Heat kernel of fractional Laplacian in cones

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November 2008

This paper is devoted to the memory of Professor Andrzej Hulanicki.

Abstract

We give sharp estimates for the transition density of the isotropic stable Lévy process killed when leaving a right circular cone.

1 Introduction

Explicit sharp estimates for the Green function of the Laplacian in $C^{1,1}$ domains were given in 1986 by Zhao [64] (see also [38, 31]). Sharp estimates of the Green function of Lipschitz domains were given in 2000 by Bogdan [11]. Explicit qualitatively sharp estimates for the classical heat kernel in $C^{1,1}$ domains were established in 2002 by Zhang [62] (see also [63, 32], and [30, 54] for further extensions). Qualitatively sharp heat kernel estimates in Lipschitz domain were given in 2003 by Varopoulos [59].

The development of the boundary potential theory of the fractional Laplacian follows an analogous path. Green function estimates were obtained in 1997 and 1998 by Kulczycki and Chen and Song for $C^{1,1}$ domains [46, 29] (see [23], Corollary 1.8) for the case of dimension one, see also [14], and in 2002 by Jakubowski for Lipschitz domains [45] (see also [51, 17]). In 2008 Chen, Kim and Song gave a sharp and explicit estimate for the heat kernel of the fractional Laplacian on $C^{1,1}$ domains [26], see (9) below. In this note we give an extension of the estimate to the right circular cones. We also conjecture, in agreement with the results [59], a likely form of the estimate for a more general class of domains:

$$p_t^D(x, y) \approx P_x(\tau_D > t)P_y(\tau_D > t)p_t(x, y).$$

Here $p_t(x, y)$ is the heat kernel of the fractional Laplacian on the whole space $\mathbb{R}^d$, and $P_x(\tau_D > t) = \int_{\mathbb{R}^d} p_t^D(x, y)dy$ is the survival probability of the corresponding isotropic $\alpha$-stable Lévy process. The main result of this paper, Theorem 1, asserts that (1) holds indeed for the right circular cones for all $t > 0$, $x, y \in \mathbb{R}^d$.

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02000 Mathematics Subject Classification: Primary 60J35, 60J50; Secondary 60J75, 31B25.

Key words and phrases: cone, fractional Laplacian, killed stable Lévy process, transition density, heat kernel.

Research partially supported by KBN (MNiI 1 P03A 026 29)
\( \mathbb{R}^d \) (see also (23) for a more explicit statement). Noteworthy, all the above-mentioned estimates for bounded \( C^{1,1} \) domains have the same form as for the ball (in this connection compare [26, Corollary 1.2] with [14, Corollary 3]; see also (9) below). We also like to note that the right circular cones are merely special Lipschitz domains, but a number of techniques and explicit formulas make them an interesting and important test case (see [5, 53, 54, 23, 22]). We hope to encourage a further study of Lipschitz and more general domains for stable and other jump-type processes [25, 41, 36, 12]. We should emphasize that generally the estimates for Lipschitz domains cannot be as explicit as those for \( C^{1,1} \) domains. For instance, the decay rate of harmonic and parabolic functions in the vertex of a cone delicately depends on the aperture of the cone, see [2, 51] (see also [11]). Nevertheless Lipschitz domains offer a natural setting for studying the boundary behavior of the Green function and the heat kernel of the Brownian motion and \( \alpha \)-stable Lévy processes \((0 < \alpha < 2)\). This is so because of scaling, the rich range of asymptotic behaviors depending on the local geometry of the domain’s boundary, connections to the boundary Harnack principle, approximate factorization of the Green function, and applications in the perturbation theory of generators, in particular via the 3G Theorem, [11, 51], 42, 43, 13, 18, 16, and 3P Theorem 18. Noteworthy, (1) is an approximate factorization of the heat kernel (see [11, 19] in this connection).

Cones are also examples of unbounded domains, which are only partially resolved by the results of [26, 27] (note that [26] is valid only for bounded times). We should note that the upper bound in (9) was proved in 2006 by Siudeja for semibounded convex domains [57, Theorem 1.6] (stated for general convex domain in [57, Remark 1.7]). It appears that the impulse for the proof of (9) was given by Siudeja and Kulczycki in [48, Theorem 4.2], see also [4, Proposition 2.9] by Kulczycki and Bañuelos. A similar but weaker upper bound was earlier given in [2, 26], see also [50, 52]. We also remark that [40, Theorem 4.4] gives a sharp explicit estimate for the survival probability of the relativistic process in a half-line. Generally, the subject is far from exhausted—and it seems manageable with the existing techniques.

For completeness we like to mention recent estimates [28, 21, 53, 60, 56, 37, 37] for transition density and potential kernel of jump-type processes. We need to point out that generally these are estimates for processes without killing. Killing is a dramatic “perturbation” analogous to Schrödinger perturbations with singular negative potentials [14, 16, 10, 12], and it strongly influences the asymptotics of the transition density and Green function. The asymptotics is crucial for solving the Dirichlet problem for the corresponding operators, see also [39, 40]. As we shall see, the heat kernel of the fractional Laplacian in the right circular cones has a power-type asymptotics at infinity, and it decays like the distance to the boundary to the power \( \alpha/2 \) except at the vertex, where it decays with the rate of \( \beta \in (0, \alpha) \).

The paper is composed as follows. Below in this section we recall basic facts about the transition density of the \( \alpha \)-stable Lévy processes killed when first leaving a domain. In Section 2 we give a sharp explicit estimate for the survival probability \( P_x(\tau_D > t) \) for \( C^{1,1} \) domains \( D \). In Section 3 we prove our
main estimates, Theorem 1 and (23), by using the ideas and results of [26] and [2]. Our general references to the boundary potential theory of the fractional Laplacian are [13] and [19]. We also refer the reader to [15] for a broad non-technical overview of the goals and methods of the theory.

In what follows, \( \mathbb{R}^d \) denotes the Euclidean space of dimension \( d \geq 1 \), \( dx \) is the Lebesgue measure on \( \mathbb{R}^d \), and \( 0 < \alpha < 2 \). For \( t > 0 \) we let \( p_t \) be the smooth real-valued function on \( \mathbb{R}^d \) with the following Fourier transform,

\[
\int_{\mathbb{R}^d} p_t(x) e^{i x \cdot \xi} \, dx = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d. \tag{2}
\]

For instance, \( \alpha = 1 \) yields

\[
p_t(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{d/2}} x^{-d/2} \left(\frac{t}{|x|^2 + t^2}\right)^{d/2},
\]

the Cauchy convolution semigroup of functions [58]. We generally have that

\[
p_t(x) = t^{-d/\alpha} p_t(t^{-1/\alpha} x), \quad x \in \mathbb{R}^d, \ t > 0. \tag{3}
\]

This follows from (2). The semigroup \( P_t f(x) = \int_{\mathbb{R}^d} f(y)p_t(y-x)dy \) has \( \Delta^{\alpha/2} \) as infinitesimal generator ([7], [61], [13], [44]), where

\[
\Delta^{\alpha/2} \varphi(x) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(-\alpha/2)} \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{\varphi(x + y) - \varphi(x)}{|y|^{d+\alpha}} dy, \quad x \in \mathbb{R}^d.
\]

Here \( \varphi \in C^\infty_c(\mathbb{R}^d) \), i.e. \( \varphi: \mathbb{R}^d \to \mathbb{R} \) is smooth and compactly supported on \( \mathbb{R}^d \). Put differently,

\[
\int_s \int_{\mathbb{R}^d} p_{u-s}(z-x) \left[ \partial_u \phi(u, z) + \Delta^{\alpha/2}_u \phi(u, z) \right] \, dz \, du = -\phi(s, x),
\]

where \( s \in \mathbb{R}, \ x \in \mathbb{R}^d, \) and \( \phi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^d) \) ([16]). We denote

\[
\nu(y) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(-\alpha/2)} |y|^{-d-\alpha},
\]

the density function of the Lévy measure of the semigroup \( \{P_t\} \) [58] [20] [15].

There is a constant \( c \) such that (see [20] or [9])

\[
c^{-1} \left( \frac{t}{|x|^{d+\alpha}} \land t^{-d/\alpha} \right) \leq p_t(x) \leq c \left( \frac{t}{|x|^{d+\alpha}} \land t^{-d/\alpha} \right), \quad x \in \mathbb{R}^d, \ t > 0. \tag{4}
\]

(4) and similar sharp estimates (i.e. such that the lower and upper bounds are comparable) will be abbreviated as follows:

\[
p_t(x) \approx t^{-d/\alpha} \land \frac{t}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d, \ t > 0. \tag{5}
\]
The standard isotropic $\alpha$-stable Lévy process $(X_t, P_x)$ on $\mathbb{R}^d$ may be constructed by specifying the following time-homogeneous transition probability:

$$P_t(x, A) = \int_A p_t(y - x)dy, \quad t > 0, \ x \in \mathbb{R}^d, \ A \subset \mathbb{R}^d,$$

and stipulating that $P_x(X(0) = x) = 1$. Thus, $P_x$, $E_x$ denote the distribution and expectation for the process starting from $x$. The distribution of the process is concentrated on right continuous functions: $[0, \infty) \rightarrow \mathbb{R}^d$ with left limits, and for all $s \geq 0, \ x \in \mathbb{R}^d$ we have that $P_x(X_s = X_{s-}) = 1$. It is well-known that $(X_t, P_x)$ is strong Markov with respect to the so-called standard filtration [3, 10].

The Lévy system (see [35, VII.68], [28, Appendix A], see also [57, Theorem 2.4], [48, Corollary 2.8] and [3, Lemma 1]) for $(X_t, P_x)$ amounts to the equality,

$$E_x \left[ \sum_{s \leq T} f(s, X_{s-}, X_s) \right] = E_x \left[ \int_0^T \left( \int_{\mathbb{R}^d} f(s, X_s, y)\nu(w - X_s)dw \right) ds \right], \quad (6)$$

where $x \in \mathbb{R}^d, \ f \geq 0$ is a Borel function on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, such that $f(s, z, w) = 0$ if $z = w$, and $T$ is a stopping time with respect to the filtration of $X$.

For open $D \subset \mathbb{R}^d$ we let $\tau_D = \inf\{t > 0 : X_t \notin D\}$, and we define

$$p_t^D(x, y) = p_t(x, y) - E_x[\tau_D < t; p_{t-\tau_D}(X_{\tau_D}, y)], \ x, y \in \mathbb{R}^d, \ t > 0,$$

see, e.g., [26, 13]. Clearly,

$$0 \leq p_t^D(x, y) \leq p_t(y - x). \quad (7)$$

By the strong Markov property, $p_t^D$ is the transition density of the isotropic stable process killed on leaving $D$, meaning that $p^D$ satisfies the Chapman-Kolmogorov equation:

$$\int_{\mathbb{R}^d} p_{s+t}^D(x, z)p_t^D(z, y)dz = p_{s+t}^D(x, y), \ x, y \in \mathbb{R}^d, \ s, t > 0,$$

and for every $x \in \mathbb{R}^d, \ t > 0$ and bounded Borel function $f$,

$$\int_{\mathbb{R}^d} f(y)p_t^D(x, y)dy = E_x[\tau_D < t; f(X_t)].$$

Furthermore, for $s \in \mathbb{R}, \ x \in \mathbb{R}^d, \ and \ \phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_{\mathbb{R}^d} \int_s^\infty p_{s-u}^D(x, z) \left[ \partial_u \phi(u, z) + \Delta_z^{\alpha/2} \phi(u, z) \right] du \ dz = -\phi(s, x),$$

which justifies calling $p^D_t$ the heat kernel of the fractional Laplacian on $D$. In analogy with [3] we have the following scaling property

$$p_t^D(x, y) = t^{-d/\alpha} p_1^{\frac{1}{1-\alpha}D(t^{-1/\alpha}x, t^{-1/\alpha}y)}, \ x, y \in \mathbb{R}^d, \ t > 0. \quad (8)$$
2 \( C^{1,1} \) domains

Let \( D \subset \mathbb{R}^d \) be a \( C^{1,1} \) domain, meaning that \( D \) is open and there is \( r_0 > 0 \) such that for every \( z \in \partial D \) there exist balls \( B_z(r_0) \subset D \) and \( B_z'(r_0) \subset D^c \) of radius \( r_0 \), tangent at \( z \). Denote \( \delta_D(x) = \text{dist}(x, D^c) \), the distance to \( D^c \). The transition density of the stable Lévy process killed off \( D \) satisfies (26)

\[
p_t^D(x, y) \approx \left(1 + \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right) \left(1 + \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right) p_t(x, y), \quad 0 < t \leq 1, \; x, y \in \mathbb{R}^d.
\]

(9)

**Corollary 1.** If \( D \) is a \( C^{1,1} \) domain then

\[
P_x(\tau_D > t) \approx 1 + \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}, \quad 0 < t \leq 1, \; x, y \in \mathbb{R}^d.
\]

(10)

**Proof.** We have

\[
P_x(\tau_D > t) = \int_{\mathbb{R}^d} p_t^D(x, y) dy.
\]

By (9),

\[
P_x(\tau_D > t) = \int_D p_t^D(x, y) dy \approx \left(1 + \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right) I_t(x), \quad 0 < t \leq 1, \; x, y \in \mathbb{R}^d,
\]

where

\[
I_t(x) = \int_D \left(1 + \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right) p_t(x, y) dy.
\]

Clearly, \( I_t(x) \leq \int_{\mathbb{R}^d} p_t(x, y) dy = 1 \). This yields the upper bound in (10). To prove the lower bound we consider \( 0 < t \leq 1 \) and we will first assume that \( \delta_D(x) > t^{1/\alpha} \). If \( |y - x| < t^{1/\alpha}/2 \), then \( p_t(x, y) \approx t^{-d/\alpha} \), and we get

\[
I_t(x) \geq c \int_{|y - x| < t^{1/\alpha}/2} t^{-d/\alpha} \; dy = c > 0.
\]

If \( \delta_D(x) \leq t^{1/\alpha} \), then let \( z \in \partial D \) be such that \( |x - z| = \delta_D(x) \), and consider the inner tangent ball \( B_z(t^{1/\alpha} \land r_0) \) for \( D \) at \( z \), with center at, say, \( w \). We have

\[
I_t(x) \geq \int_{B_z(t^{1/\alpha} \land r_0)} \left(\frac{t^{1/\alpha} - |y - w|}{\sqrt{t}}\right)^{\alpha/2} p_t(x, y) \; dy.
\]

Since \( p_t(x, y) = t^{-d/\alpha} p_1\left(\frac{r_w - w}{t^{1/\alpha}}\right) \), by changing variable \( v = \frac{r_w - w}{t^{1/\alpha}} \), we get

\[
I_t(x) \geq \int_{B(0, 1 \land r_0)} (1 - |v|)^{\alpha/2} p_1(u, v) \; dv,
\]

where \( u = t^{-1/\alpha}(x - w) \in B(0, 1) \). The latter integral is continuous and strictly positive for \( u \in \overline{B}(0, 1) \). Thus, \( \inf_{x \in D} I_t(x) > 0 \). The proof of (10) is complete. \( \square \)
Lemma 3. The following result strengthens [2, Lemma 4.2].

\[ p_t^D(x, y) \approx P_x(\tau_D > t)P_y(\tau_D > t)p_t(x, y), \quad 0 < t \leq 1, \ x, y \in \mathbb{R}^d. \]

3 Cones

For \( x \in \mathbb{R}^d \setminus \{0\} \) we denote by \( \theta(x) \) the angle between \( x \) and the point \((0, \ldots, 0, 1) \in \mathbb{R}^d\). We fix \( 0 < \theta(x) < \pi \) and consider the right circular cone \( \Gamma = \{ x \in \mathbb{R}^d \setminus \{0\} : \theta(x) < \theta \} \). Clearly, \( r\Gamma = \Gamma \) for every \( r > 0 \). By [5],

\[ p_t^\Gamma(x, y) = t^{-d/\alpha}p_t^1(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x, y \in \mathbb{R}^d, \ t > 0. \] (11)

We fix \( x_0 \in \Gamma \) and consider the Martin kernel \( M \) for \( \Gamma \) with the pole at infinity, so normalized that \( M(x_0) = 1 \). It is known that there is \( 0 < \beta < \alpha \) such that

\[ M(x) = |x|^\beta M(x/|x|), \quad x \neq 0, \]

see [2], [51], [17]. Since the boundary of \( \Gamma \) is smooth except at the origin, by [51, Lemma 3.3],

\[ M(x) \approx \delta_t(x)^{\alpha/2}|x|^\beta-\alpha/2, \quad x \in \mathbb{R}^d. \] (12)

The following result strengthens [2, Lemma 4.2].

Lemma 3. If \( \Gamma \) is a right circular cone then

\[ P_x(\tau_\Gamma > t) \approx \left( \delta_t^{\alpha/2}(t^{-1/\alpha}x) \wedge 1 \right) \left( |t^{-1/\alpha}x| \wedge 1 \right)^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d, \ t > 0. \] (13)

Proof. Since \( P_x(\tau_\Gamma > t) = P_{t^{-1/\alpha}x}(\tau_{t^{-1/\alpha}x} > 1) \), we only need to prove that

\[ P_x(\tau_\Gamma > 1) \approx \left( \delta_t^{\alpha/2}(x) \wedge 1 \right) (|x| \wedge 1)^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d. \] (14)

If \( |x| < 1 \) then (14) is a consequence of (12) and [2, Lemma 4.2]. If \( |x| \geq 1 \) then \( P_x(\tau_\Gamma > 1) \approx \delta_t(x)^{\alpha/2} \wedge 1 \). Indeed, considering \( C^{1,1} \) domains \( \Gamma' \) and \( \Gamma'' \) such that \( \Gamma' \subset \Gamma \subset \Gamma'' \) and \( \Gamma'' \setminus \Gamma' \subset B(0, 1/2) \), we see that \( \delta_{\Gamma'}(x) \leq \delta_{\Gamma}(x) \leq 2\delta_{\Gamma''}(x) \) for such \( x \). Since \( P_x(\tau_{\Gamma''} > 1) \leq P_x(\tau_{\Gamma'} > 1) \leq P_x(\tau_{\Gamma'} > 1) \), by using (14) we obtain (14).

An interesting, if trivial, consequence of (13) is that

\[ P_x(\tau_\Gamma > t) \approx P_x(\tau_\Gamma > t/2), \quad t > 0, \ x \in \mathbb{R}^d. \] (15)

Theorem 1.

\[ p_t^\Gamma(x, y) \approx P_x(\tau_\Gamma > t)P_y(\tau_\Gamma > t)p_t(x, y), \quad x, y \in \mathbb{R}^d, \ t > 0. \] (16)
Proof. We note that the right hand side, say \( R_t(x, y) \), of (16) satisfies
\[
R_t(x, y) = e^{-\frac{d}{\alpha} R_1(t^{-1/\alpha} x, t^{-1/\alpha} y)}.
\]
Thus, in view of (11), we only need to prove (16) for \( t = 1 \).
Let \( \Gamma' \) and \( \Gamma'' \) be as in the proof of Lemma 3. Then
\[
p_1^{\Gamma'}(x, y) \leq p_1^\Gamma(x, y) \leq p_1^{\Gamma''}(x, y).
\]
By (9) we have, for \(|x|, |y| \geq 1, \)
\[
p_1^{\Gamma'}(x, y) \approx \left(1 \wedge \delta_1^{\alpha/2}(x)\right) \left(1 \wedge \delta_1^{\alpha/2}(y)\right) p_1(x, y) \approx p_1^{\Gamma''}(x, y).
\]
Hence by Lemma 3 we obtain,
\[
p_1^\Gamma(x, y) \approx P_x(\tau_1 > 1)P_y(\tau_1 > 1)p_1(x, y), \quad |x|, |y| \geq 1. \quad (17)
\]
In particular, there is a constant \( c \) such that
\[
p_1^\Gamma(x, y) \leq cP_x(\tau_1 > 1)p_1(x, y), \quad |x|, |y| \geq 1. \quad (18)
\]
If \(|x| < 1 \) and \(|y| \leq 4\), then \(|x - y| < 5 \) and \( p_1(x, y) \geq c(1 \wedge |x - y|^{-d-\alpha}) \geq c \). By the semigroup property (7) and (15),
\[
p_1^\Gamma(x, y) = \int_{\Gamma} p_{1/2}^\Gamma(x, w)p_{1/2}^\Gamma(w, y)dw \leq c \int_{\Gamma} p_{1/2}^\Gamma(x, w)dw \\
= cP_x(\tau_1 > 1/2) \leq cP_x(\tau_1 > 1) \\
= cP_x(\tau_1 > 1)p_1(x, y), \quad |x| < 1, \quad |y| \leq 4. \quad (19)
\]
We next assume that \(|x| < 1 \) and \(|y| > 4\). Then \(|x - y| > 3 \) and \( p_1(x, y) \approx |x - y|^{-d-\alpha}\). Denote \( \Gamma_1 = \Gamma \cap B(0, 2), \Gamma_2 = (\Gamma \setminus \Gamma_1) \cap B(0,(|y| + 1)/2) \) and \( \Gamma_3 = \Gamma \setminus B(0,(|y| + 1)/2) \). Using the strong Markov property and the Lévy system (8) with \( f(s, z, w) = 1_{\Gamma_1}(z)1_{\Gamma_2}(w)p_{1-s}^\Gamma(w, y) \) and \( T = 1 \wedge \tau_1 \), we obtain
\[
p_1^\Gamma(x, y) = E_x[\tau_{T_1} < 1; p_{1-\tau_1}^\Gamma(X_{\tau_1}, y)] \\
= \int_0^1 \int_{\Gamma_1} p_s^\Gamma(x, z) \int_{\Gamma \setminus \Gamma_1} \nu(w - z) p_t^{1-s}(w, y) dw dz ds \\
= \int_0^1 \int_{\Gamma_1} p_s^\Gamma(x, z) \int_{\Gamma_2} \nu(w - z) p_t^{1-s}(w, y) dw dz ds \\
+ \int_0^1 \int_{\Gamma_1} p_s^\Gamma(x, z) \int_{\Gamma_3} \nu(w - z) p_t^{1-s}(w, y) dw dz ds \\
= I + II.
\]
We note that for \( w \in \Gamma_2 \),
\[
|w - y| \geq |y| - |w| \geq |y|/4 \geq |x - y|/8.
\]
Since \( p_{1-s}^\Gamma(w - y) \leq p_{1-s}(w - y) \leq c(1-s)|w - y|^{-\alpha}, \) we obtain
\[
I \leq \int_0^1 \int_{\Gamma_1} p_1^\Gamma(x, z) \int_{\Gamma_2} \nu(w - z) \frac{1 - s}{|w - y|^{d + \alpha}} \, dw \, dz \, ds
\]
\[
\leq c|x - y|^{-d - \alpha} \int_0^1 \int_{\Gamma_1} p_1^\Gamma(x, z) \int_{\Gamma_1} \nu(w - z) \, dw \, dz \, ds
\]
\[
= c|x - y|^{-d - \alpha} P_x(X_{\tau_4} \in \Gamma \setminus \Gamma_1, \tau_4 \leq 1)
\]
\[
\leq c|x - y|^{-d - \alpha} P_x(X_{\tau_4} \in \Gamma \setminus \Gamma_1) \approx M(x) p_1(x, y).
\]
In the last line we used BHP \([2]\). For \( z \in \Gamma_1 \) and \( w \in \Gamma_3 \), we have
\[
|w - z| \geq |w| - |z| \geq |y|/2 - 3/2 \geq |y|/8 \geq |x - y|/16,
\]
hence \( \nu(w - z) \leq c|x - y|^{-d - \alpha} \), and so
\[
II \leq c|x - y|^{-d - \alpha} \int_0^1 \int_{\Gamma_1} p_1^\Gamma(x, z) \int_{\Gamma_3} p_{1-s}(w, y) \, dw \, dz \, ds
\]
\[
\leq c|x - y|^{-d - \alpha} \int_0^1 \int_{\Gamma_1} p_1^\Gamma(x, z) \, dz \, ds \leq c|x - y|^{-d - \alpha} E_x \tau_4
\]
\[
\leq cp_1(x, y) M(x),
\]
where the last inequality follows from \([3]\) and \([2\) Lemma 4.6]. By Lemma \(3\)
\[
p_1^\Gamma(x, y) \leq cP_x(\tau_T > 1)p_1(x, y), \quad |x| < 1, |y| > 4. \tag{20}
\]
Combining \((18), (19)\) and \((20)\), we get
\[
p_1^\Gamma(x, y) \leq cP_x(\tau_T > 1)p_1(x, y), \quad x, y \in \mathbb{R}^d.
\]
By the symmetry, semigroup property and \((15)\) we obtain
\[
p_1^\Gamma(x, y) = \int_{\Gamma} p_1^\Gamma(x, w)p_1^\Gamma(w, y) \, dw
\]
\[
= \int_{\mathbb{R}^d} 2^{2d/\alpha} p_1^\Gamma(x2^{1/\alpha}, w2^{1/\alpha}) p_1^\Gamma(w2^{1/\alpha}, y2^{1/\alpha}) \, dw
\]
\[
\leq cP_{x2^{1/\alpha}}(\tau_T > 1)P_{y2^{1/\alpha}}(\tau_T > 1) \int_{\mathbb{R}^d} p_1/2(x, w)p_1/2(w, y) \, dw
\]
\[
\leq cP_x(\tau_T > 1)P_y(\tau_T > 1)p_1(x, y).
\]
We will now prove the lower bound in \((16)\). We first assume that \(|x| < 1\) and \(|y| \leq 2\), and we let \( \Gamma_4 = \Gamma \cap B(0, 4)\). Since \( \Gamma_4 \) is bounded, the semigroup \( p_1^\Gamma \) is intrinsically ultracontractive \((27)\). In particular,
\[
p_1^\Gamma(x, y) \geq p_1/2(x, y) \geq cE_x \tau_4 E^y \tau_4.
\]
Furthermore, by \([2\) Lemma 4.6\) and Lemma \(3\) we obtain
\[
E^y \tau_4 \geq cM(y) \geq cP_y(\tau_T > 1).
\]
We see that
\[ p_{1/2}^r(x, y) \geq c P_z(\tau T > 1) P_y(\tau T > 1) p_1(x, y), \quad |x| < 1, \ |y| \leq 2. \] (21)

If \(|x| < 1\) and \(|y| > 2\), then by the semigroup property, (17) and (21),
\[
\begin{align*}
p_{1}^r(x, y) &= \int p_{1/2}^r(x, z) p_{1/2}^r(z, y) dz \\
&\geq c P_z(\tau T > 1) P_y(\tau T > 1) \int_{\Gamma \setminus B(0, 1)} P_z(\tau T > 1)^2 p_1(x, z) p_1(z, y) dz \\
&\geq c P_z(\tau T > 1) P_y(\tau T > 1) p_1(x, y) \int_{\Gamma \setminus B(0, 1)} P_z(\tau T > 1)^2 dz.
\end{align*}
\]

Hence
\[
p_{1}^r(x, y) \geq c P_z(\tau T > 1) P_y(\tau T > 1) p_1(x, y), \quad |x| < 1, \ |y| > 2. \] (22)

By (17), (21), (22), symmetry (and scaling), we get the lower bound in (16).

We like to note that Theorem [1] strengthens [2, Corollary 4.8]. Also,
\[
p_{1}^r(x, y) \approx p_{1/2}^r(x, y), \quad x, y \in \mathbb{R}^d, \ t > 0.
\]

In view of Lemma [8] for the right circular cone \(\Gamma\), (10) is equivalent to
\[
p_{1}^r(x, y) \approx \left( \delta_{\Gamma}^{\alpha/2}(t^{-1/\alpha} x) \wedge 1 \right) \left( |t^{-1/\alpha} x| \wedge 1 \right)^{\beta - \alpha/2} \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right) \tag{23}
\]
\[
\left( \delta_{\Gamma}^{\alpha/2}(t^{-1/\alpha} y) \wedge 1 \right) \left( |t^{-1/\alpha} y| \wedge 1 \right)^{\beta - \alpha/2}, \quad t > 0, \ x, y \in \mathbb{R}^d.
\]

This is explicit except for the exponent \(\beta\) (see [2] in this connection). Recall that \(\int_0^\infty p_{1}^r(x, y) dt = G_t(x, y)\), the Green function of \(\Gamma\). By integrating (23) one can obtain sharp estimates for the Green function of the right circular cone. For \(d \geq 2\) the estimates—first given in [51, Theorem 3.10]—are the following,
\[
\begin{align*}
G_t(x, y) \approx \left( \delta_{\Gamma}^{\alpha/2}(x) \delta_{\Gamma}^{\alpha/2}(y) \right) \left( \frac{|x| \wedge |y|}{|x| \vee |y|} \right)^{\beta - \alpha/2}, \quad x, y \in \mathbb{R}^d. \tag{24}
\end{align*}
\]

We skip the details of the integration (similar calculations are given in [57] and [26]). Noteworthy, \(\beta = \alpha/2\) if \(\Gamma\) is a half-space [2]. For the case of dimension \(d = 1\), and \(\Gamma = (0, \infty)\), we refer the reader to [24], see also [26, Corollary 1.2].

As stated in Introduction, we expect [1] to be true quite generally. In particular the approximation should hold for domains above the graph of a Lipschitz function for all times \(t > 0\). Corollary [2] confirms this conjecture for \(C^{1,1}\) domains and small times, while Theorem [1] proves it for the right circular cones and all times. By inspecting the relevant proofs in [51], the reader may also verify without difficulty that Theorem [1] and (24) hold the same for all those generalized cones ([2]) in \(\mathbb{R}^d, \: d \geq 2\), which are \(C^{1,1}\) except at the origin.
On the other hand, if $D$ is a bounded $C^{1,1}$ domain, and if we denote by $-\lambda_1$ the first eigenvalue of $\Delta^{\alpha/2}$ on $D$ (i.e. when acting on functions vanishing off $D$), then by the intrinsic ultracontractivity (see, e.g., [26 Theorem 1.1]),

$$p_t^D(x,y) \approx \delta_D^{\alpha/2}(x)\delta_D^{\alpha/2}(y)e^{-\lambda_1 t}, \quad t > 1, \ x, y \in \mathbb{R}^d,$$

and so

$$P_x(\tau_D > t) \approx \delta_D^{\alpha/2}(x)e^{-\lambda_1 t}, \quad t > 1, \ x \in \mathbb{R}^d.$$

Therefore (1) fails for large times $t$ if $D$ is bounded (see also [47]).

Acknowledgments. We thank Michal Ryznar and Mateusz Kwaśnicki for discussions and suggestions.

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