Generalized Scaling for Models with Multiple Absorbing States

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Abstract

At a continuous transition into a nonunique absorbing state, particle systems may exhibit nonuniversal critical behavior, in apparent violation of hyperscaling. We propose a generalized scaling theory for dynamic critical behavior at a transition into an absorbing state, which is capable of describing exponents which vary according to the initial configuration. The resulting hyperscaling relation is supported by simulations of two lattice models.
Nonequilibrium phase transitions continue to elicit great interest in the physical and biological sciences. In hopes of better understanding nonequilibrium critical phenomena, models exhibiting phase transitions into an absorbing state are under intensive study in statistical physics. The dichotomy between an absorbing (dead, inactive) state and an active one arises naturally in such diverse areas as catalysis [1], epidemiology [2, 3, 4], and the transition to turbulence [5].

The essential features of the phase transition are typified by the contact process (CP) [2]. In the CP, each site of the lattice $\mathbb{Z}^d$ is either occupied or vacant. Occupied sites become vacant at unit rate, whilst a vacant site $i$ becomes occupied at rate $\lambda q_i$, with $q_i$ the fraction of occupied nearest neighbors of $i$. Evidently, the vacuum is absorbing. The growth rate $\lambda$ determines the ultimate survival of the population: for $\lambda < \lambda_c$ the vacuum is the unique steady state, but for $\lambda > \lambda_c$ ($\simeq 3.298$ in one dimension), there is also an active stationary state characterized by a nonzero particle density $\rho \propto (\lambda - \lambda_c)^\beta$. The transition at $\lambda_c$ is a nonequilibrium critical point, belonging to the universality class of directed percolation (DP) [4] and Reggeon field theory [8]. Indeed, studies of a host of models provide ample support for the conjecture that continuous transitions into a unique absorbing state generically fall in the DP class [3, 10].

The situation regarding models with multiple absorbing states is more complex. On one hand, studies of some two-dimensional catalysis models yield critical exponents different from those of DP [1, 2, 13]. On the other, the one-dimensional pair contact process (PCP) and dimer reaction (DR) clearly fall in the DP class, as far as static critical behavior is concerned [14, 15]. (In all of these models, the number of absorbing configurations grows exponentially with lattice size.) The dynamic critical properties of the PCP and DR were found, surprisingly, to be nonuniversal, the exponents depending upon the nature of the initial configuration [13]. While this variation is quite regular, it appears to violate a basic hyperscaling relation amongst the exponents $\delta$, $\eta$, and $z$ (defined below), suggesting a breakdown of the well-established scaling theory.

This apparent breakdown has prompted us to reexamine the scaling hypothesis for models with multiple absorbing states. We arrive at a scaling theory in which additional exponents are expected to depend upon the starting configuration, and in which the exponents satisfy a generalized hyperscaling relation. The latter is verified in simulations of the DR and of a new model called the threshold transfer process. In what follows we define the models, present the scaling theory, and report the numerical evidence supporting it.

Simulations of the threshold transfer process (TTP) permit us to study nonuniversal critical spreading in the context of a three-state model, providing a check on the robustness of earlier findings [13]. In the TTP each site may be vacant, or singly or doubly occupied, corresponding to $\sigma_i = 0, 1, 2$. In each cycle of the evolution, a site $i$ is chosen at random. If $\sigma_i = 0$, then $\sigma_i \to 1$ with probability $r$; if $\sigma_i = 1$, then $\sigma_i \to 0$ with probability $1 - r$. (“0” and “1” sites are left unchanged with probabilities $1 - r$ and $r$, respectively.) In the absence of doubly-occupied sites, we have a trivial dynamics in which a fraction $r$ of the
sites have $\sigma_i = 1$, in the steady state. However, if $\sigma_i = 2$, the particles may move to neighboring sites. If $\sigma_{i-1} < 2$, one particle moves to that site; independently, a particle moves from $i$ to $i + 1$ if $\sigma_{i+1} < 2$. $\sigma_i$ is diminished accordingly in this deterministic, particle-conserving transfer. Survival of the doubly-occupied sites hinges on the process $(1, 2, 1) \rightarrow (2, 0, 2)$ (their number decreases or remains the same in all other events), and so depends upon the parameter $r$, which controls the particle density. The set of configurations devoid of doubly-occupied sites comprises an absorbing subspace of the dynamics, which can be avoided only if $r$ is sufficiently large. Thus we identify the density of doubly-occupied sites, $\rho_2$, as the order parameter of the threshold transfer process.

We note in passing that the TTP bears some resemblance to a sandpile model devised by Manna [16], and to a forest fire model (FFM) proposed by Bak, Tang and Weisenfeld [17, 18]. Under the correspondence: $2 \leftrightarrow$ burning tree, $1 \leftrightarrow$ live tree, and $0 \leftrightarrow$ ashes, the process $1, 2, 1 \rightarrow 2, 0, 2$ describes a burning tree setting its neighbors on fire, and $r$ represents the rate at which new trees emerge from the ashes. But the TTP permits doubly occupied sites to arise only via transfer; there is no “lightning” process, as in the FFM. A further contrast is that a rule such as $0, 2, 0 \rightarrow 1, 0, 1$ has no analog in the FFM. Despite certain common aspects, our model is therefore very different from the FFM.

The dimer reaction (DR), introduced in Ref. [15], is a lattice model in which sites may be either vacant or (singly) occupied; particles may not occupy adjacent sites. In each step of the process a site $i$ is selected at random. If $i$ is occupied, or is blocked by a neighboring particle then nothing happens. But if $i$ is open (i.e., $i$ and its neighbors are vacant), a new particle appears, which may remain at $i$, or react with another particle, depending on the occupancy of the nearby sites:

(i) If at least one of the second neighbors, $i - 2$, and $i + 2$, are occupied, then with probability $1 - p$ there is a reaction between the new particle and its neighbor (chosen at random if there is a choice), which removes them both; with probability $p$ there is no reaction and the new particle remains.

(ii) If both second neighbors are vacant, but at least one of the third neighbors is occupied, then a reaction with a third-neighbor particle may occur, with probabilities as in case (i).

(iii) If none of the second or third neighbors are occupied, the new particle remains at site $i$.

In the DR, any configuration devoid of open sites is absorbing. The order parameter is the stationary open site fraction, $\rho_o$.

Both the TTP and the DR exhibit continuous phase transitions to an absorbing state marked by a vanishing order parameter at a critical value of $r$ or of $p$. In this work we are concerned with critical spreading, that is, the evolution of a critical system from a nearly-absorbing initial configuration. The exponents describing this spreading are nonuniversal, i.e., they depend upon the particle density in the initial state [15]. Before reporting our numerical results, we present a scaling theory for such processes.

Following Grassberger and de la Torre [19], we consider an ensemble of trials, all starting from the same initial configuration: a single seed in an otherwise absorbing configuration. (For the contact process, this
means one particle in an otherwise empty lattice; for the DR, one open site; for the TTP, one doubly-occupied site.) Let \( \rho(x, t) \) denote the local order-parameter density, \( \Delta \) the distance from the critical point (\( \Delta = \lambda - \lambda_c \) in the contact process). In the critical region the system is characterized by a correlation length \( \xi \propto \Delta^{-\nu} \), and relaxation time \( \tau \propto \Delta^{-\eta} \). At the critical point the asymptotic evolution is described by power-laws; for \( \Delta \neq 0 \) the power laws are modified by scaling functions which depend upon the dimensionless ratios \( x/\xi \) and \( t/\tau \). Thus the survival probability - i.e., that a trial has evaded the absorbing state, \( \rho(x) \equiv 0 \) - is expected to follow

\[
P(t) \approx t^{-\delta} \phi(\Delta t^{1/\nu}),
\]

so that \( P \propto t^{-\delta} \) at the critical point. The order parameter density (averaged over all trials) is

\[
\rho(x, t) \approx t^{\eta - dz/2} F(x^2/t^z, \Delta t^{1/\nu}),
\]

where the \( x \)-dependence reflects symmetry and power-law critical spreading from the seed at \( x = 0 \). For \( \Delta = 0 \), one finds upon integrating Eq (2) over space, that the mean population \( n(t) \propto t^\eta \), whilst the second moment implies that the mean-square spread of the population \( R^2(t) = \langle x^2 \rangle_t \propto t^z \). The exponents \( \delta, \eta \) and \( z \) characterize critical spreading; several relations connect them with other exponents.

Consider first the CP, for which the ultimate survival probability, \( P_\infty \equiv \lim_{t \to \infty} P(t) = \overline{p} \), the stationary particle density [13]. Existence of the limit requires \( \phi(x) \propto x^{\beta/\nu} \) for large \( x \), and \( \overline{p} \propto \Delta^z \) then implies the scaling relation

\[
\delta = \beta/\nu.
\]

For \( \Delta < 0 \) and (large) fixed \( t \) we expect \( \rho(x, t) \approx e^{-x/\xi} \), which implies (since \( \xi \propto \Delta^{-\nu} \)), that for \( v < 0 \), \( F(u, v) \propto \exp(-\text{const.}\sqrt{u} |v|^{\nu}) \). In order that \( \xi \) be time-independent, we must have

\[
z = 2\nu / |\nu|.
\]

Finally, note that for \( \Delta > 0 \), the local density at any fixed \( x \), in a surviving trial, must approach \( \overline{p} \) as \( t \to \infty \). Since \( \rho(x, t) \) represents an average over all trials, we have

\[
\rho(x, t) \to P_\infty \Delta^\beta \propto \Delta^{2\beta},
\]

as \( t \to \infty \), which implies that \( F(0, v) \propto v^{2\beta} \) for large \( v \). Existence of a stationary state then requires that the exponents satisfy the hyperscaling relation

\[
4\delta + 2\eta = dz.
\]

We turn now to models such as the pair contact process (PCP) [13], the dimer reaction [13], and the threshold transfer process, which possess a multitude of absorbing configurations. Absorbing configurations in the PCP and DR can have various particle densities; the analogous variable in the TTP is the density \( \rho_i \) of singly occupied sites. We refer to this aspect of the (near-absorbing) initial configuration in a critical spreading process as the “initial density,” \( \phi_i \). One value of \( \phi_i \) is special in these models: the “natural” particle density \( \phi_c \) of the quasistationary critical process. (For the DR, \( \phi_c \approx 0.418 \); for the TTP, \( \phi_c \approx 0.69 \).) Simulations indicate that in each of these one-dimensional models, the static critical behavior belongs to the directed percolation class, but that the exponents \( \delta, \eta \) and \( z \) are nonuniversal, varying continuously with initial density. (The critical point \( \lambda_c \), by contrast, does not change as \( \phi_i \) is varied.) Only
when \( \phi_i = \phi_c \) do the exponents assume DP values, and only then do they satisfy Eq (3). Rather than interpreting this as a violation of hyperscaling, we shall argue that in these models the scaling hypothesis must be modified, leading to a generalized hyperscaling relation.

We assume that as in models with a unique absorbing state, the order parameter density has the scaling form

\[
\rho(x, t) \simeq t^{-\nu' - dz'/2} G(x^2 / t^{z'}, \Delta t^{1/\nu'|}),
\]

(7)

where the primed exponents are functions of \( \phi_i \). Similarly, we suppose the survival probability follows

\[
P(t) \simeq t^{-\delta'} \Phi(\Delta t^{1/\nu|}).
\]

(8)

Since the stationary distribution is unique, we have as before that

\[
\rho(x, t) \to P_\infty(\phi_i) \Delta^\beta,
\]

(9)

as \( t \to \infty \), with \( \beta \) the usual directed percolation exponent. However there is no reason to suppose that \( P_\infty(\phi_i) \propto \rho_i \), when \( \phi_i \neq \phi_c \). In fact if this were so, we would have \( \delta' \nu'| = \beta \), implying that the primed exponents satisfy Eq (3). Since they do not, we conclude that the exponent governing the ultimate survival probability must also depend upon \( \phi_i \), i.e., \( P_\infty \propto \Delta^{\beta'} \), with \( \beta' = \delta' \nu'|'. \) By the same arguments applied to the CP we find

\[
z' = 2 \nu'| / \nu|',
\]

(10)

where we have introduced exponents \( \nu|' \) and \( \nu|' \) which govern the mean lifetime and spatial extent of a cluster grown from a single seed. The asymptotic behavior of the order parameter density is now \( \rho(x, t) \to \Delta^{\beta + \beta'} \), and \( G(0, y) \propto y^{\beta + \beta'} \) for large \( y \), which implies the generalized hyperscaling relation

\[
2(1 + \frac{\beta'}{\beta'})\delta' + 2\eta' = dz'.
\]

(11)

We have verified Eq (11) in simulations of the dimer reaction and the threshold transfer process. Using time-dependent simulations (for \( t \leq 2000 \), and samples of \( 10^5 \) to \( 8 \times 10^5 \) trials), we determined the critical point of the TTP as \( r_c = 0.6894(3) \). Analysis of steady-state data for \( P_2 \), as shown in figure 1, then yields \( \beta = 0.279(5) \), in good agreement with the value for DP in \( 1 + 1 \) dimensions, \( \beta = 0.2769(2) \) [20, 21]. The exponents \( \delta', \eta' \) and \( z' \) may be determined from simulations at \( r_c \), using an initial configuration very close to the absorbing state. We studied various initial densities, including \( \phi_i = 0.69 \), the natural value. The ultimate survival probability exponent \( \beta' \) was determined from similar studies, using \( r \) values slightly above critical. The simulations begin with one doubly-occupied site at the origin; the remaining sites are taken as occupied or vacant, independently, with probabilities \( \phi_i \) and \( 1 - \phi_i \), respectively. The dynamics is restricted to an active region defined as follows. Let \( \Lambda_i \) be the set of all sites which are doubly-occupied or have a doubly-occupied neighbor, after the \( i^{th} \) step of the trial. (\( \Lambda_0 \) comprises the origin and its neighbors.) The site to be updated at step \( i + 1 \) is selected at random from \( \bigcup_{j=0}^i \Lambda_j \). Thus the evolution proceeds on an expanding set within the “light-cone” emanating from the origin. As in the DR and the PCP, distant sites are not updated until the active region reaches their neighborhood. Figure 2 shows a local-slope analysis for \( \delta, \) i.e., a plot of \( \delta(t) \equiv \ln[P(mt)/P(t)] \ln m \) vs. \( t^{-1} \), for
various initial densities. (In this study we used $m = 5$.) In figure 3 we show typical results for the ultimate survival probability, leading to an estimate for $\beta'$. 

The simulation procedure for the DR is described in Ref. [15]. On the basis of more extensive studies of the half-life $\tau$, on lattices of up to 1000 sites, we now find $p_c = 0.26401(2)$, consistent with the earlier result of 0.26400(5). We used $p$ values slightly below critical ($p_c - p \leq 0.05$) in determinations of $\beta'$ at the four initial densities studied in Ref. [15]. According to Eq 8, in a plot of $\tilde{P} \equiv \Delta^{-\beta'} P$ vs. $\tilde{t} \equiv \Delta t^{1/\nu_i}$, data for various $\Delta$ (for a particular $\phi_i$), should fall on a single curve. Figure 4 shows a reasonably good collapse of the data for each of four initial densities.

Our results for the exponents in the TTP and DR are given in Table I, together with a test of the new hyperscaling relation, Eq (11). Evidently it is confirmed to within the precision of the data. (By contrast, the DP hyperscaling relation, Eq (3), is clearly violated.) Thus the spreading and survival exponents for transitions into a nonunique absorbing state may be described using the conventional sort of scaling theory, properly generalized to allow for a dependence upon the initial density.

The dependence of $\beta'$, $\eta'$ and $\beta'$ upon initial density is quite pronounced, that of $z$ much weaker. We have made no determination of $\nu_{\perp}$, and our results for $\nu_{||}$, which come solely from the relation $\nu_{||} = \beta'/\delta'$, show no significant variation with $\phi_i$. (We find $\nu_{||} = 1.80(6)$ and 1.76(6) for the TTP and the DR, respectively; the DP value is 1.74(1).) In light of Eq (10), it appears that $\nu_{\perp}$ is not strongly dependent upon initial density either.

Further examination of the data indicates that $\delta' + \eta'$ is also very nearly constant. This is clear from the plot $\eta'$ vs. $\delta'$ for all three models (TTP, DR, and PCP) shown in figure 5. (The slope of the linear best-fit is -0.995.) It is also worth noting that the exponents of the (two-dimensional) dimer-trimer model [12] differ from those of DP, but that $\delta + \eta$ is again the same as in DP. (Simulations of the dimer-trimer model yield $\delta = 0.40(1)$, $\eta = 0.28(1)$, compared with $0.460(6)$ and $0.214(8)$, respectively, for two-dimensional DP [22].) Now $\delta' + \eta'$ is the exponent governing the population growth in surviving critical trials. Its independence of $\phi_i$, suggests that the asymptotic properties of a surviving trial are not affected by the initial density. This conclusion is strongly supported by the absence of any detectable shift in the critical point as $\phi_i$ is varied. As further support, one may note that as $t \to \infty$, only a negligible fraction of a surviving cluster is actually in contact with the external density $\phi_i$. Deep inside the cluster, the particle density must approach the natural value $\phi_o$. This point of view also implies that $z'$, which describes surviving trials exclusively, should be constant. In fact, if $z'$ and $\delta' + \eta'$ are constant, then Eqs (10) and (11) require that $\nu_{||}$ and $\nu_{\perp}$ are as well. We are led, by this line of argument, to a more economical description of critical spreading, in which all of the exponent variation follows from a single cause: the dependence of the survival probability upon initial density. The ensuing predictions regarding exponents are consistent with our numerical results, except for a small variation in $z'$ with $\phi_i$. One may argue, however, that the results for $z'$ are affected by $\phi_i$-dependent corrections to scaling, and that a more pre-
cise numerical test is needed. In summary, we believe that the most natural and parsimonious interpretation is that the initial density influences the survival probability, but not the scaling properties of surviving events.

The pair contact process, dimer reaction, and threshold transfer process all involve a second variable, $\phi$, dynamically coupled to the order parameter. A quantitative theory of the dependence survival probability, and the associated exponents $\delta'$ and $\beta'$, upon the initial density $\phi_i$ has yet to be devised. But we can offer some intuitive basis for understanding nonuniversality by suggesting that in these processes, the initial density plays a role analogous to that of a marginal parameter. Such parameters, invariant under renormalization group transformations, often give rise to exponents which vary continuously along a line of fixed points. In the present case, $\phi_i$ represents a property of the medium into which the process grows, and which is never forgotten, since to survive, a critical process must repeatedly invade new territory. A RG transformation generally involves coarse graining (which conserves density in the large), and rescaling. Such a transformation may be expected to leave $\phi_i$, the density outside the active region, invariant, while driving the correlation length of this region to zero. Indeed, the spreading exponents are insensitive to (short range) correlations in the initial state \cite{15}. A more detailed understanding of nonuniversality in these models may emerge when a suitable renormalization group scheme is devised.

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Figure Captions

Figure 1. Steady state order parameter density, $\rho_2$, versus distance from critical point in the TTP. (Main graph: log-log plot; inset: linear plot.)

Figure 2. Local slope analysis of the survival probability data for various $\phi_i$ values in the TTP. $\delta$ is estimated from the $t \to \infty$ intercept.

Figure 3. Ultimate survival probability versus distance from the critical point in the TTP, for initial density $\phi_i = 0.4$.

Figure 4. Scaling plot of the survival probability in the DR. +: $\Delta = 0.05$; $\times$: $\Delta = 0.02$; $\square$: $\Delta = 0.01$; $\circ$: $\Delta = 0.005$; $\diamond$: $\Delta = 0.002$. Initial densities (top to bottom) $\phi_i = 1/2, 0.418, 0.38, and 1/3$.

Figure 5. $\eta'$ vs $\delta'$ for the PCP, DR, and TTP. The slope of the best-fit straight line is -0.995.
Table 1: Critical exponents of the threshold transfer process and the dimer reaction. Figures in parentheses denote uncertainties in the last figure(s).

| $\phi_i$ | $\delta'$ | $\eta'$ | $z'/2$ | $\beta'$ | $\eta' + (1 + \frac{\delta'}{\beta'})\delta' - z'/2$ |
|----------|-----------|---------|--------|---------|----------------------------------|
| 0.75     | 0.136(1)  | 0.347(4)| 0.632(7)| 0.250(5)| 0.00(1)                          |
| 0.69     | 0.161(2)  | 0.319(3)| 0.632(7)| 0.279(5)| 0.00(1)                          |
| 0.60     | 0.192(2)  | 0.288(3)| 0.630(7)| 0.356(5)| 0.00(1)                          |
| 0.50     | 0.227(2)  | 0.246(2)| 0.623(7)| 0.426(5)| 0.00(1)                          |
| 0.40     | 0.270(3)  | 0.204(2)| 0.622(7)| 0.497(5)| 0.00(1)                          |
| 0.31     | 0.299(3)  | 0.169(2)| 0.617(7)| 0.556(6)| 0.00(1)                          |
| 0.30     | 0.303(3)  | 0.170(2)| 0.621(7)| 0.567(6)| 0.00(1)                          |
| 0.25     | 0.316(3)  | 0.161(2)| 0.624(7)| 0.591(6)| 0.00(1)                          |
| 0.20     | 0.342(3)  | 0.133(1)| 0.622(7)| 0.640(6)| 0.00(1)                          |
| 0.10     | 0.371(4)  | 0.097(1)| 0.615(7)| 0.705(7)| 0.00(1)                          |

Threshold Transfer Process

| 0.333    | 0.107(2)  | 0.362(3)| 0.634(3)| 0.182(10)| 0.00(1)                          |
| 0.380    | 0.133(2)  | 0.327(3)| 0.629(5)| 0.241(6)| -0.02(1)                          |
| 0.418    | 0.158(2)  | 0.302(4)| 0.626(3)| 0.275(2)| -0.01(1)                          |
| 0.500    | 0.205(5)  | 0.250(5)| 0.620(3)| 0.357(10)| -0.01(1)                          |

Dimer Reaction