Laplacian Solitons and Symmetry in $G_2$-geometry

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Abstract
In this paper, it is shown that (with no additional assumptions) on a compact 7-dimensional manifold which admits a $G_2$-structure soliton solutions to the Laplacian flow of R. Bryant can only be shrinking or steady. We also show that the space of symmetries (vector fields that annihilate via the Lie derivative) of a torsion-free $G_2$-structure on a compact 7-manifold is canonically isomorphic to $H^1(M, R)$. Some comparisons with Ricci solitons are also discussed, along with some future directions of exploration.

1 Introduction

Let $M$ be a 7-dimensional manifold that admits a $G_2$-structure given by a non-degenerate 3-form $\varphi$. A natural geometric flow when $M$ is compact is the Laplacian flow first suggested by R. Bryant in [1]:

$$\frac{\partial \varphi}{\partial t} = -\Delta \varphi$$

for a family $\varphi = \varphi(t)$ of $G_2$ structures, where $\Delta \varphi$ denotes the Hodge Laplacian with respect to the metric induced by $\varphi(t)$ (hence nonnegative-definite). The original intention of the equation (1) is to flow $\varphi$ to a torsion-free $G_2$-structure, since $\varphi$ being torsion-free is equivalent to being harmonic with respect to the metric $g_{\varphi}$ it induces. Although it is not clear what it means in this case, when $M$ is not compact the flow (1) still makes sense. In fact, when $M$ is not compact we suspect that a more general flow is needed (see [6] for some general results in this direction). The short-time existence and uniqueness of (1) for a closed initial $G_2$-structure when $M$ is compact have been established in [2] and [10].

As in the Ricci flow, let us consider solutions of the form

$$\varphi(t) = \tau(t) f_1^* \varphi$$

1The sign in front of the Laplacian turns out to be purely a technical convention, as explained in [7].
for a fixed $G_2$-structure $\varphi$, pulled back by a smooth family family $f_t$ of diffeomorphisms of $M$, and where $|\tau(t)| > 0$ is a scaling factor. Such special solutions are called solitons, and in the present context of $G_2$ geometry it has already appeared in [7] by Karigiannis, McKay, and Tsui. We will call solutions of the form (2) Laplacian solitons, or just solitons if no confusion arises. Just as in the Ricci flow, we seek a static description of a soliton by substituting (2) into (1), and we will arrive at the equation

$$\rho \varphi + L_X \varphi = -\Delta \varphi$$

for some constant $\rho$ and vector field $X$. Thus equivalently, we can define a (Laplacian) soliton to be a $G_2$-structure $\varphi$ that satisfies (3).

Analogous to Ricci solitons, we can define:

**Definition 1.** Let $\varphi$ be a $G_2$-structure, and $X$ a vector field on $M$. We say that $(\varphi, X)$ is a Laplacian soliton if equation (3) is satisfied for some constant $\rho \in \mathbb{R}$. We say $(\varphi, X)$ is an expanding soliton if $\rho > 0$, a steady soliton if $\rho = 0$, and a shrinking soliton if $\rho < 0$.

We would like to point out that a torsion-free $G_2$-structure is steady, and is the most trivial example of a soliton. Also, just to distinguish between the dynamic and static versions of the soliton concept, we will refer to (2) as a soliton solution whereas the terminology of soliton will be reserved for the definition above.

Work on a dual equation (the Laplacian coflow) to (1) and the corresponding soliton equation analogous to (3) have been done in [7]. The original intention in [7] was to study (3), but instead they focused on the coflow version because there is available a special cohomogeneity-1 ansatz. In this paper we focus on a more detailed examination of the fundamental equation (3). In addition, we found out very recently in [9] that Weiss and Witt had also studied soliton solutions to the $L^2$-gradient flow of a Dirichlet-type functional they proposed in an earlier paper [8]. In [9] similar results to the ones in this paper and in [7] appeared, but note that the $L^2$-gradient flow of their energy functional is a different equation from the Laplacian flow (or coflow).

One of the main results of this paper is Corollary 1 which says that there are no compact expanding solitons, and no compact steady solitons except torsion-free $G_2$-structures. A similar result was proved in [7], where
the soliton was assumed to be coclosed (and closed for the original flow \( \square \)). Our result does not depend on the soliton \( \varphi \) being closed, and the proof follows directly from a fundamental identity established in Lemma \( \square \) in the same section. Although the short-time existence of the Laplacian flow (and of the coflow as well) have only been established for closed/coclosed structures, Corollary \( \square \) is still valuable because we think \( \Box \) is still an interesting equation in its own right.

The other main result of this paper is Corollary \( \Box \), which says that for a torsion-free \( G_2 \)-structure \( \varphi \) on a compact 7-manifold, the space of all vector fields \( X \) such that \( L_X \varphi = 0 \) is isomorphic to \( H^1(M, \mathbb{R}) \). We will refer to such a vector field as a symmetry of the \( G_2 \)-structure \( \varphi \), for brevity. We view Corollary \( \Box \) as a kind of rigidity result, because for \( \varphi \) fixed the soliton equation \( \Box \) is invariant only by adding such vector fields \( \Box \). Our original goal was to do the same for compact shrinking solitons as well, but at the present we do not have such an analogous result.

2 \( G_2 \)-Structures and Torsion Forms

We give a brief review of the background relating to \( G_2 \)-structures here. The standard reference for this is the book \( \Box \) by Dominic D. Joyce, although the papers \( \Box \) and \( \Box \) are also good sources.

The group \( G_2 \) is a compact, connected, simply-connected Lie group sitting in \( SO(7) \). Algebraically, it can be defined as the Automorphism group of the Octonians. It appears as one of the exceptional holonomy groups in the classification by Berger, et al. The working definition of \( G_2 \) that we will adopt is the following. Consider the differential 3-form

\[
\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}
\]

in \( \mathbb{R}^7 \), where \( dx_{ijk} = dx_i \wedge dx_j \wedge dx_k \). The group \( G_2 \) can be defined as the subgroup in \( GL(7, \mathbb{R}) \) that preserves \( \varphi_0 \), which means

\[
\varphi_0(g(v_1), g(v_2), g(v_3)) = \varphi_0(v_1, v_2, v_3)
\]

for all vectors \( v_1, v_2, v_3 \in \mathbb{R}^7 \) and every \( g \in G_2 \). The peculiar form that \( \Box \) takes reflects the combinatorial nature of permuting the Octonians.

\( ^2 \)We shall see that this is again rooted in Lemma \( \Box \)
From a principal bundle point of view, a $G_2$-structure on a 7-dimensional manifold $M$ is just a sub-bundle with structure group $G_2$, of the $GL(7, \mathbb{R})$-frame bundle over $M$. In other words, one can find local frames of the tangent bundle $TM$ such that all the transition functions value in the group $G_2$. Because $G_2 \subset SO(7)$, a $G_2$-structure induces an orientation and a unique metric (which we will write $g$) on $M$. Take any local frame $\{e_i\}_{i=1}^7$ in a $G_2$-structure, then the 3-form

$$\varphi = \omega_{123} + \omega_{145} + \omega_{167} + \omega_{246} - \omega_{257} - \omega_{347} - \omega_{356}$$

(5)

is well-defined over all of $M$, where $\omega_i$ is the local dual 1-form to $e_i$. Conversely, if a 3-form $\varphi$ on $M$ can be locally represented as (5) with respect to a frame, then the transition functions of such frames value in $G_2$ and we have a $G_2$-structure. As a result, a $G_2$-structure is equivalent to a 3-form $\varphi$ locally represented as in (5), and this is what we will refer to as a $G_2$-structure. It is well-known that a $G_2$-structure exists if and only if the 7-manifold is orientable and spin.

A $G_2$ structure $\varphi$ induces a point-wise orthogonal decomposition (with respect to $g$) of $p$-forms on $M$:

$$\Omega^2 = \{X \cdot \varphi | X \in \Gamma(TM)\} = \{\beta \in \Omega^2 | *(\varphi \wedge \beta) = -2\beta\}$$

$$\Omega^2_{14} = \{\beta \in \Omega^2 | \beta \wedge *\varphi = 0\} = \{\beta \in \Omega^2 | *(\varphi \wedge \beta) = \beta\}$$

$$\Omega^3_1 = \{f \varphi | f \in C^\infty(M)\}$$

$$\Omega^3_7 = \{X \cdot \varphi | X \in \Gamma(TM)\}$$

$$\Omega^3_{27} = \{h_{ij}g^{jl}dx_i \wedge (\partial/\partial x^l) \varphi | h \in Sym^2(T^*M), Tr_g(h) = 0\},$$

where

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$$

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}.$$ Then we can write

$$d\varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \tau_3$$

$$d \ast \varphi = 4\tau_1 \wedge \ast \varphi + \ast \tau_2,$$

where $\tau_0 \in \Omega^0_7$, $\tau_1 \in \Omega^1_7$, $\tau_2 \in \Omega^2_{14}$, and $\tau_3 \in \Omega^3_{27}$ are called the torsion forms. The fact that the same 1-form $\tau_1$ appears in the decompositions of $d \varphi$ and

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$d \ast \varphi$ is non-trivial, but can be shown via some computations (see [6]). The terminology of torsion forms comes from the following. Having a $G_2$-structure $\varphi$ does not mean the holonomy group of $g_\varphi$ is contained in $G_2$. The additional condition that is needed is the so-called torsion-free condition. We say that a $G_2$-structure is torsion-free if $\varphi$ solves the nonlinear system of partial differential equations $\nabla \varphi = 0$, where $\nabla$ is the covariant derivative induced by $g_\varphi$. It was shown in [3] that a $G_2$-structure is torsion-free if and only if it is closed and co-closed (with respect to the hodge star induced by $g_\varphi$).\footnote{Thus when $M$ is compact, $\varphi$ being torsion-free is equivalent to it being harmonic with respect to $g_\varphi$. In view of the torsion forms defined above, we see that $\varphi$ is torsion-free if and only if all four torsion forms vanish on $M$. A 7-manifold $M$ that admits a torsion-free $G_2$-structure has its Riemannian holonomy (with respect to $g_\varphi$) a subgroup of $G_2$, and such manifolds are simply known as $G_2$ manifolds.}

### 3 The Soliton Equation

From direct computations, we see that a soliton solution (2) to the Laplacian flow (1) satisfies

$$\tau(t) f_t^* \varphi + \tau(t) f_t^* (L_X(t) \varphi) = -\tau(t)^{1/3} f_t^*(\Delta \varphi),$$

which is exactly (3) and where the vector field $X(t)$ is the infinitesimal generator of the diffeomorphism $f_t$. We have also used the following fact:

**Lemma 1.** If $\varphi$ is a $G_2$-structure, then any non-zero constant multiple $c \varphi$ is also a $G_2$-structure, and $g_{c \varphi} = c^{2/3} g_\varphi$.\footnote{This is an entirely local property, it is independent of whether or not $M$ is compact.}

Note that we dropped the subscript on the Laplace operator since $\varphi$ is now fixed. Then we see that there is a soliton solution (2) to the flow (1) only if

$$\rho \varphi + L_X \varphi = -\Delta \varphi$$

for some vector field $X$ on $M$, where we have frozen at a time $t$ and the constant $\rho = \tau / \tau^{1/3}$. As in the case of Ricci Solitons, one can show that given a vector field $X$ and a $G_2$ structure $\varphi$ that satisfy (3) one can generate $c > 0$ preserves the orientation given by $\varphi$, $c < 0$ reverses it.
a solution of the form (2) to (1).

We would also like to point out that the soliton equation (3) is scale-invariant in the following sense. Note that given any $G_2$-structure $\varphi$ satisfying (3), then for any $c \varphi$, $c \neq 0$, equation (3) is again satisfied for $\tilde{\rho} = c^{-2/3} \rho$ and $\tilde{X} = c^{-2/3} X$.

4 Compact Solitons

In this section we show that

**Lemma 2.** Let $M$ be a compact 7-manifold. For any $G_2$-structure $\varphi$ on $M$, vector field $X$, and $f \in C^\infty(M)$, we have

\[ \int_M L_X \varphi \wedge * f \varphi = -3 \int_M df \wedge * X^\flat \]

**Proof.** We have

\[ L_X \varphi = X \lrcorner d \varphi + d(X \lrcorner \varphi). \]

From the decomposition of $d \varphi$ we see that

\[
\begin{align*}
(X \lrcorner d \varphi) \wedge * f \varphi &= \tau_0 f (X \lrcorner \varphi) \wedge * \varphi + 3 f (X \lrcorner (\tau_1 \wedge \varphi)) \wedge * \varphi + f (X \lrcorner \tau_3) \wedge * \varphi \\
&= 3 f (X \lrcorner (\tau_1 \wedge \varphi)) \wedge * \varphi + f (X \lrcorner \tau_3) \wedge * \varphi \\
&= -3 f (\tau_1 \wedge \varphi) \wedge (X \lrcorner \varphi) - f \tau_3 \wedge (X \lrcorner \varphi) \\
&= -3 f \tau_1 \wedge \varphi \wedge (X \lrcorner \varphi) \\
&= -3 f \tau_1 \wedge (-4 * X^\flat) \\
&= 12 f \tau_1 \wedge * X^\flat,
\end{align*}
\]

where we have used the identity $\varphi \wedge (X \lrcorner * \varphi) = -4 * X^\flat$ (see Appendix A in [6]) in the fifth equality and also the point-wise orthogonality of the $G_2$-decomposition of differential forms in the second and fourth equalities.
above. On the other hand, from the decomposition of \( d * \phi \) we have

\[
\int_M d(X \lrcorner \phi) \wedge f \phi = \int_M (X \lrcorner \phi) \wedge \delta f \phi
\]

\[
= - \int_M (X \lrcorner \phi) \wedge d \ast f \phi
\]

\[
= - \int_M (X \lrcorner \phi) \wedge (df \wedge \ast \phi + f d \ast \phi)
\]

\[
= - \int_M (X \lrcorner \phi) \wedge df \wedge \ast \phi - \int_M f (X \lrcorner \phi) \wedge (4 \tau_1 \wedge \ast \phi + \ast \tau_2)
\]

\[
= - \int_M df \wedge \ast \phi \wedge (X \lrcorner \phi) - 4 \int_M f (X \lrcorner \phi) \wedge \tau_1 \wedge \ast \phi
\]

\[
= - \int_M df \wedge \ast \phi \wedge (X \lrcorner \phi) - 4 \int_M f \tau_1 \wedge \ast \phi \wedge (X \lrcorner \phi)
\]

\[
= - \int_M df \wedge 3 \ast X^b - 4 \int_M f \tau_1 \wedge 3 \ast X^b
\]

\[
= -3 \int_M df \wedge \ast X^b - 12 \int_M f \tau_1 \wedge \ast X^b,  \tag{7}
\]

where we have also used the identity \( \ast \phi \wedge (X \lrcorner \phi) = 3 \ast X^b \) (see Appendix A in [6]). Integrating (6) and adding to (7), the lemma now follows.

Corollary 1. There are no compact expanding solitons, and there are no compact steady solitons except torsion-free \( G_2 \)-structures.

Proof. Wedging both sides of (3) by \( \ast \phi \) and integrating, we have

\[
\rho \int_M \phi \wedge \ast \phi = - \int_M \Delta \phi \wedge \ast \phi
\]

\[
= - \int_M d \phi \wedge \ast d \phi - \int_M \delta \phi \wedge \ast \delta \phi,  \tag{8}
\]

where we have used Lemma 2 with \( f \equiv 1 \). Then since \( \int_M \phi \wedge \ast \phi \) is the volume and hence non-zero, \( \rho \leq 0 \) necessarily because the right-hand side of (8) is non-positive. The case of \( \rho = 0 \) is equivalent to \( d \phi = \delta \phi = 0 \) by (5), hence torsion-free.

As an offshoot to the proof of Corollary 1 we also have the following observation.
Corollary 2. For any compact soliton $\varphi$ satisfying (3), the constant $\rho$ has the Rayleigh quotient expression:

$$\rho = -\frac{\int_M \Delta \varphi \wedge \ast \varphi}{\int_M \varphi \wedge \ast \varphi}.$$  \hspace{1cm} (9)

Equation (9) means that for a compact soliton, $\rho$ is completely determined by the $G_2$-structure. This simple observation will play a role in Section 6. Also recall from Section 3 that $\rho = \dot{\tau}/\tau^{1/3}$ for any time $t$ within solution (2)'s existence. Then (9) implies that $\dot{\tau}$ will always have the same sign at any time $t$ within the soliton solution's existence, i.e. this means that a compact soliton solution is either "always" shrinking or "always" steady.

5 Eigenforms as Shrinking Solitons

The defining equation (3) for a soliton shows that it is a kind of generalized eigenvalue equation for $\varphi$ (with respect to $g_\varphi$). In particular, when $\rho < 0$ an eigenform: $-\Delta \varphi = \rho \varphi$ solves the soliton equation with $X = 0$ ($f_t = \text{Id} \forall t$).

Eigenforms as Laplacian solitons are analogous to Einstein metrics as trivial examples of Ricci solitons. Nevertheless, it is enlightening to write out the exact solution for an eigenform in terms of (2) in the shrinking case.

Proposition 1. Suppose $\varphi$ is an eigenform then it is a shrinking soliton with a solution in the form of (2) as

$$\varphi(t) = \left(1 + \frac{2}{3} \rho t\right)^{3/2} \varphi.$$ \hspace{1cm} (10)

Proof. We assume the solution is of the form $\varphi(t) = R(t) \varphi$ for some real-valued function $R(t)$, and we set $R(0) = 1$. Then we see that

$$R'(t) \varphi = \frac{\partial \varphi(t)}{\partial t} = -\Delta \varphi(t) \varphi(t) = -R(t)^{1/3} \Delta \varphi \varphi = \rho R(t)^{1/3} \varphi.$$  \hspace{1cm} (11)

From this we must have $R'(t) = \rho R(t)^{1/3}$, which by dividing both sides by $R(t)^{1/3}$ we can rewrite as

$$\frac{3}{2} \frac{d}{dt} \left(R(t)^{2/3}\right) = \rho.$$  \hspace{1cm} (11)

By integrating both sides of (11) and using the initial condition, we get exactly the desired solution (10). \hfill \square
From (10) we see that for an eigenform the singularity time is \( t = -3/2\rho > 0 \). There are no eigen-forms for the expanding case, thus if we want to find expanding solitons (of course, only when \( M \) is noncompact) we must solve (3) with some nontrivial vector field \( X \). A natural question is when is a closed \( G_2 \)-structure an eigenform. Noting the characterization of \( \Omega_3^1 \), we in fact have the following result.

**Proposition 2.** Let \( \varphi \) be a closed \( G_2 \)-structure. Then \( \varphi \) is an eigen-form if and only if \( \Delta \varphi \in \Omega_3^1 \).

**Proof.** The only if part is trivial. For the other direction, note that if \( \Delta \varphi = f\varphi \) then since \( d\varphi = 0 \) we must have \( d(f\varphi) = 0 \) as well. In other words,

\[
d f \land \varphi = 0.
\]

Recall that the equation above, along with the special form that \( \varphi \) takes with respect to a local orthonormal frame, shows in a straight-forward way that \( f \) must be constant. Thus \( \varphi \) must be an eigenform. \( \square \)

Although we have the proposition above, it is not straight-forward to find closed eigenforms. However, there are plenty of eigenforms that are not closed. We recall the following

**Definition 2.** A \( G_2 \)-structure is called nearly parallel if its only nonzero torsion form is \( \tau_0 \). We say that a 7-manifold is a nearly \( G_2 \) manifold if it admits a nearly parallel \( G_2 \)-structure.

Thus \((M, \varphi)\) is a nearly \( G_2 \) manifold if and only if \( \delta \varphi = 0 \) and \( d\varphi = \tau_0 \ast \varphi \). We want to point out that in this case, \( \tau_0 \) is necessarily a constant because \( 0 = d(d\varphi) = d\tau_0 \land \ast \varphi \) implies so. The squashed 7-sphere is an example of a nearly \( G_2 \) manifold. By straight-forward computation, we immediately see that:

**Proposition 3.** If \((M, \varphi)\) is a nearly \( G_2 \) manifold, then \( \varphi \) is an eigenform satisfying

\[
\Delta \varphi = \tau_0^2 \varphi.
\]

In particular, \( \varphi \) is a shrinking soliton.

Thus we see that any nearly \( G_2 \) manifold admits a shrinking soliton.

Let us return to the somewhat opposite case to a nearly \( G_2 \) manifold, which is the case where a \( G_2 \)-structure \( \varphi \) is closed. In this case, we can
consider the de Rham cohomology classes. We would first like to point out that if \( \varphi \) is closed then \( f_t^* \varphi \) always stays within the cohomology class of \( f_0^* \varphi \) (\( \varphi \) is closed, hence \( f_0^* \varphi \) is too)\(^5\) which is based on the following elementary result:

**Lemma 3.** Let \( \omega \) be a closed \( p \)-form, then for any smooth family of diffeomorphisms \( f_t \) homotopic to the identity, \( f_t^* \omega \) and \( \omega \) are cohomologous for all \( t \).

Now, for a smooth family \( f_t \) of diffeomorphisms of \( M \), we define a new family \( \tilde{f}_t = f_0^{-1} \circ f_t \). Then \( f_t^* \circ f_0^* \varphi = f_t^* \varphi \), and \( f_0 = I \). Thus by the lemma above, we see that \( f_t^* \varphi \) is cohomologous to \( f_0^* \varphi \) for all \( t \). Therefore a closed\(^6\) steady soliton solution remains in the original cohomology class of \( f_0^* \varphi \) (normalizing the scaling factor to be 1) and hence can be seen as a periodic solution in a fixed cohomology class. On the other hand, for closed expanding and shrinking solitons, we see that

\[
\rho \varphi = -d(X \lrcorner \varphi + \delta \varphi). \quad (13)
\]

In other words, the \( G_2 \)-structure \( \varphi \) must be exact\(^7\).

## 6 Rigidity of Laplacian Solitons

On a compact 7-manifold, if a \( G_2 \)-structure \( \varphi \) satisfies the soliton equation \(^3\) for some \( X \) and \( \rho \), we may ask whether there are other vector fields \( X' \) and constants \( \rho' \) with which \( \varphi \) is also a soliton. We already saw at the end of Section\(^4\) that we must have \( \rho' = \rho \). On the other hand, if \( -\Delta \varphi = L_{X'} \varphi + \rho \varphi \) for some other vector field \( X' \), then subtracting it from the original soliton equation gives

\[
L_{X-X'} \varphi = 0.
\]

In other words, for any compact soliton \( \varphi \), the only symmetries of its defining equation \(^3\) are \( X \rightarrow X + Y \) for vector fields such that \( L_Y \varphi = 0 \). With the \( G_2 \)-structure fixed, this is the only change to a soliton equation that leaves it invariant. In general, a vector field \( X \) such that \( L_X \varphi = 0 \) is simply

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\(^5\)Clearly, a soliton solution remains closed for all \( t \) if \( \varphi \) is closed. On the other hand, in general one can show from \(^3\) that it preserves the closedness condition of the initial value \( \varphi(0) \).

\(^6\)Note that we are distinguishing this from compact solitons.

\(^7\)If \( M \) is compact, this immediately precludes a closed shrinking soliton from being torsion-free. See the next section for a more general derivation of this fact.
called a symmetry of the $G_2$-structure $\varphi$. One can go further in revealing the properties of these vector fields, in fact we have the following result.

**Proposition 4.** On a compact 7-manifold $M$ admitting a $G_2$-structure $\varphi$, any symmetry $X$ of $\varphi$ must satisfy $\text{div}(X) = 0$.

**Proof.** Again by Lemma 2, if $L_X \varphi = 0$, then

$$0 = \int_M L_X \varphi \wedge \ast f \varphi = -3 \int_M df \wedge \ast X^b$$

$$= -3 \int_M f \ast \delta X^b$$

for all $f \in C^\infty(M)$. This implies that $\delta X^b = 0$, or $\text{div}(X) = 0$.

We want to understand the full structure of the space of symmetries for solitons in general. However, for now we will prove a partial result, but which has immediate significance.

**Lemma 4.** If a $G_2$-structure $\varphi$ is closed, then $X \lrcorner \varphi$ is harmonic for any symmetry $X$ of $\varphi$. Moreover, if $M$ is compact and $\varphi$ is torsion-free then $X^b$ is a harmonic 1-form, and in particular if $g_\varphi$ has full $G_2$ holonomy then $X = 0$.

**Proof.** If $d\varphi = 0$, then we have

$$L_X \varphi = d(X \lrcorner \varphi) = 0.$$  

Then using the fact that $X \lrcorner \varphi \in \Omega^2_7$,

$$d \ast (X \lrcorner \varphi) = -\frac{1}{2} d[\varphi \wedge (X \lrcorner \varphi)]$$

$$= -\frac{1}{2} (d\varphi \wedge (X \lrcorner \varphi) - \varphi \wedge d(X \lrcorner \varphi))$$

$$= 0.$$  

Thus $X \lrcorner \varphi \in \Omega^2_7$ is harmonic.\(^8\)

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\(^8\)Note that the computations in (15) show that if $\varphi$ is closed, then $(X \lrcorner \varphi)$ being closed implies $(X \lrcorner \varphi)$ is coclosed as well, for any vector field $X$. 

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Now, if \( \varphi \) is torsion-free then we further have
\[
0 = d \ast (X \ast \varphi) = d(X^\flat \wedge \ast \varphi) = dX^\flat \wedge \ast \varphi.
\]
This shows that \( dX^\flat \in \Omega^2_{14} \). Then using the other characterization of \( \Omega^2_{14} \) we see that
\[
d(X^\flat \wedge \varphi \wedge dX^\flat) = dX^\flat \wedge \varphi \wedge dX^\flat
\]
Integrating both sides above and using Stoke’s theorem, we see that \( dX^\flat = 0 \) necessarily. By Proposition 3 we conclude that \( X^\flat \) is harmonic. If in addition we have full \( G_2 \) holonomy, then we know that \( H^1(M, \mathbb{R}) = \{0\} \) and so \( X^\flat = 0 \).

**Corollary 3.** Let \( \varphi \) be a torsion-free \( G_2 \)-structure on a compact 7-manifold \( M \). Then a vector field \( X \) is a symmetry of \( \varphi \) if and only if \( X^\flat \) is a harmonic 1-form. Thus the space of symmetries of \( \varphi \) is isomorphic to \( H^1(M, \mathbb{R}) \).

**Proof.** The only if part is given by the lemma above. To prove the if part, we will employ the general relation below for the Levi-Civita connection:
\[
L_X \varphi(Y_1, Y_2, Y_3) = \nabla_X \varphi(Y_1, Y_2, Y_3) + \varphi(\nabla_Y X, Y_2, Y_3) + \varphi(Y_1, \nabla_Y X, Y_3)
\]
for any vector fields \( X, Y_1, Y_2, Y_3 \). The torsion-free condition is defined by \( \nabla \varphi = 0 \). Furthermore, since any \( G_2 \) manifold must have zero Ricci curvature everywhere, by the Bochner’s Theorem we know that any harmonic 1-form must be parallel. Then we must have \( \nabla X = 0 \). Combining these facts into (16) we get the desired result. \( \square \)

Corollary 3 contains the special case stated at the end of Lemma 4. In general, we know that for a compact \( G_2 \) manifold the following holds:
\[
\text{Hol}(M) = \{1\} \iff b_1(M) = 7
\]
\[
\text{Hol}(M) = SU(2) \iff b_1(M) = 3
\]
\[
\text{Hol}(M) = SU(3) \iff b_1(M) = 1
\]
\[
\text{Hol}(M) = G_2 \iff b_1(M) = 0.
\]

\(^9\)For example, see [3].
Thus Corollary 3 shows that for a torsion-free $G_2$ structure, increasing the holonomy will decrease its symmetries - making it more and more "rigid". In particular, the $G_2$ structure of a $G_2$ manifold with full $G_2$ holonomy admits no non-trivial symmetries, hence it is "rigid". This is a rigidity result in stark contrast to that for compact manifolds with positive Ricci curvature, whose $b_1(M) = 0$ but yet typically admits a lot of Killing vector fields.

For $G_2$ manifolds that do not have full $G_2$ holonomy, the list (17) shows that there are non-trivial symmetries $X$. In other words, such $X$ generates a one-parameter family $f_t$ of diffeomorphisms homotopic to the identity such that $f_t^* \varphi = \varphi$ for all $t$. The existence of such symmetries is most visible in the following known examples:

1. $\mathbb{T} \times Y$, with $\varphi = dx \wedge \omega + Re \theta$, where $Y$ is a Calabi-Yau 3-fold with Kähler form $\omega$ and holomorphic volume form $\theta$. The holonomy group is $SU(3)$.

2. $\mathbb{T}^3 \times Y$, with $\varphi = dx_{123} + dx_1 \wedge \omega + dx_2 \wedge Re \theta - dx_3 \wedge Im \theta$, where $Y$ is a Calabi-Yau 2-fold with Kähler form $\omega$ and holomorphic volume form $\theta$. The holonomy group is $SU(2)$.

3. $\mathbb{T}^7$, inheriting the standard $G_2$ structure on $\mathbb{R}^7$. The holonomy group is $\{1\}$.

We would like to point-out that Corollary 1 and Corollary 3 together say that a compact steady soliton $(\varphi, X)$ consists of a torsion-free $G_2$-structure $\varphi$ and a symmetry of $\varphi$. As mentioned in the Introduction, it would be desirable to prove an analogous result for compact shrinking solitons. However, it seems that there are some technical difficulties.

7 Concluding Remarks and Questions

In this paper we have discussed fundamental properties of Laplacian solitons on manifolds admitting a $G_2$-structure. In particular, we have investigated solitons on compact 7-manifolds to some detail. Due to the recent flurry of interest in Ricci solitons, we want to make some comparisons to it. Recall that a Ricci soliton is a metric $g$ along with a vector field $X$ on a manifold $M$ such that

$$\text{Ric}_g = L_X g + \rho g$$  \hspace{1cm} (18)
for some constant $\rho$. The sign on $\rho$ dictates the terminologies of shrinking, steady, and expanding solitons in the same way as for our Laplacian solitons.

When $M$ is compact, we know that the only steady and expanding Ricci solitons are Einstein, i.e. $X$ must be a Killing vector field. Corollary $\Box$ can be seen as a close parallel to this result. In fact, by the works of Richard Hamilton and Thomas Ivey we also know that compact shrinking Ricci solitons in dimensions 2 and 3 have to also be Einstein. Since we mentioned that eigenforms should be viewed as an analogy to Einstein metrics in the context of solitons, it is a natural question whether or not compact shrinking Laplacian solitons also have to be eigenforms. If this conjecture is true, then we would be able to finish the classification of compact Laplacian solitons if we further classify all compact eigenforms. The present difficulty seems to be due to a lack of maximum principle-type techniques in $G_2$-geometry, since such techniques were essential in proving results in Ricci Solitons (or the Ricci flow in general).

Using his entropy functionals, when $M$ is compact G. Perelman showed that in (18) $X$ can always be chosen as a gradient vector field, i.e. compact Ricci solitons are always gradient solitons. It would be interesting to know if compact Laplacian solitons are always gradient as well, in the same sense. The entropy functionals of Perelman has a deeper implication: they turn the Ricci flow into a gradient-like flow with respect to the functionals. In particular, compact Ricci solitons appear as critical points of these functionals. Functionals associated to the Laplacian flow have been suggested in $\Box$ and $\Box$, and it would be nice to see if they (or possibly other functionals) can be used to characterize compact Laplacian solitons.

Finally, recall that on a compact manifold $M$ of nonpositive Ricci curvature the space of Killing vector fields is contained in the space of parallel vector fields, and if the Ricci curvature is negative there are no nontrivial Killing vector fields. In view of the preceding results and our intended analogy, it seems that for compact shrinking Laplacian solitons all symmetries should be parallel vector fields (with respect to $g_\phi$) as well, or even always trivial. On the other hand, it seems more tantalizing to conjecture

\begin{itemize}
\item[$\Box$] A relevant question here would be: are the only compact eigenforms nearly parallel $G_2$-structures?
\item[$\Box$] coupled to other equations
\item[$\Box$] Therefore when $M$ is Ricci-flat the space of Killing vector fields is isomorphic to a subspace of $H^1(M,\mathbb{R})$ via Bochner’s Theorem.
\end{itemize}
that the space of symmetries of a compact shrinking Laplacian soliton is simply the eigenspace of the equation $\Delta \omega = \rho \omega$ for 1-forms $\omega$. Identifying the symmetries of solitons will be important because one would like to construct the associated moduli spaces by dividing out the symmetries\footnote{A discussion on such moduli spaces have also appeared in \cite{9}.}

References

[1] Robert Bryant, *Some remarks on $G_2$-structures*, Proceedings of Gokova Geometry/Topology Conference, Gokova, 2006, pp. 75-109.

[2] Robert Bryant and Feng Xu, *Laplacian flow for closed $G_2$-structures: Short time behavior*, arXiv:1101.2004

[3] M. Fernández and A. Gray, *Riemannian manifolds with structural group $G_2$*, Ann. Mat. Pura Appl (IV) 32 (1982), 19-45.

[4] Nigel Hitchin, *The geometry of three-forms in six and seven dimensions*, \texttt{arXiv:math/0010054}

[5] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR MR1787733 (2001k:53093)

[6] Spiro Karigiannis, *Flows of $G_2$-structures I*, Q.J. Math. 60 (2009), no.4, 487-522.

[7] Spiro Karigiannis, Benjamin McKay, Mao-Pei Tsui, *Soliton solutions for the Laplacian coflow of some $G_2$ structures with symmetry*, \texttt{arXiv:1108.2192v1}

[8] Hartmut Weiss and Frederik Witt, *A heat flow for special metrics*, arXiv:0912.0421

[9] Hartmut Weiss and Frederik Witt, *Energy Functionals and Soliton Equations for $G_2$ forms*, \texttt{arXiv:1201.1208v1}

[10] Feng Xu and Rugang Ye, *Existence, convergence and limit map of the Laplacian Flow*, \texttt{arXiv:0912.0074}
