disintegration process of ^6He into three free particles. It is well known how to treat photodisintegration of ^3He in the Faddeev scheme. In essentially the same manner one can formulate photodisintegration of ^6He based on an effective three-particle picture. Let O be the photon absorption operator and |Ψ_{^6He}\rangle the ^6He ground state. The break up amplitude into nnα can then be written as an infinite series of processes

\[
\langle Φ_{0,a}|U_0|Ψ_{^6He}\rangle = \langle Φ_{0,a}|O|Ψ_{^6He}\rangle + \sum_i (Φ_{0,a}|V_iG_0 O|Ψ_{^6He}\rangle + \cdots .
\] (7.1)

Here \(V_i\) are the pair forces among the \(nn\) and \(nα\) particles, and \(G_0\) is the free propagator. This infinite series in terms of pair forces represents final state interactions (FSI). The first term is the direct break up process generated by \(O\).

Let us define

\[
\langle Φ_{0,a}|U_0|Ψ_{^6He}\rangle = \langle Φ_{0,a}|O|Ψ_{^6He}\rangle + \sum_i \langle Φ_{0,a}|U_0|Ψ_{^6He}\rangle,
\] (7.2)

where \(U_{0i}\) comprises all terms with \(V_i\) to the very left:

\[
U_{0i}|Ψ_{^6He}\rangle ≡ V_iG_0 O|Ψ_{^6He}\rangle + V_i \sum_j G_0 V_j G_0 O|Ψ_{^6He}\rangle + \cdots .
\] (7.3)

Clearly this can be summed up as

\[
U_{0i}|Ψ_{^6He}\rangle = V_i G_0 O|Ψ_{^6He}\rangle + V_i G_0 \sum_j U_{0j}|Ψ_{^6He}\rangle.
\] (7.4)

Separating the terms \(U_{0i}|Ψ_{^6He}\rangle\) to the left and introducing the \(t\)-matrices \(t_i\) leads to three coupled Faddeev equations

\[
U_{0i}|Ψ_{^6He}\rangle = t_i G_0 O|Ψ_{^6He}\rangle + t_i G_0 \sum_{j \neq i} U_{0j}|Ψ_{^6He}\rangle.
\] (7.5)

The photon absorption operator \(O\) has to be symmetric under the exchange of the two neutrons, which we number as particles 2 and 3. Thus, using the antisymmetry of \(|Ψ_{^6He}\rangle\) with respect to the two neutrons, one finds

\[
P_{23} U_{02}|Ψ_{^6He}\rangle = -U_{03}|Ψ_{^6He}\rangle.
\] (7.6)

This leads to the two coupled equations

\[
\begin{align*}
U_{01}|Ψ_{^6He}\rangle &= t_1 G_0 O|Ψ_{^6He}\rangle + t_1 G_0 (1 - P_{23} U_{02})|Ψ_{^6He}\rangle, \\
U_{02}|Ψ_{^6He}\rangle &= t_2 G_0 O|Ψ_{^6He}\rangle + t_2 G_0 (U_{01} - P_{23} U_{02})|Ψ_{^6He}\rangle,
\end{align*}
\] (7.7)

corresponding to Eqs. [2,4,12] from Sect. [11].

The complete break up amplitude is given by

\[
\langle Φ_{0,a}|U_0|Ψ_{^6He}\rangle = \langle Φ_{0,a}|O|Ψ_{^6He}\rangle + \langle Φ_{0,a}|U_{01}|Ψ_{^6He}\rangle + \langle Φ_{0,a}|(1 - P_{23}) U_{02}|Ψ_{^6He}\rangle.
\] (7.8)

Using adequate pair forces and photon absorption operators (single particle currents, two-body currents and possibly beyond) these coupled equations can be solved by standard techniques [10].

VIII. SUMMARY

The structure inherent in the continuum states of the \(n+n+α\) system has so far only been explored in the framework of the HH approach [7, 8, 26, 28, 32]. There are strong initial and final state interaction peaks, not only in the \(nn\)
subsystem but also in the $n - \alpha$ subsystem. This poses a still unsolved challenge for the expansion into the discrete set of K-harmonics as already known for the $n + d \to n + n + p$ system. This is pointed out by the authors of Ref. [7], who note, that even for a maximum $K_{\text{max}} = 20$ in their calculation, the result is not completely converged.

A corresponding investigation in the Faddeev approach is still missing. The aim of this paper is to lay the formal groundwork to do so.

In the Faddeev approach all the structures in the relative motions of the three particles are mapped out correctly, thus leading to a reliable path to the three-to-three scattering $S$-matrix, which contains the information of the resonance structure of the $^6\text{He}$ system.

We derived two coupled Faddeev equations for the three-to-three scattering amplitudes. In a partial wave decomposed representation they form a system of two-dimensional coupled equations for each fixed total angular momentum. The multiple scattering series being arranged in powers in the two-body $t$-matrices is the natural starting point for the solution of this coupled system of integral equations. The term linear in the $t$-matrices is disconnected. The next term, second order in $t$, has well established logarithmic singularities in the external momenta. Only the term of third order in $t$ is a smooth function of the external momenta and thus can serve as driving term for the consecutive application of the Faddeev kernels. This provides all higher order terms which can then summed up by Padé.

Since up to now nearly all Faddeev based investigations of the discrete structure of the $n + n + \alpha$ system are based on finite rank forces we also derived the continuum equations using this type of forces. The unitarity relations are especially interesting since the rows and columns of the $S$-matrix are not only numbered by discrete quantum numbers but also by continuously varying on-shell momenta.

Finally we provided Faddeev equations for the $n + n + \alpha$ capture process to the $^6\text{He}$ ground state. We pointed out that it is not necessary here to first evaluate the three-to-three wave function as has been done in Ref. [38], and that one directly can use a Faddeev form for the entire break-up amplitude with no disconnected terms as is done in modern calculations [39, 40], and as it was pioneered in [36, 37].

This capture process is relevant for the production rate of $^6\text{He}$ in astrophysical environments [41] characterized by high neutron and alpha densities, e.g. those related to supernova shock fronts. In Ref. [21] this three-body process is approximated by sequential two-body processes, whereas in principle a genuine three-body reactions needs to be calculated. Very recently it has been pointed out [41] that currently employed two-step mechanisms over intermediate resonances in the three-to-three scattering of the $n + n + \alpha$ system are most likely insufficient, since the time delays for those intermediate steps are comparable to the duration of the entire process. This strongly supports the need for the approach we present in this paper.

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Appendix A: Partial Wave Decomposition in the $nna$ System

Partial wave decomposition of three-body wave functions have often been documented, see for instance [39] and focus on the $nna$- system [40]. Thus we are here relatively brief.

The projection of the free state, Eq. (2.13) onto the partial wave basis states of Eq. (3.1) is given by

$$\langle p_1 q_1 \alpha_1 | \Phi_{p_1 q_1 m_{232}} \rangle = (1 + (-)^{s_1 + s_2}) \sum_{l_1 \lambda_1 J_1, \mu, M' - \mu} \left( \frac{1}{2} \gamma^{l_1 s_1' m_{232}} \right) \delta(p_1' - p_1) \delta(q_1'' - q_1) \frac{1}{q_1''} Y_{l_1 \lambda_1 J_1}^* \left( \hat{p}_1 \right) Y_{\lambda_1 M' - \mu}^* \left( \hat{q}_1 \right).$$

This leads to

$$\langle \tilde{p}_1 \tilde{q}_1 \alpha_1 | t_1 | \Phi_{p_1 q_1 m_{232}} \rangle = \int dp_1'' p_1'' t_{\alpha_1''} (p_1'' q_1'') \left( \frac{1}{2} \gamma^{l_1 s_1' m_{232}} \right) \delta(p_1' - p_1) \delta(q_1'' - q_1) \frac{1}{q_1''} Y_{l_1 \lambda_1 J_1}^* \left( \hat{p}_1 \right) Y_{\lambda_1 M' - \mu}^* \left( \hat{q}_1 \right) = \delta(q_1'' - q_1) \frac{t_{\alpha_1''}}{q_1''} (p_1, p_1', E_{q_1}) C_{\alpha_1''}^{m_{232} + m_3} (\hat{p}_1, \hat{q}_1).$$

and gives the driving term of Eq. (3.2) together with the amplitude $C$ given in Eq. (3.3).

In case of the second driving term in Eq. (2.14) it is adequate to rewrite the free state $| \Phi_{p_1 q_1 m_{232}} \rangle$ in terms of the Jacobi momenta of the type 2 (where the neutron is the spectator):

$$p_2 = -\beta p_1 - \gamma q_1,$$
$$q_2 = p_1 + \alpha q_1,$$

with

$$\alpha = \frac{1}{2} m_\alpha,$$
$$\beta = \frac{m_\alpha}{m + m_\alpha},$$
$$\gamma = \frac{m + m_\alpha}{2(m + m_\alpha)},$$

and $\alpha \beta + \gamma = 1$.

Then,

$$| \Phi_{p_1 q_1 m_{232}} \rangle = | p_2 q_2 \rangle | m_2 \rangle | m_3 \rangle_2 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) - | \tilde{p}_2 \tilde{q}_2 \rangle | 0 \rangle | m_2 \rangle | m_3 \rangle_2$$

$$| 0 \rangle \left( \frac{1}{2} \right) \left( \frac{1}{2} \right).$$

where

$$\tilde{p}_2 = \beta p_1 - \gamma q_1,$$
$$\tilde{q}_2 = -p_1 - \alpha q_1.$$
\[
\sum_{\mu}(j_1^2 J_2 J', \mu M' - \mu) (l_1^2 j_2 l', \mu - m_3, m_3) Y_{j_1, \mu-m_3}^{l_1} (\hat{p}_2)
\]
\[
(\lambda_2^2 l_2^2, M' - \mu - m_2, m_2) Y_{j_1, \mu-m_2}^{l_1} (\hat{q}_2)
\]
\[
\delta(p_2 - \hat{p}_2) \delta(q_2' - \hat{q}_2)
\]
\[
\sum_{\mu}(j_1^2 J_2 J', \mu M' - \mu) (l_1^2 j_2 l', \mu - m_2, m_2) Y_{j_1, \mu-m_2}^{l_1} (\hat{q}_2)
\]
\[
(\lambda_2^2 l_2^2, M' - \mu - m_3, m_3) Y_{j_1, \mu-m_3}^{l_1} (\hat{q}_2),
\]
and the second driving term becomes
\[
\langle p_2' q_2' \alpha_2' | t_2 | \Phi_{p_1 q_1 m_2 m_3} \rangle \equiv \frac{\delta(q_2' - q_2)}{q_2^2} \int t_{\alpha_2'} (p_2' p_2, E q_2) D_{\alpha_2'}^{m_2 m_3} (\theta_{p_2} \theta_{q_2})
\]
\[
- \frac{\delta(q_2' - q_2)}{q_2^2} \int t_{\alpha_2'} (p_2' p_2, E q_2) D_{\alpha_2'}^{m_2 m_3} (\theta_{p_2} \theta_{q_2}),
\]
with \(D\) and \(\tilde{D}\) given in Eqs. (3.4) and (3.5). For the kernel pieces we refer to [46].

**Appendix B: Avoiding logarithmic singularities in the integrals**

We illustrate the new manner to rewrite the Faddeev kernel such that only a single pole singularity appears in an example (for more details see [48, 49]). Consider the first kernel in Eq. (3.2), of which the first piece can be rewritten as
\[
\int dx \int dq_2' \sum_{\alpha_2'} dp_1' P_2' \delta(p_1' - \pi_1(q_1' q_2')) t_{\alpha_2'} (p_1' p_1, E q_1') G_0 (p_1', q_1')
\]
\[
G_{\alpha_1, \alpha_2'} (q_1' q_2') \int dp_2' P_2' \delta(p_2' - \pi_2(q_1' q_2')) \langle p_2' q_2' \alpha_2' | U_{2,a} \rangle.
\]

The two \(\delta\)-functions are then changed according to
\[
\delta(p_1' - \pi_1(q_1' q_2')) = \frac{2p_1''}{2\alpha q_1 q_2} \delta(x - x_0) \Theta(1 - |x_0|)
\]
\[
\delta(p_2' - \pi_2(q_1' q_2')) = \delta \left( p_2' - \sqrt{\gamma q_1'^2 + \frac{\beta}{\alpha} p_2'^2 - \frac{\beta}{\alpha} q_2'^2} \right)
\]
\[
\Theta \left( \gamma q_1'^2 + \frac{\beta}{\alpha} p_2'^2 - \frac{\beta}{\alpha} q_2'^2 \right),
\]
with
\[
x_0 = \frac{p_1'' - \alpha q_1^2 - q_2'^2}{2\alpha q_1^2 q_2'^2}.
\]

Inserting this into Eq. (B1) leads to
\[
\int dx \int dq_2' \sum_{\alpha_2'} dp_1' P_2' \delta(x - x_0) \Theta(1 - |x_0|) t_{\alpha_2'} (p_1' p_1, E q_1')
\]
\[
G_0 (p_1', q_1') G_{\alpha_1, \alpha_2'} (q_1' q_2') \int dp_2' P_2' \delta \left( p_2' - \sqrt{\gamma q_1'^2 + \frac{\beta}{\alpha} p_2'^2 - \frac{\beta}{\alpha} q_2'^2} \right)
\]
\[
\Theta \left( \gamma q_1'^2 + \frac{\beta}{\alpha} p_2'^2 - \frac{\beta}{\alpha} q_2'^2 \right) \langle p_2' q_2' \alpha_2' | U_{2,a} \rangle.
\]
The singularity in Eq. (6.15) yields

$$G_{\alpha'\alpha''}(q^2_1 q^2_2 x_0)\left(\sqrt{\gamma q^2_1 + \frac{2\gamma}{\alpha q^2_2} q^2_2 q^2_2} \right) U_{2,0} \Theta \left(1 - \left|\frac{p^\gamma_1 - \alpha q_2^2 - q^2_2}{2 \alpha q^2_1 q^2_2}\right|\right) \Theta \left(\gamma q^2_1 + \frac{\beta}{\alpha q^2_2} q^2_2 - \frac{\beta^2}{\alpha q^2_1} q^2_2\right).$$

(B4)

The two $\Theta$-functions define the domain D for the integrations over $p^\gamma_1$ and $q^2_2$. Thus we end up with

$$\frac{1}{\alpha q^2_1} \int dp^\gamma_1 \frac{\alpha}{p^\gamma_1} t_{\alpha_1}(p^\gamma_1 p^\gamma_2, E_{q_1}) \left(1 + \frac{\beta}{\alpha q^2_2} p^\gamma_2 - \frac{\beta^2}{\alpha q^2_1} q_2^2\right) \left|U_{2,0}\right|^2.$$

(B5)

The singularity in $G_0(p^\gamma_1, q^2_1)$ is now a single pole in $p^\gamma_1$ for a given $q^2_1$. This type of singularity does not pose any numerical problem and can be implemented with standard techniques. Note that for $q^2_1 \geq 2M_1 E$ there is no pole and one might as well keep the original form.

### Appendix C: Partial Wave Decomposed Transition Amplitude

The definition of the partial wave decomposed transition amplitude is

$$(\Phi_{p_1 q_1, m_1^2 m_3^2} | U^{00} | \Phi_{p_1 q_1, m_2^2 m_2^2}) \equiv$$

$$\sum_{\alpha_1} \int dp^\gamma_1 \frac{\beta}{\alpha q^2_2} dp^\gamma_2 \left(\Phi_{p_1 q_1, m_1^2 m_3^2} \Phi_{p_1 q_1, m_2^2 m_2^2}\right) \langle p^\gamma_1 q^2_1 \alpha_1 | U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle$$

(C1)

This inserted into Eq. (6.15) yields

$$\delta(E - E') \sum_{\alpha_1} \int \sum_{\alpha_2} \int \langle p^\gamma_1 q^2_1 \alpha_1 | \Phi_{p_1 q_1, m_2^2 m_2^2} \rangle (\Phi_{p_1 q_1, m_2^2 m_3^2} \Phi_{p_1 q_1, m_2^2 m_2^2}) \langle p^\gamma_1 q^2_1 \alpha_1 \rangle$$

$$(i \langle p^\gamma_1 q^2_1 \alpha_1 | U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^* - i \langle p^\gamma_1 q^2_1 \alpha_1 | U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle$$

$$+ 2\pi \sum_{m_2^2 m_3^2} \int d^3 p^\gamma_1 d^3 q^2_1 \delta(E'' - E)$$

$$\sum_{\alpha_1} \int \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle^* \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle \langle U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^*$$

$$\sum_{\alpha_1} \int \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle \langle U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^*$$

$$= 0,$$

(C2)

which, using the completeness relation,

$$\sum_{m_2^2 m_3^2} \int d^3 p^\gamma_1 d^3 q^2_1 \delta(E'' - E) \langle \Phi'' | p^\gamma_1 q^2_1 \alpha_1 \rangle^* \langle \Phi'' | p^\gamma_1 q^2_1 \alpha_1 \rangle^* =$$

$$\frac{\delta(p^\gamma_1 - \tilde{p}_1)}{(p^\gamma_1)^2} \delta(q^2_1 - \tilde{q}_1) \delta_{\alpha_1'' \alpha_1''} \delta(E'' - E),$$

(C3)

leads to

$$\delta(E - E') \sum_{\alpha_1} \int \sum_{\alpha_1} \int \langle p^\gamma_1 q^2_1 \alpha_1 | \Phi_{p_1 q_1, m_2^2 m_2^2} \rangle (\Phi_{p_1 q_1, m_2^2 m_2^2} \Phi_{p_1 q_1, m_2^2 m_2^2}) \langle p^\gamma_1 q^2_1 \alpha_1 \rangle$$

$$(i \langle p^\gamma_1 q^2_1 \alpha_1 | U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^* - i \langle p^\gamma_1 q^2_1 \alpha_1 | U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle$$

$$+ 2\pi \sum_{m_2^2 m_3^2} \int d^3 p^\gamma_1 d^3 q^2_1 \delta(E'' - E)$$

$$\sum_{\alpha_1} \int \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle^* \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle \langle U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^*$$

$$\sum_{\alpha_1} \int \langle \Phi'' \mid p^\gamma_1 q^2_1 \alpha_1 \rangle \langle U^{00} | p^\gamma_1 q^2_1 \alpha_1 \rangle^*$$

$$= 0.$$
\[
\begin{aligned}
&\left[ i\langle \hat{p}_1 \hat{q}_1 \hat{\alpha}_1 | U^{00} | \hat{p}_1' \hat{q}_1' \hat{\alpha}_1 ' \rangle^* - i\langle \hat{p}_1' \hat{q}_1' \hat{\alpha}_1 ' | U^{00} | \hat{p}_1 \hat{q}_1 \hat{\alpha}_1 \rangle \right] \\
&+ 2\pi \sum_{\hat{\alpha}_1 ''} \int \delta(E'' - E) \langle \hat{p}_1 '' \hat{q}_1 '' \hat{\alpha}_1 '' | U^{00} | \hat{p}_1' \hat{q}_1' \hat{\alpha}_1 ' \rangle^* \\
&\{ \hat{p}_1'' \hat{q}_1'' \hat{\alpha}_1'' | U^{00} | \hat{p}_1' \hat{q}_1' \hat{\alpha}_1' \} \\
&= 0.
\end{aligned}
\]  

(C4)

Then using Eqs. (A1) and (6.16), the orthogonality of the spherical harmonics and of the Clebsch-Gordan coefficients, one can project onto the on s-shell unitarity relation of Eq. (6.17).

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FIG. 1: Diagrammatic representation of the first few terms of the multiple scattering series for the neutron-neutron-\(\alpha\) system. Here the alpha particle (1) is indicated by the thicker line.