Research Article

Synchronization of Chaotic Systems with Dead Zones via Fuzzy Adaptive Variable-Structure Control

Yongbing Huangfu and Kaijuan Xue

Department of Mechanical and Electronic Engineering, Shanxi Engineering Vocational College, Taiyuan 030009, China

Correspondence should be addressed to Yongbing Huangfu; hf200133w@163.com

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This work is devoted to solving synchronization problem of uncertain chaotic systems with dead zones. Based on the Lyapunov stability theorems, by using fuzzy inference to estimate system uncertainties and by designing effective fuzzy adaptive controllers, the synchronization between two chaotic systems with dead zones is realized and a fuzzy variable-structure control is implemented. The stability is proven strictly, and all the states and signals are bounded in the closed-loop system. A simulation example is presented to test the theoretical results finally.

1. Introduction

It is widely known that chaos is almost everywhere in the domain of engineering and science. It is a kind of complex dynamic behavior of nonlinear dynamic systems, and chaos has many applications in mechanical, electronics, and biochemistry fields. There are many common chaotic systems, such as Chen system [1, 2], Lorenz system [3], Genesio–Tesi system [4, 5], Rössler system [6], and Lur’e system [7]. Chaotic systems, as everyone knows, have deterministic behavior; for example, they are extremely sensitive when the initial conditions alter to a small extent and difficult to predict, but their trajectories are bounded in the phase space [8, 9].

People always think that chaotic systems cannot be synchronized due to the characteristics of chaos. It was realized first on electronic circuit by Pecora et al. [10, 11]. They found that if the chaotic system can be decomposed into two subsystems, and, in the response system, all the conditional Lyapunov exponents are less than zero, there will be chaotic synchronization effect in the drive and response system [1, 12]. Chaos synchronization means that, for two chaotic systems starting from different initial points, their trajectories gradually tend to be consistent with each other over time, and this synchronization is structurally stable [13–15]. After that, synchronization study in chaotic dynamical systems has received much attention. There are some way to achieve the synchronization of some chaotic systems, for instance, time-delay feedback control [7], active control synchronization [16], impulsive synchronization and synchronization [17, 18], sliding mode synchronization [19, 20], and projective synchronization [15].

The input nonlinearities, such as dead zones and saturation, may destroy the control performance of the system, and the control problem of uncertain nonlinear systems with nonlinear input is getting more complicated and receiving more considerable attention [21, 22]. Control performance of the system may be degraded or even cause the instability of the control system due to input nonlinearities, and their synchronization problem becomes more challenging [23]. Please refer to [15, 24–30] for some works about synchronization study in chaotic dynamical systems, which are subject to input nonlinearity.

Dead zone is that the value of the output variable does not change with the change of the input variable value. The range of the input variable can be understood as dead zone [31]. Previously, many researchers have used various methods to settle the synchronization problem of the nonlinear systems, that is, the control input with dead zones. For instance, using Laplace transform approach to settle synchronization of nonlinear systems with disturbances subjected to dead-zone
and saturation characteristics in control input was studied in [32], projective synchronization of Chua’s chaotic systems that control input has dead zones was studied in [33, 34], and using adaptive fuzzy sliding mode control to deal with unknown nonlinear chaotic gyros synchronization with unknown dead-zone input was studied in [35].

In this paper, we put forward a fuzzy adaptive variable-structure synchronization scheme to manage the dead-zone nonlinearity and analyze the synchronization properties of chaotic systems. Comparing the related works, for example, [26, 32, 34], the contribution of this works consists in the following: (1) Input nonlinearity is considered in this paper. However, it is not considered in the above literature. (2) Compared with [26, 32], the assumptions of this work are more realistic.

This paper is organized as follows. In Section 2, the notation, problem statement, and preliminaries are raised, including the description of the uncertain chaotic MIMO systems, fuzzy logic system, and input nonlinearity. In Section 3, we present a fuzzy adaptive controller based on the universal approximator property. In Section 4, the effectiveness of the approach is tested and verified by a simulation. In addition, conclusions are contained in Section 5.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ represents the real numbers, $\mathbb{R}^n$ represents the real $n$-vectors, and $\mathbb{R}^{m \times n}$ represents the real $m \times n$ matrices. $\| \cdot \|$ represents any suitable vector norm.

Two uncertain chaotic MIMO systems are given as follows. The driving system is expressed by

$$\dot{x}_i = f_1(x),$$

and the response system is given as

$$\dot{y}_i = f_1(y) + \sum_{j=1}^{n} (g_{ij} \Phi_j(u_j)).$$

and (2) can be written as

$$\dot{y} = F(y) + G \Phi(u),$$

where $F(\cdot)$ and $\Phi(\cdot) \in \mathbb{R}^n$ and $G(\cdot) \in \mathbb{R}^{m \times n}$. Some simple assumptions are set forth.

Assumption 1. $G$ is an unknown positive-definite matrix, and one can find an unknown positive constant $\sigma_0$ such that $G \geq \sigma_0 I_n$, with $I_n$ being the identity matrix.

Remark 1. In fact, the above assumption is not restrictive, because many physical systems satisfy it. This assumption that devoted to adaptive control of MIMO systems is ubiquitous in the literature, and without exaggeration, we can say that the controllability of the system is assured by it.

The response system is driven by controller; however, it does not affect the behavior of the driving system. The control purpose of this paper is to put forward a control input $u$ in response system to synchronize the driving systems; that is, all signals of the driving and response systems must be bounded under the constraint.

First, we define the synchronization errors between driving and response systems as

$$e_i = y_i - x_i.$$ (5)

The filtered synchronization errors is given by

$$S = [e_1, \ldots, e_n]^T.$$ (6)

Thus, we have

$$\dot{S} = \dot{y} - \dot{x}.$$ (7)

Afterwards, (7) will be used to develop the controller and conduct the stability analysis.

2.1. Description of the Fuzzy Logic System. By and large, a fuzzy system contains four aspects, that is, fuzzifier, fuzzy rules, fuzzy inference, and defuzzifier, which is depicted in Figure 1. By using proper fuzzy rules, a fuzzy inference drives the input $y^T = [y_1, y_2, \ldots, y_n] \in \mathbb{R}^n$ becoming an output signal $f \in \mathbb{R}$. The i-th fuzzy rule has the following form:

$$\mathbb{R}^{10}: \text{if } y_1 = A_{i1}^t \text{ and } \ldots \text{ and } y_n = A_{in}^t \text{ then } f_i = f_i^t,$$

with $A_1^t, A_2^t, \ldots, A_n^t$ being fuzzy sets. The output of a fuzzy system is given by

$$\tilde{f}(y) = \frac{\sum_{i=1}^{m} f_i^t \prod_{j=1}^{n} \mu_j^f(y_j)}{\sum_{i=1}^{n} \prod_{j=1}^{n} \mu_j^f(y_j)},$$

where $\mu_j^f(y_j)$ means the membership function of $y_j$ to $A_j^t$; suppose that there are $m$ fuzzy rules involved in the fuzzy system, $\theta^T = [f^1, \ldots, f^m]^T$ can be adjusted online, and
that being the fuzzy basis function (FBF) with \( \sum_{i=1}^{m} (\prod_{j=1}^{n} \mu_{A_{j}}(y_{j})) > 0 \).

It can be noted that the fuzzy system (8) is ubiquitous in control applications. In light of the universal approximation results, fuzzy system (8), on a compact operating space, is capable of approximating any nonlinear smooth function \( f \) to an arbitrary degree of accuracy. It is extremely important to assume that the membership function parameters need to be prespecified. That being said, the structure of the fuzzy system needs the designer decision for determination, namely, the pertinent inputs. However, the parameters \( \theta \) have to be defined by learning algorithms.

2.2. Input Nonlinearity. We conduct the input nonlinearity \( \Phi_{i}(u_{i}) \) as

\[
\Phi_{i}(u_{i}) = \begin{cases} 
\phi_{+}(u_{i})(u_{i} - u_{i+}), & u_{i} > u_{i+}, \\
0, & -u_{i-} \leq u_{i} \leq u_{i+}, \\
\phi_{-}(u_{i})(u_{i} + u_{i-}), & u_{i} < -u_{i-},
\end{cases}
\]

where \( \phi_{+}(u_{i}) > 0 \) and \( \phi_{-}(u_{i}) > 0 \) are nonlinear functions with respect to \( u_{i} \) and \( u_{i+}, u_{i-} \) are positive constants.

Here, \( \Phi_{i}(u_{i}) \) has some significant properties as follows:

\[
\begin{cases} 
(u_{i} - u_{i+})\Phi_{i}(u_{i}) \geq m_{i+}^{*}(u_{i} - u_{i+})^2, & u_{i} > u_{i+}, \\
(u_{i} + u_{i-})\Phi_{i}(u_{i}) \geq m_{i-}^{*}(u_{i} + u_{i-})^2, & u_{i} < -u_{i-},
\end{cases}
\]

where \( m_{i+}^{*} \) and \( m_{i-}^{*} \) are constants known as “gain reduction tolerances.” We can define \( \eta_{i} = \min[m_{i+}^{*}, m_{i-}^{*}] \).

Then, we give some reasonable assumptions to study the properties of input nonlinearity in control problems.

Assumption 2

(a) \( m_{i+}^{*}, m_{i-}^{*} \), namely, the gain reduction tolerances, are unknown, so \( \eta_{i} \) is unknown.

(b) The explicit mathematical equation of \( \Phi_{i}(u_{i}) \) is unknown, but we know the properties (11) and assume that \( u_{i+} \) and \( u_{i-} \) are known constants.

Remark 2

(1) It can be seen from (10) and (11) that the input nonlinearity \( \Phi_{i}(u_{i}) \) can be reduced to the special sector nonlinear function if \( u_{i+} = u_{i-} = 0 \). Consequently, the MIMO system with the input nonlinearities (10), which we considered, is more universal.

(2) It can be noted that the model (10) has been widely used in the past, but it has some limitations, and we made some improvements. The limitations are as follows:

(i) The chaotic system, which they considered, is a simple SISO system, which is input with sector nonlinearities and/or dead zones

(ii) They assumed that \( m_{i+}^{*} \) and \( m_{i-}^{*} \) or \( \eta_{i} = \min[m_{i+}^{*}, m_{i-}^{*}] \) are known in their control scheme

3. Design of the Fuzzy Adaptive Controller

In this part, for the class of unknown chaotic MIMO systems, we will develop a fuzzy adaptive variable-structure control plan (3).

Substituting (4) into the (7), we get

\[
\dot{S} = F(y) + G\Phi(u) - \dot{x}.
\]

Now posing \( G_{1} = G^{-1} \), we have

\[
G_{1}\dot{S} = G_{1}(-\dot{x} + F(y)) + \Phi(u).
\]

Further, for the stability analysis and controller design, (13) can be arranged as

\[
G_{1}\dot{S} = G_{1}(-\dot{x} + F(y)) + \Phi(u),
\]

where \( \alpha(y, v) = [\alpha_{1}(y, v), \alpha_{2}(y, v), \ldots, \alpha_{n}(y, v)] = G_{1}[v + F(y)] \) and \( v = -\dot{x} \).

Assumption 3. There is an unknown continuous positive function \( \pi(y) \) satisfying \( |\alpha_{i}(y, v)| \leq \gamma \pi(y) \), where \( \gamma = \min_{i}\{\eta_{i}\} \).

Remark 3. The reasons why the above Assumption 3 is not restrictive are as follows:

(i) We assume that the upper bound \( \gamma \pi(y) \) is unknown

(ii) As \( v \) is a function about \( (y, y_{d}) \), \( y_{d} \in L_{\infty} \) and \( \alpha_{i}(y, v) \) is continuous, \( \pi(y) \) is always there

The unknown continuous positive function \( \pi(y) \) over compact set \( \Omega_{y} \) can be approximated by the fuzzy system (8) as follows:

\[
\hat{\pi}(y, \theta) = \theta^T \psi(y),
\]
where \( \psi_i(y) \) is the FBF vector, which is determined in advance by the designer, and \( \theta_i \) is the adjustable parameter vector in the fuzzy system.

Let

\[
\theta_i^* = \arg \min_{\theta_i} \left[ \sup_{y \in \Omega_i} |\tilde{\alpha}_i(y) - \hat{\alpha}_i(y, \theta_i)| \right]
\]  

be the optimal value of \( \theta_i \).

It is worth noting that, for the sake of analysis, we put forward artificial constant quantities \( \theta_i^* \), and when implementing the controller, their values are not needed.

Fix the parameter estimate error as

\[
\tilde{\theta}_i = \theta_i - \theta_i^*,
\]

and the fuzzy approximation error as

\[
e_i(y) = \tilde{\alpha}_i(y) - \hat{\alpha}_i(y, \theta_i^*),
\]

where \( \tilde{\alpha}_i(y, \theta_i^*) = \theta_i^{*T} \psi_i(y) \).

In this work, assume the compact set \( \Omega_i \), and the fuzzy systems we used do not infringe the universal approximator property, and \( \Omega_i \) is supposed to be large enough, so that it can contain the input vector of the fuzzy system in a closed-loop control system. So, it is rational to suppose that \( e_i(y) \) is bounded for all \( y \in \Omega_i \), i.e., \( |e_i(y)| \leq \bar{e}_i, \forall y \in \Omega_i \), where \( \bar{e}_i \) is an unknown constant.

Then, we have

\[
\tilde{\alpha}_i(y, \theta_i^*) - \alpha_i(y) = \tilde{\alpha}_i(y, \theta_i) - \tilde{\alpha}_i(y, \theta_i^*) + \tilde{\alpha}_i(y, \theta_i^*) - \alpha_i(y),
\]

\[
= \tilde{\alpha}_i(y, \theta_i) - \tilde{\alpha}_i(y, \theta_i^*) + e_i(y),
\]

\[
= \tilde{\theta}_i^T \psi_i(y) - e_i(y).
\]

In order to achieve the control objective, let us propose a suitable fuzzy adaptive variable-structure controller:

\[
u_i = \begin{cases} -\rho_i(t) \text{sign}(S_i) - u_i, & S_i > 0, \\ 0, & S_i = 0, \\ -\rho_i(t) \text{sign}(S_i) + u_i, & S_i < 0, \end{cases}
\]

(20)

with \( \rho_i(t) = k_{ui} + k_{\alpha} |S_i|^2 + \tilde{\theta}_i^T \psi_i(y), \forall i = 1, \ldots, n \) and

\[
\dot{k}_{ui} = -\gamma_{ui} \theta_i + \gamma_{0i} |S_i|, \quad k_{ui}(0) \geq 0,
\]

\[
\dot{\theta}_i = -\gamma_{1i} \theta_i \times \psi_i(y), \quad \theta_i(0) \geq 0,
\]

(21)

(22)

where \( \gamma_{ui}, \gamma_{0i}, \gamma_{1i}, k_{ui} \geq 0 \) are design constants and \( k_{ui} \) and \( \theta_i \) are the online estimates of the uncertain terms \( k_{ui}^* = \bar{e}_i \) and \( \theta_i^* \), respectively.

**Remark 4.** From adaptive laws (21) and (22), we can get their solutions satisfied \( k_{ui}(t) \geq 0 \) and \( \theta_i(t) \geq 0 \), for \( t > 0 \) so that \( k_{ui}(0) \geq 0 \) and \( \theta_i(0) \geq 0 \).

Multiplying (9) by \( (1/\eta)S_i^T \) and using Assumption 3, we have

\[
\frac{1}{\eta} S_i^T G_1(x) \dot{S}_i = \frac{1}{\eta} S_i^T \alpha(y, y) + \frac{1}{\eta} S_i^T \Phi(u),
\]

\[
\leq \sum_{i=1}^{n} |S_i| \tilde{\alpha}_i(y) + \frac{1}{\eta} S_i^T \Phi(u).
\]

(23)

From (19) and (23), we get

\[
\frac{1}{\eta} S_i^T G_1(x) \dot{S}_i \leq \sum_{i=1}^{n} |S_i| \tilde{\alpha}_i(y) + \frac{1}{\eta} S_i^T \Phi(u),
\]

\[
\leq - \sum_{i=1}^{n} |S_i| k_{ui} - \sum_{i=1}^{n} |S_i| \hat{\theta}_i^T \psi_i(y) + \sum_{i=1}^{n} |S_i| k_{ui}
\]

\[
+ \sum_{i=1}^{n} S_i \hat{\theta}_i^T \psi_i(y) + \frac{1}{\eta} S_i^T \Phi(u),
\]

(24)

where \( \tilde{\theta}_i = \theta_i - \theta_i^* \) and \( \tilde{k}_{ui} = k_{ui} - k_{ui}^* = k_{ui} - \bar{e}_i \).

**Theorem 1.** For system (3), if Assumptions 1–3 are satisfied, the control law (20)–(22) can guarantee the following properties:

(i) It is no exaggeration to say that, in the closed-loop system, all signals are uniformly ultimately bounded

(ii) The system enclosed is asymptotically stable

**Proof** of Theorem 1. Let the Lyapunov function be

\[
V = \frac{1}{2\eta} S_i^T G_1 S_i + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\gamma_{ui}} k_{ui}^2 + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\gamma_{1i}} \theta_i^2
\]

\[
(25)
\]

The time derivative of V is

\[
\dot{V} = \frac{1}{\eta} S_i^T G_1 S_i + \sum_{i=1}^{n} \frac{1}{\gamma_{ui}} \dot{k}_{ui} \dot{k}_{ui} + \sum_{i=1}^{n} \frac{1}{\gamma_{1i}} \dot{\theta}_i \dot{\theta}_i
\]

\[
(26)
\]

with \( \dot{G}_1 = 0 \).

It can be noticed from (24) that \( u_t < -u_{i-} \) for \( S_i > 0 \) and \( u_t > u_{i+} \) for \( S_i < 0 \). Thus, from (20) and (24), we can get that for \( S_i > 0 \),

\[
(u_t + u_{i-}) \Phi_i(u_t) = -\rho_i(t) \text{sign}(S_i) \Phi_i(u_t) \geq m_{i+}^* \rho_i^2(t) \geq \eta \rho_i^2(t),
\]

\[
(27)
\]

and for \( S_i < 0 \),

\[
(u_t - u_{i+}) \Phi_i(u_t) = -\rho_i(t) \text{sign}(S_i) \Phi_i(u_t) \geq m_{i-}^* \rho_i^2(t) \geq \eta \rho_i^2(t).
\]

(28)

Then, for \( S_i > 0 \) and \( S_i < 0 \), we have

\[
-\rho_i(t) \text{sign}(S_i) \Phi_i(u_t) \geq \eta \rho_i^2(t).
\]

(29)

Since \( S_i^2 > 0 \) and \( S_i \text{sign}(S_i) = |S_i| \), then from (29), we have

\[
-\rho_i(t) S_i^2 \text{sign}(S_i) \Phi_i(u_t) \geq \eta \rho_i^2(t) S_i^2 = \eta \rho_i^2(t) |S_i|^2.
\]

(30)

For all \( S_i \), we have

\[
\eta \rho_i^2(t) |S_i|^2 \geq \eta \rho_i^2(t) |S_i|^2
\]

(31)
\[ S_i \Phi_i(u_i) \geq -\eta \rho_i(t)|S_i|. \] (31)

Using (21), (22), (24), and (26), (31) becomes
\[ V \leq \sum_{i=1}^{n} S_i |k_i| \psi_i(y) + \frac{1}{\eta} \sum_{i=1}^{n} S_i \Phi_i(u_i) - \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} \]
\[ - \sum_{i=1}^{n} \sigma_i \tilde{\theta}_i \theta_i \leq \sum_{i=1}^{n} S_i |k_i| \psi_i(y) + \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} \]
\[ - \sum_{i=1}^{n} \sigma_i \tilde{\theta}_i \theta_i \leq \sum_{i=1}^{n} S_i |k_i| \psi_i(y) + \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} \]
\[ = - \sum_{i=1}^{n} k_i S_i - \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} - \sum_{i=1}^{n} \sigma_i \tilde{\theta}_i \theta_i. \] (32)

Obviously, we have
\[ -\sigma_{0i} \tilde{k}_{0i} k_{0i} \leq \frac{\sigma_{0i}^2}{2} \tilde{k}_{0i}^2 + \frac{\sigma_{0i}}{2} k_{0i}^2, \]
\[ -\sigma_i \tilde{\theta}_i \theta_i \leq \frac{\sigma_{\theta_i}^2}{2} \tilde{\theta}_i^2 + \frac{\sigma_{\theta_i}}{2} \theta_i^2. \] (33)

Then, (32) becomes
\[ V \leq \sum_{i=1}^{n} k_i S_i - \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} - \sum_{i=1}^{n} \sigma_{\theta_i} \tilde{\theta}_i \theta_i - \sum_{i=1}^{n} \theta_i \psi_i(y) \]
\[ + \sum_{i=1}^{n} \sigma_{0i} \tilde{k}_{0i} k_{0i} \]
\[ + \sum_{i=1}^{n} \sigma_{\theta_i} \tilde{\theta}_i \theta_i. \] (34)

Since \( G \geq \sigma_g I_n \), then we have
\[ S^T G^{-1} S \geq S^T G_1 S \leq \frac{1}{\sigma_{g0}} \|S\|^2. \] (35)

From (34) and (35), we have
\[ V \leq -\mu V + \pi, \] (36)
where
\[ \pi = \sum_{i=1}^{n} \frac{\sigma_{0i}^2}{2} \tilde{k}_{0i}^2 + \sum_{i=1}^{n} \frac{\sigma_{\theta_i}^2}{2} \tilde{\theta}_i^2, \]
\[ \mu = \min \left\{ \frac{2}{\eta} \sigma_{g0}^2, \min_i \left\{ \frac{y_0}{\sigma_{0i}}, \min_i \frac{y_i}{\sigma_{\theta_i}} \right\} \right\}. \] (37)

Multiplying (36) by \( e^{\mu t} \) yields
\[ \frac{d}{dt} (V e^{\mu t}) \leq n e^{\mu t}. \] (38)

Integrating (38) over \([0, t]\), we have
\[ 0 \leq V(t) \leq \frac{\pi}{\mu} + \left( V(0) - \frac{\pi}{\mu} \right) e^{-\mu t}. \] (39)

From the above analysis, we can get \( k_{0i}, \theta_i, S_i, E \) and \( y \) that are uniformly ultimately bounded. Thus, \( u_i \) is bounded. Then, by using (25), \( V(0) \) can be written as
\[ V(0) = \frac{1}{2\eta} S(0)^T G_1 S(0) + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{Y_{0i}} (k_{0i} - k_{0i}^*)^2 \]
\[ + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{Y_{1i}} (\theta_i(0) - \theta_i^*)^T (\theta_i(0) - \theta_i^*). \] (40)

As \( G_1 \) is symmetric positive-definite (i.e., there is an unknown positive constant \( \sigma_g \), such that \( G_1 \geq \sigma_g I_n \)), from (25) and (39), we have
\[ |S_i| \leq \left( \frac{2\eta}{\sigma_g} \left( \frac{\pi}{\mu} + \left( V(0) - \frac{\pi}{\mu} e^{-\mu t} \right) \right) \right)^{1/2}. \] (41)

We can get that the solution of \( S_i \) exponentially converges to a bounded region \( \Omega_S = [S_i]|S_i| \leq ((2\eta/\sigma_g) (\pi/\mu))^{1/2} \). This completes the proof of the theorem. \( \square \)

Remark 5. If \( \Phi_i(u_i) = u_i \) (or when \( u_{i+} = u_{i-} = 0 \), and \( \phi_i(u_i) = \phi_i(u_i) = 1 \), namely, there are neither dead zones, nor sector nonlinearities in the input, we can prove that the controller is still applicable for these MIMO chaotic systems.

Remark 6

(1) There is a special case that \( u_{i+} = u_{i-} = 0 \), and (20) can be simplified to the following form:
\[ u_i = -\rho_i(t) + u_0 \text{sign}(S_i), \] (42)
where \( \rho_i(t) = k_{0i} + k_{1i}|S_i| + \theta_i^T \psi_i(y) \).

(2) It is worth mentioning that the function \( \text{sign}(\cdot) \) can be replaced by any equivalent smooth function: \( \tanh(\cdot), \arctan(\cdot), \text{Sat}(\cdot) \), and so on. The chattering effect caused by the discontinuous control term in (20) and (42) can be removed.

4. Simulation Results

In order to demonstrate the effectiveness of the proposed adaptive fuzzy controller for uncertain chaotic MIMO systems, we consider the Lotka–Volterra system: the driving system is given as
\[ \begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2 + 2x_1^2 - 2.7x_3 x_1, \\
\dot{x}_2 &= -x_2 + x_1 x_2, \\
\dot{x}_3 &= -3x_3 + 2.7x_5 x_3.
\end{align*} \] (43)

and the response system is
\[ \begin{align*}
\dot{y}_1 &= y_1 - y_1 y_2 + 2y_2^2 - 2.7y_3 y_2^2 + \Phi_1(u_1), \\
\dot{y}_2 &= -y_2 + y_1 y_2 + \Phi_2(u_2), \\
\dot{y}_3 &= -3y_3 + 2.7y_5 y_3^2 + \Phi_3(u_3),
\end{align*} \] (44)

where \( x_1, x_2, x_3 \) and \( y_1, y_2, y_3 \) are state variables of the driving system and the response system, respectively, and \( \Phi_i(u_i), i = 1, 2, 3 \), are the inevitable input nonlinear models. Let \( x = [x_1, x_2, x_3]^T, y = [y_1, y_2, y_3]^T, u = [u_1, u_2, u_3]^T, \) and \( \Phi(u) = [\Phi_1(u_1), \Phi_2(u_2), \Phi_3(u_3)]^T \).

Then, system (43) can be written as
**Figure 2**: Simulation results in (a) $x_1$ and $x_2$; (b) $x_1$ and $x_3$; (c) $x_2$ and $x_3$; and (d) $x_1$, $x_2$, and $x_3$.

**Figure 3**: Simulation results in (a) synchronizing of $x_1$ and $y_1$; (b) synchronizing of $x_2$ and $y_2$; (c) synchronizing of $x_3$ and $y_3$; and (d) $e_1$, $e_2$, and $e_3$.  

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Complexity


\[ \dot{x} = F(x), \]

and system (44) can be written as

\[ \dot{y} = F(y) + G\Phi(u), \]

The input nonlinearities \( \Phi_i(u_i) \), for \( i = 1, 3 \), are described by

\[
\Phi_i(u_i) = \begin{cases} 
(u_i - 2)(1 - 0.3 \sin(u_i)), & u_i > 2, \\
0, & -2 \leq u_i \leq 2, \\
(u_i + 2)(0.8 - 0.3 \cos(u_i)), & u_i < -2,
\end{cases}
\]

and the input nonlinearity \( \Phi_2(u_2) \) is supposed to be

\[
\Phi_2(u_2) = \begin{cases} 
(u_2 - 5)(1 - 0.3 \sin(u_2)), & u_2 > 5, \\
0, & -5 \leq u_2 \leq 5, \\
(u_2 + 5)(0.8 - 0.3 \cos(u_2)), & u_2 < -5.
\end{cases}
\]

It is worth noting that \( F(x) \), \( G \), \( \Phi(u) \) are unknown, except for some structural natures, for example:

(i) The symmetry and sign of \( G \)

(ii) The natures (11) of \( \Phi_i(u_i) \) and an understanding of \( u_{i+} \) and \( u_{i-} \), with \( i = 1, 2, 3 \)

The initial conditions of the driving system are \( x(0) = [x_1(0), x_2(0), x_3(0)]^T = [1, 1.4, 1]^T \), and the initial conditions of the response system are \( y(0) = [y_1(0), y_2(0), y_3(0)]^T = [2, 2, 1]^T \).

The adaptive fuzzy systems, \( \theta_i^T \psi_i(x) \), with \( i = 1, 2, 3 \), have the vector \( y = [y_1, y_2, y_3]^T \) as input, and three triangular membership functions that uniformly distribute on the intervals \([-2, 2]\) are given for each entry variable of these fuzzy systems. The design parameters are taken as \( q = -61/44, \quad v = 3/4, \quad \gamma_0 = \gamma_9 = 30, \quad \gamma_2 = 80, \quad \gamma_1 = \gamma_{12} = \gamma_{13} = 4000, \quad \sigma_0 = \sigma_9 = \sigma_{10} = 0.001, \quad \sigma_{11} = \sigma_{12} = \sigma_{13} = 0.0005, \quad \lambda_1 = \lambda_2 = \lambda_3 = 2, \quad k_{11} = k_{12} = k_{13} = 2 \). The initial conditions are given as \( k_{01}(0) = k_{02}(0) = k_{03}(0) = 0, \quad \theta_{11}(0) = \theta_{21}(0) = \theta_{31}(0) = 0 \), \( j = 1, \ldots, 27 \).

It is worth noting that \( \text{sign}(S_i) \), which is a discontinuous function, has been replaced with \( \tanh(k_S S_i) \), which is a smooth function, with \( k_S = 20, i = 1, 2, 3 \).

Finally, the simulation results are shown in Figures 2–4. Figure 2 shows the chaotic phenomenon of the driving system. Figure 3 shows that the driving system and the response system are basically synchronized after 0.2 seconds, and the results indicate the effectiveness of our method. The transient behaviors of controller are presented in Figure 4.

5. Conclusion

This paper proposes a fuzzy adaptive variable-structure controller for the synchronization of the MIMO unknown chaotic system that has sector nonlinearities and dead zones. Based on the Lyapunov stability theory, the whole system can achieve asymptotic stability; namely, all the closed-loop signals and states are bounded, and the synchronization performance of two systems can be achieved. To be specific, a smooth function can reduce the chattering phenomena in the process of control. The validity of the approach has been tested and verified by means of example and simulation. The approach can be applied to settle the synchronization of a large class of chaotic system with dead zones. How to obtain accurate control performance is one of our future research directions.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest related to this article.

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