Dividend Barrier Strategies In a Renewal Risk Model With Phase-type Distributed Interclaim Times *

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Abstract In this paper, we consider the dividend problem of the renewal risk model with phase-type distributed interclaim times and exponentially distributed claim sizes. We assume that the phases of the interclaim times can be observed. We first consider the phase-wise barrier strategies and look for optimal barriers to maximize the discounted cumulative dividend. We analyze some properties of the optimal phase-wise barrier strategy and do some numerical experiments to see the optimal phase-wise barriers. From the numerical experiments, we propose a conjecture that when all the optimal phase-wise dividend barriers are not zero, then the size ranking of the optimal barrier is opposite to the size ranking of the expected time of the next claim. We prove rigorously that this conjecture holds when the interclaim times are 2-dimensional distributed and we show that the optimal phase-wise two-barrier strategy is optimal among all the dividend policies.

Keywords: Hamilton-Jacobi-Bellman equation, Phase-type distribution, optimal dividend.

1 Introduction

The optimal dividend problem can be traced back to De Finetti [8]. Asmussen and Taksar [4] studied the optimal dividend when the surplus process is modeled by a Brownian motion with drift. The dividend optimization problem under the compound Poisson model is studied in Azcue and Muler [5], Belhaj [6] and Gerber and Shiu [9]. When the Poisson model is replaced by the renewal process, Albrecher et al. [1] calculated the distribution of the discounted dividends for a barrier

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strategy when the interclaim times follow the generalized Erlang($n$)-distribution. By numerical simulation, Albrecher and Hartinger [2] showed that the horizontal dividend barrier strategy is not necessarily optimal. Mishura and Schmidli [10] showed that the phase-wise dividend barrier strategy is optimal when the interclaim times are Erlang($n$)-distributed and the claim sizes are exponentially distributed. In this paper, we will extend their work and consider a renewal risk model where the interclaim times are phase-type distributed.

Mishura and Schmidli [10] shows that when the interclaim times follow the Erlang distribution, the optimal phase-wise barriers $\{b^*_1\}_{i=1}^n$ satisfies $b^*_1 \leq b^*_2 \leq \cdots \leq b^*_n$. They also show that the phase-wise barrier strategy with $\{b^*_i\}_{i=1}^n$ is optimal among all dividend strategies. Denote $T_i$ the expected time to the next claim of phase $i$. For the Erlang distribution, the size ranking of $\{T_i\}_{i=1}^n$ is $T_1 \geq T_2 \geq \cdots \geq T_n$. The Erlang distribution is a special phase-type distribution. For the phase-type distribution, is the size ranking of the barriers related to the size ranking of the expected times to the next claim? To analyze this question, we will firstly study the optimal phase-wise barrier strategy’s value function and get some necessary conditions for the optimal phase-wise barriers. Although we get some necessary conditions of the optimal barrier, we still feel difficult to compare the size of barriers of different phases theoretically. Thus, we do some numerical experiments to see the barriers directly.

We do four different experiments: 2 different two-dimensional phase-type distribution cases, 1 three-dimensional phase-type distribution case, and 1 four-dimensional phase-type distribution case. From the numerical experiments, we find that the ranking of the barriers’ size is opposite to that of the expected time of the next claim. Due to the mathematical difficulty, we only prove rigorously the conjecture holds when the interclaim times are two-dimensional phase-type distributed.

Now we explain the main steps of the proof. Denote the optimal barrier of the $i$th phase as $b^*_i$, the expected time to the next claim of the $i$th phase as $T_i$, where $i = 1, 2$. Firstly, we assume that $0 < b^*_1 < b^*_2$. Then we show that under this assumption, the optimal barrier strategy’s value function is concave on $[b^*_1, +\infty)$. The concave property can also be shown in the two-dimensional numerical experiments. Combining the concavity with the necessary conditions of the optimal barrier, we can show that the expected time for the next claim $T_1 > T_2$. This results shows that when the interclaim times follow the two-dimensional phase-type distribution, the necessary condition of $0 < b^*_1 < b^*_2$ is $T_1 > T_2$. This explains why the size ranking of the barriers is opposite to the size ranking of the expected time of the next claim. Eventually, using the concavity, we can verify that the optimal phase-wise barrier’s value function satisfies the Hamilton-Jacobi-Bellman
(HJB) equation, which shows that the phase-wise optimal barriers is optimal among all the dividend policies.

This paper is organized as follows. In section 2, we establish the basic model and formulate the problem. In section 3, we study the optimal phase-wise barrier and get some necessary conditions for the optimal barrier. In section 4, we present the algorithm first and then we show some examples to show the correlation between the size ranking of the expected time of the next claim and the size ranking of the optimal barriers. In section 5, we prove that when the interclaim times follow the two-dimensional phase-type distribution, the necessary condition of \( 0 < b_1^* < b_2^* \) is \( T_1 > T_2 \). In section 6, we show that under the the two dimensional phase-wise distributed interclaim times, the optimal phase-wise barrier strategy is optimal among all dividend strategies.

## 2 Models and Assumption

In this section, we present the surplus process of the insurer, which includes models for aggregate claims, dividend payments to policyholders and define the value function. We work on a complete probability space \((\Omega, \mathcal{F}, P)\) on which all processes are well defined. The information at time \( t \) is given by \( \mathcal{F}_t \), in which \( \{\mathcal{F}_t : t \geq 0\} \) is the complete filtration generated by the claim and the dividend processes.

The surplus process of an insurance company with dividend payments is modeled as

\[
X^D_t = x + ct - \sum_{i=1}^{N_t} Y_i - D_t,
\]

where \( c > 0 \) is the premium rate, \( x \) is the initial wealth, \( D_t \) is the cumulative amount of dividends paid out up to time \( t \). \( N \) is a simple point process representing the frequency of the incoming claims and \( \{Y_i\} \) are iid and independent of \( N_t \). The interclaim times are independent and follow the phase-type distribution.

The phase-type distribution is the distribution of the life time of a terminating Markov process \( \{J_t\}_{t \geq 0} \) with finitely many phases and time homogenous transition rates. More precisely, let \( \{J_t\}_{t \geq 0} \) be a Markov process on the finite state space \( E_\Delta = E \cup \{\Delta\} \), where \( E \) is the state space and \( \Delta \) is the absorbing state. The terminating Markov process \( \{J_t\} \) with state space \( E \) and intensity matrix \( T \) is defined as the restriction to \( E \) of \( \{J_t\} \). The Markov process \( \{J_t\}_{t \geq 0} \) jumps from one state to another. All the states \( i \in E \) are transient and once the Markov process \( \{J_t\} \) enters the absorbing state, then it will stay in this absorbing state forever. See [3] and [7] for more
details about phase-type distribution.

In our model, the state space $E := \{\text{state 1, state 2, \ldots, state } n\}$ and the $\Delta := \text{state } n+1$, where $n$ is a positive constant. We call this phase-type distribution a $n$-dimensional phase-type distribution. Thus, the intensity matrix of $J_i$ has the form

$$\Lambda = \begin{pmatrix} T & t \\ 0 & 0 \end{pmatrix},$$

where $T$ is $n \times n$ dimensional matrix, $t$ is a $n$ dimensional column vector and $0$ is the $n$ dimensional row vector of zeros. In particular, $t = -Te$, where $e$ is the column $E$–vector with all components equal to 1, which means, the intensity of leaving state $i$ equals to the sum of the intensities of leaving state $i$ and entering the new state $j$.

The Markov process $J_t$ jumps from one state to another and stay in the state $i$ for an exponential time with parameter $\lambda_i$, $0 \leq i \leq n, \lambda_i = -\lambda_{ii} > 0$. Once it enters the absorbing state $n+1$, the claim will occur. After the claim, the Markov chain $J_t$ will restart at the state $i \in E$ with the initial probabilities $\pi_i$, $i = 1, 2, \ldots, n$. Here $\sum_{i=1}^n \pi_i = 1$. Then it continues to jump from one state to another until the next absorption (next claim).

Now we assume that the Markov process $\{J_t\}$ can be observed. The claim sizes are independent of $\{J_t\}$ and iid random variables with distribution $G(x) = 1 - e^{-\beta x}$, where $\beta > 0$.

$D_t$ is the cumulative amount of dividends paid out up to time $t$. We say that a dividend strategy $D_t$ is admissible if

- $D_t$ is predictable, nondecreasing, càglàd;
- The process $D_t$ verifies $D_t \leq x + ct - \sum_{i=1}^{N_t} Y_i$.

We denote by $\mathcal{U}_{ad}$ the set of all the admissible control strategies. For any dividend strategy $D$, the expected discounted dividend payments is defined as

$$J_i^D(x) = \mathbb{E} \left[ \int_0^{\tau} e^{-\delta t} dD_t \middle| J_0 = i, X_{0-}^D = x \right], i = 1, 2, \ldots, n,$$

where $\delta > 0$ is the discount factor, $\tau = \inf\{t : X_t^D < 0\}$ is the time of ruin. The optimal return function is defined as

$$V_i(x) = \sup_{D \in \mathcal{U}_{ad}} J_i^D(x),$$

for all $x \geq 0$. In the next paragraph, we will first study the optimal barrier strategy and then analyze the correlation between the ranking of the size of barriers and the ranking of the size of
the expected time of the next claim of \(n\) different phases.

## 3 The optimal phase-wise barrier

We focus on the phase-wise barrier strategies first, in other words, we will choose a barrier \(b_i \geq 0\) for a given phase \(i, i = 1, 2, \cdots, n\). If the Markov process \(\{J_t\}\) is in state \(J_t = i\), all the capital above \(b_i\) is paid as a dividend. If the wealth equals to \(b_i\) when \(J_t = i\), then all the incoming premium will be paid as dividends until the next jump of \(J_t\) occurs. Let

\[
\mathcal{J}_t = \{J_0, J_t, X_t, \mathcal{F}_t \}_{t \geq 0},
\]

denote the phase-wise barrier strategy \(D\)'s cost function. For simplicity, we let

\[
f_n(x) = \sum_{i=1}^{n} \pi_i \mathbb{E}[f_i(x - Y)],
\]

where \(Y\) is a random variable with distribution \(G(y)\). Standard considerations show that the functions \(f_i(x)\) are continuously differentiable on \([0, b_i]\) and fulfill

\[
c f'_i(x) + \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j(x) - (\lambda_i + \delta) f_i(x) = 0, \quad x \leq b_i, \tag{3.1}
\]

and

\[
f_i(x) = f_i(b_i) + x - b_i, \quad x \geq b_i.
\]

Conditioning on the first jump of \(\{J_t\}\),

\[
f_i(b_i) = \int_0^\infty e^{-\gamma s} \mathbb{E}[e^{-\gamma J_s} | J_0 = i, X_i = x] ds + \int_0^\infty \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j(b_i) e^{-\gamma (\lambda_i + \delta) t} dt = \frac{c + \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j(b_i)}{\lambda_i + \delta}. \tag{3.2}
\]

Combing with (3.1), we conclude that \(f'_i(b_i) = 1\). Using a martingale approach, a solution \(f_i(x)\) to equations (3.1) on \([0, b_i]\) and \(f_i(x) = f_i(b_i) + x - b_i\) for \(x > b_i\) with \(f'_i(b_i) = 1\) is the value function of the barrier strategy with barriers at \(b_i\). Now we try to find the optimal phase-wise barriers \(\{b_i\}_{i=1}^{n}\) such that \(\{f_i(x)\}_{i=1}^{n}\), become maximal for all \(x < b_i\). Denote the optimal phase-wise barrier as \(\{b_i^*\}_{i=1}^{n}\) and the corresponding value function as \(\{f_i^*\}_{i=1}^{n}\). Given \(1 \leq i \leq n\) and \(x < b_i^*\),

\[
f_i^*(x) = \int_0^{b_i^* - x} e^{-(\lambda_i + \delta) t} \left( \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j^*(x + ct) \right) dt + e^{-(\lambda_i + \delta) (b_i^* - x)} f_i^*(b_i^*). \tag{3.3}
\]
We will calculate some necessary conditions for the optimal barrier \( \{b^*_i\}_{i=1}^n \). For a given initial wealth \( x < b^*_i \), consider a special strategy: paying the incoming premium as dividend until next phase jump. After the jump, follow the “optimal” barrier strategy. Then, by the definition of the optimal barrier strategy, we see that

\[
 f^*_i(x) > \int_0^\infty \left( \int_0^t ce^{-\delta s} ds \right) \lambda_i e^{-\lambda_i t} dt + \int_0^\infty \left( \sum_{j=1}^{n+1} \lambda_{ij} f^*_j(x) \right) e^{-(\delta+\lambda_i) t} dt
 = \frac{c + \sum_{j=1,j\neq i}^{n+1} \lambda_{ij} f^*_j(x)}{\lambda_i + \delta}.
\] (3.4)

Combing with (3.1), we see that \( f^*_i(x) > 1 \) on \((0, b^*_i)\). We can also use a similar method which is used in Mishura and Schmidli [10] to show \( f^{*\prime\prime}_i(b^*_i) = 0 \). Different with Mishura and Schmidli [10], we cannot solve the barrier strategy’s value function directly due to the mathematical difficulty. Thus we use the numerical method to explore the relation between the size ranking of the expected time of the next claim and the size ranking of the barriers.

Recall that for the given phase \( i \), the expected time of the next claim \( T_i \) is, confer Asmussen and Albrecher [3] (Page 256, Theorem 1.5-(d)),

\[
 T_i = \alpha_i T^{-1} e,
\] (3.5)

where \( \alpha_i = (0, 0, \cdots, 0, -1, 0, \cdots, 0) \) is the \( n \)-dimensional vector with \(-1\) being the \( i \)th term, \( T \) is the subintensity matrix of \( \Lambda \) restricted to \( E \) and \( e \) is the column vector with all components equal to one. In the next section, we will do some numerical experiments to see the size ranking of the optimal phase-wise barrier.

4 Numerical Method

4.1 Algorithm

In this section, we present the numerical algorithm and show the results of different examples. We use iteration algorithm.

(1) Set \( k = 0 \) and the initial function \( f_i^{(0)} \) as

\[
 f_i^{(0)}(x) = x + \frac{c}{\lambda_i + \delta}, \quad i = 1, 2, \cdots, n.
\]
\( f_{n+1}^{(0)}(x) = \beta e^{-\beta x} \int_0^x e^{\beta y} \left( \sum_{i=1}^n \pi_i f_i^{(0)}(y) \right) dy. \)

(2) For any given \( \{f_i^{(k)}\}_{i=1}^{n+1} \), we choose \( b_i^{(k+1)} \) such that

\[
\arg \max_{b_i \geq 0} \left( c + \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j^{(k)}(b_i) \right), \quad i = 1, 2, \ldots, n.
\]  

(4.1)

The above equation comes from (3.2).

(3) After determining the new barrier \( b_i^{(k+1)} \), we define

\[
f_i^{(k+1)}(b_i^{(k+1)}) = c + \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j^{(k)}(b_i^{(k+1)}). \quad i = 1, 2, \ldots, n.
\]

For all \( x > b_i^{(k+1)} \),

\[
f_i^{(k+1)}(x) = f_i^{(k+1)}(b_i^{(k+1)}) + x - b_i^{(k+1)}, \quad i = 1, 2, \ldots, n.
\]

For all \( x < b_i^{(k+1)} \),

\[
f_i^{(k+1)}(x) = \int_0^{b_i^{(k+1)} - x} e^{-(\lambda_i + \delta) t} \left( \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j^{(k)}(x + ct) \right) dt + e^{-(\lambda_i + \delta) (\lambda_i^{(k+1)} - x) / c} f_i^{(k+1)}(b_i^{(k+1)}) \]

\[
= \int_x^{b_i^{(k+1)}} \frac{1}{c} e^{-\left(\lambda_{i} + \delta\right) (t - x)} \left( \sum_{j=1, j \neq i}^{n+1} \lambda_{ij} f_j^{(k)}(t) \right) dt + e^{-\left(\lambda_{i} + \delta\right) \left(\lambda_{i}^{(k+1)} - x\right) / c} f_i^{(k+1)}(b_i^{(k+1)}),
\]

\[
i = 1, 2, \ldots, n.
\]  

(4.2)

The above equation comes from (3.3). For the absorption phase \( n+1 \), we define

\[
f_{n+1}^{(k+1)}(x) = \beta e^{-\beta x} \int_0^x e^{\beta y} \left( \sum_{i=1}^n \pi_i f_i^{(k+1)}(y) \right) dy.
\]

(4) If the \( \max_{i=1,2,\ldots,n} |b_i^{(k+1)} - b_i^{(k)}| > \) tolerance, then \( k \to k + 1 \) and go to step (2), else the iteration stops.
4.2 Examples

Example 4.1. Now we try a two-dimensional phase-type distribution case first. The state 3 is the absorption state. The subintensity matrix $T$ is

$$
\begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{pmatrix} =
\begin{pmatrix}
-10 & 5 \\
4 & -12
\end{pmatrix},
$$

the premium rate $c = 15$, the discount factor $\delta = 0.1$, $\lambda_1 = 10, \lambda_2 = 12$, $\pi_1 = 0.4, \pi_2 = 0.6, \beta = 1$. Now we make a form to compare the expected time of the next claim and the barriers.

| State $i$ | 1     | 2     |
|-----------|-------|-------|
| The expected time of the next claim $T_i$ | 0.17  | 0.14  |
| The barrier $b_i^*$                        | 11.779| 12.219|

From the above form, we see that $T_1 > T_2$ and $b_1^* > b_2^*$. The $p f_1^*(x) - x$ and $f_2^*(x) - x$ is shown in Figure 4.1. In this figure, we can see that $f_1^*(x) > 1$ for all $x < b_1^*$ and $f_2^*(x) > 1$ for all $x < b_2^*$.

![Figure 4.1: The functions $f_1^*(x) - x$ and $f_2^*(x) - x$.](image)

Example 4.2. We do another experiment to see what happens if the expected times of the next claim are the same. Let $c = 15, \delta = 0.1, \lambda_1 = 8, \lambda_2 = 6, \pi_1 = 0.4, \pi_2 = 0.6, \beta = 1$. The subintensity matrix $T$ is

$$
\begin{pmatrix}
-\lambda_1 & \lambda_{12} \\
\lambda_{21} & -\lambda_2
\end{pmatrix} =
\begin{pmatrix}
-8 & 3 \\
1 & -6
\end{pmatrix},
$$
The expected time of the next claim $T_i$ and the barrier $b_i^*$ for states 1 and 2 are provided in the table:

| State $i$ | 1   | 2   |
|-----------|-----|-----|
| $T_i$     | 0.2 | 0.2 |
| $b_i^*$   | 10.738 | 10.738 |

From this example, we see that the expected times of the next claim are the same and the barriers are the same. The picture of $f_1^*(x) - x$ and $f_2^* - x$ is shown in Figure 4.2. We see that in this case, $f_1^*(x) = f_2^*(x)$.

![Figure 4.2: Two functions $f_1^*(x) - x$ and $f_2^*(x) - x$.](image)

**Example 4.3.** Now we see an example of a three-dimensional case. The subintensity matrix $\mathbf{T}$ is

$$
\begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{pmatrix} = 
\begin{pmatrix}
-10 & 5 & 2 \\
2 & -12 & 4 \\
2 & 4 & -8
\end{pmatrix},
$$

with $c = 21.4, \delta = 0.1, \lambda_1 = 10, \lambda_2 = 12, \lambda_3 = 8, \pi_1 = 0.2, \pi_2 = 0.3, \pi_3 = 0.5, \beta = 1$. After calculation, the barriers and the expected times of the next claim of different phases are shown in the following form and the picture of $f_1^*(x), f_2^*(x), f_3^*(x)$ is shown in Figure 4.3. As we can see, the ranking of barriers is $b_3^* < b_1^* < b_2^*$, the ranking of the expected time to the next claim is $T_3 > T_1 > T_2$.

| State $i$ | 1   | 2   | 3   |
|-----------|-----|-----|-----|
| $T_i$     | 0.27922 | 0.23376 | 0.31168 |
| $b_i^*$   | 9.61 | 10.26 | 9.27 |
If we change the initial probabilities \( \pi_1 = 0.2, \pi_2 = 0.3, \pi_3 = 0.5 \) to \( \pi_1 = 0.1, \pi_2 = 0.1, \pi_3 = 0.8 \), the new barriers are shown in the following form.

| State \( i \) | 1   | 2   | 3   |
|----------------|-----|-----|-----|
| The expected time of the next claim \( T_i \) | 0.27922 | 0.23376 | 0.31168 |
| The barrier \( b_i^* \) | 9.39 | 10.03 | 9.05 |

As we can see, the ranking of the new barriers are still \( b_3^* < b_1^* < b_2^* \).

**Example 4.4.** Now we show a four-dimensional phase-type case. Let \( c = 15, \pi_1 = 0.5, \pi_2 = 0.2, \pi_3 = 0.2, \pi_4 = 0.1, \delta = 0.1, \beta = 1 \), the subintensity matrix \( T \) is

\[
\begin{pmatrix}
\lambda_1 & -\lambda_2 & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\
\lambda_{31} & \lambda_{32} & -\lambda_3 & \lambda_{34} \\
\lambda_{41} & \lambda_{42} & \lambda_{43} & -\lambda_4
\end{pmatrix}
= 
\begin{pmatrix}
-10 & 5 & 2 & 1 \\
3 & -14 & 4 & 3 \\
2 & 2 & -12 & 7 \\
2 & 3 & 1 & -8
\end{pmatrix}.
\]

The optimal barriers and the expected times of the next claim of different phases are shown in the following form.

| State \( i \) | 1   | 2   | 3   | 4   |
|----------------|-----|-----|-----|-----|
| The expected time of the next claim \( T_i \) | 0.440876 | 0.399397 | 0.483191 | 0.445392 |
| The barrier \( b_i^* \) | 8.61 | 9.25 | 8.08 | 8.59 |
In this example, we can see that $b_3^* < b_4^* < b_2^* < b_1^*$ and $T_3 > T_1 > T_1 > T_2$.

**Remark 4.1.** From all the above examples: Example 4.1-4.4, we propose the following conjecture:

**Conjecture 1.** If all the barriers are not zero, then the size ranking of the barriers is opposite to that of the expected time to the next claim.

Due to mathematical complexity, we only prove that this conjecture holds when the interclaim times follow the two-dimensional phase-type distribution.

### 5 The special case of two phases

We assume that Markov process $J_t$ has two states, state 1 and state 2. The state 3 is the absorption state. Now we will introduce a lemma which will be use in the later proof.

**Lemma 5.1.** For a two-dimensional phase type distribution, $\lambda_{13} \geq \lambda_{23}$ is equivalent to $T_1 \leq T_2$, where $\lambda_{ij}$ is the element of the intensity matrix $\Lambda$ in row $i$, column $j$, $1 \leq i, j \leq 3$.

**Proof.**

\[ T_1 - T_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot T^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} \begin{pmatrix} -1 & 1 \\ -\lambda_2 - \lambda_{12} & -\lambda_{21} - \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} (\lambda_{23} - \lambda_{13}). \]

Noticing that $\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}$ is always positive, thus, we show that $\lambda_{13} \geq \lambda_{23}$ is equivalent to $T_1 \leq T_2$.

**Theorem 5.2.** If $0 < b_1^* < b_2^*$, then the optimal phase-wise barrier strategy’s value function $f_2^*(x)$ is concave on $[b_1^*, +\infty)$. 

**Proof.** Recall that $f_i^*(x)$, $i = 1, 2$, satisfy the equations

\[
\begin{align*}
\left[ c f_1^*(x) + \lambda_{12} f_2^*(x) - (\lambda_1 + \delta) f_1^*(x) + \lambda_{13} \beta e^{-\beta x} \int_0^x e^{\beta y} (\pi_1 f_1^*(y) + \pi_2 f_2^*(y)) dy \right] & = 0, x \in [0, b_1^*]. \\
\left[ c f_2^*(x) + \lambda_{21} f_1^*(x) - (\lambda_2 + \delta) f_2^*(x) + \lambda_{23} \beta e^{-\beta x} \int_0^x e^{\beta y} (\pi_1 f_1^*(y) + \pi_2 f_2^*(y)) dy \right] & = 0, x \in [0, b_2^*].
\end{align*}
\]  

(5.1)  

(5.2)  

\[ f_1^*(x) = f_1^*(b_1^*) + x - b_1^*, \quad x \in (b_1^*, +\infty). \]
Taking the derivative in (5.2), we see

\[ cf_2''(x) + \lambda_2 f_1''(x) + (\beta c - (\lambda_2 + \delta))f_2''(x) + \beta(\lambda_2 \pi_1 + \lambda_21) f_1'(x) \]
\[ + \beta(\lambda_2 \pi_2 - (\lambda_2 + \delta)) f_2'(x) = 0, \quad x \in [0, b^*_2). \] (5.3)

Using \( f_1'(x) = 1, \)

\[ cf_2'''(x) + (c\beta - (\lambda_2 + \delta))f_2''(x) + \beta(\lambda_2 \pi_2 - (\lambda_2 + \delta))f_2'(x) + \beta(\lambda_2 \pi_1 + \lambda_21) = 0, x \in [b^*_1, b^*_2). \] (5.4)

Suppose there exists a point \( x \in [b^*_1, b^*_2) \) such that \( f_2'''(x) > 0. \) If \( c\beta - (\lambda_2 + \delta) \leq 0, \) then

\[ cf_2'''(x) \geq -\beta(\lambda_2 \pi_1 + \lambda_21) + \beta(\lambda_2 + \delta - \lambda_2 \pi_2) f_2'(x). \]

Since \( f_2'(x) > 1 \) on \([b^*_1, b^*_2), \) we see that \( cf_2'''(x) > \beta \delta. \) Thus, \( f_2'''(x) > 0 \) on \((x, b^*_2], \) contradicting that \( f_2'''(b^*_2) = 0. \) If \( c\beta - (\lambda_2 + \delta) > 0, \) taking the derivative in (5.4),

\[ cf_2''''(x) + (c\beta - (\lambda_2 + \delta))f_2'''(x) + \beta(\lambda_2 \pi_2 - (\lambda_2 + \delta))f_2''(x) = 0. \]

Because \( f_2'''(b^*_2) = 0, \) the solution of \( f_2''''(x) \) is of the form

\[ f_2''''(x) = A(e^{r_1(x-b^*_2)} - e^{r_2(x-b^*_2)}), \]

where \( r_2 < 0 < r_1 \) are the two roots of \( cr^2 + (c\beta - (\lambda_2 + \delta))r + \beta(\lambda_2 \pi_2 - (\lambda_2 + \delta)) = 0. \) From the assumption that \( f_2''''(x) > 0 \) for some \( x \in [b^*_1, b^*_2), \) we conclude that \( A < 0. \) Thus, \( f_2''''(x) > 0 \) for all \( x \in [b^*_1, b^*_2). \) Then we have \( f_2''(x) < 1 \) for \( x \in [b^*_1, b^*_2), \) yielding a contradiction. Until now, we show that for all \( x \in [b^*_1, b^*_2), \) \( f_2''''(x) \leq 0. \) Because \( f_2''(x) = f_2''(b^*_2) + x - b^*_2 \) for \( x > b^*_2, \) we find that \( f_2''(x) \) is concave on \([b^*_1, +\infty). \)

**Theorem 5.3.** If \( 0 < b^*_1 < b^*_2, \) then we can deduce that the expected time to the next claim satisfies \( T_1 > T_2. \)

\[ f_2''(x) = f_2''(b^*_2) + x - b^*_2, \quad x \in (b^*_2, +\infty). \]
Proof. Denote $f_2^*(x) = \beta e^{-\beta x} \int_0^x e^{\beta y}(\sum_{i=1}^2 \pi_i f_i^*(y))dy$. From (5.2), we see that
\[cf_2''(x) + \lambda_{21} - (\lambda_2 + \delta)f_2'(x) + \lambda_{23}f_3'(x) = 0, x \in [b_1^*, b_2^*].\]
Combining $f_2^*(x)$ is concave on $[b_1^*, b_2^*]$, we see
\[\lambda_{21} - (\lambda_2 + \delta)f_2'(b_1^*) + \lambda_{23}f_3'(b_1^*) \geq 0.\]  
(5.5)
On the other hand, from (5.1), we see
\[\lambda_{12}f_2'(b_1^*) + \lambda_{13}f_3'(b_1^*) - (\lambda_1 + \delta) = 0.\]  
(5.6)
Combing (5.5) with (5.6), we see that
\[f_2'(b_1^*) \leq \frac{\lambda_{13}\lambda_{21} + (\lambda_1 + \delta)\lambda_{23}}{(\lambda_2 + \delta)\lambda_{13} + \lambda_{23}\lambda_{12}}.\]  
(5.7)
Since $f_2'(b_1^*) > 1$, we know $\frac{\lambda_{13}\lambda_{21} + (\lambda_1 + \delta)\lambda_{23}}{(\lambda_2 + \delta)\lambda_{13} + \lambda_{23}\lambda_{12}} > 1$. After simplification, we see that $\lambda_{23} > \lambda_{13}$. By Lemma 5.1, we see that $T_1 > T_2$. \qed

Remark 5.4. In practical use, this conclusion means that if the company enters a new phase which the time to the next claim is shorter, then the manager will raise the dividend barrier to reduce the risk, otherwise, the company will lower the dividend barrier.

6 Optimality

In this chapter, we will show that when the interclaim times follow the two-dimensional phase-type distribution, the optimal phase-wise barrier dividend strategy is optimal among all strategies.

Theorem 6.1. If the interclaim times follow the two-dimensional phase-type distribution, then the optimal phase-wise barrier strategy’s value function $f_i^*(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation
\[
\max \left\{ cf_i''(x) + \sum_{j=1, j \neq i}^3 \lambda_{ij}f_j'(x) - (\lambda_i + \delta)f_i^*(x), 1 - f_i'(x) \right\} = 0, i = 1, 2, \]  
(6.1)
where $f_3^*(x) = \beta e^{-\beta x} \int_0^x e^{\beta y}(\pi_1 f_1^*(y) + \pi_2 f_2^*(y))dy$. 

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Proof. Since for all \( x > b_i^* \), \( f_i^*(x) = f_i^*(b_i^*) + x - b_i^* \), \( i = 1, 2 \) and for all \( 0 \leq x < b_i^* \), \( f_i^*(x) \) satisfies (5.1) and (5.2), we only need to show that \( cf_i^*(x) + \sum_{j=1, j \neq i}^3 \lambda_{ij} f_j^*(x) - (\lambda_i + \delta) f_i^*(x) \leq 0 \) on \([b_i^*, +\infty)\). We will prove \( cf_i^*(x) + \sum_{j=2}^3 \lambda_{ij} f_j^*(x) - (\lambda_1 + \delta) f_i^*(x) \leq 0 \) on \([b_1^*, +\infty)\) first. Multiplying by \( e^{\beta x} \) gives

\[
 cf_i^*(x)e^{\beta x} + \lambda_{12} e^{\beta x} f_2^*(x) + \lambda_{13} \beta \int_0^x e^{\beta y} (\pi_1 f_1^*(y) + \pi_2 f_2^*(y)) dy - (\delta + \lambda_1) e^{\beta x} f_i^*(x).
\]

Taking the derivative yields

\[
e^{\beta x} \left[ \lambda_{12} f_2''(x) + (\lambda_{12} + \lambda_{13} \pi_2) f_2'(x) + (\lambda_{13} \pi_1 - (\delta + \lambda_1)) f_i^*(x) + c\beta - (\delta + \lambda_1) \right] := e^{\beta x} g_1(x).
\]

Noticing that \( g_1(b_1^*) = 0 \), we only need to show that

\[
g_1'(x) = \lambda_{12} f_2''(x) + \beta(\lambda_{12} + \lambda_{13} \pi_2) f_2'(x) + \beta(\lambda_{13} \pi_1 - (\delta + \lambda_1)) \leq 0, x \in [b_1^*, +\infty).
\]

Since \( f_2^*(x) \) is concave on \([b_1^*, +\infty)\), we only need to show that

\[
(\lambda_{12} + \lambda_{13} \pi_2) f_2''(b_1) \leq \delta + \lambda_1 - \lambda_{13} \pi_1.
\]

Recall that

\[
f_i^*(x) + \lambda_{12} f_2^*(x) + \lambda_{13} f_3^*(x) - (\lambda_1 + \delta) f_i^*(x) = 0, x \in (0, b_1^]. \quad (6.3)
\]

Taking the derivative of (6.3), we see that

\[
\lambda_{12} f_2''(b_1) + \lambda_{13} f_3''(b_1) - (\lambda_1 + \delta) = 0. \quad (6.4)
\]

Combing (6.2) with (6.3), we only need to show that

\[
f_3''(b_1) \geq \pi_1 + \pi_2 f_2''(b_1). \quad (6.5)
\]

On the other hand,

\[
f_2''(x) + \lambda_{21} f_1^*(x) + \lambda_{23} f_3^*(x) - (\lambda_2 + \delta) f_2^*(x) = 0, x \in (0, b_2]. \quad (6.6)
\]
Taking the derivative of (6.6),

\[ cf_2''(x) + \lambda_2 f_1'(x) + \lambda_3 f_3'(x) - (\lambda_2 + \delta) f_2'(x) = 0, \ x \in (0, b_2]. \]

Since \( f_2' \) is concave on \([b_1, b_2]\) and \( f_1''(b_1) = 1 \),

\[ \lambda_2 f_1''(b_1) + \lambda_3 f_3''(b_1) - (\lambda_2 + \delta) f_2''(b_1) \geq 0, \]

After simplification,

\[ f_2''(b_1) \geq \frac{(\lambda_2 + \delta)f_2''(b_1)}{\lambda_2} \] \hspace{1cm} (6.7)

Now we compare the right-hand side of both (6.5) and (6.7). Using the fact that \( f_2''(b_1) \geq 1 \), we can easily calculate that

\[ f_2''(b_1) \geq \frac{(\lambda_2 + \delta)f_2''(b_1)}{\lambda_2} \geq \pi_1 + \pi_2 f_2''(b_1). \] \hspace{1cm} (6.8)

Until now, we show that (6.5) holds, eventually \( cf_1''(x) + \sum_{j=2}^{3} \lambda_j f_j''(x) - (\lambda_1 + \delta) f_1''(x) \leq 0 \) on \([b_1, +\infty)\).

Now we show that \( cf_2''(x) + \lambda_2 f_1''(x) + \lambda_3 f_3''(x) - (\lambda_2 + \delta) f_2''(x) \leq 0 \) on \([b_2, +\infty)\). Multiplying by \( e^{\beta x} \) gives

\[ ce^{\beta x} f_2''(x) + \lambda_2 e^{\beta x} f_1''(x) + \lambda_3 e^{\beta x} \int_0^x e^{\beta y}(\pi_1 f_1''(y) + \pi_2 f_2''(y))dy - (\lambda_2 + \delta) e^{\beta x} f_2''(x). \]

Taking the derivative yields

\[ e^{\beta x}(c\beta + \lambda_2 \beta f_1''(x) + \lambda_2 + \lambda_3 \beta (\pi_1 f_1''(x) + \pi_2 f_2''(x)) - (\lambda_2 + \delta) \beta f_2''(x) - (\lambda_2 + \delta)) := e^{\beta x} g_2(x), \]

here we use that \( f_2''''(x) = 0, f_1''''(x) = 1 \) on \([b_2, +\infty)\). Since \( g_2(b_2) = 0 \) and \( g_2' = \lambda_2 \beta + \lambda_3 \beta - (\lambda_2 + \delta) \beta < 0 \), we see that \( e^{\beta x} g_2(x) \leq 0 \) on \([b_2, +\infty)\). Combining this with \( cf_2''(b_1) + \lambda_2 f_1''(b_1) + \lambda_3 f_3''(b_1) - (\lambda_2 + \delta) f_2''(b_1) = 0 \), we see that

\[ cf_2''(x) + \lambda_2 f_1''(x) + \lambda_3 f_3''(x) - (\lambda_2 + \delta) f_2''(x) \leq 0, \ x \in [b_2, +\infty). \]

Until now, we show that the two-barrier strategy’s value function satisfies the HJB equation. \( \square \)

**Theorem 6.2.** For exponential distributed claim sizes and two-dimensional phase-type distributed
interclaim times, the optimal barrier strategy is optimal among all dividend strategies.

Proof. Clearly $f^*_i(x) \leq V_i(x)$, where $V_i(x)$ is defined in (2.1). Let $L_t$ be an arbitrary dividend process and $X^L_t$ denote the corresponding surplus process. Then by Itô formula,

$$f^*_i(x) + \int_0^{\tau \wedge t} e^{-\delta s} [-\delta f^*_i(X^L_s) + \mathcal{L}f^*_i(X^L_s)] ds - \int_0^{\tau \wedge t} f^*_i(X^L_s)e^{-\delta s} dL_s,$$

where and $\tau$ is the ruin time and

$$\mathcal{L}f^*_i(x) = cf^*_i(x) + \sum_{j \neq i, j=1}^3 \lambda_{ij} f^*_j(x) - \lambda_i f^*_i(x)$$

is the infinitesimal generator of the process. From (6.1) we conclude that

$$\mathbb{E} \left[ f^*_i(X^L_{\tau \wedge t})e^{-\delta (\tau \wedge t)} \right] \leq f^*_i(x) - \mathbb{E} \left[ \int_0^{\tau \wedge t} f^*_i(X^L_s)e^{-\delta s} dL_s | J_0 = i, X^L_{0-} = x \right]$$

$$\leq f^*_i(x) - \mathbb{E} \left[ \int_0^{\tau \wedge t} e^{-\delta s} dL_s | J_0 = i, X^L_{0-} = x \right]. \quad (6.9)$$

Letting $t \to \infty$ gives

$$f^*_i(x) \geq \mathbb{E} \left[ \int_0^{\tau \wedge t} e^{-\delta s} dL_s | J_0 = i, X^L_{0-} = x \right].$$

Because the strategy is arbitrary, we have $f^*_i(x) \geq V_i(x)$. Now we complete the proof. \qed

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8 Conclusion

In this paper, we study the dividend problem when the interclaim times follow the $n$-dimensional phase-type distribution. We start with the optimal phase-wise barrier strategy and do some numerical experiments to see the barriers. From the numerical experiments, we propose a conjecture that if all the barriers are not zero, then the size ranking of the optimal barriers is opposite to the size ranking of the expected time to the next claim. When the interclaim times follow the two dimensional phase-type distribution, we theoretically prove this conjecture is true and we show that the optimal phase-wise ybarrier strategy is optimal among all dividend strategies. We do not
do the theoretical analysis for the $n$-dimensional case in this paper. Because it is more complex to use the inverse of $n$-dimensional matrix $T$. But it is highly possible that the conjecture also holds when the interclaim time follows the $n$-dimensional phase-type distribution. We will discuss the $n$-dimensional case in the future work.

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