Two-axes pseudo-Finsleroid metrics: general overview and angle-regular solution

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Abstract

The class of the two-axes pseudo-Finslerian metrics which is specified by the condition of the angle-separation in the involved characteristic functions is proposed and studied. The complete Total Set of algebraic and differential equations is derived in all rigor which are necessary and sufficient in order that a pseudo-Finsleroid metric function belong to the class. It proves possible to solve the equations of the set. The angle-regular solution of the Finsleroid-in-pseudo-Finsleroid type is found and described in detail.

Keywords Finsler geometry - Finsler metrics - Metric spaces

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1 Introduction

The applied ability of the methods of the Finsler Geometry is proportional to the variety of classes of the Finsler metric functions elaborated to reflect various violations of the spherical symmetry presupposed in the Euclidean and Riemannian Geometries. Among possible violations, the substitution of the axial symmetry with the spherical symmetry is the simplest case. Despite the axial symmetry is widely appeared in theoretical and applied sciences, in many patterns the axial symmetry is fulfilled but approximately. The two-axes asymmetry can well be regarded as the much capable idea to proceed with. Do the pseudo-Finsleroid metric functions possessing such an asymmetry exist when high regularity properties are assumed?

We shall make a systematic search to answer the question, following the method of introducing the angle dependence and assuming the separation of angle dependencies, that was proposed and applied in the previous work [1-2].

Historically, Minkowski [3,4] is well-known to have remarked that the Special Relativity implies introducing the pseudo-Euclidean metric to geometrize the space-time. After that, the pseudo-Euclidean geometry became raising its influence on the philosophy as well as computational methods of relativistic theories [5]. Beyond the square-root metric one may apply the ingenious methods of the Finsler geometry, including the classical methods [6-9] as well as the recent and modern methods (see [9-15] and numerous references therein).

We call a direction in the tangent space of a Finsler space geometrically distinguished if the direction makes trace in the structure of the Finsler metric function. Our goal in the present paper is to derive and develop the class of four-dimensional pseudo-Finsleroid metric functions which involve two such directions, respectively of the vertical and horizontal geometrical meaning, specifying the class by the condition of angle-separated dependence of the characteristic functions.

Like to the preceding work [1,2], we shall construct the pseudo-Finsleroid space on the four-dimensional pseudo-Riemannian space, to be denoted by \( \mathcal{R}_4 \). To this end we expand the pseudo-Riemannian metric tensor \( \{a_{ij}(x)\} \) of the space \( \mathcal{R}_4 \) with respect to an orthonormal vector frame \( \{b_i(x), i(x), j_i(x), i_{\{3\}i}(x)\} \), such that

\[
a_{ij} = b_ib_j - i_i i_j - j_i j_j - i_{\{3\}i}i_{\{3\}j}.
\]

(1.1)

Respective 1-forms will be denoted by \( b = b_i y^i, i = i_i y^i, j = j_i y^i, i_{\{3\}} = i_{\{3\}i}y^i \). The tensorial indices will be raised by means of the tensor \( \{a^{ij}(x)\} \) reciprocal to \( \{a_{ij}(x)\} \), such that \( a_{mn}a^{mj} = \delta^i_j, b^i = a^{in}b_n, etc \). The consideration will be of local nature. By \( F = F(x, y) \) we shall denote a pseudo-Finslerian metric function.

Definition 1.1 The pseudo-Finslerian cone is

\[
\mathcal{C}_x = \{ y \in T_xM : F(x, y) = 0 \},
\]

which defines the hypersurface \( \mathcal{C}_x \subset T_xM \). If a vector \( y \in T_xM \) belongs to the cone, the vector is called isotropic. The vectors \( y \in T_xM \) inside the cone \( \mathcal{C}_x \) are called b-like.

This \( \mathcal{C}_x \) generalizes the conventional pseudo-Euclidean cone.

Below we confine our consideration to the b-like region of the tangent space, always assuming \( b > 0 \). Inside the cone \( \mathcal{C}_x \), the function \( F \) is assumed to be positive and positively homogeneous with respect to tangent vectors, namely for any admissible \( y \in T_xM \) and for an arbitrary nonnegative \( t \) we have \( F(x, ty) = tF(x, y) \).
Definition 1.2 The hypersurface $\mathcal{I}_x \subset T_x M$ introduced by

$$\mathcal{I}_x = \{y \in T_x M : F(x, y) = 1\}$$

is called the pseudo-Finslerian indicatrix.

In the pseudo-Riemannian limit, the indicatrix $\mathcal{I}_x$ reduces to the pseudo-Euclidean hyperboloid.

The straight line to which the vector $b^i(x)$ belongs in the tangent space $T_x$ will play the role of the axis of the indicatrix $\mathcal{I}_x$. This axis will be interpreted as vertical. Additionally, $i^i_{\{i\}}(x)$ will be interpreted as the vector which assigns the horizontal axis of the indicatrix $\mathcal{I}_x$.

In Sect. 2, we introduce the angle representations for the Finslerian metric tensor $g_{ij}$ and unit vector $l^i$. We also construct the orthonormal frame $\{l_i, u_i, m_i, p_i\}$, obtain the tensor $C_{ijn} = (1/2)\partial g_{ij}/\partial y^n$, and propose the General Expansions of angle derivatives. These objects will play a fundamental role in the consideration performed in the subsequent sections.

In Sect. 3, after introducing the notion of the separation of angle dependence for the characteristic functions which enter the pseudo-Finsleroid metric function, we systematically derive the Total Set of algebraic and differential equations required to find solutions for the metric functions $F$ under study. To this end, the Particular Expansions of angle derivatives are derived which involve many simplifications as compared to the General Expansions introduced in Sect. 2.

For existence of the functions $\eta = \eta(x, y), \theta = \theta(x, y), \phi = \phi(x, y)$ which enter the Particular Expansions the integrability conditions should be fulfilled which involve the skew-vanishing lists $S_1, S_2, S_3$ which we shall find.

The structural conditions of the studied class of the pseudo-Finsleroid metric functions give rise to the additional equations written out in the First Group, in the Second Group, and in the Third Group.

After that, we evaluate the expansion of the tensor $C_{ijn}$ in terms of the orthonormal vector frame. For the symmetry of the tensor, the Symmetrizing Conditions should be set forth to specify the entered coefficients. Formula (3.36) represents the resultant expansion, which is truly convenient to use when evaluating the curvature tensor $\hat{R}_{jppn} = C^h_{pq}C_{hjn} - C^h_{pn}C_{hjq}$ in the tangent space. Subjecting the tensor $\hat{R}_{jppn}$ to the requirement of the curvature constancy of indicatrix entails the additional conditions, namely (3.39)-(3.41), and the value $\{-H^2\}$ for the curvature is assigned by Formulas (3.43) and (3.44). The inequality $H \geq 1$ is always implied. The Imperative Theorem is the completer assertion in Sect. 3. The angle separation (3.1)-(3.5) introduced complies with the separation presupposed in [1,2].

In Sect. 4, the method of solving the equations entered the Total Set is proposed and developed. The possibility to successfully proceed is to note that the equation triple (4.1) entails the convenient ordinary first-order partial differential equation (4.2) from which the involved key function $\hat{L}$ can readily be found (Formula (4.5)). The verification of this implication is the one page evaluation, in the process of which the differentiation of the second equality of (4.1) with respect to $\eta$ is the first step. The first member of the triple has been taken from the integrability condition $S_1$, Formula (3.14), and two other members reflect the constancy of curvature of indicatrix (see Formulae (3.43)-(3.44)). The obtained function $\hat{L}$ involves one constant of integration, which we denote by $P$.

The solutions for $\hat{L}$ are divided in three different classes: 

- Class I: $0 < P < 1$; 
- Class II: $P > 1$; 
- Class III: $P < 0$. 

Additionally, the two constants $C$ and $C_7$ arise when the separation of variables is performed in the equations of the Second Group and of the First Group obtained in Sect. 3.

Thus the pseudo-Finsleroid space under consideration involves three geometrically intrinsic scalars $P = P(x), C = C(x), C_7 = C_7(x)$ (which are constants in each tangent space $T_x$).

The made observations permit us to find dependence of the characteristic functions $V$ and $r$ on $\eta$, according to Assertion 4.2 formulated to the end of Sect. 4.

In Sect. 5, the $\eta$-regular solution of the \textit{F$^{\text{REG-FRD-IN-PSEUDO-FRD}}$}-type will be proposed and described. To this end we choose the Class II, assuming that $P > 1$. We shall use the notation $\tilde{C} = 1/C$, where $C$ is the constant of Separation of Variables that was arisen in the representation $\dot{\phi}_t \dot{\phi}_t = CZ_t/Z$ (see (4.9) in Sect. 4), and introduce the constant $T$ by the help of the equality $P = 1/T \tilde{C}$. The appropriate idea is to subject the introduced constants to the inequalities $T > 1$, $0 < \tilde{C} < 1$, $T \tilde{C} < 1$. The identification $C_7 = 1/P$ is made.

The dependence on $\eta$ can explicitly be found for the characteristic functions $V = \tilde{V}(x, \eta)$ and $r = \tilde{r}(x, \eta)$. The obtained Formulas (5.4) and (5.14) representing the functions $V = \tilde{V}(x, \eta)$ and $r = \tilde{r}(x, \eta)$ manifest clearly the property of $\eta$-regularity, namely that the functions are smooth of class $C^\infty$ with respect to $\eta$ over the total definition range $0 < \eta < \infty$.

This observation is geometrically of the vertical meaning and is not affected anyway by the geometry in the horizontal sections, namely by the form of dependence of other characteristic functions \{U = \tilde{U}(x, \theta), f = \tilde{f}(x, \theta), Z = \tilde{Z}(x, \phi), t = \tilde{t}(x, \phi)\} on $\theta$ and $\phi$. However, it proves possible to subject the geometry to the requirement of constancy of curvature of the indicatrices in the sectional horizontal spaces. To this end, we assume the particular form (5.19) for the involved function $R_2$, which leads to success. Indeed, the dependence of functions \{U = \tilde{U}(x, \theta), f = \tilde{f}(x, \theta)\} on $\theta$ is found explicitly, according to (5.21) and (5.24). Next, the simple dependence (see (5.28)) of the function $t = \tilde{t}(x, \phi)$ on $\phi$ is proposed which fulfills the property that the geometry is $i_{[3]}$-axial, thereafter derivation the dependence of the function $Z = \tilde{Z}(x, \phi)$ on $\phi$ proves to be possible (see Formula (5.29)). With these solutions at hands, the attentive evaluation of the curvature of the indicatrices in the sectional horizontal spaces lead to the striking result that the curvature is constant and has the simple value (5.56). The clear expression (5.57) is indicated for the determinant of the involved metric tensor of the horizontal sections, which reveals the properties of the $\eta$-regularity and positive-definiteness. The results of the evaluations performed are shortly summarized in Assertion 5.3, which presents the complete list (5.58) of the characteristic differential equations required.

In Conclusions, several important remarks have been made.

Appendix A has been added to show how the Skew-Vanishing Lists of Sect. 3 can be verified by performing attentive evaluation.

Structurally, the evidence of the property of constant negative curvature for the indicatrix of the pseudo-Finsleroid space under study is supported by the equalities (3.39)-(3.44) formulated in Sect. 3. In Appendix B we derive the equalities upon evaluating the respective curvature tensor.

2 Angle representation

We refer the consideration to the dimension $N = 4$ and assume the signature sign($g_{ij}$) = (+ − − −) for the Finslerian metric tensor $g_{ij}$.
Primary Theorem Given a pseudo-Finsleroid metric function $F$. The entailed Finslerian metric tensor admits the representation

$$g_{ij} = l_i l_j - \frac{1}{H^2} \left( \eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \phi_i \phi_j) \right) F^2,$$  \hspace{1cm} (2.1)

where $\eta(x,y), \theta(x,y), \phi(x,y)$ are three scalars homogeneous of degree zero with respect to the argument $y$, and $l_i = \partial F / \partial y^i$.

The notation $\{ \eta_i = \partial \eta / \partial y^i, \theta_i = \partial \theta / \partial y^i, \phi_i = \partial \phi / \partial y^i \}$, and $g_{ij} = (1/2) \partial F^2 / \partial y^i \partial y^j$ is used. The tensor $h_{ij} = g_{ij} - l_i l_j$ can be given by the formula

$$h_{ij} = -\frac{1}{H^2} \left( \eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \phi_i \phi_j) \right) F^2.$$  \hspace{1cm} (2.2)

The validity of the theorem can primarily be recognized on the basis of the differential geometry of indicatrix. Such a geometry was systematically described in [7], applying the method of parametrical representation of the indicatrix and using the respective projection factors. Namely, the Finslerian indicatrix $I_x \subset T_x$ is considered as a hypersurface defined by the equation $F(x,y) = 1$ in the tangent space $T_x$. In the present paper, we consider the $(N=4)$-dimensional case, so that the indicatrix is three-dimensional. Let $U_x \subset I_x$ be an open region on the indicatrix. Fixing a point $x \in M$ and parameterizing the region by a set of three parameters $\{U^a\} = (U^1, U^2, U^3)$ assign the parametric representation $l^i = t^i(x, U)$ to the unit tangent vectors $l^i = y^i / F$. With the help of the derivatives $t^i_a = \partial t^i / \partial U^a$ the indicatrix metric tensor $i_{ab}(x, U)$ is defined by the equality $i_{ab} = -t^i_a t^i_b h_{ij}$ (we have inserted the minus $\{-\}$ to respond to the signature (+ - - -) implied in the present paper). The parameters $U^a$ play the role of coordinates for the tensor $i_{ab}$ which can geometrically be interpreted as a Riemannian metric tensor on the indicatrix.

Therefore, if we assume that the tensor $i_{ab}(x, U)$ corresponds to the constant negative curvature, we can use various known properties of the pseudo-Riemannian geometry. Among them, there exists the possibility of the choice $\{U^1 = \eta, U^2 = \theta, U^3 = \phi\}$ with $\eta, \theta, \phi$ denoting hyperbolic angles. With this choice, the components of the tensor $i_{ab}(x, U)$ are given by the list

$$i_{11} = \frac{1}{H^2}, \hspace{0.5cm} i_{22} = \frac{1}{H^2} \sinh^2 \eta, \hspace{0.5cm} i_{33} = \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta,$$  \hspace{1cm} (2.3)

with the non-diagonal components vanished identically. The factor $1/H^2$ reflects the property that the tensor corresponds to the curvature value $\{-H^2\}$, which we assume in the present paper. Applying this method of parametrization of indicatrix, the unit tangent vectors $l^i = y^i / F$ are given parametrically as

$$l^i = t^i(x, \eta, \theta, \phi).$$  \hspace{1cm} (2.4)

Definition 2.1 Formula (2.1) assigns the angle representation of the Finslerian metric tensor $g_{ij}$, and Formula (2.4) introduces the angle representation of the unit vector $l^i$.

Also, the square of the length element on the indicatrix

$$ds^2 = \frac{1}{H^2} \left( (d\eta)^2 + \sinh^2 \eta \left( (d\theta)^2 + \sin^2 \theta (d\phi)^2 \right) \right)$$  \hspace{1cm} (2.5)

is obtained.
From (2.1) we obtain the expansion
\[ g_{ij} = l_i l_j - u_i u_j - m_i m_j - p_i p_j, \quad (2.6) \]
where the orthonormal frame \( \{l_i, u_i, m_i, p_i\} \) consists of the unit vector \( l_i \) and the vectors
\[ u_i = \frac{1}{H} F \eta_i, \quad m_i = \frac{1}{H} F \sinh \eta \theta_i, \quad p_i = \frac{1}{H} F \sinh \eta \sin \theta \phi_i. \quad (2.7) \]

By differentiation, from (2.1) we arrive at the following angle representation of tensor \( C_{ijn} = (1/2) \partial g_{ij} / \partial y^n \):
\[ 2C_{ijn} = \frac{1}{F} (h_i n l_j + l_i h_j n) - \frac{2F}{H^2} l_n \left( \eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j) \right) \]
\[- \frac{F^2}{H^2} (\eta_i n \eta_j + \eta_i \eta j n) \]
\[- \frac{F^2}{H^2} (\theta_i n \theta_j + \theta_i \theta j n + 2 \sin \theta \cos \theta \eta \phi_i \phi_j + \sin^2 \theta (\phi_i \phi_j + \phi_i \phi j n)) \sinh^2 \eta \]
\[- \frac{2F^2}{H^2} \sinh \eta \cosh \eta \eta_i (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j), \]
or
\[ 2C_{ijn} = \frac{1}{F} (h_i n l_j + l_i h_j n - 2h_i n) - \frac{F^2}{H^2} (\eta_i n \eta_j + \eta_i \eta j n) \]
\[- \frac{F^2}{H^2} (\theta_i n \theta_j + \theta_i \theta j n + 2 \sin \theta \cos \theta \eta \phi_i \phi_j + \sin^2 \theta (\phi_i \phi_j + \phi_i \phi j n)) \sinh^2 \eta \]
\[- \frac{2F^2}{H^2} \sinh \eta \cosh \eta \eta_i (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j). \quad (2.8) \]

Since the orthonormal frame \( \{l_i, u_i, m_i, p_i\} \) is complete and the partial derivatives of \( \eta_i, \theta_i, \phi_i \) with respect \( y^j \) are symmetric, we can propose the following General Expansions of angle derivatives:
\[ \eta_{ij} = -\frac{1}{F} (l_i \eta_j + l_j \eta_i) + u_1 \frac{1}{H^2} \sinh \eta \sin (\phi_i \eta_j + \phi_j \eta_i) + u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + u_3 \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j \]
\[ + u_4 \frac{1}{H^2} \sinh \eta (\theta_i \eta_j + \theta_j \eta_i) + u_5 \frac{1}{H^2} \sinh^2 \eta \sin (\phi_i \theta_j + \phi_j \theta_i) + u_6 \frac{1}{H^2} \eta_i \eta_j. \quad (2.9) \]
\[ \theta_{ij} = -\frac{1}{F} (l_i \theta_j + l_j \theta_i) + z_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta (\phi_i \theta_j + \phi_j \theta_i) + z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \phi_i \phi_j + z_3 \frac{1}{H^2} \sin^2 \eta \theta_i \theta_j \]

\[ + z_4 \frac{1}{H^2} \sinh \eta (\eta_i \theta_j + \eta_j \theta_i) + z_5 \frac{1}{H^2} \sinh \eta \sin \theta (\phi_i \eta_j + \phi_j \eta_i) + z_6 \frac{1}{H^2} \eta_i \eta_j, \quad (2.10) \]

and

\[ \phi_{ij} = -\frac{1}{F} (l_i \phi_j + l_j \phi_i) + r_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta (\phi_i \theta_j + \phi_j \theta_i) + r_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \phi_i \phi_j + r_3 \frac{1}{H^2} \sin^2 \eta \theta_i \theta_j \]

\[ + r_4 \frac{1}{H^2} \sinh \eta (\eta_i \theta_j + \eta_j \theta_i) + r_5 \frac{1}{H^2} \sinh \eta \sin \theta (\phi_i \eta_j + \phi_j \eta_i) + r_6 \frac{1}{H^2} \eta_i \eta_j. \quad (2.11) \]

The coefficients \( u_1, u_2, \ldots, r_6 \), which are functions of \( x \) and of the triple \( \{ \eta, \theta, \phi \} \) bear the essential information on the Finsler metric functions that the expansions are referred to. To be applicable, the expansions must be subjected to the conditions of integrability, which require obviously that the skew-symmetrization of partial derivatives of the right-hand parts in the expansions yields always zero.

3 Separation of angle dependence

Henceforth, we assume for the pseudo-Finslerian metric functions \( F = F(x, y) \) of the considered type the particular structure specified by the condition of separation of angle dependence

\[ V = \bar{V}(x, \eta), \quad r = \bar{r}(x, \eta), \quad U = \bar{U}(x, \theta), \quad f = \bar{f}(x, \theta), \quad Z = \bar{Z}(x, \phi), \quad t = \bar{t}(x, \phi) \quad (3.1) \]

of the characteristic functions \( \{ V, r, U, f, Z, t \} \) which enter the functions \( F \) in accordance with the following Separation Scheme:

\[ F = bV(x, r), \quad r = zU(x, f), \quad f = c_2 Z(x, t) \quad (3.2) \]

with

\[ z = w_3, \quad c_1 = \frac{w_1}{z}, \quad c_2 = \frac{w_2}{z}, \quad w_1 = \frac{i}{b}, \quad w_2 = \frac{j}{b}, \quad w_3 = \frac{i}{b}, \quad t = \frac{w_1}{w_2} = \frac{c_1}{c_2} = \frac{i}{j}. \quad (3.3) \]

Here, \( b \) is the 1-form introduced in Sect. 1.

It follows that

\[ \bar{V}(x, \eta) = V(x, \bar{r}(x, \eta)), \quad \bar{U}(x, \theta) = U(x, \bar{f}(x, \theta)), \quad \bar{Z}(x, \phi) = Z(x, \bar{t}(x, \phi)), \quad (3.4) \]

and
\[ \eta = \bar{\eta}(x, V), \quad \theta = \bar{\theta}(x, f), \quad \phi = \bar{\phi}(x, t). \] (3.5)

The structural conditions (3.2)-(3.3) specify the dependence of the characteristic functions on tangent vectors \( \{y^i\} \), and therefore the form of derivatives of the functions with respect to \( y^i \). In particular, we obtain the expansions

\[ l_i = b_i V + bV_i, \quad V_i = V_\eta \eta_i, \quad \eta_i = \eta_i V, \quad \eta_{ij} = \eta_{ij} V_i V_j + \eta_i V_{ij}, \] (3.6)

and

\[ V_i = V_i r_i, \quad V_{ij} = V_{ir} r_i r_j + V_i r_{ij}, \] (3.7)

etc, where the subscripts in \( V, \eta, r \) mean differentiations; \( l_i = \partial F / \partial y_i \).

The following assertion is of the fundamental importance for our subsequent analysis.

**Assertion 3.1** Assuming the separated angle dependence (3.1) reduces the General Expansions (2.9)-(2.11) to the following Particular Expansions of angle derivatives:

\[ \eta_{ij} = -\frac{1}{F} (l_i \eta_j + l_j \eta_i) + u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + u_3 \frac{1}{H^2} \sinh^2 \eta \theta_j + u_6 \frac{1}{H^2} \eta_i \eta_j, \] (3.8)

\[ \theta_{ij} = -\frac{1}{F} (l_i \theta_j + l_j \theta_i) + z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + z_3 \frac{1}{H^2} \sinh^2 \eta \theta_j + z_4 \frac{1}{H^2} \sinh \eta (\eta_j \theta_i + \eta_i \theta_j), \] (3.9)

and

\[ \phi_{ij} = -\frac{1}{F} (l_i \phi_j + l_j \phi_i) + r_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta (\phi_i \theta_j + \phi_j \theta_i) + r_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j \]

\[ + r_5 \frac{1}{H^2} \sinh \eta \sin \theta (\phi_i \eta_j + \phi_j \eta_i). \] (3.10)

The verification of the validity of the assertion, namely that the separation of angle dependence implies the nullifications

\[ u_1 = u_4 = u_5 = z_1 = z_5 = z_6 = r_3 = r_4 = r_6 = 0 \] (3.11)

in (2.10)-(2.12), involves calculations which are lengthy, although simple. They are not reproduced in the present paper.

The following assertion is valid.

**Assertion 3.2** For existence of the functions \( \eta = \eta(x, y), \theta = \theta(x, y), \phi = \phi(x, y) \) which obey the set (3.8)-(3.10) of Particular Expansions it is necessary and sufficient that the functions fulfill the conditions \( S_1, S_2, S_3 \) listed below.

The Skew-Vanishing Lists \( S_1, S_2, S_3 \) are obtainable respectively from the integrability conditions

\[ \frac{\partial \eta_{ij}}{\partial y^n} - \frac{\partial \eta_{jn}}{\partial y^j} = 0, \quad \frac{\partial \theta_{ij}}{\partial y^n} - \frac{\partial \theta_{jn}}{\partial y^j} = 0, \quad \frac{\partial \phi_{ij}}{\partial y^n} - \frac{\partial \phi_{jn}}{\partial y^j} = 0. \] (3.12)
Namely, when the general expansions

\[ u_{2n} = u_{2\eta} \eta_n + u_{2\theta} \theta_n + u_{2\phi} \phi_n, \quad u_{3n} = u_{3\eta} \eta_n + u_{3\theta} \theta_n + u_{3\phi} \phi_n, \quad u_{6n} = u_{6\eta} \eta_n + u_{6\theta} \theta_n + u_{6\phi} \phi_n \]

are inserted in (3.8), we come to the following list.

**The Skew-Vanishing List \( S_1 \):**

\[ u_{2\theta} = u_{2\phi} = u_{6\theta} = u_{6\phi} = 0, \quad (3.13) \]

\[ u_{2\eta} + \frac{1}{\text{H}^2} u_2(u_2 - u_6) = 1, \quad (3.14) \]

and

\[ z_{4\theta} = z_{4\phi} = 0, \quad (3.15) \]

Next, with the derivative expansions

\[ z_{2n} = z_{2\eta} \eta_n + z_{2\theta} \theta_n + z_{2\phi} \phi_n, \quad z_{3n} = z_{3\eta} \eta_n + z_{3\theta} \theta_n + z_{3\phi} \phi_n, \quad z_{4n} = z_{4\eta} \eta_n, \]

from (3.9) we obtain the following equations.

**The Skew-Vanishing List \( S_2 \):**

\[ z_{3\phi} = 0, \quad z_{2\eta} \sinh \eta + 2z_2 \cosh \eta = 0, \quad z_{3\eta} \sinh \eta + 2z_3 \cosh \eta = 0, \quad (3.16) \]

\[ z_{2\theta} - z_4u_2 \frac{1}{\text{H}^2} \sinh \eta + \frac{1}{\text{H}^2} z_2(z_2 - z_3) \sinh^2 \eta = 1, \quad (3.17) \]

and

\[ z_{4\eta} \sinh \eta + z_4 \cosh \eta + \frac{1}{\text{H}^2} z_4 \left( u_6 \sinh \eta - z_4 \sinh^2 \eta \right) = -1. \quad (3.18) \]

From (3.16) we can make the significant conclusion

\[ (z_2 \sinh^2 \eta)_\eta = (z_3 \sinh^2 \eta)_\eta = 0. \quad (3.19) \]

Finally, applying the expansions

\[ r_{1n} = r_{1\eta} \eta_n + r_{1\theta} \theta_n + r_{1\phi} \phi_n, \quad r_{2n} = r_{2\eta} \eta_n + r_{2\theta} \theta_n + r_{2\phi} \phi_n, \quad r_{5n} = r_{5\eta} \eta_n + r_{5\theta} \theta_n \]

to (3.10) leads to the following result.

**The Skew-Vanishing List \( S_3 \):**

\[ (r_2 \sinh^2 \eta)_\eta = 0, \quad (r_2 \sin^2 \theta)_\theta = 0, \quad r_{1\phi} = 0, \quad (3.20) \]
\[ r_1 \theta \sin \theta + r_1 \left( -r_1 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta + \frac{1}{H^2} z_3 \sinh \eta \sin \theta + \cos \theta \right) + r_5 u_2 \sinh \eta \sin \theta = -1, \tag{3.21} \]

and

\[ r_5 \eta \sin \eta \sin \theta + r_5 \left( u_6 \frac{1}{H^2} \sinh \eta \sin \theta - r_5 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta + \cosh \eta \sin \theta \right) = -1, \tag{3.22} \]

together with

\[ (r_2 \sin^2 \theta)_{\phi} = 0, \quad r_{1 \phi} = 0. \tag{3.23} \]

The validity of these lists \( S_1, S_2, S_3 \) is verified in Appendix A.

Next, we can draw additional valuable information by using the structural expansions (3.6)-(3.7) and their concomitants.

Namely, attentive calculations lead to the conclusion that the initial equation (3.8) for \( \eta_{ij} \) is equivalent to the following list of vanishing.

**First Group of structural equations:**

\[
\left( \eta_{\nu \nu} - u_6 \frac{1}{H^2} \eta_{\nu \nu} + \frac{2}{V} \eta_{\nu} \right) V_{r} V_{r} + \eta_{\nu} V_{rr} = 0, \tag{3.24}
\]

\[
u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_t \phi_t - \eta_{\nu} V_{r} U_{f} Z_{tt} w_{2} = 0, \tag{3.25}
\]

\[
u_3 \frac{1}{H^2} \sinh^2 \eta \phi_f \phi_f - \eta_{\nu} z U_{f f} V_{r} = 0. \tag{3.26}
\]

The same method can be applied to the expansion (3.9) obtained for \( \theta_{ij} \). We arrive at the additional equations.

**Second Group of structural equations:**

\[
\theta_{ff} - \left( z_3 \frac{1}{H^2} \sinh^2 \eta \theta_f - \frac{2}{U} \right) \theta_f = 0, \tag{3.27}
\]

\[
z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_t \phi_t - \theta_{f c} \phi_{tt} = 0, \tag{3.28}
\]

\[
\left( z_4 \frac{1}{H^2} \sinh \eta \eta_{\nu} - \frac{1}{V} \right) z V_{r} + \frac{1}{U} = 0. \tag{3.29}
\]

Lastly, considering the expansion (3.10) for \( \phi_{ij} \) leads to new series of equations.
Third Group of structural equations:

$$\phi_t = -\frac{2}{Z} Z_t \phi_t + r_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_t \phi_t, \quad (3.30)$$

$$r_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta \phi_f - \frac{1}{U} U_f + \frac{1}{f} = 0, \quad (3.31)$$

$$\left( r_5 \frac{1}{H^2} \sin \eta \sin \theta \eta \phi_f - \frac{1}{V} \right) r V_r + 1 = 0. \quad (3.32)$$

The verification of the First, Second, and Third Groups of equations is the straightforward process and is not displayed in the present paper.

Now, we can evaluate the tensor $C_{ijn}$ using the representation (2.8) prepared to convenience in Sect. 2. The tensor should be given by a symmetric representation. It can readily be verified that the following conditions should be set forth.

**Symmetrizing Conditions for the tensor** $C_{ijn}$:

$$z_4 \sinh^2 \eta = u_3 \sinh \eta - 2H^2 \cosh \eta, \quad r_5 \sinh^2 \eta \sin \theta = u_2 \sinh \eta - 2H^2 \cosh \eta, \quad (3.33)$$

$$z_2 \sinh^2 \eta \sin \theta = r_1 \sinh^2 \eta \sin^2 \theta + 2H^2 \cos \theta. \quad (3.34)$$

If we compare the last equality with (3.19), we just conclude that

$$(r_1 \sinh^2 \eta)_{\eta} = 0. \quad (3.35)$$

With the Symmetrizing Conditions assumed, simple calculations lead to the following result.

**Assertion 3.3** *The tensor $C_{ijn}$ when written in terms of the orthonormal vectors has the structure*

$$F H C_{ijn} = -u_6 u_i u_j u_n - L_3 (u_i m_j m_n + u_n m_j m_i + u_j m_i m_n) - L_2 (u_i p_j p_n + u_n p_i p_j + u_j p_i p_n)$$

$$- \sinh \eta z_3 m_i m_j m_n - L (m_j p_i p_n + m_n p_i p_j + m_i p_j p_n) - r_2 \sinh \eta \sin \theta p_i p_j p_n, \quad (3.36)$$

where

$$L_2 = u_2 - H^2 \frac{\cosh \eta}{\sinh \eta}, \quad L_3 = u_3 - H^2 \frac{\cosh \eta}{\sinh \eta}, \quad L = \left( r_1 \sinh^2 \eta \sin \theta + H^2 \frac{\cos \theta}{\sin \theta} \right) \frac{1}{\sinh \eta}. \quad (3.37)$$
The orthonormal frame \{l_i, u_i, m_i, p_i\} (see (2.7) in Sect. 2) has been used.

Finally, we evaluate the curvature tensor
\[
\hat{R}_{jpqn} = C^h_{pq}C^h_{jn} - C^h_{pn}C^h_{jq}
\]  
(3.38)
of the tangent space and set forth the requirement that the tensor manifests the property
that the indicatrix is a space of constant negative curvature. In the process of evaluation
it is getting clear that the property implies the equalities
\[
L_2 = L_3,
\]  
(3.39)
which is equivalent to
\[
u_3 = u_2
\]  
(3.40)
(see (3.37)), and
\[
L_2(u_6 - 2L_2) = L(z_3 \sinh \eta - L).
\]  
(3.41)
The result reads
\[
F^2 \hat{R}_{jpqn} = T^*(h_{pq}h_{jn} - h_{pn}h_{jq}) = -T^*(h_{pn}h_{jq} - h_{pq}h_{jn}),
\]  
(3.42)
where
\[
T^* = \frac{1}{H^2} \left( (L_2)^2 + L_2(u_6 - 2L_2) \right)
\]  
(3.43)
(see Appendix B).

To have the value \(-H^2\) for the indicatrix curvature we should take
\[
T^* = 1 - H^2.
\]  
(3.44)
Since \(u_3 = u_2\) (see (3.40)), we can conclude that the equality
\[
z_4 = r_5 \sin \theta
\]  
(3.45)
holds owing to the symmetrizing condition (3.34), from which it follows that the last
equation (3.32) in the Third Group is the same as the third equation (3.29) in the Second
Group.

From (3.41) and (3.43) it follows that
\[
Lz_3 \sinh \eta = -H^2T^* - (L_2)^2 + L^2.
\]  
(3.46)
The function \(L\) was introduced in (3.37). Using here the equality \(z_2 \sinh^2 \eta \sin \theta = r_1 \sinh^2 \eta \sin^2 \theta + 2H^2 \cos \theta\) which presents in the Symmetrizing Condition (3.34), we obtain
the convenient representation
\[
L \sinh \eta = z_2 \sinh^2 \eta - H^2 \frac{\cos \theta}{\sin \theta}.
\]
Thus, in terms of the new notation
\[
\ddot{z}_2 = z_2 \sinh^2 \eta, \quad \ddot{z}_3 = z_3 \sinh^2 \eta,
\]  
(3.47)
the equation (3.46) takes on the form of the following \{\ddot{z}_2, \ddot{z}_3\}-quadratic equation:
\[
\left( \ddot{z}_2 - H^2 \frac{\cos \theta}{\sin \theta} \right)^2 - \left( \ddot{z}_2 - H^2 \frac{\cos \theta}{\sin \theta} \right) \ddot{z}_3 - (H^2 T^* + (L_2)^2) \sinh^2 \eta = 0. \tag{3.48}
\]

Since \( \ddot{z}_2 \eta = \ddot{z}_3 \eta = 0 \) (see (3.19)), \( \dot{z}_2 \) and \( \dot{z}_3 \) are functions of \( \theta \) (and are independent of \( \eta \) or \( \phi \)).

The derivative of \( \dot{z}_2 \) can be obtained from (3.17), namely
\[
\ddot{z}_2 - z_4 \sinh^2 \eta u_2 + \frac{1}{H^2} \ddot{z}_2 (\dot{z}_2 - \dot{z}_3) = \sinh^2 \eta.
\]

The Symmetrizing Conditions (3.33) involves \( z_4 \sinh \eta = u_2 \sinh \eta - 2H^2 \cosh \eta \). We eventually obtain
\[
\ddot{z}_2 + \frac{1}{H^2} \ddot{z}_2 (\dot{z}_2 - \dot{z}_3) = \sinh^2 \eta + \frac{1}{H^2} (u_2 \sinh \eta - 2H^2 \cosh \eta) u_2 \sinh \eta. \tag{3.49}
\]

To summarize up, we introduce the following definition.

**Definition 3.1** The *Total Set* of the conditions and equations involve the Skew-Vanishing integrability conditions \( S_1, S_2, S_3 \) and the Symmetrizing Conditions, supplemented by the structural equations written out in the First Group, in the Second Group, and in the Third Group, and also the following three conditions (i), (ii), and (iii) which assure the constancy of indicatrix curvature: (i) \( u_3 = u_2 \); (ii) \( T^* = -((L_2)^2 + L_2(u_6 - 2L_2))/H^2 \) with \( T^* = 1 - H^2 \); (iii) the \( \{ \dot{z}_2, \dot{z}_3 \} \)-quadratic equation must be fulfilled. The curvature value is equal to \(-H^2\).

From the content of the present section we can make the following general conclusion.

**Imperative Theorem** *In order that a two-axes pseudo-Finsleroid metric function \( F \) be of the angle-separated type, it is necessary and sufficient that the function be compatible with each condition or equation which enters the Total Set.*

**4 Solving the derived equations**

The *Total Set* involves many differential equations for numerous unknown functions. At the first sight, the hope to resolve the set can be regarded as excessively optimistic.

However, a lucky and simple possibility is offered when we pay due attention to the following equation triple:

\[
u_{2 \eta} + \frac{1}{H^2} u_2 (u_2 - u_6) = 1, \quad L_2 = u_2 - H^2 \frac{\cosh \eta}{\sinh \eta}, \quad \frac{1}{H^2} L_2 (L_2 - u_6) = T^* \equiv 1 - H^2. \tag{4.1}\]

Here, the first member has been taken from the integrability condition \( S_1 \), Formula (3.14), and two other members reflect the constancy of curvature of indicatrix (see Formulae (3.40)-(3.44)).

Indeed, differentiation with respect to \( \eta \) yields
\[
L_{2 \eta} = -\frac{1}{H^2} u_2 (u_2 - u_6) + 1 + H^2 \frac{1}{\sinh^2 \eta} = -\frac{1}{H^2} u_2 (L_2 - u_6) - u_2 \frac{\cosh \eta}{\sinh \eta} + 1 + H^2 \frac{1}{\sinh^2 \eta},
\]
or

\[ L_{2\eta} = \frac{1}{H^2} L_2 (u_6 - L_2) + \frac{\cosh \eta}{\sinh \eta} (u_6 - L_2) - u_2 \frac{\cosh \eta}{\sinh \eta} + 1 + H^2 \frac{1}{\sinh^2 \eta}, \]

which can be written as

\[ L_{2\eta} = \frac{1}{H^2} L_2 (u_6 - L_2) + \frac{\cosh \eta}{\sinh \eta} (u_6 - L_2) - \left( L_2 + H^2 \frac{\cosh \eta}{\sinh \eta} \right) \frac{\cosh \eta}{\sinh \eta} + 1 + H^2 \frac{1}{\sinh^2 \eta}. \]

We reduce this equality to read

\[ L_{2\eta} = \frac{1}{H^2} L_2 (u_6 - L_2) + \frac{\cosh \eta}{\sinh \eta} (u_6 - L_2) - L_2 \frac{\cosh \eta}{\sinh \eta} + 1 - H^2. \]

Multiplying by \( L_2 \) yields

\[ L_2 L_{2\eta} = -L_2 T^* - \frac{\cosh \eta}{\sinh \eta} H^2 T^* - (L_2)^2 \frac{\cosh \eta}{\sinh \eta} - L_2 (H^2 - 1) = \frac{\cosh \eta}{\sinh \eta} H^2 (H^2 - 1) - (L_2)^2 \frac{\cosh \eta}{\sinh \eta}. \]

In this way, we arrive at the validity of the following assertion.

**Assertion 4.1** The function \( L_2 \) should be obtained from the following simple differential equation:

\[ L_2 L_{2\eta} = (H^2 (H^2 - 1) - (L_2)^2) \frac{\cosh \eta}{\sinh \eta}. \]  

(4.2)

The equation can readily be solved, yielding

\[ \left( \frac{1}{H} L_2 \right)^2 = H^2 - 1 - \frac{H^2}{P \sinh^2 \eta} (1 - P), \]

(4.3)

where \( P \) is an integration constant, or

\[ \left( \frac{1}{H^2} \sinh \eta L_2 \right)^2 = \hat{L}, \]

(4.4)

with

\[ \hat{L} = 1 - \frac{1}{P} + \left( 1 - \frac{1}{H^2} \right) \sinh^2 \eta \equiv \frac{1}{H^2} - \frac{1}{P} + \left( 1 - \frac{1}{H^2} \right) \cosh^2 \eta. \]

(4.5)

The solutions are divided in three different classes:  *Class I*: \( 0 < P < 1 \);  *Class II*: \( P > 1 \);  *Class III*: \( P < 0 \).

Let us evaluate the last term in the \( \{ \tilde{z}_2, \tilde{z}_3 \} \)-quadratic equation (3.48), making use of (4.4). We get

\[ \frac{1}{H^2} \sinh^2 \eta [H^2 T^* + (L_2)^2] = \frac{1}{H^2} T^* \sinh^2 \eta + \hat{L} = \]
\[
\frac{1}{H^2}(1 - H^2) \sinh^2 \eta + 1 - \frac{1}{P} + \left(1 - \frac{1}{H^2}\right) \sinh^2 \eta = 1 - \frac{1}{P}
\]

So, the considered equation (3.48) can explicitly be written as

\[
\left(\ddot{z}_2 - H^2 \frac{\cos \theta}{\sin \theta}\right)^2 - \left(\ddot{z}_2 - H^2 \frac{\cos \theta}{\sin \theta}\right) \dddot{z}_3 - \left(1 - \frac{1}{P}\right) H^4 = 0. \quad (4.6)
\]

Using the notation

\[
\ddot{L}_2 = \ddot{z}_2 - H^2 \frac{\cos \theta}{\sin \theta}, \quad (4.7)
\]

we can write the equation in the form

\[
\frac{1}{H^4} \ddot{L}_2 (\ddot{L}_2 - \dddot{z}_3) = 1 - \frac{1}{P}, \quad (4.8)
\]

which is quite similar to (3.43).

The second line of simplifications is opening up when we refer to (3.28) of the Second Group in Sect. 3 and write

\[
z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \frac{1}{f} \theta_1 f = \frac{1}{Z} \frac{1}{\phi_t \phi_t} Z_{tt},
\]

where we have used the equality \(c_2 = f/Z\) which was written in (3.2) in Sec. 3. The equation is remarkable in that the left-hand part is a function of \(\theta\) (it should be noted that \((z_2 \sinh^2 \eta)_\eta = 0\) according to (3.19) in Sect. 3), while the right-hand part does not depend on \(\theta\) and is a function of the argument \(t\). Therefore, the equation implies the separation of variables:

\[
\theta_1 f = z_2 \frac{1}{H^2} C \sinh^2 \eta \sin^2 \theta, \quad \phi_t \phi_t = C \frac{Z_{tt}}{Z}, \quad C = C(x), \quad (4.9)
\]

where \(C\) is the first constant of separation of variables.

Introducing the function

\[
R_2 = \frac{1}{H^2} z_2 \sinh^2 \eta \sin \theta, \quad (4.10)
\]

we get

\[
\theta_1 f = R_2 C \sin \theta. \quad (4.11)
\]

Keeping in mind the identification \(u_3 = u_2\) (see (3.40) in Sect. 3), on compare (3.25) with (3.26) of the First Group we can conclude

\[
\frac{1}{Z} \phi_t \phi_t Z_{tt} w_2 = \frac{1}{\theta_1 f \theta_1 f} z U_{ff} \sin^2 \theta.
\]

Since \(f = c_2 Z\) and \(w_2 = c_2 z\), we obtain

\[
U_{ff} \frac{1}{Z} \phi_t \phi_t Z_{tt} = \frac{1}{\theta_1 f \theta_1 f} U_{ff} \sin^2 \theta,
\]

or

\[
CU_{ff} \sin^2 \theta = U_{ff} \theta_1 f \theta_1 f. \quad (4.12)
\]
Also, considering (3.25) of the First Group

\[ u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta = \eta V_r U_f \frac{1}{\phi_t \phi_t} Z_{\nu} w_2, \]

we are coming to

\[ u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta = \eta V_r r \frac{1}{U_f} \frac{1}{C}. \]  \hspace{1cm} (4.13)

From the previous equality the second angle-separation line starts

\[ \frac{1}{H^2} u_2 \sinh^2 \eta = C_7 \eta r, \quad \frac{1}{U} U_f \frac{1}{C} = C_7 \sin^2 \theta, \quad C_7 = C_7(x), \]

where the vanishing \( u_2 \theta = u_2 \phi = 0 \) written in (3.12) has been taken into account. \( C_7 \) is the second constant of separation of variables.

With (4.14), the dependence of the function \( r = \tilde{r}(x, \eta) \) on \( \eta \) can be found upon integration:

\[ \ln r = C_7 \int \frac{d\eta}{H^2 u_2 \sinh^2 \eta} \]  \hspace{1cm} (4.15)

with

\[ u_2 = L_2 + H^2 \frac{\cosh \eta}{\sinh \eta} \]

and \( L_2 \) taken from (4.3).

To find also \( V_r \), we compare the third line (3.29) of the Second Group with first line (3.33) in the Symmetrizing Condition. We obtain the equation

\[ \frac{1}{H^2} \left( u_2 - 2H^2 \frac{\cosh \eta}{\sinh \eta} \right) \eta r - \frac{1}{V} V_r r + 1 = 0, \]

where \( r = zU \) has been used. Multiplying by \( C_7 \) and using \( (1/H^2) u_2 \sinh^2 \eta = C_7 \eta r \) (see (4.14)), we conclude that

\[ \frac{1}{H^2} \left( u_2 - 2H^2 \frac{\cosh \eta}{\sinh \eta} \right) \frac{1}{H^2} u_2 \sinh^2 \eta + C_7 (-\frac{1}{V} V_r r + 1) = 0, \]

or

\[ \frac{1}{V} V_r r = \frac{1}{C_7 H^2} \left( u_2 - 2H^2 \frac{\cosh \eta}{\sinh \eta} \right) \frac{1}{H^2} u_2 \sinh^2 \eta + 1. \]

Last, taking into account the equality

\[ u_2 = L_2 + H^2 \frac{\cosh \eta}{\sinh \eta}, \]

we arrive at the representation

\[ \frac{1}{V} V_r r = \frac{1}{C_7 H^2} \left( L_2 - H^2 \frac{\cosh \eta}{\sinh \eta} \right) \left( L_2 + H^2 \frac{\cosh \eta}{\sinh \eta} \right) \frac{1}{H^2} \sinh^2 \eta + 1. \]

Inserting here

\[ \frac{1}{H^2} (L_2)^2 = H^2 - 1 - \frac{H^2}{P \sinh^2 \eta} (1 - P) \]
(see (4.3)) yields the following simple result:

\[
\frac{1}{V} V_r r = - \frac{1}{C_7 H^2} \sinh^2 \eta - \frac{1}{C_7 P} + 1. \tag{4.17}
\]

It is convenient to use the function

\[
R_1 = \cosh \eta + \sqrt{\hat{L}}, \tag{4.18}
\]

such that

\[
\frac{1}{H^2} u_2 \sinh \eta = R_1; \tag{4.19}
\]

\(\hat{L}\) is the function that was introduced in (4.4)-(4.5). By the help of (4.19) and (4.14), we obtain the equations

\[
\eta_r r = \frac{1}{C_7} R_1 \sinh \eta \tag{4.20}
\]

and

\[
\frac{1}{V} V_\eta = - \frac{1}{R_1 \sinh \eta} \left( \frac{1}{H^2} \sinh^2 \eta + \frac{1}{P} - C_7 \right), \tag{4.21}
\]

where the equality \(V_r r = V_\eta \eta_r r\) has been used.

From (4.20) and (4.21) the functions \(r = \hat{r}(x, \eta)\) and \(V = \hat{V}(x, \eta)\) can be found upon integration:

\[
\ln r = C_7 \int \frac{1}{R_1 \sinh \eta} d\eta \tag{4.22}
\]

and

\[
\ln V = - \int \frac{\frac{1}{H^2} \sinh^2 \eta + \frac{1}{P} - C_7}{R_1 \sinh \eta} d\eta. \tag{4.23}
\]

**Assertion 4.2** The integrals (4.22) and (4.23) yield the general solutions for the dependence of the functions \(r = \hat{r}(x, \eta)\) and \(V = \hat{V}(x, \eta)\) on \(\eta\) compatible with the Total Set of equations.

### 5 \(\eta\)-regular solution

In the preceding Sect. 4 the equations have been derived systematically to represent the pseudo-Finsleroid metric functions of the angle-separated type. In the present section, we find an interesting and significant particular solution to the equations. To specify the solution, we introduce the following two definitions.

**Definition 5.1** The pseudo-Finslerian metric function \(F\) of the angle-separated type is called \(\eta\)-regular, if the characteristic functions \(V = \hat{V}(x, \eta)\) and \(r = \hat{r}(x, \eta)\) which enter the function \(F\) in accordance with the Separation Scheme (3.2) are smooth of class \(C^\infty\) with respect to their angle argument \(\eta\) over the total definition range \(0 < \eta < \infty\). The metric functions possessing such a property will be denoted by \(F^{R\mathbb{E}}\).
**Definition 5.2** If a metric function $F^{\text{REG}}$ possesses the property that the indicatrix is a space of constant negative curvature, then the function is called the $\eta$-regular pseudo-Finsleroid metric function, to be denoted by $F^{\text{REG-PSEUDO-FRD}}$.

To make search for a function $F^{\text{REG-PSEUDO-FRD}}$, let us choose the Class II among the three classes formulated below Formula (4.5) in Sect. 4. Accordingly, we assume that $P > 1$. Let us also denote

$$\hat{C} = \frac{1}{C},$$

(5.1)

where $C$ is the constant of Separation of Variables that was arisen in the representation $\phi_t \phi_t = CZ_t / Z$ (see (4.9) in Sect. 4), and introduce the constant $T$ by the help of the equality

$$P = \frac{1}{T \hat{C}}.$$  

(5.2)

The appropriate idea is to subject the introduced constants to the inequalities

$$T > 1, \quad 0 < \hat{C} < 1, \quad T \hat{C} < 1.$$  

(5.3)

The inequality $H \geq 1$ is always assumed.

We start with the representation

$$F = b \tilde{V}, \quad \tilde{V} = C_1 \frac{1}{R_1} J,$$

(5.4)

and make the choice

$$C_7 = \frac{1}{P},$$

(5.5)

where $R_1$ is the known function that was written out in (4.18) and $J = J(x, \eta)$ is the input function to be determined.

With the choice $C_7 = 1/P$, the equation (4.21) reduces to merely

$$\frac{1}{\tilde{V}} \tilde{V}_\eta = - \frac{1}{H^2} \frac{1}{R_1} \sinh \eta$$

(5.6)

and the equation (4.20) takes on the form

$$\eta r_r = PR_1 \sinh \eta.$$

(5.7)

The simple and useful equality

$$\frac{1}{H^2} (\eta_r)^2 = - \frac{1}{\tilde{V}} V_{rr}$$

(5.8)

can be derived.

From the equation (5.6), the dependence of the function $J = J(x, \eta)$ on $\eta$ can explicitly be found. To this end we differentiate the equality $\ln \tilde{V} = \ln(C_1) - \ln(R_1) + \ln J$ with respect to $\eta$ and compare the obtained result with (5.6). This method leads to the equation

$$(\ln J)_\eta = \left(1 - \frac{1}{H^2}\right) \frac{1}{\hat{L}} \sinh \eta,$$

(5.9)

where $\hat{L}$ is the function given by (4.5). Integrating yields

$$J = \exp \left( H_1 \arcsinh \left( \hat{L}_1 \cosh \eta \right) \right),$$

(5.10)
where

\[ H_1 = \sqrt{1 - \frac{1}{H^2}}, \quad \hat{L}_1 = \frac{H_1}{S_1}, \quad S_1 = \sqrt{1 - \frac{1}{P}}. \] (5.11)

The representation (5.10) can alternatively be written in the form

\[ J = \left( \hat{L}_1 \cosh \eta + \sqrt{(\hat{L}_1)^2 \cosh^2 \eta + 1} \right)^{H_1}. \] (5.12)

The right-hand part in (5.12) is obviously smooth of class \( C^\infty \) on the total range \( 0 < \eta < \infty \).

Next, the function \( r = \tilde{r}(x, \eta) \) can be determined from (5.7). Indeed, it is convenient to use the representation

\[ r = \tilde{C}_2 \frac{\sinh \eta}{R_1} Y_1, \quad \tilde{C}_2 = \tilde{C}_2(x). \] (5.13)

It can readily be verified that the introduced function \( Y_1 \) should be subjected to the equation

\[ (\ln Y_1)_\eta = \left( \frac{1}{P} - 1 \right) \frac{1}{\sinh \eta \sqrt{L}}. \] (5.14)

The attentive integration yields

\[ Y_1 = \left( \frac{1 - Nx + \sqrt{N + 1} \sqrt{1 + Nx^2}}{(1 + Nx + \sqrt{N + 1} \sqrt{1 + Nx^2})} \right)^{-1/2} \sqrt{1 - \frac{1}{P}}, \] (5.15)

where \( x = \cosh \eta \) and

\[ N = \frac{1 - \frac{1}{H^2}}{H^2 - \frac{1}{P}}. \] (5.16)

From (4.11) and (4.14) it follows that

\[ \bar{\theta}_f f = \frac{1}{C} R_2 \sin \bar{\theta}, \quad \frac{1}{U} U_f f = T \sin^2 \theta, \] (5.17)

which in turn entails

\[ \frac{1}{U} \bar{U}_\theta = \frac{1}{P} \frac{\sin \theta}{R_2}. \] (5.18)

**Definition 5.3** If a metric function \( F^{\text{REG-PSEUDO-FRD}} \) possesses the property that the indicatrices in horizontal sections are of constant positive curvature, then the function is called the \( \eta \)-regular Finsleroid-in-pseudo-Finsleroid metric function, to be denoted by \( F^{\text{REG-FRD-IN-PSEUDO-FRD}} \).
In close resemblance with the function $R_1$ defined by (4.18), we introduce the function

$$R_2 = \cos \theta + \sqrt{C} \sqrt{(T - 1) \sin^2 \theta + \left(\frac{1}{C} - 1\right) \cos^2 \theta},$$

which can also be written conveniently as

$$R_2 = \cos \theta + \sqrt{L_9}, \quad L_9 = T \dot{C} - \dot{C} + (1 - T \dot{C}) \cos^2 \theta. \quad (5.19)$$

Since $T > 1$ and $T \dot{C} < 1$ (according to (5.3)), the function $R_2 = R_2(x, \theta)$ is totally regular with respect to the angle argument $\theta$, namely is of the class $C^\infty$.

From (5.17) the following simple formula can be obtained:

$$U_{ff} = (\theta f)^2 T \dot{C} U. \quad (5.20)$$

With this function $R_2$, the dependence of the function $f = \tilde{f}(x, \theta)$ on $\theta$ can be found from (5.17) with the help of the substitution

$$f = C_{17} \frac{\sin \theta}{R_2} Y_2, \quad C_{17} = C_{17}(x). \quad (5.21)$$

We obtain

$$(\ln Y_2)_\theta = -(1 - \dot{C}) \frac{1}{\sqrt{L_9} \sin \theta}. \quad (5.22)$$

The integration leads to the following result:

$$Y_2 = \left(\frac{\sqrt{\tan^2 \theta + A} + \sqrt{A}}{\sqrt{\tan^2 \theta + A} - \sqrt{A}}\right)^{\frac{1}{2}} \sqrt{1 - \dot{C}}, \quad A = \frac{1 - \dot{C}}{C T - \dot{C}}. \quad (5.23)$$

It is convenient to apply such a method also to the function $U = \tilde{U}(x, \theta)$. Namely, postulating the representation

$$U = C_{39} \frac{1}{R_2} I, \quad C_{30} = C_{39}(x), \quad (5.24)$$

from (5.18) we get

$$(\ln I)_\theta = -(1 - T \dot{C}) \frac{\sin \theta}{\sqrt{L_9}}. \quad (5.25)$$

The integration shows that

$$I = \left(\sqrt{1 - T \dot{C}} \cos \theta + \sqrt{L_9}\right) \sqrt{1 - \dot{C}}. \quad (5.26)$$

The interesting simple equality

$$\left(\ln \frac{I}{Y_2}\right)_\theta = \frac{\sqrt{L_9}}{\sin \theta}. \quad (5.27)$$
can also be derived.

It remains to consider the dependence of the function \( \phi = \bar{\phi}(x, t) \) on the argument \( t \), where \( t = w_1/w_2 \) (see the definition of the variable \( t \) in (3.3)). According to (4.9), the function \( \bar{\phi} \) must fulfill the equation

\[
\bar{\phi}_t \bar{\phi}_t = \frac{1}{\sqrt{C}} \frac{Z_{tt} Z}{Z},
\]

Let us solve the equation with the help of the function

\[
\phi = \frac{1}{\sqrt{C}} \arctan t + \ast, \quad \ast = \ast(x), \quad (5.28)
\]

which inverse is \( t = \tan(\sqrt{C} \phi - \ast) \), so that \( \bar{\phi}_t = \left(1/\sqrt{C}\right) (1/(1+t^2)) \). For the function \( Z = Z(x, t) \) we obtain the equations

\[
\frac{1}{Z}Z_{tt} = \frac{1}{(1+t^2)^2}, \quad \frac{1}{Z}Z_t = \frac{t}{1+t^2} \equiv \frac{w_1w_2}{w_1 + w_2},
\]

which can be solved as follows:

\[
Z = \sqrt{1+t^2} C_{11} \equiv \frac{1}{\cos(\sqrt{C} \phi - \ast)} C_{11}, \quad C_{11} = C_{11}(x). \quad (5.29)
\]

Recollecting the definition \( f = w_2 Z/w_3 \) (see (3.2)), we arrive at the representation

\[
f = \frac{1}{w_3} w_\perp C_{11}(x), \quad (5.30)
\]

where

\[
w_\perp = \sqrt{w_1^2 + w_2^2}. \quad (5.31)
\]

This observation motivates introducing the following definition.

**Definition 5.4** The Finslerian metric function \( F \) is called \( i_{(3)} \)-axial if the the 1-forms \( i = i_i y^i \) and \( j = j_i y^i \) introduced in Sect. 1 enter the function in the sum of squares \((i)^2 + (j)^2\).

Thus, we are entitled to formulate the following implication.

**Assertion 5.1** When the function \( \phi = \phi(x, t) \) is given by Formula (5.28), the function \( f \) is \( i_{(3)} \)-axial, for the variables \( w_1, w_2 \) enter the function in the sum of squares \((w_1)^2 + (w_2)^2\).

**Remark 5.1** Formula (5.28) extends its pseudo-Riemannian precursor by presence of the factor \( 1/\sqrt{C} \). The axial nature of the function \( f \) is retained valid under the extension from the pseudo-Riemannian geometry to the pseudo-Finsleroid theory under development in the present paper. According to Formula (5.29), the form of dependence of the function \( Z \) on the argument \( t \) is exactly the same as in the pseudo-Riemannian geometry.

If we now remind the definition \( r = w_3 U(x, f) \) (see (3.2)), we conclude that the function \( r \) also is of the \( i_{(3)} \)-axial structure:

\[
r = r^*(x, w_3, w_\perp). \quad (5.32)
\]
Because of the representation chain \( \{F = bV, V = \tilde{V}(x, \eta), \eta = \eta(x, r), \theta = \tilde{\theta}(x, f)\} \), all the functions \( F, V, \eta, \) and \( \theta \) are \( i_3 \)-axial.

Differentiating the equality \( f = (1/w_3)w_\perp C_{11} \) with respect to \( w_1, w_2, \) and \( w_3 \) yields

\[
ff_1 = \frac{1}{w_3w_3}w_1(C_{11})^2, \quad ff_2 = \frac{1}{w_3w_3}w_2(C_{11})^2, \quad ff_3 = -((w_1)^2 + (w_2)^2)\frac{1}{w_3w_3w_3}(C_{11})^2,
\]

which entails \( f_1w_1 + f_2w_2 + f_3w_3 = 0. \)

We can take \( U_f \) from (5.17): \( (1/U)f = T \sin^2 \theta. \) We get

\[
\frac{\partial U}{\partial w_3} = U_f f_3 = -UT\frac{1}{f}\sin^2 \theta f \frac{1}{w_3} = -UT \sin^2 \theta \frac{1}{w_3},
\]

or

\[
\frac{\partial U}{\partial w_3} = -UT \sin^2 \theta \frac{1}{w_3}.
\]

(5.34)

Here, the right-hand part does not involve neither \( R_2 \) nor \( I. \)

Applying this Formula to \( r = w_3U \) yields

\[
r_{w_3}^* = U - UT \sin^2 \theta, \quad f = C_{17} \frac{\sin \theta}{R_2} Y_2, \quad U = C_{39} \frac{1}{R_2} I,
\]

from which it follows that

\[
r_{w_3}^* = U - UTR_2R_2 f^2 \frac{1}{(C_{17})^2(Y_2)^2} = U - \frac{1}{U} \frac{(w_\perp)^2}{(w_3)^2} \frac{1}{(Y_2)^2} \frac{1}{(C_{11})^2 T \frac{(C_{39})^2}{(C_{17})^2}}.
\]

We obtain the representation

\[
r_{w_3}^* = U - \frac{1}{U} \frac{(w_\perp)^2}{(w_3)^2} \frac{1}{(Y_2)^2} \frac{1}{(C_{11})^2 T \frac{(C_{39})^2}{(C_{17})^2}}.
\]

(5.35)

Similarly, the derivative \( \partial U/\partial w_\perp = C_{11} U_f / w_3 \) can readily transformed to the representation

\[
\frac{\partial U}{\partial w_\perp} = UT \sin^2 \theta \frac{1}{w_\perp}
\]

(5.36)

which, when applied to \( r = w_3U \), yields the result \( r_{w_\perp}^* = UT (\sin^2 \theta) w_3 / w_\perp \), or

\[
r_{w_\perp}^* = \frac{1}{U} \frac{w_\perp}{w_3} \frac{1}{(Y_2)^2} \frac{1}{(C_{11})^2 T \frac{(C_{39})^2}{(C_{17})^2}}.
\]

(5.37)

The identity \( w_3 r_{w_3}^* + r_{w_\perp}^* w_\perp = r \equiv w_3U \) is valid.

For tensorial evaluations, it is convenient to make the redefinition

\[
v^a = w_a
\]

(a=1,2,3) for variables and apply the representations

\[
r = \hat{r}(x, v), \quad f = \hat{f}(x, v), \quad U = \hat{U}(x, v), \quad \theta = \hat{\theta}(x, v), \quad \phi = \hat{\phi}(x, v), \quad v = \{v^a\}
\]

(5.38)

to use the derivatives.
Since \( r = v^3 \hat{U} \), the function \( \hat{r} \) is positively homogeneous of the first degree with respect to the variables \( v^a \), and has the geometrical meaning of the Finsler metric function in the horizontal sections. The associated Finsler metric tensor is

\[
R_{ab} = rr_{ab} + r_ar_b \quad (5.39)
\]

and \( \{ r_a \} = \{ r_1, r_2, r_3 \} \) plays the role of the respective covariant unit vector; \( r_av^a = r \).

Because of the axial structure (5.32) of \( r \), the function \( \bar{r}(x, v^3, v_\perp) \) exists such that \( r = \bar{r} \) and

\[
\hat{r} = \bar{r}(x, v^3, v_\perp),
\]

where

\[
v_\perp = \sqrt{(v^1)^2 + (v^2)^2}
\]

(which is equal to \( w_\perp \)). We derivatives

\[
r_1 = \bar{r}v^1, \quad r_2 = \bar{r}v^2, \quad r_3 = \bar{r}v^3
\]

are convenient to use in calculations.

We can readily obtain

\[
f_1f_1 + ff_{11} = \frac{1}{v^3v^3}(C_{11})^2, \quad f_1f_2 + ff_{12} = 0, \quad f_1f_3 + ff_{13} = -2\frac{1}{v^3v^3v^3}v^1(C_{11})^2,
\]

\[
f_2f_1 + ff_{21} = 0, \quad f_2f_2 + ff_{22} = \frac{1}{v^3v^3}(C_{11})^2, \quad f_2f_3 + ff_{23} = -2\frac{1}{v^3v^3v^3}v^2(C_{11})^2,
\]

and

\[
f_1f_3 + ff_{13} = -2v^1\frac{1}{v^3v^3v^3}(C_{11})^2, \quad f_2f_3 + ff_{23} = -2v^2\frac{1}{v^3v^3v^3}(C_{11})^2,
\]

together with

\[
f_3f_3 + ff_{33} = 3\frac{(v^1)^2 + (v^2)^2}{v^3v^3v^3v^3}(C_{11})^2. \quad (5.40)
\]

The following assertion is valid.

**Assertion 5.2** Given the function \( R_2 \) by the help of Formula (5.19). If the equations

\[
\frac{1}{U}U_\theta = \frac{1}{P} \sin \theta, \quad f_\theta = \hat{C}f \frac{1}{\sin \theta R_2} \quad (5.41)
\]

are fulfilled, then the indicatrices of the sectional horizontal spaces possess the property of positive curvature, and vice versa. The respective curvature value \( \mathcal{C}^{[3]}(x) \) will be indicated below in Formula (5.56).
To verify the assertion we can evaluate the tensor $C_{abc} = (1/2) \partial R_{ab}/\partial v^c$. The insertion of the metric tensor $R_{ab} = r_ar_b + rr_{ab}$ (see (5.39)) yields the symmetric representation

$$2C_{abc} = r_ar_{bc} + r_br_{ac} + rcr_{ab} + rr_{abc}. \quad (5.42)$$

Raising the indices is naturally performed by the help of the tensor $R_{fd}$ reciprocal to $R_{fd}$, as exemplified by $C^f_{be} = R^{df}C_{dbe}$. In the process of evaluating the curvature tensor

$$R^*_{bace} = C_{afe}C^f_{be} - C_{afe}C^f_{bc},$$

it is appropriate to apply Formulas (5.34)-(5.40) and their concomitants, and in the last steps use the equalities $U_{ff} = (\theta f)^2T\hat{C}U$ (see (5.20)), $(1/H^2)(\eta r)^2 = -(1/V)V_{rr}$ (see (5.8)), and

$$\sin \theta \left( \frac{1}{\theta f} \right)^2 \left( \theta ff + 2T \sin^2 \theta \frac{1}{\theta f} \right) = \sqrt{L_9} - \frac{(1 - \hat{T}C) \sin^2 \theta}{\sqrt{L_9}}$$

(this equality can be verified with the help of (5.19)). This process (which is not short) leads to the remarkable representation

$$r^2R^*_{bace} = (P - 1)(h_{bc}h_{ae} - h_{bc}h_{ac}), \quad (5.43)$$

where $h_{bc} = R_{bc} - r_br_c$.

Also, we can obtain the angular representation

$$h_{ab} = \frac{1}{P}(\theta_a \theta_b + \sin^2 \theta \phi_a \phi_b)r^2 \quad (5.44)$$

by following closely the calculation method disclosed and applied in the previous paper [1,2].

Let us apply these observations to elucidating the curvature properties of the horizontal sections.

Like to Sect. 3 in [1,2], we denote by

$$\mathcal{T}_{x;\lambda(x)} \in T_xM \quad (5.45)$$

the space orthogonal to the vertical $b$-axis at a fixed value $\lambda_x = b_x$. This 3-dimensional space $\mathcal{T}_{x;\lambda(x)}$ is the horizontal section of the tangent space $T_xM$ through the axis point assigned by a value $\lambda(x) = \lambda_x$.

The sectional horizontal space $\mathcal{T}_{x;\lambda(x)}$ consists of the endpoints of the tangent vectors $y_x \in T_x$ representable by the component expansion

$$y^i_x = \lambda(x)b^i(x) + ii^i(x) + jj^i(x) + i\{3\}i\{3\}(x), \quad (5.46)$$

where $\lambda(x), b^i(x), i\{i\}^i(x), j\{j\}^i(x), i\{3\}(x)$ are fixed and $i, j, i\{3\}$ are arbitrary. We can identify $b_x = b_i(x)y^i_x$.

Following the geometrical imagination adopted in Sect. 3 of [1,2], we introduce the notion of the 3-dimensional centered vector space

$$\mathcal{V}^{(3)}_{x;\lambda(x)} = \{O_{x;\lambda(x)}, \mathcal{T}_{x;\lambda(x)}, u \in \mathcal{T}_{x;\lambda(x)}, u = \{u^a\}, \ a = 1, 2, 3\}. \quad (5.47)$$
which is of the sectional horizontal meaning, where $u^1 = i$, $u^2 = j$, $u^3 = i_{(3)}$, and $\{u^a\}$ are geometrically interpreted as the components of the vectors $u \in T_{x;\lambda(x)}$ that are supported by the center point $O_{x;\lambda(x)}$ of the space $V_{x;\lambda(x)}^{(3)}$. The entered $O_{x;\lambda(x)}$ belongs to the vertical $b$-axis, namely $O_{x;\lambda(x)}$ is the point at which $T_{x;\lambda(x)}$ intersects this axis.

The function $r = \hat{r}(x, v)$ introduced in (5.38) determines the function

$$F_{x;\lambda(x)}^{(3)} = F_{x;\lambda(x)}^{(3)}(x, u) \text{ with } F_{x;\lambda(x)}^{(3)}(x, u) = \hat{r}_{\lambda(x)}(x, u)$$

which we shall naturally interpret as the Finsler metric function in the sectional horizontal space $T_{x;\lambda(x)}$.

If we take into account the first-order homogeneity of the function $\hat{r}(x, v)$ with respect to $v = \{v^a\}$ (which is obvious from the input definition $r = v^3 U$ (see (3.2)) for the function $r$), together with the relation $v^a = u^a/\lambda$, we establish the important equality

$$\lambda(x)\hat{r}_{\lambda(x)}(x, v) = \hat{r}_{\lambda(x)}(x, u).$$

In the tangent space $T_x$ we confine our analysis to region bounded by the indicatrix $I_x$ (introduced by Definition 1.1 in Sect. 1). Accordingly, we should restrict the consideration by the $b$-like parts

$$\mathcal{P}T_{x;\lambda(x)} = \{ y \in \mathcal{P}T_{x;\lambda(x)} : y \in T_{x;\lambda(x)} : F_{x;\lambda(x)}^{(3)} \leq R_{x;\lambda(x)} \}$$

of the horizontal sections $T_{x;\lambda(x)}$, where

$$R_{x;\lambda(x)} = \left( F_{x;\lambda(x)}^{(3)}(x, u) \right)_{x;\lambda(x);\max}$$

is the radius of the horizontal section $\mathcal{P}T_{x;\lambda(x)}$.

The maximal value $(\hat{r}(x, v))_{\lambda(x);\max}$ of the function $\hat{r}(x, v)$ in $\mathcal{P}T_{x;\lambda(x)}$ obeys obviously the equation

$$\lambda(x)V(x, (\hat{r}(x, v))_{\lambda(x);\max}) = 1.$$  

With this value, the radius can be given by the equality

$$R_{\lambda(x)} = \lambda(x) (\hat{r}(x, v))_{\lambda(x);\max},$$

because

$$\left( F_{x;\lambda(x)}^{(3)}(x, u) \right)_{\max} = (\hat{r}(x, u))_{\lambda(x);\max}$$

and the equality (5.49) holds identically.

Accordingly, we restrict the centered vector space $V_{x;\lambda(x)}^{(3)}$ by the part $\mathcal{P}V_{x;\lambda(x)}^{(3)}$ which is bounded by the indicatrix $I_x$. The space $\mathcal{P}V_{\lambda(x)}^{(3)}$ is obtained by substituting $\mathcal{P}T_{x;\lambda(x)}$ with $T_{x;\lambda(x)}$ in the definition (5.47) of $V_{x;\lambda(x)}^{(3)}$.

From the function $F_{x;\lambda(x)}^{(3)} = F_{x;\lambda(x)}^{(3)}(x, u)$ we construct the Finsler metric tensor $G_{x;\lambda(x);ab} = G_{x;\lambda(x);ab}(x, u)$ according to the ordinary Finslerian rule

$$G_{x;\lambda(x);ab} = \frac{1}{2} \frac{\partial^2 \left( F_{x;\lambda(x)}^{(3)} \right)^2}{\partial u^a \partial u^b}.$$  

The sectional horizontal space $\mathcal{P}V_{x;\lambda(x)}$ becomes the three-dimensional sectional Finsler space

$$\mathcal{F}_{x;\lambda(x)}^{(3)} = \{ \mathcal{P}V_{x;\lambda(x)}^{(3)} : G_{x;\lambda(x);ab}, F_{x;\lambda(x)}^{(3)} \leq R_{x;\lambda(x)} \}.  

(5.54)
The axis of this space is assigned by the direction of the vector \( i_{3}^{i}(x) \).

**Definition 5.5** The surface \( T_{x;\lambda(x)}^{[2]} \subset P V_{x;\lambda(x)}^{[3]} \) introduced by

\[
T_{x;\lambda(x)}^{[2]} = \{ u \in P V_{x;\lambda(x)}^{[3]}, \ F_{\lambda(x)}^{[3]}(x, u) = R_{x;\lambda(x)} \} \tag{5.55}
\]

is called the indicatrix of the sectional Finsler space.

\( R_{x;\lambda(x)} \) is the radius of this indicatrix, which is the boundary of the centered space \( P V_{x;\lambda(x)}^{[3]} \).

Considering the equalities

\[
G_{x;\lambda(x);ab} = \frac{1}{2} \frac{\partial^{2} \left( F_{x;\lambda(x)}^{[3]} \right)}{\partial u^{a} \partial u^{b}} = \frac{1}{2} \frac{\partial^{2} \left( \dot{r}_{\lambda(x)}(x, u) \right)}{\partial u^{a} \partial u^{b}} = \frac{1}{2} \frac{\partial^{2} \left( \dot{r}_{\lambda(x)}(x, v) \right)}{\partial v^{a} \partial v^{b}},
\]

where (5.49) has been used, we arrive at the identification

\[
G_{x;\lambda(x);ab} = R_{ab},
\]

in which the last tensor \( R_{ab} \) is known from (5.39). With the knowledge of the tensor, and taking into account Formulas (5.43) and (5.44), the subsequent attentive evaluation leads to

\[
C_{x;\lambda(x)}^{[3]} = \frac{P(x)}{(R_{x;\lambda(x)})^{2}}. \tag{5.56}
\]

Assertion 5.2 is valid. The obtained formula (5.56) complies with the well-known geometrical property that the curvature of the Euclidean sphere of radius \( R \) equals \( 1/R^2 \).

The evaluation of the determinant of the tensor \( R_{ab} \) yields the result

\[
\det(R_{ab}) = \frac{1}{P^{2}C^{2}} I^{6} \frac{1}{(Y_{2})^{4}} \left( \frac{C_{11}}{C_{17}} \right)^{4} (C_{39})^{6} > 0 \tag{5.57}
\]

which is positive and of the angle-regular class \( C^{\infty} \) with respect to the involved angles \( \eta \) and \( \theta \), and is independent of the angle \( \phi \) because the space is \( i_{3}-axial \).

Thus we are justified to formulate the following statements.

**Assertion 5.3** If the characteristic functions \( \{ V = \dot{V}(x, \eta), r = \dot{r}(x, \eta), U = \dot{U}(x, \theta), f = \dot{f}(x, \theta) \} \) in a pseudo-Finslerian metric function \( F(x, y) \) of the angle-separated \( i_{3} \)-axial type fulfill the ordinary differential equations

\[
\frac{1}{V} V_{\eta} = -\frac{1}{H^{2}} \frac{1}{R_{1}} \sinh \eta, \quad r_{\eta} = \frac{r}{P} \frac{1}{\sinh \eta R_{1}}, \quad \frac{1}{U} U_{\theta} = \frac{1}{P} \frac{\sin \theta}{R_{2}}, \quad f_{\theta} = \dot{C} f \frac{1}{\sin \theta R_{2}}, \tag{5.58}
\]

where the subscripts \( \eta \) and \( \theta \) mean the differentiations, then:

the function \( F \) is of the \( F^{\text{REG}}\text{-FRD-\text{IN}-\text{PSLIDO}-\text{FRD}} \)-type;

the indicatrix \( I_{x} \) is of the constant negative curvature which value is \( -H^{2} \);

in the horizontal sections, the geometry is assigned by the Finslerian metric function \( F_{x;\lambda(x)}^{[3]} \); the associated indicatrix \( I_{x;\lambda(x)}^{[3]} \) is of the constant positive curvature \( C_{x;\lambda(x)}^{[3]} \) given by (5.56), and the metric tensor is angle-regular and positive-definite.
Remark 5.2 The class $C^\infty$ regularity of the dependence of the function $\tilde{V}$ on $\eta$ over all the definition range $\eta \in (0, \infty)$ is evidenced from the right-hand part of the formula $(1/\tilde{V})\tilde{V}_\eta = -1/(H^2R_1)\sinh \eta$ displayed in (5.6), because the function $R_1$ obviously possesses such a regularity property (see the representation for $R_1$ in (4.18)). The representation (5.10) of the function $J$ also manifests the property. The regularity formulated comes actually from the perfect regularity of the right-hand part in the input representation (4.5) of the function $\tilde{L} > 0$, because we are keeping the inequalities $P > 1$ and $H > 1$ well. The similar motivation can be addressed to the function $\tilde{U} = \tilde{U}(x, \theta)$, namely we can refer to the formula $(1/\tilde{U})\tilde{U}_\eta = 1/(PR_2)\sin \theta$ shown in (5.18) and take into account the representation (5.19) which introduces the function $R_2$ with the help of the function $L_9$. Upon the restrictions $\{T > 1, T\tilde{C} < 1\}$ assumed the function $L_9$ is positive and also smooth of class $C^\infty$ with respect to the angle argument $\theta$.

Conclusions: comparisons and properties

Thus, we have at our disposal two metric functions of the Finsleroid-in-pseudo-Finsleroid-type, namely $F^{(P<1)}$ (Class I) and $F^{(P>1)}$ (Class II). By $F^{(P<1)}$ we denote the function proposed and described in the previous paper [1] (which was preceded by the publication [2]). $F^{(P>1)}$ is the function that was found and investigated in Sec. 5 of the present paper.

The angle separation (3.1)-(3.5) is equally meaningful for both the metric functions. The expansion

$$l^i = \left(b^i + w_1i^i + w_2j^i + w_3i^i_{(3)}\right) \frac{1}{V}$$

can universally be applied with

$$w_3 = r \frac{1}{U}, \quad w_1 = w_\perp \cos \phi, \quad w_2 = w_\perp \sin \phi, \quad w_\perp = \frac{1}{C_{11}}w_3f,$$

where (5.30) and (5.31) have been used.

Outwardly, the distinction of the derivative list (5.58) of Assertion 5.3 from the case developed in the previous paper [1] is only in the presence of the scalar $\hat{C} = \hat{C}(x)$ in the last member which represents $f_\theta$ (see the list of Formulas (22)-(23) in [1]). However, in each tangent space $T_x$ the constant $\hat{C}$ gives rise to essential changes in the characteristic functions, including the appearance of new function $Y_2$ which enters Formula (5.21) for $f = f(x, \theta)$. The dependence of $Y_2$ on $\theta$ is rather complicated (see (5.23)), which does not permit to obtain the inverse function $\theta = \theta(x, f)$ in an explicit algebraic form (which was possible in the paper [1], because of the presumption $\hat{C} = 1$ made in [1]).

The significant distinction between the functions $F^{(P>1)}$ and $F^{(P<1)}$ is rooted in the structure of the functions $\hat{L}$ and $L$ which enter the key function $R_1$, namely in the present paper we used

$$R_1 = \cosh \eta + \sqrt{L}, \quad \hat{L} = 1 - \frac{1}{P} + \left(1 - \frac{1}{H^2}\right)\sinh^2 \eta, \quad P > 1$$

(see Formulas (4.5) and (4.18) in Sect. 4), while in [1] the function $R_1$ was taken according to

$$R_1 = \cosh \eta + \sqrt{L}, \quad L = 1 - \frac{1}{p^2} + \left(1 - \frac{1}{H^2}\right)\sinh^2 \eta, \quad p^2 < 1$$

(Formula (12) in [1]). Indeed, the involved function $\hat{L}$, and whence the function $R_1$, is totally positive and regular over $\eta \in (0, \infty)$. This observation explains why the pseudo-Finsleroid metric function proposed in Sect. 5 is $\eta$-regular.
Let us substitute the notation $P$ with $p^2$ in the second version of $R_1$. Since in this case $P < 1$, the function $L$ vanishes at the value $\eta_0 > 0$ obtainable from the algebraic equation

$$\frac{1}{P} - 1 = \left(1 - \frac{1}{H^2}\right) \sinh^2 \eta_0,$$

so that $L < 0$ if $\eta < \eta_0$, and $L > 0$ if $\eta > \eta_0$. Negative $L$ are nonadmitted under the square root in the right-hand part of $R_1$, whence we encounter with the deficit of angle $\eta$: the region $\eta < \eta_0$ is inaccessible. The internal cone appears, because the function $F^{(P<1)}$ vanishes at $\eta = \eta_0$, so that $F^{(P<1)}$ is the two-cone pseudo-Finsleroid metric function (see [1]). In case of $F^{(P>1)}$, only one cone is evidenced.

In the process of the systematic evaluation performed in Sect. 5 to obtain the metric function $F^{(P>1)}$, we have found explicitly the dependence of all the involved functions $V, r, U, f, Z$ on the angle triple $\eta, \theta, \phi$. The knowledge of this dependence, when taken in conjunction with the representation of the unit vector $l^i$ which has been written out in the beginning of the present Section, yields the explicit dependence $y^i = \left(b^i + w_1i^i + w_2j^i + w_3k^i\right)$ on the angle triple. At the same time, the inverse explicit dependence can be found for the function $\phi = \phi(x, y)$ (see (5.28)) and cannot be obtained for the functions $\eta = \eta(x, y)$ and $\theta = \theta(x, y)$. The generating function $V$ which enters the product $F = bV$ is specified by the representation (5.4) which is of the type $V = V(x, \eta)$. Therefore, the function $V$ borrows from $\eta$ the implicit character of dependence on tangent vectors $y$. Nevertheless, the derivatives $\partial V/\partial \eta$, $\partial r/\partial \eta$, $\partial U/\partial \theta$, $\partial F/\partial \theta$, as well as $\partial V/\partial y^i$, $\partial \eta/\partial y^i$, $\partial \theta/\partial y^i$, and $\partial U/\partial y^i$ admit simple explicit algebraic representations involving the angle triple $\eta, \theta, \phi$ (see the list (5.58) in Assertion 5.3). The latter property opens up a direct way to evaluate the components of the covariant unit vector $l_i = \partial F/\partial y^i$ and metric tensor $\{g_{ij}\}$ in concise forms. With these key objects at hands, we are able to clarify the structure of the indicatrix metric tensor and many other objects.

The (Class II)-metric function $F^{(P>1)}$ shears with (Class I)-metric function $F^{(P<1)}$ the smoothness of class $C^2$ on all the subspace $\mathcal{L}F_{(x)} \subset T_xM$ bounded by the pseudo-Finsleroid. Violations of differentiability meet only on the two-axes section $S_{b,i(3)}\{x\} \subset T_xM$. The smoothness of class $C^\infty$ holds perfect on the space $\mathcal{L}F_{(x)} \setminus S_{b,i(3)}\{x\}$. If a three-dimensional hyperplane in the tangent space $T_xM$ includes the vertical $b$-axis then the hyperplane is called the vertical section of $T_xM$. The two-axes section $S_{b,i(3)}\{x\} \subset T_xM$ is the vertical section of $T_xM$ if the horizontal $i_{(3)}$-axis also belongs to the section (see more detail in [1,2]).

### A Appendix: Skew-Vanishing Lists

The Skew-Vanishing Lists $S_1, S_2, S_3$ displayed in Sect. 3 are implications from three integrability conditions entered (3.12). Let us develop the first condition

$$\frac{\partial \eta_{ij}}{\partial y^a} - \frac{\partial \eta_{in}}{\partial y^j} = 0,$$  \hspace{1cm} (A.1)

taking into account the angle representation

$$h_{ij} = -\frac{F^2}{H^2} \left(\eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j)\right) \equiv F \frac{\partial l_i}{\partial y^j}$$

indicated in Formula (2.2) of Sect. 2. Using the Particular Expansion (3.8) of $\eta_{ij}$, from (A.1) we straightforwardly obtain
\[0 = \frac{\partial \eta_{ij}}{\partial y^n} - \frac{\partial \eta_{in}}{\partial y^i} = \frac{1}{F} l_i (l_n \eta_j - l_j \eta_n) - \frac{1}{F} (h_{in} \eta_j - h_{ij} \eta_n) - \frac{1}{F} (l_j \eta_{in} - l_n \eta_{ij}) \]

\[+ 2 u_2 \frac{1}{H^2} \left( \sinh \eta \sin^2 \theta \cosh \eta \eta_n + \sinh^2 \eta \sin \theta \cos \theta \eta_n \right) \phi_i \phi_j \]

\[-2 u_2 \frac{1}{H^2} \left( \sinh \eta \sin^2 \theta \cosh \eta \eta_j + \sinh^2 \eta \sin \theta \cos \theta \eta_j \right) \phi_i \phi_n \]

\[+ u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_j \left[ - \frac{1}{F} (l_i \phi_n + l_n \phi_i) + r_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta (\phi_i \theta_n + \phi_n \theta_i) + r_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_n \right. \]

\[+ r_5 \frac{1}{H^2} \sinh \eta \sin (\phi_i \eta_n + \phi_n \eta_i) \right] \]

\[-u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_n \left[ - \frac{1}{F} (l_i \phi_j + l_j \phi_i) + r_1 \frac{1}{H^2} \sinh^2 \eta \sin \theta (\phi_i \theta_j + \phi_j \theta_i) + r_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j \right. \]

\[+ r_5 \frac{1}{H^2} \sinh \eta \sin (\phi_j \eta_n + \phi_n \eta_i) \right] \]

\[+ 2 u_3 \frac{1}{H^2} \sinh \eta \cosh \eta \theta_i (\theta_j \eta_n - \theta_n \eta_j) \]

\[+ u_3 \frac{1}{H^2} \sinh^2 \eta \phi_j \left[ - \frac{1}{F} (l_i \theta_n + l_n \theta_i) + z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_n + z_4 \frac{1}{H^2} \sinh \eta (\eta_i \theta_n + \eta_n \theta_i) \right] \]

\[-u_3 \frac{1}{H^2} \sin^2 \eta \theta_n \left[ - \frac{1}{F} (l_i \theta_j + l_j \theta_i) + z_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + z_4 \frac{1}{H^2} \sinh \eta (\eta_i \theta_j + \eta_j \theta_i) \right] \]

\[+ u_6 \frac{1}{H^2} \eta \left[ - \frac{1}{F} (l_i \eta_n + l_n \eta_i) + u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_n + u_3 \frac{1}{H^2} \sinh^2 \eta \theta_i \eta_n + u_6 \frac{1}{H^2} \eta_i \eta_n \right]\]
\[-u_6 \frac{1}{H^2} \eta_n \left[ -\frac{1}{F}(l_i \eta_j + l_j \eta_i) + u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + u_3 \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j + u_6 \frac{1}{H^2} \eta_i \eta_j \right] \]

\[+ u_{2n} \frac{1}{H^2} \sin^2 \eta \sin^2 \theta \phi_i \phi_j + u_{3n} \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j + u_{6n} \frac{1}{H^2} \eta_i \eta_j \]

\[-u_{2j} \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j - u_{3j} \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j - u_{6j} \frac{1}{H^2} \eta_i \eta_j. \]

After convenient rearrangements we get

\[\frac{1}{F^2} l_i (l_i \eta_j - l_j \eta_i) - \frac{1}{F^2} (h_{in} \eta_j - h_{ij} \eta_n) - \frac{1}{F} (l_j \eta_n - l_i \eta_j) \]

\[+ u_2 \left( -u_6 \frac{1}{H^4} \sinh^2 \eta \sin^2 \theta + 2 \frac{1}{H^2} \sinh \eta \sin^2 \theta \cosh \eta + r_5 \frac{1}{H^4} \sinh^2 \eta \sin^2 \theta \right) \phi_i (\eta_n \phi_j - \eta_j \phi_n) \]

\[+ u_2 \left( 2 \frac{1}{H^2} \sinh^2 \eta \sin \theta \cos \theta + r_1 \frac{1}{H^4} \sinh^4 \eta \sin^3 \theta \right) \phi_i (\theta_n \phi_j - \theta_j \phi_n) \]

\[+ u_2 \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \frac{1}{F} \phi_i (\phi_n l_j - \phi_j l_n) \]

\[+ u_3 \left( -u_6 \frac{1}{H^4} \sinh^2 \eta + 2 \frac{1}{H^2} \sinh \eta \cosh \eta + z_4 \frac{1}{H^4} \sinh^3 \eta \right) \theta_i (\theta_j \eta_n - \theta_n \eta_j) \]

\[-u_3 \frac{1}{H^4} z_2 \sinh^4 \eta \sin^2 \theta \phi_i (\theta_n \phi_j - \theta_j \phi_n) + u_3 \frac{1}{H^2} \sinh^2 \eta \frac{1}{F} \theta_i (\theta_n l_j - \theta_j l_n) \]

\[+ u_6 \frac{1}{H^2} \frac{1}{F} \eta_i (\eta_n l_j - \eta_j l_n) \]

\[+ u_{2n} \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + u_{3n} \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j + u_{6n} \frac{1}{H^2} \eta_i \eta_j \]

\[-u_{2j} \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j - u_{3j} \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j - u_{6j} \frac{1}{H^2} \eta_i \eta_j = 0. \]

Here we apply the Symmetrizing Condition (3.34) \( z_2 \sinh^2 \eta \sin \theta = r_1 \sinh^2 \eta \sin^2 \theta + 2H^2 \cos \theta \), such that we are left with
\[ u_2 \left( -u_6 \frac{1}{H^4} \sinh \eta + 2 \frac{1}{H^2} \cosh \eta + r_5 \frac{1}{H^4} \sinh^2 \eta \sin \theta \right) \sinh \eta \sin^2 \theta \phi_i (\eta_n \phi_j - \eta_j \phi_n) \]

\[ + u_3 \left( -u_6 \frac{1}{H^4} \sinh \eta + 2 \frac{1}{H^2} \cosh \eta + z_4 \frac{1}{H^4} \sinh^2 \eta \right) \sinh \eta \theta_i (\theta_n \eta_j - \theta_n \eta_j) \]

\[ + u_{2n} \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j + u_{3n} \frac{1}{H^2} \sinh^2 \eta \theta_i \theta_j + u_{6n} \frac{1}{H^2} \eta_i \eta_j \]

\[-u_{2j} \frac{1}{H^2} \sinh^2 \eta \sin^2 \theta \phi_i \phi_j - u_{3j} \frac{1}{H^2} \sinh^2 \eta \theta_i \eta_n - u_{6j} \frac{1}{H^2} \eta_i \eta_n = \frac{1}{F^2} (h_{in} \eta_j - h_{ij} \eta_n). \]

Inserting

\[ u_{2n} = u_{2\theta} \eta_n + u_{2\phi} \phi_n, u_{3n} = u_{3\theta} \eta_n + u_{3\phi} \phi_n, u_{6n} = u_{6\theta} \eta_n + u_{6\phi} \phi_n, \]

reduces the equality under study to

\[ u_{2\theta} = u_{2\phi} = u_{6\theta} = u_{6\phi} = 0, \]

\[ \left( u_{2n} + \frac{1}{H^2} u_2 (u_2 - u_6) \right) \sinh^2 \eta \sin^2 \theta \phi_i (\eta_n \phi_j - \eta_j \phi_n), \]

and

\[ + \left( u_{2n} + \frac{1}{H^2} u_2 (u_2 - u_6) \right) \sinh^2 \eta \theta_i (\eta_n \theta_j - \eta_j \theta_n) = \frac{H^2}{F^2} (h_{in} \eta_j - h_{ij} \eta_n). \]

The tensor \( h_{ij} \) can be expanded in accordance with the angle representation indicated below (A.1). In this way, the integrability condition (A.1) is reduced to the following set

\[ u_{2\theta} = u_{2\phi} = u_{6\theta} = u_{6\phi} = 0, \quad u_{2\eta} + \frac{1}{H^2} u_2 (u_2 - u_6) = 1, \]  

(A.2)

and

\[ z_{4\theta} = z_{4\phi} = 0, \quad (r_5 \sin \theta)_{\theta} = r_{5\phi} = 0. \]  

(A.3)

Thus the List \( S_1 \) (see (3.13)-(3.15)) is valid.

The similar method of evaluation leads to verify the Lists \( S_2 \) and \( S_3 \).

B Appendix: Evaluation of the curvature tensor
Below we show how the equalities

\[ L_2 = L_3, \quad F^2(C_{pq} C_{ijn} - C_{pm} C_{ijq}) = T^*(h_{pq} h_{jn} - h_{pn} h_{jq}), \quad (B.1) \]

\[ T^* = -\frac{1}{H^2} \left( (L_2)^2 + L_2(u_6 - 2L_2) \right), \quad (B.2) \]

and

\[ T^* = 1 - H^2, \quad L_2(u_6 - 2L_2) = L(z_3 \sinh \eta - L) \quad (B.3) \]

(see (3.39)-(3.44)) can be arrived at.

To this end we use the expansion (3.36) of the tensor \( C_{ijn} \) written in Sect. 3 in terms of the unit vectors and evaluate the contraction

\[
H^2 F^2 C_{pq} C_{ijn} = \left[ u_6 u^i u_p u_q + L_3(u^i m_p m_q + u_q m_p m^i + u_p m^i m_q) \right. \\
+ L_2(u^i p_p p_q + u_q p^i p_p + u_p p^i p_q) + z_3 \sinh \eta m^i m_p m_q \\
+ L(m_p p^i p_q + m_q p^i p_p + m^i p_p p_q) + r_2 \sinh \eta \sin \theta p^i p_p p_q \times \\
\left. \left[ u_6 u^i u_j u_n + L_3(u_i m_j m_n + u_n m_j m_i + u_j m_i m_n) \right. \\
+ L_2(u_i p_j p_n + u_n p_i p_j + u_j p_i p_n) + z_3 \sinh \eta m_i m_j m_n \\
+ L(m_j p_i p_n + m_n p_i p_j + m_i p_j p_n) + r_2 \sinh \eta \sin \theta p_i p_j p_n \right] \right].
\]

We obtain

\[
H^2 F^2 C_{pq} C_{ijn} = -(u_6 u_p u_q + L_3 m_p m_q + L_2 p_p p_q)(u_6 u_j u_n + L_3 m_j m_n + L_2 p_j p_n) \\
- \left( L_2(u_q p_p + u_p q_p) + L(m_q p_q + m_p p_q) + r_2 \sinh \eta \sin \theta p_p p_q \right) \times \\
\left( L_2(u_n p_j + u_j p_n) + L(m_j p_n + m_n p_j) + r_2 \sinh \eta \sin \theta p_j p_n \right) \\
- \left( L_3(u_q m_p + u_p m_q) + z_3 \sinh \eta m_p m_q + L p_p p_q \right) \times \\
\]
\[
(L_3(u_mj + u_jm_n) + z_3 \sinh \eta m_j m_n + Lp_j p_n),
\]

after which we get

\[
H^2 F^2 (C^i_{pq} C_{ij} - C^i_{pn} C_{ijq})
\]

\[
= -(u_6 u_p q + L_3 m_p m_q + L_2 p_p p_q) (u_6 u_j u_n + L_3 m_j m_n + L_2 p_j p_n) - [qn]
\]

\[
-L_2 u_q p_p \left( L_2(u_mj + u_j p_n) + L(m_j p_n + m_n p_j) + r_2 \sinh \eta \sin \theta p_j p_n \right) - [qn]
\]

\[
-L_2 u_p q_q \left( L_2(u_mj + u_j p_n) + L(m_j p_n + m_n p_j) + r_2 \sinh \eta \sin \theta p_j p_n \right) - [qn]
\]

\[
-L m_p q_q \left( L_2(u_mj + u_j p_n) + L(m_j p_n + m_n p_j) + r_2 \sinh \eta \sin \theta p_j p_n \right) - [qn]
\]

\[
-L m_q p_p \left( L_2(u_mj + u_j p_n) + L(m_j p_n + m_n p_j) + r_2 \sinh \eta \sin \theta p_j p_n \right) - [qn]
\]

\[
-r_2 \sinh \eta \sin \theta p_p q_q \left( L_2(u_mj + u_j p_n) + L(m_j p_n + m_n p_j) \right) - [qn]
\]

\[
-L_3 u_q m_p \left( L_3(u_mj + u_j m_n) + z_3 \sinh \eta m_j m_n + Lp_j p_n \right) - [qn]
\]

\[
-L_3 u_p m_q \left( L_3(u_mj + u_j m_n) + z_3 \sinh \eta m_j m_n + Lp_j p_n \right) - [qn]
\]

\[
-z_3 \sinh \eta m_p m_q \left( L_3(u_mj + u_j m_n) + z_3 \sinh \eta m_j m_n + Lp_j p_n \right) - [qn]
\]

\[
-L p_p q_q \left( L_3(u_mj + u_j m_n) + z_3 \sinh \eta m_j m_n \right) - [qn].
\]

Now we reduce similar terms, arriving at the representation

\[
H^2 F^2 (C^i_{pq} C_{ij} - C^i_{pn} C_{ijq})
\]
\[ = -(u_6 u_p u_q + L_3 m_p m_q + L_2 p_p p_q)(u_6 u_j u_n + L_3 m_j m_n + L_2 p_j p_n) - [q n] \]

\[-(L_2)^2(u_j u_q p_p p_n + u_p u_n p_j p_q) - L L_2 u_q p_p (m_j p_n + m_n p_j) - [q n] \]

\[-L L_2 u_p q m_n p_j - L^2 (m_n m_p p_q p_j + m_q m_j p_n p_p) - [q n] \]

\[-L L_2 m_p p_q u_n p_j - [q n] - L L_2 m_q p_p (u_n p_j + u_j p_n) - [q n] \]

\[-L_3 u_q m_p \left( L_3 u_j m_n + L p_j p_n \right) - L_3 u_p m_q \left( L_3 u_n m_j + L p_j p_n \right) - [q n] \]

\[-L z_3 \sin \eta (m_p m_q p_j p_n + m_j m_n p_p p_q) - L L_3 p_p q (u_n m_j + u_j m_n) - [q n] \]

which can be written in the simpler form:

\[ H^2 F^2 \left( C^i_{p q} C_{j i n} - C^i_{p n} C_{j i q} \right) \]

\[= -(u_6 u_p u_q + L_3 m_p m_q + L_2 p_p p_q)(u_6 u_j u_n + L_3 m_j m_n + L_2 p_j p_n) - [q n] \]

\[-(L_2)^2(u_j u_q p_p p_n + u_p u_n p_j p_q) - [q n] \]

\[-L L_2 \left( u_q p_p m_j p_n + u_p p_q m_n p_j + m_p p_q u_n p_j + m_q p_p u_j p_n \right) - [q n] \]

\[-L^2 (m_n m_p p_q p_j + m_q m_j p_n p_p) - [q n] \]

\[-(L_3)^2(u_j u_q m_p m_n + u_p u_n m_j m_q) - [q n] \]

\[-L z_3 \sin \eta (m_p m_q p_j p_n + m_j m_n p_p p_q) - [q n] \]

\[-L L_3 \left( p_p q (u_n m_j + u_j m_n) + p_j p_n (u_q m_p + u_p m_q) \right) - [q n]. \] (B.4)

From the obtained expansion (B.4) the validity of the equalities (B.1)-(B.3) just follows.

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