LOCAL BEHAVIOR OF \( p \)-HARMONIC GREEN’S FUNCTIONS IN METRIC SPACES

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This paper is dedicated to the memory of Professor Juha Heinonen

ABSTRACT. We describe the behavior of \( p \)-harmonic Green’s functions near a singularity in metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality.

1. Introduction

Holopainen and Shanmugalingam [20] constructed in the metric measure space setting a \( p \)-harmonic Green’s function, called a singular function there, having most of the characteristics of the Green function which is the fundamental solution of the Laplace operator.

In this paper we study the following question related to the local behavior of a \( p \)-harmonic Green’s function on locally doubling metric measure space \( X \) supporting a local \((1,p)\)-Poincaré inequality: Given a relatively compact domain \( \Omega \subset X \), \( x \in \Omega \), and a \( p \)-harmonic Green’s function \( G \) with a singularity at \( x \), then can we describe the behavior of \( G \) near \( x \)?

Capacitary estimates for metric rings play an important role in the study of the asymptotic behavior. Following the ideas in the works of Serrin [33], [34], (see also [28]) such estimates were used in Capogna et al. [7] to establish the local behavior of singular solutions to a large class of nonlinear subelliptic equations which arise in the Carnot–Carathéodory geometry. Sharp capacitary estimates for metric rings with unrelated radii were established in the metric measure space setting in [13].

Here, we confine ourselves to mention that a fundamental example of the spaces included in this paper is obtained by endowing a connected Riemannian manifold \( M \) with the Carathéodory metric \( d \) associated with a given subbundle of the tangent bundle, see [8]. If such subbundle generates the tangent space at every point, then thanks to the theorem

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of Chow \cite{Chow} and Rashevsky \cite{Rashevsky} \((M, d)\) is a metric space. Such metric spaces are known as sub-Riemannian or Carnot-Carathéodory (CC) spaces. By the fundamental works of Rothschild and Stein \cite{Rothschild}, Nagel, Stein and Wainger \cite{Nagel}, and of Jerison \cite{Jerison}, every CC space is locally doubling, and it locally satisfies a \((p, p)\)-Poincaré inequality for any \(1 \leq p < \infty\). Another basic example is provided by a Riemannian manifold \((M^n, g)\) with nonnegative Ricci tensor. In such case thanks to the Bishop comparison theorem the doubling condition holds globally, see e.g. \cite{Bishop}, whereas the \((1, 1)\)-Poincaré inequality was proved by Buser \cite{Buser}. An interesting example to which our results apply and that does not fall in any of the two previously mentioned categories is the space of two infinite closed cones \(X = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + ... + x_{n-1}^2 \leq x_n^2\}\) equipped with the Euclidean metric of \(\mathbb{R}^n\) and with the Lebesgue measure. This space is Ahlfors regular, and it is shown in Hajlasz–Koskela \cite{Hajlasz} Example 4.2 that a \((1, p)\)-Poincaré inequality holds in \(X\) if and only if \(p > n\). Another example is obtained by gluing two copies of closed \(n\)-balls \(\{x \in \mathbb{R}^n : |x| \leq 1\}, n \geq 3, \) along a line segment. In this way one obtains an Ahlfors regular space that supports a \((1, p)\)-Poincaré inequality for \(p > n - 1\). A thorough overview of analysis on metric spaces can be found in Heinonen \cite{Heinonen}. One should also consult Semmes \cite{Semmes} and David and Semmes \cite{David}.

Our main result in this paper is a quantitative description of the local behavior of a \(p\)-harmonic Green’s function defined in Holopainen–Shanmugalingam \cite{Holopainen}. We shall prove that a Green’s function \(G\) with a singularity at \(x_0\) in a relatively compact domain satisfies the asymptotic behavior

\[
G(x) \approx \left(\frac{d(x, x_0)^p}{\mu(B(x_0, d(x, x_0)))}\right)^{1/(p-1)},
\]

where \(x\) is uniformly close to \(x_0\). Our approach uses upper gradients à la Heinonen and Koskela \cite{Heinonen2}, and \(p\)-harmonic functions that can be characterized in terms of \(p\)-energy minimizers among functions with the same boundary values in relatively compact subsets. Following \cite{Holopainen} we adopt a definition for Green’s functions that uses an equation for \(p\)-capacities of level sets.

We want to stress the fact that even in Carnot groups of homogeneous dimension \(Q\) it is not known whether such \(p\)-harmonic Green’s function is unique when \(1 < p < Q\). However, in the conformal case, i.e. when \(p = Q\), the uniqueness for Green’s function for the \(Q\)-Laplace equation in Carnot groups was settled by Balogh et al. in \cite{Balogh}.

The paper is organized as follows. The second section gathers together the relevant background such as the definition of doubling measures, upper gradients, Poincaré inequality, Newton–Sobolev spaces, and capacity. In Section 3 we recall sharp capacitary estimates for metric rings with unrelated radii proved in Garofalo–Marola \cite{Garofalo}. In Section 4 we give the definition of Green’s functions. We establish the
local behavior of Green’s functions in Section 5, and we also prove a result on the local integrability of Green’s functions. Section 6 closes the paper with a result on the local behavior of Cheeger singular functions. In this section our approach uses Cheeger gradients (see Cheeger [10]) emerging from a differentiable structure that the ambient metric space admits. In particular, \( p \)-harmonic functions can thus be characterized in terms of a weak formulation of the \( p \)-Laplace equation.

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2. Preliminaries

We begin by stating the main assumptions we make on the metric space \( X \) and the measure \( \mu \).

2.1. General Assumptions. Throughout the paper \( X = (X, d, \mu) \) is a locally compact metric space endowed with a metric \( d \) and a positive Borel regular measure \( \mu \). We assume that for every compact set \( K \subset X \) there exist constants \( C_K \geq 1 \), \( R_K > 0 \) and \( \tau_K \geq 1 \), such that for any \( x \in K \) and every \( 0 < r \leq R_K \), \( 0 < \mu(B) < \infty \), where \( B := B(x, r) := \{ y \in X : d(y, x) < r \} \), and, in particular, one has:

(i) the closed balls \( \overline{B}(x, r) = \{ y \in X : d(y, x) \leq r \} \) are compact;
(ii) (local doubling condition) \( \mu(B(x, 2r)) \leq C_K \mu(B(x, r)) \);
(iii) (local weak \((1, p_0)\)-Poincaré inequality) there exists \( 1 < p_0 < \infty \) such that for all measurable functions \( u \) on \( X \) and all upper gradients \( g_u \) (see Section 2.3) of \( u \)

\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u - u_{B(x,r)}| \, d\mu \leq C_K r \left( \frac{\int_{B(x, \tau_K r)} g_u^{p_0} \, d\mu}{\mu(B(x, \tau_K r))} \right)^{1/p_0},
\]

where \( u_{B(x,r)} := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu := \int_{B(x,r)} u \, d\mu / \mu(B(x, r)) \).

(iv) (X is LLC, i.e. linearly locally connected) there exists a constant \( \alpha \geq 1 \) such that for all balls \( B(x, r) \subset X \), \( 0 < r \leq R_K \), each pair of distinct points in the annulus \( B(x, 2r) \setminus \overline{B}(x, r) \) can be connected by a rectifiable path in the annulus \( B(x, 2\alpha r) \setminus \overline{B}(x, r/\alpha) \).

Hereafter, the constants \( C_K, R_K \) and \( \tau_K \) will be referred to as the local parameters of \( K \). We also say that a constant \( C \) depends on the local doubling constant of \( K \) if \( C \) depends on \( C_K \).

The above assumptions encompass, e.g., all Riemannian manifolds with Ric \( \geq 0 \), but they also include all Carnot–Carathéodory spaces, and therefore, in particular, all Carnot groups. For a detailed discussion
of these facts we refer the reader to the paper by Garofalo–Nhieu [14].
In the case of Carnot–Carathéodory spaces, recall that if the Lie algebra
generating vector fields grow at infinity faster than linearly, then the
compactness of metric balls of large radii may fail in general. Consider
for instance in \( \mathbb{R} \) the smooth vector field of Hörmander type
\( X_1 = (1 + x^2) \frac{d}{dx} \). Some direct calculations prove that the distance relative
to \( X_1 \) is given by
\[
d(x, y) = | \arctan(x) - \arctan(y) |
\]
and therefore, if \( r \geq \pi / 2 \), we have
\[
B(0, r) = \mathbb{R}.
\]

2.2. Local doubling property. We note that assumption (ii) im-
plies that for every compact set \( K \subset X \) with local parameters \( C_K \) and
\( R_K \), for any \( x \in K \) and every \( 0 < r \leq R_K \), one has for \( 1 \leq \lambda \leq R_K / r \),
\[
\mu(B(x, \lambda r)) \leq C \lambda^Q \mu(B(x, r)),
\]
where \( Q = \log_2 C_K \), and the constant \( C \) depends only on the local
doubling constant \( C_K \). The exponent \( Q \) serves as a local dimension of
the doubling measure \( \mu \) restricted to the compact set \( K \).

For \( x \in X \) we define the pointwise dimension \( Q(x) \) by
\[
Q(x) = \sup\{ q > 0 : \exists C > 0 \text{ such that } \\
\lambda^q \mu(B(x, r)) \leq C \mu(B(x, \lambda r)), \\
\text{for all } 1 \leq \lambda < \text{diam } X \text{ and } 0 < r < \infty \}.
\]
The inequality (2.1) readily implies that \( Q(x) \leq Q \) for every \( x \in K \).
Moreover, it follows that
\[
\lambda^{Q(x)} \mu(B(x, r)) \leq C \mu(B(x, \lambda r))
\]
for any \( x \in K, 0 < r \leq R_K \) and \( 1 \leq \lambda \leq R_K / r \), and the constant
\( C \) depends on the local doubling constant \( C_K \). Furthermore, for all
\( 0 < r \leq R_K \) and \( x \in K \)
\[
C_1 r^Q \leq \frac{\mu(B(x, r))}{\mu(B(x, R_K))} \leq C_2 r^{Q(x)},
\]
where \( C_1 = C(K, C_K) \) and \( C_2 = C(x, K, C_K) \).

For more on doubling measures, see, e.g. Heinonen [16] and the
references therein.

2.3. Upper gradients. A nonnegative Borel function \( g \) on \( X \) is an
upper gradient of an extended real valued function \( f \) on \( X \) if for all
rectifiable paths \( \gamma \) joining points \( x \) and \( y \) in \( X \) we have
\[
|f(x) - f(y)| \leq \int_\gamma g \, ds.
\]
whenever both \( f(x) \) and \( f(y) \) are finite, and \( \int_\gamma g \, ds = \infty \) otherwise. See
Cheeger [10], Shanmugalingam [35], and Heinonen–Koskela [18] for a
discussion on upper gradients.

If \( g \) is a nonnegative measurable function on \( X \) and if (2.4) holds for
\( p \)-almost every path, then \( g \) is a weak upper gradient of \( f \). By saying
that (2.4) holds for \( p \)-almost every path we mean that it fails only for a path family with zero \( p \)-modulus (see, for example, [32]).

If \( f \) has an upper gradient in \( L^p(X) \), then it has a minimal weak upper gradient \( g_f \in L^p(X) \) in the sense that for every \( p \)-weak upper gradient \( g \in L^p(X) \) of \( f \), \( g_f \leq g \) \( \mu \)-almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [36]. The minimal weak upper gradient can be obtained by the formula

\[
g_f(x) := \inf_g \limsup_{r \to 0^+} \frac{1}{r} \int_{B(x,r)} g \, d\mu,
\]

where the infimum is taken over all upper gradients \( g \in L^p(X) \) of \( f \), see Lemma 2.3 in Björn [4].

2.4. Capacity. Let \( \Omega \subset X \) be open and \( K \subset \Omega \) compact. The relative \( p \)-capacity of \( K \) with respect to \( \Omega \) is the number

\[
\text{Cap}_p(K, \Omega) = \inf \int_{\Omega} g_u^p \, d\mu,
\]

where the infimum is taken over all functions \( u \in N^{1,p}(X) \) such that \( u = 1 \) on \( K \) and \( u = 0 \) on \( X \setminus \Omega \). If such functions do not exist, we set \( \text{Cap}_p(K, \Omega) = \infty \). When \( \Omega = X \) we simply write \( \text{Cap}_p(K) \).

Observe that the infimum above could be taken over all functions \( u \in \text{Lip}_0(\Omega) = \{f \in \text{Lip}(X) : f = 0 \text{ on } X \setminus \Omega\} \) such that \( u = 1 \) on \( K \). In addition, the relative \( p \)-capacity is a Choquet capacity and consequently for all Borel sets \( E \) we have

\[
\text{Cap}_p(E, \Omega) = \sup \{\text{Cap}_p(K) : K \subset E, K \text{ compact}\}.
\]

For other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen et al. [21], Kinnunen–Martio [25, 26], and Kallunki–Shanmugalingam [22].

Finally, we say that a property holds \( p \)-quasieverywhere if the set of points for which the property does not hold is of zero capacity.

2.5. Newtonian spaces. We define Sobolev spaces on the metric space following Shanmugalingam [33]. Let \( \Omega \subseteq X \) be nonempty and open. Whenever \( u \in L^p(\Omega) \), let

\[
\|u\|_{N^{1,p}(\Omega)} = \left( \int_{\Omega} |u|^p \, d\mu + \inf_g \int_{\Omega} g^p \, d\mu \right)^{1/p},
\]

where the infimum is taken over all weak upper gradients of \( u \). The Newtonian space on \( \Omega \) is the quotient space

\[
N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}/\sim,
\]

where \( u \sim v \) if and only if \( \|u - v\|_{N^{1,p}(\Omega)} = 0 \). The Newtonian space is a Banach space and a lattice, moreover Lipschitz functions are dense, see [35] and Björn et al. [2].
To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let $E$ be a measurable subset of $X$. The Newtonian space with zero boundary values is the space

$$N_0^{1,p}(E) = \{ u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus E \}.$$  

The space $N_0^{1,p}(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in Shanmugalingam [36].

We say that $u$ belongs to the local Newtonian space $N_{\text{loc}}^{1,p}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$ (or equivalently that $u \in N^{1,p}(E)$ for every measurable $E \Subset \Omega$).

We will also need an inequality for Newtonian functions with zero boundary values. If $f \in N_0^{1,p}(B(x,r))$, then there exists a constant $C > 0$ only depending on $p$, the local doubling constant, and the constants in the weak Poincaré inequality, such that

$$\left( \int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} \leq C r \left( \int_{B(x,r)} g^p_f \, d\mu \right)^{1/p}$$

for every ball $B(x,r)$ with $r \leq \frac{1}{3} \text{diam } X$. For this result we refer to Kinnunen and Shanmugalingam [27].

2.6. Differentiable structure. Cheeger [10] demonstrated that metric measure spaces that satisfy assumptions (ii) and (iii) admit a differentiable structure with which Lipschitz functions can be differentiated almost everywhere. This differentiable structure gives rise to an alternative definition of a Sobolev space over the given metric measure space than defined above. However, assuming (ii) and (iii) these definitions lead to the same space, see Shanmugalingam [35, Theorem 4.10]. Thanks to a deep theorem by Cheeger the corresponding Sobolev space is reflexive, see [10, Theorem 4.48].

The differentiable structure gives the notion of partial derivatives in the following theorem, see Cheeger [10] Theorem 4.38], and it is compatible with the notion of an upper gradient.

**Theorem 2.1** (Cheeger). Let $X$ be a metric measure space equipped with a doubling Borel regular measure $\mu$. Assume that $X$ admits a weak $(1,p_0)$-Poincaré inequality for some $1 < p_0 < \infty$. Then there exists measurable sets $U_\alpha$ with positive measure such that

$$\mu(X \setminus \bigcup_\alpha U_\alpha) = 0,$$

and Lipschitz “coordinate charts”

$$X^\alpha = (X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha) : X \to \mathbb{R}^{k(\alpha)}$$

such that for each $\alpha$ functions $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha$ are linearly independent on $U_\alpha$ and

$$1 \leq k(\alpha) \leq N,$$
where $N$ is a constant depending only on the doubling constant of $\mu$ and the constants in the Poincaré inequality. Moreover, if $f : X \to \mathbb{R}$ is Lipschitz, then there exist unique (up to a set of measure zero) bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that
\[
\lim_{r \to 0^+} \sup_{x \in B(x_0, r)} \frac{|f(x) - f(x_0) - d^\alpha f \cdot (X^\alpha(x) - X^\alpha(x_0))|}{r} = 0
\]
for $\mu$-a.e. $x_0 \in U_\alpha$.

We can assume that the sets $U_\alpha$ are pairwise disjoint, and extend $d^\alpha f$ by zero outside $U_\alpha$. Regard $d^\alpha f$ as vectors in $\mathbb{R}^N$ and let $Df := \sum_\alpha d^\alpha f$. By Shanmugalingam [35, Theorem 4.10] and [10, Theorem 4.47], the Newtonian space $N^{1,p_0}(X)$ is equal to the closure in the $N^{1,p_0}$-norm of the collection of (locally) Lipschitz functions on $X$, then the derivation operator $D$ can be extended to all of $N^{1,p_0}(X)$ so that there exists a constant $C > 0$ such that
\[
C^{-1} |Df(x)| \leq g_f(x) \leq C |Df(x)|
\]
for all $f \in N^{1,p_0}(X)$ and $\mu$-a.e. $x \in X$. Here the norms $|\cdot|$ can be chosen to be inner product norms. The differential mapping $Df$ satisfies the product and chain rules: if $f$ is a bounded Lipschitz function on $X$, $u \in N^{1,p_0}(X)$, and $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative, then $uf$ and $h \circ u$ both belong to $N^{1,p_0}(X)$ and
\[
D(uf) = uDf + fDu;
\]
\[
D(h \circ u) = (h \circ u)' Du.
\]
See the discussion in Cheeger [10] and Keith [23].

2.7. $p$-harmonic functions. Let $\Omega \subset X$ be a domain. A function $u \in N^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ is $p$-harmonic in $\Omega$ if for all relatively compact sets $\Omega' \subset \Omega$ and for all $\varphi \in N^1_0(\Omega')$,
\[
\int_{\Omega'} g^p_u \, d\mu \leq \int_{\Omega'} g^p_{u+\varphi} \, d\mu.
\]
It is known that nonnegative $p$-harmonic functions satisfy Harnack’s inequality and the strong maximum principle, there are no non-constant nonnegative $p$-harmonic functions on all of $X$, and $p$-harmonic functions have locally Hölder continuous representatives. See [27].

As a consequence of the LLC property of $X$ a nonnegative $p$-harmonic function on an annulus $B(y, Cr) \setminus B(y, r/C)$ satisfies Harnack’s inequality on the sphere $S(y, r) = \{x \in X : d(x, y) = r\}$ for sufficiently small $r$, see Björn et al. [5, Lemma 5.3].

We also say that a function $u \in N^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ is Cheeger $p$-harmonic in $\Omega$ if in the above definition upper gradients $g_u$ and $g_{u+\varphi}$ are replaced by $|D u|$ and $|D(u + \varphi)|$, respectively. Note that by a result in Cheeger [10], the Cheeger $p$-harmonic functions are $p$-quasiminimizers in the sense of, e.g., Kinnunen–Shanmugalingam [27]. Moreover, the
Cheeger $p$-harmonic functions can be characterized in terms of a weak formulation of the $p$-Laplace equation: $u$ is Cheeger $p$-harmonic if and only if
\[ \int_{\Omega'} |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0 \]
for all $\Omega'$ and $\varphi$ as in the above definition.

3. Capacitary estimates

The aim of this section is to recall sharp capacity estimates for metric rings with unrelated radii proved in [13]. We emphasize an interesting feature of Theorems 3.1 and 3.3 that cannot be observed in the setting of, for example, Carnot gouprs. That is the dependence of the estimates on the center of the ring. This is a consequence of the fact that in the general setting $Q(x_0) \neq Q$ where $x_0 \in X$, see Section 2. The results in this section will play an important role in the subsequent developments.

**Theorem 3.1.** (Estimates from below) Let $\Omega \subset X$ be a bounded open set, $x_0 \in \Omega$, and $Q(x_0)$ be the pointwise dimension at $x_0$. Then there exists $R_0(\Omega) > 0$ such that for any $0 < r < R \leq R_0(\Omega)$ we have
\[
\operatorname{Cap}_{p_0}(B(x_0, r), B(x_0, R)) \geq \begin{cases} 
C_1(1 - \frac{r}{R})^p_{p_0(p_0 - 1)} \frac{\mu(B(x_0, r))}{r^{p_0 - 1}} , & \text{if } 1 < p_0 < Q(x_0), \\
C_2(1 - \frac{r}{R})^p_{Q(x_0)(Q(x_0) - 1)} \left( \log \frac{R}{r} \right)^{1 - Q(x_0)} , & \text{if } p_0 = Q(x_0), \\
C_3(1 - \frac{r}{R})^p_{p_0(p_0 - 1)} \left( 2R \frac{p_0 - Q(x_0)}{p_0 - 1} - r \frac{p_0 - Q(x_0)}{p_0 - 1} \right)^{1 - p_0} , & \text{if } p_0 > Q(x_0), 
\end{cases}
\]

where
\[
C_1 = C \left( 1 - \frac{1}{2} \frac{Q(x_0) - p_0}{p_0 - 1} \right)^{p_0 - 1}, \\
C_2 = C \frac{\mu(B(x_0, r))}{r^{Q(x_0)}}, \\
C_3 = C \frac{\mu(B(x_0, r))}{r^{Q(x_0)}} \left( 2 \frac{p_0 - Q(x_0)}{p_0 - 1} - 1 \right)^{p_0 - 1},
\]
with $C > 0$ depending only on $p_0$ and the local doubling constant of $\Omega$.

**Remark 3.2.** Observe that if $X$ supports the weak $(1,1)$-Poincaré inequality, i.e. $p_0 = 1$, these estimates reduce to the capacitary estimates, e.g., in Capogna et al. [7, Theorem 4.1].

**Theorem 3.3.** (Estimates from above) Let $\Omega$, $x_0$, and $Q(x_0)$ be as in Theorem 3.1. Then there exists $R_0(\Omega) > 0$ such that for any $0 < r <
We have the following immediate corollary.

**Corollary 3.4.** If $1 < p_0 \leq Q(x_0)$, then we have
\[
\text{Cap}_{p_0}(\{x_0\}, \Omega) = 0.
\]

### 4. Green’s functions

We define a Green’s function on metric spaces following Holopainen and Shanmugalingam [20]. Note that Holopainen and Shanmugalingam referred to this function class as singular functions. We consider here a definition that uses an equation for $p$-capacities of level sets. Green’s function on a Riemannian manifold satisfies this equation, see Holopainen [19].

**Definition 4.1.** Given $1 < p_0 \leq Q(x_0)$, let $\Omega \subset X$ be a relatively compact domain, and $x_0 \in \Omega$. An extended real-valued function $G = G(\cdot, x_0)$ on $\Omega$ is said to be a Green’s function with singularity at $x_0$ if the following criteria are satisfied:

1. $G$ is $p_0$-harmonic and positive in $\Omega \setminus \{x_0\}$,
2. $G|_{X \setminus \Omega} = 0$ p-quasieverywhere and $G \in N_{1,p_0}^{1}(X \setminus B(x_0, r))$ for all $r > 0$,
3. $x_0$ is a singularity, i.e.,
   \[
   \lim_{x \to x_0} G(x) = \infty.
   \]
4. whenever $0 \leq \alpha < \beta$,
   \[
   C_1(\beta - \alpha)^{1-p_0} \leq \text{Cap}_{p_0}(\Omega^\beta, \Omega^\alpha) \leq C_2(\beta - \alpha)^{1-p_0},
   \]
where \( \Omega^\beta = \{ x \in \Omega : G(x) \geq \beta \} \), \( \Omega_\alpha = \{ x \in \Omega : G(x) > \alpha \} \), and \( C_1, C_2 > 0 \) are constants depending only on \( p_0 \).

**Remark 4.2.** (Existence) The existence of Green’s functions in the \( Q \)-regular metric space setting was first proved by Holopainen and Shanmugalingam in [20]. Being a \( Q \)-regular metric measure space means that the measure \( \mu \) satisfies, for all balls \( B(x, r) \) a double inequality

\[
C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q
\]

with a fixed constant \( Q \). There are, however, many instances where the \( Q \)-regularity condition is not satisfied. For example, systems of vector fields of Hörmander type are, in general, not \( Q \)-regular for any \( Q > 0 \).

In [13] the \( Q \)-regularity assumption was removed and the existence of this function class was proved in more general setting. For the proof of the existence, we refer to [20, Theorem 3.4], see also remarks in [13].

**Remark 4.3.** (Uniqueness) It is not known whether a Green’s function is unique in the metric space setting even in the case of Cheeger \( p \)-harmonic functions. Indeed, the uniqueness of Green’s functions is not settled in Carnot groups when \( 1 < p_0 < Q \), where \( Q \) is the homogeneous dimension attached to the non-isotropic dilations. However, Green’s function is known to be unique when \( p_0 = Q \), see Balogh et al. [1].

5. **Local behavior of \( p \)-harmonic Green’s functions**

We begin by recalling that if \( K \subset \Omega \) is closed, \( u \in N^{1,p_0}(X) \) is a \( p_0 \)-potential of \( K \) (with respect to \( \Omega \)) if

(i) \( u \) is \( p_0 \)-harmonic on \( \Omega \setminus K \);

(ii) \( u = 1 \) on \( K \) and \( u = 0 \) in \( X \setminus \Omega \).

By Lemma 3.3 in Holopainen–Shanmugalingam [20] \( p_0 \)-potentials always exist if \( \text{Cap}_{p_0}(K, \Omega) < \infty \).

From now on, we set

\[
m(r) = m_G(x_0, r) = \min_{\partial B(x_0, r)} G, \quad M(r) = M_G(x_0, r) = \max_{\partial B(x_0, r)} G,
\]

where \( G \) is a Green’s function with singularity at \( x_0 \). We can now state the following growth estimates for a Green’s function near a singularity. In what follows, \( R_0(\Omega) > 0 \) is the constant from theorems 3.1 and 3.3.

**Theorem 5.1.** Let \( \Omega \) be a relatively compact domain in \( X \), \( x_0 \in \Omega \), and \( 1 < p \leq Q(x_0) \). If \( G \) is a Green’s function with singularity at \( x_0 \) and given \( 0 < R \leq R_0(\Omega) \) for which \( \overline{B(x_0, R)} \subset \Omega \), then for every \( 0 < r < R \) we have

\[
m_G(x_0, r) \leq C_1 \left( \frac{1}{\text{Cap}_{p_0}(\overline{B(x_0, r)}, B(x_0, R))} \right)^{1/(p_0 - 1)} + M_G(x_0, R).
\]
Suppose $r_0 \in (0, R)$ is such that $m_G(x_0, r_0) \geq M_G(x_0, R)$, then for every $0 < r < r_0$ we have

$$M_G(x_0, r) \geq C_2 \left( \frac{1}{\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, r_0))} \right)^{1/(p_0 - 1)} + M_G(x_0, R),$$

where the constants $C_1$ and $C_2$ both depend only on $p_0$.

Proof. Consider a radius $R > 0$ such that $\overline{B}(x_0, R) \subset \Omega$. Since $G(x) \to \infty$ when $x$ tends to $x_0$, the maximum principle implies that

$$m(r) \geq m(\rho), \quad 0 < r < \rho < R.$$  \hspace{1cm} (5.1)

Define $w = G - M(R)$, and hence $w \leq 0$ on $\partial B(x_0, R)$. Observe that the first inequality in the theorem obviously holds true if $m(r) \leq M(R)$, thus, we might as well assume that

$$m(r) > M(R),$$

and consider the function $v$ in the annulus $B(x_0, R) \setminus \overline{B}(x_0, r)$ defined by

$$v = \begin{cases} 
0, & \text{if } G \leq M(R), \\
w, & \text{if } M(R) < G < m(r), \\
m_w(r), & \text{if } G \geq m(r).
\end{cases}$$

If we extend $v$ by letting $v = m_w(r)$ on $\overline{B}(x_0, r)$, then $v \in N_0^{1, p_0}(B(x_0, R))$. Our assumption (5.2) implies that $m_w(r) = m(r) - M(R) > 0$, so the function

$$\varphi = \frac{v}{m_w(r)},$$

which equals to 1 in $\overline{B}(x_0, r)$, is both an admissible function for the capacity of $\overline{B}(x_0, r)$ with respect to $B(x_0, R)$ and the $p_0$-potential of the set $\{x \in X : \varphi(x) \geq 1\}$ with respect to the set $\{x \in X : \varphi(x) > 0\}$. Thus one has

$$\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \leq \int_{B(x_0, R)} g_{\varphi}^{p_0} d\mu$$

$$= \text{Cap}_{p_0}(\{x \in X : \varphi(x) \geq 1\}, \{x \in X : \varphi(x) > 0\})$$

$$= \text{Cap}_{p_0}(\{x \in X : G(x) \geq m(r)\}, \{x \in X : G(x) > M(R)\})$$

$$\leq C_1 (m(r) - M(R))^{1 - p_0},$$

where we used criterion 4 from Definition 4.1 and the fact that $\varphi \geq 1$ or $\varphi > 0$ if and only if $G \geq m(r)$ or $G > M(r)$, respectively. This implies the first claim.

To prove the second inequality of the claim, let $w = G - M(R)$. Let $r_0 \in (0, R)$ be such that $m(r_0) \geq M(R)$. This implies that $w \geq 0$ on $\overline{B}(x_0, r_0)$ and also that $M(r) \geq M(R)$, for all $0 < r < r_0$. Hence, by the maximum principle we have that

$$\{x \in \Omega : G(x) \geq M(r)\} \subset \overline{B}(x_0, r)$$
and
\[ B(x_0, r_0) \subset \{ x \in \Omega : G(x) > M(R) \}. \]
Hence it follows that
\[
\begin{align*}
\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, r_0)) \\
\geq \text{Cap}_{p_0}(\{ x \in \Omega : G(x) \geq M(r) \}, B(x_0, r_0)) \\
\geq \text{Cap}_{p_0}(\{ x \in \Omega : G(x) \geq M(r) \}, \{ x \in \Omega : G(x) > M(R) \}) \\
\geq C_2(M(r) - M(R))^{1-p_0},
\end{align*}
\]
which implies the second claim and the proof is complete. \(\square\)

**Remark 5.2.** Theorem 7.1 in Capogna et al. [7] is slightly incorrect as the additional term \(M(R)\) is missing from the left-hand and the right-hand side in (ii). However, this does not affect the results in that paper since the additional term can be absorbed when establishing results on the behavior near a singularity.

We have the following result on the local behavior of a Green’s function near a singularity.

**Theorem 5.3.** Let \( \Omega \) be a relatively compact domain in \( X \), and \( x_0 \in \Omega \). If \( G \) is a Green’s function with singularity at \( x_0 \), then there exist positive constants \( C_1, C_2 \) and \( R_0 \) such that for any \( 0 < r < \frac{R_0}{2} \) and \( x \in B(x_0, r) \) we have
\[
C_1 \left( \frac{d(x, x_0)^{p_0}}{\mu(B(x, d(x, x_0)))} \right)^{1/(p_0 - 1)} \leq G(x) \\
\leq C_2 \left( \frac{d(x, x_0)^{p_0}}{\mu(B(x, d(x, x_0)))} \right)^{1/(p_0 - 1)},
\]
when \( 1 < p_0 < Q(x_0) \), whereas
\[
C_1 \log \left( \frac{R_0}{d(x, x_0)} \right) \leq G(x) \leq C_2 \log \left( \frac{R_0}{d(x, x_0)} \right),
\]
when \( p_0 = Q(x_0) \). Here the constants \( C_1 \) and \( C_2 \) depend on \( p_0 \), \( x_0 \), and the local parameters of \( \Omega \), whereas constant \( R_0 \) depends only on \( \Omega \).

**Proof.** Let \( R_0 = \min\{r_0, R_0(\Omega)\} \), where \( r_0 > 0 \) is from the second estimate in Theorem 5.1. The Harnack inequality on a sphere implies that there exists a constant \( C > 0 \) such that
\[ M(r) \leq Cm(r). \]
for every \( 0 < r < R_0 \). Let, in particular, \( r := d(x_0, x) < \frac{R_0}{2} \). From the first estimate in Theorem 5.1, the maximum principle, and the Harnack inequality on the sphere, we obtain for any \( 0 < r < \frac{R_0}{2} \)
\[
G(x) \leq M(r) \leq Cm(r) \\
\leq C \text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R_0))^{-1/(p_0 - 1)}.
\]

Thanks to Theorem 3.1 we have
\[
G(x) \leq C \left(1 - \frac{r}{R_0}\right)^{-p_0} \left(\frac{\mu(B(x_0, r))}{r^{p_0}}\right)^{1/(p_0-1)}
\leq C \left(\frac{\mu(B(x_0, r))}{r^{p_0}}\right)^{1/(p_0-1)}
\]
when \(1 < p < Q(x_0)\), and
\[
G(x) \leq C \log \left(\frac{R_0}{r}\right),
\]
when \(p = Q(x_0)\). This proves the estimate from above.

To show the estimate from below, observe that the second estimate in Theorem 5.1, the maximum principle, and the Harnack inequality on a sphere imply for \(0 < r < R_0\)
\[
G(x) \geq m(r) \geq C^{-1} M(r)
\geq C \text{Cap}_{p_0}(B(x_0, r), B(x_0, R_0))^{-1/(p_0-1)}
\]
Applying Theorem 3.3 we conclude for \(1 < p < Q(x_0)\)
\[
G(x) \geq C \left(\frac{\mu(B(x_0, r))}{r^{p_0}}\right)^{1/(p_0-1)},
\]
and for \(p_0 = Q(x_0)\) that
\[
G(x) \geq C \log \left(\frac{R_0}{r}\right).
\]
This completes the proof. \(\Box\)

**Remark 5.4.** Note that if \(1 < p_0 < Q(x_0)\) then due to (2.3), it readily follows that
\[
C_1 d(x, x_0)^{(p_0 - Q(x_0))/(p_0 - 1)} \leq G(x) \leq C_2 d(x, x_0)^{(p_0 - Q)/ (p_0 - 1)},
\]
when \(x \in B(x_0, r)\) with \(0 < r < \frac{R_0}{2}\). Here the constants \(C_1\) and \(C_2\) depend on \(p_0, x_0\) and the local parameters of \(\Omega\).

In general Green’s function \(G \notin L_{\text{loc}}^{p_0}(\Omega)\), but as a corollary of Theorem 5.3 we have the following integrability result near a singularity.

**Corollary 5.5.** Let \(1 < p_0 < Q(x_0)\). Under the assumptions of Theorem 6.4, one has
(i) \(G \in \bigcap_{0 < q < \frac{Q(x_0)(p_0-1)}{Q-p_0}} L^q(B(x_0, r))\),
(ii) \(gG \in \bigcap_{0 < q < \frac{Q(x_0)(p_0-1)}{Q-1}} L^q(B(x_0, r))\),
(iii) If $p_0 > (Q + Q(x_0) - 1)/Q(x_0)$, then

$$G \in \bigcap_{1 < q < Q(x_0)} N_0^{1,q}(B(x_0, r)).$$

Proof. The proof of (i) is an immediate consequence of the estimate from above in Theorem 6.4. To prove (ii), we note that since $1 < p_0 < Q(x_0) \leq Q$, $q^* := \frac{Q(x_0)(p_0 - 1)}{Q - 1} < p_0$.

Applying Hölder’s inequality, the Caccioppoli inequality, see Björn–Marola [3, Proposition 7.1], and again Theorem 6.4, we find for $0 < q < p$ and for $\sigma \in (0, r)$

$$\int_{B(x_0, 2\sigma) \setminus B(x_0, \sigma)} g^q_G \, d\mu \leq C \sigma^{Q(x_0) - \frac{q(Q-1)}{p_0-1}}.$$

Note that the exponent $Q(x_0) - \frac{q(Q-1)}{p_0-1}$ is strictly positive, when $0 < q < q^*$ and zero when $q = q^*$. This observation gives us that

$$\int_{B(x_0, r)} g^q_G \, d\mu = \sum_{i=0}^{\infty} \int_{B(x_0, 2^{-i}r) \setminus B(x_0, 2^{-(i+1)}r)} g^q_G \, d\mu \leq C \mu(B(x_0, r)) \sum_{i=0}^{\infty} (2^{-i}r)^{Q(x_0) - \frac{q(Q-1)}{p_0-1}} < \infty.$$

This proves (ii). Finally, (iii) follows from (ii) once we observe that the condition $p_0 > (Q + Q(x_0) - 1)/Q(x_0)$ is equivalent to $Q(x_0)(p_0 - 1)/(Q - 1) > 1$. $\square$

6. **Cheeger singular functions**

In this section we study Cheeger singular functions, i.e. functions that satisfy only conditions 1, 2 and 3 in Definition 4.1 and the notion of a $p_0$-harmonic function is replaced by that of a Cheeger $p_0$-harmonic function.

Let $G'$ be a functions that satisfies conditions 1–3. in Definition 4.1. We begin by defining $K(G)$ by

$$(6.1) \quad K(G') = \int_{\Omega} |DG'|^{p_0-2}DG' \cdot D\varphi \, d\mu,$$

where $\varphi \in N_0^{1,p_0} (\Omega)$ is such that $\varphi = 1$ in a neighborhood of $x_0$. If $\varphi_i \in N_0^{1,p_0}(\Omega)$, $i = 1, 2$, and $\varphi_i = 1$ in a neighborhood of $x_0$ then $\varphi = \varphi_1 - \varphi_2 \in N_0^{1,p_0}(\Omega \setminus \{x_0\})$. This gives us

$$\int_{\Omega} |DG'|^{p_0-2}DG' \cdot D\varphi_1 \, d\mu = \int_{\Omega} |DG'|^{p_0-2}DG' \cdot D\varphi_2 \, d\mu.$$
Thus \( K(G') = K(\gamma', p_0, \Omega) \), in particular, \( K \) does not depend on \( \varphi \). Another property of \( K(G') \) that will play an important role is that

\[ K(G') > 0, \]

see (6.2) below. We obtain the following result on the growth of Cheeger singular functions near a singularity.

**Theorem 6.1.** Let \( \Omega \) be a relatively compact domain in \( X, \ x_0 \in \Omega, \) and \( 1 < p < Q(x_0) \). If \( G' \) is a Cheeger singular function, i.e. \( G' \) satisfies conditions 1–3 in Definition 4.1, with singularity at \( x_0 \) and given \( 0 < R \leq R_0(\Omega) \) for which \( \overline{B}(x_0, R) \subset \Omega \), then for every \( 0 < r < R \) we have

\[
m_{G'}(x_0, r) \leq \left( \frac{K(G')}{\text{Cap}_{p_0}(B(x_0, r), B(x_0, R))} \right)^{1/(p_0-1)} + M_{G'}(x_0, R).
\]

Suppose \( r_0 \in (0, R) \) is such that \( m_{G'}(x_0, r_0) \geq M_{G'}(x_0, R) \), then for every \( 0 < r < r_0 \) we have

\[
M_{G'}(x_0, r) \geq C(1 - \frac{r}{r_0})^{p_0} \left( \frac{K(G')}{\text{Cap}_{p_0}(B(x_0, r), B(x_0, r_0))} \right)^{1/(p_0-1)} + M_{G'}(x_0, R),
\]

where \( C = (C_1/C_4)^{1/(p_0-1)} > 0 \), and the constants \( C_1 \) and \( C_4 \) are as in theorems 3.1 and 3.3, respectively.

**Proof.** Consider a radius \( R > 0 \) such that \( \overline{B}(x_0, R) \subset \Omega \). Define \( w = G' - M(R) \), and hence \( w \leq 0 \) on \( \partial B(x_0, R) \). Observe also that the first inequality in the theorem obviously holds true if \( m(r) \leq M(R) \), thus, we might as well assume that \( m(r) > M(R) \). Let functions \( v \) and \( \varphi = v/m_w(r) \) be defined as in the proof of Theorem 5.1, with \( G \) replaced by \( G' \). Then \( \varphi \) can be used in the definition of \( K(G') \), see (6.1). We have

\[
K(G') = \int_{B(x_0, R) \setminus B(x_0, r)} |DG'|^{p_0-2} DG' \cdot D\varphi \, d\mu
\]

\[
= \frac{1}{m_w(r)} \int_{B(x_0, R) \setminus B(x_0, r)} |DG'|^{p_0-2} DG' \cdot Dw \, d\mu.
\]

Observing that \( Dw = 0 \) whenever \( v \neq w \), whereas \( Dw = Dw = DG' \) on the set where \( v = w \), we conclude

(6.2) \[
K(G') = \frac{1}{m_w(r)} \int_{B(x_0, R)} |Dv|^{p_0} \, d\mu = m_w^{p_0-1} \int_{B(x_0, R)} |D\varphi|^{p_0} \, d\mu.
\]

Note at this point that (6.2) proves that \( K(G') > 0 \). Indeed, if, in fact, \( K(G') \leq 0 \), the Sobolev–Poincaré inequality (2.5) implies that

\[
\int_{B(x_0, R)} |v|^{p_0} \, d\mu \leq CR^{p_0} \int_{B(x_0, R)} |Dv|^{p_0} \, d\mu \leq 0,
\]
and, moreover, \( v \equiv 0 \) in \( B(x_0, R) \). This, in turn, would contradict the fact that \( G'(x) \to \infty \) when \( x \) tends to \( x_0 \). This shows that \( K(G') > 0 \).

Observing that \( \varphi = v/m_w(r) \) is an admissible function for the capacity of \( B(x_0, r) \) with respect to \( B(x_0, R) \), we obtain from (6.2) that

\[
\mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \leq \int_{B(x_0, R) \setminus \overline{B}(x_0, r)} |D(v/m_w(r))|^{p_0} d\mu
\]

\[
\leq \frac{1}{m_w(r)^{p_0}} \int_{B(x_0, R) \setminus \overline{B}(x_0, r)} |Dv|^{p_0} d\mu \leq m_w(r)^{1-p_0} K(G').
\]

This implies the first claim.

To prove the second inequality of the claim, we observe that \( w(x) \to \infty \), when \( x \) tends to \( x_0 \). As above, \( w = G' - M(R) \). Also thanks to (5.1) one has that

\[
m_w(r) \geq m_w(\rho), \quad 0 < r < \rho < R.
\]

Let \( r_0 \in (0, R) \) be such that \( m(r_0) \geq M(R) \). This implies that \( w \geq 0 \) on \( \overline{B}(x_0, r_0) \). For any \( 0 < r < r_0 \) consider the function \( \psi : \mathbb{R} \to \mathbb{R} \) defined by

\[
\psi(t) = \begin{cases} 
1, & \text{in } 0 \leq t \leq r, \\
n \frac{\rho_0 - Q(x_0)}{p_0 - 1} - \frac{\rho_0 - Q(x_0)}{p_0 - 1} r_0 - r, & \text{in } r \leq t \leq r_0, \\
n \frac{\rho_0 - Q(x_0)}{p_0 - 1} - \frac{\rho_0 - Q(x_0)}{p_0 - 1} r_0 - r_0, & \text{in } r_0 \leq t \leq R.
\end{cases}
\]

Observe that \( \psi \in L^\infty(\mathbb{R}) \), \( \text{supp}(\psi') \subset [r, r_0] \), and that \( \psi' \in L^\infty(\mathbb{R}) \), thus \( \psi \) is a Lipschitz function. Moreover, \( \psi \circ d(x_0, x) \in N^{1,p_0}(B(x_0, R)) \). As in the proof of Theorem 4.5 in Garofalo–Marola [13], we obtain

\[
\int_{B(x_0, R)} |D\psi|^p d\mu \leq C_4 \frac{\mu(B(x_0, r))}{r^{p_0}}.
\]

On the other hand, if we use Theorem 5.1 for the proof see [13], we have

\[
\int_{B(x_0, R)} |D\psi|^{p_0} d\mu \leq C_4 \frac{C_1}{C_4} \left(1 - \frac{r}{r_0}\right)^{p_0(1-p_0)} \mathrm{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, r_0)).
\]
Since \( \psi \circ d(x_0, x) \) is an admissible function for \( K(G') \), it follows from (6.1), (6.4), and Hölder’s inequality that

\[
(6.5) \quad K(G')^{p_0/(p_0-1)} \\
\leq \left( \int_{B(x_0, R)} |D\psi|^{p_0} \, d\mu \right)^{1/(p_0-1)} \int_{B(x_0, r_0) \setminus B(x_0, r)} |D\psi|^{p_0} \, d\mu \\
\leq \left( \frac{C_4}{C_1} \right)^{1/(p_0-1)} (1 - \frac{r}{r_0})^{-p_0} \text{Cap}_{p_0}(B(x_0, r), B(x_0, r_0))^{1/(p_0-1)}. \\
\int_{B(x_0, r_0) \setminus B(x_0, r)} |D\psi|^{p_0} \, d\mu.
\]

Let us introduce the function \( \xi \in N^{1,p_0}(B(x_0, R)) \) defined by

\[
\xi = \begin{cases} 
0, & \text{in } \Omega \setminus B(x_0, R), \\
\max\{w, 0\}, & \text{in } B(x_0, R) \setminus B(x_0, r_0), \\
w, & \text{in } B(x_0, r_0) \setminus B(x_0, r), \\
\min\{w, M_w(r)\}, & \text{in } B(x_0, r).
\end{cases}
\]

Observe that we have \( \xi = M_w(r) \) in a neighborhood of \( x_0 \). Let

\[
I = \{ x \in B(x_0, R) : \xi(x) = w(x) \}.
\]

Since \( |D\xi| = |Dw| = |DG'| \) on \( I \), and \( |D\xi| = 0 \) on \( B(x_0, R) \setminus I \), from (6.1) we have

\[
\int_{B(x_0, r_0) \setminus B(x_0, r)} |Dw|^{p_0} \, d\mu \leq \int_I |Dw|^{p_0-2} Dw \cdot Dw \, d\mu \\
= \int_I |Dw|^{p_0-2} Dw \cdot D\xi \, d\mu = \int_{B(x_0, R)} |Dw|^{p_0-2} Dw \cdot D\xi \, d\mu \\
= K(G') M_w(r).
\]

By plugging this in (6.5), we finally conclude that

\[
M(r) \geq \left( \frac{C_1}{C_4} \right)^{1/(p_0-1)} (1 - \frac{r}{r_0})^{p_0} \\
\cdot \left( \frac{K(G')}{\text{Cap}_{p_0}(B(x_0, r), B(x_0, r_0))} \right)^{1/(p_0-1)} + M(R).
\]

This completes the proof. \( \Box \)

**Remark 6.2.** By obvious modifications, the preceding argument holds in the case \( p_0 = Q(x_0) \) as well.

**Remark 6.3.** Observe that assuming only conditions 1–3 in Definition 4.1, factor \( K(G') \) comes up in the above estimates as opposed to the estimates in Theorem 5.1.
We have the following result on the local behavior of a Cheeger singular function near a singularity. The proof of this result is similar to that of Theorem 5.3, thus, we omit the proof.

**Theorem 6.4.** Let $\Omega$ be a relatively compact domain in $X$, and $x_0 \in \Omega$. If $G'$ is a Cheeger singular function, i.e. $G'$ satisfies conditions 1–3 in Definition 4.1, with singularity at $x_0$, then there exist positive constants $C_1, C_2$ and $R_0$ such that for any $0 < r < \frac{R_0}{2}$ and $x \in B(x_0, r)$ we have

$$C_1 \left( \frac{d(x, x_0)^{p_0}}{\mu(B(x_0, d(x, x_0)))} \right)^{1/(p_0-1)} \leq G'(x) \leq C_2 \left( \frac{d(x, x_0)^{p_0}}{\mu(B(x_0, d(x, x_0)))} \right)^{1/(p_0-1)},$$

when $1 < p_0 < Q(x_0)$, whereas

$$C_1 \log \left( \frac{R_0}{d(x, x_0)} \right) \leq G'(x) \leq C_2 \log \left( \frac{R_0}{d(x, x_0)} \right),$$

when $p_0 = Q(x_0)$. Here the constants $C_1$ and $C_2$ depend on $K(G')$, $p_0$, $x_0$, and the local parameters of $\Omega$, and $R_0$ depends only on $\Omega$.

The following lemma is well-known and we omit the proof.

**Lemma 6.5.** Let $K$ be a closed subset of a relatively compact domain $\Omega$, and let $u$ be the $p_0$-potential of $K$ with respect to $\Omega$. Then for all $0 \leq \alpha < \beta \leq 1$ one has

$$\text{Cap}_{p_0}(\Omega^\beta, \Omega^\alpha) = \frac{\text{Cap}_{p_0}(K, \Omega)}{(\beta - \alpha)^{p_0-1}}.$$

We close this paper with the following observation. The proof of Proposition 6.6 is similar to the proof of Lemma 3.16 in Holopainen [19], but we present it here for completeness.

**Proposition 6.6.** Let $G'$ be a Cheeger singular function, i.e. $G'$ satisfies conditions 1–3 in Definition 4.1. Then

$$G = K(G')^{-1/(p_0-1)}G'$$

is a (Cheeger) Green's function with equality in condition 4 in Definition 4.1.

**Proof.** Observing that the function $\varphi = \min\{G', 1\}$ can be used in (6.1), and since $G'$ is the $p_0$-potential of the set $\{x \in \Omega : G' \geq 1\}$ with respect to $\Omega$, we obtain

$$\text{Cap}_{p_0}(\{x \in \Omega : G'(x) \geq 1\}, \Omega) = K(G').$$
Let $0 \leq \alpha < \beta$ and suppose first that $\beta \leq K(G')^{-1/(p_0-1)}$. Then one has
\[
\text{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \{x \in \Omega : G(x) > \alpha\}) \\
= \text{Cap}_{p_0}(\{x \in \Omega : G'(x) \geq \beta K(G')^{1/(p_0-1)}\}, \\
\{x \in \Omega : G'(x) > \alpha K(G')^{1/(p_0-1)}\} \\
= (\beta - \alpha)^{1-p_0} K(G')^{-1} \text{Cap}_{p_0}(\{x \in \Omega : G'(x) \geq 1\}, \Omega) \\
= (\beta - \alpha)^{1-p_0}.
\]

Let then assume that $K(G')^{-1/(p_0-1)} < \beta$. Equation (6.6) implies that
\[
\text{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega) \\
\leq \frac{(\beta - \alpha)^{1-p_0} K(G')^{-1} \text{Cap}_{p_0}(\{x \in \Omega : G'(x) \geq 1\}, \Omega)}{(\beta - \alpha)^{1-p_0}} \\
= K(G'),
\]
from which it follows that
\[
\text{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega) = \beta^{1-p_0}.
\]

Then one has
\[
\text{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \{x \in \Omega : G(x) > \alpha\}) \\
= \text{Cap}_{p_0}(\{x \in \Omega : G(x)/\beta \geq 1\}, \{x \in \Omega : G(x)/\beta > \alpha/\beta\}) \\
= (1 - \alpha/\beta)^{1-p_0} \text{Cap}_{p_0}(\{x \in \Omega : G(x) \geq \beta\}, \Omega) \\
= (\beta - \alpha)^{1-p_0}.
\]

This completes the proof. \hfill \Box

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