ON EXPANSIONS OF $(\mathbb{Z},+,0)$

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Abstract. Call a (strictly increasing) sequence $R = (r_n)$ of natural numbers regular if it satisfies the following condition: $r_{n+1}/r_n \to \theta \in \mathbb{R}^{>1} \cup \{\infty\}$ and, if $\theta$ is algebraic, then $R$ satisfies a recurrence relation whose characteristic polynomial is the minimal polynomial of $\theta$. Our main result states that $\mathcal{Z}_R = (\mathbb{Z},+,0,R)$ is superstable whenever $R$ is a regular sequence. We provide two proofs of this result. One relies on a result of E. Casanovas and M. Ziegler and the other on a quantifier elimination result. Both proofs share an important ingredient: for a regular sequence, the set of solutions of homogeneous equations is either finite or controlled by (finitely many) recurrence relations. This property resembles the Mann property defined by L. van den Dries and A. Gündaydın in their work on expansions of fields by subgroups. Inspired by their work and the proofs of our main result, we show that when $M$ is the domain of a multiplicative monoid of integers with the Mann property, then $\mathcal{Z}_M$ is also superstable.

Introduction

Recently, stability properties of expansions of $\mathcal{Z} = (\mathbb{Z},+,0)$ by a predicate $R$ for a set of integers has attracted the attention of many researchers. Motivated by a question of A. Pillay on the induced structure on non-trivial centralizers in the free group on two generators, D. Palacin and R. Sklinos proved in [17, Théorème 25]) that for all natural number $q$, the structure $(\mathbb{Z},+,0,\Pi_q)$ is superstable of Lascar rank $\omega$, where $\Pi_q = \{q^n | n < \omega\}$ (this was also proved independently and using different methods by B. Poizat in [17 Théorème 25]). They also show the same result for $R = (n!)$ and more generally for sequences $(r_n)$ that are ultimately periodic modulo $m$ for all $m > 1$ and such that $r_{n+1}/r_n \to \infty$. D. Palacin and R. Sklinos used former results of E. Casanovas and M. Ziegler [3] on stable expansions by a unary predicate. In another direction, when $R$ is the set $P$ of prime numbers, I. Kaplan and S. Shelah show in [10], assuming Dickson’s Conjecture ([10 Conjecture 1.1]), that $(\mathbb{Z},+,0,P \cup -P)$ is unstable and supersimple of Lascar rank 1.

In this paper, we investigate expansions of $\mathcal{Z}$ by a unary predicate $R$ interpreting a subset of the natural numbers. We generalize the result of D. Palacin and R. Sklinos. Call a sequence $R = (r_n)$ regular if it satisfies the following condition: $r_{n+1}/r_n \to \theta \in \mathbb{R}^{>1} \cup \{\infty\}$ and, if $\theta$ is algebraic, $R$ follows a linear recurrence relation whose characteristic polynomial is the minimal polynomial of $\theta$. Our Theorem 1.4 states that $\mathcal{Z}_R := (\mathbb{Z},+,0,R)$ is superstable when $R$ is regular sequence. In order to achieve this, we apply the same techniques used in D. Palacin and R. Sklinos’ work. As we mentioned above, the main tool they used is a result of E. Casanovas and M. Ziegler which states in our context that an expansion of the form $\mathcal{Z}_R$ is superstable when $R$ is small (see Theorem 1.6) and the induced structure (see Definition 1.5) on $R$ by $\mathcal{Z}_R$ is superstable, where the induced structure on $R$ is the structure with universe $R$ and predicates for the trace on $R$ of any definable set of $\mathbb{Z}$.

Section 1 is divided in three subsections. In Subsection 1.1 we analyze the trace of equations on a regular sequence $R$. A crucial part of this analysis is Proposition 1.13 where we show that the trace of an equation on $R$ is either finite or controlled by finitely many recurrence relations.

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satisfied by almost all elements of $R$. More generally, we consider functions $f : N \to Z$, called operators, of the form $f(n) = \sum_{i=0}^{d} a_i r_{n+i}$, where $\bar{a} \in Z$, and show that such operators behave nicely: the set of roots of $f$ is either finite or cofinite in $N$. This, with Proposition 1.13, will allow us to show that any regular sequence has a superstable induced structure (Corollary 1.16).

The smallness of regular sequences is considered in Subsection 1.2 where we show that we can not bound the length of expansions in base $R$ of natural numbers. This is done by showing that any set of the form $\{z + f_1(n_1) + \cdots + f_k(n_k) \mid n \in N\}$, where $z \in Z$, is not piecewise syndetic: such a set does not contain arbitrarily long sequences of bounded gaps. We finally bring the work done in Subsections 1.1 and 1.2 together to prove Theorem 1.4.

The analysis of expansions by a regular sequence is continued in Section 2 where we axiomatize, in a natural language, the theory $T_R$ of these expansions. We show that $T_R$ has quantifier elimination and is complete. This allows us to give another proof of the superstability of $T$ directly by counting types. We end this section with a decidability result and we point out similarities with expansions of Presburger arithmetic.

The last section of this paper is concerned with other kinds of expansions. First, we slightly improve the result of Kaplan and Shelah to the set of primes and their opposites which are congruent to $r$ modulo $m$, where $r < m$ are coprime natural numbers (see Proposition 3.4). Second, we show that $Z_R$ is superstable for $R$ a finitely generated submonoid of $(Z, \cdot, 1)$. This is done using the so-called Mann property for a multiplicative subgroup of an algebraically closed field, as defined and studied by L. van den Dries and A. Günaydın in [22] and [21]. This property is comparable to the statement of Proposition 1.13 and in fact, the proof of the superstability of $Z_R$, when $R$ is a finitely generated submonoid of $(N, \cdot, 1)$, is along the same lines as the proof of Theorem 1.4.

Independently of our work [16], G. Conant released a preprint ([5]) on sparsity notions and stability for sets of integers. There he defines the notion of a geometrically sparse infinite sequence $R$ (see [5] Definition 6.2). For such sequence $R$, he proves superstability of $(Z, +, 0, R)$ and calculates its Lascar rank [5] Theorem 7.1). So there is an overlap between his result and our Theorem 1.4 (see also [11]): we give an account of this overlap at the end of Section 4. We also point out that, in the first version of this paper, our main result had an extra hypothesis on regular sequences, namely that they were eventually periodic modulo $m$ for all $m > 1$. This hypothesis was necessary to understand the trace, on $R$, of congruence relations. However, G. Conant showed that in some cases, the analysis of the trace of congruence relations is not necessary and we decided to incorporate this in Theorem 1.4. This is explained after the statement of Theorem of E. Casanovas and M. Ziegler (see Theorem 1.6).

In another, more recent, preprint ([4]), G. Conant investigated expansions of $Z$ by (subsets) of finitely generated submonoids of $(N, \cdot, 1)$ and showed that these expansions are superstable. His result ([4] Theorem 3.2) is more general than Theorem 3.6. Despite this fact, we decided to keep this section as the proof of Theorem 3.6 is short and to stress the parallel between these expansions and expansions of an algebraically closed field (respectively real closed field) by a multiplicative subgroup.

Notation and convention. In this section, we fix some notations and conventions for the rest of this paper. The set of natural numbers (including 0), of integers and of real numbers will be denoted respectively $N$, $Z$ and $R$. When $X$ is one of the above sets and $a \in X$, the notations $X^\geq a$ and $X^\geq a$ refer respectively to the sets $\{x \in X \mid x > a\}$ and $\{x \in X \mid x \geq a\}$. For a natural number $n$, the set $\{1, \ldots, n\}$ will be denoted $[n]$. The set of prime numbers will be denoted by $P$.

Capital letters $I$, $J$ and $K$ will refer to (usually non empty) sets of indices. In particular, the notion $I \subseteq [n]$ will tacitly imply that $I$ is not empty. Capital letters will refer to sets and small letters will refer to elements of a given set. For a tuple $\bar{a}$ of length $n$ and $I \subseteq [n]$, $\bar{a}_I$ refers to the tuple $(a_i \mid i \in I)$.

A first order language will be denoted by the letter $L$, possibly with a subscript. An $L$-structure will be referred to by a round letter and its domain by the corresponding capital one. For instance $\mathcal{M}$ is an $L$-structure whose domain is $M$. For an element $a$ of $M$ and $A \subseteq M$, the notations acl($a/A$), tp($a/A$) and $\text{tp}^f(a/A)$ mean respectively the algebraic closure, the type and the quantifier-free
type of a over $A$ in $\mathcal{M}$. If $R \in \mathcal{L}$ is a predicate symbol, the set $\{a \in M | \mathcal{M} \models R(a)\}$ will be denoted $R(M)$ or simply $R$ when there is no confusion.

We make the following (usual) abuse of notations. When $R$ is a unary predicate symbol, expressions of the form $\exists x \in R \varphi(x)$ and $\forall x \in R \varphi(x)$ respectively means $\exists x (R(x) \land \varphi(x))$ and $\forall x (R(x) \Rightarrow \varphi(x))$. An expression of the form $x > c$, where $c \in \mathbb{N}$, is an abbreviation for $\bigwedge_{i=0}^{\infty} x \neq i$.

For each $n \in \mathbb{N}^+$, let $D_n$ be a unary predicate. The language $\{+, -, 0, 1, D_n | n > 1\}$ will be denoted $\mathcal{L}_g$ and the language $\{S, S^{-1}, 1\}$, where $S$ and $S^{-1}$ are unary functions, will be denoted $\mathcal{L}_S$. In an $\mathcal{L}_g$-structure $\mathcal{M}$, for each $n \in \mathbb{N}^+$, the symbol $D_n$ will always be interpreted as the set $\{x \in M | \mathcal{M} \models \exists y x = ny\}$.

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1. Expansion of $(\mathbb{Z}, +, -, 0)$ by a regular sequence

This section is concerned with expansions of $(\mathbb{Z}, +, -, 0)$ by a unary predicate $R$ interpreting a (strictly increasing) sequence $(r_n)$ of natural numbers. The main result of this section is the superstability of the expansion $\mathcal{Z}_R = (\mathbb{Z}, +, -, 0, R)$ when $R$ belongs to the class of regular sequences, which we define below.

Definition 1.1. Let $R = (r_n)$ be a sequence of natural numbers that satisfy a recurrence relation: there are $a_0, \ldots, a_{k-1} \in \mathbb{Z}$, with $k \in \mathbb{N}$ minimal, such that for all $n \in \mathbb{N}$, $r_{n+k} = \sum_{i=0}^{k-1} a_i r_{n+i}$. The characteristic polynomial of $(r_n)$ is the polynomial $P_R$ defined by $P_R(X) = X^k - \sum_{i=0}^{k-1} a_i X^i$.

Definition 1.2. Let $R = (r_n)$ be a sequence of natural numbers. We say that $R$ is regular if and only if it satisfies the following property: $r_{n+1}/r_n \rightarrow \theta \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ and, if $\theta$ is algebraic, then $R$ satisfies a linear recurrence relation whose characteristic polynomial $P_R$ is the minimal polynomial of $\theta$.

Remark 1.3. Let $R = (r_n)$ be defined by a recurrence relation whose characteristic polynomial is $P$. The limit $\theta = \lim_{n \rightarrow \infty} r_{n+1}/r_n$ is known as the Kepler limit of $R$. A. Fiorenza and G. Vincenzi, in [2, 8] provide a necessary and sufficient condition on $P$ and the initial conditions of $R$ for the existence of its Kepler limit (see [7, Theorem 2.3]). In particular, their result show that the Kepler limit may exist even when $P$ does not have a unique root with highest modulus: they provide an example where the characteristic polynomial’s roots are $1, 2, \pm 2i$ and the Kepler limit is 2 (see [7, Example 4.8]).

Here is a list of examples of regular sequences:

- $(n!)$;
- $(q^n)$, where $q \in \mathbb{N}^+$;
- the Fibonacci sequence as defined by $r_0 = 1$, $r_1 = 2$ and $r_{n+2} = r_{n+1} + r_n$ for all $n \in \mathbb{N}$.

Now, let us state the main result of this section.

Theorem 1.4. Let $R$ be a regular sequence. Then $\text{Th}(\mathcal{Z}_R)$ is superstable of Lascar rank $\omega$.

The proof of this theorem follows the same strategy as D. Palacin and R. Sklinos in [14]. Essentially, we apply the following result of E. Casanovas and M. Ziegler [9].

Definition 1.5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

1. Let $A \subset M$. To each $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, we associate a new $n$-ary predicate $R_{\varphi,n}$ and we denote by $\mathcal{L}_{\text{ind}}$ the language

$$\{R_{\varphi,n} | \varphi(x_1, \ldots, x_n) \text{ is an } \mathcal{L}\text{-formula}\}.$$

The induced structure on $A$ (by $\mathcal{M}$), denoted $A_{\text{ind}}$, is the $\mathcal{L}_{\text{ind}}$-structure whose domain is $A$ and $R_{\varphi,n}(A) = \varphi(M^n) \cap A^n$. 
Let $R$ be a unary predicate not in $\mathcal{L}$ and let $\mathcal{L}_R$ be the language $\mathcal{L} \cup \{ R \}$. We say that an $\mathcal{L}_R$-formula $\varphi(\bar{x})$ is bounded (with respect to $R$) if it is equivalent (in $\mathcal{M}$) to a formula of the form $Q_1 y_1 \in R \ldots Q_n y_n \in R \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $Q_i \in \{ \exists, \forall \}$.

**Theorem 1.6** ([3] Proposition 3.1). Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subset M$ and $R$ a unary predicate not in $\mathcal{L}$. Suppose that every $\mathcal{L}_R$-formula in $\mathcal{M}_A$, the $\mathcal{L}_R$-expansion of $\mathcal{M}$ where $R(M) = A$, is equivalent to a bounded one. Then for all $\lambda \geq |\mathcal{L}|$, if $\mathcal{M}$ and $A_{\text{ind}}$ are $\lambda$-stable, then $\mathcal{M}_A$ is $\lambda$-stable.

Let $R$ be a regular sequence. Since $\mathcal{Z}$ is known to be superstable (see [18] Theorem 15.4.4]), in order to show that $\mathcal{Z}_R$ is superstable, we only need to show that $R$ is small and that $R_{\text{ind}}$ is superstable. For the latter, we can use quantifier elimination of $\mathcal{Z}$ in $\mathcal{L}$ (see [18] Theorem 15.2.1]) to reduce the study of $R_{\text{ind}}$ to the trace on $R$ of equations of the form $a_1 x_1 + \cdots + a_n x_n = 0$ or divisibility relations of the form $D_m(a_1 x_1 + \cdots + a_n x_n)$. Actually, we can further reduce the analysis of $R_{\text{ind}}$, using the following observation of G. Conant (see [5] Section 5]). Let $R^0_{\text{ind}}$ be the induced structure on $R$ by formulas that are boolean combinations of homogeneous equations. For $\mathcal{N}$ an $\mathcal{L}$-structure, let $\mathcal{N}^1$ be the expansion of $\mathcal{N}$ by predicates for all subsets of $N$. G. Conant observed that $R_{\text{ind}}$ is an expansion of $R^0_{\text{ind}}$ by unary predicates [5 Corollary 5.7]. As a consequence of this observation, if $R^0_{\text{ind}}$ is definably interpreted in a structure $\mathcal{N}$ whose expansion $\mathcal{N}^1$ is superstable, then $R_{\text{ind}}$ is superstable. We apply this by showing that $R^0_{\text{ind}}$ is definably interpreted in the structure $\mathcal{N} = (\mathbb{N}, S)$, where $S(n) = n + 1$, whose expansion $\mathcal{N}^1$ has been shown to be superstable [5 Proposition 5.9]. To achieve this, we proceed in two steps. We first show that functions of the form $f : \mathbb{N} \to R : n \mapsto a_0 r_n + a_1 r_{n+1} + \cdots + a_d r_{n+d}$, where $a \in \mathbb{Z}$, called operators on $R$, behave predictably: either $f(n) = 0$ for all sufficiently large $n \in \mathbb{N}$ for $f(n) \neq 0$ for all but finitely many $n \in \mathbb{N}$, see Proposition 1.8. Second, we show that the set of solutions in $R$ of an equation of the form $a_1 x_1 + \cdots + a_n x_n = 0$ is determined by finitely many operators, see Proposition 1.13. This work on equations will allow us to show that $R_{\text{ind}}$ is definably interpreted in $\mathcal{N} = (\mathbb{N}, S)$.

Another consequence of Propositions 1.8 and 1.13 is that $R$ is small: we deduce from them that a set of the form $a + d \mathbb{N}$ cannot be covered by finitely many sets of the form $\{ z + f_1(n_1) + \cdots + f_k(n_k) | \bar{n} \in \mathbb{N} \}$. This is done in Section 1.2.

**1.1. The induced structure on a regular sequence.** In this section, we fix a regular sequence $R = (r_n)$ and we let $\theta \in R \cup \{ \infty \}$ be $\lim_{n \to \infty} r_{n+1}/r_n$.

**1.1.1. Operators on a regular sequence.**

**Definition 1.7.** Let $\bar{a} \in \mathbb{Z}$ a $d$-tuple such that $a_d \neq 0$. The operator associated to $\bar{a}$, denoted $f_\bar{a}$ or simply $f$, is the function $f : \mathbb{N} \to R : n \mapsto a_0 r_n + \cdots + a_d r_{n+d}$.

In this section, we establish the following property of operators.

**Proposition 1.8.** Let $f$ be an operator. Then the equation $f(n) = 0$ either has finitely many solutions or cofinitely many solutions.

The proof of this Proposition follows from the following Lemmas.

**Lemma 1.9.** Suppose that $\theta = \infty$. Then for all operators $f$, $f(n) = 0$ has finitely many solutions.

**Proof.** Indeed, suppose that $f(n) = a_0 r_n + \cdots + a_d r_{n+d}$, $d \geq 1$. Then $f(n) = 0$ if, and only if, $a_0 r_n + a_1 r_{n+1} + \cdots + a_d r_{n+d} = 0$. Thus, as $r_{n+i}/r_{n+d} \to 0$ for all $0 \leq i < d$, for all $n$ sufficiently large, $f(n) \neq 0$. \hfill \Box

**Lemma 1.10.** Suppose that $\theta \in R$. Let $a_0, \ldots, a_d \in \mathbb{Z}$ and suppose that $\sum_{i=0}^d a_i \theta^i \neq 0$. Then there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $a_0 r_n + a_1 r_{n+1} + \cdots + a_d r_{n+d} \neq 0$.

**Proof.** Suppose $u = \sum_{i=0}^d a_i \theta^i > 0$. Let $\bar{u}$ denote $\sum_{i=0}^d |a_i|$. Choose $\epsilon > 0$ such that $\epsilon \bar{u} < u$ and let $k \in \mathbb{N}$ be such that for all $n \geq k$ and $i \in \{ 1, \ldots, d \}$, $r_{n+i} - \theta^i r_n < \epsilon r_n$. Then $|a_i r_{n+i} - a_i \theta^i r_n| < \epsilon |a_i| r_n$ (whenever $a_i \neq 0$). By our choice of $\epsilon$ we have $0 < r_n (u - \epsilon \bar{u}) < a_0 r_n + \cdots + a_d r_{n+d}$. \hfill \Box
Lemma 1.11. Suppose that $R$ satisfies a linear recurrence whose characteristic polynomial is the minimal polynomial of $\theta$. Then for any operator $f$, the equation $f(x) = 0$ has finitely many or cofinitely many solutions. Furthermore, $f(n) = a_0r_n + \cdots + a_dr_{n+d} = 0$ has infinitely many solutions if, and only if, $a_0 + a_1\theta + \cdots + a_d\theta^d = 0$.

Proof. Let $f(n) = a_0r_n + \cdots + a_dr_{n+d}$. Then, by assumption, $f(n)/r_n \to a_0 + a_1\theta + \cdots + a_d\theta^d$. Thus, if $a_0 + a_1\theta + \cdots + a_d\theta^d \neq 0$, $f(n) = 0$ has finitely many solutions. Otherwise, $P_R$ divides $a_0 + a_1X + \cdots + a_dX^d$ and in this case, $f(n) = 0$ has cofinitely many solutions in $R$.\]

We end this section with a remark concerning inhomogeneous equations.

Proposition 1.12. Let $f$ be an operator and $z \in \mathbb{Z}\setminus\{0\}$. Then the equation $f(n) = z$ has finitely many solutions.

Proof. Assume that $f(n) = \sum_{i=0}^d a_ir_{n+i}$ and $a_d \neq 0$. Since $R$ is regular, $\theta$ exists and is either infinite or greater than 1. This implies that $u = \lim_{n\to\infty} f(n)/r_{n+d}$ exists and is finite. Notice that if $u \neq 0$, then $f(n) = z$ has finitely many solutions since $\lim_{n\to\infty} z/r_{n+d} = 0$. Now, if $u = 0$, then $\theta$ is algebraic: $a_0 + a_1\theta + \cdots + a_d\theta^d = 0$. Thus, $R$ satisfies a linear recurrence relation, and since the minimal polynomial of $\theta$ divides $a_0 + a_1X + \cdots + a_dX^d$, we have that $f(n) = 0$ for all but finitely many $n \in \mathbb{N}$. We conclude that $f(n) = z$ has finitely many solutions.\]

1.1.2. Equations and the induced structure. Let $f_1, \ldots, f_s$ be operators, $z \in \mathbb{Z}$ and let $\varphi_z(n)$ be the equation $f_1(n_1) + \cdots + f_s(n_s) = z$. We will show that the set $\varphi_z(R)$ is determined by a finite number of operators on $R$ and that $\varphi_z(\mathbb{N})$ is finite when $z \neq 0$. We call a solution $\bar{n} \in \varphi$ non-degenerate if the following two conditions hold: (1) $n_i \neq n_j$ if $i \neq j$ and (2) for all $I \subseteq [s]$ $\sum_{i \in I} f_i(n_i) \neq 0$.

Proposition 1.13. There exists $\bar{m}_1, \ldots, \bar{m}_k \in \mathbb{Z}$ such that for all $\bar{l} \in \mathbb{N}$, if $\bar{l}$ is a non-degenerate solution of $\varphi_z$ then for some $i \in [k]$, $l_i = l_1 + m_{ij}$ for all $j \leq n$.

Proof. Let $z \in \mathbb{Z}$, assume that $f_j(n) = \sum_{i=0}^d a_{ij}r_{n+i}$ and let $P_j(X) = \sum_{i=0}^d a_{ij}X^i$. Assume, without loss of generality, that $d_s \geq d_i$ for all $i \in [s]$. Suppose, towards a contradiction, that there exists two sequences $(\bar{m}_i) \in \mathbb{N}^s$ and $(l_i) \in \mathbb{N}$ such that for all $i \in \mathbb{N}$:

1. $l_i + m_{11}, \ldots, l_i + m_{ij}$ is a non-degenerate solution of $\varphi_z$;
2. $m_{is} < m_{(i+1)s}$;
3. $m_{ij} < m_{(i+1)j}$ for all $j < s$.

Let $J \subseteq [s]$ maximal such that for all $j \in J$, $\max\{m_{is} - m_{ij} | i \in \mathbb{N}\} < \infty$. Assume, without loss of generality, that for all $j \in J$, there exists $k_j \in \mathbb{N}$ such that $m_{is} - m_{ij} = k_j$ for all $i \in \mathbb{N}$. One can further assume, up to passing to subsequences, that for all $j \notin J$, $m_{ik} - m_{ij} \to \infty$. Let $u = a_{sd}$, if $\lim_{n\to\infty} r_{n+1}/r_n = \infty$ and $\sum_{i \in J} P_i(\theta^{-k_j})$ if $\lim_{n\to\infty} r_{n+1}/r_n = \theta \geq 1$. Then,

$$\lim_{i \to \infty} \sum_{j=1}^n f_j(l_i + m_{ij})/r_{l_i+m_is} = u.$$\]

This leads to a contradiction if $u \neq 0$. If $u = 0$, then $\theta$ is algebraic and in this case, we assumed that $R$ follows a linear recurrence relation whose characteristic polynomial is the minimal polynomial of $\theta$. But in that case, we have that $\sum_{j \in J} f_j(l_i + m_{ij}) = 0$ for all $i$ sufficiently large and since $1 \notin J$, we have a contradiction.\]

To clarify the first part of the statement of this proposition, let $f_{\bar{m}_i}$ be the operator defined by $f_{\bar{m}_i}(l) = \sum_{j=1}^n f_j(l + m_{ij})$. Notice that for all $i \leq k$, $m_{i1} = 0$. Then the non-degenerated solutions of $\varphi_z(x)$ are given by the tuples of the form $(l, l + m_{i2}, \ldots, l + m_{in})$ such that $f_{\bar{m}_i}(l) = z$.

Corollary 1.14. Let $z \in \mathbb{Z}$. Then the equation $\varphi_z$ has infinitely many non-degenerate solutions if and only if $z = 0$ and $f_{\bar{m}_i}(n) = 0$ has infinitely many solutions for some $i \in [k]$.

Proof. This follows from Propositions 1.12 and 1.13.\]

As a corollary of Proposition 1.13 and the following Fact, we obtain the superstability of $R_{ind}$.\]
Proposition 1.15 ([5 Proposition 5.9]). Let $\mathcal{N}$ be the structure $(\mathbb{N}, S)$, where $S(n) = n + 1$. Then $\mathcal{N}^1$ is superstable.

Corollary 1.16. Let $R$ be a regular sequence. Then $R^0_{\text{ind}}$ is definably interpreted in $\mathcal{N}$.

Proof. We interpret the domain of $R^0_{\text{ind}}$ as $\mathbb{N}$. Let $\varphi(x)$ be the equation $a_1x_1 + \cdots + a_nx_n = 0$. Note that the set $\varphi(R)$ is equal to the set

$$\bigcup_{J \in \Psi([n])} \left\{ \bar{x} \in R \left| \sum_{i \in J} a_i x_i = 0 \text{ and } \bar{x} \text{ is non-degenerate} \right. \right\},$$

where $\Psi([n])$ is the set of partitions of $[n]$. Now let $J \in \Psi([n])$. Then, by Proposition 1.13 there exist $\bar{m}_1, \ldots, \bar{m}_k$ such that the non-degenerate solutions of the equation

$$\sum_{i \in J} a_i x_i = 0$$

can be interpreted in $\mathcal{N}$ by formulas of the form

$$\bigwedge_{i \in J \setminus \{I_0\}} x_I = S^{m_i}(x_{I_0}) \land \bigwedge_{i \in J_{m_i}} x_{I_0} \neq i \land \bigwedge_{i \in J \setminus \{I_0\}} x_I = S^{m_i}(x_{I_0}) \land \bigwedge_{i \in J'_{m_i}} x_{I_0} = i,$$

where $i \in I_0$, $J_{m_i}$ and $J'_{m_i}$ are fixed finite sets associated to $\bar{m}_i$ in the following way. Let $f_i$ be the operator $k \mapsto \sum_{j \in J} a_{ir_{k+m_i}}$, where $a_I = \sum_{i \in I} a_i$. Then

1. if $f_i(k) = 0$ has finitely many solutions, then $J'_{m_i}$ is the finite set of natural numbers $k$ such that $f_i(k) \neq 0$. We set $J'_{m_i} = \emptyset$.

2. if $f_i(k) = 0$ has finitely many solutions, then $J'_{m_i}$ is the finite set of natural numbers $k$ such that $f_i(k) = 0$. We set $J'_{m_i} = \emptyset$.

1.2. Every $\mathcal{L}_R$-formula is bounded. This section is devoted to the proof of the following Theorem.

Theorem 1.17. Let $a, d \in \mathbb{N}$. Then, the set $a + d\mathbb{N}$ cannot be covered by finitely many sets of the form $\{z + f_1(n_1) + \cdots + f_k(n_k) | \bar{n} \in \mathbb{N}\}$, where $f_i$ is an operator for all $i \in [k]$, $k \in \mathbb{N}$ and $z \in \mathbb{Z}$.

Recall that a set $A \subset \mathbb{N}$ is called piecewise syndetic if there exists $d \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, there exists $a_1 < \cdots < a_k \in A$ such that $|a_{i+1} - a_i| \leq d$ for all $i \in [k-1]$. A key property of piecewise syndetic sets is the so-called Brown’s Lemma.

Theorem 1.18 (Brown’s Lemma [12 Theorem 10.37]). Let $A \subset \mathbb{N}$ be piecewise syndetic. If $A = A_1 \cup \cdots \cup A_n$, then there exists $i \in [n]$ such that $A_i$ is piecewise syndetic.

In the next Proposition, we show that the image of arbitrary linear combinations of operators is not piecewise syndetic.

Proposition 1.19. Let $\bar{f}$ be a tuple of $k$ operators. Then the set $\{f_1(n_1) + \cdots + f_k(n_k) | \bar{n} \in \mathbb{N}\}$ is not piecewise syndetic.

Before giving a proof of Proposition 1.19 let us show how it is used to prove Theorem 1.17.

Proof of Theorem 1.17. Since $a + d\mathbb{N}$ is piecewise syndetic, if it were covered by sets of the form $\{z + f_1(n_1) + \cdots + f_k(n_k) | \bar{n} \in \mathbb{N}\}$, then one of them would also be piecewise syndetic. But this would imply that a set of the form $\{f_1(n_1) + \cdots + f_k(n_k) | \bar{n} \in \mathbb{N}\}$ is piecewise syndetic since any translate of a piecewise syndetic set is again piecewise syndetic. This contradicts Proposition 1.19. \hfill $\square$

We now prove Proposition 1.19 by induction.
Proof of Proposition 1.13. Let $f$ be an operator. We may assume that it is non trivial, that is $f$ has only finitely many roots. Let $d \in \mathbb{N}$ and consider the equation $f(n_1) - f(n_2) = d \ (*)$. Notice that all but finitely many solutions of this equation is non degenerate since $f$ is non trivial. So, by Proposition 1.13, there exists only finitely many solutions to $(\ast)$. This proves that $\text{Im} (A)$ is not piecewise syndetic and establishes the Proposition for $k = 1$.

Let $k > 1$ and assume that the Proposition holds for all tuple $\bar{f}$ of length less than $k$. Let $f_1, \ldots, f_{k+1}$ be non trivial operators such that $\text{Im} (\bar{f}) = \{f_1(n_1) + \cdots + f_{k+1}(n_{k+1}) | \bar{n} \in \mathbb{N} \}$ contains infinitely many natural numbers. Suppose, towards a contradiction that $\text{Im} (\bar{f})$ is piecewise syndetic. Assuming $d \in \mathbb{N}$ witnesses the fact that $\text{Im} (\bar{f})$ is piecewise syndetic, define, for all $i \in [d]$, $X_i$ to be the set

$$\{a \in \text{Im} (\bar{f}) | 3a' \in \text{Im} (\bar{f}) , |a - a'| = i \}.$$ 

Even though $X_1 \cup \cdots \cup X_d$ may not equal $\text{Im} (\bar{f})$, it is this subset that will play a key role in the rest of the proof, as it is the \textquoteleft \textquoteleft syndetic part of $\text{Im} (\bar{f})$ with respect to $d'$ \textquotequoteleft \textquotequoteleft. Indeed, the set $X_1 \cup \cdots \cup X_d$ is itself piecewise syndetic so that by Brown's Lemma, there exists $i \in [d]$ such that $X_i$ is also piecewise syndetic. We will show that $X_i$ is a finite union of sets of the form $z + \text{Im} (\bar{f})'$, where the length of $\bar{f}'$ is less than $k$, obtaining a contradiction with the induction hypothesis by Brown's Lemma.

The set $X_i$ corresponds to the set $N$ of solutions of the equation

$$(1) \quad f_1(n_1) + \cdots + f_{k+1}(n_{k+1}) - (f_1(n_{k+2}) + \cdots + f_{k+1}(n_{2k+2})) = i.$$ 

Recall that for any solution $\bar{n}$ of $(1)$, there exists a partition $P = (P_0, \ldots, P_{\ell})$ of $[2k + 1]$ such that, for all $i \in [\ell]$, $\bar{n}_i$ is a non-degenerate solution of the subequation of $(1)$ corresponding to $P_i$. Let us fix $P = (P_0, \ldots, P_{\ell})$, partition of $[2k + 2]$, and for all $j \leq \ell$, let $P_{j0} = P_j \cap [k + 1]$ and $P_{j1} = P_j \setminus [k + 1]$. Notice that $\ell \leq k$. Now, we look at the set of solutions $N_P$ of $(1)$ associated to $P$. Namely

$$N_P = \{\bar{n} \in \mathbb{N} | \bar{n}_P \text{ is a non degenerate solution of } \varphi_{P_j}(n) \text{ for all } j \leq \ell \},$$

where $\varphi_{P_j}$ is the equation

$$\sum_{h \in P_{j0}} f_h(n_h) - \sum_{h \in P_{j1}} f_{h-(k+1)}(n_h) = 0.$$ 

and, for $j \in [\ell]$, $\varphi_{P_j}$ is the equation

$$\sum_{h \in P_{j0}} f_h(n_h) - \sum_{h \in P_{j1}} f_{h-(k+1)}(n_h) = 0.$$ 

We know, by Proposition 1.13, that the set of non degenerate solutions of $\varphi_{P_j}$ is finite, say $N_{P_j} = \{\bar{n}_{P_{j0}}, \ldots, \bar{n}_{P_{j,k+1}}\}$. Also, Proposition 1.13 tells us that for $i \in [\ell]$, the set of solutions of the equation $\varphi_{P_j}$ is determined by a finite number of recurrence relations, that is equations of the form $f(n) = 0$, for some operator $f$ depending only on the tuple $\bar{f}$. Indeed, for all $i \in [\ell]$, there exists $(\bar{m}_{P_{j0}}, \bar{m}_{P_{j1}}), \ldots, (\bar{m}_{P_{j,k+1}}, \bar{m}_{P_{j,k+1}})$ such that, setting

$$f_{P_{j0}}(n) = \sum_{h \in P_{j0}} f_h(n + m_{P_{j0}h}) \text{ and } f_{P_{j1}}(n) = \sum_{h \in P_{j1}} f_h(n + m_{P_{j1}h}),$$

$n_{P_j}$ is a non degenerate solution of $\varphi_{P_j}$ if and only if for some $j_0 \in [k+1]$, $j_0 \in P_{j0}$ and $j_1 \in P_{j1}$,

$$f_{P_{j0}}(n_{j_0}) - f_{P_{j1}}(n_{j_1}) = 0,$$

$n_{j} = n_{j_0} + m_{P_{j0}j_0}$ for all $j \in P_{j0}$ and $n_{j} = n_{j_1} + m_{P_{j1}j_0}$ for all $j \in P_{j1}$. All this shows that

$$X_i = \bigcup_{P=(P_0, \ldots, P_{\ell}) \in \mathcal{P}([2k+2])} X_P,$$

where $X_P$ is a finite union of sets of the form

$$\left( \sum_{h \in P_{j0}} f_h(n_h) + \text{Im} \left( f_{P_{j0}}(n) \text{ with respect to } \bar{f} \right) \mid j \in [\ell] \right) \cup \left( \sum_{h \in P_{j1}} f_{h-(k+1)}(n_h) + \text{Im} \left( f_{P_{j1}}(n) \text{ with respect to } \bar{f} \right) \mid j \in [\ell] \right).$$
So, by the induction hypothesis, none of the $X_P$ is piecewise syndetic. Hence, by Brown’s Lemma, $X_t$ cannot be piecewise syndetic, a contradiction. \hfill \Box

**Corollary 1.20.** Let $R$ be regular sequence. Then in $\mathcal{X}_R$, every $\mathcal{L}_R$-formula is equivalent to a bounded one.

**Proof.** The proof is done as in the proofs of [14 Lemma 3.4 and Lemma 3.5] using Theorem 1.17. We give here a summary of the proof of R. Sklinos and D. Palacin. First, one shows, using Theorem 1.17 that any consistent set of formulas of the form $\Gamma(y) = \{ \varphi(b, y, \bar{a})|\bar{a} \in R\}$, where $b \in \mathbb{Z}$, is realized by some $c \in \mathbb{Z}$. One then shows, using the fact that $(\mathbb{Z}, +, 0)$ does not have the finite cover property, that for all $\mathcal{L}_R$-formula $\varphi(x, y, \bar{z})$, there exists $k \in \mathbb{N}$ such that

$$
\mathcal{X}_R \models \forall \bar{x} \left( \forall 0 \in R \ldots \forall \bar{z}_k \in R \exists y \left( \bigvee_{j \leq k} \varphi(\bar{x}, y, \bar{z}_j) \Rightarrow \exists y \forall \bar{z} \in R \varphi(\bar{x}, y, \bar{z}) \right) \right).
$$

Finally, one shows by induction on the number of quantifiers of $\mathcal{L}_R$ formulas that they are bounded. \hfill \Box

### 1.3. Main theorem.
We are now able to prove the main theorem of this section.

**Theorem 1.4.** Let $R$ be a regular sequence. Then $\text{Th}(\mathcal{X}_R)$ is superstable of Lascar rank $\omega$.

**Proof.** By Proposition 1.15 and Corollary 1.16, we get that $R^0_{\text{ind}}$ is superstable. Furthermore, by Corollary 1.20 any formula in $\mathcal{X}_R$ is bounded. So, we deduce from Theorem 1.6 that $\mathcal{X}_R$ is superstable. Concerning the rank, one proceeds exactly as in the proof of [14 Theorem 2], where it is shown that any forking extension of the principal generic has finite rank. Let $\mathcal{C}$ be a monster model of $\text{Th}(\mathcal{X}_R)$ and let $p$ be the principal generic of the connected component. Assume $q$ is a forking extension of $p$ over $B$, a set of parameters. One can show that any realization of $q$ is algebraic over $R(\mathcal{C}) \cup B$, so that $q$ has finite rank. \hfill \Box

1.3.1. Comments. As we mentionned in the Introduction, there is an overlap between Theorem 1.4 and [5 Theorem 7.1]. More precisely:

- the case where $r_{n+1}/r_n \to \infty$ is completely covered by [5 Theorem 7.1] (as a consequence of [5 Proposition 6.3]);
- the case where $r_{n+1}/r_n \to \theta$ and $\theta$ is algebraic is more general than [5 Theorem 7.1]: in addition to our hypotheses, $\theta$ needs to be either a Pisot number or a Salem number in order to be geometrically sparse. In fact, we can show by direct calculations that the sequence defined by $r_{n+2} = 5r_{n+1} + 7r_n$, $r_1 = 1$ and $r_0 = 0$, is is regular but not geometrically sparse;
- for the case where $r_{n+1}/r_n \to \theta$ and $\theta$ is transcendental, the overlap is less precise and we did not manage make a clear distinction between the two results. However, if $(r_n)$ is such that $r_n/\theta^n \to \tau \in \mathbb{R}^{>0}$ and is geometrically sparse (in the sense of [5 Definition 6.2]), that is $\sup_{n \in \mathbb{N}} |r_n - \tau \theta^n| < \infty$, then the sequence $(r_n + n)$ is not geometrically sparse but satisfies Theorem 1.4.

The assumption on $\theta$ when it is algebraic cannot be removed. Indeed, if $R = (a + bn)$ is an arithmetic progression, then $\mathcal{X}_R$ is unstable\footnote{This is also true for any sequence $(r_n)$ such that there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $|r_{n+1} - r_n| \leq k$.} and satisfy the linear recurrence $r_{n+2} = 2r_{n+1} - r_n$. Also, for $R = (2^n + n)$, it is known that $\mathcal{X}_R$ is unstable ([4 Theorem 4.8]). However, we have the following.

**Proposition 1.21.** Let $(r_n)$ and $(r'_n)$ be two regular sequences. Assume that $r_{n+1}/r_n \to \theta$, $r'_{n+1}/r'_n \to \theta'$ and $\theta \leq \theta'$. Then, setting $R'' = (r_n + r'_n)$, $\mathcal{X}_{R''}$ is superstable.

**Proof.** The proof is identical to that of Theorem 1.4 using the fact for any operator $f(n) = \sum_{i=0}^{d} a_i r''_{n+i}$ on $R''$, one has

$$
\frac{f(n)}{r''_n} \to \sum_{i=0}^{d} a_i \theta''^i.
$$
Using this, one proves Proposition 1.8 directly. For the analysis of equations, on proceeds as in the proof of Proposition 1.13. We work in the context of this proof. Let $\hat{A}$ be a tuple of $k$ operators and let

$$u = \lim_{i \to \infty} \sum_{j=1}^{k} f_j(l_i + m_{ij})/r''_i + m_s.$$ 

If $u = 0$, then we deduce that the sequence $(r'_n)$ is such that the equation

$$\sum_{j \in J} f'_j(l_i + m_{ij}) = 0$$

is satisfied for all but finitely many $i$, where $f'_j$ is defined on $(r'_n)$ with the same coefficients as in $f_j$. This implies that the equation

$$\sum_{j \in J} f''_j(l_i + m_{ij}) = 0,$$

where $f''_j$ is defined on $(r_n)$ with the same coefficients as in $f_j$, is satisfied for all $i$ sufficiently large. This leads to a contradiction using again the same strategy as in the proof of Proposition 1.13.

In particular, the expansion $\mathcal{Z}(\mathbb{Z}^+, \mathbb{Z}^+, \mathbb{Z}^+)$ is superstable. Notice that we can generalize Proposition 1.13 to any number of regular sequences: any sum of regular sequences gives a superstable expansion of $\mathcal{Z}$. On the other hand, we haven’t been able to prove that the expansion of $\mathcal{Z}$ by a product of two regular sequences that satisfy a recurrence relation is superstable. The main difficulty is that, given two recurrence sequences $R = (r_n)$ and $R' = (r'_n)$, the product $P_R P_{R'}$ does not necessarily divides the characteristic polynomial of $(r_n r'_n)$, while it divides the characteristic polynomial of $(r_n + r'_n)$ (see [24, 9]).

2. The theory $T_R$

2.1. Axiomatization and quantifier elimination. In this section, we axiomatize, in a reasonable language, the theory $T_R$ of a structure of the form $\mathcal{Z}_R = (\mathbb{Z}, +, - , 0, 1, R)$, where $R$ is a sequence of natural numbers that behaves like a regular sequence. We show that this theory has quantifier elimination and has a prime model (and hence is complete). Using this quantifier elimination, we then prove, by means of counting of types, that $\mathcal{Z}_R$ is superstable.

Let us define the language in which we axiomatize $\mathcal{Z}_R$. As mentioned in the introduction, $L_g$ is the language $\{+, - , 0, 1, D_n| 1 < n \in \mathbb{N}\}$ and $L_S$ is the language $\{S, S^{-1}, 1\}$. We say that an $L_g \cup L_S$-term is an operator if it is of the form $\sum_{i=0}^{d} n_i S^i(x)$ $(\ast)$, where $n_i \in \mathbb{Z}$ and $S^0(x) = x$. Notice that this notion of operator is similar to Definition 1.7, in fact, in a model of $T_R$, the restriction to $R(\mathbb{N})$ of a term of the form $(\ast)$ composed with the function $n \mapsto r_n$ will be an operator in the sense of Definition 1.7. This explain why we decided to use the same terminology. We use the letter $f$ to denote operators, possibly with subscript. Let $n, m \in \mathbb{N}$. Let $C = \{(f_i, \ell_i, k_i) | 0 \leq k_i < \ell_i, i \in [m]\}$ and let $D = \{(f_{i1}, \ldots, f_{im}) | i \in [n]\}$. For brevity, let $\varphi_C(\bar{x})$ be the formula

$$\bigwedge_{i \in [m]} D_{\ell_i}(f_i(x_i) + k_i)$$

and $\varphi_D(\bar{x}, \bar{y})$ be the formula

$$\bigwedge_{i \in [n]} \sum_{j \in [m]} f_{ij}(x_i) = y_i.$$

To $C$ and $D$, we let $\Sigma_{C,D}$ be an $n$-ary predicate, which will be interpreted as the image of the function defined by the formula $\varphi_D(\bar{x}, \bar{y}) \land \varphi_C(\bar{x})$. When $C$ is empty, we write $\Sigma_D$ instead of $\Sigma_{C,D}$ and when $D = \{f\}$, we write $\Sigma_f$ instead of $\Sigma_D$. Finally let $L$ be the language

$L_g \cup L_S \cup \{R\} \cup \{\Sigma_{C,D}|(C,D) \text{ as above}\}.$
We fix an axiomatization $T_1$ of $\text{Th}(\mathbb{Z},+,\cdot,0,1,D_n|1 < n \in \mathbb{N})$ (see [13] Chapter 15, Section 15.1) and we let $T_2$ be the following universal axiomatization of $\text{Th}(R,S,S^{-1},1)$:

$$T_2 = \{ \forall x(x \neq 1 \Rightarrow S(S^{-1}(x)) = x), \forall x(S^{-1}(S(x)) = x), \forall x(S(x) \neq 1), S^{-1}(1) = 1 \}.$$ 

We will frequently use the fact that, modulo $T_1$, a formula of the form $\neg D_n(x)$ is equivalent to

$$\Bigl( \bigvee_{k=1}^{n-1} D_n(x+k) \Bigr).$$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\bar{f}$ be a tuple of $n$ operators. We say that $\bar{a} \in M^n$ is a non degenerate solution of $\sum_{i=1}^{n} f_i(x_i) = 0$ if no proper sub-sum is equal to 0. This can be expressed by the following first-order formula

$$\varphi_l(\bar{x}) := \sum_{i=1}^{n} f_i(x_i) = 0 \land \sigma_l(\bar{x}),$$

where $\sigma_l(\bar{x})$ is the formula

$$\bigwedge_{I \subseteq [n]} \sum_{i \in I} f_i(x_i) \neq 0.$$

Let $T_R$ be the following set of axioms. We will denote by $T_R^{\bar{R}}$ the theory obtained by relativizing to the predicate $R$ the quantifiers appearing in each element of $T_2$.

(Ax.1) $T_1$,

(Ax.2) $T_R^{\bar{R}}$,

(Ax.3) $\forall x(\neg R(x) \Rightarrow S(x) = x)$,

(Ax.4) For all pair $(C,D)$ as above,

$$\forall x \left( \Sigma_{C,D}(\bar{x}) \iff \exists \bar{y} \in R. \varphi_C(\bar{y}) \land \varphi_D(\bar{x},\bar{y}) \right).$$

(Ax.5) For every operator $f$, there is a constant $c = c(f) \in \mathbb{N}$ such that either

$$\left( \forall x \in R \left( f(x) = 0 \Rightarrow \bigvee_{i < c} x = i \right) \right)$$

or

$$\left( \forall x \in R \left( \bigwedge_{i < c} x \neq i \Rightarrow f(x) = 0 \right) \right),$$

(Ax.6) For every $n$-tuple $\bar{f}$ of operators, there exist constants $c = c(\bar{f})$, $k = k(\bar{f}) \in \mathbb{N}$ and for all $j \in [k]$ there exists a finite set $E_j = \{k_{j1},\ldots,k_{jn}\}$ of integers such that

$$\forall \bar{x} \in R \left( \bigwedge_{i=1}^{n} x_i > c \land \sum_{i=1}^{n} f_i(x_i) = 0 \land \sigma_l(\bar{x}) \Rightarrow \bigvee_{j=1}^{k} \bigwedge_{i \in E_j} x_i = S^{k_{ji}}(x_1) \right).$$

Before proving quantifier elimination, we first notice that, as a consequence of the work done in Sections [1.1] and [1.2] $T_R$ is consistent whenever $R$ is a regular sequence.

**Theorem 2.1.** Assume that $R$ is a regular sequence. Then $T_R$ axiomatizes $\text{Th}(\mathcal{Z}_R)$.

Notice that the sequence $R = (2^n + n)$ does not satisfy Axiom [Ax.6]: considering the operator $f(x) = S^2(x) - 3S(x) + 2x$, one can find infinitely many (non degenerate) solutions of the equation $f(x_1) - f(x_2) = 0$. In view of Theorem 2.1 this is not surprising because the structure $(\mathbb{Z},+,\cdot,0,1,R,S)$ is known to be unstable: $\mathbb{N}$ is definable by the formula $\exists y \in R \ y \neq 1 \land (2y - S(y) = x)$. However, we do not know if there exists a sequence $R$ such that $\mathcal{Z}_R$ is (super)stable and $(\mathbb{Z},+,\cdot,0,1,R,S)$ unstable.

To establish quantifier elimination, we use the following criterion.

**Proposition 2.2** ([13 Corollary 3.1.12]). Let $T$ be an $\mathcal{L}$-theory such that
(1) \( T \) has algebraically prime models) for all \( \mathcal{M} \models T \) and all \( \mathcal{A} \subseteq \mathcal{M} \), there exists a model \( \mathcal{N} \) of \( T \) such that for all \( \mathcal{N} \models T \), any embedding \( f : \mathcal{A} \to \mathcal{N} \) extends to an embedding \( \bar{f} : \mathcal{M} \to \mathcal{N} \); 

(2) \( T \) is 1-e.c.) for all \( \mathcal{M}_0 \subseteq \mathcal{M} \) and all \( \mathcal{A} \subseteq \mathcal{M}_0 \), if \( \mathcal{M}_0 \models T \) and any definable subset of \( \mathcal{A} \) defined with parameters in \( \mathcal{M}_0 \), has a non empty intersection with \( \mathcal{M}_0 \).

Then \( T \) has quantifier elimination.

Given two models \( \mathcal{M}_0 \subseteq \mathcal{M} \) of \( T \), we say that \( \mathcal{M}_0 \) is 1-e.c. in \( \mathcal{M} \) if any definable subset of \( \mathcal{M} \), defined with parameters in \( \mathcal{M}_0 \), has a non empty intersection with \( \mathcal{M}_0 \).

Let us state the main result of this section.

**Theorem 2.3.** The theory \( T \) has quantifier elimination.

The proof of Theorem 2.3 will be a consequence of Proposition 2.2 and the work done in the following three subsections. More precisely, in Section 2.1.1, we prove several direct consequences of \( T \) regarding equations of the form \( f_1(x_1) + \cdots + f_n(x_n) = a \). Then in Section 2.1.2, we give a detailed construction of algebraically prime models of \( T \). Finally, we show in Section 2.1.3 that \( T \) is 1-e.c.

As a corollary of (the proof of) Theorem 2.3, we have that \( \mathcal{Z}_R \) is a prime model of \( T \) and that \( T \) is complete.

**Corollary 2.4.** \( \mathcal{Z}_R \) is a prime model of \( T \). Hence \( T \) is complete.

2.1.1. Equations in \( T \). We first establish that an operator induces either an injective function on \( R \) or gives a recurrence relation satisfied by all sufficiently large elements of \( R \).

**Definition 2.5.** An operator \( f \) is said to be trivial if and only if there exists \( c \in \mathbb{N} \) such that \( T \models \forall x \in R (x > c \Rightarrow f(x) = 0) \).

**Lemma 2.6.** Let \( f \) be a non trivial operator. Then there exist \( c \in \mathbb{N} \) and \( k_1, \ldots, k_l \in \mathbb{Z}_0 \) such that

\[
T \models \forall x, y \in R (x \neq y \land x > c \land y > c \Rightarrow f(x) \neq f(y)) \land \forall x \in R (x > c \Rightarrow f(x) = f(S^{k_i}(x))) .
\]

**Proof.** From axiom ([Ax.6]) applied to the 2-tuple \((f, -f)\), there exists a constant \( c \in \mathbb{N} \) and \( k_1, \ldots, k_l \in \mathbb{Z} \) such that

\[
T \models \forall x, y \in R \left( x > c \land y > c \land f(x) = f(y) \Rightarrow \bigvee_{i=1}^n y = S^{k_i}(x) \right).
\]

Next, by axiom ([Ax.5]) applied to \( f_i(x) = f(x) - f(S^{k_i}(x)) \), for each \( i \in \{1, \ldots, l\} \) such that \( k_i \neq 0 \), there exists a constant \( c_i \in \mathbb{N} \) such that

\[
T \models \left( \forall x \in R \left( f_i(x) = 0 \Rightarrow \bigvee_{i < c_i} x = i \right) \lor \left( \forall x \in R \left( \bigwedge_{i < c_i} x \neq i \Rightarrow f_i(x) = 0 \right) \right) \right).
\]

Let \( k = \max\{c, c_1, \ldots, c_l\} \).

Let \( \mathcal{M} \models T \). Assume \( \mathcal{M} \models \exists x, y \in R (x \neq y \land x > c \land y > c \land f(x) = f(y)) \) and let \( a, b \in R \) such that \( a, b > c \), \( a \neq b \) and \( f(a) = f(b) \). By (2), we have that \( M \models \bigvee_{i=1}^n b = S^{k_i}(a) \). Since \( a \neq b \), there is \( i \in \{1, \ldots, l\} \) such that \( k_i \neq 0 \) and \( f_i(a) = 0 \). In this case, since \( a, b > c \), we conclude by (3) that \( \mathcal{M} \models \forall x \in R (x > c \Rightarrow f_i(x) = 0) \). This shows that

\[
\mathcal{M} \models \bigvee_{i \in J} \forall x \in R (x > c \Rightarrow f(x) = f(S^{k_i}(x))),
\]

where \( J \) is the set of indices \( i \) for which \( k_i \neq 0 \).
where \( J = \{ i \in [l] \mid k_i \neq 0 \} \). Hence,
\[
\mathcal{M} \models (\forall x, y \in R(x \neq y \land x > c \land y > c \Rightarrow f(x) \neq f(y))
\]
\[
\lor \left( \bigwedge_{i=1}^{n} \forall x \in R(x > c \Rightarrow f(x) = f(S^{k_i}(x)) \right),
\]
and this completes the proof. \(\square\)

**Definition 2.7.** Let \( \mathcal{M} \models T_R \) and \( a, b \in R \). The orbit of \( a \) is the set \( \{ S^k(a) \mid k \in \mathbb{Z} \} \) and is denoted by \( \text{Orb}(a) \). We say that \( a \) and \( b \) are in the same orbit if and only if \( b \in \text{Orb}(a) \).

**Lemma 2.8.** Let \( \mathcal{M} \models T_R \). Let \( \bar{f} \) be a \( n \)-tuple of non trivial operators, \( n > 1 \), and let \( b_1, \ldots, b_k \in R \), \( k \leq n \), be in different orbits.

1. If \( k > n/2 \), then for all \( c_{k+1}, \ldots, c_n \in R \),
\[
\sum_{i=1}^{k} f_i(b_i) + \sum_{i=k+1}^{n} f_i(c_i) \neq 0;
\]

2. If \( k \leq n/2 \), then for all \( c_{k+1}, \ldots, c_n \in R \), the elements \( b_1, \ldots, b_k, c_{k+1}, \ldots, c_n \) do not form a non degenerate solution of the equation \( \sum_{i=1}^{n} f_i(x_i) = 0 \). Moreover, if \( \sum_{i=1}^{k} f_i(b_i) + \sum_{i=k+1}^{n} f_i(c_i) = 0 \), then for all \( i \in [k] \) there exists a non empty \( P_i \subset \{ k + 1, \ldots, n \} \) such that \( P_i \cap P_{i'} = \emptyset \) for all \( i \neq i' \in [k] \) and for all \( i \in [k] \) \( b_i, (c_j)_{j \in P_i} \) is a non degenerate solution of
\[
f_i(x_i) + \sum_{j \in P_i} f_j(x_j) = 0.
\]

**Proof.** Let \( c_{k+1}, \ldots, c_n \in R \). It is clear from [Ax.6] that the \( b_1, \ldots, b_k, c_{k+1}, \ldots, c_n \) cannot be a non degenerate solution of
\[
\sum_{i=1}^{n} f_i(x_i) = 0,
\]

since, for instance, \( b_1 \) is not in the same orbit as \( b_2 \). Since \( b_1, \ldots, b_k, c_{k+1}, \ldots, c_n \) is degenerate, one shows by induction on \( n \) that there exists a partition \( \{ P_1, \ldots, P_{\ell} \} \) of \( [n] \) such that for all \( j \in [\ell] \) \( (b_i)_{i \in P_j \cap [k]}, (c_i)_{i \in P_j \cap [k+1, \ldots, n]} \) is a non degenerate solution of
\[
\sum_{i \in P_j \cap [k]} f_i(x_i) + \sum_{i \in P_j \cap [k+1, \ldots, n]} f_i(x_i) = 0.
\]

Since \( b_1, \ldots, b_k \) are in different orbits, we must have, by [Ax.6], \( |P_j \cap [k]| \leq 1 \) for all \( j \in [\ell] \). Also, since all operators involved are non trivial, we must have \( |P_j \cap \{ k + 1, \ldots, n \}| > 0 \) for all \( j \in [\ell] \). This implies in particular that \( k \leq n/2 \) and finishes the proof of the lemma. \(\square\)

We now show that [Ax.6] is true for non-homogeneous equations.

**Proposition 2.9.** Let \( \mathcal{M} \models T_R \), \( \bar{f} \) be a \( n \)-tuple of non trivial operators and \( a \in M \). Then there exist \( b_1, \ldots, b_k \in R \) such that
\[
\mathcal{M} \models (\forall x \in R \left( \bigwedge_{i=1}^{n} f_i(x_i) = a \land \sigma(f(x)) \Rightarrow \bigvee_{j=1}^{k} \bigwedge_{i=1}^{n} x_i = b_{ij} \right)).
\]

**Proof.** This is done by induction on the number of (non trivial) operators. The case where \( n = 1 \) follows from Lemma 2.6. Assume that the Proposition holds for all tuples of operators of length \( k \) and all \( a \in M \). Let \( \bar{f} \) be a tuple of \( (k + 1) \) operators and \( a \in M \). By the induction hypothesis, there are only finitely many solutions \( \bar{b} \) of \( f_1(x_1) + \cdots + f_{k+1}(x_{k+1}) = a (\ast) \) such that for some \( i, i' \in [k + 1] \), \( b_i \) and \( b_{i'} \) are in the same orbit. So all we need to do is to show that there are only finitely many solutions \( \bar{b} \) of (\ast) such that for all \( i \neq i' \in [k + 1] \), \( b_i \) and \( b_{i'} \) are not in the same orbit.

Assume there exist infinitely many distinct solutions \( b_i, i \in \mathbb{N} \), such that for all \( i \in \mathbb{N} \), \( b_{ij} \) is not in the orbit of \( b_{ij'} \) if \( j \neq j' \). We have that for all \( i \in \mathbb{N} \), the tuple \( (b_0, \bar{b}_i) \) is a (non degenerate) solution
of $\varphi_{t,x}(\bar{x})$. By Lemma 2.8 for all $i \in \mathbb{N}$, for all $j \in [k+1]$ there exists a unique $j' \in [2k+2]\setminus[k+1]$ such that $b_{ij'}$ is in the orbit of $b_{ij}$. Since $[2k+2]$ is finite, we may assume that $j'$ is the same for all $i \in \mathbb{N}$. But by axiom (Ax.6), for all $j \in [k+1]$, there exists $k_1, \ldots, k_\ell \in \mathbb{Z}$ such that for all $i \in \mathbb{N}$, $b_{ij} = S^{k_i}(b_{ij'}) \lor \cdots \lor b_{ij'} = S^{k_\ell}(b_{ij'})$. This is a contradiction. \hfill \Box

**Proposition 2.10.** Let $\mathcal{M}, \mathcal{M}_0 \models T_R$ such that $\mathcal{M}_0 \subset \mathcal{M}$. Let $\bar{t}$ be a tuple of $n$ non trivial operators, $b_1, \ldots, b_n \in R(M) \setminus R(M_0)$ in different orbits and $a \in M_0$, $a \neq 0$. Then

$$\sum_{i=1}^{n} f_i(b_i) + a \notin R(M).$$

**Proof.** Suppose, towards a contradiction, that $\sum_{i=1}^{n} f_i(b_i) + a = b_{n+1} \in R$. Then, there exists $b_{n+1}, \ldots, b_{2n+2} \in R(M_0)$ such that $\sum_{i=1}^{n} f_i(b_i) - b_{n+1} = \sum_{i=1}^{n} f_i(b_i) - b_{n+1+i} - b_{2n+2} = -a$, since $\mathcal{M} \models \Sigma_{(\lambda,x)}(a)$ and $\mathcal{M}_0 \subset \mathcal{M}$. Let us show that this contradicts axiom (Ax.6) By lemma 2.8 applied to

$$\sum_{i=1}^{n} f_i(x_i) - x_{n+1} + \sum_{i=1}^{n} f_i(x_{n+1+i}) - x_{2n+2} = 0,$$

and $b_1, \ldots, b_n$, for all $i \in [n]$, there exists $J_i \subset \{n+1, \ldots, 2n+2\}$ such that $b_{ij}, (b_{i'}) \in J_i$, is a non degenerate solution to the corresponding equation. We furthermore have that $J_i \neq \emptyset$ for all $i \in [n]$ and $J_i \cap J_i' = \emptyset$ for all $i \neq i' \in [n]$. However, since $b_k \in M_0$ for all $k > n+1$, $k \notin J_i$ for all $i \in [n]$. This implies that $n = 1$. This in turn implies that $f_1(b_1) - b = 0$. But this contradicts the fact that $a \neq 0$. So we conclude that an expression of the form

$$\sum_{i=1}^{n} f_i(b_i) + a$$

cannot be in $R$. \hfill \Box

2.1.2. $T_R$ has algebraically prime models. Let $\mathcal{M} \models T_R$ and $\mathcal{A} \subset \mathcal{M}$. For $X \subset M$, we let $\text{div}(X)$ be the divisible closure of $X$ in $\mathcal{M}$, that is the substructure generated by $X$ and $\{d \ | \ nd \in X \text{ for some } n \in \mathbb{N}\}$. The construction of the algebraically prime model over $\mathcal{A}$, denoted $\mathcal{A}'$, is done as follows. Let $\bar{t}$ be a $n$-tuple of non-trivial operators. Call a $n$-tuple $\bar{b} \in R(M)$ $\bar{t}$-good if

1. $b_i \notin A$ for all $i \in [n]$;
2. $f_1(b_1) + \ldots + f_n(b_n) \in \mathcal{A}$;
3. $b_i \notin \text{Orb}(b_j)$ whenever $j \neq i$.

Let $\mathcal{A}'$ be the substructure generated by $\mathcal{A}$ and $\bar{t}$-good tuples of elements of $R(M)$, for all tuples $\bar{t}$ of non-trivial operators. This structure will satisfy all axioms of $T_R$ except the definition of the symbols $D_n$. So our algebraically prime model over $\mathcal{A}$ will be $\mathcal{A}' = \text{div}(\mathcal{A})$.

**Lemma 2.11.** $\mathcal{A}'$ is a model of $T_R$.

**Proof.** We begin with a description of elements in $\mathcal{A}'$. Assume $\mathcal{A}' = \langle A, (b_{\lambda})_{\lambda < \kappa} \rangle$, where $b_{\lambda} \notin \text{Orb}(b_{\lambda'})$ for all $\lambda \neq \lambda'$ and each $b_{\lambda}$ appears in a good tuple. We want to show that any $d \in \mathcal{A}'$ can be put in the form $a + \sum_{i=1}^{n} f_i(b_{\lambda_i})$ (*), where $\lambda_i \neq \lambda_j$ for all $i \neq j \in [n]$ and $a \in \mathcal{A}$. Let $t(x, y)$ be the term $y + \sum_{i=1}^{n} f_i(x_i)$. We will evaluate the expression $c = S(t(b, a))$ for $a \in A$ and $b_{\lambda_1}, \ldots, b_{\lambda_n}$ in different orbits. Assume $s \in R(M) \notin R(A)$. In this case, either there is a unique $i \in [n]$ such that $b \in \text{Orb}(b_{\lambda_i})$ or the tuple $(b, b_{\lambda_1}, \ldots, b_{\lambda_n})$ is $\bar{t}$-good, so that $b$ is in the orbit of some $b_{\lambda}$ for some $\lambda \leq \kappa$. Thus, $s$ is either in $A$, or $b_{\lambda}$ for some $\lambda < \kappa$ or $t(b, a)$. This shows that any element of $\mathcal{A}'$ can be put in the form (*).

**Claim 2.12.** Let $d \in \mathcal{A}'$. Then either $d \in \mathcal{A}$ or there exist $a \in \mathcal{A}$ and $n \in \mathbb{N}$ such that $nd = a$.

**Proof.** First, let us show that if $nd = a$ and $d \notin \mathcal{A}$, then $d \notin R(M)$. If it were not the case, then $a$ would be in the image of the operator $x \mapsto nx$. This would imply by construction of $\mathcal{A}'$ that $d$ is in the orbit of $b_{\lambda_i}$ for some $\lambda \leq \kappa$. Thus, whenever $m \in \mathbb{Z}$ and $d \in \mathcal{M}$ are such that

1. there exists $k \in \mathbb{N}$ such that $k\lambda_i d_i \in \mathcal{A}$;
2. $m \notin k_i \mathbb{Z}$,
then, for all $a \in \mathcal{A}$, $d = a + \sum b_i$ is not in $R(M)$ since $k_1 \cdots k_n d \in \mathcal{A}$.

Let us finally show that $\mathcal{A} \models T_R$. The only axiom that requires details is axiom (Ax.4). Assume that $\mathcal{A} \models \Sigma_{C,D}(d_1, \ldots, d_n)$. By Claim 2.12, we may assume that $d_1, \ldots, d_n \in \mathcal{A}$: for all $i \in [n]$, $d_i = a_i + \sum_{j=1}^{k} f_j(d_{\lambda_j})$. So we may also assume that $d \in \mathcal{A}$ and by construction of $\mathcal{A}$ we have witnesses in $\mathcal{A}$ of the fact that $\Sigma_{C,D}(d)$ holds.

Let us show that any embedding $f : \mathcal{A} \to \mathcal{N}$ extends to an embedding $\bar{f} : \mathcal{A} \to \mathcal{N}$.

**Lemma 2.13.** $f$ extends to an $\mathcal{L}$-embedding $\bar{f} : \mathcal{A} \to \mathcal{N}$.

**Proof.** Let $\mathcal{L}_0$ be the language $\{+,-,0,1,R\} \cup \Sigma_S$. We first extend $f$ to an $\mathcal{L}_0$-embedding $\bar{f} : \mathcal{A} \to \mathcal{N}$. Let $g$ be the partial type
\[
\{f_1(x_{\lambda_1}) + \cdots + f_n(x_{\lambda_n}) = f(a)f_1(b_{\lambda_1}) + \cdots + f_n(b_{\lambda_n}) = a, a \in A\}
\]
\[
\cup \{D_0(f(x_a + k)) \mid a \in A\},
\]
By axiom (Ax.4) $\mathcal{N}$ is finitely consistent in $\mathcal{N}$. Hence, $\mathcal{N}$ is realized in an elementary elementary extension $\mathcal{N'}$ of $\mathcal{N}$ by some $(b^{\lambda}_\lambda)_{\lambda<\kappa}$. Let us show that $(b^{\lambda}_\lambda)_{\lambda<\kappa}$ is in $\mathcal{N}$. Let $\lambda < \kappa$. By definition, $b^\lambda$ appears in a $f$-good tuple: there exist $b_{\lambda_1}, \ldots, b_{\lambda_n} \in R(M) \setminus A$ and $a \in A$ such that $f_1(b_1) + f_2(b_2) + \cdots + f_n(b_n) = a$. The same holds for $b^{\lambda_1}, b^{\lambda_2}, \ldots, b^{\lambda_n}$ and $f(a)$. Furthermore, we have that $\mathcal{N} \models \Sigma_i(f(a))$. Since $\mathcal{N} \models T_R$, there are $d_1, \ldots, d_n \in R(N)$ such that
\[
\sum_{i=1}^{n} f_i(d_i) = f(a).
\]
Hence, by Lemma 2.8, $b^{\lambda_i}_i$ is in the orbit of $d_i$ for some $i \in [n]$; this shows that $b^{\lambda_i}_i \in \mathcal{N}$. Let us show that for all $\lambda \neq \lambda', b^{\lambda}_\lambda \not\in \text{Orb}(b^{\lambda'}_{\lambda'})$. Assume on the contrary that $S^f(b^{\lambda}_\lambda) = b^{\lambda'}_{\lambda'}$. Let $\bar{f}$ and $a \in A$ such that $\bar{f}_1(b^{\lambda}_\lambda) + \cdots + \bar{f}_n(b^{\lambda}_\lambda) = f(a)$. By assumption, there exist $d, d_1, \ldots, d_n \in R(M)$ such that $f_1(d) + f_2(S^f(d)) + f_3(d_3) + \cdots + f_n(d_n) = a = f_1(b^{\lambda}_\lambda) + \cdots + f_n(b^{\lambda}_\lambda)$. This contradicts the fact that $(b^{\lambda_1}, \ldots, b^{\lambda_n})$ is a $f$-good tuple by Lemma 2.8. Furthermore, using a similar argument, one can show that $b^{\lambda}_\lambda \not\in f(A)$. This shows that $(b^{\lambda}_\lambda)_{\lambda<\kappa}$ realizes the quantifier-free type of $(b^{\lambda}_\lambda)_{\lambda<\kappa}$ over $A$ in $\mathcal{L}_0$. Hence the map $\bar{f}$ defined on $\mathcal{A}$ by $a + \sum_{i=1}^{n} f_i(b^{\lambda}_\lambda) \to f(a) + \sum_{i=1}^{n} f_i(b^{\lambda'}_\lambda)$ is an $\mathcal{L}_0$-embedding.

Now we extend $\bar{f}$ to an $\mathcal{L}$-embedding $\bar{f} : \mathcal{A} \to \mathcal{N}$. Recall that for all $d \in \mathcal{A} \setminus \mathcal{A}$, there exist $a \in A$, $\bar{f}$ a tuple of non trivial operators, $b^{\lambda_1}_{\lambda_1}, \ldots, b^{\lambda_n}_{\lambda_n}$ and $n \in \mathcal{N}^+$ such that $nd = a + \sum_{i=1}^{n} f_i(b^{\lambda}_\lambda)$. By construction $\bar{f}(ad)$ is divisible by $n$: there exists $d^*$ such that $\bar{f}(ad) = nd^*$. We extend $\bar{f}$ by the rule $\bar{f}(d) = d^*$.

2.1.3. $T_R$ is 1-e.c.

**Proposition 2.14.** Let $\mathcal{M}, \mathcal{M}_0 \models T_R$ such that $\mathcal{M}_0 \subseteq \mathcal{M}$. Assume that $R(M_0) = R(M)$. Then $\mathcal{M}_0$ is 1-e.c. in $\mathcal{M}$.

**Proof.** Let $\varphi(x, \bar{y})$ be a quantifier-free formula such that $\mathcal{M} \models \varphi(b, \bar{a})$ for some $b \in M \setminus M_0$ and $\bar{a} \in M_0$. We will show that there exists $b_0 \in M_0$ such that $\mathcal{M}_0 \models \varphi(b_0, \bar{a})$. Let us first simplify $\varphi$.

**Claim 2.15.** We may assume that $\varphi(x, \bar{a})$ is of the form
\[
\bigwedge_{i \in I_1} -\Sigma_{C,D}(n_1x + a_{1i}, \ldots) \land \bigwedge_{i \in I_2} n_ix + a_i \neq 0 \land \bigwedge_{i \in I_3} D_{m_i}(n_ix + a_i),
\]
where $I_j \subseteq \mathbb{N}$ is finite for all $j \in [3]$, $m_i \in \mathbb{N} \setminus \{0, 1\}$ for all $i \in I_3$ and $n_i \in Z_0$ for all $i \in I_1 \cup I_2 \cup I_3$.

**Proof.** Let $n \in Z$, $b \in M \setminus M_0$ and $a \in M_0$. Then, since $R(M) = R(M_0)$, it is clear that $\neg R(nb + a)$, unless $n = 0$ and $R(a)$. This shows that a term of the form $t(b, \bar{a})$ is equivalent to one of the form $nb + a$, for some $n \in Z$ and $a \in M_0$. Similarly, one has $\Sigma_{C,D}(n_1b + a_1, \ldots)$, unless $n = 0$ and $\Sigma_{C,D}(\bar{a})$. Also, the same holds for equations: we always have $nb + a \neq 0$, unless $n = 0$ and $a = 0$. Thus we may assume that $\varphi(x, \bar{a})$ is of the form
\[
\bigwedge_{i \in I_1} -\Sigma_{C,D}(n_1x + a_{1i}, \ldots) \land \bigwedge_{i \in I_2} n_ix + a_i \neq 0 \land \bigwedge_{i \in I_3} D_{m_i}(n_ix + a_i),
\]
By model completeness of $\text{Th}(\mathbb{Z}, +, 0, 1, D_n | 1 < n \in \mathbb{N})$, there exists $b_0 \in M_0$ such that

$$\mathcal{M}_0 \models \bigwedge_{i \in I_2} n_i b_0 + a_i \neq 0 \land \bigwedge_{i \in I_3} D_{m_i}(n_i b_0 + a_i).$$

However, $\mathcal{M}_0$ may not satisfy $\varphi(b_0, \bar{a})$. But this can be overcome in the following way. First, note that the set

$$X_1 = \left\{ m \in \mathbb{N} \bigg| \mathcal{M}_0 \models \bigwedge_{i \in I_2} n_i (b_0 + m) + a_i \neq 0 \right\}$$

is cofinite and that the set

$$X_2 = \left\{ m \in \mathbb{N} \bigg| \mathcal{M}_0 \models \bigwedge_{i \in I_2} \neg \Sigma_{C_i, D_i}(n_i (b_0 + m) + a_i, \ldots) \right\},$$

is not piecewise syndetic (this follows from Proposition 1.19 for $X_2$). So, the set

$$X = \left\{ m \in \mathbb{N} \bigg| \mathcal{M}_0 \models \bigwedge_{i \in I_3} D_{m_i}(n_i m) \right\} \cap X_1 \cap X_2$$

is non-empty: there exists $m \in X$ such that $\mathcal{M}_0 \models \varphi(b_0 + m, \bar{a})$.

**Theorem 2.16.** The theory $T_R$ is 1-e.c.

**Proof.** Let us show that for all $\mathcal{M}, \mathcal{M}_0 \models T_R$ such that $\mathcal{M}_0 \subset \mathcal{M}$, then $\mathcal{M}_0$ is 1-e.c. in $\mathcal{M}$. Let $\mathcal{M}, \mathcal{M}_0 \models T_R$ such that $\mathcal{M}_0 \subset \mathcal{M}$. Two cases are possible: either $R(M_0) = R(M)$ or $R(M_0) \subseteq R(M)$. The first case has been proved in Proposition 2.14. So let us assume that we are in the second case.

By Lemma 2.11 we may assume that $\mathcal{M} = \langle M_0, R(M) \rangle$. The strategy is as follows. By Lemma 2.11 any element of $\mathcal{M}$ is of the form $a + \sum_{i=1}^n f_i(b_i)$, where $a \in M_0$ and $b_1, \ldots, b_n \in R(M) \setminus R(M_0)$ are in different orbits. To show that $\mathcal{M}_0$ is 1-e.c. in $\mathcal{M}$, we show that for all tuple $\bar{b}$ of elements of $R(M) \setminus R(M_0)$ in different orbits, all $\bar{a} \in M_0$ and all $\varphi(\bar{x}, \bar{y})$, $\mathcal{M} \models \varphi(\bar{x}, \bar{y})$ implies $\mathcal{M}_0 \models \exists \bar{x} \in R \varphi(\bar{x}, \bar{a})$.

Let $\bar{b} \in R(M) \setminus R(M_0)$ be in different orbits. We first reduce the complexity of terms of the form $t(\bar{b}, \bar{a})$, $\bar{a} \in M_0$.

**Claim 2.17.** Any term $t(\bar{b}, \bar{a})$, where $\bar{a} \in M_0$, is equal to $\sum_{i=1}^n f_i(b_i) + a$, where $\bar{f}$ is a tuple of operators and $a \in M_0$.

**Proof.** The proof is similar to the proof of Lemma 2.11. Consider an expression of the form

$$t_0 = \sum_{i=1}^m f_i(b_i) + a,$$

where $\bar{f}$ is a tuple of operators and $a \in M_0$. We want to evaluate $S(t_0)$. We apply Proposition 2.10. If $m = 1$, then either $a = 0$ and $S(t_0) = S^k(b_1)$ for some $k \in \mathbb{Z}$ or $a \neq 0$ and $S(t_0) = t_0$, since by axiom [Ax.6] if $f(b) \in R(M)$, where $f$ is an operator and $b \in R(M)$, then there is $k \in \mathbb{Z}$ such that $f(b) = S^k(b)$. If $m > 1$, then $S(t_0) = t_0$. This implies that any term of the form $t(\bar{b}, \bar{a})$ is equal to $\sum_{i=1}^m f_i(b_i) + a$, for some tuple of operators $\bar{f}$ and $a \in M_0$.

By the claim above, is enough to prove that for any tuple $\bar{b}$ of elements of $R(M)$ not in $M_0$, in different orbits, and all quantifier free formula $\varphi(\bar{x}, \bar{a})$, if $\varphi(\bar{b}, \bar{a})$ holds, then $\varphi(M_0, \bar{a})$ is non empty. Let $\varphi(\bar{x}, \bar{a})$ be such a formula. We may assume that it is of the form

$$\bigwedge_{i \in I_2} \sum_{j=1}^m f_{ij}(x_j) + a_i \neq 0 \land \bigwedge_{i \in I_3} D_{m_i}(f_i(x_i) + k_i)$$

$$\land \bigwedge_{i \in I_3} \Sigma_{C_i, D_i} \left( \sum_{j=1}^{k_{ij}} f_{ij1}(x_j) + a_{i1}, \ldots \right) \land \bigwedge_{i \in I_4} \neg \Sigma_{C_i, D_i} \left( \sum_{j=1}^{k_{ij}} f_{ij1}(x_j) + a_{i1}, \ldots \right) \right).$$
By Axioms [Ax.6] and Lemma 2.8, one can replace a condition of the form
\[ \sum_{i=1}^{k_n} f_{ij1}(b_j) + a_i, \ldots \]
by a disjunction of conditions of the form
\[ \sum_{i=1}^{k_n} f_{ij}(b_i) = 0 \land \sum_{i=1}^{m} D_{\ell_i}(f_i'(b_i) + k_i). \]
Furthermore, \( \bigwedge_{i=1}^{n} f_i(b_i) = 0 \) can be replaced, by Axiom [Ax.5], by a conjunction of conditions of the form \( f(b_i) + k_i \neq 0 \). So we may assume that \( \varphi(x, \bar{a}) \) is of the form
\[ \bigwedge_{i \in I_1} \sum_{j=1}^{m} f_{ij}(x_j) + a_i = 0 \land \bigwedge_{i \in I_2} D_{m_i}(f_i(x_i) + k_i). \]
Thus, we have that \( \varphi(M_0, \bar{a}) \) is not empty, since by assumption \( \mathcal{M}_0 \models \Sigma_{C,D}(\bar{a}) \), where \( C = \{ f_i(m_i, k_i) \mid i \in I_2 \} \) and \( D = \{ f_i \mid i \in I_1 \} \).

2.2. Superstability. From the quantifier elimination of \( T_R \), we deduce, by means of counting of types, that it is superstable.

**Theorem 2.18.** The theory \( T_R \) is superstable.

**Proof.** Let \( \mathcal{C} \) be a monster model of \( T_R \) and let \( A \subseteq C \) be a small set of parameters. We want to show that \( |S(A)| \leq \max\{2^{\aleph_0}, |A|\} \). Without loss of generality, we may assume that \( A \) is the domain of the model. By quantifier elimination (see Theorem 2.3), any type \( p(x) \) over \( A \) is determined by the set of atomic formulas it contains. Let \( \mathcal{L}_1 = \mathcal{L}_g \cup \mathcal{L}_s \) and \( \mathcal{L}_2 \) be \( \mathcal{L} \setminus \{ D_n \mid n > 1 \} \). Let \( p_{\mathcal{L}_i} \) denote the restriction of \( p \) to \( \mathcal{L}_i \), so that \( p(x) = p_{\mathcal{L}_1}(x) \cup p_{\mathcal{L}_2}(x) \). We may assume that \( p(x) \) does not contain a formula of the form \( f(x) = a \) for some \( a \in M \). We consider two cases:

1. there exist \( m \in \mathbb{Z} \setminus \{0\} \) and \( a \in A \) such that \( R(mx + a) \in p(x) \);
2. for all \( m \in \mathbb{Z} \setminus \{0\} \) and all \( a \in A \), \( R(mx + a) \notin p(x) \).

Note that whenever we are in the first case, we may assume that \( x \) itself is in \( R \). Indeed, assume \( p(x) = \text{tp}(a/A) \) and let \( b = md + a \). Let \( q(x) \) be the type of \( b \) over \( M \). Then \( b' \models p(x) \) if and only if \( md' + b' \models q(x) \). Thus in both cases, whenever we consider a term \( t(x, \bar{a}) \), we may assume that it is of the form \( f(x) + a \) for some operator \( f \) and \( a \in A \).

**Claim 2.19.** The number of types of the form \( p_{\mathcal{L}_1}(x) \) is at most \( 2^{\aleph_0} \).

**Proof.** Indeed, any formula of the form \( D_n(f(x)+a) \) is equivalent to a formula of the form \( D_n(f(x)+m) \), where \( m \in \mathbb{Z} \) is such that \( D_n(a-m) \). Next, assume that \( f(x) + a \in p_{\mathcal{L}_1}(x) \), where \( a \in A \). Then, by axiom [Ax.4], \( \Sigma_t(a) \) holds in \( \mathcal{A} \). Thus there exists \( b \in R(A) \) such that \( f(x) = f(b) \). This implies, by axiom [Ax.5], that \( f(x) = 0 \). Hence \( a = 0 \).

By the previous claim, it remains to show that the number of types of the form \( p_{\mathcal{L}_2}(x) \) is at most \( \max\{|A|, 2^{\aleph_0}\} \).

**Claim 2.20.** Assume we are in case 2. Let \( \bar{f} \) be a tuple of operators of length \( n \) and \( f \) a non trivial operator. Then for all \( I \subseteq [n] \), there exists a finite \( E_I \subseteq \mathbb{Z}^n \) such that for all \( a \in A \), \( \Sigma_t(f(x) + a) \in p_{\mathcal{L}_2}(x) \) if and only if
\[ \bigvee_{I \subseteq [n]} \left( \Sigma_t(a) \land \bigvee_{n \in E_I} \sum_{i \in I} f_i(S^n(x)) = f(x) \right) \in p_{\mathcal{L}_2}(x). \]

**Proof of Claim.** Let’s assume that \( \Sigma_t(f(x) + a) \in p_{\mathcal{L}_2}(x) \). By Axiom [Ax.4], this implies that \( \sum_{i=1}^{n} f_i(b_i) = f(x) + a \) for some \( \bar{b} \in C \). Furthermore, there exists \( b', x' \in R(A) \) such that \( \sum_{i=1}^{n} f_i(b_i') = f(x') + a \) since \( \mathcal{A} \models \Sigma_t(-a) \). Because \( \mathcal{A} \) is a model, there exists \( I_0 \subseteq [n] \) such that \( \sum_{i \in I_0} f_i(b_i) = f(x) \). Thus, by Lemma 2.8, there is some \( I \subseteq I_0 \) such that \( \langle \bar{b}_I, x \rangle \) is a non degenerate solution of the corresponding equation. Notice that this implies that \( \mathcal{C} \models \Sigma_{t_{[n]}}(\bar{a}) \). This concludes the proof of the Claim.
A consequence of the previous Claim is that the number of types of the form $p_1 \mathcal{L}_2(x)$ that falls into case $1$ is at most $\max\{|A|, 2^{80}\}$.

**Claim 2.21.** Assume we are in case 2. Let $\bar{\mathcal{L}}$ be a tuple of $n$ operators and $m \in \mathbb{Z}\backslash\{0\}$. Then there exists at most one $a_\bar{\mathcal{L}} \in A$ such that $\sum_{i=1}^{n} f_i(x_i) = mx + a_\bar{\mathcal{L}}$ has a non degenerate solution in $R\backslash R(A)$.

**Proof of Claim.** Assume that there exists another $a' \in A$ that satisfies the claim. Then we have $\mathcal{A} \models \Sigma_{\bar{\mathcal{L}},-i}(a_i - a')$. Thus, we can find tuples $\bar{b}_1, \bar{b}_2 \in R\backslash R(A)$ and $\bar{b}'_1, \bar{b}'_2 \in R(A)$ such that

$$\sum_{i=1}^{n} f_i(b_{1i}) - f_i(b_{2i}) - (f_i(b'_{1i}) - f_i(b'_{2i})) = 0.$$

But this can happen only if $a_i = a'$ by Lemma 2.8.

As a consequence, we get that in case 2, a formula of the form $\Sigma_{\bar{\mathcal{L}},m,a}$ is in $p_1 \mathcal{L}_2(x)$ if and only if some disjunction of formulas of the form

$$\Sigma_{\bar{\mathcal{L}},m,a} \land \Sigma_{\bar{\mathcal{L}},n\backslash i}(a - a_i)$$

is in $p_1 \mathcal{L}_2(x)$. This proves that the number of types of the form $p_1 \mathcal{L}_2(x)$ in case 2 is at most $\max\{|A|, 2^{80}\}$. We conclude that $|S(A)| \leq \max\{|A|, 2^{80}\}$.

2.3. Decidability. As a consequence of the fact that the theory of $\mathcal{Z}_R$ is axiomatized by $T_R$ when $R$ is regular, we get the following decidability result.

**Theorem 2.22.** Let $R = (r_n)$ be a regular sequence. Assume that

1. the limit $\theta = \lim_{n \to \infty} \frac{r_{n+1}}{r_n}$ can be computed effectively;
2. $R$ is effectively congruence periodic: for all $k \in \mathbb{N}^{>1}$, there exist effective constants $m, p \in \mathbb{N}$ such that the sequence $(r_n)_{n \geq m}$ is periodic modulo $k$ with period $p$.

Then $\text{Th} (\mathcal{Z}_R)$ is decidable.

**Proof.** Indeed, under these assumptions, the constants that appear in axioms [(Ax.5), (Ax.6)] can be computed effectively, using the proofs of Propositions 1.8 and 1.12. Furthermore, axiom [(Ax.4)] becomes effective thanks to the effective periodicity of $R$. Thus, $T_R = \text{Th} (\mathcal{Z}_R)$ is recursively axiomatizable. And since $T_R$ is complete, we conclude that it is decidable.

Examples of regular sequences that satisfy Theorem 2.22 are $(q^n)$, $(n!)$ and the Fibonacci sequence. It is worth pointing out that the corresponding expansions of Presburger arithmetic are also tame: they remain decidable and have NIP. This was first established by A. L. Semenov in 19. In fact, 19 Theorem 3 states that $\mathcal{Z}_{<,R}$ is model complete whenever $R$ is a sparse sequence and decidable whenever $R$ is an effectively sparse sequence. Recall that a sequence $R = (r_n)$ is sparse if it satisfies the following properties:

1. for all operator $f$ on $R$, either $f(n) = 0$ for all $n \in \mathbb{N}$, or $\{n \in \mathbb{N} | f(n) \leq 0\}$ is finite or the set $\{n \in \mathbb{N} | f(n) \geq 0\}$ is finite;
2. for all operator $f$ on $R$, if $f$ has finitely many roots, then there exists a natural number $\Delta$ such that $f(n + \Delta) - r_n > 0$ for all $n \in \mathbb{N}$.

A sparse sequence $R$ is called effectively sparse if the above conditions are effective. Another proof of 19 Theorem 3 can be found in 15. There, in addition to the decidability of $\mathcal{Z}_{<,R}$ when $R$ is effectively sparse, a quantifier elimination result is proved 15 Proposition 9] and an explicit class of recurrence sequences for which $\theta$ is effective (and thus, effectively congruence periodic) is given 15 Proposition 11]. Furthermore, one can use the quantifier elimination result in 15 to prove that $\mathcal{Z}_{<,R}$ is NIP whenever $R$ is sparse.
3. Additional results

3.1. Expansions by sets of prime numbers. Let us first recall Dickson’s conjecture.

**Conjecture 1 (D).** Let $k \geq 1$, $a_i, b_i$ be integers such that $a_i \geq 1$ and $b_i \geq 0$ for all $i < k$. Let $f_i(x)$ be the polynomial $a_i x + b_i$. Assume that the following condition holds:

- there does not exist any integer $n > 1$ dividing $\prod_{i<k} f_i(s)$ for all $s \in \mathbb{N}$.

Then, there exist infinitely many $m \in \mathbb{N}$ such that $f_i(m)$ is prime for all $i < k$.

Under the assumption that Dickson’s conjecture is true, P. T. Bateman, C. G. Jockusch and A. R. Woods showed that the theory of $\mathcal{L}_{<P} = (\mathbb{Z},+,0,<,P)$ is undecidable and, in fact, that the multiplication is definable (see [10, Theorem 1]). This result was slightly improved in [12] by M. Boffa, who obtained the same results for $\mathcal{L}_{<P_{m,r}}$, where, for coprime natural numbers $r < m$, $P_{m,r}$ is the set $\{ p | p \equiv_m r \text{ and } p \in P \}$.

In this spirit, I. Kaplan and S. Shelah proved in [10, Theorem 1.2] that the structure $(\mathbb{Z},+,0,P \cup \neg P)$ is unstable, supersimple, of rank 1 and decidable (which is in contrast with [11, Theorem 1]), provided that Dickson’s conjecture is true. Notice that the structure $\mathcal{L}_{<P}$ has the order property which is why the consider the expansion $(\mathbb{Z},+,0,P \cup \neg P)$. The purpose of this section is to show [10, Theorem 1.2] is still true when we replace $P \cup \neg P$ by $P_{r,m}^\pm = P_{r,m} \cup \neg P_{r,m}$, for coprime $r < m$.

A key part in the proof of [10, Theorem 1.2] is the following fact.

**Fact 3.1 ([10, Lemma 2.3]).** Assuming (D), given $f_i(x) = a_i x + b_i$ with $a_i, b_i$ intergers, $a_i \geq 1$ for all $i < k$, and $g_j(x) = c_j x + d_j$ with $c_j, d_j$ intergers, $c_j \geq 1$ for all $j < k'$, if $\ast_f$ holds and $(a_i, b_i) \neq (c_j, d_j)$ for all $i, j$, then there are infinitely many natural numbers $m$ for which $f_i(m)$ is prime and $g_i(x)$ is composite for all $i < k$ and $j < k'$.

A careful inspection of the proof of [10, Theorem 1.2] shows that the only modification needed to prove that $(\mathbb{Z},+,0,P_{m,r}^\pm)$ is superstable of rank one is the following improvement of Fact 3.1.

**Lemma 3.2.** Assume (D). Fix coprime $m, n$ such that $r < m$. Let $f_i(x) = a_i x + b_i$ with $a_i, b_i$ integers, $a_i \geq 1$ for all $i < k$, and $g_j(x) = c_j x + d_j$ with $c_j, d_j$ integers, $c_j \geq 1$ for all $j < k'$. Suppose that there exists $0 < \varepsilon < m$ such that $a_i x + b_i \equiv_m r$, $\ast_f$ holds and $(a_i, b_i) \neq (c_j, d_j)$ for all $i, j$. Then there are infinitely many natural numbers $m$ for which $f_i(m)$ is prime and $g_i(x)$ is composite for all $i < k$ and $j < k'$.

This lemma is proved using the following lemma and the proof of fact 5.1.

**Lemma 3.3.** Let $f_i(x) = a_i x + b_i$, $a_i, b_i$ natural numbers such that $a_i \geq 1, i < k$. Let $m, r$ be fixed coprime natural numbers such that $r < m$. Suppose that $\ast_f$ holds and that there exists $0 < \varepsilon < m$ such that $a_i \varepsilon + b_i \equiv_m r$. Then, assuming (D), there exists infinitely many natural $s$ such that $f_i(s) \in P_{m,n}$ for all $i \leq k$.

**Proof.** Add to the family $\tilde{f}$ the polynomial $f_k(x) = m x + r$. Then, by our assumption, we have that $\ast_{f, f_k}$ still holds. By Dickson’s conjecture, we obtain infinitely many $s$ such that $f_i(s) \in P$ for all $i \leq k$. In particular, for infinitely many such $s$, $f_i(s) \in P_{m,n}$.

**Proposition 3.4.** Let $m > n$ be coprime natural numbers. Then, $(\mathbb{Z},+,0,P_{m,r}^\pm)$ is unstable, supersimple of Lascar rank 1.

**Proof.** The proof follows exactly the proof of [10, Theorem 1.2] using Lemma 5.2 instead of [10, Lemma 2.3].

3.2. Expansion by a finitely generated submonoid of $(\mathbb{Z}, \cdot, 1)$. Assume $K$ has characteristic zero and consider equations of the form $\sum_{i=1}^n q_i x_i = 1$, where $q_i \in \mathbb{Q}^*$. Let $A \subseteq K$. A solution $\bar{a}$ in $A^n$ is non-degenerate if each sub-sum $\sum_{j \in J} q_j a_j \neq 0$, for any subset $J$ of $[n]$. The set $X$ has the Mann property if any such equation has only finitely many non-degenerate solutions.

\[\text{For instance, a Theorem of Tao (see [20]) states that every natural number greater than 1 is the sum of at most five prime numbers.}\]
This terminology comes from the work of L. van den Dries and A. G"unaydın on expansions of algebraically closed fields or real-closed fields $K$ by a small (in the sense of [21 Section 2]) subgroup $G$ of the multiplicative group of the field. Their paper [21] is concerned with the model theory of pairs $(K, G)$, where $K$ is either algebraically closed of characteristic 0 or real closed, and where $G$ has the Mann property. One of their results is a characterization of elementary equivalence between those structures (see [21, Theorems 1.2 and 1.3]).

In this section, we will consider expansions of the form $(\mathbb{Z}, +, 0, M)$ where $(\mathbb{Z}, 1)$ is a submonoid of $(\mathbb{Z}, 1)$ with the Mann property. Let $G$ be the subgroup of $(\mathbb{Q}_0, 1)$ generated by $M$, then it has the Mann property. An example of such monoid is $(\mathbb{Z}, 3 \mathbb{Z}) \cap \mathbb{N} = (P_2, P_3)$. More generally, any finitely generated submonoid $(\mathbb{Z}, 1)$ of $(\mathbb{Z}, 1)$ has the Mann property, since the corresponding group $(\mathbb{Z}, 1)$ has finite rank (as an abelian group, that is $\dim_\mathbb{Q} G \otimes_\mathbb{Z} \mathbb{Q}$ is finite, see [23, 6]). As mentioned in the Introduction, this result of the section is a special case of [11 Theorem 3.2], but with a short proof.

Let $(\mathbb{Z}, 1)$ be a submonoid of $(\mathbb{Z}, 1)$. Let $\mathcal{L}_M$ be the language $\{1, s | s \in M\}$, where $s$ is a unary function interpreted as $s(m) = s \cdot m$. Let $\mathcal{M}$ be the $\mathcal{L}_M$-structure $(\mathbb{Z}, 1, s | s \in M)$. Finally let $\mathcal{M}'$ be the expansion of $\mathcal{M}$ by unary predicates for all subsets of $M$.

**Lemma 3.5.** Th$(\mathcal{M}')$ has quantifier elimination and is superstable.

**Proof.** Let $\varphi(x, y)$ be a quantifier-free formula. One can assume that it is of the form

$$\bigwedge_{i \in I_1} s_i(x) = s_i'(y_i) \land \bigwedge_{i \in I_2} s_i(x) \neq s_i'(y_i) \land \psi(x),$$

where $\psi(x)$ is a quantifier-free formula in the language of $\mathcal{M}'$.

Since, for all $s \in M$, $\mathcal{M} \models \forall x, y \ x = y \iff s(x) = s(y)$, one can further assume that for all $i \in I_1 \cup I_2$, $s_i = s$. Now, if $I_1 \neq \emptyset$, $\exists x \varphi(x, y)$ is equivalent to

$$\bigwedge_{i,j \in I_1} s_i'(y_i) = s_j'(y_j) \land \bigwedge_{i \in I_2} s_i'(y_i) \neq s'_i(y_i) \land \psi(s_i(y_i)),$$

for some $i_0 \in I_1$. So let us assume that $I_1 = \emptyset$, so that $\varphi(x, y)$ has the form

$$\bigwedge_{i \in I_2} s_i(x) \neq s_i'(y_i) \land \psi(x).$$

We then distinguish two cases. First, assume that $\mathcal{M}' \not\models \exists x \ \psi(x)$. In this case, we have that $\exists x \varphi(x, y)$ is equivalent to $1 \neq 1$. Second, assume that $\mathcal{M}' \models \exists x \ \psi(x)$. We again distinguish two cases. First, assume that the set $\psi(M)$ is infinite. Thus $\exists x \varphi(x, y)$ is equivalent to $1 = 1$. Second, assume that $\psi(M)$ is finite. Then there exists $s_1''', \ldots, s_n'' \in M$ such that

$$\mathcal{M}' \models \forall x \ \psi(x) \iff \bigvee_{i=1}^n x = s_i''(1).$$

So, in that case, $\exists x \varphi(x, y)$ is equivalent to

$$\bigvee_{j=1}^n \bigwedge_{i \in I_2} s(s_j''(1)) \neq s'_i(y_i) \land \psi(s_j''(1)).$$

This finishes the first part of the proof. Superstability follows easily by counting types. \hfill $\square$

**Theorem 3.6.** Let $(\mathbb{Z}, 1)$ be a submonoid of $(\mathbb{Z}, 1)$ with the Mann Property. Then $\mathcal{Z}_M$ is superstable.

**Proof.** We apply the same strategy as in the proof of Theorem 1.4. We first need to show that $M$ is small. Let $G$ be the subgroup of $(\mathbb{Q}_0, 1)$ generated by $M$. By [21 Corollary 6.3], the sequence of sets $(G^n + r)$ is strictly increasing, so that no set of the form $a \mathbb{Z} + b$ can be covered by a set of the form $n_1 M + \cdots + n_k M$. This shows that $M$ is small. To conclude the proof, we only need to show that $M$ is superstable. We will in fact show that it is definably interpreted in $\mathcal{M}$.

Let $\varphi(x)$ be the equation $a_1 x_1 + \cdots + a_n x_n = 0$. We have to show that the set $\varphi(M)$ corresponds to a definable subset of $\mathcal{M}$. As in the proof of Corollary 1.10, we only have to show that the set
of non degenerate solutions of $\phi(\bar{x})$ corresponds to a definable subset of $\mathcal{M}$. This is an adaptation of the work done in [21, Section 5]. Indeed, one can show that the set of non degenerate solutions of $\phi(\bar{x})$ (in $M$) is

$$\bigcup_{(g_2,\ldots,g_n) \in S'} (1,g_2,\ldots,g_n)M,$$

where $S'$ is the (finite) set of non degenerate solutions of the equation

$$a_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

\[\square\]

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