Quantum tetrahedra and simplicial spin networks

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A new link between tetrahedra and the group SU(2) is pointed out: by associating to each face of a tetrahedron an irreducible unitary SU(2) representation and by imposing that the faces close, the concept of quantum tetrahedron is seen to emerge. The Hilbert space of the quantum tetrahedron is introduced and it is shown that, due to an uncertainty relation, the “geometry of the tetrahedron" exists only in the sense of “mean geometry”.

A kinematical model of quantum gauge theory is also proposed, which shares the advantages of the Loop Representation approach in handling in a simple way gauge- and diff-invariances at a quantum level, but is completely combinatorial. The concept of quantum tetrahedron finds a natural application in this model, giving a possible interpretation of SU(2) spin networks in terms of geometrical objects.

I. INTRODUCTION

The links between geometric objects and angular momenta in Quantum Mechanics have been observed in many ways since long ago [1], and the properties of some invariants which can be obtained from SU(2) representations have been used by Ponzano and Regge to build a quantum gravity model in three dimensions [2]. In the PR model a partition function is defined for a given 3-dimensional simplicial complex (it has been proved later, by deforming SU(2) to a quantum group [3], that the partition function depends only on the topology of the manifold which is triangulated by the simplicial complex) by means of the following procedure: to each edge of the complex is associated a spin $j_e$ (that is, an irreducible unitary SU(2) representation, characterized only by its dimension $d = 2j_e + 1$).

The “exponential of the action" in such a configuration is a suitable product of the 6$j$-symbols associated to the 3-simplices and the partition function is obtained by taking the sum of this products over the possible associations of spins to the edges.

This construction has been inspired by the following property: ruling out the trivial 1-dimensional representations, the 6$j$-symbols are zero unless the values of the representations’ spins can represent lengths of the edges of a tetrahedron (i.e. the triangular inequality is satisfied in each face), which is, in a loose sense, the “building block” of 3-dimensional simplicial complexes. The main feature of the PR model is that the partition function is defined in a purely algebraic fashion, and in the classical limit (i.e. when the spins $j_e$ are large) it resembles the Feynman path integral for (Euclidean) General Relativity in 3-dimensions: one could say that there is a classical geometry (that defined by the edges’ lengths) obeying a quantum dynamics.

In this paper we highlight another link between tetrahedra and SU(2) by associating to each face of the tetrahedron an irreducible unitary SU(2) representation. This construction is, in some sense, dual to that of PR and, after the imposition of a “closure” condition analogous to the “triangular inequality” condition seen above, it leads to the notion of a quantum geometry by itself. In section II we introduce the Hilbert space of the quantum tetrahedron and we discuss some of its properties. We shall see that, due to the noncommutativity of some operators simultaneously needed in order to define a classical geometry, an uncertainty relation arises; it is however possible to obtain some geometrical informations by taking the mean values of the relevant operators and we have computed such “mean geometries" in some simple cases.

A strict connection exists between quantum tetrahedra and 4-valent vertices of SU(2) spin networks. Since their first appearence in the pioneering Penrose’s work [4], aiming to a combinatorial description of some kind of quantum geometry, spin networks have found applications in many branches of mathematics and physics; in particular, it is now a firm result that spin networks embedded into a manifold are “the skeleton” of the kinematical structure of quantum gauge theories with compact gauge groups [5,6]. When diffeomorphism invariance holds, as is the case in the loop quantization of (Euclidean, if one wants a compact gauge group) General Relativity using Ashtekar connection variables [7–9], one must factor out all the classes of “equivalent" spin networks under diffeomorphisms (i.e. one must solve the “momentum constraints" which arise in the corresponding functional quantization), obtaining what are usually called S-knots. One of the main difficulties with this approach is the rather “knotted” structure of the set of arbitrary S-knots, manifesting itself e.g. in the fact that this set is not countable [10].
In section III we propose a model of quantum gauge theory in which the only allowed spin networks are the simplest they could be in order to be nontrivial: they are defined on the (4-valent) dual graphs $\Gamma_M$ of the simplicial complexes $M$ which triangulate the manifold. We shall see in this way that both gauge- and diff-invariances are obtained, and that the resulting structure is purely combinatorial. (We must remark that, while completing this work, we came across a similar construction of combinatorial quantum gauge theory using simplicial complexes, appeared in recent works of other authors [11,12].)

We conclude with some speculations about the possible links between our model and some “quantum version” of Regge Calculus [13] and its possible applications to the problem of Quantum Gravity.

II. QUANTUM TETRAHEDRA

A. Classical geometry of a tetrahedron

A tetrahedron can be understood as the convex envelope of four points in 3-dimensional Euclidean space $E^3$. With reference to figure 1, we see that a triad $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of independent vectors (nine numbers) defines completely the tetrahedron. If one is interested only in its properties independently from its space orientation (we may consider these properties as defining the “geometry” of the tetrahedron), the relevant independent parameters, due to the factorization of the rotation group, become six and can be taken to be

$$\vec{e}_1 \cdot \vec{e}_1, \quad \vec{e}_2 \cdot \vec{e}_2, \quad \vec{e}_3 \cdot \vec{e}_3, \quad \vec{e}_1 \cdot \vec{e}_2, \quad \vec{e}_2 \cdot \vec{e}_3, \quad \vec{e}_3 \cdot \vec{e}_1. \tag{1}$$

Given these six numbers, it is possible to reproduce the original tetrahedron apart from its original orientation in space. (The upper bound in this hierarchy is given by the twelve cartesian coordinates of the four points and the degrees of freedom match: three parameters for the translations group plus three for that of rotations plus six of “geometry”.)

Let us now consider the vectorial areas of the tetrahedron in figure 1, given by

$$\vec{n}_1 \equiv -\vec{e}_1 \times \vec{e}_2, \tag{2}$$
$$\vec{n}_2 \equiv -\vec{e}_2 \times \vec{e}_3, \tag{3}$$
$$\vec{n}_3 \equiv -\vec{e}_3 \times \vec{e}_1, \tag{4}$$
$$\vec{n}_4 \equiv \vec{e}_4 \times \vec{e}_5 = -\vec{n}_1 - \vec{n}_2 - \vec{n}_3. \tag{5}$$

The last equation, which is simply the “closure” condition (in the form $\int_S \vec{n} \, da = 0$ it holds for every closed surface $S$), shows that only three of these vectors, which will be called, somewhat improperly, normals, are independent; it seems now natural to see whether the tetrahedron’s geometry can be reconstructed from the normals rather than from the edges.

The independent parameters must belong to the set of the invariants which can be obtained from $\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4$, that is, their squares (four times the squared areas of the faces) and their mutual scalar products (properly normalized, the cosines of the dihedral angles associated to the edges). We should add to these quantities the triple product $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3$, which is also invariant; this quantity will play a rôle later.

In all we have ten numbers which, owing to eqn. (1), are not independent; by taking the scalar products of this expression with the normals we obtain four independent equations:

$$n_{11}^2 + n_{12} + n_{13} + n_{14} = 0, \tag{6}$$
$$n_{12}^2 + n_{12} + n_{23} + n_{24} = 0, \tag{7}$$
$$n_{13}^2 + n_{13} + n_{23} + n_{34} = 0, \tag{8}$$
$$n_{14}^2 + n_{14} + n_{24} + n_{34} = 0, \tag{9}$$

where $n_{ij} \equiv \vec{n}_i \cdot \vec{n}_j$ and $n_{ij} \equiv \vec{n}_i \cdot \vec{n}_j$ (the square in the first symbol is part of the notation).

It is easy to verify that independent parameters are the four squared areas $n_{ij}$ and two dihedral angles associated to edges sharing a vertex, e.g. $n_{12}$ and $n_{23}$. The relations between couples of angles associated to opposite edges are the following:

$$n_{12}^2 + n_{23}^2 + 2n_{12} = n_{23}^2 + n_{34}^2 + 2n_{34}, \tag{6'}$$
$$n_{23}^2 + n_{34}^2 + 2n_{23} = n_{12}^2 + n_{34}^2 + 2n_{14}, \tag{7'}$$
$$n_{12}^2 + n_{34}^2 + 2n_{12} = n_{23}^2 + n_{34}^2 + 2n_{24}. \tag{8'}$$

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and eqn. (7), taken into account the last of these expressions, gives the following relation between $n_{13}$ and the chosen variables:

$$n_{13} = \frac{1}{2} [n^2_2 - n^2_1 - n^2_2 - n^2_3] - n_{12} - n_{23}. \quad (10)$$

By taking the various scalar products between the definitions (2-5), with the parameters (1) considered as unknowns, we obtain a system of algebraic equations with just quadratic and constant (i.e. functions of $n^2_i$ and $n_{ij}$) terms. This system, provided that some geometrical non-holonomic restrictions such as $n^2_i > 0$ and $|n_{ij}| < (n^2_i n^2_j)^{1/2}$ are satisfied, has two sets of opposite real roots; the correct solution is clearly the one with $\vec{e}_i \cdot \vec{e}_j > 0$, while the other one can be imagined to correspond to purely imaginary edges ($\vec{e}_i \to i\vec{e}_i$).

To summarize, the values of the areas and of two “non opposite” dihedral angles actually define completely the tetrahedron’s geometry.

B. Quantum geometry

As we have mentioned in the introduction, let us now associate to the four faces of the tetrahedron four unitary irreducible representations of $SU(2)$ acting on the spaces $\mathcal{H}_j$ ($j$ is the spin of the representation, while $i$ labels the faces). The “quantum versions” of the vectorial areas are assumed to be the generators $\mathbf{J}_i$ acting on the tensor product

$$\mathcal{H}_{j_1,j_2,j_3,j_4} \equiv \bigotimes_{i=1}^{4} \mathcal{H}_j.$$ 

In the PR model, the spins associated to the edges have to obey triangular inequalities for each face; in our case, we must impose the quantum normals to obey the closure condition (3):

$$\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \mathbf{J}_4 = 0, \quad (11)$$

that is, the space $\mathcal{H}_{j_1,j_2,j_3,j_4}$ must contain a subspace $\mathcal{H}^0_{j_1,j_2,j_3,j_4}$ of spherically symmetric vectors. In such a space the operators

$$\mathbf{n}^2_i = \mathbf{J}_i \cdot \mathbf{J}_i, \quad \mathbf{n}_{ij} = \mathbf{J}_i \cdot \mathbf{J}_j \quad (12)$$

are well defined and they obey a set of operatorial equations identical to (13). If, in order not to single out particular values of the $j$’s, we consider now the orthogonal sum

$$\mathcal{H}_\tau \equiv \bigoplus_{\{J\}} \mathcal{H}^0_{\{J\}}, \quad (13)$$

where $\{J\}$ runs over the set of ordered 4-tuples of integers or half-integers such that $\mathcal{H}^0_{\{J\}}$ is nonempty, the operators $\mathbf{n}^2_i$ and $\mathbf{n}_{ij}$ can be defined in $\mathcal{H}_\tau$ as hermitean operators, and, since they are blockwise diagonal, they continue to obey equations (14); we thus may say that $\mathbf{n}^2_i$ and $\mathbf{n}_{ij}$, acting on $\mathcal{H}_\tau$, define a quantum tetrahedron.

Condition (12) may be considered to implement in this context the fact that the set of the faces must form the boundary of the tetrahedron and as such it must have no boundary. The topological principle that “the boundary of a boundary is zero” usually manifests itself in physics via the Stokes Theorem as an “inducer” of automatic conservation laws [18]; in our case it induces “automatic” invariance under rotations imposed onto the full tetrahedron; the faces by themselves transform non trivially (since they carry a non-zero spin), but the quantum tetrahedron is invariant (the spin “created inside it” must be zero). As we shall see in section (13) this fact is in intimate connection with gauge invariance.

It is worth noting that, since e.g. $\{\mathbf{n}^2_1, \mathbf{n}^2_2, \mathbf{n}^2_3, \mathbf{n}^2_4, \mathbf{n}_{12}\}$ is easily seen to be a complete set of commuting operators in $\mathcal{H}_\tau$, the set of operators which is needed in order to determine the geometry is non-commuting. Indeed, if we take $\mathbf{n}_{23}$ as the last parameter we find

$$[\mathbf{n}_{12}, \mathbf{n}_{23}] = -i \mathbf{J}_1 \cdot \mathbf{J}_2 \times \mathbf{J}_3 = i \mathbf{U}, \quad (14)$$

where the r.h.s., being a scalar quantity, is well defined in $\mathcal{H}_\tau$ (the $\mathbf{n}^2_i$ are proportional to the identity in each subspace $\mathcal{H}^0_{\{J\}}$, therefore they commute with all the $\mathbf{n}_{ij}$). This equation implies the uncertainty relation

$$(\Delta n_{12})(\Delta n_{23}) \geq \frac{1}{2} |\langle \mathbf{U} \rangle|, \quad (15)$$

where $(\Delta n_{ij})^2 \equiv \langle \mathbf{n}^2_{ij} \rangle - \langle \mathbf{n}_{ij} \rangle^2$ is the mean square deviation of $\mathbf{n}_{ij}$. 

3
Permutations and Parity transformation

Despite its non symmetric definition, the operator $U$ is, except for the sign, unique: if we take as independent variables e.g. $n_{12}$ and $n_{24}$, the commutator is then

$$[n_{12}, n_{24}] = -i\hat{J}_1 \cdot \hat{J}_2 \times \hat{J}_4 = -iU,$$

where the last equality follows from the fundamental equation (1), and a little reflection shows that $\hat{J}_i \cdot \hat{J}_j \times \hat{J}_k$ is the same of $U$ (recall the definition (4) with the minus sign) if and only if $\{ijk\}$ are the first three elements of an odd permutation of $\{1234\}$, while it is opposite if the permutation is even. The latter property has an interesting connection with the notion of parity.

The description we have given from the beginning of the geometry of tetrahedra has skipped the issue of parity: from the numerical values alone of the parameters that we have chosen, it is by no means possible to distinguish one tetrahedron from its $P$-transformed: both the parameters $\mathcal{H}$ and $n_{ij}$, $n_{ij}$ are invariant under the transformation $\vec{e}_i \to -\vec{e}_i$. In the description in term of the edges this ambiguity can be resolved by the choice of a sign for $\vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3$ (where $V$ is the volume of the tetrahedron): it is positive if the triad is right-handed, and negative if it is left-handed. Essentially the same thing can be done in the description in terms of the normals, but there is a subtlety.

Classically, the triple product of the normals $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, as defined in (3), is given by

$$\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = -(\vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3)^2 = -36V^2,$$

and it is always negative, whatever $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is right or left-handed: a closer inspection on the geometrical settings reveals however that they point outwards the tetrahedron only in the first case. The reason for this is that the vector products are defined in terms of the initial basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ which we have supposed right-handed and to maintain the geometrical settings we must change sign to all the normals, with the result that the l.h.s. of eqn. (16) also changes its sign: we may distinguish the original and its $P$-transformed tetrahedra by means of the sign we give to $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3$. This prescription, however, requires an additional information: an ordering modulo even permutations of the normals. Indeed if the normals are non ordered it is not possible to decide whether a given triplet should be left- or right-handed.

These considerations about the link between parity and permutations can be transposed in the quantum context. If we define in $\mathcal{H}_\tau$ a representation of the group $\Pi^{(4)}$ (permutations of four elements) in the following way

$$\pi \in \Pi^{(4)} \rightarrow \pi|j_1j_2j_3j_4m\rangle = |j_\pi(1)j_\pi(2)j_\pi(3)j_\pi(4)\text{sgn}(\pi)m\rangle,$$

where sgn($\pi$) = ±1 if $\pi$ is an even (odd) permutation, we see that the situation is exactly the same found for the classical geometry: we may define parity on the quantum tetrahedron’s Hilbert space in the following way

$$\mathcal{P}|j_1j_2j_3j_4m\rangle = |j_1j_2j_3j_4 - m\rangle,$$

so that we have

$$\mathcal{P}U \mathcal{P} = -U,$$

but without an ordering modulo even permutations of the faces we cannot decide if “positive $u$” means right- or left-handed, since

$$\pi^\dagger U \pi = \text{sgn}(\pi)U.$$
The maximum dimensionality $d_{\text{max}} = 2j + 1$ is achieved by the maximally symmetric space $\mathcal{H}_{j_1j_2j_3j_4}^0$, while the minimum $d_{\text{min}} = 1$ is achieved when any of the $j_j$ is zero or it holds $j_4 = j_1 + j_2 + j_3$. However, difficulties arise if one tries to interpret the results in a classical way; for instance, when $j_4 = j_1 + j_2 + j_3$ corresponds to the “quantum sum” of three parallel vectors, that is, to a “flat” (contained in a 2-dimensional plane) tetrahedron, while the quantum geometry does not lead to this description, since e.g. the sum of the first three areas should be equal to the fourth, and one can check that it is not the case. In the generic case, eqn. (13) implies that the geometry of the quantum tetrahedron cannot be defined exactly, because of “quantum fluctuations”: we can continue to speak about geometry only “in the mean”.

Let now $j_1$, $j_2$, $j_3$, $j_4$ be “allowed” values for the external spins, and $\{j_{12}\}$ be the orthonormal basis in $\mathcal{H}_{j_1j_2j_3j_4}^0$ diagonalizing $(\mathbf{J}_1 + \mathbf{J}_2)^2$ (with the standard choice of the phase factors); the matrix element of eqn. (14) between the states $|j_{12}\rangle$ and $|j'_{12}\rangle$ gives:

$$\langle j'_{12} | U | j_{12}\rangle = \frac{i}{2} |j_{12} (j_{12} + 1) - j'_{12} (j'_{12} + 1)\rangle |j'_{12} | n_{23} | j_{12}\rangle,$$

thus the diagonal elements in this basis are zero. Concerning the off-diagonal elements, we have found the selection rule

$$\langle j'_{12} | U | j_{12}\rangle \neq 0 \Rightarrow j'_{12} = j_{12} \pm 1, \tag{20}$$

but to prove it we need the commutator between $n_{12}$ and $U$:

$$[n_{12}, U] = -i \left[ [n_{12}^2 + n_{12}] n_{23} - n_{31} (n_{22}^2 + n_{12}) \right].$$

From the matrix element of this expression between the states $|j_{12}\rangle$ and $|j'_{12}\rangle$, with $j_{12} \neq j'_{12}$, one obtains

$$\langle j'_{12} | U | j_{12}\rangle = \frac{i}{j_{12} (j_{12} + 1) - j'_{12} (j'_{12} + 1)} \left\{ [j'_{12} (j'_{12} + 1) + j_{12} (j_{12} + 1) - j_{12} (j_{12} + 1) |j'_{12} | n_{23} | j_{12}\rangle - |j_{12} (j_{12} + 1) + j_{12} (j_{12} + 1) - j_{12} (j_{12} + 1) |j'_{12} | n_{31} | j_{12}\rangle \right\},$$

and, by substituting in the above expression the operatorial analogue of eqn. (14), it is found

$$\langle j'_{12} | U | j_{12}\rangle = \frac{i j_{12} (j_{12} + 1) + j'_{12} (j'_{12} + 1)}{j_{12} (j_{12} + 1) - j'_{12} (j'_{12} + 1)} \langle j'_{12} | n_{23} | j_{12}\rangle.$$

By comparing this expression with eqn. (19), we see that if $\langle j'_{12} | n_{23} | j_{12}\rangle$ is different from zero, then it follows that

$$\frac{j_{12} (j_{12} + 1) - j'_{12} (j'_{12} + 1)}{2} \geq \frac{j_{12} (j_{12} + 1) + j'_{12} (j'_{12} + 1)}{j_{12} (j_{12} + 1) - j'_{12} (j'_{12} + 1)},$$

but it is not difficult to show that this equation actually coincides with eqn. (20).

$U$ being self-adjoint, we may restrict our computation to $j'_{12} = j_{12} + 1$, obtaining

$$\langle j_{12} + 1 | U | j_{12}\rangle = -i (j_{12} + 1) |j_{12} + 1 | n_{23} | j_{12}\rangle.$$

(21)
the independent matrix elements are then \(d-1\), where \(d\) is the dimension of \(\mathcal{H}_{j_1,j_2,j_3,j_4}^0\).

The matrix element \(\langle 21 \rangle\) can be expressed in a general form by means of the \(6j\)-symbols; indeed, the insertion in the r.h.s. of eqn. (21) of the identity in \(\mathcal{H}_{j_1,j_2,j_3,j_4}^0\) in the form

\[
1 = \sum_{j_{23}} |j_{23}\rangle \langle j_{23}|
\]

produces the final result

\[
\langle j_{12} + 1|U|j_{12}\rangle = -\frac{i}{2} (j_{12} + 1) \sum_{j_{23}} j_{23} (j_{23} + 1) (j_{12} + 1) j_{23} \langle j_{23}|j_{12}\rangle,
\]

where the scalar products are given by the expression [1]

\[
\langle j_{12}|j_{23}\rangle = (-1)^{j_1+j_2+j_3+j_4} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \left\{ j_1 \quad j_2 \quad j_1 \quad j_2 \right\}.
\]

In order to verify if, in practice, the exposed construction produces consistent results, we have computed in some simple cases the quantum geometries of the quantum tetrahedron via the following procedure:

- given the values \(j_1, j_2, j_3, j_4\) of the four external spins, we have computed the eigenvalues and the (normalized) eigenvectors of the operator \(U\) in \(\mathcal{H}_{j_1,j_2,j_3,j_4}^0\);

- we have then computed, using the mean values of \(n_{12}\) and \(n_{23}\) in each eigenstate of \(U\) (the choice of these states is motivated by reasons of symmetry), the “mean” geometry in the form of the “mean” edges’ squared lengths [2].

The results obtained are shown in the appendix.

From the few cases explored we can infer some general properties of the mean geometries. Firstly, it seems that actually \(U\) is non-degenerate in \(\mathcal{H}_{j_1}^0\); secondly, the mean geometries corresponding to states with opposite eigenvalues of \(U\) are the same. Furthermore, when there are four or three equal values of the external spins, corresponding respectively to equilateral tetrahedra and right pyramids with equilateral basis, the mean geometries do not depend on the eigenvalues, but this “degeneration” disappears for less symmetric cases.

The case \((1/2,1/2,1/2,1/2)\), which in some sense is the first nontrivial one, is suggestive for the following reason. Since \(\mathbf{n}_4\) must have the dimensions of an area, we may write

\[
\mathbf{n}_i = l^2 \mathbf{J}_i,
\]

where \(l\) is a constant with the dimensions of a length, and we see that the mean length is exactly the unit length \(l\).

In order to have a better understanding of these results it seems necessary to have a consistent interpretation of the operator \(U\). As shown in eqn. (14), classically, the triple product \(\mathbf{n}_1 \cdot \mathbf{n}_2 \times \mathbf{n}_3\) is (minus) 36 times the squared volume of the tetrahedron. For this reason the operator \(|U|^1/2\) could be interpreted as (a multiple of) the quantum analogue of the volume. It is not obvious, however, that this identification is consistent, because (14) has been derived from eqns. (2-5) assuming that the edges \(\mathbf{e}_i\) commute.

C. Another dimension?

The fact that in the highly symmetric cases the mean geometry does not depend on \(m\), suggests that \(U\) could be interpreted as an object with higher dimensional nature, as in Minkowski space an invariant under rotations may have different 4-dimensional origins. Indeed, in the cases explored, we have noticed that the maximum eigenvalue of \(U\) is always less than \(36\langle V\rangle^2\), where \(\langle V\rangle\) is the volume computed using the mean geometry.

This property may be used to introduce the concept of a timelike direction. (We discuss the timelike case because it is more intriguing, but if one wishes the dimension more can be imagined as spacelike; it does not change much.) If we imagine the tetrahedron as imbedded in 3+1 Minkowski spacetime, the requirement that all the edges are spacelike is equivalent to the requirement that the 4-dimensional “normal” of the tetrahedron (i.e. the 1-form \(n_a\) dual to the

\[1\] All the computations have been performed by Mathematica.
trivector \((\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3)_{abc}\) lies inside the light cone. Assuming a privileged time axis, the immersion properties of our spacelike tetrahedron can be characterized (in an invariant way under spatial rotations) by the “angle” formed by its 4-velocity and the time axis, or, simply, by its velocity. Notice that, due to the condition that \(n_a\) lies inside the light cone, the velocity is necessarily less than one (\(c = 1\)).

We then see that, if the inequality \(|U| < 36(V)^2\) is actually obeyed in all cases, it would be natural to interpret \(|U|/36(V)^2\) as some power of the “\(\beta\)” of the quantum tetrahedron with respect to some “background” rest frame. In this light, the symmetry \(U \sim -U\) could be also interpreted as the possibility for \(n_a\) to lie inside the future or past light cone.

It is clear that if one considers only one quantum tetrahedron, the possibility of defining its velocity with respect to nothing else, can be of little use; one the other hand, if we collect together some quantum tetrahedra joining them by the faces (i.e. we consider suitable tensor products of \(H_e\) with itself), it may be then possible to define “spatial slices” of 4-dimensional “quantum simplicial complexes”. In the next section we will try to formalize in a general framework these ideas.

### III. SIMPLICIAL GAUGE THEORY

The connection between quantum tetrahedra and the spin networks used in the Loop Representation framework of quantum General Relativity is fairly obvious: condition [14], translated in terms of the representations spins, is nothing but the compatibility condition for the spins of the edges adjacent to a 4-valent vertex. Furthermore the (eigen)values of the areas of the faces are one half those obtained in the Loop Representation framework and \(|U|^{1/2}\) is \(2^{1/2}\) times the volume operator of that model, restricted to 4-valent vertices (see [9]). This analogy cannot be pushed farther, because in the Loop Representation spin networks have vertices whose valence is arbitrary, and for valence different from four it does not seem possible to find a geometric interpretation in terms perhaps of more complicated polyhedra. Anyway, 4-valent vertices seem to play a particular rôle for the following conjunct two reasons: first, the volume operator always acts trivially on vertices with valence less than four; second, diffeomorphism-equivalence classes of embedded spin networks with vertices with valence higher than four are labeled by continuous parameters [10], thus failing to have a purely combinatorial structure.

In this section we shall sketch a model of quantum gauge theory which is a kind of hybrid between the Loop Representation and lattice gauge theories, and in which the only allowed vertices are 4-valent. This model is not complete in an essential way, since it lacks a dynamics: all the considerations below are limited to the kinematical structure, but the framework itself seems to suggest how to introduce the concept of a dynamical evolution. We will return to this point at the end.

#### A. Kinematics

The whole construction has been inspired by the following simple observation (found in [14]): let \(M\) be a 3-dimensional Regge simplicial complex with positive metric; then the set

\[
M' \equiv M - \Sigma_1(M),
\]

where \(\Sigma_1(M)\) is the 1-skeleton of \(M\), is (isomorphic to) a multiply connected Riemann manifold, which we will call a Regge manifold, whose fundamental group is isomorphic to the fundamental group of the 1-graph \(\Gamma_M\) dual to \(M\) (\(\Gamma_M\) is always 4-valent). This manifold is everywhere locally flat and the information about its “curvature” is contained into a representation of \(\pi_1(\Gamma_M)\) in \(SO(3)\) (or \(SU(2)\)).

Turning to gauge theories, all the information about the gauge field is contained [15] into a representation of the group \(L_0\) of base-pointed loops in the gauge group \(G\), given by the holonomy of the connection (which can be considered as a lagrangian coordinate for a 3-dimensional theory as well as a canonical coordinate on the phase space of a 4-dimensional one; the underlying manifold \(\mathcal{M}\) is in both cases 3-dimensional). If however the connection is flat, we remain again with the representations of \(\pi_1(M)\) in \(G\).

The idea for what should be called a simplicial gauge theory is to “substitute” the configuration space for a non-flat gauge theory, with the set of flat connections which can be defined over the multiply connected manifolds obtained by triangulating the “original” manifold \(\mathcal{M}\) and by removing the 1-skeletons: in this way each configuration is given by a couple \((M, h_M)\), where \(M\) is a simplicial complex which triangulates \(\mathcal{M}\) and \(h_M\) is a representation of \(\pi_1(\Gamma_M)\) in \(G\). Intuitively one sees that there is no significant “loss of configurations”, as happens when one goes from a non-flat to a flat connection over the same manifold. Furthermore the configuration space has a far more manageable form: the part that substitutes the local degrees of freedom, that is, the structure of the triangulation, is purely combinatorial.
Finally, as any reference to $\mathcal{M}$ is contained in the properties of the simplicial complexes that triangulate it, and is thus of a purely topological nature, diffeomorphism invariance (or rather covariance) is, despite the “lattice-like” formulation, automatically obtained.

It should be pointed out that this construction is simply meaningless if interpreted in a classical fashion; in this respect it is not a “quantized” model but it has to be “quantum” from the beginning. The space of states should be taken as the set of the functionals over the configuration space; if the gauge group $G$ is compact, and if the set of simplicial complexes triangulating a given manifold is countable (which is surely the case if $\mathcal{M}$ is compact), this space can be given a natural measure:

$$\langle \phi | \psi \rangle \equiv \sum_{\mathcal{M}} \int_{G^{N(\Gamma_M)}} \phi^*(M, g_1, \cdots, g_{N(\Gamma_M)}) \psi(M, g_1, \cdots, g_{N(\Gamma_M)}) \prod_{i=1}^{N(\Gamma_M)} \, d\mu_i^H,$$

where $N(\Gamma_M)$ is the number of edges of $\Gamma_M$, and $d\mu^H$ is the Haar measure for $G$ (the representation $h_M$ is assigned by means of the $g_i \in G$ in 1-1 correspondence with the oriented edges of $\Gamma_M$).

This measure however is not fully satisfying, because it leaves little space to impose additional relations or constraints to handle issues like the introduction of a dynamical evolution, the refinements of the triangulation and the limit in which one should “see the continuum”. Regarding this problem, we are looking for a generalization of the projective techniques used for integration over the gauge groups in continuum gauge theories [16], using the fundamental groups of the manifolds

$$\mathcal{M}_M \equiv \mathcal{M} - T(\Sigma_1(M))$$

($T$ represents a triangulation of $\mathcal{M}$ “isomorphic” to $M$) instead of the tame groups. The main obstacle is that the $\Gamma_M$ do not form a group in a trivial way.

Let us now turn to gauge invariance: in perfect analogy with lattice gauge theories, gauge transformations are defined over each graph of each triangulation by assigning an element $U(v) \in G$ to each vertex $v$; for each edge $e_i$ we have

$$g_i \rightarrow g_i' \equiv U(v^+(e_i))g_iU^{-1}(v^-(e_i)),$$

where $v^+(e_i)$ is the final vertex of $e_i$ and the opposite holds for $v^-(e_i)$.

A basis in the space of invariant vectors under gauge transformations is provided (see [8] for the proof) by the spin networks defined over the graphs $\Gamma_M$, of which we recall the definition in the case of an arbitrary (compact) group $G$; to each edge $e$ one assigns a unitary irreducible representation $\rho_e$ of $G$, in such a way that for each vertex $v$ the following property is satisfied: let $T(v)$ and $S(v)$ be the set of the edges having $v$ respectively as the final and initial vertex; then in the set

$$\bigotimes_{e \in T(v)} \rho_e \otimes \bigotimes_{e \in S(v)} \rho_e^*,$$

decomposed into a sum of irreducible representations, are contained trivial representations. A spin network is identified by such an assignment plus a choice of a trivial representation for each vertex.

**B. Anything to do with Quantum Gravity?**

When the gauge group is $SU(2)$, a spin network can be interpreted as a collection of quantum tetrahedra joined by the faces and the concept of a simplicial quantum geometry for the manifold seems to emerge. The reason why the “geometrical content” of the model has appeared in the $SU(2)$ case can be explained by means of the mentioned observation about Regge manifolds: gauge transformations are, in this context, simply frame rotations within each tetrahedron, so that there should be no surprise that, by handling invariants under these transformations, one gets geometrical quantities. Indeed, our belief that the construction exposed has, for the $SU(2)$ case, more than something to do with some “quantum version” of Regge Calculus has been recently enforced by some observations found in [17], where also it is suggested that the natural “site” of spin networks in a loop-quantization of Regge Calculus is the dual graph rather than the 1-skeleton.

Another feature of the $SU(2)$ case is that the space of states is a subset of the space of states of the Loop Representation of quantum General Relativity (each graph $\Gamma_M$ can be imbedded into the manifold by means of a triangulation), but, as mentioned in the beginning of this section, it is the subset that behaves, in the sense specified
above, better. The fact that the model provides just this subset is somewhat gratifying and gives it some mathematical appeal.

A posteriori, one may ask if it is really necessary to have a classical manifold “in the background”; after all, its only rôle is to provide some constraints on the combinatorial structure of the simplicial complexes one must consider. Actually, from the point of view of consistency, it does not seem to make a lot of difference considering all possible (perhaps compact) simplicial complexes in the space of states, so that the model logically decouples from the manifold’s concept itself, which, instead, may arise as a suitable “semiclassical limit”.

Obviously, whether the $SU(2)$ simplicial gauge theory has something to do with Quantum Gravity or not, cannot be said before the introduction of some kind of dynamics; regarding this latter point, we think there are two possible ways that can be pursued. The first one is the introduction of a “scalar constraint” to keep track of the fact that there is not a background temporal structure; if one tries to mimic the form of the scalar constraint of General Relativity a problem arises with the definition of an extrinsic curvature: the gauge group being $SU(2)$ rather than the Lorentz group, it is not clear how to introduce the boosts. One way out could be something analogous to the generalized Wick rotation proposed for the Ashtekar formalism [19]; we think that such a construction could be linked to the property of the operator $U$ which we exposed at the end of the last section. The second possibility, maybe closest to the “simplicial spirit”, is the definition of transition amplitudes following steps similar to those in [23], using perhaps the topological moves that generate the (classical) evolution in Regge Calculus.

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Note added

A group-theoretical dimensional extension of the PR(TV) model, meanwhile appeared in [21], seems to confirm some of the speculations of the last unit. It is argued in [22] that this model have a chance to reproduce the euclidean Einstein-Hilbert action in the classical limit.

On the “minkowskian” side (that taking into account causality) I have to point out that what I called “second possibility” above was indeed already contained in [12] (I cannot explain myself my blindness) and is used in [23] to interpret a string worldsheet as a perturbation of evolving $\mathcal{G}$ simplicial spin networks, where $\mathcal{G}$ is any (quantum) gauge group. It is argued in [23] that the action of such a perturbation with respect to a background which tends (in a still unprecised sense) to flat spacetime tends to the spacetime area of the worldsheet.
## APPENDIX A: SOME MEAN GEOMETRIES

| $j_1$ $j_2$ $j_3$ $j_4$ | $\Upsilon$ eigenvalues | $l^2_i$ |
|--------------------------|-------------------------|---------|
| $1/2$ $1/2$ $1/2$ $1/2$ | $\pm 3^{1/2}/4$ | $l_1^2 = \cdots = l_6^2 = 1$ |
| $1$ $1$ $1$ $1$ | $0$, $\pm 3^{1/2}$ | $l_1^2 = \cdots = l_6^2 = 2 \cdot (2/3)^{1/2}$ |
| $3/2$ $3/2$ $3/2$ $3/2$ | $\pm 3 \cdot 3^{1/2}/4$, $\pm 3 \cdot 35^{1/2}/4$ | $l_1^2 = \cdots = l_6^2 = 5^{1/2}$ |
| $2$ $2$ $2$ $2$ | $0$, $\pm 3(11 \pm 57^{1/2}/2)^{1/2}/21^{1/2}$ | $l_1^2 = \cdots = l_6^2 = 2 \cdot 21^{1/2}$ |
| $5/2$ $5/2$ $5/2$ $5/2$ | $\pm 1155^{1/2}/34$, $\pm (2211 \pm 96 \cdot 481^{1/2}/2)^{1/2}/4$ | $l_1^2 = \cdots = l_6^2 = (35/3)^{1/2}$ |

| $j_1$ $j_2$ $j_3$ $j_4$ | $\Upsilon$ eigenvalues | $l^2_i$ |
|--------------------------|-------------------------|---------|
| $1$ $1$ $1$ $2$ | $\pm 3^{1/2}$ | $l_1^2 = l_2^2 = l_3^2 = 25^{3/2}$ |
| $3/2$ $3/2$ $3/2$ $1/2$ | $\pm 3^{1/2}$ | $l_1^2 = l_2^2 = l_3^2 = 4$ |
| $3/2$ $3/2$ $3/2$ $5/2$ | $0$, $\pm 3^{3/2}$ | $l_1^2 = l_2^2 = l_3^2 = 4 \cdot (5/21)^{1/2}$ |
| $3/2$ $3/2$ $3/2$ $7/2$ | $\pm 3^{3/2}/4$ | $l_1^2 = l_2^2 = l_3^2 = 3 \cdot (3/7)^{1/2}$ |
| $2$ $2$ $2$ $1$ | $0$, $\pm 3^{3/2}$ | $l_1^2 = l_2^2 = l_3^2 = 5 \cdot (2/3)^{1/2}$ |
| $2$ $2$ $2$ $3$ | $\pm 2 \cdot 3^{1/2}$, $\pm 2 \cdot 3^{1/2}$ | $l_1^2 = l_2^2 = l_3^2 = 5/2$ |
| $2$ $2$ $2$ $4$ | $0$, $\pm 6 \cdot 3^{1/2}$ | $l_1^2 = l_2^2 = l_3^2 = 19 \cdot 15^{1/2}/2$ |
| $2$ $2$ $2$ $5$ | $\pm 4 \cdot 3^{1/2}$ | $l_1^2 = l_2^2 = l_3^2 = 4 \cdot (2/5)^{1/2}$ |

| $j_1$ $j_2$ $j_3$ $j_4$ | $\Upsilon$ eigenvalues | $l^2_i$ |
|--------------------------|-------------------------|---------|
| $1/2$ $1/2$ $1/1$ $1$ | $\pm 2^{1/2}$ | $l_2^2 = l_4^2 = l_6^2 = 23/192^{1/2}$ |
| $1/2$ $1/2$ $3/2$ $3/2$ | $\pm 15^{1/2}/4$ | $l_2^2 = l_4^2 = l_6^2 = 11/28^{1/2}$ |
| $1/2$ $1/2$ $2$ $2$ | $\pm (3/2)^{1/2}$ | $l_2^2 = l_4^2 = l_6^2 = 71/736^{1/2}$ |
| $1$ $1$ $3/2$ $3/2$ | $0$, $\pm(13/2)^{1/2}$ | $l_2^2 = l_4^2 = l_6^2 = 232/2368^{1/2}$ |
| $1$ $1$ $2$ $2$ | $0$, $\pm 11^{1/2}$ | $l_2^2 = l_4^2 = l_6^2 = 59/440^{1/2}$ |
| $1/2$ $1/2$ $3/2$ $2$ | $\pm 3^{/2}$ | $l_2^2 = l_4^2 = l_6^2 = 347/8932^{1/2}$ |

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