Lie Groupoids in Classical Field Theory II: 
Gauge Theories, Minimal Coupling and Utiyama’s Theorem

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Abstract

In the two papers of this series, we initiate the development of a new approach to implementing the concept of symmetry in classical field theory, based on replacing Lie groups/algebras by Lie groupoids/algebroids, which are the appropriate mathematical tools to describe local symmetries when gauge transformations are combined with space-time transformations. In this second part, we shall adapt the formalism developed in the first paper to the context of gauge theories and deal with minimal coupling and Utiyama’s theorem.
1 Introduction

In the first paper of this series [6], we have initiated an investigation of how to handle symmetries – or more precisely, local symmetries – in classical field theories using the language of Lie groupoids and their actions. However, the formalism developed there is perhaps a bit too general because it allows us to leave the nature of the underlying Lie groupoids and their actions completely unspecified, whereas there can be no doubt that the motivation for the entire program comes predominantly from one single (class of) example(s), namely, gauge theories. Spelling out the details for this case is the main goal of the present paper and is necessary not only because it provides us with a class of examples whose importance can hardly be overestimated but also because it leads to a substantial clarification of the general structure of the theory. Moreover, the results will generalize those of earlier work [10] by extending them from internal symmetries to space-time symmetries.

Let us begin with a few comments on the already traditional geometric formulation of gauge theories (as classical field theories) over a general space-time manifold \( M \); more details can be found in textbooks such as [3,8,12]. The basic input data one has to fix right at the start are an internal symmetry group, which is a Lie group \( G_0 \) with Lie algebra \( g_0 \) together with a principal bundle \( P \) over \( M \) with structure group \( G_0 \) and bundle projection \( \rho : P \to M \): then gauge fields are described in terms of connections in \( P \), which can be viewed as sections of an affine bundle over \( M \), namely, the connection bundle \( CP = JP/G_0 \) of \( P \). Moreover, if the theory is to contain not only gauge fields (as in “pure” Yang-Mills theories) but also matter fields, one also has to fix a vector space \( V \) equipped with a representation of \( G_0 \) or, more generally, a manifold \( Q \) equipped with an action of \( G_0 \): then matter fields are described by sections of the associated vector bundle \( E = P \times_{G_0} V \) (for scalar matter fields) or of its tensor product with some tensor or spinor bundle over \( M \) (for tensor or spinor matter fields) or of the associated fiber bundle \( E = P \times_{G_0} Q \) (for nonlinear scalar matter fields such as in the nonlinear sigma models). Finally, there is gravity, described by yet another and very special kind of field, namely, a metric tensor \( g \) on \( M \). (Some discussion of what sets the metric tensor apart from all other fields can be found in Ref. [15].)

Symmetries in this approach are traditionally described in terms of automorphisms of the principal bundle \( P \) and the induced automorphisms of its connection bundle and its associated bundles. To set the stage, recall that an automorphism of \( P \) is a diffeomorphism of \( P \) as a manifold which is \( G_0 \)-equivariant, i.e., which commutes with the right action of the structure group \( G_0 \) on \( P \): since the orbits of this action are precisely the fibers of \( P \), it then follows that it takes points in the same fiber to points in the same fiber and hence induces a diffeomorphism of the base manifold \( M \). Moreover, the automorphism is said to be strict if it preserves the fibers, or equivalently, if the induced diffeomorphism on the base manifold is the identity. Automorphisms of \( P \) form a group \( \text{Aut}(P) \) and strict automorphisms of \( P \) form a normal subgroup \( \text{Aut}_s(P) \) which is the kernel of a natural group homomorphism

\[
\text{Aut}(P) \longrightarrow \text{Diff}(M)
\]

that projects each automorphism of \( P \) to the diffeomorphism of \( M \) it induces. In physics

\footnote{Note that we perform a slight change of notation as compared to Ref. [10], where we have denoted the internal symmetry group by \( G \) and its Lie algebra by \( g \): here, we want to reserve these symbols for the basic Lie groupoid and Lie algebroid of the theory.}
language, strict automorphisms are also called *gauge transformations* and the group \( \text{Aut}_s(P) \) is often called the *gauge group* and denoted by \( \text{Gau}(P) \), but we prefer the more precise term *group of gauge transformations* so as to avoid the confusion whether by “gauge group” one means the infinite-dimensional group \( \text{Gau}(P) \) or the finite-dimensional structure group \( G_0 \). Thus strict automorphisms, or gauge transformations, are internal symmetries since they do not move points in space-time, whereas general automorphisms will in what follows be referred to as *space-time symmetries*\(^2\). At any rate, all such symmetry transformations, being represented by automorphisms of \( P \), can be lifted to automorphisms of its jet bundle \( JP \) and hence act naturally on the connection bundle \( CP = JP/G_0 \) of \( P \) as well as on any associated vector bundle or fiber bundle \( E \), its jet bundle \( JE \) and any tensor or spinor bundle over \( M \), thus providing the appropriate setting for deciding which of them are symmetries of the field theoretical model under consideration.

The main mathematical difficulty within this approach comes from the fact that one is dealing here with infinite-dimensional groups which are notoriously hard to handle from the point of view of Lie theory. Therefore, it is desirable to recast the property of invariance of a field theory under such local symmetries into a form where one deals exclusively with finite-dimensional objects. This program has been initiated in Ref. [10] and implemented there for strict automorphisms (gauge transformations), where it leads naturally to replacing Lie groups by Lie group bundles (and similarly Lie algebras by Lie algebra bundles), making use of the well-known fact that there is a natural isomorphism between the group of strict automorphisms of \( P \) and the group of sections of the gauge group bundle of \( P \), which is the Lie group bundle \( P \times_{G_0} G_0 \) associated to \( P \) via the action of \( G_0 \) on itself by conjugation:

\[
\text{Aut}_s(P) \cong \Gamma(P \times_{G_0} G_0).
\]

In order to extend the resulting analysis from strict automorphisms to general automorphisms, we have to go one step further and replace Lie groups or Lie group bundles by Lie groupoids (and similarly Lie algebras or Lie algebra bundles by Lie algebroids). In this case, the basic observation is that there is a natural isomorphism between the group of automorphisms of \( P \) and the group of bisections of the gauge groupoid of \( P \), which is the Lie groupoid \( (P \times P)/G_0 \) obtained as the quotient of the cartesian product of two copies of \( P \) by the “diagonal” right action of \( G_0 \):

\[
\text{Aut}(P) \cong \text{Bis}((P \times P)/G_0).
\]

Thus our task in what follows will be to extend the results of Ref. [10] by applying the general formalism of Ref. [6] to this specific situation.

When we replace Lie groups by Lie groupoids, or to put it a bit more precisely, actions of Lie groups on manifolds by actions of Lie groupoids on fiber bundles (over the same base manifold), we have to face one important novel feature, namely, that the construction of induced actions will involve changing the Lie groupoid as well. For example, while an action of a Lie group \( G_0 \)

\[^2\]There is some abuse of language in this simplified terminology because general automorphisms always represent a mixture of “pure” space-time symmetries with internal symmetries. The problem here is that there is in general no natural notion of a “pure” space-time symmetry, since that would require a *lifting* of the group \( \text{Diff}(M) \) (or at least of an appropriate subgroup thereof) to realize it as a subgroup (and not only as a quotient group) of \( \text{Aut}(P) \), whose elements would then represent the “pure” space-time symmetries. However, such a lifting may not even exist, and even if it does (which happens, e.g., when the principal bundle \( P \) is trivial), it is far from unique, so what one means by a “pure” space-time transformation still depends on which lifting is chosen.
on a manifold \( X \) induces an action of the same Lie group \( G_0 \) on its tangent bundle \( TX \), an action of a Lie groupoid \( G \) on a fiber bundle \( E \) (both over the same base manifold \( M \)) induces an action not of the original Lie groupoid \( G \) but rather of its jet groupoid \( JG \) on the jet bundle \( JE \) of \( E \). (A similar phenomenon already occurs for Lie group bundles, as observed in Ref. [10].) As it turns out, properly dealing with this feature is the key to make the entire theory work out smoothly.

Let us pass to briefly describe the contents of the paper. In Section 2, we present the minimal coupling prescription and the curvature map that enters the formulation of Utiyama’s theorem in a very general context, and we show that these constructions are invariant (or perhaps it might be better to say, equivariant) under any action of any Lie groupoid over space-time on the bundle of field configurations over space-time, provided we employ the correct induced actions of the pertinent Lie groupoids derived from the former on the pertinent bundles derived from the latter. We conclude with a series of comments intended to show why, from the point of view of field theory, this approach is excessively general and needs to be adapted to a setting where all bundles are derived from some principal bundle and all connections are derived from principal connections in that principal bundle – which is the standard setup for gauge theories anyway. In Section 3, we collect the technical tools needed to perform this adjustment and to state the main results. The first step here is to recall the definition of the gauge groupoid \( G \) of a principal bundle \( P \) and of its natural actions on any bundle \( E \) associated to \( P \) (including \( P \) itself). Next, we introduce the (first order) jet groupoid \( JG \) of \( G \) and use the results of the previous section and of Ref. [6] to write down natural actions of \( JG \) on various derived bundles such as the jet bundle \( JP \) and the connection bundle \( CP \) of \( P \) or the jet bundle \( JE \) of any bundle \( E \) associated to \( P \). We also show how iterating this procedure provides induced actions of the second order jet groupoid \( J^2G \) and, more generally, the semiholonomic second order jet groupoid \( J^2G \) of \( G \) on the semiholonomic second order jet bundle \( J^2P \) and on the (first order) jet bundle \( J(CP) \) of the connection bundle \( CP \) of \( P \). In Section 4, we then prove the main theorems concerning the invariance (or perhaps it might be better to say, the equivariance) of the minimal coupling prescription and the curvature map under the actions of the pertinent Lie groupoids introduced in the previous section, thus providing the desired extension of the results of Ref. [10] from the setting of Lie group bundles (internal symmetries) to that of Lie groupoids (space-time symmetries).

In an appendix, we present an interesting result that links some of our constructions to analogous constructions using jet prolongations of principal bundles. This subject is treated in great generality in Ref. [17] at the level of principal bundles and their associated bundles, but is not addressed at the level of Lie groupoids; in fact, the concept of Lie groupoid does not appear there at all. The basic ingredient is the (first order) jet prolongation \( P^{(1)} \) of the given principal bundle \( P \), which is a principal bundle over the same base manifold and whose structure group \( G_0^{(1)} \) is the jet group of the structure group \( G_0 \) of \( P \), as defined, e.g., in Ref. [17]. This allows us not only to show that various bundles derived from a bundle \( P \times_{G_0} Q \) associated to \( P \) (such as its jet bundle \( J(P \times_{G_0} Q) \) and the tangent bundle \( T(P \times_{G_0} Q) \) of its total space) or even just from \( P \) itself (such as its connection bundle \( CP = JP/G_0 \)) are bundles associated to \( P^{(1)} \) (which is not new), but also that the jet groupoid \( J((P \times P)/G_0) \) of the gauge groupoid \((P \times P)/G_0 \) of \( P \) is canonically isomorphic to the gauge groupoid \((P^{(1)} \times P^{(1)})/G_0^{(1)} \) of \( P^{(1)} \), or to put it more bluntly: jet groupoids of gauge groupoids are gauge groupoids! However, we have not explored all consequences of this approach, since this is not needed to derive our results.
2 Minimal coupling and Utiyama’s theorem I

As stated in the introduction, our main goal in this paper is to extend the results of Ref. [10] about invariance of the minimal coupling prescription and of the curvature map (Utiyama’s theorem) from the context of Lie group bundles to that of Lie groupoids. To do so, let us begin by recalling the general definition of these two constructions.

The term “minimal coupling” is widely used in mathematical physics to denote a procedure for converting ordinary derivatives to covariant derivatives. Such derivatives apply to “matter fields” on space-time M which in a general geometric framework are sections of some fiber bundle E over M: then their ordinary derivatives are sections of its (first order) jet bundle \( J_E \), as a fiber bundle over M, while their covariant derivatives are sections of its linearized (first order) jet bundle

\[
\tilde{J}E \cong L(\pi^*(TM),VE) \cong \pi^*(T^*M) \otimes VE,
\]

as a fiber bundle over M, where \( \pi \) is the bundle projection from E to M, \( \pi^*(TM) \) resp. \( \pi^*(T^*M) \) is the pull-back of the tangent resp. cotangent bundle of M to E, VE is the vertical bundle of E and \( L(\pi^*(TM),VE) \) denotes the bundle of fiberwise linear maps from \( \pi^*(TM) \) to VE. Within this context, the minimal coupling prescription states that the covariant derivative \( D\varphi \) of a section \( \varphi \) of E is obtained from its ordinary derivative \( \partial\varphi \) by using a connection in E to decompose the tangent bundle TE of (the total space of) E into the direct sum of the vertical bundle VE and horizontal bundle HE and then projecting onto the vertical part. Now if we think of that connection as being given by its horizontal lifting map, which is a section \( \Gamma \) of \( JE \) as an affine bundle over E, so that at each point \( e \in E \) with \( \pi(e) = x \), \( \Gamma(e) \) is a linear map from \( T_xM \) to \( T_xE \) whose image is the horizontal space \( H_xE \) at \( e \) of the connection, then that projection onto the vertical part is precisely \( 1 - \Gamma(e) \circ T_x\pi \). Thus if \( \varphi \in \Gamma(M, E) \), so that \( \partial\varphi \in \Gamma(M, JE) \) and \( D\varphi \in \Gamma(M, \tilde{J}E) \), then as maps from M to JE, or equivalently, as fiberwise linear maps from TM to TE, \( \partial\varphi \) is just the first order jet (or tangent map) of \( \varphi \), while \( D\varphi \) is the difference

\[
D\varphi = \partial\varphi - \Gamma \circ \varphi.
\]

This rule can be recast in a purely algebraic form, namely, by viewing it as the result of inserting \( \partial\varphi \) and \( \Gamma \circ \varphi \) into the difference map for (first order) jet bundles, i.e., the bundle map

\[
- : J_E \times_E J_E \longrightarrow L(\pi^*(TM),VE) \cong \pi^*(T^*M) \otimes VE
\]

over E, explicitly constructed as follows: given any point \( e \in E \) with \( \pi(e) = x \) and any two jets \( u_e^1, u_e^2 \in J_eE \subset L(T_xM,T_xE) \), we have \( T_e\pi \circ u_e^i = \text{id}_{T_xM} \), for \( i = 1, 2 \), and hence the difference \( u_e^1 - u_e^2 \) (in the vector space \( L(T_xM,T_xE) \)) takes values in the kernel of \( T_e\pi \), that is, the vertical space \( V_eE \) of E, so it becomes a linear map from \( T_xM \) to \( V_eE \).

The construction of the “curvature map” for connections in a given fiber bundle E over M is similar but somewhat more complicated because it involves its semiholonomous second order jet bundle \( \tilde{J}^2E \). To see how that goes, we proceed as in Ref. [6] by first constructing the iterated jet bundle \( J(JE) \) of E and noting that this allows two projections to \( J_E \), namely, the iterated jet target projection \( \pi_{J(JE)} : J(JE) \longrightarrow J_E \) as well as the jet prolongation \( J\pi_{JE} : J(JE) \longrightarrow J_E \) of the jet target projection \( \pi_{JE} : J_E \longrightarrow E \): then by definition, \( \tilde{J}^2E \) is the subset of \( J(JE) \) where these two projections coincide. Concretely, for \( e \in E \), \( u_e \in J_eE \) and \( u_{ue} \in J_{ue}(JE) \),

\[
(\pi_{J(JE)})_{ue}(u'_{ue}) = u_e, \quad (J\pi_{JE})_{ue}(u'_{ue}) = T_{ue}\pi_{JE} \circ u'_{ue}.
\]
As it turns out [20, Theorem 5.3.4, p. 174], \( \mathcal{J}^2 E \) is an affine bundle over \( JE \) which decomposes naturally into a symmetric part and an antisymmetric part: the former is precisely the usual second order jet bundle \( J^2 E \) of \( E \) (sometimes also called the holonomous second order jet bundle of \( E \)) and is an affine bundle over \( JE \), with difference vector bundle equal to the pullback to \( JE \) of the vector bundle \( \pi^*(\bigwedge^2 T^*M) \otimes V E \) over \( E \) by the jet target projection \( \pi_{JE} \), whereas the latter is a vector bundle over \( JE \), namely the pullback to \( JE \) of the vector bundle \( \pi^*(\bigwedge^2 T^*M) \otimes V E \) over \( E \) by the jet target projection \( \pi_{JE} \):

\[
\begin{align*}
\mathcal{J}^2 E & \cong J^2 E \times_{JE} \pi_{JE}^* \left( \pi^*(\bigwedge^2 T^*M) \otimes V E \right), \\
\mathcal{J}^2 E & \cong \pi_{JE}^* \left( \pi^*(\bigwedge^2 T^*M) \otimes V E \right).
\end{align*}
\]

(5)

Now the proofs of these statements given in Ref. [20] and elsewhere in the literature all involve local coordinate representations, so it may be of some interest to provide a more direct, global argument. To this end, consider what we shall call the difference map for semiholonomous second order jet bundles, i.e., the bundle map

\[ - : \mathcal{J}^2 E \times_{JE} \mathcal{J}^2 E \to L^2(\pi^*(TM), V E) \cong \pi^*(\bigwedge^2 T^*M) \otimes V E \]

(6)

over \( \pi_{JE} \), where \( L^2(\pi^*(TM), V E) \) denotes the bundle of fiberwise bilinear maps from \( \pi^*(TM) \) to \( V E \), explicitly constructed as follows: given any point \( e \in E \) with \( \pi(e) = x \), any jet \( u_e \in J_e E \) and any two semiholonomous second order jets \( u'^1_e, u'^2_e \in \mathcal{J}^2 E \subset J^2 E \subset L(T_x M, T u_e (JE)) \), we have \( T_{u_e} \pi_{JE} \circ u'^i_e = u_e \), for \( i = 1, 2 \), and hence the difference \( u'^1_e - u'^2_e \) takes values in the kernel of \( T_{u_e} \pi_{JE} \), that is, the vertical space \( V^h_{u_e}(JE) \) of \( JE \) with respect to the jet target projection \( \pi_{JE} \) from \( JE \) to \( E \). But with respect to this projection, \( JE \) is an affine bundle with difference vector bundle \( \mathcal{J}E \), so this vertical space is canonically isomorphic to the corresponding difference vector space,

\[ V^h_{u_e}(JE) \cong L(T_x M, V_e E), \]

and thus the difference \( u'^1_e - u'^2_e \) becomes a linear map from \( T_x M \) to this vector space, which can be identified with a bilinear map from \( T_x M \) to \( V_e E \). Obviously, any such bilinear map can be canonically decomposed into its symmetric and its antisymmetric part, and the restriction of the difference map for semiholonomous second order jet bundles to the symmetric part will provide the difference map for second order jet bundles, i.e., the bundle map

\[ - : J^2 E \times_{JE} J^2 E \to L^2_s(\pi^*(TM), V E) \cong \pi^*(\bigwedge^2 T^*M) \otimes V E \]

(7)

over \( \pi_{JE} \), where \( L^2_s(\pi^*(TM), V E) \) denotes the bundle of fiberwise symmetric bilinear maps from \( \pi^*(TM) \) to \( V E \). Moreover, it will provide an alternator or antisymmetrizer for semiholonomous second order jets, which is an affine bundle map

\[ \text{Alt} : \mathcal{J}^2 E \to L^2_a(\pi^*(TM), V E) \cong \pi^*(\bigwedge^2 T^*M) \otimes V E \]

(8)

over \( \pi_{JE} \), where \( L^2_a(\pi^*(TM), V E) \) denotes the bundle of fiberwise antisymmetric bilinear maps from \( \pi^*(TM) \) to \( V E \), as follows: given any point \( e \in E \) with \( \pi(e) = x \), any jet \( u_e \in J_e E \) and any semiholonomous second order jet \( u'_e \in \mathcal{J}^2 E \), choose any holonomous second order jet \( u'^0_e \in J^2 E \) and define \( \text{Alt}(u'_e) \) to be the antisymmetric part of the difference \( u'_e - u'^0_e \), which obviously does not depend on the choice of \( u'^0_e \). It is this construction that we shall
use to define the curvature of a connection in $E$, given, say, in terms of its horizontal lifting map, which is a section $\Gamma$ of $J^2E$ as a bundle over $E$: observing that its jet prolongation $j\Gamma$ will then be a section not just of $J(E)$ but actually of $J^2E$, again as a bundle over $E$, since $T\pi_{J^2E} \circ j\Gamma = T(\pi_{JE} \circ \Gamma) = T\text{id}_E = \text{id}_{T^*E}$, and noting that it will therefore be a section of $J^2E$ along $\Gamma$ when $J^2E$ is considered as a bundle over $JE$ instead, we can compose it with the alternator to produce a section of $\pi^*(\Lambda^2 T^*M) \otimes VE$ and (possibly up to a sign which is a matter of convention) is the curvature
\[
\text{curv}(\Gamma) = \text{Alt} \circ j\Gamma
\] of the given connection.

The main statement we want to prove in this section is that the two constructions are invariant (or perhaps it might be better to say, equivariant) under any action of any Lie groupoid $G$ over $M$ on the bundle $E$ over $M$, provided we employ the correct induced actions of the pertinent Lie groupoids derived from $G$ on the pertinent bundles derived from $E$.

Thus assume we are given a Lie groupoid $G$ over $M$, with source projection $\sigma_G : G \rightarrow M$ and target projection $\tau_G : G \rightarrow M$, together with an action $\Phi_E : G \times_M E \rightarrow E$
\[
(g, e) \mapsto g \cdot e
\] of $G$ on $E$. (Cf. equation (44) of Ref. [6].) Then we obtain an induced action
\[
\Phi_{VE} : G \times_M VE \rightarrow VE
\]
\[
(g, v_e) \mapsto g \cdot v_e
\] of $G$ on the vertical bundle $VE$ of $E$, defined by
\[
g \cdot v_e = T_e L_g(v_e),
\] where $TL_g$ denotes the tangent map to $L_g$; in other words, left translation by $g$ in $VE$ is just the derivative of left translation by $g$ in $E$. (Cf. equations (89) and (90) of Ref. [6].) Combining this with the natural action of the linear frame groupoid $GL(TM)$ of the base manifold $M$ on the cotangent bundle $T^*M$ of $M$, we obtain an induced action of the Lie groupoid $GL(TM) \times_M G$ on the linearized jet bundle $\tilde{JE}$ of $E$,
\[
(GL(TM) \times_M G) \times_M \tilde{JE} \rightarrow \tilde{JE}
\]
\[
((a, g), \tilde{u}_e) \mapsto (a, g) \cdot \tilde{u}_e
\] as suggested by the isomorphism of equation (11), defined by
\[
(a, g) \cdot \tilde{u}_e = T_e L_g \circ \tilde{u}_e \circ a^{-1}.
\] (Cf. equations (96) and (98) of Ref. [6].) On the other hand, applying the jet functor to all structural maps that appear in the original action (11), we obtain an induced action
\[
\Phi_{JE} : JG \times_M JE \rightarrow JE
\]
\[
(u_g, u_e) \mapsto u_g \cdot u_e
\]
of the jet groupoid $JG$ of $G$ on the jet bundle $JE$ of $E$, defined by

$$u_g \cdot u_e = T_{(g,e)}\Phi_E \circ (u_g, u_e) \circ \pi^{\fr}_JG(u_g)^{-1},$$

(16)

where $T\Phi_E$ denotes the tangent map to $\Phi_E$ and $\pi^{\fr}_JG : JG \rightarrow GL(TM)$ is the natural projection of $JG$ to the linear frame groupoid $GL(TM)$ of the base manifold $M$ defined by

$$\pi^{\fr}_JG(u_g) = T_g \tau_G \circ u_g,$$

(17)

whereas $\pi_JG : JG \rightarrow G$ is the usual jet target projection. (Cf. equations (51), (93) and (94) of Ref. [6] or [7].) This definition can also be phrased in terms of (b)sections, as follows: given any bisection $\beta$ of $G$ and any section $\varphi$ of $E$, concatenate them into a map $(\beta, \varphi)$ from $M$ to $G \times_M E$ and compose that with the action $\Phi_E$ of $G$ on $E$ to produce a map from $M$ to $E$ which, when precomposed with the inverse of the diffeomorphism $\tau_G \circ \beta$ of $M$ induced by $\beta$, gives a new section $\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}$ of $E$, and $\Phi_{JE}$ is then fully characterized by the property that, upon taking the jet prolongations of all these (b)sections,

$$\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1} = j(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}).$$

(18)

Indeed, for any $y \in M$, putting $x = (\tau_G \circ \beta)^{-1}(y) \in M$, we have $(\beta(x), \varphi(x)) \in G \times_M E$, $(j\beta(x), j\varphi(x)) \in JG \times_M JE$ and

$$\pi^{\fr}_JG(j\beta(x))^{-1} = (T_{\beta(x)}\tau_G \circ T_x\beta)^{-1} = (T_x(\tau_G \circ \beta))^{-1} = T_y((\tau_G \circ \beta)^{-1}),$$

so

$$\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1}(y) = \Phi_{JE}(j\beta(x), j\varphi(x))$$

$$= T_{(\beta(x), \varphi(x))}\Phi_E \circ (T_x\beta, T_x\varphi) \circ T_y((\tau_G \circ \beta)^{-1})$$

$$= T_y(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}) = j(\Phi_E \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1})(y).$$

Now we have the following statement about compatibility between these various actions:

**Proposition 1** The difference map of equation (3) is equivariant, i.e., the diagram

$$\begin{array}{ccc}
JG \times_M (JE \times_E JE) & \longrightarrow & JE \times_E JE \\
(\pi^{\fr}_G \times \pi_JG, -) & \downarrow & - \\
(GL(TM) \times_M G) \times_M JE & \longrightarrow & JE
\end{array}$$

(19)

commutes.

**Proof:** Given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y, e \in E$ with $\pi(e) = x, u_g \in J_g G$ and $u_e^1, u_e^2 \in J_e E \subset L(T_x M, T_e E)$, we want to prove that

$$u_g \cdot u_e^2 - u_g \cdot u_e^1 = (\pi^{\fr}_JG(u_g), g) \cdot (u_e^2 - u_e^1).$$

Fixing some tangent vector $v \in T_x M$, choose a vertical curve $e(t)$ in $E$ ($\pi(e(t)) = x$) such that

$$e(t)\big|_{t=0} = e, \quad \frac{d}{dt} e(t)\big|_{t=0} = (u_e^2 - u_e^1)(v).$$

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Then
\[ T_{(g,e)}\Phi_E(u_g(v), u_e^2(v)) - T_{(g,e)}\Phi_E(u_g(v), u_e^1(v)) = T_{(g,e)}\Phi_E \left( 0, (u_e^2 - u_e^1)(v) \right) \]
\[ = \frac{d}{dt} \Phi_E(g,e(t)) \bigg|_{t=0} = \frac{d}{dt} L_g(e(t)) \bigg|_{t=0} = T_e L_g \left( (u_e^2 - u_e^1)(v) \right), \]

or using that \( v \) was arbitrary,
\[ T_{(g,e)}\Phi_E \circ (u_g, u_e^2) - T_{(g,e)}\Phi_E \circ (u_g, u_e^1) = T_{(g,e)}\Phi_E \circ (0, u_e^2 - u_e^1) = T_e L_g \circ (u_e^2 - u_e^1). \]

Precomposing with \( \pi_{fr}^{-1}(u_g) \) proves the claim. \( \square \)

To deal with the second part, we begin by iterating the procedure of applying the jet functor to obtain an induced action
\[ \Phi_{J(E)} : \ J(G) \times_M J(E) \longrightarrow J(E) \]
\[ (u_{u_g}, u_{u_e}) \longmapsto u_{u_g} \cdot u_{u_e} \quad (20) \]
of the iterated jet groupoid \( J(G) \) of \( G \) on the iterated jet bundle \( J(E) \) of \( E \), defined by
\[ u_{u_g} \cdot u_{u_e} = T_{(u_g, u_e)}\Phi_{J(E)} \circ (u_{u_g}, u_{u_e}) \circ \pi_{fr}^{-1}(u_g), \quad (21) \]

with the same notation as before; in particular, the definition can again be phrased in terms of (bi)sections. Namely, given any bisection \( \beta \) of \( JG \) and any section \( \varphi \) of \( JE \) which (by composition with \( \pi_{fr}^{-1} \)) project to a bisection \( \beta \) of \( G \) and to a section \( \varphi \) of \( E \), respectively, so that \( \tau_{JG} \circ \beta = \tau_G \circ \beta \), we have, just as in equation (18) above,
\[ \Phi_{J(E)} \circ (j\bar{\beta}, j\bar{\varphi}) \circ (\tau_G \circ \beta)^{-1} = j \left( \Phi_{J(E)} \circ (\bar{\beta}, \bar{\varphi}) \circ (\tau_G \circ \beta)^{-1} \right). \quad (22) \]

This iterated action admits restrictions to several subgroupoids and subbundles, among which the following will become important to us at some point or another: the natural induced actions
\[ \Phi_{J2E} : \ J^2G \times_M J^2E \longrightarrow J^2E \]
\[ (u_{u_g}, u_{u_e}) \longmapsto u_{u_g} \cdot u_{u_e} \quad (23) \]
of the semiholonomous second order jet groupoid \( J^2G \) of \( G \) and
\[ \Phi_{J2E} : \ J^2G \times_M J^2E \longrightarrow J^2E \]
\[ (u_{u_g}, u_{u_e}) \longmapsto u_{u_g} \cdot u_{u_e} \quad (24) \]
of the second order jet groupoid \( J^2G \) of \( G \) on the semiholonomous second order jet bundle \( J^2E \) of \( E \), as well as the action
\[ \Phi_{J2E} : \ J^2G \times_M J^2E \longrightarrow J^2E \]
\[ (u_{u_g}, u_{u_e}) \longmapsto u_{u_g} \cdot u_{u_e} \quad (25) \]
of the second order jet groupoid \( J^2G \) of \( G \) on the second order jet bundle \( J^2E \) of \( E \), all defined by the same formula:
\[ u_{u_g} \cdot u_{u_e} = T_{(u_g, u_e)}\Phi_{J(E)} \circ (u_{u_g}, u_{u_e}) \circ \pi_{fr}^{-1}(u_g). \quad (26) \]
Here, the simplification in the last term on the rhs of equation (26), as compared to that of equation (21), stems from the fact that when $u'_{ug} \in J^2_ug$, i.e., $T_{ug} \tau_{JG} \circ u'_{ug} = u_g$, then since $\tau_{JG} = \tau_G \circ \tau_{JG}$, we get

$$\pi_{JG}^t(u'_{ug}) = T_{ug} \tau_{JG} \circ u'_{ug} = T_g \tau_G \circ u'_{ug} = T_g \tau_G \circ u_g = \pi_{JG}^t(u_g).$$

Moreover, if $u'_{ug}$ and $u'_{ue}$ are both semiholonomous, then so is $u'_{ug} \cdot u'_{ue}$, i.e., we have

$$u'_{ug} \in J^2_ug, u'_{ue} \in J^2_ue \implies u'_{ug} \cdot u'_{ue} \in J^2_{ug \cdot ue},$$

since in this case, $T_{ug} \tau_{JG} \circ u'_{ug} = u_g$ and $T_{ue} \tau_{JE} \circ u'_{ue} = u_e$, and using the equality $\pi_{JE} \circ \Phi_{JE} = \Phi_{E} \circ (\pi_{JG} \times_M \pi_{JE})$, we get

$$T_{ug \cdot ue} \tau_{JE} \circ (u'_{ug} \cdot u'_{ue}) = T_{ug} \tau_{JG} \circ u'_{ug} \circ T_{ue} \tau_{JE} \circ u'_{ue} \circ \pi_{JG}(u_g)^{-1} = T_{(g,e)} \Phi_{E} \circ (T_{ug} \tau_{JG} \circ u'_{ug}, T_{ue} \tau_{JE} \circ u'_{ue}) \circ \pi_{JG}(u_g)^{-1} = T_{(g,e)} \Phi_{E} \circ (u_g, u_e) \circ \pi_{JG}(u_g)^{-1} = u_g \cdot u_e.$$

Similarly, it is clear that if $u'_{ug}$ and $u'_{ue}$ are both holonomous, then so is $u'_{ug} \cdot u'_{ue}$, i.e., we have

$$u'_{ug} \in J^2_ug, u'_{ue} \in J^2_ue \implies u'_{ug} \cdot u'_{ue} \in J^2_{ug \cdot ue},$$

since in this case there will exist a local bisection $\beta$ of $G$ and a local section $\varphi$ of $E$, both defined in some open neighborhood $U$ of $x$, satisfying $g = \beta(x), e = \varphi(x), u_g = j\beta(x) = T_x \beta, u_e = j\varphi(x) = T_x \varphi, u'_{ug} = j(j\beta)(x) = T_x (j\beta), u'_{ue} = j(j\varphi)(x) = T_x (j\varphi)$ and hence, putting $y = (\tau_G \circ \beta)(x)$ and using equation (26), equation (22) with $\tilde{\beta} = j\beta, \tilde{\varphi} = j\varphi$ and equation (18),

$$u'_{ug} \cdot u'_{ue} = \Phi_{J(JE)}(j(j\beta)(x), j(j\varphi)(x)) = (\Phi_{J(JE)} \circ (j(j\beta), j(j\varphi)) \circ (\tau_G \circ \beta)^{-1})(y) = j(\Phi_{JE} \circ (j\beta, j\varphi) \circ (\tau_G \circ \beta)^{-1})(y) = j(j(\Phi_{E} \circ (\beta, \varphi) \circ (\tau_G \circ \beta)^{-1}))(y).$$

Finally, observe that, just like the (first order) jet groupoid $JG$ of $G$, its iterated jet groupoid $J(JG)$ and, by restriction, its semiholonomous second order jet groupoid $J^2G$ and second order jet groupoid $J^2G$ all admit natural projections both to $GL(TM)$ and to $G$, which are just given by composition of those for $JG$ with the natural projection $\pi_{J(JG)} : J(JG) \longrightarrow JG$ and its respective restrictions $\pi_{J^2G} : J^2G \longrightarrow JG$ and $\pi_{J^2G} : J^2G \longrightarrow JG$:

$$\pi_{J(JG)}^t = \pi_{JG}^t \circ \pi_{J(JG)} : J(JG) \longrightarrow GL(TM), \quad \pi_{J(JG),G} = \pi_{JG} \circ \pi_{J(JG)} : J(JG) \longrightarrow G$$

$$\pi_{J^2G}^t = \pi_{JG}^t \circ \pi_{J^2G} : J^2G \longrightarrow GL(TM), \quad \pi_{J^2G,G} = \pi_{JG} \circ \pi_{J^2G} : J^2G \longrightarrow G$$

With this notation, we can now formulate the following statement about compatibility between these various actions:
Proposition 2 The difference maps of equations (6) and (7) are equivariant, i.e., the diagrams

\[
\begin{array}{ccc}
\mathcal{J}^2 G \times_M (\mathcal{J}^2 E \times_{\mathcal{J}E} \mathcal{J}^2 E) & \longrightarrow & \mathcal{J}^2 E \times_{\mathcal{J}E} \mathcal{J}^2 E \\
(\pi^\mathcal{J}_2 G \times \pi^\mathcal{J}_2 G, -) & \downarrow & - \\
(GL(TM) \times_M G) \times_M \left(\pi^*(\bigotimes^2 T^* M) \otimes V E\right) & \longrightarrow & \pi^*(\bigotimes^2 T^* M) \otimes V E
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{J}^2 G \times_M (\mathcal{J}^2 E \times_{\mathcal{J}E} \mathcal{J}^2 E) & \longrightarrow & \mathcal{J}^2 E \times_{\mathcal{J}E} \mathcal{J}^2 E \\
(\pi^\mathcal{J}_2 G \times \pi^\mathcal{J}_2 G, -) & \downarrow & - \\
(GL(TM) \times_M G) \times_M \left(\pi^*(V^2 T^* M) \otimes V E\right) & \longrightarrow & \pi^*(V^2 T^* M) \otimes V E
\end{array}
\]

commute. Similarly, the alternator or antisymmetrizer map of equation (9) is also equivariant, i.e., the diagram

\[
\begin{array}{ccc}
\mathcal{J}^2 G \times_M \mathcal{J}^2 E & \longrightarrow & \mathcal{J}^2 E \\
(\pi^\mathcal{J}_2 G \times \pi^\mathcal{J}_2 G, \text{Alt}) & \downarrow & \text{Alt} \\
(GL(TM) \times_M G) \times_M \left(\pi^*(\bigwedge^2 T^* M) \otimes V E\right) & \longrightarrow & \pi^*(\bigwedge^2 T^* M) \otimes V E
\end{array}
\]

commutes.

Proof: First of all, the statements about commutativity of the last two diagrams are trivial consequences of that about commutativity of the first, together with the fact that the decomposition of rank 2 tensors into their symmetric and antisymmetric parts is obviously invariant under the action of \(GL(TM) \times_M G\). To deal with the first diagram, we shall find it convenient to keep track of the identifications made in the definition of the difference map in equation (6) by momentarily (i.e., just for the remainder of this proof) denoting that difference map by \(\delta\). Thus given \(g \in G\) with \(\sigma_G(g) = x\) and \(\tau_G(g) = y\), \(e \in E\) with \(\pi(e) = x\), \(u_g \in J_x G\), \(u_e \in J_e E\), \(u^0_{u_g} \in J^2_{u_g} G\) and \(u^1_{u_e}, u^2_{u_e} \in \mathcal{J}^2_{u_e} E \subset J_{u_e}(\mathcal{J}E) \subset L(T_x M, T_{u_e}(\mathcal{J}E))\), we want to show that

\[
\delta(u^2_{u_e}, u^1_{u_e}, u^0_{u_g}) = (\pi^\mathcal{J}_G(u_g), g) \cdot \delta(u^2_{u_e}, u^1_{u_e}).
\]

Note that \(\delta(u^2_{u_e}, u^1_{u_e}) \in L^2(T_x M, V_e E)\) can be defined explicitly by stating that, for any tangent vector \(v \in T_x M\), the standard difference \(u^2_{u_e} - u^1_{u_e}\), when evaluated on \(v\), gives a tangent vector in \(T_{u_e}(\mathcal{J}E)\) which, being vertical with respect to the jet target projection \(\pi_{J_E}\), can be realized as that of a straight line in \(J_e E\) through \(u_e\), whose direction is \(\delta(u^2_{u_e}, u^1_{u_e})(v, \cdot) \in L(T_x M, V_e E)\):
projection $\pi_{JE}$, can be realized as that of a straight line in $J_{gE}E$ through $u_g \cdot u_e$, whose direction is $\delta(u_{u_g} \cdot u_{u_e}^2, u_{u_g}^2, u_{u_g}^1)(w, \cdot) \in L(T_y M, V_{gE})$:

$$(u'_{u_g} \cdot u_{u_e}^2 - u'_{u_g} \cdot u_{u_e}^1)(w) = \frac{d}{dt} (u_g \cdot u_e + t \delta(u_{u_g} \cdot u_{u_e}^2, u_{u_g}^2, u_{u_g}^1)(w, \cdot)) \big|_{t=0}.$$ 

On the other hand, putting $v = \pi_{JG}^{-1}(u_g)$, we have

$$(u'_{u_g} \cdot u_{u_e}^2 - u'_{u_g} \cdot u_{u_e}^1)(w) = T_{(u_g, u_e)} \Phi_{JE}(u'_{u_g}(v), u_{u_e}^2(v)) - T_{(u_g, u_e)} \Phi_{JE}(u'_{u_g}(v), u_{u_e}^1(v))$$

$$= T_{(u_g, u_e)} \Phi_{JE}(0, (u_{u_e}^2 - u_{u_e}^1)(v))$$

$$= \frac{d}{dt} \Phi_{JE}(u_g, u_e + t (-u_{u_e}^2, u_{u_e}^1))(v, \cdot) \big|_{t=0}$$

$$= \left( (\pi_{JG}^{-1}(u_g), g) \cdot \delta(u_{u_e}^2, u_{u_e}^1)(v, \cdot) \right),$$

where in the last step we have used the fact that, as shown in Ref. [6], the action $\Phi_{JE}$ is affine along the fibers of $JE$ over $E$, together with Proposition [1].

Returning to the formalization of the minimal coupling prescription and the curvature map, we want to emphasize that the context outlined above is a little bit too broad to fit into the theoretical setting of field theory, since general connections in general fiber bundles are not fields! This is so because they are not sections of bundles over space-time but rather sections of bundles over some “extended space-time” which is itself the total space of some fiber bundle over ordinary space-time. As such, when expressed in local coordinates and local trivializations, such sections correspond to multiplets of functions which, apart from being functions on space-time, depend on extra “vertical” variables, namely, the local coordinates along the fibers of this bundle, and in the absence of stringent restrictions on that dependence will produce infinite multiplets of fields when expanded in an appropriate basis. This situation is familiar from “Kaluza-Klein” type theories, which have been proposed long ago as models for unifying gravity with the other fundamental interactions and where the extended space-time is assumed to be the total space of some principal bundle over ordinary space-time, so that one can use the representation theory of the underlying structure group to control and restrict the dependence of functions on the extra vertical variables [3]. The main problem with these models is that the aforementioned stringent restrictions, needed to weed out the large number of (often unwanted) extra fields, are usually quite artificial and imposed more or less “ad hoc”, without any convincing argument as to how they should arise from the dynamics of a fundamental theory in higher dimensions.

Here, these remarks serve merely as a guide to what should be done and what not: we shall completely avoid all these problems by working not with general connections but only with connections that do have a natural interpretation as fields in physics: these are connections whose behavior along the fibers is fixed by some condition, such as linear connections in vector bundles.
or affine connections in affine bundles, where the connection coefficients are required to be linear
or affine functions along the fibers, respectively, or more generally, principal connections, which
are required to be equivariant under the action of the structure group on the fibers of the
principal bundle and are therefore completely fixed along the entire fiber once they are known
at a single point in that fiber.

Thus from this point onward and throughout the rest of the paper, we shall assume that
\( E \) is not just a general fiber bundle but rather a fiber bundle with structure group, which is
a Lie group \( G_0 \), with Lie algebra \( \mathfrak{g}_0 \), say, so there is a principal \( G_0 \)-bundle \( P \) to which \( E \) is
associated (this, by the way, includes the case where \( E \) is \( P \) itself), and any connection in \( E \) to
be considered is associated to a principal connection in \( P \). As a result, we have to adapt our
formalism to this situation, and of course the Lie groupoid \( G \) that appears above, as well as in
Ref. [6], but has so far been left unspecified, will now be the gauge groupoid of \( P \).

3 Gauge groupoids, jet groupoids and induced actions

In order to implement the program outlined in the last paragraph of the previous section, we
shall first introduce the gauge groupoid of a principal bundle and some of its actions (more
specifically, on the principal bundle itself and on any of its associated bundles, as well as on the
respective vertical bundles) and then investigate how some of these lift when taking first and
second order jet prolongations.

3.1 The gauge groupoid and its actions

To begin with, let us recall the definition of the gauge groupoid of a principal bundle [18]:

**Proposition 3** Given a principal bundle \( P \) over a manifold \( M \) with structure group \( G_0 \), whose
bundle projection will be denoted by \( \rho : P \to M \), let

\[
G = (P \times P)/G_0
\]

denote the orbit space of the cartesian product of \( P \) with itself under the diagonal action of \( G_0 \) (we
shall write its elements as classes \([p_2, p_1]\) of pairs \((p_2, p_1)\) in \( P \times P \), where \([p_2, g_0, p_1, g_0] = [p_2, p_1]\)).

Then \( G \) is a Lie groupoid over \( M \), called the **gauge groupoid** of \( P \), with source projection
\( \sigma_G : G \to M \), target projection \( \tau_G : G \to M \), multiplication map \( \mu_G : G \times_M G \to G \), unit
map \( 1_G : M \to G \) and inversion \( \iota_G : G \to G \) defined as follows:

- for \([p_2, p_1] \in G\),
  \[
  \sigma_G([p_2, p_1]) = \rho(p_1) , \quad \tau_G([p_2, p_1]) = \rho(p_2) ;
  \]

- for \([p_2, p_1], [p_3, p_2] \in G\),
  \[
  [p_3, p_2][p_2, p_1] \equiv \mu_G([p_3, p_2], [p_2, p_1]) = [p_3, p_1] ;
  \]

- for \( x \in M \),
  \[
  (1_G)_x = [p, p] ,
  \]

where \( p \) is any element of \( \rho^{-1}(x) \);
• for \([p_2, p_1] \in G\),
  \[ [p_2, p_1]^{-1} = \iota_G([p_2, p_1]) = [p_1, p_2]. \]

Observe that the gauge group bundle associated with \(P\) employed in Ref. [10], also known as the adjoint bundle \(\text{Ad}P = P \times_G G_0\) (where \(G_0\) acts on itself by conjugation), is (up to a canonical isomorphism) just the isotropy subgroupoid of \(G\), that is,
\[
P \times_{G_0} G_0 \cong G_{\text{iso}}. \tag{30}\]

This isomorphism can be constructed explicitly by noting that the map
\[
P \times G_0 \longrightarrow P \times P
(p, g_0) \longmapsto (p, p \cdot g_0)
\]
is equivariant under the right action of \(G_0\) on both sides (since it takes \((p \cdot g_0', (g_0')^{-1}g_0g_0')\) to \((p \cdot g_0', p \cdot g_0g_0')\)) and hence factors to the respective quotients to yield a map
\[
P \times_{G_0} G_0 \longrightarrow (P \times P)/G_0
[p, g_0] \longmapsto [p, p \cdot g_0]
\]
which is the desired isomorphism onto its image
\[
G_{\text{iso}} = \{ [p_2, p_1] \in G \mid \tau_G([p_2, p_1]) = \sigma_G([p_2, p_1]) \} = \{ [p_2, p_1] \in G \mid \rho(p_2) = \rho(p_1) \}. \tag{31}\]

Moreover, it is well known that the group of bisections of the gauge groupoid \(G = (P \times P)/G_0\) is isomorphic to the group of automorphisms of \(P\),
\[
\text{Bis}(G) \cong \text{Aut}(P), \tag{32}\]
while the group of sections of the gauge group bundle \(G_{\text{iso}} \cong P \times_{G_0} G_0\) is isomorphic to the group of strict automorphisms of \(P\),
\[
\Gamma(G_{\text{iso}}) \cong \text{Aut}_s(P). \tag{33}\]

Next, let us specify how the gauge groupoid of a principal bundle acts naturally on the principal bundle itself and on any of its associated bundles. To this end, some authors find it convenient to introduce the “difference map” for \(P\), which is the smooth map
\[
\delta_P : P \times_M P \longrightarrow G_0
\]
defined implicitly by the condition that given any two points \(p\) and \(p'\) in the same fiber of \(P\), \(\delta_P(p, p')\) is the unique element of \(G_0\) that transforms \(p\) into \(p'\):
\[
p \cdot \delta_P(p, p') = p'.
\]
Note that, obviously, \(\delta_P(p, p) = 1\) and
\[
\delta_P(p \cdot g_0, p' \cdot g_0) = g_0^{-1}\delta_P(p, p')g_0.
\]
Here, we use this map to write down a natural action

\[ \Phi_P : G \times_M P \longrightarrow P \]

\[ ([p_2, p_1], p) \mapsto [p_2, p_1] \cdot p \]

of the gauge groupoid \( G = (P \times P)/G_0 \) on the principal bundle \( P \) itself, defined as follows: given \([p_2, p_1] \in G \) and \( p \in P \) such that \( \rho(p_1) = \sigma_G([p_2, p_1]) = \rho(p) \), put

\[ [p_2, p_1] \cdot p = p_2 \cdot \delta_P(p_1, p) \]

Note, however, that we can always adapt the second component in the pair \((p_2, p_1)\) representing the class \([p_2, p_1]\) to be equal to \( p \), which allows us to rewrite the previous two equations in the simplified form

\[ \Phi_P : G \times_M P \longrightarrow P \]

\[ ([p', p], p) \mapsto [p', p] \cdot p \]

where

\[ [p', p] \cdot p = p' \]  \hspace{1cm} (35)

In the sequel, when defining other actions of the gauge groupoid, we shall already perform this kind of simplification right from the start and without further notice, thus dispensing the need to deal with the difference map \( \delta_P \) altogether. Of course, as the total space of a principal bundle, \( P \) also carries a right action of the structure group \( G_0 \), and remarkably, these two actions commute,

\[ [p', p] \cdot (p \cdot g_0) = ([p', p] \cdot p) \cdot g_0 , \]

because both sides are equal to \([p' \cdot g_0, p \cdot g_0] \cdot (p \cdot g_0) = p' \cdot g_0 \). Thus using the natural projection of \( G \) to the pair groupoid \( M \times M \) of the base manifold \( M \), we get a commutative diagram:

\[ \begin{array}{ccc}
G \times_M P & \longrightarrow & P \\
(M \times M) \times_M M & \longrightarrow & M 
\end{array} \]  \hspace{1cm} (37)

This procedure can be generalized as follows. First, given any manifold \( Q \), we can introduce a natural action

\[ \Phi_{P \times Q} : G \times_M (P \times Q) \longrightarrow P \times Q \]

\[ ([p', p], (p, q)) \mapsto [p', p] \cdot (p, q) \]

of the gauge groupoid \( G = (P \times P)/G_0 \) on the product manifold \( P \times Q \) (as a fiber bundle over \( M \)), defined by letting \( G \) act as above on the first factor and trivially on the second factor,

\[ [p', p] \cdot (p, q) = (p', q) . \]

Now suppose we are also given a left action

\[ G_0 \times Q \longrightarrow Q \]

\[ (g_0, q) \mapsto g_0 \cdot q \]

(40)
of $G_0$ on the manifold $Q$, which according to the standard definition of the total space of an associated bundle is extended to a “diagonal” right action
\[
G_0 \times (P \times Q) \longrightarrow P \times Q
\]
\[
(g_0, (p, q)) \mapsto (p \cdot g_0^{-1}, q)
\]
of $G_0$ on the product manifold $P \times Q$, and once again, these two actions commute,
\[
[p', p] \cdot ((p, q) \cdot g_0) = ([p', p] \cdot (p, q)) \cdot g_0,
\]
because both sides are equal to $[p' \cdot g_0, p \cdot g_0] \cdot (p \cdot g_0^{-1}, q) = (p' \cdot g_0, g_0^{-1} \cdot q)$. This implies that the action $\Phi_{P \times Q}$ of $G$ on $P \times Q$ in equation (38) passes to the quotient $P \times G_0 Q$, and so we get a natural induced action
\[
\Phi_{P \times G_0 Q} : G \times_M (P \times G_0 Q) \longrightarrow P \times G_0 Q
\]
\[
([p', p], [p, q]) \mapsto [p', p] \cdot [p, q]
\]
of the gauge groupoid $G = (P \times P)/G_0$ on the associated bundle $P \times G_0 Q$, defined by
\[
[p', p] \cdot [p, q] = [p', q].
\]
It will be convenient to visualize this construction in terms of the “magical square” for associated bundles, i.e., the commutative diagram
\[
P \times Q \xrightarrow{\rho_Q} P \times G_0 Q \xrightarrow{\pi} M
\]
in which the horizontal projections define principal $G_0$-bundles while the vertical projections provide fiber bundles with typical fiber $Q$ (the first of which is of course just the trivial bundle over $P$) such that $\rho_Q$ is an isomorphism on each fiber and, by definition, is $G$-equivariant. And again, using the natural projection of $G$ to the pair groupoid $M \times M$ of the base manifold $M$, we get a commutative diagram:
\[
G \times_M (P \times G_0 Q) \longrightarrow P \times G_0 Q
\]
\[
(M \times M) \times_M M \longrightarrow M
\]
Of course, these actions extend the actions of the gauge group bundle $P \times G_0 G_0$ on the principal bundle $P$ itself and on the associated bundle $P \times G_0 Q$, respectively, considered in Ref. [10].

As a first example of induced actions, consider those of the gauge groupoid of a principal bundle on the vertical bundle of the principal bundle itself and on the vertical bundle of any of its associated bundles, constructed according to the prescription specified in equations (10)–(12) above. These actions can be simplified by making use of the fact that the vertical bundle of
a principal bundle is trivial and that the vertical bundle of an associated bundle is again an associated bundle, i.e., we have canonical isomorphisms
\[ VP \cong P \times \mathfrak{g}_0, \] (47)
and
\[ V(P \times_{G_0} Q) \cong P \times_{G_0} TQ, \] (48)
both as fiber bundles over \( M \) and as vector bundles over the respective total spaces \( P \) and \( P \times_{G_0} Q \), where in the second case, the action of \( G_0 \) on the tangent bundle \( TQ \) of \( Q \) is the one induced from that on \( Q \). Similarly, we also have canonical isomorphisms
\[ \tilde{\mathcal{J}}P \cong L(\pi^*(TM), (P \times \mathfrak{g}_0)) \cong \pi^*(T^*M) \otimes (P \times \mathfrak{g}_0), \] (49)
and
\[ \tilde{\mathcal{J}}(P \times_{G_0} Q) \cong L(\pi^*(TM), P \times_{G_0} TQ) \cong \pi^*(T^*M) \otimes (P \times_{G_0} TQ), \] (50)
in the same sense. The statement is then that these bundle isomorphisms are equivariant under the action of the gauge groupoid \( G \), in the first two cases, and of the Lie groupoid \( GL(TM) \times_M G \), in the last two cases.

For the proof, we need only consider the statements for the vertical bundles, since the corresponding ones for the linearized jet bundles follow directly from them by combining the corresponding actions of the gauge groupoid with that of the linear frame groupoid \( GL(TM) \times_M G \) on the cotangent bundle \( T^*M \) of \( M \). To this end, consider the fundamental vector fields \((X_0)_P\) on \( P \) associated to the generators \( X_0 \in \mathfrak{g}_0 \) through the right action of \( G_0 \) on \( P \), and for later use, also the fundamental vector fields \((X_0)_Q\) on \( Q \) associated to the generators \( X_0 \in \mathfrak{g}_0 \) through the left action of \( G_0 \) on \( Q \), defined by
\[ (X_0)_P(p) = \frac{d}{dt}(p \cdot \exp(tX_0)) \bigg|_{t=0}, \] (51)
and by
\[ (X_0)_Q(q) = \frac{d}{dt}(\exp(-tX_0) \cdot q) \bigg|_{t=0}, \] (52)
respectively. Then the isomorphism in equation (47) is given by the mapping that takes the pair \((p, X_0)\) to the vertical vector \((X_0)_P(p)\), and that this is equivariant follows immediately from the following simple calculation:
\[
[p', p] \cdot (X_0)(p) = T_pL_{[p', p]} \left( \frac{d}{dt}(p \cdot \exp(tX_0)) \bigg|_{t=0} \right) = \left. \frac{d}{dt}([p', p] \cdot (p \cdot \exp(tX_0))) \right|_{t=0} \\
= \left. \frac{d}{dt}([p' \cdot \exp(tX_0), p \cdot \exp(tX_0)] \cdot (p \cdot \exp(tX_0))) \right|_{t=0} \\
= \left. \frac{d}{dt}(p' \cdot \exp(tX_0)) \right|_{t=0} = (X_0)(p').
\]

\[\]We recall that the correspondence in equation (47) establishes a canonical linear isomorphism between the Lie algebra \( \mathfrak{g}_0 \) and the vertical space \( \mathfrak{v}_P \) of \( P \) at \( p \), whereas the extra minus sign in equation (48) is introduced merely for convenience, so as to guarantee consistency of the formulas when we switch between left and right actions.
Similarly, the isomorphism in equation (48) is given by the mapping (momentarily denoted by $\phi$) that takes $[p, \frac{d}{dt}q(t)|_{t=0}] \in (P \times G_0)TQ$ to $\frac{d}{dt}|[p, q(t)|_{t=0}] \in V_{[p,q]}(P \times G_0Q)$, and that this is equivariant follows immediately from the following simple calculation:

$$
[p', p] \cdot \phi\left([p, \frac{d}{dt}q(t)|_{t=0}]\right) = [p', p] \cdot \left(\frac{d}{dt}|[p, q(t)|_{t=0}]\right)
$$

$$
= \frac{d}{dt}\left(\left[\frac{d}{dt}|[p', p] \cdot [p, q(t)|_{t=0}]\right]\right) = \phi\left(\left[\frac{d}{dt}|[p', p] \cdot [p, q(t)|_{t=0}]\right]\right).
$$

Similar simplifications occur for the other induced actions considered in the previous section, and this will be discussed in the next two subsections.

### 3.2 First order jet groupoids and induced actions

To begin with, we apply the general procedure developed in Ref. [6] of “differentiating” actions of Lie groupoids on fiber bundles to the natural actions of the gauge groupoid $G = (P \times P)/G_0$ on the principal bundle $P$ itself and on any associated bundle $P \times G_0Q$ to obtain natural induced actions

$$
\Phi_{JP}: JG \times M JP \to JP
$$

$$(u_{[p', p]}, u_p) \mapsto u_{[p', p]} \cdot u_p \quad (53)$$

and

$$
\Phi_{J(P \times G_0Q)}: JG \times M J(P \times G_0Q) \to J(P \times G_0Q)
$$

$$(u_{[p', p]}, u_{[p, q]}) \mapsto u_{[p', p]} \cdot u_{[p, q]} \quad (54)$$

derived from the actions $\Phi_P$ in equation (34) and $\Phi_{P \times G_0Q}$ in equation (43) by applying the general formula in equation (16) of the previous section.

A more profound understanding of the situation can be obtained by extending the “magical square” for associated bundles in equation (45) to the corresponding jet bundles, considering the commutative diagram

$$
\begin{array}{ccc}
J(P \times Q) & \xrightarrow{\rho_Q} & J(P \times G_0Q) \\
\pi_{J(P \times Q)} \downarrow & & \downarrow \pi_{J(P \times G_0Q)} \\
P \times Q & \xrightarrow{\rho_Q} & P \times G_0Q \\
\text{pr}_1 \downarrow & & \downarrow \pi \\
P & \xrightarrow{\rho} & M
\end{array}
$$

(55)

and noting that, just like there is a natural action of $G$ on $P \times Q$ derived from that on $P$ such that $\rho_Q$ is an isomorphism on each fiber and is $G$-equivariant, as discussed in the previous subsection, there is also a natural action of $JG$ on $J(P \times Q)$ derived from that on $JP$ such
that $J\rho_Q$, although no longer an isomorphism on each fiber (it is still onto but has a kernel), is $JG$-equivariant.

To prove these statements, let us pick points $p \in P$ and $q \in Q$ with $\rho(p) = x$ and take tangent maps to the commutative diagram in equation (45) to obtain the commutative diagram

$$
\begin{array}{ccc}
T_pP \oplus T_qQ & \stackrel{T_{(p,q)}\rho_Q}{\longrightarrow} & T_{[p,q]}(P \times G_0 Q) \\
pr_1 \downarrow & & \downarrow T_{[p,q]}\pi \\
T_pP & \stackrel{T_p\rho}{\longrightarrow} & T_xM
\end{array}
$$

(56)

Since $\rho_Q$ is a submersion and hence its tangent maps are surjective, this means that the tangent spaces $T_{[p,q]}(P \times G_0 Q)$ of the orbit space $P \times G_0 Q$ can be realized as quotient spaces, namely, the linear maps

$$
T_{(p,q)}\rho_Q : T_pP \oplus T_qQ \longrightarrow T_{[p,q]}(P \times G_0 Q)
$$

(57)

induce isomorphisms

$$
T_{[p,q]}(P \times G_0 Q) \cong (T_pP \oplus T_qQ)/\ker T_{(p,q)}\rho_Q,
$$

(58)

and noting that

$$
J_{(p,q)}(P \times Q) = J_pP \oplus L(T_xM, T_qQ),
$$

(59)

this leads to an analogous realization of the jet spaces $J_{[p,q]}(P \times G_0 Q)$ of the orbit space $P \times G_0 Q$ as quotient spaces, namely, the affine maps

$$
J_{(p,q)}\rho_Q : J_pP \oplus L(T_xM, T_qQ) \longrightarrow J_{[p,q]}(P \times G_0 Q)
$$

(60)

defined by

$$
J_{(p,q)}\rho_Q(u_p, u_q) = T_{(p,q)}\rho_Q \circ (u_p, u_q)
$$

(61)

induce isomorphisms

$$
J_{[p,q]}(P \times G_0 Q) \cong (J_pP \oplus L(T_xM, T_qQ))/L(T_xM, \ker T_{(p,q)}\rho_Q).
$$

(62)

Now using the $G$-equivariance of $\rho_Q$, which means that $\Phi_{P \times G_0 Q} \circ (\id_G \times_M \rho_Q) = \rho_Q \circ (\Phi_p \times \id_Q)$ (where in the last equality we have applied the identity $G \times_M (P \times Q) = (G \times M) \times Q$), we can prove the $JG$-equivariance of $J\rho_Q$. To this end, let us also pick a point $[p', q] \in G$, a jet $u_{[p', p]} \in J_{[p', p]}G$ and another jet $u_p \in J_pP$ together with a linear map $u_q \in L(T_xM, T_qQ)$, and calculate

$$
u_{[p', p]} \cdot J_{(p,q)}\rho_Q(u_p, u_q)
$$

$$
= T_{([p', p],[p,q])}(\Phi_{P \times G_0 Q} \circ (u_{[p', p]} \cdot T_{(p,q)}\rho_Q \circ (u_p, u_q))) \circ \pi^G_{JG}(u_{[p', p]})^{-1}
$$

$$
= T_{([p', p],[p,q])}(\Phi_{P \times G_0 Q} \circ T_{([p', p],[p,q])}(\id_G \times_M \rho_Q) \circ (u_{[p', p]}, (u_p, u_q)) \circ \pi^G_{JG}(u_{[p', p]})^{-1}
$$

$$
= T_{([p', q], [p,q])}(\Phi_{P \times G_0 Q} \circ (u_{[p', p]}, (u_p, u_q)) \circ \pi^G_{JG}(u_{[p', p]})^{-1}
$$

$$
= J_{([p', q], [p,q])}(u_{[p', p]} \cdot (u_p, u_q)) = J_{([p', q], [p,q])}(u_{[p', p]} \cdot u_p, u_q).
$$

Note that here, $J(P \times Q)$ is meant to be the jet bundle of $P \times Q$ as a bundle over $M$, i.e., with respect to the projection $\rho \circ pr_1$, whereas the previous statement that $P \times Q$ is a trivial bundle refers to its structure as a bundle over $P$, i.e., to the projection $pr_1$. 

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This implies that the action $\Phi_{JP}$ which is an affine bundle over $M$ corresponds precisely to the $G$-equivariant sections of the principal bundle $P$ and of the associated bundle $P \times G_0 Q$ are related by

$$V_{[p,q]}(P \times G_0 Q) \cong (V_p P \oplus T_q Q)/\ker T_{(p,q)}\rho_Q,$$

while, with respect to any principal connection in $P$ and its associated connection in $P \times G_0 Q$, the corresponding horizontal spaces of the principal bundle $P$ and of the associated bundle $P \times G_0 Q$ are related by

$$H_{[p,q]}(P \times G_0 Q) \cong (H_p P \oplus \{0\})/\ker T_{(p,q)}\rho_Q.$$

At the end of this subsection, we shall see how to express the correspondence between principal connections on $P$ and their associated connections in $P \times G_0 Q$ in terms of jets.

Another important property of the action of $JG$ on $JP$ in equation (53) is that it commutes with the right action of the structure group $G_0$ on $JP$: this is essentially obvious because they are induced from an action of $G$ on $P$ and a right action of $G_0$ on $P$ which commute. But since this is an important fact, let us give a quick formal proof of the pertinent formula,

$$u_{[p',p]} \cdot (w_p \cdot g_0) = (u_{[p',p]} \cdot w_p) \cdot g_0. \quad (66)$$

Indeed, according to equations (13) and (30) (the second of which can be reformulated as stating that $\Phi_P \circ (\text{id}_G \times R_{g_0}) = R_{g_0} \circ \Phi_P$, where $R_{g_0}$ denotes right translation by $g_0$ in $P$),

$$u_{[p',p]} \cdot (w_p \cdot g_0) = T_{([p',p],p,g_0)} \Phi_P \circ (u_{[p',p]} \cdot T_p R_{g_0} \circ w_p) \circ \pi_{JG}^{-1}(u_{[p',p]})^{-1}
= T_{([p',p],p,g_0)} \Phi_P \circ T_{([p',p],p)}(\text{id}_G \times M R_{g_0}) \circ (u_{[p',p]} \cdot w_p) \circ \pi_{JG}^{-1}(u_{[p',p]})^{-1}
= T_p R_{g_0} \circ T_{([p',p],p)} \Phi_P \circ (u_{[p',p]} \cdot w_p) \circ \pi_{JG}^{-1}(u_{[p',p]})^{-1}
= (u_{[p',p]} \cdot w_p) \cdot g_0.$$

This implies that the action $\Phi_{JP}$ of $JG$ on $JP$ in equation (53) passes to the quotient

$$CP = JP/G_0,$$

which is an affine bundle over $M$ called the connection bundle of $P$ because its sections correspond precisely to the $G_0$-equivariant sections of $JP$ (as an affine bundle over $P$), which are exactly the principal connections on $P$. Thus we get a natural induced action

$$\Phi_{CP} : JG \times_M CP \rightarrow CP
\begin{array}{lcr}
(w_{[p',p]} \cdot [w_p]) & \mapsto & u_{[p',p]} \cdot [w_p]
\end{array} \quad (68)$$
of $JG$ on $CP$. It will be convenient to visualize this construction in terms of the “magical square” for connection bundles, i.e., the commutative diagram

$$\begin{array}{ccc}
JP & \xrightarrow{\rho_C} & CP \\
\pi_{JP} \downarrow & & \downarrow \pi_{CP} \\
P & \xrightarrow{\rho} & M
\end{array}$$ (69)

in which the horizontal projections define principal $G_0$-bundles while the vertical projections provide affine bundles such that $\rho_C$ is an isomorphism on each fiber and, by definition, is $JG$-equivariant.

Now we can formulate the rule that to each principal connection in $P \times_{G_0} Q$ assigns its associated bundle map over $P \times_{G_0} Q$, namely:

$$\pi^*(CP) \rightarrow J(P \times_{G_0} Q),$$

$$(p, q, [w_p]) \mapsto J_{(p, q)}\rho_Q(w_p, 0)$$ (70)

To see that it is well defined, we have to check that, given any point $x \in M$, the result remains unchanged if we pick any $g_0 \in G_0$ to replace the representative $(p, q) \in (P \times Q)_x$ of $\gamma \in P \times G_0$ by another representative $(f, g_0, g^{-1} \cdot q)$ and the representative $w_p \in J_p P$ of $[w_p] \in C_x P$ by another representative $w'_{p, g_0}$: writing $R_{g_0}^P$ for right translation by $g_0$ in $P$ and $L_{g_0}^Q$ for left translation by $g_0^{-1}$ in $Q$, we have $w_{p, g_0} = T_{g_0}^P \circ w_p$ and get

$$J_{(p, g_0, g^{-1} \cdot q)}\rho_Q(w'_{p, g_0}, 0) = T_{(p, g_0, g^{-1} \cdot q)}\rho_Q \circ T_{p}^P \circ w_p, 0) = T_{(p, g_0, g^{-1} \cdot q)}\rho_Q \circ T_{(p, q)}(R_{g_0}^P \times L_{g_0}^Q) \circ (w_p, 0) = T_{(p, q)}\rho_Q \circ (w_p, 0) = J_{(p, q)}\rho_Q(w_p, 0).$$

Moreover, this bundle map is also $JG$-equivariant: this follows trivially from the definition of the action of $JG$ on the spaces involved and the $JG$-equivariance of $J\rho_Q$ that was proved above. And finally, we observe that this bundle map does capture the essence of passing from a principal connection to its associated connection, since if the former is given by a section $\Gamma^P : M \rightarrow CP$ and the latter by a section $\Gamma^P \times_{G_0} Q : P \times_{G_0} Q \rightarrow J(P \times_{G_0} Q)$, then $\Gamma^P \times_{G_0} Q$ is simply the push-forward of the section $\Gamma^P \circ \pi : P \times_{G_0} Q \rightarrow \pi^*(CP)$ with this bundle map. Note also that the prescription corresponds precisely to that given in equation (65) at the level of horizontal bundles.

### 3.3 Second order jet groupoids and induced actions

In this subsection, we apply the general procedure developed in Ref. 6 of “differentiating” actions of Lie groupoids on fiber bundles once more, namely, to the natural actions of the jet groupoid $JG$ of the gauge groupoid $G = (P \times P)/G_0$ on the jet bundle $JP$ and the connection bundle $CP$ of the principal bundle $P$ itself, to obtain natural induced actions\footnote{In this subsection, we often write $g = [p', p]$ for points in the gauge groupoid $G = (P \times P)/G_0$.}

$$\Phi_{J(JP)} : J(JG) \times_M J(JP) \rightarrow J(JP),$$

$$(u_{u_p}, u'_{u_p}) \mapsto u'_{u_p} \cdot u_{u_p}$$ (71)
and
\[
\Phi_{J(CP)} : J(J^2G) \times_M J(CP) \rightarrow J(CP)
\]
\[
(u'_g, u'[w_p]) \mapsto u'_g \cdot u'[w_p]
\]
(72)
derived from the actions \(\Phi_{JP}\) in equation (63) and \(\Phi_{CP}\) in equation (68) by applying the general formula in equation (10) of the previous section. Explicitly, we have
\[
u'_g \cdot u'_p = T_{(u_g, u_p)} \Phi_{JP} \circ (u'_g, u'_p) \circ \pi_{J(J^2G)}(u'_g)^{-1},
\]
(73)
and
\[
u'_g \cdot u'[w_p] = T_{(u_g, [w_p])} \Phi_{CP} \circ (u'_g, u'[w_p]) \circ \pi_{J(J^2G)}(u'_g)^{-1},
\]
(74)
respectively. These actions admit restrictions to several subgroupoids and subbundles, among which the following will become important to us at some point or another: the natural induced actions
\[
\Phi_{J^2P} : J^2G \times_M J^2P \rightarrow J^2P
\]
\[
(u'_g, u'_p) \mapsto u'_g \cdot u'_p
\]
(75)
of the semiholonomous second order jet groupoid \(J^2G\) of \(G\) and
\[
\Phi_{J^2P} : J^2G \times_M J^2P \rightarrow J^2P
\]
\[
(u'_g, u'_p) \mapsto u'_g \cdot u'_p
\]
(76)
of the second order jet groupoid \(J^2G\) of \(G\) on the semiholonomous second order jet bundle \(J^2P\) of \(P\), as well as the action
\[
\Phi_{J^2P} : J^2G \times_M J^2P \rightarrow J^2P
\]
\[
(u'_g, u'_p) \mapsto u'_g \cdot u'_p
\]
(77)
of the second order jet groupoid \(J^2G\) of \(G\) on the second order jet bundle \(J^2P\) of \(P\), all defined by the same formula,
\[
u'_g \cdot u'_p = T_{(u_g, u_p)} \Phi_{JP} \circ (u'_g, u'_p) \circ \pi_{J(J^2G)}(u'_g)^{-1},
\]
(78)
and similarly, the natural induced actions
\[
\Phi_{J(CP)} : J^2G \times_M J(CP) \rightarrow J(CP)
\]
\[
(u'_g, u'[w_p]) \mapsto u'_g \cdot u'[w_p]
\]
(79)
of the semiholonomous second order jet groupoid \(J^2G\) of \(G\) and
\[
\Phi_{J(CP)} : J^2G \times_M J(CP) \rightarrow J(CP)
\]
\[
(u'_g, u'[w_p]) \mapsto u'_g \cdot u'[w_p]
\]
(80)
of the second order jet groupoid \(J^2G\) of \(G\) on the jet bundle \(J(CP)\) of the connection bundle \(CP\) of \(P\), defined by
\[
u'_g \cdot u'[w_p] = T_{(u_g, [w_p])} \Phi_{CP} \circ (u'_g, u'[w_p]) \circ \pi_{J(J^2G)}(u'_g)^{-1}.
\]
(81)
As noted in the discussion preceding Proposition [2] in the previous section, the simplification in the last term on the rhs of equations (78) and (81), as compared to equations (73) and (74),
comes from the assumption that $u'_{u_g}$ is semiholonomous, and the definition of the actions in equations (75) and (77) relies on the fact that when $u'_{u_g}$ and $u'_{u_p}$ are both semiholonomous or both holonomous, then so is $u'_{u_g} \cdot u'_{u_p}$.

A more profound understanding of the situation can be obtained by extending the “magical square” for connection bundles in equation (69) to the corresponding jet bundles, considering the commutative diagram

$$
\begin{array}{ccc}
J(JP) & \xrightarrow{J\rho_C} & J(CP) \\
\downarrow \pi_{J(JP)} & & \downarrow \pi_{J(CP)} \\
JP & \xrightarrow{\rho_C} & CP \\
\downarrow \pi_{JP} & & \downarrow \pi_{CP} \\
P & \xrightarrow{\rho} & M
\end{array}
$$

and noting that $J\rho_C$, although no longer an isomorphism on each fiber (it is still onto but has a kernel), is $J(JG)$-equivariant. Even more importantly, by restricting to the semiholonomous second order jet bundle of $P$, we arrive at a “magical square” for jet bundles of connection bundles, i.e., the commutative diagram

$$
\begin{array}{ccc}
\bar{J}^2P & \xrightarrow{\bar{J}\rho_C} & J(CP) \\
\downarrow \pi_{\bar{J}^2P} & & \downarrow \pi_{J(CP)} \\
JP & \xrightarrow{\rho_C} & CP \\
\downarrow \pi_{JP} & & \downarrow \pi_{CP} \\
P & \xrightarrow{\rho} & M
\end{array}
$$

in which all three horizontal projections define principal $G_0$-bundles while the vertical projections provide affine bundles such that $\rho_C$ and $J\rho_C$ are both isomorphisms on each fiber, $\rho_C$ is $JG$-equivariant and $J\rho_C$ is $\bar{J}^2G$-equivariant.

To prove these statements, let us pick a point $p \in P$ with $\rho(p) = x$ and a jet $w_p \in J_p P$ and take tangent maps to the commutative diagram in equation (69) to obtain the commutative diagram

$$
\begin{array}{ccc}
T_{w_p}(JP) & \xrightarrow{T_{w_p}\rho_C} & T_{[w_p]}(CP) \\
\downarrow T_{w_p}\pi_{JP} & & \downarrow T_{[w_p]}\pi_{CP} \\
T_pP & \xrightarrow{T_{p\rho}} & T_xM
\end{array}
$$

Since $\rho_C$ is a submersion and hence its tangent maps are surjective, this means that the tangent spaces $T_{[w_p]}(CP)$ of the orbit space $CP$ can be realized as quotient spaces, namely, the linear maps

$$
T_{w_p}\rho_C : T_{w_p}(JP) \rightarrow T_{[w_p]}(CP)
$$

induce isomorphisms

$$
T_{[w_p]}(CP) \cong T_{w_p}(JP)/\ker T_{w_p}\rho_C,
$$
and this leads to an analogous realization of the jet spaces $J_{[w_p]}(CP)$ of the orbit space $CP$ as quotient spaces, namely, the affine maps

$$J_{[w_p]} \rho_C : J_{[w_p]}(JP) \longrightarrow J_{[w_p]}(CP) \quad (87)$$

defined by

$$J_{[w_p]} \rho_C(u'_{w_p}) = T_{w_p} \rho_C \circ u'_{w_p} \quad (88)$$

induce isomorphisms

$$J_{[w_p]}(CP) \cong J_{[w_p]}(JP) / L(T_p \rho_C) \quad (89)$$

Now using the $JG$-equivariance of $\rho_C$, which means that $\Phi_{GP} \circ (id_{JG} \times M \rho_C) = \rho_C \circ \Phi_{JP}$, we can prove the $J(JG)$-equivariance of $J\rho_C$. To this end, let us also pick a point $g = [p', p] \in G$ and a jet $u_g \in J_g G$, together with iterated jets $u'_{u_g} \in J_{u_g} (JG)$ and $u''_{w_p} \in J_{w_p} (JP)$, and calculate

$$u'_{u_g} \cdot J_{w_p} \rho_C(u'_{w_p})$$

$$= T_{(u_g, [w_p])} \Phi_{GP} \circ (u'_{u_g}, T_{w_p} \rho_C \circ u'_{w_p}) \circ \pi_{fr}^{J(JG)}(u'_{u_g})^{-1}$$

$$= T_{(u_g, [w_p])} \Phi_{GP} \circ T_{(u_g, w_p)} (id_{JG} \times M \rho_C) \circ (u'_{u_g}, u'_{w_p}) \circ \pi_{fr}^{J(JG)}(u'_{u_g})^{-1}$$

$$= T_{w_p} \rho_C \circ T_{(u_g, w_p)} \Phi_{JP} \circ (u'_{u_g}, u'_{w_p}) \circ \pi_{fr}^{J(JG)}(u'_{u_g})^{-1}$$

$$= J_{w_p} \rho_C(u'_{u_g} \cdot u'_{w_p}).$$

But here we can actually do better if we replace iterated jets by semiholonomous second order jets because that will eliminate the need of passing to a quotient and convert the commutative diagram in equation [82] to the one in equation [83]. To show this, we first note that, as before,

$$\ker T_{w_p} \rho_C = \{(X_0)_{JP}(w_p) \mid X_0 \in g_0 \} \cong g_0 \cong V_p P, \quad (90)$$

where $(X_0)_{JP}$ denotes the fundamental vector field on $JP$ associated to a generator $X_0 \in g_0$ via the pertinent action of $G_0$, defined by the appropriate analogue of equation [51] above. Here, we shall need a more explicit form of this isomorphism between the spaces $\ker T_{w_p} \rho_C$ and $V_p P$: it is simply the restriction

$$T_{w_p} \pi_{JP} : \ker T_{w_p} \rho_C \cong V_p P \quad (91)$$

of the linear map

$$T_{w_p} \pi_{JP} : T_{w_p}(JP) \longrightarrow T_p P \quad (92)$$

that appears in the definition of semiholonomous second order jets. (Indeed, the right action of $G_0$ on $JP$ being induced from that on $P$, the tangent map $T_{w_p} \pi_{JP}$ will of course take any fundamental vector field $(X_0)_{JP}$ at $w_p$ to the corresponding fundamental vector field $(X_0)_P$ at $p$.) This in turn implies that the restriction of the (affine) map in equation [87] to the (affine) subspace $J^2_{w_p} P$ of the (affine) space $J_{w_p}(JP)$ will establish an isomorphism

$$J_{w_p} \rho_C : J^2_{w_p} P \cong J_{[w_p]}(CP) \quad (93)$$

so we can replace equation [89] by the much simpler equation

$$J_{[w_p]}(CP) \cong J^2_{w_p} P. \quad (94)$$
To prove this statement, we have to show that the affine map in equation (87), when restricted to the affine subspace $J^2_w P$, (a) becomes injective and (b) remains surjective. For (a), assume we are given two semiholonomic second order jets $u^1_{wp}, u^2_{wp} \in J^2_w P$ which under $J_{wp} \rho_C$ have the same image; then their difference is a linear map from $T_x M$ to $T_w (JP)$ satisfying two conditions, namely that its composition with $T_w \rho_C$ is zero, so it takes value in $\ker T_w \rho_C$, and that its composition with $T_{wp} \pi J P$ is also zero, since $u^1_{wp}$ and $u^2_{wp}$ are both semiholonomic. But this implies that it must itself be zero since according to equation (91), $T_{wp} \pi J P$ is injective on $\ker T_w \rho_C$. For (b), assume we are given a general iterated jet $u'_{wp} \in J_{wp} (JP)$ and consider the difference $T_{wp} \pi J P \circ u'_{wp} - w_p$, which is a linear map from $T_x M$ to $V_p P$, so that according to equation (91), there is a unique linear map $\tilde{u}'_{wp}$ from $T_x M$ to $\ker T_{wp} \rho_C \subset T_{wp} (JP)$ satisfying $T_{wp} \pi J P \circ \tilde{u}'_{wp} = T_{wp} \pi J P \circ u'_{wp}$. But this implies that the difference $u'_{wp} = u'_{wp} - \tilde{u}'_{wp}$ is a semiholonomic second order jet, $u'_{wp} \in J^2_w P$, which under $J_{wp} \rho_C$ has the same image as the original iterated jet $u'_{wp} \in J_{wp} (JP)$.

4 Minimal coupling and Utiyama’s theorem II

In the context of the formalism adopted in the previous section, the minimal coupling prescription and the curvature map can be viewed as stemming from bundle maps

$$D : CP \times_M J(P \times \mathfrak{g}_0 Q) \rightarrow \bar{J}(P \times \mathfrak{g}_0 Q),$$

and

$$F : J(CP) \rightarrow \bigwedge^2 T^* M \otimes (P \times \mathfrak{g}_0),$$

over $M$, which have already appeared in Ref. [10] (see the diagrams in equations (52) and (57) there). What we want to show here is that, and in precisely what sense, these bundle maps are equivariant under the action not only of the pertinent Lie group bundles but also of the pertinent Lie groupoids. To this end, it turns out to be convenient to “lift” all bundles to the space appearing in the upper left hand corner of the appropriate “magical square”, that is, the space $P \times Q$ in the first case (see equation (45)) and the space $JP$ in the second case (see equation (93)), where these bundle maps take a much simpler form.

4.1 Minimal coupling

To deal with the minimal coupling prescription, we observe that the bundle map $D$ in equation (87) fits into the following commutative diagram

$$
(JP \times Q) \times_{P \times Q} J(P \times Q) \xrightarrow{D} \bar{J}(P \times Q)

\downarrow_{(\rho_C \circ \pi_1, J \rho_Q)}

CP \times_M J(P \times \mathfrak{g}_0 Q) \xrightarrow{D} \bar{J}(P \times \mathfrak{g}_0 Q)
$$

where the bundles in the top row are over $P \times Q$ while those in the bottom row are over $M$. (Here, we have identified the pull-back of $JP$ by the projection from $P \times Q$ to $P$ with the
cartesian product $JP \times Q$.) In fact, it is convenient to expand this to a commutative diagram

\[
\begin{array}{ccc}
(JP \times Q) \times_{P \times Q} J(P \times Q) & \xrightarrow{D} & \tilde{J}(P \times Q) \\
\pi^*(CP) \times_{P \times G_0} J(P \times G_0 \times Q) & \xrightarrow{D} & \tilde{J}(P \times G_0 \times Q) \\
CP \times_M J(P \times G_0 \times Q) & \xrightarrow{D} & \tilde{J}(P \times G_0 \times Q)
\end{array}
\]

(98)

where the bundles in the middle row are over the quotient space $P \times G_0 \times Q$, i.e., the total space of the corresponding associated bundle. (Here, we omit the labels on the vertical maps, which are either the same as in the previous diagram or else are obvious.) Then the bundle map $D$ in the middle row is the composition of the difference map already introduced at the beginning of this paper (see equation (3)) and the canonical bundle map of equation (70) in the first factor, up to a sign that can be taken care of by switching the two factors. Continuing to use the same notation as in Section 3.1, we see that this corresponds to the bundle map $D$ in the bottom row being given in terms of that in the top row according to

\[
D_x([w_p], J_{(p,q)}\rho_Q(u_p, u_q)) = \tilde{J}_{(p,q)}\rho_Q\big(D_{(p,q)}(w_p, (u_p, u_q))\big),
\]

(99)

whereas the latter is simply defined by

\[
D_{(p,q)}(w_p, (u_p, u_q)) = (u_p - w_p, u_q).
\]

(100)

(This follows from equations (99)–(102) together with the same equations with $J$ replaced by $\tilde{J}$.) To show that $D_x$ is well defined, note first that if we replace the point $p$ in $P$ by another point in $P$ in the same fiber over $x$, which is of the form $p \cdot g_0$ for some (unique) $g_0 \in G$, then we must replace $w_p$ by $w_p \cdot g_0 = T_pr_{g_0} \circ w_p$ and similarly $u_p$ by $u_p \cdot g_0 = T_pr_{g_0} \circ u_p$, as well as $u_q$ by $u_q \cdot g_0 = T_qL_{g_0^{-1}} \circ u_q$, so as to guarantee that $J_{(p,q)}\rho_Q(u_p, u_q)$ remains unaltered:

\[
\begin{align*}
J_{(p_0, g_0)}(u_p \cdot g_0, u_q \cdot g_0) &= T_{(p_0, g_0)}(R_{g_0} \times L_{g_0^{-1}}) \circ (u_p, u_q) \\
&= T_{(p_0)}(\rho_Q \circ (R_{g_0} \times L_{g_0^{-1}})) \circ (u_p, u_q) = T_{(p_0)}(\rho_Q \circ (u_p, u_q) \\
&= J_{(p_0)}(\rho_Q(u_p, u_q)).
\end{align*}
\]

But then $\tilde{J}_{(p_0)}(u_p - w_p, u_q)$ will remain unaltered as well:

\[
\begin{align*}
\tilde{J}_{(p_0, g_0)}(u_p - w_p, u_q) &= T_{(p_0, g_0)}(R_{g_0} \times L_{g_0^{-1}}) \circ (u_p - w_p, u_q) \\
&= T_{(p_0)}(\rho_Q \circ (R_{g_0} \times L_{g_0^{-1}})) \circ (u_p - w_p, u_q) = T_{(p_0)}(\rho_Q \circ (u_p - w_p, u_q) \\
&= \tilde{J}_{(p_0)}(u_p - w_p, u_q).
\end{align*}
\]

Moreover, even if we leave $p$ fixed, we may still modify the second component in the argument of $D_{(p,q)}$, i.e., the pair $(u_p, u_q) \in J_p P \times L(T_x M, T_q Q)$, without changing its image under $J_{(p,q)}\rho_Q$. 

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namely, by adding a pair \((\vec{u}_p, \vec{u}_q)\) \(\in L(T_x M, \text{ker} T_{(p,q)}^\rho Q)\). But then since \(w_p \in J_p P\) remains unaltered, the expression \((u_p - w_p, u_q) \in \tilde{J}_p P \times L(T_x M, T_q Q)\) will be modified in the same way and, in particular, without changing its image under \(\tilde{J}_{(p,q)}^\rho Q\).

Now we are ready to formulate the first main theorem in this paper, which extends the left part of the commutative diagram in equation (52) of Ref. [10], as follows.

**Theorem 1**  The minimal coupling map \(D\) in equation (95) is equivariant under the actions of the pertinent Lie groupoids, i.e., the diagram

\[
\begin{array}{ccc}
JG \times M (CP \times M J(P \times G_0 Q)) & \longrightarrow & CP \times M J(P \times G_0 Q) \\
\downarrow (\pi^*_J G \times \pi_J G) \times_M D & & \downarrow D \\
(GL(TM) \times_M G) \times_M \tilde{J}(P \times G_0 Q) & \longrightarrow & \tilde{J}(P \times G_0 Q)
\end{array}
\]

(101)

commutes.

**Proof:** This follows immediately from equivariance of \(J \rho Q\) under \(JG\) (which as we have seen implies equivariance of the canonical bundle map in equation (70) under \(JG\)) and equivariance of \(\tilde{J} \rho Q\) under \(GL(TM) \times_M G\) (which can be shown in precisely the same way), in combination with Proposition 11 to prove that the bundle maps \(D\) in the top and middle rows of the diagram in equation (98) are equivariant in the same sense, the former obviously being equivariant under the right action of \(G_0\) as well.

To complete the discussion, let us specify in what sense the map \(D\) in equation (95) captures the essence of the minimal coupling prescription. Abbreviating \(P \times G_0 Q\) to \(E\), assume that \(\Gamma : M \longrightarrow CP\) is a section of \(CP\) representing a principal connection in \(P\), \(\Gamma^E : E \longrightarrow JE\) is the section of \(JE\) (as a bundle over \(E\)) representing the resulting associated connection in \(E\), obtained by push-forward with the canonical bundle map in equation (70), \(\varphi : M \longrightarrow E\) is a section of \(E\) and \(\partial \varphi : M \longrightarrow JE\) is its derivative (also denoted by \(j \varphi\) and called its jet prolongation); then \(D \circ (\Gamma, \partial \varphi) : M \longrightarrow \tilde{J}E\) is indeed the covariant derivative of \(\varphi\) with respect to that connection, because it is elementary to see that equation (99) combined with equation (100) will boil down to the formula in equation (2).

### 4.2 Utiyama’s theorem

To deal with the curvature map, we observe that the bundle map \(F\) of equation (96) fits into the following commutative diagram

\[
\begin{array}{ccc}
\tilde{J}^2 P & \longrightarrow & \pi^*_J P \left( \rho^* (\wedge^2 T^* M) \otimes VP \right) \\
\downarrow \quad J \rho C & & \downarrow \quad J (CP) \\
J (CP) & \longrightarrow & \wedge^2 T^* M \otimes (P \times G_0 \mathfrak{g}_0)
\end{array}
\]

(102)

where the bundles in the upper row are over \(JP\) while those in the lower row are over \(M\). (Here, we have identified the vertical bundle \(VP\) of \(P\) with the trivial vector bundle \(P \times \mathfrak{g}_0\) over \(P\).
then the second tensor factor in the vertical map on the rhs of this diagram is just the map \( \rho_{g_0} \) in the “magical square” of equation (45) for the adjoint bundle \( P \times G_0 \mathfrak{g}_0 \), pulled back to \( JP \).

Again, it is convenient to expand this to a commutative diagram

\[
\begin{array}{ccc}
J^2P & \xrightarrow{F} & \pi_J^* \left( \rho^* (\Lambda^2 T^* M) \otimes VP \right) \\
\downarrow & & \downarrow \\
J^2P & \xrightarrow{F} & \rho^* (\Lambda^2 T^* M) \otimes VP \\
\downarrow & & \downarrow \\
J(CP) & \xrightarrow{F} & \Lambda^2 T^* M \otimes (P \times G_0 \mathfrak{g}_0)
\end{array}
\]

where the bundles in the middle row are over the total space \( P \) of the principal bundle. (And again, we omit the labels on the vertical maps, which are either the same as in the previous diagram or else are obvious.) Then the bundle map \( F \) in the middle row is the alternator or antisymmetrizer already introduced at the beginning of this paper (see equation (5)). Continuing to use the same notation as in Section 3.2, we see that this corresponds to the bundle map \( F \) in the bottom row being given in terms of that in the top row according to

\[
F_x(J_{w_p} \rho_C(u_{w_p}))(v_1, v_2) = \rho_{g_0}(F_{w_p}(u_{w_p})(v_1, v_2)),
\]

for \( v_1, v_2 \in T_xM \), whereas the latter, as we recall from Section 2, is explicitly defined as follows: given a semiholonomous second order jet \( u_{w_p}^0 \in J^2_{w_p}P \), we arbitrarily choose some holonomous second order jet \( u_{w_p}^0 \in J^2_{w_p}P \) (this choice will ultimately drop out under the antisymmetrization) to form the difference \( u_{w_p}^0 - u_{w_p}^0 \), which is a linear map from \( T_xM \) to the vertical space \( V_{w_p}^k(JP) \) of \( JP \) with respect to the jet target projection \( \pi_{JP} \); then we can apply the canonical isomorphism

\[
V_{w_p}^k(JP) = \ker T_{w_p} \pi_{JP} = T_{w_p}(J_{w_p}P) \cong J_{w_p}P = L(T_xM, V_{w_p}P)
\]

(105)

to identify it with a linear map from \( T_xM \) to \( L(T_xM, V_{w_p}P) \), that is, with an element of \( L^2(T_xM, V_{w_p}P) \), and obtain \( F_{w_p}(u_{w_p}^0) \in L^2_{g_0}(T_xM, V_{w_p}P) \) by antisymmetrizing in the usual sense.

The last step then consists in applying the additional canonical isomorphism

\[
V_{w_p}P \cong \mathfrak{g}_0.
\]

(106)

To show that \( F_x \) is well defined, note that if we replace the point \( p \) in \( P \) by another point in \( P \) in the same fiber over \( x \), which is of the form \( p \cdot g_0 \) for some (unique) \( g_0 \in G \), then we must replace \( w_p \) by \( w_p \cdot g_0 = T_p R_{g_0}^P \circ w_p \); \( u_{w_p}^0 \) by \( u_{w_p}^0 \cdot g_0 = T_w R_{g_0}^J \circ u_{w_p}^0 \) and similarly \( u_{w_p}^0 \) by \( u_{w_p}^0 \cdot g_0 = T_w R_{g_0}^J \circ u_{w_p}^0 \), where \( R_{g_0}^P \) and \( R_{g_0}^J \) denote right translation by \( g_0 \) in \( P \) and in \( JP \), respectively, so as to guarantee that \( J_{w_p} \rho_{C}(u_{w_p}^0) \) remains unaltered:

\[
J_{w_p} g_0 \rho_{C}(u_{w_p}^0) = T_{w_p} g_0 \rho_{C} \circ T_{w_p} R_{g_0}^{JP} \circ u_{w_p}^0 = T_{w_p} (\rho_{C} \circ R_{g_0}^{JP}) \circ u_{w_p}^0 = T_{w_p} \rho_{C} \circ u_{w_p}^0 = J_{w_p} \rho_{C}(u_{w_p}^0).
\]

But then

\[
u_{w_p}^0 \cdot g_0 - u_{w_p}^0 \cdot g_0 = T_{w_p} R_{g_0}^{JP} \circ u_{w_p}^0 - T_{w_p} R_{g_0}^{JP} \circ u_{w_p}^0,
\]

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so that applying the isomorphism in equation (105), we get
\[(u'_w - a'_w) \cdot g_0 = T_g R_{g_0} \circ (u'_w - a'_w),\]
and applying the additional isomorphism in equation (106), we get
\[(u'_w - a'_w) \cdot g_0 = \text{Ad}(g_0^{-1}) \circ (u'_w - a'_w),\]
implying that
\[[p \cdot g_0, (u'_w - a'_w) \cdot g_0] = [p, u'_w - a'_w].\]
(To justify this conclusion, note that the linear isomorphism \(T_w R_{g_0} : T_w (JP) \to T_{w'g_0} (JP)\), when restricted to the vertical space of \(JP\) with respect to the jet target projection \(\pi_{JP}\), reduces to the tangent map \(T_w R_{g_0} : T_w (JP) \to T_{w'g_0} (JP)\) to the restricted right translation \(R_{g_0} : J_p P \to J_{p'g_0} P\) by \(g_0\). But this is an affine map between affine spaces, so under the isomorphism in equation (105), its tangent map at each point becomes the corresponding difference map, which is a linear map \(R_{g_0} : J_p P \to J_{p'g_0} P\), and that is just composition with \(T_p R_{g_0} : V_p JP \to V_{p'g_0} P\). Finally, it is well known that under the isomorphism in equation (106), this becomes \(\text{Ad}(g_0^{-1}) : g_0 \to g_0\).

Now we are ready to formulate the second main theorem in this paper, which extends the left part of the commutative diagram in equation (57) of Ref. [10], as follows.

**Theorem 2**  The curvature map \(F\) in equation (106) is equivariant under the actions of the pertinent Lie groupoids, i.e., the diagram
\[
\begin{array}{ccc}
J^2G \times_M J(CP) & \xrightarrow{(\pi_{jG} \times \pi_{jG}) \circ \pi_{j2G} \circ F} & J(CP) \\
(GL(TM) \times_M G) \times_M (\Lambda^2 T^*M \otimes (P \times G_0 g_0)) & \xrightarrow{F} & \Lambda^2 T^*M \otimes (P \times G_0 g_0)
\end{array}
\]
commutes.

**Proof:**  This follows immediately from Proposition 2 together with the fact that, as shown in Section 3.1, the canonical isomorphism \(VP \cong P \times g_0\) and the projection \(\rho_{g_0} : P \times g_0 \to P \times G_0 g_0\) are both \(G\)-equivariant.

To complete the discussion, let us specify in what sense the map \(F\) in equation (106) captures the essence of the prescription for defining the curvature of a principal connection. Assume that \(\Gamma : M \to CP\) is a section of \(CP\) representing a principal connection in \(P\) and \(\partial \Gamma : M \to J(CP)\) is its derivative (also denoted by \(\partial \Gamma\) and called its jet prolongation); then \(F \circ \partial \Gamma : M \to \Lambda^2 T^*M \otimes (P \times G_0 g_0)\) is a 2-form on \(M\) with values in the adjoint bundle \(P \times G_0 g_0\) which is precisely the curvature form of that connection, because it is elementary to see that equation (101) will boil down to the formula in equation (89).
5 Conclusions and Outlook

The equivariance statements formulated in the two theorems in this paper are very general, in that this equivariance holds for the full jet groupoid $JG$ of the gauge groupoid $G$, in the case of Theorem 1, and for the full second order jet groupoid $J^2G$ of the gauge groupoid $G$, in the case of Theorem 2. But this does of course not mean that a concrete field theoretical model will have such a huge amount of symmetry – quite to the contrary! Any such model will be subject to restrictions on what are its allowed symmetries coming from the dynamics, which is governed, say, by its Lagrangian: such a Lagrangian will typically be invariant not under the pertinent jet groupoid but rather only under a certain Lie subgroupoid thereof. The generic situation here, which prevails for all standard Lagrangians in gauge theories, is that when $M$ comes equipped with some metric $g$, this Lie subgroupoid will be the inverse image of the corresponding orthonormal frame groupoid $O(TM,g) \subset GL(TM)$ under the “frame” projection from the pertinent jet groupoid to the linear frame groupoid $GL(TM)$ of $M$. Thus what the two theorems in the previous section really prove is that there are no other restrictions, so this is in fact the correct Lie groupoid for hosting the symmetries of any such theory, and remarkably, it is large enough to accommodate not only its gauge symmetries but also its space-time symmetries, including isometries as well as orthonormal frame transformations, unifying them all within a single mathematical object. Finally, the formalism can also be adapted to handle symmetry breaking, as has been discussed in Ref. [16] (even though only at the level of Lie group bundles and not of full Lie groupoids, which is however enough to deal with that subject).

With this picture in mind, we hope to have demonstrated, in the two papers of this series, that Lie groupoids provide a much wider and more flexible mathematical framework than Lie groups for describing symmetries in physics, and in some cases such as that of gauge theories, we would venture to say they provide the “right” one. What remains to be seen is how this approach will evolve when one tries to extend it from classical to quantum field theories.

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Appendix: Jet prolongations and gauge groupoids

Our goal in this appendix is to prove a fact which is not used directly in the main text (and that is why it has been relegated to an appendix) but provides important additional insight into the way how Lie groupoid theory is applied to gauge theories and has actually played a rather important role in the development of the ideas underlying our work. Briefly, the statement is
that, (a) passing from a principal bundle first to its jet prolongation and then to the gauge
groupoid of that, or (b) passing from a principal bundle first to its gauge groupoid and then to
the jet groupoid of that, gives the same result, up to a canonical isomorphism; we may abbreviate
this by saying that the processes of building gauge groupoids and of taking jet prolongations
commute, provided the latter are interpreted correctly, each one in its category. To show this,
we must first explain the concept of jet prolongation of a principal bundle.

Jet prolongations of principal bundles and associated bundles

The main obstacle against an entirely trivial compatibilization between the jet functor and the
passage from principal bundles to associated bundles resides in the fact that, although the (first
order) jet bundle of a fiber bundle is again a fiber bundle, the (first order) jet bundle $JP$ of a
principal bundle $P$ is, by itself, not a principal bundle. However, there is a simple way to remedy
this defect, namely by taking the fiber product with the linear frame bundle $Fr(M, GL(n, R))$ of
the base manifold. Indeed, it follows from the general constructions presented in [17, Chapter 4]
that if $P$ is a principal bundle over $M$ with structure group $G_0$, then

$$P^{(1)} = Fr(M, GL(n, R)) \times_M JP$$

is again a principal bundle over $M$, called the (first order) jet prolongation of $P$, with structure
group

$$G^{(1)}_0 = (GL(n, R) \times G_0) \ltimes L(R^n, g_0)$$

called the (first order) jet group of $G_0$: this is simply the semidirect product of the direct product
$GL(n, R) \times G_0$ with the vector space $L(R^n, g_0)$ of linear maps from $R^n$ to the Lie algebra $g_0$,
which in this context is viewed as an Abelian Lie group, where the semidirect product is taken
with respect to the natural (left) action

$$(GL(n, R) \times G_0) \times L(R^n, g_0) \rightarrow L(R^n, g_0)$$

$$((a_0, g_0), \xi_0) \mapsto (a_0, g_0) \cdot \xi_0$$

given by

$$(a_0, g_0) \cdot \xi_0 = Ad(g_0) \circ \xi_0 \circ a_0^{-1},$$

so the product in $G^{(1)}_0$ is explicitly given by

$$(a_{0,1}^0, g_{0,1}^0; \xi_{0,1}^0)(a_{0,2}^0, g_{0,2}^0; \xi_{0,2}^0) = (a_{0,1}^0 a_{0,2}^0, g_{0,1}^0 g_{0,2}^0; \xi_{0,1}^0 + (a_{0,1}^0, g_{0,1}^0) \cdot \xi_{0,2}^0).$$

To write an explicit formula for the (right) action of $G^{(1)}_0$ on $P^{(1)}$, we introduce the following
notation: given any point $p$ of $P$, the isomorphism from the Lie algebra $g_0$ onto the vertical
space $V_p P$ given by associating to every $X_0$ in $g_0$ the value of the corresponding fundamental

---

Our notation for the linear frame bundle of a manifold may at first sight look a bit clumsy, but it pays off by
becoming almost self-evident when we consider $G$-structures, which are principal subbundles of the linear frame
bundle with structure groups that are closed subgroups $G$ of $GL(n, R)$ and can with this notation simply be
denoted by $Fr(M, G)$: a typical example would be the orthonormal frame bundle $Fr(M, O(n))$ induced by some
Riemannian metric. (We apologize for the momentary change of meaning of the symbol $G$ in this footnote).
vector field \((X_0)_p\) at \(p\) (see equation (51)) is, for any finite-dimensional real vector space \(W\), extended to an isomorphism

\[
L(W; g_0) \cong W^* \otimes g_0 \quad \xrightarrow{\xi_0} \quad W^* \otimes V_p P \cong L(W; V_p P)
\]

simply by taking the tensor product with the identity on \(W^*\) (i.e., \((\xi_0)_p(p)(w) \equiv (\xi_0(w))_p(p)\)). Note that \(G_0\)-equivariance of fundamental vector fields implies that, denoting right translation by elements \(g_0\) of \(G_0\) on \(P\) as well as on \(TP\) by \(R_{g_0}\) (so that \(R_{g_0}: T_p P \rightarrow T_{g_0 \cdot p} P\) is the derivative at \(p\) of \(R_{g_0}: P \rightarrow P\)), we have

\[
R_{g_0} \circ (\xi_0)_p(p) = (\text{Ad}(g_0)^{-1} \circ \xi_0)_p(p \cdot g_0).
\]

Similarly, if we are given a (left) action \(G_0 \times Q \rightarrow Q\) of \(G_0\) on some manifold \(Q\), then for any point \(q\) of \(Q\), we consider the linear map from the Lie algebra \(g_0\) into the tangent space \(T_q Q\) given by associating to every \(X_0\) in \(g_0\) the value of the corresponding fundamental vector field \((X_0)_Q\) at \(q\) (see equation (52)) and, for any finite-dimensional real vector space \(W\), extend it to a linear map

\[
L(W; g_0) \cong W^* \otimes g_0 \quad \xrightarrow{\xi_0} \quad W^* \otimes T_q Q \cong L(W; T_q Q)
\]

simply by taking the tensor product with the identity on \(W^*\) (i.e., \((\xi_0)_Q(q)(w) \equiv (\xi_0(w))_Q(q)\)). Again, \(G_0\)-equivariance of fundamental vector fields implies that, denoting left translation by elements \(g_0\) of \(G_0\) on \(Q\) as well as on \(TQ\) by \(L_{g_0}\) (so that \(L_{g_0}: T_q Q \rightarrow T_{g_0 \cdot q} Q\) is the derivative at \(q\) of \(L_{g_0}: Q \rightarrow Q\)), we have

\[
L_{g_0} \circ (\xi_0)_Q(q) = (\text{Ad}(g_0) \circ \xi_0)_Q(g_0 \cdot q).
\]

Then for \(a_x \in \text{Fr}_x(M, GL(n, \mathbb{R})) = GL(\mathbb{R}^n, T_x M), u_p \in J_p P \subset L(T_x M, T_p P), a_0 \in GL(n, \mathbb{R}), g_0 \in G_0\) and \(\xi_0 \in L(\mathbb{R}^n, g_0)\),

\[
(a_x, u_p) \cdot (a_0, g_0; \xi_0) = (a_x \circ a_0, R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})_p(p))).
\]
Let us check explicitly that this formula does define a (right) action:

\[
((a_x, u_p) \cdot (a_{0,1}, g_{0,1}; \xi_{0,1})) \cdot (a_{0,2}, g_{0,2}; \xi_{0,2}) = (a_x \cdot a_{0,1}, R_{g_{0,1}} \cdot (u_p + (\xi_{0,1} \cdot a_{0,1}^{-1}) p(p))) \cdot (a_{0,2}, g_{0,2}; \xi_{0,2})
\]

\[
= ((a_x \cdot a_{0,1}) \cdot a_{0,2}, R_{g_{0,2}} \cdot R_{g_{0,1}} \cdot (u_p + (\xi_{0,1} \cdot a_{0,1}^{-1}) p(p) + (\text{Ad}(g_{0,1}) \circ \xi_{0,2} \circ a_{0,1}^{-1} \circ a_{x}^{-1}) p(p)))
\]

\[
= (a_x \cdot a_{0,1}, R_{g_{0,1}g_{0,2}} \cdot ((u_p + ((\xi_{0,1} + (a_{0,1} \cdot g_{0,1}) \cdot \xi_{0,2}) \circ a_{x}^{-1}) p(p)))
\]

\[
= (a_x, u_p) \cdot (a_{0,1}a_{0,2}, g_{0,1}g_{0,2}; \xi_{0,1} + (a_{0,1}, g_{0,1}) \cdot \xi_{0,2})
\]

\[
= (a_x, u_p) \cdot ((a_{0,1}, g_{0,1}; \xi_{0,1})(a_{0,2}, g_{0,2}; \xi_{0,2})).
\]

Further evidence that, in the case of principal bundles, the jet prolongation in this sense – rather than just the usual jet bundle – is the correct object to consider can be accumulated by noting that (a) the tangent bundle (of the total space), the jet bundle and the linearized jet bundle of an associated bundle for \(P\) are all associated bundles for \(P^{(1)}\) and (b) the connection bundle \(CP\) of \(P\) is also an associated bundle for \(P^{(1)}\), i.e., there are canonical bundle isomorphisms

\[
T(P \times G_0 Q) \cong P^{(1)} \times_{G_0} (\mathbb{R}^n \times TQ),
\]

(116)

\[
J(P \times G_0 Q) \cong P^{(1)} \times_{G_0} L(\mathbb{R}^n, TQ),
\]

(117)

\[
\tilde{J}(P \times G_0 Q) \cong P^{(1)} \times_{G_0} L(\mathbb{R}^n, TQ),
\]

(118)

and

\[
CP \cong P^{(1)} \times_{G_0} L(\mathbb{R}^n, g_0),
\]

(119)

which preserve any invariant additional structures if such are present (such as, for example, that of a vector bundle over \(P\) in the first and third case or that of an affine bundle over \(P\) in the second and fourth case). Here, the relevant (left) actions of the structure group to be employed in the definition of the associated bundles on the rhs of these equations are

\[
G_0^{(1)} \times (\mathbb{R}^n \times TQ) \rightarrow (\mathbb{R}^n \times TQ)
\]

((a_0, g_0; \xi_0), (v, v_q)) \rightarrow (a_0, g_0; \xi_0) \cdot (v, v_q)
\]

(120)

with

\[
(a_0, g_0; \xi_0) \cdot (v, v_q) = (a_0 v, L_{g_0}(v_q) - (\xi_0(a_0 v)) q(g_0 \cdot q))
\]

(121)

in the first case,

\[
G_0^{(1)} \times L(\mathbb{R}^n, TQ) \rightarrow L(\mathbb{R}^n, TQ)
\]

((a_0, g_0; \xi_0), u_q) \rightarrow (a_0, g_0; \xi_0) \cdot u_q
\]

(122)
with 
\[(a_0, g_0; \xi_0) \cdot u_q = L_{g_0} \circ u_q \circ a_0^{-1} - (\xi_0)_Q(g_0 \cdot q)\]  
(123)
in the second case, 
\[G_{0}^{(1)} \times L(\mathbb{R}^n, TQ) \rightarrow L(\mathbb{R}^n, TQ)\]  
(124)
with 
\[(a_0, g_0; \xi_0) \cdot \tilde{u}_q = L_{g_0} \circ \tilde{u}_q \circ a_0^{-1}\]  
(125)
in the third case, and 
\[G_{0}^{(1)} \times L(\mathbb{R}^n, g_0) \rightarrow L(\mathbb{R}^n, g_0)\]  
(126)
with 
\[(a_0, g_0; \xi_0) \cdot A_0 = \text{Ad}(g_0) \circ A_0 \circ a_0^{-1} + \xi_0\]  
(127)
in the last case. (That these formulas do indeed define group actions follows by elementary calculations, which we leave to the reader, using equations (111)-(114).

In order to explicitly construct the isomorphisms in equations (116)-(119), we resort to the “magical square” for associated bundles, i.e., the commutative diagram in equation (69), to handle the last case. More specifically, for the first case, consider the map 
\[P^{(1)} \times (\mathbb{R}^n \times TQ) \rightarrow TP \times TQ\]  
(128)
and observe that it takes 
\[(a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0, g_0; \xi_0)^{-1} \cdot (v, v_q))\]  
\[= ((a_x, u_p) \cdot (a_0, g_0; \xi_0), a_0^{-1} g_0^{-1} - \text{Ad}(g_0)^{-1} \circ \xi_0 \circ a_0) \cdot (v, v_q))\]  
(129)
between the quotient spaces which is the desired isomorphism. Similarly, for the second case, consider the map

\[
P^{(1)} \times L(\mathbb{R}^n, TQ) \rightarrow J P \times L(TM, TQ)
\]

\[
((a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0, g_0; \xi_0)^{-1} \cdot u_q) \mapsto (u_p, u_q \circ a_x^{-1})
\]

and observe that it takes

\[
((a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0, g_0; \xi_0)^{-1} \cdot u_q)
\]

\[
= ((a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0^{-1}, g_0^{-1}; -\text{Ad}(g_0)^{-1} \circ \xi_0 \circ a_0) \cdot u_q)
\]

\[
= ((a_x \circ a_0 \cdot R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})p(p)), L_{g_0^{-1}} \circ u_q \circ a_0 + (\text{Ad}(g_0)^{-1} \circ \xi_0 \circ a_0)Q(g_0^{-1} \cdot q))
\]

which in the quotient \( P^{(1)} \times G_0 \times L(\mathbb{R}^n, TQ) \) represents the same class as \((a_x, u_p), u_q)\), to

\[
(R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})p(p)), L_{g_0^{-1}} \circ u_q \circ a_x^{-1} + (\text{Ad}(g_0)^{-1} \circ \xi_0 \circ a_x^{-1})Q(g_0^{-1} \cdot q))
\]

which in the quotient \( J(P \times G_0 Q) \) represents the same class as \((R_{g_0} \circ u_p, L_{g_0^{-1}} \circ u_q \circ a_x^{-1})\), since their difference belongs to \(L(T_xM, \text{ker} T_{(p, g_0; g_0^{-1}, q)}\mathcal{P}Q)\); therefore, the map in equation (130) induces a well defined map

\[
P^{(1)} \times G_0 \times L(\mathbb{R}^n, TQ) \rightarrow J(P \times G_0 Q)
\]

between the quotient spaces which is the desired isomorphism. For the third case, the argument is entirely analogous but somewhat simpler since some terms drop out; we leave it to the reader to fill in the details. Finally, for the last case, consider the map

\[
P^{(1)} \times L(\mathbb{R}^n, g_0) \rightarrow JP
\]

\[
((a_x, u_p), A_0) \mapsto u_p + (A_0 \circ a_x^{-1})p(p)
\]

and observe that it takes

\[
((a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0, g_0; \xi_0)^{-1} \cdot A_0)
\]

\[
= ((a_x, u_p) \cdot (a_0, g_0; \xi_0), (a_0^{-1}, g_0^{-1}; -\text{Ad}(g_0)^{-1} \circ \xi_0 \circ a_0) \cdot A_0)
\]

\[
= ((a_x \circ a_0 \cdot R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})p(p)), \text{Ad}(g_0)^{-1} \circ (A_0 - \xi_0) \circ a_0)
\]

which in the quotient \( P^{(1)} \times G_0 \times L(\mathbb{R}^n, g_0) \) represents the same class as \((a_x, u_p), A_0)\), to

\[
R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})p(p)) + (\text{Ad}(g_0)^{-1} \circ (A_0 - \xi_0) \circ a_x^{-1})p(p)
\]

\[
= R_{g_0} \circ (u_p + (\xi_0 \circ a_x^{-1})p(p)) + R_{g_0} \circ ((A_0 - \xi_0) \circ a_x^{-1})p(p)
\]

\[
= R_{g_0} \circ (u_p + (A_0 \circ a_x^{-1})p(p))
\]

which in the quotient \( CP = JP/G_0 \) represents the same class as \(R_{g_0} \circ u_p\); therefore, the map in equation (132) induces a well defined map

\[
P^{(1)} \times G_0 \times L(\mathbb{R}^n, g_0) \rightarrow JP/G_0 = CP
\]

between the quotient spaces which is the desired isomorphism.
Higher order jet prolongations can be constructed similarly, but as in our previous work, we shall only use jet prolongations up to second order, which can be constructed by iterating the first order construction once and then performing an appropriate reduction.

The jet groupoid of a gauge groupoid

We begin with a more explicit description of the jet groupoid of the gauge groupoid of a principal bundle $P$, which is based on the “magical square” for gauge groupoids, i.e., the commutative diagram

\[
\begin{array}{ccc}
P \times P & \xrightarrow{\rho \rho} & (P \times P)/G_0 \\
p_2 \downarrow & & \downarrow \sigma_G \\
P & \xrightarrow{\rho} & M
\end{array}
\] (134)

in which the horizontal projections define principal $G_0$-bundles while the vertical projections provide Lie groupoids (the first of which is of course just the pair groupoid of $P$) such that $\rho_P$ is an isomorphism on each (source or target or double) fiber. Once more, we may seek to gain a more profound understanding of the situation by extending this diagram to include the corresponding jet groupoids, but a direct approach is not feasible here since we cannot simply apply the jet functor to this diagram as we did before (see equations (55) and (82)), the reason being that $P \times P$ is a Lie groupoid over $P$ but not over $M$. Instead, we shall also consider the “magical square” for gauge groupoids at the next level, which is the commutative diagram

\[
\begin{array}{ccc}
P^{(1)} \times P^{(1)} & \xrightarrow{\rho_P^{(1)}} & (P^{(1)} \times P^{(1)})/G_0^{(1)} \\
p_{2(1)} \downarrow & & \downarrow \sigma_{G^{(1)}} \\
P^{(1)} & \xrightarrow{\rho^{(1)}} & M
\end{array}
\] (135)

and construct a canonical map

\[
P^{(1)} \times P^{(1)} \rightarrow J((P \times P)/G_0)
\] (136)

which we will show to be $G_0^{(1)}$-invariant, so it factors through the projection $\rho_{P^{(1)}}$ to yield a canonical map

\[
(P^{(1)} \times P^{(1)})/G_0^{(1)} \rightarrow J((P \times P)/G_0)
\] (137)

which will turn out to be an isomorphism (see Theorem 3 below).

To see how this construction goes, let us pick points $p_1, p_2 \in P$ with $\rho(p_1) = x_1$ and $\rho(p_2) = x_2$ and take tangent maps to the commutative diagram in equation (134) to obtain the commutative diagrams

\[
\begin{array}{ccc}
T_{p_2}P \oplus T_{p_1}P & \xrightarrow{T_{(p_2,p_1)}\rho \rho} & T_{(p_2,p_1)}((P \times P)/G_0) \\
p_2 \downarrow & & \downarrow T_{(p_2,p_1)}\sigma_G \\
T_{p_1}P & \xrightarrow{T_{p_1}\rho} & T_{x_1}M
\end{array}
\] (138)
referring to the source projection and

\begin{equation}
T_{p_2}P \oplus T_{p_1}P \xrightarrow{T(p_2,p_1)\rho_p} T_{[p_2,p_1]}((P \times P)/G_0)
\end{equation}

referring to the target projection. Since \( \rho_p \) is a submersion and hence its tangent maps are surjective, this means that the tangent spaces \( T_{[p_2,p_1]}((P \times P)/G_0) \) of the orbit space \( (P \times P)/G_0 \) can be realized as quotient spaces, namely, the linear maps

\begin{equation}
T_{[p_2,p_1]}\rho_p : T_{p_2}P \oplus T_{p_1}P \rightarrow T_{[p_2,p_1]}((P \times P)/G_0)
\end{equation}

induce isomorphisms

\begin{equation}
T_{[p_2,p_1]}((P \times P)/G_0) \cong (T_{p_2}P \oplus T_{p_1}P)/\ker T_{[p_2,p_1]}\rho_p,
\end{equation}

with

\begin{equation}
\ker T_{[p_2,p_1]}\rho_p = \{((X_0)_{p_2}, (X_0)_P(p_1)) \mid X_0 \in \mathfrak{g}_0\} \cong \mathfrak{g}_0,
\end{equation}

where as before, \( (X_0)_p \) denotes the fundamental vector field on \( P \) associated to a generator \( X_0 \in \mathfrak{g}_0 \) via the pertinent action of \( G_0 \), as defined in equation (51) above.

With this notation, we can define the map in equation (136) above, or more explicitly, its restriction to the fiber over the pair \( (p_2, p_1) \in P \times P \), that is, the map

\begin{equation}
P^{(1)}_{p_2} \times P^{(1)}_{p_1} = J_{p_2}P \times GL(\mathbb{R}^n, T_{x_2}M) \times GL(\mathbb{R}^n, T_{x_1}M) \times J_{p_1}P \rightarrow J_{[p_2,p_1]}((P \times P)/G_0)
\end{equation}

by setting

\begin{equation}
u_{[p_2,p_1]} = T_{[p_2,p_1]}\rho_p \circ (u_{p_2} \circ a_{x_2} \circ a_{x_1}^{-1} \circ u_{p_1}).
\end{equation}

We claim that this map is onto. Indeed, any linear map \( u_{[p_2,p_1]} \) from \( T_{x_1}M \) to \( T_{[p_2,p_1]}((P \times P)/G_0) \) can be represented in the form

\begin{equation}
u_{[p_2,p_1]} = T_{[p_2,p_1]}\rho_p \circ (\bar{u}_{p_2}, u_{p_1})
\end{equation}

with linear maps \( u_{p_1} \) from \( T_{x_1}M \) to \( T_{p_1}P \) and \( \bar{u}_{p_2} \) from \( T_{x_1}M \) to \( T_{p_2}P \), where the pair \( (\bar{u}_{p_2}, u_{p_1}) \) in \( L(T_{x_1}M, T_{p_2}P \oplus T_{p_1}P) \) is determined up to addition of a linear map from \( T_{x_1}M \) to \( \ker T_{[p_2,p_1]}\rho_p \). Moreover, if we assume \( u_{[p_2,p_1]} \) to be a jet in \( J_{[p_2,p_1]}((P \times P)/G_0) \) and to project to some \( a_{x_2,x_1} \in GL(T_{x_1}M, T_{x_2}M) \), which we recall means that \( T_{[p_2,p_1]}\sigma((P \times P)/G_0) \circ u_{[p_2,p_1]} = \text{id}_{T_{x_1}M} \) while \( T_{[p_2,p_1]}\tau((P \times P)/G_0) \circ u_{[p_2,p_1]} = a_{x_2,x_1} \), then we conclude that \( u_{p_1} \) will be a jet in \( J_{p_1}P \) and \( u_{p_2} = \bar{u}_{p_2} \circ a_{x_2,x_1}^{-1} \) will be a jet in \( J_{p_2}P \). Finally, we may write \( a_{x_2,x_1} = a_{x_2} \circ a_{x_1}^{-1} \) with \( a_{x_1} \in GL(\mathbb{R}^n, T_{x_1}M) \) and \( a_{x_2} \in GL(\mathbb{R}^n, T_{x_2}M) \). It is then clear that the map in equation (143) takes

\begin{equation}
(u_{p_2}, a_{x_2}, a_{x_1}, u_{p_1}) \cdot (a_0, g_0; \xi_0)
= (R_{g_0} \circ (u_{p_2} + (\xi_0 \circ a_{x_2}^{-1})(p_2)), a_{x_2} \circ a_0, a_{x_1} \circ a_0, R_{g_0} \circ (u_{p_1} + (\xi_0 \circ a_{x_1}^{-1})(p_1)))
\end{equation}
which in the quotient \((P^{(1)} \times P^{(1)})/G_0^{(1)}\) represents the same class as \((u_{p_2}, a_{x_2}, a_{x_1}, u_{p_1})\), to
\[
T_{(p_2, 0, 0, p_1)}\rho_P \circ (R_{g_0} \circ u_{p_2} \circ a_{x_2} \circ a_{x_1}^{-1} \circ R_{g_0} \circ u_{p_1}) = T_{(p_2, p_1)}\rho_P \circ (u_{p_2} \circ a_{x_2} \circ a_{x_1}^{-1} \circ u_{p_1})
\]
in \(J_{[p_2, 0, 0, p_1]}((P \times P)/G_0) = J_{[p_2, p_1]}((P \times P)/G_0)\). This proves that the map in equation (137) is well defined, and it is now easy to see that it induces an isomorphism of Lie groupoids over \(M\); we leave the details of the remainder of the proof to the reader and just state the result as a

**Theorem 3** Up to a canonical isomorphism, the (first order) jet groupoid of the gauge groupoid of a principal bundle \(P\) is equal to the gauge groupoid of its (first order) jet prolongation \(P^{(1)}:\)
\[
J((P \times P)/G_0) \cong (P^{(1)} \times P^{(1)})/G_0^{(1)}.
\]