ON HOMOTOPICAL AND HOMOLOGICAL $Z_n$-SETS

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Abstract. A closed subset $A \subset X$ is called a homological (homotopical) $Z_n$-set if for any $k < n+1$ and any open set $U \subset X$ the relative homology (homotopy) group $H_k(U, U \setminus A)$ vanishes. A closed subset $A$ of an LC$^n$-space $X$ is a homotopical $Z_n$-set if and only if each point $a \in A$ is a homological $Z_n$-point in $X$. Applying Hurewicz Isomorphism Theorem we prove that a homotopical $Z_2$-subset of an LC$^n$-space is a homotopical $Z_n$-set if and only if it is a homological $Z_n$-set in $X$. From the Künneth Formula we derive Multiplication, Division and $k$-Root Formulas for homological $Z_n$-sets. We prove that the set $Z_n^0(X)$ of homological $Z_n$-points of a metrizable separable lc$^n$-space $X$ is of type $G_δ$ in $X$. We introduce and study the classes $Z_n^0$ (resp. $\mathcal{Z}_n^0$) of topological spaces $X$ with $Z_n^0(X) = X$ (resp. $\mathcal{Z}_n^0(X) = X$) and prove Multiplication, Division and $k$-Root Formulas for such classes. We also show that a (locally compact lc$^n$)-space $X \in \mathcal{Z}_n^0$ has Steinke dimension $\delta(X) \geq n+1$ (has cohomological dimension $\dim_G(X) \geq n+1$ for any coefficient group $G$).

A locally compact ANR-space $X \in \mathcal{Z}_n^0$ is not a $G$-space and has extension dimension $e\dim X \leq L$ for any non-contractible CW-complex $L$.

In this paper we focus on applications of homological methods to studying $Z_n$-sets in topological spaces. Being higher-dimensional counterparts of closed nowhere dense subsets, $Z_n$-sets are of crucial importance in infinite-dimensional and geometric topologies [BRZ, BP, Dac, DW, vM, BV] and play a role also in Dimension Theory [Ba] and Theory of Selections [Us]. $Z_n$-Sets were introduced by H. Toruńczyk in [To1]. He defined a closed subset $A$ of a topological space $X$ to be a $Z_n$-set if any map $f : I^n \rightarrow X$ from the $n$-dimensional cube $I^n = [0,1]^n$ can be uniformly approximated by maps into $X \setminus A$. Actually, $Z_n$-sets work properly only in LC$^n$-spaces where they coincide with so-called homotopical $Z_n$-sets. By definition, a closed subset $A$ of a topological space $X$ is a homotopical $Z_n$-set if for any open cover $U$ of $X$ every map $f : I^n \rightarrow X$ can be approximated by a map $f : I^n \rightarrow X \setminus A$, $U$-homotopic to $f$. In fact, homotopical $Z_n$-sets are nothing else but closed locally $n$-negligible sets in the sense of H. Toruńczyk [To1].

In Section 3 we apply the Hurewicz Isomorphism Theorem to characterize homotopical $Z_n$-sets $A$ in Tychonov LC$^1$-spaces $X$ as homotopical $Z_{\min(n,\infty)}$-sets such that the relative homology groups $H_k(U, U \setminus A)$ vanish for all $k < n+1$ and all open sets $U \subset X$. Having in mind this characterization of homotopical $Z_n$-sets, we define a closed subset $A$ of a topological space $X$ to be a homotopical $Z_n$-set (more generally, a $G$-homotopical $Z_n$-set for a coefficient group $G$) if $H_k(U, U \setminus A) = 0$ (resp. $H_k(U, U \setminus A; G) = 0$) for all $k < n+1$ and all open sets $U \subset X$. Therefore, a homotopical $Z_2$-set in a Tychonov LC$^1$-space $X$ is a homotopical $Z_n$-set if and only if it is a homological $Z_n$-set. It should be mentioned that under some restrictions on the space $X$ this characterization of $Z_n$-sets has been exploited in mathematical literature [DW 4.2], [Dob], [Kr]. The homological characterization of $Z_n$-sets makes possible to apply powerful tools of Algebraic Topology for studying $Z_n$-sets. Homological $Z_n$-sets behave like usual $Z_n$-sets: the union of two $G$-homotopical $Z_n$-sets is a $G$-homotopical $Z_n$-set and so is each closed subset of a $G$-homotopical $Z_n$-set.

In Section 4 applying the technique of irreducible homological barriers, we prove that a closed subset $A \subset X$ is a homological $Z_n$-set in $X$ if each point $a \in A$ is a homological $Z_n$-point in $X$ and each closed subset $B \subset A$ with $|B| > 1$ can be separated by a homological $Z_{n+1}$-set. This characterization

2010 Mathematics Subject Classification. Primary 57N20; Secondary 54C50; 54C55; 54F35; 54F45; 55M10; 55M15; 55M20; 55N10.

Key words and phrases. $Z_n$-set, homological $Z_n$-set.

The substantial part of this paper was written during the stay of the first author in Nipissing University (North Bay, Canada).
makes it possible to apply Steinke’s separation dimension \(t(\cdot)\) and its transfinite extension \(\text{trt}(\cdot)\) to studying homological \(Z_n\)-sets. In particular, we prove that a closed subset \(A \subset X\) with finite separation dimension \(d = t(A)\) is a \(G\)-homological \(Z_n\)-set in \(X\) if each point \(a \in A\) is a \(G\)-homological \(Z_{n+d}\)-point in \(X\). An infinite version of this result asserts that a closed subset \(A \subset X\) having transfinite separation dimension \(\text{trt}(A)\) is a \(G\)-homological \(Z_{\infty}\)-set in \(X\) if and only if each point \(a \in A\) is a \(G\)-homological \(Z_{\infty}\)-point in \(X\).

In Section 5 we develop the Bockstein theory for \(G\)-homological \(Z_n\)-sets. The main result is Theorem 5.5 asserting that a subset \(A \subset X\) is a \(G\)-homological \(Z_n\)-set in \(X\) if and only if \(A\) is an \(H\)-homological \(Z_n\)-set for all groups \(H \in \sigma(G)\), where \(\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p\) is prime\} is the Bockstein family of \(G\).

The main result of Section 6 is Multiplication Theorem 6.1 for a homological (homotopical) \(Z_n\)-set \(A\) in a space \(X\) and a homological (homotopical) \(Z_m\)-set in \(Y\) the product \(X \times B\) is a homological (homotopical) \(Z_{n+m+1}\)-set in \(X \times Y\). For ANRs the homotopical version of this result has been proved by T.Banakh and Kh.Trushchak in [BT]. It is interesting that the Multiplicative Theorem for homological \(Z_n\)-sets can be partly reversed, which leads to Division and \(k\)-Root Theorems proved in Section 8.

In Section 9 we apply the obtained results on \(Z_n\)-sets to study \(Z_n\)-points. We show that the set of \(Z_n\)-points in a metrizable separable space \(X\) always is a \(G_\delta\)-set. Moreover, if \(X\) is an \(LC^n\)-space, then the sets of homological and homotopical points also are \(G_\delta\) in \(X\).

In Sections 10 and 11 we introduce and study the classes \(Z_n, Z^n\) of spaces whose all points are homotopical (resp. homological) \(Z_n\)-points, and classes \(\overline{Z}_n\) (resp. \(\overline{Z}_n\)) of spaces containing dense sets of homotopical (resp. homological) \(Z_n\)-points. Applying the results from the preceding sections, we prove Multiplication, Division and \(k\)-Root Formulas for these classes.

In Sections 12 and 13 we study dimension properties of spaces from the class \(Z^n\) (i.e., spaces whose all points are homological \(Z_n\)-points). We show that for any space \(X \in Z^n\) the transfinite separation dimension \(\text{trt}(X) \geq 1+n\). If, in addition, \(X\) is a locally compact \(LC^n\)-space, then \(X\) has cohomological dimension \(\dim G(X) \geq n+1\) for any group \(G\). Also the inequality \(e\text{-dim}(X) \leq L\) for a CW-complex \(L\) implies that \(\pi_k(L) = 0\) for all \(k \leq n\). Each locally compact locally contractible space \(X \in Z^n\) is infinite-dimensional in a rather strong sense: \(X\) fails to be a \(C\)-space and has extension dimension \(e\text{-dim}X \not\leq L\) for any non-contractible CW-complex \(L\), see Theorem 13.1.

1. Preliminaries

All topological spaces considered in this paper are Tychonov; \(I\) stands for the closed interval \([0,1]\), \(n\) will denote a non-negative integer or infinity. In the paper we use singular homology \(H_*(X;G)\) with coefficients in a non-trivial abelian group \(G\). If \(G = \mathbb{Z}\) is the group of integers, we omit the symbol \(\mathbb{Z}\) and write \(H_*(X)\) instead of \(H_*(X;\mathbb{Z})\). By \(\tilde{H}_*(X;G)\) we denote the singular homology of \(X\), reduced in dimension zero.

Let \(\mathcal{U}\) be a cover of a space \(X\). Two maps \(f, g : Z \to X\) are called

- \(\mathcal{U}\)-near (denoted by \((f, g) \prec \mathcal{U}\)) if for any \(z \in Z\) there is \(U \in \mathcal{U}\) with \(\{f(z), g(z)\} \subset U\);
- \(\mathcal{U}\)-homotopic (denoted by \(f \sim \mathcal{U} g\)) if there is a homotopy \(h : Z \times [0,1] \to X\) such that \(h(z,0) = f(z), h(z,1) = g(z)\) and \(h(\{z\} \times [0,1]) \subset U \in \mathcal{U}\) for all \(z \in Z\).

There is also a (pseudo)metric counterpart of these notions. Let \(\rho\) be a continuous pseudometric on a space \(X\) and \(\varepsilon > 0\) be a real number. Two maps \(f, g : Z \to X\) are called

- \(\varepsilon\)-near if \(\text{dist}(f,g) < \varepsilon\) where \(\text{dist}(f,g) = \sup_{z \in Z} \rho(f(z),g(z))\);
- \(\varepsilon\)-homotopic if there is a homotopy \(h : Z \times [0,1] \to X\) such that \(h(z,0) = f(z), h(z,1) = g(z)\) and \(\text{diam}_Z h(\{z\} \times [0,1]) < \varepsilon\) for all \(z \in Z\).

The following easy lemma helps to reduce the “cover” version of (homotopical) nearness to the “pseudometric” one.
Lemma 1.1. For any open cover $\mathcal{U}$ of a Tychonov space $X$ and any compact set $K \subset X$ there is a continuous pseudometric $\rho$ on $X$ such that each 1-ball $B(x, 1) = \{x' \in X : \rho(x, x') < 1\}$ centered at a point $x \in K$ lies in some set $U \in \mathcal{U}$.

Proof. Embed the Tychonov space $X$ into a Tychonov cube $I^\kappa$ for a suitable cardinal $\kappa$. For each $x \in K$ find a finite index set $F(x) \subset \kappa$ and an open subset $W_x \subset I^{F(x)}$ whose preimage $V_x = \text{pr}^{-1}_{F(x)}(W_x)$ under the projection $\text{pr}_{F(x)} : X \to I^{F(x)}$ contains the point $x$ and lies in some $U \in \mathcal{U}$. By the compactness of $K$, the open cover $\{V_x : x \in K\}$ of $K$ contains a finite subcover $\{V_{x_1}, \ldots, V_{x_m}\}$. Now consider the finite set $F = \bigcup_{i=1}^m F(x_i)$ and note that each set $V_{x_i}$ is the preimage of the some open set $W_i \subset I^F$ under the projection $\text{pr}_F : X \to I^F$. Let $d$ be any metric on the finite-dimensional cube $I^F$. By the compactness of $C = \text{pr}_F(K) \subset \bigcup_{i=1}^m W_i$ there is $\varepsilon > 0$ such that each $\varepsilon$-ball centered at a point $z \in C$ lies in some $W_i$. Finally, define the pseudometric $\rho$ on $X$ letting $\rho(x, x') = \frac{1}{\varepsilon} \cdot d(\text{pr}_F(x), \text{pr}_F(x'))$ for $x, x' \in X$. It is easy to see that each 1-ball centered at any point $x \in K$ lies in some $U \in \mathcal{U}$. □

Let us recall that a space $X$ is called an $\text{LC}^n$-space if for each point $x \in X$, each neighborhood $U$ of $x$, and each $k < n + 1$ there is a neighborhood $V \subset U$ of $x$ such that each map $f : \partial I^k \to V$ from the boundary of the $k$-dimensional cube extends to a map $f : I^k \to U$.

The following homotopy approximation theorem for $\text{LC}^n$-spaces is well-known, see [Hu] p.159 or [BV].

Lemma 1.2. For any open cover $\mathcal{U}$ of a paracompact $\text{LC}^n$-space $X$ and any $k < n + 1$ there is an open cover $\mathcal{V}$ of $X$ such that any two $\mathcal{V}$-near maps $f, g : K \to X$ from a simplicial complex $K$ of dimension $\dim K \leq k$ are $\mathcal{U}$-homotopic.

This lemma has a homological counterpart. A space $X$ is defined to be an $\text{lec}^n$-space if for each point $x \in X$, each neighborhood $U$ of $x$, and each $k < n + 1$ there is a neighborhood $V \subset U$ of $x$ such that the homomorphism $i_* : H_k(K) \to H_k(U)$ of singular homologies induced by the inclusion map $i : V \hookrightarrow U$ is trivial.

It is known that each $\text{LC}^n$-space is an $\text{lec}^n$-space, see [Un]. The converse is true for $\text{LC}^1$-spaces, see [Un]. The proof of the following lemma can be found in [Bow 5.4]

Lemma 1.3. For any paracompact $\text{lec}^n$-space $X$ and any $k < n + 1$ there is an open cover $\mathcal{U}$ of $X$ such that any two $\mathcal{U}$-near maps $f, g : K \to X$ defined on a simplicial complex $K$ of dimension $\dim K \leq k$ induce the same homomorphisms $f_*, g_* : H_*(K) \to H_*(X)$ on homologies.

In the sequel we shall need another three homological properties of $\text{lec}^n$-spaces. First, we recall one well-known fact from homological algebra, see Lemma 16.3 in [Br1].

If the commutative diagram in the category of abelian groups

\[
\begin{array}{ccc}
A_2 & \longrightarrow & A_3 \\
\downarrow{i_2} & & \downarrow{i_3} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \\
\downarrow{i_1} & & \downarrow{i_4} & & \\
C_1 & \longrightarrow & C_2
\end{array}
\]

has exact middle row, then the finite generacy (or triviality) of the groups $i_1(B_1)$ and $i_3(A_3)$ implies the finite generacy (triviality) of the group $i_4 \circ i_2(A_2)$.

A subset $A$ of a topological space $X$ is called precompact if $A$ has compact closure in $X$.

Lemma 1.4. Any precompact set $C$ in a Tychonov $\text{lc}^n$-space $X$ with $n < \infty$ has an open neighborhood $U$ such that the inclusion homomorphisms $H_k(U) \to H_k(X)$ have finitely-generated image for all $k \leq n$.

Proof. We shall prove this lemma by induction on $n$. For $n = 0$ the assertion of the lemma follows from the local connectedness of $\text{lc}^0$-spaces. Assume that for some $n$ the lemma has been proved for all
Lemma 1.5. Let $X$ be a locally compact $\mathcal{C}^n$-space and $V \subset U$ be open subsets of $X$ such that $\overline{V} \subset U$ and $\overline{U}$ is compact. Then for any $k \leq n$ the inclusion homomorphism $H_k(X, X \setminus \overline{U}) \rightarrow H_k(X, X \setminus V)$ has finitely generated range.

Proof. Take open sets $W_1 \subset W_2 \subset W_3 \subset X$ such that $W_3$ has compact closure in $X$ and $\overline{U} \subset W_1 \subset W_2 \subset W_2 \subset W_3$. The excision property for singular homology (see Theorem 2.20 of [Hat]) implies that the inclusion homomorphisms $H_k(W_1, W_1 \setminus \overline{U}) \rightarrow H_k(X, X \setminus U)$ and $H_k(W_3, W_3 \setminus V) \rightarrow H_k(X, X \setminus V)$ are isomorphisms. Thus it suffices to check that the inclusion homomorphism $H_k(W_1, W_1 \setminus \overline{U}) \rightarrow H_k(W_3, W_3 \setminus V)$ has finitely generated range. This will be done with help of the commutative diagram whose rows are the exact sequences of the pairs $(W_1, W_1 \setminus \overline{U})$, $(W_2, W_2 \setminus \overline{V})$, $(W_3, W_3 \setminus \overline{V})$ and columns are inclusion homomorphisms in homologies:

\[
\begin{array}{cccc}
H_k(W_1, W_1 \setminus \overline{U}) & \longrightarrow & H_k(W_1, W_1 \setminus \overline{U}) \\
\downarrow i_2 & & \downarrow i_3 \\
H_k(W_2) & \longrightarrow & H_k(W_2, W_2 \setminus \overline{V}) & \longrightarrow & H_k(W_2, W_2 \setminus \overline{V}) \\
\downarrow i_1 & & \downarrow i_4 \\
H_k(W_3) & \longrightarrow & H_k(W_3, W_3 \setminus \overline{V}) \\
\end{array}
\]

Lemma 14 implies that the homomorphisms $i_1$ and $i_3$ have finitely generated ranges (because $W_2$ and $W_1 \setminus \overline{U}$ have compact closures in $W_3$ and $W_3 \setminus \overline{V}$, respectively). Consequently, the inclusion homomorphism $i_4 \circ i_2 : H_k(W_1, W_1 \setminus \overline{U}) \rightarrow H_k(W_3, W_3 \setminus \overline{V})$ also has finitely generated range. \qed
Lemma 1.6. Let $X$ be a locally compact lc$^0$-space and $x$ be a point with $H_k(X, X \setminus \{x\}) = 0$ for some $k \leq n$. Then for any neighborhood $U \subset X$ of $x$ there is a neighborhood $V \subset U$ of $x$ such that the inclusion homomorphism $H_k(X, X \setminus U) \to H_k(X, X \setminus V)$ is trivial.

Proof. Without loss of generality the neighborhood $U$ has compact closure in $X$. By Lemma 1.5 there is a neighborhood $W \subset U$ of $x$ such that the inclusion homomorphism $i_W : H_k(X, X \setminus W) \to H_k(X, X \setminus V)$ has finitely generated range $\text{im}(i_W)$. Pick up finitely many generators $g_1, \ldots, g_m$ of the group $\text{im}(i_W)$. The triviality of the homotopy group $H_k(X, X \setminus \{x\})$ implies the triviality of the inclusion homomorphism $j : H_k(X, X \setminus W) \to H_k(X, X \setminus \{x\})$. Then for every $i \leq m$ the image $j(g_i) = 0$ and we can find a neighborhood $V_i \subset W$ of $x$ such that the image of $g_i$ under the inclusion homomorphism $H_k(X, X \setminus \{x\}) \to H_k(X, X \setminus V_i)$ is trivial. For the neighborhood $V = \bigcap_{i \leq m} V_i$ of $x$ the elements $g_1, \ldots, g_m$ have trivial images under the homomorphism $i_W : H_k(X, X \setminus \{x\}) \to H_k(X, X \setminus V)$. Since these elements generate the group $\text{im}(i_W)$, the inclusion homomorphism $i_W : H_k(X, X \setminus U) \to H_k(X, X \setminus V)$ is trivial.

2. Characterizing locally $n$-negligible sets

In this section we present a homological characterization of locally $n$-negligible sets. Following H. Toruńczyk [To1] we define a subset $A$ of a space $X$ to be locally $n$-negligible if given $x \in X$, $k < n + 1$, and a neighborhood $U$ of $x$ there is a neighborhood $V \subset U$ of $x$ such that for each $f : (I^k, \partial I^k) \to (X, X \setminus A)$ there is a homotopy $(h_t) : (I^k, \partial I^k) \to (X, X \setminus A)$ with $h_0 = f$ and $h_1(I^k) \subset U \setminus A$. Following [Spa], we shall say that a pair $(X, A)$ is $n$-connected if $\pi_i(X, A) = 0$ for all $i \leq n$.

Theorem 2.1. For a subset $A$ of a Tychonov space $X$ the following conditions are equivalent:

1. $A$ is locally $n$-negligible;
2. given: a simplicial pair $(K, L)$ with $\dim(K) \leq n$, a continuous pseudometric $\rho$ on $X$ and maps $\varepsilon : [K] \to (0, \infty)$ and $f : [K] \times \{0\} \cup [L] \times I \to X$ with $\rho(f(x, 0), f(x, t)) < \varepsilon(x)$ and $f(x, 1) \notin A$ for all $(x, t) \in [L] \times I$ there is $\bar{f} : [K] \times I \to X$ which extends $f$ and satisfies $\rho(\bar{f}(x, 0), f(x, 0)) < \varepsilon(x)$ and $\bar{f}(x, 1) \notin A$ for all $(x, t) \in [K] \times I$;
3. for each open set $U \subset X$ and $k < n + 1$ the relative homotopy group $\pi_k(U, U \setminus A)$ vanishes;
4. each $x \in X$ has a basis $\Omega_x$ of open neighborhoods with $\pi_k(U, U \setminus A) = 0$ for all $U \in \Omega_x$ and $k < n + 1$.

If, in addition, $X$ is an LC$^1$-space and $A$ is locally $2$-negligible in $X$, then the conditions (1)–(4) are equivalent to

5. for each open $U \subset X$ and $k < n + 1$ the relative homotopy group $H_k(U, U \setminus A)$ vanishes;
6. each $x \in X$ has a basis $\Omega_x$ of open neighborhoods with $H_k(U, U \setminus A) = 0$ for all $U \in \Omega_x$ and $k < n + 1$.

Proof. The equivalence (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) have been proved by Toruńczyk [To1, 2.3] in case of a normal space $X$. But because of Lemma [Spa, 7.4] his proof works also for any Tychonov $X$.

Now assume that $A$ is a locally $2$-negligible set in an LC$^1$-space $X$.

The implication (5) $\implies$ (6) is trivial while (3) $\implies$ (5) easily follows from the Hurewicz Isomorphism Theorem 4 in [Spa, §7.4] (see also Theorem 4.37 [Hall]) because $X$ is an LC$^0$-space and $A$ is locally $1$-negligible in $X$.

It remains to prove the implication (6) $\implies$ (1). Take any point $x \in X$ and a neighborhood $U \subset X$ of $x$. We lose no generality assuming that $U$ is connected.

The LC$^1$-property of $X$ yields a neighborhood $V \subset U$ of $x$ such that any map $f : \partial I^2 \to V$ extends to a map $\bar{f} : I^2 \to U$. Moreover, by (6) we can choose the neighborhood $V$ so that $H_k(V, V \setminus A) = 0$ for all $k < n + 1$. Replacing $V$ by a connected component containing the point $x$, if necessary, we may assume that $V$ is connected. Then the LC$^0$-property of $X$ implies that $V$ is path connected and the local 1-negligibility of $A$ in $V$ implies $V \setminus A$ is path-connected too.
We claim that every map \( f : (I^k, \partial I^k) \to (V, V \setminus A) \) with \( k < n + 1 \) is homotopic in \( U \) to a map \( \hat{f} : (I^k, \partial I^k) \to (U \cap A, U \cap A) \) which will imply the local \( n \)-negligibility of \( A \) in \( X \). This will be done by induction. For \( k \leq 2 \) the local \( k \)-negligibility of \( A \) in \( X \) follows from the local 2-negligibility and there is nothing to prove. So we assume that the local \((k - 1)\)-negligibility of \( A \) has been proved for some \( k > 2 \). Then the implication \((1) \Rightarrow (3)\) yields \( \pi_i(V, V \setminus A) = 0 \) for all \( i < k \). This means that the pair \((V, V \setminus A)\) is \((k - 1)\)-connected, which makes legal applying the Relative Hurewicz Isomorphism Theorem 4 from [Spa §7.5] to this pair.

Fix any point \(* \in \partial I^k\) and let \( x_0 = f(*)\). Then the map \( f \) can be considered as an element of the relative homotopy group \( \pi_k(V, V \setminus A, *) \). The Relative Hurewicz Isomorphism Theorem applied to the pair \((V, V \setminus A)\) implies that \( \pi_k(V, V \setminus A, x_0) = 0 \), where \( \pi_k(V, V \setminus A, x_0) \) is the quotient group of the homotopy group \( \pi_k(V, V \setminus A, x_0) \) by the normal subgroup \( G \) generated by the elements \([\gamma] - \{\alpha\}, [\alpha] \in \pi_k(V, V \setminus A, x_0), [\gamma] \in \pi_1(V, V \setminus A, x_0)\), where \( : \pi_1(V, V \setminus A, x_0) \times \pi_k(V, V \setminus A, x_0) \to \pi_k(V, V \setminus A, x_0) \) is the left action of the fundamental group on the relative \( k \)-th homotopy group, see [Spa §7.3] or [Hat]. It follows from \( 0 = \pi_k'(V, V \setminus A, x_0) = \pi_k(V, V \setminus A, x_0)/G \) that the subgroup \( G \) coincides with \( \pi_k(V, V \setminus A, x_0) \). Now consider the homomorphism \( i_* : \pi_k(V, V \setminus A, x_0) \to \pi_k(U, U \setminus A, x_0) \) induced by the inclusion of pairs \( i : (V, V \setminus A) \to (U, U \setminus A) \). We claim that \( i_* \) is a null-homomorphism. Since the group \( \pi_k(V, V \setminus A, x_0) \) is generated by the elements \([\gamma] - \{\alpha\}, [\alpha] \in \pi_k(V, V \setminus A, x_0), [\gamma] \in \pi_1(V, V \setminus A, x_0)\), it suffices to check that \( i_*([\gamma] - \{\alpha\}) = 0 \) for any such \([\alpha], [\gamma]\). Let \( j_* : \pi_1(V, V \setminus A, x_0) \to \pi_1(U, U \setminus A, x_0) \) be the homomorphism induced by the inclusion \( j : (V, x_0) \to (U, x_0) \). The local 2-negligibility of \( A \) in \( U \) and the choice of the set \( V \) implies that \( j_* = 0 \) and thus the action of the element \( j_*([\gamma]) \in \pi_1(U \setminus A, x_0) \) on \( \pi_k(U, U \setminus A, x_0) \) is trivial. The naturality of the action of the fundamental group on the homotopy groups (see Lemma 1 in [Spa §7.3]) implies that

\[
i_*([\gamma] - \{\alpha\}) = j_*([\gamma]) \cdot i_*([\alpha]) - i_*([\alpha]) = i_*([\alpha]) - i_*([\alpha]) = 0.
\]

Thus the homomorphism \( i_* : \pi_k(V, V \setminus A, x_0) \to \pi_k(U, U \setminus A, x_0) \) is trivial, which means that the map \( f : (I^k, \partial I^k, *) \to (V, V \setminus A, x_0) \) is null homotopic in \((U, U \setminus A, x_0)\) and this completes the proof of the local \( k \)-negligibility of \( A \) in \( X \).

3. Homotopical and Homological \( Z_n \)-sets

In this section we apply the characterization of locally \( n \)-negligible sets established in the preceding section to study \( Z_n \)-sets. We recall the definition of a \( Z_n \)-set and its homotopical and homological versions.

**Definition 3.1.** A closed subset \( A \) of a topological space \( X \) is defined to be

- a **\( Z_n \)-set** if for any open cover \( U \) of \( X \) and any map \( f : I^n \to X \) there is a map \( f' : I^n \to X \setminus A \) which is \( U \)-near to \( f \);
- a **homotopical \( Z_n \)-set** in \( X \) if for any open cover \( U \) of \( X \) and any map \( f : I^n \to X \) there is a map \( f' : I^n \to X \setminus A \) which is \( U \)-homotopic to \( f \);
- a **\( G \)-homological \( Z_n \)-set** in \( X \), where \( G \) is a coefficient group, if for any open set \( U \subset X \) and any \( k < n + 1 \) the relative homology group \( H_k(U \cap A, G) = 0 \);
- a **homological \( Z_n \)-set** in \( X \) if it is a \( Z \)-homological \( Z_n \)-set in \( X \).

A point \( x \) in a space \( X \) is called a **\( Z_n \)-point** if the singleton \( \{x\} \) is a \( Z_n \)-set in \( X \). By analogy we define homotopical and homological \( Z_n \)-points.

The following theorem reveals interplay between various versions of \( Z_n \)-sets.

**Theorem 3.2.** Let \( G \) be a non-trivial Abelian group and \( X \) be a Tychonov space.

1. A subset \( A \) of \( X \) is a homotopical \( Z_n \)-set in \( X \) iff \( A \) is closed and locally \( n \)-negligible set in \( X \).
2. Each homotopical \( Z_n \)-set in \( X \) is a \( Z_n \)-set in \( X \).
3. If \( X \) is an \( LC^n \)-space, then each \( Z_n \)-set in \( X \) is a homotopical \( Z_n \)-set.
4. Each homotopical \( Z_n \)-set in \( X \) is a \( G \)-homological \( Z_n \)-set.
5. Each \( G \)-homological \( Z_0 \)-set in \( X \) is a homotopical \( Z_0 \)-set.
(6) Each $G$-homological $Z_1$-set in $X$ is a $Z_1$-set.

Proof. The first item follows immediately from Theorem 2.1 while the second is trivial. The third item was proved in [16, 3.3] and follows from Lemma 1.2. The fourth item can be easily derived from Corollary 10.6 [Br2] (of the Relative Hurewicz Theorem). The fifth item follows from the fact that a relative homology group $H_0(U, U \setminus A; G)$ vanishes if and only if each path-connected component of $U$ meets the set $U \setminus A$.

To prove the sixth item assume that $A$ is a $G$-homological $Z_1$-set in $X$. Being a $G$-homological $Z_0$-set, $A$ is a homotopical $Z_0$-set in $X$. To show that $A$ is a $Z_1$-set in $X$, fix an open cover $U$ of $X$ and a map $f: I \to X$. Consider the open cover $f^{-1}(U) = \{f^{-1}(U) : U \in \mathcal{U}\}$ of the interval $I = [0, 1]$ and find a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for each $i \leq m$ the interval $[t_{i-1}, t_i]$ lies in $f^{-1}(U_i)$ for some set $U_i \in \mathcal{U}$. Let $W_m = U_m$ and $W_i = U_i \cap U_{i+1}$ for $i < m$.

Since $H_0(W_i, W_i \setminus A; G) = 0$, the path-connected component of $W_i$ containing the point $f(t_i)$ meets the set $W_i \setminus A$ at some point $x_i$. We claim that the points $x_{i-1}, x_i$ lie in the same path-connected component of $U_i \setminus A$. Assuming the converse we would get a nontrivial cycle $\alpha = g \cdot x_{i-1} - g \cdot x_i$ in $H_0(U_i \setminus A; G)$ with $g \in G$ being any non-zero element. On the other hand, this cycle is the boundary of an obvious 1-chain $\beta$ in $U_i$ and thus vanishes in the homology group $H_0(U_i; G)$. But this contradicts the exact sequence

$$0 = H_1(U_i, U_i \setminus A; G) \to H_0(U_i \setminus A; G) \to H_0(U_i; G)$$

for the pair $(U_i, U_i \setminus A)$.

Therefore $x_{i-1}, x_i$ lie in the same path-connected component of $U_i \setminus A$, ensuring the existence of a continuous map $g_i : [t_{i-1}, t_i] \to U_i \setminus A$ with $g_i(t_{i-1}) = x_{i-1}$ and $g_i(t_i) = x_i$. The maps $g_i$, $i \leq m$, compose a single continuous map $g : [0, 1] \to X \setminus A$ which is $\mathcal{U}$-near to $f$, witnessing the $Z_1$-set property of $A$. \hfill $\square$

Combining the first item of Theorem 3.2 with Theorem 2.1 we get the following important characterization of homotopical $Z_n$-sets.

**Theorem 3.3.** A homotopical $Z_2$-set $A$ in a Tychonov LC$^1$-space $X$ is a homotopical $Z_n$-set in $X$ if and only if $A$ is a homotopical $Z_n$-set in $X$.

**Remark 3.4.** Theorem 3.3 has been known as a folklore and its particular cases have appeared in literature, see e.g. [Kro, Dob, DW]. It should be also mentioned that a substantial part of [DW] is devoted to closed sets of infinite codimension (coinciding with our homological $Z_\infty$-sets).

Next, we establish some elementary properties of $G$-homological $Z_n$-sets. From now on, $G$ is a non-trivial abelian group.

**Proposition 3.5.** Let $A, B$ be $G$-homological $Z_n$-sets in a topological space $X$.

1. Any closed subset $F \subset A$ is a $G$-homological $Z_n$-set in $X$.
2. The union $A \cup B$ is a $G$-homological $Z_n$-set in $X$.

Proof. 1. We should check that $H_k(U, U \setminus F; G) = 0$ for all open sets $U \subset X$ and all $k < n + 1$. The $G$-homology $Z_n$-set property of $A$ yields $H_k(U, U \setminus A; G) = 0$. Since $(U \setminus F, U \setminus A) = (U \setminus F, (U \setminus F) \setminus A)$, we get also $H_{k-1}(U \setminus F, U \setminus A; G) = 0$. Writing the exact sequence of the triple $(U, U \setminus F, U \setminus A)$ gives us $H_k(U, U \setminus F; G) = 0$.

2. Given any $k < n + 1$ and any open set $U \subset X$ consider the exact sequence of the triple $(U, U \setminus A, U \setminus (A \cup B))$:

$$0 = H_k(U \setminus A, U \setminus (A \cup B); G) \to H_k(U, U \setminus (A \cup B); G) \to H_k(U, U \setminus A; G) = 0$$

and conclude that $H_k(U, U \setminus (A \cup B); G) = 0$. \hfill $\square$

Next, we show that in the definition of a $G$-homological $Z_n$-set we can require that $U$ runs over some base for $X$. 


Proposition 3.6. Let \( \mathcal{B} \) be a base of the topology of a topological space \( X \). A closed set \( A \subseteq X \) is a \( G \)-homological \( \mathbb{Z}_n \)-set in \( X \) if and only if \( H_k(U, U \setminus A; G) = 0 \) for all \( k < n + 1 \) and \( U \in \mathcal{B} \).

Proof. The “only if” part of the theorem is trivial. To prove the “if” part, assume that \( H_k(U, U \setminus A; G) = 0 \) for all \( k < n + 1 \) and all sets \( U \in \mathcal{B} \). Let \( \mathcal{U}_k \) be the family of all open subsets \( U \subseteq X \) such that \( H_k(U, U \setminus A; G) = 0 \). By induction on \( k < n + 1 \) we shall show that \( \mathcal{U}_k \) consists of all open subsets in \( X \).

First we verify the case \( k = 0 \). Take any non-empty open set \( U \subseteq X \). The equality \( H_0(U, U \setminus A; G) = 0 \) will follow as soon as we show that each path-connected component \( C \) of \( U \) intersects the set \( U \setminus A \). Fix any point \( c \in C \) and find a neighborhood \( V \in \mathcal{B} \) of \( x \) lying in \( U \). The equality \( H_0(V, V \setminus A; G) = 0 \) implies that \( V \setminus A \) meets each path-connected component of the set \( V \), in particular the path-connected component \( C' \) of the point \( c \) in \( V \). Since \( C' \subseteq C \), we conclude that \( U \setminus A \supseteq V \setminus A \) meets the component \( C \), which completes the proof of the case \( k = 0 \).

Assuming that for some positive \( k < n + 1 \) the family \( \mathcal{U}_{k-1} \) consists of all open subsets of \( X \), we first show that for every \( k \) \( \mathcal{U}_k \) is closed under finite unions. Indeed, given any two sets \( U, V \in \mathcal{U}_k \) we can write down the piece of the relative Mayer-Vietoris exact sequence

\[
H_k(U, U \setminus A; G) \oplus H_k(V, V \setminus A; G) \to H_k(U \cup V, (U \cup V) \setminus A; G) \to H_{k-1}(U \cap V, U \cap V \setminus A; G)
\]

and conclude that \( U \cup V \in \mathcal{U}_k \) because \( U \cap V \in \mathcal{U}_{k-1} \). Since \( \mathcal{U}_k \) contains the base \( \mathcal{B} \), it contains all possible finite unions of sets of the base. Because the singular homology theory has compact support, \( \mathcal{U} \) contains all possible unions of sets from the base \( \mathcal{B} \) and consequently, \( \mathcal{U}_k \) consists of all open sets in \( X \).

Corollary 3.7. A subset \( A \) of a topological space \( X \) is a \( G \)-homological \( \mathbb{Z}_n \)-set in \( X \) if and only if there is an open cover \( \mathcal{U} \) of \( X \) such that for every \( U \in \mathcal{U} \) the intersection \( U \cap A \) is a \( G \)-homological \( \mathbb{Z}_n \)-set in \( U \).

4. Detecting \( G \)-homological \( \mathbb{Z}_n \)-sets with help of partitions

In this section we apply the technique of irreducible homological barriers to detect \( G \)-homological \( \mathbb{Z}_n \)-sets with help of their partitions.

A closed subset \( B \) of a topological space \( X \) is called an irreducible barrier for a non-zero homology element \( \alpha \in H_n(X, X \setminus B; G) \) if for every closed subset \( A \subseteq B \) with \( A \neq B \) the image \( i^H_A(\alpha) \) under the inclusion homomorphism \( i^H_A : H_n(X, X \setminus B; G) \to H_n(X, X \setminus A; G) \) is trivial. We shall say that a subset \( A \) of a topological space \( X \) is separated by a subset \( B \) of \( X \) if the complement \( A \setminus B \) is disconnected.

In Dimension Theory closed separating sets also are referred to as partitions, see [En] 1.1.3.

We shall need two elementary properties of irreducible barriers, which were also exploited in [Ca] and [BC1].

Lemma 4.1. Let \( X \) be a topological space and \( G \) be a non-trivial abelian group.

1. Each closed subset \( A \) of \( X \) with \( H_n(X, X \setminus A; G) \neq 0 \) for some \( n \geq 0 \) contains an irreducible barrier \( B \subseteq A \) for some element \( \alpha \in H_n(X, X \setminus B; G) \).

2. If \( A \) is an irreducible barrier for some element \( \alpha \in H_n(X, X \setminus A; G) \), then \( H_{n+1}(X, X \setminus B; G) \neq 0 \) for any closed subset \( B \subseteq A \) separating \( A \).

Proof. 1. Given a closed subset \( A \subseteq X \) and a non-zero element \( \alpha \in H_n(X, X \setminus A; G) \neq 0 \), consider the family \( \mathcal{B} \) of closed subsets of \( A \) such that for every \( B \in \mathcal{B} \) the image \( i^H_A(\alpha) \) is not trivial. We claim that \( \mathcal{B} \) contains a minimal element. This will follow from Zorn Lemma as soon as we prove that the intersection \( \cap \mathcal{C} \) of any linearly ordered subfamily \( \mathcal{C} \subseteq \mathcal{B} \) belongs to \( \mathcal{B} \). Since singular homology has compact support, the homology group \( H_n(X, X \setminus \cap \mathcal{C}; G) \) is the direct limit of the groups \( H_n(X, X \setminus C; G), C \in \mathcal{C} \), see [Br2] IV.8.13. Since all the elements \( i^H_C(\alpha), C \in \mathcal{C} \), are not trivial, so is the element \( i^H_{\cap \mathcal{C}}(\alpha) \), which means that \( \cap \mathcal{C} \in \mathcal{B} \). Now, the Zorn Lemma yields a minimal element \( B \) in \( \mathcal{B} \). Let \( \beta = i^H_B(\alpha) \neq 0 \).
The minimality of $B$ implies that for any closed subset $C \subseteq B$ with $C \neq B$ we get $i^C_2(\alpha) = i^C_2(\beta) = 0$, which means that $B$ is an irreducible barrier for $\beta$.

2. Assume that $A$ is an irreducible barrier for some element $\alpha \in H_n(X, X \setminus A; G)$ and let $B$ be a closed subset separating $A$. Write $A \setminus B = U \cup V$ as the union of two disjoint non-empty open sets $U, V \subset A$. Let $C = V \cup B = A \setminus U$ and $D = U \cup B = A \setminus V$. The irreducibility of $A$ for $\alpha$ yields $i^A_2(\alpha) = i^D_2(\alpha) = 0$. Now writing a Mayer-Vietoris exact sequence for the pair $(X, X \setminus B) = (X, X \setminus C \cup X \setminus D)$, we get

$$H_{n+1}(X, X \setminus B; G) \xrightarrow{i^A_2} H_n(X, X \setminus A; G) \xrightarrow{i^C_2} H_n(X, X \setminus C; G) \oplus H_n(X, X \setminus D; G).$$

Since $f(\alpha) = (i^A_2(\alpha), -i^D_2(\alpha)) = (0, 0), 0 \neq \alpha = \partial(\beta)$ for some nontrivial element $\beta \in H_{n+1}(X, X \setminus B; G)$.

Irreducible barriers help us to prove the following theorem detecting $G$-homological $Z_n$-sets.

**Theorem 4.2.** A closed subset $A$ of a topological space $X$ is a $G$-homological $Z_n$-set in $X$ if each point of $A$ is a $G$-homological $Z_n$-point in $X$ and each closed subset $B$ of $A$ with $|B| > 1$ can be separated by a $G$-homological $Z_{n+1}$-set.

**Proof.** Assume that each point of a closed subset $A \subseteq X$ is a $G$-homological $Z_n$-point and each closed subset $B \subseteq A$ with $|B| > 1$ can be separated by a $G$-homological $Z_{n+1}$-set $B \subseteq A$ in $X$. Assuming that $A$ fails to be a $G$-homological $Z_n$-set, find an open subset $U \subset X$ and $k < n + 1$ with $H_k(U, U \setminus A; G) \neq 0$. By Lemma 4.1, the set $A \cap U$ contains an irreducible barrier $B$ for a non-zero element $\alpha \in H_k(U, U \setminus A; G)$. The set $B$ must contain more than one point, since singletons are $G$-homological $Z_n$-sets in $X$. By our assumption the closure $\overline{B}$ in $X$ can be separated by a $G$-homological $Z_{n+1}$-set $C$. Then $C \cap U$ separates $\overline{B} \cap U = B$ and hence $H_{k+1}(U, U \setminus C; G) \neq 0$ by Lemma 4.1(2). But this is not possible because $C$ is a $G$-homological $Z_{n+1}$-set in $X$.

Theorem 4.2 will be applied to show that a subset $A \subset X$ is a $G$-homological $Z_n$-set in $X$ if each point $a \in A$ is a $Z_{n+d}$-point in $X$ where $d = \text{trt}(A)$ is the separation dimension of $A$. The separation dimension $\text{trt}(\cdot)$ was introduced by G. Steinke [St] and later was extended to the transfinite separation dimension $\text{trt}(\cdot)$ by Arenas, Chatyrok and Puertas [ACP] as follows: given a topological space $X$ we write

- $\text{trt}(X) = -1$ iff $X = \emptyset$;
- $\text{trt}(X) \leq \alpha$ for an ordinal $\alpha$ if any closed subset $B \subset X$ with $|B| \geq 2$ can be separated by a closed subset $P \subset B$ with $\text{trt}(P) < \alpha$.

A space $X$ is defined to be $\text{trt}$-dimensional if $\text{trt}(X) \leq \alpha$ for some ordinal $\alpha$. In this case the ordinal

$$\text{trt}(X) = \min\{\alpha : \text{trt}(X) \leq \alpha\}$$

is called the (transfinite) separation dimension of $X$. If $X$ is not $\text{trt}$-dimensional, then we write $\text{trt}(X) = \infty$ and assume that $\alpha < \infty$ for all ordinals $\alpha$.

By transfinite induction one can show that $\text{trt}(X) \leq \text{trind}(X)$ where $\text{trind}(X)$ is the transfinite extension of the small inductive dimension $\text{ind}(X)$, see [ACP 2.9]. This implies that each countable-dimensionally completely-metrizable space is $\text{trt}$-dimensional (because $\text{trind}(X) < \omega_1$ [En 7.1.9]). On the other hand, each $\text{trt}$-dimensional compact space is a $C$-space, see [ACP 4.7]. Observe that $\text{trt}(X) \leq 0$ if and only if $X$ is hereditarily disconnected space. The strongly infinite-dimensional totally disconnected Polish space $X$ constructed in [En 6.2.4] has $\text{trt}(X) = 0$ and $\text{trind}(X) = \infty$ while a compatification $c(X)$ of $X$ with strongly countable-dimension remainder (the famous Pol’s example) has $\text{trt}(X) = \omega$ and $\text{trind}(X) = \infty$. Thus (even on the compact level) the gap between $\text{trt}(X)$ and $\text{trind}(X)$ can be huge. However, for finite-dimensional metrizable compacta $X$, the separation dimension $\text{trt}(X)$ coincides with the usual dimension $\text{dim}(X)$ (and hence with $\text{trind}(X)$), see [St].

**Theorem 4.3.** A closed subspace $A$ of a space $X$ with $m = \text{trt}(A) < \omega$ is a $G$-homological $Z_n$-set in $X$ provided each point $a \in A$ is a $G$-homological $Z_{n+m}$-point in $X$. 
Proposition 5.3. The proof is by induction of the number \( m = \text{trt}(A) \). The assertion is trivial if \( m = -1 \) (which means that \( A \) is empty). Assume that for some number \( m \) the theorem has been proved for all sets \( A \) with \( \text{trt}(A) < m \). Take a closed subset \( A \subset X \) with \( \text{trt}(A) = m \) and all points \( a \in A \) being \( G \)-homological \( Z_{n+m} \)-points in \( X \). Assuming that \( A \) fails to be a \( Z_n \)-set in \( X \), find an open set \( U \subset X \) and a number \( k < n+1 \) with \( H_k(U, U \setminus A; G) \neq 0 \). By Lemma 4.1(1), the set \( A \cap U \) contains an irreducible barrier \( B \subset A \cap U \) for some non-zero element \( \beta \in H_k(U, U \setminus B; G) \). Since singletons are \( G \)-homological \( Z_n \)-sets in \( X \), \(|B| > 1 \). Since \( \text{trt}(A) \leq m \), there is a partition \( C \subset B \) of \( B \) with \( d = \text{trt}(C) < m \). Then all points of the set \( C \) are \( G \)-homological \( Z_{n+d+1} \)-points. Now the inductive assumption guarantees that \( C \) is a \( G \)-homological \( Z_{n+1} \)-set in \( U \), which contradicts Lemma 4.1(2). The assertion is trivial if \( m = -1 \).

By the same method one can prove an infinite version of this theorem.

Theorem 4.4. A closed \( \text{trt} \)-dimensional subspace \( A \) of a space \( X \) is a \( G \)-homological \( Z_\infty \)-set in \( X \) if and only if each point \( a \in A \) is a \( G \)-homological \( Z_\infty \)-point in \( X \).

5. Bockstein Theory for \( G \)-homological \( Z_n \)-Sets

In this section, we study the interplay between \( G \)-homological \( Z_n \)-sets for various coefficient groups \( G \). Our principal instrument here is the Universal Coefficients Formula expressing the homology with respect to an arbitrary coefficient group via homology with respect to the group \( \mathbb{Z} \) of integers. The following its form is taken from [Hat, 3A.4].

Lemma 5.1 (Universal Coefficients Formula). For each pair \((X, A)\) and all \( n \geq 1 \) there is a natural exact sequence

\[
0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to H_{n-1}(X, A) \ast G \to 0
\]

and this sequence splits (though non-naturally).

Here \( G \otimes H \) and \( G \ast H \) stand for the tensor and torsion products of the groups \( G, H \), respectively. We need some information of those products.

At first some notation. By \( \Pi \) we denote the set of prime numbers. We recall that a group \( G \) is divisible if it is divisible by each number \( n \in \mathbb{N} \). The latter means that for any \( g \in G \) there is \( x \in G \) with \( n \cdot x = g \). By \( \text{Tor}(G) = \{ x \in G : \exists n \in \mathbb{N} \text{ with } nx = 0 \} \) we denote the torsion part of \( G \). It is well-known that \( \text{Tor}(G) \) is the direct sum of \( p \)-torsion parts

\[
p\text{-Tor}(G) = \{ x \in G : \exists k \in \mathbb{N} \text{ } p^k x = 0 \}
\]

where \( p \) runs over prime numbers.

The following useful result can be found in [Dyer].

Lemma 5.2. Let \( G_0, G_1 \) be non-trivial abelian groups and \( p \) be a prime number.

(1) The tensor product \( G_0 \otimes G_1 \) contains an element of infinite order if and only if both groups \( G_0 \) and \( G_1 \) contain elements of infinite order.

(2) The torsion product \( G_0 \ast G_1 \) contains an element of order \( p \) if and only if \( G_0 \) and \( G_1 \) contain elements of order \( p \).

(3) The tensor product \( G_0 \otimes G_1 \) contains an element of order \( p \) if and only if for some \( i \in \{ 0, 1 \} \) either \( G_i \) is not divisible by \( p \) and \( p\text{-Tor}(G_{1-i}) \) is divisible by \( p \) or else \( G_i/p\text{-Tor}(G_i) \) is not divisible by \( p \) and \( p\text{-Tor}(G_{1-i}) \neq 0 \) is divisible by \( p \).

The following fact follows immediately from the Universal Coefficients Formula.

Proposition 5.3. Each homological \( Z_n \)-set in a space \( X \) is a \( G \)-homological \( Z_n \)-set in \( X \) for any coefficient group \( G \).

Next, we show that the study of \( G \)-homological \( Z_n \)-sets for an arbitrary coefficient group \( G \) can be reduced to studying \( H \)-homological \( Z_n \)-sets for coefficient groups \( H \) from the countable family of so-called Bockstein groups.
The are many (more or less standard) notations for Bockstein groups. We follow those of [Kuz] and Dyer:

- \( \mathbb{Q} \) is the group of rational numbers;
- \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) is the cyclic group of a prime order \( p \);
- \( R_p \) is the group of rational numbers whose denominator is not divisible by \( p \);
- \( \mathbb{Q}_p = \mathbb{Q}/R_p \) the quasicyclic \( p \)-group.

By \( \mathcal{B} = \{ \mathbb{Q}, \mathbb{Z}, \mathbb{Q}_p, R_p : p \in \Pi \} \) we denote the family of all Bockstein groups.

To each abelian group \( G \) assign the Bockstein family \( \sigma(G) \subset \mathcal{B} \) containing the group:

- \( \mathbb{Q} \) if and only if the group \( G/\text{Tor}(G) \neq 0 \) is divisible;
- \( R_p \) if and only if \( G/p\text{-Tor}(G) \) is not divisible by \( p \);
- \( \mathbb{Z}_p \) if and only if \( p\text{-Tor}(G) \) is not divisible by \( p \);
- \( \mathbb{Q}_p \) if and only if \( p\text{-Tor}(G) \neq 0 \) is divisible by \( p \).

In particular, \( \sigma(G) = \{ G \} \) for every Bockstein group \( G \in \mathcal{B} \).

The Bockstein families are helpful because of the following fact which can be easily derived from Lemma 5.2:

**Lemma 5.4.** For abelian groups \( G, H \) the tensor product \( G \otimes H \) is trivial if and only if \( G \otimes B \) is trivial for every group \( B \in \sigma(H) \).

Combining this lemma with the Formula of Universal Coefficients we will obtain the principal result of this section.

**Theorem 5.5.** A closed subset \( A \) of a topological space \( X \) is a \( G \)-homological \( \mathbb{Z}_n \)-set in \( X \) if and only if \( A \) is an \( H \)-homological \( \mathbb{Z}_n \)-set in \( X \) for every group \( H \in \sigma(G) \).

**Proof.** Assume that a closed subset \( A \subset X \) fails to be a \( G \)-homological \( \mathbb{Z}_n \)-set in \( X \) and find \( k < n + 1 \) and an open set \( U \subset X \) with \( H_k(U, U \setminus A; G) \neq 0 \). The Universal Coefficients Formula implies that either \( H_k(U, U \setminus A) \otimes G \neq 0 \) or \( H_{k-1}(U, U \setminus A) * G \neq 0 \).

In the latter case, the group \( H_{k-1}(U, U \setminus A) * G \) contains an element of a prime order \( p \) and then both the groups \( H_{k-1}(U, U \setminus A) \) and \( G \) contain elements of order \( p \) by Lemma 5.2(2). Consequently, \( \sigma(G) \) contains a (quasi)cyclic \( p \)-group \( H \in \{ \mathbb{Z}_p, \mathbb{Q}_p \} \) and then \( H_{k-1}(U, U \setminus A) * H \neq 0 \). Applying the Formula of Universal Coefficients, we conclude that \( H_k(U, U \setminus A; H) \neq 0 \).

Next, we assume that \( H_k(U, U \setminus A) \otimes G \neq 0 \). Then Lemma 5.2 implies that \( H_k(U, U \setminus A) \otimes H \neq 0 \) for some group \( H \in \sigma(G) \). Applying the Formula of Universal Coefficients, we conclude that \( H_k(U, U \setminus A; H) \neq 0 \), which means that \( A \) fails to be an \( H \)-homological \( \mathbb{Z}_n \)-set in \( X \). This proves the “if” part of the theorem.

To prove the “only if” part, assume that \( A \) fails to be a \( H \)-homological \( \mathbb{Z}_n \)-set in \( X \) for some group \( H \in \sigma(G) \). Then for some \( k < n + 1 \) and an open set \( U \subset X \) the group \( H_k(U, U \setminus A; H) \) is not trivial. The Formula of Universal Coefficients yields that either \( H_k(U, U \setminus A) \otimes H \neq 0 \) or \( H_{k-1}(U, U \setminus A) * H \neq 0 \).

In the first case the tensor product \( H_k(U, U \setminus A) \otimes G \neq 0 \) by Lemma 5.2. In the second case, \( H_{k-1}(U, U \setminus A) * H \) contains an element of a prime order \( p \) and so do the groups \( H_{k-1}(U, U \setminus A) \) and \( H \). The inclusion \( H \in \sigma(G) \) implies then that \( G \) contains an element of order \( p \) and hence \( H_{k-1}(U, U \setminus A) * G \neq 0 \). In both cases the Formula of Universal Coefficients implies that \( H_k(U, U \setminus A; G) \neq 0 \), which means that \( A \) fails to be a \( G \)-homological \( \mathbb{Z}_n \)-set in \( X \).

Next, we study the interplay between \( G \)-homological \( \mathbb{Z}_n \)-sets for various Bockstein groups \( G \).

**Theorem 5.6.** Let \( A \) be a closed subset of a space \( X \) and \( p \) be a prime number.

(1) If \( A \) is a \( R_p \)-homological \( \mathbb{Z}_n \)-set in \( X \), then \( A \) is a \( \mathbb{Q} \)-homological and \( \mathbb{Z}_p \)-homological \( \mathbb{Z}_n \)-set in \( X \).
(2) If \( A \) is a \( \mathbb{Z}_p \)-homological \( \mathbb{Z}_n \)-set in \( X \), then \( A \) is a \( \mathbb{Q}_p \)-homological \( \mathbb{Z}_n \)-set in \( X \).
(3) If \( A \) is a \( \mathbb{Q}_p \)-homological \( \mathbb{Z}_{n+1} \)-set in \( X \), then \( A \) is a \( \mathbb{Z}_p \)-homological \( \mathbb{Z}_n \)-set in \( X \).
(4) If \( A \) is a \( R_p \)-homological \( \mathbb{Z}_n \)-set in \( X \) provided \( A \) is a \( \mathbb{Q} \)-homological \( \mathbb{Z}_n \)-set in \( X \) and a \( \mathbb{Q}_p \)-homological \( \mathbb{Z}_{n+1} \)-set in \( X \).
Proof. 1. Assuming that $A$ is not a $\mathbb{Q}$-homological $Z_n$-set in $X$, find a number $k < n+1$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Q}) \neq 0$. Since $\mathbb{Q}$ is torsion free, the formula of universal coefficients implies that $H_k(U, U \setminus A)$ contains an element of infinite order and then $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$. Applying the Formula of Universal Coefficients once more, we obtain that $H_k(U, U \setminus A; \mathbb{Q}_p) \neq 0$, which means that $A$ is not a $R_p$-homological $Z_n$-set in $X$.

Next, assume that $A$ is not a $Z_p$-homological $Z_n$-set in $X$ and find an integer number $k \leq n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$. The Formula of Universal Coefficients implies that either $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$ or $H_{k-1}(U, U \setminus A; \mathbb{Z}_p) \neq 0$. In the latter case, the group $H_{k-1}(U, U \setminus A) \otimes \mathbb{Z}_p$ contains an element of order $p$ and by Lemma 5.2, $H_{k-1}(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$. Now the Formula of Universal Coefficients implies that $H_{k-1}(U, U \setminus A; \mathbb{Q}_p) \neq 0$, which means that $A$ fails to be a $R_p$-homological $Z_{k-1}$-set in $X$.

So assume that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$. If $H_k(U, U \setminus A)$ contains an element of order $p$, then we can proceed as in the preceding case to prove that $A$ fails to be a $R_p$-homological $Z_k$-set in $X$. So we can assume that $p$-Tor($H_k(U, U \setminus A)) = 0$, which implies that the torsion part Tor($H_k(U, U \setminus A))$ is divisible by $p$. Taking into account that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$, we can apply Lemma 5.2 to find an element $a \in H_k(U, U \setminus A)$, not divisible by $p$. This element cannot belong to the torsion part of $H_k(U, U \setminus A)$. Hence the group $H_k(U, U \setminus A)$ contains an element of infinite order and so does the tensor product $H_k(U, U \setminus A) \otimes \mathbb{Z}_p = H_k(U, U \setminus A; \mathbb{Z}_p)$. This means that $A$ fails to be a $R_p$-homological $Z_k$-set in $X$.

2. Now assume that $A$ fails to be a $\mathbb{Q}_p$-homological $Z_n$-set in $X$ and find $k \leq n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Q}_p) \neq 0$. Applying the Formula of Universal Coefficients, we get $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$ or $H_{k-1}(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$. In the latter case, $H_{k-1}(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$ and hence $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$. If $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$, then by Lemma 5.2, the group $H_k(U, U \setminus A)$ contains an element not divisible by $p$ and then $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$. In both cases, the Formula of Universal Coefficients implies that $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$, which means that $A$ fails to be a $Z_p$-homological $Z_n$-set in $X$.

3. Assume that $A$ fails to be a $Z_p$-homological $Z_n$-set in $X$ and find $k \leq n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$. The Formula of Universal Coefficients yields $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$ or $H_{k-1}(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$. In the latter case $H_{k-1}(U, U \setminus A)$ contains an element of order $p$ and then $H_{k-1}(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$. Applying the Formula of Universal Coefficients once more, we get $H_k(U, U \setminus A; \mathbb{Q}_p) \neq 0$, which means that $A$ fails to be a $\mathbb{Q}_p$-homological $Z_k$-set.

Now assume that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$. If $H_k(U, U \setminus A)$ contains an element of order $p$, then $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$ and $H_{k+1}(U, U \setminus A; \mathbb{Q}_p) \neq 0$ by the Formula of Universal Coefficients. This means that $A$ fails to be a $\mathbb{Q}_p$-homological $Z_{k+1}$-set in $X$. So it remains to consider the case when the group $H = H_k(U, U \setminus A)$ has trivial $p$-torsion $p$-Tor($H))$. Since $H \otimes \mathbb{Z}_p \neq 0$, the group $H = H/p$-Tor($H))$ contains an element not divisible by $p$, see Lemma 5.2. Then $H_k(U, U \setminus A) \otimes \mathbb{Q}_p = H \otimes \mathbb{Q}_p \neq 0$ according to Lemma 5.2. Now the Formula of Universal Coefficients implies that $H_k(U, U \setminus A; \mathbb{Q}_p) \neq 0$, which means that $A$ fails to be a $\mathbb{Q}_p$-homological $Z_k$-set in $X$.

4. Assume that $A$ fails to be a $R_p$-homological $Z_n$-set in $X$. Then for some $k \leq n$ and an open set $U \subset X$ the homology group $H_k(U, U \setminus A; \mathbb{Z}_p) = H_k(U, U \setminus A) \otimes \mathbb{Z}_p$ is not trivial. By Lemma 5.2 the group $H_k(U, U \setminus A)$ contains an element of infinite order or an element of order $p$. In the first case the group $H_k(U, U \setminus A; \mathbb{Q}) = H_k(U, U \setminus A) \otimes \mathbb{Q}$ is not trivial, which means that $A$ fails to be a $\mathbb{Q}$-homological $Z_k$-set. In the second case the subgroup $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \subset H_{k+1}(U, U \setminus A; \mathbb{Q}_p)$ is not trivial, which means that $A$ fails to be a $\mathbb{Q}_p$-homological $Z_{k+1}$-set in $X$.

Combining Theorems 5.5 and 5.6, we get

**Corollary 5.7.** Let $A$ be a closed subset of a topological space $X$.

1. If $A$ is a $G$-homological $Z_n$-set in $X$ for some coefficient group $G$, then $A$ is an $H$-homological $Z_n$-set in $X$ for some divisible group $H \in \{\mathbb{Q}, \mathbb{Q}_p : p \in \mathbb{P}\}$.

2. $A$ is an $F$-homological $Z_n$-set in $X$ for a field $F$ if and only if $A$ is a $G$-homological $Z_n$-set in $X$ for the field $G \in \{\mathbb{Q}, \mathbb{Q}_p : p \in \mathbb{P}\}$ with $\{G\} = \sigma(F)$. 

6. Multiplication Theorem for homotopical and homological $\mathbb{Z}_n$-sets

In this section we discuss so-called Multiplication Theorems for homotopical and $R$-homological $\mathbb{Z}_n$ sets with $R$ being a principal ideal domain. The latter means that $R$ is a commutative ring with unit and without zero divisors in which each proper ideal is generated by a single element. A typical example of a principal ideal domain is the ring $\mathbb{Z}$ of integers.

According to the Künneth Formula [Spa, Th.5.3.10], for closed subsets $A \subset X$, $B \subset Y$ in topological spaces $X,Y$ and a principal ideal domain $R$ the relative homology group $H_n(X \times Y, X \times Y, A \times B; R)$ is isomorphic to the direct sum of the $R$-modules

$$[H_*(X, X \setminus A; R) \otimes_R H_*(Y, Y \setminus B; R)]_n = \oplus_{i+j=n} H_i(X, X \setminus A; R) \otimes_R H_j(Y, Y \setminus B; R)$$

and

$$[H_*(X, X \setminus A; R) \ast_R H_*(Y, Y \setminus B; R)]_n = \oplus_{i+j=n-1} H_i(X, X \setminus A; R) \ast_R H_j(Y, Y \setminus B; R).$$

Here $G \otimes_R H$ and $G \ast_R H$ stand for the tensor and torsion products of $R$-modules $G,H$ over $R$. If $R = \mathbb{Z}$, then we omit the subscript and write $G \otimes H$ and $G \ast H$. It is known that the torsion product over a field $F$ is always trivial. In this case,

$$H_n(X \times Y, X \times Y \setminus A \times B; F) = \left[ H_*(X, X \setminus A; F) \otimes_F H_*(Y, Y \setminus B; F) \right]_n.$$

With help of the Künneth Formula and Theorem 5.5 we shall prove Multiplication Formulas for homological and homotopical $\mathbb{Z}_n$-sets.

**Theorem 6.1.** Let $A \subset X$, $B \subset Y$ be closed subsets in Tychonov spaces $X,Y$, and $G$ be a coefficient group.

1. If $A$ is a $G$-homological $\mathbb{Z}_n$-set in $X$ and $B$ is a $G$-homological $\mathbb{Z}_m$-set in $Y$, then $A \times B$ is a $G$-homological $\mathbb{Z}_{n+m}$-set in $X \times Y$. Moreover, if $\sigma(G) \subset \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \}$, then $A \times B$ is a $G$-homological $\mathbb{Z}_{n+m+1}$-set in $X \times Y$.

2. If $A$ is a homotopical $\mathbb{Z}_n$-set in $X$ and $B$ is a homotopical $\mathbb{Z}_m$-set in $Y$, then $A \times B$ is a homotopical $\mathbb{Z}_{n+m+1}$-set in $X \times Y$.

**Proof.** 1. In light of Theorem 5.5 it suffices to prove the first item only for a Bockstein group $G \in \{ \mathbb{Q}, \mathbb{Z}_p, R_p, \mathbb{Q}_p : p \in \Pi \}$. Assume that $A$ is a $G$-homological $\mathbb{Z}_n$-set in $X$ and $B$ is a $G$-homological $\mathbb{Z}_m$-set in $Y$.

First, we prove the second part of the first item, assuming that $G \in \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \}$ is a principal ideal domain. We have to check that $A \times B$ is a $G$-homological $\mathbb{Z}_{n+m+1}$-set in $X \times Y$, which means that $H_k(W, W \setminus A \times B; G) = 0$ for every $k \leq n + m + 1$ and every open set $W \subset X \times Y$. By Lemma 5.9 it suffices to consider the case $W = U \times V$ for some open sets $U \subset X$ and $V \subset Y$.

By the Künneth Formula, the homology group $H_k(U \times V, U \times V \setminus A \times B; G)$ is isomorphic to the direct sum of the groups

$$\bigoplus_{i+j=k} H_i(U, U \setminus A; G) \otimes_G H_j(V, V \setminus B; G) \quad \text{and} \quad \bigoplus_{i+j=k+1} H_i(U, U \setminus A; G) \ast_G H_j(V, V \setminus B; G).$$

Observe that for every $i, j$ with $i + j \leq n + m + 1$ either $i \leq n$ or $j \leq m$. In the first case the group $H_i(U, U \setminus A; G)$ is trivial (because $A$ is a $G$-homological $\mathbb{Z}_n$-set in $X$), in the second case the group $H_j(V, V \setminus B; G) = 0$. Hence the above tensor and torsion products are trivial and so is the group $H_k(U \times V, U \times V \setminus A \times B; G)$.

Next, assume that $G = \mathbb{Q}_p$ for a prime $p$. We need to prove that $A \times B$ is a $G$-homological $\mathbb{Z}_{n+m+1}$-set in $X \times Y$. As in the preceding case this reduces to showing that for every $k \leq n + m$ and open sets $U \subset X$, $V \subset Y$ the homology group $H_k(U \times V, U \times V \setminus A \times B; \mathbb{Q}_p)$ is trivial. By the Formula of Universal Coefficients, this group is isomorphic to the direct sum of the groups $H_k(U \times V, U \times V \setminus A \times B) \otimes \mathbb{Q}_p$ and $H_{k-1}(U \times V, U \times V \setminus A \times B) \ast \mathbb{Q}_p$. So it suffices to prove the triviality of these two groups.

The triviality of the group $H_{k-1}(U \times V, U \times V \setminus A \times B) \ast \mathbb{Q}_p$ will follow as soon as we prove that the group $H_{k-1}(U \times V, U \times V \setminus A \times B)$ has no $p$-torsion. Assuming the converse and using the Künneth
Formula, we conclude that either the group
\[ \bigoplus_{i+j=k-1} H_i(U, U \setminus A) \otimes H_j(V, V \setminus B) \] or the group
\[ \bigoplus_{i+j=k-2} H_i(U, U \setminus A) * H_j(V, V \setminus B) \]
contains an element of order \( p \).

In the latter case there are \( i, j \) with \( i + j = k - 2 \) such that \( H_i(U, U \setminus A) * H_j(V, V \setminus B) \) contains an element of order \( p \). By Lemma 5.2 both the groups \( H_i(U, U \setminus A) \) and \( H_j(V, V \setminus B) \) contain elements of order \( p \) and then \( H_i(U, U \setminus A) \otimes \mathbb{Q}_p \) and \( H_j(V, V \setminus B) \otimes \mathbb{Q}_p \) are not trivial and so are the homology group \( H_{i+1}(U, U \setminus A; \mathbb{Q}_p) \) and \( H_{j+1}(V, V \setminus B; \mathbb{Q}_p) \) by the Formula of Universal Coefficients. Since \( A \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set in \( X \) and \( B \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set in \( Y \), the non-triviality of the two latter homology groups implies that \( i \geq n \) and \( j \geq m \) and hence \( k - 2 = i + j \geq n + m \geq k \), which is a contradiction.

Next, we consider the case when for some \( i, j \) with \( i + j = k - 1 \) the group \( H_i(U, U \setminus A) \otimes H_j(V, V \setminus B) \) contains an element of order \( p \). To simplify notation let \( H_i = H_i(U, U \setminus A) \), \( H_j = H_j(V, V \setminus B) \). Since \( H_i \otimes H_j \) contains an element of order \( p \), we can apply Lemma 5.2(3) to conclude that either \( H_i \) or \( H_j \) contains an element of order \( p \). Without loss of generality, \( H_i \) contains an element of order \( p \). It follows from the Formula of Universal Coefficients and the fact that \( A \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set in \( X \) that \( i \geq n \). Then \( j = k - 1 - i \leq n - m - 1 - i < m \). Since \( B \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set in \( Y \), the Formula of Universal Coefficients implies that \( H_j = H_j(V, V \setminus B) \) has no \( p \)-torsion. Taking into account that \( H_i \) has \( p \)-torsion and \( H_i \otimes H_j \) contains an element of order \( p \), we can apply Lemma 5.2(3) to conclude that \( H_j = H_j/p\text{-Tor}(H_i) \) is not divisible by \( p \) and hence \( H_j(V, V \setminus B) \otimes \mathbb{Q}_p = H_j \otimes \mathbb{Q}_p \neq 0 \) by Lemma 5.2. The Formula of Universal Coefficients implies now that \( H_j(V, V \setminus B; \mathbb{Q}_p) \neq 0 \), which contradicts the fact that \( B \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set (because \( j \leq m \)). This completes the proof of the triviality of the group \( H_{k-1}(U \times V, U \times V \setminus A \times B) \otimes \mathbb{Q}_p \).

Next, we check the triviality of the group \( H_k(U \times V, U \times V \setminus A \times B) \otimes \mathbb{Q}_p \). By the Künneth Formula, the group \( H_k(U \times V, U \times V \setminus A \times B) \) is isomorphic to the direct sum of the groups
\[ \bigoplus_{i+j=k} H_i(U, U \setminus A) \otimes H_j(V, V \setminus B) \] and
\[ \bigoplus_{i+j=k-1} H_i(U, U \setminus A) * H_j(V, V \setminus B). \]
The second sum is a torsion group and hence its tensor product with \( \mathbb{Q}_p \) is trivial. So, it suffices to prove that \( (H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)) \otimes \mathbb{Q}_p = 0 \) for any \( i, j \) with \( i + j = k \). Since \( k \leq n + m \) either \( i \leq n \) or \( j \leq m \). If \( i \leq n \), we can use the fact that \( A \) is a \( \mathbb{Q}_p \)-homological \( Z_m \)-set in \( X \) to conclude that the homology group \( H_i(U, U \setminus A; \mathbb{Q}_p) \) is trivial and so is the tensor product \( H_i(U, U \setminus A) \otimes \mathbb{Q}_p \) according to the Formula of Universal Coefficients. The associativity of the tensor product implies that \( (H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)) \otimes \mathbb{Q}_p \) is trivial as well. If \( j \leq m \), then the triviality of the above tensor product follows from the \( \mathbb{Q}_p \)-homological \( Z_m \)-set property of \( B \).

2. Let \( A \) be a homotopical \( Z_n \)-set in \( X \) and \( B \) be a homotopical \( Z_m \)-set in \( Y \). It will be convenient to uniformize the notations and put \( X_0 = X \), \( X_1 = Y \), \( A_0 = A \), \( A_1 = B \), \( k_0 = n \), \( k_1 = m \) and \( k = k_0 + k_1 + 1 \). So we need to prove that the product \( A_0 \times A_1 \) is a homotopical \( Z_k \)-set in \( X_0 \times X_1 \) provided \( A_1 \) is a homotopical \( Z_{k_1} \)-set in \( X_1 \) for \( i \in \{0, 1\} \).

Let \( \mathcal{U} \) be an open cover of \( X_0 \times X_1 \) and \( f = (f_0, f_1) : I^k \to X_0 \times X_1 \) be a map of the \( k \)-dimensional cube. Then the sets \( f_i(I^k) \subset X_i \), \( i \in \{0, 1\} \), are compact and so is their product. A standard compactness argument yields finite covers \( \mathcal{U}_i \) of \( f_i(I^k) \) by open subsets of \( X_i \) for \( i \in \{0, 1\} \) such that for any sets \( U_0 \in \mathcal{U}_0 \), \( U_1 \in \mathcal{U}_1 \) the product \( U_0 \times U_1 \) lies in some set \( U \in \mathcal{U} \).

By Lemma 1.1 each space \( X_i \) has a continuous pseudometric \( \rho_i \) such that any 1-ball \( B(x, 1) = \{x' \in X_i : \rho_i(x, x') < 1\} \) centered at a point \( x \in f_i(I^k) \) lies in some set \( U \in \mathcal{U}_i \). Then the pseudometric \( \rho = \max\{\rho_0, \rho_1\} \)
\[ \rho((x_0, x_1), (x'_0, y'_1)) = \max\{\rho_0(x_0, x'_0), \rho_1(x_1, y'_1)\} \]
on \( X_0 \times X_1 \) has a similar feature: any 1-ball \( B(a, 1) \) centered at a point \( a \in f_0(I^k) \times f_1(I^k) \) lies in some set \( U \in \mathcal{U} \).
Let $T$ be a triangulation of $I^k$ so fine that the image $f(\sigma)$ of any simplex $\sigma \in T$ has $\rho$-diameter $< 1/3$. Let $K_0$ be the $k_0$-dimensional skeleton of the triangulation $T$ and $K_1$ be the dual skeleton consisting of all simplexes of the barycentric subdivision of $T$ that do not meet the skeleton $K_0$. It is well-known (and easy to see) that $K_1$ has dimension $k_1 = k - k_0 - 1$ and each point $z \in I^k$ lying in a simplex $\sigma \in T$ can be written as $z = (1 - \lambda(z))z_0 + \lambda(z)z_1$ for some points $z_i \in K_i$, $i \in \{0, 1\}$, and some real number $\lambda(z) \in [0, 1]$. This number is uniquely determined by the point $z$ and is equal to zero iff $z \in K_0$ and equal to 1 iff $z \notin K_1$. Moreover, the point $z_i$ is uniquely determined by $z$ iff $z \notin K_{1-i}$ for $i \in \{0, 1\}$. This means that the cube $I^k$ has the structure of a subset of the join $K_0 * K_1$. This structure allows to write the cube $I^k$ as $I^k = K_{\leq 1/2} \cup K_{\geq 1/2}$, where $K_{\leq 1/2} = \{z \in I^k : \lambda(z) \leq 1/2\}$ and $K_{\geq 1/2} = \{z \in I^k : \lambda(z) \geq 1/2\}$.

Let $\ell : [0, 1] \to [0, 1]$ be the piecewise linear map determined by the conditions $\ell(0) = 0 = \ell(1)$ and $\ell(1) = 1$. Combined with the joint structure $K_0 * K_1$, the map $\ell$ induces two piece-linear maps $h_i : I^k \to I^k$, $i \in \{0, 1\}$, assigning to each point $z = \lambda_0 z_0 + \lambda_1 z_1$ where $z_i \in K_i$, $\lambda_0 + \lambda_1 = 1$ the point $h_i(z) = \lambda_0 z_0 + \lambda_1 z_1$ where $\lambda_i = \ell(\lambda_i)$ and $\lambda_i = 1 - \lambda_i$. The crucial property of the maps $h_i$ is that $h_0(K_{\leq 1/2}) \subset K_0$, $h_1(K_{\geq 1/2}) \subset K_1$ and both $h_0$ and $h_1$ are $S$-homotopic to the identity map of $I^k$ with respect to the cover $S$ of $I^k$ by maximal simplexes of the triangulation $T$.

Applying Theorem 2.1 to the homotopical $Z_{k_i}$-set $A_i \subset X_i$ find a map $g_i : K_i \to X_i \setminus A_i$, $1/6$-homotopic to the map $f_i|K_i : K_i \to X_i$ with respect to the pseudometric $\rho_i$. Since $K_i$ is a subcomplex of $I^k$, we may apply Borsuk Homotopy Extension Theorem (see [Spa 1.1D]) and extend the map $g_i : K_i \to X_i$ to a map $\tilde{g}_i : I^k \to X_i$, $1/6$-homotopic to $f_i$. Finally consider the map $\tilde{f} = (\tilde{f}_0, \tilde{f}_1) : I^k \to X_0 \times X_1$, where $\tilde{f}_i = \tilde{g}_i \circ h_i$. We claim that $\tilde{f}$ is $U$-homotopic to $f$ and $\tilde{f}(I^k) \cap (A_0 \times A_1) = \emptyset$.

The $S$-homotopy of the maps $h_i$ to the identity implies the $\tilde{g}_i(S)$-homotopy of $\tilde{g}_i \circ h_i$ to $\tilde{g}_i$. Now observe that for each simplex $\sigma$ of the triangulation $T$ we get

$$\text{diam}(\tilde{g}_i(\sigma)) \leq \text{diam}(f_i(\sigma)) + 2\text{dist}(f_i, \tilde{g}_i) < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$ 

Consequently, $\tilde{f}_i$ is $2/3$-homotopic to $\tilde{g}_i$. Since $\tilde{g}_i$ is $1/6$-homotopic to $f_i$, we get that $\tilde{f}_i$ is $1$-homotopic to $f_i$ and consequently, $\tilde{f}$ is $1$-homotopic to $f$. Now the choice of the pseudometric $\rho$ implies that $\tilde{f}$ is $U$-homotopic to $f$.

So it remains to prove that $\tilde{f}(z) \notin A_1 \times A_2$ for every point $z \in I^k$. Indeed, if $z \in K_{\leq 1/2}$, then $h_0(z) \in K_0$ and $\tilde{f}_0(z) = \tilde{g}_0(h_0(z)) \in \tilde{g}_0(K_0) \subset X_0 \setminus A_0$. A similar argument yields $\tilde{f}_1(z) \notin A_1$ provided $z \in K_{\geq 1/2}$.

7. Functions $z(A, X)$ and $z^G(A, X)$

It will be convenient to write Theorem 6.1 in terms of the functions

$z(A, X) = \sup\{n \in \omega : A \text{ is a homotopical } Z_n\text{-set in } X\}$ and

$z^G(A, X) = \sup\{n \in \omega : A \text{ is a } G\text{-homological } Z_n\text{-set in } X\}$

defined for a closed subset $A$ of a space $X$ and a coefficient group $G$. In this definition we put $\sup \emptyset = -1$. So $z^G(A, X) = -1$ iff $A$ is not a $G$-homotopical $Z_0$-set in $X$. For a point $x$ in $X$ of a topological space $X$ we write $z^G(x, X)$ instead of $z^G(\{x\}, X)$.

Rewriting Theorems 3.2, 3.3, 4.3, 5.5, and 5.6 in the terms of the functions $z(A, X)$ and $z^G(A, X)$ we obtain

**Theorem 7.1.** Let $A$ be a closed subset of a topological space $X$, and $G$ be a coefficient group. Then

1. $z(A, X) \leq z^G(A, X) \leq z^G(\{x\}, X) = \min_{H \in \sigma(G)} z^H(A, X)$;
2. $z(A, X) = z^G(A, X)$ if $X$ is an LC$_1$-space and $z(A, X) \geq 2$;
3. $z^G(A, X) + \text{trt}(A) \geq \min_{A \in A} z^G(A, X)$;
4. $\min\{z^G(A, X), z^G(p, A, X) - 1\} \leq z^R_p(A, X)$ and $z^R_p(A, X) \leq \min\{z^G(A, X), z^G_p(A, X)\}$;
5. $z^G_p(A, X) \leq z^G_p(A, X) \leq z^G_p(A, X) + 1$.

Also, Theorem 6.1 can be rewritten in the form of Multiplication Formulas.
Theorem 7.2 (Multiplication Formulas). Let $A, B$ be closed subsets in Tychonov spaces $X, Y$, and $G$ be a coefficient group. Then

1. $z^G(A \times B, X \times Y) \geq z^G(A, X) + z^G(B, Y)$;
2. $z^G(A \times B, X \times Y) \geq z^G(A, X) + z^G(B, Y) + 1$ if $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \mathbb{P}\}$;
3. $z(A \times B, X \times Y) \geq z(A, X) + z(B, Y) + 1$.

8. Division and $k$-Root Formulas for homological $\mathbb{Z}_n$-sets

It turns out that inequalities in the Multiplication Formulas [7.2] can be partly reversed, which leads to so-called Division and $k$-Root Formulas. We start with Division Formulas for Bockstein coefficient groups.

Lemma 8.1. Let $A \subset X$, $B \subset Y$, $C \subset \mathbb{Z}$ be closed subsets in topological spaces $X, Y, \mathbb{Z}$, and $p$ be a prime number. Then

1. $z^F(A \times B, X \times Y) = z^F(A, X) + z^F(B, Y) + 1$ for a field $F$;
2. $z^{Q_p}(A \times B, X \times Y) \leq z^{Q_p}(A, X) + z^{Q_p}(B, Y) + 2;
3. z^{R_p}(A, X) + 1 \geq \min\{z^{R_p}(A \times B, X \times Y) - z^{Q_p}(B, Y), z^{R_p}(A, C, X \times Z) - z^{Q_p}(C, Z)\}$ if $\max\{z^{Q_p}(B, Y), z^{Q_p}(C, Z)\} < \infty$.

Proof. 1. Let $F$ be a field. By the Multiplication Theorem [6.1(1)], $z^F(A \times B, X \times Y) \geq z^F(A, X) + z^F(B, Y) + 1$. So it remains to prove the reverse inequality, which is trivial if one of the numbers $n = z^F(A, X)$ or $m = z^F(B, Y)$ is infinite. So assume that $n, m < \infty$ and find open sets $U \subset X$ and $V \subset Y$ such that the homology groups $H_{n+1}(U, U \setminus A; F)$ and $H_{m+1}(V, V \setminus B; F)$ are not trivial. Their tensor product $H_{n+1}(U, U \setminus A; F) \otimes F H_{m+1}(V, V \setminus B; F)$ over the field $F$ is not trivial as well. Now the Künneth Formula implies that the group $H_{n+m+2}(U \times V, U \times V \setminus A \times B; F)$ is not trivial, which means that $A \times B$ is not an $F$-homological $Z_{n+m+2}$-set in $X \times Y$ and hence $z^F(A \times B, X \times Y) \leq n + m + 1 = z^F(A, X) + z^F(B, Y) + 1$.

2. The second item follows from the first one and Theorem [7.1(5)]:

$z^{Q_p}(A \times B, X \times Y) \leq z^{Q_p}(A \times B, X \times Y) + 1 = z^{Q_p}(A, X) + z^{Q_p}(B, Y) + 2 \leq z^{Q_p}(A, X) + z^{Q_p}(B, Y) + 2$.

3. To prove the third item, let $n = z^{Q_p}(A, X)$ and find an open set $U \subset X$ and $i \leq n + 1$ with $H_i(U, U \setminus A; R_p) \neq 0$. The Formula of Universal Coefficients yields $H_i(U, U \setminus A) \otimes R_p \neq 0$. By Lemma [5.2], the group $H_i(U, U \setminus A)$ contains an element of infinite order or of order $p$.

In the first case, use the fact that $B$ is not a $Q$-homological $Z_{m+1}$-set in $Y$ for $m = z^{Q_p}(B, Y) < \infty$ to find an open set $V \subset Y$ with $H_{m+1}(V, V \setminus B; Q) \neq 0$. By the Formula of Universal Coefficients, $H_{m+1}(V, V \setminus B) \otimes Q \neq 0$ and hence $H_{m+1}(V, V \setminus B)$ contains an element of infinite order and so does the tensor product $H_i(U, U \setminus A) \otimes H_{m+1}(V, V \setminus B)$, which lies in the homology group $H_{i+m+1}(U \times V, U \times V \setminus A \times B)$ according to the Künneth Formula. Then the tensor product

$H_{i+m+1}(U \times V, U \times V \setminus A \times B) \otimes R_p = H_{i+m+1}(U \times V, U \times V \setminus A \times B; R_p)$

is not trivial. This means that $A \times B$ fails to be an $R_p$-homological $Z_{i+m+1}$-set in $X \times Y$ and thus

$z^{R_p}(A \times B, X \times Y) \leq i + m \leq 1 + n + m$. Consequently,

$z^{R_p}(A, X) + 1 = n + 1 \geq z^{R_p}(A \times B, X \times Y) - z^{Q_p}(B, Y)$.

Next, assume that $H_i(U, U \setminus A)$ contains an element of order $p$ and use the fact that the set $C$ fails to be a $Q_p$-homological $Z_{m+1}$-set in $Z$ for the number $m = z^{Q_p}(C, Z) < \infty$ to find an open set $V \subset Z$ with $H_{m+1}(V, V \setminus C; Q_p) \neq 0$. Then either $H_{m+1}(V, V \setminus C) \otimes Q_p \neq 0$ or $H_{m+1}(V, V \setminus C) \ast Q_p \neq 0$. In the latter case $H_{m+1}(V, V \setminus C)$ contains an element of order $p$ and so does the torsion product $H_i(U, U \setminus A) \ast H_{m}(V, V \setminus C)$ which lies in $H_{i+m+1}(U \times V, U \times V \setminus A \times C)$ by the Künneth Formula. Applying Lemma [5.2] we see that

$H_{i+m+1}(U \times V, U \times V \setminus A \times C) \otimes R_p = H_{i+m+1}(U \times V, U \times V \setminus A \times C; R_p)$

is not trivial. This means that $A \times C$ is not a $R_p$-homological $Z_{i+m+1}$-set in $X \times Z$. 

Finally assume that \( H_{m+1}(V, V \setminus C) \otimes \mathbb{Q}_p \neq 0 \). By Lemma 5.2, the group \( H_{m+1}(V, V \setminus C) / p \cdot \text{Tor}(H_{m+1}(V, V \setminus C)) \) is not divisible by \( p \) and hence the tensor product \( H_i(U, U \setminus A) \otimes H_{m+1}(V, V \setminus C) \) contains an element of order \( p \). By the Künneth Formula, the latter tensor product lies in \( H_{i+m+1}(U \times V, U \times V \setminus A \times C) \).

Applying Lemma 5.2 again, we obtain that the tensor product

\[
H_{i+m+1}(U \times V, U \times V \setminus A \times C) \otimes R_p = H_{i+m+1}(U \times V, U \times V \setminus A \otimes C; R_p)
\]

is not trivial. In both the cases \( A \times C \) is not a \( R_p \)-homological \( Z_{i+m+1} \)-set in \( X \times Z \) and hence

\[
z^{R_p}(A \times C, X \times Z) \leq i + m \leq n + 1 + z^{Q_p}(C, Z).
\]

Then \( z^{R_p}(A, X) + 1 = n + 1 \geq z^{R_p}(A \times C, X \times Z) - z^{Q_p}(C, Z). \)

\[\square\]

Now we can establish Division Formulas in the general case. First, to each coefficient group \( G \) assign two families \( d(G), \varphi(G) \subset \{ \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi \} \) as follows. Put

- \( d(\mathbb{Q}) = \varphi(\mathbb{Q}) = \{ \mathbb{Q} \} \),
- \( d(\mathbb{Z}_p) = \varphi(\mathbb{Z}_p) = \{ \mathbb{Z}_p \} \),
- \( d(\mathbb{Q}_p) = \varphi(\mathbb{Q}_p) = \{ \mathbb{Z}_p \} \),
- \( d(R_p) = \{ \mathbb{Q}, \mathbb{Q}_p \}, \varphi(R_p) = \{ \mathbb{Q}, \mathbb{Z}_p \} \)

and let

\[
d(G) = \bigcup_{H \in \sigma(G)} d(H) \quad \text{and} \quad \varphi(G) = \bigcup_{H \in \sigma(G)} \varphi(H).
\]

In particular, \( d(\mathbb{Z}) = \{ \mathbb{Q}, \mathbb{Q}_p : p \in \Pi \} \) is the family of divisible Bockstein groups and \( \varphi(\mathbb{Z}) = \{ \mathbb{Q}, \mathbb{Z}_p : p \in \Pi \} \) is the family of Bockstein fields.

**Theorem 8.2** (Division Formulas). *Let \( A \subset X, B \subset Y \) be closed subsets in topological spaces and \( G \) be a coefficient group. Then*

1. \( z^G(A \times B, X \times Y) \leq 2 + z^G(A, X) + \sup_{F \in \varphi(G)} z^F(B, Y); \)
2. \( z^G(A \times B, X \times Y) \leq 1 + z^G(A, X) + \sup_{H \in d(G)} z^H(B, Y) \) provided \( \sigma(G) \subset \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \} \).

**Proof.** 1. The inequality \( z^G(A \times B, X \times Y) \leq 2 + z^G(A, X) + \sup_{F \in \varphi(G)} z^F(B, Y) \) is trivial if \( n = z^G(A, X) \) is infinite. So assume that \( n \) is finite and using Theorem 7.11, find a Bockstein group \( D \in \sigma(G) \) with \( z^D(A, X) = n \). Theorem 5.5 implies that \( z^G(A \times B, X \times Y) \leq z^D(A \times B, X \times Y) \). If \( D \) is a field, then \( \{ D \} = \varphi(D) \subset \varphi(G) \) and

\[
z^G(A \times B, X \times Y) \leq z^D(A \times B, X \times Y) = 1 + z^D(A, X) + z^D(B, Y) \leq 1 + z^G(A, X) + \sup_{H \in \varphi(G)} z^H(B, Y)
\]

by Lemma 8.1.

If \( D = R_p \) for some \( p \), then \( \varphi(R_p) = \{ \mathbb{Q}, \mathbb{Z}_p \} \subset \varphi(G) \) and by Lemma 8.1 we get

\[
z^G(A \times B, X \times Y) \leq z^{R_p}(A \times B, X \times Y) \leq 1 + z^{R_p}(A, X) + \max\{ z^{Q}(B, Y), z^{Q_p}(B, Y) \} \leq
\]

\[
1 + z^{R_p}(A, X) + \max\{ z^{Q}(B, Y), z^{Q_p}(B, Y), 1 \} \leq
\]

\[
2 + \max\{ z^{Q}(B, Y), z^{Q_p}(B, Y) \} \leq 2 + z^G(A, X) + \sup_{F \in \varphi(G)} z^F(B, Y).
\]

If \( D = \mathbb{Q}_p \) for some \( p \), then

\[
z^G(A \times B, X \times Y) \leq z^{Q_p}(A \times B, X \times Y) \leq 2 + z^{Q_p}(A, X) + z^{Q_p}(B, Y) \leq 2 + z^G(A, X) + \sup_{F \in \varphi(G)} z^F(B, Y)
\]

according to Lemma 8.1(2).

2. If \( \sigma(G) \subset \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \} \) then the last case in the preceding item is excluded and we can repeat the preceding argument to prove the inequality

\[
z^G(A \times B, X \times Y) \leq 1 + z^G(A, X) + \sup_{H \in d(G)} z^H(B, Y).
\]

\[\square\]
Theorem 8.3. Let $A \subset X$, $B \subset Y$ be closed nowhere dense subsets in Tychonov spaces $X, Y$. Then

1. $1 + z(A, X) + z(B, Y) \leq z(A \times B, X \times Y) \leq z(A \times B, X \times Y)$;
2. $z(A \times B, X \times Y) = z^2(A \times B, X \times Y)$ provided $X, Y$ are LC$_1$-spaces.

Proof. The first item follows from Theorem 6.1(2) and 7.1(1).

To prove the second item assume that $X, Y$ are LC$_1$-spaces, consider three cases.

i) $z^2(A, X) = z^2(B, X) = 0$. In this case $z(A, X) = z^2(A, X) = z^2(B, X)$. So $z(A, X) + z(B, Y) + 1 = z^2(A, X) + z^2(B, Y) + 1 = z^2(A \times B, X \times Y) \geq z(A \times B, X \times Y) \geq z(A, X) + z(B, Y) + 1$ by Lemma 8.1(1) and Theorem 6.1(2).

ii) $z^2(A, X) > 0$. In this case $A$ is a homological $Z_1$-set in $X$ and a homotopical $Z_1$-set in the LC$_1$-space $X$ by Theorem 8.1(2). The set $B$, being nowhere dense in the LC$_1$-space $Y$, is a homotopical $Z_0$-set in $Y$. Then $A \times B$, being the product of a homological $Z_1$-set and a homotopical $Z_0$-set, is a homotopical $Z_2$-set in $X \times Y$ by Theorem 6.1(2). By Theorem 8.3, $A \times B$ is a homotopical $Z_n$-set in $X \times Y$ if and only if it is a homological $Z_n$-set in $X \times Y$, which implies the desired equality $z(A \times B, X \times Y) = z^2(A \times B, X \times Y)$.

iii) The case $z^2(B, Y) > 0$ can be considered by analogy. \hfill $\Box$

Theorem 8.3 implies

Corollary 8.4. A subset $A$ of a Tychonov LC$_1$-space $X$ is a homological $Z_n$-set in $X$ if and only if $A \times \{0\}$ is a homotopical $Z_{n+1}$-set in $X \times [-1, 1]$.

Next we turn to the $k$-Root Theorem.

Theorem 8.5 (k-Root Theorem). Let $A$ be a closed subset in a topological space $X$, $k \in \mathbb{N}$, and $G$ be a coefficient group. Then

1. $k \cdot z^G(A, X) \leq z^G(A^k, X^k) \leq k \cdot z^G(A, X) + 2k - 2$
2. $k \cdot z^G(A, X) + k - 1 \leq z^G(A^k, X^k)$ provided $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$;
3. $z^G(A^k, X^k) \leq k \cdot z^G(A, X) + k$, provided $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$;
4. $z^G(A^k, X^k) = k \cdot z^G(A, X) + k - 1$, provided $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$;

Proof. The items of this theorem will be proved in the following order: 2, 4, 3, 1. There is nothing to prove if $k = 1$. So we assume that $k \geq 2$.

2. The second item follows by induction from the Multiplication Formula 7.2(2).

4. Assume that $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$. By Theorem 7.1(1), there is a Bockstein group $F \in \sigma(G)$ with $z^F(A, X) = z^G(A, X)$. Since $F$ is a field, we may apply Lemma 8.1(1) $k - 1$ times and get the equality $z^F(A^k, X^k) = k \cdot z^F(A, X) + k - 1$. Then

$$k \cdot z^G(A, X) + k - 1 = k \cdot z^F(A, X) + k - 1 = z^F(A^k, X^k) \geq z^G(A^k, X^k) \geq k \cdot z^G(A, X) + k - 1$$

the last inequality follows from the preceding item which yields the equality from the fourth item of the theorem.

3. Assume that $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$ and using Theorem 7.1(1), find a Bockstein group $H \in \sigma(G)$ with $z^G(A, X) = z^H(A, X)$. For this group we also get $z^G(A^k, X^k) \leq z^H(A^k, X^k)$ according to Theorem 7.1(1). If $H \in \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$, then

$$z^G(A^k, X^k) \leq z^H(A^k, X^k) = k \cdot z^H(A, X) + k - 1 = k \cdot z^G(A, X) + k - 1$$

by the preceding case.

So it remains to consider the case $H = \mathbb{Q}_p$ for a prime $p$. Applying Theorem 7.1(5) and the second item of this theorem, we get

$$z^G(A^k, X^k) \leq z^{\mathbb{Q}_p}(A^k, X^k) \leq z^{\mathbb{Z}_p}(A^k, X^k) + 1 = k \cdot z^{\mathbb{Z}_p}(A, X) + k \leq k \cdot z^{\mathbb{Q}_p}(A, X) + k = k \cdot z^G(A, X) + k.$$ 

1. The inequality $k \cdot z^G(A, X) \leq z^G(A^k, X^k)$ follows by induction from Theorem 7.2(1). To prove the inequality $z^G(A^k, X^k) \leq k \cdot z^G(A, X) + 2k - 2$, apply Theorem 7.1(1) to find a group $H \in \sigma(G)$...
such that $z^H(A, X) = z^G(A, X)$. If $H \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$, then

$$z^G(A^k, X^k) \leq z^H(A^k, X^k) \leq k \cdot z^H(A, X) + k = k \cdot z^G(A, X) + k \leq z^G(A, X) + 2k - 2.$$ 

So it remains to consider the case of $H = R_p$ for a prime $p$ and prove that $z^G(A^k, X^k) \leq k \cdot z^G(A, X) + 2k - 2$. Assuming the converse, we conclude that $A^k$ is a $G$-homological $Z_{kn+k-1}$-set in $X^k$ for $n = z^G(A, X) + 1 = z^{R_p}(A, X) + 1$. It follows from the definition of $z^{R_p}(A, X) = n - 1$ that the homology group $H_n(U, U \setminus A; R_p) = H_n(U, U \setminus A) \otimes R_p$ is not trivial for some open set $U \subset X$. Applying Lemma 5.2, we get that $H_n(U, U \setminus A)$ contains an element $a$ that has either infinite order or order $p$.

If $a$ has infinite order then the tensor product $H_n(U, U \setminus A) \otimes H_n(U, U \setminus A)$ contains an element of infinite order and so does group $H_{2n}\left(U^2, U^2 \setminus A^2\right)$ according to the K"unneth Formula. Now, we can show by induction that the group $H_{kn}(U^k, U^k \setminus A^k)$ contains an element of infinite order, which implies that $H_{kn}(U^k, U^k \setminus A^k) \otimes R_p \neq 0$ and hence $A^k$ fails to be a $R_p$-homological $Z_{kn}$-set in $X^k$. Then we get a contradiction:

$$z^G(A^k, X^k) \leq z^{R_p}(A^k, X^k) < kn \leq kn + k - 1 \leq z^G(A^k, X^k).$$

If $a$ is of order $p$, then the tensor product $H_n(U, U \setminus A) \otimes H_n(U, U \setminus A)$ contains an element of order $p$ and so does group $H_{2n+1}\left(U^2, U^2 \setminus A^2\right)$ according to the K"unneth Formula. Proceeding by induction, we can show that the homology group $H_{i(n+1)-1}\left(U^i, U^i \setminus A^i\right)$ contains an element of order $p$. For $i = k$, the group $H_{kn+k-1}(U^k, U^k \setminus A^k)$ contains an element of order $p$ and hence $H_{kn+k-1}(U^k, U^k \setminus A^k) \otimes R_p = H_{kn+k-1}(U^k, U^k \setminus A^k; R_p)$ is not trivial by Lemma 5.2. This means that $A^k$ is not a $R_p$-homological $Z_{kn+k-1}$-set in $X$ and thus

$$z^G(A^k, X^k) \leq z^{R_p}(A^k, X^k) \leq kn + k - 2 < z^G(A^k, X^k),$$

which is a contradiction. 

9. $Z_n$-POINTS

A point $x$ of a space $X$ is defined to be a $Z_n$-point if its singleton $\{x\}$ is a $Z_n$-set in $X$. By analogy we define homotopical and $G$-homological $Z_n$-points. It is clear that all results proved in the preceding sections for $Z_n$-sets concern also $Z_n$-points. For $Z_n$-points there are however some simplifications. In particular, the Excision Property for Singular Homology Theory (see Theorem 4 in [Spa, 4.6]) allows us to characterize homological $Z_n$-points as follows.

Proposition 9.1. For a point $x$ of a regular topological space $X$ and a coefficient group $G$ the following conditions are equivalent:

1. $x$ is a $G$-homological $Z_n$-point in $X$;
2. $H_k(X, X \setminus \{x\}; G) = 0$ for all $k < n + 1$;
3. there is an open neighborhood $U \subset X$ of $x$ such that $H_k(U, U \setminus \{x\}; G) = 0$ for all $k < n + 1$.

In this section, given a topological space $X$ and a coefficient group $G$ we shall study the Borel complexity of the sets:

- $Z_n(X)$ of all homotopical $Z_n$-points in $X$, and
- $Z^n(X)$ of all $G$-homological $Z_n$-points in $X$.

Theorem 9.2. Let $X$ be a metrizable separable space and $G$ be a coefficient group. Then

1. the set of $Z_n$-points in $X$ is a $G_\delta$-set in $X$;
2. the set $Z_n(X)$ of homotopical $Z_n$-points is a $G_\delta$-set in $X$ provided $X$ is an LC$^n$-space;
3. the set of $G$-homological $Z_n$-points is a $G_\delta$-set in $X$ if $|H_k(U; G)| \leq \aleph_0$ for all open sets $U \subset X$ and all $k < n + 1$;
4. the set $Z^n(X)$ of $G$-homological $Z_n$-points is a $G_\delta$-set in $X$ provided is an LC$^n$-space;
5. the set of homotopical $Z_n$-points is a $G_\delta$-set in $X$ if $X$ is an LC$^2$-space and $|H_k(U)| \leq \aleph_0$ for all open subsets $U \subset X$ and all $k < n + 1$. 

Proof. 1. Let $\rho$ be a metric on $X$ generating the topology of $X$. Since the space $X$ is metrizable and separable, so is the function space $C(I^n, X)$ endowed with the sup-metric
\[
\hat{\rho}(f, g) = \sup_{z \in I^n} \rho(f(z), g(z)).
\]
So, we may fix a countable dense set \( \{f_i\}_{i=1}^\infty \) in $C(I^n, X)$. Observe that a closed subset $A \subset X$ fails to be a $\mathbb{Z}_n$-set if and only if there is a function $f \in C(I^k, X)$ and $\varepsilon > 0$ such that for each function $g \in C(I^n, X)$ with $\hat{\rho}(f, g) < \varepsilon$ the image $g(I^n)$ meets the set $A$. The density of \( \{f_i\} \) implies that 
\[
\hat{\rho}(f, f_i) < \varepsilon/2 \quad \text{for some } i \in \omega.
\]
Consequently, $g(I^k) \cap A \neq \emptyset$ for all $g \in C(I^n, X)$ with $\hat{\rho}(g, f_i) < \varepsilon/2$.

Now for each $i, k \in \mathbb{N}$ consider the set
\[
F_{i,k} = \{x \in X : \forall g \in C(I^n, X) \quad \hat{\rho}(g, f_i) < 1/k \Rightarrow x \in g(I^n)\}.
\]
It is easy to see that the set $F_{i,k}$ is closed in $X$. Moreover the preceding discussion implies that $F = \bigcup_{i=1}^\infty F_{i,k}$ is the set of all points of $X$ that fail to be $\mathbb{Z}_n$-points in $X$. Then its complement is a $G_\delta$-set consisting with the set of all $\mathbb{Z}_n$-points in $X$.

2. If $X$ is an LC$^n$-space, then each $\mathbb{Z}_n$-point is a homotopical $\mathbb{Z}_n$-point in $X$ and consequently the set $\mathbb{Z}_n(X)$ of homotopical $\mathbb{Z}_n$-points coincides with the $G_\delta$-set $X \setminus F$ of all $\mathbb{Z}_n$-points.

3. Assume that the homology groups $H_k(U; G)$ are countable for all $k < n + 1$ and all open sets $U \subset X$. Let $\mathcal{B} = \{U_i : i \in \omega\}$ be a countable base of the topology for $X$ with $U_0 = \emptyset$. For every $i \in \omega$ and $k < n + 1$ use the countability of the groups $H_k(X \setminus \overline{U}_i; G)$ to find a countable sequence $(\alpha_{i,j})_{j \in \omega}$ of cycles in $X \setminus \overline{U}_i$ whose representatives $[\alpha_{i,j}]$ exhaust all non-zero elements of the groups $H_k(X \setminus \overline{U}_i; G)$, $k < n + 1$.

For every $j \in \omega$ consider the open set $W_{0,j} = \{x \in X : \alpha_{0,j} \text{ is homologous to some cycle in } X \setminus \{x\}\}$. Also for every $i \in \mathbb{N}$ and $j \in \omega$ consider the open set $W_{i,j} = \{x \in U_i : \alpha_{i,j} \text{ is null-homologous in } X, \text{then it is null-homologous in } X \setminus \{x\}\}$. It remains to prove that the $G_\delta$-set $Z = \bigcap_{i,j \in \omega} W_{i,j}$ coincides with the set $Z^G_n(X)$ of all $G$-homologous $\mathbb{Z}_n$-points in $X$. It is clear that $Z$ contains all $G$-homologous $\mathbb{Z}_n$-points of $X$. Now take any point $x \in Z$. Assuming that $x$ is not a $G$-homologous $\mathbb{Z}_n$-point, find $k < n + 1$ such that $H_k(X, X \setminus \{x\}; G) \neq 0$. The exact sequence
\[
H_k(X \setminus \{x\}; G) \to H_k(X; G) \to H_k(X, X \setminus \{x\}; G) \to H_{k-1}(X \setminus \{x\}; G) \to H_{k-1}(X; G)
\]
of the pair $(X, X \setminus \{x\})$ now implies that for $m = k$ or $m = k - 1$ the inclusion homomorphism $i : H_m(X \setminus \{x\}; G) \to H_m(X; G)$ fails to be an isomorphism.

If $i$ is not onto, then for some $j \in \omega$ the element $\alpha_{0,j}$ is homologous to no cycle in $X \setminus \{x\}$. This means that $x \notin W_{0,j}$ and thus $x \notin Z$ which is a contradiction.

If $i$ is not injective, then there is a $k$-cycle $\alpha$ in $X \setminus \{x\}$ which is homologous to zero in $X$ but not in $X \setminus \{x\}$. Since $\alpha$ has compact support, there is a basic neighborhood $U_i$ of $x$ such that $\alpha$ is supported by the set $X \setminus \overline{U}_i$. Find $j \in \omega$ such that the cycle $\alpha_{i,j}$ is homologous to $\alpha$ in $X \setminus \overline{U}_i$. It follows that $\alpha_{i,j}$ is homologous to zero in $X$ but not in $X \setminus \{x\}$. Then $x \notin W_{i,j}$ and hence $x \notin Z$. This contradiction completes the proof of the third item.

4. Assuming that $X$ is an LC$^n$-space we shall show that the set $Z^G_n(X)$ is of type $G_\delta$ in $X$. Since $Z^G_n(X) = \bigcap_{H \in \mathfrak{S}(G)} Z^H_n(X)$, it suffices to check that $Z^H_n(X)$ is a $G_\delta$-set in $X$ for every countable group $H$. This will follows from the preceding item as soon as we show that for every open set $U \subset X$ the homology groups $H_i(U; H)$, $i \leq n$, are at most countable. In its turn, this will follow from the Formula of Universal Coefficients as soon as we check that the homology groups $H_i(U)$, $i \leq n$, are at most countable.

Fix a countable family $\mathcal{K}$ of compact polyhedra containing a topological copy of each compact polyhedron. For each polyhedron $K \in \mathcal{K}$ fix a countable dense set $\mathcal{F}_K$ in the function space $C(K, U)$. Note that for each polyhedron $K \in \mathcal{K}$ the homology group $H_*(K)$ is finitely generated and hence at most countable.

For every homology element $\alpha \in H_i(U)$ with $i \leq n$ there is a continuous map $f : K \to U$ of a compact polyhedron $K \in \mathcal{K}$ such that $\alpha \in f_*(H_i(K))$. Moreover, according to Lemma 1.3 we can
assume that $f \in \mathcal{F}_K$. Then the homology group

$$H_i(U) = \bigcup_{K \in K} \bigcup_{f \in \mathcal{F}_K} f_*(H_i(K))$$

is countable, being the countable union of finitely generated groups.

5. The fifth item follows from items (2), (4), and the characterization of homotopical $Z_n$-sets given by Theorem 3.3. \hfill \Box

10. ON SPACES WHOSE ALL POINTS ARE $Z_n$-POINTS

In this section we introduce three classes of Tychonov spaces related to $Z_n$-points:

- $Z_n$ the class of spaces $X$ with $X = Z_n(X)$,
- $Z_n^G$ the class of spaces $X$ with $X = Z_n^G(X)$;
- $\cup G Z_n^G$ the union of classes $Z_n^G$ over all coefficient groups.

For example, $\mathbb{R}^{n+1}$ belongs to all of these classes while $\mathbb{R}^n$ belongs to none of them. The classes $Z_n$ play an important role in studying the general position properties from [BV].

By $LC^n$ (resp. $lc^n$) we shall denote the class of metrizable $LC^n$-spaces (resp. $lc^n$-spaces).

The following corollary describes the relation between the introduced classes and can be easily derived from Theorems 3.2, 3.3, 5.5 and 5.6.

**Corollary 10.1.** Let $n \in \omega \cup \{ \infty \}$ and $G$ be a coefficient group. Then

1. $Z_n \subset Z_n^Z \subset Z_n^G = \bigcap_{H \in \sigma(G)} H_n^H$;
2. $Z_0 = Z_0^Z = Z_0^G$;
3. $LC^1 \cap Z_n^G \subset Z_1$;
4. $LC^1 \cap Z_2 \cap Z_3^G \subset Z_n$;
5. $\cup G Z_n^G = Z_n^Q \cup \bigcup_{p \in \mathbb{P}} Z_n^{Q_p}$;
6. $Z_n^Z = \bigcap_{p \in \mathbb{P}} Z_n^{R_p}$;
7. $Z_n^{R_p} \subset Z_n^Q \cap Z_n^{Q_p}$, $Z_n^{Q_p} \subset Z_n^{Q_p} \cap Z_n^{Q_p}$, $Z_n \cap Z_n^{Q_p} \subset Z_n^{R_p}$ for every prime number $p$.

For a better visual presentation of our subsequent results, let us introduce the following operations on subclasses $A, B \subset \text{Top}$ of the class $\text{Top}$ of topological spaces:

$A \times B = \{ A \times B : A \in A, B \in B \}$,

$$\frac{A}{B} = \{ X \in \text{Top} : \exists B \in B \text{ with } X \times B \in A \},$$

$$A^k = \{ A^k : A \in A \} \text{ and } \sqrt[k]{A} = \{ A \in \text{Top} : A^k \in A \}.$$

Multiplication Formulas 7.2 imply the following three Multiplication Formulas for the classes $Z_n$ and $Z_n^G$.

**Theorem 10.2 (Multiplication Formulas).** Let $n, m \in \omega \cup \infty$, $X, Y$ be Tychonov spaces.

1. If $X \in Z_n$ and $Y \in Z_m$, then $X \times Y \in Z_{n+m+1}$:

   $Z_n \times Z_m \subset Z_{n+m+1}$

2. If $X \in Z_n^G$, $Y \in Z_m^G$ for a coefficient group $G$, then $X \times Y \in Z_{n+m}^G$:

   $Z_n^G \times Z_m^G \subset Z_{n+m}^G$

3. If $X \in Z_n^{R_p}$, $Y \in Z_m^{R_p}$ for a coefficient group $R$ with $\sigma(R) \subset \{ Q, Z_p, R_p : p \in \mathbb{P} \}$, then $X \times Y \in Z_{n+m+1}^{R_p}$:

   $Z_n^{R_p} \times Z_m^{R_p} \subset Z_{n+m+1}^{R_p}$

The Multiplication Formulas for the classes $Z_n^G$ can be reversed.
Theorem 10.3 (Division Formulas). Let \( n, m \in \omega \cup \infty \), \( X, Y \) be Tychonov spaces.

1. If \( X \times Y \in Z^{G}_{n+m+1} \) for a coefficient group \( G \), then either \( X \in Z^{G}_{n} \) or \( Y \in \bigcup_{F \in \varphi(G)} Z^{F}_{m} \). This can be written as

\[
\frac{Z^{G}_{n+m+1}}{\text{Top}} \setminus \frac{Z^{G}_{n}}{\text{Top}} \subset \bigcup_{F \in \varphi(G)} \frac{Z^{F}_{m}}{\text{Top}}
\]

2. If \( X \times Y \in Z^{R}_{n+m} \) for a coefficient group \( R \) with \( \sigma(R) \subset \{ \mathbb{Q}, \mathbb{Z}_{p}, R_{p} : p \in \Pi \} \), then either \( X \in Z^{R}_{n} \) or \( Y \in \bigcup_{H \in d(R)} Z^{H}_{m} \). This can be written as

\[
\frac{Z^{R}_{n+m}}{\text{Top}} \setminus \frac{Z^{R}_{n}}{\text{Top}} \subset \bigcup_{H \in d(R)} \frac{Z^{H}_{m}}{\text{Top}}
\]

Proof. 1. Assume that \( X \notin Z^{G}_{n} \) and \( Y \notin \bigcup_{F \in \varphi(G)} Z^{F}_{m} \). Then there is a point \( x \in X \) with \( z^{G}(x, X) < n \) and for every field \( F \in \varphi(G) \) there is a point \( y_{F} \in Y \) with \( z^{F}(y_{F}, Y) < m \). By Theorem 5.5, \( z^{G}(x, X) = z^{H}(x, X) \) for some group \( H \in \sigma(G) \). If \( H \) is a field, then \( H \in \varphi(H) \subset \varphi(G) \) and by Lemma 8.1, \( z^{H}(x, y_{H}, X \times Y) = z^{H}(x, X) + z^{H}(y_{H}, Y) + 1 \leq (n-1) + (m-1) + 1 = n + m - 1 \), which means that \( (x, y_{H}) \) is not an \( H \)-homological \( Z_{n+m} \)-set in \( X \times Y \) and thus \( X \times Y \notin Z^{H}_{n+m} \subset Z^{G}_{n+m+1} \).

If \( H = R_{p} \) for some \( p \), then \( (\mathbb{Q}, \mathbb{Q}_{p}) = d(R_{p}) \) and \( \mathbb{Z}_{p} \) in \( \varphi(G) \). By Lemma 8.1, \( z^{H}(x, y_{H}, X \times Y) = z^{H}(x, X) + z^{H}(y_{H}, Y) + 1 = (n-1) + (m-1) + 1 = n + m \). In both cases, \( X \times Y \notin Z^{H}_{n+m+1} \).

2. Assume that \( X \notin Z^{R}_{n} \) and \( Y \notin \bigcup_{F \in \varphi(R)} Z^{F}_{m} \) for a group \( R \) with \( \sigma(R) \subset \{ \mathbb{Q}, \mathbb{Z}_{p}, R_{p} : p \in \Pi \} \). Then there is a point \( x \in X \) with \( z^{R}(x, X) < n \) and for every group \( H \in d(R) \) there is a point \( y_{H} \in Y \) with \( z^{H}(y_{H}, Y) < m \). By Theorem 5.5, \( z^{R}(x, X) = z^{H}(x, X) \) for some group \( H \in \sigma(R) \subset \{ \mathbb{Q}, \mathbb{Z}_{p}, R_{p} : p \in \Pi \} \). Theorem 5.5 implies that \( Z^{R}_{n+m} \subset Z^{H}_{n+m} \).

If \( H \) is a field, then repeating the reasoning from the preceding item, we can prove that

\[
z^{H}(x, y_{H}, X \times Y) = z^{H}(x, X) + z^{H}(y_{H}, Y) + 1 \leq (n-1) + (m-1) + 1 = n + m - 1
\]

which means that \( X \times Y \notin Z^{H}_{n+m} \).

If \( H = R_{p} \) for some \( p \), then \( (\mathbb{Q}, \mathbb{Q}_{p}) = d(R_{p}) \subset d(G) \). By Lemma 8.1, \( z^{H}(x, y_{H}, X \times Y) = z^{H}(x, X) + z^{H}(y_{H}, Y) + 1 = (n-1) + (m-1) + 1 = n + m - 1 \).

Therefore, either

\[
z^{H}(x, y_{H}, X \times Y) \leq 1 + z^{H}(x, X) + z^{H}(y_{H}, Y) \leq (n-1) + (m-1) + 1 = n + m - 1 \text{ or }
z^{H}(x, y_{H}, X \times Y) \leq 1 + z^{H}(x, X) + z^{H}(y_{H}, Y) \leq (n-1) + (m-1) + 1 = n + m - 1.
\]

In both cases, \( X \times Y \notin Z^{H}_{n+m} \subset Z^{R}_{n+m} \).

Finally, we prove \( k \)-Root Formulas for the classes \( Z^{G}_{n} \).

Theorem 10.4 (\( k \)-Root Formulas). Let \( n \in \omega \cup \infty \), \( k \in \mathbb{N} \), \( X \) be a topological space, and \( G \) be a coefficient group.
(1) If \( X^k \in \mathbb{Z}^G_{kn+k-1} \), then \( X \in \mathbb{Z}^G_n \):
\[
\sqrt[n]{\mathbb{Z}^G_{nk+k-1}} \subset \mathbb{Z}^G_n
\]

(2) If \( \sigma(G) \subset \{ Q, Z_p, Q_p : p \in \Pi \} \) and \( X^k \in \mathbb{Z}^G_{kn+1} \), then \( X \in \mathbb{Z}^G_n \):
\[
\sigma(G) \subset \{ Q, Z_p, Q_p : p \in \Pi \} \Rightarrow \sqrt[n]{\mathbb{Z}^G_{nk+1}} \subset \mathbb{Z}^G_n
\]

(3) If \( \sigma(G) \subset \{ Q, Z_p : p \in \Pi \} \) and \( X^k \in \mathbb{Z}^G_{kn} \), then \( X \in \mathbb{Z}^G_n \):
\[
\sigma(G) \subset \{ Q, Z_p : p \in \Pi \} \Rightarrow \sqrt[n]{\mathbb{Z}^G_{nk}} = \mathbb{Z}^G_n
\]

Proof. Assume that \( X \notin \mathbb{Z}^G_n \) and find a point \( x \in X \) with \( z^G(\{x\}, X) < n \).

1. Applying Theorem 8.5(1), we get \( z^G(\{x\}, X^k) < k \cdot z^G(x, X) + 2k - 1 \leq k(n-1) + 2k - 1 = kn + k - 1 \) and hence \( X^k \notin \mathbb{Z}^G_{kn+k-1} \).

2. If \( \sigma(G) \subset \{ Q, Z_p, Q_p : p \in \Pi \} \), then we can apply Theorem 8.5(3) to conclude that \( z^G(\{x\}, X^k) \leq k \cdot z^G(x, X) + k \leq k(n-1) + k = kn \) and \( X^k \notin \mathbb{Z}^G_{kn+1} \).

3. If \( \sigma(G) \subset \{ Q, Z_p : p \in \Pi \} \), then we can apply Theorem 8.5(4) to conclude that \( z^G(\{x\}, X^k) = k \cdot z^G(x, X) + k - 1 \leq k(n-1) + k - 1 = kn - 1 \) and hence \( X^k \notin \mathbb{Z}^G_{kn} \).

11. On spaces containing a dense set of (homological) \( Z_n \)-points

In this section we consider classes of Tychonov spaces containing dense sets of (homological) \( Z_n \)-points. More precisely, let

- \( \mathbb{Z}_n \) be the class of spaces \( X \) with dense set \( \mathbb{Z}_n(X) \) of homotopical \( Z_n \)-points in \( X \);
- \( \mathbb{Z}_n^G \) be the class of spaces \( X \) with dense set \( \mathbb{Z}_n^G(X) \) of \( G \)-homological \( Z_n \)-points;
- \( \cup G \mathbb{Z}_n \) be the union of classes \( \mathbb{Z}_n^G \) over all coefficient groups.

The classes \( \mathbb{Z}_n \) play an important role in the paper [BV] devoted to general position properties. It is clear that \( \mathbb{Z}_n \subset \mathbb{Z}_n^G \). On the other hand, a dendrite \( D \) with dense set of end-points belongs to \( \mathbb{Z}_\infty \) but not to \( \mathbb{Z}_1 \). By \( \mathbb{B} \) we shall denote the class of metrizable separable Baire spaces.

The following proposition describes relation between the introduced classes.

Proposition 11.1. Let \( n \in \omega \cup \{ \infty \} \) and \( G \) be a coefficient group. Then

\begin{align*}
(1) & \; \mathbb{Z}_n \subset \mathbb{Z}_n^G \subset \mathbb{Z}_n^G \subset \bigcap_{H \in \sigma(G)} \mathbb{Z}_n^H ; \\
(2) & \; \mathbb{B} \cap \mathbb{L} \cap \bigcap_{H \in \sigma(G)} \mathbb{Z}_n^H \subset \mathbb{Z}_n^G ; \\
(3) & \; \mathbb{Z}_0^G = \mathbb{Z}_0^G ; \\
(4) & \; \mathbb{L} \cap \mathbb{I} \subset \mathbb{Z}_1^G ; \\
(5) & \; \mathbb{L} \cap \mathbb{I} \subset \mathbb{Z}_n^G \subset \mathbb{Z}_n^G ; \\
(6) & \; \mathbb{L} \cap \mathbb{B} \cap \mathbb{I} \subset \mathbb{Z}_n^G \subset \mathbb{Z}_n^G .
\end{align*}

Proof. 1. The first item follows from Theorem 3.2(4), Proposition 5.3 and Theorem 5.5.

2. If \( X \in \bigcap_{H \in \sigma(G)} \mathbb{Z}_n^H \) is a metrizable separable Baire \( \mathbb{L} \)-space, then for every group \( H \in \sigma(G) \) the set \( \mathbb{Z}_n^H(X) \) of \( H \)-homological \( Z_n \)-points is a dense \( G_\delta \)-set in \( X \) according to Theorem 9.2(4). By Theorem 5.3 the intersection \( \bigcap_{H \in \sigma(G)} \mathbb{Z}_n^H(X) \) coincides with \( \mathbb{Z}_n^G(X) \) and is dense in \( X \), being the countable intersection of dense \( G_\delta \)-sets in the Baire space \( X \). Hence \( X \in \mathbb{Z}_n^G \).

3-5. The items (3)–(5) follow immediately from Theorems 3.2 and 3.3.
6. Assume that \( X \in \mathcal{Z}_2 \cap \mathcal{Z}_\infty \) is a metrizable separable Baire LC\( ^p \)-space. By Theorem 3.2 the sets \( \mathcal{Z}_2(X) \) and \( \mathcal{Z}_\infty(X) \) are dense \( G_\delta \) in \( X \). Since \( X \) is Baire, the intersection \( \mathcal{Z}_2(X) \cap \mathcal{Z}_\infty(X) \) is dense \( G_\delta \) in \( X \). By Theorem 3.3, the latter intersection consists of homotopical \( Z_n \)-points. So \( X \in \mathcal{Z}_n \). □

Multiplication Formulas for the classes \( \mathcal{Z}_n \) and \( \mathcal{Z}_n^G \) follow immediately from Multiplication Theorem 6.1 for homotopical and homological \( Z_n \)-sets.

**Theorem 11.2** (Multiplication Formulas). Let \( n, m \in \omega \cup \{ \infty \} \) and \( X, Y \) be Tychonov spaces.

1. Assume that

\[
\exists (\text{Division Formulas})
\]

Theorem 11.3

2. By Lemma 8.1(1) either \( U \) or else some non-empty open set \( U \in Y \) belongs to the class \( \mathcal{Z}_n \).

**Theorem 11.3** (Division Formulas). Let \( X, Y \) be topological spaces.

1. If \( X \times Y \in \mathcal{Z}_n^{F} \) for some field \( F \), then either \( X \in \mathcal{Z}_n^{F} \) or \( Y \in \mathcal{Z}_m^{F} \):

\[
\mathcal{Z}_n^{F} \times \mathcal{Z}_m^{F} \subset \mathcal{Z}_{n+m+1}^{F}
\]

2. Assume that \( X \times Y \in \mathcal{Z}_n^{R} \) for a coefficient group \( R \) with \( \sigma(R) \subset \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \} \) and either \( \sigma(R) \) is finite or \( X \) is a metrizable separable Baire LC\( ^p \)-space. Then either \( X \in \mathcal{Z}_n^{R} \) or else some non-empty open set \( U \subset Y \) belongs to the class \( \bigcup_{H \in \mathcal{d}(R)} \mathcal{Z}_m^{H} \):

\[
\mathcal{Z}_n^{R} \times \mathcal{Z}_m^{R} \subset \mathcal{Z}_{n+m+1}^{R}
\]

3. Assume that \( X \times Y \in \mathcal{Z}_n^{G} \) for a coefficient group \( G \) and either \( \sigma(G) \) is finite or \( X \) is a metrizable separable Baire LC\( ^p \)-space. Then either \( X \in \mathcal{Z}_n^{G} \) or else some non-empty open set \( U \subset Y \) belongs to the class \( \bigcup_{F \in \mathcal{d}(G)} \mathcal{Z}_m^{F} \):

\[
\mathcal{Z}_n^{G} \times \mathcal{Z}_m^{G} \subset \mathcal{Z}_{n+m+1}^{G}
\]

**Proof.** 1. Assume that \( X \times Y \in \mathcal{Z}_n^{F} \) for some field \( F \), but \( X \notin \mathcal{Z}_n^{F} \) and \( Y \notin \mathcal{Z}_m^{F} \). Let \( U \) be the interior of the set \( X \setminus \mathcal{Z}_n^{F}(X) \) and \( V \) be the interior of the set \( Y \setminus \mathcal{Z}_m^{F}(Y) \). The product \( U \times V \) is a non-empty open subset of \( X \times Y \) and thus contains some \( F \)-homological \( Z_{n+m} \)-point \((x, y)\) in \( X \times Y \). By Lemma 8.1(1) either \( x \in \mathcal{Z}_n^{F}(X) \) or \( y \in \mathcal{Z}_m^{F}(Y) \). Both cases are not possible by the choice of \( U, V \). This contradiction shows that \( X \in \mathcal{Z}_n^{F} \) or \( Y \in \mathcal{Z}_m^{F} \).
2. Assume that \( X \times Y \in \mathcal{Z}_{n+m}^R \) for a group \( R \) with \( \sigma(R) \subset \{ \mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi \} \) but \( X \notin \mathcal{Z}_{n}^R \). The latter means that \( X \) contains a non-empty open set \( W \subset X \) disjoint with the set \( \mathcal{Z}_{n}^R (X) \) of \( R \)-homological \( Z_n \)-points of \( X \). It is necessary to find an open set \( U \in \bigcup_{H \in \sigma(R)} \mathcal{Z}_{m}^H \). Assume conversely that no such a set \( U \) exists. This means that for every \( H \in d(G) \) the set \( \mathcal{Z}_{m}^H (Y) \) is nowhere dense in \( Y \).

If \( \sigma(R) \) is finite, then so is the set \( d(R) \) and we can find a non-empty open set \( U \subset X \) disjoint with \( \bigcup_{H \in d(R)} \mathcal{Z}_{m}^H (Y) \).

By Theorem 8.2(2), no point \( (x, y) \in W \times U \) is an \( R \)-homological \( Z_{n+m} \)-point in \( X \times Y \). But this contradicts the density of the set \( \mathcal{Z}_{n+m}^R (X \times Y) \) in \( X \times Y \).

Next, we consider the case when \( \sigma(R) \) is infinite and \( X \) is a metrizable separable Baire \( lc^n \)-space. By Theorem 9.2, for every group \( H \in \sigma(R) \) the set \( \mathcal{Z}_{n}^H (X) \) is of type \( G_\delta \) in \( X \) and by Theorem 5.5 \( \mathcal{Z}_{n}^G (X) = \bigcap_{H \in \sigma(R)} \mathcal{Z}_{m}^H (X) \). Assuming that \( \mathcal{Z}_{n}^G (X) \) is not dense in the Baire space \( X \), we conclude that for some group \( H \in \sigma(R) \) the set \( \mathcal{Z}_{m}^H (X) \) is not dense in \( X \). Since \( \sigma(H) = \{ H \} \) is finite, we may apply the preceding case to conclude that some nonempty open set \( U \subset Y \) belongs to the class

\[
\bigcup_{D \in d(H)} \mathcal{Z}_{m}^D \subset \bigcup_{D \in d(G)} \mathcal{Z}_{m}^D.
\]

3. Assume that \( X \times Y \in \mathcal{Z}_{n+m+1}^G \) for a coefficient group \( G \) but \( X \notin \mathcal{Z}_{n}^G \). The latter means that \( X \) contains a non-empty open set \( W \subset X \) disjoint with the set \( \mathcal{Z}_{n}^G (X) \) of \( G \)-homological \( Z_n \)-points of \( X \). It is necessary to find an open set \( U \in \bigcup_{F \in \varphi(G)} \mathcal{Z}_{m}^F \). Assume conversely that no such a set \( U \) exists. This means that for every \( F \in \varphi(G) \) the set \( \mathcal{Z}_{m}^F (Y) \) is nowhere dense in \( Y \).

If \( \sigma(G) \) is finite, then so is the set \( \varphi(G) \), and we can find a non-empty open set \( U \subset X \) disjoint with \( \bigcup_{F \in \varphi(G)} \mathcal{Z}_{m}^F (Y) \). By Theorem 8.2(1), no point \( (x, y) \in W \times U \) is a \( G \)-homological \( Z_{n+m+1} \)-point in \( X \times Y \). But this contradicts the density of the set \( \mathcal{Z}_{n+m+1}^G (X \times Y) \) in \( X \times Y \).

The case of infinite \( \sigma(G) \) can be reduced to the previous case by the same argument as in the item 2.

\[\square\]

**Theorem 11.4 (k-Root Formulas).** Let \( n \in \omega \cup \{ \infty \} \), \( k \in \mathbb{N} \), \( X \) be a topological space, and \( G \) be a coefficient group.

1. \( X \in \mathcal{Z}_{n}^F \) for some field \( F \) if and only if \( X^n \in \mathcal{Z}_{nk}^F \):

\[ \sqrt[k]{\mathcal{Z}_{nk}^F} = \mathcal{Z}_{n}^F \]

2. If \( X \) is a metrizable separable Baire \( lc^n \)-space and \( X^n \in \mathcal{Z}_{nk+k}^G \), then \( X \in \mathcal{Z}_{n}^G \):

\[ \text{Br} \cap lc^n \cap \sqrt[k]{\mathcal{Z}_{nk+k}} \subset \mathcal{Z}_{n}^G \]

3. If \( X \) is a metrizable separable Baire \( lc^{nk+k-1} \)-space and \( X^n \in \mathcal{Z}_{nk+k-1}^G \), then \( X \in \mathcal{Z}_{n}^G \):

\[ \text{Br} \cap lc^{nk+k-1} \cap \sqrt[k]{\mathcal{Z}_{nk+k-1}} = \mathcal{Z}_{n}^G \]

**Proof.** 1. The first item can be derived by induction from Theorems 11.2(2) and 11.3(1).

2. Assume that \( X \) is a metrizable separable Baire \( lc^n \)-space with \( X^n \in \mathcal{Z}_{kn+k}^G \). It follows from Theorem 9.2 that for every group \( H \in \sigma(G) \) the set \( \mathcal{Z}_{n}^H (X) \) is of type \( G_\delta \) in \( X \). Since \( \mathcal{Z}_{n}^G (X) = \bigcap_{H \in \sigma(G)} \mathcal{Z}_{m}^H (X) \), the density of \( \mathcal{Z}_{n}^G (X) \) in the Baire space \( X \) will follow as soon as we prove the density of \( \mathcal{Z}_{n}^H (X) \) in \( X \) for each group \( H \in \sigma(G) \).

If \( H \) is a field, then the inclusion \( X^n \in \mathcal{Z}_{kn+k}^G \subset \mathcal{Z}_{kn+k}^H \) combined with the first item implies that \( X \in \mathcal{Z}_{n+1}^H \subset \mathcal{Z}_{n}^H \), which means that the set \( \mathcal{Z}_{n}^H (X) \) is dense in \( X \).
If $H = \mathbb{R}_p$ for some $p$, then Theorem 5.6(1) implies that $X^k \in \mathbb{Z}_{nk+k}^{\mathbb{R}_p} \subset \mathbb{Z}_{nk+k}^\mathbb{Q} \cap \mathbb{Z}_{nk+k}$ and the first item yields $X \in \mathbb{Z}_{n+1}^\mathbb{Q} \cap \mathbb{Z}_{n+1}^\mathbb{Q}_p$. Applying Theorem 5.6(2,4), we get

$$X \in \mathbb{Z}_{n+1}^\mathbb{Q} \cap \mathbb{Z}_{n+1}^{\mathbb{Q}_p} \subset \mathbb{Z}_n^\mathbb{Q} \cap \mathbb{Z}_n^{\mathbb{Q}_p} \subset \mathbb{Z}_n^\mathbb{R}_p = \mathbb{Z}_n^{H}.$$

If $H = \mathbb{Q}_p$ for some $p$, then Theorem 5.6(3) implies that $X^k \in \mathbb{Z}_{nk+k}^\mathbb{R}_p \subset \mathbb{Z}_{nk+k-1}^\mathbb{Q}_p$ and the first item combined with Theorem 5.6(2) implies $X \in \mathbb{Z}_n^{\mathbb{Q}_p} \subset \mathbb{Z}_n^{\mathbb{Q}_p} = \mathbb{Z}_n^H$.

3. Assume that $X$ is a metrizable separable Baire lc$^\alpha_{nk+k-1}$-space and $X^k \in \mathbb{Z}_n^{G_{nk+k-1}}$. Arguing as in the preceding case, we can reduce the problem to the case $G = \mathbb{R}_p$ for some prime number $p$. By Theorem 9.2 $\mathbb{Z}_{nk+k-1}^\mathbb{R}_p(X^k)$ is a dense $G_\delta$-set in $X^k$. Assuming that $\mathbb{Z}_{n}^{\mathbb{R}_p}(X)$ is not dense in $X$, find a non-empty open set $U \subset X$ disjoint with $\mathbb{Z}_n^{\mathbb{R}_p}(X)$. Since $\mathbb{Z}_n^{\mathbb{R}_p}(X) \subset \mathbb{Z}_{nk+k-1}^\mathbb{Q}_p \subset \mathbb{Z}_{nk}^\mathbb{Q}$, the first item implies that $X \in \mathbb{Z}_n^\mathbb{Q}$. Then $D = U \cap \mathbb{Z}_n^\mathbb{Q}(X)$ is a dense $G_\delta$-set in $U$ by Theorem 9.2 and hence $D^k$ is a dense $G_\delta$ set in $U^k$. Since $X^k$ is a Baire space, there is a point $\vec{x} \in D^k \cap \mathbb{Z}_{nk+k-1}^{\mathbb{R}_p}(X^k)$ which can be written as $\vec{x} = (x_1, \ldots, x_k)$.

Each point $x_i$ is a $\mathbb{Q}$-homological but not $\mathbb{R}_p$-homological $Z_n^\mathbb{Q}$-point in $X$. This means that for some $n_i \leq n$ the group $H_n(X, X \setminus \{x_i\}; \mathbb{R}_p) = H_n(X, X \setminus \{x_i\}) \cap \mathbb{R}_p$ is not trivial. Since $x_i$ is a $\mathbb{Q}$-homological $Z_n$-set in $X$, the group $H_n(X, X \setminus \{x_i\})$ cannot contain an element of infinite order. Since $H_n(X, X \setminus \{x_i\}) \cap \mathbb{R}_p \neq 0$, the group $H_n(X, X \setminus \{x_i\})$ contains an element of order $p$.

Let $m_i = i - 1 + \sum_{j=1}^{i-1} n_j$ for $i \leq k$. The torsion product $H_{n_1}(X, X \setminus \{x_1\}) \ast H_{n_2}(X, X \setminus \{x_2\})$ contains an element of order $p$ and so does the group $H_{n_k}(X^2, X^2 \setminus \{(x_1, x_2)\})$ by the Künneth Formula. Now by induction we can show that for every $i \leq k$ the homology group $H_{m_k}(X^i, X^i \setminus \{(x_1, \ldots, x_i)\})$ contains an element of order $p$. For $i = k$, we get that the group $H_{m_k}(X^k, X^k \setminus \{\vec{x}\})$ contains an element of order $p$ and hence the tensor product

$$H_{m_k}(X^k, X^k \setminus \{\vec{x}\}) \cap \mathbb{R}_p = H_{m_k}(X^k, X^k \setminus \{\vec{x}\}; \mathbb{R}_p)$$

is not trivial which is not possible because $m_k \leq nk + k - 1$ and $\vec{x}$ is a $\mathbb{R}_p$-homological $Z_n^{nk+k-1}$-point in $X^k$.

\begin{remark}
An example of a space in which $Z_\infty$-points form a dense $G_\delta$-set is a dendrite $D$ with dense set of end-points. Being large in sense of Baire category, the set of $Z_\infty$-point of $D$ is small in geometric sense: it is locally $\infty$-negligible in $D$. On the other hand, W.Kuperberg [Kup] has constructed a finite polyhedron $P$ in which the set of all $Z_\infty$-points fails to be locally $\infty$-negligible.

Yet, according to I.Namioka [Na], the set $Z$ of $Z_\infty$-points in a finite-dimensional LC$^\infty$-space $X$ is small in a homological sense: $H_k(U, U \setminus Z) = 0$ for any open set $U \subset X$ and any $k \in \omega$. By its spirit this Namioka’s result is near to our Theorem 4.4 asserting that a closed $\text{trt}$-dimensional subspace $A \subset X$ consisting of $G$-homological $Z_\infty$-points of $X$ is a $G$-homological $Z_\infty$-set in $X$.
\end{remark}

12. $Z_n$-POINTS AND DIMENSION

In this section we study the dimension properties of spaces whose all points are homological $Z_n$-points. First we note a simple corollary of Theorem 4.3.

\begin{theorem}
If a (separable metrizable) topological space $X \in \bigcup_G Z_n^G$, then $\text{trd}(X) \geq \text{trt}(X) \geq 1 + n$ (and hence $\text{dim}(X) \geq 1 + n$).
\end{theorem}

A similar lower bound holds also for the cohomological and extension dimensions. Given a space $X$ and a CW-complex $L$ we write $e\text{dim}X \leq L$ if any map $f : A \to L$ defined on a closed subset $A \subset X$ extends to a map $\tilde{f} : X \to L$. The Extension Dimension $e\text{dim}$ generalizes both the usual covering dimension $\text{dim}$ and the cohomological dimension $\text{dim}_G$ because:

- $\text{dim} X \leq n$ iff $e\text{dim}X \leq S^n$ and
- $\text{dim}_G X \leq n$ iff $e\text{dim}X \leq K(G, n)$,
where $K(G,n)$ is an Eilenberg-MacLane complex, i.e., a CW-complex $K$ with a unique non-trivial homotopy group $\pi_n(K) = G$.

The main (and technically most difficult) result of this section is

**Theorem 12.2.** If $X \in L^n_n$ is a locally compact lc^n-space, then \( \dim_X X \geq n + 1 \) for any non-trivial abelian group $G$.

**Proof.** Assume that $\dim_X X = n < \infty$ for some abelian group $G$. By Theorem 2 of [Kuz] (or Theorem 1.8 of [Dr1]), the space $X$ contains a point $x$ having an open neighborhood $U \subset X$ with compact closure such that for any smaller neighborhood $V \subset U$ of $x$ the homomorphism in the relative Čech cohomology groups $i_{V,U} : \check{H}^n(X, X \setminus V; G) \to \check{H}^n(X, X \setminus U; G)$, induced by the inclusion $(X, X \setminus U) \subset (X, X \setminus V)$, is non-trivial. The complete regularity of the locally compact space $X$ allows us to find a compact $G_\delta$-set $K_1 \subset U$ containing $x$ in its interior.

It is well-known (see [Spa, VI, §9]) that in paracompact lc^n-spaces Čech cohomology coincide with singular cohomology. Singular cohomology relates to singular homology via the following exact sequence, see [Hat, §3.1]:

$$0 \to \text{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \to \text{Hom}(H_n(X, A), G) \to 0.$$  

This sequence will be applied to the pairs

$$(X, X \setminus K_1) \subset (X, X \setminus K_2) \subset (X, X \setminus K_3)$$

where $K_3 \subset K_2 \subset K_1$ are compact $G_\delta$-neighborhoods of $x$ so small that the inclusion homomorphisms

$$H_k(X, X \setminus K_1) \to H_k(X, X \setminus K_2) \to H_k(X, X \setminus K_3)$$

are trivial for all $k \leq n$ (the existence of such neighborhoods $K_2, K_3$ follows from Lemma 1.6).

These trivial homomorphisms induce trivial homomorphisms

$$e_{2,1} : \text{Ext}(H_{n-1}(X, X \setminus K_2), G) \to \text{Ext}(H_{n-1}(X, X \setminus K_1), G)$$

and

$$h_{3,2} : \text{Hom}(H_n(X, X \setminus K_3), G) \to \text{Hom}(H_n(X, X \setminus K_2), G).$$

Now consider the commutative diagram

$$\begin{array}{ccc}
\text{Ext}(H_{n-1}(X, X \setminus K_3), G) & \longrightarrow & H^n(X, X \setminus K_3; G) \\
\downarrow & & \downarrow \text{i}_{3,2} \\
\text{Ext}(H_{n-1}(X, X \setminus K_2), G) & \longrightarrow & H^n(X, X \setminus K_2; G) \\
\downarrow & & \downarrow \text{i}_{2,1} \\
\text{Ext}(H_{n-1}(X, X \setminus K_1), G) & \longrightarrow & H^n(X, X \setminus K_1; G)
\end{array}$$

The exactness of rows of the diagram and the triviality of the homomorphisms $e_{2,1}$ and $h_{3,2}$ imply the triviality of the homomorphism $\text{i}_{3,1} = e_{2,1} \circ \text{i}_{3,2} : H^n(X, X \setminus K_3; G) \to H^n(X, X \setminus K_1; G)$. The local compactness of $X$ allows us to find an open $\sigma$-compact subset $W \subset X$ containing the compact set $K_1$. Since the sets $K_i$, $i \in \{1,3\}$, are of type $G_\delta$ in $X$, the spaces $W \setminus K_i$ are $\sigma$-compact and thus paracompact.

The Excision Axiom for singular cohomology (see [Spa, V, §4]) implies that $H^n(X, X \setminus K_i; G) = H^n(W, W \setminus K_i; G)$ for $i \in \{1,3\}$. This observation implies that the inclusion homomorphism $\text{i}_{3,1} : H^n(W, W \setminus K_3; G) \to H^n(W, W \setminus K_1; G)$ is trivial. Since $W, W \setminus K_i, i \in \{1,3\}$, are paracompact lc^n-spaces, the singular cohomology group $H^n(W, W \setminus K_i; G)$ coincides with the Čech cohomology group $H^n(W, W \setminus K_i; G)$. Consequently, the inclusion homomorphism $\check{H}^n(W, W \setminus K_3; G) \to \check{H}^n(W, W \setminus K_1; G)$ is trivial. By Excision Axiom for Čech cohomology, $\check{H}^n(W, W \setminus K_i; G) = H^n(X, X \setminus K_i; G)$ for $i \in \{1,3\}$. Consequently, the inclusion homomorphism $\check{H}^n(X, X \setminus K_3; G) \to \check{H}^n(X, X \setminus K_1; G)$ is trivial and so is the inclusion homomorphism $\check{H}^n(X, X \setminus V; G) \to \check{H}^n(X, X \setminus U; G)$, where $V$ is the interior of $K_3$. But this contradicts the choice of the neighborhood $U$. \qed
Theorem 12.2 will help us to evaluate the extension dimension of a locally compact LC\(^n\)-space whose all points are homological \(Z_\infty\)-points.

**Theorem 12.3.** If a metrizable locally compact LC\(^0\)-space \(X \in Z_\infty^Z\) has e-dim\((X) \leq L\) for some CW-complex \(L\), then \(\pi_k(L) = 0\) for all \(k \leq n\).

**Proof.** Separately we shall consider the cases of \(n = 0, 1\) and \(n \geq 2\).

0. If \(n = 0\), then it suffices to check that the CW-complex \(L\) is connected. Since each point of the LC\(^0\)-space \(X\) is a homological \(Z_0\)-point, \(X\) contains no isolated point and thus \(X\) contains an arc \(J\) connecting two distinct points \(a, b \in X\). Assuming that the complex \(L\) is disconnected, consider any map \(f: \{a, b\} \to L\) sending the points \(a, b\) to different components of \(L\). Because of the connectedness of \(J \subset X\) the map \(f\) does not extend to \(X\), which contradicts e-dim\(X \leq L\).

1. For \(n = 1\) we should prove the simple-connectedness of \(L\). Theorem 12.1 implies that dim\((X) > 1\). Then there is a point \(x \in X\) whose any neighborhood \(U \subset X\) has dimension dim\(U > 1\). Since \(X\) is an LC\(^1\)-space, the point \(x\) has a closed neighborhood \(N\) such that any map \(f: \partial U^2 \to N\) is null-homotopic in \(X\). Moreover, we can assume that \(N\) is a Peano continuum. Since dim\(N > 1\), the continuum \(N\) is not a dendrite and consequently, contains a simple closed curve \(S \subset N\). Assuming that the CW-complex \(L\) is not simply-connected, we can find a map \(f: S \to L\) that is not homotopic to a constant map. Then the map \(f\) cannot be extended over \(X\) since the identity map of \(S\) is null-homotopic in \(X\). But this contradicts e-dim\(X \leq L\).

2. Finally, we consider the case of \(n \geq 2\). Suppose that \(\pi_k(L) \neq 0\) for some \(k \leq n\). We can assume that \(k\) is the smallest number with \(\pi_k(L) \neq 0\). The simple connectedness of \(L\) implies that \(k > 1\). Applying Hurewicz Isomorphism Theorem, we conclude that \(H_k(L) = \pi_k(L) \neq 0\). Since e-dim\(X \leq L\), we may apply Theorem 7.14 of [Dyd] to conclude that dim\(H_n(L), X \leq n\). But this contradicts Theorem 12.2.

\[\Box\]

13. Dimension of spaces whose all points are \(Z_\infty\)-points

In this section we study the dimension properties of spaces whose all points are homological \(Z_\infty\). We shall show that locally compact ANR’s with this property are infinite-dimensional in a rather strong sense: they cannot be \(C\)-spaces and have infinite cohomological dimension.

We recall that a topological space \(X\) is defined to be a \(C\)-space if for any sequence \(\{V_n: n \in \omega\}\) of open covers of \(X\) there exists a sequence \(\{U_n: n \in \omega\}\) of disjoint families of open sets in \(X\) such that each \(U_n\) refines \(V_n\) and \(\bigcup\{U_n: n \in \omega\}\) is a cover of \(X\). By Theorem 6.3.8 [En] each metrizable countable-dimensional space is a \(C\)-space.

**Theorem 13.1.** If \(X \in Z_\infty^Z\) is a locally compact metrizable LC\(^0\)-space, then

1. \(X\) fails to be \(u\)-dimensional and is not countable-dimensional;
2. if \(X\) is an \(lc^\infty\)-space, then dim\(G \cdot X = \infty\) for any abelian group \(G\);
3. if \(X\) is an \(LC^\infty\)-space, then e-dim\(X \leq L\) for any non-contractible CW-complex \(L\);
4. \(X\) is locally contractible, then \(X\) fails to be a \(C\)-space.

**Proof.** 1–3. The first three items follow from Theorems 4.4 [12.2] and 12.3 respectively.

4. First we prove the fourth item under a stronger assumption that all points of \(X\) are homotopical \(Z_\infty\)-points. Assume that \(X\) is a \(C\)-space. By Gresham’s Theorem [Grc], the \(C\)-space \(X\), being locally contractible, is an ANR. Then the product \(X \times Q\) is a Hilbert cube manifold according to the Edwards’ ANR-theorem, see [Chap 44.1]. The product \(X \times Q\), being a \(Q\)-manifold, contains an open subset \(U \subset X \times Q\) whose closure \(\overline{U}\) is homeomorphic to the Hilbert cube \(Q\) whose boundary \(\partial U\) is a \(Z_\infty\)-set in \(\overline{U}\). Now consider the multivalued map \(\Phi: X \to U\) assigning to each point \(x \in X\) the set \(\Phi(x) = \overline{U} \setminus \{x\} \times Q\). Our goal is to show that the map \(\Phi\) has a continuous selection \(s: X \to U\).

Assuming that this is done, consider the map \(s \circ \text{pr}: U \to \overline{U}\) where \(\text{pr}: \overline{U} \to X\) stands for the natural projection of \(\overline{U} \subset X \times Q\) onto the first factor. This map is continuous and has no fixed point, which is a contradiction.
So, it remains to construct the continuous selection $s$ of the multivalued map $\Phi$. The existence of such a selection will follow from Uspešnički’s Selection Theorem [US] as soon as we shall verify that

1. for each $x \in X$ the complement $\overline{U} \setminus \Phi(x) = \overline{U} \cap (\{x\} \times Q)$ is a $Z_\infty$-set in $\overline{U}$;
2. for any compact set $K \subseteq \overline{U}$ the set $\{x \in X : \Phi(x) \supseteq K\}$ is open in $K$.

The first condition holds since each point of $X$ is a $Z_\infty$-point in $X$ and the boundary $\partial U$ is a $Z_\infty$-set in $\overline{U}$. The second condition holds because $\{x \in X : F(x) \supseteq K\} = X \setminus \text{pr}(K)$.

This completes the proof of the special case when all points of $X$ are $Z_\infty$-points. Now assume that all points of $X$ are merely homological $Z_\infty$-points in $X$. Then the points of the product $X \times [0,1]$ are (homotopical) $Z_\infty$-points by Corollary [5.3]. Now the preceding discussion implies that $X \times [0,1]$ fails to be a $C$-space. Taking into account that the product of a metrizable $C$-spaces with the interval is a $C$-space [AG, Theorem 2.2.3], we conclude that $X$ is not a $C$-space.

**Remark 13.2.** The last item of Theorem [13.1] is true in a bit stronger form: each locally compact locally contractible space $X \in \cup_G Z_\infty^G$ fails to be a $C$-space, see [BC2]. However, the proof of this stronger result requires non-elementary tools like homological version of Uspešnički’s Selection Theorem combined with a homological version of the Brouwer Fixed Point Theorem. On the other hand, this stronger result would follow from Theorem 4.4 if each compact $C$-space were trt-dimensional. However we are not sure that this is true.

**Remark 13.3.** In light of Theorem [13.1] it is interesting to mention that a compact AR whose all points are $Z_\infty$-points need not be homeomorphic to the Hilbert cube. A suitable counterexample was constructed in [DW] 9.3.

**Problem 13.4.** Let $X$ be a compact absolute retract whose all points are $Z_\infty$-points. Is $X$ strongly infinite-dimensional? Is $X \times I$ homeomorphic to the Hilbert cube?

In this respect let us mention the following characterization of Hilbert cube manifolds [BR] which can be deduced from Theorem [4.3] and the homological characterization of $Q$-manifolds due to R. Daverman and J. Walsh [DW], see also [BV].

**Theorem 13.5.** A locally compact ANR-space $X$ is a Hilbert cube manifold if and only if

1. $X$ has the Disjoint Disk Property;
2. each point of $X$ is a homological $Z_\infty$-point;
3. each map $f : K \to X$ of a compact polyhedron can be approximated by a map with trt-dimensional image.

We recall that a space $X$ has the Disjoint Disk Property if any two maps $f, g : I^2 \to X$ can be approximated by maps with disjoint images.

Theorem 4.4 implies that each closed trt-dimensional subspace of the Hilbert cube $Q$ is a homological $Z_\infty$-set in $Q$. By Proposition 4.7 of [ACP], each compact trt-dimensional space is a $C$-space.

**Question 13.6.** Is a closed subset $A \subseteq Q$ a homological $Z_\infty$-set in $Q$ if $A$ is weakly infinite-dimensional or a $C$-space?

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