COMPOSITIONS OF BELYÏ MAPS AND THEIR EXTENDED MONODROMY GROUPS

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Abstract. Given a composition of Belyï maps \( \beta \circ \gamma : X \to Z \), paths between edges of \( \beta \) are extended to form loops, then lifted by \( \gamma \). These liftings are then studied to understand how loops in \( Z \) act on edges of \( \beta \circ \gamma \), demonstrating the group operation in \( \text{Mon} \beta \circ \gamma \subseteq \text{Mon} \gamma \setminus \text{Mon} \beta \). Abstracting away the specific Belyï map \( \gamma \) and finding the image of \( \pi_1(Z) \) in \( \pi_1(Y) \setminus \text{Mon} \beta \) instead allows subsequently determining \( \text{Mon} \beta \circ \gamma \), for any \( \gamma \), using only the monodromy representation of \( \gamma \).

1. Introduction

A Belyï map is a meromorphic function \( \beta \) from a Riemann surface \( X \) to \( \mathbb{P}^1(\mathbb{C}) \) which is unbranched outside of \( \{0, 1, \infty\} \). The pair \((X, \beta)\) is called a Belyï pair. Let \( \mathbb{P}^1(\mathbb{C})_* := \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \), \( X_* := X \setminus \beta^{-1}(\{0, 1, \infty\}) \).

Associated to \((X, \beta)\) is its monodromy representation \( \rho : \pi_1(\mathbb{P}^1(\mathbb{C})_*, z) \to S_d \), where \( z \in \mathbb{P}^1(\mathbb{C}) \) is a fixed base point and \( d \) is the degree of \( \beta \). Let \( \text{Mon} \beta \) denote the monodromy group of \( \beta \), which is defined as the image of \( \rho \).

Historically, the monodromy group was considered to carry the important group theoretic information about the ramification of the Belyï map. However, the monodromy group alone is insufficient for determining the monodromy of a composition of Belyï maps. For example, let

\[
\beta_1(x) = x^3, \quad \beta_2(x) = (1 - x)^3, \quad \gamma(x) = x^2.
\]

Then \( \text{Mon} \beta_1 \approx \text{Mon} \beta_2 \approx C_3 \), but

\[
\text{Mon}(\beta_1 \circ \gamma) \approx C_6 \times (6T1) \neq C_2 \times A_4 \times (6T6) \approx \text{Mon}(\beta_2 \circ \gamma).
\]

This shows that although the monodromy groups of \( \beta_1 \) and \( \beta_2 \) are isomorphic, composing each \( \beta_i \) with \( \gamma \) results in distinct monodromy groups.

To this end, the extended monodromy group will be introduced for Belyï maps satisfying \( \beta(\{0, 1, \infty\}) \subseteq \{0, 1, \infty\} \). An object called the extending pattern of \( \beta \) will be defined which will provide a map, the extended monodromy representation,

\[
\pi_1(\mathbb{P}^1(\mathbb{C})_*, z) \to \pi_1(X_*, x) \mid_{E_\beta} \text{Mon} \beta,
\]

where \( E_\beta \) is the set of edges of \( \beta \) and \( x \in X \) is a fixed base point. The extended monodromy group, denoted \( \text{EMon} \beta \), will then be defined as the image of the extended monodromy representation. Note that \( \text{Mon} \beta \) can be recovered from \( \text{EMon} \beta \) through projection onto the second component of \( \text{EMon} \beta \).

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Finally, for any Belyı́ map $\gamma$, $\text{Mon}_\beta \circ \gamma$ is easily recovered from $\text{EMon}_\beta$ through postcomposition of the first component of $\text{EMon}_\beta$ by the monodromy representation of $\gamma$. In particular, it will be shown that given the extending pattern of $\beta$, as well as the monodromy representations of $\beta$ and $\gamma$, the monodromy representation of $\beta \circ \gamma$ can be efficiently determined. Although not effectively computable for larger examples, a group theoretic description of the extended monodromy group is also given.

2. Background

Consideration of compositions of Belyı́ maps has many motivations in light of their correspondence with dessins d’enfants, called dessins for short. Shabat & Zvonkin [10, Example 6.1] refer to composition as “a manifestation of the hidden symmetries” of a dessin. When a dessin decomposes as a composition of dessins, the Belyı́ map of the composition can be determined by computing the Belyı́ map of each piece of the composition. In the opposite direction, increasingly complex pairs of Belyı́ maps and dessins can be established through composition of simpler pairs. Belyı́ maps have even been considered for use in cryptography by way of composition. [2, Chapter 5]

2.1. Setup.

The rigid nature of Belyı́ maps means that compositions of Belyı́ maps do not always result in another Belyı́ map. However, in the case that $\beta(\{0,1,\infty\}) \subseteq \{0,1,\infty\}$ for a Belyı́ map $\beta$, then for any Belyı́ map $\gamma$, $\beta \circ \gamma$ is again a Belyı́ map.

**Definition.** [7, Section 2.5.5] A Belyı́ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a dynamical Belyı́ map if $\beta(\{0,1,\infty\}) \subseteq \{0,1,\infty\}$.

**Lemma 2.1.** [7, cf. Prop. 2.5.17] If $\gamma : X \to \mathbb{P}^1(\mathbb{C})$ is a Belyı́ map and $\beta$ is a dynamical Belyı́ map, then $\beta \circ \gamma$ is a Belyı́ map.

In the case of a dynamical Belyı́ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, let the domain of $\beta$ be denoted by $Y$ and the codomain by $Z$. Further, fix basepoints $y \in Y$ and $z \in Z$ and let $\pi^Y_1 := \pi_1(Y, y)$ and $\pi^Z_1 := \pi_1(Z, z)$.

Associated to a Belyı́ map $\gamma$ is a dessin d’enfant, $\Delta_\gamma$, obtained from the preimage of the interval $[0,1]$ by neglecting the complex structure of the preimage.

**Definition.** A dessin d’enfant is a graph with a fixed bipartite structure and a labelling of the edges which specifies a cyclic ordering of the edges around each vertex.

The correspondence of a preimage $\gamma^{-1}([0,1])$ with a dessin is made by associating $\gamma^{-1}(0,1)$ with the edges of the dessin and the points $\gamma^{-1}(\{0,1\})$ with the bipartition of the vertices according to whether a point lies over 0 or 1. The cyclic
ordering of the edges around the vertices in each subset of the bipartition yields a pair of permutations \( \sigma_0, \sigma_1 \), arising from the vertices lying over 0 and 1, respectively. Setting \( \sigma_{\infty} := (\sigma_1 \sigma_0)^{-1} \) produces a 3-constellation \( \{ \sigma_0, \sigma_1, \sigma_{\infty} \} \): a triple that acts transitively on \( \{1, \ldots, n\} \) and whose product is the identity. \([7, \text{Section 1.1}]\)

**Notation.** Group actions will be applied on the right as \( x^g \). This agrees with the standard notation for path composition where \( p_1 \circ p_2 \) traverses \( p_1 \) followed by \( p_2 \). Additionally, computer algebra systems, such as SageMath \([9]\) and GAP \([11]\), multiply permutations from left to right.

### 2.2. Related Work.

Shabat & Zvonkin \([10, \text{Section 6}]\) observe that the operation of composition of two plane trees \( \Delta_{\beta}, \Delta_{\gamma} \) is tantamount to substituting \( \Delta_{\beta} \) for every edge of \( \Delta_{\gamma} \). Adrianov & Zvonkin \([1, \text{Theorem 3.3}]\) refine this notion, obtaining a description of the permutations \( \sigma_0, \sigma_1 \) defining the plane tree \( \Delta_{\beta \circ \gamma} \), although they stop short of identifying the group \( \langle \sigma_0, \sigma_1 \rangle \) or determining when it is a proper subgroup of \( \text{Mon} \beta \lhd \text{Mon} \gamma \). Lando & Zvonkin \([7, \text{Prop. 1.7.10}]\) reiterate that the monodromy group “can be represented as a subgroup of” \( \text{Mon} \beta \lhd \text{Mon} \gamma \). Wood \([12, \text{Section 3.3}]\) obtains a similar description as \([1]\) for \( \sigma_0, \sigma_1 \) in the case of Bely\’i maps defined over \( \mathbb{R} \) through a geometric approach.

### 2.3. Organization.

In Section 3, the extending pattern of a Bely\’i map will be established by tracking the way in which it lifts the canonical triangulation \([7, \text{Section 1.5.4}]\) and the way in which the preimages of \([0, 1] \) intersect the lifted structure. Section 4 will use the extending pattern of a Bely\’i map to construct the extended monodromy group \( \text{EMon} \beta \). Having established \( \text{EMon} \beta \), Section 5 will determine generators for \( \text{Mon} \beta \circ \gamma \), and Section 6 will describe the structure of \( \text{Mon} \beta \circ \gamma \) as a subgroup of \( \text{Mon} \gamma \lhd \text{EMon} \beta \), concluding with an example.

### 3. Constructing the Extending Pattern

Being an unramified covering map away from \( \{0, 1, \infty\} \), the preimage of a Bely\’i map \( \gamma^{-1}(\mathbb{P}^1(\mathbb{C}),_*) \) decomposes into disjoint sheets \( \{S_i\}_{i=1}^n \), where \( \deg \gamma = n \). The approach taken in determining the monodromy of a composition uses sheets of covering maps to track traversals around the points 0, 1, and \( \infty \). For this reason, it is important to have a precise specification of the sheets lying over \( \mathbb{P}^1(\mathbb{C}) \), which also enables a correspondence between the edges \( E_{\beta \circ \gamma} \) of a composition and the Cartesian product of edges of each function \( E_{\beta} \times E_{\gamma} \). \([12, \text{Prop. 3.5}]\)

Key to the determination of the monodromy group of a composition will be consideration of the effect traversing paths between edges of \( \beta \) has on the sheets of \( \gamma \) when the paths are lifted by \( \gamma \). To evaluate this effect, an object referred to as the extending pattern, which is constructed by examining the paths between edges of \( \beta \), will be utilized.

### 3.1. Covering Maps, Sheets, & Edges.

For notational convenience, let \( [-\infty, 0] \) denote the nonpositive real axis together with \( \infty \) as a subset of \( \mathbb{P}^1(\mathbb{C}) \). Further, let \( \mathbb{H} \) denote the upper half-plane of \( \mathbb{C} \) embedded into \( \mathbb{P}^1(\mathbb{C}) \) and \( \overline{\mathbb{H}} \) denote the closure of \( \mathbb{H} \) in \( \mathbb{P}^1(\mathbb{C}) \).
Let \( \{ B_i \}_{i=1}^n \) be the path components of 
\[ \gamma^{-1}\left( \mathbb{P}^1(\mathbb{C}) \setminus ([-\infty,0] \cup [1,\infty]) \right). \]
Let \( \{ T_i \}_{i=1}^n \) be the path components of \( \gamma^{-1}(\mathbb{H}) \), ordered so that \( T_i \) is the unique path component of \( \gamma^{-1}(\mathbb{H}) \) intersecting \( B_i \). Then \( \gamma \) maps \( B_i \cup T_i \) homeomorphically onto \( \mathbb{P}^1(\mathbb{C}). \) [Prop. 13.3]

**Convention 3.1.** The sheets of the Belyi map \( \gamma \), regarded as a covering map of \( \mathbb{P}^1(\mathbb{C}), \) will be defined to be \( \{(B_i \cup T_i) \setminus \{0,1,\infty\}\}_{i=1}^n \), as presented above.

The various sets just introduced are illustrated in Figure 3.2, in which
- a set \( B_i \) is the interior of a pair of dark and light triangles together with the edge of the dessin between them,
- a set \( T_i \) is a dark triangle together with its boundary, and
- a sheet of \( \gamma \) is a set \( B_i \) together with the boundary of dark triangle contained in \( B_i \).

**Figure 3.2.** A fundamental domain for a dessin \( \Delta \). Image from [8, 5].

Note that this convention is related to the concept of the canonical triangulation [7, Section 1.5.4] and coincides with the approach used by Wood [12].

The edges of the dessins and their Belyi maps will play a critical role, so that it is important to have an explicit definition for the edges of a Belyi map. Let \( I = [0,1] \subset \mathbb{R} \), which, by abuse of notation, may also be considered as a subset of \( \mathbb{C} \) or \( \mathbb{P}^1(\mathbb{C}) \). Analogously, let \( I^\circ \) denote the interior of \( I \) in \( \mathbb{R} \), though it may be embedded in \( \mathbb{P}^1(\mathbb{C}) \) as well.

**Definition 3.3.** An edge of a Belyi map \( \gamma \) is a lifting of \( \text{id}_I : x \mapsto x \) by \( \gamma \). That is, an edge of \( \gamma \) is a continuous function \( e : I \to \gamma^{-1}(I) \) satisfying \( \gamma \circ e = \text{id}_I \).
An edge will often be implicitly identified with its image in \( P^1(\mathbb{C}) \).

Let \( E_\gamma \) denote the set of edges of \( \gamma \). Since each sheet \( S_\gamma \) of \( \gamma \) maps homeomorphically onto \( P^1(\mathbb{C}) \), there is a bijection between edges \( E_\gamma \) and sheets \( \{ S_\gamma \} \).

**Proposition 3.4.** There is a bijection \( E_{\beta \circ \gamma} \leftrightarrow E_\gamma \times E_\beta \) defined as follows:

For an edge \( e \in E_{\beta \circ \gamma} \), let \( S_e \) be the sheet of \( \gamma \) containing \( e(1/2) \) and let \( e_\gamma \) be the unique element of \( E_\gamma \) with \( e_\gamma \in S_e \). Then

\[
e \leftrightarrow (e_\gamma, \gamma(e)).
\]

**Note.** There is nothing special about the point 1/2 in Proposition 3.4 as any point in the interval (0, 1) would suffice. Its sole purpose is to assign a unique sheet to each edge \( e_{\beta \circ \gamma} \in E_{\beta \circ \gamma} \).

**Proof.** Given an edge \( e_{\beta \circ \gamma} \) of \( \beta \circ \gamma \), \( \gamma(e_{\beta \circ \gamma}) \) is an edge of \( \beta \) as

\[
\beta \circ \gamma(e_{\beta \circ \gamma}) = \text{id}_1 \implies \beta(\gamma \circ e_{\beta \circ \gamma}) = \text{id}_1.
\]

To see that the mapping is surjective, let \((e_\gamma, e_\beta) \in E_\gamma \times E_\beta \). Then \( \gamma^{-1}(e_\beta) \) consists of \( \deg \gamma \) distinct preimages of \( e_\beta \), each constituting an edge of \( \beta \circ \gamma \). In particular, each of the \( \deg \gamma \) points in \( A := \gamma^{-1}(e_\beta(1/2)) \) is a preimage of 1/2 by \( \beta \circ \gamma \), as

\[
\beta \circ \gamma \left( \gamma^{-1}(e_\beta \left( \frac{1}{2} \right)) \right) = \frac{1}{2}.
\]

Finally, each point of \( A \) must lie in a distinct sheet of \( \gamma \) because \( \gamma \) maps each sheet injectively onto \( P^1(\mathbb{C}) \), so that exactly one point \( a \) of \( A \) lies in the sheet containing \( e_\gamma \). The edge \( e \) of \( E_{\beta \circ \gamma} \) containing \( a \) maps to \((e_\gamma, e_\beta)\).

By the cardinality of the involved sets from degree considerations, the mapping is a bijection. \( \square \)

The bijection of Proposition 3.4 is illustrated in Figure 3.5 for

\[
(1) \quad \beta(x) = (1 - \mu(x))^3, \quad \gamma(x) = 3x^2 - 2x^3, \quad \mu(x) = \frac{-21x}{5i\sqrt{3}(x - 1) - 11x - 10}.
\]

Henceforth, \( E_{\beta \circ \gamma} \) will be identified with \( E_\gamma \times E_\beta \) by this bijection.

### 3.2. Action of Paths on Sheets

As the edges of \( \beta \circ \gamma \) are liftings of \( E_\beta \) by \( \gamma \), a path \( p : [0, 1] \to Y \) between edges of \( \beta \) will lift to \( \deg \gamma \) paths between edges of \( \beta \circ \gamma \). It is clear how these paths act on the \( E_\beta \) component of \( E_\gamma \times E_\beta \), but the question remains as to how they act on the \( E_\gamma \) component. Analyzing the action on the sheets of \( \gamma \) by a path traversing between edges of \( \beta \) will provide the answer using the bijection between \( E_\gamma \) and the sheets of \( \gamma \). The action is summarized nicely by Wood in [12, Remark 3.6], paraphrased here:

The path \( p \) can be viewed as an element of \( \pi_1(\mathbb{P}^1(\mathbb{C})_*, S) \), where the base-point is the sheet \( S \). However, \( S \) can be canonically identified with the interval \((0, 1)\), hence the point 1/2, since \((0, 1)\) lies inside \( S \). This allows us to identify the path \( p \) with an element \( p^\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C})_*, 1/2) \). Then \( \sigma_{p^\gamma} \) is just the image of \( p^\gamma \) in \( \text{Mon} \gamma \).
What is being described is that just as the monodromy of a Belyĭ map $\gamma$ is determined according to how the sets of points comprising the edges $E_\gamma$ are mapped onto one another by the lifting a loop in $\pi_1(\mathbb{P}^1(\mathbb{C})_\ast, 1/2)$, so too the action of a path $p$ on the $E_\gamma$ component of $E_\beta \circ \gamma$ can be determined according to how the sets of points comprising the sheets of of $\gamma$ are mapped onto one another.

It is instructive to make this correspondence more explicit.

**Construction 3.6.** Let $p$ be a path in $\mathbb{P}^1(\mathbb{C})_\ast$. For $j = 0, 1$, let $s_j := -\text{sgn}(\text{Re } p(j))$ and let

$$\alpha_{p,j}(t) := \begin{cases} 
\frac{1}{2} + \frac{1}{2} - \frac{p(j)}{2} e^{s_j \pi i t} & \text{if } p(j) \in (-\infty, 0) \cup (1, \infty), \\
(1 - t)p(j) + \frac{1}{2} t & \text{else}, 
\end{cases}$$

Finally, let

$$p^\circ := \alpha_{p,0}^{-1} \ast p \ast \alpha_{p,1}.$$
If \( p(j), \ j = 0, 1 \), lies in either interval \((-\infty, 0)\) or \((1, \infty)\), \( \alpha_j \) is a circular arc from \( p(j) \) through \( \mathbb{H}^+ \) to \( \frac{1}{2} \). Otherwise, it is the straight-line path from \( p(j) \) to \( \frac{1}{2} \). The result of Construction 3.6 is a homomorphism of groupoids.

**Lemma 3.7.** The function
\[
\mathcal{C} : \text{paths in } \mathbb{P}^1(\mathbb{C})_* / \simeq_p \rightarrow \pi_1(\mathbb{P}^1(\mathbb{C})_* , 1/2),
\]
\( p \mapsto p^\mathcal{C} \),
where \( \simeq_p \) is equivalence under path homotopy, is a surjective homomorphism of groupoids, and the action of \( p \) on the sheets of a Belyï map \( \gamma \) is identical to the action of \( p^\mathcal{C} \).

**Proof.** First, if \( H : p_1 \simeq_p p_2 \), then because \( \alpha_{p_j,i} \), \( j = 0, 1 \), depends only on \( p_j(i) \), \( \text{id} * H * \text{id} \) is a path homotopy between \( p^\mathcal{C}_1 \) and \( p^\mathcal{C}_2 \), so that \( \mathcal{C} \) is well-defined. Let \( p_1, p_2 \) be paths in \( \mathbb{P}^1(\mathbb{C})_* \) with \( p_1(1) = p_2(0) \). Again because \( \alpha_{p_j,i} \) depends only on \( p_j(i) \), \( \alpha_{p_1,i} = \alpha_{p_2,j} \) and \((p_1 p_2)^\mathcal{C} \simeq_p p_1^\mathcal{C} p_2^\mathcal{C} \). Finally, because \( \alpha_{p_j,i} \) does not cross \( \mathbb{P}^1(\mathbb{R}) \), \( p^\mathcal{C}(i) \) lies in the same sheet as \( p(i) \) and the action of \( p^\mathcal{C} \) is the same as \( p \). \( \square \)

As \( \mathcal{C} \in \pi_1(\mathbb{P}^1(\mathbb{C})_*) \), it has an image \( \sigma_{p^\mathcal{C}} \) under the monodromy representation of \( \gamma \). By the preceding lemma, the action of complicated paths can be constructed from the action of simpler paths. To this end, it is useful to consider the following regions of \( \mathbb{P}^1(\mathbb{C})_* \), containing certain segments of \( \mathbb{P}^1(\mathbb{R}) \), in order to describe crossing of \( \mathbb{P}^1(\mathbb{R}) \):
\[
\mathcal{R}_{-1/2} := \mathbb{P}^1(\mathbb{C}) \setminus [0, \infty], \quad \mathcal{R}_{1/2} := \mathbb{P}^1(\mathbb{C}) \setminus ([-\infty, 0] \cup [1, \infty]),
\]
\[ \mathcal{R}_{3/2} := \mathbb{P}^1(\mathbb{C}) \setminus [-\infty, 1], \]
where the subscript indicates the unique real point in the region lying on either \( e^{2\pi it}/2 \) or \( 1 - e^{2\pi it}/2 \), \( t \in [0, 1] \). More complicated paths can then be partitioned into finitely many paths, each lying in one of these regions, by the Heine-Borel theorem. In particular, the following basic cases serve as building blocks in the determination of \( \sigma_{p^\mathcal{C}} \) for more complex paths \( p \) in \( \mathbb{P}^1(\mathbb{C})_* \), with an example of each case being illustrated in Figure 3.9.

**Description 3.8.**

1. If \( p \subseteq \mathcal{R}_{1/2} \), then \( p^\mathcal{C} \simeq_p 1 \).
2. If either \( p(0), p(1) \in \mathbb{H}^+ \) or \( p(0), p(1) \in \mathbb{H}^- \) and either \( p \subseteq \mathcal{R}_{-1/2} \) or \( p \subseteq \mathcal{R}_{3/2} \), then \( p^\mathcal{C} \simeq_p 1 \).
3. If \( p(0) \in \mathbb{H}^+, \ p(1) \in \mathbb{H}^- \), and \( p \subseteq \mathcal{R}_{-1/2} \), then \( p^\mathcal{C} \simeq_p e^{2\pi it}/2 \).
4. If \( p(0) \in \mathbb{H}^+, \ p(1) \in \mathbb{H}^- \), and \( p \subseteq \mathcal{R}_{3/2} \), then \( p^\mathcal{C} \simeq_p 1 - e^{-2\pi it}/2 \).
5. If \( p(0) \in \mathbb{H}^-, \ p(1) \in \mathbb{H}^+ \), and \( p \subseteq \mathcal{R}_{-1/2} \), then \( p^\mathcal{C} \simeq_p e^{-2\pi it}/2 \).
6. If \( p(0) \in \mathbb{H}^-, \ p(1) \in \mathbb{H}^+ \), and \( p \subseteq \mathcal{R}_{3/2} \), then \( p^\mathcal{C} \simeq_p 1 - e^{2\pi it}/2 \).

**3.3. Action of Loops on Edges.**

Now that the identification of a path \( p \subseteq \mathbb{P}^1(\mathbb{C})_* \) with an element \( p^\mathcal{C} \in \pi_1(\mathbb{P}^1(\mathbb{C})_*) \) has been made, it is possible to determine, for a given \( \beta \), how a loop \( \lambda \in \pi_1(\mathbb{P}^1(\mathbb{C})_*) \) would permute the edges of an arbitrary Belyï map \( \beta \circ \gamma \). At this point, the extending pattern of a Belyï map \( \beta \) (cf. [12, Section 3.2]) is introduced in order to describe the action of a loop \( \lambda \) on an edge \((e, e_\beta)\).

**Definition 3.10.** Let \( \beta : Y \rightarrow Z \) be a dynamical Belyï map.
The extending pattern of $\beta$ from $\lambda \in \pi_1(Z*, 1/2)$ is the function $f_\lambda : E_\beta \to \pi_1(Y*, 1/2)$ defined as follows. For $e_\beta \in E_\beta$, lift $\lambda$ by $\beta$ to $\lambda_Y$ with $\lambda_Y(0) = e_\beta(1/2)$ and form $\lambda_Y^0$ following Construction 3.6. Then

$$f_\lambda(e_\beta) := \lambda_Y^0 \in \pi_1(Y*, 1/2).$$

The extending pattern of $\beta$ is the pair $(f_0, f_1) := (f_{\lambda_0}, f_{\lambda_1})$, where $\lambda_0$, respectively $\lambda_1$, is a loop with winding number one around $0$, respectively $1$, and winding number zero around both $1$ and $\infty$, respectively $0$ and $\infty$.

Example 3.12. Define $\lambda_0 := e^{2\pi it}/2$ and $\lambda_1 := 1 - e^{2\pi it}/2$, and let their homotopy classes be denoted by $a$ and $b$, respectively. An example of determining the extending pattern is shown in Figure 3.11. The only lifting of $\lambda_0$ which crosses $\mathbb{P}^1(\mathbb{R})$ outside of $(0, 1)$ is the lifting beginning at Edge 4. As the lifting crosses $(-\infty, 0)$ from $\mathbb{H}^+$ to $\mathbb{H}^-$, Description 3.8 prescribes that $f_0(4) = a$, while $f_0(i) = 1$ for $i \neq 4$. On the other hand, liftings of $\lambda_1$ beginning at Edge 4 and Edge 5 cross $(-\infty, 0)$ in opposite directions, so that $f_1(4) = a$ and $f_1(5) = a^{-1}$. Further, a
lifting of \( \lambda_1 \) beginning at Edge 2 crosses \((1, \infty)\) from \( \mathbb{H}^- \) to \( \mathbb{H}^+ \), hence \( f_1(2) = b \). Finally, \( f_1(1) = f_1(3) = 1 \).

**Example 3.13.** For \( \beta \) defined in (3.1) and shown in Figure 3.5(A), the extending pattern is \( f_0 = (1, 1, b) \), \( f_1 = (a, 1, 1) \) because the path from Edge 3 to Edge 1 crosses \((1, \infty)\) from \( \mathbb{H}^- \) to \( \mathbb{H}^+ \), while the loop around the white vertex of Edge 1 crosses \((\infty, 0)\) from \( \mathbb{H}^+ \) to \( \mathbb{H}^- \).

**Theorem 3.14.** Let \( \beta : Y \to Z \) be a dynamical Belyi map. For any Belyi map \( \gamma : X \to Y \), the action of the loop \( \lambda \in \pi_1(Z, 1/2) \) on the edge \((e_\gamma, e_\beta) \in E_{\beta \circ \gamma} \) is given by

\[
(2) \quad (e_\gamma, e_\beta)^{\lambda} = (e_{\gamma^{\lambda}(e_\beta)}, e_\beta).
\]

where each action is the monodromy action of the respective Belyi map.

**Proof.** Let \( \lambda \in \pi_1(Z, 1/2) \), let \( e = (e_\gamma, e_\beta) \in E_{\beta \circ \gamma} \), let \( \lambda_Y \) as in the statement of the theorem, and let \( \lambda_Y' = f_\lambda(e_\beta) \). Further, let \( \lambda_X \subseteq X \) be the lifting of \( \lambda \) by \( \beta \circ \gamma \) with \( \lambda_X(0) = e(1/2) \). From Proposition 3.4, to show that the first component of \( e_\lambda \) is \( e_\gamma \lambda_Y' \), it is necessary to show that \( e_\gamma \lambda_Y' \) is the edge of \( \gamma \) lying in the sheet containing \( e_\lambda(1/2) = \lambda_X(1) \).

By Proposition 3.4, \( \gamma(e) = e_\beta \), so that \( \gamma(\lambda_X(0)) = e_\beta(1/2) = \lambda_Y(0) \). Moreover, \( \beta(\gamma \circ \lambda_X) = \lambda \) shows that \( \gamma \circ \lambda_X \) is a lifting of \( \lambda \) by \( \beta \) and it follows that \( \gamma(\lambda_X) = \lambda_Y \). Then \( \lambda_Y \) acts on the sheets of \( \gamma \) by

\[
\lambda_Y : \text{sheet of } \gamma \text{ containing } \lambda_X(0) \longrightarrow \text{sheet of } \gamma \text{ containing } \lambda_X(1).
\]

By Lemma 3.7, \( \lambda_Y' \) permutes the sheets, hence edges, of \( \gamma \) in the same way and \( e_\gamma \lambda_Y' \) lies in the sheet containing \( \lambda_X(1) \).

Finally, because \( \gamma(\lambda_X) = \lambda_Y \),

\[
\gamma \left( e_\lambda \left( \frac{1}{2} \right) \right) = \gamma \left( \lambda_X(1) \right) = \lambda_Y(1) = e_\beta \left( \frac{1}{2} \right),
\]

and the second component of \( e_\lambda \) is given by \( e_\beta \), completing the result. \( \square \)

**Example 3.16.** Theorem 3.14 is illustrated in Figure 3.15. The loop \( \lambda \subset Z \) around 1 is lifted to \( \lambda_Y \subset Y \), spanning between Edge 2 and Edge 1 of \( \beta \). Next, \( \lambda_Y \) is used to form \( \lambda_Y' \) according to Construction 3.6, as represented by the dashed segments. Being a loop in \( Y \) based at \( 1/2, \lambda_Y' \), which includes \( \lambda_Y \) as a subset, is lifted by \( \gamma \) to a path between edges of \( \gamma \), which simultaneously lifts \( \lambda_Y \) to a path \( \lambda_X \subset X \) between edges of \( \beta \circ \gamma \). The path \( \lambda_X \) demonstrates the action of \( \lambda_Y \) on \( E_{\beta \circ \gamma} \), while the lifting of \( \lambda_Y' \) creates a correspondence between the endpoints of \( \lambda_X \) and edges of \( \gamma \). In the notation of Theorem 3.14, let \( \lambda_1 \subset Z \) be a loop around 1 and let \( b \subset Y \) also be a loop around 1. Since \( f_{\lambda_1} = (1, 1, b) \), where \( b \) acts on the edges of \( \gamma \) by \( (1, 3) \in \text{Mon } \gamma \),

\[
(1, 3)^{\lambda_1} = (f_{\lambda_1}(3), 3^{\lambda_1}) = (1^b, 1) = (3, 1).
\]

From [2], it becomes clear that the action of a loop \( \lambda \) on an edge \( e \in E_{\beta \circ \gamma} \) can be determined from the monodromy of \( \beta \), the monodromy of \( \gamma \), and \( f_\lambda \). Perhaps
10 COMPOSITIONS OF BELYI MAPS AND THEIR EXTENDED MONODROMY GROUPS

(a) A loop $\lambda$ around 0 on the Belyi map $f(x) = x$

(b) A lifting $\lambda_Y$ of $\lambda$ and the creation of a loop $\lambda_Y^L$

(c) A lifting $\lambda_X$ of $\lambda_Y$ and a lifting of $\lambda_Y^L$

Figure 3.15. Lifting a loop in $\mathbb{Z}$ by $\beta \circ \gamma$ determines a path between edges of $\gamma$

the most interesting observation from Theorem 3.14 is that $f_\lambda$ is independent of $\gamma$. Thus, the effect of composition with the Belyi map $\beta$ on the monodromy of any Belyi map $\gamma$ can be established once and for all by determining $f_\lambda$ for loops $\lambda_0$ and $\lambda_1$ around 0 and 1, respectively, then specializing to the function $\gamma$ at hand.

4. Monodromy as a Wreath Product

Theorem 3.14 enables the development of the group structure of $\text{Mon} \beta \circ \gamma$ as a wreath product. The fact that $\text{Mon} \beta \circ \gamma$ is a subgroup of $\text{Mon} \gamma l_{E_\beta}$ $\text{Mon} \beta$ begins to appear in (2), in which the action of $\lambda$ on $e_\gamma$ is a function of $e_\beta$. As the codomain of $f_\lambda$ is $\pi^1_1$, rather than $\text{Mon} \gamma$, $\pi^Z_1 l_{E_\beta}$ $\text{Mon} \beta$ will be considered first, reflecting the independence from $\gamma$ of the effect of composition with $\beta$ on the monodromy action. Then $\pi^Z_1 l_{E_\beta}$ $\text{Mon} \beta$ will be mapped onto $\text{Mon} \gamma l_{E_\beta}$ $\text{Mon} \beta$.

4.1. The Extended Monodromy Group.
Theorem 4.1. Let $\beta : Y \to Z$ be a dynamical Belyï map, let

$$\pi_Y^1 := \pi_1(Y, 1/2), \quad \pi_Y^2 := \pi_1(Z, 1/2),$$

let $\rho_\beta$ be the monodromy representation of $\beta$, let $\tau_\lambda := \rho_\beta(\lambda)$, and let $f_\lambda$ be defined as in Definition 3.10. Define the action of $\text{Mon}_\beta$ on $\text{Fun}(E_\beta, \pi_Y^1)$ by $f^\tau(e_\beta) = f(e_\beta^{-1})$. For $\lambda \in \pi_Y^2$, define

$$\varphi_\beta(\lambda) := f_\lambda \times \tau_\lambda.$$

Then $\varphi_\beta$ is a homomorphism

$$\varphi_\beta : \pi_Y^2 \to \pi_Y^1 \mid_{E_\beta} \text{Mon}_\beta = \text{Fun}(E_\beta, \pi_Y^1) \rtimes \text{Mon}_\beta$$

whose image is an extension by $\text{Mon}_\beta$ of

$$\ker \left( \text{proj}_\beta \mid_{\varphi_\beta(\pi_Y^2)} \right) = \varphi_\beta(\ker \rho_\beta) \approx \ker \rho_\beta / \ker \varphi_\beta,$$

where $\text{proj}_\beta : \varphi_\beta(\pi_Y^2) \to \text{Mon}_\beta$ is restricted to $\varphi_\beta(\pi_Y^2)$.

Proof. The situation in question is illustrated by

$$\pi_Y^2 \xrightarrow{\varphi_\beta} \pi_Y^1 \mid_{E_\beta} \text{Mon}_\beta \xrightarrow{\text{proj}_\beta} \text{Mon}_\beta.$$

To begin, because $\rho_\beta$ and $\cdot \circ$ are well-defined, $\varphi_\beta$ is well-defined. Now, let $\lambda_1, \lambda_2 \in \pi_Y^2$. To see that $\varphi_\beta$ is a homomorphism, first note that because $\rho_\beta$ is a homomorphism, $\tau_\lambda \cdot \lambda_2 = \tau_{\lambda_1 \cdot \lambda_2}$ and

$$\varphi_\beta(\lambda_1 \cdot \lambda_2) = (f_{\lambda_1}, \tau_{\lambda_1})(f_{\lambda_2}, \tau_{\lambda_2}) = (f_{\lambda_1} \cdot f_{\lambda_2}^{\tau_1^{-1}}, \tau_{\lambda_1 \cdot \lambda_2}).$$

As $\varphi_\beta(\lambda_1 \cdot \lambda_2) = (f_{\lambda_1 \cdot \lambda_2}, \tau_{\lambda_1 \cdot \lambda_2})$, it remains to show that $f_{\lambda_1 \cdot \lambda_2} = f_{\lambda_1} \cdot f_{\lambda_2}^{\tau_1^{-1}}$.

Let $e_\beta \in E_\beta$ and lift $\lambda_1$ by $\beta$ to $\tilde{\lambda}_1$ so that $\tilde{\lambda}_1(0) \in e_\beta$. Further, lift $\lambda_2$ by $\beta$ to $\tilde{\lambda}_2$ so that $\tilde{\lambda}_2(0) = \tilde{\lambda}_1(1)$. The uniqueness of liftings guarantees that the $\tilde{\lambda}_1 \cdot \tilde{\lambda}_2$ is identical to the lifting of $\lambda_1 \cdot \lambda_2$ beginning at $e_\beta(1/2)$. By definition, $e_\beta^{\tau_1^{-1}}$ is the edge containing $\tilde{\lambda}_1(1) = \tilde{\lambda}_2(0)$, and

$$(f_{\lambda_1} \cdot f_{\lambda_2}^{\tau_1^{-1}})(e_\beta) = f_{\lambda_1}(e_\beta) \ast f_{\lambda_2}(e_\beta^{\tau_1})$$

$$= \tilde{\lambda}_1^\circ \ast \tilde{\lambda}_2^\circ$$

$$\simeq_{p} (\tilde{\lambda}_1 \ast \tilde{\lambda}_2)^\circ \quad \text{(by Lemma 3.7)}$$

$$= f_{\lambda_1 \cdot \lambda_2}(e_\beta).$$

As $e_\beta$ was arbitrary, this shows that $\varphi_\beta$ is a homomorphism.

As $\lambda \in \ker \rho_\beta \iff \lambda \in \ker(\text{proj}_\beta \circ \varphi_\beta) \iff \varphi_\beta(\lambda) \in \ker \text{proj}_\beta$ and $\varphi_\beta(\ker \rho_\beta) \approx \ker \rho_\beta / \ker \varphi_\beta$ by the first isomorphism theorem, the result follows. \qed

The homomorphism $\varphi_\beta$, and its image, serve to capture the important information regarding the effect that $\beta$ has on the monodromy of a Belyï map $\gamma$ when composing the functions.
Definition 4.2. The extended monodromy representation of the dynamical Belyi map $\beta : Y \to Z$ is defined by

$$
\varphi_\beta : \pi_1^Z \longrightarrow \pi_1^Y \triangleright \text{Mon}_\beta,
\lambda \mapsto f_\lambda \rtimes \rho_\beta(\lambda),
$$

where $f_\lambda$ is the extending pattern of $\beta$ from $\lambda$ and $\rho_\beta$ is the monodromy representation of $\beta$. Further, the extended monodromy group of $\beta$, $\text{EMon}_\beta$, is the image of the extended monodromy representation, $\varphi_\beta(\pi_1^Z)$.

Corollary 4.3. The extended monodromy group of $\beta$, $\text{EMon}_\beta$, is given as the extension

$$
1 \to \ker \rho_\beta / \ker \varphi_\beta \to \text{EMon}_\beta \to \text{Mon}_\beta \longrightarrow 1,
$$

where $\rho_\beta$ and $\varphi_\beta$ are the monodromy representation and extended monodromy representation of $\beta$, respectively.

Definition 4.4. The monodromy extending group of $\beta$, $\text{MonExt}_\beta$, is the complement of $\text{Mon}_\beta$ in $\text{EMon}_\beta$ and is given by

$$
\text{MonExt}_\beta = \ker \rho_\beta / \ker \varphi_\beta.
$$

Example 4.5. In computing the monodromy extending group of $\beta(x)$ from Example 3.13, writing $a,b$ for $\lambda_0, \lambda_1$, $\ker \rho_\beta$ is computed first as

``` gap
gap> MonBeta := Group((1,2,3), ());;
gap> rho := GroupHomomorphismByImages(FreeGroup("a", "b"), MonBeta,);
gap> GeneratorsOfGroup(Kernel(rho));
[ b, a^3, a*b*a^-1, a^-1*b*a ]
```

From Example 3.13, $f_{\lambda_0} = (1,1,b)$ and $f_{\lambda_1} = (a,1,1)$, so that

$$
\varphi_\beta(\lambda_0) = (1,1,b) \rtimes (1 2 3), \quad \varphi_\beta(\lambda_1) = (a,1,1) \rtimes 1.
$$

Then from $\varphi_\beta(\lambda) \approx \ker \rho_\beta / \ker \varphi_\beta$, applying $\varphi_\beta$ to the generators of $\ker \rho_\beta$ above results in

$$
\text{MonExt}_\beta \approx \langle (a,1,1) \rtimes 1, (b,b,b) \rtimes 1, (1,1,a) \rtimes 1, (1,a,1) \rtimes 1 \rangle.
$$

4.2. The Monodromy Group of a Composition.

Having established the extended monodromy group of $\beta$ allows for determining $\text{Mon}_\beta \circ \gamma$, for a Belyi map $\gamma$, simply through postcomposition of the extending patterns of $\beta$ by the monodromy representation of $\gamma$.

Theorem 4.6. Let $\beta$ be a dynamical Belyi map. For any Belyi map $\gamma$, let $\rho_\gamma$ be its monodromy representation and define

$$
\rho_{\gamma*} : \text{Fun}(E_\beta, \pi_1^Y) \to \text{Fun}(E_\beta, \text{Mon}_\gamma)
$$

by $\rho_{\gamma*}(f) = \rho_\gamma \circ f$. Then

$$
\varphi_\gamma := (\rho_{\gamma*} \times \text{id}) \circ \varphi_\beta : \pi_1^Z \to \text{Mon}_\gamma \triangleright \text{Mon}_\beta
$$

coincides with $\rho_{\beta\gamma}$.

Proof. Showing that $\ker \rho_{\beta\gamma} = \ker \varphi_\gamma$ will imply that $\text{Mon}_\beta \circ \gamma \approx \varphi_\gamma(\pi_1^Z)$ by the first isomorphism theorem.

If $\lambda \in \ker \rho_{\beta\gamma}$, then by (2), $\rho_\beta(\lambda) = 1$ and $\rho_\gamma \circ f_\lambda = 1$, so that $\lambda \in \ker \varphi_\gamma$. On the other hand, if $\lambda \in \ker \varphi_\gamma$, then $\rho_\beta = 1$ and $\rho_\gamma \circ f_\lambda = 1$, and by (2), $\lambda \in \ker \rho_{\beta\gamma}$. □
Example 4.7. For $\gamma$ as in Example 3.1, applying $\rho_{\gamma^*}$ to $\text{MonExt} \beta$ as computed in Example 4.5 yields
\[
\left\{(1 2), (1 1) \times 1, (1 3), (1 3), (1 3) \times 1, (1 1), (1 2) \times 1, (1, (1 2), 1) \times 1\right\} \approx S_3 \times S_3 \times S_3.
\]
As such, $\text{Mon} \beta \circ \gamma$ satisfies the short exact sequence
\[
1 \rightarrow S_3 \times S_3 \times S_3 \rightarrow \text{Mon} \beta \circ \gamma \rightarrow C_3 \rightarrow 1.
\]

Corollary 4.8. \cite{1, Theorem 3.3, 2, Section 4.3} The monodromy group $\text{Mon} \beta \circ \gamma$ of the composition of a dynamical Belyï map $\beta$ and a Belyï map $\gamma$ is isomorphic to a subgroup of the wreath product $\text{Mon} \gamma \wr \text{Mon} \beta$. Moreover, this isomorphism is given by
\[
\begin{align*}
\text{Mon} \beta \circ \gamma & \rightarrow \text{Mon} \gamma \wr \text{Mon} \beta \\
\rho_{\beta \circ \gamma}(\lambda) & \mapsto (\rho_{\gamma} \circ f_\lambda, \rho_{\beta}(\lambda)),
\end{align*}
\]
where $f_\lambda$ is the extending pattern of $\beta$ from $\lambda$.

Corollary 4.9. \cite{2, Theorem 4.20} Let $\beta$ be a dynamical Belyï map with monodromy $(\tau_0, \tau_1)$ and extending pattern $(f_0, f_1)$. Then for any Belyï map $\gamma$,
\[
\text{Mon} \beta \circ \gamma = \left\{(\rho_{\gamma} \circ f_0, \tau_0), (\rho_{\gamma} \circ f_1, \tau_1)\right\}.
\]

Example 4.10. Applying Corollary 4.9 to the extending pattern from Example 3.13 for $\beta, \gamma$ defined in Example 3.1,
\[
\text{Mon} \beta \circ \gamma = \left\{(1, 1, \sigma_1) \times \tau_0, (\sigma_0, 1, 1) \times \tau_1\right\}
\]
\[
= \left\{(1, 1, (1 3)) \times (1 2 3), ((1 2), 1, 1) \times 1\right\}
\]
\[
= (S_3 \times S_3 \times S_3) \rtimes C_3,
\]
in agreement with Example 4.7.

5. Working with the Extended Monodromy Group

In this section, let $a, b$ be the generators of $\pi_1(\mathbb{P}^1(\mathbb{C})_*)$, where $a \simeq_\rho e^{2\pi i t}/2$ and $b \simeq_\rho 1 - e^{2\pi i t}/2$.

5.1. Computing the Extended Monodromy Group.

Based on Convention 3.1 and Lemma 3.7 determining the extending pattern of a dynamical Belyï map $\beta$ amounts to determining where the liftings $\beta^{-1}(e^{2\pi it}/2)$ and $\beta^{-1}(1 - e^{2\pi it}/2)$ cross $(-\infty, 0)$ or $(1, \infty)$ and in which direction. One approach is to use differential equations to lift the loops $e^{2\pi it}/2$ and $1 - e^{2\pi it}/2$ \cite{4, Section 9.2.5}, then sample points along the lifted paths and identify consecutive points whose imaginary parts have distinct signs. The extending pattern is then determined by Description 3.8 according to the identified sign change.

Let $|E_\beta| = n$ and let $\gamma$ be a Belyï map with $|E_\gamma| = m$. Labeling the edges of $\beta$, $\gamma$, and $\beta \circ \gamma$ beginning with 0, the identification $(s, t) \mapsto s \cdot n + t$ gives a mapping $E_\gamma \times E_\beta \rightarrow E_{\beta \circ \gamma}$. In this way, \cite{2} specifies the action of a loop $\lambda$ as
\[
(s \cdot n + t)^\lambda \leftrightarrow (s, t)^\lambda = \left(s^{f_\lambda(t)}, t^\lambda\right) \leftrightarrow s^{f_\lambda(t)} \cdot n + t^\lambda.
\]
As such, Algorithm 5.1 constructs the monodromy of $\beta \circ \gamma$ as arrays $(\eta_0, \eta_1)$ with $\eta_i[j] = \eta_i(j)$ from the extending pattern $(f_0, f_1)$ of $\beta$ and the monodromies $(\tau_0, \tau_1)$ and $(\sigma_0, \sigma_1)$ of $\beta$ and $\gamma$, respectively.

Algorithm 5.1 Obtaining the monodromy of $\beta \circ \gamma$ from $\beta$ and $\gamma$

\begin{verbatim}
function CompositeMonodromy(\{\tau_0, \tau_1, f_0, f_1\}, \{\sigma_0, \sigma_1\})
\begin{align*}
\rho_{\gamma*} &\leftarrow (a \mapsto \sigma_0, b \mapsto \sigma_1) \\
f_0, f_1 &\leftarrow \rho_{\gamma*}(f_0), \rho_{\gamma*}(f_1) \\
n &\leftarrow |E_\beta|, m &\leftarrow |E_\gamma| \\
\eta_0 &\leftarrow [], \eta_1 &\leftarrow [] \\
for 0 \leq s < m do \\
\text{append } f_0(t)(s) \cdot n + \tau_0(t) \text{ to } \eta_0 \\
\text{append } f_1(t)(s) \cdot n + \tau_1(t) \text{ to } \eta_1 \\
\end{align*}
\end{verbatim}

5.2. Extended Monodromy Group Examples.

![Figure 5.2. The dessins of $\beta_1$ (left) and $\beta_2$ (right).](image)

Example 5.3. Consider $\beta_1$ and $\beta_2$ as in the introduction, which have monodromy $(\tau_0, \tau_1) = ((1 2 3), (1))$. The extending patterns are given by

\[
\begin{align*}
    f_0^{(1)} &= (1, a, 1), & f_1^{(1)} &= (b, 1, 1), & f_0^{(2)} &= (1, b, 1), & f_1^{(2)} &= (a, 1, 1),
\end{align*}
\]

as can be seen from Figure 5.2. For $\gamma, (\sigma_0, \sigma_1) = ((1 2), (1))$, so that in the case of $\beta_1$,

\[
\begin{align*}
    \rho_{\gamma*} \circ \varphi_{\beta_1}(a) &= \rho_{\gamma*}((1, a, 1) \rtimes \tau_0) = (1, \sigma_0, 1) \rtimes \tau_0, \\
    \rho_{\gamma*} \circ \varphi_{\beta_1}(b) &= \rho_{\gamma*}((b, 1, 1) \rtimes \tau_1) = (1, 1, 1) \rtimes 1.
\end{align*}
\]

Computing powers of $(1, \sigma_0, 1) \rtimes \tau_0$ shows that

\[
\text{Mon } \beta_1 \circ \gamma = \langle (1, \sigma_0, 1) \rtimes \tau_0 \rangle \approx C_6.
\]

On the other hand, for $\beta_2$,

\[
\begin{align*}
    \rho_{\gamma*} \circ \varphi_{\beta_2}(a) &= \rho_{\gamma*}((1, b, 1) \rtimes \tau_0) = (1, 1, 1) \rtimes \tau_0, \\
    \rho_{\gamma*} \circ \varphi_{\beta_2}(b) &= \rho_{\gamma*}((a, 1, 1) \rtimes \tau_1) = (\sigma_0, 1, 1) \rtimes 1,
\end{align*}
\]

and $\text{Mon } \beta_2 \circ \gamma \approx C_2 \times A_4$. 

Example 5.4. Let
\[ \beta_0(z) := \frac{(z^4 + 228z^3 + 494z^2 - 228z + 1)^3}{1728z(z^2 - 11z - 1)^5}, \]
\[ \mu(z) := \frac{55\sqrt{5} + 123}{5\sqrt{5} + 11} z = (11 + \alpha^{-1})z, \]
where \( \alpha \) is the positive root of \( z^2 - 11z - 1 \). Let \( \beta(z) = \beta_0 \circ \mu(z) \), and consider the family of Belyi maps \( \gamma_m(z) = z^m \). Because the pole of \( \beta_0 \) lying in the face of \( \beta \) containing \( z = 1 \) does not lie over 1, \( \mu \) is needed to move the pole to \( z = 1 \) to make \( \beta \) dynamical. The monodromy and extending pattern of \( \beta \) under the ordering of edges given in Figure 5.5 is given by
\[
\begin{align*}
\tau_0 &= (1 \ 2 \ 3)(4 \ 5 \ 6)(7 \ 9 \ 8)(10 \ 11 \ 12), \quad f_0 = [a, a^{-1}, 1, a, a^{-1}, 1, 1, 1, 1, 1, 1, 1], \\
\tau_1 &= (1 \ 2)(3 \ 4)(5 \ 8)(6 \ 7)(9 \ 10)(11 \ 12), \quad f_1 = [b^{-1}, b, 1, 1, 1, 1, 1, 1, 1, a, a^{-1}].
\end{align*}
\]

It follows that \( \text{Mon} \beta \circ \gamma_m \) is an extension of
\[
\text{MonExt} \beta \approx \begin{cases} 
C_{m^6}^6 & \text{if } 5 \nmid m, \\
C_{m^6/5}^6 & \text{if } 5 \mid m.
\end{cases}
\]
by \( \text{Mon} \beta = A_5 \). Note that by the Schur-Zassenhaus theorem, if \( \gcd(m, 30) = 1 \) or if \( m = 5 \), then this extension is guaranteed to be split.

Example 5.6. Consider once again \( \beta \) from the previous example and let \( \gamma \) be a Belyi map with monodromy given by \( (\sigma_0, \sigma_1) = ((2 \ 3 \ 4), (1 \ 2)(3 \ 4)) \). Then
\[
\text{MonExt} \beta = C_2^6 \rtimes C_3, \quad \text{Mon} \beta = C_3, \quad \text{Mon} \beta \circ \gamma = C_2^6 \rtimes C_9.
\]
But since $C_9$ is not a split extension of $C_3$ by $C_3$, this shows that $\text{Mon} \beta \circ \gamma$ is not in general a split extension of $\text{MonExt} \beta$.

6. Conclusion

By using Theorem 3.14 to characterize the action of $\pi_1^2$ on sheets lying over $Y$ in terms of functions into $\pi_1^1$, it is possible to capture the information about $\beta$ required to determine $\text{Mon} \beta \circ \gamma$ for any Belyı́ map $\gamma$. The concept of the extending pattern of a dynamical Belyı́ map, a pair of functions $f_0, f_1 : E_\beta \to \pi_1^Y$, is introduced to express the action of $\pi_1^2$, when lifted by $\beta$, on sheets lying over $Y$. Finally, the extending pattern is used to determine the group which $\text{Mon} \beta$ extends to $\text{Mon} \beta \circ \gamma$.

The dynamical nature of $\beta$ is primarily applied in the use of sheets of $\gamma$ of a prescribed form to identify how paths between edges of $\beta \circ \gamma$ affect the edge of $\gamma$ corresponding to an edge of $\beta \circ \gamma$. Extending the approach of Section 3.3 to arbitrary Belyı́ maps would allow for applying Section 4 to the determination of $\text{Mon} \beta \circ \gamma$ for non-dynamical Belyı́ maps, as well as additional classes of covering maps such as origamis.

Finally, although the extending pattern of a dynamical Belyı́ map $\beta$ permits determination of $\text{MonExt} \beta$, so that once a Belyı́ map is specified, the groups which extend to $\text{Mon} \beta \circ \gamma$ can be determined, it would be useful to devise conditions on $\beta$ and $\gamma$ to determine when $\text{Mon} \beta \circ \gamma$ is a split extension of $\text{MonExt} \beta$ and $\text{Mon} \beta$, as well as when $\text{Mon} \beta \circ \gamma$ is a proper subgroup of $\text{Mon} \gamma \wr \text{Mon} \beta$.

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A.1. Computing $\text{MonExt} \beta$.

To compute $\text{EMon} \beta$, along with $\text{Mon} \beta \circ \gamma_m$, in Example 5.3, SageMath’s \cite{9} interface to GAP \cite{11} is used. First, $\text{EMon} \beta$ is constructed using the monodromy and extending pattern of $\beta$.

```sage
sage: F2 = libgap.FreeGroup('a', 'b'); A,B = F2.GeneratorsOfGroup()
sage: tau0 = libgap.eval('(1,2,3)(4,6,5)(7,9,8)(10,11,12)')
sage: tau1 = libgap.eval('(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)')
sage: MonBeta = libgap.Group(tau0, tau1)
sage: rho = F2.GroupHomomorphismByImages(MonBeta)
sage: wr = F2.WreathProduct(MonBeta)
sage: a_gens = [A^wr.Embedding(j) for j in range(1, 13)]
sage: b_gens = [B^wr.Embedding(j) for j in range(1, 13)]
```

Because $\rho_\gamma(b) = 1$, $f_1(1)$ and $f_1(2)$ are set to 1, rather than $b$ and $b^{-1}$, respectively, prior to application of $\rho_\gamma \ast$ in order to simplify the computations. Specifically, the goal in this computation is the determination of the kernel of $\text{proj}_\beta : \text{Mon} \beta \circ \gamma_m \rightarrow \text{Mon} \beta$ in the case that $\rho_\gamma(a) = (1 \cdots m)$ and $\rho_\gamma(b) = 1$. But $\rho_\gamma \ast \text{id} = (\rho_\gamma \ast \text{id}) \circ (a \mapsto a, b \mapsto 1)$, so that their kernels are equal.

```sage
sage: f0 = a_gens[0] * a_gens[1]^-1 * a_gens[3] * a_gens[5]^-1
sage: f1 = a_gens[10] * a_gens[11]^-1
sage: tau0, tau1 = wr.Embedding(13).Image().GeneratorsOfGroup()
sage: EMonBeta = libgap.Group(f0*tau0, f1*tau1)
```

Next, $\text{MonExt} \beta$ is computed using Lemma \cite{4.3}. As a result of a lack of efficient methods for finding generators in a wreath product not expressed as a permutation group, GAP is unable to find a minimal generating set for $\text{KerProjBeta}$, instead finding a set of 61 generators. However, removing duplicates from this set reduces the generating set to 15 generators.

```sage
sage: phi = F2.GroupHomomorphismByImages(EMonBeta)
sage: KerProjBeta = phi.RestrictedMapping(rho.Kernel()).Image()
sage: gens = list(KerProjBeta.GeneratorsOfGroup().Unique())
```

Finally, from a set $\{g_i\}_{i=1}^6$ of six generators, all 15 unique generators can be shown to have the form $\prod_{i=1}^6 g_i^{e_i}$, with $e_i \in \{0, -1, 1\}$. Manual inspection of $\{g_i\}_{i=1}^6$ shows them to be a minimal generating set.\footnote{The output of the final command was formatted to improve readability.}

```sage
sage: exponents_0_pm_1 = ( (wr.One(), gen, gen^-1) for gen in gens[1:7] )
sage: gen_and_inv_prods = map(prod, itertools.product(*exponents_0_pm_1))
sage: set(gens).issubset(gen_and_inv_prods)
True
```
sage: gens[1:7]

[WreathProductElement(<id>, a^5, <id>, <id>, <id>, <id>,
  <id>, <id>, <id>, <id>, a^-5, ()),
WreathProductElement(a^-5, <id>, <id>, <id>, <id>, <id>,<id>, <id>, <id>, a^5, <id>, ()),
WreathProductElement(<id>, <id>, a^-5, <id>, <id>, <id>,
  <id>, <id>, <id>, a^5, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, a^-5, <id>, <id>, <id>,
  <id>, <id>, <id>, a^5, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, <id>, <id>, a^-5,
  a^5, <id>, <id>, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, <id>, a^-5, <id>,
  <id>, a^5, <id>, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, <id>, a^-5, <id>,
  <id>, <id>, a^5, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, <id>, a^-5, <id>,
  <id>, <id>, <id>, a^5, <id>, <id>, ()),
WreathProductElement(<id>, <id>, <id>, <id>, a^-5, <id>,
  <id>, <id>, <id>, a^5, <id>, <id>, ())]

It is worth noting that in the case EMon_\beta can be viewed as a subgroup of \Z_{\tilde{L}_\beta} \rtimes \text{Mon}_\beta, as in this example where EMon_\beta is specialized under the assumption \sigma_1^\gamma = 1 or in the case that the edges of \beta cross only one of (-\infty,0) and (1,\infty), the use of Residue-Class-Wise Affine groups \cite{6} provides an efficient method for determination of generators of MonExt_\beta.