True Concurrency Can Be Easy

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Abstract. Step net bisimulation is a coinductive behavioral relation for finite Petri nets, which is a smooth generalization of the definition of standard step bisimulation [25] on finite Petri nets. Its induced equivalence offers an alternative, much simpler characterization of causal-net bisimilarity [13,19], as it does not resort to any causality structure, and of structure-preserving bisimilarity [13], as it does not require bijective mappings between related markings. We show that step net bisimilarity can be characterized logically by means of a suitable modal logic, called NML (acronym of net modal logic): two markings are step net bisimilar if and only if they satisfy the same NML formulae.

1 Introduction

Truly concurrent behavioral equivalences, such as history-preserving bisimilarity [29,8,11] or, equivalently, fully-concurrent bisimilarity [4], have been advocated by many as suitable relations to compare the behavior of distributed systems, in particular of Petri nets [28,9], because they respect not only the branching structure of systems (being based on the concept of bisimulation [27,23]), but also the causal dependencies among the performed actions/events. However, their formulation is usually very complex, so cumbersome that they are of very little practical help: in fact, not even one single case study of systems with never-ending behavior has been developed on them. Moreover, their usability seems rather limited, as all the behavioral equivalences in the spectrum ranging from interleaving bisimilarity (i.e., standard bisimilarity on the reachability graph of the marked net) to fully-concurrent bisimilarity are undecidable [22,10] on finite (unbounded) Petri nets. This situation is very frustrating.

However, this paper wants to offer a possible solution to this problem, by proposing for Petri nets, as a first approximation, a new bisimulation-like, very simple relation on markings that is a smooth generalization of ordinary bisimulation on Labeled Transition Systems [27,23,14] (LTSs, for short). This relation $R$, we call net bisimulation, essentially states that if two markings $m_1$ and $m_2$ are related by $R$, then (i) they must have the same size, i.e., $|m_1| = |m_2|$; moreover, (ii) for each transition $t_1$ executable from $m_1$ reaching $m'_1$, there exists a transition $t_2$ executable from $m_2$ reaching $m'_2$ such that, not only the labels of $t_1$ and $t_2$ are the same and that $(m'_1,m'_2) \in R$ (as for ordinary bisimulation), but also that the pre-sets of $t_1$ and $t_2$ are related by $R$ (i.e., $(\bullet t_1, \bullet t_2) \in R$); finally, (iii), the symmetric condition if $m_2$ moves first. Note that the two matching transitions $t_1$ and $t_2$ in the net bisimulation game must have the same shape
because \((i)\) \((r_1, r_2) \in R\) implies that \(|r_1| = |r_2|\), then \((ii)\) \(l(r_1) = l(r_2)\) and, finally,  
\((iii)\) from \(m_1 = m_2\), \(|m_1| = |m_2|\) and \(|r_1| = |r_2|\) we derive that \(|r_1^i| = |r_2^i|\), because \(m_i' = m_i \oplus r_i \oplus t_i^*\) for \(i = 1, 2\).

Net bisimulation is, indeed, a smooth generalization of ordinary bisimulation on LTSs [23][14], as on the subclass of Petri nets called sequential finite-state machines (which are essentially LTSs, as each reachable marking is a singleton), its definition reduces to standard bisimulation, as the additional side conditions \(|m_1| = |m_2|\) and \((r_1, r_2) \in R\) are redundant.

One may wonder if this apparently simplistic definition induces some actually sensible behavioral equivalence. As a matter of fact, even if net bisimilarity \(\sim_n\) may distinguish many systems with different causal structure, it may equate systems with different degree of parallelism. For this reason, we propose step net bisimulation as a similar enhancement of ordinary step bisimulation [25] on finite Petri nets. A step net bisimulation \(R\) is such that if two markings \(m_1\) and \(m_2\) are related by \(R\), then \((i)\) they must have the same size, i.e., \(|m_1| = |m_2|\); moreover, \((ii)\) for each step (i.e., multiset of concurrently enabled transitions) \(G_1\) executable from \(m_1\) reaching \(m_1'\), there exists a step \(G_2\) of the same size executable from \(m_2\) reaching \(m_2'\) such that, not only the labels of \(G_1\) and \(G_2\) are the same and that \((m_1', m_2') \in R\) (as for ordinary step bisimulation), but also that the pre-sets of \(G_1\) and \(G_2\) are related by \(R\) (i.e., \((G_1, G_2) \in R\)); finally, \((iii)\), the symmetric condition if \(m_2\) moves first. Note that a single enabled transition forms a singleton step, so that a step (net) bisimulation is also a (net) bisimulation, i.e., step (net) bisimilarity is finer than (net) bisimilarity.

The first main contribution of this paper is to show that, surprisingly enough, the very simple step net bisimilarity \(\sim_s\) coincides with a well-known, truly-concurrent behavioral relation, proposed some years ago by R. van Glabbeek, namely causal-net bisimilarity [13] (further studied in [19], who called it by this name), whose decidability on unbounded Petri nets is an open problem and which is intuitively very appealing. Causal-net bisimilarity \(\sim_{cn}\) is essentially a bisimulation-like relation over the causal nets (sometimes called nonsequential processes) [12][3][26] originating from the two initial markings of interest. Hence, its definition is very complex and cumbersome. However, as step net bisimilarity coincides with causal-net bisimilarity, we get that our extremely simple definition of step net bisimilarity is slightly finer than fully-concurrent bisimilarity [4] in that it relates markings that not only generate the same the partial orders of events, but that also have the same size.

To be precise, van Glabbeek defined in [13] an alternative characterization of causal-net bisimilarity, called structure-preserving bisimilarity, which is defined over multisets of so-called linkings or, equivalently, over triples of type (marking, bijection, marking); however, also this characterization is more complex than step net bisimilarity, as the latter is defined simply on pairs of type (marking, marking), as for ordinary bisimulations.

The second main contribution of this paper is the logical characterization of step net bisimilarity \(\sim_s\) in terms of a new modal logic, called NML (acronym of net modal logic), which can be seen as a generalization of Hennessy-Milner Logic [21] with an operator of parallel composition of formulae (e.g., \(F_1 \otimes F_2\)), and with a generalization of the HML modal operator \(\langle a \rangle\) to \(\langle a \rangle_N\), which stands for the executability of a net
transition labeled \( a \) with pre-set of size \( h \) and post-set of size \( k \). We prove that two markings are step net bisimilar if and only if they satisfy the same NML formulae.

The paper is organized as follows. Section 2 introduces the basic definitions about Petri nets, including interleaving bisimilarity, step bisimilarity and causal-net bisimilarity. Section 3 introduces first the coinductive definition of net bisimulation, showing that in some cases it may be inadequate, then introduces step net bisimulation, proves that the largest step net bisimulation is an equivalence relation, discusses some illustrative examples, in particular a tiny case study. Section 4 proves that step net bisimilarity and causal-net bisimilarity coincide. Section 5 presents the modal logic NML and proves that two markings are step net bisimilar if and only if they satisfy the same NML formulae. At last, Section 6 is devoted to some final remarks.

2 Basic Definitions

**Definition 1.** (Multiset) Let \( \mathbb{N} \) be the set of natural numbers. Given a finite set \( S \), a multiset over \( S \) is a function \( m : S \to \mathbb{N} \). The support set \( \text{dom}(m) \) of \( m \) is \( \{ s \in S \mid m(s) \neq 0 \} \). The set of all multisets over \( S \), denoted by \( \mathcal{M}(S) \), is ranged over by \( m \). We write \( s \in m \) if \( m(s) > 0 \). The multiplicity of \( s \) in \( m \) is given by the number \( m(s) \). The size of \( m \), denoted by \( |m| \), is the number \( \sum_{s \in S} m(s) \), i.e., the total number of its elements. A multiset \( m \) such that \( \text{dom}(m) = \emptyset \) is called empty and is denoted by \( \emptyset \). We write \( m \subseteq m' \) if \( m(s) \leq m'(s) \) for all \( s \in S \). We also write \( m \preceq m' \) if \( m \subseteq m' \) and \( m(s) < m'(s) \) for some \( s \in S \).

Multiset union \( _{\oplus} \) is defined as follows: \( (m \oplus m')(s) = m(s) + m'(s) \); it is commutative, associative and has \( \emptyset \) as neutral element. Multiset difference \( _{-} \) is defined as follows: \( (m_1 \ominus m_2)(s) = \max\{m_1(s) - m_2(s), 0\} \). The scalar product of a number \( j \) with \( m \) is the multiset \( j \cdot m \) defined as \( (j \cdot m)(s) = j \cdot (m(s)) \). By \( s_i \) we also denote the multiset with \( s_i \) as its only element. Hence, a multiset \( m \) over \( S = \{s_1, \ldots, s_n\} \) can be represented as \( k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \ldots \oplus k_n \cdot s_n \), where \( k_j = m(s_j) \geq 0 \) for \( j = 1, \ldots, n \). \( \Box \)

**Definition 2.** (Place/Transition net) A labeled Place/Transition Petri net (P/T net for short) is a tuple \( N = (S, A, T) \), where

- \( S \) is the finite set of places, ranged over by \( s \) (possibly indexed),
- \( A \subseteq \text{Lab} \) is the finite set of labels, ranged over by \( \ell \) (possibly indexed), and
- \( T \subseteq (\mathcal{M}(S) \setminus \{\emptyset\}) \times A \times \mathcal{M}(S) \) is the finite set of transitions, ranged over by \( t \) (possibly indexed).

Given a transition \( t = (m, \ell, m') \), we use the notation:

- \( \bullet t \) to denote its pre-set \( m \) (which cannot be an empty multiset) of tokens to be consumed;
- \( l(t) \) for its label \( \ell \), and
- \( \ell \cdot t \) to denote its post-set \( m' \) of tokens to be produced.

Hence, transition \( t \) can be also represented as \( \bullet t \xrightarrow{l(t)} \ell \cdot t \). We also define pre-sets and post-sets for places as follows: \( \bullet s = \{ t \in T \mid s \in \bullet t \} \) and \( s' = \{ t \in T \mid s \in \ell \cdot t \} \). Note that while the pre-set (post-set) of a transition is, in general, a multiset, the pre-set (post-set) of a place is a set. \( \Box \)
In the graphical description of finite P/T nets, places (represented as circles) and transitions (represented as boxes) are connected by directed arcs. The arcs may be labeled with the number representing how many tokens are to be removed from (or produced into) that place; no label on the arc is interpreted as the number one, i.e., one token flowing on the arc. This numerical label of the arc is called its weight.

**Definition 3. (Marking, P/T net system)** Given a P/T net system \( N = (S,A,T) \), a multiset over \( S \) is called a marking. Given a marking \( m \) and a place \( s \), we say that the place \( s \) contains \( m(s) \) tokens, graphically represented by \( m(s) \) bullets inside place \( s \). A P/T net system \( N(m_0) \) is a tuple \((S,A,T,m_0)\), where \((S,A,T)\) is a P/T net and \( m_0 \) is a marking over \( S \), called the initial marking. We also say that \( N(m_0) \) is a marked net. \( \Box \)

### 2.1 Sequential Semantics

**Definition 4. (Enabling, firing sequence, reachable marking)** Given a P/T net \( N = (S,A,T) \), a transition \( t \) is enabled at \( m \), denoted by \( m[t] \), if \( ^*t \subseteq m \). The execution (or firing) of \( t \) enabled at \( m \) produces the marking \( m' = (m \oplus ^*t) \ominus t^* \). This is written \( m[t]m' \). A firing sequence starting at \( m \) is defined inductively as follows:

- \( m[\varepsilon]m \) is a firing sequence (where \( \varepsilon \) denotes an empty sequence of transitions) and
- if \( m[\sigma]m' \) is a firing sequence and \( m'[t]m'' \), then \( m[\sigma t]m'' \) is a firing sequence.

The set of reachable markings from \( m \) is denoted by \( |m| \) and \( |m| = \{ m' \mid \exists \sigma. m[\sigma]m' \} \).

Note that the reachable markings can be countably infinitely many when the net is not bounded, i.e., when the number of tokens in some places can grow unboundedly (cf. Definition 5).

**Definition 5. (Classes of P/T Petri nets)** A P/T Petri net \( N = (S,A,T) \) is

- **statically acyclic** if there exists no sequence \( x_1x_2\ldots x_n \), such that \( n \geq 3 \), \( x_i \in S \cup T \) for \( i = 1, \ldots, n \), \( x_1 = x_n \), \( x_1 \in S \) and \( x_i \in ^*x_{i+1} \) for \( i = 1, \ldots, n-1 \), i.e., the arcs of the net do not form any cycle.
- **a finite-state machine (FSM, for short)** if \( N \) is finite and for all \( t \in T \), \( |t| = 1 \) and \( |^*t| = 1 \);
- **a BPP net** if \( N \) is finite and every transition has exactly one input place, i.e., for all \( t \in T \), \( |t| = 1 \);

A P/T net system \( N(m_0) \) is

- **dynamically acyclic** if there exists no \( m_1 \in |m_0| \) with a nonempty (i.e., with \( n \geq 2 \)) firing sequence \( m_1[t_1]m_2\ldots[t_{n-1}]m_n \) such that \( m_1 \subseteq m_0 \);
- **a sequential FSM** if \( N \) is an FSM and \( m_0 \) is a singleton, i.e., \( |m_0| = 1 \);
- **a concurrent FSM** if \( N \) is an FSM and \( m_0 \) is arbitrary;
- **\( k \)-bounded** if any place contains at most \( k \) tokens in any reachable marking, i.e., for all \( s \in S \), \( m(s) \leq k \) for all \( m \in |m_0| \);
- **safe** if it is 1-bounded.
- **bounded** if there exists \( k \geq 1 \) such that \( N(m_0) \) is \( k \)-bounded.
- **unbounded** if it is not bounded. \( \Box \)
We now define standard bisimilarity over Petri nets, called *interleaving* bisimilarity because it equates a parallel system, such as that in Figure 3(b), to the sequential/nondeterministic system performing its interleavings, such as that in Figure 3(a).

**Definition 6. (Interleaving Bisimulation)** Let \( N = (S,A,T) \) be a P/T net. An interleaving bisimulation is a relation \( R \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \) such that if \( (m_1,m_2) \in R \) then

1. \( \forall t_1 \text{ such that } m_1(t_1)m_1', \exists t_2 \text{ such that } m_2(t_2)m_2' \text{ with } l(t_1) = l(t_2) \text{ and } (m_1',m_2') \in R \),
2. \( \forall t_2 \text{ such that } m_2(t_2)m_2', \exists t_1 \text{ such that } m_1(t_1)m_1' \text{ with } l(t_1) = l(t_2) \text{ and } (m_1',m_2') \in R \).

Two markings \( m_1 \) and \( m_2 \) are interleaving bisimilar (or interleaving bisimulation equivalent), denoted by \( m_1 \sim_{\text{int}} m_2 \), if there exists an interleaving bisimulation \( R \) such that \( (m_1,m_2) \in R \).

Interleaving bisimilarity was proved undecidable in [22], with a proof based on the comparison of two sequential P/T nets, where a P/T net is sequential if it does not offer any concurrent behavior. Hence, interleaving bisimulation equivalence is undecidable even for the subclass of sequential finite P/T nets. Esparza observed in [10] that all the non-interleaving bisimulation-based equivalences – in the spectrum ranging from interleaving bisimilarity to fully-concurrent bisimilarity [4] – collapse to interleaving bisimilarity over sequential P/T nets. Hence, the proof in [22] applies to all these non-interleaving bisimulation equivalences as well.

### 2.2 Concurrent Semantics

Given a net \( N = (S,A,T) \) and a marking \( m \), we say that two transitions \( t_1,t_2 \in T \) are **concurrently enabled** at \( m \) if \( \bullet t_1 \oplus \bullet t_2 \subseteq m \). The **concurrent firing** of these two transitions produces the marking \( m' = (m \oplus (\bullet t_1 \oplus \bullet t_2)) \oplus (t_1^* \oplus t_2^*) \). We denote this fact by \( m(t_1,t_2)m' \). It is also possible that the same transition is **self-concurrent** at some marking \( m \), meaning that two or more occurrences of it are concurrently enabled at \( m \).

We can generalize the definition of concurrently enabled transitions to a finite, nonempty multiset \( G \) over the set \( T \), called a step. A step \( G : T \to \mathbb{N} \) is enabled at marking \( m \) if \( \bullet G \subseteq m \), where \( \bullet G = \bigoplus_{t \in T} G(t) \cdot \bullet t \) and \( G(t) \) denotes the number of occurrences of transition \( t \) in the step \( G \). The execution of a step \( G \) enabled at \( m \) produces the marking \( m' = (m \oplus \bullet G) \oplus G^* \), where \( G^* = \bigoplus_{t \in T} G(t) \cdot t^* \). This is written \( m(G)m' \).

We sometimes refer to this as the **concurrent token game**, in opposition to the sequential token game of Definition 4. The label \( I(G) \) of a step \( G \) is the multiset \( I(G) : A \to \mathbb{N} \) defined as follows: \( I(G)(a) = \sum_{t \in \text{dom}(G), l(t) = a} G(t) \).

Now we define a notion of bisimulation based on the firing of steps, rather than single transitions (as for interleaving bisimulation), originally proposed in [25].

**Definition 7. (Step Bisimulation)** Let \( N = (S,A,T) \) be a P/T net. A step bisimulation is a relation \( R \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \) such that if \( (m_1,m_2) \in R \) then

1. \( \forall G_1 \text{ such that } m_1(G_1)m_1', \exists G_2, m_2(G_2)m_2' \text{ with } I(G_1) = I(G_2) \text{ and } (m_1',m_2') \in R \),
2. \( \forall G_2 \text{ such that } m_2(G_2)m_2', \exists G_1, m_1(G_1)m_1' \text{ with } I(G_1) = I(G_2) \text{ and } (m_1',m_2') \in R \).

Two markings \( m_1 \) and \( m_2 \) are step bisimilar (or step bisimulation equivalent), denoted by \( m_1 \sim_{\text{step}} m_2 \), if there exists a step bisimulation \( R \) such that \( (m_1,m_2) \in R \).

Of course, \( \sim_{\text{step}} \) is finer than \( \sim_{\text{int}} \); moreover, also step bisimilarity is undecidable for P/T nets having at least two unbounded places [10].
2.3 Causal Semantics

However, now we present a recently proposed [13,19] behavioral equivalence, called *causal-net bisimilarity*, whose decidability is open, as the proof in [22,10] does not apply to it. In order to define it, we need some auxiliary definitions, adapting those in, e.g., [12,3,4,26,13].

The concurrent semantics of a marked P/T net is defined by a particular class of acyclic safe nets, where places are not branched (hence they are essentially deterministic) and all arcs have weight 1. This kind of net is called *causal net*. We use the name $C$ (possibly indexed) to denote a causal net, the set $B$ to denote its places (called *conditions*), the set $E$ to denote its transitions (called *events*), and $L$ to denote its labels.

**Definition 8.** (Causal net) A causal net is a marked net $C(m_0) = (B, L, E, m_0)$ satisfying the following conditions:

1. $C$ is statically acyclic;
2. $\forall b \in B\ |b| \leq 1 \wedge |b^*| \leq 1$ (i.e., the places are not branched);
3. $\forall b \in B\ m_0(b) = \begin{cases} 1 & \text{if } b = \emptyset \\ 0 & \text{otherwise} \end{cases}$;
4. $\forall e \in E\ e(b) \leq 1 \wedge e^*(b) \leq 1$ for all $b \in B$ (i.e., all the arcs have weight 1).

We denote by $\text{Min}(C)$ the set $m_0$, and by $\text{Max}(C)$ the set $\{b \in B \mid b^* = \emptyset\}$. □

Note that any reachable marking of a causal net is a set, i.e., this net is *safe*; in fact, the initial marking is a set and, assuming by induction that a reachable marking $m$ is a set and enables $e$, i.e., $m[e]m'$, then also $m' = (m \ominus e) \oplus e^*$ is a set, as the net is acyclic and because of the condition on the shape of the post-set of $e$ (weights can only be 1).

As the initial marking of a causal net is fixed by its shape (according to item 3 of Definition 8), in the following, in order to make the notation lighter, we sometimes omit the indication of the initial marking, so that $C(m_0)$ is simply denoted by $C$.

**Definition 9.** (Moves of a causal net) Given two causal nets $C = (B, L, E, m_0)$ and $C' = (B', L, E', m_0)$, we say that $C$ moves in one step to $C'$ through $e$, denoted by $C[e]C'$, if $e \subseteq \text{Max}(C)$, $E' = E \cup \{e\}$ and $B' = B \cup e^*$.

**Definition 10.** (Folding and Process) A folding from a causal P/T net $C = (B, L, E, m_0)$ into a P/T net system $N(m_0) = (S, A, T, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:

- $L = A$ and $l(e) = l(\rho(e))$ for all $e \in E$;
- $\rho(m_0) = m_0$, i.e., $m_0(s) = |\rho^{-1}(s) \cap m_0|$;
- $\forall e \in E, \rho^*(e) = \star \rho(e)$, i.e., $\rho^*(e)(s) = |\rho^{-1}(s) \cap e|$ for all $s \in S$;
- $\forall e \in E, \rho(e^*) = \rho(e)$, i.e., $\rho(e^*)(s) = |\rho^{-1}(s) \cap e^*|$ for all $s \in S$.

A pair $(C, \rho)$, where $C$ is a causal net and $\rho$ a folding from $C$ to a net system $N(m_0)$, is a process of $N(m_0)$.
Definition 11. (Moves of a process) Let \( N(m_0) = (S, A, T, m_0) \) be a net system and let \((C_i, \rho_i)\) for \( i = 1, 2 \), be two processes of \( N(m_0) \). We say that \((C_i, \rho_i)\) moves in one step to \((C_2, \rho_2)\) through \( e \), denoted by \((C_i, \rho_i) \xrightarrow{e} (C_2, \rho_2)\), if \( C_i[e]C_2 \) and \( \rho_1 \subseteq \rho_2 \). \( \square \)

The recently proposed novel behavioral equivalence is the following causal-net bisimulation \([13, 19]\).

Definition 12. (Causal-net bisimulation) Let \( N = (S, A, T) \) be a P/T net. A causal-net bisimulation is a relation \( R \), composed of triples of the form \((p_1, C, p_2)\), where, for \( i = 1, 2 \), \((C, \rho_i)\) is a process of \( N(m_0) \) for some \( m_0 \), such that if \((p_1, C, p_2) \in R\) then

\[
\begin{align*}
&i) \quad \forall \gamma_1, C', \rho_1' \text{ such that } (C, \rho_1) \xrightarrow{\gamma_1} (C', \rho_1') \text{ with } \rho_1'(e) = \gamma_1, \text{ there exist } \gamma_2, \rho_2' \text{ such that } (C, \rho_2) \xrightarrow{\gamma_2} (C', \rho_2') \text{ with } \rho_2'(e) = \gamma_2, \text{ and } (\rho_1', C', \rho_2') \in R; \\
&ii) \quad \text{and symmetrically, } \forall \gamma_2, C', \rho_2' \text{ such that } (C, \rho_2) \xrightarrow{\gamma_2} (C', \rho_2') \text{ with } \rho_2'(e) = \gamma_2, \text{ there exist } \gamma_1, \rho_1' \text{ such that } (C, \rho_1) \xrightarrow{\gamma_1} (C', \rho_1') \text{ with } \rho_1'(e) = \gamma_1, \text{ and } (\rho_1', C', \rho_2') \in R. 
\end{align*}
\]

Two markings \( m_1 \) and \( m_2 \) of \( N \) are cn-bisimilar (or cn-bisimulation equivalent), denoted by \( m_1 \sim_{cn} m_2 \), if there exists a causal-net bisimulation \( R \) containing a triple \((\rho_1^0, C^0, \rho_2^0)\), where \( C^0 \) contains no events and \( \rho_i^0(\text{Min}(C^0)) = \rho_i^0(\text{Max}(C^0)) = m_i \) for \( i = 1, 2 \). \( \square \)

Of course, \( \sim_{cn} \) is an equivalence relation \([19]\) and, moreover, it coincides with structure-preserving bisimilarity \([13]\).

3 Net Bisimulation and Step Net Bisimulation

Definition 13. (Net Bisimulation) Let \( N = (S, A, T) \) be a P/T net. A net bisimulation is a relation \( R \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \) such that if \((m_1, m_2) \in R\) then

- \( |m_1| = |m_2| \),
- \( \exists t_1 \text{ such that } m_1[t_1]m_1' \) and \( \exists t_2 \text{ such that } m_2[t_2]m_2' \) with \( l(t_1) = l(t_2) \), \( (t_1, t_2) \in R \) and \( (m_1', m_2') \in R \),
- \( \exists t_2 \text{ such that } m_2[t_2]m_2' \) and \( \exists t_1 \text{ such that } m_1[t_1]m_1' \) with \( l(t_1) = l(t_2) \), \( (t_1, t_2) \in R \) and \( (m_1', m_2') \in R \).

Two markings \( m_1 \) and \( m_2 \) are net bisimilar, denoted by \( m_1 \sim_n m_2 \), if there exists a net bisimulation \( R \) such that \((m_1, m_2) \in R\). \( \square \)

Net bisimulation is a smooth generalization of ordinary bisimulation on LTSs (see, e.g., \([23, 14]\)). In fact, on sequential finite-state machines (which are essentially labeled transition systems, as each reachable marking is a singleton; cf. Definition \([5]\), its definition reduces to standard bisimulation, as the side conditions \(|m_1| = |m_2| \) and \((t_1, t_2) \in R \) are redundant.

Note that the two matching transitions \( t_1 \) and \( t_2 \) in the net bisimulation game must have the same shape because \( (i) \) \( (t_1, t_2) \in R \) implies that \( |t_1| = |t_2| \), then \( (ii) \) \( l(t_1) = l(t_2) \) and, finally, \( (iii) \) from \(|m_1| = |m_2| \), \( |m_1'| = |m_2'| \) and \(|t_1| = |t_2| \) we derive that \( |t_1'| = |t_2'| \), because \( m'_i = m_i \oplus t_i \oplus t_i^* \) for \( i = 1, 2 \).
Example 1. Consider the two nets in Figure 1 representing two unbounded producer-consumer systems, where prod is the action of producing an item, del of delivering an item, cons of consuming an item. It is easy to realize that relation
\[ R_1 = \{ (P_1 \oplus C_1, P_2 \oplus C_2) \} \cup \{ (P_1', \oplus C_1 \oplus D_1, P_2 \oplus C_2 \oplus k_1 \cdot D_1' \oplus k_2 \cdot D_2') \mid k, k_1, k_2 \geq 0, k = k_1 + k_2 \} \cup \{ (P_1', \oplus C_1' \oplus k \cdot D_1, P_2 \oplus C_2' \oplus k_1 \cdot D_1' \oplus k_2 \cdot D_2') \mid k, k_1, k_2 \geq 0, k = k_1 + k_2 \} \]
is an interleaving bisimulation, containing the pair \((P_1 \oplus C_1, P_2 \oplus C_2)\), where each pair \((m_1, m_2) \in R_1\) is such that \(|m_1| = |m_2|\). Moreover, relation
\[ R_2 = R_1 \cup \{ (P_1, P_2), (C_1 \oplus D_1, C_2 \oplus D_2') \}, (C_1 \oplus D_1, C_2 \oplus D_2'), (C_1', C_2'), (C_1, C_2) \} \]
\[ \cup \{ (P_1', \oplus D_1, P_2 \oplus k_1 \cdot D_1' \oplus k_2 \cdot D_2') \mid k, k_1, k_2 \geq 0, k = k_1 + k_2 \} \]
is a net bisimulation. Note that if \(P_1 \oplus C_1 \xrightarrow{\text{prod}} P_1' \oplus D_1 \oplus C_1\), then \(P_1 \oplus C_2\) can respond with \(P_2 \oplus C_2 \xrightarrow{\text{prod}} P_2 \oplus D_2' \oplus C_2\) and the two required pairs belong to \(R_2\), because \((P_1' \oplus D_1 \oplus C_1, P_2 \oplus D_2' \oplus C_2) \in R_1\) (and so also in \(R_2\)) and \((P_1, P_2) \in R_2\); the latter pair requires that both \((P_1' \oplus D_1, P_2 \oplus D_2')\) and \((P_1' \oplus D_1, P_2 \oplus D_2')\) belong to \(R_2\) and, by iterating the firing of the transitions \(\text{prod}\), that all the pairs in the last group belong to \(R_2\).

Example 2. By the previous example, one may be induced to think that an interleaving bisimulation \(R_1\), with the property that each pair \((m_1, m_2) \in R_1\) is such that \(|m_1| = |m_2|\), may be completed with additional pairs to become a net bisimulation \(R_2\). This is false. For instance, consider the nets in Figure 2. Clearly, \(R_1 = \{ (s_1 \oplus s_2, s_5 \oplus s_7), (s_1 \oplus s_3, s_6 \oplus s_7), (s_1 \oplus s_3, s_5 \oplus s_6), (s_1 \oplus s_4, s_6 \oplus s_8) \} \) is an interleaving bisimulation relating markings of the same size only. However, it is not possible to build a net bisimulation because, to the move \(s_1 \oplus s_2 \xrightarrow{\text{a}} s_1 \oplus s_3\) (due to transition \(t_1 = (s_2, a, s_3)\)), marking \(s_5 \oplus s_7\) can respond by reaching either \(s_6 \oplus s_7\) (due to transition \(t_2 = (s_5, a, s_6)\)) or \(s_5 \oplus s_8\) (due to transition \(t_2' = (s_7, a, s_8)\)). In any case, \(s_2\) is not net bisimilar to \(s_5\) or \(s_7\), because \(s_2\) can perform the sequence \(aa\), while \(s_5\) and \(s_7\) cannot.

Net bisimilarity \(\sim_n\) seems a truly concurrent behavioral equivalence, and, indeed, some canonical examples are correctly classified by it. For instance, consider the three nets in Figure 3 which are the semantics of the CCS-like [23] terms \(a.b.0 + b.a.[0,1]\),
a.0|b.0 and a.C|b.0 (with C \cong 0), respectively, according to the net semantics in [15]. Of course, s₁ is not net bisimilar to s₄ ⊕ s₅ because these two markings have different size; as a matter of fact, these two markings generate different partial orders of actions, where the former may perform one of the two causally related sequences ab or ba, while for the latter a and b are causally independent. Moreover, s₄ ⊕ s₅ is not net bisimilar to s₆ ⊕ s₇ because s₄ \approxₜ s₆: after the execution of a, the reached markings (namely, \( \emptyset \) and s₈) are not net bisimilar because they have different size. So, net bisimilarity not only observes the partial order of the actions performed, but also the size of the distributed state. However, net bisimilarity, being an enhancement of interleaving bisimilarity, is not accurate enough, as it may (wrongly) equate systems with different degree of parallelism, as illustrated in the following example.

**Example 3.** Consider Figure 4, where we call by \( t_i \) the transition \((s_i, a, s_j)\) for \( i = 1, \ldots, 3 \). It is easy to realize that relation \( R = \{(s_1 \oplus s_2, s_3 \oplus s_4), (s_1, s_3), (s_2, s_3)\} \) is a net bisimulation. Consider the two markings s₁ ⊕ s₂ and s₃ ⊕ s₄; even if these two markings are net bisimilar, they are not causal-net bisimilar because they generated different causal nets (and different partial orders). Actually, they are not even step bisimilar because s₁ ⊕ s₂ can perform the step \( \{t_1, t_2\} \), while s₃ ⊕ s₄ cannot perform a step of size two. \( \square \)
This example shows that net bisimilarity $\sim_n$, being based on interleaving bisimilarity, may fail to distinguish systems with different degree of parallelism. For this reason, we now introduce step net bisimulation, in order to enhance step bisimulation with similar features as those of net bisimulation.

**Definition 14. (Step Net Bisimulation)** Let $N = (S, A, T)$ be a P/T net. A step net bisimulation is a relation $R \subseteq \mathcal{M}(S) \times \mathcal{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $|m_1| = |m_2|$,
- $\forall G_1$ such that $m_1 \mid G_1 \mid m_1'$, $\exists G_2$ such that $m_2 \mid G_2 \mid m_2'$, with $(\cdot G_1, \cdot G_2) \in R$, $l(G_1) = l(G_2)$ and $(m_1', m_2') \in R$,
- $\forall G_2$ such that $m_2 \mid G_2 \mid m_2'$, $\exists G_1$ such that $m_1 \mid G_1 \mid m_1'$, with $(\cdot G_1, \cdot G_2) \in R$, $l(G_1) = l(G_2)$ and $(m_1', m_2') \in R$.

Two markings $m_1$ and $m_2$ are step net bisimilar, denoted by $m_1 \sim_s m_2$, if there exists a step net bisimulation $R$ such that $(m_1, m_2) \in R$.

**Remark 1. (Same shape of matching transitions)** When the steps $G_1$ and $G_2$ are singletons (e.g., $\{t_1\}$ and $\{t_2\}$, respectively), we get that the two matching transitions $t_1$ and $t_2$ in the step net bisimulation game must have the same shape, as already observed for net bisimulation. In general, it is possible to prove that, given a step $G_1 = \{t_1^1, \ldots, t_1^k\}$, if there exists a matching step $G_2$ (i.e., such that $l(G_1) = l(G_2)$, $(\cdot G_1, \cdot G_2) \in R$ and the reached pair of markings $(m_1', m_2')$ are related by $R$), then there exists also a matching step $G_2 = \{t_2^1, \ldots, t_2^k\}$, such that, additionally, $l(t_1^i) = l(t_2^i)$ and $(\cdot t_1^i, \cdot t_2^i) \in R$ (and so $t_1^i$ and $t_2^i$ must have the same shape) for $i = 1, \ldots, k$, and moreover that the reached pair of markings $(m_1', m_2')$ are related by $R$. This property is very important in the initial step of the proof of Proposition 5 (stating that step net bisimilarity implies causal net bisimilarity) to ensure that if two markings are step net bisimilar, then it is possible to match their initial transitions by putting in correspondence those with the same shape. This is illustrated by the following example.

**Example 4.** Consider the nets in Figure 5 where we call the ten transitions by

$t_1 = (s_1, a, s_3)$ $t_2 = (s_1, a, s_3)$ $t_3 = (s_1 \oplus s_2, a, s_6)$
$u_1 = (s_1, b, s_4)$ $u_2 = (s_1 \oplus s_2, b, s_7)$
$t_4 = (s_8, a, s_{11})$ $t_5 = (s_8 \oplus s_9, a, s_{12})$ $t_6 = (s_8 \oplus s_9, a, s_{13})$
$u_3 = (s_8, b, s_{10})$ $u_4 = (s_8 \oplus s_9, b, s_{14})$

It is not difficult to prove that $2 \cdot s_1 \oplus s_2 \sim_s 2 \cdot s_8 \oplus s_9$. Note that $2 \cdot s_1 \oplus s_2$ can perform, e.g., the step $\{u_1, t_3\}$ and that $2 \cdot s_8 \oplus s_9$ can respond with $\{u_4, t_4\}$, where $u_1$ ($t_3$) has not the same shape of $u_4$ ($t_4$). However, in order to respect the step net bisimulation game,
transition \( u_1 \) must be matched by some transition (and in fact, \( u_3 \) is the required one) and also \( t_5 \) must be matched (and in fact, \( t_5 \) or \( t_6 \) can be the correct match); in all, the step \( \{ u_3, t_5 \} \) is ‘equivalent’ to the step \( \{ u_4, t_4 \} \), but the former has the property that the matched step is composed of matched transitions with the same shape. And this holds in general: whenever a step \( G_1 \) is matched by some step \( G_2 \), then by inspecting the single components of \( G_1 \), we can reconstruct another matching step \( \overline{G_2} \) with the property that there is a shape-preserving bijection between the transitions of \( G_1 \) and those of \( G_2 \).

Fig. 5. Two step net bisimilar nets

In order to prove that \( \sim_n \) and \( \sim_s \) are equivalence relations, we now list some useful properties of (step) net bisimulation relations.

**Proposition 1.** For each P/T net \( N = (S, A, T) \), the following hold:

1. The identity relation \( \mathcal{I} = \{ (m, m) \mid m \in \mathcal{M}(S) \} \) is a (step) net bisimulation;
2. the inverse relation \( \mathcal{R}^{-1} = \{ (m', m) \mid (m, m') \in \mathcal{R} \} \) of a (step) net bisimulation \( \mathcal{R} \) is a (step) net bisimulation;
3. the relational composition \( \mathcal{R}_1 \circ \mathcal{R}_2 = \{ (m, m'') \mid \exists m'. (m, m') \in \mathcal{R}_1 \land (m', m'') \in \mathcal{R}_2 \} \) of two (step) net bisimulations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) is a (step) net bisimulation;
4. given a family \( \{ \mathcal{R}_i \}_{i \in I} \) of (step) net bisimulations, their union \( \bigcup_{i \in I} \mathcal{R}_i \) is a (step) net bisimulation.

**Proof.** The proof is standard, following the same steps of the analogous proof for ordinary bisimulation on LTSs (see, e.g., [23][24]).

**Proposition 2.** For each P/T net \( N = (S, A, T) \), relations \( \sim_n \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \) and \( \sim_s \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \) are equivalence relations.

**Proof.** The proof, based on the observations in Proposition 1, is standard.
Fig. 6. Two step bisimilar nets, not step net bisimilar, with different causal semantics

\[ \sim_n = \bigcup \{ R \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \mid R \text{ is a net bisimulation} \} \]
\[ \sim_s = \bigcup \{ R \subseteq \mathcal{M}(S) \times \mathcal{M}(S) \mid R \text{ is a step net bisimulation} \} \]
is the largest (step) net bisimulation.

The following proposition follows trivially from the definitions.

**Proposition 3.** For each P/T net \( N = (S, A, T) \), the following hold:

(i) if \( m_1 \sim_\text{step} m_2 \), then \( m_1 \sim_\text{int} m_2 \).
(ii) if \( m_1 \sim_n m_2 \), then \( m_1 \sim_\text{int} m_2 \).
(iii) if \( m_1 \sim_s m_2 \), then \( m_1 \sim_\text{step} m_2 \).
(iv) if \( m_1 \sim_s m_2 \), then \( m_1 \sim_n m_2 \).

Example 5. We want to show that each implication in the proposition above is strict, by showing a counterexample for the converse implication.

For cases (i) and (iv), let us consider again Example 3. Of course, relation \( R \) is a net bisimulation (and also an interleaving bisimulation) but not a step net bisimulation (and not even a step bisimulation) because only \( s_1 \oplus s_2 \) can perform a step of size two.

For case (ii), the counterexample is discussed in Example 2.

For case (iii), consider the nets in Figure 6 where we denote the five transitions by \( t_1 = (s_1, a, \theta) \), \( t_2 = (s_2, b, \theta) \), \( t_3 = (s_3, a, \theta) \), \( t_4 = (s_4, b, \theta) \) and \( t_5 = (s_3 \oplus s_4, a, s_4) \). It is easy to see that relation \( R = \{(s_1 \oplus s_2, s_3 \oplus s_4), (s_1, s_3), (s_2, s_4)\} \) is a step bisimulation such that the related markings have always the same size. However, \( R \) is not a step net bisimulation (and cannot be completed with additional pairs to become such), because if \( s_3 \oplus s_2[t_5]s_4 \), then \( s_1 \oplus s_2 \) can only try to respond with \( t_1 \), but the presets of \( t_5 \) and \( t_1 \) have different size and so cannot be related by a step net bisimulation.

Example 6. By the examples above, it seems that if two markings are not step net bisimilar, then they are not causal-net bisimilar. However, we want to argue also that if two markings are step net bisimilar, then they are causal-net bisimilar. Consider Example 1. It is easy to note that \( R_1 \) is not only an interleaving bisimulation, but also a step bisimulation. Moreover, \( R_2 \) is not only a net bisimulation, but also a step net bisimulation. Not surprisingly, the two markings \( P_1 \oplus C_1 \) and \( P_2 \oplus C_2 \) are causal-net bisimilar.

4 Step Net Bisimilarity and Causal-net Bisimilarity Coincide

Now we want to prove that step net bisimilarity \( \sim_s \) and causal-net bisimilarity \( \sim_{\text{cn}} \) coincide by proving the two implications separately.
Proposition 4. For each P/T net N = (S,A,T), if \( m_1 \sim_{cn} m_2 \), then \( m_1 \sim_s m_2 \).

Proof. If \( m_1 \sim_{cn} m_2 \), then a causal-net bisimulation \( R \) exists such that it contains a triple of the form \( (\rho^0_1(C^0), \rho^0_2(C^0)) \), where \( C^0 \) is the causal net without events and \( \rho^0_1, \rho^0_2 \) are such that \( \rho^0_i(\operatorname{Max}(C^0)) = \rho^0_i(\operatorname{Max}(C^0)) = m_i \) for \( i = 1, 2 \).

Let us consider

\[
R_2 = \{ (\rho_1(\operatorname{Max}(C)), \rho_2(\operatorname{Max}(C))) | \exists R_1 \text{ causal net bisimulation such that } (\rho_1, C, \rho_2) \in R_1 \}.
\]

We want to prove that \( R_2 \) is a step net bisimulation. First of all, consider the triple \( (\rho^0_1, C^0, \rho^0_2) \in R \) mentioned above. Hence, the pair \( (\rho^0_1(\operatorname{Max}(C^0)), \rho^0_2(\operatorname{Max}(C^0))) \) must belong to \( R_2 \), so that, if \( R_2 \) is a step net bisimulation, this fact ensures that \( m_1 \sim_s m_2 \).

Now, assume \( (\rho_1(\operatorname{Max}(C)), \rho_2(\operatorname{Max}(C))) \in R_2 \). In order for \( R_2 \) to be a step net bisimulation, we must prove that

1. \( |\rho_1(\operatorname{Max}(C))| = |\rho_2(\operatorname{Max}(C))| \)
2. \( \forall G_1 \text{ such that } \rho_1(\operatorname{Max}(C))[G_1]m^1, \exists G_2 \text{ such that } \rho_2(\operatorname{Max}(C))[G_2]m^2 \text{ with } l(G_1) = l(G_2), (G_1, \{G_1, \star G_2\}) \in R_2 \text{ and } (m^1, m^2) \in R_2 \).
3. symmetrically, if \( \rho_2(\operatorname{Max}(C)) \) moves first.

The first item, i.e., \( |\rho_1(\operatorname{Max}(C))| = |\rho_2(\operatorname{Max}(C))| \), is trivial. To prove the second item, note that the move \( \rho_1(\operatorname{Max}(C))[G_1]m^1 \) ensures that there exists a sequence \( t_1^k \ldots t_k^1 \) (with \( k \geq 1 \)) of concurrently enabled transitions (such that \( G = \{t_1^1, \ldots, t_k^1\} \)) and markings \( m_1^1 \ldots m_k^1 \) such that

\[
\rho_1(\operatorname{Max}(C))[t_1^1]m_1^1[t_2^1]m_2^1 \ldots m_{k-1}^1[t_k^1]m_k^1 = m_1'.
\]

This ensures that there exist causally independent events \( e_1, \ldots, e_k \) such that

\[
(C, \rho_1) \xrightarrow{e_1} (C_1, \rho_1^1) \ldots (C_{k-1}, \rho_1^{1-k}) \xrightarrow{e_k} (C_k, \rho_1^1) = (C', \rho_1^1)
\]

with \( \rho_1^1(e_i) = t_i^1 \) and \( \rho_1^1(\operatorname{Max}(C_i)) = m_i^1 \) for \( i = 1, \ldots, k \). Since \( (\rho_1, C, \rho_2) \in R_1 \), for sure we have that

\[
(C, \rho_2) \xrightarrow{e_1} (C_1, \rho_2^1) \ldots (C_{k-1}, \rho_2^{1-k}) \xrightarrow{e_k} (C_k, \rho_2^1) = (C', \rho_2^1)
\]

with \( \rho_2^1(e_i) = t_i^2 \) and \( \rho_2^1(\operatorname{Max}(C_i)) = m_i^2 \) and \( (\rho_1^1, C_i, \rho_2^1) \in R_1 \) for \( i = 1, \ldots, k \) and so also that \( (\rho_1^1, C', \rho_2^1) \in R_1 \). Therefore, also

\[
\rho_2(\operatorname{Max}(C))[t_1^1]m_1^1[t_2^1]m_2^1 \ldots m_{k-1}^1[t_k^1]m_k^1 = m_2'
\]

is derivable. Note that \( G_2 = \{t_2^1, \ldots, t_k^1\} \) is also a step and so \( \rho_2(\operatorname{Max}(C))[G_2]m_2' \) is derivable, with \( (m_1', m_2') \in R_2 \) trivially following from \( (\rho_1^1, C', \rho_2^1) \in R_1 \).

Now, consider the set \( E_G = \{e_1, \ldots, e_k\} \) and observe that \( (\rho_1^{E_G}, C, \rho_2^{E_G}) \), where \( \rho_i^{E_G} \) is the restriction of \( \rho_i \) to the elements in \( \star E_G \), is a triple that must belong to some causal-net bisimulation \( R'_i \) (simply the projection of \( R_i \) on the relevant elements reachable from \( \star E_G \)), hence proving that \( (\star G_1, \star G_2) \in R_2 \) as required.

The third item is symmetric, and so omitted.

Thus, \( R_2 \) is a step net bisimulation and, since \( (\rho_1^0(\operatorname{Max}(C^0)), \rho_2^0(\operatorname{Max}(C^0))) \in R_2 \), we have \( m_1 \sim_s m_2 \).

More intriguing is to prove the reverse implication. In order to help understanding the idea, we present a simple example.
Example 7. Consider the net in Figure 7 where we call by \( t_i \) the transition \((s_i, a, s_i)\) for \( i = 1, \ldots, 4 \). Consider also the relation

\[
R = \{ (s_1 \oplus s_2, s_3 \oplus s_4), (s_1, s_3), (s_1, s_4), (s_2, s_3), (s_2, s_4) \},
\]

which is a step net bisimulation, indeed. Note also that the markings \( s_1 \oplus s_2 \) and \( s_3 \oplus s_4 \) are causal-net bisimilar.

It is interesting to observe that it may happen that matched computations w.r.t. the step net bisimulation game originate different causal nets. E.g., if \( s_1 \oplus s_2(t_1)s_1 \oplus s_2 \) then \( s_3 \oplus s_4 \) can reply also with \( s_3 \oplus s_4(t_4)s_3 \oplus s_4 \). Now, if \( s_1 \oplus s_2(t_2)s_1 \oplus s_2 \), then \( s_3 \oplus s_4 \) can reply also with \( s_3 \oplus s_4(t_4)s_3 \oplus s_4 \). However, note that \( s_1 \oplus s_2 \) has performed a computation where the two transitions \( t_1 \) and \( t_2 \) are causally independent, while \( s_3 \oplus s_4 \) has responded with a computation where the two occurrences of transition \( t_4 \) are causally dependent. However, among the many different matched transitions/computations, there is also the one that generates the same causal net. In fact, to the move \( s_1 \oplus s_2(t_1)s_1 \oplus s_2 \), \( s_3 \oplus s_4 \) can reply, e.g., with \( s_3 \oplus s_4(t_3)s_3 \oplus s_4 \). Then, if \( s_1 \oplus s_2(t_2)s_1 \oplus s_2 \), then \( s_3 \oplus s_4 \) can reply only with \( s_3 \oplus s_4(t_4)s_3 \oplus s_4 \) in order to ensure that the matched computations generate the same causal net. Of course, there is some freedom in choosing the initial mapping from conditions \( b_1 \) and \( b_2 \) of the causal net (with no events) to places of the net; in the matched computations above, we have assumed to map \( b_1 \) to \( s_1 \), \( b_2 \) to \( s_2 \), on one side, as well as \( b_1 \) to \( s_3 \) and \( b_2 \) to \( s_4 \), on the other side; but also other mappings are possible (with a different choice of the matching transitions).

Note that it is not possible to perform a similar initial mapping from conditions to places for the net in Figure 7 since in this case, \( s_1 \oplus s_2 \) is not step net bisimilar to \( s_3 \oplus s_4 \); it is incorrect to map \( s_4 \) to \( b_2 \), because \( s_2 \) can move, while \( s_4 \) cannot. So, in order to ensure that with the initial mapping we map conditions to places in such a way that transitions are preserved, it is necessary that the two initial markings are step net bisimilar, and not simply net bisimilar.

Even more strikingly, consider (on the same figures) the step net bisimulation

\[
R' = \{ (2 \cdot s_1, 2 \cdot s_2), (s_1, s_2) \}.
\]

Note that \( 2 \cdot s_1 \sim cn 2 \cdot s_2 \). If \( 2 \cdot s_1(t_1)2 \cdot s_1 \), then \( 2 \cdot s_2 \) can respond with \( 2 \cdot s_2(t_2)2 \cdot s_2 \). Now, if \( 2 \cdot s_1(t_1)2 \cdot s_1 \), where this new occurrence of transition \( t_1 \) is causally dependent on the previous one (i.e., it consumes the token in \( s_1 \) produced by the previous occurrence of \( t_1 \)), then \( 2 \cdot s_2 \) can respond with \( 2 \cdot s_2(t_2)2 \cdot s_2 \), where \( t_2 \) can be either independent of the previous occurrence of \( t_2 \) or causally dependent on it. In any case, one of the two matching computations, according to the step net bisimulation game, is the causally correct one. And this is enough to be able to construct a causal-net bisimulation proving \( 2 \cdot s_1 \sim cn 2 \cdot s_2 \).
In other words, a step net bisimulation is more nondeterministic than a causal-net bisimulation, but for any computation performed according to the step net bisimulation definition, there is always one matching computation, among many others, in the step net bisimulation game that generates exactly the same causal net. The crucial point in the proof that follows is that we can always choose, by construction, the correct one w.r.t. the definition of causal-net bisimulation. □

**Proposition 5.** For each P/T net \( N = (S, A, T) \), if \( m_1 \sim_s m_2 \), then \( m_1 \sim_{cn} m_2 \).

**Proof.** Let us consider

\[
R \overset{\text{def}}{=} \{ (\rho_1, C, \rho_2) | (C, \rho_1) \text{ is a process of } N(m_1) \text{ and } (C, \rho_2) \text{ is a process of } N(m_2) \text{ and } \rho_1(\text{Max}(C)) \sim_s \rho_2(\text{Max}(C)) \text{ and } \forall B \subseteq \text{Max}(C) \forall G_1, \rho_1(B) = \ast G_1, \exists G_2, \rho_2(B) = \ast G_2 \text{ with } \rho_1(\text{Max}(C))[G_1]m_1', \rho_2(\text{Max}(C))[G_2]m_2', \ast G_1 \sim_s \ast G_2, \text{ and } m_1' \sim_s m_2' \}
\]

We want to prove that \( R \) is a causal-net bisimulation.

First of all, consider a triple of the form \((\rho_1^0, C^0, \rho_2^0)\), where \( C^0 \) is the causal net without events and \( \rho_1^0, \rho_2^0 \) are such that \( \rho_1^0(\text{Min}(C^0)) = \rho_2^0(\text{Max}(C^0)) = m_i \) for \( i = 1, 2 \). Since \( m_1 \sim_s m_2 \), the mappings \( \rho_1^0 \) and \( \rho_2^0 \) can be chosen in such a way that for all \( B \subseteq \text{Max}(C^0) \) \( \forall G_1, \rho_1(B) = \ast G_1, \exists G_2, \rho_2(B) = \ast G_2 \) with \( \rho_1(\text{Max}(C^0))[G_1]m_1' \), \( \rho_2(\text{Max}(C^0))[G_2]m_2' \), \( \ast G_1 \sim_s \ast G_2 \), \( l(G_1) = l(G_2) \) and \( m_1' \sim_s m_2' \) (and, symmetrically, for all \( G_2 \) there exists \( G_1 \) with the required properties). In fact, w.l.o.g., if \( m_1[G_1]m_1' \) with \( \rho_1(B) = \ast G_1 \), then, as \( m_1 \sim_s m_2 \), there exists \( G_2 \) such that \( m_2[G_2]m_2' \), with \( \ast G_1 \sim_s \ast G_2 \) and \( m_1' \sim_s m_2' \). This means that we can choose \( \rho_2(B) = \ast G_2 \). Of course, by Remark 7 this can be scaled down to single transitions: if \( G_1 = \{ t_1^1, \ldots, t_k^1 \} \), then (we may have chosen a suitable) \( G_2 = \{ t_1^2, \ldots, t_k^2 \} \) in such a way that there exists a partition of \( B \) in disjoint subsets \( B_i \), for \( i = 1, \ldots, k \), such that \( \rho_1(B_i) = \ast t_i^1 \sim_s \ast t_i^2 = \rho_2(B_i) \), and the reached markings by firing corresponding transitions are related by step net bisimilarity.

Of course, if the size of \( B \) is larger than 1, we may have some freedom in defining these mappings, that is unavoidable if \( \rho_1^0(B) \) does not enable (or is not involved in the enabling of) any other transition (and so any choice is ok), or can be limited and handled otherwise.

This can be done for all the subsets of \( \text{Max}(C^0) \) and all the initially enabled step of transitions from \( m_1 \) or \( m_2 \). At the end, when all initially enabled transitions have been considered, we get submarkings \( m_1 \subseteq m_1' \) and \( m_2 \subseteq m_2' \) of places not involved in any initial transition, and a set \( B \subseteq \text{Max}(C^0) \), so that \( \rho_1^0(B) = m_1' \) and \( \rho_2^0(B) = m_2' \).

In case the size of \( B \) is larger than 1, we may have some freedom that is not always possible to fix in advance. In fact, in some simple cases, we can make a choice; for instance, if a place \( s_1 \) in \( m_1 \) has empty post-set and a place \( s_2 \) in \( m_2 \) has empty post-set, then it is possible to choose \( \rho_1^0(b) = s_1 \) and \( \rho_2^0(b) = s_2 \) for some \( b \in B \). However,
if $s_1$ in $m_1$ has nonempty post-set, then it will be possible to assign a corresponding place $s_2$ in $m_2$ with nonempty post-set only when it will be necessary for enabling a future transition; and, in case no future transition will be ever enabled by $s_1$, then the association can be taken freely. Hence, the choice of $\rho_1^0$ and $\rho_2^0$ can be eventually taken in such a way that for all $B \subseteq \text{Max}(C^0)$ such that $\exists G, \rho_1^0(B) = \ast G$ for $i = 1$ or $i = 2$, the step net bisimulation game holds, i.e., for all $G_1$ such that $\ast G_1 = \rho_1^0(B)$, there exists $G_2$ such that $\ast G_1 = \rho_1^0(B) \sim \ast G_2$ and the reached markings are step net bisimilar (and vice versa). Therefore, $(\rho_1^0, C^0, \rho_2^0)$ must belong to $R$, because $(C^0, \rho_1^0)$ is a process of $N(m_1)$, for $i = 1, 2$ and, by hypothesis, $m_1 \sim_{m_2}$ $m_2$. Hence, if $R$ is a causal-net bisimulation, then the triple $(\rho_1^0, C^0, \rho_2^0) \in R$ ensures that $m_1 \sim_{\text{cn}} m_2$.

Now, assume $(\rho_1, C, \rho_2) \in R$. In order for $R$ to be a causal-net bisimulation, we must prove that

1. $\forall i, C, \rho_i^1$ such that $(C, \rho_1) \overset{e}{\rightarrow} (C', \rho_i^1)$, where $\rho_1(e) = t_1$, there exist $t_2, \rho_i^2$ such that $(C, \rho_2) \overset{e}{\rightarrow} (C', \rho_i^2)$, where $\rho_2(e) = t_2$, and $(\rho_i^1, C, \rho_i^2) \in R$;
2. symmetrically, if $(C, \rho_2)$ moves first.

Assume $(C, \rho_1) \overset{e}{\rightarrow} (C', \rho_i^1)$, where $\rho_1(e) = t_1$. Recall that if $(\rho_1, C, \rho_2) \in R$, then $\rho_1(\text{Max}(C)) \sim \rho_2(\text{Max}(C))$ and, by taking $B = \ast e$, we know that there exists $t_2$ with the property that, for the move $\rho_1(\text{Max}(C))[t_1]m_1'$, with $m_1' = \rho_1'([\text{Max}(C)])$, we have $\rho_2(\text{Max}(C))[t_1]m_2'$, with $l(t_1) = l(t_2)$, $t_2 = \rho_2(\ast e)$, $t_1 \sim t_2$ and $m_1 \sim_{m_2} m_2'$. This means that $t_2$ has exactly the same shape of $t_1$ and so it is possible to derive $(C, \rho_2) \overset{e}{\rightarrow} (C', \rho_i^2)$, with $\rho_2(e) = t_2$ and $m_2' = \rho_2'([\text{Max}(C)])$, so that $\rho_1'(\text{Max}(C')) \sim \rho_2'(\text{Max}(C'))$, as required. The mappings $\rho_1'$ and $\rho_2'$ on $e^*$ are defined in such a way that for all $B$ such that $B \cap e^* \neq \emptyset$ and $\exists G, \rho_1'(\ast G) = B$ for $i = 1$ or $i = 2$, the step net bisimulation game holds, i.e., for all $G_1$ such that $\ast G_1 = \rho_1'(B)$ and $\rho_1'(\text{Max}(C'))[G_1]m_1'$, there exists $G_2$ such that $\rho_2'(\text{Max}(C'))[G_2]m_2'$, $\ast G_1 = \rho_1'(B) \sim \ast G_2$, $l(G_1) = l(G_2)$ and $m_1' \sim_{m_2'} m_2'$ (which is really possible as explained for the base case case of $\rho_1^0$ and $\rho_2^0$), and, symmetrically, for all $G_2$ such that $\rho_2'(\text{Max}(C'))[G_2]m_2'$, there exists $G_1$ with the required properties.

It remains to note that $\forall B \subseteq \text{Max}(C') \forall G_1, \rho_1'(B) = \ast G_1$, $\exists G_2, \rho_2'(B) = \ast G_2$ such that $\rho_1'(\text{Max}(C'))[G_1]m_1', \rho_2'(\text{Max}(C'))[G_2]m_2'$, $\ast G_1 \sim \ast G_2$, $l(G_1) = l(G_2)$, $m_1' \sim_{m_2'} m_2'$ (and also the symmetric condition). Of course, this is true, by induction, for all $B \subseteq \text{Max}(C) \cap \text{Max}(C')$ such that $\exists G, \rho_1'(B) = \ast G$ for $i = 1$ or $i = 2$, because in this case, as $\rho_1'(B) = \rho_1(B)$, the two matching steps are such that $\ast G_1 = \rho_1(B) \sim \rho_2(B) = \ast G_2$, and so also $\rho_1'(B) \sim \rho_2'(B)$. The other case to consider is when a step $G$ is enabled, w.l.o.g., by $\rho_1'(B)$, for $B \subseteq e^*$ or for $B \cap (\text{Max}(C) \cap \text{Max}(C')) \neq \emptyset$ and $B \cap e^* \neq \emptyset$. However, by construction (as discussed above), we have chosen mappings $\rho_1'$ and $\rho_2'$ on $e^*$ in such a way that this condition is satisfied. Hence, as a consequence, we have $(\rho_i^1, C', \rho_i^2) \in R$.

The case when $(C, \rho_1)$ moves first is symmetrical and therefore omitted. Thus, $R$ is a causal-net bisimulation and so, since $(\rho_1^0, C^0, \rho_2^0) \in R$, we have $m_1 \sim_{\text{cn}} m_2$.

Theorem 1. (Step net bisimilarity and causal-net bisimilarity coincide) For each $P/T$ net $N = (S, A, T)$, $m_1 \sim_{\text{m}_2}$ $m_2$ if and only if $m_1 \sim_{\text{cn}} m_2$.

Proof. Direct consequence of Propositions 4 and 5.

$\square$
5 Net Modal Logic

In this section we propose a new modal logic, called NML (acronym of Net Modal Logic). We will prove that model checking is coherent with equivalence checking: two markings are step net bisimilar if and only if they satisfy the same NML formulae.

The NML formulae are generated from the finite set $A$ of actions by the following abstract syntax:

$$
F ::= \text{nn} \mid \text{vv} \mid t \mid f \mid F \land F \mid \neg F \mid F \otimes F \mid H.F
$$

$$
H ::= \langle a \rangle^h_k \mid m \otimes H \mid H \otimes m \mid H \otimes H
$$

where $a$ is any action in $A$, $\text{nn}$ and $\text{vv}$ denote two atomic propositions (which are not $t$, for $true$, and $f$, for $false$), $\land$ is the operator of logical conjunction, $\neg$ is logical negation and $\otimes$ is the operator of parallel composition of formulae. A special formula is $H.F$, where $H$ is a formula composed of instances of a generalization of the diamond operator of Hennessy-Milner Logic \cite{21} – parametrized on two natural numbers, $h$ and $k$ – possibly composed in parallel with a number of $m$ subformulae. The intuitive meaning of the formula $\langle a \rangle^h_k.F$ is that, for the current marking of size $h$, it is possible to do a transition labeled $a$, with pre-set of size $h$ and post-set of size $k$, and then to reach a marking where $F$ holds. In general, a formula $H.F$ is satisfied by the current marking if a suitable step $G$ is executable, so that the reached marking satisfies $F$.

We denote by $\mathcal{F}_A$ the set of all NML formulae, built from the finite set $A$ of actions, and with post-set of size $h$ and pre-set of size $k$. The semantics of any formula $F$ is the set of markings that satisfy it; hence, the semantic function is parametrized with respect to some given P/T net $N = (S, A, T)$. Let $[\cdot] : \mathcal{F}_A \rightarrow \mathcal{P}(\mathcal{M}(S))$ be the denotational semantics function, defined in Table \cite{11}

**Definition 15.** (NML satisfaction relation) We say that a marking $m$ satisfies a formula $F$, written $m \models F$, if $m \in [F]$.

The semantics of any formula $F$ is a set of markings. The semantics of $\text{nn}$ is $S$: all (and only) the places satisfy $\text{nn}$. The semantics of $\text{vv}$ is $\{\emptyset\}$: only the empty marking satisfies $\text{vv}$. The semantics of $tt$ is $\mathcal{M}(S)$: every marking satisfies $tt$. The semantics of $ff$ is $\emptyset$: no marking satisfies $ff$.

The logical operator of conjunction $\land$ is interpreted as intersection $\cap$ of the set of markings satisfied by the two subformulae. The semantics of $\neg F$ is the set of all the markings that do not satisfy $F$, i.e., the complement of $[F]$. The semantics of $F_1 \otimes F_2$ is the set of markings of the form $m_1 \otimes m_2$ such that $m_1 \models F_1$ and $m_2 \models F_2$.

The semantics of $\langle a \rangle^h_k$ is the set of all the pre-sets of transitions of size $h$, labeled $a$ and with post-set of size $k$. The semantics of $m \otimes G$ is the set of all the markings of the form $s \otimes m$ such that $s \in S$ and $m \in [G]$; symmetrically for the semantics of $G \otimes m$. The semantics of $G_1 \otimes G_2$ is the set of markings $m_1 \otimes m_2$ such that $m_i \in [G_i]$, for $i = 1, 2$.

The semantics of $H.F$ is the set of markings $m$ that satisfy $H$, such that there exists a reachable marking $m'$ by “performing” $H$ (this is the intuitive meaning of $\text{End}(H, m)$) such that $m'$ satisfies $F$. 

\[ [mn] = S \quad [vv] = \{\theta\} \quad [tt] = \mathcal{M}(S) \quad [ff] = \emptyset \]

\[ [F_{1} \land F_{2}] = [F_{1}] \land [F_{2}] \quad [\neg F] = \mathcal{M}(S) \setminus [F] \]

\[ [F_{1} \otimes F_{2}] = \{m_{1} \oplus m_{2} \mid m_{1} \in [F_{1}], m_{2} \in [F_{2}]\} \]

\[ [(a)_{k}^{h}] = \{m \in \mathcal{M}(S) \mid \exists T. l(t) = a, t = m, \forall i \in m, |r^{i}| = k\} \]

\[ [nn \otimes H] = \{s \oplus m \mid s \in S, m \in [H]\} \]

\[ [H \otimes nn] = \{m \oplus s \mid s \in S, m \in [H]\} \]

\[ [H_{1} \otimes H_{2}] = \{m_{1} \oplus m_{2} \mid m_{1} \in [H_{1}], m_{2} \in [H_{2}]\} \]

\[ [H:F] = \{m \in \mathcal{M}(S) \mid m \in [H], \forall m' \in \text{End}(H,m). m' \in [F]\} \]

\[ \text{End}(1)_{k}^{h}; m = \{m' \mid \exists t \in T. t = m, |m| = h, l(t) = a, t^{*} = m', |m'| = k\} \]

\[ \text{End}(nn \otimes H, s \oplus m) = \{s \oplus m' \mid m' \in \text{End}(H,m)\} \]

\[ \text{End}(H \otimes nn, m \oplus s) = \{m' \oplus s \mid m' \in \text{End}(H,m)\} \]

\[ \text{End}(H_{1} \otimes H_{2}, m_{1} \oplus m_{2}) = \{m'_{1} \oplus m'_{2} \mid m_{i} \in [H_{i}], m'_{i} \in \text{End}(H_{i}, m_{i}), i = 1,2\} \]

**Table 1. Denotational semantics**

We are now ready to prove the coherence theorem: two markings are step net bisimilar if and only if they satisfy the same NML formulae. We prove the two implications separately. In the proof of the first implication it is necessary the following interesting property of step net bisimilarity \(\sim_{s}\).

**Lemma 1. (Decomposition)** Let \(N = (S, A, T)\) be a P/T net. Assume \(m_{1} \sim_{s} m_{2}\). Let \(m'_{1}\) and \(m''_{1}\) be such that \(m_{1} = m'_{1} \oplus m''_{1}\). Then there exist \(m'_{2}\) and \(m''_{2}\) such that \(m_{2} = m'_{2} \oplus m''_{2}\), \(m'_{1} \sim_{s} m'_{2}\) and \(m''_{1} \sim_{s} m''_{2}\).

**Proof.** If \(m_{1}\) is split into two separate submarkings \(m'_{1}\) and \(m''_{1}\), then, since \(m_{1} \sim_{s} m_{2}\), we can split \(m_{2}\) in such a way that the corresponding transitions in the step net bisimulation game are matched. In fact, if \(m'_{1} \mid G_{1} \mid m'_{1}\), then also \(m_{1} \mid G_{1} \mid m_{1}\) and, as \(m_{1} \sim_{s} m_{2}\), there exists \(G_{2}\) such that \(m_{2} \mid G_{2} \mid m_{2}\), with \(m'_{1} \sim_{s} G_{2}\) (and \(m''_{1} \sim_{s} m_{2}\)) and so we put in \(m'_{2}\) the tokens in \(G_{2}\). This can be done for all the steps of transitions enabled by \(m'_{1}\) (or \(m''_{1}\)) also in case some transition in \(G_{1}\) is self-concurrent. This can be done for all the initially enabled steps of transitions from \(m_{1}\) or \(m_{2}\). At the end, when all initially enabled transitions have been considered, what remains is a set of places/tokens not involved in any initial transition. However, by looking at the step net bisimulation game, we can make a choice; for instance, if a place \(s_{1}\) in \(m'_{1}\) has empty post-set and a place \(s_{2}\) in \(m_{2}\) has empty post-set, then it is possible to add \(s_{2}\) to \(m'_{2}\). Or if \(s_{1}\) in \(m'_{1}\) has nonempty post-set, then it will be possible to assign a corresponding place \(s_{3}\) in \(m'_{2}\) with nonempty post-set only when it will be necessary for enabling a future transition; and, in case no future transition will be ever enabled by using \(s_{1}\), then the association can be taken freely.

\[\square\]
Proposition 6. Let $N = (S, A, T)$ be a P/T net. If $m_1 \sim s m_2$, then $m_1$ and $m_2$ satisfy the same NML formulae, i.e., $\{F_1 \in \mathcal{F}_A \mid m_1 \models F_1\} = \{F_2 \in \mathcal{F}_A \mid m_2 \models F_2\}$.

Proof. Let us assume that $m_1 \sim s m_2$. We will prove that, for any $F \in \mathcal{F}_A$, if $m_1 \models F$ then also $m_2 \models F$. This is enough because, by a symmetric argument, we can also prove that if $m_2 \models F$ then also $m_1 \models F$, and so $m_1$ and $m_2$ satisfy the same NML formulae.

The proof is by induction on the structure of $F$, where the first four cases are the base cases of the induction.

- $F = nm$: if $m_1 \models nm$, then $m_1$ is a place; since two step net bisimilar markings have the same size, also $m_2$ is a single place and so also $m_2 \models nm$.
- $F = vv$: if $m_1 \models vv$, then $m_1 = \theta$: since two step net bisimilar markings have the same size, also $m_2 = \theta$ and so $m_2 \models vv$.
- $F = tt$: since all the markings satisfy true, also $m_2 \models tt$.
- $F = ff$: since no marking satisfies false, $m_1 \not\models ff$ and also $m_2 \not\models ff$.
- $F = F_1 \land F_2$: since $m_1 \models F_1$ and $m_1 \models F_2$, it follows that $m_1 \models F_1 \land F_2$; by induction, we can assume that also $m_2 \models F_1$ and $m_2 \models F_2$; hence, also $m_2 \models F_1 \land F_2$, as required.
- $F = \neg F'$: since $m_1 \models \neg F'$, it follows that $m_1 \not\models F'$. By induction, as $F'$ is a subformula, if $m_1$ does not satisfy $F'$, then also $m_2$ does not satisfy $F'$, and so $m_2 \models \neg F'$, as required.
- $F = F_1 \otimes F_2$: if $m_1 \models F_1 \otimes F_2$ only if there exist $m_1'$ and $m_2'$ such that $m_1 = m_1' \oplus m_1'$, $m_1' \models F_1$ and $m_1'' \models F_2$. As $m_1 \sim s m_2$, by Lemma 1, there exist $m_2'$ and $m_2''$ such that $m_2 = m_2' \oplus m_2''$ and $m_2' \sim s m_2''$. By induction, $m_2' \models F_1$ and $m_2'' \models F_2$; therefore, also $m_2 \models F_1 \otimes F_2$, as required.
- $F = H.F'$: by additional induction on the shape of $H$.

If $H = \langle a \rangle_k^l$, then, since $m_1 \models \langle a \rangle_k^l.F'$, we have that $m_1 \in [\langle a \rangle_k^l]$ and that there exist a transition $t_1$ and a marking $m_1'$ such that $m_1[t_1]m_1', l(t_1) = a, \bullet t_1 = m_1, |m_1| = k$, $t_1 = m_1', l(m_1') = l \bullet t_1$, i.e., $m_1' \in \text{End}(\langle a \rangle_k^l, m_1)$ and, finally, $m_1 \models [F']$. As $m_1 \sim s m_2$, there exist a transition $t_2$ and a marking $m_2'$ such that $m_2[t_2]m_2', l(t_2) = a, \bullet t_2 = m_2$ and $t_2 = m_2'$ (i.e., $m_2' \in \text{End}(\langle a \rangle_k^l, m_2)$) and, finally, $m_1 \sim s m_2$, so that $m_1' \models [F']$ by induction. Summing up, $m_2 \models \langle a \rangle_k^l.F'$ as required.

If $H = mn \otimes H'$, then we assume that $m_1 \models H.F'$. This means that $m_1 \in [H]$ and so there exist a place $s$ and a marking $m_1$ such that $m_1 = s \oplus m_1$ and $m_1 \in [H]$. This means that there exist a step $G_1$ and a marking $m_1'$ such that $m_1'[G_1]m_1$, where $m_1' \in \text{End}(H', m_1)$. Therefore, also $m_1[G_1]m_1'$, where $m_1' = s \oplus m_1'$, such that $m_1' \in \text{End}(mn \otimes H', m_1)$ and $m_1' \in [F']$. Since $m_1 \sim s m_2$, there exists a step $G_2$ and a marking $m_2'$ such that $m_2[G_2]m_2', l(G_1) = l(G_2), *G_1 \sim s G_2$ and $m_2 \sim s m_2'$. This means that there exist a place $s'$ and a marking $m_2$ such that $m_2 = s' \oplus m_2$ and $m_2 \in [H']$. Therefore, also $m_2'[G_2]m_2'$, where $m_2' \in \text{End}(H', m_2)$, and $m_2' \in [mn \otimes H', m_2')$, where $m_2' = s' \oplus m_2$. As $m_1 \sim s m_1'$, and $m_1' \in [F']$, by induction it follows that $m_2' \in [F']$. Summing up, $m_2 \models H.F'$ as required.

The case when $H = H' \otimes mn$ is symmetric and so omitted.

If $H = H_1 \otimes H_2$, then, since $m_1 \models H.F'$, there exists $m_1'$ and $m_1''$ such that $m_1 = m_1' \oplus m_1''$, $m_1' \models H_1$ and $m_1'' \models H_2$: this means that there exist two steps $G_1$ and $G_2$ such that $m_1'[G_1]m_1'$ and $m_1''[G_2]m_1''$, with $m_1' \in \text{End}(H_1, m_1')$, $m_1'' \in \text{End}(H_2, m_1'')$ and, moreover, $m_1'[G_1]m_1' = m_1' \in [F']$. Since $m_1 \sim s m_2$, by Lemma 1 there exists $m_1'$ and $m_1''$ such that $m_2 = m_2' \oplus m_2''$, $m_2' \sim s m_2''$ and $m_1'' \sim s m_2''$. By induction, $m_2' \models H_1$. 


and \(m''_1 \models H_2\); moreover, these conditions ensure that there exist two steps \(G_1^2\) and \(G_2^2\) such that \(m'_1[G_1^2]\bar{m}_1\) and \(m'_2')[G_2^2]\bar{m}_2'\), with \(\bar{m}_2' \in \text{End}(H_1, m'_2)\), \(\bar{m}_1' \in \text{End}(H_2, m''_1)\) and, moreover, \(\bar{m}_1' \sqcup \bar{m}_1 = \bar{m}_1 \sim \bar{m}_2 = \bar{m}_2' \oplus \bar{m}_1'\). By induction, since \(\bar{m}_1 \sim \bar{m}_2\) and \(\bar{m}_1' \in [F']\), it follows that \(\bar{m}_2' \in [F']\), so that \(m_2' \models H.F'\).

As no other cases are possible, the proof is complete. \(\Box\)

**Proposition 7.** Let \(N = (S, A, T)\) be a P/T net. If \(m_1\) and \(m_2\) satisfy the same NML formulae, i.e., \(\{F_1 \in \mathcal{F}_A \mid m_1 \models F_1\} = \{F_2 \in \mathcal{F}_A \mid m_2 \models F_2\}\), then \(m_1 \sim_\Delta m_2\).

**Proof.** We want to prove that \(\Delta = \{\langle m_1, m_2 \rangle \mid \text{m_1 and m_2 satisfy the same NML formulae}\}\) is a step bisimulation, hence proving that two markings that satisfy the same formulae are step net bisimilar.

First of all, note that if \((m_1, m_2) \in \Delta\), then \(|m_1| = |m_2|\). In fact, if this is not the case, it is very easy to find a formula \(F\) that is satisfied by \(m_1\) but not by \(m_2\) (or vice versa). Assume, w.l.o.g., that \(|m_1| = d > |m_2|\). Then, \(F = m^d\), where \(m^d = mn \otimes \ldots \otimes mn\), is the distinguishing formula.

Now, assume \((m_1, m_2) \in \Delta\) and \(m_1[G_1]m'_1\). We will prove that there exist a step \(G_2\) and a marking \(m'_2\) such that \(m_2[G_2]m'_2\) with \(l(G_1) = l(G_2)\), \((G_1, G_2) \in \Delta\) and \((m'_1, m'_2) \in \Delta\). Since \(\Delta\) is symmetric, this is enough for proving that \(\Delta\) is a step net bisimulation.

Assume, towards a contradiction, that there exist no \(G_2\) or \(m'_2\) satisfying the step net bisimulation conditions. This means that one of the following conditions must hold:

1. either there is no step \(G_2\) labeled \(l(G_1)\) enabled by \(m_2\);
2. or each step \(G_2\) (labeled \(l(G_1)\) and enabled by \(m_2\)) has some transition with pre-set or post-set of size different from those of \(G_1\);
3. or each step \(G_2\) (labeled \(l(G_1)\), enabled by \(m_2\) and with coherent size of pre-sets and post-sets of the transitions) is such that \((G_1, G_2) \notin \Delta\);
4. or for each markings \(m'_2\) reachable by performing some step \(G_2\) (labeled \(l(G_1)\), enabled by \(m_2\) such that \((G_1, G_2) \in \Delta\)), it happens that \((m'_1, m'_2) \notin \Delta\).

We want to prove that in each case we are able to construct a NML formula that is satisfied by \(m_1\) but not by \(m_2\) (or vice versa), hence contradicting the assumption that \((m_1, m_2) \in \Delta\). This is enough to conclude that \(\Delta\) is a step net bisimulation.

Case 1 and 2 can be treated easily. W.l.o.g., assume \(G_1 = \{t_1, \ldots, t_k\}\) with \(l(t_i) = a_i\), \(\|t_i\| = h_i\), \(\|t_i\| = k_i\) for \(i = 1, \ldots, l\). Then, if \(d\) is the size of \(m_1\) and \(m_2\), and \(h = \sum_{i=1}^{l} h_i\), the distinguishing formula is

\[
F = \langle a_1^{h_1} \otimes \ldots \otimes a_l^{h_l} \otimes m^d-h, t \rangle,
\]

which is satisfied by \(m_1\) but not by \(m_2\).

For case 3, assume that \(\{G_1^2, \ldots, G_j^2\}\) is the finite set of all the steps that \(m_2\) can perform, labeled \(l(G_1)\) and with coherent size of pre-sets and post-sets for all the transitions. Since \((G_1^2, G_2^2) \notin \Delta\), we know that there exists a formula \(F_i\) such that \(G_1 \models F_i\) and \(G_2 \not\models F_i\) for \(i = 1, \ldots, j\). Hence, the distinguishing formula is
\[ F = ((\langle a_1 \rangle_{h_1} \otimes \ldots \otimes \langle a_l \rangle_{h_l}) tt \land F_1 \land \ldots \land F_j) \otimes nn^{d-h}, \]
which is satisfied by \( m_1 \) but not by \( m_2 \).

Finally, for case 4, assume that \( \{ m_1^1, \ldots, m_j^2 \} \) is the finite set of all the markings that \( m_2 \) can reach by performing some \( G_2 \) satisfying all the other conditions. Since \( (m_1^1, m_2^j) \not\in R \), we know that there exists a formula \( F_i \) such that \( m_1^i \models F_i \) and \( m_2^j \not\models F_i \) for \( i = 1, \ldots, j \). Hence, the distinguishing formula is
\[ F = ((\langle a_1 \rangle_{h_1} \otimes \ldots \otimes \langle a_l \rangle_{h_l}) \otimes nn^{d-h}),(F_1 \land \ldots \land F_j), \]
which is satisfied by \( m_1 \) but not by \( m_2 \).

\[ \square \]

**Theorem 2. (Coherence)** Let \( N = (S,A,T) \) be a P/T net. It holds that \( m_1 \sim_s m_2 \) if and only if \( \{ F_1 \in \mathcal{F}_A \mid m_1 \models F_1 \} = \{ F_2 \in \mathcal{F}_A \mid m_2 \models F_2 \} \).

**Proof.** Direct consequence of Propositions 6 and 7. \( \square \)

## 6 Conclusion

Step net bisimulation is a very simple relation, extending ordinary step bisimulation on Petri nets with two additional conditions: the related markings must have the same size and the pre-sets of the two matching steps (i.e., multisets of concurrently enabled transitions) must be related as well. With these minimal additions, its definition is still very manageable and also very sensible as we proved to coincide with causal-net bisimilarity \[13,19\] (or, equivalently, structure-preserving bisimilarity \[13\]), a truly concurrent behavioral equivalence fully respecting causality and the branching structure of systems.

It is sometimes argued that a truly concurrent semantics should abstract from the observation of the distributed state of a system, as fully-concurrent bisimilarity \[4\] (or, equivalently, history-preserving bisimilarity \[29,11,8\]) does. However, we think that causal-net bisimilarity (or step net bisimilarity, or structure-preserving bisimilarity) is more accurate than fully-concurrent bisimilarity \[4\] because it is resource-aware. In fact, in the implementation of a system, a token is an instance of a sequential process, so that a processor is needed to execute it. If two markings have different size, then a different number of processors is necessary. Hence, a semantics such as causal-net bisimilarity, which relates markings of the same size only, is more accurate as it equates distributed systems only if they require the same amount of execution resources. Moreover, \[17\] offers, in the area of information flow security, further arguments in favor of this more concrete equivalence, that observes the size of the distributed state.

Van Glabbeek \[13\] argued that structure-preserving bisimilarity is the most appropriate behavioral equivalence for Petri nets, as it is the only one respecting a list of desirable requirements he proposed. Among these, there is the observation that it is the coarsest semantics respecting inevitability \[24\], i.e., if two systems are equivalent, and in one the occurrence of a certain action is inevitable, then so is it in the other one. One further criterion in \[13\] is ‘compositionality’, up to structure-preserving bisimilarity, of the operators of the process algebra CCSP, that Olderog proposed in his monograph \[26\] and equipped with a safe net semantics. In the companion paper \[20\] we have extended his result by proving that step net bisimilarity can be used to give a compositional semantics of the process algebra FNM \[15\], which truly represents all (and only) the finite
P/T nets, up to isomorphism. In this way, we have obtained for the first time a compositional semantics, fully respecting causality and the branching structure of systems, for the class of all the finite Petri nets.

It is possible to prove that step net bisimilarity $\sim_s$ coincides with team bisimilarity over BPP nets [16], which in turn coincides with place bisimilarity [1] on this class of nets. Interestingly enough, the definition of step net bisimulation has some similarities with that of place bisimulation, as also this relation requires that the size of the related markings is the same and that the pre-sets of the matching transitions be related. However, on the one hand, on general Petri nets place bisimilarity is strictly finer than causal-net bisimilarity, as claimed in [13] and proved in [18]. On the other hand, place bisimilarity is decidable [18], while the decidability of causal-net bisimilarity (and so of step net bisimilarity) is an open problem. Step net bisimilarity is decidable on bounded nets, because causal-net bisimilarity has been proved decidable in (large) exponential time on that class of nets [17]. However, we expect that our simpler characterization of $\sim_{cn}$ in term of step net bisimilarity $\sim_s$ may offer some hints towards a more efficient algorithm, yet exponential as well. Step net bisimilarity is decidable on BPP nets, because on this class of nets it coincides with team bisimilarity [16,19], which is decidable in polynomial time.

The modal logic NML is an extension of TML [16], a modal logic characterizing team bisimilarity over BPP nets, which for the first time featured the atomic propositions $vv$ and $nn$ needed to count the size of a marking. The main difference is the introduction of the modality $\langle a \rangle_h^k$, which generalizes the modality $\langle a \rangle$ with the parameters $h$ and $k$, expressing the size of the pre-set and of the post-set, respectively, of the net transition of interest. In turn, TML is an extension of Hennessy-Milner Logic [21] with an operator $\otimes$ of parallel composition of formulae, in the style of Caires’ and Cardelli’s spatial logic [6]. Despite its apparent complexity, NML is rather simple and intuitive, as its operators have a natural operational interpretation on the net model. Many other truly concurrent modal logics have been proposed for other truly concurrent behavioral equivalences (see, e.g., [5,2] and the references therein).

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