Asymptotic Behavior of Multigroup SEIR Model with Nonlinear Incidence Rates under Stochastic Perturbations

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In this paper, the asymptotic behavior of a multigroup SEIR model with stochastic perturbations and nonlinear incidence rate functions is studied. First, the existence and uniqueness of the solution to the model we discuss are given. Then, the global asymptotical stability in probability of the model with $R_0 < 1$ is established by constructing Lyapunov functions. Next, we prove that the disease can die out exponentially under certain stochastic perturbation while it is persistent in the deterministic case when $R_0 > 1$. Finally, several examples and numerical simulations are provided to illustrate the dynamic behavior of the model and verify our analytical results.

1. Introduction

The history of human beings is full of struggle against diseases which cause great disaster to humans. At present, many countries and people around the world are suffering from the COVID-19, which has seriously affected people’s lives and brought huge losses to the economy. Epidemiology is the subject to study the spread of diseases and formulate the strategies and measures for controlling and eliminating diseases. Mathematical modeling has been widely used in epidemiology to depict the mechanism of disease transmission and study the behavior of disease. One of the classic epidemic models is the SIR model which divides the host population into three parts, the susceptible, the infective, and the removed, and records their sizes by $S(t)$, $I(t)$, and $R(t)$ at time $t$, respectively. However, many diseases do not break out immediately, and there will be a latent period of time, so SEIR models with latent period have been widely studied. In SEIR models, the size of the exposed individuals is labeled by $E(t)$ at time $t$.

Many models have considered the case of only one group; however, groups in different communities, regions, or with different cultural backgrounds have various lifestyles, dietary habits, and so on, which will make the disease have different ways of transmission. Therefore, considering different contact patterns, transmission, or geographic distributions, it is more reasonable to divide the host population into several subgroups and study the disease interactions among different subgroups. This is known as the multigroup model. One of the earliest works on the multigroup disease model was done by Lajmanovich and Yorke [1], who discussed a class of SIS multigroup models for the transmission of gonorrhea and used Lyapunov functions to prove the stability of the unique endemic equilibrium. Since then, there has been a great quantity of literature on the multigroup model, such as [2–8].

In the classic SEIR models, the incidence function takes the bilinear form. A premise for this form is that the host population is homogeneously mixed, and everyone has the same possibility to be infected when the infectives are introduced to the group. In real life, however, the population may not be homogeneously mixed, and the immunity of each person may be different such that the chances of being infected are disparate, so extending bilinear incidence to nonlinear functions can conform to the actual situation better. Many scholars have studied the epidemic models with nonlinear incidence rate, such as [4, 8–11] and the reference therein. Also, many scholars investigated the epidemic...
models with time delays, such as [12, 13]. In [4], the authors discussed the global stability of the multigroup epidemic model with nonlinear incidence rates of the form \( f_{kj}(S_k, I_j) \), which satisfies the following assumptions:

(i) \((H1)\) for \(0 < S_k \leq S^0_k \), it has \(0 < \lim_{t \to +0} \frac{dS_k}{dt} < d^S_k \), where \(d^S_k\) is the positive solution of certain function.

(ii) \((H2)\) \(f_{kj}(S_k, I_j) \leq C_{kj}(S_k)I_j\) for any \(I_j > 0\).

(iii) \((H3)\) \(C_{kj}(S_k) \leq C_{kj}(S^0_k)\) for \(0 < S_k \leq S^0_k\).

This research intends to study this general form of incidence function and assumes further that \((C_{kj}(S_k)/S_k) \leq K\), where \(K\) is a positive constant. The above incidence rate functions \(f_{kj}(S_k, I_j)\) include some special cases which can be seen in some literature, for example,

\[
\begin{align*}
  f_{kj}(S_k, I_j) &= S_k I_j, \\
  f_{kj}(S_k, I_j) &= S^I_k I_j, \quad q \geq 1, \\
  f_{kj}(S_k, I_j) &= \frac{S_k I_j}{1 + d^q_j}, \\
  f_{kj}(S_k, I_j) &= \frac{S_k I_j}{\phi(I_j)}.
\end{align*}
\]

The multigroup SEIR model with above incidence functions can be obtained:

\[
\begin{align*}
  \frac{dS_k}{dt} &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj}f_{kj}(S_k, I_j) - d^S_k S_k, \\
  \frac{dE_k}{dt} &= \sum_{j=1}^{n} \beta_{kj}f_{kj}(S_k, I_j) - (\epsilon_k + d^E_k) E_k, \\
  \frac{dI_k}{dt} &= \epsilon_k E_k - (\alpha_k + d^I_k + \gamma_k) I_k, \\
  \frac{dR_k}{dt} &= \gamma_k I_k - d^R_k R_k.
\end{align*}
\]

What the parameters mean can be summarized in the following list:

- \(\Lambda_k\): the influx of individuals in the \(k\)th group.
- \(\beta_{kj}\): the transmission rate between \(S_k\) and \(I_j\).
- \(d^S_k, d^I_k, d^E_k,\) and \(d^R_k\): the natural death rate of \(S, E, I,\) and \(R\) in the \(k\)th group, respectively.
- \(\epsilon_k\): the rate of becoming infectious in the \(k\)th group.
- \(\alpha_k\): the death rate caused by disease in the \(k\)th group.
- \(\gamma_k\): the cure rate in the \(k\)th group.

The parameters above are all nonnegative. In particular, when \(\beta_{kj} = 0\), it means that there is no disease transmission between \(S_k\) and \(I_j\). The matrix \(B = (\beta_{kj})_{n \times n}\) reflects the transmission mechanism of disease among different subgroups built in the model. In this paper, we assume that the matrix \(B\) is irreducible.

Since that \(R_k, k = 1, 2, \ldots, n\), do not appear in the first three equations of model (2) but only in the fourth equation, their properties and behaviors can be solved easily if \(I_k, k = 1, 2, \ldots, n\), are known; they can be omitted when analyzed. Therefore, the model can be simplified into the following form:

\[
\begin{align*}
  \frac{dS_k}{dt} &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj}f_{kj}(S_k, I_j) - d^S_k S_k, \\
  \frac{dE_k}{dt} &= \sum_{j=1}^{n} \beta_{kj}f_{kj}(S_k, I_j) - (\epsilon_k + d^E_k) E_k, \\
  \frac{dI_k}{dt} &= \epsilon_k E_k - (\alpha_k + d^I_k + \gamma_k) I_k.
\end{align*}
\]

In the epidemic models, the basic reproduction number \(R_0\), which represents the number of second generations produced by a single infected individual, plays an important role in the spread of disease for the long time. According to [4, 14], \(R_0 = \rho(M_0)\), where \(M_0 = (\beta_{kj}\epsilon_k C_{kj}(S^0_k)/(\alpha_k + d^I_k + \gamma_k)(\epsilon_k + d^E_k)_{n \times n}) S^0_k = \Lambda_k/d^S_k\), and \(\rho\) is the spectral radius of the matrix \(M_0\). If \(R_0 < 1\), there is only disease-free equilibrium \(P_0\), where \(P_0 = ((\Lambda_I/d^I_k), 0, \ldots, (\Lambda_n/d^I_n), 0, 0)\). When \(R_0 > 1\), then \(P_0\) is unstable, and the model has an endemic equilibrium \(P^*\) which means the disease will be persistent. In this situation, our concern is whether there is a way to exterminate the disease.

The reality is filled with randomness, and the epidemic models are often influenced by random environments. For example, there are a lot of natural disasters in reality, such as storm and earthquake. If these randomnesses happen, the parameters and the transmission mechanism in the model are likely to be affected. Thus, the deterministic model has some limitations to fully describe transmission of disease. Many scholars have studied the epidemic model with stochastic perturbations described by Brownian motion, and a lot of literature studies have been published; we refer the readers to [5, 7, 10, 12, 13, 15–17]. In [18–20], the authors studied the SIR or SIRS model with Markovian switching, and they gave some conditions on extinction or ergodicity of the model.

Influenced by the work of predecessors, we use the similar method of Dalal et al. and Witbooi [21, 22] to construct stochastic perturbations, that is, we replace the parameters \(d^E_k\) and \(d^I_k\) by \(d^E_k - \sigma_{1k}dB_k\) and \(d^I_k - \sigma_{2k}dW_k\), where the stochastic perturbations \(B_k\) and \(W_k\) are independent standard Brownian motions. The reason that not all parameters but only some of them are disturbed by stochastic perturbations may be the uncertainty of stochastic factors and the change of behavior of the infected.

For all we know, the papers that discuss asymptotic behaviors of stochastic multigroup SEIR models with nonlinear incidence rate functions are relatively few. In this paper, we will study the following stochastic multigroup SEIR model:
\[
\begin{aligned}
\dot{S}_k &= \left(\Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - d_k S_k\right) dt, \\
\dot{E}_k &= \left(\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - (\epsilon_k + d_k^2) E_k\right) dt + \sigma_{1k} E_k dB_k(t), \\
\dot{I}_k &= (\epsilon_k E_k - (\alpha_k + d_k^2 + \gamma_k) I_k) dt + \sigma_{2k} I_k dW_k(t),
\end{aligned}
\]

where \(\sigma_{ik}, i = 1, 2\), are the intensities of stochastic perturbation.

Because the incidence rate functions \(f_{kj}(S_k, I_j)\) are general and can be of different types in one model, which increase the difficulty of research, we will overcome it by some inequality techniques. This paper is organized as follows. Section 2 presents some background knowledge and lemmas which will be used afterwards. In Section 3, we prove that there is a unique positive solution to the model for any initial value. Section 4 proves that the disease-free equilibrium is globally asymptotically stable in probability when \(R_0 < 1\) by constructing Lyapunov functions. In Section 5, the disease will die out exponentially under certain stochastic perturbations when \(R_0 > 1\), and in Section 6, we provide some numerical simulations of the model to verify our analytical results.

### 2. Preliminaries

Throughout the paper, unless otherwise specified, \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) denotes a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is right continuous, and \(\mathcal{F}_0\) contains all \(P\)-null sets). Denote

\[\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}.
\]

In general, let \(X\) be a regular homogeneous Markov process in \(\mathbb{R}_+^n\); consider the stochastic differential equation

\[dX(t) = b(X(t))dt + \sum_{k=1}^{d} \sigma_k(X(t)) dB_k(t),\]

with initial value \(X(t_0) = x_0 \in \mathbb{R}_+^n\) and \(B_k(t), 1 \leq k \leq d\), are standard Brownian motions. Define the differential operator \(\mathcal{L}\) associated with the above equation by

\[\mathcal{L} = \sum_{k=1}^{n} b_k(x) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sigma^T(x) \sigma(x)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.
\]

If \(\mathcal{L}\) acts on a function \(V \in C^{2,1}(E' \times \mathbb{R}_+ ; \mathbb{R}_+)\), then by Itô\'s formula,

\[dV(X, t) = \mathcal{L}V(X, t) dt + \sum_{r=1}^{d} V_x(X, t) \sigma_r(X(t)) dB_r(t),\]

where

\[\mathcal{L}V(X, t) = V_t(X, t) + \sum_{k=1}^{n} b_k(x) \frac{\partial V}{\partial x_k}
\]

\[+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sigma^T(x) \sigma(x)\right]_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}.
\]

Next, we introduce some definitions about stability and lemmas which will be used latter. Assume that \(b(0) = 0\) and \(\sigma_k(0) = 0, k = 1, 2, \ldots, d\); then, \(X(t) \equiv 0\) is the trivial solution to (6).

**Definition 1.** The trivial solution is called to be

1. Stable in probability if for any \(\epsilon > 0\) and the solution \(X(t, x_0)\) with initial value \(X(0) = x_0\), then

\[\lim_{x_0 \to 0} \mathbb{P}(\sup_{t \geq 0} |X(t, x_0)| \geq \epsilon) = 0.\]

2. Globally asymptotically stable in probability if it is stable in probability, and for any \(x_0 \in \mathbb{R}_+^n\),

\[\mathbb{P}\left(\lim_{t \to \infty} X(t, x_0) = 0\right) = 1.\]

**Lemma 1** (cf. [23]). If there is a positive definite function \(V(t, x) \in C^2(\mathbb{R}_+^n)\) with an infinitesimal upper limit such that the function \(\mathcal{L}V\) is negative definite, then the trivial solution is globally asymptotically stable in probability.

**Lemma 2** (Perron–Frobenius). If \(A = (a_{ij})_{n \times n}\) is irreducible and nonnegative, then the spectral radius \(\rho(A)\) of \(A\) is a single eigenvalue, and there is a positive eigenvector \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)\) corresponding to \(\rho(A)\) of \(A\). Moreover, \(\rho(A)\) satisfies the inequality

\[\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_j \sum_i a_{ij}.
\]

**Remark 1.** From our previous description in Introduction, we know that \(R_0 = \rho(M_0) < 1\) will lead to the extinction of disease in deterministic model (3). Combining the expression of \(R_0\) with the estimation of \(\rho(M_0)\) in (12), we can infer that if transmission rate \(\beta_{kj}\) decreases, \(\rho(M_0)\) will become smaller, which provides the possibility of eliminating disease. A very important way to reduce \(\beta_{kj}\) is to isolate people at home and restrict them from going out. This measure is being taken in many countries to combat COVID-19.

**Lemma 3** (cf. [24]). Let \(M = \{M_t\}_{t \geq 0}\) be a real-valued continuous local martingale vanishing at \(t = 0\). Then,
\[ \lim_{t \to \infty} \frac{\langle M, M \rangle_t}{M_t} = 0 \quad \text{a.s.} \]
\[ \limsup_{t \to \infty} \frac{\langle M, M \rangle_t}{t} < \infty \quad \text{a.s.} \]
\[ \text{for } t \geq 0 \text{ almost surely.} \]

(13)

### 3. The Existence and Uniqueness of the Solution to Model (4)

The first question we concern is whether the system has a solution or not. In this section, we prove that the system has a global and positive solution for any initial value.

**Theorem 1.** Given any initial value \((S_1(0), E_1(0), I_1(0) \cdots, S_n(0), E_n(0), I_n(0)) \in \mathbb{R}^{3n}_+\), then model (4) has a unique solution on \(t \geq 0\), and the solution will remain in \(\mathbb{R}^{3n}_+\) with probability one, that is, \((S_1(t), E_1(t), I_1(t) \cdots, S_n(t), E_n(t), I_n(t)) \in \mathbb{R}^{3n}_+\) for \(t \geq 0\) almost surely.

**Proof.** Since the coefficients of the model are locally Lipschitz continuous, there exists a unique local solution \((S_1(t), E_1(t), I_1(t) \cdots, S_n(t), E_n(t), I_n(t)) \text{ on } t \in [0, \tau_e]\), where \(\tau_e\) is the explosion time (cf. [24]). In order to illustrate the solution is global, we only need to prove \(\tau_e = \infty\). Assume \(c_0 \) is sufficiently large so that 
\[ S_1(0), E_1(0), I_1(0) \cdots, S_n(0), E_n(0), I_n(0) \text{ lie within the interval } [(1/c_0), c_0]. \]

For \(c \geq c_0\), define the stopping time
\[ \tau_c = \inf \left\{ t \in [0, \tau_e], \min_{1 \leq k \leq n} \left\{ S_k(t), E_k(t), I_k(t) \right\} \leq \frac{1}{c} \right\} \]
\[ \text{or } \max_{1 \leq k \leq n} \left\{ S_k(t), E_k(t), I_k(t) \right\} \geq c. \]

(14)

We set \(\inf \emptyset = \infty\) (where \(\emptyset\) denotes the empty set). Clearly, \(\tau_c\) is increasing as \(c \to \infty\). Let \(\tau_{c_0} = \lim_{c \to \infty} \tau_c\), and \(\tau_{c_0} \leq \tau_e\) a.s. If we can prove \(\tau_{c_0} = \infty\) a.s., then equality \(\tau_e = \infty\) holds true, and the conclusion can be obtained. If the assertion is false, then there exist two constants \(T > 0\) and \(\epsilon \in (0, 1)\) such that
\[ \mathbb{P}(\tau_{c_0} \leq T) > \epsilon. \]

(15)

Hence, there exists a positive integer \(c_1 \geq c_0\) such that
\[ \mathbb{P}(\tau_e \leq T) > \epsilon, \quad \text{for all } c \geq c_1. \]

(16)

Then, we define a function \(V: \mathbb{R}^{3n}_+ \to \mathbb{R}\) by
\[ V(S_k, E_k, I_k) := \sum_{k=1}^{n} \left( S_k - a_k - a_k \ln \frac{S_k}{a_k} \right) \]
\[ + \left( E_k - 1 - \ln E_k \right) \]
\[ + \left( I_k - 1 - \ln I_k \right) \] 

(17)

where \(a_k, k = 1, 2, \cdots, n\) are constants which will be determined later. Using Itô’s formula, we can get

\[ dV = \sum_{k=1}^{n} \left( 1 - \frac{a_k}{S_k} \right) \left[ \left( \Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_k(S_k, I_j) - d_k^E S_k \right) dt \right] \]
\[ + \frac{1}{2} \sum_{k=1}^{n} \left[ \sigma_{1k}^2 + \sigma_{2k}^2 \right] \]
\[ + \sum_{k=1}^{n} \left( 1 - \frac{1}{E_k} \right) \left[ \left( \sum_{j=1}^{n} \beta_{kj} f_k(S_k, I_j) - (a_k + d_k^E + \gamma_k) E_k \right) dt \right] \]
\[ + \sigma_{1k} E_k dB_k(t) \]
\[ + \sum_{k=1}^{n} \left( 1 - \frac{1}{I_k} \right) \left[ (\epsilon_k E_k - (a_k + d_k^I + \gamma_k) I_k) dt \right] \]
\[ + \sigma_{2k} I_k dW_k(t) \]
\[ = \mathcal{D}V dt + \sum_{k=1}^{n} \left[ \sigma_{1k} (E_k - 1) dB_k(t) + \sigma_{2k} (I_k - 1) dW_k(t) \right]. \]

(18)

where

\[ \mathcal{D}V = \sum_{k=1}^{n} \left[ \Lambda_k - d_k^E S_k - \frac{a_k}{S_k} - \frac{a_k}{S_k} + \sum_{j=1}^{n} \beta_{kj} f_k(S_k, I_j) \right. \]
\[ + a_k d_k^E - d_k^E E_k \]
\[ - \frac{1}{E_k} \sum_{j=1}^{n} \beta_{kj} f_k(S_k, I_j) - (a_k + d_k^I + \gamma_k) E_k - \frac{\epsilon_k E_k}{I_k} \]
\[ + \epsilon_k + d_k^E + a_k + d_k^I + \gamma_k + \frac{1}{2} \sum_{k=1}^{n} \left[ \sigma_{1k}^2 + \sigma_{2k}^2 \right] \]
\[ \leq \sum_{k=1}^{n} \left[ \Lambda_k + a_k \sum_{j=1}^{n} \beta_{kj} K I_j + a_k d_k^E - (a_k + d_k^I + \gamma_k) I_k \right] \]
\[ + \epsilon_k + d_k^E + a_k + d_k^I + \gamma_k + \frac{1}{2} \sum_{k=1}^{n} \left[ \sigma_{1k}^2 + \sigma_{2k}^2 \right]. \]

(19)

Notice that

\[ \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} K I_j - \sum_{k=1}^{n} (a_k + d_k^I + \gamma_k) I_k \]
\[ = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} a_k \beta_{kj} K I_j \right)\]
\[ - \sum_{j=1}^{n} (a_j + d_j^I + \gamma_j) I_j \]
\[ = \sum_{j=1}^{n} \left[ \sum_{k=1}^{n} K a_k \beta_{kj} - (a_j + d_j^I + \gamma_i) \right] I_j. \]

(20)
We choose appropriate numbers \( a_k, 1 \leq k \leq n \), such that
\[ \sum_{k=1}^{n} K_k \beta_k - (\alpha_j + d_j^i + \gamma_k) = 0; \]
then, \( \mathcal{Z} V \leq M \), where \( M \) is a positive constant. Therefore,
\[
dV \leq Mdt + \sum_{k=1}^{n} \left[ \sigma_{2k}(I_k - 1)dW_k(t) + \sigma_{2k}(I_k - 1)dW_k(t) \right].
\]
Integrate both sides of (21) from 0 to \( \tau_c \wedge T \) and take expectation; then,
\[
\mathbb{E} V(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)) \\
\leq V(S_k(0), E_k(0), I_k(0)) + \mathbb{E} \int_{0}^{\tau_c \wedge T} Mdt \\
\leq V(S_k(0), E_k(0), I_k(0)) + MT.
\]

\[ (21) \]

Let \( \Omega_\omega = \{ \tau_c \leq T \} \); we have \( \mathbb{P}(\Omega_\omega) \geq c \). Notice that, for every \( \omega \in \Omega_\omega \), there exists at least one of \( S(\tau_c, \omega), E(\tau_c, \omega), I(\tau_c, \omega) \) which equals either \( c \) or \( 1/c \). Therefore,
\[
V(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)) \\
\geq \min_{1 \leq k \leq n} \left\{ c - a_k - a_k \ln \frac{c}{a_k} - a_k \ln \frac{1}{a_k c} \right\} + (c - 1 - \ln c) \left( \frac{1}{c} - 1 - \ln \frac{1}{c} \right).
\]
Combining (22) with (23), we can obtain that
\[
\mathbb{E} [1_{\Omega_\omega}] V(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)) \\
\geq c \left\{ \min_{1 \leq k \leq n} \left( c - a_k - a_k \ln \frac{c}{a_k} - a_k \ln \frac{1}{a_k c} \right) \right\} + (c - 1 - \ln c) \left( \frac{1}{c} - 1 - \ln \frac{1}{c} \right).
\]

\[ (24) \]

The assumption \( S_k(0) \leq \Lambda_k/d_k^i \) will be used in the rest of the paper. \( \square \)

4. The Behavior of the Model with \( R_0 < 1 \)

In the deterministic SEIR model, \( P_0 \) is the disease-free equilibrium, and it is globally stable which means that the disease will die out with any initial value when \( R_0 < 1 \). In this section, we will discuss the asymptotic behavior of the stochastic model with \( R_0 < 1 \).

Theorem 2. Let \((S_1(t), E_1(t), I_1(t), \ldots, S_n(t), E_n(t), I_n(t))\) be the solution to model (4) with the initial value initial
\( (S_1(0), E_1(0), I_1(0), \ldots, S_n(0), E_n(0), I_n(0)) \in \mathbb{R}_+^{3n} \). If \( B = \left( b_{kj} \right)_{n \times n} \) is irreducible and \( R_0 = \rho(M_0) < 1 \), then \( P_0 \) is the unique equilibrium of model (4), and it is globally asymptotically stable in probability.

Proof. According to the assumption, \( B \) is irreducible and nonnegative; then, by Lemma 2, \( M_0 \) has a single eigenvalue \( \rho(M_0) \) and a positive eigenvector \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) corresponding to \( \rho(M_0) \) such that
Let $V_1 = \sum_{k=1}^{n}(1/2)\alpha_k ((\Lambda_k/d_k^S)-S_k)^2$ and $V_2 = \sum_{k=1}^{n}\beta_k (E_k + ((\epsilon_k + d_k^S)/\epsilon_k)I_k)$, where $\epsilon_k = \omega_k\epsilon_k/(d_k^S + \epsilon_k)(a_k + d_k^S + \gamma_k)$ and $a_k$ will be determined later. Using Itô’s formula, we can obtain that

$$\mathcal{L}V_1 = -\sum_{k=1}^{n}a_k \left( \frac{\Lambda_k}{d_k^S} - S_k \right) \left[ \Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \frac{\Lambda_k}{d_k^S} - S_k \right]$$

$$\leq -\sum_{k=1}^{n}a_k \left( \frac{\Lambda_k}{d_k^S} - S_k \right)^2 + \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj} (S_k) I_j \left( S_k - \frac{\Lambda_k}{d_k^S} \right)$$

$$\leq -\sum_{k=1}^{n}a_k \left( \frac{\Lambda_k}{d_k^S} - S_k \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj} (S_k) I_j^2$$

$$\leq -\sum_{k=1}^{n}a_k \left( \frac{\Lambda_k}{d_k^S} - S_k \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj} (S_k) I_j^2.$$

Here, the second inequality holds true because of the inequality $ab \leq (\epsilon/2)a^2 + (1/2\epsilon)b^2$. Similarly, we use Itô’s formula to $V_2$ to get

$$\mathcal{L}V_2 = \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \frac{(\epsilon_k + d_k^S)(a_k + d_k^S + \gamma_k)}{\epsilon_k} \right]$$

$$\leq \sum_{k=1}^{n} \sum_{j=1}^{n} \omega_k \left( \epsilon_k + d_k^S \right) \left( a_k + d_k^S + \gamma_k \right) I_j - \sum_{k=1}^{n} \omega_k I_k$$

$$= (\rho_0 - 1) \sum_{k=1}^{n} \omega_k I_k.$$

5. The Influence of Large Noise on Disease

In this section, we will discuss the influence of large noises on disease when $R_0 > 1$. Before we give the theorem, an inequality is presented first.

**Lemma 4.** For $a_k$, $b_k$, $c_k$, $d_k$, $k = 1, 2, \ldots, n$, the following inequality holds true:

$$\left( \sum_{k=1}^{n} (a_k b_k + c_k d_k) \right) \leq \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} c_k \right).$$

**Proof.** We prove it by transforming it into an inner product in space $\mathbb{R}^n$. Let $a = (a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n$, and the vectors $b$, $c$, and $d$ are defined in a similar way. Then,

$$(a^T \mathbf{b} + c^T \mathbf{d})^2 = ((a, b) + (c, d))^2 \leq \|a\|^2 \|b\|^2 + \|c\|^2 \|d\|^2$$

$$+ 2 \|a\| \|b\| \|c\| \|d\|$$

$$\leq \|a\|^2 \|b\|^2 + \|c\|^2 \|d\|^2 + \|a\|^2 \|c\|^2 + \|b\|^2 \|d\|^2$$

$$\leq \|a\|^2 + \|c\|^2 \left( \|b\|^2 + \|d\|^2 \right).$$

The proof is completed.

**Theorem 3.** If $B = (\beta_{kj})_{1 < k, j < n}$ is irreducible, then we have
\[
\max_{1 \leq k \leq n} \left\{ \lim_{t \to \infty} \frac{1}{t} \ln E_k(t), \lim_{t \to \infty} \frac{1}{t} \ln I_k(t) \right\} \\
\leq (R_0 - 1) \max_{1 \leq k \leq n} \{ \alpha_k + d_k^l + \gamma_k \} \\
- \frac{1}{2} \sum_{k=1}^{n} \left( \frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right) \text{ a.s.}
\]  

**Proof.** We define a \( C^2 \) function \( V(E_k, I_k) \) by
\[
V = \sum_{k=1}^{n} \left( c_k \left( E_k + \frac{d_k^E + e_k}{e_k} I_k \right) \right),
\]
where \( c_k = \omega_k e_k / (d_k^E + e_k) (\alpha_k + d_k^l + \gamma_k) \). By calculation, we can get that
\[
dV = \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \beta_{kj} f_{kj}(S_k, I_j) - \sum_{k=1}^{n} \omega_k I_k \ \text{dt}
\]
\[
+ \sum_{k=1}^{n} c_k \sigma_{1k} E_k dB_k(t)
\]
\[
+ \sum_{k=1}^{n} c_k \sigma_{2k} I_k dW_k(t).
\]
Using Ito’s formula, we arrive at
\[
d\ln V = \frac{1}{V} \left[ \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \beta_{kj} f_{kj}(S_k, I_j) - \sum_{k=1}^{n} \omega_k I_k \right] \ \text{dt}
\]
\[
- \frac{1}{2V^2} \left[ \sum_{k=1}^{n} \sigma_{1k}^2 E_k^2 + \sigma_{2k}^2 \left( \frac{d_k^E}{e_k} + e_k \right)^2 I_k \right] \ \text{dt}
\]
\[
+ \frac{1}{V} \left[ \sum_{k=1}^{n} c_k \sigma_{1k} E_k dB_k(t) \right] + \frac{1}{V} \left[ \sum_{k=1}^{n} c_k \sigma_{2k} \frac{d_k^E + e_k}{e_k} I_k dW_k(t) \right].
\]
\[
\leq \frac{1}{V} \left[ \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \beta_{kj} f_{kj}(S_k, I_j) - \sum_{k=1}^{n} \omega_k I_k \right] \ \text{dt}
\]
\[
- \frac{1}{2V^2} \left[ \sum_{k=1}^{n} \sigma_{1k}^2 E_k^2 + \sigma_{2k}^2 \left( \frac{d_k^E}{e_k} + e_k \right)^2 I_k \right] \ \text{dt}
\]
\[
+ \frac{1}{V} \left[ \sum_{k=1}^{n} c_k \sigma_{1k} E_k dB_k(t) \right] + \frac{1}{V} \left[ \sum_{k=1}^{n} c_k \sigma_{2k} \frac{d_k^E + e_k}{e_k} I_k dW_k(t) \right].
\]
\[
= V_1(t) \ \text{dt} + V_2(t) \ \text{dt} + V_3(t) + V_4(t).
\]

For \( V_1(t) \), from the expression of eigenvector of \( R_0 \), i.e., \( (\omega_1, \omega_1 \cdots \omega_n) M_0 = R_0 (\omega_1, \omega_1 \cdots \omega_n) \), we obtain that
\[
V_1(t) = \frac{1}{V} (R_0 - 1) \sum_{k=1}^{n} \omega_k I_k
\]
\[
\leq \frac{R_0 - 1}{\sum_{k=1}^{n} c_k (d_k^E + e_k/e_k) I_k} \sum_{k=1}^{n} \omega_k I_k
\]
\[
\leq \max_{1 \leq k \leq n} \{ \alpha_k + d_k^l + \gamma_k \} (R_0 - 1).
\]

According to the expression of \( V \), utilizing Lemma 4 yields
\[
V^2 = \left[ \sum_{k=1}^{n} \left( c_k \sigma_{1k} E_k \frac{d_k^E + e_k}{e_k} I_k \right) \right]^2
\]
\[
\leq \left[ \sum_{k=1}^{n} c_k \left( \sigma_{1k}^2 E_k^2 + \sigma_{2k}^2 \left( \frac{d_k^E + e_k}{e_k} \right)^2 I_k \right) \right] \left[ \sum_{k=1}^{n} \left( \frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right) \right].
\]

Hence, \( V_2(t) \) satisfies the inequality
\[
V_2(t) \leq \frac{1}{2 \sum_{k=1}^{n} \left( \frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right)}.
\]

Because
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \sum_{k=1}^{n} c_k \sigma_{1k} E_k dB_k(t) < \infty,
\]
applying Lemma 3 to \( V_3(t) \) yields
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \sum_{k=1}^{n} V^{-1} c_k \sigma_{1k} E_k dB_k(t) = 0.
\]

\( V_4(t) \) can be done in the same way. Therefore,
\[
\limsup_{t \to \infty} \frac{\ln V(t)}{t} \leq \max_{1 \leq k \leq n} \{ d_k^l + \gamma_k \} (R_0 - 1)
\]
\[
- \frac{1}{2 \sum_{k=1}^{n} \left( \frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right)}.
\]

Since
\[
\frac{1}{t} \ln \left( \sum_{k=1}^{n} c_k E_k \right) \leq \frac{1}{t} \left[ \ln \left( \max_{1 \leq k \leq n} \{ c_k \} \right) \max_{1 \leq k \leq n} \{ E_k \} \right]
\]
\[
\leq \frac{1}{t} \left[ \ln \left( \max_{1 \leq k \leq n} \{ c_k \} \right) + \ln \left( \max_{1 \leq k \leq n} \{ E_k \} \right) \right],
\]
\[
\frac{1}{t} \ln \left( \sum_{k=1}^{n} c_k E_k \right) \geq \frac{1}{t} \left[ \ln \left( \min_{1 \leq k \leq n} \{ c_k \} \right) \max_{1 \leq k \leq n} \{ E_k \} \right]
\]
\[
\geq \frac{1}{t} \left[ \ln \left( \min_{1 \leq k \leq n} \{ c_k \} \right) + \ln \left( \max_{1 \leq k \leq n} \{ E_k \} \right) \right],
\]
taking the upper limit yields
Making use of the same method, we can obtain that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left( \sum_{k=1}^{n} c_k E_k \right) = \max_{1 \leq k \leq n} \left\{ \limsup_{t \to \infty} \frac{1}{t} \ln E_k \right\}. \tag{43}
\]

Making use of the same method, we can obtain that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left( \sum_{k=1}^{n} \frac{c_k \left( d_k E_k + e_k \right)}{e_k} I_k \right) = \max_{1 \leq k \leq n} \left\{ \limsup_{t \to \infty} \frac{1}{t} \ln I_k \right\}. \tag{44}
\]

Combining (43) and (44) yields
\[
\limsup_{t \to \infty} \frac{1}{t} \ln V(t) \geq \max_{1 \leq k \leq n} \left\{ \limsup_{t \to \infty} \frac{1}{t} \ln E_k, \limsup_{t \to \infty} \frac{1}{t} \ln I_k \right\}. \tag{45}
\]

Along with (41), we arrive at the desired assertion. The proof is complete. \qed
**Corollary 2.** For the solution to model (4), $E_k(t)$ and $I_k(t)$, $k = 1, 2 \cdots n$, decay exponentially to zero almost surely if

$$R_0 - 1 > \max_{1 \leq k \leq n} \left\{ \alpha_k + d_k + \gamma_k \right\} \frac{1}{2 \sum_{k=1}^{n} \left( \frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right)}.$$  \hfill (46)

**Remark 2.** From (46), we know that the right side of the inequality increases with the increase of $\sigma_{1k}$ and $\sigma_{2k}$; therefore, the inequality above holds true for certain $\alpha_k$, $d_k$, $\gamma_k$, and sufficiently large $\sigma_{1k}$ and $\sigma_{2k}$ even if $R_0 > 1$, which makes the disease extinct. It reflects that stochastic perturbations play an important role in disease control. Compared with the deterministic model in [4], the SEIR model with stochastic perturbations can show more properties and different behaviors.

**Remark 3.** We can see from many literature studies that the incidence function of the multigroup SEIR model is single one, such as $S_k(t)I_j(t)$ in [5, 7] and $S_k(t)I_j(t)/(1 + \alpha_kI_j(t))$ in [9]. These may have some limitations and cannot reflect the actual situation well. Incidence functions $f_{kj}(S_k(t), I_j(t))$ in this paper can be expressed in different forms, which can better
describe the reality of life. We will provide different examples to illustrate the results in Section 6.

6. Examples and Numerical Simulations

In this section, we give some simulations of model (4) to confirm the analytical results above. By using Milstein’s higher-order method [25], we obtain the corresponding discretization equation:

![Figure 3: The trajectories with $R_0 > 1$ and two different incidence functions: (a) the trajectory without stochastic perturbation; (b, c) the trajectories with stochastic perturbations where parameters are shown in Example 3.](image)
\[
S_{i,k+1} = S_{i,k} + \left( \Lambda_i - \sum_{j=1}^{n} \beta_{ij} f_{kj}(S_{i,k}, I_{j,k}) - d_{i}^{S} S_{i,k} \right) \Delta t,
\]
\[
E_{i,k+1} = E_{i,k} + \left( \sum_{j=1}^{n} \beta_{ij} f_{kj}(S_{i,k}, I_{j,k}) - (\epsilon_i + d_{i}^{E}) E_{i,k} \right) \Delta t + \sigma_{1i} E_{i,k} \eta_{ik} \sqrt{\Delta t} + \frac{1}{2} \sigma_{1i}^2 E_{i,k} (\eta_{ik}^2 \Delta t - \Delta t),
\]
\[
I_{i,k+1} = I_{i,k} + \left( \epsilon_i E_{i,k} - (\alpha_i + d_{i}^{I} + \gamma_i) I_{i,k} \right) \Delta t + \sigma_{2i} I_{i,k} \rho_{ik} \sqrt{\Delta t} + \frac{1}{2} \sigma_{2i}^2 I_{i,k} (\rho_{ik}^2 \Delta t - \Delta t),
\]

where $\eta_{ik}, \rho_{ik}$ are Gaussian random variables which follow the distribution $N(0, 1)$. Let $n = 2$, i.e., we consider the interactions of diseases in two groups.

First, we give an example to verify Theorem 2.

**Example 1.** Assume that $f_{kj} = S_{i,k} I_{j,k}/(1 + 2 I_{j,k}^2)$. We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \gamma_1 = 0.5, \gamma_2 = 0.6; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.02, \beta_{12} = 0.05, \beta_{21} = 0.04, \beta_{22} = 0.02, d_{1}^{S} = 0.05, d_{1}^{E} = 0.1, d_{1}^{I} = 0.08, d_{2}^{S} = 0.12, d_{2}^{E} = 0.1, d_{2}^{I} = 0.15$ such that $R_0 = 0.4835 < 1$, which satisfies the condition of Theorem 2. Moreover, let $\sigma_{11} = 1, \sigma_{12} = 0.5, \sigma_{21} = 0.8$, and $\sigma_{22} = 1.5$. Its trajectory is shown in Figure 1.

From Figure 1(a), we can see that the diseases are extinct when stochastic perturbations are absent. From Figures 1(b) and 1(c), we can see the diseases in two groups are globally asymptotically stable.

Now, we move forward to verify Theorem 3. We will present two examples to illustrate the two cases of incidence functions. In Example 2, we give the same incidence function for two groups, and in Example 3, two different incidence functions are presented.

**Example 2.** Assume that $f_{kj} = S_{i,k} I_{j,k}/(1 + 2 I_{j,k}^2)$. We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \epsilon_1 = 0.5, \epsilon_2 = 0.6; \gamma_1 = 0.4, \gamma_2 = 0.3; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.04, \beta_{12} = 0.05, \beta_{21} = 0.04, \beta_{22} = 0.02, d_{1}^{S} = 0.1, d_{1}^{E} = 0.08, d_{1}^{I} = 0.12, d_{2}^{S} = 0.1, d_{2}^{E} = 0.08, d_{2}^{I} = 0.12, d_{1}^{E} = 0.1, d_{2}^{E} = 0.15$ such that $R_0 = 1.685 > 1$, which satisfies the condition of Theorem 3. Moreover, let $\sigma_{11} = 1, d_{12} = 0.5, \sigma_{31} = 0.8, \sigma_{32} = 1.5$ so that $(R_0 - 1) \max_{1 \leq k \leq n} [\alpha_k + d_{k}^{I} + \gamma_k] < 1/2 \sum_{i=1}^{n} [(1/ \sigma_{ik}) + (1/ \sigma_{ik}^2)]$ is satisfied. Its trajectory is shown in Figure 2. From Figure 2(a), we can see that the diseases are persistent because of $R_0 > 1$ when stochastic perturbation is absent. We can see in Figures 2(b) and 2(c) that the diseases in two groups die out under certain stochastic perturbations and the exposed are the same results.

**Example 3.** Assume that $f_{1j} = S_{i,j}/(1 + 2 I_{j}^2)$, $f_{2j} = S_{i,j}/(1 + 2 I_{j}^2)$, $j = 1, 2$, such that

\[
M_0 = \begin{pmatrix}
\beta_{11} \epsilon_1 (\Lambda_1 / d_{1}^{E})^2 & \beta_{12} \epsilon_1 (\Lambda_1 / d_{1}^{E})^2 \\
(\alpha_1 + d_{1}^{I} + \gamma_1) (\alpha_1 + d_{1}^{I} + \gamma_1) & (\alpha_2 + d_{2}^{I} + \gamma_2) (\alpha_2 + d_{2}^{I} + \gamma_2)
\end{pmatrix}
\]

We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \epsilon_1 = 0.5, \epsilon_2 = 0.6; \gamma_1 = 0.4, \gamma_2 = 0.3; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.04, \beta_{12} = 0.05, \beta_{21} = 0.04, \beta_{22} = 0.02, d_{1}^{S} = 0.1, d_{1}^{E} = 0.08, d_{1}^{I} = 0.12, d_{2}^{S} = 0.1, d_{2}^{E} = 0.08, d_{2}^{I} = 0.12, d_{1}^{E} = 0.1, d_{2}^{E} = 0.15$ so that $R_0 = 3.61 > 1$ can be obtained. Moreover, let $\sigma_{11} = 4, \sigma_{12} = 4.5, \sigma_{21} = 2.8, \sigma_{22} = 5$; then, the conditions in Theorem 3 are satisfied. Its trajectory is shown in Figure 3. From Figure 3(a), we can see that the diseases are persistent because $R_0 > 1$ without stochastic perturbation. We can see in Figures 3(b) and 3(c) that the exposed and infected in two groups die out under certain stochastic perturbations, which conform to the results of Theorem 3.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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