GLOBAL WELL-POSEDNESS AND BLOW-UP FOR THE DAMPED PERTURBATION OF CAMASSA-HOLM TYPE EQUATIONS

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ABSTRACT. In this paper, we prove global well-posedness of strong solutions to a class of perturbed Camassa-Holm type equations in Besov spaces. It is shown that the existence of global solutions depends only on the $L^1$-integrability of the time-dependent parameters, but not on the shape or smoothness conditions on initial data. As a by-product, we obtain a new global-in-time result for the weakly dissipative Camassa-Holm type equations in Besov spaces, which considerably improves the results in [11, 21, 22]. Moreover, we derive two kinds of precise blow-up criteria for a certain perturbed Camassa-Holm type equations in Sobolev space, which in some sense tell us how the time-dependent parameters affect the singularity formation of strong solutions.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following Cauchy problem to the weakly dissipative two-component $b$-family equations

$$
\begin{aligned}
m_t + um_x + bu_xm + \kappa \sigma \sigma_x + \lambda m &= 0, \quad t > 0, \ x \in \mathbb{K}, \\
m &= u - u_{xx}, \quad t > 0, \ x \in \mathbb{K}, \\
\sigma_t + (u \sigma)_x + \lambda \sigma &= 0, \quad t > 0, \ x \in \mathbb{K}, \\
m(0,x) &= m_0(x), \ \sigma(0,x) = \sigma(x), \quad x \in \mathbb{K},
\end{aligned}
$$

(1.1)

with $\mathbb{K} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$, where $\lambda > 0$ is the dissipative parameter, $\kappa = \pm 1$, and the real dimensionless constant $b \in \mathbb{R}$ is a parameter that provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching; it is also the number of covariant dimensions associated with the momentum density $m$. The unknown $u(t,x)$ stands for the horizontal velocity of the fluid, and $\sigma(t,x)$ is related to the free surface elevation from equilibrium with the boundary assumption, $u \to 0$ and $\sigma \to 1$ as $|x| \to \infty$. The system (1.1) without the dissipative terms has been studied by several authors, such as [6,7,15,26,27].

Note that the system (1.1) can be reduced to the damping perturbation of some well-known Camassa-Holm type equations. For example, the system (1.1) reduces to the weakly dissipative Camassa-Holm equation when $b = 2$ and $\sigma(t,x) \equiv 0$ [10,18,19,22]. The local well-posedness in $H^s$ with $s > \frac{3}{2}$ was established in [22], and the global existence and blow-up results were derived in [22]. When $b = 3$ and $\sigma(t,x) \equiv 0$, it reduces to the weakly dissipative Degasperis-Procesi equation [9,11,20,21]. If the second component $\sigma(t,x)$ is allowed to be nonzero, the choices $b = 2$ and $b = 3$...
in (1.1) give rise to the weakly dissipative two-component Camassa-Holm and Degasperis-Procesi equations, respectively [12, 21].

The damped system (1.1) was considered by Lenells and Wunsch [14], in which the authors proved that, up to a simple change of variables, the non-dissipative and dissipative versions of these equations are equivalent. To be more precise, the pair \((u, \sigma)\) is a solution to the system (1.1) if and only if the pair \((v, \rho)\) defined by

\[
\begin{align*}
  u(t, x) &= e^{-\lambda t} v \left( \frac{1 - e^{-\lambda t}}{\lambda}, x \right), \\
  \sigma(t, x) &= e^{-\lambda t} \rho \left( \frac{1 - e^{-\lambda t}}{\lambda}, x \right)
\end{align*}
\]

is a solution to the corresponding non-dissipative system. The similar result also holds for the Novikov equation with cubic nonlinearity [14]. In terms of the exponentially time-dependent scaling (1.2), one can directly obtain the local and global existence result for the damped system (1.1) in Sobolev spaces by corresponding results for non-dissipative equations. Within our knowledge, the existence of global strong solutions are usually proved under the reasonable shape conditions on initial data, such as the sign condition for initial momentum density \(m(t, x)\), and the proof is strongly based on the blow-up results at hand and the invariant property of solutions along the characteristic curves.

Our first aim in present paper is to prove the global Hadamard well-posedness for the damped system (1.1) in Besov spaces, without assuming shape conditions on initial data. Instead of using (1.2), we shall take the following change of variables

\[
\begin{align*}
  \tilde{m}(t, x) &= e^{\lambda t} m(t, x), \\
  \tilde{\sigma}(t, x) &= e^{\lambda t} \sigma(t, x).
\end{align*}
\]

Then the Cauchy problem (1.1) can be reformulated as

\[
\begin{align*}
  \tilde{m}_t + e^{-2\lambda t} \tilde{u} \tilde{m}_x + be^{-2\lambda t} \tilde{u}_x \tilde{m} + \kappa e^{-2\lambda t} \tilde{\sigma} \tilde{\sigma}_x = 0, \\
  \tilde{\sigma}_t + e^{-2\lambda t} (\tilde{u} \tilde{\sigma})_x = 0, \\
  \tilde{m}(0, x) = m_0(x), \\n  \tilde{\sigma}(0, x) = \sigma(x),
\end{align*}
\]

Observe that the system (1.4) can be regarded as a perturbed two-component \(b\)-family equations with variable coefficients \(e^{-2\lambda t}\). Indeed, our approach is motivated by the classical ODE theory, for example one can consider the initial value problem

\[
\frac{df(t)}{dt} + \lambda f(t) = g(t), \quad f(0) = f_0.
\]

If one takes the change of variable \(\tilde{f}(t) = e^{\lambda t} f(t)\), then the last equation becomes \(\frac{d\tilde{f}(t)}{dt} = e^{\lambda t} g(t)\), which can be solved immediately. However, the transformed system (1.4) is a coupling of nonlinear PDEs, which makes the solvability of the equations more difficult. In this paper we shall overcome this difficulty by applying the Littlewood-Paley decomposition theory and transport equations theory in Besov spaces.
To investigate the system (1.4), we would like to go further to consider the following generalized perturbation of the two-component \(b\)-family equations

\[
\begin{align*}
    m_t + \alpha(t)um_x + \beta(t)u_xm + \gamma(t)\sigma\sigma_x &= 0, & t > 0, & x \in \mathbb{R}, \\
    \sigma_t + \xi(t)(u\sigma)_x &= 0, & t > 0, & x \in \mathbb{R}, \\
    m(0,x) &= m_0(x), & \sigma(0,x) &= \sigma_0(x), & x \in \mathbb{R},
\end{align*}
\]

(1.5)

where \(\alpha, \beta, \gamma\) and \(\xi\) are arbitrary time-dependent parameters.

Note that if \(p(x) = \frac{1}{2}e^{-|x|}\), then \((1 - \partial_x^2)^{-1}f = p \ast f\) for all \(f \in L^2(\mathbb{R})\) and \(p \ast m = u\). Here we denote by \(\ast\) the convolution. Using these two identities, the system (1.5) can be rewritten as follows:

\[
\begin{align*}
    u_t + \alpha(t)uu_x &= -\partial_x p \ast \left( \frac{\beta(t)}{2}u^2 + \frac{\gamma(t)}{2}\sigma^2 + \frac{\beta(t)-3\alpha(t)}{2}u_x^2 \right), & t > 0, & x \in \mathbb{R}, \\
    \sigma_t + \xi(t)u\sigma_x &= -\xi(t)\sigma u_x, & t > 0, & x \in \mathbb{R}, \\
    u(0,x) &= u_0(x), & \sigma(0,x) &= \sigma_0(x), & x \in \mathbb{R}.
\end{align*}
\]

(1.6)

To state the main result, let us introduce the following space

\[
E_{p,r}^s(\infty) \overset{\text{def}}{=} \bigcap_{T > 0} E_{p,r}^s(T)
\]

\[
= \bigcap_{T > 0} C([0,T];B_{p,r}^s(\mathbb{R}) \times B_{p,r}^{s-1}(\mathbb{R})) \cap C^1([0,T];B_{p,r}^{s-1}(\mathbb{R}) \times B_{p,r}^{s-2}(\mathbb{R})).
\]

The main result of the global well-posedness for the Cauchy problem (1.6) (or (1.5)) in Besov spaces is enunciated by the following theorem.

**Theorem 1.1.** Let \(s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}\) and \(p, r \in [0, \infty]\). Assume that \((u_0, \rho_0) \in B_{p,r}^{s}(\mathbb{R}) \times B_{p,r}^{s-1}(\mathbb{R})\), and the time-dependent parameters \(\alpha, \beta, \gamma, \xi \in L^1([0, \infty); \mathbb{R})\) such that

\[
\int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|)dt' \leq \frac{\ln 2}{6C^2h(\|u_0\|_{B_{p,r}^s} + \|\rho_0\|_{B_{p,r}^{s-1}})}
\]

where \(h(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+\) is a modulus of continuity, given by

\[
h(x) \overset{\text{def}}{=} e^{2C^2x} \int_0^\infty (|\alpha(t')| + |\xi(t')|)dt' \left( x + 4C^2x^2 \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|)dt' \right).
\]

Then the Cauchy problem (1.6) has a unique global strong solution \((u, \sigma) \in E_{p,r}^s(\infty)\), and the data-to-solution map \((u_0, \sigma_0) \mapsto (u, \sigma)\) is continuous from a neighborhood of \((u_0, \sigma_0)\) in \(B_{p,r}^{s}(\mathbb{R}) \times B_{p,r}^{s-1}(\mathbb{R})\) to

\[
(u, \sigma) \in C([0, \infty);B_{p,r}^{s'}(\mathbb{R}) \times B_{p,r}^{s'-1}(\mathbb{R})) \cap C^1([0, \infty);B_{p,r}^{s'-1}(\mathbb{R}) \times B_{p,r}^{s'-2}(\mathbb{R})),
\]

for \(s' < s\) when \(r = \infty\) and \(s' = s\) whereas \(r < \infty\).

Regarding the above Theorem 1.1, several remarks are listed below.
Remark 1.2. 1) Observing that the function
\[ f(x) \equiv 6C^2xe^{2C^2(||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}})x} \left( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} + 4C^2(||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}})^2 \right) \]
is an increasing and continuous function for all \( x \in [0, \infty) \) with \( f(0) = 0 \). The integrability condition provided in Theorem 1.1 holds true as long as the \( L^1 \)-integral \( ||\alpha||_{L^1} + ||\beta||_{L^1} + ||\gamma||_{L^1} + ||\xi||_{L^1} \) is sufficiently small. On the other hand, if the parameters \( \alpha, \beta, \gamma, \xi \in L^1([0,\infty); \mathbb{R}) \) are given, then the integrability condition still holds true provided that the norm \( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} \) is sufficiently small.

2) The global existence result in Theorem 1.1 is independent of the shape or smooth conditions for the initial data, and it only depends on the \( L^1 \)-integrability of the time-dependent parameters \( \alpha, \beta, \gamma \) and \( \xi \), which is quite different with the well-known results in [2–4, 16, 17]. It is also worth pointing out that Theorem 1.1 seems to be the first time to obtain a global result in Besov spaces, which considerably improves the corresponding results in Sobolev spaces.

3) The classic method for proving the uniform bound and the strong convergence of approximate solutions is inapplicable in present case. We overcome this difficulty by utilizing a new iterative method (cf. Step 1 in next section) and establishing a weighed iterative estimate (cf. Lemma 3.2). The method used in this paper can be applied to other shallow water wave equations with high order nonlinearities, such as the Novikov equation [13, 23], the modified Camassa-Holm equation (or the FORQ/MCH equation) [8].

Let us now come back to the Cauchy problem for the system (1.4). If we take
\[ \alpha(t) = e^{-2\lambda t}, \quad \beta(t) = be^{-2\lambda t}, \quad \gamma(t) = \kappa e^{-2\lambda t}, \quad \xi(t) = e^{-2\lambda t}, \]
then we immediately conclude from Theorem 1.1 the following global existence theorem for the damped two-component \( b \)-family equations in Besov spaces.

**Theorem 1.3.** Let \( s > \max\{1 + \frac{1}{p}, 3\} \) and \( p, r \in [0, \infty] \). Assume that \( (u_0, \rho_0) \in B^p_{\rho,r} \times B^{r-1}_{p,r} \), and dissipative constant \( \lambda > 0 \) is sufficiently large such that
\[
\frac{2C^2}{e} \left( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} \right) \left( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} + 2C^2(2 + |b| + |\kappa|) \right) \left( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} \right)^2 \leq \frac{\lambda \ln 2}{3C^2(2 + |b| + |\kappa|)}.
\]
Then the Cauchy problem (1.4) has a unique global strong solution \( (u, \sigma) \in B^s_{p,r}(\infty) \).

**Remark 1.4.** As an example, we provide a sufficient condition which satisfies the inequality in Theorem 1.3:
\[ \lambda \geq \frac{8C^2(2 + |b| + |\kappa|)}{\ln 2} \left( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} \right). \]
Observing that, for any given \( \lambda > 0 \), Theorem 1.3 with above inequality actually gives the small data global-in-time result for the system (1.4) in Besov spaces, where the initial data satisfies \( ||u_0||_{B^p_{\rho,r}} + ||\sigma_0||_{B^{r-1}_{p,r}} \leq \frac{\lambda \ln 2}{8C^2(2 + |b| + |\kappa|)} \). More importantly, Theorem 1.3 casted off the shape conditions of initial data, cf. [20–22].
Our second goal in the present paper is to investigate the finite time blow-up regime for the system (1.5) (when $\beta \equiv 3\alpha$) in Sobolev spaces, which in some sense tell us how the time-dependent parameters affect the singularity formation. The first blow-up criteria is given in the following theorem.

**Theorem 1.5.** Assume that $s > \frac{3}{2}$, $\beta(t) = 3\alpha(t)$ in (1.6), and the parameters $\alpha, \gamma, \xi \in L^2_{loc}(\mathbb{R})$. Let $T^* > 0$ be the maximum existence time of solution $(u, \sigma)$ with respect to the initial data $(u_0, \sigma_0) \in H^s \times H^{s-1}$. If $T^* < \infty$, then

$$T^* < \infty \Rightarrow \int_0^{T^*} (|\alpha(t)| + |\gamma(t)| + |\xi(t)|) \|\partial_x u(t')\|_{L^\infty} dt' = \infty,$$

and the blow-up time $T^*$ is estimated as follows

$$T^* \geq \sup \left\{ t > 0; \|\alpha\|_{L^1(0,t)} + \|\gamma\|_{L^1(0,t)} + \|\xi\|_{L^1(0,t)} < \frac{1}{C(\|u_0\|_{H^s} + \|\sigma_0\|_{H^{s-1}})} \right\}.$$

In addition, if the parameters $\alpha, \gamma, \xi \in L^1([0, \infty); \mathbb{R})$ satisfies

$$\|\alpha\|_{L^1(0,\infty)} + \|\gamma\|_{L^1(0,\infty)} + \|\xi\|_{L^1(0,\infty)} = \frac{1}{2C(\|u_0\|_{H^s} + \|\sigma_0\|_{H^{s-1}})};$$

then $(u, \sigma)$ is a global strong solution.

Theorem 1.5 indicates that if the time-dependent parameters $\alpha, \gamma$ and $\xi$ are locally integrable functions, then the strong solutions to the system (1.6) can still blow-up in finite time. Moreover, it is seen from Theorem 1.5 that the blow-up of solutions only depends on the parameters $\alpha, \gamma, \xi$ as well as the slope of $u$, but not on the slope of $\sigma$. Similar blow-up criteria has been established for the non-isospectral peakon system [24][25].

The next theorem provides a precise blow-up criteria for the strong solution with suitable conditions on parameters.

**Theorem 1.6.** Let $s > \frac{3}{2}$ and $\beta(t) = 3\alpha(t)$ in (1.6). Assume that the parameters $\alpha, \gamma, \xi$ belong to $L^2_{loc}(\mathbb{R})$ and satisfy the following sign conditions:

$$\xi(t) \geq 0, \quad \alpha(t) + \gamma(t) + \xi(t) \geq 0, \quad \forall t \in [0, \infty).$$

If $(u, \sigma)$ solves the system (1.6) with initial data $(u_0, \sigma_0) \in H^s \times H^{s-1}$, then

$$T^* < \infty \iff \liminf_{t \to T^*, x \in \mathbb{R}} u_x(t, x) = -\infty.$$

Theorem 1.6 demonstrate that the blow-up in the first component $u$ must occur before that in the second component $\sigma$ in finite time, which is similar to the well-known Camassa-Holm equation. However, the present result strongly depends on the integrability and sign conditions for the time-dependent parameters $\alpha, \gamma$ and $\xi$. Note that Theorem 1.6 can be applied to the damped two-component $b-$family system (1.1) (or (1.4)).

The remaining of this paper is organized as follows. In Section 2, we introduce the Littlewood-Paley theory, and some well-known results of the transport theory in Besov spaces. The Section 3 is devoted to the proof of the global Hadamard well-posedness for the Besov spaces. The Section 4 is devoted to the proof of the global Hadamard well-posedness for the Besov spaces.
Besov spaces. In Section 4, we first prove a crucial lemma regarding the $L^2$- and $L^\infty$-bound for solution $(u, \sigma)$, and then give the proofs of the blow-up criteria in Theorem 1.5 and Theorem 1.6.

**Notation.** Throughout the paper, since all spaces of functions are over $\mathbb{R}$, we will drop them in the notation of function spaces if there is no any specific to be clarified.

2. Preliminaries

In this section, we recall some well-known facts of the Littlewood-Paley decomposition theory and the linear transport theory in Besov spaces. We introduce the Besov spaces defined on $\mathbb{R}$, the periodic case can be defined similarly.

**Lemma 2.1 ([P]).** Denote by $\mathcal{C}$ the annulus of centre 0, short radius $3/4$ and long radius $8/3$. Then there exits two positive radial functions $\chi$ and $\phi$ belonging respectively to $C^\infty_c(B(0,4/3))$ and $C^\infty_c(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \phi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}.$$  

For the functions $\chi$ and $\phi$ satisfying Lemma 2.1, we write $h = F^{-1}\phi$ and $\tilde{h} = F^{-1}\chi$, where $\mathcal{F}^{-1}u$ denotes the inverse Fourier transformation of $u$. The nonhomogeneous dyadic blocks $\Delta_j$ are defined by

$$\Delta_q u \overset{\text{def}}{=} 0, \quad q \leq -2; \quad \Delta_{-1} u \overset{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}} u(x-y)\tilde{h}(y)dy, \quad q = -1;$$

$$\Delta_q u \overset{\text{def}}{=} \phi(2^{-j}D)u = \int_{\mathbb{R}^d} u(x-y)h(2^{-j}y)dy, \quad q \geq 0.$$  

The nonhomogeneous Littlewood-Paley decomposition of $u \in \mathcal{S}'$ is given by

$$u = \sum_{q \geq -1} \Delta_q u.$$  

We also define the high-frequency cut-off operator as

$$S_q u \overset{\text{def}}{=} \sum_{p \leq q-1} \Delta_p u, \quad \forall q \in \mathbb{N}.$$  

**Definition 2.2 ([P]).** For any $s \in \mathbb{R}$ and $p, r \in [1, \infty]$, the $d$-dimension nonhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} \overset{\text{def}}{=} \left\{ u \in \mathcal{S}'; \|u\|_{B^s_{p,r}} = \|(2^q\|\Delta_q u\|_{L^p})_{l \geq -1}\|_{r} < \infty \right\}. $$

If $s = \infty$, $B^\infty_{p,r} \overset{\text{def}}{=} \bigcap_{s \in \mathbb{R}} B^s_{p,r}$.  

Now let us state some useful results in the transport equation theory in Besov spaces, which are crucial to the proofs of our main theorems.
Lemma 2.3 ([1]). Assume that \( p, r \in [1, \infty] \) and \( s > -\frac{d}{p} \). Let \( v \) be a vector field such that \( \nabla v \) belongs to \( L^1_T(B^{s-1}_{p,r}) \) if \( s > 1 + \frac{d}{p} \) or to \( L^1_T(B^{s}_{p,r} \cap L^\infty) \) otherwise. Suppose also that \( f_0 \in B^s_{p,r} \), \( F \in L^1_T(B^s_{p,r}) \) and that \( f \in L^\infty_T(B^s_{p,r}) \cap C_T(\mathcal{S}) \) solves the linear transport equation
\[
(T) \quad \begin{cases}
\partial_t f + v \cdot \nabla f = F, \\
|_{t=0} = f_0.
\end{cases}
\]
Then there is a constant \( C \) depending only on \( s, p \) and \( r \) such that the following statements hold:
1) If \( r = 1 \) or \( s \neq 1 + \frac{d}{p} \), then
\[
\|f(t)\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t V(\tau) \|f(\tau)\|_{B_{p,r}^s} d\tau,
\]
or
\[
\|f(t)\|_{B_{p,r}^s} \leq e^{CV(t)} \left( \|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau \right)
\]
hold, where \( V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}^d d\tau \) if \( s < 1 + \frac{d}{p} \) and \( V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}^{p-1} d\tau \) otherwise.
2) If \( s \leq \frac{d}{p} \) and, in addition, \( \nabla f_0 \in L^\infty, \nabla f \in L^1_T(L^\infty) \) and \( \nabla F \in L^1_T(L^\infty) \), then
\[
\|f(t)\|_{B_{p,r}^s} + \|\nabla f(t)\|_{L^\infty} \leq e^{CV(t)} \left( \|f_0\|_{B_{p,r}^s} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)} (\|F(\tau)\|_{B_{p,r}^s} + \|\nabla F(\tau)\|_{L^\infty}) d\tau \right).
\]
3) If \( f = v \), then for all \( s > 0 \), the estimate (2.1) holds with \( V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \).
4) If \( r < \infty \), then \( f \in C_T(B^s_{p,r}) \). If \( r = \infty \), then \( f \in C_T(B^s_{p,1}) \) for all \( s' < s \).

Lemma 2.4 ([1]). Let \( (p, p_1, r) \in [1, \infty]^3 \). Assume that \( s > -d \min\{\frac{1}{p_1}, \frac{1}{p}\} \) with \( p' = (1 - \frac{1}{p})^{-1} \).

Let \( f_0 \in B^s_{p,r} \) and \( F \in L^1_T(B^s_{p,r}) \). Let \( v \in L^p_T(B_{p,1}^{s-M}) \) for some \( p > 1, M > 0 \) and \( \nabla v \in L^1_T(B_{p,1}^{s-M}) \) if \( s < 1 + \frac{d}{p_1} \), and \( \nabla v \in L^1_T(B_{p,1}^{s-M}) \) if \( s > 1 + \frac{d}{p_1} \) or \( s = 1 + \frac{d}{p_1} \) and \( r = 1 \). Then \((T)\) has a unique solution \( f \in L^\infty_T(B^s_{p,r}) \cap (\cap_{s' < s} C_T(B^{s'}_{p,1})) \) and the inequalities in Lemma 2.3 hold true. If, moreover, \( r < \infty \), then we have \( f \in C_T(B^s_{p,r}) \).

The following a priori estimates for the transport equation \( (T) \) is crucial in the study of wave-breaking criteria.

Lemma 2.5. ([7]) Let \( 0 \leq s < 1 \). Suppose that \( f \in L^\infty_T(H^s) \cap C_T(\mathcal{S}) \) is a solution to \((T)\) with velocity \( v, \partial_s v \in L^1_T(L^\infty) \), initial data \( f_0 \in H^s, F \in L^1_T(H^s) \). Then, there exists a constant \( C > 0 \) such that
\[
\|f(t)\|_{H^s} \leq \|f_0\|_{H^s} + C \int_0^t \|F(\tau)\|_{H^s} d\tau + C \int_0^t V(\tau) \|f(\tau)\|_{H^s} d\tau,
\]
where
\[
V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_s v(\tau)\|_{L^\infty}) d\tau.
\]
3. Global well-posedness in Besov spaces

In this section, we mainly focus on the proof of Theorem 1.1 which will be divided into the following several steps. Since the proof of Theorem 1.3 is completely same to Theorem 1.1, we shall omit the details here.

**Step 0 (Approximate solutions):** We construct the approximate solutions via the Friedrichs iterative method. Starting from \((u^{(1)}, \sigma^{(1)}) \overset{\text{def}}{=} (S_1 u_0, S_1 \sigma_0)\), we recursively define a sequence of functions \((u^{(n)}, \sigma^{(n)})_{n \geq 1}\) by solving the following linear transport equations

\[
\begin{aligned}
& \left\{
\begin{array}{ll}
& u^{(n+1)}_t(t) + \alpha(t) u^{(n)}_x(t) = -\partial_x \rho \left( \frac{B(t)}{2} (u^{(n)}(t))^2 + \frac{\gamma(t)}{2} (\sigma^{(n)}(t))^2 + \frac{\beta(t) - 3 \alpha(t)}{2} (u^{(n)}_x(t))^2 \right), \\
& \sigma^{(n+1)}_t(t) + \xi(t) u^{(n)}_x(t) = -\xi(t) \sigma^{(n)}_x(t), \\
& u^{(n+1)}(t)|_{t=0} = S_{n+1} u_0, \quad \sigma^{(n+1)}(t)|_{t=0} = S_{n+1} \sigma_0.
\end{array}
\right.
\end{aligned}
\tag{3.1}
\]

Assuming that \((u^{(n)}, \sigma^{(n)}) \in E^s_{p,r}(T)\) for any positive \(T\), it is seen that the right hand side of the system (3.1) belongs to \(L^{1}_{loc}([0, \infty); B^\infty_{p,r} \times B^s_{p,r})\) since the parameters \(\alpha, \beta, \gamma, \xi \in L^1([0, \infty); \mathbb{R})\). Hence, applying Lemma 2.4 ensures that the Cauchy problem (3.1) has a global solution \((u^{(n+1)}, \sigma^{(n+1)})\) which belongs to \(E^s_{p,r}(T)\) for any positive \(T\).

**Step 1 (Uniform bound):** Applying the Lemma 2.3 to (3.1), we get

\[
\begin{aligned}
& \|u^{(n+1)}(t)\|_{B^s_{p,r}} \leq \|S_{n+1} u_0\|_{B^s_{p,r}} + C \int_0^t \|\alpha(t') \partial_x u^{(n)}(t')\|_{B^\infty_{p,r}} \|u^{(n+1)}(t')\|_{B^s_{p,r}} dt' \\
& \quad + \int_0^t \left\| \partial_x \rho \left( \frac{B(t')}{2} (u^{(n)}(t')^2 + \frac{\gamma(t')}{2} (\sigma^{(n)}(t'))^2 + \frac{\beta(t') - 3 \alpha(t')}{2} (u^{(n)}_x(t'))^2 \right) \right\|_{B^s_{p,r}} dt',
\end{aligned}
\tag{3.2}
\]

and

\[
\begin{aligned}
& \|\sigma^{(n+1)}(t)\|_{B^s_{p,r}} \leq \|S_{n+1} \sigma_0\|_{B^s_{p,r}} + C \int_0^t \|\xi(t') \partial_x u^{(n)}(t')\|_{B^\infty_{p,r}} \|\sigma^{(n+1)}(t')\|_{B^s_{p,r}} dt' \\
& \quad + \int_0^t \|\xi(t') \sigma^{(n)}(t') u^{(n)}_x(t')\|_{B^s_{p,r}} dt'.
\end{aligned}
\tag{3.3}
\]

Using the fact that \((1 - \partial_x^2)^{-1}\) is a \(S^{-2}\)-multiplier and the Besov space \(B^s_{p,r}\) is a Banach algebra for any \(s > \frac{1}{p}\), we deduce from (3.2) and (3.3) that

\[
\begin{aligned}
& \|u^{(n+1)}(t)\|_{B^s_{p,r}} + \|\sigma^{(n+1)}(t)\|_{B^s_{p,r}} \leq C \left( \|u_0\|_{B^s_{p,r}} + \|\sigma_0\|_{B^s_{p,r}} \right) \\
& \quad + C \int_0^t \left( \|\alpha(t')\| + \|\xi(t')\| \right) \|u^{(n)}(t')\|_{B^s_{p,r}} \left( \|u^{(n+1)}(t')\|_{B^s_{p,r}} + \|\sigma^{(n+1)}(t')\|_{B^s_{p,r}} \right) dt' \\
& \quad + C \int_0^t \left( \|\alpha(t')\| + \|\beta(t')\| + \|\gamma(t')\| + \|\xi(t')\| \right) \left( \|u^{(n)}(t')\|_{B^s_{p,r}}^2 + \|\sigma^{(n)}(t')\|_{B^s_{p,r}}^2 \right) dt'.
\end{aligned}
\]
An application of the Gronwall lemma to above inequality leads to
\[
\|u^{(n+1)}(t)\|_{B_{p,r}^s} + \|\sigma^{(n+1)}(t)\|_{B_{p,r}^{s-1}} \\
\leq Ce^{C\int_0^t(|\alpha(t')|+|\xi(t')|)dt'}\left[|u_0|_{B_{p,r}^s} + \|\sigma_0\|_{B_{p,r}^{s-1}}
+ C\int_0^t(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)\left(\|u^{(n)}(t')\|_{B_{p,r}^s} + \|\sigma^{(n)}(t')\|_{B_{p,r}^{s-1}}^2\right)dt'\right].
\]
(3.4)

Define
\[
H_0 \overset{\text{def}}{=} H^{(n)}(0) = \|u_0\|_{B_{p,r}^s} + \|\sigma_0\|_{B_{p,r}^{s-1}},
\]
and
\[
H^{(n)}(t) \overset{\text{def}}{=} \|u^{(n)}(t)\|_{B_{p,r}^s} + \|\sigma^{(n)}(t)\|_{B_{p,r}^{s-1}}, \quad \text{for all } n \geq 1.
\]

It follows from the inequality (3.4) that
\[
H^{(n+1)}(t) \leq Ce^{C\int_0^t(|\alpha(t')|+|\xi(t')|)H^{(n)}(t')dt'}\times \left(H_0 + \int_0^t(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)(H^{(n)}(t'))2dt'\right).
\]
(3.5)

Notice that the iterative inequality (3.5) contains additional factor \(|\alpha(t)| + |\gamma(t)| + |\gamma(t)| + |\xi(t)|\), the classic method used in [5] is inapplicable in present case. To overcome this difficulty, we use another iterative method to derive the uniform bound.

For \(n = 1\), it follows from (3.5) that
\[
\sup_{t \in [0,\infty)} H^{(1)}(t) \leq 2Ce^{C^2H_0\int_0^\infty(|\alpha(t')|+|\xi(t')|)dt'}
\times \left(H_0 + 4C^2H_0^2\int_0^\infty(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)dt'\right) = 2Ch(H_0),
\]
where
\[
h(x) \overset{\text{def}}{=} e^{2C^2x\int_0^\infty(|\alpha(t')|+|\xi(t')|)dt'}\left(x + 4C^2x^2\int_0^\infty(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)dt'\right).
\]

It is clear that \(h(0) = 0\) and the function \(h(x)\) is a modulus of continuity defined on \(\mathbb{R}^+\), which is independent of the initial data \((m_0,n_0)\).

For \(k = 2\), we deduce from (3.5) that
\[
\sup_{t \in [0,\infty)} H^{(2)}(t) \leq Ce^{C^2h(H_0)\int_0^\infty(|\alpha(t')|+|\xi(t')|)dt'}
\times \left(H_0 + 4C^2h(H_0)\int_0^\infty(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)dt'\right)
\leq Ch(H_0)e^{2C^2h(H_0)\int_0^\infty(|\alpha(t')|+|\xi(t')|)dt'}
\times \left(1 + 4C^2h(H_0)\int_0^\infty(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)dt'\right)
\leq Ch(H_0)e^{6C^2h(H_0)\int_0^\infty(|\alpha(t')|+|\beta(t')|+|\gamma(t')|+|\xi(t')|)dt'}.
\]
(3.6)
In the last inequality of (3.6), we have used the facts of \( h(H_0) \geq H_0 \) and \( 1 + x \leq e^x \) for any \( x \geq 0 \). To estimate the right hand side of (3.6), we assume that

\[
\int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt' \leq \frac{\ln 2}{6C^2e^{2C^2H_0} \int_0^\infty (|\alpha(t')| + |\xi(t')|) dt'} (H_0 + 4C^2H_0^2 \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt')
\]

which combined with (3.6) imply that

\[
(3.7) \quad \sup_{t \in [0, \infty)} H^{(2)}(t) \leq Ch(H_0)e^{6C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt'} \leq 2Ch(H_0).
\]

Assuming for any given \( n \geq 3 \) that

\[
\sup_{t \in [0, \infty)} H^{(n)}(t) \leq 2Ch(H_0).
\]

For \( H^{(n+1)}(t) \), it follows from (3.5) and (3.7) that

\[
\sup_{t \in [0, \infty)} H^{(n+1)}(t) \leq Ce^{\int_0^\infty (|\alpha(t')| + |\xi(t')|) dt'} \cdot \sup_{t' \in [0, \infty)} H^{(n)}(t')
\]

\[
\leq Ce^{2C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\xi(t')|) dt'} \cdot \left( H_0 + \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt' \cdot \sup_{t' \in [0, \infty)} (H^{(n)}(t'))^2 \right)
\]

\[
\leq Ce^{2C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\xi(t')|) dt'} \cdot \left( H_0 + 4C^2h^2(H_0) \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt' \right)
\]

\[
\leq Ch(H_0)e^{2C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\xi(t')|) dt'} \cdot \left( 1 + 4C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt' \right)
\]

\[
\leq Ch(H_0)e^{6C^2h(H_0) \int_0^\infty (|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|) dt'} \leq 2Ch(H_0).
\]

Using the mathematical induction with respect to \( n \), we get

\[
\sup_{n \geq 1} \sup_{t \in [0, \infty)} H^{(n+1)}(t) \leq 2Ch(H_0),
\]

which implies the uniform bound

\[
(3.8) \quad \sup_{n \geq 1} \left( \|u^{(n)}\|_{L^\infty([0, \infty); B_{p,r}^s)} + \|\sigma^{(n)}\|_{L^\infty([0, \infty); B_{p,r}^{s-1}} \right) \leq 2Ch(H_0).
\]

As a result, the approximate solutions \((u^{(n)}, \sigma^{(n)})\) \(n \geq 1\) is uniformly bounded in \( C([0, \infty); B_{p,r}^s \times B_{p,r}^{s-1})\). Moreover, using the system (3.1) itself, one can verify that the sequence \((\partial_t u^{(n)}, \partial_t \sigma^{(n)})\) \(n \geq 1\)
is uniformly bounded in $C((0,\infty);B^{s-1}_{p,r} \times B^{s-2}_{p,r})$. Therefore we obtain that $(u^{(n)},\sigma^{(n)})_{n \geq 1}$ is uniformly bounded in $E^s_{p,r}(\infty)$.

**Step 2 (Convergence):** We are going to show that $(u^{(n)},\sigma^{(n)})_{n \geq 1}$ is a Cauchy sequence in the space $C((0,\infty);B^{s-1}_{p,r} \times B^{s-2}_{p,r})$. Indeed, according to (3.1), we obtain that for any $n,m \geq 1$

$$
\partial_t(u^{(n+1)} - u^{(n+m+1)}) + \alpha(t)u^{(n)}\partial_x(u^{(n+1)} - u^{(n+m+1)})
= -\alpha(t)(u^{(n)} - u^{(n+m)})u_x^{(n+m+1)} + \frac{\beta(t)}{2}\partial_x p * [(u^{(n+m)} + u^{(n)})(u^{(n+m)} - u^{(n)})]
\quad + \frac{\gamma(t)}{2}\partial_x p * [(\sigma^{(n+m)} + \sigma^{(n)})(\sigma^{(n+m)} - \sigma^{(n)})]
\quad + \frac{\beta(t) - 3\alpha(t)}{2}\partial_x p * [(u_x^{(n+m)} + u_x^{(n)})(u_x^{(n+m)} - u_x^{(n)})],
$$

(3.9)

and

$$
\partial_t(\sigma^{(n+1)} - \sigma^{(n+m+1)}) + \xi(t)u^{(n)}\partial_x(\sigma^{(n+1)} - \sigma^{(n+m+1)})
= -\xi(t)(u^{(n)} - u^{(n+m)})\sigma_x^{(n+m+1)} + \xi(t)(\sigma^{(n+m)} - \sigma^{(n)})u_x^{(n+m)}
\quad + \xi(t)\sigma^{(n)}(u_x^{(n+m)} - u_x^{(n)}),
$$

(3.10)

with the initial conditions

$$(u^{(n+1)} - u^{(n+m+1)})(0,x) = S_{n+1}u_0(x) - S_{n+m+1}u_0(x),$$

$$(\sigma^{(n+1)} - \sigma^{(n+m+1)})(0,x) = S_{n+1}\sigma_0(x) - S_{n+m+1}\sigma_0(x).$$

Applying the Lemma 2.3 to (3.9), we deduce that

$$
-C\int_0^t \|\alpha(t')\partial_x u^{(n)}(t')\|_{B^{s-1}_{p,r}} dt' + e^{-C\int_0^t \|\alpha(t')\|_{B^{s-1}_{p,r}} dt'}
\leq \|S_{n+1}u_0 - S_{n+m+1}u_0\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C\int_0^t \|\alpha(t')\|_{B^{s-1}_{p,r}} dt'}
\quad \times \left( \|\alpha(t')(u^{(n)} - u^{(n+m)})u_x^{(n+m+1)}\|_{B^{s-1}_{p,r}} + \|\beta(t')\partial_x p * [(u^{(n+m)} + u^{(n)})(u^{(n+m)} - u^{(n)})]\|_{B^{s-1}_{p,r}}
\quad + \|\frac{\gamma(t')}{2}\partial_x p * [(\sigma^{(n+m)} + \sigma^{(n)})(\sigma^{(n+m)} - \sigma^{(n)})]\|_{B^{s-1}_{p,r}}
\quad + \|\frac{\beta(t') - 3\alpha(t')}{2}\partial_x p * [(u_x^{(n+m)} + u_x^{(n)})(u_x^{(n+m)} - u_x^{(n)})]\|_{B^{s-1}_{p,r}} \right) dt'.
$$

(3.11)

For $s > 1 + \frac{1}{p}$, the space $B^{s-1}_{p,r}$ is a Banach algebra and $B^{s-1}_{p,r}$ continuously embedded into $B^{\frac{s-1}{p}}_{p,r} \cap L^\infty$, we get from the uniform bound (3.8) that

$$
\int_0^t \|\alpha(t')\partial_x u^{(n)}(t')\|_{B^{\frac{s-1}{p}}_{p,r} \cap L^\infty} dt' \leq C \int_0^t \|\alpha(t')\|_{B^{s-1}_{p,r}} dt' \leq C \int_0^t |\alpha(t')| dt',
$$

\[\text{for } t \leq T.\]
\[ \left\| \alpha(t')(u^{(n)} - u^{(n+m)})u_x^{(n+m+1)} \right\|_{B_{p,r}^{-1}} \leq C \left\| \alpha(t) \right\| \left\| u^{(n+m+1)} - u^{(n)} \right\|_{B_{p,r}^p} \]

and

\[ \left\| \frac{\beta(t')}{2} \partial_x p \left( u^{(n+m)} + u^{(n)} \right) (u^{(n+m)} - u^{(n)}) \right\|_{B_{p,r}^{-1}} \leq C \left\| \beta(t') \right\| \left\| u^{(n+m)} \right\|_{B_{p,r}^p} + \left\| u^{(n)} \right\|_{B_{p,r}^p} \left\| u^{(n+m)} - u^{(n)} \right\|_{B_{p,r}^{-1}} \]

\[ \leq C \left\| \beta(t') \right\| \left\| u^{(n+m)} - u^{(n)} \right\|_{B_{p,r}^{-1}}. \]

- For \( \max \{ 1 + \frac{1}{p}, \frac{3}{2} \} < s \leq 2 + \frac{1}{p} \), we first recall the following conclusion.

**Lemma 3.1 (Moser estimate [5]).** Let \( s_1 \leq 1/p < s_2 \) (\( s_2 \geq 1/p \) if \( r = 1 \)) and \( s_1 + s_2 > 0 \), then

\[ \left\| f g \right\|_{B_{p,r}^{s_1}} \leq C \left\| f \right\|_{B_{p,r}^{s_2}} \left\| g \right\|_{B_{p,r}^{s_2}}. \]

If we take

\[ s_1 = s - 2 \leq \frac{1}{p} < s_2 = s - 1, \]

then the last two terms in (3.11) can be estimated as follows

\[ \left\| \frac{\gamma(t')}{2} \partial_x p \left( (\sigma^{(n+m)} + \sigma^{(n)})(\sigma^{(n+m)} - \sigma^{(n)}) \right) \right\|_{B_{p,r}^{-1}} \leq C \left\| \gamma(t') \right\| \left\| \sigma^{(n+m)} \right\|_{B_{p,r}^p} \left\| \sigma^{(n)} \right\|_{B_{p,r}^{-1}} \left\| \sigma^{(n+m)} - \sigma^{(n)} \right\|_{B_{p,r}^{-2}} \]

and

\[ \left\| \frac{\beta(t') - 3\alpha(t')}{2} \partial_x p \left( u_x^{(n+m)} + u_x^{(n)} \right) (u_x^{(n+m)} - u_x^{(n)}) \right\|_{B_{p,r}^{-1}} \leq \left\| \beta(t') - 3\alpha(t') \right\| \left\| u^{(n+m)} \right\|_{B_{p,r}^p} + \left\| u^{(n)} \right\|_{B_{p,r}^p} \left\| u_x^{(n+m)} - u_x^{(n)} \right\|_{B_{p,r}^{-2}} \]

\[ \leq C \left\| \beta(t') - 3\alpha(t') \right\| \left\| u^{(n+m)} - u^{(n)} \right\|_{B_{p,r}^{-1}}. \]

- If \( s > 2 + \frac{1}{p} \), the space \( B_{p,r}^{s_2} \) is a Banach algebra, then it is not difficult to obtain the similar estimates by using the uniform bound (3.8).

As a consequence, we deduce from (3.11) that

\[ \left\| (u^{(n+1)}(t) - u^{(n+m+1)}(t)) \right\|_{B_{p,r}^{-1}} \]

\[ \leq e^{C \int_0^t |\alpha(t')| |\beta(t')|} \left( \left\| S_{n+1}u_0 - S_{n+m+1}u_0 \right\|_{B_{p,r}^{-1}} \right. \]

\[ + \left. \int_0^t \left( \left\| \alpha(t') \right\|_p + \left\| \beta(t') \right\|_p \right) \left\| u^{(n+m)} - u^{(n)} \right\|_{B_{p,r}^p} + \left\| \gamma(t') \right\| \left\| \sigma^{(n+m)} - \sigma^{(n)} \right\|_{B_{p,r}^{-2}} \right) dt'. \]
Next, we apply the Lemma 2.3 to (3.10) to find
\[
\| (\sigma^{(n+1)} - \sigma^{(n+m+1)})(t) \|_{B^r_{p,r}} \leq e^{C_{j_0} |\xi(t')|} \| u^{(n)} \|_{B^r_{p,r}} \left( \left\| S_{n+1} \sigma_0 - S_{n+m+1} \sigma_0 \right\|_{B^r_{p,r}} + \int_0^t |\xi(t')| \left( \| u^{(n)} - u^{(n+m)} \|_{B^r_{p,r}} + \| \sigma^{(n)} \|_{B^r_{p,r}} + \| \sigma^{(n+m)} \|_{B^r_{p,r}} \right) dt' \right) 
\]
(3.13)

\[
\leq e^{C_{j_0} |\xi(t')|} \left( \left\| S_{n+1} \sigma_0 - S_{n+m+1} \sigma_0 \right\|_{B^r_{p,r}} \right) + \int_0^t |\xi(t')| \left( \| u^{(n)} - u^{(n+m)} \|_{B^r_{p,r}} + \| \sigma^{(n+m)} - \sigma^{(n)} \|_{B^r_{p,r}} \right) dt' 
\]

According to the definition of the Littlewood-Paley blocks \( \Delta_p \) and the almost orthogonal property \( \Delta_i \Delta_j = 0 \) for \( |i - j| \geq 2 \), we have
\[
\| S_{n+1} u_0 - S_{n+m+1} u_0 \|_{B^r_{p,r}} = \sum_{j \geq 1} 2^{jr} \left\| \Delta_j \sum_{n+1 \leq i \leq n+m} \Delta_i u_0 \right\|_{L^p} 
\]
(3.14)

\[
\leq C \sum_{n \leq j \leq n+m+1} \left( 2^{-jr} \sum_{n+1 \leq i \leq n+m} \left\| \Delta_i \Delta_j u_0 \right\|_{L^2} \right) \leq C 2^{-m} \sum_{n \leq j \leq n+m+1} 2^{jr} \left\| \Delta_j u_0 \right\|_{L^2} \leq C 2^{-m} \| u_0 \|_{B^r_{p,r}}, 
\]

and similarly,
\[
\| S_{n+1} \sigma_0 - S_{n+m+1} \sigma_0 \|_{B^r_{p,r}} \leq C 2^{-m} \| \sigma_0 \|_{B^r_{p,r}} 
\]
(3.15)

For simplicity, we set
\[
\mathcal{H}^{(n,m)}(t) \overset{\text{def}}{=} \| (u^{(n)} - u^{(n+m)})(t) \|_{B^r_{p,r}} + \| \sigma^{(n)} - \sigma^{(n+m)}(t) \|_{B^r_{p,r}} 
\]

Putting the estimates (3.12)-(3.15) together and using the inequality \( e^x + e^y \leq 2e^{x+y} \) for any \( x, y \geq 0 \), we obtain
\[
\mathcal{H}^{(n+1,m)}(t) \leq C \left( 2^{-n} + \int_0^t \left( |\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')| \right) \mathcal{H}^{(n,m)}(t') dt' \right) 
\]
(3.16)

To derive an estimate for \( \mathcal{H}^{(n+1,m)}(t) \), we first prove the following useful lemma.

**Lemma 3.2.** Let \( a \) be a positive constant, and the nonnegative function \( \mu(t) \in L^1(0, \infty) \). Assume that the sequence of nonnegative functions \( (g_n(t))_{n \geq 1} \) satisfies the inequality
\[
g_{n+1}(t) \leq a_n + \int_0^t \mu(t') g_n(t') dt',
\]
where \( g_0 \) is a nonnegative number. Then we have
\[
\sup_{t \in [0, \infty)} g_{n+1}(t) \leq \sum_{k=0}^n \frac{a_{n-k} \| \mu \|_{L^1(0, \infty)}^k}{k!} + \frac{g_0 \| \mu \|_{L^1(0, \infty)}^{n+1}}{(n+1)!}.
\]
Proof. Using the iterative inequality, we first have
\[
g_{n+1}(t) \leq a_n + \int_0^t \mu(t_2) \left( a_{n-1} + \int_0^{t_2} \mu(t_1) g_{n-1}(t_1) dt_1 \right) dt_2
\]
(3.17)
\[
\leq a_n + a_{n-1} \int_0^t \mu(t_2) dt_2 + \int_0^t \int_0^{t_2} \mu(t_1) g_{n-1}(t_1) \mu(t_2) dt_1 dt_2
\]
\[
= a_n + a_{n-1} \int_0^t \mu(t_2) dt_2 + \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right) g_{n-1}(t_1) dt_1.
\]
Inserting the inequality \( g_{n-1}(t_1) \leq a_{n-2} + \int_0^{t_1} \mu(t_3) g_{n-2}(t_3) dt_3 \) into (3.17), we have
\[
g_{n+1}(t) \leq a_n + a_{n-1} \int_0^t \mu(t_2) dt_2 + a_{n-2} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right) dt_1
\]
(3.18)
\[
+ \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right) \left( \int_0^{t_1} \mu(t_3) g_{n-2}(t_3) dt_3 \right) dt_1.
\]
For the last two terms on the right-hand side of (3.18), we have
\[
\int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right) dt_1 = \frac{1}{2} \left( \int_0^t \mu(t_2) dt_2 \right)^2,
\]
and
\[
\int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right) \left( \int_0^{t_1} \mu(t_3) g_{n-2}(t_3) dt_3 \right) dt_1
\]
\[
= -\frac{1}{2} \int_0^t \left( \int_0^{t_1} \mu(t_3) g_{n-2}(t_3) dt_3 \right) dt_1 \left[ \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^2 \right]
\]
\[
= \frac{1}{2} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^2 g_{n-2}(t_1) dt_1.
\]
Hence we get
\[
g_{n+1}(t) \leq a_n + a_{n-1} \int_0^t \mu(t_2) dt_2 + \frac{a_{n-2}}{2} \left( \int_0^t \mu(t_2) dt_2 \right)^2
\]
(3.19)
\[
+ \frac{1}{2} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^2 g_{n-2}(t_1) dt_1.
\]
For the integrand \( g_{n-2}(t_1) \) in (3.19), we use the iterative inequality again and obtain
\[
g_{n+1}(t) \leq a_n + a_{n-1} \int_0^t \mu(t_2) dt_2 + \frac{a_{n-2}}{2} \left( \int_0^t \mu(t_2) dt_2 \right)^2 + \frac{a_{n-3}}{2} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^2 dt_1
\]
(3.20)
\[
+ \frac{1}{2} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^2 \left( \int_0^{t_1} \mu(t_4) g_{n-3}(t_4) dt_4 \right) dt_1
\]
\[
= \sum_{k=0}^{n-k} \frac{a_{n-k}}{k!} \left( \int_0^t \mu(t_2) dt_2 \right)^k + \frac{1}{3!} \int_0^t \mu(t_1) \left( \int_{t_1}^t \mu(t_2) dt_2 \right)^3 g_{n-3}(t_1) dt_1.
\]
Following the similar procedure for several times, we finally obtain
\[
g_{n+1}(t) \leq \sum_{k=0}^{n} \frac{a_{n-k}}{k!} \left( \int_{0}^{t} \mu(t_2)dt_2 \right)^k + \frac{g_0}{n!} \int_{0}^{t} \mu(t_1) \left( \int_{t_1}^{t} \mu(t_2)dt_2 \right)^n dt_1
\]
\[
= \sum_{k=0}^{n} \frac{a_{n-k}}{k!} \left( \int_{0}^{t} \mu(t_2)dt_2 \right)^k + \frac{g_0}{(n+1)!} \left( \int_{0}^{t} \mu(t_2)dt_2 \right)^{n+1}
\]
\[
\leq \sum_{k=0}^{n} \frac{a_{n-k} \| \mu \|^k_{L^1(0,\infty)}}{k!} + \frac{g_0 \| \mu \|^{n+1}_{L^1(0,\infty)}}{(n+1)!}.
\]

The proof of Lemma 3.2 is completed. \(\square\)

By taking \(a_n = 2^{-n}\) and \(\mu(t) = C(\| \alpha(t) \| + \| \beta(t) \| + \| \gamma(t) \| + \| \xi(t) \|)\), we get from Lemma 3.2 that
\[
\sup_{t \in [0,\infty)} \mathcal{H}^{(n+1,m)}(t) \leq \frac{1}{2^n} \sum_{k=0}^{n} \frac{C}{k!} \left( \frac{\ln 2}{3C^2 h(H_0)} \right)^k + \frac{g_0}{(n+1)!} \left( \frac{\ln 2}{6C^2 h(H_0)} \right)^{n+1},
\]
which implies that
\[
\lim_{n \to \infty} \sup_{t \in [0,\infty)} \mathcal{H}^{(n+1,m)}(t) = 0.
\]

Therefore, the sequence \((u^{(n)}, \sigma^{(n)})_{n \geq 1}\) converges strongly in the Banach space \(C([0,\infty); B^{s-1}_{p,r} \times B^{s-2}_{p,r})\), and we denote the limit function by \((u, \sigma)\).

**Step 3 (Existence):** In this step we shall verify that the limit function \((u, \sigma)\) indeed belongs to \(E_{p,r}^{s}([0,\infty))\) and is a strong solution to the system (1.6). Since the sequence \((u^{(n)}, \sigma^{(n)})_{n \geq 1}\) is uniformly bounded in \(L^\infty([0,\infty); B^{s-1}_{p,r} \times B^{s-1}_{p,r})\), and \((u^{(n)}, \sigma^{(n)}) \to (u, \sigma)\) strongly in \(B^{s-1}_{p,r} \times B^{s-2}_{p,r} \to \mathcal{Q}_1 \times \mathcal{Q}_2\) as \(n \to \infty\), it follows from the Fatou’s lemma (cf. [1]) that \((u, \sigma) \in L^\infty([0,\infty); B^{s}_{p,r} \times B^{s-1}_{p,r})\).

On the other hand, as \((u^{(n)}, \sigma^{(n)})_{n \geq 1}\) converges strongly to \((u, \sigma)\) in \(C([0,\infty); B^{s-1}_{p,r} \times B^{s-2}_{p,r})\), an interpolation argument insures that the convergence holds in \(C([0,\infty); B^{s'}_{p,r} \times B^{s'-1}_{p,r})\), for any \(s' < s\).

It is then easy to pass to the limit in the system (3.1) and to conclude that \((u, \sigma)\) is indeed a solution to (1.6). Thanks to the fact that \((u, \sigma)\) belongs to \(C([0,\infty); B^{s}_{p,r} \times B^{s-1}_{p,r})\), we know that the right-hand side of the equation
\[
u_t + \alpha(t)\nu u_x = -\partial_x p \ast \left( \frac{\beta(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2 + \frac{\beta(t) - 3\alpha(t)}{2} u_x^2 \right),
\]
belongs to \(L^1([0,\infty); B^{s}_{p,r})\), and the right-hand side of the equation
\[
\sigma_t + \xi(t) u \sigma_x = -\xi(t) \sigma u_x,
\]
belongs to \(L^1([0,\infty); B^{s-1}_{p,r})\). In particular, for the case \(r < \infty\), applying Lemma 2.4 implies that \((u, \sigma) \in C([0,\infty); B^{s'}_{p,r} \times B^{s'-1}_{p,r})\), for any \(s' < s\). Finally, using the system (1.6) again, we see that \((\partial_t u, \partial_t \sigma) \in C([0,\infty); B^{s-1}_{p,r} \times B^{s-2}_{p,r})\) for \(r < \infty\), and in \(L^\infty([0,\infty); B^{s-1}_{p,r} \times B^{s-2}_{p,r})\) otherwise. Therefore, \((u, \sigma)\) belongs to \(E_{p,r}^{s}([0,\infty))\). Moreover, a standard use of a sequence of viscosity approximate
solutions \((u_ε, σ_ε)_{ε>0}\) for system (3.6) which converges uniformly in

\[ C([0, ∞); B^{s}_{p,r} × B^{s-1}_{p,r}) \cap C^1([0, ∞); B^{s-1}_{p,r} × B^{s-2}_{p,r}) \]

gives the continuity of solution \((u, σ)\) in \(E^{s}_{p,r}(∞)\).

**Step 4 (Uniqueness and stability):** Let \((u^{(i)}, σ^{(i)})\) is the solution to system (1.6) with respect to the initial data \((u_0^{(i)}, σ_0^{(i)}), i = 1, 2\). Setting

\[ u^{(12)} = u^{(1)} - u^{(2)}, \quad σ^{(12)} = σ^{(1)} - σ^{(2)}. \]

Then the pair \((u^{(12)}, σ^{(12)})\) satisfies the following system

\[
\begin{align*}
\frac{∂_t u^{(12)}}{∂_t} + α(t) u^{(1)} ∂_t u^{(12)} &= -α(t) u^{(12)} u_x^{(2)} + f(u^{(12)}, u^{(1)}, u^{(2)}, σ^{(1)} + σ^{(2)}), \\
\frac{∂_t σ^{(12)}}{∂_t} + ξ(t) u^{(1)} ∂_t σ^{(12)} &= -ξ(t) u^{(12)} σ_x^{(2)} - ξ(t) σ^{(12)} u_x^{(2)} - ξ(t) σ^{(1)} u_x^{(12)}, \\
\end{align*}
\]

(3.21)

where

\[
f(u^{(12)}, u^{(1)}, u^{(2)}, σ^{(1)} + σ^{(2)}) \equiv \frac{∂_t x_p}{2} [u^{(12)} u^{(2)} + u^{(1)}] - \frac{γ(t)}{2} [σ^{(12)}(σ^{(2)} + σ^{(1)})] - \frac{β(t) - 3α(t)}{2} p [u^{(12)} (u_x^{(2)} + u^{(1)})].
\]

According to Lemma 2.23 we have the following estimates for (3.21):

\[
e^{-C \int_0^t |α(t)| ||u^{(1)}(t')||_{B^{s}_{p,r},r} dt'} |u^{(12)}(t)||_{B^{s-1}_{p,r}} \leq ||u^{(12)}_0||_{B^{s-1}_{p,r}} + \int_0^t e^{-C \int_0^t |α(t)| ||u^{(1)}(t)||_{B^{s}_{p,r},r} dτ} \times \left( ||α(t') u^{(12)} u_x^{(2)}||_{B^{s-1}_{p,r}} + ||f(u^{(12)}, u^{(1)}, u^{(2)}, σ^{(1)} + σ^{(2)})||_{B^{s-1}_{p,r}} \right) dt',
\]

(3.22)

and

\[
e^{-C \int_0^t |ξ(t')||u^{(1)}(t')||_{B^{s}_{p,r},r} dt'} ||σ^{(12)}(t)||_{B^{s-2}_{p,r}} \leq ||σ^{(12)}_0||_{B^{s-2}_{p,r}} + \int_0^t e^{-C \int_0^t |ξ(t)| ||u^{(1)}(t)||_{B^{s}_{p,r},r} dτ} \times \left( ||ξ(t') u^{(12)} σ_x^{(2)}||_{B^{s-2}_{p,r}} + ||ξ(t') σ^{(12)} u_x^{(2)}||_{B^{s-2}_{p,r}} + ||ξ(t') σ^{(1)} u^{(12)}||_{B^{s-2}_{p,r}} \right) dt',
\]

(3.23)
Since $B_{p,r}^{s-1}$ is a Banach algebra for any $s > 1 + \frac{1}{p}$, so we have

\[
\|\alpha(t')u^{(12)}(n+m+1)\|_{B_{p,r}^{s-1}} \leq C|\alpha(t')|\|u_x^{(n+m+1)}\|_{B_{p,r}^{s-1}},
\]

\[
\|\frac{\beta(t')}{2}\partial_s p \times [(u^{(n+m)} + u^{(n)}u^{(12)})\|_{B_{p,r}^{s-1}} \leq C|\beta(t')|\|u^{(n+m)} + u^{(n)}\|_{B_{p,r}^{s-1}},
\]

\[
\|\frac{\beta(t') - 3\alpha(t')}{2}\partial_s p \times [(u^{(n+m)} + u^{(n)}u^{(12)})\|_{B_{p,r}^{s-1}} \leq C|\beta(t') - 3\alpha(t')|\|u^{(n+m)} + u^{(n)}\|_{B_{p,r}^{s-1}},
\]

\[
\|\frac{\gamma(t')}{2}\partial_s p \times [(\sigma^{(n+m)} + \sigma^{(n)})\|_{B_{p,r}^{s-1}} \leq C|\gamma(t')|\|\sigma^{(n+m)} + \sigma^{(n)}\|_{B_{p,r}^{s-1}}.
\]

Therefore, one can deduce that

\[
\|f(u^{(12)}, u^{(1)}, u^{(2)}, \sigma^{(1)} + \sigma^{(2)})\|_{B_{p,r}^{s-1}}
\leq C(|\alpha(t')| + |\beta(t')| + |\gamma(t')|)(\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(1)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(2)}\|_{B_{p,r}^{s-1}})
\times (\|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(12)}\|_{B_{p,r}^{s-1}}).
\]

For max$\{1 + \frac{1}{p}, \frac{3}{2}\} < s \leq 2 + \frac{1}{p}$, we get by using the Lemma 2.3

\[
\|\xi(t')u^{(12)}\|_{B_{p,r}^{s-2}} \leq C|\xi(t')|\|u^{(12)}\|_{B_{p,r}^{s-2}},
\]

\[
\|\xi(t')\sigma^{(12)}u_x^{(2)}\|_{B_{p,r}^{s-2}} \leq C|\xi(t')|\|\sigma^{(12)}\|_{B_{p,r}^{s-2}},
\]

\[
\|\xi(t')\sigma^{(1)}u_x^{(12)}\|_{B_{p,r}^{s-2}} \leq C|\xi(t')|\|\sigma^{(1)}\|_{B_{p,r}^{s-2}}.
\]

Similar estimates also hold for the case of $s > 2 + \frac{1}{p}$. Putting above estimates together, we get from (3.22) and (3.23) that

\[
e^{-C\int_0^t(|\alpha(t')| + |\xi(t')|)|u^{(1)}(t')|_{B_{p,r}^{s-2}} dt'} (\|u^{(12)}(t)\|_{B_{p,r}^{s-1}} + \|\sigma^{(12)}(t)\|_{B_{p,r}^{s-2}})
\leq \|u_0^{(12)}\|_{B_{p,r}^{s-1}} + \|\sigma_0^{(12)}\|_{B_{p,r}^{s-2}} + \int_0^t e^{-C\int_0^\tau^t (|\alpha(\tau)| + |\xi(\tau)|)|u^{(1)}(\tau)|_{B_{p,r}^{s-2}} d\tau}
\times (\|u^{(12)}(t')\|_{B_{p,r}^{s-1}} + \|\sigma^{(12)}(t')\|_{B_{p,r}^{s-2}})(|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|)
\times (\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(1)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(2)}\|_{B_{p,r}^{s-1}})dt'.
\]

Applying the Gronwall lemma to above inequality leads to

\[
\|u^{(12)}(t)\|_{B_{p,r}^{s-1}} + \|\sigma^{(12)}(t)\|_{B_{p,r}^{s-2}} \leq (\|u_0^{(12)}\|_{B_{p,r}^{s-1}} + \|\sigma_0^{(12)}\|_{B_{p,r}^{s-2}})
\times e^{-C\int_0^t(|\alpha(t')| + |\beta(t')| + |\gamma(t')| + |\xi(t')|)(\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(1)}\|_{B_{p,r}^{s-1}} + \|\sigma^{(2)}\|_{B_{p,r}^{s-1}})dt'},
\]

which implies the uniqueness and continuity with respect to the initial data. The proof of Theorem 1.1 is now completed.
4. BLOW-UP PHENOMENA

In this section, we derive a precise blow-up criteria of strong solutions to the system (1.6) (when \( \beta(t) = 3\alpha(t) \)) with initial data in Sobolev spaces, and gives specific characterization for the lower bound of the blow-up time. Additionally, with sufficient conditions on the time-dependent parameters \( \alpha, \gamma \) and \( \xi \), we provide a precise wave-breaking criteria for the strong solution, which is shown to be independent of the second component.

In the case of \( \beta(t) = 3\alpha(t) \), the system (1.5) reduces to the following form:

\[
\begin{align*}
\left\{ \begin{array}{l}
m_t + \alpha(t)um_x + 3\alpha(t)u,m + \gamma(t)\sigma \xi_x = 0, \\
\sigma_t + \xi(t)u\sigma_x = 0, \\
m(0,x) = m_0(x), \quad \sigma(0,x) = \sigma(x),
\end{array} \right. \\
t > 0, \quad x \in \mathbb{K},
\end{align*}
\]

(4.1)

If the parameters \( \alpha, \gamma \) and \( \xi \) are merely locally integrable, then Theorem 1.1 implies that, given \((u_0, \sigma_0) \in H^s \times H^{s-1} (s > \frac{3}{2})\), the system (4.1) has a unique local-in-time solution \((u, \sigma) \in C([0, T], H^s \times H^{s-1})\) for some \( T > 0 \).

Now we consider the flow \( t \rightarrow \psi(t,x) \) generated by \( \xi(t)u(t,x) \):

\[
\left\{ \begin{array}{l}
\frac{d\psi(t,x)}{dt} = \xi(t)u(t, \psi(t,x)), \\
\psi(0,x) = x,
\end{array} \right. \quad t \geq 0, \quad x \in \mathbb{K}.
\]

By the regularity of \( u(t,x) \) in \( x \)-variable and the classic ODE theory, the previous initial value problem has a unique solution \( \psi \in C^1([0, T]; \mathbb{R}) \). Moreover, direct calculation leads to

\[
\left\{ \begin{array}{l}
\frac{d\psi(t,x)}{dt} = \xi(t)u_t(t, \psi(t,x)) \psi_x(t,x), \\
\psi_x(0,x) = 1,
\end{array} \right. \quad t \geq 0, \quad x \in \mathbb{K}.
\]

Hence for any \( t \in [0, T], x \in \mathbb{K} \), we find that

\[
(4.2) \quad \psi_x(t,x) = e^{\int_0^t \xi(t')u_t(t', \psi(t',x))dt'} > 0,
\]

which implies that \( \psi(t, \cdot) : \mathbb{K} \rightarrow \mathbb{K} \) is a diffeomorphism on \( \mathbb{K} \) for every \( t \in [0, T] \).

Using the second component of (1.6), it is easy to find that

\[
\frac{d}{dt} (\sigma(t, \psi(t,x)) \psi_x(t,x))
\]

\[
= \sigma_t(t, \psi(t,x)) \psi_x(t,x) + \sigma_x(t, \psi(t,x)) \psi_t(t,x) \psi_x(t,x) + \sigma(t, \psi(t,x)) \psi_{xx}(t,x)
\]

\[
= [\sigma_t + \xi(t)\sigma_u + \xi(t)\sigma_x](t, \psi(t,x)) \psi_x(t,x) = 0,
\]

which implies

\[
\sigma(t, \psi(t,x)) \psi_x(t,x) = \sigma(0, \psi(0,x)) \psi_x(0,x) = \sigma_0(x).
\]

As a result, we have the following \( L^\infty \)-bound

\[
(4.3) \quad \|\sigma(t, \cdot)\|_{L^\infty} = \|\sigma(t, \psi(t,x))\|_{L^\infty}
\]

\[
\leq \|\sigma_0\|_{L^\infty} \left\| e^{\int_0^t \xi(t')u_t(t', \psi(t',\cdot))dt'} \right\|_{L^\infty}
\]

\[
\leq \|\sigma_0\|_{L^\infty} e^{\int_0^t \inf_{x \in \mathbb{K}} \{\xi(t')u(t',x)\}dt'}.
\]
and the $L^2$-bound
\[
\|\sigma(t, \cdot)\|_{L^2} = \left( \int_{\mathbb{R}} |\sigma(t, \psi(t,x))|^2 \psi_x(t,x) \, dx \right)^{\frac{1}{2}}
\]
(4.4)
\[
= \left( \int_{\mathbb{R}} |\sigma_0(x)|^2 \psi_x(t,x) \, dx \right)^{\frac{1}{2}} \leq \|\sigma_0\|_{L^2} e^{-\frac{1}{2} \int_0^t inf_{x \in \mathbb{R}} \{\xi(t') u_x(t', x)\} \, dt'}.
\]

The following lemma provides the $L^\infty$-bound and $L^2$-bound for the first component $u$ of the solution, which is crucial in the proof of the following blow-up criteria.

**Lemma 4.1.** Assume that the parameters $\alpha, \gamma, \xi \in L^1_{loc}([0, \infty); \mathbb{R})$. Let $(u, \sigma)$ be the solution to the system (4.1) with initial data $(u_0, \sigma_0) \in H^5 \times H^{s-1}$ with $s > \frac{3}{2}$, then we have the $L^2$-bound
\[
\|u(t)\|_{L^2} \leq 2\|u_0\|_{L^2} \exp \left\{ 2\|\sigma_0\|^4_{H^{-1}} \|\gamma\|_{L^1(0,t)} e^{-3 \int_0^t inf_{x \in \mathbb{R}} \{\xi(t') u_x(t', x)\} \, dt'} \right\},
\]
and the $L^\infty$-bound
\[
\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t \mathcal{J}(t),
\]
where
\[
\mathcal{J}(t) \frac{\text{def}}{\text{def}} = 4\|u_0\|^2_{L^2} |\alpha(t)| \exp \left\{ 4\|\sigma_0\|^4_{H^{-1}} \|\gamma\|_{L^1(0,t)} e^{-3 \int_0^t inf_{x \in \mathbb{R}} \{\xi(t') u_x(t', x)\} \, dt'} \right\}
+ \|\sigma_0\|^2_{L^2} |\gamma(t)| e^{-\frac{1}{2} \int_0^t inf_{x \in \mathbb{R}} \{\xi(t') u_x(t', x)\} \, dt'}.
\]

**Proof.** By a standard density argument, it is sufficient to prove the lemma by assuming $s \geq 3$. Define $\chi = (4 - \partial_x^2)^{-1} u$, then we have $m = (1 - \partial_x^2)(4 - \partial_x^2)\chi$. Moreover, by the Plancheral’s theorem, we have
\[
(m, \chi)_{L^2} = (\mathcal{F} m, \mathcal{F} \chi)_{L^2} = ((1 + \eta^2)(4 + \eta^2)\mathcal{F} \chi_t, \mathcal{F} \chi)_{L^2}
= ((\mathcal{F} \chi_t, (1 + \eta^2)(4 + \eta^2)\mathcal{F} \chi)_{L^2} = (\chi_t, m)_{L^2}.
\]
In terms of the first component of (4.1), we have
\[
\frac{d}{dt} (m, \chi)_{L^2} = (m_t, \chi)_{L^2} + (m, \chi_t)_{L^2} = 2(m_t, \chi)_{L^2}
= -(2\alpha(t)(um)_x, \chi)_{L^2} - (4\alpha(t)u_t m, \chi)_{L^2} + (\gamma(t) \sigma^2, \chi_x)_{L^2}.
\]
Note that
\[
-(2\alpha(t)(um)_x, \chi)_{L^2} = (2\alpha(t)u^2, \chi_x)_{L^2} + (2\alpha(t)u_{xx}, u \chi_x)_{L^2}
= (2\alpha(t)u^2, \chi_x)_{L^2} + (2\alpha(t)u_x^2, \chi_x)_{L^2} - (\alpha(t)u^2, \chi_{xx})_{L^2}
= (2\alpha(t)u^2, \chi_x)_{L^2} + (2\alpha(t)u_x^2, \chi_x)_{L^2} - (\alpha(t)u^2, 4\chi - u_x)_{L^2}
= - (2\alpha(t)u^2, \chi_x)_{L^2} + (2\alpha(t)u_x^2, \chi_x)_{L^2},
\]
and
\[
-(4\alpha(t)u_t m, \chi)_{L^2} = (2\alpha(t)u^2, \chi_x)_{L^2} - (2\alpha(t)u_x^2, \chi_x)_{L^2}.
\]
Therefore we get from (4.5) that
\[
\begin{align*}
(m(t), \chi(t))_{L^2} &= (m_0, \chi_0)_{L^2} + \int_0^t (\gamma(t') \sigma^2(t'), \chi(t'))_{L^2} dt'.
\end{align*}
\]
(4.6)

Since the Fourier transformation is an isometric map from \(L^2\) into \(L^2\), we have
\[
\|u(t)\|_{L^2}^2 \geq (m(t), \chi(t))_{L^2} = (\mathcal{F}m(t), \mathcal{F}\chi(t))_{L^2}
\]
\[
= ((1 + \eta^2) \mathcal{F}u(t, \eta), (4 + \eta^2)^{-1} \mathcal{F}u(t, \eta))_{L^2}
\]
\[
\geq \frac{1}{4} \|\mathcal{F}u(t, \eta)\|_{L^2}^2 = \frac{1}{4} \|u(t)\|_{L^2}^2,
\]

and it follows from the Hölder inequality that
\[
\int_0^t (\gamma(t') \sigma^2(t'), \chi(t'))_{L^2} dt' \leq \int_0^t |\gamma(t')| \|\sigma^2(t')\|_{L^2} \|\chi(t')\|_{L^2} dt'
\]
\[
= \int_0^t |\gamma(t')| \|\sigma(t')\|_{L^2}^2 |\eta(4 + \eta^2)^{-1} \mathcal{F}u(t, \eta)|_{L^2} dt'
\]
\[
\leq \int_0^t |\gamma(t')| \|\sigma(t')\|_{L^2}^2 \|\sigma(t')\|_{L^2} \|u(t')\|_{L^2} dt'
\]
\[
\leq \|\sigma_0\|_{H^{-1}}^2 \int_0^t |\gamma(t')| e^{-3 \int_0^t \inf_{\xi \in \xi} \{\xi(t')u_0(t', x)\}} dt' \|u(t')\|_{L^2} dt',
\]

Inserting above two estimates into (4.6), we get
\[
\|u(t)\|_{L^2}^2 \leq 4\|u_0\|_{L^2}^2 + 4\|\sigma_0\|_{H^{-1}}^2 \int_0^t |\gamma(t')| e^{-3 \int_0^t \inf_{\xi \in \xi} \{\xi(t')u_0(t', x)\}} dt' \|u(t')\|_{L^2} dt',
\]

which combined with the Gronwall’s lemma yields the \(L^2\)-bound for \(u\).

To derive the \(L^\infty\)-bound for \(u\), we utilize the equivalent form of the first component:

\[
u_t + \alpha(t) u_x = -\partial_x p \ast \left( \frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2 \right).
\]

Along the characteristic \(\psi(t, x)\), we have
\[
\left\| \frac{du(t, \psi(t, x))}{dt} \right\|_{L^\infty} = \left\| \partial_x p \ast \left( \frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2 \right) \right\|_{L^\infty}
\]
\[
\leq \left\| \partial_x p \right\|_{L^\infty} \left\| \frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2 \right\|_{L^1}
\]
\[
\leq |\alpha(t)| \|u\|_{L^2}^2 + |\gamma(t)| \|\sigma\|_{L^2}^2
\]
\[
\leq 4\|u_0\|_{L^2}^2 |\alpha(t)| \exp \left\{ 4 \|\sigma_0\|_{H^{-1}}^2 \|\gamma\|_{L^1(0, t)} e^{-3 \int_0^t \inf_{\xi \in \xi} \{\xi(t')u_0(t', x)\}} dt' \right\}
\]
\[
+ \|\sigma_0\|_{L^2}^2 |\gamma(t)| e^{-3 \int_0^t \inf_{\xi \in \xi} \{\xi(t')u_0(t', x)\}} dt'
\]
\[
= \mathcal{J}(t).
\]

Integrating with respect to \(t\), we get from the above estimate that
\[
- \int_0^t \mathcal{J}(t') dt' \leq u(t, \psi(t, x)) - u_0 \leq \int_0^t \mathcal{J}(t') dt'.
\]
By using the homeomorphism property of $\psi(t, x)$ in $x$-variable, we obtain the $L^\infty$-bound for $u$. The proof of Lemma is completed.

Based on the above properties for $u$ and $\sigma$, we can now give the proof of Theorem 1.6.

**Proof of Theorem 1.5** We prove the theorem by an inductive argument with respect to the space regularity index $s$.

**Step 1 ($\frac{3}{2} < s < 2$):** By applying Lemma 2.5 to the transport equation with respect to $\sigma$:

\[
\sigma_t + \xi(t) u \sigma_x + \xi(t) \sigma u_x = 0,
\]

we have (for every $1 < s < 2$, indeed)

\[
\|\sigma(t)\|_{H^{s-1}} \leq \|\sigma_0\|_{H^{s-1}} + C \int_0^t \|\xi(t')\|_{H^{s-1}} dt' + C \int_0^t \|\xi(t')\|_{H^{s-1}} \|\sigma\|_{H^{s-1}} (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) dt'.
\]

Thanks to the Moser estimate (cf. Corollary 2.8 in [1]), we have

\[
\|\xi(t) \sigma u_x\|_{H^{s-1}} \leq \|\xi(t)\| (\|\sigma\|_{L^\infty} \|u_x\|_{H^{s-1}} + \|u_x\|_{L^\infty} \|\sigma\|_{H^{s-1}}).
\]

Therefore, we have

\[
\|\sigma(t)\|_{H^{s-1}} \leq \|\sigma_0\|_{H^{s-1}} + C \int_0^t \|\xi(t')\| (\|\sigma\|_{L^\infty} \|u\|_{H^s} + \|u_x\|_{L^\infty} \|\sigma\|_{H^{s-1}}) dt' 
+C \int_0^t \|\xi(t')\| (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \|\sigma\|_{H^{s-1}} dt'.
\]

On the other hand, applying Lemma 2.3 to the equation

\[
u_t + \alpha(t) uu_x = -\partial_x p \ast \left(\frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2\right)
\]

implies (for every $s > 1$, indeed)

\[
\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C \int_0^t \left\|\partial_x p \ast \left(\frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2\right)(t')\right\|_{H^s} dt' 
+C \int_0^t |\alpha(t')| \|u_x(t')\|_{L^\infty} \|u(t')\|_{H^s} dt'.
\]

Thanks to the Moser estimate for $s - 1 > 0$, one has

\[
\left\|\partial_x p \ast \left(\frac{3\alpha(t)}{2} u^2 + \frac{\gamma(t)}{2} \sigma^2\right)\right\|_{H^s} \leq C(|\alpha(t)||u|_{L^\infty} ||u||_{H^{s-1}} + |\gamma(t)||\sigma||_{L^\infty} ||\sigma||_{H^{s-1}}).
\]

From this, we reach

\[
\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C \int_0^t (|\alpha(t')| \|u(t')\|_{L^\infty} \|u(t')\|_{H^{s-1}} + |\gamma(t')| \|\sigma(t')\|_{L^\infty} \|\sigma(t')\|_{H^{s-1}}) dt' 
+C \int_0^t |\alpha(t')| \|u_x(t')\|_{L^\infty} \|u(t')\|_{H^s} dt'.
\]

\[
(4.10)
\]
which together with (4.8) yields that
\begin{equation}
\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1} \leq \| u_0 \|_{H^s} + \| \sigma_0 \|_{H^s-1} + C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| \sigma \|_{L^\infty})(\| u \|_{H^s} + \| \sigma \|_{H^s-1}) dt' .
\end{equation}

An application of the Gronwall’s lemma leads to
\begin{equation}
\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1} \leq (\| u_0 \|_{H^s} + \| \sigma_0 \|_{H^s-1}) e^{C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| \sigma \|_{L^\infty}) dt'} .
\end{equation}

In terms of the estimate (4.4) and Lemma 4.1, we see that
\begin{equation}
\| \sigma(t) \|_{L^\infty} \leq \| \sigma_0 \|_{H^s-1} e_0^t \| \xi(t') \| \| u_x(t') \|_{L^\infty} dt' ,
\end{equation}
and
\begin{equation}
\| u(t) \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + 4 \| u_0 \|_{L^2}^2 \| \alpha(t) \| \exp \left\{ 4 \| \sigma_0 \|_{H^s-1}^4 \| \gamma \|_{L^1(0,t)} e^3 \int_0^t \| \xi(t') \| \| u_x(t') \|_{L^\infty} dt' \right\}
+ \| \sigma_0 \|_{L^2}^2 \| \gamma(t) \| e_0^t \| \xi(t') \| \| u_x(t') \|_{L^\infty} dt' .
\end{equation}

If \( \int_0^{T^*} (|\alpha(t')| + |\xi(t')| + |\gamma(t')|) \| u_x(t') \|_{L^\infty} dt' < \infty \), then it follows from (4.12) and the fact of \( \alpha, \gamma, \xi \in L^2_{loc}([0,\infty); \mathbb{R}) \) that
\[ \limsup_{t \to T^*} (\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1}) < \infty, \]
which is incompatible with the assumption that \( T^* \) is the maximum existence time of solutions.

Step 2 \((2 \leq s < \frac{3}{2})\): Applying Lemma [2,3] to Eq.(4.7), we get
\begin{equation}
\| \sigma(t) \|_{H^s-1} \leq \| \sigma_0 \|_{H^s-1} + C \int_0^t \| \xi(t') \| \| \sigma \|_{L^\infty} \| u \|_{H^s} dt' + C \int_0^t \| \xi(t') \| \| u_x \|_{L^\infty \cap H^\frac{1}{2}} \| \sigma \|_{H^s-1} dt' ,
\end{equation}
which together with (4.10) yields
\begin{equation}
\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1} \leq \| u_0 \|_{H^s} + \| \sigma_0 \|_{H^s-1} + C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{H^{\frac{3}{2}+\varepsilon}} + \| \sigma \|_{L^\infty})(\| u \|_{H^s} + \| \sigma \|_{H^s-1}) dt',
\end{equation}
for any \( \varepsilon \in (0, \frac{1}{2}) \), where we used the fact of \( H^{\frac{3}{2}+\varepsilon} \hookrightarrow L^\infty \cap H^{\frac{1}{2}} \).

Applying the Gronwall’s lemma to above inequality leads to
\begin{equation}
\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1} \leq (\| u_0 \|_{H^s} + \| \sigma_0 \|_{H^s-1}) e^{C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{H^{\frac{3}{2}+\varepsilon}} + \| \sigma \|_{L^\infty}) dt' .
\end{equation}

Therefore, thanks to the uniqueness of solution in Theorem 1.1, we get that: if the maximal existence time \( T^* \) satisfies \( \int_0^{T^*} (|\alpha(t')| + |\xi(t')| + |\gamma(t')|) \| u_x(t') \|_{L^\infty} dt' < \infty \), then Step 1 with \( \frac{3}{2} < \frac{3}{2} + \varepsilon < 2 \) and the fact of \( \| \sigma(t) \|_{L^\infty} \leq \| \sigma_0 \|_{H^s-1} e_0^t \| \xi(t') \| \| u_x(t') \|_{L^\infty} dt' \), imply that
\[ \limsup_{t \to T^*} (\| u(t) \|_{H^s} + \| \sigma(t) \|_{H^s-1}) < \infty \]
contradicts the assumption that \( T^* \) is the maximum existence time of solutions.
Step 3 ($2 < s < 3$): Differentiating the Eq. (4.7) with respect to $x$, we have
\[ \partial_t \sigma_x + \xi(t)u \partial_x \sigma_x + 2\xi(t)u_x \sigma_x + \xi(t) \sigma u_{xx} = 0. \]

Applying Lemma 2.5 to the above equation implies that
\begin{align*}
\| \sigma_x(t) \|_{H^{s-2}} & \leq \| \partial_x \sigma_0 \|_{H^{s-2}} + C \int_0^t \| \xi(t') \| \| (2u_x \sigma_x + \sigma u_{xx})(t') \|_{H^{s-2}} dt' \\
& \quad + C \int_0^t \| \xi(t') \| (\| u \|_{L^\infty} + \| u_x \|_{L^\infty}) \| \sigma \|_{H^{s-2}} dt' \\
& \leq C \int_0^t \| \xi(t') \| (\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| \sigma \|_{L^\infty}) (\| u(t') \|_{H^s} + \| \sigma(t') \|_{H^{s-1}}) dt',
\end{align*}
(4.15)
where we used the following Moser-type estimates:
\[ \| u_x \sigma_x \|_{H^{s-2}} \leq C (\| \sigma \|_{L^\infty} \| u_x \|_{H^{s-1}} + \| \sigma \|_{H^{s-2}} \| u_x \|_{L^\infty}), \]
and
\[ \| \sigma u_{xx} \|_{H^{s-2}} \leq C (\| \sigma \|_{H^{s-1}} \| u_x \|_{L^\infty} + \| u_{xx} \|_{H^{s-2}} \| \sigma \|_{L^\infty}). \]
(4.15), together with (4.18) and (4.16) (where $s - 1$ is replaced by $s - 2$), implies that
\[ \| u(t) \|_{H^s} + \| \sigma(t) \|_{H^{s-1}} \leq \| u_0 \|_{H^s} + \| \sigma_0 \|_{H^{s-1}} + C \int_0^t (| \alpha(t') | + | \xi(t') | + | \gamma(t') |)(\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| \sigma \|_{L^\infty})(\| u \|_{H^s} + \| \sigma \|_{H^{s-1}}) dt'. \]

Applying the Gronwalls inequality again gives (4.10). Hence, using the arguments as in Step 1, it completes the proof of Theorem 4.1 for $2 < s < 3$.

Step 4 ($s = k \in \mathbb{N}, k \geq 3$): Differentiating the Eq. (4.15) $k - 2$ times with respect to $x$, we get
\begin{align*}
\partial_t \partial_x^{k-2} \sigma + \xi(t)u \partial_x \partial_x^{k-2} \sigma + \xi(t) \sum_{l=1}^{k-2} C_{k-l}^l \partial_x^l u \partial_x^{k-l-1} \sigma + \sigma \partial_x^{k-1} u = 0.
\end{align*}
(4.16)

Applying Theorem 3.1 to the transport equation (4.24), we have
\begin{align*}
\| \partial_x^{k-2} \sigma(t) \|_{H^1} & \leq \| \partial_x^{k-2} \sigma_0 \|_{H^1} + C \int_0^t \| \partial_x^{k-2} \sigma(t') \|_{H^1} \| \xi(t) \partial_x u(t') \|_{L^\infty \cap H^1 \subset H^2} dt' \\
& \quad + C \int_0^t \| \xi(t') \| \sum_{l=1}^{k-2} C_{k-l}^l \partial_x^l u \partial_x^{k-l-1} \sigma + \sigma \partial_x^{k-1} u \|_{H^1} dt' \\
& \leq \| \sigma_0 \|_{H^{k-1}} + C \int_0^t \| \xi(t') \| (\| u(t') \|_{H^s} + \| \sigma(t') \|_{H^{s-1}})(\| u(t') \|_{H^{s-1}} + \| \sigma(t') \|_{H^s}) dt',
\end{align*}
(4.17)
where we have used the following facts (as $H^1$ is an algebra):

\[
\left\| \xi(t) \sum_{l=1}^{k-2} C^l_{k-1} \partial_x^l u \partial_x^{k-l-1} \sigma \right\|_{H^1} \\
\leq |\xi(t)| \sum_{l=1}^{k-2} C^l_{k-1} \left\| \partial_x^l u \right\|_{H^1} \left\| \partial_x^{k-l-1} \sigma \right\|_{H^1} \leq C |\xi(t)| \left\| u \right\|_{H^{1-1}} \left\| \sigma \right\|_{H^{2-1}},
\]

and

\[
\left\| \sigma \partial_x^{k-1} u \right\|_{H^1} \leq \left\| \sigma \right\|_{H^1} \left\| \partial_x^{k-1} u \right\|_{H^1} \leq C \left\| \sigma \right\|_{H^1} \left\| u \right\|_{H^s}.
\]

(4.18), together with (4.10) and (4.8) (where $s - 1$ is replaced by 1), implies that

\[
\left\| u(t) \right\|_{H^s} + \left\| \sigma(t) \right\|_{H^{s-1}} \leq \left\| u_0 \right\|_{H^s} + \left\| \sigma_0 \right\|_{H^{s-1}} + C \int_0^t \left( |\alpha(t')| + |\xi(t')| + |\gamma(t')| \right) \left\| u(t') \right\|_{H^{s-1}} + \left\| \sigma(t') \right\|_{H^1} \left( \left\| u(t') \right\|_{H^s} + \left\| \sigma(t') \right\|_{H^{s-1}} \right) dt'.
\]

Applying the Gronwall’s lemma to the above inequality leads to

(4.18) \[
\left\| u(t) \right\|_{H^s} + \left\| \sigma(t) \right\|_{H^{s-1}} \leq \left( \left\| u_0 \right\|_{H^s} + \left\| \sigma_0 \right\|_{H^{s-1}} \right) e^{C \int_0^t \left( \left\| u(t') \right\|_{H^{s-1}} + \left\| \sigma(t') \right\|_{H^1} \right) dt'}.
\]

If the maximal existence time $T^*$ satisfies $\int_0^{T^*} \left( |\alpha(t')| + |\xi(t')| + |\gamma(t')| \right) \left\| u(t') \right\|_{L^\infty} dt' < C$ for some positive constant $C$, thanks to the uniqueness of solution in Theorem 4.11 we get that $\left\| u(t) \right\|_{H^{s-1}} + \left\| \sigma(t) \right\|_{H^1}$ is uniformly bounded by the induction assumption, which together with (4.18) and the fact of $L^2_{loc}(0, \infty; \mathbb{R}) \subset L^1_{loc}(0, \infty; \mathbb{R})$ implies

\[
\limsup_{t \to T^*} \left( \left\| u(t) \right\|_{H^s} + \left\| \sigma(t) \right\|_{H^{s-1}} \right) < \infty.
\]

This leads to a contradiction.

**Step 5** $(k < s < k + 1, k \in \mathbb{N}, k \geq 3)$: Differentiating the Eq. (4.15) $k - 1$ times with respect to $x$, we get

(4.19) \[
\partial_t \partial_x^{k-1} \sigma + \xi(t) u \partial_x \partial_x^{k-1} \sigma + \xi(t) \sum_{l=1}^{k-1} C^l_k \partial_x^l u \partial_x^{k-l-1} \sigma + \xi(t) \sigma \partial_t \partial_x^{k-1} u = 0.
\]

Applying Lemma 2.5 to (4.19) implies that

\[
\left\| \partial_x^{k-1} \sigma(t) \right\|_{H^{s-k}} \leq \left\| \partial_x^{k-1} \sigma_0 \right\|_{H^{s-k}} + C \int_0^t \left( \left\| \xi(t) \right\| \sum_{l=1}^{k-1} C^l_k \partial_x^l u \partial_x^{k-l-1} \sigma + \xi(t) \sigma \partial_t \partial_x^{k-1} u \right)_{H^{s-k}} dt' + C \int_0^t \left\| \xi(t') \right\| \left\| \partial_x^{k-1} \sigma \right\|_{H^{s-k}} \left( \left\| u \right\|_{L^\infty} + \left\| u_x \right\|_{L^\infty} \right) dt'.
\]
Using the Moser-type estimate and the Sobolev embedding theorem, we have for $0 < \varepsilon < \frac{1}{2}$

$$
\left\| \xi(t) \sum_{i=1}^{k-1} C_i \partial_x^i u \partial_x^{k-i} \sigma \right\|_{H^{s-k}} \\
\leq C |\xi(t)| \sum_{i=1}^{k-1} (\| \partial_x^i u \|_{H^{s-k+i}} + \| \partial_x^{k-i} \sigma \|_{L^\infty}) + \| \partial_x^{k-1} \sigma \|_{H^{s-k}} \| \partial_x^i u \|_{L^\infty}) \\
\leq C |\xi(t)| \sum_{i=1}^{k-1} (\| \partial_x^i u \|_{H^{s-k+i}} + \| \partial_x^{k-i} \sigma \|_{H^{\frac{k}{2}+\varepsilon}} + \| \partial_x^{k-1} \sigma \|_{H^{\frac{k}{2}+\varepsilon}} \| \partial_x^i u \|_{H^{\frac{k}{2}+\varepsilon}}) \\
\leq C |\xi(t)| (\| u \|_{H^{s-1}} + \| \sigma \|_{H^{s-1}} \| u \|_{H^{k-\frac{1}{2}+\varepsilon}}),
$$

and

$$
\left\| \xi(t) \sigma \partial_x \partial_x^{k-1} u \right\|_{H^{s-k}} \leq C |\xi(t)| (\| \partial_x^{k-1} u \|_{H^{s-k}} + \| \sigma \|_{H^{s-1}} \| \partial_x^{k-1} u \|_{L^\infty}) \\
\leq C |\xi(t)| (\| u \|_{H^{s-1}} + \| \sigma \|_{H^{s-1}} \| u \|_{H^{k-\frac{1}{2}+\varepsilon}}).
$$

Hence, we get

$$
\left\| \partial_x^{k-1} \sigma(t) \right\|_{H^{s-k}} \leq \| \sigma_0 \|_{H^{s-1}} + C \int_0^t |\xi(t')(\| u \|_{H^s} + \| \sigma \|_{H^{s-1}})(\| u \|_{H^{k-\frac{1}{2}+\varepsilon}} + \| \sigma \|_{H^{k-\frac{1}{2}+\varepsilon}}) dt',
$$

which together with (4.10) and (4.8) (where $s - 1$ is replaced by $s - k$) implies that

$$
\left\| u(t) \right\|_{H^s} + \| \sigma(t) \|_{H^{s-k}} \leq \| u_0 \|_{H^s} + \| \sigma_0 \|_{H^{s-1}} \\
+ C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{H^s} + \| \sigma \|_{H^{s-1}})(\| u \|_{H^{k-\frac{1}{2}+\varepsilon}} + \| \sigma \|_{H^{k-\frac{1}{2}+\varepsilon}}) dt'.
$$

Applying Gronwall’s inequality then gives that

$$
\left\| u(t) \right\|_{H^s} + \| \sigma(t) \|_{H^{s-k}} \leq (\| u_0 \|_{H^s} + \| \sigma_0 \|_{H^{s-1}}) e^{C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u \|_{H^{k-\frac{1}{2}+\varepsilon}} + \| \sigma \|_{H^{k-\frac{1}{2}+\varepsilon}}) dt'}.
$$

If $\int_0^T (|\alpha(t')| + |\xi(t')| + |\gamma(t')|) u_0(t') L^\infty dt' < \infty$, thanks to the uniqueness of solution in Theorem [1] we get that $\| u \|_{H^{k-\frac{1}{2}+\varepsilon}} + \| \sigma \|_{H^{k-\frac{1}{2}+\varepsilon}}$ is uniformly bounded by the induction assumption, which together with (4.20) and the fact of $L^2_{loc}((0,\infty); \mathbb{R}) \subset L^1_{loc}((0,\infty); \mathbb{R})$ implies

$$
\lim_{t \to T^*} \| u(t) \|_{H^s} + \| \sigma(t) \|_{H^{s-1}} < \infty.
$$

which leads to a contradiction. Therefore, from Step 1 to Step 5, we complete the proof of the blow-up criteria.

To derive a lower bound for the blow-up time, it follows from the above analysis and the Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{1}{2}$ that

$$
\left\| u(t) \right\|_{H^s} + \| \sigma(t) \|_{H^{s-k}} \leq \| u_0 \|_{H^s} + \| \sigma_0 \|_{H^{s-1}} \\
+ C \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)(\| u(t') \|_{H^s} + \| \sigma(t') \|_{H^{s-1}}) dt'.
$$
We find by solving the above inequality that

\[(4.21)\]
\[
\|u(t)\|_{H^r} + \|\sigma(t)\|_{H^{r-k}} \leq \left(1 - C(\|u_0\|_{H^r} + \|\sigma_0\|_{H^{r-1}}) \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)dt'\right)^{-1}.
\]

From the right hand side of (4.21), we have the lower bound of blow-up time \(T^*\)

\[
T^* \geq T(u_0, \sigma_0) \overset{\text{def}}{=} \sup_{t>0} \left\{t > 0; \int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)dt' \leq \frac{1}{C(\|u_0\|_{H^r} + \|\sigma_0\|_{H^{r-1}})}\right\}.
\]

Therefore, if \(\alpha, \beta, \gamma \in L^1([0, \infty); \mathbb{R})\) satisfies

\[
\int_0^t (|\alpha(t')| + |\xi(t')| + |\gamma(t')|)dt' < \frac{1}{C(\|u_0\|_{H^r} + \|\sigma_0\|_{H^{r-1}})},
\]

we get

\[
T^* = T(u_0, \sigma_0) = \infty,
\]

which together with (4.21) implies that

\[
\sup_{t \in [0, \infty)} (\|u(t)\|_{H^r} + \|\sigma(t)\|_{H^{r-1}}) < \infty.
\]

In other words, the solution \((u, \sigma)\) exists globally. The proof of Theorem is now completed. \(\square\)

**Proof of Theorem 1.6** Assume that the solution \((u, \sigma)\) blows up in finite time \((T^* < \infty)\) and there exists an \(M > 0\) such that \(\inf_{x \in \mathbb{R}} u_x(t, x) \geq -M, \ \forall t \in [0, T^*).\) Since \(\xi(t) \geq 0\) and \(\alpha(t) + \gamma(t) + \xi(t) \geq 0,\) we have

\[(4.22)\]
\[
- \inf_{x \in \mathbb{R}} \{\xi(t)u_x(t, x)\} \leq M\xi(t), \ \forall t \in [0, T^*),
\]

and

\[(4.23)\]
\[
- \inf_{x \in \mathbb{R}} \{(\alpha(t) + \gamma(t) + \xi(t))u_x(t, x)\} \leq M(\alpha(t) + \gamma(t) + \xi(t)), \ \forall t \in [0, T^*).
\]

It then follows from (4.22) and Lemma 4.1 that

\[
\|\sigma(t, \cdot)\|_{L^\infty} \leq \|\sigma_0\|_{H^{r-1}} e^{M\|\xi\|_{L^1(0,t)}},
\]

and

\[
\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + 4\|u_0\|_{L^2(\mathbb{R})}^2 t |\alpha(t)| \exp\left\{4\|\sigma_0\|_{H^{r-1}}^4 \|\gamma\|_{L^1(0,t)} e^{3M\|\xi\|_{L^1(0,t)}}\right\}
\]

\[
+ \|\sigma_0\|_{L^2(\mathbb{R})}^2 t |\gamma(t)| e^{M\|\xi\|_{L^1(0,t)}}.
\]

Differentiating the first component of system (1.6), we have

\[(4.24)\]
\[
\partial_t u_x + \alpha(t)u_x = -p \left(\frac{3\alpha(t)}{2}u^2 + \frac{\gamma(t)}{2} \sigma^2\right) + \frac{3\alpha(t)}{2}u^2 + \frac{\gamma(t)}{2} \sigma^2 - \alpha(t)u_x^2.
\]
Along the characteristic curve $\psi(t,x)$, it is seen that
\[
\frac{du_x(t, \psi(t,x))}{dt} = -p \left( \frac{3\alpha}{2} u^2 + \frac{\gamma}{2} \sigma^2 \right) (t, \psi(t,x)) + \left( \frac{3\alpha}{2} u^2 + \frac{\gamma}{2} \sigma^2 - \alpha u_x^2 \right) (t, \psi(t,x))
\]
\[
\leq \left( \frac{3\alpha}{2} u^2 + \frac{\gamma}{2} \sigma^2 \right) (t, \psi(t,x))
\]
\[
\leq \frac{3}{2} |\beta(t)| \|u(t)\|_{L^\infty} + \frac{\gamma(t)}{2} \|\sigma(t)\|_{L^\infty}
\]
\[
\leq \frac{3}{2} \|u_0\|_{H^s} |\beta(t)| + 6 \|u_0\|_{L^2(t)}^2 |\beta(t)| \|\alpha(t)\| + 4 \|\sigma_0\|_{H^{s-1}} \|\gamma(t)\|_{L^1(t)} e^{3M\|\xi\|_{L^1(t)}}
\]
\[
+ \frac{3}{2} \|\sigma_0\|_{H^{s-1}} \|\gamma(t)\|_{L^1(t)} e^{M\|\xi\|_{L^1(t)}} + \frac{\|\sigma_0\|_{H^{s-1}}}{2} |\gamma(t)| e^{M\|\xi\|_{L^1(t)}}.
\]

Integrating the above inequality on $[0,t]$, we obtain
\[(4.25)\]
\[
u_x(t, \psi(t,x)) - u_{0,x} \leq \frac{3}{2} \|u_0\|_{H^s} \|\beta\|_{L^1(t)} + \frac{3}{2} \|\sigma_0\|_{H^{s-1}} e^{M\|\xi\|_{L^1(t)}} (\|\beta\|_{L^1(t)}^2 + \|\gamma\|_{L^1(t)}^2)
\]
\[
+ 6 \|u_0\|_{L^2(t)}^2 \exp \left\{ 4 \|\sigma_0\|_{H^{s-1}} \|\gamma\|_{L^1(t)} e^{3M\|\xi\|_{L^1(t)}} \right\} (\|\beta\|_{L^1(t)}^2 + \|\gamma\|_{L^1(t)}^2)
\]
\[
+ \frac{\|\sigma_0\|_{H^{s-1}}}{2} \|\gamma\|_{L^1(t)} e^{M\|\xi\|_{L^1(t)}}
\]
\[
= \mathcal{L}(t) \leq \mathcal{L}(T^*) < \infty.
\]

Note that $\mathcal{L}(t)$ is an increasing and absolutely continuous function. Since $\alpha, \gamma, \xi \in L^2_{loc}([0,\infty); \mathbb{R}) \subset L^1_{loc}([0,\infty); \mathbb{R})$, it follows from (4.25) that
\[
\sup_{x \in \mathbb{K}} u_x(t, x) = \sup_{x \in \mathbb{K}} u_x(t, \psi(t,x)) \leq \|u_{0,x}\|_{L^\infty} + \mathcal{L}(T^*), \quad \forall t \in [0,T^*),
\]
which together with the fact of $\alpha(t) + \gamma(t) + \xi(t) \geq 0$ implies that
\[(4.26)\]
\[
\sup_{x \in \mathbb{K}} \{(\alpha(t) + \gamma(t) + \xi(t)) u_x(t, x)\} \leq (\|u_0\|_{H^s} + \mathcal{L}(T^*)) (\alpha(t) + \gamma(t) + \xi(t)),
\]
for any $t \in [0,T^*)$. As a result, we get from (4.23) and (4.26) that
\[
\int_0^{T^*} (\alpha(t') + \gamma(t') + \xi(t')) \|u_x(t')\|_{L^\infty} dt'
\]
\[
\leq (M + \|u_0\|_{H^s} + \mathcal{L}(T^*)) \int_0^{T^*} (\alpha(t') + \gamma(t') + \xi(t')) dt' < \infty.
\]

Theorem 1.6 implies that the maximal existence time $T^* = \infty$, which contradicts the assumption on the maximal existence time $T^* < \infty$.

Conversely, the Sobolev embedding $H^s \hookrightarrow L^\infty$ with $s > \frac{1}{2}$ implies that if $\lim_{t \to T^*} \inf_{x \in \mathbb{K}} u_x(t, x) = -\infty$ holds, then the corresponding solution blows up in finite time, which completes the proof of Theorem 1.6. \qed
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REFERENCES

[1] H. Bahouri, J. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Vol. 343, Springer Science & Business Media, 2011.

[2] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Annali della Scuola normale superiore di Pisa. Classe di scienze 26 (1998), no. 2, 303–328.

[3] ———, *Global existence and blow-up for a shallow water equation*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 26 (1998), no. 2, 303–328.

[4] ———, *Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Communications on Pure and Applied Mathematics 51 (1998), no. 5, 475–504.

[5] R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential and Integral Equations 14 (2001), no. 8, 953–988.

[6] C. Guan and Z. Yin, *Global existence and blow-up phenomena for an integrable two-component camassa–holm shallow water system*, Journal of Differential Equations 248 (2010), no. 8, 2003–2014.

[7] G. Gui and Y. Liu, *On the global existence and wave-breaking criteria for the two-component camassa–holm system*, Journal of Functional Analysis 258 (2010), no. 12, 4251–4278.

[8] G. Gui, Y. Liu, P.J. Olver, and C. Qu, *Wave-breaking and peakons for a modified camassa–holm equation*, Communications in Mathematical Physics 319 (2013), no. 3, 731–759.

[9] Y. Guo, S. Lai, and Y. Wang, *Global weak solutions to the weakly dissipative degasperis–procesi equation*, Nonlinear Analysis: Theory, Methods & Applications 74 (2011), no. 15, 4961–4973.

[10] Z. Guo, *Blow up, global existence, and infinite propagation speed for the weakly dissipative camassa–holm equation*, Journal of Mathematical Physics 49 (2008), no. 3, 033516.

[11] ———, *Some properties of solutions to the weakly dissipative degasperis–procesi equation*, Journal of Differential Equations 246 (2009), no. 11, 4332–4344.

[12] Q. Hu, *Global existence and blow-up phenomena for a weakly dissipative periodic 2-component camassa-holm system*, Journal of mathematical physics 52 (2011), no. 10, 103701.

[13] S. Lai, *Global weak solutions to the novikov equation*, Journal of Functional Analysis 265 (2013), no. 4, 520–544.

[14] J. Lenells and M. Wunsch, *On the weakly dissipative camassa–holm, degasperis–procesi, and novikov equations*, Journal of Differential Equations 255 (2013), no. 3, 441–448.

[15] J. Liu and Z. Yin, *On the cauchy problem of a two-component b-family system*, Nonlinear Analysis: Real World Applications 12 (2011), no. 6, 3608–3620.

[16] Y. Liu, *Global existence and blow-up solutions for a nonlinear shallow water equation*, Mathematische Annalen 335 (2006), no. 3, 717–735.

[17] Y. Liu and Z. Yin, *Global existence and blow-up phenomena for the degasperis-procesi equation*, Communications in mathematical physics 267 (2006), no. 3, 801–820.

[18] E. Novruzov, *On blow-up phenomena for the weakly dissipative camassa–holm equation*, Journal of Engineering Mathematics 1 (2012), no. 77, 187–195.

[19] C. Shen, *Time-periodic solution of the weakly dissipative camassa-holm equation*, International Journal of Differential Equations 2011 (2011).

[20] S. Wu, J. Escher, and Z. Yin, *Global existence and blow-up phenomena for a weakly dissipative degasperis-procesi equation*, Discrete & Continuous Dynamical Systems-B 12 (2009), no. 3, 633.
[21] S. Wu and Z. Yin, *Blow-up and decay of the solution of the weakly dissipative degasperis–procesi equation*, SIAM Journal on Mathematical Analysis 40 (2008), no. 2, 475–490.

[22] ———, *Global existence and blow-up phenomena for the weakly dissipative camassa–holm equation*, Journal of Differential Equations 246 (2009), no. 11, 4309–4321.

[23] W. Yan, Y. Li, and Y. Zhang, *Global existence and blow-up phenomena for the weakly dissipative novikov equation*, Nonlinear Analysis: Theory, Methods & Applications 75 (2012), no. 4, 2464–2473.

[24] L. Zhang and Z. Qiao, *Global-in-time solvability and blow-up for a non-isospectral two-component cubic camassa-holm system in a critical besov space*, In press (2020).

[25] ———, *The periodic cauchy problem for a two-component non-isospectral cubic camassa-holm system*, Journal of Differential Equations 268 (2020), no. 3, 1270–1305.

[26] M. Zhu and J. Xu, *On the cauchy problem for the two-component b-family system*, Mathematical Methods in the Applied Sciences 36 (2013), no. 16, 2154–2173.

[27] X. Zong, *Properties of the solutions to the two-component b-family systems*, Nonlinear Analysis: Theory, Methods & Applications 75 (2012), no. 17, 6250–6259.

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