\( \theta \) renormalization, electron-electron interactions and super universality in the quantum Hall regime

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The renormalization theory of the quantum Hall effect relies primarily on the non-perturbative concept of \( \theta \) renormalization by instantons. Within the generalized non-linear \( \sigma \) model approach initiated by Finkelstein we obtain the physical observables of the interacting electron gas, formulate the general (topological) principles by which the Hall conductance is robustly quantized and derive - for the first time - explicit expressions for the non-perturbative (instanton) contributions to the renormalization group \( \beta \) and \( \gamma \) functions. Our results are in complete agreement with the recently proposed idea of super universality which says that the fundamental aspects of the quantum Hall effect are all generic features the instanton vacuum concept in asymptotically free field theory.

PACS numbers: 73.43.-f, 73.43.Nq

I. INTRODUCTION

One of the long standing mysteries in the theory of the plateau transitions in the quantum Hall regime is the apparently insignificant or subdominant role that is played by the long ranged Coulomb interaction between the electrons. The pioneering experiments on quantum criticality in the quantum Hall regimes by H. P. Wei et al. for example, are in many ways a carbon copy of the scaling predictions based on the field theory of Anderson localization in strong magnetic fields. The initial success of the free electron theory has primarily led to a widely spread belief in Fermi liquid type of ideas as well as an extended literature on scaling and critical exponent phenomenology. Except for experimental considerations, however, there exists absolutely no valid (microscopic) argument that would even remotely justify any of the different kinds of free (or nearly free) electron scenarios that have frequently been proposed over the years. In fact, Fermi liquid principles are fundamentally in conflict with the novel insights that have more recently emerged from the development of a microscopic theory on interaction effects. These developments are naturally based on the topological concept of an instanton vacuum which is very well known to be the fundamental mechanism by which the free electron gas de-localizes in two spatial dimensions and in strong magnetic fields. The outstanding and difficult problem that one is faced with is whether or not the topological concepts in quantum field theory retain their significance also when the electron-electron interactions are taken into account.

For a variety of reasons, however, it has taken a very long time before the subject matter gained the physical clarity that it now has. Perhaps the most awkward obstacles were provided by the historical controversies in QCD where the idea of an instanton parameter \( \theta \) arose first but its exact meaning remained rather obscure. These controversies have mainly set the stage for the wrong physical ideas and the wrong mathematical objectives. For example, in sharp contrast to the general expectations in the field, the fundamental problems do not reside in the conventional aspects of disordered systems such as the replica method or “exact” critical exponent values. A more fundamental issue has emerged, the massless chiral edge excitations that dramatically change the way in which the \( \theta \) parameter is generally being perceived. A detailed understanding of the physics of the edge has resolved, amongst many other things, the long standing controversies that historically have spanned the subject such as the quantization of topological charge, the meaning of instantons and instanton gases, etc. As a result of all this we can now state that the instanton angle \( \theta \) generically displays all the basic features of the quantum Hall effect, independent of the details such as the replica limit. This includes not only the appearance of gapless excitations at \( \theta = \pi \) but also the most fundamental and much sought after aspect of the theory, the existence of robust topological quantum numbers that explain the precision and observability of the quantum Hall effect.

A second major complication in dealing with interaction effects is the notorious complexity of the underlying theory. Although Finkelstein’s original ideas in the field have been very illuminating, it has nevertheless taken herculean efforts to understand how the generalized non-linear \( \sigma \) model approach can be studied as a field theory. This includes not only the theory of perturbative expansion but also such basic aspects like the global symmetries of the problem (\( F \) invariance), electrodynamics \( U(1) \) gauge invariance as well as the physical observables of the theory. Advances along these lines are absolutely necessary if one wants to extend the perturbative theory of localization and interaction effects to include the highly non-trivial consequences of the \( \theta \) vacuum.

It obviously makes an enormously big difference to know that the instanton vacuum theory of the quantum Hall effect is NOT merely an isolated critical exponent
problem that exists in replica field theory or “super symmetric” extensions of free electron approximations alone. Contrary to this widely spread misconception in the literature, the fundamental features of the quantum Hall effect actually reveal themselves as a super universal consequence of topological principles in quantum field theory that until to date have not been well understood. The concept of super universality makes it easier and more natural to comprehend why the basic phenomena of scaling are retained by the electron gas also when the Coulomb interaction between the electrons is taken into account. Moreover, it facilitates the development of a unifying theory that includes completely different phenomena such as the fractional quantum Hall regime. Unlike Fermi liquid ideas, however, super universality does not necessarily imply that the quantum critical details at $\theta = \pi$ remain the same. The various different applications of the $\theta$ vacuum concept do in general have different exponent values at $\theta = \pi$ and, hence, they belong to different universality classes.

In this paper we revisit the problem of topological excitations (instantons) and $\theta$ renormalization in the theory of the interacting electron gas. The results of an early analysis of instanton effects have been reported in a short paper by Pruisken and Baranov. However, much of the conceptual structure of the theory was not known at that time, in particular the principle of $F$ invariance and the appearance of the massless edge excitations that together elucidate the fundamental aspects of the $\theta$ vacuum on the strong coupling side. These novel insights unequivocally define the physical observables (i.e. the conductance parameters $\sigma_{xx}$ and $\sigma_{xy}$) that control the dynamics of the $\theta$ vacuum at low energies. These physical observables should therefore quite generally be regarded as some of the most fundamental quantities of the theory.

A detailed knowledge of instanton effects on the physical observables of the theory has fundamental significance since it bridges the gap that exists between the weak coupling Goldstone singularities at short distances, and the super universal features of the quantum Hall effect that generally appear at much larger distances only. The theory of observable parameters, as it now stands, provides the general answer to the “arena of bloody controversies” that historically arose because of a complete lack of any physical objectives of the theory. A prominent and exactly solvable example of these statements is given by the large $N$ expansion of the $CP^{N-1}$ model that, unlike the previous expectations, sets the stage for all the non-perturbative features of the $\theta$ parameter that one is interested in Ref. 24.

The main objective of the present work is to review the instanton methodology, provide the technical details of the computation and extend the analysis in several ways. Our study of the interacting electron gas primarily relies on the procedure of spatially varying masses that very recently has been applied, with great success, in the context of the ordinary $U(M + N)/U(M) \times U(N)$ non-linear $\sigma$ model. The important advantage of this procedure is that it facilitates non-perturbative computations of the renormalization group $\beta$ and $\gamma$ functions of the theory. These computations, together with the new insights into the strong coupling features and symmetries of the problem, lay out the complete phase and singularity structure of the disordered electron gas. The results of this paper, which include the non-Fermi liquid behavior of the Coulomb interaction problem, obviously cannot be obtained in any different manner.

This paper is organized as follows. We start out in Section II with a brief introduction to the formalism and recall the effective action procedure for massless chiral edge excitations. In Section II C we briefly elaborate on the general topological principles that explain the robust quantization of the Hall conductance. The general argument is deeply rooted in the methods of quantum field theory and relies on the relation that exists between the conductances on the one hand, and the sensitivity of the interacting electron gas to infinitesimal changes in the boundary conditions on the other. The argument is furthermore based on the relation between Kubo formalism, the background field methodology and the effective action for chiral edge excitations which is described in Appendix A.

In Section II D we give the complete list of physical observables which then serves as the basic starting point for the remainder of this paper. We show the general relationship between the physical observables and the renormalization group $\beta$ and $\gamma$ functions and briefly discuss the results of the theory in $2 + \epsilon$ spatial dimensions.

In Section III we recall the various different aspects associated with instanton matrix field configurations and embark on the problem of quantum fluctuations. We introduce the method of spatially varying masses and end the Section with the complete action for the quantum fluctuations in Table III.

In Section IV together with Appendices B and C we present the results of detailed computations that deal, amongst many other things, with the technical difficulties associated with the theory in Pauli-Villars regularization, the replica method as well as the infinite sums over Matsubara frequency indices that are inherent to the problem of electron-electron interactions.

In Section V we address the various different aspects associated with the integration over zero modes and embark on the general problem of transforming the Pauli-Villars masses in curved space back into flat space following the methodology introduced by ‘t Hooft. This finally leads to the most important advances of this paper, the renormalization-group $\beta$ and $\gamma$ functions which are evaluated at a non-perturbative level. These final results provide a unified theory of the disordered electron gas that includes the effects of both finite range electron-electron interactions and infinite range interactions such as the Coulomb potential. We end this paper with a discussion in Section VII.
II. FORMALISM

A. The action

The generalized replica non-linear sigma model involves unitary matrix field variables $Q_{nm}^\alpha (r)$ that obey the following constraints

$$Q = Q^I, \quad \text{tr} \, Q = 0, \quad Q^2 = 1. \quad (1)$$

The superscripts $\alpha, \beta = 1, \ldots, N_r$ represent the replica indices and the subscripts $n, m$ are the indices of the Matsubara frequencies $\omega_k = \pi T(2k + 1)$ with $k = n, m$. A convenient representation in terms of unitary matrices $T(r)$ is obtained by writing

$$Q(r) = T^{-1}(r) \Lambda T(r), \quad \Lambda_{nm}^{\alpha\beta} = \text{sign}(\omega_n) \delta^{\alpha\beta} \delta_{nm}. \quad (2)$$

The effective action for the two-dimensional interacting electron gas in the presence of disorder and a perpendicular magnetic field can be written as follows:

$$Z = \int D[Q] \exp S, \quad S = S_\sigma + S_F. \quad (3)$$

Here, $S_\sigma$ is the free electron action:

$$S_\sigma = -\frac{\sigma_{xx}}{8} \int dr \, \text{tr}(\nabla Q)^2 + \frac{\sigma_{xy}}{8} \int dr \, \varepsilon_{jk} Q \nabla_j Q \nabla_k Q. \quad (4)$$

The quantities $\sigma_{xx}$ and $\sigma_{xy}$ represent the mean field values for the longitudinal and Hall conductances in units $e^2/h$ respectively. The symbol $\varepsilon_{jk} = -\varepsilon_{kj}$ stands for the antisymmetric tensor. Next, $S_F$ contains the singlet interaction term:

$$S_F = \pi T z \int dr \, \mathcal{O}_F(Q) \quad (5)$$

where

$$\mathcal{O}_F(Q) = c \sum_{\alpha\beta} \text{tr} \, I_n^\alpha Q \text{tr} \, I_n^\beta Q + 4 \text{tr} \, \eta Q - 6 \text{tr} \, \eta \Lambda. \quad (6)$$

Here, $z$ is the so-called singlet interaction amplitude, $T$ the temperature and $c$ the crossover parameter which allows the theory to be interpolated between the case of electrons with Coulomb interaction ($c = 1$) and the free electron case ($c = 0$). The singlet interaction term involves a matrix

$$(I_n^\alpha)_{km} = \delta^{\alpha\beta} \delta^{\gamma\delta} \delta_{k,n+m} \quad (7)$$

which is the Matsubara representation of the $U(1)$ generator $\exp(-i\omega_n \tau)$ with $\tau$ being imaginary time. Matrix

$$\eta_{nm}^{\alpha\beta} = n \delta^{\alpha\beta} \delta_{nm} \quad (8)$$

is used to represent the set of the Matsubara frequencies $\omega_n$.

B. $\mathcal{F}$ invariance and $\mathcal{F}$ algebra

Unlike the free particle problem ($c = 0$), the theory with electron-electron interactions ($0 < c \leq 1$) is mainly complicated by the fact that the range of Matsubara frequency indices $m, n$ must be taken from $-\infty$ to $+\infty$, along with the replica limit $N_r \to 0$. Under these circumstances one can show that the singlet interaction term fundamentally affects the ultra violet singularity structure of the theory (the renormalization group $\beta$ and $\gamma$ functions) which is one of the peculiar features of the theory of electron-electron interactions. Moreover, the problem with infinite ranged interactions ($c = 1$) such as the Coulomb interaction displays an exact global symmetry named $\mathcal{F}$ invariant. This means that $S_F$ is invariant under dynamo $U(1)$ gauge transformations which are spanned by the matrices $I_n^\alpha$. This symmetry is broken by the problem with finite ranged interactions ($0 < c < 1$). In order to retain the $U(1)$ algebra in a truncated frequency space with a cut-off $N_m$, a set of algebraic rules has been developed named $\mathcal{F}$ algebra. These rules permit one to proceed in finite frequency space where the index $n$ runs from $-N_m$ to $N_m - 1$, i.e. the matrix field variables $Q$ have a finite size

$$Q(r) = T^{-1}(r) \Lambda T(r), \quad T(r) \in U(2N) \quad (9)$$

where $N = N_r N_m$. The two limits of the theory, $N_r \to 0$ and then $N_m \to \infty$ respectively, are taken at the end of all computations. The main purpose of $\mathcal{F}$ algebra is to ensure that dynamo $U(1)$ gauge invariance as well as $\mathcal{F}$ invariance are preserved by the renormalization group, both perturbatively and at a non-perturbative level.

C. Quantization of the Hall conductance

The robust quantization of the Hall conductance can be demonstrated on the basis of very general principles such as mass generation and the fact that the conductances can be expressed in terms of the response of the system to changes in the boundary conditions. The subtleties of the argument involve a novel and previously unexpected ingredient of the instanton vacuum concept, however, which has been recognized very recently only. The main problem resides in the $\sigma_{xy}$ term in Eq. (1) which is formally identified as the topological charge $C[Q]$ associated with the matrix field configuration $Q$. Assuming for simplicity the geometry of a square of size $L \times L$ then we can express the topological charge in terms of both a bulk integral and an edge integral as follows

$$C[Q] = \frac{1}{16\pi i} \int dr \, \varepsilon_{ab} Q \nabla_a Q \nabla_b Q$$

$$= \frac{1}{4\pi i} \oint dx \, \nabla_z T^{-1} \Lambda. \quad (10)$$
The remarkable thing that is usually overlooked is that the matrix field \( Q \) generally splits up into distinctly different components, each with a distinctly different topological significance and very different physical properties. For this purpose we introduce a change of variables

\[
Q = t^{-1} Q_0 t.
\]  

(11)

Here, the \( Q_0 \) is an arbitrary matrix field with boundary conditions \( Q_0 = \Lambda \) at the edge (or, equivalently, \( T_0 \) equals an arbitrary \( U(N) \times U(N) \) gauge at the edge). The unitary matrix field \( t \) generally represents the fluctuations about the special boundary conditions. This change of variables is just a formal way of splitting the topological charge \( C(Q) \) of an arbitrary matrix field configuration \( Q \) into an integral piece \( C(Q_0) \) and a fractional piece \( C[q] \),

\[
C(Q) = C(Q_0) + C[q], \quad q = t^{-1} \Lambda t.
\]  

(12)

Without a loss in generality we can write

\[
C(Q_0) \in \mathbb{Z}, \quad -\frac{1}{2} < C[q] \leq \frac{1}{2}.
\]  

(13)

The main new idea is that the matrix field \( t \) or \( q \) should be taken as a dynamical variable in the problem, rather than being a fixed quantity that one can choose freely. The reason is that one can generally associate massless chiral edge excitations with the fluctuating matrix fields \( q \). These so-called edge modes \( q \) are distinctly different from the bulk modes \( Q_0 \) which usually (i.e. for arbitrary values of \( \sigma_{xy} \)) generate dynamically a mass gap in the bulk of the system. These various statements immediately suggest that the low energy dynamics of the strong coupling phase is described by an effective action of the matrix field variable \( q \) obtained by formally eliminating the bulk modes \( Q_0 \). This effective action procedure is furthermore based on the fact that the mean field quantity \( \sigma_{xy} \) (which is equal to the filling fraction \( \nu \) of the Landau levels) can in general be split into an integral edge part \( k(\nu) \) and a fractional bulk piece \( \theta(\nu) \) as follows

\[
\sigma_{xy} = \nu = k(\nu) + \frac{\theta(\nu)}{2\pi}
\]  

(14)

where

\[
k(\nu) \in \mathbb{Z}, \quad -\pi < \theta(\nu) \leq \pi.
\]  

(15)

In what follows we shall separately consider the theory with \( c = 0 \) (free particles) and \( c = 1 \) (Coulomb interactions) both of which are invariant under the action of renormalization group.

1. Free particles \((c = 0)\)

In the absence of external frequencies and at \( T = 0 \) we can write the action for the free electron gas as follows

\[
S = S^\text{edge}_\sigma[q] + S^\text{bulk}_\sigma[Q]
\]  

(16)

Provided the matrix field variable \( t \) satisfies the classical equations of motion we can obtain an effective action for \( q \) by eliminating the bulk matrix field \( Q_0 \)

\[
S^\text{eff}_\sigma[q] = S^\text{edge}_\sigma[q] + S^\text{bulk}_\sigma[Q_0].
\]  

(19)

where

\[
S^\text{bulk}_\sigma[Q] = -\frac{\sigma_{xx}}{8} \int d\theta \text{tr}(\nabla Q)^2 + i\theta(\nu)C[Q].
\]  

(18)

Here the subscript \( \partial V \) indicates that the functional integral has to be performed with \( Q_0 = \Lambda \) at the edge. The effective action for the bulk can be written as

\[
S^\text{bulk}_\sigma[q] = -\frac{\sigma'_{xx}}{8} \int d\theta \text{tr}(\nabla q)^2 + i\theta' C[Q].
\]  

(21)

Here, \( \sigma'_{xx} = \sigma_{xx}(L) \) and \( \theta' = \theta(L) \) play the role of response parameters that measure the sensitivity of the system to an infinitesimal change in the boundary conditions. For exponentially localized states these parameters are expected to vanish for large enough \( L \) and the effective action is now given by the edge piece (Eq. 17) alone. This one dimensional action is known to describe massless chiral edge excitations. To obtain a suitably regulated action for the edge we may proceed by stacking many blocks of size \( L \times L \) on top of one another to form an infinite strip (see Fig. 1). The action for the quantum Hall state is then defined along infinite edges
and can be written as

\[ S_{\text{eff}}[q] = \frac{k(\nu)}{2} \oint \! dx \, \text{tr} \, t \nabla_x t^{-1} \Lambda + \pi T \rho_{\text{edge}} \oint \! dx \, \text{tr} \, \eta q. \]  

(22)

Here we have introduced a frequency term to regulate the infrared. The quantity \( \rho_{\text{edge}} \) stands for the density of edge states and the integer \( k(\nu) = \sigma'_{xy} \) indicates that the Hall conductance is robustly quantized.

At this stage several remarks are in order. First of all, from an explicit (non-perturbative) computation of the response parameters \( \sigma'_{xx} \) and \( \theta' \) we know that the argument generally fails for \( \theta(\nu) = \theta' = \pi \) where the mass gap vanishes and the system is quantum critical. This happens at the center of the Landau bands where a transition takes place between adjacent quantum Hall plateaus.

Secondly, it is important to keep in mind that the aforementioned argument for an exact quantization of the Hall conductance is entirely based on the fact that the edge modes \( q \) are massless. The beauty of the effective action procedure is that it unequivocally demonstrates that the so-called spherical boundary conditions (i.e. \( Q_0 = \Lambda \) at the edge) are dynamically generated by the system itself, independent of any weak coupling arguments such as finite action requirements and independent of \( N_r \) and \( N_m \). The bulk components \( Q_0 \) have mathematically very interesting properties in that they are a realization of the formal homotopy theory result

\[ \pi_2(G/H) = \pi_1(H) = \mathbb{Z}. \]  

(23)

The integer \( \mathbb{Z} \) is equal to the topological charge \( C[Q_0] \) which is identified as the jacobian for the mapping of the manifold \( U(2N)/U(N) \times U(N) \) onto the two-dimensional plane. Physically the quantization of \( C[Q_0] \) represents the quantization of flux and the integer \( k(\nu)C[Q_0] \) can be interpreted in terms of a discrete number of electrons that have crossed the Fermi energy at the edge of the system.

Notice that except for the massless chiral edge modes there exists no compelling reason to believe why the topological charge \( C[Q_0] \) and, hence, the Hall conductance is robustly quantized. In fact, the quantization of topological charge has been one of the longstanding and controversial issues in quantum field theory\(^\text{22} \) that have fundamentally complicated the development of a microscopic theory of the quantum Hall effect.

2. Coulomb interaction \((c = 1)\)

An extension of the effective action procedure to the problem with the long ranged Coulomb interaction is by no means obvious. The argument relies, to a major extent, on the detailed knowledge obtained from an explicit analysis of the Finkelstein approach which shows that the theory undergoes structural changes in the limit where \( N_r \to 0 \) and \( N_m \to \infty \). The action is more complicated and now given by

\[ S = S^{\text{edge}}[\sigma] + S^{\text{bulk}}[Q] + S_F[Q] \]  

(24)

where \( c = 1 \) is inserted in the expression for \( S_F[Q] \). Elimination of the matrix field variable \( Q_0 \) leads to the definition of the effective action

\[ e^S_{\text{eff}}[q] = \int \! \delta V \! D[Q_0] e^{S^{\text{bulk}}[t^{-1}Q_0]+S_F[Q_0]}. \]  

(25)

On the basis of symmetries one can write down the following explicit result

\[ S^{\text{bulk}}[q] = -\frac{\sigma'_{xx}}{8} \int \! dx \, (\nabla q)^2 + i\theta'C[q]. \]  

(26)

Here, the response parameters \( \sigma'_{xx} = \sigma_{xx}(L) \) and \( \theta' = \theta(L) \) are evaluated in the limit where \( T \to 0 \). It is important to emphasize that \( S_F \) cannot be omitted from Eqs. \( \text{[24]-[26]} \). The reason is, as we already mentioned before, that this term fundamentally affects the ultra violet singularity structure of the theory\(\text{24,25,43,44,45}\).

The remaining part of the argument proceeds along similar lines as before. Provided the system with Coulomb interactions generates a mass gap, both parameters \( \sigma'_{xx} \) and \( \theta' \) should vanish for \( L \) large enough. A suitably regulated action for the quantum Hall state has been obtained previously and the result is as follows:

\[ S_{\text{eff}}[q] = \frac{k(\nu)}{2} \oint \! dx \, \text{tr} \, t \nabla_x t^{-1} \Lambda + \frac{\pi^2}{2} T \rho_{\text{edge}} \oint \! dx \, \mathcal{O}_F[q] \]

\[ - \frac{\pi}{4} T k(\nu) \oint \! dx \oint \! dy \, \text{tr} \, I_{n}^o q(x) v_{\text{eff}}^{-1}(x-y) \times \text{tr} \, I_{n}^o q(y). \]  

(27)

As before we have \( \sigma'_{xy} = k(\nu) \). Here, the quantity \( v_{\text{eff}}(x-y) \) contains the Coulomb interaction \( U_0(x-y) = 1/|x-y| \). The Fourier transform is given by

\[ v_{\text{eff}}(p) = \frac{k(\nu)}{2\pi \rho_{\text{edge}}} \left( 1 + \rho_{\text{edge}} U_0(p) \right). \]  

(28)

D. Physical observables

Next, for a detailed understanding of interaction effects it is clearly necessary to develop a quantum theory for the observable parameters \( \sigma'_{xx} \), \( \sigma'_{xy} \) or \( \theta' \), \( \theta' \), \( c' \) and \( c' \). At the same time it is extremely important to show that the response quantities defined by the effective action procedure are precisely the same as those obtained from ordinary linear response theory. This will be done in Appendix A where we embark on some of the principal results of \( \mathcal{F} \) algebra.

In this Section we recollect the \( \mathcal{F} \) invariant expressions for the observable parameters that will be used in the remainder of this paper. As pointed out in the original
papers the main advantage of working with invariant quantities is that they facilitate renormalization group computations at finite temperatures and frequencies. They are furthermore valid in the entire range $0 \leq c \leq 1$ and simpler to work with in general. In the second part of this Section we briefly recall the results of the theory in $2 + \epsilon$ spatial dimensions.

1. **Kubo formula**

The response quantities $\sigma'_{xx}$ and $\sigma'_{xy}$ for arbitrary values of $c$ can be expressed in terms of current-current correlations according to \cite{24,47}:

$$
\sigma'_{xx} = -\frac{\sigma_{xx}}{4nL^2} \int dr \langle [I_n^\alpha, Q(r)][I_{-n}^\alpha, Q(r)] \rangle + \frac{\sigma^2_{xx}}{8nL^2} \int dr \int dr' \langle \nabla Q(r) \nabla Q(r') \rangle \tag{29}
$$

$$
\sigma'_{xy} = \sigma_{xy} + \frac{\sigma^2_{xx}}{8nL^2} \int dr \int dr' \langle [I_n^\alpha, Q(r)] \nabla_j Q(r) [I_{-n}^\alpha, Q(r')] \nabla_k Q(r') \rangle \tag{30}
$$

Here and from now onward the expectations are defined with the respect to the theory of Eq. (3) - (6) and we assume spherical boundary conditions.

2. **Specific heat**

A natural definition of the observable quantity $z'$ is obtained through the derivative of the thermodynamic potential with respect to temperature which is directly related to the specific heat of the electron gas \cite{24,25}. Write

$$
\frac{\partial \ln \Omega}{\partial \ln T} = \pi T z \int dr \langle O_F(Q) \rangle = \pi T z' \int dr O_F(\Lambda) \tag{31}
$$

then the expression for $z'$ becomes

$$
z' = z \frac{\langle O_F(Q) \rangle}{O_F(\Lambda)}. \tag{32}
$$

The expression for remaining observable $c'$ is determined by the general condition imposed on the static response of the system which says that the quantity $z' = z(1-c)$ remains unaffected by the quantum fluctuations \cite{23,24,43,44,45}. The second equation therefore reads as follows

$$
z'(1-c') = z(1-c) \tag{33}
$$

or $z'a' = za$. Eq. \cite{33} has been explicitly verified in the theory of perturbative expansions. In what follows we proceed and employ Eqs. \cite{32} and \cite{33} for non-perturbative computational purposes as well. A justification of this procedure is given in Section \cite{45} where we embark on the various different subtleties associated with instanton calculus.

3. **$\beta$ and $\gamma$ functions**

The expressions of the previous Sections facilitate renormalization group studies that include not only ordinary perturbative expansions but also the non-perturbative effects of instantons. Since much of the analysis is based on the theory in $2 + \epsilon$ spatial dimensions we shall first recapitulate some of the results of the perturbative renormalization group in two dimensions \cite{24,25,45}. Let $\mu'$ denote the momentum scale associated with the observable theory then the quantities $\sigma'_{xx} = \sigma_{xx}(\mu')$, $z' = z(\mu')$ and $c' = c(\mu')$ can be expressed in terms of the renormalization group $\beta$ and $\gamma$ functions according to (see Ref. \cite{42})

$$
\sigma'_{xx} = \sigma_{xx} + \int_{\mu_0}^{\mu'} d\mu \frac{d\mu}{\mu} \beta_\sigma(\sigma_{xx}, c) \tag{34}
$$

$$
z' = z - \int_{\mu_0}^{\mu'} d\mu \frac{d\mu}{\mu} \gamma_z(\sigma_{xx}, c)z \tag{35}
$$

$$
z'a' = za \tag{36}
$$

where

$$
\beta_\sigma(\sigma_{xx}, c) = \beta_0(c) + \beta_1(c) \sigma_{xx}^{-2} + O(\sigma_{xx}^{-3}) \tag{37}
$$

$$
\gamma_z(\sigma_{xx}, c) = \frac{c\gamma_0}{\sigma_{xx}^2} + \frac{c\gamma_1(c)}{\sigma_{xx}^2} + O(\sigma_{xx}^{-3}). \tag{38}
$$

The one-loop results are known for arbitrary value of the crossover parameter $c$ and are given by \cite{24,45}

$$
\beta_0(c) = \frac{2}{\pi} \left( 1 + \frac{\alpha}{c} \ln c \right), \quad \gamma_0 = -\frac{1}{\pi} \tag{39}
$$

whereas the two-loop results were obtained for $c = 1$ and $c = 0$ only. In the case of electrons with the Coulomb
interaction ($c = 1$) the results are as follows:

$$\beta_1(1) = \frac{4}{\pi^2} \left[ 50 + \frac{1}{6} - 3\pi^2 + \frac{\pi^4}{12} + \frac{19}{2} \zeta(3) + 16G ight]$$

$$- 8\text{li}_4 \left( \frac{1}{2} \right) + \left( \frac{\pi^2}{2} - 44 - 7\zeta(3) \right) \ln 2$$

$$+ \left( 16 + \frac{\pi^2}{3} \right) \ln^2 2 - \frac{1}{3} \ln^4 2 - 8\text{li}_4 \left( \frac{1}{2} \right) \right]$$

$$\approx 0.66$$ (40)

$$\gamma_1(1) = -\frac{3}{\pi^2} - \frac{1}{6} \approx 0.47$$ (41)

where $\zeta(n)$ denotes the Riemann zeta function, $G = 0.915 \ldots$ the Catalan constant, and $\text{li}_n(x) = \sum_{k=1}^{\infty} x^k/k^n$ the polylogarithmic function. For free electrons ($c = 0$) we have:

$$\beta_1(0) = \frac{1}{2\pi^2}$$

$$\gamma_1(0) = 0.$$ (43)

The main objective of the present paper is to extend the results for the observable theory Eqs. $[35, 37]$ to include the effect of instantons and $\theta$ renormalization.

4. RG flows in $2 + \epsilon$ dimensions

For a general understanding of the problem it is important to spell out the consequences of the theory in $2 + \epsilon$ dimension. In this case the $\beta$ function is given by

$$\frac{d\sigma_{xx}}{d\ln \mu} = -c\sigma_{xx} + \beta_\sigma(\sigma_{xx}, c).$$ (44)

Following Eq. (49) we can express the renormalization of $c$ as follows

$$\frac{dc}{d\ln \mu} = \beta_c(\sigma_{xx}, c) = (1 - c)\gamma_c(\sigma_{xx}, c).$$ (45)

The renormalization group flow lines in the $(\sigma_{xx}, c)$ plane are sketched in Fig. 2. We see that there are two critical fixed points describing a quantum phase transition between a metal and an insulator. Along the Coulomb line ($c = 1$) the fixed point value is $\sigma^*_x = O(\epsilon^{-1})$ and along the Fermi liquid line ($c = 0$) we have $\sigma^*_x = O(\epsilon^{-1/2}).$

The results show that the problem with finite range interactions $0 < c < 1$ and the Coulomb interaction problem $c = 1$ belong to different universality classes.

In two spatial dimensions the metallic phases ($\sigma_{xx} > \sigma^*_x$) disappear altogether indicating that all the states of the (spin polarized or spinless) electron gas are now Anderson localized, independent of the presence of electron-electron interactions. This means that as far as the quantum Hall effect is concerned one is generally faced with identically the same fundamental difficulties as those previously encountered in the scaling theory of the free electron gas.

Let us see how the perturbative theory of localization and interaction effects manages to describe an insensitivity of the system to changes in the boundary conditions as outlined in the previous Section. We consider for simplicity the problem with the Coulomb interaction problem ($c = 1$) in two dimensions ($\epsilon = 0$). Since the response parameter $\sigma^*_x$ is independent of the arbitrary momentum scale $\mu_0$ that defines the “renormalized” theory $\sigma_{xx}(\mu_0)$ we immediately obtain from Eq. (51) the general scaling result

$$\sigma^*_x = \sigma_{xx}(\mu') = f_\sigma(\mu' \xi)$$ (46)

where $\mu'$ is related to the linear dimension $L$ of the system according to $\mu' = L^{-1}$. The $\xi$ obeys the differential equation

$$\left( \frac{\partial}{\partial \ln \mu_0} + \beta_\sigma \frac{\partial}{\partial \sigma_{xx}} \right) \xi = 0$$ (47)

and can be identified with a dynamically generated correlation length (localization length) of the system

$$\xi = \mu_0^{-1} \sigma_{xx}^{-\beta_1(1)/\beta_0(1)} e^{\sigma_{xx}/\beta_0(1)}.$$ (48)

Next, comparison of Eqs. (49) and (48) with the expression of Eq. (51) leads to the following explicit (weak coupling) result for the scaling function $f_\sigma(X)$ with $X = (\mu' \xi) \beta_0(1)$

$$f_\sigma(X) \approx \ln X + \frac{\beta_1(1)}{\beta_0(1)} \ln \ln X + \frac{\beta_1(1)}{\beta_0(1)} \frac{\ln \ln X}{\ln X}, \quad X \gg 1.$$ (49)

The statement of exponential localization can now be formulated by saying that in the regime of strong coupling the scaling function $f(X)$ vanishes according to

$$f_\sigma(X) \approx \exp \left( -X^{-1/\beta_0(1)} \right) = \exp \left( -1/(\mu' \xi) \right), \quad X \ll 1.$$ (50)
These naive expectations are fundamentally modified by the $\theta(\nu)$ dependence of the theory which is invisible in perturbation theory. In anticipation of the results of the present paper we can say that the fixed point structure of the theory in $2 + \epsilon$ dimension (Fig. 2) is reminiscent of what happens in the theory in two dimensions at $\theta(\nu) = \pi$. Although the physics is very different in both cases, it is nevertheless important to keep in mind that the renormalization is determined, to a major extend, by the global symmetries of the problem. In particular, since $\mathcal{F}$ invariance is retained along the Coulomb line $c = 1$ only and broken otherwise one generally expects, like the theory in $2 + \epsilon$ dimensions, that the problem with finite range interactions $0 < c < 1$ lies in the domain of attraction of the Fermi liquid line $c = 0$ whereas the Coulomb interaction problem $c = 1$ describes a distinctly different, non-Fermi liquid universality class. Armed with the insights obtained from the perturbative renormalization group we next embark - for the remainder of this paper - on the problem of instantons.

### III. INSTANTONS

In this Section we recapitulate the instanton analysis for the Grassmannian non-linear $\sigma$ model (Sections IIIA and IIIB). We introduce the methodology of spatially varying masses which essentially adapts the interaction part of the action $S_F$ to the metric of a sphere (Section IIICC). In Section IIIIB we derive the complete action for the small oscillator problem that will be used as a starting point for the remainder of this paper.

#### A. Introduction

1. The action $S_\sigma$

On the basis of the Polyakov-Schwartz inequality\textsuperscript{46}

$$\frac{1}{8} \int dr \text{tr}(\nabla Q)^2 \geq 2\pi |\mathcal{C}[Q]|$$

(51)

one can construct stable matrix field configurations (instantons) for each of the discrete topological sectors labelled by the integer $\mathcal{C}[Q]$. The classical action $S_\sigma$ is finite

$$S_\sigma^{\text{inst}} = -2\pi \sigma_{xx} |\mathcal{C}[Q]| + i \theta \mathcal{C}[Q].$$

(52)

The single instanton configuration with the topological charge $\mathcal{C}[Q] = \pm 1$ which is of interest to us can be represented as follows\textsuperscript{27,46}

$$Q_{\text{inst}}(r) = T_0^{-1} \Lambda_{\text{inst}}(r) T_0, \quad \Lambda_{\text{inst}}(r) = \Lambda + \rho(r).$$

(53)

Here, the matrix $\rho_{\alpha\beta}^{nm}(r)$ has four non-zero matrix elements only

$$\rho_{00}^{11} = -\rho_{11}^{00} = -\frac{2\lambda^2}{|z - z_0|^2 + \lambda^2}$$

$$\rho_{01}^{10} = \rho_{10}^{01} = \frac{2\lambda(z - z_0)}{|z - z_0|^2 + \lambda^2}$$

with $z = x + iy$. The manifold of instanton parameters consists the quantity $z_0$ denoting the position of the instanton, the parameter $\lambda$ which equals the scale size as well as the global unitary rotation $T_0$ which describes the orientation in the coset space $U(2N)/U(N) \times U(N)$. These parameters do not change the value of the classical action $S_\sigma^{\text{inst}}$. The anti-instanton with $\mathcal{C}[Q] = -1$ is simply obtained by complex conjugation.

2. The action $S_F$

In the presence of mass terms like the singlet interaction term $S_F$ the idea of stable topologically non-trivial field configurations becomes generally more complicated. The minimum action requirement, for example, immediately tells us that the global matrix $T_0$ is now restricted to run over the subgroup $U(N) \times U(N)$ only.\textsuperscript{27} Instead of Eq. (53) we therefore write

$$Q_{\text{inst}}(r) = U^{-1} \Lambda_{\text{inst}}(r) U = \Lambda + U^{-1} \rho(r) U$$

(55)

with $U \in U(N) \times U(N)$. Next, by substituting Eq. (55) into Eq. (51) one can split $S_F$ into a topologically trivial piece and an instanton peace as follows

$$S_F[Q_{\text{inst}}] = S_F[\Lambda] + S_F^{\text{inst}}$$

(56)

where

$$S_F[\Lambda] = -2\pi T z \int dr \text{tr} \eta \Lambda$$

(57)

and

$$S_F^{\text{inst}}[U] = \pi T z \int dr \left[ c \sum_{\alpha n} \text{tr} \Pi_\alpha^\ast U^{-1} \rho U \text{tr} \Pi_\alpha U^{-1} \rho U + 4 \text{tr} \eta U^{-1} \rho U \right].$$

(58)

Similarly we can write the classical contribution to the thermodynamic potential as the sum of two pieces

$$\Omega^{\text{class}} = \Omega_0^{\text{class}} + \Omega_{\text{inst}}^{\text{class}}$$

(59)

where $\Omega_0^{\text{class}}$ is the contribution of the trivial vacuum

$$\Omega_0^{\text{class}} = S_F[\Lambda] = -2\pi T z \int dr \text{tr} \eta \Lambda$$

(60)

and $\Omega_{\text{inst}}^{\text{class}}$ is the instanton peace

$$\Omega_{\text{inst}}^{\text{class}} = \int_{\text{inst}} \text{exp} \left( -2\pi \sigma_{xx} \pm i \theta + S_F^{\text{inst}}[U] \right).$$

(61)
up on the idea of
this respect a true advantage to be gained if one follows
at best and useless for practical purposes.

The subscript “inst” indicates that the integral is over
the manifold of instanton parameters $z_0$, $\lambda$ and $U$.

One of the main complications next is that the action $S_{\text{inst}}[U]$ is not finite but, rather, it diverges logarithmi-
cally in the size of the system. Although these and other
complications associated with mass terms are quite well
known, the resolution that has been proposed is formal
at best and useless for practical purposes.22 There is in
this respect a true advantage to be gained if one follows
up on the idea of spatially varying masses which has re-
cently been introduced and analyzed in great detail by
the authors.23 This methodology not only extends the
formalism developed for the massless theory in a natural
fashion, but also lends itself to a non-perturbative analy-
sis of the renormalization group $\beta$ and $\gamma$ functions of the
theory. Before embarking on the specific problem of the
interacting electron gas it is necessary to first recapitu-
late some of the main results obtained for the ordinary
Grassmannian manifold.24 This will be done in the Sec-
tions below where we generalize the harmonic oscillator
problem to include an arbitrary range of Matsubara fre-
quencies. The most important results are written in Sec-
tion [111][111] Table [111] which contains the complete action
of quantum fluctuations about the single instanton.

\subsection{Quantum fluctuations}

\subsubsection{Preliminaries}

To obtain the most general matrix field variable $Q$ with
topological charge equal to unity we first rewrite the inst-
ton solution $A_{\text{inst}}$ in Eqs. (54) and (55) as a unitary
rotation $R$ about the trivial vacuum $A$

$$A_{\text{inst}} = R^{-1} A R.$$  \hfill (62)

From now onward we use the following notation for an
arbitrary matrix $A$

$$A_{mn}^{\alpha \beta} = \begin{pmatrix}
A_{n_1 n_3}^{\alpha \beta} & A_{n_1 n_2}^{\alpha \beta} \\
A_{n_2 n_1}^{\alpha \beta} & A_{n_2 n_4}^{\alpha \beta}
\end{pmatrix}. \hfill (63)$$

Here, the $n_i$ with odd subscripts $i$ denote the indices
for positive Matsubara frequencies. Similarly, the even
subscripts $i$ refer to the negative Matsubara frequencies.
Hence, the indices $n_1$ and $n_3$ run over the set of non-
negative integers \{0, 1, 2, \ldots\}. The indices $n_2$ and $n_4$
run over the set of negative integers \{-1, -2, -3, \ldots\}. Fully
written out the different frequency blocks of the unitary
matrix $R_{mn}^{\beta}$ now become

$$R_{n_1 n_3}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{n_1 n_3} \left[ 1 + (e_1 - 1) \delta^{\alpha 1} \delta_{n_1,0} \right]$$  \hfill (64)

$$R_{n_2 n_4}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{n_2 n_4} \left[ 1 + (e_1 - 1) \delta^{\alpha 1} \delta_{n_2,-1} \right]$$  \hfill (65)

$$R_{n_1 n_2}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{n_1,0} \delta_{n_2,-1} [e_0]$$  \hfill (66)

$$R_{n_2 n_1}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{n_1,0} \delta_{n_2,-1} [-e_0] = -R_{n_1 n_2}^{\alpha \beta}$$  \hfill (67)

where the quantities $e_0$ and $e_1$ are defined by

$$e_0 = \frac{\lambda}{\sqrt{|z - z_0| + \lambda^2}}$$  \hfill (68)

$$e_1 = \frac{z - z_0}{\sqrt{|z - z_0|^2 + \lambda^2}}.$$  \hfill (69)

The structure of the matrix $R_{mn}^{\beta}$ is illustrated in Fig. 3.

It is a simple matter next to generalize Eq. (62) and the
result is

$$Q = T_0^{-1} R^{-1} V R T_0.$$  \hfill (70)

Here, $T_0$ denotes a global $U(2N)$ rotation and the matrix
$V$ with $V^2 = 1$ represents the small fluctuations about the
one instanton. Write

$$V = w + A \sqrt{1 - w^2}$$  \hfill (71)

with

$$w = \begin{pmatrix}
0 \\
v
\end{pmatrix}$$  \hfill (72)

then the matrix $V$ can formally be written as a series expansion in powers of the $N \times N$ complex matrices $v$, $v^\dagger$ which are taken as the independent field variables in the
problem.

\subsubsection{Stereographic projection}

Eq. (61) lends itself to an exact analysis of the small
oscillator problem. First we recall the results obtained
for the free electron theory.23

$$\frac{\sigma_{xx}}{8} \int \text{d}r \text{tr}(\nabla_j Q)^2 = \frac{\sigma_{xx}}{8} \int \text{d}r \text{tr}[\nabla_j + A_j, V]^2$$  \hfill (73)
where the matrix $A_j$ contains the instanton degrees of freedom

$$A_j = R \mathcal{T}_0 \nabla_j T_0^{-1} R^{-1} = R \nabla_j R^{-1}.$$  \hspace{1cm} (74)

$$\frac{\sigma_{xx}}{8} \int dr \, tr[\nabla_j + A_j, \mathcal{V}]^2 =$$

$$= \frac{\sigma_{xx}}{4} \int dr \mu^2(r) \left( \sum_{\alpha=2}^{N_r} \sum_{\beta=2}^{N_r} \sum_{n_1 n_2} \delta_{\alpha \beta} O^{(0)}(0) v^{(0)}_{n_2 n_1} + \sum_{n_1 n_2} \sum_{\lambda=1}^{N_r} \left( \sum_{n_1 n_2} \left( \sum_{\lambda=1}^{N_r} \sum_{n_1 n_2} \delta_{\alpha \beta} O^{(0)}(0) v^{(0)}_{n_2 n_1} \right) \right) \right) \left. \right|_{\lambda = 1}^{N_r}.$$  \hspace{1cm} (75)

The “prime” on the summation signs are defined as follows

$$\sum'_{n_1} = \sum_{n_1 = 1}^{N_r - 1}, \quad \sum'_{n_2} = \sum_{n_2 = -1}^{N_r - 1}.$$  \hspace{1cm} (76)

The three different operators $O^{(a)}$ with $a = 0, 1, 2$ are given as

$$O^{(a)} = \left( \frac{r^2 + \lambda^2}{4\lambda^2} \right)^2 \left( \nabla_j + \frac{ia}{r^2 + \lambda^2} \epsilon_{jk} R_k \right)^2 + \frac{a}{2}.$$  \hspace{1cm} (77)

The introduction of a measure $\mu^2(r)$ for the spatial integration in Eq. (75),

$$\mu(r) = \frac{2\lambda}{r^2 + \lambda^2}$$  \hspace{1cm} (78)

indicates that the quantum fluctuation problem is naturally defined on a sphere with radius $\lambda$. It is convenient to employ the stereographic projection

$$\eta = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}, \quad -1 < \eta < 1$$  \hspace{1cm} (79)

$$\theta = \tan^{-1} \frac{y}{x}, \quad 0 \leq \theta < 2\pi.$$  \hspace{1cm} (80)

In terms of $\eta, \theta$ the integration can be written as

$$\int dr \mu^2(r) = -\int d\eta d\theta.$$  \hspace{1cm} (81)

Moreover,

$$e_0 = \sqrt{\frac{1 - \eta}{2}}, \quad e_1 = \sqrt{\frac{1 + \eta}{2}} e^{i\theta}.$$  \hspace{1cm} (82)

By expanding the $\mathcal{V}$ in Eq. (73) to quadratic order in the quantum fluctuations $v, v^\dagger$ we obtain the following results

| Operator | Number of fields $v^{\alpha \beta}_{n_1 n_2}$ | Degeneracy |
|----------|------------------------------------------|------------|
| $O^{(0)}$ | $(N - 1)^2$ | 1 |
| $O^{(1)}$ | $2(N - 1)$ | 2 |
| $O^{(2)}$ | 1 | 3 |

and the operators become

$$O^{(a)} = \frac{\partial}{\partial \eta} \left( 1 - \eta^2 \right) \frac{\partial}{\partial \eta} + \frac{1}{1 - \eta^2} \frac{\partial^2}{\partial \theta^2} - i a \frac{\partial}{\partial \theta} - \frac{a^2}{4} \frac{1 + \eta}{1 - \eta} \frac{\partial}{\partial \theta} + \frac{a}{2}$$  \hspace{1cm} (83)

with $a = 0, 1, 2$. Finally, using Eq. (78) we can count the total number of fields $v^{\alpha \beta}$ on which each of the operators $O^{(a)}$ act. The results are listed in Table I.

3. Energy spectrum

We are interested in the eigenvalue problem

$$O^{(a)} \Phi^{(a)}(\eta, \theta) = E^{(a)} \Phi^{(a)}(\eta, \theta)$$  \hspace{1cm} (84)

where the set of eigenfunctions $\Phi^{(a)}$ are taken to be orthonormal with respect to the scalar product

$$\langle \Phi^{(a)}_1, \Phi^{(a)}_2 \rangle = \int d\eta d\theta \, \bar{\Phi}^{(a)}_1(\eta, \theta) \Phi^{(a)}_2(\eta, \theta).$$  \hspace{1cm} (85)
The Hilbert space of square integrable eigenfunctions is given in terms of Jacobi polynomials,

\[ P_n^{\alpha, \beta}(\eta) = \frac{(-1)^n (1 - \eta)^{-\alpha}}{2^n n!} \frac{d^n}{d\eta^n} (1 - \eta)^{n+\alpha}. \]  

(86)

Introducing the quantum number \( J \) to denote the discrete energy levels

\[
E^{(0)}_J = J(J + 1), \quad J = 0, 1, \ldots \\
E^{(1)}_J = (J - 1)(J + 1), \quad J = 1, 2, \ldots \\
E^{(2)}_J = (J - 1)(J + 2), \quad J = 1, 2, \ldots
\]

(87)

then the eigenfunctions are labelled by \((J, M)\) and can be written as follows

\[
\Phi^{(0)}_{J,M} = C^{(0)}_{J,M} e^{iM\theta} (1 - \eta^2)^{M/2} P^M_{J-M} (\eta) \\
C^{(0)}_{J,M} = \sqrt{\frac{\Gamma(J - M + 1) \Gamma(J + M + 1)}{2^{M+1} \pi \Gamma(J + 1)}} M = -J, \ldots, J
\]

(88)

\[
\Phi^{(1)}_{J,M} = C^{(1)}_{J,M} e^{iM\theta} (1 - \eta^2)^{M/2} (1 - \eta) P^M_{J-M+1} (\eta) \\
C^{(1)}_{J,M} = \sqrt{\frac{\Gamma(J - M) \Gamma(J + M + 1)}{2^{M+1} \pi \Gamma(J)}} M = -J, \ldots, J - 1
\]

(89)

\[
\Phi^{(2)}_{J,M} = C^{(2)}_{J,M} e^{iM\theta} (1 - \eta^2)^{M/2} P^M_{J+M-1} (\eta) \\
C^{(2)}_{J,M} = \sqrt{\frac{\Gamma(J - M) \Gamma(J + M + 2)}{2^{M+2} \pi \Gamma(J) \sqrt{J(J+1)}}} M = -J - 1, \ldots, J - 1.
\]

(90)

4. Zero modes

From Eq. (86) we see that the operators \( O^{(0)} \) has a zero frequency mode \( E^{(0)}_J = 0 \) for \( J = 0 \). Similarly, we have \( E^{(1)}_J = E^{(2)}_J = 0 \) for \( J = 1 \). The corresponding eigenfunctions can be written as follows

\[
O^{(0)} \rightarrow \Phi^{(0)}_{0,0} = 1 \\
O^{(1)} \rightarrow \Phi^{(1)}_{1,-1} = \frac{1}{\sqrt{2\pi}} e_1, \quad \Phi^{(1)}_{1,0} = \frac{1}{\sqrt{2\pi}} e_0 \\
O^{(2)} \rightarrow \Phi^{(2)}_{1,-2} = \sqrt{\frac{3}{4\pi}} e_2, \quad \Phi^{(2)}_{1,-1} = \sqrt{\frac{3}{2\pi}} e_0 e_1 \\
\Phi^{(2)}_{1,0} = \sqrt{\frac{3}{4\pi}} e_0.
\]

(91)

Here, the quantities \( e_0 \) and \( e_1 \) are defined in Eqs. (88) and (89) (see also Eq. (82)). The number of the zero modes of each \( O^{(0)} \) is listed in Table I. The total we find \( 2(N^2 + 2N) \) zero modes in the problem.

Next, it is important to show that these zero modes precisely correspond to all the instanton degrees of freedom contained in the matrices \( R \) and \( T_0 \) of Eq. (74). For this purpose we write the instanton solution as follows

\[
Q_{\text{inst}} (\xi_i) = U_{\text{inst}}^{(1)} (\xi_i) \Lambda U_{\text{inst}} (\xi_i).
\]

(92)

Here, \( U_{\text{inst}} = R T_0 \) and the \( \xi_i \) stand for the parameters \( \omega_0, \lambda \) and the generators of \( T_0 \). An infinitesimal change in the instanton parameters \( \xi_i \rightarrow \xi_i + \varepsilon_i \) can be written in the form of Eq. (92) as follows

\[
Q_{\text{inst}} (\xi_i + \varepsilon_i) = U_{\text{inst}}^{(1)} (\xi_i) \varepsilon_i U_{\text{inst}} (\xi_i)
\]

(93)

where to linear order in \( \varepsilon_i \) we can write

\[
\varepsilon_i = \Lambda - \varepsilon_i [U_{\text{inst}} \partial_i U_{\text{inst}}^{-1}, \Lambda]
\]

(94)

We have written \( \partial_i = \partial_i \partial_\xi_i \). By comparing this expression with Eq. (70) we see that the fluctuations tangential to the instanton manifold can be expressed in terms of the matrix field variables \( v, v^\dagger \) according to

\[
v^{\alpha\beta}_{n_1 n_2} = 2 \varepsilon_i [U_{\text{inst}} \partial_i U_{\text{inst}}^{-1}]^{\alpha\beta}_{n_1 n_2}
\]

(95)

\[
[v^\dagger]^{\alpha\beta}_{n_1 n_2} = -2 \varepsilon_i [U_{\text{inst}} \partial_i U_{\text{inst}}^{-1}]^{\alpha\beta}_{n_1 n_2}.
\]

(96)

To obtain explicit expressions it suffices to expand \( T_0 \) about unity

\[
T_0 = 1 + i t
\]

(97)

and write

\[
R(\lambda + \delta \lambda, \omega_0 + \delta \omega_0) = R(\lambda, \omega_0) + \delta \lambda \partial_\lambda R + \delta \omega_0 \partial_{\omega_0} R.
\]

(98)

The expression for \( v \) now becomes

\[
v^{\alpha\beta}_{n_1 n_2} = 2 i [R t_{\text{inst}} R^{-1}]^{\alpha\beta}_{n_1 n_2} + 2 \delta \lambda [R \partial_\lambda R^{-1}]^{\alpha\beta}_{n_1 n_2} + 2 \delta \omega_0 [R \partial_{\omega_0} R^{-1}]^{\alpha\beta}_{n_1 n_2}.
\]

(99)

Notice that \( v^\dagger \) is just the hermitian conjugate of \( v \) as it should be. In Table I we present the complete list of zero energy modes \( v^{\alpha\beta}_{n_1 n_2} \) written in terms of \( t^{\alpha\beta}_{n_1 n_2} \), \( \delta \lambda \) and \( \delta \omega_0 \) as well as the eigenfunctions \( \Phi^{(a)}_{J,M} \).

In these expressions \( t^{\alpha\beta}_{n_1 n_2} \) and \( t^{\alpha\beta}_{n_2 n_1} \) denote the generators of \( U(2N) \times U(N) \times U(N) \). The \( t^{\alpha\beta}_{n_1 n_0} \) and \( t^{\alpha\beta}_{0 n_1} \), with \( n_1 \neq 0 \) and \( \alpha = 1 \) are the generators of a \( U(N) / U(N - 1) \) rotation. The same holds for \( t^{\alpha\beta}_{n_2 n_{-1}} \) and \( t^{\alpha\beta}_{n_{-1} n_2} \) with \( n_2 \neq -1 \) and \( \alpha = 1 \). Finally, \( t^{\alpha\beta}_{00} - t^{\alpha\beta}_{11} - 1 \) denotes the \( U(1) \) generator corresponding to rotations of the \( O(3) \) instanton in the \( xy \) plane. The number of instanton degrees of freedom adds up to \( 2(N^2 + 2N) \) which is the same as the number of zero modes in the problem. The various different generators \( t \) of the instanton manifold is illustrated in Fig. 4.
TABLE II: Zero energy modes $v_{n_1n_2}^{\alpha\beta}$ expressed in terms of $t_{mn}^{\alpha\beta}$, $\delta\lambda$, $\delta z_0$ and $\Phi_{\lambda\mu}^{(n)}$, see text.

| $\alpha$ $\beta$ | $n_1$ $n_2$ | $O^{(0)}$ | $O^{(1)}$ | $O^{(2)}$ |
|------------------|-------------|-----------|-----------|-----------|
| $\alpha > 1$, $\beta > 1$ | $n_1 \geq 0$, $n_2 \leq -1$ | $2i t_{n_1n_2}^{\alpha\beta} \Phi_{\lambda\mu}^{(0)}$ | | |
| $\alpha > 1$, $\beta = 1$ | $n_1 \geq 0$, $n_2 = -1$ | $2i \sqrt{2} \pi (t_{n_1n_2}^{\alpha\beta} \Phi_{\lambda\mu}^{(1)} - t_{n_1n_2}^{01} \Phi_{\lambda\mu}^{(1)})$ | | |
| | $n_1 \geq 0$, $n_2 < -1$ | $2i t_{n_1n_2}^{\alpha\beta} \Phi_{\lambda\mu}^{(0)}$ | | |
| $\alpha = 1$, $\beta > 1$ | $n_1 > 0$, $n_2 \leq -1$ | $2i t_{n_1n_2}^{1\beta} \Phi_{\lambda\mu}^{(0)}$ | | |
| | $n_1 = 0$, $n_2 < -1$ | $2i \sqrt{2} \pi (t_{0n_2}^{1\beta} \Phi_{\lambda\mu}^{(1)} + t_{-1n_2}^{1\beta} \Phi_{\lambda\mu}^{(1)})$ | | |
| | $n_1 > 0$, $n_2 = -1$ | $2i \sqrt{2} \pi (t_{n_1n_2}^{1\beta} \Phi_{\lambda\mu}^{(1)} - t_{n_1n_2}^{01} \Phi_{\lambda\mu}^{(1)})$ | | |
| | $n_1 = 0$, $n_2 = -1$ | $4i \sqrt{2} \pi \left[ t_{-1,0}^{11} \Phi_{\lambda\mu}^{(2)} + t_{1,0}^{11} \Phi_{\lambda\mu}^{(2)} \right] (t_{-1,1}^{01} \Phi_{\lambda\mu}^{(2)} - t_{0,0}^{11} \Phi_{\lambda\mu}^{(2)}) \right]$ | | |

C. Spatially varying masses

In the previous Section we have seen that the instanton problem naturally acquires the geometry of a sphere. This clearly complicates the problem of mass terms in the theory which are usually written in flat space. To deal with this problem we shall modify the definition of the singlet interaction term and introduce a spatially varying momentum scale $\mu(r)$ as follows

$$z \rightarrow z \mu^2(r), \quad z c \rightarrow z c \mu^2(r) \quad (100)$$

such that the action $S_F$ is now finite and can be written as

$$S_F[Q] \rightarrow \pi T z \int d\mu^2(r) \left( c \sum_{\alpha n} \text{tr} I_n^{\alpha} Q \text{tr} I_n^{\alpha} Q + 4 \text{tr} \eta Q - 6 \text{tr} \eta \Lambda \right). \quad (101)$$

As we will show below, in Sections 4.2 and 4.4 the introduction of a spatially varying momentum scale $\mu(r)$ permits the development of a complete quantum theory of the interacting electron gas that is defined on a
sphere. Although the philosophy so far proceeds along similar lines as those employed in the ordinary Grassmannian model\[24\] it is important to keep in mind that the presence of $S_F$ is itself affecting the ultraviolet singularity structure of the theory. This means that both the physics and the conceptual structure of the problem with interactions are fundamentally different from what one is used to. Moreover, in view of the mathematical peculiarities of the theory, in particular those associated with the limits $N_r \to 0$ and $N_m \to \infty$, it must be shown explicitly that instantons are well defined at a quantum level and that the aforementioned ultraviolet behavior of the interacting electron gas does not depend on the specific geometry that one chooses, i.e. the introduction of $\mu(r)$ in Eq. (101). In this respect, we shall in what follows greatly benefit from our theory of observables since it provides an appropriate framework for a general understanding of the theory at short distances. To study the ultraviolet we first address the problem of quantum fluctuations for the special case where unitary matrix $T_0$ in Eq. (11) is equal to unity. We will come back to the general case not until Section IVV where embark on the infrared of the system, in particular the various different steps that are needed in order to change the geometry of the system from curved space to flat space.

D. Action for the quantum fluctuations

Keeping the remarks of the previous section in mind we obtain the complete action as the sum of a classical part $S^{\text{inst}}$ and a quantum part $\delta S$ as follows

$$S = S_F[A] + S^{\text{inst}} + \delta S$$

where

$$S^{\text{inst}} = -2\pi \sigma_{xx} + i\theta + S_F$$

and

$$\delta S = \delta S^{(0)} + \delta S^{(1)} + \delta S^{(2)} + \delta S^{\text{linear}}.$$  \hspace{1cm} (104)

Here $S^{\text{inst}}_F$ stands for the classical action of the modified singlet interaction term, Eq. (25), with $U = 1$ and is given by

$$S^{\text{inst}}_F = \pi T \int dr \mu^2(r) \left( e \sum_{\alpha} \alpha \rho \alpha \rho + 4 \text{tr} \eta \rho \right) = 16\pi^2 T \left( \frac{C}{3} - 1 \right).$$  \hspace{1cm} (105)

Next, the results for $\delta S$ in Eq. (114) are classified in four different parts. The complete list of contributions is presented in Table III. We use the following notations $n_{12} = n_1 - n_2$ and $\kappa^2 = 8\pi T/\sigma_{xx}$ from now onward. We will first briefly comment on the different parts of $\delta S$.

1. $\delta S^{(0)}$

This term contains all the fluctuations $v^{\alpha \beta}_{mn}$ with replica indices $\alpha, \beta > 1$ that do not couple to the instanton. $\delta S^{(0)}$ has therefore the same form as the fluctuations about the trivial vacuum.

2. $\delta S^{(1)}$, $\delta S^{(2)}$

The terms $\delta S^{(1)}$ and $\delta S^{(2)}$ contain all the fluctuations $v^{\alpha \beta}_{mn}$ with either $\alpha = 1$ or $\beta = 1$. $\delta S^{(2)}$ only contains the fluctuations in the first replica channel $v^{11}_{mn}$ and the remaining contributions are collected in $\delta S^{(1)}$. In both $\delta S^{(1)}$ and $\delta S^{(2)}$ we distinguish between the "diagonal" contributions that mainly originate from $S_\sigma$ (first four lines in Table III) and the "off-diagonal" ones originating from $S_F$ (fifth and subsequent lines).

3. $\delta S^{\text{linear}}$

The contributions linear in $v$ and $v^1$ originate from the singlet interaction term $S_F$ and are written in the bottom line of Table III. They can be written in terms of the eigenfunctions $\Phi^{(i)}_{JM}$ as follows

$$\int d\eta d\theta \left( e_0^2 v^{11}_{00,-2} + e_0^2 v^{11}_{00,-2} \right) \propto \int d\eta d\theta \left( \Phi^{(1)}_{2,1} v^{11}_{00,-2} + \Phi^{(1)}_{2,1} v^{11}_{00,-2} \right) \hspace{1cm} (106)$$

$$\int d\eta d\theta \left( e_0^2 v^{11}_{11,-1} + e_0^2 v^{11}_{11,-1} \right) \propto \int d\eta d\theta \left( \Phi^{(1)}_{2,1} v^{11}_{11,-1} + \Phi^{(1)}_{2,1} v^{11}_{11,-1} \right) \hspace{1cm} (107)$$

$$\int d\eta d\theta \left( e_0^2 v^{11}_{00,-2} + e_0^2 v^{11}_{00,-2} \right) \propto \int d\eta d\theta \left( \Phi^{(2)}_{2,1} v^{11}_{00,-1} + \Phi^{(2)}_{2,1} v^{11}_{00,-1} \right) \hspace{1cm} (108)$$

Since the $\Phi^{(1)}_{2,1}$ and $\Phi^{(2)}_{2,1}$ do not correspond to the zero modes of the operators $O^{(1)}$ and $O^{(2)}$ one can eliminate these terms by performing a simple shift in $v$, $v^1$. This leads to an insignificant contribution to the classical action of the order $O(T^2)$. Next,

$$\int d\eta d\theta \left( e_0^2 v^{11}_{00,-1} + e_0^2 v^{11}_{00,-1} \right) \propto \int d\eta d\theta \left( \Phi^{(2)}_{1,1} v^{11}_{00,-1} + \Phi^{(2)}_{1,1} v^{11}_{00,-1} \right) \hspace{1cm} (109)$$

This means that the fluctuations tangential to the instanton parameter $\lambda$ are the only unstable fluctuations in the problem. As will be discussed further below, these fluctuations will be treated separately and we will proceed.
\[ \delta S^{(0)} = - \frac{\sigma_x}{4} \int d\eta d\theta \sum_{\alpha, \beta=2}^{N_0} \sum_{n_1 \ldots n_4} \delta_{n_1 \ldots n_4} v_{n_1 n_2} \left[ (O^{(0)} + \kappa^2 zn_{12}) \delta_{n_1 n_3} - \kappa^2 z e \delta_{n_2 n_3} \right] \] 

\[ \delta S^{(1)} = - \frac{\sigma_x}{4} \int d\eta d\theta \sum_{\alpha=2}^{N_0} \left\{ \sum_{n_1 n_2} \left[ \nu_{n_1 n_2}^{(1)} (O^{(0)} + \kappa^2 zn_{12}) \nu_{n_2 n_1}^{(1)} + \sum_{n_1 n_3} \nu_{n_1 n_2}^{(2)} (O^{(0)} + \kappa^2 zn_{12}) \nu_{n_2 n_1}^{(1)} \right] \right. 

\left. + \sum_{n_1} \nu_{n_1 n_1}^{(1)} (O^{(1)} + \kappa^2 zn_{11} + 1) \nu_{n_1 n_1}^{(1)} + \sum_{n_2} \nu_{n_2 n_2}^{(1)} (O^{(0)} - \kappa^2 zn_{22}) \nu_{n_2 n_2}^{(1)} \right\} \] 

\[ \delta S^{(2)} = - \frac{\sigma_x}{4} \int d\eta d\theta \left\{ \sum_{n_1 \ldots n_4} \left[ \nu_{n_1 n_2}^{(1)} (O^{(0)} + \kappa^2 zn_{12}) \delta_{n_1 n_3} \delta_{n_2 n_4} - \kappa^2 z e \delta_{n_1 n_3} \right] \right\} \] 

\[ \delta S^{(2)}_{\text{linear}} = \frac{\sigma_x}{2} \kappa^2 z \int d\eta d\theta \left\{ \nu_{n_1 n_2}^{(1)} (O^{(0)} + \kappa^2 zn_{12}) \right\}
by formally evaluating the quantum theory to first order in the temperature $T$ only.

4. Trivial vacuum

For completeness we give the expression for the quantum fluctuations about the trivial vacuum. The result can be written as follows

$$S_0 = S_F[\Lambda] + \delta S_0$$

where

$$\delta S_0 = - \frac{\sigma_{xx}}{4} \int \frac{d\eta d\theta}{N} \sum_{\alpha,\beta=1}^{N_c} \sum_{n_1,\ldots,n_4} \delta_{n_1,2,3} \delta_{n_2,1,4} \xi_{n_1 n_2}^{\alpha\beta} \times \left[ \left( O^{(0)}(k) + \kappa^2 \eta_{12}(k) \right) \delta_{n_1 n_3} - \kappa^2 \zeta \delta_{\alpha\beta} \right] \right]_{n_4 n_3}$$

IV. DETAILS OF COMPUTATIONS

In this Section we present the detailed computations of the harmonic oscillator problem. In the first part we address the thermodynamic potential which is in many ways standard. The complications primarily arise from the infinite sums over Matsubara frequencies which fundamentally alter the ultraviolet singularity structure of the theory. We set up a systematic series expansion of the thermodynamic potential in powers of the temperature $T$. To perform the algebra we make use of the complete set of eigenvalues and eigenfunctions obtained in the previous Section as well as certain mathematical identities in Appendix C. The regularized theory is then defined as

$$\delta S_{\text{reg}} = \delta S_0 + \sum_{f=1}^{K} e_f \delta S_f.$$ (114)

Here, action $\delta S_f$ is the same as $\delta S$ except that the operators $O^{(a)}(k)$ are all replaced by $O^{(a)}(k) + M_f^2$. Our task is to evaluate Eq. (114) to lowest orders in a series expansion in powers of $T$. This expansion still formally diverges due to the zero modes of the operators $O^{(a)}(k)$. These zero modes, however, shall be treated separately by employing the collective coordinate formalism introduced in Ref. 16.

To simplify the notation we will next present the results while omitting the alternating metric and the Pauli-Villars masses. This can be done since in each case we consider one easily recognizes how the metric and masses should be included. Consider the ratio

$$\frac{Z_{\text{inst}}}{Z_0} = \frac{\int D[v, v^\dagger] \exp S}{\int D[v, v^\dagger] \exp S_0} = \exp \left[ -2\pi \sigma_{xx} + i\theta + S_{\text{inst}}^0 + \Delta S_\sigma + \Delta S_F \right].$$ (115)

Here, the quantum corrections denoted by $\Delta S_\sigma$ and $\Delta S_F$ can be expressed in terms of the propagators

$$G_\sigma(\omega) = \frac{1}{O^{(a)}(k) + \omega} = \sum_{JM} \frac{|JM\rangle_{a} \langle JM|}{E_{j}^{(a)} + \omega}$$ (116)

$$G_\sigma^\dagger(\omega) = \frac{1}{O^{(a)}(k) + \omega} = \sum_{JM} \frac{|JM\rangle_{a} \langle JM|}{E_{j}^{(a)} + \omega}$$ (117)

where $a = 0, 1, 2$. These expressions are directly analogous to those that appear in flat space (see Ref. 24). It is important to emphasize that even at a Gaussian level the integration over the field variables $v, v^\dagger$ in Eq. (115) is not simple and straightforward. The main reason is that some of the frequency sums can be written as an integral in the limit $T \to 0$ and, along with that, they absorb a factor of $T$. It is therefore not always obvious how the series expansion in powers of $T$ should be evaluated. The simplest way to proceed is to expand the functional integrals of Eq. (115) in non-diagonal terms which are proportional to $\kappa^2 \sim T$. By inspection one can then convince oneself that in the replica limit $N_r \to 0$, the expansion in the non-diagonal terms can be truncated beyond third order only. We shall next summarize the various different contributions to $\Delta S_\sigma$ as well as $\Delta S_F$.  

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**Pauli-Villars regulators**

1. Introduction

Recall that after integration over the quantum fluctuations one is in general left with two sources of divergences. First, there are the ultraviolet singularities which eventually result in a renormalization of the coupling constant or $\sigma_{xx}$. At present we wish to extend the analysis to include the renormalization of the $z$ and $zc$ fields. The ultraviolet of the theory can be dealt with in a standard manner by employing Pauli-Villars regulator fields with masses $M_f (f = 0, 1, \ldots, K)$ and an alternating metric $e_f$. We assume $e_0 = 1$, $M_0 = 0$ and large masses $M_f \gg 1$ for $f > 1$. The following constraints are imposed

$$\sum_{f=0}^{K} e_f M_f^2 = 0, \quad 0 \leq k < K$$ (112)

$$\sum_{f=1}^{K} e_f \ln M_f = - \ln M.$$ (113)

The regularized theory is defined as

$$\delta S_{\text{reg}} = \delta S_0 + \sum_{f=1}^{K} e_f \delta S_f.$$ (114)
2. $\Delta S_o$

The quantum correction $\Delta S_o$ is obtained by expanding the non-diagonal terms of Table III up to second order. The results in the limit $T \to 0$, $N_r \to 0$ and $N_m \to \infty$ can be written as follows

$$
\Delta S_o = 2 \text{tr} [\ln G_1(0) - \ln G_0(0)] - \text{tr} [\ln G_2(0) - \ln G_0(0)] + 2c \int_0^\infty d\omega \text{tr} [G_1(\omega) - G_0(\omega)] + 2c^2 \int_0^\infty d\omega \omega \text{tr} [e_1 G_0^c(\omega) e_1 G_1(\omega) + c_0 G_0^c(\omega) c_0 G_1(\omega) - G_0^c(\omega) G_0^c(\omega)].
$$

(118)

In these expressions the trace is taken with respect to the complete set of eigenfunctions of the operators $O^{(n)}$. To evaluate these expressions we need the help of the identities (B.3) and (B.4) of Appendix B. After elementary algebra we obtain

$$
\Delta S_o = 2\alpha D^{(1)}_c - D^{(2)}_c - 2c \left( H^{(1)} \ln \alpha - H^{(2)} - cH^{(3)} \right).
$$

(120)

Here the quantities

$$
D^{(r)} = \sum_{J=1}^{\infty} (2J+1) \ln E^{(r)}_J - \sum_{J=1}^{\infty} (2J+1) \ln E^{(0)}_J
$$

(121)

with $r = 1, 2$ originate from Eq. (115). The quantities $H^{(i)}$ originate from Eq. (119) and are defined by

$$
H^{(1)} = \sum_{J=0}^{J+1} \sum_{J=j}^{\infty} J_1 \frac{E^{(0)}_J - E^{(1)}_{J_1}}{E^{(0)}_{J_1} - \alpha E^{(1)}_{J_1}}
$$

(122)

3. $\Delta S_F$

To obtain the quantum correction $\Delta S_F$ we need to carry out the expansion in the non-diagonal terms of Table III up to the third order. By taking the appropriate limits as discussed earlier we find the following results

$$
\Delta S_F = 2\kappa^2 \left\{ \text{tr} \left[ (\alpha + e^2)(G_1(0) - G_0(0)) - \frac{\alpha}{2} (G_2(0) - G_0(0)) \right] \\
+ \text{tr} \left[ \alpha (2|e_1|^2 - 1) c_0^2 G_0(0) \right] - (3c|e_1|^2 - 1) c_0^2 G_0(0) - 2c^2 c_0^2 G_1(0) \right\}
$$

(125.1)

$$
+ c^3 \int_0^\infty d\omega \omega \text{tr} [e_1 G_0^c(\omega) e_1 G_1(\omega) + c_0 G_0^c(\omega) c_0 G_1(\omega) - G_0^c(\omega) G_0^c(\omega)]
$$

(125.2)

$$
- c^3 \int_0^\infty d\omega \omega \text{tr} [e_1 G_0^c(\omega) e_1 G_1(\omega) + c_0 G_0^c(\omega) c_0 G_1(\omega) - G_0^c(\omega) G_0^c(\omega)]
$$

(125.3)

$$
+ 2c^2 \int_0^\infty d\omega \omega \text{tr} [e_0^2 G_1(\omega) e_0^2 G_1(\omega)]
$$

(125.4)

$$
+ 5c^2 \int_0^\infty d\omega \omega \text{tr} [c_0 e_1 G_1(\omega) c_0 e_1 G_1(\omega)]
$$

(125.5)

$$
- c^2 \int_0^\infty d\omega \omega \text{tr} [c_0 e_1 G_0(\omega) c_0 e_1 G_0(\omega)]
$$

(125.6)

$$
- c^2 \int_0^\infty d\omega \omega \text{tr} [e_0 G_0^c(\omega) e_0 G_1^c(\omega)]
$$

(125.7)

$$
- 4c^3 \int_0^\infty d\omega \omega \text{tr} [e_0 e_1 G_1(\omega) e_0 G_0^c(\omega) e_1 G_1(\omega)]
$$

(125.8)

$$(125.9)$$
To evaluate these expressions we use the identities \(K\) (see Appendix B). After some algebra we find
\[
\Delta S_F = 2\kappa^2 z^2 \sum_{i=1}^{9} B^{(i)}.
\]
Here, the nine contributions \(B^{(i)}\), \(i = 1, \ldots, 9\) correspond to the nine equations \[125.1\]–\[125.9\]. The first two of them are given by
\[
B^{(1)} = (\alpha + c^2)(Y^{(1)} - Y^{(0)}) - \frac{\alpha}{2}(Y^{(2)} - Y^{(0)})
\]
\[
B^{(2)} = \frac{\alpha}{2} \left( \frac{2c}{3} - 1 \right) Y^{(1)} + \frac{\alpha}{2} Y^{(2)} - c^2 Y^{(1)}
\]
where we have introduced
\[
Y^{(s)} = \sum_{j=1}^{\infty} \frac{2J + (s-1)^2}{E_j^{(r)}}, \quad s = 0, 1, 2.
\]
The next two terms can be written as
\[
B^{(3)} = c^3 \sum_{J=0}^{\infty} \sum_{J_1=J}^{J+1} J_1 K_\alpha(E_j^{(0)}, E_{j_1}^{(1)})
\]
\[
- c^3 \sum_{J=0}^{\infty} (2J + 1) K_\alpha(E_j^{(0)}, E_{j_1}^{(1)})
\]
and
\[
B^{(4)} = -\frac{c^2}{2} \left( \frac{2c}{3} - 1 \right) \sum_{J=0}^{\infty} \sum_{J_1=J}^{J+1} J_1 K_\alpha(E_j^{(0)}, E_{j_1}^{(1)}).
\]
The function \(K_\alpha(x, y)\) is defined as
\[
K_\alpha(x, y) = \frac{x}{(x - \alpha y)^2} \ln \frac{x}{\alpha y} - \frac{1}{x - \alpha y}.
\]
Notice that \(K_\alpha(x, x) = -\ln(\alpha + c)/(c^2 x)\). The next three terms are given by
\[
B^{(5)} = 2c^2 \sum_{J=1}^{\infty} \left[ \frac{J(6J^2 - 1)}{3(4J^2 - 1)} \frac{1}{E_j^{(1)}} + \frac{J(J + 1)}{3(2J + 1)} \right]
\]
\[
\times L(E_j^{(1)}, E_{j+1}^{(1)})
\]
\[
B^{(6)} = \frac{5c^2}{3} \sum_{J=1}^{\infty} \left[ \frac{J}{4J^2 - 1} \frac{1}{E_j^{(1)}} + 2 \frac{J(J + 1)}{2J + 1} \right]
\]
\[
\times L(E_j^{(1)}, E_{j+1}^{(1)})
\]
\[
B^{(7)} = -\frac{c^2}{3} \sum_{J=0}^{\infty} (J + 1) L(E_j^{(1)}, E_{j+1}^{(1)}).
\]
We have introduced the function
\[
L(x, y) = \frac{\ln x - \ln y}{x - y}
\]
such that \(L(x, x) = 1/x\). Finally, the last to terms are
\[
B^{(8)} = -\frac{c^2}{2} \sum_{J=0}^{\infty} \sum_{J_1=J}^{J+1} J_1 K_\alpha(E_j^{(0)}, E_{j_1}^{(1)})
\]
and
\[
B^{(9)} = \frac{c^3}{3} \sum_{J=0}^{\infty} \frac{(2J + 1)^2 + 2}{2J + 1} F_\alpha(E_j^{(0)}, E_{j+1}^{(1)}).
\]
Here the function \(F_\alpha(x, y, z)\) is defined as follows
\[
F_\alpha(x, y, z) = \frac{1}{y - z} \left[ \frac{y}{x - \alpha y} \ln \frac{x}{\alpha y} - \frac{z}{x - \alpha z} \ln \frac{x}{\alpha z} \right].
\]
such that \(F_\alpha(x, y, y) = K_\alpha(x, y)\).

B. Regularized expressions

To obtain the regularized theory one has to include the alternating metric \(e_f\) and add the masses \(M_f\) to the energies \(E_j^{(a)}\) in the expressions for \(D^{(r)}, H^{(i)}, Y^{(s)}\) and \(B^{(i)}\) respectively. We will proceed by discussing the regularization of \(\Delta S_\sigma\) and \(\Delta S_F\) separately.

1. \(\Delta S_\sigma\)

To start let us define the function
\[
\Phi^{(A)}(p) = \sum_{J=p}^{\Lambda} 2J \ln(J^2 - p^2).
\]
According to Eq. \[134\] the regularized function \(\Phi^{(A)}(p)\) is
\[
\Phi^{(\text{reg})}(p) = \sum_{J=1}^{K} e_f \sum_{J=p}^{\Lambda} 2J \ln(J^2 - p^2 + M_f^2)
\]
\[
+ \sum_{J=p+1}^{\Lambda} 2J \ln(J^2 - p^2).
\]
where we assume that the cut-off \(\Lambda\) is much larger than \(M_f\). In the presence of a large mass \(M_f\) we may consider the logarithm to be a slowly varying function of the discrete variable \(J\). We may therefore approximate the summation by means of the Euler-Maclaurin formula
\[
\sum_{J=p+1}^{\Lambda} g(J) = \int g(x) dx + \frac{1}{2} g(x) \bigg|_{p}^{\Lambda} + \frac{1}{12} g'(x) \bigg|_{p}^{\Lambda}.
\]
After some algebra we find that Eq. (141) can be written as follows:

\[
\Phi_{\text{reg}}^{(\Lambda)}(p) = -2\Lambda(\Lambda + 1) \ln \Lambda + \Lambda^2 - \frac{\ln e\Lambda}{3} + 4 \sum_{J=1}^{\Lambda} J \ln J
\]

\[
+ \frac{1 - 6p}{3} \ln M + 2p^2 - 2 \sum_{J=1}^{2p} (J - p) \ln J.
\]

(143)

The regularized expression for \(D^{(r)}\) can now be obtained as

\[
D_{\text{reg}}^{(r)} = \lim_{\Lambda \to \infty} \left[ \Phi_{\text{reg}}^{(\Lambda)} \left( \frac{1 + r}{2} \right) - \Phi_{\text{reg}}^{(\Lambda)} \left( \frac{1}{2} \right) \right].
\]

(144)

The final results are obtained as follows:

\[
D_{\text{reg}}^{(1)} = -\ln M + \frac{3}{2} \ln 2
\]

(145)

\[
D_{\text{reg}}^{(2)} = -2 \ln M + 4 - 3 \ln 3 - \ln 2.
\]

(146)

The evaluation of \(H_{i,\text{reg}}^{(i)}\) is somewhat more subtle but proceeds along similar lines. The results can be written as follows:

\[
H_{i,\text{reg}}^{(1)} = -\alpha / c^2 \left[ 2 \ln M + 1 - \psi \left( \frac{3c - 1}{c} \right) - \psi \left( \frac{1}{c} \right) \right].
\]

(147)

The Euler digamma function \(\psi(z)\) appears as a result of the following summation:

\[
\sum_{J=0}^{\infty} \left[ (J + 1)^{-1} - (J + z)^{-1} \right] = \psi(z) - \psi(1).
\]

Similarly we find

\[
H_{i,\text{reg}}^{(2)} = -\lim_{\Lambda \to \infty} \Phi_{\text{reg}}^{(\Lambda)} \left( \frac{1}{2} \right) - \alpha / c^2 \left[ 2 \ln M + 1 + 2 \ln^2 M \right]
\]

\[
+ 4\gamma_S + f \left( \frac{\alpha}{c} \right) + f \left( 1 - \frac{\alpha}{c} \right)
\]

(148)

\[
H_{i,\text{reg}}^{(3)} = \frac{1}{c} \lim_{\Lambda \to \infty} \Phi_{\text{reg}}^{(\Lambda)}(1) + \frac{1}{c^3} \left( 2 \ln M + 1 \right) + \frac{1}{c^3} \left[ 2 \ln^2 M \right]
\]

\[
+ 4\gamma_S + f \left( \frac{1}{c} \right) + f \left( 1 - \frac{\alpha}{c} \right) - 2c^2 \frac{\ln 2}{1 - 2\alpha},
\]

(149)

where \(\gamma_S \approx -0.0728\) is the Stieltjes constant and

\[
f(z) = 2z^2 \sum_{J=2}^{\infty} \frac{\ln J}{J(J^2 - z^2)}.
\]

(150)

We finally have the following total result for the quantum correction \(\Delta S_F\):

\[
\exp \Delta S_{F,\text{reg}} = \frac{27}{8} \hat{D}(c) \exp \left\{ 4 \left( 1 + \frac{\alpha \ln \alpha}{c} \right) \ln M + 1 \right\}
\]

(151)

where

\[
\ln \hat{D}(c) = -2\alpha / c \left[ \psi \left( \frac{3c - 1}{c} \right) + \psi \left( \frac{1}{c} \right) - 1 \right] \ln \alpha
\]

\[
- f \left( \frac{1 - c}{c} \right) + f \left( \frac{1}{c} \right) - 2 \ln 2 \frac{c^2}{2c - 1}.
\]

(152)

Notice that according to Eq. (152) the quantity \(\hat{D}(c)\) depends on the crossover parameter \(c\) in a highly non-trivial fashion. Some of contributions diverge at the points \(c_k = 1/k\) with \(k = 1, 2, 3, \ldots\) but the final total answer remains finite for all values of \(c\) in the interval \(0 \leq c \leq 1\). A plot of the function \(\hat{D}(c)\) with varying \(c\) is shown in Fig. 5.

2. \(\Delta S_F\)

Notice that in contrast to the expression for \(\Delta S_{F,\text{reg}}\) where the numerical constants play an important role, the expression for \(\Delta S_{F,\text{reg}}\) can only be determined up to the logarithmic singularity in the Pauli-Villars mass \(M\). In the latter case the constant terms should actually be considered to be of order \(1/\sigma_{x}\) which is beyond the level of approximation as considered in this paper. Keeping this in mind we proceed and define the following function

\[
Y^{(\Lambda)}(p) = \sum_{J=p}^{\Lambda} \frac{2J}{J^2 - p^2}.
\]

(153)

According to Eq. (114) the regularized function \(Y_{\text{reg}}^{(\Lambda)}(p)\) is given by

\[
Y_{\text{reg}}^{(\Lambda)}(p) = \sum_{J=p}^{K} \frac{2J}{J^2 - p^2 + M_j^2} + \sum_{J=p+1}^{\Lambda} \frac{2J}{J^2 - p^2}
\]

(154)

where as before we assume that \(\Lambda \gg M_f\). Proceeding along similar lines as discussed earlier we now find

\[
Y_{\text{reg}}^{(\Lambda)}(p) = 2 \ln M + 2\gamma_E - \sum_{J=1}^{2p} \frac{1}{J}.
\]

(155)
where \( \gamma_E \approx 0.577 \) is the Euler constant. The regularized expressions for \( Y^{(s)} \) can be written as

\[
Y^{(s)}_{\text{reg}} = \lim_{\Lambda \to \infty} Y^{(A)}_{\text{reg}} \left( \frac{1 + s}{2} \right).
\] (156)

We finally obtain

\[
Y^{(s)}_{\text{reg}} = 2 \ln \mathcal{M} + 2\gamma_E - 1 - \sum_{j=2}^{s+1} \frac{1}{j}.
\] (157)

Within the same logarithmic accuracy we can substitute \( K_a(x, y) \) for the functions \( K_a(x, y) \) and \( F_a(x, y, z) \) in Eqs. (134) - (137). With the help of Eq. (157) we then find

\[
\begin{align*}
B^{(1)} &= 0 \ln \mathcal{M} \\
B^{(2)} &= \left( \frac{2c(1 - c)}{3} - 2c^2 \right) \ln \mathcal{M} \\
B^{(3)} &= 0 \ln \mathcal{M} \\
B^{(4)} &= \left( \frac{2c}{3} - 1 \right) (\ln \alpha + c) \ln \mathcal{M} \\
B^{(5)} &= \frac{4c^2}{3} \ln \mathcal{M} \\
B^{(6)} &= \frac{5c^2}{3} \ln \mathcal{M} \\
B^{(7)} &= -\frac{c^2}{3} \ln \mathcal{M} \\
B^{(8)} &= (\ln \alpha + c) \ln \mathcal{M} \\
B^{(9)} &= -\frac{2c}{3} (\ln \alpha + c) \ln \mathcal{M}.
\end{align*}
\] (158 - 166)

The final total result for \( \Delta S^{\text{reg}}_F \) can now be written as follows

\[
\Delta S^{\text{reg}}_F = \frac{32}{3\sigma_{xx}} \pi T z \epsilon (\ln \mathcal{M} e^{\gamma_E} - 1/2 + \text{const}).
\] (167)

3. Regularized \( \frac{Z_{\text{inst}}}{Z_0} \)

We next collect the various different contributions together and obtain the following result for the instanton contribution to the thermodynamic potential

\[
\ln \left[ \frac{Z_{\text{inst}}}{Z_0} \right]_{\text{reg}} = 3 \ln 3 - 7 \ln 2 - \ln \pi + \ln D(c) + i\theta
\]

\[
-2\pi \sigma_{xx} \left[ 1 - \frac{2}{\pi \sigma_{xx}} \left( 1 + \frac{\alpha}{c} \ln \mathcal{M} e^{\gamma_E} \right) \right] + \frac{16\pi^2}{3} T z \epsilon \left[ 1 - \frac{1}{\pi \sigma_{xx}} \ln \mathcal{M} e^{\gamma_E} - 1/2 \right]
\]

\[
-16\pi^2 T z \left[ 1 - \frac{c}{\pi \sigma_{xx}} \ln \mathcal{M} e^{\gamma_E} - 1/2 \right].
\] (168 - 171)

We have introduced new function \( D(c) \) which is defined as

\[
D(c) = 16\pi \tilde{D}(c) \exp \left[ 1 - 4 \left( 1 + \frac{\alpha}{c} \ln \mathcal{M} e^{\gamma_E} \right) \right].
\] (172)

A plot of \( D(c) \) with varying \( c \) is shown in Fig. 6.

C. Observable theory in Pauli-Villars regularization

The most important result next is that the quantum corrections to the parameters \( \sigma_{xx}, z c, \gamma_E \) and \( z' \) in Eqs. (109) - (111) are all identically the same as those obtained from a perturbative expansion of the observable parameters \( \sigma_{xx}, \gamma' \gamma' \), and \( z' \) introduced in Section 11D. In Appendix C we give the details of the computation. Denoting the results for \( \sigma_{xx}, \gamma' \gamma' \) and \( c' \) by \( \sigma_{xx}(\mathcal{M}), \gamma(\mathcal{M}) \) and \( c(\mathcal{M}) \) respectively then we have

\[
\begin{align*}
\sigma_{xx}(\mathcal{M}) &= \sigma_{xx} \left[ 1 - \frac{\beta_0(c)}{\sigma_{xx}} \ln \mathcal{M} e^{\gamma_E} \right] \\
z(\mathcal{M}) &= z \left[ 1 + \frac{\gamma(\mathcal{M})}{\sigma_{xx}} \ln \mathcal{M} e^{\gamma_E} - 1/2 \right] \\
\end{align*}
\] (173 - 174)

The results of Eqs. (168 - 171) can therefore be written as follows

\[
\ln \left[ \frac{Z_{\text{inst}}}{Z_0} \right]_{\text{reg}} = \frac{27\tilde{D}(c)}{128\pi} \exp \left( -2\pi \sigma_{xx}(\mathcal{M}) + i\theta + S^\text{inst}_{\tilde{F}}(\rho) \right).
\] (176)
where
\[ S_{\text{inst}}[\rho] = \pi T \zeta(\mathcal{M}) \int d\mu^2(r) \left( c(\mathcal{M}) \sum_{\alpha n} \text{tr} I^\alpha_n \rho \text{tr} I^{-\alpha}_n \rho ight) + 4 \text{tr} \eta \rho. \] (177)

Notice that the expression in the exponent is similar to the classical action with the rotation matrix \( \mathcal{T}_0 \) put equal to unity. The main difference is in the expressions for \( \sigma_{xx}(\mathcal{M}), z(\mathcal{M})c(\mathcal{M}) \) as well as \( z(\mathcal{M}) \) which are all precisely the radiative corrections as obtained from the observable theory.

At this stage of the analysis several remarks are in order. First of all, it is important to stress that our result for the observable theory, Eq. (177), uniquely fixes the amplitude \( \mathcal{D}(c) \) of the thermodynamic potential which is left unresolved otherwise. This aspect of the problem is going to play a significant role when extracting the non-perturbative renormalization behavior of the theory. In fact, we shall see later on, in Section V D 2, that the most important features of the theory, notably the values of \( \mathcal{D}(c) \) at \( c = 0 \) and \( c = 1 \) respectively, are universal in the sense that they are independent of the specific regularization scheme that one uses to define the renormalized theory. Secondly, our results demonstrate that the idea of spatially varying masses does not alter the ultraviolet singularity structure of the instanton theory. In particular, Eqs. (168) - (177) display exactly the same logarithms as found previously in flat space and by employing dimensional regularization. The detailed computations of Appendix C provide a deeper understanding of this aspect of the problem, especially where it says that the Pauli-Villars regularization scheme retains the translational invariance of the electron gas.

V. TRANSFORMATION FROM CURVED SPACE TO FLAT SPACE

A. Instanton manifold

1. Integration over zero frequency modes

We are now in a position to extend the results for the thermodynamic potential to include the integration over the zero modes. The complete expression for \( Z_{\text{inst}}/Z_0 \) can be written as follows:

\[
\left[ \frac{Z_{\text{inst}}[\mathcal{Q}_{\text{inst}}]}{Z_0[Q_0]} \right]_{\text{reg}} = \frac{2 \mathcal{D}(c)}{128\pi} \exp \left\{ -2\pi \sigma_{xx}(\mathcal{M}) + i \theta + z(\mathcal{M}) \int d\eta d\theta \left( c(\mathcal{M}) \sum_{\alpha n} \text{tr} I^\alpha_n Q_{\text{inst}} \text{tr} I^{-\alpha}_n Q_{\text{inst}} + 4 \text{tr} \eta Q_{\text{inst}} \right) \right\} \exp \left\{ z(\mathcal{M}) \int d\eta d\theta \left( c(\mathcal{M}) \sum_{\alpha n} \text{tr} I^\alpha_n Q_0 \text{tr} I^{-\alpha}_n Q_0 + 4 \text{tr} \eta Q_0 \right) \right\}.
\] (179)

On the other hand, the \( Q_0 \) are the zero modes associated with the trivial vacuum

\[
\int \mathcal{D}[Q_0] = \int \mathcal{D} \left[ \frac{U(2N)}{U(N) \times U(N)} \right].
\] (184)

The numerical factors \( A_{\text{inst}} \) and \( A_0 \) are given by

\[
A_{\text{inst}} = \langle \epsilon^4 \rangle \langle |\epsilon_1|^4 \rangle \langle \epsilon_2^2 |\epsilon_1|^2 \rangle \langle \epsilon_0^2 \rangle \langle |\epsilon_2|^2 \rangle \langle |\epsilon_1|^2 \rangle^{2N-2} 
\times \langle 1 \rangle^{(N-1)(N-1)} \pi^{-N^2 - 2N}
\] (185)

\[
A_0 = \langle 1 \rangle^{N^2} \pi^{-N^2}
\] (186)

where the average \( \langle \ldots \rangle \) is with respect to the surface of a sphere

\[
(f) = \sigma_{xx} \int_1^2 \int_0^{2\pi} d\eta d\theta f(\eta, \theta). \tag{187}
\]
2. \textit{U} rotation

We have already mentioned earlier that the fluctuations in the Goldstone modes $t_0, Q_0 \in U(2N)/U(N) \times U(N)$ have an infinite action in flat space and eventually drop out. We can therefore write the result of Eq. \ref{eq:main_result} as follows

\[
\left[ \frac{Z_{\text{inst}}}{Z_0} \right]^{\text{reg}} = \frac{27}{128\pi} \int dx_0 \int \frac{d\lambda}{\lambda^3} \int \mathcal{D}[U] \frac{A_{\text{inst}}D(c)}{A_0} e^{S_{\text{inst}}}. \tag{188}
\]

Here,

\[
S'_{\text{inst}} = -2\pi \sigma_{xx}(\mathcal{M}) \pm i\theta + \tilde{S}_F[U^{-1}\rho U] \tag{189}
\]

with $\tilde{S}_F$ defined by Eq. \ref{eq:tildeSF}. Next, by making use of the identity\cite{footnote1}

\[
\int \mathcal{D} \left[ \frac{U(k)}{U(1) \times U(k-1)} \right] = \frac{\pi^{k-1}}{\Gamma(k)} \tag{190}
\]

can write the result for the thermodynamic potential in the limit $N_r \to 0$ in a more compact fashion as follows

\[
\left[ \frac{Z_{\text{inst}}}{Z_0} \right]^{\text{reg}} = \frac{N^2}{8\pi^2} \int dx_0 \int \frac{d\lambda}{\lambda^3} \mathcal{D}(c) \langle \epsilon^{S_{\text{inst}}} \rangle_U \tag{191}
\]

where the average is defined according to

\[
\langle X \rangle_U = \frac{\int \mathcal{D}[U]|X| \int \mathcal{D}[U]}{\int \mathcal{D}[U]} \tag{192}
\]

3. \textit{Curved space versus flat space}

Our final result of Eq. \ref{eq:main_result} still involves a spatially varying momentum scale $\mu(r)$ and our task next is to express the final answer in quantities that are defined in flat space, rather than curved space. The first step is to rewrite the integral $\int d\eta d\theta$ in $\tilde{S}_F$ as an integral over flat space following the substitution

\[
\int d\eta d\theta = \int d\mu^2(r) \rightarrow \int d\mu. \tag{193}
\]

The expression for $\tilde{S}_F$ now reads

\[
\tilde{S}_F[U^{-1}\rho U] = \pi T \int dz(\mathcal{M}) \left[ \varepsilon(\mathcal{M}) \sum_{\alpha n} \text{tr} I_n U^{-1} \rho U \times \text{tr} I^\alpha_n U^{-1} \rho U + 4\text{tr} q U^{-1} \rho U \right] \tag{194}
\]

where the “prime” on the integral sign reminds us of the fact that the expression for $\tilde{S}_F$, as it now stands, still diverges logarithmically in the sample size. What remains, however, is to perform the next step which is to express the Pauli-Villars masses $\mathcal{M}$ in terms of the appropriate quantities that are defined in flat space. Notice here to that $\mathcal{M}$ actually describes a \textit{spatially varying momentum scale} $\mu(r)$.$\mathcal{M}$. In Section \ref{section:4.3} below as well as in the remainder of this paper we will embank on the general problem of how to translate a momentum scale in \textit{curved space} into a quantity $\mu_0$ that is defined in \textit{flat space}. As an extremely important consequence of this procedure we shall show in what follows that the final expression for the interaction term $\tilde{S}_F$ is finite in the infrared. This remarkable result is the primary reason as to why one can proceed and obtain the non-perturbative corrections to the renormalization of the quantities $z$ and $c$.

B. Physical observables

1. Linear response

Our results for the thermodynamic potential are easily extended to include the quantities $\sigma'_{xx}$ and $\theta'$ defined by Eqs. \ref{eq:linear_response1} and \ref{eq:linear_response2}. To leading order in $\sigma_{xx}$ we obtain the following result (see also Ref. \cite{footnote2})

\[
\sigma'_{xx} = \sigma_{xx}(\mathcal{M}) + \int \frac{d\lambda}{\lambda} \mathcal{D}(c) \langle \left( J_{xx}[Q_{\text{inst}}] e^{i\theta} + c.c. \right) \rangle_U \tag{195}
\]

\[
\frac{\theta'}{2\pi} = \frac{\theta}{2\pi} + \int \frac{d\lambda}{\lambda} \mathcal{D}(c) \langle \left( J_{xy}[Q_{\text{inst}}] e^{i\theta} + c.c. \right) \rangle_U. \tag{196}
\]

Here, we have introduced the quantity $J_{ab}[Q_{\text{inst}}]$ which is given as

\[
J_{jk}[Q_{\text{inst}}] = N^2 \frac{\sigma_{xx}}{32\pi^2 n\lambda^2} \int dr \text{tr} I_n^a U \rho N_j \rho U^{-1} \times \int dr' \text{tr} I_n^a U \rho N_k \rho U^{-1}. \tag{197}
\]

The interaction term $\tilde{S}_F$ in Eqs. \ref{eq:main_result} and \ref{eq:main_result2} does not contribute in the limit $T \to 0$ and can be dropped. By using the normalization conditions

\[
\sum_{n_1,\alpha} (U^{-1})_{0,n_1}^{\alpha} (U)^{\alpha}_{n_1,0} = 1 \tag{198}
\]

\[
\sum_{n_2,\alpha} (U^{-1})_{-1,n_2}^{\alpha} (U)^{\alpha}_{n_2,-1} = 1 \tag{199}
\]

we find the following results for the expressions bilinear in the $U$

\[
\langle U \rangle_{n_1,0}^{\alpha_1} (U^{-1})_{0,n_3}^{\beta_2} = \frac{1}{N} \delta_{n_1 n_2} \delta_{\alpha_3} \tag{200}
\]

\[
\langle U \rangle_{n_2,-1}^{\alpha_1} (U^{-1})_{-1,n_4}^{\beta_3} = \frac{1}{N} \delta_{n_3 n_4} \delta_{\alpha_3}. \tag{201}
\]
For the quartic combinations we find
\[\left\langle \langle U \right\rangle_{n_1,0}^\dagger U^{-1}\right\rangle_{0, n_3} \right\rangle_{n_5, 0} U^{-1}\right\rangle_{0, n_1} = \frac{\delta n_1 \delta n_2 \delta n_3 \delta n_5}{N(1 + N)} \] (202)
\[\left\langle \langle U \right\rangle_{n_2, -1}^\dagger U^{-1}\right\rangle_{1, n_4} \right\rangle_{n_5, -1} U^{-1}\right\rangle_{1, n_3} = \frac{\delta n_2 \delta n_3 \delta n_4 \delta n_5}{N(1 + N)} \] (203)

We have used the shorthand notation \(\delta n_1 \delta n_3 = \delta n_1 \delta n_3 \delta n_2 \delta n_5\).

In the limit where \(N \to 0\) we obtain
\[\langle J_{jk} | Q_{\text{inst}} \rangle | U \rangle = \frac{\sigma_{xx}^2}{32 \pi^2 \lambda^2} \int d\rho \left( \delta_{jk} - i \varepsilon_{jk} \right) \] (204)

The expressions for \(\sigma'_{xx}\) and \(\theta'\) can now be written as follows
\[\sigma'_{xx} = \sigma_{xx}(M) - \int' \frac{d\lambda}{\lambda} D(c) \sigma_{xx} e^{-2\pi \sigma_{xx}(M)} \cos \theta \] (205)
\[\frac{\theta'}{2\pi} = \frac{\theta}{2\pi} - \int' \frac{d\lambda}{\lambda} D(c) \sigma_{xx} e^{-2\pi \sigma_{xx}(M)} \sin \theta \] (206)

2. Specific heat

The simplest way of obtaining the parameters \(z'\) and \(z'\) is by using the definitions in Section V D and expand the instanton result in powers of \(\tilde{S}_F[U]\). This leads to the expression
\[z' = z(M) - \int' \frac{d\lambda}{\lambda} D(c) e^{-2\pi \sigma_{xx}(M)} \cos \theta \] (207)
\[z' = z(M) c(M) - \int' \frac{d\lambda}{\lambda} D(c) e^{-2\pi \sigma_{xx}(M)} \cos \theta \times \frac{N^2}{8 \pi^3 \lambda^2} \eta \tilde{S}_F[U] \] (208)

The expectations can be evaluated along the same lines as was done in the previous Section. It should be mentioned, however, that there are certain subtleties associated with the limit \(T = 0\) in this case and these will be addressed in detail in Section V B. Here we just state the result
\[\left\langle \langle \tilde{S}_F[U] \rangle \right\rangle_U = \frac{2\pi T}{N^2} \int d\tau z(M) c(M) \frac{(11)}{\rho_{00}(0)}(r) \right\rangle \] (209)

Eqs. 208 and 207 therefore greatly simplify and we obtain
\[z' = z(M) + \frac{\gamma_0}{4\pi} \int' \frac{d\lambda}{\lambda} D(c) e^{-2\pi \sigma_{xx}(M)} \cos \theta \times \int' \frac{d\lambda}{\lambda} z(M)(M) \mu(r) \] (210)
\[z' = z(M) c(M) + \frac{\gamma_0}{4\pi} \int' \frac{d\lambda}{\lambda} D(c) e^{-2\pi \sigma_{xx}(M)} \cos \theta \times \int' \frac{d\lambda}{\lambda} z(M)(M) \mu(r) \] (211)

The most important feature of these results is that the non-perturbative contributions to the observable parameters \(\sigma'_{xx}, \theta'\) and \(z'\) are all unambiguously expressed in terms of the perturbative quantities \(\sigma_{xx}(M), \theta'(r), c(M)\) and \(z(M)\).

C. Transformation \(\mu^2(r) M \to \mu_0\)

As a last step in the development of a quantum theory we wish to express the Pauli-Villars masses which carry the metric of a sphere \(\mu^2(r) M^2\) in terms of a mass or momentum scale in flat space, say \(\mu_0\). By changing the momentum scale from \(\mu(r) M\) to \(\mu_0\) one changes the renormalized theory according to
\[\sigma_{xx}(M) \to \sigma_{xx}(M) \left[ 1 + \frac{\beta_0(c)}{\sigma_{xx}} \ln \frac{\mu(r) M}{\mu_0} \right] \] (212)
\[c(M) \to c(M) \left[ 1 + \alpha \gamma_0 \ln \frac{\mu(r) M}{\mu_0} \right] \] (213)
\[z(M) \to z(M) \left[ 1 + \gamma_0 \ln \frac{\mu(r) M}{\mu_0} \right] \] (214)

The introduction of spatially varying parameters \(\sigma_{xx}(\mu(r)), c(\mu(r))\) and \(z(\mu(r))\) means that the action \(S'_{\text{inst}}\) gets modified according to the prescription
\[S'_{\text{inst}} \to - \int d\tau \sigma_{xx}(\mu(r)) \right\rangle \text{tr}(\nabla Q_{\text{inst}}(r))^2 + i \theta + \tilde{S}_F[W] \] (215)

where
\[\tilde{S}_F[U] = \pi T \int d\tau z(\mu(r)) [c(\mu(r)) \sum \text{tr} I_{\alpha} U^{-1} \rho U \times \text{tr} I_{\alpha} U^{-1} \rho U + 4 \text{tr} \eta U^{-1} \rho U] \] (216)

Notice that in these expressions the instanton quantity \(\rho\) depends explicitly on \(r\) and should be read as \(\rho = \rho(r)\).
D. The quantities $\sigma_{xx}$ and $\sigma'_{xx}$ in flat space

1. Transformation

Let us first evaluate the first spatial integral in Eq. 216, which can be written as

$$
\int d\mathbf{r} \sigma_{xx}(\mu(\mathbf{r})) \text{tr}(\nabla Q_{\text{inst}}(\mathbf{r}))^2 = \int d\mu^2(\mathbf{r})\sigma_{xx}(\mu(\mathbf{r})) = 2\pi \sigma_{xx}(\zeta) \quad (217)
$$

where

$$
\sigma_{xx}(\zeta\lambda) = \sigma_{xx} - \beta_0(\epsilon) \ln \zeta \lambda \mu_0 e^{\gamma E}, \quad \zeta = e^2/4. \quad (218)
$$

We have introduced the quantity $\zeta$ that from now onward denotes the different numerical factors that one in general can associate with each of the different regularization schemes that one uses. Notice that the expression for $\sigma'_{xx}$, Eq. 216, now becomes

$$
\sigma'_{xx} = \sigma_{xx}(\mathcal{M}) - \int d\lambda \frac{D(c) \sigma_{xx}^2 e^{-2\pi \sigma_{xx}(\zeta) \cos \theta}}{\zeta}. \quad (219)
$$

To complete the transformation from curved space to flat space we still have to perform similar operations on the observable theory. Write

$$
\sigma'_{xx}(\mathcal{M}) \to \sigma'_{xx}(\mu(\mathbf{r}))
$$

then completely analogous to the definition of Eq. 216, we obtain the observable parameter $\sigma'_{xx}$ in flat space according to the prescription

$$
\sigma'_{xx}(\zeta\lambda) = \frac{1}{2\pi} \int d\mu(\mathbf{r})^2 \sigma'_{xx}(\mu(\mathbf{r})). \quad (220)
$$

One can think of the quantity $\mu'(\mathbf{r}) = 2\lambda'(r^2 + \lambda'^2)$ as being the result of a background instanton with a large scale size $\lambda'$. The result for $\sigma'_{xx}$ and $\theta'$ in flat space can now be written as follows

$$
\sigma'_{xx}(\zeta\lambda') = \sigma_{xx}(\zeta\lambda') - \int d\lambda \frac{D(c) \sigma_{xx}^2 e^{-2\pi \sigma_{xx}(\zeta\lambda)}}{\zeta \lambda \cos \theta} \quad (221)
$$

$$
\theta'(\zeta\lambda') = \theta - 2\pi \int d\lambda \frac{D(c) \sigma_{xx}^2 e^{-2\pi \sigma_{xx}(\zeta\lambda)}}{\zeta \lambda \sin \theta}. \quad (222)
$$

2. Integration over scale sizes $\lambda$

Notice that the expression for $\sigma_{xx}(\zeta\lambda')$ has precisely the same meaning as Eq. 218 with $\lambda$ replaced by $\lambda'$. In Table IV we compare this expression with the result obtained in dimensional regularization. To discuss the effect of the arbitrary factor $\zeta$ it is convenient to write the quantity $\sigma_{xx}(\zeta\lambda')$ as an integral over scale sizes

$$
\sigma_{xx}(\zeta\lambda') = \sigma_{xx}^0 - \int_{1/\mu_0 e^{\gamma E}}^\zeta \frac{d[\zeta\lambda]}{\zeta \lambda} \beta_0(\epsilon) \quad (223)
$$

where $\sigma_{xx}^0 = \sigma_{xx}(1/\mu_0 e^{\gamma E})$. We thus obtain the following natural expression for the observable theory

$$
\sigma'_{xx}(\zeta\lambda') = \sigma_{xx}^0 - \int_{1/\mu_0 e^{\gamma E}}^{\zeta\lambda'} \frac{d[\zeta\lambda]}{\zeta \lambda} \left[ \beta_0(\epsilon) + D(c) \sigma_{xx}^2 e^{-2\pi \sigma_{xx}(\zeta\lambda) \cos \theta} \right] \quad (224)
$$

$$
\theta'(\zeta\lambda') = \theta - 2\pi \int_{1/\mu_0 e^{\gamma E}}^{\zeta\lambda'} \frac{d[\zeta\lambda]}{\zeta \lambda} \left[ 0 + D(c) \sigma_{xx}^2 e^{-2\pi \sigma_{xx}(\zeta\lambda) \sin \theta} \right]. \quad (225)
$$

Notice that the contributions from instantons are finite in the ultraviolet and the limit $\mu_0 \to \infty$ was taken implicitly in the computation of the original expressions of Eqs. 221 and 222. By comparing Eqs. 221-222 with the result of Eq. 224 obtained from the theory in dimensional regularization one clearly sees that the integral over scale sizes $\lambda$ should be interpreted in terms of the integral over momentum scales that generally defines the relation between the observable and renormalized theories. Hence, we have found the natural meaning for the instanton parameter $\lambda$. This meaning obviously does not emerge from free energy considerations alone. The results of this paper therefore fundamentally resolve the infrared controversies that historically were associated with the problem of instantons and instanton gases in scale invariant theories.55,56,57

Eqs. 221 and 222 show furthermore that the factor $\zeta$ can be absorbed in a redefinition of $\lambda$. Different values of $\zeta$ simply amount to different values of the momentum scale that one associates with the bare parameters $\sigma'_{xx}$ and $\theta$. These differences, however, do not affect the expressions $\ldots$ of the integrand which are therefore in-
dependent of the specific regularization scheme that has been used to define the renormalized theory. This aspect of universality has recently been exploited for the purpose of making detailed comparisons between the quantum critical predictions of the free electron theory \((c = 0)\) and the results known from numerical experiments.

Before embarking on the renormalization of the theory with interactions we shall first address the various difficulties associated with the observable parameters \(z\) and \(z'c\). This will be done in the Sections below and we will come back to the \(\beta\) and \(\gamma\) functions of the theory in Section VI.

### E. The quantities \(z\), \(zc\) and \(z'c'\) in flat space

In this Section we extend the various steps of Eqs. 217 and 218 and translate the parameters \(z(M)\) and \(z'(M)\) as well as \(z(M)c(M)\) and \(z'(M)c'(M)\) into the appropriate quantities that are defined in flat space. As an important check upon the procedure we make sure that the relation \(z'c' = z\alpha\) is satisfied at different stages of the analysis. For the main part, however, the present Section proceeds along the similar lines as those presented in the study of the ordinary Grassmannian theory.

#### 1. Transformation

Let us first introduce the spatially varying momentum scales \(\mu(r)\) and \(\mu'(r)\) according to Eqs. 218 and 218,

\[
z' = z(\mu'(r)) + \frac{\gamma_0}{4\pi} \int \frac{d\lambda}{\lambda} D(c) A_1 e^{-2\pi\sigma_{xx}(\lambda')} \cos \theta \tag{226}
\]

\[
z'c' = z(\mu'(r))c(\mu'(r)) + \frac{\gamma_0}{4\pi} \int \frac{d\lambda}{\lambda} D(c) A_1 e^{-2\pi\sigma_{xx}(\lambda')} \cos \theta. \tag{227}
\]

The amplitude \(A_1\) is given as

\[
A_1 = \int d\mu r \frac{\mu(r)}{\lambda} z(\mu(r)) c(\mu(r)). \tag{228}
\]

By using exactly the same procedure as in Eqs. 218 and 218, we next define the quantities in flat space \(z(\lambda)\) and \(z(\lambda)c(\lambda)\) according to

\[
z(\lambda) = \frac{1}{2\pi} \int dr \mu^2(r) z(\mu(r)) \tag{229}
\]

\[
z(\lambda)c(\lambda) = \frac{1}{2\pi} \int dr \mu^2(r) z(\mu(r)) c(\mu(r)). \tag{230}
\]

From this one obtains the explicit results

\[
z(\lambda) = z \left[ 1 - \frac{\gamma_0}{\sigma_{xx}} \ln \lambda \mu_0 e^{\gamma e^{-1/2}} \right] \tag{231}
\]

\[
z(\lambda)c(\lambda) = z c \left[ 1 - \frac{\gamma_0}{\sigma_{xx}} \ln \lambda \mu_0 e^{\gamma e^{-1/2}} \right]. \tag{232}
\]

In Table IV we show that these results are precisely consistent with those of the theory in dimensional regularization. To proceed let us first apply the transformations to obtain the observable parameters in flat space. Completely analogous to Eq. 218, we have

\[
z'(\lambda') = z(\lambda') + \frac{\gamma_0}{4\pi} \int \frac{d\lambda}{\lambda} D(c) A_1 e^{-2\pi\sigma_{xx}(\lambda')} \cos \theta \tag{233}
\]

\[
z'c'(\lambda') = z(\lambda')c(\lambda') + \frac{\gamma_0}{4\pi} \int \frac{d\lambda}{\lambda} D(c) A_1 e^{-2\pi\sigma_{xx}(\lambda')} \cos \theta. \tag{234}
\]

Here, the \(z(\lambda')\) and \(z(\lambda')c(\lambda')\) are defined by Eqs 231 and 232 with \(\lambda\) replaced by \(\lambda'\).

The problem that clearly remains is how to express the amplitude \(A_1\), Eq. 228, in terms of the spatially flat quantities defined in Eqs 231 and 232.

#### 2. Amplitude \(A_1\)

To evaluate \(A_1\) further it is convenient to introduce the quantity \(M_1(r)\) according to

\[
A_1 = z(\mu(0))c(\mu(0)) M_1 \tag{235}
\]

\[
M_1 = -2\pi \int_{\mu(0)}^{\mu(L')} d[\ln \mu] M_1(r) \tag{236}
\]

\[
M_0(r) = \frac{z(\mu(r))c(\mu(r))}{z(\mu(0))c(\mu(0))}. \tag{237}
\]

Since the anomalous dimension \(\gamma_{zc} = \gamma_z/c\) is negative the quantity \(M_1(r)\) is in all respects like a spatially varying spontaneous magnetization in the classical Heisenberg ferromagnet. The associated momentum scale \(\mu(r)\) strongly varies from large values \(O(\lambda^{-1})\) at short distances \(|r| \ll \lambda\) to small values \(O(\lambda/(L')^2)\) at very large distances \(|r| \approx L' \gg \lambda\). This means that at distances sufficiently far from the center of the instanton the system is effectively in the symmetric or strong coupling phase where \(M_1(r)\) vanishes. Hence we expect the amplitude \(M_1\) to remain finite as \(L' \to \infty\). This is in spite of the fact that the amplitude \(A_1\) diverges at a classical level.

#### 3. Details of computation

The expression for \(M_1\) can be written in terms of the \(\gamma_{zc}\) function as follows

\[
M_1 = -2\pi \int_{\ln \mu(0)}^{\ln \mu(L')} d[\ln \mu] \times \exp \left\{ -\int_{\ln \mu(0)}^{\ln \mu(r)} d[\ln \mu] \gamma_{zc} \right\}. \tag{238}
\]
Taking the derivative with respect to \( \ln \lambda \) we find that \( M_1 \) obeys the following differential equation
\[
\left( -\frac{d}{d\ln \lambda} + \gamma_{zc} \right) M_1 = 2\pi (1 + M_1(L')).
\] (239)

We can safely take the limit \( L' = \infty \) and put \( M_1(L') = 0 \) from now onward. At the same time one can solve Eq. (239) in the weak coupling limit where \( \lambda \to 0 \), \( \mu(0) \to \infty \) and \( \sigma_{xx}(\mu(0)) \to \infty \). Under these circumstances it suffices to insert the perturbative expressions of Eqs. (37), (38) and (45) for the \( \gamma_{zc}, \beta_{e} \) and \( \beta_{c} \) functions such that the quantity \( M_1 = M_1(\sigma_{xx}(\mu(0)), c(\mu(0))) \) is obtained as the solution of the differential equation
\[
\left( \beta_{o} \frac{\partial}{\partial \sigma_{xx}} + \beta_{c} \frac{\partial}{\partial c} + \gamma_{zc} \right) M_1 = 2\pi,
\] (240)
where to leading order \( \beta_{o} = \beta_{o}(c) \), \( \beta_{c} = (1-c)\gamma_{o}/\sigma_{xx} \) and \( \gamma_{zc} = -\gamma_{o}/\sigma_{xx} \). The result for \( M_1 \) can generally be expanded in powers of \( \sigma_{xx}^{-1}(\mu(0)) \)
\[
M_1 = 2\pi^{2} \sigma_{xx} m_{1}^{(1)}(c) + m_{0}^{(1)}(c) + \sigma_{xx}^{-1} m_{-1}^{(1)}(c) + \ldots
\] (241)

We are interested in the leading order quantity \( m_{1}^{(1)}(c) \) which obeys the following differential equation
\[
\left( -\gamma_{c}(1-c) \frac{d}{dc} + (\beta_{o}(c) - \gamma_{o}) \right) m_{1}^{(1)}(c) = \frac{1}{\pi}.
\] (242)

The solution can be written as
\[
m_{1}^{(1)}(c) = \frac{\alpha}{c} \exp \left[ \frac{2}{c} \ln \alpha \right] \int_{0}^{c} ds (1-s)^{-2-2/s}.
\] (243)

The quantity \( m_{1}^{(1)}(c) \) varies between the Fermi liquid value \( m_{1}^{(1)}(0) \) and the Coulomb interaction value \( m_{1}^{(1)}(1) \) which are obtained as
\[
m_{1}^{(1)}(0) = 1, \quad m_{1}^{(1)}(1) = 1/3.
\] (244)

The result for \( A_1 \) becomes
\[
A_1 = -2\pi^{2} z(\mu(0)) c(\mu(0)) \sigma_{xx}(\mu(0)) m_{1}^{(1)}(c(\mu(0))).
\] (245)

As a final step we wish to express \( \sigma_{xx}(\mu(0)), c(\mu(0)) \) and \( z(\mu(0)) \) in terms of the spatially flat quantities \( \sigma_{xx}(\lambda), c(\lambda) \) and \( z(\lambda) \) respectively. The following relations are obtained
\[
\sigma_{xx}(\mu(0)) = \sigma_{xx}(\lambda) \left[ 1 + \frac{\beta_{o}(c)}{\sigma_{xx}(\lambda)} \ln 2\zeta \right] \]
\[
c(\mu(0)) = c(\lambda) \left[ 1 + \frac{\alpha \gamma_{o}}{\sigma_{xx}(\lambda)} \ln 2\zeta \right] \]
\[
z(\mu(0)) = z(\lambda) \left[ 1 + \frac{\gamma_{o}}{\sigma_{xx}(\lambda)} \ln 2\zeta \right].
\] (246) (247) (248)

For our purposes the correction terms \( O(\sigma_{xx}^{-1}) \) are unimportant. Hence we obtain the final result for the amplitude \( A_1 \) which can be written as follows
\[
A_1 = -2\pi^{2} z(\lambda) c(\lambda) \sigma_{xx}(\lambda) m_{1}^{(1)}(c(\lambda)).
\] (249)

The function \( m_{1}^{(1)}(c) \) is given by Eq. (243). The complete expressions for the quantities \( z' \) and \( z'c' \) now become
\[
z'(\lambda') = z(\lambda') - \int_{1/\mu_{c}}^{\lambda} \frac{d[\lambda]}{\lambda} z c \mathcal{D}_{\gamma}(c) \sigma_{xx} e^{-2\pi\sigma_{xx} \cos \theta}
\] (250)
\[
z'(\lambda') c'(\lambda') = z(\lambda') c(\lambda') - \int_{1/\mu_{c}}^{\lambda} \frac{d[\lambda]}{\lambda} z c \mathcal{D}_{\gamma}(c) \sigma_{xx} e^{-2\pi\sigma_{xx} \cos \theta}
\] (251)

where
\[
\mathcal{D}_{\gamma}(c) = -\frac{\gamma_{o} \pi}{2} \mathcal{D}(c) m_{1}^{(1)}(c).
\] (252)

In Fig. 6 we plot the function \( \mathcal{D}_{\gamma}(c) \) with varying \( c \). It has the Fermi liquid value \( \mathcal{D}_{\gamma}(0) = 1/2 \) and the Coulomb interaction value \( \mathcal{D}_{\gamma}(1) = 1/6 \).

4. Integration over scale sizes \( \lambda \)

As before we can write the renormalized parameters \( z(\lambda') \) and \( z(\lambda') c(\lambda') \) as an integral over scale sizes. This leads to the more general expression for the observable theory

\[
z'(\lambda') = z_{0} - \int_{1/\mu_{c}}^{\lambda'} \frac{d[\lambda]}{\lambda} z c \left( \frac{\gamma_{o}}{\sigma_{xx}} + \mathcal{D}_{\gamma}(c) \sigma_{xx} e^{-2\pi\sigma_{xx} \cos \theta} \right)
\] (253)
\[
z'(\lambda') c'(\lambda') = z_{0} c_{0} - \int_{1/\mu_{c}}^{\lambda'} \frac{d[\lambda]}{\lambda} z c \left( \frac{\gamma_{o}}{\sigma_{xx}} + \mathcal{D}_{\gamma}(c) \sigma_{xx} e^{-2\pi\sigma_{xx} \cos \theta} \right)
\] (254)

where the parameters \( z_{0} \) and \( z_{0} c_{0} \) are defined for a fixed microscopic length scale \( 1/\mu_{c} e^{\gamma_{e}} \). Again we compare the results with those obtained from the theory in dimensional regularization, Eqs. (38) and (39). This compar-
ison further demonstrates the validity of the statement made earlier which says that the significance of the instanton parameter $\lambda$ is primarily found in the fundamental relation that exists between the observable and renormalized theories. At the same time we conclude that the numerical factor $\zeta$ has exactly the same meaning as discussed earlier and is immaterial.

F. Thermodynamic potential

As an important general check on the consistency of the procedure we next reconstruct the thermodynamic potential $\Omega$ of the electron gas in the limit where $T$ goes to zero. It turns out that the integration over the zero modes $U \in U(N) \times U(N)$ is not always as trivial as one might expect on the basis of the previous Sections. For example, there is an ambiguity in evaluating the expectation of $S_F[U]$ as given by Eq. \ref{eq:210} and the answer depends on cut-off procedure that one uses in the summation over the $I_n$ matrices in the definition of $S_F$. To obtain an unambiguous result we must take the limit $T = 0$ in a more careful fashion. As we next shall see, this aspect of the problem has direct consequences for the statement of $F$ invariance as well as the statement made in the beginning which says that the quantity $z \alpha$ is unrenormalized.

1. $t_0 = 1$

We start from the expression for the singlet interaction term $\hat{S}_F[U]$ as given by Eq. \ref{eq:210} which still contains the spatially varying momentum scale $\mu(r)$. In order to separate the spatial integrals from the global matrices $U$ we introduce the matrices $\Lambda$ and $I$

\[
\Lambda_{nm}^{\alpha\beta} = \delta^{\alpha1} \delta^{\beta1} \delta_{nm} [\delta_{n0} - \delta_{n,-1}] \quad \text{(255)}
\]

\[
\hat{I}_{nm}^{\alpha\beta} = \delta^{\alpha1} \delta^{\beta1} \delta_{nm} [\delta_{n0} + \delta_{n,-1}] \quad \text{(256)}
\]

Eq. \ref{eq:210} can then be written as follows

\[
\hat{S}_F(U) = \hat{S}_i(U) + \hat{S}_q(U) \quad \text{(257)}
\]

where

\[
\hat{S}_i = -\frac{\pi}{2} T \lambda^2 \left( A_1 - \frac{5}{2} A_2 \right) \sum_{\alpha n} \text{tr} \left( \Lambda_n^U U^{-1} \Lambda_n U U^{-1} \Lambda_n^U \right)
\]

\[
-\frac{\pi}{2} T \lambda^2 \left( A_1 - \frac{1}{2} A_2 \right) \sum_{\alpha n} \text{tr} \left( \Lambda_n^U U^{-1} \hat{I}_n U U^{-1} \hat{I}_n \right)
\]

\[
\hat{S}_q = -4 \pi T \lambda^2 A_3 \text{tr} \eta U^{-1} \hat{I}_n U^{-1} \hat{I}_n \quad \text{(258)}
\]

Here, the spatial integrals are all contained in the quantities $A_i$. $A_1$ is defined in Eq. \ref{eq:210} whereas $A_i$ are given as

\[
A_2 = \int d \mu^2(r) z(\mu(r)) \frac{c(\mu(r))}{\lambda} \quad \text{(259)}
\]

\[
A_3 = \int d \mu(r) \frac{\mu(r)}{\lambda} z(\mu(r)) \quad \text{(260)}
\]

Notice that $\hat{S}_F[U]$ has the same form as the classical expression $S_F^{\text{inst}}[U]$ except that the amplitudes $A_i$ are replaced by $A_i^{\text{inst}}$ according to

\[
A_1^{\text{inst}} = z c \int dr \frac{\mu(r)}{\lambda} \quad \text{(261)}
\]

\[
A_2^{\text{inst}} = z c \int dr \mu^2(r) \quad \text{(262)}
\]

\[
A_3^{\text{inst}} = z \int dr \frac{\mu(r)}{\lambda} \quad \text{(263)}
\]

We have already mentioned earlier that the classical expression $S_F^{\text{inst}}[U]$, in particular the amplitudes $A_1^{\text{inst}}$ and $A_2^{\text{inst}}$, diverge logarithmically in the sample size. By following the same procedure as discussed in Section \ref{sec:21e} however, we find that the final expressions for the amplitudes $A_2$ and $A_3$ are finite

\[
A_2 = -2 \pi z(\lambda) \quad \text{(264)}
\]

\[
A_3 = -2 \pi^2 z(\lambda) \sigma_{xx}(\lambda) m_1^{(3)}(c(\lambda)) \quad \text{(265)}
\]

Here, the quantity $m_1^{(3)}$ is given by

\[
m_1^{(3)}(c) = \alpha \exp \left[ \frac{2}{c} \ln \alpha \right] \int_0^c \frac{ds}{s(1-s)^2} \times \exp \left[ -\frac{2}{s} \ln(1-s) \right] \quad \text{(266)}
\]

Notice that we can neglect the amplitude $A_2$ relative to the quantities $A_1$ and $A_3$ which are of order $\sigma_{xx}(\lambda)$. On the other hand, in Fig. \ref{fig:7} we plot of the function $m_1^{(3)}(c)$ that defines the quantity $A_3$. We see that $m_1^{(3)}(c)$ diverges as $c$ tends to 0. This means that for $c = 0$ the leading term in $A_3$ is proportional to $\sigma_{xx}^2$ rather than $\sigma_{xx}$. Keeping these remarks in mind we finally obtain the instanton contribution to the thermodynamic potential

\[FIG. 7: m^{(3)}(c) \text{ versus } c, \text{ see text.} \]
as follows

$$\Omega_{\text{inst}} = \frac{N^2}{4\pi^2} \int_0^1 \frac{d\lambda}{\lambda^3} D(c(\zeta)) e^{-2\pi \sigma_{xx}(\zeta)} \cos \theta$$

\[
\times \langle e^{S_\eta(U)} (1 + \hat{S}_\eta(U) + \ldots) \rangle_U. \tag{267}
\]

2. Expansion in $T$

Next, in a naive expansion of the thermodynamic potential $\Omega$ in powers of the temperature $T$ one would proceed by replacing the quantity $\hat{S}_\eta(U)$ by its expectation with respect to the matrix $U$. In the limit where $N = N_r N_m \to 0$ this expectation is given by

$$\langle \hat{S}_\eta(U) \rangle_U = \frac{2\pi \lambda^2 T A_1}{N^2} \text{tr} \eta \Lambda$$

$$\langle \hat{S}_\eta(U) \rangle_U = -\frac{4\pi \lambda^2 T A_2}{N} \text{tr} \eta \Lambda. \tag{268}$$

To the lowest order in $T$ only the quantity $\langle \hat{S}_\eta(U) \rangle_U$ survives in Eq. (268), whereas the term $\langle \hat{S}_\eta(U) \rangle_U$ vanishes in the limit where $N \to 0$. We have already mentioned, however, that the expectation of $S_F$, in particular Eq. (268), is complicated and cut-off dependent. These as well as other complications disappear once it is recognized that the frequency term $S_F(U)$ in the action is actually not a perturbative quantity at all and should generally be retained in the exponential of $\Omega_{\text{inst}}$. The correct series expansion in powers of $T$ therefore has the following general form

$$\Omega_{\text{inst}} = \frac{N^2}{4\pi^2} \int_0^1 \frac{d\lambda}{\lambda^3} D(c(\zeta)) e^{-2\pi \sigma_{xx}(\zeta)} \cos \theta$$

\[
\times \langle e^{S_\eta(U)} (1 + \hat{S}_\eta(U) + \ldots) \rangle_U. \tag{269}
\]

The problem that remains is to evaluate expectations of the type

$$\langle X \rangle_U = \langle X e^{-\epsilon \text{tr} \eta U^{-1} \hat{\Lambda} U} \rangle_U$$

where we have written $\epsilon = -4\pi \lambda^2 T A_3$. For our purposes the only expectations that we shall need are the following results which are valid in the limit $N \to 0$

$$\langle (U^{-1} \hat{\Lambda} U)_{nm} \rangle_U = \frac{1}{N} \delta_{nm} e^{-\epsilon |n|} \tag{272}$$

$$\langle (U^{-1} \hat{1} U)_{nm} \rangle_U = \frac{1}{N} \delta_{nm} e^{-\epsilon |n|}. \tag{273}$$

We see that the main effect of $\epsilon$ is to exponentially suppress the large Matsubara frequency components. To justify Eqs. (272) and (273) we proceed as follows. Since the averaging over positive and negative frequency blocks is independent of one another we first introduce for brevity the symbol $P^{\alpha}_{\beta_n}(U)_{nm} = \langle (U^{-1} \hat{1} U)_{nm} \rangle_{U_{\beta_n}}$ where $n$ is limited to, say, positive frequency indices only. Equations (272) and (273) can then be expressed in terms of an infinite series expansion in powers of $\epsilon$ with coefficients of the type

$$\langle P^{\beta_1}_{m_1} \cdots P^{\beta_k}_{m_k} \rangle_U. \tag{274}$$

The lowest order coefficients we already have, in particular

$$\langle P^{\alpha}_{n_1} \rangle_U = \frac{1}{N}, \quad \langle P^{\alpha}_{n_1} P^{\beta}_{n_3} \rangle_U = \frac{1 + \delta^{\alpha \beta}_{n_1 n_3}}{1 + N}. \tag{275}$$

The second of these equations simplifies in the limit $N \to 0$ and can be replaced by the following expression

$$\langle P^{\alpha}_{n_1} P^{\beta}_{n_3} \rangle_U = \frac{1}{N} \delta^{\alpha \beta}_{n_1 n_3}. \tag{276}$$

It is clear that the terms that have been left out are all of higher order in $N_r$ and therefore insignificant. Proceeding along the same lines one can prove by induction that the general expression can be written as

$$\langle P^{\beta_1}_{m_1} \cdots P^{\beta_k}_{m_k} \rangle_U = \frac{1}{N} \delta^{\beta_1 \cdots \beta_k}_{m_1 \cdots m_k}. \tag{277}$$

Using this result one can re-exponentiate the series in powers of $\epsilon$ and the result can be written as follows

$$\langle P^{\alpha}_{n} \exp\left(-\epsilon \sum_{\beta, m > 0} m P^{\beta}_{m} \right) \rangle_U = \frac{1}{N} \exp(-\epsilon |n|). \tag{278}$$

| TABLE IV: Observable theory using different regularization schemes, see text |
|---------------------------------|---------------------------------|---------------------------------|
| Pauli-Villars regularization (curved space) | Pauli-Villars regularization (flat space) | Dimensional regularization |
| $\sigma'_{xx}(M) = \sigma_{xx} - \beta_0(c) \ln M e^{\gamma}$ | $\sigma'_{xx}(\zeta) = \sigma_{xx} - \beta_0(c) \ln \zeta \mu_0 e^{\gamma}$ | $\sigma'_{xx}(\mu) = \sigma_{xx} - \beta_0(c) \ln \frac{\mu}{\mu}$ |
| $z' = z \left(1 - \frac{\gamma}{\sigma_{xx}} \ln M e^{\gamma} - \frac{1}{2} \right)$ | $z' = z \left(1 - \frac{\gamma}{\sigma_{xx}} \ln \zeta \mu_0 e^{\gamma} - \frac{1}{2} \right)$ | $z' = z \left(1 - \frac{\gamma}{\sigma_{xx}} \ln \frac{\mu}{e} - \frac{1}{2} \right)$ |

Note: The table entries are not fully transcribed here for brevity.
This, then, directly leads to the result of Eqs. (272) and (273).

3. Effective action for \( t_0 \)

On the basis of Eqs. (272) and (273) one can write the expectation \( \langle \hat{S}_f(U) \rangle_\epsilon \) as follows

\[
\langle \hat{S}_f(U) \rangle_\epsilon = -\frac{\pi^2 T A_1}{2 N^2} \sum_{\alpha n} |n| e^{-\epsilon |n|} \quad \epsilon \Rightarrow \epsilon_\Lambda e^{-\epsilon \eta \Lambda}.
\]

This expression is important for a variety of reasons. First of all, a finite value of \( \epsilon \) permits us to take the frequency cut-off \( N_\epsilon \) appearing in the size of the matrices \( [I_n, U^{-1}\hat{\Lambda} U] \) and \( [I_n, U^{-1}U] \) to infinity first. An explicit computation leads to

\[
\langle \hat{S}_f(U) \rangle_\epsilon = \frac{2 \pi^2 T A_1}{N^2} \sum_{\alpha n} |n| e^{-\epsilon |n|} = \frac{2 \pi^2 T A_1}{N^2} \text{tr} \left( \eta \Lambda e^{-\epsilon \eta \Lambda} \right).
\]

This indicates that the large frequency components \( n \gtrsim \epsilon^{-1} \) are being suppressed by the theory itself and the results are clearly independent of the arbitrary cut-off \( N_\epsilon \) as they should be.

Secondly, we can now proceed and extend the result of Eq. (278) and, hence, the thermodynamic potential \( \Omega_{\text{inst}} \) to include the zero modes \( t_0 \) or \( q_0 \in U(2N)/U(N) \times U(N) \). Although we have seen that these zero modes do not appear in the final answer, they can nevertheless be used as an important check on the general statement which says that the quantity \( \alpha \eta \) is unrenormalized. Notice that as a general prescription for inserting the rotation \( t_0 \) we can use the procedure of Appendix A which shows how to deal with the electrodynamic \( U(1) \) gauge invariance of the theory. Replacing in Eq. (279)

\[
U^{-1}\hat{\Lambda} U \rightarrow t_0^{-1}U^{-1}\hat{\Lambda} U t_0
\]

then one should think of the matrix \( t_0 \) as being a “small” \( U(2n)/U(n) \times U(n) \) rotation with \( n = N_\epsilon N_\eta \) much smaller than \( N = N_\epsilon N_\eta \). Indeed, according to the rules of \( \mathcal{F} \) algebra one considers the different cut-offs \( n_\epsilon \ll \epsilon^{-1} \ll n_\eta \) as a general prescription that should be followed before taking the limit to infinite frequency space, i.e. \( n_\epsilon \ll \epsilon^{-1} \ll n_\eta \rightarrow \infty \). Evaluating the theory of Eq. (279) in the presence of the matrix field \( t_0 \) we can write

\[
\langle \hat{S}_f(U(t_0)) \rangle_\epsilon = -\frac{\pi^2 T A_1}{2 N^2} \Gamma[t_0]
\]

To appreciate the subtleties that are associated with the “finiteness” of \( \epsilon \) as well as the “smallness” of the background field \( t_0 \) we next analyze the result of Eq. (283) in some detail. First, \( n_m \ll \epsilon^{-1} \) means that we can write

\[
[e^{-\epsilon \eta \Lambda}, t_0^{-1}] \approx 0, \quad [e^{-\epsilon \eta \Lambda}, t_0] \approx 0.
\]

Hence,

\[
\Gamma[q_0] = \sum_{\alpha n} \text{tr}[I^n, q_0][e^{-\epsilon \eta \Lambda}I^n - e^{-\epsilon \eta \Lambda}, q_0]
\]

where \( q_0 = t_0^{-1}\Lambda t_0 \). Eq. (283) shows that the results correctly display \( U(N) \times U(N) \) invariance as it should be. Next we split the matrix \( q_0 \) into “small” components \( q_0 - \Lambda \) and “large” components \( \Lambda \) and write

\[
\Gamma[q_0] = \sum_{\alpha n} \text{tr}[I^n, \Lambda][e^{-\epsilon \eta \Lambda}I^n, \Lambda]e^{-\epsilon \eta \Lambda}
\]

\[
+ 2 \sum_{\alpha n} \text{tr}[I^n, (q_0 - \Lambda)][e^{-\epsilon \eta \Lambda}I^n, \Lambda]e^{-\epsilon \eta \Lambda}
\]

\[
+ \sum_{\alpha n} \text{tr}[I^n, (q_0 - \Lambda)][I^n, (q_0 - \Lambda)].
\]

By using the following identity

\[
e^{-\epsilon \eta \Lambda}[I^n - \Lambda]e^{-\epsilon \eta \Lambda} = e^{-\epsilon |n|}[I^n - \Lambda]
\]

we finally obtain two equivalent expressions for the quantity \( \Gamma \)

\[
\Gamma[q_0] = \sum_{\alpha n} e^{-\epsilon |n|} \text{tr}[I^n, q_0][I^n, q_0]
\]

\[
\Gamma[q_0] = 2 \left( \sum_{\alpha n} \text{tr}[I^n, q_0][I^n, q_0] + 4 \text{tr} \eta (q_0 - \Lambda) \right)
\]

Notice that for all practical purposes we can represent the results in terms of a reduced matrix space of size \( N_\epsilon \times N_\eta \) as follows

\[
\Gamma[q_0] = \sum_{\alpha n} ' \text{tr}[I^n, q_0][I^n, q_0]
\]

\[
\Gamma[q_0] = 2 \left( \sum_{\alpha n} \text{tr}[I^n, q_0][I^n, q_0] + 4 \text{tr} \eta q_0 - 6 \text{tr} \eta \Lambda \right).
\]
Here, the size $N_m$ of the matrices $f_n$ and $\lambda$ is such that $n_m \ll N_m \ll e^{-1}$. The prime on the summation sign in Eq. (221) denotes the restriction $-N_m < n < N_m$. In summary we can say that the background field quantity $\Gamma'_0[\eta_0]$ has the familiar $F$ invariant form.$^{24}$

The result for the thermodynamic potential, Eq. (220), in the presence of a global background field $t$ becomes

$$\Omega_{\text{inst}} = \frac{N^2}{4\pi^2} \int dr_0 \int d\lambda \frac{D(c(\lambda))}{\lambda^2} e^{-2\pi \sigma_{xx}(\zeta)} \cos \theta \times (1 + \frac{2\pi \lambda^2 T A_1}{N^2} \Gamma[\eta_0] + O(T^2)).$$  (292)

To obtain the final total expression for the effective action $S_{\text{eff}}[\eta_0]$ we have to add the results obtained for the trivial vacuum. Splitting the thermodynamic potential in $T = 0$ and $T \neq 0$ parts

$$\Omega = \Omega(T = 0) + \Omega(T)$$  (293)

then in the limit where $N \to 0$ we obtain

$$\Omega(T) = \ln \int D[\eta_0] e^{S_{\text{eff}}[\eta_0]}$$

$$S_{\text{eff}}[\eta_0] = (L^2 T) \zeta' \sum_{\alpha n} \lambda_\alpha \lambda_\alpha \ln \left[ T_\alpha^\beta, \eta_0 \right] \left[ T_\alpha^\beta, \eta_0 \right] + (L^2 T) z \alpha \left( \frac{1}{4} \text{tr} \eta_0 - \frac{1}{6} \text{tr} \eta \right).$$  (294)

These expressions are all well defined with $L$ denoting the linear dimensions of the system. The most important result is that quantity $z' c'$ is given precisely by Eq. (251) whereas $z \alpha$ is unrenormalized. Notice that in the limit where $L \to \infty$ only the classical value $\eta_0 = \Lambda$ contributes as mentioned before. $\Omega(T)$ therefore reduces to $S_{\text{eff}}[\Lambda] = -2(L^2 T) \zeta' \theta$ to $\eta$ which by itself does not determine the renormalization of $z$ and/or $c$. On the basis of Eq. (293) we conclude, however, that the observable parameters $z'$ and $z' \alpha'$ are correctly given by the definitions of Eqs. (282) and (283). At the same time we have explicitly verified the $T$ dependent part of the effective action as presented in Appendix A, Eq. (A.22).

VI. THE $\beta'$ AND $\gamma'$ FUNCTIONS

We next summarize the results obtained for the observable parameters and derive expressions for the renormalization group $\beta'$ and $\gamma'$ functions of the interacting electron gas. The final expressions that we obtain in this Section are amongst the most important results of the present paper.

A. Observable and renormalized theories

Introducing an arbitrary scale size $\lambda_0$ we can rewrite Eqs. (221), (222), (223) and (224) in the following manner

$$\sigma'_{xx}(\zeta') = \sigma'_{xx}(\zeta_0)$$

$$- \int_{\zeta_0}^{\zeta'} \frac{d[\zeta]}{\zeta'} \beta'_a(\sigma_{xx}, \theta, c)$$  (295)

$$\theta'(\zeta') = \theta'(\zeta_0)$$

$$- \int_{\zeta_0}^{\zeta'} \frac{d[\zeta]}{\zeta'} \beta'_\theta(\sigma_{xx}, \theta, c)$$  (296)

$$z'(\zeta') = z'(\zeta_0)$$

$$+ \int_{\zeta_0}^{\zeta'} \frac{d[\zeta]}{\zeta'} \zeta' \gamma'_{\zeta}(\sigma_{xx}, \theta, c, c')$$  (297)

$$z'(\zeta') c'(\zeta') = z'(\zeta_0) c'(\zeta_0)$$

$$+ \int_{\zeta_0}^{\zeta'} \frac{d[\zeta]}{\zeta'} \zeta' c' \gamma'_{\zeta}(\sigma_{xx}, \theta, c, c')$$  (298)

where

$$\beta'_a(\sigma_{xx}, \theta, c) = \frac{-d \sigma'_{xx}}{d \ln \lambda}$$

$$\beta'_\theta(\sigma_{xx}, \theta, c) = \frac{-d \theta'}{d \ln \lambda}$$

$$\gamma'_{\zeta}(\sigma_{xx}, \theta, c, c') = \frac{d \ln \zeta'}{d \ln \lambda} \left( \frac{1 - c'}{1 - c} \right) \left[ \frac{\eta_0}{\sigma_{xx}} + \frac{D(\gamma)(\sigma_{xx}, \theta, c, c')}{D(\gamma)(\sigma_{xx}, \theta, c, c')} \right]$$  (299)

$$\gamma'_{\zeta}(\sigma_{xx}, \theta, c, c') = \frac{d \ln \zeta'}{d \ln \lambda} \left( \frac{1 - c'}{1 - c} \right) \left[ \frac{\eta_0}{\sigma_{xx}} + \frac{D(\gamma)(\sigma_{xx}, \theta, c, c')}{D(\gamma)(\sigma_{xx}, \theta, c, c')} \right]$$  (300)

The difference between the observable theory $\sigma'_{xx}$, $\theta'$, $c'$ and $z'$ and the renormalized theory $\sigma_{xx}$, $\theta$ and $c$ can be expressed in terms of the renormalization group functions as follows

$$\beta_a(\sigma_{xx}, c) \Rightarrow \beta'_a(\sigma_{xx}, \theta, c)$$  (301)

$$\beta_\theta(\sigma_{xx}, \theta, c) \Rightarrow \beta'_\theta(\sigma_{xx}, \theta, c)$$  (302)

$$\gamma_{\zeta}(\sigma_{xx}, \theta, c, c') \Rightarrow \gamma'_{\zeta}(\sigma_{xx}, \theta, c, c')$$  (303)

$$\gamma_{\zeta}(\sigma_{xx}, \theta, c, c') \Rightarrow \gamma'_{\zeta}(\sigma_{xx}, \theta, c, c')$$  (304)

Our final task is to express the $\beta'$ and $\gamma'$ functions of the observable theory in terms of the observable parameters $\sigma'_{xx}$, $\theta'$ and $c'$ alone, rather than the renormalized quantities $\sigma_{xx}$, $\theta$ and $c$. To ensure that this can be done in a legitimate fashion we proceed as follows. First of all it is important to notice that the following general relations hold

$$\gamma_{\zeta}(\sigma_{xx}, c) = c \gamma_{\zeta}(\sigma_{xx}, c)$$  (305)

$$\gamma'_{\zeta}(\sigma_{xx}, \theta, c, c') = c \gamma'_{\zeta}(\sigma_{xx}, \theta, c, c')$$  (306)
This means that both quantities $z\alpha$ and $z'\alpha'$ are unrenormalized as they should be. Next, we compare the renormalization behavior of the quantities $c$ and $c'$

$$\beta_c(\sigma_{xx}, c) = -\frac{d c}{d \ln \lambda} = (1 - c)c\gamma_{2c}(\sigma_{xx}, c)$$

$$\beta'_c(\sigma_{xx}, \theta, c', c') = -\frac{d c'}{d \ln \lambda} = (1 - c')c'\gamma'_{2c}(\sigma_{xx}, \theta, c', c').$$

We see that the Fermi liquid plane $c = c' = 0$ and the Coulomb interaction plane $c = c' = 1$ correspond to zero's of both the $\beta_c$ and $\beta'_c$ functions provided the $\gamma'_{2c}$ is well behaved.

**B. The $\beta'$ and $\gamma'$ functions**

The relation between the observable and renormalized theories can be obtained by solving the following differential equations

$$\beta_\sigma(\sigma_{xx}, c) \frac{\partial \sigma'}{\partial \sigma_{xx}} + \beta_c(\sigma_{xx}, c) \frac{\partial c}{\partial c} = \beta'_\sigma(\sigma_{xx}, \theta, c)$$

$$\beta_\theta(\sigma_{xx}, c) \frac{\partial \theta'}{\partial \sigma_{xx}} + \beta_c(\sigma_{xx}, c) \frac{\partial c}{\partial c} = \beta'_\theta(\sigma_{xx}, \theta, c)$$

$$\beta_\gamma(\sigma_{xx}, c) \frac{\partial \gamma'}{\partial \sigma_{xx}} + \beta_c(\sigma_{xx}, c) \frac{\partial c}{\partial c} = \beta'_\gamma(\sigma_{xx}, \theta, c, c').$$

To obtain solutions that are meaningful in the entire range $0 \leq c \leq 1$ we must work with the two loop results for the $\beta_\sigma$ function as in Eq. (311). It is next a matter of simple algebra to show that the results can be expressed in terms of an infinite double series in powers of $\exp(-2\pi\sigma'_{xx})$ and the trigonometric functions of $\theta'$. The first few terms in the series can be written as follows

$$\beta'_\sigma(\sigma'_{xx}, \theta', c', c') = \left\{ \beta_\sigma(\sigma'_{xx}, c') + F'_0 e^{-4\pi\sigma'_{xx}} \right\} + \left\{ D(c') (\sigma'_{xx})^2 e^{-2\pi\sigma'_{xx}} \cos \theta' + \left[ H'_2 e^{-4\pi\sigma'_{xx}} \right] \cos 2\theta' + \ldots \right\} + \left\{ D(c') (\sigma'_{xx})^2 e^{-2\pi\sigma'_{xx}} \sin \theta' + \left[ H'_2 e^{-4\pi\sigma'_{xx}} \right] \sin 2\theta' + \ldots \right\}$$

$$\beta'_\theta(\sigma'_{xx}, \theta', c', c') = \left\{ \beta_\theta(\sigma'_{xx}, c) + F'_0 e^{-4\pi\sigma'_{xx}} \right\} + \left\{ D(c') (\sigma'_{xx})^2 e^{-2\pi\sigma'_{xx}} \cos \theta' + \left[ H'_2 e^{-4\pi\sigma'_{xx}} \right] \cos 2\theta' + \ldots \right\}$$

$$\beta'_\gamma(\sigma'_{xx}, \theta', c', c') = \left\{ \gamma_{2c}(\sigma'_{xx}, c') + H'_0 e^{-4\pi\sigma'_{xx}} \right\} + \left\{ D_\gamma(c') (\sigma'_{xx})^2 e^{-2\pi\sigma'_{xx}} \cos \theta' + \left[ H'_2 e^{-4\pi\sigma'_{xx}} \right] \cos 2\theta' + \ldots \right\}$$

Eqs. (311) - (313) generally become important when multi instanton configurations are taken into account. In particular, the terms with $H'_0$ and $F'_0$ indicate that the trivial vacuum is affected by instanton and anti instanton combinations. Similarly, the terms proportional to $F'_2$ and $H'_2$ are recognized as the disconnected pieces that appear in the contributions from instantons of topological charge $\pm 2$. It is not difficult to see that a consistent procedure for multi instantons is likely to involve the effects of merons.

To summarize the main results of this paper we can say that the theory of observable parameters can be expressed as follows

$$\sigma'_{xx}(\zeta' \lambda') = \sigma_{xx}(\zeta_0 \lambda_0) - \int_{\zeta_0 \lambda_0}^{\zeta' \lambda'} \frac{d[\zeta \lambda]}{\zeta \lambda} \beta'_\sigma(\sigma_{xx}, \theta', c')$$

$$\frac{\theta'(\zeta \lambda')}{2\pi} = \frac{\theta'(\zeta_0 \lambda_0)}{2\pi} - \int_{\zeta_0 \lambda_0}^{\zeta' \lambda'} \frac{d[\zeta \lambda]}{\zeta \lambda} \beta'_\theta(\sigma_{xx}, \theta', c')$$

$$z'(\zeta \lambda') = z'(\zeta_0 \lambda_0) - \int_{\zeta_0 \lambda_0}^{\zeta' \lambda'} \frac{d[\zeta \lambda]}{\zeta \lambda} z' c' \gamma'_{2c}(\sigma'_{xx}, \theta', c')$$
Here, $\beta'$, $\beta''_0$ and $\gamma'$, are given to the appropriate order by Eqs. (314)-(316). These final results generalize the perturbative expressions of Eqs. (314)-(316).

VII. DISCUSSION

In this paper we have extended the perturbative theory of localization and interaction effects to include the highly non-trivial effects of the $\theta$ term. The analysis that we have presented is an important technical as well as conceptual advance since it directly relates to some of the most fundamental and the long standing problems of the interacting electron gas on the strong coupling side.

We have seen, first of all, that the appearance of massless chiral edge excitations has important consequences for the low energy dynamics of the instanton vacuum and can be used, amongst many other things, to formulate a Thouless-like criterion for the quantum Hall effect. Our introduction of an effective action for the edge excitations resolves the previously encountered ambiguities in the Kubo formulae and renormalization group, in particular the general problem of boundary conditions as well as the quantization of topological charge. The effective action procedure for edge excitations uniquely defines the response parameters or physical observables $\sigma'_{xx}$ and $\theta'$. Moreover, by recognizing the differences between the edge excitations and bulk excitations we have fundamentally explained the various different aspects of symmetry in the problem, notably particle-hole symmetry and periodicity in $\sigma'_{xy}$. Furthermore, the conditions for the quantum Hall effect can now quite generally be expressed by saying that $\sigma'_{xx} = \theta' = 0$. This means that the bulk of the system renders insensitive to changes in the boundary conditions. This generally happens when the bulk excitations of the system generate a mass gap. These general statements have motivated us to develop a unified microscopic theory for the physical observables $\sigma'_{xx}$ and $\theta'$ of the electron gas in the presence of electron-electron interactions. The complete list of observable parameters includes also the parameter $c'$ which distinguishes between finite range electron-electron interactions $(0 < c' < 1)$ and infinite range interactions $(c' = 1)$, as well as the parameter $z'$ which controls the temperature and frequency dependence of the electron gas. The most important results of this work are given by Eqs. (321)-(324) expressing how the observable parameters are related to the renormalization group $\beta$ and $\gamma$ functions of the theory. The closed set of renormalization group functions $\beta''_0$, $\beta''_0$ and $\beta''_0$ that we have obtained (Eqs. (314)-(316)) controls the low energy dynamics of the electron gas at $T = 0$ and zero external frequency. The principal features of this theory are encapsulated in the three dimensional renormalization group flow diagram as sketched in Fig. 8. The regime of finite range electron-electron interactions $0 < c < 1$, like the theory in $2 + \epsilon$ dimensions, lies the domain of attraction of the Fermi liquid plane $c = 0$ which is stable in the infrared. These results are in accordance with the principle of $F$ invariance which states the distinctly different problems of the Coulomb interaction $c = 1$ and finite range electron-electrons interactions $0 \leq c < 1$ are preserved separately under the action of the renormalization group.

A. Robust quantization of Hall conductance

We are now in a position to elaborate on the quantum Hall effect which is represented in Fig. 8 by the infrared fixed points located at precise values of $\sigma'_{xx} = k(\nu)$ or $\theta' = 0$ and $\sigma'_{xx} = 0$. For this purpose let us consider the renormalization group equations along the lines $\sigma'_{xy} = k(\nu)$ or $\theta' \approx 0$. Specializing to the most interesting case $c = 1$ then we can write

$$\frac{d\ln \sigma'_{xx}}{d\ln \lambda} = \beta_\sigma(\sigma'_{xx}) = \frac{2}{\pi \sigma'_{xx}} + \frac{\beta_1(1)}{\sigma'_{xx}} - D(1)\sigma'_{xx} e^{-2\pi \sigma''_{xx}}$$

$$\frac{d\ln |\theta'|}{d\ln \lambda} = \beta_\theta(\sigma'_{xx}) = - \frac{2}{\pi D(1)}(\sigma'_{xx})^2 e^{-2\pi \sigma''_{xx}}.$$  

These results are clearly consistent with the Thouless-like criterion presented in Section II C which tells us that along the lines $\theta' \approx 0$ both quantities $\sigma'_{xx}$ and $\theta'$ should become exponentially small for large scale sizes $\lambda$. Recall
from the discussion in Section II D 4 that the perturbative $\beta_\sigma$ function usually indicates that the response parameter $\sigma'_{xx}$ scales from $-(2/\pi)\ln(\lambda/\xi)$ for small values of $\lambda$ to $\exp(-\lambda/\xi)$ for large values of $\lambda$. Here, $\xi$ is the dynamically generated correlation length (localization length), see Eqs. (325) and (326). From Eq. (325) we see that the instanton contribution generally enhances the tendency of the electron gas to localize at large distances. In Fig. 9 we sketch the overall behavior of the $\tilde{\beta}_\sigma$ function which is given by the weak coupling result of Eq. (325) for large values of $\sigma'_{xx}$ and the strong coupling result

$$\tilde{\beta}_\sigma = \ln \sigma'_{xx}$$

(327)

as $\sigma'_{xx}$ goes to zero. These results give rise to the well known scaling scenario of localization in two spatial dimensions. However, Eq. (326) shows that $|\theta'|$ deceases at a much slower rate with increasing values of $\lambda$ which means that the quantum Hall regime is generally confined to the regime of “bad conductors” $\sigma'_{xx} \lesssim 1$ only. Similar to $\sigma'_{xx}$ there seems to be something remarkably universal about the exponential form with which $|\theta'|$ vanishes in the strong coupling regime. The experiments on the quantum Hall effect, for example, indicate that $\theta' \propto (\sigma'_{xx})^a$ with some positive value for the exponent $a$ which is presumably equal to two. The same behavior has recently been found in strong coupling studies of closely related two dimensional models of the instanton vacuum. Analogous to Eq. (327) one therefore expects that

$$\tilde{\beta}_\theta = a \ln \sigma'_{xx}$$

(328)

in the limit where $\sigma'_{xx}$ goes to zero. In Fig. 9 we compare the scaling results for the Hall conductance $\tilde{\beta}_\theta$ with those for the longitudinal conductance $\tilde{\beta}_\sigma$. These scaling results indicate that the quantization phenomenon is a (super) universal strong coupling feature of the instanton vacuum concept, independent of the specific application of this concept or, for that matter, independent of the presence of electron-electron interactions.

B. Fermi liquid versus non-Fermi liquid theory

The most important features next are the quantum critical fixed points that are located at $\theta = \pi$ or half-integer values of $\sigma_{xy}$. Fig. 8 shows that the Fermi liquid fixed point located at $c = 0$ is distinctly different from the Coulomb interaction fixed point at $c = 1$. Like the mobility edge problem in $2 + \epsilon$ dimensions, the quantum critical behavior of the transitions between adjacent quantum Hall plateaus is very different for finite range electron-electron interactions and the Coulomb potential, each involving different exponent values as well as a fundamentally different dynamical behavior. The results of this paper therefore completely invalidate any attempt to explain the experimentally observed exponent values on the basis of Fermi liquid type of ideas.

C. Super universality

The results of this paper explain, at the same time, why the scaling behavior of the free electron gas and the Coulomb interaction problem in strong magnetic fields look so similar. In spite of the fact that the underlying theories are fundamentally different they have nevertheless important features in common such as asymptotic freedom, instantons, massless edge excitations etc. Since in both cases the topological concepts are the same it is natural to expect that the basic phenomena are the same, in particular the existence of robust topological quantum numbers that explain the observability and precision of the quantum Hall effect, as well as quantum criticality at $\theta = \pi$ that generally facilitates a transition to take place between different quantum Hall plateaus. Finally,
by recognizing the fact that quantum Hall physics actually reveals itself as a generic, super universal feature of the instanton vacuum in asymptotically free field theory one has essentially laid the foundation for a more ambitious unifying theory that includes besides integral quantum Hall regime also the scaling behavior of completely different physical phenomena such as the abelian quantum Hall states.

Acknowledgments

This research was funded in part by the Dutch National Science Foundations NWO and FOM. One of us (IB) is indebted to the Russian Foundation for Basic Research (RFBR), the Russian Ministry of Science and Russian Science Support Foundation for financial support.

APPENDIX A: LINEAR RESPONSE VERSUS BACKGROUND FIELD PROCEDURE

With the introduction of $\mathcal{F}$ algebra it has become possible to show that observable quantities $\sigma'_{xx}$, $\sigma'_{xy}$, $\zeta'$ and $\zeta''$ which are usually obtained by means of background field procedures or momentum shell procedures are, in fact, precisely the same as the expressions for the conductances at zero temperature that one derives from ordinary linear response theory in the external vector potential. In this Appendix we briefly repeat the argument for the special case where the infrared of the system is regulated by a finite size $L \times L$. We show in particular that the linear response formulae given by Eqs. (29) and (30) are the same expressions for $\sigma'_{xx}$ and $\sigma'_{xy}$ as those appearing in the effective action for the edge modes, Eq. (31).

1. Linear response theory

Specializing to the theory of Eqs. (A.1) with spherical boundary conditions on the field variables $Q$ then the response of the system to an external vector potential $A$ can generally be written in terms of an effective action $S_{\text{eff}}[A]$ according to

$$\exp S_{\text{eff}}[A] = \int_{\partial V} D[Q_0] \exp \left( S_{\sigma}[Q_0, A] + S_{F}[Q_0] \right).$$

(A.1)

The vector potential $A$ couples the free electron part of the action $S_{\sigma}$ only

$$S_{\sigma}[Q_0, A] = -\frac{\sigma_{xx}}{8} \int d^2 r \text{tr}[D_j, Q_0][D_j, Q_0]$$

$$+ \frac{\sigma_{xy}}{8} \int d^2 r \varepsilon_{jk} Q_0[D_j, Q_0][D_k, Q_0]$$

(A.2)

with $D_j$ standing for the covariant derivative

$$D_j = \nabla_j - i \hat{A}_j, \quad \hat{A}_j = \sum_{\alpha n} A^\alpha_j(\nu_n) t^\alpha_n.$$  \hspace{1cm} (A.3)

Since we are interested in the global response at zero temperature and frequency it suffices to take a spatially independent $\mathbf{A}^\alpha(\nu_n)$ and consider a small range of values $\nu_n = 2\pi T n \approx 0$ only. The response parameters $\sigma'_{xx}$ and $\sigma'_{xy}$ are then defined by the following general form of the effective action

$$S_{\text{eff}}[A] = -L^2 \sum_{\alpha n > 0} n \left[ \sigma'_{xx} \delta_{jk} + \sigma'_{xy} \varepsilon_{jk} \right] A_j(\nu_n) A_k(-\nu_n).$$

(A.4)

By using this expression for the left hand side of Eq. (A.1) it is easy to derive the results of Eqs. (29) and (30) for $\sigma'_{xx}$ and $\sigma'_{xy}$ respectively which are the main objectives of this paper. These formulae are some of the most fundamental quantities of the theory since they can generally be used for studies at finite temperature and frequency rather than finite sample sizes. Moreover, they facilitate an analysis of mesoscopic fluctuations as well as important self-consistency checks in practical computations such as the replica limit $N_r = 0$ and $N_m \to \infty$.

However, the complications primarily arise if one wants to make sure that the Finkelstein formalism preserves the fundamental symmetries of the interacting electron gas, in particular the electrodynamic $U(1)$ gauge invariance as well as $\mathcal{F}$ invariance which are properly defined in infinite Matsubara frequency space only. As we shall see next, these complications automatically arise in the attempt to lay the bridge between linear response theory and the effective action for the edge modes.

2. $\mathcal{F}$ invariance

To deal with electrodynamic gauge invariance in finite frequency space we start out by embedding the matrix variables $Q_0$ of size $2N_r N_m \times 2N_r N_m$ in a much larger matrix space of size $2N_r N'_m \times 2N_r N'_m$ with $1 \ll N_m \ll N'_m$. All matrix manipulations will be carried out from now onward in the space of large matrices whereas the unitary rotations $Q_0$ effectively retain their size $2N_r N_m \times 2N_r N_m$ which we term small.

Let us next introduce the quantity $\varphi^\alpha_0(\mathbf{r}) = \mathbf{A}^\alpha(\nu_n) \cdot \mathbf{r}$. We can then express the vector potential $\tilde{\mathbf{A}}$ in terms of the large unitary matrix $\hat{\varphi} = \hat{\varphi}(\mathbf{r})$ according to

$$\tilde{\mathbf{A}} = \nabla \hat{\varphi} = i W^{-1} \nabla W, \quad W = \exp(-i \hat{\varphi}).$$  \hspace{1cm} (A.5)

Following the rules of $\mathcal{F}$ algebra the unitary matrix $W$ just stands for an electrodynamic $U(1)$ gauge transformation in Matsubara frequency notation. The free electron part of the action $S_{\text{eff}}[\mathbf{A}]$ can be expressed in terms of the $W$ rotation on the matrix field variable $Q_0$ according to
\[ S_\sigma[Q_0, A] = S_\sigma[W^{-1}Q_0W] = -\frac{\sigma_{xx}}{8} \int d\tau \text{tr}[\nabla(W^{-1}Q_0W)]^2 + \frac{\sigma_{xy}}{8} \int d\tau \varepsilon_{jk} Q_0 \nabla_j(W^{-1}Q_0W) \nabla_k(W^{-1}Q_0W). \] 

(A.6)

Next we split the quantity \( O_F(Q_0) \) into an \( \mathcal{F} \) invariant part \( O_s(Q_0) \) and a symmetry breaking part

\[ O_F(Q_0) = O_s(Q_0) + O_\eta(Q_0) \]

where

\[ O_s(Q_0) = zc \left( \sum_{\alpha} \text{tr} I_n^\alpha Q_0 \text{tr} I_n^{-\alpha} Q_0 + 4 \text{tr} \eta Q_0 \right) - 6 \text{tr} \eta \Lambda \]

\[ = zc \sum_{\alpha} \text{tr}[I_n^\alpha Q_0][I_n^{-\alpha} Q_0] \]

(A.8)

\[ O_\eta(Q_0) = zc \left( 4 \text{tr} \eta Q_0 - 6 \text{tr} \eta \Lambda \right). \]

(A.9)

The statement of \( \mathcal{F} \) invariance now says that \( O_s(Q_0) \) is gauge invariant

\[ O_s(Q_0) = O_s(W^{-1}Q_0W). \]

(A.10)

On the other hand, as long as one evaluates the theory at zero temperature and finite system sizes, the response parameters \( \sigma'_{xx} \) and \( \sigma'_{xy} \) remain unchanged if one inserts the \( W \) rotation into the quantity \( O_\eta(Q) \), i.e. the replacement

\[ O_\eta(Q_0) \rightarrow O_\eta(W^{-1}Q_0W) \]

(A.11)

does not affect the statement of Eq. (A.4) where the \( \sigma'_{xx} \) and \( \sigma'_{xy} \) depend on the system size \( L \). Linear response theory at zero temperature and finite system sizes is therefore formally the same thing as evaluating the theory in the presence of a gauge field \( W \)

\[ e^{S_{\text{eff}}[A]} = \int_{\partial V} \mathcal{D}[Q_0] e^{S_s[W^{-1}Q_0W] + S_F[W^{-1}Q_0W]}. \]

(A.12)

The main reason for introducing the two different cut-offs \( 1 \ll N_m \ll N'_m \) in finite Matsubara frequency space is to ensure that Eqs. (A.10), (A.11) and (A.12) display the exact same symmetries that are known to exist in the theory where \( N_m \) and \( N'_m \) are being sent off to infinity.

3. Background field formalism

It is clear that the statement of Eq. (A.12) is non-trivial only due to the fact that that we work at zero temperature and with fixed boundary conditions on the matrix field variable \( Q_0 \). If on the other hand we were to work with finite temperatures and infinite system sizes \( L \) then Eq. (A.12) is merely a statement of electrodynamic \( U(1) \) gauge invariance which is clearly very different from Eq. (A.4).

Notice that Eq. (A.12) is not yet quite the same as the back ground field methodology that previously has been studied intensively for renormalization group purposes. This is because the quantities \( Q_0 \) and \( W^{-1}Q_0W \) by construction belong to different manifolds for any finite value of \( N_m \) and \( N'_m \). However, in order for the \( W \) rotation in Eqs. (A.10), (A.11) and (A.12) to represent an exact electrodynamic \( U(1) \) gauge transformation it is imperative that the results do not fundamentally depend on the details of how the frequency cut-offs \( N_m \) and \( N'_m \) go to infinity. Moreover, the statement of Eq. (A.12) renders highly non-trivial if one recognizes that the unitary matrix \( W \) can in general be written as the product of two distinctly different matrices \( t \) and \( U_0 \)

\[ W = \exp(-i\hat{\varphi}) = U_0 t, \quad U_0 \in U(N') \times U(N') \] (A.13)

where \( N' = N_rN'_m \). Here, \( t \) is a “small” background matrix field in the true sense of the word

\[ t = \exp \left( \frac{i}{2} [\hat{\varphi}, \Lambda] + \ldots \right) \]

(A.14)

whereas the “large” generators of \( W \) are all collected together in the \( U(N') \times U(N') \) gauge \( U_0 \) which can be written as

\[ U_0 = \exp \left( \frac{i}{2} (\hat{\varphi}, \Lambda) \right). \]

(A.15)

Next we consider the change of variables

\[ U_0^{-1}Q_0U_0 \rightarrow Q_0. \]

(A.16)

It is clear that this transformation preserves the spherical boundary conditions and leaves the measure of the functional integral unchanged. Equation (A.12) can therefore be represented as follows

\[ \exp \tilde{S}_{\text{eff}}[A] = \int_{\partial V} \mathcal{D}[Q] \exp \left( S_s[t^{-1}Q_0t] + S_F[t^{-1}Q_0t] \right) \]

(A.17)

which precisely corresponds to the background field methodology with the “small” matrix field \( t \) given explicitly by Eq. (A.13). This, then, leads to the principle result of this Appendix which says that Eq. (A.17) in the limit where \( N_m, N'_m \rightarrow \infty \) and \( T = 0 \) is identically the same as linear response theory Eqs. (29) and (30).

Eq. (A.12) together with Eq. (A.17) can be used to derive different or alternative expressions for the quantities \( \sigma'_{xx} \) and \( \sigma'_{xy} \), which are completely equivalent to those given by Eqs. (29) and (30). Here we do not list these expressions but instead we simply verify the correctness of the effective action of Eq. (A.1). Since Eq. (A.17) has the
the normalization condition
\[ e \] can immediately write down the following general result
\[ \langle \bar{S}_{\text{eff}}[A] \rangle = e^{\bar{S}_{\text{eff}}[A]} (1 + T z' c') \int d\tau \sum_{\alpha_n} \Gamma[\tau^\alpha_n, \tau^\alpha_n] \]
(A.18)
where the superscript “0” denotes the result at \( T = 0 \). Eq. (A.18) can be obtained, as before, by expanding in the gradients of the slowly varying matrix field \( q = f^{-1} A \).

By inserting the expression for Eq. (A.4).

The following identities have been used
\[ \text{tr}[\hat{I}^\alpha_n, \Lambda][\hat{I}^\alpha_n, \Lambda] = -4 n \]
(A.20)
\[ \text{tr} \Lambda[\hat{I}^\alpha_n, \hat{I}^\alpha_n] = 2 n. \]
(A.21)

We see that we recover the same results as those in Eq. (A.3).

4. The quantities \( z' \) and \( c' \)

For completeness we next extend the results of the background field methodology to include the terms obtained by expanding to lowest order in \( T \)

\[ e^{\bar{S}_{\text{eff}}[A]} = e^{\bar{S}_{\text{eff}}[A]} \left( 1 + T z' c' \int d\tau \sum_{\alpha_n} \Gamma[\tau^\alpha_n, \tau^\alpha_n] \right) \]

These results indicate that the quantity \( zc \) is renormalized whereas the statement \( z_0 = z' a' \) is a physical constraint that should in general be imposed upon the theory. Eq. (A.22) has been verified in the theory of perturbative expansions. In Section VI of this paper we explicitly check the validity of this statement at a non-perturbative level. As a final remark, it should be mentioned that by taking \( q = \Lambda \) in Eq. (A.22), one immediately obtains the expression for \( z' \), Eq. (32).

APPENDIX B: MATRIX ELEMENTS

The matrix elements of a function \( f(\eta, \theta) \) are defined as follows

\[ \langle a | J, M | f(\eta, \theta) | J', M' \rangle = \int d\eta d\theta \Phi_{J, M}(\eta, \theta) f(\eta, \theta) \Phi_{J', M'}(\eta, \theta) \]

where \( a, b = 0, 1, 2 \). By using the following identity for the Jacobi polynomials

\[ (2n + \alpha + \beta)P_n^{(\alpha, \beta)}(x) = (n + \alpha + \beta)P_n^{(\alpha, \beta)}(x) - (n + \beta)P_n^{(\alpha, \beta)}(x) \]
and the normalization condition

\[ \int_{-1}^{1} dx x^\alpha (1 + x)^\beta P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) = \delta_{n,m} 2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1) \]
(B.1)

we find that the matrix elements for \( e_0 \) and \( e_1 \) are given as

\[ \langle 0 | J, M | e_0^1 | M - 1, J \rangle_{(1)} = \frac{1}{\sqrt{2}} \sqrt{\frac{J + M}{2J + 1}}, \quad \langle 0 | J, M | e_0^1 | M - 1, J \rangle_{(1)} = \frac{1}{\sqrt{2}} \sqrt{\frac{J - M + 1}{2J + 1}} \]
(B.3)
\[ \langle 0 | J, M | e_0 | M, J \rangle_{(1)} = -\frac{1}{\sqrt{2}} \sqrt{\frac{J - M}{2J + 1}}, \quad \langle 0 | J, M | e_0 | M, J + 1 \rangle_{(1)} = \frac{1}{\sqrt{2}} \sqrt{\frac{J + M + 1}{2J + 1}} \]
(B.4)

Next for \( e_0^2 \) we have

\[ \langle 0 | J, M | e_0^2 | M - 1, J \rangle_{(1)} = -\sqrt{(J - M - 1)(J + M)} \frac{1}{2(2J - 1)}, \quad \langle 0 | J, M | e_0^2 | M - 1, J \rangle_{(1)} = \frac{1}{2} \left[ 1 + \frac{2M + 1}{J^2 - 1} \right] \]
(B.5)
\[ \langle 0 | J, M | e_0^2 | M, J \rangle_{(1)} = -\sqrt{(J + M + 1)(J - M)} \frac{1}{2(2J + 1)}, \quad \langle 0 | J, M | e_0^2 | M, J \rangle_{(2)} = \frac{1}{2} \left[ M + 1 \right] \]
(B.6)
The matrix elements of $e_0 e_1$ are as follows

\[
\begin{align*}
(1) \langle J, M - 1 | e_0 e_1 | M, J - 1 \rangle_{(1)} &= \frac{\sqrt{(J - M)(J - M - 1)}}{2(2J - 1)}, \\
(1) \langle J, M - 1 | e_0 e_1 | M, J \rangle_{(1)} &= \frac{\sqrt{(J - M)(J + M + 1)}}{4J^2 - 1}, \\
(1) \langle J, M - 1 | e_0 e_1 | M, J + 1 \rangle_{(1)} &= \frac{\sqrt{(J + M)(J + M + 1)}}{2(-2J - 1)}, \\
(0) \langle J, M | e_0 e_1^* | M - 1, J - 1 \rangle_{(0)} &= \sqrt{(J + M - 1)(J + M)} / 4(2J - 1)(2J + 1), \\
(0) \langle J, M + 1 | e_0 e_1^* | M, J + 1 \rangle_{(0)} &= -\sqrt{(J - M)(J - M + 1)} / 4(2J + 1)(2J + 3).
\end{align*}
\]

Finally, the following summation theorems are of interest

\[
\sum_{M = -J}^{J-1} (1) \langle J, M | e_0^2 | e_1^2 | M, J \rangle_{(1)} = \frac{J}{3}, \quad \sum_{M = -J}^{J-1} (2) \langle J, M | e_0^2 | e_1^2 | M, J \rangle_{(2)} = \frac{2J + 1}{6}.
\]

**APPENDIX C: PERTURBATIVE EXPANSIONS OF OBSERVABLE THEORY USING PAULI-VILLARS REGULARIZATION**

1. Renormalization of $\sigma_{xx}$

For ordinary perturbation theory we use the expression for the matrix field variable $Q$ as in Eqs. (10) - (12) but with the matrices $T$ and $R$ put equal to the unit matrix. Evaluating Eq. (29) to the second order in the independent field variables $v, v'$ we then obtain

\[
\sigma'_{xx} = \sigma_{xx} + \frac{\sigma_{xx}^2}{2n} \int d^4r \left( \mathrm{tr} I^a_n v(r) \nabla v^\dagger(r) \times \mathrm{tr} I^a_n v'(r') \nabla v^\dagger(r') \right).
\]

Notice that in flat space one can choose the point $r'$ arbitrarily due to translational invariance. In curved space, however, we must evaluate Eq. (C.1) in terms of the propagators of Eqs. (14) and (15) in which case translational invariance is no longer obvious. In terms of the energies and eigenfunctions in curved space Eq. (C.1) reads as follows

\[
\sigma'_{xx} = \sigma_{xx} - 4c \int_0^\infty d\omega \sum_J \frac{E_j^0}{(E_j^0 + \omega)(E_j^0 + \alpha\omega)} \times \sum_{M = -J}^J \Phi_{JM}(\eta', \theta') \Phi_{JM}(\eta, \theta')
\]

where $\eta', \theta'$ denote the spherical coordinates of the point $r'$. Since the eigenfunction $\Phi_{JM}$ is proportional to the Jacobi polynomial $P_{J-M}(\eta)$ which itself is proportional to the Gegenbauer polynomial $C_{J-M}^{M+1/2}(\eta)$ we can use the well known summation theorem for Gegenbauer polynomials and obtain

\[
\sum_J \Phi_{JM}^0(\cos \phi, \theta) \Phi_{JM}^0(\cos \phi', \theta) = \frac{2J + 1}{4\pi} C_{J}^{1/2}(\cos(\phi - \phi')).
\]

We recognize this identity as a projection operator statement which means that Eq. (C.1) is in fact independent of $\eta', \theta'$.

Next, introducing the Pauli-Villars masses as well as the alternating metric, using $C_{J}^{1/2}(1) = 1$ and after integrating over $\omega$ we obtain

\[
\sigma'_{xx} = \sigma_{xx} - \frac{\beta_0(c)}{2} \lim_{\Lambda \to \infty} \sum_{J=3/2}^\Lambda \frac{2J(J^2 - \frac{1}{4})}{(J^2 - \frac{1}{4})^2} + \sum_{J=1/2}^\Lambda \frac{2J(J^2 - \frac{1}{4})}{(J^2 - \frac{1}{4} + M_J^2)^2}.
\]

Evaluating the sums we finally have

\[
\sigma'_{xx} = \sigma_{xx} - \frac{\beta_0(c)}{2} \left( Y_{\text{reg}}(0) + 1 \right) = \sigma_{xx} \left( 1 - \frac{\beta_0(c)}{\sigma_{xx}} \ln M e^{\gamma_E} \right).
\]

2. Renormalization of $\zeta c'$

The expression for $\zeta'c'$ can be expanded in a similar fashion. To lowest order in $v, v'$ we can write the con-
tributing terms as follows

\[ z'c' = zc \left( 1 - \frac{1}{\eta\lambda} \sum_{\alpha, n > 0} \langle \text{tr} I_n^\alpha u(r) \text{ tr} I_n^\alpha u^\dagger(r) \rangle \right). \]  

(C.6)

In curved space this expression becomes

\[ z'c' = zc \left[ 1 + \frac{2\pi \gamma_0}{\sigma_{xx}} \sum_J \frac{1}{E_J(0)} \sum_{M=-J}^J \Phi_J(\eta, \theta) \bar{\Phi}_{-J}(\eta, \theta) \right] \]  

(C.7)

with \( \eta, \theta \) denoting the point \( r \). Next, using Eq. (C.8) as well as [157] we finally obtain

\[ z'c' = zc \left( 1 + \frac{\gamma_0}{2\sigma_{xx}} \ln m \right). \]  

(C.8)

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