ON THE STABILITY OF SETS FOR REACTION–DIFFUSION COHEN–GROSSBERG DELAYED NEURAL NETWORKS

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ABSTRACT. In this paper, we introduce the notion of stability of sets for reaction-diffusion Cohen–Grossberg neural networks with time-varying delays. The Lyapunov–Razumikhin technique and a comparison principle are adapted to prove the new stability criteria. In addition, the obtained results are extended to the uncertain case, and the robust stability notion is also investigated. Examples are considered to demonstrate the effectiveness of our results.

1. Introduction. The so-called Cohen–Grossberg neural networks (CGNNs) introduced by Cohen and Grossberg in 1983 [5] are attracted more research interest, due to their extensive applications in signal transmission, pattern recognition, optimization. This class of neural network systems is also very advantageous in global pattern formation and partial memory storage and generalizes a number of cellular neural network models, including Hopfield-type neural network models. In addition, the delayed CGNNs have been extensively investigated. See, for example, [6, 12, 13, 18, 44, 46] and the references therein. Indeed, fixed and time-varying delays ineluctably exist in network models and working networks, and they may lead to oscillation, instability, bifurcation, or chaos of a network [14, 15, 42]. The topic of delayed CGNNs does not lose its actuality today, but the centrality of such neural network models for theory and applications is witnessed by the current persistency of new contributions to the topic [1, 3, 25].

On the other hand, reaction-diffusion differential equations serve as mathematical models to investigate the dynamics of processes that not merely dependent on the evolution time, but also intensively dependent on the space position of their variables. Since diffusion factors and delays can greatly affect the dynamics of a neural network model, their effects have been broadly studied in the literature [4, 7, 16, 21, 26, 27, 32, 35, 41], including results on delayed reaction-diffusion CGNNs [17, 24, 40].

2020 Mathematics Subject Classification. Primary: 92B20, 34K20; Secondary: 34K60.

Key words and phrases. Cohen–Grossberg neural networks, Time delays, Reaction-diffusion terms, Stability of sets analysis, Lyapunov-type functions.

This paper is supported in part by the European Regional Development Fund through the Operational Programme “Science and Education for Smart Growth” under Contract UNTe No. BG05M2OP001–1.001–0004 (2018–2023).

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The stability of neural networks is an important issue in order to understand and/or predict the behavior in time, and excellent stability results for reaction-diffusion CGNNs have been reported, see \[9, 10, 23, 38, 39\]. In the existing studies, the authors mainly investigated the exponential stability behavior of equilibria. However, in many applications of neural networks some extensions of the stability notion are more appropriate. In this paper we will further expand the stability analysis of reaction-diffusion CGNNs with a particular focus on addressing the so-called stability of sets.

The origins of the stability of sets problem go back to the question: How far can initial data be allowed to vary without disrupting the stability properties established in the immediate vicinity of equilibrium states of neural networks? On this problem, researchers proposed to study stable sets (or manifolds) \[8\]. The notion of stability of sets, which includes as a special case stability of an equilibrium, stability of invariant sets, stability of moving manifolds, etc., is one of the most important notions in the stability theory. Similar ideas for different types of delay differential models are presented in \[31, 33, 34, 43, 45\]. Some proved applications for the use of stability of sets are maneuvering systems \[30\], adaptive systems, and observer designs \[11\]. The stable and unstable manifolds are also key notions in differential dynamics \[2, 29\]. Indeed, stability of sets is a more realistic, more general and complicated concept than exponential stability. However, despite the high importance of the stability of sets notion for theory and applications it is not developed for reaction-diffusion CGNNs and this paper’s aim is mainly to fill the gap.

In addition, we extend our results to the uncertain case. Indeed, uncertain factors due to modeling errors, external disturbances, and parameter fluctuations can often sabotage the qualitative behavior of complex systems \[16, 19, 45\]. Note that some authors investigated the effect of uncertain terms on the exponential stability and synchronization of CGNNs \[6, 9, 10, 37, 44\]. Uncertain CGNNs with reaction-diffusion terms are hardly considered and investigated \[28, 36\]. However, in order to establish robust stability or synchronization criteria, all existing results are related to separate equilibria of CGNNs and do not consider a set of solutions. Compared with all previous works, we investigate the effects on some uncertain terms on the stability of sets in our robust stability analysis. A new notion of robust stability of sets with respect to reaction-diffusion CGNNs is also introduced.

In the present paper, motivated by the above considerations, we introduce the notion of stability of sets for CGNNs with time-varying delays and reaction-diffusion terms. Based on the comparison principle and Razumikhin technique \[4, 22, 31, 34\] for delayed systems and suitable Lyapunov function method, some sufficient conditions for stability with respect to sets of a general type for CGNNs under consideration are introduced. In addition, we introduce the notion of robust stability of sets for the neural network model under consideration, and consider uncertainty in our dynamic analysis.

The remaining part of the paper is organized as follows. In section 2, we introduce the model of CGNNs with time-varying delays and reaction-diffusion terms. Some notations, definitions, and preliminary results which will be used later are also presented. Section 3 is devoted to our main stability of sets results. We establish new sufficient conditions that ensure stability, uniform stability and asymptotic stability of sets with respect to the reaction-diffusion delayed CGNN system. In section 4 we extend our main stability of sets results to the uncertain case. By employing the
Inequality technique, we propose robust stability of sets criteria CGNNs with time-varying delays, reaction-diffusion terms and parametric uncertainties. Examples are given to validate the effectiveness of the derived results in Section 5. Finally, conclusion remarks are drawn in Section 6.

2. Preliminaries. Let \( \mathbb{R}_+ = [0, \infty) \), \( ||x|| = \left( \sum_{k=1}^{n} x_k^2 \right)^{1/2} \) be the norm of \( x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n \). Let \( \Omega \) be an open and bounded set in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and the measure expressed by \( \text{mes} \Omega > 0 \), \( 0 = (0, 0, ..., 0)^T \in \Omega \). For \( u(t, x) = (u_1(t, x), u_2(t, x), ..., u_m(t, x))^T \in \mathbb{R}^m \), we also consider the following norm:

\[
||u(t, x)||_2 = \left[ \int_{\Omega} \sum_{i=1}^{m} u_i^2(t, x) \, dx \right]^{1/2}.
\]

We note that, the space \( L^2(\Omega) \) of all real functions on \( \Omega \), which are \( L^2 \) for the Lebesgue measure is a Banach space with respect to the above norm \([23, 38, 40]\).

In the present paper we will investigate the following delayed reaction-diffusion CGNN

\[
\frac{\partial u_i(t, x)}{\partial t} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i(u_i(t, x)) \left[ b_i(u_i(t, x)) \right. \\

\left. -I_i(t, x) - \sum_{j=1}^{m} c_{ij}(t)f_j(u_j(t, x)) \right. \\

\left. - \sum_{j=1}^{m} w_{ij}(t)g_j(u_j(t - \tau_j(t), x)) \right],
\]

where \( i = 1, 2, ..., m, m \geq 2 \), \( t > 0, x = (x_1, x_2, ..., x_n)^T \in \Omega \), \( u_i(t, x) \) denotes the state of the \( i \)-th neural unit at time \( t \) and in space \( x \), \( \tau_j(t) \) is the transmission time-varying delay \((0 \leq \tau_j(t) \leq \tau, \tau = \text{const.})\), \( a_i(u_i(t, x)) \) is an amplification function, \( b_i(u_i(t, x)) \) is an appropriately behaved function, \((c_{ij})_{m \times m}(t) \in \mathbb{R}^{m \times m}\) and \((w_{ij})_{m \times m}(t) \in \mathbb{R}^{m \times m}\) are, respectively, the connection weight and time-varying delay connection weight matrices of \( j \)-th neuron on the \( i \)-th neuron, \( I_i(t, x) \) is the external input of the \( i \)-th neural unit, \( f_j(\cdot) \) and \( g_j(\cdot) \) are the activation functions of the \( j \)-th neuron, the smooth functions \( D_{ik} = D_{ik}(t, x) \geq 0 \) are the transmission diffusion coefficients along the \( i \)th neuron.

Let \( \mathcal{C} = C([-\tau, 0] \times \Omega, \mathbb{R}^m) \) be the space of all continuous functions from \([-\tau, 0] \times \Omega \) into \( \mathbb{R}^m \) and \( \varphi_0 = (\varphi_0, \varphi_2, ..., \varphi_m)^T \in \mathcal{C} \). In our paper, we will denote by \( u(t; \varphi_0) \), \( u = (u_1, u_2, ..., u_m)^T \in \mathbb{R}^m \), the solution of system (1) which satisfies the following boundary and initial conditions:

\[
u_i(t, x) = 0, \ t \in [-\tau, \infty), \ x \in \partial \Omega, \ (2)
\]

\[
u_i(s, x) = \varphi_i(s, x), \ s \in [-\tau, 0], \ x \in \Omega, \ (3)
\]

where \( i = 1, 2, ..., m \).

For any function \( \varphi = (\varphi_1, \varphi_2, ..., \varphi_m)^T \in \mathcal{C} \) we define

\[
||\varphi||_\tau = \max_{-\tau \leq s \leq 0} ||\varphi(s, x)||_2.
\]
Remark 1. The classical delayed CGNNs [1, 3, 5, 18, 39] are particular cases of the neural network model (1) for $D_{ik} = 0$, $i = 1, 2, ..., m$, $k = 1, 2, ..., n$. If, in addition, $a_i = 1$, $i = 1, 2, ..., m$ the system (1) becomes a well-known reaction-diffusion neural network system [4, 7, 16, 20, 21, 27, 32, 35, 41]. Therefore, the system (1) generalizes many useful types of neural network models.

Throughout the paper, we make the following assumptions:

A1. The functions $a_i$, $i = 1, 2, ..., m$, are positive, continuous and bounded, i.e. there exist constants $a_i$ and $\bar{a}_i$ such that $1 < a_i(\chi) \leq \bar{a}_i$ for $\chi \in \mathbb{R}$.

A2. The functions $b_i$, $i = 1, 2, ..., m$, are continuous and there exist positive constants $B_i$ with

$$\frac{b_i(\chi_1) - b_i(\chi_2)}{\chi_1 - \chi_2} \geq B_i > 0$$

for $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$.

A3. The activation functions $f_i$ and $g_i$ are continuous and Lipschitz, i.e. there exist positive constants $L_i, M_i$, $i = 1, 2, ..., m$, with

$$|f_i(\chi_1) - f_i(\chi_2)| \leq L_i|\chi_1 - \chi_2|,$$

$$|g_i(\chi_1) - g_i(\chi_2)| \leq M_i|\chi_1 - \chi_2|$$

for all $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$.

A4. The activation functions $f_i$ and $g_i$, $i = 1, 2, ..., m$, are bounded in $\mathbb{R}$, and $f_i(0) = g_i(0) = 0$, $i = 1, 2, ..., m$.

A5. The functions $c_{ij}$, $w_{ij}$ and $L_i$, $i, j = 1, 2, ..., m$ are continuous on their domains.

A6. For any $i = 1, 2, ..., m$ and $k = 1, 2, ..., n$ there exist constants $d_{ik} \geq 0$ such that

$$D_{ik}(t, x) \geq d_{ik}, \ t > 0, \ x \in \Omega.$$

Remark 2. Assumptions A1-A6 guarantee the existence and uniqueness of solutions of the system (1) under boundary and initial conditions (2), (3). See, for example, [23, 24] and the references therein. For more results from the fundamental theory of CGNNs with delays and reaction-diffusion terms we refer to [17, 39, 40].

In the following, we introduce the stability of sets notion for reaction-diffusion delayed CGNNs. For this reason, we will need the new notations. Let $M \subset [-\tau, \infty) \times \Omega \times \mathbb{R}^m$.

$$M(t, x) = \{u \in \mathbb{R}^m : (t, x, u) \in M(t, x) \in \mathbb{R}^+ \times \Omega\};$$

$$M_0(t, x) = \{z \in \mathbb{R}^m : (t, x, z) \in M, \ (t, x) \in [-\tau, 0] \times \Omega\};$$

$$d(u, M(t, x)) = \inf_{v \in M(t, x)} ||u - v||_2$$ is the distance between $u \in \mathbb{R}^m$ and $M(t, x)$;

$$M(t, x)(\varepsilon) = \{u \in \mathbb{R}^m : d(u, M(t, x)) < \varepsilon\}$$ ($\varepsilon > 0$) is an $\varepsilon$- neighborhood of $M(t, x)$;

$$d_0(\varphi, M_0(t, x)) = \max_{s \in [-\tau, 0]} d(\varphi(s, x), M_0(s, x)), \varphi \in \mathcal{C};$$

$$M_0(t, x)(\varepsilon) = \{\varphi \in \mathcal{C} : d_0(\varphi, M_0(t, x)) < \varepsilon\}$$ is an $\varepsilon$- neighborhood of $M_0(t, x)$;
\[ M(t,x)(\varepsilon) = \{u \in \mathbb{R}^m : d(u, M(t,x)) \leq \varepsilon\}; \]

\[ M_0(t,x)(\varepsilon) = \{\varphi \in \mathcal{C} : d_0(\varphi, M_0(t,x)) \leq \varepsilon\}; \]

\[ S_\alpha = \{u \in \mathbb{R}^m : ||u||_2 \leq \alpha\}; \quad S_\alpha^\prime(\mathcal{C}) = \{\varphi \in \mathcal{C} : ||\varphi||_\tau \leq \alpha\}. \]

**Definition 2.1.** We say that the solutions of system (1) are:

(a) *equi-M-bounded*, if for any positive constants \(\eta > 0\) and \(\alpha > 0\) there exists a constant \(\beta = \beta(\eta, \alpha) > 0\) such that \(x \in \Omega\) and \(\varphi_0 \in S_\alpha^\prime(\mathcal{C}) \cap M_0(t,x)(\beta)\) imply

\[ u(t,x;\varphi_0) \in M(t,x)(\beta), \quad t \geq 0; \]

(b) *uniformly M-bounded*, if the number \(\beta\) from (a) depends only on \(\eta\).

**Definition 2.2.** The set \(M\) is said to be:

(a) *uniformly stable* with respect to system (1), if for any positive constants \(\alpha > 0\) and \(\varepsilon > 0\) there exists a constant \(\delta = \delta(\alpha, \varepsilon) > 0\) such that \(x \in \Omega\) and \(\varphi_0 \in S_\alpha^\prime(\mathcal{C}) \cap M_0(t,x)(\delta)\) imply

\[ u(t,x;\varphi_0) \in M(t,x)(\varepsilon), \quad t \geq 0; \]

(b) *uniformly globally attractive* with respect to system (1), if for any positive constants \(\eta > 0\), \(\varepsilon > 0\) and \(\alpha > 0\) there exists a constant \(\sigma = \sigma(\eta, \varepsilon) > 0\) such that \(x \in \Omega\) and \(\varphi_0 \in S_\alpha^\prime(\mathcal{C}) \cap M_0(t,x)(\eta)\) imply

\[ u(t,x;\varphi_0) \in M(t,x)(\varepsilon), \quad t \geq \sigma; \]

(c) *uniformly globally asymptotically stable* with respect to system (1), if \(M\) is a uniformly stable and uniformly globally attractive set of system (1), and if the solutions of system (1) are uniformly \(M\)-bounded.

**Remark 3.** In the case \(M = [-\tau, \infty) \times \Omega \times \{u \in \mathbb{R}^m : u = 0\}\), where \(0 = (0,0,...,0)^T\) is the zero equilibrium of (1), Definition 2.2 is reduced to the definition of the Lyapunov-like stability of the zero equilibrium of reaction-diffusion CGNNs [38, 39], and Definition 2.1 is a generalization of the boundedness definitions for the solutions of different classes of reaction-diffusion neural network models [20, 40].

In the case \(M = \{[-\tau, \infty) \times \Omega \times \mathbb{R}^m : u = u^*\}\), where \(u^* = (u_1^*, u_2^*,..., u_m^*)^T\) is a non-zero equilibrium of (1), Definition 2.2 is reduced to the definition of the Lyapunov-like stability of the non-zero equilibrium of reaction-diffusion CGNNs [23, 24].

Following the works [4, 22, 31, 32, 34], we will give some definitions and comparison results from the Lyapunov–Razumikhin method.

To prove our main results we will use Lyapunov-type functions from the class \(C_0 = \{V : [0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}_+ : V \text{ is continuous on } \mathbb{R}_+ \times \mathbb{R}^m, V \text{ is locally Lipschitz with respect to its second argument}\}\).

Let \(F : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}^m,\)

\[ F(t, \tilde{\varphi}) = (F_1(t, \tilde{\varphi}), F_2(t, \tilde{\varphi}), ..., F_m(t, \tilde{\varphi}))^T \]

be a continuous on \([0, \infty) \times \mathcal{C}\) function that is locally Lipschitz in \(\tilde{\varphi} \in \mathcal{C}\).

**Definition 2.3.** For a function \(V \in C_0\), the *upper right-hand derivative* of \(V\) along the solutions of the system

\[ \frac{d u(t,\cdot)}{d t} = F(t, u(t-\tau,\cdot)), t > 0 \]
is defined by

\[ D^+ V(t, \tilde{\varphi}(0, .)) = \lim_{\chi \to 0^+} \sup_{\chi} \frac{1}{\chi} [V(t + \chi, \tilde{\varphi}(0, .) + \chi F(t, \tilde{\varphi})) - V(t, \tilde{\varphi}(0, .))] \]

The next result is similar to the comparison results in [31, 32, 34].

**Lemma 2.4.** Let the function \( V \in C_0 \) be such that:

1. For \( V(t + s, \tilde{\varphi}(s, .)) \leq V(t, \tilde{\varphi}(0, .)), -\tau \leq s \leq 0, t > 0 \) and \( \tilde{\varphi} \in C \), the following inequality is satisfied

\[ D^+ V(t, \tilde{\varphi}(0, .)) \leq F(t, \tilde{\varphi}(0, .)) \]

2. The maximal solution \( y^+(t, .) \) of the comparison problem (4) is defined in \( \mathbb{R}_+ \times \Omega \).

Then

\[ V(t, u(t, .)) \leq y^+(t, .), \quad t > 0. \]

**Remark 4.** For more comparison lemmas with Razumikhin conditions of the type \( V(t + s, \tilde{\varphi}(s, .)) \leq V(t, \tilde{\varphi}(0, .)), -\tau \leq s \leq 0, t > 0 \) and \( \tilde{\varphi} \in C \), we refer to [4, 21, 30, 31, 33, 34].

In the case when \( F(t, u) = 0 \) for \( (t, u) \in [0, \infty) \times \mathbb{R}^m \), we deduce the following corollary from Lemma 2.4.

**Corollary 1.** Assume that the function \( V \in C_0 \) is such that for \( V(t + s, \tilde{\varphi}(s, .)) \leq V(t, \tilde{\varphi}(0, .)), -\tau \leq s \leq 0, t > 0 \) and \( \tilde{\varphi} \in C \), the following inequality is satisfied

\[ D^+ V(t, \tilde{\varphi}(0, .)) \leq 0. \]

Then

\[ V(t, u(t, .)) \leq V(0, u(0, .)), \quad t > 0. \]

Finally, we provide the following lemma.

**Lemma 2.5.** [21] Assume that the set \( \Omega \) is defined by \( |x_k| < l_k \) \( (k = 1, 2, ..., n) \), the function \( v : \Omega \to \mathbb{R} \) is continuously differentiable on \( x \in \Omega \) and such that \( v(x)|_\Omega = 0 \). Then

\[ \int_{\Omega} v^2(x) dx \leq l_k^2 \int_{\Omega} \left| \frac{\partial v(x)}{\partial x_k} \right|^2 dx. \]

In our main theorems, we will also use the Hahn class \( K = \{ w \in C[\mathbb{R}_+, \mathbb{R}_+] : w \text{ is strictly increasing and } w(0) = 0 \} \) of functions \( w \), which are called wedges.

**3. Main results.**

**Theorem 3.1.** Assume that:

1. Assumptions A1-A6 hold.

2. \( M(t, x) \neq \emptyset \) for \( t \in [0, \infty) \times \Omega \) and \( M_0(t, x) \neq \emptyset \) for \( t \in [-\tau, 0) \times \Omega \).

3. The assumptions of Lemma 2.4 are satisfied for \( F(t, u) = -p(t)w_3(d(u(t, .), M(t, .))) \) for \( (t, u) \in \mathbb{R}_+ \times \mathbb{R}^m \), where the continuous function \( p : \mathbb{R}_+ \to (0, \infty) \) and \( w_3 \in K \).

4. The function \( V \in C_0 \) is such that \( V(t, u(t, .)) = 0 \) for \( (t, u) \in M, t \geq 0 \) and \( V(t, u(t, .)) > 0 \) for \( (t, x, u) \in \{ \mathbb{R}_+ \times \Omega \times \mathbb{R}^m \} \setminus M \).

5. The inequalities

\[ w_1(d(u, M(t, x))) \leq V(t, u) \leq w_2(d(u, M(t, x))), \]

hold, where \( w_1(\xi) \to \infty \) as \( \xi \to \infty \), \( (t, x) \in \mathbb{R}_+ \times \Omega \), \( u \in \mathbb{R}^m \), \( w_1, w_2 \in K \).

6. \( \int_0^\infty p(\theta)w_2(w_2^{-1}(\eta))d\theta = \infty \) for each sufficiently small value of \( \eta > 0 \).
Then the set \( M \) is uniformly globally asymptotically stable with respect to system (1).

**Proof.** Let \( \varepsilon > 0 \). Choose \( \delta = \delta(\varepsilon) > 0 \), \( \delta < \varepsilon \) so that \( w_2(\delta) < w_1(\varepsilon) \).

Let \( \alpha > 0 \) be arbitrary, \( \varphi_0 \in \overline{S}(C) \cap M_0(t, x)(\delta), x \in \Omega \) and \( u(t, x) = u(t, x; \varphi_0) \) be the solution of (1), (2), (3).

Since all conditions of Corollary 1 are satisfied, we have

\[
V(t, u(t, x)) \leq V(0, u(0, x)), \quad t > 0, \quad x \in \Omega.
\]

From (6) and condition 5 of Theorem 3.1, it follows that for \( t \in \mathbb{R}_+ \) the following inequalities are valid

\[
w_1(d(u(t, x; \varphi_0), M(t, x))) \leq V(t, u(t, x)) \leq V(0, u(0, x))
\]

\[
\leq w_2(d(0, x), M(0, x))) \leq w_2(d_0(0, M_0(t, x)) < w_2(\delta) < w_1(\varepsilon).
\]

Hence, \( u(t, x; \varphi_0) \in M(t, x)(\varepsilon) \) for all \( t \geq 0 \). This proves the uniform stability of the set \( M \) with respect to the reaction-diffusion CGNN (1).

Next, we will prove that the solutions of (1) are uniformly \( M \)-bounded.

Let \( \eta > 0 \). Choose the number \( \beta = \beta(\eta) > 0 \) so that \( w_2(\eta) < w_1(\beta), \beta > \eta \).

Let \( \alpha > 0 \) and \( \varphi_0 \in \overline{S}_a(C) \cap M_0(t, x)(\eta) \). Using again (6) and condition 5 of Theorem 3.1, we obtain

\[
w_1(d(u(t, x; \varphi_0), M(t, x))) \leq V(t, u(t, x)) \leq V(0, u(0, x))
\]

\[
\leq w_2(d(0, x), M(0, x))) \leq w_2(d_0(\varphi_0, M_0(t, x))) < w_2(\eta) < w_1(\beta)
\]

for \( t \in \mathbb{R}_+ \). Hence, \( u(t, x; \varphi_0) \in M(t, x)(\beta) \) for \( t \geq 0 \).

Finally, let \( \eta > 0 \) and \( \varepsilon > 0 \) be given and let the number \( \sigma = \sigma(\eta, \varepsilon) > 0 \) be chosen so that

\[
\int_0^\sigma p(\vartheta)w_3[w_2^{-1}(\frac{w_1(\varepsilon)}{2})]d\vartheta > w_2(\eta).
\]

(This is possible in view of condition 6 of Theorem 3.1.)

Let \( \alpha > 0 \) be arbitrary, \( \varphi_0 \in \overline{S}_a(C) \cap M_0(t, x)(\eta) \) and \( u(t, x) = u(t, x; \varphi_0) \).

Assume that for any \( t \in [0, \sigma] \) the following inequality holds

\[
d(u(t, x), M(t, x)) \geq w_2^{-1}(\frac{w_1(\varepsilon)}{2}).
\]

Then, by condition 6 of Theorem 3.1, it follows that

\[
\int_0^\sigma D^+(V(\vartheta, u(\vartheta, .)))d\vartheta \leq -\int_0^\sigma p(\vartheta)w_3[w_2^{-1}(\frac{w_1(\varepsilon)}{2})]d\vartheta < -w_2(\eta).
\]

On the other, we obtain

\[
\int_0^\sigma D^+(V(\vartheta, u(\vartheta, .)))d\vartheta = V(\sigma, u(\sigma, .)) - V(0, \varphi_0(0, .)),
\]

whence, in view of (9), it follows that \( V(\sigma, u(\sigma, .)) < 0 \), which contradicts (5).

The contradiction obtained shows that there exists \( t^* \in [0, \sigma] \), such that

\[
d(u(t^*, x), M(t^*, x)) < w_2^{-1}(\frac{w_1(\varepsilon)}{2}), \quad x \in \Omega.
\]

Then for \( t \geq t^* \) (hence for any \( t \geq \sigma \) as well) and \( x \in \Omega \) the following inequalities are valid

\[
w_1(d(u(t, x), M(t, x))) \leq V(t, u(t, x)) \leq V(t^*, u(t^*, x))
\]

\[
\leq w_2(d(u(t^*, x), M(t^*, x))) < \frac{w_1(\varepsilon)}{2} < w_1(\varepsilon).
\]
Hence, \( u(t, x) \in M(t, x)(\epsilon) \) for \( t \geq \sigma \), i.e. the set \( M \) is uniformly globally attractive with respect to the reaction-diffusion CGNN (1).

In the next result we will use integrally positive functions defined by the following definition.

**Definition 3.2.** A measurable function \( \lambda(t) : [0, \infty) \rightarrow \mathbb{R}_+ \), is **integrally positive**, if

\[
\int_J \lambda(t) \, dt = \infty
\]

whenever \( J = \bigcup_{k=1}^{\infty} [\alpha_k, \beta_k], \alpha_k < \beta_k < \alpha_{k+1}, \text{ and } \beta_k - \alpha_k \geq \theta > 0, k = 1, 2, \ldots. \)

**Theorem 3.3.** Assume that:

1. Conditions 1, 2, 4, 5 of Theorem 3.1 hold.
2. The assumptions of Lemma 2.4 are satisfied for \( F(t, u) = -\lambda(t)w_3(d(u(t, .), M(t, .))) \), \( (t, u) \in \mathbb{R}_+ \times \mathbb{R}^m \),

where \( \lambda(t) \) is an integrally positive function and \( w_3 \in K \).

Then the set \( M \) is uniformly globally asymptotically stable with respect to system (1).

**Proof.** The uniform stability of the set \( M \) with respect to (1) and the uniform \( M \)-boundedness of the solutions of (1) follow from Theorem 3.1.

Now, we will prove the uniform global attractivity of the set \( M \) with respect to the reaction-diffusion CGNN (1).

Let again \( \epsilon > 0 \) and \( \eta > 0 \) be given. Choose the number \( \delta = \delta(\epsilon) > 0 \) so that \( w_2(\delta) < w_1(\epsilon) \).

We will prove that there exists \( \sigma = \sigma(\epsilon, \eta) > 0 \) such that for any solution \( u(t, x) = u(t, x; \varphi_0) \) of the reaction-diffusion CGNN (1) for which \( \varphi_0 \in \mathcal{S}_\alpha(\mathcal{C}) \cap \mathcal{M}_\theta(t, x)(\eta) \) (\( \alpha > 0 \) - arbitrary), and there exists \( t^* \in [0, \sigma] \) such that the following inequality is valid

\[
d(u(t^*, x), M(t^*, x)) < \delta(\epsilon), \ x \in \Omega. \tag{11}
\]

Suppose that this is not true. Then, for any \( \sigma > 0 \) there exists a solution \( u(t, x) = u(t, x; \varphi_0) \) of (1) for which \( \varphi_0 \in \mathcal{S}_\alpha(\mathcal{C}) \cap \mathcal{M}_\theta(t, x)(\eta) \), such that

\[
d(u(t, x), M(t, x)) \geq \delta(\epsilon), \ t \in [0, \sigma], \ x \in \Omega. \tag{12}
\]

For \( t \geq 0 \) it follows from condition 2 of Theorem 3.3 that

\[
V(t, u(t, .)) - V(0, u(0, .)) \leq \int_0^t D^+ V(\vartheta, u(\vartheta, .)) d\vartheta \leq -\int_0^t \lambda(\vartheta) w_3(d(u(\vartheta, .), M(\vartheta, .))) d\vartheta. \tag{13}
\]

The properties of the function \( V(t, u(t, .)) \) on \( \mathbb{R}_+ \) imply the existence of the finite limit

\[
\lim_{t \to \infty} V(t, u(t, .)) = v_0 \geq 0. \tag{14}
\]

Then, condition 2 of Theorem 3.3, (12), (13) and (14), imply

\[
\int_0^\infty \lambda(t) w_3(d(u(t, .), M(t, .))) dt \leq w_2(\eta) - v_0.
\]
From the integral positivity of the function \( \lambda(t) \), it follows that the number \( \sigma \) can be chosen so that
\[
\int_0^\sigma \lambda(t)dt \geq \frac{w_2(\eta) - v_0 + 1}{w_3(\delta(\varepsilon))}.
\]
Then, we obtain
\[
w_2(\eta) - v_0 \geq \int_0^\infty \lambda(t)w_3(d(u(t,.), M(t,.)))dt
\geq \int_0^\sigma \lambda(t)w_3(d(u(t,.), M(t,.)))dt \geq w_3(\delta(\varepsilon)) \int_0^\sigma \lambda(t)dt > w_2(\eta) - v_0 + 1.
\]
The contradiction obtained proves the existence of a positive constant \( \sigma = \sigma(\varepsilon, \eta) \) and a \( t^* \in [0, \sigma] \) such that for any solution \( u(t, x) = u(t, x; \varphi_0) \) of the reaction-diffusion CGNN (1) with \( \varphi_0 \in \mathcal{S}_\alpha(\mathcal{C}) \cap \mathcal{M}_0(t, x)(\eta) \) the inequality (11) holds.

Then for \( t \geq t^* \) (hence for any \( t \geq \sigma \) as well) and \( x \in \Omega \), we have
\[
w_1(d(u(t, x), M(t, x))) \leq V(t, u(t, x)) \leq V(t^*, u(t^*, x)) \leq w_2(d(u(t^*, x), M(t^*, x))) < w_2(\delta) < w_1(\varepsilon),
\]
which proves that the set \( M \) is uniformly globally attractive with respect to the reaction-diffusion CGNN (1).

**Remark 5.** The results established in Theorem 3.1 and Theorem 3.3 generalize many existing stability and boundedness results for reaction-diffusion CGNNs. For example, the results in [38, 39] on exponential stability of zero equilibriums can be received as corollaries for \( M = [-\tau, \infty) \times \Omega \times \{u \in \mathbb{R}^m : u = 0\} \), where \( 0 = (0, 0, ..., 0)^T \) is the zero equilibrium of (1), the stability results in [23, 24] for non-zero equilibriums \( u^* \) of (1) can be obtained from theorems 3.1 and 3.3 for \( M = [-\tau, \infty) \times \Omega \times \{u \in \mathbb{R}^m : u = u^*\} \). Thus, our results generalize and extend some results in the previous literature.

In the next result, we will consider the \( n \)-dimensional set \( \Omega \) of points \( x, x = (x_1, x_2, ..., x_n)^T \) such that \( |x_k| < l_k, k = 1, 2, ..., n \), where \( l_k, k = 1, 2, ..., n \), are positive constants. Denote by \( \tilde{d}_i = \sum_{k=1}^n \frac{d_{ik}l_k}{T_k} \), \( i = 1, 2, ..., m \), \( \tilde{c}_{ij}^+ = \sup_{t \in \mathbb{R}^+} c_{ij}(t) \), \( \tilde{w}_{ij}^+ = \sup_{t \in \mathbb{R}^+} w_{ij}(t) \).

We will assume that \( \tilde{u}^* = (\tilde{u}^*_1, \tilde{u}^*_2, ..., \tilde{u}^*_m)^T \in \mathbb{R}^m_+ \) and \( \tilde{\pi}^* = (\tilde{\pi}^*_1, \tilde{\pi}^*_2, ..., \tilde{\pi}^*_m)^T \in \mathbb{R}^m_+ \) are two constant solutions of the reaction-diffusion CGNN (1).

**Theorem 3.4.** Let assumptions A1-A6 hold and the following condition
\[
\min_{1 \leq i \leq m} \left[ 2(\tilde{d}_i + a_iB_i) - \tilde{a}_i \sum_{j=1}^m (L_j \tilde{c}_{ij}^+ + M_j \tilde{w}_{ij}^+ + L_i \tilde{c}_{ij}^+) \right] > \max_{1 \leq i \leq m} \left( M_i \sum_{j=1}^m \tilde{\pi}_j \tilde{w}_{ji}^+ \right) > 0
\]
be fulfilled.

Then the set \( M = [-\tau, \infty) \times \Omega \times \{u^* \in \mathbb{R}^m : u^*_i \leq u_i, i = 1, 2, ..., m\} \) is uniformly globally asymptotically stable with respect to the reaction-diffusion CGNN (1).
Proof. Consider a solution \( u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x))^T \) of the CGNN (1) with initial function \( \varphi_0 \in C, \varphi_0 = (\varphi_{01}, \varphi_{02}, \ldots, \varphi_{0m})^T \), i.e. \( u(t, x) = u(t, x; \varphi_0) \).

Let \( v \in M(t, x), v = v(t, x), (t, x) \in [-\tau, \infty) \times \Omega. \)

We define a Lyapunov function

\[
V(u(t, \cdot), v(t, \cdot)) = \frac{1}{2} \inf_{v \in M(t, x)} \|u - v\|_2^2.
\]

We have that

\[
\frac{1}{2} \frac{d}{dt} \|u(t, \cdot) - v(t, \cdot)\|_2^2 \leq \int_{\Omega} \sum_{i=1}^{m} \left| u_i(t, x) - v_i(t, x) \right| \frac{\partial (u_i(t, x) - v_i(t, x))}{\partial t} dx. \tag{16}
\]

Suppose, without loss of generality, that \( u_i(t, x) \geq v_i(t, x) \) for any \( i = 1, 2, \ldots, m \) and \((t, x) \in [-\tau, \infty) \times \Omega.\)

It follows, then, from the definition of the set \( M \), that

\[
\frac{1}{2} \frac{d}{dt} \|u(t, \cdot) - v(t, \cdot)\|_2^2 \leq \sum_{i=1}^{m} \int_{\Omega} (u_i(t, x) - u_i^*) \frac{\partial (u_i(t, x) - u_i^*)}{\partial t} dx. \tag{17}
\]

Since \( u^* = (u_1^*, u_2^*, \ldots, u_m^*)^T \) is a constant solution of (1), from A1, we have

\[
(u_i(t, x) - u_i^*) \frac{\partial (u_i(t, x) - u_i^*)}{\partial t} \\
\leq (u_i(t, x) - u_i^*) \left( \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i(t, x) - u_i^*)}{\partial x_k} \right) \\
- a_i [b_i(u_i(t, x)) - b_i(u_i^*)] \\
+ \overline{a_i} \sum_{j=1}^{m} c_{ij}^+ |f_j(u_j(t, x)) - f_j(u_j^*)| \\
+ \overline{a_i} \sum_{j=1}^{m} w_{ij}^+ |g_j(u_j(t - \tau_j(t, x)) - g_j(u_j^*)| \right)
\tag{18}
\]

or, after the integration over \( \Omega \), we get

\[
\int_{\Omega} (u_i(t, x) - u_i^*) \frac{\partial (u_i(t, x) - u_i^*)}{\partial t} dx \\
\leq \int_{\Omega} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i(t, x) - u_i^*)}{\partial x_k} \right) (u_i(t, x) - u_i^*) dx \\
- \int_{\Omega} a_i (u_i(t, x) - u_i^*) [b_i(u_i(t, x)) - b_i(u_i^*)] dx \\
+ \overline{a_i} \int_{\Omega} (u_i(t, x) - u_i^*) \sum_{j=1}^{m} c_{ij}^+ |f_j(u_j(t, x)) - f_j(u_j^*)| dx \\
+ \overline{a_i} \int_{\Omega} (u_i(t, x) - u_i^*) \sum_{j=1}^{m} w_{ij}^+ |g_j(u_j(t - \tau_j(t, x)) - g_j(u_j^*)| dx. \tag{19}
\]
Applying the Green’s theorem, the Dirichlet-type boundary conditions, A6 and Lemma 2.5, we obtain

\[
\int_{\Omega} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial (u_i(t,x) - u_i^*)}{\partial x_k} \right) (u_i(t,x) - u_i^*) \, dx
\]

\[
= - \sum_{k=1}^{n} \int_{\Omega} D_{ik} \left( \frac{\partial (u_i(t,x) - u_i^*)}{\partial x_k} \right)^2 \, dx
\]

\[
\leq - \sum_{k=1}^{n} \int_{\Omega} d_{ik} \left( \frac{\partial (u_i(t,x) - u_i^*)}{\partial x_k} \right)^2 \, dx
\]

\[
\leq - \sum_{k=1}^{n} \int_{\Omega} \frac{d_{ik}}{l_k^2} (u_i(t,x) - u_i^*)^2 \, dx = - \tilde{d}_i \int_{\Omega} (u_i(t,x) - u_i^*)^2 \, dx. \tag{20}
\]

Next, from A1-A4 it follows that

\[
\int_{\Omega} a_i (u_i(t,x) - u_i^*) [b_i (u_i(t,x)) - b_i (u_i^*)] \, dx \geq a_i B_i \int_{\Omega} (u_i(t,x) - u_i^*)^2 \, dx, \tag{21}
\]

\[
\tilde{a}_i \int_{\Omega} (u_i(t,x) - u_i^*) \sum_{j=1}^{m} c_{ij}^+ |f_j (u_j(t,x)) - f_j (u_j^*)| \, dx
\]

\[
\leq \tilde{a}_i \int_{\Omega} \sum_{j=1}^{m} c_{ij}^+ L_j |u_k(t,x) - u_k^*| |u_j(t,x) - u_j^*| \, dx \tag{22}
\]

\[
\leq \frac{1}{2} \tilde{a}_i \sum_{j=1}^{m} \int_{\Omega} c_{ij}^+ L_j \left[ (u_i(t,x) - u_i^*)^2 + (u_j(t,x) - u_j^*)^2 \right] \, dx,
\]

and

\[
\tilde{a}_i \int_{\Omega} (u_i(t,x) - u_i^*) \sum_{j=1}^{m} w_{ij}^+ |g_j (u_j(t - \tau_j(t),x)) - g_j (u_j^*)| \, dx
\]

\[
\leq \tilde{a}_i \sum_{j=1}^{m} \int_{\Omega} w_{ij}^+ M_j |u_k(t,x) - u_k^*| |u_j(t - \tau_j(t),x) - u_j^*| \, dx \tag{23}
\]

\[
\leq \frac{1}{2} \tilde{a}_i \sum_{j=1}^{m} \int_{\Omega} w_{ij}^+ M_j \left[ (u_i(t,x) - u_i^*)^2 + (u_j(t - \tau_j(t),x) - u_j^*)^2 \right] \, dx.
\]
Taking the inequalities (19)-(23) into account, we obtain
\[
\frac{1}{2} \frac{d}{dt} ||u(t,.) - v(.,.)||_2^2 \\
\leq \sum_{i=1}^{m} \left[ - (\bar{d}_i + a_i B_i) \int_{\Omega} (u_i(t,x) - u^*_i)^2 dx \\
+ \frac{1}{2} \bar{a}_i \sum_{j=1}^{m} \int_{\Omega} c_{ij}^+ L_j [(u_i(t,x) - u^*_i)^2 + (u_j(t,x) - u^*_j)^2] dx \\
+ \frac{1}{2} \bar{a}_i \sum_{j=1}^{m} \int_{\Omega} w_{ij}^+ M_j [(u_i(t,x) - u^*_i)^2 + (u_j(t-\tau_j(t),x) - u^*_j)^2] dx \right] \\
\leq - \frac{1}{2} \sum_{i=1}^{m} \left[ 2(\bar{d}_i + a_i B_i) \\
- \bar{a}_i \sum_{j=1}^{m} (L_j c_{ij}^+ + M_j w_{ij}^+ + L_i c_{ji}^+) \int_{\Omega} [(u_i(t,x) - u^*_i)^2 \\
+ \max_{1 \leq i \leq m} (M_i \sum_{j=1}^{m} \bar{a}_j w_{ji}^+) \frac{1}{2} ||u - u^*||_2^2 = -\alpha_1 \frac{1}{2} ||u - u^*||_2^2 + \alpha_2 \frac{1}{2} ||u - u^*||^2, \right]
\]

where
\[
\alpha_1 = \min_{1 \leq i \leq m} \left[ 2(\bar{d}_i + a_i B_i) - \bar{a}_i \sum_{j=1}^{m} (L_j c_{ij}^+ + M_j w_{ij}^+ + L_i c_{ji}^+) \right], \\
\alpha_2 = \max_{1 \leq i \leq m} \left( M_i \sum_{j=1}^{m} \bar{a}_j w_{ji}^+ \right).
\]

It follows from condition (15) of Theorem 3.4 that there exists a positive \( \alpha > 0 \) \((0 < \alpha \leq \alpha_1 - \alpha_2)\) such that
\[
\frac{1}{2} \frac{d}{dt} ||u(t,.) - v(.,.)||_2^2 \leq -\frac{1}{2} \alpha \left( ||u - u^*||_2^2 + ||u - u^*||^2 \right) \\
\leq -\frac{1}{2} \alpha \left( ||u - v||_2^2 + ||u - v||_2^2 \right). \quad (25)
\]

The proof of the inequality (25) in the case when \( u_i(t,x) < v_i(t,x) \) for any \( i = 1, 2, ..., m \) and \( (t,x) \in [-\tau, \infty) \times \Omega \) is similar. Instead of \( u^* \) the solution \( \bar{u}^* \) is used. The inequality (25) can be proved similarly in any other case, when \( u_i(t,x) < v_i(t,x) \) for some \( i = 1, 2, ..., m \) and \( u_i(t,x) > v_i(t,x) \) for the rest of the variables of \( u(t,x) \) and \( v(t,x) \), \( (t,x) \in [-\tau, \infty) \times \Omega \).

Computing the upper right-hand derivative of \( V \) along the solutions of system (1), we get for \( \hat{\varphi} \in C, \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, ..., \hat{\varphi}_m)^T \)
\[
D^+ V(t, \hat{\varphi}(0,.)) \leq -\alpha V(t, \hat{\varphi}(0,.))
\]
when \( V(t+s, \hat{\phi}(s, .)) \leq V(t, \hat{\phi}(0, .)) \) for \( -\tau \leq s \leq 0, \hat{\phi} \in C, t \geq 0 \).

Since all conditions of Theorem 3.1 are satisfied, we conclude that the set \( M = [-\tau, \infty) \times \Omega \times \{ \mathbb{R}^m : u^*_i \leq u_i \leq \bar{u}^*_i, i = 1, 2, ..., m \} \) is uniformly globally asymptotically stable with respect to the reaction-diffusion CGNN (1).

**Remark 6.** For \( u^* = \bar{u}^* \), Theorem 3.4 generalizes some exponential stability results for single equilibrium states of reaction-diffusion CGNNs obtained in [23, 24, 38, 39]. Therefore, with this research we extend and improve the existing results to the stability of sets case. Also, since CGNNs involve numerous important classes of artificial neural network models, our results have a universal significance and can be easily applied to many complex networks, and, in addition, provide a new design process for network designers.

4. **The uncertain case.** In this section, we will consider the effect of some uncertain terms on the stability of sets with respect to reaction-diffusion CGNNs.

We will investigate the following uncertain reaction-diffusion delayed CGNN corresponding to the system (1)

\[
\frac{\partial u_i(t,x)}{\partial t} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t,x)}{\partial x_k} \right) - \left( a_i(u_i(t,x)) + \bar{a}_i(u_i(t,x)) \right) \left( b_i(u_i(t,x)) + \bar{b}_i(u_i(t,x)) \right)
- I_i(t,x) - \bar{I}_i(t,x)
- \sum_{j=1}^{m} \left( c_{ij}(t) + \bar{c}_{ij}(t) \right) \left( f_j(u_j(t,x)) + \bar{f}_j(u_j(t,x)) \right)
- \sum_{j=1}^{m} \left( w_{ij}(t) + \bar{w}_{ij}(t) \right) \left( g_j(u_j(t-\tau_j(t),x)) + \bar{g}_j(u_j(t-\tau_j(t),x)) \right),
\]

where \( \bar{a}_i, \bar{b}_i, \bar{c}_{ij}, \bar{w}_{ij}, \bar{f}_j, \bar{g}_j, \bar{I}_i, i, j = 1, ..., m \) are all continuous functions in their domains, and represent the uncertain terms in the system (26) [19, 28, 36, 37, 44]. In the case when all of these functions equivalent to zeros, we will recover the “nominal system” (1) [19].

It is well known that, uncertain terms are often fluctuating within some scopes in engineering applications, and uncertainties often break the stability of systems. Thus, it is meaningful to consider uncertainty in the dynamic analysis of CGNNs.

We will introduce the new notion of robust asymptotic stability of a set with respect to system (1) by the following definition.

**Definition 4.1.** The set \( M \) is said to be **robustly uniformly globally asymptotically stable** with respect to system (1) if for any functions \( \bar{a}_i, \bar{b}_i, \bar{c}_{ij}, \bar{w}_{ij}, \bar{f}_j, \bar{g}_j, \bar{I}_i, i, j = 1, ..., m \), the set \( M \) is uniformly globally asymptotically stable with respect to system (26).

Introduce the following conditions:

**A7.** For \( \bar{a}_i^+ = \sup_{\chi \in \mathbb{R}} \bar{a}_i(\chi), i = 1, 2, ..., m \), we have

\[
\bar{a}_i^+ \in [\underline{a}_i - a_i, \overline{a}_i - a_i].
\]
A8. The functions $\tilde{b}_i, i = 1, 2, ..., m$, are continuous and
\[ \frac{b_i(\chi_1) + \tilde{b}_i(\chi_1) - (b_i(\chi_2) + \tilde{b}_i(\chi_2))}{\chi_1 - \chi_2} \geq B_i \]
for $\chi_1, \chi_2 \in \mathbb{R}, \chi_1 \neq \chi_2$.

A9. The functions $\tilde{c}_{ij}, \tilde{I}_i, \tilde{w}_{ij}$, $i, j = 1, 2, ..., m$, are continuous on their domains, and
\[ \sup_{t \in \mathbb{R}} \tilde{c}_{ij}(t) = \tilde{c}_{ij}^+, \quad \sup_{t \in \mathbb{R}} \tilde{w}_{ij}(t) = \tilde{w}_{ij}^+. \]

A10. There exist positive constants $\tilde{L}_i, \tilde{M}_i$, $i = 1, 2, ..., m$, with
\[ |\tilde{f}_i(\chi_1) - \tilde{f}_i(\chi_2)| \leq \tilde{L}_i|\chi_1 - \chi_2|, \]
\[ |\tilde{g}_i(\chi_1) - \tilde{g}_i(\chi_2)| \leq \tilde{M}_i|\chi_1 - \chi_2| \]
for all $\chi_1, \chi_2 \in \mathbb{R}, \chi_1 \neq \chi_2$, and $\tilde{f}_i(0) = \tilde{g}_i(0) = 0$.

**Theorem 4.2.** Let assumptions A1-A10 hold and the following condition
\[ \min_{1 \leq i \leq m} \left[ 2(\tilde{d}_i + \tilde{a}_i B_i) \right. \]
\[ -\pi_i \sum_{j=1}^{m} \left( (L_j + \tilde{L}_j)(c_{ij}^+ + \tilde{c}_{ij}) + (M_j + \tilde{M}_j)(w_{ij}^+ + \tilde{w}_{ij}^+) + (L_i + \tilde{L}_i)(c_{ji}^+ + \tilde{c}_{ji}) \right) \]
\[ \left. > \max_{1 \leq i \leq m} \left( (M_i + \tilde{M}_i) \sum_{j=1}^{m} \pi_j w_{ji}^+ \right) > 0 \right] \]
be fulfilled.

Then the set $M = [-\tau, \infty) \times \Omega \times \{ \mathbb{R}^m : y_i^* \leq u_i \leq y_i^+ \}$ is robustly uniformly globally asymptotically stable with respect to the reaction-diffusion CGNN (1).

**Proof.** The fact that the set $M$ is uniformly globally asymptotically stable with respect to system (26) for any values of the uncertain terms follows in the same way as in the proof of Theorem 3.1 using the assumptions A1-A10. Therefore, from Definition 4.1, it follows that it is robustly uniformly globally asymptotically stable with respect to system (1).

5. **Examples.** In this section, fractional reaction-diffusion delayed neural networks of type (1) are proposed as examples to demonstrate the effectiveness of our sufficient conditions.

**Example 5.1.** Let the set $\Omega$ be two-dimensional and the constants $l_k = 1, k = 1, 2$. Consider the following model of a delayed reaction-diffusion CGNN represented
by

\[
\frac{\partial u_i(t, x)}{\partial t} = \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i(u_i(t, x)) \left[ b_i(u_i(t, x)) \right]
\]

\[
\begin{align*}
&\quad - I_i(t, x) - \sum_{j=1}^{2} c_{ij}(t) f_j(u_j(t, x)) \\
&\quad - \sum_{j=1}^{2} w_{ij}(t) g_j(u_j(t - \tau_j(t), x)) \right], \\
&u_i(t, x) = 0, \ t \in [-\tau, \infty), \ x \in \partial \Omega, \\
&u_i(s, x) = \varphi_i(s, x), \ s \in [-\tau, 0], \ x \in \Omega,
\end{align*}
\]

where \( t > 0, m = n = 2, I_1 = I_2 = 0, f_i(u_i) = g_i(u_i) = \frac{1}{2}(|u_i + 1| - |u_i - 1|), \)
\( \tau_1(t) = \tau_2(t) = e^t/(1 + e^t), 0 \leq \tau_i(t) \leq \tau \ (\tau = 1), a_i(u_i) = 1, b_1(u_i) = u_i, \)
\( b_2(u_i) = 2u_i, i = 1, 2, \)
\( (c_{ij})_{2 \times 2} = \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.7 - 0.3 \sin(t) & 0.1 - 0.4 \cos(t) \\ 0.4 - 0.2 \cos(t) & 0.2 - 0.3 \sin(t) \end{pmatrix}, \)
\( (w_{ij})_{2 \times 2} = \begin{pmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.2 \sin(t) & 0.3 \cos(t) \\ 0.2 \cos(t) & 0.3 \sin(t) \end{pmatrix}, \)
\( (D_{ik})_{2 \times 2} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} 1 + \sin t & 0 \\ 0 & 2 + \cos t \end{pmatrix}. \)

We can easily derived that all assumptions A1-A6 are satisfied for \( a_i = \tilde{a}_i = 1, \)
\( i = 1, 2, B_1 = 1, B_2 = 2, L_1 = L_2 = M_1 = M_2 = 1 \) and
\( (d_{ik})_{2 \times 2} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \)

In addition, we can prove in a similar way as in [22] that the system (28) has a unique equilibrium point \( u^* = (u^*_1, u^*_2)^T. \)

Since, condition (15) of Theorem 3.4 is also satisfied for
\[
\alpha_1 = \min_{1 \leq i \leq 2} \left[ 2(\tilde{d}_i + a_i B_i) - \tilde{a}_i \sum_{j=1}^{2} (L_j c_{ij}^+ + M_j w_{ij}^+ + L_i c_{ji}^+) \right] = 2.4
\]
and
\[
\alpha_2 = \max_{1 \leq i \leq 2} \left( M_i \sum_{j=1}^{2} a_j w_{ji}^+ \right) = 0.6,
\]
then, by Theorem 3.4 we conclude that the set \( M = [-\tau, \infty) \times \Omega \times (\mathbb{R}^2 : u_i \leq u^*_i, i = 1, 2, \ldots, m) \) is uniformly globally asymptotically stable with respect to the reaction-diffusion CGNN (28).

**Remark 7.** Example 5.1 again shows that our results improve and extend some existing results in the literature to the stability of sets case. See, for example, [23, 24].
**Remark 8.** In our Example 5.1 we consider a case when $a_i(u_i) = 1$ for $i = 1, 2$. In this particular case the reaction-diffusion CGNN (28) is a Hopfield-type reaction-diffusion neural network studied by some authors [26] and has a lot of applications in science and technology. Since, system (28) includes many other models from neurobiology, population biology, and evolution theory, our results will be efficient in a wide application fields.

**Example 5.2.** Consider the reaction-diffusion CGNN model (28) with uncertain terms as follows

$$
\frac{\partial u_i(t, x)}{\partial t} = \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - (a_i(u_i(t, x)) + \tilde{a}_i(u_i(t, x))) \left[ (b_i(u_i(t, x)) + \tilde{b}_i(u_i(t, x))) \right.
$$

$$
- I_i(t, x) - \tilde{I}_i(t, x)
$$

$$
- \sum_{j=1}^{2} \left( c_{ij}(t) + \tilde{c}_{ij}(t) \right) \left( f_j(u_j(t, x)) + \tilde{f}_j(u_j(t, x)) \right)
$$

$$
- \sum_{j=1}^{2} \left( w_{ij}(t) + \tilde{w}_{ij}(t) \right) \left( g_j(u_j(t - \tau_j(t), x)) + \tilde{g}_j(u_j(t - \tau_j(t), x)) \right)
$$

where the continuous functions $\tilde{a}_i, \tilde{b}_i, \tilde{c}_{ij}, \tilde{w}_{ij}, \tilde{f}_j, \tilde{g}_j, \tilde{I}_i, i, j = 2$ are the uncertain terms.

If all uncertain terms are bounded so that all conditions of Theorem 4.2 are satisfied, then set $M = [-\tau, \infty) \times \Omega \times \{ \mathbb{R}^2 : u_i \leq u_i^*, i = 1, 2, \ldots, m \}$ is robustly uniformly globally asymptotically stable with respect to the reaction-diffusion CGNN (28).

6. **Conclusions.** In this paper, we extend the existing stability results for delayed reaction-diffusion Cohen–Grossberg neural networks to the stability of sets case. The new notion includes as a special case stability of an equilibrium, stability of invariant sets, stability of moving manifolds, etc., and is one of the most important notions in the stability theory. In addition, we introduce the notion of robust stability of sets for the neural network model under consideration, and consider uncertainty in our dynamic analysis. Since Cohen–Grossberg neural networks are generalizations of numerous neural network models, including Hopfield-type neural network models, with this research we open the door for the development of the stability of sets theory for many practical problems of diverse interest.

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Received September 2019; revised January 2020.

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