1 Introduction

The theory of groupoid C*-algebras [Re] has proved to be a very useful tool and provide a nearly universal context for the study of operator algebras, especially in connection with geometric structures. There have been many successful examples of such uses of groupoid C*-algebras [Cd, CuM, MRd, SaShU, Sh1].

Recently interesting and important examples of C*-algebras of quantum groups $G_q$ and quantum spaces $M_q$ arise from the quantization [Dr, RTF, So, Wo1, Wo2, Pt, Ri3, Ri2, VaSo, Sh2, Na2] of Poisson Lie groups $G$ and their Poisson homogeneous spaces $M = H \backslash G$ with $H$ a Poisson Lie subgroup of $G$. Known results [So, Sh2] show that they are closely related to the underlying singular symplectic foliation [We, LuWe1]. It is shown in [Sh4] that the C*-algebra of compact quantum groups can be effectively described and studied in the context of groupoid C*-algebras with the groupoid summarizing the underlying singular symplectic foliation. In fact, the structures of the algebras of quantum groups $SU(n)_q$ and quantum spheres $S^{2n-1}_q$ are analyzed in detail in [Sh4, Sh5].

More recently, nonstandard quantum complex projective spaces $\mathbb{CP}^n_{q,c}$ have been studied by Dijkhuizen and Noumi [DiNo]. In this paper, we show how the C*-algebras of quantum complex projective spaces (standard or nonstandard) are related to groupoids.

2 Quantum groups $SU(n)_q$

In this section, we briefly recall the notations and results needed from [Sh4]. For more complete information, we refer to the paper [Sh4].
First we recall that the transformation group groupoid \( \mathcal{T}_n := \mathbb{Z}^n \times \mathbb{Z}^n \) (with \( \mathbb{Z}^n \) acting on \( \mathbb{Z}^n \) by translation) when restricted to the positive cone \( \mathbb{Z}^n_\geq \) gives an important (Toeplitz) groupoid

\[
\mathcal{T}_n := \mathbb{Z}^n \times \mathbb{Z}^n \bigg|_{\mathbb{Z}^n_\geq} = \{(j, k) \in \mathbb{Z}^n \times \mathbb{Z}^n_\geq | j + k \in \mathbb{Z}^n_\geq\}
\]

where \( \mathbb{Z} = \mathbb{Z} \cup \{+\infty\} \) and \( \mathbb{Z}^n_\geq := \{0, 1, 2, 3, \ldots\} \cup \{+\infty\} \).

Recall that the C*-algebra \( C(SU(n+1)_q) \) is generated by \( u_{ij}, 1 \leq i, j \leq n+1 \), satisfying \( u^*u = uu^* = I \) and some other relations [Wo3, Sc]. In particular, the C*-algebra \( C(SU(2)_q) \) is generated by \( u_{ij}, 1 \leq i, j \leq 2 \), satisfying \( u_{22} = u_{11}^*, u_{12} = q^{-1}u_{21}^* \), and \( u^*u = uu^* = 1 \) [Wo1, VaSo]. An important irreducible (non-faithful) *-representation \( \pi_0 \) of \( C(SU(2)_q) \), \( q > 1 \), on \( \ell^2(\mathbb{Z}_q) \) is given by

\[
\pi_0(u) = \begin{pmatrix}
\alpha & -q^{-1} \gamma \\
\gamma & \alpha^*
\end{pmatrix}
\]

where \( \alpha(e_j) = (1 - q^{-2})^{1/2} e_{j-1} \) and \( \gamma(e_j) = q^{-j} e_j \) for \( j \geq 0 \). Here \( \pi_0 \) is applied to \( u = (u_{ij}) \) entrywise.

Irreducible one-dimensional *-representations of \( C(SU(n+1)_q) \) are defined by \( \tau_t(u_{ij}) = \delta_{ij} t_j \) for \( t \in \mathbb{T}^n \) (with \( t_{n+1} = t_1^{-1} t_2^{-1} \cdots t_n^{-1} \)). The \( \mathbb{T}^n \)-family \( \{\tau_t\}_{t \in \mathbb{T}^n} \) of one-dimensional irreducible *-representations of \( C(SU(n+1)_q) \) can be viewed as a C*-algebra homomorphism \( \tau_{n+1} = \tau : C(SU(n+1)_q) \to C(\mathbb{T}^n) \cong C^*(\mathbb{Z}^n) \). There are \( n \) fundamental *-representations \( \pi_i = \pi_0 \phi_i \) with \( \phi_i : C(SU(n+1)_q) \to C(SU(2)_q) \) given by \( \phi_i(u_{jk}) = u_{j-i+1,k-i+1} \) if \( \{j, k\} \subseteq \{i, i+1\} \) and \( \phi_i(u_{jk}) = \delta_{jk} \) if otherwise.

The unique maximal element in the Weyl group of \( SU(n+1) \) can be expressed in the reduced form

\[
s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n s_{n-1} \cdots s_2 s_1.
\]

So by Soibelman’s classification of irreducible *-representations [Sc], \( C(SU(n+1)_q) \) can be embedded into

\[
C^* (\mathcal{G}^n) \subseteq C^* (\mathbb{Z}^n) \otimes \mathcal{B} (\ell^2 (\mathbb{Z}_q^N))
\]

by

\[
(\tau_{n+1} \otimes \pi_{121321 \cdots n(n-1)21}) \Delta^N
\]

where \( N = n(n+1)/2 \),

\[
\pi_{121321 \cdots i_m} := \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_m},
\]

\( \Delta \) is the comultiplication on \( C(SU(n+1)_q) \) with \( \Delta^m \) defined recursively as \( \Delta^k = (\Delta \otimes \mathrm{id}) \Delta^{k-1} \), and \( \mathcal{G}^n \) is the groupoid \( \mathbb{Z}^n \times \mathbb{Z}^N \times \mathbb{Z}_q^N \) with \( \mathbb{Z}^n \) acting trivially on \( \mathbb{Z}^N \), and \( \mathbb{Z}^N \) acts by translation on \( \mathbb{Z}_q^N \).
3 Quantum spheres $S^2_{q}$

Recall that the C*-algebra of the quantum sphere $S^2_{q} = SU(n)q \setminus SU(n+1)q$ (as a homogeneous quantum space) is defined by

$$C(S^2_{q}) = \{ f \in C(SU(n+1)q) : (\Phi \otimes id)(\Delta f) = 1 \otimes f \}$$

where $\Phi : C(SU(n+1)q) \to C(SU(n)q)$ is the quotient map defined by $\Phi(u_{ij}) = u_{ij}$, $\Phi(u_{n+1,k}) = \Phi(u_{k,n+1}) = 0$, and $\Phi(u_{n+1,n+1}) = 1$, for $1 \leq i,j,k \leq n$. From

$$u_{n+1,m} = (h_n \otimes 1)(1 \otimes u_{n+1,m}) = (h_n \otimes 1)(\Phi \otimes 1)(\Delta(u_{n+1,m})),$$

we have

$$u_{n+1,m} \in (h_n \otimes 1)(\Phi \otimes 1)\Delta(C(SU(n+1)q)) = C(S^2_{q})$$

by Nagy’s result [Na1], where $h_n$ is the Haar functional on $C(SU(n)q)$. [Wo2] [Sh4] such that $h_n(1) = 1$.

As discussed in [Sh4], we can prove that $C(S^2_{q})$ is generated by $u_{n+1,m}$’s, by verifying that the monomials

$$p^{i,j,k} = (u^{*}_{n+1,1})^{i_1} \ldots (u^{*}_{n+1,1})^{i_n} y^{j_1}_{n+1} \ldots y^{j_{n+1}}_{n+1}(u_{n+1,1})^{k_1} \ldots (u_{n+1,n})^{k_n}$$

are linearly independent, where $y_m = u_{n+1,m}u_{n+1,m}$ and $i_m,j_m,k_m \geq 0$ with $i_m,k_m = 0$ (this last condition was accidentally missing in [Sh4]) for $1 \leq m \leq n$.

So we have

$$C(S^2_{q}) = C^*(\{u_{n+1,m} | 1 \leq m \leq n+1\}) \subset C(SU(n+1)q)$$

(used as the definition of $S^2_{q}$ in [VaSo]) which can be embedded into the groupoid C*-algebra $C^*(F^n)$, where $F^n$ is the augmented $n$-dimensional Toeplitz groupoid $\mathbb{Z} \times T_n = \mathbb{Z} \times (\mathbb{Z}^n \times \mathbb{Z}^n)$, through the faithful representation

$$\pi_{n+1} \otimes \pi_{n+1,1, \ldots, 21} \Delta^n$$

where $\pi_{k} \otimes \pi_{1, \ldots, 1, \ldots, 21} \otimes 1 \in C(T) \cong C^*(\mathbb{Z})$ for $1 \leq k \leq n+1$.

It turns out that $C(S^2_{q})$ is actually isomorphic to the C*-algebra of a subquotient groupoid of $F^n$ [Sh5]. In fact, let

$$\tilde{\mathcal{G}}_n := \{(z, x, w) \in F^n | w_i = \infty \implies x_i = -z - x_1 - x_2 - \ldots - x_{i-1} \text{ and } x_{i+1} = \ldots = x_n = 0\}$$

be a subgroupoid of $F^n$. Define $\mathcal{G}_n := \tilde{\mathcal{G}}_n / \sim$ where $\sim$ is the equivalence relation generated by

$$(z, x, w) \sim (z, x, w_1, \ldots, w_i = \infty, \infty, \ldots, \infty)$$

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for all \((z, x, w)\) with \(w_i = \infty\) for an \(1 \leq i \leq n\). Then we have \(C(S^2_{q}^{2n+1}) \simeq C^*(\tilde{\mathfrak{F}}_n)\).

Since \(SU(n)\) is a Poisson Lie subgroup of \(SU(n+1)\), the multiplicative Poisson structure on \(SU(n+1)\) induces a natural covariant Poisson structure on the homogeneous space \(S^2_{q}^{2n+1} = SU(n) \backslash SU(n+1)\), i.e. the \(SU(n+1)\)-action

\[ S^2_{q}^{2n+1} \times SU(n+1) \to S^2_{q}^{2n+1} \]
on \(S^2_{q}^{2n+1}\) is a Poisson map \(\text{[LuWe1]}\). Endowed with this covariant Poisson structure, the sphere \(S^2_{q}^{2n+1}\) induces a natural covariant Poisson structure on the homogeneous space \(S^2_{q}^{2n+1} = SU(n) \backslash SU(n+1)\), i.e. the \(SU(n+1)\)-action \(S^2_{q}^{2n+1} \times SU(n+1) \to S^2_{q}^{2n+1}\) is a Poisson map \(\text{[LuWe1]}\). Endowed with this covariant Poisson structure, the sphere \(S^2_{q}^{2n+1}\) decomposes into symplectic leaves \(\text{[We]}\) which are the symplectic leaves in the canonically embedded Poisson \(S^2_{q}^{2n-1}\) plus a circle family of leaves symplectically isomorphic to the complex space \(C^n\). Analyzing the above groupoid structure, we can easily see that this Poisson foliation structure is reflected on the quantum algebras, namely, there is a short exact sequence of \(C^*\)-algebras

\[ 0 \to C(\mathbb{T}) \otimes K\left(\ell^2\left(\mathbb{Z}^n_{\geq 1}\right)\right) \to C(S^2_{q}^{2n+1}) \to C(S^2_{q}^{2n-1}) \to 0. \]

In fact, \(\tilde{\mathfrak{F}}_{n-1}\) can be identified with

\[ \left\{ ((z, x, w)) : (z, x, w) \in \tilde{\mathfrak{F}}_n \text{ and } w_n = \infty \right\}, \]
i.e. \(\mathfrak{F}_n\) restricted to the invariant closed subset \(X\) consisting of \([w] \in Z\) with \(w_n = \infty\) in the unit space \(Z\) of \(\mathfrak{F}_n\), through the groupoid monomorphism sending \((z, x', w') \in \tilde{\mathfrak{F}}_{n-1}\) to

\[ (z, x', -z - x_1' - x_2' - \ldots - x_{n-1}', w', \infty) \in \tilde{\mathfrak{F}}_n, \]
and \(\tilde{\mathfrak{F}}_n \backslash \tilde{\mathfrak{F}}_{n-1} = \tilde{\mathfrak{F}}_n|Z \backslash X\) (the restriction of \(\tilde{\mathfrak{F}}_n\) to the invariant open set \(Z \backslash X\)) is isomorphic to the groupoid \(\mathbb{Z}^1 \times \left(\mathbb{Z}^n \times \mathbb{Z}^n_{\geq 1}\right)\). It is well known \(\text{[Re]}\) that there is a short exact sequence of groupoid \(C^*\)-algebras

\[ 0 \to C^*(\tilde{\mathfrak{F}}_n \backslash \tilde{\mathfrak{F}}_{n-1}) \to C^*(\tilde{\mathfrak{F}}_n) \to C^*(\tilde{\mathfrak{F}}_{n-1}) \to 0 \]
which leads to the above short exact sequence for the quantum spheres since

\[ C^*(\tilde{\mathfrak{F}}_n \backslash \tilde{\mathfrak{F}}_{n-1}) = C^*(\mathbb{Z}^1 \times \left(\mathbb{Z}^n \times \mathbb{Z}^n_{\geq 1}\right)) \cong C(\mathbb{T}) \otimes K\left(\ell^2\left(\mathbb{Z}^2_{\geq 1}\right)\right). \]

4 Standard quantum projective spaces \(CP^n_q\)

The \(C^*\)-algebra of the standard quantum complex projective space

\[ CP^n_q := U(n)_q \backslash SU(n+1)_q, \]
corresponding to the Poisson Lie subgroup \(U(n)\) of \(SU(n+1)\), can be identified with

\[ C(CP^n_q) = C^*\left(\{u^*_{n+1,i} u_{n+1,j} \mid 1 \leq i, j \leq n + 1\}\right) \subset C(S^2_{q}^{2n+1}). \]
In fact, as a quantum homogeneous space, $\mathbb{C}P^n_q$ is defined by

$$C(\mathbb{C}P^n_q) = \{ f \in C(SU(n+1)_q) : (\Phi' \otimes id)(\Delta f) = 1 \otimes f \}$$

where $\Phi' : C(SU(n+1)_q) \to C(U(n)_q)$ is the quotient map defined by $\Phi(u_{ij}) = y_{ij}$, $\Phi(u_{n+1,k}) = \Phi(u_{k,n+1}) = 0$, and

$$\Phi(u_{n+1,n+1}) = t_{n+1} = (t_1 \cdots t_n)^{-1} \in C \left( U(n)_q \right) \subset C^* \left( \mathbb{Z}^n \times \mathcal{T}(n-1)/2 \right)$$

for $1 \leq i, j, k \leq n$. By Nagy’s result \cite{Na}, we have

$$(h'_n \otimes 1)(\Phi' \otimes 1)\Delta(C(SU(n+1)_q)) = C(\mathbb{C}P^n_q)$$

where $h'_n$ is the Haar functional on $C(U(n)_q)$ \cite{Wo2} such that $h'_n(1) = 1$. From

$$(h'_n \otimes 1)(\Phi' \otimes 1)(\Delta(u_{n+1,i}u_{n+1,j})) = (h'_n \otimes 1)(t_{n+1}t_{n+1} \otimes u_{n+1,i}u_{n+1,j})$$

$$= (h'_n \otimes 1)(1 \otimes u_{n+1,i}u_{n+1,j}) = u_{n+1,i}u_{n+1,j},$$

we get $u_{n+1,i}u_{n+1,j} \in C(\mathbb{C}P^n_q)$ for $1 \leq i, j \leq n+1$. On the other hand, similar to the discussion in \cite{Sh}, we can prove that $C(\mathbb{C}P^n_q)$ is generated by $u_{n+1,i}u_{n+1,j}$, by verifying that the monomials

$$P^{r,i,j,m} = z_{i1}^{r_1} \cdots z_{in}^{r_n} z_{i1} \cdots z_{im}^{r_m}$$

(i) are linearly independent in the quantum case (i.e. when $q > 1$) and (ii) linearly span the *-algebra generated by $u_{n+1,i}u_{n+1,j}$, or the *-subalgebra of $\mathbb{T}$-invariant polynomials in $u_{n+1,i}$’s and $u_{n+1,j}$’s, in the classical case (i.e. when $q = 1$), where $m, r_1, \ldots, r_n \geq 0$, $z_{i,j} = u_{n+1,i}u_{n+1,j}$, and the two sets of indices $i_1 \geq i_2 \geq \ldots \geq i_m$ and $j_1 \leq j_2 \leq \ldots \leq j_m$ are disjoint subsets of $\{1, 2, \ldots, n+1\}$. Note that if we allow all $u_{n+1,i}$’s and $u_{n+1,j}$’s in $P^{r,i,j,m}$ to commute, then we can rewrite it, by simply permuting such factors, in the form

$$p^{r,i,j,k} = (u_{n+1,n})^{i_1} \cdots (u_{n+1,1})^{i_m} y_{1}^{j_1} \cdots y_{n}^{j_m} y_{n+1}^{j_{n+1}} (u_{n+1,1})^{k_1} \cdots (u_{n+1,n})^{k_n}$$

where $y_{m} = u_{n+1,m}u_{n+1,n}$ and $i_m, j_m, k_m \geq 0$ with $i_mk_m = 0$ for $1 \leq m \leq n$, $y_{n+1}^{j_{n+1}} = (u_{n+1,n+1})^{j_{n+1}}$ if $j_{n+1} \geq 0$, and $y_{n+1}^{j_{n+1}} = (u_{n+1,n+1})^{-j_{n+1}}$ if $j_{n+1} \leq 0$. So condition (ii) can be verified easily by integrating $\mathbb{T}$-action on polynomials $p^{r,i,j,k}$ which form a linear basis of polynomials in $u_{n+1,i}$’s and $u_{n+1,j}$’s on $\mathbb{S}^{2n+1}$. On the other hand, condition (i) can be verified by considering the weight functions and the orders of the weighted shifts of

$$(\tau'_{n+1} \otimes \pi_{n(n-1),21}) \Delta^n (P^{r,i,j,m})$$

which can be determined by

$$(\tau'_{n+1} \otimes \pi_{n(n-1),21}) \Delta^n (u_{n+1,i}) = \text{id}_T \otimes \gamma \otimes \ldots \otimes \gamma \otimes \alpha^* \otimes 1 \otimes \ldots \otimes 1.$$
Under the faithful \(*\)-representation $(\tau'_{n+1} \otimes \pi_{n(n-1)\ldots 21}) \Delta^n$, we have

$$(\tau'_{n+1} \otimes \pi_{n(n-1)\ldots 21}) \Delta^n \left( u^*_{n+1,i} u_{n+1,j} \right)$$

$$= \left[ (\tau'_{n+1} \otimes \pi_{n(n-1)\ldots 21}) \Delta^{n-1} (\Delta u_{n+1,i} \otimes \pi_{n(n-1)\ldots 21}) (\Delta u_{n+1,j}) \right]^*$$

$$= (\text{id}_T \otimes \pi_{n(n-1)\ldots 21} \Delta^{n-1} u_{n+1,i})^* (\text{id}_T \otimes \pi_{n(n-1)\ldots 21} \Delta^{n-1} u_{n+1,j})$$

$$= 1 \otimes (\pi_{n(n-1)\ldots 21} \Delta^{n-1} u_{n+1,i})^* (\pi_{n(n-1)\ldots 21} \Delta^{n-1} u_{n+1,j})$$

$$= 1 \otimes \pi_{n(n-1)\ldots 21} \Delta^{n-1} (u^*_{n+1,i} u_{n+1,j}).$$

So it is clear that the \(*\)-representation

$$\pi_{n(n-1)\ldots 21} \Delta^{n-1} = (\pi_n \otimes \pi_{n-1} \otimes \ldots \otimes \pi_1) \Delta^{n-1}$$

gives an embedding of $C(\mathbb{C}P^n_q)$ into the groupoid $C^*(T_n)$, where

$$T_n = \mathbb{Z}^n \times \mathbb{Z}_n^{\geq 1}.$$

Applying a similar analysis as used in [Sh5], we can get the following results.

Let

$$\tilde{\mathbb{T}}_n := \{ (j,k) \in T_n | k_i = \infty \implies j_i = -j_1 - j_2 - \ldots - j_{i-1} \text{ and } j_{i+1} = \ldots = j_n = 0 \}$$

be a subgroupoid of $T_n$. Define a subquotient groupoid $\mathbb{T}_n := \tilde{\mathbb{T}}_n / \sim$ where $\sim$ is the equivalence relation generated by

$$(j,k) \sim (j,k_1,\ldots,k_i = \infty,\infty,\ldots,\infty)$$

for all $(j,k)$ with $k_i = \infty$ for an $1 \leq i \leq n$.

**Theorem 1** $C(\mathbb{C}P^n_q) \simeq C^*(\mathbb{T}_n)$ and hence is independent of $q$.

**Corollary 2** There is a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow C(\mathbb{C}P^n_k) \rightarrow C(\mathbb{C}P^n_{k+1}) \rightarrow 0$$

for $k \geq 1$ with $C(\mathbb{C}P^n_0) \simeq \mathbb{C}$.

**Corollary 3** The $C^*$-algebra $C(\mathbb{C}P^n_q)$ has the following composition sequence,

$$C(\mathbb{C}P^n_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \ldots \supseteq \mathcal{I}_n \supseteq \mathcal{I}_{n+1} := 0,$$

with

$$\mathcal{I}_k/\mathcal{I}_{k+1} \simeq \mathcal{K}(\ell^2(\mathbb{Z}^k))$$

for $k > 0$ and $\mathcal{I}_0/\mathcal{I}_1 \simeq \mathbb{C}$.

The above short exact sequences reflect faithfully the underlying singular foliation defined by the canonical $SU(n+1)$-covariant Poisson structure on $

\mathbb{C}P^n = U(n) \setminus SU(n+1)$.
5 Nonstandard quantum projective spaces $\mathbb{C}P^m_{q,c}$

Recently Korogodsky and Vaksman studied nonstandard quantum projective spaces $\mathbb{C}P^m_{q,c}$ [KoVa]. Here we use the approach and result of Dijkhuizen and Noumi [DiNo]. They define $\mathbb{C}P^m_{q,c}$ by

$$C(\mathbb{C}P^m_{q,c}) := \{v \in C(SU(n+1)_q) : v \cdot \mathfrak{t}^c = 0\}$$

for some coideal $\mathfrak{t}^c$ of the quantized universal enveloping algebra $U_q \frak{su}(n+1)$, $c \in (0, \infty)$, and show that

$$C(\mathbb{C}P^m_{q,c}) \cong C^*([x_i^* x_j | 1 \leq i, j \leq n+1]) \subset C(SU(n+1)_q)$$

where

$$x_i = \sqrt{c}u_{1,i} + u_{n+1,i}$$

We remark that when $c = 0$, we have $C(\mathbb{C}P^m_{q,c}) \cong C(\mathbb{C}P^m_{q})$ the standard case, but for $c \in (0, \infty)$, $C(\mathbb{C}P^m_{q,c})$ does not come from $SU(n+1)_q$ modulo a quantum subgroup.

To embed $C(\mathbb{C}P^m_{q,c})$ into a groupoid $C^*$-algebra, we first note that the unique maximal element in the Weyl group of $SU(n+1)$ can also be expressed in the reduced form

$$s_{283}s_{284}s_{382}...s_ns_{n-1}...s_{281}s_{82}...s_{n-1}s_n.$$  

So by Soibelman’s classification of irreducible $*$-representations [Se],

$$(\tau_{n+1} \otimes \pi_{232432...n(1-n)...2123...-(n-1)n})\Delta^N$$

is a faithful $*$-representation of $C(SU(n+1)_q)$.

Note that

$$(\tau_{n+1} \otimes \pi_{232432...n(1-n)...2123...-(n-1)n})\Delta^N (u_{k,i})$$

$$= t_k \otimes 1 \otimes ... 1 \otimes (\pi_{n(1-n)...2123...(n-1)n} \Delta^{2n-2} (u_{k,i}))$$

when $k = 1$ or $k = n + 1$. Thus $(\tau_{n+1} \otimes \pi_{n(1-n)...2123...(n-1)n})\Delta^{2n-1}$ is a faithful $*$-representation of $C^*([u_{1,i}, u_{n+1,i}]_{i=1}^{n+1}) \supset C(\mathbb{C}P^m_{q,c})$. Furthermore, since

$$(\tau_{n+1} \otimes \pi_{n(1-n)...2123...(n-1)n})\Delta^{2n-1} (x_i) = t_1 \otimes \sqrt{c}\pi_{n(1-n)...2123...(n-1)n} \Delta^{2n-2} (u_{1,i})$$

$$+ (t_1 t_2...t_n)^{-1} \otimes \pi_{n(1-n)...2123...(n-1)n} \Delta^{2n-2} (u_{n+1,i})$$

$$\in \{t_1^l (t_2...t_n)^m : l - 2m = 1\} \otimes C^* \left(\mathbb{Z}^{2n-1} \times \mathbb{Z}_q^{2n-1} \right)\right.$$
we have
\[(\tau_{n+1} \otimes \pi_{n(n-1)\ldots2123\ldots(n-1)n}) \Delta^{2n-1} (x^*_i x_j)\]
\[\in \{ t^l_1 (t_2 \ldots t_n)^m : l - 2m = 0 \} \otimes C^* \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right)\]
\[= \left\{ (t_1^l t_2 \ldots t_n)^m : m \in \mathbb{Z} \right\} \otimes C^* \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right) .\]
Thus \((\tau_{n+1} \otimes \pi_{n(n-1)\ldots2123\ldots(n-1)n}) \Delta^{2n-1}\) embeds \(C(CP^n_{qc})\) into
\[C^* (\mathbb{Z}) \otimes C^* \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right)\]
\[\cong C^* \left( \left\{ (t_1^l t_2 \ldots t_n) \right\} \otimes C^* \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right)\right)\]
\[\subset C (T^n) \otimes C^* \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right) .\]

**Theorem 4** For \(0 < c < \infty\), \(C(CP^n_{qc})\) can be embedded into the groupoid \(C^*\)-algebra \(C^* (\mathfrak{G}_n)\) where
\[\mathfrak{G}_n := \mathbb{Z} \times \left( \mathbb{Z}^{2n-1} \times \mathbb{Z}^{2n-1}_{\mathbb{Z}_2} \right) .\]

When \(n = 1\), we have \(CP^1 = S^2\) in the classical case, while in the quantum case, \(CP^1_{qc}\) is indeed a Podles’ quantum sphere \(S^2_{\mu c}\) \(\left( \mathbb{P}^3 \right)\) (with \(\mu = q^{-1}\) for some (different) \(c\). In [LuWe2] Lu and Weinstein classified all \(SU (2)\)-covariant Poisson structures on \(S^2\) by a real parameter and showed that each ‘nonstandard’ \(SU (2)\)-covariant Poisson sphere (corresponding to a suitable parameter \(c \in (0, \infty)\)) contains the trivial Poisson 1-sphere \(S^1\) (consisting of a circle family of 0-dimensional symplectic leaves) and exactly two 2-dimensional symplectic leaves. This geometric structure is again reflected faithfully in the algebraic structure of the algebra \(C (S^2_{\mu c})\) of the nonstandard quantum spheres \(S^2_{\mu c}\) as follows.

**Theorem 5** For \(0 < c < \infty\), \(C(S^2_{\mu c}) \cong C^* (\mathfrak{G}')\), where
\[\mathfrak{G}' = \left\{ (j, j, k_1, k_2) \mid k_1 = \infty \text{ or } k_2 = \infty \right\}\]
is a subgroupoid of the groupoid \(\mathbb{Z}^2 \times \mathbb{Z}^2_{\mathbb{Z}_2}\), and there is a short exact sequence
\[0 \to K \oplus K \to C(S^2_{\mu c}) \to C(S^1) \to 0.\]
On higher dimensional projective spaces $\mathbb{C}P^n$, we also have a one-parameter family of nonstandard $SU(n+1)$-covariant Poisson structures suitably parametrized by $c \in (0, \infty)$, and similar to the case of $n = 1$, each such nonstandard Poisson $\mathbb{C}P^n$ contains an embedded copy of a standard Poisson $S^{2n-1}$ [Sh6]. On the quantum level, one would then expect that $C(\mathbb{C}P^n)$ should have $C(S_q^{2n-1})$ as a quotient to reflect this geometric fact. Using the above groupoid description of $C(\mathbb{C}P^n)$, we can indeed show that this is the case.

**Theorem 6** For $0 < c < \infty$, there is a short exact sequence

$$0 \to \mathcal{I} \to C(\mathbb{C}P^n) \to C(S_q^{2n-1}) \to 0$$

for some ideal $\mathcal{I}$.

**Proof.** First we note that since $(\tau_{n+1} \otimes \pi_{n(n-1), 212... (n-1)n}) \Delta^{2n-1}$ is faithful on $C(S_q^{2n+1})$,

$$\Delta^{2n-1}(u_{n+1, i}) = t_{n+1} \otimes \pi_{n(n-1), 2123... (n-1)n} \Delta^{2n-2} u_{n+1, i}$$

$$= (1 - \delta_{n+1, i}) t_{n+1} \otimes \gamma \otimes i \otimes 1 \otimes \cdots \otimes 1 \otimes \gamma \otimes 1 \otimes \cdots \otimes 1 + (1 - \delta_{1, i}) (1 - \delta_{2, i})$$

$$+ \left[ \sum_{k=n+1-i}^{n-2} t_{n+1} \otimes \gamma \otimes k \otimes \alpha^* \otimes 1 \otimes \cdots \otimes 1 \otimes \gamma \otimes 1 \otimes \cdots \otimes 1 \right]$$

with $1 \leq i \leq n + 1$ generate a $C^*$-algebra isomorphic to $C(S_q^{2n+1})$.

Now let $\rho : C^*(Z \times \mathbb{Z}_{\mathbb{Z}}) \to \mathbb{C}$ be the composition of the $C^*$-homomorphism

$$C^*(Z \times \mathbb{Z}_{\mathbb{Z}}) \to C^*(Z \times \mathbb{Z}_{\mathbb{Z}}|\{\infty\}) = C^*(\mathbb{Z}) \cong C(T)$$

induced by the restriction to the invariant closed subset $\{\infty\}$ of the unit space $\mathbb{Z}_{\mathbb{Z}}$ and the evaluation map $C(T) \to \mathbb{C}$ at $1 \in T$, and let

$$\tilde{\rho} := \underbrace{1 \otimes \cdots \otimes 1}_{n-1} \otimes \rho \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-2} : C^*(T_{2n-1}) \to C^*(T_{2n-3})$$

be the homomorphism that 'removes' the middle two tensor factors. Then by direct calculation, we get

$$(1 \otimes \tilde{\rho})(\tau_{n+1} \otimes \pi_{n(n-1), 212... (n-1)n}) \Delta^{2n-1}(\sqrt{c} u_{1,1} + u_{n+1,1})$$

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= \sqrt{c} t_1 \otimes 1 \otimes \ldots \otimes 1
while
(1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (\sqrt{c} u_{1,i} + u_{n+1,i})
= t_{n+1} \otimes \pi_{(n-1)\ldots 323\ldots (n-1)n} \Delta^{2n-4} (u_{n+1,i})
for \ i > 1. So
(1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (x_1^* x_1) = c,
and for \ i > 1,
[(1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (x_1^* x_1)]^* \n= (1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (x_1^* x_1) \n= \sqrt{c} (t_1^2 t_2 \ldots t_n)^{-1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n} \Delta^{2n-4} (u_{n+1,i}) \n= \sqrt{c} (\tau'' \otimes \pi_{(n-1)\ldots 323\ldots (n-1)n}) \Delta^{2n-3} (u_{n+1,i})
while for \ i, j > 1,
(1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (x_i^* x_j) \n= 1 \otimes \pi_{(n-1)\ldots 323\ldots (n-1)n} \Delta^{2n-4} (u_{n+1,i}^* u_{n+1,j}) \n= \tau'' \otimes \pi_{(n-1)\ldots 323\ldots (n-1)n}) \Delta^{2n-3} (u_{n+1,i}^* u_{n+1,j}),
where \ \tau'' (u_{ij}) = \delta_{i,j} \delta_{n+1,i} (t_1^2 t_2 \ldots t_n)^{-1}. It is straight forward to verify that
(\tau'' \otimes \pi_{(n-1)\ldots 323\ldots (n-1)n}) \Delta^{2n-3} (u_{n+1,i}),
with \ 1 < i \leq n + 1, coincides with the operators
(\tau_n \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-3} (u_{n+1,i})^\ast
whose formula is explicitly written above for the case of dimension \ n + 1. So
(1 \otimes \tilde{\rho}) (\tau_{n+1} \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-1} (C(\mathbb{C} P_{n-q}^\ast)) \n\cong (\tau_n \otimes \pi_{(n-1)\ldots 212\ldots (n-1)n}) \Delta^{2n-3} (C(\mathbb{S}_{n-q}^{2n-1})) \cong C(\mathbb{S}_{q}^{2n-1})
and hence \ C(\mathbb{S}_{q}^{2n-1}) \ is \ a \ quotient \ of \ C(\mathbb{C} P_{n-q}^\ast). \ \blacksquare
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