On $\Pi$-supplemented subgroups of a finite group

Xiaoyu Chen, Wenbin Guo†
Department of Mathematics, University of Science and Technology of China,
Hefei 230026, P. R. China
E-mail: jelly@mail.ustc.edu.cn, wbguo@ustc.edu.cn

Abstract

A subgroup $H$ of a finite group $G$ is said to satisfy $\Pi$-property in $G$ if for every chief factor $L/K$ of $G$, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. A subgroup $H$ of $G$ is called to be $\Pi$-supplemented in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq I \leq H$, where $I$ satisfies $\Pi$-property in $G$. In this paper, we investigate the structure of a finite group $G$ under the assumption that some primary subgroups of $G$ are $\Pi$-supplemented in $G$. The main result we proved improves a large number of earlier results.

1 Introduction

Throughout this paper, all groups mentioned are finite, $G$ always denotes a finite group and $p$ denotes a prime. Let $\pi$ denote a set of some primes, $\pi(G)$ denote the set of all prime divisors of $|G|$, and $|G|_p$ denote the order of the Sylow $p$-subgroups of $G$. An integer $n$ is called a $\pi$-number if all prime divisors of $n$ belong to $\pi$. For a subgroup $H$ of $G$, let $H^G$ denote the normal closure of $H$ in $G$, that is, $H^G = \langle H^g : g \in G \rangle$.

Recall that a class of groups $\mathcal{F}$ is called a formation if $\mathcal{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is said to be saturated (resp. solubly saturated) if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$ (resp. $G/\Phi(N) \in \mathcal{F}$ for a soluble normal subgroup $N$ of $G$). A chief factor $L/K$ of $G$ is said to be $\mathcal{F}$-central in $G$ if $(L/K) \rtimes (G/C_G(L/K)) \in \mathcal{F}$.

*Research is supported by a NNSF grant of China (grant #11371335) and Research Fund for the Doctoral Program of Higher Education of China (Grant 20113402110036).
†Corresponding author.

Keywords: $\Pi$-property, $\Pi$-supplemented subgroups, supersolubility, Sylow subgroups.

Mathematics Subject Classification (2000): 20D10, 20D15, 20D20.
A normal subgroup \( N \) of \( G \) is called to be \( \mathcal{F} \)-hypercentral in \( G \) if every chief factor of \( G \) below \( N \) is \( \mathcal{F} \)-central in \( G \). Let \( Z_{\mathcal{F}}(G) \) denote the \( \mathcal{F} \)-hypercentre of \( G \), that is, the product of all \( \mathcal{F} \)-hypercentral normal subgroups of \( G \). We use \( \mathcal{U} \) (resp. \( \mathcal{U}_p \)) to denote the class of finite supersoluble (resp. \( p \)-supersoluble) groups and \( \mathcal{G}_\pi \) to denote the class of all finite \( \pi \)-groups.

Recall that \( G \) is said to be quasinilpotent if for every chief factor \( L/K \) of \( G \) and every element \( x \in G \), \( x \) induces an inner automorphism on \( L/K \). The generalized Fitting subgroup \( F^*(G) \) of \( G \) is the quasinilpotent radical of \( G \) (for details, see \cite{21} Chapter X). All notations and terminology not mentioned above are standard, as in \cite{9, 14, 20}.

In \cite{23}, Li introduced the concepts of \( \Pi \)-property and \( \Pi \)-supplemented subgroup as follows:

**Definition 1.1.** \cite{23} A subgroup \( H \) of \( G \) is said to satisfy \( \Pi \)-property in \( G \) if for every chief factor \( L/K \) of \( G \), \( |G/K : N_{G/K}(HK/K \cap L/K)| \) is a \( \pi \)-(\( HK/K \cap L/K \))-number.

A subgroup \( H \) of \( G \) is called to be \( \Pi \)-supplemented in \( G \) if there exists a subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq I \leq H \), where \( I \) satisfies \( \Pi \)-property in \( G \).

As we showed in Section 4 below, the concept of \( \Pi \)-supplemented subgroup generalizes many known embedding properties. However, besides \cite{24}, this concept has not been deeply investigated. In this paper, we will continue to study the properties of \( \Pi \)-supplemented subgroups, and arrive at the following main result.

**Theorem A.** Let \( \mathcal{F} \) be a solubly saturated formation containing \( \mathcal{U} \) and \( E \) a normal subgroup of \( G \) with \( G/E \in \mathcal{F} \). Let \( X \leq G \) such that \( F^*(E) \leq X \leq E \). For every prime \( p \in \pi(X) \) and every non-cyclic Sylow \( p \)-subgroup \( P \) of \( X \), suppose that \( P \) has a subgroup \( D \) such that \( 1 \leq |D| < |P| \) and every proper subgroup \( H \) of \( P \) with \( |H| = p^n|D| \) (\( n = 0, 1 \)) either is \( \Pi \)-supplemented in \( G \) or has a \( p \)-supersoluble supplement in \( G \). If \( P \) is not quaternion-free and \( |D| \neq 1 \), suppose further that every cyclic subgroup of \( P \) of order 4 either is \( \Pi \)-supplemented in \( G \) or has a 2-supersoluble supplement in \( G \). Then \( G \in \mathcal{F} \).

Recall that a subgroup \( H \) of \( G \) is said to be \( c \)-supplemented \cite{5} in \( G \) if there exists a subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_G \), where \( H_G \) denotes the largest normal subgroup of \( G \) contained in \( H \). It is easy to find that all \( c \)-supplemented subgroups of \( G \) are \( \Pi \)-supplemented in \( G \), and the converse does not hold. For example, let \( G = \langle a, b \mid a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle \) and \( H = \langle b^2 \rangle \). Then \( H \) is \( \Pi \)-supplemented, but not \( c \)-supplemented in \( G \). In \cite{2}, M. Asaad proved the following excellent theorem.

**Theorem 1.2.** \cite{2} Theorems 1.5 and 1.6 Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{U} \) and \( E \) a normal subgroup of \( G \) with \( G/E \in \mathcal{F} \). Let \( X \leq G \) such that \( X = E \) or \( X = F^*(E) \). For any Sylow subgroup \( P \) of \( X \), let \( D \) be a subgroup of \( P \) such that \( 1 \leq |D| < |P| \). Suppose that every subgroup \( H \) of \( P \) with \( |H| = p^n|D| \) (\( n = 0, 1 \)) is \( c \)-supplemented in \( G \). If \( P \) is a non-abelian 2-group and \( |D| = 1 \), suppose further that every cyclic subgroup of \( P \) of order 4
is $c$-supplemented in $G$. Then $G \in \mathcal{F}$.

One can see that Theorem A can be viewed as a large improvement of M. Asaad’s result. The following theorems are the main stages of the proof of Theorem A.

**Theorem B.** Let $P$ be a normal $p$-subgroup of $G$. Suppose that $P$ has a subgroup $D$ such that $1 \leq |D| < |P|$ and every proper subgroup $H$ of $P$ with $|H| = p^n|D|$ $(n = 0, 1)$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. If $P$ is not quaternion-free and $|D| = 1$, suppose further that every cyclic subgroup of $P$ of order 4 either is $\Pi$-supplemented in $G$ or has a 2-supersoluble supplement in $G$. Then $P \leq Z_4(G)$.

**Theorem C.** Let $E$ be a normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $E$ with $(|E|, p - 1) = 1$. Suppose that $P$ has a subgroup $D$ such that $1 \leq |D| < |P|$ and every proper subgroup $H$ of $P$ with $|H| = p^n|D|$ $(n = 0, 1)$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. If $P$ is not quaternion-free and $|D| = 1$, suppose further that every cyclic subgroup of $P$ of order 4 either is $\Pi$-supplemented in $G$ or has a 2-supersoluble supplement in $G$. Then $E$ is $p$-nilpotent.

Finally, the following corollaries can be deduced immediately from Theorem A.

**Corollary 1.3.** Let $\mathcal{F}$ be a solubly saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of $G$ with $G/E \in \mathcal{F}$. Let $X \unlhd G$ such that $F^*(E) \leq X \leq E$. For every prime $p \in \pi(X)$ and every non-cyclic Sylow $p$-subgroup $P$ of $X$, suppose that every maximal subgroup of $P$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. Then $G \in \mathcal{F}$.

**Corollary 1.4.** Let $\mathcal{F}$ be a solubly saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of $G$ with $G/E \in \mathcal{F}$. Let $X \unlhd G$ such that $F^*(E) \leq X \leq E$. For every prime $p \in \pi(X)$ and every non-cyclic Sylow $p$-subgroup $P$ of $X$, suppose that every cyclic subgroup of $P$ of prime order or order 4 (when $P$ is not quaternion-free) either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. Then $G \in \mathcal{F}$.

## 2 Basic Properties

**Lemma 2.1.** [23] Proposition 2.1] Let $H \leq G$ and $N \leq G$.

1. If $H$ satisfies $\Pi$-property in $G$, then $HN/N$ satisfies $\Pi$-property in $G/N$.

2. If either $N \leq H$ or $(|H|, |N|) = 1$ and $H$ is $\Pi$-supplemented in $G$, then $HN/N$ is $\Pi$-supplemented in $G/N$.

**Lemma 2.2.** Let $H \leq G$, $N \unlhd G$ such that $N \leq H$ and $P$ be a Sylow $p$-subgroup of $H$. Suppose that $P$ has a subgroup $D$ such that $|N|_p \leq |D| < |P|$ and every subgroup of $P$ of order $|D|$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. Then every
subgroup of \( PN/N \) of order \(|D|/|N|_p \) either is \( \Pi \)-supplemented in \( G/N \) or has a \( p \)-supersoluble supplement in \( G/N \).

**Proof.** Let \( X/N \) be a subgroup of \( PN/N \) of order \(|D|/|N|_p \). Then \( X = (P \cap X)N \), and so \( X/N \cong P \cap X/P \cap N \). Hence \(|P \cap X| = |D|\). By hypothesis, \( P \cap X \) either is \( \Pi \)-supplemented in \( G \) or has a \( p \)-supersoluble supplement in \( G \). Then \( P \cap X \) has a supplement \( T \) in \( G \) such that either \( T \) is \( p \)-supersoluble or \( P \cap X \cap T \leq I \leq P \cap X \), where \( I \) satisfies \( \Pi \)-property in \( G \). Obviously, \( G/N = (X/N)(TN/N) \). Since \(|N : P \cap N|, |N : T \cap N| = 1 \), \( N = (P \cap N)(T \cap N) \).

This deduces that \( X \cap TN = (P \cap X)N \cap TN = (P \cap X \cap T)N \). Therefore, either \( TN/N \) is \( p \)-supersoluble or \( X/N \cap TN/N = (P \cap X \cap T)N/N \leq IN/N \leq X/N \), where \( IN/N \) satisfies \( \Pi \)-property in \( G/N \) by Lemma 2.1(1). Consequently, \( X/N \) either is \( \Pi \)-supplemented in \( G/N \) or has a \( p \)-supersoluble supplement in \( G/N \). \( \square \)

For any function \( f: \mathbb{P} \cup \{0\} \longrightarrow \{\text{formations of groups}\} \). Following [27], let

\[
CF(f) = \{G \text{ is a group } | G/C_G(H/K) \in f(0) \text{ for each non-abelian chief factor}
\]

\[
H/K \text{ of } G \text{ and } G/C_G(H/K) \in f(p) \text{ for each abelian } p\text{-chief factor } H/K \text{ of } G\}.
\]

**Lemma 2.3.** [27] For any non-empty solubly saturated formation \( \mathcal{F} \), there exists a unique function \( F: \mathbb{P} \cup \{0\} \longrightarrow \{\text{formations of groups}\} \) such that \( \mathcal{F} = CF(F), F(p) = G_pF(p) \subseteq \mathcal{F} \) for all \( p \in \mathbb{P} \) and \( F(0) = \mathcal{F} \).

The function \( F \) in Lemma 2.3 is called the canonical composition satellite of \( \mathcal{F} \).

**Lemma 2.4.** [18 Lemma 2.14] Let \( \mathcal{F} \) be a saturated \( (\text{resp. solubly saturated}) \) formation and \( F \) the canonical local \( (\text{resp. the canonical composition}) \) satellite of \( \mathcal{F} \) \( (\text{for the details of canonical local satellite, see [9, Chapter IV, Definition 3.9]}\) \). Let \( E \) be a normal \( p \)-subgroup of \( G \). Then \( E \leq Z_F(G) \) if and only if \( G/C_G(E) \in F(p) \).

**Lemma 2.5.** [12 Lemma 2.4] Let \( P \) be a \( p \)-group. If \( \alpha \) is a \( p' \)-automorphism of \( P \) which centralizes \( \Omega_1(P) \), then \( \alpha = 1 \) unless \( P \) is a non-abelian 2-group. If \( [\alpha, \Omega_2(P)] = 1 \), then \( \alpha = 1 \) without restriction.

**Lemma 2.6.** [10 Lemma 2.15] If \( \sigma \) is an automorphism of odd order of the quaternion-free 2-group \( P \) and \( \sigma \) acts trivially on \( \Omega_1(P) \), then \( \sigma = 1 \).

**Lemma 2.7.** [13 Chapter 5, Theorem 3.13] A \( p \)-group \( P \) possesses a characteristic subgroup \( C \) (which is called a Thompson critical subgroup of \( P \)) with the following properties:

1. The nilpotent class of \( C \) is at most 2, and \( C/Z(C) \) is elementary abelian.
2. \([P, C] \leq Z(C)\).
3. \( C_P(C) = Z(C)\).
(4) Every nontrivial $p'$-automorphism of $P$ induces a nontrivial automorphism of $C$.

If $P$ is either an odd order $p$-group or a quaternion-free 2-group, then let $\Omega(P)$ denote the subgroup $\Omega_1(P)$, otherwise $\Omega(P)$ denotes $\Omega_2(P)$. The following lemma is a generalization of [6, Lemma 2.12], which is attributed to A. N. Skiba.

**Lemma 2.8.** Let $\mathcal{F}$ be a solubly saturated formation, $P$ a normal $p$-subgroup of $G$ and $C$ a Thompson critical subgroup of $P$. If either $P/\Phi(P) \leq Z_\mathcal{F}(G/\Phi(P))$ or $\Omega(C) \leq Z_\mathcal{F}(G)$, then $P \leq Z_\mathcal{F}(G)$.

**Proof.** Let $F$ be the canonical composition satellite of $\mathcal{F}$. Suppose that $P/\Phi(P) \leq Z_\mathcal{F}(G/\Phi(P))$. Then by Lemma 2.4, $G/C_G(P/\Phi(P)) \in \mathcal{F}(p)$. Note that by [13] Chapter 5, Theorem 1.4, $C_G(P/\Phi(P))/C_G(P)$ is a $p$-group. This implies that $G/C_G(P) \in \mathcal{G}_pF(p) = F(p)$. Hence by Lemma 2.4 again, $P \leq Z_\mathcal{F}(G)$.

Now assume that $\Omega(C) \leq Z_\mathcal{F}(G)$. Then by Lemma 2.4, $G/C_G(\Omega(C)) \in \mathcal{F}(p)$. Since $C_G(\Omega(C))/C_G(C)$ is a $p$-group by Lemmas 2.5 and 2.6, we have that $G/C_G(C) \in \mathcal{G}_pF(p) = F(p)$. It follows from Lemma 2.7(4) that $C_G(C)/C_G(P)$ is a $p$-group, and so $G/C_G(P) \in \mathcal{G}_pF(p) = F(p)$. Thus by Lemma 2.4 again, $P \leq Z_\mathcal{F}(G)$. \hfill $\square$

**Lemma 2.9.** [28] Lemma 3.1 Let $G$ be a non-abelian quaternion-free 2-group. Then $G$ has a characteristic subgroup of index 2.

**Lemma 2.10.** Let $C$ be a Thompson critical subgroup of a nontrivial $p$-group $P$.

1. If $p$ is odd, then the exponent of $\Omega_1(C)$ is $p$.
2. If $P$ is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.
3. If $p = 2$, then the exponent of $\Omega_2(C)$ is at most 4.

**Proof.** (1) Since the nilpotent class of $C$ is at most 2 by Lemma 2.7(1), the statement (1) directly follows from [13] Chapter 5, Lemma 3.9(i)].

Statement (2) is clear.

(3) Let $x$ and $y$ be elements of $C$ of order 4. Then by Lemma 2.7(1) and [13] Chapter 2, Lemma 2.2], we have that $[x, y]^2 = [x^2, y] = 1$, and so $(yx)^4 = [x, y]^6y^4x^4 = 1$. This shows that the order of $yx$ is at most 4, and thus the exponent of $\Omega_2(C)$ is at most 4. \hfill $\square$

Recall that $G$ is said to be $\pi$-closed if $G$ has a normal Hall $\pi$-subgroup. Also, $G$ is said to be a $C_\pi$-group if $G$ has a Hall $\pi$-subgroup and any two Hall $\pi$-subgroups of $G$ are conjugate in $G$.

**Lemma 2.11.** Let $p$ be a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Suppose that $G$ has a Hall $p'$-subgroup. Then $G$ is a $C_{p'}$-group.

**Proof.** If $p > 2$, then $2 \nmid |G|$. By Feit-Thompson Theorem, $G$ is soluble, and so $G$ is a $C_{p'}$-group. If $p = 2$, then by [8] Theorem A], $G$ is also a $C_{p'}$-group. \hfill $\square$
Lemma 2.12. [18, Corollary 3.7] Let $P$ be a $p$-subgroup of $G$. Suppose that $G$ is a $C_\pi$-group for some set of primes $\pi$ with $p \notin \pi$. If every maximal subgroup of $P$ has a $\pi$-closed supplement in $G$, then $G$ is $\pi$-closed.

The next lemma is well-known.

Lemma 2.13. Let $p$ be a prime divisor of $|G|$ with $(|G|, p - 1) = 1$.

1. If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.
2. If $E$ is a normal subgroup of $G$ such that $|E|_p \leq p$ and $G/E$ is $p$-nilpotent, then $G$ is $p$-nilpotent.
3. If $H$ is a subgroup of $G$ such that $|G:H| = p$, then $H \trianglelefteq G$.

Lemma 2.14. [1, Lemma 2.6] If $G$ possesses two subgroups $K$ and $T$ such that $|G:K| = 2^r$ and $|G:T| = 2^{r+1}$ ($r \geq 3$) and $T$ is not a 2'-Hall subgroup of $G$, then $G$ is not a non-abelian simple group.

Recall that a subgroup $H$ of $G$ is said to be complemented in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T = 1$. In this case, $T$ is called a complement of $H$ in $G$.

Lemma 2.15. Let $P$ be a Sylow $p$-subgroup of $G$ with $(|G|, p - 1) = 1$. If every subgroup of $P$ of order $p$ is complemented in $G$, then $G$ is $p$-nilpotent.

Proof. Let $H$ be a subgroup of $P$ of order $p$ and $T$ a complement of $H$ in $G$. Then by Lemma 2.13(3), $T \trianglelefteq G$. If $p \nmid |T|$, then $G$ is $p$-nilpotent. Thus $p \mid |T|$. Clearly, $P \cap T$ is a Sylow $p$-subgroup of $T$ and every subgroup of $P \cap T$ of order $p$ is complemented in $T$. Then by induction, $T$ is $p$-nilpotent. Since the normal $p$-complement of $T$ is the normal $p$-complement of $G$, $G$ is also $p$-nilpotent. □

Lemma 2.16. [4, Lemma 3.1] Let $\mathcal{F}$ be a saturated formation of characteristic $\pi$ and $H$ a subnormal subgroup of $G$ containing $O_\pi(\Phi(G))$ such that $H/O_\pi(\Phi(G)) \in \mathcal{F}$. Then $H \in \mathcal{F}$.

Lemma 2.17. [26, Theorem B] Let $\mathcal{F}$ be any formation. If $E \trianglelefteq G$ and $F^*(E) \leq Z_{\mathcal{F}}(G)$, then $E \leq Z_{\mathcal{F}}(G)$.

Lemma 2.18. [18, Lemma 2.13] Let $\mathcal{F} = CF(F)$ be a solubly saturated formation, where $F$ is the canonical composition satellite of $\mathcal{F}$. Let $H/K$ be a chief factor of $G$. Then $H/K$ is $\mathcal{F}$-central in $G$ if and only if $G/C_G(H/K) \in F(p)$ in the case where $H/K$ is a $p$-group, and $G/C_G(H/K) \in F(0) = \mathcal{F}$ in the case where $H/K$ is non-abelian.

3 Proofs of Theorems

Proof of Theorem B. Suppose that the result is false and let $(G, P)$ be a counterexample for which $|G| + |P|$ is minimal. We proceed via the following steps.
(1) \(|D| \geq p^2\).

If \(|D| \leq p\), we may assume that \(|D| = 1\) (in the conditions of the theorem, the case \(|D| = p\) can be viewed as a special case of \(|D| = 1\)). Then:

(i) \(G\) has a unique normal subgroup \(N\) such that \(P/N\) is a chief factor of \(G\), \(N \leq Z_u(G)\) and \(|P/N| > p\).

Let \(P/N\) be a chief factor of \(G\). Then \((G, N)\) satisfies the hypothesis of this theorem. By the choice of \((G, P)\), we have that \(N \leq Z_u(G)\). If \(P/N \leq Z_u(G/N)\), then \(P \leq Z_u(G)\), which is impossible. Hence \(P/N \not\leq Z_u(G/N)\), and so \(|P/N| > p\). Now let \(P/R\) be a chief factor of \(G\), which is different from \(P/N\). Then we can obtain that \(R \leq Z_u(G)\) similarly as above. This implies that \(P/N \leq Z_u(G/N)\) by \(G\)-isomorphism \(P/N = NR/N \cong R/N \cap R\), a contradiction.

(ii) Let \(C\) be a Thompson critical subgroup of \(P\). Then \(P = \Omega(C)\).

If not, then \(\Omega(C) \leq N \leq Z_u(G)\) by (i). Thus by Lemma 2.8, \(P \leq Z_u(G)\), which is absurd.

(iii) The exponent of \(P\) is 4 (when \(P\) is not quaternion-free).

If \(P\) is a non-abelian quaternion-free 2-group, then \(P\) has a characteristic subgroup \(T\) of index 2 by Lemma 2.9. It follows from (i) that \(T \leq N\), and so \(|P/N| = 2\), which is impossible.

Hence by (ii) and Lemma 2.10, the exponent of \(P\) is \(p\) or 4 (when \(P\) is not quaternion-free).

(iv) Final contradiction of (1).

Let \(G_p\) be a Sylow \(p\)-subgroup of \(G\). Since \(P/N \cap Z(G_p/N) > 1\), we may take a subgroup \(V/N\) of \(P/N \cap Z(G_p/N)\) of order \(p\). Let \(l \in V \setminus N\) and \(H = \langle l \rangle\). Then \(V = HN\) and \(H\) is a group of order \(p\) or 4 (when \(P\) is not quaternion-free) by (iii). By hypothesis, \(H\) either is \(\Pi\)-supplemented in \(G\) or has a \(\Pi\)-supersoluble supplement in \(G\). Let \(X\) be any supplement of \(H\) in \(G\). If \(P \not\leq X\), then \(P \cap X < P\). Since \((P \cap X)^G = (P \cap X)^P < P\), we have that \(P \cap X \leq N\) by (i). This implies that \(P/N\) is cyclic for \(P/P \cap X \cong H/H \cap X\) is cyclic, and so \(|P/N| = p\), which contradicts (i). Therefore, \(P \leq X\), and thereby \(X = G\). Consequently, \(G\) is the unique supplement of \(H\) in \(G\). If \(H\) has a \(\Pi\)-supersoluble supplement in \(G\), then \(G\) is \(p\)-supersoluble. It follows that \(P \leq Z_u(G)\), which is impossible. Hence \(H\) is \(\Pi\)-supplemented in \(G\), and so \(H\) satisfies \(\Pi\)-property in \(G\). Then \(|G : N_G(V)| = |G : N_G(HN)|\) is a \(p\)-number. This induces that \(V \leq G\). Then by (i), \(P = V\), and so \(|P/N| = p\), a contradiction. This completes the proof of (1).

(2) \(\Phi(P) = 1\), and so \(P\) is an elementary abelian \(p\)-group.

Suppose that \(\Phi(P) > 1\). If \(|\Phi(P)| > |D|\), then \((G, \Phi(P))\) satisfies the hypothesis of this theorem. By the choice of \((G, P)\), we have that \(\Phi(P) \leq Z_u(G)\). Let \(L\) be a minimal normal subgroup of \(G\) contained in \(\Phi(P)\). Then \(|L| = p\). Since \(|D| > |L| = p\) by (1), \((G/L, P/L)\) satisfies the hypothesis of this theorem by Lemma 2.1(2). By the choice of \((G, P)\), \(P/L \leq Z_u(G/L)\). It follows that \(P \leq Z_u(G)\), which is absurd.

Hence \(|\Phi(P)| \leq |D|\). Now we shall show that \(P/\Phi(P) \leq Z_u(G/\Phi(P))\). If \(|\Phi(P)| < |D|\),
then by Lemma 2.1(2), \((G/\Phi(P), P/\Phi(P))\) satisfies the hypothesis of this theorem. The choice of \((G, P)\) implies that \(P/\Phi(P) \leq Z_\ell(G/\Phi(P))\). Hence we may consider that \(|\Phi(P)| = |D|\). If \(p|D| = |P|\), then clearly, \(P/\Phi(P) \leq Z_\ell(G/\Phi(P))\). If \(p|D| < |P|\), then by Lemma 2.1(2), every subgroup of \(P/\Phi(P)\) of order \(p\) either is \(\Pi\)-supplemented in \(G/\Phi(P)\) or has a \(p\)-supersoluble supplement in \(G/\Phi(P)\). This shows that \((G/\Phi(P), P/\Phi(P))\) satisfies the hypothesis of this theorem, and so \(P/\Phi(P) \leq Z_\ell(G/\Phi(P))\) by the choice of \((G, P)\). Then by Lemma 2.8, \(P \leq Z_\ell(G)\), which is impossible. Therefore, \(\Phi(P) = 1\).

(3) \(G\) has a unique minimal normal subgroup \(N\) contained in \(P\), \(P/N \leq Z_\ell(G/N)\) and \(p < |N| \leq |D|\).

Let \(G_p\) be a Sylow \(p\)-subgroup of \(G\) and \(N\) a minimal normal subgroup of \(G\) contained in \(P\). If \(N = P\), then \(P\) is a minimal normal subgroup of \(G\). Let \(H\) be a subgroup of \(P\) of order \(|D|\) such that \(H \leq G_p\). By hypothesis, \(H\) either is \(\Pi\)-supplemented in \(G\) or has a \(p\)-supersoluble supplement in \(G\). For any supplement \(X\) of \(H\) in \(G\), we have that \(P \cap X \leq G\). If \(P \cap X = 1\), then \(H = P\), which is impossible. This induces that \(P \cap X = P\), and so \(X = G\). Therefore, \(G\) is the unique supplement of \(H\) in \(G\). Since \(G\) is not \(p\)-supersoluble, \(H\) satisfies \(\Pi\)-property in \(G\). It follows that \(|G : N_G(H)|\) is a \(p\)-number. Hence \(H \leq G\), a contradiction. Consequently, \(N < P\). If \(|N| > |D|\), then \((G, N)\) satisfies the hypothesis of this theorem. By the choice of \((G, P)\), we have that \(N \leq Z_\ell(G)\). This shows that \(|N| = p > |D|\), which contradicts (1). Therefore, \(|N| \leq |D|\).

Now we claim that \(P/N \leq Z_\ell(G/N)\). If \(|N| < |D|\), then by Lemma 2.1(2), \((G/N, P/N)\) satisfies the hypothesis of this theorem. By the choice of \((G, P)\), \(P/N \leq Z_\ell(G/N)\). Hence we may assume that \(|N| = |D|\). If \(|N| = |D|\), then clearly, \(P/N \leq Z_\ell(G/N)\). If \(|N| > |D|\), then by Lemma 2.1(2), every subgroup of \(P/N\) of order \(p\) either is \(\Pi\)-supplemented in \(G/N\) or has a \(p\)-supersoluble supplement in \(G/N\). Since \(P\) is abelian, \((G/N, P/N)\) satisfies the hypothesis of this theorem. Then by the choice of \((G, P)\), we also have that \(P/N \leq Z_\ell(G/N)\). Consequently, our claim holds. If \(|N| = N\), then \(N \leq Z_\ell(G)\), and so \(P \leq Z_\ell(G)\), which is absurd. Thus \(|N| > p\). If \(G\) has a minimal normal subgroup \(R\) contained in \(P\), which is different from \(N\), then we get that \(G/R \leq Z_\ell(G/R)\) similarly as above. It follows that \(NR/R \leq Z_\ell(G/R)\), and so \(N \leq Z_\ell(G)\) for \(G\)-isomorphism \(N \cong NR/R\). This implies that \(P \leq Z_\ell(G)\), a contradiction. Hence (3) holds.

(4) \(|p|D| = |P|\).

If \(|p|D| < |P|\), then since \(P/N \leq Z_\ell(G/N)\), \(G\) has a normal subgroup \(K\) properly contained in \(P\) such that \(N \leq K\) and \(|K| = p|D|\). Then \((G, K)\) satisfies the hypothesis of this theorem. By the choice of \((G, P)\), we have that \(K \leq Z_\ell(G)\), and thus \(|N| = p\), which contradicts (3). This shows that (4) holds.

(5) Final contradiction.
Since \( \Phi(P) = 1 \), \( N \) has a complement \( S \) in \( P \). Let \( L \) be a maximal subgroup of \( N \) such that \( L \leq G_p \). Then \( L \neq 1 \) and \( H = LS \) is a maximal subgroup of \( P \). By hypothesis and (4), \( H \) either is \( \Pi \)-supplemented in \( G \) or has a \( p \)-supersoluble supplement in \( G \). For any supplement \( X \) of \( H \) in \( G \), since \( P \) is abelian, we have that \( P \cap X \leq G \). If \( P \cap X = 1 \), then \( H = P \), which is impossible. Hence \( P \cap X > 1 \), and so \( N \leq X \) by (3). Suppose that \( H \) is \( \Pi \)-supplemented in \( G \). Then \( H \) has a supplement \( T \) in \( G \) such that \( H \cap T \leq I \leq H \), where \( I \) satisfies \( \Pi \)-property in \( G \). Since \( H \cap T = I \cap T \) and \( N \leq T \), \( L = H \cap N = I \cap N \). It follows that \( |G : N_G(L)| = |G : N_G(I \cap N)| \) is a \( p \)-number. As \( L \leq G_p \), we have that \( L \leq G \).

Then by (3), \( L = 1 \), and so \( |N| = p \), a contradiction. We may therefore, assume that \( H \) has a \( p \)-supersoluble supplement \( T \) in \( G \). Let \( F \) be the canonical local satellite of \( U_p \) such that \( F(p) = G_p F(p) = U_p \cap G_p A(p − 1) \), where \( A(p − 1) \) denotes the class of finite abelian groups of exponent \( p − 1 \) and \( F(q) = U_p \) for all primes \( q \neq p \). By Lemma 2.4, \( T/C_T(N) \in F(p) \).

Since \( P \leq C_G(N) \), we have that \( G/C_G(N) \cong T/C_T(N) \in F(p) \). Then by Lemma 2.4 again, \( N \leq Z_{U_p}(G) \), and so \( |N| = p \). The final contradiction ends the proof. \( \square \)

**Proof of Theorem C.** Suppose that the result is false and let \( (G, E) \) be a counterexample for which \( |G| + |E| \) is minimal. We proceed via the following steps.

1. \( O_{p'}(G) = 1 \).

   If not, by Lemma 2.1(2), \( (G/O_{p'}(G), EO_{p'}(G)/O_{p'}(G)) \) satisfies the hypothesis of this theorem. By the choice of \( (G, E) \), we have that \( EO_{p'}(G)/O_{p'}(G) \) is \( p \)-nilpotent, and so \( E \) is \( p \)-nilpotent, a contradiction.

2. \( O_p(E) > 1 \).

   Suppose that \( O_p(E) = 1 \) and let \( N \) be a minimal normal subgroup of \( G \) contained in \( E \).

   Since \( O_{p'}(G) = 1 \), \( p \mid |N| \). Then we discuss three possible cases below:

   - **Case 1:** \( |N|_p < |D| \).

     In this case, by Lemma 2.2, \( (G/N, E/N) \) satisfies the hypothesis of this theorem. By the choice of \( (G, E) \), \( E/N \) is \( p \)-nilpotent. Let \( A/N \) be the normal \( p \)-complement of \( E/N \). Then obviously, \( A \leq G \) and \( |A|_p = |N|_p < |D| \). By Lemma 2.2, \( (G/A, E/A) \) satisfies the hypothesis of Theorem B. Therefore, \( E/A \leq Z_{U_p}(G/A) \). If \( p|D| < |P| \), then we may take a normal subgroup \( L \) of \( G \) such that \( A \leq L < E \) and \( |L|_p = p|D| \). Clearly, \( G/L \) satisfies the hypothesis of this theorem. Then by the choice of \( (G, E) \), \( L \) is \( p \)-nilpotent, and so \( N \) is \( p \)-nilpotent. Since \( O_{p'}(G) = 1 \), \( N \) is a \( p \)-group. Hence \( N \leq O_p(E) \), which is absurd.

   Thus we have that \( p|D| = |P| \). Then by hypothesis, every maximal subgroup of \( P \) either is \( \Pi \)-supplemented in \( G \) or has a \( p \)-supersoluble supplement in \( G \). If every maximal subgroup of \( P \) has a \( p \)-supersoluble supplement in \( E \), then since \( (|E|, p − 1) = 1 \), every maximal subgroup of \( P \) has a \( p \)-nilpotent supplement in \( E \). By Lemmas 2.11 and 2.12, \( E \) is \( p \)-nilpotent, a contradiction. Hence \( P \) has a maximal subgroup \( P_1 \) such that \( P_1 \) is \( \Pi \)-supplemented in \( G \).
and $P_1$ does not have a $p$-supersoluble supplement in $E$. Then $P_1$ has a supplement $T$ in $G$ such that $T \cap E$ is not $p$-supersoluble and $P_1 \cap T \leq I \leq P_1$, where $I$ satisfies $\Pi$-property in $G$. This implies that $|G : N_G(I \cap N)|$ is a $p$-number, and so $I \cap N \leq O_p(E) = 1$. It follows that $P_1 \cap T \cap N = I \cap N = 1$. As $|T \cap E : P \cap T| = |E : P|$, $P \cap T$ is a Sylow $p$-subgroup of $T \cap E$. This induces that $P \cap T \cap N$ is a Sylow $p$-subgroup of $T \cap N$. Note that $|P \cap T \cap N| = |P \cap T \cap N : P_1 \cap T \cap N| = |P_1(P \cap T \cap N) : P_1| \leq p$. Hence $|T \cap N|_p \leq p$. Since $T \cap E/T \cap N \cong (TN \cap E)/N \leq E/N$ is $p$-nilpotent, by Lemma 2.13(2), $T \cap E$ is $p$-nilpotent, a contradiction.

(ii) Case 2 : $|N|_p > |D|$.

In this case, if $N < E$, then $(G, N)$ satisfies the hypothesis of this theorem. By the choice of $(G, E)$, $N$ is $p$-nilpotent. Since $O_p(G) = 1$, $N$ is a $p$-group, which is absurd. Hence $N = E$. By hypothesis, for every proper subgroup $H$ of $P$ with $|H| = p^n|D|$ ($n = 0, 1$) or $4$ (when $|D| = 1$ and $P$ is not quaternion-free), $H$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. If $H$ is $\Pi$-supplemented in $G$, then $H$ has a supplement $T$ in $G$ such that $H \cap T \leq I \leq H$, where $I$ satisfies $\Pi$-property in $G$. It follows that $|G : N_G(I)|$ is a $p$-number, and so $I \leq O_p(E) = 1$. Hence $H$ either is complemented in $G$ or has a $p$-supersoluble supplement in $G$. If $E < G$, then clearly, $H$ either is complemented in $E$ or has a $p$-supersoluble supplement in $E$. This shows that $(G, E)$ satisfies the hypothesis of this theorem. By the choice of $(G, E)$, $E$ is $p$-nilpotent, a contradiction. Thus $G = E$ is a non-abelian simple group. By Feit-Thompson Theorem, $p = 2$.

If every maximal subgroup of $P$ has a 2-supersoluble supplement in $G$, then by Lemmas 2.11 and 2.12, $G$ is 2-nilpotent, which is impossible. This shows that $P$ has a maximal subgroup which does not have a 2-supersoluble supplement in $G$. Suppose that $2|D| < |P|$. Then $P$ has subgroups $H_1$ and $H_2$ with $|H_1| = |D|$ and $|H_2| = 2|D|$ such that $H_1$ and $H_2$ are complemented in $G$. Let $T_1$ and $T_2$ be complements of $H_1$ and $H_2$ in $G$, respectively. Then $|G : T_1| = 2^r$ and $|G : T_2| = 2^{r+1}$ such that $T_2$ is not a $2^r$-Hall subgroup of $G$. If $T_1 = G$, then $|G : T_2| = 2$, and so $T_2 \leq G$, which is absurd. Hence $r \geq 1$. If $r \leq 2$, then $G \cong G/(T_1)_G \leq S_4$, where $S_4$ denotes the symmetric group of degree 4, and so $G$ is soluble, a contradiction. Thus $r \geq 3$. By Lemma 2.14, $G$ is not a non-abelian simple group, which is impossible. Now assume that $2|D| = |P|$. Then $P$ has a maximal subgroup $H$ such that $H$ is complemented in $G$ such that every complement $T$ of $H$ in $G$ is not 2-supersoluble. However, since $|T|_2 = 2$, $T$ is 2-supersoluble, which contradicts our assumption.

(iii) Case 3 : $|N|_p = |D|$.

In this case, if $p|D| = |P|$, then $|E/N|_p = p$. Hence by Lemma 2.13(2), $E/N$ is $p$-nilpotent. With a similar argument as in the proof of Case 1 of (2), we can get a contradiction. Now assume that $p|D| < |P|$. Let $E/A$ be a chief factor of $G$ such that $N \leq A$. If $|A|_p > |N|_p = |D|$,
then \((G,A)\) satisfies the hypothesis of this theorem. By the choice of \((G,E)\), \(A\) is \(p\)-nilpotent. Since \(O_\nu\cdot(G) = 1\), \(A\) is a \(p\)-group, a contradiction. Hence \(|A|_p = |N|_p = |D|\). By hypothesis, every subgroup of \(P\) of order \(|H| = p|D|\) either is \(\Pi\)-supplemented in \(G\) or has a \(p\)-supersoluble supplement in \(G\). Then by Lemma 2.2, every subgroup of \(PA/A\) of order \(p\) either is \(\Pi\)-supplemented in \(G/A\) or has a \(p\)-supersoluble supplement in \(G/A\).

Suppose that there exists a subgroup \(H\cdot(A)\) of \(PA/A\) of order \(p\) such that \(H/A\) is \(\Pi\)-supplemented, but not complemented in \(G/A\). Then clearly, \(H/A\) satisfies \(\Pi\)-property in \(G/A\). This implies that \(|G/A : N_{G/A}(H/A)|\) is a \(p\)-number, and so \(H/A \leq O_p(E/A)\). Hence \(E/A = O_p(E/A)\) for \(E/A\) is a chief factor of \(G\). Consequently, \(E/A\) is an elementary abelian \(p\)-group. Then \((G/A,E/A)\) satisfies the hypothesis of Theorem B. Thus \(E/A \leq Z_d(G/A)\). This induces that \(|E/A| = p\), and so \(p|D| = p|A|_p = |P|\), which is contrary to our assumption. Therefore, every subgroup of \(PA/A\) of order \(p\) either is complemented in \(G/A\) or has a \(p\)-supersoluble supplement in \(G/A\). Now we will show that \(E/A\) is \(p\)-nilpotent. If \(PA/A\) has a subgroup of order \(p\) which has a \(p\)-supersoluble supplement in \(G/A\), but is not complemented in \(G/A\), then clearly, \(G/A\) is \(p\)-supersoluble, and so is \(E/A\). Since \((|E/A|, p - 1) = 1\), \(E/A\) is \(p\)-nilpotent. Now consider that every subgroup of \(PA/A\) of order \(p\) is complemented in \(G/A\). Then by Lemma 2.15, \(E/A\) is also \(p\)-nilpotent. Since \(p \mid |E/A|\), \(E/A\) is an elementary abelian \(p\)-group. As discussed above, we can obtain that \(|E/A| = p\), and thus \(p|D| = p|A|_p = |P|\). The final contradiction shows that (2) holds.

(3) Final contradiction.

Since \(O_p(E) > 1\), let \(N\) be a minimal normal subgroup of \(G\) contained in \(O_p(E)\). Then we discuss three possible cases as follows:

(i) Case 1 : \(|N| < |D|\).

In this case, by Lemma 2.1(2), \((G/N,E/N)\) satisfies the hypothesis of this theorem. By the choice of \((G,E)\), \(E/N\) is \(p\)-nilpotent. Let \(A/N\) be the normal \(p\)-complement of \(E/N\). Since \(|A|_p = |N| < |D|\), by Lemma 2.2, \((G/A,E/A)\) satisfies the hypothesis of Theorem B, and so \(E/A \leq Z_d(G/A)\). If \(p|D| < |P|\), then we may take a normal subgroup \(L\) of \(G\) such that \(A \leq L < E\) and \(|L|_p = p|D|\). It is easy to see that \((G,L)\) satisfies the hypothesis of this theorem. Then by the choice of \((G,E)\), \(L\) is \(p\)-nilpotent. Since \(O_{p'}(G) = 1\), \(L\) is a \(p\)-group. It follows that \(E\) is a \(p\)-group, a contradiction.

We may, therefore, assume that \(p|D| = |P|\). By hypothesis, every maximal subgroup of \(P\) either is \(\Pi\)-supplemented in \(G\) or has a \(p\)-supersoluble supplement in \(G\). Since \(E/N\) is \(p\)-nilpotent, by Lemma 2.16, \(N \notin \Phi(G)\). Thus \(G\) has a maximal subgroup \(M\) such that \(N \notin M\). Obviously, \(M \cap N = 1\). As \(E/N\) is \(p\)-nilpotent, \(M \cap E\) is \(p\)-nilpotent. Let \(G_p\) be a Sylow \(p\)-subgroup of \(G\) and \(G_{p_1}\) a maximal subgroup of \(G_p\) containing \(G_p \cap M\). Then \(G_p = G_{p_1}(N)\). Let \(P_1 = G_{p_1} \cap P\). Since \(|P : P_1| = |G_p : G_{p_1}| = p\), \(P_1\) is a maximal subgroup of \(P\) such
that $P = P_1 N$. If $P_1$ is $\Pi$-supplemented in $G$, then $P_1$ has a supplement $T$ in $G$ such that $P_1 \cap T \leq I \leq P_1$, where $I$ satisfies $\Pi$-property in $G$. It follows that $|G : N_G(I \cap N)|$ is a $p$-number. If $I \cap N > 1$, then $N = (I \cap N)^G = (I \cap N)^G_{p_1} \leq G_{p_1}$, a contradiction. Hence $I \cap N = 1$, and thus $P_1 \cap T \cap N = 1$. Since $T \cap E/T \cap N \cong (TN \cap E)/N$ is $p$-nilpotent and $|T \cap N| = |T \cap N : P_1 \cap T \cap N| = |P_1(T \cap N) : P_1| \leq p$, $T \cap E$ is $p$-nilpotent by Lemma 2.13.(2). Consequently, no matter $P_1$ is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$, $P_1$ has a $p$-nilpotent supplement $T_1$ in $E$ for $(|E|, p - 1) = 1$. Let $(M \cap E)_{p'}$ and $(T_1)_{p'}$ be the normal $p$-complements of $M \cap E$ and $T_1$, respectively. Then $(M \cap E)_{p'}$ and $(T_1)_{p'}$ are $p'$-Hall subgroups of $E$. By Lemma 2.11, $E$ is a $C_{p'}$-group. This implies that $E$ has an element $g$ such that $(T_1)_{p'}^g = (M \cap E)_{p'}$. Considering the fact that $T_1 \leq N_E((T_1)_{p'})$, we may let $g \in P_1$. It follows that $E = P_1 N_E((T_1)_{p'})^g = P_1 N_E((M \cap E)_{p'})$. Since $O_{p'}(G) = 1$ and $M \leq N_G((M \cap E)_{p'})$, we have that $N_G((M \cap E)_{p'}) = M$. This implies that $E = P_1(M \cap E)$. As $P \cap M \leq G_{p_1} \cap P = P_1$, $P = P_1(P \cap M) = P_1$, which is impossible.

(ii) Case 2: $|N| > |D|$.

In this case, $(G, N)$ satisfies the hypothesis of Theorem B. Hence $N \leq Z_{d}(G)$, and so $|N| = p$. It follows that $|D| = 1$. As $(G, O_p(E))$ satisfies the hypothesis of Theorem B, $O_p(E) \leq Z_d(G) \leq Z_d(E)$. Since $(|E|, p - 1) = 1$, it is easy to see that $O_p(E) \leq Z_{\infty}(E)$. Let $A/O_p(E)$ be a chief factor of $G$ below $E$. If $A < E$, then $(G, A)$ satisfies the hypothesis of this theorem. By the choice of $(G, E)$, $A$ is $p$-nilpotent. Then since $O_{p'}(G) = 1$, $A$ is a $p$-group. This shows that $A \leq O_p(E)$, which is absurd. Hence $E/O_p(E)$ is a chief factor of $G$. If $p \nmid |E/O_p(E)|$, then $E$ is $p$-nilpotent for $O_p(E) \leq Z_{\infty}(E)$, a contradiction. Thus $p \mid |E/O_p(E)|$. Obviously, $E$ is not soluble. Thus by Feit-Thompson Theorem, $p = 2$.

Now let $V$ be a minimal non-2-nilpotent group contained in $E$. By [20, Chapter IV, Satz 5.4], $V$ is a minimal non-nilpotent group such that $V = V_2 \times V_q$, where $V_2$ is the Sylow 2-subgroup of $V$ and $V_q$ is a Sylow $q$-subgroup of $V$ with $q > 2$. Without loss of generality, we may let $V_2 \leq P$. Then by [9, Chapter VII, Theorem 6.18], $V_2/\Phi(V_2)$ is a $V$-chief factor; $\Phi(V) = Z_{\infty}(V)$; $\Phi(V_2) = V_2 \cap \Phi(V)$; and $V_2$ has exponent 2 or 4 (when $V_2$ is non-abelian). It follows that $O_2(E) \cap V_2 \leq Z_{\infty}(E) \cap V_2 \leq Z_{\infty}(V) \cap V_2 = \Phi(V) \cap V_2 = \Phi(V_2)$. Therefore, $V_2$ has an element $x$ which is not contained in $O_2(E)$. Let $H = \langle x \rangle$. Then $|H| = 2$ or 4 (when $V_2$ is non-abelian). If $V_2$ is non-abelian and quaternion-free, then $V_2$ has a characteristic subgroup of index 2 by Lemma 2.9. This implies that $|V_2/\Phi(V_2)| = 2$, and so $V_2$ is cyclic, which contradicts our assumption. Therefore, $|H| = 2$ or 4 (when $V_2$ is non-nilpotent). If $V_2$ is non-abelian and quaternion-free, then $V_2$ has a characteristic subgroup of index 2 by Lemma 2.9. This implies that $|V_2/\Phi(V_2)| = 2$, and so $V_2$ is cyclic, which contradicts our assumption. Therefore, $|H| = 2$ or 4 (when $V_2$ is non-nilpotent). By hypothesis, $H$ either is $\Pi$-supplemented in $G$ or has a $p$-supersoluble supplement in $G$. Let $X$ be any supplement of $H$ in $G$. Suppose that $X < G$. Then $G/X_2 \leq S_4$ for $|G : X| \leq 4$, where $S_4$ denotes the symmetric group of degree 4. Thus $E/X_2 \cap E$ is soluble. Since $X_2 \cap E < E$ and $(G, X_2 \cap E)$ satisfies the hypothesis of this theorem, $X_2 \cap E = 2$-nilpotent by the choice
Therefore, we obtain that $X$ is soluble, which is impossible. Therefore, $G$ is the unique supplement of $H$ in $G$. Since $G$ is not 2-supersoluble, $H$ is $\Pi$-supplemented in $G$, and so $H$ satisfies $\Pi$-property in $G$. Then $|G : N_G(HO_2(E))|$ is a 2-number. This implies that $H \leq O_2(E)$, a final contradiction of (ii).

(iii) Case 3: $|N| = |D|$.

In this case, if $p|D| = |P|$, then $|E/N|_p = p$, and thus $E/N$ is $p$-nilpotent by Lemma 2.13(2). With a similar discussion as in the proof of Case 1 of (3), we can get a contradiction. Hence $p|D| < |P|$. Let $E/A$ be a chief factor of $G$ such that $N \leq A$. If $|A|_p = |N| = |D|$, then a contradiction can be derived in a similar way as in Case 3 of (2). Now we may assume that $|A|_p > |N| = |D|$. Then $(G, A)$ satisfies the hypothesis of this theorem. By the choice of $(G, E)$, $A$ is $p$-nilpotent. Since $O_{p'}(G) = 1$, $A$ is a $p$-group. It follows that $(G, A)$ satisfies the hypothesis of Theorem B. Hence $A \leq Z_{p}(G)$, and so $|N| = |D| = p$. This case can be viewed as a special case of Case 2 of (3) (we may take $|N| = p$ and $|D| = 1$), and this fact yields a contradiction. The theorem is thus proved. \hfill $\square$

**Proof of Theorem A.** Let $p$ be the smallest prime divisor of $|X|$ and $P$ a Sylow $p$-subgroup of $X$. If $P$ is cyclic, then by Lemma 2.13(1), $X$ is $p$-nilpotent. Now assume that $P$ is not cyclic. Then by Theorem C, $X$ is also $p$-nilpotent. Let $X_{p'}$ be the normal $p$-complement of $X$. Then $X_{p'} \unlhd G$. If $P$ is cyclic, then $X/X_{p'} \leq Z_{p'}(G/X_{p'})$. Now consider that $P$ is not cyclic. Then by Lemma 2.1(2), $(G/X_{p'}, X/X_{p'})$ satisfies the hypothesis of Theorem B. Hence we also have that $X/X_{p'} \leq Z_{p'}(G/X_{p'})$.

Let $q$ be the second smallest prime divisor of $|X|$ and $Q$ a Sylow $q$-subgroup of $X$. With a similar argument as above, we can get that $X_{p'}$ is $q$-nilpotent and $X_{p'}/X_{(p,q)'} \leq Z_{p'}(G/X_{(p,q)'}))$, where $X_{(p,q)'}$ is the normal $q$-complement of $X_{p'}$. The rest may be deduced by analogy. Therefore, we obtain that $X \leq Z_{p'}(G) \leq Z_{\mathcal{F}}(G)$. It follows from Lemma 2.17 that $E \leq Z_{\mathcal{F}}(G)$. Then by Lemma 2.18, $G \in \mathcal{F}$ as desired. \hfill $\square$

### 4 Further Applications

In this section, we will show that the subgroups of $G$ which satisfy a certain known embedding property mentioned below are all $\Pi$-supplemented in $G$. For the sake of simplicity, we only focus on most recent embedding properties.

Recall that a subgroup $H$ of $G$ is called to be a **CAP-subgroup** if $H$ either covers or avoids every chief factor of $G$. Let $\mathcal{F}$ be a saturated formation. A subgroup $H$ of $G$ is said to be $\mathcal{F}$-hypercentrally embedded [1] in $G$ if $H^G/H_G \leq Z_{\mathcal{F}}(G/H_G)$. A subgroup $H$ of $G$ is called to be **$S$-quasinormal** (or $S$-permutable) in $G$ if $H$ permutes with every Sylow subgroup of $G$. A subgroup $H$ of $G$ is said to be **$S$-semipermutable** [7] in $G$ if $H$ permutes with every Sylow
A subgroup $H$ of $G$ is called to be $S$-quasinormally embedded \[3\] in $G$ if every Sylow subgroup of $H$ is a Sylow subgroup of some $S$-quasinormal subgroup of $G$. A subgroup $H$ of $G$ is said to be $S$-conditionally permutable \[19\] in $G$ if $H$ permutes with at least one Sylow $p$-subgroup of $G$ for every $p \in \pi(G)$.

**Proposition 4.1.** Let $H$ be a subgroup of $G$. Then $H$ satisfies $\Pi$-property in $G$ if one of the following holds:

1. $H$ is a CAP-subgroup of $G$.
2. $H$ is $U$-hypercentrally embedded in $G$.
3. $H$ is $S$-quasinormal in $G$.
4. $H$ is a $p$-group and $H$ is $S$-semipermutable in $G$.
5. $H^G$ is soluble and $H$ is $S$-quasinormally embedded in $G$.
6. $H^G$ is soluble and $H$ is $S$-conditionally permutable in $G$.

**Proof.** Statements (1)-(3) and (5)-(6) were proved in \[23\], and the proof of \[23, Proposition 2.4\] still works for statement (4).

Recall that a subgroup $H$ of $G$ is called to be a CAS-subgroup \[29\] if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is a CAP-subgroup of $G$. Let $F$ be a saturated formation. A subgroup $H$ of $G$ is said to be $F$-supplemented \[15\] in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_G / H_G \leq Z_F(G/H_G)$. A subgroup $H$ of $G$ is called to be weakly $s$-supplemented \[25\] in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ denotes the subgroup generated by all those subgroups of $H$ which are $S$-quasinormal in $G$. A subgroup $H$ of $G$ is said to be weakly $\bar{s}$-supplemented \[30\] in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ denotes the subgroup generated by all those subgroups of $H$ which are $S$-semipermutable in $G$. A subgroup $H$ of $G$ is called to be weakly $s$-supplementally embedded \[31\] in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{se}$, where $H_{se}$ denotes an $S$-quasinormally embedded subgroup of $G$ contained in $H$. A subgroup $H$ of $G$ is said to be completely $c$-permutable \[17\] in $G$ if for every subgroup $T$ of $G$, there exists some $x \in \langle H, T \rangle$ such that $HT^x = T^xH$. A subgroup $H$ of $G$ is called to be weakly $c$-permutable \[16\] in $G$ if there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is completely $c$-permutable in $G$.

**Proposition 4.2.** Let $H$ be a subgroup of $G$. Then $H$ is $\Pi$-supplemented in $G$ if one of the following holds:

1. $H$ is a CAS-subgroup of $G$.
2. $H$ is $U$-supplemented in $G$.
3. $H$ is weakly $s$-supplemented in $G$.
4. $H$ is a $p$-group and $H$ is weakly $\bar{s}$-supplemented in $G$.
5. $H^G$ is soluble and $H$ is weakly $s$-supplementally embedded in $G$. 


(6) $H^G$ is soluble and $H$ is weakly $c$-permutable in $G$.

Proof. Note that by [22, Satz 2], $H_{sG}$ is $S$-quasinormal in $G$, and if $H$ is a $p$-group, then by definition, $H_{sG}$ is $S$-semipermutable in $G$. Also, every completely $c$-permutable subgroup of $G$ is clearly $S$-conditionally permutable in $G$. Then Proposition 4.2 directly follows from Proposition 4.1.

By the above proposition, a large number of previous results are immediate consequences of our theorems. We omit further details, and readers may refer to the relevant literature for more information.

References

[1] M. Asaad. Finite groups with certain subgroups of Sylow subgroups complemented. *J. Algebra*, 323:1958–1965, 2010.
[2] M. Asaad. On $c$-supplemented subgroups of finite groups. *J. Algebra*, 362:1–11, 2012.
[3] A. Ballester-Bolinches and M. C. Pedraza-Aguilera. Sufficient conditions for supersolubility of finite groups. *J. Pure Appl. Algebra*, 127:113–118, 1998.
[4] A. Ballester-Bolinches and M. D. Pérez-Ramos. On $\mathfrak{S}$-subnormal subgroups and Frattini-like subgroups of a finite group. *Glasgow Math. J.*, 36:241–247, 1994.
[5] A. Ballester-Bolinches, Y. Wang, and X. Guo. $c$-supplemented subgroups of finite groups. *Glasgow Math. J.*, 42:383–389, 2000.
[6] X. Chen, W. Guo, and A. N. Skiba. Some conditions under which a finite group belongs to a Baer-local formation. *Comm. Algebra*, 2013. (accepted).
[7] Z. Chen. On a theorem of Srinivasan (in Chinese). *J. Southwest Normal Univ. Nat. Sci.*, 12(1):1–4, 1987.
[8] F. Cross. Conjugacy of odd order Hall subgroups. *Bull. London Math. Soc.*, 19:311–319, 1987.
[9] K. Doerk and T. Hawkes. *Finite Soluble Groups*. Walter de Gruyter, Berlin/New York, 1992.
[10] L. Dornhoff. M-groups and 2-groups. *Math. Z.*, 100:226–256, 1967.
[11] L. M. Ezquerro and X. Soler-Escrivà. Some permutability properties related to $\mathcal{F}$-hypercentrally embedded subgroups of finite groups. *J. Algebra*, 264:279–295, 2003.
[12] T. M. Gagen. *Topics in Finite Groups*. Cambridge, London/New York/Melbourne, 1976.
[13] D. Gorenstein. *Finite Groups*. Harper & Row Publishers, New York/Evanston/London, 1968.
[14] W. Guo. *The Theory of Classes of Groups*. Kluwer, Dordrecht, 2000.
[15] W. Guo. On $F$-supplemented subgroups of finite groups. *Manuscripta Math.*, 127:139–150, 2008.

[16] W. Guo and S. Chen. Weakly $c$-permutable subgroups of finite groups. *J. Algebra*, 324:2369–2381, 2010.

[17] W. Guo, K. P. Shum, and A. N. Skiba. Conditionally permutable subgroups and supersolubility of finite groups. *Southeast Asian Bull. Math.*, 29:493–510, 2005.

[18] W. Guo and A. N. Skiba. On $\mathcal{F}\Phi^*$-hypercentral subgroups of finite groups. *J. Algebra*, 372:275–292, 2012.

[19] J. Huang and W. Guo. S-conditionally permutable subgroups of finite groups (in Chinese). *Chin. Ann. Math. Ser. A*, 28(1):17–26, 2007.

[20] B. Huppert. *Endliche Gruppen I*. Springer-Verlag, 1968.

[21] B. Huppert and N. Blackburn. *Finite groups III*. Springer-Verlag, Berlin/Heidelberg, 1982.

[22] O. H. Kegel. Sylow-gruppen und subnormalteiler endlicher gruppen. *Math. Z.*, 78:205–221, 1962.

[23] B. Li. On II-property and II-normality of subgroups of finite groups. *J. Algebra*, 334:321–337, 2011.

[24] B. Li. Finite groups with II-supplemented minimal subgroups. *Comm. Algebra*, 41:2060–2070, 2013.

[25] A. N. Skiba. On weakly $s$-permutable subgroups of finite groups. *J. Algebra*, 315:192–209, 2007.

[26] A. N. Skiba. On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups. *J. Group Theory*, 13:841–850, 2010.

[27] A. N. Skiba and L. A. Shemetkov. Multiply $\mathcal{L}$-composition formations of finite groups. *Ukr. Math. J.*, 52(6):898–913, 2000.

[28] H. N. Ward. Automorphisms of quaternion-free 2-groups. *Math. Z.*, 112:52–58, 1969.

[29] H. Wei and Y. Wang. On CAS-subgroups of finite groups. *Israel J. Math.*, 159:175–188, 2007.

[30] Y. Xu and X. Li. Weakly $s$-semi-permutable subgroups of finite groups. *Front. Math. China*, 6(1):161–175, 2011.

[31] T. Zhao, X. Li, and Y. Xu. On weakly $s$-supplementally embedded subgroups of finite groups. *Proc. Edinburgh Math. Soc.*, 54:799–807, 2011.