Infinite products of $2 \times 2$ matrices
and the Gibbs properties of Bernoulli convolutions

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Abstract.– We consider the infinite sequences $(A_n)_{n \in \mathbb{N}}$ of $2 \times 2$ matrices with nonnegative entries, where the $A_n$ are taken in a finite set of matrices. Given a vector $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $v_1, v_2 > 0$, we give a necessary and sufficient condition for $A_1 \ldots A_n V / \|A_1 \ldots A_n V\|$ to converge uniformly. In application we prove that the Bernoulli convolutions related to the numeration in Pisot quadratic bases are weak Gibbs.

Key-words: Infinite products of matrices, weak Gibbs measures, Bernoulli convolutions, Pisot numbers, $\beta$-numeration.

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Introduction

Let $\mathcal{M} = \{M_0, \ldots, M_{s-1}\}$ be a finite subset of the set – stable by matrix multiplication – of nonnegative and column-allowable $d \times d$ matrices (i.e., the matrices with nonnegative entries and without null column). We associate to any sequence $(\omega_n)_{n \in \mathbb{N}}$ with terms in $S := \{0, 1, \ldots, s-1\}$, the sequence of product matrices

$$P_n(\omega) = M_{\omega_1} M_{\omega_2} \ldots M_{\omega_n}.$$ 

Experimentally, in most cases each normalized column of $P_n(\omega)$ converges when $n \to \infty$ to a limit-vector, which depends on $\omega \in S^\mathbb{N}$ and may depend on the index of the column. Nevertheless the normalized rows of $P_n(\omega)$ in general do not converge: suppose for instance that all the matrices in $\mathcal{M}$ are positive but do not have the same positive normalized left-eigenvector, let $L_k$ such that $L_k M_k = \rho_k L_k$. For any positive matrix $M$, the normalized rows of $MM_0^n$ converge to $L_0$ and the ones of $MM_1^n$ to $L_1$. Consequently we can choose the sequence $(n_k)_{k \in \mathbb{N}}$ sufficiently increasing such that the normalized rows of $M_0^{n_1} M_1^{n_2} \ldots M_0^{n_{2k-1}}$ converge to $L_0$ while the ones of $M_0^{n_1} M_1^{n_2} \ldots M_0^{n_{2k-1}} M_1^{n_{2k}}$ converge to $L_1$. This proves – if $L_0 \neq L_1$ – that the normalized rows in $P_n(\omega)$ do not converge when $\omega = 0^{n_1}1^{n_2}0^{n_3}1^{n_4} \ldots$.
Now in case $M$ is a set of positive matrices it is clear that, if both normalized columns and normalized rows in $P_n(\omega)$ converge then – after replacing each matrix $M_k$ by $\frac{1}{\rho_k}M_k$ – the matrix $P_n(\omega)$ itself converges: the previous counterexample proves that the matrices $P_n(\omega)$ have a common left-eigenvector for any $n$, and a straightforward computation (using the limits of the normalized columns in $P_n(\omega)$) proves the existence of $\lim_{n \to \infty} P_n(\omega)$.

The existence of a common left-eigenvector is settled in a more general context by L. Elsner and S. Friedland ([5, Theorem 1]), in case $M$ is a finite set of matrices with entries in $\mathbb{C}$. This theorem means (after transposition of the matrices) that if $P_n(\omega)$ converges to a non-null limit, then there exists $N \in \mathbb{N}$ such that the matrices $M_{\omega_n}$ for $n \geq N$ have a common left-eigenvector for the eigenvalue 1. Now, L. Elsner & S. Friedland (in [5, Main Theorem]) and I. Daubechies & J. C. Lagarias (in [2, Theorem 5.1] (resp. [1, Theorem 4.2])) give necessary and sufficient conditions for $P_n(\omega)$ to converge for any $\omega \in \mathbb{S}_N$ (resp., to converge to a continuous map).

By these theorems we see that the problem of the convergence of the normalized columns in $P_n(\omega)$ is very different from the problem of the convergence of $P_n(\omega)$ itself. Let for instance $M_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$; then the normalized columns in

$$P_n(\omega) = \begin{cases} \frac{1}{10} \begin{pmatrix} 4 + 6 \cdot 6^{-n} & 6 - 6 \cdot 6^{-n} \\ 4 - 4 \cdot 6^{-n} & 6 + 4 \cdot 6^{-n} \end{pmatrix} & \text{if } \omega_1 \ldots \omega_n = 0 \ldots 0 \\ \frac{1}{10} \begin{pmatrix} 4 + 6^{-h} & 6 - 6^{-h} \\ 4 + 6^{-h} & 6 - 6^{-h} \end{pmatrix} & \text{if } \omega_1 \ldots \omega_n = \omega_1 \ldots \omega_{n-h-1}10 \ldots 0 \end{cases}$$

converge to $\left(\begin{array}{c} 1/2 \\ 1/2 \end{array}\right)$ for any $\omega \in \{0,1\}^N$, but $P_n(\omega)$ diverges (although it is bounded) if $\omega$ is not eventually constant.

In Section 1 we study the uniform convergence – in direction – of $P_n(\omega)V$ in case the $M_k$ are $2 \times 2$ nonnegative column-allowable matrices and $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ a positive vector (Theorem 1.1). Notice that the convergence in direction of the columns of $P_n(\omega)$, to a same vector, implies the ones of $P_n(\omega)V$, but the converse is not true: see for instance the case $M = \left\{ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right\}$.

The second section is devoted to the Bernoulli convolutions [4], which have been studied since the early 1930’s (see [8] for the other references). We give a matricial relation for such measures.
In the third section we apply more precisely Theorem 1.1 to prove that certain Bernoulli convolutions are weak Gibbs in the following sense (see [10]): given a system of affine contractions $S_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that the intervals $S_\varepsilon([0, 1])$ make a partition of $[0, 1]$ for $\varepsilon \in \mathcal{S} = \{0, 1, \ldots, s-1\}$, a measure $\eta$ supported by $[0, 1]$ is weak Gibbs w.r.t. $\{S_\varepsilon\}_{\varepsilon=0}^{s-1}$ if there exists a map $\Phi : S^N \to \mathbb{R}$, continuous for the product topology, such that
\[
\lim_{n \to \infty} \left( \frac{\eta[\xi_1 \ldots \xi_n]}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi)\right)} \right)^{1/n} = 1 \quad \text{uniformly on } \xi \in \mathcal{S}^N, \tag{1}
\]
where $[\xi_1 \ldots \xi_n] := S_{\xi_1} \circ \ldots \circ S_{\xi_n}([0, 1])$ and $\sigma$ is the shift on $\mathcal{S}^N$. Let us give a sufficient condition for $\eta$ to be weak Gibbs. For each $\xi \in \mathcal{S}^N$ we put $\phi_1(\xi) = \log \eta[\xi_1]$ and for $n \geq 2$, $\phi_n(\xi) = \log \left( \frac{\eta[\xi_1 \ldots \xi_n]}{\eta[\xi_2 \ldots \xi_n]} \right)$. \tag{2}

The continuous map $\phi_n : \mathcal{S}^N \to \mathbb{R}$ ($n \geq 1$) is the $n$-step potential of $\eta$. Assume the existence of the uniform limit $\Phi = \lim_{n \to \infty} \phi_n$; it is then straightforward that for $n \geq 1$,
\[
\frac{1}{K_n} \leq \frac{\eta[\xi_1 \ldots \xi_n]}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi)\right)} \leq K_n \quad \text{with } K_n = \exp\left(\sum_{k=1}^{n} \|\Phi - \phi_n\|_\infty\right). \tag{3}
\]

By a well known lemma on the Cesàro sums, $K_1, K_2, \ldots$ form a subexponential sequence of positive real numbers, that is $\lim_{n \to \infty} (K_n)^{1/n} = 1$ and thus, (3) means $\eta$ is weak Gibbs w.r.t. $\{S_\varepsilon\}_{\varepsilon=0}^{s-1}$.

Now the weak Gibbs property can be proved for certain Bernoulli convolutions by computing the $n$-step potential by means of products of matrices (see [6] for the Bernoulli convolution associated with the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$ – called the Erdős measure – and the application to the multifractal analysis). In Theorem 3.1 we generalize this result in case $\beta > 1$ is a quadratic number with conjugate $\beta' \in ]-1, 0[.$

1 Infinite product of $2 \times 2$ matrices

From now the vectors $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we consider are supposed to be nonnegative and column-allowable that is, $x_1, x_2, a, b, c, d$ are nonnegative.
and \( x_1 + x_2, a + c, b + d \) are positive. In particular we suppose that the matrices in \( \mathcal{M} = \{M_0, \ldots, M_{s-1}\} \) satisfy these conditions. We associate to \( X \) the normalized vector:

\[
N(X) := \left( \frac{x_1}{x_1 + x_2}, \frac{x_2}{x_1 + x_2} \right) = \left( \frac{n(X)}{1 - n(X)} \right)
\]

where \( n(X) := \frac{x_1}{x_1 + x_2} \)

and define the distance between the column of \( A \) (or the rows of \( tA \)):

\[
d_{\text{columns}}(A) := \left| n \left( \begin{pmatrix} a \\ c \end{pmatrix} \right) - n \left( \begin{pmatrix} b \\ d \end{pmatrix} \right) \right| = \frac{|\det A|}{(a + c)(b + d)} =: d_{\text{rows}}(tA).
\]

**Theorem 1.1** Given \( V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) with \( v_1, v_2 > 0 \), the sequence of vectors \( N(P_n(\omega)V) \) converges uniformly for \( \omega \in S^N \) only in the five following cases:

**Case 1:** \( \begin{pmatrix} a \\ b \\ 0 \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d; \quad \begin{pmatrix} a \\ 0 \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d; \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow bc' \geq b'c; \quad \text{no matrix in } \mathcal{M} \text{ has the form } \begin{pmatrix} a \\ 0 \\ d \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ b \\ c' \\ d' \end{pmatrix} \).

**Case 2:** \( \begin{pmatrix} a \\ b \\ 0 \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d; \quad \begin{pmatrix} a \\ 0 \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d; \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow bc' < b'c. \)

**Case 3:** \( \begin{pmatrix} a \\ b \\ 0 \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d; \quad \begin{pmatrix} a \\ 0 \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d; \quad \text{no matrix in } \mathcal{M} \text{ has the form } \begin{pmatrix} 0 \\ b \\ c \\ d \end{pmatrix} \).

**Case 4:** \( \begin{pmatrix} a \\ b \\ 0 \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d; \quad \begin{pmatrix} a \\ 0 \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d; \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathcal{M} \Rightarrow b'c \geq bc' \).
no matrix in $\mathcal{M}$ has the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$;

**Case 5:** $V$ is an eigenvector of all the matrices in $\mathcal{M}$.

**Corollary 1.2** If $\mathbb{N}(P_n(\cdot)V)$ converges uniformly on $S^N$, the limit do not depend on the positive vector $V$, except in the fifth case of Theorem 1.1.

**Proof.** Suppose that $\mathcal{M}$ satisfies the conditions of the case 1,2,3 or 4 in Theorem 1.1 and let $V, W$ be two positive vectors. Then the following set $\mathcal{M}'$ also do:

$$\mathcal{M}' := \mathcal{M} \cup \{ M_s \},$$

where $M_s$ is the matrix whose both columns are $W$.

Denoting by $\omega' = \omega_1 \ldots \omega_n$ the sequence defined by

$$\omega'_i = \begin{cases} 
\omega_i & \text{if } i \leq n \\
\mathbb{S} & \text{if } i > n 
\end{cases}$$

for any $\omega \in S^N$, we have

$$\mathbb{N}(P_n(\omega)V) - \mathbb{N}(P_n(\omega)W) = \mathbb{N}(P_n(\omega')V) - \mathbb{N}(P_{n+1}(\omega')V)$$

and this tends to 0, according to the uniform Cauchy property of the sequence $\mathbb{N}(P_n(\cdot)V)$. $\blacksquare$

Nevertheless, this limit may depend of $V$ if one assume only that $V$ is nonnegative. For instance, if $\mathcal{M} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ then $\lim_{n \to \infty} \mathbb{N} \left( P_n(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ differs from $\lim_{n \to \infty} \mathbb{N} \left( P_n(\omega') \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ iff $\omega = \mathbb{S}$ (implying the second limit is not uniform on $S^N$).

### 1.1 Geometric considerations

We follow the ideas of E. Seneta about products of nonnegative matrices in Section 3 of [9], or stochastic matrices in Section 4. In what follows we denote the matrices by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ or $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for $n \in \mathbb{N}$, and we suppose they are nonnegative and column-allowable. We define the coefficient

$$\tau(A) := \sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')}.$$  

The straightforward formula

$$d_{\text{columns}} \left( \prod_{k=1}^{n} (A_k) \right) \leq d_{\text{columns}}(A_1) \prod_{k=2}^{n} \tau(A_k)$$  \hspace{1cm} (4)

is of use to prove Theorem 1.1 because, according to the following proposition one has $\tau(A) < 1$ if $A$ is positive.
Proposition 1.3
\[
\tau(A) = \begin{cases} 
\sqrt{ad} - \sqrt{bc} & \text{if } A \text{ do not have any null row} \\
\sqrt{ad} + \sqrt{bc} & \text{otherwise.} 
\end{cases}
\]

Proof. \[
\frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')} = \frac{|\det A|}{(a + c/x)(b + d)} \left( \text{where } x = \frac{a' + c'}{b' + d'} \right) \text{ is maximal for } x = \sqrt{cd/ab}. \]

Remark 1.4 One can consider instead of \(d_{\text{columns}}\) the angle between the columns of \(A\):
\[
\alpha(A) := \left| \arctan \frac{a}{c} - \arctan \frac{b}{d} \right|,
\]
or the Hilbert distance between the columns of a positive matrix \(A\):
\[
d_{\text{Hilbert}}(A) := \left| \log \frac{a}{c} - \log \frac{b}{d} \right|.
\]
This last can be interpreted either as the distance between the columns or the rows of \(A\), because \(d_{\text{Hilbert}}(A) = d_{\text{Hilbert}}(A')\). The Birkhoff coefficient \(\tau_{\text{Birkhoff}}(A) := \sup_{d_{\text{Hilbert}}(A') \neq 0} \frac{d_{\text{Hilbert}}(A'A)}{d_{\text{Hilbert}}(A')}\) has – from Theorem (3.12) – the same value as \(\tau(A)\) in Proposition 1.3, and probably as a large class of coefficients defined in this way.

In the following proposition we list the properties of \(d_{\text{columns}}\) that are required for proving Theorem 1.1.

Proposition 1.5
(i) \[
\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(AA')}{d_{\text{columns}}(A')} = \frac{|\det A|}{\min((a + c)^2, (b + d)^2)} =: \tau_1(A).
\]
(ii) If \(A\) is positive then \[
\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{rows}}(AA')}{d_{\text{columns}}(A')} \leq \frac{|\det A|}{\min(a, b) \cdot \min(c, d)} =: \tau_2(A).
\]
(iii) If \(\lim_{n \to \infty} d_{\text{columns}}(A_n) = 0\) then \(\lim_{n \to \infty} d_{\text{columns}}(AA_nA') = 0\) and, assuming that \(A\) is positive, \(\lim_{n \to \infty} d_{\text{rows}}(AA_nA') = 0\).

(iv) Suppose the matrices \(A_n\) are upper-triangular. If \(\inf_{k \in \mathbb{N}} \frac{a_k}{d_k} \geq 1\) and \(\sum_{k \in \mathbb{N}} \frac{b_k}{d_k} = \infty\) then \[
\lim_{n \to \infty} d_{\text{columns}}(A_1 \ldots A_n) = \lim_{n \to \infty} d_{\text{columns}}(A_n \ldots A_1) = 0.
\]

(v) Suppose the matrices \(A_n\) are lower-triangular. If \(\inf_{k \in \mathbb{N}} \frac{d_k}{a_k} \geq 1\) and \(\sum_{k \in \mathbb{N}} \frac{c_k}{a_k} = \infty\) then \[
\lim_{n \to \infty} d_{\text{columns}}(A_1 \ldots A_n) = \lim_{n \to \infty} d_{\text{columns}}(A_n \ldots A_1) = 0.
\]
Proof. (i) and (ii) are obtained from the formula
\[ d_{\text{columns}}(AA') = \frac{\det A \cdot \det A'}{((a+c)a' + (b+d)c') \cdot ((a+c)b' + (b+d)d')} \]
and the relation \( d_{\text{rows}}(AA') = d_{\text{columns}}(t'A' tA) \).

(iii) is due to the fact that the inequalities of items (i), (ii) and (iv) imply \( d_{\text{columns}}(A_n A') \leq \tau_1(A)n(A_n A') \) and if \( A \) is positive \( d_{\text{rows}}(A_n A') \leq \tau_2(A)n(A_n A') \).

(iv) follows from the formula
\[ A_1 \ldots A_n = \begin{pmatrix} a_1 \ldots a_n & s_n \\ 0 & d_1 \ldots d_n \end{pmatrix}, \]
where \( s_n = \sum_{k=1}^{n} a_1 \ldots a_{k-1}b_k d_{k+1} \ldots d_n \geq d_1 \ldots d_n \sum_{k=1}^{n} \frac{b_k}{d_k} \).

(v) can be deduced from (iv) by using the relation
\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ 0 & a \end{pmatrix}. \]

We need also the following:

**Proposition 1.6** Let \( V_A \) be a nonnegative eigenvector associated to the maximal eigenvalue of \( A \), and \( C \) a cone of nonnegative vectors containing \( V_A \). If \( \det A \geq 0 \) then \( C \) is stable by left-multiplication by \( A \).

Proof. The discriminant of the characteristic polynomial of \( A \) is \((a-d)^2 + 4bc\). In case this discriminant is null the proof is obtained by direct computation, because \( A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \) or \( \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \). Otherwise \( A \) has two eigenvalues \( \lambda > \lambda' \) and, given a nonnegative vector \( X \), there exists a real \( \alpha \) and an eigenvector \( W_A \) (associated to \( \lambda' \)) such that
\[ X = \alpha V_A + W_A \quad \text{and} \quad AX = \lambda \alpha V_A + \lambda' W_A = \lambda' X + (\lambda - \lambda')\alpha V_A. \]

Notice that \( \alpha \geq 0 \) (because the nonnegative vector \( A^nX = \lambda^n \alpha V_A + \lambda'^nW_A \) converges in direction to \( \alpha V_A \)) and \( \lambda' \geq 0 \) (from the hypothesis \( \det A \geq 0 \)). Hence \( AX \) is a nonnegative linear combination of \( X \) and \( V_A \); if \( X \) belongs to \( C \) then \( AX \) also do. \( \blacksquare \)

1.2 How pointwise convergence implies uniform convergence

Let \( m \) and \( M \) be the bounds of \( n(P_n(\omega)V) \) for \( n \in \mathbb{N} \) and \( \omega \in \mathcal{S}_N \), and let \( M_V := \begin{pmatrix} m & M \\ 1-m & 1-M \end{pmatrix} \). Each real \( x \in [m, M] \) can be written \( x = mx_1 + Mx_2 \) with \( x_1, x_2 \geq 0 \)
and $x_1 + x_2 = 1$; in particular the real $x = n(P_n(\omega)V)$ can be written in this form, hence
\[ \forall \omega \in \mathcal{S}^N, \exists t_1, t_2 \geq 0, \quad P_n(\omega)V = MV \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \tag{5} \]

**Proposition 1.7** If $d_{\text{columns}}(P_n(\cdot)V)$ converges pointwise to 0 when $n \to \infty$, then $\mathbb{N}(P_n(\cdot)V)$ converges uniformly on $\mathcal{S}^N$.

**Proof.** Suppose the pointwise convergence holds. Given $\omega \in \mathcal{S}^N$ and $\varepsilon > 0$, there exists the integer $n = n(\omega, \varepsilon)$ such that $d_{\text{columns}}(P_n(\omega)V) \leq \varepsilon$. The family of cylinders $C(\omega, \varepsilon) := [\omega_1 \cdots \omega_n(\omega, \varepsilon)]$, for $\omega$ running over $\mathcal{S}^N$, is a covering of the compact $\mathcal{S}^N$; hence there exists a finite subset $X \subset \mathcal{S}^N$ such that $\mathcal{S}^N = \bigcup_{\omega \in X} C(\omega, \varepsilon)$. For any $\xi \in \mathcal{S}^N$, there exists $\zeta \in X$ such that $\xi \in C(\zeta, \varepsilon)$ that is, $\xi_k = \zeta_k$ for any $k \leq n = n(\zeta, \varepsilon)$. From (5) there exists two nonnegative vectors $V_p$ and $V_q$ such that $P_p(\xi)V = P_n(\zeta)V_p$ and $P_q(\xi)V = P_n(\zeta)V_q$. Denoting by $M(p, q)$ the column-allowable matrix whose columns are $V_p$ and $V_q$ we have – in view of (4)
\[ |n(P_p(\xi)V) - n(P_q(\xi)V)| = d_{\text{columns}}(P_n(\zeta)V_mM(p, q)) \leq d_{\text{columns}}(P_n(\zeta)V_m) \leq \varepsilon, \]
implying the uniform Cauchy property for $\mathbb{N}(P_n(\cdot)V)$. ■

### 1.3 Proof of the uniform convergence of $\mathbb{N}(P_n(\cdot)V)$

According to Proposition 1.7 it is sufficient to prove that $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)V) = 0$ for each $\omega \in \mathcal{S}^N$. This convergence is obvious in the following cases:

- If there exists $N$ such that $M_{\omega_N}$ has rank 1, then $P_n(\omega)V$ has rank 1 for $n \geq N$ and $\forall n \geq N$, $d_{\text{columns}}(P_n(\omega)V) = 0$.

- If there exists infinitely many integers $n$ such that $M_{\omega_n}$ is a positive matrix, one has $\tau(M_{\omega_n}) \leq \rho := \max_{M \in \mathcal{M}, M > 0} \tau(M) < 1$, and the formula (4) implies
\[ \lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)V) = 0. \]

- Similarly, this limit is null also in case there exists infinitely many integers $n$ such that $M_{\omega_n}M_{\omega_{n+1}}$ is a positive matrix.
So we can make from now the following hypotheses on the sequence $\omega$ under consideration:

(H): $\det M_{\omega_n} \neq 0$ for any $n \in \mathbb{N}$, and there exists $N$ such that the matrix $M_{\omega_n}M_{\omega_{n+1}}$ has at least one null entry for any $n > N$.

Proof in the case 1: Since the couples of matrices

$$\left( \begin{array}{cc} a & b \\ c & 0 \end{array} \right), \left( \begin{array}{cc} 0 & b' \\ c' & d' \end{array} \right) \in \mathcal{M}$$

satisfy $b/c \geq b'/c'$, there exists a real $\alpha$ such that

$$\forall \left( \begin{array}{cc} a & b \\ c & 0 \end{array} \right), \left( \begin{array}{cc} 0 & b' \\ c' & d' \end{array} \right) \in \mathcal{M}, \quad \frac{b}{c} \geq \alpha \geq \frac{b'}{c'}.$$  

Let $\Delta = \left( \begin{array}{cc} 0 & \alpha \\ 1 & 0 \end{array} \right)$. We denote by $P$ the set of $2 \times 2$ matrices with nonnegative determinant and by $\mathcal{M}$ the subset of $P$ defined as follows:

$$\mathcal{M} := \{ \Delta^{-1}M, M\Delta ; M \in \mathcal{M} \setminus P \} \cup \{ M, \Delta^{-1}M \Delta ; M \in \mathcal{M} \cap P \}.$$  

This set of matrices also satisfies the conditions mentioned in the case 1: for instance if

$$\left( \begin{array}{cc} a & b \\ c & 0 \end{array} \right) \in \mathcal{M},$$

the matrix $\Delta^{-1}\left( \begin{array}{cc} a & b \\ c & 0 \end{array} \right)$ satisfies $c \leq b/\alpha$, and so one.

For any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of elements of $\{0, 1\}$ such that $\varepsilon_0 = 0$ we can write

$$P_n(\omega) = M_{\omega_1}M_{\omega_2} \ldots M_{\omega_n}$$

$$= (\Delta^{-\varepsilon_0}M_{\omega_1}\Delta^{\varepsilon_1}) \cdot (\Delta^{-\varepsilon_1}M_{\omega_2}\Delta^{\varepsilon_2}) \cdot \ldots \cdot (\Delta^{-\varepsilon_{n-1}}M_{\omega_n}\Delta^{\varepsilon_n}) \cdot \Delta^{-\varepsilon_n} \quad (6)$$

where $A_n := \Delta^{-\varepsilon_{n-1}}M_{\omega_n}\Delta^{\varepsilon_n}$ for any $n \in \mathbb{N}$. By the following choice of the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, the matrices $A_n$ belong to $\tilde{\mathcal{M}}$:

$$\varepsilon_n = \begin{cases} \varepsilon_{n-1} & \text{if } \det M_{\omega_n} > 0 \\ 1 - \varepsilon_{n-1} & \text{otherwise.} \end{cases}$$

The hypotheses (H) imply that either all the matrices $A_n$ for $n > N$ are upper-triangular, or all of them are lower-triangular (otherwise $M_{\omega_n}M_{\omega_{n+1}} = \Delta^{\varepsilon_{n-1}}A_nA_{n+1}\Delta^{-\varepsilon_{n+1}}$ is positive for some $n > N$). By Proposition 1.5 (iv) and (v),

$$\lim_{n \to \infty} d_{\text{columns}}(A_{N+1} \ldots A_n) = 0.$$  

From (6) and Proposition 1.5 (iii), $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$.

Proof in the case 2: We use the matrix $\Delta$ and the set of matrices $\tilde{\mathcal{M}}$ defined in the previous case; here the real $\alpha$ is supposed such that $\frac{b}{c} \leq \alpha \leq \frac{b'}{c'}$ for any

$$\left( \begin{array}{cc} a & b \\ c & 0 \end{array} \right), \left( \begin{array}{cc} 0 & b' \\ c' & d' \end{array} \right) \in \mathcal{M},$$
and consequently $\tilde{M}$ satisfies the hypotheses of the case 2. This imply that each matrix in $\tilde{M}$ has a positive eigenvector. Let $C$ be the (minimal) cone containing $V$, $\Delta^{-1}V$ and the positive eigenvectors of the matrices in $\tilde{M}$. From (6) and Proposition 1.6 $P_n(\omega)V$ belongs to this cone for any $\omega \in S^N$ hence $M_V$ is positive.

Using again the relation (2) we have

$$d_{\text{columns}}(P_n(\omega)M_V) = d_{\text{rows}}(t^\top M_V \ t^\top \Delta^{-\varepsilon_n} t A_n \ldots t A_1).$$  \(7\)

Each matrix $t^\top A_n$ for $n > N$ satisfy $a > d$ if $t^\top A_n$ is upper-triangular, and $a < d$ if it is lower-triangular. By Proposition 1.5 (iv) and (v), $\lim_{n \to \infty} d_{\text{columns}}(t^\top A_n \ldots t A_{N+1}) = 0$. This implies $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ by applying Proposition 1.5 (iii) to the r.h.s. in (7).

Proof in the case 3: Let $C'$ be the (minimal) cone containing $V$, the nonnegative eigenvectors (associated to the maximal eigenvalues) of the matrices in $M \cap P$, and the column-vectors of the matrices in $M \setminus P$. All the vectors delimiting $C'$ are distinct from $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and Proposition 1.6 implies that $P_n(\omega)V \in C'$ for any $\omega \in S^N$. Hence $m$ and $M$ that is, the bounds of $n(P_n(\omega)V))$, are positive.

Suppose first that $M_{\omega_n}$ is lower-triangular for any $n \in \mathbb{N}$ and let $\begin{pmatrix} \alpha_n \\ 0 \\ \gamma_n \end{pmatrix} = P_n(\omega)$. The hypotheses of the case 3 imply $\lim_{n \to \infty} \frac{\delta_n}{\alpha_n} = 0$. A simple computation gives $d_{\text{columns}}(P_n(\omega)M_V) \leq \frac{\delta_n}{\alpha_n} \frac{M - m}{Mm}$ hence $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$. This conclusion remains valid if $M_{\omega_n}$ is eventually lower-triangular.

Suppose now $M_{\omega_n}$ is not lower-triangular for infinitely many $n$. The hypotheses mentioned in the case 3 and (H) imply that $M_{\omega_n}$ is upper-triangular for any $n > N$ (because for each $n$ such that $M_{\omega_n}$ is lower-triangular or has the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, (H) implies that $M_{\omega_{n+1}}$ is lower-triangular). Proposition 1.5 (iii) and(iv) implies that $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$.

Proof in the case 4: Let $\mathcal{M}'$ be the set of matrices $M'_k = \Delta^{-1}M_k \Delta$ for $k = 0, \ldots, s-1$, and let $V' = \Delta^{-1}V$ (here we can choose $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). The set $\mathcal{M}'$ satisfies the hypotheses of the case 3 hence $\lim_{n \to \infty} d_{\text{columns}}(P_n(\omega)V') = \lim_{n \to \infty} d_{\text{columns}}(\Delta M'_1 \ldots M'_\omega V') = 0$.  

10
1.4 Proof of the converse assertion in Theorem 1.1

Now we suppose the existence of the uniform limit $V(\cdot) := \lim_{n \to \infty} N(P_n(\cdot)V)$ and we want to check the conditions contained in one of the five cases involved in Theorem 1.1. Let $\mathcal{M}^2$ be the set of matrices $MM'$ for $M, M' \in \mathcal{M}$, and let $\mathcal{U}$ (resp. $\mathcal{L}$) be the set of upper-triangular (resp. lower-triangular) matrices $M \in \mathcal{M} \cup \mathcal{M}^2$.

We first prove that $\mathcal{U}$ cannot contain a couple of matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$ such that $a \geq d$ and $a' < d'$: suppose that $\mathcal{U}$ contain such matrices let, for simplicity, $M_0 = A$ and $M_1 = A'$. One has $V(0) = \lim_{n \to \infty} N(A^n V)$, and this limit is also the normalized nonnegative right-eigenvector of $A$ associated to its maximal eigenvalue, hence $V(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, $V(1)$ is colinear to $\begin{pmatrix} b' \\ d' - a' \end{pmatrix}$ (eigenvector of $A'$) hence distinct from $V(0)$. Moreover, for fixed $N \in \mathbb{N}$

$$V(1^N0) = \lim_{n \to \infty} N(A^{1N} A^n V) = \lim_{n \to \infty} N(A^{1N} N(A^n V)) = N(A^{1N} V(0)) = V(0).$$

Since $1^N0$ tends to $1$ when $N \to \infty$, the inequality $V(0) \neq V(1)$ contradicts the continuity of the map $V$. This proves that the couple of matrices $A, A' \in \mathcal{U}$ such that $a \geq d$ and $a' < d'$ do not exist. Similarly, the couple of matrices $A, A' \in \mathcal{L}$ such that $a \leq d$ and $a' > d'$ do not exist.

- Suppose that all the matrices in $\mathcal{U}$ satisfy $a \geq d$ and all the ones in $\mathcal{L}$ satisfy $a \leq d$. If $\mathcal{M}$ contains a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, it is necessarily an homothetic matrix. If it contains a matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, the square of this matrix is homothetic. So in both cases $\mathcal{M} \cup \mathcal{M}^2$ contains an homothetic matrix, let $H$. We use the same method as above: since the map $V$ is continuous, the vector $\lim_{n \to \infty} N(H^n V)$ must be equal to $\lim_{N \to \infty} \left( \lim_{n \to \infty} N(H^N M H^n V) \right)$ for any $M \in \mathcal{M}$. But the first is $N(V)$ and the second $N(MV)$, hence $V$ is an eigenvector of all the matrices in $\mathcal{M}$. Suppose now that $\mathcal{M}$ do not contain matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ nor $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$: all the conditions of the case 1 are satisfied.

- Suppose that all the matrices in $\mathcal{U}$ satisfy $a < d$ and all the ones in $\mathcal{L}$ satisfy $a > d$; then the conditions of the case 2 are satisfied.
• Suppose that all the matrices \( A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in U \) satisfy \( a \geq d \) and all the matrices \( A' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in L \) satisfy \( a' > d' \). If there exists \( A \in U, A' \in L \) and \( M = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{M} \), the map \( V \) is discontinuous because

\[
\lim_{n \to \infty} N \left( A'^n M A^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \lim_{n \to \infty} N \left( \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}^n \begin{pmatrix} \beta & 0 \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

differs from \( \lim_{n \to \infty} N \left( A'^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \) which is colinear to \( \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \). Hence, either \( \mathcal{M} \) do not contain a matrix of the form \( \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \) and we are in the case 3, or \( U = \emptyset \) and we are in the case 2, or \( L = \emptyset \) and we are in the case 1.

• The case when all the matrices \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in U \) satisfy \( a < d \) and all the matrices \( \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in L \) satisfy \( a' \leq d' \) is symmetrical to the previous, by using the set of matrices \( \mathcal{M}' := \{ \Delta^{-1} M \Delta : M \in \mathcal{M} \} \) and the vector \( V' = \Delta^{-1} V \), where \( \Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

2 Some properties of the Bernoulli convolutions in base \( \beta > 1 \)

Given a real \( \beta > 1 \), an integer \( d > \beta \) and a \( d \)-dimensional probability vector \( p := (p_i)_{i=0}^{d-1} \), the \( p \)-distributed \( (\beta, d) \)-Bernoulli convolution is by definition the probability distribution \( \mu_p \) of the random variable \( X \) defined by

\[
\forall \omega \in D^N := \{0, \ldots, d-1\}^N, X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k},
\]

where \( \omega \mapsto \omega_k \ (k = 1, 2, \cdots) \) is a sequence of i.i.d. random variables assuming the values \( i = 0, 1, \ldots, d-1 \) with probability \( p_i \).

Denoting by \( \overline{\omega} \) the sequence such that \( \overline{\omega}_k = d-1 - \omega_k \) for any \( k \), one has the relation \( X(\omega) + X(\overline{\omega}) = \alpha := \frac{d-1}{\beta-1} \). Hence, setting \( \overline{p}_i = p_{d-1-i} \) for any \( i = 0, 1, \ldots, d-1 \), the following symmetry relation holds for any Borel set \( B \subset \mathbb{R} \):

\[
\mu_p(B) = \mu_{\overline{p}}(\alpha - B) \quad (9)
\]

(notice that the support of \( \mu_p \) is a subset of \( [0, \alpha] \)).
The measure \( \mu_p \) also satisfy the following selfsimilarity relation: denoting by \( \sigma \) the shift on \( \mathcal{D}^\mathbb{N} \) one has – for any Borel set \( B \subset \mathbb{R} \)

\[
X(\omega) \in B \iff X(\sigma \omega) \in (\beta B - \omega_0)
\]
hence, using the independance of the random variables \( \omega \mapsto \omega_k \),

\[
\mu_p(B) = \sum_{k=0}^{d-1} p_k \cdot \mu_p(\beta B - k) \quad \text{for any Borel set } B \subset \mathbb{R} \tag{10}
\]
and in particular

\[
\mu_p(B) = p_0 \cdot \mu_p(\beta B) \quad \text{if } \beta B \subset [0, 1]. \tag{11}
\]

The following proposition is proved in [3, Theorem 2.1 and Proposition 5.4] in case the probability vector \( p \) is uniform:

**Proposition 2.1** The 1-periodic map \( H : ] - \infty, 0[ \to \mathbb{R} \) defined by

\[
H(x) = (p_0)^x \cdot \mu_p([0, \beta^x])
\]
is continuous and a.e. differentiable. Moreover \( H \) is not differentiable on a certain continuum of points if \( \beta \) is an irrational Pisot number or an integer and – in this latter case – if \( \beta \) do not divide \( d \).

Let us give also the matricial form of the relation (10) (from [7, §2.1]). We define the (finite or countable) set \( \mathcal{I}(\beta, d) = \{0 = i_0, i_1, \cdots\} \) as follows (where \( \mathcal{B} \) is the alphabet \( \{0, 1, \ldots, b - 1\} \) such that \( b - 1 < \beta \leq b \)):

**Definition 2.2** \( \mathcal{I}(\beta, d) \) is the set of \( i \in ] - 1, \alpha[ \) for which there exists \( -1 < i_1, \cdots, i_n < \alpha \) with \( 0 > i_1 > \cdots > i_n > i \), where \( x > y \) means that exists \( (\varepsilon, k) \in \mathcal{B} \times \mathcal{D} \) such that \( y = \beta x + (\varepsilon - k) \).

Let \( \varepsilon \in \mathcal{B} \); the entries of the matrix \( M_\varepsilon \) are – for the row index \( i \) and the column index \( j \), with \( i, j \in \mathcal{I}(\beta, d) \),

\[
M_\varepsilon(i, j) = \begin{cases} 
  p_k & \text{if } k = \varepsilon + \beta i_i - i_j \in \mathcal{D} \\
  0 & \text{otherwise.}
\end{cases}
\]

Setting \( R_\varepsilon(x) = \frac{x + \varepsilon}{\beta} \) for any \( \varepsilon \in \mathcal{B} \) and \( x \in \mathbb{R} \), we have the following
Proposition 2.3 ([7, Lemma 2.2]) If $\mathcal{I}_{(\beta,d)} = \{i_0, \cdots, i_{r-1}\}$ then, for any Borel set $B \subset [0,1]$ and any $\varepsilon \in \mathcal{B}$ such that $\mathbb{R}_\varepsilon^{-1}(B) \subset [0,1],$

\[
\begin{pmatrix}
\mu_p(B + i_0) \\
\vdots \\
\mu_p(B + i_{r-1})
\end{pmatrix} = M_\varepsilon
\begin{pmatrix}
\mu_p(\mathbb{R}_\varepsilon^{-1}(B) + i_0) \\
\vdots \\
\mu_p(\mathbb{R}_\varepsilon^{-1}(B) + i_{r-1})
\end{pmatrix}.
\]

Remark 2.4 The finiteness of $\mathcal{I}_{(\beta,d)}$ is assured, according to [7, §2.2], if $\beta$ is an irrational Pisot number or an integer.

We shall use also the probability distribution of the fractionnal part of the random variable $X$, that we denote by $\mu_p^*$. Suppose that $\alpha$ belongs to $]1,2[$, or equivalently that $\beta < d < 2\beta - 1$. Then $\mu_p^*$ – which has support $[0,1]$ – satisfy the following relation for any Borel set $B \subset [0,1]$: $\mu_p^*(B) = \mu_p(B) + \mu_p(B + 1)$

and, if $B \subset [\alpha - 1, 1]$,

\[\mu_p^*(B) = \mu_p(B). \tag{12}\]

The following proposition points out that in certain cases, the restriction of $\mu_p$ (or $\mu_p^*$) to the interval $[\alpha - 1, 1]$ is "representative" of $\mu_p$ itself.

Proposition 2.5 Suppose $\beta < d \leq \beta + 1 - \frac{1}{\beta}$.

(i) The interval $]0, \alpha[\subset I_k := \left[\frac{1}{\beta^{k+1}}, \frac{1}{\beta^k}\right]$ and $I'_k := \left[\alpha - \frac{1}{\beta^k}, \alpha - \frac{1}{\beta^{k+1}}\right]$ for $k \in \mathbb{N} \cup \{0\}$

(ii) Let $B \subset \mathbb{R}$ be a Borel set. If $B \subset I_k$ (or equivalently if $\alpha - B \subset I'_k$), then $\beta^k B$ and $\alpha - \beta^k B$ are two subsets of $[\alpha - 1, 1]$ such that

\[
\begin{align*}
\mu_p(B) &= p^k_0 \cdot \mu_p^*(\beta^k B) \\
\mu_p(\alpha - B) &= p^k_{d-1} \cdot \mu_p^*(\alpha - \beta^k B).
\end{align*}
\]

Proof. (i) The hypothesis on $d$ implies $\alpha < 2$ hence $]0, \alpha[\subset$ is the reunion of $]0,1]$ and $[\alpha - 1, \alpha].$

(ii) $B \subset I_k \Rightarrow \beta^k B \subset \left[\frac{1}{\beta}, 1\right] \subset [\alpha - 1, 1]$. The equality $\mu_p(B) = p^k_0 \cdot \mu_p^*(\beta^k B)$ results from (11) and (12).
Since $\beta^k B \subset [\alpha - 1, 1]$ one has $\alpha - \beta^k B \subset [\alpha - 1, 1]$. The equality $\mu_p(\alpha - B) = p_{d-1}^k \cdot \mu_p(\alpha - \beta^k B)$ follows from (9), (11) and (12).

3 Bernoulli convolution in Pisot quadratic bases

In this section $\beta > 1$ is solution of the equation $x^2 = ax + b$ (with integral $a$ and $b$), and we suppose that the other solution belongs to $]-1,0[$. This implies $1 \leq b \leq a \leq \beta - \frac{1}{\beta} < \beta < a + 1$. Let $p = (p_0, \ldots, p_a)$ be a positive probability vector; the Bernoulli convolution $\mu_p$ has support $[0, \alpha]$, where $\alpha = \frac{a}{\beta - 1}$ belongs to $]1, 2[$. The condition in Proposition 2.5 is satisfied hence it is sufficient to study the Gibbs properties of $\mu_p^*$ on its support $[0, 1]$, to get the local properties of $\mu_p$ on $[0, \alpha]$ (see [6] for the multifractal analysis of the weak Gibbs measures).

With the notations of the previous subsection one has $B = D = \{0, \ldots, a\}$, $I(\beta, a+1) = \{0, 1, \beta - a\}$ and – for any $\varepsilon \in B$

$$M_\varepsilon = \begin{pmatrix} p_\varepsilon & p_{\varepsilon - 1} & 0 \\ 0 & 0 & p_{a+\varepsilon} \\ p_{b+\varepsilon} & p_{b+\varepsilon - 1} & 0 \end{pmatrix},$$

where, by convention, $p_i = 0$ if $i \not\in D$.

Notice that the intervals $\mathbb{R}_\varepsilon([0, 1])$ do not make a partition of $[0, 1]$ for $\varepsilon \in B$ but, setting

$$S_\varepsilon := \begin{cases} \mathbb{R}_\varepsilon & \text{for } 0 \leq \varepsilon \leq a - 1 \\ \mathbb{R}_a \circ \mathbb{R}_{\varepsilon - a} & \text{for } a \leq \varepsilon \leq a + b - 1 \end{cases},$$

the intervals $S_\varepsilon([0, 1])$ make such a partition for $\varepsilon \in A := \{0, \ldots, a + b - 1\}$.

**THEOREM 3.1** The measure $\mu_p^*$ is weak Gibbs w.r.t. $\{S_\varepsilon\}_{\varepsilon=0}^{a+b-1}$ if and only if $p_0^2 \geq p_a p_{b-1}$ and $p_0^2 \geq p_{a-b+1} \geq p_a^2$.

**Proof.** The $n$-step potential of $\mu_p^*$ can be computed by means of the matrices

$$M_\varepsilon^* := \begin{cases} M_\varepsilon & \text{for } 0 \leq \varepsilon \leq a - 1 \\ M_a M_{\varepsilon - a} & \text{for } a \leq \varepsilon \leq a + b - 1. \end{cases}$$

Indeed by applying Proposition 2.3 to the sets $B = [\xi_1 \ldots \xi_n]$ and $B' = [\xi_2 \ldots \xi_n]$, one has

$$\exp(\phi_n(\xi)) = \log \begin{pmatrix} (1 & 1 & 0) M_{\xi_1}^* \ldots M_{\xi_n}^* V \\ (1 & 1 & 0) M_{\xi_2}^* \ldots M_{\xi_n}^* V \end{pmatrix},$$

where $V := \frac{\mu_p([0, 1])}{\mu_p([0, 1] + \beta - a)}$.

(13)
Now the matrices $M_\varepsilon^*$ are $3 \times 3$ and we shall use $2 \times 2$ ones. The matrices defined – for any $\varepsilon \in A' := \{0, \ldots, 2a\}$ – by

$$M_\varepsilon^* := \begin{cases} M_0M_\varepsilon & \text{if } \varepsilon \leq a \\ M_{\varepsilon-a} & \text{if } \varepsilon > a \end{cases}$$

satisfy the commutation relation $YM_\varepsilon^* = P_\varepsilon Y$, where

$$Y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_\varepsilon := \begin{cases} \begin{pmatrix} \varepsilon & \varepsilon-1 \\ \varepsilon & \varepsilon-1 \end{pmatrix} & \text{if } \varepsilon \leq a \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \varepsilon > a \end{cases}$$

Let $\xi \in A^N$ such that $\sigma \xi \neq \emptyset$. There exists an integer $k \geq 0$ and $\varepsilon \in A \setminus \{0\}$ such that

$$M_{\varepsilon_2} \cdots M_{\varepsilon_{k+2}} = M_0^kM_\varepsilon.$$ 

One can associate to the sequence $\xi$, the sequence $\zeta \in A^N$ such that

$$\forall n \geq k + 4, \exists k(n) \in \mathbb{N}, \quad M_{\varepsilon_{k+3}}^* \cdots M_{\varepsilon_{n}}^* = M_{\zeta_1}^* \cdots M_{\zeta_{k(n)}}^* \quad \text{or} \quad M_{\zeta_1}^* \cdots M_{\zeta_{k(n)}}^* M_0.$$

Now – according to (13) and the commutation relation

$$n \geq k + 4 \Rightarrow \exp(\phi_n(\xi)) = \frac{(1 \ 1 \ 0) M_{\zeta_1}^* M_0^k Q_\varepsilon N(P_{\zeta_1} \cdots P_{\zeta_{k(n)}} W)}{(1 \ 1 \ 0) M_0^k Q_\varepsilon N(P_{\zeta_1} \cdots P_{\zeta_{k(n)}} W)}$$

(14)

where $Q_\varepsilon := \begin{pmatrix} \varepsilon & \varepsilon-1 \\ 0 & 0 \\ \varepsilon & \varepsilon-1 \end{pmatrix}$ and $W = YV$ or $YM_0V$.

If $p_0 p_{a-b+1} \geq p_a^2$, the uniform convergence – on $A^{2N}$ – of the sequence $N(P_{\zeta_1} \cdots P_{\zeta_k} YV)$ and $N(P_{\zeta_1} \cdots P_{\zeta_k} YM_0V)$ to the same vector $V(\zeta)$ is insured by Theorem 1.[1] and Corollary 1.[2] When $n \to \infty$ the numerator in (14) converges to $V_1(\xi) := (1 \ 1 \ 0) M_{\zeta_1}^* M_0^k Q_\varepsilon V(\zeta)$, and the denominator to $V_2(\xi) := (1 \ 1 \ 0) M_0^k Q_\varepsilon V(\zeta)$; this convergence is uniform on each cylinder $[\varepsilon^0 k^\varepsilon]$. Since the first entry in $Q_\varepsilon V(\zeta)$ is at least $\min\{p_\varepsilon, p_{\varepsilon-1}\} > 0$, $V_1(\xi)$ and $V_2(\xi)$ are positive and consequently $\phi_n(\xi)$ converges uniformly to $\log \frac{V_1(\xi)}{V_2(\xi)}$. This is also true on any finite reunion of such cylinders; let us denote by $X(k_0)$ the reunion of $[\varepsilon^0 k^\varepsilon]$ for $k < k_0, \varepsilon \in A \setminus \{0\}$ and $\varepsilon' \in A$; then

$$\forall \eta > 0, \exists n_0 \in \mathbb{N}, \quad n \geq n_0 \quad \text{and} \quad \xi \in X(k_0) \Rightarrow \left| \phi_n(\xi) - \log \frac{V_1(\xi)}{V_2(\xi)} \right| \leq \eta.$$ 

(15)
We consider now a sequence \( \xi \in \mathcal{A}^N \setminus X(k_0) \). By using the left and right eigenvectors of \( M_0 \) – for the eigenvalue \( p_0 \) – we obtain

\[
\lim_{k \to \infty} A_k = \lambda_0 \begin{pmatrix} p_0^2 - p_ap_b & 0 & 0 \\ p_ap_b & 0 & 0 \\ p_b^2 & 0 & 0 \end{pmatrix}
\]

where \( \lambda_0 > 0 \), \( A_k := \begin{cases} p_0^{-k}M_0^k & \text{if } p_ap_b < p_0^2 \\ k^{-1}p_0^{-k}M_0^k & \text{if } p_ap_b = p_0^2. \end{cases} \)

The entries \( p_ap_b \) and \( p_b^2 \) being positive, there exists \( \lambda(\varepsilon') > 0 \) such that

\[
\lim_{k \to \infty} (1 \ 1 \ 0)M^*_\varepsilon A_k Q_\varepsilon = \lambda(\varepsilon')(p_\varepsilon \ p_{\varepsilon-1}).
\]

Moreover the convergence of \((1 \ 1 \ 0)M^*_\varepsilon A_k Q_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}\) to \( \lambda(\varepsilon')(p_\varepsilon x + p_{\varepsilon-1}y) \) is uniform on the set of normalized nonnegative column-vectors \( \begin{pmatrix} x \\ y \end{pmatrix} \). Similarly, there exists \( \lambda_1 > 0 \) such that \((1 \ 1 \ 0)A_k Q_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}\) converges uniformly to \( \lambda_1(p_\varepsilon x + p_{\varepsilon-1}y) \). Both limits are positive if \( \varepsilon \neq 0 \), implying that the ratio converges uniformly to \( \frac{\lambda(\varepsilon')}{\lambda_1} \). Hence, using (14) one can choose \( k_0 \) such that – if we assume \( \xi \in \mathcal{A}^N \setminus X(k_0) \) and \( \sigma \xi \neq 0 \)

\[
n \geq k_0 + 4 \Rightarrow \left| \phi_n(\xi) - \log \frac{\lambda(\varepsilon')}{\lambda_1} \right| \leq \eta. \tag{16}
\]

The uniform convergence of \( \phi_n(\xi) \) on \( \mathcal{A}^N \) follows from (15) and (16) since, in the remaining case \( \sigma \xi = 0 \) one has \( \lim_{n \to \infty} \phi_n(\xi) = 0 \).

Conversely, suppose \( p_0 p_{b-1} > p_0^2 \). If \( \mu_\varepsilon^* \) is weak Gibbs w.r.t. \( \{S_\varepsilon\}_{\varepsilon=0}^{\varepsilon-1} \) then, from (1) and (2) one has \( \phi_n(\xi) = o(n) \) for any \( \xi \in \mathcal{A}^N \). But this is not true: \( \phi_{2n+1}(10) \sim n \log \frac{p_0 p_{b-1}}{p_0^2} \).

Suppose now \( p_0 p_{a-b+1} < p_0^2 \). If \( b = 1 \) we have \( p_0 < p_a \) hence \( p_0 p_{b-1} > p_0^2 \); that is, we are in the previous case. If \( b \neq 1 \), \( \mu_\varepsilon^* \) is no more weak Gibbs w.r.t. \( \{S_\varepsilon\}_{\varepsilon=0}^{\varepsilon-1} \) because there exists a contradiction between the limit in (1) and the following:

\[
\lim_{n \to \infty} \left( \frac{\mu_\varepsilon^*[0(a-b+1)^n]}{\mu_\varepsilon^*[0(a-b+1)^n]} \cdot \mu_\varepsilon^*[1^n] \right)^{1/n} = \frac{p_0 p_{a-b+1}}{p_0^2} < 1.
\]

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