Waves of large amplitudes in geophysical and technical applications

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Abstract. The paper addresses the asymptotic models that explicitly describe the structure of solitary waves with recirculation zones for some special but important incident stream configurations, which occur in geophysical and industrial applications. The key feature that allows building a theoretical model is that shear and/or stratification of the media in geophysical applications or axisymmetric swirling incoming flows in technical applications have an almost linear equation for vorticity. It happens when the upstream flow is almost linearly stratified in the case of geophysical applications and a swirling flow has almost solid body rotation if industrial applications are considered. It is shown that the resulting equation for the stationary amplitude function of the solitary waves is described by a generalized Korteweg-de Vries equation in the outer zone, consistently matched with a new complicated equation in the inner zone, which contains the recirculation core. This recirculation zone exists for wave amplitudes that are slightly greater than a certain critical amplitude, for which a wave breaking initially occurs. The recirculation zones have a universal structure, so that their width increases up to the semi-infinite bore as the amplitude of the wave increases from a critical amplitude to a certain maximum amplitude. This amplitude may be sensitive to the actual configuration of the incident flow. Regardless of the physical nature, solitary waves are described by two types of equations with different nonlinearities. The type of nonlinearity in the inner zone where the vortex core is located is then caused by the position of the vortex core at the outer boundary or in the middle of the flow.

1. Introduction
The majority of theoretical studies for solitary waves in inviscid, incompressible dispersive media within weakly nonlinear regimes are conducted for waves of small albeit finite amplitudes. For long waves in shallow fluids the theories lead to the Korteweg-de Vries equation. However, observations often show that these waves can have large amplitudes and even contain a vortex core, that is, an area of recirculation flow. Solitary waves in stratified flows confined to a channel of finite depth have been the subject of many studies, beginning with the pioneering work of Dubreil-Jacotin [1] and Long [2]. For this special but important class of flows, they showed that the fully nonlinear, steady, two-dimensional, inviscid and incompressible equations can be reduced to a single nonlinear elliptic equation for the streamfunction. It is usually derived on the assumption that all streamlines originate upstream, thus in particular excluding regions of closed streamlines, and can then be used to establish the existence of solitary waves of permanent form. An important special case arises when the stratification is nearly...
uniform and the upstream flow is also nearly uniform. In this limit the equation is almost linear for finite-amplitude waves, and this circumstance has been exploited by Benney and Ko [3], Derzho [4] and Makarenko et al. [16] amongst others.

However, these theories are usually limited to amplitudes (see review [5]) less than a certain critical amplitude for which there is incipient flow reversal. Derzho and Grimshaw [6] showed that asymptotic solutions could be constructed for solitary waves with amplitudes just greater than this critical amplitude, provided that the waves incorporated a small vortex core containing recirculating fluid. Their results were generally in agreement with several experimental and numerical investigations, see Aigner et al. [7]. Recent account of the problem was given by Yangxin, Lamb and Lien [8].

An analogous situation exists for the axisymmetric, steady, rotating flow of an inviscid, incompressible fluid contained in a circular tube. Here again a single nonlinear elliptic equation can be derived for the streamfunction, namely the Bragg-Hawthorne equation [9], based on the assumption that all streamlines originate upstream. This equation has been the basis for several studies of solitary waves and the phenomena of vortex breakdown (see, for instance, Benjamin [12], Leibovich and Kribus [13] and Keller [14]). Derzho and Grimshaw [15] pointed out that generally these theories were limited to amplitudes less than a certain critical amplitude for which there is incipient flow reversal. They then used an asymptotic construction to find solitary waves with a small recirculating zone on the tube axis. The recent account of the problem is given Rusak et al. [10] and Vaniershot [11].

In the next sections we describe the problem for stratified and then swirling flows with recirculation cores.

2. Stratified Flows

Let us first consider the steady, two-dimensional flow of an inviscid, incompressible density-stratified fluid. The coordinates are \((x, z)\) with \(x\) directed horizontally and \(z\) vertically upwards, \((u, w)\) are the corresponding velocity components, \(\rho\) is the density. The streamfunction \(\psi\) is defined in the usual way,

\[-\psi_z = u, \quad w = \psi_x.\]  \(1\)

It can be shown that the density is constant on streamlines, so that \(\rho = \rho(\psi)\). Here the functional form of \(\rho(\psi)\) is yet to be determined. For those streamlines which originate upstream, it follows that it is determined from the upstream density profile, \(\rho_0(z)\) where the streamfunction is given by \(\psi_0(z)\). Finally, the vorticity equation can be derived

\[\psi_{xx} + \psi_{zz} + \frac{1}{\rho} \frac{d\rho}{d\psi} \left[ g z + \frac{1}{2} (\psi_x^2 + \psi_z^2) \right] = G(\psi). \]  \(2\)

As for \(\rho(\psi)\) the functional form of \(G(\psi)\) is found from the upstream boundary conditions for those streamlines which originate upstream, but otherwise it is yet undetermined. The boundary conditions are that

\[\psi \to \psi_0(z) \quad \text{and} \quad \rho \to \rho_0(z), \quad \text{as} \quad x \to \infty, \]  \(3\)

while on the channel walls,

\[\psi_x = 0, \quad \text{on} \quad z = 0, h. \]  \(4\)

Equation (2), together with the boundary conditions (3) and (4), is a nonlinear elliptic boundary value problem. Our aim here is to construct, asymptotically for long waves, a family of finite-amplitude solitary wave solutions including those containing vortex cores, that is a region of recirculating fluid. More details can be found in [6].

The basis of our approach is the observation that for nearly uniform, weak stratification and nearly uniform flow, then equation (2) is nearly linear without any restriction in wave amplitude. Hence we assume that

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\[ N^2(z) = -\frac{g}{\rho_0} \frac{d\rho}{dz} = N_0^2 + \sigma \Omega(z), \quad \text{and} \quad \psi_0(z) = cz + \kappa \Psi(z). \] 

Here \( N_0 \) is a constant value characterising the weak stratification, and \( c \) is the wave speed to be determined, while the functions \( \Omega(z) \) and \( \Psi(z) \) characterise the small but significant departures from the uniform state, so that \( \sigma \) and \( \kappa \) are two small parameters, both of \( O(\epsilon^2) \) where the small parameter \( \epsilon \) characterizes the ratio of the vertical scale \( h \) to the long horizontal length scale. Thus we introduce the new horizontal variable \( X = \epsilon x/h \), and seek solutions for which \( \psi \) is a function of \( X \) and \( z \). An asymptotic solution is sought as a power series in \( \epsilon^2 \), whose leading term is given by  

\[ \psi_0(z) = \epsilon^2 z + A(X) \sin(\pi z/h), \quad \text{and} \quad c(0) = N_0 h/\pi. \] 

At the next order in the asymptotic expansion an inhomogeneous linear equation is obtained whose compatibility condition leads to the required equation for the wave amplitude \( A(X) \),  

\[ A_{XX} - \Delta A + M(A) = 0. \] 

Here \( \Delta \) is proportional to the speed correction term \( \epsilon^1 \), while \( M(A) \) is a nonlinear term related in quite a complicated way to the functions \( \Omega(z) \) and \( \Psi(z) \) in (5). Equation (7) has the form of a generalised KdV equation. When, all streamlines originate upstream, solutions of (7) such that \( A \to 0 \) as \( |X| \to \infty \) describe the required solitary waves. 

Next we observe that a sufficient condition for all streamlines to originate upstream is that \( \psi_x > 0 \). Here we are concerned with the situation when this condition is violated at some point on the wave profile. Hence we suppose that \( \psi_x = 0 \) when \( |X| = X_0 \) for some critical amplitude \( A_* \), so that equation (7) holds for \( |X| > X_0 \) and \( |A| < A_* \). To leading order \( A_* = N_0 h^2/\pi^2 \) when the breakdown occurs at the upper channel wall. 

For \( |X| < X_0 \) we construct an asymptotic solution which contains a vortex core attached to the upper channel wall, within the region defined by \( h - \eta(x) < z < h \). For the flow within the vortex core, equation (2) again holds, but the density \( \rho(\psi) \) and the function \( G(\psi) \) cannot be determined from the upstream conditions. Instead, we assume that the density is a constant, equal to its value on the vortex core boundary which is a streamline; this is to ensure that there is static stability. Further it can be shown that within the vortex core the vorticity \( G(\psi) \) is sufficiently small that the flow is stagnant to leading order in the asymptotic analysis. 

The vortex core and the inner region (\( |X| < X_0 \) and \( 0 < z < h - \eta \)) are characterised by a different length scale from the outer region (\( |X| > X_0 \)), and here we put \( \xi = \beta x/h \). Further we assume that the height of the vortex core is \( O(\delta) \) so that \( \eta = \delta f(\xi) \). In the inner region the asymptotic solution is again given by (6) to leading order, but now we set  

\[ A = A_* + \mu B. \] 

Here \( \beta, \delta \) and \( \mu \) are small parameters. An optimal balance shows that the required scaling is given by  

\[ \beta = \epsilon^{1/3}, \quad \delta = \epsilon^{2/3}, \quad \text{and} \quad \mu = \epsilon^{4/3}. \] 

A compatibility condition applied to the next order equation in the asymptotic expansion, together with the use of matching conditions that the streamfunction and the velocity field are continuous across the vortex core boundary, yields the desired equation for \( B \),  

\[ B_{\xi \xi} - \Delta A_* + M(A_*) = \frac{2}{3} \nu B^{3/2}. \] 

Here \( \nu \) is a supercriticality parameter, which is of the order of unity, and given by \( \nu = 2^{1/2} \pi^2 / N_0^{3/2} h \). The interface with the vortex core is given by \( f \approx (2B/N_0)^{1/2} \). This equation is to be solved in the
region $|\xi| < \xi_0$ subject to the matching conditions that $B = 0$ at $|\xi| = \xi_0$ (so that $A = A_*$ there) and that $A_X = \mu^{1/2} B_\xi$ also at $|\xi| = \xi_0$. Of course the solution in the outer zone, $|X| > X_0$, must also be found from (7) simultaneously. Both equations can be integrated once, and in particular (10) yields

$$B^2_\xi = R(A_*)(B_0 - B) - \frac{8\nu}{15}(B_0^{5/2} - B^{5/2}),$$
where

$$R(A_*) = 2M(A_*) - 4\int_0^{A_*} M(A)dA.$$ (11)

Here we have used the matching conditions at $|\xi| = \xi_0$ and the normalisation condition that $B = B_0$ at the wave maximum at $|\xi| = 0$, where $B_0 = N_0 h^2/\pi$. In particular it follows that solutions only exist when $4\nu/3 < R(A_*)$, implying a maximum possible amplitude above $A_*$ for these waves with a vortex core. Further, as this maximum amplitude is approached, the vortex core width increases indefinitely. The structure of equation (10) indicates that the inner zone and the vortex core have a universal structure, and depend on the upstream conditions only through the single parameter $R(A_*)$. However, whether or not these solitary waves can exist at all depends in general on the details of the nonlinear function $M(A)$ in (7).

### 3. Swirling flows

Let us consider the steady, axisymmetric flow of an inviscid, incompressible fluid, confined to a circular tube.

Here the coordinates are $(x, r, \theta)$ with $x$ in the axial direction, $r$ in the radial direction, and $\theta$ in the azimuthal direction, $(u, v, w)$ are the corresponding velocity components, and $p$ is the pressure. The streamfunction $\psi$ is defined in the usual way,

$$ru = -\psi_r, \quad rv = \psi_x.$$ (12)

The circulation is constant along a streamline, so that $rw = C(\psi)$. Here the functional form of $C(\psi)$ is yet to be determined. For those streamlines which originate upstream, it is determined from the upstream angular velocity profile, $\Omega_0(r)$ where $w = r\Omega$. Finally, the procedure yields the vorticity equation,

$$\psi_{xx} + \psi_{rr} - \frac{\psi_x}{r} + C(\psi)C'(\psi) = G(\psi).$$ (13)

As for $C(\psi)$ the functional form of $G(\psi)$ is found from the upstream boundary conditions for those streamlines which originate upstream, but otherwise is as yet undetermined. The boundary conditions are that

$$\psi \to \psi_0(z) \quad \text{and} \quad \Omega \to \Omega_0(z), \quad \text{as} \ x \to \infty,$$ (14)
while at the tube centre, and on the tube wall,

$$\psi_x = 0, \quad \text{on} \ r = 0, \ a.$$ (15)

Equation (13), together with the boundary conditions (14) and (15), is a nonlinear elliptic boundary value problem. The analogy with the stratified flow formulation in the previous section is clear. As for that case, our aim is to construct, asymptotically for long waves, a family of finite-amplitude solitary wave solutions, including those containing a zone of recirculating fluid. The account given here is necessarily brief, and more details can be found in [15].

As for the stratified flow problem of the previous section, the basis of our approach is the identification of the upstream conditions which lead to a nearly linear equation without any restriction in wave amplitude. Thus here we observe that this is achieved when the upstream axial flow and the upstream angular velocity are both nearly uniform. Hence we assume that
Here \( \Omega_1 \) is a constant value characterizing the nearly uniform angular velocity, and \( c \) is the wave speed to be determined, while the functions \( \Omega(z) \) and \( \Psi(z) \) characterise the small but significant departures from the uniform state, so that \( \sigma \) and \( \kappa \) are two small parameters, both of the order of \( O(\epsilon^2) \) where the small parameter \( \epsilon \) characterises the ratio of the radial scale \( a \) to the long axial length scale. Thus we introduce the new horizontal variable \( X = \epsilon x/a \), and seek solutions for which \( \psi \) is a function of \( X \) and \( r \). An asymptotic solution is sought as a power series in \( \epsilon^2 \), whose leading term is given by

\[
\psi^{(0)} = \frac{1}{2} c \epsilon^0 r^2 + A(X)r J_1(\lambda^{(0)} r/a), \quad \text{and} \quad J_1(\lambda^{(0)}) = 0.
\]  

Here \( \lambda = |2\Omega_1/c| \) and \( J_1 \) is the Bessel function of order 1. At the next order in the asymptotic expansion an inhomogeneous linear equation is obtained whose compatibility condition leads to the required equation for the wave amplitude \( A(X) \), which has precisely the same form and meaning as (7). When all streamlines originate upstream, solutions of (7) such that \( A \to 0 \) as \( |X| \to \infty \) describe solitary waves.

Next we observe that a sufficient condition for all streamlines to originate upstream is that \( \psi_\tau > 0 \). Here we are concerned with the situation when this condition is violated at some point on the wave profile. Hence we suppose that \( \psi_\tau = 0 \) when \( |X| = X_0 \) for some critical amplitude \( A_\ast \), so that equation (7) holds for \( |X| > X_0 \) and \( |A| < A_\ast \). To leading order \( A_\ast = 2\epsilon a/\lambda^{(0)} \) when the breakdown occurs on the tube axis.

The analysis now closely parallels that for the stratified flow case. Thus, for \( |X| < X_0 \) we construct an asymptotic solution which contains a recirculation zone located now on the tube axis, within the region defined by \( 0 < r < \eta(x) \). For the flow within the recirculation zone, equation (13) again holds, but the circulation \( C(\psi) \) and the function \( G(\psi) \) cannot be determined from the upstream conditions. Instead, we assume that the circulation is zero, or at least asymptotically small; this is to ensure that the flow satisfies the Rayleigh stability criterion, namely that \( (u-c)C(\psi)C'(\psi) > 0 \). Further it can be shown that within the recirculation zone the vorticity \( G(\psi) \) is sufficiently small that the flow is stagnant to leading order in the asymptotic analysis.

The recirculation zone and the inner region \( (|X| < X_0 \text{ and } \eta < r < a) \) are characterised by a different length scale from the outer region \( (|X| > X_0) \), and here we put \( \xi = \beta x/a \). Further we assume that the height of the vortex core is \( O(\delta) \) so that \( \eta = \delta f(\xi) \). In the inner region the asymptotic solution is again given by (17) to leading order, but now we set \( A = A_\ast + \mu B \), just as in (8). Here \( \beta, \delta \) and \( \mu \) are again small parameters. An optimal balance shows that the required scaling is now given by

\[
\beta = \epsilon^{1/2}, \quad \delta = \epsilon^{1/2}, \quad \text{and} \quad \mu = \epsilon.
\]

It is noticeable, this scaling differs from that in (9) for the stratified flow case. As before, a compatibility condition applied to the next order equation in the asymptotic expansion, together with the use of matching conditions that the streamfunction and the velocity field are continuous across the vortex core boundary, yields the desired equation for \( B \),

\[
B \xi \xi - \Delta A_\ast + M(A_\ast) = \nu B^2.
\]

Note that while this is similar in form to the analogous equation (10) in the stratified flow case, the power of \( B \) on the right-hand side is now \( 2 \) in place of \( 3/2 \). Here again \( \nu \) is a supercriticality parameter, which is of the order of unity, and given now by \( \nu = 1/IA_\ast \), where \( I = J_0(\lambda^{(0)})^2/2 \). The interface with the recirculation zone is given by \( f = 2a (B/A_\ast)^{1/2}/\lambda^{(0)} \). As in the stratified flow case, this equation is to be solved in the region \( |\xi| < \xi_0 \) subject to the same matching conditions. In place of (11) we now get
\[ B_\xi^2 = R(A_*) (B_0 - B) - \frac{2\nu}{3} (B_0^3 - B^3), \]  

(20)

and \( R(A_*) \) is defined exactly as in (11). Analogously to the stratified flow case, it follows that solutions only exist when \( 2\nu < R(A_*) \), implying a maximum possible amplitude above \( A_* \) for these waves with a recirculation zone. Further, as this maximum amplitude is approached, the width of the recirculation zone increases indefinitely. The structure of equation (19) indicates that the inner zone and the recirculation zone have a universal structure, and depend on the upstream conditions only through the single parameter \( R(A_*) \). However, whether or not these solitary waves can exist at all depends in general on the details of the nonlinear function \( M(A) \) in (7). Further details and the implications for the phenomenon of vortex breakdown are discussed in [15].

**Conclusion**

The presented models are used to describe internal waves and inertial waves leading to the disintegration of the vortex. The key feature which enables this construction is that, both for stratified shear flows and for axisymmetric swirling flows, the steady state vorticity equation is almost linear when the upstream flow is almost uniform. That is, for stratified shear flows the upstream flow and the upstream stratification are almost constant, while for rotating flows the upstream axial flow and angular velocity are almost constant. This feature enables the asymptotic construction of solitary waves described by a steady-state generalised Korteweg-de Vries equation in an outer zone, matched to an inner zone containing a recirculation zone. The main effect of the recirculation zones is independent on the physical nature of the flow. It has a universal structure such that their width increases without limit as the wave amplitude increases from the critical amplitude to a certain maximum amplitude, but their existence can be sensitive to the actual upstream flow configuration. The main result is that a description of a wave with a small recirculation zone is presented whenever a wave amplitude of a wave slightly exceeds

![Figure 1](image-url)

**Figure 1.** The results for the case when the recirculation core is located near the outer boundary are shown by solid lines. Dashed lines correspond to the location of the core near the center line. \( \beta \) scales the lengths and \( \delta \) the height of the flow around the core (inner zone), while \( \epsilon \) scales the length in the outer zone where there is no recirculation core.
the critical amplitude value at which the wave reverses. The obtained equations for the inner and outer zones are of the Korteweg-de Vries type, where the weak dispersion is balanced by weak nonlinearity. Note that while the inner region equations are similar in form, the near boundary recirculation core leads to the $3/2$ power of the difference between wave amplitude function and the critical amplitude but the near the center-line recirculation core results in $2$ power law instead. The maximal amplitude for steady solutions corresponds to a semi-infinite bore. For the value of the phase velocity of the wave, it was shown in [6, 15] that a wave with a vortex core propagates slightly faster than the Korteweg de Vries theory predicts for the same amplitude. The major difference between flows of various physical origins lies in the fact where the recirculation zone is located. Near the boundary, recirculation zones and flows around them are generally smaller in height and shorter in length compared to the case of the near the center-line location of the vortex core. It is shown in Figure according to (9) and (18). The size of the separation zone could affect the heat transfer processes in the problem. It is worth noting that the problem allows multisaled structures and more complicated geometrical configurations as discussed in [17, 18, 20]. Another problem of geophysical interest, namely, Rossby waves on a shear flow, leads to the near boundary recirculation core [19].

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