Quantum repeated games

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December 23, 2021

Abstract

In a two-stage repeated classical game of prisoners’ dilemma the knowledge that both players will defect in the second stage makes the players to defect in the first stage as well. We find a quantum version of this repeated game where the players decide to cooperate in the first stage while knowing that both will defect in the second.

1 Introduction

The well known simultaneous-move bimatrix game of prisoners’ dilemma (PD) has attracted early attention in the recent studies in quantum game theory. In classical game theory a two-stage repeated version of this game consists of the two players playing the game twice, observing the outcome of the first play before the second play begins. The payoffs for the entire game are simply taken as the sum of the payoffs from the two stages. Generally a two-stage repeated game has more complex strategic structure than its one-stage counterpart and players’ strategic choices in the second stage are affected by the outcome of their moves in the first stage. For the classical one-stage PD game the strategy of ‘defection’ by both the players is well known as a unique Nash Equilibrium (NE). In its two-stage version the same NE appears again at the second stage because the first stage payoffs are added as constants to the second stage. In fact in all of the finitely repeated versions of the PD game the strategy of ‘defection’ by both the players appears as unique NE at every stage.

Recent interesting and important study of the one-stage quantum PD game by Eisert, Wilkens, and Lewenstein makes one to ask a question: what can be a possible role for quantum mechanics when the game is played twice? It appears that this role should be relevant to the new feature showing itself in the game i.e. the two-stages. A role for quantum mechanics exists if it inter-links the two stages of the game in some way of interest. Classically both the players ‘defect’ at each stage and the strategic choices remain the same because of the uniqueness of the NE at each stage. In our search for the quantum role we found
useful to study the idea of subgame-perfect outcome (SGPO) [2] in a two-stage repeated bimatrix game in its quantum form. For a two-stage repeated game the idea of a SGPO is the natural analog of the backwards-induction outcome (BIO) [2] studied in games of complete and perfect information. In a recent paper [3] we considered the BIO idea in a quantum form of duopoly game and showed how entanglement can give an outcome corresponding to static form of the duopoly even when the game is played dynamically. In the present paper we study the natural analogue of BIO for a two-stage repeated PD quantum game, i.e., the idea of SGPO in a situation that can be said to lie in quantum domain. We solve the two-stage PD quantum game in the spirit of backwards induction studied in ref. [3], but now the first step in working backwards from the end of the game involves solving a real game rather than solving a single-person optimization problem as done in ref. [3]. Classically the idea of SGPO comes out as a stronger solution concept especially when multiple NE appear in a stage. Our motivation is the observation that a quantization scheme for the PD game is known where the NE in a stage does not remain unique, thus making relevant a consideration of the concept of SGPO in the two-stage game played in a quantum setting. For the purpose of completeness, we will first describe how SGPO works for the classical two-stage PD game. Afterwards, we quantize the game using a known scheme, and then, show how a SGPO can exist that is counter-intuitive compared to the classical SGPO for the two-stage repeated PD game.

2 Two-stage games of complete but imperfect information

Like the dynamic game of complete and perfect information — for example the Stackelberg duopoly analyzed in ref. [3] — the play in a two-stage game of complete but imperfect information proceeds in a sequence of two stages, with the moves in the first-stage observed before the next stage begins. The new feature is that within each stage there are now simultaneous moves. The simultaneity of moves within each stage means that information is imperfect in the game. A two-stage game of complete but imperfect information consists of the following steps [2]:

1. Players $A$ and $B$ simultaneously choose actions $p$ and $q$ from feasible sets $P$ and $Q$, respectively.

2. Players $A$ and $B$ observe the outcome of the first stage, $(p, q)$, and then simultaneously choose actions $p_1$ and $q_1$ from feasible sets $P$ and $Q$, respectively.

3. Payoffs are $P_i(p, q, p_1, q_1)$ for $i = A, B$.

A usual approach to solve a game from this class uses the method of backwards induction. In ref. [3] the first step in working backwards involves solving
a single-person optimization problem. Now the first step involves solving the
real simultaneous-move game between players A and B in the second stage,
given the outcome from stage one. If the players A and B anticipate that their
second-stage behavior will be given by \((p^*_1(p, q), q^*_1(p, q))\), then the first-stage in-
teraction between players A and B amounts to the following simultaneous-move
game:

1. Players A and B simultaneously choose actions \(p\) and \(q\) from feasible sets \(P\) and \(Q\), respectively.
2. Payoffs are \(P_i(p, q, p^*_1(p, q), q^*_1(p, q))\) for \(i = A, B\).

When \((p^*, q^*)\) is the unique NE of this simultaneous-move game, the
\((p^*, q^*, p^*_1(p, q), q^*_1(p, q))\) is known as the SGPO \(^3\) of this two-stage game.
This outcome is the natural analog of the BIO in games of complete and perfect
information.

3 Two-stage prisoner’s dilemma

3.1 Classical form

We use a normal form of the PD game given by the following matrix \(^1\)

| Alice’s strategy | \(C\) | \(D\) |
|------------------|-----|-----|
| Bob’s strategy   |     |     |
| \(C\)            | (3, 3) | (0, 5) |
| \(D\)            | (5, 0) | (1, 1) |

where \(C\) and \(D\) are for the strategies of ‘cooperation’ and ‘defection’ respectively.
The players play this simultaneous-move game twice. The outcome of the first
play is observed before the second stage begins. The payoff for the entire game
is simply the sum of the payoffs from the two stages. It is a two-stage game of
complete but imperfect information \(^2\). Suppose \(p\) and \(q\) are the probabilities
with which the pure strategy \(C\) is played by the players A and B, respectively,
in the stage 1. Similarly, \(p_1\) and \(q_1\) are the probabilities with which the pure
strategy \(C\) is played by the players A and B, respectively, in the stage 2. We
write \([P_{A1}]_{cl}\) and \([P_{B1}]_{cl}\) as the payoffs to players A and B, respectively, in
the stage 1; where the symbol \(cl\) is for “classical”. These payoffs can be found from
the matrix \(^4\) as

\[
[P_{A1}]_{cl} = -pq + 4q - p + 1, \quad [P_{B1}]_{cl} = -pq + 4p - q + 1 \tag{2}
\]

The NE conditions for this stage are

\[
[P_{A1}(p^*, q^*) - P_{A1}(p, q^*)]_{cl} \geq 0, \quad [P_{B1}(p^*, q^*) - P_{B1}(p^*, q)]_{cl} \geq 0 \tag{3}
\]
giving $p^* = q^* = 0$ (i.e. defection for both the players) as the unique NE in this stage. Likewise, in the second stage the payoffs to players $A$ and $B$ are written as $[P_{A2}]_{cl}$ and $[P_{B2}]_{cl}$ respectively, where

$$[P_{A2}]_{cl} = -p_1 q_1 + 4q_1 - p_1 + 1, \quad [P_{B2}]_{cl} = -p_1 q_1 + 4p_1 - q_1 + 1 \quad (4)$$

and once again the strategy of defection, i.e. $p^*_1 = q^*_1 = 0$, comes out as a unique NE in the second stage. To compute the SGPO of this two-stage game, we analyze the first stage of this two-stage PD game by taking into account the fact that the outcome of the game remaining in the second stage will be the NE of that remaining game —namely, $p^*_1 = q^*_1 = 0$. At this NE the payoffs for the second stage are

$$[P_{A2}(0,0)]_{cl} = 1, \quad [P_{B2}(0,0)]_{cl} = 1 \quad (5)$$

Thus, the players’ first-stage interaction in the two-stage PD amounts to a one-shot game in which the payoff pair $(0, 0)$ for the second stage is added to each first-stage payoff pair. Writing this observation as

$$[P_{A(1+2)}]_{cl} = [P_{A1} + P_{A2}(0,0)]_{cl} = -pq + 4q - p + 2$$
$$[P_{B(1+2)}]_{cl} = [P_{B1} + P_{B2}(0,0)]_{cl} = -pq + 4p - q + 2 \quad (6)$$

It has again $(0,0)$ as the unique NE. Therefore, the unique SGPO of the two-stage PD game is $(0,0)$ in the first stage, followed by $(0,0)$ in the second stage. The strategy of defection in both stages appears as the SGPO for two-stage classical PD game.

We now see how it becomes possible —in a quantum form of this two-stage PD game— to achieve a SGPO in which the players decide to cooperate in the first stage while knowing that they will both defect in the second. The quantum form of the two-stage PD game is played using a system of four qubits. Players’ moves are given by manipulation of these qubits by two unitary and Hermitian operators (identity and inversion operator) in Marinatto and Weber’s scheme [4] to play a quantum form of a matrix game.

### 3.2 Quantum form

A quantum version of a two-stage game must have the corresponding classical two-stage game as a subset [5]. A scheme where this requirement is satisfied via a control of the initial state is the Marinatto and Weber’s idea of playing a quantum version of a matrix game [4]. The scheme was proposed originally to play a quantum form of a one-stage bimatrix game of the battle of sexes. The fundamental idea can be extended to play a two-stage version of a bimatrix game. For example, the two-stage quantum version of the PD game starts by
making available a 4-qubit pure quantum state to the players. This state can be written as

\[ |\psi_{\text{ini}}\rangle = \sum_{i,j,k,l=1,2} c_{ijkl} |ijkl\rangle \quad \text{where} \quad \sum_{i,j,k,l=1,2} |c_{ijkl}|^2 = 1 \quad (7) \]

where \( i, j, k, \) and \( l \) are identifying symbols for four qubits. The upper and lower states of a qubit are 1 and 2 respectively and \( c_{ijkl} \) are complex numbers. It is a quantum state in \( 2 \otimes 2 \otimes 2 \otimes 2 \) dimensional Hilbert space. We suppose the qubits \( i \) and \( j \) are manipulated by the players in the first stage of the game and, similarly, the qubits \( k \) and \( l \) are manipulated in the second stage. Let \( \rho_{\text{ini}} \) denote the density matrix for the initial state (7). Suppose during their moves in the first stage of the game, the players \( A \) and \( B \) apply the identity operator \( I \) on \( |\psi_{\text{ini}}\rangle \) with probabilities \( p \) and \( q \) respectively. The inversion operator \( C \) is, then, applied with probabilities \( (1-p) \) and \( (1-q) \) respectively. The players’ action in the first stage changes \( \rho_{\text{ini}} \) to

\[ \rho_{\text{fin}} = pq I_A \otimes I_B \rho_{\text{ini}} I_A^\dagger \otimes I_B^\dagger + p(1-q) I_A \otimes C_B \rho_{\text{ini}} I_A^\dagger \otimes C_B^\dagger + \\
q(1-p) C_A \otimes I_B \rho_{\text{ini}} C_A^\dagger \otimes I_B^\dagger + (1-p)(1-q) C_A \otimes C_B \rho_{\text{ini}} C_A^\dagger \otimes C_B^\dagger \quad (8) \]

We suppose that the actions of the players in this stage are simultaneous and they remember their moves (i.e. the numbers \( p \) and \( q \)) in the next stage also. In the second stage the players \( A \) and \( B \) apply the identity operator with the probabilities \( p_1 \) and \( q_1 \), respectively, on \( \rho_{\text{fin}} \). The inversion operator \( C \) is, then, applied with probabilities \( (1-p_1) \) and \( (1-q_1) \) on \( \rho_{\text{fin}} \), respectively. Fig. 1 shows the overall idea of playing the two-stage game. One notices that the moves or actions of the players in the two stages of the game are done on two different pairs of qubits.

After the moves of the second stage the quantum state changes to

\[ \rho_{\text{ffin}} = p_1 q_1 I_A \otimes I_B \rho_{\text{fin}} I_A^\dagger \otimes I_B^\dagger + p_1(1-q_1) I_A \otimes C_B \rho_{\text{fin}} I_A^\dagger \otimes C_B^\dagger + \\
q_1(1-p_1) C_A \otimes I_B \rho_{\text{fin}} C_A^\dagger \otimes I_B^\dagger + (1-p_1)(1-q_1) C_A \otimes C_B \rho_{\text{fin}} C_A^\dagger \otimes C_B^\dagger \quad (9) \]

The state \( \rho_{\text{ffin}} \) is now ready for a measurement, giving payoffs for the two stages of the game. If classically the bimatrix game (1) is played at each stage, the possession of the following four payoff operators by the ‘measuring agent’ corresponds to a quantum version of the two-stage game:
Figure 1: Playing a two-stage quantum game of prisoner’s dilemma. \( I \) and \( C \) are unitary and Hermitian operators.

\[
\begin{align*}
\left[(P_A)_{\text{oper}}\right]_1 &= \sum_{k,l=1,2} \{3|11kl\rangle \langle 11kl| + 5|21kl\rangle \langle 21kl| + |22kl\rangle \langle 22kl|\} \\
\left[(P_A)_{\text{oper}}\right]_2 &= \sum_{i,j=1,2} \{3|ij11\rangle \langle ij11| + 5|ij21\rangle \langle ij21| + |ij22\rangle \langle ij22|\} \\
\left[(P_B)_{\text{oper}}\right]_1 &= \sum_{k,l=1,2} \{3|11kl\rangle \langle 11kl| + 5|12kl\rangle \langle 12kl| + |22kl\rangle \langle 22kl|\} \\
\left[(P_B)_{\text{oper}}\right]_2 &= \sum_{i,j=1,2} \{3|ij11\rangle \langle ij11| + 5|ij12\rangle \langle ij12| + |ij22\rangle \langle ij22|\}
\end{align*}
\]

(10)

The corresponding payoffs are, then, obtained as mean values of these operators \( \mathbb{E} \). For example, Alice’s payoff in stage 1 is

\[
[P_{A1}]_{\text{qu}} = \text{Trace} \left\{ \left[(P_A)_{\text{oper}}\right]_1 \rho_{\text{fin}} \right\}
\]

(11)

We consider a two-stage quantum PD game played with an initial state in the form \( |\psi_{\text{ini}}\rangle = c_1|1111\rangle + c_2|1122\rangle + c_3|2211\rangle + c_4|2222\rangle \) with \( \sum_{l=1}^4 |c_l|^2 = 1 \). For this state the payoffs to the players \( A \) and \( B \) in the two stages are found as
from eq. (16) when decide to defect in the second stage:

Similar to the classical analysis, one then finds the players’ payoffs when both

quantum PD amounts to a one-shot game in which the payoff pair $3(p_1^*, q_1^*)$ in the second stage of the quantum game, if

stage, then

recovered from the eq. (12) by making the initial state unentangled and fixing $|c_1|^2 = 1$. The classical game is, therefore, a subset of its quantum version.

One now proceeds —in the spirit of backwards-induction— to find a NE in

the second stage of the quantum game. Suppose $(p_1^*, q_1^*)$ is a NE in the second stage, then

\[ [P_{A1}]_{qu} = (|c_1|^2 + |c_2|^2)(-pq - p + 4q + 1) + \\
( |c_3|^2 + |c_4|^2)(-pq + 2p - 3q + 3) \]

\[ [P_{A2}]_{qu} = (|c_1|^2 + |c_3|^2)(-p_1 q_1 - p_1 + 4q_1 + 1) + \\
( |c_2|^2 + |c_4|^2)(-p_1 q_1 + 2p_1 - 3q_1 + 3) \]

\[ [P_{B1}]_{qu} = (|c_1|^2 + |c_2|^2)(-pq - q + 4p + 1) + \\
( |c_3|^2 + |c_4|^2)(-pq + 2q + 3p + 3) \]

\[ [P_{B2}]_{qu} = (|c_1|^2 + |c_3|^2)(-p_1 q_1 - q_1 + 4p_1 + 1) + \\
( |c_2|^2 + |c_4|^2)(-p_1 q_1 + 2p_1 - 3q_1 + 3) \] (12)

The players’ payoffs in the classical two-stage PD game of eqs. (2,4) can now be recovered from the eq. (13) by making the initial state unentangled and fixing $|c_1|^2 = 1$. The classical game is, therefore, a subset of its quantum version.

With the players’ payoffs of the two stages given by eq. (12), the Nash inequalities (3) can be written as

\[ (p_1^* - p_1) \left\{ -q_1^* + 2(|c_2|^2 + |c_4|^2) - (|c_1|^2 + |c_3|^2) \right\} \geq 0 \]

\[ (q_1^* - q_1) \left\{ -p_1^* + 2(|c_2|^2 + |c_4|^2) - (|c_1|^2 + |c_3|^2) \right\} \geq 0 \] (14)

and the strategy of defection by both the players, i.e. $p_1^* = q_1^* = 0$, becomes a NE in the second stage of the quantum game, if

\[ \left\{ 2(|c_2|^2 + |c_4|^2) - (|c_1|^2 + |c_3|^2) \right\} \leq 0 \] (15)

Similar to the classical analysis, one then finds the players’ payoffs when both decide to defect in the second stage:

\[ [P_{A2}(0,0)]_{qu} = [P_{B2}(0,0)]_{qu} = 3(|c_2|^2 + |c_4|^2) + (|c_1|^2 + |c_3|^2) \] (16)

The classical payoffs -when both players defect- of the eq. (3) can be recovered from eq. (16) when $|c_1|^2 = 1$, i.e. the initial state becomes unentangled.

Like the classical case, the players’ first-stage interaction in the two-stage quantum PD amounts to a one-shot game in which the payoff pair $3(|c_2|^2 + |c_4|^2) + (|c_1|^2 + |c_3|^2) = 0$. 

7
\( |c_4|^2 + (|c_1|^2 + |c_3|^2) \) for the second stage is added to each first-stage payoff pair i.e.

\[
\begin{align*}
[P_{A(1+2)}]_{qu} &= [P_{A1} + P_{A2}(0,0)]_{qu} = |c_1|^2 (-pq + 4q - p + 2) + |c_2|^2 (-pq + 4q - p + 4) + |c_3|^2 (-pq - 3q + 2p + 4) + |c_4|^2 (-pq - 3q + 2p + 6) \\
[P_{B(1+2)}]_{qu} &= [P_{B1} + P_{B2}(0,0)]_{qu} = |c_1|^2 (-pq + 4p - q + 2) + |c_2|^2 (-pq + 4p - q + 4) + |c_3|^2 (-pq - 3p + 2q + 4) + |c_4|^2 (-pq - 3p + 2q + 6)
\end{align*}
\]

(17)

Now the strategy of cooperation \((p^*_1 = q^*_1 = 1)\) becomes a NE for the first-stage interaction in this quantum game, if

\[
\left\{ 2(|c_1|^2 + |c_2|^2) - (|c_3|^2 + |c_4|^2) \right\} \leq 0
\]

(18)

The inequalities (15) and (18) define the conditions to be satisfied when players will decide to cooperate in their first-stage interaction and both will defect in the next stage. These conditions can be rewritten as

\[
|c_1|^2 + |c_2|^2 \leq \frac{1}{3}, \quad |c_2|^2 + |c_4|^2 \leq \frac{1}{3}
\]

(19)

For example, at \(|c_1|^2 = |c_2|^2 = |c_4|^2 = \frac{1}{6}\) and \(|c_3|^2 = \frac{1}{2}\) these conditions hold.

Because for the classical game the inequalities (19) cannot hold together, it shows why classically it is not possible that players cooperate in the first stage knowing that they will both defect in the second.

4 Discussion and conclusion

Classical analysis tells that the repeated games differ from one-shot games because players’ current actions can depend on the past behavior of the other players. In a repeated bimatrix game the same matrix game is played repeatedly, over a number of stages that represent the passing of time. The payoffs are accumulated over time. The accumulation of information about the “history” of the game changes the structure of the game with time. With each new stage the information at the disposal of the players changes and, since strategies transform this information into actions, the players’ strategic choices are affected. If a game is repeated twice, the players’ moves at the second stage depend on the outcome of the first stage. This situation becomes more and more complex as the number of stages increases, since the players can base their decisions on histories represented by sequences of actions and outcomes observed over increasing number of stages.
Recent interesting findings in quantum game theory motivate a study of repeated games in the new quantum settings, because an extensive as well as useful analysis of repeated games already exists in the literature of classical game theory. In present paper —to look for a quantum role in repeated games—we consider a quantum form of a well known bimatrix game called prisoners’ dilemma (PD). The classical analysis of the PD game has been developed in many different formats, including its finitely and infinitely repeated versions. In the history of quantum games the PD game became a focus of an early and important study [1] telling how to play a quantum form of a bimatrix game. We selected a quantum scheme to play this bimatrix game where the players actions or moves consist of selecting positive numbers in the range [0, 1], giving the probabilities with which they apply two quantum mechanical (unitary and Hermitian) operators on an initial 4-qubit pure quantum state [4]. The players’ actions in each stage are done on two different pairs of qubits. The classical two-stage PD game corresponds to an unentangled initial state and the classical SGPO consists of players defecting in both the stages. Our results show that a SGPO where the players go for cooperation in a stage is a non-classical feature that can be made to appear in quantum settings. The argument presented here is based on the assumption that all games, resulting from a play starting with a 4-qubit quantum state of the form of the eq. (7), are ‘quantum forms’ of the classical two-stage game. This assumption originates from the fact that the classical game corresponds to a particular 4-qubit quantum state which is also unentangled. The assumption makes possible to translate the desired appearance of cooperation in a stage to certain conditions on the parameters of the initial state; giving a SGPO where players decide to cooperate in their first-stage interaction while knowing that they both will defect in the next stage.

We are thankful to the anonymous referee who asked about the compelling reason to choose a $2 \otimes 2 \otimes 2 \otimes 2$ dimensional Hilbert space instead of a $2 \otimes 2$ dimensional one. A $2 \otimes 2$ dimensional treatment of this problem, in the same quantization scheme, involves denominator terms in the expressions for payoff operators when these are obtained under the condition that classical game corresponds to an unentangled initial state. It then leads to many ‘if-then’ conditions before one gets finally the payoffs. On the contrary, a treatment in $2 \otimes 2 \otimes 2 \otimes 2$ dimensions becomes much smoother. Also a study of the concept of SGPO in a two-stage repeated quantum game, then, becomes a logical extension of the backwards-induction procedure proposed in the ref. [3].

In conclusion, we found how cooperation in two-stage PD game can be achieved by quantum means. In infinitely repeated versions of the classical PD game it is established that cooperation can occur in every stage of a SGPO, even though the only NE in the stage game is defection [4]. In two-stage PD game to get a SGPO where players cooperate in the first stage is a result with no classical analogue. We have also indicated a possible way to study the concept of SGPO in repeated quantum games.
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