Particle-hole symmetry in algebraic Bethe Ansatz for the XXX model

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Abstract. It is well known that the space of all quantum states of the XXX model for a magnetic ring of \(N\) nodes, each with the spin \(1/2\), decomposes into sectors with \(r\) spin deviations, \(r = 0, 1, 2, ..., N\). The sectors \(r\) and \(N - r\) are related mutually by the particle-hole transformation which exchanges the signs + and − on each node. We discuss here effects of this transformation on the formalism of algebraic Bethe Ansatz, in particular on the form of the monodromy matrix, the main tool of this formalism. We derive explicitly appropriate transformation rules for \(C_N\)-orbits of magnetic configurations and the corresponding Fourier transformations within the bases of wavelets. In particular, we point out some important phase relations between orbits on both sides of the equator.

1. Introduction
In this paper we discuss the particle-hole symmetry within the algebraic Bethe Ansatz [5], which consists in interchange of + and − signs at each node. We are therefore not restricted to the sectors \(r = 1, \ldots, \text{up to the equator}\), but consider the total space of quantum states of the magnet.

We use here a notation in which \(\mathbb{N} = \{j = 1, 2, \ldots, N\}\) denotes the set of nodes, labelled by consecutive integers \(j, 1 \leq j \leq N\), and each magnetic configuration is described by the mapping:

\[
J : \mathbb{N} \mapsto \mathbb{N}, J(f) = \sum_{j \in \mathbb{N}} i_j 2^{N-j} + 1.
\]

Thus the lexical order of magnetic configurations is defined in terms of \(J\) as: \(f < f'\) if \(J(f) < J(f')\). The mapping \(J\) is in fact a slightly modified "binary → decimal" conversion, and "+1" shift arises from the presence of ferromagnetic vacuum.
In terms of $J$, the action of the transitional symmetry group of the ring ($C_N$ group) on any magnetic configuration can be defined by:

$$C_N(J(f)) := (J(f) - 1)/2 + 2^{N-1}(J(f) - 1) \mod 2) + 1$$

(3)

It is well known that each configuration can be also displayed by its number of spin deviations $r = \{0, 2, ..., r\}$, the relative position vector $t = (t_1, t_2, ..., t_r)$ where $t_\alpha$ is the distance between the $\alpha$'th and $(\alpha + 1) \mod r$ spin deviation on the chain, and $j$ is the consecutive number of configuration within the $C_N$-orbit labelled by $t$.

Relative position vector $t$ specifies the distribution of Bethe pseudoparticles within the $C_N$-orbit [6]. The bases of $C_N$-orbits is constructed with respect to:

- the lexical order of the parameter $r = 0, 1, 2, ..., N$
- the first element of each $C_N$-orbit has the property $J(f_1) = \max\{J(f)\}$ for $f$-s within this orbit.
- for a fixed $r$, the orbits are ordered according to descending values of $J(f_1)$ of the corresponding initial element
- consecutive elements in each orbit are generated by the action of $C_N$

According to this convention the transform matrix $G$ from the lexical bases to that of $C_N$-orbits for magnetic ring of $N = 4$ nodes has the form:

$$G = \begin{pmatrix}
J(i_1, ..., i_4) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
J(r, t, j) & 1 & 9 & 5 & 3 & 2 & 13 & 7 & 4 & 10 & 11 & 6 & 15 & 8 & 12 & 14 & 16
\end{pmatrix}$$

Since the bases of $C_N$-orbits is just a reordered lexical one, the $G$ operator shows a bijection between those two bases.

3. Particle-hole symmetry

The particle-hole symmetry $Q$ exchanges the signs of spin "+" and "+" on each node. Operation $Q$ can be formally defined as

$$Q : \tilde{2}^N \mapsto \tilde{2}^N, \quad Q|f > = |(i_1 + 1) \mod 2, ..., (i_N + 1) \mod 2 >$$

(4)

The operation $Q$ has the following properties:

- no fixed point: $Qf \neq f$
- $Q^2 = id_{\tilde{2}^N}$
- $Q$ is a bijection

In the lexical bases, the matrix of $Q$ has the antidiagonal form $Q_{ij} = \delta_{i,j-1}\mod 2 \mod 2$. To investigate the action of particle-hole symmetry on $C_N$-orbits it is convenient to write matrix of operator $Q$ in the bases of orbits.
The action of $Q$ on the $C_N$-orbits of Heissenberg magnet with $N = 4$ nodes can be presented as:

| $r$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $t$ | (4) | (1,3) | (2,2) | (1,1,2) | (1,1,1,1) |

| $i$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $t$ | (4) | 0 1 0 0 | 0 0 1 0 | 0 0 0 1 | 0 0 0 0 |
| 1 | 2 | 3 | 4 |

The particle-hole symmetry usually exchanges the orbit. Case of $N = 4$ is exceptional since the orbits on "the equator" i.e. $r = N/2$ are preserved. Unimportant 0's inside the matrix are omitted.

In order to provide the momentum representation of the space $H = lC2^N$ of all quantum states of the magnetic ring e.g. in the bases of wavelets, the set $\mathcal{B}_N$ should be replaced by the Brillouin zone $B$ [7].

$$|B, k > = \frac{1}{\sqrt{N}} \sum_{j \in \mathcal{B}_N} \exp[-\frac{2i\pi jk}{N}]|\mathcal{B}_N, j >$$

where $k$ is the quasimomentum. Transformation to the bases of wavelets can be written as:

$$\hat{F}|\mathcal{B}_N, j > = |B, \beta(j) >,$$

where the Fourier operator $\hat{F}$ for a regular orbit of $C_N$ is defined by the formula

$$< \mathcal{B}_N, j_1 | \hat{F} | \mathcal{B}_N, j_2 > = \frac{1}{\sqrt{N}} \omega^{-j_1 j_2}, \ \omega = \exp[\frac{2i\pi}{N}]$$

and $\beta$ is a bijection defined by:

$$\beta : \mathcal{B}_N \mapsto B, \beta(j) = j \mod B$$

the bijection $\beta$ for the $N = 4$ is:

| $j$ | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| $\beta(j)$ | 1 | 2 | -1 | 0 |
An example of the $Q$ operator matrix in the wavelets bases for $N=4$

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | () | (1) | 2 | 3 | 4 |
| 1 | (4) | 2 | 3 | 4 |
| 2 | (1,3) | 2 | 3 | 4 |
| 3 | (1,1,2) | 2 | 3 | 4 |
| 4 | (1,1,1,1) | 2 | 3 | 4 |

The action of $Q$ operation preserves the minimal and maximal pseudomomentum $k$, just adding the phase $\pm 1$, pseudomomenta of other states usually changes and may have imaginary phases.

To achieve the exact solutions of Bethe Ansatz (rigged string configurations) let us choose the Hamilton operator in the form required by the Monodromy matrix formalism [5]

$$H_N = \frac{1}{4} \sum_{j \in \mathcal{N}} (\sigma^x_{j} \sigma^x_{j+1} + \sigma^y_{j} \sigma^y_{j+1} + \sigma^z_{j} \sigma^z_{j+1} - I_N)$$

(7)

where

$$\sigma^a_j = I \otimes ...I \otimes \sigma^a_j \otimes I... \otimes I, \ a=x,y,z$$

(8)

$$I_N = I \otimes ... \otimes I$$

and $\sigma^x, \sigma^y, \sigma^z$ are standard Pauli matrices and $I$ is the $2 \times 2$ unit matrix.

It is convenient to transform that Hamiltonian into the bases of orbits or wavelets. The block form of $H_N$ in these bases allows us to find solutions for each block separately. Thus allows to use rigged configurations bases, and the corresponding transformation matrix for $N = 4$ has the form:

$$6G =$$

The transformation matrix consists of eigenvectors, here for convenience multiplied by 6.

Once the eigenproblem is solved the action of $Q$ on the exact solutions of XXX Heisenberg magnet can be easily explored.
For ring consisting of 4 nodes the corresponding $Q$ operator can be written in form:

\[
\begin{array}{ccccccccc}
0 & 2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & g & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 2g & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 2 & 2g & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2g & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2g & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & ggg & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & ggg & 1 & 0 & i & 0 & 0 & 0 \\
1 & 2 & ggg & 2 & 0 & 0 & -1 & 0 & 0 \\
1 & 2 & ggg & -1 & 0 & 0 & 0 & -i & 0 \\
4 & 2 & ggg & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where $g$ denotes the Goldstone boson. Particle-hole operator in rigged string bases consists of diagonal submatrices placed on antidiagonal. Operation $Q$ replaces the state with the one of the same type. Allowed phases on “the equator” are $\pm 1$. When acting on the state (below/beyond) the equator $Q$ (adds/substracts) some goldston bosons, and the allowed phases are $\{\pm 1, \pm i\}$.

4. Conclusions

Despite its simplicity, the particle-hole symmetry operator has a non trivial form in various useful bases of Bethe Ansatz. We have discussed here its action on: the bases of magnetic configuration in the lexical order, $C_N$-orbits, wavelets and the exact Bethe Ansatz solutions.

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