Nonzero-sum optimal stopping games and generalised Nash equilibrium

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Abstract

In the nonzero-sum setting, we establish a connection between Nash equilibria in games of optimal stopping (Dynkin games) and generalised Nash equilibrium problems (GNEP). In the Dynkin game this reveals novel equilibria of threshold type and of more complex types, and leads to novel uniqueness and stability results.

1 Introduction

In this paper we establish a connection between Nash equilibria in two different types of game. The first type is the two-player, nonzero-sum Dynkin game of optimal stopping (for general background on optimal stopping problems the reader is referred to [26]). Player \( i \in \{1, 2\} \) chooses a stopping time \( \tau_i \) for a strong Markov process \( X = (X_t)_{t \geq 0} \) defined on the interval \((x_\ell, x_r)\). Reward functions \( f_i, g_i, h_i \) are given and the reward or payoff to player \( i \) is

\[
J_i(\tau_1, \tau_2) := f_i(X_{\tau_1})\mathbb{1}_{\{\tau_1 < \tau_2 - i\}} + g_i(X_{\tau_2})\mathbb{1}_{\{\tau_2 < \tau_1 - i\}} + h_i(X_{\tau_1})\mathbb{1}_{\{\tau_1 = \tau_2 - i\}},
\]

for each player \( i \in \{1, 2\} \) the subscript \( -i \) denotes the other player. In this context equilibrium strategies \((\tau_1, \tau_2)\) of the form

\[
\tau_1 = \inf\{t \geq 0 : X_t \leq \ell\} \quad \text{and} \quad \tau_2 = \inf\{t \geq 0 : X_t \geq r\},
\]

for constants \( \ell, r \in (x_\ell, x_r) \) with \( \ell < r \), are referred to as threshold-type equilibria. A recent example is in [11], in which the thresholds \( \ell, r \) are drawn from the disjoint strategy spaces \( S_1 \) and \( S_2 \) respectively where

\[
S_1 := [x_\ell, a], \quad S_2 := [b, x_r],
\]

for some constants \( a, b \) with \( x_\ell < a < b < x_r \).

The second type of game is a deterministic generalised game [15] (or abstract economy [1]) with \( n \geq 2 \) players, where \( n \) will depend on the structure of the equilibrium studied in the Dynkin game. Since the examination of all cases \( n \geq 2 \) is reserved for future work, however, we focus on \( n = 2 \) and simply provide an example with \( n = 3 \).

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The connection yields novel equilibria in the Dynkin game. This novelty is threefold. Firstly the reward functions are not required to be differentiable. Secondly we obtain novel equilibria of threshold type, since both cases \( a < b \) and \( a \geq b \) are permitted. Thirdly, while threshold-type equilibria correspond to the case \( n = 2 \), the cases \( n > 2 \) yield equilibria with more complex structures. To the best of our knowledge, these complex equilibrium structures in the Dynkin game have not been previously studied.

In the threshold-type case, we obtain the uniqueness of the equilibria among Markovian strategies, and results about their local and global stability.

1.1 Background

The structure of Nash equilibria in nonzero-sum Dynkin games has recently been investigated in [3] and [11], where sufficient conditions for the existence and uniqueness of threshold-type equilibria are obtained. A key difference between the case \( n = 2 \) of the present paper and the latter work is that there, the functions \( f_i \) in (1.1) are twice differentiable and have unique points of inflexion \( a \) and \( b \) respectively with \( a < b \), conditions which may all be relaxed in the present approach. Appendix A contains remarks on the inclusion of time discounting, and the use of other Markov processes \( X \), in our setup.

Our results on stability relate to an iterative approximation scheme for Nash equilibria, which has been previously studied outside the Markovian framework in [17] and, in the Markovian framework, in [6], [9], [19] and [25]. In [19] it is assumed that \( f_i = g_i \) and in [6], [9] and [25] a condition related to superharmonicity is imposed for the \( g_i \). The latter conditions ensure monotone convergence over the iteration, whereas the approach via stability in Section 5 does not rely on monotonicity.

The special case of zero-sum Dynkin games, in which \( f_i = -g_{-i} \) and \( h_i = -h_{-i} \), has received particular attention in the literature. Thorough analyses of the zero-sum game for a large class of driving Markov processes can be found in [14] and [27] and in that context Assumption 1, which relates the game to a war of attrition [16, Section 4.5.2], is sufficient for the existence of a Nash equilibrium among pure strategies. We adopt the same setting, as is also common in the nonzero-sum context (see for example [6, 9, 25]).

Assumption 1. For \( i = 1, 2 \) the functions \( f_i, g_i \) and \( h_i \) are bounded and continuous on \([x_L, x_R]\), and satisfy \( f_i \leq h_i \leq g_i \).

In Remark 4.6 we discuss how Assumption 1 can be weakened without affecting the main results.

1.2 Preliminaries

In this section we recall necessary background on subprocesses, superharmonic and quasi-concave functions, which should be familiar.

1.2.1 Subprocesses of a Brownian motion

Let \( W = (W_t)_{t \geq 0} \) be a one-dimensional standard Brownian motion defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \hat{\mathbb{P}})\), where \( \mathbb{F} \) is the universally completed filtration [7] p. 27. We will write the probability measure as \( \hat{\mathbb{P}}^x \) in the case \( \hat{\mathbb{P}}(\{W_0 = x\}) = 1 \), and denote the expectation operator with respect to \( \hat{\mathbb{P}}^x \) by \( \hat{\mathbb{E}}^x \). From \( W \) we derive subprocesses in the sense of [7] Chapter III]. More precisely, for each subset \( E \) of \([0, 1]\) we define a subprocess \( X^E = X = (X_t)_{t \geq 0} \)
such that $X$ and its (almost surely finite) lifetime $\zeta$ satisfy,

$$\zeta = \inf\{t \geq 0 : W_t \notin E\},$$

(1.4)

$$X_t = \begin{cases} W_t, & 0 \leq t < \zeta, \\ \Delta, & t \geq \zeta. \end{cases}$$

(1.5)

Here $\Delta$ is a cemetery state and the state space of $X$ is $E$ equipped with its Borel sigma-algebra $\mathcal{B}(E)$ and augmented by $\Delta$. We set $\phi(\Delta) = 0$ for every measurable function $\phi$ on $E_\Delta := E \cup \Delta$, showing in Section 3.1 below that this choice involves no loss of generality. For $x \in E$ let $\mathbb{P}^x$ and $\mathbb{E}^x$ denote the probability measure and expectation operator corresponding to $X$ when $\mathbb{P}^x(\{X_0 = x\}) = 1$. For every measurable function $\phi$ vanishing outside $E$ and every $t \geq 0$ we have [2] p. 105:

$$\mathbb{E}^x[\phi(X_t)] = \mathbb{E}^x[\phi(W_t)1_{\{t<\zeta\}}].$$

(1.6)

For each measurable set $A$ we write the associated first entrance time of $X$ as

$$D_A := \inf\{t \geq 0 : X_t \in A\} = \inf\{t > 0 : X_t \in A\} \quad \text{a.s.,}$$

(1.7)

where we take $\inf \emptyset = \zeta$ (the second equality follows since every point is regular for Brownian motion, see for example [24, Remark 8.2]).

1.2.2 Superharmonic functions

We will use the fact that value functions of various optimal stopping problems for these subprocesses are superharmonic (see e.g. [10] and Proposition 3.1). Let $E \subseteq [0, 1]$, $A \in \mathcal{B}(E_\Delta)$, and write $T$ for the set of all $\mathbb{P}$-stopping times with values in $\mathbb{R}_+ \cup \{\infty\}$.

**Definition 1.** A measurable function $\phi : E_\Delta \to \mathbb{R}$ is said to be superharmonic (resp. harmonic) on $A$ if for every $x \in E$ and $\tau \in T$:

$$\phi(x) \geq (\text{resp. } =) \mathbb{E}^x[\phi(X_{\tau \wedge D_A})].$$

A measurable function $\phi : E_\Delta \to \mathbb{R}$ is said to be subharmonic on $A$ if $-\phi$ is superharmonic on $A$, and harmonic on $A$ if it is both superharmonic and subharmonic on $A$. If $A = E$ then the term superharmonic, subharmonic, or harmonic is used as appropriate.

Using the strong Markov property, one can show that (see [27] p. 561) for details): if $\phi$ is superharmonic then

$$\phi(X_\nu) \geq \mathbb{E}^x[\phi(X_\rho)|\mathcal{F}_\nu], \quad \text{a.s. } \forall \rho, \nu \in T \text{ such that } \nu \leq \rho.$$

(1.8)

In other words, $\phi$ is a superharmonic function if and only if $(\phi(X_t))_{t \geq 0}$ is a strong supermartingale with respect to $\mathbb{F}$. Here the qualifier ‘strong’ refers to the extension of the supermartingale property to stopping times. In our setup the superharmonic functions on $E$ are also equivalent to the strongly supermedian functions on $E$ (see for example [9], [20] and [23]), as follows. Taking $A = E$ and $\tau = \zeta$ in Definition 1 the convention $\phi(\Delta) = 0$ implies that the superharmonic functions $\phi$ on $E$ are non-negative and are therefore strongly supermedian. Moreover, since $X$ is a subprocess of Brownian motion, superharmonic (respectively subharmonic and harmonic) functions are concave (resp. convex, linear) on convex subsets of $E$ (see [10] p. 179).

We will make repeated use of the following transformation.

**Definition 2.** Given $A \in \mathcal{B}(E_\Delta)$ and a bounded measurable function $\phi : E_\Delta \to \mathbb{R}$, and recalling the first entrance time defined in (1.7), define $\phi_A : E_\Delta \to \mathbb{R}$ by

$$\phi_A(x) := \mathbb{E}^x[\phi(X_{\tau_A})].$$

(1.9)

It is not difficult to show (using the strong Markov property) that for any measurable function $\phi$, the function $\phi_A$ is harmonic on $A^c$, and is superharmonic if $\phi$ is superharmonic. Moreover, it is continuous whenever $\phi$ is continuous and $A$ is closed in $E$ [30].
1.2.3 Quasi-concavity

For use in the existence results below, we recall the definition and some properties of quasi-concave functions (see e.g. [8, Chapter 3.4]). The extended real line will be denoted by $\mathbb{R} = [-\infty, +\infty]$.

**Definition 3.** Let $D \subseteq \mathbb{R}$ be convex. A function $F: D \rightarrow \bar{\mathbb{R}}$ is said to be quasi-concave if for every $\alpha \in \mathbb{R}$ the superlevel sets $L_\alpha^+$ defined by

$$L_\alpha^+ = \{ x \in D: F(x) \geq \alpha \}$$

are convex. If the same statement holds but with the sets $\{ x \in D: F(x) > \alpha \}$ then $F$ is said to be strictly quasi-concave. A function $F$ is said to be (strictly) quasi-convex on a convex domain $D$ if and only if $-F$ is (strictly) quasi-concave.

All concave functions are quasi-concave. Moreover a function $F: D \rightarrow \bar{\mathbb{R}}$ is quasi-concave if and only if $D$ is convex and for any $x_1, x_2 \in D$ and $0 \leq \theta \leq 1$ we have

$$F(\theta x_1 + (1-\theta) x_2) \leq \min(F(x_1), F(x_2)). \quad (1.10)$$

If (1.10) holds with strict inequality then $F$ is strictly quasi-concave.

The remainder of this paper is organised as follows. In Section 2 the two game settings are presented and connected. Useful alternative expressions for the expected payoffs in the Dynkin game, as optimal stopping problems for subprocesses, are developed in Section 3, and our main existence and uniqueness results follow in Sections 4 and 5. Finally, in Section 6 we present an extension for a more complex equilibrium structure.

2 Two games

2.1 Generalised Nash equilibrium

In the $n$-player generalised game each player’s set of available strategies, or feasible strategy space, depends on the strategies chosen by the other $n-1$ players. The case $n=2$ is as follows. Player $i \in \{1, 2\}$ has a strategy space $S_i$ and a set-valued map $K_i: S_{-i} \rightharpoonup S_i$ determining their feasible strategy space. Denoting a generic strategy for player $i$ by $s_i$, a strategy pair $(s_1, s_2)$ is then feasible if $s_i \in K_i(s_{-i})$ for $i = 1, 2$. Setting $S_1 = [0, a]$ and $S_2 = [b, 1]$, the pair of mappings $K_1: [b, 1] \rightharpoonup [0, a]$ and $K_2: [0, a] \rightharpoonup [b, 1]$ will be given by

$$K_1(y) = [0, y \wedge a],$$

$$K_2(x) = [x \lor b, 1], \quad (2.1)$$

where $a$ and $b$ are given constants lying in the interval $(0, 1)$. That is, the feasible strategy pairs are given by the convex, compact set

$$C = \{ (x, y) \in [0, a] \times [b, 1]: x \leq y \}. \quad (2.2)$$

This choice of $C$ will be appropriate for equilibria of the threshold form (1.2) in the Dynkin game. (The set $C$ will be modified in Section 6 below, where an example of a more complex equilibrium is studied). Writing $U_i: C \rightarrow \mathbb{R}$ for the utility function of player $i$, the generalised Nash equilibrium problem is then given by:

**Definition 4** (GNEP, $n=2$). Find $s^* = (s_1^*, s_2^*) \in C$ which is a Nash equilibrium, that is:

$$\begin{align*}
U_1(s^*) &= \sup_{(s_1, s_2) \in C} U_1(s_1, s_2), \\
U_2(s^*) &= \sup_{(s_1^*, s_2) \in C} U_2(s_1^*, s_2). \quad (2.3)
\end{align*}$$
In the proofs below it will be convenient to write $S := S_1 \times S_2$. We will also make use of the following definition:

**Definition 5.** Let $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ and $w \in \mathbb{R}$. Then for each $i \in \{1, \ldots, n\}$ we will write $(w, s_{-i})$ for the vector $s$ modified by replacing its $i$th entry with $w$.

### 2.2 Optimal stopping

We also consider a Dynkin game in which two players observe the Brownian motion subprocess $X$ of Section 1.2.1. Each player can stop the game and receive a reward (which may be positive or negative) depending on the process value and on who stopped the game first. More precisely we consider only pure strategies: that is, each player $i \in \{1, 2\}$ chooses a stopping time $\tau_i$ lying in $T$ as their strategy. Let $f_i$, $g_i$, and $h_i$ be real-valued reward functions on $E$ which respectively determine the reward to player $i$ from stopping first, second, or at the same time as the other player. For convenience we will refer to the $f_i$ as the leader reward functions and to the $g_i$ as the follower reward functions. Given a pair of strategies $(\tau_1, \tau_2)$ and recalling the payoff defined in (1.1), we denote the expected payoff to player $i$ by

$$M^x_i(\tau_1, \tau_2) = \mathbb{E}_x[\mathcal{J}_i(\tau_1, \tau_2)].$$  \hspace{1cm} (2.4)

The problem of finding a Nash equilibrium for this Dynkin game is then:

**Definition 6 (DP).** Find a pair $(\tau_1^*, \tau_2^*) \in T \times T$ such that for every $x \in E$ we have:

$$\begin{align*}
M^x_1(\tau_1^*, \tau_2^*) &= \sup_{\tau_1 \in T} M^x_1(\tau_1, \tau_2^*) \\
M^x_2(\tau_1^*, \tau_2^*) &= \sup_{\tau_2 \in T} M^x_2(\tau_1^*, \tau_2).
\end{align*}$$  \hspace{1cm} (2.5)

If $\tau_1^* = D_{S_1}$ and $\tau_2^* = D_{S_2}$ with $S_1, S_2 \in \mathcal{B}(E_{\Delta})$, then the Nash equilibrium $(D_{S_1}, D_{S_2})$ is said to be Markovian.

### 2.3 Linking the games

We now present the link between the games in the case $n = 2$ and $E = (0, 1)$, which is the setting used in the rest of the paper (with the exception of Section 6 where $n = 3$). The idea is that threshold-type solutions to the DP can be characterised by the slopes $U_1(x, y)$ and $U_2(x, y)$ of certain secant lines. This gives nothing else than a deterministic game, which may be studied in the above generalised setting in order to discover additional novel equilibria. We will close this section by illustrating that this link between the DP and GNEP does not preserve the zero-sum property.

#### 2.3.1 Construction of utility functions for the GNEP

For $(x, y) \in [0, 1]^2$ we define

$$U_1(x, y) = \begin{cases} f_1(x) - g_1(\max(y, 1) - y), & x < y, \\ -\infty, & \text{otherwise,} \end{cases}$$  \text{and}  \hspace{1cm} U_2(x, y) = \begin{cases} f_2(y) - g_2(\max(0, x) - y), & x < y, \\ -\infty, & \text{otherwise,} \end{cases}$$

where for $A \in \mathcal{B}(E_{\Delta})$, the function $g_i, A$ is obtained by taking $\phi = g_i$ in Definition 2. To ensure that these utility functions are continuous and bounded above on $C$ we strengthen Assumption 1 to:

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Assumption 1' If $b \leq a$ then $g_i > f_i$ on $[b, a]$ for $i = 1, 2$.

We now record comments on this choice of utility functions in the GNEP:

(i) The rationale for the form (2.6) of $U_1$ and $U_2$ is as follows. Lemma 3.2 below will confirm that in equilibrium, player 1’s strategy is characterised by an optimal stopping problem with obstacle $f_1 - g_1$, whose geometry determines the solution (in the sense of [10], for example). In particular we show in Theorem 4.5 that, for threshold strategies, the function $U_1$ characterises the solution. Similar comments of course apply to player 2.

(ii) The GNEP characterisation does not assume smoothness but, if the reward functions are differentiable, then the double smooth fit condition (that is, the differentiability of the players’ equilibrium payoffs across the thresholds $\ell$ and $r$ respectively) follows as a corollary.

(iii) Later, in Section 6, we show how additional functions $U_i$ may be added to characterise more complex equilibria than the threshold type, leading to GNEPs with more than two players.

2.3.2 Remark on the zero-sum property

It is interesting to note that the zero-sum property in the DP does not imply the same for the GNEP and vice versa. Suppose that the GNEP (2.6) has zero sum: that is,

$$\sum_{i=1}^{2} U_i(x, y) = 0, \quad \forall (x, y) \in S. \quad (2.7)$$

By Definition 2 the functions $g_{1,[y,1]}$ and $g_{2,[0,x]}$ are given by:

$$g_{1,[y,1]}(x) = \begin{cases} g_1(y) \cdot \frac{x}{y}, & \forall x \in [0, y) \\ g_1(x), & \forall x \in [y, 1] \end{cases}, \quad (2.8)$$

$$g_{2,[0,x]}(y) = \begin{cases} g_2(y), & \forall y \in [0, x) \\ g_2(x) \cdot \frac{1-y}{1-x}, & \forall y \in (x, 1] \end{cases}, \quad (2.9)$$

and we recall that $f_1(0) = g_2(0) = g_1(1) = f_2(1) = 0$. Then considering separately the case $x = 0$, $y \in [b, 1]$ in (2.7) and the case $y = 1$, $x \in [0, a]$, we conclude that $f_1(x) = f_2(y) = 0, \forall (x, y) \in S$. Then in the DP, any nonzero choice of the reward functions $g_i$ satisfying Assumption 1 results in a game with $f_i \neq -g_{-i}$ and hence is nonzero sum.

On the other hand, suppose that $a < b$ and consider the zero-sum DP with reward functions

$$f_1(x) = \begin{cases} x(a - x), & x \in [0, a] \\ (1-x)(a-x), & x \in (a, 1], \end{cases}$$

$$g_1(x) = \begin{cases} x(b - x), & x \in [0, b) \\ (1-x)(b-x), & x \in [b, 1], \end{cases}$$

$$f_2 = -g_1, \quad g_2 = -f_1, \quad h_1 = -h_2.$$

Then for $(x, y) \in S$ the sum of the payoffs in the GNEP is

$$\sum_{i=1}^{2} U_i(x, y) = x \left( \frac{a-x}{y-x} \right) \left( 1 + \frac{1-y}{1-x} \right) - \left( \frac{(1-y)(b-y)}{y-x} \right) \left( \frac{y+x}{y} \right),$$

which is strictly positive for $(x, y) \in \{0, a\} \times (b, 1)$, and so the GNEP is not zero sum.
3 Optimal stopping of a subprocess

In this section we provide three equivalent expressions for expected payoffs in the Dynkin game, as optimal stopping problems for subprocesses. These will be used repeatedly to establish the existence and uniqueness results of Sections 4 and 5.

3.1 Preliminaries

We begin by confirming that without loss of generality all reward functions may be set equal to zero on \( \Delta \). Suppose instead that the reward is to be nonzero on \( \Delta \). This could be accommodated by taking the following modified form for the expected payoffs:

\[
M_E^f(\tau, \sigma) = \mathbb{E}^x\left[ \{ f(W_{\tau}) \mathbf{1}_{\{\tau < \sigma\}} + g(W_{\sigma}) \mathbf{1}_{\{\tau > \sigma\}} + h(W_{\sigma}) \mathbf{1}_{\{\tau = \sigma\}} \} \mathbf{1}_{\{\tau \wedge \sigma < D_E^c\}} \right]
\]

(3.1)

where \( \tau, \sigma \) denote the players’ stopping times and \( H \) specifies the reward received at the boundaries of \( E \). This is for example the approach taken in [3], where it is assumed that \( f(x) = g(x) = H(x) \), \( x \in E^c \). Note also that by construction only the values of \( H \) on \( E^c \) are relevant. Now taking \( \phi = H \) and \( A = E^c \) in Definition 2 and using the strong Markov property we can show that,

\[
M_E^f(\tau, \sigma) - H_{E^c}^c(x) = \mathbb{E}^x\left[ \{ f - H_{E^c}^c \}|W_{\tau}) \mathbf{1}_{\{\tau < \sigma\}} + [g - H_{E^c}^c]|W_{\sigma}) \mathbf{1}_{\{\tau > \sigma\}}
\]

\[
+ [h - H_{E^c}^c]|X_{\sigma}) \mathbf{1}_{\{\tau = \sigma\}} \} \mathbf{1}_{\{\tau \wedge \sigma < D_E^c\}} \right],
\]

(3.2)

where the second equality comes from (3.1) above. The right-hand side of (3.2) is equal to the expected payoff (3.1) when the reward functions \( f, g, h \) and \( H \) are taken to be \( \tilde{f} = f - H_{E^c} \), \( \tilde{g} = g - H_{E^c} \), \( \tilde{h} = h - H_{E^c} \) and \( \tilde{H} \equiv 0 \) respectively. It may therefore be assumed without loss of generality in the proofs below that the reward functions are zero on \( \Delta \) (and indeed on \( E^c \)).

We will be interested in optimally stopping the subprocess \( X^{A^c} \). For this, define the set of stopping times \( T_{0,DA} := \{ \tau \in T : 0 \leq \tau \leq D_A \} \). The proof of the following useful result can be found in, for example, [6] and [14]:

**Proposition 3.1.** For \( A \in B(E_\Delta) \) and functions \( f, g \) and \( h \) satisfying Assumption 1 the map

\[
x \mapsto \tilde{V}(x) := \sup_{\tau \in T} \mathbb{E}^x\left[ f(X_{\tau}) \mathbf{1}_{\{\tau < D_A\}} + g(X_{D_A}) \mathbf{1}_{\{D_A < \tau\}} + h(X_{D_A}) \mathbf{1}_{\{\tau = D_A\}} \right],
\]

is measurable and satisfies:

\[
\forall \rho \in T_{0,DA} : \mathbb{E}^x[\tilde{V}(X_{\rho})] \leq \tilde{V}(x) \quad \forall x \in E.
\]

(3.3)

In other words, \( x \mapsto \tilde{V}(x) \) is superharmonic on \( A^c \).

3.2 Single player problem

Suppose that in the Dynkin game, the strategy of player \(-i\) is specified by a set \( A \in B(E_\Delta) \) on which that player stops. The next lemma expresses the resulting optimisation problem for player \( i \) in terms of optimal stopping problems of different kinds for the subprocess \( X^{A^c} \).
Lemma 3.2. For \( x \in E \) consider the problems
\[
V^A(x) := \sup_{\tau \in \mathcal{T}} M^A(\tau, D_A), \quad \tau \in \mathcal{T} 
\]
\[
\bar{V}^A(x) := \sup_{\tau \in \mathcal{T}} \bar{M}^A(\tau, D_A), \quad \tau \in \mathcal{T} 
\]
\[
\bar{V}^A(x) := \sup_{\tau \in \mathcal{T}} \bar{M}^A(\tau, D_A), \quad \tau \in \mathcal{T} 
\]
where for \( \tau \in \mathcal{T} \) we have
\[
M^A(\tau, D_A) := \mathbb{E}^\tau \left[ f(X_\tau) \mathbf{1}_{\{\tau < D_A\}} + g(X_{\tau,D_A}) \mathbf{1}_{\{\tau = D_A\}} + h(X_{\tau}) \mathbf{1}_{\{\tau = D_A\}} \right], \quad \tau \in \mathcal{T} 
\]
\[
\bar{M}^A(\tau, D_A) := \mathbb{E}^\tau \left[ f(X_\tau) \mathbf{1}_{\{\tau < D_A\}} + g(X_{\tau,D_A}) \mathbf{1}_{\{\tau = D_A\}} \right], \quad \tau \in \mathcal{T} 
\]
and \( f, g \) and \( h \) are functions satisfying Assumption \( \mathcal{J} \). Then, recalling Definition \( \mathcal{J} \) we have
\[
V^A(x) = \bar{V}^A(x) = g_A(x) + \bar{V}^A(x). \quad (3.10)
\]

Proof. Let \( \tau \in \mathcal{T}, x \in E \) be arbitrary. We have \( \bar{M}^A(\tau, D_A) \geq M^A(\tau, D_A) \) and therefore \( \bar{V}^A(x) \geq V^A(x) \). To show the reverse inequality, first recall that \( x \mapsto V^A(x) \) is measurable. By assumption we have \( V^A \geq f \) on \( E \), so that \( V^A(X_\tau) \mathbf{1}_{\{\tau < D_A\}} \geq f(X_\tau) \mathbf{1}_{\{\tau < D_A\}} \) a.s., while from the strong Markov property we have \( V^A(X_{\tau,D_A}) = g(X_{\tau,D_A}) \) a.s.. It follows from (3.8) and superharmonicity that
\[
\bar{M}^A(\tau, D_A) \leq \mathbb{E}^\tau \left[ V^A(X_{\tau,D_A}) \right] \leq V^A(x),
\]
and taking the supremum over \( \tau \) we have \( \bar{V}^A(x) = V^A(x) \). Finally, recalling Definition \( \mathcal{J} \) we have
\[
\bar{M}(\tau, D_A) - g_A(x) = \mathbb{E}^\tau \left[ \{ f - g_A \} (X_\tau) \mathbf{1}_{\{\tau < D_A\}} \right]. \quad (3.11)
\]

Remark 3.3. It follows from (3.10) that
\[
V^A(x) = f(x) \iff \bar{V}^A(x) = f(x) - g_A(x).
\]
That is, defining the stopping region to be the subset of \( A^c \) on which the obstacle equals the value function, the optimal stopping problems \( V^A(x) \) and \( \bar{V}^A(x) \) have identical stopping regions. An easy consequence is that if \( x \in A^c \) lies in either stopping region then \( f(x) \geq g_A(x) \), and that if \( f \leq g_A \) on \( A^c \) then \( \tau = D_A \) is optimal in (3.6).

4 Existence of equilibria

In this section we exploit the link between the games to show, firstly, that the existence of a solution to the GNEP with utility functions given by \( (2.6) \) implies the existence of a threshold-type solution to the DP (Theorems 4.4 and 4.5). These results are then applied to show the existence of novel Nash equilibria in the DP. More precisely we will show that the following condition on the geometry of the reward functions is sufficient for the existence of an equilibrium:

Condition G1. There exist points \( a \in (0, 1) \) and \( b \in (0, 1) \) such that

(i) \( f_1 \) is concave on \( [0, a] \) and is convex on \( [a, 1] \)
(ii) \( f_2 \) is convex on \( [0, b] \) and is concave on \( [b, 1] \)
(iii) If \( b \leq a \), then \( f_i < g_i \) on \( [b, a] \) for \( i = 1, 2 \).
The case $a > b$ is novel when compared with the existing literature. It is interesting to note that in the case $a \leq b$, which is analysed in [3] and [11], the generalised problem (2.3) reduces to a classical game (that is, where each player’s strategy space does not depend on the other player’s chosen strategy). This is because the dependency between the players’ strategies (which is specified by the choice of $C$) allows some additional control on the equilibria, which is required when $a > b$. The case when at least one of the functions $f_i$ is not differentiable is also novel.

4.1 Preliminaries

A solution to the GNEP is known to exist under the following condition (see for example [1] and [15]):

**Condition U.**

(i) For each fixed $s_2 \in \mathcal{S}_2$, the mapping $s_1 \mapsto U_1(s_1, s_2)$ is quasi-concave on $K_1(s_2)$. For each fixed $s_1 \in \mathcal{S}_1$, the mapping $s_2 \mapsto U_2(s_1, s_2)$ is quasi-concave on $K_2(s_1)$.

(ii) The utility functions $s \mapsto U_i(s)$ for $i = 1, 2$ are continuous in $s = (s_1, s_2)$.

For convenience we record the necessary argument here:

**Lemma 4.1.** Suppose Assumption 1’ and Condition U hold. Then there exists a solution $(s_1^*, s_2^*) \in \mathcal{C}$ to the GNEP (2.3) satisfying $s_1^* < s_2^*$.

**Proof.** For $i = 1, 2$ the correspondence $K_i$ is compact and convex valued. Furthermore, using the notion of continuity for set-valued maps in [28], we can confirm that $K_1$ and $K_2$ are continuous. Under the present hypotheses, $U_i$ is continuous on $S$ and has the quasi-concavity property specified in Condition U. Therefore by Lemma 2.5 in [1], there exists a solution $s^*$ to (2.3). From the construction (2.6), this solution must satisfy $s_1^* < s_2^*$.

**Remark 4.2.** For possible extensions of Lemma 4.1 see also [15, 18] and references therein.

Before presenting the main result of this section we need the following fact:

**Lemma 4.3.** Suppose $D \subseteq \mathbb{R}$ is convex, $f : D \to \bar{\mathbb{R}}$ is (strictly) concave, and $\varphi : D \to (0, \infty)$ is linear. Then the function $\frac{f}{\varphi} : D \to \bar{\mathbb{R}}$ is (strictly) quasi-concave.

**Proof.** In the case of concavity, for each $\alpha \in \mathbb{R}$ define a function $F_\alpha : D \to \bar{\mathbb{R}}$ by $F_\alpha(x) = f(x) - \alpha \varphi(x)$. This function is concave on $D$, and therefore quasi-concave, which means the superlevel set $\{x \in D : F_\alpha(x) \geq 0\}$ is convex for every $\alpha \in \mathbb{R}$. The function $\frac{f}{\varphi}$ is quasi-concave on $D$ since for every $\alpha \in \mathbb{R}$,

$$\left\{ x \in D : \left(\frac{f}{\varphi}\right)(x) \geq \alpha \right\} = \left\{ x \in D : f(x) \geq \alpha \varphi(x) \right\} = \left\{ x \in D : F_\alpha(x) \geq 0 \right\}.$$

The proof for strictly concave $f$ follows in the same way.

4.2 Existence results

The GNEP is used in this section to establish the existence of equilibria in the DP. This both allows $a > b$ in Condition G1 and avoids the need for smoothness assumptions.

**Theorem 4.4.** Under Condition G1, there exists a pair $(\ell_*, r_*) \in [0, a] \times [b, 1]$ such that $(D_{[0, \ell_*]}, D_{[r_*, 1]})$ is a solution to the DP.
Proof. We begin by noting that for each $r \in [0,1]$ and $\ell \in [0,1]$,
\[
\sup_{x \in [0,r]} U_1(x, r) \leq \sup_{x \in [0,a]} U_1(x, r), \tag{4.1}
\]
\[
\sup_{x \in (\ell,1]} U_2(\ell, x) \leq \sup_{x \in [b,1]} U_2(\ell, x). \tag{4.2}
\]
For $r \in (a,1]$, eq. \((4.1)\) follows from the convexity of $f_1 - g_{1,[r,1]}$ on $[a,r]$ and the fact that $f_1(r) \leq g_1(r) = g_{1,[r,1]}(r)$:
\[
\frac{f_1(x) - g_{1,[r,1]}(x)}{r-x} \leq \frac{f_1(a) - g_{1,[r,1]}(a)}{r-a} + \left(\frac{f_1(r) - g_{1,[r,1]}(r)}{r-a}\right) \left(\frac{x-a}{r-x}\right) \\
\leq \frac{f_1(a) - g_{1,[r,1]}(a)}{r-a}, \quad \forall x \in (a,r).
\]
Similar reasoning establishes \((4.2)\).

Using Condition G1 and Lemma 4.3 we can verify the hypotheses of Lemma 4.1 and assert the existence of a pair $(\ell, r) \in [0,a] \times [b,1]$ with $\ell < r$ such that
\[
\begin{cases}
U_1(x, r) \leq U_1(\ell, r), & \forall x \in [0,r \wedge a], \\
U_2(\ell, y) \leq U_2(\ell, r), & \forall y \in [\ell \vee b, 1].
\end{cases} \tag{4.3}
\]
The pair $(\ell, r)$ that satisfies \((4.3)\) therefore also satisfies
\[
\begin{align*}
U_1(x, r) & \leq U_1(\ell, r), & \forall x \in [0,r), \\
U_2(\ell, y) & \leq U_2(\ell, r), & \forall y \in (\ell, 1],
\end{align*} \tag{4.4}
\]
and the result follows from the next theorem. \(\Box\)

**Theorem 4.5.** For every $r \in [b,1]$, a point $\ell_r \in [0,a]$ with $\ell_r < r$ satisfies \((4.1)\) if and only if
\[
V_1^{[r,1]}(x) := \sup_{\tau_1 \in T} M_1^\tau(\tau_1, D_{[r,1]}) = M_1^r(D_{[0,\ell_r]}, D_{[r,1]}), \quad \forall x \in [0,1]. \tag{4.6}
\]
Similarly, for every $\ell \in [0,a]$, a point $r_\ell \in [b,1]$ with $\ell < r_\ell$ satisfies \((4.2)\) if and only if
\[
V_2^{[0,\ell]}(x) := \sup_{\tau_2 \in T} M_2^\tau(D_{[0,\ell]}, \tau_2) = M_2^\ell(D_{[0,\ell]}, D_{[r_\ell,1]}), \quad \forall x \in [0,1]. \tag{4.7}
\]
**Proof.** We only show \((4.4) \iff (4.6)\), since \((4.5) \iff (4.7)\) follows by similar arguments. Let $r \in [b,1]$ and $\ell_r \in [0,a]$ with $\ell_r < r$ be given. We will make repeated use of the function
\[
u_r(x) := M_1^r(D_{[0,\ell_r]}, D_{[r,1]}) - g_{1,[r,1]}(x) = \begin{cases} f_1(x) - g_{1,[r,1]}(x), & x \in [0,\ell_r), \\
(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)) \frac{x-\ell_r}{r-\ell_r}, & x \in [\ell_r, r), \\
0, & x \in [r, 1], \end{cases} \tag{4.8}
\]
where the middle line is a straightforward consequence of the identities in Appendix B and the fact that, for $x \in [0, r]$, we have
\[
g_1(r) \frac{x - \ell_r}{r - \ell_r} = g_{1,[r,1]}(x) = g_1(r) \left(\frac{x - \ell_r}{r - \ell_r} + \frac{x - \ell_r}{r - \ell_r}\right) = g_1(r) \left(\frac{r(x - \ell_r)}{r(r - \ell_r)} - 1\right) = -g_1(r) \left(\frac{r - x}{r - \ell_r}\right). \tag{4.9}
\]
Nonzero-sum games of optimal stopping and generalised Nash equilibrium

 Sufficiency ($
\iff$
).
Suppose that (4.6) is satisfied. Substituting this in (4.9), dividing both sides of (4.8) by $r - x$ (when $x < r$), and using the definition (2.6) of $U_1$, we obtain

$$
\frac{V_1^{[r,1]}(x) - g_{1,[r,1]}(x)}{r - x} = \begin{cases} 
U_1(x, r), & \forall x \leq \ell_r \\
U_1(\ell_r, r), & \forall \ell_r < x < r.
\end{cases}
$$

(4.10)

It is easy to see that $V_1^{[r,1]}(r) = g_1(r) = g_{1,[r,1]}(r)$ and $V_1^{[r,1]}(x) \geq f_1(x)$ for all $x \in [0, r]$. Therefore when $x \in (\ell_r, r)$ we have

$$
U_1(\ell_r, r) \geq U_1(x, r).
$$

To treat the case $x \in [0, \ell_r]$, note from Lemma 3.2 that $x \mapsto V_1^{[r,1]}(x) - g_{1,[r,1]}(x)$ is the value function of an optimal stopping problem for a subprocess as in, for example, [10], and, as such, is non-negative and superharmonic in $(0, r)$. For $0 \leq x < y \leq 1$ define $\tau_{x,y} = D_{x} \land D_{y}$. Using superharmonicity and the fact that $X$ is a positively recurrent diffusion, for every $0 \leq x \leq \ell_r$ we have,

$$
V_1^{[r,1]}(\ell_r) - g_{1,[r,1]}(\ell_r) \geq E^{\ell_r} \left[ V_1^{[r,1]}(X_{\tau_{x,\ell_r}}) - g_{1,[r,1]}(X_{\tau_{x,\ell_r}}) \right]
= \left(V_1^{[r,1]}(x) - g_{1,[r,1]}(x)\right) E^{\ell_r} \left[ 1_{D_x \land D_{\ell_r}} \right]
= \left(V_1^{[r,1]}(x) - g_{1,[r,1]}(x)\right) \frac{r - \ell_r}{r - x}.
$$

(4.11)

Since for all $0 \leq x \leq \ell_r$ we have $V_1^{[r,1]}(x) = f_1(x)$, (4.11) gives

$$
U_1(x, r) \leq U_1(\ell_r, r), \quad \forall x \in [0, \ell_r],
$$

establishing (4.4) with $\ell = \ell_r$.

Necessity ($\implies$).

Suppose that the pair $(\ell_r, r)$ satisfies (4.4) with $\ell = \ell_r$. We will establish (4.6) by showing that

$$
u_r(x) = V_1^{[r,1]}(x) - g_{1,[r,1]}(x), \quad \forall x \in [0, 1].
$$

(4.12)

By construction (4.12) holds for $x \in [r, 1]$, and so we restrict attention to the domain $[0, r]$. By Lemma 3.2 it is sufficient to show that $u_r$ is the value function of the optimal stopping problem on $[0, r]$ with the obstacle $\vartheta := f_1 - g_{1,[r,1]}$. Therefore using Proposition 3.2 in [10], it is enough to show that $u_r$ is the smallest non-negative concave majorant of $\vartheta$ on $[0, r]$. The majorant property on $[\ell_r, r)$ follows from (4.4), which gives

$$
f_1(x) - g_{1,[r,1]}(x) \leq \left( f_1(\ell_r) - g_{1,[r,1]}(\ell_r) \right) \left( \frac{r - x}{r - \ell_r} \right), \quad \forall x \in [0, r],
$$

(4.13)

and the majorant property at $x = r$ follows from recalling that $f_1(r) \leq g_1(r)$. For nonnegativity we first recall that the reward functions are null at the boundaries, so taking $x = 0$ in (4.13) gives $0 \leq f_1(\ell_r) - g_{1,[r,1]}(\ell_r) = u_r(\ell_r)$. Combining this with the fact that $u_r$ equals the obstacle on $[0, \ell_r]$, and hence is concave there, establishes nonnegativity. For concavity we note that $u_r$ is a straight line on $[\ell_r, r]$, so it remains only to consider any $x_1 \in [0, \ell_r)$ and $x_2 \in (\ell_r, r]$. Then
we have
\[\frac{x_2 - \ell_r}{x_2 - x_1} u_r(x_1) + \frac{x_2 - \ell_r}{x_2 - x_1} u_r(x_2) = \frac{x_2 - \ell_r}{x_2 - x_1} \left[ f_1(x_1) - g_{1,[r,1]}(x_1) \right] + \frac{x_2 - \ell_r}{x_2 - x_1} \left[ f_1(\ell_r) - g_{1,[r,1]}(\ell_r) \right] \left( \frac{r - x_1}{r - \ell_r} \right) \]
\[\leq \frac{x_2 - \ell_r}{x_2 - x_1} \left[ f_1(\ell_r) - g_{1,[r,1]}(\ell_r) \right] \left( \frac{r - x_1}{r - \ell_r} \right) + \frac{x_2 - \ell_r}{x_2 - x_1} \left[ f_1(\ell_r) - g_{1,[r,1]}(\ell_r) \right] \left( \frac{r - x_2}{r - \ell_r} \right) = f_1(\ell_r) - g_{1,[r,1]}(\ell_r) = u_r(\ell_r),\]
where the inequality follows from (4.4). Finally, since \(u_r\) equals the obstacle on \([0, \ell_r]\) and is a straight line on \([\ell_r, r]\), it is smaller than any other nonnegative concave majorant on \([0, r]\).

**Remark 4.6.** Theorem 4.4 remains valid under Condition G1 and the following condition which is weaker than Assumption 1:

\[f_i \leq g_i \text{ on } S^{-i}.\]

## 5 Stability and uniqueness results

In this section we exploit the above connection to obtain additional novel results for Nash equilibria in the DP. We define a concept of stability and provide a sufficient condition under which it holds locally (Corollary 5.2), showing in Theorem 5.4 that this condition always holds in the particular case of zero-sum Dynkin games. By establishing global stability, Theorem 5.5 provides sufficient conditions for uniqueness of the threshold-type equilibrium of Theorem 4.4 among the Markovian strategies. Finally, Theorem 5.8 transfers another uniqueness result for the GNEP to the DP.

### 5.1 Policy iteration

We will apply the *Gauss-Seidel policy iteration* or *tâtonnement process* [5, 16] to the GNEP. This iteration scheme has previously been used for Dynkin games in [9] and [19] and, outside the Markovian framework, in [17]. Throughout Section 5 for convenience we will assume the following Condition G1’, rather than G1:

**Condition G1’.** Condition G1 holds, with:

1) \(a < b\),

2) strict convexity and strict concavity,

3) \(f_i, g_i \in C^2[0, 1]\), and

4) For all \((x, y) \in [0, a] \times [b, 1]\) there exists \((\hat{x}, \hat{y}) \in (0, a] \times [b, 1)\) with \(f_1(\hat{x}) > g_1(\hat{y}) \cdot \frac{\hat{x}}{\hat{y}}\) and \(f_2(\hat{y}) > g_2(\hat{x}) \cdot \frac{1 - \hat{y}}{1 - \hat{x}}\).

We emphasise that these assumptions are for ease of exposition. Parts 1) and 3) imply that the GNEP utility functions are finite and smooth on \(S\), which is convenient for the transfer of results from generalised games. Part 2) says that \(f_1\) is strictly concave on \([0, a]\) and strictly convex on \([a, 1]\), and \(f_2\) is strictly convex on \([0, b]\) and strictly concave on \([b, 1]\). This ensures that iteration (i) below is well defined. Part 4) removes the need to consider the points 0 and 1 as candidate thresholds, which is convenient since the principle of smooth fit (used below) may break down there. Recalling the equality (3.10), this is straightforward to see from (3.6), (3.9).
and \((2.8)-(2.9)\). Part 4) similarly ensures that threshold-type equilibria have their thresholds in \((0, 1)\) and not at either boundary 0 or 1.

Taking \(\ell(1) \in [0, a]\), we consider the following two iteration schemes:

(i) **In the GNEP**: taking \(r(1) = \arg\max_{y \in [b, 1]} U_2(\ell(1), y)\), for \(n \geq 2\) define
\[
\ell(n) = \arg\max_{x \in [0, a]} U_1(x, r^{(n-1)}), \quad r^{(n)} = \arg\max_{y \in [b, 1]} U_2(\ell(n), y). \tag{5.1}
\]

(ii) **In the DP**: taking \(A_1 = [0, \ell(1)]\), for \(n \geq 1\) define
\[
(i) \quad V_{2n}(x) = \sup_{\tau} \mathbb{E}^x [f_2(X_{\tau}) \mathbb{1}_{\{\tau < D_{A_{2n-1}}\}} + g_2(X_{D_{A_{2n-1}}}) \mathbb{1}_{\{D_{A_{2n-1}}\}}];
(ii) \quad A_{2n} = \{x \in [0, 1] \setminus A_{2n-1} : V_{2n}(x) = f_2(x)\},
(iii) \quad V_{2n+1}(x) = \sup_{\tau} \mathbb{E}^x [f_1(X_{\tau}) \mathbb{1}_{\{\tau < D_{A_{2n}}\}} + g_1(X_{D_{A_{2n}}}) \mathbb{1}_{\{D_{A_{2n}}\}}], \tag{5.2}
(iv) \quad A_{2n+1} = \{x \in [0, 1] \setminus A_{2n} : V_{2n+1}(x) = f_1(x)\}.
\]

We will call a solution \(s^* = (\ell^*, r^*)\) to the GNEP (2.3) **globally stable** if for any \(\ell(1) \in [0, a]\) the iteration (5.1) satisfies \(\ell(n) \to \ell^*\) and \(r(n) \to r^*\), and **locally stable** if this convergence holds only for \(\ell(1)\) in a neighbourhood of \(\ell^*\). Similarly we call a threshold-type solution \(s' = (D_{[0, \ell]}, D_{[\ell, 1]})\) to the DP (2.5) globally stable if for any \(\ell(1) \in [0, a]\) the iteration (5.2) satisfies
\[
\liminf_{n \to \infty} A_{2n-1} = \limsup_{n \to \infty} A_{2n-1} = [0, \ell'],
\]
and locally stable if convergence holds only for \(\ell(1)\) in a neighbourhood of \(\ell'\).

### 5.2 Local stability

We will appeal to the following local stability result for the GNEP:

**Proposition 5.1** (Theorem 1.2.3, [21]). Suppose that Condition G1' holds and that \((\ell_*, r_*) \in (0, a) \times (b, 1)\) is a solution to the GNEP. For \(w \in S_1\) set
\[
\bar{y} = \bar{y}(w) = \arg\max_{y \in S_2} U_2(w, y), \quad \bar{x} = \bar{x}(w) = \arg\max_{x \in S_1} U_1(x, \bar{y}(w)), \tag{5.3}
\]
and
\[
T(w, \bar{x}, \bar{y}) := \frac{\partial_{xy} U_1(\bar{x}, \bar{y})}{\partial_{xx} U_1(\bar{x}, \bar{y})} \frac{\partial_{xy} U_2(w, \bar{y})}{\partial_{yy} U_2(w, \bar{y})},
\]

If it is true that
\[
\rho_0 = |T(\ell_*, \ell_*, r_*)| < 1, \tag{5.4}
\]
then there exists \(\delta > 0\) such that \(\forall \ell(1) \in [0, a]\) satisfying \(|\ell(1) - \ell_*| < \delta\), the sequence \(\{\ell^{(n)}\}_{n \geq 1}\) in (5.1) converges to \(\ell_*\). The convergence is exponential: for any \(\varepsilon > 0\) there exists a positive constant \(c(\ell(1); \varepsilon)\) such that
\[
|\ell(n) - \ell_*| \leq c(\ell(1); \varepsilon)(\rho_0 + \varepsilon)^n. \tag{5.5}
\]

Our next result translates this into a local stability result for the DP.
Corollary 5.2. Suppose Condition G1’ holds and that \((D_{[0, \ell_*]}, D_{[r_*, 1]})\) is a solution to the DP such that \((\ell_*, r_*) \in (0, a) \times (b, 1)\) and (5.4) holds. Then the equilibrium \((D_{[0, \ell_*]}, D_{[r_*, 1]})\) in the DP is locally stable.

Proof. We have from Theorem 4.5 that \((\ell_*, r_*) \in (0, a) \times (b, 1)\) is a solution to the GNEP. Applying Proposition 5.1 take \(\ell'(1) \in [0, a]\) satisfying \(|\ell'(1) - \ell_*| < \delta\) and consider the iteration given by (5.1). This yields sequences \((\ell(n)) \to \ell_*\) and \((r(n)) \to r_*\), taking values respectively in \((0, a)\) and \((b, 1)\). Lemma 3.2 and Theorem 4.5 then show that the stopping time \(\ell(n)\) is optimal in (5.2-iii). Similarly, the stopping time \(\ell(n+1)\) is optimal in (5.2-iii).

Next we establish that the stopping region \(A_2\) is given by \([r(1), 1]\). From Remark 3.3 we may study the optimal stopping problem (5.2-i) in either of its equivalent forms (3.4) or (3.6) (taking \(f = f_2\), \(g = g_2\) and \(A = A_1 = [0, \ell(1)]\)). Using (5.4), it is immediate from the strict convexity of the obstacle \(f_2\) on \([\ell(1), b]\) and Dynkin’s formula that \(A_2 \cap [\ell(1), b]\) = ∅. On the other hand, considering problem (3.6) it follows from the strict concavity of the obstacle \(f_2 - g_2, A_1\) on \([b, 1]\) and the smooth fit principle that the obstacle lies strictly below the value function on \([b, r(1)]\), establishing that \(A_2 = [r(1), 1]\). Arguing similarly for \(A_3\) and then proceeding inductively we obtain \(A_{2n+1} = [0, \ell(n+1)]\) and \(A_{2n+2} = [r(n+1), 1]\) for all \(n\).

Remark 5.3. The fact that \(A_1\) is an interval plays no role in the above proof, which only uses the inclusion \(A_1 \subseteq [0, a]\).

Local stability in the zero-sum DP. The link to GNEPs also provides the following result on local stability of equilibria in the zero-sum DP, that is, when \(f_i = -g_{-i}\), \(i \in \{1, 2\}\). The result is novel to the best of our knowledge.

Theorem 5.4. Under Condition G1’ every threshold-type solution of the zero-sum DP is locally stable.

Proof. Let a threshold-type solution \((D_{[0, \ell_*]}, D_{[r_*, 1]})\) be given for the DP. We have \(V_1^{[r_*, 1]} + V_2^{[0, \ell_*]} = 0\). Using the principle of smooth fit we get,

\[
-g_2'(\ell_*) = f_1'(\ell_*) = \frac{g_1(r_*) - f_1(\ell_*)}{r_* - \ell_*} = \frac{-f_2(r_*) + g_2(\ell_*)}{r_* - \ell_*} = -f_2'(r_*) = g_1'(r_*).
\]

Using the expressions for \(U_1\) and \(U_2\) in (2.6), the general expressions for the partial derivatives of the utility functions in Appendix C and the smooth fit principle at \((w, \bar{y})\) and \((\bar{x}, \bar{y})\), one can show that

\[
T(w, \bar{x}, \bar{y}) = \begin{pmatrix} f_1'(\bar{x}) - g_1'(\bar{y}) g_2'(w) - f_2'(\bar{y}) \\ f_1'(\bar{x})(\bar{y} - \bar{x}) f_2'(\bar{y})(\bar{y} - w) \end{pmatrix}.
\] (5.6)

In this zero-sum context we therefore have \(T(\ell_*, \ell_*, r_*) = 0\), and the local stability of the equilibrium point now follows from Proposition 5.1.

5.3 Global stability and uniqueness

There is a stronger version of the criterion (5.4) that guarantees the iteration scheme to converge irrespective of player 1’s initial strategy \(\ell(1) \in [0, a]\). Furthermore, the equilibrium strategy \((\ell_*, r_*)\) thus obtained is unique.

Theorem 5.5. Suppose that Condition G1’ holds and that the reward functions \(f_i\) and \(g_i\), \(i = 1, 2\), satisfy

\[
\sup_{w \in S_1} \left| \frac{f_1'(\bar{x}) - g_1'(\bar{y})}{f_1'(\bar{x})(\bar{y} - \bar{x})} \left( g_2'(w) - f_2'(\bar{y}) \right) \right| < 1, \quad (5.7)
\]
where \( \bar{y} = \bar{y}(w) \) and \( \bar{x} = \bar{x}(w) \) are defined by (5.3). Then there exists \( (\ell^*, r^*_a) \in S \) such that \((D_{[0,\ell]}, D_{[r,1]})\) is a solution to the DP. This solution is stable, and is unique in the class of Markovian strategies \((D_{S_1}, D_{S_2})\) for closed stopping sets \(S_1 \subseteq [0,a] \) and \(S_2 \subseteq [b,1] \).

**Proof.** Under Condition G1’ every solution \((\ell^*, r^*_a)\) to the GNEP lies in \((0, a) \times (b, 1)\). A standard contraction argument then shows that under (5.7), there exists a unique solution \((\ell^*, r^*_a)\) to the GNEP and, further, that it is globally stable (see for example Theorem 1 in [22] or Proposition 4.1 in [5]; see also Theorem 1.2.1 in [21]).

Thus from Theorem 4.5 \((D_{[0,\ell]}, D_{[r,1]})\) is a solution to the DP. The fact that it is stable follows from the corresponding property in the GNEP. Suppose that the DP has another solution \((\ell^*, r^*_a)\) with \(\ell < r\). Again arguing as in Corollary 5.2, the reward function geometry gives \(\ell \in [0, a]\) and \(r \in [b, 1]\). Therefore, \((\ell, r)\) is a solution to the GNEP and we have \(\ell = \ell^*_a\) and \(r = r^*_a\) by uniqueness.

Suppose that \((D_{S_1}, D_{S_2})\) is an equilibrium with closed stopping sets \(S_1 \subseteq [0, a]\) and \(S_2 \subseteq [b, 1]\). Recalling Remark 5.3 now consider applying the iteration (ii) above, modified by choosing \(A_1 = S_1\), to obtain \(A_2 = [r, 1]\), say. Then by optimality \(S_2 \subseteq A_2\). Finally it is not difficult to see from a standard ‘small ball’ argument that the strict concavity of \(f_2\) on \([b, 1]\) implies that \(A_2 \setminus S_2 = \emptyset\). We conclude similarly that \(A_1\) has the form \([0, \ell]\), completing the proof.

**Remark 5.6.** The sets \(S_1\) and \(S_2\) in Theorem 5.5 are closed in order to avoid trivialities, since every point is regular for standard Brownian motion. Note that the theorem establishes uniqueness among the Markovian strategies, rather than uniqueness among the subset of threshold-type strategies (cf. [11]).

### 5.4 Examples

We begin this section by constructing an example DP satisfying the global stability condition (5.7). This example is then used to derive a second DP for which local stability, but not global stability, holds. Finally, we discuss local stability of the zero-sum DP.

**Global stability.** Suppose that \(b - a > \frac{1}{2}\) and that \(F_i, G_i\) are functions satisfying Condition G1’ and furthermore,

\[
F_1(x) = x(\frac{a}{2} - x), \quad x \in [0, \frac{a}{2}].
\]

It follows from Condition G1’ that \(F_1\) is negative on \([\frac{a}{2}, 1]\). Therefore, for every \(w \in S_1\) the ‘best response’ \(\bar{x}(w)\) to \(\bar{y}(w)\) takes values in \([0, \frac{a}{2}]\), where we have the inequality

\[
\left| \frac{F_1'(x)}{F_1''(x)} \right| = \left| x - \frac{a}{4} \right| \leq \frac{1}{4}.
\]

Since \(G_i'\) is bounded on \([0, a]\) by Condition G1’, and recalling that \(\bar{y} \in [b, 1]\) by definition, for a sufficiently large constant \(R_1 > 0\) we have:

\[
\left| \frac{F_1'(\bar{x}) - \frac{1}{R_1} G_1'(\bar{y})}{F_1'(\bar{x})(\bar{y} - \bar{x})} \right| \leq 2 \cdot \frac{1}{4} \cdot \frac{1}{b - a} < 1.
\]

Therefore if player 1’s reward functions in the DP are \(f_1 = F_1\) and \(g_1 = \frac{1}{R_1} G_1\) (which clearly satisfy Condition G1’), then the left hand parenthesis in (5.7) has absolute value less than 1. Similarly if we take \(F_2(x) = (x - \frac{b + 1}{2})(1 - x)\) for all \(x \in [\frac{b + 1}{2}, 1]\) and let player 2’s reward functions be \(f_2 = F_2\) and \(g_2 = \frac{1}{R_2} G_2\) for a sufficiently large constant \(R_2\), the right hand parenthesis in (5.7) has absolute value less than 1 and so the global stability condition (5.7) holds.
Remark 5.7. Under Assumption [4.6] the reward functions in the DP must satisfy $f_i \leq g_i$ on $[0, 1]$. Given the choice of $g_i$ in the example above, $f_i \leq g_i$ implies that the rather strong condition $G_i \geq R_i F_i$ on $[0, 1]$ must hold. Although Remark [4.6] shows that $G_i \geq R_i F_i$ is only needed on $S_{-i}$, there are alternative choices for $g_i$ that satisfy Assumption [4.6] and lead to a conclusion similar to that of the example above. More specifically, in the case $i = 1$, take any $G_1 \geq \max(0, F_1)$ which is in $C^2[0, 1]$ and define $g_1$ to be a suitable restriction of $G_1$ to $[0, \frac{a}{2}]$ such that $g_1$ is in $C^2[0, 1]$, and on $[b, 1]$, $g_1$ is nonnegative and $g_1'$ is sufficiently small. For example, let $x \mapsto \eta(x)$ be the standard mollifier,

$$
\eta(x) = \begin{cases} 
C \exp\left(\frac{1}{x^2-1}\right), & |x| < 1 \\
0, & |x| \geq 1
\end{cases}
$$

where $C > 0$ is chosen so that $\int_{\mathbb{R}} \eta(x) dx = 1$. For $\epsilon > 0$ define $\eta_\epsilon(x) := \frac{1}{\epsilon} \eta(\frac{x}{\epsilon})$, $H_\epsilon(x) = \int_{-\infty}^{x} \eta_\epsilon(y) dy$ and set $g_1(x; \epsilon) = H_\epsilon(\frac{a}{2} - x + \epsilon) G_1(x)$. For $x \leq \frac{a}{2}$ we have $g_1(x; \epsilon) = G_1(x) \geq F_1(x) = f_1(x)$. For $x \geq \frac{a}{2} + 2\epsilon$ we have $g_1(x; \epsilon) = 0 \geq F_1(x) = f_1(x)$ and, for an appropriate choice of $\epsilon$, $g_1'(x) = 0$ on $[b, 1]$.

Local stability only. Global stability implies that the local stability condition (5.4) holds at the unique Nash equilibrium $(\ell_*, r_*)$ in the DP we have just constructed. Taking the same reward functions in the DP, suppose now that player 1’s strategy is $w_0 \in S_1$ and that player 2’s best response is $r_*$. Then from the smooth fit condition for player 2, the point $(w_0, g_2(w_0))$ must lie on the straight line tangent to $f_2$ at $(r_*, f_2(r_*))$. We may therefore conclude that if $g_2$ is not linear on $S_1$, then there exists a strategy $w_0 \in S_1 \setminus \{\ell_*\}$ for player 1 to which player 2’s best response is $y_0 \in S_2 \setminus \{r_*\}$. It is also not difficult to see that $y_0 \in (\frac{b+1}{2}, 1)$, and hence smooth fit holds at $y_0$, provided that $g_2$ is bounded above by the tangent to $f_2$ at $(1, f_2(1))$.

Next we remark that the function $f_2$ may be arbitrarily ‘flattened’ in a small neighbourhood of $y_0$ without violating Condition G1’. That is, let $N_0$ be an open neighbourhood of $y_0$ whose closure does not contain $r_*$ and let $\epsilon \in (f_2''(y_0), 0)$. Then $f_2$ may be modified on $N_0$ to produce a new function $\tilde{f}_2$ with

$$
\tilde{f}_2(y) = f_2(y), \quad y \in \{y_0\} \cup N_0^c, \\
\tilde{f}_2''(y_0) = f_2''(y_0), \\
\tilde{f}_2''(y) = \epsilon,
$$

and such that Condition G1’ holds for the reward functions $f_1$, $\tilde{f}_2$ and $g_i$. By construction, the smooth fit condition continues to hold at $y_0$ when $f_2$ is replaced by $\tilde{f}_2$, so that $y_0$ remains player 2’s best response to $w_0$. In this way the right hand multiplicand in (5.7) may be made arbitrarily large in absolute value when $w = w_0$ (provided the numerator is non-zero, a mild condition). We thus obtain a DP satisfying Condition G1’ which has local, but not global, stability.

5.5 Uniqueness of Nash equilibria

We close this section with a final result on uniqueness of equilibria in the DP by applying a well known condition in [29] for uniqueness of a solution to the GNEP.

Theorem 5.8. Suppose that Condition G1’ holds,

$$
f_1''(x) \leq -2 \frac{f_1(x) + f_1'(x)(y - x) - g_1(y)}{(y - x)^2}, \quad \forall (x, y) \in (0, a) \times [b, 1], \\
f_2''(y) \leq -2 \frac{f_2(y) - f_2'(y)(y - x) - g_2(x)}{(y - x)^2}, \quad \forall (x, y) \in [0, a] \times (b, 1),
$$

$$
(5.8) \\
(5.9)
$$
and \( \exists (r_1, r_2) \in [0, \infty) \times [0, \infty) \) such that \( \forall (x, y) \in S \),
\[
4r_1r_2H_1(x, y)H_2(x, y) - (r_1H_3(x, y) + r_2H_4(x, y))^2 > 0,
\]
(5.10)
where \( H_1, \ldots, H_4 \) are given by,
\[
\begin{align*}
H_1(x, y) &= f_1'(x)(y - x)^2 + 2[f_1(x) + f_1'(x)(y - x) - g_1(y)] \\
H_2(x, y) &= f_2''(y)(y - x)^2 + 2[f_2(y) - f_2'(y)(y - x) - g_2(x)] \\
H_3(x, y) &= 2[g_1(y) - f_1(x)] - (f_1'(x) + g_1'(y))(y - x) \\
H_4(x, y) &= 2[g_2(x) - f_2(y)] + (g_2'(x) + f_2'(y))(y - x).
\end{align*}
\]
(5.11)
Then there exists a unique solution \((\ell_*, r_*) \in S\) to the GNEP (2.3), and therefore \((D_{[0, \ell_*]}, D_{[r_*, 1]})\) is the unique solution to the DP in the class of Markovian strategies \((D_{S_1}, D_{S_2})\) for closed stopping sets \(S_1 \subseteq [0, a] \) and \(S_2 \subseteq [b, 1]\).

**Proof.** Conditions (5.8)–(5.9) ensure that each utility function \(s_i \mapsto U_i(s_i, s_{-i}), i \in \{1, 2\},\) is concave on \(S_i\) for each \(s_{-i} \in S_{-i}\). The condition (5.11) is sufficient for strict diagonal concavity according to Theorem 6 of [29]. The uniqueness result for the GNEP is an application of Theorem 2 in [29], whereas uniqueness for the DP follows from the proof of Theorem 5.5.

**Remark 5.9.** For possible extensions of Theorem 5.8 to quasi-concave utility functions see, for example, [2]. A comment on the relationship between the sufficient conditions for uniqueness of Nash equilibria used in Theorems 5.5 and 5.8 can be found in Remark 3.3 of [22].

### 6 Complex strategies and multiplayer GNEPs

The study of appropriate generalised games with \( n \geq 2 \) players yields equilibria for the two-player Dynkin problem of Definition 2.5 with more complex structures than the threshold type which has been previously studied. The systematic study of all cases \( n \geq 2 \) is beyond the scope of this paper and so in this section we provide an example with \( n = 3 \).

This example uses the following relaxation of Condition G1, under which the reward function \( f_1 \) has an additional convex portion:

**Condition G2.** There exist points \( a_1 \) and \( a_2 \) with \( 0 < a_1 \leq a_2 < b < 1 \) such that:

(i) \( f_1 \) is convex on \([0, a_1]\), concave on \([a_1, a_2]\) and convex on \([a_2, 1]\),

(ii) \( f_2 \) is convex on \([0, b]\) and concave on \([b, 1]\).

Define sets \( \hat{S}_1 = \hat{S}_2 = [a_1, a_2], \hat{S}_3 = [b, 1] \) and \( \hat{S} = \prod_{i=1}^3 \hat{S}_i \). Let the utility functions \( \hat{U}_i : [0, 1]^3 \to \mathbb{R}, i \in \{1, 2, 3\} \) be defined by
\[
\begin{align*}
\hat{U}_1(x, y, z) &= \frac{f_1(x) - g_1(x, 1)(x)}{x} \\
\hat{U}_2(x, y, z) &= \frac{f_1(y) - g_1(x, 1)(y)}{z - y} \\
\hat{U}_3(x, y, z) &= \frac{f_2(z) - g_2(0, y)(z)}{z - y}.
\end{align*}
\]
(taking \( \hat{U}_2(x, y, z) = \hat{U}_3(x, y, z) = -\infty \) if \( y \geq z \)). Define the players’ feasible strategy spaces by the set-valued maps \( \hat{K}_i : \hat{S}_{-i} \to \hat{S}_i \), where
\[
\hat{K}_1(y, z) = [a_1, y \land a_2], \quad \hat{K}_2(x, z) = [x \lor a_1, a_2], \quad \hat{K}_3(x, y) = [b, 1],
\]
(6.2)
so that the feasible strategy triples belong to the convex, compact set \( \hat{C} \) defined by
\[
\hat{C} = \{(x, y, z) \in [a_1, a_2] \times [a_1, a_2] \times [b, 1] : x \leq y\}.
\]
(6.3)
Theorem 6.1. Suppose that the DP satisfies Condition G2. Then:

(a) there exists \( s^* = (\ell_1, \ell_2, r) \in \hat{\mathcal{C}} \) with

\[
\hat{U}_i(s^*) = \sup_{(s_i, s^*_{-i}) \in \hat{\mathcal{C}}} \hat{U}_i(s_i, s^*_{-i}), \quad i \in \{1, 2, 3\},
\]

(b) a solution \( s^* = (\ell_1, \ell_2, r) \in \hat{\mathcal{C}} \) to (6.4) satisfies \( \hat{U}_2(s^*) \geq 0 \) if and only if \( (D_{[\ell_1, \ell_2]}, D_{[r, 1]}) \) is a Nash equilibrium for the DP.

Proof. Part (a) follows as in the proof of Lemma 4.1. For part (b), we claim that the pair \((\ell_1, \ell_2)\) solves the following problem:

Problem: Find two points \( \ell_1, \ell_2 \) satisfying

\[
i)
\quad a_1 \leq \ell_1 \leq \ell_2 \leq a_2, \\
\quad \hat{U}_1(x, \ell_1, \ell_2, r) \leq \hat{U}_1(\ell_1, \ell_2, r), \quad \forall x \in (0, r), \quad (P)
\]

To establish part iii) note that the function \( y \mapsto f_1(y) - g_1_{[r, 1]}(y) \) is zero at \( y = 0 \), convex for \( y \in [0, a_1] \), concave for \( y \in [a_1, a_2] \), convex for \( y \in [a_2, r] \), nonnegative at \( y = \ell_2 \) and negative at \( y = r \). It is then a straightforward exercise in convex analysis, similar to that in the proof of Theorem 4.4, to show that the maximum of the function \( y \mapsto \hat{U}_2(\ell_1, y, r) \) on \([0, r]\) must be attained at a point in \([a_1, a_2]\). Taking \( i = 2 \) in (6.4) then establishes the claim. Part ii) follows similarly.

The necessity and sufficiency claim for the Nash equilibrium in stopping strategies then follows by applying Propositions D.1 and D.2 in the Appendix. \(\square\)

A Other Markov processes and discounting

Let \( X = (X_t)_{t \geq 0} \) be a continuous strong Markov process defined on an interval \( E = (\ell, r) \). Suppose that the rewards in the DP are discounted by a factor \( \lambda \geq 0 \), so that (1.1) becomes

\[
\mathcal{J}_i(\tau_1, \tau_2) := e^{-\lambda(\tau_1 \wedge \tau_2)} \left\{ f_i(X_{\tau_1}) \mathbb{I}_{\{\tau_1 < \tau_2\}} + g_i(X_{\tau_2}) \mathbb{I}_{\{\tau_1 > \tau_2\}} + h_i(X_{\tau_2}) \mathbb{I}_{\{\tau_1 = \tau_2\}} \right\}, \quad i \in \{1, 2\}.
\]

Lemma 3.2 has a straightforward extension to the case \( \lambda > 0 \). Extending the concept of superharmonic functions in Definition 1, we say that a measurable function \( \phi: E_\Delta \to \mathbb{R} \) is \( \lambda \)-superharmonic on a set \( A \in \mathcal{B}(E_\Delta) \) if for every \( x \in E \) and \( \tau \in \mathcal{T} \),

\[
\phi(x) \geq \mathbb{E}^{\tau} \left[ e^{-\lambda(\tau \wedge D_A)} \phi(X_{\tau \wedge D_A}) \right].
\]

The function \( \phi_A \) introduced in Definition 2 is given more generally by,

\[
\phi_A(x) := \mathbb{E}^{\tau} \left[ e^{-\lambda D_A} \phi(X_{D_A}) \right].
\]

It was noted in Section 1.2.2 that \( \phi_A \) is continuous when \( \lambda = 0 \), \( g \) is continuous and \( A \) is closed, since \( X = X^F \) is a subprocess of a Brownian motion. This same property, which is important for ensuring that the obstacle in problem (3.6) is continuous, also holds for \( \lambda \geq 0 \) when \( X \) is a more general diffusion with strictly positive diffusion coefficient. Furthermore, when \( X \) is a subprocess of a regular diffusion \( Z = (Z_t)_{t \geq 0} \), the results in Sections 4.5 hold under
an appropriate modification of Condition G1. We now briefly discuss this extension when \( Z \) satisfies the stochastic differential equation,

\[
dZ_t = \mu(Z_t)dt + \sigma(Z_t)dW_t,
\]

where \( W = (W_t)_{t \geq 0} \) is a standard Brownian motion, \( \mu: E \to \mathbb{R} \) and \( \sigma: E \to (0, \infty) \) are measurable functions, \( \mu \) bounded and \( \sigma \) continuous, satisfying the following condition: for every \( x \in E \),

\[
\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < \infty \text{ for some } \varepsilon > 0.
\]

Let \( \mathcal{G} = \frac{1}{2} \sigma^2(\cdot) \frac{d^2}{dx^2} + \mu(\cdot) \frac{d}{dx} \) denote the infinitesimal generator corresponding to \( Z \).

### A.1 Undiscounted rewards

For the case \( \lambda = 0 \), we first recall from [10] that there is a continuous increasing function \( S \) on \( E \), the scale function, which satisfies \( \mathcal{G}S(\cdot) = 0 \). Let \( \tilde{\ell} = S(\ell) \), \( \tilde{r} = S(r) \) and \( \tilde{X} = (\tilde{X}_t)_{t \geq 0} \) be a Brownian motion on \( \tilde{E} = (\tilde{\ell}, \tilde{r}) \). Then, it follows from Proposition 3.3 of [10] that the DP corresponding to the process \( X \) and rewards \( f_i \), \( g_i \) and \( h_i \) on \( E \) can be studied by an equivalent DP corresponding to \( \tilde{X} \) with reward functions \( \tilde{f}_i(\cdot) = f_i(S^{-1}(\cdot)), \tilde{g}_i(\cdot) = g_i(S^{-1}(\cdot)), \tilde{h}_i(\cdot) = h_i(S^{-1}(\cdot)) \) on \( \tilde{E} \).

### A.2 Discounted rewards

For the case \( \lambda > 0 \), we first let \( \psi^\lambda \) and \( \phi^\lambda \) denote the fundamental solutions to the diffusion generator equation \( \mathcal{G}w = \lambda w \), where \( \psi^\lambda \) is strictly increasing and \( \phi^\lambda \) is strictly decreasing [10 p. 177]. Let \( F(\cdot) = \frac{\psi^\lambda(\cdot)}{\phi^\lambda(\cdot)}, \tilde{\ell} = F(\ell), \tilde{r} = F(r) \) and \( \tilde{X} = (\tilde{X}_t)_{t \geq 0} \) be a Brownian motion on \( \tilde{E} = (\tilde{\ell}, \tilde{r}) \). Then, it follows from Proposition 4.3 of [10] that the DP corresponding to the process \( X \) and rewards \( f_i \), \( g_i \) and \( h_i \) on \( E \) discounted by \( \lambda > 0 \) can be studied by an equivalent DP corresponding to \( \tilde{X} \) with reward functions \( \tilde{f}_i(\cdot) = \frac{f_i}{\phi^\lambda}(F^{-1}(\cdot)), \tilde{g}_i(\cdot) = \frac{g_i}{\phi^\lambda}(F^{-1}(\cdot)), \tilde{h}_i(\cdot) = \frac{h_i}{\phi^\lambda}(F^{-1}(\cdot)) \) on \( \tilde{E} \) without discounting.

### B Expected payoffs for threshold strategies

If players 1 and 2 use the strategies \( D_{[0,\ell]} \) and \( D_{[r,1]} \) respectively, where \( 0 \leq \ell < r \leq 1 \), then the expected payoff \( M^\ell_1(D_{[0,\ell]}, D_{[r,1]}) \) for player 1 (cf. (2.4)) satisfies,

\[
M^\ell_1(D_{[0,\ell]}, D_{[r,1]}) = E^x\left[f_1(X_{D_{[0,\ell]}})\mathbb{I}_{D_{[0,\ell]} < D_{[r,1]}} + g_1(X_{D_{[r,1]}})\mathbb{I}_{D_{[r,1]} < D_{[0,\ell]}}\right] + E^x\left[h_1(X_{D_{[0,\ell]}})\mathbb{I}_{D_{[0,\ell]} = D_{[r,1]}}\right]
\]

\[
= \begin{cases}
  f_1(x), & \forall x \in [0, \ell] \\
  f_1(\ell) \cdot P^x(\{D_{[0,\ell]} < D_{[r,1]}\}) + g_1(r) \cdot P^x(\{D_{[0,\ell]} > D_{[r,1]}\}), & \forall x \in (\ell, r) \\
  g_1(x), & \forall x \in [r, 1]
\end{cases}
\]

\[
= \begin{cases}
  f_1(x), & \forall x \in [0, \ell] \\
  f_1(\ell) \cdot \frac{x-\ell}{\ell-r} + g_1(r) \cdot \frac{r-x}{r-\ell}, & \forall x \in (\ell, r) \\
  g_1(x), & \forall x \in [r, 1]
\end{cases}
\]
Analogously, the expected payoff $M^x_2(D_{[0,\ell]}, D_{[r,1]})$ for player 2 satisfies,

$$M^x_2(D_{[0,\ell]}, D_{[r,1]}) = \begin{cases} g_2(x), & \forall x \in [0, \ell] \\ g_2(\ell) \cdot \frac{r-x}{\ell-x} + f_2(r) \cdot \frac{x-r}{x-\ell}, & \forall x \in (\ell, r) \\ f_2(x), & \forall x \in [r, 1]. \end{cases}$$

### C Derivatives of utility functions

Throughout this section we assume Condition G1’ holds. We first provide general formulas for the first and second partial derivatives of a utility function $U(x, y)$ which is of the form $U(x, y) = \frac{F(x,y)}{y-x}$.

\[
\partial_x U(x, y) = \frac{\partial_x F(x,y)(y-x) + F(x,y)}{(y-x)^2}, \quad \partial_y U(x, y) = \frac{\partial_y F(x,y)(y-x) - F(x,y)}{(y-x)^2} \tag{C.1}
\]

\[
\partial_{xx} U(x, y) = \frac{\partial_{xx} F(x,y)(y-x)^2 + 2[\partial_x F(x,y)(y-x) + F(x,y)]}{(y-x)^3} \tag{C.2}
\]

\[
\partial_{yy} U(x, y) = \frac{\partial_{yy} F(x,y)(y-x)^2 - 2[\partial_y F(x,y)(y-x) - F(x,y)]}{(y-x)^3} \tag{C.3}
\]

\[
\partial_{xy} U(x, y) = \frac{\partial_{xy} F(x,y)(y-x) + \partial_x F(x,y) + \partial_y F(x,y)}{(y-x)^2} - 2[\partial_x F(x,y)(y-x) + F(x,y)] \tag{C.4}
\]

\[
= \frac{\partial_{xy} F(x,y)(y-x) - \partial_y F(x,y) - \partial_x F(x,y)}{(y-x)^2} + 2[\partial_y F(x,y)(y-x) - F(x,y)] \tag{C.4}
\]

Using equation (2.6) for the utility functions gives the following expressions for their partial derivatives,

\[
\partial_x U_1(x, y) = \frac{f_1(x) + f'_1(x)(y-x) - g_1(y)}{(y-x)^2}, \quad \partial_y U_2(x, y) = \frac{g_2(x) + f'_2(y)(y-x) - f_2(y)}{(y-x)^2} 
\]

\[
\partial_{xx} U_1(x, y) = \frac{f''_1(x)(y-x)^2 + 2[f'_1(x) + f'_1(x)(y-x) - g_1(y)]}{(y-x)^3} 
\]

\[
\partial_{yy} U_2(x, y) = \frac{f''_2(y)(y-x)^2 + 2[f''_2(y)(y-x) - g_2(x)]}{(y-x)^3} 
\]

\[
\partial_{xy} U_1(x, y) = \frac{2[g_1(y) - f_1(x)] - (f'_1(x) + g'_1(y))(y-x)}{(y-x)^3} 
\]

\[
\partial_{xy} U_2(x, y) = \frac{2[g_2(x) - f'_2(y)] + (f'_2(x) + g'_2(y))(y-x)}{(y-x)^3} 
\]

### D A verification theorem

**Proposition D.1.** Under Condition G2 and given $r \in (a_2, 1]$, $(\ell^1, \ell^2)$ is a solution to Problem $\mathbb{P}$ if and only if

\[
V_1^{[\tau,1]}(x) := \sup_{\tau_1 \in \mathcal{T}} M^x_1(\tau_1, D_{[\tau,1]}) = M^x_2(D_{[\ell^1,\ell^2]}, D_{[r,1]}), \quad \forall x \in [0, 1]. \tag{D.1}
\]

**Proof.** The arguments are more or less the same as those establishing Theorem 4.5. For the sake of brevity we therefore only show the proof of necessity (Problem $\mathbb{P} \implies$ (D.1)).
Define \( u_r \) on \([0, 1]\) by,

\[
u_r(x) = M^r_1(D_{[\ell^1, \ell^2]}, D_{[r, 1]}, g_{1, [r, 1]}(x)) = \begin{cases} 
  (f_1(\ell^1) - g_{1, [r, 1]}(\ell^1)) \frac{x}{\ell^1}, & x \in [0, \ell^1), \\
  f_1(x) - g_{1, [r, 1]}(x), & x \in [\ell^1, \ell^2), \\
  (f_1(\ell^2) - g_{1, [r, 1]}(\ell^2)) \frac{r-x}{\ell^2-r}, & x \in [\ell^2, r), \\
  0, & x \in [r, 1].
\end{cases} \tag{D.2}
\]

Suppose \((\ell^1, \ell^2)\) is a solution to Problem \([P]\). Similarly to Theorem 4.5, we will prove \((D.1)\) by showing that \( u_r \) is the smallest non-negative concave majorant of \( f_1 - g_{1, [r, 1]} \) on \([0, r]\). Initially we will analyse \( u_r \) separately on \([0, \ell^1]\) and \([\ell^1, \ell^2]\). Observe firstly that the function \( f_1 - g_{1, [r, 1]} \) is nonnegative when evaluated at the points \( \ell^1 \) and \( \ell^2 \) and hence, by concavity, on \([\ell^1, \ell^2]\). Recalling \((6.1)\), this follows from \([P]\), since \( f_1(0) = g_{1, [r, 1]}(0) \) and so \( f_1(\ell^2) - g_{1, [r, 1]}(\ell^2) \geq 0 \). Also

\[
f_1(x) - g_{1, [r, 1]}(x) \leq \left( f_1(\ell^1) - g_{1, [r, 1]}(\ell^1) \right) \frac{x}{\ell^1}, \quad \forall x \in (0, r),
\]

and taking \( x = \ell^2 \) shows that \( f_1(\ell^1) - g_{1, [r, 1]}(\ell^1) \geq 0 \). Therefore \( u_r \) is a non-negative majorant of \( f_1 - g_{1, [r, 1]} \) on \([0, \ell^1]\). This is also true on \([\ell^1, r]\), since \( f_1(r) \leq g_1(r) \) and so

\[
f_1(x) - g_{1, [r, 1]}(x) \leq \left( f_1(\ell^2) - g_{1, [r, 1]}(\ell^2) \right) \left( \frac{r-x}{r-\ell^2} \right), \quad \forall x \in [0, r]. \tag{D.3}
\]

Concavity holds for \( u_r \) on the three intervals \([0, \ell^1]\), \([\ell^1, \ell^2]\) and \([\ell^2, r]\) separately and, arguing as in the proof of Theorem 4.5, we can show that \( u_r \) is continuous and concave on the entire interval \([0, r]\), completing the proof. □

\section*{Proposition D.2}

Under Condition \( G2 \), for every \( \ell^1, \ell^2 \) satisfying \( 0 < \ell^1 \leq \ell^2 < b \), a point \( r \in [b, 1] \) satisfies \( [4.5] \) with \( \ell = \ell^2 \) and \( U_2 = \hat{U}_3 \) if and only if

\[
V^2_2(\ell^1, \ell^2)(x) := \sup_{\tau_2 \in \mathcal{T}} M^r_2(D_{[\ell^1, \ell^2]}, \tau_2) = M^r_2(D_{[\ell^1, \ell^2]}, D_{[r, 1]}), \quad \forall x \in [0, 1]. \tag{D.4}
\]

\textit{Proof.} By Lemma 3.3 it is sufficient merely to consider the optimal stopping problem on the set \([0, \ell^1] \cup [\ell^2, 1]\) with obstacle \( f_2 - g_{2, [\ell^1, \ell^2]} \), and we will only sketch the solution. Note that since \( f_2 \leq g_2 \) it is clearly suboptimal to stop in \([\ell^1, \ell^2]\). From Dynkin's formula it is also suboptimal to stop on \([0, \ell^1]\), since \( f_2 - g_{2, [\ell^1, \ell^2]} \) is convex there and \( f_2(x) - g_{2, [\ell^1, \ell^2]}(x) \leq 0 \) for \( x \in (0, \ell^1) \). The solution is nontrivial only on \([\ell^2, 1]\), where the arguments used for Theorem 4.5 are sufficient to complete the proof. □

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