STABILITY ANALYSIS OF POSITIVE SEMI-MARKOVIAN JUMP LINEAR SYSTEMS WITH STATE RESETS

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Abstract. This paper studies the mean stability of positive semi-Markovian jump linear systems. We show that their mean stability is characterized by the spectral radius of a matrix that is easy to compute. In deriving the condition we use a certain discretization of a semi-Markovian jump linear system that preserves stability. Also we show a characterization for the exponential mean stability of continuous-time positive Markovian jump linear systems. Numerical examples are given to illustrate the results.

Key words. Semi-Markovian jump linear systems, mean stability, positive systems, Markovian renewal processes

AMS subject classifications. 60K15, 93E15, 15B48, 93C05

1. Introduction. The stability analysis of switched systems, a class of dynamical systems whose mathematical structure experiences abrupt changes, is one of the most fundamental problems in mathematical systems theory [13, 29, 37]. In particular, the stability of positive switched systems, whose state variables are constrained to be in positive orthants, has received considerable attentions over the past decade [3, 17, 27, 32, 35]. The study of positive switched systems is motivated by their possible application in pharmacokinetics. In the modern treatment of human immunodeficiency virus (HIV) infection, multiple drug regimens are employed to prevent the emergence of drug-resistant virus [39]. The authors in [21] solve the minimization problem of such virus mutation by its reduction to the optimal control problem of a positive switched system under simplifying assumptions. The reduction was made possible by the switching nature of HIV treatments and the positivity constraint naturally placed on the population of virus. The importance of this class of switched systems also stems from the fact that such positivity constraints naturally arise in broad areas including communication systems [36], formation flying [24] and multi agent systems [33].

The stability of positive switched linear systems has been mainly studied by co-positive Lyapunov functions [3, 17, 19, 27, 40]. A co-positive Lyapunov function is a non-negative linear form of state variables, which makes a contrast with general cases where the non-negativity of Lyapunov functions forces us to use quadratic functionals of state variables. Its linearity often reduces the stability analysis of positive switched linear systems to linear problems. For example, a positive switched linear system is stable for all switching signals if a family of square matrices associated with the given system consists of matrices whose eigenvalues have only negative real parts [17, 27, 40]. However, once a switched system is modeled as a stochastic switched system [28, 34], the above mentioned Lyapunov function approach fails to take the probability distribution into account appropriately because it treats any sample path in a rather equal manner. One of the natural notions of stability in this case is mean stability [28], which requires that the norm of state variables converges to 0 in expectation. One of the earliest results along this line is by Feng et al. [16], where they give a necessary and sufficient condition for the exponential mean square stability of continuous-time

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Markovian jump linear systems. This result has been extended to switched linear systems with various stochastic structures including switching Markovian jump linear systems [4], Markovian jump linear systems with disturbances [18], and stochastic hybrid systems with renewal transitions [1]. Stochastic Lyapunov function approaches for the stability analysis of Markovian jump nonlinear systems with disturbances can be found in [26, 30].

The aim of this paper is to give criteria for the mean stability of positive stochastic switched systems. We assume that the switched systems are semi-Markovian jump linear systems [23] (also called stochastic hybrid systems with renewal transitions [1]), whose switching signal is a Markovian renewal process [25]. We show that their exponential mean stability is characterized by the spectral radius of a matrix that is easy to compute. We also allow their state to be reset [31] by random linear mappings at the switching instances.

One of the difficulties in analyzing semi-Markovian jump linear systems is that its transition rate of discrete-modes is not time-invariant [23]. Instead of employing Volterra integral equations used in [1], we avoid this difficulty by introducing a discretization of semi-Markovian jump linear systems that admits a certain time-invariant expression (Proposition 3.8). That discretization turns out to preserve stability properties and hence the system matrix of the discretization exactly determines the stability of the original system.

This paper is organized as follows. After preparing necessary mathematical notations, in Section 2 we give the definition of continuous-time positive semi-Markovian jump linear systems and state the main result. Section 3 gives the stability analysis of discrete-time positive semi-Markovian jump linear systems. Based on the analysis Section 4 gives the proof of the main result. The exponential mean stability of positive Markovian jump linear systems is studied in Section 5.

1.1. Mathematical Preliminaries. Let $(\Omega, \mathcal{M}, P)$ be a probability space. For an integrable random variable $X$ on $\Omega$ its expected value is denoted by $E[X]$. Without being explicitly stated, the random variables that appear in this paper will be assumed to be integrable. If $\mathcal{M}_1 \subset \mathcal{M}$ is a $\sigma$-algebra then $E[X | \mathcal{M}_1]$ denotes the conditional expectation of $X$ given $\mathcal{M}_1$. It is well known (see, e.g., [5]) that, if $\mathcal{M}_2 \subset \mathcal{M}_1 \subset \mathcal{M}_3$ then

$$E[E[X | \mathcal{M}_2] | \mathcal{M}_1] = E[E[X | \mathcal{M}_1] | \mathcal{M}_2] = E[X | \mathcal{M}_2].$$

The $\sigma$-algebra generated by random variables $X_1, \ldots, X_\ell$ is defined [5] as the smallest $\sigma$-algebra on which $X_1, \ldots, X_\ell$ are measurable and is denoted by $\mathcal{M}(X_1, \ldots, X_\ell)$. For a function $f$ on $\mathbb{R}$ its limit at $t$ from the left, if it exists, is denoted by $f(t^-)$.

**Lemma 1.1.** Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \subset \mathcal{M}$ be $\sigma$-algebras on $\Omega$ and $X$ be a random variable on $\Omega$. If $E[X | \mathcal{M}_1] = E[X | \mathcal{M}_3]$ and $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ then $E[X | \mathcal{M}_1] = E[X | \mathcal{M}_2]$.

**Proof.** By (1.1) and the assumption,

$$E[X | \mathcal{M}_1] = E[E[X | \mathcal{M}_1] | \mathcal{M}_2] = E[E[X | \mathcal{M}_3] | \mathcal{M}_2] = E[X | \mathcal{M}_2].$$

This completes the proof. $\Box$
A real matrix $A$ is said to be **nonnegative** if it has only nonnegative entries and we write $A \geq 0$. Let $A$ be square. $A$ is said to be **Metzler** if its off-diagonal entries are nonnegative. Since any Metzler matrix $A$ can be written as $A - \alpha I$ where $A \geq 0$, $\alpha \in \mathbb{R}$, and $I$ is the identity matrix, the exponential matrix $e^{At}$ ($t \geq 0$) is nonnegative. The Kronecker product \cite{6} of $A$ and another matrix $B$ is denoted by $A \otimes B$. The spectral radius of $A$ is denoted by $\rho(A)$. We say that $A$ is Schur stable if $\rho(A) < 1$. Also the spectral abscissa of $A$, denoted by $\eta(A)$, is defined as the largest real part of the eigenvalues of $A$. We say that $A$ is Hurwitz stable if $\eta(A) < 0$. The next lemma gives basic facts about nonnegative matrices (see, e.g., \cite{38}).

**Lemma 1.2.** Let $A \geq 0$ be square. Then $\rho(A)$ is an eigenvalue of $A$ and there exists a nonnegative eigenvector to the eigenvalue $\rho(A)$. Moreover if $A \geq B \geq 0$ then $\rho(A) \geq \rho(B)$.

The next corollary readily follows from the lemma.

**Corollary 1.3.** Let $A$ and $B$ be nonnegative square matrices with the same dimensions. If $A_{ij} > B_{ij}$ whenever $B_{ij} > 0$ then $\rho(A) > \rho(B)$.

**Proof.** Define $r = \min_{i,j:B_{ij}\neq 0} \frac{A_{ij}}{B_{ij}} > 1$. Then $A \geq rB \geq 0$ so that, by Lemma 1.2, $\rho(A) \geq r\rho(B) > \rho(B)$. \square

The $m$-norm of $x \in \mathbb{R}^n$ is defined by $\|x\|_m = (\sum_{i=1}^n |x_i|^m)^{1/m}$. The symbol $1_\ell$ denotes the column vector of length $\ell$ whose entries are all 1. The 1-norm is linear on the positive orthant $\mathbb{R}_+^n$ because if $x \geq 0$ then $\|x\|_1 = 1_n^T x$. By $e_i$ we denote the $i$-th standard unit vector in $\mathbb{R}^N$ defined by

$$
[e_i]_j = \begin{cases} 
1 & j = i, \\
0 & \text{otherwise.}
\end{cases}
$$

(1.2)

It is easy to see that, for all $i$, $m \geq 1$, and $x \in \mathbb{R}^n$,

$$\|x\|_m = \|e_i \otimes x\|_m,$$

where the $m$-norm on the right hand side is defined on $\mathbb{R}^{nN}$.

For $x \in \mathbb{R}^n$ and a positive integer $m$ the vector $x^{[m]}$ is defined \cite{2, 8} as the real vector of length

$$n_m = \left(\begin{array}{c} n + m - 1 \\ m \end{array}\right)$$

whose elements are all the lexicographically ordered monomials of degree $m$ in $x_1$, \ldots, $x_n$. Their nonzero coefficients are chosen in such a way that $\|x\|_2^m = \|x^{[m]}\|_2$. From (1.2) it follows that

$$\|e_i \otimes x^{[m]}\|_2 = \|x\|_2^m$$

(1.3)

for any standard unit vector $e_i$. For $A \in \mathbb{R}^{n \times n}$ we define the $n_m \times n_m$ matrices $A^{[m]}$ and $A_{[m]}$ as the unique matrices \cite{2, 8} satisfying

$$ (Ax)^{[m]} = A^{[m]}x^{[m]}$$

(1.4)

for every $x \in \mathbb{R}^n$ and

$$\begin{bmatrix} dx \\ dt \end{bmatrix} = Ax \Rightarrow \begin{bmatrix} dx^{[m]} \\ dt \end{bmatrix} = A_{[m]}x^{[m]}$$

(1.5)
for every $\mathbb{R}^n$-valued differentiable function $x$ on $\mathbb{R}$. We need the next two lemmas about the vector $x^{[m]}$.

**Lemma 1.4.** Let $x$ be an $\mathbb{R}^n$-valued random variable. If $x \geq 0$ with probability 1 then

$$E[||x||^{m}] \leq n\|E[x^{[m]}]\|_1. \tag{1.6}$$

**Proof.** By the assumption we have $E[||x||^{m}] = E[\sum_{i=1}^{n} |x_i^{[m]}|] = \sum_{i=1}^{n} E[x_i^{m}]$. Since $x^{[m]}$ has the entry $x_i^{m}$ it holds that $E[x_i^{m}] = |E[x_i^{m}]| \leq \|E[x^{[m]}]\|_1$. Thus (1.6) is true. \[\square\]

**Lemma 1.5.** The set $\{e_i \otimes x^{[m]} : 1 \leq i \leq N, \ x \in \mathbb{R}^n\} \subset \mathbb{R}^{n \times N}$ spans $\mathbb{R}^{n \times N}$ over $\mathbb{R}$.

**Proof.** Since each of $e_i$ is a standard unit vector, it is sufficient to show that the set $S = \{x^{[m]} : x \in \mathbb{R}^n\}$ spans $\mathbb{R}^{n \times m}$. Assume that $v \in \mathbb{R}^{n \times m}$ is orthogonal to span $S$. Then, for every $x \in \mathbb{R}^n$ we have

$$v^\top x^{[m]} = 0. \tag{1.7}$$

Since $x^{[m]}$ has all the monomials of degree $m$ with nonzero coefficient, comparing the coefficient of each monomial in (1.7) shows $v = 0$, which implies that span $S$ is the entire space $\mathbb{R}^{n \times m}$. \[\square\]

2. Continuous-time Positive Semi-Markovian Jump Linear Systems.

This section introduces continuous-time positive semi-Markovian jump linear systems. Then we define their exponential and stochastic mean stability. After that we state our main result, which gives the characterization of the mean stability of continuous-time positive semi-Markovian jump linear systems. A numerical example is given to illustrate the result.

Let $A_1, \ldots, A_N$ be $n \times n$ real matrices. Throughout this paper we fix a probability space $(\Omega, \mathcal{M}, P)$. Let $\{\sigma_k\}_{k=0}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$, and $\{J_k\}_{k=0}^{\infty}$ be stochastic processes on $\Omega$ taking values in $\{1, \ldots, N\}$, $\mathbb{R}_+$, and $\mathbb{R}^{n \times n}$, respectively. We assume that $\{t_k\}_{k=0}^{\infty}$ is non-decreasing. We let

$$h_k = t_{k+1} - t_k, \ k \geq 0.$$ 

Assume that $t_0 = 0$ and $\sigma_0$ is a constant. Let $\Sigma$ be the stochastic switched system defined by

$$\Sigma : \begin{cases} \frac{dx}{dt} = A_{\sigma_k} x(t), \ t_k \leq t < t_{k+1} \\ x(t_{k+1}) = J_k x(t^{-}_{k+1}), \ k \geq 0 \end{cases} \tag{2.1}$$

where $x(0) = x_0 \in \mathbb{R}^n$ is a constant vector.

**Definition 2.1.** We say that $\Sigma$ is a continuous-time semi-Markovian jump linear system if the following two conditions hold for every $i, j \in \{1, \ldots, N\}$, $t \geq 0$, and every Borel subset $B$ of $\mathbb{R}^{n \times n}$.

CI. (Markovian property) It holds that

$$P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k, \ldots, \sigma_0, t_k, \ldots, t_0, J_{k-1}, \ldots, J_0) = P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k).$$
C2. (Time homogeneity) The probability

\[ P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k = i) \]

is independent of \( k \).

Furthermore we say that \( \Sigma \) is positive if

C3. (Positivity) The matrices \( A_1, \ldots, A_N \) are Metzler and, for each \( k \geq 0 \), \( J_k \) is nonnegative with probability 1.

The conditions C1 and C2 in particular show that the process \( \{(\sigma_k, t_k)\}_{k=0}^{\infty} \) is a time-homogeneous Markovian renewal process and therefore \( \{\sigma_k\}_{k=0}^{\infty} \) is a time-homogeneous Markov chain [25]. We let \( [p_{ij}]_{ij} \in \mathbb{R}^{N \times N} \) be the transition matrix of the Markov chain \( \{\sigma_k\}_{k=0}^{\infty} \). The condition C3 implies that \( x(t) \geq 0 \) for every \( t \geq 0 \) with probability 1 provided \( x_0 \geq 0 \). Without being explicitly stated, throughout this paper \( \Sigma \) denotes a positive continuous-time semi-Markovian jump linear system. The aim of this paper is to study the stability of \( \Sigma \) defined as follows.

**Definition 2.2.** Let \( m \) be a positive integer.

- \( \Sigma \) is said to be exponentially \( m \)-th mean stable if there exist \( C > 0 \) and \( \beta > 0 \) such that, for every \( x_0 \) and \( \sigma_0 \),

\[ E[\|x(t)\|^m] \leq Ce^{-\beta t}\|x_0\|^m. \]

- \( \Sigma \) is said to be stochastically \( m \)-th mean stable if, for any \( x_0 \) and \( \sigma_0 \),

\[ \int_0^\infty E[\|x(t)\|^m] \, dt < \infty. \]

**Remark 2.3.** The stability notions in Definition 2.2 are independent of the norms used in (2.3) and (2.4) by the equivalence of the norms on a finite-dimensional normed vector space. Actually we can even use two different norms in (2.3).

We now state the next assumption.

**Assumption 2.4.**

1. For every \( k \geq 0 \),

\[ h_k > 0. \]

2. There exists \( T > 0 \) such that, for every \( k \geq 0 \),

\[ h_k \leq T. \]

3. There exists \( R > 0 \) such that, for every \( k \geq 0 \),

\[ \|J_k\| \leq R. \]

In this assumption, only the second condition, which is also used in [1], is essential. The first condition 1 in Assumption 2.4 is not restrictive because most of semi-Markovian jump linear systems can be rewritten as a semi-Markovian jump linear system that satisfies the condition 1. Also, though in [1] they use constant jump matrices, this paper allows them to be uniformly bounded random variables.

The next theorem is the main result of this paper.

**Theorem 2.5.** \( \Sigma \) is exponentially \( m \)-th mean stable if and only if the block matrix \( A_m(\Sigma) \in \mathbb{R}^{(n_mN) \times (n_mN)} \) whose \((i,j)\)-block is defined by

\[ (A_m(\Sigma))_{ij} = p_{ji}E[(J_ke^{A_{\sigma_k}h_k})^m \mid \sigma_k = j, \sigma_{k+1} = i] \in \mathbb{R}^{n_m \times n_m} \]
is Schur stable.

Let us see an example.

**Example 2.6.** Let \( \Sigma \) be a positive semi-Markovian jump linear system with the two subsystems \( \Sigma_1: dx/dt = A_1 x \) and \( \Sigma_2: dx/dt = A_2 x \) given by

\[
A_1 = \begin{bmatrix} -2 & 0.2 \\ 0.1 & -2.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.1 & 0.9 \\ 0.2 & 0.3 \end{bmatrix}.
\]

We assume that \( J_k = I \) with probability 1, \( p_{11} = p_{22} = 0 \), and \( p_{12} = p_{21} = 1 \). Suppose that the transition probability (2.2) is given by

\[
P(\sigma_{k+1} = 1, h_k \leq t \mid \sigma_k = 2) = \begin{cases} 0, & t \leq a \\ (t - a)/2a, & a \leq t \leq 3a \\ 1, & t \geq 3a \end{cases}
\]

where \( a > 0 \) is a constant and

\[
P(\sigma_{k+1} = 2, h_k \leq t \mid \sigma_k = 1) = \begin{cases} F(t; k, \lambda), & t \leq t_p \\ 1, & t \geq t_p \end{cases}
\]

where \( F(\cdot; k, \lambda) \) denotes the probability distribution function of the Weibull distribution with the shape parameter \( k > 0 \) and the scale parameter \( \lambda > 0 \), i.e.,

\[
F(t; k, \lambda) = 1 - e^{-t/\lambda^k} \quad \text{for} \ t \geq 0 \quad \text{and} \quad F(t; k, \lambda) = 0 \quad \text{for} \ t < 0, \quad \text{and} \ t_p > 0 \text{ is the unique number satisfying} \ F(t_p; k, \lambda) = 1 - p \quad (p > 0) .
\]

\( \Sigma \) is clearly positive and satisfies Assumption 2.4 with \( T = \max(3a, t_p) \) and \( R = 1 \).

We can regard \( \Sigma \) as a controlled system with controller failures. The stable subsystem \( \Sigma_1 \) models the controlled dynamics while the unstable one \( \Sigma_2 \) models the open dynamics without a controller in effect. The occurrence of a failure, whose probability is modeled by the Weibull-like distribution (2.9), gives rise to the switching from \( \Sigma_1 \) to \( \Sigma_2 \). The time it takes for the controller to be repaired is modeled by the uniform distribution (2.8).

We check the first mean and mean square stability of \( \Sigma \) using Theorem 2.5 for the parameters \( \lambda = 3, k = 10, \) and \( p = 0.1 \). Fig. 2.1 shows the graph of \( \rho(A_1(\Sigma)) \) and \( \sqrt{\rho(A_2(\Sigma))} \) as the constant \( a \) moves over \([0.8, 1.2] \). We can see that \( \Sigma \) is exponentially first mean stable if \( a < 1.035 \) while \( \Sigma \) is exponentially mean square stable only
when $a < 0.908$. Notice that the result in [1] does not check the first mean stability because it deals with only even exponents $m$. A sample path of $\Sigma$ for $a = 1$ is shown in Fig. 2.2.

We will prove Theorem 2.5 by first investigating the stability of its discretization. For the trajectory $x$ of $\Sigma$ we define the discrete-time stochastic process $\{x_d(k)\}_{k=0}^\infty$ by

$$x_d(k) := x(t_k), \quad k \geq 0.$$ 

Then, by (2.5), $\{x_d(k)\}_{k=0}^\infty$ satisfies

$$\mathcal{S}\Sigma : x_d(k + 1) = J_k e^{A_k h_k} x_d(k), \quad k \geq 0.$$ 

In the next section we analyze the stability of a class of stochastic discrete-time switched systems that include the above defined $\mathcal{S}\Sigma$. Based on the analysis Section 4 gives the proof of the main result of Theorem 2.5.

3. Discrete-time Positive Semi-Markovian Jump Linear Systems. Let $\{\sigma_k\}_{k=0}^\infty$ be a time-homogeneous Markov chain taking values in $\{1, \ldots, N\}$ with the probability transition matrix $[p_{ij}]_{ij}$. We assume that $\sigma_0$ is a constant. Let $\{F_k\}_{k=0}^\infty$ be another stochastic process on $\Omega$ taking values in $\mathbb{R}^{n \times n}$. Define the discrete-time switched system $\Sigma_d$ by

$$\Sigma_d : x_d(k + 1) = F_k x_d(k), \quad x_d(0) = x_0.$$ 

**Definition 3.1.** We say that $\Sigma_d$ is a discrete-time semi-Markovian jump linear system if the following two conditions hold for every $k \geq 0$, $i, j \in \{1, \ldots, N\}$, and every Borel subset $B$ of $\mathbb{R}^{n \times n}$.

**D1.** (Markovian property) It holds that

$$P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k, \ldots, \sigma_0, F_{k-1}, \ldots, F_0) = P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k).$$ 

**D2.** (Time homogeneity) The expected probability

$$P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k = i)$$

does not depend on $k$. 
Furthermore we say that $\Sigma_d$ is positive if

D3. (Positivity) $F_k$ is nonnegative with probability 1.

Each of the conditions D1, D2, and D3 corresponds to C1, C2, and C3 in Definition 2.2, respectively. Throughout this section $\Sigma_d$ denotes a positive discrete-time semi-Markovian jump linear system.

The stability of discrete-time semi-Markovian jump linear systems is defined in a similar way as that of continuous-time ones. As mentioned in Remark 2.3, any combination of norms can be also used in the following definition.

**Definition 3.2.** Let $m$ be a positive integer.

- $\Sigma_d$ is said to be exponentially $m$-th mean stable if there exist $C > 0$ and $\beta > 0$ such that, for any $x_0$ and $\sigma_0$,

  \[ E[\|x_d(k)\|^m] \leq Ce^{-\beta k}\|x_0\|^m. \]  

- $\Sigma_d$ is said to be stochastically $m$-th mean stable if, for any $x_0$ and $\sigma_0$,

  \[ \sum_{k=0}^{\infty} E[\|x_d(k)\|^m] < \infty. \]

We place the following assumption that corresponds to the conditions 2 and 3 of Assumption 2.4.

**Assumption 3.3.** The expected value

\[ E[F_k^m | \sigma_k = i, \sigma_{k+1} = j] \]

exists for all $i, j \in \{1, \ldots, N\}$.

Notice that, by D2, the expected value (3.2) does not depend on $k$. The next theorem gives a characterization of the stability of $\Sigma_d$ and is used in Section 4 to prove Theorem 2.5.

**Theorem 3.4.** The following statements are equivalent:

1. $\Sigma_d$ is exponentially $m$-th mean stable.
2. $\Sigma_d$ is stochastically $m$-th mean stable.
3. The block matrix $F_m \in \mathbb{R}^{(n_mN) \times (n_mN)}$ whose $(i, j)$-block is defined by

   \[ [F_m]_{ij} = p_{ji}E[F_k^m | \sigma_k = j, \sigma_{k+1} = i] \in \mathbb{R}^{n_m \times n_m} \]

   is Schur stable.

**Remark 3.5.** Theorem 3.4 extends the stability characterizations of discrete-time Markovian jump linear systems given in [12, 14] to semi-Markovian jump linear systems.

The rest of this section is devoted to the proof of Theorem 3.4. Let us first observe that, by the positivity of $\Sigma_d$, the initial state $x_0$ can be assumed to be nonnegative without loss of generality.

**Lemma 3.6.** $\Sigma_d$ is exponentially $m$-th mean stable if and only if there exist $C > 0$ and $\beta > 0$ such that (3.1) holds for any $x_0 \geq 0$ and $\sigma_0$.

**Proof.** The necessity part is obvious. Let us prove the sufficiency part. Assume that there exist $C > 0$ and $\beta > 0$ such that (3.1) holds for any $x_0 \geq 0$ and $\sigma_0$. Let $x_0 \in \mathbb{R}^n$ be arbitrary. Define $x_0^+, x_0^- \in \mathbb{R}_+^n$ as the entry-wise maximum and minimum $x_0^+ := \max(x_0, 0)$ and $x_0^- := \max(-x_0, 0)$. Let $x_d^+(k)$ and $x_d^-(k)$ denote the solutions of $\Sigma_d$ with the initial states $x_0^+$ and $x_0^-$, respectively. Since $x_0 = x_0^+ - x_0^-$ we have
\( x_d(k) = x^+_d(k) - x^-_d(k) \). Then the general inequalities \((a + b)^m \leq 2^m(a^m + b^m)\) \((a, b \geq 0)\) and \(\|x^+_0\|_2^m + \|x^-_0\|_2^m \leq 2\|x_0\|_2^m\) show that

\[
E[\|x_d(k)\|_2^m] \leq E \left[ (\|x_d(k; x^+_0)\|_2 + \|x_d(k; x^-_0)\|_2)^m \right] \\
\leq 2^m E \left[ \|x_d(k; x^+_0)\|_2^m + \|x_d(k; x^-_0)\|_2^m \right] \\
\leq 2^m e^{-\beta k} (\|x^+_0\|_2^m + \|x^-_0\|_2^m) \\
\leq 2^{m+1} e^{-\beta k} \|x_0\|_2^m.
\]

Therefore \(\Sigma_d\) is exponentially \(m\)-th mean stable. \(\Box\)

Let us introduce the stochastic process \(\{\zeta(k)\}_{k=0}^\infty\) taking values in the set of standard unit vectors of \(\mathbb{R}^N\) and defined by

\[
\zeta(k) = e_{\sigma_k}.
\]

We will need the next technical lemma.

**Lemma 3.7.** Let \(k \geq 0\) and \(i, j \in \{1, \ldots, N\}\) be arbitrary. Assume \(P(\sigma_k = i) \neq 0\).

Define the probability space \((\Omega', \mathcal{M}', P')\) by

\[
\Omega' = \{\omega \in \Omega : \sigma_k = i\},
\]

\[
\mathcal{M}' = \{M' \subset \Omega' : M' \in \mathcal{M}\},
\]

\[
P'(M') = P(M')/P(\Omega').
\]

Then the random variables \(\zeta(k+1), F_k^{[m]}\) and \(x_d(k)^{[m]}\) are independent on \(\Omega'\).

**Proof.** See Appendix A. \(\Box\)

This lemma proves the next proposition, which plays the key role in the proof of Theorem 3.4.

**Proposition 3.8.** The matrix \(\mathcal{F}_m\) is nonnegative. Moreover, for every \(k \geq 0\),

\[
E \left[ \zeta(k+1) \otimes x_d(k+1)^{[m]} \right] = \mathcal{F}_m E \left[ \zeta(k) \otimes x_d(k)^{[m]} \right].
\]

**Proof.** The nonnegativity of \(\mathcal{F}_m\) is clear from D3. To show (3.5) let us fix arbitrary \(x_0\) and \(\sigma_0\). By the definition of Kronecker products,

\[
\zeta(k+1) \otimes x_d(k+1)^{[m]} = \begin{bmatrix} \zeta(k+1)_1 x_d(k+1)^{[m]} \\ \vdots \\ \zeta(k+1)_N x_d(k+1)^{[m]} \end{bmatrix}.
\]

Now we recall \(\sum_{i=1}^N \zeta(k)_i = 1\). Since the equation (1.4) gives \(x_d(k+1)^{[m]} = F_k^{[m]} x_d(k)^{[m]}\) we have \(x_d(k+1)^{[m]} = \sum_{i=1}^N \zeta(k)_i F_k^{[m]} x_d(k)^{[m]}\). Therefore, for a fixed \(j\),

\[
E \left[ \zeta(k+1)_j x_d(k+1)^{[m]} \right] = \sum_{i=1}^N E \left[ \zeta(k+1)_j \zeta(k)_i F_k^{[m]} x_d(k)^{[m]} \right].
\]

Let us temporally fix also \(i\) and assume that \(P(\sigma_k = i) \neq 0\). Let \((\Omega', \mathcal{M}', P')\) be the probability space defined by (3.4) and let \(E'[:\cdot]\) denote the expected value on \(\Omega'\). Notice that, for a random variable \(X\) on \(\Omega\), we have

\[
E[\zeta(k) X] = E[1_{\{\sigma_k = i\}} X] = E'[X] P(\sigma_k = i).
\]
Then, Lemma 3.7 yields that

\begin{equation}
E \left[ \zeta(k+1) \zeta(k) F_k^{[m]} x_d(k)^{[m]} \right] = P(\sigma_k = i) E' \left[ \zeta(k+1) \zeta(k) F_k^{[m]} x_d(k)^{[m]} \right] \\
= P(\sigma_k = i) E' \left[ \zeta(k+1) \right] E' \left[ x_d(k)^{[m]} \right]
\end{equation}

(3.8)

Since

\begin{equation}
E \left[ \zeta(k+1) \zeta(k) F_k^{[m]} \right] = E \left[ F_k^{[m]} \right] E \left[ \zeta(k) \right] E \left[ x_d(k)^{[m]} \right],
\end{equation}

by (3.8) we have

\begin{equation}
E \left[ \zeta(k+1) \zeta(k) F_k^{[m]} x_d(k)^{[m]} \right] = p_{ij} E \left[ F_k^{[m]} \right] E \left[ \zeta(k) \right] E \left[ x_d(k)^{[m]} \right] = [\mathcal{F}_m]_{ji} E \left[ \zeta(k) \right] E \left[ x_d(k)^{[m]} \right],
\end{equation}

where $[\mathcal{F}_m]_{ji}$ is defined by (3.3). Notice that this equation is valid even when $P(\sigma_k = i) = 0$ because, in this case, the left and right hand sides of the equation are both 0. Hence, by (3.7),

\begin{equation}
E[\zeta(k+1) x_d(k+1)^{[m]}] = [\mathcal{F}_m]_{j1} \cdots [\mathcal{F}_m]_{jN} E[\zeta(k) \otimes x_d(k)^{[m]}].
\end{equation}

This equation and (3.6) prove (3.5). □

Proof of Theorem 3.4. Let us show the cycle $[1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1]$.

[1 $\Rightarrow$ 2]: If $\Sigma_d$ is exponentially $m$-th mean stable then $\Sigma_d$ is clearly stochastically $m$-th mean stable.

[2 $\Rightarrow$ 3]: Suppose that $\Sigma_d$ is stochastically $m$-th mean stable. Assume $\rho(\mathcal{F}_m) \geq 1$ to derive a contradiction. Since $\mathcal{F}_m$ is nonnegative it has an eigenvector $v$ corresponding to the eigenvalue $\rho(\mathcal{F}_m)$ by Lemma 1.2. For this vector $v$, by Lemma 1.5 there exist $c_1, \ldots, c_\ell \in \mathbb{R}$, $\sigma_{0}^{(1)}, \ldots, \sigma_{0}^{(\ell)} \in \{1, \ldots, N\}$, and $x_{0}^{(1)}, \ldots, x_{0}^{(\ell)} \in \mathbb{R}^n$ such that $v = \sum_{i=1}^{\ell} c_i \sigma_{0}^{(i)} \otimes (x_{0}^{(i)})^{[m]}$. Let $x_d^{(i)}(k)$ be the solution of $\Sigma_d$ with the initial state $x_{0}^{(i)}$ and mode $\sigma_{0}^{(i)}$. Then, by (3.5),

\begin{equation}
\sum_{i=1}^{\ell} c_i E[\zeta^{(i)}(k) \otimes x_d^{(i)}(k)^{[m]}] = \mathcal{F}_m^k v = \rho(\mathcal{F}_m)^k v.
\end{equation}

Therefore, by (1.3),

\begin{equation}
\rho(\mathcal{F}_m)^k \|v\|_2 \leq \sum_{i=1}^{\ell} c_i E \left[ \|\zeta^{(i)}(k) \otimes x_d^{(i)}(k)^{[m]}\|_2 \right] = \sum_{i=1}^{\ell} c_i E \left[ \|x_d^{(i)}(k)\|_2^2 \right]
\end{equation}
because $\zeta^{(i)}(k)$ takes its value in the set of standard unit vectors. Thus, the stochastic $m$-th mean stability of $\Sigma_d$ shows that

$$
\sum_{k=0}^{\infty} \rho(F_m)^k \|v\|_2 \leq \sum_{i=1}^{\ell} c_i \sum_{k=0}^{\infty} E \left[ \|x_d^{(i)}(k)\|_2^m \right] < \infty.
$$

This gives a contradiction because $\rho(F_m) \geq 1$ and $v \neq 0$.

[3 $\Rightarrow$ 1]: Suppose $\rho(F_m) < 1$. Then we can take [22, Lemma 5.6.10] a norm $\|\|$ on $\mathbb{R}^{n \times N}$ such that

$$
\|F_m\| < 1.
$$

Let $x_0 \geq 0$ and $\sigma_0$ be arbitrary. Since $x_d(k)$ is nonnegative, (1.6) shows that

$$
(3.9) \quad E \left[ \|x_d(k)\|_m^m \right] \leq \|E[x_d(k)^m]\|_1.
$$

Since $\sum_{i=1}^{N} \zeta(k)_i = 1$ and the 1-norm is linear on a positive orthant,

$$
\|E[x_d(k)^m]\|_1 = \left\| E \left[ \sum_{i=1}^{m} \zeta(k)_i x_d(k)^m \right] \right\|_1
$$

$$
= \sum_{i=1}^{m} \left\| E[\zeta(k)_i x_d(k)^m] \right\|_1
$$

$$
= \left\| E[\zeta(k)_1 x_d(k)^m] \right\|_1
$$

$$
= \left\| E[\zeta(k)_N x_d(k)^m] \right\|_1
$$

$$
= \|E[\zeta(k) \otimes x_d(k)^m]\|_1
$$

$$
= \|F_m^k (\zeta_0 \otimes x_0)^m]\|_1. \quad \text{(by (3.5))}
$$

By the equivalence of the norms on a finite-dimensional vector space, there exist positive constants $C_1$ and $C_2$ such that

$$
\|F_m^k (\zeta_0 \otimes x_0)^m]\|_1 \leq C_1 \left\| F_m^k (\zeta_0 \otimes x_0)^m]\right\|_1
$$

$$
\leq C_1 \|F_m\|^k \|\zeta_0 \otimes x_0^m\|_1
$$

$$
\leq C_1 C_2 \|F_m\|^k \|\zeta_0 \otimes x_0^m\|_2
$$

$$
= C_1 C_2 \|F_m\|^k \|x_0^m\|_2. \quad \text{(by (1.3))}
$$

This inequality together with (3.9) and (3.10) proves the exponential convergence (3.1) because $\|F_m\| < 1$. Thus, since $x_0 \geq 0$ and $\sigma_0$ were arbitrary, Lemma 3.6 shows the exponential $m$-th mean stability of $\Sigma_d$. 

4. Proof of the Main Result. This section gives the proof of Theorem 2.5. We separately prove sufficiency and necessity. Let $\Sigma$ be a continuous-time positive semi-Markovian jump linear system satisfying Assumption 2.4 and let $\mathcal{D}\Sigma$ be its discretization defined by (2.10).
4.1. Proof of Sufficiency. Let us begin by checking that the discrete-time system $SS$ is a discrete-time positive semi-Markovian jump linear system.

**Lemma 4.1.** $SS$ is a discrete-time positive semi-Markovian jump linear system satisfying Assumption 3.3.

**Proof.** Let $\{\sigma_k\}_{k=0}^{\infty}, \{t_k\}_{k=0}^{\infty}$ and $\{J_k\}_{k=0}^{\infty}$ be the stochastic processes defining $\Sigma$ and define $F_k = J_ke^{A_kh_k}$. Let us show the first condition D1. Define the $\sigma$-algebras $M_1, M_2$, and $M_3$ on $\Omega$ by

$$
M_1 = M(\sigma_k),
M_2 = M(\sigma_k, \ldots, \sigma_0, F_{k-1}, \ldots, F_0),
M_3 = M(\sigma_k, \ldots, \sigma_0, t_k, \ldots, t_0, J_{k-1}, \ldots, J_0).
$$

Then one can see that

$$
M_1 \subset M_2 \subset M_3.
$$

The first part of this inclusion is obvious. To show the second part it is sufficient to show that the function $F_\ell$ is measurable on $M_3$ for every $\ell = 0, \ldots, k - 1$. Let us fix an $\ell$. Since both $\sigma_t$ and $h_t$ are measurable on $M_3$ so is $A_{\sigma_t}h_t$. Since the matrix exponential function on $\mathbb{R}^{n \times n}$ is continuous, the mapping $e^{A_{\sigma_t}h_t} : \Omega \to \mathbb{R}^{n \times n}$ is measurable on $M_3$. Thus the measurability of $J_\ell$ actually proves that $F_\ell$ is measurable on $M_3$, which completes the proof of $M_2 \subset M_3$. Now let $j \in \{1, \ldots, N\}$ and a Borel set $B \subset \mathbb{R}^{n \times n}$ be arbitrary and define $f = \chi_{\{\sigma_{k+1} = j, F_k \in B\}}$. Since C1 shows $E[f \vert M_3] = E[f \vert M_1]$, from Lemma 1.1 and (4.1) we can see $E[f \vert M_2] = E[f \vert M_1]$, which immediately proves D1.

The second condition D2 is obviously true by C2 because $F_k$ is a measurable function of $\sigma_k$, $h_k$, and $J_k$. D3 follows from the positivity condition C3. Finally Assumption 3.3 holds by the conditions 2 and 3 of Assumption 2.4. $\square$

The next corollary immediately follows from the definition (2.7) of the matrix $A_m(\Sigma)$, Theorem 3.4, and Lemma 4.1.

**Corollary 4.2.** The following statements are equivalent.

- $SS$ is exponentially $m$-th mean stable.
- $SS$ is stochastically $m$-th mean stable.
- $A_m(\Sigma)$ is Schur stable.

The next lemma helps us to relate the stability of $\Sigma$ and $SS$ and will be used repeatedly.

**Lemma 4.3.** There exist $0 < C_1 < C_2 < \infty$ such that, for every sample path $x$ of $\Sigma$ and $k \geq 0$,

$$
C_1 \|x(t_k)\| \leq \|x(t)\|, \quad t_k \leq t < t_{k+1}
$$

and

$$
\|x(t)\| \leq C_2 \|x(t_k)\|, \quad t_k \leq t \leq t_{k+1}.
$$

**Proof.** First let $t \in [t_k, t_{k+1})$ be arbitrary. Then there exist $h \in [0, T]$ and $i \in \{1, \ldots, N\}$ such that $x(t) = e^{A_i h}x(t_k)$ and therefore $x(t_k) = e^{-A_i h}x(t)$. Hence

$$
\|x(t_k)\| = e^{\|A_i\|h} \|x(t)\|
\leq e^{\max_{1 \leq i \leq N} \|A_i\|T} \|x(t)\|
$$
so that $C_1 := e^{-\max_{1 \leq i \leq N} \|A_i\| T}$ satisfies (4.2).

Then let $t \in [t_k, t_{k+1}]$. Then there exist $h \in [0, T]$, $i \in \{1, \ldots, N\}$, and $J \in \mathbb{R}^{n \times n}$ such that $x(t) = Je^{A_i h} x(t_k)$. By Assumption 2.4,

$$\|x(t)\| \leq \|J\| e^{\|A_i\| h} \|x(t_k)\| \leq Re^{\max_{1 \leq i \leq N} \|A_i\| T} \|x(t_k)\|.$$  

Thus the inequality (4.3) holds for $C_2 = Re^{\max_{1 \leq i \leq N} \|A_i\| T}$. □

Let us prove the sufficiency part of Theorem 2.5.

**Proof of the sufficiency part of Theorem 2.5.** Assume that $A_m(\Sigma)$ is Schur stable.

Then, by Corollary 4.2, $S\Sigma$ is exponentially $m$-th mean stable. We shall show that $\Sigma$ is exponentially $m$-th mean stable. To this end, for $t \geq 0$, define the random variable $k_t$ by

$$k_t(\omega) = \max\{k \in \mathbb{N} : t_k(\omega) \leq t\}.$$

Notice that (2.6) shows $t < t_{k+1} \leq T(k+1)$ and therefore

$$k_t > T^{-1} t - 1.$$

Let $x_0$, $\sigma_0$, and $t \geq 0$ be arbitrary. Let $x$ be the trajectory of $\Sigma$ and define $x_d(k) = x(t_k)$. Since $t_{k+1} \leq t < t_{k+1}$, the inequality (4.3) gives $\|x(t)\| \leq C_2 \|x(t_k)\| = C_2 \|x_d(k)\|$. Therefore

$$E[\|x(t)\|^m] \leq C_2 \int_{\Omega} \|x_d(k_t)\|^m dP$$

$$= C_2 \sum_{t>T^{-1} t-1} \int_{\{\omega : k_t = t\}} \|x_d(\ell)\|^m dP$$

$$\leq C_2 \sum_{t>T^{-1} t-1} E[\|x_d(\ell)\|^m]$$

$$\leq C_2 \sum_{t>T^{-1} t-1} Ce^{-\beta \ell} \|x_0\|^m$$

$$\leq \frac{CC_2 e^\beta}{1 - e^{-\beta}} e^{-\beta T^{-1} t} \|x_0\|^m.$$

Thus $\Sigma$ is exponentially $m$-th mean stable. □

**4.2. Proof of Necessity.** Then let us prove necessity for Theorem 2.5. For the proof we use a family of continuous-time semi-Markovian jump linear systems $\Sigma^{(\tau)} (\tau > 0)$ defined by

$$\Sigma^{(\tau)} : \begin{cases} \frac{dx}{dt} = A_{\sigma_k} x(t), \quad t_k \leq t < t_{k+1} \\ x(t_{k+1}) = J^{(\tau)}_k x(t_{k+1}^-), \quad k \geq 0 \end{cases}$$

where

$$J^{(\tau)}_k := \chi_{\{h_k \geq \tau\}} J_k$$

for each $k$. Roughly speaking, $\Sigma^{(\tau)}$ makes its state jump to 0 whenever it observes a dwell time $h_k$ less than $\tau$ (see Fig. 4.1).
The next proposition shows that the switched system $\mathcal{S}\Sigma^{(\tau)}$ inherits the stability of $\Sigma$.

**Proposition 4.4.** Let $\tau > 0$ be arbitrary. If $\Sigma$ is stochastically $m$-th mean stable then $\mathcal{S}\Sigma^{(\tau)}$ is also stochastically $m$-th mean stable.

**Proof.** Assume that $\Sigma$ is stochastically $m$-th mean stable. Let us show that $\mathcal{S}\Sigma^{(\tau)}$ is stochastically $m$-th mean stable. Let $x$ be a sample path of $\Sigma$ and define $x_d(k) = x(t_k)$. First assume that there exists $k \geq 0$ such that $h_k < \tau$. Let $k_0$ be the minimum of such $k$. Then $x_d(k) = 0$ for every $k > k_0$ by the definition of $J^{(\tau)}$. Therefore, by (4.3),

$$
\sum_{k=0}^{\infty} \|x_d(k)\|^m = \|x(t_{k_0})\|^m + \sum_{k=0}^{k_0-1} \|x(t_k)\|^m \\
\leq C_2^m \|x(t_{k_0-1})\|^m + \sum_{k=0}^{k_0-1} \|x(t_k)\|^m \\
\leq (C_2^m + 1) \sum_{k=0}^{k_0-1} \|x(t_k)\|^m.
$$

Since $h_k \geq \tau$ and therefore $1 \leq \tau^{-1} \int_{t_k}^{t_{k+1}} \|x(t)\|^m dt$ for every $k = 0, \ldots, k_0 - 1$, using (4.2) we can show that

$$
\sum_{k=0}^{k_0-1} \|x(t_k)\|^m \leq \tau^{-1} \sum_{k=0}^{k_0-1} \int_{t_k}^{t_{k+1}} \|x(t)\|^m dt \\
\leq \tau^{-1} C_1^{-m} \sum_{k=0}^{k_0-1} \int_{t_k}^{t_{k+1}} \|x(t)\|^m dt \\
\leq \tau^{-1} C_1^{-m} \int_0^{\infty} \|x(t)\|^m dt.
$$

Therefore

$$
\sum_{k=0}^{\infty} \|x_d(k)\|^m \leq (C_2^m + 1) \tau^{-1} C_1^{-m} \int_0^{\infty} \|x(t)\|^m dt.
$$

Next consider the case when $h_k \geq \tau$ for every $k$. In a similar way as (4.4) we can
show that
\[
\sum_{k=0}^{\infty} \|x_d(k)\|^m \leq \tau^{-1} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \|x_d(t)\|^m dt \\
\leq \tau^{-1} C_1^{-m} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \|x(t)\|^m dt \\
= \tau^{-1} C_1^{-m} \int_{0}^{\infty} \|x(t)\|^m dt.
\]

This inequality and (4.5) imply that there exists \( C > 0 \) such that, for every sample path \( x \) of \( \Sigma \), it holds that
\[
\sum_{k=0}^{\infty} \|x_d(k)\|^m \leq C \int_{0}^{\infty} \|x(t)\|^m dt.
\]

Since \( \Sigma \) is stochastically \( m \)-th mean stable, taking expectations in this inequality and using Fubini’s theorem show that \( \mathcal{S} \Sigma^{(r)} \) is stochastically \( m \)-th mean stable. \( \square \)

The next proposition shows the continuity of the matrix \( \mathcal{A}_m(\Sigma^{(r)}) \) at \( r = 0 \).

**Proposition 4.5.** It holds that
\[
\lim_{r \to 0} \mathcal{A}_m(\Sigma^{(r)}) = \mathcal{A}_m(\Sigma).
\]

**Proof.** By (2.5) the function \( \chi_{\{h_k(\omega) \geq r\}} \) on \( \Omega \) increasingly converges to the constant 1 point-wise as \( r \to 0 \) with probability 1. Hence, since each \( A_i \) is Metzler, the random variable \( (J_{k r}^{(r)} e^{A_{\sigma_k} h_k})^{[m]} = \chi_{\{h_k(\omega) \geq r\}} (J_k e^{A_{\sigma_k} h_k})^{[m]} \) increasingly converges to \( (J_k e^{A_{\sigma_k} h_k})^{[m]} \) as \( r \to 0 \) point-wise with probability 1. Then the conditional monotone convergence theorem (see, e.g., [5]) immediately shows (4.6). \( \square \)

Before showing the proof of the necessity part we need to introduce another semi-Markovian jump system given by
\[
\Sigma_\alpha : \begin{cases} \\
\frac{dx_\alpha}{dt} = (A_{\sigma_k} + \alpha I)x_\alpha(t), & t_k \leq t < t_{k+1} \\
x_\alpha(t_{k+1}) = J_k x_\alpha(t_k^{-}), & k \geq 0 \end{cases}
\]

for \( \alpha \in \mathbb{R} \). Notice that \( \Sigma_0 \) equals \( \Sigma \). About this system \( \Sigma_\alpha \) we will need the next proposition.

**Proposition 4.6.** If \( \Sigma \) is exponentially \( m \)-th mean stable then there exists \( \alpha > 0 \) such that \( \Sigma_\alpha \) is also exponentially \( m \)-th mean stable.

**Proof.** Assume that there exist \( C > 0 \) and \( \beta > 0 \) such that (2.3) holds. Let \( x_0 \) and \( \sigma_0 \) be arbitrary. Notice that, if \( x(\omega, \cdot) \) and \( x_\alpha(\omega, \cdot) \) denote the sample paths of \( \Sigma \) and \( \Sigma_\alpha \) for a fixed \( \omega \in \Omega \) with the common initial data \( x_0 \) and \( \sigma_0 \), respectively, then we have \( x_\alpha(\omega, t) = e^{\alpha t} x(\omega, t) \) because \( \Sigma \) and \( \Sigma_\alpha \) share the same underlying stochastic processes \( \{\sigma_k\}_{k=0}^\infty \), \( \{t_k\}_{k=0}^\infty \), and \( \{J_k\}_{k=0}^\infty \). Therefore we have \( E[\|x_\alpha(t)\|^m] = e^{\alpha t} E[\|x(t)\|^m] \leq C e^{(m-\beta)t} \|x_0\|^m \) for every \( t \geq 0 \). Hence \( \Sigma_\alpha \) is exponentially \( m \)-th mean stable if \( 0 < \alpha < \beta/m \). This completes the proof. \( \square \)

We will also use the next proposition.

**Proposition 4.7.** \( \rho(\mathcal{A}_m(\Sigma)) < \rho(\mathcal{A}_m(\Sigma_\alpha)) \) for every \( \alpha > 0 \).

**Proof.** Let \( i, j \in \{1, \ldots, N\} \) and \( r, s \in \{1, \ldots, n_m\} \) be arbitrary. Assume that
\[
[\mathcal{A}_m(\Sigma)]_{ij} = \int_{\Omega} [(J_k e^{A_{\sigma_k} h_k})^{[m]}]_{rs} dP_j > 0
\]
where $P_{ji}$ denotes the conditional probability distribution given by $P_{ji}(\cdot) = P(\cdot \mid \sigma_k = i, \sigma_{k+1} = j)$. By Lemma 1.3 it is sufficient to show that, for these $i$, $j$, $r$, and $s$, we have $[A_m(S_\alpha)]_{ij}rs > [A_m(S)]_{ij}rs$. Define $\Omega_\tau := \{\omega \in \Omega : h_k \geq \tau\}$. Since the family of the measurable sets $\{\Omega_\tau\}_{\tau > 0}$ covers $\Omega$ minus a null set by (2.5), there exists $\tau > 0$ such that $\int_{\Omega_\tau} [(J_k e^{A_j h_k})^{[m]}]_rs dP_{ji} > 0$. Therefore

$$
[A_m(S_\alpha)]_{ij}rs - [A_m(S)]_{ij}rs = \int_{\Omega} [(J_k e^{A_j h_k})^{[m]}]_rs (e^{\alpha h_k} - 1) dP_{ji} \\
\geq \int_{\Omega_\tau} [(J_k e^{A_j h_k})^{[m]}]_rs (e^{\alpha h_k} - 1) dP_{ji} \\
\geq (e^{\alpha \tau} - 1) \int_{\Omega_\tau} [(J_k e^{A_j h_k})^{[m]}]_rs dP_{ji} > 0,
$$

as desired. $\square$

Now we are at the position to prove the necessity part of Theorem 2.5. Notice that the two mappings $\Sigma \mapsto \Sigma^{(\tau)}$ and $\Sigma \mapsto \Sigma_\alpha$ obviously commute so that we can without confusion denote $\Sigma^{(\tau)}_\alpha = (\Sigma_\alpha)^{(\tau)}$ by $\Sigma^{(\tau)}_\alpha$.

**Proof of the necessity part of Theorem 2.5.** Assume that $\Sigma$ is exponentially $m$-th mean stable. Then, by Proposition 4.6, the system $\Sigma_\alpha$ is exponentially $m$-th mean stable for some $\alpha > 0$ as well. Since $\Sigma_\alpha$ is clearly stochastically $m$-th mean stable, by Proposition 4.4 the discrete-time system $SS_\alpha^{(\tau)}$ is stochastically $m$-th mean stable for every $\tau > 0$. Then Corollary 4.2 shows $\rho(A_m(S_\alpha^{(\tau)})) < 1$. Hence, by Proposition 4.7 and Proposition 4.5,

$$
\rho(A_m(S)) < \rho(A_m(S_\alpha)) = \lim_{\tau \to 0} \rho(A_m(S_\alpha^{(\tau)})) \\
\leq 1
$$

because the spectral radius is a continuous function of a matrix. $\square$

Finally let us remark that, by proving Theorem 2.5, we have actually shown the next corollary.

**Corollary 4.8.** $\Sigma$ is exponentially $m$-th mean stable if and only if $SS_\alpha$ is exponentially $m$-th mean stable.

5. **Continuous-time Positive Markovian Jump Linear Systems.** Though Theorem 2.5 gives the condition for the mean stability of positive semi-Markovian jump linear systems, the condition 2 in Assumption 2.4 excludes Markovian jump linear systems from the class of systems we can deal with. The aim of this section is to give a stability characterization of continuous-time Markovian jump linear systems.

Let us recall the definition of Markovian jump linear systems (see, e.g., [16]). Let $\{r(t)\}_{t \geq 0}$ be a continuous-time homogeneous Markov process taking values in the set $\{1, \ldots, N\}$. The transition probabilities of the process $r(t)$ is given by, for every $h > 0$,

$$
P(r(t + h) = j \mid r(t) = i) = \begin{cases} q_{ij} + o(h), & i \neq j, \\ 1 + q_{ii} + o(h), & i = j \end{cases}
$$

where $q_{ij}$ ($1 \leq i, j \leq N$) are constants such that $q_{ij} \geq 0$ if $i \neq j$ and $q_{ii} = -\sum_{j:j \neq i} q_{ij}$. The matrix $Q = [q_{ij}]_{ij}$ is called the infinitesimal generator of the process $\{r(t)\}_{t \geq 0}$.
Let \( A_1, \ldots, A_N \) be \( n \times n \) real matrices. Then the differential equation

\[
\Sigma : \frac{dx}{dt} = A_r(t)x(t)
\]

is said to be a Markovian jump linear system. We assume that \( x(0) = x_0 \in \mathbb{R}^n \) and \( r(0) = r_0 \in \{1, \ldots, N\} \) are constants. Notice that the semi-Markovian jump linear system (2.1) is a Markovian jump linear system if \( J_k = I \) and

\[
P(\sigma_{k+1} = j, h_k \leq t \mid \sigma_k = i) = \begin{cases} 
q_{ij}(1 - e^{-q_i t}), & (i \neq j) \\
0, & (i = j)
\end{cases}
\]

Finally we say that \( \Sigma \) is positive if the matrices \( A_1, \ldots, A_N \) are Metzler.

Since the condition 2 of Assumption 2.4 is not satisfied we cannot use Theorem 2.5 to check the stability of the Markovian jump linear system (5.1). The aim of this section is to give the next alternative characterization of the mean stability of positive Markovian jump linear systems.

**Theorem 5.1.** \( \Sigma \) is exponentially \( m \)-th mean stable if and only if the matrix

\[
T_m := Q^\top \otimes I_{n_m} + \text{diag}((A_1)[m], \ldots, (A_N)[m]) \in \mathbb{R}^{(n_m N \times n_m N)}
\]

is Hurwitz stable.

**Remark 5.2.** This theorem extends the well-known characterization [16] of the mean square stability of Markovian jump linear systems to positive Markovian jump linear systems.

The proof follows a similar way as that of Theorem 3.4. We introduce the stochastic process \( \{\delta(t)\}_{t \geq 0} \) that takes its value in the set of standard unit vectors in \( \mathbb{R}^N \) and is defined by

\[
\delta(t) = e_{r(t)}.
\]

Then \( x \) satisfies the differential equation

\[
dx = \left( \sum_{i=1}^{N} \delta_i A_i \right) x dt,
\]

which is a generalization of the system studied in [20]. Applying the operator \( (\cdot)^{[m]} \) to this equation yields, by (1.5),

\[
dx^{[m]} = \left( \sum_{i=1}^{N} \delta_i (A_i)[m] \right) x^{[m]} dt.
\]

We can derive a differential equation also for \( \delta \) as follows. Let \( \Pi_{ij} \) denote the Poisson process of the rate \( q_{ij} \) for every distinct pair \((i, j)\). Also define

\[
E_{ij} := e_i e_j^\top.
\]

Then it can be seen that [7]

\[
d\delta = \sum_{i,j: i \neq j} (E_{ji} - E_{ii}) \delta d\Pi_{ij}.
\]
Using the differential equations (5.3) and (5.5) we can prove the next proposition, which is similar to Proposition 3.8.

PROPOSITION 5.3. It holds that

\[ \frac{d}{dt} E[\delta \otimes x[m]] = T_m E[\delta \otimes x[m]]. \]

Proof. First let us consider the case \( m = 1 \). The Itô rule for jump processes [7] applied to the variable \( \delta \otimes x \) shows that, by (5.3) and (5.5),

\[ \frac{d}{dt} E[\delta \otimes x] = E \left[ \delta \otimes \sum_{i=1}^{N} \delta_i A_i x \right] + E \left[ \sum_{i,j \neq j} \left( (E_{ji} - E_{ii}) \delta q_{ij} \right) \otimes x \right]. \]

Since \( \delta_i \delta_j = 0 \) if \( i \neq j \) and \( \delta_i^2 = \delta_i \) for every \( i \), from the definition of Kronecker products it follows that

\[ \delta \otimes \sum_{i=1}^{N} \delta_i A_i x = \begin{bmatrix} \delta_1 A_1 x \\ \vdots \\ \delta_N A_N x \end{bmatrix} = \text{diag} (A_1, \ldots, A_N) (\delta \otimes x). \]

Moreover, since \( \left( (E_{ji} - E_{ii}) \delta q_{ij} \right) \otimes x = (q_{ij} (E_{ji} - E_{ii}) \otimes I_n) (\delta \otimes x) \) and

\[ \sum_{i,j \neq j} q_{ij} (E_{ji} - E_{ii}) = \sum_{i,j \neq j} q_{ij} E_{ji} + \sum_{i=1}^{N} \sum_{j \neq i} (-q_{ij}) E_{ii} \]

\[ = \sum_{i,j \neq j} q_{ij} E_{ji} + \sum_{i=1}^{N} q_{ii} E_{ii} \]

\[ = Q^{\top}, \]

the second term of the right hand side in (5.7) equals \( (Q^{\top} \otimes I_n) E[\delta \otimes x] \). This shows (5.6) for \( m = 1 \). For a general \( m \) notice that for \( y := x[m] \) we have \( dy = \left( \sum_{i=1}^{N} \delta_i (A_i)[m] \right) y dt \) by (5.4). Then we can proceed in the same way as above to prove \( \frac{d}{dt} E[\delta \otimes y] = T_m E[\delta \otimes y] \), which is (5.6). \( \square \)

Let us prove Theorem 5.1.

Proof of Theorem 5.1. Notice that, by (5.6),

\[ E[\delta(t) \otimes x(t)[m]] = e^{T_m t} (\delta_0 \otimes x_0[m]). \]

First assume that \( T_m \) is Hurwitz stable. Then there exist \( C > 0 \) and \( \beta > 0 \) such that, for every \( y \in \mathbb{R}^{N} \) and \( t \geq 0 \),

\[ \| e^{T_m t} y \|_1 \leq Ce^{-\beta t} \| y \|_2. \]

Let us show that \( \Sigma \) is exponentially \( m \)-th mean stable. Let \( x_0 \) and \( r_0 \) be arbitrary. As in Lemma 3.6 we can assume \( x_0 \geq 0 \) without loss of generality. Then, by the
equivalence of the norms on $\mathbb{R}^{n \times m}$, there exists $C_1 > 0$ such that

$$E[\|x(t)\|_2^m] = E[\|x(t)\|_2^m]$$

$$\leq C_1 E[\|x(t)\|_1^m]$$

$$= C_1 E[\|\delta(t) \otimes x(t)\|_1^m]$$

$$= C_1 E[\|\delta(t) \otimes x(t)\|_1^m]$$

by the linearity of $\|\cdot\|_1$ on a positive orthant. This inequality shows the exponential $m$-th mean stability of $\Sigma$ because (5.8) and (5.9) gives

$$\|E[\delta(t) \otimes x(t)\|_1^m]\|_1 \leq C e^{-\beta t} \|\delta_0 \otimes x_0\|_2^m.$$ 

On the other hand assume that $\Sigma$ is exponentially $m$-th mean stable. Let $C > 0$ and $\beta > 0$ be constants satisfying (2.3). By (5.8),

$$\|e^{T_m t}(\delta_0 \otimes x_0)\|_2^m \leq E[\|\delta(t) \otimes x(t)\|_2^m]$$

$$= E[\|x(t)\|_2^m]$$

$$\leq C e^{-\beta t} \|x_0\|_2^m$$

for every $x_0 \in \mathbb{R}^n$ and $\delta_0$. Therefore, by Lemma 1.5, $e^{T_m t}y$ converges to 0 for every $y \in \mathbb{R}^{n \times m}$, which proves that $T_m$ is Hurwitz stable. 

### 5.1. Output Feedback Stabilization

As an illustration of Theorem 5.1 this section studies the stabilization of positive Markovian jump linear systems in the first mean, following the setting in [15] where the authors study the stabilization of Markovian jump linear systems in the mean square. Consider the Markovian jump linear system with input and output defined by

$$\begin{align*}
\frac{dx}{dt} &= A_{r(t)}x(t) + B_{r(t)}u(t) \\
y &= C_{r(t)}x(t),
\end{align*}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, and $A_i$, $B_i$, and $C_i$ ($i = 1, \ldots, N$) are real matrices with appropriate dimensions. We study the stabilization of the system (5.10) via the output feedback

$$u(t) = K_{r(t)}y(t),$$

where $K_1, \ldots, K_N \in \mathbb{R}^{n_u \times n_y}$. Then the controlled system is described by

$$\Sigma_K : \frac{dx}{dt} = A_{K,r(t)}x(t), \quad A_{K,i} = A_i + B_i K_i C_i$$

which is again a Markovian jump linear system.

We consider the following problem. Let $T_1(\Sigma_K)$ denote the matrix $T_1$ given by (5.2) for the system $\Sigma_K$.

**Problem 5.4.** Assume that the matrices $A_1, \ldots, A_N$ are Metzler. Find feedback gain matrices $K_1, \ldots, K_N$ and a transition matrix $Q$ such that the controlled system $\Sigma_K$ is positive and $\eta(T_1(K))$ is minimized.
This problem aims to stabilize a given positive Markovian jump linear system in the first mean while keeping its positivity. Notice that not only the gain matrices but also the transition matrix can be designed. We can reduce the problem to an optimization without constraints as follows. For $A \in \mathbb{R}^{n \times n}$ define its distance from the set of $n \times n$ Metzler matrices $\mathcal{M}_n$ by $d(A,\mathcal{M}_n) := \sum_{i=1}^{n} \sum_{j \neq i} \max(-A_{ij},0)$. Clearly $A$ is Metzler if and only if $d(A,\mathcal{M}_n) = 0$. Then define

$$f(K_1, \ldots, K_N, Q) := \eta(T_1(\Sigma_K)) + \Gamma \left( \sum_{i=1}^{N} d(A_{K,i},\mathcal{M}_n) + \sum_{i \neq j} \max(-q_{ij},0) \right)$$

where $\Gamma > 0$ is a constant. When $\Gamma$ is sufficiently large, Problem 5.4 is almost equivalent to the minimization of $f$ with respect to the free parameters $K_1, \ldots, K_N \in \mathbb{R}^{n \times n}$ and $q_{ij} \in \mathbb{R}$ ($i \neq j$) because the second term of $f$ forces $\Sigma_K$ to be positive and $Q$ be an admissible transition matrix. One can solve this minimization problem using the gradient sampling algorithm proposed in [11], which has been applied to the design of low-order and fixed-order controllers successfully [10,11].

**Example 5.5.** Consider the Markovian jump linear system (5.10) with the two subsystems given by the triples

$$(A_1, B_1, C_1) = \left( \begin{bmatrix} 0 & 0.2 \\ 0.9 & 0.9 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}, \begin{bmatrix} 0.3 & 0.1 \end{bmatrix} \right),$$

$$(A_2, B_2, C_2) = \left( \begin{bmatrix} 0.1 & 0.4 \\ 0.6 & -0.3 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \begin{bmatrix} -0.8 & 1 \end{bmatrix} \right).$$

We set $x_0 = [1 \ 1]^T$ and $r_0 = 1$. We minimize the function (5.11) using a MATLAB implementation [9] of the gradient sampling algorithm [11] with $\Gamma = 10^5$ and obtain

$$K_1 = -3.3333, \quad K_2 = -2.0000, \quad Q = \begin{bmatrix} -2068.3 & 2068.3 \\ 3123.1 & -3123.1 \end{bmatrix}$$

that make $\Sigma_K$ positive and achieve $\eta(T_1(\Sigma_K)) = -0.1936$. A sample path of $\Sigma_K$ is shown in Fig. 5.1.

Though the above obtained parameters achieve stabilization in the first mean, the transition matrix yields rather fast switchings that may not be desirable in practice.
To achieve stabilization with slower switching rates let us consider the next modified objective function
\[ g(K_1, \ldots, K_N, Q) = f(K_1, \ldots, K_N, Q) + \Gamma \sum_{i \neq j} \max(q_{ij} - \bar{q}, 0) \]
where \( f \) is defined in (5.11) and \( \bar{q} > 0 \) is a constant. The second term of \( g \) aims to confine each off-diagonal entry of \( Q \) less than \( \bar{q} \). Minimizing this function with \( \Gamma = 10^5 \) and \( \bar{q} = 2 \) gives
\[ K_1 = -3.3308, \quad K_2 = -1.9998, \quad Q = \begin{bmatrix} -1.9997 & 1.9997 \\ 1.9817 & -1.9817 \end{bmatrix}. \]

With these parameters \( \Sigma_K \) is still positive and first mean stable as \( \eta(\mathcal{T}_1(\Sigma_K)) = -0.02251 \). A sample path of the stabilized system is shown in Fig. 5.2.

6. Conclusion. This paper studied the mean stability of positive semi-Markovian jump linear systems. We showed that the mean stability is determined by the spectral radius of an associated matrix that is easy to compute. For deriving the condition we used a discretization of a semi-Markovian jump linear system that preserves stability. Also we have given a characterization for the exponential mean stability of continuous-time positive Markovian jump linear systems. We illustrated the obtained results with numerical examples.

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Appendix A. Proof of Lemma 3.7.

Let $B_1 \subset \mathbb{R}^{n \times n}$ and $B_2 \subset \mathbb{R}^n$ be arbitrary Borel sets. We need to show

\[(A.1) \quad P'(\zeta(k+1), F_k \in B_1, x_d(k) \in B_2) = P'(\zeta(k+1), F_k \in B_1) \cdot P'(x_d(k) \in B_2).\]

The left-hand side of this equation can be computed as

\[(A.2) \quad P'(\zeta(k+1), F_k \in B_1, x_d(k) \in B_2) = \frac{1}{P(\Omega)} P(\zeta(k+1), F_k \in B_1, x_d(k) \in B_2, \sigma(k) = i) = P(\zeta(k+1), F_k \in B_1 | x_d(k) \in B_2, \sigma(k) = i) \cdot P(x_d(k) \in B_2 | \sigma(k) = i) = P(\zeta(k+1), F_k \in B_1 | \sigma(k) = i) \cdot P(x_d(k) \in B_2 | \sigma(k) = i),\]

where in the last equation we used the condition D1 of Assumption 3.1. Then it is easy to show that

\[(A.3) \quad P(\zeta(k+1), F_k \in B_1 | \sigma(k) = i) = \frac{1}{P(\Omega)} P(\zeta(k+1), F_k \in B_1, \sigma(k) = i) = P'(\zeta(k+1), F_k \in B_1).\]

In a similar way we can see that

\[(A.4) \quad P(x_d(k) \in B_2 | \sigma(k) = i) = P'(x_d(k) \in B_2).\]

The equations (A.2) to (A.4) prove (A.1). $\Box$