ABSTRACT. It is shown that a function $u$ satisfying $|\partial_t u + \sum_{i,j} \partial_i (a^{ij} \partial_j u)| \leq N(|u| + |\nabla u|), |u(x, t)| \leq N e^{N|x|^2}$ in $\mathbb{R}_+^n \times [0, T]$ and $u(x, 0) = 0$ in $\mathbb{R}_+^n$ under certain conditions on $\{a^{ij}\}$ must vanish identically in $\mathbb{R}_+^n \times [0, T]$. The main point of the result is that the conditions imposed on $\{a^{ij}\}$ are of the type: $\{a^{ij}\}$ are Lipschitz and $|\nabla_x a^{ij}(x, t)| \leq E |x|$, where $E$ is less than a given number, and the conditions are in some sense optimal.

1. Introduction

Let $U$ be a domain in $\mathbb{R}^n$ and $P$ be a backward parabolic operator on $U \times [0, T]$,

$$P = \partial_t + \sum_{i,j} \partial_i (a^{ij} \partial_j) = \partial_t + \nabla \cdot (A \nabla),$$

where $A(x, t) = (a^{ij}(x, t))_{i,j=1}^{n}$ is a real symmetric matrix such that for some $\Lambda \geq \lambda > 0$,

$$\lambda |\xi|^2 \leq \sum_{i,j} a^{ij}(x, t) \xi_i \xi_j \leq \Lambda |\xi|^2, \forall \xi \in \mathbb{R}^n.$$  \hfill (1)

Consider a function $u$ which satisfies

$$\begin{cases} |Pu| \leq N(|u| + |\nabla u|) \text{ in } U \times [0, T], \\ |u(x, t)| \leq N e^{N|x|^2} \text{ in } U \times [0, T], \\ u(x, 0) = 0 \text{ in } U. \end{cases}$$

Where $N$ is a given positive constant.

The backward uniqueness (BU) problem is: does $u$ vanish identically in $U \times [0, T]$? If so, we say that $U$ is a BU domain for the operator $P$.

We should point out that there is no boundary condition about $u$ on the boundary of the domain $U$ which is recently discussed by Escauriaza, Seregin and Šverák on $\mathbb{R}^n \backslash B_R$ in [7]. The backward uniqueness problem appeared in many problems, for example, in the control theory for PDEs and the regularity theory of parabolic equations, in particular the Navier-Stokes equation by Escauriaza, Seregin and Šverák [9]. When $P$ is the backward heat operator, there are many results already on various domains, such as, on the whole space $\mathbb{R}^n$ by Poon [6], on the exterior of a ball $\mathbb{R}^n \backslash B_R$ by Escauriaza, Seregin and Šverák [7], on half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n > 0\}$ by Escauriaza, Seregin and Šverák [8, 9] and

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on some cones by Li and Šverák [16]. Those are all proved to be BU domains for the backward heat operator. On another hand, any bound domain is not BU domain [2, 3].

When $P$ is in general, variable coefficients, there are few results have been proved. While some related results have already been obtained. In particular, L. Escauriara and F. J. Fernández proved a unique continuation property when $\{a^{ij}\}$ are Lipschitz in [10]. Then it implies immediately that if $U \subset V$ and if $U$ is a BU domain, so is $V$. Recently, Tu A. Nguyen in [18] proved a conjecture of E. M. Landis and O. A. Oleinik which implies that $\mathbb{R}^n$ and $\mathbb{R}^n_+$ are BU domains under the conditions that $|\nabla_x a^{ij}(x, t)|$ and $|\partial_t a^{ij}(x, t)|$ are bounded and the decay at infinity conditions that

$$|\nabla_x a^{ij}(x, t)| \leq M(x)^{-1-\varepsilon}, \quad |a^{ij}(x, t) - a^{ij}(x, s)| \leq M(x)^{-1}|t-s|^{1/2},$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$ and $\varepsilon > 0$.

This paper can be regarded as a continuation of the above results. Since $\mathbb{R}^n \setminus B(R) \subset \mathbb{R}^n_+$ can be treated as a subset of $\mathbb{R}^n \setminus B(R)$, by the unique continuation property, we could only consider the case of $\mathbb{R}^n_+$. Also general simply connected domains may be mapped onto $\mathbb{R}^n_+$. Then we focus on operator $P$ with variable coefficients on the domain $\mathbb{R}^n_+$. Our main result is the following.

**Theorem 1.1.** Suppose $\{a^{ij}\}$ satisfy [1], and for some constants $E, M, N > 0$,

$$|\nabla_x a^{ij}(x, t)| + |\partial_t a^{ij}(x, t)| \leq M, \quad \forall (x, t) \in \mathbb{R}^n_+ \times [0, T],$$

and

$$|\nabla_x a^{ij}(x, t)| \leq \frac{E}{|x|}, \quad \forall (x, t) \in \mathbb{R}^n_+ \times [0, T].$$

Assume that $u$ satisfies

$$
\begin{cases}
|Pu| \leq N(|u| + |\nabla u|) \text{ in } \mathbb{R}^n_+ \times [0, T], \\
|u(x, t)| \leq Ne^{N|x|^2} \text{ in } \mathbb{R}^n_+ \times [0, T], \\
u(x, 0) = 0 \text{ in } \mathbb{R}^n_+.
\end{cases}
$$

Then there exists a constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, $u(x, t) \equiv 0$ in $\mathbb{R}^n_+ \times [0, T]$.

We remark that our assumptions are in some sense optimal. From the counterexamples constructed by A. Plis [20], K. Miller [21] and N. Mandache [22], we can see that to ensure BU, certain regularity of the coefficients should be required. Moreover, Tu A. Nguyen proved in [18] that the regularity conditions (3) and the decay at infinity conditions (2) will be sufficient to ensure BU. However, are the decay at infinity conditions necessary? Or, is condition (3) alone enough to guarantee BU? Here, we show that condition (3) is not enough and the decay at infinity condition (4) in Theorem 1.1 $|\nabla_x a^{ij}(x, t)| \leq \frac{E}{|x|}$, where $E$ is small, is some what optimal.

First, we copy the example given by N. Mandache in [22].

**Proposition 1.2.** There exist smooth functions $u, b_{11}, b_{12}, b_{22}$ and continuous functions $d_1, d_2$ defined on $\mathbb{R}^3 \ni (s, x, y)$, with the following properties:

i) $u$ is the solution of the equation

$$\partial^2_{xx} u + \partial_x((b_{11} + d_1)\partial_x u) + \partial_y(b_{12}\partial_x u) + \partial_x(b_{12}\partial_y u) + \partial_y((b_{22} + d_2)\partial_y u) = 0.$$

ii) There is a $T > 0$ such that $\text{supp } u = (-\infty, T] \times \mathbb{R}^2$.

iii) $u, b_{ij}$ and $d_i$ are periodic in $x$ and in $y$ with period $2\pi$. 

iv) \( d_1 \) and \( d_2 \) do not depend on \( x \) and \( y \) and are Hölder continuous of order \( \alpha \) for all \( \alpha < 1 \).

v) \( \frac{1}{2} < \left( \begin{array}{cc} d_1 + b_{11} & b_{12} \\ b_{12} & d_2 + b_{22} \end{array} \right) < 2 \) on \( \mathbb{R}^3 \).

Furthermore, there are also functions as above, satisfying conditions i)-v) except that (6) is replaced with the parabolic equation:

\[
\partial_s u = \partial_x ((b_{11} + d_1) \partial_x u) + \partial_y (b_{12} \partial_x u) + \partial_x (b_{12} \partial_y u) + \partial_y ((b_{22} + d_2) \partial_y u).
\]

The solution of (7) implies that the Hölder regularity in the time variable is not enough for BU. Hence it is reasonable for us to assume that \( |\partial_t a^{ij}(x,t)| \) are bounded in Theorem 1.1.

Next we consider the requirement of the regularity in the space variable. Assume that \( u \) is the solution of (6). We denote that

\[
v(t,s,x,y) = u(T + s + t,x,y); \quad \tilde{b}_{ij}(t,s,x,y) = b_{ij}(T + s + t,x,y); \quad \tilde{d}_i(t,s) = d_i(T + s + t).
\]

in

\([-1,0] \times \mathbb{R}_+^3 = \{(t,s,x,y)|t \in [-1,0], s > 0, x \in \mathbb{R}, y \in \mathbb{R}\}.\]

Then

\[
\partial_t v - [\partial_s^2 v + \partial_x ((\tilde{b}_{11} + \tilde{d}_1) \partial_x v) + \partial_y (\tilde{b}_{12} \partial_x v) + \partial_x (\tilde{b}_{12} \partial_y v) + \partial_y ((\tilde{b}_{22} + \tilde{d}_2) \partial_y v)] - \partial_s v = 0.
\]

By ii) of Proposition 1.2 \( v(0,s,x,y) = 0 \) and \( v \) is nonzero in \([-1,0] \times \mathbb{R}_+^3 \), thus BU fails. It shows that the Hölder regularity in the space variable is not enough for BU, hence it is also reasonable for us to assume that \( |\partial_t a^{ij}(x,t)| \) are bounded in Theorem 1.1.

Now we consider the decay condition at infinity. We could construct an example as follows. Consider a cone \( C_{\theta_0} \) with opening angle \( \theta_0 \) and the system

\[
\begin{cases}
\partial_t u + \Delta u = 0 \text{ in } C_{\theta_0} \times [0,T], \\
|u(x,t)| \leq N \text{ in } C_{\theta_0} \times [0,T], \\
u(x,0) = 0 \text{ in } C_{\theta_0}.
\end{cases}
\]

In [16], L. Escauriaza gave an example to show that the above system has a nonzero solution when \( \theta_0 < \frac{\pi}{2} \) and Lu Li and V. Šverák proved that the system has only zero solution when \( \pi > \theta_0 > 2 \arccos(1/\sqrt{3}) \approx 109.5° \). Now we consider a cone of dimension 2,

\[
C_{\theta_0} = \{(r,\theta)|0 < \theta < \theta_0\}, \ (0 < \theta_0 < \pi)
\]

and \( u(x_1,x_2,t) \) is the solution of system (8) in dimension 2, where

\[
\begin{cases}
x_1 = r \cos \theta \\
x_2 = r \sin \theta.
\end{cases}
\]

Let

\[
\tilde{\theta} = l\theta, \text{ with } l = \frac{\pi}{\theta_0} > 1,
\]
A direct calculation gives us
\[
\begin{aligned}
y_1 &= r \cos \bar{\theta} \\
y_2 &= r \sin \bar{\theta},
\end{aligned}
\]
then \((y_1, y_2) \in \mathbb{R}^2_+\). We denote

\[
v(y_1, y_2, t) = u(x_1, x_2, t),\ (y_1, y_2, t) \in \mathbb{R}^2_+ \times [0, T].
\]

By simple calculation,
\[
\begin{aligned}
\partial^2_{x_1} + \partial^2_{x_2} &= \partial^2_r + \frac{\partial_r}{r} + \frac{\partial^2_{\theta}}{r^2} = \partial^2_r + \frac{\partial_r}{r} + l^2 \frac{\partial^2_{\theta}}{r^2} \\
&= (\partial^2_r + \frac{\partial_r}{r} + \frac{\partial^2_{\theta}}{r^2}) + (l^2 - 1) \frac{\partial^2_{\theta}}{r^2} \\
&= \partial^2_{y_1} + \partial^2_{y_2} + (l^2 - 1) \left( \frac{y_1^2}{r^2} \partial^2_{y_1} + \frac{y_2^2}{r^2} \partial^2_{y_2} - 2 \frac{y_1 y_2}{r^2} \partial_{y_1} \partial_{y_2} - \frac{y_1}{r^2} \partial_{y_1} - \frac{y_2}{r^2} \partial_{y_2} \right) \\
&= [1 + (l^2 - 1) \frac{y_1^2}{r^2}] \partial^2_{y_1} + [1 + (l^2 - 1) \frac{y_2^2}{r^2}] \partial^2_{y_2} - 2 \frac{y_1 y_2}{r^2} \partial_{y_1} \partial_{y_2} - (l^2 - 1) \left( \frac{y_1}{r^2} \partial_{y_1} + \frac{y_2}{r^2} \partial_{y_2} \right).
\end{aligned}
\]

Together with the equation in \((8)\) we can deduce that

\[
|\partial_r v + \nabla \cdot (A \nabla v)| \leq \frac{C(l)}{r} |\nabla v|,
\]
where
\[
A(y_1, y_2) = \begin{pmatrix}
1 + (l^2 - 1) \frac{y_1^2}{r^2} & -\frac{y_1 y_2}{r^2} \\
-\frac{y_1 y_2}{r^2} & 1 + (l^2 - 1) \frac{y_2^2}{r^2}
\end{pmatrix}.
\]

\(A(y_1, y_2)\) is positive since \(l > 1\).

Denote
\[
w(y_1, y_2, t) = v(y_1, y_2 + 1, t),\ (y_1, y_2, t) \in \mathbb{R}^2_+ \times [0, T],
\]
and
\[
B(y_1, y_2) = A(y_1, y_2 + 1) = \begin{pmatrix}
1 + (l^2 - 1) \frac{(y_1 + 1)^2}{y_1^2 + (y_1 + 1)^2} & -\frac{y_1 (y_2 + 1)}{y_1^2 + (y_1 + 1)^2} \\
-\frac{y_1 (y_2 + 1)}{y_1^2 + (y_1 + 1)^2} & 1 + (l^2 - 1) \frac{y_2^2}{y_1^2 + (y_1 + 1)^2}
\end{pmatrix} = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}.
\]

A direct calculation gives us
\[
|\nabla b_{ij}| \leq \max \{l^2 - 1, 1\} \ \text{and} \ |\nabla b_{ij}| \leq \frac{\max \{l^2 - 1, 1\}}{r} \equiv \frac{E_1}{r}.
\]

By \((10)\) we have
\[
|\partial_r w + \nabla \cdot (B \nabla w)| \leq \frac{C(l)}{\sqrt{y_1^2 + (y_2 + 1)^2}} |\nabla w| \leq C(l) |\nabla w|.
\]

By the notations of \(u\) and \(w\), we see that \(w\) is a solution of the following system:

\[
\begin{aligned}
|\partial_r w + \nabla \cdot (B \nabla w)| &\leq C(l) |\nabla w| \ \text{in} \ \mathbb{R}^2_+ \times [0, T], \\
|w(y, t)| &\leq N \ \text{in} \ \mathbb{R}^2_+ \times [0, T], \\
w(y, 0) &= 0 \ \text{in} \ \mathbb{R}^2_+.
\end{aligned}
\]

By the result of Li and Šverák \cite{16}, we conclude that when
\[
E_1 < \left(\frac{\pi}{2 \text{arccos}(1/\sqrt{3})}\right)^2 - 1,
\]
we have
\[ 1 < l < \frac{\pi}{2 \arccos(1/\sqrt{3})}, \quad 2 \arccos(1/\sqrt{3}) < \theta_0 < \pi, \]
and then \( u \equiv 0 \) and thus \( w \equiv 0 \); when \( E_1 > 3 \), we have \( l > 2 \), \( \theta_0 < \frac{\pi}{2} \), and then (11) has a nonzero solution and thus (11) also has a nonzero solution, that is BU fails, although \( |\nabla b^{ij}| \) are bounded and \( |\nabla b^{ij}| \leq \frac{E_1}{r} \).

The example above shows that the decay at infinity condition, that is assumption in (4), where \( E \) is small than a given constant, is somewhat optimal.

To prove Theorem 1.1 we need to obtain the corresponding Carleman inequalities. Now we introduce two Carleman inequalities for the case of variable coefficients. They are generalizations of the two Carleman inequalities for the case of constant coefficients, as shown in [8, 9].

**Proposition 1.3.** Suppose \( \{a^{ij}\} \) satisfy (1) and
\[
|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \leq M, \quad |\nabla_x a^{ij}(x,t)| \leq \frac{E}{|x|}, \quad \forall (x,t) \in \mathbb{R}^n \times (0,2).
\]
Then there exists a constant \( K = K(n,\Lambda,\lambda,M,E) \), such that for any \( u \in C_c^\infty(\mathbb{R}^n \times (0,2)) \) and any number \( \gamma > 0 \),
\[
\int_{\mathbb{R}^n \times (0,2)} e^{2\gamma(t-K-1)-\frac{b|x|^2}{t}} (|u|^2 + |\nabla u|^2) dx dt \\
\leq \int_{\mathbb{R}^n \times (0,2)} e^{2\gamma(t-K-1)-\frac{2b|x|^2+K}{t}} |Pu|^2 dx dt,
\]
where \( b = \frac{1}{8\Lambda} \).

**Proposition 1.4.** Suppose \( \{a^{ij}\} \) satisfy (1) and
\[
|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \leq M, \quad |\nabla_x a^{ij}(x,t)| \leq \frac{E}{|x|}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,1).
\]
Let \( Q = \{(x,t) | x_1 \geq 1, t \in (0,1)\} \) and
\[
\psi(x) = |x|^2 - 2\frac{\Lambda}{\lambda} |x|x_1 + 2\left(\frac{\Lambda}{\lambda}\right)^2 x_1^2.
\]
Then there exist positive constants \( E_0 = E_0(n,\Lambda,\lambda) \), \( \alpha = \alpha(n,\Lambda,\lambda,E) \in (1,2) \), \( b = b(\Lambda,\lambda) \) and \( K = K(n,\Lambda,\lambda,M,E) \) such that when \( E < E_0 \), for any function \( u \in C_c^\infty(Q) \) and any number \( \gamma > 0 \), we have
\[
\int_Q e^{2\gamma(t-K-1)x_1^{\alpha} - \frac{b\psi(x)+K}{t}} (|u|^2 + |\nabla u|^2) dx dt \\
\leq \int_Q e^{2\gamma(t-K-1)x_1^{\alpha} - \frac{2b\psi(x)+K}{t}} |Pu|^2 dx dt.
\]

**Remark 1.5.** In fact, we can take \( E_0 = \frac{\lambda}{16n^2 \alpha^{n+1}} \), \( \alpha = 1 + \frac{E}{E_0} \) and \( b = \frac{1}{64M(\alpha+1)^4} \) in Proposition 1.4, which can be seen from the proof.

Carleman inequality (13) is the key results in this paper. Assuming it, there is only a standard argument by following the corresponding parts of Escauriza, Seregin, and Šverák in [7, 8] to prove Theorem 1.1. In the establishment of Carleman inequality (13), the construction of the function \( \psi \) is crucial.
Remark 1.6. It is worthwhile to note that Carleman inequality (12) does not require the smallness of $E$, while Carleman inequality (13) does, which is stronger.

Moreover, the Carleman inequality (13) for the parabolic operators with variable coefficients in a half space is stronger than the one for the case of constant coefficients, as shown in [3, 9]. When $P$ is the backward heat operator, there are two Carleman inequalities to prove BU. The first one implies an exponential decay of the solution, which enable us to apply the second one to prove BU. And here, we just need one Carleman inequality (13) to prove BU. We list Carleman inequality (12) here just for comparison with the case of constant coefficients.

The paper organized as follows. We first make use of Carleman inequality (13) to prove Theorem 1.1 in next section. Then we prove the two Carleman inequalities Proposition 1.3 and Proposition 1.4 in the last section.

2. Proof of Theorem 1.1

In this section, we prove the main theorem by assuming Proposition 1.4 first. Then we shall prove the Carleman inequalities in next section.

We always assume that $T = 1$ and extend $u$ and $a^{ij}$ by the following way:

\[ u(x,t) = 0, \text{ if } t < 0; \]
\[ a^{ij}(x,t) = a^{ij}(x,0), \text{ if } t < 0. \]

We denote $e_n = (0, 0, ..., 0, 1)$.

The next lemma implies Theorem 1.1 immediately.

Lemma 2.1. Suppose $\{a^{ij}\}$ and $u$ satisfy assumptions (1), (3)-(5). Then there exists $T_1 = T_1(\Lambda, \lambda, N) \in (0, \frac{1}{2})$, such that

\[ u(x,t) \equiv 0 \]

in $\mathbb{R}^n_+ \times (0, T_1)$.

Proof. We make use of Carleman inequality (13) to prove this lemma. We mainly follow the arguments of corresponding parts of Escauriaza, Seregin and Šverák in [7, 8]. By the regularity theory for solutions of parabolic equations, we have

\[ |u(x,t)| + |\nabla u(x,t)| \leq C(n, \Lambda, \lambda, M, N)e^{2N|x|^2} \]

for $(x,t) \in (\mathbb{R}^n_+ + e_n) \times (0, \frac{1}{2})$. Let

\[ T_1 = \min\left\{ \frac{b}{32N}, \frac{1}{12N^2}, \frac{1}{2} \right\}, \]

where $b$ is the one in Proposition 1.4. Let $\tau = \sqrt{2T_1}$. We denote

\[ v(y, s) = u(\tau y, \tau^2 s - T_1) \]

and

\[ \tilde{a}^{ij}(y, s) = a^{ij}(\tau y, \tau^2 s - T_1) \]

for $(y, s) \in \mathbb{R}^n_+ \times (0, 1)$. Then it is easy to see

\[ |\nabla_y \tilde{a}^{ij}(y, s)| + |\partial_s \tilde{a}^{ij}(y, s)| \leq \tau M \leq M, \]
and
\[ |\nabla_y \tilde{a}^{ij}(y, s)| = \tau |\nabla a^{ij}(\tau y, \tau^2 s - T_1)| \leq \tau \frac{E}{|\tau y|} = \frac{E}{|y|}. \]

We denote
\[ \tilde{P} v = \partial_s v + \sum_{ij} \partial_{ij} (\tilde{a}^{ij} \partial_{y^i} v), \]
by our notation and (5),
\[ (16) \quad |\tilde{P} v| \leq \tau N(|v| + |\nabla v|), \]
for \((y, s) \in \mathbb{R}_+^n \times (0, 1)\). From (14), we have
\[ (17) \quad |v(y, s)| + |\nabla v(y, s)| \leq C(n, \Lambda, \lambda, M, N)e^{2N\tau^2|y|^2}; \]
for \((y, s) \in (\mathbb{R}_+^n + \frac{1}{\tau} e_n) \times (0, 1)\); and
\[ (18) \quad v(y, s) = 0, \]
for \((y, s) \in \mathbb{R}_+^n \times (0, \frac{1}{2})\).

In order to apply Carleman inequality (13), we choose two smooth cut-off functions such that
\[ \eta_1(p) = \begin{cases} 0, & \text{if } p < \frac{1}{\tau} + 1; \\ 1, & \text{if } p > \frac{1}{\tau} + 2. \end{cases} \]
And
\[ \eta_2(q) = \begin{cases} 0, & \text{if } q < -\frac{3}{4}; \\ 1, & \text{if } q > -\frac{3}{4}. \end{cases} \]
All functions take values in \([0, 1]\) and \(|\eta_1'|, |\eta_1''|, |\eta_2'|\) and \(|\eta_2''|\) are all bounded. Denote
\[ f(s) = s^{-K} - 1 \]
and
\[ C_* = 1 + \sup_{\frac{1}{\tau + 1} < s < 1} \{ f(s)y_n^\alpha \} = 1 + f(\frac{1}{2})(\frac{1}{\tau} + 2)^\alpha. \]
Set
\[ \eta(y, s) = \eta_1(y_n)\eta_2(2C_*/\eta_1^\alpha - 1), \]
and \(w = \eta v\). Then \(\text{supp } w \subset Q\), and
\[ |\tilde{P} w| = |\eta \tilde{P} v + v \tilde{P} \eta + 2\tilde{a}^{ij} \partial_i \eta \partial_j v| \leq |\eta \tilde{P} v| + C(n, \Lambda, M)\chi(|v| + |\nabla v|)(|\partial_s \eta| + |\nabla \eta| + |\nabla^2 \eta|), \]
where \(\chi\) is the characteristic function of the set
\[ \Omega = \{(y, s) | \frac{1}{2} < s < 1, 0 < \eta < 1\}. \]
By (16), we have
\[ |\tilde{P} w| \leq \tau N(|v| + |\nabla v|) + C(n, \Lambda, M)\chi(|v| + |\nabla v|)(|\partial_s \eta| + |\nabla \eta| + |\nabla^2 \eta|) \leq \tau N(|v| + |\nabla w|) + C(n, \Lambda, M, N)\chi(|v| + |\nabla v|)(|\partial_s \eta| + |\nabla \eta| + |\nabla^2 \eta|). \]
Notice that \(\frac{1}{2} < s < 1\) in \(\Omega\), and when \(\frac{1}{2} < s < 1\),
\[ |\partial_s \eta| + |\nabla \eta| + |\nabla^2 \eta| \leq C(n, \Lambda, \lambda, M, E)y_n^\alpha \leq C(n, \Lambda, \lambda, M, E)y_n^2, \]
Moreover,\]
\begin{align*}
\Omega = & \{(y, s) \mid \frac{1}{2} < s < 1, \eta_1 > 0, 0 < \eta_2 < 1\} \\
& \cup \{(y, s) \mid \frac{1}{2} < s < 1, 0 < \eta_1 < 1, \eta_2 = 1\} \\
= & \{(y, s) \mid \frac{1}{2} < s < 1, y_n > \frac{1}{\tau}, \frac{1}{2} < \frac{f(s)y_n^\alpha}{C} < 1\} \\
& \cup \{(y, s) \mid \frac{1}{2} < s < 1, \frac{1}{\tau} + 1 < y_n < \frac{1}{\tau} + 2, \frac{f(s)y_n^\alpha}{C} \geq 1\}.
\end{align*}

By the choice of $C_*$ we obtain that the second set of the right side of the above identity is empty, then
\begin{align*}
\Omega = & \{(y, s) \mid \frac{1}{2} < s < 1, y_n > \frac{1}{\tau}, \frac{1}{2} < \frac{f(s)y_n^\alpha}{C} < 1\}.
\end{align*}

By (17), in the support of $w$ we have
\begin{align*}
e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} (|v| + |\nabla v|)^2 & \leq Ce^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s} + 4N\tau^2|y|^2}.
\end{align*}

Notice that $\psi(y) \geq \frac{|y|^2}{2}$ and $\tau = \sqrt{2T_1}$, then we have
\begin{align*}
e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} (|v| + |\nabla v|)^2 & \leq Ce^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s} + \frac{1}{2}|y|^2 + 8NT_1|y|^2}.
\end{align*}

By (15), we know that $T_1 \leq \frac{1}{2\Omega}$, then in $\text{supp } w$ we have
\begin{align*}
e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} (|v| + |\nabla v|)^2 & \leq Ce^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s} + \frac{1}{2}|y|^2}.
\end{align*}

Although $\text{supp } w$ may be unbounded, $\text{supp } w \subset Q$ and (21) allow us to claim the validity of Proposition 1.4 for $w$. Then by Carleman inequality (13), together with (19), we have
\begin{align*}
J & \equiv \int_Q e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} (|w|^2 + |\nabla w|^2) dy ds \\
& \leq \int_Q e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} |\bar{P}w|^2 dy ds \\
& \leq 3\tau^2 N^2 J + C \int_Q e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} \chi(|v| + |\nabla v|)^2 y_n^4 dy ds.
\end{align*}

By (15), we know that $3\tau^2 N^2 = 6T_1 N^2 \leq \frac{1}{2}$. We deduce from the above inequality that
\begin{align*}
J & \leq C \int_{\Omega} e^{2\gamma f(s)y_n^\alpha - \frac{b(y)+K}{s}} (|v| + |\nabla v|)^2 y_n^4 dy ds.
\end{align*}

By (21), we have
\begin{align*}
J & \leq C \int_{\Omega} e^{2\gamma f(s)y_n^\alpha - \frac{b(y)}{2}} y_n^4 dy ds.
\end{align*}

By (20) we have that $f(s)y_n^\alpha < C_*$ in $\Omega$, then
\begin{align*}
J & \leq Ce^{2\gamma C_*} \int_{\Omega} e^{-\frac{b(y)}{2}} y_n^4 dy ds \leq Ce^{2\gamma C_*}.
\end{align*}
On the other hand, we denote
\[ \Omega_1 = \{(y, s) | 0 < s < 1, \eta = 1\} = \{(y, s) | 0 < s < 1, y_n \geq \frac{1}{\tau} + 2, \frac{f(s)y_n^\alpha}{C_*} \geq 1\}, \]
(23)
\[ \Omega_2 = \{(y, s) | 0 < s < 1, y_n \geq \frac{1}{\tau} + 2, \frac{f(s)y_n^\alpha}{C_*} \geq 2\}. \]
Obviously \( \Omega_2 \subset \Omega_1 \) and \( w = v \) in \( \Omega_1 \). Then
\[ J \geq \int_{\Omega_1} e^{2\gamma f(s)y_n^\alpha} b_0(y_1 + K) \left(|v|^2 + |\nabla v|^2\right) dyds \]
\[ \geq \int_{\Omega_2} e^{2\gamma f(s)y_n^\alpha} b_0(y_1 + K) \left(|v|^2 + |\nabla v|^2\right) dyds \]
By (23), we know that \( f(s)y_n^\alpha \geq 2C_* \) in \( \Omega_2 \). Hence
\[ J \geq e^{4\gamma C_*} \int_{\Omega_2} e^{-\frac{b_0(y_1 + K)}{s}} \left(|v|^2 + |\nabla v|^2\right) dyds. \]
Combining (22) and (24), we have
\[ \int_{\Omega_2} e^{-\frac{b_0(y_1 + K)}{s}} \left(|v|^2 + |\nabla v|^2\right) dyds \leq Ce^{-2\gamma C_*}. \]
Passing to the limit as \( \gamma \to +\infty \), we obtain \( v(y, s) = 0 \) in \( \Omega_2 \). Using unique continuation though spatial boundaries (see [10]), we obtain that \( v(y, s) \equiv 0 \) in \( \mathbb{R}_+^n \times (0, 1) \). That is, \( u(x, t) \equiv 0 \) in \( \mathbb{R}_+^n \times (0, T_1) \). Thus we proved this lemma.

3. PROOF OF CARLEMAN INEQUALITIES

In this section, we shall prove two Carleman Inequalities which is the crucial part of the whole argument. The main idea is to choose a proper weighted functions \( G \). We denote
\[ \tilde{\Delta}u = \partial_i(a^{ij}\partial_j u). \]
Here and in the following argument, we use the summation convention on the repeated indices. We shall make use of the following lemma which is due to L. Escauriza and F. J. Fernández in [10] (see also [18]).

**Lemma 3.1.** Suppose \( \sigma(t) : \mathbb{R}_+ \to \mathbb{R}_+ \) is a smooth function, \( \alpha \) is a real number, \( F \) and \( G \) are differentiable functions and \( G > 0 \). Then the following identity holds for any \( u \in C^\infty_0(\mathbb{R}_+^n \times (0, T)) \)
\[ 2 \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} |L u|^2 G dxdt + \frac{1}{2} \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} u^2 MG dxdt \]
\[ + \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} \langle A \nabla u, \nabla u \rangle \left[(\log \frac{\sigma}{\sigma'})' + \frac{\partial \sigma G - \tilde{\Delta} G}{G} - F \right] G dxdt \]
\[ + 2 \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} (D_G \nabla u, \nabla u) G dxdt - \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} u \langle A \nabla u, \nabla F \rangle G dxdt \]
\[ = 2 \int_{\mathbb{R}^n_+ \times (0, T)} \frac{\sigma^{1-\alpha}}{\sigma'} LuPu G dxdt \]
(25)
where

\[ Lu = \partial_t u - \langle A \nabla \log G, \nabla u \rangle + \frac{F u}{2} - \frac{\alpha \sigma'}{2\sigma} u, \]

\[ M = (\log \frac{\sigma}{\sigma_0})^t F + \partial_t F + (F - \frac{\sigma'}{2})(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) - \langle A \nabla F, \nabla \log G \rangle, \]

and

\[ D_G^{ij} = a^{ik} \partial_{kl} (\log G) a^{lj} + \frac{\partial_t (\log G)}{2} \left( a^{ki} \partial_k a^{lj} + a^{kj} \partial_l a^{i} - a^{kl} \partial_j a^{ij} \right) + \frac{1}{2} \partial_t a^{ij}. \]

We first give a modification of this lemma which will be used in our proof. Letting \( \alpha = 0 \) and \( \sigma(t) = e^t \) in Lemma 3.1, we obtain the following identity for \( u \in C_0^\infty(\mathbb{R}^n \times (0,T)) \)

\[ \frac{1}{2} \int_{\mathbb{R}^n \times (0,T)} u^2 M G dx dt + \int_{\mathbb{R}^n \times (0,T)} [2 \langle D_G \nabla u, \nabla u \rangle + \langle A \nabla u, \nabla u \rangle (\frac{\partial_t G - \tilde{\Delta} G}{G} - F)] G dx dt \]

\[ - \int_{\mathbb{R}^n \times (0,T)} u \langle A \nabla u, \nabla F \rangle G dx dt = 2 \int_{\mathbb{R}^n \times (0,T)} Lu(Pu - Lu) G dx dt. \]

If \( \nabla F \) is differentiable, we can integrate by parts to obtain

\[ - \int_{\mathbb{R}^n \times (0,T)} u \langle A \nabla u, \nabla F \rangle G dx dt \]

\[ = \frac{1}{2} \int_{\mathbb{R}^n \times (0,T)} u^2 \tilde{\Delta} F G dx dt + \frac{1}{2} \int_{\mathbb{R}^n \times (0,T)} u^2 \langle A \nabla F, \nabla \log G \rangle G dx dt. \]

The function \( \nabla F \) may not be differentiable. To overcome this difficulty, we follow Tu’s idea in [18]. We approximate \( F \) by some smooth function \( F_0 \) and use the above identity with \( F_0 \) in place of \( F \). Thus we obtain the following result.

**Corollary 3.2.** Suppose \( F \) and \( G \) are differentiable functions and \( G > 0 \). Then the following identity holds for any \( u \in C_0^\infty(\mathbb{R}^n \times (0,T)) \)

\[ \frac{1}{2} \int_{\mathbb{R}^n \times (0,T)} u^2 M_0 G dx dt + \int_{\mathbb{R}^n \times (0,T)} [2 \langle D_G \nabla u, \nabla u \rangle + \langle A \nabla u, \nabla u \rangle (\frac{\partial_t G - \tilde{\Delta} G}{G} - F)] G dx dt \]

\[ - \int_{\mathbb{R}^n \times (0,T)} u \langle A \nabla u, \nabla (F - F_0) \rangle G dx dt = 2 \int_{\mathbb{R}^n \times (0,T)} Lu(Pu - Lu) G dx dt, \]

where

\[ Lu = \partial_t u - \langle A \nabla u, \nabla \log G \rangle + \frac{F u}{2}, \]

\[ M_0 = \partial_t F + F (\frac{\partial_t G - \tilde{\Delta} G}{G} - F) + \tilde{\Delta} F_0 - \langle A \nabla (F - F_0), \nabla \log G \rangle, \]

and

\[ D_G^{ij} = a^{ik} \partial_{kl} (\log G) a^{lj} + \frac{\partial_t (\log G)}{2} \left( a^{ki} \partial_k a^{lj} + a^{kj} \partial_l a^{i} - a^{kl} \partial_j a^{ij} \right) + \frac{1}{2} \partial_t a^{ij}. \]
3.1. **Proof of Proposition 1.3.** We use identity (26) to prove Proposition 1.3. In (26), we let
\[ G = e^{2γ(t−K−1)−\frac{b|x|^2+K}{t}}, \]
then
\[ \frac{∂_tG − \tilde{Δ}G}{G} = \frac{b|x|^2 − 4b^2a^{ij}x_ix_j + K}{t^2} \frac{2ba^{ii} + 2b∂_ka^{kl}x_l}{t} − 2γKt^{−K−1}. \]
Let
\[ F = \frac{b|x|^2 − 4b^2a^{ij}x_ix_j + K}{t^2} \frac{2ba^{ii}}{t} − 2γKt^{−K−1} − d\left(\frac{1}{t} + 1\right), \]
where \( d \) is a positive constant to be determined. Set
\[ F_0 = \frac{b|x|^2 − 4b^2a^{ij}x_ix_j + K}{t^2} \frac{2ba^{ii}}{t} − 2γKt^{−K−1} − d\left(\frac{1}{t} + 1\right), \]
where
\[ a^{ij}_ε(x, t) = \int_{\mathbb{R}^n} a^{ij}(x−y, t)φ_ε(y) dy, \]
\( φ \) is a mollifier, and \( ε = \frac{1}{2} \).

We denote by \( I_n \) the identity matrix of \( \mathbb{R}^n \), \( C \) are generic constants depending on \( n, Λ, λ, M \) and \( E \) in the following arguments. We need some estimates which we list in the following lemma.

**Lemma 3.3.** Set \( b = \frac{1}{8Λ} \) and \( K = 12d \). For \( d = d(n, Λ, λ, M, E) \) large enough, we have

\[ \text{(27)} \quad 2DG + A\left(\frac{∂_tG − \tilde{Δ}G}{G} − F\right) ≥ \left(\frac{1}{t} + 1\right)I_n; \]

\[ \text{(28)} \quad ∂_tF + F\left(\frac{∂_tG − \tilde{Δ}G}{G} − F\right) ≥ db(|x|^2 + 1); \]

\[ \text{(29)} \quad \tilde{Δ}F_0 ≥ −\frac{C(|x|^2 + 1)}{t^3}; \]

\[ \text{(30)} \quad |∇(F − F_0)| ≤ \frac{C(|x| + 1)}{t^2}. \]

We will prove this lemma later.

First by applying (27) we have
\[ \int_{\mathbb{R}^n \times (0, 2)} \left[ 2\langle DG∇u, ∇u⟩ + A∇u(\frac{∂_tG − \tilde{Δ}G}{G} − F)\right]Gdxdt ≥ \int_{\mathbb{R}^n \times (0, 2)} \left(\frac{1}{t} + 1\right) ∇u^2Gdxdt. \]

Next we estimate \( M_0 \). By applying (30) we have
\[ |⟨A∇(F − F_0), ∇logG⟩| ≤ Λ|∇(F − F_0)||∇logG| ≤ \frac{C(|x| + 1)|x|}{t^3} ≤ \frac{C(|x|^2 + 1)}{t^3}. \]
Then by (28), (29) and (32) we have
\[ M_0 = \partial_t F + F\left(\frac{\partial G - \Delta G}{G}\right) + \Delta F_0 - \langle A\nabla(F - F_0), \nabla\log G \rangle \]
\[ \geq \left(\frac{db}{4} - C\right)\frac{|x|^2 + 1}{t^3}, \]
thus
\[ \frac{1}{2} \int_{\mathbb{R}^n \times (0,2)} u^2 M_0 G \, dx \, dt \geq \left(\frac{db}{8} - C\right) \int_{\mathbb{R}^n \times (0,2)} \frac{|x|^2 + 1}{t^3} u^2 G \, dx \, dt. \]
By (30) and the Cauchy inequality we have
\[ |\int_{\mathbb{R}^n \times (0,2)} u (A \nabla u, \nabla (F - F_0)) G \, dx \, dt| \]
\[ \leq \Lambda \int_{\mathbb{R}^n \times (0,2)} |\nabla (F - F_0)||u| |\nabla u| G \, dx \, dt \]
\[ \leq C \int_{\mathbb{R}^n \times (0,2)} \frac{|x| + 1}{t^2} |u| |\nabla u| G \, dx \, dt \]
\[ \leq \int_{\mathbb{R}^n \times (0,2)} |u|^2 G \, dx \, dt + \int_{\mathbb{R}^n \times (0,2)} |\nabla u|^2 G \, dx \, dt. \]
Finally, by (26), (31), (33), (34) and the Cauchy inequality, we have
\[ \int_{\mathbb{R}^n \times (0,2)} |Pu|^2 G \, dx \, dt \geq \left(\frac{db}{8} - C\right) \int_{\mathbb{R}^n \times (0,2)} \frac{|x|^2 + 1}{t^3} |u|^2 G \, dx \, dt + \int_{\mathbb{R}^n \times (0,2)} |\nabla u|^2 G \, dx \, dt, \]
if we choose \(d\) large enough, we obtain
\[ \int_{\mathbb{R}^n \times (0,2)} |Pu|^2 G \, dx \, dt \geq \int_{\mathbb{R}^n \times (0,2)} \left(|u|^2 + |\nabla u|^2\right) G \, dx \, dt. \]
Thus we proved Carleman inequality (12).

Proof of Lemma 3.3. We estimate them one by one.

Estimate of \(2D_G + A\left(\frac{\partial G - \Delta G}{G}\right) - F\).

By direct calculations we have
\[ 2D_G + A\left(\frac{\partial G - \Delta G}{G}\right) - F \]
\[ = -\frac{4b}{t} A^2 - \frac{2bxx_l}{t}(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij}) + \partial_i a^{ij} + A\left(\frac{d + 2b\partial_k a^{kl}x_l}{t} + d\right) \]
\[ = -\frac{4b}{t} A^2 - \frac{2bxx_l}{t}(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij} - a^{ij}) + \partial_i a^{ij} + A\left(\frac{d}{t} + 2\right) \]
\[ \geq -\frac{4b}{t} \Lambda^2 I_n - \frac{2bxx_l}{t}(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij} - a^{ij}) + \partial_i a^{ij} + \lambda d\left(\frac{1}{t} + 1\right) I_n. \]
Next we estimate the lower bounds of the matrices in the right side of the above inequality.
We just need to estimate matrix $x_ia^{ki}\partial_k a^{lj}$ and $\partial_t a^{ij}$. For any $\xi \in \mathbb{R}^n$,

$$|x_ia^{ki}\partial_k a^{lj}\xi_i\xi_j| \leq n^2|x|\lambda E \sum_{i,j} |\xi_i||\xi_j| \leq n^3\lambda E|\xi|^2,$$

then

$$-n^3\lambda EI_n \leq x_ia^{ki}\partial_k a^{lj} \leq n^3\lambda EI_n;$$

and

$$|\partial_t a^{ij}\xi_i\xi_j| \leq M\sum_{i,j} |\xi_i||\xi_j| \leq Mn|\xi|^2,$$

then

$$\partial_t a^{ij} \geq -MnI_n.$$

Thus we have

$$2D_G + A(\partial_G - \tilde{\Delta}G - F) \geq \left[-\frac{4b}{t}\lambda^2 - \frac{8b}{t}n^3\Lambda E - Mn + \lambda d\left(\frac{1}{t} + 1\right)\right]I_n,$$

if we choose $d = d(n, \Lambda, \lambda, M, E)$ large enough, then

$$2D_G + A(\partial_G - \tilde{\Delta}G - F) \geq (\frac{1}{t} + 1)I_n.$$

Estimate of $\partial_t F + F(\frac{\partial G - \tilde{\Delta}G}{G} - F)$.

By direct calculations we have

$$\partial_t F + F(\frac{\partial G - \tilde{\Delta}G}{G} - F) = (d - 2 + 2b\partial_ia^{ij}x_j)(b|x|^2 - 4b^2a^{ij}x_ix_j + K)$$

$$+ \frac{(db|x|^2 - 4db^2a^{ij}x_ix_j - 4b^2\partial_ia^{ij}x_ix_j + Kd - (d - 2ba^{ii})(d - 1 + 2b\partial_ia^{ij}x_j)}{t^3}$$

$$- \frac{d(2d - 2ba^{ii} + 2b\partial_ia^{ij}x_j) - 2b\partial_ia^{ii}}{t^2} - d^2$$

$$+ 2\gamma Kt^{-K-2}(K + 1 - d - td - 2b\partial_ia^{ij}x_j).$$

Noticing that

$$|\partial_ia^{ij}x_j| \leq n^2\frac{E}{|x|}|x| = n^2E,$$

$$a^{ij}x_ix_j \leq \Lambda|x|^2, |\partial_t a^{ij}x_ix_j| \leq M\sum_{i,j} |x_i||x_j| \leq Mn|x|^2,$$

then we have

$$\partial_t F + F(\frac{\partial G - \tilde{\Delta}G}{G} - F) \geq \frac{(d - C)(b - 4b^2\Lambda)|x|^2 + K}{t^3} + \frac{(db - 4db^2\Lambda - C)|x|^2 + Kd - (d + C)^2}{t^2}$$

$$- \frac{d(2d + C) + C}{t} - d^2 + 2\gamma Kt^{-K-2}(K + 1 - 3d - C).$$
Recall that \( b = \frac{1}{8\kappa} \), and we choose \( d \) large enough, then
\[
\partial_t F + F\left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) \\
\geq \frac{d(b\frac{|x|^2 + K)}{2t^3} + \left(\frac{db}{2-C}\right)|x|^2 + Kd - 2d^2}{t^2} - \frac{3d^2}{t} - d^2 + 2\gamma K t^{-K-2}(K - 4d) \\
\geq \frac{d(b|\nabla|^2 + K)}{4t^3} + \left(\frac{db}{2-C}\right)|x|^2 + Kd - 12d^2}{t^2} + 2\gamma K t^{-K-2}(K - 4d),
\]
Since \( K = 12d \), then
\[
\partial_t F + F\left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) \geq \frac{db(|x|^2 + 1)}{4t^3}.
\]

**Estimate of \( \tilde{\Delta} F_0 \).**

In order to estimate \( \tilde{\Delta} F_0 \) and \( |\nabla(F - F_0)| \), we need some estimates about \( \{a^{ij}\} \) which we put in Appendix A.

In fact, \( \{a^{ij}\} \) satisfy the following properties:

\( i \) \( \lambda |\xi|^2 \leq a^{ij}_\epsilon(x, t)\xi_i\xi_j \leq \Lambda |\xi|^2, \forall \xi \in \mathbb{R}^n; \)

\( ii \) \( |\nabla a^{ij}_\epsilon(x, t)| \leq M; |\nabla a^{ij}_\epsilon(x, t)| \leq \frac{2E}{|x|} \) when \( |x| \geq 1; \)

\( iii \) \( |a^{ij}_\epsilon(x, t) - a^{ij}(x, t)| \leq 2\Lambda; |a^{ij}_\epsilon(x, t) - a^{ij}(x, t)| \leq \frac{E}{|x|} \) when \( |x| \geq 1; \)

\( iv \) \( |\partial_{kl} a^{ij}_\epsilon(x, t)| \leq c(n)M; |\partial_{kl} a^{ij}_\epsilon(x, t)| \leq \frac{c(n)E}{|x|} \) when \( |x| \geq 1. \)

Direct calculations show that
\[
\tilde{\Delta} F_0 = \frac{b}{t^2} \tilde{\Delta}(|x|^2) - \frac{4b^2}{t^2} \tilde{\Delta}(a^{ij}_\epsilon x_i x_j) + \frac{2b}{t} \tilde{\Delta}(a^{ij}_\epsilon)
\]
\[
= \frac{2b}{t^2} (a^{ij} + \partial_i a^{ij} x_j) - \frac{4b^2}{t^2} \left[ (a^{kl} \partial_{kl} a^{ij}_\epsilon + \partial_k a^{kj} \partial_l a^{ij}_\epsilon) x_i x_j \right] + 2(\partial_h a^{kj} a^{ij}_\epsilon + 2a^{kj} \partial_h a^{ij}_\epsilon) x_i + 2a^{ij} a^{ij}_\epsilon \right] + \frac{2b}{t} (a^{kl} \partial_{kl} a^{ij}_\epsilon + \partial_h a^{kl} \partial_\epsilon a^{ij}_\epsilon).
\]

Now we estimate the terms in the right side of the above identity. By (35) we have \( |a^{ij}_\epsilon|, \ |\nabla a^{ij}_\epsilon| \) and \( |\nabla^2 a^{ij}_\epsilon| \) are all bounded, then
\[
|a^{ij}_\epsilon + \partial_i a^{ij}_\epsilon x_j| \leq C(1 + |x|);

|a^{kl} \partial_{kl} a^{ij}_\epsilon + \partial_k a^{kj} \partial_l a^{ij}_\epsilon) x_i x_j| \leq C|x|^2;

|\partial_h a^{kj} a^{ij}_\epsilon + 2a^{kj} \partial_h a^{ij}_\epsilon) x_i| \leq C|x|;

|a^{kl} \partial_{kl} a^{ij}_\epsilon + \partial_k a^{kl} \partial_\epsilon a^{ij}_\epsilon| \leq C.
\]

Thus
\[
\tilde{\Delta} F_0 \geq -\frac{C(|x| + 1)}{t^2} - \frac{C(|x|^2 + |x| + 1)}{t^2} - \frac{C}{t} \geq -\frac{C(|x|^2 + 1)}{t^3}.
\]

**Estimate of \( |\nabla(F - F_0)| \).**
Since
\[ F - F_0 = \frac{4b^2(a_e^{ij} - a^{ij})x_i x_j}{t^2} - \frac{2b(a_e^{ii} - a^{ii})}{t}, \]
then
\[
|\nabla (F - F_0)| = \left|\frac{4b^2}{t^2} (\nabla a_e^{ij} - \nabla a^{ij}) x_i x_j + \frac{8b^2}{t^2} (a_e^{ij} - a^{ij}) x_i \nabla x_j - \frac{2b}{t} (\nabla a_e^{ii} - \nabla a^{ii})\right|
\leq \frac{4b^2}{t^2} |\nabla a_e^{ij} - \nabla a^{ij}||x_i||x_j| + \frac{8b^2}{t^2} \sum_j |a_e^{ij} - a^{ij}||x_i| + \frac{2b}{t} |\nabla a_e^{ii} - \nabla a^{ii}|.
\]

We now estimate the terms in the right side of the above inequality. By \( ii) \) of (35), when \(|x| < 1\),
\[
|\nabla a_e^{ij} - \nabla a^{ij}||x_i||x_j| \leq 2M \sum_{i,j} |x_i||x_j| \leq 2M n|x|^2 \leq 2M n,
\]
and when \(|x| \geq 1\),
\[
|\nabla a_e^{ij} - \nabla a^{ij}||x_i||x_j| \leq \left(\frac{2E}{|x|} + \frac{E}{|x|}\right) \sum_{i,j} |x_i||x_j| \leq \frac{3E}{|x|} n|x|^2 = 3nE|x|.
\]

Then we have
\[
|\nabla a_e^{ij} - \nabla a^{ij}||x_i||x_j| \leq C(|x| + 1).
\]

By \( ii) \) and \( iii) \) of (35) we have
\[
\sum_j |a_e^{ij} - a^{ij}||x_i| \leq 2n^2 \Lambda |x|,
\]
\[
|\nabla a_e^{ii} - \nabla a^{ii}| \leq 2M n.
\]

With the above three estimates we have
\[
|\nabla (F - F_0)| \leq \frac{C(|x| + 1)}{t^2} + \frac{C|x|}{t^2} + \frac{C}{t} \leq \frac{C(|x| + 1)}{t^2}.
\]

Thus we proved Lemma 3.3.

3.2. Proof of Proposition 1.4. Before we prove proposition 1.4 we need to prove another result as an alternative version of Corollary 3.2.

In (26), we let \( \Phi = \gamma (t^{-K} - 1) x_n^\alpha - \frac{b \psi + K}{2\epsilon} \), \( G = e^{2\Phi}, v = e^{\Phi} u \) and we denote
\[
B = 2D_G + A(\frac{\partial_G G - \Delta G}{G} - F).
\]

Then the third term of the left hand side of (26) is
\[
- \int_Q u \langle A\nabla (F - F_0), \nabla u \rangle e^{2\Phi} dx dt
= - \int_Q v \langle A\nabla (F - F_0), \nabla v - \nabla \Phi v \rangle dx dt
= - \int_Q v \langle A\nabla (F - F_0), \nabla v \rangle dx dt + \int_Q \langle A\nabla (F - F_0), \nabla \Phi \rangle v^2 dx dt.
\]
We use the above identity and rewrite (26) as

\[ \frac{1}{2} \int_Q M_1 v^2 dx dt + \int_Q \langle B \nabla u, \nabla u \rangle e^{2\Phi} dt - \int_Q v \langle A \nabla (F - F_0), \nabla v \rangle e^{2\Phi} dt = 2 \int_Q Lu (Pu - Lu) e^{2\Phi} dt, \]

where

\[ M_1 = \partial_t F + F \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) + \tilde{\Delta} F_0, \]

\[ B = 4AD^2 \Phi A + 2\partial_t \Phi (a^{ki} \partial_k a^{ij} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij}) + \partial_t a^{ij} + A \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right). \]

We rewrite \( \Phi \) as the following:

\[ \Phi = \Phi_1 + \Phi_2, \]

\[ \Phi_1 = \gamma f(t) x_n^a, \quad f(t) = t^{-K} - 1, \]

\[ \Phi_2 = -\frac{b\psi + K}{2t}. \]

The function \( \psi \) has the following properties which we will prove in Appendix B:

\[ i) \quad \psi \geq \frac{|x|^2}{2}; \]

\[ ii) \quad D^2 \psi \leq C \left( \frac{\Lambda}{\lambda} \right) I_n; \]

\[ iii) \quad |\nabla \psi| \leq 4 \left( \frac{\Lambda}{\lambda} + 1 \right)^2 |x|, \quad |\nabla^k \psi| \leq \frac{C(n, \Lambda)}{|x|^{k-2}}, \quad k = 2, 3, 4; \]

\[ iv) \quad a^{ni} \partial_i \psi \leq C(\Lambda, \lambda) x_n. \]

By direct calculations we have

\[ \frac{\partial_t G - \tilde{\Delta} G}{G} = 2\partial_t \Phi - 2a^{ij} \partial_{ij} \Phi - 2a^{ij} \partial_j \Phi - 4\langle A \nabla \Phi, \nabla \Phi \rangle. \]

Let

\[ (39) \quad F = 2\partial_t \Phi - 2a^{ij} \partial_{ij} \Phi - 4\langle A \nabla \Phi, \nabla \Phi \rangle - H, \]

where \( H \) is a positive smooth function to be determined. Let

\[ F_0 = 2\partial_t \Phi - 2a^{ij} \partial_{ij} \Phi - 4a^{ij} \partial_i \Phi \partial_j \Phi - H. \]

We estimate matrix \( B \) first. Direct calculations show that

\[ B = 4AD^2 \Phi A + 2\partial_t \Phi (a^{ki} \partial_k a^{ij} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij}) + \partial_t a^{ij} + A \left( -2\partial_k a^{kl} \partial_l \Phi + H \right) \]

\[ = 4AD^2 \Phi A + 2\partial_t \Phi (a^{ki} \partial_k a^{ij} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij} - a^{ij} \partial_k a^{kl}) + \partial_t a^{ij} + HA. \]

We estimate the lower bounds of the matrices in the right side of the above identity. First, by \( ii) \) of (38) we have

\[ AD^2 \Phi A = AD^2 \Phi_1 A - \frac{b}{2t} AD^2 \psi A \geq \partial_{nn} \Phi_1 a^{in} a^{nj} - \frac{C}{t} I_n. \]
Second, we estimate matrix $\partial_t \Phi a^{ki} \partial_k a^{lj}$ and $\partial_t a^{ij}$.

For any $\xi \in \mathbb{R}^n$,

$$|\partial_t \Phi a^{ki} \partial_k a^{lj} \xi_i \xi_j| \leq n \Lambda \frac{E}{|x|} \sum_{i,j} |\partial_t \Phi| |\xi_i||\xi_j|$$

$$\leq n^2 \Lambda \frac{E}{|x|} |\xi|^2 \sum_{l} |\partial_l \Phi|$$

$$\leq n^2 \Lambda \frac{E}{|x|} (\partial_n \Phi_1 + \frac{C}{t} |\nabla \psi|) |\xi|$$

and by $iii$ of (38), $|\nabla \psi| \leq C|x|$, then

$$|\partial_t \Phi a^{ki} \partial_k a^{lj} \xi_i \xi_j| \leq (n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) |\xi|^2,$$

thus

$$-(n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) I_n \leq \partial_t \Phi a^{ki} \partial_k a^{lj} \leq (n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) I_n;$$

again,

$$|\partial_t a^{ij} \xi_i \xi_j| \leq M \sum_{i,j} |\xi_i||\xi_j| \leq Mn|\xi|^2,$$

then

$$\partial_t a^{ij} \geq -MnI_n.$$  

Consequently,

$$B \geq 4\partial_{nn} \Phi_1 a^{in} a^{nj} - \frac{C}{t} I_n - 8(n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) I_n - MnI_n + HA$$

(40)

$$\geq 4\partial_{nn} \Phi_1 a^{in} a^{nj} - (8n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) I_n + HA + \frac{1}{t} I_n$$

$$\equiv \tilde{B} + \frac{1}{t} I_n,$$

where

(41)  

$$\tilde{B} = 4\partial_{nn} \Phi_1 a^{in} a^{nj} - (8n^2 \Lambda \frac{E}{|x|} \partial_n \Phi_1 + \frac{C}{t}) I_n + HA.$$  

To make $\tilde{B}$ positive, we choose

(42)  

$$H = 16n^2 \Lambda \frac{E}{|x|} + \frac{d}{t},$$

where $d$ is a positive constant to be determined.

Since $\tilde{B}$ is differentiable, then by (41) we have

$$\int_Q \langle B \nabla u, \nabla u \rangle e^{2\Phi} dx dt$$

(43)

$$\geq \int_Q \langle \tilde{B} \nabla u, \nabla u \rangle e^{2\Phi} dx dt + \int_Q e^{2\Phi} \frac{|\nabla u|^2}{t} dx dt$$

$$= \int_Q \langle \tilde{B} \nabla v, \nabla v \rangle dx dt + \int_Q \langle (\tilde{B} \nabla \Phi, \nabla \Phi) + div(\tilde{B} \nabla \Phi) \rangle |v|^2 dx dt + \int_Q e^{2\Phi} \frac{|\nabla u|^2}{t} dx dt.$$
By (37), (43) and the Cauchy inequality, we have
\[
\int_Q e^{2\Phi} \frac{|\nabla u|^2}{t} \, dx \, dt + \int_Q \langle \tilde{B} \nabla v, \nabla v \rangle \, dx \, dt + \int_Q M_2 v^2 \, dx \, dt \\
- \int_Q v \langle A \nabla (F - F_0), \nabla v \rangle \, dx \, dt \leq \int_Q e^{2\Phi} |Pu|^2 \, dx \, dt,
\] (44)
where
\[
M_2 = \langle \tilde{B} \nabla \Phi, \nabla \Phi \rangle + \text{div} (\tilde{B} \nabla \Phi) + \frac{1}{2} \partial_t F + \frac{1}{2} F \left( \frac{\partial G - \tilde{\Delta} G}{G} - F \right) + \frac{1}{2} \tilde{\Delta} F_0.
\]

We use inequality (44) to prove Proposition 1.4. We also need some estimates which we list in the following lemma. We will prove this lemma later.

Lemma 3.4. Set \( b = \frac{1}{64\Lambda(\Lambda + 1)} \), \( E_0 = \frac{\lambda}{16n^2(\Lambda + 1)} \), \( \alpha = 1 + \frac{E}{E_0} \) and \( K = \frac{13}{\lambda} d \). We take \( d = d(n, \Lambda, \lambda, M, E) \) large enough, when \( E < E_0 \), we have
\[
\tilde{B} \geq 8n^2 \Lambda E \left( \frac{\partial_n \Phi_1}{|x|} + \frac{1}{t} \right) I_n;
\] (45)
\[
M_2 \geq 2 [(\alpha - 1) \lambda^2 - (16n^2 \Lambda \lambda + 8n^2 + 4n) \Lambda E] \left( \frac{\partial_n \Phi_1}{|x|} \right)^3 + \frac{bd|x|^2}{16t^3} + \frac{1}{t^3};
\] (46)
\[
|\nabla (F - F_0)| \leq 32nE \left[ \frac{(\partial_n \Phi_1)^2}{|x|} + \frac{C|x|}{t^2} \right].
\] (47)

Then by applying (45) in Lemma 3.4, we have
\[
\int_Q \langle \tilde{B} \nabla v, \nabla v \rangle \, dx \, dt \geq 8n^2 \Lambda E \int_Q \left( \frac{\partial_n \Phi_1}{|x|} + \frac{1}{t} \right) |\nabla v|^2 \, dx \, dt.
\] (48)

By (46) we have
\[
\int_Q M_2 v^2 \, dx \, dt \geq 2 \int_Q \left[ (\alpha - 1) \lambda^2 - (16n^2 \Lambda \lambda + 8n^2 + 4n) \Lambda E \right] \left( \frac{\partial_n \Phi_1}{|x|} \right)^3 v^2 \, dx \, dt \\
+ \int_Q \left( \frac{bd|x|^2}{16t^3} + \frac{1}{t^3} \right) v^2 \, dx \, dt.
\] (49)

By (47)
\[
|\int_Q v \langle A \nabla (F - F_0), \nabla v \rangle \, dx \, dt| \leq \Lambda \int_Q |\nabla (F - F_0)||v||\nabla v| \, dx \, dt \\
\leq 32n\Lambda E \int_Q \left[ \frac{(\partial_n \Phi_1)^2}{|x|} + \frac{C|x|^2}{t^2} \right] |v||\nabla v| \, dx \, dt.
\]

Using the Cauchy inequality, we have
\[
|\int_Q v \langle A \nabla (F - F_0), \nabla v \rangle \, dx \, dt| \leq 32\Lambda E \int_Q \left[ \frac{(\partial_n \Phi_1)^3}{|x|} + \frac{C|x|^2}{t^3} \right] v^2 \, dx \, dt \\
+ 8n^2 \Lambda E \int_Q \left( \frac{\partial_n \Phi_1}{|x|} + \frac{1}{t} \right) |\nabla v|^2 \, dx \, dt.
\] (50)
Because of (44), (48), (49) and (50), we have
\[
\int_Q e^{2\Phi} |Pu|^2 dxdt \geq \int_Q 2[(\alpha - 1)\lambda^2 - (16n^2\frac{\Lambda}{\lambda} + 8n^2 + 4n + 16)\Lambda E] \frac{(\partial_n \Phi_1)^3}{|x|} v^2 dxdt
+ \int_Q \left[ \frac{(bd - C)|x|^2}{16t^3} + \frac{1}{t^3} \right] v^2 dxdt + \int_Q e^{2\Phi} \frac{\nabla u^2}{t} dxdt,
\]
Since
\[
16n^2\frac{\Lambda}{\lambda} + 8n^2 + 4n + 16 \leq 16n^2(\frac{\Lambda}{\lambda} + 1),
\]
and if we take \(d\) large enough, then
\[
\int_Q e^{2\Phi} |Pu|^2 dxdt \geq \int_Q 2[(\alpha - 1)\lambda^2 - 16n^2(\frac{\Lambda}{\lambda} + 1)\Lambda E] \frac{(\partial_n \Phi_1)^3}{|x|} v^2 dxdt
+ \int_Q e^{2\Phi} \left( \frac{|u|^2}{t^3} + \frac{|
abla u|^2}{t} \right) dxdt.
\]
Notice that
\[
(\alpha - 1)\lambda^2 - 16n^2(\frac{\Lambda}{\lambda} + 1)\Lambda E = 0,
\]
thus
\[
\int_Q e^{2\Phi} |Pu|^2 dxdt \geq \int_Q e^{2\Phi} \left( \frac{|u|^2}{t^3} + \frac{|
abla u|^2}{t} \right) dxdt.
\]
Thus we proved Carleman inequality (13).

**Proof of Lemma 3.4**

**Estimate of \(\tilde{B}\).**

By (11) and (12) we have
\[
\tilde{B} \geq [-8n^2\Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}] + \lambda H] I_n = (8n^2\Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{\lambda d - C}{t}) I_n,
\]
if we take \(d = d(n, \Lambda, \lambda, M, E)\) large enough, then
\[
\tilde{B} \geq 8n^2\Lambda E \left( \frac{\partial_n \Phi_1}{|x|} + \frac{1}{t} \right) I_n.
\]

**Estimate of \(M_2\).**

In order to estimate \(M_2\), we have to divide \(M_2\) into several parts and estimate each of them:
\[
M_2 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6,
\]
where

\[ J_1 = 4 \partial_{nn} \Phi_1 (a^{ni} \partial_i \Phi)^2; \]
\[ J_2 = - (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t} |\nabla \Phi|^2 - (H - 4 \partial_i a^{ij} \partial_j \Phi) (A \nabla \Phi, \nabla \Phi) \]
\[ + \langle A \nabla H, \nabla \Phi \rangle - 8n^2 \Lambda E \langle \nabla (\frac{\partial_n \Phi_1}{|x|}), \nabla \Phi \rangle - (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}) \Delta \Phi; \]
\[ (51) \]

\[ J_3 = 4 \partial_{nn} \Phi_1 \partial_i (a^{in} a^{nj}) \partial_j \Phi + 4 \partial_i^2 \Phi_1 a^{nn} a^{nj} \partial_j \Phi + 4 \partial_{nn} \Phi_1 a^{mn} a^{nj} \partial_j \Phi; \]
\[ J_4 = \partial_i \Phi + \partial_i (H - 2 \partial_i a^{ij} \partial_j \Phi) - \partial_i a^{ij} \partial_j \Phi - a^{ij} \partial_{ij} \Phi \]
\[ + 2 \partial_i a^{ij} \partial_j \Phi (H + a^{ij} \partial_j \Phi) - \frac{1}{2} \partial_i H - \frac{1}{2} H^2; \]
\[ J_5 = - 2 \partial_i (A \nabla \Phi, \nabla \Phi); \]
\[ J_6 = \frac{1}{2} \tilde{\Delta} F_0. \]

**Estimate of** \( J_1. \)

\[ J_1 = 4 \partial_{nn} \Phi_1 (a^{nn} \partial_n \Phi_1 + a^{ni} \partial_i \Phi_2)^2, \]

By the Cauchy inequality \((a + b)^2 \geq \delta a^2 - \frac{\delta}{1 - \delta} b^2, 0 < \delta < 1,\) we have

\[ J_1 \geq 4 \partial_{nn} \Phi_1 [\frac{x_n}{2|x|} (a^{nn} \partial_n \Phi_1)^2 - \frac{x_n}{2|x|} (a^{ni} \partial_i \Phi_2)^2] \]
\[ \geq 4 \partial_{nn} \Phi_1 [\frac{x_n}{2|x|} \lambda^2 (\partial_n \Phi_1)^2 - \frac{x_n}{|x|} (a^{ni} \partial_i \Phi_2)^2] \]
\[ = 2(\alpha - 1) \frac{\partial_n \Phi_1}{|x|} [\lambda^2 (\partial_n \Phi_1)^2 - 2 (a^{ni} \partial_i \Phi_2)^2]. \]

Again by the Cauchy inequality we have

\[ (a^{ni} \partial_i \Phi_2)^2 \leq \sum_i (a^{ni})^2 |\nabla \Phi_2|^2 \leq \frac{\Lambda^2 b^2}{4t^2} |\nabla \psi|^2, \]

then

\[ J_1 \geq 2(\alpha - 1) \frac{\partial_n \Phi_1}{|x|} [\lambda^2 (\partial_n \Phi_1)^2 - \frac{\Lambda^2 b^2}{2t^2} |\nabla \psi|^2] \]
\[ = 2(\alpha - 1) \lambda^2 \frac{(\partial_n \Phi_1)^3}{|x|} - \frac{C}{t^2} \frac{\partial_n \Phi_1}{|x|} |\nabla \psi|^2. \]

By iii) of (38), \(|\nabla \psi| \leq C|x|,\) then

\[ J_1 \geq 2(\alpha - 1) \lambda^2 \frac{(\partial_n \Phi_1)^3}{|x|} - \frac{C}{t^2} \partial_n \Phi_1 |x|. \]

Using the Cauchy inequality, we obtain

\[ J_1 \geq 2(\alpha - 1) \lambda^2 \frac{(\partial_n \Phi_1)^3}{|x|} - \frac{C}{t} (\partial_n \Phi_1)^2 - \frac{C}{t^3} |x|^2. \]

**Estimate of** \( J_2. \)
Let’s estimate $\partial_i a^{ij} \partial_j \Phi$ first. We will use this estimate afterwards again.

$$|\partial_i a^{ij} \partial_j \Phi| \leq \frac{nE}{|x|} \sum_j |\partial_j \Phi| \leq \frac{nE}{|x|} (\partial_n \Phi_1 + \frac{C}{t} |\nabla \psi|).$$

Recall that $|\nabla \psi| \leq C|x|$, then

$$|\partial_i a^{ij} \partial_j \Phi| \leq nE \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}. \quad (53)$$

By $(51)$ we have

$$J_2 \geq - (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}) |\nabla \Phi|^2 - [(16n^2 \frac{\Lambda}{\lambda} + 4n)E \frac{\partial_n \Phi_1}{|x|} + \frac{d + C}{t}] \Lambda |\nabla \Phi|^2$$

$$- \Lambda |\nabla \Phi||\nabla \Phi| - 8n^2 \Lambda E |\nabla (\frac{\partial_n \Phi_1}{|x|})||\nabla \Phi| - (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}) \Delta \Phi$$

$$- \Lambda |\nabla \Phi||\nabla \Phi| - 8n^2 \Lambda E |\nabla (\frac{\partial_n \Phi_1}{|x|})||\nabla \Phi| - (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}) \Delta \Phi.$$

Because

$$|\nabla (\frac{\partial_n \Phi_1}{|x|})| \leq \frac{\partial_n \Phi_1}{x_n|x|} \leq \partial_n \Phi_1,$$

and we have by $iii$ of $(38)$ that

$$\Delta \Phi = \partial_{nn} \Phi_1 - \frac{b}{2t} \Delta \psi \leq \partial_{nn} \Phi_1 + \frac{C}{t},$$

then

$$J_2 \geq - [\Lambda (16n^2 \frac{\Lambda}{\lambda} + 8n^2 + 4n)E \frac{\partial_n \Phi_1}{|x|} + \frac{d \Lambda + C}{t}] |\nabla \Phi|^2 - C \partial_n \Phi_1 |\nabla \Phi|$$

$$- (8n^2 \Lambda E \frac{\partial_n \Phi_1}{|x|} + \frac{C}{t}) (\partial_{nn} \Phi_1 + \frac{C}{t})$$

$$\geq - [\Lambda (16n^2 \frac{\Lambda}{\lambda} + 8n^2 + 4n)E \frac{\partial_n \Phi_1}{|x|} + \frac{d \Lambda + C}{t}] |\nabla \Phi|^2 - C \partial_n \Phi_1 |\nabla \Phi|$$

$$- C (\partial_n \Phi_1)^2 - \frac{C}{t} \partial_n \Phi_1 - \frac{C}{t^2}.$$

By the Cauchy inequality we have

$$C \partial_n \Phi_1 |\nabla \Phi| \leq C (\partial_n \Phi_1)^2 + C |\nabla \Phi|^2, \quad \frac{C}{t} \partial_n \Phi_1 \leq C (\partial_n \Phi_1)^2 + \frac{C}{t^2}.$$
Using the Cauchy inequality, we have

\[
J_2 \geq - \left[ (16n^2 \frac{\Delta}{\lambda} + 8n^2 + 4n) \Delta E \frac{\partial_n \Phi_1}{|x|} + \frac{d\Delta + C}{t} \right] |\nabla \Phi|^2 - C(\partial_n \Phi_1)^2 - \frac{C}{t^2}
\]

\[
\geq - 2[(16n^2 \frac{\Delta}{\lambda} + 8n^2 + 4n) \Delta E \frac{\partial_n \Phi_1}{|x|} + \frac{d\Delta + C}{t}][(\partial_n \Phi_1)^2 + \frac{b^2}{4t^2} |\nabla \psi|^2] - C(\partial_n \Phi_1)^2 - \frac{C}{t^2}
\]

\[
\geq - 2(16n^2 \frac{\Delta}{\lambda} + 8n^2 + 4n) \Delta E \frac{(\partial_n \Phi_1)^3}{|x|} - \frac{2d\Delta + C}{t} (\partial_n \Phi_1)^2 - \frac{C \partial_n \Phi_1 |\nabla \psi|^2}{t^2} - \frac{(d\Delta + C)b^2}{2t^3} |\nabla \psi|^2 - \frac{C}{t^2}.
\]

Taking in account that \( |\nabla \psi| \leq C|x| \) and using the Cauchy inequality, we have that

\[
\frac{C \partial_n \Phi_1}{t^2} |\nabla \psi|^2 \leq C \frac{\partial_n \Phi_1}{t^2} |x| \leq C \frac{(\partial_n \Phi_1)^2}{t} + \frac{C}{t^3} |x|^2,
\]

then

\[
J_2 \geq - 2(16n^2 \frac{\Delta}{\lambda} + 8n^2 + 4n) \Delta E \frac{(\partial_n \Phi_1)^3}{|x|} - \frac{2d\Delta + C}{t} (\partial_n \Phi_1)^2 - \frac{C \partial_n \Phi_1 |\nabla \psi|^2}{t^2} - \frac{C}{t^2}.
\]

(54)

In the following, we always use the fact that

\[
|\nabla^k \psi| \leq \frac{C}{|x|^{k-2}}, \quad k = 1, 2, 3, 4.
\]

Estimate of \( J_3 \).

\[
J_3 = 4 \partial_{nn} \Phi_1 (\partial_i a^{in} a^{nj} + a^{in} \partial_i a^{nj}) \partial_j \Phi + 4 \partial^3 \Phi_1 a^{in} a^{nj} \partial_j \Phi + 4 \partial_{nn} \Phi_1 a^{in} a^{nj} \partial_j \Phi.
\]

Next we estimate the terms of \( J_3 \).

\[
|\partial_i a^{in} a^{nj} + a^{in} \partial_i a^{nj}| \partial_j \Phi| \leq C|\nabla \Phi| \leq C(\partial_n \Phi_1 + \frac{|\nabla \psi|}{t}) \leq C(\partial_n \Phi_1 + \frac{|x|}{t});
\]

\[
|a^{in} a^{nj} \partial_j \Phi| \leq C|\nabla \Phi| \leq C(\partial_n \Phi_1 + \frac{|\nabla \psi|}{t}) \leq C(\partial_n \Phi_1 + \frac{|x|}{t});
\]

\[
|a^{in} a^{nj} \partial_j \Phi| \leq C|\nabla^2 \Phi| \leq C(\partial_{nn} \Phi_1 + \frac{|\nabla^2 \psi|}{t}) \leq C(\partial_{nn} \Phi_1 + \frac{1}{t}).
\]

Combining the above estimates, we have

\[
J_3 \geq - C \partial_{nn} \Phi_1 (\partial_n \Phi_1 + \frac{|x|}{t}) - C|\partial^3 \Phi_1|(\partial_n \Phi_1 + \frac{|x|}{t}) - C \partial_{nn} \Phi_1 (\partial_{nn} \Phi_1 + \frac{1}{t})
\]

\[
\geq - C[(\partial_n \Phi_1)^2 + \frac{1}{t} \partial_n \Phi_1 + \partial_n \Phi_1 |\frac{x}{t}|].
\]

Using the Cauchy inequality, we have

(55)

\[
J_3 \geq - C[(\partial_n \Phi_1)^2 + \frac{1}{t^2} + \frac{|x|^2}{t^2}].
\]
Estimate of $J_4$.

$$J_4 = \partial_t \Phi + \partial_t \Phi(H - 2\partial_t a^{ij} \partial_j \Phi) - \partial_t a^{ij} \partial_j \Phi - a^{ij} \partial_{ij} \Phi$$

$$+ 2\partial_t a^{ij} \partial_j \Phi(H + a^{ij} \partial_j \Phi) - \frac{1}{2} \partial_t H - \frac{1}{2} H^2.$$  

We estimate the terms of $J_4$.

In fact

$$\partial_t \Phi = \partial_t \Phi_1 + \partial_t \Phi_2 = \partial_t \Phi_1 - \frac{b\psi + K}{t^3},$$

$$\partial_t \Phi(H - 2\partial_t a^{ij} \partial_j \Phi) = \partial_t \Phi_1(H - 2\partial_t a^{ij} \partial_j \Phi) + \frac{b\psi + K}{2t^2}(H - 2\partial_t a^{ij} \partial_j \Phi).$$

Recall (53)

$$|\partial_t a^{ij} \partial_j \Phi| \leq nE\frac{\partial_n \Phi_1}{|x|} + \frac{C}{t},$$

and notice that $\partial_t \Phi_1 < 0$, then we have

$$\partial_t \Phi(H - 2\partial_t a^{ij} \partial_j \Phi) \geq \partial_t \Phi_1[(16n^2 \Lambda \lambda + 2n)E\frac{\partial_n \Phi_1}{|x|} + \frac{d + C}{t} + \frac{b\psi + K}{2t^2}[\partial_n \Phi_1] - \frac{n}{2} \partial_t \Phi_1 + \frac{(d - C)(b\psi + K)}{2t^3}].$$

Because $\frac{\partial_n \Phi_1}{|x|} \geq \frac{t'}{f} \partial_n \Phi_1$, and if we choose $d$ large enough, then

$$\partial_t \Phi(H - 2\partial_t a^{ij} \partial_j \Phi) \geq (16n^2 \Lambda \lambda + 2n)E\frac{f(t) \partial_n \Phi_1}{f'(\partial_n \Phi_1)^2} + \frac{2d}{t} \partial_t \Phi_1 + \frac{(d - C)}{2} \frac{b\psi + K}{t^3}.$$  

And

$$|\partial_t a^{ij} \partial_j \Phi| \leq C|\nabla^2 \Phi| \leq C(\partial_n \Phi_1 + \frac{\nabla^2 \Phi}{t} \leq C(\partial_n \Phi_1 + \frac{1}{t}).$$

Also

$$-a^{ij} \partial_{ij} \Phi = -a^{nn} \partial_{nn} \Phi_1 - \frac{a^{ij} a^{ij}}{2t^2} |\partial_{ij} \psi| \geq -a^{nn} \partial_{nn} \Phi_1 - \frac{C}{t^2} |\nabla^2 \psi|,$$

because $\partial_{nn} \Phi_1 < 0$, $|\nabla^2 \psi| \leq C$, then

$$-a^{ij} \partial_{ij} \Phi \geq -\frac{C}{t^2}.$$  

By (53) we have

$$|2\partial_t a^{ij} \partial_j \Phi(H + a^{ij} \partial_j \Phi)| \leq 2(\partial_n \Phi_1 + \frac{C}{t})[(16n^2 \Lambda \lambda + 2n)E\frac{\partial_n \Phi_1}{|x|} + \frac{d + a^{nn} \partial_{nn} \Phi_1 + \frac{C}{t} |\nabla^2 \psi|}].$$

Notice that $|\nabla^2 \psi| \leq C$, and if we choose $d$ large enough, then

$$|2\partial_t a^{ij} \partial_j \Phi(H + a^{ij} \partial_j \Phi)| \leq C(\partial_n \Phi_1 + \frac{1}{t})(\partial_n \Phi_1 + \frac{d}{t})$$

$$\leq C[(\partial_n \Phi_1)^2 + \frac{d}{t} |\partial_n \Phi_1 + \frac{d}{t^2}].$$
Using the Cauchy inequality, we obtain

$$|2\partial_t a^{ij} \partial_j \Phi(H + a^{ij} \partial_j \Phi)| \leq C(\partial_n \Phi_1)^2 + \frac{d^2}{t^2}.$$ 

And

$$-\frac{1}{2} \partial_t H - \frac{1}{2} H^2 = -8n^2 \frac{\lambda}{\chi} \frac{\partial_{nt} \Phi_1}{|x|} + \frac{d}{2t^2} - \frac{1}{2}(16n^2 \frac{\lambda}{\chi} \frac{\partial_n \Phi_1}{|x|} + \frac{d}{t})^2$$

$$\geq -8n^2 \frac{\lambda}{\chi} \frac{\partial_{nt} \Phi_1}{|x|} + \frac{d}{2t^2} - (16n^2 \frac{\lambda}{\chi} \frac{\partial_n \Phi_1}{|x|})^2 - \frac{d^2}{t^2}$$

$$\geq -8n^2 \frac{\lambda}{\chi} \frac{\partial_{nt} \Phi_1}{|x|} - C(\partial_n \Phi_1)^2 - \frac{d^2}{t^2}.$$ 

Take in account that $\partial_{nt} \Phi_1 < 0$, then

$$-\frac{1}{2} \partial_t H - \frac{1}{2} H^2 \geq -C(\partial_n \Phi_1)^2 - \frac{d^2}{t^2}.$$ 

Combining them together, we obtain

$$J_4 \geq [(16n^2 \frac{\lambda}{\chi} + 2n) E \frac{f'}{f} - C](\partial_n \Phi_1)^2 + \partial_t \Phi_1 + \frac{2d}{t} \partial_t \Phi_1 - C \partial_n \Phi_1$$

$$+ \frac{b \psi + K}{t^2}(\frac{d}{2} - C) - \frac{2d^2 + C}{t^2}$$

$$\geq [(16n^2 \frac{\lambda}{\chi} + 2n) E \frac{f'}{f} - C](\partial_n \Phi_1)^2 + \partial_t \Phi_1 + \frac{2d}{t} \partial_t \Phi_1 - C \partial_n \Phi_1$$

$$+ \frac{b \psi}{t^2}(\frac{d}{2} - C) + \frac{Kd}{4t^3} - \frac{2d^2 + C}{t^2}. 
$$

Estimate of $J_5$.

$$J_5 = -2 \partial_t \langle A \nabla \Phi_1, \nabla \Phi_1 \rangle - 2 \partial_t \langle A \nabla \Phi_2, \nabla \Phi_2 \rangle - 4 \partial_t \langle A \nabla \Phi_1, \Phi_2 \rangle$$

$$= -2 \partial_t [a^{nn}(\partial_n \Phi_1)^2] - \frac{b \psi}{2} \partial_t (\frac{a^{ij} \partial_i \psi \partial_j \psi}{t^2}) + 2b \partial_t (\frac{\partial_n \Phi_1}{t} a^{ni} \partial_i \psi)$$

$$= -2 \partial_t [a^{nn}(\partial_n \Phi_1)^2] - 4a^{nn} \frac{f'}{f} (\partial_n \Phi_1)^2 - \frac{b^2 \partial_t a^{ij} \partial_i \psi \partial_j \psi}{2t^2} + \frac{b^2 a^{ij} \partial_i \psi \partial_j \psi}{t^3}$$

$$+ 2b \frac{\partial_n \Phi_1}{t} \partial_t a^{ni} \partial_i \psi + 2b \partial_t (\frac{\partial_n \Phi_1}{t}) a^{ni} \partial_i \psi.$$ 

Because $f' < 0$, $a^{nn} \geq \lambda$, $|\partial_t a^{nn}| \leq M$, and

$$|\partial_t a^{ij} \partial_i \psi \partial_j \psi| \leq M \sum_{i,j} |\partial_i \psi||\partial_j \psi| \leq M n|\nabla \psi|^2,$$

$$a^{ij} \partial_i \psi \partial_j \psi \geq \lambda|\nabla \psi|^2 \geq 0,$$

$$|\partial_t a^{ni} \partial_i \psi| \leq M \sum_i |\partial_i \psi| \leq M \sqrt{n}|\nabla \psi|,$$
then
\[
J_5 \geq -2M(\partial_n \Phi_1)^2 - 4\lambda \frac{f'}{f}(\partial_n \Phi_1)^2 - \frac{b^2 M n \nabla \psi}{2t^2} + \frac{2b}{t} M \sqrt{n} \partial_n \Phi_1 |\nabla \psi| + 2\alpha b(\frac{f'}{t} - \frac{f}{t^2}) \gamma x_n^{\alpha-1} a^{ni} \partial_i \psi.
\]
By the Cauchy inequality we have
\[
\frac{2b}{t} M \sqrt{n} \partial_n \Phi_1 |\nabla \psi| \leq 2M(\partial_n \Phi_1)^2 + \frac{b^2 M n |\nabla \psi|^2}{2t^2},
\]
then
\[
J_5 \geq -4M(\partial_n \Phi_1)^2 - 4\lambda \frac{f'}{f}(\partial_n \Phi_1)^2 - \frac{b^2 M n |\nabla \psi|^2}{t^2} + 2\alpha b(\frac{f'}{t} - \frac{f}{t^2}) \gamma x_n^{\alpha} a^{ni} \partial_i \psi.
\]
By (38), \(|\nabla \psi| \leq C|x|\), \(a^{ni} \partial_i \psi \leq Cx_n\), and notice that \(f' < 0\), then we have
\[
J_5 \geq -4M(\partial_n \Phi_1)^2 - 4\lambda \frac{f'}{f}(\partial_n \Phi_1)^2 - \frac{C|x|^2}{t^2} + C(\frac{f'}{t} - \frac{f}{t^2}) \gamma x_n^{\alpha}
\]
(57)
\[
= -4\lambda \frac{f'}{f} - 4M(\partial_n \Phi_1)^2 - \frac{C|x|^2}{t^2} + C(\frac{f'}{tf} - \frac{1}{t^2}) \Phi_1.
\]

**Estimate of J_6.**

Recall that
\[
F_0 \equiv 2\partial_t \Phi - 2a^{ij}_e \partial_j \Phi - 4a^{ij}_e \partial_i \Phi \partial_j \Phi - 16n^2 \frac{\Lambda}{\lambda} E \partial_n \Phi_1 - \frac{d}{t}.
\]
Direct calculations show that
\[
J_6 = \tilde{\Delta}(\partial_t \Phi) - \tilde{\Delta}[\epsilon^{ij}(\partial_j \Phi + 2\partial_i \Phi \partial_j \Phi) - 8n^2 \frac{\Lambda}{\lambda} E \Delta(\frac{\partial_n \Phi_1}{|x|})]
\]
\[
= \frac{f'}{f} \tilde{\Delta} \Phi + \frac{b}{2\epsilon^2} \tilde{\Delta} \psi - (a^{kl} \partial_k a^{ij}_e + \partial_k a^{ij} \partial_i \partial_j \Phi)(\partial_j \Phi + 2\partial_i \Phi \partial_j \Phi)
\]
\[
- (\partial_k a^{ij} - 2a^{kl} \partial_k a^{ij}_e)(\partial_{ij} \Phi + 4\partial_i \Phi \partial_j \Phi)
\]
\[
- a^{kl} \partial_i \phi(\partial_{ijkl} \Phi + 4\partial_{ik} \Phi \partial_j \Phi + 4\partial_{ik} \Phi \partial_j \Phi)
\]
\[
- 8n^2 \frac{\Lambda}{\lambda} E[\partial^2 \Phi_1 \frac{a^{mn}}{|x|} + \partial_m \Phi_1 (\frac{\partial_k a^{kn}}{|x|^2} - \frac{2a^{kn} x_k}{|x|^3})
\]
\[
+ \partial_n \Phi_1 \frac{3a^{kl} x_k x_l}{|x|^5} - \frac{\partial_k a^{kl} x_l}{|x|^3} - \frac{a^{kk}}{|x|^3})].
\]
Next we estimate the terms of \(J_6\). By (35) and (38) we have
\[
|a^{ij}_e|, \ |\nabla a^{ij}_e| \text{ and } |\nabla^2 a^{ij}_e| \text{ are all bounded;}
\]
\[
|\nabla^k \psi| \leq \frac{C}{|x|^{k-2}}, \ k = 1, 2, 3, 4.
\]
Then it is easy to see
\[
|\tilde{\Delta} \Phi_1| = |\partial_i a^{in} \partial_n \Phi_1 + a^{nn} \partial_{mn} \Phi_1| \leq C\partial_n \Phi_1;
\]
\[
|\tilde{\Delta} \psi| = |\partial_i a^{ij} \partial_j \psi + a^{ij} \partial_{ij} \psi| \leq C(|\nabla \psi| + |\nabla^2 \psi|) \leq C|x|;
\]
and

\[
(a^{kl} \partial_{kl} \alpha^i_k + \partial_k a^{kl} \partial_l \alpha^i_j) (\partial_{ij} \Phi + 2 \partial_i \Phi \partial_j \Phi) \\
\leq C(|\nabla^2 \Phi| + |\nabla \Phi|^2) \\
\leq C[\partial_{nn} \Phi_1 + \frac{|\nabla^2 \psi|}{t} + (\partial_n \Phi_1)^2 + \frac{|\nabla \psi|^2}{t^2}] \\
\leq C[(\partial_n \Phi_1)^2 + \partial_n \Phi_1 + \frac{1}{t} + \frac{|x|^2}{t^2}] ;
\]

also

\[
(\partial_k a^{kl} \alpha^i_k + 2 a^{kl} \partial_k \alpha^i_k) (\partial_{ij} \Phi + 4 \partial_i \Phi \partial_j \Phi) \\
\leq C(|\nabla^3 \Phi| + |\nabla \Phi|^2 + |\nabla^2 \Phi|^2) \\
\leq C[|\partial_n^3 \Phi_1| + \frac{|\nabla^3 \psi|}{t} + (\partial_n \Phi_1)^2 + \frac{|\nabla \psi|^2}{t^2} + (\partial_{nn} \Phi_1)^2 + \frac{|\nabla^2 \psi|^2}{t^2}] \\
\leq C[(\partial_n \Phi_1)^2 + \partial_n \Phi_1 + \frac{1}{t} + \frac{|x|^2}{t^2}] ;
\]

\[
a^{kl} \alpha^i_k (\partial_{ijkl} \Phi + 4 \partial_{ikl} \Phi \partial_j \Phi + 4 \partial_{ik} \Phi \partial_{jl} \Phi) \\
\leq C(|\nabla^4 \Phi| + \sum_{k=1}^3 |\nabla^k \Phi|^2) \\
\leq C[|\partial_n^4 \Phi_1| + \frac{|\nabla^4 \psi|}{t} + \sum_{k=1}^3 (\partial_n^k \Phi_1)^2 + \sum_{k=1}^3 \frac{|\nabla^k \psi|^2}{t^2}] \\
\leq C[(\partial_n \Phi_1)^2 + \partial_n \Phi_1 + \frac{1}{t} + \frac{|x|^2}{t^2}] ;
\]

\[
\partial_n^3 \Phi_1 \frac{a^{nn}}{|x|} + \partial_{nn} \Phi_1 (\frac{\partial_k a^{kn}}{|x|^2} - \frac{2 a^{kn} x_k}{|x|^3}) + \partial_n \Phi_1 \frac{3 a^{kl} x_k x_l}{|x|^5} - \frac{\partial_k a^{kl} x_k + a^{kk}}{|x|^3} \\
\leq C(|\partial_n^3 \Phi_1| + \partial_{nn} \Phi_1 + \partial_n \Phi_1) \\
\leq C \partial_n \Phi_1 .
\]

Combining them together, we have

\[
(58) \quad J_6 \geq C \frac{f'}{f} \partial_n \Phi_1 - C [(\partial_n \Phi_1)^2 + \partial_n \Phi_1 + \frac{1}{t} + \frac{|x|^2}{t^2}] .
\]

Combining (52), (54), (55), (56), (57) and (58), we have

\[
M_2 \geq 2[\alpha - 1 - (16 n^2 \frac{\Lambda}{\lambda} + 8 n^2 + 4 n) \Lambda E] \frac{(\partial_n \Phi_1)^3}{|x|} \\
+ [-4(\Lambda - 16 n^2 \frac{\Lambda}{\lambda} + 2 n) E] \frac{f'}{f} - \frac{2 d \Lambda + C}{t} [(\partial_n \Phi_1)^2 \\
+ [\partial_n \Phi_1 + \frac{2d}{t} \partial_l \Phi_1 + C \frac{f'}{f} \partial_n \Phi_1 - C \partial_n \Phi_1 + C \frac{f'}{f} \partial_n \Phi_1] \\
+ \frac{b \psi}{t^3} \frac{d}{2} - C] - \frac{C}{t^3} |x|^2 - \frac{d b^2 \Lambda + C}{2 t^3} |\nabla \psi|^2 + \frac{K d}{4 t^3} - \frac{2 d^2 + C}{t^2} .
\]
Next we estimate the terms of the right side of the above inequality. We always choose $d$ large enough.

Notice that $(16n^2\Lambda + 2n)E < 16n^2(\Lambda + 1)E_0 < \lambda$, then

$$-(4\lambda - (16n^2\Lambda + 2n)E)\frac{f'}{f} - \frac{2d\Lambda + C}{t} \geq -3\lambda\frac{f'}{f} - \frac{3d\Lambda}{t}$$

$$\geq \frac{3\lambda K}{t(1-t^K)} - \frac{3d\Lambda}{t} \geq \frac{3\lambda K - 3d\Lambda}{t} \geq 0.$$

And

$$\partial_u\Phi_1 + \frac{2d}{t}\partial_t\Phi_1 + C\frac{f'}{f}\partial_n\Phi_1 - C\partial_n\Phi_1 + C\left(\frac{f'}{tf} - \frac{1}{t^2}\right)\Phi_1 \geq \Phi_1\left[\frac{f''}{f} + \frac{2d}{t}\frac{f'}{f} + C\frac{f'}{f} - C\left(\frac{f'}{tf} - \frac{1}{t^2}\right)\right]$$

$$\geq \Phi_1\left[\frac{f''}{f} + \frac{3d}{t}\frac{f'}{f} - \frac{d}{t^2}\right]$$

$$= \Phi_1\left[K(K + 1 - 3d) - \frac{d}{t^2}\right]$$

$$\geq \Phi_1\frac{K(K + 1 - 3d) - d}{t^2}$$

$$\geq 0.$$

By (38)

$$\psi \geq \frac{|x|^2}{2}, \quad |\nabla \psi| \leq 4(\Lambda + 1)^2|x|,$$

then

$$\frac{b\psi}{t^3}\left(\frac{d}{2} - C\right) - \frac{C}{t^3}|x|^2 - \frac{db^2\Lambda + C}{2t^3}|\nabla \psi|^2 \geq \left(\frac{db}{4} - C\right)|x|^2 - C\frac{|x|^2}{t^3} - 8(db^2\Lambda + C)(\Lambda + 1)^4\frac{|x|^2}{t^3}$$

$$= \frac{|x|^2}{t^3}\left[\frac{db}{4} - 8db^2\Lambda(\Lambda + 1)^4 - C\right]$$

$$= \frac{|x|^2}{t^3}\left[db\left(\frac{1}{4} - 8b\Lambda(\Lambda + 1)^4\right) - C\right].$$

Since

$$b = \frac{1}{64\Lambda(\Lambda + 1)^4},$$

then

$$\frac{b\psi}{t^3}\left(\frac{d}{2} - C\right) - \frac{C}{t^3}|x|^2 - \frac{db^2\Lambda + C}{2t^3}|\nabla \psi|^2 \geq \frac{|x|^2}{t^3}\left(\frac{db}{8} - C\right) \geq \frac{db|x|^2}{16t^3}. $$
Finally,\[ \frac{Kd}{4t^3} - \frac{2d^2 + C}{t^2} \geq \frac{Kd}{4t^3} - \frac{3d^2}{t^2} \geq \frac{Kd - 12d^2}{4t^3} \geq \frac{1}{t^3}. \]

Combining them together, we have\[ M_2 \geq 2[(\alpha - 1)\lambda^2 - (16n^2\frac{\Lambda}{\lambda} + 8n^2 + 4n)\Lambda E]\left(\frac{\partial_n\Phi_1}{|x|}\right)^3 + \frac{bd|x|^2}{16t^3} + \frac{1}{t^3}. \]

**Estimate of $|\nabla (F - F_0)|$.**

Recall\[ F - F_0 = 2(a_{ij}^e - a_{ij})(\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi), \]
and then\[ |\nabla (F - F_0)| = 2|\nabla a_{ij}^e - \nabla a_{ij}^j| (\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi) \]
\[ + (a_{ij}^e - a_{ij}^j)(\nabla \partial_j\Phi + 4\partial_i\Phi\nabla \partial_j\Phi)| \]
\[ \leq 2|\nabla a_{ij}^e - \nabla a_{ij}^j|(\partial_j\Phi) + |\partial_i\Phi|^2 + |\partial_j\Phi|^2 \]
\[ + 2|a_{ij}^e - a_{ij}^j|(\nabla \partial_j\Phi + 2|\partial_i\Phi|^2 + 2|\nabla \partial_j\Phi|^2). \]

Since $|x| \geq 1$ in $Q$, by $\Box$ we have\[ |\nabla a_{ij}^e| \leq \frac{2E}{|x|}, \quad |a_{ij}^e - a_{ij}| \leq \frac{E}{|x|}, \]
then\[ |\nabla (F - F_0)| \leq \frac{6E}{|x|}(|\partial_n\Phi_1| + \frac{C}{t}(|\nabla^2\psi| + 2n|\nabla\Phi|^2) \]
\[ + \frac{2E}{|x|}(|\partial_n^2\Phi_1| + \frac{C}{t}(|\nabla^3\psi| + 2n|\nabla\Phi|^2 + 2n|\nabla^2\Phi|^2) \]
\[ \leq \frac{E}{|x|}[8\partial_n\Phi_1 + \frac{C}{t}(|\nabla^2\psi| + |\nabla^3\psi|) + 16|\nabla\Phi|^2 + 4n|\nabla^2\Phi|^2]. \]

Next we estimate the terms of the right side of the above inequality.

By the inequality $(a + b)^2 \leq \frac{3}{2}a^2 + 3b^2$, we have
\[ |\nabla\Phi|^2 = |\nabla\Phi_1 + \nabla\Phi_2|^2 \leq \frac{3}{2}|\nabla\Phi_1|^2 + 3|\nabla\Phi_2|^2 \leq \frac{3}{2}(\partial_n\Phi_1)^2 + \frac{C|x|^2}{t^2}; \]
\[ |\nabla^2\Phi|^2 = |\nabla^2\Phi_1 + \nabla^2\Phi_2|^2 \leq \frac{3}{2}|\nabla^2\Phi_1|^2 + 3|\nabla^2\Phi_2|^2 \leq \frac{3}{2}(\partial_{nn}\Phi_1)^2 + \frac{C}{t^2}. \]

Then
\[ |\nabla (F - F_0)| \leq \frac{E}{|x|}[8\partial_n\Phi_1 + 24n(\partial_n\Phi_1)^2 + 6n(\partial_{nn}\Phi_1)^2 + \frac{C|x|^2}{t^2}] \]
\[ \leq \frac{E}{|x|}[8\partial_n\Phi_1 + 30n(\partial_n\Phi_1)^2 + \frac{C|x|^2}{t^2}]. \]

Because
\[ 8\partial_n\Phi_1 \leq 2n(\partial_n\Phi_1)^2 + \frac{8}{n} \leq 2n(\partial_n\Phi_1)^2 + \frac{8|x|^2}{t^2}, \]
then
\[ |\nabla (F - F_0)| \leq \frac{E}{|x|}[32n(\partial_\nu \Phi_1)^2 + \frac{C|x|^2}{t^2}] = 32nE[\frac{(\partial_\nu \Phi_1)^2}{|x|} + \frac{C|x|}{t^2}].\]

Thus we proved Lemma 3.4.

### 4. Appendix

**Appendix A: The properties of \( \{a_{ij}^\epsilon\} \).**

\( a_{ij}^\epsilon(x, t) = \int_{\mathbb{R}^n} a_{ij}(x-y, t) \phi_\epsilon(y) dy \), where \( \phi \) is a mollifier and \( \epsilon = \frac{1}{2} \), and

\( (x, t) \in \begin{cases} \mathbb{R}^n \times (0, 2), & \text{Under the assumptions of Proposition 1.3;} \\ (\mathbb{R}^n_+ + e_n) \times (0, 1), & \text{Under the assumptions of Proposition 1.4.} \end{cases} \)

Then \( \{a_{ij}^\epsilon\} \) satisfy:

1. \( \lambda |\xi|^2 \leq a_{ij}^\epsilon(x, t) \xi_i \xi_j \leq \Lambda |\xi|^2; \forall \xi \in \mathbb{R}^n; \)
2. \( |\nabla a_{ij}^\epsilon(x, t)| \leq M; |\nabla a_{ij}^\epsilon(x, t)| \leq \frac{2E}{|x|} \text{ when } |x| \geq 1; \)
3. \( |a_{ij}^\epsilon(x, t) - a_{ij}(x, t)| \leq 2\Lambda; |a_{ij}^\epsilon(x, t) - a_{ij}(x, t)| \leq \frac{E}{|x|} \text{ when } |x| \geq 1; \)
4. \( |\partial_{kl}a_{ij}^\epsilon(x, t)| \leq c(n)M; |\partial_{kl}a_{ij}^\epsilon(x, t)| \leq \frac{c(n)E}{|x|} \text{ when } |x| \geq 1. \)

**Proof.**

1. Obvious.
2. \[
|\nabla a_{ij}^\epsilon(x, t)| \leq \int_{\mathbb{R}^n} |\nabla a_{ij}(x-y, t)| \phi_\epsilon(y) dy \leq M \int_{\mathbb{R}^n} \phi_\epsilon(y) dy = M,
\]

and when \( |x| \geq 1, \)

\[
|\nabla a_{ij}^\epsilon(x, t)| \leq \int_{\mathbb{R}^n} |\nabla a_{ij}(x-y, t)| \phi_\epsilon(y) dy \leq \int_{\mathbb{R}^n} \frac{E}{|x-y|} \phi_\epsilon(y) dy \leq \int_{\mathbb{R}^n} \frac{E}{|x| - \frac{1}{2}} \phi_\epsilon(y) dy \leq \frac{2E}{|x|}.
\]

3. The first part is obvious. We only need to prove the second one.

\[
|a_{ij}^\epsilon(x, t) - a_{ij}(x, t)| \leq \int_{\mathbb{R}^n} |a_{ij}(x-y, t) - a_{ij}(x, t)| \phi_\epsilon(y) dy \\
\leq \int_{\mathbb{R}^n} |\nabla a_{ij}(x-\theta y, t)||y| \phi_\epsilon(y) dy, \ (0 < \theta < 1)
\]

and when \( |x| \geq 1, \)

\[
|a_{ij}^\epsilon(x, t) - a_{ij}(x, t)| \leq \int_{\mathbb{R}^n} \frac{E}{2|x-\theta y|} \phi_\epsilon(y) dy \leq \int_{\mathbb{R}^n} \frac{E}{2(|x| - \frac{1}{2})} \phi_\epsilon(y) dy \leq \frac{E}{|x|}.
\]
\( iv \)

\[
|\partial k a^ij(x, t)| \leq \int_{\mathbb{R}^n} |\partial_k a^ij(x - y, t)||\partial_l \phi| dy
\]

\[
\leq \epsilon^{-n-1} \int_{\mathbb{R}^n} |\partial_k a^ij(x - y, t)||(\partial_l \phi)(\frac{y}{\epsilon})| dy
\]

\[
\leq \frac{M}{\epsilon}||\partial_l \phi||_1
\]

\[
\leq 2||\nabla \phi||_1 M,
\]

and when \(|x| \geq 1\),

\[
|\partial k a^ij(x, t)| \leq \epsilon^{-n-1} \int_{\mathbb{R}^n} |\partial_k a^ij(x - y, t)||(\partial_l \phi)(\frac{y}{\epsilon})| dy
\]

\[
\leq \epsilon^{-n-1} \int_{\mathbb{R}^n} \frac{E}{|x - y|}||(\partial_l \phi)(\frac{y}{\epsilon})| dy
\]

\[
\leq \frac{2E}{\epsilon|x|}||\partial_l \phi||_1
\]

\[
\leq \frac{4||\nabla \phi||_1 E}{|x|}.
\]

**Appendix B: The properties of \( \psi \).**

Recall that \( \psi(x) = |x|^2 - 2\frac{\Lambda}{\lambda}|x|x_n + 2(\frac{\Lambda}{\lambda})^2 x_n^2 \). Then \( \psi \) satisfies

\( i \) \( \psi \geq |x|^2/2 \);

\( ii \) \( D^2 \psi \leq C(\frac{\Lambda}{\lambda}) I_n \);

\( iii \) \( |\nabla \psi| \leq 4(\frac{\Lambda}{\lambda} + 1)^2|x|, \quad |\nabla^k \psi| \leq \frac{C(n, \frac{\Lambda}{\lambda})}{|x|^{k-2}}, k = 2, 3, 4; \)

\( iv \) \( a^{ni} \partial_i \psi \leq C(\Lambda, \lambda)x_n. \)

**Proof.** Because \( i \), the first part of \( iii \) and \( iv \) play more important role in the proof of Proposition 1.4 we just prove them. The others can be proved by direct calculations.

In fact

\[
\psi = \frac{|x|^2}{2} + \frac{1}{2}(|x| - 2\frac{\Lambda}{\lambda}x_n)^2 \geq \frac{|x|^2}{2}.
\]

And

\[
\partial_i \psi = \begin{cases} 
2(1 - \frac{\Lambda}{\lambda})|x_i|, & \text{if } 1 \leq i \leq n - 1; \\
-2\frac{\Lambda}{\lambda}|x| - 2\frac{\Lambda}{\lambda}x_n + [4(\frac{\Lambda}{\lambda})^2 + 2]x_n, & \text{if } i = n.
\end{cases}
\]

Then

\[
|\nabla \psi| \leq 2|1 - \frac{\Lambda}{\lambda} \frac{x_n}{|x|}| |x'| + | - 2\frac{\Lambda}{\lambda} |x| - 2\frac{\Lambda}{\lambda} \frac{x_n}{|x|} x_n + [4(\frac{\Lambda}{\lambda})^2 + 2]x_n|
\]

\[
\leq 2(1 + \frac{\Lambda}{\lambda})|x| + [4(\frac{\Lambda}{\lambda})^2 + 4\frac{\Lambda}{\lambda} + 2]|x|
\]

\[
\leq 4(\frac{\Lambda}{\lambda} + 1)^2|x|.
\]
Notice that
\[ a^{ni} \partial_i \psi = 2 \sum_{i=1}^{n-1} a^{ni} x_i - 2 \frac{\Lambda}{\lambda} a^{nn} |x| - 2 \frac{\Lambda}{\lambda} x_n \sum_{i=1}^{n} a^{ni} x_i + [4(\frac{\Lambda}{\lambda})^2 + 2] a^{nn} x_n. \]

By the Cauchy inequality we have
\[ \left| \sum_{i=1}^{n} a^{ni} x_i \right| \leq \sqrt{\sum_{i=1}^{n} (a^{ni})^2 \sum_{i=1}^{n} x_i^2} \leq \Lambda |x|. \]

Similarly, we have
\[ \left| \sum_{i=1}^{n-1} a^{ni} x_i \right| \leq \Lambda |x'|. \]

Since \( \lambda \leq a^{nn} \leq \Lambda \),
then
\[ a^{ni} \partial_i \psi \leq 2\Lambda |x'| - 2\Lambda |x| + \frac{2\Lambda^2}{\lambda} x_n + [4(\frac{\Lambda}{\lambda})^2 + 2] \Lambda x_n \]
\[ \leq [4(\frac{\Lambda}{\lambda})^2 + 2 \frac{\Lambda}{\lambda} + 2] \Lambda x_n. \]

REFERENCES

[1] N. Aronszajn, A. Krzywicki and J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Ark. Mat. 4(1962), 417-453.

[2] B. F. Jones, *A fundamental solution of the heat equation which is supported in a strip*, J. Math. Anal. Appl. 60 (1977), 314C324.

[3] W. Littman, *Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients*, Annali Scuola Norm. Sup. Pisa Serie IV, 3 (1978), 567C580.

[4] L. Hörmander, *Uniqueness theorems for second order elliptic differential equations*, Comm. Partial Differential Equations 8(1983), 21-64.

[5] X. Y. Chen, *A strong unique continuation theorem for parabolic equations*, Math. Ann. 311(1996), 603-630.

[6] C. C. Poon, *Unique continuation for parabolic equations*, Comm. Partial Differential Equations 21(1996), 521-539.

[7] L. Escauriaza, G. A. Seregin and V. Šverák, *Backward uniqueness for parabolic equations*, Arch. Ration. Mech. Anal., 169(2003), no. 2, 147-157.

[8] L. Escauriaza, G. A. Seregin and V. Šverák, *Backward uniqueness for the heat operator in half-space*, Algebra i Analiz 15 (2003), no. 1, 201-214; translation in St. Petersburg Math. J. 15(2004), no. 1, 139-148.

[9] L. Escauriaza, G. A. Seregin and V. Šverák, *L^{3,\infty} solutions to the Navier-Stokes equations and backward uniqueness*, Russ. Math. Surv., 58(2003), no. 2, 211-250.

[10] L. Escauriaza and F. J. Fernández, *Unique continuation for parabolic operators*, Ark. Mat., 41(2003), no. 1, 35-60.

[11] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *Decay at infinity of caloric functions within characteristic hyperplanes*, Math. Res. Lett. 13(2006), no. 3, 441-453.

[12] L. Escauriaza, *Carleman inequalities and the heat operator*, Duke Math. J. 104(2000), 113-127.

[13] L. Escauriaza and L. Vega, *Carleman inequalities and the heat operator II*, Indiana Univ. Math. J. 50(2001), 1149-1169.

[14] J. C. Saut and B. Scheurer, *Unique continuation for evolution equations*, J. Diff. Equ., 66(1987), 118-137.
[15] C. D. Sogge, *A unique continuation theorem for second order parabolic differential operators*, Ark. Mat. 28(1990), 159-182.

[16] Lu Li and V. Šverák, *Backward uniqueness for the heat equation in cones*, arXiv:1011.2790.

[17] E. M. Landis and O. A. Oleinik, *Generalized analyticity and some related properties of solutions of elliptic and parabolic equations*, Uspekhi Mat. Nauk 29(176): 2(1974), 190-206(Russian). English transl.: Russian Math. Surveys 29(1974), 195-212.

[18] Tu A. Nguyen, *On a question of Landis and Oleinik*, Tran. Amer. Math. Soc., 362(2010), no. 6, 2875-2899.

[19] F. H. Lin, *A uniqueness theorem for parabolic equations*, Comm. Pure Appl. Math. 42(1988), 125-136.

[20] A. Plis, *On non-uniqueness in Cauchy problems for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Math. Astronom. Phys. 11(1963), 95-100.

[21] K. Miller, *Non-unique continuation for certain ode’s in Hilbert space and for uniformly parabolic and elliptic equations in self-adjoint divergence form*, Arch. Rational Mech. Anal. 54(1963), 105-117.

[22] N. Mandache, *On a counterexample concerning unique continuation for elliptic equations in divergence form*, Mathematical Physics, Analysis and Geometry. 1(1998), 273C292.

[23] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., 1998.

[24] O. A. Ladyženskaja and V. A. Solonnikov, *Linear and quasilinear equations of parabolic type*, Translations of mathematical monographs, Amer. Math. Soc., 1968.

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