Pebble minimization: the last theorems

Gaëtan Douéneau-Tabot

1 Université Paris Cité, CNRS, IRIF, F-75013, Paris, France
2 Direction générale de l’armement - Ingénierie des projets, Paris, France
doueneau@irif.fr

Abstract Pebble transducers are nested two-way transducers which can
drop marks (named “pebbles”) on their input word. Such machines can
compute functions whose output size is polynomial in the size of their
input. They can be seen as simple recursive programs whose recursion
height is bounded. A natural problem is, given a pebble transducer, to
compute an equivalent pebble transducer with minimal recursion height.
This problem has been open since the introduction of the model.
In this paper, we study two restrictions of pebble transducers, that can-not
see the marks (“blind pebble transducers” introduced by Nguyên et
al.), or that can only see the last mark dropped (“last pebble transducers”
introduced by Engelfriet et al.). For both models, we provide an effective
algorithm for minimizing the recursion height. The key property used in
both cases is that a function whose output size is linear (resp. quadratic,
cubic, etc.) can always be computed by a machine whose recursion height
is 1 (resp. 2, 3, etc.). We finally show that this key property fails as soon
as we consider machines that can see more than one mark.

Keywords: Pebble transducers · Polyregular functions · Blind pebble
transducers · Last pebble transducers · Factorization forests.

1 Introduction

Transducers are finite-state machines obtained by adding outputs to finite aut-
omata. They are very useful in a lot of areas like coding, computer arithmetic,
language processing or program analysis, and more generally in data stream
processing. In this paper, we consider deterministic transducers which compute
functions from finite words to finite words. In particular, a deterministic two-
way transducer is a two-way automaton with outputs. This model describes
the class of regular functions, which is often considered as one of the func-
tional counterparts of regular languages. It has been intensively studied for its
properties such as closure under composition [5], equivalence with logical trans-
ductions [12] or regular expressions [7], decidable equivalence problem [14], etc.

Pebble transducers and polyregular functions. Two-way transducers can
only describe functions whose output size is at most linear in the input size.
A possible solution to overcome this limitation is to consider nested two-way
transducers. In particular, the model of \textit{k-pebble transducer} has been studied for a long time \cite{13}. For \( k = 1 \), a 1-pebble transducer is just a two-way transducer. For \( k \geq 2 \), a \( k \)-pebble transducer is a two-way transducer that, when on any position \( i \) of its input word, can call a \(( k-1)\)-pebble transducer. The latter takes as input the original input where position \( i \) is marked by a “pebble”. The main two-way transducer then outputs the concatenation of all the outputs produced along its calls. The intuitive behavior of a 3-pebble transducer is depicted in fig. 1. It can be seen as a recursive program whose recursion stack has height 3.

The class of functions computed by pebble transducers is known as \textbf{polyregular functions}. It has been intensively studied due to its properties such as closure under composition \cite{11}, equivalence with logical interpretations \cite{4}, etc.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Behavior of a 3-pebble transducer.}
\end{figure}

\textbf{Optimization of pebble transducers.} Given a \( k \)-pebble transducer computing a function \( f \), a very natural problem is to compute the least possible \( 1 \leq \ell \leq k \) such that \( f \) can be computed by an \( \ell \)-pebble transducer. Furthermore, we can be interested in effectively building an \( \ell \)-pebble transducer for \( f \). Both questions are open, but they are meaningful since they ask whether we can optimize the recursion height (i.e. the running time) of a program.

It is easy to observe that if \( f \) is computed by a \( k \)-pebble transducer, then \( |f(u)| = \mathcal{O}(|u|^k) \). It was first claimed in a LICS 2020 paper that the minimal recursion height \( \ell \) of \( f \) (i.e. the least possible \( \ell \) such that \( f \) can be computed by an \( \ell \)-pebble transducer) was exactly the least possible \( \ell \) such that \( |f(u)| = \mathcal{O}(|u|^\ell) \). However, Bojańczyk recently disproved this statement in \cite[Theorem 6.3]{3}: the function \textit{inner-squaring} \( : u_1 \# \cdots \# u_n \rightarrow (u_1 \#)^n \cdots (u_n \#)^n \) can be computed by a 3-pebble transducer and is such that \( |\text{inner-squaring}(u)| = \mathcal{O}(|u|^2) \), but it cannot be computed by a 2-pebble transducer. Other counterexamples were given in \cite{16} using different proof techniques. Therefore, computing the minimal recursion height of \( f \) is believed to be hard, since this value not only depends on the output size of \( f \), but also on the word combinatorics of this output.
Optimization of blind pebble transducers. A subclass of pebble transducers, named blind pebble transducers, was recently introduced in [17]. A blind $k$-pebble transducer is somehow a $k$-pebble transducer, with the difference that the positions are no longer marked when making recursive calls. The behavior of a blind 3-pebble transducer is depicted in fig. 2. The class of functions computed by blind pebble transducers is strictly included in polyregular functions [10,17]. The main result of [17] shows that for blind pebble transducers, the minimal recursion height for computing a function only depends on the growth of its output. More precisely, if $f$ is computed by a blind $k$-pebble transducer, then the least possible $1 \leq \ell \leq k$ such that $f$ can be computed by a blind $\ell$-pebble transducer is the least possible $\ell$ such that $|f(u)| = \mathcal{O}(|u|^\ell)$.

![Figure 2: Behavior of a blind 3-pebble transducer.](image)

Contributions. In this paper, we first give a new proof of the connection between minimal recursion height and growth of the output for blind pebble transducers. Furthermore, our proof provides an algorithm that, given a function computed by a blind $k$-pebble transducer, builds a blind $\ell$-pebble transducer which computes it, for the least possible $1 \leq \ell \leq k$. This effective result is not claimed in [17], and our proof techniques significantly differ from theirs. Indeed, we make a heavy use of factorization forests, which have already been used as a powerful tool in the study of pebble transducers [2,8,10].

Secondly, the main contribution of this paper is to show that the (effective) connection between minimal recursion height and growth of the output also holds for the class of last pebble transducers (introduced in [13]). Intuitively, a last $k$-pebble transducer is a $k$-pebble transducer where a called submachine can only see the position of its call, but not the full stack of the former positions. The behavior of a last 3-pebble transducer is depicted in fig. 3. Observe that a blind $k$-pebble transducer is a restricted version of a last $k$-pebble transducer. Formally, we show that if $f$ is computed by a last $k$-pebble transducer, then the least possible $\ell$ such that $f$ can be computed by a last $\ell$-pebble transducer is the least possible $\ell$ such that $|f(u)| = \mathcal{O}(|u|^\ell)$. Furthermore, our proof gives an algorithm that effectively builds a last $\ell$-pebble transducer computing $f$. 

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**Figure 2:** Behavior of a blind 3-pebble transducer.
As a third theorem, we show that our result for last pebble transducers is tight, in the sense that the connection between minimal recursion height and growth of the output does not hold for more powerful models. More precisely, we define the model of \textit{last-last $k$-pebble transducers}, which extends last $k$-pebble transducers by allowing them to see the two last positions of the calls (and not only the last one). We show that for all $k \geq 1$, there exists a function $f$ such that $|f(u)| = O(|u|^2)$ and that is computed by a last-last $(2k+1)$-pebble transducer, but cannot be computed by a last-last $2k$-pebble transducer. The proof of this result relies on a counterexample presented by Bojańczyk in [2].

Outline. We introduce two-way transducers in section 2. In section 3 we describe blind pebble transducers and last pebble transducers. We also state our main results that connect the minimal recursion height of a function to the growth of its output. Their proof goes over sections 4 to 6. In section 7, we finally show that these results cannot be extended to two visible marks.

2 Preliminaries on two-way transducers

Capital letters $A, B$ denote alphabets, i.e. finite sets of letters. The empty word is denoted by $\varepsilon$. If $u \in A^*$, let $|u| \in \mathbb{N}$ be its length, and for $1 \leq i \leq |u|$ let $u[i]$ be its $i$-th letter. If $i \leq j$, we let $u[i:j]$ be $u[i]u[i+1]\cdots u[j]$ (empty if $j < i$). If $a \in A$, let $|u|_a$ be the number of letters $a$ occurring in $u$. We assume that the reader is familiar with the basics of automata theory, in particular two-way automata and monoid morphisms. The type of total (resp. partial, i.e. possibly undefined on some inputs) functions is denoted $S \to T$ (resp. $S \twoheadrightarrow T$).

The machines described in this paper are always \textbf{deterministic}.

\textbf{Definition 2.1.} A \textit{two-way transducer} $\mathcal{T} = (A, B, Q, q_0, F, \delta, \lambda)$ consists of:

- an input alphabet $A$ and an output alphabet $B$;
- a finite set of states $Q$ with $q_0 \in Q$ initial and $F \subseteq Q$ final;
- a transition function $\delta : Q \times (A \cup \{\text{	exttt{\textdagger}}, \text{	exttt{\textdaggerdbl}}\}) \to Q \times \{\triangleright, \triangleleft\}$;
- an output function $\lambda : Q \times (A \cup \{\text{	exttt{\textdagger}}, \text{	exttt{\textdaggerdbl}}\}) \to B^*$ with same domain as $\delta$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{transducer.png}
\caption{Behavior of a last 3-pebble transducer.}
\end{figure}
The semantics of a two-way transducer $\mathcal{T}$ is defined as follows. When given as input a word $u \in A^*$, $\mathcal{T}$ disposits of a read-only input tape containing $\vdash u \dashv$. The marks $\vdash$ and $\dashv$ are used to detect the borders of the tape, by convention we denote them by positions 0 and $|u|+1$ of $u$. Formally, a configuration over $\vdash u \dashv$ is a tuple $(q, i)$ where $q \in Q$ is the current state and $0 \leq i \leq |u|+1$ is the position of the reading head. The transition relation $\rightarrow$ is defined as follows. Given a configuration $(q, i)$, let $(q', \star) := \delta(q, u[i])$. Then $(q, i) \rightarrow (q', i')$ whenever either $\star = \leftarrow$ and $i' = i-1$ (move left), or $\star = \rightarrow$ and $i' = i+1$ (move right), with $0 \leq i' \leq |u|+1$. A run is a sequence of configurations $(q_1, i_1) \rightarrow \cdots \rightarrow (q_n, i_n)$. Accepting runs are those that begin in $(q_0, 0)$ and end in a configuration of the form $(q, |u|+1)$ with $q \in F$ (and never visit such a configuration before).

The partial function $f : A^* \rightarrow B^*$ computed by the two-way transducer $\mathcal{T}$ is defined as follows: for $u \in A^*$, if there exists an accepting run on $\vdash u \dashv$, then it is unique, and $f(u)$ is defined as $\lambda(q_1, (\vdash u \dashv)[i_1]) \cdots \lambda(q_n, (\vdash u \dashv)[i_n]) \in B^*$. The class of functions computed by two-way transducers is called regular functions.

**Example 2.2.** Let $\tilde{u}$ be the mirror image of $u \in A^*$. Let $\# \not\in A$ be a fresh symbol. The function $\text{map-reverse} : u_1 \# \cdots \# u_n \rightarrow \tilde{u}_1 \# \cdots \# \tilde{u}_n$ can be computed by a two-way transducer, that reads each factor $u_j$ from right to left.

It is well-known that the domain of a regular function is always a regular language (see e.g. [18]). From now on, we assume without losing generalities that our two-way transducers only compute total functions (in other words, they have exactly one accepting run on each $\vdash u \dashv$). Furthermore, we assume that $\lambda(q, \vdash) = \lambda(q, \dashv) = \varepsilon$ for all $q \in Q$ (we only lose generality for the image of $\varepsilon$).

In the rest of this section, $\mathcal{T}$ denotes a two-way transducer with input alphabet $A$, output alphabet $B$ and output function $\lambda$. Now, we define the crossing sequence in a position $1 \leq i \leq |u|$ of input $\vdash u \dashv$. Intuitively, it regroups the states of the accepting run which are visited in this position.

**Definition 2.3.** Let $u \in A^*$ and $1 \leq i \leq |u|$. Let $(q_1, i_1) \rightarrow \cdots \rightarrow (q_n, i_n)$ be the accepting run of $\mathcal{T}$ on $\vdash u \dashv$. The crossing sequence of $\mathcal{T}$ in $i$, denoted $\text{cross}_\mathcal{T}^\lambda(i)$, is defined as the sequence $(q_j)_{1 \leq j \leq n}$ and $i_j = i$.

If $\mu : A^* \rightarrow M$ is a monoid morphism, we say that any $m, m' \in M$ and $a \in A$ define a $\mu$-context that we denote by $m[a]m'$. It is well-known that the crossing sequence in a position of the input only depends on the context of this position, for a well-chosen monoid, as claimed in proposition 2.4 (see e.g. [7]).

**Proposition 2.4.** One can build a finite monoid $\mathbb{T}$ and a monoid morphism $\mu : A^* \rightarrow \mathbb{T}$, called the transition morphism of $\mathcal{T}$, such that for all $u \in A^*$ and $1 \leq i \leq |u|$, $\text{cross}_\mathcal{T}^\lambda(i)$ only depends on $\mu(u[1:i-1]), u[i]$ and $\mu(u[i+1:|u|])$. Thus we denote it $\text{cross}_\mathcal{T}^\mu(\mu(u[1:i-1])[u[i]]\mu(u[i+1:|u|]))$.

Finally, let us define “the output produced below position $i$”.

**Definition 2.5.** Let $u \in A^*$ and $1 \leq i \leq |u|$ and $q_1 \cdots q_n := \text{cross}_\mathcal{T}^\mu(i)$. We define the production of $\mathcal{T}$ in $i$, denoted $\text{prod}_\mathcal{T}^u(i)$, as $\lambda(q_1, u[i]) \cdots \lambda(q_n, u[i])$.

By proposition 2.4, it also makes sense to define $\text{prod}_\mathcal{T}(m[a]m') \in B^*$ to be $\text{prod}_\mathcal{T}(i)$ whenever $m = \mu(u[1:i-1])$, $m' = \mu(u[i+1:|u|])$ and $a = u[i]$.
3 Blind and last pebble transducers

Now, we are ready to define formally the models of blind pebble transducers and last pebble transducers. Intuitively, they correspond to two-way transducers which make a tree of recursive calls to other two-way transducers.

Definition 3.1 (Blind pebble transducer [17]). For $k \geq 1$, a blind $k$-pebble transducer with input alphabet $A$ and output alphabet $B$ is:
- if $k = 1$, a two-way transducer with input alphabet $A$ and output $B$;
- if $k \geq 2$, a tree $\mathcal{T}(B_1, \ldots, B_p)$ where the subtrees $B_1, \ldots, B_p$ are blind $(k-1)$-pebble transducers with input $A$ and output $B$; and the root label $\mathcal{T}$ is a two-way transducer with input $A$ and output alphabet $\{B_1, \ldots, B_p\}$.

The (total) function $f : A^* \to B^*$ computed by the blind $k$-pebble transducer of definition 3.1 is built in a recursive fashion, as follows:
- for $k = 1$, $f$ is the function computed by the two-way transducer;
- for $k \geq 2$, let $u \in A^*$ and $(q_1, i_1) \to \cdots \to (q_n, i_n)$ be the accepting run of $\mathcal{T} = (A, B, Q, q_0, F, \delta, \lambda)$ on $\lambda(u)$. For all $1 \leq j \leq n$, let $f_j : A^* \to B^*$ be the concatenation of the functions recursively computed by the sequence $\lambda(q_j, (\lambda(u))|i_j|) \in \{B_1, \ldots, B_p\}^*$. Then $f(u) := f_1(u) \cdots f_n(u)$.

The behavior of a blind 3-pebble transducer is depicted in fig. 2.

Example 3.2. The function $\text{unmarked-square} : A^* \to A^* \cup \{\#\}, u \mapsto (u\#)^{|u|}$ can be computed by a blind 2-pebble transducer. This machine has shape $\mathcal{T} \cup \mathcal{T}'$: $\mathcal{T}$ calls $\mathcal{T}'$ on each position $1 \leq i \leq |u|$ of its input $u$, and $\mathcal{T}'$ outputs $u\#$.

The class of functions computed by a blind $k$-pebble transducer for some $k \geq 1$ is called polyblind functions [10]. They form a strict subclass of polyregular functions [8,10,17] which is closed under composition [17, Theorem 6.1].

Now, let us define last pebble transducers. They corresponds to blind pebble transducers enhanced with the ability to mark the current position of the input when doing a recursive call. Formally, this position is underlined and we define $u \cdot i := u[1] \cdots u[i-1]u[i]u[i+1] \cdots u[|u|]$ for $u \in A^*$ and $1 \leq i \leq |u|$.

Definition 3.3 (Last pebble transducer [13]). For $k \geq 1$, a last $k$-pebble transducer with input alphabet $A$ and output alphabet $B$ is:
- if $k = 1$, a two-way transducer with input alphabet $A \cup \overline{A}$ and output $B$;
- if $k \geq 2$, a tree $\mathcal{T} \cup \mathcal{T}'$ where the subtrees $\mathcal{L}_1, \ldots, \mathcal{L}_p$ are last $(k-1)$-pebble transducers with input $A$ and output $B$; and the root label $\mathcal{T}$ is a two-way transducer with input $A \cup \overline{A}$ and output alphabet $\{\mathcal{L}_1, \ldots, \mathcal{L}_p\}$.

The (total) function $f : (A \cup \overline{A})^* \to B^*$ computed by the last $k$-pebble transducer of definition 3.3 is defined in a recursive fashion, as follows:
- for $k = 1$, $f$ is the function computed by the two-way transducer;
- for $k \geq 2$, let $u \in A^*$ and $(q_1, i_1) \to \cdots \to (q_n, i_n)$ be the accepting run of $\mathcal{T} = (A \cup \overline{A}, B, Q, q_0, F, \delta, \lambda)$ on $\lambda(u)$. For all $1 \leq j \leq n$, let $f_j : A^* \to B^*$ be the concatenation of the functions recursively computed by $\lambda(q_j, (\lambda(u)|i_j|) \in \{\mathcal{L}_1, \ldots, \mathcal{L}_p\}^*$. Let $\tau : (A \cup \overline{A})^* \to A^*$ be the morphism which erases the underlining (i.e. $\tau(\overline{a}) = a$), then $f(u) := f_1(\tau(u) \cdot i_1) \cdots f_n(\tau(u) \cdot i_n)$.
The behavior of a last 3-pebble transducer is depicted in fig. 3. Observe that our definition builds a function of type \((A\uplus A^*)^* \rightarrow B^*\), but we shall in fact consider its restriction to \(A^*\) (the marks are only used within the induction step).

**Example 3.4 ([1]).** The function \(\text{square} : u \mapsto (u \cdot 1)^# \cdots (u \cdot |u|)^#\) can be computed by a last 2-pebble transducer, which successively marks and makes recursive calls in positions 1, 2, etc. However this function is not polyblind [17].

We are ready to state our main result. Its proof goes over sections 4 to 6.

**Theorem 3.5 (Minimization of the recursion height).** Let \(1 \leq \ell \leq k\). Let \(f : A^* \rightarrow B^*\) be computed by a blind \(k\)-pebble transducer (resp. by a last \(k\)-pebble transducer). Then \(f\) can be computed by a blind \(\ell\)-pebble transducer (resp. by a last \(\ell\)-pebble transducer) if and only if \(|f(u)| = O(|u|^{\ell})\).

This property is decidable and the construction is effective.

As an easy consequence, the class of functions computed by last pebble transducers form a strict subclass of the polyregular functions (because theorem 3.5 does not hold for the full model of pebble transducers [3, Theorem 6.3]) and therefore it is not closed under composition (because any polyregular function can be obtained as a composition of regular functions and squares [1]).

Even if a (non-effective) theorem 3.5 was already known for blind pebble transducers [17, Theorem 7.1], we shall first present our proof of this case. Indeed, it is a new proof (relying on factorization forests) which is simpler than the original one. Furthermore, understanding the techniques used is a key step for understanding the proof for last pebble transducers presented afterwards.

### 4 Factorization forests

In this section, we introduce the key tool of factorization forests. Given a monoid morphism \(\mu : A^* \rightarrow M\) and \(u \in A^*\), a \(\mu\)-factorization forest of \(u\) is an unranked tree structure defined as follows. We use the brackets \((\cdots)\) to build a tree.

**Definition 4.1 (Factorization forest [19]).** Given a morphism \(\mu : A^* \rightarrow M\) and \(u \in A^*\), we say that \(\mathcal{F}\) is a \(\mu\)-forest of \(u\) if:

- either \(u = \varepsilon\) and \(\mathcal{F} = \varepsilon\); or \(u = (a) \in A\) and \(\mathcal{F} = a\);
- or \(\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_n), u = u_1 \cdots u_n,\) for all \(1 \leq i \leq n, \mathcal{F}_i\) is a \(\mu\)-forest of \(u_i \in A^+\), and if \(n \geq 3\) then \(\mu(u) = \mu(u_1) = \cdots = \mu(u_n)\) is idempotent.

We use the standard tree vocabulary of height, child, sibling, descendant and ancestor (a node being itself one of its ancestors/descendants), etc. We denote by \(\text{Nodes}^\mathcal{F}\) the set of nodes of \(\mathcal{F}\). In order to simplify the statements, we identify a node \(t \in \text{Nodes}^\mathcal{F}\) with the subtree rooted in this node. Thus \(\text{Nodes}^\mathcal{F}\) can also be seen as the set of subtrees of \(\mathcal{F}\), and \(\mathcal{F} \in \text{Nodes}^\mathcal{F}\). We say that a node is idempotent if it has at least 3 children. We denote by \(\text{Forests}_\mu^\mathcal{F}(u)\) (resp. \(\text{Forests}_\mu^d(u)\)) the set of \(\mu\)-forests of \(u \in A^*\) (resp. \(\mu\)-forests of \(u \in A^*\) of height at most \(d\)). We write \(\text{Forests}_\mu\) and \(\text{Forests}_\mu^d\) of all forests (of any word).
A $\mu$-forest of $u \in A^*$ can also be seen as “the word $u$ with brackets” in definition 4.1. Therefore Forests$_\mu$ can be seen as a language over $A := A \cup \{(,\})$. In this setting, it is well-known that $\mu$-forests of bounded height can effectively be computed by a rational function, i.e. a particular case of regular function that can be computed by a non-deterministic one-way transducer (see e.g. [8]).

**Theorem 4.2 (Simon [19,6]).** Given a morphism $\mu : A^* \to M$ into a finite monoid $M$, one can effectively build a rational function $\text{forest}_\mu : A^* \to (\hat{A})^*$ such that for all $u \in A^*$, $\text{forest}_\mu(u) \in \text{Forests}_\mu M(u)$.

Building $\mu$-forests of bounded height is especially useful for us, since it enables to decompose any word in a somehow bounded way. This decomposition will be guided by the following definitions, that have been introduced in [8,10]. First, we define iterable nodes as the middle children of idempotent nodes.

**Definition 4.3.** Let $F \in \text{Forests}_\mu(u)$. Its iterable nodes, denoted $\text{Iter}^F$, are:

- if $F = (a) \in A$ or $F = \varepsilon$, then $\text{Iter}^F := \emptyset$;
- otherwise if $F = (F_1, \ldots, F_n)$, then:

$$\text{Iter}^F := \{F_i : 2 \leq i \leq n-1\} \cup \bigcup_{1 \leq i \leq n} \text{Iter}^{F_i}.$$

Now, we define the notion of skeleton of a node $t$, which contains all the descendants of $t$ except those which are iterable.

**Definition 4.4 (Skeleton, frontier).** Let $F \in \text{Forests}_\mu(u)$, $t \in \text{Nodes}^F$, we define the skeleton of $t$, denoted $\text{Skel}^F(t)$, by:

- if $t = (a) \in A$ is a leaf, then $\text{Skel}^F(t) := \{t\}$;
- otherwise if $t = (F_1, \ldots, F_n)$, then $\text{Skel}^F(t) := \{t\} \cup \text{Skel}^F(F_1) \cup \text{Skel}^F(F_n)$.

The frontier of $t$ is the set $\text{Fr}^F(t) \subseteq [1:|u|]$ containing the positions of $u$ which belong to $\text{Skel}^F(t)$ (when seen as leaves of the $\mu$-forest $F$ over $u$).

**Example 4.5.** Let $M := \{\{-1, 1, 0\}, \times\}$ and $\mu : M^* \to M$ the product. A $\mu$-forest $F$ of the word $(-1)(-1)0(-1)000000$ is depicted in Figure 4. Double lines denote idempotent nodes. The set of blue nodes is the skeleton of the topmost blue node.

![Figure 4: $F \in \text{Forests}_\mu((-1)(-1)0(-1)000000)$ and a skeleton.](image-url)
\{\text{Skel}^F(t) : t \in \text{Iter}^F \cup \{F\}\} is a partition of \text{Nodes}^F [8, Lemma 33]. As a consequence, the set of frontiers \{\text{Fr}^F(t) : t \in \text{Iter}^F \cup \{F\}\} is a partition of $[1;|u|]$. Given a position $1 \leq i \leq |u|$, we can thus define the \text{origin} of $i$ in $F$, denoted \text{origin}^F(i), as the unique $t \in \text{Iter}^F \cup \{F\}$ such that $i \in \text{Fr}^F(t)$.

**Definition 4.6 (Observation).** Let $F \in \text{Forests}_\mu$ and $t, t' \in \text{Nodes}^F$. We say that $t \in \text{Nodes}^F$ \textbf{observes} $t' \in \text{Nodes}^F$ if either $t'$ is an ancestor of $t$, or $t'$ is the immediate right or left sibling of an ancestor of $t$.

![Figure 5: Nodes that observe \bullet and that \bullet observes](image)

The intuition behind the notion of observation (which is \emph{not} symmetrical) is depicted in fig. 5. Note that in a forest of bounded height, the number of nodes that some $t$ observes is bounded. This will be a key argument in the following. We say that $t$ and $t'$ are \textbf{dependent} if either $t$ observes $t'$ or the converse. Given $F$, we can translate these notions to the positions of $u$: we say that $i$ \textbf{observes} (resp. \textbf{depends} on) $i'$ if \text{origin}^F(i) observes (resp. depends on) \text{origin}^F(i').

## 5 Height minimization of blind pebble transducers

In this section, we show theorem 3.5 for blind pebble transducers. We say that a two-way transducer $\mathcal{T}$ is a \textbf{submachine} of a blind pebble transducer $\mathcal{B}$ if $\mathcal{T}$ labels a node in the tree description of $\mathcal{B}$. If $\mathcal{B} = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle$, we say that the submachine $\mathcal{T}$ is the \textbf{head} of $\mathcal{B}$. We let the \textbf{transition morphism} of $\mathcal{B}$ be the cartesian product of all the transition morphisms of all the submachines of $\mathcal{B}$. Observe that it makes sense to consider the production of a submachine $\mathcal{T}$ in a context defined using the transition morphism of $\mathcal{B}$.

### 5.1 Pumpability

We first give a sufficient condition, named \textbf{pumpability}, for a blind $k$-pebble transducer to compute a function $f$ such that $|f(u)| \neq O(|u|^{k-1})$. The behavior of a pumpable blind 2-pebble transducer is depicted in fig. 6 over a well-chosen
input: it has a factor in which the head $\mathcal{R}_1$ calls a submachine $\mathcal{R}_2$, and a factor in which $\mathcal{R}_2$ produces a non-empty output. Furthermore both factors can be iterated without destroying the runs of these machines (due to idempotents).

**Definition 5.1.** Let $\mathcal{B}$ be a blind $k$-pebble transducer whose transition morphism is $\mu : A^* \to T$. We say that the transducer $\mathcal{B}$ is pumpable if there exists:
- submachines $\mathcal{R}_1, \ldots, \mathcal{R}_k$ of $\mathcal{B}$, such that $\mathcal{R}_1$ is the head of $\mathcal{B}$;
- $m_0, \ldots, m_k, \ell_1, \ldots, \ell_k, r_1, \ldots, r_k \in \mu(A^*)$;
- $a_1, \ldots, a_k \in A$ such that for all $1 \leq j \leq k$, $e_j := \ell_j \mu(a_j) r_j$ is an idempotent;
- a permutation $\sigma : [1:k] \to [1:k]$;

such that if $M^i_j := m_i e_{i+1} m_{i+1} \cdots e_j m_j$ for all $0 \leq i \leq j \leq k$, and if we define the following context for all $1 \leq j \leq k$:

$$C_j := M^\sigma(j)-1_j e_{\sigma(j)} \ell_{\sigma(j)} \prod_{j \neq \sigma(j)} r_{\sigma(j)} e_{\sigma(j)} M^{k}_{\sigma(j)}$$

then for all $1 \leq j \leq k-1$, $|\prod_{\mathcal{J}_j}(C_j)|_{\mathcal{J}_{j+1} \neq 0}$, and $\prod_{\mathcal{J}_k}(C_k) \neq \varepsilon$.

![Figure 6: Pumpability in a blind 2-pebble transducer.](image)

**Lemma 5.2.** Let $f$ be computed by a pumpable blind $k$-pebble transducer. There exists words $v_0, \ldots, v_k, u_1, \ldots, u_k$ such that $|f(v_0 u_1^X \cdots u_k^X v_k)| = \Theta(X^k)$.

Now, we use pumpability as a key ingredient for showing theorem 3.5, which directly follows by induction from the more precise theorem 5.3.

**Theorem 5.3 (Removing one layer).** Let $k \geq 2$ and $f : A^* \to B^*$ be computed by a blind $k$-pebble transducer $\mathcal{B}$. The following are equivalent:

1. $|f(u)| = O(|u|^{k-1})$;
2. $\mathcal{B}$ is not pumpable;
3. $f$ can be computed by a blind $(k-1)$-pebble transducer.

Furthermore, this property is decidable and the construction is effective.

**Proof.** Item 3 $\Rightarrow$ item 1 is obvious. Item 1 $\Rightarrow$ item 2 is lemma 5.2. Furthermore, pumpability can be tested by an enumeration of $\mu(A^*)$ and $A$. It remains to show item 2 $\Rightarrow$ item 3 (in an effective fashion): this is the purpose of section 5.2.
5.2 Algorithm for removing a recursion layer

Let \( k \geq 2 \) and \( \mathcal{U} \) be a blind \( k \)-pebble transducer that is not pumpable, and that computes \( f : A^* \rightarrow B^* \). We build a blind \((k-1)\)-pebble transducer \( \overline{\mathcal{U}} \) for \( f \).

Let \( \mu : A^* \rightarrow T \) be the transition morphism of \( \mathcal{U} \). We shall consider that, on input \( u \in A^* \), the submachines of \( \mathcal{U} \) can in fact use forest\(_\mu\)(\( u \)) \( \subseteq (\hat{A})^* \) as input. Indeed forest\(_\mu\) is a rational function (by theorem 4.2), hence its information can be recovered by using a lookaround. Informally, the lookaround feature enables a two-way transducer to choose its transitions not only depending on its current state and current letter \( u[i] \) in position \( 1 \leq i \leq |u| \), but also on a regular property of the prefix \( u[1:i-1] \) and the suffix \( u[i+1:|u|] \). It is well-known that given a two-way transducer \( T \) with lookarounds, one can build an equivalent \( T' \) that does not have this feature (see e.g. [15,12]). Furthermore, even if the accepting runs of \( T \) and \( T' \) may differ, they produce the same outputs from the same positions (this observation will be critical for last pebble transducers, in order to ensure that the marked positions of the recursive calls will be preserved).

Now, we describe the two-way transducers that are the submachines of \( \overline{\mathcal{U}} \). First, it has submachines old-\( \mathcal{T} \) for \( \mathcal{T} \) a submachine of \( \mathcal{U} \), which are described in algorithm 1. Intuitively, old-\( \mathcal{T} \) is just a copy of \( \mathcal{T} \). It is clear that if \( \mathcal{T} \) is a submachine of \( \mathcal{U} \), then old-\( \mathcal{T} \)(\( u \)) is the concatenation of the outputs produced by (the recursive calls of) \( \mathcal{T} \) along its accepting run on \( \vdash u \vdash \).

Algorithm 1: Submachines that behave as the original ones

1. Submachine old-\( \mathcal{T} \)(\( u \))
2. \( \rho : = \) accepting run of \( \mathcal{T} \) over \( \vdash u \vdash \); \( \lambda : = \) output function of \( \mathcal{T} \);
3. for \( (q, i) \in \rho \) do
4.  if \( \mathcal{T} \) is a leaf of \( \mathcal{U} \) then
5.    Output \( \lambda(q, (\vdash u \vdash )[i]) \); /* \( \mathcal{T} \) has output in \( B^* \) */
6.  else
7.    for \( B' \in \lambda(q, (\vdash u \vdash )[i]) \) do
8.      \( \mathcal{T}' : = \) head of \( B' \);
9.      Call old-\( \mathcal{T} \)'(\( u \)); /* \( \mathcal{T} \) makes recursive calls */
10. end
11. end
12. end

\( \overline{\mathcal{U}} \) also has submachines accelerate-\( \mathcal{T} \) for \( \mathcal{T} \) a submachine of \( \mathcal{U} \), which are described in algorithm 2. Intuitively, accelerate-\( \mathcal{T} \) simulates \( \mathcal{T} \) while trying to inline recursive calls in its own run. More precisely, let \( u \in A^* \) be the input and \( \mathcal{F} : = \text{forest}_\mu(u) \). If \( \mathcal{T} \) calls \( B' \) in \( 1 \leq i \leq |u| \) that belongs to the frontier of the root node \( \mathcal{F} \) of \( \mathcal{T} \), then accelerate-\( \mathcal{T} \) inlines the behavior of the head of \( B' \). Otherwise it makes a recursive call, except if \( B' \) is a leaf of \( \mathcal{U} \). Hence if \( \mathcal{T} \) is a submachine of \( \mathcal{U} \) which is not a leaf, accelerate-\( \mathcal{T} \)(\( u \)) is the concatenation of the outputs produced by the calls of \( \mathcal{T} \) along its accepting run.
Algorithm 2: Submachines that try to simulate their recursive calls

1. Submachine `accelerate-$\mathcal{T}$` ($u$)
2. /* $\mathcal{T}$ is not a leaf of $\mathcal{U}$ (i.e. it makes calls); */
3. $\rho :=$ accepting run of $\mathcal{T}$ over $\vdash u \downarrow$; $\mathcal{F} := \text{forest}_u(u)$; $\lambda :=$ output fun. of $\mathcal{T}$;
4. for $(q, i) \in \rho$ do
5. for $B' \in \lambda(q, (\vdash u \downarrow)[i])$ do
6. $\mathcal{T}' :=$ head of $B'$;
7. if $i \in \text{Fr}_\mathcal{F}(\mathcal{F})$ then
8. /* We can inline the call since $|\text{Fr}_\mathcal{F}(\mathcal{F})|$ is bounded; */
9. Inline the code of old-$\mathcal{T}'$ ($u$) /* (see explanations); */
10. else if $B'$ is a leaf of $\mathcal{U}$ then
11. /* Then $B' = \mathcal{T}'$ and we can inline the call because the output of $\mathcal{T}'$ on input $u$ is bounded; */
12. Inline the code of old-$\mathcal{T}'$ ($u$) /* (see explanations); */
13. else
14. /* It is not possible to inline the call to $B'$, so we make a recursive call; */
15. Call `accelerate-$\mathcal{T}'$` ($u$);
16. end
17. end
18. end

Finally, the transducer $\overline{\mathcal{U}}$ is obtained by defining accelerate-$\mathcal{T}$ to be its head, where $\mathcal{T}$ is the head of $\mathcal{U}$. Furthermore, we remove the submachines old-$\mathcal{T}$ or accelerate-$\mathcal{T}$ which are never called. Observe that $\overline{\mathcal{U}}$ indeed computes the function $f$. Furthermore, we observe that $\overline{\mathcal{U}}$ has recursion height (i.e. the number of nested Call instructions, plus 1 for the head) $k - 1$, since each inlining of lines 9, 10 and 12 in algorithm 2 removes exactly one recursion layer of $\mathcal{U}$.

It remains to justify that each accelerate-$\mathcal{T}$ can be implemented by a two-way transducer (i.e. with lookarounds but a bounded memory). We represent variable $i$ by the current position of the transducer. Since it has access to $\mathcal{T}$, the lookahead can be used to check whether $i \in \text{Fr}_\mathcal{F}(\mathcal{F})$ or not (since the size of $\text{Fr}_\mathcal{F}(\mathcal{F})$ is bounded). It remains to explain how the inlinings are performed:

- if $i \in \text{Fr}_\mathcal{F}(\mathcal{F})$, the two-way transducer inlines old-$\mathcal{T}'$ by executing the same moves and calls as $\mathcal{T}'$ does. Once its computation is ended, it has to go back to position $i$. This is indeed possible since belonging to $\text{Fr}_\mathcal{F}(\mathcal{F})$ is a property that can be detected by using the lookahead, hence the machine only needs to remember that $i$ was the $\ell$-th position of $\text{Fr}_\mathcal{F}(\mathcal{F})$ ($\ell$ being bounded);
- else if $B' = \mathcal{T}'$ is a blind 1-pebble transducer, we produce the output of $\mathcal{T}'$ without moving. This is possible since for all $i' \not\in \text{Fr}_\mathcal{F}(\mathcal{F})$, $\text{prod}^{\mathcal{F}}_{\mathcal{T}',i'}(i') = \varepsilon$ (hence the output of $\mathcal{T}'$ on $u$ is bounded, and its value can be determined without moving, just by using the lookahead). Indeed, if $\text{prod}^{\mathcal{F}}_{\mathcal{T}',i'}(i') \neq \varepsilon$ for such an $i' \not\in \text{Fr}_\mathcal{F}(\mathcal{F})$ when reaching line 12 of algorithm 2, then the conditions of lemma 5.4 hold, which yields a contradiction. This lemma is the key argument of this proof, relying on the non-pumpability of $\mathcal{U}$. 
Lemma 5.4 (Key lemma). Let \( u \in A^* \) and \( F \in \text{Forests}_u(u) \). Assume that there exists a sequence \( \mathcal{T}_1, \ldots, \mathcal{T}_k \) of submachines of \( \mathcal{W} \) and a sequence of positions \( 1 \leq i_1, \ldots, i_k \leq |u| \) such that:
- \( \mathcal{T}_1 \) is the head of \( \mathcal{W} \);
- for all \( 1 \leq j \leq k-1 \), \(|\text{prod}^u_{\mathcal{T}_j}(i_j)|_{\mathcal{T}_{j+1}} \neq 0 \) and \( \text{prod}^u_{\mathcal{T}_k}(i_k) \neq \varepsilon \);
- for all \( 1 \leq j \leq k \), \( i_j \notin \text{Fr}^F(F) \) (i.e. \( \text{origin}^F(i_j) \in \text{iter}^F \)).

Then \( \mathcal{B} \) is pumpable.

Proof (idea). We first observe that pumpability follows as soon as the nodes \( \text{origin}^F(i_j) \) are pairwise independent. We then show that this independence condition can always be obtained, up to duplicating some iterable subtrees of \( F \) (and some factors of \( u \)), because the behavior of a submachine in a blind pebble transducer does not depend on the positions of the above recursive calls.

6 Height minimization of last pebble transducers

In this section, we show theorem 3.5 for last pebble transducers. The notions of submachine, head and transition morphism for a last pebble transducer are defined as in section 5. The transition morphism is now defined over \((A \cup \prod A)^*\).

6.1 Pumpability

The sketch of the proof is similar to section 5. We first give an equivalent of pumpability for last pebble transducers. The intuition behind this notion is depicted in fig. 7. The formal definition is however more cumbersome, since we need to keep track of the fact that the calling position is marked.

Definition 6.1. Let \( \mathcal{L} \) be a last \( k \)-pebble transducer whose transition morphism is \( \mu : (A \cup \prod A)^* \rightarrow \mathbb{T} \). We say that the transducer \( \mathcal{L} \) is pumpable if there exists:
- submachines \( \mathcal{T}_1, \ldots, \mathcal{T}_k \) of \( \mathcal{L} \), such that \( \mathcal{T}_1 \) is the head of \( \mathcal{L} \);
- \( m_0, \ldots, m_k, \ell_1, \ldots, \ell_k, r_1, \ldots, r_k \in \mu(A^*) \);
- \( a_1, \ldots, a_k \in A \) such that for all \( 1 \leq i \leq k \), \( e_j := \ell_j \mu(a_i) r_j \) is idempotent;
- a permutation \( \sigma : [1:k] \rightarrow [1:k] \);

such that if we let \( \mathcal{M}_i := m_i e_{i+1} m_{i+1} \cdots e_j m_j \) for all \( 0 \leq i \leq j \leq k \), and if we define the following context:

\[
C_1 := \mathcal{M}_0^{(1)} \ell_{\sigma(1)} [a_{\sigma(1)}] \mathcal{M}_{\sigma(1)}^{k} \mathcal{M}_{\sigma(1)}^{k}
\]

and for all \( 1 \leq j \leq k-1 \) the context:

\[
C_{j+1} := \mathcal{M}_0^{(j)} \ell_{\sigma(j)} [a_{\sigma(j)}] \mu(a_{\sigma(j)}) r_{\sigma(j)} e_{\sigma(j)} \mathcal{M}_{\sigma(j)}^{k} \mathcal{M}_{\sigma(j)}^{k} \quad \text{if } \sigma(j) < \sigma(j+1);
\]

\[
C_{j+1} := \mathcal{M}_0^{(j)} \ell_{\sigma(j+1)} [a_{\sigma(j+1)}] r_{\sigma(j+1)} e_{\sigma(j+1)} \mathcal{M}_{\sigma(j+1)}^{k} \mathcal{M}_{\sigma(j+1)}^{k} \quad \text{otherwise};
\]

then for all \( 1 \leq j \leq k-1 \), \(|\text{prod}^u_{\mathcal{T}_j}(C_j)|_{\mathcal{T}_{j+1}} \neq 0 \), and \( \text{prod}^u_{\mathcal{T}_k}(C_k) \neq \varepsilon \).
We obtain lemma 6.2 by a proof which is similar to that of lemma 5.2.

**Lemma 6.2.** Let \( f \) be computed by a pumpable last \( k \)-pebble transducer. There exists words \( v_0, \ldots, v_k, u_1, \ldots, u_k \) such that \( |f(v_0u_1^X \ldots u_k^Xv_k)| = \Theta(X^k) \).

**Theorem 6.3 (Removing one layer).** Let \( k \geq 2 \) and \( f : A^* \rightarrow B^* \) be computed by a last \( k \)-pebble transducer \( L \). The following are equivalent:
1. \( |f(u)| = O(|u|^{k-1}) \);
2. \( L \) is not pumpable;
3. \( f \) can be computed by a last \( (k-1) \)-pebble transducer.
Furthermore, this property is decidable and the construction is effective.

**Proof.** Item 3 \( \Rightarrow \) item 1 is obvious. Item 1 \( \Rightarrow \) item 2 is lemma 6.2. Furthermore, pumpability can be tested by an enumeration of \( \mu(A^*) \) and \( A \). It remains to show item 2 \( \Rightarrow \) item 3 (in an effective fashion): this is the purpose of section 6.2.

### 6.2 Algorithm for removing a recursion layer

Let \( k \geq 2 \) and \( \mathcal{W} \) be a last \( k \)-pebble transducer that is not pumpable, and that computes \( f : A^* \rightarrow B^* \). We build a last \( (k-1) \)-pebble transducer \( \mathcal{W} \) for \( f \). Let \( \mu : (A \cup \Delta)^* \rightarrow T \) be the transition morphism of \( \mathcal{W} \). As before (using a lookahead), the submachines of \( \mathcal{W} \) have access to forest,_{\mu}(u) on input \( u \in A^* \).

Now, we describe the submachines of \( \mathcal{W} \). It has submachines \texttt{old-\mathcal{F}-along-\rho} for \( \mathcal{F} \) a submachine of \( \mathcal{W} \) and \( \rho \) a run of \( \mathcal{F} \), which are described in algorithm 1. Intuitively, these machines mimics the behavior of \( \mathcal{F} \) along the run \( \rho \) (which is not necessarily accepting) of \( \mathcal{F} \) over \( +\cdot v-\cdot \; \) with \( v \in (A \cup \Delta)^* \).

Since they are indexed by a run \( \rho \), it may seem that we create an infinite number of submachines, but it will not be the case. Indeed, a run \( \rho \) will be represented by its first configuration \( (q_1, i_1) \) and last configuration \( (q_n, i_n) \). This information is sufficient to simulate exactly the two-way moves of \( \rho \), but there is still an unbounded information: the positions \( i_1 \) and \( i_n \). In fact, the input will be of the form \( v = u \cdot i \) and we shall guarantee that the \( i_1 \) and \( i_n \) can be detected by the lookahead if \( i \) is marked. Hence the run \( \rho \) will be represented in a bounded
way, independently from the input $v$, and so that its first and last configurations can be detected by the lookaround of the submachine.

It follows from algorithm 3 that if $\mathcal{T}$ is a submachine of $\mathcal{W}$, then for all $v \in (A \cup \mathcal{A})^*$ and $\rho$ run of $\mathcal{T}$ on $\tau v \downarrow$, old-$\mathcal{T}$-along-$\rho$ $(v)$ is the concatenation of the outputs produced by (the recursive calls of) $\mathcal{T}$-along-$\rho$.

We also define a submachine normal-$\mathcal{T}$-along-$\rho$-pebble-i that is similar to old-$\mathcal{T}$-along-$\rho$, except that it ignores the mark of its input and acts as if it was in position $i$ (as above for $\rho$, $i$ will be encoded by a bounded information).

### Algorithm 3: Submachines that behave like the original ones

```
1 Submachine old-$\mathcal{T}$-along-$\rho$($v$)
2 /* $v \in (A \cup \mathcal{A})^*$; $\rho$ is a run of $\mathcal{T}$ over $\tau v \downarrow$ */
3 $\lambda :=$ output function of $\mathcal{T}$;
4 for $(q,i) \in \rho$ do
5     if $\mathcal{T}$ is a leaf of $\mathcal{W}$ then
6         Output $\lambda(q, (\tau v \downarrow)[i])$; /* $\mathcal{T}$ has output in $B^*$ */
7     else
8         for $\mathcal{L}' \in \lambda(q, (\tau v \downarrow)[i])$ do
9             $\mathcal{T}' :=$ head of $\mathcal{L}'$; $\rho' :=$ accepting run of $\mathcal{T}'$ on $\tau(v) \downarrow$;
10            Call old-$\mathcal{T}'$-along-$\rho'$($\tau(v) \downarrow$); /* Recursive call */
11         end
12     end
13 end
14 Submachine normal-$\mathcal{T}$-along-$\rho$-pebble-i($v$)
15 /* $v \in (A \cup \mathcal{A})^*$; $\rho$ is a run of $\mathcal{T}$ over $\tau(v) \downarrow$ */
16 Simulate old-$\mathcal{T}$-along-$\rho$ ($\tau(v) \downarrow$);
```

$\mathcal{W}$ also has submachines accelerate-$\mathcal{T}$-along-$\rho$ for $\mathcal{T}$ a submachine of $\mathcal{W}$, which are described in algorithm 4. Intuitively, accelerate-$\mathcal{T}$-along-$\rho$ simulates $\mathcal{T}$ along $\rho$ while trying to inline some recursive calls. Whenever it is in position $i$ and needs to call recursively $\mathcal{L}'$ whose head is $\mathcal{T}'$, it first slices the accepting run $\rho'$ of $\mathcal{T}'$ on $\tau u \downarrow$, with respect to forest$_\rho$($u$) and $i$, as explained in definition 6.4 and depicted in fig. 8. Intuitively, this operation splits $\rho'$ into a bounded number of runs whose positions either all observe $i$, or $i$ observes all of them, or none of these cases occur (the positions are either 0, $|u|+1$ or independent of $i$).

### Definition 6.4 (Slicing).

Let $u \in A^*$, $\mathcal{F} \in \text{Forests}_\rho$($u$) and $1 \leq i \leq |u|$. We let $\uparrow i$ (resp. $\downarrow i$) be the set of positions that $i$ observes (resp. that observe $i$).

Let $\rho = (q_1,i_1) \rightarrow \cdots \rightarrow (q_n,i_n)$ be a run of a two-way transducer $\mathcal{T}$ on $\tau u \downarrow$.

We build by induction a sequence $\ell_1, \ldots, \ell_{N+1}$ with $\ell_1 := 1$ and:
- if $\ell_j = n+1$ then $j := N$ and the process ends;
- else if $i_{\ell_j} \in \uparrow i$ (resp. $i_{\ell_j} \in \downarrow i \uparrow i$, resp. $i_{\ell_j} \in \{0;|u|+1\} \setminus (\uparrow i \cup \downarrow i)$), then $\ell_{j+1}$ is the largest index such that for all $\ell_j \leq \ell \leq \ell_{j+1}-1$, $i_{\ell} \in \uparrow i$ (resp. $i_{\ell} \in \downarrow i \uparrow i$, resp. $i_{\ell} \in \{0;|u|+1\} \setminus (\uparrow i \cup \downarrow i)$).
Finally the slicing of ρ, with respect to F and i, is the sequence of runs ρ₁, . . . , ρ₉ where ρ_j := (q_{t_j}, i_{t_j}) → (q_{t_{j+1}}, i_{t_{j+1}}) → · · · → (q_{t_{j+1}-1}, i_{t_{j+1}-1}).

Now, let ρ₁, . . . , ρ₉ be slicing of the run ρ' of T' on the input u•i. For all 1 ≤ j ≤ N, there are mainly two cases. Either the positions of ρ'_j all are in ↑ i or ↓ i. In this case, accelerate-T'-along-ρ directly inlines old-T'-along-ρ'_j within its own run (i.e. without making a recursive call). Otherwise, it makes a recursive call to accelerate-T'-along-ρ'_j, except if L' is a leaf of F (thus L' = T').

Finally, F is described as follows: on input u ∈ A*, its head is the submachine 2-way transducer T'-along-ρ (u), where T is the head of F and ρ is the accepting run of T on ↑ u•i (represented by the bounded information that it is both initial and final). As before, we remove the submachines which are never called in F. Observe that we have created a machine with recursion height k−1 (because line 17 in algorithm 4 prevents from calling a k-th layer).

Let us justify that each accelerate-T'-along-ρ can indeed be implemented by a two-way transducer. First, let us observe that since F has bounded height, the number N of slices given in line 7 of algorithm 4 is bounded. Furthermore, we claim that the first and last positions of each ρ'_j belong to a given set of bounded size, which can be detected by a lookaround which has access to i. For the ρ'_j whose positions are in ↑ i, this is clear since ↑ i is bounded (because the frontier of any node is bounded). For ↓ i \ ϖ ↑ i we use lemma 6.5, which implies that this set is a bounded union of intervals. The last case is very similar.

**Lemma 6.5.** Let 1 ≤ i ≤ |u|, t := origin_F(i) and t₁ (resp. t₂) be its immediate left (resp. right) sibling (they exist whenever t ∈ Iter_F, i.e. here t \not\in F). Then:

\[ ↓ i \cup ↑ i = \{\min(\text{Fr}_F(t₁)) : \max(\text{Fr}_F(t₂))\} \setminus \{\text{Fr}_F(t₁), \text{Fr}_F(t), \text{Fr}_F(t₂)\} \]

This analysis justifies why each ρ'_j can be encoded in a bounded way. Now, we show how to implement the inlinings while using i as the current position:

- if i₁, . . . , iₙ ∈ ↑ i, then n is bounded (because ↑ i is bounded). We can thus inline old-T'-along-ρ'_j (u•i) while staying in position i. However, when T' calls some L'' (of head T'') on position iₖ, we would need to call old-T''-along-ρ''(u•iₖ) (where ρ'' is the accepting run of T'' along ↑ u•iₖ). But we cannot do this operation, since we are in position i and not in iₖ.

The solution is that the inlined code calls normal-T''-along-ρ''-pebble-iₖ(u•i)

![Figure 8: Slicing of a run ρ with respect to i and F.](image-url)
Algorithm 4: Submachines that try to simulate their recursive calls

```plaintext
Submachine accelerate-F-along-ρ (v)
/* F is not a leaf of W (i.e. it makes calls); */
/* v ∈ (A ∪ A)*; ρ is a run of F over ⊑v; */
u := τ(v); F := forest_u(u); λ := output function of F;
for (q, i) ∈ ρ do
  for L' ∈ λ(q, (⊑v)[i]) do
    ρ' := accepting run of F over ⊑u ⊔ i;
    ρ_1', ..., ρ_N := slicing of ρ' with respect to F and i;
    for j = 1 to N do
      (q_1, i_1) → ••• (q_n, i_n) := ρ_j;
      if i_1, ..., i_n ∈ ↑i then
        /* We inline the call because n is bounded; */
        Inline the code of old-F'-along-ρ_j (u ⊔ i);
      else if i_1, ..., i_n ∈ ↓i then
        /* We can inline the call because the positions
         i_1, ..., i_n are 'below' i in F; */
        Inline the code of old-F'-along-ρ_j (u ⊔ i);
      else if L' is a leaf of W then
        /* The output of L' = F' along ρ_j is empty; */
        else
          /* It is not possible to inline the call to L', so
          we make a recursive call; */
          Call accelerate-F'-along-ρ_j (u ⊔ i);
  end
end
```

instead, which simulates an accepting run ρ" of F on u ⊔ i, even if its input is u ⊔ i. Note that i_κ can be represented as a bounded information and recovered by a lookahead given u ⊔ i as input, since i observes i_κ;
- if i_1, ..., i_n ∈ ↓i ∧ ↑i, then the nodes origin_F(i_1), ..., origin_F(i_n) are roughly below origin_F(i) in F (see fig. 5). We inline old-F'-along-ρ_j (u ⊔ i), by moving along i_1, ..., i_n as ρ_j does. We can keep track of the height of origin_F(i) above the current origin_F(i_κ) (it is a bounded information). With
the lookahead, we can detect the end of ρ_j, and go back to position i.

It remains to justify that W is correct. For this, we only need to show that when it reaches line 18 in algorithm 4, the output of F' along ρ_j is indeed empty. Otherwise, the conditions of lemma 6.6 would hold (since we never execute two successive recursive calls in dependent positions). It provides a contradiction.

Lemma 6.6 (Key lemma). Let u ∈ A* and F ∈ Forests_u (u). Assume that there exists a sequence T_1, ..., T_k of submachines of W and a sequence of positions 1 ≤ i_1, ..., i_k ≤ |u| such that:
- T_1 is the head of W;
– $|\text{prod}_{\mathcal{F}_1}^{\mathcal{F}_2}(i_1)|_{\mathcal{F}_2} \neq 0$ and $\text{prod}_{\mathcal{F}_k}^{\mathcal{F}_{k-1}}(i_k) \neq \varepsilon$;
– for all $2 \leq j \leq k-1$, $|\text{prod}_{\mathcal{F}_j}^{\mathcal{F}_{j-1}}(i_j)|_{\mathcal{F}_{j+1}} \neq 0$;
– for all $1 \leq j \leq k-1$, $\text{origin}_{\mathcal{F}}(i_j)$ and $\text{origin}_{\mathcal{F}}(i_{j+1})$ are independent;

Then $\mathcal{U}$ is pumpable.

Proof (idea). As for lemma 5.4, the key observation is that pumpability follows as soon as the nodes $\text{origin}_{\mathcal{F}}(i_j)$ are pairwise independent. Furthermore, this condition can be obtained by duplicating some nodes in $\mathcal{F}$.

7 Making the two last pebbles visible

We can define a similar model to that of last $k$-pebble transducer, which sees the two last calling positions instead of only the previous one. Let us name this model a last-last $k$-pebble transducer. A very natural question is to know whether we can show an analog of theorem 3.5 for these machines.

Note that for $k = 1, 2$ and 3, a last-last $k$-pebble transducer is exactly the same as a $k$-pebble transducer. Hence the function $\text{inner-squaring}$ of page 2 is such that $|\text{inner-squaring}(u)| = O(|u|^2)$ and can be computed by a last-last 3-pebble transducer, but it cannot be computed by a last-last 2-pebble transducer. It follows that the connection between minimal recursion height and growth of the output fails. However, this result is somehow artificial. Indeed, a last-last 2-pebble transducer is a degenerate case, since it can only see one last pebble. More interestingly, we show that the connection fails for arbitrary heights.

**Theorem 7.1.** For all $k \geq 2$, there exists a function $f : A^* \rightarrow B^*$ such that $|f(u)| = O(|u|^2)$ and that can be computed by a last-last $(2k+1)$-pebble transducer, but not by a last-last $2k$-pebble transducer.

Proof (idea). We re-use a counterexample introduced by Bojańczyk in [2] to show a similar failure result for the model of $k$-pebble transducers.

8 Outlook

This paper somehow settles the discussion concerning the variants of pebble transducers for which the minimal recursion height only depends on the growth of the output. As soon as two marks are visible, the combinatorics of the output also has to be taken into account, hence minimizing the recursion height in this case (e.g. for last-last pebble transducers) seems hard with the current tools.

As observed in [13], one can extend last pebble transducers by allowing the recursion height to be unbounded (in the spirit of marble transducers [9]). This model enables to produce outputs whose size grows exponentially in the size of the input. A natural question is to know whether a function computed by this model, but whose output size is polynomial, can in fact be computed with a recursion stack of bounded height (i.e. by a last $k$-pebble transducer).

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References

1. Bojańczyk, M.: Polyregular functions. arXiv preprint arXiv:1810.08760 (2018)
2. Bojańczyk, M.: The growth rate of polyregular functions. arXiv preprint arXiv:2212.11631 (2022)
3. Bojańczyk, M.: Transducers of polynomial growth. In: Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science. pp. 1–27 (2022)
4. Bojańczyk, M., Kiefer, S., Lhote, N.: String-to-string interpretations with polynomial-size output. In: 46th International Colloquium on Automata, Languages, and Programming, ICALP 2019 (2019)
5. Chytil, M.P., Jákl, V.: Serial composition of 2-way finite-state transducers and simple programs on strings. In: 4th International Colloquium on Automata, Languages, and Programming, ICALP 1977. pp. 135–147. Springer (1977)
6. Colcombet, T.: Green’s relations and their use in automata theory. In: International Conference on Language and Automata Theory and Applications. pp. 1–21. Springer (2011)
7. Dave, V., Gastin, P., Krishna, S.N.: Regular transducer expressions for regular transformations. In: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science. pp. 315–324. ACM (2018)
8. Douéneau-Tabot, G.: Pebble transducers with unary output. In: 46th International Symposium on Mathematical Foundations of Computer Science, MFCS 2021 (2021)
9. Douéneau-Tabot, G., Filiot, E., Gastin, P.: Register transducers are marble transducers. In: 45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020 (2020)
10. Douéneau-Tabot, G.: Hiding pebbles when the output alphabet is unary. In: 49th International Colloquium on Automata, Languages, and Programming, ICALP 2022 (2022)
11. Engelfriet, J.: Two-way pebble transducers for partial functions and their composition. Acta Informatica 52(7-8), 559–571 (2015)
12. Engelfriet, J., Hoogeboom, H.J.: MSO definable string transductions and two-way finite-state transducers. ACM Transactions on Computational Logic (TOCL) 2(2), 216–254 (2001)
13. Engelfriet, J., Hoogeboom, H.J., Samwel, B.: Xml transformation by tree-walking transducers with invisible pebbles. In: Proceedings of the twenty-sixth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems. pp. 63–72. ACM (2007)
14. Gurari, E.M.: The equivalence problem for deterministic two-way sequential transducers is decidable. SIAM Journal on Computing 11(3), 448–452 (1982)
15. Hopcroft, J.E., Ullman, J.D.: An approach to a unified theory of automata. The Bell System Technical Journal 46(8), 1793–1829 (1967)
16. Kiefer, S., Nguyên, L.T.D., Pradic, C.: Revisiting the growth of polyregular functions: output languages, weighted automata and unary inputs. arXiv preprint arXiv:2301.09234 (2023)
17. Nguyên, L.T.D., Noûs, C., Pradic, C.: Comparison-free polyregular functions. In: 48th International Colloquium on Automata, Languages, and Programming, ICALP 2021 (2021)
18. Shepherdson, J.C.: The reduction of two-way automata to one-way automata. IBM Journal of Research and Development 3(2), 198–200 (1959)
19. Simon, I.: Factorization forests of finite height. Theor. Comput. Sci. 72(1), 65–94 (1990)
A Omitted proofs of section 5

A.1 Proof of lemma 5.2

Let \( w_i \in A^* \) be such that \( \mu(w_i) = \ell_i \), \( w'_i \in A^* \) be such that \( \mu(w'_i) = r_i \) and 
\( u_i := w_i a_i w'_i \), for \( 1 \leq i \leq k \). Let \( v_0, \ldots, v_k \in A^* \) such that \( \mu(v_i) = m_i \) for all 
\( 1 \leq i \leq k \). Let \( f_i \) be the function computed by \( \mathcal{T}_i \) for all \( 1 \leq i \leq k \).

It is easy to see that \( |f_k(v_0 u_1^X \cdots u_k^X v_k)| \geq (X-2) \). Indeed, \( \mathcal{T}_k \) must produce 
at least one letter when reading the letter \( a_{\sigma(k)} \) of each factor \( u_{\sigma(k)} \) (with possibly 
an exception for the borders, hence the \( -2 \)). We then show by induction on 
k \( \geq i \geq 1 \) that \( |f_i(v_0 u_1^X \cdots u_k^X v_k)| \geq (X-2)^{k-i+1} \). The upper bound follows 
since \( \mathcal{B} \) is a blind \( k \)-pebble transducer, thus |\( f(v_0 u_1^X \cdots u_k^X v_k) \)| = \( \Theta(X^k) \).

A.2 Additional arguments in section 5.2

We first justify that the construction of \( \overline{W} \) in section 5.2 indeed reduces the 
recursion height of \( \mathcal{W} \) by 1. This statement was claimed on page 12.

Claim. The machine \( \overline{W} \) described in section 5.2 has recursion height \( k-1 \).

Proof. Recall that the recursion height corresponds to the number of nested 
\textbf{Call} instructions, plus 1 (for the submachine which is the head). We first show 
by (decreasing) induction on \( 1 \leq \ell \leq k \) that if \( \mathcal{T} \) is the head of a subtree of \( \mathcal{W} \) 
whose recursion height is \( 1 \leq \ell \leq k \), then old-\( \mathcal{T} \) has recursion height \( \ell \) as well.

Then, we show by (decreasing) induction on \( 1 \leq \ell \leq k \) that if \( \mathcal{T} \) is the head 
of a subtree of \( \mathcal{W} \) whose recursion height is \( \ell \), then accelerate-\( \mathcal{T} \) has recursion 
height \( \ell-1 \). Indeed, the base case \( \ell = 2 \) is justified by line 12 in algorithm 2 
(there are no calls since we inline all the computations). For \( \ell > 2 \) the function 
accelerate-\( \mathcal{T} \) inlines old-\( \mathcal{T}' \) of recursion height \( \ell-1 \) and makes a recursive call 
to accelerate-\( \mathcal{T}' \) whose height is \( \ell-2 \) by induction hypothesis.

The result follows since the head of \( \mathcal{W} \) has recursion height \( k \) by definition of a 
blind \( k \)-pebble transducer.

Since we have justified in the main paper how each function of \( \overline{W} \) can be 
implemented by a two-way transducer, then \( \overline{W} \) is indeed a blind \( (k-1) \)-pebble 
transducer. Now, we justify a claim of page 12, that is used to show that the 
output in line 12 of algorithm 2 must be bounded.

Claim. Assume that, in the execution of \( \overline{W} \) on input \( u \in A^* \), we reach line 12 
in algorithm 2 and that \( \text{prod}^u_{\mathcal{T}_j}(i') \neq \varepsilon \) for \( i' \notin \text{Fr}^\mathcal{F} \). Then the conditions of 
lemma 5.4 hold, that is there exists \( \mathcal{F} \in \text{Forests}_\mu(u) \), a sequence \( \mathcal{T}_1, \ldots, \mathcal{T}_k \) of 
submachines of \( \mathcal{W} \) and a sequence of positions \( 1 \leq i_1, \ldots, i_k \leq |u| \) such that:

- \( \mathcal{T}_1 \) is the head of \( \mathcal{W} \);
- for all \( 1 \leq j \leq k-1 \), \( |\text{prod}^u_\mathcal{T}_j(i_j)\mathcal{T}_{j+1} \neq 0 \) and \( \text{prod}^u_\mathcal{T}_j(i_k) \neq \varepsilon \);
- for all \( 1 \leq j \leq k \), \( i_j \notin \text{Fr}^\mathcal{F} \) (i.e. origin\( ^\mathcal{F} \)(\( i_j \)) \in \text{iter}^\mathcal{F} \).
Proof. Let $T$ be a submachine of $U$ which is not a leaf (i.e. it labels the head of a subtree of height $> 1$). We claim that for accelerate-$T(u)$ to be called when executing $U$ on input $u \in A^*$, there must exist a sequence $T_1, \ldots, T_\ell$ of submachines of $U$ and a sequence of positions $1 \leq i_1, \ldots, i_{\ell-1} \leq |u|$ such that:

- $T_1$ is the head of $U$ and $T_\ell = T$;
- for all $1 \leq j \leq \ell-1$, $|\prod_{T_j}(i_j)|_{T_{j+1}} \neq 0$;
- for all $1 \leq j \leq \ell-1$, $i_j \not\in Fr^\mathcal{F}(F)$ where $F := forest_u(u)$.

This result can be checked by induction. Intuitively, it means that to systematically avoid inlinings, we have to make recursive calls in a sequence of positions which are never in the frontier on the root.

Finally, by considering $T$ the head of a subtree of height 2, we see that the conditions of lemma 5.4 must hold if we reach line 12 in accelerate-$T$ and if $\prod_{T}(i') \neq \varepsilon$ for $i' \not\in Fr^\mathcal{F}(F)$.

To conclude about the omitted proofs in section 5.2, it remains to show lemma 5.4. This is the purpose of appendix A.3.

### A.3 Proof of lemma 5.4

Assume that the conditions of lemma 5.4 hold and let $t_j := origin^\mathcal{F}(i_j)$ for all $1 \leq j \leq k$. If the $t_j$ are pairwise independent, then each $t_j$ is surrounded by two nodes whose frontiers cannot contain a position $i_{j'}$ for some $1 \leq j' \leq k$. The image of the factor of $u$ which is below these nodes provides an idempotent $e_j$. It can easily be concluded that $U$ is pumpable (see also [10, Lemma E.5]).

![Figure 9: Duplicating a subtree in $\mathcal{F}$ so that $t'_{\ell_1}$ and $t'_{\ell_2}$ are not dependent.](image)

Now, we suppose that the $t_j$ are not necessarily pairwise independent. Let us show how to make the number of dependent couples of $(t_{j_1}, t_{j_2})$ decrease strictly, while preserving the properties of lemma 5.4. Indeed, repeating this process will enable us to make all the nodes pairwise independent. Assume that $t_{\ell_1}$ observes $t_{\ell_2}$ for some $1 \leq \ell_1 \neq \ell_2 \leq k$. To simplify the proof, we assume that $t_{\ell_2}$ is an ancestor of $t_{\ell_1}$ (the case of the immediate sibling of an ancestor is similar). Let $\mathcal{F}'$ be $\mathcal{F}$ in which the subtree $t_{\ell_2}$ has been copied 3 times (since $t_{\ell_2}$ is an iterable node, then $\mathcal{F}'$ still a $\mu$-forest), see fig. 9. We define for $1 \leq j \leq k$ the nodes $t'_j \in Nodes^{\mathcal{F}'}$ as follows:
- if $j = \ell_2$, then $t'_j$ is (the root of) the third copy of $t_j$;
- else if $j$ is such that $t_j$ was a descendant of $t_{e_1}$ (including $t_{e_1}$), then we let $t'_j$ be the corresponding node in the first copy of $t_j$;
- else $t_j$ was in the rest of $\mathcal{F}$, and we let $t'_j$ be the corresponding node in $\mathcal{F}'$.

Observe that now, $t'_{e_1}$ and $t'_{e_2}$ are not dependent. Furthermore if $t_{j_1}$ and $t_{j_2}$ were independent, then $t'_{j_1}$ and $t'_{j_2}$ are also independent. Let $u' \in A^*$ be the word such that $\mathcal{F}' \in \text{Forests}_{\mu}(u')$. We also define $1 \leq i_1' \leq \ldots \leq i_k' \leq |u'|$ as the positions which correspond to the former $1 \leq i_1, \ldots, i_k \leq |u|$ in the frontiers of $t'_{i_1}, \ldots, t'_{i_k}$ in the new $\mu$-forest $\mathcal{F}'$. The conditions of lemma 5.4 still hold, because $\text{prod}_{\mu}^{u'}(i_j) = \text{prod}_{\mu}^{u}(i'_j)$ (indeed, we have only duplicated an iterable node, which does neither modify the context around $i'_j$ nor its crossing sequence).

\section*{B Omitted proofs of section 6}

\subsection*{B.1 Proof of lemma 6.2}

The proof is similar to that of lemma 5.2. Let $w_i \in A^*$ be such that $\mu(w_i) = \ell_i$, $w'_i \in A^*$ be such that $\mu(w'_i) = r_i$, $u_i := w_ia_iw'_i$ and $u_i := w_ia_iw'_i$, for $1 \leq i \leq k$. Let $v_0, \ldots, v_k \in A^*$ such that $\mu(v_i) = m_i$ for all $1 \leq i \leq k$. Let $f_i$ be the function computed by the call $\mathcal{T}_i$ for all $1 \leq i \leq k$.

To simply the proof, we assume that $\sigma : [1:k] \to [1:k]$ is the identity function. We then observe that for all $X \geq 2$, for all $1 \leq Y \leq X - 2$:

$$f_k(v_0u_1^{X} \cdots v_{k-2}(u_{k-1}^{X-1}u_{k-1}^{X-1})u_{k-1}^{X}) \geq (X - 2).$$

Observe that the use of $u_{k-1}^{X-1}u_{k-1}^{X-1}$ means that the result holds independently from the factor in which the call (i.e. the mark) to $\mathcal{T}_k$ was done. Finally, we conclude by induction in a similar way to lemma 5.2.

\subsection*{B.2 Additional arguments in section 6.2}

We first justify that the construction of $\overline{\mathcal{W}}$ in section 6.2 indeed reduces the recursion height of $\mathcal{W}$ by 1. This statement was claimed on page 16.

Claim. The machine $\overline{\mathcal{W}}$ described in section 6.2 has recursion height $k - 1$.

Proof. Recall that the recursion height corresponds to the number of nested Call instructions, plus 1 (due to the head). We first show by (decreasing) induction on $1 \leq \ell \leq k$ that if $\mathcal{T}$ is the head of a subtree of $\mathcal{W}$ whose recursion height is $1 \leq \ell \leq k$, then old-$\mathcal{T}$-along-$\rho$ has recursion height $\ell$ as well.

Then, we show by (decreasing) induction on $2 \leq \ell \leq k$ that if $\mathcal{T}$ is the head of a subtree of $\mathcal{W}$ whose recursion height is $\ell$, then accelerate-$\mathcal{T}$-along-$\rho$ has recursion height $\ell - 1$. Indeed, the base case $\ell = 2$ is justified by line 18 in algorithm 4 (there are no calls). For $\ell > 2$ the function accelerate-$\mathcal{T}$-along-$\rho$ inlines some old-$\mathcal{T}$-along-$\rho'_j$ of recursion height $\ell - 1$ and makes recursive calls to accelerate-$\mathcal{T}'$-along-$\rho'_j$, whose height is $\ell - 2$ by induction hypothesis.

The result follows since the head of $\mathcal{W}$ has recursion height $k$ by definition of a last $k$-pebble transducer.
Now, let us justify a claim of page 17, that is used to show that the output of line 18 in algorithm 4 is indeed empty.

**Claim.** Assume that, in the execution of $\mathcal{W}$ on input $u \in A^*$, we reach line 18 in algorithm 4 and that the output of $\mathcal{T}'$ along $\rho_j'$ is not empty. Then the conditions of lemma 6.6 hold, that is there exists $\mathcal{T} \in \text{Forests}_u(u)$, a sequence $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of submachines of $\mathcal{W}$ and a sequence of positions $1 \leq i_1, \ldots, i_k \leq |u|$ such that:

- $\mathcal{T}_1$ is the head of $\mathcal{W}$;
- $|\prod_{\mathcal{T}_1}^{\mathcal{T}_1}(i_1)|_{\mathcal{T}_2} \neq 0$ and $|\prod_{\mathcal{T}_1}^{\mathcal{T}_1}(i_k)|_{\mathcal{T}_k} \neq \varepsilon$;
- for all $2 \leq j \leq k-1$, $|\prod_{\mathcal{T}_1}^{\mathcal{T}_1}(i_j)|_{\mathcal{T}_j+1} \neq 0$;
- for all $1 \leq j \leq k-1$, $\text{origin}^\mathcal{T}(i_j)$ and $\text{origin}^\mathcal{T}(i_{j+1})$ are independent;

**Proof.** Let $\mathcal{T}$ be a submachine of $\mathcal{W}$ which is neither a leaf (i.e. it is the head of a subtree of height $> 1$) nor the head of $\mathcal{W}$ (i.e. not the root of $\mathcal{W}$). We claim that for accelerate-$\mathcal{T}$-along-$\rho$ $(u)$ to be called within the execution of $\mathcal{W}$ on input $u \in A^*$, there must exist a sequence $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of submachines of $\mathcal{W}$ and a sequence of positions $1 \leq i_1, \ldots, i_k \leq |u|$ such that:

- $\mathcal{T}_1$ is the head of $\mathcal{W}$;
- $u = u^{i_{k-1}}$;
- $|\prod_{\mathcal{T}_1}^{\mathcal{T}_1}(i_1)|_{\mathcal{T}_2} \neq 0$ and for all $2 \leq j \leq k-1$, $|\prod_{\mathcal{T}_1}^{\mathcal{T}_1}(i_j)|_{\mathcal{T}_{j+1}} \neq 0$;
- for all $1 \leq j \leq k-2$, $\text{origin}^\mathcal{T}(i_j)$ and $\text{origin}^\mathcal{T}(i_{j+1})$ are independent, where we define $\mathcal{F} := \text{forest}_u(u)$;
- for all $(i, q) \in \rho$, $\text{origin}^\mathcal{T}(i_{\ell-1})$ and $\text{origin}^\mathcal{T}(i)$ are independent

This result can be checked by induction. Intuitively, the two crucial last point follows from the fact that we only make recursive calls in portions of runs whose positions are not dependent on the calling position.

Finally, by considering $\mathcal{T}$ the head of a subtree of height 2, we see that the conditions of lemma 6.6 must hold if we reach line 18 in some accelerate-$\mathcal{T}$-along-$\rho$ called in $\mathcal{W}$ and if the output of $\mathcal{T}'$ along $\rho_j'$ is not empty.

To conclude about the omitted proofs in section 6.2, it remains to show lemmas 6.5 and 6.6. This is the purpose of appendices B.3 and B.4.

### B.3 Proof of Lemma 6.5

Let $1 \leq i \leq |u|$, $t := \text{origin}^\mathcal{T}(i)$ and $t_1$ (resp. $t_2$) be its immediate left (resp. right) sibling (they exist whenever $t \in \text{Iter}^\mathcal{T}$, i.e. here $t \neq \mathcal{T}$). We show that:

$$\downarrow i \uparrow i = \{ \text{Fr}^F(t_1) : \text{Fr}^F(t_2) \}$$

Let us assume that $t_1$ and $t_2$ are iterable nodes of $\mathcal{F}$ (the other cases are similar). By considering the forest of fig. 5, it can be noted that $\downarrow i$ is the interval $[\min(\text{Fr}^F(t_1)) : \max(\text{Fr}^F(t_2))]$. We conclude since $t, t_1$ and $t_2$ are the only iterable nodes that both observe $t$ and that $t$ observes.

Therefore $\downarrow i \uparrow i$ is the union of a bounded number of intervals (since the frontiers have bounded size). It is easy to observe that the “borders” of these intervals can easily be recovered by a lookaround, if $t$ (or $i$) is given.
B.4 Proof of lemma 6.6

The proof is similar to that of lemma 5.4. The goal is to show that the for $1 \leq j \leq k$, the $t_j := \text{origin}^F(i_j)$ can be chosen pairwise independent (in the hypothesis, it is only assumed for the consecutive pairs $(t_j, t_{j+1})$).

For this, we show once more how to make the number of dependent nodes decrease strictly, while preserving the properties of lemma 6.6. Assume that $t_{i_j}$ observes $t_{i_k}$ for some $1 \leq i_1 \neq i_2 \leq k$ (note that $i_1$ and $i_2$ are not consecutive).

To simplify the proof, we assume that $t_{i_2}$ is an ancestor of $t_{i_1}$ (the case of the immediate sibling of an ancestor is similar). We build $F' \in \text{Forests}_\mu(u')$ as in the proof of lemma 5.4 (see fig. 9), and define the new nodes $t'_1, \ldots, t'_k \in \text{Nodes}^{F'}$ in the same way. We also define $1 \leq i'_1, \ldots, i'_k \leq \left|u'\right|$ as the positions which correspond to the former $1 \leq i_1, \ldots, i_k \leq \left|u\right|$ adapted to the new nodes $t'_1, \ldots, t'_k$.

Now, we justify that $\prod_{i'_j}^{u_i'j^{-1}}(i_j) = \prod_{i_j}^{u_i^{-1}j^{-1}}(i'_j)$ for all $2 \leq j \leq k$. This is the only difference with the proof of of lemma 5.4 that we need to treat:

- if both $i_{j-1}$ and $i_j$ belong to the subtree rooted in $t_{i_2}$, then $j \neq i_2$ (since otherwise $i_2$ and $i_{j-1}$ would be dependent) and similarly $j-1 \neq i_2$. The result holds because we only iterate an iterable node;
- if both $i_{j-1}$ and $i_j$ do not belong to this subtree, the argument is similar;
- if $i_{j-1}$ is in the subtree but not $i_j$ (the converse is similar), we use the fact that these two nodes are independent. Indeed, it implies that $i_j$ cannot be "below" an immediate sibling of $t_{i_2}$. Hence duplicating this iterable node will not change the monoid value between positions $i_{j-1}$ and $i_j$.

C Proof of theorem 7.1

Let $A$ be an alphabet and $k \geq 1$. We define the tree language $T_A^k$ as the set of trees such that all root-to-leaf branches have exactly $k$ nodes (hence the tree has height $k$), whose leaves are labelled by words of $A^*$ and whose inner nodes have no labels. As observed for factorization forests, $T_A^k$ can be seen as a regular word language over the alphabet $\hat{A} := A \cup \{\langle,\rangle\}$. Given such a tree, we say that the root has height 1, its children height 2, etc. and the leaves have height $k$.

Now, we describe for all $k \geq 1$ a function $\text{alternating-square}^k : T_A^{k+1} \to T_A^{2k+1}$ which goes from words to words (i.e. it works on the word representation of the trees). This function was introduced by Bojańczyk in [2]. Intuitively, it produces a tree whose leaves labels are tuples $uv\#vw$ for $u, v$ labels of the original tree, but the ordering of these tuples is very specific.

Let us first describe the function $\text{alternating-square}^1$. It takes as input a tree of height 2 of shape $\langle\langle u_1\rangle, \langle u_2\rangle, \ldots, \langle u_n\rangle\rangle$ and it produces a tree of height 3 whose $i$-th child of the root is $\langle\langle u_i\#u_1\rangle, \ldots, \langle u_i\#u_n\rangle\rangle$ (i.e. the tuples are ordered lexicographically). This function be implemented by a 3-pebble transducer (i.e. a last-last 3-pebble transducer) which uses its two first pebble to see which leaves have to be produced, and the last layer to indeed output these leaves. Observe that alternating-square$^1$ can be seen as a variant of the function unmarked-square presented in example 3.2.
Now, the function alternating-square$^2$ is described formally in algorithm 5.

\begin{algorithm}[h]
\centering
\caption{Computing the alternating-square$^2$ function}
\begin{algorithmic}[1]
\Function{alternating-square$^2(u)$}
\State $u \in A^*$ represents a tree of depth $k+1$;
\State $i_0 := j_0 :=$ the root of $u$;
\For {$i_1$ ranging from left to right on the children of $i_0$}
\State Output $();$
\For {$j_1$ ranging from left to right on the children of $j_0$}
\State Output $();$
\EndFor
\For {$i_2$ ranging from left to right on the children of $i_1$}
\State Output $();$
\For {$j_2$ ranging from left to right on the children of $j_1$}
\Comment{Depth 3: $i_2$ and $j_2$ are leaves;}
\State $u :=$ label of $i_2$; $v :=$ label of $j_2$;
\State Output $(u\#v)$
\EndFor
\EndFor
\EndFor
\EndFunction
\end{algorithmic}
\end{algorithm}

Observe that alternating-square$^2$ no longer produces the tuples $u\#v$ in a lexicographic ordering. Indeed, it corresponds to two ranges over the leaves of the original tree (one with $i_1, i_2$ and one with $j_1, j_2$) which are highly entangled. It is easy to guess how to extend algorithm 5, in order to define alternating-square$^k$ using $2k$ nested loops.

Originally, the alternating-square$^k$ functions were used in order to show that the minimal number of layers and the growth of the output do not coincide for pebble transducers, as claimed in theorem C.1.

**Theorem C.1** ([2, section 3]). *For all $k \geq 1$, the function alternating-square$^k$ is such that $|\text{alternating-square}^k(u)| = O(|u|^2)$. Furthermore, it can be computed by a $(2k+1)$-pebble transducer but not by a $2k$-pebble transducer.*

As a consequence, alternating-square$^k$ cannot be computed by a last-last $2k$-pebble transducer. To show theorem 7.1, it is thus sufficient for us to justify that alternating-square$^k$ can be computed by last-last $(2k+1)$-pebble transducer. This is indeed the case: it uses its $2k$ first layers to describe the nested loops on $i_1, i_2, j_2, \ldots, i_k, j_k$ and the last one to range over the labels of the tuple of leaves and output them (see algorithm 5). The key observation is that it only needs to see the two last loop indexes, since this information is sufficient to find their children.