Abstract We obtain uniqueness and nondegeneracy results for ground states of Choquard equations \(-\Delta u + u = (|x|^{-1} * |u|^p) |u|^{p-2}u\) in \(\mathbb{R}^3\), provided that \(p > 2\) and \(p\) is sufficiently close to 2.

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1 Introduction and main results

1.1 Introduction

In this paper, we study the nonlinear elliptic problem

\[- \Delta u + \lambda u = \left( |x|^{-1} \ast |u|^p \right) |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \quad (1.1)\]

where \( \lambda > 0, 1 < p < \infty \) are positive constants, \( \Delta = \sum_{i=1}^{3} \partial_{x_i}^2 \) is the usual Laplacian operator in \( \mathbb{R}^3 \) and \( u \) is a real-valued measurable function.

Equation (1.1) is usually called the nonlinear Choquard or Choquard–Pekar equation. It is closely related to the focusing time-dependent Choquard equation

\[ i \psi_t = -\Delta \psi - \left( |x|^{-1} \ast |\psi|^p \right) |\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+. \quad (1.2) \]

This can be seen from the fact that the function \( \psi(x, t) = e^{i\lambda t} u(x) \) gives a solitary wave for Eq. (1.2) whenever \( u \) solves Eq. (1.1). In this context, Eq. (1.1) is known as the stationary nonlinear Choquard equation. In the case \( p = 2 \), Eq. (1.1) is reduced to

\[- \Delta u + \lambda u = \left( |x|^{-1} \ast |u|^2 \right) u \quad \text{in } \mathbb{R}^3. \quad (1.3)\]

Equation (1.3) is also called the nonlinear Hartree or Schrödinger–Newton equation. It was used to describe the quantum mechanics of a polaron at rest in the work of Pekar [18]. It was also used by Choquard to describe an electron trapped in its own hole in a certain approximating to Hartree–Fock theory of one component plasma in 1976, see e.g. Lieb [10]. For more mathematical and physics background for problems (1.1)–(1.3), we refer the readers to e.g. [7,10,12,13,16] and the references therein.

In this paper, we study ground state solutions (see below) of Eq. (1.1). By a solution to Eq. (1.1), we mean a function \( u \in H^1(\mathbb{R}^3) \cap L^{6p/5}(\mathbb{R}^3) \) such that for any function \( \varphi \) belonging to \( C^0_0(\mathbb{R}^3) \), the space of infinitely differentiable functions in \( \mathbb{R}^3 \) with compact support, there holds

\[ \int_{\mathbb{R}^3} \left( \nabla u \cdot \nabla \varphi + \lambda u \varphi - \left( |x|^{-1} \ast |u|^p \right) |u|^{p-2}u \varphi \right) \mathrm{d}y = 0. \]

The solution is well defined due to the Hardy–Littlewood–Sobolev inequality

\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v(x)|^p |v(y)|^p}{|x-y|} \mathrm{d}x \mathrm{d}y \leq A \left( \int_{\mathbb{R}^3} |v|^{\frac{6p}{5}} \mathrm{d}x \right)^{\frac{5}{6p}} \quad \text{for } \ v \in L^{\frac{6p}{5}}(\mathbb{R}^3), \]

where \( A > 0 \) is a constant independent of \( v \in L^{6p/5}(\mathbb{R}^3) \). We are concerned about the uniqueness and the so called nondegeneracy (see below) of ground state solutions of problem (1.1). Before giving our results, let us first give the definition of ground state solutions to problem (1.1), and then summarize some known results about ground state solutions.

It is well known [13,14,17] that Eq. (1.1) is variational. So solutions to Eq. (1.1) can be found by investigating critical points of related variational functionals. For instance, Moroz and Van Schaftingen [17] proved the existence of positive radial solutions to Eq. (1.1) in \( H^1(\mathbb{R}^3) \) by exploring minimizers of the variational problem

\[ \inf \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2 \mathrm{d}y}{\left( \int_{\mathbb{R}^3} \left( |x|^{-1} \ast |u|^p \right) |u|^p \mathrm{d}y \right)^{1/p}} : u \in H^1(\mathbb{R}^3), u \neq 0 \right\}. \]
In fact, in the same way, Moroz and Van Schaftingen [17] studied much more general problems than (1.1). From a physical point of view [1,9,10,13,14], the most interesting critical points are minimizers of the problem

\[ m(N, p) = \inf \left\{ E_p(u) : u \in \mathcal{A}_N \right\}, \tag{1.4} \]

where \( E_p : H^1(\mathbb{R}^3) \to \mathbb{R} \) is an energy functional defined by

\[ E_p(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} \, dx \, dy, \]

and \( \mathcal{A}_N \) is the admissible set defined as

\[ \mathcal{A}_N = \{ u \in H^1(\mathbb{R}^3) : \|u\|_2 = N \} \]

for a given number \( N > 0 \). Here \( \|u\|_2 = (\int_{\mathbb{R}^3} |u|^2 \, dy)^{1/2} \) denotes the norm of the space \( L^2(\mathbb{R}^3) \). Following the convention of Cazenave and Lions [1] (see also [5,6,9]), we call any minimizer \( Q \) of problem (1.4) a ground state solution, or simply a ground state, of problem (1.1) in \( \mathcal{A}_N \).

We summarize the existence result of ground states of problem (1.1) along with a list of basic properties as follows.

**Theorem 1.1** Assume that \( 5/3 < p < 7/3 \). Then for any given number \( N, N > 0 \), the following results hold.

(1) **(Existence)** There exits at least one ground state for problem (1.1) in \( \mathcal{A}_N \).

(2) **(Symmetry)** For any ground state \( Q \in \mathcal{A}_N \) of problem (1.1), there exists a strictly decreasing positive function \( v : [0, \infty) \to (0, \infty) \) such that \( Q = v(| \cdot - y |) \) for a point \( y \in \mathbb{R}^3 \).

(3) **(Regularity)** Let \( Q \in \mathcal{A}_N \) be an arbitrary ground state of problem (1.1). Then \( Q \) solves Eq. (1.1) with \( \lambda \) being a positive Lagrange multiplier. Moreover, \( Q \in W^{2,s}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \) holds for any \( s > 1 \).

(4) **(Decay)** For any radial ground state \( Q \in \mathcal{A}_N \) of problem (1.1) with \( 2 \leq p < 7/3 \), there exists a constant \( \gamma > 0 \) such that \( Q(x) = O(e^{-\gamma |x|}) \) holds for \( |x| \) sufficiently large.

For a complete proof of Theorem 1.1, we refer the readers to e.g. Moroz and Van Schaftingen [17]. See also Lions [14] for the existence of ground states for problem (1.3). For the sake of completeness, we give a sketch of the proof of Theorem 1.1 in Appendix 1.

**1.2 Main results**

In this paper, we are concerned about the uniqueness and the so called nondegeneracy (see below) of ground states of problem (1.1). The motivation comes from the well known fact that the uniqueness and nondegeneracy of ground states \( Q \) of problem (1.1) play a fundamental role in the stability and blow up analysis for the corresponding solitary wave solutions \( \psi(x, t) = e^{i\lambda t} Q(x) \) of the focusing time-dependent Hartree equation (1.2), see e.g. Lenzmann [9] and the references therein. We also refer the interested readers to e.g. [2,8,23] for studies on the uniqueness and nondegeneracy of ground states for nonlinear Schrödinger equations with local nonlinearities, and to e.g. [4–6,9] for studies on the same topics for nonlocal problems.

However, in striking contrast to the questions of existence, it seems fair to say that extremely little is known about uniqueness and nondegeneracy of ground states for problem (1.1), except in the isolated case \( p = 2 \), the uniqueness result of which is due to Lieb.
Theorem 1.2 There exists a number $0 < \delta < 1/3$ such that for any $p$, $2 < p < 2 + \delta$, and for any $N > 0$, there exists a unique ground state $Q \in H^1(\mathbb{R}^3)$ for problem (1.1) with $\|Q\|_2 = N$ up to translations. In particular, there exists a unique positive radial ground state $Q = Q(|x|) > 0$ for problem (1.1) with $\|Q\|_2 = N$.

As already mentioned, Theorem 1.2 is also true for $p = 2$, which is due to Lieb [10]. Recall that ground states of problem (1.1) are minimizers of problem (1.4). However, note that the functional $E_p$ in problem (1.4) is not convex. So the conventional way to prove uniqueness for minimizers does not work. On the other hand, since ground states are positive radial solutions to Eq. (1.1) in $H^1(\mathbb{R}^3)$, a natural idea to derive uniqueness of ground states of problem (1.1) is to show that Eq. (1.1) admits a unique positive radial solution in $H^1(\mathbb{R}^3)$.

Indeed, as one of his main results, Lieb [10] proved the quite strong result that in the case $p = 2$ Eq. (1.1), namely Eq. (1.3), admits a unique positive radial solution in $H^1(\mathbb{R}^3)$. One may try to extend his arguments to derive uniqueness for positive radial solutions to Eq. (1.1) in the general case $p \neq 2$. Unfortunately, this does not seem to work. The arguments of Lieb depend heavily on the particular nonlinearity of Eq. (1.3). In the general case when $p \neq 2$, the strong nonlinearity of the term $(|x|^{-1} * |u|^p) |u|^{p-2} u$ in Eq. (1.1) prevents one from using the arguments of Lieb [10]. For details of the arguments of Lieb [10], we refer to e.g. Lieb [10] or Lenzmann [9, Appendix A]. Therefore new ideas to derive uniqueness of ground states for problem (1.1) are in need.

Inspired by the recent works [4–6,9], in the present paper we will apply a combination of compactness argument and local uniqueness argument to prove Theorem 1.2. Let $N > 0$ be given and let $Q$ be the unique positive radial solution to Eq. (1.3) in $H^1(\mathbb{R}^3)$ with $\|Q\|_2 = N$. First we prove a compactness result (see Theorem 2.1 in Sect. 2), which states that positive radial ground states for problem (1.1) converges to $Q$ as $p$ tends to 2. Then we derive a local uniqueness result (see Proposition 3.1 in Sect. 3), which states that Eq. (1.1) has a unique positive radial solution in a sufficiently small neighborhood of $Q$ when $p > 2$ and $p$ is sufficiently close to 2. Finally, arguing by contradiction, we conclude Theorem 1.2. To prove the compactness result, apriori estimates for the corresponding Lagrange multipliers are in need, which requires a careful analysis on the Eq. (1.1) and the problem (1.4). We will also deduce a uniform estimate for the sequence of positive radial ground states of problem (1.1). To derive the local uniqueness result, we need a deeper knowledge on the unique positive radial ground state $Q$ to Eq. (1.3). Precisely, we need to know that the linearized operator for Eq. (1.3) associated to $Q$ is nondegenerate. Fortunately, this fact has been confirmed by Lenzmann [9, Theorem 1.4], see also Tod-Moroz [21] and Wei-Winter [22]. Then we apply an implicit function argument to derive the local uniqueness result.

Before proceeding further, we would like to remark a recent progress on the strong uniqueness result of Lieb [10]. By applying the moving plane method of Chen et al. [3], Ma and Zhao [15] proved that every positive solution to Eq. (1.3) in $H^1(\mathbb{R}^3)$ is radially symmetric about some point in $\mathbb{R}^3$. Thus in view of the uniqueness result of Lieb [10], Ma and Zhao [15] concluded that the Hartree equation (1.3) admits a unique positive solution in $H^1(\mathbb{R}^3)$ up to translations. We refer the readers to Ma and Zhao [15] for more symmetry results on positive solutions to nonlinear equations.

Now we move to our next topic in this paper. Since the ground states of problem (1.1) are unique by Theorem 1.2, it is natural to ask whether they are nondegenerate. To be precise, let $\delta > 0$ be defined as in Theorem 1.2, and let $Q = Q(|x|) > 0$ be the unique positive
radial ground state for problem (1.1) with \( \|Q\|_2 = N \) and \( 2 < p < 2 + \delta \). Define the linear operator \( \mathcal{L}_{+, p} \) associated to \( Q \) by
\[
\mathcal{L}_{+, p} \xi = -\Delta \xi + \lambda \xi - (p - 1) (|x|^{-1} * Q^p) Q^{p-2} \xi - p \left( |x|^{-1} * (Q^{p-1} \xi) \right) Q^{p-1} \tag{1.5}
\]
acting on \( L^2(\mathbb{R}^3) \) with form domain \( H^1(\mathbb{R}^3) \). We have the following result.

**Theorem 1.3** Let \( \delta > 0 \) be defined as in Theorem 1.2 and \( 2 < p < 2 + \delta \). Consider the unique positive radial ground state \( Q \) for problem (1.1) with \( \|Q\|_2 = N \). Then there exists a number \( 0 < \delta' \leq \delta \) such that for all \( p, 2 < p < 2 + \delta' \), the operator \( \mathcal{L}_{+, p} \) defined as in (1.5) is nondegenerate around \( Q \). That is,
\[
\text{Ker} \mathcal{L}_{+, p} = \text{span} \left\{ \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q \right\}.
\]

In the case \( p = 2 \), Theorem 1.3 is proved by Tod-Moroz [21], Lenzmann [9] and Wei-Winter [22], respectively. The nondegeneracy argument of Lenzmann [9] is an extension of Lieb’s uniqueness argument. We will follow the arguments of Lenzmann [9] to prove Theorem 1.3. It is open whether Theorems 1.2 and 1.3 hold for (1.1) in the complex valued Sobolev space \( H^1(\mathbb{R}^3; \mathbb{C}) \).

Let us now give some remarks on Theorem 1.3 before we close this section.

**Remark 1.4** (1) Let \( Q, \lambda \) and \( p \) satisfy the assumptions in Theorem 1.3. Consider problem (1.1) in the complex valued Sobolev space \( H^1(\mathbb{R}^3; \mathbb{C}) \). Then the linearized operator \( \mathcal{L} \) at \( Q \) is given by
\[
\mathcal{L} \xi = -\Delta \xi + \lambda \xi - (|x|^{-1} * Q^p) Q^{p-2} \xi - \frac{p - 2}{2} (|x|^{-1} * Q^p) Q^{p-2} (\xi + \bar{\xi}) - \frac{p}{2} (|x|^{-1} * (Q^{p-1} (\xi + \bar{\xi}))) Q^{p-1}.
\]

Easy to note that \( \mathcal{L} \) is even not \( \mathbb{C} \)-linear. To study the kernel of \( \mathcal{L} \), we view \( \mathcal{L} \) as a combination of operators \( \mathcal{L}_{+, p} \) and \( \mathcal{L}_{-, p} \), which act on the real part \( \text{Re} \xi \) and the imaginary part \( \text{Im} \xi \) of \( \xi \) respectively. That is,
\[
\mathcal{L} \xi = \mathcal{L}_{+, p} \text{Re} \xi + i \mathcal{L}_{-, p} \text{Im} \xi.
\]

Here \( \mathcal{L}_{+, p} \) is the linear operator defined as in (1.5), and \( \mathcal{L}_{-, p} \) is defined as
\[
\mathcal{L}_{-, p} = -\Delta + \lambda - (|x|^{-1} * Q^p) Q^{p-2}.
\]

Since \( Q \) does not change sign in \( \mathbb{R}^3 \), it is standard (see e.g. Lieb-Loss [11]) to verify that
\[
\text{Ker} \mathcal{L}_{-, p} = \text{span} \{ Q \}
\]
holds. Hence, Theorem 1.3 implies that
\[
\text{Ker} \mathcal{L} = \left\{ \left( \sum_{k=1}^{3} a_k \partial_{x_k} Q \right) + i b Q : a_1, a_2, a_3, b \in \mathbb{R} \right\}.
\]

(2) An immediate application of Theorem 1.3 that is of importance in the stability and blowup analysis of solitary waves for the focusing time-dependent Hartree equation (1.2) is given in terms of a coercivity estimate of \( \mathcal{L}_{+, p} \). Precisely, set \( M = \text{span} \{ \phi, \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q \} \subset L^2(\mathbb{R}^3) \), where \( \phi \) denotes a first eigenfunction of \( \mathcal{L}_{+, p} \) acting on \( L^2(\mathbb{R}^3) \). Then we can use Theorem 1.3 to derive the estimate
\[
\langle \mathcal{L}_{+, p} \eta, \eta \rangle \geq c \| \eta \|^2_{H^1(\mathbb{R}^3)} \quad \text{for} \ \eta \in M^\perp,
\]
where \( c > 0 \) is a constant independent of \( \eta \), and \( \langle \mathcal{L} + p \cdot \cdot, \cdot \rangle \) denotes the associated quadratic form on \( H^1 \).

The rest of the paper is organized as follows. Section 2 is devoted to the compactness result of ground states for problem (1.4) when \( p \) tends to 2. Section 3 is devoted to the proofs of Theorem 1.2 and Theorem 1.3. For the sake of completeness, we briefly prove Theorem 1.1 in Appendix 1. We also give in Appendix 1 a short proof of the regularity of the functional \( F \) used in Sect. 3.

Our notations are standard. We denote by \( B_r(0) \) the ball centered at the origin in \( \mathbb{R}^3 \) with radius \( r \). For \( 1 \leq q \leq \infty \), we use \( L^q(\mathbb{R}^3) \) to denote the Banach space of real-valued Lebesgue measurable functions \( u \) such that the norm

\[
\|u\|_q = \begin{cases} 
\left(\int_{\mathbb{R}^3} |u|^q \, dx\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\
\text{esssup}_{\mathbb{R}^3} |u| & \text{if } q = \infty
\end{cases}
\]

is finite. A function \( u \) belongs to the Sobolev space \( H^1(\mathbb{R}^3) \) if and only if \( u \in L^2(\mathbb{R}^3) \) and its first order weak partial derivatives also belong to \( L^2(\mathbb{R}^3) \). We equip \( H^1(\mathbb{R}^3) \) with the norm

\[
\|u\|_{H^1(\mathbb{R}^3)} = \|u\|_2 + \|\nabla u\|_2.
\]

We denote by \( L^q_{\text{rad}}(\mathbb{R}^3) \) and \( H^1_{\text{rad}}(\mathbb{R}^3) \) the subspaces of radial functions in \( L^q(\mathbb{R}^3) \) and \( H^1(\mathbb{R}^3) \) respectively. By the usual abuse of notation, we write \( u(r) = u(x) \) with \( r = |x| \) whenever \( u \) is a radial function in \( \mathbb{R}^3 \).

## 2 Compactness analysis

In this section, our aim is to prove the following compactness result.

**Theorem 2.1** Let \( \{p_n\} \subset (2, 7/3) \) be an arbitrary sequence with \( \lim_{n \to \infty} p_n = 2 \) and let \( N > 0 \) be given. Let \( Q_{p_n} = Q_{p_n}(|x|) > 0 \) be a positive radial ground state for problem (1.1) with \( p = p_n \) and \( \|Q_{p_n}\|_2 = N \) for all \( n \in \mathbb{N} \). Then we have that

\[
Q_{p_n} \rightarrow Q_2 \quad \text{in } H^1(\mathbb{R}^3).
\]

Here \( Q_2 = Q_2(|x|) > 0 \) is the unique positive radial ground state for problem (1.3) with \( \|Q_2\|_2 = N \).

The proof of Theorem 2.1 is elementary but long. We divide the proof into several lemmas. For simplicity, we introduce the notations

\[
K(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \quad \text{and} \quad D_p(u) = \frac{1}{2p} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x - y|} \, dx \, dy,
\]

so that the energy functional \( E_p \) can be written as

\[
E_p(u) = K(u) - D_p(u).
\]

### 2.1 Apriori estimates for Lagrange multipliers.

Recall that any ground state of problem (1.4) satisfies Euler–Lagrange equation (1.1) with a Lagrange multiplier depending on \( p \). This subsection is devoted to prove apriori estimates for these Lagrange multipliers. To this end, we will give a series of elementary lemmas. First we have an equivalent variational characterization for the constrained problem (1.4).
Lemma 2.2 Assume that $5/3 < p < 7/3$. Then for any $N > 0$, we have
\[
m(N, p) = -C_1(p) \sup \left\{ \left( \frac{D_p(u)^2}{K(u)^{3p-5}} \right)^{\frac{1}{3p-3}} : u \in A_N \right\},
\tag{2.1}
\]
where $C_1(p)$ is a positive constant given by
\[
C_1(p) = \frac{7 - 3p}{3p - 5} \left( \frac{3p - 5}{2} \right)^{\frac{2}{3p}}.
\tag{2.2}
\]

Proof For any $u \in A_N$, we have $t^{3/2}u(tx) \in A_N$ for all $t > 0$. An elementary calculation gives that
\[
\inf_{t>0} E_p \left( t^{3/2}u(tx) \right) = -C_1(p) \left( \frac{D_p(u)^2}{K(u)^{3p-5}} \right)^{\frac{1}{3p}},
\]
with $C_1(p)$ given by (2.2). Then Lemma 2.2 follows from above easily. \hfill \Box

Lemma 2.2 can also be found in Moroz and Van Schaftingen [17] in a more general context. It is easy to infer from (2.1) that if $Q \in A_N$ is a minimizer of problem (1.4), then
\[
m(N, p) = E_p(Q) = -C_1(p) \left( \frac{D_p(Q)^2}{K(Q)^{3p-5}} \right)^{\frac{1}{3p}}.
\]

Next we have the following observation.

Lemma 2.3 Assume that $5/3 < p < 7/3$. Then for any $N > 0$, we have
\[
m(N, p) = m(1, p) N^{\frac{10-2p}{3-3p}}.
\]

Proof Note that $u \in A_1$ if and only if $u_N \equiv N^{5/2}u(Nx) \in A_N$. An elementary calculation gives that
\[
\left( \frac{D_p(u_N)^2}{K(u_N)^{3p-5}} \right)^{\frac{1}{3p}} = \left( \frac{D_p(u)^2}{K(u)^{3p-5}} \right)^{\frac{1}{3p}} N^{\frac{10-2p}{3-3p}}.
\]
Thus we deduce from (2.1) and above equality that
\[
m(N, p) = -C_1(p) \sup \left\{ \left( \frac{D_p(u_N)^2}{K(u_N)^{3p-5}} \right)^{\frac{1}{3p}} : u \in A_1 \right\}
= -N^{\frac{10-2p}{3-3p}} C_1(p) \sup \left\{ \left( \frac{D_p(u)^2}{K(u)^{3p-5}} \right)^{\frac{1}{3p}} : u \in A_1 \right\}
= m(1, p) N^{\frac{10-2p}{3-3p}}.
\]

The proof of Lemma 2.3 is complete. \hfill \Box

Lemma 2.3 implies that it is sufficient to prove Theorem 2.1 in the case $N = 1$. In the rest of this paper, we shall only consider the case $N = 1$. For simplicity we write
\[
m(p) = m(1, p).
\]
Then (2.1) gives that
\[ m(p) = -C_1(p) \sup \left\{ \left( \frac{D_p(u)^2}{K(u)^{3p-5}} \right)^{\frac{1}{3p-5}} : u \in A_1 \right\} \]  
(2.3)
with \( C_1(p) \) given by (2.2).

We will need a lower bound for \( m(p) \). Recall that by the classical Hardy–Littlewood–Sobolev inequality (see e.g. [11]), there exists an absolute constant \( A > 0 \) such that, for any \( 5/3 < p < 5 \), we have that
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} \, dx \, dy \leq A \|u\|^{2p}_{6p} \|u\|^{2p}_{6p}, \quad \forall u \in L^{6p} (\mathbb{R}^3). \]  
(2.4)
Then by interpolation inequality, we have that
\[ \|u\|_{6p} \leq \|u\|_2^{\theta} \|\nabla u\|_6^{1-\theta}, \quad \forall u \in L^2 (\mathbb{R}^3) \cap L^6 (\mathbb{R}^3), \]
with \( 0 < \theta = (5 - p)/2p < 1 \). By Sobolev inequality, there exists an absolute constant \( B > 0 \) such that
\[ \|u\|_6 \leq B \|\nabla u\|_2, \quad \forall u \in H^1(\mathbb{R}^3). \]  
(2.5)
Hence combining above three inequalities we obtain that
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} \, dx \, dy \leq AB^{3p-5} \|u\|^{5-5p}_{2p} \|\nabla u\|^{3p-5}_{2p} \]
for all \( u \in H^1(\mathbb{R}^3) \). In particular, we deduce from (2.6) that
\[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|} \, dx \, dy \leq AB^{3p-5} \|\nabla u\|^{3p-5}_{2p}, \quad \forall u \in A_1. \]  
(2.7)
Hence combining (2.3) and (2.7) yields the following a priori estimate for \( m(p) \).

**Lemma 2.4** Assume that \( 5/3 < p < 7/3 \). Then we have
\[ m(p) \geq -C_2(p), \]  
(2.8)
where \( C_2(p) \) is a positive constant given by
\[ C_2(p) = \frac{7 - 3p}{3p-5} \left( \frac{(3p-5)A(\sqrt{2}B)^{3p-5}}{4p} \right)^{\frac{1}{3p-5}}, \]  
(2.9)
and \( A, B \) are the positive absolute constants given by (2.4) and (2.5) respectively.

Now let us consider Lagrange multipliers associated to minimizers of problem (1.4).

**Lemma 2.5** Assume that \( 5/3 < p < 7/3 \). Let \( Q \) be an arbitrary minimizer of problem (1.4) with \( N = 1 \). Consider the Lagrange multiplier \( \lambda_p \) corresponding to \( Q \) such that Eq. (1.1) is satisfied by \( Q \) with \( \lambda = \lambda_p \). Then we have
\[ \lambda_p = -\frac{2(5-p)}{7-3p} m(p). \]  
(2.10)
Proof The proof is based on a Pohozaev type identity for solutions to Eq. (1.1). Note that $Q$ satisfies

$$- \Delta Q - (|x|^{-1} \ast Q^p) Q^{p-1} = -\lambda_p Q \quad \text{in } \mathbb{R}^3. \quad (2.11)$$

Multiplying each side of Eq. (2.11) by $x \cdot \nabla Q$, we obtain by integrating by parts that

$$K(Q) - 5D_p(Q) = -\frac{3}{2} \lambda_p \int_{\mathbb{R}^3} |Q|^2 \, dx.$$ 

For details of the proof of above identity, we refer to Moroz and Van Schaftingen [17, Proposition 3.1]. Multiplying each side of Eq. (2.11) by $Q$, we obtain that

$$2K(Q) - 2pD_p(Q) = -\lambda_p \int_{\mathbb{R}^3} |Q|^2 \, dx.$$ 

Recall that $\|Q\|_2 = 1$. Combining above two identities yields that

$$K(Q) = \frac{(3p - 5)\lambda_p}{2(5 - p)} \quad \text{and} \quad D_p(Q) = \frac{\lambda_p}{5 - p}. \quad (2.12)$$

On the other hand, (2.3) gives us that

$$m(p) = -C_1(p) \left( \frac{D_p(Q)^2}{K(Q)^{3p-5}} \right)^{\frac{1}{7-3p}},$$

where $C_1(p)$ is defined as in (2.2). Hence we derive from (2.12) and above equation that

$$m(p) = -\frac{7 - 3p}{2(5 - p)} \lambda_p.$$ 

This proves (2.10). The proof of Lemma 2.5 is complete now. \qed

We remark that Lemma 2.5 implies that the Lagrange multiplier $\lambda_p$ is independent of the choice of minimizers of problem (2.3) (with $N = 1$). Furthermore, we have the following apriori estimates for $\lambda_p$.

**Lemma 2.6** Let $\lambda_p$ be defined as in (2.10) for $5/3 < p < 7/3$. Then for any compact subset $K \subset (5/3, 7/3)$, we have

$$0 < \inf_{p \in K} \lambda_p \leq \sup_{p \in K} \lambda_p < \infty.$$ 

Proof By (2.10), it is equivalent to prove that for any compact subset $K \subset (5/3, 7/3)$, there holds

$$-\infty < \inf_{p \in K} m(p) \leq \sup_{p \in K} m(p) < 0.$$ 

The lower bound $\inf_{K} m > -\infty$ follows from (2.8) of Lemma 2.4, since the function $p \mapsto C_2(p)$ is a positive continuous function for $5/3 < p < 7/3$.

So it remains to prove that $\sup_{p \in K} m(p) < 0$. By (2.3), it is easy to see that $m(p) < 0$ for all $5/3 < p < 7/3$. Hence we always have $\sup_{p \in K} m(p) \leq 0$. To obtain the strict inequality, we claim that the function $p \mapsto m(p)$ is upper semicontinuous. Indeed, note the fact that for any $u \in H^1(\mathbb{R}^3)$ fixed, the function $p \mapsto D_p(u)$ is continuous. This fact can be proved by the same argument as that of Lieb and Loss [11, Section 8.14]. We omit the details. Thus the function $p \mapsto E_p(u)$ is continuous for any $u \in H^1(\mathbb{R}^3)$ fixed. Now we can conclude that
the function $p \mapsto m(p)$ is upper semicontinuous, since $m(p)$ is the infimum of a family of continuous functions $p \mapsto E_p(u)$ with $u \in A_1$. Then we known that for any compact subset $K \subset (5/3, 7/3)$, there exists $p_0 \in K$ such that
\[
\sup_{p \in K} m(p) = m(p_0) < 0.
\]
The proof of Lemma 2.6 is complete. ☐

2.2 Uniform estimates for ground states.

Let $\{p_n\} \subset (2, 7/3)$ be an arbitrary sequence such that $\lim_{n \to \infty} p_n = 2$. Let $Q_{p_n}(|x|) > 0$ be a positive radial ground state for problem (1.1) with $p = p_n$ and $\|Q_{p_n}\|_2 = 1$ for all $n \in \mathbb{N}$. Then $Q_{p_n}$ satisfies
\[
-\Delta Q_{p_n} - (|x|^{-1} * Q_{p_n}^{p_n}) Q_{p_n}^{p_n - 1} = -\lambda_{p_n} Q_{p_n} \quad \text{in } \mathbb{R}^3,
\] (2.13)
where $\lambda_{p_n}$ is defined as in (2.10). In this subsection, we prove the following uniform estimate for the sequence $\{Q_{p_n}\}$.

**Proposition 2.7** There exists a positive function $F \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, such that
\[
Q_{p_n} \leq F \quad \text{in } \mathbb{R}^3
\] (2.14)
holds for all $n \in \mathbb{N}$.

It has been shown by Theorem 1.1 that each $Q_{p_n}$ is bounded and decays exponentially to zero at infinity. However, we can not find a literature where a uniform estimate for $Q_{p_n}$ with respect to $n$ is given. Hence we derive Proposition 2.7 to gives a uniform estimate for all $Q_{p_n}$. We remark that our estimates are only precise enough for use and far from optimal.

To prove Proposition 2.7, first we derive the following uniform boundedness estimate for $Q_{p_n}$.

**Lemma 2.8** There exists a constant $M > 0$ such that
\[
\sup_n \|Q_{p_n}\|_\infty \leq M < \infty.
\]

In the rest of this section, we denote, for all $n \in \mathbb{N}$,
\[
V_n = |x|^{-1} * Q_{p_n}^{p_n}.
\]

**Proof of Lemma 2.8.** To prove Lemma 2.8, we need the following estimate
\[
\sup_n \|V_n\|_\infty < \infty. \tag{2.15}
\]
Note that $-\Delta V_n = 4\pi Q_{p_n}^{p_n}$ and that $V_n$ is positive, radial and decreasing with respect to $|x|$. Hence $0 \leq V_n(x) \leq V_n(0)$. We only need to show that
\[
\sup_n V_n(0) < \infty. \tag{2.16}
\]
By definition, we have that
\[
V_n(0) = \int_{\mathbb{R}^3} |x|^{-1} Q_{p_n}^{p_n}(x) \, dx = \int_{B_1(0)} |x|^{-1} Q_{p_n}^{p_n}(x) \, dx + \int_{\mathbb{R}^3 \setminus B_1(0)} |x|^{-1} Q_{p_n}^{p_n}(x) \, dx.
\]
Since $2 < p_n < 7/3$, there exists $r > 1$ such that $r' = r/(r - 1) < 3$ and $p_n r \leq 6$. Then Hölder’s inequality gives that

$$
\int_{B_1(0)} |x|^{-1} Q_{p_n}^n(x) dx \leq \left( \int_{B_1(0)} |x|^{-r'} dx \right)^{1/r'} \left( \int_{B_1(0)} Q_{p_n}^{p_n} dx \right)^{1/r}.
$$

Note that $Q_{p_n} \in H^1(\mathbb{R}^3)$ is uniformly bounded. Thus Sobolev inequality implies that

$$
\int_{B_1(0)} Q_{p_n}^{p_n} dx \leq C \text{ holds for a constant } C > 0 \text{ independent of } n.
$$

Therefore there exists a constant $M > 0$ for a constant $C > 0$ independent of $n$. Therefore

$$
\sup_n \int_{B_1(0)} |x|^{-1} Q_{p_n}^n(x) dx < \infty.
$$

On the other hand, $2 < p_n < 7/3$ implies that $2 < 4p_n/3 < 6$. Combining Hölder’s inequality and Sobolev inequality yields that

$$
\int_{\mathbb{R}^3 \setminus B_1(0)} |x|^{-1} Q_{p_n}^n(x) dx \leq \left( \int_{\mathbb{R}^3 \setminus B_1(0)} |x|^{-4} dx \right)^{1/4} \left( \int_{\mathbb{R}^3 \setminus B_1(0)} Q_{p_n}^{4p_n} dx \right)^{1/4} \leq C' < \infty
$$

for a constant $C' > 0$ independent of $n$. Combining above two estimates completes the proof of (2.16). Thus (2.15) holds.

Now we prove Lemma 2.8 as follows. Note that $Q_{p_n}$ satisfies

$$
- \Delta Q_{p_n} + Q_{p_n} = (1 - \lambda_{p_n})Q_{p_n} + V_n Q_{p_n}^{p_n - 1}.
$$

(2.17)

Since $2 < p_n < 7/3$, we have $9/2 < 6/(p_n - 1) < 6$. Since $p_n$ tends to 2 as $n \to \infty$, $\{p_n\}$ is contained in a compact subset of $(5/3, 7/3)$. Therefore, by Lemma 2.6, $\lambda_{p_n}$ is bounded uniformly for all $n \in \mathbb{N}$. Thus we easily deduce that

$$
\sup_n \| (1 - \lambda_{p_n}) Q_{p_n} \|_{9/2} \leq M_1 < \infty
$$

for a constant $M_1 > 0$, and that

$$
\sup_n \| Q_{p_n}^{p_n - 1} \|_{9/2} \leq \sup_n \| Q_{p_n} \|_{2}^{(p_n - 1)\theta} \| Q_{p_n} \|_{6}^{(p_n - 1)(1 - \theta)} \leq M_2 < \infty
$$

for a constant $M_2 > 0$, since $Q_{p_n} \in H^1(\mathbb{R}^3)$ is uniformly bounded, where $\theta \in (0, 1)$ is a constant. Thus we deduce from (2.15) and above estimate that

$$
\sup_n \| V_n Q_{p_n}^{p_n - 1} \|_{9/2} \leq \left( \sup_n V_n \right) \sup_n \| Q_{p_n}^{p_n - 1} \|_{9/2} < \infty.
$$

Therefore there exists a constant $M_3 > 0$ such that we have

$$
\sup_n \| (1 - \lambda_{p_n}) Q_{p_n} + V_n Q_{p_n}^{p_n - 1} \|_{9/2} \leq M_3 < \infty.
$$

Thus by elliptic regularity theory, Eq. (2.17) gives that $Q_{p_n} \in W^{2,9/2}(\mathbb{R}^3)$ and

$$
\| Q_{p_n} \|_{W^{2,9/2}(\mathbb{R}^3)} \leq CM_1 + CM_3 < \infty,
$$

for some constant $C > 0$ independent of $n$. By Sobolev embedding theorem, we have that $W^{2,9/2}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. Therefore, by setting $M = CM_1 + CM_3$, we complete the proof of Lemma 2.8.

Next we prove that $V_n$ decays to zero at infinite in a uniform way.
Lemma 2.9 We have that
\[ V_n(x) \leq C_0|x|^{-3/4} \text{ for } x \in \mathbb{R}^3. \]
for all \( n \in \mathbb{N} \), where \( C_0 > 0 \) is a constant independent of \( n \).

Proof Since \( p_n > 2 \), it is easy to deduce from Lemma 2.8 that \( Q_{p_n}^{p_n} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) and \( \sup_n (\| Q_{p_n}^{p_n} \|_1 + \| Q_{p_n}^{p_n} \|_\infty) < \infty \). By the Riesz potential theory (see e.g. [19]), we obtain that \( V_n \in L^4(\mathbb{R}^3) \) and
\[ \| V_n \|_4 \leq C_1 \| Q_{p_n}^{p_n} \|_{\frac{12}{11}} < \infty \]
for all \( n \in \mathbb{N} \), where \( C_1 > 0 \) is a constant independent of \( n \). Note that \( V_n \) is symmetric decreasing. We obtain that
\[ C_1 \geq \int_{B_r(0)} V_n^4 \, dx \geq V_n^4(r) \frac{4\pi}{3} r^3 \]
for all \( r > 0 \) and for all \( n \in \mathbb{N} \). By setting \( C_0 = 3C_1/(4\pi) \), we complete the proof of Lemma 2.9. \( \square \)

Note that \( \| Q_{p_n} \|_2 = 1 \) holds for all \( n \) and that \( Q_{p_n}(|x|) \) is decreasing with respect to \( |x| \), we derive as above that
\[ Q_{p_n}(x) \leq C|x|^{-3/2} \text{ for } x \in \mathbb{R}^3 \] (2.18)
for all \( n \in \mathbb{N} \), where \( C > 0 \) is a constant independent of \( n \). The estimate (2.18) can be improved as follows.

Lemma 2.10 For any \( \gamma > 1 \), there exists a constant \( C = C(\gamma) > 0 \) such that for all \( n \in \mathbb{N} \), we have that
\[ Q_{p_n}(x) \leq C|x|^{-\gamma} \text{ for } |x| \geq 1. \]

Proof Note that Eq. (2.13) gives that
\[ Q_{p_n}(x) = \frac{1}{-\Delta + \lambda_{p_n}} V_n Q_{p_n}^{p_n-1} = G_n \ast \left( V_n Q_{p_n}^{p_n-1} \right), \]
where \( G_n(x) = \exp\left(-\sqrt{-\lambda_{p_n}}|x|\right)/(4\pi|x|) \) is the integral kernel of the operator \( \frac{1}{-\Delta + \lambda_{p_n}} \).
Since \( \inf_n \lambda_{p_n} > 0 \) by Lemma 2.6, there exists \( \delta > 0 \) such that
\[ G_n(x) \leq \frac{e^{-\delta|x|}}{4\pi|x|} \text{ for all } x \in \mathbb{R}^3 \text{ and all } n \in \mathbb{N}. \]
(2.20)

Now we estimate \( Q_{p_n} \) by virtue of (2.19). Fix \( |x| \geq 1 \). Then by (2.15) and Lemma 2.8, we obtain that
\[ \int_{B_{|x|/2}(0)} G_n(x - y) V_n(y) Q_{p_n}^{p_n-1}(y) \, dy \leq C_1 G_n(|x|/2)|x|^3. \]
(2.21)

Here we used the fact that \( G_n \) is monotone decreasing with respect to \( |x| \). On the other hand, by (2.18), Lemmas 2.8, 2.9 and the fact that \( p_n > 2 \), we deduce that
\[ \int_{\mathbb{R}^3 \setminus B_{|x|/2}(0)} G_n(x - y) V_n(y) Q_{p_n}^{p_n-1}(y) \, dy \leq C|x|^{-9/4} \int_{\mathbb{R}^3} G_n \, dy = C|x|^{-9/4}. \]
(2.22)
Combining (2.21) and (2.22) and noticing (2.20), we conclude that
\[ Q_{pn}(x) \leq C|x|^{-9/4} \quad \text{for all } |x| \geq 1, \tag{2.23} \]
for all \( n \in \mathbb{N} \), where \( C > 0 \) is independent of \( n \). Note that (2.23) is an improvement of (2.18).

Finally, for any given constant \( \gamma > 9/4 \), we can substituting (2.23) into (2.22) and iterating finitely many times to deduce that
\[ Q_{pn}(x) \leq C|x|^{-\gamma} \quad \text{uniformly for all } n \in \mathbb{N}. \tag{2.24} \]

The proof of Lemma 2.10 is complete.

Now we can prove Proposition 2.7.

**Proof of Proposition 2.7** We need to find the function \( F \). This follows easily from Lemma 2.8 and Lemma 2.10. Indeed, set
\[
F(x) = \begin{cases} 
\sup_n \|Q_{pn}\|_{\infty} & \text{for } |x| \leq 1, \\
C|x|^{-\gamma} & \text{for } |x| > 1,
\end{cases}
\]
where \( C = C(4) > 0 \) is given as in Lemma 2.10. Then \( Q_{pn}(x) \leq F(x) \) holds for all \( x \in \mathbb{R}^3 \) and for all \( n \in \mathbb{N} \). Furthermore, it is easy to see that \( F \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \). The proof of Proposition 2.7 is complete.

### 2.3 Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1. Let \( p_n, Q_{pn} \) be defined as in Theorem 2.1. We assume that \( N = 1 \) so that \( \|Q_{pn}\|_2 = 1 \) for all \( n \in \mathbb{N} \). Let \( \lambda_{pn} \) be defined as in (2.10) such that \( Q_{pn} \) satisfies Eq. (2.13). Since \( \lambda_{pn} \) is bounded uniformly for all \( n \in \mathbb{N} \), (2.12) implies that \( \{Q_{pn}\} \) is a bounded sequence in \( H^1(\mathbb{R}^3) \) since \( \|Q_{pn}\|_2 = 1 \) for all \( n \in \mathbb{N} \). Therefore we can assume, after possibly passing to a subsequence, that \( Q_{pn} \) converges weakly to a nonnegative radial function \( Q_{\infty} \in H^1(\mathbb{R}^3) \), that is,
\[ Q_{pn} \rightharpoonup Q_{\infty} \quad \text{in } H^1(\mathbb{R}^3). \tag{2.24} \]
Moreover, by the compact embedding \( H^1_{\text{rad}}(\mathbb{R}^3) \subset C(\mathbb{R}^3) \) for any \( 2 < q < 6 \) (see Strauss [20]), we can assume that
\[ Q_{pn} \to Q_{\infty} \quad \text{in } L^q(\mathbb{R}^3) \tag{2.25} \]
for any \( 2 < q < 6 \), and
\[ Q_{pn} \to Q_{\infty} \quad \text{a.e. in } \mathbb{R}^3. \tag{2.26} \]
Furthermore, by Lemma 2.6, we can extract a subsequence of \( \{p_n\} \), still denote by \( \{p_n\} \), such that
\[ \lim_{n \to \infty} \lambda_{pn} = \mu \tag{2.27} \]
for some \( 0 < \mu < \infty \).

We claim that \( \mu \) is independent of the choice of the subsequence \( \{p_n\} \). Indeed, by Proposition 2.7, we easily deduce that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} (|x|^{-1} * Q_{pn}^{p_n}) Q_{pn}^{p_n-1} \varphi \, dy = \int_{\mathbb{R}^3} (|x|^{-1} * Q_{\infty}^2) Q_{\infty} \varphi \, dy \tag{2.28}
\]
for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, and that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} (|x|^{-1} \ast Q_{pn}^2) \ Q_{pn}^2 \, dy = \int_{\mathbb{R}^3} (|x|^{-1} \ast Q_{\infty}^2) \ Q_{\infty}^2 \, dy. \tag{2.29}
\]
Then by passing to limit in Eq. (2.13), we derive from (2.24) and (2.28) that $Q_{\infty}$ is a solution to equation
\[
- \Delta Q_{\infty} - (|x|^{-1} \ast Q_{\infty}^2) \ Q_{\infty} = -\mu \ Q_{\infty} \quad \text{in } \mathbb{R}^3. \tag{2.30}
\]
We show that $\|Q_{\infty}\|_2 = 1$. By multiplying $Q_{\infty}$ on each side of above equation and combining (2.24), (2.29), we deduce that
\[
- \mu \int_{\mathbb{R}^3} |Q_{\infty}|^2 \, dy = \int_{\mathbb{R}^3} \left| \nabla Q_{\infty} \right|^2 \, dx - \int_{\mathbb{R}^3} (|x|^{-1} \ast Q_{\infty}^2) \ Q_{\infty}^2 \, dy \leq \lim_{n \to \infty} \left( \int_{\mathbb{R}^3} \left| \nabla Q_{pn} \right|^2 \, dx - \int_{\mathbb{R}^3} (|x|^{-1} \ast Q_{pn}^2) \ Q_{pn}^2 \, dy \right) = -\lim_{n \to \infty} \lambda_{pn} = -\mu. \tag{2.31}
\]
Then $\|Q_{\infty}\|_2 \geq 1$ follows from (2.31) since $\mu > 0$. On the other hand, since $Q_{pn} \to Q_{\infty}$ in $L^2(\mathbb{R}^3)$, we have that $\|Q_{\infty}\|_2 \leq 1$ holds. Therefore, we conclude that $\|Q_{\infty}\|_2 = 1$ holds.

Then by the uniqueness result of Lieb [10], we find that $Q_{\infty} = Q_2$ is the unique positive radial solution to Eq. (2.30) and $\mu$ is determined uniquely by $Q_{\infty}$ with $\|Q_{\infty}\|_2 = 1$. This proves the claim.

Furthermore, $\|Q_{\infty}\|_2 = 1$ implies that $Q_{pn} \to Q_{\infty}$ in $L^2(\mathbb{R}^3)$ and that the inequality in (2.31) is in fact an equality. Hence we obtain that $\|\nabla Q_{pn}\|_2 \to \|\nabla Q_{\infty}\|_2$ as $n \to \infty$, from which we deduce that $Q_{pn} \to Q_{\infty}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ in view of (2.24).

Finally, to complete the proof of Theorem 2.1, we note that we have convergence along every subsequence due to the uniqueness of the limit point $Q_{\infty} \in \mathcal{A}_1$.

We remark that above proof also implies that $Q_p \to Q_2$ in $H^1(\mathbb{R}^3)$ and $\lambda_p \to \lambda_2$ as $p \to 2$ since $Q_2$ and $\lambda_2$ are unique.

3 Proofs of main results

In this section, we prove our main results Theorems 1.2 and 1.3. Since our parameter $p$ varies between 2 and 6, the critical Sobolev exponent for $H^1(\mathbb{R}^3)$, it is convenient to consider our problems in the function space
\[
\mathcal{X} = L^2_{rad}(\mathbb{R}^3) \cap L^6_{rad}(\mathbb{R}^3),
\]
equipped with norm
\[
\|u\|_{\mathcal{X}} = \|u\|_2 + \|u\|_6.
\]
Note that $H^1_{rad}(\mathbb{R}^3)$ is continuously embedded into $\mathcal{X}$ by Sobolev embedding theorem. First we derive a local uniqueness result.

**Proposition 3.1** Let $Q_2 \in \mathcal{A}_1$ be the unique positive radial ground state to Eq. (1.3) with $\lambda = \lambda_2$ given by (2.10) with $p = 2$. Then there exists a small number $\delta > 0$ and a map $(\tilde{Q}, \tilde{\lambda}) \in C^1(I; X \times \mathbb{R}_+)$ defined on the interval $I = [2, 2 + \delta)$ such that the following holds, where we denote $(\tilde{Q}_p, \tilde{\lambda}_p) = (\tilde{Q}(p), \tilde{\lambda}(p))$ in the sequel.
There exists $\epsilon > 0$ such that $(\tilde{Q}_p, \tilde{\lambda}_p)$ is the unique solution to Eq. (1.1) with $\lambda = \tilde{\lambda}_p$ for all $p \in I$.

(2) For all $p \in I$, the neighborhood $\mathcal{N}_\epsilon = \{(u, \mu) \in \mathbb{R} \times \mathbb{R}_+ : \|u - Q_2\|_X + |\mu - 2| \leq \epsilon\}$.

In particular, we have that $(\tilde{Q}_p, \tilde{\lambda}_p) = (Q_2, \lambda_2)$ holds.

(3) For all $p \in I$, we have

$$\|\tilde{Q}_p\|_2 = \|Q_2\|_2 = 1.$$

The proof of Proposition 3.1 will be given later. With the help of Proposition 3.1, we are able to prove Theorem 1.2 now.

**Proof of Theorem 1.2** It suffices to assume that $N = 1$. We argue by contradiction. Suppose, on the contrary, that there exist a sequence $\{p_n\} \subset (2, 7/3)$ with $p_n \to 2$ as $n \to \infty$, such that for every $p_n$, there exist at least two distinct positive radial ground states $Q_{p_n} \in \mathcal{A}_1$ and $\bar{Q}_{p_n} \in \mathcal{A}_1$ for problem (1.1) with $p = p_n$. Let $I = [2, 2 + \delta)$ be the interval defined as in Proposition 3.1. With no loss of generality, we can assume that $\{p_n\} \subset I$. Let $\lambda_{p_n}$ be defined as in (2.10) with $p = p_n$ for all $n \in \mathbb{N}$. Then both $(Q_{p_n}, \lambda_{p_n})$ and $(\bar{Q}_{p_n}, \lambda_{p_n})$ solves Eq. (1.1) with $p = p_n$ and $\lambda = \lambda_{p_n}$.

Applying Theorem 2.1 to both sequences $\{Q_{p_n}\}$ and $\{\bar{Q}_{p_n}\}$, we deduce that both $Q_{p_n}$ and $\bar{Q}_{p_n}$ converge strongly in $H^1(\mathbb{R}^3)$ to the unique positive radial ground state $Q_2 \in \mathcal{A}_1$ of problem (1.3). In particular, by Sobolev embedding theorem, $Q_{p_n} \to Q_2$ and $\bar{Q}_{p_n} \to Q_2$ in $\mathbb{R}$ hold. Moreover, (2.27) implies $\lambda_{p_n} \to \lambda_2$ as well.

Now we apply Proposition 3.1 to deduce a contradiction as follows. Let $\epsilon > 0$ and the neighborhood $\mathcal{N}_\epsilon$ be given as in Proposition 3.1. Let $(\tilde{Q}_{p_n}, \tilde{\lambda}_{p_n})$ be defined as in Proposition 3.1 such that $(\tilde{Q}_{p_n}, \tilde{\lambda}_{p_n})$ is the unique solution to Eq. (1.1) with $p = p_n$ and $\lambda = \tilde{\lambda}_{p_n}$ in the neighborhood $\mathcal{N}_\epsilon$ for all $n \in \mathbb{N}$. Recall that both $(Q_{p_n}, \lambda_{p_n})$ and $(\tilde{Q}_{p_n}, \tilde{\lambda}_{p_n})$ are positive radial solutions to Eq. (1.1) with $p = p_n$ and $\lambda = \lambda_{p_n}$ for all $n \in \mathbb{N}$. Recall also that $\lambda_{p_n}$ is independent of the choice of $Q_{p_n}$ or $\tilde{Q}_{p_n}$ for all $n \in \mathbb{N}$ in view of (2.10). Since both $Q_{p_n} \to Q_2$ and $\tilde{Q}_{p_n} \to Q_2$ in $\mathbb{R}$ and $\lambda_{p_n} \to \lambda_2$ hold as $n \to \infty$, we find that $(Q_{p_n}, \lambda_{p_n}) \in \mathcal{N}_\epsilon$ and $(\tilde{Q}_{p_n}, \tilde{\lambda}_{p_n}) \in \mathcal{N}_\epsilon$ hold for all sufficiently large $n$. Therefore by Proposition 3.1, we deduce for all sufficiently large $n$ that

$$Q_{p_n} = \tilde{Q}_{p_n} = \bar{Q}_{p_n} \quad \text{and} \quad \lambda_{p_n} = \tilde{\lambda}_{p_n}.$$

We reach a contradiction since we assumed that $Q_{p_n} \neq \tilde{Q}_{p_n}$ for all $n \in \mathbb{N}$. The proof of Theorem 1.2 is complete now.

We remark that from above proof of Theorem 1.2, we find that $(Q_p, \lambda_p) = (\tilde{Q}_p, \tilde{\lambda}_p)$ for $2 < p < 2 + \delta$, where $Q_p \in \mathcal{A}_1$ is the unique positive radial ground state for problem (1.1) and $\lambda_p$ is the corresponding Lagrange multiplier. Thus $p \mapsto (Q_p, \lambda_p)$ is $C^1$ for $2 < p < 2 + \delta$.

It remains to prove Proposition 3.1. We use an implicit function argument. We follow the line of Frank and Lenzmann [5, Proposition 5.2].

**Proof of Proposition 3.1** Observe that $u \in \mathbb{R}$ is a solution to Eq. (1.1) if and only if

$$u - \frac{1}{-\Delta + \lambda} \left(|x|^{-1} * |u|^p\right) |u|^{p-2} u = 0 \quad \text{in} \quad \mathbb{R}^3.$$
Here \( \frac{1}{-\Delta + \lambda} \) denotes the bounded inverse operator of \(-\Delta + \lambda\) on \( L^2(\mathbb{R}^3) \) for \( \lambda > 0 \). For \( 0 < \delta < 1/3 \) sufficiently small, we define the map \( F : \mathbb{R} \times \mathbb{R}_+ \times [2, 2 + \delta) \to \mathbb{R} \times \mathbb{R} \) by
\[
F(u, \lambda, p) = \left( u - \frac{1}{-\Delta + \lambda} \left( |x|^{-1} * |u|^p \right) |u|^{p-2}u \right).
\]
By Lemma B.1, \( F \) is continuously Fréchet differentiable, and \( \partial_{(u, \lambda)} F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \) is given by
\[
\partial_{(u, \lambda)} F = \begin{pmatrix}
\text{Id} + K_p, & W_p \\
2\langle u, \cdot \rangle, & 0
\end{pmatrix},
\]
where \( K_p \) is given by
\[
K_p \xi = -\frac{1}{-\Delta + \lambda} \left( (p - 1) \left( |x|^{-1} * |u|^p \right) |u|^{p-2} \xi + p \left( |x|^{-1} * (|u|^{p-2} u \xi) \right) |u|^{p-2} u \right),
\]
\( W_p \) is given by
\[
W_p = \frac{1}{(-\Delta + \lambda)^2} \left( |x|^{-1} * |u|^p \right) |u|^{p-2} u,
\]
and \( \langle u, \cdot \rangle : \mathbb{R} \to \mathbb{R} \) denotes the inner product \( \langle u, v \rangle = \int_{\mathbb{R}^3} u v dx \).

Consider the derivative \( \partial_{(u, \lambda)} F \) at the point \( (u, \lambda, p) = (Q_2, \lambda_2, 2) \). For simplicity, we write \( T = \partial_{(u, \lambda)} F |_{(u, \lambda, p) = (Q_2, \lambda_2, 2)} \). We claim that the inverse of \( T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \) exists. That is, we have to show that for any \( (f, \alpha) \in \mathbb{R} \times \mathbb{R} \) given, there exists a unique \( (g, \beta) \in \mathbb{R} \times \mathbb{R} \) such that the following system is satisfied:
\[
(Id + K_2)g + W_2 \beta = f,
\]
\[
2(Q_2, g) = \alpha.
\]
Note that \( K_2 \) is given by
\[
K_2 \xi = -\frac{1}{-\Delta + \lambda_2} \left( \left( |x|^{-1} * Q_2^2 \right) \xi + 2 \left( |x|^{-1} * (Q_2 \xi) \right) Q_2 \right),
\]
and \( W_2 \) is given by
\[
W_2 = \frac{1}{(-\Delta + \lambda_2)^2} \left( |x|^{-1} * Q_2^2 \right) Q_2.
\]
It is straightforward to verify that \( K_2 \) satisfies the identity
\[
(Id + K_2) = (-\Delta + \lambda_2)^{-1} \mathcal{L}_{+2},
\]
where \( \mathcal{L}_{+2} \) is defined as in (1.5) with \( p = 2 \), and \( W_2 \) satisfies the identity
\[
(-\Delta + \lambda_2) W_2 = Q_2.
\]
We claim that \( Id + K_2 \) has a bounded inverse on \( L^2_{\text{rad}}(\mathbb{R}^3) \). Otherwise, \(-1\) belongs to the spectrum of \( K_2 \). Since \( K_2 \) is a compact operator on \( L^2_{\text{rad}}(\mathbb{R}^3) \), we know that \(-1\) is an eigenvalue of \( K_2 \). Thus there exists \( v \in L^2_{\text{rad}}(\mathbb{R}^3) \), \( v \neq 0 \), such that \( (Id + K_2)v = 0 \). But then (3.3) gives that \( \mathcal{L}_{+2} v = 0 \). However, by the nondegeneracy result of Lenzmann [9] (that is, Theorem 1.3 with \( p = 2 \)), we have that \( v = 0 \). We obtain a contradiction. This proves the claim. Moreover, since \( K_2 : \mathbb{R} \to \mathbb{R} \) holds (see the proof of Lemma B.1 for details), we
deduce that \((\text{Id} + K_2)^{-1}\) exists on the space \(X\) as well. Hence we can solve Eq. (3.1) for \(g\) uniquely by

\[
g = (\text{Id} + K_2)^{-1} (f - W_2 \beta).
\]

Combining this equation together with (3.2) yields

\[
2(Q_2, (\text{Id} + K_2)^{-1} W_2)\beta = 2(Q_2, (\text{Id} + K_2)^{-1} f) - \alpha.
\]

Thus, to solve \(\beta\) uniquely, it is equivalent to show that \(2(Q_2, (\text{Id} + K_2)^{-1} W_2) \neq 0\). To see this, we use the fact that

\[
\mathcal{L}_{+2}R = -2\lambda_2 Q_2,
\]

where \(R = 2Q_2 + x \cdot \nabla Q_2\) (see (4–28) of Lenzmann [9]). Then using the identities (3.3) (3.4) and (3.5) gives us that

\[
2(Q_2, (\text{Id} + K_2)^{-1} W_2) = -\frac{1}{2\lambda_2} \int_{\mathbb{R}^3} |Q_2|^2 dx \neq 0.
\]

This proves the claim that \(T\) has an inverse mapping. Finally, applying the implicit function theorem to the map \(F\) at \((Q_2, \lambda_2, 2)\) as that of Frank and Lenzmann [5, Proposition 5.2], we derive the assertions (1)–(3) provided that \(\delta > 0\) is sufficiently small. The proof of Proposition 3.1 is complete.

Next we prove Theorem 1.3. We follow the argument of Lenzmann [9, Theorem 3].

**Proof of Theorem 1.3** Assume that \(N = 1\). Let \((Q_2, \lambda_2)\) be the unique positive radial ground state to Eq. (1.1) with \(\lambda = \lambda_2\) and consider the linear operator \(\mathcal{L}_{+2}\) associated to \(Q_2\) defined as in (1.5) with \(p = 2\). Then it was given by Lenzmann [9, Theorem 4] that

\[
\text{Ker}\mathcal{L}_{+2} = \text{span}\{\partial_{x_1} Q_2, \partial_{x_2} Q_2, \partial_{x_3} Q_2\}.
\]

On the other hand, let \(\delta > 0\) be defined as in Theorem 1.2 and consider \(2 < p < 2 + \delta\). Let \((Q_p, \lambda_p)\) be the unique positive ground state to Eq. (1.1) with \(\lambda = \lambda_p\). Consider the linear operator \(\mathcal{L}_{+,p}\) associated to \(Q_p\) defined as in (1.5). By differentiation, we deduce that

\[
\text{span}\{\partial_{x_1} Q_p, \partial_{x_2} Q_p, \partial_{x_3} Q_p\} \subseteq \text{Ker}\mathcal{L}_{+,p}.
\]

Our aim is to show that above equality is attained. The idea is to show that the dimension of \(\text{Ker}\mathcal{L}_{+,p}\) is at most three. We use the following standard perturbation argument.

As pointed out by Lenzmann [9], 0 is an isolated eigenvalue of the spectrum of \(\mathcal{L}_{+,2}\). Consider the integral of the resolvent of \(\mathcal{L}_{+,2}\)

\[
P_0 = \frac{1}{2\pi i} \oint_{\partial D_r} (\mathcal{L}_{+,2} - z)^{-1} dz,
\]

the so called Riesz projection \(P_0\) of \(\mathcal{L}_{+,2}\) on its kernel \(\text{Ker}\mathcal{L}_{+,2}\), where \(D_r = \{z \in \mathbb{C} : |z| < r\}\). Here we choose \(r\) sufficiently small such that 0 is the unique eigenvalue of \(\mathcal{L}_{+,2}\) on the closed ball \(\bar{D}_r\). We claim that the projection

\[
P_{0,p} = \frac{1}{2\pi i} \oint_{\partial D_r} (\mathcal{L}_{+,p} - z)^{-1} dz
\]

exists for \(2 < p < 2 + \delta'\), where \(0 < \delta' \leq \delta\) is a sufficiently small number, and satisfies

\[
\|P_{0,p} - P_0\|_{L^2 \to L^2} \to 0 \quad \text{as} \quad p \to 2.
\]

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Indeed, we conclude by the remark that follows the proof of Theorem 2.1 and Lemma B.1 that
\[
\| (L_{+} + z)^{-1} - (L_{+} - z)^{-1} \|_{L^2 \to L^2} \leq C \| (L_{+} - z)^{-1} \|_{L^2 \to L^2}
\]
for \( p > 2 \) and sufficiently close to 2 and \( z \in \partial D_r \), where \( C > 0 \) is a constant, and that
\[
\| (L_{+} + z)^{-1} - (L_{+} - z)^{-1} \|_{L^2 \to L^2} \to 0 \quad \text{as} \quad p \to 2.
\]
This shows that \( P_{0, p} \) exists for \( 2 < p < 2 + \delta' \) and (3.7) holds, provided that \( 0 < \delta' \leq \delta \) is sufficiently small. Since \( rank P_{0, p} = 3 \) and \( P_{0, p} \) remains constant for \( 2 < p < 2 + \delta' \), we deduce by (3.7) that \( P_{0, p} \) has at most 3 eigenvalues (counted with their multiplicity) on \( D_r \) for \( 2 < p < 2 + \delta' \). In particular, we obtain that
\[
\dim \ker L_{+, p} \leq 3
\]
for \( 2 < p < 2 + \delta' \). Therefore the equality in (3.6) must hold for \( 2 < p < 2 + \delta' \). This finishes the proof of Theorem 1.3.

\[\square\]

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**Appendix A: Proof of Theorem 1.1**

In this section, we give a short proof for the existence part of Theorem 1.1 for the sake of completeness. A complete proof of Theorem 1.1 can be found in Moroz and Van Schaftingen [17]. We start the proof with the following observation.

**Lemma A.1** Assume that \( 5/3 < p < 7/3 \). Then \(-\infty < m(N, p) < 0 \) for any \( N > 0 \).

**Proof** It is a consequence of Lemmas 2.2 and 2.4. \( \square \)

By Lemma 2.3, we have the following conclusion.

**Lemma A.2** The infimum \( m(N, p) \) is strictly decreasing with respect to \( N \).

To prove the existence of minimizers of problem (1.4), we apply the rearrangement technique. For any given function \( u \in H^1(\mathbb{R}^3) \), we denote by \( u^* \) the symmetric-decreasing rearrangement of \( u \). The following properties hold for all \( u \in H^1(\mathbb{R}^3) \)
\[
\| u^* \|_q = \| u \|_q \quad \forall \; 2 \leq q \leq 6; \quad (A.1)
\]
\[
K(u^*) \leq K(u); \quad (A.2)
\]
\[
D_p(u^*) \geq D_p(u). \quad (A.3)
\]
Equality of (A.3) is attained if and only if \( u(x) = u^*(x - x_0) \) for some \( x_0 \in \mathbb{R}^3 \). For the precise definition of \( u^* \) and the proof of above properties, we refer to e.g. Lieb and Loss [11].

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** By Lemma 2.3, it suffices to prove Theorem 1.1 in the case \( N = 1 \). Recall that we denote \( m(p) = m(1, p) \). That is,
\[
m(p) = \inf \{ E_p(u) : u \in A_1 \}. \quad (A.4)
\]
Our first aim is to show that \(m(p)\) is attained. Let \(\{u_n\} \subset \mathcal{A}_1\) be a minimizing sequence of problem (A.4). Consider the symmetric-decreasing rearrangement \(u_n^{\ast}\) for all \(n \in \mathbb{N}\). By (A.1) (A.2) and (A.3), we deduce that the sequence \(\{u_n^\ast\}\) is also a minimizing sequence of problem (A.4). On the other hand, since \(0 < 3p - 5 < 2\), we deduce from (2.7) that

\[
E_p(u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - C_p \quad \forall u \in \mathcal{A}_1,
\]

for a constant \(C_p > 0\) depending only on \(p\). Therefore we conclude by using above estimate that \(\{u_n^\ast\}\) is a bounded sequence in \(H^1(\mathbb{R}^3)\). Hence there exists a function \(u \in H^1(\mathbb{R}^3)\) such that

\[
u_n^\ast \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^3).
\]

Moreover, we have that \(u \in H^1_{\text{rad}}(\mathbb{R}^3)\) and \(u \geq 0\) hold since \(u_n^\ast\) are nonnegative radial functions for all \(n \in \mathbb{N}\). Furthermore, by the compactness embedding \(H^1_{\text{rad}}(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)\) for any \(2 < q < 6\) (see Strauss [20]), we can assume by passing to a subsequence that

\[
u_n^\ast \to u \quad \text{in} \quad L^{6p/5}(\mathbb{R}^3).
\]

Then it follows from (A.5) that

\[
\int_{\mathbb{R}^3} |\nabla u_n^\ast|^2 \, dx = \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + o(1),
\]

where \(v_n = u_n^\ast - u\), and it follows from (A.6) that

\[
\int_{\mathbb{R}^3} (I_2 * u_n^{\ast p}) u_n^{\ast p} \, dx = \int_{\mathbb{R}^3} (I_2 * u^{p}) u^p \, dx + o(1)
\]

as \(n \to \infty\). Therefore, combining above two equalities gives us that

\[
E_p(u_n^\ast) = K(v_n) + E_p(u) + o(1)
\]

as \(k \to \infty\). Since \(K(v_n) \geq 0\), we obtain that \(E_p(u) \leq m(p) < 0\). In particular, we obtain that \(u \neq 0\).

To show that \(E_p(u) = m(p)\), it suffices to show that \(u \in \mathcal{A}_1\). Suppose that \(u \notin \mathcal{A}_1\) holds. Since \(u_n^\ast \rightharpoonup u\) in \(L^2(\mathbb{R}^3)\), we have \(\|u\|_2 \leq \liminf_n \|u_n^\ast\|_2 = 1\). Hence there exists \(\alpha \in (0, 1)\) such that \(\|u\|_2 = \alpha\), that is, \(u \in \mathcal{A}_\alpha\). Then \(m(\alpha, p) \leq E_p(u)\) holds. Note that \(m(\alpha, p) > m(1, p) = m(p)\) by Lemma A.2. We obtain that

\[
\begin{align*}
m(p) < m(\alpha, p) & \leq E_p(u) \leq m(p),
\end{align*}
\]

which is impossible! Hence \(u \in \mathcal{A}_1\) holds. Then we obtain \(E_p(u) = m(p)\). This shows that \(u\) is a minimizer of problem (A.4).

Suppose now \(Q \in \mathcal{A}_1\) is an arbitrary minimizer of problem (A.4). Then we have \(Q^\ast \in \mathcal{A}_1\) by (A.1), which implies that \(E_p(Q^\ast) \geq E_p(Q)\). On the other hand, by (A.2) and (A.3) we derive \(E_p(Q^\ast) \leq E_p(Q)\). Hence, we have \(E_p(Q^\ast) = E_p(Q)\), from which we infer that \(Q^\ast\) is also a minimizer of problem (A.4), and that the equalities in (A.2) (A.3) are attained at \(u = Q\). Therefore, there exists a point \(x_0 \in \mathbb{R}^3\) such that \(Q(x) = Q^\ast(x - x_0)\). This proves that every minimizer of problem (A.4) is a nonnegative radial function with respect to every point \(x_0 \in \mathbb{R}^3\) and symmetric-decreasing with respect to \(r = |x - x_0|\).

Next we prove that \(Q\) is positive everywhere. It is well known that \(Q\) solves Eq. (1.1) with a positive Lagrange multiplier \(\lambda > 0\). Thus we obtain from Eq. (1.1) that \(Q\) satisfies

\[
Q = \frac{1}{-\Delta + \lambda} \left(|x|^{-1} * Q^p\right) Q^{p-1}.
\]
As the integral kernel of $\frac{1}{-\Delta + \lambda}$ is positive everywhere and $Q$ is nonnegative nontrivial, we infer from above formula that $Q$ is strictly positive in $\mathbb{R}^3$.

Now applying Proposition 4.1 of Moroz and Schaftingen [17], we obtain that $Q \in W^{2,s}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ for any $s > 1$. By the maximum principle, we conclude that $Q'(|x|) < 0$ for $|x| \neq 0$. The last assertion of Theorem 1.1 is covered by Theorem 4 of Moroz and Schaftingen [17]. We omit the details. The proof of Theorem 1.1 is complete. \hfill \Box

### Appendix B: Regularity of $F$

Recall that in Sect. 3 we denote

$$\mathbb{X} = L^2_{\text{rad}}(\mathbb{R}^3) \cap L^6_{\text{rad}}(\mathbb{R}^3)$$

equipped with norm $\|u\|_\mathbb{X} = \|u\|_2 + \|u\|_6$. Define the map $F : \mathbb{X} \times \mathbb{R}_+ \times (2, 7/3) \to \mathbb{X} \times \mathbb{R}$ by

$$F(u, \lambda, p) = \left( u - \frac{1}{-\Delta + \lambda} (|x|^{-1} * |u|^p) |u|^{p-2}u \right),$$

for $u \in \mathbb{X}$, $\lambda \in \mathbb{R}_+$ and $p \in (2, 7/3)$. Here $c_0$ is a fixed constant. For simplicity, we write $I = (2, 7/3)$ below. Since we will make a series of estimates, it is convenient to use the standard notation $A \lesssim B$ to denote $A \leq CB$ for some constant $C > 0$ that only depends on some fixed quantities. We also write $A \lesssim_{a,b,...} B$ to underline that $C$ depends on the fixed quantities $a, b, \ldots$ etc.

#### Lemma B.1

_The map $F : \mathbb{X} \times \mathbb{R}_+ \times I \to \mathbb{X} \times \mathbb{R}$ is $C^1$._

**Proof** First, we prove that $F : \mathbb{X} \times \mathbb{R}_+ \times I \to \mathbb{X} \times \mathbb{R}$ is well defined. Note that there holds

$$\left\| \frac{1}{-\Delta + \lambda} v \right\|_{H^2(\mathbb{R}^3)} \lesssim_\lambda \|v\|_2 \quad \text{for } v \in L^2(\mathbb{R}^3). \quad (B.1)$$

Since $H^2(\mathbb{R}^3)$ is continuously embedded in $L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $H^2(\mathbb{R}^3)$ is continuously embedded in $\mathbb{X}$ continuously as well. Thus we only need to show that $(|x|^{-1} * |u|^p) |u|^{p-2}u \in L^2_{\text{rad}}(\mathbb{R}^3)$ holds for any $u \in \mathbb{X}$ and $p \in I$. For notational simplicity, write

$$g(u, p) = (|x|^{-1} * |u|^p) |u|^{p-2}u.$$

Set $q = 6(2p - 1)/7$. Then we have $2 \leq p < q < 6$ since $p \in I$. By interpolation, it is elementary to compute that

$$\|u\|_q \leq \|u\|_\mathbb{X} \quad (B.2)$$

for $u \in \mathbb{X}$. Then Young’s inequality gives us that

$$\left\| |x|^{-1} * |u|^p \right\|_r \lesssim_\rho \|u\|_q^p \lesssim_\rho \|u\|_\mathbb{X}^p \quad (B.3)$$

where $r$ is given by

$$\frac{1}{r} + 1 = \frac{1}{2} + \frac{p}{q}. \quad (B.4)$$

It is elementary to obtain that

$$\|u|^{p-2}u\|_{\frac{q}{p-1}} = \|u\|_{\frac{q-1}{q}}^{p-1} \leq \|u\|_{\mathbb{X}}^{p-1}. \quad (B.5)$$

\[\square\] Springer
Hence Combining (B.3) and (B.5) yields that
\[ \|g(u, p)\|_2 \lesssim_p \|u\|_\infty^{2p-1}. \] (B.6)

Thus, by (B.6) and (B.1) we deduce that
\[ \left\| \frac{1}{-\Delta + \lambda} g(u, p) \right\|_\infty \lesssim \left\| \frac{1}{-\Delta + \lambda} g(u, p) \right\|_{H^2} \lesssim_{\lambda, p} \|u\|_\infty^{2p-1}. \]

This proves that \( F \) is well defined.

Next we turn to the Fréchet differentiability of \( F \). It is straightforward to verify that the second component \( F_2 \) of \( F \) is continuously Fréchet differentiable and its Fréchet derivative at \( u \) is given by \( F_2'(u) = 2(u, \cdot) \), where \( (u, \cdot) \) denotes the map \( g \mapsto \langle u, g \rangle \). Let us now turn to consider the Fréchet differentiability the first component
\[ F_1 = u - \frac{1}{-\Delta + \lambda} g(u, p). \]

We claim that \( F_1 \in C^1 \) and its partial derivatives are given by
\[ \frac{\partial F_1}{\partial u} = \frac{1}{1 - \frac{1}{-\Delta + \lambda} g_u(u, p)}, \quad \frac{\partial F_1}{\partial \lambda} = \frac{1}{(-\Delta + \lambda)^2} g(u, p) \]
and
\[ \frac{\partial F_1}{\partial p} = -\frac{\partial g(u, p)}{\partial p}, \]
where \( g_u(u, p) = \partial_u g(u, p) : \mathbb{X} \to \mathbb{X} \) is given by
\[ g_u(u, p) f = (|x|^{-1} * (p|u|^{p-2}u f)) |u|^{p-2}u + (|x|^{-1} * |u|^p) (p - 1)|u|^{p-2}f, \]
and
\[ g_p(u, p) = \frac{\partial g(u, p)}{\partial p} = (|x|^{-1} * (|u|^p \log |u|)) |u|^{p-2}u + (|x|^{-1} * |u|^p)|u|^{p-2}u \log |u|. \]

This claim follows in a standard way by using Sobolev inequalities, Hölder’s inequality and estimates such as (B.2) (B.3) (B.5) and the regularity of functions such as \( t \mapsto t^{p-1} \) with \( p \geq 2 \). In the following we prove the claim of \( \partial F_1/\partial u \). The claims of other two partial derivatives \( \partial_i F_1, \partial_p F_1 \) can be proved similarly.

First we prove that \( \partial F_1/\partial u \) exists and be given as above. So we have to show that for any \( h \in \mathbb{X} \),
\[ F_1(u + h, \lambda, p) - F_1(u, \lambda, p) - \frac{\partial F_1}{\partial u}(u, \lambda, p)h = o(1)h, \] (B.7)
where \( o(1) \to 0 \) as \( \|h\|_\infty \to 0 \). By a direct calculation, we obtain that
\[ F_1(u + h, \lambda, p) - F_1(u, \lambda, p) - \frac{\partial F_1}{\partial u}(u, \lambda, p)h = -\frac{1}{-\Delta + \lambda} \sum_{i=1}^{3} M_i, \]
where \( M_i, i = 1, 2, 3 \), are given by
\[ M_1 = (|x|^{-1} * (|u + h|^p - |u|^p - p|u|^{p-2}uh)) |u + h|^{p-2}(u + h), \]
\[ M_2 = (|x|^{-1} * |u|^p) (|u + h|^{p-2}(u + h) - |u|^{p-2}u - (p - 1)|u|^{p-2}h), \]
\[ M_3 = (|x|^{-1} * p|u|^{p-2}uh) (|u + h|^{p-2}(u + h) - |u|^{p-2}u). \]
respectively. We always assume that $\|h\|_X \leq 1$ since we let $\|h\|_X$ tend to zero in the end. Note that

$$|u + h|^p - |u|^p - p|u|^{p-2}uh| \lesssim_p (|u|^{p-2} + |h|^{p-2})|h|^2, \quad (B.8)$$

Then by (B.3) (B.8) and (B.2) we deduce that

$$\|x|^{-1} \star (|u + h|^p - |u|^p - p|u|^{p-2}uh)\|_r \lesssim_p \|u\|_X \|h\|_X^2,$$

where $r$ is defined as in (B.4) with $q = 6(2p - 1)/7$. Then combining above estimate and (B.5) as before implies that

$$\|M_1\|_2 \lesssim_p \|u\|_X \|h\|_X^2.$$

Similarly, since $p \geq 2$, we deduce that

$$\|M_2\|_2 \lesssim_p \|u\|_X \circ(1) |h|_X,$$

where $\circ(1) \to 0$ as $\|h\|_X \to 0$, and that

$$\|M_3\|_2 \lesssim_p \|u\|_X \|h\|_X^2.$$

Therefore, we obtain by combining above three estimates together that

$$\left\| F_1(u + h, \lambda, p) - F_1(u, \lambda, p) - \frac{\partial F_1}{\partial u}(u, \lambda, p) \right\|_{X} \lesssim_p \|u\|_X \circ(1) |h|_X.$$

This proves (B.7) and thus $\partial_{u} F_1$ exists and be given as claimed.

Next we prove that $\partial_{u} F_1$ depends continuously on $(u, \lambda, p)$. Fix an arbitrary point $(u, \lambda, p) \in \mathbb{X} \times \mathbb{R}_+ \times I$ and let $\epsilon > 0$. We have to find $\delta > 0$ such that

$$\left\| \left( \frac{\partial F_1}{\partial u}(u, \lambda, p) - \frac{\partial F_1}{\partial u}(\tilde{u}, \tilde{\lambda}, \tilde{p}) \right) f \right\|_{X} \leq \epsilon \|f\|_{X}, \quad (B.9)$$

whenever $\|u - \tilde{u}\|_X + |\lambda - \tilde{\lambda}| + |p - \tilde{p}| \leq \delta$ holds for $(\tilde{u}, \tilde{\lambda}, \tilde{p}) \in \mathbb{X} \times \mathbb{R}_+ \times I$.

Note that

$$\left\| \left( \frac{\partial F_1}{\partial u}(u, \lambda, p) - \frac{\partial F_1}{\partial u}(\tilde{u}, \tilde{\lambda}, \tilde{p}) \right) f \right\|_{X} \leq \left\| \frac{1}{-\Delta + \lambda} \left( g_u(\tilde{u}, \tilde{p}) - g_u(u, p) \right) f \right\|_{X} + \left\| \frac{1}{-\Delta + \tilde{\lambda}} - \frac{1}{-\Delta + \lambda} \right\| \left( \frac{1}{-\Delta + \tilde{\lambda}} g_u(\tilde{u}, \tilde{p}) f \right)_{X}$$

$$=: J_1 + J_2.$$

In the following we only prove (B.9) for the first term $J_1$. That is, there exists $\delta > 0$, such that whenever $\|u - \tilde{u}\|_X + |\lambda - \tilde{\lambda}| + |p - \tilde{p}| \leq \delta$ holds for $(\tilde{u}, \tilde{\lambda}, \tilde{p}) \in \mathbb{X} \times \mathbb{R}_+ \times I$, then

$$J_1 \equiv \left\| \frac{1}{-\Delta + \lambda} \left( g_u(\tilde{u}, \tilde{p}) - g_u(u, p) \right) f \right\|_{X} \leq \epsilon \|f\|_{X}. \quad (B.10)$$

The estimate of $J_2$ can be derived in the same way as that of Frank and Lenzmann [9, Appendix E], where even more general operators are considered. For example, the $C^1$ continuity about the parameters $s$ and $\lambda$ of the operator $((-\Delta)^s + \lambda)^{-1}$ is proven in Frank and Lenzmann [9, Lemma E.1].

To prove (B.10), (B.1) implies that it is sufficient to prove

$$\left\| \left( g_u(\tilde{u}, \tilde{p}) - g_u(u, p) \right) f \right\|_{2} \lesssim_{\lambda, p} \epsilon \|f\|_{X} \quad (B.11)$$
whenever \( \|u - \tilde{u}\|_X + |p - \tilde{p}| \leq \delta \) holds 1 > \( \delta > 0 \) small enough and \((\tilde{u}, \tilde{p}) \in \mathbb{X} \times I\). Note that

\[
\|(g_a(\tilde{u}, \tilde{p}) - g_a(u, p)) f\|_2 \leq \left\| \left( g_a(\tilde{u}, \tilde{p}) - g_a(u, \tilde{p}) \right) f \right\|_2 + \left\| \left( g_a(u, \tilde{p}) - g_a(u, p) \right) f \right\|_2.
\]

Denote

\[
L_1 = \left( g_a(\tilde{u}, \tilde{p}) - g_a(u, \tilde{p}) \right) f \quad \text{and} \quad L_2 = \left( g_a(u, \tilde{p}) - g_a(u, p) \right) f.
\]

We show that

\[
\|L_1\|_2 \lesssim_{p, \|u\|_X} \|f\|_X
\]

and that

\[
\|L_2\|_2 \lesssim_{p, \|u\|_X} \|f\|_X
\]

hold. In the sequel we estimate \(L_1\) and \(L_2\) one by one.

First we estimate \(L_1\), it is easy to obtain that \(L_1 = \sum_{i=1}^{4} L_{i1}\), where

\[
L_{11} = \left( |x|^{-1} * |u|^{\tilde{p}} \right) (\tilde{p} - 1) \left( |\tilde{u}|^{\tilde{p}} - 2 - |u|^{\tilde{p}} - 2 \right) f,
\]

\[
L_{12} = \left( |x|^{-1} * (|\tilde{u}|^{\tilde{p}} - |u|^{\tilde{p}}) \right) (\tilde{p} - 1)|\tilde{u}|^{\tilde{p}} - 2 f,
\]

\[
L_{13} = \left( |x|^{-1} * \tilde{p} \left( |\tilde{u}|^{\tilde{p}} - 2 \tilde{u} - |u|^{\tilde{p}} - 2 u \right) f \right) |u|^{\tilde{p}} - 2 u,
\]

and

\[
L_{14} = \left( |x|^{-1} * \tilde{p} |\tilde{u}|^{\tilde{p}} - 2 \tilde{u} f \right) \left( |\tilde{u}|^{\tilde{p}} - 2 \tilde{u} - |u|^{\tilde{p}} - 2 u \right),
\]

respectively. We will estimate \(L_{11}, L_{12}\) for instance and leave the estimates of \(L_{13}, L_{14}\) for the interested readers. We assume that \(\tilde{p} > 2\), for otherwise \(L_{11} \equiv 0\), we are done. Note that

\[
|u|^{\tilde{p}} - 2 - |\tilde{u}|^{\tilde{p}} - 2 \leq |u - \tilde{u}|^{\tilde{p}} - 2 \text{ since } 0 < \tilde{p} - 2 < 1.
\]

Set \(\tilde{q} = 6(2\tilde{p} - 1)/7\). Thus (B.5) gives that

\[
\| \left( |\tilde{u}|^{\tilde{p}} - 2 - |u|^{\tilde{p}} - 2 \right) f \|_{\tilde{p}^{-1}} \leq \|u - \tilde{u}\|_{\tilde{q}}^{\tilde{p} - 2} \|f\|_{\tilde{q}} \leq \delta^{\tilde{p} - 2} \|f\|_X.
\]

Combining above inequality together with (B.3) yields that

\[
\|L_{11}\|_2 \lesssim_{p, \|u\|_X} \delta^{\tilde{p} - 2} \|f\|_X.
\]

Here we used the assumption \(\|u - \tilde{u}\|_X + |\tilde{p} - p| < \delta\), which implies that \(\|\tilde{u}\|_X \leq \|u\|_X + 1\) and \(\tilde{p} \leq p + 1\). To estimate \(L_{12}\), note that \(|u|^{\tilde{p}} - |\tilde{u}|^{\tilde{p}} \lesssim_p \left( |u|^{\tilde{p} - 1} + |\tilde{u}|^{\tilde{p} - 1} \right) |u - \tilde{u}|\). Then (B.3) implies that

\[
\left\| \left( |x|^{-1} * \left( |u|^{\tilde{p}} - |\tilde{u}|^{\tilde{p}} \right) \right) \right\|_{\tilde{p}} \lesssim_{p, \|u\|_X} \|u - \tilde{u}\|_{\tilde{q}} \lesssim_{p, \|u\|_X} \delta
\]

since \(\|u - \tilde{u}\|_{\tilde{q}} \leq \|u - \tilde{u}\|_X \leq \delta\), where \(r\) is given by \(1/\tilde{r} + 1 = 1/3 + \tilde{p}/\tilde{q}\) with \(\tilde{q} = 6(2\tilde{p} - 1)/7\). Thus combining above inequality together with (B.5) gives us that

\[
\|L_{12}\|_2 \lesssim_{p, \|u\|_X} \delta \|f\|_X.
\]

Since we can obtain similar estimates for \(L_{13}, L_{14}\) as above, it becomes obvious from e.g. (B.14) and (B.15) that we can choose \(\delta > 0\) sufficiently small such that (B.12) holds. This gives the estimate of \(L_1\).
Next we estimate $L_2$. We have $L_2 = \sum_{i=1}^{4} L_{2i}$, where

\[
L_{21} = (|x|^{-1} * |u|^p) \left((\tilde{p} - 1)|u|^{\tilde{p}-2} - (p - 1)|u|^{p-2}\right) f,
\]
\[
L_{22} = (|x|^{-1} * (|u|^{\tilde{p}} - |u|^p)) \left((\tilde{p} - 1)|u|^{\tilde{p}-2} f,\right.
\]
\[
L_{23} = (|x|^{-1} * (p|u|^{p-2} - 2f)) \left(|u|^{\tilde{p}-2}u - |u|^{p-2}u\right),
\]
and
\[
L_{24} = (|x|^{-1} * \left(\tilde{p}|u|^{\tilde{p}-2}u - p|u|^{p-2}f\right)) |u|^{\tilde{p}-2}u
\]
respectively. We estimate the first term $L_{21}$ for instance, and leave the estimates of $L_{22}$, $L_{23}$ and $L_{24}$ for the interested readers. Note that

\[
|p - 1||u|^{p-2} - (\tilde{p} - 1)|u|^{\tilde{p}-2} \leq |p - \tilde{p}||u|^{p-2} + (\tilde{p} - 1)\left(|u|^{p-2} - |u|^{\tilde{p}-2}\right). \quad \text{(B.16)}
\]

Set $q = 6(2p - 1)/7$. Then by (B.3), (B.5) and above inequality we deduce that

\[
\|L_{21}\|_{2} \lesssim_{p, \|u\|_{\infty}} \left(\|p - \tilde{p}\| + \|u|^{p-2} - |u|^{\tilde{p}-2}\right) \|f\|_{\infty}.
\]

We estimate the second term in the bracket of above inequality as follows. We only consider the case $\tilde{p} > p$. The case $\tilde{p} < p$ can be considered similarly. By an elementary calculation, we obtain that

\[
|u|^{p-2} - |u|^{\tilde{p}-2} \lesssim_{p, \tilde{p}} M |p - \tilde{p}| |u|^{p-2} \quad \text{on } \{1/M \leq |u| \leq M\},
\]
for any given constant $M > 1$, and

\[
|u|^{p-2} - |u|^{\tilde{p}-2} \leq 2|u|^{\tilde{p}-2} \chi_{\{|u| > M\}} \quad \text{on } \{|u| > M\},
\]
where $\chi_{\{|u| > M\}}(x) = 1$ if $|u(x)| > M$ and $\chi_{\{|u| > M\}}(x) = 0$ if $|u(x)| \leq M$, and

\[
|u|^{p-2} - |u|^{\tilde{p}-2} \leq 2|u|^{p-2} \chi_{\{|u| < 1/M\}} \quad \text{on } \{|u| < 1/M\}.
\]

Thus we can obtain that

\[
\left\|u|^{p-2} - |u|^{\tilde{p}-2}\right\|_{p=2}^{q} \leq C(M, p, \|u\|_{\infty}) \|p - \tilde{p}\| + C_{p} \left(\int_{\mathbb{R}^{3}} |u|^{\tilde{p}-2q} \chi_{\{|u| > M\}} \right)^{(p-2)/q}
\]
\[
+ \left\|u \chi_{\{|u| < 1/M\}}\right\|_{p}^{p-2}.
\]

It is elementary to show that

\[
\left(\int_{\mathbb{R}^{3}} |u|^{\tilde{p}-2q} \chi_{\{|u| > M\}} \right)^{(p-2)/q} \rightarrow 0 \quad \text{as } M \rightarrow \infty \text{ and } \tilde{p} \rightarrow p
\]
and that

\[
\left\|u \chi_{\{|u| < 1/M\}}\right\|_{q}^{p-2} \rightarrow 0 \quad \text{as } M \rightarrow \infty.
\]

Thus for given $\epsilon > 0$, we first choose $M > 1$ sufficiently large such that

\[
\left(\int_{\mathbb{R}^{3}} |u|^{\tilde{p}-2q} \chi_{\{|u| > M\}} \right)^{(p-2)/q} + \left\|u \chi_{\{|u| < 1/M\}}\right\|_{p}^{p-2} \lesssim_{p} \epsilon,
\]
and then fix such $M$ and choose $\delta > 0$ sufficiently small enough such that $C(M, \|u\|_{\infty}, p) |p - \tilde{p}| \lesssim \epsilon$. This proves that
\[ \|L_{21}\|_2 \lesssim_p, \|u\|_{\infty}, \epsilon \|f\|_{\infty}. \] (B.17)

Since we can derive above estimates for $L_{22}$, $L_{23}$ and $L_{24}$, we conclude that (B.13) holds. This gives the estimate for $L_2$.

Finally, combining (B.1), (B.11), (B.12) and (B.13) gives us the estimate (B.10), and thus follows the continuity of $\partial_u F_1$. As we can prove similarly the continuity of the derivatives $\partial_\lambda F_1$, $\partial_p F_1$, the proof of Lemma B.1 is complete. $\blacksquare$

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