Are the Muonic Hydrogen and Electron Scattering Experiments Measuring the Same Observable?

T.W. Donnelly, D.K. Hasell, R.G. Milner

Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics
Massachusetts Institute of Technology, Cambridge, MA 02139

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Abstract. Elastic scattering of relativistic electrons from the nucleon yields Lorentz invariant form factors that describe the fundamental distribution of charge and magnetism. The spatial dependence of the nucleon’s charge and magnetism is typically interpreted in the Breit reference frame which is related by a Lorentz boost from the laboratory frame, where the nucleon is at rest. We construct a model to estimate how the Sachs electric and magnetic form factors can be corrected for the effects of relativistic recoil. When the corrections are applied, the ratio of the proton’s Sachs form factors is approximately flat with $Q^2$, i.e. the spatial distributions of the proton’s intrinsic charge and magnetization are similar. Further, we estimate the correction due to recoil that must be applied to the determination of the proton charge radius from elastic electron scattering before it can be compared to the value determined using the Lamb shift in hydrogen. Application of the correction brings the two values of the proton charge radius into significantly closer agreement. Predicted corrections based on the model are provided for the rms charge radii of the deuteron, the triton, and the helium isotopes.

Key words. proton radius – proton form factors – elastic electron scattering – Lamb shift – few body nuclei

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Of great current interest is the proton radius puzzle, namely that the charge radius of the proton as determined from precision elastic electron-proton scattering \cite{1} disagrees with a high precision determination obtained from the Lamb shift in muonic hydrogen \cite{2, 3}. This discrepancy has sparked considerable interest \cite{4, 5}. In this Letter we question the fundamental and widely-accepted ansatz that they are measuring the same quantity.

Quantum electrodynamics describes the energy levels of hydrogen (both electronic and muonic) with great accuracy. In particular, the energies of $S$-states in hydrogen are given by

$$E(nS) \approx -\frac{R_{\infty}}{n^2} + \frac{L_{1S}}{n^3},$$

where $n$ is the principal quantum number and $L_{1S}$ denotes the Lamb shift of the $1S$ ground state, which depends on the proton charge radius $r_p$. For electronic hydrogen, $L_{1S} \approx (8.712 + 1.56r_p^2)$ MHz when $r_p$ is expressed in femtometers. Thus, the finite size effect is of order 1.2 MHz. We emphasize that this analysis of the hydrogen energy levels is carried out in coordinate space, i.e. the hydrogen atom is solved exactly (fully relativistically) using the Dirac equation. We note that recent work \cite{6} in hydrogen spectroscopy reports a value of the Rydberg constant in tension with the world’s data.

Recent technical advances have made stopped muon beams of unprecedented intensity and quality available. These have allowed precision measurements of the Lamb shift in muonic hydrogen which have yielded a determination of the proton charge radius of 0.84087(39) fm.

Consider relativistic elastic electron scattering from the nucleon. In single photon exchange approximation, the unpolarized elastic $eN$ scattering cross section in the lab. system (nucleon at rest) can be written

$$\frac{d\sigma}{d\Omega} = \sigma_M f_{\text{rec}}^{-1} \cdot \frac{1}{(1+\tau)} \left[\epsilon G_E^2 + \tau G_M^2\right],$$

where the Mott cross section is

$$\sigma_M = \left[\frac{\alpha \cos \theta / 2}{2E_e \sin^2 \theta / 2}\right]^2,$$

$G_E(Q^2)$ and $G_M(Q^2)$ are the nucleon electric and magnetic Sachs form factors, $e^{-1} = 1 + 2(1 + \tau)\tan^2 \theta / 2$, $\tau = Q^2 / 4m_N^2 \geq 0$ and the recoil factor is $f_{\text{rec}} = E_e / E_e'$. Here the incident electron has energy $E_e$, the scattered electron has energy $E_e'$ and the scattering is through angle...
\( \theta \); also \( \alpha \) is the fine-structure constant. In all expressions, the extreme relativistic limit is taken, namely the electron mass is ignored with respect to its energy.

In analysis of elastic electric-proton scattering data, the proton charge radius (denoted \( r_{E_p}^{\text{scatt}} \)) is determined through the derivative of \( G_E^p(Q^2) \) with respect to the invariant 4-momentum transfer, \( Q^2 \), namely via

\[
-6 \frac{dG_E^p(Q^2)}{dQ^2} \bigg|_{Q^2=0} \equiv \left( r_{E,p}^{\text{scatt}} \right)^2 . \tag{4}
\]

Since both \( G_E^p(Q^2) \) and \( Q^2 \) are Lorentz invariants, the quantity \( r_{E,p}^{\text{scatt}} \) is also, and we point out that this type of proton charge radius is determined in momentum space. It is not the RMS charge radius of the proton, \( r_p \), above in coordinate space, is different from that extracted via electron scattering \( r_{E,p}^{\text{scatt}} \) in momentum space. While \( r_p \) is determined using an essentially static proton, \( r_{E,p}^{\text{scatt}} \) involves a process where the proton must recoil after absorbing the momentum transfer from the exchanged virtual photon.

We stress that this last point was recognized in the earliest work on electron scattering. For example, Yennie et al. pointed out [7] in 1957 that there would be a \( Q^2 \) dependence of the form factors determined in electron scattering which was independent of structure and “which would be kinematic in origin.” Fundamentally, this arises from the relativistic recoil which is unavoidable and means that “inuitive concepts of static charge and current distributions are no longer valid” [7]. We will show that, at the precision demanded by the proton charge radius comparison, these kinematic effects are non-negligible.

Consider the Breit reference frame, defined by zero energy transfer of the virtual photon and reversal of the 3-momentum of the target between the initial and final states. Thus, one has \( \omega_B = 0 \) and \( q_B = \sqrt{Q^2} \). Since in that frame (with \( z \)-axis along the momentum transfer vector) one has the nucleon entering with 3-momentum \(-p_B\) and leaving with 3-momentum \( +p_B\), one has \( p_B = q_B/2 \). Thus, the relativistic \( \gamma \)-factor relative to the lab. frame for the nucleon in that frame is

\[
\gamma = \frac{1}{\sqrt{1 + \tau}} . \tag{5}
\]

We note that, no matter what reference frame is adopted, the nucleon before and after the scattering must have different momenta.

We now formulate a non-relativistic model of eN scattering. Assume that one puts a single nucleon into the lowest level in a very deep harmonic oscillator (HO) potential to avoid any recoil problem. In this model, the nucleon is bound and held essentially at rest. Physically, this is similar to the case of a nucleon bound in a heavy nucleus and not allowed to carry the full recoil momentum when an electron scatters from it. Using, the tables of [8] or the review article of [9], one can compute the multipole matrix elements of the C0 (Coulomb monopole) and M1 (magnetic dipole) elastic scattering operators. Using the non-relativistic limit for the current operators, together with 1s-1/2 harmonic oscillator wave functions, one obtains the following:

\[
\langle 1s_{1/2} \| M_0^{\text{Coul}} \| 1s_{1/2} \rangle = F_1 \langle 1s_{1/2} \| M_0 \| 1s_{1/2} \rangle \tag{6}
\]

\[
\langle 1s_{1/2} \| iT_1^{\text{mag}} \| 1s_{1/2} \rangle = \frac{q}{m_N} F_1 \langle 1s_{1/2} \| \Delta_1(qx) \| 1s_{1/2} \rangle \tag{7}
\]

Here, following [9], \( F_1 \) and \( F_2 \) are the Dirac and Pauli single-nucleon form factors. Either working directly with the harmonic oscillator wave functions and the explicit forms for the current multipole operators (see [9]) or, using the tables of [8], one finds for the three required reduced matrix elements

\[
\sqrt{4\pi} \langle 1s_{1/2} \| M_0(qx) \| 1s_{1/2} \rangle = 2e^{-\gamma} \tag{8}
\]

\[
\sqrt{4\pi} \langle 1s_{1/2} \| \Delta_1(qx) \| 1s_{1/2} \rangle = 0 \tag{9}
\]

\[
\sqrt{4\pi} \langle 1s_{1/2} \| \Sigma_1(qx) \| 1s_{1/2} \rangle = 2e^{-\gamma} . \tag{10}
\]

Each is proportional to \( \exp(-\gamma) \) where \( y = (bq/2)^2 \) with \( b \) the HO parameter. However, one must multiply by the center-of-mass correction which in the non-relativistic HO shell model can be computed: it is a multiplicative factor of \( f_{cm} = \exp(+y/A) = \exp(+y) \) for \( A = 1 \) and cancels the above factor, leaving only the remaining factors obtained by using the above-cited tables.

Using this model, one can then correct the Sachs form factors for the effect of recoil to obtain elastic electric and magnetic form factors \( (G^{\text{int}}) \) that are dominated by the intrinsic charge and magnetic structure

\[
G_E^{\text{int}} \equiv \frac{1}{1 + \tau} G_E \tag{11}
\]

\[
G_M^{\text{int}} \equiv \frac{1}{\sqrt{1 + \tau}} G_M . \tag{12}
\]

The usual proton form factor ratio is defined as follows using the Sachs form factors

\[
R_p \equiv \frac{G_E^p}{G_M^p} \tag{13}
\]

and is shown in Fig. [1] see [10] for references to the data and to the so-called GKex vector meson based model [11] shown as a solid line in the figure. The ratio based on the intrinsic form factors is then immediately given by

\[
R_p^{\text{int}} = (1 + \tau)R_p \tag{14}
\]

and is shown in Fig. [2]. Note that this has the boost factor squared (going as \( 1 + \tau \) and shown in the right panel as a red line) rather than just linearly as in the individual form factors. Clearly this introduces large modifications at high momentum transfers. Indeed, the intrinsic results are relatively flat as functions of \( Q^2 \) and differ from unity by less
than roughly 20%. This is consistent with the physically reasonable expectation that the proton’s intrinsic charge and magnetization spatial distributions are similar. Finally, we note that the \( Q^2 \) dependence arising largely from \((1 + \tau)\) in our model is the basis for the observed discrepancy \([12]\) between the elastic form factors measured via the Rosenbluth technique and recoil polarization method and widely understood to be due to two photon contributions to the radiative corrections to the elastic electron-proton cross section \([13]\).

Sachs form factors modified by factors of \((1 + \tau)^{\pm n/2}\) have been proposed long ago \([7]\) and more recently \([14]\) but were never adopted. However, the particular modification arising from our model is unique in explaining the measured \( Q^2 \) dependence of \( R_p \) as predominantly a relativistic boost effect, as illustrated in Fig. 2.

With the Sachs elastic form factors corrected for recoil effects, we are now ready to correct the RMS charge and magnetic radii using the same procedure; we call the corrected radii \( r_{EM}^{nr} \). The usual definition is obtained by expanding the \(Jo_3\) spherical Bessel function for low momentum transfer to obtain

\[
r_{EM}^{scatt} = \sqrt{\frac{3}{2m_N}} \left( \frac{d}{d\tau} G_E \right)_{\tau \to 0};
\]

(15)

this is simply a re-writing of Eq. \((3)\). We then proceed to correct for recoil effects using our HO model. For this one must include the \(\sqrt{1 + \tau}\) factor in Eq. \((11)\), obtaining at small momentum transfer

\[
G_E^{nt} = \left( 1 + \frac{1}{2} \tau + \ldots \right)
\]

\[
\times \left( 1 - \frac{2}{3} \frac{m^2}{m_N^2} \left[ r_{EM}^{scatt} \right]^2 + \ldots \right)
\]

(16)

\[
= 1 - \frac{2}{3} \frac{m^2}{m_N^2} \left[ r_{EM}^{scatt} \right]^2 - \frac{1}{2} \tau + \ldots
\]

(17)

\[
\equiv 1 - \frac{2}{3} \frac{m^2}{m_N^2} \left[ r_{EM}^{nr} \right]^2 + \ldots
\]

(18)

and leading to the relationship

\[
r_{EM}^{nr} = \sqrt{\left[ r_{EM}^{scatt} \right]^2 - \Delta_N},
\]

(19)

where one has

\[
\Delta_p = \frac{3}{4m_p} = 0.0332 \text{ fm}^2
\]

(20)

\[
\Delta_n = \frac{3}{4m_n} = 0.0331 \text{ fm}^2
\]

(21)

for protons and neutrons, respectively. The same arguments for the magnetic form factor where the required boost factor is now \(1/\sqrt{1 + \tau}\) (see Eq. \((12)\)) leads to the expression

\[
r_{M}^{nr} = \sqrt{\left[ r_{M}^{scatt} \right]^2 + \Delta_N}.
\]

(22)

We note that the same result for the proton charge form factor was obtained in \([15]\), arguing from a very different point of view; see also \([16]\), on which that work is based. \([17]\) also derives the result within a factor of two.

It is to be noted that the Darwin-Foldy correction to the proton charge radius discussed in \([13]\) and in the appendix to \([19]\) has a similar form to \(\Delta_N\) but has a very different physical origin and is opposite in sign.

It is important to understand that these effects due to recoil do not go away if electron scattering data are obtained at ever smaller values of the momentum transfer. As the above expressions clearly show, the relativistic boost factor arising from \((1 + \tau)^{\pm 1/2}\) deviates from unity at order \(Q^2\); however, that is the order needed to extract the charge or magnetic radii. In other words, being locked together at the same order when expanding in powers of \(Q^2\), the effects can never be separated, no matter how small the momentum transfer becomes.

Frequently, it is argued \([3]\) that the effective value of \(Q \cdot r_p\) is very small for these atomic systems \((\sim 10^{-5})\) and that the coordinate space and momentum space values converge. However, the above argument on the “locking” of the boost factor with the radius shows that the smallness of this product is not sufficient to make relativistic and non-relativistic radii effectively the same. It is not the scale of momentum transfer that is critical (as long as it is small enough to allow only terms of quadratic order to be
considered), but the fact that a scattering process is analyzed in momentum space and measurements of an atomic system are studied in coordinate space.

Specifically, using the Bernauer value for the proton rms charge radius [1], and the PDG values [20] for the proton magnetic, neutron charge, and neutron magnetic rms radii, one has the following:

\[ r_{E,p}^{nr} = \sqrt{(0.879)^2 - 0.0332} \]
\[ = 0.860 \pm 0.008 \text{ fm} \]  \hspace{1cm} (23)
\[ r_{M,p}^{nr} = \sqrt{(0.777)^2 + 0.0332} \]
\[ = 0.7981 \pm 0.013 \pm 0.010 \text{ fm} \]  \hspace{1cm} (24)
\[ r_{M,n}^{nr} = \sqrt{(0.862)^2 + 0.0331} \]
\[ = 0.8810 + 0.009 - 0.009 \text{ fm} \]  \hspace{1cm} (25)
\[ [r_{E,n}^{nr}]^2 = -0.1161 - 0.0331 \]
\[ = -0.1492 \pm 0.0022 \text{ fm}^2 \].  \hspace{1cm} (26)

Here the uncertainties are taken from Bernauer [1] and from the PDG compilation [20], respectively. Note that the electric result for the neutron is traditionally expressed as the square of the radius, which is negative.

The proton charge radius discrepancy has arisen from the different values resulting from a precise determination using the Lamb shift in muonic hydrogen \((0.84087 \pm 0.00026 \pm 0.00029)\) [3] disagreeing with the CODATA 2010 value \((0.8775 \pm 0.0051)\) [21], largely determined by the most precise value resulting from elastic electron proton scattering [1]. This amounts to more than 4% difference, whereas the stated total uncertainty in the Bernauer electron scattering result is quoted as 0.9%. The corrected proton charge radius \(r_{E,p}^{nr}\) resulting from the boost between Breit and lab. frames as calculated in our model decreases the electron scattering determination of \(r_{E}^{scatt}\) towards the muonic hydrogen value. The resulting discrepancy in the different determinations of \(r_{E,p}^{nr}\) is halved using the corrected value and now differs from the muonic Lamb shift value by only about 2%. Further, the corrected value here for the proton charge radius is not inconsistent with the value determined using the hydrogen atom, within experimental uncertainty. Fig. 3 shows a selection of determinations of the proton rms charge radius since the year 2000 including all the work cited here. In particular, it shows the electron scattering determination from Bernauer and our corrected value with both 1σ and 2σ uncertainties. If one accepts the ideas we have developed in this paper, then one is left with a discrepancy of order \(\alpha\), which could be explained by other model dependences in the problem (experimental systematic uncertainties, radiative corrections, etc.)

The \(Q^2\) independent recoil correction to the rms charge radius derived from elastic electron scattering can also be evaluated for the deuteron, the triton, and the helium isotopes. For heavier nuclei, the correction factor above is replaced as follows:

\[ \Delta_N = \frac{3}{4m_N^2} \rightarrow \Delta_N \cdot \left( \frac{m_N}{m_A} \right)^2, \]  \hspace{1cm} (27)

where \(m_A\) is the mass of that heavier system. In Table 1 we summarize the results based on our model. We note that the corrected deuteron rms charge radius from electron scattering is in excellent agreement with the recent high precision value obtained from measurements of the Lamb shift in muonic deuterium. We provide predictions for the rms charge radii of the helium isotopes, which are being determined to high precision in ongoing experiments that employ measurement of the Lamb shift in muonic atoms. New high precision measurements of the Lamb shift in electronic hydrogen are in progress and results are expected soon.

We have also considered how the conclusions here can be validated by experiment. We point out that the correction we derive cannot be separated by going to lower \(Q^2\) in electron scattering experiments nor by comparison of elastic electron and muon scattering on the proton. However, based on the estimates made here, a precision comparison of electron scattering from the proton with electron scattering from the deuteron at low \(Q^2\) should deviate at the level of about 1.5%, since the corrections we derive differ at this level. For highest precision, such measurements should be carried out using the apparatus and systematics must be minimized. We note that internal radiative corrections, which arise mainly from the incident and scattered electrons, should be quite similar for proton and deuteron targets.

In summary, we argue that corrections between the Breit and lab. frames are important in interpreting the form factors of the nucleon as determined in relativistic electron scattering. We have constructed a model to estimate these corrections. In this model, the observed significant decrease of the ratio of the proton elastic form factors as a function of \(Q^2\) is understood as a predominantly relativistic effect. Furthermore, in this model, the proton charge radius as determined in electron scattering has a correction that reduces its value towards that resulting from the precision determination using the Lamb shift in muonic hydrogen. The model provides predictions for the rms charge radii of the deuteron, triton, and helium isotopes.

The work here underlines the importance of having a model of the nucleon that allows boosting between different reference frames. For example, in models where relativistic quarks are confined via a “bag” one must confront the problem of how to boost the latter in going between the frames that inevitably enter in electron scattering. One possible, but different, related case that could be studied to test the ideas is that of relativistic (covariant, boostable) modeling of the deuteron [31]. There, one could directly compute the elastic form factor, while in parallel also computing the ground-state charge distribution and then Fourier transforming it to momentum space. Upon comparing the two results it is likely that differences will be found that relate to the boost issues raised in the present study.

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Fig. 3. Timeline of recent determinations [1, 4, 21–28] of the proton charge radius $r_{E,p}$. For earlier experimental results see [2]. The Bernauer value for the proton charge radius, namely \( r_{mom}^{E,p} = r_{rel}^{E,p} \), is indicated with a black cross lying at 0.879, while its corrected value \( r_{nr}^{E,p} = r_{E,p}^{corr} \) is indicated by the white cross at 0.860. The shaded bands show the 1σ and 2σ uncertainties about the latter and may then be compared with the Lamb shift value indicated by a star and lying at about 0.84.

Table 1. Comparison of charge radii measured by electron scattering from various light nuclei and the corrected values following the procedure proposed herein. The muonic determinations for the proton and the deuteron [30] are also given. We note that high precision values for the helium isotopes will be forthcoming from muonic atom experiments.

| Nucleus | \( r_g \) (fm) | \( r_g^{corr} \) (fm) | \( r_g^{muonic} \) (fm) |
|---------|----------------|------------------------|------------------------|
| \(^1\)H | 0.879 ± 0.008 [1] | 0.860 ± 0.008          | 0.84087 ± 0.00039       |
| \(^2\)H | 2.13 ± 0.01 [29]   | 2.12 ± 0.01            | 2.12562 ± 0.00078       |
| \(^3\)H | 1.755 ± 0.087 [29] | 1.751 ± 0.087          |                        |
| \(^3\)He | 1.959 ± 0.034 [29] | 1.955 ± 0.034          |                        |
| \(^4\)He | 1.680 ± 0.005 [29] | 1.678 ± 0.005          |                        |

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