Towards a grand unified picture for neutrino and quark mixings

Zurab Berezhiani a and Anna Rossi b

a Università dell’Aquila, I-67010 Coppito, L’Aquila, Italy, and Institute of Physics, Georgian Academy of Sciences, 380077 Tbilisi, Georgia

b Università di Padova and INFN Sezione di Padova, I-35131 Padova, Italy

The comparison of the CKM mixing angles with the leptonic mixings implied by the recent atmospheric and solar neutrino data exhibits an interesting complementarity. This pattern can be understood in the context of the SU(5) grand unification, assuming that the fermion mass matrices have Fritzsch-like structures but are not necessarily symmetric. (The present contribution is based on the paper in ref. [1].)

1. Introduction

One of the mysteries of particle physics is the manifest hierarchy in the fermion spectrum and mixing angles. The masses of the quarks and charged leptons are spread over five orders of magnitude, from MeVs to hundreds of GeVs and the quark mixing angles are:

\[ \theta_{23}^q = (2.3 \pm 0.2)^\circ, \quad \theta_{12}^q = (12.7 \pm 0.1)^\circ, \quad \theta_{13}^q = (0.18 \pm 0.04)^\circ \] (1)

As for the neutrinos, the recent data from the atmospheric and solar neutrino (AN and SN) experiments [2] providing information on their masses and mixings, have made the mystery of “flavour” even more intriguing. On the one hand, the ranges of \( \delta m^2_{\text{atm}} \) and \( \delta m^2_{\text{sol}} \) needed for the explanation of the AN and SN anomalies, can be translated directly into values of the neutrino masses. Namely, assuming the mass hierarchy \( m_3 > m_2 > m_1 \) for the neutrino mass eigenstates \( \nu_{1,2,3} \) we find a mass hierarchy \( m_2/m_3 \) similar to that of the charged leptons:

\[ m_3 = (5.7^{+2.7}_{-2.2}) \times 10^{-2} \text{ eV}, \quad m_2 = (2.5^{+0.7}_{-0.5}) \times 10^{-3} \text{ eV} \] (2)

On the other hand, the magnitudes of the neutrino mixing angles:

\[ \theta_{23}^l = (45 \pm 11)^\circ, \quad \theta_{12}^l = (2.0 \pm 1.2)^\circ, \quad \theta_{13}^l < (13 - 20)^\circ \] (3)

are in clear contrast with the corresponding quark angles [1]. In short: the AN anomaly points to maximal 23 mixing in the leptonic sector to be compared with the very small 23 mixing of quarks, and on the contrary, the MSW solution implies a very small 12 lepton mixing angle versus the reasonably large value of the Cabibbo angle.

In the standard model (SM) or in its supersymmetric extension the masses of the charged fermions \( q_i = (u_i, d_i), \quad l_i = (\nu_i, e_i) \) \( (i = 1, 2, 3 \) is a family index) emerge from the Yukawa terms:

\[ \phi_2 u_i^c \mathbf{Y}_{u}^{ij} q_j + \phi_1 d_i^c \mathbf{Y}_{d}^{ij} q_j + \phi_1 e_i^c \mathbf{Y}_{e}^{ij} l_j \] (4)

\[ \text{Below we concentrate on the small-mixing angle MSW solution for the SN problem [4], barring other possibilities such as the large-mixing angle MSW or vacuum oscillation solutions.} \]

\[ \text{For } \delta m^2_{\text{atm}} > 2 \times 10^{-3} \text{ eV}^2 \text{ the limit } \theta_{13}^l < 13^\circ \text{ follows from the CHOOZ experiment. Moreover, taking into account all the experimental data, } \theta_{13}^l = 0 \text{ provides the best data fit both for AN and SN cases [4].} \]
where $\phi_{1,2}$ are the Higgs doublets: $\langle \phi_{1,2} \rangle = v_{1,2}$,
\begin{equation}
(v_1^2 + v_2^2)^{1/2} = v_w = 174 \text{ GeV}
\end{equation}
and $Y_{u,d,e}$ are arbitrary matrices of coupling constants. The neutrino masses emerge only from the higher order effective operator 3:
\begin{equation}
\frac{\phi_2 \phi_2}{M} l_i Y_{\nu}^{ij} l_j,
\end{equation}
where $M \gg v_w$ is some cutoff scale and $Y_{\nu}$ is a matrix of dimensionless coupling constants. The fermion mass eigenstates are identified by diagonalizing the Yukawa matrices $Y_{u,d,e,\nu}$ by bi-unitary transformations:
\begin{equation}
U_f^T Y_f U = Y_f', \quad f = u, d, e, \nu
\end{equation}
(for the neutrinos it is $U_\nu' \equiv U_\nu$). In this way the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_q = U_u^T U_d$ and the leptonic mixing matrix $V_l = U_e^T U_\nu$, describing the neutrino oscillation phenomena, are also determined:

\begin{equation}
V_q =
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\end{equation}

\begin{equation}
V_l =
\begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix}
\end{equation}

For both mixing matrices, we adopt the “standard” parametrization utilizing the angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a CP-phase $\delta^\pm$ in the following, we distinguish the quark and lepton mixing angles in $V_q$ and $V_l$ by the subscripts ‘q’ and ‘l’, respectively.

As already mentioned, the SM does not provide any theoretical hints to constrain the matrices $Y_{u,d,e}$ and $Y_{\nu}$, leaving the issue of the fermion mass hierarchy and mixing pattern unexplained. Concerning the neutrinos, also the mass scale $M$ remains a free parameter. One can only conclude that if the maximal constant in $Y_{\nu}$ is of order the top Yukawa constant, $Y_t \sim Y_t \sim 1$, then the mass value $m_3$ in (3) points to the scale $M \sim 10^{15}$ GeV, rather close to the grand unified scale.

In this respect, the grand unified theories can be very useful. In these theories, as a consequence of the larger gauge group, relationships between quark and lepton masses or between CKM angles and quark mass ratios can emerge naturally. Moreover, the assumption of further symmetries in the Yukawa sector — well known examples being the “horizontal” or “family” symmetries — implies further predictions and thus a possible clue to discern the “flavour” mystery [8].

A popular Yukawa texture is that suggested by Fritzsch [8]:
\begin{equation}
Y_{u,d,e} = 
\begin{pmatrix}
0 & A_{u,d,e} & 0 \\
A_{u,d,e} & 0 & B_{u,d,e} \\
0 & B_{u,d,e} & C_{u,d,e}
\end{pmatrix}
\end{equation}

where all elements are generically complex and obey the additional condition:
\begin{equation}
|A_f| = |A_f|, \quad |B_f| = |B_f|; \quad f = u, d, e
\end{equation}

where $\delta^\mp$ is in good agreement with the data. Unfortunately, this texture implies cannot account at the same time for the small value of $V_{cb}$ and the large top mass.

However, this shortcoming can be cured just by embedding the ansatz in a $SU(5)$ grand unified theory and breaking the symmetry condition in the 23-family sector with $b_1 = |B_{u,d,e}^\prime / B_{u,d,e}| > 1$ and $b_2 = |B_{u,d,e}^\prime / B_{u,d,e}| > 1$. The $SU(5)$ symmetry ensures the following product rule for the mixing angles:
\begin{equation}
\tan \theta^\prime_{23} \tan \theta^\prime_{23} \sim \left(\frac{m_\mu m_\tau}{m_\mu m_\tau}\right)^{1/2}
\end{equation}

This rule is certainly exact when the down-quark and charged-lepton matrices have the symmetric Fritzsch texture from which one derives $\tan \theta^\prime_{23} = (m_\mu / m_\tau)^{1/2}$ and $\tan \theta_{23} = (m_\mu / m_\tau)^{1/2}$. However, these two relations are unsatisfactory as $|V_{cb}| < (m_\mu / m_\tau)^{1/2}$ and $\sin \theta_{atm} < (m_\mu / m_\tau)^{1/2}$.

The need of such an asymmetry was invoked in the context of $SO(10)$ models [8].
On the other hand, whenever the symmetry condition is broken, the rule \( \tan^2 \theta_{13} \) is only approximate since none of those angles can be predicted in terms of mass ratios. Indeed their values now depend on the amount of asymmetry between the 23 and 32 entries, i.e. on the factors \( b_d \) and \( b_c \). One can easily realize that the increasing of \( b_c \) goes in parallel with that of \( b_d \) since in SU(5) the Yukawa matrices are related as \( Y_e = Y_d^T \), modulo certain Clebsch factors. As a result the 23 mixing becomes larger in the leptonic sector and smaller in the quark sector. Therefore, if \( \tan \theta_{23}^d \) decreases below \( (m_s/m_b)^{1/2} \), then \( \theta_{23}^e \) should correspondingly increase above \( (m_u/m_c)^{1/2} \), and when the former reaches the value \( |V_{cb}| \simeq 0.05 \), the latter becomes \( \sim 1 \) (this happens for \( b_{d,c} \sim 8 \)). Though these estimates are not precise, they qualitatively demonstrate the ‘seesaw’ correspondence between the quark and lepton mixing angles whenever their magnitudes are dominated by the rotation angles coming from the down fermions. A similar argument can be applied also to the 12 mixing:

\[
\tan \theta_{12}^d \tan \theta_{12}^e \sim \left( \frac{m_u m_d}{m_c m_s} \right)^{1/2} \tag{12}
\]

The relation \( V_{us} \simeq (m_d/m_s)^{1/2} \) suggests that the 12 block of \( Y_d \) should be nearly symmetric, and hence we expect that \( \sin \theta_{12} \sim (m_u/m_c)^{1/2} \).

The above discussion is the key-point that will be extensively developed and discussed in the next section.

### 2. Modifying the Fritzsch ansatz in SU(5)

In the SU(5) model the masses of the fermions \( 5_i = (d^c, l)_i, \ 10_i = (u^c, c, q)_i \) arise from the following Yukawa terms:

\[
\tilde{H} 10_i G^{ij}_c \bar{5}_j + H 10_i G^{ij}_u \bar{10}_j + \frac{HH}{M} \bar{5}_i G^{ij}_c \bar{5}_j \tag{13}
\]

where \( H = (T, \phi_2) \sim 5 \) and \( \tilde{H} = (\tilde{T}, \phi_1) \sim \bar{5} \) are the Higgses. The Yukawa constant matrices \( G_u \) and \( G_c \) are symmetric due to SU(5) symmetry reasons while the form of \( G \) is not constrained. Upon breaking the SU(5) symmetry, we recover the SM Yukawa couplings \( Y \) with

\[
Y_e = G, \quad Y_d = G^T, \quad Y_u = G_u, \quad Y_\nu = G_\nu \tag{14}
\]

To simplify the discussion we shall assume, without loss of generality, that the matrices \( G_u \) and \( G_\nu \) are diagonal. Then the weak mixing matrices in the quark and leptonic sectors are just \( V_q = U_d \) and \( V_l = U_\nu \). On the other hand, since \( Y_d = Y_d^T \), we get that \( U_d = U'_d \) and \( U_\nu = U'_\nu \), so that the rotation angles of the left down quarks (charged leptons) are related to the unphysical angles rotating the right states of the charged leptons (down quarks). In the minimal SU(5) model the entries of the matrix \( G \) are just constants and one faces the well-known problem of the down-quark and charged-lepton degeneracy at the GUT scale. While the \( Y_b = Y_\tau \) unification is a success of the SUSY SU(5) GUT, the other predictions \( Y_{s,d} = Y_{\mu,e} \) are clearly wrong.

A more satisfactory picture emerges if the terms \( \tilde{H} 10_i G^{ij}_c \bar{5}_j \) are understood as effective cubic couplings originating from higher-order operators, such as \( \tilde{H} 10_i (\frac{2}{M} \varepsilon^{ij} G_{ij}) \bar{5}_j \), where \( \Phi \) is the SU(5) adjoint and \( M_s \) is some fundamental scale larger than the GUT scale. As a consequence, the corresponding entries in \( Y_e \) and \( Y_d \) can be distinguished by Clebsch coefficients.

In this perspective the matrices \( Y_e \) and \( Y_d \) can assume the asymmetric form given in Eq. (14). Phenomenological arguments impose these further relations:

\[
\begin{align*}
C_d &= C_e (= C), \\
A_d &= A'_d = A'_e = A_e (= A), \\
B_d' &= k'B_c, \\
B_d &= k B_c 
\end{align*}
\tag{15}
\]

where the coefficients \( k \) and \( k' \) are nontrivial SU(5) Clebsches breaking the quark and lepton symmetry. Introducing the 23-sector asymmetry parameters \( b_c = B_c / B_e \) and \( b_d = B_d' / B_d = k / k' b_c \), we finally end up with the following textures:

\[
\begin{pmatrix}
Y_e = \\
Y_d =
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 \\
A & 0 & \frac{1}{2}B \\
0 & B & C \\
0 & k'B & C
\end{pmatrix}
\tag{16}
\tag{17}
\]

This ansatz depends on six parameters: three Yukawa entries \( A, B, C \) and three Clebsch factors \( k, k' \) and \( b \). Through these parameters we
we have to determine six eigenvalues – $\mathcal{Y}_{e,\mu,\tau}$ and $\mathcal{Y}_{d,s,b}$ – and six mixing angles – $s_{12}^q, s_{23}^q, s_{13}^q$ and $s_{12}, s_{23}, s_{13}$. Hence at the GUT scale we are left with six relations between the physical observables. The leptonic mixing angles can be then expressed in terms of the lepton mass parameters $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$. Figure 1 illustrates the $b$-dependence of the leptonic mixing angles and of the corresponding parameters $\sin^2 2\theta_{23} = 4|V_{\mu 3}|^2(1 - |V_{\mu 3}|^2)$ and $\sin^2 2\theta_{12} = 4|V_{cb}|^2(1 - |V_{cb}|^2)$.

For $b = 1$ the 23 mixing angle is rather small for explaining the AN anomaly, while the 12 mixing is somewhat above the upper limit obtained by the MSW fit of the SN data (c.f. (3)). However, for larger $b$, $|V_{\mu 3}|$ increases roughly as $\sqrt{b}$ and becomes maximal around $b = 8.4$, while $|V_{cb}|$ slowly decreases (roughly as $\sqrt{c_{23}^2}$). Thus, the AN bound, $\sin^2 2\theta_{23} > 0.86$, requires $6 < b < 12$, while the SN data favour $b > 7$, when $\sin^2 2\theta_{12}$ drops below $1.5 \cdot 10^{-2}$.

Analogously the quark masses and mixing angles can be expressed in terms of the lepton mass ratios and of the three parameters $b, k, k'$. Then we show the behaviour of the mixings (Fig. 2), of the masses $m_d, m_b$ and the ratio $m_s/m_d$ (Fig. 3) for several values of $k \cdot k'$. For $k \sim k'$ and large values of $b$, ($b = 7 - 12$ as required from the lepton mixing) we achieve quite a satisfactory description also of the quark sector. The pattern with $k = k' = 1/2$ looks somehow favoured. We also learn from Fig. 3 that rather small values of $Y_1 \sim 0.5 - 1$ are needed to obtain the correct bottom mass for $b \gtrsim 7$. This special feature arises from the substantial correction to the $b - \tau$ Yukawa unification due to the large $b$.

In a more general case, we have to expect also $\mathcal{X}_u, \mathcal{X}_\nu$ to have a Fritzsch-like form. This would occur in the presence of some underlying horizontal symmetry. Such a scenario would provide some different features. In this case smaller values of $b_{e,d}$ can suffice since now the mixing angles will be contributed also by the unitary matrices $U_u$ and $U_\nu$: $V_q = U_u^\dagger U_d$ and $V_\ell = U_\nu^\dagger U_\ell$. For the CKM mixing angles we have:

$$|V_{cb}| = s_{23}^q \approx |s_{23}^q - e^{i\phi} s_{23}^u|,$$

$$|V_{us}| = s_{12}^q \approx |s_{12}^q - e^{i\delta} s_{12}^u|, \quad \frac{V_{ub}}{V_{cb}} \approx s_{12}^u \quad (18)$$

where the phases $\phi, \delta$ etc are combinations of the independent phases in the Yukawa matrices. The

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The re-normalization scaling has been taken into account for the bottom mass.
5

Figure 3. The dependence of the down quark masses on $b_e = b$. $m_{s}(1 \text{ GeV})$ is shown in units of 100 MeV and the ratio $m_s/m_d$ in units of 20. We also show iso-contours for $m_b(m_b) = 4.25 \text{ GeV}$ in the $b - Y_t$ plane (for $\alpha_s = 0.118$).

$\theta_{23}^u$, $\theta_{12}^u$ are the analogous angles diagonalizing $Y_u$: $\tan \theta_{23}^u = \sqrt{Y_c/Y_t}$ and $\tan \theta_{12}^u = \sqrt{m_u/m_c}$. By varying the phase $\phi$ from 0 to $\pi$, the value of the 23 mixing angle in the CKM matrix can vary between its minimal and maximal possible values:

$$\theta_{23}^{(\mp)} = \theta_{23}^d \pm \theta_{23}^u$$

(19)

Analogously, for the leptonic mixing we have

$$\theta_{23}^{(\mp)} = \theta_{23}^e \pm \theta_{23}^\nu$$

(20)

where $\tan \theta_{23}^\nu = \sqrt{m_2/m_3}$. Thus, for the range of the neutrino masses indicated in (2), we obtain $\theta_{23}^\nu = (11.8^{+5.0}_{-3.0})^\circ$. In case of moderate asymmetry in $Y_{d,e}$, the entries in (14) are big as compared to the experimental value of $\theta_{23}^u$ while each of the entries in (20) is too small for the magnitude of $\theta_{23}^l$ required by the AN oscillation. However, by properly tuning the phases, $\theta_{23}^u$ can get close to $\theta_{23}^{(\mp)} = \theta_{23}^d \pm \theta_{23}^u$ while $\theta_{23}^l$ can approach $\theta_{23}^{(\pm)} = \theta_{23}^e \pm \theta_{23}^\nu$. Therefore, even for small values $b_{e,d} \approx 2$, one could achieve a proper fit of the mixing angles. In ref. \[7\] an example of realization of such a scenario, implementing the $U(2)$ horizontal symmetry, is illustrated.

3. Conclusions

We have discussed how the present pattern of the leptonic mixing angles, characterized by a maximal mixing between the second and third generation, can be linked to the CKM mixing angles in the $SU(5)$ grand unification thanks to the fermion multiplet structure. In particular, this has been shown assuming the fermion Yukawa matrices to have a Fritzsch-like form with an asymmetric 23-block and (essentially) symmetric 12-block.

We remark that alternative and realistic ansätze – with diagonal $Y_{u,\nu}$ – (accounting e.g. for CP-violation) can be motivated in the context of $U(3)$ horizontal symmetry [1].

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