The distribution of path lengths of self avoiding walks on Erdős-Rényi networks

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Abstract. We present an analytical and numerical study of the paths of self avoiding walks (SAWs) on random networks. Since these walks do not retrace their paths, they effectively delete the nodes they visit, together with their links, thus pruning the network. The walkers hop between neighboring nodes, until they reach a dead-end node from which they cannot proceed. Focusing on Erdős-Rényi networks we show that the pruned networks maintain a Poisson degree distribution, \( p_t(k) \), with an average degree, \( \langle k \rangle_t \), that decreases linearly in time. We enumerate the SAW paths of any given length and find that the number of paths, \( n_T(\ell) \), increases dramatically as a function of \( \ell \). We also obtain analytical results for the path-length distribution, \( P(\ell) \), of the SAW paths which are actually pursued, starting from a random initial node. It turns out that \( P(\ell) \) follows the Gompertz distribution, which means that the termination probability of an SAW path increases with its length.

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1. Introduction

Random walk models [1, 2] are useful for the study of a large variety of stochastic processes such as Brownian motion and diffusion [3, 4], polymer structure and dynamics [5, 6, 7], and random search [8, 9]. The models were studied extensively in geometries including continuous space [10], regular lattices of different spatial dimensions [11], fractals [12] and random networks [13]. In the context of complex networks [14, 15], random walks can be used for either probing the network structure itself [16] or to model dynamical processes such as the spreading of rumors, opinions and epidemics [17, 18]. Recent studies focusing on the properties of random walks on random networks have
produced analytical results for the mean first passage time \[19\] between two distinct nodes \[20\], the average trapping time, namely the average time to reach a specific node from any starting node \[20\], the average number of distinct nodes visited throughout the walk \[21\] and the average cover time \[22\].

A special type of random walk which has been studied extensively on regular lattices is the self avoiding walk (SAW), also referred to as the kinetic growth self-avoiding walk \[23\], or true or myopic self-avoiding walk \[24\]. This is a random walk which does not visit the same node more than once \[25\]. At each step, the walker chooses its next move randomly from the neighbors of its present node, excluding nodes that were already visited. The walk terminates when it reaches a dead end node, namely a node which does not have any yet unvisited neighbors. The length of the path, \(\ell\), is given by the number of steps made until the walk is terminated. A large number of studies of SAWs on regular lattices were devoted to the enumeration of paths as a function of their length \[26, 27, 28, 29, 30, 31\]. These studies provided much insight on the structure and thermodynamics of polymers \[6, 7\]. However, SAWs on networks have not attracted much attention \[32, 33, 34, 23, 35\].

SAWs on networks may describe agents or robots propagating and damaging a network of computers, such that damaged nodes are effectively wiped out from the network. The path length of an SAW on a connected network of size \(N\) can take values between 1 and \(N - 1\). The latter case corresponds to a Hamiltonian path \[36\]. More specifically, the SAW path lengths between a given pair of nodes, \(i\) and \(j\), are distributed in the range bounded from below by the shortest path length between these nodes \[37\] and from above by the longest non-overlapping path between them \[38\].

From a theoretical point of view, the SAW path length corresponds to the attrition length, also referred to as the last hitting time of the SAW \[33\]. This is in contrast to the first hitting time of random walks on networks \[21\]. Both studies show that the behavior of random walks on random networks exhibits common properties with those of random walks on regular lattices of high dimension. In particular, they are highly effective in exploring the space without retracing their steps, in contrast to the case of low dimensional lattices \[39\].

In this paper we study SAWs on Erdős-Rényi (ER) networks \[40, 41, 42\], above the percolation threshold. These walks can be viewed as random walks which delete the nodes they visit, thus reducing the network size one node at a time. Surprisingly, we find that the remaining network is still an ER network with connectivity that decreases linearly in time. We enumerate all the possible SAW paths, \(n_T(\ell)\), of any given length, \(\ell\) on an ER network, providing closed form expressions. We also study the distribution \(P(\ell)\) of the path lengths of the SAWs which are actually pursued, starting from a random node in the network. The analytical results are found to be in excellent agreement with numerical simulations.

The paper is organized as follows. In Sec. 2 we present the SAW model on an ER network. In Sec. 3 we study the evolution of the network structure as it is pruned by the SAW. In Sec. 4 we enumerate the SAW paths. In Sec. 5 we study the distribution of
path lengths. In Sections 6, 7 and 8 we present central measures, dispersion measures and extreme value statistics of this distribution. The results are summarized and discussed in Sec. 9.

2. The self avoiding walk

Consider a random walk on a random network of \( N \) nodes. Each time step the walker chooses randomly one of the neighbors of its current node, and hops to the chosen node. Such walks can go on without limit, visiting many nodes multiple times. Many interesting questions have been studied in this context. For example, the number of distinct nodes visited by the random walker as a function of the path length has been calculated in Ref. [21]. A related property is the average cover time, which is the average number of steps required for the random walk to visit all the nodes in the network at least once [22]. Another interesting question is how many steps the walker makes until the first time it enters a node which was already visited. This time is referred to as the first hitting time, while the path length up to this point is referred to as the first intersection length. The distribution of first hitting times was recently studied using the cavity approach [21].

Self avoiding walkers are random walkers which hop only to neighboring nodes which have not been visited before. Here we study SAWs on random networks. Since the nodes already visited become inaccessible, the self avoidance condition is equivalent to a process in which the walkers delete the nodes they visit. More precisely, the visited node is deleted promptly after the walker has moved to the next node. Thus, the network size is reduced by 1 at each step. The edges connected to the deleted node are also removed. As a result, the degree of each node which was connected to the visited node is reduced by 1. Eventually, the walker may reach a node which does not have any unvisited neighbors. At this point, the path of the random walker is terminated. We choose as the initial network an \( ER(N, p) \) network, namely an ER network which consists of \( N \) nodes, where each pair of nodes is connected with probability \( p \). The SAW path starts from a randomly chosen node, which is not isolated. The path length, \( \ell \), is given by the number of steps taken until it terminates.

3. Evolution of the network structure

Consider an \( ER(N, p) \) network. The degree \( k_i \) of node \( i = 1, \ldots, N \) is the number of links connected to this node. The degree distribution \( p(k) \) of the ER network is a binomial distribution, which in the sparse limit (\( p \ll 1 \)) is approximated by a Poisson distribution of the form

\[
p(k) = \frac{c^k}{k!} e^{-c},
\]

where \( c = (N - 1)p \) is the average degree. In the asymptotic limit (\( N \to \infty \)), the ER network exhibits a phase transition at \( c = 1 \) (a percolation transition), such that for
c < 1 the network consists only of small clusters and isolated nodes, while for c > 1 there is a giant cluster which includes a macroscopic fraction of the network, in addition to the small clusters and isolated nodes. At a higher value of the connectivity, namely at \( c = \ln N \), there is a second transition, above which the entire network is included in the giant cluster and there are no isolated components. Here we focus on the regime above the percolation transition, namely c > 1, where the network includes a giant component.

Considering the SAW as a node deleting walk, it effectively tears up the network, removing one node and its associated links at every step. Clearly, the network size after \( t \) steps is given by \( N(t) = N - t \). The degree distribution evolves in time and is denoted by \( p_t(k), k = 0, \ldots, N(t) - 1 \), where \( p_0(k) = p(k) \). The average degree

\[
\langle k \rangle_t = \sum_{k=0}^{N(t)-1} kp_t(k)
\]

evolves accordingly. We denote it by \( c(t) = \langle k \rangle_t \), where \( c(0) = c \).

For random walks on random networks, there is a higher probability for the walker to visit nodes with high degrees. More precisely, the probability to visit a node of degree \( k \) in the next step is given by \( kp_t(k)/c(t) \), namely it is proportional to the degree of the node. A special property of the Poisson distribution is that the probability \( kp_t(k)/c(t) = p_t(k - 1) \). This means that, in fact, the probability of a node of degree \( k \) to be visited by the random walker is proportional to \( p_t(k - 1) \). However, by the time the walker enters the next node, the previous node is deleted, together with the edge connecting the two nodes. Therefore, when the walker enters a node of degree \( k \), the degree of this node is reduced to \( k - 1 \). The outcome of this reasoning is that the probability of the SAW to visit a node of degree \( k \) at time \( t \) is simply \( p_t(k) \), as if it makes a random choice of a node in the smaller network.

We now examine the evolution of the network in terms of the average number of links that are removed at each step. Deleting a node along the SAW path removes, on average, \( c(t) \) edges, namely \( 2c(t) \) half-edges from the node and its neighbors. The average degree of the network at time \( t \) is given by

\[
c(t) = \frac{\sum_{i=1}^{N} k_i(t)}{N(t)},
\]

where \( k_i(t) \) is the degree of node \( i \) at time \( t \), and \( k_i(0) = k_i \). The degrees of all the nodes already deleted are counted as \( k_i(t) = 0 \). The average degree can be expressed as

\[
c(t) = \frac{\sum_{i=1}^{N} k_i(0) - 2 \sum_{t' = 0}^{t-1} c(t')}{N - t}.
\]

Note that \( \sum_i k_i(0) = Nc \). Therefore, we obtain the recursion equation
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\[ c(t) = \left(1 - \frac{1}{N-t}\right)c(t-1). \]  
\[ (5) \]

Solving this equation we obtain

\[ c(t) = \left(1 - \frac{t}{N-1}\right)c. \]  
\[ (6) \]

The correctness of Eq. (6) can also be demonstrated by considering the case of the complete network, \( ER(N,p = 1) \). In this case, the SAW visits the entire network with probability 1, and \( c(t) = N(t) - 1 \) for all values of \( t \). This result is consistent with Eq. (6), where for a complete network \( c = N - 1 \).

In Fig. 1 we present the average degree \( c(t) \) vs. time for \( N = 1000 \) and different initial values of \( c \). The results obtained from Eq. (6) are compared to numerical simulations, finding excellent agreement. We now extend the discussion to the temporal evolution of the entire degree distribution \( p_t(k) \). In Fig. 2 we present the degree distribution \( p_t(k) \) vs. \( k \), for \( t = 0, 350 \) and \( 700 \), where the initial network is \( ER(1000, 20/1000) \). Clearly, the degree distribution of the initial network is a Poisson distribution with \( c = 20 \) [Eq. (1)]. Interestingly, the degree distribution \( p_t(k) \) remains a Poisson distribution and its average degree \( \langle k \rangle_t \) coincides with \( c(t) \) given by Eq. (6).

In order to understand these results we digress to the simpler process of random node deletion, in which at each time step a randomly chosen node is deleted. Thus, the probability that the deleted node at time \( t \) has degree \( k \) is given by \( p_t(k) \). The random node deletion actually maintains the ER character of the network, with a Poisson degree distribution and the same value of \( p \). This property can be easily understood from the fact that the ER network can be constructed by starting from a single node and at each time step adding one node and connecting it to any existing node with probability \( p \) [36]. Repeating this node addition step \( N - 1 \) times we obtain an ER network of \( N \) nodes. The node deletion process is simply the time reversal of this construction.

In random node deletion, the probability \( p \) remains unchanged and thus \( c(t) = [N(t) - 1]p \). Since \( p = c/(N - 1) \) and \( N(t) = N - t \), we find that in random node deletion

\[ c(t) = \frac{N(t) - 1}{N - 1}c, \]  
\[ (7) \]

which coincides with Eq. (6), describing SAW deletion. However, these two processes are different. The fact that the expressions for \( c(t) \) coincide is a result of two opposing effects which cancel each other. On the one hand, the probability of a node of degree \( k \) to be visited by the SAW is proportional to its degree and given by \( kp_t(k)/c(t) \). Therefore, highly connected nodes are more likely to be removed from the network compared to the random deletion process. On the other hand, once the SAW enters a node the link along which it entered is already deleted, reducing the degree of the node by 1. As mentioned above, it so happens that in the case of a Poisson distribution \( kp_t(k)/c(t) = p_t(k - 1) \), so the net result is that a node with degree \( k \) is visited with
probability $p_t(k)$. The conclusion is that although the SAW and the random deletion are two different processes, the degree distribution of the remaining network is the same. Therefore, at all times, the degree distribution is

$$p_t(k) = \frac{c(t)^k}{k!}e^{-c(t)},$$

(8)

where $c(t)$ is given by Eq. (6).

When node removal processes such as random deletion or node-deleting walks are inflicted on a network with $c > 1$, they drive the network towards the percolation transition. This transition takes place at time $t_p$ for which $c(t_p) = 1$. Using Eq. (6) one can evaluate the time $t_p$, which is given by $t_p = (N - 1)(1 - 1/c)$. It is important to note that in the process of random node deletion the percolation transition is always reached. On the other hand, the SAW path is likely to terminate long before the percolation threshold is reached. In fact, as the network approaches the percolation transition, the termination rate of the SAW paths quickly increases.

Random node deletion and node-deleting walks can be considered in the context of network attacks. While random node deletion is an example of a random attack which has been studied extensively in the literature [14], the node-deleting walk belongs to the class of localized attacks. A model of localized attacks which has been studied in Ref. [43], the attack is initiated at a random node and deletes entire shells of neighbors around the initial node, one shell at a time. It was found that properties of percolation in this model on the ER network are identical to those obtained by random removal of nodes. It turns out that node deleting walks on ER networks share this property. In other networks, such as the scale-free network or the random regular graph, localized attacks affect the network differently than random attacks [43].

4. The number of SAW paths

In this section we study the combinatorics of the SAW paths starting from a random node, $i$. More precisely, we are interested in the expected number of possible SAW paths of length $\ell$ (not necessarily terminating), which will be denoted by $n(\ell)$, where $\ell = 1, \ldots, N - 1$. Starting at node $i$, in the $ER[N, c/(N - 1)]$ network, the initial step of the SAW can be chosen from the $k_i$ nearest neighbors of $i$. The average number of nearest neighbors is $\langle k \rangle = c$. Hence, $n(1) = c$. As the SAW proceeds, the network remains an ER network with a decreasing value of the mean degree $c(t)$, given by Eq. (6). Therefore, each step, $t$, can be considered as the first SAW step on the smaller network, $ER[N(t), c(t)/(N(t) - 1)]$. As a result, the number of SAW paths of length $\ell$ is simply

$$n(\ell) = \prod_{t=0}^{\ell-1} c(t),$$

(9)

where $c(0) = c$. Plugging in the expression for $c(t)$ from Eq. (6) we obtain
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\[ n(\ell) = \frac{(N-1)!}{(N-\ell)!} \left( \frac{c}{N-1} \right)^\ell. \]  

Identifying \( p = c/(N-1) \) and rearranging the last expression yields

\[ n(\ell) = \left( \frac{N-1}{\ell} \right) \ell! p^\ell. \]  

Expressing \( n(\ell) \) in this form highlights the fact that it accounts for all the possible ordered choices of \( \ell \) nodes from the \( N-1 \) nodes in the network, apart from the initial node, \( i \). The combinatorial factor is multiplied by the probability that all the links along such a path exist, which is given by \( p^\ell \). Using the Stirling approximation we obtain

\[ n(\ell) = \left( \frac{N}{N-\ell} \right)^{N-\ell+1/2} \left( \frac{c}{e} \right)^\ell. \]  

For short SAW paths, for which \( \ell \ll N \), this can be further approximated by

\[ n(\ell) = e^{-\frac{\ell^2}{2N}+\ell \ln c}. \]  

In the limit \( N \to \infty \) this expression reduces to

\[ n(\ell) = c^\ell. \]  

This resembles the results obtained for SAWs on infinite regular lattices of a finite dimension, \( D \), in which the number of SAW paths on length \( \ell \) is

\[ n(\ell) \sim \mu^\ell \ell^\alpha, \]  

where \( \mu \) is the connective constant and the exponent \( \alpha \) provides a sub-leading correction. It was found that for \( D < 4 \) the exponent \( \alpha > 0 \), while for \( D \geq 4 \) it satisfies \( \alpha = 0 \). Comparing Eqs. (14) and (15) we conclude that an SAW on an ER network is consistent with \( \alpha = 0 \), and thus resembles an SAW on a regular lattice of dimension \( D \geq 4 \). On regular lattices, the connective constant satisfies \( \mu \leq z-1 \), where \( z \) is the coordination number of the lattice (for the hyper-cubic lattice \( z = 2D \)). This is due to the fact that the SAW does not backtrack its path. In high dimensions the connective constant \( \mu \to z-1 \). A similar result is obtained for a regular graph in which all nodes are of degree \( c \). Interestingly, in the ER network the number of paths of length \( \ell \) scales like \( c^\ell \), where \( c \) is the average coordination number. This is different from the case of regular lattices of high dimension, where \( n(\ell) \) scales like \( (z-1)^\ell \). The reason is that the SAW path tends to visit nodes of high degree more often, in a way that compensates for the loss of the backtracking link.

In Fig. 3 we present the number of SAW paths, \( n(\ell) \), obtained from Eq. (10) for an ER network of size \( N = 100 \) and three values of \( c \) (dashed lines). The function \( n(\ell) \) has a well defined and highly symmetric peak, which shifts to the right as \( c \) is increased.
To obtain the location of the peak, we solve for the derivative \( dn(\ell)/d\ell = 0 \), where \( n(\ell) \) in taken from Eq. (10). We obtain

\[
\ell_{\text{peak}} \simeq N - \frac{1}{2W\left[\frac{c}{2(N-1)}\right]},
\]

where \( W(x) \) is the Lambert W function, also referred to as the ProductLog function \(^{44}\). In the limit of large and dilute networks, this expression can be simplified to

\[
\ell_{\text{peak}} \simeq N - \frac{N - 1}{c} - \frac{1}{2}.
\]

Expanding \( \ln n(\ell) \) around \( \ell_{\text{peak}} \) to second order in \( \ell \) leads to a Gaussian approximation of the form

\[
n(\ell) \simeq \frac{n^\text{tot}}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ell-\ell_{\text{peak}})^2}{2\sigma^2}},
\]

where

\[
\sigma^2 = \frac{1}{\psi^{(1)}\left(\frac{N-1}{c} + \frac{1}{2}\right)},
\]

and \( \psi^{(m)}(x) \) is the PolyGamma function of order \( m \) \(^{44}\). For dilute networks, this converges to \( \sigma^2 = N/c \). The prefactor, \( n^\text{tot} \), represents the total number of SAW paths of all possible lengths, namely

\[
n^\text{tot} = \sum_{\ell=1}^{N-1} n(\ell).
\]

It is given by

\[
n^\text{tot} = \frac{\sqrt{2\pi\sigma^2}(N-1)!}{\Gamma\left(\frac{N-1}{c} + \frac{1}{2}\right)} \left(\frac{c}{N-1}\right)^{N-1/2 - \frac{1}{2}}.
\]

The Gaussian approximation of Eq. (18), for the number of SAW paths, \( n(\ell) \), is shown in Fig. 3 for three values of \( c \) (solid lines). They are found to be in excellent agreement with the exact results (dashed lines).

The expressions presented above for \( n(\ell) \) enumerate all the SAW paths of length \( \ell \) (starting from a random node \( i \)), regardless of whether they terminate after \( \ell \) steps or continue to form longer paths. To enumerate only the paths which terminate after \( \ell \) steps, one needs to multiply \( n(\ell) \) by the termination probability, which is given by \( p_\ell(k = 0) = \exp[-c(\ell)] \). Therefore, the number of SAW paths which terminate after \( \ell \) steps is

\[
n_T(\ell) = n(\ell)e^{-c(\ell)}.
\]

We find that the peak of this function is at
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\[ \ell_{\text{peak}} \approx N - \frac{1}{2W \left[ \frac{c \exp(c/N)}{2(N-1)} \right]} . \]  

(23)

For large and dilute networks, it can be approximated by

\[ \ell_{\text{peak}} \approx N - \frac{N - 1}{c} e^{-c/N} - \frac{1}{2} . \]  

(24)

The function \( n_T(\ell) \) can be approximated by a Gaussian of the form

\[ n_T(\ell) \approx \frac{n_{\text{tot}}}{\sqrt{2\pi \sigma_T^2}} e^{-\frac{(\ell - \ell_{\text{peak}})^2}{2\sigma_T^2}} , \]  

(25)

with

\[ \sigma_T^2 = \frac{1}{\psi^{(1)} \left( \frac{N-1}{c} e^{-c/N} + \frac{1}{2} \right)} . \]  

(26)

The pre-factor

\[ n_{\text{tot}} = \sum_{\ell=1}^{N-1} n_T(\ell) \]  

(27)

is given by

\[ n_{\text{tot}} = \sqrt{\frac{2\pi \sigma_T^2}{\Gamma \left( \frac{N-1}{c} e^{-c/N} + \frac{1}{2} \right)}} \left( \frac{c}{N-1} \right)^{N-1} e^{-\frac{c}{2}} e^{-\frac{c}{2(N-1)}} \exp(-c/N) . \]  

(28)

We now compare the results for the number of SAW paths which terminate after \( \ell \) steps, \( n_T(\ell) \), vs. the total number, \( n(\ell) \), of paths of length \( \ell \), given by Eqs. (22) and (10), respectively. Clearly, \( n_T(\ell) \leq n(\ell) \), for \( \ell = 1, \ldots, N - 1 \), with equality at \( \ell = N - 1 \). Since the termination probability increases with \( \ell \), the ratio \( n_T(\ell)/n(\ell) \) is a monotonically increasing function. As a result, the peak of \( n_T(\ell) \) is shifted to the right with respect to the peak of \( n(\ell) \), namely \( \ell_{\text{peak}}^T > \ell_{\text{peak}} \). This can be easily confirmed by a comparison between Eqs. (17) and (24), using the fact that \( W(x) \) is a monotonically increasing function for \( x > 0 \).

In case of a dilute network, both functions can be approximated by Gaussian forms, given by Eqs. (25) and (18), respectively. Comparing between Eqs. (19) and (26), and using the fact that \( \psi^{(1)}(x) \) is a monotonically decreasing function for \( x > 0 \), one can show that the peak of \( n_T(\ell) \) is narrower than the peak of \( n(\ell) \), namely \( \sigma_T < \sigma \).

5. The distribution of SAW path lengths

Consider an SAW on an ER network, which starts from a node with degree \( k \geq 1 \) (non-isolated node). The SAW hops between nearest neighbor nodes until it reaches a node whose all neighbors have already been visited. At that stage the SAW has
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no exit link and it is terminated. In the complementary picture of a pruned network, the walker hops until it reaches a node which becomes isolated upon its arrival. The probability that a node is isolated at time $t$ is $p_t(k = 0)$. Therefore, the probability that the SAW will proceed from time $t$ to time $t+1$ is given by the conditional probability $P(d > t|d > t - 1) = 1 - p_t(k = 0)$, where the random variable $d$ denotes the distance pursued along the SAW path. Thus, the probability that the path length of the SAW will be longer than $\ell$ is given by

$$P(d > \ell) = P(d > 0) \prod_{t=1}^{\ell} P(d > t|d > t - 1),$$

(29)

where $P(d > 0) = 1$ (since the initial node is not isolated). Thus, this tail distribution takes the form

$$P(d > \ell) = \prod_{t=1}^{\ell} [1 - p_t(k = 0)].$$

(30)

While Eq. (30) applies to any network, in the case of ER networks we have an explicit expression for the probability of a node to become isolated, namely $p_t(k = 0) = \exp[-c(t)]$. Therefore, the tail distribution takes the form

$$P(d > \ell) = \prod_{t=1}^{\ell} [1 - e^{-c(t)}].$$

(31)

The validity of this expression relies on the validity of the equation

$$P(d > \ell|d > \ell - 1) = 1 - e^{-(1 - \frac{\ell}{N - 1})c}$$

(32)

for an ER network. In Fig. 4 we present the conditional probability $P(d > \ell|d > \ell - 1)$ vs. $\ell$ for a network of size $N = 1000$ and for three values of $c$. The analytical results (solid lines) obtained from Eq. (32) are found to be in good agreement with numerical simulations (symbols), confirming the validity of this equation. Note that the numerical results become more noisy as $\ell$ increases, due to diminishing statistics, and eventually terminate. This is particularly apparent for the smaller values of $c$.

To obtain a closed form expression for the tail distribution, $P(d > \ell)$, we take the natural logarithm on both sides of Eq. (31). This leads to

$$\ln [P(\ell > \ell)] = \sum_{t=1}^{\ell} \ln \left[ 1 - \exp \left( \frac{ct}{N-1} - c \right) \right].$$

(33)

Approximating this sum by an integral we obtain

$$\ln [P(\ell > \ell)] \approx \int_{1/2}^{\ell+1/2} \ln \left[ 1 - \exp \left( \frac{ct}{N-1} - c \right) \right] dt.$$  

(34)
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This integral is in fact a partial Bose-Einstein integral, which can be expressed in terms of the Polylogarithm $\text{Li}_n(x)$ function \[44\]

$$P(d > \ell) \simeq \exp \left\{ \frac{N - 1}{c} \left[ \text{Li}_2 \left( e^{-\left(1 - \frac{\ell}{N-1}\right)c} \right) - \text{Li}_2 \left( e^{-\left(1 - \frac{\ell+1}{N-1}\right)c} \right) \right] \right\}. \quad (35)$$

The probability distribution $P(\ell)$ is obtained from the tail distribution by

$$P(\ell) = P(d > \ell - 1) - P(d > \ell). \quad (36)$$

In the analysis below, we replace the difference in Eq. (36) by a derivative. This replacement is justified either for very smooth functions or for large values of $\ell$. Indeed, the function $P(d > \ell)$ satisfies these conditions for both sparse and dense networks. For small values of $c$, it is smooths over its entire range, while for larger values of $c$ it exhibits a sharp variation only in the range of large $\ell$.

Replacing the difference by a derivative we obtain

$$P(\ell) = -\frac{dP(d > \ell)}{d\ell}. \quad (37)$$

Plugging Eq. (35) into Eq. (37) we obtain

$$P(\ell) \simeq -\ln \left[ 1 - e^{-\left(1 - \frac{\ell+1}{N}\right)c} \right] \cdot P(d > \ell). \quad (38)$$

For large networks ($N \gg 1$) one can further approximate Eqs. (35) and (38) by

$$P(d > \ell) \simeq \exp \left[ -\frac{N}{c} e^{-c} \left( e^{\frac{c}{N}\ell} - 1 \right) \right] \quad (39)$$

and

$$P(\ell) \simeq \exp \left[ -\frac{N}{c} e^{-c} \left( e^{\frac{c}{N}\ell} - 1 \right) - \left( 1 - \frac{\ell + 1}{N} \right) c \right], \quad (40)$$

respectively. In Fig. 5 we present the distributions of path lengths of SAWs on ER networks of size $N = 1000$, for different values of $c$. The tail distributions, $P(d > \ell)$, are shown in the top row and the corresponding probability density functions, $P(\ell)$, are shown in the bottom row. The analytical results (solid lines), obtained from Eqs. (35) and (38), are found to be in excellent agreement with numerical simulations (circles). In fact, the approximated expressions of Eqs. (39) and (40) provide results which are practically indistinguishable from the more accurate expressions presented in Fig. 5.

In the numerical simulations, the initial node of the SAW is chosen randomly among the nodes on the largest connected cluster. This is justified because for $c \geq 3$ less than one percent of the nodes reside on small isolated clusters. Fig. 5 reveals three different qualitative behaviors of $P(\ell)$. For small values of $c$ (sparse networks), $P(\ell)$ is a monotonically decreasing function. As $c$ is increased, $P(\ell)$ forms a peak and becomes broader and more symmetric. In the limit of dense networks the peak becomes
narrower as it shifts to the right. As \( c/(N - 1) \to 1 \), it approaches a delta function at \( \ell = N - 1 \). Further insight about \( P(\ell) \) is given below in the context of central and dispersion measures.

It is interesting to note that the expressions for the distribution of SAW path lengths for large networks, presented in Eqs. (39) and (40), coincide with the corresponding equations of the Gompertz distribution [45]. In particular, the tail distribution of the Gompertz distribution for a random variable \( X \) takes the form

\[
P(X > x) = \exp \left[ -\eta \left( e^{bx} - 1 \right) \right]
\]

for \( x \geq 0 \). Inserting the scale parameter

\[
b = \frac{c}{N},
\]

and the shape parameter

\[
\eta = \frac{N}{c} e^{-c}
\]

into Eq. (41) gives rise to Eq. (39).

The Gompertz law describes the distribution of adult lifespans [46, 47] as well as various other survival probabilities, such as the failure rates of computer codes [48]. The very old observation, attributed to Halley [49] and Euler [50], is that an exponential life expectancy of the form \( S(t) = \exp(-t/t_0) \), where \( S(t) \) is the survival probability of an individual, and \( t_0 \) being a characteristic life span (say 70), would entail millions of people with the age of 200. Gompertz suggested that the mortality rate is not a constant, as implied by the exponential law, but rather increases exponentially with age, which explains why the longest recorded human life did not exceed 123 years [51].

In our case, this observation provides an interesting narrative for the life expectancy of an SAW on the network - the termination rate increases exponentially with time as a result of the fact that the SAW prunes the network along its walk and makes it sparser. There are however two important differences between the Gompertz distribution \( P(X > x) \) of Eq. (41) and the tail distribution \( P(d > \ell) \) of Eq. (39). The first difference is that \( P(d > \ell) \) describes a discrete distribution over integer values of \( \ell \) while the Gompertz distribution describes a continuous random variable. The second difference is that the Gompertz law is unbounded (valid for any \( x \geq 0 \)), while the longest possible SAW path on a network is of length \( \ell = N - 1 \). The first difference is important in the limit of sparse networks, where the SAW path lengths are small and the discrete nature of \( \ell \) is apparent. The second difference is important in the limit of dense networks, where the SAW path lengths approach their maximal value.

6. Central measures of the SAW path length distribution

In order to characterize the distribution of path lengths of the SAW we derive expressions for the mean, median and mode of this distribution. The mean of the distribution can
The distribution of path lengths of SAWs on ER networks

be obtained from the tail-sum formula

$$\ell_{\text{mean}}(N,c) = \sum_{\ell=0}^{N-2} P(d > \ell).$$

(44)

Assuming that the initial node is not isolated, this sum can be written in the form

$$\ell_{\text{mean}}(N,c) = 1 + \sum_{\ell=1}^{N-2} P(d > \ell).$$

(45)

Expressing the sum as an integral we obtain

$$\ell_{\text{mean}}(N,c) = 1 + \int_{1/2}^{N-3/2} P(d > \ell) d\ell.$$  

(46)

Plugging in $P(d > \ell)$ from Eq. (39), the resulting integral has the closed form expression

$$\ell_{\text{mean}}(N,c) \simeq 1 + \left[ Ei \left( -\frac{N}{c} e^{-\frac{3c}{2N}} \right) - Ei \left( -\frac{N}{c} e^{-\left(1-\frac{1}{2N}\right)c} \right) \right] \exp \left( \frac{N}{c} e^{-c} \right),$$

(47)

where $Ei(x)$ is the exponential integral [44]. In the limit of large $N$ one can write an approximated expression using only elementary functions, by expanding in the parameter $\eta = (N/c)e^{-c}$, resulting in two regimes. In sparse networks, where $\eta > 1$, the mean path length is given by

$$\ell_{\text{mean}}(N,c) \simeq 1 + e^c.$$  

(48)

In dense networks, where $\eta < 1$, the mean path length is

$$\ell_{\text{mean}}(N,c) \simeq 1 + \left[ N - \frac{N}{c} \left( \ln \frac{N}{c} + \gamma \right) + \left( \frac{N}{c} \right)^2 e^{-c} - \frac{1}{4} \left( \frac{N}{c} \right)^3 e^{-2c} \right] \exp \left( \frac{N}{c} e^{-c} \right),$$

(49)

where $\gamma$ is the Euler-Mascheroni constant [44].

The median of $P(\ell)$ is obtained by equating the right hand side of Eq. (39) to $1/2$ and solving for $\ell$. The resulting expression is

$$\ell_{\text{median}}(N,c) \simeq \frac{N}{c} \ln \left( 1 + e^c \cdot \frac{c}{N} \ln 2 \right).$$

(50)

The mode of the distribution of path lengths is the value of $\ell$ which maximizes $P(\ell)$. For $c < \ln N$ the distribution is monotonically decreasing and the maximum is obtained for $\ell = 1$. For larger values of $c$, the distribution develops a peak, where the derivative of $P(\ell)$ vanishes. Using $dP(\ell)/d\ell = 0$ in Eq. (40) we obtain

$$\ell_{\text{mode}}(N,c) \simeq \begin{cases} 
1 & \text{if } c \leq c_0 \\
\left[ N - \frac{N}{c} \ln \left( \frac{N}{c} \right) \right] & \text{if } c > c_0,
\end{cases}$$

(51)
where \([x]\) is the integer part of \(x\). The transitional \(c_0\) between these two regimes of the probability density function is obtained by equating \(\ell_{\text{mode}}\) to 1 and solving for \(c\), given approximately by \(\ln c + c = \ln N\). Solving this equation we find that

\[
c_0 = W(N),
\]

where \(W(x)\) is the Lambert W function \([44]\). Note that \(c_0\) is also the point at which the shape parameter \(\eta\) of the Gompertz distribution, given by Eq. (43), which also appears in the discussion that follows Eq. (47), is equal to 1. It separates the small \(c\) regime, where \(\eta > 1\), from the large \(c\) regime, where \(\eta < 1\).

Interestingly, in the regime \(c \gg c_0\) we find that all the three central measures presented above converge to the same asymptotic expression given by

\[
\ell_{\text{m}}(N, c) \simeq N \left[1 - \frac{1}{c} \ln \left(\frac{N}{c}\right)\right].
\]

This means that the path lengths of typical SAWs converge towards \(N\) as \(c\) is increased. Thus, a typical SAW path becomes a Hamiltonian path as \(c \to N\).

In Fig. 6 we present the central measures \(\ell_{\text{mean}}(N, c)\) (a) \(\ell_{\text{median}}(N, c)\) (b) and \(\ell_{\text{mode}}(N, c)\) (c), vs. \(c\), for ER networks of size \(N = 1000\). The solid lines represent the analytical results, obtained from Eqs. (47), (50), and (51), respectively. They are found to be in excellent agreement with numerical simulations (circles).

7. Measures of dispersion of the SAW path length distribution

The width of the path length distribution can be characterized by the variance \(\sigma^2_\ell = \langle \ell^2 \rangle - \langle \ell \rangle^2\), where \(\langle \ell^n \rangle\), is given by the tail-sum formula \([52]\)

\[
\langle \ell^n \rangle = \sum_{\ell=0}^{N-2} [(\ell + 1)^n - \ell^n] P(d > \ell).
\]

Neglecting exponential corrections we can set the upper limit of the sum to \(\infty\) and replace the sum by an integral. Plugging in \(P(d > \ell)\) from Eq. (39) results in the following closed-form expression for the variance

\[
\sigma^2_\ell = \frac{N^2}{c^2} e^\eta \left[\frac{\gamma^2}{6} - e^\eta [Ei(-\eta)]^2 - 2\eta \cdot 3F_3 \left(\frac{1,1,1}{2,2,2}, -\eta\right) + 2 \gamma \ln \eta + (\ln \eta)^2\right],
\]

where \(\eta = \frac{N}{c} e^{-c}\) is the shape parameter of the corresponding Gompertz distribution, given by Eq. (43), and \(3F_3\) is a generalized hypergeometric function \([44]\). It can be expressed by

\[
3F_3 \left(\frac{1,1,1}{2,2,2}, x\right) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)^3 k!}.
\]
For small values of $c$, namely $c < c_0$, expanding the right hand side of Eq. (55) in powers of the small parameter $1/\eta$ we obtain

$$
\sigma^2_\ell \simeq e^{2c} \left[ 1 - \frac{4}{\eta} + \frac{17}{\eta^2} + O\left( \frac{1}{\eta^3} \right) \right].
$$

(57)

For large values of $c$, namely $c > c_0$, we use $\eta$ as a small parameter to express the right hand side of Eq. (55) as

$$
\sigma^2_\ell \simeq \frac{\pi^2 N^2}{6 c^2} \left[ 1 + \eta \left( \frac{\pi^2 - 12 + 12\gamma - 6\gamma^2}{\pi^2} + 12 (1 - \gamma) \ln \eta - 6(\ln \eta)^2 \right) + O(\eta^2) \right].
$$

(58)

Another way to characterize the width of the path length distribution is to calculate the interquartile range (IQR). It is defined as $IQR = \ell_{3/4} - \ell_{1/4}$, where $\ell_{3/4}$ is the upper quartile [namely, $P(d \leq \ell_{3/4}) = 3/4$] and $\ell_{1/4}$ is the lower quartile [namely, $P(d \leq \ell_{1/4}) = 1/4$]. The upper (lower) quartile is obtained by equating Eq. (39) to $1/4$ ($3/4$) and solving for $\ell_{3/4}$ ($\ell_{1/4}$). The result is

$$
IQR(N, c) \simeq \frac{N c}{c} \ln \left( 1 + \frac{\ln 3}{N c e^{-c} + \ln \frac{4}{3}} \right).
$$

(59)

In Fig. [7] we present $\sigma_\ell$ and $IQR$ as a function of $c$. The analytical results for both measures (solid lines) are found to be in excellent agreement with the numerical results (symbols). Both measures indicate a maximal dispersion around $c = c_0$, which marks the crossover between the regimes of sparse and dense networks.

8. Extreme value statistics of the SAW path length

Another way to characterize the dispersion of the distribution is to express it in terms of extreme value statistics. To this end, we examine the lengths $\ell_{\text{max}}$ and $\ell_{\text{min}}$ of the longest and shortest paths among $r$ independent realizations of the SAW on an $ER[N, c/(N-1)]$ network. The expected value of $\ell_{\text{max}}$ is given by the condition $P(d > \ell) < 1/r$ while the expected value of $\ell_{\text{min}}$ is given by $P(d < \ell) < 1/r$.

Using the expression for $P(d > \ell)$, given by Eq. (39), we obtain

$$
\ell_{\text{max}} = \frac{N}{c} \ln \left( 1 + \frac{c}{N e^{-c} \ln r} \right)
$$

(60)

and

$$
\ell_{\text{min}} = \frac{N}{c} \ln \left[ 1 + \frac{c}{N e^{-c} \ln \left( \frac{r}{r-1} \right)} \right].
$$

(61)

In the limit of a large number of realizations, $r \gg 1$, the expression for $\ell_{\text{min}}$ can be simplified to

$$
\ell_{\text{min}} \simeq \frac{N}{c} \ln \left( 1 + \frac{c}{N r e^{-c}} \right).
$$

(62)
In Fig. 8 we present the lengths $\ell_{\text{max}}$ and $\ell_{\text{min}}$ of the longest and shortest paths among $r = 10^4$ independent SAW realizations vs. the initial connectivity, $c$, on an ER network of size $N = 1000$. The theoretical results for the longest paths (solid line), obtained from Eq. (60), are in excellent agreement with the numerical simulations (circles). The theoretical results for the shortest paths (solid line), obtained from Eq. (61), are also in excellent agreement with the numerical simulations (squares). The average lengths $\ell_{\text{mean}}$, obtained from the simulations (crosses) and from Eq. (49) (solid line), are shown for comparison.

In Fig. 9 we present the lengths $\ell_{\text{max}}$ and $\ell_{\text{min}}$ of the longest and shortest paths as a function of the number of independent realizations, $r$, of the SAW on an ER network with $N = 1000$ and $c = 10$. The theoretical results for the longest paths (solid line), obtained from Eq. (60), are in excellent agreement with numerical simulations (circles). The theoretical results for the shortest paths (solid line), obtained from Eq. (61), are also in excellent agreement with the numerical simulations (squares).

9. Summary and Discussion

We have studied SAW paths on finite ER networks. In practice, SAWs on networks delete the nodes they visit, thus gradually reducing the network size. We have shown that the pruned network maintains its ER character, with a linearly decreasing average degree, $c(t)$. We obtained an exact formula for the number of SAW paths, $n_T(\ell)$, which terminate after $\ell$ steps and analyzed its behavior as a function of the initial connectivity, $c$. We also studied the distribution of path lengths $P(\ell)$ for SAW paths which are actually pursued by a self-avoiding random walker starting from a random initial node, $i$. We obtained an analytical expression for $P(\ell)$ as well as a large $N$ approximation valid for large networks. It was found that for low initial connectivity, $P(\ell)$ is a monotonically decreasing function, while for larger values of $c$ it exhibits a well rounded peak, which shifts to the right as $c$ is increased. To characterize the distribution $P(\ell)$ we obtained analytical expressions for its central measures, namely the mean, median and mode. We also derived measures for the dispersion of $P(\ell)$, namely the standard deviation and the inter-quartile range. Studying the extreme value statistics of the SAW path lengths, we obtained the expectation value of the longest and shortest paths among $r$ independent SAW paths.

In Fig. 10 we present the number of SAW paths, $n_T(\ell)$ of length $\ell$ (a) and the distribution of path lengths $P(\ell)$ (b) for an ER network of size $N = 100$ and $p = 0.05$. It is observed that the peak of $P(\ell)$ takes place at a smaller $\ell$ value than the peak of $n_T(\ell)$. This reflects the fact that the SAW paths which are actually pursued are typically shorter than SAW paths chosen at random from the list of all SAW paths. The former SAW paths are often referred to as kinetic growth self-avoiding walks [23], or true self-avoiding walks [24], in contrast to the SAW paths which are uniformly sampled among all possible self avoiding paths of a given lengths.

The reason that true self avoiding walks are typically shorter is that the probability
of an SAW path to be pursued is a decreasing function of its length, $\ell$. More specifically, the number of paths $n_T(\ell)$ proliferates at rate determined by $c(\ell)$, and thus keeps increasing as long as $c(\ell) > 1$. On the other hand, $P(\ell)$ is sensitive to the termination rate, given by $p_1(k = 0)$, which increases as a function of $\ell$. Thus, $P(\ell)$ reaches its maximum earlier than $n_T(\ell)$. This is demonstrated in Fig. 11 in which we present the locations of the peaks, $\ell_{\text{peak}}^T$ and $\ell_{\text{mode}}$, as a function of $c$. Both curves increase monotonically with $c$. For dilute networks, it is shown that $\ell_{\text{mode}}$ is much smaller than $\ell_{\text{peak}}^T$, while for dense networks they are comparable. A similar effect was observed long ago for SAWs on regular lattices [53].

From another perspective, the distribution $P(\ell)$ follows the Gompertz law, where the termination rate increases with the number of steps already pursued by the SAW. Therefore, the fact that the number of available SAW paths, $n_T(\ell)$ increases with the length is overridden by the super-exponential decay of the Gompertz distribution.

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Figure 1. The average degree, \( c(t) \), of the remaining network vs. time, \( t \), for an ER network of initial size \( N = 1000 \) and for different initial values of the average connectivity, \( c \). The analytical results (solid lines), obtained from Eq. (6), are in excellent agreement with numerical simulations for \( c = 5 \) (circles), for \( c = 10 \) (triangles) and for \( c = 20 \) (diamonds). The numerical results were obtained from \( 10^4 \) realizations of the random walk for each value of \( c \).

Figure 2. The evolution of the degree distribution \( p_t(k) \) at three different times, \( t = 0, 350 \) and 700 (represented by circles, crosses and triangles respectively), on a network of size \( N = 1000 \). The initial value of the average degree is \( c = 20 \). The analytical results (lines), obtained from a Poisson distribution [Eq. (8)], using the predicted value of \( c(t) \) presented in Eq. (6), are found to be in excellent agreement with numerical simulations.
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Figure 3. The number, \( n(\ell) \), of SAW paths of length \( \ell \), starting at a randomly chosen node, for a network of size \( N = 100 \) and different values of \( c \): \( c = 2 \) (a), \( c = 5 \) (b) and \( c = 7 \) (c). The dashed lines (blue) are obtained from exact enumeration of the paths, using Eq. (10). The solid lines (red) are obtained from an asymptotic expression, namely the Gaussian approximation given by Eq. (18), showing excellent agreement with the exact enumeration. The function \( n(\ell) \) exhibits a well defined and highly symmetric peak, which shifts to the right as \( c \) is increased.

Figure 4. The conditional probability \( P(d > \ell|d > \ell - 1) \) vs. \( \ell \), obtained from Eq. (32) (solid lines) and from numerical simulations of SAWs (symbols) on ER networks of size \( N = 1000 \) and initial connectivities \( c = 5, 10 \) and \( 20 \) (squares, circles and crosses, respectively). The analytical and numerical results are found to be in good agreement.
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Figure 5. The distributions of path lengths of SAWs on ER networks of size $N = 1000$ and $c = 3, 5, 10$ and 50. The tail distributions $P(d > \ell)$, obtained from Eq. (35) (solid lines) and from numerical simulations (circles) are presented in the top row, with excellent agreement between the two. The corresponding probability density functions $P(\ell)$, obtained from Eq. (38) are shown in the bottom row. The agreement with the numerical results is already established in the top row and therefore the numerical data is not shown in the bottom row.

Figure 6. The mean (a), median (b) and mode (c) of the distribution of path lengths of SAWs vs. the initial connectivity, $c$ for ER networks of size $N = 1000$. The analytical results (solid lines) for the mean, median and mode are obtained from Eqs. (47), (50), and (51), respectively. The results are in excellent agreement with numerical simulations (circles).
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Figure 7. The standard deviation $\sigma_\ell$ (triangles) and the inter quartile interval, IQR (circles), for the distribution of path lengths of SAWs as a function of the initial connectivity, $c$, for ER networks of size $N = 1000$. The analytical results (solid lines), obtained from Eqs. (55) and (59) respectively, are in excellent agreement with numerical simulations (symbols).

Figure 8. The lengths, $\ell_{\text{max}}$ and $\ell_{\text{min}}$ of the longest (circles) and shortest (squares) SAW paths, respectively, as a function of $c$, among $r = 10^4$ independent SAW realizations on an ER network of size $N = 1000$. The solid lines are obtained from Eq. (60) for the longest path and from Eq. (61) for the shortest path, and both are in excellent agreement with the numerical simulations. The average lengths (crosses) are shown for comparison.
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Figure 9. The lengths, $\ell_{\text{max}}$ and $\ell_{\text{min}}$ of the longest (circles) and shortest (squares) SAW paths, respectively, as a function of the number of independent realizations, $r$, of the SAW on an ER network with $N = 1000$ and $c = 10$. The solid lines are obtained from Eq. (60) for the longest path and from Eq. (61) for the shortest path, and both are in excellent agreement with the numerical simulations.

Figure 10. (a) The number of SAW paths, $n_T(\ell)$, vs. $\ell$ for an ER network of $N = 100$ and $p = 5/100$, obtained from Eq. (22). The peak is at $\ell \simeq 80$, in agreement with Eq. (24). (b) The distribution of SAW path lengths, $P(\ell)$, vs. $\ell$, for the same network, obtained from Eq. (38). The peak is at $\ell \simeq 40$, in agreement with Eq. (51). The difference between the locations of the two peaks reflects the fact that long SAW paths are less likely to be pursued than shorter ones.
Figure 11. The location of the peaks, $\ell_T^{\text{peak}}$ (dashed line), and $\ell_{\text{mode}}$ (solid line) of $n_T(\ell)$ and $P(\ell)$, respectively vs. $c$. Both curves increase monotonically as a function of $c$. For small values of $c$, $\ell_{\text{mode}}$ is much smaller than $\ell_T^{\text{peak}}$, which means that the huge number of long SAW paths are rarely pursued.