A novel observable for $CP$ violation in multi-body decays and its application potential to charm and beauty meson decays

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Abstract

A novel observable measuring the $CP$ asymmetry in multi-body decays of heavy mesons, which is called the forward-backward asymmetry induced $CP$ asymmetry (FBI-$CP_A$), $A_{FB}^{CP}$, is introduced. This observable has the dual advantages that 1) it can isolate the $CP$ asymmetry associated with the interference of the $S$- and $P$-wave amplitude from that associated with the $S$- or $P$-wave amplitude alone; 2) it can effectively almost double the statistics comparing to the conventionally defined regional $CP$ asymmetry. We also suggest to perform the measurements of FBI-$CP_A$ in some three-body decay channels of charm and beauty mesons.

Keywords: $CP$ violation, multi-body decay, heavy meson, B meson, D meson

1. Introduction

$CP$ violation is an important ingredient of the Standard Model of particle physics [1], also one of the necessary conditions for the dynamical generation of the baryonic asymmetry of the Universe [2]. It was first discovered in the neutral kaon system in the year 1964 [3], and its discoveries in $B$ meson decay processes by the $B$ factories confirmed the Cabibbo-Kobayashi-Maskawa mechanism of SM [4, 5, 6, 7, 8, 9, 10]. Recently, $CP$ violation was also discovered in the charmed meson decay processes [11].
Intensive studies on \( CP \) violations in multi-body decays of beauty and charmed hadrons have been performed both theoretically \([12, 13, 14, 15, 16, 17, 18, 19, 20]\) and experimentally \([21, 22, 23, 24, 25, 26, 27, 28]\) during the last ten years. One advantage for multi-body decays is that \( CP \) violation can be studied through the phase space distribution of the decay, namely, the regional \( CP \) asymmetries distributed in the phase space. The total decay amplitudes can be expressed as a superposition of various amplitudes, which can allow the presence of different strong phases. Because of the interference effects of these amplitudes, the regional \( CP \) asymmetries in certain places of the phase space can be very large. Up to now, the regional \( CP \) asymmetry is one of the most important and extensively studied observables associated with \( CP \) violation in multi-body decays, other than the integrated \( CP \) asymmetry. Although for four-body decay channels and baryon three-body decay channels, one can also study the \( CP \) violation associated with triple product asymmetry \([29, 30, 31, 32]\).

The disadvantage of the regional \( CP \) asymmetry in multi-body decays is also obvious. Once focusing on a small region of the phase space, the experimental study of regional \( CP \) asymmetries will suffer from low statistics.

In this paper, other than the regional \( CP \) asymmetry, we are going to introduce an observable to measure the \( CP \) violation in multi-body decays of heavy mesons, which according to our analysis below, can almost effectively double the statistics comparing to the conventionally defined regional \( CP \) asymmetries. Furthermore, this observable could potentially promote the discovery of \( CP \) violation in multi-body decays of beauty and charmed mesons.

2. The Forward-Backward asymmetry induced \( CP \) asymmetry

Consider a multi-body decay \( H \rightarrow h_1 h_2 h_3 \cdots h_n \), where \( H \) is a heavy meson, and \( h_1, h_2, \cdots, h_n \) are light ones. We will focus on the phase space in the vicinity of a \( P \)-wave intermediate resonance \( X \), where, the decay will be dominated by the cascade decay \( H \rightarrow X h_3 \cdots h_n, X \rightarrow h_1 h_2 \). The part of the phase space which we focus on satisfies \( (m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2 \), where \( s_{12} \) is the invariant mass squared of \( h_1 \) and \( h_2 \), \( m_X \) is the mass of \( X \), \( \sigma_X \) is of the same order with the decay width of \( X \), \( \Gamma_X \). Let us denote the relative angle between the momenta of \( h_1 \) and \( H \) in the rest frame of \( h_1 \) and \( h_2 \) system (hence, of \( X \)) as \( \theta_1^* \). Then, the part of phase space that we focus on can be further divided into two parts according to whether \( \theta_1^* \) is
larger or smaller than $\pi/2$. An observable, describing the forward-backward asymmetry in these two parts of the phase space, can be defined as

$$A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB} = \frac{\Gamma_H(c_{q_1} > 0) - \Gamma_H(c_{q_1} < 0)}{\Gamma_H(c_{q_1} > 0) + \Gamma_H(c_{q_1} < 0)}. \quad (1)$$

where $c_{q_1} \equiv \cos \theta_1^*$, $\Gamma_H(c_{q_1} > 0)$ and $\Gamma_H(c_{q_1} < 0)$ are the regional decay widths of $H \rightarrow h_1 h_2 h_3 \cdots h_n$ in the aforementioned two part of the phase space.\(^1\)

The nonzero of $A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB}$ indicates that the decay amplitude of $H \rightarrow h_1 h_2 h_3 \cdots h_n$ in the region of phase space $(m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2$ is not only just dominated by the cascade decay $H \rightarrow X(\rightarrow h_1 h_2)h_3$, but other contributions, usually $S$-wave amplitude, are also comparable. This can be seen as follows. Suppose that the amplitude of $H \rightarrow h_1 h_2 h_3 \cdots h_n$ in the region of phase space $(m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2$ is dominated by the cascade decay $H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n$, plus an $S$-wave amplitude, so that it can be expressed as a coherent sum:

$$\mathcal{M}_{H \rightarrow h_1 h_2 h_3 \cdots h_n} = \mathcal{M}_{H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n} + \mathcal{M}_{S\text{-wave}}, \quad (2)$$

where, the amplitude of the cascade decay $H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n$ can be parameterized as $\mathcal{M}_{H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n} = a_p c_{q_1}^\ast$,\(^2\) while the $S$-wave amplitude can be parameterized as $\mathcal{M}_{S\text{-wave}} = a_S$. Then, the differential decay width will take the form

$$d\Gamma \propto |\mathcal{M}_{H \rightarrow h_1 h_2 h_3 \cdots h_n}|^2 ds_{12} dc_{q_1} d\tau \equiv \left[|a_p|^2 c_{q_1}^2 + |a_S|^2 + 2\text{Re}(a_p a_S^\ast)c_{q_1}^\ast\right] ds_{12} dc_{q_1} d\tau, \quad (3)$$

where a Jacobi factor corresponding to the variable transformation from $s_{13}$ to $c_{q_1}^\ast$ is omitted. After integrating over $ds_{12}$, $d\tau$ and $c_{q_1}^\ast$, one has

$$\Gamma_{H \rightarrow h_1 h_2 h_3 \cdots h_n}(c_{q_1}^\ast \geq 0) \propto \langle |a_p|^2 \rangle / 3 + \langle |a_S|^2 \rangle \pm \text{Re}(\langle a_p a_S^\ast \rangle), \quad (4)$$

where the angle brackets represent the phase space integration over all variables $-s_{12}$ is integrated from $(m_X - \sigma_X)^2$ to $(m_X + \sigma_X)^2$, except $c_{q_1}^\ast$, i.e.,

\(^1\)Note that $\Gamma_{H \rightarrow h_1 h_2 h_3 \cdots h_n}(c_{q_1}^\ast \geq 0)$ is in fact abbreviation for $\Gamma_{H \rightarrow h_1 h_2 h_3 \cdots h_n}((m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2, c_{q_1}^\ast \geq 0)$, where the constraint $(m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2$ has been omitted.

\(^2\)Note that there is a Breit-Wigner factor $1/(s_{12} - m_X^2 + im_X \Gamma_X)$ in $a_p$. 

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\[
(\cdot \cdot \cdot) = \int_{(m_X - \sigma_X)^2}^{(m_X + \sigma_X)^2} (\cdot \cdot \cdot) d\tau, 
\]
\(c_{g_1}^*\) is integrated from -1 to 0 or from 0 to 1, respectively. By substituting Eq. (4) into Eq. (1), the forward-backward asymmetry can be expressed as
\[
A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB} = \frac{\Re(\langle a_P a_S^* \rangle)}{\langle |a_P|^2 \rangle / 3 + \langle |a_S|^2 \rangle}.
\] (5)

From the above equation one can clearly see that the presence of both the \(P\)- and \(S\)-wave amplitudes \(M_{H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n}\) and \(M_{S\text{-wave}}\) results in nonzero forward-backward asymmetry \(A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB}\). On the other hand, if only the \(P\)-wave amplitude \(M_{H \rightarrow X(\rightarrow h_1 h_2)h_3 \cdots h_n}\) contributes, \(A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB}\) would simply be zero.

Up to this point, nothing is mentioned about the \(CP\) conjugate process \(\bar{H} \rightarrow \bar{h}_1 \bar{h}_2 \bar{h}_3 \cdots \bar{h}_n\), and hence, the \(CP\) asymmetry. It is easy to see that if \(CP\) symmetry is respected, one would simply have \(A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB} = A_{\bar{H} \rightarrow \bar{h}_1 \bar{h}_2 \bar{h}_3 \cdots \bar{h}_n}^{FB}\). On the other hand, if the \(CP\) symmetry is violated, \(A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB}\) and \(A_{\bar{H} \rightarrow \bar{h}_1 \bar{h}_2 \bar{h}_3 \cdots \bar{h}_n}^{FB}\) would not equal to each other. Consequently, one can introduce a new observable measuring the \(CP\) asymmetry of multi-body decay \(H \rightarrow h_1 h_2 h_3 \cdots h_n\), which will be called the forward-backward asymmetry \(CP\) asymmetry (FBI-CPA) hereafter and is defined as
\[
A_{CP}^{FB} = \frac{1}{2} (A_{H \rightarrow h_1 h_2 h_3 \cdots h_n}^{FB} - A_{\bar{H} \rightarrow \bar{h}_1 \bar{h}_2 \bar{h}_3 \cdots \bar{h}_n}^{FB}).
\] (6)

From the definition of FBI-CPA one can easily see that its nonzero value indeed represents the violation of \(CP\).

3. Discussions on FBI-CPA

One of the motivations for the introduction of \(A_{CP}^{FB}\) can be explained as follows. When the \(S\)-wave amplitude are comparable with the \(P\)-wave one in the vicinity of the resonance \(X\), the regional \(CP\) asymmetries for \(c_{g_1}^* > 0\) and \(c_{g_1}^* < 0\), which are conventionally defined as
\[
A_{CP}^{reg}(c_{g_1}^* \geq 0) = \frac{\Gamma_H(c_{g_1}^* \geq 0) - \Gamma_H(c_{g_1}^* \geq 0)}{\Gamma_H(c_{g_1}^* \geq 0) + \Gamma_H(c_{g_1}^* \geq 0)},
\] (7)
are correlated with each other. To see this, one just needs to reexpressed $A_{CP}(c_{\theta_1}^* \gtrsim 0)$ as

$$A_{CP}^{\text{reg}}(c_{\theta_1}^* \gtrsim 0) = \frac{1}{3} A_{CP}^P + \frac{1}{3} A_{CP}^S + \frac{2}{3} A_{CP}^S \pm \frac{\Re(\langle a_P a_0^* \rangle - \langle a_P a_0^* \rangle)}{\langle |a_P|^2 |a_S|^2 \rangle},$$  \hspace{1cm} (8)

by substituting Eq. (4) into Eq. (7), where $A_{CP}^{S/P} = \frac{\langle |a_{S/P}|^2 \rangle - \langle |a_{S/P}|^2 \rangle}{\langle |a_{S/P}|^2 \rangle + \langle |a_{S/P}|^2 \rangle}$. It can be seen that there are three terms in the numerator of the above equation, corresponding to three origins of the regional CP asymmetry, $A_{CP}(c_{\theta_1}^* \gtrsim 0)$: the CP asymmetry associated with the S- and P-wave alone, and that associated with the interference effect between the S- and P-waves. Among these three terms, the first two are the same for $A_{CP}(c_{\theta_1}^* > 0)$ and $A_{CP}(c_{\theta_1}^* < 0)$, except for the difference in the denominator, while the last one changes signs. It is easy to see that the last origin of CP asymmetry for $A_{CP}(c_{\theta_1}^* \gtrsim 0)$ is proportional to the sine of the relative strong angle between the S- and P-wave amplitudes, which, according to Watson’s theorem [33], comes from the final state interaction, so that it can be large because of its nonperturbative attribute. As a consequence, the last term in the numerator of Eq. (8) can be comparable with –some times it can even dominate over– the first two terms, resulting in a substantial difference between $A_{CP}^{\text{reg}}(c_{\theta_1}^* > 0)$ and $A_{CP}^{\text{reg}}(c_{\theta_1}^* < 0)$. In fact, it has a good chance that the signs of $A_{CP}^{\text{reg}}(c_{\theta_1}^* > 0)$ and $A_{CP}^{\text{reg}}(c_{\theta_1}^* < 0)$ are opposite because of the presence of the last term in the numerator of Eq. (8). Indeed, this kind of behaviour has already been observed in $B^\pm \rightarrow \pi^+\pi^- K^\pm$ and $B^\pm \rightarrow \pi^+\pi^- \pi^\pm$ [24], and has been studied extensively in the literature. One interesting property of the newly defined FBI-CPA is that it is capable of isolating the CP asymmetry associated with the interference of the S- and P-waves, which can be seen by expressing $A_{CP}^{FB}$ as

$$A_{CP}^{FB} = \frac{\Re(\langle \bar{a}_P a_0^* \rangle)}{\langle |a_P|^2 \rangle /3 + \langle |a_S|^2 \rangle} - \frac{\Re(\langle a_P \bar{a}_0^* \rangle)}{\langle |\bar{a}_P|^2 \rangle /3 + \langle |\bar{a}_S|^2 \rangle}.$$  \hspace{1cm} (9)

It is this property which motivates the introduction of FBI-CPA.\footnote{One can see that in contrast to the conventionally defined CP asymmetry, there are extra factors $1/\left(\langle |a_P|^2 \rangle /3 + \langle |a_S|^2 \rangle\right)$ and $1/\left(\langle |\bar{a}_P|^2 \rangle /3 + \langle |\bar{a}_S|^2 \rangle\right)$ in the above ex-}
One important issue, which has to do with the integration interval of $s_{12}$, should be pointed out here. In the above discussion, the integral of $s_{12}$ is performed symmetrically around the $X$ resonance, i.e. $(m_X - \sigma_X)^2 < s_{12} < (m_X + \sigma_X)^2$. This may result in some cancellation when obtaining the FBI-CPA or regional CPAs. This has to do with the fact that the interference term may flip its sign at $s_{12} \sim m_X^2$. To see this in more detail, let us first write out the Breit-Wigner in the $P$-wave amplitude explicitly:

$$a_P = \frac{\tilde{a}_P}{s_{12} - m_X^2 + im_X \Gamma_X},$$

where $\tilde{a}_P$ is introduced to isolate the Breit-Wigner factor from $a_P$, so that the interference term in $A^{FB}$, $A^{FB}_{CP}$, and $A^{reg}_{CP}$’s can be expressed as

$$\Re(a_P a_S^*) = \frac{2c_{\rho s}}{s_{12} - m_X^2 + im_X \Gamma_X} \left[ (s_{12} - m_X^2)\Re(a_S^* \tilde{a}_P) + m_X \Gamma_X \Im(a_S^* \tilde{a}_P) \right],$$

where the first term implies a sign flip at $s = m_X^2$. Roughly speaking, when the phases of $a_S$ and $\tilde{a}_P$ are about the same, the first term would dominate, and there will be large cancellation when $s_{12}$ is integrated from $(m_X - \sigma_X)^2$ to $(s_{12} + \sigma_X)^2$, resulting in a much smaller FBI-CPA or regional CPA for the region $(m_\rho - \sigma_\rho)^2 < s_{12} < (m_\rho + \sigma_\rho)^2$. In fact, the this kind of behaviour has already been observed by LHCb in the decay $B^\pm \rightarrow \pi^+\pi^-\pi^\pm$, in which it is shown that the regional $CP$ asymmetries for $s_{12}$ below and above the $\rho^0(770)$ mass squared, $(m_\rho - \sigma_\rho)^2 < s_{12} < m_\rho^2$ and $m_\rho^2 < s_{12} < (m_\rho + \sigma_\rho)^2$, tend to take opposite signs [26, 27]. The aforementioned cancellation can be circumvented by choosing different interval of $s_{12}$. For example, for the case of $B^\pm \rightarrow \pi^+\pi^-\pi^\pm$, one can measure the FBI-CPAs defined on $(m_\rho - \sigma_\rho)^2 < s_{12} < m_\rho^2$ and $m_\rho^2 < s_{12} < (m_\rho + \sigma_\rho)^2$, respectively, instead of that of the combined interval $(m_\rho - \sigma_\rho)^2 < s_{12} < (m_\rho + \sigma_\rho)^2$.

Besides the aforementioned motivation, another important advantage of FBI-CPA is that it can almost effectively double the statistics in the experiments. Consequently, as a complement to the reginal $CP$ asymmetries, FBI-CPA can be used in searching for $CP$ violations in some three-body decays of beauty or charmed meson, in which the $CP$ violations are expected...
to be small so that higher statistics is essential. To see this, one first needs to notice that FBI-CPA can be approximated to an experimentally useful expression, which is

$$A_{CP}^{FB} \approx \frac{[\Gamma_H(c_{\theta^*} > 0) + \Gamma_H(c_{\theta^*} < 0)] - [\Gamma_H(c_{\theta^*} < 0) + \Gamma_H(c_{\theta^*} > 0)]}{[\Gamma_H(c_{\theta^*} > 0) + \Gamma_H(c_{\theta^*} < 0)] + [\Gamma_H(c_{\theta^*} < 0) + \Gamma_H(c_{\theta^*} > 0)]},$$

(12)

if the CP violation is small. By comparing to the conventionally defined regional CP asymmetries $A_{CP}^{reg}(c_{\theta^*} \geq 0)$ in Eq. (7), it can be clearly seen that the statistics has indeed almost doubled.

It would be useful to further compare FBI-CPA with the regional CP asymmetry $A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0)$, which is defined as

$$A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0) = \frac{[\Gamma_H(c_{\theta^*} > 0) + \Gamma_H(c_{\theta^*} < 0)] - [\Gamma_H(c_{\theta^*} < 0) + \Gamma_H(c_{\theta^*} > 0)]}{[\Gamma_H(c_{\theta^*} > 0) + \Gamma_H(c_{\theta^*} < 0)] + [\Gamma_H(c_{\theta^*} < 0) + \Gamma_H(c_{\theta^*} > 0)]}.$$

(13)

Just as the cases of $B^\pm \rightarrow \pi^+\pi^-\pi^\pm$ and $B^\pm \rightarrow \pi^+\pi^-K^\pm$, although the regional CP asymmetries $A_{CP}^{reg}(c_{\theta^*} > 0)$ and $A_{CP}^{reg}(c_{\theta^*} < 0)$ around the vicinity of $\rho^0$ can be large, they tend to take opposite signs because of the interference of the S- and P-waves, hence there is cancellation when summing up the event yields to obtain the regional CP asymmetry $A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0)$. In fact, the CP asymmetry originated from the interference of the S- and P-waves is totally cancelled, which can be seen from the expression:

$$A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0) = \frac{A_{CP}^P + 3\left[\langle|a_S|^2\rangle + \langle|a_P|^2\rangle\right]A_{CP}^S}{1 + 3\left[\langle|a_S|^2\rangle + \langle|a_P|^2\rangle\right]}.$$

(14)

Consequently, FBI-CPA may take larger values than $A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0)$, making the former easier to observe.

From the above analysis, one can see that FBI-CPA serves as a complementary observable for CP asymmetries around the vicinity of $X$, along with $A_{CP}^{reg}(c_{\theta^*} > 0)$, $A_{CP}^{reg}(c_{\theta^*} < 0)$, and $A_{CP}^{reg}(c_{\theta^*} < 0 \& c_{\theta^*} > 0)$. Moreover, for some multi-body decays of $B$ and $D$ mesons, it has a good chance that CP violation could be first confirmed through the measurement of FBI-CPA.
4. Application potential to multi-body decays of charm and beauty mesons

There are a lot of channels which are suitable to perform the measurements of FBI-CP. In the $B$ meson sector, for channels such as $B^\pm \rightarrow \pi^+\pi^-K^\pm$ and $B^\pm \rightarrow \pi^+\pi^-\pi^\pm$ [24], there are very clear interference effect between $S$- and $P$-wave when the invariant mass of $\pi^+\pi^-$ lies around the vicinity of the vector resonance $\rho^0(770)$. The regional $CP$ asymmetries has already been measured by LHCb. We suggest to perform measurements of FBI-CP around $\rho^0(770)$ in these channels. For channels such as $B^\pm \rightarrow K^+K^-K^\pm$ or $B^\pm \rightarrow K^+K^-\pi^\pm$ [24], FBI-CPV around the $P$-wave resonances such as $\phi(1020)$ are also worth measuring.

Similarly, measurements of FBI-CP could potentially find evidence of $CP$ violations in $D^\pm \rightarrow K^+K^-\pi^\pm$ [34] and $D^{\pm(0)}_s \rightarrow \pi^+\pi^-\pi^\pm$ [35]. For $D^\pm \rightarrow K^+K^-\pi^\pm$, the resonances $K^*(892)$ and $\phi(1020)$ are clearly visible in the Dalitz plot. The forward-backward asymmetries for these two $P$-wave resonances are also visible. It would be interesting to check weather the $CP$ violation shows up in FBI-CPA around these resonances. For $D^\pm \rightarrow \pi^+\pi^-\pi^\pm$, the vector resonance $\rho^0(770)$ and its forward-backward asymmetry is also visible.

For an illustration, we consider the decay process $D^\pm \rightarrow K^+K^-\pi^\pm$. From FIG. 2 of Ref. [34] one can see that the forward-backward asymmetry around the resonance $K^*(892)^0$ is quite clear, indicating an interference effect. This interference is probably caused by the S-wave resonance $K^{*0}(700)$. If this is the case, the decay amplitudes in the phase space region around the vicinity of the resonance $K^*(892)^0$ can then be expressed as

$$M = \frac{1}{s_K} M_{D^\pm \rightarrow K^+K^-\pi^\pm} M_{K^\pm \rightarrow K^-\pi^\pm} + \frac{1}{s_{K^*}} M_{D^\pm \rightarrow K^+K^*0} M_{K^*0 \rightarrow K^-\pi^\pm}$$

(15)

where $s_X = s - m_X^2 + im_X\Gamma_X$ ($X = K^*$ or $K^*_0$), and the decay amplitudes can be parameterized as

$$M_{D^\pm \rightarrow K^+K^-} M_{K^- \rightarrow K^-\pi^\pm} = c_{\theta^\ast} (\lambda_s\eta_s + \lambda_d\eta_d + \lambda_b\eta_b),$$

(16)

$$M_{D^\pm \rightarrow K^+K^*0} M_{K^*0 \rightarrow K^-\pi^\pm} = \lambda_s \xi_s + \lambda_d \xi_d + \lambda_b \xi_b,$$

(17)

where $\lambda_q = V_{uq}V_{cq}^\ast$. Since $\lambda_s \gg \lambda_b$, we can rewrite the above two amplitudes into the form

$$M_{D^\pm \rightarrow K^+K^-} M_{K^\pm \rightarrow K^-\pi^\pm} = c_{\theta^\ast} (\lambda_s \tilde{\eta}_s + \lambda_b \tilde{\eta}_b),$$

(18)
\[ M_{D^+ \to K^+ K^0} M_{K^0 \to K^- \pi^+} = \lambda_s \tilde{\xi}_s + \lambda_b \tilde{\xi}_b, \quad (19) \]

where \( \tilde{\eta}_{s/b} = \eta_{s/b} - \eta_d, \) \( \tilde{\xi}_{s/b} = \xi_{s/b} - \xi_d, \) and \( \phi \) is the phase difference between \( \lambda_s \) and \( \lambda_b. \) The FBI-CPA is then approximated to be

\[ A_{FB}^{CP} \approx \frac{6 \Im \left( \left( \frac{\xi_b}{\eta_b} - \frac{\tilde{\eta}_b \tilde{\xi}_b}{\eta_b^2} \right)^* \left\langle \frac{1}{s_{K^0} s_{K^0}} \right\rangle \right) \lambda_b}{\lambda_s} \sin \phi, \quad (20) \]

The relative strong phase between the amplitudes corresponding to these two resonances can be large because of the non-perturbative effect. As a consequence, FBI-CPA will be roughly of the order \( A_{FB}^{CP} = \frac{\lambda_b}{\lambda_s} \sin \phi \sim 0.1\%, \) which is just about the same order with the regional CPAs. In order to distinct from zero for such a small value, the statistics should be large enough. In this sense, the measurement of FBI-CPA is better than that of the regional CPAs, as the former can make use of the data more efficiently. Although FBI-CPA is defined through forward-backward asymmetry, one does not need to obtain FBI-CPA by means of the measurement of forward-backward asymmetry at all for this situation. Since the CP asymmetry is quite small, one just needs to measure FBI-CPA in \( D^\pm \to K^\pm K^- \pi^\pm \) around \( K^- (892)^0 \) according to Eq. (12), from which one can see that the statistics are indeed almost doubled comparing to the regional CP asymmetry in Eq. (7).

In fact, besides the above suggested decay channels, both the measurements of the forward-backward asymmetry and FBI-CPA are meaningful in other multi-body decay channels of charm and beauty meson, provided that a \( P \)-wave resonances is presented in the Dalitz plot.

5. Conclusion

To sum up, we introduce an observable for CP violations in multi-body decays of heavy meson, the forward-backward asymmetry induced CP asymmetry, FBI-CPA. We suggest to perform the measurements of FBI-CPA in some decay channels of charm and beauty mesons.

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