Numerical solution of unsteady state fractional advection–dispersion equation

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1. Introduction

Fractional calculus is known to have played a very important role recently in various fields of science. In fact, it has many advantages over integer order models, including nonlinear ones, which in many cases do not work adequately. Mostly, fractional operators using various approaches turn to be nonlocal in nature describing long term memory effects and asymptotic scaling. Although most of them are already well-studied, some of the usual features concerning the differentiation of functions fail, like the Leibniz rule, quotient rule, chain rule, the semigroup property, to name a few. These inconsistencies lead to constantly evolving the fractional derivative so that the best choice of definitions of fractional operators depends on empirical data that fit best in the theoretical model, and for this we find already a vast number of definitions for fractional derivative, including Caputo, Atangana, Riemann–Liouville, Hadamard, Caputo–Fabrizio, Grünwald derivative, and some others. Thus, it becomes unnecessary to worry about the definition of the function away from the point of interest. There are many of these definitions in the literature nowadays, but few of them are commonly used. The most popular ones are the Riemann–Liouville and the Caputo derivatives. It is very important to point out that all these fractional derivatives definitions have their advantages and disadvantages. Although these fractional derivative display great advantages, they are not applicable in all the situations. For example, the Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. The Riemann–Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag–Leffler functions. Theses disadvantages reduce the field of application of the Riemann–Liouville fractional derivative. Caputo’s derivative demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense. Typically, one deals with functions admitting some non-integrable singularities on a discrete set of isolated points located at finite distances from the origin. For the past 300 years, researchers have been studying fractional calculus, which generalizes the integration and differentiation of integer order to arbitrary order. Because of their non-local behaviour, fractional differential equations are well suited to describe various phenomena in the fields of engineering and science, and researchers’ growing interest in this field has led to the solution of real-world problems in this field. Furthermore, fractional derivatives can be used to...
mathematically model (Ali, Osman, Baskonus, Elazabb, & Ilhan, 2020; Alshabanat, Jeli, Kumar, & Samet, 2020; Gao, Veeresha, Prakash, & Baskonus, 2022; Ghanbari, Kumar, & Kumar, 2020; Kumar, Chauhan, Osman, & Mohiuddine, 2021; Mohammadi, Kumar, Rezapour, & Etemad, 2021; Veeresha, Prakash, & Kumar, 2020) a variety of processes that exhibit memory and hereditary properties. The Newell–Whitehead–Segel and Allen–Cahn equations provide analytical and numerical solutions to mathematical biology models (Inan, Osman, Ak, & Baleanu, 2020). The application of the generalized differential transform method (Ramani, Khan, & Suthar, 2019b) to solve nonlinear partial differential equations with fractional space and time derivatives. The singularly perturbed boundary value issues for semilinear reaction-diffusion equations were explored in Yamac and Erdogan (2020). The Tikhonov regularization method (Djennadi et al., 2021) is used to solve the inverse source problem of the time fractional heat equation using the ABC-fractional technique. Using the reproducing kernel discretization method (Omar, Osman, Abdel-Aty, Mohamed, & Momani, 2020), a numerical algorithm for the solutions of ABC singular Lane–Emden type models arising in astrophysics. Now recently published papers on a novel numerical method, study (Ali, Abd El Salam, et al., 2020; Arslan, 2020; Mistry, Khan, Suthar, & Kumar, 2019; Mundewadi & Kumbinarasaih, 2019; Rahaman, Kamruel Hasan, Ali, & Shamsul Alam, 2021; Sabir Wahab, Javeed, & Baskonus, 2021) for solving non-linear fractional differential equations and generalized nonlinear fractional integro-differential equations with linear functional arguments.

The advection–dispersion equation (ADE) is used in the study of solute transport (Ghosh, Kundu, & Kumar, 2021) or Brownian motion of particles in a fluid that occurs when advection and particle dispersion occur at the same time. In anomalous diffusion (Kumar, Ghosh, Jeli, & Araci, 2020), the solute transport is faster or faster than the time’s inferred square root given by Baeumer, Benson, and Meerschaert (2005). Fractional ADE better characterizes the phenomenon of anomalous diffusion of particles in transport processes. Advection–dispersion describes how pollutants flow through groundwater at seepage velocity, with Advection referring to the movement of contaminants with accompanying groundwater and Dispersion referring to a process triggered by velocity changes. The preservation of mass (Doungmo Goufo, Kumar, & Mugisha, 2020; Saw & Kumar, 2018) principle is used to investigate the ADE for solute transport. As a result, many scientists have been interested in fractional ADE. As a result, researchers are interested in solving the FADE in order to determine the solute concentration at a specific time and space. Pareek, Gupta, Agarwal, and Suthar (2021) investigated the use of the natural transform in conjunction with the HPM approach to solve fractional ADE.

Now a days, the usage of spectral and Haar wavelet methods (Bhrawy, 2014; Kumar, Ghosh, Kumar, & Jeli, 2021; Kumar, Kumar, Agarwal, & Samet, 2020; Kumar, Kumar, Osman, & Samet, 2021; Xiao-Yong & Junlin, 2015) is growing rapidly. While using spectral collocation methods for solving a certain problem, we can have a boost in our solution; since these methods are global in nature, have high accuracy, and exponential rate of convergence, for further reading in spectral methods (Bhrawy, Zaky, & Baleanu, 2015; Boyd, 2001; Canuto et al., 2006; Diethelm, 2010; Tatari & Haghighi, 2014). The primary goal of this work is to solve the space FADE using the Laguerre collocation method and the finite difference scheme.

The present paper is structured as follows: we began with classical definitions, concepts and notations of fractional calculus in Section 2; consequently, in Section 3, we introduced fundamental properties of the proposed method i.e. Laguerre spectral collocation method. In Section 4, we have given numerical examples to testify the efficiency of our method. Finally, some concluding remarks are presented in Section 5.

2. Preliminaries and notations

Definition 1. The Caputo fractional derivative operator $D^\gamma$ of order $\gamma$ is given by Kilbas, Srivastava, and Trujillo (2006) as follows:

$$D^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x f^{(n)}(t) (x-t)^{\gamma-n+1} dt, \quad (n-1 < \gamma \leq n, n \in \mathbb{N}, x > 0).$$

(1)

Some of the properties for the fractional derivative in Caputo sense are provided by:

$$D^\gamma (\gamma_1 p(x) + \gamma_2 q(x)) = \gamma_1 D^\gamma p(x) + \gamma_2 D^\gamma q(x),$$

(2)

where $\gamma_1$ and $\gamma_2$ are constants, and the notation $\lceil \gamma \rceil$ denotes the smallest integer greater than or equal to $\gamma$, and $\mathbb{N}_0 = \{0,1,2,...\}$.

$$D^\gamma C = 0,$$

(3)

$C$ is a constant.

$$D^\gamma x^\sigma = \begin{cases} 0, & \text{for } \sigma \in \mathbb{N}_0, \text{ and } \sigma < \lceil \gamma \rceil, \\ \Gamma(\lceil \sigma + 1 \rceil) x^{\sigma-\gamma}, & \text{for } \sigma \in \mathbb{N}_0 \text{ and } \sigma \geq \lceil \gamma \rceil. \end{cases}$$

(4)

The primary goal of this work is to find an approximation of an unsteady state space fractional ADE with source $f(x, t)$ defined on (5) using the idea of Laguerre collocation method.

$$\frac{\partial u(x, t)}{\partial t} = \lambda \frac{\partial^2 u(x, t)}{\partial x^2} - \mu \frac{\partial^\gamma u(x, t)}{\partial x^\gamma} + f(x, t),$$

(0 < x < 1, 0 < t \leq T),

(5)
where $1 < \gamma \leq 2, \ 0 < \eta \leq 1$, $\mu$ is the average fluid velocity and $\lambda$ belongs to the dispersion coefficient.

### 3. The properties and fractional derivatives of Laguerre polynomials

The generalized Laguerre polynomial $\mathcal{L}_m^{(\nu)}(x)_{m=0}^{\infty}$ $\nu > -1$ defined on the interval $[0, \infty)$ and can be expressed by the recurrence relation (see also Khader, Talaat, Danaf, & Hendy, 2012) as:

$$(n + 1)\mathcal{L}_m^{(\nu)}(x) + (x - 2n - \nu - 1)\mathcal{L}_{m-1}^{(\nu)}(x) + (n + \nu)\mathcal{L}_{m-1}^{(\nu)}(x) = 0, \quad n = 1, 2, 3, \ldots$$

with $\mathcal{L}_0^{(\nu)}(x) = 1$ and $\mathcal{L}_1^{(\nu)}(x) = x + \nu - 1 - x$.

The explicit formula of degree ‘$n$’ Laguerre polynomial is:

$$\mathcal{L}_n^{(\nu)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n + \nu}{n} \right) \sum_{i=0}^{n-k} \frac{(-n)^i}{(v+1)^i} x^k,$$

i.e., $(\nu)_0 = 1, (\nu)_k = \nu(\nu+1)(\nu+2) \ldots (\nu+k-1), \ k = 1, 2, 3, \ldots$ and $\mathcal{L}_n^{(\nu)}(0) = \left( \frac{n + \nu}{n} \right)$.

These degree ‘$n$’ Laguerre polynomials with respect to the weight function $\phi(x) = \frac{1}{(1+x)^\nu} x^me^{-x}$ are orthogonal on the interval $[0, \infty)$ and its orthogonality relation is defined as:

$$\frac{1}{\Gamma(1+\nu)} \int_0^\infty x^m e^{-x} \mathcal{L}_m^{(\nu)}(x) \mathcal{L}_n^{(\nu)}(x) dx = \left( \frac{n + \nu}{n} \right) \delta_{mn}.$$

### Error analysis

**Theorem 3.1.** Let $\gamma > 0$ and $u_m(x)$ be an approximate function based on Laguerre polynomials, then:

$$D^\gamma (u_m(x)) \equiv \sum_{i=0}^{m} \sum_{k=0}^{i} \omega_i \psi_{i,k} x^{i-k},$$

where

$$\psi_{i,k} = \frac{(-1)^k}{\Gamma(k+1-\gamma)} \left( \frac{i + \nu}{i - k} \right).$$

**Proof.** Based on linearity property of Caputo’s fractional derivative, we arrive at:

$$D^\gamma (u_m(x)) = \sum_{i=0}^{m} \omega_i D^\gamma \left( \mathcal{L}_i^{(\nu)}(x) \right).$$

Using Equations (3) and (4), we get:

$$D^\gamma \left( \mathcal{L}_i^{(\nu)}(x) \right) = 0, i = 0, 1, \ldots, [\gamma] - 1, \gamma > 0.$$

Also, for $i = [\gamma], \ldots, m$, by using Equations (3) and (4), we get:

$$D^\gamma \left( \mathcal{L}_i^{(\nu)}(x) \right) = \sum_{k=0}^{i} \frac{(-1)^k}{k!} \left( \frac{i + \nu}{i - k} \right) D^\gamma \left( x^k \right)$$

From Equations (17)–(19):

$$D^\gamma (u_m(x)) = \sum_{k=0}^{m} \sum_{i=0}^{k} \omega_i \frac{(-1)^k}{\Gamma(k+1-\gamma)} \left( \frac{i + \nu}{i - k} \right) x^{i-k}.$$

Rearranging (20), we arrive at (15). $\square$

### 2.1. Error analysis

**Theorem 3.2.** Suppose the $(m+1)$ continuously differentiable function $h(x)$ is defined on the semi-infinite interval $[0, \infty)$ and $u_m(x) = \sum_{i=0}^{m} \omega_i \mathcal{L}_i^{(0.5)}(x)$ be best approximate function square of $h(x)$, then:

$$||h(x) - u_m(x)|| \leq \frac{MS^{m+1}}{(m+1)!}$$

Where $M = \max_{x \in [0, \infty]} h^{(m+1)}(x)$ and $S = \max_{x_0, x - x_0}$.

**Proof.** Consider the Taylor polynomial:

$$h(x) = h(x_0) + h'(x_0) \frac{(x - x_0)}{1!} + \cdots + h^{(m)}(x_0) \frac{(x - x_0)^m}{m!} + h^{(m+1)}(\xi) \frac{(x - x_0)^{m+1}}{(m+1)!},$$

where $x_0 \in [0, \infty)$ and $\xi \in [x_0, x]$. 

$$u_m(x) = \sum_{i=0}^{m} \omega_i \mathcal{L}_i^{(0.5)}(x).$$
We assume that
\[ p_m(x) = h(x_0) + h'(x_0) \frac{(x - x_0)}{1!} + \cdots + h^{(m)}(x_0) \frac{(x - x_0)^m}{m!}. \]  
Hence,
\[ ||h(x) - p_m(x)||^2 = \int_0^\infty (\phi(x)) ||h(x) - p_m(x)||^2 dx \]
\[ = \int_0^\infty \frac{\phi(x)}{(m + 1)!} \frac{(x - x_0)^m}{m!} dx \]
\[ \leq \frac{M^2}{[(m + 1)]^2} \int_0^\infty \frac{\phi(x)}{(x - x_0)^{2m+2}} dx \]
\[ = \frac{M^2}{[(m + 1)]^2} \int_0^\infty \frac{1}{\Gamma(0.5)} x^{-0.5} e^{-x} dx. \]

Taking square root both sides, we get
\[ ||h(x) - u_m(x)|| \leq \frac{M^{m+1}}{(m + 1)!}. \]

4. Numerical examples

Example 1. Take a look at the space fractional ADE below.
\[ \frac{\partial u(x,t)}{\partial t} = \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} - \frac{\partial^\eta u(x,t)}{\partial x^\eta} + f(x,t), \]
\[ (0 < x < 1, \ 0 < t \leq T, \ 1 < \gamma \leq 2, \ 0 < \eta \leq 1), \]
\[ u(0,t) = u(1,t) = 0, \]
\[ u(x,0) = x^\gamma - x^\eta. \]

For comparison, the exact solution of our proposed problem is
\[ u(x,t) = e^{-2t}(x^\gamma - x^\eta). \]

**Table 1.** The approximate solution, exact solution and their comparison using \( T = 0.4, \ m = 4, \ \gamma = 2, \ \eta = 1, \ \Delta t = 0.0005. \)

| x   | Exact | Laguerre | Error |
|-----|-------|----------|-------|
| 0   | 0     | 1.25e-15 | 1.25e-15 |
| 0.1 | -4.04e-02 | -4.04e-02 | 6.51e-06 |
| 0.2 | -7.19e-02 | -7.19e-02 | 1.43e-05 |
| 0.3 | -9.44e-02 | -9.44e-02 | 2.22e-05 |
| 0.4 | -1.08e-01 | -1.08e-01 | 2.93e-05 |
| 0.5 | -1.12e-01 | -1.12e-01 | 3.44e-05 |
| 0.6 | -1.09e-01 | -1.09e-01 | 3.66e-05 |
| 0.7 | -9.44e-02 | -9.44e-02 | 3.51e-05 |
| 0.8 | -7.19e-02 | -7.19e-02 | 2.89e-05 |
| 0.9 | -4.04e-02 | -4.05e-02 | 1.75e-05 |
| 1   | 0     | -5.28e-09 | 5.28e-09 |

To use the Laguerre collocation method, we approximate it with \( m = 4. \)
\[ u_4(x,t) = \sum_{i=0}^{4} u_i(t) L_i^{(4)}(x). \]

Adopting Theorem 3.1 and Equation (29) on (25), we get:
\[ \sum_{i=0}^{4} \frac{d u_i(t)}{dt} L_i^{(4)}(x_p) \]
\[ = \sum_{i=0}^{4} \sum_{k=0}^{i} u_i(t) \psi_k^{(i)} x_p^{k-\gamma} + \sum_{i=0}^{4} \sum_{k=0}^{i} u_i(t) \psi_k^{(i)} x_p^{k-\eta} + f(x_p,t), \]
\[ \quad p = 0, 1, 2, \]
\[ u_0(t) + R_1 u_1(t) + R_2 u_2(t) + R_3 u_3(t) + R_4 u_4(t) = G_1 u_1(t) + G_2 u_2(t) + G_3 u_3(t) + G_4 u_4(t) + f_1(t), \]
\[ u_0(t) + K_1 u_1(t) + K_2 u_2(t) + K_3 u_3(t) + K_4 u_4(t) = H_1 u_1(t) + H_2 u_2(t) + H_3 u_3(t) + H_4 u_4(t) + f_1(t), \]
\[ u_0(t) + S_1 u_1(t) + S_2 u_2(t) + S_3 u_3(t) + S_4 u_4(t) = T_1 u_1(t) + T_2 u_2(t) + T_3 u_3(t) + T_4 u_4(t) + f_2(t), \]
\[ u_0(t) + (v+1) u_1(t) + \frac{1}{2} (v+2)(v+1) u_2(t) + \frac{1}{6} (v+3)(v+2)(v+1) u_3(t) + \frac{1}{24} (v+4)(v+3)(v+2)(v+1) u_4(t) = 0, \]
\[ u_0(t) + L_1^{(4)}(1) u_1(t) + L_2^{(4)}(1) u_2(t) + L_3^{(4)}(1) u_3(t) + L_4^{(4)}(1) u_4(t) = 0, \]

such that:
\[ R_1 = L_1^v(x_0), R_2 = L_2^v(x_0), R_3 = L_3^v(x_0), R_4 = L_4^v(x_0), \]
\[ G_1 = -\psi_{1,1}^v x_0^{-1}, \]
\[ G_2 = \left\{ \psi_{2,2}^v x_0^{-3} - \left( \psi_{2,2}^v x_0^{-1} + \psi_{2,2}^v x_0^{-2} \right) \right\}, \]
\[ G_3 = \left\{ \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} - \left( \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} + \psi_{3,3}^v x_0^{-1} \right) \right\}, \]
\[ G_4 = \left\{ 1 - \psi_{4,4}^v x_0^{-4} - \psi_{4,4}^v x_0^{-3} + \psi_{4,4}^v x_0^{-2} - \left( \psi_{4,4}^v x_0^{-2} + \psi_{4,4}^v x_0^{-1} \right) \right\}, \]
\[ K_1 = L_1^v(x_1), K_2 = L_2^v(x_1), K_3 = L_3^v(x_1), K_4 = L_4^v(x_1), \]
\[ H_1 = -\psi_{1,1}^v x_0^{-1}, H_2 = \left\{ \psi_{2,2}^v x_0^{-2} - \left( \psi_{2,2}^v x_0^{-1} + \psi_{2,2}^v x_0^{-2} \right) \right\}, \]
\[ H_3 = \left\{ \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} - \left( \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} + \psi_{3,3}^v x_0^{-1} \right) \right\}, \]
\[ H_4 = \left\{ 1 - \psi_{4,4}^v x_0^{-4} - \psi_{4,4}^v x_0^{-3} + \psi_{4,4}^v x_0^{-2} - \left( \psi_{4,4}^v x_0^{-2} + \psi_{4,4}^v x_0^{-1} \right) \right\}, \]
\[ S_1 = L_1^v(x_2), S_2 = L_2^v(x_2), S_3 = L_3^v(x_2), S_4 = L_4^v(x_2), \]
\[ T_1 = -\psi_{1,1}^v x_0^{-1}, T_2 = \left\{ \psi_{2,2}^v x_0^{-2} - \left( \psi_{2,2}^v x_0^{-1} + \psi_{2,2}^v x_0^{-2} \right) \right\}, \]
\[ T_3 = \left\{ \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} - \left( \psi_{3,3}^v x_0^{-3} + \psi_{3,3}^v x_0^{-2} + \psi_{3,3}^v x_0^{-1} \right) \right\}, \]
\[ T_4 = \left\{ 1 - \psi_{4,4}^v x_0^{-4} - \psi_{4,4}^v x_0^{-3} + \psi_{4,4}^v x_0^{-2} - \left( \psi_{4,4}^v x_0^{-2} + \psi_{4,4}^v x_0^{-1} \right) \right\}, \]
\[ a_1 = 1 - \psi_{1,1}^v x_0^{-1}, a_2 = \frac{1}{2} (v + 2)(v + 1), a_3 = \frac{1}{2} (v + 3)(v + 2)(v + 1), a_4 = \frac{1}{24} (v + 4)(v + 3)(v + 2)(v + 1), \]
\[ b_1 = L_1^v(1), b_2 = L_2^v(1), b_3 = L_3^v(1), b_4 = L_4^v(1), \]

Consider \( T = T_{\text{final}}, 0 < t_m < T \) and \( \Delta t = T/N, t_m = m\Delta t, \) for \( m = 0, 1, 2, \ldots, N. \) Now, in order to solve Equations (31)–(35). We discretize the first derivatives using the idea of finite difference method (FDM).

\[
\frac{u_0^m - u_0^{m-1}}{\Delta t} + R_1 \frac{u_1^m - u_1^{m-1}}{\Delta t} + R_2 \frac{u_2^m - u_2^{m-1}}{\Delta t} + R_3 \frac{u_3^m - u_3^{m-1}}{\Delta t} + R_4 \frac{u_4^m - u_4^{m-1}}{\Delta t} = G_1 u_1(t) + G_2 u_2(t) + G_3 u_3(t) + G_4 u_4(t) + f_0(t),
\]

\[
\frac{u_0^m - u_0^{m-1}}{\Delta t} + K_1 \frac{u_1^m - u_1^{m-1}}{\Delta t} + K_2 \frac{u_2^m - u_2^{m-1}}{\Delta t} + K_3 \frac{u_3^m - u_3^{m-1}}{\Delta t} + K_4 \frac{u_4^m - u_4^{m-1}}{\Delta t} = H_1 u_1(t) + H_2 u_2(t) + H_3 u_3(t) + H_4 u_4(t) + f_1(t),
\]

\[
\frac{u_0^m - u_0^{m-1}}{\Delta t} + S_1 \frac{u_1^m - u_1^{m-1}}{\Delta t} + S_2 \frac{u_2^m - u_2^{m-1}}{\Delta t} + S_3 \frac{u_3^m - u_3^{m-1}}{\Delta t} + S_4 \frac{u_4^m - u_4^{m-1}}{\Delta t} = T_1 u_1(t) + T_2 u_2(t) + T_3 u_3(t) + T_4 u_4(t) + f_2(t),
\]

\[
\frac{u_0^m + a_1 u_1^m + a_2 u_2^m + a_3 u_3^m + a_4 u_4^m}{\Delta t} = 0;
\]

\[
\frac{u_0^m + b_1 u_1^m + b_2 u_2^m + b_3 u_3^m + b_4 u_4^m}{\Delta t} = 0.
\]

Re-arranging Equations (36)–(40), we arrive at the following matrix form:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & a_1 & b_1 \\
R_1 - \Delta t G_1 & R_2 - \Delta t G_2 & R_3 - \Delta t G_3 & R_4 - \Delta t G_4 & 1 & b_1 \\
1 - \Delta t H_1 & K_2 - \Delta t H_2 & K_3 - \Delta t H_3 & K_4 - \Delta t H_4 & 1 & b_2 \\
S_1 - \Delta t T_1 & S_2 - \Delta t T_2 & S_3 - \Delta t T_3 & S_4 - \Delta t T_4 & 1 & b_3 \\
1 & a_2 & a_3 & a_4 & 1 & b_4 \\
\end{pmatrix}
\begin{pmatrix}
u_0^m \\
u_1 \\
u_2 \\
u_3 \\
u_4 \\
\end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 & a_1 & b_1 \\
R_1 & R_2 & R_3 & R_4 & a_1 & b_1 \\
K_1 & K_2 & K_3 & K_4 & a_2 & b_2 \\
S_1 & S_2 & S_3 & S_4 & a_3 & b_3 \\
1 & a_4 & 1 & b_4 \\
\end{pmatrix}
\begin{pmatrix}
\nu_0^{m-1} \\
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\end{pmatrix}
+ \Delta t
\begin{pmatrix}
f_0^m \\
f_1 \\
f_2 \\
f_3 \\
f_4 \\
\end{pmatrix}.
\]

Or, it can be written as:

\[
AU^m = BU^{m-1} + F^m, A^{-1}AU^m = A^{-1}BU^{m-1} + A^{-1}F^m \iff U^m = A^{-1}BU^{m-1} + A^{-1}F^m,
\]

where \( U^m = \begin{pmatrix}
u_0^m & \nu_1^m & \nu_2^m & \nu_3^m & \nu_4^m
\end{pmatrix}^T \) and \( F^m = \begin{pmatrix}
\Delta t f_0^m & \Delta t f_1^m & \Delta t f_2^m & 0 & 0
\end{pmatrix}^T. \)
Using the Laguerre collocation points $L_{m+1-\gamma}$, we find the unknown coefficients (Table 1)

\[ x_0 = 0.1902, x_1 = 1.7845, x_2 = 5.5253 \]

Here, we represent approximate solution of Example 1 for different values of parameters in Figures 1–9.
Example 2. Find an approximate solution for the given unsteady state space fractional ADE subject to the initial and boundary conditions.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} - 2 \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad (0 < x < 1, \ 0 < t \leq T), \quad (42)$$

where $f(x,t) = x(x-1)(2t-1) + 2t(t-1) \ (2x - 1) - \frac{4\sqrt{\pi t(t-1)}}{\sqrt{\pi}}$, subject to

$$u(x,0) = 0, \quad (43)$$

$$u(0,t) = u(1,t) = 0. \quad (44)$$

For comparison, the exact solution of our proposed problem is $u(x,t) = xt(x-1)(t-1)$ (Table 2).

Here, we represent approximate solution of Example 2 for parameters in Figure 11.

5. Conclusion

In this study, we used the Laguerre collocation method to solve a time-dependent fractional order ADE, which is described in a Caputo sense. To better illustrate our method, we provided two examples, and the numerical results, as well as its numerical simulation, show that the proposed method has high agreement and suitability with the exact solution. Furthermore, it is easy to see that the accuracy of the method is also dependent on the order of the proposed partial differential equation. Matlab2020a’s
high accuracy algorithm is used to generate all tabular expressions and numerical simulations.

The proposed approach could be used to issues like two-dimensional fractional Zakharov, Cable and Schrödinger equations. Two-sided space-time Caputo PDEs subject to Dirichlet, Robin, and/or non-local conditions could also be considered. This is one area where further research could be conducted.

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Authors’ contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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