Optimization of Controlled Free-Time Sweeping Processes with Applications to Marine Surface Vehicle Modeling

Tan H. Cao\(^1\), Nathalie T. Khalil\(^2\), Boris S. Mordukhovich\(^3\), Dao Nguyen\(^3\), Trang Nguyen\(^3\), Fernando Lobo Pereira\(^2\)

Abstract—The paper is devoted to a free-time optimal control problem for sweeping processes. We develop a constructive finite-difference approximation procedure that allows us to establish necessary optimality conditions for discrete optimal solutions and then show how these optimality conditions are applied to solving a controlled marine surface vehicle model.

I. INTRODUCTION AND PROBLEM FORMULATION

The sweeping process was introduced by Moreau in the 1970s (see [12]) in the form of the differential inclusion form
\[
\begin{cases}
  \dot{x}(t) \in -N(x(t); C) & \text{a.e. } t \in [0, T], \\
  x(0) = x_0 \in C \subset \mathbb{R}^n.
\end{cases}
\]

The sweeping process and its modifications have been extensively studied and applied to various fields including aerospace engineering, process control, robotics, bioengineering, chemistry, biology, economics, finance, management science, and engineering. Moreover, the sweeping dynamics play an important role in the theory of variational inequalities and complementarity problems. Among applications in mechanical and electrical engineering, we mention mechanical impact, Coulomb friction, diodes and transistors, queues and resource limits, etc.; see, e.g., the recent survey in [2].

Optimal control problems for various types of sweeping processes have been formulated much more recently (see [7] and the references therein), while being realized as very challenging control theory due to high discontinuity of the controlled sweeping dynamics and the unavoidable presence of hard state constraints. Nevertheless, within a rather short period of time, many important results have been obtained on necessary optimality conditions for controlled sweeping processes with valuable applications to friction and plasticity, robotics, traffic equilibria, ferromagnetism, hysteresis, economics, and other fields of engineering and applied sciences; see, e.g., [1], [3], [7], [8], [17] with more references and discussions. Let us mention to this end the recent papers [15], [16], where optimal control problems for linear complementarity systems have been studied and applied to practical models that are highly important in the area of Automatic Control. Such problems can be written in a form of controlled sweeping processes, where $C$ is an orthant in $\mathbb{R}^n$. However, there are many great unsolved problems in optimal control theory for sweeping processes with strong requirements for further applications. Some of these issues, from both viewpoints of theory and applications, are addressed in this paper.

Here, we consider the following free-time optimal control problem labeled as $(P)$: minimize the Mayer-type cost functional which depends explicitly on the final time
\[
J[x, u, T] := \varphi(x(T), T)
\]
over control functions $u(\cdot)$ and the corresponding trajectories $x(\cdot)$ satisfying the system
\[
\begin{cases}
  \dot{x}(t) \in -N(x(t); C) + g(x(t), u(t)) & \text{a.e. } t \in [0, T], \\
  x(0) = x_0 \in C \subset \mathbb{R}^n, \\
  u(t) \in U \subset \mathbb{R}^d & \text{a.e. } t \in [0, T],
\end{cases}
\]
where the set $C$ is a convex polyhedron given by
\[
C := \bigcap_{j=1}^s C^j
\]
with $C^j := \{x \in \mathbb{R}^n \mid (x^j_1, x^j_2) \leq c_j\}$, (2) with $N(x; C)$ standing for the normal cone of convex analysis. From (1) we automatically have the state constraints $x(t) \in C$, i.e., $(x^j_1, x^j_2) \leq c_j$ for all $t \in [0, T]$ (with different $T$) and $j = 1, \ldots, s$.

Note that defining state constraints implicitly via the domain of the normal cone in (1) is significantly different from the formulation of pure state constraints in standard control theory. In what follows, we identify the arc $x : [0, T] \to \mathbb{R}^n$ with its extension to $(0, \infty)$ defined by
\[
x_\epsilon(t) := x(T) \text{ for all } t > T.
\]

Given $x(\cdot) \in W^{1,2}([0, T], \mathbb{R}^n)$ with the norm
\[
\|x\|_{W^{1,2}} := \|x(0)\| + \|\dot{x}\|_{L^2},
\]
we specify the notion of local minimizers studied below. For simplicity, suppose that the set $g(x; U)$ is convex, which actually does not much restrict the generality; cf. [7].

Definition 1.1: A feasible solution $(\tilde{x}(\cdot), \tilde{u}(\cdot), T)$ for (P) is a $W^{1,2} \times L^2$-local minimizer to this problem if there exists $\epsilon > 0$ such that $J[\tilde{x}, \tilde{u}, T] \leq J[x, u, T]$ for all feasible solutions $(x(\cdot), u(\cdot), T)$ satisfying the constraints of (P) and
\[
\int_0^T \left( \|\dot{x}(t) - \tilde{x}(t)\|^2 + \|u(t) - \tilde{u}(t)\|^2 \right) dt + (T - T)^2 < \epsilon.
\]
Our approach to investigate problem (P) in order to establish necessary optimality conditions for its local minimizers is based on the method of discrete approximations developed in [10], [11] for Lipschitzian differential inclusions and then extended in [3], [4], [5], [6], [7], [8] to various kinds of controlled sweeping processes. This method consists of constructing well-posed discrete approximations of (P) whose solutions strongly converge to the prescribed local minimizers of (P), then deriving necessary optimality conditions for discrete-time problems, and finally establishing by passing to the limit, with the discretization step decreasingly converging to zero, necessary optimality conditions for local minimizers of (P). Due to the text size limitation, we present here only optimality conditions for discrete approximations, which give us sufficient information to solve some applied optimal control problems that arising in marine surface vehicle modeling and control.

The rest of the paper is organized as follows. In Section II we present the standing assumptions on the problem data. Section III reviews the tools of variational analysis used below. Section IV is devoted to the formulation of necessary optimality conditions for discrete approximations of (P). A proof outline with the key ideas is given in Section V. Section VI contains applications of the obtained necessary optimality condition to the controlled marine surface vehicle model with providing numerical calculations in typical settings for two marine surface vehicles. We finish with Section VII that contains concluding remarks and discussions of some topics of our future research.

II. STANDING ASSUMPTIONS

First we present the following standing assumptions:

(H1) The set \( U \neq \emptyset \) is closed and bounded in \( \mathbb{R}^d \).

(H2) The positive linear independence constraint qualification (PLICQ) holds at \( x(t) \) on \( [0, T] \) with varying \( T \):

\[
\left[ \sum_{j \in I(x)} \lambda_j x^i_j = 0, \lambda_j \in \mathbb{R}_+ \right] \implies \lambda_j = 0 \text{ for all } j \not\in I(x),
\]

where the active index set \( I(x) \), \( x \in C \), is defined by

\[
I(x) := \left\{ j \in \{1, \ldots, s\} \mid \langle x^i_j, x \rangle = c_j \right\}.
\]

(H3) The perturbation mapping \( g : \mathbb{R}^n \times U \to \mathbb{R}^n \) is Lipschitz continuous with respect to \( x \) uniformly on \( U \) whenever \( x \) belongs to a bounded subset of \( \mathbb{R}^n \) and satisfies there the sublinear growth condition

\[
\|g(x, u)\| \leq \beta (1 + \|x\|) \text{ for all } u \in U
\]

with some positive constant \( \beta \).

Define the set-valued mapping \( F : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n \) by

\[
F(x, u) := N(x; C) - g(x, u)
\]

deduce from the theorem of the alternative that

\[
F(x, u) = \left\{ \sum_{i \in I(x)} \lambda^i x^i_\ast \mid \lambda^i \geq 0 \right\} - g(x, u).
\]

III. TOOLS OF VARIATIONAL ANALYSIS

Let us recall the tools of variational analysis employed below; see [11] and [13] for more details. The (Painlevé-Kuratowski) outer limit of a set-valued mapping/multifunction \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) at \( x \) with \( F(x) \neq \emptyset \) is

\[
\text{Lim sup } F(x) := \{ y \in \mathbb{R}^m \mid \exists \text{ sequences } x_k \to x, y_k \to y \text{ such that } y_k \in F(x_k), k \in \mathbb{N} \}.
\]

The (basic, limiting, Mordukhovich) normal cone to subsets \( \Omega \subset \mathbb{R}^n \) that are locally closed around \( \bar{x} \) is given by

\[
N(x; \Omega) = N_{\Omega}(\bar{x}) := \text{Lim sup } \{ \text{cone}(x - \Pi(x; \Omega)) \}, \quad (6)
\]

where \( \Pi(x; \Omega) := \{ u \in \Omega \mid \|x - u\| = \text{dist}(x; \Omega) \} \) is the Euclidean projection of \( x \) to \( \Omega \), and where “cone” stands for the (generally nonconvex) conic hull of the set.

We consider a set-valued mapping \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) locally closed around a graph point \((\bar{x}, \bar{y}) \in gph F\). The coderivative of \( F \) at \((\bar{x}, \bar{y})\) is defined by

\[
D^* F(\bar{x}, \bar{y})(u) := \{ v \in \mathbb{R}^n \mid \langle v, -u \rangle \in N((\bar{x}, \bar{y}); \text{gph } F) \},
\]

where \( u \in \mathbb{R}^m \). If \( F : \mathbb{R}^n \to \mathbb{R}^m \) is single-valued and smooth around \( \bar{x} \), then we have

\[
D^* F(\bar{x})(u) = \{ \nabla F(\bar{x})^* u \}
\]

for all \( u \in \mathbb{R}^m \), where \( \nabla F(x)^* \) is the adjoint/transposed Jacobian matrix, and where \( \bar{y} = F(\bar{x}) \) is omitted.

Given an extended-real-valued and lower semicontinuous function \( \varphi : \mathbb{R}^n \to (-\infty, \infty] \) with \( \varphi(\bar{x}) < \infty \), the subdifferential of \( \varphi \) at \( \bar{x} \in \text{dom } \varphi \) is defined via the normal cone (6) to the epigraph of \( \varphi \) by

\[
\partial \varphi(\bar{x}) := \{ v \in \mathbb{R}^m \mid \langle v, -1 \rangle \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi) \}.
\]

These normal cone, coderivative, and subdifferential enjoy full calculus, which can be found in [11], and [13].

IV. NECESSARY OPTIMALITY CONDITIONS

We first construct a sequence of discrete approximation problems with a varying grid whose optimal solutions strongly converge in \( W^{1,2} \times L^2 \) to a prescribed local minimizer \( \{\bar{x}(\cdot), \bar{u}(\cdot), \overline{T}\} \) of the original problem (P). For simplicity we replace the derivative \( \dot{x}(t) \) in (1) by

\[
\dot{x}(t) \approx \frac{x(t + h) - x(t)}{h}
\]

as \( h \downarrow 0 \).

Whenever \( k \in \mathbb{N} \), take \( T_k \) close to \( T \) and set the grid

\[
\left\{ \begin{array}{l}
t_0^k = 0, \quad t_k^k = T_k, \\
t_{k+1}^k = t_k^k + h^k, \quad i = 0, \ldots, k - 1.
\end{array} \right. \quad (7)
\]

Let \( (\bar{x}(\cdot), \bar{u}(\cdot), \overline{T}) \) be a \( W^{1,2} \times L^2 \)-local minimizer of problem (P) for the differential inclusion (1). Define the discrete approximation problem (P_k) by:

\[
\begin{align*}
\text{minimize } & J_k[x^k, u^k, T_k] := \varphi(x_k^k, T_k) + (T_k - \overline{T})^2 + \\
& \sum_{i=0}^{k-1} \int_{t_i^k}^{t_{i+1}^k} \left( \left\| \frac{x_{i+1}^k - x_i^k}{h_i^k} - \dot{x}(t) \right\|^2 + \left\| u_i^k - \bar{u}(t) \right\|^2 \right) \, dt.
\end{align*}
\]
over \((x^k, u^k, T_k) := (x^k_0, x^k_1, \ldots, x^k_{k-1}, u^k_0, u^k_1, \ldots, u^k_{k-1}, T_k)\) satisfying the following constraints:

\[ x^k_{i+1} - x^k_i \in -h^k_i F(x^k_i, u^k_i) \quad \text{for } i = 0, \ldots, k - 1, \]

\[ x^k_0 = \bar{x}_0 \in C, \quad u^k_0 := \bar{u}(0), \]

\[ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left( \frac{x^k_{i+1} - x^k_i}{h^k_i} - \dot{x}(t) \right)^2 + \left( u^k_i - \bar{u}(t) \right)^2 \, dt \leq \epsilon, \]

\[ u^k_i \in U \quad \text{for } i = 0, \ldots, k - 1, \quad |T_k - \bar{T}| \leq \epsilon, \]

\[ \| (x^k_i, u^k_i) - (\bar{x}^i_k, \bar{u}(t_i)) \| \leq \epsilon \quad \text{for } i = 0, \ldots, k - 1, \]

\[ \langle x^k_j, x^k_k \rangle \leq c_j \quad \text{for } j = 1, \ldots, s, \]  

(8)

where \(\epsilon > 0\) as in Definition 1.1. In each problem \((P_k)\), the final time \(T_k\) and the discretization step \(h^k_i\) are variable for any fixed \(k \in \mathbb{N}\).

Now, we present necessary optimality conditions for \((P_k)\) expressed entirely via the given data. Having \(I(x)\) from (3) and \(y \in \mathbb{R}^n\), consider \(j \in I(x)\) for which

\[ I_0(y) := \{ j \mid x^j_0, y \} = c_j \quad \text{and} \quad I_2(y) := \{ j \mid x^j_0, y \geq c_j \}. \]

**Theorem 4.1:** Let \((x^k(\cdot), u^k(\cdot), T_k)\) be an optimal solution to problem \((P_k)\), where the cost function \(\varphi\) is locally Lipschitzian around \((x^k(T_k), T_k)\) in addition to the standing assumptions. Then, there exist dual elements \((\mu^k_0, q^k, p^k)\) together with vectors \(v^k_i \in \mathbb{R}^n\) for \(i = 0, \ldots, k\), and \(\gamma^k_i \in \mathbb{R}^s\) for \(i = 0, \ldots, k - 1\) satisfying the following:

1. **The nontriviality conditions.** In general we have

\[ \mu^k_0 + \| v^k_i \| + \sum_{i=0}^{k-1} \| p^k_i \| + \| q^k \| \neq 0. \]

If the matrices \(\nabla u g(x^k_i, u^k_i)\) are of full rank as \(i = 0, \ldots, k - 1\), the enhanced nontriviality condition holds:

\[ \mu^k_0 + \| v^k_i \| + \| p^k_i \| + \| q^k \| \neq 0. \]

2. **The primal-dual relationship.**

   a. **The primal arc representation**

\[ -\frac{x^k_{i+1} - x^k_i}{h^k_i} + g(x^k_i, u^k_i) = \sum_{j \in I(t^k_i)} \eta^k_{ij} x^j_i. \]

   b. **The adjoint dynamic systems**

\[ \frac{p^k_{i+1} - p^k_i}{h^k_i} = -\nabla^*_x g(x^k_i, u^k_i) + \gamma^k_i x^k_i + \sum_{j \in I(t^k_i)} \eta^k_{ij} x^j_i, \]

where

\[ \mathcal{I} := I_0 \left( \frac{\mu^k_0}{h^k_i} + p^k_i \right) \cup I_2 \left( \frac{\mu^k_0}{h^k_i} + p^k_i \right), \]

\[ -\frac{p^k_0}{h^k_i} - \sum_{j=1}^s \eta^k_{ij} x^j_i \in -h^k_i F(x^k_i, u^k_i) \quad \text{for } i = 0, \ldots, k - 1, \]

\[ \frac{\mu^k_0}{h^k_i} + p^k_i \in \mathcal{I}, \quad \mu^k_0 > 0. \]

\[ \partial \left( \mu^k_0 \varphi \right)(x^k(T_k), T_k), \]

where we use the notation

\[ H^k := \frac{1}{k} \sum_{i=0}^{k-1} (p^k_{i+1}, y^k_i), \]

\[ \varrho^k := \frac{1}{k} \sum_{i=0}^{k-1} \frac{i}{|i|} \left( \frac{x^k_{i+1} - x^k_i}{h^k_i} - \dot{x}(t_i), \bar{u}(t_i) - \bar{u}(t_i) \right)^2, \]

\[ \xi^k_i := \left( \xi^k_{ti}, \xi^k_{ty} \right) \text{ with } \xi^k_{ti} := \int_{t_i}^{t_{i+1}} \left( \bar{u}^k_t - \bar{u}(t) \right) dt \]

and \(\xi^k_{ty} := \int_{t_i}^{t_{i+1}} \left( \bar{u}^k_t - \bar{u}(t) \right) dt\).

3. **The local maximum principle:** \(q^k_i \in N(\bar{u}^k_i; U)\) as \(i = 0, \ldots, k - 1\) with

\[ -\frac{\mu^k_0 \xi^k_{ti}}{h^k_i} - \frac{q^k_i}{h^k_i} = -\nabla u g(x^k_i, u^k_i) \left( -\frac{\mu^k_0 \xi^k_{ti}}{h^k_i} + p^k_i \right), \]

which yields the linearized global form of the maximum principle when \(U\) is convex.

4. **The complementarity slackness conditions.** The following implications hold:

\[ \left( [x^k_i, x^k_k] \right) \implies \eta^k_{ij} = 0, \]

\[ \{ j \in I_0 \left( -\frac{\mu^k_0 \xi^k_{ti}}{h^k_i} + p^k_i \right) \} \implies \gamma^k_{ij} \geq 0, \]

\[ \{ j \notin I \} \implies \gamma^k_{ij} = 0, \]

\[ \langle x^k_i, x^k_k \rangle \leq c_j \implies \gamma^k_{ij} = 0, \]

\[ \langle x^k_i, x^k_k \rangle < c_j \implies \gamma^k_{ij} = 0. \]

where \(i = 0, \ldots, k - 1\) and \(j = 1, \ldots, s\).

Finally, imposing the linear independence of the vectors \([x^k_i] \mid j \in I(x^k_i)\) ensures the implication

\[ \eta^k_{ij} > 0 \implies \left( [x^k_i, \frac{\mu^k_0 \xi^k_{ti}}{h^k_i} + p^k_i \right) = 0. \]

V. **Brief outline of key ideas of the proof**

For any fixed \(k \in \mathbb{N}\) and \(\epsilon > 0\), consider the following problem \((P_k)\) with respect to variables \(z := (x^k_0, x^k_1, \ldots, x^k_{k-1}, y^k_0, \ldots, y^k_{k-1}, \theta)\):

\[ \text{minimize } \phi_0(z) := \varphi(x^k_0, \theta) + (\theta - T)^2 + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left( y^k_i - \bar{u}(t), u^k_i - \bar{u}(t) \right)^2 dt \quad \text{s.t.} \]
that the constraint $D_{ij} (x(t_k + h)) \geq 0$ should be satisfied. To verify this, employ the first-order Taylor expansion at $x_k \neq 0$ and reduce the constraint on the velocity vector to $$D_{ij} (x(t_k + h)) = D_{ij} (x(t_k)) + h \nabla D_{ij} (x(t_k)) \dot{x}(t_k) + o(h)$$ for small $h > 0$. This constraint will be used to construct the next configuration in order to avoid the collision with static and/or dynamic obstacles. In this regard, set the vector $V(x)$ to be the desired velocity of all USVs. The admissible velocities preventing collisions during the navigation of USVs are defined as $$C_h (x) := \{ V(x) \in \mathbb{R}^m | D_{ij} (x) + h \nabla D_{ij} (x) V(x) \geq 0 \}$$ for all $i, j \in \{1, \ldots, n\}, i \neq j$ and $x \in \mathbb{R}^m$. Taking the admissible velocity $\dot{x}(t_k) \in C_h (x_k)$ gives us $$D_{ij} (x_k) + h(\nabla D_{ij} (x_k), \dot{x}(t_k)) \geq 0.$$ Skipping the term $o(h)$ for small $h$, we deduce that $D_{ij} (x(t_k + h)) \geq 0$, i.e., $x(t_k + h) \in A$.

In the absence of obstacles, the desired velocities are given by $V(x) = g(x) \in \mathbb{R}^m$. In the presence of obstacles, the algorithm in [9] seeks for optimal velocities to escape from surrounding obstacles by solving the following convex constrained optimization problem:

$$\text{minimize} \quad \|g(x, u) - V(x)\|^2 \quad \text{s.t.} \quad V(x) \in C_h (x),$$

(9)

where the control $u$ is involved into the desired velocity term to adjust the actual velocities of the USVs and make sure that they do not overlap. The velocities can be modeled as $$g(x(t), u(t)) = (s_1 \|u_1(t)\| \cos \theta_1^i (t), s_1 \|u_1(t)\| \sin \theta_1^i (t), \ldots, s_n \|u_n(t)\| \cos \theta_n^i (t), s_n \|u_n(t)\| \sin \theta_n^i (t)),$$

where $s_i$ denotes the speed of the USV $i$, and $\theta_i^i$ stands for the corresponding constant direction which is the smallest positive angle in standard position formed by the positive $x$-axis and vectors $u^i(t)$, with practically motivated control constraints represented by $$u(t) = (u^1(t), \ldots, u^n(t)) \in U \quad \text{for a.e.} \quad t \in [0, T],$$

(10)

where the control set $U \subset \mathbb{R}^n$ will be specified below.

The algorithmic design in (9) means that $V_k + 1$ is selected as the unique element from the set of admissible velocities as the one closest to the desired velocity $g(x, u)$ while avoiding overlapping. Consequently, the proposed scheme seeks for new directions $V(x)$ of USVs close to the desired direction $g(x, u)$ in order to bypass the surrounding obstacles. The desired position of the next configuration of marine vessels is generated as $x_{ref}(t + h) = (x_{ref}, y_{ref}) = x(t) + hV(x)$ and the desired via-point posture position of the marine craft is

$$\eta_{ref} = (x_{ref}, y_{ref}, \psi_{ref}), \quad \psi_{ref} = \tan^{-1} \left( \frac{y_{ref} - y_i(t)}{x_{ref} - x_i(t)} \right).$$

To proceed, for any $k$ consider $T_k$ close to $T$ and the grid as in (7) with $x^k_i := x^i_{k+1}$ for $i = 1, \ldots, k$. Denote $t^k_0 := 0,$
of the convex polyhedron. Consider the cost functional that can be treated as a continuous-time counterpart of the sweeping process. The perturbed sweeping process. The controlled perturbed sweeping process. The controlled model under consideration here is significantly more involved. For all $x(t) \in \mathbb{R}^{2n}$, define the set

$$K(x(t)) := \{ y(t) \in \mathbb{R}^{2n} \mid D_{ij}(x(t)) + \nabla D_{ij}(x(t))(y(t) - x(t)) \geq 0 \text{ whenever } i < j \},$$

which allows us to represent the algorithm in (11), (12) as $x^{k}_{i+1} = \Pi(x^{k}_{i} + h^{k}_{i}g(x^{k}_{i}, u^{k}_{i}); K(x^{k}_{i}))$ for $ i = 0, \ldots, k - 1$. It can be equivalently rewritten in the form

$$x^{k}(\tilde{t}^{k}(t)) = \Pi(x^{k}(\tau^{k}(t)) + h^{k}_{k}g(x^{k}(\tau^{k}(t)), u^{k}(\tau^{k}(t))); K(x^{k}(\tau^{k}(t))))$$

for all $t \in [0, T]$ with $\tau^{k}(t) := t^{k}_{i}$ and $\tilde{t}^{k}(t) := t^{k}_{i+1}$ for all $t \in I^{k}_{i}$. Taking into account the construction of $K(x)$ in (14) together with (13), we arrive at the sweeping inclusions

$$\dot{x}^{k}(t) \in -N(x^{k}(\tilde{t}^{k}(t)); K(x^{k}(\tilde{t}^{k}(t)))) + g(x^{k}(\tilde{t}^{k}(t)), u^{k}(\tilde{t}^{k}(t))) \quad \text{a.e. } t \in [0, T],$$

where $x^{k}(0) = x_{0} \in K(x_{0}) = A$ and $x^{k}(\tilde{t}^{k}(t)) \in K(x^{k}(\tilde{t}^{k}(t)))$ on $[0, T]$. To formalize (15) as a controlled perturbed sweeping process, define the convex polyhedron

$$C := \bigcap \{ x \in \mathbb{R}^{2n} \mid \langle x_{j}^{*}, x \rangle \leq c_{j}, j = 1, \ldots, n - 1 \}$$

with $c_{j} := -(R_{USV} + R_{obs})$, where $R_{USV}, R_{obs}$ are the radii of the considered USV and the obstacle, respectively, and the $n - 1$ vertices of the polyhedron

$$x^{*}_{j} := e_{j1} + e_{j2} - e_{(j+1)1} - e_{(j+1)2}, j = 1, \ldots, n - 1,$$

where $e_{kl} := (e_{11}, e_{12}, e_{21}, e_{22}, \ldots, e_{k1}, e_{k2}, \ldots, e_{n1}, e_{n2}) \in \mathbb{R}^{2n}$, $k = 1, \ldots, n$ and $l = 1, 2$, with 1 at only one position of $e_{kl}$ and 0 at all other positions.

Let us formulate the *sweeping optimal control problem* (P) that can be treated as a continuous-time counterpart of the discrete algorithm to optimize the *controlled marine surface vehicle model*. Consider the cost functional

$$\min J[x, u, T] := \frac{1}{2} \| x(T) \|^{2},$$

which reflects the model goal to minimize the distance and the time of the USV from the admissible configuration set to the target. We describe the continuous-time dynamics by the controlled sweeping process

$$\begin{cases}
-\dot{x}(t) \in N(x(t); C) + g(x(t), u(t)), \\
x(0) = x_{0} \in C, u(t) \in U \text{ a.e. } t \in [0, T],
\end{cases}$$

where $C$ is taken from (16), the control constraints reduce to (10), and the dynamic nonoverlapping condition $\| x(t) - x(t) \| \geq R_{i} + R_{j}$ is equivalent to the pointwise state constraints

$$x(t) \in C \iff \langle x_{j}, x(t) \rangle \leq c_{j}, t \in [0, T], j = 1, \ldots, n - 1.$$

Now, we present the applications of the optimality conditions from Theorem 4.1 to the sweeping optimal control problem in (17) and (18) with two moving marine crafts MC 1 and MC 2. The marine surface vehicles are represented by triangle shapes immersed in discs (see Fig. 1). The objective is to move MC 1 and MC 2 to the target without colliding with each other. However, in the presence of MC 2, after the contacting time $t^{*}$ the vehicle USV 1 pushes MC 2 to the target with the same velocity. The mathematical USV's model is taken from the physical ship called Cyber-Ship [14] with the mass 23.8 kg and the length 1.255 m.

The initial configuration (positions of MC 1 and MC 2) is $x(0) = (x_{1}(0), x_{2}(0))$, and the target is the origin. The radii of the discs used in this model are $R_{1} = R_{2} = 3.5$ m. Then, we have the model in (17) and (18) with the data

$$\begin{align*}
&n = 2, x_{a} = (1, 1, -1, -1), c = -7, \\
g(x, u) := u, \varphi(x, T) := \frac{1}{2} \| x(T) \|^{2}, \\
&U := \{ u = (u^{1}, u^{2}) \in \mathbb{R}^{2} \mid u^{1} \in [-2, 2]; u^{2} \in [-2, 2] \}, \\
x_{1}(0) = (-25, -25), x_{2}(0) = (-15, -15), \\
&\theta_{0} := \theta_{1} = \theta_{2} = 45^\circ, s_{1} = s_{2} = 1.
\end{align*}$$

The set $C$ in (18) is described by

$$C = \{ x \in \mathbb{R}^{4} \mid \langle x_{a}, x \rangle \leq c \} = \{ x \in \mathbb{R}^{4} \mid x^{11} + x^{12} - x^{21} - x^{22} \leq -7 \} = \{ x \in \mathbb{R}^{4} \mid |x^{21} - x^{11}| + |x^{22} - x^{12}| \geq 7 \}$$

(under the imposed assumptions $x^{21} > x^{11}$ and $x^{22} > x^{12}$) for all $t \in [0, T]$.

The structure of the problem suggests that the object only changes its velocity when it hits the boundary at some time $t_{c}$ with $c \in \{ 0, 1, \ldots, k \}$. Moreover, if $t_{c} < T$, the object slides on the boundary of $C$ for the whole interval $[t_{c}, T]$. In this case, by construction of $t_{c}$, it must be one of the mesh points $t_{k}^{i}$ of some partition $\Delta_{k}$ in Theorem 4.1. It is easy to see that all the assumptions of Theorem 4.1 are satisfied for (19), and we can employ the obtained necessary optimality conditions, where the superscript “$k$” is dropped, and where $g_{k}$, $(\xi_{iv}, \xi_{ov})$ are supposed to be 0 for large $k$ due to the convergence of discrete optimal solutions. Applying all the conditions in Theorem 4.1 and using calculations in
MATLAB, we get that the USV reaches the target at the minimum ending time $T \approx 26.003$, and the hitting time is $t_c \approx 7.8201$. In this way we arrive at the optimal velocity $(\tilde{u}^1, \tilde{u}^2) \approx (1.67547, 0.49999)$ and the optimal trajectory on $[0,T]$ with the different expressions before and after the contacting time:

$$\begin{align*}
\begin{cases}
\tilde{x}^1(t) &\approx (-25 + 1.18474t, -25 + 1.18474t), \\
\tilde{x}^2(t) &\approx (-15 + 0.35355t, -15 + 0.35355t),
\end{cases}
\text{for } t \in [0, 7.8201],
\end{align*}
$$

and

$$\begin{align*}
\begin{cases}
\tilde{x}^1(t) &\approx (-21.7500 + 0.76914t, -21.7500 + 0.76914t), \\
\tilde{x}^2(t) &\approx (-18.2500 + 0.76914t, -18.2500 + 0.76914t),
\end{cases}
\text{for } t \in [7.8201, 26.003].
\end{align*}$$

VII. Concluding Remarks

In this paper we formulated and studied a new class of optimal control problems governed by free-time controlled sweeping processes, where the duration of the process is also included to optimization. Developing the method of discrete approximations and using the generalized differential tools of variational analysis, we derive efficient necessary conditions for discrete optimal solutions that approximate a prescribed local minimizer of the continuous-time problem. The obtained results are applied to optimizing a controlled version of the marine surface vehicle model with static and dynamic obstacles, which is formulated in this paper based on the sweeping dynamics.

In our future research, we intend to furnish the limiting procedure of deriving necessary optimality conditions for the free-time continuous sweeping dynamics and provide further applications (qualitative and algorithmic) to more general versions of the controlled marine surface vehicle model dealing with many vessels.

Acknowledgment

T.H. Cao acknowledges the support of the National Research Foundation of Korea grant funded by the Korea Government (MIST) NRF-2020R1F1A1A01071015.

N.T. KHALIHI AND F.L. PEREIRA acknowledge the support of FCT R&D Unit SYSTEC-POCI-01-0145-FEDER-006933 funded by ERDFCOMPETE2020FCT/MECPT2020. Project STRIDE-NORTE-01-0145-FEDER-000033 funded by ERDFNORTE 2020, and Project MAGIC-POCI-01-0145-FEDER-032485 funded by FEDER-COMPETE2020-POCI and PIDDAC through FCT/MCTES.

B.S. Mordukhovich, D. Nguyen, and T. Nguyen acknowledges the support of the US National Science Foundation under grants DMS-1007132 and DMS-1512846, by the US Air Force Office of Scientific Research grant #15RT0462.

References

[1] C. E. Arroud and G. Colombo, A Maximum Principle of the Controlled Sweeping Process, Set-Valued Var. Anal., vol. 26, 2018, pp 607–629.
[2] B. Brogliato and A. Tanwani, Dynamical Systems Coupled with Monotone Set-Valued Operators: Formalisms, Applications, Well-Posedness, and Stability, SIAM Rev., vol. 62, 2020, pp 3–129.
[3] T.H. Cao and B.S. Mordukhovich, Optimal Control of a Nonconvex Perturbed Sweeping Process, J. Diff. Eqns., vol. 266, 2019, pp 1003–1050.
[4] T.H. Cao and B.S. Mordukhovich, Applications of Optimal Control of a Nonconvex Sweeping Processes to Optimization of the Planar Crowd Motion Model, Discrete Contuin. Dyn. Syst., Ser. B, vol. 24, 2019, pp 4191–4216.
[5] G. Colombo, R. Henrion, N.D. Hoang and B.S. Mordukhovich, Optimal Control of the Sweeping Process over Polyhedral Controlled Sets, J. Diff. Eqns., vol. 60, 2016, pp 3397–3447.
[6] G. Colombo, B.S. Mordukhovich and D. Nguyen, Optimal Control of Sweeping Processes in Robotics and Traffic Flow Models, J. Optim. Theory Appl., vol. 182, 2019, pp 439–472.
[7] G. Colombo, B.S. Mordukhovich and D. Nguyen, Optimization of a Perturbed Sweeping Process by Discontinuous Controls, SIAM J. Control Optim., vol. 58, 2020, pp 2679–2709.
[8] M.d.R. de Pinho, M.M.A. Ferreira and G.V. Smirnov, Optimal Control Involving Sweeping Processes, Set-Valued Var. Anal., vol. 27, 2019, pp 523–548.
[9] R. Hedjar and M. Bounkhel, An Automatic Collision Avoidance Algorithm for Multiple Marine Surface Vehicles, Int. J. Appl. Math. Comput. Sci., vol. 29, 2019, pp 759–768.
[10] B.S. Mordukhovich, Discrete Approximations and Refined Euler-Lagrange Conditions for Differential Inclusions, SIAM J. Control Optim., vol. 33, 1995, pp 882–915.
[11] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications, Springer, Berlin; 2006.
[12] J.J. Moreau, On Unilateral Constraints, Friction and Plasticity, in: New Variational Techniques in Mathematical Physics (G. Capriz and G. Stampacchia, eds.), Proc. C.I.M.E. Summer Schools, Cremonese, Rome, 1974, pp 173–322.
[13] R.T. Rockafellar and R.J-B. Wets, Variational Analysis, Springer, Berlin; 1998.
[14] R. Skjetne, T.I. Fossen abd P.V. Kokotovic, Adaptive Maneuvering, with Experiments, for a Model Ship in Marine Control Laboratory, Automatica, vol. 41, 2005, pp 289–298.
[15] A. Vieira, B. Brogliato and C. Prieur, Optimality Conditions for the Minimal Time Problem for Complementarity Systems, IFAC PapOnLine, vol. 52–16, 2019, pp 239–244.
[16] A. Vieira and B. Brogliato and C. Prieur, Quadratic Optimal Control of Linear Complementarity Systems: First order necessary conditions and numerical analysis, IEEE J. Automat. Control, vol. 65, 2020, pp 2743–2750.
[17] V. Zeidan, C. Nour and H. Saoud, A Nonsmooth Maximum Principle for a Controlled Nonconvex Sweeping Process, J. Diff. Eqns., vol. 269, 2020, pp 9531–9582.
This figure "CDC_B1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2105.03490v1
This figure "CDC_B2.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2105.03490v1