Noncommutative reciprocity laws on algebraic surfaces: the case of tame ramification

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Abstract. We prove noncommutative reciprocity laws on an algebraic surface defined over a perfect field. These reciprocity laws establish that some central extensions of globally constructed groups split over certain subgroups constructed by points or projective curves on a surface. For a two-dimensional local field with a last finite residue field, the local central extension which is constructed is isomorphic to the central extension which comes from the case of tame ramification of the Abelian two-dimensional local Langlands correspondence suggested by Kapranov.

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§ 1. Introduction

The goal of this paper is to prove some noncommutative reciprocity laws on algebraic surfaces. Let $X$ be a normal irreducible algebraic surface over a perfect field $k$. For any element $a$ from the multiplicative group of the field $k(X)$ of rational functions on $X$ we construct a central extension $\hat{\text{GL}}_{n,a}(A_X)$ of the group $\text{GL}_n(A_X)$ by the group $k^*$. Here $A_X$ is the Parshin-Beilinson adelic ring of the surface $X$. By definition, we have

$$A_X \subset \prod_{x \in C} K_{x,C},$$

where the product is taken of all pairs: a point $x \in X$ and an irreducible curve $C \subset X$ which contains the point $x$. The ring $K_{x,C}$ is canonically constructed by such a pair $\{x \in C\}$ and this ring is a finite product of two-dimensional local fields.

Let $K_{x,C} = \prod_{i=1}^{l} K_i$, where $K_i$ is a two-dimensional local field. Let $K = K_i = k'((u))((t))$ for some $0 \leq i \leq l$. The restriction of the central extension $\hat{\text{GL}}_{n,a}(A_X)$ (for $n \geq 2$) to the subgroup $\text{GL}_n(K)$ of the group $\text{GL}_n(A_X)$ is described by some element of the group $\text{Hom}(K_2(K), k^*)$. In Proposition 2 we calculate this element,
which is given as the following map:

\[ K_2(K) \ni (f, g) \mapsto \text{Nm}_{k'/k}(f, g, a)_K \in k^*, \quad (1) \]

where \((\cdot, \cdot, \cdot)_K\) is the two-dimensional tame symbol.

If \(k' = \mathbb{F}_q\), then the map (1) coincides with the two-dimensional local reciprocity map for the Abelian Kummer extension \(K(a^{1/q-1})\) of the field \(K\).

We prove noncommutative reciprocity laws in Theorem 1 for the central extension \(c_{\text{GL}_n(a)}(\mathbb{A}_X)\). Let \(x\) be a point on \(X\) and let \(K_x\) be a subring of the field \(\text{Frac} \hat{\mathcal{O}}_x\) such that \(K_x = k(X) \cdot \hat{\mathcal{O}}_x\). The central extension \(\hat{\text{GL}}_{n,a}(\mathbb{A}_X)\) splits over the subgroup \(\text{GL}_n(K_x)\) of the group \(\text{GL}_n(\mathbb{A}_X)\). Let \(C\) be a projective irreducible curve on \(X\) and let the field \(K_C\) be the completion of the field \(k(X)\) with respect to the discrete valuation given by the curve \(C\). The central extension \(\hat{\text{GL}}_{n,a}(\mathbb{A}_X)\) splits over the subgroup \(\text{GL}_n(K_C)\) of the group \(\text{GL}_n(\mathbb{A}_X)\). Besides, if \(X\) is a projective surface, then the central extension \(\hat{\text{GL}}_{n,a}(\mathbb{A}_X)\) splits over the subgroup \(\text{GL}_n(k(X))\) of the group \(\text{GL}_n(\mathbb{A}_X)\).

We note that the main technical tool used in this paper comes from [1]. In particular, we use categorical central extensions of groups by the Picard groupoid of \(\mathbb{Z}\)-graded one-dimensional \(k\)-vector spaces. These groups act on 2-Tate \(k\)-vector spaces.

The paper is organized as follows. In §2 we recall some explicit formulae from the two-dimensional local class field theory for Kummer extensions of two-dimensional local fields. In §3 we recall some calculations of the group \(H^2(\text{GL}_n(K), k^*)\) (for \(n \geq 2\)) and the relationship which central extensions of this group have with the algebraic \(K\)-theory of the field \(K\). In §4 we recall some categorical notions and constructions from [1]. In §5 we construct the central extension \(\hat{\text{GL}}_{n,a}(\mathbb{A}_X)\) and study its main properties (see Propositions 1 and 2). We also prove the noncommutative reciprocity laws (see Theorem 1).

§2. Some explicit formulae from two-dimensional class field theory

Let \(K = \mathbb{F}_q((u))((t))\) be a two-dimensional local field, where \(\mathbb{F}_q\) is a finite field. We suppose that \(\mu_m \subset \mathbb{F}_q^*,\) where the group \(\mu_m\) is the group of all roots of unity of degree \(m\). We consider an element \(a \in K^*\) and an extension \(L = K(a^{1/m})\). By Kummer theory, the field \(L\) is a Galois extension of the field \(K\) and

\[ \text{Gal}(L/K) \cong \mu_l, \]

where \(\mu_l \subset \mu_m\) and \(l \mid m\).

By local class field theory for the two-dimensional local field \(K\) (see [2]) there exists a surjective reciprocity map

\[ \varphi_{L/K} : K_2(K) \longrightarrow \text{Gal}(L/K). \]

The map \(\varphi_{L/K}\) is explicitly written as

\[ \varphi_{L/K}((f, g)) = (f, g, a)_K^{(q-1)/m}, \quad (2) \]
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where \( f \) and \( g \) are elements in \( K^* \) and

\[
(\cdot, \cdot, \cdot)_K : K_3^M(K) \rightarrow \mathbb{F}_q^*
\]

is the two-dimensional tame symbol. The map \((\cdot, \cdot, \cdot)_K\) is the composition of the boundary maps in Milnor \(K\)-theory and it is defined for any field \( K = k((u))(t) \), where we have changed the field \( \mathbb{F}_q \) to a field \( k \) and values of \((\cdot, \cdot, \cdot)_K\) are in \( k^* \). An explicit formula for the map \((\cdot, \cdot, \cdot)_K\) is as follows:

\[
(f, g, h)_K = \text{sgn}(f, g, h)f^{\nu_K(g,h)}g^{\nu_K(h,f)}h^{\nu_K(f,g)} \mod m_K \mod m_K^*, \tag{3}
\]

where the elements \( f, g \) and \( h \) are from \( K^* \), \( m_K \) is the maximal ideal of the discrete valuation ring of the field \( K \) and \( m_K^* \) is the maximal ideal of the discrete valuation ring of the field \( \overline{K} = k((u)) \). Furthermore, \( \text{sgn}(f, g, h) = (-1)^A \), where

\[
A = \nu_K(f, g)\nu_K(f, h) + \nu_K(g, h)\nu_K(g, f) + \nu_K(h, f)\nu_K(h, g) + \nu_K(f, g)\nu_K(g, h)\nu_K(h, f)
\]

and the map \( \nu_K(\cdot, \cdot) : K_2(K) \rightarrow \mathbb{Z} \) is also a composition of some boundary maps in Milnor \(K\)-theory given by the explicit formula:

\[
\nu_K(f, g) = \nu_{\overline{K}}\left(f^{\nu_K(g)}g^{\nu_K(h)} \mod m_K\right),
\]

where \( \nu_K : K^* \rightarrow \mathbb{Z} \) and \( \nu_{\overline{K}} : \overline{K}^* \rightarrow \mathbb{Z} \) are discrete valuations of the corresponding fields.

§ 3. A reminder of some calculations

We change the ground field \( \mathbb{F}_q \) to a perfect field \( k \). We recall some calculations from [3], §2.2. For any \( n \geq 2 \) from Remark 2 in [3] we have the following formula:

\[
H^2(\text{GL}_n(K), k^*) = H^2(K^*, k^*) \oplus \text{Hom}(K_2(K), k^*). \tag{4}
\]

We will be interested in central extensions of the group \( \text{GL}_n(K) \) by the group \( k^* \) such that these central extensions (up to isomorphism) come from elements of the group \( \text{Hom}(K_2(K), k^*) \) via formula (4). These central extensions are in one-to-one correspondence with central extensions which are isomorphic to the trivial central extension after a central extension has been restricted to the subgroup \( K^* \) of the group \( \text{GL}_n(K) \), where the group \( K^* \) is embedded in the upper left corner of the matrix group \( \text{GL}_n(K) \).

Remark 1. In this paper we consider the subgroup \( K^* \) embedded into the upper left corner of the group \( \text{GL}_n(K) \). Since an inner automorphism of the group \( \text{GL}_n(K) \) does not change the isomorphism class of any central extension of this group, we could fix an embedding of the group \( K^* \) into any other place of the diagonal. We will obtain the same results.

If a central extension

\[
1 \rightarrow k^* \rightarrow \hat{G} \rightarrow \text{GL}_n(K) \rightarrow 1
\]
splits (that is, is isomorphic to the trivial one) over the subgroup $K^* \hookrightarrow \text{GL}_n(K)$, then the corresponding element from $\text{Hom}(K_2(K), k^*)$, which we call the symbol, is obtained as follows:

$$\{x, y\} = \langle \text{diag}(y, 1, \ldots, 1), \text{diag}(1, x, 1, \ldots, 1) \rangle,$$

where $x$ and $y$ are from $K^*$, $\{x, y\} \in k^*$ and $\langle \cdot, \cdot \rangle$ is the commutator of the lifting of two commuting elements from $\text{GL}_n(K)$ to $G$.

Our goal in this paper is to construct a central extension of the group $\text{GL}_n(K)$ without using algebraic $K$-theory such that this central extension will correspond to the symbol given by the map (2) (when $m = q - 1$, then the map (2) makes sense for the field $k$, because this map comes from the two-dimensional tame symbol, and the map (2) depends on an element $a \in K^*$). This construction also has to be valid if we change from the field $K$ to the ring $\mathbb{A}_X$ of Parshin-Beilinson adeles of an algebraic surface $X$ defined over the field $k$. We will also prove the reciprocity laws for central extensions constructed in this way.

**Remark 2.** When $n = 2$ and $k = \mathbb{F}_q$, starting from a symbol given by the map (2) we obtain a central extension which corresponds to the Abelian two-dimensional local Langlands correspondence suggested by Kapranov, see [4] and also [3].

§ 4. Some facts about categories

We recall some ideas from [1].

By definition, a Picard groupoid $\mathcal{P}$ is a symmetric monoidal group-like groupoid. We will consider the two following examples of Picard groupoids.

- $\mathcal{P} = \text{Pic}$ is the groupoid of one-dimensional $k$-vector spaces where $k$ is a field. (Here groupoid means that we consider only isomorphisms in the category of one-dimensional $k$-vector spaces.)
- $\mathcal{P} = \text{Pic}^\mathbb{Z}$ is the groupoid of $\mathbb{Z}$-graded one-dimensional $k$-vector spaces. In other words, $X \in \text{Ob}(\text{Pic}^\mathbb{Z})$ if and only if $X = (l, n)$, where $l \in \text{Ob}(\text{Pic})$, $n \in \mathbb{Z}$, and

$$\text{Hom}_{\text{Pic}^\mathbb{Z}}((l_1, n_1), (l_2, n_2)) = \begin{cases} \text{Hom}_{\text{Pic}}(l_1, l_2) = \text{Hom}_k(l_1, l_2) \setminus 0 & \text{if } n_1 = n_2, \\ \emptyset & \text{if } n_1 \neq n_2. \end{cases}$$

We put $(l_1, n_1) \otimes (l_2, n_2) = (l_1 \otimes l_2, n_1 + n_2)$.

The main difference between the Picard groupoids $\text{Pic}$ and $\text{Pic}^\mathbb{Z}$ is commutativity constraints: $X \otimes Y \rightarrow Y \otimes X$ (where $X$ and $Y$ are either objects of $\text{Pic}$ or objects of $\text{Pic}^\mathbb{Z}$). Indeed,

- if $X = l_1$ and $Y = l_2$ are objects from $\text{Pic}$, then the commutativity constraint $c_{l_1, l_2}: l_1 \otimes l_2 \rightarrow l_2 \otimes l_1$ is the usual isomorphism;
- if $X = (l_1, n_1)$ and $Y = (l_2, n_2)$ are objects from $\text{Pic}^\mathbb{Z}$, then the commutativity constraint is equal to

$$(-1)^{n_1 n_2} c_{l_1, l_2} \in \text{Hom}_{\text{Pic}^\mathbb{Z}}((l_1 \otimes l_2, n_1 + n_2), (l_2 \otimes l_1, n_1 + n_2)).$$
We introduce a natural Picard groupoid $\mathbf{Z}$. Objects of $\mathbf{Z}$ are elements of the group $\mathbb{Z}$, that is, integers. The only morphisms in $\mathbf{Z}$ are the identity morphisms. In other words, for any integers $m$ and $n$ we have

$$\text{Hom}_\mathbf{Z}(m, n) = \begin{cases} \text{id} & \text{if } m = n, \\ \emptyset & \text{if } m \neq n. \end{cases}$$

We note that there is a symmetric monoidal functor between Picard groupoids:

$$\psi: \mathcal{Pic}^\mathbb{Z} \longrightarrow \mathbf{Z}, \quad \psi((l, n)) = n.$$ 

In addition, there is a monoidal functor

$$F_{\mathcal{Pic}}: \mathcal{Pic}^\mathbb{Z} \longrightarrow \mathcal{Pic}, \quad F_{\mathcal{Pic}}((l, n)) = l.$$ 

The functor $F_{\mathcal{Pic}}$ is not symmetric, that is, this functor does not preserve the commutativity constraints.

Let $G$ be a group and $\mathcal{P}$ a Picard groupoid. We recall (see [1], §2D) that a Picard groupoid $H^1(BG, \mathcal{P})$ is the groupoid of monoidal functors from the discrete monoidal category $G$ to the monoidal category $\mathcal{P}$. In other words, let $f$ be an object from $H^1(BG, \mathcal{P})$; then for any element $g \in G$ we have $f(g) \in \text{Ob}(\mathcal{P})$ and for any $g_1, g_2 \in G$ we have an isomorphism $f(g_1g_2) \simeq f(g_1) + f(g_2)$, compatible with the associativity condition in $\mathcal{P}$.

If $\mathcal{P} = \mathcal{Pic}$, then the groupoid $H^1(BG, \mathcal{P})$ is equivalent to the groupoid of central extensions of the group $G$ by the group $k^*$. Hence, the group of isomorphism classes of objects in $H^1(BG, \mathcal{P})$ is isomorphic to the group $H^2(G, k^*)$. We note that the functor $F_{\mathcal{Pic}}$ induces a functor (which is not monoidal):

$$H^1(BG, \mathcal{Pic}^\mathbb{Z}) \longrightarrow H^1(BG, \mathcal{Pic}): f \mapsto F_{\mathcal{Pic}} \circ f. \quad (6)$$

Let $f$ be an element from the set $\text{Ob}(H^1(BG, \mathcal{P}))$, where $\mathcal{P}$ is any Picard groupoid. For any $g_1, g_2 \in G$ such that $[g_1, g_2] = 1$ there is an element

$$\text{Comm}(f)(g_1, g_2) \in \text{Aut}_\mathcal{P}(e) = \text{End}_\mathcal{P}(e),$$

where $e$ is a unit object in $\mathcal{P}$ (see [1], Lemma-Definition 2.5). The map $\text{Comm}(f)$ is a map from the set of pairs of commuting elements of $G$ to the Abelian group $\text{Aut}_\mathcal{P}(e)$. This map is a bimultiplicative antisymmetric map with respect to the elements $g_1$ and $g_2$.

If $\mathcal{P} = \mathcal{Pic}$, then $\text{Comm}(f)(g_1, g_2) = \langle g_1, g_2 \rangle \in k^*$, where $\langle g_1, g_2 \rangle$ is the commutator of the lifting of elements $g_1$ and $g_2$ to the central extension of the group $G$ by the group $k^*$ such that this central extension corresponds to $f$.

If $\mathcal{P} = \mathcal{Pic}^\mathbb{Z}$, then $\text{Comm}(f)(g_1, g_2) \in k^*$. Moreover, from the construction of the map $\text{Comm}(f)$ the following property holds. If $f(g_1) = (l_1, n_1) \in \text{Ob}(\mathcal{Pic}^\mathbb{Z})$, $f(g_2) = (l_2, n_2) \in \text{Ob}(\mathcal{Pic}^\mathbb{Z})$, where $n_i \in \mathbb{Z}$, then after taking the composition of $f$ with the functor $F_{\mathcal{Pic}}$ we obtain

$$\text{Comm}(f)(g_1, g_2) = (-1)^{n_1n_2} \text{Comm}(F_{\mathcal{Pic}} \circ f)(g_1, g_2) = (-1)^{n_1n_2} \langle g_1, g_2 \rangle, \quad (7)$$

Here and in what follows we use also the notation $+$ for the monoidal product in a Picard groupoid.
where \( \langle g_1, g_2 \rangle \) is the commutator of the lifting of the elements \( g_1 \) and \( g_2 \) to the central extension of the group \( G \) by the group \( k^* \) such that this central extension corresponds to \( F_{\text{Pic}} \circ f \).

Let \( G \) be any group and \( \mathcal{P} \) any Picard groupoid. We recall (see [1], §2E) that objects of the Picard groupoid \( H^2(BG, \mathcal{P}) \) are (categorical) central extensions \( \mathcal{L} \) of the group \( G \) by the Picard groupoid \( \mathcal{P} \):

\[
1 \longrightarrow \mathcal{P} \xrightarrow{i} \mathcal{L} \xrightarrow{\pi} G \longrightarrow 1,
\]

where \( \mathcal{L} \) is a group-like monoidal groupoid, \( G \) is a discrete group-like monoidal groupoid constructed from the group \( G \), \( i \) and \( \pi \) are monoidal functors. More precisely, a central extension \( \mathcal{L} \) can be given in the following way. For any \( g \in G \) we have a \( \mathcal{P} \)-torsor \( \mathcal{L}_g = \pi^{-1}(g) \) and for any \( g_1, g_2 \in G \) we have a natural equivalence of \( \mathcal{P} \)-torsors:

\[
\mathcal{L}_{g_1 g_2} \simeq \mathcal{L}_{g_1} + \mathcal{L}_{g_2}
\]

together with some further isomorphisms between equivalences and compatibility conditions on these isomorphisms. We have \( \text{Ob}(\mathcal{L}) = \bigcup_{g \in G} \text{Ob}(\mathcal{L}_g) \). If \( X \in \text{Ob}(\mathcal{L}_{g_1}) \) and \( Y \in \text{Ob}(\mathcal{L}_{g_2}) \), then

\[
\text{Hom}_{\mathcal{L}}(X, Y) = \begin{cases} 
\text{Hom}_{\mathcal{L}_{g_1}}(X, Y) & \text{if } g_1 = g_2, \\
\emptyset & \text{if } g_1 \neq g_2.
\end{cases}
\]

**Remark 3.** If \( \mathcal{P} = \mathcal{P}_{\text{Pic}} \), then the group of isomorphism classes of objects of \( H^2(BG, \mathcal{P}) \) is isomorphic to the group \( H^3(G, k^*) \) (see, for example, [5]).

We note that the functor \( \psi: \mathcal{P}_{\text{Pic}} \xrightarrow{\mathbb{Z}} \mathbb{Z} \) induces a symmetric monoidal functor \( \Psi \) from the Picard 2-groupoid of \( \mathcal{P}_{\text{Pic}}^{\mathbb{Z}} \)-torsors to the Picard groupoid of \( \mathbb{Z} \)-torsors by the rule

\[
\Psi(\mathcal{M}) = \mathbb{Z} \mathcal{P}_{\text{Pic}}^{\mathbb{Z}} \otimes \mathcal{M},
\]

where \( \mathcal{M} \) is a \( \mathcal{P}_{\text{Pic}}^{\mathbb{Z}} \)-torsor (for details see [6], §6.7). The Picard groupoid of \( \mathbb{Z} \)-torsors is equivalent to the Picard groupoid of \( \mathbb{Z} \)-torsors. Using the functor \( \Psi \) we obtain a symmetric monoidal functor \( \tilde{\Psi} \) from the Picard groupoid \( H^2(BG, \mathcal{P}_{\text{Pic}}^{\mathbb{Z}}) \) to the Picard groupoid of central extensions of the group \( G \) by the group \( \mathbb{Z} \)

\[
\mathcal{L} \in H^2(BG, \mathcal{P}_{\text{Pic}}^{\mathbb{Z}}), \quad \tilde{\Psi}(\mathcal{L})(g) = \Psi(\mathcal{L}_g) \quad \forall g \in G.
\]

(We note that the group of isomorphism classes of objects of the Picard groupoid of central extensions of the group \( G \) by the group \( \mathbb{Z} \) is isomorphic to \( H^2(G, \mathbb{Z}) \).)

For any object \( \mathcal{L} \) of the Picard groupoid \( H^2(BG, \mathcal{P}) \), for any \( g_1, g_2 \in G \) such that \( [g_1, g_2] = 1 \) there is an object \( C_2^\mathcal{P}(g_1, g_2) \) of the Picard groupoid \( \mathcal{P} \) (see [1], Lemma-Definition 2.13). The morphism \( C_2^\mathcal{P}(g_1, g_2) \) is a bimultiplicative and antisymmetric (with respect to \( g_1 \) and \( g_2 \)) morphism from the set of pairs of commuting elements of \( G \) to the Picard groupoid \( \mathcal{P} \). An object \( C_2^\mathcal{P}(g_1, g_2) \) comes from an auto-equivalence of the \( \mathcal{P} \)-torsor \( \mathcal{L}_{g_1 g_2} \). This auto-equivalence is the composition of the following equivalences of \( \mathcal{P} \)-torsors:

\[
\mathcal{L}_{g_1 g_2} \simeq \mathcal{L}_{g_1} + \mathcal{L}_{g_2} \simeq \mathcal{L}_{g_2} + \mathcal{L}_{g_1} \simeq \mathcal{L}_{g_2 g_1} = \mathcal{L}_{g_1 g_2}.
\]
Remark 4. If $\mathcal{P} = \mathcal{Pic}$, then the morphism $C_2^\mathcal{L}$ is a weak biextension of the set of pairs of commuting elements of the group $G$ by the group $k^\ast$. This weak biextension is described by partial symmetrizations of the 3-cocycle corresponding to the category of central extensions $\mathcal{L}$ (see [5]).

If $\mathcal{P} = \mathcal{Pic}^\mathcal{L}$ and $\mathcal{L} \in \text{Ob}(H^2(BG, \mathcal{P}))$, then it is easy to see from formula (8) that applying the functor $\psi: \mathcal{Pic}^\mathcal{L} \to \mathbb{Z}$ we obtain

$$\psi(C_2^\mathcal{L}(g_1, g_2)) = -\langle g_1, g_2 \rangle \in \mathbb{Z},$$

(9)

where $\langle g_1, g_2 \rangle$ is the commutator of the liftings of elements $g_1$ and $g_2$ to the central extension $\Psi(\mathcal{L})$ of the group $G$ by the group $\mathbb{Z}$.

Let $G$ be any group and $\mathcal{P}$ any Picard groupoid. We consider a central extension $\mathcal{L} \in \text{Ob}(H^2(BG, \mathcal{P}))$. We fix an element $g \in G$. We denote by $Z_G(g) \subset G$ the subgroup which is the centralizer of the element $g$. From the morphism $C_2^\mathcal{L}$ we obtain an object $C_2^\mathcal{L}_g$ of $H^1(BZ_G(g), \mathcal{P})$ by the rule

$$\forall h \in Z_G(g) \quad C_2^\mathcal{L}_g(h) = C_2^\mathcal{L}(g, h) \in \text{Ob}(\mathcal{P}).$$

(10)

For any $g_1, g_2, g_3 \in G$ such that $[g_i, g_j] = 1, 1 \leq i, j \leq 3$, there is an element

$$C_3^\mathcal{L}(g_1, g_2, g_3) = \text{Comm}(C_2^\mathcal{L}_g)(g_2, g_3) \in \text{Aut}_\mathcal{P}(e).$$

(11)

The map $C_3^\mathcal{L}$ is a map from the set of triples of pairwise commuting elements of $G$ to the Abelian group $\text{Aut}_\mathcal{P}(e)$. This map is trimultiplicative and antisymmetric with respect to the elements $g_1, g_2$ and $g_3$ (see [1], Proposition 2.17).

§ 5. Central extensions

Let $X$ be a normal irreducible algebraic surface over a perfect field $k$ (in particular, $X$ is an integral scheme). We will try to keep the notation similar to that in [3], §3.2.

Let $\Delta$ be some subset of all pairs $\{x \in C\}$ where $x \in X$ is a point and $C \subset X$ is an irreducible curve which contains the point $x$ (in other words, $C$ is an integral one-dimensional subscheme of $X$). Let $K_{x, C}$ be the ring canonically associated with a pair $\{x \in C\}$ from $\Delta$ (see, for example, [7], §2.2). The ring $K_{x, C}$ is a finite product of two-dimensional local fields such that every two-dimensional local field from this product corresponds to a branch of the curve $C$ restricted to the formal neighbourhood of the point $x$. In particular, if $x$ is a regular point on $C$ and on $X$, then $K_{x, C}$ is a two-dimensional local field isomorphic to $k(x)((u))((t))$, where $k(x)$ is the residue field of the point $x$, $t = 0$ is a local equation of the curve $C$ in some affine neighbourhood of the point $x$ on $X$, $u = 0$ is a local equation of a curve which is defined in the same neighbourhood and is transversal to $C$ in $x$. If $K_{x, C} = \prod_i K_i$, where $K_i$ is a two-dimensional local field, then we denote $\mathcal{O}_{K_{x, C}} = \prod_i \mathcal{O}_{K_i}$, where $\mathcal{O}_{K_i}$ is the discrete valuation ring of the field $K_i$. In particular, if $x$ is a regular point on $C$ and on $X$, then the ring $\mathcal{O}_{K_{x, C}}$ is isomorphic to the ring $k(x)((u))[[t]]$.

Let $\mathbb{A}_X$ be the Parshin-Beilinson adelic ring of $X$ (see, for example, the survey article [7]). We have an embedding

$$\mathbb{A}_X \subset \prod_{\{x \in C\}} K_{x, C}.$$
Inside the ring $\prod_{\{x \in C\}} K_{x,C}$ we define subrings $\mathbb{A}_\Delta$ and $\mathcal{O}_{\mathbb{A}_\Delta}$ (which depend on the set $\Delta$):

$$\mathbb{A}_\Delta = \mathbb{A}_X \cap \prod_{\{x \in C\} \in \Delta} K_{x,C}, \quad \mathcal{O}_{\mathbb{A}_\Delta} = \mathbb{A}_X \cap \prod_{\{x \in C\} \in \Delta} \mathcal{O}_{K_{x,C}}.$$ 

If $\Delta_1$ is a subset of the set $\Delta$, then we have

$$\mathbb{A}_\Delta = \mathbb{A}_{\Delta_1} \times \mathbb{A}_{\Delta \setminus \Delta_1}, \quad \mathcal{O}_{\mathbb{A}_\Delta} = \mathcal{O}_{\mathbb{A}_{\Delta_1}} \times \mathcal{O}_{\mathbb{A}_{\Delta \setminus \Delta_1}}.$$ 

Hence for any $n \geq 1$ we obtain

$$\text{GL}_n(\mathbb{A}_\Delta) = \text{GL}_n(\mathbb{A}_{\Delta_1}) \times \text{GL}_n(\mathbb{A}_{\Delta \setminus \Delta_1}). \tag{12}$$

Let $i_{\Delta_1,\Delta,n}: \text{GL}_n(\mathbb{A}_{\Delta_1}) \hookrightarrow \text{GL}_n(\mathbb{A}_\Delta)$ and $j_{\Delta_1,\Delta,n}: \text{GL}_n(\mathbb{A}_\Delta) \twoheadrightarrow \text{GL}_n(\mathbb{A}_{\Delta_1})$ be the embedding and the surjection of groups with respect to the decomposition (12).

We know that $\mathbb{A}_\Delta^n$ is a $C_2$-space over the field $k$ (see [8], Theorem 1). Furthermore, the group $\text{GL}_n(\mathbb{A}_\Delta)$ acts on the space $\mathbb{A}_\Delta^n$ and this action is given by automorphisms of the $C_2$-space. Moreover, $\mathbb{A}_\Delta^n$ is a complete $C_2$-vector space, that is, it is a 2-Tate vector space over the field $k$. The subspace $\mathcal{O}_{\mathbb{A}_\Delta}^n \subset \mathbb{A}_\Delta^n$ is a lattice in $\mathbb{A}_\Delta^n$. We fix this lattice. Now as in [1], §4C, we canonically construct (using Kapranov’s graded determinantal theories) a categorical central extension $\mathcal{D}et_{\mathcal{A}_\Delta^n}$ of the group $\text{GL}_n(\mathbb{A}_\Delta)$ by the Picard groupoid $\mathcal{P}ic^\mathbb{A}$, that is, we construct

$$\mathcal{D}et_{\mathcal{A}_\Delta^n} \in \text{Ob}(H^2(B\text{GL}_n(\mathbb{A}_\Delta), \mathcal{P}ic^\mathbb{A})).$$

We fix some element $a \in \mathbb{A}_\Delta^*$. The element $\text{diag}(a, \ldots, a)$ belongs to the centre of $\text{GL}_n(\mathbb{A}_\Delta)$. Therefore, $Z_{\text{GL}_n(\mathbb{A}_\Delta)}(\text{diag}(a, \ldots, a)) = \text{GL}_n(\mathbb{A}_\Delta)$. Using (10), we obtain the following object from $H^1(B\text{GL}_n(\mathbb{A}_\Delta), \mathcal{P}ic^\mathbb{A})$:

$$\mathcal{D}et_{a,n} = C_{\text{diag}(a, \ldots, a)}^{\mathcal{D}et_{\mathcal{A}_\Delta^n}}. \tag{13}$$

We apply the functor $F_{\mathcal{P}ic^\mathbb{A}}: \mathcal{P}ic^\mathbb{A} \rightarrow \mathcal{P}ic$ and as in (6), we obtain

$$\mathcal{D}et_{a,n} = F_{\mathcal{P}ic^\mathbb{A}} \circ \mathcal{D}et_{a,n} \in \text{Ob}(H^1(B\text{GL}_n(\mathbb{A}_\Delta), \mathcal{P}ic)).$$

The object $\mathcal{D}et_{a,n}$ is given by the central extension

$$1 \rightarrow k^* \rightarrow \text{GL}_{n,a}(\mathbb{A}_\Delta) \xrightarrow{\theta} \text{GL}_n(\mathbb{A}_\Delta) \rightarrow 1. \tag{14}$$

The central extension (14) defines an element of the group $H^2(\text{GL}_n(\mathbb{A}_\Delta), k^*)$.

We know that $\text{GL}_n(\mathbb{A}_\Delta) = \text{SL}_n(\mathbb{A}_\Delta) \rtimes \mathbb{A}_\Delta^*$, where the group $\mathbb{A}_\Delta^*$ is embedded into the upper left corner of the matrix group $\text{GL}_n(\mathbb{A}_\Delta)$ and acts by conjugations. We define a group

$$\text{GL}_{n,a}(\mathbb{A}_\Delta) = \theta^{-1}(\text{SL}_n(\mathbb{A}_\Delta)) \rtimes \mathbb{A}_\Delta^*. \tag{15}$$

\(^{2}\text{From [7], Propositions 3.4 and 3.8 we see that the ring } \mathcal{O}_{\mathbb{A}_\Delta}\text{ coincides with the adelic ring defined by the Beilinson construction (see [7], §3.2) by means of the set of 2-simplices } (C,x) \text{ (where } \{x \in C\} \in \Delta) \text{ of the simplicial set which is defined by scheme points of the surface } X. \text{ Analogously, the ring } \mathbb{A}_\Delta\text{ coincides with the adelic ring defined by the set of 3-simplices } (X,C,x) \text{ (where } \{x \in C\} \in \Delta). \text{ In our notation we identify irreducible varieties and the corresponding generic scheme points.} \)
The group $\widehat{\text{GL}}_{n,a}(A_\Delta)$ is a central extension of the group $\text{GL}_n(A_\Delta)$ by the group $k^*$. This central extension splits over the subgroup $A_\Delta^* \subset \text{GL}_n(A_\Delta)$. We note that similarly to Remark 1 we have that embedding the group $A_\Delta^*$ into another place on the diagonal of the matrix group $\text{GL}_n(A_\Delta)$ does not change the isomorphism class of the central extension $\widehat{\text{GL}}_{n,a}(A_\Delta)$ (cf. also Remark 3 in [3]).

**Proposition 1.** Fix some element $a \in A_\Delta^*$, where $\Delta$ is a subset of the set of all pairs $\{x \in C\}$ on $X$ given at the start of this section.

1. Let $\Delta_1 \subset \Delta$ be a subset. The restriction of the central extension $\widehat{\text{GL}}_{n,a}(A_\Delta)$ to the subgroup $\text{GL}_n(A_{\Delta_1})$ embedded in the group $\text{GL}_n(A_\Delta)$ by the homomorphism $i_{\Delta_1,\Delta,n}$ is isomorphic to the central extension $\widehat{\text{GL}}_{m,j_{\Delta_1,\Delta,n},a}(A_{\Delta_1})$ of the group $\text{GL}_n(A_{\Delta_1})$ by the group $k^*$.

2. Let $1 \leq m \leq n$. Then the restriction of the central extension $\widehat{\text{GL}}_{n,a}(A_\Delta)$ to the subgroup $\text{GL}_m(A_\Delta)$ of the group $\text{GL}_n(A_\Delta)$ embedded into the upper left corner is isomorphic to the central extension $\widehat{\text{GL}}_{m,a}(A_\Delta)$ of the group $\text{GL}_m(A_\Delta)$ by the group $k^*$.

**Proof.** Clearly, it is enough to prove the proposition for the central extensions $\text{GL}_{n,a}(A_\Delta)$. Since the functor given by (6) commutes with the restriction on a subgroup $H \subset G$, it is enough to prove the analogous statement for $\text{Det}_{a,n} \in \text{Ob}(H^1(B\text{GL}_n(A_\Delta), \mathcal{P}ic^Z))$. In the latter case the statements of the proposition follow from Lemma 2.15 in [1], which states that the morphism $C_2^{L_1 + L_2}$ is isomorphic to the morphism $C_2^{L_1} + C_2^{L_2}$ for any $L_1, L_2 \in \text{Ob}(H^2(BG, \mathcal{P}ic^Z))$, and from Lemma 4.9 in [1], which states that if a group $G$ acts on 2-Tate vector spaces $V_1$ and $V_2$, then

$$\text{Det}_{V_1} + \text{Det}_{V_2} \simeq \text{Det}_{V_1 \oplus V_2}$$

in $H^2(BG, \mathcal{P}ic^Z)$. The proposition is proved.

We now consider the case when the set $\Delta$ is a singleton, that is, it corresponds to a pair $\{x \in C\}$ on $X$. We have

$$A_\Delta = K_{x,C} = \prod_{i=1}^l K_i,$$

where $K_i$ is a two-dimensional local field. Let $K = K_i = k^i((u))((t))$ for some $i$, where $k^i \supset k$ is a finite extension. Let the central extension $\widehat{\text{GL}}_{n,a}(K)$ of the group $\text{GL}_n(K)$ by the group $k^*$ be a restriction of the central extension $\widehat{\text{GL}}_{n,a}(A_\Delta)$ of the group $\text{GL}_n(A_\Delta)$ by the group $k^*$ with respect to the morphism $\text{GL}_n(K) \rightarrow \text{GL}_n(A_\Delta)$. We note that, by analogy with statement 1 of Proposition 1, the central extension $\text{GL}_{n,a}(K)$ depends on the image of the element $a \in K_{x,C}^*$ in $K^*$. Therefore we will assume that $a \in K^*$ in the notation of the central extension $\widehat{\text{GL}}_{n,a}(K)$.

**Proposition 2.** We fix an integer $n \geq 2$ and an element $a \in K^*$. Then the central extension $\widehat{\text{GL}}_{n,a}(K)$ of the group $\text{GL}_n(K)$ by the group $k^*$ corresponds to the following element of $\text{Hom}(K_2(K), k^*)$ (with respect to the decomposition (4)):

$$K_2(K) \ni (f, g) \mapsto \text{Nm}_{k^i/k}(f, g, a)_{K} \in k^*,$$

where $(\cdot, \cdot, \cdot)_{K}$ is the two-dimensional tame symbol (see formula (3)).
Proof. Since, by construction, the central extension $\tilde{GL}_{n,a}(K)$ splits over the subgroup $K^* \subset GL_n(K)$, we have to calculate the expression given by (5). We have
\[
\{f, g\} = \langle \text{diag}(g, 1, \ldots, 1), \text{diag}(1, f, 1, \ldots, 1) \rangle
\]
\[
= \langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle \langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle.
\]
Since the central extension $\tilde{GL}_{n,a}(K)$ splits over the subgroup $K^* \subset GL_n(K)$, we have $\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle = 1$. Therefore, we have to calculate the value of $\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle$. Since the group $\tilde{GL}_{n,a}(K)$ is a semidirect product, the value of the last expression coincides with the value of the same expression but calculated in the central extension $\tilde{GL}_{n,a}(K)$ (where the central extension $\tilde{GL}_{n,a}(K)$ of the group $GL_n(K)$ by the group $k^*$ is the restriction of the central extension $\tilde{GL}_{n,a}(K_{x,C})$ to the subgroup $GL_n(K) \hookrightarrow GL_n(K_{x,C})$).

Hence from formula (7) we have
\[
\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle
\]
\[
= \text{Comm}(\triangledown et_{a,n})(\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle)
\]
\[
= (-1)^B \text{Comm}(\triangledown et_{a,n})(\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle),
\]
where
\[
B = \psi(\triangledown et_{a,n}(\langle \text{diag}(g, 1, \ldots, 1) \rangle)) \cdot \psi(\triangledown et_{a,n}(\langle \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle)).
\]
From formulae (13), (10) and (9) we have
\[
\psi(\triangledown et_{a,n}(\langle \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle)) = \psi(C^{\triangledown et_{a,n}}_{\text{diag}(a, \ldots, a)}(\langle \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle))
\]
\[
= \psi(C_2^{\triangledown et_{a,n}}(\langle \text{diag}(a, \ldots, a), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle))
\]
\[
= -\langle \text{diag}(a, \ldots, a), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle_Z,
\]
where $\langle \cdot, \cdot \rangle_Z$ is the commutator of the lifting of commuting elements from the group $GL_n(K)$ to the central extension $\Psi(\triangledown et_{a,n})$ of the group $GL_n(K)$ by the group $Z$.

From (16) we see that
\[
\langle \text{diag}(a, \ldots, a), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle_Z
\]
\[
= \langle \text{diag}(a, 1, \ldots, 1), \text{diag}(f^{-1}, 1, \ldots, 1) \rangle_Z
\]
\[
+ \langle \text{diag}(1, a, 1, \ldots, 1), \text{diag}(1, f, 1, \ldots, 1) \rangle_Z = 0.
\]
Therefore, $B = 0$. Now using (11) and (16) we have
\[
\text{Comm}(\triangledown et_{a,n})(\langle \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle)
\]
\[
= C_3^{\triangledown et_{a,n}}(\langle \text{diag}(a, \ldots, a), \text{diag}(g, 1, \ldots, 1), \text{diag}(f^{-1}, f, 1, \ldots, 1) \rangle)
\]
\[
= C_3^{\triangledown et_{K}}(a, g, f^{-1}) = C_3^{\triangledown et_{K}}(f, g, a) = N_{k'/k}(f, g, a)_K,
\]
where the last equality follows from [1], Theorem 4.11. From the above calculations we obtain $\{f, g\} = N_{k'/k}(f, g, a)_K$. The proposition is proved.
Let $X$ be an algebraic surface as given at the start of §5. Let $k(X)$ be the field of rational functions on $X$. For any irreducible curve $C$ on $X$, $K_C$ denotes a field which is the completion of the field $k(X)$ with respect to a discrete valuation given by the curve $C$. For any point $x \in X$, $\hat{O}_x$ denotes the completion of the local ring $\hat{O}_x$ of the point $x$ on $X$ with respect to the maximal ideal. We introduce a ring $K_x = k(X) \cdot \hat{O}_x$, where the product is taken in the field $\text{Frac}(\hat{O}_x)$. We fix an integer $n \geq 1$. There is a diagonal embedding of the group $GL_n(k(X))$ into the group $GL_n(\mathbb{A}_X)$ (through the embedding of the field $k(X)$ into the ring $K_{x,C}$ for any pair $\{x \in C\}$). For any irreducible curve $C$ on $X$ there is an embedding of the group $GL_n(K_C)$ into the group $GL_n(\mathbb{A}_X)$ (through the diagonal embedding of the field $K_C$ into the ring $K_{x,C}$ for any point $x$ on $C$ and we set $1 \in GL_n(K_{y,F})$ for any pair $\{y \in F\}$ when $F \neq C$). For any point $x$ on $X$ there is an embedding of the group $GL_n(K_x)$ into the group $GL_n(\mathbb{A}_X)$ (through the diagonal embedding of the ring $K_x$ into the ring $K_{x,C}$ for any irreducible curve $C \ni x$ and we put $1 \in GL_n(K_{y,F})$ for any pair $\{y \in F\}$ when $y \neq x$).

**Theorem 1** (Noncommutative reciprocity laws). Consider a normal irreducible algebraic surface $X$ over a perfect field $k$. Fix some element $a \in k(X)^* \subset \mathbb{A}_X^*$ and an integer $n \geq 1$. The central extension $\hat{GL}_{n,a}(\mathbb{A}_X)$ of the group $GL_n(\mathbb{A}_X)$ by the group $k^*$ splits canonically over the following subgroups of the group $GL_n(\mathbb{A}_X)$:

1) over the subgroup $GL_n(K_x)$ for any point $x$ on $X$;
2) over the subgroup $GL_n(K_C)$ for any irreducible projective curve $C$ on $X$;
3) over the subgroup $GL_n(k(X))$ when $X$ is a projective surface.

**Proof.** The method of proof of this theorem is similar to the method of proof for Theorem 1 in [3].

We note that if $n = 1$ then from the construction of the central extension $\hat{GL}_{1,a}(\mathbb{A}_X)$ we see that it splits over the group $\mathbb{A}_X^*$. Therefore we can assume that $n \geq 2$.

We fix a point $x \in X$. Let $\hat{\mathbb{A}}_x$ be the adelic ring of the scheme $\text{Spec} \hat{O}_x$. We have

$$\hat{\mathbb{A}}_x = \left\{ b = (b_F) \in \prod_F K_F : b_F \in \hat{O}_K \text{ for all but finitely many } F \right\},$$

where $F$ runs over all prime ideals of height 1 of the ring $\hat{O}_x$ and $K_F$ is the two-dimensional local field constructed by $F$. Clearly, $\hat{\mathbb{A}}_x$ is a 2-Tate vector space over the field $k$. Therefore, as in [1], §4C we construct a categorical central extension $\mathcal{D}et_{\hat{\mathbb{A}}^n_\Delta} \in \text{Ob}(H^2(BGL_n(\hat{\mathbb{A}}_x), \mathcal{Pic}^0))$. Now, the restriction of the categorical central extension $\mathcal{D}et_{\hat{\mathbb{A}}_n^\Delta}$ to the subgroup $GL_n(K_x)$ of the group $GL_n(\hat{\mathbb{A}}_x)$ is isomorphic to the restriction of the categorical central extension $\mathcal{D}et_{\hat{\mathbb{A}}^n_\Delta}$ to the subgroup $GL_n(K_x)$ of the group $GL_n(\mathbb{A}_X)$. We consider a (usual) central extension $\hat{GL}_{n,a}(\mathbb{A}_X)$ of the group $GL_n(\mathbb{A}_X)$ by the group $k^*$ which is constructed using formulae analogous to (13)–(15), but where we have to change the 2-Tate vector space $\mathbb{A}_\Delta^a$ to the 2-Tate vector space $\hat{\mathbb{A}}_x^n$. Now, using the embedding $GL_n(K_x) \subset GL_n(\text{Frac} \hat{O}_x)$ and the above arguments we see that the first statement of the theorem will follow if the central extension $\hat{GL}_{n,a}(\mathbb{A}_X)$ splits over the subgroup $GL_n(\text{Frac} \hat{O}_x)$ of the group $GL_n(\hat{\mathbb{A}}_x)$. To see this last splitting we calculate \{f, g\}.
for any $f, g \in (\text{Frac} \widehat{\mathcal{O}}_x)^*$ according to formula (5). As in the calculations in the proof of Proposition 2 we obtain \( \{f, g\} = C_3^{\text{et} \widehat{\mathcal{O}}_x}(f, g, a) \). Using Theorem 4.11 in [1] and considering only a finite number of height 1 prime ideals $F$ of the ring $\widehat{\mathcal{O}}_x$ such that all the poles and zeros of the elements $f$, $g$ and $a$ are among these ideals $F$ we find that

$$C_3^{\text{et} \widehat{\mathcal{O}}_x}(f, g, a) = \prod_F \text{Nm}_{k_F/k}(f, g, a)_{K_F},$$

where $K_F$ is the two-dimensional local field corresponding to $F$ with last residue field $k_F$, which is a finite extension of the field $k$. From the reciprocity law around a point for the two-dimensional tame symbol we have

$$\prod_F \text{Nm}_{k_F/k}(f, g, a)_{K_F} = 1.$$

Hence, using formula (4), we see that the central extension $\widehat{\text{GL}}_{n,a} (\widehat{\mathbb{A}}_x)$ splits over the subgroup $\text{GL}_n(\text{Frac} \widehat{\mathcal{O}}_x)$ of the group $\text{GL}_n(\widehat{\mathbb{A}}_x)$. Thus, we have proved the first statement of the theorem.

We fix a projective irreducible curve $C$ on $X$. Let $\Delta_C$ be the set of all pairs \( \{x \in C\} \) when the curve $C$ is fixed. We note that $\mathbb{A}_\Delta_C = \mathbb{A}_C((t_C))$, where $\mathbb{A}_C$ is the adelic ring of the curve $C$ and $t_C = 0$ is a local equation of the curve $C$ on some open affine subset of $X$. From statement 1 of Proposition 1 we know that to prove the central extension $\widehat{\text{GL}}_{n,a} (\mathbb{A}_X)$ splits over the subgroup $\text{GL}_n(K_C)$ of the group it is enough to prove the central extension $\widehat{\text{GL}}_{n,a} (\mathbb{A}_\Delta_C)$ splits over the subgroup $\text{GL}_n(K_C)$ of the group $\text{GL}_n(\widehat{\mathbb{A}}_x)$. Moreover, we will assume that $a \in K_C^*$ (recall that $k(X)^* \subset K_C^*$).

To prove $\widehat{\text{GL}}_{n,a} (\mathbb{A}_\Delta_C)$ splits over the subgroup $\text{GL}_n(K_C)$ of the group $\text{GL}_n(\mathbb{A}_\Delta_C)$ we will use (5). According to (4) it is enough to show that $\{f, g\} = 1$ for any $f, g \in K_C^*$. Similarly to the calculations in the proof of Proposition 2 we obtain $\{f, g\} = C_3^{\text{et} \mathbb{A}_C}(f, g, a)$. From Steinberg’s relation we have $\{h, h\} = \{-1, h\}$ for any $h \in K_C^*$. Also, $C_3^{\text{et} \mathbb{A}_C}(\cdot, \cdot, \cdot)$ is a trimultiplicative antisymmetric map. Therefore, to prove

$$C_3^{\text{et} \mathbb{A}_C}(f_1, f_2, f_3) = 1 \quad \text{for any } f_1, f_2, f_3 \in K_C^* \quad (17)$$

it is enough to consider the following two cases: 1) $f_i \in \mathbb{O}_K^*$ for any $i$; 2) $f_1, f_2 \in \mathbb{O}_K^*$, $f_3 = t_C$ (we recall that $\mathbb{O}_K^*$ is the discrete valuation ring of the field $K_C$). The first case follows because $f_i \cdot \mathbb{A}_C[[t_C]] = \mathbb{A}_C[[t_C]]$ and $\mathbb{A}_C[[t_C]]$ is a lattice in the 2-Tate vector space $\mathbb{A}_C((t_C))$ (see Lemma 4.10 in [1]). Using formula (11) and analogues of Lemma 4.12 and diagram (4-8) in [1] for the 2-Tate vector space $\mathbb{A}_C((t_C))$, the second case reduces to the following:

$$C_3^{\text{et} \mathbb{A}_C}(f_1, f_2, t_C) = C_3^{\text{et} \mathbb{A}_C}(t_C, f_1, f_2) = \text{Comm} (C_{t_C}^{\mathbb{A}_C}(f_1, f_2) = \text{Comm} (\mathbb{A}_C(f_1, f_2))^{-1},$$
where $f_1, f_2 \in k(C)^*$, that is, reduces to the case of the projective curve $C$. Now the equality $\text{Comm}(\mathcal{D}et_{\mathbf{k}C}) (f_1, f_2) = 1$ follows from [1], §5A. Thus, we have proved the second statement of the theorem.

The proof that the central extension $\widetilde{\text{GL}}_{n,a}(\mathbf{k}_X)$ splits over the subgroup $\text{GL}_n(k(X))$ is analogous to the proof of the second statement of the theorem. For any $f, g \in k(X)^*$ we calculate $\{f, g\} = C_3^{\mathcal{D}et_{kX}}(f, g, a)$. Now we have

$$C_3^{\mathcal{D}et_{kX}}(f, g, a) = \prod_{i=1}^l C_3^{\mathcal{D}et_{k\Delta E_i}}(f, g, a),$$

where the product is taken over a finite set of irreducible curves $E_i$ on $X$ such that the union of the supports of divisors $(f)$, $(g)$ and $(a)$ is a subset of the union of these curves $E_i$. Now from (17) we obtain that $C_3^{\mathcal{D}et_{k\Delta E_i}}(f, g, a) = 1$ for any $1 \leq i \leq l$. Thus $\{f, g\} = 1$ and we have proved the third statement of the theorem. The theorem is proved.

Remark 5. We also obtain the following two statements from the proof of the above theorem.

1. Let $x$ be a point on $X$ and let $\Delta_x$ be the set of all pairs $\{x \in C\}$ where $x$ is fixed. We fix an element $a \in K_x^*$. Then the central extension $\widetilde{\text{GL}}_{n,a}(\mathbf{k}_{\Delta_x})$ of the group $\text{GL}_n(\Delta_x)$ by the group $k^*$ splits over the subgroup $\text{GL}_n(k_X)$ of the group $\text{GL}_n(\Delta_x)$.

2. Let $C$ be an irreducible projective curve on $X$ and let $\Delta_C$ be the set of all pairs $\{x \in C\}$ where $C$ is fixed. We fix an element $a \in K_C^*$. Then the central extension $\widetilde{\text{GL}}_{n,a}(\mathbf{k}_{\Delta_C})$ of the group $\text{GL}_n(\Delta_C)$ by the group $k^*$ splits over the subgroup $\text{GL}_n(k_C)$ of the group $\text{GL}_n(\Delta_C)$.

Remark 6. We note that for partial symmetrized cocycles reciprocity laws with values in cohomology groups were proved in [9] for a two-dimensional complex analytic space by topological methods. Therefore, the reciprocity laws in Theorem 1 are analogues of the reciprocity laws in [9] for an algebraic surface defined over a field of any characteristic.

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