Non-equilibrium 1D many-body problems and asymptotic properties of Toeplitz determinants

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Abstract
Non-equilibrium bosonization technique facilitates the solution of a number of important many-body problems out of equilibrium, including the Fermi-edge singularity, the tunneling spectroscopy and full counting statistics of interacting fermions forming a Luttinger liquid. We generalize the method to non-equilibrium hard-core bosons (Tonks–Girardeau gas) and establish interrelations between all these problems. The results can be expressed in terms of Fredholm determinants of the Toeplitz type. We analyze the long time asymptotics of such determinants, using Szegő and Fisher–Hartwig theorems. Our analysis yields dephasing rates as well as power-law scaling behavior, with exponents depending not only on the interaction strength but also on the non-equilibrium state of the system.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
There is a number of quantum many-body problems that are of paramount importance for condensed matter physics and, at the same time, possess an exact solution. These are the Anderson orthogonality catastrophe [1], Fermi edge singularity (FES) [2], Luttinger liquid (LL) [3] zero-bias anomaly [4], and Kondo problems [5]. It has been realized long ago that these problems are, in fact, inter-connected, both vis-à-vis the underlying physics and the mathematics involved. Such connections have been used, e.g. for the representation of the dynamics of the Kondo problem as an infinite sequence of Fermi-edge-singularity events [6].
Table 1. Non-equilibrium correlation functions of many-body problems: Fermi edge singularity (G\textsubscript{FES}), Green’s functions of right- and left-moving fermions in a LL (G\textsubscript{FR} and G\textsubscript{FL}), full counting statistics of a LL (\chi\textsubscript{τ}(λ)), Green’s function of the Tonks–Girardeau gas (G\textsubscript{B}). All these correlation functions can be cast in the form \(G=\langle e^{-iO_\tau}e^{iO_0}\rangle\), where \(O=\sum_{\eta=R,L}c_\eta\phi_\eta\) and \(\phi_\eta\) are free bosonic fields. The coefficients \(c_\eta\) are shown in the second and third columns of the table. Evaluation of these correlation functions yields the results in the form of Fredholm–Toeplitz determinants \(\Delta_{R}\) \(\Delta_{L}\). The corresponding phases \(\delta_{R,L}\) are presented in the last two columns for the case where the LL is adiabatically connected to the reservoirs. These phases are expressed through LL parameter \(K\), counting field \(\lambda\), and phase shift in the FES problem \(\delta_0\).

| Function | \(c_R\) | \(c_L\) | \(\delta_R\) | \(\delta_L\) |
|----------|---------|---------|-------------|-------------|
| G\textsubscript{FES} | \(-1+\frac{\delta_0}{\pi}\) | 0 | 2(\(\pi-\delta_0\)) | 0 |
| G\textsubscript{FR} | -1 | 0 | \(2\pi\frac{1-K}{2\sqrt{K}}\) | \(2\pi\frac{1-K}{2\sqrt{K}}\) |
| G\textsubscript{FL} | 0 | -1 | \(2\pi\frac{1-K}{2\sqrt{K}}\) | \(2\pi\frac{1-K}{2\sqrt{K}}\) |
| \chi\textsubscript{τ}(λ) | -\(\frac{1}{\pi}\) | \(\frac{1}{\pi}\) | \(λ\sqrt{K}\) | -\(λ\sqrt{K}\) |
| G\textsubscript{B} | \(-\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{\sqrt{K}}\) | \(\frac{1}{\sqrt{K}}\) |

In our recent works [7, 8], we have addressed some of these problems away from equilibrium. To achieve this goal, we have developed a non-equilibrium bosonization technique generalizing the conventional bosonization [9–13] onto problems with non-equilibrium distribution functions. We have shown that the relevant correlation functions can be expressed through Fredholm determinants of a ‘counting’ operator. The information on the specific type of the problem, as well as on different aspects of the interaction, is encoded in the time-dependent scattering phase of the counting operator. The results of [7, 8] have demonstrated that those classic many-body problems are more closely connected than has been previously understood, extending the interrelations into the non-equilibrium regime.

The purpose of this paper is twofold. First, we present the relation between the LL tunneling spectroscopy, the full counting statistics (FCS), and the Fermi-edge singularity problems in a systematic way. We also include in this scheme the generalization of the LL problem to a non-equilibrium gas of strongly repulsive bosons (Tonks–Girardeau gas). These results are summarized in table 1. Second, we apply the general theory of asymptotic behavior of Toeplitz determinants to the present problems. This allows us to extract important information on the many-body physics of these models, including non-equilibrium dephasing rates and the modification of power-law exponents due to non-equilibrium conditions.

The structure of this paper is the following. In section 2.1, we summarize the results of our earlier works [7, 8] for the FES problem, as well as the tunneling spectroscopy and the counting statistics problems in the LL, all underlined by interacting fermions. Section 2.2 is devoted to the tunneling spectroscopy problem for a non-equilibrium Tonks–Girardeau gas. In section 3, we study the long-time behavior of emerging Fredholm determinants, employing Szegő and Fisher–Hartwig theorems. For completeness, properties of the Tonks–Girardeau gas at equilibrium are summarized in appendix A, while relevant mathematical theorems from the theory of Fredholm determinants are presented in appendix B.

2. Many-body problems as Fredholm determinants

2.1. Fermionic problems

2.1.1. Free fermions. We first discuss non-interacting electrons. We consider Green’s functions of free fermions

\[
G_{0,0}(\tau) = -i \langle \psi_\eta(\tau) \bar{\psi}_\eta(0) \rangle, \tag{1}
\]
The function $G^>$ describes the evolution of an electron, created at time $t = 0$ and annihilated at $t = \tau$, while the function $G^<$ characterizes the evolution of a hole, created at time $t = \tau$ and annihilated at $t = 0$ [14]. To simplify the notations, we focus on the case of coinciding spatial coordinates. The subscript $\eta$ is the chirality index ($\eta = L$ for left and $\eta = R$ for right movers). The functions $G^\eta$ can easily be calculated within a fermionic description:

$$G^\eta_{0,0}(\tau) = \frac{i}{v}[1 - n_\eta(\epsilon)],$$

(3)

$$G^\eta_{0,0}(\epsilon) = \frac{i}{v}n_\eta(\epsilon),$$

(4)

where $n_\eta(\epsilon)$ is a fermionic distribution function, and $v$ is the Fermi velocity.

It is insightful to recalculate Green’s functions employing the non-equilibrium bosonization [7]. The result reads

$$G^\eta_{0,0}(\tau) = -\frac{1}{2\pi v} \frac{1}{\tau \mp i/\Lambda_1}[\delta_\eta(t)].$$

(5)

Here $\Lambda_1$ is an ultraviolet cutoff, the phase $\delta_\eta(t) = \lambda\omega_\tau(t, 0)$, where

$$w_\tau(t, \tilde{t}) = \theta(\tilde{t} - t) - \theta(\tilde{t} - t - \tau)$$

(6)

is a ‘window function’, and $\lambda = 2\pi$. The finite time window originates from events of the creation and annihilation of an electron at times $t = 0, \tau$. Further, $\Delta_\eta[\delta_\eta(t)]$ is the Fredholm determinant of a counting operator

$$\Delta_\eta[\delta_\eta(t)] = \det[1 + (e^{i\delta_\eta} - 1)\hat{n}_\eta],$$

(7)

and we denote by $\overline{\Delta}_\eta$ the determinant normalized to its value for zero-$T$ equilibrium distribution. As we see, in the bosonic description, a free fermion is represented by a $2\pi$ phase soliton entering the counting operator. The latter consists of a fermionic distribution function $n_\eta(\epsilon)$, and a time-dependent scattering phase $\delta_\eta(t)$, with $\epsilon$ and $t$ to be understood as canonically conjugate variables. Since the matrices $\delta$ and $\hat{n}$ can not be diagonalized simultaneously, the task of calculating $\Delta_\eta[\delta_\eta(t)]$ is non-trivial. In most cases the analysis of such determinants can be done only numerically. However, the long time asymptotic properties can be studied analytically, by means of Szegő and Fisher–Hartwig theorems, as discussed in section 3.

When read from right to left, equation (5) can be viewed as a remarkable identity for Fredholm determinants. It yields an exact value of such a determinant for an arbitrary distribution function $n_\eta(\epsilon)$, and for the phase function $\delta(\tau)$ given by a window function with amplitude $2\pi$. It is worth mentioning that the determinant should be considered as analytically continued from the region of small phases $\delta$.

2.1.2. Fermi edge singularity. The FES problem describes the scattering of conduction electrons off a localized hole, which is left behind by an electron excited into the conduction band. Historically, the FES problem was first solved by exact summation of an infinite diagrammatic series [2]. While in the FES problem there is no interaction between electrons in the conducting band, it has many features characteristic of genuine many-body physics. Despite the fact that conventional experimental realizations of FES are three-dimensional, the problem can be reduced (due to the local and isotropic character of the interaction with the core hole) to that of one-dimensional chiral fermions. For this reason, bosonization technique can be effectively applied, leading to an alternative and very elegant solution [15].
One can consider the FES out of equilibrium [16], with an arbitrary electron distribution function \( n(\epsilon) \). This problem can be solved within the framework of non-equilibrium bosonization [7], with the following results for the emission/absorption rates:

\[
G_{\text{FES}}(\tau) = \mp \frac{i\Lambda\overline{\Delta}_R(2\pi - 2\delta_0)}{2\pi v(1 \pm i\tau)^{(1-\delta_0)/\pi\tau}}.
\]

Here \( \delta_0 \) is the s-wave electronic phase shift due to the scattering of conduction electrons off the core hole.

2.1.3. Luttinger liquid: tunneling spectroscopy. The tunneling spectroscopy technique allows one to explore experimentally Keldysh Green’s functions of an interacting system that carry information about both tunneling density of states and energy distribution. Recent experiments on carbon nanotubes and quantum Hall edges have proved the efficiency of this technique in the context of 1D systems [17, 18]. The technological and experimental advances motivate the theoretical interest in the tunneling spectroscopy of strongly correlated 1D structures away from equilibrium [7, 19–25].

In the case of a LL made of 1D interacting fermions, the Keldysh Green’s function may be evaluated theoretically via the non-equilibrium bosonization technique. Assuming that a long LL conductor is coupled to two reservoirs (modeled as non-interacting 1D wires [26–28]) with distribution functions \( n_R(\epsilon) \) and \( n_L(\epsilon) \) respectively, one obtains [7]

\[
G_R(\tau) = \mp \frac{i\Lambda}{2\pi u} \overline{\Delta}_R[(\delta_R(t))\overline{\Delta}_L(\delta_L(t))] \left[ 1 \pm \frac{i\Lambda}{\pi\tau} \right]^{1+\gamma},
\]

where \( u = v/K \) is the sound velocity,

\[
\gamma = (1 - K)^2/2K,
\]

and

\[
K = (1 + g/\pi v)^{-1/2}
\]

is the standard LL parameter in the interacting region. The phase \( \delta_0(t) \) is found to be a superposition of rectangular pulses:

\[
\delta_0(t) = \sum_{n=0}^{\infty} \delta_n u_t(t, t_n),
\]

where

\[
t_n = (n + 1/2 - 1/2K)L/u
\]

and

\[
\delta_{n,2m} = \pi^{-1} t_0 r_{L,R}^{m+1} \frac{(1 + \eta K)}{\sqrt{K}},
\]

\[
\delta_{n,2m+1} = -\pi^{-1} t_0 r_{L,R}^{m+1} \frac{(1 - \eta K)}{\sqrt{K}}.
\]

Here \( r_0, t_0 \) are reflection and transmission coefficients of plasmons at the left (\( \eta = -1 \)) and right (\( \eta = +1 \)) boundaries. For \( \tau \ll L/u \) the coherence of plasmon scattering may be neglected and the result assumes the form of a product

\[
\overline{\Delta}_R(\delta_0(t)) \simeq \prod_{n=0}^{\infty} \overline{\Delta}_{\eta R}(\delta_{n,0}).
\]

The plasmon scattering causes a fractionalization (cf [27, 29–34]) of the phase soliton, splitting it into an infinite series of pulses. As a result, the Fredholm determinant of a counting operator
takes the form of an infinite product of determinants, each calculated for a rectangular pulse with a corresponding scattering phase $\delta_{\eta,n}(t) = \delta_{\eta,n}\omega T(t, 0)$.

In the case of smooth boundaries with the leads we have $r_\eta \approx 0$, which implies that only the first ($n = 0$) pulse survives in both $\delta_R$ and $\delta_L$. The corresponding amplitudes are depicted in table 1.

The above results correspond to the case where the tunneling spectroscopy point is located inside the interacting part of the wire. The case of tunneling into one of non-interacting leads is studied in the same way; the results can be found in [7].

So far we have discussed Green’s functions at coinciding spatial points having in mind tunneling spectroscopy experiments. Green’s functions at different spatial coordinates are of interest, in particular, in the context of Aharonov–Bohm interferometry [7, 30]. (A similar problem in the context of chiral edge states has been considered in [35–37].) Equation (9) expressing the non-equilibrium LL Green’s function in terms of Fredholm determinants can be generalized for this case as well [7].

2.1.4. Luttinger liquid: counting statistics. In the problem of counting statistics of non-equilibrium LL one is interested in the generating function $\chi(\lambda) = \sum_{n=-\infty}^{\infty} p_\tau(n) e^{i\lambda n}$, where $p_\tau(n)$ is the probability for $n$ electrons to pass through a given cross-section during the time interval $\tau$. On the experimental side, the second moment (shot noise) has been measured in correlated 1D systems [38, 39]; an experimental analysis of higher moments and of the FCS remains a challenging issue.

For the non-interacting case the FCS generating function $\chi(\lambda)$ has been calculated in [40] by means of Landauer scattering-state approach. For an ideal quantum wire (with no scattering inside the wire) with distributions $n_\eta(\epsilon)$, the generating function of FCS is given by

$$\chi(\lambda) = \Delta_R[\delta_R(t)]\Delta_L[\delta_L(t)],$$

with the phases $\delta_\eta(t) = \lambda \eta \omega T(t, 0)$. For an interacting wire (LL) the result [8] retains the form of a product of Fredholm determinants, equation (16), but with the scattering phase $\delta_\eta(t)$ turning into the following sequence of pulses:

$$\delta_\eta(t) = \sum_{n=0}^{\infty} \delta_{\eta,n}\omega T(t, t_0),$$

with partial phase shifts

$$\delta_{\eta,2n} = \eta \lambda T_\eta \sqrt{K} r_\eta^n t_{-\eta} \equiv \eta \lambda e_{\eta,2n},$$

$$\delta_{\eta,2n+1} = \eta \lambda T_\eta \sqrt{K} r_\eta^{n+1} t_{-\eta} \equiv \eta \lambda e_{\eta,2n+1}.$$  

The time moment $t_n = (n + 1/2 - 1/2K)L/u$ corresponds to the beginning of the $n$th pulse.

The result is non-trivial when the measurement interval $\tau$ is small compared to the length of the LL conductor, $\tau \ll L/u$. (In the opposite limit the pulses overlap and one recovers the results of the FCS of non-interacting fermions [40].) As for the tunneling spectroscopy problem, the determinants then split into a product of determinants for individual pulses. The FCS of the LL then becomes a superposition of FCS of non-interacting electrons with fractional charges $e_{\eta,n}^\pm$. For the case of smooth boundaries we obtain only one fractional charge, $e_{\eta,0}^\pm = \sqrt{K}$ (table 1). In the opposite limit of sharp boundaries, we obtain a sequence of fractional charges of the form $e_{\eta,n}^\pm = 2K(1 - K)^n/(1 + K)^{n+1}$.

These results describe current fluctuations in the interacting part of the wire. The FCS measured in the non-interacting part (keeping the assumption $\tau \ll L/u$) has a similar structure, but the values of the fractional charges are different, see [8].
2.2. Luttinger liquid of bosons: Tonks–Girardeau gas

We now consider the problem of a non-equilibrium LL formed by bosons with strong repulsion. Interacting bosons out of equilibrium attract currently a great deal of attention, in particular, in connection with experiments on cold atoms [41].

To make the connection between the fermionic and the bosonic LL particularly transparent, we will adopt a toy model with the same setup, and with the same behavior of the LL interaction constant $K(x)$ as the one assumed for the fermionic models considered above. Specifically, we will assume that $K(x)$ takes the value 1 in the reservoirs, and a value $K$ in the central part of the setup. For a gas of fermions $K = 1$ implies the absence of interaction; now, in the present context, it corresponds to hard-core bosons (whose many-body wavefunction vanishes when the coordinates of two particles coincide); this system is known as the Tonks–Girardeau gas [42]. The wavefunction of this system is related to that of non-interacting fermions via the transformation

$$\psi_B(x_1,\ldots,x_N) = s(x_1,\ldots,x_N)\psi_F(x_1,\ldots,x_N),$$

(20)

where $s(x_1,\ldots,x_N)$ is a sign factor ($\pm 1$) counting the parity of the number of permutations of coordinates:

$$s(x_1,\ldots,x_N) = \prod_{i>j} \text{sgn}(x_i - x_j).$$

(21)

If the fermionic wavefunction is real, this simply yields $\psi_B = |\psi_F|$. We will further assume that after this boson-to-fermion transformation, the fermions in the reservoirs are characterized by distribution functions $n_\eta(\epsilon)$, as in the above fermionic models. While very natural in the fermionic language, this requirement is in general quite artificial for bosons. We do not know whether it can be realized in an experiment and consider this as a theoretical toy model. A more realistic situation arises in the case of partial non-equilibrium, where each of the reservoirs is at equilibrium but the temperatures of the two reservoirs are different.

We will analyze the single-particle bosonic Green’s functions

$$G^>_B(x, t) = -i\langle\Psi_1B(x, t)\Psi_1^\dagger(0, 0)\rangle,$$

(22)

$$G^<_B(x, t) = -i\langle\Psi_1^\dagger(0, 0)\Psi_1B(x, t)\rangle.$$  

(23)

that carry information about spectral properties (density of states and distribution functions) of the system. To calculate them, we proceed, in analogy with fermionic systems, via the non-equilibrium bosonization technique. The term ‘bosonization’ here is, perhaps, not optimal since the original system is bosonic to begin with. What actually happens is a transformation from the original bosonic fields $\Psi_B$ to new bosonic fields $\phi, \theta$, the latter describing density fluctuations in the system. The original field operator is expressed in term of the new fields as [43]

$$\Psi_1^\dagger_B(x) = \sqrt{\rho_0 + \Pi(x)}\left\{ \sum_{m \text{ even}} e^{i m \phi(x)} \right\} e^{-i \theta(x)}.$$  

(24)

Here the field $\phi(x)$ is related to the smeared density $\rho(x) = \rho_0 + \Pi(x)$ (where $\rho_0$ is the average density) via $\rho(x) = -\partial_x \phi(x)/\pi$. The bosonic fields ($\phi, \theta$) satisfy the commutation relation

$$[\phi(x), \theta(x')] = \frac{i \pi}{2} \text{sgn}(x - x').$$

(25)

The bosonization prescription (24) may be further simplified by discarding fast oscillating terms (corresponding to $m \neq 0$ in equation (24)) and neglecting $\Pi(x)$ in comparison to $\rho_0$ in the pre-exponential factor. We thus obtain

$$\Psi_1^\dagger_B(x) \simeq \sqrt{\rho_0} e^{-i \theta(x)}.$$  

(26)
The goal of the discussion above was to present the correlators needed for $G_r^> (x, t)$ in terms of the bosonic fields $\phi, \theta$. We next discuss the pertinent action.

Since the boson-to-fermion transformation (20) preserves the density $|\psi|^2$, and in view of our assumption about the density matrix in the reservoirs, the Keldysh action of the non-equilibrium Tonks–Girardeau gas has exactly the same form as for the corresponding problem of free fermions [7],

$$S_0[\rho, \bar{\rho}] = \sum_\eta S_0[\rho_\eta, \bar{\rho}_\eta].$$

(27)

where $\rho$ and $\bar{\rho}$ are the classical and quantum components in Keldysh representation, and the action for excitations with a given chirality is

$$S_0[\rho_\eta, \bar{\rho}_\eta] = -\rho_\eta \Pi_\eta^{-1} \bar{\rho}_\eta - i \ln Z_\eta[\bar{x}_\eta].$$

(28)

Here $\Pi_\eta$ is the advanced component of the free-fermion polarization operator equal to $\Pi_\eta = nq/2\pi (n \nu_\eta - \omega + i0)$ in the frequency-momentum representation. The information about the non-equilibrium state of the problem is encoded in the infinite sum of vacuum loops

$$i \ln Z_\eta[\bar{x}_\eta] = \sum_{n=2}^{\infty} \frac{i^{n+1}}{n!} \bar{x}_\eta^n S_{n, \eta},$$

(29)

representing a partition function of free fermions subject to the external quantum field

$$\bar{x}_\eta = \Pi_\eta^{-1} \bar{\rho}_\eta.$$  

(30)

Here $S_{n, \eta}$ is the $n$th order density cumulant of free fermions in the given non-equilibrium state. The right and left density components are related to the fields $\phi$ and $\theta$ via

$$\rho_\eta(x) = \frac{\eta}{2\pi} \partial_x \phi_\eta,$$

(31)

where

$$\phi_R = \theta - \phi,$$

$$\phi_L = \theta + \phi.$$  

(32)

In order to find Green’s functions of the original bosons one therefore needs to calculate the functional integral

$$G_r^> (x, t) = -i \rho_0 \int D\phi \, e^{iS[\rho, \bar{\rho}]} \, e^{(-i/\sqrt{2})(\theta[0, 0] - \theta[x, t] + \bar{\theta}[0, 0] + \bar{\theta}(x, t))},$$

(33)

and similarly for $G_r^<$. This functional integral is fully analogous to the one we had to evaluate while studying Green’s functions of free fermions (see [7] for details). The only difference is in the source part, i.e. in the exponent that contains terms linear in bosonic fields. This difference has a rather transparent meaning: creation of the fermion $\psi_\eta$ corresponds to the vertex operator $\Psi_{F, \eta} \propto e^{-i\phi_\eta}$, that generates a soliton (a step-like plasmon wave with amplitude $2\pi$) that propagates in the direction $\eta$. By contrast, bosons represented by the operators $\Psi_B$ do not have chirality; the corresponding creation operator $\Psi_{B, \eta} \propto e^{-i\theta} = e^{-i(\phi_R + \phi_L)}$ generates both right and left moving waves. This results in two Fredholm determinants—one corresponding to the left and another to the right reservoir—both with scattering phases $\pi$ (as opposed to a single determinant with a scattering phase $2\pi$ for free fermions):

$$G_r^> (x, \tau) = -i \rho_0 \Delta_{\tau-x/v}(\pi) \Delta_{\tau+x/v}(\pi) \, e^{-i\pi/2 [\text{sgn}(\tau+x/v) + \text{sgn}(\tau-x/v)]}.$$  

(34)

Here the numerical prefactor has been fixed by comparison with the equilibrium case, see appendix A. At zero temperature $\Delta_{\tau}(\delta) \sim (1 + \Lambda^2 \tau^2)^{-1/2},$ which reproduces the well-known $x^{-1/2}$ (for $\tau = 0$) behavior corresponding to the $k^{-1/2}$ momentum distribution of particles in the equilibrium $T = 0$ Tonks–Girardeau gas.
Equation (34) yields Green’s function of the non-equilibrium Tonks–Girardeau gas with a LL constant $K = 1$. We now assume that the interaction is different in the central part of the system, so that $K(x)$ there is equal to $K \neq 1$. Performing the analysis in analogy with the fermionic problem, we find

$$G_B(x, \tau) = -i \frac{\rho_0}{\Delta_1} \frac{1}{R} \left[ \delta_R(t) \right] \frac{1}{\Delta_1} \left[ \delta_L(t) \right] e^{-\pi \tau \text{sgn}(\tau)} ,$$

(35)

where each of the phases $\delta_n(t)$ is given by the arithmetic mean of the corresponding phase for fermionic Green’s functions $G_{FR}$ and $G_{FL}$. In general we obtain again an infinite sequence of pulses, as for the fermionic problem. In the case of smoothly varying $K(x)$ only the first pulse survives in each of the phases $\delta_n(t)$; its amplitude is equal to $\pi/\sqrt{K}$, as shown in table 1.

2.3. Summary

Let us briefly summarize the results presented above. All the problems considered (Fermi edge singularity, tunneling spectroscopy of interacting fermions, their FCS, and spectral properties of interacting bosons) can be solved by the non-equilibrium bosonization approach, with the results expressed in terms of Fredholm determinants of counting operators. In these expressions, all differences between the problems are encoded in the values of scattering phases $\delta_n(t)$. These scattering phases consist either of one pulse (for the FES problem and LL problem with smooth boundaries), or of a sequence of well-separated pulses (for a sufficiently long LL sample with sharp boundaries). In the latter case the determinant can be split into a product of determinants, each of which corresponding to a single pulse. Therefore, physical properties of a number of many-body problems are governed by the behavior of Fredholm determinants with a single phase pulse. The analysis of this behavior is presented in the next section.

It is worth mentioning that there is a vast literature on the connection of Fredholm determinants to counting statistics as well as to quantum and classical integrable models and free-fermion problems; see, in particular, [40, 44–57].

3. Asymptotic properties of Fredholm determinants

3.1. Ultraviolet regularization and reduction to Toeplitz form

We consider a Fredholm determinant of the counting operator for scattering phase exhibiting a single pulse of an amplitude $\delta$ and duration $\tau$:

$$\delta(t) = \delta \times w_\tau(t, 0).$$

(36)

In this case the Fredholm determinant (7) is of the Toeplitz type. To show this, one defines a projection operator $\hat{P}$ that acts on a function $y(t)$ by restricting it to the time interval $[0, \tau]$:

$$\hat{P} y(t) = \begin{cases} y(t), & \text{for } t \in [0, \tau] \\ 0, & \text{otherwise}. \end{cases}$$

(37)

For the single-pulse scattering phase (36) equation (7) can then be rewritten in the form

$$\Delta[\delta(t)] = \det[1 + \hat{P} (e^{-\delta \hat{n}} - 1) \hat{n}].$$

(38)

Since $\hat{P}^2 = \hat{P}$, we can bring determinant (38) to the form

$$\Delta[\delta(t)] = \det[1 + \hat{P} (e^{-\delta \hat{n}} - 1) \hat{n} \hat{P}].$$

(39)

This determinant still requires an ultraviolet regularization. A possible way to introduce it is to discretize the coordinate $t$ by introducing an elementary unit of the size $\Delta t = \pi/\Lambda$, such that
t_j = j \Delta t. This corresponds to restricting the energy \( \epsilon \) variable to the range \([-\Lambda, \Lambda]\). In the formulations of Szegő and Hartwig–Fisher theorems that will be applied below the function \( f(z) \), that generates the matrix is defined in the complex plane of variable \( z \), and its values on the unit circle \( z = e^{i\theta} \) parameterized by the angle \( \theta \in [-\pi, \pi] \) are important. In equation (39) this function is \( f(\epsilon) = 1 + n(\epsilon)(e^{-i\delta} - 1) \). The correspondence between \( \epsilon \) and \( \theta \) is established by rescaling the energy \( \pi \epsilon / \Lambda = \theta \). Further, we need to eliminate the jump in \( n(\theta) \) at \( \theta = \pm\pi \) that results from a hard cutoff and would generate an additional, unphysical contribution of the Fermi-edge type. This is done by introducing a phase factor,

\[
f(\epsilon) = [1 + n(\epsilon)(e^{-i\delta} - 1)]e^{-i\frac{\pi}{2}\frac{\epsilon}{\Lambda}},
\]

that makes \( f(\epsilon) \) periodic on \([-\Lambda, \Lambda]\). Fourier transforming the periodic function \( f(\epsilon) \), we obtain \( f(t_j) \) with \( t_j = j\pi / \Lambda \). Equation (39) then reduces to a determinant of a large \((N \times N)\), where \( N = \tau \Lambda / \pi \) but finite matrix

\[
\Delta_N[f] = \det[f(t_j - t_k)], \quad 0 \leq j, k \leq N - 1.
\]

In the above derivation we assumed \( \tau > 0 \); the result for \( \tau < 0 \) follows from the property \( \Delta_{-\tau}(\delta) = \Delta_{\tau}(-\delta) \).

The matrix \( f(t_j - t_k) \) with \( 0 \leq j, k \leq N - 1 \) is of a Toeplitz form. Below this will allow us to apply known mathematical results concerning the asymptotic properties of its determinant \( \Delta_N \) in the limit of large \( N \). Physically, this corresponds to the regime of long time \( \tau \), i.e. to infrared asymptotics of correlation functions under interest. For arbitrary times \( \tau \) equations (41) and (40) can be directly used for numerical evaluation of the determinant \( \Delta_N[f] \).

The following important point is also worth stressing. Green’s functions of LL tunneling spectroscopy and FES problems require evaluation of such determinants \( \Delta(\delta) \) at phases \( \delta \) that are not small. For example, in the case of relatively weak LL interaction \( (K \text{ close to unity}) \) or small phase shift \( \delta_0 \) for the scattering on the core hole in the FES problem, one needs to know the determinant at \( \delta \) close to \( 2\pi \), see table 1. For strong LL interaction or large phase shift \( \delta_0 \) the value of \( \delta \) can be, in principle, arbitrarily large. As was discussed in our papers [7] the determinant \( \Delta(\delta) \) should be then understood as analytically continued from the region of small \( \delta \). We emphasize now that the present regularization (discretization of time and introduction of the phase factor ensuring periodicity in energy) implements the required analytic continuation. Indeed, in the original form of the determinant, equation (38), the information about the integer part of \( \delta / 2\pi \) was not explicit, which made the analytic continuation necessary. On the other hand, in the present regularization the integer part of \( \delta / 2\pi \) enters explicitly via the last phase factor in equation (40). As we will demonstrate below, this allows one to directly compute the determinant at arbitrary large \( \delta \).

3.2. Simplified analysis of asymptotics via Szegő formula

The long time behavior of determinants of Toeplitz matrices can be found using the Szegő theorem and its extension known as Fisher–Hartwig conjecture. The condition of applicability of Szegő theorem requires that \( f(z) \) is a sufficiently smooth function. This condition is not fulfilled in our case. Indeed, already at equilibrium (and at zero temperature) \( f(\epsilon) \) has a jump at the Fermi energy. In non-equilibrium situations we are interested in \( f(\epsilon) \) will have two (‘double-step distribution’) or more such jumps. We will see, however, that the Szegő formula nevertheless yields correctly the main ingredients of the result (dephasing rate and modified power-law exponents). A more accurate treatment will be performed below in section 3.3 in the framework of the Fisher–Hartwig formula.
Here we assume $|\delta| < \pi$. (The accurate consideration in section 3.3 will be performed for arbitrary $\delta$.) We first consider a simple case of thermal equilibrium, when determinant (39) can be calculated explicitly

$$
\Delta_\tau (\delta) \simeq \frac{1}{(1 + \Lambda^2 \tau^2)/4 \pi^2} \left( \frac{\pi T \tau}{\sinh \pi T \tau} \right)^{(\delta/2\pi)^2}.
$$

Note that the precise behavior of the functional determinant at the ultraviolet scale, $\tau \sim \Lambda^{-1}$, depends on the regularization procedure. Equation (42) corresponds to a smooth cut-off $e^{-|\epsilon|/\Lambda}$ in the energy space [7] that is different from the regularization we use in this work. We are interested, however, in energy scales much less than $\Lambda$, i.e., $\tau \gg \Lambda^{-1}$, where the determinant does not depend on the regularization scheme, up to an overall prefactor independent on the distribution function $n(\epsilon)$. At $T = 0$ one readily finds from equation (42)

$$
\Delta_\tau (\delta) \simeq (\tau \Lambda)^{-\delta(2\pi)^2}, \quad \tau \gg \Lambda^{-1}.
$$

To apply the Szegő formula (see appendix B), we have to calculate the Fourier transform $V(t_j)$ of $\ln f(\epsilon)$,

$$
V(t_j) = \int_{-\Lambda}^{\Lambda} \frac{d\epsilon}{2\pi} e^{-i\epsilon t_j} \ln f(\epsilon),
$$

where $f(\epsilon)$ is given by equation (40). For the case of the Fermi–Dirac distribution with $T = 0$ one finds

$$
V(t_j) = \frac{\delta}{2\pi} \times \begin{cases} 1/t_j, & t_j \neq 0; \\ -i\Lambda, & t_j = 0. \end{cases}
$$

According to the (strong) Szegő theorem (equation (B.4)), the large-$N$ behavior of the determinant reads, in the present notations

$$
\Delta_N \sim \exp \left\{ N \Delta t V(0) + \Delta t \sum_{j=0}^{N} t_j V(t_j) V(-t_j) \right\}.
$$

The first term in the exponent $N \Delta t V(0) = \tau V(0) = -i\delta \tau \Lambda/2\pi$ is purely imaginary and yields just a phase factor. Calculating the second term, we find

$$
\Delta t \sum_{j=0}^{N} t_j V(t_j) V(-t_j) \simeq -\left( \frac{\delta}{2\pi} \right)^2 \int_{\Lambda^{-1}}^{\tau} \frac{dt}{t} \ln(t \Lambda).
$$

Here, we assumed that the time is sufficiently long (compared to the ultraviolet scale), $\tau \Lambda \gg 1$, which is exactly the condition of applicability of the Szegő theorem ($N \gg 1$). Exponentiation of equation (47) according to equation (46) reproduces the result (43).

It is easy to verify that the Szegő formula yields the correct behavior of the determinant also at finite temperature $T$. (In fact, in the long-time regime $\tau T \gg 1$ the Szegő theorem becomes rigorously applicable.)

We turn now to the non-equilibrium situation and focus on a double step distribution function:

$$
n(\epsilon) = (1 - a)n_0(\epsilon - \epsilon_0) + an_0(\epsilon - \epsilon_1).
$$

Here, $\epsilon_0 = -aU$, $\epsilon_1 = (1 - a)U$, and $n_0(\epsilon)$ is the zero-temperature Fermi–Dirac function, $n_0(\epsilon) = \theta(-\epsilon)$. 

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The function $f(\epsilon)$ has now the form

$$f(\epsilon) = e^{-i\delta} \begin{cases} e^{-i\delta}, & -\Lambda < \epsilon < \epsilon_0 \\ 1 + (e^{-i\delta} - 1)a, & \epsilon_0 < \epsilon < \epsilon_1 \\ 1, & \epsilon_1 < \epsilon < \Lambda. \end{cases} \quad (49)$$

The Fourier transform of $\ln f$, equation (44), reads

$$V(t_j) = \begin{cases} -\frac{1}{t_j} (\beta_0 e^{-i\delta t_j} + \beta_1 e^{i\delta t_j}), & t_j \neq 0 \\ -\frac{\Lambda}{2\pi} + iU (a\frac{\Lambda}{2\pi} + \beta_1), & t_j = 0, \end{cases} \quad (50)$$

where we defined

$$\beta_1 = -\frac{i}{2\pi} \ln \left[ 1 + (e^{-i\delta} - 1)a \right],$$

$$\beta_0 = -\frac{\delta}{2\pi} - \beta_1. \quad (51)$$

Now we apply the Szegő theorem. The leading, linear-in-$t$, term in the asymptotics of $\ln \Delta$ is governed by $V(0)$ given by the second line in equation (50). Particularly important is the real part of $V(0)$ that leads to the exponential decay of the determinant with time, $\Delta \propto \exp(-\tau/2\tau_\phi)$. The corresponding decay rate is given by

$$\tau_\phi^{-1} = 2 \text{Re} V(0) = -\frac{U}{2\pi} \ln \left[ 1 - 4a(1 - a) \sin^2 \frac{\delta}{2} \right]. \quad (52)$$

To find the subleading term in the Szegő formula (46), we have to evaluate the sum $\Delta t \sum_{k=1}^N t_k V(t_k) V(-t_k)$. We get

$$\Delta t \sum_{k=1}^N t_k V(t_k) V(-t_k) \simeq -\int_{-\Lambda^{-1}}^{\Lambda^{-1}} \frac{dt}{2\pi} \left( \beta_0^2 + \beta_1^2 + 2\beta_0\beta_1 \cos Ut \right) \quad (53)$$

$$\simeq \begin{cases} -\left( \frac{\delta}{2\pi} \right)^2 \ln \Lambda \tau, & \tau \ll U^{-1} \\ -\left( \frac{\delta}{2\pi} \right)^2 \ln \frac{\Lambda}{U} - (\beta_0^2 + \beta_1^2) \ln U \tau, & \tau \gg U^{-1}. \end{cases} \quad (54)$$

In the short-time regime, $U\tau \ll 1$, we simply reproduce the equilibrium result. On the other hand, for long times, $U\tau \gg 1$, a different behavior emerges,

$$\Delta \tau(\delta) \sim e^{-\delta/2\pi} \left( \Lambda/U \right)^{-\delta/2\pi} (U \tau)^{-(\beta_0^2 + \beta_1^2)}, \quad (55)$$

where $\beta_0, \beta_1$ are given by equation (51) and $\tau_\phi$ by equation (52).

As has been already mentioned, the present problem goes, strictly speaking, beyond the range of applicability of the Szegő theorem, since the function $f(\epsilon)$ has discontinuities. This results in the (correct) $\ln N$ behavior of the second term in the exponent of (46), while it should have a constant limit as $N \to \infty$ under the conditions of applicability of the Szegő theorem. It turns out, however, that the key results obtained above—dephasing rate and modified power-law exponents—are correct. This will be shown in section 3.3 by using recent mathematical results on the Fisher–Hartwig conjecture which treats Toeplitz determinants of exactly the type we have encountered. The application of the Fisher–Hartwig formalism will allow us not just to confirm the above results but also to go considerably further. First, we will calculate the asymptotics of the determinants exactly, including prefactors $\sim N^0$. Second, we
will obtain results for an arbitrary phase \( \delta \). As we have already emphasized, this is important for the analysis of the many-body problems considered above. Third, we will obtain not only the leading contribution but also subleading terms. As we show below, various contributions have very transparent physical meaning in the problems of FES and tunneling spectroscopy, corresponding to power-law behavior at multiple Fermi edges.

### 3.3. Accurate analysis of asymptotics via Fisher–Hartwig conjecture

It is instructive to begin again by considering the zero-temperature equilibrium case, \( n(\epsilon) = \theta(-\epsilon) \). In the case of a single singular point, the Fisher–Hartwig generating function \( f(z) \) has the form

\[
f(z) = e^{V(z)} |z - z_0|^\beta_0 \left( \frac{z}{z_0} \right)^{\beta_0} g_{z_0, \beta_0}(z),
\]

where \( g_{z_0, \beta_0}(z) \) is a 'jump function', equation (B.6). Comparing equation (56) with equation (40) (where we set \( \theta = \pi \epsilon / \Lambda \) and \( z = e^{i\theta} \)), we identify the parameters: \( z_0 = 1, \alpha_0 = 0, \beta_0 = -\delta / 2\pi, \) and \( V(z) = -i \delta / 2 \). Using the Fisher–Hartwig formula (B.8), we obtain the asymptotics

\[
\Delta_N(\delta) = e^{-\frac{i\delta}{2\pi}} N^{-i(\delta/2\pi)^2} G \left( 1 - \frac{\delta}{2\pi} \right) G \left( 1 + \frac{\delta}{2\pi} \right),
\]

where \( G(z) \) is the Barnes \( G \)-function. In physical notations \( N = \tau \Lambda / \pi \). It is now easy to see that the Fisher–Hartwig formula (B.8) reproduces correctly the power-law behavior (43) of the determinant.

Result (57) is valid for arbitrary \( \delta \). Consider an important case of \( \delta \) in the vicinity of \( 2\pi \), \( \delta = 2\pi + \delta' \) with \( |\delta'| \ll 1 \), that is relevant to the FES problem with small scattering phase and to tunneling spectroscopy of LL with weak interaction. Using \( G(1) = 1 \) and \( G(z) \simeq z \) for small \( z \), we get

\[
\Delta_\tau(\delta) \simeq - \frac{\delta'}{2\pi} e^{-i\tau \Lambda (1+\delta'/2\pi)} \left( \frac{\tau \Lambda}{\pi} \right)^{-i(\delta'/2\pi)^2}.
\]

It was shown earlier [7] that \( \Delta_\tau(\delta) \) at \( \delta \to 2\pi \) should yield, up to a proportionality factor, the free fermion Green’s function \( G_0(\tau) \), see equation (5). As we see from equation (57), the exact correspondence in the present ultraviolet regularization of the determinant is

\[
G_0^Z(\tau) = e^{i\tau \Lambda / \pi} \frac{\Lambda}{\pi v} \frac{1}{1 + i \Lambda \tau} \lim_{\delta' \to 0} \frac{1}{\delta'} \Delta_\tau(2\pi + \delta').
\]

We will demonstrate below how this general formula works for the case of a multiple-step distribution.

We now turn to the case of a double-step distribution (48). The singular points are \( z_j = e^{i\pi \epsilon_j / \Lambda} (j = 0, 1), \) where \( \epsilon_0 = -aU, \epsilon_1 = \epsilon_0 + U \). We consider first the case of \( |\delta| < \pi \). It is easy to see that equation (49) is of Fisher–Hartwig form of generating function with two jump-type singularities:

\[
f(z) = e^{V(z)} \left( \frac{z}{z_0} \right)^{\beta_0} \left( \frac{z}{z_1} \right)^{\beta_1} g_{z_0, \beta_0}(z) g_{z_1, \beta_1}(z),
\]

with \( \beta_j (j = 0, 1) \) given by equation (51) and

\[
V(z) = \text{const} = -\frac{i\delta}{2} + \frac{iU \pi}{\Lambda} \left( \frac{a \delta}{2\pi} + \beta_1 \right).
\]
The logarithm in equation (51) and in analogous formulas for $\beta'_j$ below is understood in the sense of its main branch (with imaginary part between $-\pi$ and $\pi$). We further note that under the condition $U \ll \Lambda$ we can approximate $|z_1 - z_0| \approx \pi U / \Lambda$. Applying equation (B.8), we thus get

$$
\Delta_r(\delta) \approx \exp \left\{ -\frac{i\delta}{2\pi} \tau \Lambda - i\tau \mu - \tau / 2 \tau_\phi \right\} \left( \frac{\tau \Lambda}{\pi} \right)^{-\delta_1 + \beta_1} \left( \frac{\Lambda}{\pi U} \right)^{-2\beta_0 \beta_1} \times G(1 + \beta_0)G(1 - \beta_0)G(1 + \beta_1)G(1 - \beta_1),
$$

(62)

where $1/\tau_\phi$ is the exponential decay (dephasing) rate, equation (52), and $\mu = -U(\text{Re} \beta_1 + a\delta/2\pi)$. This confirms the long-time ($U\tau \gg 1$) behavior of the determinant obtained above from the Szegő formula, equation (55), and yields the exact value of the corresponding prefactor.

Consider now the more general situation, when $\delta$ is not small: $\delta = 2\pi M + \delta'$ with some integer $M$ and $|\delta'| < \pi$. The exponents $\beta_0$ and $\beta_1$ can now be chosen to be

$$
\beta_1 = -\frac{1}{2\pi} \ln(1 - a + a e^{-i\delta}) \equiv \beta_1',
$$

(63)

$$
\beta_0 = -\frac{\delta}{2\pi} - \beta_1 = -M - \frac{\delta'}{2\pi} - \beta_1' \equiv -M + \beta_0'.
$$

(64)

We have introduced here $\beta_0'$ and $\beta_1'$ satisfying $|\text{Re} \beta_0'| < 1/2$, $\beta_0' + \beta_1' = -\delta'/2\pi$. The exponents $\beta_j$ (that may differ from $\beta_j'$ by an integer only) satisfy $\beta_0 + \beta_1 = -\delta/2\pi \equiv -M - \delta'/2\pi$. Equations (63) and (64) represent one possible choice; the final result will not depend on a particular choice in view of the summation over integers $n_j$ in equation (B.9). We obtain from equation (B.9):

$$
\Delta_r(\delta) \approx \exp \left\{ -\frac{i\delta}{2\pi} \tau \Lambda - i\tau \mu - iM\epsilon_0 \tau - \tau / 2 \tau_\phi \right\} \sum_{n=-\infty}^{\infty} \left( \frac{\tau \Lambda}{\pi} \right)^{-M+n+\delta-n} \times \left( \frac{\Lambda}{\pi U} \right)^{-2\beta_0 (M+n)+\beta_1' (\delta-n)} \times \exp \left\{ -i\pi U \tau (1 - a) e^{i(\delta - \beta_1') \tau} \right\} \times G(1 + \beta_0')G(1 - \beta_0')G(1 + \beta_1')G(1 - \beta_1'),
$$

(65)

where $\mu' = -U(\text{Re} \beta_1 + a\delta'/2\pi)$. When $M = 0$, the dominant term (that is characterized by the smallest power-law exponent $\text{Re}[(\beta_0' - M + n)^2 + (\beta_1' - n)^2]$) is the one with $n = 0$, reproducing equation (62).

In the particularly interesting case of $M = 1$, when $\delta$ is in the vicinity of $2\pi$, the two leading terms are those with $n = 0$ and $n = 1$. Retaining only these terms, we find

$$
\Delta_r(\delta) \approx \exp \left\{ -\frac{i\delta}{2\pi} \tau \Lambda - i\tau \mu' - \tau / 2 \tau_\phi \right\} \left( \frac{\tau \Lambda}{\pi} \right)^{-\beta_0 (M+n)-\beta_1 (\delta-n)^2} \left( \frac{\Lambda}{\pi U} \right)^{-2\beta_0 \beta_1} \left( -\frac{\delta'}{2\pi} \right) \times \left( 1 - a \right) e^{i(\delta - \beta_1') \tau} \left( \frac{\tau \Lambda}{\pi} \right)^{2\beta_0} \left( \frac{\Lambda}{\pi U} \right)^{2\beta_1} + a e^{i(\delta - \beta_1') \tau} \left( \frac{\tau \Lambda}{\pi} \right)^{-\beta_0} \left( \frac{\Lambda}{\pi U} \right)^{-\beta_1'},
$$

(66)

In the limit $\delta' \to 0$, substituting equation (66) into equation (59), we correctly reproduce the free-electron Green’s function for the double-step distribution, $G_0^\delta(\tau) = [(1 - a) e^{i(\delta - \beta_1') \tau} + a e^{i(-a) (\delta - \beta_1') \tau}] G_{0,T=0}^\delta(\tau)$, where $G_{0,T=0}^\delta(\tau)$ is the equilibrium, $T = 0$ value of $G_0^\delta(\tau)$. 


Figure 1. Distribution functions with multiple edges. Two examples of three-step distributions are shown: \((A)\) with monotonously decreasing occupation and \((B)\) with population inversion.

These results can be generalized to a multi-step distribution, figure 1. Consider a distribution function of the form

\[
n(\epsilon) = \begin{cases} 
1 \equiv a_0, & \epsilon < \epsilon_0 \\
a_1, & \epsilon_0 < \epsilon < \epsilon_1 \\
\ldots \\
a_m, & \epsilon_{m-1} < \epsilon < \epsilon_m \\
0 \equiv a_{m+1}, & \epsilon_m < \epsilon .
\end{cases} \tag{67}
\]

Here, all \(a_j\) with \(j = 1, \ldots, m\) satisfy \(0 \leq a_j \leq 1\) without any further restrictions. In particular, no requirement of monotonicity is imposed: the distribution \(n(\epsilon)\) can describe inversion of population in some regions of energy, with \(a_{j+1} > a_j\). Using equation (B.9), we obtain for \(\delta = 2\pi M + \delta'\) where, as before, \(M\) is the integer closest to \(\delta/2\pi\):

\[
\Delta_\tau (\delta) \simeq \exp \left\{ -i \frac{\delta}{2\pi} \tau \Lambda - i \tau \mu' - \tau/2\tau_\phi \right\} \sum_{n_{0}+\ldots+n_{m}=-M} \exp \left\{ i \tau \sum_j n_j \epsilon_j \right\} \left( \frac{\tau \Lambda}{\pi} \right)^{-\sum_j \beta'_j} \times \prod_{j<k} \left( \frac{\Lambda}{\pi U_{jk}} \right)^{-2\beta'_j \beta'_k} \prod_j G(1+\beta'_j)G(1-\beta'_j) \bigg|_{\beta'_j=\beta'_j+n_j}. \tag{68}
\]

Here, the exponents \(\beta'_j\) (satisfying \(|\text{Re} \beta'_j| < 1/2\)) are

\[
\beta'_j = -\frac{i}{2\pi} \left[ \ln(1 - a_i + a_j e^{-i\delta}) - \ln(1 - a_{j+1} + a_{j+1} e^{-i\delta}) \right]. \tag{69}
\]

the dephasing rate reads

\[
\frac{1}{\tau_\phi} = 2 \text{Im} \sum_j \beta'_j \epsilon_j = -\frac{1}{2\pi} \sum_{j=1}^m (\epsilon_j - \epsilon_{j-1}) \ln \left[ 1 - 4a_j(1 - a_j) \sin^2 \frac{\delta}{2} \right], \tag{70}
\]

\(U_{jk} = |\epsilon_j - \epsilon_k|\), and \(\mu' = -\text{Re} \sum_j \beta'_j \epsilon_j\).

The remarkable periodicity in the dependence of the dephasing rate for a multi-step distribution, as a function on the phase \(\delta\), should be emphasized. When applied to LL spectroscopy, this results in the periodic dependence of \(1/\tau_\phi\) on the interaction strength [7]. The dephasing rate manifests itself in a broadening of the singularities in the energy space.
(see section 3.4), as well as in an exponential damping of Aharonov–Bohm oscillations in out-of-equilibrium interferometry [7].

While the above results are obtained in time representation, the experimental measurements of Green’s functions describing the FES, equation (8) and the LL tunneling spectroscopy, equation (9), are normally performed in energy space. It is thus important to see what the implications of the above findings in the energy representation are.

### 3.4. Singularities in energy representation

After Fourier transforming from time into energy space, equation (68) yields multiple power-law singularities of the type $|\epsilon + \sum_{j} n_j \epsilon_j - \mu'|^{-1 + \sum_{j} \beta'_j}$, where $\beta'_j = \beta'_j + n_j$. Positions of the singularities are given by linear combinations of the singular points $\epsilon_j$ of the distribution function, with a small overall shift $\mu'$. All singularities are broadened by the dephasing rate $1/2\tau_\phi$.

As an important example, consider the case $M = 1$, where $\delta$ is close to $2\pi$. The leading singularities then correspond to all $n_l$ being equal to 0 except for $1$, $n_k = -1$. The position of such a singularity is close to $\epsilon = \epsilon_k$ and the exponent is $\sum_{j} \beta'_j^2 = 1 - 2\beta'_k + \sum_{j} (\beta'_j)^2$. There is such a singularity for each of the singular points of the original distribution function (i.e. for each $k = 0, 1, \ldots, m$) as expected. Further singularities are much weaker. The next ones correspond to all $n_l$ being zero except for $n_k = n_l = 1$ and $n_p = 1$. Such a singularity is located at $\epsilon = \epsilon_k + \epsilon_j - \epsilon_p$, with an exponent $\sum_{j} \beta'_j^2 = 3 - 2\beta'_k - 2\beta'_p + 2\beta'_j + \sum_{j} (\beta'_j)^2$. The next ones are generated by $n_k = -2, n_l = 1$, located at $\epsilon = 2\epsilon_k - \epsilon_j$ and are characterized by exponents $\sum_{j} \beta'_j^2 = 5 - 4\beta'_k + 2\beta'_j + \sum_{j} (\beta'_j)^2$, and so on. These subleading singularities can be understood as resulting from inelastic processes. For example, the edge $\epsilon = \epsilon_k + \epsilon_j - \epsilon_p$ results from a creation of an electron near $\epsilon_k$ accompanied by creation of a particle-hole pair with energies near $\epsilon_j$ and $\epsilon_p$, respectively.

Let us present results for dominant singularities for the FES Green’s function (8) in the energy space in an explicit form. (The LL tunneling spectroscopy Green’s function is analyzed in the same way.) The behavior of Green’s functions $G^\infty(\epsilon)$ for energies close to singular points $\epsilon_k + \mu'$ is given by (up to an additive contribution that can be considered as constant near $\epsilon_k + \mu'$)

$$G^\infty(\epsilon) \simeq \pm \frac{i}{2^v} \prod_{j \neq k} \left( \frac{\Lambda}{\pi U_{jk}} \right)^{2\beta'_j} \prod_{j < l} \left( \frac{\Lambda}{\pi U_{jl}} \right)^{-2\beta'_j} (\epsilon - \epsilon_k - \mu' - i/2\tau_\phi)^{\gamma_k}$$

$$\times \begin{cases} 1 \pm (\alpha_k - \alpha_{k+1}), & \epsilon < \epsilon_k + \mu' \\ 1 \pm (\alpha_k - \alpha_{k+1}), & \epsilon_k + \mu' < \epsilon \end{cases}$$  

(71)

where the exponents $\gamma_k$ are given by $\gamma_k = -2\beta'_k + \sum_{j} (\beta'_j)^2$.

Let us assume for simplicity that all distances between consecutive singular points are of the same order, $U_{j,j-1} \sim U$. The region of validity of the behavior (71) is then $|\epsilon - \epsilon_k - \mu'| \lesssim U$. Let us emphasize that the power-law singularity in equation (71) is smeared by the dephasing $1/2\tau_\phi$. In a generic situation, when the phase $\delta' = \delta - 2\pi$ is of order $\pi$ (i.e. not small), the dephasing rate is of order $U$. Then the smearing is strong, and the power law essentially does not have room to develop. On the other hand, when the phase $\delta = 2\pi + \delta'$ is close to $2\pi$ (which corresponds to a weak interaction in LL or to small phase shift for scattering on core hole in the FES problem), the dephasing rate is small as $1/2\tau_\phi \sim (\delta')^2 U$, which yields a parametrically broad interval for the power-law behavior, $(\delta')^2 U \lesssim |\epsilon - \epsilon_k - \mu'| \lesssim U$. Note, though, that the power-law exponents $\gamma_k$ in this situation are also small, $\gamma_k \sim \delta'$, so that the power law essentially reduces to a logarithmic correction.
Figure 2. Schematic results for TDOS in a LL with not-too-strong interaction and with multiple-step distributions. The distributions of electrons from both reservoirs are assumed to be equal and of the type shown in figure 1(A) (upper pane) or figure 1(B) (lower panel). The exponents $\gamma_i$ characterizing ZBA at multiple edges are indicated. All singularities are broadened by the non-equilibrium dephasing rate $1/2\tau_\phi$.

Another case, where the dephasing is absent, is non-monotonous distribution with alternating values $a_{2k} = 1$ and $a_{2k+1} = 0$, as one that is shown in figure 1(B).

When applied to the problem of split FES, see equation (8), our result agrees with that obtained by Abanin and Levitov [16] for a double-step distribution. A related result for FES in a system with inverse population (similar to figure 1(B) but with $a_0 = 0$) was obtained by Tanguy and Combescot [58].

It should be emphasized that our approach is different from the one used in [16]. While we work in the non-equilibrium bosonization framework and present Green’s function in terms of a single determinant $\Delta(\delta)$ at the phase $\delta = 2\pi - 2\delta_0$, Abanin and Levitov used the fermionic FES theory and obtained the result in the form of a product of a determinant $\Delta(-2\delta_0)$ and Green’s function, and then analyzed both terms by an approximate solution of the corresponding Riemann–Hilbert problem.

Applying these results to the LL tunneling spectroscopy, see equation (9), we obtain split power-law singularities, with modified (compared to the equilibrium regime) exponents and with broadening by the non-equilibrium dephasing rate $1/2\tau_\phi$. This is illustrated in figure 2 where we show the behavior of the tunneling density of states (TDOS), $\nu(\epsilon) = [G^-(\epsilon) - G^+(\epsilon)]/2\pi i$. As discussed above, for a weak interaction the non-equilibrium power laws reduce to logarithmic corrections and are weakly smeared (by small $1/\tau_\phi$). For an arbitrary strength of interaction the scale for smearing of singularities becomes comparable to
the distance between the singular points. The profile of TDOS in a general situation can be obtained by numerical evaluation of the Toeplitz determinant.

It is worth mentioning some further recent works that addressed the non-equilibrium LL spectroscopy. Influence of non-equilibrium conditions on exponents was also found within the functional renormalization group approach (by using approximations justified for weak interaction) in [19]; dephasing was discarded there. Qualitatively similar results (modification of exponents and the oscillatory dependence of dephasing rate on the interaction strength) were also obtained in [23]. The difference is due to the fact that the non-equilibrium setup of [23] (a biased quantum wire with an impurity inside the interacting region) is different from that of our work, where the non-equilibrium distribution is assumed to be formed by scattering outside of the interacting LL region.

If the phase $\delta$ is close to $4\pi$, $M = 2$, the leading singularities are given by $n_k = n_l = -1$, located at $\epsilon = \epsilon_k + \epsilon_l$, with an exponent $\sum_j \beta_j^2 = 2 - 2\beta_k^2 - 2\beta_l^2 + \sum_j (\beta_j')^2$. The next singularities are produced by $n_k = -2$, located at $\epsilon = 2\epsilon_k$, with an exponent $\sum_j \beta_j^2 = 4 - 4\beta_k^2 + \sum_j (\beta_j')^2$, etc. For the case of a double-step distribution and $\delta = 4\pi$ this agrees with the exact result for $\Delta \tau$ obtained in [7].

4. Conclusion

To conclude, we have considered several many-body problems out of equilibrium, including the FES, the counting statistics, and the tunneling spectroscopy in LL of fermions as well as of bosons with hard-core repulsion. We have shown that the correlation function in all these problems may be expressed in terms of Fredholm determinants of counting operators. The operators are controlled by the (non-equilibrium) distribution function, as well as by the value of the scattering phase depending on the interaction strength. Our non-equilibrium bosonization approach allows us to solve these problems and to establish connections among them.

We have performed an analysis of the long-time asymptotics of the relevant Fredholm determinants (which are of Toeplitz form). In the interesting case of double-step (or, more generally, multiple-step) distribution functions the corresponding generating functions possess Fisher–Hartwig singularities induced by Fermi edges. When transformed from time into energy representation, the results reveal power-law behavior, associated with multi-particle processes at various discontinuities of distribution function (edges). The power laws thus obtained differ from equilibrium one; in addition, the singular behavior is broadened by non-equilibrium dephasing rate.

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Appendix A. Hard-core bosons at equilibrium

At equilibrium the action is quadratic and to find Green’s function one needs to calculate Gaussian functional integrals.
\[ G_B^>(\tau) = -i\rho_0 \int D\phi \exp \left( iS[\phi_L] + \frac{i}{2\sqrt{2}} \sum_{\omega, q} [\phi_L(\omega, q)(1 - e^{-i\omega\tau}) + \bar{\phi}_L(\omega, q)(1 + e^{-i\omega\tau})] + (L \leftrightarrow R) \right). \]  

(A.1)

The action at equilibrium is given by

\[ S[\phi_L] = -\sum_{\omega, q} \left( \frac{q^2}{2\pi} \right)^2 \left[ \phi_L(-\omega, -q)\Pi_L^{-1}(\omega, q)\bar{\phi}_L(\omega, q) + \frac{1}{2} \bar{\phi}_L(-\omega, -q)\Pi_L^{-1}(\omega, q)\phi_L(\omega, q) \right]. \]  

(A.2)

Performing the Gaussian integration over bosonic fields, one finds Green’s functions

\[ G_B^>(\tau) = -i\rho_0 \exp \left( -\frac{i\pi^2}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{\sqrt{\pi T\tau}}{\sinh \pi T\tau} \exp \left( \frac{\omega^2 T}{2} \right) \right) \left[ \prod_{\omega, q} \left( 1 - \cos \omega\tau \right) - 2i \sin \omega\tau \Pi_L(\omega, q) + (L \leftrightarrow R) \right]. \]  

(A.3)

The integrals over momentum \( q \) can be easily calculated

\[ \int \frac{dq}{2\pi q^2} \Pi_L(\omega, q) = -\frac{i}{4\pi\omega}, \]  

\[ \int \frac{dq}{2\pi q^2} \Pi_L(\omega, q) = \frac{i}{2\pi\omega} \coth \frac{\omega}{2T}. \]  

(A.4)

Performing the standard integrals over \( \omega \) one obtains

\[ G_B^>(\tau) = -i\rho_0 \sqrt{\frac{\pi T\tau}{\sinh \pi T\tau}} \exp \left( \frac{-\omega^2 T}{2} \right). \]  

(A.5)

Switching to the frequency domain one obtains

\[ G_B^>(\omega) = -\frac{i\rho_0}{\pi T} \left( \frac{T}{2\pi} \right)^{\frac{1}{2}} \left| \Gamma \left( \frac{1}{4} + \frac{i\omega}{2\pi T} \right) \right|^2 \exp \left( \frac{-\omega^2 T}{2} \right). \]  

(A.6)

Similar calculation for the \( G^< \) component yield

\[ G_B^<(\omega) = -\frac{i\rho_0}{\pi T} \left( \frac{T}{2\pi} \right)^{\frac{1}{2}} \left| \Gamma \left( \frac{1}{4} + \frac{i\omega}{2\pi T} \right) \right|^2 \exp \left( \frac{-\omega^2 T}{2} \right). \]  

(A.7)

The ratio between Green’s functions \( G^> \) and \( G^< \) is equal to

\[ \frac{G_B^>(\omega)}{G_B^<(\omega)} = \exp \frac{\omega T}{2}. \]  

(A.8)

in agreement with the fluctuation–dissipation theorem.

**Appendix B. Mathematical background: Szegő and Fisher–Hartwig formulas**

For completeness we present here the short summary of known mathematical results concerning the theory of Toeplitz determinants, see [59] and references therein. A Toeplitz matrix \( \{ f_{ij} \} \), \( 0 \leq i, j \leq N - 1 \) is generated by a complex-valued function \( f(z) \) on the unit circle \( z = e^{i\theta} \), where \( \theta \) is the polar angle \( \theta \in [0, 2\pi] \). Entries of the Toeplitz matrix are the Fourier coefficients

\[ f_j = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f(e^{i\theta}) e^{-i\theta j}. \]  

(B.1)
The Szegő theorem is formulated for the case when \( f(z) = e^{V(z)} \) is non-zero and sufficiently smooth on the unit circle. The function \( V(z) \) may be described by its Fourier harmonics

\[
V(z) = \sum_{k=\infty}^{\infty} V_k z^k, 
\]

where

\[
V_k = \int_{-\pi}^{\pi} \left( \frac{d\theta}{2\pi} \right) V(z) z^{-k}. 
\]

The smoothness condition requires that \( \sum_{k=\infty}^{\infty} |k||V_k| \) converges. It is further assumed the \( \arg V(z) \) returns to its original value (rather than picking up a \( 2\pi n \) contribution with non-zero integer \( n \)) when \( z \) goes around the unit circle. According to the (strong) Szegő theorem, the large-\( N \) asymptotic behavior of the determinant \( \Delta_N[f] \) of the corresponding matrix is

\[
\Delta_N[f] = \exp \left( NV_0 + \sum_{k=\infty}^{\infty} k V_k V_{-k} \right). 
\]

The Fisher–Hartwig formula deals with Toeplitz matrices of a more general form, with generating function \( f(z) \) having \( m + 1 \) singularities \( (m = 0, 1, 2, \ldots) \),

\[
f(z) = e^{V(z)} \sum_{j=0}^{m} \prod_{0 \leq j < k \leq m} |z - z_j|^{2\beta_j} \beta_j(z) z^{-\beta_j}, 
\]

where

\[
g_{\alpha_j, \beta_j}(z) = \begin{cases} e^{i\pi \alpha_j}, & -\pi < \arg z < \theta_j \\ e^{-i\pi \beta_j}, & \theta_j < \arg z < \pi. \end{cases} 
\]

The singularities are located at points \( z_j = e^{i\theta_j} \) with \( j = 0, \ldots, m \); for definiteness, they can be assumed to be ordered as follows:

\[
-\pi = \theta_0 < \theta_1 < \cdots < \theta_m < \pi. 
\]

The strength of singularities is controlled by a set of parameters \( \alpha_j, \beta_j \), satisfying \( \Re \alpha_j > -\frac{1}{2}, \beta_j \in \mathbb{C} \).

Derivation of the asymptotic behavior of Toeplitz determinant with Hartwig–Fisher singularities as well as overview of previous literature can be found in the recent work [59]. In this paper, we are interested in a particular case of \( \alpha_j = 0 \) and \( V(z) = \const \equiv V_0 \). Indeed, function \( (40) \), with distribution \( n(\epsilon) \) having double-step or multiple-step form (superposition of two or more zero-temperature Fermi distributions with different chemical potentials) belongs exactly to this class of function. We thus present the results for this particular case only, referring the reader to [59] for general results. If all \( \beta_j \) are sufficiently close to each other, such that \( |\Re \beta_j - \Re \beta_k| < 1 \) for all \( j, k = 0, \ldots, m \), the asymptotic behavior of the determinant reads

\[
\Delta_N = e^{NV_0} N^{-\sum_{j=0}^{m} \beta_j^2} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2\beta_j \beta_k} \prod_{j=0}^{m} G(1 + \beta_j) G(1 - \beta_j). 
\]

where \( G(z) \) is the Barnes \( G \)-function. A more general results that is valid for any values of \( \beta_j \) and yields also subleading contributions has the following form:

\[
\Delta_N = e^{NV_0} \sum_{n_1 + \cdots + n_m = 0} \prod_{j=0}^{m} \xi_j^{n_j} \left[ N^{-\sum_{j=0}^{m} \beta_j^2} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2\beta_j \beta_k} \right] \prod_{j=0}^{m} G(1 + \beta_j) G(1 - \beta_j) \beta_j \rightarrow \beta_j + n_j. 
\]
The summation in equation (B.9) goes over all sets of integers \(n_0, n_1, \ldots, n_m\) satisfying \(\sum_{j=0}^{m} n_j = 0\). This formula plays a central role in our analysis of the asymptotics of determinants governing Green’s functions of many-body problems in section 3.3. As we show there, the shifts by integers \(n_j\) in equation (B.9) generate contributions corresponding to multiple Fermi edges to these Green’s functions.

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