Abstract. A triangulation of a simplicial complex $\Delta$ is called uniform if the $f$-vector of its restriction to a face of $\Delta$ depends only on the dimension of that face. This paper proves that the entries of the $h$-vector of a uniform triangulation of $\Delta$ can be expressed as nonnegative integer linear combinations of those of the $h$-vector of $\Delta$, where the coefficients depend only on the dimension of $\Delta$ and the $f$-vectors of the restrictions of the triangulation to simplices of various dimensions. Moreover, it provides information about these coefficients, including formulas, recurrence relations and various interpretations, and gives a criterion for the $h$-polynomial of a uniform triangulation to be real-rooted. These results unify and generalize several results in the literature about special types of triangulations, such as barycentric, edgewise and interval subdivisions.

1. Introduction

The study of triangulations of simplicial complexes from a face enumeration point of view was pioneered by Stanley [25, 27, Section III.10]. The main objective of [25] was to understand the effect that various types of subdivision, including triangulations, have on the $h$-vector (a certain linear transformation of the face vector) of a simplicial complex; see [27, Chapter II] for the importance of $h$-vectors on the face enumeration of simplicial complexes.

The transformation of the $h$-vector has been studied for specific triangulations since then, beginning with the work of Brenti–Welker [15] on barycentric subdivisions. These authors showed that the $h$-vector of the barycentric subdivision of a simplicial complex $\Delta$ is given by a nonnegative integer linear transformation of the $h$-vector of $\Delta$, which depends only on the dimension of $\Delta$. They also provided a combinatorial interpretation of the coefficients in terms of permutation enumeration. Analogous results have been proven for edgewise subdivisions [16], partial barycentric subdivisions [11], interval subdivisions [2] and antiprism triangulations [9]. The main result of [15] states that the $h$-polynomial (the generating polynomial for the $h$-vector) of the barycentric subdivision of $\Delta$ has only real roots (in particular, log-concave and unimodal coefficients) for every simplicial complex $\Delta$ with nonnegative $h$-vector. An analogous statement for edgewise subdivision follows from a result of Jochemko [21] on the Veronese construction for rational formal power series, which improved earlier results by Brenti–Welker [16] and Beck–Stapledon [10].
The present paper aims to provide a common framework to explain and generalize these results. Given \( d \in \mathbb{N} \cup \{\infty\} \), a triangular array \( \mathcal{F} \) of numbers \( f_{\mathcal{F}}(i, j) \) for \( 0 \leq i \leq j \leq d \) and a simplicial complex \( \Delta \) of dimension less than \( d \), we will say that a triangulation \( \Delta' \) of \( \Delta \) is \( \mathcal{F} \)-uniform if for all \( i, j \), the restriction of \( \Delta' \) to any \((j-1)\)-dimensional face of \( \Delta \) has exactly \( f_{\mathcal{F}}(i, j) \) faces of dimension \( i-1 \). We will refer to the array \( \mathcal{F} \) as an \( f \)-triangle of size \( d \) and will call \( \Delta' \) uniform, if it is \( \mathcal{F} \)-uniform for some \( \mathcal{F} \). All aforementioned examples of triangulations studied in the literature are uniform, since they have the stronger property that their restrictions to faces of \( \Delta \) of the same dimension are combinatorially isomorphic.

The following statement is the first main contribution of this paper. We use the convention that \( \{0, 1, \ldots, d\} := \mathbb{N} = \{0, 1, 2, \ldots\} \), when \( d = \infty \).

**Theorem 1.1.** Let \( \mathcal{F} \) be an \( f \)-triangle of size \( d \). There exist nonnegative integers \( p_{\mathcal{F}}(n, k, j) \) for \( n \in \{0, 1, \ldots, d\} \) and \( k, j \in \{0, 1, \ldots, n\} \), such that for all \( n \leq d \),

\[
(1) \quad h_j(\Delta') = \sum_{k=0}^{n} p_{\mathcal{F}}(n, k, j) h_k(\Delta)
\]

for every \((n-1)\)-dimensional simplicial complex \( \Delta \), every \( \mathcal{F} \)-uniform triangulation \( \Delta' \) of \( \Delta \) and all \( j \in \{0, 1, \ldots, n\} \).

Given the main results of [15, 21] on real-rootedness, it seems natural to ask which uniform triangulations transform \( h \)-polynomials with nonnegative coefficients into polynomials with only real (necessarily negative) roots. Our second main contribution is a partial answer to this question which easily applies to barycentric and edgewise subdivisions (see Section 7). Given an \( f \)-triangle \( \mathcal{F} \), we will denote by \( h_{\mathcal{F}}(\sigma_n, x) \) and \( h_{\mathcal{F}}(\partial \sigma_n, x) \) the \( h \)-polynomial of any \( \mathcal{F} \)-uniform triangulation of the \((n-1)\)-dimensional simplex \( \sigma_n \) and its boundary complex, respectively (by Theorem 1.1 these polynomials depend only on \( n \) and \( \mathcal{F} \)).

**Theorem 1.2.** Let \( \mathcal{F} \) be an \( f \)-triangle of size \( d \in \mathbb{N} \) and assume the following:

(i) \( h_{\mathcal{F}}(\sigma_n, x) \) is a real-rooted polynomial for all \( n < d \).

(ii) \( h_{\mathcal{F}}(\sigma_n, x) - h_{\mathcal{F}}(\partial \sigma_n, x) \) is either identically zero, or a real-rooted polynomial of degree \( n - 1 \) with nonnegative coefficients which is interlaced by \( h_{\mathcal{F}}(\sigma_{n-1}, x) \), for all \( n \leq d \).

Then, for every \((d-1)\)-dimensional simplicial complex \( \Delta \) with nonnegative \( h \)-vector, the polynomial \( h(\Delta', x) \) is real-rooted for every \( \mathcal{F} \)-uniform triangulation \( \Delta' \) of \( \Delta \).

We now provide some more details about the content, methods and structure of this paper. Section 2 includes preliminaries on simplicial complexes, triangulations and their face enumeration. Section 3 discusses uniform triangulations, their basic properties and motivating examples, given by barycentric and edgewise subdivisions and their variations and generalizations. A similar concept was introduced in [17, Section 5] in order to study the asymptotics of the roots of the \( h \)-polynomial of a simplicial complex after iterated simplicial subdivision.
Theorem 1.1 is proven in Section 4 and some immediate consequences are drawn (see Corollary 4.5). An explicit formula (Equation (12)) and further information (see Proposition 4.6), including a symmetry property and a universal recurrence relation, are given there for the coefficients which appear in the transformation (1) of the \( h \)-vector of a simplicial complex \( \Delta \) under uniform triangulation. Perhaps not surprisingly, their nonnegativity follows from that of the local \( h \)-polynomials \cite[Section 4]{25} of the restrictions of the triangulation to the faces of \( \Delta \). In particular, the results of this paper are valid more generally for the class of quasi-geometric simplicial subdivisions \cite{25}, which includes that of geometric simplicial subdivisions (triangulations), discussed here. One should note that for the special types of triangulations treated in \cite{1, 2, 15, 16}, the nonnegativity of the coefficients in (1) is proven there by finding explicit combinatorial interpretations by various techniques (a task which has required considerable effort in each case). This is achieved in Section 4 for the \( r \)-colored barycentric subdivision by exploiting the universal recurrence (see Proposition 4.7).

Section 5 extends the subdivision operator on polynomials, associated with barycentric subdivisions (see \cite[Section 4]{11}, \cite[Section 7.3.3]{12} and references therein), to arbitrary uniform triangulations. The main properties of this operator, developed there, yield two interpretations of the coefficients in transformation (1) (see Propositions 5.5 and 5.8). One of them is exploited in order to give a short proof of the recurrence relation for these coefficients (see Corollary 5.6). The other expresses them as entries of \( h \)-vectors of Cohen–Macaulay relative simplicial complexes and results in a new proof of their nonnegativity (see Remark 5.9).

Section 6 proves Theorem 1.1 using the theory of interlacing polynomials. Section 7 includes applications. Among others, it recovers the main result of Brenti–Welker \cite{15} on barycentric subdivisions and (partially) that of Jochemko \cite{21} on edgewise subdivisions and the Veronese construction for formal power series (see Examples 7.1 and 7.2). For an approach via shellability which applies to these two situations, as well as to barycentric subdivisions of certain cubical polytopes, see \cite{19}. Section 7 also deduces that certain polynomials which have appeared in the combinatorial literature have nonnegative, real-rooted and interlacing symmetric decompositions. Section 8 concludes with some comments and directions for further research.

2. Preliminaries

This section recalls definitions and background on the face enumeration of (abstract) simplicial complexes and their triangulations. Any undefined terminology can be found in \cite{23, 27}. All simplicial complexes we consider will be finite. Throughout this paper, we denote by \( \sigma_n \) the (simplicial complex of faces of the) standard \((n - 1)\)-dimensional simplex in \( \mathbb{R}^n \). We also set \( \mathbb{N} = \{0, 1, 2, \ldots \} \), we recall our convention from the introduction that \( \{0, 1, \ldots, \infty\} = \mathbb{N} \) and for \( d \in \mathbb{N} \cup \{\infty\} \), we denote by \( \mathbb{R}_d[x] \) the real vector space of polynomials of degree at most \( d \).

The \textit{f-polynomial} of an \((n - 1)\)-dimensional (abstract) simplicial complex \( \Delta \) is defined as \( f(\Delta, x) := \sum_{i=0}^{n} f_{i-1}(\Delta)x^i \), where \( f_i(\Delta) \) is the number of \( i \)-dimensional faces of \( \Delta \).
The numerical sequence \( f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{n-1}(\Delta)) \) is called the \( f \)-vector. The \( h \)-polynomial of \( \Delta \) is then defined by the formula

\[
h(\Delta, x) := (1 - x)^n f(\Delta, \frac{x}{1-x}) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1 - x)^{n-i}
\]

and the sequence \( h(\Delta) := (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta)) \) is called the \( h \)-vector of \( \Delta \). The polynomial \( h(\Delta, x) \) has nonnegative coefficients for every \( \Delta \) which is Cohen–Macaulay over some field. This happens, in particular, if \( \Delta \) triangulates a ball or sphere (meaning that its geometric realization is homeomorphic to a ball or sphere). Moreover, \( h(\Delta, x) \) is symmetric, with center of symmetry \( n/2 \) (meaning that \( h_i(\Delta) = h_{n-i}(\Delta) \) for \( 0 \leq i \leq n \)), if \( \Delta \) triangulates a sphere. For more information on these topics, we refer the reader to [24, Chapter II].

A relative simplicial complex [27, Section III.7] is any pair \((\Delta, \Gamma)\), where \( \Delta \) is a simplicial complex and \( \Gamma \) is a subcomplex of \( \Delta \). The definitions of the \( f \)- and \( h \)-polynomial extend naturally to this setting. Following [13, Section 2], we say that \( h(x) := (1-x)^n f(x/(1-x)) \) is the \( h \)-polynomial associated to \( f(x) \in \mathbb{R}_n[x] \) (with respect to \( n \)). Equivalently, \( f(x) = (1 + x)^n h(x/(1 + x)) \) is the \( f \)-polynomial associated to \( h(x) \in \mathbb{R}_n[x] \) (with respect to \( n \)). The \( h \)-polynomial \( h(\Delta/\Gamma, x) \) of the relative complex \((\Delta, \Gamma)\) is then defined as the \( h \)-polynomial associated to the \( f \)-polynomial \( f(\Delta/\Gamma, x) := \sum_{i=0}^{n} f_{i-1}(\Delta/\Gamma) x^i \), where \( n-1 \) is the dimension of \( \Delta \) and \( f_i(\Delta/\Gamma) \) is the number of \( i \)-dimensional faces of \( \Delta \) which do not belong to \( \Gamma \). In particular, this defines the \( h \)-polynomial of the interior \( \text{int}(\Delta) = \Delta \setminus \partial \Delta \) of \( \Delta \), when \( \Delta \) triangulates a ball and \( \partial \Delta \) is its boundary complex.

The following statement is a special case of [24, Lemma 6.2].

**Proposition 2.1.** ([21]) Let \( \Delta \) be a triangulation of an \((n-1)\)-dimensional ball. Let \( \Gamma \) be a subcomplex of \( \partial \Delta \) which is homeomorphic to an \((n-2)\)-dimensional ball or sphere and \( \bar{\Gamma} \) be the subcomplex of \( \partial \Delta \) whose facets are those of \( \partial \Delta \) which do not belong to \( \Gamma \). Then,

\[
x^n h(\Delta/\Gamma, 1/x) = h(\Delta/\bar{\Gamma}, x).
\]

In particular, \( x^n h(\Delta, 1/x) = h(\text{int}(\Delta), x) \).

Consider two simplicial complexes \( \Delta \) and \( \Delta' \). We say that \( \Delta' \) is a triangulation of \( \Delta \) if there exist geometric realizations \( K' \) and \( K \) of \( \Delta' \) and \( \Delta \), respectively, such that \( K' \) geometrically subdivides \( K \). Given a simplex \( L \in K \) with corresponding face \( F \in \Delta \), the triangulation \( K' \) naturally restricts to a triangulation \( K'_L \) of \( L \). The subcomplex \( \Delta'_F \) of \( \Delta' \) corresponding to \( K'_L \) is a triangulation of the abstract simplex \( 2^F \), called the restriction of \( \Delta' \) to \( F \). The carrier of a face \( G \in \Delta' \) is defined as the smallest face \( F \in \Delta \) such that \( G \in \Delta'_F \).

Associated to the restrictions \( \Delta'_F \) are certain enumerative invariants, called local \( h \)-polynomials. Given any triangulation \( \Gamma \) of an \((n-1)\)-dimensional simplex with vertex set
V, the local $h$-polynomial of $\Gamma$ (with respect to $V$) is defined \cite[Definition 2.1]{25} by the formula

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x).$$

By the principle of inclusion-exclusion, we have

$$h(\Gamma, x) = \sum_{F \subseteq V} \ell_F(\Gamma_F, x).$$

The polynomial $\ell_V(\Gamma, x)$ is symmetric, with center of symmetry $n/2$, and has nonnegative coefficients; it plays a fundamental role in the enumerative theory of simplicial subdivisions \cite{25,27} Section III.10] and in the proof of Theorem \ref{thm1} as well.

3. Uniform triangulations

This section introduces uniform triangulations, fixes related notation and terminology and discusses basic properties and the main examples of interest.

Let us fix a number $d \in \mathbb{N} \cup \{\infty\}$ once and for all; for the main applications, one can always take $d = \infty$. An $f$-triangle of size $d$ will be a triangular array $F = (f_F(i, j))_{0 \leq i \leq j \leq d}$ of nonnegative integers (where $i, j$ are finite numbers). We will say that

$$f_F(\sigma_n, x) := \sum_{i=0}^{n} f_F(i, n)x^i$$

is the $n$th $f$-polynomial associated to $F$.

**Definition 3.1.** Let $F$ be an $f$-triangle of size $d$ and $\Delta$ be a simplicial complex of dimension less than $d$. A triangulation $\Delta'$ of $\Delta$ is called $F$-uniform if $f(\Delta'_F, x) = f_F(\sigma_n, x)$ for every $(n-1)$-dimensional face $F \in \Delta$ and all $n \leq d$.

Equivalently, we require that for all $0 \leq i \leq j \leq d$, the restriction of $\Delta'$ to any face $F$ of $\Delta$ of dimension $j-1$ has exactly $f_F(i, j)$ faces of dimension $i-1$. This paper is not concerned with the problem to determine necessary and sufficient conditions on $F$, so that $F$-uniform triangulations of $(n-1)$-dimensional simplices exist for all $n \leq d$. We will then say that $F$ is feasible. Most statements in this paper are vacuously true for non-feasible triangles $F$. As the notation in Equation (2) suggests, when we talk about the face enumeration of an $F$-uniform triangulation of $\sigma_n$, we will mean that of the restriction of $\Delta'$ to any $(n-1)$-dimensional face of $\Delta$. Clearly, if a triangulation $\Delta'$ of $\Delta$ is $F$-uniform, then so is the restriction of $\Delta'$ to any subcomplex of $\Delta$. 
3.1. **Face triangles.** A feasible $f$-triangle $F$ of size $d$ gives rise to the $h$-polynomials, interior $f$-polynomials, interior $h$-polynomials and local $h$-polynomials

$$
h_F(\sigma_n, x) = \sum_{i=0}^{n} h_F(i, n)x^i,
$$

$$
f_F^\circ(\sigma_n, x) = \sum_{i=0}^{n} f_F^\circ(i, n)x^i,
$$

$$
h_F^\circ(\sigma_n, x) = \sum_{i=0}^{n} h_F^\circ(i, n)x^i,
$$

$$
\ell_F(\sigma_n, x) = \sum_{i=0}^{n} \ell_F(i, n)x^i
$$

for $0 \leq n \leq d$, respectively. These are defined as the $h$-polynomial, interior $f$-polynomial, interior $h$-polynomial and local $h$-polynomial of any $F$-uniform triangulation of the simplex $\sigma_n$. They can be represented by the corresponding $h$-triangle $H = (h_F(\sigma_n, x))$, interior $f$-triangle $F^\circ = (f_F^\circ(i, j))$, interior $h$-triangle $H^\circ = (h_F^\circ(i, j))$ and local $h$-triangle $L = (\ell_F(i, j))$ of size $d$, respectively. For example, $f_F^\circ(i, n)$ is equal to the number of interior $(i-1)$-dimensional faces of any $F$-uniform triangulation of $\sigma_n$ (or of any other simplex of the same dimension). These triangles determine each other, since the corresponding polynomials are related by the invertible transformations

$$
h_F(\sigma_n, x) = (1 - x)^n f_F(\sigma_n, \frac{x}{1 - x}),
$$

$$
h_F^\circ(\sigma_n, x) = x^n h_F(\sigma_n, \frac{1}{x}),
$$

$$
f_F^\circ(\sigma_n, x) = (1 + x)^n h_F^\circ(\sigma_n, \frac{x}{1 + x}),
$$

$$
\ell_F(\sigma_n, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} h_F(\sigma_k, x).
$$

These equalities define the triangles $H$, $F^\circ$, $H^\circ$ and $L$ even when $F$ may not be feasible. The second follows from the last sentence of Proposition 2.1 and yields the relation

$$
f_F^\circ(\sigma_n, -1 - x) = (-1)^n f_F(\sigma_n, x)
$$

between the $f$-polynomials and interior $f$-polynomials associated to $F$.

**Example 3.2.** Let $d = 2$ and suppose that $F$ comes from the uniform triangulation which subdivides any 1-simplex into $r$ such simplices, by inserting $r - 1$ interior vertices. Then, the various triangles we have defined are given by:

- $f_F(\sigma_0, x) = 1, f_F(\sigma_1, x) = 1 + x, f_F(\sigma_2, x) = 1 + (r + 1)x + rx^2$,
- $f_F^\circ(\sigma_0, x) = 1, f_F^\circ(\sigma_1, x) = x, f_F^\circ(\sigma_2, x) = (r - 1)x + rx^2$. 

Figure 1. The 4-fold edgewise subdivision of the 2-simplex

- $h_F(\sigma_0, x) = h_F(\sigma_1, x) = 1, h_F(\sigma_2, x) = 1 + (r - 1)x$,
- $h^0_F(\sigma_0, x) = 1, h^0_F(\sigma_1, x) = x, h^0_F(\sigma_2, x) = (r - 1)x + x^2$, and
- $\ell_F(\sigma_0, x) = 0, \ell_F(\sigma_1, x) = 0, \ell_F(\sigma_2, x) = (r - 1)x$.

3.2. Examples. We now briefly review important examples of uniform triangulations which have already been studied in the literature, in terms of their face enumeration, and have provided much of the motivation behind this paper. The corresponding $f$-triangles can be considered to be of infinite size.

At the core of our examples lie barycentric and edgewise subdivisions. Let $\Delta$ be a simplicial complex of dimension $n - 1$ with vertex set $V(\Delta)$ and $r$ be a positive integer. The **barycentric subdivision** of $\Delta$ is denoted by $\text{sd}(\Delta)$ and defined as the simplicial complex of all chains in the poset of nonempty faces of $\Delta$. The edgewise subdivision depends on $r$ and a linear ordering of $V(\Delta)$ (although its face vector is independent of the latter). Given such an ordering $v_1, v_2, \ldots, v_m$, denote by $V_r(\Delta)$ the set of maps $f : V(\Delta) \to \mathbb{N}$ such that $\text{supp}(f) \in \Delta$ and $f(v_1) + f(v_2) + \cdots + f(v_m) = r$, where $\text{supp}(f)$ is the set of all $v \in V(\Delta)$ for which $f(v) \neq 0$. For $f \in V_r(\Delta)$, let $\iota(f) : V(\Delta) \to \mathbb{N}$ be the map defined by setting $\iota(f)(v_j) = f(v_1) + f(v_2) + \cdots + f(v_j)$ for $j \in \{1, 2, \ldots, m\}$. The **$r$-fold edgewise subdivision** of $\Delta$, denoted by $\text{esd}_r(\Delta)$, is the simplicial complex on the vertex set $V_r(\Delta)$ of which a set $E \subseteq V_r(\Delta)$ is a face if the following two conditions are satisfied:

- $\bigcup_{f \in E} \text{supp}(f) \in \Delta$ and
- $\iota(f) - \iota(g) \in \{0, 1\}^{V(\Delta)}$, or $\iota(g) - \iota(f) \in \{0, 1\}^{V(\Delta)}$, for all $f, g \in E$.

The simplicial complexes $\text{sd}(\Delta)$ and $\text{esd}_r(\Delta)$ can be realized as triangulations of $\Delta$. This is elementary and well known for the former, but slightly less obvious for the latter; see [8, Section 5] and references therein. The 4-fold edgewise subdivision is shown on Figure 1; it will be used as a running example in this paper.

Barycentric and edgewise subdivisions can be combined to form the **$r$-colored barycentric subdivision** of $\Delta$. This triangulation is defined as the $r$-fold edgewise subdivision of $\text{sd}(\Delta)$; it was introduced in [3] in order to partially interpret geometrically the derangement polynomial for the colored permutation group $\mathbb{Z}_r \wr S_n$. The enumerative combinatorics of the $r$-colored barycentric subdivision relates to that of $r$-colored permutations, just as the enumerative combinatorics of barycentric subdivision relates to that of usual (uncolored) permutations; see [3, 8, Section 5]. Clearly, it reduces to $\text{sd}(\Delta)$ for $r = 1$. 
As explained in [4, Remark 4.5], for $r = 2$ it has the same $f$-vector as another interesting triangulation of $\Delta$, namely the *interval triangulation*. As a simplicial complex, the latter consists of all chains of nonempty closed intervals in the poset of nonempty faces of $\Delta$; it can also be described as a cubical (or signed) analogue of barycentric subdivision. The face enumeration of the interval triangulation was studied in [23], when $\Delta$ is a simplex (see also [4, Section 4] [6, Section 3.3] [7, Section 5]), and in [2] for arbitrary $\Delta$.

The transformation of the $h$-vector of $\Delta$ under barycentric, edgewise and interval subdivision was studied in [15] [16] [2], respectively, where combinatorial interpretations of the coefficients which appear in Equation (11) were found. By a simple application of the recurrence (see Proposition 4.6) for these coefficients, we generalize the interpretations in the first and third cases to that of the $r$-colored barycentric subdivision in Section 4 (see Proposition 4.7). Useful formulas for the $h$-polynomial of the barycentric and edgewise subdivisions of an $(n - 1)$-dimensional simplicial complex $\Delta$ are

\begin{equation}
\sum_{m \geq 0} \left( \sum_{i=0}^{n} h_i(\Delta)m^i(m+1)^{n-i} \right)x^m = \frac{h(sd(\Delta), x)}{(1 - x)^{n+1}}
\tag{8}
\end{equation}

(see [15, Equation (3.5)]) and

\begin{equation}
h(esd_r(\Delta), x) = \left((1 + x + x^2 + \cdots + x^{r-1})^n h(\Delta, x)\right)_{(r,0)}^\langle r \rangle
\tag{9}
\end{equation}

(see [3, Section 4] and references therein), where we have used the standard notation

\[ g(x) = g^{(r,0)}(x^r) + x g^{(r,1)}(x^r) + \cdots + x^{r-1} g^{(r,r-1)}(x^r) \]

for $g(x) \in \mathbb{R}[x]$. The main results of [15] and [21], respectively, show that barycentric subdivision transforms $h$-polynomials with nonnegative coefficients to polynomials with only real roots and that the $r$-fold edgewise subdivision has this property, provided $r$ is larger than the dimension of $\Delta$. We deduce these results from Theorem 1.2 in Section 7.

The fact that the $r$-colored barycentric subdivision has the same property for every $r$ follows from the result of [15] for barycentric subdivisions and [14, Theorem 4.5.6] or [30, Corollary 3.4] (see Proposition 7.5).

All triangulations of $\Delta$ we have discussed here are uniform, since their restrictions to faces of $\Delta$ are triangulations of the same kind. For more information on their combinatorics (in particular, for combinatorial interpretations of their local $h$-polynomials), see [4, Section 4] [6, Section 3.3] and references therein. Another very interesting and motivating example of uniform triangulation, namely the antiprism triangulation [20, Section Appendix A], has been studied more recently in [9].

### 4. Transformation of the $h$-vector

This section proves the following more detailed version of Theorem 1.1, which describes the transformation of the $h$-vector of a simplicial complex under uniform triangulation, and studies the coefficients which appear in Equation (11). A combinatorial interpretation of these coefficients is deduced from the main recurrence they satisfy, in the special case of the $r$-colored barycentric subdivision.
Theorem 4.1. Let $F$ be an $f$-triangle of size $d \in \mathbb{N} \cup \{\infty\}$. For $n, k \in \{0, 1, \ldots, d\}$ with $k \leq n$, there exist polynomials

$$p_{F,n,k}(x) = \sum_{j=0}^{n} p_{F}(n, k, j)x^{j}$$

with nonnegative integer coefficients, such that the following holds for all $n \leq d$: the $h$-polynomial of any $F$-uniform triangulation of any $(n-1)$-dimensional simplicial complex $\Delta$ is equal to

$$h_{F}(\Delta, x) := \sum_{k=0}^{n} h_{k}(\Delta)p_{F,n,k}(x).$$

The explicit formula

$$p_{F,n,k}(x) = \sum_{r=0}^{n} \ell_{F}(\sigma_{r}, x) \sum_{i=0}^{r} \left(\frac{n-k}{i}\right)\left(\frac{k}{r-i}\right)x^{k-r+i}$$

holds for all $n, k$.

Example 4.2. For $d \geq 2$ we have $\ell_{F}(\sigma_{0}, x) = 1$, $\ell_{F}(\sigma_{1}, x) = 0$ and $\ell_{F}(\sigma_{2}, x) = (r-1)x$, where $r - 1$ is the number of interior vertices of any $F$-uniform triangulation of the 1-simplex. Equation (12) yields that $p_{F,0,0}(x) = p_{F,1,0}(x) = 1$, $p_{F,1,1}(x) = x$ and

$$p_{F,2,k}(x) = \begin{cases} 1 + (r - 1)x, & \text{if } k = 0 \\ rx, & \text{if } k = 1 \\ (r - 1)x + x^{2}, & \text{if } k = 2. \end{cases}$$

Letting $d = 3$ and assuming that $F$ is the $f$-triangle for the edgewise subdivision of Figure 1, we also have $\ell_{F}(\sigma_{2}, x) = 3x$ and $\ell_{F}(\sigma_{3}, x) = 3x + 3x^{2}$ and compute that

$$p_{F,3,k}(x) = \begin{cases} 1 + 12x + 3x^{2}, & \text{if } k = 0 \\ 10x + 6x^{2}, & \text{if } k = 1 \\ 6x + 10x^{2}, & \text{if } k = 2 \\ 3x + 12x^{2} + x^{3}, & \text{if } k = 3. \end{cases}$$

Before proceeding with the proof of Theorem 4.1 we establish the following combinatorial identity which will be used there. We adopt the standard convention about binomial coefficients that $\binom{n}{k} := 0$, if $k$ is not in the range $0 \leq k \leq n$.

Lemma 4.3. We have

$$\sum_{m=0}^{n} \binom{m}{r} \binom{n-k}{m-k} x^{m-r}(1-x)^{n-m} = \sum_{i=0}^{r} \binom{n-k}{i} \binom{k}{r-i} x^{k-r+i}$$

for every $n \in \mathbb{N}$ and all $r, k \in \{0, 1, \ldots, n\}$.
Proof. Let us denote the left-hand side by \( L(n, k, r) \). Shifting the index \( m \) to \( m + k \) and using the identity \( \binom{m+k}{r} = \sum_{i=0}^{r} \binom{m}{i} \binom{k}{r-i} \), we get

\[
L(n, k, r) = \sum_{m=0}^{n-k} \binom{m+k}{r} \binom{n-k}{m} x^{m+k-r} (1-x)^{n-k-m} \\
= x^{k-r} \sum_{m=0}^{n-k} \left( \sum_{i=0}^{r} \binom{m}{i} \binom{k}{r-i} \right) \binom{n-k}{m} x^m (1-x)^{n-k-m} \\
= x^{k-r} \sum_{i=0}^{r} \binom{k}{r-i} \sum_{m=0}^{n-k} \binom{m}{i} \binom{n-k}{m} x^m (1-x)^{n-k-m}.
\]

Using the identity \( \binom{m}{i} \binom{n-k}{m} = \binom{n-k-i}{m-i} \) and applying the binomial theorem shows that the inner sum equals \( \binom{n-k}{i} x^i \) and the proof follows. \( \square \)

Proof of Theorem 4.1. Let \( \Delta' \) be an \( \mathcal{F} \)-uniform triangulation of an \( (n-1) \)-dimensional simplicial complex \( \Delta \). Using the notation of Section 3.1, since \( \Delta' \) is \( \mathcal{F} \)-uniform, for every \( (m-1) \)-dimensional face \( F \in \Delta \) there exist exactly \( f_{\mathcal{F}}(j, m) \) faces of \( \Delta' \) of dimension \( j-1 \) with carrier \( F \). Therefore,

\[
(13) \quad f_{j-1}(\Delta') = \sum_{m=j}^{n} f_{m-1}(\Delta) \cdot f_{\mathcal{F}}(j, m)
\]

for every \( j \in \{0, 1, \ldots, n\} \) and hence

\[
f(\Delta', x) = \sum_{j=0}^{n} f_{j-1}(\Delta') x^j = \sum_{j=0}^{n} \left( \sum_{m=j}^{n} f_{m-1}(\Delta) \cdot f_{\mathcal{F}}(j, m) \right) x^j \\
= \sum_{m=0}^{n} f_{m-1}(\Delta) \left( \sum_{j=0}^{m} f_{\mathcal{F}}(j, m) x^j \right) \\
= \sum_{m=0}^{n} f_{m-1}(\Delta) \cdot f_{\mathcal{F}}(\sigma_m, x).
\]

Applying the transformations between \( f \)-polynomials and \( h \)-polynomials and Equation (4), we conclude that
\[
\begin{align*}
h(\Delta', x) &= (1 - x)^n f(\Delta', \frac{x}{1 - x}) \\
&= (1 - x)^n \sum_{m=0}^{n} f_{m-1}(\Delta) \cdot f_F(\sigma_m, \frac{x}{1 - x}) \\
&= \sum_{m=0}^{n} (1 - x)^{n-m} f_{m-1}(\Delta) \cdot h_F(\sigma_m, x) \\
&= \sum_{m=0}^{n} x^m (1 - x)^{n-m} f_{m-1}(\Delta) \cdot h_F(\sigma_m, 1/x) \\
&= \sum_{m=0}^{n} x^m (1 - x)^{n-m} h_F(\sigma_m, 1/x) \cdot \sum_{k=0}^{m} h_k(\Delta) \binom{n-k}{m-k} \\
&= \sum_{k=0}^{n} h_k(\Delta) p_{F,n,k}(x),
\end{align*}
\]

where

\[
p_{F,n,k}(x) := \sum_{m=k}^{n} \binom{n-k}{m-k} x^m (1 - x)^{n-m} h_F(\sigma_m, 1/x).
\]

Expressing \(h_F(\sigma_m, 1/x)\) in terms of local \(h\)-polynomials and using the symmetry of the latter, we conclude further that

\[
p_{F,n,k}(x) = \sum_{m=k}^{n} \binom{n-k}{m-k} x^m (1 - x)^{n-m} h_F(\sigma_m, 1/x) \cdot \ell_F(\sigma_r, 1/x) \\
= \sum_{r=0}^{n} \ell_F(\sigma_r, x) \sum_{m=\max\{r,k\}}^{n} \binom{m}{r} \binom{n-k}{m-k} x^{m-r} (1 - x)^{n-m} \\
= \sum_{r=0}^{n} \ell_F(\sigma_r, x) \sum_{i=0}^{r} \binom{n-k}{i} \binom{k}{r-i} x^{k-r+i},
\]

where the last equality follows from Lemma 4.3. This computation, together with the nonnegativity of the coefficients of the polynomials \(\ell_F(\sigma_r, x)\), imply all claims in the statement of the theorem. \(\square\)

**Remark 4.4.** We will write \(f_F(\Delta, x) := (1 + x)^n h_F(\Delta, x/(1 + x))\) for the \(f\)-polynomial corresponding to \(h_F(\Delta, x)\). Thus, \(f_F(\Delta, x)\) is equal to the \(f\)-polynomial of any \(F\)-uniform triangulation of any \((n - 1)\)-dimensional simplicial complex \(\Delta\) and the coefficient of \(x^j\) in \(f_F(\Delta, x)\) is equal to the right-hand side of Equation (13). \(\square\)

The following corollary generalizes analogous statements for barycentric, edgewise and interval subdivisions [15 Section 2] [16 Section 1] [2 Section 3] to uniform triangulations.
Corollary 4.5. Let $\mathcal{F}$ be an $f$-triangle of size $d$ and $\Delta$ be an $(n-1)$-dimensional simplicial complex, for some $n \leq d$.

(a) If $h(\Delta, x)$ is symmetric with center of symmetry $n/2$, then so is $h(\Delta', x)$ for every $\mathcal{F}$-uniform triangulation $\Delta'$ of $\Delta$.

(b) If $h(\Delta, x)$ has nonnegative coefficients, then the inequality $h(\Delta, x) \leq h(\Delta', x)$ holds coefficientwise for every $\mathcal{F}$-uniform triangulation $\Delta'$ of $\Delta$.

Proof. To prove part (a), suppose that $h_k(\Delta) = h_{n-k}(\Delta)$ for every $k \in \{0, 1, \ldots, n\}$. In view of Theorem 4.1, we need to show that $x^n h_{\mathcal{F}}(\Delta, 1/x) = h_{\mathcal{F}}(\Delta, x)$. Our assumption and Equation (11) imply that

$$x^n h_{\mathcal{F}}(\Delta, 1/x) = \sum_{k=0}^{n} h_k(\Delta) x^n p_{\mathcal{F}, n, k}(1/x) = \sum_{k=0}^{n} h_{n-k}(\Delta) x^n p_{\mathcal{F}, n, n-k}(1/x)$$

and thus, it suffices to verify that $x^n p_{\mathcal{F}, n, n-k}(1/x) = p_{\mathcal{F}, n, k}(x)$. Indeed, using Equation (12) and the symmetry of the polynomials $\ell_{\mathcal{F}}(\sigma_r, x)$, we get

$$x^n p_{\mathcal{F}, n, n-k}(1/x) = x^n \sum_{r=0}^{n} \sum_{i=0}^{r} \binom{k}{i} \binom{n-k}{r-i} (1/x)^{n-k-r+i}$$

$$= \sum_{r=0}^{n} x^r \ell_{\mathcal{F}}(\sigma_r, 1/x) \sum_{i=0}^{r} \binom{k}{i} \binom{n-k}{r-i} x^{k-i}$$

$$= \sum_{r=0}^{n} \ell_{\mathcal{F}}(\sigma_r, x) \sum_{i=0}^{r} \binom{k}{r-i} \binom{n-k}{i} x^{k-r+i} = p_{\mathcal{F}, n, k}(x).$$

For part (b), suppose that $h_k(\Delta) \geq 0$ for all $k$. By Theorem 4.1, it suffices to show that the coefficient of $x^k$ in $p_{\mathcal{F}, n, k}(x)$ is positive for every such $k$. Indeed, since $\ell_{\mathcal{F}}(\sigma_0, x) = 1$, the summand of the right-hand side of Equation (12) corresponding to $r = 0$ is equal to $x^k$ and the proof follows.

The following statement lists the main properties of the coefficients $p_{\mathcal{F}}(n, k, j)$.

Proposition 4.6. Let $\mathcal{F}$ be a feasible $f$-triangle of size $d$ and $n \in \{0, 1, \ldots, d\}$.

(a) We have $p_{\mathcal{F}}(n, k, j) = p_{\mathcal{F}}(n, n-k, n-j)$ for all $k, j \in \{0, 1, \ldots, n\}$.

(b) The recurrence

$$p_{\mathcal{F}}(n, k, j) = p_{\mathcal{F}}(n, k-1, j) + p_{\mathcal{F}}(n-1, k-1, j-1) - p_{\mathcal{F}}(n-1, k-1, j)$$

holds for all $k, j \in \{0, 1, \ldots, n\}$ with $k \geq 1$. 

We have 
\[ p_{\mathcal{F}}(n, 0, j) = h_{\mathcal{F}}(j, n) \] for every \( j \in \{0, 1, \ldots, n\} \). Equivalently,
\[ \sum_{j=0}^{n} p_{\mathcal{F}}(n, 0, j)x^j = h_{\mathcal{F}}(\sigma_n, x). \]

(d) For every \( k \in \{0, 1, \ldots, n\} \),
\[ \sum_{j=0}^{n} p_{\mathcal{F}}(n, k, j) = h_{\mathcal{F}}(\sigma_n, 1). \]

Equivalently, the sum on the left-hand side is independent of \( k \) and equal to the number of facets of any \( \mathcal{F} \)-uniform triangulation of the simplex \( \sigma_n \).

(e) For every \( j \in \{0, 1, \ldots, n\} \),
\[ \sum_{k=0}^{n} p_{\mathcal{F}}(n, k, j) = (h_{\mathcal{F}})_j(\partial \sigma_{n+1}). \]

Proof. Part (a) can be restated as \( p_{\mathcal{F}, n,k}(x) = x^n p_{\mathcal{F}, n,n-k}(1/x) \), which has already been shown in the proof of Corollary 4.5. Part (b) can be verified by direct computation; see the comment following this proof. Since a more conceptual argument is given in Section 5 (see Corollary 5.6), we postpone the proof of (b) until then.

Parts (c) and (e) are direct consequences of Equation (11), applied when \( \Delta = \sigma_n \) and \( \Delta = \partial \sigma_{n+1} \), respectively: we have \( h_0(\Delta) = 1 \) and \( h_k(\Delta) = 0 \) for \( 1 \leq k \leq n \) in the former case, and \( h_k(\Delta) = 1 \) for all \( 0 \leq k \leq n \) in the latter. For part (d), we may compute directly from the formula (12) that
\[ \sum_{j=0}^{n} p_{\mathcal{F}}(n, k, j) = \sum_{r=0}^{n} \ell_{\mathcal{F}}(\sigma_r, 1) \cdot \frac{k!(n-k)!}{r!(n-r)!} \cdot \sum_{i=0}^{r} \binom{n-r}{i} \binom{n}{n-k} = \sum_{r=0}^{n} \binom{n}{r} \ell_{\mathcal{F}}(\sigma_r, 1) = h_{\mathcal{F}}(\sigma_n, 1). \]

Alternatively, one can sum the recurrence of part (b) with respect to \( j \) and use induction on \( k \), together with the result of part (c).

The recurrence of part (b) was discovered by J-M. Brunink and M. Juhnke-Kubitzke in the special case of antiprism triangulations (see [9]) and proven by direct computation, based on an explicit expression for the coefficients \( p_{\mathcal{F}}(n, k, j) \) in terms of the \( \mathcal{F} \)-triangle.
We now derive a combinatorial interpretation of $p_F(n, k, j)$ from this recurrence in the special case of the $r$-colored barycentric subdivision. Recall that an $r$-colored permutation $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is defined as a pair $(\tau, \varepsilon)$, where $\tau = (\tau(1), \tau(2), \ldots, \tau(n))$ is a permutation of $\{1, 2, \ldots, n\}$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0, 1, \ldots, r-1\}^n$ and $\varepsilon_i$ is thought of as the color assigned to $\tau(i)$. An index $1 \leq i \leq n$ is a descent of $w$ if either $\varepsilon_i > \varepsilon_{i+1}$, or $\varepsilon_i = \varepsilon_{i+1}$ and $\tau(i) > \tau(i+1)$, where $\tau(n+1) := n+1$ and $\varepsilon_{n+1} := 0$ (in particular, $n$ is a descent of $w$ if and only if $\tau(n)$ has nonzero color). Our interpretation reduces to those of [15] Theorem 2.2 and [2, Theorem 3.1] for $r = 1$ and $r = 2$, respectively (which are obtained in [2, 15] by different methods).

**Proposition 4.7.** Let $F$ be the $f$-triangle for the $r$-colored barycentric subdivision. Then, $p_F(n, k, j)$ is equal to the number of $r$-colored permutations $w \in \mathbb{Z}_r \wr \mathfrak{S}_{n+1}$ which have first coordinate of zero color, $j$ descents and last coordinate of zero color and equal to $n + 1 - k$.

**Proof.** Let $q(n, k, j)$ be the number of $r$-colored permutations, described in the proposition. By Proposition 4.6 (c), $p_F(n, 0, j)$ is equal to the coefficient of $x^j$ in $h_F(\sigma_n, x)$. This is known [8] Proposition 5.1 (c)] to equal the number of colored permutations $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ which have first coordinate of zero color and $j$ descents. This number is equal to $q(n, 0, j)$. Thus, it suffices to show that $q(n, k, j)$ satisfies recurrence (14) for $k \geq 1$. Indeed, this can rewritten as

$$q(n, k, j) - q(n - 1, k - 1, j - 1) = q(n, k - 1, j) - q(n - 1, k - 1, j).$$

We leave it to the reader to verify that: (a) the left-hand side equals the number of colored permutations $w \in \mathbb{Z}_r \wr \mathfrak{S}_{n+1}$ which have first coordinate of zero color, $j$ descents, last coordinate of zero color and equal to $n + 1 - k$ and next to last coordinate not of zero color and equal to $n + 2 - k$; (b) the right-hand side equals the number of colored permutations $w \in \mathbb{Z}_r \wr \mathfrak{S}_{n+1}$ which have first coordinate of zero color, $j$ descents, last coordinate of zero color and equal to $n + 2 - k$ and next to last coordinate not of zero color and equal to $n + 1 - k$; and (c) swapping the positions of $n + 1 - k$ and $n + 2 - k$ (while preserving the colors) sets up a bijection between the two sets of permutations in (a) and (b). \qed

5. **Polynomial operators**

The subdivision operator (see [11] Section 4, [12] Section 7.3.3 and references therein) is a linear operator on polynomials, closely related to the barycentric subdivision operation on simplicial complexes, which is important in the study of roots of real polynomials. This section generalizes this operator in the framework of uniform triangulations and employs this concept as a tool to prove the main recurrence and derive new interpretations of the coefficients $p_F(n, k, j)$.

**Definition 5.1.** Given an $f$-triangle $F$ of size $d$, the $F$-subdivision operator is defined as the linear operator $E_F : \mathbb{R}_d[x] \to \mathbb{R}_d[x]$ for which

$$E_F(x^n) := \sum_{k=0}^n f_F^0(k, n)x^k = f_F^0(\sigma_n, x)$$
for every $n \in \{0, 1, \ldots, d\}$.

By Example 3.2 we have $E_F(1) = 1$, $E_F(x) = x$ and $E_F(x^2) = (r - 1)x + rx^2$, where $r - 1$ is the number of interior vertices of any $F$-uniform triangulation of a 1-simplex. For the barycentric subdivision we have the explicit formula $f_F^\circ(k, n) = k!S(n, k)$, where $S(n, k)$ is a Stirling number of the second kind, and hence $E_F$ coincides with the usual subdivision operator [12, Section 7.3.3].

Recall that $f_F(\Delta, x)$ stands for the $f$-polynomial of any $F$-uniform triangulation of $\Delta$. The following proposition generalizes [12, Lemma 7.3.11] and [11, Lemma 4.3] to uniform triangulations.

**Proposition 5.2.** Let $F$ be an $f$-triangle of size $d$ and $n \in \{0, 1, \ldots, d\}$.

(a) We have $f_F(\Delta, x) = E_F(f(\Delta, x))$ for every $(n - 1)$-dimensional simplicial complex $\Delta$. In particular, $E_F((x + 1)^n) = f_F(\sigma_n, x)$.

(b) Suppose that $F$ is feasible. Then, $E_F$ is invertible and commutes with the restriction on $\mathbb{R}_d[x]$ of the algebra automorphism $I : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $I(x) = -1 - x$.

**Proof.** The coefficient of $x^j$ in $f_F(\Delta, x)$ is equal to the right-hand side of Equation (13); see Remark 5.3. As a result,

$$
\begin{align*}
    f_F(\Delta, x) &= \sum_{j=0}^{n} \left( \sum_{m=j}^{n} f_{m-1}(\Delta) \cdot f_F^\circ(j, m) \right) x^j \\
    &= \sum_{m=0}^{n} f_{m-1}(\Delta) \cdot E_F(x^m) \\
    &= E_F(\sum_{m=0}^{n} f_{m-1}(\Delta) x^m) = E_F(f(\Delta, x)).
\end{align*}
$$

This verifies the first assertion of part (a). The second case is special case $\Delta = \sigma_n$.

For (b), we have to show that $(E_F \circ I)(x^n) = (I \circ E_F)(x^n)$ for every $n \in \{0, 1, \ldots, d\}$. By the second assertion of Proposition 5.2 (a) we have $(E_F \circ I)(x^n) = E_F((-1 - x)^n) = (-1)^n E_F((x + 1)^n) = (-1)^n f_F(\sigma_n, x)$. Since $(I \circ E_F)(x^n) = f_F^\circ(-1 - x)$, the desired equality is equivalent to Equation (7) and the proof follows. Clearly, $E_F$ is invertible since it has triangular form in the standard basis of $\mathbb{R}_d[x]$ with nonzero diagonal entries $f_F^\circ(n, n)$. □

**Remark 5.3.** Just as in the proof of [12, Lemma 7.3.11] for barycentric subdivision, one can use Proposition 5.2 to obtain a new proof of part (a) of Corollary 4.5. Indeed, assume that $h(\Delta, x)$ is symmetric, with center of symmetry $n/2$. Then, $(-1)^n f_F(\Delta, -1 - x) = f(\Delta, x)$ and applying the operator $E_F$, we get

$$
(-1)^n f_F(\Delta, -1 - x) = (-1)^n (I \circ E_F) f(\Delta, x) = (-1)^n (E_F \circ I) f(\Delta, x) \\
= (-1)^n E_F(f(\Delta, -1 - x)) = E_F((-1)^n f(\Delta, -1 - x)) \\
= E_F(f(\Delta, x)) = f_F(\Delta, x).
$$

This means that $h_F(\Delta, x)$ is symmetric, with center of symmetry $n/2$. □
We have shown that $\mathcal{E}_F$ is the linear operator which describes the transformation of the $f$-polynomial of a simplicial complex under $F$-uniform triangulation. We now consider the corresponding transformation of the $h$-polynomial.

**Definition 5.4.** Given an $f$-triangle $F$ of size $d$ and $n \in \{0, 1, \ldots, d\}$, we define the linear operator $\mathcal{D}_{F,n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ by setting $\mathcal{D}_{F,n}(x^k) := p_{F,n,k}(x)$ for every $k \in \{0, 1, \ldots, n\}$. Equivalently,

$$\mathcal{D}_{F,n}(h(x)) := \sum_{k=0}^{n} h_k p_{F,n,k}(x)$$

for every $h(x) = \sum_{k=0}^{n} h_k x^k \in \mathbb{R}_n[x]$.

The following proposition lists basic properties of $\mathcal{D}_{F,n}$ and interprets the polynomial $p_{F,n,k}(x)$ by means of the subdivision operator $\mathcal{E}_F$.

**Proposition 5.5.** Let $F$ be an $f$-triangle of size $d$ and $n \in \{0, 1, \ldots, d\}$.

(a) $h_F(\Delta, x) = \mathcal{D}_{F,n}(h(\Delta, x))$ for every $(n-1)$-dimensional simplicial complex $\Delta$.

(b) We have $\mathcal{D}_{F,n} = J_n^{-1} \circ \mathcal{E}_F \circ J_n$, where $J_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ is the invertible linear operator which maps a polynomial $h(x) \in \mathbb{R}_n[x]$ to the associated $f$-polynomial and $\mathcal{E}_F$ is the restriction of $\mathcal{E}_F$ to $\mathbb{R}_n[x]$.

(c) The polynomial $p_{F,n,k}(x)$ is equal to the $h$-polynomial associated to $f_{F,n,k}(x) := \mathcal{E}_F(\Delta^{k+1}(1 + x)^{n-k})$, so that

$$p_{F,n,k}(x) = (1 - x)^n f_{F,n,k} \left( \frac{x}{1-x} \right),$$

for every $k \in \{0, 1, \ldots, n\}$.

**Proof.** Part (a) follows directly from Theorem 4.1. Moreover, the proof of this theorem shows that given any polynomial $h(x) \in \mathbb{R}_n[x]$ with associated $f$-polynomial $f(x)$, the $h$-polynomial which corresponds to $\mathcal{E}_F(f(x))$ equals $\mathcal{D}_{F,n}(h(x))$. Thus, part (b) holds as well. Part (c) follows by applying (b) to $h(x) = x^k$, whose associated $f$-polynomial is $f(x) = x^k(1 + x)^{n-k}$.

**Corollary 5.6.** For any $f$-triangle $F$ of size $d$ we have

$$(15) \quad p_{F,n,k}(x) = p_{F,n,k-1}(x) + (x-1)p_{F,n-1,k-1}(x)$$

for $1 \leq k \leq n \leq d$. Equivalently, the recurrence \[(14)\] holds for the coefficients $p_F(n, k, j)$.

**Proof.** Under the notation of Proposition 5.5 (c),

$$f_{F,n,k}(x) = \mathcal{E}_F(\Delta^{k+1}(1 + x)^{n-k}) = \mathcal{E}_F(\Delta^{k+1}(1 + x)^{n-k-1} + x^k(1 + x)^{n-k-1})$$

$$= \mathcal{E}_F(\Delta^{k+1}(1 + x)^{n-k-1}) + \mathcal{E}_F(x^k(1 + x)^{n-k-1})$$

$$(16) \quad = f_{F,n,k+1}(x) + f_{F,n-1,k}(x).$$

Thus, by the same proposition,
\[
p_{F,n,k+1}(x) = (1-x)^n f_{F,n,k+1}(x/(1-x))
= (1-x)^n f_{F,n,k}(x/(1-x)) - (1-x)^n f_{F,n-1,k}(x/(1-x))
= p_{F,n,k}(x) - (1-x)p_{F,n-1,k}(x)
\]
and the proof follows. \(\square\)

\textbf{Example 5.7.} By Proposition 5.5 and Equation (9), for the \(r\)-fold edgewise subdivision the operator \(D_{F,n}\) has the explicit form
\[
D_{F,n}(h(x)) = \left( (1 + x + x^2 + \cdots + x^{r-1}) h(x) \right)^{(r,0)}.
\]
For the relation of this operator to the Veronese construction for rational formal power series, see \cite{10, Section 3}. \(\square\)

The second interpretation of the polynomials \(p_{F,n,k}(x)\) offered in this section is as follows.

\textbf{Proposition 5.8.} Let \(F\) be an \(f\)-triangle of size \(d\) and \(\Gamma_n\) be an \(F\)-uniform triangulation of the simplex \(\sigma_n\) for some \(n \leq d\). Then, \(p_{F,n,k}(x) = h(\Gamma_n,k,x)\) for every \(k \in \{0,1,\ldots,n\}\), where \(\Gamma_n,k\) is the relative simplicial complex obtained from \(\Gamma_n\) by removing all faces carried by any \(k\) chosen facets of \(\sigma_n\).

\textbf{Proof.} Since \(\Gamma_{n,0} = \Gamma_n\), we have \(h(\Gamma_{n,0},x) = h(\Gamma_n,x) = h_F(\sigma_n,x) = p_{F,n,0}(x)\) for all \(n \leq d\). Thus, it suffices to show that the polynomials \(h(\Gamma_n,k,x)\) satisfy recurrence (15) or, equivalently, that the polynomials \(f(\Gamma_n,k,x)\) satisfy recurrence (16). Indeed, this is true because, by construction, \(\Gamma_{n,k+1}\) is the complex obtained from \(\Gamma_n,k\) by removing a relative subcomplex combinatorially isomorphic to \(\Gamma_{n-1,k}\). \(\square\)

For example, let \(d = 3\) and suppose again that \(F\) is the \(f\)-triangle for the edgewise subdivision of Figure 1. By counting directly the faces of the relative simplicial complexes \(\Gamma_{3,k}\), we find that
\[
f(\Gamma_{3,k},x) = \begin{cases} 
1 + 15x + 30x^2 + 16x^3, & \text{if } k = 0 \\
10x + 26x^2 + 16x^3, & \text{if } k = 1 \\
6x + 22x^2 + 16x^3, & \text{if } k = 2 \\
3x + 18x^2 + 16x^3, & \text{if } k = 3 
\end{cases}
\]
We leave it to the reader to verify that the corresponding \(h\)-polynomials are exactly those we computed as the polynomials \(p_{F,3,k}(x)\) in Example 4.2.

\textbf{Remark 5.9.} Proposition 5.8 implies the nonnegativity of the polynomials \(p_{F,n,k}(x)\), via relative Stanley–Reisner theory, as well as the symmetry property \(x^np_{F,n,k}(1/x) = p_{F,n,n-k}(x)\). Indeed, the former follows since \(\Gamma_n,k\) is relatively Cohen–Macaulay, by \cite{27, Corollary III.7.3 (iii)}, and hence has nonnegative \(h\)-vector. The latter follows from Proposition 2.1.
6. REAL-ROOTEDNESS OF THE $h$-POLYNOMIAL

This section proves an expanded version of Theorem 1.2.

We recall that a polynomial $g(x) \in \mathbb{R}[x]$ is real-rooted if all complex roots of $g(x)$ are real, or $g(x)$ is the zero polynomial. A real-rooted polynomial, with roots $\alpha_1 \geq \alpha_2 \geq \cdots$, is said to interlace a real-rooted polynomial, with roots $\beta_1 \geq \beta_2 \geq \cdots$, if

$$\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1.$$ 

By convention, the zero polynomial interlaces and is interlaced by every real-rooted polynomial and constant polynomials interlace all polynomials of degree at most one. We refer to [12, Section 7.8] and the references given there for background on the theory of interlacing and for any related undefined terminology.

Every polynomial $g(x) \in \mathbb{R}_n[x]$ can be written uniquely in the form $g(x) = a(x) + xb(x)$, where $a(x) \in \mathbb{R}_n[x]$ is symmetric, with center of symmetry $n/2$ and $b(x) \in \mathbb{R}_{n-1}[x]$ is symmetric, with center of symmetry $(n-1)/2$. Following [13], we will say that $g(x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$, if $a(x)$ and $b(x)$ are real-rooted polynomials with nonnegative coefficients. We will also say that such a decomposition is interlacing if the following equivalent (by [13, Theorem 2.6]) conditions hold: (i) $a(x)$ interlaces $g(x)$; (ii) $b(x)$ interlaces $g(x)$; (iii) $b(x)$ interlaces $a(x)$; and (iv) $x^ng(1/x)$ interlaces $g(x)$.

We can now state the main result of this section.

**Theorem 6.1.** Let $\mathcal{F}$ be a feasible $f$-triangle of size $d \in \mathbb{N}\cup\{\infty\}$ and assume the following for some $n \in \{1, 2, \ldots, d\}$:

(i) $h_\mathcal{F}(\sigma_m, x)$ is a real-rooted polynomial for all $2 \leq m < n$.

(ii) $h_\mathcal{F}(\sigma_m, x) - h_\mathcal{F}(\partial \sigma_m, x)$ is either identically zero, or a real-rooted polynomial of degree $m - 1$ with nonnegative coefficients which is interlaced by $h_\mathcal{F}(\sigma_{m-1}, x)$, for all $2 \leq m \leq n$.

Then:

(a) The polynomial $D_{\mathcal{F}, n}(h(x))$ is real-rooted for every polynomial $h(x) \in \mathbb{R}_n[x]$ with nonnegative coefficients. In particular, $h_\mathcal{F}(\Delta, x)$ is real-rooted for every $(n - 1)$-dimensional simplicial complex $\Delta$ with nonnegative $h$-vector.

(b) The polynomials $h_\mathcal{F}(\sigma_n, x)$ and $h_\mathcal{F}(\partial \sigma_n, x)$ are real-rooted and are interlaced by $h_\mathcal{F}(\sigma_{n-1}, x)$. Moreover, $h_\mathcal{F}(\sigma_n, x)$ has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to $n - 1$.

Some comments on the hypotheses of the theorem are in order. We recall the fact [29, Proposition 3.3], to be used in the sequel, that if two real-rooted polynomials with positive leading coefficients interlace (respectively, are interlaced by) a nonzero real-rooted polynomial $g(x)$, then their sum is also real-rooted and interlaces (respectively, is interlaced by) $g(x)$.

**Remark 6.2.** (a) The degree of $h_\mathcal{F}(\sigma_m, x)$ is at most $m - 1$ for every positive integer $m$. The last sentence of Proposition 2.1 implies that the coefficient of $x^{m-1}$ in $h_\mathcal{F}(\sigma_m, x)$ is...
equal to the number \( f^*_F(1,m) \) of interior vertices of any \( F \)-uniform triangulation of \( \sigma_m \). Therefore, the coefficient of \( x^{m-1} \) in \( h_F(\sigma_m, x) - h_F(\partial \sigma_m, x) \) is one less than this number. As a consequence, a necessary condition for assumption (ii) to be valid is that there exists at least one such interior vertex (this is also sufficient for \( m = 2 \)). No such vertex exists for the \( r \)-fold edgewise subdivision of \( \sigma_m \), unless \( r \geq m \). This matches the range of values of \( r \) for which [21, Theorem 1.1] holds.

(b) A sufficient condition for \( h_F(\sigma_m, x) - h_F(\partial \sigma_m, x) \) to have nonnegative coefficients is that no facet of some \( F \)-uniform triangulation of \( \sigma_m \) has all vertices on the boundary of \( \sigma_m \). This is a consequence of [26, Theorem 2.1].

(c) Assumption (ii) cannot be dropped from the hypotheses, even if the assumption that \( h_F(\sigma_m, x) \) is interlaced by \( h_F(\sigma_{m-1}, x) \) for all \( m < n \) is added. Indeed, let \( d = \infty \) and suppose that \( F \) is the \( f \)-triangle for 2-fold edgewise subdivision. Equation (9) implies that

\[
h_F(\sigma_n, x) = \sum_{k=0}^{\infty} \binom{n}{2k} x^k.
\]

Setting

\[
g_n(x) := \sum_{k=0}^{\infty} \binom{n}{2k+1} x^k,
\]

it is well known (see, for instance, [18, Theorem 7.64] [21, Proposition 3.4]) that these two polynomials are real-rooted and that \( g_{n}(x) \) interlaces \( h_F(\sigma_n, x) \) for every \( n \). As a result, the latter interlaces \( xg_n(x) \). Since, by the standard recurrence for binomial coefficients, we have \( h_F(\sigma_{n+1}, x) = h_F(\sigma_n, x) + xg_n(x) \), it follows that \( h_F(\sigma_n, x) \) interlaces \( h_F(\sigma_{n+1}, x) \) for every \( n \) as well. However, as noted in [16, p. 552], the polynomial \( h_F(\partial \sigma_5, x) \) is not real-rooted and hence conclusion (a) of Theorem 6.1 fails in this case.

To prepare for the proof of Theorem 6.1, we focus on the polynomials \( p_{F,n,k}(x) \). We have already shown that they have nonnegative coefficients and that:

\[
\begin{align*}
\text{(17)} & \quad p_{F,n,0}(x) = h_F(\sigma_n, x), \\
\text{(18)} & \quad p_{F,n,k}(x) = p_{F,n,k-1}(x) + (x - 1)p_{F,n-1,k-1}(x), \quad \text{for } k \geq 1, \\
\text{(19)} & \quad x^n p_{F,n,k}(1/x) = p_{F,n,n-k}(x), \\
\text{(20)} & \quad \sum_{k=0}^{n} p_{F,n,k}(x) = h_F(\partial \sigma_{n+1}, x).
\end{align*}
\]

Given that \( h_F(\sigma_n, x) \) has nonnegative coefficients and constant term equal to 1 for every \( n \in \mathbb{N} \), recurrence [18] shows that the polynomials \( p_{F,n,k}(x) \) are nonzero and that for fixed \( n \in \mathbb{N} \), their degrees are increasing in \( k \).

Throughout this section, we set \( p_{F,n-1,n}(x) := h_F(\sigma_n, x) - h_F(\partial \sigma_n, x) \) for \( n \geq 1 \). This polynomial has zero constant term and, as explained in [22, Remark 4.8], it is always
symmetric, with center of symmetry \( n/2 \) (but it does not necessarily have nonnegative coefficients).

**Lemma 6.3.** For every \( f \)-triangle \( \mathcal{F} \) of size \( d \), the recurrence

\[
(21) 
\quad p_{\mathcal{F},n,k}(x) = x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) + \sum_{i=k}^{n} p_{\mathcal{F},n-1,i}(x)
\]

holds for \( 0 \leq k \leq n \leq d \), where \( p_{\mathcal{F},n-1,n}(x) = h_\mathcal{F}(\sigma_n, x) - h_\mathcal{F}(\partial\sigma_n, x) \).

**Proof.** For \( k \geq 1 \), replacing \( k \) by \( i \) in the recurrence (18) and summing for \( 1 \leq i \leq k \), we get

\[
\quad p_{\mathcal{F},n,k}(x) = p_{\mathcal{F},n,0}(x) + x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) - \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x)
\]

(which holds trivially for \( k = 0 \)). Equation (21) follows from this equality, when combined with Equations (17) and (20).

A sequence \((g_0(x), g_1(x), \ldots, g_m(x))\) of real-rooted polynomials is called **interlacing** if \( g_i(x) \) interlaces \( g_j(x) \) for all \( 0 \leq i < j \leq m \). For that to happen, it suffices to require that \( g_{i-1}(x) \) interlaces \( g_i(x) \) for all \( 1 \leq i \leq m \) and \( g_0(x) \) interlaces \( g_m(x) \) (see [11, Lemma 2.3] [29, Proposition 3.3]). We are now in position to prove Theorem 6.1.

**Proof of Theorem 6.1.** The proof is motivated by [12, Example 7.8.8] (essentially, the special case of barycentric subdivision). Let us consider the sequences

\[
\mathcal{P}_{\mathcal{F},n} := (p_{\mathcal{F},n-1,0}(x), p_{\mathcal{F},n-1,1}(x), \ldots, p_{\mathcal{F},n-1,n}(x)) \quad \text{and} \quad \mathcal{Q}_{\mathcal{F},n} := (p_{\mathcal{F},n,0}(x), p_{\mathcal{F},n,1}(x), \ldots, p_{\mathcal{F},n,n}(x)).
\]

By Theorem 4.1 and assumption (ii), all polynomials in these sequences have nonnegative coefficients. We will first show by induction on \( n \) that these sequences are interlacing (in particular, their elements are real-rooted). This is true for \( n = 1 \) since \( p_{\mathcal{F},0,0}(x) = p_{\mathcal{F},1,0}(x) = 1, p_{\mathcal{F},0,1}(x) = 0, p_{\mathcal{F},1,1}(x) = x, p_{\mathcal{F},1,1}(x) = (r - 2)x \), where \( r \) has the same meaning as in Example 4.2 and \( r \geq 2 \) by assumption (ii).

For the inductive step, let us assume that \( n \geq 2 \) and that the sequences \( \mathcal{P}_{\mathcal{F},n-1} \) and \( \mathcal{Q}_{\mathcal{F},n-1} \) are interlacing. We will show that so are \( \mathcal{P}_{\mathcal{F},n} \) and \( \mathcal{Q}_{\mathcal{F},n} \). As in [12, Example 7.8.8], Lemma 6.3 shows, by an application of [12, Corollary 7.8.7], that the interlacing property for \( \mathcal{P}_{\mathcal{F},n} \) implies that for \( \mathcal{Q}_{\mathcal{F},n} \). Therefore, we only need to show that \( \mathcal{P}_{\mathcal{F},n} \) is interlacing. Since the first \( n \) terms of \( \mathcal{P}_{\mathcal{F},n} \) form \( \mathcal{Q}_{\mathcal{F},n-1} \), we already know that \( p_{\mathcal{F},n-1,i}(x) \) interlaces \( p_{\mathcal{F},n-1,j}(x) \) for \( 0 \leq i \leq j \leq n - 1 \). Moreover, by assumption (ii), \( p_{\mathcal{F},n-1,0}(x) = h_\mathcal{F}(\sigma_{n-1}, x) \) interlaces \( p_{\mathcal{F},n-1,1}(x) = h_\mathcal{F}(\sigma_n, x) - h_\mathcal{F}(\partial\sigma_n, x) \). Thus, by the weak transitivity property [11, Lemma 2.3] of interlacing, mentioned earlier, it suffices to show that \( p_{\mathcal{F},n-1,n-1}(x) \) interlaces \( p_{\mathcal{F},n-1,n}(x) \). We may assume that the latter is nonzero, in which case it must have degree \( n - 1 \) by assumption (ii). Since \( p_{\mathcal{F},n-1,n}(x) \) has zero constant term, \( p_{\mathcal{F},n-1,0}(x) \) is interlaced by \( p_{\mathcal{F},n-1,n}(x)/x \). Since both polynomials have degree \( n - 2 \) (see Remark 6.2, the recurrence (21) follows from this equality, when combined with Equations (17) and (20)).
(a)) and the latter is symmetric, with center of symmetry \((n - 2)/2\), the polynomial \(x^{n-2}p_{F,n-1,0}(1/x)\) interlaces \(p_{F,n-1,n}(x)/x\) and therefore \(p_{F,n-1,n-1}(x) = x^{n-1}p_{F,n-1,0}(1/x)\) interlaces \(p_{F,n-1,n}(x)\). This completes the induction.

Suppose now that \(h(x) \in \mathbb{R}[x]\) has nonnegative coefficients. Then, by Definition 5.4 \(D_{F,n}(h(x))\) can be written as a nonnegative linear combination of the elements of \(Q_{F,n}\). Since the latter is interlacing, \(D_{F,n}(h(x))\) is real-rooted [12 Theorem 7.8.2]. This proves part (a). We have already shown that \(h_F(σ_n, x) = p_{F,n,0}(x)\) is real-rooted. The equality

\[
(22) \quad h_F(σ_n, x) = h_F(∂σ_n, x) + (h_F(σ_n, x) - h_F(∂σ_n, x)),
\]

combined with assumption (ii), shows that it also has a nonnegative symmetric decomposition with respect to \(n-1\). Since, as we have already shown, \(h_F(σ_n, x) = p_{F,n,0}(x)\) interlaces \(p_{F,n,n}(x) = x^n h_F(σ_n, 1/x)\), we also have that \(h_F(σ_n, x)\) is interlaced by \(x^{n-1}h_F(σ_n, 1/x)\).

These facts and [13 Theorem 2.6] imply that \(h_F(σ_n, x)\) has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to \(n-1\). This proves the last statement of part (b) and that \(h_F(∂σ_n, x)\) is real-rooted as well.

Finally, recall that \(h_F(σ_n-1, x)\) interlaces all elements of \(P_{F,n}\) and \(Q_{F,n-1}\). As a result [29 Proposition 3.3], it also interlaces the sums of the elements of these sequences, which equal \(h_F(σ_n, x)\) and \(h_F(∂σ_n, x)\), respectively. This completes the proof of part (b).

7. Applications

We now discuss how Theorem [12] applies to the motivating examples of Section 6.2.

Example 7.1. Consider the barycentric subdivision and let \(F\) be the corresponding \(f\)-triangle of infinite size. Then, \(h_F(σ_m, x) = h_F(∂σ_m, x)\) is the \(m\)th Eulerian polynomial (see [28 Section 1.4]) and thus, assumption (ii) of Theorem [6.1] is satisfied trivially. Given Theorem 6.1 a straightforward induction on \(n\) then shows that \(D_{F,n}(h(x))\) is real-rooted for every polynomial \(h(x) \in \mathbb{R}_n[x]\) with nonnegative coefficients. In particular, \(h(\text{sd}(Δ), x)\) is real-rooted for every simplicial complex \(Δ\) with nonnegative \(h\)-vector. Hence, Theorem 6.1 recovers the main result of [15] in this case.

Example 7.2. Let \(F\) be the \(f\)-triangle of infinite size which corresponds to the \(r\)-fold edgewise subdivision. Setting \(c_{n,r}(x) := (1 + x + x^2 + \cdots + x^{r-1})^n\), from Equation (9) we deduce that \(h_F(σ_n, x) = (c_{n,r}(x))^{(r,0)}\) and

\[
h_F(∂σ_n, x) = (c_{n-1,r}(x)(1 + x + x^2 + \cdots + x^{n-1}))^{(r,0)}.
\]

It is well known [18 Example 3.76] [21 Proposition 3.4] that the sequence \((c_{n,r}(x))_{1 \leq i \leq r}^{(r,r-i)}\) is interlacing. We deduce that, for \(r \geq m\),

\[
h_F(σ_m, x) - h_F(∂σ_m, x) = (c_{m,r}(x))^{(r,0)} - (c_{m-1,r}(x)(1 + x + x^2 + \cdots + x^{m-1}))^{(r,0)}
\]

\[
= (c_{m-1,r}(x)(x^m + x^{m+1} + \cdots + x^{r-1}))^{(r,0)}
\]

\[
= \sum_{i=m}^{r-1} x(c_{m-1,r}(x))^{(r,r-i)}
\]
is interlaced by \((c_{m-1,r}(x))^{(r,0)} = h_F(\sigma_{m-1}, x)\), since the latter is interlaced by all of the \((c_{m-1,r}(x))^{(r,r-1)}\). Hence, all assumptions of Theorem 6.1 are satisfied when \(r \geq n\) and the theorem recovers the case \(i = 0\) of [21, Theorem 1.1].

It was observed in [5, p. 15] that
\[
(23) \quad h_F(\sigma_{n+1}, x) = \sum_{w \in \{0, 1, \ldots, r-1\}^n} x^\text{asc}(w),
\]
where for \(w = (w_1, w_2, \ldots, w_n) \in \{0, 1, \ldots, r-1\}^n\), \(\text{asc}(w)\) denotes the number of indices \(i \in \{0, 1, \ldots, n-1\}\) such that \(w_i < w_{i+1}\), with the convention that \(w_0 := 0\). Thus, part (b) of Theorem 6.1 yields the following corollary.

**Corollary 7.3.** For every positive integer \(n\), the polynomial on the right-hand side of (23) has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to \(n\), for every \(r \geq n+1\). □

**Example 7.4.** To illustrate the applicability of Theorem 6.1 consider the following generalization of the two previous examples. Fix integers \(1 \leq s < r\), think of \(\Delta\) as a geometric simplicial complex and construct a triangulation \(\text{sd}_{r,s}(\Delta)\) of \(\Delta\) as follows. First triangulate the \(s\)-skeleton of \(\Delta\) with the \(r\)-fold edgewise subdivision. Then, for \(j \geq s\) and assuming the triangulation of the \(j\)-skeleton of \(\Delta\) has been defined, triangulate each \((j+1)\)-dimensional face of \(\Delta\) by inserting one point in the relative interior of that face and coning over its boundary, which is already triangulated by induction.

This process defines a triangulation \(\text{sd}_{r,s}(\Delta)\) of \(\Delta\) which reduces to \(\text{sd}(\Delta)\) for \(s = 1\) and \(r = 2\), and to \(\text{esd}_r(\Delta)\) when \(s\) is at least as large as the dimension of \(\Delta\). As verified in Example 7.2, assumption (ii) of Theorem 6.1 is satisfied for \(m \leq s\). Moreover, it is trivially satisfied for \(m > s\), since then \(h_F(\sigma_m, x) - h_F(\partial \sigma_m, x) = 0\) (the operation of coning for simplicial complexes leaves the \(h\)-polynomial invariant). A straightforward induction on \(n\) then shows that all conclusions in the statement of Theorem 6.1 hold. It would be interesting to interpret combinatorially the coefficients \(p_F(n, k, j)\) for this example. □

For the \(r\)-colored barycentric subdivision (in particular, for the interval triangulation), the conclusion (a) of Theorem 6.1 can be deduced from known results. It would still be interesting to decide whether assumption (ii) of the theorem is valid in this case; this question will be answered in a forthcoming paper.

**Proposition 7.5.** Let \(r\) be any positive integer and \(F\) be the \(f\)-triangle of infinite size for the \(r\)-colored barycentric subdivision. Then, the polynomial \(D_{F,n}(h(x))\) is real-rooted for every \(h(x) \in \mathbb{R}_n[x]\) with nonnegative coefficients.

In particular, the \(h\)-polynomial of the \(r\)-colored barycentric subdivision of \(\Delta\) is real-rooted for every simplicial complex \(\Delta\) with nonnegative \(h\)-vector.

**Proof.** This is because, by [15, Theorem 3.1], barycentric subdivision transforms any \(h\)-polynomial with nonnegative coefficients to one with only real nonpositive roots and \(r\)-fold edgewise subdivision (for instance, by [14, Theorem 4.5.6] or [30, Corollary 3.4]) transforms any \(h\)-polynomial with only real nonpositive roots to one with the same property. □
As already mentioned in the proof of Proposition 4.7, for the \( r \)-colored barycentric subdivision the polynomial \( h_F(\sigma_n, x) \) is known to equal

\[
A_{n,r}^+(x) := \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_n)^+} x^{\des(w)},
\]

where \((\mathbb{Z}_r \wr \mathfrak{S}_n)^+\) is the set of colored permutations \( w \in \mathbb{Z}_r \wr \mathfrak{S}_n \) with first coordinate of zero color and \( \des(w) \) stands for the number of descents of \( w \in \mathbb{Z}_r \wr \mathfrak{S}_n \). This polynomial is real-rooted by Proposition 7.5. The following statement partially confirms conclusion (b) of Theorem 6.1 in this case.

**Proposition 7.6.** For all positive integers \( n, r \), the polynomial \( A_{n,r}^+(x) \) has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to \( n - 1 \).

**Proof.** Recall that (22) expresses the symmetric decomposition of \( h_F(\sigma_n, x) = A_{n,r}^+(x) \), with respect to \( n - 1 \). The polynomial \( h_F(\partial \sigma_n, x) \) has nonnegative coefficients, being the \( h \)-polynomial of a triangulation of a sphere, and only real roots, by Proposition 7.5. To show that \( h_F(\sigma_n, x) - h_F(\partial \sigma_n, x) \) has the same properties, we work as in Example 7.2. Using the notation adopted there, and since the \( h \)-polynomial of the barycentric subdivision of \( \sigma_n \) or its boundary is given by the \( n \)th Eulerian polynomial \( A_n(x) \) (see [28, Section 1.4]), we find that

\[
h_F(\sigma_n, x) - h_F(\partial \sigma_n, x) = (c_{n,r}(x)A_n(x))^{(r,0)} - (c_{n-1,r}(x)A_n(x))^{(r,0)} = \sum_{i=1}^{r-1} x(g(x))^{(r, r-i)},
\]

where \( g(x) := c_{n-1,r}(x)A_n(x) = (1 + x + \cdots + x^{r-1})^{n-1}A_n(x) \). Since \( A_n(x) \) is real-rooted, an application of [30, Corollary 3.4] shows that the sequence \((g(x))^{(r, r-i)}\) is interlacing. Therefore, \((h_F(\sigma_n, x) - h_F(\partial \sigma_n, x))/x \) is real-rooted and interlaces \( h_F(\partial \sigma_n, x) = (g(x))^{(r,0)} \) and the proof follows. \( \square \)

8. Further directions

Several interesting classes of polynomials, occurring in algebraic and geometric combinatorics, were shown to have nonnegative real-rooted symmetric decompositions in [13]. The following statement exhibits another such class.

**Proposition 8.1.** The polynomial \( h(\sd(\Delta), x) \) has a nonnegative real-rooted symmetric decomposition with respect to \( n - 1 \) for every triangulation \( \Delta \) of the \((n - 1)\)-dimensional ball.

**Proof.** We recall Equation (8). Shifting \( m \) to \( m - 1 \) and replacing \( i \) in the inner summation with \( n - i \) first, and then replacing \( m \) with \(-m\) and applying [28, Proposition 4.2.3], this
The equation may be rewritten successively as
\[
\sum_{m \geq 1} \left( \sum_{i=0}^{n} h_{n-i}(\Delta) m^i (m-1)^{n-i} \right) x^m = \frac{x h(\text{sd}(\Delta), x)}{(1-x)^{n+1}} \iff \\
\sum_{m \geq 1} \left( \sum_{i=0}^{n} h_{n-i}(\Delta) m^i (m+1)^{n-i} \right) x^m = \frac{x^n h(\text{sd}(\Delta), 1/x)}{(1-x)^{n+1}} \iff \\
\sum_{m \geq 0} \left( \sum_{i=0}^{n} h_{n-i}(\Delta) m^i (m+1)^{n-i} \right) x^m = \frac{x^n h(\text{sd}(\Delta), 1/x)}{(1-x)^{n+1}}.
\]

The last equivalence holds because \( h_n(\Delta) = 0 \). As a consequence of [26, Lemma 2.3], we have
\[
h_n(\Delta) + h_{n-1}(\Delta) + \cdots + h_{n-i}(\Delta) \leq h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta)
\]
for all \( 0 \leq i \leq n/2 \). An application of [13, Theorem 2.13] shows that \( x^n h(\text{sd}(\Delta), 1/x) \) has a nonnegative real-rooted symmetric decomposition with respect to \( n \). Since \( x^n h(\text{sd}(\Delta), 1/x) \) has zero constant term, this is equivalent to the desired statement. \( \square \)

It seems natural to ask for conditions on an \( f \)-triangle \( \mathcal{F} \) which guarantee that \( h_{\mathcal{F}}(\Delta, x) \) has a nonnegative, real-rooted symmetric decomposition for reasonably broad classes of simplicial complexes \( \Delta \). A forthcoming paper will show that, in fact, this is the case under the same conditions on \( \mathcal{F} \), as those of Theorem 6.1, and the same conditions on \( h(\Delta, x) \), as those in the special case of barycentric subdivision [13, Theorem 2.13]. It may also be interesting to study the ‘deranged map’ of [13, Section 3.2] in the framework of uniform triangulations.

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References

[1] S. Ahmad and V. Welker, On partial barycentric subdivision, Results Math. 73 (2018), Article 21, 20pp.
[2] I. Anwar and S. Nazir, The \( f \)- and \( h \)-vectors of interval subdivisions, J. Combin. Theory Series A 169 (2020), Article 105124, 22pp.
[3] C.A. Athanasiadis, Edgewise subdivisions, local \( h \)-polynomials and excedances in the wreath product \( \mathbb{Z}_r \wr S_n \), SIAM J. Discrete Math. 28 (2014), 1479–1492.
[4] C.A. Athanasiadis, A survey of subdivisions and local \( h \)-vectors, in The Mathematical Legacy of Richard P. Stanley (P. Hersh, T. Lam, P. Pylyavskyy, V. Reiner, eds.), Amer. Math. Society, Providence, RI, 2016, pp. 39–51.
[5] C.A. Athanasiadis, The local \( h \)-polynomial of the edgewise subdivision of the simplex, Bull. Hellenic Math. Soc. (N.S.) 60 (2016), 11–19.
[6] C.A. Athanasiadis, Gamma-positivity in combinatorics and geometry, Sém. Lothar. Combin. 77 (2018), Article B77i, 64pp (electronic).
[7] C.A. Athanasiadis, Some applications of Rees products of posets to equivariant gamma-positivity, Algebr. Comb. 3 (2020), 291–300.
[8] C.A. Athanasiadis, Binomial Eulerian polynomials for colored permutations, J. Combin. Theory Series A 173 (2020), Article 105214, 38pp.
[9] C.A. Athanasiadis, J-M. Brunink and M. Juhnke-Kubitzke, Combinatorics of antiprism triangulations, arXiv:2006.10789.
[10] M. Beck and A. Stapledon, On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series, Math. Z. 264 (2010), 195–207.
[11] P. Brändén, On linear transformations preserving the Pólya frequency property, Trans. Amer. Math. Soc. 358 (2006), 3697–3716.
[12] P. Brändén, Unimodality, log-concavity, real-rootedness and beyond, in Handbook of Combinatorics (M. Bona, ed.), CRC Press, 2015, pp. 437–483.
[13] P. Brändén and L. Solus, Symmetric decompositions and real-rootedness, Int. Math. Res. Not. (to appear).
[14] F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc., no. 413, 1989.
[15] F. Brenti and V. Welker, f-vectors of barycentric subdivisions, Math. Z. 259 (2008), 849–865.
[16] F. Brenti and V. Welker, The Veronese construction for formal power series and graded algebras, Adv. in Appl. Math. 42 (2009), 545–556.
[17] E. Delucchi, A. Pixton and L. Sabalka, Face vectors of subdivided simplicial complexes, Discrete Math. 312 (2012), 248–257.
[18] S. Fisk, Polynomials, roots, and interlacing, arXiv:0612833.
[19] M. Hlavacex and L. Solus, Subdivisions of shellable complexes, arXiv:2003.07328.
[20] I. Izmestiev and M. Joswig, Branching coverings, triangulations and 3-manifolds, Adv. Geom. 3 (2003), 191–225.
[21] K. Jochemko, On the real-rootedness of the Veronese construction for rational formal power series, Int. Math. Res. Not. 2018 (2018), 4780–4798.
[22] M. Juhnke-Kubitzke, S. Murai and R. Sieg, Local h-vectors of quasi-geometric and barycentric subdivisions, Discrete Comput. Geom. 61 (2019), 364–379.
[23] C. Iamvédous, Barycentric subdivisions, clusters and permutation enumeration (in Greek), Doctoral Dissertation, University of Athens, 2013.
[24] R.P. Stanley, Generalized h-vectors, intersection cohomology of toric varieties and related results, in Commutative Algebra and Combinatorics (N. Nagata and H. Matsumura, eds.), Adv. Stud. Pure Math. 11, Kinokuniya, Tokyo and North-Holland, Amsterdam-New York, 1987, pp. 187–213.
[25] R.P. Stanley, Subdivisions and local h-vectors, J. Amer. Math. Soc. 5 (1992), 805–851.
[26] R.P. Stanley, A monotonicity property of h-vectors and h*-vectors, European J. Combin. 14 (1993), 251–258.
[27] R.P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhäuser, Basel, 1996.
[28] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, second edition, Cambridge, 2011.
[29] D.G. Wagner, Total positivity of Hadamard products, J. Math. Anal. Appl. 163 (1992), 459–483.
[30] P.B. Zhang, Interlacing polynomials and the Veronese construction for rational formal power series, Proc. Roy. Soc. Edinburgh Sect. A (to appear).

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