Harmonic polynomials for expanding the fluctuations of the Cosmic Microwave Background:
The Poincaré and the 3-sphere model.

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1 Abstract.

Fluctuations of the Cosmic Microwave Background CMB are observed by the WMAP. When expanded into the harmonic eigenmodes of the space part of a cosmological model, they provide insight into the large-scale topology of space. All harmonic polynomials on the multiply connected dodecahedral Poincaré space are constructed. Strong and specific selection rules are given by comparing the polynomials to those on the 3-sphere, its simply connected cover.

2 Motivation from cosmic topology.

The global topology of 3-space is not fixed by Einstein general relativity, since this is formulated in terms of local differential equations. Einstein's first static cosmological models used for the space-part of the universe a simply connected sphere $S^3$. With present-day cosmological information it becomes possible to test multiply-connected topologies for the space-part of the universe. J-P Luminet et al. [7] 2003 and J Weeks [10] 2004 propose to explore the topology of 3-space from temperature fluctuations of the cosmic microwave background (CMB). These fluctuations are measured by the Wilkinson Microwave Anisotropy Probe (WMAP) with very high precision.

A way to test the topology is to expand the temperature fluctuations of the CMB into harmonic polynomials of the chosen topological 3-manifold, for example the Poincaré dodecahedral manifold $\mathcal{P}$. The topology will be verified if the harmonic polynomials of the manifold suffice to expand these fluctuations. The simply connected space parts of cosmological models are the 3-sphere $S^3$ for positive, and the hyperbolic
space $H^3$ for negative curvature. The manifold $\mathcal{P}$ has $S^3$ as its universal cover and so has positive curvature.

As the backbone for such an expansion, in what follows we characterize an orthogonal basis of harmonic polynomials on $\mathcal{P}$. The details of this characterization by a novel operator were given in P. Kramer [5] and gr-qc/0410094. Here we describe the main steps in group and representation theory for the analysis.

The present construction allows to compare the harmonic polynomials of both manifolds. By restriction of the topology from $S^3$ to $\mathcal{P}$ we derive strong and specific selection rules for the harmonic polynomials.

3 General notions from topology.

For general notions of topology we refer to the classical monograph by Seifert and Threlfall [8], 1934. The topology of a manifold $\mathcal{M}$ is characterized by its homotopy group $\pi_1(\mathcal{M})$ [8] pp. 149-80. This group operates on $\mathcal{M}$ by loop composition. If the manifold is multiply connected, the homotopy group is non-trivial. Associated to $\mathcal{M}$ is its simply-connected universal cover $\tilde{\mathcal{M}}$. The topological manifold $\mathcal{M}$ appears on its universal cover $\tilde{\mathcal{M}}$ in the form of a tiling into copies of $\mathcal{M}$. There is a group of deck transformations $\text{deck}(\tilde{\mathcal{M}})$ [8] pp. 195-97. It acts fixpoint-free on $\tilde{\mathcal{M}}$ and produces the tiling. This group is isomorphic to the homotopy group,

$$\text{deck}(\tilde{\mathcal{M}}) \sim \pi_1(\mathcal{M}).$$

These relations allow to work out the topology on the universal cover and to view the topological manifold $\mathcal{M}$ as the quotient space

$$\mathcal{M} = \tilde{\mathcal{M}}/\text{deck}(\tilde{\mathcal{M}}).$$

4 Topology of the Poincaré dodecahedral 3-manifold $\mathcal{P}$.

H Poincaré in 1895 introduced the dodecahedral manifold $\mathcal{P}$. C Weber and H Seifert in 1933 [9] gave a gluing prescription for $\mathcal{P}$: Glue all pairs of opposite faces of a spherical dodecahedron, after rotation by $\pi/5$, to get the topological manifold $\mathcal{P}$. H Seifert and W Threlfall [8] pp. 214-19 derived from the gluing prescription the homotopy group $\pi_1(\mathcal{P})$. Their proof requires non-trivial steps in combinatorial group theory to transform from the original gluing generators and their relations to new ones. These new generators are then shown to belong to the binary icosahedral group $\mathcal{H}_3$ of order $|\mathcal{H}_3| = 120$ without reflections,

$$\pi_1(\mathcal{P}) \sim \mathcal{H}_3 < SU(2, C).$$
The group $\mathcal{H}_3$ consists of the preimages in $SU(2)$ of all the rotations of the familiar icosahedral group, which is isomorphic to the alternating group $A_5$ of five objects. We now wish to view the topology on the universal cover. The Poincaré dodecahedral manifold $\mathcal{P}$ has as universal cover the 3-sphere, $\tilde{\mathcal{P}} = S^3$ of constant positive curvature $\kappa = +1$. By the isomorphism eq. (1) there is an action
\[ \text{deck}(\tilde{\mathcal{P}}) \times \tilde{\mathcal{P}} \to \tilde{\mathcal{P}} \]
such that $\tilde{\mathcal{P}}$ is tiled by images under $\mathcal{H}_3$ of a prototile $\mathcal{P}$, presently a spherical dodecahedron. Conversely from eq. (2) the Poincaré manifold may be taken as the quotient
\[ \mathcal{P} = S^3/\text{deck}(\tilde{\mathcal{P}}) = S^3/\mathcal{H}_3. \]
We shall work on the universal cover $S^3$.

5 Coordinates and Lie group actions on $S^3$.

The sphere $S^3$ itself is a homogeneous space,
\[ S^3 := SO(4, R)/SO(3, R). \] (6)
Moreover $S^3$ as a manifold is in one-to-one correspondence to $SU(2)$, so that the pair $(z_1, z_2)$ of complex numbers may serve as its coordinates,
\[ u := \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad z_1\bar{z}_1 + z_2\bar{z}_2 = 1. \] (7)
In the coordinates eq. (7) $S^3$ admits the following left and right actions:
\[ u \in S^3, (g_l, g_r) \in SU(2) : \] (8)
\[ ((g_l, g_r) \times u) = g_l^{-1}ug_r. \]
The left and right actions $(g_l, e), (e, g_r)$ commute. Moreover the full group $SO(4, R)$ of isometries of $S^3$ has the direct product form
\[ SO(4, R) = SU^1(2) \times SU^r(2)/Z_2. \] (9)
Here $Z_2 = \{e, -e\}$ is the group consisting of the unit $2 \times 2$ matrix and its negative.

6 Klein’s fundamental invariant of $\mathcal{H}_3$.

F Klein [4], 1884 in his monograph - Vorlesungen über das Ikosaeder- implements the E Galois 1847 theory of $A_5$. He lets the binary icosahedral group $\mathcal{H}_3 < SU(2)$ act
by linear fractional transforms on two complex projective coordinates \((z_1, z_2)\), \(\zeta = (z_1/z_2)\) as

\[
\zeta \rightarrow \zeta' = \frac{a\zeta - b}{b\zeta + a},
\]

(10)

\[
\begin{pmatrix}
  a & b \\
  -b & a
\end{pmatrix}
= g_r \in SU(2, \mathbb{C}), a\bar{a} + b\bar{b} = 1.
\]

Eq. \(9\) rewritten in terms of \((z_1, z_2)\) is exactly the right action in eq. \(8\) of \(g_r\) from eq. \(9\) on \(u \in S^3\) from eq. \(7\) Klein constructs a \(H_3\)-invariant complex polynomial,

\[
f_k(z_1, z_2) := (z_1 z_2) \left[ (z_1 z_1)^5 + 11(z_1 z_2)^5 - (z_2 z_2)^5 \right],
\]

(11)

of degree 12 from the coordinates of the midpoints of the twelve dodecahedral faces. We shall see below that this invariant is in fact a lowest degree harmonic polynomial on \(P\). Moreover we shall build from this particular invariant polynomial an invariant operator-valued polynomial which quantizes any harmonic polynomial on \(P\).

### 7 The group of deck transformations.

For the construction, we use a Coxeter group, H S M Coxeter \([2]\) pp. 187-212, which can be placed, eq. \(14\), in between the continuous group of isometries of \(SO(4, \mathbb{R})\) eq. \(9\) and the discrete group \(H_3\) of deck transformations which produces the dodecahedral tiling. A Coxeter group is finitely generated by involutive generators and relations, coded in a Coxeter-Dynkin diagram. The relevant spherical Coxeter group with icosahedral subgroup has the Dynkin diagram, the four involutive generators and non-trivial relations

\[
\circ 5 \circ 3 \circ 3 \circ :=
\]

(12)

\[
\langle R_1, R_2, R_3, R_4 | (R_1)^2 = (R_2)^2 = (R_3)^2 = (R_4)^2 = (R_1 R_2)^5 = (R_2 R_3)^3 = (R_3 R_4)^3 = I \rangle
\]

Any pair of generators unlinked in the Dynkin diagram commutes. A Coxeter group has a linear isometric representation by Weyl reflections \([5]\) pp. 3520-2), one for each generator. The first three generators in eq. \(12\) form the icosahedral Coxeter group \(\circ 5 \circ 3 \circ\) and generate from a 3-simplex in \(S^3\) a dodecahedron.

This dodecahedron \(P\) is the prototile of eq. \(4\) under the group deck(\(\tilde{P}\)) of deck transformations. We find the following results \([5]\):

(A1): The group of deck transformations is a **subgroup of the Coxeter group** eq. \(12\)
We construct a first Weber-Seifert gluing generator $C_1$ according to Fig. 1. 
(A2): We can express explicitly the Weber-Seifert gluing generator $C_1$, lifted to $\text{deck}(\hat{P})$, as a product of Coxeter group elements,

$$C_1 = R_4 \, 5_1^{-2} \, I = \text{even}, \{5_1, I\} \in \circ \circ \circ \circ \circ.$$  \hspace{1cm} (13)

Here $5_1$ is a 5fold rotation around the vertical axis in Fig. 1, and $I$ the inversion in the center of the dodecahedron.

By conjugation of $C_1$ with icosahedral rotations we get five more gluings, and so find (A3): The group $\text{deck}(\hat{P}) = \mathcal{H}_3$ has the subgroup embedding

$$\text{deck}(\hat{P}) < S(\circ \circ \circ \circ \circ) < \text{SO}(4, R).$$  \hspace{1cm} (14)

Here $S()$ denotes the unimodular restriction of the Coxeter group to $\text{SO}(4, R)$.

By computing from eq. 13 the action of $C_1 = R_4 \, 5_1^{-2} \, I$ eq. 13 on the complex coordinates eq. 7 of $S^3$ we find

$$C_1 : (z_1, z_2) \rightarrow (z_1, z_2) \begin{bmatrix} \epsilon^{-2} & 0 \\ 0 & \epsilon^2 \end{bmatrix}, \epsilon := \exp(2\pi i/5).$$  \hspace{1cm} (15)

The matrix in eq. 15 belongs to $\mathcal{H}_3$ (\text{[5]} p. 3524).

Fig 1. Gluing prescription for $P$ by Weber and Seifert: Two opposite pentagonal faces of the dodecahedral prototile in a projection along a 2fold axis. (1) The inversion $I$ maps the shaded triangle from the bottom pentagon face to the white
triangle on the top face. (2) The operation \(5_1^{-2}\) rotates the white triangle on the top face to the shaded position. (3) The Weyl generator \(R_4\), a reflection in the top pentagon face, glues the dodecahedron to its top face neighbour.

\((A4):\) The group deck(\(\hat{\mathcal{P}}\)) \(< SU^r(2, \mathbb{C})\) acts from the right on \(u \in S^3\), therefore commutes with the continuous left action of \(SU^l(2, \mathbb{C})\).

\((A5):\) The group deck(\(\hat{\mathcal{P}}\)) is normal in the unimodular restriction of the Coxeter group and forms with the icosahedral rotation group \(S(\sigma \bar{2} \sigma \bar{2})\) the semidirect product:

\[
S(\sigma \bar{2} \sigma \bar{2}) = \text{deck}\(\hat{\mathcal{P}}\) \times S(\sigma \bar{2} \sigma \bar{2}).
\]

(16)

8 From group actions to representation spaces: Harmonic polynomials.

Suppose we would adopt the simply connected 3-sphere \(S^3\) as the space part of the cosmological model. Then we need the harmonic polynomials on \(S^3\).

Harmonic polynomials \(P\) on \(S^3\), homogeneous of degree \(\lambda\), obey \(\Delta P = 0\) where \(\Delta\) is the Laplacian on \(S^3\). We find:

The harmonic polynomials on \(S^3\) in the complex coordinates eq. 7 are identical with Wigner’s standard irrep matrices of \(SU(2)\), [3] pp. 53-67, taken as polynomials of degree \(\lambda = 2j\) in the elements of the matrix eq. 7 \(u \in SU(2)\):

\[
P^\lambda(z_1, z_2, \bar{z}_1, \bar{z}_2) = D^j_{m,m'}(z_1, z_2, \bar{z}_1, \bar{z}_2) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2} \sum_{\sigma} \frac{(j+m)!(j-m)!}{(j+m' - \sigma)!(m-m' + \sigma)!(j-m - \sigma)!} z_1^{j+m'-\sigma} (-z_2)^{m-m'+\sigma} (\bar{z}_1)^{j-m-\sigma} \bar{z}_2^\sigma,
\]

\(-j \leq (m, m') \leq j,
\]

For given degree \(\lambda = 2j\), there are \((2j+1)^2\) orthogonal harmonic polynomials.

Proof: (i) The degree of \(D^j_{m,m'}(z_1, z_2, \bar{z}_1, \bar{z}_2)\) is \(\lambda = 2j\). (ii) For \(m = j\), \(D^j\) is analytic in \((z_1, z_2)\), hence \(\Delta D^j_{j,m'} = 0\). (iii) Since \(m\) can be lowered by the operator \(L^j\) commuting with \(\Delta\),

\[
\Delta D^j_{m,m'} \sim \Delta(L^j)^{j-m}D^j_{j,m'} = (L^j)^{j-m} \Delta D^j_{j,m'}, = 0.
\]

If the space part of the cosmology is modelled by \(S^3\), the basis eq. 17 should allow to expand the CMB fluctuations. In contrast, the harmonic polynomials for the Poincaré model \(\mathcal{P}\) must be a subset of those in eq. 17 since they must repeat their functional values on all copies of the spherical Poincaré dodecahedron which tile \(S^3\).
In terms of group representations, the problem of finding the harmonic polynomials on the Poincare manifold $\mathcal{P}$ can be formulated as follows: From all the harmonic polynomials eq. 17 on $S^3$, we must select the subset which belongs to the identity irrep $D_{\alpha_0} \equiv 1$ of $\mathcal{H}_3 \in SU^r(2)$.

For fixed irrep $j$ of $SU^r(2)$ with character $\chi^j$ we can compute the multiplicity $m(j, \alpha_0)$ of $\mathcal{H}_3$-invariant polynomials from a scalar product of characters,

$$m(j, \alpha_0) := \frac{1}{|\mathcal{H}_3|} \sum_{g \in \mathcal{H}_3} \chi^j(g)^* \chi^{\alpha_0}(g), \, \chi^{\alpha_0}(g) = 1,$$

with the following result:

The multiplicity $m(j, \alpha_0)$ of invariant harmonic polynomials is zero for degree $2j = \text{odd}$. For $2j = \text{even}$ it is given by

(i) the starting values $m(j, \alpha_0) = 1$ for $j < 30$ : $j = 0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28$, $m = 0$ otherwise,

(ii) the recursion relation from the characters : $m(j + 30, \alpha_0) = m(j, \alpha_0) + 1$.

The multiplicity eq. 20 characterizes the subduction $SU^r(2) > \mathcal{H}_3$. In addition, since $\mathcal{H}_3$ and $SU^l(2)$ commute, there is for fixed $j$ an additional multiplicative degeneracy $(2j + 1)$ of harmonic polynomials, see B4 below.

From (ii) we conclude that the relative fraction up to integer $j$ of harmonic polynomials for $(\mathcal{P}/S^3)$ on average is $\frac{\sum_{j'} m(j', \alpha_0)}{(\sum_{j'}(2j' + 1))} \sim 1/30$.

**There are strong selection rules and a low mode suppression** for $2j < 12$ of harmonic polynomials on $\mathcal{P}$ versus those on its universal cover $S^3$. Weeks [10] phrases this as the *Mystery of the missing fluctuations*. The Poincaré model would be verified if the expansion of the CMB fluctuations in terms of the harmonic polynomials eq. 17 would display the selectivity stated in eq. 20.

### 9 Group/subgroup subduction of irreps by a generalized Casimir operator.

We proceed to the explicit determination of the invariant polynomials. For this purpose we first extend the problem and find the full group/subgroup subduction in $SU(2) > \mathcal{H}_3$ for all irreps of $\mathcal{H}_3$, and then from these select the identity irrep $D_{\alpha_0}$.

We follow the procedure from V Bargmann and M Moshinsky [1] 1960, exemplified by them for $SU(3, C) > SO(3, R)$:

A **generalized Casimir operator** $\Omega$ determines the irrep subduction $G > H$ iff (i,ii,iii) hold:

(i) $\Omega$ is from the enveloping Lie algebra $\text{Env}(l_G)$ and so preserves irrep spaces under
\( G \)

(ii) \( \Omega \) is invariant under \( H \) but not under \( G \),

(iii) \( \Omega \) is non-degenerate.

Part (iii) of this definition excludes the Casimir or projection operators of \( H \) constructed in \( Env(l_G) \), since they cannot distinguish between repeated irreps!

As tools we determine the right action generators of \( SU^r(2) \) acting on \( S^3 \) eq. 8.

The right Lie generators of \( SU^r(2) \) from eq. 8 act on functions of \((z_1, z_2, \bar{z}_1, \bar{z}_2)\) as first order differential operators:

\[
L_+ := L_1^r + iL_2^r = [z_1 \partial z_2 - \bar{z}_2 \partial \bar{z}_1], \\
L_- := L_1^r - iL_2^r = [z_2 \partial z_1 - \bar{z}_1 \partial \bar{z}_2], \\
L_3 := L_3^r = (1/2) [z_1 \partial z_1 - z_2 \partial z_2 - \bar{z}_1 \partial \bar{z}_1 + \bar{z}_2 \partial \bar{z}_2], \\
[L_3, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2L_3.
\]

The left Lie generators of \( SU_l(2) \) from eq. 8 look similar but commute with all the right Lie generators (\([5]\) pp. 3525-6).

The next tool is Klein’s homomorphism \( SU^r(2) \to SO(3, R) \):

Under the right action

\[
(z_1, z_2) \to (z_1, z_2) g_r, \quad g_r \in SU(2),
\]

the vector

\[
((x + iy)/\sqrt{2}, z, (x - iy)/\sqrt{2})
\]

\( \equiv (2z_1\bar{z}_2, z_1\bar{z}_1 - z_2\bar{z}_2, 2\bar{z}_1z_2), \)

and under conjugation

\[
(L_+, L_3, L_-) \to U_{g_r} (L_+, L_3, L_-) U_{g_r}^{-1}
\]

the right action generators eq. 21 transform linearly with the same representation \( D^1(g_r) \in SO(3, R) \)!

In the following steps we apply the noncommutative geometry of operators from the enveloping algebra \( Env(su(2)) \), similar to the Penrose length quantization in spin networks and to quantum gravity.

10 Lie-algebraic results on harmonic polynomials of \( \mathcal{P} \).

To find harmonic polynomials we proceed [5] as follows:

(B1): Construct a \( H_3 \)-invariant polynomial

\[
\mathcal{K}' = P(2z_1\bar{z}_2, z_1\bar{z}_1 - z_2\bar{z}_2, 2\bar{z}_1z_2)
\]

\( \equiv P((x + iy)/\sqrt{2}, z, (x - iy)/\sqrt{2}). \)
Klein’s invariant polynomial eq. 11 cannot be written in terms of the vector components eq. 23 which then would allow to pass to the generators \((L_+, L_3, L_-)\). Fortunately we can generate other invariant polynomials by applying the left lowering generator \(L_l^-\) from \(SU_l^l(2)\) to Klein’s fundamental invariant eq. 11. Applying the power \((L_l^-)^6\) to Klein’s invariant one obtains ([5] pp. 3526-8) a polynomial in which the vector components of the homomorphism eq. 22 can be substituted:

\[
K' \sim (L_l^-)^6 f_k(z_1, z_2) \\
\sim (x + iy)^5 z + z(x - iy)^5 + P_2(r^2, z) := P.
\]  

(B2): Use the Klein homomorphism eq. 23 substitute \(K'(x + iy, z, x - iy) \rightarrow K(L_+, L_3, L_-)\).

Since now we are dealing with homogeneous polynomials of degree \(n\), we run into the \textit{noncommutativity problem} of operator-valued polynomials in \((L_+, L_3, L_-)\): To assure the same transformation of the polynomial operator \(K\) as for \(K'\), we must apply the operator of symmetrization defined by

\[
\text{Sym } P(A_1, \ldots, A_n) := \frac{1}{n!} \sum_{(i_1, \ldots, i_n) \in S_n} P(A_{i_1}, \ldots, A_{i_n}).
\]

In this way we get from eqs. 26, 27 the \(H_3\)-invariant \textbf{hermitian} generalized Casimir operator

\[
K := \text{Sym } [K(L_+, L_3, L_-)] \in Env(su^r(2)).
\]

By construction \(K\) commutes with the Casimir operator \(\Lambda^2\) of \(SO(4, R)\). Note that Sym has \(6! = 720\) terms. The symmetrization eq. 28 is performed in the appendix of [6].

(B3): Quantize the spherical harmonics \(P^{2j}\) by diagonalizing the right action of \(K\). The eigenspaces are characterized by irreps \(D^\alpha\) of \(\mathcal{H}_3\).

(B4): \(K\) commutes with \(SU^l(2, C)\), so the degeneracy of any eigenvalue \(\kappa\) of \(K\) is \((2j + 1)dim(\alpha)\), and the harmonic polynomials on \(S^3\) are now characterized by

\[
P = P^{2j}_{m, \kappa}(z_1, z_2, \overline{z}_1, \overline{z}_2) : \\
\Delta P^{2j}_{m, \kappa} = 0, (x \cdot \nabla) P^{2j}_{m, \kappa} = (2j) P^{2j}_{m, \kappa}, \\
\Lambda^2 P^{2j}_{m, \kappa} = 4j(j + 1) P^{2j}_{m, \kappa}, \\
K P^{2j}_{m, \kappa} = \kappa P^{2j}_{m, \kappa}, L_l^l P^{2j}_{m, \kappa} = m P^{2j}_{m, \kappa}.
\]

(B5): Harmonic polynomials on \(P\) must belong to the \textbf{identity irrep} \(\alpha_0\) of \(\mathcal{H}_3\).
(B6): The Spectrum of $\mathcal{K}$ ([5] pp. 3530-1): $\mathcal{K}$ by eq. 28 acts in linear subspaces $L^\mu \ni m \equiv \mu \mod 5$ and in these subspaces is tridiagonal. By general reasons given in [1], it follows from tridiagonality that the operator $\mathcal{K}$ in any linear subspace $L^\mu$ is non-degenerate (no repeated eigenvalues). Moreover it is shown in ([5] p.3531) that the identity irrep $D^{\alpha_0}$ of $\mathcal{H}_3$ can occur only in $L^0$.

The hermiticity of $\mathcal{K}$ yields the orthogonality of the eigenstates. The properties B1 ... B6 taken together show that the operator $\mathcal{K}$ by its eigenvalues and eigenstates completely characterizes the harmonic polynomials on the dodecahedral Poincaré manifold $\mathcal{P}$. The unique characterization by $\mathcal{K}$ extends beyond the identity representation to any irrep $\mathcal{D}^{\alpha}$ of $\mathcal{H}_3$.

For the full analysis of the subduction and detailed results for degrees $2j = 0, \ldots, 12$ we refer to [5]. As an example for the diagonalization of $\mathcal{K}$ we give case $j = 6$. The values of $\mu$ and corresponding values of $m$ are given by

$$j = 6: \quad \mu \quad m$$

$$0 \quad -5, 0, 5$$
$$1 \quad -4, 1, 6$$
$$2 \quad -3, 2$$
$$3 \quad -2, 3$$
$$4 \quad -6, -1, 4$$

From B6, we expect harmonic polynomials only in $L^0$ of dimension 3. Evaluation of the operator $\mathcal{K}$ in this subspace gives its matrix form and eigenvalues,

$$\mathcal{K} V^\mu = V^\mu \mathcal{K}^{\mu, diag}$$

$$j = 6, \mu = 0, m = (-5, 0, 5):$$

$$\kappa = (-51975, \frac{-51975}{2}, \frac{-51975}{2}).$$

$$\mathcal{K} = \begin{bmatrix}
-51975 & \frac{4725\sqrt{77}}{2} & 0 \\
\frac{4725\sqrt{77}}{2} & -18900 & \frac{-4725\sqrt{77}}{2} \\
0 & \frac{-4725\sqrt{77}}{2} & \frac{51975}{2}
\end{bmatrix},$$

$$V^0 = \begin{bmatrix}
\sqrt{\frac{11}{25}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{11}{50}} \\
\sqrt{\frac{7}{25}} & 0 & -2\sqrt{\frac{11}{50}} \\
\sqrt{\frac{11}{25}} & \frac{1}{2} & \sqrt{\frac{11}{50}}
\end{bmatrix},$$

$$\mathcal{K}^{0, diag} = \begin{bmatrix}
-51975 & \frac{-51975}{2} \\
\frac{-51975}{2} & \frac{14175}{2}
\end{bmatrix}.$$
with the same eigenvalue $\kappa$ are obtained by applying powers of the lowering operator $L_l$.

The other two eigenstates are not harmonic polynomials of $\mathcal{P}$ for the following reason: It turns out ([5] p. 3534) that they have companions with the same eigenvalue $\kappa$, but in subspaces $L^\mu$, $\mu \neq 0$. Together with these they span irreps $D^\alpha$, $\alpha \neq \alpha_0$ of $H_3$ and therefore are not harmonic polynomials of $\mathcal{P}$.

The explicit diagonalization of $K$ for degrees $2j \leq 12$ is given in ([5] pp. 3532-6). There is no problem in going on to any higher degree harmonic polynomials as eigenstates of $K$. Since these harmonic polynomials are orthogonal, the expansion coefficients of the observed CMB fluctuations in terms of harmonic polynomials are given by the scalar products between the observed fluctuations and these polynomials. The strict validity of the Poincaré manifold as a model for the space-part of the cosmos would imply that all scalar products between the fluctuations and harmonic polynomials not belonging to the identity irrep $D^{\alpha_0}$ of $H_3$ must vanish.

11 Conclusion.

• The subduction $SO(4, R) > H_3$ for any irrep of $H_3$ is explicitly resolved by the operator $K$. Harmonic polynomials on $\mathcal{P}$ become (non-degenerate) eigenstates of $K$ eq. [28]
• For degree $2j = 12$, only Felix Klein’s invariant harmonic polynomial $f_k$ eq. [11] plus twelve orthogonal companions belong to the non-degenerate eigenvalue $\kappa = -51975$ of $K$. All other eigenstates have degenerate companions in subspaces $L^{\mu'}$, $\mu' \neq 0$ belonging to irreps $\alpha \neq \alpha_0$.
• There is an additional controlled degeneracy $(2j+1)$ from invariance under $SU^j(2, C)$.
• If 3-space has the topology of $\mathcal{P}$, we can expand the temperature fluctuations of CMB exclusively in invariant eigenmodes of $K$!
• A similar analysis can be done for the topological 3-manifolds $S^3/T^*$, $S^3/O^*$ with $T^*$, $O^*$ the binary tetrahedral, octahedral group. All these 3-manifolds share $S^3$ as their universal cover.
• What about hyperbolic 3-manifolds like the Weber-Seifert manifold [6]? Here the universal cover is the hyperbolic space of dimension 3. All hyperbolic 3-manifolds have negative curvature $\kappa = -1$.

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