FILIFORM LIE ALGEBRAS OF ORDER 3

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Abstract. The aim of this work is to generalize a very important type of Lie algebras and superalgebras, i.e. filiform Lie (super)algebras, into the theory of Lie algebras of order $F$. Thus, the concept of filiform Lie algebras of order $F$ is obtained. In particular, for $F = 3$ it has been proved that in order to obtain all the class of filiform elementary Lie algebras of order 3 it is only necessary determine the infinitesimal deformations of the associated model elementary Lie algebra, analogously as that occurs into the theory of Lie algebras (Verge, 1970). Also we give the dimension, using an adaptation of the $\mathfrak{sl}(2,\mathbb{C})$-module Method, and a basis of such infinitesimal deformations in some generic cases.

1. Introduction

A classical use of Lie algebras (identification and classification) is in the study of symmetries in physics. Nowadays such symmetries are not limited to the geometrical ones of space-time, arising thus the concept of supersymmetry and consequently Lie superalgebra. Among others, the possible generalizations of Lie superalgebras that have been proven to be physically relevant are color Lie superalgebras ([13], [14], [15], [18]) and Lie algebras of order $F$ ([20], [21], [2]).

In this paper we shall consider Lie algebras of order $F$, that constitutes the underlying algebraic structure associated to fractional supersymmetry ([3], [4], [10], [17]), (note that a different point of view can be seen in [9]). Thus, Lie algebras of order 3 (or more generally Lie algebras of order $F$) were introduced as a possible generalisation of Lie superalgebras, in order to implement non-trivial extensions of the Poincaré symmetry which are different than the usual supersymmetric extension. A Lie algebra of order $F$ admits a $\mathbb{Z}_F$-grading, the zero-graded part being a Lie algebra. An $F$-fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part. This new structure was subsequently applied within the framework of the Poincaré algebra, and a Quantum Field Theory with a non-trivial extension, different from supersymmetry.

In particular, we will focus our study in generalize a very important type of nilpotent Lie superalgebra, i.e. filiform Lie superalgebras, obtaining the notion of filiform Lie algebras of order $F$.

The concept of filiform Lie algebras was firstly introduced in [22] by Vergne. This type of nilpotent Lie algebra has important properties; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra.

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In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$ \[1\], \[6\] and \[12\].

In this paper we generalize this concept obtaining filiform Lie algebras of order $F$ and the model filiform Lie algebra of order 3. We have proved that in order to obtain all the class of filiform elementary Lie algebras of order 3 it is only necessary to determine the infinitesimal deformations of the model elementary algebra (see Theorem 2).

We have given too the dimension, using an adaptation of the $sl(2,\mathbb{C})$-module Method, and a basis of such infinitesimal deformations in some generic cases (see Theorems 3 and 4). Also, we have given some properties of the algebraic variety of elementary Lie algebras of order 3.

We do assume that the reader is familiar with the standard theory of Lie algebras. All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{C}$-vector spaces with finite dimension.

2. Preliminaries

The vector space $V$ is said to be $\mathbb{Z}_F$-graded if it admits a decomposition in direct sum, $V = V_0 \oplus V_1 \oplus \cdots V_{F-1}$, with $F \in \mathbb{N}^*$. An element $X$ of $V$ is called homogeneous of degree $\gamma$ ($deg(X) = d(X) = \gamma$), $\gamma \in \mathbb{Z}_F$, if it is an element of $V_\gamma$.

Let $V = V_0 \oplus V_1 \oplus \cdots V_{F-1}$ and $W = W_0 \oplus W_1 \oplus \cdots W_{F-1}$ be two graded vector spaces. A linear mapping $f : V \rightarrow W$ is said to be homogeneous of degree $\gamma$ ($deg(f) = d(f) = \gamma$), $\gamma \in \mathbb{Z}_F$, if $f(V_\alpha) \subset W_{\alpha+\gamma(mod\ F)}$ for all $\alpha \in \mathbb{Z}_F$. The mapping $f$ is called a homomorphism of the $\mathbb{Z}_F$-graded vector space $V$ into the $\mathbb{Z}_F$-graded vector space $W$ if $f$ is homogeneous of degree 0. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_F$-graded vector spaces.

A superalgebra $g$ is just a $\mathbb{Z}_2$-graded algebra $g = g_0 \oplus g_1$, \[10\] and \[19\]. That is, if we denote by $[\ , \ ]$ the bracket product of $g$, we have $[g_\alpha, g_\beta] \subset g_{\alpha+\beta(mod2)}$ for all $\alpha, \beta \in \mathbb{Z}_2$.

Definition 2.1. Let $g = g_0 \oplus g_1$ be a superalgebra whose multiplication is denoted by the bracket product $[\ , \ ]$. We call $g$ a Lie superalgebra if the multiplication satisfies the following identities:

1. $[X,Y] = -(-1)^{\alpha\beta}[Y,X]$ for all $X \in g_\alpha, Y \in g_\beta$.
2. $(-1)^{\gamma\alpha}[X,Y,Z] + (-1)^{\alpha\beta}[Y,[Z,X]] + (-1)^{\beta\gamma}[Z,[X,Y]] = 0$

for all $X \in g_\alpha, Y \in g_\beta, Z \in g_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

Identity 2 is called the graded Jacobi identity and it will be denoted by $J_g(X,Y,Z)$.

We observe that if $g = g_0 \oplus g_1$ is a Lie superalgebra, we have that $g_0$ is a Lie algebra and $g_1$ has structure of $g_0$-module.

Next we recall the definition and some basic properties of Lie algebras of order $F$ introduced in \[20\], \[21\] and \[7\].
Definition 2.2. [3] Let $F \in \mathbb{N}^*$. A $\mathbb{Z}_F$-graded $\mathbb{C}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex Lie algebra of order $F$ if the following hold:

1. $\mathfrak{g}_0$ is a complex Lie algebra.
2. For all $i = 1, \ldots, F - 1$, $\mathfrak{g}_i$ is a representation of $\mathfrak{g}_0$. If $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_i$, then $[X, Y]$ denotes the action of $X \in \mathfrak{g}_0$ on $Y \in \mathfrak{g}_i$ for all $i = 1, \ldots, F - 1$.
3. For all $i = 1, \ldots, F - 1$, there exists an $F$-Linear, $\mathfrak{g}_0$-equivariant map, $\{\cdots\} : S^F(\mathfrak{g}_i) \rightarrow \mathfrak{g}_0$, where $S^F(\mathfrak{g}_i)$ denotes the $F$-fold symmetric product of $\mathfrak{g}_i$.
4. For all $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_k$, the following “Jacobi identities” hold:

\begin{align*}
(2.2.1) & \quad [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0. \\
(2.2.2) & \quad [[X_1, X_2], Y_3] + [[X_2, Y_3], X_1] + [[Y_3, X_1], X_2] = 0. \\
(2.2.3) & \quad [X, \{Y_1, \ldots, Y_F\}] = \{[X, Y_1], \ldots, Y_F\} + \cdots + \{Y_1, \ldots, [X, Y_F]\}. \\
(2.2.4) & \quad \sum_{j=1}^{F+1} [Y_j, \{Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{F+1}\}] = 0.
\end{align*}

Remark 2.3. It can be seen that a Lie algebra of order 1 ($F = 1$) is a Lie algebra and a Lie algebra of order 2 ($F = 2$) is a Lie superalgebra. Therefore, Lie algebras of order $F$ appear as some kind of generalizations of Lie algebras and superalgebras.

Proposition 2.4. [3] Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order $F$, with $F > 1$. For any $i = 1, \ldots, F - 1$, the $\mathbb{Z}_2$-graded vector spaces $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ inherits the structure of a Lie algebra of order $F$. We call these type of algebras elementary Lie algebras of order $F$.

We will restrict our study to elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Examples of elementary Lie algebras of order 3 can be seen in [7].

Definition 2.5. A representation of an elementary Lie algebra of order $F$ is a linear map $\rho : \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \rightarrow End(V)$, such that for all $X_i \in \mathfrak{g}_0$, $Y_j \in \mathfrak{g}_1$,

\begin{align*}
\rho([X_1, X_2]) &= \rho(X_1)\rho(X_2) - \rho(X_2)\rho(X_1) \\
\rho([X_1, Y_2]) &= \rho(X_1)\rho(Y_2) - \rho(Y_2)\rho(X_1) \\
\rho(Y_1, \ldots, Y_F) &= \sum_{\sigma \in S_F} \rho(Y_{\sigma(1)}) \cdots \rho(Y_{\sigma(F)})
\end{align*}

$S_F$ being the symmetric group of $F$ elements.

By construction, the vector space $V$ is graded $V = V_0 \oplus \cdots \oplus V_{F-1}$, and for all $a = \{0, \ldots, F - 1\}$, $V_a$ is a $\mathfrak{g}_0$-module. Further, the condition $\rho(\mathfrak{g}_1)(V_a) \subseteq V_{a+1}$ holds.

3. Filiform Lie algebras of order $F$

In this section we will focus our study in generalize a very important type of nilpotent Lie algebras, i.e. filiform Lie algebras obtaining the notion of filiform Lie algebras of order $F$.

To do that we will star with a previous concept “filiform module”. 


Definition 3.1. Let \( g = g_0 \oplus g_1 \oplus \cdots \oplus g_{F-1} \) be a Lie algebra of order \( F \). \( g_i \) is called a \( g_0\)-filiform module if there exists a decreasing subsequence of vectorial subspaces in its underlying vectorial space \( V \), \( V = V_m \supset \cdots \supset V_1 \supset V_0 \), with dimensions \( m, m-1, \ldots 0 \), respectively, \( m > 0 \), and such that \( [g_0, V_{i+1}] = V_i \).

Definition 3.2. Let \( g = g_0 \oplus g_1 \oplus \cdots \oplus g_{F-1} \) be a Lie algebra of order \( F \). Then \( g \) is a filiform Lie algebra of order \( F \) if the following conditions hold:

1. \( g_0 \) is a filiform Lie algebra.
2. \( g_i \) has structure of \( g_0\)-filiform module, for all \( i, 1 \leq i \leq F-1 \).

From now on we will restrict our study to \( F = 3 \). Thus, if we take a homogeneous basis \( \{X_0, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p\} \) of a Lie algebra of order 3 \( g = g_0 \oplus g_1 \oplus g_2 \) with \( X_i \in g_0, Y_j \in g_1 \) and \( Z_k \in g_2 \), then the Lie algebra of order 3 will be completely determined by its structure constants, which is, by the set of constants \( \{C^k_{ij}, D^k_{ij}, E^k_{ij}, F^k_{ijl}, G^k_{ijl} \}_{i,j,l,k} \) that verify

\[
\begin{align*}
[X_i, X_j] &= \sum_{k=0}^{n} C^k_{ij} X_k, & 0 \leq i < j \leq n, \\
[X_i, Y_j] &= \sum_{k=1}^{p} D^k_{ij} Y_k, & 0 \leq i \leq n, 1 \leq j \leq m, \\
[X_i, Z_j] &= \sum_{k=1}^{p} E^k_{ij} Z_k, & 0 \leq i \leq n, 1 \leq j \leq p, \\
\{Y_i, Y_j, Y_l\} &= \sum_{k=0}^{n} F^k_{ijl} X_k, & 1 \leq i \leq j \leq l \leq m, \\
\{Z_i, Z_j, Z_l\} &= \sum_{k=0}^{n} G^k_{ijl} X_k, & 1 \leq i \leq j \leq l \leq p,
\end{align*}
\]

with

\[
\begin{align*}
C^k_{ij} &= -C^k_{ji}, & D^k_{ij} &= -D^k_{ji}, & E^k_{ij} &= -E^k_{ji}, \\
F^k_{ijl} &= F^k_{lij}, & F^k_{lij} &= F^k_{ijl}, & F^k_{ijl} &= F^k_{lij}. \\
G^k_{ijl} &= G^k_{ijl}, & G^k_{ijl} &= G^k_{lij}, & G^k_{ijl} &= G^k_{lij}.
\end{align*}
\]

By the Jacobi identity we would have some polynomial equations that the structure constants have to verify. All these equations give to the set of Lie algebras of order 3, denoted by \( L_{n,m,p} \), the structure of algebraic variety.

We denote by \( F_{n,m,p} \) the subset of \( L_{n,m,p} \) composed of all filiform Lie algebras of order 3.

If we consider \( A = (g_0 \wedge g_0) \oplus (g_0 \wedge g_1) \oplus (g_0 \wedge g_1) \oplus S^3(g_1) \oplus S^3(g_2) \) and the multiplication of the Lie algebra of order 3 \( g = g_0 \oplus g_1 \oplus g_2 \) as the linear map

\[
\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) : A \rightarrow g \text{ where,}
\]

\[
\begin{align*}
\psi_1 : g_0 \wedge g_0 &\rightarrow g_0, & \psi_2 : g_0 \wedge g_1 &\rightarrow g_1, & \psi_3 : g_0 \wedge g_2 &\rightarrow g_2, \\
\psi_4 : S^3(g_1) &\rightarrow g_0, & \psi_5 : S^3(g_2) &\rightarrow g_0.
\end{align*}
\]
Usually \( \psi_1, \psi_2 \) and \( \psi_3 \) are represented by \([,]\), and \( \psi_4 \) and \( \psi_5 \) by \(\{,\}\). If we consider the action of the group \(GL(n + 1, m, p) \cong GL(n + 1) \times GL(m) \times GL(p)\) on \(L_{n,m,p}\) we would have the following action with \((f_0, f_1, f_2) \in GL(n + 1, m, p)\)

\[
(f_0, f_1, f_2)(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \rightarrow (\psi_1', \psi_2', \psi_3', \psi_4', \psi_5'),
\]

where \(\psi_1'(X_1, X_2) = f_0^{-1}\psi_1(f_0(X_1), f_0(X_2)), \psi_2'(X_1, Y_2) = f_1^{-1}\psi_2(f_0(X_1), f_1(Y_2)),\)

\[
\psi_3'(X_1, Z_2) = f_2^{-1}\psi_3(f_0(X_1), f_2(Z_2)), \quad \psi_4'(Y_1, Y_2, Y_3) = f_0^{-1}\psi_4(f_1(Y_1), f_1(Y_2), f_1(Y_3)),
\]

and \(\psi_5'(Z_1, Z_2, Z_3) = f_0^{-1}\psi_5(f_2(Z_1), f_2(Z_2), f_2(Z_3))\).

The group \(GL(n + 1, m, p)\) can be embedded in \(GL(n + 1 + m + p)\) and it can be
seen as the subgroup of \(GL(n + 1 + m + p)\) which let the subspaces \(g_0, g_1\) and \(g_2\) invariant. If we denote by \(O_\psi\) the orbit of \(\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)\) with respect to this action, then the algebraic variety \(L_{n,m,p}\) is fibered by these orbits. The quotient set is the set of isomorphism classes of \((n + 1 + m + p)\)-dimensional Lie algebras of order 3.

Prior to studying general classes of Lie algebras of order 3 it is convenient to solve
the problem of finding a suitable basis; a so-called adapted basis. This question is not trivial for Lie algebras of order 3 and it is very difficult to prove the general existence of such a basis. However, for the class of filiform Lie algebras of order 3, analogously as for “filiform color Lie superalgebras” \([13]\), it can be obtained that there always exists an adapted basis. Thus we have the following result.

**Theorem 1. (Adapted basis)** Let \(g = g_0 \oplus g_1 \oplus g_2\) be a Lie algebra of order 3. If \(g\) is a filiform Lie algebra of order 3, then there exists an adapted basis of \(g\),

namely \(\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p\}\) with \(\{X_0, X_1, \ldots, X_n\}\) a basis of \(g_0\),

\(\{Y_1, \ldots, Y_m\}\) a basis of \(g_1\) and \(\{Z_1, \ldots, Z_p\}\) a basis of \(g_2\), such that:

\[
\begin{align*}
[X_0, X_i] &= X_{i+1}, \quad 1 \leq i \leq n - 1, \\
[X_0, X_n] &= 0, \\
[X_0, Y_j] &= Y_{j+1}, \quad 1 \leq j \leq m - 1, \\
[X_0, Y_m] &= 0 \\
[X_0, Z_k] &= Z_{k+1}, \quad 1 \leq k \leq p - 1, \\
[X_0, Z_p] &= 0.
\end{align*}
\]

\(X_0\) will be called the characteristic vector.

**Remark 3.3.** We observe that \(g_1\) is a \(g_0\)-filiform module, then the subspaces \(V = V_m \supset \cdots \supset V_1 \supset V_0\) with dimensions \(m, \ldots, 0\) such that \([g_0, V_{i+1}] = V_i\) there will be

\(V_m = \langle Y_1, \ldots, Y_m \rangle, V_{m-1} = \langle Y_2, \ldots, Y_m \rangle, \ldots, V_1 = \langle Y_m \rangle, V_0 = 0\)

Analogously for \(g_2\): \(V_m = \langle Z_1, \ldots, Z_m \rangle, V_{m-1} = \langle Z_2, \ldots, Z_m \rangle, \ldots, V_1 = \langle Z_m \rangle, V_0 = 0\)

**Remark 3.4.** An adapted basis is composed by homogeneous elements.
Definition 3.5. The **model filiform Lie algebra of order** 3, is the simplest filiform Lie algebra of order 3. It will be defined in an adapted basis \( \{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p\} \) by the following non-null bracket products:

\[
\mu_0 = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n - 1 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m - 1 \\
[X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p - 1
\end{cases}
\]

We observe that all the three-brackets are null.

**Examples.** Other examples of filiform Lie algebras of order 3 distinct from the model are easy to obtain. Thus,

1. \( \mu_{n,m,p}^1 \) is a family of non model filiform Lie algebras of order 3 and it can be expressed in an adapted basis \( \{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p\} \) by the following non-null bracket and 3-bracket products:

\[
\mu_{n,m,p}^1 = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n - 1 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m - 1 \\
[X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p - 1 \\
\{Y_1, Y_1, Y_1\} = X_n \\
\{Z_1, Z_1, Z_1\} = X_n
\end{cases}
\]

2. If \( m \geq n \) then we have all a family of non model filiform Lie algebras of order 3: \( \mu_{n,m,p}^2 \) that can be expressed in an adapted basis \( \{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p\} \) by the following non-null bracket and 3-bracket products:

\[
\mu_{n,m,p}^2 = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n - 1 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m - 1 \\
[X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p - 1 \\
\{Y_1, Y_1, Y_1\} = 3X_1 \\
\{Y_1, Y_1, Y_2\} = X_2 \\
\{Y_1, Y_1, Y_3\} = X_3 \\
\{Y_1, Y_1, Y_4\} = X_4 \\
\vdots \\
\{Y_1, Y_1, Y_n\} = X_n
\end{cases}
\]

From now on we are going to restrict our study to elementary Lie algebras of order 3 due to its physical applications.

4. The algebraic variety of elementary Lie algebras of order 3

By the Jacobi identity we have some polynomial equations that the structure constants hold. All these equations give to the set of elementary Lie algebras of
order 3 the structure of algebraic variety (analogously as for Lie algebras of order 3).

We denote by $L_{n,m}$ the mentioned algebraic variety and by $F_{n,m}$ the subset of $L_{n,m}$ composed of all filiform elementary Lie algebra of order 3.

Next, we are going to define some descending sequences of ideals.

**Definition 4.1.** Let $g = g_0 \oplus g_1$ be an elementary Lie algebra of order 3. Then, we define the descending sequences of ideals $C^k(g_0)$ and $C^k(g_1)$, as follows:

\[
C^0(g_0) = g_0, \quad C^{k+1}(g_0) = [g_0, C^k(g_0)], \quad k \geq 0
\]

and

\[
C^0(g_1) = g_1, \quad C^{k+1}(g_1) = [g_0, C^k(g_1)], \quad k \geq 0
\]

Using the descending sequences of ideals defined above we give an invariant of elementary Lie algebras of order 3 called **order-nilindex**.

**Definition 4.2.** If $g = g_0 \oplus g_1$ is an elementary Lie algebra of order 3, then $g$ has order-nilindex $(p_0, p_1)$, if the following conditions holds:

\[
(C^{p_0-1}(g_0))(C^{p_1-1}(g_1)) \neq 0
\]

and

\[
C^{p_0}(g_0) = C^{p_1}(g_1) = 0
\]

**Remark 4.3.** Note that are equivalent the definitions of “filiform” elementary Lie algebras of order 3 and “maximal order-nilindex”, maximal in the sense of lexicographic order. That is, if $g = g_0 \oplus g_1$, with $dim(g_0) = n + 1$ and $dim(g_1) = m$, is a filiform elementary Lie algebra of order 3 then $g$ has maximal order-nilindex $(n, m)$ and reciprocally.

We note by $N_{p_0, p_1}^{n, m}$ the subset of $L_{n,m}$ composed of all the elementary Lie algebras of order 3 with order-nilindex $(r, s)$ where $r \leq p_0$ and $s \leq p_1$.

**Proposition 4.4.** $N_{p_0, p_1}^{n, m}$ is an algebraic subvariety of $L_{n,m}$.

**Proof.** The set $N_{p_0, p_1}^{n, m}$ of $L_{n,m}$ is defined by the restrictions $C^{p_0}(g_0) = 0$ and $C^{p_1}(g_1) = 0$, but these restrictions are polynomial equations of the structure constants. Thus $N_{p_0, p_1}^{n, m}$ is closed for the Zariski topology and it will have the structure of an algebraic subvariety. We denote by $N_{n,m}^{p_0, p_1}$ the corresponding affine variety. □

For simplicity we will refer to $N_{n,m}^{p_0, p_1}$ as $N_{n,m}$.

**Proposition 4.5.** Each component of $F_{n,m}$ determines a component of $N_{n,m}$.

**Proof.** $F_{n,m} = N_{n,m} - N_{n-1,m-1}$ is a Zariski open subset of $N_{n,m}$. □

**Corollary 4.5.1.** For any $g \in F_{n,m}$ the Zariski closure of the orbit $O(g)$, $\overline{O(g)}^Z$, will be an irreducible component of $N_{n,m}$.
5. The class of filiform Lie algebras of order 3

Recall that the concept of filiform Lie algebras was firstly introduced in [22] by Vergne. This type of nilpotent Lie algebra has important properties as it has been seen in section above; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra $L_n$. In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$. In this paper we generalize this result for filiform Lie algebras of order 3. In particular, for elementary Lie algebras of order 3.

Next, we are going to generalize the concept of infinitesimal deformations for elementary Lie algebras of order 3. For more details of deformations of elementary Lie algebras of order 3 see [7].

**Definition 5.1.** Let $g = g_0 \oplus g_1$ be an elementary Lie algebra of order 3 and let $A = (g_0 \wedge g_0) \oplus (g_0 \wedge g_1) \oplus S^3(g_1)$. The linear map $\psi : A \rightarrow g$ is called an infinitesimal deformation of $g$ if it satisfies

$$\mu \circ \psi + \psi \circ \mu = 0$$

with $\mu$ representing the law of $g$.

If we consider the restrictions of $\mu$ and $\psi$ to each of the terms of $A$, i.e. $\psi = \psi_1 + \psi_2 + \psi_3$ and $\mu = \mu_1 + \mu_2 + \mu_3$ respectively, with

$\psi_1, \mu_1 : g_0 \wedge g_0 \rightarrow g_0$,

$\psi_2, \mu_2 : g_0 \wedge g_1 \rightarrow g_1$,

$\psi_3, \mu_3 : S^3(g_1) \rightarrow g_0$,

then the condition to be an infinitesimal deformation can be decomposed into 4 equations:

1. $\mu_1(\psi_1(X_i, X_j), X_k) + \psi_1(\mu_1(X_i, X_j), X_k) + \mu_1(\psi_1(X_k, X_i), X_j) + \psi_1(\mu_1(X_k, X_i), X_j) + \mu_1(\psi_1(X_k, X_i), X_j) + \psi_1(\mu_1(X_k, X_i), X_j) = 0$

2. $\mu_2(\psi_1(X_i, X_j), Y) + \psi_2(\mu_1(X_i, X_j), Y) + \mu_2(\psi_2(X_i, Y), X_j) + \psi_2(\mu_2(X_i, Y), X_j) + \mu_2(\psi_2(Y_i, X_j), X_j) + \psi_2(\mu_2(Y_i, X_j), X_j) = 0$

3. $\mu_1(X_i, \psi_3(Y_i, Y_j, Y_k)) + \psi_3(X_i, \mu_3(Y_i, Y_j, Y_k)) + \psi_3(\mu_2(X_i, Y_j), Y_k) + \mu_3(\psi_3(Y_i, Y_j, Y_k)) + \mu_3(\mu_2(Y_i, Y_j), Y_k) + \psi_3(Y_i, \mu_3(Y_i, Y_j, Y_k)) = 0$

4. $\mu_2(Y_i, \psi_3(Y_i, Y_j, Y_k)) + \psi_3(Y_i, \mu_3(Y_i, Y_j, Y_k)) + \mu_2(Y_i, \psi_3(Y_j, Y_k, Y_i)) + \psi_3(Y_j, \mu_3(Y_j, Y_k, Y_i)) + \mu_2(Y_j, \psi_3(Y_j, Y_k, Y_i)) + \psi_2(Y_k, \mu_3(Y_i, Y_j, Y_k)) = 0$

for all $X, X_i, X_j, X_k \in g_0$ and $Y, Y_i, Y_j, Y_k, Y_l \in g_1$.

**Theorem 2.** (1) Any filiform elementary Lie algebra of order 3 law $\mu$ is isomorphic to $\mu_0 + \psi$ where $\mu_0$ is the law of the model filiform elementary Lie algebra of order 3 and $\psi$ is an infinitesimal deformation of $\mu_0$ verifying that $\psi(X_0, X) = 0$ for all $X \in \mu_0$, with $X_0$ the characteristic vector of the model one.
(2) Conversely, if \( \psi \) is an infinitesimal deformation of a model filiform elementary Lie algebra of order 3 law \( \mu_0 \) with \( \psi(X_0, X) = 0 \) for all \( X \in \mu_0 \), then the law \( \mu_0 + \psi \) is a filiform Lie algebra of order 3 law iff \( \psi \circ \psi = 0 \).

If we consider the restrictions of \( \psi \) to each of the terms of \( A \), i.e. \( \psi = \psi_1 + \psi_2 + \psi_3 \), then the condition \( \psi \circ \psi = 0 \) can be decomposed into four equations, i.e.

1. \( \psi_1(\psi_1(X_i, X_j), X_k) + \psi_1(\psi_1(X_k, X_i), X_j) + \psi_1(\psi_1(X_j, X_k), X_i) = 0 \)
2. \( \psi_2(\psi_1(X_i, X_j), Y) + \psi_2(\psi_2(Y, X_i), X_j) + \psi_2(\psi_2(Y, X_j), X_i) = 0 \)
3. \( \psi_1(X, \psi_3(Y_i, Y_j, Y_k)) - \psi_3(\psi_2(X, Y_i), Y_j, Y_k) - \psi_3(\psi_2(X, Y_i), Y_j, Y_k) - \psi_3(\psi_2(X, Y_i), Y_j, Y_k) = 0 \)
4. \( \psi_2(Y_i, \psi_3(Y_j, Y_k, Y_l)) + \psi_2(Y_j, \psi_3(Y_i, Y_k, Y_l)) + \psi_2(Y_k, \psi_3(Y_i, Y_j, Y_l)) + \psi_2(Y_l, \psi_3(Y_i, Y_j, Y_k)) = 0 \)

Remark 5.2. Note that these last equations are equivalent to Jacobi identities \( (2.2.1) - (2.2.4) \) for \( \psi \).

**Proof of the theorem.**

1. Let \( g = g_0 \oplus g_1 \) be a filiform elementary Lie algebra of order 3 and let \( A = (g_0 \wedge g_0) \oplus (g_0 \wedge g_1) \oplus S^2(g_1) \). Thanks to the theorem of adapted basis, its law \( \mu \) can be expressed in a basis \( \{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \) by \( \mu = \mu_0 + \psi \), with \( \mu_0 \) the law of the corresponding model filiform Lie algebra of order 3:

\[
\mu_0 = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n - 1 \\
[X_0, X_n] = 0 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m - 1, \\
[X_0, Y_m] = 0
\end{cases}
\]

and \( \psi \) a linear map \( \psi : A \to g \) with \( \psi(X_0, X) = 0 \) for all \( X \in g \).

As \( \mu = \mu_0 + \psi \) is a filiform elementary Lie algebra of order 3 it verifies the Jacobi identities \( (2.2.1) - (2.2.4) \), but \( \mu \circ \mu \) in the sense of Jacobi can be decomposed as \( \mu \circ \mu = \mu_0 \circ \mu_0 + \mu_0 \circ \psi + \psi \circ \mu_0 + \psi \circ \psi = 0 \).

Taking into account that \( \mu_0 \) verifies the Jacobi identity we have that \( \mu_0 \circ \mu_0 = 0 \) leading to \( \mu_0 \circ \psi + \psi \circ \mu_0 + \psi \circ \psi = 0 \). At this point we are going to consider two possibilities in the election of the basis vectors \( X_i, Y_j \):

1. (1.1) If one of the vectors is equal to the characteristic vector \( X_0 \). In this case and considering that \( \psi(X_0, X) = 0 \) \( \forall X \in g \) we have that \( \psi \circ \psi = 0 \) and the above equation rests

\[
\mu_0 \circ \psi + \psi \circ \mu_0 = 0
\]

Then \( \psi \) is an infinitesimal deformation of \( \mu_0 \).

1. (1.2) In this case all vectors are different of the characteristic vector. As \( \mu = \mu_0 + \psi \), it can be decomposed into \( \mu = \mu_1 + \mu_2 + \mu_3 \) considering the restrictions of \( \mu \) to each
of the terms of $A$. Then, $\mu = \mu_0 + \psi = \mu_1 + \mu_2 + \mu_3$ is the law of a filiform elementary Lie algebra of order 3 and so we have the Jacobi identities (2.2.1), i.e. $$\mu_1(X_1, X_j, X_k) + \mu_1(X_k, X_i, X_j) + \mu_1(X_j, X_k, X_i) = 0$$

(2.2.2) $$\mu_2(X_1, X_j, Y) + \mu_2(Y, X_i, Y) + \mu_2(Y, X, Y_i) = 0$$

(2.2.3) $$\mu_1(X, \mu_3(Y_i, Y_j, Y_k)) - \mu_3(\mu_2(X, Y_i), Y_j, Y_k) - \mu_3(Y_i, \mu_2(X, Y_j), Y_k) -$$

$$ \mu_3(Y_j, Y_k, \mu_2(X, Y_i)) = 0$$

(2.2.4) $$\mu_2(Y_i, \mu_3(Y_j, Y_k, Y_l)) + \mu_2(Y_j, \mu_3(Y_k, Y_l, Y_i)) + \mu_2(Y_k, \mu_3(Y_l, Y_i, Y_j)) +$$

$$ \mu_2(Y_l, \mu_3(Y_i, Y_j, Y_k)) = 0$$

also $\mu_1$ is the law of a filiform Lie algebra with $X_0$ as characteristic vector; and the vector space $\langle Y_1, Y_2, \ldots, Y_m \rangle$ has the structure of a $g_0$-filiform module with $g_0 = \langle X_0, X_1, \ldots, X_n \rangle$, i.e. $\mu_2(g_0, \langle Y_1, \ldots, Y_m \rangle) \subset \langle Y_{i+1}, \ldots, Y_m \rangle$.

Next, we are going to prove that $X_0 \notin \text{Im}\mu$ and so neither in $\text{Im}\psi$, i.e. $X_0 \notin \text{Im}\mu$. In fact, as $\mu_1$ is the law of a filiform Lie algebra being $X_0$ as characteristic vector, then it is easy to prove using Lie theory of nilpotent algebras that $X_0 \notin \text{Im}\mu_1$. Thus, only remains to be seen that $X_0 \notin \text{Im}\mu_3$.

However, if there exist any $Y_i, Y_j$ and $Y_k$ such that $\mu_3(Y_i, Y_j, Y_k) = \lambda X_0 + \sum_{i=1}^n \lambda_i X_i$ with $\lambda \neq 0$, then we always arrive to a contradiction. To do that we distinguish some cases. Without loss of generality we can suppose $\mu_3(Y_i, Y_j, Y_k) = X_0$ with $i \leq j \leq k$.

Case 1. If $i = j = k$. In this case $\mu_3(Y_i, Y_i, Y_i) = X_0$. For $i, 1 \leq i \leq m - 1$ using the equation (2.2.4) for the vectors $Y_i, Y_i, Y_i$ we have

$$4\mu_2(Y_i, \mu_3(Y_i, Y_i, Y_i)) = 0 \Rightarrow \mu_2(Y_i, X_0) = 0 \Rightarrow \mu_0(Y_i, X_0) = -Y_{i+1} = 0 \ (\rightarrow \leftarrow)$$

we note that $\mu_2(Y_i, X_0) = \mu_0(Y_i, X_0)$ because $\psi(X_0, -) = 0$ and $\mu = \mu_0 + \psi$.

For $i = m$, using the equation (2.2.4) for the vectors $X_1, Y_m, Y_m, Y_m$ we have

$$\mu_1(X_1, \mu_3(Y_m, Y_m, Y_m)) = 3\mu_3(\mu_2(X_0, Y_m), Y_m, Y_m)$$

but as $\psi(X_0, -) = 0$ and $\mu = \mu_0 + \psi$ then, $\mu_2(X_0, Y_m) = \mu_0(X_0, Y_m) = 0$ and thus

$$0 = \mu_1(X_1, \mu_3(Y_m, Y_m, Y_m)) = \mu_1(X_1, X_0) = \mu_0(X_1, X_0) = -X_2 \ (\rightarrow \leftarrow)$$

We observe that if $m = 1$, then we would use the subcase $i = m$ for prove the result. However, we always have $n > 1$, that is, $g_0$ is an authentic filiform Lie algebra, because for $n \leq 1 g_0$ is an abelian Lie algebra.

Case 2. If $i < j \leq k$. In this case $\mu_3(Y_i, Y_j, Y_k) = X_0$ with $j, k > i$. Thanks to the equation (2.2.4) for the vectors $Y_i, Y_j, Y_k$ we have

$$\mu_2(Y_i, \mu_3(Y_j, Y_k)) + \mu_2(Y_i, \mu_3(Y_k, Y_j)) + \mu_2(Y_j, \mu_3(Y_i, Y_k)) +$$

$$+\mu_2(Y_k, \mu_3(Y_i, Y_j)) = 0$$

$$\Rightarrow \psi(X_0, -) = 0 \text{ and } \mu = \mu_0 + \psi, \text{ then } \mu_2(X_0, Y_i) = \mu_0(X_0, Y_i) \text{ and thus}$$

$$-2Y_{i+1} + a + b = 0$$
This constitutes a contradiction taking into account that \( <Y_1, Y_2, \ldots, Y_m> \) has the structure of a \( g_0 \)-filiform module, i.e. \( \mu_2(g_0, <Y_i, \ldots, Y_m>) \subset <Y_{i+1}, \ldots, Y_m> \) and thus \( a \subset <Y_{j+1}, \ldots, Y_m> \) and \( b \subset <Y_{k+1}, \ldots, Y_m> \).

**Case 3.** If \( i = j < k \). In this case \( \mu_3(Y_i, Y_j, Y_k) = X_0 \) with \( k > i \). Thanks to the equation (2) for the vectors \( Y_k, Y_i, Y_j \) we have

\[
\mu_2(Y_i, \mu_3(Y_i, Y_j, Y_k)) + \mu_2(Y_i, \mu_3(Y_i, Y_k)) + \mu_2(Y_i, \mu_3(Y_j, Y_k)) + \\
\mu_2(Y_k, \mu_3(Y_i, Y_j)) = 0
\]

Using a similar reasoning than in the precedent case we arrive to

\[-3Y_{i+1} + a = 0 \text{ with } a \subset <Y_{k+1}, \ldots, Y_m>\]

which constitutes a contradiction.

Thus, thanks to the precedent reasoning we have that \( X_0 \notin \text{Im} \mu \) and so neither in \( \text{Im} \psi \). Then, we have that \( \mu_0 \circ \psi + \psi \circ \mu_0 = 0 \) thanks to \( \mu_0 \) is the null map, i.e. if \( X \neq X_0 \neq Y \) then \( \mu_0(X, Y) = 0 \). Therefore, \( \psi \) is an infinitesimal deformation of \( \mu_0 \).

(2) Let \( \psi \) be an infinitesimal deformation of \( \mu_0 \), with \( \mu_0 \) a model filiform elementary Lie algebra of order 3. We have too that \( \psi(X_0, X) = 0 \forall X \in L \).

For obtaining the Jacobi identity, that is \( \mu_0 \circ \mu_0 + \mu_0 \circ \psi + \psi \circ \mu_0 + \psi \circ \psi = 0 \) as \( \mu_0 \) represents a filiform elementary Lie algebra of order 3 it verifies \( \mu_0 \circ \mu_0 = 0 \). As \( \psi \) is an infinitesimal deformation we have \( \mu_0 \circ \psi + \psi \circ \mu_0 = 0 \) and thus \( \mu_0 + \psi \) verifies the Jacobi identity iff \( \psi \circ \psi = 0 \).

\[\square\]

**Corollary 5.2.1.** Any filiform elementary Lie algebra of order 3 is a linear deformation of the corresponding model filiform elementary Lie algebra of order 3.

Thus, in order to study the class of filiform elementary Lie algebras of order 3 it will be enough to determine all the infinitesimal deformations \( \psi \) of the corresponding model \( \mu_0 \) that verify \( \psi \circ \psi = 0 \) (these type of infinitesimal deformations will be called linear integrable).

Recall that \( \mu_0 \) is the law of the model filiform elementary Lie algebra of order 3. Then, we denote by \( Z(\mu_0) \) the vector space composed by all the infinitesimal deformations of \( \mu_0 \), \( \psi \), verifying that \( \psi(X_0, X) = 0 \), \( \forall X \in \mu_0 \). Next we will see that this vector space can be seen as direct sum of three subspaces of infinitesimal deformations what facilitates its study.

**Proposition 5.3.** Let \( Z(\mu_0) \) be the vector space composed by all the infinitesimal deformations of \( \mu_0 \) that vanish on the characteristic vector \( X_0 \). Then, if we note by \( g_0 \oplus g_1 \) the underlying vector space of \( \mu_0 \), i.e. \( g_0 = <X_0, X_1, \ldots, X_n> \) and \( g_1 = <Y_1, \ldots, Y_m> \), we have that

\[
Z(\mu_0) = Z(\mu_0) \cap \text{Hom}(g_0 \wedge g_0, g_0) \oplus Z(\mu_0) \cap \text{Hom}(g_0 \wedge g_1, g_1) \\
\oplus Z(\mu_0) \cap \text{Hom}(S^3(g_1), g_0) \\
:= A \oplus B \oplus C
\]
Proof. Let $\psi$ be such that $\psi \in Z(\mu_0)$ and $\psi(X_0, X) = 0$, $\forall X \in \mu_0$. It is not difficult to see that $\psi = \psi_1 + \psi_2 + \psi_3$ with $\psi_1 \in Hom(g_0 \wedge g_0, g_0)$, $\psi_2 \in Hom(g_0 \wedge g_1, g_1)$ and $\psi_3 \in Hom(S^3(g_1), g_0)$. To complete the proof it only remains to verify that each of the above homomorphisms is also an infinitesimal deformation.

As $\psi = \psi_1 + \psi_2 + \psi_3$ is an infinitesimal deformation it will verify the equations of Definition 5.1. Taking into account the law of $\mu_0$ and that $\psi(X_0, X) = 0$, these equations remain as follows

1. $\psi_1(X_{j+1}, X_k) + \psi_1(X_j, X_{k+1}) + [\psi_1(X_j, X_k), X_0] = 0$
2. $[\psi_1(X_i, X_j), Y_k] = 0$, $1 \leq i, j$
3. $\psi_2(X_{j+1}, Y_k) + [\psi_2(X_j, Y_k), X_0] + \psi_2(X_j, Y_{k+1}) = 0$
4. $[\psi_2(Y_i, Y_j, Y_k)] + [Y_j, \psi_3(Y_i, Y_k, Y_i)] + [Y_k, \psi_3(Y_i, Y_j, Y_i)] + [Y_l, \psi_3(Y_i, Y_j, Y_k)] = 0$

for all $X_i, X_j, X_k, X_l \in g_0$ and $Y_i, Y_j, Y_k, Y_l \in g_1$. From equations (2.1) and (3.1) we can obtain that $Im\psi_1 \subset \langle X_1, \ldots, X_n \rangle$ and $Im\psi_3 \subset \langle X_1, \ldots, X_n \rangle$ respectively. In fact, the ideal $\langle X_1, \ldots, X_n \rangle := V_0$ is equal to its own centralizer in $g_0$ and from equations (2.1) and (3.1) we obtain that $\psi_1(X_i, Y_j)$ and $\psi_3(Y_i, Y_j, Y_k)$ centralize $V_0$, and are thus elements of $V_0$. Then the equation (4) disappears, and each of the remaining equations corresponds to the condition that has to verify each $\psi_i$ for be an infinitesimal deformation.

Then, as the vector space of infinitesimal deformations called $Z(\mu_0)$ is equal to $A \oplus B \oplus C$ we will restrict our study to each vector subspace. Of all of them, the most important vector subspace will be $C$ because any infinitesimal deformation $\psi$ belonging to $C$ verifies that $\psi \circ \psi = 0$, i.e. $\psi$ is linear integrable (see the conditions for being integrable in the statement of Theorem 2). Thus, $\mu_0 + \psi$ will be a filiform elementary Lie algebra of order 3 with $\psi \in C$.

6. $\mathfrak{sl}(2, \mathbb{C})$-module Method

In this section we are going to explain the $\mathfrak{sl}(2, \mathbb{C})$-module method to compute the dimensions of $C$. For the subspace $A$ it can be seen the paper [11], page 1160 subsection dimension of $C_1$; and for the subspace $B$ it can be seen the paper [12], page 134 subsection Dimension of $B_1$.

Recall the following well-known facts about the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and its finite-dimensional modules, see e.g. [2], [8]:
\(\mathfrak{sl}(2, \mathbb{C}) = \langle X_-, H, X_+ \rangle\) with the following commutation relations:

\[
\begin{align*}
[X_+, X_-] &= H, \\
[H, X+] &= 2X_+, \\
[H, X_-] &= -2X_-. \\
\end{align*}
\]

Let \(V\) be a \(n\)-dimensional \(\mathfrak{sl}(2, \mathbb{C})\)-module, \(V = \langle e_1, \ldots, e_n \rangle\). Then, up to isomorphism there exists a unique structure of an irreducible \(\mathfrak{sl}(2, \mathbb{C})\)-module in \(V\) given in a basis \(e_1, \ldots, e_n\) as follows [2]:

\[
\begin{align*}
X_+ \cdot e_i &= e_{i+1}, & 1 \leq i \leq n - 1, \\
X_+ \cdot e_n &= 0, \\
H \cdot e_i &= (-n + 2i - 1)e_i, & 1 \leq i \leq n.
\end{align*}
\]

It is easy to see that \(e_n\) is the maximal vector of \(V\) and its weight, called the highest weight of \(V\), is equal to \(n - 1\).

Let \(V_0, V_1, \ldots, V_k\) be \(\mathfrak{sl}(2, \mathbb{C})\)-modules, then the space \(\text{Hom}(\otimes_{i=1}^k V_i, V_0)\) is a \(\mathfrak{sl}(2, \mathbb{C})\)-module in the following natural manner:

\[
(\xi \cdot \varphi)(x_1, \ldots, x_k) = \xi \cdot \varphi(x_1, \ldots, x_k) - \sum_{i=1}^{i=k} \varphi(x_1, \ldots, \xi \cdot x_i, x_{i+1}, \ldots, x_n)
\]

with \(\xi \in \mathfrak{sl}(2, \mathbb{C})\) and \(\varphi \in \text{Hom}(\otimes_{i=1}^k V_i, V_0)\).

Any element \(\varphi \in \text{Hom}(V_1 \otimes V_1 \otimes V_0)\) is said to be invariant if \(X_+ \cdot \varphi = 0\), that is

\[(6.0.1)\]

\[X_+ \cdot \varphi(x_1, x_2, x_3) - \varphi(x_1, x_2, x_3) - [\varphi(x_1, x_2) - \varphi(x_1, x_2, x_3)] = 0, \forall x_1, x_2, x_3 \in V_1.
\]

Note that \(\varphi \in \text{Hom}(V_1 \otimes V_1 \otimes V_0)\) is invariant if and only if \(\varphi\) is a maximal vector.

On the other hand, we are going to consider the model filiform elementary Lie algebra of order 3 \(\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) with basis \(\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m\}\). By definition, an infinitesimal deformation \(\varphi\) belonging to \(C\) will be a symmetric multi-linear map:

\[
\varphi : S^3(\mathfrak{g}_1) \rightarrow \mathfrak{g}_0/\mathfrak{g}_0 X_0
\]

such that

\[(6.0.2)\]

\([X_0, \varphi(Y_i, Y_j, Y_k)] - \varphi([X_0, Y_i], Y_j, Y_k) - \varphi(Y_i, [X_0, Y_j], Y_k) - \varphi(Y_i, Y_j, [X_0, Y_k]) = 0
\]

with \(1 \leq i \leq j \leq k \leq m\).

We are going to consider the structure of irreducible \(\mathfrak{sl}(2, \mathbb{C})\)-module in \(V_0 = \langle X_1, \ldots, X_n \rangle = \mathfrak{g}_0/\mathfrak{g}_0 X_0\) and in \(V_1 = \langle Y_1, \ldots, Y_m \rangle = \mathfrak{g}_1\), thus in particular:

\[
\begin{align*}
X_+ \cdot X_j &= X_{i+1}, & 1 \leq i \leq n - 1, \\
X_+ \cdot X_n &= 0, \\
X_+ \cdot Y_j &= Y_{j+1}, & 1 \leq j \leq m - 1, \\
X_+ \cdot Y_m &= 0.
\end{align*}
\]

We identify the multiplication of \(X_+\) and \(X_1\) in the \(\mathfrak{sl}(2, \mathbb{C})\)-module \(V_0 = \langle X_1, \ldots, X_n \rangle\), with the bracket \([X_0, X_1]\) in \(\mathfrak{g}_0\). Analogously, we identify \(X_+ \cdot Y_j\) and \([X_0, Y_j]\).

Thanks to these identifications, the expressions \((6.0.1)\) and \((6.0.2)\) are equivalent, so we have the following result:
Proposition 6.1. Any symmetric multi-linear map \( \varphi : S^3V_1 \to V_0 \) will be an element of \( C \) if and only if \( \varphi \) is a maximal vector of the \( \mathfrak{sl}(2, \mathbb{C}) \)-module \( \text{Hom}(S^3V_1, V_0) \), with \( V_0 = \langle X_1, \ldots, X_n \rangle \) and \( V_1 = \langle Y_1, \ldots, Y_m \rangle \).

Corollary 6.1.1. As each irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of \( C \) is equal to the number of summands of any decomposition of \( \text{Hom}(S^3V_1, V_0) \) into the direct sum of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-modules.

Thanks to the symmetric structure of the weights, instead of to sum the maximal vectors it is possible, and easier, to sum the vectors of weight 0 or 1.

Corollary 6.1.2. The dimension of \( C \) is equal to the dimension of the subspace of \( \text{Hom}(S^3V_1, V_0) \) spanned by the vectors of weight 0 or 1.

7. Computation of the dimension and a basis of \( Z(\mu_0) \cap \text{Hom}(S^3g_1, g_0) \)

In this section we are going to apply the \( \mathfrak{sl}(2, \mathbb{C}) \)-module method above to \( C = Z(\mu_0) \cap \text{Hom}(S^3g_1, g_0) \).

Firstly, we consider a natural basis \( B \) of \( \text{Hom}(S^3g_1, g_0) \) consisting of the following maps where \( 1 \leq s \leq n \) and \( 1 \leq i, j, k, l, r, s \leq m \):

\[
\varphi_{i,j,k}^s(Y_1, Y_r, Y_s) = \begin{cases} 
X_s & \text{if } (i, j, k) = (l, r, s) \\
0 & \text{in all other cases}
\end{cases}
\]

Thanks to Corollary 6.1.2, it will be enough to find the basis vectors \( \varphi_{i,j,k}^s \) with weight 0 or 1. The weight of an element \( \varphi_{i,j,k}^s \) (with respect to \( H \)) is

\[
\lambda(\varphi_{i,j,k}^s) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j) - \lambda(Y_k) = 3m - n + 2(s - i - j - k + 1).
\]

In fact,

\[
(H \cdot \varphi_{i,j,k}^s)(Y_j, Y_j, Y_k) = H \cdot \varphi_{i,j,k}^s(Y_i, Y_j, Y_k) - \varphi_{i,j,k}^s(H \cdot Y_i, Y_j, Y_k) - \\
\varphi_{i,j,k}^s(Y_i, H \cdot Y_j, Y_k) - \varphi_{i,j,k}^s(Y_i, Y_j, H \cdot Y_k) \\
= H \cdot X_s - \varphi_{i,j,k}^s((-m - 1 + 2i)Y_i, Y_j, Y_k) - \\
\varphi_{i,j,k}^s(Y_i, (-m - 1 + 2j)Y_j, Y_k) - \\
\varphi_{i,j,k}^s(Y_i, Y_j, (-m - 1 + 2k)Y_k) \\
= (-n + 1 + 2s)X_s - (-m + 1 + 2i)X_s - \\
(-m + 1 + 2j)X_s - (-m + 1 + 2k)X_s \\
= [3m - n + 2(s - i - j - k + 1)]X_s.
\]

Remark 7.1. If \( m - n \) is even then \( \lambda(\varphi) \) is even, and if \( m - n \) is odd then \( \lambda(\varphi) \) is odd. So, if \( m - n \) is even it will be sufficient to find the elements \( \varphi_{i,j,k}^s \) with weight 0 and if \( m - n \) is odd it will be sufficient to find those of them with weight 1.

In order to find the elements with weight 0 or 1, we can consider the four sequences that correspond with the weights of \( V_1 = \langle Y_1, Y_2, \ldots, Y_{m-1}, Y_m \rangle \) (considered three times) and \( V_0 = \langle X_1, X_2, \ldots, X_{n-1}, X_n \rangle \):

\[
-m + 1, -m + 3, \ldots, m - 3, m - 1; \\
-m + 1, -m + 3, \ldots, m - 3, m - 1; \\
-m + 1, -m + 3, \ldots, m - 3, m - 1; \\
-n + 1, -n + 3, \ldots, n - 3, n - 1.
\]

We shall have to count the number of all possibilities to obtain 1 (if \( m - n \) is odd) or 0 (if \( m - n \) is even). Remember that \( \lambda(\varphi_{i,j,k}^s) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j) - \lambda(Y_k) \),
where \( \lambda(X_s) \) belongs to the last sequence, and \( \lambda(Y_i), \lambda(Y_j), \lambda(Y_k) \) belong to the first, second and third sequences respectively.

For example, if \( m - n \) is even we have to obtain 0, so we can fix an element (a weight) of the last sequence and then to count the possibilities to sum the same quantity between the three first sequences, taking into account also the symmetry of \( \varphi_{i,j,k}^s \). In particular, we will apply this procedure to the cases \( m = 3 \) and odd \( n \).

Thus we have

**Theorem 3.** If \( C = Z(\mu_0) \cap \text{Hom}(S^3 g_1, g_0) \) and \( m = 3 \) and \( n \) is odd, then we have the following values for the dimension of \( C \):

\[
\dim C = \begin{cases} 
2 & \text{if } n = 1 \\
6 & \text{if } n = 3 \\
8 & \text{if } n = 5 \\
10 & \text{if } n = 2k + 1, \ k \geq 3
\end{cases}
\]

### 7.2. Basis.

In this section we are going to calculate a basis of \( C \) with \( m = 3 \) and odd \( n \). Firstly, we are going to introduce a simpler weight of an element \( \varphi \in C \). It corresponds to the action of the diagonalizable derivation \( d, d \in \text{Der}(\mu_0) \), defined by:

\[
d(X_0) = X_0, \ d(X_i) = iX_i, \ d(Y_j) = jY_j; \quad 1 \leq i \leq n, \ 1 \leq j \leq m.
\]

This weight will be denoted by \( p(\varphi) \). We have that

\[
p(\varphi_{i,j,k}^s) = s - i - j - k.
\]

We have the following relationships between the two weights:

\[
\lambda(\varphi) = 2p(\varphi) + 3m - n + 2,
\]

\[
p(\varphi) = \frac{1}{2}(\lambda(\varphi) - 3m + n - 2).
\]

It is clear that two \( \varphi \) with different weights \( p \) they will be linearly independents.

Next, we are going to define some symmetric maps \( \varphi \) and so we will consider \( \varphi(Y_i, Y_j, Y_k) \) with \( i \leq j \leq k \) and for any other reordering of the vectors, \( \varphi \) will have the same value by symmetry. Thus, let \( \varphi_{1,s} \) and \( \varphi_{3,s} \) be elements of \( \text{Hom}(S^3 V_1, V_0) \) with weights \( p(\varphi_{1,s}) = s - 3 \) and \( p(\varphi_{3,s}) = s - 5 \), and defined by

\[
\varphi_{1,s} = \begin{cases} 
\varphi_{1,s}(Y_1, Y_1, Y_1) = X_s \\
\varphi_{1,s}(Y_1, Y_1, Y_3) = 0
\end{cases}
\]

\[
\varphi_{3,s} = \begin{cases} 
\varphi_{3,s}(Y_1, Y_1, Y_3) = X_s \\
\varphi_{3,s}(Y_1, Y_1, Y_1) = 0
\end{cases}
\]

with \( 1 \leq s \leq n \) and satisfying the equations

\[
(7.2.1) \quad [X_0, \varphi(Y_i, Y_j, Y_k)] = \varphi(Y_{i+1}, Y_j, Y_k) + \varphi(Y_i, Y_{j+1}, Y_k) + \varphi(Y_i, Y_j, Y_{k+1}),
\]

with \( 1 \leq i \leq j \leq k \leq 3 \) and \( i, j \neq 3 \).

Thanks to the equations \( 6.0.2 \) we observe that neither \( \varphi_{1,s} \) nor \( \varphi_{3,s} \) are always elements of \( C \). In particular, they will be elements of \( C \) if and only if they satisfy the equations
Proposition 7.4. where gously, it can be proved the result for \( \phi \).

\[
[X_0, \varphi(Y_i, Y_3, Y_3)] = \varphi(Y_{i+1}, Y_3, Y_3), \quad \text{with } 1 \leq i \leq 3.
\]

By induction it can be proved the following formula for \( \varphi_{1,s} \) and \( \varphi_{3,s} \):

\[
\varphi_{1,s} = \begin{cases} 
\varphi_{1,s}(Y_1, Y_1, Y_1) = X_s \\
\varphi_{1,s}(Y_1, Y_2, Y_2) = \frac{1}{6}X_{s+2} \\
\varphi_{1,s}(Y_2, Y_2, Y_2) = \frac{1}{3}X_{s+3} \\
\varphi_{1,s}(Y_1, Y_3, Y_3) = -\frac{1}{18}X_{s+4} \\
\varphi_{1,s}(Y_2, Y_3, Y_3) = \frac{1}{30}X_{s+5} \\
\varphi_{1,s}(Y_3, Y_3, Y_3) = \frac{1}{54}X_{s+6}
\end{cases}
\]

\[
\varphi_{3,s} = \begin{cases} 
\varphi_{3,s}(Y_1, Y_1, Y_1) = X_s \\
\varphi_{3,s}(Y_1, Y_2, Y_2) = -\frac{1}{3}X_s \\
\varphi_{3,s}(Y_1, Y_3, Y_3) = \frac{1}{18}X_{s+1} \\
\varphi_{3,s}(Y_2, Y_2, Y_2) = -\frac{1}{6}X_s \\
\varphi_{3,s}(Y_2, Y_3, Y_3) = -\frac{1}{18}X_{s+2} \\
\varphi_{3,s}(Y_3, Y_3, Y_3) = X_{s+2}
\end{cases}
\]

where \( \varphi_{1,s}(Y_3, Y_3, Y_3) \) and \( \varphi_{3,s}(Y_3, Y_3, Y_3) \) have been completed in a natural way from \( \varphi_{1,s}(Y_2, Y_3, Y_3) \) and \( \varphi_{3,s}(Y_2, Y_3, Y_3) \) respectively. Also, we suposete that if \( s+i > n \) then \( X_{s+i} = 0 \).

Proposition 7.3. The symmetric multi-linear maps \( \varphi_{1,s} \) and \( \varphi_{1,s} \) defined above are elements of \( C \) iff

\[
p(\varphi_{1,s}) = s - 3 \geq n - 7 \quad \text{and} \quad p(\varphi_{3,s}) = s - 5 \geq n - 7
\]

Proof. We only have to check whether \( \varphi_{1,s} \) and \( \varphi_{3,s} \) satisfy or not the equations

\[
[X_0, \varphi(Y_i, Y_3, Y_3)] = \varphi(Y_{i+1}, Y_3, Y_3), \quad \text{with } 1 \leq i \leq 3.
\]

If \( p(\varphi_{1,s}) = n - 7 \), then \( s = n - 4 \) and \( \varphi_{1,s}(Y_1, Y_3, Y_3) = -\frac{1}{18}X_s \). Thus, \( \varphi_{1,s} = \varphi_{1,s}(Y_2, Y_3, Y_3) = \varphi_{1,s}(Y_3, Y_3, Y_3) = 0 \) which clearly satisfy the above equations. If \( p(\varphi_{1,s}) > n - 7 \), then \( \varphi_{1,s}(Y_1, Y_3, Y_3) = \varphi_{1,s}(Y_2, Y_3, Y_3) = \varphi_{1,s}(Y_3, Y_3, Y_3) = 0 \) and also satisfies the above equations.

If \( p(\varphi_{1,s}) < n - 7 \), then \( \varphi_{1,s}(Y_1, Y_3, Y_3) = -\frac{1}{18}X_{s+4} \) and \( \varphi_{1,s}(Y_2, Y_3, Y_3) = \varphi_{1,s}(Y_3, Y_3, Y_3) = \varphi_{1,s}(Y_3, Y_3, Y_3) = \frac{1}{30}X_{s+5} \) with \( s + 5 \leq n \). But from the equation

\[
[X_0, \varphi_{1,s}(Y_1, Y_3, Y_3)] = \varphi_{1,s}(Y_2, Y_3, Y_3)
\]

we would have that \( = \frac{1}{30} \) which clearly constitutes a contradiction. Analogously, it can be proved the result for \( \varphi_{3,s} \).

Proposition 7.4. Let \( \varphi \in C \) with weight \( p = p(\varphi) \leq n - 8 \). Then

\[
\varphi = a_1\varphi_{1,p+3} + a_3\varphi_{3,p+5}
\]

for some numbers \( a_k \).

Proof. Let \( \varphi \in C \) an infinitesimal deformation with weight \( p \). Then \( \varphi(Y_1, Y_1, Y_1) = a_1X_{p+3} \) and \( \varphi(Y_1, Y_1, Y_3) = a_3X_{p+5} \). We are going to consider the difference

\[
\Psi = \varphi - a_1\varphi_{1,p+3} + a_3\varphi_{3,p+5}
\]

It is easy to check that \( \Psi \) is a symmetric multi-linear map such that

\[
\Psi(Y_1, Y_1, Y_1) = \Psi(Y_1, Y_1, Y_3) = 0.
\]

As \( \varphi_{1,s} \) and \( \varphi_{3,s} \) satisfy the equations \( \varphi_{1,s} \) satisfies them too, it is not difficult to see that \( \Psi \) vanishes which proves the result. \( \square \)

Without lose of generality we can consider \( a_1 = 1 \), then we have
**Proposition 7.5.** If we define $\varphi_{1,3,s}$ by $\varphi_{1,3,s} = \varphi_{1,s} + A\varphi_{3,s+2}$ and we consider $p = s - 3 \leq n - 8$, then $\varphi_{1,3,s}$ is an infinitesimal deformation of $C$ iff $A = \frac{1}{15}$ and $p \geq n - 9$.

**Proof.** As $\varphi_{1,s}$ and $\varphi_{3,s}$ satisfy the equations $\eqref{22.1}$, $\varphi_{1,3,s}$ satisfies too. On the other hand, by construction $\varphi_{1,s}$ and $\varphi_{3,s}$ verify too the equation

$$[X_0, \varphi(Y_2, Y_3, Y_3)] = \varphi(Y_3, Y_3, Y_3)$$

and so $\varphi_{1,3,s}$. Thus, $\varphi_{1,3,s}$ will be an infinitesimal deformation belonging to $C$ iff it verifies the equation

$$[X_0, \varphi_{1,3,s}(Y_1, Y_3, Y_3)] = \varphi_{1,3,s}(Y_2, Y_3, Y_3)$$

which leads to $-\frac{1}{15} + A = \frac{1}{15} - \frac{1}{5}$, that is $A = \frac{1}{15}$. Finally, as $\varphi_{1,3,s}(Y_3, Y_3, Y_3) = \frac{1}{15} X_{s+6}$ for not to obtain a contradiction it is necessary that $s + 6 = p + 9 \geq n$. In fact, if $s + 6 < n$ then $[X_0, \varphi_{1,3,s}(Y_3, Y_3)] = 0 = \frac{1}{15} X_{s+7}$ which is a contradiction. □

**Remark 7.6.** From now on, we will consider $A = \frac{1}{15}$ and thus $\varphi_{1,3,s}$ would be

$$\varphi_{1,3,s} = \begin{cases} \varphi_{1,3,s}(Y_1, Y_1, Y_2) = X_s, & \varphi_{1,3,s}(Y_1, Y_1, Y_3) = \frac{1}{15} X_{s+2} \\ \varphi_{1,3,s}(Y_1, Y_1, Y_2) = \frac{1}{3} X_{s+1}, & \varphi_{1,3,s}(Y_1, Y_2, Y_2) = \frac{1}{15} X_{s+2} \\ \varphi_{1,3,s}(Y_2, Y_2, Y_2) = \frac{1}{3} X_{s+3}, & \varphi_{1,3,s}(Y_1, Y_3, Y_3) = \frac{1}{3} X_{s+3} \\ \varphi_{1,3,s}(Y_2, Y_2, Y_3) = \frac{1}{3} X_{s+4}, & \varphi_{1,3,s}(Y_1, Y_3, Y_3) = \frac{1}{3} X_{s+4} \\ \varphi_{1,3,s}(Y_2, Y_3, Y_3) = \frac{1}{3} X_{s+5}, & \varphi_{1,3,s}(Y_3, Y_3, Y_3) = \frac{1}{3} X_{s+6} \end{cases}$$

Thanks to the precedent results we can give a basis of $C$.

**Theorem 4.** If $C = Z(\mu_0) \cap \text{Hom}(S^3 \mathfrak{g}_1, \mathfrak{g}_0)$ and $m = 3$ and $n$ is odd, then we have the following vector basis of $C$

- $\{\varphi_{1,1}, \varphi_{3,1}\}$ if $n = 1$
- $\{\varphi_{1,3}, \varphi_{1,2}, \varphi_{1,1}, \varphi_{3,3}, \varphi_{3,2}, \varphi_{3,1}\}$ if $n = 3$
- $\{\varphi_{1,5}, \varphi_{1,4}, \varphi_{1,3}, \varphi_{1,2}, \varphi_{1,1}, \varphi_{3,5}, \varphi_{3,4}, \varphi_{3,3}\}$ if $n = 5$
- $\{\varphi_{1,n}, \varphi_{1,n-1}, \varphi_{1,n-2}, \varphi_{1,n-3}, \varphi_{1,n-4}, \varphi_{3,n}, \varphi_{3,n-1}, \varphi_{3,n-2}, \varphi_{1,3,n-5}, \varphi_{1,3,n-6}\}$ if $n \geq 7$

**Remark 7.7.** As we have already noted before, for any $\psi \in C$ we will have a filiform elementary Lie algebra of order 3: $\mu_0 + \psi$.

8. OTHER FAMILIES OF FILIFORM ELEMENTARY LIE ALGEBRAS OF ORDER 3

In this section we are going to give some others families of filiform elementary Lie algebras of order 3 by considering families of infinitesimal deformations $\psi$ that verify $\psi \circ \psi = 0$ (linear integrable).

To find infinitesimal deformations integrable we search in those of type $C$. Thus, we consider a family of maps, $\{\psi\}_k$ with $k \geq 1$ that verifies:

$$\psi_k(Y_k, Y_m, Y_m) = X_1 \text{ and } \psi_k(Y_i, Y_j, Y_l) = 0, \quad \forall (i, j, l) / (i, j, l) \leq (k, m, m)$$
with \( \leq \) the lexicographic order. As \( \psi \) is symmetric we consider \( i \leq j \leq k \) and any other reordering will have the same value by symmetry.

By applying the conditions to be an infinitesimal deformation, i.e.

\[
[X_0, \psi_k(Y_i, Y_m, Y_m)] = \psi_k(Y_{i+1}, Y_m, Y_m), \quad k \leq i \leq m - 1
\]

we have the final expression for \( \{\psi\}_k \) with \( k \geq 1 \):

\[
\psi_k = \begin{cases} 
\psi_k(Y_{k+i}, Y_m, Y_m) = X_{1+i} & 0 \leq i \leq \min\{n-1, m-k\} \\
\psi_k(Y_i, Y_j, Y_l) = 0 & \text{in any other case}
\end{cases}
\]

Therefore we have the following Proposition

**Proposition 8.1.** The family of elementary Lie algebras of order 3, in \( \mathcal{L}_{n,m} \), that follows

\[
\{\mu_0 + \psi_k\}_k \quad 1 \leq k \leq m
\]

is a family of filiform elementary Lie algebras of order 3.

**Remark 8.2.** Recall the expression of the model filiform elementary Lie algebra of order 3, \( \mu_0 \), that is the simplest filiform elementary Lie algebra. It is defined in an adapted basis \( \{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \) by the following non-null bracket products

\[
\mu_0 = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1
\end{cases}
\]

and note that in the Proposition above we always have \( n > 1 \), that is, \( g_0 =< X_0, X_1, \ldots, X_n > \) is an authentic filiform Lie algebra, because for \( n \leq 1 \) \( g_0 \) is an abelian Lie algebra.

Next, we present another family of infinitesimal deformations integrable of \( \mu_0, \psi_t \). Then \( \mu_0 + \psi_t \) will be a family of filiform elementary Lie algebras.

**Proposition 8.3.** The family of elementary Lie algebras of order 3, in \( \mathcal{L}_{n,m} \) with \( n \geq m \), that follows

\[
\{\mu_0 + \psi_t\}_t \quad n - m \leq t \leq n
\]

with

\[
\psi_t = \begin{cases} 
\psi_t(Y_1, Y_1, Y_1) = 3X_t \\
\psi_t(Y_1, Y_1, Y_{1+i}) = X_{t+i} & 1 \leq i \leq n - t \\
\psi_t(Y_i, Y_j, Y_l) = 0 & \text{in any other case}
\end{cases}
\]

is a family of filiform elementary Lie algebras of order 3.
FILIFORM LIE ALGEBRAS OF ORDER 3

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