Complementary Romanovski-Routh polynomials: From orthogonal polynomials on the unit circle to Coulomb wave functions∗

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Abstract

We consider properties and applications of a sequence of polynomials known as complementary Romanovski-Routh polynomials (CRR polynomials for short). These polynomials, which follow from the Romanovski-Routh polynomials or complexified Jacobi polynomials, are known to be useful objects in the studies of the one-dimensional Schrödinger equation and also the wave functions of quarks. One of the main results of this paper is to show how the CRR-polynomials are related to a special class of orthogonal polynomials on the unit circle. As another main result, we have established their connection to a class of functions which are related to a subfamily of Whittaker functions that includes those associated with the Bessel functions and the regular Coulomb wave functions. An electrostatic interpretation for the zeros of CRR-polynomials is also considered.

Keywords: Romanovski-Routh polynomials, Second order differential equations, Orthogonal polynomials on the unit circle, Para-orthogonal polynomials, Coulomb wave functions.

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1 Introduction

The Romanovski-Routh polynomials are defined by the Rodrigues formula

$$R_n^{(\alpha,\beta)}(x) = \frac{1}{\omega^{(\alpha,\beta)}(x)} \frac{d^n}{dx^n} \left[ \omega^{(\alpha,\beta)}(x)(1+x^2)^n \right], \quad n \geq 1,$$

(1.1)

(see [23]), where \(\omega^{(\alpha,\beta)}(x) = (1+x^2)^{\beta-1}e^{-\arccot x} x^{\alpha}\). These polynomials, first appeared in Routh [27], were rediscovered by Romanovski [26] in his work regarding probability distributions. They are found to be solutions of the second order differential equations

$$(1+x^2)y'' + (2\beta x + \alpha)y' - n(2\beta + n - 1)y = 0, \quad n \geq 1.$$  

(1.2)

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Since $R_n^{(a,b)}(x) = (-2i)^n n! P_n^{(b-1+\frac{1}{2}a, b-1-\frac{1}{2}a)}(ix)$, $n \geq 0$, the polynomials $R_n^{(a,b)}$ are also known as complexified Jacobi polynomials. The above connection formula between the polynomials $R_n^{(a,b)}$ and the Jacobi polynomials $P_n^{(a,b)}$ can be directly verified from Eq. (1.1) and the Rodrigues formula (see, for example, Eq. (4.2.8)) for the Jacobi polynomials.

The Romanovski-Routh polynomials $R_n^{(a,b)}$ are not orthogonal polynomials in the usual sense. When $\beta$ is chosen to be negatively large enough (see, for example, [25]) then there holds a finite orthogonality. Precisely,

$$\int_{-\infty}^{\infty} R_n^{(a,b)}(x) R_m^{(a,b)}(x) \omega^{(a,b)}(x) dx = 0 \quad \text{for} \quad m \neq n,$$

only when $m + n - 1 < -2\beta$.

In this paper we will focus on a study of the so called complementary Romanovski-Routh polynomials (CRR polynomials for short). These polynomials were defined in Weber [36] by considering a variation of the Rodrigues formula (1.1). From the definition given in Weber [36], the CRR polynomials are

$$Q_n^{(a,b)}(x) = \frac{(1 + x^2)^n}{\omega^{(a,b)}(x)} \frac{d^n}{dx^n} \omega^{(a,b)}(x), \quad n \geq 1. \quad (1.3)$$

The idea of looking at such complementary polynomials started in the work [35] of the same author.

As already observed in [25], one can easily verify that $Q_n^{(a,b)}(x) = R_n^{(a,-b-n)}(x)$, $n \geq 1$. Moreover, as shown also in [25] and [36], the polynomials $Q_n^{(a,b)}$ satisfy the three term recurrence formula

$$Q_{n+1}^{(a,b)}(x) = [\alpha + 2(\beta - n - 1)x]Q_n^{(a,b)}(x) n(-2\beta + n + 1)(1 + x^2)Q_{n-1}^{(a,b)}(x), \quad (1.4)$$

for $n \geq 1$, with $Q_0^{(a,b)}(x) = 1$ and $Q_1^{(a,b)}(x) = \alpha + 2(\beta - 1)x$. It is also not difficult to verify from the differential equation (1.2) that

$$Q_n^{(a,b)}(x) = (-2i)^n \left(\beta - n + i\frac{\alpha}{2}\right)_n {}_2F_1\left(-n, 2\beta - 1 - n; \beta - n + i\frac{\alpha}{2}; \frac{1 - ix}{2}\right). \quad (1.5)$$

The CRR polynomials are known to be (see [25]) important ingredients in some studies concerning wave functions of quarks in accord with QCD (quantum chromodynamics) quark-gluon dynamics. Moreover, these polynomials also play an important role in the studies of (one-dimensional) Schrödinger equation with hyperbolic Scarf potential.

The main objective in the present work is to consider some further properties of these CRR polynomials. However, for convenience, we will view the CRR polynomials using the modified notation $P_n(b;x)$, where $b = \lambda + i\eta$ and

$$P_n(b;x) = \frac{(-1)^n}{2^n(\lambda)_n} Q_n^{(2n-\lambda+1)} B_n^{(2n-\lambda+1)}(x), \quad n \geq 1. \quad (1.6)$$

We will also assume that $\lambda = \text{Re}(b) > 0$ and refer to the polynomials $P_n(b;x)$ as modified CRR polynomials or simply CRR polynomials.

One of the reasons for adopting the notation $P_n(b;x)$, instead of $P_n^{(\lambda,\eta)}$ for example, is to avoid confusion with the notation used for Jacobi polynomials.

The notation $\{P_n(b;x)\}_{n \geq 0}$ is also somewhat interesting to work with. For example, in the three term recurrence formula (1.7) for $\{P_n(b;x)\}_{n \geq 0}$ given below, the sequence of coefficients $\{d_{n+1}\}_{n \geq 1}$ is exactly the same as the sequence of coefficients that appear in the three term recurrence formula for the monic Gegenbauer (i.e., ultraspherical) polynomials $\{C_n^{(\lambda)}\}_{n \geq 0}$.

In the following theorem we have gathered some of the basic properties of the CRR polynomials.
Theorem 1.1. For \( b = \lambda + i\eta, \lambda > 0 \), the complementary Romanovski-Routh polynomials \( \mathcal{P}_n(b,.) \) can be given by the hypergeometric expression

\[
\mathcal{P}_n(b;x) = \frac{(x-i)^n}{2^n} \binom{2\lambda}{n} \binom{-\lambda}{n} {}_2F_1 \left( -n, b+\delta; \frac{-2i}{x-i} \right), \quad n \geq 1.
\]

They satisfy the three term recurrence

\[
\mathcal{P}_{n+1}(b;x) = (x - c_{n+1}^{(b)}) \mathcal{P}_n(b;x) - d_{n+1}^{(b)}(x^2 + 1) \mathcal{P}_{n-1}(b;x), \quad n \geq 1,
\]

with \( \mathcal{P}_0(b;x) = 1 \) and \( \mathcal{P}_1(b;x) = x - c_1^{(b)} \), where

\[
c_n^{(b)} = \frac{\eta}{\lambda + n - 1} \quad \text{and} \quad d_n^{(\lambda)} = d_{n+1}^{(\lambda)} = \frac{n(2\lambda + n - 1)}{4(\lambda + n - 1)(\lambda + n)}, \quad n \geq 1.
\]

Moreover, if \( \lambda > 1/2 \) then they also satisfy the orthogonality

\[
\int_{-\infty}^{\infty} x^m \mathcal{P}_n(b;x) \nu^{(\lambda,\gamma)}(x) dx = \gamma_n^{(\lambda)} \delta_{m,n}, \quad m = 0, 1, \ldots, n,
\]

where

\[
\nu^{(\lambda,\gamma)}(x) = \frac{2^{2\lambda-1} \Gamma(b)^2 e^{\gamma \eta}}{\Gamma(2\lambda - 1)} 2\pi \omega^{(2\eta, -\lambda + 1)}(x) = \frac{2^{2\lambda-1} \Gamma(b)^2 e^{\gamma \eta}}{\Gamma(2\lambda - 1)} \frac{(e^{-\text{arccot} x})^{2\eta}}{(1 + x^2)^\lambda}.
\]

Here, \( \gamma_0^{(\lambda)} = \int_{-\infty}^{\infty} \nu^{(\lambda,\gamma)}(x) dx = 1 \) and \( \gamma_n^{(\lambda)} = (1 - L_n^{(\lambda)}) \gamma_{n-1}^{(\lambda)}, \quad n \geq 1, \)

with

\[
L_n^{(\lambda)} = \frac{12 \lambda + n - 2}{2(\lambda + n - 1)}, \quad n \geq 1.
\]

Remark 1.1. Here, we have assumed \( \text{arccot}(x) \) to be a continuous function that decreases from \( \pi \) to 0 as \( x \) increases from \( -\infty \) to \( \infty \).

Part of the proof of Theorem 1.1 directly follows from some of the known results stated above. However, we have given a proof of this theorem in Section 2 which follows as a consequence of some recently known results on orthogonal polynomials on the unit circle.

From the three term recurrence (1.7) for \( \{\mathcal{P}_n(b,.)\}_{n \geq 0} \) one can easily observe that the leading coefficient of \( \mathcal{P}_n(b,.) \) is positive. Precisely, if \( \mathcal{P}_n(b;x) = p_n^{(b)} x^n + \text{lower order terms} \), then \( p_0^{(b)} = 1 \) and \( p_n^{(b)} = (1 - \ell_n^{(\lambda)}) p_{n-1}^{(b)}, \quad n \geq 1, \)

where

\[
1 - \ell_n^{(\lambda)} = \frac{(2\lambda)_n}{2n(\lambda)_n}, \quad n \geq 1.
\]

The sequence \( \{d_n^{(\lambda)}\}_{n \geq 1} \) in (1.7) is a so called positive chain sequence with the sequence \( \{\ell_n^{(\lambda)}\}_{n \geq 0} \), as given above, its minimal parameter sequence. That is,

\[
(1 - \ell_n^{(\lambda)}) \ell_{n+1}^{(\lambda)} = d_n^{(\lambda)}, \quad n \geq 1, \quad \text{with} \quad \ell_1^{(\lambda)} = 0 \quad \text{and} \quad 0 < \ell_n^{(\lambda)} < 1, \quad n \geq 2.
\]

Any sequence \( \{g_n\}_{n \geq 0} \) such that \( (1 - g_n) g_{n+1} = d_n^{(\lambda)}, \quad n \geq 1, \) with \( 0 \leq g_1 < 1 \) and \( 0 < g_n < 1 \) for \( n \geq 2 \), can be referred to as a parameter sequence of the positive chain sequence \( \{d_n^{(\lambda)}\}_{n \geq 1} \).

When \( 1/2 \geq \lambda > 0 \), the sequence \( \{\ell_n^{(\lambda)}\}_{n \geq 0} \) is the only parameter sequence of \( \{d_n^{(\lambda)}\}_{n \geq 1} \).

When \( \lambda > 1/2 \), the sequence \( \{L_n^{(\lambda)}\}_{n \geq 1} \) given by (1.10) is also such that

\[
(1 - L_n^{(\lambda)}) L_{n+1}^{(\lambda)} = d_n^{(\lambda)}, \quad n \geq 1, \quad \text{with} \quad 0 < L_n^{(\lambda)} < 1, \quad n \geq 1.
\]
Hence, when \( \lambda > 1/2 \), the sequence \( \{ \mathcal{L}^{(\lambda)}_{n+1} \}_{n \geq 0} \) is also a parameter sequence of the positive chain sequence \( \{ d^{(\lambda)}_{n+1} \}_{n \geq 1} \). It turns out \( \{ \mathcal{L}^{(\lambda)}_{n+1} \}_{n \geq 0} \) is the so called maximal parameter sequence of this positive chain sequence. For definitions and for many of the basic results concerning positive chain sequences we refer to [9].

We can verify from (1.2) that the differential equation satisfied by \( \mathcal{P}_n(b;.) \) is

\[
A(x) \mathcal{P}_n''(b; x) - 2B(n, \lambda, \eta; x) \mathcal{P}_n'(b; x) + C(n, \lambda) \mathcal{P}_n(b; x) = 0, \tag{1.13}
\]

where \( A(x) = x^2 + 1, B(n, \lambda, \eta; x) = (\lambda + n - 1)x - \eta, C(n, \lambda) = n(n - 1 + 2\lambda) \).

Many other properties of \( \mathcal{P}_n(b;.) \) (i.e. of the polynomials \( Q_{n(\alpha,\beta)} \)) have been explored in [31] and [25]. Perhaps one of the most interesting and simplest of these properties is

\[
\frac{d \mathcal{P}_n(b; x)}{dx} = n(1 - \ell_n) \mathcal{P}_{n-1}(b; x), \quad n \geq 1, \tag{1.14}
\]

which can be verified from (1.3). With this property, clearly (1.7) and (1.13) are equivalent statements. The results given in Section 4 of this paper are developed as a consequence of the property (1.14).

The contents in the remaining sections of this paper are as follows.

- In Section 2 as one of the main results of this paper, we show how the CRR polynomials \( \mathcal{P}_n(b;.) \) are related to a special class of orthogonal polynomials on the unit circle. A proof of Theorem 1.1 is also given in this section.

- In Section 3 we also give an electrostatic interpretation for the zeros of CRR polynomials.

- Another main contribution of this paper, considered in Section 4, is concerned with the generating function of the form \( e^{xw} \zeta(b; w) \) for the monic CRR polynomials. It turns out that the functions \( \zeta(b;.) \) are closely related to the subfamily \( M_{-\eta,\lambda-1/2} \) of Whittaker functions (see [2]). Thus, special cases of the function \( \zeta(b;.) \) are also related to the Bessel functions and the regular Coulomb wave functions. We have referred to this subfamily of Whittaker functions as extended regular Coulomb wave functions.

## 2 Orthogonal polynomials on the unit circle

We now consider the connection the CRR polynomials \( \mathcal{P}_n(b;.) \) have with the polynomials \( \Phi_n(b;.) \) which are orthogonal on the unit circle with respect to the probability measure

\[
d\mu^{(b)}(e^{i\theta}) = \frac{4Re(b)|\Gamma(b + 1)|^2}{\Gamma(b + b + 1)} \frac{1}{2\pi} [e^{i\theta}]^{Im(b)} [\sin^2(\theta/2)]^{Re(b)} d\theta. \tag{2.1}
\]

Observe that the above measure presents a Fischer-Hartwig type singularity. The monic orthogonal polynomials \( \Phi_n(b;.) \) and the associated orthogonality relation, which exist for \( \lambda > -1/2 \), are explicitly given by (see [31])

\[
\Phi_n(b; z) = \frac{(2\lambda + 1)_n}{(b + 1)_n} \binom{\alpha}{\beta}(-n, b + 1; b + b + 1; 1 - z), \quad n \geq 0, \tag{2.2}
\]

and, for \( n, m = 0, 1, 2, \ldots \),

\[
\int_0^{2\pi} \Phi_m(b; e^{i\theta}) \Phi_n(b; e^{i\theta}) d\mu^{(b)}(e^{i\theta}) = \frac{(2\lambda + 1)_n n!}{|(b + 1)_n|^2} \delta_{m,n}.
\]
For general information on orthogonal polynomials on the unit circle we refer to the monographs Szegő [31], Simon [28], Simon [29] and Ismail [16].

Let the polynomials \( \{ R_n(b; \cdot) \}_{n \geq 0} \) be such that

\[
R_n(b; \zeta) = \frac{2^n}{(x - i)^n} P_n(b; x), \quad n \geq 0,
\]

(2.3)

where \( \zeta = (x + i)/(x - i) \) or equivalently \( x = i(\zeta + 1)/(\zeta - 1) \). This transformation maps the real line \((-\infty, \infty) \) onto the cut unit circle \( \mathbb{T} \setminus [1] : = \{ \zeta = e^{i\theta}; 0 < \theta < 2\pi \} \). With known results about the polynomials \( R_n(b; \cdot) \) on the unit circle, which we will point out as needed, we now look into a proof of Theorem 1.1.

**Proof of Theorem 1.1.** From (1.13) and (2.3), it is not difficult to see that

\[
z(1 - z) \frac{d^2 R_n(b; z)}{dz^2} - [b + \bar{b} - (n + b + 1)(1 - z)] \frac{dR_n(b; z)}{dz} + nbR_n(b; z) = 0,
\]

for \( n \geq 1 \). From this differential equation we can easily identify the polynomials \( R_n(b; \cdot) \) to be

\[
R_n(b; z) = \frac{(2\lambda)^n}{(\lambda)^n} _2F_1(-n, b + \bar{b}; 1 - z), \quad n \geq 0.
\]

(2.4)

This result, together with (2.3), gives the proof of the hypergeometric expression given in Theorem 1.1. However, the hypergeometric expression in Theorem 1.1 can also be obtained from (1.5) using two transformations of Pfaff, namely the ones given by (2.2.6) and (2.3.14) in [2].

Now, if the three term recurrence in Theorem 1.1 is true then one must also have

\[
R_{n+1}(b; z) = \left[ (1 + i c_{n+1}^{(b)})z + (1 - i c_{n+1}^{(b)}) \right] R_n(b; z) - 4d_{n+1}^{(b)} z R_{n-1}(b; z),
\]

(2.5)

for \( n \geq 1 \), with \( R_0(b; z) = 1 \) and \( R_1(b; z) = (1 + i c_1^{(b)})z + (1 - i c_1^{(b)}) \), where the coefficients \( c_n^{(b)} \) and \( d_{n+1}^{(b)} \) are as in (1.8).

From results given in [10] and [31] we also find that the polynomials \( R_n(b; \cdot) \) given by (2.4) actually satisfy the above three term recurrence. Thus, from (2.3) the three term recurrence relation in Theorem 1.1 is also verified. Once again, the three term recurrence relation in Theorem 1.1 can also be easily derived from the three term recurrence relation (1.4) for the polynomials \( C_n^{(2)}(\lambda - \eta + 1) \).

The polynomials \( R_n(b; \cdot) \) (see [31]) are also the para-orthogonal polynomials

\[
R_n(b; z) = \frac{(b)}{(\lambda)^n} \left[ z \Phi_{n-1}(b; z) + \frac{(b)}{(\lambda)^n} \Phi^{\ast}_{n-1}(b; z) \right], \quad n \geq 1,
\]

(2.6)

associated with the orthogonal polynomials on the unit circle \( \Phi_n(b; \cdot) \) with respect to the measure \( \mu^{(b)} \) given above.

The polynomials \( R_n(b; \cdot) \) and the associated orthogonal polynomials \( \Phi_n(b; \cdot) \), in addition to have been studied in [31], they have been used as examples in a sequence of papers [3, 4, 5, 10, 12, 18, 19], without knowing anything about their connection to the CRR polynomials. The results obtained in [18] are focused on the three term recurrence of the type (1.7) and the associated generalized eigenvalue problem (with these respect see also [17] and [40]). From what we can observe from [4, p. 304], the polynomials \( R_n(b; \cdot) \) and \( \Phi_n(b; \cdot) \) were also have been observed as belonging to a class of hypergeometric biorthogonal polynomials. But, again in [4], no such connection to the Romanovski-Routh polynomials has been mentioned.

However, the connection between the circular Jacobi polynomials and a subfamily of the CRR polynomials is somewhat known in the literature (see [37]). We recall that the circular Jacobi polynomials are the subclass of the polynomials \( \Phi_n(b; \cdot) \) in which \( \text{Im}(b) = \eta = 0 \).
Even though the polynomials $\mathcal{R}_n(b;\cdot)$ and their associated CRR polynomials $\mathcal{P}_n(b;\cdot)$ are defined for $\text{Re}(b) = \lambda > 0$, the orthogonal polynomials $\Phi_n(b;\cdot)$ themselves can be considered for $\lambda > -1/2$.

We now consider the reproducing kernels
\[
K_n(b; z, w) = \frac{\varphi_{n+1}(b; w) \varphi_{n+1}(b; z) - \varphi_{n+1}^*(b; w) \varphi_{n+1}^*(b; z)}{wz - 1}, \quad n \geq 0.
\]

Here, $\varphi_n(b;\cdot)$ are the orthonormal version of $\Phi_n(b;\cdot)$.

The above kernels are also known as Christoffel-Darboux kernels or simply CD Kernels (see, for example, [23]). For any fixed $w$, the kernel $K_n(b; z, w)$ is a polynomial in $z$ of degree $\leq n$ and, in particular, if $|w| \geq 1$ then it is a polynomial in $z$ of exact degree $n$.

When $\lambda > 1/2$, the polynomials $\mathcal{R}_n(b;\cdot)$ are also modified kernel polynomials of the orthogonal polynomials $\Phi_n(b - 1;\cdot)$, evaluated at $w = 1$. That is,
\[
\mathcal{R}_n(b; z) = \xi_n^{(b-1)} K_n(b - 1; z, 1), \quad n \geq 0,
\]
where $\xi_0^{(b-1)} = \int_T d\mu^{(b-1)}(\xi)$ and $\xi_n^{(b-1)} = \xi_{b_n}^{(b-1)} \prod_{j=1}^n (1 - L_j^{(b)})$, $n \geq 1$. Here, $\{L_n^{(b)}\}_{n \geq 1}$ is the sequence given by (1.10) and, since the measure $\mu^{(b-1)}$ being a probability measure, $\xi_0^{(b-1)} = 1$.

Known results on Kernel polynomials also give the orthogonality (see [10])
\[
\int_T \zeta^{-k} \mathcal{R}_n(b; \zeta) (1 - \zeta^{-1}) d\mu^{(b-1)}(\zeta) = 0, \quad 0 \leq k \leq n - 1.
\]

From this, by using the transformation (2.3) we obtain the orthogonality (1.9) in Theorem 1.1 where to be precise $\nu^{(\lambda, n)}(x) dx = -d\mu^{(b-1)}(\zeta)$.

Finally, to obtain the formula for $\gamma_n^{(\lambda)}$ in Theorem 1.1 from (1.7) we have
\[
\frac{x^{n-1} \mathcal{P}_{n+1}(b; x)}{(x^2 + 1)^n} = (x - c_n^{(b)}) \frac{x^{n-1} \mathcal{P}_n(b; x)}{(x^2 + 1)^n} - d_{n+1}^{(\lambda)} \frac{x^{n-1} \mathcal{P}_{n-1}(b; x)}{(x^2 + 1)^{n-1}},
\]
for $n \geq 1$. Hence, integration with respect to $\nu^{(\lambda, n)}$ gives $\gamma_{n+1}^{(\lambda)} = \gamma_n^{(\lambda)} - d_{n+1}^{(\lambda)} \gamma_{n-1}^{(\lambda)}$, $n \geq 1$, which can be written in the alternative form
\[
\frac{\gamma_n^{(\lambda)}}{\gamma_{n-1}^{(\lambda)}} \left( 1 - \frac{\gamma_n^{(\lambda)}}{\gamma_{n+1}^{(\lambda)}} \right) = d_{n+1}^{(\lambda)}, \quad n \geq 1.
\]

Thus from (1.12), what one has to verify is $\gamma_1^{(\lambda)} / \gamma_0^{(\lambda)} = (1 - L_1^{(\lambda)}) = (2\lambda)^{-1}$. Clearly, $\gamma_0^{(\lambda)} = \int_{-\infty}^{\infty} \nu^{(\lambda, n)}(x) dx = \int_T d\mu^{(b-1)}(\zeta) = 1$. However,
\[
\gamma_1^{(\lambda)} = \int_{-\infty}^{\infty} \frac{\mathcal{P}_1(b; x)}{(1 + x^2)} \nu^{(\lambda, n)}(x) dx = \frac{1}{4} \int_T \frac{\zeta + 1}{\zeta} R_1(b; \zeta) d\mu^{(b-1)}(\zeta)
\]
\[
= (1 + ic_1^{(b)}) \mu^{(b-1)}_{-1} + 2 + (1 - ic_1^{(b)}) \mu^{(b-1)}_1.
\]

Thus, using the expression for $c_1^{(b)}$ in Theorem 1.1 together with $\mu^{(b-1)}_1 = \mu^{(b-1)}_{-1} = (-b + 1)/\sqrt{b}$, we find $\gamma_1^{(\lambda)} = (2\lambda)^{-1}$. This completes the proof of Theorem 1.1. \hfill \blacksquare
3 Electrostatic interpretation for the zeros of CRR polynomials

Consider the electrostatic field which obeys the logarithmic potential law composed of two fixed negative charges of size $\lambda_m$ at $i$ and $-i$, and $m$ movable positive unit charges along the x-axis. In addition, assume that there is also a background energy filed of type arctan effecting the movable charges. To be precise, we consider the electrostatic field where the energy $E = E(x_1, x_2, \ldots, x_m)$ is given by

$$E = \sum_{1 \leq j < i \leq m} \ln \frac{1}{|x_j - x_i|} - \frac{\lambda_m}{2} \sum_{j=1}^{m} \left[ \ln \frac{1}{|x_j - i|} + \ln \frac{1}{|x_j + i|} \right] - \eta \sum_{j=1}^{m} \arctan(x_j).$$

With $\lambda_m$ taken to be large enough, we need to find the set of locations of the movable charges that minimizes this energy.

**Theorem 3.1.** Let $\lambda_m = \lambda + m - 1$. Then the set of values $x_1^{(m)}(b), x_2^{(m)}(b), \ldots, x_m^{(m)}(b)$ that minimizes the the energy $E(x_1, x_2, \ldots, x_m)$ are the zeros of $P_m(b; \cdot)$.

**Proof.** The problem of minimizing $E$ is equivalent to maximizing the positive function

$$F = F(x_1, x_2, \ldots, x_m) = e^{-2E(x_1, x_2, \ldots, x_m)}.

Clearly,

$$F = \prod_{1 \leq j < i \leq m} (x_j - x_i)^2 \prod_{j=1}^{m} (1 + x_j^2)^{-\lambda_m} \prod_{j=1}^{m} e^{2\eta \arctan x_j}.

With $\lambda_m > m - 1$, one can easily verify that $F$ is bounded from above. We write

$$F = f_m(\hat{x} \setminus x_k) \prod_{j \neq k} (x_k - x_j)^2 (1 + x_k^2)^{-\lambda_m} e^{2\eta \arctan x_k},

where $f_m(\hat{x} \setminus x_k)$ is the part of $F$ that is independent of $x_k$. With $r_k(x) = r(x)/(x - x_k)$, where $r(x) = \prod_{j=1}^{m} (x - x_j)$, we can also write

$$F = f_m(\hat{x} \setminus x_k) r_k^2(x_k) (1 + x_k^2)^{-\lambda_m} e^{2\eta \arctan x_k},$$

and our problem is now to find a polynomial $r(x)$ of degree $m$ that maximizes $F$.

Differentiating $F$ with respect to $x_k$ gives

$$\frac{\partial F}{\partial x_k} = f_m(\hat{x} \setminus x_k) \left[ 2r_k(x_k)r'_k(x_k)(1 + x_k^2)^{-\lambda_m} e^{2\eta \arctan x_k} - 2\lambda_m x_k (1 + x_k^2)^{-\lambda_m - 1} r_k^2(x_k) e^{2\eta \arctan x_k} + \frac{2\eta}{1 + x_k^2} e^{2\eta \arctan x_k} r_k^2(x_k) (1 + x_k^2)^{-\lambda_m} \right]

= f_m(\hat{x} \setminus x_k) r_k(x_k)(1 + x_k^2)^{-\lambda_m - 1} e^{2\eta \arctan x_k}

\times \left[ 2(1 + x_k^2)r'_k(x_k) - 2(\lambda_m x_k - \eta) r_k(x_k) \right].$$

Using $r'(x_k) = r_k(x_k)$ and $r''(x_k) = 2r'_k(x_k)$, we then have

$$\frac{\partial F}{\partial x_k} = f_m(\hat{x} \setminus x_k) r_k(x_k)(1 + x_k^2)^{-\lambda_m - 1} e^{2\eta \arctan x_k}

\times \left[ (1 + x_k^2)r''(x_k) - 2(\lambda_m x_k - \eta) r'(x_k) \right].$$

The expression $s(x) = (1 + x^2)r''(x) - 2(\lambda_m x - \eta)r'(x)$ is a polynomial of degree $m$. We must choose $r(x)$ such that at the maximum of $F$ the polynomial $s(x)$ also vanishes at the zeros of $r$. That is, $s(x) = \text{const} \times r(x)$.

Thus, if we take $\lambda_m = \lambda + m - 1$, where $\lambda > 0$, then from the differential equation (1.13) we find $r(x) = \text{const} \times P_m(b; x)$ and $x_k$ are the zeros of $P_m(b; x)$.

\[\square\]
4 Generating functions

Generating functions have been known to be an important tool in the theory of special functions. The following generating function for the CRR polynomials $Q_n^{(2\eta, -\lambda+1)}$ is given in \[36\] Thm. 1.9:

$$\frac{(x^2+1)\lambda e^{2\eta \arccot(x)}}{[x + w(x^2 + 1)]^\lambda e^{2\eta \arccot[x + w(x^2 + 1)]}} = \sum_{n=0}^{\infty} Q_n^{(2\eta, -\lambda+1)}(x) \frac{w^n}{n!}.$$  

From a simplification of the above left hand side and then using (1.6) we can state the following result.

**Theorem 4.1.** For $b = \lambda + i\eta$ and $\lambda > 0$,

$$e^{2\eta \arccot(x)} \left[ (xw - 1)^2 + w^2 \right]^{\lambda} e^{2\eta \arccot[x - w(x^2 + 1)]} = \sum_{n=0}^{\infty} (2\lambda)_n \widehat{P}_n(b; x) \frac{w^n}{n!},$$

where $\widehat{P}_n(b; x)$ are the monic CRR polynomials given by

$$\widehat{P}_n(b; x) = \frac{1}{P_{n}(b)} P_n(b; x) = \frac{2^{n}(\lambda)_n}{(2\lambda)_n} P_n(b; x), \quad n \geq 1. \quad (4.1)$$

### 4.1 Generating function as an Appell sequence

In order to see the importance of the new generating function that we obtain for $\{\widehat{P}_n(b; x)\}_{n \geq 0}$, we consider the functions $M(b; w)$ and $N(b; w)$ given by

$$M(b; w) = \mathfrak{C}(b) w^{\lambda} N(b; w) \quad \text{and} \quad N(b; w) = e^{-iw} {}_1F_1(b; b + \bar{b}; 2iw), \quad (4.2)$$

where

$$\mathfrak{C}(b) = 2^{\lambda-1}e^{\pi \eta/2} \frac{|\Gamma(b)|}{\Gamma(2\lambda)} \quad (4.3)$$

and, as we have assumed so far, $b = \lambda + i\eta$ and $\lambda > 0$. Here, the notation ${}_1F_1$ denotes Kummer’s confluent hypergeometric function.

Clearly, from the ${}_1F_1$ hypergeometric representation

$$M(b; w) = (i2)^{-\lambda} \mathfrak{C}(b) M_{-i\eta, \lambda-1/2}(2iw), \quad \lambda > 0, \quad (4.4)$$

where $M_{-i\eta, \lambda-1/2}$ is a subclass of the Whittaker functions (see, for example, \[2\] Pg. 195)). The function $M(b; w)$ becomes familiar if one considers the alternative notations

$$M(b; w) = F_{\lambda-1}(\eta, w) \quad \text{and} \quad \mathfrak{C}(b) = C_{\lambda-1}(\eta). \quad (4.5)$$

When $\lambda$ takes positive integer values, the resulting functions $F_L(\eta, w)$, $L = 0, 1, \ldots$, are the so called regular Coulomb wave functions. Precisely, with our definitions of $M(b; w)$ and $N(b; w)$ the regular Coulomb wave functions can be given by

$$F_L(\eta, w) = M(L + 1 - i\eta; w) = C_L(\eta) w^{L+1} N(L + 1 - i\eta; w), \quad L = 0, 1, 2, \ldots. \quad (4.6)$$

It is known that the regular Coulomb wave functions satisfy the differential equation

$$F''_L(\eta, w) + \left[ 1 - \frac{2\eta}{w} - \frac{L(L + 1)}{w^2} \right] F_L(\eta, w) = 0, \quad (4.7)$$
where \( F''_L(\eta, w) = d^2 F_L(\eta, w) / dw^2 \). Moreover, it is also known that

\[
F_{L+1}(\eta, w) = \frac{(2L + 1)}{L(L + 1 + i\eta)} \left[ \frac{L(L + 1)}{w} + \eta \right] F_L(\eta, w) - \frac{(L + 1)(L + i\eta)}{L(L + 1 + i\eta)} F_{L-1}(\eta, w),
\]

which holds for \( L = 1, 2, 3, \ldots \). This three term recurrence relation associated with the Coulomb wave functions was first given by Powel in [24].

As stated in [14], the Coulomb wave functions are of great importance in the study of nuclear interactions. They arise when Schrödinger’s equation for a charged particle in the Coulomb field of a fixed charge is separated in polar coordinates. For some of the earliest studies on these functions we cite [33] and references therein.

Numerical evaluation of regular Coulomb wave functions have been the subject of many contributions including [11, 13, 14, 20, 33]. Except for [20, 33], they are mainly based on the use of the above three term recurrence relation. Moreover, the derivation of some of the basic properties of the zeros of these regular Coulomb wave functions and also the evaluation of these zeros have been based on an eigenvalue problem that follows from this three term recurrence relation (see [15, 22]).

By examining the differential equations, the three term recurrence relation and also the associated eigenvalue problems satisfied by the regular Coulomb wave functions, it is somewhat evident that the function \( F_\lambda(\eta, w) = \mathcal{M}(\bar{b} + 1; w) \), obtained by extending the integer parameter \( L \) to the real parameter \( \lambda \), will preserve many of the properties satisfied by the Coulomb wave function \( F_L(\eta, w) \), in particular with regards to the zeros. This is clearly true in the case \( \eta = 0 \) and the resulting functions lead to the Bessel functions. With such a knowledge, the author of [6] looks at some Turán type inequalities associated with these extended regular Coulomb wave functions \( F_\lambda(\eta, w) \) and obtains also some information regarding the zeros of these functions. With a different objective, the authors of [30] study the orthogonal polynomials that follow from the extended three term recurrence relation, which they call orthogonal polynomials associated with Coulomb wave functions. For some other contributions regarding the functions \( F_\lambda(\eta, w) \) for non-integer values of \( \lambda \), and even complex values of the parameter \( \lambda \), we cite, for example, [11, 21] and references therein.

In view of the above observations, in the present manuscript we will refer to the functions \( \mathcal{M}(b; w) = F_{\lambda-1}(-\eta, w) \) as extended regular Coulomb wave functions (ERCW functions for short).

From the recurrence for the regular Coulomb wave functions there follows

\[
\mathcal{M}(b + 2; w) = \frac{(2\lambda + 1)}{\lambda|b + 1|} \left[ \frac{\lambda(\lambda + 1)}{w} - \eta \right] \mathcal{M}(b + 1; w) - \frac{(\lambda + 1)|b|}{\lambda|b + 1|} \mathcal{M}(b; w), \tag{4.8}
\]

which holds for \( \lambda > 0 \). This result can be verified from (4.4) and from well known contiguous relations (see [32, pg. 27]) satisfied by Whittaker functions.

The following theorem gives the the role played by the ERCW function \( \mathcal{M}(b; w) \) as part of a generating function for the monic CRR polynomials \( \tilde{P}_n(b; x) \).

**Theorem 4.2.** Let \( b = \lambda + i\eta \), where \( \lambda > 0 \). Then the sequence \( \{\tilde{P}_n(b; .)\}_{n \geq 0} \) of monic complementary Romanovski-Routh polynomials is an Appell sequence and satisfy

\[
\frac{1}{\mathcal{C}(b)} e^{xw} w^{-\lambda} \mathcal{M}(b; w) = e^{xw} \mathcal{N}(b; w) = \sum_{n=0}^{\infty} \tilde{P}_n(b; x) \frac{w^n}{n!}. \tag{4.9}
\]

**Proof.** The property [1, 14] of the CRR polynomials \( P_n(b; x) \) shows us that the monic polynomial sequence \( \{\tilde{P}_n(b; x)\}_{n \geq 0} \) is an Appell [3] sequence. Thus, there holds a generating function of the form \( e^{xw} F(w) \). We need to prove \( F(w) = \mathcal{N}(b; w) \).
By setting $F(w) = e^{-iw}H(w)$, where $H(w) = \sum_{j=0}^{\infty} h_j^{(b)} w^j$, therefore one needs to prove that (4.9) holds if $H(w) = {}_1F_1(b; b+\bar{b}; 2iw)$. That is,

$$h_j^{(b)} = \frac{(b)_j (2i)^j}{(2\lambda)_j j!}, \quad j \geq 0. \quad (4.10)$$

We have

$$e^{xw} F(w) = e^{(x-i)w} \sum_{j=0}^{\infty} h_j^{(b)} w^j = \left( \sum_{k=0}^{\infty} \frac{(x-i)^k}{k!} w^k \right) \left( \sum_{j=0}^{\infty} h_j^{(b)} w^j \right).$$

Hence,

$$e^{xw} F(w) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} h_{n-l}^{(b)} \frac{(x-i)^l}{l!} \right) w^n.$$

From the hypergeometric expression for $P_n(b, \cdot)$ in Theorem 1.1 we can easily verify that if

$$\sum_{l=0}^{n} h_{n-l}^{(b)} \frac{(x-i)^l}{l!} = \frac{1}{n!} \hat{P}_n(b; x), \quad n \geq 0,$$

then $h_j^{(b)}$ are as in (4.10). This completes the proof of the Theorem.

**Remark 4.1.** The function $H(w)$ considered in the proof of Theorem 4.2 is a $_1F_1$ confluent hypergeometric function and hence it is an entire function. Thus, the function $F(w) = H(b; w)$ and also, for each $x$, the generating function $e^{xw} H(b; w)$ are entire functions. Thus, the right hand side of (4.9) is absolutely convergent for all $w$ on the complex plane.

By letting $b = \lambda + i\eta = 1$ in Theorem 4.2 it is not difficult to verify the following.

**Corollary 4.2.1.**

$$e^{xw} w^{-1} \sin(w) = e^{xw} H(1; w) = \sum_{n=0}^{\infty} \hat{P}_n(1; x) \frac{w^n}{n!}.$$ 

The ECWF function $M(b; w)$, when $\text{Im}(b) = \eta = 0$, is related to the Bessel function of order $\lambda - 1/2$. To be precise,

$$\frac{2\Gamma(2\lambda)}{\Gamma(\lambda)} (2w)^{-\lambda} M(\lambda; w) = H(\lambda; w) = \Gamma(\lambda + 1/2) \left( \frac{w}{2} \right)^{-\lambda+1/2} J_{\lambda-1/2}(w).$$

For more information on Bessel functions see, for example, [1], [2] and [38]. We can now state the following.

**Corollary 4.2.2.** For $\alpha > -1/2$ the following expansion formula holds with respect to the Bessel function $J_\alpha(w)$:

$$e^{xw} J_\alpha(w) = \frac{1}{\Gamma(\alpha+1)} \left( \frac{w}{2} \right)^\alpha \sum_{n=0}^{\infty} \hat{P}_n(\alpha + 1/2; x) \frac{w^n}{n!}.$$

With the connection (4.6) with the regular Coulomb wave functions we can state:

**Corollary 4.2.3.** The following expansion formula holds with respect to the regular Coulomb wave function $F_L(\eta, w)$:

$$e^{xw} M(\lambda + 1 - i\eta; w) = e^{xw} F_L(\eta, w) = C_L(\eta) w^{L+1} \sum_{n=0}^{\infty} \hat{P}_n(L + 1 - i\eta; x) \frac{w^n}{n!},$$

for $L = 0, 1, 2, \ldots$, where the so called Gamow-Sommerfeld factor $C_L(\eta)$ is as in (4.3) and (4.5).
Letting $x = 0$ in Corollary 4.2.3 gives the series expansion

$$F_L(\eta, w) = C_L(\eta) w^{L+1} \sum_{k=L+1}^{\infty} A_k^L(\eta) w^{k-L-1},$$

where $(k - L - 1)! A_k^L(\eta) = \hat{P}_{k-L-1}(L+1-i\eta; 0)$. This expansion formula, together with the three term recurrence relation (1.7) satisfied by $\{\hat{P}_n(L+1-i\eta; 0)\}_{n \geq 0}$, is exactly the expansion result found in [11, Pg. 538]. (This expansion result for $F_L(\eta, w)$ has first appeared in Yost, Wheeler and Breit [9].) Now with the hypergeometric expression in Theorem 4.4 for $\mathcal{P}_n(b; x)$ we can give the following closed form expression for $A_k^L(\eta)$:

$$A_{k+L+1}^L(\eta) = \frac{(-i)^k}{k!} 2F_1(-k, L + 1 - i\eta; 2L + 2; 2), \quad k = 0, 1, 2, \ldots.$$ 

One can also verify that the result corresponding to $x = -\eta/(L+1)$ in Corollary 4.2.3 is the expansion formula presented in [20].

We now give some further expansion formulas associated with the functions $\mathcal{X}(b; w)$ and the corresponding ERCW functions $\mathcal{M}(b; w)$.

**Theorem 4.3.** Let $b = \lambda + i\eta$, where $\lambda > 0$. Let the real sequences $\{a_n\}_{n \geq 0} = \{a_n(\lambda, \eta)\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0} = \{b_n(\lambda, \eta)\}_{n \geq 0}$ be given by

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \frac{2}{2\lambda + n} \begin{bmatrix} -\eta & (\lambda + n) \\ -(\lambda + n) & -\eta \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \quad n \geq 0,$$

with $a_0 = 1$ and $b_0 = 0$. Then

$$\frac{1}{\mathcal{E}(b)} w^{-\lambda} \cos(w) \mathcal{M}(b; w) = \cos(w) \mathcal{X}(b; w) = \sum_{n=0}^{\infty} \frac{a_n}{n!} w^n,$$

$$\frac{1}{\mathcal{E}(b)} w^{-\lambda} \sin(w) \mathcal{M}(b; w) = \sin(w) \mathcal{X}(b; w) = w \sum_{n=0}^{\infty} \frac{1}{n + 1} b_{n+1} \frac{w^n}{n!}.$$ 

Moreover, if $c_n(w) = a_n \cos(w) + \frac{1}{n+1} b_{n+1} w \sin(w)$, $n \geq 0$, then

$$\mathcal{X}(b; w) = \sum_{n=0}^{\infty} c_n(w) \frac{w^n}{n!}.$$ 

**Proof.** In (4.9), letting $x = i$ and $x = -i$, and then, respectively, summing and subtracting the resulting equations, we get

$$2 \cos(w) \mathcal{X}(b; w) = \sum_{n=0}^{\infty} \left[ \hat{P}_n(b; i) + \hat{P}_n(b; -i) \right] \frac{w^n}{n!},$$

$$i2 \sin(w) \mathcal{X}(b; w) = \sum_{n=0}^{\infty} \left[ \hat{P}_n(b; i) - \hat{P}_n(b; -i) \right] \frac{w^n}{n!}.$$ 

Hence, we set

$$2a_n = \left[ \hat{P}_n(b; i) + \hat{P}_n(b; -i) \right] \quad i2b_n = \left[ \hat{P}_n(b; i) - \hat{P}_n(b; -i) \right], \quad n \geq 0.$$ 

Clearly, $a_0 = 1$ and $b_0 = 0$. To obtain the recurrence formula for $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, we observe from (4.7) and (4.11) that $\hat{P}_n(b; i) = \hat{P}_n(b; -i) = \frac{2^n i^n(b_n)}{(2\lambda)_n}$, $n \geq 0$. Hence,

$$2a_n = \frac{2^n}{(2\lambda)_n} \left[ (i)^n(b)_n + (-i)^n(b)_n \right] \quad i2b_n = \frac{2^n}{(2\lambda)_n} \left[ (i)^n(b)_n - (-i)^n(b)_n \right].$$
for $n \geq 1$. Hence, all one needs to verify is that if

\[ 2a_n = [(i)^n(b)_n + (-i)^n(\overline{b})_n] \quad \text{and} \quad i2\tilde{b}_n = [(i)^n(b)_n - (-i)^n(\overline{b})_n], \]

then there hold

\[ \tilde{a}_{n+1} = -\eta \tilde{a}_n - (\lambda + n)\tilde{b}_n \quad \text{and} \quad \tilde{b}_{n+1} = -\eta \tilde{b}_n + (\lambda + n)\tilde{a}_n, \quad n \geq 0. \]

This is easily verified with the substitutions $\eta = [(b + n) - (\overline{b} + n)]/(2i)$ and $(\lambda + n) = [(b + n) + (\overline{b} + n)]/2$. This completes the proof of the formulas in (4.11).

Now to prove the latter part of the theorem we just multiply the first formula in (4.11) by $\cos(w)$ and the second formula in (4.11) by $\sin(w)$, and add the resulting formulas.

In the case of the Coulomb wave function Theorem 4.3 becomes:

**Corollary 4.3.1.** For $L \geq 0$, let the real sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be given by $a_0 = 1$, $b_0 = 0$ and

\[
\begin{bmatrix}
a_{n+1} \\
b_{n+1}
\end{bmatrix} = \frac{2}{2L + n + 2} \begin{bmatrix}
\eta & (L + n + 1) \\
-(L + n + 1) & \eta
\end{bmatrix} \begin{bmatrix}
a_n \\
b_n
\end{bmatrix}, \quad n \geq 0,
\]

Then

\[
\cos(w) F_L(\eta, w) = C_L(\eta) w^{L+1} \sum_{n=0}^{\infty} a_n \frac{w^n}{n!},
\]

\[
\sin(w) F_L(\eta, w) = C_L(\eta) w^{L+2} \sum_{n=0}^{\infty} \frac{1}{n+1} b_{n+1} \frac{w^n}{n!},
\]

where $C_L$ is as in Corollary 4.2. Moreover, if $\{c_n(w)\}_{n \geq 0}$ is such that

\[ c_n(w) = a_n \cos(w) + \frac{b_{n+1}}{n+1} w \sin(w), \quad n \geq 0, \]

then

\[ F_L(\eta, w) = C_L(\eta) w^{L+1} \sum_{n=0}^{\infty} c_n(w) \frac{w^n}{n!}. \]

When $\eta = 0$, the formulas for $a_n$ and $b_n$ in Theorem 4.3 is much simpler. It is easily verified that

\[ a_1 = b_0 = 0, \quad b_1 = \frac{2}{2\lambda} (-\lambda) a_0 = -1. \]

Hence, by taking $a_n = a_n(\lambda-1/2,0)$ and $b_n = b_n(\lambda-1/2,0)$ we can state:

**Corollary 4.3.2.** For $\alpha > -1/2$, let

\[ a_{2n} = (-1)^n \frac{2^{2n}(\alpha + 1/2)_{2n}}{(2\alpha + 1)_{2n}}, \quad b_{2n+1} = (-1)^{n+1} \frac{2^{2n}(\alpha + 3/2)_{2n}}{(2\alpha + 2)_{2n}}, \quad n \geq 0. \]

Then

\[
\cos(w) J_\alpha(w) = \frac{1}{\Gamma(\alpha + 1)} \left( \frac{w}{2} \right)^\alpha \sum_{n=0}^{\infty} a_{2n} \frac{w^{2n}}{(2n)!},
\]

\[
\sin(w) J_\alpha(w) = \frac{w}{\Gamma(\alpha + 1)} \left( \frac{w}{2} \right)^\alpha \sum_{n=0}^{\infty} \frac{1}{2n+1} b_{2n+1} \frac{w^{2n}}{(2n)!}.
\]

Moreover, if $c_{2n}(w) = a_{2n} \cos(w) + \frac{1}{2n+1} b_{2n+1} w \sin(w)$, $n \geq 0$, then

\[ J_\alpha(w) = \frac{1}{\Gamma(\alpha + 1)} \left( \frac{w}{2} \right)^\alpha \sum_{n=0}^{\infty} c_{2n}(w) \frac{w^{2n}}{(2n)!}. \]
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