Properties of K-Isomorphism on K-Algebra

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Abstract. K-Algebra \((G, \ast, \circ, e)\) is a system that is built on a group with unit element \((G, \ast)\) and binary operation \(\circ\), that defined \(x \circ y = x \ast y^{-1}\forall x, y \in G\) and satisfies certain axioms. Let \(K_1 = (G_1, \ast, \circ, e_1)\) and \(K_2 = (G_2, \ast, \circ, e_2)\) are K-Algebra. K-Homomorphism on K-Algebra that is a mapping from \(K_1\)-Algebra to \(K_2\)-Algebra that satisfies \(\mu(x_1 \circ y_1) = \mu(x_1) \circ \mu(y_1), \forall x_1, y_1 \in K_1\). Based on the result, K-Isomorphism on K-Algebra is \(\mu\) that satisfies bijective function. By adopting a concept of isomorphism group, it has been proven that some concepts such as theorems and propositions also apply on K-Isomorphism on K-Algebra. If \(\mu : K_1 \rightarrow K_2\) is K-Isomorphism, then \(\mu^{-1} : K_2 \rightarrow K_1\) also K-Isomorphism. Then, apply \(\mu(e_1) = e_2\), and \(\mu(x^n) = [\varphi(x)]^n\forall x \in K_1\) and \(n \in Z^+\). Order of any element in K-Algebra that is positive integer \(n\), so that \(g^n = e\).

1. Introduction

Abstract algebra or commonly called algebraic structure is a non-empty set with binary operations. Some systems in algebra are often studied and known, among others, groups, rings, fields, etc. Besides that, there are still many systems in the algebraic structure, known as K-Algebra. K-Algebra is an algebraic structure built from a group with identity element \(G\) \[1\][2]. K-Algebra is an algebraic structure that is built from a group, so the some properties that apply to the group will also apply to K-Algebra \[3\][4][5]. If in the group there are concepts of Subgroups, Homomorphism, and Isomorphism, then in K-Algebra a similar concept will also apply\[6\]. Dar and Akram regarding K-Algebra built on a group \[1\], K-Algebra characterization\[7\][8], K-Subalgebra \[2\][9] and K -Homomorphism on K-Algebra.

The concept of K-algebra can also be applied to fuzzy, as in the research conducted by Dar and Akram about K-Algebra in the value of the fuzzy interval \[10\] and applied to the topology space\[11\]. In addition, there is also research on the characteristics of K-Homomorphism in K-Algebra which discusses the traits contained in theorems and propositions adopted from the concept of group homomorphism \[12\][13]. However, until now the research that discussed the new Algebra came to the concept of K-Subalgebra which was a concept adopted from the subgroup, and in other studies only reached K-Homomorphism and its characteristics.

Therefore, this paper aims to examine and prove the concept of K-Isomorphism in K-Algebra which is examined from several data in the form of journals, books, and other references related to algebra. The purpose of this research is to find out the definitions and some of the characteristics and theorems related to K-Isomorphism on K-Algebra to complement the algebraic structure of K-Algebra.
2. Literature Study

2.1. Homomorphism

Let \((G, \ast)\) and \((G', \cdot)\) are a group. A Function \(f : G \rightarrow G'\) called group homomorphism, function \(f\) satisfies \(f(x \ast y) = f(x) \cdot f(y), \forall x, y \in G[6]\).

According to [6] there are several definitions of the concept of group homomorphism in the function \(f\), i.e.:

- The mapping \(f\) is called group monomorphism if and only if \(f(x) = f(y)\) then \(x = y, \forall x, y \in G\).
- Function \(f\) is called group epimorphism if for all \(g \in G\) there is \(g \in G\) so \(f(g) = g'\).
- Homomorphism \(f : G \rightarrow G'\) is called isomorphism if \(f\) injective and bijective.

2.2. K-Algebra

Let \((G, \ast)\) is a group. Group \(G\) is defined as operations \(\circ\) such that for all \(x, y \in G, x \circ y = x \ast y^{-1}\) will form a new algebraic structure, \((G, \ast, \circ, e)\), is called K-Algebra if \(G\) is not a group with order-2 and and \(\forall x, y, z \in G\) applies [1][4]:

- \((x \circ y) \circ (x \circ z) = x \circ [(e \circ z) \circ (e \circ y)] \circ x\),
- \(x \circ (x \circ y) = [x \circ (e \circ y)] \circ x\),
- \(x \circ e = e\),
- \(x \circ e = x\),
- \(e \circ x = x^{-1}, \forall x, y, z \in G\).

A non-empty subset \(H\) of K-Algebra \((G, \ast, \circ, e)\) is called K-Subalgebra if [4][14][15]:

- \(e \in H\),
- \(h_1 \circ h_2 \in H, \forall h_1, h_2 \in H\).

2.3. K-Homomorphism and K-Algebra

If there are \(K_1 = (G_1, \ast, \circ, e_1)\) and \(K_2 = (G_2, \ast, \circ, e_2)\) as K-Algebra, then the mapping can be defined from \(K_1\) to \(K_2\). The mapping includes K-Homomorphism in K-Algebra which is explained in the following definition. Mapping \(\varphi\) from K-Algebra \(K_1\) to \(K_2\) is called K-Homomorphism if for every \(x_1, y_1 \in K_1, \varphi(x_1 \circ y_1) = \varphi(x_1) \circ \varphi(y_1)\), with \(\varphi(x_1), \varphi(y_1) \in K_2\) [15][2].

2.4. Example

Let \((G, \circ)\) is K-Algebra and \(H = \{g \circ (g \circ x) | x \in G\}\), where \(H\) is K-Subalgebra from \(G\). Defined mapping \(\varphi : (G, \circ) \rightarrow (H, \circ)\), with \(\varphi(x) = g \circ (g \circ x) \forall x \in G\). It will be shown that \(\varphi : (G, \circ) \rightarrow (H, \circ)\) is a K-homomorphism, because for any element \(x, y \in G, x \circ y \in G\) and

\[
\varphi(x \circ y) = g \circ (g \circ (x \circ y)) = [g \circ e \circ (x \circ y)] \circ g = [g \circ ((e \circ y) \circ (e \circ x))] \circ g = (g \circ (g \circ x)) \circ (g \circ (g \circ y)) = \varphi(x) \circ \varphi(y)
\]

Because of \(\varphi(x \circ y) = \varphi(x) \circ \varphi(y)\), so \(\varphi : (G, \circ) \rightarrow (H, \varphi)\) is a K-Homomorphism.
2.5. Proposition
Let $K_1 = (G_1, *, \circ, e_1)$ and $K_2 = (G_2, *, \circ, e_2)$ and $\varphi : K_1 \rightarrow K_2$ is a K-Homomorphism. If $K_1$ is a K-Algebra, then $\forall x_1, x_2 \in K_1$ applies [15][2]:

- $\varphi(e_1) = e_2$
- $\varphi(x_1^{-1}) = \varphi(x_1)^{-1}$
- $\varphi(x_1 \circ x_1) = e_2 \circ \varphi(x_1)$
- $\varphi(x_1 \circ x_2) = e_2 \leftrightarrow \varphi(x_1) = \varphi(x_2)$

If $H_1$ is a subalgebra from $K_1$, then $\varphi(H_1)$ is a subalgebra from $K_2$.

3. Main Result
The discussion on K-Isomorphism is inseparable from several concepts, including K-Homomorphism, K-Monomorphism, and K-Epimorphism in K-Algebra. Included are injective functions, surjective functions, and wise functions that can be used in constructing a concept of K-Isomorphism in K-Algebra.

The initial concept that must be studied is the concept of K-Monomorphism, K-Epimorphism, the concept is related to the function of injective and surjective, the following is given the definition of K-Monomorphism and K-Epimorphism in K-Algebra.

Homomorphism $\varphi$ from K-Algebra $K_1$ to $K_2$ is called K-Monomorphism in K-Algebra, because it fulfills two conditions, namely K-Homomorphism in K-Algebra and mapping $\varphi$ is injective mapping.

3.1. Example
For example $(G, \ast, \circ, e)$ is a K-Algebra. Given the set $H = \{g \circ (g \circ x) | x \in G\}$, where $H$ is K-Subalgebra from $G$. Defined mapping $\varphi : (G, \circ) \rightarrow (H, \circ)$, with $\varphi(x) = g \circ (g \circ x), \forall x \in G$ is a K-Monomorphism in K-Algebra.

In Example 2.4, it has been shown that mapping $\varphi : (G, \circ) \rightarrow (H, \circ)$ with $\varphi(x) = g \circ (g \circ x), \forall x \in G$ is a K-Homomorphism. Futhermore it must be shown that $\varphi$ is an injective mapping, that is for each $x, y \in G$ with $\varphi(x) = \varphi(y)$ applies $x = y$.

$$\varphi(x) = \varphi(y)$$
$$g \circ (g \circ x) = g \circ (g \circ y)$$

With regard to the definition of K-Algebra, and binary operations on K-Algebra are obtained

$$(g \ast x) \ast g^{-1} = (g \ast y) \ast g^{-1}$$

Next, by operating $g$ with binary operations $\ast$ from the right side of $g^{-1}$ from the left, it is obtained

$$e \ast x = e \ast y$$

Furthermore, taking into account the definition of identity in the group is obtained

$$x = y$$

because it can be shown for every $x, y \in G$ with $\varphi(x) = \varphi(y)$ applies $x = y$, it is proved that $\varphi : (G, \circ) \rightarrow (H, \circ)$ is an injective function.

Mapping $\varphi : (G, \circ) \rightarrow (H, \circ)$ is K-Monomorphism in K-Algebra, because it fulfills two conditions, namely K-Homomorphism in K-Algebra and mapping $\varphi$ is injective mapping.

Homomorphism $\varphi$ from K-Algebra $K_1$ to $K_2$ is called K-Epimorphism if $\varphi$ is an injective mapping.
3.2. Example
For example \((G, \ast, \odot, e)\) a K-Algebra. Given the set \(H = \{g \odot (g \odot x) | x \in G\}\), where \(H\) is K-Subalgebra from \(G\). Next, is defined mapping \(\varphi : (G, \odot) \rightarrow (H, \odot)\), with \(\varphi(x) = g \odot (g \odot x), \forall x \in G\). It will be shown the mapping \(\varphi : (G, \odot) \rightarrow (H, \odot)\) is a K-Epimorphism.

In Example 2.4 and 3.1, it has been shown that mapping \(\varphi : (G, \odot) \rightarrow (H, \odot)\) with \(\varphi(x) = g \odot (g \odot x), \forall x \in G\) is a K-Homomorphism, and \(\varphi\) is a injective function can be referred to as K-Monomorphism. Next to show that the mapping \(\varphi : (G, \odot) \rightarrow (H, \varphi)\) is a K-Epimorphism, then it must be shown that is a surjective function.

Take any element \(y \in H\). To be shown for each \(y \in H, \exists x \in G \ni \varphi(x) = y\).

\[
\begin{align*}
\varphi(x) &= y \\
g \odot (g \odot x) &= y
\end{align*}
\]

Using the K-Algebra definition is obtained

\[
(g \ast x) \ast g^{-1} = y
\]

Furthermore, operating \(g\) with binary operations \(\ast\) from the right obtained

\[
g \ast x = y \ast g
\]

And \(g^{-1}\) from the left so that they are obtained \(e \ast x = g^{-1} \ast (y \ast g)\) Then using the definition of K-Algebra, the following results are obtained.

\[
\begin{align*}
x &= g^{-1} \odot (g^{-1} \ast y^{-1}) \\
x &= g^{-1} \odot (g^{-1} \odot y)
\end{align*}
\]

Because \(H\) is K-subalgebra from K-Algebra \(G\), then we have that \(x = g^{-1} \odot (g^{-1} \odot y)\) is an element of \(H\). Therefore, \(\varphi\) is K-Epimorphism. Homomorphism from K-Algebra \(K_1\) to \(K_2\) is called K-Isomorphism if \(\varphi\) is a wise (injective and surjective) mapping.

In Example 2.4 and 3.1, it has been shown that mapping \(\varphi : (G, \odot) \rightarrow (H, \odot)\) with \(\varphi(x) = g \odot (g \odot x), \forall x \in G\) is a K-Homomorphism, and \(\varphi\) is an injective function. Furthermore, in Example 3.2 it’s a K-Homomorphism, and \(\varphi\) is a wise function, it is proven that \(\varphi : (G, \ast, \odot, e) \rightarrow (H, \ast, \odot, e)\) with \(\varphi(x) = g \odot (g \odot x), \forall x \in G\) is a K-Isomorphism

3.3. Definition
Given K-Algebra \((G, \ast, \odot, e)\). The order element \(g \in (G, \odot)\), is the smallest positive integer \(n\), such that \(g^n = e\), with \(g^n = g \odot g \odot \cdots \odot g\).

3.4. Characteristic of K-Isomorphism in K-Algebra
In K-Isomorphism there are several theorems and traits derived from the concept of Isomorphism in the Group. The following is the theorem and nature of K-Isomorphism in K-Algebra.

3.4.1. Theorem
Given a K-Algebra \(K_1 = (G_1, \ast, \odot, e_1)\) and \(K_2 = (G_2, \ast, \odot, e_2)\). If \(\varphi : K_1 \rightarrow K_2\) is K-Isomorphism, then \(\varphi^{-1} : K_2 \rightarrow K_1\) is also K-Isomorphism.

Proof
Given that \(\varphi : K_1 \rightarrow K_2\) is K-isomorphism so \(\varphi\) is a wise function, such that \(\varphi^{-1}\) also wise function. Furthermore, it must be shown that \(\varphi^{-1}\) is K-Homomorphism.

Because of \(\varphi : K_1 \rightarrow K_2\) is a wise function, if taken any \(u, v \in K_2\), then there are \(a, b \in K_1\), such
that $\varphi(a) = u$ and $\varphi(b) = v$. As a result $\varphi^{-1} : K_2 \to K_1$, obtained $\varphi^{-1}(u) = a$ and $\varphi^{-1}(v) = b$. Note that

$$u \odot v = u * v^{-1} = \varphi(a) * (\varphi(b))^{-1} = \varphi(a) \odot \varphi(b) = \varphi(a \odot b)$$

Because of $\varphi(a \odot b) = u \odot v$, it result in $\varphi^{-1}(u \odot v) = a \odot b$. By considering $\varphi^{-1}(u) = a$ and $\varphi(v) = b$, obtained

$$\varphi^{-1}(u \odot v) = a \odot b = \varphi^{-1}(u) \odot \varphi^{-1}(v)$$

Because it has been shown that $\varphi^{-1}$ wise mapping and $\varphi^{-1} : K_2 \to K_1$ is K-Homomorphism, it is proved that $\varphi^{-1}$ is K-Isomorphism.

3.4.2. Theorem Suppose $K_1 = (G_1, *, \odot, e_1)$ and $K_2 = (G_2, *, \odot, e_2)$ are K-Algebra. If $\varphi : K_1 \odot K_2$ is a K-Isomorphism, then

(i) $\varphi(e_1) = e_2$
(ii) $\varphi(x^n) = (\varphi(x))^n$, for $n$ positive integer.

Proof
Mapping $\varphi : K_1 \to K_2$ is a K-Isomorphism such that $\varphi$ K-Homomorphism in K-Algebra. Suppose that $e_1$ dan $e_2$ successively declare the identity from $K_1$ and $K_2$ against binary operation $\odot$. Will shown $\varphi(e_1) = e_2$.
Take any $x \in K_1$. Because it is known that $\varphi$ is K-Isomorphism, then for each $y \in K_2$ applies that

$$\varphi(x) = y$$

Note that $x \odot e_1 = x$ and

$$\varphi(x \odot e_1) = y$$

Because of $\varphi : K_1 \odot K_2$ is a K-Homomorphism, then obtained

$$\varphi(x) \odot \varphi(e_1) = \varphi(x)$$
$$\varphi(x) * \varphi(e_1)^{-1} = \varphi(x)$$
$$\varphi(x)^{-1} * \varphi(x) = \varphi(e_1)^{\{1} - 1 \right) = \varphi(x)^{-1} * \varphi(x)$$
$$\left[ \varphi(x)^{-1} * \varphi(x) \right] * \varphi(e_1)^{\{1} - 1 \right) = \varphi(x)^{-1} * \varphi(x)$$

Because of $\varphi(x)^{-1}$ and $\varphi(x)$ in $K_2$, then

$$[\varphi(x)^{-1} * \varphi(x)] * \varphi(e_1)^{-1} = e_2$$
$$e_2 * \varphi(e_1)^{-1} = e_2$$
$$\varphi(e_1)^{-1} = e_2$$
$$\varphi(e_1)^{-1} = e_2$$
$$\varphi(e_1) = (e_2)^{-1}$$
$$\varphi(e_1) = e_2$$
So it is proved that $\varphi(e_1) = e_2$.

It will be proved $\varphi(x^n) = \varphi(x)^n$ by using mathematical induction.

For $n = 1$, the statement is true, because

$$\varphi(x^1) = (\varphi(x))^1$$

For example, it is true for $n = k$, i.e

$$\varphi(x^k) = \varphi(x)^k$$

It will also be shown correctly for $n = (k + 1)$, that is

$$\varphi(x^{(k + 1)}) = \varphi(x^k \odot x) \quad \text{(Definition)}$$

$$= \varphi(x^k) \odot \varphi(x) \quad \text{(K - Homomorphism)}$$

$$= \varphi(x)^k \odot \varphi(x) \quad \text{(Proposition)}$$

$$= \varphi(x)^{(k + 1)} \quad \text{(Definition)}$$

So, it is proven $\varphi(x^n) = (\varphi(x))^n$.

3.4.3. **Theorem** Given K-Algebra $(G, \ast, \odot, e)$. For any $g \in G$, order $g$ is an even positive integer $n$, such that $g^n = e$.

**Proof**

For example, even integer $n$ is positive, it can be written with $n = 2k$ with $k \in Z$. Take any $g \in G$, mathematical induction will be used to show that the order of $g$ is an even positive integer $n$, such that $g^n = e$.

i For $k = 1, n = 2k = 2$, is true because $g^n = g^2 = e$.

ii Suppose that is true for $k = k, n = 2k$ that is $g^n = g^{2k} = e$.

iii Will be shown correctly for $k = m + 1, n = 2(m + 1)$, note that

$$g^n = g^{2(m+1)}$$

$$= g^{2m} \odot g^2$$

$$= e \odot e$$

$$= e$$

4. **Conclusion**

Based on the explanation in the discussion it can be concluded that K-Homomorphism $\varphi$ from K-Algebra $K_1$ to $K_2$ is called K-Isomorphism if $\varphi$ is a injective and surjective function. A K-Algebra $K_1 = (G_1, \ast, \odot, e_1)$ and $K_2 = (G_2, \ast, \odot, e_2)$ is a K-Algebra, if $\varphi : K_1 \rightarrow K_2$ K-Isomorphism, then $\varphi^{-1} : K_2 \rightarrow K_1$ is also K-Isomorphism. K-Isomorphism also has properties, i.e. let $K_1 = (G_1, \ast, \odot, e_1)$ and $K_2 = (G_2, \ast, \odot, e_2)$ are K-Algebra. If $\varphi : K_1 \rightarrow K_2$ is a K-Isomorphism, then for all $x \in K_1$ applies: $\varphi(e_1) = e_2$, and $\varphi(x^n) = (\varphi(x))^n$. As a result, there is an order theorem in K-Isomorphism, which given K-Algebra $(G, \ast, \odot, e)$. For any $g \in (G, \odot)$, order of $g$ is a positive integer $n$ such that $g^n = e$.

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