A slowly rotating perfect fluid body in an ambient vacuum

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Abstract

A global model of a slowly rotating perfect fluid ball in general relativity is presented. To second order in the rotation parameter, the junction surface is an ellipsoidal cylinder. The interior is given by a limiting case of the Wahlquist solution, and the vacuum region is not asymptotically flat. The impossibility of joining an asymptotically flat vacuum region has been shown in a preceding work.

1 Introduction

In a preceding paper [1], henceforth called I, the impossibility of matching the Wahlquist metric to an asymptotically flat vacuum domain was shown. This result is not too surprising in the light of investigations in [2], where the tendency of the matching conditions to be overdetermined has been pointed out. However, it would be very embarrassing from the point of view of general relativity if this matching turned out to be impossible to any vacuum region. In this paper, the problem of matching of the slowly rotating Wahlquist metric to a more general vacuum exterior is investigated to a precision of quadratic order in the angular velocity.

An approximation scheme for slow rotation was first introduced by Brill and Cohen [3, 4] to first order in the small angular velocity. In our paper we are using the the formalism developed by Hartle [5], taking into account quadratic-order terms in the power series expansion in the angular velocity $\Omega$ of the fluid. The metric of both regions has the form
\[ds^2 = (1 + 2h)A^2dt^2 - (1 + 2m)B^2dr^2 - (1 + 2k)C^2\left\{d\theta^2 + \sin^2\theta \left[ d\varphi + (\Omega - \omega)dt \right]^2 \right\}\] (1)

where the static, spherically symmetric state is described by the functions \(A\), \(B\) and \(C\) depending only on the radial coordinate \(r\), while the functions \(\omega\), \(h\), \(m\) and \(k\) can, in general, depend both on \(r\) and \(\theta\). The rotation potential \(\omega\) is of first order in the angular velocity \(\Omega\), and the functions \(h\), \(m\) and \(k\) are of order \(\Omega^2\).

In the Wahlquist interior domain, the rotation potential \(\omega\) is a function of the radial coordinate alone. Hence, from the junction conditions it follows, that \(\omega\) is independent of \(\theta\) in the vacuum region as well. In our model of the spacetime we drop the condition of asymptotic flatness, and we perform the matching with the most general vacuum metric with \(\omega = \omega(r)\). Likewise the expansions of the second-order metric functions in Legendre polynomials are sought in the form \(h = h_0 + h_2P_2(\cos\theta), m = m_0 + m_2P_2(\cos\theta), k = k_2P_2(\cos\theta)\), where the functions \(h_0, h_2, m_0, m_2\) and \(k_2\) depend only on the radial coordinate \(r\), and \(P_2(\cos\theta) = (3\cos^2\theta - 1)/2\).

In Sec. 2 of this paper, we present the form of these functions for the vacuum domain, and investigate the effect of the non asymptotically flat part of the perturbations on the curvature. In Sec. 3., the junction conditions are calculated, and the constants determining the exterior metric as functions of the unperturbed radius \(r_1\) are given in full detail.

2 The Vacuum Exterior

In this section we consider the form of the vacuum metric to the required accuracy. The unperturbed metric is described by the Schwarzschild solution, \(A^2 = 1/B^2 = 1 - 2M/r\) and \(C = r\). The solution for the perturbed metric is of the form (1), with \(\Omega = 0\) and

\[\omega = \frac{2aM}{r^3},\] (2)

where the additive constant of integration has been removed by a rigid rotation. Integration of the second-order metric functions yields

\[h_0 = \frac{1}{r - 2M} \left( \frac{a^2M^2}{r^3} + \frac{r}{2M}c_2 \right)\] (3)

\[m_0 = \frac{1}{2M - r} \left( \frac{a^2M^2}{r^3} + c_2 \right)\] (4)

\[h_2 = 3c_1r(2M - r)\log\left(1 - \frac{2M}{r}\right) + a^2\frac{M}{r^3}(M + r) + 2c_1\frac{M}{r}(3r^2 - 6Mr - 2M^2)\left(1 - \frac{2M}{r}\right) + \left(1 - \frac{2M}{r}\right)r^2q_1\] (5)
\[ k_2 = 3c_1(r^2 - 2M^2) \log \left(1 - \frac{2M}{r}\right) - a_2^2 \frac{M}{r^4}(2M + r) \]
\[-2c_1 \frac{M}{r}(2M^2 - 3Mr - 3r^2) + (2M^2 - r^2) q_1 \]
\[ m_2 = 6a_2^2 \frac{M^2}{r^4} - h_2 . \]

In this approximation, the slowly rotating solution is characterized by the mass \( M \), the first order small rotation parameter \( a \), and the second order small constants \( c_1, c_2 \), and \( q_1 \).

With the special choice \( q_1 = 0 \), the metric is asymptotically flat, and this form was used in [1] to show the impossibility of matching the Wahlquist solution to an asymptotically flat vacuum. Since the asymptotic behaviour changes completely when \( q_1 \) becomes nonzero, the far field behaviour cannot be treated in a perturbative way, and the present series expansion is certain to hold only within an open neighborhood of the junction surface. This is also indicated by the fact that the terms containing the integration constant \( q_1 \) tend to \( r^2 \) as \( r \to \infty \).

To explore the effects of the \( q_1 \) perturbations on the curvature, we use a canonical locally nonrotating Lorentz tetrad for the metric (9):

\[ e_0 = \left(1 - h + \frac{r^2}{2A^2} \omega^2 \sin^2 \vartheta \right) \frac{1}{A} \frac{\partial}{\partial t} \]
\[ e_1 = (1 - m)A \frac{\partial}{\partial r} \]
\[ e_2 = (1 - k)r^{-1} \frac{\partial}{\partial \vartheta} \]
\[ e_3 = -\frac{r}{A^2} \omega \sin \vartheta \frac{\partial}{\partial \vartheta} - \left(1 - k + \frac{r^2}{2A^2} \omega^2 \sin^2 \vartheta \right) \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi} . \]

In this tetrad, the gravitoelectric \( (E_i) \) and gravitomagnetic \( (H_i) \) fields are defined in terms of the Riemann tensor as follows [6]:

\[ E_1 = R_{0101} \quad E_2 = R_{0202} \quad E_3 = R_{0102} \]
\[ H_1 = R_{0123} \quad H_2 = -R_{0213} \quad H_3 = R_{0223} . \]

In the asymptotic region, \( r \to \infty \), the gravimagnetic part of the curvature goes to zero. The gravielectric components tend to the finite values

\[ \lim_{r \to \infty} E_1 = (1 - 3 \cos^2 \vartheta) q_1 \]
\[ \lim_{r \to \infty} E_2 = (1 - 3 \sin^2 \vartheta) q_1 \]
\[ \lim_{r \to \infty} E_3 = 3 \sin \vartheta \cos \vartheta q_1 . \]

The algebraic structure of the Weyl tensor at infinity, (14)-(16), guarantees that the components cannot all be transformed to zero, and hence the space-time
cannot be asymptotically flat. However, these values are obtained by means of perturbative calculations, and higher-order corrections may contribute by divergent terms to the limiting values.

The angular behavior of the gravielectric field indicates that a quadrupolar mass distribution at large distance may act as the source of the deviations from asymptotic flatness. To show this, let us consider a large sphere with radius $R$, and with a non-uniform surface density distribution

$$\mu = \sum_{i=0}^{\infty} \mu_{2i} P_{2i}(\cos \vartheta) ,$$  \hspace{1cm} (17)

where $\mu_{2i}$ are constants. In the weak-field approximation, linearizing around the background Minkowski metric and expanding in powers of $1/R$, the metric near the center of the sphere can be written as

$$ds^2 = (1 + 2\psi)dt^2 - (1 - 2\psi)(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2)$$  \hspace{1cm} (18)

where

$$\psi = -\frac{1}{2} \mu_0 R - \frac{1}{10R} \mu_2 r^2 P_2(\cos \vartheta) + O\left(\frac{1}{R^3}\right) .$$  \hspace{1cm} (19)

(With the conventions used in I the gravitational constant is $\frac{1}{16\pi}$.) The $\mu_0$ terms can be absorbed by rescaling the $t$ and $r$ coordinates, while the $\mu_2$ terms have exactly the same angular and radial dependence as the $q_1$ terms in the far field exterior vacuum region in (5)-(7). The higher than quadrupole mass distribution coefficients can be arbitrary, which shows that they cannot be determined from a slow rotation formalism which takes into account only up to quadratic order terms in the angular velocity $\Omega$. Instead of using the linearized gravity approximation, an exact static but not spherically symmetric Weyl-class vacuum metric could also be constructed. But since for slow enough rotation there is always a region where the $r^2 q_1$ terms are small whereas the other terms in the metric are negligible, no new insight would result from that analysis.

In a general stationary axisymmetric vacuum exterior solution the suitable hypersurfaces for matching to an interior fluid region are determined in the paper of Roos[7]. In the limit of no rotation, the matching surface is the history of the sphere $r = r_1$. For slow rotation the surface becomes an ellipsoid, and its deformation is described by

$$r = r_1 + a^2 [\chi_0 + \chi_2 P_2(\cos \vartheta)]$$  \hspace{1cm} (20)

where $a$ is the small rotational parameter and $\chi_0$ and $\chi_2$ are constants to be determined by the matching conditions. The expressions for the normal vector and the extrinsic curvature of the matching surface have the same forms as in I.

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1We remind the reader that the asymptotic region may lie outside the domain of convergence of the power series expansion in the angular velocity.
3 Matching with the Wahlquist solution

In I we have computed the second order form of the Wahlquist metric [8] in Hartle’s coordinates. The metric functions and the zero-pressure matching surface is expressed in I in terms of the constants $\mu_0$ and $\kappa$ characterizing the static configuration and in terms of the the small parameter $r_0$ which is proportional to the angular velocity. These results will be used now for matching with the vacuum solution of the previous section. The radial coordinate in the fluid region is denoted by $x$ instead of $r$. In the limit of no rotation the fluid region is described by the Whittaker metric [9], and the matching surface is the sphere characterized by $x = x_1$, where $x_1 \cot x_1 = \kappa^2$ (cf. I).

For slow rotation the matching surface $S$ is an ellipsoidal cylinder characterized by a vanishing pressure and by the embedding condition (20). We equate with each other the respective induced extrinsic curvatures $K_{(V)}$ and $K_{(W)}$ of $S$, in the vacuum and in the Wahlquist region. Hence, the equations of matching are

$$ds^2_{(V)}|_S = ds^2_{(W)}|_S \quad K_{(V)}|_S = K_{(W)}|_S .$$

(21)

where $ds^2|_S$ is the induced metric. The values of the metric coefficients and their derivatives on $S$ are given by a power series expansion in $r_0$ in the fluid and in $a$ in the vacuum regions, respectively. We apply a rigid rotation in the fluid region by setting $\varphi \rightarrow \varphi + \Omega t$ where $\Omega$ is a constant. Then we re-scale the interior time coordinate $t$ by

$$t \rightarrow c_4 (1 + r_0^2 c_3) t$$

(22)

with further constants $c_3$ and $c_4$ to be determined from the matching conditions.

Substituting in the matching conditions (21), and Taylor expanding to second order in the angular velocity, we get a set of linear equations for the parameters $\chi_0$, $\chi_2$, $c_1$, $c_2$, $c_3$ and $q_1$. Here we list the solution of the matching equations for the constant $c_3$ of the interior time scaling and for the constants $q_1$, $c_1$, $c_2$, $x_0$ and $x_2$ of the vacuum domain in terms of the radius $x_1$ of the Whittaker fluid ball, the rotation parameter $a$ given by

$$a = \frac{r_0}{3 \cos x_1} \frac{2x_1 \cos^2 x_1 - 3 \sin x_1 \cos x_1 + x_1}{\sin x_1 \cos x_1 - x_1}$$

(23)

and the density $\mu_0$. The results of I for the matching to zero order in the angular velocity

$$\Omega = \frac{\mu_0 x_1 r_0}{6 \sin x_1 \cos x_1}$$

(24)

hold without any change,

$$M = \frac{r_1}{2 \kappa^2} (\kappa^2 - \cos^2 x_1)$$

(26)

$$r_1 = \frac{2^{1/2}}{\kappa \mu_0^{1/2}} \sin x_1$$

(27)

$$c_4 = \cos x_1 .$$

(28)
The second-order matching conditions have the solution

\[
\begin{align*}
    q_1 &= \frac{a^2 \cos^2 x_1 \mu_0^2 x_1^3}{2 f_3^2 \sin^3 x_1} \\
    c_1 &= \frac{a^2 \cos^5 x_1 \mu_0^5 x_1^5}{f_3^2 \sin^3 x_1} \\
    c_2 &= \frac{-a^2 l_3 \cos^{3/2} x_1 \sin^{1/2} x_1 \mu_0^{1/2}}{2^{3/2} f_2^2 x_1^{3/2}} \\
    \chi_0 &= \frac{3 l_3 \cos^{3/2} x_1 \mu_0^{1/2}}{2^{3/2} x_1^{1/2} l_2^2 \sin^{3/2} x_1} (x_1^2 \cos x_1 - 2 x_1 \cos^4 x_1 \sin x_1) \\
    \chi_2 &= \frac{-3 l_3 \cos^{3/2} x_1 \mu_0^{1/2}}{2^{3/2} x_1^{1/2} l_2^2 \sin^{3/2} x_1} (x_1^2 \sin x_1 + 4 x_1^3 \cos^3 x_1 - 8 x_1 \cos^3 x_1 \sin^2 x_1)
\end{align*}
\]

where

\[
\begin{align*}
    l_1 &= 3 x_1^2 + 6 x_1 \cos x_1 x_1 - \cos^2 x_1 \sin^2 x_1 - 8 \sin^2 x_1 \\
    l_2 &= x_1 (3 \cos^2 x_1 + \sin^2 x_1) - 3 \sin x_1 \cos x_1 \\
    l_3 &= x_1 - \sin x_1 \cos x_1
\end{align*}
\]

It is easy to show that \(q_1\) is non-zero in the allowed range of \(x_1\), \(0 < x_1 < \frac{\pi}{2}\).

### 4 Conclusions

In this paper we have shown that there exist configurations where a Wahlquist fluid ball in an ambient vacuum domain is kept in equilibrium. In our model space-time the components of the curvature tensor tend to finite values at infinity. A far-away quadrupole mass distribution can ensure the required non-asymptotically flat nature of the exterior vacuum. The solution is linearization
stable in the sense that there exists a family of exact solutions corresponding to our approximate solution, which is valid in the fluid and in at least an open vacuum region surrounding it. This follows from the work of Roos [7, 10] showing the existence and uniqueness in a neighbourhood of the matching surface in the general axisymmetrical and stationary case.

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