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Part 7. Stochastic geometry

CONTINUUM AB PERCOLATION AND AB RANDOM GEOMETRIC GRAPHS

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CONTINUUM AB PERCOLATION AND AB RANDOM
 GEOMETRIC GRAPHS

By MATHEW D. PENROSE

Abstract

Consider a bipartite random geometric graph on the union of two independent homogeneous Poisson point processes in $d$-space, with distance parameter $r$ and intensities $\lambda$ and $\mu$. We show for $d \geq 2$ that if $\lambda$ is supercritical for the one-type random geometric graph with distance parameter $2r$, there exists $\mu$ such that $(\lambda, \mu)$ is supercritical (this was previously known for $d = 2$). For $d = 2$, we also consider the restriction of this graph to points in the unit square. Taking $\mu = \tau \lambda$ for fixed $\tau$, we give a strong law of large numbers as $\lambda \to \infty$ for the connectivity threshold of this graph.

Keywords: Bipartite geometric graph; continuum percolation; connectivity threshold

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1. Introduction and statement of results

The continuum AB percolation model, introduced by Iyer and Yogeshwaran [3], goes as follows. Particles of two types, A and B, are scattered randomly in Euclidean space as two independent Poisson processes, and edges are added between particles of opposite type that are sufficiently close together. This provides a continuum analogue of lattice AB percolation which is discussed in, e.g. [2]. Motivation for considering continuum AB percolation is discussed in detail in [3]; the main motivation comes from wireless communications networks with two types of transmitter.

Another type of continuum percolation model with two types of particle is the secrecy random graph [9] in which the type-B particles (representing eavesdroppers) inhibit percolation; each type-A particle may send a message to every other type-A particle lying closer than its nearest neighbour of type B. See also [7]. Such models are not considered here; they are complementary to ours.

To describe continuum AB percolation more precisely, we make some definitions. Let $d \in \mathbb{N}$. Given any two locally finite sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$, and given $r > 0$, let $G(\mathcal{X}, \mathcal{Y}, r)$ be the bipartite graph with vertex sets $\mathcal{X}$ and $\mathcal{Y}$, and with an undirected edge $\{X, Y\}$ included for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with $\|X - Y\| \leq r$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$ (our parameter $r$ would be denoted $2r$ in the notation of [3]). Also, let $G(\mathcal{X}, r)$ be the graph with vertex set $\mathcal{X}$ and with an undirected edge $\{X, X'\}$ included for each $X, X' \in \mathcal{X}$ with $\|X - X'\| \leq r$.

For $\lambda, \mu > 0$, let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\mu$ be independent homogeneous Poisson point processes in $\mathbb{R}^d$ of intensity $\lambda$ and $\mu$, respectively, where we view each point process as a random subset of $\mathbb{R}^d$. Our first results are concerned with the bipartite graph $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$.

Let $I$ be the class of graphs having at least one infinite component. By a version of the Kolmogorov zero–one law, given parameters $r, \lambda, \mu$ (and $d$), we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in I] \in \{0, 1\}$. Provided $r, \lambda$, and $\mu$ are sufficiently large, we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in I] = 1$; see [3],
Theorem 1.1.

Let \( \mu_c(r, \lambda) := \inf \{ \mu : \mathbb{P}(G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}) = 1 \} \),

with the infimum of the empty set interpreted as +\( \infty \). Also, for the more standard one-type continuum percolation graph \( G(\mathcal{P}_\lambda, r) \), define

\[ \lambda_c(2r) := \inf \{ \lambda : \mathbb{P}(G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}) = 1 \} , \]

which is well known to be finite for \( d \geq 2 \) [2, 5], but is not known analytically. By scaling (see Proposition 2.11 of [5]), \( \lambda_c(2r) = r^{-d} \lambda_c(2) \), and explicit bounds for \( \lambda_c(2) \) are provided in [5]. Simulation studies indicate that \( 1 - e^{-\pi \lambda_c(2)} \approx 0.67635 \) for \( d = 2 \) [8] and \( 1 - e^{-(4\pi/3)\lambda_c(2)} \approx 0.28957 \) for \( d = 3 \) [4].

Obviously, if \( G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I} \) then also \( G(\mathcal{P}_\lambda, 2r) \in \mathcal{I} \), and, therefore, a necessary condition for \( \mu_c(r, \lambda) \) to be finite is that \( \lambda \geq \lambda_c(2r) \). In other words, for any \( r > 0 \), we have

\[ \lambda_c(AB)(r) := \inf \{ \lambda : \mu_c(r, \lambda) < \infty \} \geq \lambda_c(2r). \] (1.1)

For \( d = 2 \) only, Iyer and Yogeshwaran [3] showed that the inequality in (1.1) is in fact an equality. For general \( d \geq 2 \), they also provided an explicit finite upper bound, here denoted by \( \tilde{\lambda}_c(AB)(r) \), and established explicit upper bounds on \( \mu_c(r, \lambda) \) for \( \lambda > \tilde{\lambda}_c(AB)(r) \). Note that even for \( d = 2 \), their explicit upper bounds for \( \mu_c(r, \lambda) \) are given only when \( \lambda > \tilde{\lambda}_c(AB)(r) \), with \( \tilde{\lambda}_c(AB)(r) > \lambda_c(2r) \) for all \( d \geq 2 \); for the case with \( d = 2 \) and \( \lambda_c(2r) < \lambda \leq \tilde{\lambda}_c(AB)(r) \), their proof that \( \mu_c(r, \lambda) < \infty \) does not provide an explicit upper bound on \( \mu_c(r, \lambda) \).

In our first result, proved in Section 2, we establish for all dimensions (and all \( r > 0 \) that the inequality in (1.1) is an equality, and provide explicit asymptotic upper bounds on \( \mu_c(r, \lambda) \) as \( \lambda \) approaches \( \lambda_c(2r) \) from above. Let \( \pi_d \) denote the volume of the ball in \( d \) dimensions with unit radius.

**Theorem 1.1.** Let \( d \geq 2 \) and \( r > 0 \). Then

(i) \( \lambda_{c, AB}(r) = \lambda_c(2r) \), and

(ii) with \( \lambda_c = \lambda_c(2r) \),

\[ \limsup_{\delta \downarrow 0} \left( \frac{\mu_c(r, \lambda_c + \delta)}{\delta - 2d |\log \delta|} \right)^{\frac{1}{2}} \leq \left( \frac{4\lambda_c^2}{r^2} \right)^{\frac{1}{2}} \pi_d \] (1.2)

Our proof (see Section 2) is based on the classic elementary continuum percolation techniques of discretization, coupling, and scaling. We also indicate how, for any given \( \lambda > \lambda_c(2r) \), we can compute an explicit upper bound for \( \mu_c(r, \lambda) \) (see (2.7) below).

It would be interesting to try to find complementary lower bounds for \( \mu_c(r, \lambda) \). An analogous problem in the lattice is mixed bond-site percolation, which similarly has two parameters. For that model, similar questions have been studied by Chayes and Schonman [1], but it is not clear to what extent their methods can be adapted to the continuum.

Our second result concerns full connectivity for the **AB random geometric graph**, i.e. the restriction of the AB percolation model to points in a bounded region of \( \mathbb{R}^d \). For \( \lambda > 0 \), let \( \mathcal{P}_\lambda^F := \mathcal{P}_\lambda \cap [0, 1]^d \) and \( \mathcal{Q}_\mu^F := \mathcal{Q}_\mu \cap [0, 1]^d \) (these are finite Poisson processes of intensity \( \lambda \); hence, the superscript \( F \)). Given also \( \tau > 0 \) and \( r > 0 \), let \( \mathbb{G}^F(\lambda, \tau, r) \) be the graph on the vertex set \( \mathcal{P}_\lambda^F \), with an edge between each pair of vertices sharing at least one common neighbour in \( G(\mathcal{P}_\lambda^F, \mathcal{Q}_\mu^F, r) \).
Let $\mathcal{G}^2(\lambda, \tau, r)$ be the graph on the vertex set $Q_n^F$, with an edge between each pair of vertices sharing at least one common neighbour in $G(P_s, Q_n^F)$. Then $G(P_s, Q_n^F)$ is connected, if and only if both $\mathcal{G}^1(\lambda, \tau, r)$ and $\mathcal{G}^2(\lambda, \tau, r)$ are connected.

Let $\mathcal{K}$ be the class of connected graphs, and let

$$\rho_n(\tau) = \min\{r : \mathcal{G}^1(n, \tau, r) \in \mathcal{K}\},$$

which is a random variable determined by the configuration of $(P_s, Q_n)$. It is a connectivity threshold for the AB random geometric graph. Let us assume that $P_s$ and $Q_n^F$ are coupled for all $\lambda, \mu > 0$ as follows. Let $(X_1, Y_1, X_2, Y_2, \ldots)$ be a sequence of independent uniform random $d$-vectors uniformly distributed over $[0, 1]^d$. Independently, let $(N_t, t \geq 0)$ and $(N'_t, t \geq 0)$ be independent Poison counting processes of rate 1. Let $P_s^F = \{X_1, \ldots, X_{N_t}\}$ and $Q_n^F = \{Y_1, \ldots, Y_{N'_t}\}$.

In Section 3 we prove the following result, with 'a.a.s.' denoting almost-sure convergence as $n \to \infty$ (with $n \in \mathbb{N}$).

**Theorem 1.2.** Assume that $d = 2$. Let $\tau > 0$. Then

$$\frac{n\pi(\rho_n(\tau))^2}{\log n} \xrightarrow{\text{a.s.}} \max\left(\frac{1}{\tau}, \frac{1}{4}\right).$$

(1.3)

**Remark 1.1.** The restriction to $d = 2$ arises because boundary effects become more important in higher dimensions (and $d = 1$ is a different case). It should be possible to adapt the proof to obtain a similar result to (1.3) in the unit torus in arbitrary dimensions $d \geq 2$, namely, $n\pi_d(\rho_n(\tau))^d/\log n \xrightarrow{\text{a.s.}} \max(1/\tau, 2^{-d})$, although we have not checked the details.

**Remark 1.2.** Iyer and Yogeshwaran [3, Theorem 3.1] gave a.s. lower and upper bounds for $\rho_n(\tau)$ in the torus. The extension of our result mentioned in Remark 1.1 would show that the lower bound of [3] is sharp for $\tau \leq 2^d$, and improve on their upper bound.

**Notation.** Given a countable set $\mathcal{X}$, we write $|\mathcal{X}|$ for the number of elements of $\mathcal{X}$ and if also $\mathcal{X} \subset \mathbb{R}^d$, given $A \subset \mathbb{R}^d$, we write $\mathcal{X}(A)$ for $|\mathcal{X} \cap A|$. Also, for $a > 0$, we write $a\mathcal{A}$ for $\{ay : y \in \mathcal{A}\}$. Let $\mathbb{D}^d$ denote the Minkowski addition of sets (see, e.g. [6]).

2. Percolation: proof of Theorem 1.1

Fix $r > 0$, and let $\lambda > \lambda_c(2r)$. We first prove that $\mu_c(r, \lambda) < \infty$; combined with (1.1) this shows that $\lambda_c(2r) = \lambda_c(2r)$, which is part (i) of the theorem. Later we shall quantify the estimates in our argument, thereby establishing part (ii).

Choose $s < r$ and $v < \lambda$ such that $\Pr[G(P_s, 2s) \in I] = 1$. This is possible because decreasing the radius slightly is equivalent to decreasing the Poisson intensity slightly, by scaling (see [5]; also the first equality of (2.5) below). Set $t = (r + s)/2$, and let $\varepsilon > 0$ be chosen small enough so that any cube of side length $\varepsilon$ has Euclidean diameter at most $t - s = \frac{1}{2}(r - s)$. For $a > 0$, let $\rho_a := 1 - \exp(-\varepsilon^d a)$, the probability that a given cube of side length $\varepsilon$ contains at least one point of $P_s$.

Consider Bernoulli site percolation on the graph $(\varepsilon\mathbb{Z}^d, \sim)$, where, for $u$ and $v \in \varepsilon\mathbb{Z}^d$, $u \sim v$ if and only if there exists $w \in \varepsilon\mathbb{Z}^d$ with $\|w - u\| \leq t$ and $\|w - v\| \leq t$. Given $p > 0$, suppose that each site $u \in \varepsilon\mathbb{Z}^d$ is independently occupied with probability $p$. Let $D_1$ be the event that there is an infinite path of occupied sites in the graph, and let $\Pr_p[D_1]$ be the probability that this event occurs.
Divide \( \mathbb{R}^d \) into cubes \( Q_u, u \in \varepsilon \mathbb{Z}^d \), defined by \( Q_u := [u] \oplus [0, \varepsilon)^d \). For \( x \in \mathbb{R}^d \), let \( z_x \in \varepsilon \mathbb{Z}^d \) be such that \( x \in z_x \). The Poisson process \( \mathcal{P}_v \) may be coupled to a realization of the site percolation process with parameter \( p_v \), by deeming each \( z \in \varepsilon \mathbb{Z}^d \) to be occupied if and only if \( \mathcal{P}_v(Q_z) \geq 1 \). By the choice of \( \varepsilon \), for \( X, Y \in \mathcal{P}_v \), if \( \|X - Y\| \leq 2\varepsilon \) then \( \|z_X - z(X + Y)/2\| \leq t \) and \( \|z_Y - z(X + Y)/2\| \leq t \), and, hence, \( z_x \sim z_y \). Therefore, with this coupling, if \( G(\mathcal{P}_v, 2\varepsilon) \in I \) then there is an infinite path of occupied sites in \( (\varepsilon \mathbb{Z}^d, \sim) \). Because we chose \( v \) and \( s \) in such a way that \( \mathbb{P}[G(\mathcal{P}_v, 2\varepsilon) \in I] = 1 \), we must have \( \mathbb{P}_{p_v}[D_1] = 1 \).

Now consider a form of lattice AB percolation on \( \varepsilon \mathbb{Z}^d \) with parameter pair \( (p, q) \in [0, 1]^2 \) (not necessarily the same as any of the lattice AB percolation models in the literature). Let each \( \{V_{u,v}, u \in \varepsilon \mathbb{Z}^d\} \) and \( \{W_{u,v}, u \in \varepsilon \mathbb{Z}^d\} \) be a family of independent Bernoulli random variables, with parameters \( p \) and \( q \), respectively. Let \( D_2 \) be the event that there is an infinite sequence \( u_1, u_2, \ldots \) of distinct elements of \( \varepsilon \mathbb{Z}^d \) and an infinite sequence \( v_1, v_2, \ldots \) of elements of \( \varepsilon \mathbb{Z}^d \) such that, for each \( i \in \mathbb{N} \), we have \( V_{u_i}W_{v_i} = 1 \) and max \( \|u_i - v_i\|, \|v_i - u_{i+1}\| \leq t \). Let \( \mathbb{P}_{p,q}[D_2] \) be the probability that event \( D_2 \) occurs, given the parameter pair \( (p, q) \).

Since \( \mathbb{P}_{p_v}[D_1] = 1 \), clearly \( \mathbb{P}_{p_v,q}[D_2] = 1 \). Increasing \( p \) slightly and decreasing \( q \) slightly, we shall show that there exists \( q < 1 \) such that

\[
\tilde{\mathbb{P}}_{p,q}[D_2] = 1.
\]

This is enough to demonstrate that \( \mu_\varepsilon(r, \lambda) < \infty \). Indeed, suppose that such a \( q \) exists and choose \( \mu \) such that \( p_\mu = q \). Then, for \( u \in \varepsilon \mathbb{Z}^d \), set \( V_{u,v} = 1 \) if and only if \( \mathcal{P}_\mu(Q_u) \geq 1 \) and \( W_{u,v} = 1 \) if and only if \( \mathcal{Q}_\mu(Q_u) \geq 1 \). Suppose that \( D_2 \) occurs, and let \( u_1, v_1, u_2, v_2, \ldots \) be as in the definition of the event \( D_2 \). Then, for each \( i \in \mathbb{N} \), we have \( V_{u_i} = 1 \), so we can pick a point \( X_i \in \mathcal{P}_\lambda \cap Q_{u_i} \), and \( W_{v_i} = 1 \), so we can pick a point \( Y_i \in \mathcal{Q}_\mu \cap Q_{v_i} \). Then, by the choice of \( \varepsilon \), for each \( i \in \mathbb{N} \), we have

\[
\max(\|X_i - Y_i\|, \|Y_i - X_{i+1}\|) \leq t + (t - s) = r,
\]

and, hence, \( G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in I \). Hence, by (2.1) we have \( \mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in I] = 1 \). Therefore, \( \mu_\varepsilon(r, \lambda) \leq \mu < \infty \), as asserted.

To complete the proof of part (i), it remains to prove that (2.1) holds for some \( q < 1 \). Let \( \{T_u, u \in \varepsilon \mathbb{Z}^d\} \) be independent Bernoulli variables with parameter \( p_v \). For each ordered pair \((u, v) \in (\varepsilon \mathbb{Z}^d)^2\) with \( 0 < \|u - v\| \leq t \), let \( U_{u,v} \) be independent Bernoulli random variables with parameter \((p_v/p_\mu)^{\Delta/\lambda}\), where we set

\[
\Delta := \{u \in \varepsilon \mathbb{Z}^d : 0 < \|u\| \leq t\}.
\]

Assume that the variables \( U_{u,v} \) and \( T_u \) are all mutually independent, and, for \( u, v \in \varepsilon \mathbb{Z}^d \), define the Bernoulli variables

\[
V_u := T_u \prod_{\{v \in \varepsilon \mathbb{Z}^d : 0 < \|v-u\| \leq t\}} U_{u,v},
\]

\[
W_v := 1 - \prod_{\{u \in \varepsilon \mathbb{Z}^d : 0 < \|v-u\| \leq t\}} (1 - U_{u,v}).
\]

Then each of \( \{V_u\}_{u \in \varepsilon \mathbb{Z}^d} \) and \( \{W_v\}_{v \in \varepsilon \mathbb{Z}^d} \) is a family of independent Bernoulli variables, with respective parameters \( p_v \) and

\[
q := 1 - \left(1 - \left(\frac{p_\mu}{p_v}\right)^{\Delta/\lambda}\right) \Delta < 1,
\]

and each is independent of \( \{T_u, u \in \varepsilon \mathbb{Z}^d\} \).
Since \( \mathbb{P}_{p_u}[D_1] = 1 \), with probability 1, there exists an infinite sequence \( u_1, u_2, \ldots \) of distinct elements of \( \mathbb{Z}^d \) with \( u_i \sim u_{i+1} \) for all \( i \in \mathbb{N} \), and with \( V_{u_i} = 1 \) for each \( i \in \mathbb{N} \). By the definition of the relation ‘~’, we can choose a sequence \( v_1, v_2, \ldots \) of elements of \( \mathbb{Z}^d \) such that, for each \( i \in \mathbb{N} \), we have \( \max(\|v_i - u_i\|, \|v_i - u_{i+1}\|) \leq t \). Then, for each \( i \), since \( V_{u_i} = 1 \), we have \( U_{u_i, v_i} = 1 \), and, therefore, \( W_{u_i} = 1 \); also, \( T_{u_i} = 1 \). Hence, (2.1) holds as required, establishing that \( \mu_c(r, \lambda) < \infty \). We have proved part (i).

To prove part (ii), we need to quantify the preceding argument. First note that the value of \( \mu \) associated with \( q \) given by (2.3) (i.e. with \( p_{\mu} = q \)) has \( \exp(-\mu \varepsilon^d) = (1 - (p_{\nu}/p_{\lambda})^{1/\Delta})^\Delta \), so that, since \( \varepsilon^d \Delta \leq \pi_d r^d \) by (2.2), we have

\[
\mu_c(r, \lambda) \leq \mu = \varepsilon^{-d} \Delta \log \left( \frac{1}{1 - (p_{\nu}/p_{\lambda})^{1/\Delta}} \right) \leq \varepsilon^{-2d} \pi_d r^d \log \left( \frac{1}{1 - (p_{\nu}/p_{\lambda})^{(\varepsilon/r)^d/\pi_d}} \right).
\]

(2.4)

From now on, set \( \lambda_c := \lambda_c(2r) \) and \( \lambda = \lambda_c + \delta \) for some \( \delta > 0 \). We need to choose \( s < r \) and \( \nu < \lambda \) such that \( \mathbb{P}[G(\mathcal{P}_c, 2s) \in \mathcal{I}] = 1 \). Choose \( \alpha, \beta > 0 \) with \( \alpha + \beta < 1 \) and also let \( \alpha' \in (\alpha, \alpha) \) and \( \beta' \in (0, \beta) \). Set

\[
s := r \left( 1 + \frac{\alpha \delta}{\lambda_c} \right)^{-1/d} \text{ and } \nu := \lambda_c + (1 - \beta)\delta.
\]

By scaling (see [5, Proposition 2.11]) and our choice of \( s \), we have

\[
\lambda_c(2s) = \left( \frac{r}{s} \right)^d \lambda_c(2r) = \lambda_c + \alpha \delta,
\]

and, hence, \( \nu > \lambda_c(2s) \), so \( \mathbb{P}[G(\mathcal{P}_c, 2s) \in \mathcal{I}] = 1 \), as required.

Our choice of \( \varepsilon \) in the discretization needs to satisfy

\[
\varepsilon \leq \frac{r - s}{2 \sqrt{d}} = \frac{r}{2 \sqrt{d}} \left( 1 - \left( 1 + \frac{\alpha \delta}{\lambda_c} \right)^{-1/d} \right),
\]

(2.6)

and the right-hand side of (2.6) is asymptotic to \( \alpha r \delta/(2d^{1/2} \lambda_c) \) as \( \delta \to 0 \). Hence, taking \( \varepsilon = \alpha' r \delta/(2d^{1/2} \lambda_c) \), we have (2.6) provided \( \delta \leq \delta_1 \) for some fixed \( \delta_1 > 0 \). Also,

\[
\frac{p_u}{p_\lambda} \leq e^{d \lambda} \exp(-e^{d \lambda}) = \left( \frac{\lambda_c + (1 - \beta)\delta}{\lambda_c + \delta} \right) \exp(e^{d \lambda}),
\]

and so, by Taylor’s expansion, there is some \( \delta_2 > 0 \) such that, provided \( 0 < \delta \leq \delta_2 \), taking \( \varepsilon = \alpha' r \delta/(2d^{1/2} \lambda_c) \) we have

\[
\left( \frac{p_u}{p_\lambda} \right)^{(\varepsilon/r)^d/\pi_d} \leq 1 - \beta' \delta \varepsilon^d \pi_d d^{d \lambda_c} = 1 - \frac{\beta' \delta^{d+1}}{\pi_d \lambda_c (2d^{1/2} \lambda_c/\alpha')^d}.
\]

Therefore, by (2.4), for \( 0 < \delta \leq \min(\delta_1, \delta_2) \), we have

\[
\mu_c(r, \lambda) \leq \left( \frac{2d^{1/2} \lambda_c}{r \delta \alpha'} \right)^{2d} \pi_d d^{d \lambda_c} \log \left( \frac{\pi_d \lambda_c (2d^{1/2} \lambda_c/\alpha')^d}{\beta' \delta^{d+1}} \right)
\]

and since we can take \( \alpha' \) arbitrarily close to 1, (1.2) follows, completing the proof.
For a given value of $\lambda$ with $\lambda = \lambda_c(2r) + \delta$ for some $\delta > 0$, an explicit upper bound for $\mu_c(r, \lambda)$ could be computed as follows. Choose $\alpha, \beta > 0$ with $\alpha + \beta < 1$, and let $\epsilon$ be given by the right-hand side of (2.6). Then a numerical upper bound for $\mu_c(r, \lambda)$ can be obtained by computing the right-hand side of (2.4). To make this bound as small as possible (given $\alpha$), we make $\nu$ as small as we can, i.e. make $\beta$ approach $1 - \alpha$ and $\nu$ approach $\lambda_c + \alpha \delta$. Taking this limit and then optimizing further over $\alpha$ gives us the upper bound

$$\mu_c(r, \lambda) \leq \inf_{\alpha \in (0,1)} \epsilon(\alpha) - \frac{2}{\pi} \frac{d}{dr} \log \left( \frac{1}{1 - (p_{\lambda c + \alpha \delta}/p_{\lambda})^{\epsilon(\alpha)/r} \pi d} \right),$$

with $\epsilon = \epsilon(\alpha)$ given by the right-hand side of (2.6).

### 3. Connectivity: proof of Theorem 1.2

Throughout this section, we assume that $d = 2$. All asymptotics are as $n \to \infty$. Given $a, b \in \mathbb{R}$, we sometimes write $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. Fix $\tau > 0$. Given $\tau$ and $r_n$, let $\delta_n$ denote the minimum degree of $G_1(n, \tau, r_n)$.

**Lemma 3.1.** Let $\alpha \in (0, 1/\tau)$. If $n \pi r_n^2/\log n = \alpha$ for $n \geq 2$ then, almost surely, $\delta_n = 0$ for all but finitely many $n$.

**Proof.** See [3, Proposition 5.1].

**Lemma 3.2.** Let $\alpha \in (0, 1/4)$. If $n \pi r_n^2/\log n = \alpha$ for $n \geq 2$ then, almost surely, $\delta_n = 0$ for all but finitely many $n$.

**Proof.** By [6, Theorem 7.8], for this choice of $r_n$, almost surely, the minimum degree of the (one-type) geometric graph $G(P_n^F, 2r_n)$ is 0 for all but finitely many $n$, and, therefore, so is the minimum degree of $\mathcal{G}^1(n, \tau, r_n)$.

**Corollary 3.1.** Let $d = 2$. Given $\epsilon > 0$, almost surely, $n \pi (\rho_n(\tau))^2/\log n > (1 - \epsilon) \max\{1/4, 1/\tau\}$ for all but finitely many $n$.

**Proof.** Assume that $\epsilon < 1$. For $n \geq 2$, set $r_n = [(1 - \epsilon)(1/4 \lor 1/\tau) \log n/(n \pi)]^{1/2}$, so $n \pi r_n^2/\log n = (1 - \epsilon)[1/4 \lor 1/\tau]$. Let $\delta_n$ be the minimum degree of $\mathcal{G}^1(n, \tau, r_n)$. If the minimum degree of a graph of order greater than 1 is zero, then it is not connected; hence,

$$\left\{ \frac{n \pi (\rho_n(\tau))^2}{\log n} \leq (1 - \epsilon) \left( \frac{1}{4} \lor \frac{1}{\tau} \right) \right\} = \{\mathcal{G}^1(n, \tau, r_n) \in \mathcal{K}\}
\subset \{\delta_n > 0\} \cup \{\mathcal{G}^F_n([0, 1]^2) \leq 1\},$$

and, by Lemmas 3.1 and 3.2, this occurs for only finitely many $n$ almost surely.

To complete the proof of Theorem 1.2, it suffices to prove the following result.

**Theorem 3.1.** Suppose for some fixed $\alpha$ that $\{r_n\}_{n \in \mathbb{N}}$ is such that, for all $n \geq 2$,

$$\frac{n \pi r_n^2}{\log n} = \alpha > \max\left\{ \frac{1}{\tau}, \frac{1}{4} \right\}.$$

Then, almost surely, $\mathcal{G}^1(n, \tau, r_n) \in \mathcal{K}$ for all but finitely many $n$. 

Our proof of this theorem requires a series of lemmas and proceeds by discretization of space. Assume that $\alpha$ and $r_0$ are given, satisfying (3.1). Let $\varepsilon_0 \in (0, \frac{1}{99})$ be chosen in such a way that, for $\varepsilon = \varepsilon_0$, we have both
\begin{align*}
\alpha \tau (1 - 12\varepsilon) &> 1 + \varepsilon \tag{3.2} \\
\alpha (4 - 12\varepsilon (3 + \tau)) &> 1 + \varepsilon. \tag{3.3}
\end{align*}
Given $n$, partition $[0, 1]^2$ into squares of side $\varepsilon_n r_0$ with $\varepsilon_n$ chosen so that $\varepsilon_0 \leq \varepsilon_n < \frac{1}{99}$ and $1/(\varepsilon_n r_0) \in \mathbb{N}$, and $\varepsilon = \varepsilon_n$ satisfies (3.2) and (3.3); this is possible for all large enough $n, n \geq n_0$ say. In the sequel we assume that $n \geq n_0$ and often write just $\varepsilon$ for $\varepsilon_n$.

Let $\mathcal{L}_n$ be the set of centres of the squares in this partition (a finite lattice). Then $|\mathcal{L}_n| = \Theta(n / \log n)$. List the squares as $Q_i$, $1 \leq i \leq |\mathcal{L}_n|$, and the corresponding centres of squares (i.e. the elements of $\mathcal{L}_n$) as $q_i$, $1 \leq i \leq |\mathcal{L}_n|$.

Given a set $\mathcal{X} \subset [0, 1]^2$, define the projection of $\mathcal{X}$ onto $\mathcal{L}_n$ to be the set of $q_i \in \mathcal{L}_n$ such that $\mathcal{X} \cap Q_i \neq \emptyset$. Given also $\mathcal{Y} \subset [0, 1]^2$, define the projection of $(\mathcal{X}, \mathcal{Y})$ onto $\mathcal{L}_n$ to be the pair $(\mathcal{X}', \mathcal{Y}')$, where $\mathcal{X}'$ is the projection of $\mathcal{X}$ onto $\mathcal{L}_n$ and $\mathcal{Y}'$ is the projection of $\mathcal{Y}$ onto $\mathcal{L}_n$. We refer to $|\mathcal{X}'| + |\mathcal{Y}'|$ (respectively $|\mathcal{X}'|, |\mathcal{Y}'|$) as the order of the projection of $(\mathcal{X}, \mathcal{Y})$ (respectively of $\mathcal{X}$, of $\mathcal{Y}$) onto $\mathcal{L}_n$.

**Lemma 3.3.** Let $n \in \mathbb{N}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are finite subsets of $[0, 1]^2$ such that $G(\mathcal{X}, \mathcal{Y}, r_n)$ is connected. Let $(\mathcal{X}', \mathcal{Y}')$ be the projection of $(\mathcal{X}, \mathcal{Y})$ onto $\mathcal{L}_n$. Then the bipartite geometric graph $G(\mathcal{X}', \mathcal{Y}', r_n, 1 + 2\varepsilon_n))$ is connected.

**Proof.** If $q_i, q_j \in \mathcal{L}_n$, and $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with $\|X - Y\| \leq r_n$, then by the triangle inequality we have

$$
\|q_i - q_j\| \leq \|X - q_i\| + \|X - Y\| + \|Y - q_j\| \leq r_n (1 + 2\varepsilon),
$$

and, therefore, since $G(\mathcal{X}, \mathcal{Y}, r_n)$ is connected, so is $G(\mathcal{X}', \mathcal{Y}', r_n, 1 + 2\varepsilon))$.

Given $n, m \in \mathbb{N}$, let $\mathcal{A}_{n, m}$ denote the set of pairs $(\sigma_1, \sigma_2)$ with each $\sigma_j \in \mathcal{L}_n$, with $|\sigma_1| + |\sigma_2| = m$ and $|\sigma_1| \geq 1$, such that $G(\sigma_1, \sigma_2, r_n, 1 + 2\varepsilon_n))$ is connected; these may be viewed as ‘bipartite lattice animals’.

Let $\mathcal{A}_2$ be the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n, m}$ such that all elements of $\sigma_1 \cup \sigma_2$ are distances at least $2r_n$ from the boundary of $[0, 1]^2$.

Let $\mathcal{A}_{1, n, m}$ be the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n, m}$ such that $\sigma_1 \cup \sigma_2$ is a distance less than $2r_n$ from just one edge of $[0, 1]^2$.

Let $\mathcal{A}_{0, n, m} := \mathcal{A}_{n, m} \setminus (\mathcal{A}_2 \cup \mathcal{A}_{1, n, m})$, the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n, m}$ such that $\sigma_1 \cup \sigma_2$ is a distance less than $2r_n$ from two edges of $[0, 1]^2$ (i.e. near a corner of $[0, 1]^2$).

**Lemma 3.4.** Given $m \in \mathbb{N}$, there exists a constant $C = C(m)$ such that, for all $n \geq n_0$,

$$
|\mathcal{A}_{n, m}| \leq C \left(\frac{n}{\log n}\right), \quad |\mathcal{A}_{1, n, m}| \leq C \left(\frac{n}{\log n}\right)^{1/2}, \quad |\mathcal{A}_{0, n, m}| \leq C.
$$

**Proof.** Fix $m$. Consider how many ways there are to choose $\sigma \in \mathcal{A}_{n, m}$.

For the first element of $\sigma_1$ in the lexicographic ordering, there are at most $|\mathcal{L}_n|$ choices, and, hence, $O(n / \log n)$ choices. Having chosen the first element of $\sigma_1$, there are a bounded number of ways to choose the rest of $\sigma$.

We now consider how many ways there are to choose $\sigma \in \mathcal{A}_{n, m}$. There are $O(r_0^{-1}) = O((n / \log n)^{1/2})$ ways to choose the first element of $\sigma_1$ (a distance at most $2r_n$ from the boundary of $[0, 1]^2$), and then a bounded number of ways to choose the rest of $\sigma$. 

Finally, consider how many ways there are to choose \( \sigma \in A_{n,m}^0 \). There are \( O(1) \) ways to choose the first element of \( \sigma_1 \), and then a bounded number of ways to choose the rest of \( \sigma \).

For \( n \in \mathbb{N} \), set \( v(n) := n^{[4/10]} \). Note that \( v(n + 1) \sim v(n) \) and \( r_{v(n+1)} \sim r_{v(n)} \) as \( n \to \infty \), and that \( r_{\nu(n)} \) is monotone decreasing in \( n \) for \( n \geq 3 \).

Given \( n \in \mathbb{N} \) with \( v(n) \geq n_0 \), and given \( \sigma_1 \in \mathcal{L}_{v(n)} \) and \( \sigma_2 \in \mathcal{L}_{v(n)} \), let \( E_{\sigma_1, \sigma_2} \) be the event that there exists some \( n' \in \mathbb{N} \cap [v(n), v(n + 1)) \) such that there is a component \((U, V)\) of \( G(p_n^F, Q_n^F, r_n^\nu) \) such that \((\sigma_1, \sigma_2)\) is the projection of \((U, V)\) onto \( \mathcal{L}_{v(n)} \).

For \( x \in \mathbb{R}^2 \) and \( r > 0 \), let \( B(x, r) := \{y \in \mathbb{R}^2 : \|y - x\| \leq r\} \). Also, let \( B_+(r) \) be the right half of \( B((0, 0), r) \), and let \( B_-(r) \) be the left half of \( B((0, 0), r) \). Let \( \nu_2(\cdot) \) denote the Lebesgue measure, defined on Borel subsets of \( \mathbb{R}^2 \).

**Lemma 3.5.** There exists \( n_1 \in \mathbb{N} \) such that, for all \( m \in \mathbb{N} \) and \( n \geq n_1 \),

\[
\sup_{\sigma \in A_{v(n),m}^0} \mathbb{P}[E_\sigma] \leq v(n)^{-1/20}.
\]

**Proof.** Choose \( n_1 \) so that \( v(n_1) \geq n_0 \) and also \((1 - \epsilon_0) r_{v(n)} < r_{v(n+1)} \) for \( n \geq n_1 \). Assume from now on \( n \geq n_1 \).

Given \( \sigma = (\sigma_1, \sigma_2) \in A_{v(n),m} \), let \( q_i \) and \( q_j \) respectively be the lexicographically first and last elements of \( \sigma_1 \). Let \( \sigma_2^\circ \) be the set of \( q_k \in \sigma_2 \cap B(q_i, r_{v(n)}(1 - 4\epsilon)) \) lying strictly to the left of \( q_i \) (in this proof, \( \epsilon := \epsilon_{v(n)} \)). Let \( \sigma_2^\circ \) be the set of \( q_k \in \sigma_2 \cap B(q_j, r_{v(n)}(1 - 4\epsilon)) \) lying strictly to the right of \( q_j \). Let \( \tilde{\sigma}_2^\circ := \sigma_2^\circ \oplus [\epsilon r_{v(n)}/2, \epsilon r_{v(n)}/2]^2 \) and \( \tilde{\sigma}_2^\circ := \sigma_2^\circ \oplus [\epsilon r_{v(n)}/2, \epsilon r_{v(n)}/2]^2 \) (see Figure 1).

Let \( B_{\tilde{\sigma}}^- \) be the part of \( B(q_i, r_{v(n)}(1 - 5\epsilon)) \) lying strictly to the left of \( Q_i \). Let \( B_{\tilde{\sigma}}^+ \) be the part of \( B(q_j, r_{v(n)}(1 - 5\epsilon)) \) lying strictly to the right of \( Q_j \).

Given \( \sigma \), define the events \( A_{\sigma}^+ \) and \( A_{\sigma}^- \) by

\[
A_{\sigma}^+ := (Q_{v(n)+1}^F \setminus \tilde{\sigma}_2^\circ) = 0 \cap (\mathcal{F}_{v(n)+1}^F(\tilde{\sigma}_2^\circ \oplus B_+ (r_{v(n)}(1 - 3\epsilon))) = 0),
\]

\[
A_{\sigma}^- := (Q_{v(n)+1}^F \setminus \tilde{\sigma}_2^\circ) = 0 \cap (\mathcal{F}_{v(n)+1}^F(\tilde{\sigma}_2^\circ \oplus B_- (r_{v(n)}(1 - 3\epsilon))) = 0).
\]

See Figure 1 for an illustration of the event \( A_{\sigma}^+ \). Note that the events \( A_{\sigma}^+ \) and \( A_{\sigma}^- \) are independent.

Suppose that \( k \) is such that \( Q_k \cap B_{\tilde{\sigma}}^+ \neq \emptyset \). Then, by the triangle equality,

\[
\|q_k - q_l\| \leq r_{v(n)}(1 - 5\epsilon) + \epsilon r_{v(n)} = r_{v(n)}(1 - 4\epsilon).
\]

Similarly, if \( Q_k \cap B_{\tilde{\sigma}}^- \neq \emptyset \) then \( \|q_k - q_l\| \leq r_{v(n)}(1 - 4\epsilon) \).

By our coupling of Poisson processes, for \( v(n) \leq n' < v(n + 1) \), we have \( \mathcal{F}_{v(n)} \subset \mathcal{F}_{v(n+1)} \). Also, if \( x \in Q_k \) and \( y \in Q_i \) with \( \|q_i - q_k\| \leq r_{v(n)}(1 - 3\epsilon) \), then, by the triangle inequality and our condition on \( n_1 \), we have \( \|x - y\| \leq r_{v(n)}(1 - \epsilon) \leq r_{\nu(n)} \) for all \( n' \in [v(n), v(n + 1)) \). Hence, by the argument at (3.7), for any \( \sigma \in A_{n,m} \), we have \( E_{\sigma} \subset A_{\sigma}^+ \cap A_{\sigma}^- \).
Figure 1: The dots are the points of $\sigma_1$, and the crosses are the points of $\sigma_2$. The grey squares are the set $\tilde{\sigma} + 2$ (since $\varepsilon = \varepsilon_n < \frac{1}{10^9}$, they should really be smaller). The event $A_{\tilde{\sigma}}^+$ says that the black region contains no points of $Q_{\nu(n+1)}^F$ and the grey region (partly obscured by the black region) contains no points of $P_{\nu(n+1)}^F$.

and in this case we have

$$\mathbb{P}[A_{\tilde{\sigma}}^+] \leq \exp\left(\frac{1}{2} - \tau \nu(n)(\pi(r_{\nu(n)}(1 - 5\varepsilon))^2 - 2\varepsilon r_{\nu(n)}^2)\right)$$

$$\leq \exp\left(\frac{1}{2} - \tau \alpha(\log \nu(n))(1 - 12\varepsilon)\right)$$

$$\leq \nu(n)^{-(1+\varepsilon)/2},$$

where the last inequality comes from (3.2). This proves (3.5) for this case.

Suppose instead that $\sigma_2 + 2 \neq \emptyset$. Then $\tilde{\sigma}_2 + B_+(r_{\nu(n)}(1 - 3\varepsilon))$, so that $v_2(\tilde{\sigma}_2 +) \leq \pi r_{\nu(n)}^2(1 - 3\varepsilon)^2/2$. Let $s \in [0, 1]$ be chosen such that $v_2(\tilde{\sigma}_2 +) = s^2 \pi r_{\nu(n)}^2(1 - 3\varepsilon)^2/2$. Then, by the Brunn–Minkowski inequality (see, e.g. [6]),

$$v_2(\tilde{\sigma}_2 + B_+(r_{\nu(n)}(1 - 3\varepsilon))) \geq \frac{\pi r_{\nu(n)}^2}{2}(1 - 3\varepsilon)^2(1 + s)^2,$$

and also $v_2(B_+^\nu(1 - 3\varepsilon))^2/2$, so that

$$\mathbb{P}[A_{\tilde{\sigma}}^+] \leq \exp\left(-\tau \nu(n)v_2(B_+^\nu(1 - 3\varepsilon))^2/2\right)$$

$$\leq \exp\left(-\frac{1}{2}(1 + \tau)(1 - 12\varepsilon) + s^2(1 - 3\varepsilon)^2(1 - \tau)\right)$$

$$\leq \exp\left(-\frac{1}{2}\alpha(\log \nu(n))g_t(s)\right),$$

where we set $g_t(s) := (\tau + 1 + 2s)(1 - 12\varepsilon) + s^2(1 - 3\varepsilon)^2(1 - \tau)$. If $\tau \leq 1$ then $g_t(s)$ is minimised over $s \in [0, 1]$ at $s = 0$. If $\tau > 1$ then $g_t(s)$ is concave, so its minimum over $[0, 1]$ is achieved at $s = 0$ or $s = 1$; also in this case $g_t(1) \geq (3 + \tau)(1 - 12\varepsilon) + 1 - \tau$. Hence, using (3.2) and (3.3), we obtain

$$\mathbb{P}[A_{\tilde{\sigma}}^+] \leq \exp\left(-\frac{1}{2}\alpha(\log \nu(n))\min((1 + \tau)(1 - 12\varepsilon), 4 - 12\varepsilon(3 + \tau))\right)$$

$$\leq \nu(n)^{-(1+\varepsilon)/2},$$

completing the proof of (3.5).
Now we prove (3.4). If \( \sigma \in A^0_{n,m} \), then \( \mathbb{P}[A^+_{n}] \leq v(n)^{-1-\varepsilon/2} \) by (3.8) and (3.9), and \( \mathbb{P}[A^-_n] \leq v(n)^{-(1+\varepsilon/2)} \) similarly. Therefore, \( \mathbb{P}[E_{\sigma}] \leq \mathbb{P}[A^+_n \cap A^-_n] \leq v(n)^{-1-\varepsilon} \), completing the proof of (3.4).

Finally, to prove (3.6), let \( \sigma \in A^0_{n,m} \). Assume that \( \sigma \) is near the lower-left corner of \([0, 1]^2\) (the other cases are treated similarly). First suppose that \( \sigma^2 = \emptyset \). Then \( \mathbb{P}[E_{\sigma}] \leq \mathbb{P}[\mathcal{Q}_{\nu(n+1)}^F(B^+_n) = 0] \) and since the upper half of \( B^+_n \) is contained in \([0, 1]^2\), in this case

\[
\mathbb{P}[E_{\sigma}] \leq \exp\left(-\tau v(n)\pi r^2(n)\left[\frac{1}{2}(1 - 5\varepsilon)^2 - \frac{\varepsilon}{2}\right]\right)
\leq v(n)^{-\alpha(1-\varepsilon)/4}
\leq v(n)^{-1+\varepsilon/4}.
\] (3.10)

Now suppose that \( \sigma^2 \neq \emptyset \). Let \( q_\ell \) be the last element (in the lexicographic order) of \( \sigma^2 \). Then

\[
\mathbb{P}[E_{\sigma}] \leq \mathbb{P}[\mathcal{Q}_{\nu(n+1)}^F(\{q_\ell\} \oplus B_+(r_\nu(n)(1 - 3\varepsilon))) = 0] \leq \exp\left(-\tau v(n)\pi r^2(n)(1 - 3\varepsilon)^2\right)
\leq v(n)^{-\alpha(1-6\varepsilon)/4}
\leq v(n)^{-1/20},
\]

where, for the last inequality, we used the facts that \( \alpha > \frac{1}{2} \) and \( \varepsilon < \frac{1}{100} \). Together with (3.10) this demonstrates (3.6).

For \( m, n \in \mathbb{N} \) and \( r > 0 \), let \( \mathcal{K}_{n,m}(r) \) be the class of bipartite point sets \( (\mathcal{X}, \mathcal{Y}) \) in \([0, 1]^2\) such that \( G(\mathcal{X}, \mathcal{Y}, r) \) has at least one component, the vertex set of which has projection onto \( \mathcal{L}_m \) of order \( m \) and contains at least one element of \( \mathcal{X} \).

**Lemma 3.6.** Let \( m \in \mathbb{N} \). Almost surely, for all but finitely many \( n \in \mathbb{N} \), we have \( (\mathcal{Q}_{n}^F, \mathcal{Q}_{n}^F) \notin \mathcal{K}_{\nu(n),m}(r_n') \) for all \( n' \in \mathbb{N} \cap [\nu(n), \nu(n+1)) \).

**Proof.** By Lemmas 3.3 and 3.5, for \( n \geq n_1 \), we have

\[
\mathbb{P}\left[ \bigcup_{\nu(n) \leq n' < \nu(n+1)} (\mathcal{Q}_{n}^F, \mathcal{Q}_{n}^F) \in \mathcal{K}_{\nu(n),m}(r_n') \right]
\leq \sum_{\sigma \in A_{\nu(n),m}} \mathbb{P}[E_{\sigma}]
\leq |A^2_{\nu(n),m}| \times v(n)^{-(1+\varepsilon)} + |A^1_{\nu(n),m}| \times v(n)^{-(1+\varepsilon)/2} + |A^0_{\nu(n),m}| \times v(n)^{-1/20}.
\]

Using Lemma 3.4 and the definition \( v(n) := n^{[d/\varepsilon]} \), and recalling that \( \varepsilon = \varepsilon_0 \geq \varepsilon_0 \) as described just after (3.3), this probability is \( O(n^{-2}) \), so it is summable in \( n \). Then the result follows by the Borel–Cantelli lemma.

**Lemma 3.7.** (See [6, Lemma 9.1].) For any two closed connected subsets \( A \) and \( B \) of \([0, 1]^2\) with union \( A \cup B = [0, 1]^2 \), the intersection \( A \cap B \) is connected.

Given \( n \in \mathbb{N} \), let \( k(n) \) be the choice of \( k \in \mathbb{N} \) satisfying \( v(k) \leq n < v(k+1) \). Also, given \( K \in \mathbb{N} \), let \( F_K(n) \) be the event that \( G(\mathcal{Q}_{n}^F, \mathcal{Q}_{n}^F, r_n) \) has two or more components with projections onto \( \mathcal{L}_{v(k(n))} \) of order greater than \( K \).

**Lemma 3.8.** There exists \( K \in \mathbb{N} \) such that, with probability 1, the event \( F_K(n) \) occurs for only finitely many \( n \).
Proof. Suppose that $F_K(n)$ occurs. Then there exist distinct components $U = (U_1, U_2)$ and $V = (V_1, V_2)$ in $G(\mathcal{P}_n, \mathcal{Q}_r)$, both with projections onto $\mathcal{L}_v$ of order greater than $K$. Let $U'$ be the union of closed Voronoi cells in $[0, 1]^2$ (relative to $\mathcal{P}_n \cup \mathcal{Q}_r$) of vertices of $U$, and let $V'$ be the union of closed Voronoi cells in $[0, 1]^2$ of vertices of $V$.

The interior of $U'$ and the interior of $V'$ are disjoint subsets of $[0, 1]^2$, and we now show that they are connected sets. Suppose that $X \in U_1$ and $Y \in U_2$ with $\|X - Y\| \leq r_n$; then we claim that the entire line segment $[X, Y]$ is contained in the interior of $U'$, and indeed, let $z \in [X, Y]$, and suppose that $z$ lies in the closed Voronoi cell of some $W \in \mathcal{P}_n \cup \mathcal{Q}_r$. If $W \in \mathcal{P}_n$ then

$$\|W - Y\| \leq \|W - z\| + \|z - Y\| \leq \|X - z\| + \|z - Y\| = \|X - Y\| \leq r_n,$$

so $W \in U$. Similarly, if $W \in \mathcal{Q}_r$ then $\|W - X\| \leq r_n$, so again $W \in U$. Hence, the interior of $U'$ is connected, and likewise for $V'$.

Let $\tilde{V}$ be the closure of the component of $[0, 1]^2 \setminus U'$, containing the interior of $V'$, and let $\tilde{U}$ be the closure of $[0, 1]^2 \setminus \tilde{V}$ (essentially, this is the set obtained by filling in the holes of $U'$ that are not connected to $V'$).

Then $\tilde{U}$ and $\tilde{V}$ are connected closed sets, whose union is $[0, 1]^2$. Therefore, by Lemma 3.7, the set $\partial U := \tilde{U} \cap \tilde{V}$ is connected. Note that $\partial U$ is part of the boundary of $U'$ (it is the ‘exterior boundary’ of $U'$ relative to $\tilde{V}$).

Let $T$ be the set of cube centres $q_i \in \mathcal{L}_v(k(n))$ such that $Q_i \cap (\partial U) \neq \emptyset$. Then $T$ is $\ast$-connected in $\mathcal{L}_v(k(n))$, i.e. for any $x, y \in T$, there is a path $(x_0, x_1, \ldots, x_k)$ with $x_0 = x, x_k = y, x_i \in \mathcal{L}_v(k(n))$, and $\|x_i - x_{i-1}\| = \epsilon r_v(k(n))$ for $1 \leq i \leq k$ (here $\epsilon = \epsilon_v(k(n))$).

Also, for each $q_i \in T$, we claim that $\mathcal{P}_n(Q_i) \mathcal{Q}_r(Q_i) = 0$. Indeed, suppose on the contrary that $\mathcal{P}_n(Q_i) \mathcal{Q}_r(Q_i) > 0$. Then all points of $(\mathcal{P}_n \cup \mathcal{Q}_r) \cap Q_i$ lie in the same component of $G(\mathcal{P}_n, \mathcal{Q}_r, r_n)$. If they are all in $U$ then $Q_i$ and all neighbouring $Q_j$ (including diagonal neighbours) are contained in $U'$. If all points of $(\mathcal{P}_n \cup \mathcal{Q}_r) \cap Q_i$ are not in $U$ then $Q_i$ and all neighbouring $Q_j$ (including diagonal neighbours) are disjoint from $U'$. Therefore, $(\partial U) \cap Q_i = \emptyset$.

We now prove the isoperimetric inequality

$$|T| \geq \left(\frac{K}{2}\right)^{1/2}.$$ (3.11)

To see this, define the width of a nonempty closed set $A \subset [0, 1]^2$ to be the maximum difference between $x$-coordinates of points in $A$, and the height of $A$ to be the maximum difference between $y$-coordinates of points in $A$.

We claim that either the height or the width of $\partial U$ is at least $(K/2)^{1/2} \epsilon r_v(k(n))$. Indeed, if not then $\partial U$ is contained in some square of side $(K/2)^{1/2} \epsilon r_v(k(n))$, and then either $U'$ or $V'$ is contained in that square, so either $U'$ or $V'$ is contained in that square, contradicting the assumption that the projections of $U$ and $V$ onto $\mathcal{L}_v(k(n))$ have order greater than $K$. For example, if the projection of $U$ has order greater than $K$ then at least one of $U_1$ and $U_2$, say $U_1$, has projection of order greater than $K/2$, and then the union of squares of side $\epsilon r_v(k(n))$ centred at vertices in the projection of $U_1$ has total area greater than $(K/2) \epsilon^2 r_v^2(k(n))$, so is not contained in any square of side $(K/2)^{1/2} \epsilon r_v(k(n))$. Thus, the claim holds, and so (3.11) follows by the $\ast$-connectivity of $T$.

For $v, m \in \mathbb{N}$, let $\mathcal{A}_{v,m}$ be the set of $\ast$-connected subsets of $\mathcal{L}_v$ with $m$ elements. By a similar argument as used in the proof of Lemma 3.4 (see also [6, Lemma 9.3]), there are finite
constants $\gamma$ and $C$ such that, for all $\nu, m \in \mathbb{N}$,

$$|A_{\nu,m}^v| \leq C \left(\frac{\nu}{\log \nu}\right)^\gamma m. \quad (3.12)$$

Set $\phi_n := \mathbb{P}[\mathcal{F}_n(Q_i) \mathcal{Q}_{\tau n}(Q_i) = 0]$; this does not depend on $i$. By the union bound and (3.1),

$$\phi_n \leq \exp(-n(\theta r_v(k(n)))^2) + \exp(-\tau n(\theta r_v(k(n)))^2)$$

$$\leq 2 \exp\left(-\frac{\alpha n \log v(k(n))}{v(k(n))}\right)$$

$$\leq 2v(k(n))^\frac{\alpha}{(\tau \wedge 1)^2}$$

$$\leq 3n^{-\frac{(\tau \wedge 1)^2}{2}},$$

where the last inequality holds for all large enough $n$. Using (3.11) and (3.12), we obtain

$$\mathbb{P}[F_K(n)] \leq \sum_{m \geq (K/2)^{1/2}} C \left(\frac{v(k(n))}{\log v(k(n))}\right)^m \phi_n^m \leq 2Cn(3Yn^{-\frac{\alpha}{2}}(\tau \wedge 1)/\pi)^{(K/2)^{1/2}},$$

which is summable in $n$ provided $K$ is chosen so that $\varepsilon^2 \pi^{-1}(\tau \wedge 1)(K/2)^{1/2} > 3$. The result then follows by the Borel–Cantelli lemma.

**Proof of Theorem 3.1.** Choose $K \in \mathbb{N}$ as in Lemma 3.8. Writing ‘i.o.’ for ‘for infinitely many $n$’ (i.e. infinitely often), we have

$$\mathbb{P}[\mathcal{G}^1(n, \tau, r_n) \notin \mathcal{K} \ i.o.] \leq \sum_{m = 1}^{K} \mathbb{P}[\mathcal{F}_n^F, \mathcal{Q}_{\tau n}^F \in \mathcal{K}_{v(k(n)), m} (r_n) \ i.o.] + \mathbb{P}[F_K(n) \ i.o.].$$

By Lemmas 3.6 and 3.8, this is 0.

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