Neural networks of the brain form one of the most complex systems we know. Many qualitative features of the emerging collective phenomena, such as correlated activity, stability, response to inputs, chaotic and regular behavior, can, however, be understood in simple models that are accessible to a treatment in statistical mechanics, or, more precisely, classical statistical field theory.

This tutorial presents the fundamentals behind contemporary developments in the theory of neural networks of rate units [e.g. 1–3] that are based on methods from statistical mechanics of classical systems with a large number of interacting degrees of freedom. In particular we will focus on a relevant class of systems that have quenched (time independent) disorder. In neural networks, the main source of disorder arises from random synaptic couplings between neurons. These systems are in many respects similar to spin glasses [4]. The tutorial therefore also explains the methods for these disordered systems as far as they are applied in neuroscience.

The presentation consists of two parts. In the first part we introduce stochastic differential equations (in the Ito-formulation and Stratonovich formulation) and present their treatment in the Martin–Siggia–Rose-De Dominicis path integral formalism [5, 6], reviewed in [7–9]. In the second part we will employ this language to derive the dynamic mean-field theory for deterministic random networks [10]. To our knowledge, a detailed presentation of the methods behind the results of this seminal paper is still lacking in the literature. Any inaccuracies in the present manuscript should therefore not be attributed to the authors of the original work [10], but to those of this tutorial. In deriving the formalism, we will follow the De Dominicis approach [6], that was also employed to obtain the dynamic mean-field theory of spin glasses [11–13]. The formalism in particular explains the statistics of the fluctuations in these networks and the emergence of different phases with regular and chaotic dynamics [10]. We will also cover a recent extension of the model to stochastic units [14].
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References
I. FUNCTIONAL FORMULATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

We here follow Chow and Buice [15] to derive the Martin-Siggia-Rose [5-8, 16, 17] path integral representation of a stochastic differential equation and Wio et al. [18] to obtain the Onsager-Machlup path integral. We generalize the notation to also include the Stratonovich convention as in [18]. Hertz et al. [9] also provide a pedagogical survey of the Martin-Siggia-Rose path integral formalism for the dynamics of stochastic and disordered systems.

The presented functional formulation of dynamics is advantageous in several respects. First, it recasts the dynamical equations into a path-integral, where the dynamic equations give rise to the definition of an “action”. In this way, the known tools from theoretical physics, such as perturbation expansions with the help of Feynman diagrams or the loopwise expansions to obtain a systematic treatment of fluctuations [19], can be applied. Within neuroscience, the recent review [8] illustrates the first, the work by [20] the latter approach. Moreover, this formulation will be essential for the treatment of disordered systems in Section II, following the spirit of the work by De Dominicis and Peliti [6] to obtain a generating functional that describes an average system belonging to an ensemble of systems with random parameters.

Many dynamic phenomena can be described by differential equations. Often, the presence of fluctuations is represented by an additional stochastic forcing. We therefore consider the stochastic differential equation (SDE)

\[ dx(t) = f(x) \, dt + dW(t) \]
\[ x(0+) = a, \]  

where \( a \) is the initial value and \( dW \) a stochastic increment. Stochastic differential equations are defined as the limit \( h \to 0 \) of a dynamics on a discrete time lattice of spacing \( h \). For discrete time \( t_i = lh, \, l = 0, \ldots, M \), the solution of the SDE consists of the discrete set of points \( x_i = x(t_l) \). For the discretization there are mainly two conventions used, the Ito and the Stratonovich convention [21]. Since we only consider additive noise, i.e. the stochastic increment in (1) does not depend on the state, both conventions yield the same continuous-time limit. However, as we will see, different discretization conventions of the drift term lead to different path integral representations. The Ito convention defines the symbolic notation of (1) to be interpreted as

\[ x_{i+1} - x_i = f(x_i) \, h + a \delta \delta_{00} + W_i, \]

where \( W_i \) is a stochastic increment that follows a probabilistic law. A common choice for \( W_i \) is a normal distribution \( \rho(W_i) = \mathcal{N}(0, hD) \), called a Wiener increment. Here the parameter \( D \) controls the variance of the noise. The term \( a \delta \delta_{00} \) ensures that the solution obeys the stated initial condition, assuming that \( x_{i0} = 0 \) in the absence of noise \( W_0 = 0 \). If the variance of the increment is proportional to the time step \( h \), this amounts to a \( \delta \)-distribution in the autocorrelation of the noise \( \xi = \frac{dW}{dt} \). The Stratonovich convention, also called mid-point rule, instead interprets the SDE as

\[ x_{i+1} - x_i = f \left( \frac{x_{i+1} + x_i}{2} \right) h + a \delta \delta_{00} + W_i, \]

Both conventions can be treated simultaneously by defining

\[ x_{i+1} - x_i = f(\alpha x_{i+1} + (1-\alpha)x_i) \, h + a \delta \delta_{00} + W_i \]
\[ \alpha \in [0, 1]. \]  

Here \( \alpha = 0 \) corresponds to the Ito convention and \( \alpha = \frac{1}{2} \) to Stratonovich. If the noise is drawn independently for each time step, i.e. if it is white, the probability density of the path \( x(t) \), i.e. a distribution in the points \( x_1, \ldots, x_M \), can be written as

\[ p(x_1, \ldots, x_M | 0) \equiv \int \prod_{i=0}^{M-1} dW_i \rho(W_i) \delta(x_{i+1} - y_{i+1}(W_i, x_i)), \]  

where, by (2), \( y_{i+1}(W_i, x_i) \) is understood as the solution of (2) at time point \( i + 1 \) given the noise realization \( W_i \) and the solution until the previous time point \( x_i \); The solution of the SDE starts at \( i = 0 \) with \( x_0 = 0 \) so that \( W_0 \) and \( a \) together determine \( x_1 \). In the next time step, \( W_1 \) and \( x_1 \) together determine \( x_2 \), and so on. In the Ito-convention \( (\alpha = 0) \) we have an explicit solution \( y_{i+1}(W_i, x_i) = x_i + f(x_i) \, h + a \delta \delta_{00} + W_i \), while the Stratonovich convention yields an implicit equation, since \( x_{i+1} \) appears as an argument of \( f \). We will see in (4) that the latter gives rise to a non-trivial normalization factor \((1-\alpha f^2 h)\) for \( p \), while for the former this factor is unity.
The notation \( y_{i+1}(W_i, x_i) \) indicates that the solution only depends on the last time point \( x_i \), but not on the history longer ago, which is called the \textbf{Markov property} of the process. This form also shows that the density is correctly normalized, because integrating over all paths

\[
\int dx_1 \cdots \int dx_M p(x_1, \ldots, x_M | a) = \int \prod_{i=0}^{M-1} dW_i \rho(W_i) \int dx_{i+1} \delta(x_{i+1} - y_{i+1}(W_i, x_i))
\]

\[= \prod_{i=0}^{M-1} dW_i \rho(W_i) = 1 \]

yields the normalization condition of \( \rho(W_i) \), \( i = 0, \ldots, M - 1 \), the distribution of the stochastic increments. In the limit \( M \to \infty \), we therefore define the probability functional as \( p(x[a]) = \lim_{M \to \infty} p(x_1, \ldots, x_M | a) \).

Using (3) and the substitution \( \delta(y) dy = \delta(x_{i+1}) \delta^i dx_{i+1} \) with \( y = \phi(x_{i+1}) = W_i(x_{i+1}) \) obtained by solving (2) for \( W_i \)

\[
W_i(x_{i+1}) = x_{i+1} - x_i - f(\alpha x_{i+1} + (1 - \alpha)x_i) h - a\delta_0
\]

\[
\frac{\partial W_i}{\partial x_{i+1}} = \delta' = 1 - \alpha f'h
\]

we obtain

\[
p(x_1, \ldots, x_M | a) = \int \prod_{i=0}^{M-1} dW_i \rho(W_i) \times
\]

\[
\times \delta(W_i - x_{i+1} - x_i - f(\alpha x_{i+1} + (1 - \alpha)x_i) h - a\delta_0) (1 - \alpha f'h).
\]

\[
= \prod_{i=0}^{M-1} \rho(x_{i+1} - x_i - f(\alpha x_{i+1} + (1 - \alpha)x_i) h - a\delta_0) (1 - \alpha h f'(\alpha x_{i+1} + (1 - \alpha)x_i)).
\]

In section Section I A we will look at the special case of Gaussian noise and derive the so called Onsager-Machlup path integral [22]. This path integral has a square in the action, originating from the Gaussian noise. For many applications, this square complicates the analysis of the system. The formulation presented in Section I B removes this square on the expense of the introduction of an additional field, the so called response field. This formulation has the additional advantage that responses of the system to perturbations can be calculated in compact form, as we will see below.

\[\text{A. Onsager-Machlup path integral}\]

For the case of a Gaussian noise \( \rho(W_i) = \mathcal{N}(0, Dh) = \frac{1}{\sqrt{2\pi Dh}} e^{-\frac{W^2}{2Dh}} \) the variance of the increment is

\[
(W_i W_j) = \begin{cases} D h & i = j \\ 0 & i \neq j \end{cases}
\]

\[= \delta_{ij} Dh. \]

Using the Gaussian noise and then taking the limit \( M \to \infty \) of eq. (5) with \( 1 - \alpha f'h \to \exp(-\alpha f'h) \) we obtain

\[
p(x_1, \ldots, x_M | a) = \prod_{i=0}^{M-1} \rho(x_{i+1} - x_i - f(\alpha x_{i+1} + (1 - \alpha)x_i) h - a\delta_0) (1 - \alpha f'h) + O(h^2)
\]

\[
= \prod_{i=0}^{M-1} \frac{1}{\sqrt{2\pi Dh}} \exp \left[ -\frac{1}{2Dh} (x_{i+1} - x_i - f(\alpha x_{i+1} + (1 - \alpha)x_i) h - a\delta_0)^2 - \alpha f'h \right] + O(h^2)
\]

\[= \left( \frac{1}{\sqrt{2\pi Dh}} \right)^M \exp \left[ -\frac{1}{2D} \sum_{i=0}^{M-1} \left( \frac{x_{i+1} - x_i}{h} - f(\alpha x_{i+1} + (1 - \alpha)x_i) - a\frac{\delta_0}{h} \right)^2 - \alpha f'h \right] h + O(h^2). \]
We will now define a symbolic notation by recognizing \( \lim_{h \to 0} \frac{x_{i+1} - x_i}{h} = \partial_t x(t) \) as well as \( \lim_{h \to 0} \frac{h}{\pi} = \delta(t) \) and \( \lim_{h \to 0} \sum_i f(h_i) h = \int f(t) \, dt \)

\[
p[x|x(0) = a] \mathcal{D}_{\sqrt{2\pi Dh}} x = \exp \left( -\frac{1}{2D} \int_0^T \left( \partial_t x - f(x) - a\delta(t) \right)^2 - \alpha f' dt \right) \mathcal{D}_{\sqrt{2\pi Dh}} x
\]

\[
:= \lim_{M \to \infty} p(x_1, \ldots, x_M|a) \frac{dx_1}{\sqrt{2\pi Dh}} \cdots \frac{dx_M}{\sqrt{2\pi Dh}},
\]

where we defined the integral measure \( \mathcal{D}_{\sqrt{2\pi Dh}} x := \Pi_{i=1}^M \frac{dx_i}{\sqrt{2\pi Dh}} \) to obtain a normalized density \( 1 = \int \mathcal{D}_{\sqrt{2\pi Dh}} x p[x|x(0) = a] \).

**B. Martin-Siggia-Rose-De Dominicis-Janssen (MSRDJ) path integral**

The square in the action (7) sometimes has disadvantages for analytical reasons, for example if quenched averages are to be calculated, as we will do in Section II. To avoid the square we will here introduce an auxiliary field, the **response field** \( \tilde{x} \) (the name will become clear in Section I D). This field enters the probability functional (5) by representing the \( \delta \)-distribution by its Fourier integral

\[
\delta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \, e^{i \tilde{x} x}.
\]

Replacing the \( \delta \)-distribution at each time slice by an integral over \( \tilde{x}_i \) at the corresponding slice, eq. (5) takes the form

\[
p(x_1, \ldots, x_M|a) = \Pi_{i=0}^{M-1} \left\{ \int dW_i \rho(W_i) \int_{-\infty}^{\infty} \frac{d\tilde{x}_i}{2\pi i} \exp \left( \tilde{x}_i (x_{i+1} - x_i) - f(\alpha x_{i+1} + (1-\alpha)x_i)h - W_i - a\delta(t) \right) - \alpha f'(h) \right\}
\]

\[
= \Pi_{i=0}^{M-1} \left\{ \int_{-\infty}^{\infty} \frac{d\tilde{x}_i}{2\pi i} \exp \left( \tilde{x}_i (x_{i+1} - x_i) - f(\alpha x_{i+1} + (1-\alpha)x_i)h - a\delta(t) \right) - \alpha f'(h) \right\} Z_W(\tilde{x}_i)
\]

\[
Z_W(\tilde{x}) \equiv \int dW_i \rho(W_i) e^{-\tilde{x}_i W_i} = \langle e^{-\tilde{x}_i W_i} \rangle_W.
\]

Here \( Z_W(\tilde{x}) \) is the moment generating function [23] also known as the characteristic function of the noise process, which is identical to the Fourier transform of the density (with \( i\omega = -\tilde{x} \)). Note the index \( i \) of the field \( \tilde{x}_i \) is the same as the index of the noise variable \( W_i \), which allows the definition of the characteristic function \( Z_W \). Hence the distribution of the noise only appears in the probability functional in the form of \( Z_W(\tilde{x}) \). For Gaussian noise (6) the characteristic function is

\[
Z_W(\tilde{x}) = \frac{1}{\sqrt{2\pi Dh}} \int dW e^{-\frac{W^2}{2dh}} e^{-\tilde{x}W} = \frac{1}{\sqrt{2\pi Dh}} \int dW e^{-\frac{W^2}{2dh}(W+D\tilde{x})^2} e^{-\frac{Dh}{2} \tilde{x}^2}
\]

\[
= e^{-\frac{Dh}{2} \tilde{x}^2}.
\]

**C. Moment generating functional**

The probability distribution (9) is a distribution for the random variables \( x_1, \ldots, x_M \). We can alternatively describe the probability distribution by the moment-generating functional by adding the terms \( \sum_{i=1}^{M} j_i x_i h \) to the action and integrating over all paths

\[
Z(j_1, \ldots, j_M) := \Pi_{i=1}^{M} \left\{ \int_{-\infty}^{\infty} dx_1 \exp \left( j_i x_i h \right) \right\} p(x_1, \ldots, x_M|a).
\]

Moments of the path can be obtained by taking derivatives (writing \( j = (j_1, \ldots, j_M) \))

\[
\frac{\partial}{\partial(h_{jk})} Z(j) \bigg|_{j=0} = \Pi_{i=1}^{M} \left\{ \int_{-\infty}^{\infty} dx_i \right\} p(x_1, \ldots, x_M|a) x_k \equiv \langle x_k \rangle.
\]
For $M \to \infty$ and $h \to 0$ the additional term $\exp \left( \sum_{i=1}^{M} j_i x_i h \right) \to \exp (\int j(t) x(t) dt)$. So the derivative on the left hand side of (12) turns into the functional derivative

$$\frac{\partial}{\partial(h_jk)} Z(j) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( Z(j_1, \ldots, j_k + \epsilon, \ldots, j_M) - Z(j_1, \ldots, j_k, \ldots, j_M) \right) \to_0 \frac{\delta}{\delta j(t)} Z[j],$$

and the moment becomes $(x(t))$ at time point $t = \hbar k$. The generating functional takes the explicit form

$$Z(j) = \Pi_{i=1}^{M} \left\{ \int_{-\infty}^{\infty} dx \exp (j_i x_i h) \right\} \times \exp \left( \sum_{i=0}^{M-1} \hat{x}_i (x_{i+1} - x_i) - f(\alpha x_{i+1} + (1 - \alpha) x_i) h - a\delta_{i0} \right) - \alpha f'h. \right) \right. \right) \right).$$

Note the different index ranges for the path coordinates $x_1, \ldots, x_M$ and the response field $\hat{x}_0, \ldots, \hat{x}_{M-1}$. Letting $h \to 0$ we now define the path integral as the generating functional (13) and introduce the notations $\Pi_{i=1}^{M} \int_{-\infty}^{\infty} dx_i \to \int Dx$ as well as $\Pi_{i=0}^{M-1} \int_{-\infty}^{\infty} d\hat{x}_i \to \int D\hat{x}$. Note that the different index ranges and the different integral boundaries are implicit in this notation, depending on whether we integrate over $x(t)$ or $\hat{x}(t)$. We hence write symbolically for the probability distribution (9)

$$p[x(t)|x(0+) = a] = \mathcal{D}\pi \hat{x} \exp \left( \int_{-\infty}^{\infty} \hat{x}(t)(\partial_t x - f(x) - a\delta(t)) - \alpha f' dt \right) Z_W[-\hat{x}]$$

$$Z_W[-\hat{x}] = \left\{ \exp \left( -\int_{-\infty}^{\infty} \hat{x}(t) dW(t) \right) \right\} W$$

where the respective second lines use the definition of the inner product on the space of functions

$$x^T y := \int_{-\infty}^{\infty} x(t)y(t) dt. \tag{15}$$

This vectorial notation also reminds us of the discrete origin of the path integral. Note that the lattice derivative appearing in (14) follows the definition $\partial_t x = \lim_{h \to 0} \frac{1}{h} (x(t+h) - x(t))$. We compactly denote the generating functional (13) as

$$Z[j] = \int D\pi \hat{x} \exp \left( \int \hat{x}(t)(\partial_t x - f(x) - a\delta(t)) - \alpha f' + j(t) x(t) dt \right) Z_W[-\hat{x}] \tag{16}.$$

For Gaussian white noise we have with (10) the moment generating functional $Z_W[-\hat{x}] = \exp \left( \frac{D}{2} \hat{x}^T \hat{x} \right)$. If in addition, we adopt the Ito convention, i.e. setting $\alpha = 0$, we get

$$Z[j] = \int D\pi \hat{x} \exp \left( \hat{x}^T (\partial_t x - f(x) - a\delta(t)) + \frac{D}{2} \hat{x}^T \hat{x} + j^T x \right). \tag{17}$$

### D. Response function in the MSRDJ formalism

The path integral (9) can be used to determine the response of the system to an external perturbation. To this end we consider the stochastic differential equation (1) that is perturbed by a time-dependent drive $-j(t)$

$$dx(t) = (f(x) - j(t)) dt + dW(t)$$

$$x(0+) = a.$$  

In the following we will only consider the Ito convention and set $\alpha = 0$. We perform the analogous calculation that leads from (1) to (13) with the additional term $-j(t)$ due to the perturbation. In the sequel we will see that, instead
of treating the perturbation explicitly, it can be expressed with the help of a second source term. The generating functional including the perturbation is

\[
Z(j, \tilde{j}) = \Pi_{i=1}^{M} \left\{ \int_{-\infty}^{\infty} dx_{i} \right\} \Pi_{k=0}^{M-1} \left\{ \int_{-\infty}^{\infty} \frac{d\tilde{x}_{k}}{2\pi i} Z_{W}(-\tilde{x}_{k}) \right\} \times \\
\times \exp \left( \sum_{j=0}^{M-1} \tilde{x}_{j} (x_{j+1} - x_{j} - f(x_{j}) \delta_{t,0} + j_{j+1} x_{j+1} \delta_{t,0} + \tilde{x}_{j} \delta_{t,0}) \right) \\
= \int \mathcal{D}x \int \mathcal{D}_{2\pi i} \tilde{x} Z_{W}[-\tilde{x}] \exp \left( \int_{-\infty}^{\infty} \tilde{x}(t)(\partial_{t} x - f(x) - a\delta(t)) + j(t)x(t) + \tilde{j}(t)\tilde{x}(t)\,dt \right),
\]

where we moved the \( \tilde{j} \)-dependent term out of the parenthesis.

Note that the external field \( j \) is indexed from 1, \ldots, \( M \) (as \( x_{i} \)) whereas \( \tilde{j} \) is indexed 0, \ldots, \( M-1 \) (as \( \tilde{x} \)). As before, the moments of the process follow as functional derivatives (12) \( \frac{\delta}{\delta j(t)} Z[j, \tilde{j}] \big|_{j=\tilde{j}=0} = \langle x(t) \rangle \). Higher order moments follow as higher derivatives.

The additional dependence on \( \tilde{j} \) allows us to investigate the response of arbitrary moments to a small perturbation localized in time, i.e. \( \tilde{j}(t) = -\epsilon\delta(t-s) \). In particular, we characterize the average response of the first moment with respect to the unperturbed system by the response function \( \chi(t, s) \)

\[
\chi(t, s) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \langle x(t) \rangle_{j=\epsilon\delta(-s)} - \langle x(t) \rangle_{j=0} \right) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \mathcal{D}x \, x(t) \left( p[x, j = -\epsilon\delta(t-s)] - p[x, j = 0] \right) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \frac{\delta}{\delta j(t)} \left( Z[j, \tilde{j} - \epsilon\delta(t-s)] - Z[j, \tilde{j}] \right) \right)_{j=\tilde{j}=0} \\
= -\langle x(t) \, \tilde{x}(s) \rangle,
\]

where we used the definition of the functional derivative from the third to the fourth line. So instead of treating a small perturbation explicitly, the response of the system to a perturbation can be obtained by a functional derivative with respect to \( \tilde{j} \): \( \tilde{j} \) couples to \( \tilde{x} \), \( j \) contains perturbations, therefore \( \tilde{x} \) measures the response and is the so called response field. The response function \( \chi(t, s) \) can then be used as a kernel to obtain the mean response of the system to a small external perturbation of arbitrary temporal shape.

There is an important difference for the response function between the Ito and Stratonovich formulation, that is exposed in the time-discrete formulation. For the perturbation \( \tilde{j}(t) = -\epsilon\delta(t-s) \), we obtain the perturbed equation, where \( \frac{\delta}{\delta j(t)} \) denotes the discretized time point at which the perturbation is applied. The perturbing term must be treated analogously to \( f \), so

\[
x_{i+1} - x_{i} = f(\alpha x_{i+1} + (1-\alpha) x_{i}) \, h + \epsilon \left( \alpha \delta_{i+1} + (1-\alpha) \delta_{i} \right) + W_{i} \quad \alpha \in [0, 1].
\]

Consequently, the value of the response function \( \chi(s, s) \) at the time of the perturbation depends on the choice of \( \alpha \). We denote as \( x_{s}^{\epsilon} \) the solution after application of the perturbation, as \( x_{s}^{0} \) the solution without; for \( i < j \) the two are identical and the equal-time response is

\[
\chi(s, s) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( x_{s}^{\epsilon} - x_{s}^{0} \right) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( f(\alpha x_{s}^{\epsilon} + (1-\alpha) x_{s-1}) - f(\alpha x_{s}^{0} + (1-\alpha) x_{s-1}) \right) \, h + \alpha \delta_{s} + (1-\alpha) \delta_{s-1}.
\]

because the contribution of the deterministic evolution vanishes due to the factor \( h \). So for \( \alpha = 0 \) (Ito convention) we have \( \chi(s, s) = 0 \), for \( \alpha = \frac{1}{2} \) (Stratonovich) we have \( \chi(s, s) = \frac{1}{2} \). The Ito-convention is advantageous in this respect, because it leads to vanishing contributions in Feynman diagrams with response functions at equal time points [8].
We also observe that the initial condition contributes a term $-a\delta_{t,0}$. Consequently, the initial condition can alternatively be included by setting $a = 0$ and instead calculate all moments from the generating functional $Z[j, \tilde{j} - a\delta]$ instead of $Z[j, \tilde{j}]$. In the following we will therefore skip the explicit term ensuring the proper initial condition as it can be inserted by choosing the proper value for the source $\tilde{j}$. See also [9, Sec. 5.5].

For the important special case of Gaussian white noise (6), the generating functional, including the source field $\tilde{j}$ coupling to the response field, takes the form

$$Z[j, \tilde{j}] = \int \mathcal{D}x \int \mathcal{D}_{2\pi i} \tilde{x} \exp \left( \tilde{x}^T (\partial_t x - f(x)) + \frac{D}{2} \tilde{x}^T \tilde{x} + j^T x + \tilde{j}^T \tilde{x} \right),$$

where we again used the definition of the inner product (15).
II. DYNAMIC MEAN-FIELD THEORY FOR RANDOM NETWORKS

Systems with many interacting degrees of freedom present a central quest in physics. While disordered equilibrium systems show fascinating properties such as the spin-glass transition \cite{11,24}, new collective phenomena arise in non-equilibrium systems: Large random networks of neuron-like units can exhibit chaotic dynamics \cite{10,25,26} with important functional consequences. In particular, information processing capabilities show optimal performance close to the onset of chaos \cite{27,29}.

Until today, the seminal work by Sompolinsky et al. \cite{10} has a lasting influence on the research field of random recurrent neural networks, presenting a solvable random network model with deterministic continuous-time dynamics that admits a calculation of the transition to a chaotic regime and a characterization of chaos by means of Lyapunov exponents. Many subsequent studies have built on top of this work \cite{1,3,14,30,32}.

The presentation in the original work \cite{10}, published in Physical Review Letters, summarizes the main steps of the derivations and the most important results. In this chapter, we would like to show the formal calculations that, in our view, reproduce the most important results. In lack of an extended version of the original work, we do not know if the calculations by the original authors are identical to the presentation here. However, we hope that the didactic presentation given here may be helpful to provide an easier access to the original work.

Possible errors in this document should not be attributed to the original authors, but to the authors of this manuscript. In deriving the theory, we also present a recent extension of the model to stochastic dynamics due to additive uncorrelated Gaussian white noise \cite{14}. The original results of \cite{10} are obtained by setting the noise amplitude $D=0$ in all expressions.

A. Definition of the model and generating functional

We study the coupled set of first order stochastic differential equations
\begin{equation}
    dx(t) + x(t) \, dt = J \phi(x(t)) \, dt + dW(t),
\end{equation}
where
\begin{equation}
    J_{ij} \sim \begin{cases} 
        \mathcal{N}(0, \frac{\sigma^2}{N}) \text{ i.i.d.} & \text{for } i \neq j \\
        0 & \text{for } i = j
    \end{cases}
\end{equation}
are i.i.d. Gaussian random couplings, $\phi$ is a non-linear gain function applied element-wise, the $dW_i$ are pairwise uncorrelated Wiener processes with $\langle dW_i^2(t) \rangle = D \, dt$. For concreteness we will use
\begin{equation}
    \phi(x) = \tanh(x),
\end{equation}
as in the original work \cite{10}.

We formulate the problem in terms of a generating functional from which we can derive all moments of the activity as well as response functions. Introducing the notation $\tilde{x}^T x = \sum_i \int \tilde{x}_i(t) x_i(t) \, dt$, we obtain the moment-generating functional (cf. eq. (20))
\begin{equation}
    Z[j, \tilde{j}](J) = \int \mathcal{D}x \int \mathcal{D}\tilde{x} \exp \left( S_0[x, \tilde{x}] - \tilde{x}^T J \phi(x) + j^T x + \tilde{j}^T \tilde{x} \right)
\end{equation}
with $S_0[x, \tilde{x}] = \tilde{x}^T (\partial_x + 1) x + \frac{D}{2} \tilde{x}^T \tilde{x}$, (24)
where the measures are defined as $\int \mathcal{D}x = \lim_{M \to \infty} \prod_{i=1}^N \Pi_{k=1}^M \int_{-\infty}^{\infty} dx_i^k$ and $\lim_{M \to \infty} \int \mathcal{D}\tilde{x} = \Pi_{i=1}^N \Pi_{k=0}^{M-1} \int_{-\infty}^{i\infty} \frac{dx_i^k}{2\pi i}$. Here the superscript $k$ denotes the $k$-th time slice and we skip the subscript $\mathcal{D}_{2\pi i}$, as introduced in (7) in Section 1A, in the measure of $\mathcal{D}x$. The action $S_0$ is defined to contain all single unit properties, therefore excluding the coupling term $-\tilde{x}^T J \phi(x)$, which is written explicitly.

B. Average over the quenched disorder

The dynamics of (21) shows invariant features independent of the actual realization of the couplings, only dependent on their statistics, here parameterized by $g$. To capture these properties that are generic to the ensemble of the models,
we introduce the averaged functional

$$\tilde{Z}[\tilde{j}, \tilde{\phi}] := \langle Z[\tilde{j}, \tilde{\phi}] \rangle_J$$

$$= \int \Pi_{ij} dJ_{ij} N \left(0, \frac{g^2}{N}, J_{ij} \right) Z[\tilde{j}, \tilde{\phi}] (J).$$

We use that the coupling term $\exp(-\sum_i J_{ij} \int \tilde{x}_i(t) \phi(x_j(t)) \, dt)$ in (24) factorizes into $\Pi_{is} \exp(-J_{is} \int \tilde{x}_i(t) \phi(x_j(t)) \, dt)$ as does the distribution over the couplings (due to $J_{ij}$ being independently distributed). We make use of the couplings appearing linearly in the action and complete the square (in each $J_{ij}$ separately) to obtain for $i \neq j$

$$\int dJ_{ij} N \left(0, \frac{g^2}{N}, J_{ij} \right) \exp \left(-J_{ij} \int \tilde{x}_i(t) \phi(x_j(t)) \, dt \right)$$

$$= \exp \left( \frac{g^2}{2N} \left( \int \tilde{x}_i(t) \phi(x_j(t)) \, dt \right)^2 \right).$$

We reorganize the last term including the sum $\Sigma_{is}$ as

$$\exp \left( \frac{g^2}{2N} \sum_{i \neq j} \left( \int \tilde{x}_i(t) \phi(x_j(t)) \, dt \right)^2 \right)$$

$$= \exp \left( \frac{g^2}{2N} \sum_{i \neq j} \int \int \tilde{x}_i(t) \phi(x_j(t)) \tilde{x}_i(t') \phi(x_j(t')) \, dt \, dt' \right)$$

$$= \exp \left( \frac{1}{2} \int \int \left( \sum_i \tilde{x}_i(t) \tilde{x}_i(t') \right) \left( \frac{g^2}{2N} \sum_j \phi(x_j(t)) \phi(x_j(t')) \right) \, dt \, dt' \right)$$

$$\exp \left( \frac{-g^2}{2N} \int \int \sum_i \tilde{x}_i(t) \tilde{x}_i(t') \phi(x_i(t)) \phi(x_i(t')) \, dt \, dt' \right),$$

where we used $(\int f(t) dt)^2 = \int f(t) f(t') \, dt \, dt'$ in the first step and $\Sigma_{ij} x_i y_j = \Sigma_i x_i \Sigma_j y_j$ in the second. The last term is the diagonal element that is to be taken out of the double sum. It is a correction of order $N^{-1}$ and will be neglected in the following. The disorder-averaged generating functional (25) therefore takes the form

$$\tilde{Z}[\tilde{j}, \tilde{\phi}] = \int \mathcal{D}x \int \mathcal{D}\tilde{x} \exp \left( S_0 [x, \tilde{x}] + \tilde{j}^T x + \tilde{\phi}^T \tilde{x} \right) \times$$

$$\times \exp \left( \frac{1}{2} \int \int_{-\infty}^{\infty} \sum_i \tilde{x}_i(t) \tilde{x}_i(t') \left( \frac{g^2}{2N} \sum_j \phi(x_j(t)) \phi(x_j(t')) \right) \, dt \, dt' \right).$$

The coupling term in the last line contains quantities that depend on four fields. We now aim to decouple these terms into terms of products of pairs of fields. The aim is to make use of the central limit theorem, namely that the quantity $Q_1$ indicated by the curly braces in (27) is a superposition of a large ($N$) number of (weakly correlated) contributions, which will hence approach a Gaussian distribution. The outcome of the saddle point approximation to lowest order will be the replacement of $Q_1$ by its expectation value, as we will see in the following steps. We define

$$Q_1(t, s) := \frac{g^2}{N} \sum_j \phi(x_j(t)) \phi(x_j(s))$$

and enforce this condition by inserting the Dirac-$\delta$ functional

$$\delta \left[ -\frac{N}{g^2} Q_1(s, t) + \sum_j \phi(x_j(s)) \phi(x_j(t)) \right]$$

$$= \int \mathcal{D}Q_2 \exp \left( \int \int Q_2(s, t) \left[ -\frac{N}{g^2} Q_1(s, t) + \sum_j \phi(x_j(s)) \phi(x_j(t)) \right] \, ds \, dt \right).$$

We here note that as for the response field, the field $Q_2 \in i\mathbb{R}$ is purely imaginary due to the Fourier representation (8) of the $\delta$. 
Figure 1. Finding saddle point by maximizing contribution to probability: The contribution to the overall probability mass depends on the value of the parameter $Q$, i.e. we seek to maximize $b(Q) := \int Dx \, p[x; Q]$ (32). The point at which the maximum is attained is denoted as $Q^*$, the value $b(Q^*)$ is indicated by the hatched area.

We aim at a set of self-consistent equations for the auxiliary fields. We therefore introduce one source term for each of the fields to be determined. Extending our notation by defining $Q_1^TQ_2 := \int Q_1(s, t) Q_2(s, t) \, ds \, dt$ and $\hat{x}^TQ_1\hat{x} := \int \hat{x}(s) Q_1(s, t) \hat{x}(t) \, ds \, dt$ we hence rewrite (27) as

$$
\begin{align*}
\tilde{Z}[j, \hat{j}] &= \int \mathcal{D}Q_1 \int \mathcal{D}Q_2 \exp\left(-\frac{N}{g^2} Q_1^T Q_2 + N \ln Z[Q_1, Q_2] + j^T Q_1 + \hat{j}^T Q_2\right) \\
Z[Q_1, Q_2] &= \int \mathcal{D}x \int \mathcal{D}\hat{x} \exp\left(S_0[x, \hat{x}] + \frac{1}{2} \hat{x}^T Q_1 \hat{x} + \phi(x)^T Q_2 \phi(x)\right),
\end{align*}
$$

where the integral measures $\mathcal{D}Q_{1,2}$ must be defined suitably. In writing $N \ln Z[Q_1, Q_2]$ we have used that the auxiliary fields couple only to sums of fields $\sum_x \phi^2(x_i)$ and $\sum_x \hat{x}_i^2$, so that the generating functional for the fields $x$ and $\hat{x}$ factorizes into a product of $N$ factors $Z[Q_1, Q_2]$. The latter only contains functional integrals over the two scalar fields $x$, $\hat{x}$. This shows that we have reduced the problem of $N$ interacting units to that of a single unit exposed to a set of external fields $Q_1$ and $Q_2$.

The remaining problem can be considered a field theory for the auxiliary fields $Q_1$ and $Q_2$. The form (30) clearly exposes the $N$ dependence of the action for these latter fields in (30): It is of the form $\int dQ \exp(N f(Q)) \, dQ$, which, for large $N$, suggests a saddle point approximation.

In the saddle point approximation [12] we seek the stationary point of the action determined by

$$
0 = \frac{\delta S[Q_1, Q_2]}{\delta Q_{1,2}} = \frac{\delta}{\delta Q_{1,2}} \left(-\frac{N}{g^2} Q_1^T Q_2 + N \ln Z[Q_1, Q_2]\right) = 0.
$$

We here set the value for the source fields $j = \hat{j} = 0$ to zero. This corresponds to finding the point in the space $(Q_1, Q_2)$ which provides the dominant contribution to the probability mass. This can be seen by writing the probability functional as $p[x] = \int \mathcal{D}Q_1 \mathcal{D}Q_2 \, p[x; Q_1, Q_2]$ with

$$
p[x; Q_1, Q_2] = \exp\left(-\frac{N}{g^2} Q_1^T Q_2 + \sum_i \ln \int D\hat{x} \exp\left(S_0[x_i, \hat{x}] + \frac{1}{2} \hat{x}^T Q_1 \hat{x} + \phi(x_i)^T Q_2 \phi(x_i)\right)\right)$$

$$
b[Q_1, Q_2] := \int \mathcal{D}x \, p[x; Q_1, Q_2],
$$

where we defined $b[Q_1, Q_2]$ as the contribution to the entire probability mass for a given value of the auxiliary fields $Q_1, Q_2$. Maximizing $b$ therefore amounts to the condition (31), illustrated in Figure 1. We here used the convexity of the exponential function.

A more formal argument to obtain (31) proceeds by introducing the Legendre-Fenchel transform of $\ln \tilde{Z}$ as

$$
\Gamma(q_1, q_2) := \sup_{j, \hat{j}} j^T q_1 - \ln \tilde{Z}[j, \hat{j}],
$$
called the vertex generating functional or effective action [19, 33]. It holds that \( \frac{\delta \Gamma}{\delta q_1} = j \) and \( \frac{\delta \Gamma}{\delta q_2} = \tilde{j} \), called equations of state. The leading order mean-field approximation amounts to the approximation \( \Gamma[q_1, q_2] \approx -S[q_1, q_2] \). The equations of state, for vanishing sources \( j = \tilde{j} = 0 \), therefore yield the saddle point equations

\[
0 = \frac{\delta \Gamma}{\delta q_1} = \frac{\delta S}{\delta q_1}, \\
0 = \frac{\delta \Gamma}{\delta q_2} = -\frac{\delta S}{\delta q_2},
\]

identical to (31). This more formal view has the advantage of being straightforwardly extendable to loopwise corrections.

The functional derivative in the stationarity condition (31) applied to \( \ln Z[q_1, q_2] \) produces an expectation value with respect to the distribution (32): the fields \( Q_1 \) and \( Q_2 \) here act as sources. This yields the set of two equations

\[
0 = -\frac{N}{g^2} Q_1^*(t, t') + \frac{N}{Z} \frac{\delta Z[q_1, q_2]}{\delta Q_1(s, t)} \bigg|_{Q^*} \rightarrow Q_1^*(s, t) = g^2 \langle \phi(x(s))\phi(x(t)) \rangle_{Q^*} = g^2 C_{\phi(x)\phi(x)}(t, t') \tag{33}
\]

\[
0 = -\frac{N}{g^2} Q_2^*(t, t') + \frac{N}{Z} \frac{\delta Z[q_1, q_2]}{\delta Q_2(s, t)} \bigg|_{Q^*} \rightarrow Q_2^*(s, t) = g^2 \langle \tilde{x}(s)\tilde{x}(t) \rangle_{Q^*} = 0,
\]

where we defined the average autocorrelation function \( C_{\phi(x)\phi(x)}(t, t') \) of the non-linearly transformed activity of the units. The second saddle point \( Q_2^* = 0 \) vanishes, as it would otherwise alter the normalization of the generating functional through mixing of retarded and non-retarded time derivatives which then yield acausal response functions [12].

The expectation values \( \langle \rangle_{Q^*} \) appearing in (33) must be computed self-consistently, since the values of the saddle points, by (30), influence the statistics of the fields \( x \) and \( \tilde{x} \), which in turn determines the functions \( Q_1^* \) and \( Q_2^* \) by (33).

Inserting the saddle point solution into the generating functional (30) we get

\[
Z^* \propto \int D\phi(x) \int D\phi(x) \exp \left( S_0[x, \tilde{x}] + \frac{g^2}{2} \tilde{x}^T C_{\phi(x)\phi(x)} \tilde{x} \right). \tag{34}
\]

As the saddle points only couple to the sums of fields, the action has the important property that it decomposes into a sum of actions for individual, non-interacting units that feel a common field with self-consistently determined statistics, characterized by its second cumulant \( C_{\phi(x)\phi(x)} \). Hence the saddle-point approximation reduces the network to \( N \) non-interacting units, or, equivalently, a single unit system. The second term in (34) is a Gaussian noise with a two point correlation function \( C_{\phi(x)\phi(x)}(t, t') \). The physical interpretation is the noisy signal each unit receives due to the input from the other \( N \) units. Its autocorrelation function is given by the summed autocorrelation functions of the output activities \( \phi(x_i(t)) \) weighted by \( g^2 N^{-1} \), which incorporates the Gaussian statistics of the couplings. This intuitive picture is shown in Figure 2.

The interpretation of the noise can be appreciated by explicitly considering the moment generating functional of a Gaussian noise with a given autocorrelation function \( C(t, t') \), which leads to the cumulant generating functional...
\[ \ln Z_\zeta[\tilde{x}] = \ln(\exp(\int \tilde{x}(t) \zeta(t) dt)) \]
\[ = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{x}(t) C(t, t') \tilde{x}(t') dt dt' \]
\[ = \frac{1}{2} \tilde{x}^T C \tilde{x}. \]

Note that the effective noise term only has a non-vanishing second cumulant. This means the effective noise is Gaussian, as the cumulant generating function is quadratic. It couples pairs of time points that are correlated.

This is the starting point in [10, eq. (3)], stating that the effective mean-field dynamics of the network is given by that of a single unit

\[ (\partial_t + 1) x(t) = \eta(t) \]

(35)
driven by a Gaussian noise \( \eta = \zeta + \frac{dW}{dt} \) with autocorrelation \( \langle \eta(t) \eta(s) \rangle = g^2 C_{\phi(x)\phi(x)}(t, s) + D \delta(t - s) \). In the cited paper the white noise term \( \propto D \) is absent, though.

We multiply the equation (35) for time points \( t \) and \( s \) and take the expectation value with respect to the noise \( \eta \) on both sides, which leads to

\[ (\partial_t + 1) (\partial_s + 1) C_{xx}(t, s) = g^2 C_{\phi(x)\phi(x)}(t, s) + D \delta(t - s), \]

(36)
where we defined the covariance function of the activities \( C_{xx}(t, s) := \langle x(t)x(s) \rangle \). In the next section we will rewrite this equation into an equation of a particle in a potential.

C. Stationary statistics: Self-consistent autocorrelation of as motion of a particle in a potential

We are now interested in the stationary statistics of the system, i.e. \( C_{xx}(t, s) = c(t - s) \). The inhomogeneity in (36) is then also time-translation invariant, \( C_{\phi(x)\phi(x)}(t + \tau, t) \) is only a function of \( \tau \). Therefore the differential operator \( (\partial_t + 1)(\partial_s + 1) c(t - s) \), with \( \tau = t - s \), simplifies to \( -\partial_\tau^2 + 1 \) \( c(\tau) \) so we get

\[ (-\partial_\tau^2 + 1) c(\tau) = g^2 C_{\phi(x)\phi(x)}(t + \tau, t) + D \delta(\tau). \]

(37)
Once (37) is solved, we know the covariance function \( c(\tau) \) between two time points \( \tau \) apart as well as the variance \( c(0) := c_0 \). Since by the saddle point approximation in Section II B the expression (34) is the generating functional of a Gaussian theory, the \( x_i \) are zero mean Gaussian random variables. Consequently the second moment completely determines the distribution. We can therefore obtain \( C_{\phi(x)\phi(x)}(t, s) = g^2 f_\Phi(c(\tau), c_0) \) with

\[ f_u(c, c_0) = \int u \left( \sqrt{c_0 - \frac{c^2}{c_0}} \right) \left( \sqrt{c_0} \right) Dz_1 Dz_2 \]

(38)
with the Gaussian integration measure \( Dz = \exp(-z^2/2)\sqrt{2\pi} dz \) and for a function \( u(x) \). Here, the two different arguments of \( u(x) \) are by construction Gaussian with zero mean, variance \( c(0) = c_0 \), and covariance \( c(\tau) \). Note that (38) reduces to one-dimensional integrals for \( f_u(c_0, c_0) = \langle u(x)^2 \rangle \) and \( f_u(0, c_0) = \langle u(x) \rangle^2 \), where \( x \) has zero mean and variance \( c_0 \).

We note that \( f_u(c(\tau), c_0) \) in (38) only depends on \( \tau \) through \( c(\tau) \). We can therefore obtain from the “potential” \( g^2 f_\Phi(c(\tau), c_0) \) by

\[ C_{\phi(x)\phi(x)}(t + \tau, t) = \frac{\partial}{\partial c} g^2 f_\Phi(c(\tau), c_0) \]

(39)
where \( \Phi \) is the integral of \( \phi \), i.e. \( \Phi(x) = \int_0^x \phi(x) dx = \ln \cosh(x) \). The property \( \frac{\partial}{\partial c} g^2 f_\Phi(c, c_0) = g^2 f_\Phi(c(\tau), c_0) \) (Price’s theorem [34]) is shown in the supplementary calculation in Section III A. Note that the representation in (38) differs from the one used in [10, eq. (7)]. The expression used here is also valid for negative \( c(\tau) \) in contrast to the original formulation. We can therefore express the differential equation for the autocorrelation with the definition of the potential \( V \)

\[ V(c; c_0) := -\frac{1}{2} c^2 + g^2 f_\Phi(c(\tau), c_0) - g^2 f_\Phi(0, c_0), \]

(40)
where the subtraction of the last constant term is an arbitrary choice that ensures that \( V(0; c_0) = 0 \). The equation of motion (37) therefore takes the form

\[
\partial^2_c c(\tau) = -V'(c(\tau); c_0) - D \delta(\tau),
\]

so it describes the motion of a particle in a (self-consistent) potential \( V \) with derivative \( V' = \frac{\partial}{\partial c} V \). The \( \delta \)-distribution on the right hand side causes a jump in the velocity that changes from \( \frac{\partial}{\partial c} c(\tau) \) to \(-\frac{\partial}{\partial c} c(\tau)\) at \( \tau = 0 \), because \( c \) is symmetric \((c(\tau) = c(-\tau))\) and hence \( \dot{c}(\tau) = -\dot{c}(-\tau) \) and moreover the term \(-V'(c(\tau); c_0)\) does not contribute to the kink. The equation must be solved self-consistently, as the initial value \( c_0 \) determines the effective potential \( V(\cdot; c_0) \) via (40). The second argument \( c_0 \) indicates this dependence.

The gain function \( \phi(x) = \tanh(x) \) is shown in Figure 3a, while Figure 3b shows the self-consistent potential for the noiseless case \( D = 0 \).

The potential is formed by the interplay of two opposing terms. The downward bend is due to \(-\frac{1}{2} c^2\). The term \( g^2 f_\phi(c; c_0) \) is bent upwards. We get an estimate of this term from its derivative \( g^2 f_\phi'(c; c_0) \): Since \( \phi(x) \) has unit slope at \( x = 0 \) (see Figure 3a), for small amplitudes \( c_0 \) the fluctuations are in the linear part of \( \phi \), so \( g^2 f_\phi(c, c_0) \equiv g^2 c \) for all \( c \leq c_0 \). Consequently, the potential \( g^2 f_\phi(c, c_0) = \int_0^c g^2 f_\phi(c', c_0) \, dc' \) \( c_{c_0} < 1 \) \( g^2 \frac{c}{2} c^2 \) has a positive curvature at \( c = 0 \).

For \( g < 1 \), the parabolic part dominates for all \( c_0 \), so that the potential is bent downwards and the only bounded solution in the noiseless case \( D = 0 \) of (41) is the vanishing solution \( c(\tau) \equiv 0 \).

For \( D > 0 \), the particle may start at some point \( c_0 > 0 \) and, due to its initial velocity, reach the point \( c(\infty) = 0 \). Any physically reasonable solution must be bounded. In this setting, the only possibility is a solution that starts at a position \( c_0 > 0 \) with the same initial energy \( V(c_0; c_0) + E_{\text{kin}}^{(0)} \) as the final potential energy \( V(0; c_0) = 0 \) at \( c = 0 \). The initial kinetic energy is given by the initial velocity \( \dot{c}(0+) = -\frac{D}{2} \) as \( E_{\text{kin}}^{(0)} = \frac{1}{2} \dot{c}(0+)^2 = \frac{D^2}{4} \). This condition ensures that the particle, starting at \( \tau = 0 \) at the value \( c_0 \) for \( \tau \to \infty \) reaches the local maximum of the potential at \( c = 0 \); the covariance function decays from \( c_0 \) to zero.

For \( g > 1 \), the term \( g^2 f_\phi(c; c_0) \) can start to dominate the curvature close to \( c = 0 \): the potential in Figure 3b is bent upwards for small \( c_0 \). For increasing \( c_0 \), the fluctuations successively reach the shallower parts of \( \phi \), hence the slope of \( g^2 f_\phi(c; c_0) \) diminishes, as does the curvature of its integral, \( g^2 f_\phi(c; c_0) \). With increasing \( c_0 \), the curvature of the potential at \( c = 0 \) therefore changes from positive to negative.

In the intermediate regime, the potential assumes a double well shape. Several solutions exist in this case. One can show that the only stable solution is the one that decays to 0 for \( \tau \to \infty \) [10]. In the presence of noise \( D > 0 \) this assertion is clear due to the decorrelating effect of the noise, but it remains true also in the noiseless case.
By the argument of energy conservation, the corresponding value $c_0$ can be found numerically as the root of

$$V(c_0; c_0) + E_{\text{kin}}^{(0)} \frac{1}{2} = 0$$

$$E_{\text{kin}}^{(0)} = \frac{D^2}{8},$$

for example with a simple bisectioning algorithm.

The corresponding shape of the autocovariance function then follows a straightforward integration of the differential equation (41). Rewriting the second order differential equation into a coupled set of first order equations, introducing $\partial_\tau c = y$, we get for $\tau > 0$

$$\partial_\tau \begin{pmatrix} y(\tau) \\ c(\tau) \end{pmatrix} = \begin{pmatrix} c - g^2 f_\phi(c, c_0) \\ y(\tau) \end{pmatrix}$$

with initial condition

$$\begin{pmatrix} y(0) \\ c(0) \end{pmatrix} = \begin{pmatrix} -D \\ c_0 \end{pmatrix}. $$

The solution of this equation in comparison to direct simulation is shown in Figure 4. Note that the covariance function of the input to a unit, $C_{\phi\phi}(\tau) = g^2 f_\phi(c(\tau), c_0)$, bares strong similarities to the autocorrelation $c$, shown in Figure 4c: The suppressive effect of the non-linear, saturating gain function is compensated by the variance of the connectivity $g^2 > 1$, so that a self-consistent solution is achieved.
D. Assessing chaos by a pair of identical systems

We now aim to study whether the dynamics is chaotic or not. To this end, we consider a pair of identically prepared systems, in particular with identical coupling matrix $\mathbf{J}$ and, for $D > 0$, also the same realization of the Gaussian noise. We distinguish the dynamical variables $x^\alpha$ of the two systems by superscripts $\alpha \in \{1, 2\}$.

Let us briefly recall that the dynamical mean-field theory describes empirical population-averaged quantities for a single network realization (due to self-averaging). Hence, for large $N$ we expect that

$$\frac{1}{N} \sum_{i=1}^{N} x^\alpha_i(t)x^\beta_i(s) \approx c^{\alpha\beta}(t, s)$$

holds for most network realizations. To study the stability of the dynamics with respect to perturbations of the initial conditions we consider the population-averaged (mean-)squared distance between the trajectories of the two copies of the network:

$$\frac{1}{N} \|x^1(t) - x^2(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} (x^1_i(t) - x^2_i(t))^2 \quad (44)$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} (x^1_i(t))^2 + \frac{1}{N} \sum_{i=1}^{N} (x^2_i(t))^2 - 2 \frac{1}{N} \sum_{i=1}^{N} x^1_i(t)x^2_i(t)$$

$$\approx c^{11}(t, t) + c^{22}(t, t) - 2c^{12}(t, t).$$

This idea has also been employed in [35]. Therefore, we define the mean-field mean-squared distance between the two copies:

$$d(t, s) := c^{11}(t, s) + c^{22}(t, s) - c^{12}(t, s) - c^{21}(t, s), \quad (45)$$

which gives for equal time arguments the actual mean-squared distance $d(t) := d(t, t)$. Our goal is to find the temporal evolution of $d(t, s)$. The time evolution of a pair of systems in the chaotic regime with slightly different initial conditions is shown in Figure 5. Although the initial displacement between the two systems is drawn independently for each of the four shown trials, the divergence of $d(t)$ has a stereotypical form, which seems to be dominated by one largest Lyapunov exponent. The aim of the remainder of this section is to find this rate of divergence.

To derive an equation of motion for $d(t, s)$ it is again convenient to define a generating functional that captures the joint statistics of two systems and in addition allows averaging over the quenched disorder [see also 19, Appendix 23, last remark].

The generating functional is defined in analogy to the single system (24)

$$Z[\{\tilde{j}^\alpha, \tilde{j}^\beta\}_{\alpha \in \{1, 2\}}](\mathbf{J}) = \Pi_{\alpha=1}^2 \{ \int \mathcal{D}x^\alpha \int \mathcal{D}x^\alpha \exp \left( \tilde{x}^\alpha \mathcal{T} \left( (\partial_t + 1) x^\alpha - \sum_j \mathbf{J} \phi(x^\alpha) \right) + j^\alpha T \mathbf{x}^\alpha + \tilde{j}^\alpha T \tilde{\mathbf{x}}^\alpha \right) \times \exp \left( \frac{D}{2} (\tilde{x}^1 + \tilde{x}^2)^T (\tilde{x}^1 + \tilde{x}^2) \right) \}, \quad (46)$$

where the last term is the moment generating functional due to the white noise that is common to both subsystems. We note that the coupling matrix $\mathbf{J}$ is the same in both subsystems as well. Using the notation analogous to (24) and collecting the terms that affect each individual subsystem in the first, the common term in the second line, we get

$$Z[\{j^\alpha, \tilde{j}^\alpha\}_{\alpha \in \{1, 2\}}](\mathbf{J}) = \Pi_{\alpha=1}^2 \{ \int \mathcal{D}x^\alpha \int \mathcal{D}x^\alpha \exp \left( S_0[\tilde{x}^\alpha, \tilde{\mathbf{x}}^\alpha] - \tilde{x}^\alpha T \mathbf{J} \phi(x^\alpha) + j^\alpha T \mathbf{x}^\alpha + \tilde{j}^\alpha T \tilde{\mathbf{x}}^\alpha \right) \times \exp \left( D \tilde{x}^1 + \tilde{x}^2 \right) \}. \quad (47)$$

Here the term in the last line appears due to the mixed product of the response fields in (46).

We will now perform the average over realizations in $\mathbf{J}$, as in Section II B eq. (26). We therefore need to evaluate
Figure 5. Chaotic evolution. 

a Dynamics of two systems starting at similar initial conditions for chaotic case with \( g = 2, \ N = 5000, \ D = 0.01 \). Trajectories of three units shown for the unperturbed (black) and the perturbed system (gray).

b Absolute average squared distance \( d(t) \) given by (44) of the two systems.

c Difference \( x_1 - x_2 \) for the first three units. The second system is reset to the state of the first system plus a small random displacement as soon as \( d(t) > 0.1 \). Other parameters as in Figure 4.

the Gaussian integral

\[
\int dJ_{ij} \mathcal{N}(0, \frac{g^2}{N}, J_{ij}) \exp \left( -J_{ij} \sum_{\alpha=1}^2 \tilde{x}_i^\alpha T \phi(x_j^\alpha) \right) = \exp \left( \frac{g^2}{2N} \sum_{\alpha=1}^2 \left( \tilde{x}_i^\alpha T \phi(x_j^\alpha) \right)^2 \right) \times \exp \left( \frac{g^2}{N} \tilde{x}_i^{1T} \phi(x_j^1) \tilde{x}_i^{2T} \phi(x_j^2) \right).
\]

(48)

Similar as for the Gaussian integral over the common noises that gave rise to the coupling term between the two systems in the second line of (47), we here obtain a coupling term between the two systems, in addition to the terms that only include variables of a single subsystem in the second last line. Note that the two coupling terms are different in nature. The first, due to common noise, represents common temporal fluctuations injected into both systems. The second is static in its nature, as it arises from the two systems having the same coupling \( J \) in each of their realizations that enter the expectation value. The terms that only affect a single subsystem are identical to those in (27). We treat these terms as before and here concentrate on the mixed terms, which we rewrite (including the \( \sum_{i < j} \) in (47)
and using our definition \( \bar{x}_i^{aT} \phi(x_i^a) = \int dt \bar{x}_i^a(t) \phi(x_i^a(t)) \) dt as

\[
\exp \left( \frac{g^2}{N} \sum_{i,j} \bar{x}_i^{1T} \phi(x_i^1) \bar{x}_i^{2T} \phi(x_i^2) \right) = \exp \left( \iint \sum_i \bar{x}_i^1(s) \bar{x}_i^2(t) \frac{g^2}{N} \sum_j \phi(x_j^1(s)) \phi(x_j^2(t)) \, ds \, dt \right) + O(N^{-1}),
\]

(49)

where we included the self coupling term \( i = j \), which is only a subleading correction of order \( N^{-1} \).

We now follow the steps in Section II B and introduce three pairs of auxiliary variables. The pairs \( Q_1^a, Q_2^a \) are defined as before in (28) and (29), but for each subsystem, while the pair \( T_1, T_2 \) decouples the mixed term (49) by defining

\[
T_1(s,t) := \frac{g^2}{N} \sum_j \phi(x_j^1(s)) \phi(x_j^2(t)),
\]

as indicated by the curly brace in (49).

Taken together, we can therefore rewrite the generating functional (47) averaged over the couplings as

\[
\bar{Z}([j^a, j^b]_{\alpha \in \{1, 2\}}) := \langle Z([j^a, j^b]_{\alpha \in \{1, 2\}})](J) \rangle_J = \Pi^2_{\alpha=1} \left\{ \int \mathcal{D}Q_1^a \int \mathcal{D}Q_2^a \int \mathcal{D}T_1 \int \mathcal{D}T_2 \exp \left( \Omega([Q_1^a, Q_2^a]_{\alpha \in \{1, 2\}}, T_1, T_2) \right) \right\}
\]

\[
\Omega([Q_1^a, Q_2^a]_{\alpha \in \{1, 2\}}, T_1, T_2) = - \sum_{\alpha=1}^2 Q_1^{aT} Q_2^a - T_1^T T_2 + \ln Z^{12}([Q_1^a, Q_2^a]_{\alpha \in \{1, 2\}}, T_1, T_2)
\]

\[
Z^{12}([Q_1^a, Q_2^a]_{\alpha \in \{1, 2\}}, T_1, T_2) = \Pi^2_{\alpha=1} \left\{ \int \mathcal{D}x^a \int \mathcal{D}x^b \exp \left( \Omega([Q_1^a, Q_2^a]_{\alpha \in \{1, 2\}}, T_1, T_2) \right) \right\}
\]

\[
\times \exp \left( \bar{x}^{1T} (T_1 + D) \bar{x} + \frac{g^2}{2N} \phi(x^1) T_2 \phi(x^2) \right) \right) .
\]

We now determine, for vanishing sources, the fields \( Q_1^a, Q_2^a, T_1, T_2 \) at which the contribution to the integral is maximal by requesting \( \frac{\delta \Omega}{\delta Q_1^a} = \frac{\delta \Omega}{\delta Q_2^a} = 0 \) for the exponent \( \Omega \) of (50). Here again the term \( \ln Z^{12} \) plays the role of a cumulant generating function and the fields \( Q_1^a, Q_2^a, T_1, T_2 \) play the role of sources, each bringing down the respective factor they multiply. We denote the expectation value with respect to this functional as \( \langle \phi \rangle_{Q^a, T^a} \) and obtain the self-consistency equations

\[
Q_1^{a*}(s,t) = \frac{1}{Z^{12}} \frac{\delta Z^{12}}{\delta Q_1^a(s,t)} = \frac{g^2}{2N} \sum_j \langle \phi(x_j^a) \phi(x_j^a) \rangle_{Q^a, T^a}
\]

(51)

\[
Q_2^{a*}(s,t) = 0
\]

\[
T_1^*(s,t) = \frac{1}{Z^{12}} \frac{\delta Z^{12}}{\delta T_1(s,t)} = \frac{g^2}{N} \sum_j \langle \phi(x_j) \phi(x_j^a) \rangle_{Q^a, T^a}
\]

\[
T_2^*(s,t) = 0.
\]

The generating functional at the saddle point is therefore

\[
\bar{Z}^*([j^a, j^b]_{\alpha \in \{1, 2\}}) = \iint \Pi^2_{\alpha=1} \mathcal{D}x^a \mathcal{D}x^b \exp \left( \sum_{\alpha=1}^2 S_0[\bar{x}^\alpha, \bar{x}^\alpha] + j^{aT} \bar{x}^\alpha + \bar{x}^{\alpha T} j^{a} + \bar{x}^{\alpha T} Q_1^{a*} \bar{x}^\alpha \right) \times
\]

\[
\times \exp \left( \bar{x}^{\alpha T} (T^*_1 + D) \bar{x} \right) .
\]

(52)

We make the following observations:

1. The two subsystems \( \alpha = 1, 2 \) in the first line of (52) have the same form as in (34). This has been expected, because the absence of any physical coupling between the two systems implies that the marginal statistics of the activity in one system cannot be affected by the mere presence of the second, hence also their saddle points \( Q_1^{a, 2} \) must be the same as in (34).

2. The entire action is symmetric with respect to interchange of any pair of unit indices. So we have reduced the system of 2N units to a system of 2 units.

3. If the term in the second line of (52) was absent, the statistics in the two systems would be independent. Two sources, however, contribute to the correlations between the systems: The common Gaussian white noise that gave rise to the term $\propto D$ and the non-white Gaussian noise due to a non-zero value of the auxiliary field $T_1^\tau(s,t)$.

4. Only products of pairs of fields appear in (52), so that the statistics of the $x^a$ is Gaussian.

As for the single system, we can express the joint system by a pair of dynamic equations

$$\left(\partial_t + 1\right) x^\alpha(t) = \eta^\alpha(t) \quad \alpha \in \{1, 2\} \quad (53)$$

together with a set of self-consistency equations for the statistics of the noises $\eta^\alpha$ following from (51)

$$\langle \eta^\alpha(s) \eta^\beta(t) \rangle = D \delta(t-s) + g^2 \langle \phi(x^\alpha(s)) \phi(x^\beta(t)) \rangle. \quad (54)$$

Obviously, this set of equations (53) and (54) marginally for each subsystem admits the same solution as determined in Section II C. Moreover, the joint system therefore also possesses the fixed point $x^1(t) = x^2(t)$, where the activities in the two subsystems are identical, i.e. characterized by $c^{12}(t,s) = c^{11}(t,s) = c^{22}(t,s)$ and consequently $d(t) \equiv 0 \forall t \quad (45)$.

We will now investigate if this fixed point is stable. If it is, this implies that any perturbation of the system will relax such that the two subsystems are again perfectly correlated. If it is unstable, the distance between the two systems may increase, indicating chaotic dynamics.

We already know that the autocorrelation functions in the subsystems are stable and each obey the equation of motion (41). We could use the formal approach, writing the Gaussian action as a quadratic form and determine the correlation and response functions as the inverse, or Green’s function, of this bilinear form. Here, instead we employ a simpler approach: we multiply the equation (53) for $\alpha = 1$ and $\alpha = 2$ and take the expectation value on both sides, which leads to

$$\left(\partial_t + 1\right) \left(\partial_s + 1\right) \langle x^\alpha(t)x^\beta(s) \rangle = \langle \eta^\alpha(t)\eta^\beta(s) \rangle,$$

so we get for $\alpha, \beta \in \{1, 2\}$

$$\left(\partial_t + 1\right) \left(\partial_s + 1\right) c^{\alpha\beta}(t,s) = D \delta(t-s) + g^2 F_\phi(c^{\alpha\beta}(t,s), c^{\alpha\alpha}(t,t), c^{\beta\beta}(s,s)), \quad (55)$$

where the function $F_\phi$ is defined as the Gaussian expectation value

$$F_\phi(c^{12}, c^1, c^2) := \langle \phi(x^1) \phi(x^2) \rangle$$

for the bi-variate Gaussian

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \sim \mathcal{N}_2 \left( 0, \begin{pmatrix} c^1 & c^{12} \\ c^{12} & c^2 \end{pmatrix} \right).$$

First, we observe that the equations for the autocorrelation functions $c^{\alpha\alpha}(t,s)$ decouple and can each be solved separately, leading to the same equation (41) as before. As noted earlier, this formal result could have been anticipated, because the marginal statistics of each subsystem cannot be affected by the mere presence of the respective other system. Their solutions

$$c^{11}(s,t) = c^{22}(s,t) = c(t-s)$$

then provide the “background”, i.e., the second and third argument of the function $F_\phi$ on the right-hand side, for the equation for the crosscorrelation function between the two copies. Hence it remains to determine the equation of motion for $c^{12}(t,s)$.

We first determine the stationary solution $c^{12}(t,s) = k(t-s)$. We see immediately that $k(\tau)$ obeys the same equation of motion as $c(\tau)$, so $k(\tau) = c(\tau)$. The distance (45) therefore vanishes. Let us now study the stability of this solution. We hence need to expand $c^{12}$ around the stationary solution

$$c^{12}(t,s) = c(t-s) + \epsilon k^{(1)}(t,s), \quad \epsilon \ll 1.$$
We develop the right hand side of (55) into a Taylor series using eq. (64) and (38)
\[ F_\phi(c^{12}(t, s), c_0, c_0) = f_\phi(c^{12}(t, s), c_0) = f_\phi(c(t-s), c_0) + \epsilon f_\phi'(c(t-s), c_0) k^{(1)}(t, s) + O(\epsilon^2). \]

Inserted into (55) and using that \( c \) solves the lowest order equation, we get the linear equation of motion for the first order deflection
\[ (\partial_t + 1)(\partial_s + 1) k^{(1)}(t, s) = g^2 f_\phi'(c(t-s), c_0) k^{(1)}(t, s). \]

In the next section we will determine the growth rate of \( k^{(1)} \) and hence, by (45)
\[ d(t) = c^{11}(t, t) + e c^{22}(s, s) - c^{12}(t, t) - c^{21}(t, t) \]
\[ = -\epsilon k^{(1)}(t, t) \]
the growth rate of the distance between the two subsystems. The negative sign makes sense, since we expect in the chaotic state that \( c^{12}(t, s) \xrightarrow{t,s\to\infty} 0 \), so \( k^{(1)} \) must be of opposite sign than \( c > 0 \).

### E. Schrödinger equation for the maximum Lyapunov exponent

We here want to reformulate the equation for the variation of the cross-system correlation (56) into a Schrödinger equation, as in the original work [10, eq. 10].

First, noting that \( C_{\phi\phi}(t, s) = f_\phi'(c(t-s), c_0) \) is time translation invariant, it is advantageous to introduce the coordinates \( T = t + s \) and \( \tau = t - s \) and write the covariance \( k^{(1)}(t, s) \) as \( k(T, \tau) \) with \( k^{(1)}(t, s) = k(t+s, t-s) \). The differential operator \( (\partial_t + 1)(\partial_s + 1) \) with the chain rule \( \partial_t \rightarrow \partial_T + \partial_\tau \) and \( \partial_s \rightarrow \partial_T - \partial_\tau \) in the new coordinates is \((\partial_T + 1)^2 - \partial_\tau^2\). A separation ansatz \( k(T, \tau) = e^{\frac{\kappa}{2} T} \psi(\tau) \) then yields the eigenvalue equation
\[ \left( \frac{\kappa}{2} + 1 \right)^2 \psi(\tau) - \partial_\tau^2 \psi(\tau) = g^2 f_\phi'(c(\tau), c_0) \psi(\tau) \]
for the growth rates \( \kappa \) of \( d(t) = -k^{(1)}(t, t) = -k(2t, 0) \). We can express the right hand side by the second derivative of the potential (40) \( V(c(\tau); c_0) \) so that with
\[ V''(c(\tau); c_0) = -1 + g^2 f_\phi'(c(\tau), c_0) \]
we get the time-independent Schrödinger equation
\[ (-\partial_\tau^2 - V''(c(\tau); c_0)) \psi(\tau) = \left( 1 - \left( \frac{\kappa}{2} + 1 \right)^2 \right) \psi(\tau). \]

The eigenvalues ("energies") \( E_n \) determine the exponential growth rates \( \kappa_n \) the solutions \( k(2t, 0) = e^{\kappa_n t} \psi_n(0) \) at \( \tau = 0 \) with
\[ \kappa_n^\pm = 2 \left( -1 \pm \sqrt{1 - E_n} \right). \]

We can therefore determine the growth rate of the mean-square distance of the two subsystems in Section II.D by (57). The fastest growing mode of the distance is hence given by the ground state energy \( E_0 \) and the plus in (60). The deflection between the two subsystems therefore grows with the rate
\[ \Lambda_{\text{max}} = \frac{1}{2} \kappa_0^+ \]
\[ = -1 + \sqrt{1 - E_0}, \]
where the factor 1/2 in the first line is due to \( d \) being the squared distance, hence the length \( \sqrt{d} \) growth with half the exponent as \( d \).

Energy conservation (42) determines \( c_0 \) also in the case of non-zero noise \( D \neq 0 \), as shown in Figure 6a. The autocovariance function obtained from the solution of (43) agrees well to the direct simulation Figure 6b. The quantum potential appearing in (59) is shown in Figure 6c.
Figure 6. **Dependence of the self-consistent solution on the noise level** $D$. a Potential that determines the self-consistent solution of the autocorrelation function (40). Noise amplitude $D > 0$ corresponds to an initial kinetic energy $E_{\text{kin}} = \frac{D^2}{2}$. The initial value $c_0$ is determined by the condition $V(c_0; c_0) + E_{\text{kin}} = 0$, so that the “particle” starting at $c(0) = c_0$ has just enough energy to reach the peak of the potential at $c(\tau \to \infty) = 0$. In the noiseless case, the potential at the initial position $c(0) = c_0$ must be equal to the potential for $\tau \to \infty$, i.e. $V(c_0; c_0) = V(0) = 0$, indicated by horizontal dashed line and the corresponding potential (black). b Resulting self-consistent autocorrelation functions given by (43). The kink at zero time lag $\dot{c}(0^-) - \dot{c}(0^+) = \frac{D}{2}$ is indicated by the tangential dotted lines. In the noiseless case the slope vanishes (horizontal dotted line). Simulation results shown as light gray underlying curves. c Quantum mechanical potential appearing in the Schrödinger equation (59) with dotted tangential lines at $\tau = \pm 0$. Horizontal dotted line indicates the vanishing slope in the noiseless case. Other parameters as in Figure 3.

**F. Condition for transition to chaos**

We can construct an eigensolution of (59) from (41). First we note that for $D \neq 0$, $c$ has a kink at $\tau = 0$. This can be seen by integrating (41) from $-\epsilon$ to $\epsilon$, which yields

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \partial^2_\tau c d\tau = \dot{c}(0^+) - \dot{c}(0^-) = D.$$ 

Since $c(\tau) = c(-\tau)$ is an even function it follows that $\dot{c}(0^+) = -\dot{c}(0^-) = -\frac{D}{2}$. For $\tau \neq 0$ we can differentiate (41) with respect to time $\tau$ to obtain

$$\partial_\tau \partial^2_\tau c(\tau) = \partial^2_\tau \dot{c}(\tau) = -\partial_\tau V'(c(\tau)) = -V''(c(\tau)) \dot{c}(\tau).$$

Comparing the right hand side expressions shows that $(\partial^2_\tau + V''(c(\tau))) \dot{c}(\tau) = 0$, so $\dot{c}$ is an eigensolution for eigenvalue $E_n = 0$ of (59).

Let us first study the case of vanishing noise $D = 0$ as in [10]. The solution then $\dot{c}$ exists for all $\tau$. Since $c$ is a symmetric function, $\Psi_0 = \dot{c}$ has single node. The single node of this solution implies there must be a state with zero nodes that has even lower energy, i.e. $E_0 < 0$. This, in turn, indicates a positive Lyapunov exponent $\Lambda_{\text{max}}$ according
to (61). This is the original argument in [10], showing that at $g = 1$ a transition from a silent to a chaotic state takes place.

Our aim is to find the parameter values for which the transition to the chaotic state takes place in the presence of noise. We know that the transition takes place if the eigenvalue of the ground state of the Schrödinger equation is zero. We can therefore explicitly try to find a solution of (59) for eigenenergy $E_n = 0$, i.e. we seek the homogeneous solution that satisfies all boundary conditions, i.e. continuity of the solution as well as its first and second derivative. We already know that $\dot{c}(\tau)$ is one homogeneous solution of (59) for positive and for negative $\tau$. For $D \neq 0$, we can construct a continuous solution from the two branches by defining

$$y_1(\tau) = \begin{cases} \dot{c}(\tau) & \tau \geq 0 \\ -\dot{c}(\tau) & \tau < 0 \end{cases},$$

which is symmetric, consistent with the search for the ground state. In general $y_1$ does not solve the Schrödinger equation, because the derivative at $\tau = 0$ is not necessarily continuous, since by (37) $\partial_\tau y_1(0+) - \partial_\tau y_1(0-) = \ddot{c}(0+) + \ddot{c}(0-) = 2(c_0 - g^2 f_\phi(c_0; c_0))$. Therefore $y_1$ is only an admissible solution, if the right hand side vanishes. The criterion for the transition to the chaotic state is hence

$$0 = \partial_\tau^2 c(0\pm) = c_0 - g^2 f_\phi(c_0; c_0) = -V'(c_0; c_0).$$

The latter condition therefore shows that the curvature of the autocorrelation function vanishes at the transition. In the picture of the motion of the particle in the potential the vanishing acceleration at $\tau = 0$ amounts to a potential with a flat tangent at $c_0$.

The criterion for the transition can be understood intuitively. The additive noise increases the peak of the autocorrelation at $\tau = 0$. In the large noise limit, the autocorrelation decays as $e^{-|\tau|}$, so the curvature is positive. The decay of the autocorrelation is a consequence of the uncorrelated external input. In contrast, in the noiseless case, the autocorrelation has a flat tangent at $\tau = 0$, so the curvature is negative. The only reason for its decay is the decorrelation due to the chaotic dynamics. The transition between these two forces of decorrelation hence takes place at the point at which the curvature changes sign, from dominance of the external sources to dominance of the intrinsically generated fluctuations. For a more detailed discussion please see [14].

III. APPENDIX

A. Price's theorem

We here provide a derivation of Price’s theorem [34], which, for the Gaussian integral (38) takes the form

$$\frac{\partial}{\partial c} f_u(c, c_0) = f_u'(c, c_0).$$

We here provide a proof using the Fourier representation of $u(x) = \frac{1}{2\pi} \int U(\omega) e^{i\omega x} d\omega$. Alternatively, integration by parts can be used to obtain the same result by a slightly longer calculation. We write the integral as

$$f_u(c, c_0) = \int \int U(\omega) e^{\frac{i\omega}{\sqrt{c_0^2 + c^2}} \sqrt{z_1}} U(\omega') e^{\frac{i\omega'}{\sqrt{c_0^2 + c^2}} \sqrt{z_2}} Dz_1 Dz_2 d\omega d\omega'$$

where we used the characteristic function $e^{\frac{i\omega^2}{2\omega^2}}$ of the unit variance Gaussian contained in the measures $Dz$. The derivative by $c$ with the product rule yields

$$\frac{\partial}{\partial c} f_u(c, c_0) = \int \int \frac{c}{c_0} U(\omega) e^{\frac{i\omega}{\sqrt{c_0^2 + c^2}} \sqrt{z_1}} U(\omega') e^{\frac{i\omega'}{\sqrt{c_0^2 + c^2}} \sqrt{z_2}} d\omega d\omega'$$

where we used $u'(x) = \frac{1}{2\pi} \int i\omega U(\omega) e^{i\omega x} d\omega$ in the last step, proving the assertion.
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