Abstract

We study the survival bandit problem, a variant of the multi-armed bandit problem with a constraint on the cumulative reward; at each time step, the agent receives a reward in $[-1, 1]$ and if the cumulative reward becomes lower than a preset threshold, the procedure stops, and this phenomenon is called ruin. To our knowledge, this is the first paper studying a framework where the ruin might occur but not always. We first discuss that no policy can achieve a sublinear regret as defined in the standard multi-armed bandit problem, because a single pull of an arm may increase significantly the risk of ruin. Instead, we establish the framework of Pareto-optimal policies, which is a class of policies whose cumulative reward for some instance cannot be improved without sacrificing that for another instance. To this end, we provide tight lower bounds on the probability of ruin, as well as matching policies called EXPLOIT. Finally, using a doubling trick over an EXPLOIT policy, we display a Pareto-optimal policy in the case of $\{-1, 0, 1\}$ rewards, giving an answer to the open problem by Perotto et al. (2019).

1. Introduction

The multi-armed bandit problem (MAB) (see, e.g., Lattimore and Szepesvari, 2020 or Bubeck and Cesa-Bianchi, 2012 for an introduction) is an online decision-making problem in which an agent sequentially selects one of $K$ unknown distributions called arms. More precisely, at each time step $t$, until a horizon $T$, the agent selects an arm $\pi_t \in [K] := \{1, \ldots, K\}$ and observes a reward drawn from a distribution associated to arm $\pi_t$, denoted $X_{\pi_t}^t$. The objective of the agent is to minimize the expected cumulative regret $\max_{k \in [K]} \mathbb{E} \left[ \sum_{t=1}^{T} X_k^t \right] - \mathbb{E} \left[ \sum_{t=1}^{T} X_{\pi_t}^t \right]$ as a function of the horizon $T$ and ideally to make it sublinear in $T$, where $\mathbb{E}$ denotes the expectation. This problem has been extensively studied and both a lower bound on the regret (see Lai and Robbins, 1985 and Burnetas and Katehakis, 1996) and policies matching this lower bound (see Honda and Takemura, 2010, Cappé et al., 2013 or more recently Riou and Honda, 2020) have been derived. The MAB illustrates a principle called the exploration-exploitation dilemma which consists of pulling all the arms many times in order to collect information on them (exploration) and at the same time, pulling more often the arm which seems to have a higher expectation (exploitation). Naturally, real life problems which involve such a dilemma are related to the MAB, including medicine testing (Villar et al., 2015, Aziz et al., 2021), advertising and recommender systems (Chapelle and Li, 2011), to name a few.

In many applications though, the agent has an initial budget $B$, and every time he observes a reward $X_{\pi_t}^t$, his budget increases (or decreases) by $X_{\pi_t}^t$. The agent plays until the horizon $T$, unless its budget reaches 0 before $T$ (in that case, he has to stop playing). An example of such an application
is in portfolio selection in finance, where recent works have demonstrated the strong performance of classic MAB strategies (Hoffman et al., 2011, Shen et al., 2015, Shen and Wang, 2016), and in particular of risk-aware MAB strategies (Huo and Fu, 2017). In the setting we consider, an investor has an initial budget $B$ to invest sequentially on $K$ securities (the arms). At every time step $t \leq T$, it selects a security $\pi_t$, receives a payoff $X_{\pi_t}^t$, and updates its budget by $X_{\pi_t}^t$. While an unsuccessful agent may blow its budget and stop playing after $O(B)$ time steps, a successful agent may play until the horizon $T$ and achieve a reward as large as $\Omega(T)$.

Another example is in clinical trials (Villar et al., 2015, Aziz et al., 2021). We are given $K$ medicines designed to cure one disease and $T$ infected patients to be sequentially treated. To each patient $t \leq T$, a doctor prescribes a medicine $\pi_t$ and observes $X_{\pi_t}^t$, the effect of medicine $\pi_t$ on patient $t$. However, in such a critical application, it is necessary to be able to stop the trials early in order to “reduce the number of patients exposed to the unnecessary risk of an ineffective investigational treatment and allow subjects the opportunity to explore more promising therapeutic alternatives”, as pointed out in Food and Administration (2019, p. 4). For example, we can decide to stop the procedure if the number of sick people exceeds the number of cured people by a certain value $B$ chosen in advance. In both examples given above, the procedure is formulated as a traditional MAB with the additional constraint that the agent has to stop at the time of ruin $\tau$, that is, the first time $\tau \leq T$ such that $B + \sum_{t=1}^\tau X_{\pi_t}^t \leq 0$ if such a $\tau$ exists.

In the bandit literature, variants of the MAB with a budget constraint have received considerable attention recently. In the settings considered, the horizon $T$ is replaced by a budget $B$, and at each time step, pulling an arm decreases the budget by a (positive) cost which may be different from the reward. Examples of such frameworks include the so-called budget-limited MAB (Tran-Thanh et al., 2012), the budgeted-MAB (Ding et al., 2013, Xia et al., 2017 or Zhou and Tomlin, 2018), the bandits with knapsacks (Badanidiyuru et al., 2013, Immorlica et al., 2019, Li et al., 2021), the conservative bandits (Wu et al., 2016), or MAB with cost subsidy (Sinha et al., 2021). In such frameworks, the ruin happens necessarily. This is in stark contrast with the applications described above, where it is crucial for the agent to minimize the risk of ruin.

Alternatively, a reasonable way to minimize the risk of ruin is to consider risk-averse strategies. The risk-averse bandits literature typically studies risk-averse measures of the performance as alternatives to the expectation, including the mean-variance (Sani et al., 2012, Vakili and Zhao, 2016, Zhu and Tan, 2020), functions of the expectation and the variance (Zimin et al., 2014), the moment-generating function at some parameter $\lambda$ (Maillard, 2013), the value at risk (Galichet et al., 2013), or other general measures encompassing the value at risk (Cassel et al., 2018). The objective then differs from the maximization of the cumulative reward, from the aforementioned applications.

The MAB problem with a risk of ruin is called the survival MAB (S-MAB). It is actually an open problem from the Conference on Learning Theory 2019 to define the problem, establish a (tight) bound on the best achievable performance, and provide policies which achieve that bound (see Perotto et al., 2019). To our knowledge, this paper is the first one to provide answers to this problem.

2. Problem Setup and Structure of the Paper

Let $T$ be the maximum time step, called the horizon, and let $K \geq 1$ be the number of arms, whose indices belong to the set $[K] := \{1, \ldots, K\}$. The arm distributions are denoted by $F_1, \ldots, F_K$ (of respective expectations $\mu_1, \ldots, \mu_K$) and are assumed to be bounded in $[-1, 1]$. We denote by $P_F$ the probability under $F = (F_1, \ldots, F_K)$ (and $P$ in the absence of ambiguity). Let $B > 0$ be an
initial budget. Note that we regard $B$ as a fixed problem parameter and do not have to assume it to be sufficiently large when the asymptotics in $T$ is considered.

The S-MAB procedure is as follows: at every time step $t \leq T$, (i) the agent selects an arm $\pi_t \in [K]$, then (ii) it observes a reward $X_{t,\pi_t}$ drawn from $F_{\pi_t}$, and eventually (iii) the agent updates its budget as $B_t := B_{t-1} + X_{t,\pi_t}$, where the initial budget is $B_0 = B$. If $B_t \leq 0$ at some $t \leq T$, the procedure stops. If such a time instant $t$ exists, it is called time of ruin and it is formally defined as

$$\tau(B, \pi) := \inf \left\{ t \geq 1 : B + \sum_{s=1}^{t} X_{s,\pi_s} \leq 0 \right\}.$$  

For any $k \in [K]$, we denote by $\tau(B, k)$ the time of ruin of the constant policy $\pi_t = k$ for any $t \geq 1$.

**Definition 1** The probability of survival (resp. of ruin) of a policy $\pi = (\pi_t)_{t \in \mathbb{N}}$ is defined as

$$P(\tau(B, \pi) = \infty) \text{ (resp. } P(\tau(B, \pi) < \infty)) \text{).}$$

In the classic MAB, the objective is to derive a policy $\pi$ maximizing the classic reward $\mathbb{E} \left[ \sum_{t=1}^{T} X_{t,\pi_t} \right]$. This notion extends to the S-MAB, where we define the cumulative reward of a policy $\pi$ as

$$\text{Rew}_T(\pi) := \mathbb{E} \left[ S_T \right] \text{ where } S_T = \sum_{t=1}^{T} X_{t,\pi_t} \mathbb{1}_{\tau(B, \pi) \geq t-1}.$$  

Please note that a policy with a high cumulative reward must also have a low probability of ruin, because in case of ruin, the agent receives no more rewards. For that reason, minimizing the probability of ruin is crucial even in the perspective of maximizing the reward.

Yet, if $B + \sum_{s=1}^{t} X_{s,\pi_s} > T - t$ for some $t \geq 1$, then there is no more risk of ruin and $\pi$ can perform exploration without risk. Hence, a policy $\pi^T$ aware of the time horizon $T$ has a significant advantage over one which is not. For that reason, we will consider, in all generalities, sequences of policies $\pi = (\pi_T)_{T \geq 1}$ where $\pi_T = (\pi^T_t)_{1 \leq t \leq T}$. In the absence of ambiguity, we will call “policy” such a sequence of policies. A sequence of policies $\hat{\pi} = (\hat{\pi}_T)_{T \geq 1}$ is called anytime if there exists $\pi$ such that $\hat{\pi}_T = \pi_t$ for any $1 \leq t \leq T$ and in that case, we identify the sequence $\hat{\pi}$ with the policy $\pi$.

**Definition 2** Given any policies $\pi$ and $\hat{\pi}$, the relative regret rate of $\pi$ with respect to $\hat{\pi}$ is defined as

$$\text{Reg}_T(\pi || \hat{\pi}) := \lim_{T \to \infty} \sup_{\pi_t} \frac{\text{Rew}_T(\hat{\pi}_T) - \text{Rew}_T(\pi_T)}{T}.$$  

Ideally, we would like to find a policy $\pi$ which is optimal in the worst case, in other words, such that $\sup_{\pi_t} \sup_{\pi'} \text{Reg}_T(\pi || \pi') = 0$. Such policies do exist in the standard MAB framework and a finer dependency on $T$ such as $O(\log T)$ has even been considered (see, e.g., Auer et al., 2002). On the other hand, in the S-MAB setting, no policy can achieve the above bound, as shown in the following proposition, whose proof is given in Appendix B.

**Proposition 3** For any policy $\pi$, $\sup_{\pi_t} \sup_{\pi'} \text{Reg}_T(\pi || \pi') > 0$.

The idea underlying Proposition 3 is that, for any $T \geq 1$, a single pull of a bad arm (e.g., with a negative expectation) will increase the probability of ruin of $\pi^T$, and decrease every subsequent term of the sum $\text{Rew}_T(\pi^T) = \sum_{t=1}^{T} \mathbb{E} \left[ X_{t,\pi_t} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right]$. The reward will then decrease by $\Omega(T)$, resulting in an increase of the regret rate. In such a case, a typical approach is to look for a Pareto-optimal policy, which is formalized below.
Algorithm 1: EXPLOIT-UCB-DOUBLE

**Input:** parameter $n \in \mathbb{N}$

```plaintext
j := 0; t₀ := 0.

for $t = 1, \ldots, T$ do
  if $B + \sum_{s=1}^{t-1} X_{π_s} > (j + 1)nB^2$ then
    Set $j := j + 1$ and then, set $t_j := t - 1$.
  
  Set $A_t := \left\{ k \in [K] : \sum_{s=j+1}^{t-1} X_{π_s} \mathbb{1}_{π_s=k} \geq -\frac{B + \sum_{s=1}^{j} X_{π_s}}{K} + 1 \right\}$;

  ∀$k \in [K]$, $N_k(t-1) := \sum_{s=1}^{t-1} \mathbb{1}_{π_s=k}$; $\hat{X}_{t-1}^k := \frac{1}{N_k(t-1)} \sum_{s=1}^{t-1} X_{π_s} \mathbb{1}_{π_s=k}$.

  if $A_t \neq \emptyset$ then
    Pull arm $\text{arg max}_{k \in A_t} \hat{X}_{t-1}^k + \sqrt{\frac{6 \log(t-1)}{N_k(t-1)}}$.
  else
    Pull arm $\text{arg max}_{k \in [K]} \hat{X}_{t-1}^k$.
```

### Definition 4

A policy $π$ is said to be (regret-wise) Pareto-optimal if, for any policy $π'$,

$$
\sup_{F} \text{Reg}_F(π, π') > 0 \implies \inf_{F} \text{Reg}_F(π, π') < 0.
$$

Intuitively, a policy $π$ is (regret-wise) Pareto-optimal if we cannot increase its cumulative reward for some arm distributions $F$ without decreasing it for some others. Finding a (regret-wise) Pareto-optimal policy is a challenging problem and it is the object of the following theorem, which is one of the two main results of this paper.

### Theorem 5

Assume that the arm distributions are multinomial of support $\{-1, 0, 1\}$. Then, the policy EXPLOIT-UCB-DOUBLE described in Algorithm 1 with the input parameter $n = \log T$ is (regret-wise) Pareto-optimal.

When dealing with MAB variants, most papers choose the standard approach to start from existing MAB algorithms such as UCB (Upper Confidence Bound, see, e.g., Auer et al., 2002) or Thompson Sampling (see, e.g., Agrawal and Goyal, 2013) and to adapt them to the new setting and objective. In this paper, we choose the original approach to start from the other objective in hand, the probability of ruin, find optimal policies with respect to this objective, and adapt them to our S-MAB problem. The rest of the paper is described as a pathway to EXPLOIT-UCB-DOUBLE, with some major results of independent interest paving the way.

### Structure of the Paper:

In Section 3, we derive the second main result of this paper: a tight non-asymptotic lower bound on the probability of ruin, which we relate to the probability of ruin of stochastic processes with i.i.d. increments. In Section 4, we introduce EXPLOIT, a framework of policies which achieve the lower bound on the probability of ruin but are unfortunately not regret-wise Pareto-optimal. In Section 5, we introduce the policy EXPLOIT-UCB-DOUBLE, which performs a doubling trick over an EXPLOIT policy and is regret-wise Pareto-optimal in the case of rewards in $\{-1, 0, 1\}$. In Section 6, we show the experimental performance of EXPLOIT-UCB-DOUBLE.
3. Study of the Probability of Ruin

As mentioned in the setup section, we start our study by analyzing the probability of ruin of anytime policies. In this section, we only consider anytime policies. First, we derive one of the two main results of our paper: a tight lower bound on the probability of ruin in the case of multinomial arm distributions of support \([-1, 0, 1]\) (Theorem 8), which we generalize to distributions bounded in \([-1, 1]\) (Corollary 9). We further relate this bound to the probability of ruin of i.i.d. random walks on \([-1, 0, 1]\) in Lemma 11.

In the case where there is an arm that only yields nonnegative rewards, the optimal bound \(P(\tau(B, \pi) < \infty) = 0\) is trivial. Therefore, in this section, we assume that there is no positive or zero arm, as defined below.

**Definition 6** An arm \(k \in [K]\) is called a zero arm (resp. a positive arm) if \(P_{X \sim F_k}(X = 0) = 1\) (resp. if \(P_{X \sim F_k}(X \geq 0) = 1\) and \(P_{X \sim F_k}(X > 0) > 0\)). We say that distribution \(F_k\) is zero (resp. positive) if arm \(k\) is zero (resp. positive).

3.1. Main Results on the Probability of Ruin

Let \(\mathcal{F}_{[-1,0,1]}\) (resp. \(\mathcal{F}_{[-1,1]}\)) be the set of multinomial distributions of support \([-1, 0, 1]\) (resp. of distributions bounded in \([-1, 1]\) a.s.) which are not positive or zero (see Definition 6). For any arm distributions \(F = (F_1, \ldots, F_K) \in \mathcal{F}_{[-1,1]}^K\), we define

\[
\gamma(F_k) := \inf_{Q: \mathbb{E}_X \sim Q[X] < 0} \frac{\text{KL}(Q||F_k)}{\mathbb{E}_X \sim Q[-X]} \geq 0. \tag{1}
\]

For any policies \(\pi, \pi'\), let \(P_{\text{ruin}}(\pi||\pi') := P_F(\tau(B, \pi) < \infty) - P_F(\tau(B, \pi') < \infty)\) be the relative ruin of \(\pi\) with respect to \(\pi'\) (where we omit the dependency on \(F\)). Similarly to Proposition 3, we can prove that no policy \(\pi\) achieves \(\sup_{\pi'} \sup_F P_{\text{ruin}}(\pi||\pi') \leq 0\), and for that reason, we focus on a policy which is Pareto-optimal in the sense of the probability of ruin, which is formalized as follows.

**Definition 7** A policy \(\pi\) is said to be ruin-wise Pareto-optimal if, for any policy \(\pi'\),

\[
\sup_F P_{\text{ruin}}(\pi||\pi') > 0 \implies \inf_F P_{\text{ruin}}(\pi||\pi') < 0.
\]

The main result of this section is a Pareto-type lower bound on the probability of ruin.

**Theorem 8** Let \((\alpha_k)_{k \in [K]}\) be such that for any \(k\), \(\alpha_k > 0\) and \(\sum_{k=1}^K \alpha_k = 1\). For any policy \(\pi\),

\[
\inf_{F \in \mathcal{F}_{[-1,0,1]}^K} \left\{ P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \infty) - \exp \left( -B \sum_{k=1}^K \alpha_k \gamma(F_k) \right) \right\} < 0 \implies \sup_{F \in \mathcal{F}_{[-1,0,1]}^K} \left\{ P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \infty) - \exp \left( -B \sum_{k=1}^K \alpha_k \gamma(F_k) \right) \right\} > 0. \tag{2}
\]

This result can be generalized to distributions bounded in \([-1, 1]\) as follows.
Corollary 9  Let \((\alpha_k)_{k \in [K]}\) be as in Theorem 8. For any policy \(\pi\),

\[
\inf_{F \in \mathcal{F}_{-1,1}^K} \left\{ P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \infty) - \exp \left( -(B + 1) \sum_{k=1}^K \alpha_k \gamma(F_k) \right) \right\} < 0
\]

\[
\implies \sup_{F \in \mathcal{F}_{-1,1}^K} \left\{ P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \infty) - \exp \left( -(B + 1) \sum_{k=1}^K \alpha_k \gamma(F_k) \right) \right\} > 0. \tag{3}
\]

Remark 10  We can also obtain refined bounds such that \(P(\tau(B, \pi) < \infty)\) in Theorem 8 and Corollary 9 is replaced with \(P(\tau(B, \pi) < \frac{3B}{F})\), where \(\Delta_F\) depends on the arm distributions \(F\).

From Theorem 8, we can regard \(\exp \left(-B \sum_{k=1}^K \alpha_k \gamma(F_k)\right)\) as a kind of lower bound on the probability of ruin, since a policy \(\pi\) whose probability of ruin is equal to this exponential term is ruin-wise Pareto-optimal. Furthermore, it is straightforward that if such a policy \(\pi\) satisfies

\[
\forall F \in \mathcal{F}^K_{\{-1,0,1\}}, \quad \text{Rew}_T(\pi) \geq \left( 1 - \exp \left( -B \sum_{k=1}^K \alpha_k \gamma(F_k) \right) \right) \times \max_{k \in [K]} \mu_k T + o(T), \tag{4}
\]

then \(\pi\) is regret-wise Pareto-optimal (see Appendix I.1). However, this by no means proves that any ruin-wise Pareto-optimal policy is regret-wise optimal, as we will see in Section 4.2.

The main ingredient of the proof of Theorem 8 is given in Section 3.2. In Appendix D, we give the rest of the proof of the refined version of Theorem 8 described in Remark 10. We conclude this subsection by a lemma providing an insightful interpretation of the bounds (2) and (3).

Lemma 11  For any \(k \in [K]\),

\[
\forall F_k \in \mathcal{F}_{\{-1,0,1\}}, \quad \frac{1}{B} \log P_{F_k}(\tau(B, \pi) < \infty) = -\gamma(F_k);
\]

\[
\forall F_k \in \mathcal{F}_{\{-1,1\}}, \quad \frac{1}{B} \log P_{F_k}(\tau(B, k) < \infty) \leq \liminf_{B' \to \infty} \frac{1}{B'} \log P_{F_k}(\tau(B', k) < \infty) = -\gamma(F_k).
\]

This trivially implies, for \(F \in \mathcal{F}^K_{\{-1,0,1\}}\) and \((\alpha_k)_{k \in [K]}\) such that \(\alpha_k B \in \mathbb{N}\) for any \(k\),

\[
\exp \left( -B \sum_{k=1}^K \alpha_k \gamma(F_k) \right) = \prod_{k=1}^K P_{F_k}(\tau(\alpha_k B, k) < \infty). \tag{5}
\]

This lemma means that \(\gamma(F_k)\) can be related to the probability of ruin \(P_{F_k}(\tau(B, k) < \infty)\) of the stochastic process with increments i.i.d. from \(F_k\). Whereas the statements of Lemma 11 use the KL divergence through the definition of \(\gamma(F_k)\) in (1), the probability of ruin of a stochastic process is usually analyzed using the moment generating function, which is also found in the proof of this lemma given in Appendix E. The relation between them is discussed in Lemma 22 in Appendix C, and we interchangeably use both representations.
3.2. Sketch and Main Ingredient of the Proof of Theorem 8

The proof of the non-asymptotic bound of Theorem 8 is given in Appendix D. This proof consists of (i) derivation of an asymptotic bound given in Lemma 12 below, and (ii) turning it into a non-asymptotic bound by using a sub-additivity argument on the probability of ruin. The proof of Corollary 9 follows the same path until the sub-additivity, for which formulas differ. For that reason, the proof below is conducted in the general case of arm distributions in \( F_k \).

**Lemma 12** Fix an arbitrary policy \( \pi \) and distributions \( (Q_1, \ldots, Q_K) \) such that \( \mathbb{E}_{X \sim Q_k}[X] < 0 \) for all \( k \in [K] \). Then, there exists a probability vector \( \beta(Q) = (\beta_1(Q), \ldots, \beta_K(Q)) \) such that for any distributions \( (F_1, \ldots, F_K) \),

\[
\liminf_{B \to +\infty} \frac{1}{B} \log P_{(F_1, \ldots, F_K)} \left( \tau(B, \pi) < \frac{3B}{\Delta} \right) \geq - \sum_{k=1}^{K} \beta_k(Q) \log \frac{KL(Q_k \| F_k)}{\mathbb{E}_{X \sim Q_k}[-X]},
\]

where \( \Delta = \min_{k \in [K]} \mathbb{E}_{X \sim Q_k}[-X] > 0 \).

**Proof** Let \( Q = (Q_1, \ldots, Q_K) \) be a vector of distributions such that \( \mathbb{E}_{X \sim Q_k}[X] < 0 \) for all \( k \in [K] \), and let \( \Delta := \min_{k \in [K]} \mathbb{E}_{X \sim Q_k}[-X] > 0 \). We denote by \( N_k(\tau) \) the number of pulls of arm \( k \) until \( \tau(B, \pi) \), and by \( h_k \) its realization. Denoting by \( Y^n_k \) the reward of the \( n \)-th pull of arm \( k \), let \( h_t = \left( Y_1^n, Y_2^n, \ldots, Y_n^{N_1(\tau)} \right), \left( Y_1^n, Y_2^n, \ldots, Y_n^{N_2(\tau)} \right), \ldots, \left( Y_K^n, Y_K^n, \ldots, Y_K^{N_K(\tau)} \right) \), and \( h_t \) be its realization. Please note that for any realization \( h_t, \left| B + \sum_{k \in [K]} \sum_{m=[n_k]} y_k^m \right| \leq 1 \).

We further denote by \( T(Q) \) the set of “typical” realizations \( h_t \) satisfying

\[
\begin{align*}
\left| \sum_{k=1}^{K} \left( n_k KL(Q_k \| F_k) - \sum_{m=1}^{n_k} \log \frac{dQ_k}{dF_k}(y_k^m) \right) \right| &\leq \frac{t}{B^2}, \\
\sum_{k=1}^{K} (n_k \mathbb{E}_{X \sim Q_k}[X] - \sum_{m=1}^{n_k} y_k^m) &\leq \frac{\Delta t}{B^2}, \\
\sum_{k=1}^{K} n_k \mathbb{E}_{X \sim Q_k}[X] - \sum_{k=1}^{K} \sum_{m=1}^{n_k} y_k^m &\leq -\frac{t \Delta}{B^2}.
\end{align*}
\]

Such realizations are “typical” under \( Q \) in the sense that \( \lim_{B \to +\infty} Q(h_t \in T(Q)) = 1 \) (shown by, e.g., Hoeffding’s inequality, see Appendix D.1 for details). Note that any typical \( h_t \) satisfies

\[
t \leq \frac{3B}{\Delta} \quad \text{and} \quad \left| \sum_{k=1}^{K} \frac{n_k}{B} \mathbb{E}_{X \sim Q_k}[-X] - 1 \right| \leq \frac{4}{B^2}.
\]

In particular, denoting \( r(h_t) := \frac{(n_1, \ldots, n_K)}{B} \), this implies that \( r(h_t) \) can take at most \( O(\text{poly}(B)) \) values, and hence there exists \( \tilde{r} \) such that

\[
\lim_{B \to +\infty} \frac{1}{B} \log Q(r(H_t) = \tilde{r} | H_t \in T(Q)) = 0.
\]

By performing a change of distribution and using (6), we can bound

\[
P_{(F_1, \ldots, F_K)} \left( \tau(B, \pi) < \frac{3B}{\Delta} \right)
\]
For any $k \in [K]$, we denote by $\beta_k(Q) := \frac{\tilde{r}_k}{\sum_j \tilde{r}_j}$ the normalized version of $\tilde{r}$ which, by definition, satisfies $\sum_{k=1}^K \beta_k(Q) \mathbb{E}_{X \sim Q_k} [-X] = 1$. Eq. (6) implies that $|\sum_j \tilde{r}_j - 1| \leq \frac{1}{B^4}$, and hence
\[
\frac{1}{B} \log P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) \geq - \left(1 + \frac{4}{B^4}\right) \sum_{k=1}^K \beta_k(Q) KL(Q_k \| F_k) - \frac{3}{\Delta B^4} + \frac{1}{B} \log Q(\mathcal{H}_r \in T(Q), r(\mathcal{H}_r) = \tilde{r})
\]
\[
\overset{B \to +\infty}{\longrightarrow} - \sum_{k=1}^K \beta_k(Q) KL(Q_k \| F_k),
\]
which concludes the proof of Lemma 12. 

4. The EXPLOIT framework

In this section, we introduce a framework of anytime policies, called EXPLOIT, which achieve the lower bound on the probability of ruin in Theorem 8.

4.1. Definition of the EXPLOIT Framework

In the case of rewards in $\{-1, 0, 1\}$, (5) shows that the lower bound of Theorem 8 can be written as $\prod_{k=1}^K P(\tau(\alpha_k B, k) < \infty)$, which means that an optimal policy to minimize the probability of ruin is to preset fractions of the budget $\{\alpha_k\}_{k \in [K]}$ and allocate to each arm $k$ its preset budget share $\alpha_k B$. Decomposing $B = n_K K + b$ for an integer $n_K$ and $0 \leq b < K$, we choose $\alpha_k = \frac{n_K + 1 + b + k}{B}$ for any $k$, and generalize it to rewards in $[-1, 1]$ with the EXPLOIT framework, defined as follows.

**Definition 13** For any $k \in [K]$, denoting by $(Y_k^s)_{s \geq 1}^t$ the rewards from arm $k$, let $\tau_k^\infty := \inf\{ t \geq 1 : \sum_{s=1}^t Y_k^s < -\frac{B}{K} + 1 \}$. We say that a policy $\pi$ belongs to the EXPLOIT framework if:

- for any $t < \sum_{k=1}^K \tau_k^\infty$, $\pi_t \in \{ k \in [K] : \sum_{s=1}^t 1_{\pi_s = k} < \tau_k^\infty \}$, and

- for any $t \geq \sum_{k=1}^K \tau_k^\infty$, $\pi_t = \arg\max\{ \sum_{s=1}^t X_s^\pi_s 1_{\pi_s = k} \}$ (in case of tie, $\pi_t$ is the smallest arm index).
We call this framework EXPLOIT because a policy which selects the arm with the highest cumulative reward, i.e., \( \pi_t \in \arg \max_{k \in [K]} \left\{ \sum_{s=1}^{t-1} X_{s}^k \mathbb{1}_{\tau_{s}=k} \right\} \), belongs to EXPLOIT. By the nature of this framework, all the policies in EXPLOIT have the same probability of ruin, which we denote by \( p_{\text{EX}}^B \) (\( p_{\text{EX}} \) if no ambiguity). The next proposition directly follows from the definition of EXPLOIT.

**Proposition 14** It holds that

\[
\exp \left( -B \sum_{k=1}^{K} \alpha_k \gamma(F_k) \right) \overset{(*)}{\leq} p_{\text{EX}}^B \leq \exp \left( -B \sum_{k=1}^{K} \left( \alpha_k - \frac{1}{B} \right) \gamma(F_k) \right),
\]

where the inequality \( (*) \) holds with equality if the rewards are in \( \{-1, 0, 1\} \) and \( \alpha_k B \in \mathbb{N} \) for any \( k \).

In the case of rewards in \( \{-1, 0, 1\} \), the probability of ruin of EXPLOIT policies matches the bound of Theorem 8, proving the ruin-wise Pareto-optimality of the EXPLOIT framework.

In the general case of rewards in \([-1, 1]\), we see from Proposition 14 and Lemma 11 that the probability of ruin of EXPLOIT policies with an initial budget of \( B + K + 1 \) is bounded by

\[
p_{\text{EX}}(B + K + 1) \leq \exp \left( -(B + 1) \sum_{k=1}^{K} \alpha_k \gamma(F_k) \right),
\]

which matches the lower bound in Corollary 9. Therefore, the looseness of the ruin probability in the general case (w.r.t. the bound of Corollary 9) corresponds to a budget loss of at most \( K + 1 \).

### 4.2. Expected Cumulative Reward of Policies in EXPLOIT

In this subsection, we show, in the particular case of rewards in \( \{-1, 0, 1\} \) and a budget which is an integer multiple of \( K \), that EXPLOIT policies have a fairly low cumulative reward, upper-bounded as shown in the following proposition, whose proof is in Appendix F.

**Proposition 15** We assume, w.l.o.g., that \( \mu_1 \geq \cdots \geq \mu_K \). Further assume that the rewards are in \( \{-1, 0, 1\} \) and that the budget \( B \) is an integer multiple of \( K \). Then, for any policy \( \pi \) in EXPLOIT,

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^\pi \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] \leq \left( 1 - p_{\text{EX}}^B \right) \sum_{k=1}^{K} w_k \mu_k \times T + o(T) \overset{(*)}{\leq} \left( 1 - p_{\text{EX}}^B \right) \max_{k \in [K]} \mu_k \times T + o(T),
\]

where for any \( k \in [K] \), \( w_k = \frac{P(\tau(B, k) = \infty) \prod_{j=1}^{k-1} P(\tau(B, j) < \infty)}{1 - p_{\text{EX}}^B} \). Besides, when two arms have positive and different expectations, \( (*) \) is a strict inequality.

This result implies that EXPLOIT policies cannot satisfy the sufficient condition (4) for the regret-wise Pareto-optimality. This is because they may stop the exploration of an arm after a constant (w.r.t. \( T \)) number of pulls. On the other hand, we will show in Section 5 that EXPLOIT-UCB-DOUBLE fixes that problem and achieves the bound (4), proving along the way that no EXPLOIT policy is regret-wise Pareto-optimal.
Algorithm 2: EXPLOIT-UCB

for $t = 1, \ldots, T$ do
  Set $A_t := \{ k \in [K] : \sum_{s=1}^{t-1} X_\pi^s 1_{\pi_s = k} \geq -\frac{B}{K} + 1 \}$.
  if $A_t \neq \emptyset$ then
    Pull arm $\text{arg max}_{k \in A_t} \hat{X}_t^k + \sqrt{\frac{6 \log(t-1)}{N_k(t-1)}}$.
  else
    Pull arm $\text{arg max}_{k \in [K]} \hat{X}_{t-1}^k$.

4.3. Bandit Algorithms in the EXPLOIT framework

In this subsection, we exhibit a policy within EXPLOIT achieving the first inequality of (7) with equality. For any arm $k \in [K]$ and any time step $t$, we introduce $N_k(t) := \sum_{s=1}^{t} 1_{\pi_s = k}$ as the number of pulls of arm $k$ until time $t$, and $\hat{X}_t^k := \frac{1}{N_k(t)} \sum_{s=1}^{t} 1_{\pi_s = k} X_\pi^s$ as the empirical mean of arm $k$ at time $t$.

Starting from the classic bandit algorithm UCB (Upper Confidence Bound, see Auer et al., 2002), we define EXPLOIT-UCB in Algorithm 2 as the policy which, at each time step $t \geq 1$, performs UCB among the arms whose cumulative reward is at least $-\frac{B}{K} + 1$. Please note that the choice of the constant 6 in the square root is different from the original UCB and is only for simplicity of the proof. As EXPLOIT-UCB is in EXPLOIT, it naturally achieves the optimal bound on the probability of ruin. In addition, it also achieves the middle bound in (7) on the reward, which is asymptotically optimal among EXPLOIT policies, as shown in the next proposition, whose proof is in Appendix G.

Proposition 16  Under the hypotheses of Proposition 15, EXPLOIT-UCB satisfies the first inequality of (7) with equality.

5. A Pareto-optimal Policy and Open Questions

In this section, we give an answer to the open problem in Perotto et al. (2019) with the policy EXPLOIT-UCB-DOUBLE (given in Algorithm 1) which we prove to be regret-wise Pareto-optimal if the rewards are in $\{-1, 0, 1\}$ and with the prior knowledge of the horizon $T$. We next give some insights on whether or not this result might be improved and if so, on the technicality of doing it.

5.1. A (Regret-wise) Pareto-optimal policy: EXPLOIT-UCB-DOUBLE

We start from EXPLOIT-UCB (Algorithm 2). As this policy belongs to EXPLOIT, it is ruin-wise Pareto-optimal. However, as shown in Proposition 15, its cumulative reward is rather low, because it may stop the exploration of an arm after a constant (w.r.t. $T$) number of pulls. We tackle this issue by performing a doubling trick (see, e.g., Cesa-Bianchi and Lugosi, 2006) on the budget, which relaunches the exploration when the cumulative reward is large enough.

Let $n \in \mathbb{N}$ be a parameter fixed in advance. For any integer $j \geq 0$, let

$$t_j := \inf \left\{ t \in \{0, \ldots, \min(\tau(B, \pi), T)\} : B + \sum_{s=1}^{t} X_\pi^s > jnB^2 \right\},$$

where $\tau(B, \pi)$ is the horizon of arm $\pi$ in EXPLOIT-UCB-DOUBLE.
with the convention that \( t_j = \min(\tau(B, \pi), T) + 1 \) if the above set is empty. We define EXPLOIT-UCB-DOUBLE in Algorithm 1 as the policy which, at each time step \( t \), performs EXPLOIT-UCB pretending that the initial budget is \( B \) if \( t < t_1 \), and \( jnB^2 \) if \( t_j \leq t < t_{j+1} \) for some \( j \geq 1 \). The underlying principle of EXPLOIT-UCB-DOUBLE is that, as long as the cumulative reward is low, it performs safely like an EXPLOIT policy in order to minimize the probability of ruin, and once the cumulative reward becomes large, it starts exploring more in order to start the reward maximization, and behaves more similarly to UCB.

The next proposition, whose proof is given in Appendix H.1, shows that the probability of ruin of EXPLOIT-UCB-DOUBLE is close to the one of an EXPLOIT policy.

**Proposition 17** Given \( n \geq 1 \), for EXPLOIT-UCB-DOUBLE \( \pi \), the ruin probability is bounded by

\[
P(\tau(B, \pi) < \infty) \leq p_{EX} + (p_{EX})^{nB} \frac{1}{1 - (p_{EX})^{nB}} \overset{(\ast)}{=} p_{EX} + o_{T \to \infty}(1),
\]

where \((\ast)\) holds for \( n = \log T \).

Finally, the following proposition shows that the cumulative reward given no ruin is asymptotically equal to the best possible expected reward, whose proof can be found in Appendix H.2.

**Proposition 18** The reward of EXPLOIT-UCB-DOUBLE given no ruin is bounded from below by

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^\pi \mathbb{1}_{\tau(B, \pi) > t-1} \mid \tau(B, \pi) \geq T \right] \geq \max_{k \in [K]} \mu_k T + o(T). \tag{8}
\]

Recall from Theorem 8 that, in the case of rewards in \(-1, 0, 1\), \( p_{EX} \) is the probability of ruin of EXPLOIT policies, which are ruin-wise Pareto-optimal. In that case, if the horizon \( T \) is known and we set \( n = \Theta(\log T) \), Propositions 17 and 18 directly imply Theorem 5 (see the details in Appendix I). In the general case of rewards in \([-1, 1]\) and without the prior knowledge of \( T \), EXPLOIT-UCB-DOUBLE \( \pi \) still achieves, for any policy \( \pi' \) such that \( \sup_F \text{Reg}_F(\pi \parallel \pi') > 0 \),

\[
\inf_F \text{Reg}_F(\pi \parallel \pi') < \frac{(p_{EX})^{nB}}{1 - (p_{EX})^{nB}} \max_{k \in [K]} \mu_k,
\]

which is exponentially small in \( B \), as justified in Appendix I.3.

### 5.2. Discussion, Open Problems and Technicalities

In this subsection, we sum up the theoretical strengths and limitations of EXPLOIT-UCB-DOUBLE, and give some possible research directions as well as the technical challenges associated to them.

In the case of rewards in \(-1, 0, 1\), with the prior knowledge of the horizon \( T \), EXPLOIT-UCB-DOUBLE is (regret-wise) Pareto-optimal, which is the desired result. On the other hand, without the prior knowledge of the horizon \( T \), we suspect that there is no Pareto-optimal policy. In other words, we believe that the regret grows linearly with \( T \) if we consider an anytime policy \( \pi \) rather than a sequence of policies \( \pi^T \) for various time horizons \( T \in \mathbb{N} \).

In the general case of rewards in \([-1, 1]\), EXPLOIT-UCB-DOUBLE is almost ruin-wise Pareto-optimal, up to a term exponentially small in \( B \). However, it is not proven to be regret-wise Pareto-optimal. The main difficulty is that in general, for any arm \( k \), \( \sum_{s=1}^{t_k^s} Y^s_k \) is not deterministic (using
the notations of Definition 13). As a result, even for EXPLOIT policies, the probability of ruin cannot be decomposed as a product of independent ruin probabilities of the arms. The same reason leads to the looseness in the subadditivity bound (see Lemma 25), leading to the \((B + 1)\) factor in the bound of Corollary 9 instead of \(B\). For that reason, we believe that the bound of Corollary 9 is not tight and conjecture that EXPLOIT-UCB-DOUBLE is regret-wise Pareto-optimal even in the general case.

Finally, note that in either case, the reward given no ruin of EXPLOIT-UCB-DOUBLE (see Proposition 18) is asymptotically optimal and equal to \(\max_{k \in [K]} \mu_k T\), which makes it worth applying to more standard bandit settings where an algorithm with a stronger exploitation component is desired.

6. Experiments

In order to evaluate the practical performance of the algorithms introduced in this paper, we define the survival regret as

\[
S\text{-Reg}_{T}(\pi) = \left( 1 - \exp \left( - \frac{B}{K} \sum_{k=1}^{K} \gamma(F_k) \right) \right) \max_{k \in [K]} \mu_k T - \text{Rew}_{T}(\pi),
\]

which is a hypothetical regret with respect to a policy that achieves the Pareto-optimal ruin probability and always pulls the arm with the highest expected reward given no ruin.

We compare the survival regret of EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE (with various parameters \(n \in \{1, \lceil \log T \rceil, 100\}\), where \(\lceil \log T \rceil = 10\)) to the classic bandit algorithms UCB (Auer et al., 2002) and Multinomial Thompson Sampling (MTS) (Riou and Honda, 2020). The setting chosen consists of \(K = 3\) multinomial arms of common support \([-1, 0, 1]\) and parameters \((P(X = -1), P(X = 0), P(X = 1))\) respectively equal to \((0.4, 0.1, 0.5), (0.05, 0.85, 0.1)\) and \((0.6, 0, 0.4)\), a horizon \(T = 20000\) and a budget \(B = 9\). Please note that this is an example such that arm 1 has the largest expected reward, while arm 2 has the largest probability of survival, that is, the lowest \(\gamma(F_k)\). The curves are averages over 200 simulations. Simulation results for other settings are given in Appendix J. As expected, EXPLOIT-UCB-DOUBLE with the parameter \(n = \lceil \log T \rceil\) clearly outperforms all the other algorithms. Nevertheless, EXPLOIT-UCB-DOUBLE with other values of \(n \in \{1, 100\}\) still has a decent performance and are, at least in practice, good anytime algorithms. On the other hand, MTS has the worst performance due to frequent ruins: its proportion of ruins culminates at around 0.25 among the 200 simulations with an average time of ruin slightly below 15000, which is in stark contrast to EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE’s average time of ruin, around 19000. EXPLOIT-UCB has a performance which is comparable to MTS, and while its average time of ruin is very high (above 19000), its quasi absence of exploration makes it frequently pull a suboptimal arm until the horizon \(T\), inducing a large regret.

7. Conclusion

In this paper, we introduced the survival MAB, an extension of the MAB with a risk of ruin, which naturally follows from many practical applications but is considerably more difficult to study. For example, contrary to the MAB, no policy can achieve a sublinear regret in the standard sense, because every single pull increases considerably the probability of ruin. We started by providing a Pareto-type lower bound on the probability of ruin, as well as policies achieving this bound. Building upon a doubling trick on such policies, we finally derived a Pareto-optimal policy in the sense of the regret.
(at least in the case of rewards in $\{-1, 0, 1\}$), giving an answer to an open problem from COLT 2019 (see Perotto et al., 2019).

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Appendix A. General Classic Results

A.1. A Classic Lemma in the Theory of Stochastic Processes

The following result is classic in the theory of stochastic processes and we state it without proof. It will be used throughout the paper.

**Lemma 19** Let \( k \in [K] \) and assume \( F_k \) is multinomial of support \( \{-1, 0, 1\} \). If \( \mathbb{E}_{X \sim F_k}[X] > 0 \),

\[
P(\tau(B, k) < \infty) = \left( \frac{P_{X \sim F_k}(X = -1)}{P_{X \sim F_k}(X = 1)} \right)^{[B]}.\]

Appendix B. Proof of the Linearity of the Classic Regret (Proposition 3)

In this appendix, we prove a stronger version of Proposition 3, namely that Proposition 3 even holds in the case where the supremum on \( F \) is taken on Bernoulli arm distributions of support \( \{-1, 1\} \). Let \( F_K \) be the set of \( K \)-tuples of Bernoulli arm distributions of support \( \{-1, 1\} \).

**Proposition 20** Assume that the initial budget \( B > 2 \). For any policy \( \pi \), it holds that

\[
\sup_{F \in F_K} \sup_{\tilde{\pi}} \text{Reg}_F(\pi \| \tilde{\pi}) > 0.
\]

**Proof** Let \( p_1, p_2, \ldots, p_K \) be the parameters of the \( K \) Bernoulli arms, i.e., for any \( k \in [K] \), \( p_k = P_{X \sim F_k}(X = +1) = 1 - P_{X \sim F_k}(X = -1) \). Then, there exists \( k_0 \in [K] \) such that

\[
|S_{k_0}| = \infty, \text{ where } S_{k_0} = \{ T \geq 1 : \pi_T^1 = k_0 \}.
\]

W.l.o.g., we assume that \( k_0 = 2 \) and denote \( S_2 := S_{k_0} \). Let \( T \in S_2 \). Now, let \( 1 > p_1 > \frac{1}{2} > p_2 > \cdots > p_K \) and we define \( F_k \sim \text{Ber}_{\{-1, 1\}}(p_k) \) for any \( k \in [K] \). Let \( \Delta := \frac{1 - p_1}{p_1} \). Denoting by \( \text{Rew}_T(1) \) the reward of the (optimal) policy \( \pi_1 = 1 \) for any \( t \geq 1 \), it holds that

\[
\text{Rew}_T(1) - \text{Rew}_T(\pi_T^1) = \mathbb{E} \left[ \sum_{t=1}^{T} X_t^1 \mathbb{1}_{\tau(B, 1) \geq t-1} \right] - \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_1} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right].
\]

Let us then compute the cumulative expected reward of the policy pulling only arm 1:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^1 \mathbb{1}_{\tau(B, 1) \geq t-1} \right] = \mu_1 \sum_{t=1}^{T} P(\tau(B, 1) \geq t - 1)
\]

\[
= \mu_1 P(\tau(B, 1) \geq T) T + o(T)
\]

\[
= \mu_1 P(\tau(B, 1) = \infty) T + o(T).
\]

Then, using Lemma 19, we deduce

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^1 \mathbb{1}_{\tau(B, 1) \geq t-1} \right] = \mu_1 \left( 1 - \Delta^{[B]} \right) T + o(T)
\]

\[
= (2p_1 - 1) \left( 1 - \Delta^{[B]} \right) T + o(T).
\]
Recall that $T \in S_2$. Then, the cumulative reward of $\pi^T$ is upper-bounded by

$$\mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi T} 1_{\tau(B, \pi^T) \geq t-1} \right] = \mathbb{E} \left[ X_1^2 + \sum_{t=2}^{T} X_t^{\pi T} 1_{\tau(B, \pi^T) \geq t-1} \right]$$

$$\leq \mathbb{E} \left[ X_1^2 + \sum_{t=2}^{T} X_t^{1} 1_{\tau(B, \pi^T) \geq t-1} \right]$$

$$= (2p_2 - 1) + \mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=1}^{s} x_r^T > 0} \right]$$

$$= (2p_2 - 1) + p_2 \mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=1}^{s} x_r^T > 0} \bigg| X_1^2 = 1 \right]$$

$$+ (1 - p_2) \mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=1}^{s} x_r^T > 0} \right]$$

$$= (2p_2 - 1) + p_2 \mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=2}^{s} x_r > 0} \right]$$

$$+ (1 - p_2) \mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=2}^{s} x_r > 0} \right].$$

But then, using (9), we know that

$$\mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=2}^{s} x_r > 0} \right] = (2p_1 - 1) \left( 1 - \Delta^{[B] + 1} \right) T + o(T)$$

and

$$\mathbb{E} \left[ \sum_{t=2}^{T} X_t^{1} 1_{\forall s \leq t-1, B + \sum_{r=2}^{s} x_r > 0} \right] = (2p_1 - 1) \left( 1 - \Delta^{[B] - 1} \right) T + o(T).$$

Therefore, we can deduce that the agent’s total cumulative reward is upper-bounded by

$$\mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi T} 1_{\forall s \leq t-1, B + \sum_{r=1}^{s} x_r^T > 0} \right] \leq$$

$$(2p_1 - 1) \left( p_2 \left( 1 - \Delta^{[B] + 1} \right) + (1 - p_2) \left( 1 - \Delta^{[B] - 1} \right) \right) T + o(T).$$

We can then deduce the lower bound

$$\text{Rew}_T(1) - \text{Rew}_T(\pi^T)$$

$$= \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{1} 1_{\forall s \leq t-1, \sum_{r=1}^{s} x_r > 0} \right] - \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi T} 1_{\forall s \leq t-1, B + \sum_{r=1}^{s} x_r^T > 0} \right]$$

$$\geq (2p_1 - 1) \left( \left( 1 - \Delta^{[B]} \right) - p_2 \left( 1 - \Delta^{[B] + 1} \right) - (1 - p_2) \left( 1 - \Delta^{[B] - 1} \right) \right) T + o(T).$$
\[ = (2p_1 - 1) \left[ p_2 \left( \Delta^{[B]} + 1 - \Delta^{[B]} \right) + (1 - p_2) \left( \Delta^{[B]} - 1 \right) \right] T + o(T) \]
\[ = (2p_1 - 1) \left[ p_2 \Delta^{[B]} (\Delta - 1) + (1 - p_2) \Delta^{[B]} (1 - \Delta) \right] T + o(T) \]
\[ = (2p_1 - 1)(1 - \Delta) \Delta^{[B]}^{-1} \left[ (1 - p_2) - p_2 \Delta \right] T + o(T) \]
\[ = (2p_1 - 1)(1 - \Delta) \Delta^{[B]}^{-1} (1 - p_2) (1 + \Delta)] T + o(T). \]

By definition, \( \Delta = \frac{1-p_1}{p_1} \), therefore \( p_2 (1 + \Delta) = \frac{p_2}{p_1} \). By the assumption \( p_1 > p_2, 1 - p_2 (1 + \Delta) = 1 - \frac{p_2}{p_1} > 0 \). We can then conclude that:
\[
\text{Rew}_T(\pi) - \text{Rew}_T(\pi^T) \geq (2p_1 - 1)(1 - \Delta) \Delta^{[B]}^{-1} \left( 1 - \frac{p_2}{p_1} \right) T + o(T)
\]
\[
= (2p_1 - 1)(2 - (1 + \Delta)) \Delta^{[B]}^{-1} \left( 1 - \frac{p_2}{p_1} \right) T + o(T)
\]
\[
= (2p_1 - 1) \left( 2 - \frac{1}{p_1} \right) \Delta^{[B]}^{-1} \left( 1 - \frac{p_2}{p_1} \right) T + o(T)
\]
\[
= \frac{1}{p_1} \left( 2p_1 - 1 \right)^2 \Delta^{[B]}^{-1} \left( 1 - \frac{p_2}{p_1} \right) T + o(T).
\] (10)

By taking the limit sup when \( T \to +\infty \), it is straightforward that
\[ \text{Reg}_F(\pi\|1) > 0, \]

which concludes the proof.

Remark 21 \( p_1 < 1 \) implies \( \Delta > 0 \) and thus the strict positivity of the bound in front of \( T \) in (10). In the case \( p_1 = 1 \) or more generally when there exists a positive arm, it is possible to achieve zero regret if the initial budget is \( B > K - 1 \) with, for instance, the policy \( \pi \) which performs the classic bandit algorithm UCB (see, e.g., Bubeck and Cesa-Bianchi, 2012) when \( B_{t-1} > K - 1 \), and pulls one of the arms which has not yielded a negative reward otherwise.

Appendix C. Properties of KL Divergence

In this appendix we prepare lemmas on the KL divergence used for the analysis of the ruin probability.

Lemma 22 Let \( P \in \mathcal{F}_{[-1,1]} \) be a distribution over \([-1,1]\) with a positive expectation and let \( \Lambda(\lambda) := \log \mathbb{E} [e^{\lambda X}] \) be its logarithmic moment-generating function. Then, there exists \( \lambda' < 0 \) such that \( \Lambda(\lambda') = 0 \) and it satisfies
\[
\inf_{Q : E_X \sim Q \mathbb{E} \underset{X \sim Q}{} X \sim Q} \text{KL}(Q\|P) = -\lambda'.
\]

Proof First we have
\[
\lim_{\lambda \to -\infty} \Lambda(\lambda) = \infty
\]
since \( P[X < 0] > 0 \) because the distribution \( P \) is not positive or zero by definition of \( \mathcal{F}^K_{[-1,1]} \). Therefore, by the continuity of \( \Lambda(\lambda) \) there exists \( \lambda' < 0 \) such that \( \Lambda(\lambda') = 0 \) since \( \Lambda(0) = 0 \) and \( \Lambda'(0) = \mathbb{E}_{X \sim P}[X] > 0 \).

Now, let

\[
\Lambda^*(x) := \sup_{\lambda} \{ \lambda x - \Lambda(\lambda) \}
\]

be the Fenchel-Legendre transform of \( \Lambda(\lambda) \) and define \( x' = \Lambda'(\lambda') \). Then we have \( \Lambda^*(x') = \lambda' x' - \Lambda(\lambda') = \lambda' x' \). Therefore,

\[
\inf_{x < 0} \frac{\Lambda^*(x)}{-x} \leq \frac{\Lambda^*(x')}{-x'} = \frac{\lambda' x'}{-x'} = -\lambda'.
\]

On the other hand,

\[
\inf_{x < 0} \frac{\Lambda^*(x)}{-x} = \inf_{x < 0} \sup_{\lambda} \frac{\lambda x - \Lambda(\lambda)}{-x} \geq \inf_{x < 0} \frac{\lambda' x - \Lambda(\lambda')}{-x} = \inf_{x < 0} \frac{\lambda' x}{-x} = -\lambda'.
\]

Therefore we see that

\[
\inf_{x < 0} \frac{\Lambda^*(x)}{-x} = -\lambda'.
\]

It is well-known as the relation between Cramer’s theorem and Sanov’s theorem that for \( x < \mathbb{E}_{X \sim P}[X] \),

\[
\Lambda^*(x) = \inf_{Q: \mathbb{E}_{X \sim Q}[X] \leq x} \text{KL}(Q\|P),
\]

which concludes the proof of the lemma. □

**Lemma 23** Let \( Q \) be an arbitrary distribution such that \( \mathbb{E}_{X \sim Q}[X] < 0 \) and fix \( \epsilon > 0 \). Then, there exists \( P \) such that \( \mathbb{E}_{X \sim P}[X] > 0 \) and

\[
\frac{\text{KL}(Q\|P)}{\mathbb{E}_{X \sim Q}[X]} \leq \inf_{Q': \mathbb{E}_{X \sim Q'}[X] < 0} \frac{\text{KL}(Q'\|P)}{\mathbb{E}_{X \sim Q'}[X]} (1 + \epsilon). \tag{11}
\]

**Proof** Let \( p \in \left( 0, \min \left\{ \frac{\mathbb{E}_{X \sim Q}[X]}{1 + \mathbb{E}_{X \sim Q}[X]}, 1 - \exp \left( -\frac{(\mathbb{E}_{X \sim Q}[X])^2}{2(\frac{1}{4} + 1)} \right) \right\} \right) \), and let \( Q_p = (1 - p)Q + p\delta_{\{1\}} \) be the mixture of \( Q \) and the point mass at \( X = 1 \). Let \( \Lambda_p(\lambda) = \log \mathbb{E}_{X \sim Q_p}[e^{\lambda X}] \) be the logarithmic moment generating function of \( Q_p \) and \( \lambda^* > 0 \) be such that \( \Lambda_p(\lambda^*) = 0 \). Such \( \lambda^* \) exists and satisfies \( \Lambda'_p(\lambda^*) > 0 \) since \( \Lambda_p(0) = 0 \) and \( \Lambda'_p(0) = (1 - p)\mathbb{E}_{X \sim Q_p}[X] + p < 0 \) by \( p \leq \frac{\mathbb{E}_{X \sim Q_p}[X]}{1 + \mathbb{E}_{X \sim Q_p}[X]} \).
Let $P_p$ be the distribution such that $dP_p/dQ_p(x) = e^{\lambda^* x - \Lambda_p(\lambda^*)} = e^{\lambda^* x}$. Then we have $E_{X \sim P_p}[X] = \Lambda_p(\lambda^*) > 0$. Here note that $E_{X \sim P_p}[e^{-\lambda^* X}] = E_{X \sim Q_p}[e^{-\lambda^* x} e^{\lambda^* X - \Lambda_p(\lambda^*)}] = 1$. Therefore, by Lemma 22 we have

$$\inf_{Q' : E_{X \sim Q'}[X] < 0} KL(Q' || P_p) = \lambda^*. \tag{12}$$

On the other hand,

$$KL(Q || P) = E_{X \sim Q} \left[ 1_{X < 1} \log \frac{dQ}{dP_p}(X) + 1_{X = 1} \log \frac{dQ}{dP_p}(X) \right]$$

$$= E_{X \sim Q} \left[ 1_{X < 1} \log \frac{1}{1 - p} \frac{dQ_p}{dP_p}(X) \right] + Q(X = 1) \log \frac{Q(X = 1)}{P_p(X = 1)}$$

$$\leq \log \frac{1}{1 - p} + E_{X \sim Q} \left[ 1_{X < 1} \log \frac{dQ_p}{dP_p}(X) \right] + Q(X = 1) \log \frac{Q(X = 1)}{Q_p(X = 1)}$$

$$= \log \frac{1}{1 - p} + \lambda^* E_{X \sim Q} [-X] + Q(X = 1) \log \frac{Q(X = 1)}{Q_p(X = 1)}$$

$$\leq \log \frac{1}{1 - p} + \lambda^* E_{X \sim Q} [-X],$$

which, combined with (12), implies that

$$\frac{KL(Q || P_p)}{E_{X \sim Q} [-X]} \leq \inf_{Q' : E_{X \sim Q'}[X] < 0} \frac{KL(Q' || P)}{E_{X \sim Q} [-X]} + \log \frac{1}{1 - p} \tag{13}$$

Comparing (12) and (13) with (11), we see that it is sufficient to show that

$$\frac{\log \frac{1}{1 - p}}{E_{X \sim Q} [-X]} \leq \epsilon \lambda^*. \tag{14}$$

to show (11). Note that we obtain from Pinsker’s inequality that

$$\lambda^* \geq \frac{KL(Q || P_p) - \log \frac{1}{1 - p}}{E_{X \sim Q} [-X]}$$

$$\geq \frac{2 \left( \frac{E_{X \sim Q'}[X]}{2} - \frac{E_{X \sim P_p}[X]}{2} \right)^2 - \log \frac{1}{1 - p}}{E_{X \sim Q} [-X]}$$

$$\geq \frac{(E_{X \sim Q}[X])^2}{2} - \log \frac{1}{1 - p} \frac{1}{E_{X \sim Q} [-X]}.$$ 

Recalling that $P_p$ and $Q$ are supported over $[-1, 1]$, and have positive and negative expectations, respectively. Therefore we obtain (14) since

$$\frac{1}{\lambda^* E_{X \sim Q} [-X]} \geq \frac{\log \frac{1}{1 - p}}{(E_{X \sim Q}[X])^2} - \log \frac{1}{1 - p} \leq \epsilon,$$

where the last inequality follows from since $p \leq 1 - \exp \left( -\frac{(E_{X \sim Q}[X])^2}{2(1 + \epsilon)} \right)$. 

Appendix D. Detailed Proof of the Lower Bound on the Probability of Ruin (Theorem 8 and Corollary 9)

In this section, we give a detailed proof of Theorem 8 and Corollary 9. The proof of the lower bound both in the case of multinomial arms of support \{-1, 0, 1\} (Theorem 8) and in the general case of rewards bounded in \([-1, 1]\) (Corollary 9) stems from the asymptotic lower bound, which is common to both aforementioned cases and is given in Theorem 24. The passage from the asymptotic to the non-asymptotic bound relies on sub-additivity properties, which is given in Lemma 25 and for which formulas differ depending on the case considered.

If for all the arms \(k \in [K]\),
\[
\inf_{Q_k : \mathbb{E}_{X \sim Q_k} [X] < 0} \frac{\text{KL}(Q_k \| F_k)}{\mathbb{E}_{X \sim Q_k} [-X]} = 0,
\]
then the result becomes trivial. This is why we are going to make the following assumption in the proof:

**Assumption 1** There exists an arm \(k \in [K]\) such that \(P(\tau(B, k) = \infty) > 0\).

D.1. Details of the Proof of Lemma 12

In this subsection, we provide the justification for
\[
\lim_{B \to +\infty} Q(\mathcal{H}_\tau \in T(Q)) = 1,
\]
which was omitted in the main text.

For any \(t \geq 1\) and any \(n = (n_1, \ldots, n_K)\) such that \(\sum_{k=1}^{K} n_k = t\), we introduce the following probability events:
\[
U(n, t) := \left\{ \sum_{k=1}^{K} n_k \text{KL}(Q_k \| F_k) - \sum_{m=1}^{n_k} \log \frac{dQ_k}{dF_k}(y^m_k) \leq \frac{t}{B^4} \right\},
\]
\[
V(n, t) := \left\{ \sum_{k=1}^{K} (n_k \mathbb{E}_{X \sim Q_k} [X] - \sum_{m=1}^{n_k} y^m_k) \leq \frac{t\Delta}{B^4} \right\},
\]
\[
W(n, t) := \left\{ \sum_{k=1}^{K} \sum_{m=1}^{n_k} y^m_k \leq \frac{t\Delta}{2} \right\}.
\]

Let \(h_t\) be a realization of \(\mathcal{H}_\tau\). Then, please note that the probability of the event \(\{h_t \in T(Q)\}\) is uniformly bounded independently of the policy \(\pi\) by the probability of the following event:
\[
\forall n = (n_1, \ldots, n_K) \text{ s.t. } \sum_{k=1}^{K} n_k = t, \quad \left\{ U(n, t), V(n, t), W(n, t) \right\}.
\]

For any \(k \in [K]\), let
\[
d_k := \max_{y_1 \in [-1, 1]} \log \frac{dQ_k}{dF_k}(y_1) - \min_{y_2 \in [-1, 1]} \log \frac{dQ_k}{dF_k}(y_2) \quad \text{and} \quad D := \max_{k \in [K]} d_k.
\]
Then, a direct application of Hoeffding’s inequality gives the bounds

\[
Q(U(n,t)^c) \leq 2 \exp\left(-\frac{2t}{D\sqrt{B}}\right),
\]
\[
Q(V(n,t)^c) \leq 2 \exp\left(-\frac{t\Delta^2}{2\sqrt{B}}\right),
\]
\[
Q(W(n,t)^c) \leq \exp\left(-\frac{t\Delta^2}{2}\right).
\]

Let \( C := \max\left\{ \frac{D}{2}, \frac{2\Delta^2}{2} \right\} \), this implies

\[
\max\{Q(U(n,t)^c), Q(V(n,t)^c), 2Q(W(n,t)^c)\} \leq 2 \exp\left(-\frac{t}{C\sqrt{B}}\right).
\]

Using this result, as well as a union bound, we can then bound the probability

\[
Q(h_t \notin T(Q)) \leq Q(\exists n = (n_1, \ldots, n_K) : U(n,t)^c \text{ or } V(n,t)^c \text{ or } W(n,t)^c)
\]
\[
\leq \sum_{n=(n_1,\ldots,n_K)} Q(U(n,t)^c \text{ or } V(n,t)^c \text{ or } W(n,t)^c)
\]
\[
\leq \sum_{n=(n_1,\ldots,n_K)} \{Q(U(n,t)^c) + G(V(n,t)^c) + Q(W(n,t)^c)\}
\]
\[
\leq 5(t+1)^K \exp\left(-\frac{t}{C\sqrt{B}}\right). \tag{15}
\]

We can now bound the desired probability by first using the decomposition

\[
Q(\mathcal{H}_\tau \notin T(Q)) = Q\left(\tau > \frac{3B}{\Delta}, \mathcal{H}_\tau \notin T(Q)\right) + Q\left(\tau \leq \frac{3B}{\Delta}, \mathcal{H}_\tau \notin T(Q)\right).
\]

Then, the first term can be easily bounded using Hoeffding’s inequality again:

\[
Q\left(\tau > \frac{3B}{\Delta}, \mathcal{H}_\tau \notin T(Q)\right) \leq Q\left(\tau > \frac{3B}{\Delta}\right)
\]
\[
= Q\left(\forall t \in \left\{1, \ldots, \frac{3B}{\Delta}\right\}, \exists n = (n_1, \ldots, n_K) : W(n,t)^c\right)
\]
\[
\leq Q(\exists n = (n_1, \ldots, n_K) : W(n,B)^c)
\]
\[
\leq \sum_{(n_1,\ldots,n_K)} Q(W(n,B)^c)
\]
\[
\leq (B+1)^K \exp\left(-\frac{t}{C\sqrt{B}}\right).
\]
The second term in the decomposition is decomposed using a union bound and (15):

\[
Q \left( \tau \leq \frac{3B}{\Delta}, \mathcal{H}_\tau \notin T(Q) \right) \leq Q \left( \exists t \in \left\{ B, \ldots, \frac{3B}{\Delta} \right\}: h_t \notin T(Q) \right)
\]
\[
\leq \sum_{t=B}^{\frac{3B}{\Delta}} Q(h_t \notin T(Q))
\]
\[
\leq \sum_{t=B}^{\frac{3B}{\Delta}} 5(t+1)^K \exp \left( -\frac{t}{2C\sqrt{B}} \right).
\]

Then, for any \( t \geq B \geq (2KC)^2 \), it holds that \( 5(t+1)^K \exp \left( -\frac{t}{2C\sqrt{B}} \right) \leq 5 \exp \left( -\frac{t}{2C\sqrt{B}} \right) \) and therefore

\[
Q \left( \tau \leq \frac{3B}{\Delta}, \mathcal{H}_\tau \notin T(Q) \right) \leq 5 \sum_{t=B}^{\frac{3B}{\Delta}} \exp \left( -\frac{t}{2C\sqrt{B}} \right)
\]
\[
= 5 \times \frac{e^{-\frac{\sqrt{B}}{2C}} - e^{-\frac{3B+1}{2C\sqrt{B}}}}{1 - e^{-\frac{1}{2C\sqrt{B}}}}.
\]

We then deduce that, for \( B \geq (2KC)^2 \),

\[
Q(\mathcal{H}_\tau \notin T(Q)) \leq (B + 1)^K e^{-\frac{\sqrt{B}}{2C}} + 5 \times \frac{e^{-\frac{\sqrt{B}}{2C}} - e^{-\frac{3B+1}{2C\sqrt{B}}}}{1 - e^{-\frac{1}{2C\sqrt{B}}}},
\]

and therefore, that

\[
\lim_{B \to +\infty} Q(\mathcal{H}_\tau \in T(Q)) = 1.
\]

\[\blacksquare\]

**D.2. Asymptotic Lower Bound**

The main result of this subsection is the asymptotic lower bound on the probability of ruin. This result will serve as a basis in the proof of the non-asymptotic lower bound of both Theorem 8 and Corollary 9, and for that reason, it is conducted in the general case of arm distributions in \( F^K_{[-1,1]} \).

**Theorem 24** Let \((\alpha_k)_{k \in [K]}\) such that for any \( k \in [K], \alpha_k > 0 \) and \( \sum_{k=1}^{K} \alpha_k = 1 \). There exists no policy \( \pi \) such that, for any set of arms \((F_1, \ldots, F_K)\),

\[
\liminf_{B \to +\infty} \frac{1}{B} \log p_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) \leq -\sum_{k=1}^{K} \alpha_k \inf_{Q_k: \mathbb{E}_{X \sim Q_k} [X] < 0} KL(Q_k \parallel F_k)
\]

with a strict inequality for some \((F_1, \ldots, F_K)\) and where

\[
\forall k \in [K], P^*_k := \arg \min_{Q_k: \mathbb{E}_{X \sim Q_k} [X] < 0} KL(Q_k \parallel F_k) \text{ and } \Delta = \min_{k \in [K]} \mathbb{E}_{X \sim P^*_k} [-X] > 0.
\]
We define, for any distributions $F = (F_1, \ldots, F_K)$,

$$A_F(\alpha_1, \ldots, \alpha_K) := \sum_{k=1}^{K} \alpha_k \inf_{Q_k \forall X \sim Q_k[X] < 0} \frac{\text{KL}(Q_k\|F_k)}{E_{X \sim Q_k}[-X]},$$

which we write $A_F$ in the absence of ambiguity on the $(\alpha_k)_{k \in [K]}$. The previous inequality implies that

$$\liminf_{B \to +\infty} \sup_F \frac{\log P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta})}{BA_F} \geq -1. \quad (16)$$

**Proof** Recall from Lemma 12 that for any $Q = (Q_1, \ldots, Q_K)$ such that $E_Q[X] < 0$ for any $i \in [K]$,

$$\liminf_{B \to +\infty} \frac{1}{B} \log P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) \geq -\sum_{k=1}^{K} \beta_k(Q) \frac{\text{KL}(Q_k\|F_k)}{E_{X \sim Q_k}[-X]}, \quad (17)$$

where $\beta(Q) = (\beta_1(Q), \ldots, \beta_K(Q))$ satisfy

$$\forall k \in [K], \beta_k(Q) \geq 0 \text{ and } \sum_{k=1}^{K} \beta_k(Q) = 1. \quad (18)$$

Let us fix $(\alpha_k)_{k \in [K]}$ such that for any $k \in [K]$, $\alpha_k > 0$ and $\sum_{k=1}^{K} \alpha_k = 1$. We are going to show that no policy $\pi$ can achieve both

$$\forall(F_1, \ldots, F_K), \liminf_{B \to +\infty} \frac{1}{B} \log P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) \leq -\sum_{k=1}^{K} \alpha_k \inf_{Q_k \forall X \sim Q_k[X] < 0} \frac{\text{KL}(Q_k\|F_k)}{E_{X \sim Q_k}[-X]} \quad (19)$$

and

$$\exists(F_1, \ldots, F_K), \liminf_{B \to +\infty} \frac{1}{B} \log P_{(F_1, \ldots, F_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) < -\sum_{k=1}^{K} \alpha_k \inf_{Q_k \forall X \sim Q_k[X] < 0} \frac{\text{KL}(Q_k\|F_k)}{E_{X \sim Q_k}[-X]}, \quad (20)$$

Let us then fix a policy $\pi$ such that there exists a distribution $\bar{P} = (\bar{P}_1, \ldots, \bar{P}_K)$ and $\epsilon > 0$ such that

$$\liminf_{B \to +\infty} \frac{1}{B} \log P_{(\bar{P}_1, \ldots, \bar{P}_K)}(\tau(B, \pi) < \frac{3B}{\Delta}) \leq -\sum_{k=1}^{K} \alpha_k \bar{\gamma}_k - \epsilon, \quad (21)$$

where we denoted, for any $k \in [K]$,

$$\bar{\gamma}_k := \inf_{Q : E_{X \sim Q}[X] < 0} \frac{\text{KL}(Q\|\bar{P}_k)}{E_{X \sim Q}[-X]} \text{ and } \bar{\gamma}_{\max} := \max_{k \in [K]} \bar{\gamma}_k > 0 \text{ and } \bar{\Delta} = \min_{k \in [K]} E_{X \sim P_k}[-X] > 0$$
We define  
\[ R_k := \arg \min_{Q \in P_k} \frac{\text{KL}(Q \| P_k)}{E_{X \sim Q}[-X]} \quad \text{and} \quad \bar{\Delta} = \min_{k \in [K]} \mathbb{E}_{X \sim R_k}[-X] > 0. \]

Please note that the positivity of \( \bar{\gamma}_\text{max} \) relies on Assumption 1. We are going to show that there exists \( \bar{\gamma}_\text{max} = (\bar{\gamma}_1, \ldots, \bar{\gamma}_K) \) such that, denoting

\[ \bar{\gamma}_k := \inf_{Q \in P_k} \frac{\text{KL}(Q \| P_k)}{E_{X \sim Q}[-X]}, \quad \bar{\gamma}_k^* := \min \{ \bar{\gamma}_k : k \in [K], \bar{\gamma}_k > 0 \} \quad \text{and} \quad \epsilon' := \frac{\epsilon \alpha_{\text{min}} \bar{\gamma}_\text{max}}{4(K-1) \bar{\gamma}_\text{max}}. \]

and

\[ T_k := \arg \min_{Q \in P_k} \frac{\text{KL}(Q \| P_k)}{E_{X \sim Q}[-X]} \quad \text{and} \quad \bar{\Delta}^* = \min_{k \in [K]} \mathbb{E}_{X \sim T_k}[-X] > 0, \]

the following holds:

\[ \lim_{B \to +\infty} \frac{1}{B} \log P_{(P_1, \ldots, P_K)} \left( \tau(B, \pi) < \frac{3B}{\bar{\Delta}} \right) = - \sum_{k=1}^{K} \alpha_k \bar{\gamma}_k^* + \epsilon'. \]

We define \( Q = (\bar{Q}_1, \ldots, \bar{Q}_K) \) such that, for any \( k \in [K], P_{X \sim \bar{Q}_k}[X] \leq 0 \) and

\[ \frac{\text{KL}(\bar{Q}_k \| P_k)}{E_{X \sim \bar{Q}_k}[-X]} \leq \bar{\gamma}_k + \frac{\epsilon}{2}. \] (22)

Denoting \( \alpha_{\text{min}} := \min_{k \in [K]} \alpha_k > 0 \), we then introduce the set

\[ \mathcal{K} := \left\{ k \in [K] : \beta_k(\bar{Q}) \leq \frac{1}{K} - \frac{\alpha_{\text{min}} \epsilon}{2(K-1) \bar{\gamma}_\text{max}} \right\}. \]

Let us prove that \( \mathcal{K} \) is not empty. Indeed, (21) can be re-written as

\[ \lim_{B \to +\infty} \frac{1}{B} \log P_{(P_1, \ldots, P_K)} \left( \tau(B, \pi) < \frac{3B}{\bar{\Delta}} \right) \leq - \sum_{k=1}^{K} \alpha_k \bar{\gamma}_k - \epsilon \]
\[ \leq - \sum_{k=1}^{K} \left( \alpha_k \bar{\gamma}_k + \alpha_k \epsilon \right) \]
\[ = - \sum_{k=1}^{K} \left( \alpha_k + \frac{\alpha_k \epsilon}{\bar{\gamma}_k} \right) \bar{\gamma}_k. \] (23)

Then, applying (17) to \( Q = \bar{Q} \) and \( F = \bar{P} \) and using (22), we deduce that

\[ \lim_{B \to +\infty} \frac{1}{B} \log P_{(P_1, \ldots, P_K)} \left( \tau(B, \pi) < \frac{3B}{\bar{\Delta}} \right) \geq - \sum_{k=1}^{K} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k \| P_k)}{E_{X \sim \bar{Q}_k}[-X]} \]
\[ \geq - \sum_{k=1}^{K} \beta_k(\bar{Q}) \left( \bar{\gamma}_k + \frac{\epsilon}{2} \right) \]
\[ = - \sum_{k=1}^{K} \left( \beta_k(\bar{Q}) + \alpha_k \frac{\epsilon}{2\bar{\gamma}_k} \right) \bar{\gamma}_k. \] (24)
Then, we deduce from (23) and (24) that
\[ -\sum_{k=1}^{K} \left( \alpha_k + \frac{\alpha_k e}{\gamma_k} \right) \tilde{\gamma}_k \geq -\sum_{k=1}^{K} \left( \beta_k(\bar{Q}) + \frac{\alpha_k e}{2\gamma_k} \right) \tilde{\gamma}_k, \]
in other words, that
\[ \sum_{k=1}^{K} \left( \beta_k(\bar{Q}) - \alpha_k - \frac{\alpha_k e}{2\gamma_k} \right) \tilde{\gamma}_k \geq 0. \]
This is equivalent to
\[ \sum_{k: \gamma_k > 0} \left( \beta_k(\bar{Q}) - \frac{1}{K} - \frac{\alpha_k e}{2\gamma_k} \right) \tilde{\gamma}_k \geq 0. \]
We then deduce that there exists \( k_0 \in [K] \) such that \( \beta_{k_0}(\bar{Q}) \geq \alpha_k + \frac{\alpha_k e}{2\gamma_k} \). With (18), it implies that
\[ \sum_{j \neq k_0} \beta_j(\bar{Q}) = 1 - \beta_{k_0}(\bar{Q}) \leq 1 - \alpha_{k_0} - \frac{\alpha_{k_0} e}{2\gamma_{k_0}} \leq \sum_{j \neq k_0} \alpha_j - \frac{\alpha_{\min} e}{2\gamma_{\max}}. \]
We deduce that there exists \( j \in [K] \) such that \( \beta_j(\bar{Q}) \leq \alpha_j - \frac{\alpha_{\min} e}{2(K-1)\gamma_{\max}} \), proving that \( \mathcal{K} \) is not empty. Then, we define the distribution \( \bar{P}^* = (\bar{P}_1^*, \ldots, \bar{P}_K^*) \) as follows:
- for any \( k \notin \mathcal{K} \), let \( \bar{P}_k^* := \bar{Q}_k \), and please note that \( \mathbb{E}_{X \sim \bar{P}_k^*}[X] < 0 \);
- for any \( k \in \mathcal{K} \), let \( \bar{P}_k^* \) a distribution such that \( \mathbb{E}_{X \sim \bar{P}_k^*}[X] > 0 \) and
\[ \frac{\text{KL}(\bar{Q}_k||\bar{P}_k^*)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}} \leq \gamma_k^* \left( 1 + \frac{\alpha_{\min} e}{4(K-1)\gamma_{\max}} \right) = \gamma_k^* + \frac{\alpha_{\min} e \gamma_k^*}{4(K-1)\gamma_{\max}} \]
where \( \gamma_k^* = \inf_{Q: \mathbb{E}_{X \sim Q}[X] < 0} \frac{\text{KL}(Q||\bar{P}_k^*)}{\mathbb{E}_{X \sim Q[-X]}} \) and \( \gamma_{\min}^* = \min\{\gamma_k^* : k \in \mathcal{K}, \gamma_k^* > 0\} \). Note that this distribution \( \bar{P}_k^* \) indeed exists by Lemma 23.
Since \( \mathcal{K} \neq \emptyset \) and by definition of \( \bar{P}^* \), we have
\[ \sum_{k=1}^{K} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k||\bar{P}_k^*)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}} = \sum_{k \notin \mathcal{K}} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k||\bar{Q}_k)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}} + \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k||\bar{P}_k^*)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}} \]
\[ = \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k||\bar{P}_k^*)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}}. \]
Then, (25) implies that
\[ \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \frac{\text{KL}(\bar{Q}_k||\bar{P}_k^*)}{\mathbb{E}_{X \sim \bar{Q}_k[-X]}} \leq \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \gamma_k^* + \frac{\alpha_{\min} e}{4(K-1)\gamma_{\max}} \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \gamma_k^* \]
\[ \leq \sum_{k \in \mathcal{K}} \beta_k(\bar{Q}) \gamma_k^* + \frac{\alpha_{\min} e}{4(K-1)\gamma_{\max}} \sum_{k \in \mathcal{K}} \gamma_k^*. \]
By definition of $K$, for any $k \in K$, $\beta_k(Q) \leq \alpha_k - \frac{\alpha_{\min} \epsilon}{2(K-1)\gamma_{\max}}$ and we deduce that

$$
\sum_{k \in K} \beta_k(Q) \text{KL}(Q_k||P^*_k) \leq \sum_{k \in K} \alpha_k \bar{\gamma}_k - \frac{\alpha_{\min} \epsilon}{2(K-1)} \sum_{k \in K} \bar{\gamma}_k + \frac{\alpha_{\min} \epsilon}{4(K-1)} \sum_{k \in K} \gamma_{\max}
$$

where the inequality (26) comes from the fact that $K$ is not empty. Injecting (26) in (17) (with $P = P^*$ and $Q = \bar{Q}$), we have:

$$
\liminf_{B \to +\infty} \frac{1}{B} \log P\left(\tau(B, \pi) < \frac{3B}{\Delta^*}\right) \geq - \sum_{k=1}^K \alpha_k \bar{\gamma}_k + \frac{\alpha_{\min} \epsilon \bar{\gamma}_{\min}}{4(K-1)\gamma_{\max}}.
$$

Recall that, by definition,

$$
\epsilon' = \frac{\epsilon \alpha_{\min} \bar{\gamma}_{\min}}{4(K-1)\gamma_{\max}}.
$$

We deduce the following

$$
\liminf_{B \to +\infty} \frac{1}{B} \log P(P_1, ..., P_K) \left(\tau(B, \pi) < \frac{3B}{\Delta^*}\right) \geq - \sum_{k=1}^K \alpha_k \bar{\gamma}_k + \epsilon',
$$

which concludes the proof of the theorem.

\[ \square \]

### D.3. Sub-additivity of the optimal log probability of ruin

The passage from the asymptotic to the non-asymptotic lower bound on the probability of ruin relies on sub-additivity properties described in the next lemma, which is the main result of this subsection.

**Lemma 25** Let $\bar{t} \in R^+ \cup \{+\infty\}$. For any $B_1, B_2 > 0$, we have

$$
\inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_1 + B_2 + 1, \pi) < \bar{t}\right)}{A_F} \leq \inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_1, \pi) < \bar{t}\right)}{A_F} + \inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_2, \pi) < \bar{t}\right)}{A_F}.
$$

In the case of multinomial arm distributions of support $\{-1, 0, 1\}$, if $B_1$ and $B_2$ are positive integers, the previous bound can be refined as

$$
\inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_1 + B_2, \pi) < \bar{t}\right)}{A_F} \leq \inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_1, \pi) < \bar{t}\right)}{A_F} + \inf_{\pi} \sup_{F} \frac{\log P\left(\tau(B_2, \pi) < \bar{t}\right)}{A_F}.
$$
Proof Let

\[
\pi_1^* \in \arg \min_{\pi} \sup_{\tau} \log \frac{P(\tau(B_1, \pi) < \tilde{t})}{A_F},
\]

\[
\pi_2^* \in \arg \min_{\pi} \sup_{\tau} \log \frac{P(\tau(B_2, \pi) < \tilde{t})}{A_F}.
\]

Besides, we denote by \(\tilde{\pi}\) the policy such that

\[
\tilde{\pi}_t := \begin{cases} 
  (\pi_1^*_t) & \text{if } t < \min (\tau(B_1, \tilde{\pi}), \tilde{t}) , \\
  (\pi_2^*_{t-\tau(B_1, \tilde{\pi})}) & \text{(ignoring the previously observed rewards)} \text{ otherwise}.
\end{cases}
\]

Let \(B'_2 \in \{B_2, B_2 + 1\}\). Then, it is clear that

\[
P(\tau(B_1 + B'_2, \tilde{\pi}) < \tilde{t})
\]

\[
= P\left( \exists 1 \leq t_{1+2} \leq \tilde{t} : B_1 + B'_2 + \sum_{s=1}^{t_{1+2}} X_{s}^{\tilde{\pi}_s} < 0 \right)
\]

\[
= P\left( \exists 1 \leq t_1, t_{1+2} \leq \tilde{t} : B_1 + \sum_{s=1}^{t_1} X_{s}^{\tilde{\pi}_s} < 0, B_1 + B'_2 + \sum_{s=1}^{t_{1+2}} X_{s}^{\tilde{\pi}_s} < 0 \right)
\]

\[
= P\left( \exists 1 \leq t_1 \leq \tilde{t} : B_1 + \sum_{s=1}^{t_1} X_{s}^{\tilde{\pi}_s} < 0 \right)
\]

\[
\times P\left( \exists \tau(B_1, \tilde{\pi}) \leq t_{1+2} \leq \tilde{t} : B_1 + B'_2 + \sum_{s=1}^{t_{1+2}} X_{s}^{\tilde{\pi}_s} < 0 \left| \tau(B_1, \tilde{\pi}) < \tilde{t} \right. \right)
\]

\[
= P(\tau(B_1, \pi_1^*) < \tilde{t}) \times P\left( \exists \tau(B_1, \tilde{\pi}) \leq t_{1+2} \leq \tilde{t} : B_1 + B'_2 + \sum_{s=1}^{t_{1+2}} X_{s}^{\tilde{\pi}_s} < 0 \left| \tau(B_1, \pi_1^*) < \tilde{t} \right. \right)
\]

\[
= P(\tau(B_1, \pi_1^*) < \tilde{t}) \times P\left( \exists \tau(B_1, \tilde{\pi}) \leq t_{1+2} \leq \tilde{t} : B_1 + \sum_{s=1}^{\tau(B_1, \pi_1^*)} X_{s}^{(\pi_1^*)}_s + B'_2 + \sum_{s=\tau(B_1, \pi_1^*)+1}^{t_{1+2}} X_{s}^{(\pi_2^*)}_s < 0 \left| \tau(B_1, \pi_1^*) < \tilde{t} \right. \right).
\]

From there, we are going to study separately the general case and the case of multinomial distributions of support \([-1, 0, 1]\).

First case: in the general case, we choose \(B'_2 = B_2 + 1\), and since the rewards are bounded in \([-1, 1]\), then

\[
\tau(B_1, \pi_1^*) < \tilde{t} \implies B_1 + \sum_{s=1}^{\tau(B_1, \pi_1^*)} X_{s}^{(\pi_1^*)}_s + 1 \geq 0.
\]
Hence,
\[
P(\tau(B_1 + B_2 + 1, \pi) < \tilde{t}) \\
\leq P(\tau(B_1, \pi_1^*) < \tilde{t}) \\
\times P \left( \exists \tau(B_1, \pi_1^*) \leq t_{1+2} \leq \tilde{t}: B_2 + \sum_{s=\tau(B_1, \pi_1^*)+1}^{t_{1+2}} X_s^{(\pi_2^*)} < 0 \bigg| \tau(B_1, \pi_1^*) < \tilde{t} \right) \\
\leq P(\tau(B_1, \pi_1^*) < \tilde{t}) \\
\times P \left( \exists \tau(B_1, \pi_1^*) \leq t_{1+2} \leq \tilde{t} + \tau(B_1, \pi_1^*): B_2 + \sum_{s=\tau(B_1, \pi_1^*)+1}^{t_{1+2}} X_s^{(\pi_2^*)} < 0 \bigg| \tau(B_1, \pi_1^*) < \tilde{t} \right) \\
= P(\tau(B_1, \pi_1^*) < \tilde{t}) \times P(\tau(B_2, \pi_2^*) < \tilde{t}).
\]

This yields
\[
\inf_{\pi} \sup_F \log \frac{P \left( \tau(B_1 + B_2 + 1, \pi) < \tilde{t} \right)}{A_F} \\
\leq \sup_F \log \frac{P \left( \tau(B_1 + B_2 + 1, \pi) < \tilde{t} \right)}{A_F} \\
\leq \sup_F \log \frac{P \left( \tau(B_1, \pi_1^*) < \tilde{t} \right)}{A_F} + \sup_F \log \frac{P \left( \tau(B_2, \pi_2^*) < \tilde{t} \right)}{A_F} \\
= \inf_{\pi} \sup_F \log \frac{P \left( \tau(B_1, \pi) < \tilde{t} \right)}{A_F} + \inf_{\pi} \sup_F \log \frac{P \left( \tau(B_2, \pi_2^*) < \tilde{t} \right)}{A_F},
\]

which concludes the general case.

**Second case:** in the case of multinomial arm distributions of support \{-1, 0, 1\},
\[
\tau(B_1, \pi_1^*) < \tilde{t} \implies B_1 + \sum_{s=1}^{\tau(B_1, \pi_1^*)} X_s^{(\pi_1^*)} = 0.
\]

Therefore, choosing \(B_2' = B_2\), we have
\[
P \left( \tau(B_1 + B_2, \tilde{\pi}) < \tilde{t} \right) \\
\leq P(\tau(B_1, \pi_1^*) < \tilde{t}) \\
\times P \left( \exists \tau(B_1, \pi_1^*) \leq t_{1+2} \leq \tilde{t}: B_2 + \sum_{s=\tau(B_1, \pi_1^*)+1}^{t_{1+2}} X_s^{(\pi_2^*)} < 0 \bigg| \tau(B_1, \pi_1^*) < \tilde{t} \right) \\
\leq P(\tau(B_1, \pi_1^*) < \tilde{t}) \\
\times P \left( \exists \tau(B_1, \pi_1^*) \leq t_{1+2} \leq \tilde{t} + \tau(B_1, \pi_1^*): B_2 + \sum_{s=\tau(B_1, \pi_1^*)+1}^{t_{1+2}} X_s^{(\pi_2^*)} < 0 \bigg| \tau(B_1, \pi_1^*) < \tilde{t} \right) \\
= P(\tau(B_1, \pi_1^*) < \tilde{t}) \times P(\tau(B_2, \pi_2^*) < \tilde{t}).
\]

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This yields
\[
\inf_{\pi} \sup_{P} \frac{\log P(\tau(B_1 + B_2, \pi) < \tilde{t})}{A_F} \\
\leq \sup_{P} \frac{\log P(\tau(B_1 + B_2, \pi) < \tilde{t})}{A_F} \\
\leq \sup_{P} \frac{\log P(\tau(B_1, \pi_1) < \tilde{t})}{A_F} + \sup_{P} \frac{\log P(\tau(B_2, \pi_2) < \tilde{t})}{A_F} \\
= \inf_{\pi} \sup_{P} \frac{\log P(\tau(B_1, \pi) < \tilde{t})}{A_F} + \inf_{\pi} \sup_{P} \frac{\log P(\tau(B_2, \pi_2) < \tilde{t})}{A_F},
\]
which concludes the multinomial case and the proof of the lemma.

**D.4. Proof of Theorem 8 and Corollary 9**

Let \((\alpha_k)_{k \in [K]}\) such that for any \(k \in [K]\), \(\alpha_k > 0\) and \(\sum_{k=1}^{K} \alpha_k = 1\). Recall that, by definition,
\[
A_F = \sum_{k=1}^{K} \alpha_k \inf_{Q_k: \mathbb{E}_{X \sim Q_k}[X] < 0} \frac{\text{KL}(Q_k\|F_k)}{\mathbb{E}_Q[-X]}. 
\]

Let \(\pi\) be any policy, and \(B_0 > 0\) an initial budget. For any \(n \geq 1\), let us denote by \(\pi^B\) the policy defined recursively on \(\{B \geq B_0\}\), such that \(\pi^B_0 = \pi\) and for any \(B \geq B_0, \pi^B_t = \pi_t\) for \(t \leq \tau(B_0, \pi)\) and then \(\pi^B_t = \pi^{B'}_t\) for \(t \geq \tau(B_0, \pi) + 1\), where \(B' = B + \sum_{s=1}^{\tau(B, \pi)} \). Concretely, \(\pi^B\) restarts \(\pi\) every time it exhausts the budget \(B_0\). Let
\[
\Delta_F := \min \left\{ \mathbb{E}_{X \sim R_k}[-X] \middle| k \in [K], \forall i \in [K], R_i = \arg \min_{Q: \mathbb{E}_Q[X] < 0} \frac{\text{KL}(Q\|F_i)}{\mathbb{E}_Q[-X]} \right\},
\]
and \(\bar{\Delta} := \Delta_F\). From now, we are going to study separately the general case of rewards bounded in \([-1, 1]\) and the case of multinomial arms of support \(\{-1, 0, 1\}\).

**First case:** in the case of multinomial arm distributions in \(\mathcal{F}^K_{\{-1,0,1\}}\), we assume that, for any arm distributions \(F = (F_1, \ldots, F_K)\),
\[
\frac{\log P_F(\tau(B_0, \pi) < \frac{3B_0}{\Delta_F})}{A_F} \leq -B_0,
\]
and that there exist some arm distributions \(\bar{F} = (\bar{F}_1, \ldots, \bar{F}_K)\) and \(C_F > 0\) such that
\[
\frac{\log P_F(\tau(B_0, \pi) < \frac{3B_0}{\bar{\Delta}})}{A_F} \leq -(B_0 + C_F),
\]
and show that there is contradiction.
By Lemma 25, for any arm distributions $F$ and for any $n \geq 1$,
\[
\log P \frac{P_F}{A_F} \left( \tau (nB_0, \pi^{nB_0}) < \frac{3B_0}{\Delta_F} \right) \leq n \times \log P \frac{P_F}{A_F} \left( \tau (B_0, \pi^{B_0}) < \frac{3B_0}{\Delta_F} \right) \leq -n B_0.
\]

Consequently, for any arm distributions $F$,
\[
\limsup_{n \to +\infty} \frac{1}{n B_0} \log P \left( \tau (nB_0, \pi) < \frac{3B_0}{\Delta_F} \right) \leq -A_F. \tag{27}
\]

Furthermore, the same computation applied to $\bar{F}$ gives
\[
\log P \frac{P_{\bar{F}}}{A_{\bar{F}}} \left( \tau (nB_0, \pi^{nB_0}) < \frac{3B_0}{\Delta_F} \right) \leq n \times \log P \frac{P_{\bar{F}}}{A_{\bar{F}}} \left( \tau (B_0, \pi^{B_0}) < \frac{3B_0}{\Delta_F} \right) \leq -n (B_0 + C_{\bar{F}}),
\]

which in turn, implies,
\[
\limsup_{n \to +\infty} \frac{1}{n B_0} \log P \left( \tau (nB_0, \pi) < \frac{3B_0}{\Delta} \right) \leq -\frac{B_0 + C_{\bar{F}}}{B_0} A_{\bar{F}} < -A_{\bar{F}}. \tag{28}
\]

Eq. (27) and (28) contradict Theorem 24. Therefore, we deduce that if there exist some arm distributions $\bar{F}$ such that
\[
\log P \frac{P_{\bar{F}}}{A_{\bar{F}}} \left( \tau (B_0, \pi) < \frac{3B_0}{\Delta} \right) < -B_0,
\]
then there also exist some arm distributions $F$ such that
\[
\log P \frac{P_F}{A_F} \left( \tau (B_0, \pi) < \frac{3B_0}{\Delta_F} \right) > -B_0,
\]
concluding the multinomial case and the proof of Theorem 8.

**Second case:** in the general case of arm distributions in $\mathcal{F}_K^{[-1,1]}$, we assume that, for any arm distributions $F = (F_1, \ldots, F_K)$,
\[
\log P \frac{P_F}{A_F} \left( \tau (B_0, \pi) < \frac{3B_0}{\Delta_F} \right) \leq -(B_0 + 1),
\]
and that there exist some arm distributions $\bar{F} = (\bar{F}_1, \ldots, \bar{F}_K)$ and $C_{\bar{F}} > 0$ such that
\[
\log P \frac{P_{\bar{F}}}{A_{\bar{F}}} \left( \tau (B_0, \pi) < \frac{3B_0}{\Delta} \right) \leq -(B_0 + 1 + C_{\bar{F}}),
\]
and show that there is contradiction.
By Lemma 25, for any arm distributions $F$ and for any $n \geq 1$, 

$$
\frac{\log P_F \left( \tau(nB_0 + (n-1), \pi^{nB_0}) < \frac{3B_0}{\Delta_F} \right)}{A_F} \leq n \frac{\log P \left( \tau(B_0, \pi^{B_0}) < \frac{3B_0}{\Delta} \right)}{A_F} \\
\leq -n(B_0 + 1).
$$

Consequently, for any arm distributions $F$, 

$$
\limsup_{n \to +\infty} \frac{1}{nB_0 + (n-1)} \log P \left( \tau(nB_0 + (n-1), \pi) < \frac{3B_0}{\Delta_F} \right) \leq -A_F. \quad (29)
$$

Furthermore, the same computation applied to $\bar{F}$ gives 

$$
\frac{\log P_{\bar{F}} \left( \tau(nB_0 + (n-1), \pi^{nB_0}) < \frac{3B_0}{\bar{\Delta}} \right)}{A_{\bar{F}}} \leq n \frac{\log P \left( \tau(B_0, \pi^{B_0}) < \frac{3B_0}{\bar{\Delta}} \right)}{A_{\bar{F}}} \\
\leq -n(B_0 + 1 + C_{\bar{F}}),
$$

which in turn, implies, 

$$
\limsup_{n \to +\infty} \frac{1}{nB_0 + (n-1)} \log P \left( \tau(nB_0, \pi) < \frac{3B_0}{\Delta} \right) \leq -\frac{B_0 + 1 + C_{\bar{F}}}{B_0 + 1} < -1. \quad (30)
$$

Eq. (29) and (30) contradict Theorem 24. Therefore, we deduce that if there exist some arm distributions $\bar{F}$ such that 

$$
\frac{\log P_{\bar{F}} \left( \tau(B_0, \pi) < \frac{3B_0}{\bar{\Delta}} \right)}{A_{\bar{F}}} < -(B_0 + 1),
$$

then there also exist some arm distributions $F$ such that 

$$
\frac{\log P_F \left( \tau(B_0, \pi) < \frac{3B_0}{\Delta} \right)}{A_F} > -(B_0 + 1),
$$

concluding the general case and the proof of Corollary 9. ■

**Appendix E. Proof of Lemma 11**

Please note that applying (17) to the case of one single arm ($K = 1$) gives that, for any $\epsilon_0 \in \left(0, \frac{1}{3}\right)$ and any distribution $Q$ which has a negative expectation, 

$$
\liminf_{B \to +\infty} \frac{1}{B} \log P(\tau(B, 1) < \infty) \geq -\frac{\text{KL}(Q\|F_1)}{\mathbb{E}_{X \sim Q}[X]},
$$

By taking $\epsilon_0 \downarrow 0$, we deduce that 

$$
\liminf_{B \to +\infty} \frac{1}{B} \log P(\tau(B, 1) < \infty) \geq -\inf_{F : \mathbb{E}_{X \sim F}[X] < 0} \frac{\text{KL}(F\|F_1)}{\mathbb{E}_{X \sim F}[-X]}.
$$
It thus remains to prove that for any $B > 0$,

$$\frac{1}{B} \log P(\tau(B, 1) < \infty) \leq -\inf_{F : \mathbb{E}_{X \sim F}[X] < 0} \frac{\text{KL}(F \| F_1)}{\mathbb{E}_{X \sim F}[-X]}.$$ 

The result being trivial if $\mathbb{E}_{X \sim F_1}[X] \leq 0$, we assume that $\mathbb{E}_{X \sim F_1}[X] > 0$. Let us define the logarithmic moment generating function of $X$ by $\Lambda(\lambda) := \log \mathbb{E}[e^{\lambda X}]$. By Lemma 22, there exists $\lambda' < 0$ such that $\Lambda(\lambda') = 0$ and it satisfies

$$\inf_{F : \mathbb{E}_{X \sim F}[X] < 0} \frac{\text{KL}(F \| F_1)}{\mathbb{E}_{X \sim F}[-X]} = -\lambda'.$$

Now, let $X \sim F_1$ and let $X_1, X_2 \cdots \sim F_1$ be i.i.d copies of $X$. We write $S_n = \sum_{i=1}^n X_i$. We define $\tau := \inf\{n \geq 1 : S_n \leq -B\}$ and $\tau_T := \min(\tau, T)$ for any $T \in \mathbb{N}$. Since $\tau_T$ is a bounded stopping time, by the optional stopping theorem, it holds for any $T$ that

$$\mathbb{E}[e^{\lambda' S_{\tau_T}}] = 1.$$

On the other hand,

$$1 = \mathbb{E}[e^{\lambda' S_{\tau_T}}] = \mathbb{E}\left[\mathbb{1}_{\tau_T < T} e^{\lambda' S_{\tau_T}}\right] + \mathbb{E}\left[\mathbb{1}_{\tau_T = T} e^{\lambda' S_{\tau_T}}\right] \geq \mathbb{E}\left[\mathbb{1}_{\tau_T < T} e^{\lambda' S_{\tau_T}}\right] \geq e^{-\lambda' B} P(\tau_T < T),$$

which implies that

$$P(\tau < \infty) = \lim_{T \to \infty} P(\tau_T < T) \leq e^{\lambda' B}.$$

Therefore we obtain

$$\frac{1}{B} \log P(\tau < \infty) \leq \lambda' = -\inf_{F : \mathbb{E}_{X \sim F}[X] < 0} \frac{\text{KL}(F \| F_1)}{\mathbb{E}_{X \sim F}[-X]},$$

which completes the proof.

---

**Appendix F. Proof of the Upper Bound on the Reward of EXPLOIT Policies**

(Proposition 15)

We introduce the following notation. Let $K_+$ be the number of arms such that $P\left(\tau\left(\frac{B}{K}, k\right) = \infty\right) > 0$ for $k \in [K]$. Recall that we ordered the arms in order of decreasing expectation, and therefore $\mu_1 \geq \cdots \geq \mu_{K_+} > 0$. Then, by definition of $K_+$,

$$\forall k \in [K_+], P\left(\tau\left(\frac{B}{K}, k\right) = \infty\right) > 0 ; \forall j \in \{K_+ + 1, \ldots, K\} , P\left(\tau\left(\frac{B}{K}, j\right) = \infty\right) = 0.$$
F.1. Preliminary Lemma

In this subsection, we express the expected cumulative reward of any policy in EXPLOIT as the product between \( p^E \) and a convex combination of \( \mu_1, \ldots, \mu_{K+} \), and we explicitly give the coefficients of the convex combination. This decomposition will be useful both as a first step in the proof of Proposition 15 and in the proof of the reward bound of EXPLOIT-UCB, as a particular instance of policy in EXPLOIT.

For any \( S \subseteq [K+] \), we define the event

\[
\Pi_S := \left\{ \forall j \in S, \tau \left( \frac{B}{K}, j \right) \geq T \quad \text{and} \quad \forall j \in [K+] \setminus S, \tau \left( \frac{B}{K}, j \right) < \sqrt{T} \right\}.
\]

Given a policy \( \pi \), an arm \( k \in [K+] \) and a set \( S \subseteq [K+] \), we define the coefficient \( n_{\pi,k}(S) \) as

\[
n_{\pi,k}(S) := \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbbm{1}_{\pi_t = k, \tau(B,\pi) \geq t-1} \middle| \Pi_S \right].
\]

When there is no ambiguity on the policy \( \pi \), we simply write \( n_k(S) \). Please note that for any fixed policy \( \pi \) and any set \( S \subseteq [K+] \),

\[
\sum_{k \in S} n_k(S) \leq \sum_{k=1}^{K} n_k(S) = \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbbm{1}_{\tau(B,\pi) \geq t-1} \middle| \Pi_S \right] \leq 1.
\]

The following result holds.

**Lemma 26**  For any policy \( \pi \) within the framework EXPLOIT, the expected cumulative reward satisfies

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_{\pi_t} \mathbbm{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{k=1}^{K} \left( \sum_{S \subseteq [K+] \setminus k \in S} P(\Pi_S)n_k(S) \right) \mu_k \times T + o(T). \quad (31)
\]

**Proof**  First, we write the reward as a sum over the arms of positive probability of survival:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_{\pi_t} \mathbbm{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{k=1}^{K} \mu_k \mathbb{E} \left[ \sum_{t=1}^{T} \mathbbm{1}_{\pi_t = k, \tau(B,\pi) \geq t-1} \right] = \sum_{k=1}^{K} \mu_k \mathbb{E} \left[ \sum_{t=1}^{T} \mathbbm{1}_{\pi_t = k} \mathbbm{1}_{\tau(B,\pi) \geq t-1} \right] + o(T). \quad (32)
\]

Then, we examine the term \( \mathbb{E} \left[ \sum_{t=1}^{T} \mathbbm{1}_{\pi_t = k} \mathbbm{1}_{\tau(B,\pi) \geq t-1} \right] \). In order to analyse it, we will introduce the following events, for any \( S, S' \subseteq [K+] \) such that \( S \cap S' = \emptyset \):

\[
\Pi_{S,S'} := \left\{ \forall j \in S, \tau \left( \frac{B}{K}, j \right) \geq T; \forall j \in S', \sqrt{T} \leq \tau \left( \frac{B}{K}, j \right) < T; \right. \\
\left. \forall j \in [K+] \setminus (S \cup S'), \tau \left( \frac{B}{K}, j \right) < \sqrt{T} \right\}.
\]
Please note that $\Pi_S = \Pi_{S,0}$. We can then decompose, for any $k \in \{1, \ldots, K_+\}$,

$$
E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{S \subseteq [K_+]} P(\Pi_S) \sum_{s \subseteq [K_+]} P(\Pi_{S,S'}) \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \Pi_{S,S'}.
$$

Consider the case $S' \neq \emptyset$ and let $k \in S'$. We can bound

$$
P(\Pi_{S,S'}) \leq P\left( \sqrt{T} \leq \tau \left( \frac{B_{TB}}{K}, k \right) < T \right)
$$

Indeed, the sequence $\{P(\tau(k))\}_{n \geq 1}$ is increasing and upper-bounded by 1, and thus it has a limit and it implies that

$$
P\left( \tau \left( \frac{B_{TB}}{K}, k \right) \geq \sqrt{T} \right) - P\left( \tau \left( \frac{B_{TB}}{K}, k \right) \geq T \right) = o(1).
$$

We deduce that, for any $S, S' \subseteq [K_+]$ such that $S \cap S' = \emptyset$,

$$
S' \neq \emptyset \implies P(\Pi_{S,S'}) = o(1).
$$

This implies

$$
E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{S \subseteq [K_+]} P(\Pi_S) \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \Pi_S + o(T).
$$

Re-injecting in (32), we have

$$
E \left[ \sum_{t=1}^{T} X_{B_t} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{k=1}^{K_+} \mu_k \sum_{S \subseteq [K_+]} P(\Pi_S) \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \Pi_S + o(T)
$$

$$
= \sum_{S \subseteq [K_+]} P(\Pi_S) \sum_{k=1}^{K_+} \mu_k E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \Pi_S + o(T).
$$

Besides, it is clear, by definition of $\Pi_S$, that any policy $\pi$ in EXPLOIT satisfies

$$
k \notin S \implies E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] = o(T).
$$

We deduce that

$$
E \left[ \sum_{t=1}^{T} X_{B_t} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] = \sum_{S \subseteq [K_+]} P(\Pi_S) \sum_{k \in S} \mu_k E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \Pi_S + o(T)
$$

$$
= \sum_{S \subseteq [K_+]} P(\Pi_S) \sum_{k \in S} \mu_k n_k(S) T + o(T)
$$

$$
= \sum_{k=1}^{K_+} \left( \sum_{S \subseteq [K_+] : k \in S} P(\Pi_S) n_k(S) \right) \mu_k T + o(T),
$$

which concludes the proof of the lemma. 











F.2. Proof of Proposition 15

Then, in order to complete the proof of Proposition 15, it remains to provide an upper bound to the right hand of the equality (31) in Lemma 26. In order to maximize the right term in (31), we should solve the following maximization problem:

$$\max_{k=1}^{K_+} \left( \sum_{S \subseteq [K_+]: k \in S} P(\Pi_S) n_k(S) \right) \mu_k \quad \text{s.t.} \quad \sum_{k \in S} n_k(S) \leq 1 \quad \text{and} \quad \forall k \notin S, n_k(S) = 0.$$ 

By hypothesis, $\mu_1 \geq \mu_2 \geq \ldots \mu_{K_+} > 0$ and the solution is well-known. It is obtained for

$$n_k^*(S) = \begin{cases} 1 & \text{if } k = \min S \\ 0 & \text{otherwise}, \end{cases}$$

in which case, for any given $k \in \{1, \ldots, K_+\}$,

$$\sum_{S \subseteq [K_+]: k \in S} P(\Pi_S) n_k^*(S)$$

$$= \sum_{S \subseteq \{k+1, \ldots, K_+\}} P(\Pi_{\{k\} \cup S})$$

$$= \sum_{S \subseteq \{k+1, \ldots, K_+\}} P\left(\forall j \in \{k\} \cup S, \tau\left(\frac{B}{K}, j\right) \geq T; \forall j \in [K_+] \setminus \{k\} \cup S, \tau\left(\frac{B}{K}, j\right) < \sqrt{T}\right)$$

$$= P\left(\forall j \in \{k-1\}, \tau\left(\frac{B}{K}, j\right) < \sqrt{T}; \tau\left(\frac{B}{K}, k\right) \geq T\right) + o(1)$$

$$= (1 - p^{EX}) \frac{1}{1 - p^{EX}} \prod_{j=1}^{k-1} P\left(\tau\left(\frac{B}{K}, j\right) < \infty\right) P\left(\tau\left(\frac{B}{K}, k\right) = \infty\right) + o(1).$$

We deduce that

$$\mathbb{E} \left[ \sum_{t=1}^{T} X^t_{\pi^t \uparrow \tau(B, \pi) \geq t-1} \right] \leq (1 - p^{EX}) \sum_{k=1}^{K_+} w_k u_k T + o(T),$$

which concludes the proof of the proposition.

Appendix G. Proof of the Reward Bound of EXPLOIT-UCB (Proposition 16)

In this appendix and in the next one, we will use the following notations. For any $k \in [K]$, let $Y^k_1, \ldots, Y^k_T \sim F_k$ be i.i.d. rewards drawn from arm $k$ and for any $t \geq 1$, and we denote by $N_k(t) := \sum_{s=1}^{t} 1_{\pi_s = k}$ the number of times arm $k$ has been pulled until time step $t$. Please note that $X^t_{\pi^t} = \sum_{k=1}^{K} \hat{Y}^k_{N_k(t)} 1_{\pi_t = k}$. Given that arm $k$ has been pulled $n_k$ times, we also introduce

$$\hat{Y}^k_{n_k} := \frac{1}{n_k} \sum_{n=1}^{n_k} Y^k_n$$

as the empirical average of arm $k$ at time step $t$. Please note that for any $k \in [K]$ and any time step $t$,

$$\hat{Y}^k_{N_k(t)} = \hat{X}^k_t.$$
For any \( t \geq 1 \), let
\[
C_k(t) := \hat{Y}_{N_k(t)}^k + \sqrt{\frac{6 \log(t)}{N_k(t)}}.
\]

**G.1. Preliminary Lemma**

EXPLOIT-UCB is based on the classic bandit algorithm UCB1 (which has a sublinear regret in the classic stochastic MAB), and therefore has the following characteristic which is going to be useful in the proof of the reward bound of both EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE. For the sake of clarity, we assume the arms are ordered in decreasing expectation: \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \).

**Lemma 27**  Let \((\pi_t)_{t \geq 1}\) be the policy associated to EXPLOIT-UCB. Assume that \( \mu_1 > \mu_k \), then
\[
E \left[ \sum_{t=1}^T \mathbb{1}_{\pi_t = k, C_k(t) \geq C_1(t)} \right] = o(T).
\]

**Proof**  This proof completely follows the proof of Theorem 1 in Auer et al. (2002). Let \( r \leq T \), the quantity to bound can be written as
\[
E \left[ \sum_{t=1}^T \mathbb{1}_{\pi_t = k, C_k(t) \geq C_1(t)} \right] = E \left[ \sum_{t=1}^T \mathbb{1}_{\pi_t = k, \hat{Y}_{N_k(t)}^k(t) + \sqrt{\frac{6 \log(t)}{N_k(t)}} \geq \hat{Y}_{N_1(t)}^1(t) + \sqrt{\frac{6 \log(t)}{N_1(t)}}} \right] = o(T)
\]

where \( \Delta_{T,r} = \{ t \in \{1, \ldots, T\} : \sum_{s=1}^t \mathbb{1}_{\pi_s = k} \geq r \} \). Similarly as in Auer et al. (2002), we use the fact that the probability event \( \left\{ \hat{Y}_{N_k}^k + \sqrt{\frac{6 \log(t)}{N_k}} \geq \hat{Y}_{N_1}^1 + \sqrt{\frac{6 \log(t)}{N_1}} \right\} \) implies at least one of the
following:

\[ \hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log t}{n_1}} \]

\[ \hat{Y}_{n_k}^\ast \geq \mu_k + \sqrt{\frac{6 \log t}{n_k}} \]

\[ \mu_1 < \mu_k + 2\sqrt{\frac{6 \log t}{n_k}}. \]

Therefore, we can write

\[
\mathbb{E} \left[ \sum_{t=1}^{T} 1_{\pi_t=k, C_k(t) \geq C_1(t)} \right] \\
\leq r + \mathbb{E} \left[ \sum_{t \in \Delta T, n_1=1, n_k=r} \sum_{t=1}^{t-1} \sum_{n_1=1}^{t-1} \hat{Y}_{n_k}^\ast \geq \mu_k + \sqrt{\frac{6 \log t}{n_k}} \right] + o(T)
\]

\[
\leq r + \mathbb{E} \left[ \sum_{t \in \Delta T, n_1=1, n_k=r} \sum_{t=1}^{t-1} \sum_{n_1=1}^{t-1} \left( 1_{\hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log t}{n_1}}} + 1_{\hat{Y}_{n_1}^\ast \geq \mu_1 + \sqrt{\frac{6 \log t}{n_1}}} + 1_{\hat{Y}_{n_k}^\ast \geq \mu_k + 2\sqrt{\frac{6 \log t}{n_k}}} \right) \right] + o(T).
\]

The choice

\[ r = \left\lceil \frac{24 \log T}{(\mu_1 - \mu_k)^2} \right\rceil \]

ensures that, for any \( n_k \geq r \),

\[ \mu_1 - \mu_k - 2\sqrt{\frac{6 \log T}{n_k}} \geq 0, \]

which implies

\[
\mathbb{E} \left[ \sum_{t=1}^{T} 1_{\pi_t=k, C_k(t) \geq C_1(t)} \right] \\
\leq \left\lceil \frac{24 \log T}{(\mu_1 - \mu_k)^2} \right\rceil + \mathbb{E} \left[ \sum_{t \in \Delta T, n_1=1, n_k=r} \sum_{t=1}^{t-1} \sum_{n_1=1}^{t-1} \left( 1_{\hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log t}{n_1}}} + 1_{\hat{Y}_{n_k}^\ast \geq \mu_k + \sqrt{\frac{6 \log t}{n_k}}} \right) \right] + o(T)
\]

\[
= T \times \sum_{n_1=1}^{T} \sum_{n_k=r}^{T} \mathbb{E} \left[ \left( 1_{\hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log T}{n_1}}} + 1_{\hat{Y}_{n_k}^\ast \geq \mu_k + \sqrt{\frac{6 \log T}{n_k}}} \right) \right] + o(T)
\]

\[
= T \times \sum_{n_1=1}^{T} \sum_{n_k=r}^{T} \left( P \left( \hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log T}{n_1}} \right) + P \left( \hat{Y}_{n_k}^\ast \geq \mu_k + \sqrt{\frac{6 \log T}{n_k}} \right) \right) + o(T).
\]

Using Hoeffding’s inequality, for any \( n_1 \in \{1, \ldots, T\} \), we have

\[ P \left( \hat{Y}_{n_1}^\ast \leq \mu_1 - \sqrt{\frac{6 \log T}{n_1}} \right) \leq \frac{1}{T^3}. \]
Similarly, for any \( n_k \in \{r, \ldots, T\} \), we have

\[
P \left( \hat{Y}_{n_k}^k \geq \mu_k + \sqrt{\frac{6 \log T}{n_k}} \right) \leq \frac{1}{T^3}.
\]

We can then replace in (33):

\[
E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, C_k(t) \geq C_1(t)} \right] \leq T \sum_{n_1=1}^{T} \sum_{n_k=r}^{n_k} \frac{2}{T^3} + o(T)
\]

\[
= 2 + o(T)
\]

\[
= o(T),
\]

which concludes the proof of the lemma.

G.2. Proof of Proposition 16

Let \( S \subseteq [K_+] \). Recall that for any arm \( k \in [K_+] \),

\[
n_k(S) = \frac{1}{T} E \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, k(B, \pi) \geq t} \right].
\]

Given that EXPLOIT-UCB is in EXPLOIT, by Lemma 26, it suffices to show that for any \( S \subseteq [K_+] \) and any \( k \in S \),

\[
n_k(S) = \begin{cases} 
 1 + o(1) & \text{if } k = \min S \\
 0(1) & \text{otherwise.}
\end{cases}
\]

Actually, by Proposition 15, it is sufficient to prove the above property for any \( S \subseteq [K_+] \) such that there is no positive or zero arm \( k \in [K_+] \setminus S \). Let then \( S \subseteq [K_+] \) satisfying such a property.

Then, please note that \( \sum_{k=1}^{K} n_k(S) = 1 \). Therefore, it suffices to prove that for any \( k \in S \setminus \{\min S\}, n_k(S) = o(1) \). Thus, let \( k \in S \setminus \{\min S\} \). Then, on the one hand, we are going to provide a lower bound \( P(\Pi_S) \) which is independent of \( T \). Indeed, since there is no positive or zero arm \( k \in [K_+] \setminus S \), we know that there exists \( \epsilon > 0 \) such that

\[
\forall k \in [K_+], P_{X \sim F_k}(X \leq -\epsilon) > 0.
\]

We fix such an \( \epsilon \) and we deduce that

\[
\prod_{k \in [K_+] \setminus S} P \left( \tau \left( \frac{B}{K}, k \right) \leq \frac{B}{cK} \right) > 0.
\]

We can therefore provide the following lower bound, independent of \( T \) and positive by definition of \( K_+ \). For any \( T \geq \left( \frac{B}{cK} \right)^2 \),

\[
P(\Pi_S) = \prod_{k \in S} P \left( \tau \left( \frac{B}{K}, k \right) \geq T \right) \prod_{k \in [K_+] \setminus S} P \left( \tau \left( \frac{B}{K}, k \right) < \sqrt{T} \right)
\]

\[
\geq \prod_{k \in S} P \left( \tau \left( \frac{B}{K}, k \right) = \infty \right) \prod_{k \in [K_+] \setminus S} P \left( \tau \left( \frac{B}{K}, k \right) < \frac{B}{cK} \right)
\]

\[
> 0.
\]
On the other hand, an upper bound to
\[ \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, \Pi S} \right] \]

is obtained by
\[ = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, \forall j \in S, \tau \left( \frac{B}{\alpha}, j \right) \geq T} \right] \]
\[ \leq \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, C_k(t-1) \geq C_{\min s}(t-1)} \right] \]
\[ = o(T) \]

by Lemma 27. We deduce the following bound on \( n_k(S) \):
\[ n_k(S) = \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t = k, \Pi S} \right] \]
\[ \leq \frac{1}{T} \prod_{k \in S} P \left( \tau \left( \frac{B}{K}, k \right) = \infty \right) \prod_{k \notin \{K+1\} \setminus S} P \left( \tau \left( \frac{B}{K}, k \right) < \frac{B}{cK} \right) \]
\[ = o(1), \]

which concludes the proof of the proposition.

\[ \Box \]

**Appendix H. The Performance of EXPLOIT-UCB-DOUBLE**

Recall the following notation from the previous section, for any \( t \geq 1 \),
\[ C_k(t) := \frac{k}{N_k(t)} + \sqrt{\frac{6 \log(t)}{N_k(t)}}. \]

**H.1. Proof of Proposition 17**

For any policy \( \pi \) and any budget \( B' \), we will denote \( \pi \in \text{EXPLOIT}(B') \) if at time step \( t \), \( \pi \) only pulls arms \( k \in [K] \) such that \( \sum_{s=1}^{t} X_s^\pi \mathbb{1}_{\pi_s = k} \geq -\frac{B'}{K} + 1 \).

The probability of ruin of EXPLOIT-UCB-DOUBLE can be decomposed as
\[ P(\tau(B, \pi) < T) = \sum_{j=0}^{\infty} P(\tau(B, \pi) < T \cap t_j \leq \tau(B, \pi) < t_{j+1}). \quad (34) \]

Let us first examine the term in \( j = 0 \). Then,
\[ P(\tau(B, \pi) < T \cap \tau(B, \pi) < t_1) \]
\[ \leq P \left( \tau(B, \pi) < T \text{ and } \forall t \leq T, B + \sum_{s=1}^{t} X_s^\pi \mathbb{1}_{\tau(B, \pi) \geq s-1} < nB^2 \right) \]
\[ \leq P \left( \tau(B, \pi) < T \text{ and } \pi \in \text{EXPLOIT}(B) \right) \]
\[ \leq p^\text{EX}. \]

40
Then, let us examine the other terms in the sum. Let \( j \geq 1 \). For any \( t \geq 1 \), we will denote \( \tilde{\pi}_t := \pi_{t_j + t} \).

Let us re-write each of the terms in the sum as

\[
\begin{align*}
P(\tau(B, \pi) < T \text{ and } t_j \leq \tau(B, \pi) < t_{j+1}) &= \\
&= P\left(t_j \leq \tau(B, \pi) < t_{j+1}; B + \sum_{t=1}^{T} X^\pi_t \mathbb{1}_{\tau(B, \pi) \geq t-1} < 0 \right).
\end{align*}
\]

This is re-written as

\[
\begin{align*}
P(\tau(B, \pi) < T \text{ and } t_j \leq \tau(B, \pi) < t_{j+1}) &= \\
&= P\left(t_j \leq \tau(B, \pi) < t_{j+1}; B + \sum_{t=1}^{T} X^\pi_t \mathbb{1}_{t \leq t-1, B + \sum_{r=1}^{s} X^\pi_r > 0} < 0 \right).
\end{align*}
\]

But then, by definition of \( t_j \), under the condition that \( t_j < T \), we have that

\[
B + \sum_{t=1}^{t_j} X^\pi_t \geq jnB^2,
\]

which implies that, for any \( t \geq t_j + 1 \),

\[
B + \sum_{s=1}^{t} X^\pi_s \geq jnB^2 + \sum_{s=t_j + 1}^{t} X^\pi_s.
\]

We can then replace in the previous equation:

\[
\begin{align*}
P(\tau(B, \pi) < T \text{ and } t_j \leq \tau(B, \pi) < t_{j+1}) &\leq \\
&= P\left(t_j \leq \tau(B, \pi) < t_{j+1}; jnB^2 + \sum_{t=t_j+1}^{T} X^\pi_t \mathbb{1}_{t \leq t-1, jnB^2 + \sum_{r=1}^{s} X^\pi_r > 0} < 0 \right).
\end{align*}
\]

This is re-written as

\[
\begin{align*}
P(\tau(B, \pi) < T \text{ and } t_j \leq \tau(B, \pi) < t_{j+1}) &\leq \\
&= P\left(t_j \leq \tau(B, \pi) < t_{j+1}; jnB^2 + \sum_{t=t_j+1}^{T-t_j} X^\pi_{t+t_j} \mathbb{1}_{t \leq t-1, jnB^2 + \sum_{r=1}^{s} X^\pi_{r+t_j} > 0} < 0 \right),
\end{align*}
\]

and then

\[
\begin{align*}
P(\tau(B, \pi) < T \text{ and } t_j \leq \tau(B, \pi) < t_{j+1}) &\leq P(\tau(B, \pi) < t_{j+1}; \tau(B, \tilde{\pi}) < T - t_j) \\
&\leq P(\tau(B, \tilde{\pi}) < \infty; \tilde{\pi} \in \text{EXPLOIT}(jnB^2)) \\
&\leq (p_{\text{EX}})^{jnB}.
\end{align*}
\]
We can then replace in (34):

\[
P(\tau(B, \pi) < T) = \sum_{j=0}^{\infty} P(\tau(B, \pi) < T \cap t_j \leq \tau(B, \pi) < t_{j+1})
\]

\[
\leq p^{\text{EX}} + \sum_{j=1}^{\infty} (p^{\text{EX}})^{jB}
\]

\[
= p^{\text{EX}} + \frac{(p^{\text{EX}})^{nB}}{1 - (p^{\text{EX}})^{nB}},
\]

which gives the desired result. Let \( \epsilon > 0 \), then

\[
n \geq \log \frac{\epsilon}{1+\epsilon} B \log p^{\text{EX}} \implies \frac{(p^{\text{EX}})^{nB}}{1 - (p^{\text{EX}})^{nB}} \leq \epsilon \leq \epsilon \mu^*,
\]

hence

\[
P(\tau(B, \pi) < \infty) \leq p^{\text{EX}} + \frac{\epsilon}{\mu^*},
\]

which concludes the proof of the proposition. 

\[\blacksquare\]

**H.2. Proof of Proposition 18**

We will assume that the arm with the biggest expectation is arm 1 and that it is unique for the sake of clarity. Let \( j := \lceil \frac{T^{1/4}}{nB^2} - 1 \rceil \), we still denote \( \tilde{\pi}_t := \pi_{t+j} \) for any \( t \geq 1 \). We can then decompose the reward as follows:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\tilde{\pi}_t} 1_{\tau(B, \pi) \geq t-1} \right]
\]

\[
= \sum_{k=1}^{K} \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\bar{\pi}_t} 1_{\pi_t=k} 1_{\tau(B, \pi) \geq t-1} \right]
\]

\[
= \mu_1 \mathbb{E} \left[ \sum_{t=1}^{T} 1_{\pi_t=1} 1_{\tau(B, \pi) \geq t-1} \right] + \sum_{k=2}^{K} \mu_k \mathbb{E} \left[ \sum_{t=1}^{T} 1_{\pi_t=k} 1_{\tau(B, \pi) \geq t-1} \right]
\]

\[
= \mu_1 \mathbb{E} \left[ \sum_{t=1}^{T} 1_{\pi_t=1} 1_{\tau(B, \pi) \geq t-1} \right] + \sum_{k=2}^{K} \mu_k \mathbb{E} \left[ \sum_{t=t_j+1}^{T} 1_{\pi_t=k} 1_{\tau(B, \pi) \geq t-1} \right] + O \left( \mathbb{E} \left[ t_j \right] \right).
\]

If we prove that

\[
(A) = P(\tau(B, \pi) = \infty)T + o(T), \quad \forall k \in \{2, \ldots, K\}, (B_k) = o(T), \quad \mathbb{E} \left[ t_j \right] = o(T),
\]

then, we can write that

\[
\mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\tilde{\pi}_t} 1_{\tau(B, \pi) \geq t-1} \right] = \mu_1 P(\tau(B, \pi) = \infty)T + o(T),
\]

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and using Proposition 17 gives the result:

\[
\mathbb{E}\left[ \sum_{t=1}^{T} X_t^\pi \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \geq \mu_1 \left( 1 - p^{EX} - \frac{(p^{EX})^{nB}}{1 - (p^{EX})^{nB}} \right) T + o(T).
\]

Let \( \epsilon > 0 \), then by Proposition 17,

\[
n \geq \frac{\log \frac{\epsilon}{B \log p^{EX}}}{1 - T B \log p^{EX}} \implies P(\tau(B,\pi) = \infty) \geq 1 - p^{EX} - \epsilon.
\]

In particular, the choice of \( \epsilon = \frac{T B \log p^{EX}}{1 - T B \log p^{EX}} = o_T(1) \) gives \( \log \frac{\epsilon}{B \log p^{EX}} = \log T \), hence

\[
n \geq \log T \implies \mathbb{E}\left[ \sum_{t=1}^{T} X_t^\pi \mathbb{1}_{\tau(B,\pi) \geq t-1} \right] \geq \mu_1 (1 - p^{EX}) T + o(T),
\]

which concludes the proof. It only remains to study each of the terms \((A), (B_k)\) for \( k \geq 2 \) and \( \mathbb{E}[t_j] \).

**Study of \((B_k)\)**

Let \( k \in \{2, \ldots, K\} \). We can decompose the term \((B_k)\) as follows:

\[
(B_k) = \mathbb{E}\left[ \sum_{t=t_j+1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right]
\]

\[
= \mathbb{E}\left[ \sum_{t=t_j+1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t_j} \quad \forall s \leq t_{j-1}, B + \sum_{r=1}^{s} X_r^\pi > 0 \right]
\]

\[
= \mathbb{E}\left[ \sum_{t=t_j+1}^{T} \mathbb{1}_{\pi_t = k} \mathbb{1}_{\tau(B,\pi) \geq t_j} \quad \forall s \in \{t_j+1, \ldots, t-1\}, B + \sum_{r=1}^{t_j} X_r^\pi + \sum_{r=t_j+1}^{s} X_r^\pi > 0 \right].
\]

But then, by definition of \( t_j \), if \( t_j \leq \min(\tau(B,\pi), T) \),

\[
B + \sum_{r=1}^{t_j} X_r^\pi < jnB^2 + 1 = T^{1/4} + 1.
\]

This implies that, if \( t_j \leq \min(\tau(B,\pi), T) \), denoting \( \tilde{\pi}_s := \pi_{s+t_j} \) for any \( s \geq 1 \),

\[
\mathbb{1}_{\forall s \in \{t_j+1, \ldots, t-1\}, B + \sum_{r=1}^{t_j} X_r^\pi + \sum_{r=t_j+1}^{s} X_r^\pi > 0} \leq \mathbb{1}_{\forall s \in \{t_j+1, \ldots, t-1\}, T^{1/4} + 1 + \sum_{r=t_j+1}^{s} X_r^\pi > 0}
\]

\[
= \mathbb{1}_{\forall s \in \{1, \ldots, t-t_j-1\}, T^{1/4} + 1 + \sum_{r=1}^{s-t_j} X_r^\pi > 0}
\]

\[
= \mathbb{1}_{\tau(T^{1/4} + 1, \tilde{\pi}) \geq t-t_j-1}.
\]
and thus, if \( t_j \leq \min(\tau(B, \pi), T) \),

\[
(B_k) = \mathbb{E} \left[ \sum_{t=t_j+1}^{T} \mathbbm{1}_{\pi_t=k} \mathbbm{1}_{\tau(B, \pi) > t_j} \mathbbm{1}_{\forall s \in \{t_j+1, \ldots, t-1\}, B + \sum_{r=1}^{t_j} X_{\pi_r} + \sum_{d=t_j+1}^{T} X_{\pi_d} > 0} \right] \\
\leq \mathbb{E} \left[ \sum_{t=t_j+1}^{T} \mathbbm{1}_{\pi_t=k} \mathbbm{1}_{\tau(B, \pi) > t_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-t_j-1} \right] \\
= \mathbb{E} \left[ \sum_{t=1}^{T-t_j} \mathbbm{1}_{\pi_{t+j}=k} \mathbbm{1}_{\tau(B, \pi) \geq t_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-1} \right].
\]

This inequality being a trivial equality if \( t_j = T + 1 \) (because the sum is a sum on an empty set) or if \( t_j = \tau(B, \pi) + 1 \) (0 ≤ 0), we deduce that in any case,

\[
(B_k) \leq \mathbb{E} \left[ \sum_{t=1}^{T-t_j} \mathbbm{1}_{\pi_{t+j}=k} \mathbbm{1}_{\tau(B, \pi) \geq t_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-1} \right].
\]

Then, denoting by \( s_j \) the realized value of \( t_j \) and decomposing classically, we have

\[
(B_k) \leq \mathbb{E} \left[ \sum_{t=1}^{T-t_j} \mathbbm{1}_{\pi_{t+j}=k} \mathbbm{1}_{\tau(B, \pi) \geq t_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-1} \right] \\
= \sum_{s_j=T^{1/4}}^{T} \mathbb{E} \left[ \sum_{t=1}^{T-s_j} \mathbbm{1}_{\pi_{t+s_j}=k} \mathbbm{1}_{\tau(B, \pi) \geq s_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-1} \mathbbm{1}_{t=j} \right] \\
= \sum_{s_j=T^{1/4}}^{T} \sum_{t=1}^{T-s_j} \mathbb{E} \left[ \mathbbm{1}_{\pi_{t+s_j}=k} \mathbbm{1}_{\tau(B, \pi) \geq s_j} \mathbbm{1}_{\tau(T^{1/4}+1, \tilde{\pi}) \geq t-1} \mathbbm{1}_{t=j} \right] \\
= \sum_{s_j=T^{1/4}}^{T} \sum_{t=1}^{T-s_j} P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4}+1, \tilde{\pi}) \geq t-1, t_j = s_j, \pi_{t+s_j} = k \right).
\]

Let us then study the probability \( P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4}+1, \tilde{\pi}) \geq t-1, t_j = s_j, \pi_{t+s_j} = k \right) \) and decompose it as

\[
P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4}+1, \tilde{\pi}) \geq t-1, t_j = s_j, \pi_{t+s_j} = k \right) = \]

\[
P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4}+1, \tilde{\pi}) \geq t-1, t_j = s_j, \pi_{t+s_j} = k \right) = \]

\[
\forall t \geq s_j, \sum_{s=1}^{t} X_{\pi_s} > \frac{T^{1/4} + 1}{K} \\
+ P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4}+1, \tilde{\pi}) \geq t-1, t_j = s_j, \pi_{t+s_j} = k \right).
\]
The second term on the right, though, can easily be bounded as follows:

\[
P\left( \tau(B, \pi) \geq s_j, \tau(T^{1/4} + 1, \tilde{\pi}) \geq t - 1, t_j = s_j, \pi_{t+s_j} = k, \right.
\]

\[
\exists t \geq s_j, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K}, \quad (35)
\]

\[
\leq P\left( \exists t \geq T^{1/4}, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K} \right).
\]

Then, let us denote by \( \delta_1^\pi < \delta_2^\pi < \ldots \) the time steps \( t \geq T^{1/4} \) at which \( \pi_t = 1 \), in other words, denoting \( \delta_0^\pi := T^{1/4} \),

\[
\forall j \geq 1, \delta_j^\pi := \inf\{t \geq \delta_{j-1}^\pi : \pi_t = 1\}.
\]

Then, we can bound the probability as

\[
P\left( \exists t \geq T^{1/4}, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K} \right) = P\left( \exists j \geq 0, \sum_{s=1}^{\delta_j^\pi} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K} \right)
\]

\[
= \sum_{n_1=T^{1/4}/K}^{\infty} P\left( \sum_{s=1}^{n_1} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K}, \sum_{s=1}^{T^{1/4}} \mathbb{1}_{\pi_s = 1} = n_1 \right)
\]

\[
\leq \sum_{n_1=T^{1/4}/K}^{\infty} \sum_{j=0}^{\infty} P\left( \sum_{s=1}^{\delta_j^\pi} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K}, \sum_{s=1}^{T^{1/4}} \mathbb{1}_{\pi_s = 1} = n_1 \right),
\]

because the rewards are bounded in \([-1, 1]\). Then, using Hoeffding’s inequality, for any \( n_1 \geq \frac{T^{1/4}}{K} \) and \( j \geq 0 \),

\[
P\left( \sum_{s=1}^{\delta_j^\pi} X_s^1 \mathbb{1}_{\pi_s = 1} \leq - \frac{T^{1/4} + 1}{K}, \sum_{s=1}^{T^{1/4}} \mathbb{1}_{\pi_s = 1} = n_1 \right)
\]

\[
\leq P\left( \sum_{s=1}^{\delta_j^\pi} X_s^1 \mathbb{1}_{\pi_s = 1} - \mu_1 \leq -\mu_1, \sum_{s=1}^{T^{1/4}} \mathbb{1}_{\pi_s = 1} = n_1 \right)
\]

\[
\leq \exp\left( -\frac{(n_1 + j)\mu_1^2}{2} \right).\]
Summing over $n_1$ and $j$ gives

$$\sum_{n_1=T^{1/4}/K}^{\infty} \sum_{j=0}^{\infty} P \left( \sum_{s=1}^{\delta_j} X_s^1 \mathbb{1}_{\pi_s=1} \leq -\frac{T^{1/4}+1}{K}, \sum_{s=1}^{\delta_j} \mathbb{1}_{\pi_s=1} = n_1 \right)$$

$$\leq \sum_{n_1=T^{1/4}/K}^{\infty} \sum_{j=0}^{\infty} \exp \left( - \frac{(n_1 + j)\mu_1^2}{2} \right)$$

$$= \exp \left( - \frac{T^{1/4} \mu_1^2}{2K} \right) \left( 1 - e^{-\frac{\mu_1^2}{2}} \right)^2 .$$

We then deduce that

$$P \left( \tau(B, \pi) \geq s, \tau(T^{1/4} + 1, \tilde{\pi}) \geq t - 1, t_j = s, \pi_{t+s} = k, \forall t \geq s, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s=1} \leq -\frac{T^{1/4}+1}{K} \right) = O \left( \exp \left( - \frac{T^{1/4} \mu_1^2}{2K} \right) \right) ,$$

and thus, plugging it in (35),

$$P \left( \tau(B, \pi) \geq s, \tau(T^{1/4} + 1, \tilde{\pi}) \geq t - 1, t_j = s, \pi_{t+s} = k, \forall t \geq s, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s=1} > -\frac{T^{1/4}+1}{K} \right) \leq P \left( \pi_{t+s} = k, t_j = s, C_k(t+s) \geq C_1(t+s) \right) .$$

Let us also bound the first term in the decomposition in (35):

$$P \left( \tau(B, \pi) \geq s, \tau(T^{1/4} + 1, \tilde{\pi}) \geq t - 1, t_j = s, \pi_{t+s} = k, \forall t \geq s, \sum_{s=1}^{t} X_s^1 \mathbb{1}_{\pi_s=1} \leq -\frac{T^{1/4}+1}{K} \right)$$

$$\leq P \left( \pi_{t+s} = k, t_j = s, C_k(t+s) \geq C_1(t+s) \right) .$$
We can then re-write the previous term in the sum, as
\[
\sum_{s_j = T^{1/4}}^{T} \sum_{t=1}^{T - s_j} P \left( \tau(t + s_j, \pi) \geq C_1(t + s_j) \right)
\]
\[
= \sum_{s_j = T^{1/4}}^{T} \sum_{t=1}^{T - s_j} E \left[ 1_{\tau(t + s_j, \pi) = k, C_k(t + s_j) \geq C_1(t + s_j)} \right]
\]
\[
= E \left[ \sum_{t=1}^{T - t_j} \sum_{j} 1_{\tau(t + s_j, \pi) = k, C_k(t + s_j) \geq C_1(t + s_j)} \right]
\]
\[
\leq \sum_{t=1}^{T} E \left[ 1_{\tau(t, \pi) = k, C_k(t) \geq C_1(t)} \right]
\]
\[
= o(T),
\]
by Lemma 27. Overall, we can replace in (36) and deduce that
\[
\sum_{s_j = T^{1/4}}^{T} \sum_{t=1}^{T - s_j} P \left( \tau(B, \pi) \geq s_j, \tau(T^{1/4} + 1, \tilde{\pi}) \geq t - 1, t_j = s_j, \pi_{t + s_j} = k \right) = o(T),
\]
which straightforwardly implies that
\[
(B_k) = o(T).
\]

**Study of** \( E[t_j] \)

We can decompose
\[
E[t_j] = E \left[ t_j 1_{t_j = \min(\tau(B, \pi), T) + 1} \right] + E \left[ t_j 1_{t_j \leq \min(\tau(B, \pi), T)} \right]
\]
\[
= (T + 1) P(t_j = T + 1) + E \left[ (\tau(B, \pi) + 1) 1_{t_j = \tau(B, \pi) + 1} \right] + E \left[ t_j 1_{t_j \leq \min(\tau(B, \pi), T)} \right] .
\]

We can first bound the term \((D)\), as
\[
(D) \leq \left( \sqrt{T} + 1 \right) P \left( \tau(B, \pi) \geq \sqrt{T} \right) + (T + 1) P \left( \sqrt{T} < \tau(B, \pi) \leq T \right).
\]
But since \((P(\tau(B, \pi) \leq n))_{n \geq 1}\) is a sequence which converges to \(P(\tau(B, \pi) \leq \infty)\), we deduce that
\[
P \left( \sqrt{T} < \tau(B, \pi) \leq T \right) = P(\tau(B, \pi) \leq T) - P \left( \tau(B, \pi) \leq \sqrt{T} \right) \to_{T \to \infty} 0,
\]
and therefore,
\[
(D) = o(T).
\]
We then deduce that
\[
E[t_j] = (C) + (E) + o(T).
\]
Then, let us study the term \((C)\) and bound it by:

\[
(C) \leq P \left( \tau(B, \pi) \geq T, \forall t \leq T, B + \sum_{s=1}^{t} X_{s}^{\pi_{s}} \leq T^{\frac{3}{4}}, t_{j} = T + 1 \right)
\]

\[
\leq P \left( \tau(B, \pi) \geq T, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} = T + 1 \right)
\]

\[
\leq P \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right). \tag{37}
\]

Then, let us study the term \((E)\). Actually,

\[
P(t_{j} \geq T^{3/4}, t_{j} \leq \min(\tau(B, \pi), T))
\]

\[
= P \left( \tau(B, \pi) \geq T^{3/4}, \forall t \leq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right)
\]

\[
\leq P \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right). \tag{38}
\]

We then deduce that

\[
(E) \leq TP \left( t_{j} \geq T^{3/4}, t_{j} \leq \min(\tau(B, \pi), T) \right) + T^{3/4} P \left( t_{j} < T^{3/4}, t_{j} \leq \min(\tau(B, \pi), T) \right)
\]

\[
= TP \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right) + o(T). \tag{38}
\]

Using (37) and (38), we deduce that

\[
\mathbb{E}[t_{j}] \leq 2TP \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right) + o(T).
\]

Then, consider the set of conditions \(\left\{ \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \leq T^{1/4}, t_{j} \geq T^{2} \right\}\) and assume there exists an arm \(k_{0} \in [K]\) such that \(\sum_{s=1}^{T^{3/4}} X_{s}^{\pi_{s}} \mathbbm{1}_{\pi_{s}=k_{0}} > T^{1/4}\). Since \(T^{3/4} \leq t_{j}\), we know that for any \(k \in [K]\),

\[
\sum_{t=1}^{T^{3/4}} X_{t}^{\pi_{t}} \mathbbm{1}_{\pi_{t}=k} \geq -\frac{T^{1/4}}{K} - 1,
\]

hence

\[
B + \sum_{t=1}^{T^{3/4}} X_{t}^{\pi_{t}} = B + \sum_{t=1}^{T^{3/4}} X_{t}^{\pi_{t}} \mathbbm{1}_{\pi_{t}=k_{0}} + \sum_{k \neq k_{0}} \sum_{t=1}^{T^{3/4}} X_{t}^{\pi_{t}} \mathbbm{1}_{\pi_{t}=k}
\]

\[
\geq T^{3/4} - \frac{K-1}{K} T^{1/4} - (K-1) = \Omega(T^{1/4}),
\]

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which contradicts the hypothesis $B + \sum_{t=1}^{T^{3/4}} X^\pi_t \leq T^{1/4}$. We deduce that

$$P \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} \leq T^{1/4}, t_j \geq T^{3/4} \right)$$

$$\leq P \left( \tau(B, \pi) \geq T^{3/4}, \forall k \in [K], \sum_{t=1}^{T^{3/4}} X_t^{\pi_t} 1_{\pi_t = k} \leq T^{3/4} \right)$$

$$\leq P \left( \exists k \in [K]: -\frac{T^{1/4}}{K} - 1 < \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k} < T^{1/3}, \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right)$$

$$\leq \sum_{k=1}^{K} P \left( \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k} < T^{1/3}, \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right).$$

Then, for any arm $k \in [K]$, there are 2 cases: either $\mathbb{E}[X^k_1] \neq 0$, in which case we can use Hoeffding’s inequality to bound the above probability:

$$P \left( \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k} < T^{1/3}, \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right)$$

$$\leq P \left( \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k} - \mathbb{E}[X^k_1] < \frac{K}{T^{5/12}} - \mathbb{E}[X^k_1], \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right)$$

$$\leq \exp \left( \frac{\left( \frac{K}{T^{5/12}} - \mathbb{E}[X^k_1] \right)^2}{4} \right)$$

$$= o(1).$$

Or, $\mathbb{E}[X^k_1] = 0$, in which case, we can bound this probability as follows:

$$P \left( \sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k} < T^{1/3}, \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right)$$

$$\leq P \left( \frac{\sum_{s=1}^{T^{3/4}} X_s^{\pi_s} 1_{\pi_s = k}}{\sqrt{\sum_{s=1}^{T^{3/4}} 1_{\pi_s = k}}} < \frac{K}{T^{1/21}}, \sum_{s=1}^{T^{3/4}} 1_{\pi_s = k} \geq \frac{T^{3/4}}{K} \right).$$
Then, under the assumption that there is no zero arm, \( \text{Var}(X^k_1) > 0 \) and
\[
\sum_{s=1}^{T^{3/4}} X^s_{\pi_s} \mathbb{1}_{\pi_s=k} \xrightarrow{d} \mathcal{N}(0, \text{Var}(X^k_1))
\]
and since \( \frac{1}{T^{1/4}} \xrightarrow{T \to +\infty} 0 \), we deduce that
\[
P \left( t_j \geq T^{3/4} \right) = o(1).
\]
In any case, we have that
\[
P \left( \sum_{s=1}^{T^{3/4}} X^s_{\pi_s} \mathbb{1}_{\pi_s=k} < T^{1/3}, \sum_{s=1}^{T^{3/4}} \mathbb{1}_{\pi_s=k} \geq \frac{T^{3/4}}{K} \right) = o(1),
\]
hence,
\[
P \left( \tau(B, \pi) \geq T^{3/4}, B + \sum_{s=1}^{T^{3/4}} X^s_{\pi_s} \leq T^{1/4}, t_j \geq T^{3/4} \right) = o(1).
\]
We then deduce that
\[
\mathbb{E}[t_j] = o(T).
\]

**Study of (A)**

This term is the main one in the previous decomposition.
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] + \sum_{k=2}^{K} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\pi_t=k} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] = (A) + \sum_{k=2}^{K} (B_k) + O(\mathbb{E}[t_j]).
\]
But then, using the previous bounds on \((B_k)\) and \(\mathbb{E}[t_j]\), we deduce that
\[
(A) = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] + o(T).
\]
Then, we can simply replace the factor with the expectation by the probability of survival, as
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}_{\tau(B, \pi) \geq t-1} \right] = \sum_{t=1}^{T} P(\tau(B, \pi) \geq t-1) = T P(\tau(B, \pi) = \infty) + o(T).
\]
Hence,
\[(A) = \mu_1 P(\tau(B, \pi) = \infty)T + o(T),\]
which concludes the proof of the proposition. ■

Appendix I. Proof of the Pareto-optimality of EXPLOIT-UCB-DOUBLE (Theorem 5)

The main objective of this section is to prove that EXPLOIT-UCB-DOUBLE is regret-wise Pareto-optimal in the case of rewards in \{−1, 0, 1\} and with parameter \(n = \log T\).

The first subsection provides a preliminary lemma, useful for the proof of the Pareto-optimality exposed in the second subsection. The last subsection makes use of the preliminary lemma to derive an upper bound on the relative regret of EXPLOIT-UCB-DOUBLE in the general case.

I.1. Preliminary Lemma

Lemma 28 Let \(\pi\) be any policy. Then, it holds that
\[
\text{Rew}_T(\pi) \leq P(\tau(B, \pi) \geq \sqrt{T}) \times \max_{k \in [K]} \mu_k T + o(T). \tag{39}
\]

Furthermore, if \(\pi\) is an anytime policy, it holds that
\[
\text{Rew}_T(\pi) = P(\tau(B, \pi) = \infty) \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B, \pi) \geq T} \right] + o(T).
\]

Proof In order to prove the first statement of the lemma, we decompose the expected cumulative reward as follows:
\[
\text{Rew}_T(\pi) = \mathbb{E} \left[ \sum_{t=1}^{\tau(B, \pi)} X_t^{\pi_t} \mathbb{1}_{\tau(B, \pi) < \sqrt{T}} \right] + \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B, \pi) \geq \sqrt{T}} \right]
\leq -B + P(\tau(B, \pi) \geq \sqrt{T}) \times \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B, \pi) \geq t-1} \mathbb{1}_{\tau(B, \pi) \geq \sqrt{T}} \right]
\leq -B + \sqrt{T} + P(\tau(B, \pi) \geq \sqrt{T}) \times \mathbb{E} \left[ \sum_{t=\sqrt{T}+1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B, \pi) \geq t-1} \mathbb{1}_{\tau(B, \pi) \geq \sqrt{T}} \right]
\leq -B + \sqrt{T} + P(\tau(B, \pi) \geq \sqrt{T}) \times \max_{k \in [K]} \mu_k (T - \sqrt{T})
\leq P(\tau(B, \pi) \geq \sqrt{T}) \times \max_{k \in [K]} \mu_k T + 2\sqrt{T},
\]
which gives the desired result. For the second statement, we start by writing the reward as

\[
\text{Rew}_T(\pi) = \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq t-1} \right]
\]

\[
= \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq t-1 \tau(B,\pi) < T} \right] + \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq T} \right]
\]

By definition of \( \tau(B,\pi) \),

\[
\mathbb{E} \left[ \sum_{t=1}^{\tau(B,\pi)} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) < T} \right] \leq -B < \mathbb{E} \left[ \sum_{t=1}^{\tau(B,\pi)-1} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) < T} \right],
\]

and since the rewards are bounded in \([-1, 1]\), we deduce that

\[-(B + 1) < \mathbb{E} \left[ \sum_{t=1}^{\tau(B,\pi)} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) < T} \right] \leq -B,
\]

which implies

\[
\text{Rew}_T(\pi) = \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq T} \right] + o(T).
\]

It is trivial that, for any anytime policy \( \pi \),

\[
P(\tau(B,\pi) \geq T) = P(\tau(B,\pi) = \infty) + o_{T \to +\infty}(1).
\]

This implies

\[
\text{Rew}_T(\pi) = \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq T} \right] + o(T)
\]

\[
= P(\tau(B,\pi) \geq T) \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq T} \right] + o(T)
\]

\[
= P(\tau(B,\pi) = \infty) \mathbb{E} \left[ \sum_{t=1}^{T} X_t^{\pi_t} \mathbb{1}_{\tau(B,\pi) \geq T} \right] + o(T),
\]

which concludes the proof of the lemma.
I.2. Proof of the Pareto-optimality

We denote by $\pi^n$ the anytime policy EXPLOIT-UCB-DOUBLE with parameter $n \geq 1$. Then, Propositions 17 and 18, along with Lemma 28 give the cumulative reward of EXPLOIT-UCB-DOUBLE:

$$\text{Rew}_T(\pi^n) \geq \left(1 - p^{\text{EX}}\right) \max_{k \in [K]} \mu_k T + o(T).$$

In particular, we deduce the reward of the (non-anytime) policy EXPLOIT-UCB-DOUBLE with parameter $n = \log T$:

$$\text{Rew}_T(\pi^{\log T}) = \left(1 - p^{\text{EX}}\right) \max_{k \in [K]} \mu_k T + o(T).$$

Recall from Lemma 28 that, for any policy $\tilde{\pi}$,

$$\text{Rew}_T(\tilde{\pi}^T) \leq P\left(\tau(B, \tilde{\pi}^T) \geq \sqrt{T}\right) \max_{k \in [K]} \mu_k T + o(T),$$

and as a result,

$$\frac{\text{Rew}_T(\tilde{\pi}^T) - \text{Rew}_T(\pi^{\log T})}{T} \leq \left(P\left(\tau(B, \tilde{\pi}^T) \geq \sqrt{T}\right) - \left(1 - p^{\text{EX}}\right)\right) \max_{k \in [K]} \mu_k + o(1).$$

For $T \geq \frac{9B^2}{\Delta_F}$, it holds that

$$\frac{\text{Rew}_T(\tilde{\pi}^T) - \text{Rew}_T(\pi^{\log T})}{T} \leq \left(P\left(\tau(B, \tilde{\pi}^T) \geq \frac{3B}{\Delta_F}\right) - \left(1 - p^{\text{EX}}\right)\right) \max_{k \in [K]} \mu_k + o(1),$$

and taking the limit gives

$$\text{Reg}_F(\pi \parallel \tilde{\pi}) \leq \left(p^{\text{EX}} - P\left(\tau(B, \tilde{\pi}^T) < \frac{3B}{\Delta_F}\right)\right) \max_{k \in [K]} \mu_k,$$

where $\pi$ denotes $(\pi^{\log T})_{T \geq 1}$ the optimally-tuned EXPLOIT-UCB-DOUBLE. Then, assume that

$$\sup_{\tilde{F}} \text{Reg}_F(\pi \parallel \tilde{\pi}) > 0,$$

which implies that there exists some arm distributions $\tilde{F}$ such that

$$p^{\text{EX}} - P_{\tilde{F}}\left(\tau(B, \tilde{\pi}^T) < \frac{3B}{\Delta_{\tilde{F}}}\right) > 0.$$

Then, by Theorem 8, there exist some other arm distributions $F$ such that

$$p^{\text{EX}} - P_F\left(\tau(B, \tilde{\pi}^T) < \frac{3B}{\Delta_F}\right) < 0,$$

and since $\max_{k \in [K]} \mu_k > 0$, this implies

$$\inf_{\tilde{F}} \text{Reg}_F(\pi \parallel \tilde{\pi}) < 0,$$

proving that $(\pi^{\log T})_{T \geq 1}$ is regret-wise Pareto-optimal.
I.3. Relative Regret of EXPLOIT-UCB-DOUBLE in the general case

In the general case, Propositions 17 and 18 and Lemma 28 imply that the cumulative reward of EXPLOIT-UCB-DOUBLE $\pi^n$ with parameter $n \geq 1$ is

$$\text{Rew}_T(\pi^n) \geq \left(1 - p_{\text{EX}} - \frac{(p_{\text{EX}})^{nB}}{1 - (p_{\text{EX}})^{nB}}\right) \max_{k \in [K]} \mu_k T + o(T).$$

(41)

Let $\pi'$ be any policy. The previous result implies that there exist some arm distributions $F$ such that

$$\lim_{T \to +\infty} \frac{\text{Rew}_T(\pi') - (1 - p_{\text{EX}}) \max_{k \in [K]} \mu_k T}{T} < 0,$$

With (41), this implies that EXPLOIT-UCB-DOUBLE $\pi^n$ achieves, for any policy $\pi'$,

$$\inf_F \text{Reg}_F(\pi^n || \pi') = \inf_F \lim_{T \to +\infty} \frac{\text{Rew}_T(\pi') - \text{Rew}_T(\pi^n)}{T} < \frac{(p_{\text{EX}})^{nB}}{1 - (p_{\text{EX}})^{nB}} \max_{k \in [K]} \mu_k.$$

Appendix J. Practical Performance of the Algorithms Introduced

In this appendix, we provide some additional experimental results on the performance of the algorithms EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE introduced in this paper. More precisely, we compare their practical performance to the classic bandit algorithms UCB (Auer et al., 2002) and Multinomial Thompson Sampling (MTS, Riou and Honda, 2020), the latter of which is chosen for its optimality in the classic MAB of multinomial arms of given support. We used the UCB index of form $\overline{x}_i + \sqrt{\frac{\log T}{2n_i}}$ for UCB, EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE, where $\overline{x}_i$ and $n_i$ are the mean reward and the number of samples from arm $i$, respectively. We will further look at the impact of the hyperparameter $n$ on the survival regret of the algorithm EXPLOIT-UCB-DOUBLE, and therefore, we will consider EXPLOIT-UCB-DOUBLE for various values of $n$, including the case where $n$ is properly tuned as $n = \lceil \log T \rceil$.

For all the experiments performed, we consider a bandit setting with $K = 3$ multinomial arms of common support $\{-1, 0, 1\}$ and distributions $F^{(i_1)}, F^{(i_2)}$ and $F^{(i_3)}$, where $i_1, i_2, i_3 \in \{1, \ldots, 10\}$. The distributions $F^{(i)}$ for $i \in \{1, \ldots, 10\}$ are described below:

$$F^{(1)} = \text{Mult}(0.4, 0.12, 0.48); \quad F^{(2)} = \text{Mult}(0.04, 0.48, 0.08); \quad F^{(3)} = \text{Mult}(0.5, 0.1, 0.4);$$
$$F^{(4)} = \text{Mult}(0.48, 0, 0.52); \quad F^{(5)} = \text{Mult}(0.04, 0.91, 0.05); \quad F^{(6)} = \text{Mult}(0.45, 0, 0.55);$$
$$F^{(7)} = \text{Mult}(0.05, 0.85, 0.1); \quad F^{(8)} = \text{Mult}(0.5, 0, 0.5); \quad F^{(9)} = \text{Mult}(0.495, 0, 0.505);$$
$$F^{(10)} = \text{Mult}(0.049, 0.9, 0.051).$$

We set the horizon $T$ equal to 20000 and we further assume that the initial budget $B$ is a multiple of 3, the number of arms. As a consequence, in this framework, EXPLOIT-UCB-DOUBLE with the parameter $n = \lceil \log T \rceil = 10$ is the theoretically recommended parameter. We will also study EXPLOIT-UCB-DOUBLE with parameters $n = 1$ and $n = 100$ and see if their survival regret differs much from EXPLOIT-UCB-DOUBLE with parameter $n = \lceil \log T \rceil$ in practice. The average time of ruin, as well as the proportion of ruins (i.e. the proportion of simulations for which the ruin occurred) of each algorithm in every setting considered, are gathered in Tables 1 and 2, where EX-UCB denotes
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EXPLOIT-UCB, and EX-D denotes EXPLOIT-UCB-DOUBLE. They are the averages over 200 trials, except for the last setting with $B = 30$ and arm distributions $\{F^{(9)}, F^{(10)}, F^{(3)}\}$ which is the average over 500 trials.

| Setting | Average Time of Ruin |
|---------|----------------------|
| $B = 9$ {F(1), F(2), F(3)} | 17128 13957 18210 18510 18712 18613 |
| $B = 9$ {F(4), F(5), F(3)} | 8856 5218 12160 10016 9176 9680 |
| $B = 9$ {F(6), F(7), F(8)} | 18211 16427 18617 19305 18408 18910 |
| $B = 30$ {F(1), F(2), F(3)} | 20000 19904 20000 20000 20000 20000 |
| $B = 30$ {F(9), F(10), F(3)} | 8931 6708 11912 11358 11136 11273 |

Table 1: Average Time of Ruin of the Algorithms.

| Setting | Proportions of Ruins on 200 Trials |
|---------|-----------------------------------|
| $B = 9$ {F(1), F(2), F(3)} | 0.15 0.31 0.09 0.08 0.07 0.07 |
| $B = 9$ {F(4), F(5), F(3)} | 0.57 0.75 0.4 0.52 0.56 0.54 |
| $B = 9$ {F(6), F(7), F(8)} | 0.09 0.18 0.07 0.04 0.08 0.06 |
| $B = 30$ {F(1), F(2), F(3)} | 0 0.01 0 0 0 0 |
| $B = 30$ {F(9), F(10), F(3)} | 0.68 0.76 0.5 0.56 0.57 0.58 |

Table 2: Proportion of Ruin of the Algorithms.

Figure 2: Survival regret for $B = 9$ and arms $\{F^{(1)}, F^{(2)}, F^{(3)}\}$. 

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In Figure 2, we first consider the setting where the (multinomial) arm distributions are \( \{F^{(1)}, F^{(2)}, F^{(3)}\} \) and the budget is set to \( B = 9 \). It is very clear that, in this setting, EXPLOIT-UCB-DOUBLE, whatever the value of the parameter \( n \in \{1, \lceil \log T \rceil, 100\} \), outperforms UCB, MTS and EXPLOIT-UCB. The performance of EXPLOIT-UCB-DOUBLE does not seem to be so significantly affected by the value of the parameter \( n \), as the proportion of ruins is very similar for those three algorithms, around 0.07 (see Table 2). On the other hand, UCB and MTS suffer from more frequent ruins than EXPLOIT-UCB and EXPLOIT-UCB-DOUBLE, and hence have a poor performance.

In the classic stochastic bandit setting, MTS is theoretically and practically optimal and its performance is much better than the one of UCB (see Riou and Honda, 2020). On the other hand, in the survival bandit setting, its survival regret is much larger than the one of UCB, showing that the randomization factor in Thompson sampling is the source of more frequent ruins: MTS suffers a ruin for a proportion of 0.305 of the simulations, while UCB suffers a ruin only for a proportion of 0.15 of the simulations (see Table 2). On the other hand, EXPLOIT-UCB has a poor performance, comparable to the one of MTS in this setting. While the risk of ruin of EXPLOIT-UCB is very low and comparable to the one of the various EXPLOIT-UCB-DOUBLE algorithms considered, it suffers from its lack of exploration.

In Figure 3, the budget is still set to \( B = 9 \) while the arm distributions are \( \{F^{(4)}, F^{(5)}, F^{(3)}\} \). In this setting, UCB performs surprisingly well, at a comparable level to the three EXPLOIT-UCB-DOUBLE algorithms. Please note that EXPLOIT-UCB-DOUBLE with the properly tuned parameter \( n = \lceil \log T \rceil \), as well as EXPLOIT-UCB-DOUBLE with \( n = 100 \), perform better than UCB. The reason behind the strong performance of UCB is that in this setting, the arms of distributions \( F^{(4)} \) and \( F^{(5)} \) are hard to set apart, and therefore, UCB performs a strong exploitation between these two arms. Please note that the average time of ruin and the proportion of ruins of UCB are still larger than the ones of the EXPLOIT-UCB-DOUBLE algorithms for \( n \in \{1, \lceil \log T \rceil, 100\} \) (see Tables 1 and 2).

On the other hand, EXPLOIT-UCB has a much larger survival regret than its doubling-tricked counterparts. Looking carefully at the average time of ruin of the algorithms however, it is clear...
that, as expected, EXPLOIT-UCB has a smaller risk of ruin (see Tables 1 and 2). The reason behind the larger survival regret of EXPLOIT-UCB is again its lack of exploration. Indeed, in this setting, the arms expectations are quite low, being respectively 0.04, 0.01 and −0.1. Therefore, even when the ruin does not occur, it is frequent that the arm with the largest expectation 0.04 has exhausted its budget share \( B_K = 3 \), while the other arm of positive expectation 0.01 has not, implying a large survival regret. The closeness of the expectations of the arms in this setting also explains the low performance of MTS, which suffers frequent ruins, with a very low average time of ruin around \( \tau = 5000 \) and a ruin which occurs on 3/4 of the simulations (see Tables 1 and 2).

In Figure 4, the budget is still set to \( B = 9 \) while the arm distributions are \( \{F(6), F(7), F(8)\} \). While the results in this case are very similar to the ones of Figure 2, there are two notable differences we would like to point out. The first one is that in this setting, while the survival regret of UCB is close to the one of EXPLOIT-UCB-DOUBLE with parameter \( n = 1 \), it is much larger than the one of EXPLOIT-UCB-DOUBLE with parameter \( n = 100 \) or \( n = \lceil \log T \rceil \). Theoretically, \( n \) is the exploration parameter of EXPLOIT-UCB-DOUBLE: the larger the parameter \( n \), the later the exploration. Therefore, if \( n \) is very small, the algorithm will start the exploration at a very early stage, and its behavior will not be so different from the one of UCB, explaining their similar survival regret. On the other hand, if \( n \) is very large, then the exploration will be delayed and this will result in a lower risk of ruin, explaining the better performance of EXPLOIT-UCB-DOUBLE for \( n = \lceil \log T \rceil \) and \( n = 100 \).

The second phenomenon we wish to point out is that in this setting, EXPLOIT-UCB has a survival regret which is even larger than the one of MTS, contrary to the setting of Figure 2, where both algorithms had a similar survival regret. The reason behind this is that this setting is easier compared to the one of Figure 2, in the sense that the arm expectations are quite different. As a result, having a stronger exploration component yields a better regret than being too conservative.

Overall, this phenomenon can be easily observed looking carefully at the three previous figures. In Figure 3, the arm expectations are very small (respectively 0.04 and 0.01 for arms 1 and 2), making this setting is very difficult in practice. As a result, algorithms with a strong exploitation component
perform better and in particular, EXPLOIT-UCB performs much better than MTS. Then, the setting of Figure 2 is a little easier to handle, with arms whose expectations are slightly larger (respectively 0.08 and 0.04 for arms 1 and 2), and in this setting, EXPLOIT-UCB and MTS perform comparably. Eventually, in Figure 4, the arms expectations are very distinct, being respectively 0.1 and 0.05 for arms 1 and 2, and in this setting, MTS outperforms EXPLOIT-UCB.

The reason underlying this phenomenon is that, while the risk of ruin of MTS is always higher than the one of EXPLOIT-UCB, the difference in the risk of ruin between the two algorithms becomes lower as the arm expectations grow: Table 2 shows that the difference between the proportion of ruins of MTS and EXPLOIT-UCB goes from 0.35 for the setting of Figure 3, to 0.22 for the setting of Figure 2, to only 0.11 for the setting of Figure 4. The lesson to remember from this is that, depending on the arm parameters, the strong performance of a good exploration may sometimes compensate for frequent ruins.

In Figures 5 and 6, we eventually study if some better performance can be achieved by traditional algorithms when the budget is large, i.e. of the order of $\log T$. It comes as no surprise that in some cases, like in Figure 5, with such a large budget, the ruin never happens, as we can see in Table 2. Therefore, it is natural that MTS, which has the best theoretical and practical performance in this case, outperforms the other algorithms. Nevertheless, please note that the survival regret of EXPLOIT-UCB-DOUBLE, whatever the parameter $n$, is comparable to the performance of UCB.

Another fact is that, the lower the parameter $n$ of EXPLOIT-UCB-DOUBLE, the more it explores and as a consequence, EXPLOIT-UCB-DOUBLE performs better in this case when the parameter $n$ is small.

Please note that in some settings though, even such a large budget cannot prevent the ruin from happening. Looking carefully at Figure 6, for which the budget is still set to $B = 30$ and the arm distributions are $\{F^{(9)}, F^{(10)}, F^{(3)}\}$, it is clear that the stronger the exploration, the larger the survival regret. In this setting, the arm expectations are very small, and therefore, a lack of exploitation leads to very frequent ruins. In this setting, EXPLOIT-UCB is the only algorithm which suffers a ruin less than for half of the simulations, while the proportion of ruins of EXPLOIT-UCB-DOUBLE is
Figure 6: Survival regret for $B = 30$ and arms $\{F^{(9)}, F^{(10)}, F^{(3)}\}$.

around 0.57 (whatever the parameter $n$). This is in stark contrast to UCB and MTS, which suffer a proportion of ruins around 0.68 and 0.76 respectively.