Fourth Quantization

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Abstract

In this paper we will analyse the creation of the multiverse. We will first calculate the wave function for the multiverse using third quantization. Then we will fourth quantize this theory. We will show that there is no single vacuum state for this theory. Thus, we can end up with a multiverse, even after starting from a vacuum state. This will be used as a possible explanation for the creation of the multiverse. We also analyse the effect of interactions in this fourth quantized theory.

1 Introduction

The existence of the multiverse first appeared in the many-worlds interpretation of quantum theory [1]. This idea has now resurfaced in the landscape of string theory [2, 3]. As this landscape is populated by $10^{500}$ vacuum states [4], the possibility of all of them being real vacuum states for different universes remains an open one [5]. As a matter of fact, this model of the multiverse has also been used as an explanation for inflation in chaotic inflationary multiverse [6, 7, 8]. In fact, it is expected that the number of universes in the chaotic inflationary multiverse will be even more than the number of different string theory vacuum states. This is because even for a single string theory vacuum state, the large-scale structure and the matter content in each of the locally Friedman parts will be different [9]. Thus, we should in principle be studying a collection of multiverse, each for a different string theory vacuum.

It may be noted that the big bang can be explained as the collision of two universes to form two new universes in the multiverse [10, 11]. Thus, to explain the cause of big bang, we require a theory of many universes, where these universes can interact with each other. Third quantization forms a natural formalism to study such a model of the multiverse [12, 13]. This is because it is well known that we cannot explain the creation and annihilation of particles using a single particle quantum mechanics and we have to resort to a second quantized theory for that purpose. In the second quantized formalism, the Schroedinger equation is treated as a classical field equation and it is quantized one more time. The creation and annihilation of particles is explained, by adding interaction terms to it, The Wheeler-DeWitt equation acts like the Schroedinger equation for quantum gravity [14]. So, in analogy with the single particle quantum mechanics, we cannot explain the creation and annihilation of universes using a second quantized Wheeler-DeWitt equation, and we have to use a third quantized formalism for doing that purpose [15, 16]. In doing
so the Wheeler-DeWitt equation is treated as a classical field equation and interactions are added to it. These interaction terms cause the universes to get created and annihilated. The third quantization of the Brans-Dicke theories \[17\] and Kaluza-Klein theories \[18\] have been already thoroughly studied. Virtual black holes in third quantized $f(R)$ gravity has also been studied \[19\]. In doing so a solution to the problem of time has also been proposed. The uncertainty principle for third quantization of $f(R)$ gravity has also been analysed \[20\]. If we go to fourth quantization we will go to a theory of multi-multiverses. This seems to be a natural structure that will occur in chaotic inflationary multiverse because there will be a multiverse for each different string theory vacuum state.

In this paper we will analyse the Wheeler-DeWitt equation in minisuperspace approximation. We will first obtain an explicit expression for the wave function of the multiverse. Then we will analyse the creation of multiverse from a fourth vacuum state. Finally, we will analyse the effect of interaction of different multiverses. It may be noted just like in the third quantized formalism, the creation of universe can be explained, for the creation of the multiverse, we will have to quantize the third quantized theory one more time. Thus, in this paper fourth quantization is studied for the first time.

2 Third Quantization

In the first quantized formalism, both the non-relativistic Schroedinger equation and the relativistic Klein-Gordon equation are viewed as quantum mechanical equations. These equations describe the evolution of the wave function for a single particle. However, when we go to the second quantized formalism, both these first quantized equations are no longer viewed as quantum mechanical equations. They are rather viewed as classical field equations and the first quantized wave function is viewed as a classical field, in the second quantized formalism. A Lagrangian can be constructed which generates these classical field equations. This Lagrangian is used to calculate the momentum conjugate to the classical field. After that a second quantized equal time commutator is imposed between the operator corresponding to this momentum variable and the operator corresponding to the field variable. This is how a second quantized the Fock space is constructed.

The Wheeler-DeWitt equation can be viewed as the Schroedinger’s equation for a single universe. It is already a second quantized equation. However, just as we could go from first quantization to second quantization, by viewing the first quantized equation as a classical field equation, we can go from second quantization to third quantization, by viewing the Wheeler-DeWitt equation as a classical field equation. We can also construct a Lagrangian which generates this classical field equation. Furthermore, from this Lagrangian a momentum conjugate to this new field variable can be calculated. This momentum can be used to third quantize this theory, by imposing equal time commutator with this new field variable. This way we will obtain a Fock space for the multiverse.

Now we will analyse third quantization of Wheeler-DeWitt equation in the minisuperspace approximation. Thus, we start with the Friedman-Robertson-Walker metric, which is given by

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2,$$  \hspace{1cm} (1)
where $d\Omega_2^3$ is the usual line element on the three sphere and $N$ is the lapse function and here we have set $k = 1$. In this background, we have $R_{ij} = 2\gamma_{ij}/a^2$ and $R = 6/a^2$. The Hamiltonian constraint in this minisuperspace approximation takes the following form, $[\pi_a^2 - \omega^2(a)] = 0$, where $\omega^2(a) = -3a^2(3a^2 - \Lambda a^4)/4G$. Now we promote the canonical momentum $\pi_a$ to an operator, and so we have $\pi_a = -i\partial_a$. Thus, the Wheeler-DeWitt equation corresponding to this classical Hamiltonian constraint is given by

$$\left[\frac{\partial^2}{\partial a^2} + \omega^2(a)\right] \phi[a] = 0. \tag{2}$$

We interpret Eq. (2), as a classical field equation of a classical field $\phi(a)$, whose classical action is given by

$$S[\phi(a)] = \frac{1}{2} \int da \left( \left(\frac{\partial\phi}{\partial a}\right)^2 - \omega^2(a)\phi^2 \right). \tag{3}$$

Now obviously the variation of this action for third quantization $S[\phi(a)]$, given by Eq. (3), will lead to the Wheeler-DeWitt equation (2). If we interpret the scaling factor $a$ as the time variable, then the momentum conjugate to $\phi(a)$ will be $p = \dot{\phi}$, where $\dot{\phi}$ is the derivative of $\phi$ with respect to $a$. Now the third quantized Hamiltonian obtained by the Legendre transformation, can be written as

$$H(\phi(a), p(a)) = \frac{1}{2} p^2 + \frac{\omega^2(a)}{2}\phi^2. \tag{4}$$

This Hamiltonian given by Eq. (4) is the Hamiltonian for the harmonic oscillator with time-dependent frequency $\omega(a)$ [21, 22, 23, 24]. We first third quantize this Hamiltonian by imposing equal time commutation relation between $\phi(a)$ and $p(a)$, $[\phi(a), p(a)] = i$. Then we go to the Schroedinger’s picture and diagonalize the operator version of $\phi(a)$. In the Schroedinger’s picture this diagonalized operator version of $\phi(a)$ is represented by a time independent field $\phi$, and the operator version of $p(a)$ is represented by $-i\partial/\partial \phi$. Now we can write the third quantized Schroedinger equation for the multiverse as [20]

$$O_0(\phi, a)\Phi(\phi, a) = \left[ -i\frac{\partial}{\partial a} + \mathcal{H}(\phi, p) \right] \Phi(\phi, a) \tag{5}$$

$$= \left[ -i\frac{\partial}{\partial a} - \frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \frac{\omega^2(a)}{2}\phi^2 \right] \Phi(\phi, a) = 0. \tag{5}$$

Here $\Phi(\phi, a)$ is the wave function for the multiverse, and different modes correspond to different universes.

If we denote the two linearly independent solutions to the third quantized Schroedinger as $u_0(a)$ and $v_0(a)$, and define $\rho_0(a) = \sqrt{u_0^2 + v_0^2}$, then we have

$$\frac{d^2}{da^2}\rho_0 + \omega^2(a)\rho_0 - \frac{\Omega_0^2}{\rho_0^2} = 0, \tag{6}$$

where $\Omega_0$ is given by $\Omega_0 = v_0 u_0 - u_0 v_0$. Let the third quantized wave functions $\Phi^*(\phi, b)$ be the solution to the following equation,

$$O_s(b, \phi)\Phi^*(\phi, b) = \left[ -i\frac{\partial}{\partial b} + \mathcal{H}^*(\phi, p) \right] \Phi^*(\phi, b)$$
This is the equation of a simple harmonic oscillator system of unit mass and frequency. Here \( b \) denotes the time and \( \Phi^*(\phi, b) \) denote wave functions of this simple harmonic oscillator. This time \( b \) is related to \( a \) through the relation, \( \rho_0 \delta db = \Omega_0 da \). Now we define \( U_\omega(\rho_0, \Omega_0) \) as

\[
U_\omega(\rho_0, \Omega_0) = \exp \left( \frac{i \rho_0 \phi^2}{2 \rho_0} \right) \exp \left[ -\frac{i}{2} \left( \ln \frac{\rho_0}{\sqrt{\Omega_0}} \right) \left( \phi p + p\phi \right) \right],
\]

where \( \Omega_0 U_\omega O_x(b, \phi) U_\omega^\dagger |_{b=b(a)} = \rho_0^2 O_0(a, \phi) \), and \( \Phi(\phi, a) = U_\omega \Phi^*(a) \). Furthermore, the wave function for the simple harmonic oscillator can be written as

\[
\Phi^*(\phi, b) |_{b=b(a)} = \frac{1}{\sqrt{2^n n! \sqrt{\Phi}}} e^{-i(n+\frac{1}{2})b} \exp \left[ -\frac{\phi^2}{2} \right] H_n(\phi) |_{b=b(a)}
\]

which \( \Phi(\phi, a) \) is a linear combination of \( u_0(a) \) and \( v_0(a) \). Furthermore, \( U_f \) can be written as \( U_f = \exp[i(\hat{u}_1 \phi + \delta_{u_1}(a)) \exp(-iu_1p)] \), where \( U_f O_0 U_f^\dagger = O_0 \). Therefore, the wave functions \( \Phi(\phi, a) \) of the multiverse is given by [20]

\[
\Phi(\phi, a) = U_f U_\omega \Phi^*(\phi, b) |_{b=b(a)} = \frac{1}{\sqrt{2^n n! \rho_0(a)}} \left( \frac{\Omega_0}{\pi} \right)^{1/4} \left( \frac{u_0(t) - iv_0(t)}{\rho_0(a)} \right)^{n+1/2} \times \exp \left[ i(\hat{u}_1(a) \phi + \delta_{u_1}(a)) \right] H_n \left( \frac{\sqrt{\Omega_0} \phi - u_1(a)}{\rho_0(a)} \right) \times \exp \left[ \frac{(\phi - u_1(a))^2}{2} \left( \frac{\Omega_0}{\rho_0^2(a)} + i \frac{\rho_0}{\rho_0} \right) \right].
\]

Thus, we have obtained the wave function for the full multiverse. The different modes here correspond to different universes.

### 3 Fourth Quantization

In the previous section we obtained the third quantized Schrödinger’s equation for the multiverse. Now we can repeat the procedure of going from first quantization to second quantization or going from second quantization to third quantization, to go from third quantization to fourth quantization. In order to do that we can view the third quantized Schrödinger’s equation for the multiverse, as a classical field equation and construct Lagrangian which generates
this classical field equation. We can use this Lagrangian to calculate the momentum conjugate to this field variable. Then we can use it to fourth quantize this theory and construct a fourth quantized Fock space. In this Fock space the creation and annihilation operators will create and annihilate multiverses. Thus, we will obtain a theory of multi-multiverses.

So, to analyse the fourth quantization of the Wheeler-DeWitt equation, we will view Eq. (5) as a classical field equation. In Eq. (5), \( \phi \) acts like the space coordinate, \( a \) acts like the time coordinate, and \( \omega^2(a)\phi^2 \) acts like a spacetime dependent mass. Thus, this equation looks like the classical field equation for a quantum field theory in two dimensions with a spacetime dependent mass. So, we define a fourth quantized action for it as

\[
S = \int d\phi da \Phi(\phi, a)O_0(a, \dot{\phi})\Phi(\phi, a).
\] (11)

It may be noted that instead of a Schroedinger’s like equation for the third quantized multiverse, we can use a Klein-Gordon like equation for third quantized multiverse. For a Klein-Gordon like equation for third quantized multiverse the time dependence of \( O_0(a, \dot{\phi}) \) will be given by \( \partial^2/\partial a^2 \). Now we define the following symplectic product

\[
(\Phi, \Phi') = -i\int d\phi[\Phi^*(\phi, a)\dot{\Phi}'(\phi, a) - \Phi'(\phi, a)\dot{\Phi}^*(\phi, a)].
\] (12)

We let \( \{\Phi_n\} \) and \( \{\Phi^*_n\} \) form a complete set of solutions to the third quantized equation for the multiverse, and suppose

\[
(\Phi_n, \Phi_m) = M_{nm},
\] (13)

\[
(\Phi_n, \Phi^*_m) = 0,
\] (14)

\[
(\Phi^*_n, \Phi^*_m) = -M_{nm}.
\] (15)

The condition given in Eq. (14) does not hold in general and is imposed here as a requirement on the complete set of solutions to the third quantized equation for the multiverse. We will also require \( M_{nm} \) to only have positive eigenvalues as this does not also hold in general. This matrix \( M_{nm} \) is Hermitian because

\[
M_{nm} = -i\int d\phi[\Phi^*_n(\phi, a)\dot{\Phi}_m(\phi, a) - \Phi_m(\phi, a)\dot{\Phi}^*_n(\phi, a)]
= \left[-i\int d\phi[\Phi^*_m(\phi, a)\dot{\Phi}_n(\phi, a) - \Phi_n(\phi, a)\dot{\Phi}^*_m(\phi, a)]\right]^*
= M^*_{mn}.
\] (16)

Now in analogy with second quantized quantum field theory, we promote \( \Phi \) and \( \dot{\Phi} \) to Hermitian operators, and impose the following,

\[
[\Phi(\phi, a), \dot{\Phi}(\phi', a)] = i\delta(\phi, \phi'),

[\Phi(\phi, a), \Phi(\phi', a)] = 0,

[\dot{\Phi}(\phi, a), \dot{\Phi}(\phi', a)] = 0.
\] (17)

Now we can express \( \Phi(\phi, a) \) as,

\[
\Phi(\phi, a) = \sum_n [a_n \Phi_n + a_n^* \Phi^*_n].
\] (18)
because \( \{ \Phi_n \} \) and \( \{ \Phi^*_n \} \) form a complete set of solutions to the third quantized equation for the multiverse. We define the fourth quantized vacuum state \( |0\rangle \), as the state that is annihilated by \( a_n, a^\dagger_n |0\rangle = 0 \). This state is a purely mathematical object. It contains neither matter nor spacetime. However, we can create matter and spacetime from this vacuum state using \( a^\dagger_n \). Thus, multiverse can be built from this vacuum state.

It may be noted that the division between \( \{ \Phi_n \} \) and \( \{ \Phi^*_n \} \) is not unique even after imposing conditions given by Eqs. (13)-(15). Due to this non-uniqueness in division between \( \{ \Phi_n \} \) and \( \{ \Phi^*_n \} \), there is non-uniqueness in the definition of the fourth quantized vacuum state also. This can be seen by considering \( \{ \Phi'_n \} \) and \( \{ \Phi'^*_{n'} \} \) as another complete set of solutions to the third quantized equation for the multiverse. Now we express \( \Phi(\phi, a) \) as

\[
\Phi(\phi, a) = \sum_n [a'_n \Phi'_n + a'^*_n \Phi'^*_n].
\]

(19)

Here \( \{ \Phi'_n \} \) and \( \{ \Phi'^*_{n'} \} \) also satisfying conditions similar to the conditions given in Eqs. (13)-(15). We define a corresponding fourth quantized vacuum state \( |0'\rangle \) as the state annihilated by \( a'_n, a'^*_n |0'\rangle = 0 \). The spacetime and matter can again be built by repeated action of \( a'^\dagger_n \) on \( |0'\rangle \). As \( \Phi_n \) and \( \Phi^*_n \) form a complete set of solutions to the third quantized equation for the multiverse, we can express \( \Phi'_n \) as a linear combination of \( \Phi_n \) and \( \Phi^*_n \),

\[
\Phi'_n = \sum_m [\alpha_{nm} \Phi_m + \beta_{nm} \Phi^*_m].
\]

(20)

Thus, we have

\[
a_n = \sum_m [\alpha_{nm} a'_m + \beta^*_{nm} a'^*_m],
\]

(21)

\[
a^\dagger_n = \sum_m [\alpha^*_{nm} a'^\dagger_m + \beta_{nm} a^\dagger_m].
\]

(22)

As long as \( \beta_{nm} \neq 0 \), these two Fock spaces based on different complete set of solutions to the third quantized equation for the multiverse are different. So, \( a_n |0\rangle = 0 \), however

\[
a_n |0'\rangle = \sum_m [\alpha_{nm} a'_m + \beta^*_{nm} a'^*_m] |0'\rangle = \sum_m \beta^*_{nm} a'^\dagger_m |0'\rangle \neq 0.
\]

(23)

So, \( a_n |0'\rangle \) is a multiverse state given by

\[
\langle 0' | a^\dagger_n a_n |0' \rangle = \sum_m \sum_k \beta_{nm} \beta^*_{nk} M_{mk}.
\]

(24)

Thus, even though we started from a vacuum state, we ended up with the multiverse.

## 4 Interactions

It is well know that in a second quantized theory, the interaction terms in the Lagrangian create and annihilate particles. Furthermore, in a third quantized
theory, the interaction terms in the Lagrangian create and annihilate universes \[11, 27\]. Similarly, in a fourth quantized theory, the interaction terms in the Lagrangian create and annihilate multiverses. So, in this section we will analyse the effect of interaction for the multi-multiverses.

To analyse the interaction of these multiverses, we first start from the free fourth quantized action
\[
S = \int d\phi da \Phi(\phi, a) O_0(a, \phi) \Phi(\phi, a),
\]
and then we can add interaction terms \(S_I[\Phi(\phi, a)]\) to it,
\[
S_T[\Phi(\phi, a)] = S[\Phi(\phi, a)] + S_I[\Phi(\phi, a)].
\]
In order to analyse the perturbation theory of these multiverses, we first define
\[
J \Phi = \int d\phi da J(\phi, a) \Phi(\phi, a)
\]
and then let \(a \to ia\). Now we can write the Euclidean partition function as
\[
Z[J] = \int D\Phi \exp -S[\Phi]_{ET} + J\Phi,
\]
where \(S[\Phi]_{ET}\) is the Euclidean version of \(S[\Phi]_T\) obtained by letting \(a \to ia\).

Now we can define
\[
W[J] = -\ln Z[J].
\]
We can also define the effective action as
\[
\Gamma[\Phi_b] = W[J] - \Phi_b J,
\]
where
\[
\Phi_b = \frac{\delta W[J]}{\delta J}.
\]
The full quantum equation of motion will be given by
\[
\frac{\delta \Gamma}{\delta \Phi_b} = 0.
\]

We will now analyse the formation of two multiverses, \(M_3\) and \(M_4\), from the collision of two other multiverses, \(M_1\) and \(M_2\). The interaction term which generates this process is given by
\[
S_{EI} = \frac{\lambda}{4!} \int d\phi da \Phi^4(\phi, a).
\]
If we represent matter and gauge fields collectively by \(\chi\) and include the contribution from \(\chi\) in our formalism, then \(\phi\) would also depend on \(\chi\), and so, \(\Phi\) would also depend on \(\chi\). Let \(B\) be a third quantized charge that remains conserved when two universes collide \[28\]. Also let the total number of universes (with a positive value of \(B\) ) and anti-universes (with a negative value of \(B\) ) in multiverse \(M_k\) ( with \(k = 1, 2, 3, 4\) ) be \(n_k\) and \(m_k\), respectively. Now if the multiverse \(M_1\) and \(M_2\) have formed from nothing without violating the conservation of \(B\), then we have, \(Bn_1 - Bm_1 + Bn_2 - Bm_2 = 0\). Here \(Bn_1 + Bn_2\)
represent the total \( B \) number of the universes and \(-B_{m1} - B_{m2}\) represent the total \( B \) number of the anti-universes. Furthermore, after the collision, if the \( B \) number is conserved, we will have \( B_{n3} - B_{m3} + B_{n4} - B_{m4} = 0 \). However the \( B \) number in the multiverse \( M_3 \) or \( M_4 \) need not be separately conserved, so we have \( B_{n3} - B_{m3} \neq 0 \), and \( B_{n4} - B_{m4} \neq 0 \). Thus, after the collision some multiverse can have more positive \( B \) than if the other multiverse has more negative \( B \). Previously, it was proposed that an initial universe broke into two universes and this was proposed as an explanation for the dominance of matter over anti-matter in our universe \[11\]. In this paper this model is generalized to include the full multiverse.

5 Conclusion

In this paper we analysed third quantization of Wheeler-Dewitt equation for a Friedman-Robertson-Walker universe with a cosmological constant. We analyse the multiverse in this formalism and calculated the wave function of the multiverse. We also constructed the Fock space for the multi-multiverses. As there was no single vacuum state, we can ended up with a multiverse state, even after starting from a vacuum state. This was used as an possible explanation of the creation of multiverse. We also analysed the effect of interactions for the multi-multiverses.

It is known that in second quantized quantum field theory, we can construct conserved charges which remain conserved, even when the particle number is not conserved. These conserved charges are generated by the invariance of the second quantized Lagrangian under some symmetry. In analogy with second quantized quantum field theory, we can also construct a third quantized Noether’s theorem. Thus, if there is a symmetry which will leave a third quantized Lagrangian invariant, then the charge corresponding to it will be conserved even if the number of universes is not conserved. Here we argued that a third quantized Noether’s charge will not remain conserved for individual multiverse, in case of a fourth quantized interacting theory. It will be interesting to construct third quantized Noether’s charges for different quantum cosmological models and analyse the conservation of these Noether’s charges in a fourth quantized theory.

It may be noted that canonical quantum gravity has evolved into loop quantum gravity \([29, 30, 31, 32]\). Third quantization of loop quantum gravity has led to the development of group field theory \([33, 34, 35, 36]\). Recently, group field cosmology has also been developed \([37, 38]\). It will be interesting to analyse the creation of multiverse from a vacuum state in group field cosmology. Furthermore, the group field cosmology has also been supersymmetrized \([39]\). In this supergroup field cosmology, there are universes with both bosonic and fermionic distributions in the multiverse. It is hoped that by considering a supersymmetric distribution of universes, we might get better understanding of the cosmological constant. Thus, it will be interesting to include supersymmetry in this present work and analyse multi-multiverses containing both bosonic and fermionic multiverses.
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