A geometric description of the intermediate behaviour for spatially homogeneous models

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Abstract
A new approach is suggested for the study of geometric symmetries in general
relativity, leading to an invariant characterization of the evolutionary behaviour
for a class of spatially homogeneous (SH) vacuum and orthogonal \(\gamma\)-law perfect
fluid models. Exploiting the \(1 + 3\) orthonormal frame formalism, we express
the kinematical quantities of a generic symmetry using expansion-normalized
variables. In this way, a specific symmetry assumption leads to geometric
constraints that are combined with the associated integrability conditions,
coming from the existence of the symmetry and the induced expansion-
normalized form of Einstein’s field equations (EFE), to give a close set of
compatibility equations. By specializing to the case of a kinematic conformal
symmetry (KCS), which is regarded as the direct generalization of the concept
of self-similarity, we give the complete set of consistency equations for the
whole SH dynamical state space. An interesting aspect of the analysis of the
consistency equations is that, at least for class A models which are locally
rotationally symmetric or lying within the invariant subset satisfying \(N^\alpha_{\alpha} = 0\),
a proper KCS always exists and reduces to a self-similarity of the first or second
kind at the asymptotic regimes, providing a way for the ‘geometrization’ of the
intermediate epoch of SH models.

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1. Introduction

Due to their appealing geometric, kinematic and dynamical structure, spatially homogeneous
(SH or Bianchi) models have received considerable attention in the last three decades.
Apart from the obvious gain of a direct generalization of the standard Friedmann–Lemaître
cosmological model, one of the main reasons for this interest is the fact that Einstein’s field equations (EFE) are reduced to a coupled system of ordinary differential equations. Then by introducing the orthonormal frame formalism and expansion-normalized variables, in order to scale away the influence of the expansion from the overall evolution of the corresponding models, one exploits methods from the theory of dynamical systems to examine their behaviour at early, late and intermediate periods of their history. This approach has shown the significant role of the transitively self-similar SH models since they represent the past and future (equilibrium) states for the majority of evolving vacuum and $\gamma$-law perfect fluid models [1, 2].

However, one can supplement this discussion with a broader consideration of the issues concerning the evolutionary era of SH models. The equilibrium state, whenever it exists, of evolving SH models is geometrically described by a model that admits a homothetic vector field (HVF or self-similarity of the first kind) or a kinematic self-similarity (KSS or self-similarity of the second kind) $X$ which is defined according to

$$L_X u_a = \delta u_a, \quad L_X h_{ab} = 2\alpha h_{ab},$$

(1.1)

where $h_{ab} = g_{ab} + u_a u_b$ is the projection operator normal to the timelike unit vector field $u^a u_a = -1$ and $\alpha, \beta = const$ essentially represent the (constant) time amplification and space dilation [3–7].

Therefore, it will be enlightening to complete the geometric picture and find a relevant way to invariantly (although not uniquely) characterize the intermediate behaviour of the associated models, or to put it equivalently, one is tempted to ask: what was the nature of the generator of the self-similarity (either of the first or second kind) at the time of the intermediate evolution, i.e. what kind of symmetry invariably describes the intermediate behaviour of evolving vacuum or perfect fluid SH models? The natural and intuitive answer (although using heuristic arguments) is that one should expect that this type of symmetry must involve a generalization of the self-similarity. Consequently, the symmetry will represent the smooth transition mechanism between the evolving and the equilibrium states of SH models, i.e. its ‘asymptotic behaviour’ (into the past and the future) will be one of the well-known symmetries.

In order to achieve the above goal one could consider and study general symmetries taking into account the associated local diffeomorphisms and the intrinsic geometric structure of the spacetime manifold [8]. However, from a dynamical point of view, it appears natural to pursue a different direction by making use of the full set of EFE in order to augment and enrich the study of geometric symmetries. Clearly, this approach will provide us the necessary set of compatibility equations which can be studied in each subclass of SH (or even less symmetric) models. Motivated by the above discussion and the great success of studying SH models using elements from the theory of dynamical systems, in the present paper we propose an alternative technique to study geometric symmetries that meets the aforementioned scope. This method fully incorporates the expansion-normalized orthonormal frame approach and give a transparent picture of how a specific symmetry ‘assumption’ can be consistently endowed within a class of models or will produce further constraints, thus losing certain features of the general case. An illustrative and interesting example to which the preceding discussion applies is a recently presented new type of symmetry, namely the concept of kinematic conformal symmetry (KCS). This type of symmetry is regarded as a consistent, with geometry, generalization of the KSS or the case of conformal transformations preserving at the same time the causal structure of the spacetime manifold [9, 10]. Therefore, as a first step towards an invariant and geometric description of SH models, it seems adequate to study the implications from the existence of a KCS.
2. The generic symmetry in SH models

2.1. The 1 + 3 orthonormal frame formalism

Spatially homogeneous models are specified in geometric terms by requiring the existence of a $G_3$ Lie algebra of Killing vector fields (KVFs) $X_\alpha$ with three-dimensional spacelike orbits $S$. This implies the existence of a uniquely defined unit timelike vector field $u_a$ ($u_a u_a = -1$) normal to the spatial foliations $S$:

$$u_{[a;b]} = 0 = u_{a;b} u^b \iff \frac{1}{2} \mathcal{L}_u g_{ab} = u_{a;b} = \sigma_{ab} + \frac{\theta}{3} h_{ab},$$  \hspace{1cm} (2.1)

where $\sigma_{ab}, \theta = u_{a;b} u^b, h_{ab} = g_{ab} + u_a u_b$ are the kinematical quantities associated with the $u^a$ according to the standard 1 + 3 decomposition of an arbitrary timelike congruence [11]. Because $u^a$ is irrotational and geodesic, there exists a time function $t(x^a)$ such that $u^a = \delta^a_0$, i.e. each value of $t$ essentially represents the hypersurfaces $S$.

As far as the dynamical structure is concerned, from a cosmological point of view, it is sufficient to focus our study on SH models with an orthogonal perfect fluid assuming a $\gamma$-law equation of state. Therefore the EFE become

$$R_{ab} = \frac{3\gamma - 2}{2} \rho u_a u_b + \frac{2 - \gamma}{2} \rho h_{ab}. \hspace{1cm} (2.2)$$

It follows that the timelike vector field $u^a$ is identified with the average fluid flow velocity.

On using the orthonormal frame formalism in SH cosmologies the starting point is to introduce a set of four linearly-independent vector fields $\{e_a\}$ and their dual 1-forms $\{\omega^a\}$ which are invariant under the three-parameter group of isometries:

$$[X_\alpha, e_a] = 0, \hspace{1cm} [X_\alpha, \omega^a] = 0. \hspace{1cm} (2.3)$$
Then, by performing a time-dependent rescaling of \([e_a]\), we can write (locally) the metric tensor in a manifestly Minkowskian form:

\[
d s^2 = \eta_{ab} \omega^a(t, x^\alpha) \omega^b(t, x^\alpha).
\] (2.4)

In this case the invariant vector fields \([e_a]\) and the connection forms \(\Gamma^a_{bc} \omega^c\) (where \(\nabla_{e_a} e_b = \Gamma^a_{bc} e_c\)) satisfy the commutation and the first Cartan structure equations:

\[
[e_a, e_b] = \gamma_c^{ab}(t) e_c, \quad d\omega^a = \Gamma^a_{bc} \omega^b \wedge \omega^c = -\frac{1}{2} \gamma^a_{bc} \omega^b \wedge \omega^c, \quad (2.5)
\]

where \(d\) and \(\wedge\) are the usual exterior derivative and exterior product of 1-forms respectively.

It follows that the commutation functions \(\gamma^a_{bc}\) and the connection coefficients \(\Gamma^a_{bc}\) are related via

\[
\Gamma^a_{bc} = \frac{1}{2} \left[ \eta_{ad} \gamma^d_{cb} + \eta_{bd} \gamma^d_{ac} - \eta_{cd} \gamma^d_{ba} \right] \Leftrightarrow \gamma^a_{bc} = -\left( \Gamma^a_{bc} - \Gamma^a_{cb} \right). \quad (2.6)
\]

Furthermore, the requirement of the constancy of the metric under covariant differentiation implies that \(\Gamma^a_{bc} = 0\) where \(\Gamma^a_{bc} = \eta_{da} \Gamma^d_{bc}\).

The above definitions suggest that the tetrad form of the covariant derivative of every tensor field is written in the well-known way [11]:

\[
\nabla_{e^c} K^{a}_{i...} = e^c_{i...} + \Gamma^a_{dc} K^{d}_{i...} + \cdots - \frac{1}{2} \Gamma^d_{ic} K^{a}_{d...} - \cdots, \quad (2.7)
\]

where \(e^c_{i...}\) is regarded as the directional derivative of the functions \(K^{a}_{i...}\) along the vector field \(e_c^i\).

Because in SH (vacuum or perfect fluid) models there always exists the preferred and well-defined timelike vector field \(u^a\) associated with a congruence of curves normal to the three-dimensional spacelike orbits \(S\) of homogeneity, it is natural to select it as the timelike frame vector, i.e. \(e_0 = u\). It follows from equation (2.5) that the kinematical quantities of the timelike congruence \(u\) are directly related to the commutation functions \(\gamma^a_{bc}\) according to [11, 12]

\[
\gamma^0_{0 \alpha} = \dot{u}_\alpha = 0, \quad \gamma^0_{\alpha \beta} = -2 \epsilon_{\alpha \beta \gamma} \omega^\gamma = 0, \quad (2.8)
\]

\[
\gamma^\alpha_{0 \alpha} = -\frac{1}{3} \theta \delta^\alpha_{\alpha} + \epsilon^\alpha_{\alpha \beta} \Omega^\beta, \quad (2.9)
\]

where \(\Omega^\gamma\) is the local angular velocity of the spatial frame with respect to a Fermi-propagated frame along \(e_0\). On the other hand, the spatial components of \(\gamma^a_{bc}\) are decomposed as

\[
\gamma^a_{\beta \gamma} = \alpha_{\beta \gamma} - \epsilon^\beta_{\alpha \mu} n^\mu, \quad (2.10)
\]

leading to Bianchi class A and B models according to whether the quantity \(\alpha_{\beta}\) vanishes or not.

Combining equations (2.6) and (2.8)–(2.10) we easily find

\[
\Gamma^a_{\beta \alpha} = \frac{\theta}{3} \delta^a_{\alpha \beta} + \sigma_{\alpha \beta}, \quad \Gamma^a_{0 \alpha} = 0, \quad \Gamma^a_{\alpha 0} = \epsilon_{\alpha \beta \gamma} \Omega^\gamma \quad (2.11)
\]

\[
\Gamma^a_{\beta \gamma} = 2 \alpha_{[\alpha \beta \gamma]} + \epsilon_{\gamma \beta \alpha} n^\mu + \frac{1}{2} \epsilon_{\beta \alpha \mu} n^\mu. \quad (2.12)
\]

The complete set of the gravitational field equations is expressed in terms of the shear components \(\sigma_{\alpha \beta}\), the expansion \(\theta\) and the spatial curvature quantities \(\alpha_{\beta}, n_{\alpha \beta}\), by utilizing the Ricci identity for \(u^a\), the Jacobi identities for the frame vector fields and the Bianchi identities.
2.2. Expansion normalized variables

Of particular importance in the exploration of the asymptotic dynamics of SH models, is the reformulation of the complete set of orthonormal frame equations, as an autonomous system of first-order ordinary differential equations. This can be done by defining a set of expansion-normalized (dimensionless) variables which results in the decoupling of the evolution equation of $H = \theta / 3$ from the rest of the evolution equations:

$$\frac{dt}{d\tau} = \frac{1}{H}, \quad \frac{dH}{d\tau} = -(1 + q)H$$

(2.13)

where $q$, $H$ are the deceleration and Hubble parameter respectively and $\tau$ is the dimensionless time variable.

The complete set of equations can be written in the form [13]

$$\Sigma_{\alpha\beta} = -(2 - q)\Sigma_{\alpha\beta} + 2e^{\mu\nu}_{(\alpha}\Sigma_{\beta)\mu}R_{\nu} - S_{\alpha\beta}$$

(2.14)

$$N_{\alpha\beta} = qN_{\alpha\beta} + 2\Sigma^{\mu}_{(\alpha}N_{\beta)\mu} + 2e^{\nu\mu}_{(\alpha}N_{\beta)\mu}R_{\nu}$$

(2.15)

$$A_{\alpha} = QA_{\alpha} - \Sigma^{\mu}_{\alpha}A_{\mu} + e^{\nu\mu}_{\alpha}A_{\mu}R_{\nu}$$

(2.16)

$$\Omega' = \Omega[2q - (3\gamma - 2)]$$

(2.17)

where

$$S_{\alpha\beta} = 2\Sigma_{\alpha\gamma}N_{\beta\gamma} - \Sigma_{\alpha\gamma}N_{\beta\gamma} - \frac{1}{3}(2\Sigma_{\gamma\alpha}N_{\beta\gamma} - \Sigma_{\gamma\alpha}N_{\beta\gamma})\delta^{\alpha\beta} - 2e^{\mu\nu}_{(\alpha}N_{\beta)\mu}A_{\nu}$$

(2.18)

and a prime denotes a derivative w.r.t. $\tau$.

The above system is subjected to the algebraic constraints:

$$\Omega = 1 - \frac{1}{6}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} - K$$

(2.19)

$$3\Sigma^{\beta}_{\alpha}A_{\beta} - e^{\mu\nu}_{\alpha}\Sigma^{\beta}_{\mu}N_{\beta\nu} = 0$$

(2.20)

where

$$K = \frac{1}{12}(2\Sigma^{\gamma}_{\alpha}N_{\beta\gamma} - \Sigma^{\gamma}_{\alpha}N_{\beta\gamma})\delta^{\alpha\beta} + A_{\gamma}A'$$

(2.21)

and the deceleration parameter is given by the relation

$$q = \frac{1}{3}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} + \frac{1}{3}\Omega[(3\gamma - 2)]$$

$$= 2(1 - K) + \frac{1}{3}\Omega[3(\gamma - 2)].$$

(2.22)

2.3. The generic symmetry in expansion-normalized variables

The folklore for investigating the implications of the existence of geometric symmetries in general relativity can be divided into two main categories. The first category is a geometric approach in which we study symmetries taking into account the holonomy group structure of the spacetime manifold together with the associated local diffeomorphisms [8]. In the second category one formulates the necessary and sufficient conditions, coming from the existence of the symmetry, in a covariant way and study their consequences in the kinematics and dynamics of the corresponding model [14]. Of course one could also deal directly with the resulting system of partial differential equations (pdes), presupposing a specific geometrical and dynamical configuration which renders the symmetry equations more tractable. Obviously, this approach undergoes many disadvantages and pathologies. One of the serious stumbling blocks is the fact that as the generality of a model is increased (i.e. the underlying geometric structure of the model is less symmetric than the SH geometry) the symmetry pdes are
progressively nonlinear and very often lead to solutions of the EFE which are immediately ruled out physically.

Here, we suggest an alternative approach for the study of geometric symmetries which fully exploits the well-established orthonormal frame formalism in terms of expansion-normalized variables. Although this technique will be applied to a specific symmetry ‘assumption’ (as we shall see in the next section this is not really an assumption, at least for a class of models, but a consequence of the complete set of EFE) it can be used in a more general context for the study of other types of important symmetries [15].

Let us consider an arbitrary vector field $X$ and express its components in terms of the frame vector fields $e_i$:

$$ X = X^a e_a = \lambda e_0 + X^a e_a, $$

where $\lambda$, $X^a$ are continuously differential functions of the spacetime manifold.

The first derivatives of $X$ are decomposed into irreducible symmetry kinematical parts $\{\psi, H_{ab}, F_{ab}\}$ in the standard way:

$$ \nabla_b X_a = \psi g_{ab} + H_{ab} + F_{ba}, $$

where

$$ 4\psi \equiv \nabla_k X^k, $$

$$ H_{ab} = \left[ \nabla_b X_a - \frac{1}{4} (\nabla_k X^k) g_{ab} \right], $$

$$ F_{ab} = - \nabla_b X_a $$

are the conformal factor, the traceless symmetric part and the antisymmetric part, respectively.

Using the definition (2.7) and equations (2.11), (2.12), (2.23)–(2.27), we find the tetrad analogue of the kinematical quantities. If we further invoke the expansion-normalized differential operators $\partial_a \equiv e_a/H$ in the general expressions we finally obtain

$$ 4\psi = H[\partial_0(\lambda) + \partial_a(X^a) + 3\lambda - 2A_\alpha X^\alpha] $$

(2.28)

$$ H_{00} = \frac{H}{4} \left[ \partial_0(\lambda) + \partial_a(X^a) + 3\lambda - 2A_\alpha X^\alpha \right] $$

(2.29)

$$ 2H_{0a} = H \left[ -3\partial_0(\lambda) + \partial_a(X^a) - (\delta_{ay} + \Sigma_{ay})X^y + \epsilon_{a\beta y} R^\gamma X^\beta \right] $$

(2.30)

$$ H_{a\beta} = H \left[ \partial_\beta(X_a) + (\delta_{a\beta} + \Sigma_{a\beta})\lambda - [A_\gamma \delta_{a\beta} - A_\beta \delta_{a\gamma}] X^\gamma + \epsilon_{a\beta y} N^\gamma X^\gamma \right] - \psi \delta_{a\beta} $$

(2.31)

$$ 2F_{0a} = H \left[ \partial_0(\lambda) + \partial_a(X^a) + (\delta_{a\gamma} + \Sigma_{a\gamma})X^\gamma - \epsilon_{a\beta y} R^\gamma X^\beta \right] $$

(2.32)

$$ F_{a\beta} = H \left\{ -\partial_\beta(X_a) - \left[ A_\alpha \delta_{\beta\gamma} + \frac{1}{2} \epsilon_{\alpha\beta y} N^\gamma \right] X^\gamma \right\}. $$

(2.33)

Equations (2.28)–(2.33) represent the *symmetry kinematical quantities in terms of expansion-normalized variables* in SH geometries and can be used in order to have a first hint of how the dynamics affects the geometry of SH models (or vice versa). We note that one could choose to define expansion-normalized symmetry kinematical quantities in a similar way as we do for the shear and spatial curvature variables. This would be convenient for high symmetries where first or second derivatives of $\{\psi, H_{ab}, F_{ab}\}$ are involved. However, for the cases in which we are interested in the present paper the use of (2.28)–(2.33) will be satisfactory.

We conclude this section by pointing out that, because any type of symmetry assumption is described in terms of geometric constraints, these are inherited by the dynamics through the EFE (2.2). Therefore in order to visualize how the symmetry further interacts with the
dynamics, it is necessary to determine the effect of the former on the Ricci tensor. By using the well-known commutation relation between the connection and the Lie derivative \[16\]:
\[
\mathcal{L}_X^{\Gamma^r_{bc}} = \frac{1}{2} \mathcal{G}^{ar} [\nabla_r (\mathcal{L}_X g_{bc}) + \nabla_b (\mathcal{L}_X g_{cr}) - \nabla_c (\mathcal{L}_X g_{br})],
\]
\[
\mathcal{L}_X R_{ab} = \nabla_c [\Gamma^r_{cab}] - \nabla_b [\Gamma^r_{ac}] \]
and the defining equation (2.24), we can show after a straightforward calculation that
\[
\mathcal{L}_X R_{ab} = -2 \nabla_b \nabla_a \psi - g_{ab} \nabla_c \nabla^c \psi + 2 \nabla_k \nabla^k H^a_b - \nabla_c \nabla^c H_{ab}. \tag{2.34}
\]
Essentially, the last equation represents a set of integrability conditions which can be used to check the consistent existence of every type of symmetry assumption.

3. Generalized conformal symmetries in SH models

Although there exist (and can be defined) a sufficiently large number of symmetries, the most important type of them (to date) appears to concern the constant scale invariance of the metric represented by the existence of a proper HVF \([6, 7, 17]\). For SH vacuum and perfect fluid models this is indeed the case due to the profound relevance of homothetic models with the equilibrium points of the SH state space \([18–21]\).

Recently, a new type of symmetry has been suggested, the so-called bi-conformal transformations which can be interpreted as generalizing the concepts of the self-similarity and the conformal motions. In the present work we will be concerned with an interesting subcase, that is, the so-called kinematic conformal symmetry (KCS). In particular, a smooth vector field \(X\) is the generator of a KCS \(\text{iff}\) the following relations hold \([9]\)
\[
\mathcal{L}_X u_a = \delta u_a, \quad \mathcal{L}_X h_{ab} = 2 \alpha h_{ab} \tag{3.1}
\]
or, in terms of the metric:
\[
\mathcal{L}_X g_{ab} = 2 \alpha g_{ab} + 2 (\alpha - \delta) u_a u_b, \tag{3.2}
\]
where \(\alpha, \delta\) are smooth functions that we shall call symmetry scales and \(h_{ab} = g_{ab} + u_a u_b\) is the projection operator perpendicular to the timelike congruence \(u_a\).

Combining equations (2.24) and (3.2) we express the symmetry kinematical parts in the form
\[
\psi = \frac{3 \alpha + \delta}{4}, \quad H_{ab} = \frac{\alpha - \delta}{4} (g_{ab} + 4 u_a u_b). \tag{3.3}
\]
It can be easily observed that when \(\alpha = \delta\) the KCS reduces to a conformal vector field (CVF) which, due to the equation (3.1), is necessarily inheriting, i.e. the integral curves of \(u^a\) are mapped conformally by the CVF \(X\) \([22]\). As a result the Lie algebra \(I\) of inheriting CVFs is always a subalgebra of the Lie algebra of KCS which shall be denoted as \(B\). Moreover, when the symmetry scales \(\alpha, \delta\) are both (different) constants we recover the case of a kinematic self-similarity.

3.1. Symmetry constraints

Clearly there exists a direct dependence between the existence of a KCS, as well as any other type of symmetry assumption, and a specific cosmological model. This mutual influence is reflected in the induced geometric, kinematic and dynamic constraints which are imposed due to the intrinsic nature of the symmetry and/or as a consequence of the specific geometric and dynamical structure of the spacetime manifold. In the case of a KCS these constraints are
derived from the general relations (2.28)–(2.31) and the symmetry assumptions (3.3):

\[3\alpha = H[\partial_\alpha (X^\alpha) + 3\lambda - 2A_\alpha X^\alpha]\]  

(3.4)

\[-4H \partial_\alpha (\lambda) + 3\alpha + \delta = 3(\alpha - \delta) \Rightarrow \delta = H \partial_\alpha (\lambda)\]  

(3.5)

\[-\partial_\alpha (\lambda) + \partial_\alpha (X_\alpha) - (\delta_{\alpha\gamma} + \Sigma_{\alpha\gamma})X^\gamma + \epsilon_{\alpha\beta\gamma} R^\gamma X^\beta = 0\]  

(3.6)

\[H[\partial_\beta (X_\alpha) + (\delta_{\alpha\beta} + \Sigma_{\alpha\beta})\lambda - [A_{\gamma}(\alpha\delta\beta) - A_{(\alpha\delta\beta)}^\gamma]X^\gamma + \epsilon_{\gamma\delta\beta}(\alpha) N_{\delta\beta}]X^\gamma + \alpha \delta_{\alpha\beta} = 0.\]  

(3.7)

We should emphasize that the above set of constraints must be augmented with the associated Jacobi identities satisfied by the generators \([X_\alpha, X_\beta]\) that constitute the Lie algebra \(B\) of KCS. This will require the determination of the dimension of \(B\) and the assumption that the former is finite since there are cases for which \(B\) is infinite dimensional [9]. Nevertheless we shall not pursue the problem in full generality and we will restrict our considerations to the case where the dimension of \(B\) is finite and equal to 4 (together with the \(\varphi_7\) of KVFs which can be seen as ‘trivial’ KCS). In fact this assumption seems reasonable and not even restrictive, from a geometrical point of view, because we intend to describe the ‘intermediate behaviour’ of a (proper) self-similarity of the first or second kind in a way that is identified with the presence of a KCS. In this case it can be shown from Jacobi identities [23] that the Lie bracket of a KCS with the KVFs is always a linear combination of the latter. It turns out that the scalar \(\lambda\) and the symmetry scales \(\alpha, \delta\) (due to equation (3.1)) are spatially homogeneous:

\[\partial_\alpha \lambda = \partial_\alpha \alpha = \partial_\alpha \delta = 0.\]  

(3.8)

In addition we point out that equation (3.2) is a consequence of (3.1) since the former suppresses the scale amplification of both time and space which is essentially represented by the latter. However, the spatial homogeneity condition (3.8) ensures that the definitions (3.2) and (3.1) are equivalent.

### 3.2. Integrability conditions

Due to the geometric character of the KCS to preserve the causal structure of the spacetime manifold, it is natural to expect that the associated transformation group will respect, to some level, the intrinsic properties of SH models. Therefore, one should expect that the presence of a KCS will induce weaker constraints than those in the case of CVFs or the KSS. In order to confirm this, it will be required to determine the integrability conditions adapted to the case of SH models filled, in general, with a \(\gamma\)-law non-tilted perfect fluid.

As we have shown, the scale functions and the trace \(\psi\) are both spatially homogeneous which implies that equation (2.34), due to (3.3), can be written as

\[L_X R_{ab} = 3[H(\delta - 2\alpha) - \bar{\alpha}]u_a u_b + [2(\alpha - \delta)(3H^2 + \bar{H}) + \bar{\alpha} + H(6\alpha - \bar{\delta})]h_{ab} + [(3\alpha - \delta)\sigma_{ab} + (\alpha - \delta)(6H\sigma_{ab} + 2\alpha \sigma_{ab})]\]  

(3.9)

where a dot ‘·’ denotes differentiation w.r.t. \(u^a\).

On the other hand, the EFE (2.2) and the relations (3.1) give

\[L_X R_{ab} = \frac{3y - 2}{2}[X(\rho) + 2\delta \rho]u_a u_b + \frac{2 - y}{2}[X(\rho) + 2\delta \rho]h_{ab}.\]  

(3.10)

The complete set of integrability conditions, coming from the existence of a KCS, follows by equating (3.9) and (3.10):

\[3[H(\delta - 2\alpha) - \bar{\alpha}] = \frac{3y - 2}{2}[X(\rho) + 2\delta \rho].\]  

(3.11)
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\[ 2(\alpha - \delta)(3H^2 + \dot{H}) + \ddot{\alpha} + H(6\dot{\alpha} - \dot{\delta}) = \frac{2}{\rho}[X(\rho) + 2\alpha\rho] \]  
\[ (3\dot{\alpha} - \dot{\delta})\sigma_{ab} + (\alpha - \delta)(6H\sigma_{ab} + 2\sigma_{ab}) = 0. \]

We observe that in the case of a CVF where \( \alpha = \delta \), the last equation implies the well-known result \( \dot{\alpha}\sigma_{ab} = 0 \), i.e. either the (necessary inheriting) CVF reduces to a HVF or the spacetime is the Robertson–Walker spacetime [24].

The system of equations (3.11)–(3.13) can be conveniently reformulated in expansion-normalized variables in order to append them in the autonomous set (2.14)–(2.17) and the symmetry equations (3.4)–(3.7). For simplification purposes we define the dimensionless symmetry scale functions:

\[ \tilde{\alpha} = \frac{\alpha}{H}, \quad \tilde{\delta} = \frac{\delta}{H}. \]

Then, taking into account equations (2.14) and (2.17), an appropriate combination of (3.11)–(3.13) eliminates the second-order time derivatives and give evolution equations for \( \tilde{\alpha} \) and \( \tilde{\delta} \):

\[ 4[\ddot{\alpha} - (q + 1)\dot{\alpha}] = -2(\ddot{\alpha} - \ddot{\delta})(2 - q) + [(3\gamma - 2)\ddot{\delta} + 3(2 - \gamma)\ddot{\alpha} - 6\lambda\gamma] \Omega \]
\[ [(3\dot{\alpha} - \dot{\delta}')(q + 1)(3\ddot{\alpha} - \ddot{\delta})] + 2(\ddot{\alpha} - \ddot{\delta})S_{\alpha\beta} = 0. \]

It is interesting to note that equation (3.16) excludes the existence of a proper KSS in Bianchi vacuum or perfect fluid cosmologies with \( S_{\alpha\beta} \neq 0 \). The constraint \( S_{\alpha\beta} = 0 \) is identically satisfied in Kasner type I models which are well known to admit a (proper) self-similarity of the second kind [5, 25]. This result implies that a KSS fails to be considered as a generic candidate to describe the intermediate behaviour of general SH models, but rather one should explore the possibility for a symmetry, representing a direct generalization of the conformal motions, in such a way that its asymptotic behaviour is the self-similarity of the first or second kind. We shall demonstrate in the next section that the case of a KCS provides evidence towards a satisfactory (but not conclusive) answer to this question, for a significant subclass of evolving SH vacuum and perfect fluid models.

4. Application to SH models of class A

Spatially homogeneous cosmologies of class A are defined by the condition \( A^\alpha = 0 \), which due to equation (2.20) implies that the shear and spatial curvature matrices \( \Sigma_{\alpha\beta}, N_{\alpha\beta} \) commute. Therefore, \( \Sigma_{\alpha\beta}, N_{\alpha\beta} \) have a common eigenframe and we can write

\[ \Sigma_{\alpha\beta} = \text{diag}(\Sigma_{11}, \Sigma_{22}, \Sigma_{22}), \quad N_{\alpha\beta} = \text{diag}(N_{11}, N_{22}, N_{22}). \]

Furthermore, the evolution equations (2.14), (2.15) and equation (4.1) show that \( R^\alpha = 0 \), which means that the common eigenframe of \( \Sigma_{\alpha\beta}, N_{\alpha\beta} \) is also Fermi-propagated. Under these conditions the study of the class A models is considerably simplified and permits one to investigate the set of consistency equations constituting of the EFE (2.14)–(2.17), the geometric constraints (3.4)–(3.7) and the integrability conditions (3.15)–(3.16) in a straightforward way. To illustrate the method that could be used for the consistency checking we outline some applications by restricting our study to type I, II and VI0 models.

However, before we proceed, it will be useful to give the corresponding analysis for the flat Friedmann–Lemaître model \( \mathcal{F} \) since it represents the past or future attractor for several SH models of class A.
4.1. Flat Friedmann–Lemaître $\gamma$-law perfect fluid models

It is well known that the (flat) Robertson–Walker spacetime
\[
\mathbf{ds}^2 = -dt^2 + S(t)^2 (dx^2 + dy^2 + dz^2)
\]
(4.2)
has a variety of ways for an invariant characterization of its structure. For example, kinematically, it is defined by the vanishing of the shear, vorticity and acceleration of the preferred timelike congruence $u^a$ which, due to equations (2.19) and (2.22), implies
\[
\Omega = 1, \quad q = \frac{3\gamma - 2}{2}.
\]
(4.3)
On the other hand, one can use geometric terms and describe the Friedmann–Lemaître model by the existence of nine proper CVFs, one of which is parallel to $u^a$ [26, 27]. Although in the case of a $\gamma$-law perfect fluid model a proper HVF and KSS exist [5] (lowering the dimension of the Lie algebra of conformal motions to eight) this does not mean that we have exhausted all the possible (geometric) ways for the description of Friedmann–Lemaître models. As a result one should expect that a KCS will also exist without imposing extra geometrical or dynamical restrictions (which is often the case for other types of symmetries). Indeed from the constraints (3.4)–(3.7) we find
\[
3(\tilde{\alpha} - \lambda) = \partial_a (X^a), \quad \tilde{\delta} = \lambda',
\]
(4.4)
\[
\partial_0 X_a - X_a = 0, \quad \partial_\beta X_{\alpha} + (\lambda - \tilde{\alpha}) \delta_{\alpha\beta} = 0.
\]
(4.5)
Using the set of equations (4.3)–(4.5) and the integrability condition (3.15) we can show that a KCS always exists in a Friedmann–Lemaître model and the symmetry scale $\tilde{\alpha}$ is given in terms of the temporal component
\[
(\lambda - \tilde{\alpha})' = (q + 1)(\lambda - \tilde{\alpha}) = \frac{3\gamma}{2}(\lambda - \tilde{\alpha}).
\]
(4.6)
From the above equations we determine the exact form of the KCS in the Robertson–Walker spacetime:
\[
X = \lambda(t)\partial_t + c(x\partial_x + y\partial_y + z\partial_z)
\]
(4.7)
where
\[
\alpha(t) = \frac{2\lambda(t) + 3ct}{3\gamma t}
\]
(4.8)
and the scale factor is given by $S(t) = t^{2/3\gamma}$. We note that, after a suitable change of the basis of the KCS Lie algebra, we can set the constant $c = 0$, which implies that $X$ is also parallel to the fluid velocity $u^a$.

4.2. Type I models

In this case, $N_{\alpha\beta} = 0 = S_{\alpha\beta}$ and equation (3.16) gives
\[
3\tilde{\alpha} - \tilde{\delta} = \tilde{c} \equiv \frac{c}{H} \iff 3\alpha - \delta = c,
\]
(4.9)
where $c$ is an arbitrary constant.

In addition, equation (3.5) implies that
\[
\tilde{\delta} = \lambda'.
\]
(4.10)
We should point out that the symmetry constraints are necessary to ensure the existence of a KCS. However, this does not imply that they will be preserved along the integral curves of the
timelike vector field $e_0$. Therefore, we must propagate equations (3.4)–(3.7) in order to retain the existence of a KCS in every spacelike hypersurface $S$. After a short calculation and the use of the commutator relations (2.9), we obtain

$$\lambda' - (q + 1)\lambda = \tilde{\alpha}' - (q + 1)\tilde{\alpha}$$

(4.11)

$$[(q - 2)\lambda' + \lambda' - (q + 1)\lambda]\Sigma_{\alpha\beta} + \Lambda_{\alpha\beta} = 0,$$

(4.12)

where

$$\Lambda_{\alpha\beta} = \Sigma_{\gamma(\alpha} X_{\beta)}^{\gamma} - \Sigma_{\gamma(\alpha} X_{\beta)}^{\gamma}$$

(4.13)

and $X_{\gamma,\beta} \equiv \partial_{\beta}X_{\gamma}$.

From the $\alpha\beta$-component of (4.12) it follows that

$$(q - 2)\lambda + \lambda' - (q + 1)\lambda = 0 \quad \Leftrightarrow \quad \lambda' = 3\lambda,$$

(4.14)

while the equation $\Lambda_{\alpha\beta} = 0$ expresses the spatial components of the KCS in terms of the shear variables.

In summary we have shown that every type I $\gamma$-law perfect fluid model always admits a four-dimensional group of kinematic conformal symmetries. An interesting feature of this result concerns the past ‘asymptotic behaviour’ of the KCS. In particular, from equation (4.14) we observe that $\lambda' - (q + 1)\lambda = (2 - q)\lambda$; hence at the equilibrium point $(q = 2)$ we have $\lambda' - (q + 1)\lambda = 0$, i.e. $\alpha, \delta = \text{const}$ and the KCS reduces to a proper KSS with $\lambda = \text{const} \times H^{-1}$.

The future state of the KCS is treated similarly. The type I models approach at late times the Friedmann–Lemaître model $F$, so equation (4.14) is trivially satisfied and the temporal component is given in equation (4.14). This implies that $\alpha' - (q + 1)\alpha \neq 0 \Rightarrow \alpha, \delta \neq \text{const}$ as expected. Therefore, one could argue that the existence of a proper KCS describes, in a geometric fashion, the intermediate behaviour of the evolving type I models.

We conclude the type I case by giving the exact form of the KCS in local coordinates. The general $\gamma$-law perfect fluid solution can be conveniently written in the form [1]

$$ds^2 = -A^{2(\gamma - 1)}dt^2 + \sum_{\alpha} l^{2p_\alpha} A^{2(2/3 - p_\alpha)}(dx^\alpha)^2,$$

(4.15)

where the function $A^{2\gamma} = k + m^2t^2 - \gamma$ and $k, m$ are constants.

It follows that the (local) coordinate form of the KCS is

$$X = \lambda(t)\partial_t + (ck + c_1)x\partial_x + c_2y\partial_y + c_3z\partial_z$$

(4.16)

where

$$\lambda(t) = \frac{tkeA}{(p_2 + 2p_3 - 1)(A - tA_f)}$$

(4.17)

and the constant $c_2$ is given by

$$c_2 = \frac{c_3(p_2 + 2p_3 - \gamma) + kc(p_3 - p_2)}{p_2 + 2p_3 - 1}$$

(4.18)

where we have used the well-known relations

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1$$

(4.19)

satisfied by the Kasner exponents $p_\alpha$.

We also note that at early times $\lambda(t) \approx t$ and $\alpha, \delta = \text{const}$ whereas at late times $k = 0$ and $\alpha, \delta \neq \text{const}$, which confirm the reduction of the KCS to a proper KSS [25] and to the KCS (4.7), respectively.
4.3. Type II models

The Bianchi type II invariant set is characterized by the conditions $N_1 > 0$ and $N_2 = N_3 = 0$. We find it convenient to collect the consistency equations as follows from (2.14)–(2.17), (3.4)–(3.7) and (3.15)–(3.16):

\[ \lambda' - (q + 1)\lambda = \tilde{\alpha}' - (q + 1)\tilde{\alpha} \quad (4.20) \]

\[ 0 = (\lambda' - 3\lambda)\Sigma_{\alpha\beta} + \Delta_{\alpha\beta} + \left[ 2\Sigma_{11}X' + \Sigma'_{11}X' \right]N_1\varepsilon_{11}(\alpha\delta_{\beta}) \]

\[ - \left( \frac{2}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta \right)N_1^2\lambda \quad (4.21) \]

\[ 4[\alpha' - (q + 1)\tilde{\alpha}] = -2(\tilde{\alpha} - \delta)(2 - q) + [(3\gamma - 2)\delta + 3(2 - \gamma)\tilde{\alpha} - 6\lambda\gamma]\Omega \quad (4.22) \]

\[ 0 = [(3\tilde{\alpha} - \delta)' - (1 + q)(3\tilde{\alpha} - \delta)]\Sigma_{\alpha\beta} - \left( \frac{2}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta \right)N_1^2(\tilde{\alpha} - \delta) \quad (4.23) \]

\[ \delta = \lambda' \quad (4.24) \]

\[ 1 - \frac{1}{12}N_1^2 - \Sigma^2 - \Omega = 0, \quad q = 2\Sigma^2 + \frac{3\gamma - 2}{2}\Omega \quad (4.25) \]

\[ \Sigma'_{\alpha\beta} = (q - 2)\Sigma_{\alpha\beta} - \left( \frac{2}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta - \frac{1}{3}\delta^3_\alpha\delta^3_\beta \right)N_1^2 \quad (4.26) \]

\[ N'_1 = (q + 2\Sigma_{11})N_1. \quad (4.27) \]

Note that, in complete analogy with the type I models, equations (4.20) and (4.21) are the result of the propagation of the symmetry constraints (3.4) and (3.7) along $\partial_0$.

A quick observation can be made due to the form of (4.21) or (4.23). For example, the first equation implies

\[ (\lambda' - 3\lambda)\Sigma_{22} = (\lambda' - 3\lambda)\Sigma_{33} = -\frac{1}{4}N_1^2\lambda, \quad (4.28) \]

where for $\Sigma_{22} = 0$ the KCS turns to the KVF of Bianchi type II models.

Equation (4.28) means that the existence of a KCS is compatible only with type II models which are LRS, i.e., only within the invariant subset $S^*_1(II)$. In order to determine the symmetry scale $\tilde{\alpha}$ we make use of the $\alpha\beta$-components of equation (4.21). Then, from the spatial commutators (2.10) we get $X_{2,1} = X_{3,1} = 0$ and the consistency of the remaining set of equations is assured provided that

\[ \tilde{\alpha} = \lambda(4\Sigma_{22} + 1). \quad (4.29) \]

We have proved that every type II evolving vacuum or $\gamma$-law perfect fluid model that belongs to the invariant subset $S^*_1(II)$ can be invariantly characterized by the existence of a four-dimensional group of kinematic conformal symmetries.

Regarding the ‘asymptotic behaviour’ of the KCS in type II models, it is straightforward to show that in $S^*_1(II)|_{\Omega=0}$ the following relations hold:

\[ \tilde{\alpha}' = (q + 1)\tilde{\alpha} \propto \Sigma^3_{22} - 1 \]

\[ (3\tilde{\alpha}' + \tilde{\delta}') - (q + 1)(3\tilde{\alpha}' + \tilde{\delta}) \propto \Sigma^3_{22} - 1. \]

Therefore, at the equilibrium points $\Sigma_{22} = \pm 1$ and $N_1 = 0$, the KCS becomes a proper KSS as expected, since at the asymptotic regimes, all models within $S^*_1(II)|_{\Omega=0}$ approach some vacuum Kasner model [28].

1 We recall the traceless property of $\Sigma_{\alpha\beta}$, i.e. $\Sigma_{11} + \Sigma_{22} + \Sigma_{33} = 0$. 


On the other hand, it is well known that non-vacuum models \( S^1_1(\text{II})|_{\Omega>0} \) are all future asymptotic to the homothetic Collins model \( P^*_t(\text{II}) \) for which we have proved that does not admit a proper KSS, and past asymptotic to a Kasner model \( K \) or to the Friedmann–Lemaître model \( F \). Consequently, every orbit in \( S^1_1(\text{II})|_{\Omega>0} \) joins two (first and second kind) self-similar equilibrium points; hence we expect that the KCS will be reduced to a proper HVF and a KSS, except for the case where the orbit lies in the one-dimensional unstable manifold of \( F \). This model has as past attractor the point \( F \) and the KCS will reduce to the associated KCS of the Robertson–Walker spacetime (equation (4.7)).

Indeed, using equations (4.24)–(4.29) we find
\[
\tilde{a} - \tilde{\delta} = \frac{\lambda[N^1_1 + 6\Sigma_{22}(2\Sigma_{22} - 1)]}{3\Sigma_{22}}.
\]
From the last equation we can show easily that \((\tilde{a} - \tilde{\delta})|_{P^*_t(\text{II})} = 0\), which implies that
\[
\lim_{t \to +\infty} (\tilde{a} - \tilde{\delta}) = 0.
\]
Similarly, we can show that at the equilibrium point \( K \) the symmetry scales \( a, \delta \) become constants with \( \alpha|_K = \delta|_K \) and at \( F \) we have \( \alpha = \lambda H, \lambda' = 3\lambda \).

The exact form of the KCS in local coordinates is found to be
\[
X = \lambda(t)\tilde{a} + (2cx + by)\tilde{\alpha}_x + cy\tilde{\alpha}_y + (cz - 1)\tilde{\alpha}_z,
\] (4.30)
where \( c \) is constant and
\[
\lambda(t) = \frac{2cBC}{BC_{,\lambda} - CB_{,\lambda}}.
\] (4.31)
The LRS type II metric is
\[
dx^2 = -A^2\, dt^2 + B(dx + b_2\, dy)^2 + C(dy^2 + dz^2)
\] (4.32)
for smooth functions \( B(t), C(t) \) of their argument.

For example, in the LRS vacuum model (a special case of the general solution found by Taub [29]) the metric is [1]
\[
dx^2 = -A^2\, dt^2 + t^{2p_1}A^{-2}(dx + 4p_1b_2\, dy)^2 + t^{2p_3}A^2(dy^2 + dz^2),
\] (4.33)
where \( A = (1 + b_2^2t^{4p_1})^{1/2} \) and \( p_1 \) satisfy (4.19).

The temporal component of the KCS is given by
\[
\lambda(t) = \frac{ta}{2tA_{,\lambda} - (p_1 - p_3)A}
\] (4.34)
and reduces to a KSS for small and large values of the time coordinate (the KSS in plane symmetric Bianchi models has also been found in [30]).

As a final remark we note that by defining the new time coordinate \( \tilde{t} - \tilde{t}_0 = C(t)/B(t) \), the KCS becomes \( X = 2c\tilde{a}\tilde{\alpha}_x + (2cx + by)\tilde{\alpha}_x + cy\tilde{\alpha}_y + (cz - 1)\tilde{\alpha}_z \). Then at \( \tilde{t} = 0 \) we have \( \sigma_{00} = 0 \) and equation (3.2) implies \( \epsilon = 0 \), i.e. the KCS is reduced to the KVF of type II models.

### 4.4. Type VI\(_0\) models

Let us consider now the Bianchi type VI\(_0\) invariant set which is defined by \( N_1 = 0 \) and \( N_2N_3 < 0 \). Propagating equation (3.7) and using (2.9) we obtain
\[
0 = (\lambda' - 3\lambda)\Sigma_{\alpha\beta} + \Lambda_{\alpha\beta} + \left[2\Sigma_{22}X^r + \Sigma_{2r}X^r\right]N_2\epsilon_{2\lambda\alpha\beta}^2 + \left[2\Sigma_{33}X^r + \Sigma_{3r}X^r\right]N_3\epsilon_{3\lambda\alpha\beta}^2 + \lambda S_{\alpha\beta},
\] (4.35)
where \( S_{\alpha\beta} \) and \( \Lambda_{\alpha\beta} \) are given in (2.18) and (4.13) respectively.

The \( \alpha\beta \)-components of (4.35), after a short calculation, give
\[
(S_{22} - S_{33})[2\delta_{23}X_2] + X_1(N_2 + N_3) = 0
\] (4.36)
Note that for $\Sigma_1$ We recall here that the time coordinate $T$ reduces to an isometry. Moreover, it is not difficult to show that equilibrium points $\Sigma_1$ due to equations (2.14)–(2.17), (3.4)–(3.7) and (3.15)–(3.16) is identically satisfied, provided that

$$\tilde{\alpha} = \lambda(1 - 2\Sigma_{22})$$

and the function $\lambda$ is given in terms of the shear and spatial curvature variables ($\Sigma_{22} \neq 0$)

$$\lambda' = \frac{\lambda(2N_2^2 + 9\Sigma_{22})}{3\Sigma_{22}}.$$ 

Note that for $\Sigma_{22} = 0$ the above system of equations implies $\lambda = 0 = \alpha = \delta$ and the KCS reduces to an isometry. Moreover, it is not difficult to show that

$$\tilde{\alpha} - \delta)_{|\Sigma_{22} = 0} = -\frac{2\lambda(\Sigma_{22} + 1)}{\Sigma_{22}}$$

$$\tilde{\alpha} - \delta)_{|\Sigma_{22} = 0} = -\frac{2\lambda[N_2^2 + 3\Sigma_{22}(\Sigma_{22} + 1)]}{3\Sigma_{22}}.$$ 

Due to equations (4.40) and (4.41), the analysis of the asymptotic behaviour of the KCS is straightforward. The past and future attractors of the vacuum invariant subset are the equilibrium points $\Sigma_{22} = 1$, $N_2 = 0$ (the arc $K_1$) and $\Sigma_{22} = -1$, $N_2 = 0$ (the Taub point $T_1$), respectively [1], and the KCS reduces to a proper KSS (with $3\alpha + \delta = 4\psi = 0$) and a proper HVF. Regarding the non-vacuum models it has been shown that the future attractor is the Collins homothetic model $P^+_1(VL_0)$ for which $(\tilde{\alpha} - \delta)_{|P^+_1(VL_0)} = 0$ and the KCS reduces to a proper HVF. The past equilibrium state of the invariant set $S^-_1(VL_0)$ is either the LRS Kasner point $Q_1$ where the KCS becomes a proper KSS or the Friedmann–Lemaître model $F$ and the KCS is given in equation (4.7) with $\lambda' = 3\lambda$.

These results suggest that every type VI$_0$ evolving vacuum or $\gamma$-law perfect fluid model lying in the invariant subset $S^-_1(VL_0)$ always admits a four-dimensional group of KCS that reduces, at the asymptotic regimes, to a self-similarity group of the first or second kind except from a set of measure zero for which the KCS preserves its nature.

For completeness we also give the form of the KCS for the general vacuum solution satisfying $N^a = 0$ (an arbitrary multiplication constant has been omitted) [31, 32]

$$dx^2 = r^{-1/2} e^r (-dr^2 + dx^2) + t(e^{2s} dy^2 + e^{-2s} dz^2)$$

$$X = \frac{4t(c_1 + c_2)}{4t^2 - 3}\partial_t + c_1 \partial_x + c_2 y \partial_y + (2c_1 + c_2)z \partial_z.$$ 

where the symmetry scales are as follows:

$$\alpha = \frac{(c_1 + c_2)(4t^2 - 1)}{(4t^2 - 3)}, \quad \delta = \frac{(c_1 + c_2)(4t^2 - 9)(4t^2 + 1)}{(4t^2 - 3)^2}.$$ 

We can verify that, at early and late times, we have $3\alpha + \delta \approx 0$ and $\alpha - \delta \approx 0$, so the expected limiting cases of the KCS is a self-similarity of the second or first kind. Also to be noted is the apparent ‘singular’ behaviour (due to the specific time gauge) of the KCS as $t \to \sqrt{3}/2$ $(\Rightarrow a_{ab} \to 0)$. However, similar to the previous case, we can show that at this value, the temporal component of $X$ vanishes and the symmetry equations imply that $c_1 + c_2 = 0$, i.e. the KCS $X \to \partial_t - y \partial_y + z \partial_z$ reduces to the KVF of Bianchi type VI$_0$ models.

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2 We recall here that the time coordinate $t$ does not represent the proper clock time as $t \to 0^+$ or $t \to +\infty$. However, the freedom of choosing the time gauge ensures similar asymptotic behaviour for the symmetry scales.
5. Discussion

As we have mentioned, it is customary to study a specific symmetry assumption from a geometrical point of view without taking into consideration the kinematical and dynamical structure of the corresponding model. As a consequence, the effects on the dynamics of the symmetry constraints are hidden and usually produce unphysical results. To date, self-similarity of the first kind appears to be the only symmetry condition with a transparent physical nature since it represents a geometrization of the asymptotic (equilibrium) state of general models. Therefore, in order to complete the dynamical picture, it was of interest to seek and find a symmetry that could be used effectively as a consistent tool for the invariant description of the intermediate behaviour of general models.

In the present paper we have proposed a new technique of studying geometric symmetries by fully exploiting the 1 + 3 orthonormal frame scheme and the introduction of expansion-normalized variables. The analysis of the complete set of consistency equations (2.14)–(2.17), (3.4)–(3.7) and (3.15), (3.16) has revealed a novel feature of a large class of SH models that is summarized in the following.

**Proposition 1.** Evolving SH models of Bianchi type I, LRS models of Bianchi type II and type VI$_0$ models within the associated invariant subset $N^a_{\alpha} = 0$, are geometrically described by the existence of a four-dimensional group of KCS that is reduced to a self-similarity group of the first or second kind at the asymptotic regimes, except from a set of measure zero for which the properness of the KCS is preserved.

However, as we have seen, even in the exceptional cases the corresponding equilibrium state is the Friedmann–Lemaître model $F$ which we have proved that always admits a proper KCS, supplementing the geometric properties of the standard cosmological model.

At first sight, the existence of a proper KCS in SH models appears somewhat surprising, at least from a dynamical point of view. This is mainly because the interaction mechanism between the geometric ‘assumption’ of a KCS and the dynamical behaviour of SH models is not, conceptually, apparent. However, a closer look on the structural properties of SH models indicates a possible qualitative interpretation of this interaction. In particular, the full set of nonlinear EFE can be seen as the perturbed version of the associated linearization of equations (2.14)–(2.17), in the vicinity of a hyperbolic equilibrium point. Accordingly we may interpret the generator of a KCS as representing the perturbation (to some order) of the corresponding generator of the self-similarity transformation group of the first or second kind. Eventually, this observation will enable us to geometrize the majority of the concepts and techniques that are used in the theory of dynamical systems with a view to optimize and efficiently elaborate the results from the qualitative study of general cosmological models.

Although the existence of a KCS signifies the physical ground for a first promising attempt towards a ‘geometrization’ of the evolutionary behaviour of SH models, we do not allege that the KCS (uniquely) characterizes the intermediate epoch of the totality of SH models. In fact, because a KCS possess two arbitrary spatially homogeneous functions (the symmetry scales), we expect that only SH models with two essential degrees of freedom will exhibit a proper KCS. This conjecture is confirmed by proposition 1 in which all the exact solutions found to admit a proper KCS belong to this class. The main reason that provides an interpretation for the possible connection between exact solutions and the existence of KCS appears to involve the so-called hidden symmetries of the SH cosmologies. Indeed, using the Hamilton–Jacobi reformulation of the EFE in which all the dynamical picture is encoded in one geometric object, it has been shown that all known exact solutions with two degrees of freedom are associated with a specific kind of ‘hidden’ symmetry, namely the existence of a Killing tensor...
symmetry of the Jacobi metric that generalizes the corresponding cyclic variables and the Hamilton–Jacobi separability [33, 34]. Therefore it will be of interest to extend the analysis to the rest of the Bianchi models in order to see if the existence of a KCS is a general feature of the two-dimensional SH invariant subset, i.e. if it is related to the above type of ‘hidden’ symmetry of the Bianchi cosmological models [15].

We should remark that, although we have mainly focused our study on Bianchi types with clear and simple past and future equilibrium states, one could apply the approach presented in this paper to models with more complicated dynamical structure, e.g. models with oscillating or diverging asymptotic behaviour near to the past or future attractor. The non-vacuum Bianchi VII\(_0\) invariant subset provides an interesting example since it is well known that the associated kinematical variables are unbounded and do not approach any equilibrium point in the future, i.e. non-vacuum Bianchi type VII\(_0\) models are not asymptotically self-similar [1, 35]. As a consequence one should expect that a KCS does not exist in those models due to the non-existence of a self-similar model as future attractor. However, a preliminary analysis has shown that a KCS does exist in LRS type VII\(_0\) models which is never reduced to a HVF or a KSS [15], suggesting that the concept of the KCS represents not simply a perturbed version of the self-similarity group but a generic property (in the spirit of [33, 34]) of the two-dimensional invariant subset of the SH cosmological models.

Clearly, the case of higher dimensional invariant subsets requires further investigation. Assuming the existence of several proper KCS will not solve the problem for models with three or more essential degrees of freedom, since the condition (3.1) implies that the symmetry scales are always spatially homogeneous restricting the dimension of the Lie algebra of KCS in SH models to 4. Therefore, the question of determining the symmetry which invariantly describes the whole set of SH models is still open. Nevertheless, the implications of the above results enforce the important role which may play a specific (still unknown) general symmetry as an effective geometric implement for the invariant description of general cosmological models and not only as a simplification rule towards the determination of exact solutions with ambiguous (or even without any) physical meaning. A closely related issue is how the constraints, coming from the presence of the general symmetry, could reveal a path to constructing the general (whenever possible) solution of the corresponding cosmological model. We expect that the approach of studying generic geometric symmetries in spacetime developed in this paper can be applied to more general geometric setups leading to a more efficient qualitative and analytical study of general vacuum and perfect fluid models.

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