The Gospel according to DeWitt revisited: quantum effective action in braneworld models*

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Abstract

We construct quantum effective action in spacetimes with branes (boundaries) and establish its relation to the "cosmological wave function" of the bulk – the solution of the corresponding Wheeler-DeWitt equation which can be considered as a means of the holographic description of braneworld models. We show that for a special type of the bulk-brane gauge fixing procedure the one-loop part of the action decouples into the additive sum of brane-to-brane and bulk-to-bulk effective actions, and this decomposition proliferates in a special way in higher orders of the Feynman diagrammatic expansion. This property is based on a special duality relation between the Dirichlet and Neumann boundary value problems when applied to the functional determinants of wave operators and the field-theoretic version of the well-known semiclassical Van Vleck-Morette determinant. It facilitates the gauge-independent way of treating the strong-coupling and VDVZ problems in brane induced gravity models. Importance of this technique in various implications of braneworld theory and infrared modifications of Einstein theory is briefly discussed.

1 Introduction

This talk is dedicated to the memory of Bryce DeWitt. No need to say that he laid foundations of quantum gravity and theory of gauge fields. But it is worth emphasizing that he was a moving spirit behind what later has materialized as Becchi-Rouet-Stora-Tyutin symmetry, Batalin-Fradkin-Vilkovisky formalism and Batalin-Vilkovisky

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quantization method — fundamental landmarks in modern high-energy physics and the whole epoch in the life of the Theory Department whose seventy years jubilee we are celebrating now. Moreover, for many years Bryce DeWitt was linked with bonds of deep affection to Professor Fradkin, Professor Markov, Igor Batalin and, especially, Gregory Vilkovisky by sharing with them infectious enthusiasm in challenging and attacking fundamental problems of theoretical physics and beyond. He often visited Russia, and one can hardly find another place here where an American professor would walk more frequently than Bryce did in one of the streets of Moscow. It is hard to think that he will never ever walk there again.

Huge scope of his interests and works – the Gospel according to DeWitt as Gregory Vilkovisky once called it [1] – embraced foundations of quantum mechanics, measurement theory, quantum field theory and cosmology, and other speakers at this session – Richard Woodard, Bill Unruh and Slava Mukhanov – are going to discuss these subjects. As for me, I am going to dwell on his pioneering ideas and methods of quantum theory of gauge fields, and the way these methods work in the new context — gravitational brane models with extra dimensions. I will begin with the peculiarities of Feynman-DeWitt-Faddeev-Popov gauge fixing procedure [2, 3] in brane theory, demonstrate its relation to the cosmological wavefunction of the multidimensional spacetime bulk as a solution of the corresponding Wheeler-DeWitt equation [4], consider its semiclassical expansion and a nontrivial interplay between the Dirichlet and Neumann boundary value problems for propagators of the covariant background field method in brane theory. Finally, I will briefly discuss the significance of these methods in infrared modifications of Einstein theory that recently got strongly motivated by the challenge of the cosmological constant and cosmological acceleration problems [5].

2 Gauge fixing in braneworld models

Motivated by this challenge, as well as by the D-brane idea in string theory [6] and hierarchy problem in high-energy physics [7], the new concept of extra dimensions suggests the braneworld picture of our Universe as a timelike surface embedded into the fundamental higher-dimensional spacetime. In the simplest 5-dimensional case its canonical description can be represented as a transformation of the familiar picture of time evolution from an initial 3-dimensional Cauchy surface to the picture of a timelike 4-dimensional brane with the fifth coordinate playing the role of time. Principal difference is that the Cauchy surface is an imaginary manifold carrying only the initial data while the brane is an actual bearer of 4-dimensional tension and matter spanned
by its timelike trajectories which are subject to dynamical laws.

Generically, the action in brane models contains the 5-dimensional (bulk) and 4-dimensional (brane) parts depending on the corresponding metrics $G_{AB}(X)$ and $g_{\alpha\beta}(x)$, $A = 0, 1, 2, 3, 5$, $\alpha = 0, 1, 2, 3$, and involves matter sources on the brane which we do not specify here. They contain the 5-dimensional and possibly 4-dimensional Einstein terms with or without relevant cosmological terms, and can be characterized by different gravitational constant scales $G_4 = 1/M_4^2$ and $G_5 = 1/M_5^3$ (like in brane induced gravity models),

$$S = S_4[g_{\alpha\beta}(x)] + S_5[G_{AB}(X)]$$

(1)

Quantum effective action results from functional integration over the bulk and brane metrics in which, however, one should factor out by gauge-fixing procedure spacetime diffeomorphisms,

$$e^{i\Gamma} = \int DG_{AB}(X) e^{iS[G_{AB}(X)]} \times \text{(gauge-fixing)}.$$  

(2)

These diffeomorphisms, which leave the system invariant, $G_{AB} \rightarrow G_{AB} + \mathcal{L}_{\Xi}G_{AB}$, are generated by the bulk vector fields $\Xi^A(X)$ which do not move the brane, that is have on the brane zero normal component

$$\Xi_\perp = 0, \quad \Xi^\mu \equiv \xi^\mu(x) \neq 0$$

(3)

(we shall denote by the vertical bar the restriction of the quantity to the brane). Tangential diffeomorphisms do not move the brane and, therefore, are included in the gauge group to be factored out.

A gauge-fixing procedure begins with adding to the action a gauge-breaking term quadratic in gauge conditions $F^A(G, \partial_B G)$. When gauges are relativistic, that is admit residual transformations whose parameters form the spacetime propagating modes, the complete gauge-fixing should include extra gauge conditions $\chi^\mu(g, \partial_\mu g)$ imposed on the brane to remove the degeneracy in boundary conditions for these modes. This leads to extra gauge-breaking term as a brane-surface integral quadratic in $\chi^\mu$, and the gauge-fixed action takes the form

$$S \Rightarrow S_{gf} = S + \frac{1}{2} F^A C_{AB} F^B + \frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu.$$  

(4)

Here $C_{AB}$ and $c_{\mu\nu}$ are some gauge-fixing matrices and we use condensed DeWitt notations which accumulate in the index not only discrete labels but also spacetime
coordinates, their contraction implying also spacetime integration. Since $A$ is a 5-dimensional index, its condensed version $A = (A, X)$ implies integration over the full 5D bulk, and the second term in (4) represents the bulk gauge-breaking term $S_{gb}^5$, while with the 4D index $\mu = (\mu, x)$ the third term is the brane surface integral $S_{gb}^4$,

$$S_{gb}^5 = \frac{1}{2} F^A C_{AB} F^B = \int d^5X F^A(X) C_{AB}(X) F^B(X),$$

$$S_{gb}^4 = \frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu = \frac{1}{2} \int d^4x \chi^\mu(x) c_{\mu\nu}(x) \chi^\nu(x).$$

An example of such gauge conditions is given by the linear harmonic gauges on metric fluctuations $H_{AB} = G_{AB} - G_{AB}^0$, $h_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\beta}^0$, on some background (with all covariant derivatives and contractions with respect to a background metrics)

$$F_A = \nabla^B H_{AB} - \frac{1}{2} \nabla_A H, \quad \chi^\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{2} \nabla_\mu h.$$ (7)

The Faddeev-Popov operators for these gauges represent correspondingly the 5D and 4D covariant d’Alembertians (modified by Ricci curvature terms)

$$F^A \Rightarrow Q_A^B = \Box_5 \delta^A_B + R_A^B, \quad \chi^\mu \Rightarrow J_\nu^\mu = \Box_4 \delta^\mu_\nu + R_\nu^\mu,$$ (8)

and thus describe the residual transformations as modes propagating respectively in the bulk and on the brane.

More generally, if we introduce condensed notations for bulk and brane metric fields

$$G^a = G_{AB}(X), \quad g^i = g_{\alpha\beta}(x)$$ (9)

and denote the generators of their respective 5D and 4D diffeomorphisms (Lie derivatives acting respectively in the bulk and on the brane) by $R^a_A$ and $R^i_\mu$

$$\mathcal{L}_z G^a = R^a_A z^A, \quad \mathcal{L}_\xi g^i = R^i_\mu \xi^\mu,$$ (10)

then these operators read

$$Q_A^B = \frac{\delta F^A}{\delta G^a} R^a_B,$$

$$J_\nu^\mu = \frac{\partial \chi^\mu}{\partial g^i} R^i_\nu.$$ (12)

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1There is a joke that in the history of the twentieth century there were basically two achievements in theoretical physics. One is due to Einstein who suggested to omit the summation sign and another due to Bryce DeWitt who proposed to drop the sign of integration. And maybe this joke is very close to a deep truth, because it is hard to overestimate how these suggestions expedited research in high-energy and gravitational physics.
where to distinguish the 5D and 4D variational derivatives we use correspondingly variational and partial-derivative notations.

Remarkable feature of the relativistic gauge-fixing procedure (4) that does not mix bulk and brane metrics is the bulk-brane factorization of the gauge-fixing factor in (2). More precisely, let the brane gauges $\chi^\mu(g, \partial_\mu g)$ among the full set of gauge conditions

$$\chi^\mu = \chi^\mu(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}),$$

$$F^A = F^A(G_{CD}, \partial_B G_{CD})$$

not involve lapse-shift functions $N_A \sim G_{5A}$ and the derivatives $\partial_5 g_{\alpha\beta}$ and let the bulk gauges be relativistic ones from the viewpoint of the fifth-time canonical formalism

$$\det \frac{\partial F^A}{\partial N^B} \neq 0, \quad \tilde{N}^B \equiv \partial_5 N^B.$$ (15)

This property guarantees that the matrix-valued operator $Q_B^A$, (11), has a non-degenerate term of second order in $\partial_5$ (cf. Eq. (8)) and, thus, describes dynamical propagation of the modes of residual transformations (gauge ghosts) in the fifth "time". From the viewpoint of the canonical formalism, the brane gauges (13) play the role of "unitary" gauge conditions with the ghost operator (12) ultralocal in time $x^5$, as opposed to to relativistic gauge conditions (14).

Then the full gauge-fixing factor in the path integral (2) factorizes into the product of bulk and brane pieces

$$(\text{gauge-fixing}) = \left( e^{\frac{i}{4} F^A C_{AB} F^B} \text{Det}_D Q_B^A \right) \times \left( e^{\frac{i}{2} \chi^{\mu} c_{\mu\nu} \chi^{\nu} \det J_{\nu}^\mu} \right),$$ (16)

where we again distinguish between the notations for five dimensional (Det) and four dimensional (det) functional determinants of ghost operators, and the subscript $D$ implies that the functional determinant of the bulk operator is calculated subject to Dirichlet boundary conditions on the brane. This factorization allows one to rewrite the effective action path integral (2) in a somewhat different form, provided the integration over brane metric is reserved for the last, while the bulk integration (subject to fixed induced metric on the brane) is done first according to

$$\int D G_{AB}(X) = \int d g_{\alpha\beta}(x) \int_{g_{\alpha\beta}=g_{\alpha\beta}(x)} DG_{AB}(X).$$ (17)

$^2$In its turn, this follows from the fact that the gauge transformations of Lagrange multipliers contains the time derivative of the gauge parameter $L_\Xi N^A = R_B^A \Xi^B \sim \partial_5 \Xi^A + ...$.

$^3$We are not particularly concerned with the boundary conditions for the 4D Faddeev-Popov determinant, because they are determined by concrete (in-out or in-in) setting on the brane and in the Euclidean case reduce to trivial Dirichlet conditions at infinity.
The result looks as a purely 4-dimensional Feynman-DeWitt-Faddeev-Popov functional integral \[8\]

\[ e^{i\Gamma} = \int dg_{\alpha\beta}(x) e^{iS_4[g_{\alpha\beta}]} \left( e^{\frac{1}{2}X^{\mu}c_{\mu\nu}X^{\nu}} \det J^{\mu}_{\nu} \right) \Psi_{\text{Bulk}}(g_{\alpha\beta}) \] (18)

for the system with the brane action \(S_4[g_{\alpha\beta}]\) but with the insertion of the, so to say, cosmological wavefunction of the bulk spacetime

\[ \Psi_{\text{Bulk}}(g_{\alpha\beta}) = \int DG_{AB}(X) e^{iS_5[G_{AB}]} \left( e^{\frac{i}{2}F^{A}C_{AB}F^{B}} \det DQ^{A}_{B} \right). \] (19)

3 Braneworld picture vs quantum cosmology

This is a functional of brane metric which has a number of important properties. First it is determined only by the bulk part of the action \(S_5[G_{AB}]\). As was shown in context of the 4-dimensional quantum cosmology \[9\], the path integral (19) is independent of the choice of bulk gauge conditions for any (off-shell) value of its argument \(g_{\alpha\beta}\) and satisfies the Wheeler-DeWitt equations \[4\] — the set of first-class Dirac constraints

\[ H_A(g_{\alpha\beta}, p^{\alpha\beta}) \Psi_B(g_{\alpha\beta}) = 0. \] (20)

These properties can be directly checked in the one-loop approximation, when

\[ \Psi^{1\text{-loop}}_{\text{Bulk}}(g) = e^{iS_5[G_0(g)]} \frac{\det DQ^{A}_{B}}{(\det F_{ab})^{1/2}}. \] (21)

is given by the tree-level exponential \(S_5[G_0(g)]\) — the classical bulk action on its mass shell \(G_0(g)\) (the solution of classical equations in the bulk subject to boundary data on the brane, \(G_0^{\alpha\beta}(g) = g_{\alpha\beta}(x)\)) — and the one-loop prefactor composed of functional determinants of the ghost \(Q^{A}_{B}\) and metric field \(F_{ab}\) operators \[9, 10\]

\[ F_{ab} = \frac{\delta^2}{\delta G^{a\delta}G^{b}\delta} \left( S_5 + S^{gb}_5 \right). \] (22)

\[ ^4\text{The functional determinant of } F_{ab} \text{ is taken subject to combined set of Dirichlet-Neumann boundary conditions [9] which originate from the fact that part of metric variables } G_{\mu\nu} \text{ are fixed at the brane, while the lapse and shift functions } N_A \sim G_{5A} \text{ are integrated over in infinite limits at the brane.} \]
In context of quantum cosmology, when the role of $x^5$ is played by a real physical time $x^0 = t$, $g_{\alpha\beta}$ goes over into $g^i = g_{mn}(x)$, $m = 1, 2, 3$ and the gauge index $A$ shrinks to the 4-dimensional one, $A \to \mu$, the consideration of a one-argument wavefunction can be replaced by the two-point kernel $U(g, g')$,

$$\Psi_{\text{Bulk}}(g) \to U(g, g'),$$

(23)

amplitude of transition between the 3-metric configurations of two spacelike surfaces (playing the role of two spacelike branes). As a solution of the Wheeler-DeWitt equations with respect to $g$ (or $g'$) it has the following semiclassical expression

$$U_{1\text{-loop}}(g, g') = e^{iS(g, g')} \left( \frac{\det D_{ik'}}{\det J_\nu^\mu \det J_\mu^\nu} \right)^{1/2}$$

(24)

(derived in [12] by directly solving the Wheeler-DeWitt equations and later shown in [10] to be equal to the one-loop part (21) of the corresponding full path integral). Here $S(g, g')$ is a Hamilton-Jacobi function (the action calculated on classical solution interpolating between two 3-geometries $g'$ and $g$, analogous to $S_5[G_0(g_{\mu\nu})]$) and $D_{ik'}$ is the matrix of its derivatives with respect to initial and final metrics modified by a gauge-breaking term

$$D_{ik'} = \frac{\partial^2 S(g, g')}{\partial g^i \partial g'^{k'}} + \frac{\partial \chi^\mu(g)}{\partial g^i} e_{\mu\nu} \frac{\partial \chi^\nu(g')}{\partial g'^{k'}}.$$

(25)

The preexponential factor here can have the name of Morette-DeWitt determinant, because it makes the synthesis of the well-known Van Vleck-Morette formula for the semiclassical kernel of the Schroedinger evolution in a non-gauge theory [13]

$$\left[ \det \frac{\delta^2 S[g(t)]}{\delta g(t) \delta g(t')} \right]^{-1/2} = \text{const} \left[ \det \frac{\partial^2 S(g, g')}{\partial g \partial g'} \right]^{1/2}$$

(26)

with the Feynman-DeWitt-Faddeev-Popov gauge-fixing procedure in a gauge one. From the viewpoint of the latter the Hessian matrix of the Hamilton-Jacobi function $S(g, g')$ is degenerate with the gauge generators as left and right zero vectors

$$R^i_{\mu} \frac{\partial^2 S}{\partial g^i \partial g'^{k'}} = 0, \quad \frac{\partial^2 S}{\partial g^i \partial g'^{k'}} R_{\nu}^{k'} = 0.$$  

(27)

[^5]In context of Euclidean quantum gravity the replacement opposite to (23) gives rise to the well-known no-boundary cosmological wavefunction [11] when the "initial" 3-geometry $g'$ is identified with a regular internal point (the center) of the Euclidean spacetime ball.
The gauge-breaking term in (25), built of gauge conditions associated with these invariances, makes this matrix invertible, but requires introduction of the ghost factors in (24) with

\[ J_\nu^\mu = \frac{\partial \chi^\mu (g)}{\partial g^i} R^i_\nu, \quad J'_\nu^\mu = \frac{\partial \chi^\nu (g')}{\partial g'^k} R'^k_\nu \]

in order to preserve Ward identities for the one-loop solution (in the form of its independence of the choice of gauges \( \chi^\mu \)). Thus, the equality of one-loop (21) and semiclassical (24) representations is a direct gauge-theory analogue of the non-gauge Van Vleck-Morette formula (26).

Semiclassical solutions of the Wheeler-DeWitt equations \( \Psi_{1,2}(g) \)

\[ H_\mu \left( g, \frac{i}{\partial g} \right) \Psi_{1,2}(g) = 0 \] (29)

propagated by the two-point kernel (24) conserve the inner product involving a non-trivial integration measure which in the semiclassical approximation reads as [12]

\[ \langle \Psi_1, \Psi_2 \rangle = \int dg \Psi_1^*(g) \left( \delta(\chi(g)) \det J_\mu^\nu \right) \Psi_2. \] (30)

This measure is located on the surface of unitary gauge conditions via the delta-function, \( \delta(\chi(g)) = \prod_\mu \delta(\chi^\mu (g)) \), or can be identically transformed to the Gaussian distribution

\[ \delta(\chi(g)) \rightarrow (\det c_{\mu\nu})^{1/2} e^{\frac{i}{2} \chi^\mu c_{\mu\nu} \chi^\nu} \] (31)

smeared near \( \chi^\mu = 0 \).

Therefore, returning to the braneworld context and comparing (30) with (18), one can say the brane effective action (18) can be viewed as matrix element between the wave function of the brane \( \Psi_{\text{brane}}(g) = \exp[-i S_4(g)] \) and that of the bulk (19)

\[ e^{i \Gamma} = \langle \Psi_{\text{brane}}, \Psi_{\text{Bulk}} \rangle. \] (32)

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6 One might say that the Morette-DeWitt formula (24) unifies the research efforts of Bryce DeWitt and his wife Cecile Morette who, as she joked when being awarded the Marcel Grossmann Prize on General Relativity, never collaborated on any research project but their four daughters.

7 Generally the transformation (31) is identical only in the singular limit \( c_{\mu\nu} \rightarrow \infty \), but in virtue of the gauge-independence properties of the solutions of Wheeler-DeWitt equations (29) this replacement is identical for any invertible \( c_{\mu\nu} \). Everywhere above we neglected the determinants \( \det J_\mu^\nu \) and \( \text{Det} C_{AB} \) – the contributions of the so-called Nielsen-Kallosh ghosts, which in field theories are trivial in the class of ultralocal gauge-fixing matrices.
It is important to notice, though, that while in 4D quantum cosmology all four diffeomorphisms are factored out by gauge conditions in the measure, in 5D braneworld model only 4-dimensional diffeomorphisms are gauged away in (18). This corresponds to the fact that the motion of the brane is a dynamical process, in contrast to quantum cosmology where local deformations of the spacelike surface is a gauge transformation.

The practical meaning of the representation (32) is still to be comprehended, but apparently it can be utilized in nonperturbative methods based on solving the Wheeler-DeWitt equation for $\Psi_{\text{Bulk}}(g_{\alpha\beta})$. At the moment, I know only of the tree-level application of this equation by Verlindes [14] who analyzed in the language of the Einstein-Hamilton-Jacobi function the AdS-flow in the holographic description of the Randall-Sundrum model. Also it can arise in Gutperle-Strominger context of S-branes [15] — the situation most closely related to chronologically ordered mediation by quantum bulk of the interaction between S-brane states (spacelike branes) in the low-energy limit of string field theory.\footnote{One should notice the unusual sign of phase in the definition of $\Psi_{\text{brane}}(g)$ above. Point is that in contrast to conserved inner product in quantum cosmology (30) no unitarity holds in the fictitious propagation in spacelike $x^5$-time, so that the inner-product representation (32) looks currently more as a calculational trick, rather than an ample bulk-to-brane transition amplitude. In Euclidean braneworld theory no such problem arises, but the formal link with unitarity and probability conservation disappears at fundamental level. However, in S-brane context unitary transition amplitudes between various spacelike branes (mediated by the bulk) should make sense. Future string field theory will apparently upgrade its D and S-brane descriptions to a unified unitary framework in Lorentzian spacetime. Related discussion of this point can be found in [16].}

Postponing, however, these nonperturbative aspirations till, hopefully, not so distant future, let us focus on the semiclassical expansion of brane effective action.

4 Semiclassical expansion and duality of boundary value problems

Brane action was intensively studied at the tree level in various models like Randall-Sundrum [17], Gregory-Rubakov-Sibiryakov [18], brane induced gravity models of the Dvali-Gabadadze-Porrati type [19] for the purpose of finding a consistent infrared modification of Einstein theory, that could account for the phenomenon of the recently observed cosmological acceleration [5]. These models were shown to suffer from a number of problems like van Dam-Veltman-Zakharov discontinuity violating the correspondence principle with the Einstein theory [20], low strong-coupling scale precluding
from consistent weak-field perturbation theory \cite{21, 8}, presence of ghosts and tachyons, etc., and their eradication at the tree level did not guarantee the consistency of the theory at the quantum level. On the other hand, naive Feynman loop calculations lead to uncontrollable gauge dependence of quantum effects, the lack of their manifest covariance and other difficulties \cite{8}.

The way to their resolution was, however, very well presented in lessons given to us many years ago by Bryce DeWitt. Not only did he suggest the relativistic gauge-fixing procedure of the above type, but also invented the method of background covariant gauges which make the background (mean) field method manifestly covariant \cite{22}. Interestingly, applying the combination of these methods within a special gauge fixing procedure discussed above leads to the following one-loop approximation for brane effective action

\[
e^{i\Gamma_{1-loop}(g_{\alpha\beta})} = e^{iS[G_0(g_{\alpha\beta})]} \frac{\text{Det}Q}{(\text{Det}F_{\text{BB}})^{1/2}} \frac{\text{det}J}{(\text{det}F_{\text{bb}})^{1/2}}, \tag{33}
\]

where \(F_{\text{BB}}\) and \(F_{\text{bb}}\) schematically denote the bulk-to-bulk and brane-to-brane inverse propagators of the theory given respectively by (22) and

\[
(F_{\text{bb}})_{ik} = \frac{\delta^2}{\delta g^i \delta g^k} \left( S_{4}^{gf}(g) + S_{5}^{gf}[G_0(g)] \right) \tag{34}
\]

Remarkable property of this answer is a factorization of the prefactor into bulk-to-bulk and brane-to-brane parts which takes place not only in the ghost sector, like in (16), but in the metric field sector as well. As a result the one-loop action becomes an additive sum of bulk and brane pieces. They both contain their respective 5D and 4D metric field and ghost contributions with 5D relativistic Faddeev-Popov operator \(Q\) vs its "unitary" counterpart on the brane \(J\). Moreover, they both are separately gauge-independent. The gauge independence of the bulk part is universal, because in the braneworld setting there are no sources in the bulk, and the bulk part is always on shell. The brane part is gauge-independent in the usual sense, when the background (mean) field on the brane \(g_{\alpha\beta}(x)\) is on shell — that is for scattering problems on the brane.\footnote{The mean (background) field dependence of \(\Gamma(g_{\alpha\beta})\) in (33) results from introducing the sources to quantum fields on the right-hand side of Eqs. (2) and (18) and making the Legendre transform from the resulting generating functional of connected Green's functions to the effective action of the mean field \(g_{\alpha\beta}(x)\) — the generating functional of one-particle irreducible diagrams. For brevity we did not do that in equations throughout previous sections.} The gauge-independent off-shell extension of this part of the action can be done along the lines of the so-called unique effective action by G.Vilkovisky \cite{23, 24}.

\[
\]
The factorization of metric-field contributions in (33) is a corollary of an interesting duality between the Dirichlet and Neumann boundary value problems [26] for the bulk operator \( F \). The latter problem naturally arises in the original definition of the brane effective action (2). The integration in (2) runs also over the boundary metric, which technically results in the Israel junction conditions on the brane [25] — the generalized Neumann or Robin type boundary condition for quantum perturbations, their Neumann Green’s function \( G_N \) and the corresponding one-loop functional determinant \( \text{Det}_N F \).

It turns out that the Dirichlet-Neumann duality relates this Green’s function to the brane-to-brane operator \( F_{bb} \) and allows one to reduce the functional determinants subject to Neumann conditions to that of the Dirichlet ones

\[
F_{bb} = \left( G_N || \right)^{-1}, \\
\text{Det}_N F = \text{Det}_D F \times \text{det} F_{bb}.
\]

The double vertical bar here implies the restriction of the both arguments of \( G_N(X, X') \) to the boundary, after which it becomes a kernel of the nonlocal operation on the surface, \( G_N(X, X')|| = F_{bb}^{-1}(x, x') \), which is just the brane-to-brane propagator [26]. Again, this relation can be regarded as an extension of the Van Vleck-Morette formula (26), and its application leads to the factorization (33) in question.

Such factorization does not literally extend beyond one-loop order, but its elements proliferate to multi-loop Feynman diagrams as well. They turn out to be composed of purely bulk loops, built of Dirichlet-type bulk propagators, brane-to-brane loops and special propagators joining them — the structure subject to manageable analysis in which manifest covariance and gauge-independence are strictly kept under control.

In fact, the presence of the brane contents (characterized by the brane part \( S_4[\text{g}_{\alpha\beta}(x)] \) of the full action (1)) enters the full quantum effective action as brane-dependent insertions in the purely bulk Feynman diagrams. The latter contain as metric and ghost field propagators only the Green’s functions subject to Dirichlet boundary conditions\(^{10}\) independent of \( S_4[\text{g}_{\alpha\beta}(x)] \). This is very important in models with two different scales, like brane induced gravity models with brane and bulk Planckian scales \( M_4^2 \gg M_5^2 \) [19], because the action decouples the purely bulk part which does not feel additional scale \( M_4 \) associated with the brane. This property can be very helpful in infrared modifications of Einstein gravity theory and other models encumbered with the strong-coupling problem [8].

\(^{10}\)In essence, the boundary conditions in metric sector are slightly more complicated, cf. footnote 4, but they anyway do not depend on the brane part of the full classical action.
On the other hand, duality relations (35)-(36) allow one to systematically reduce complicated boundary conditions of the generalized Robin type to much simpler Dirichlet ones and apply the well-known Schwinger-DeWitt technique of the curvature expansion in the background field formalism of the brane effective action — the programme to be realized in a forthcoming paper [27]. In particular, these relations are expected to simplify the technique for boundary surface contributions to Schwinger-DeWitt coefficients, which can be very complicated for Robin-type boundary conditions [28] and, especially, for the so-called oblique boundary conditions involving the derivatives of fields tangential to the boundary [29] and often resulting in the strong ellipticity problem [30, 31].

To summarize, the methods of quantum brane effective action, discussed above, are expected to be very productive in infrared modifications of Einstein theory, stringy D-branes, (boundary) string field theory and so on. There are many, many more revelations from the Gospel according to Bryce DeWitt in our coming scientific endeavors!

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