Spectral theory, the holomorphic functional calculus, and frames

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Abstract

We may need for some applications to construct one or more frames. To do so, we need to think of some family of vectors that satisfies the frame condition. In the present article, we will discuss a spectral criterion allowing us to check this condition for sequences constructed from an orthonormal basis, a bounded operator, and a holomorphic function.

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1 Introduction

Duffin and Shaeffer introduced the notion of a frame in a Hilbert space in 1952 [5] to study nonharmonic Fourier series. Basically, frames behave like redundant bases: in practice, they allow redundant linear decompositions of vectors. The electrical engineer and physicist Gabor had already in mind this last idea in 1946 [7] in the context of signal decompositions. Starting from 1986 [4], Daubechies, Grossmann, and Meyer published a series of articles on frames and wavelets that highlighted their importance and made them a popular present-day topic of study. Nowadays, frames are being applied in signal processing, image processing, fault-tolerant data transmission, data compression,
sampling theory, and many other fields. Frames and their generalizations intervene even in purely mathematical areas such as Banach space theory. A general introduction to frame theory can be found in ([1],[2]).

The aim of this article is to present a spectral criterion for checking the frame condition for the specific type of Bessel sequences which internal synthesis operator is of the form \( f(T) \) where \( f \) is a holomorphic function and \( T \) an operator. Specifically, there may be situations where, in a separable Hilbert space \( K \), we are interested in sequences of the form \((f(T)(e_n))_{n\in\mathbb{N}}\) where \((e_n)_{n\in\mathbb{N}}\) is an orthonormal basis, \( T \) is a bounded operator in \( K \) and \( f \) is a holomorphic function. In this article, we will see a simple spectral criterion allowing us to check that such sequences are Riesz bases whenever they are frames. Since checking the Riesz basis condition is easy in this case due to the holomorphic spectral theorem, our main proposition 3.1 is a helpful tool for asserting or rejecting the frame property.

**Plan of the article.** In section 2, we set some notations and define continuous frames in Hilbert spaces (of which discrete frames are a particular case). Section 3 is the main section of this paper. We state in this section our main proposition allowing an easy check of the frame condition, and develop in some detail the ingredients of its proof. Example 3.1 is a remarkable illustration showing the usefulness of our approach.

2 Preliminaries

2.1 Notation

The following notations are used throughout this article. We denote by \( \mathbb{F} \) one of the fields \( \mathbb{R} \) or \( \mathbb{C} \). \( \mathbb{N} \) denotes the set \( \{0, 1, 2, \cdots \} \) of natural numbers including 0, and \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \).

If \( A \) is a subset of a topological space \( X \), we denote by \( \text{cl}(A) \) the closure of \( A \) in \( X \). If \( T \) is an operator in a Hilbert space \( K \), the notations \( \sigma_p(T), \sigma_{ap}(T), \) and \( \sigma(T) \) refer to the point spectrum, approximate point spectrum and spectrum of \( T \) respectively. \( \sigma_{ap}(T) \) is defined as \( \{\lambda \in \mathbb{C} : \inf_{\|x\|=1}\|Tx - \lambda x\| = 0\} \). We have \( \sigma_p(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T) \) and \( \sigma_{ap}(T) \) and \( \sigma(T) \) are non-empty compact subsets of \( \mathbb{C} \).

If \( (X, \Sigma) \) is a measurable, \( L^2(X, \mu; \mathbb{F}) \) refers to the classical Lebesgue space of square integrable measurable functions modulo the equivalence relation of equality \( \mu \)-almost everywhere. We will frequently abuse notation by equating a function with its class under this equivalence relation.

2.2 Continuous frames with values in a Hilbert space

Let \( K \) be a Hilbert space and \( (X, \Sigma, \mu) \) a measure space.

**Definition 2.1.** [2] We say that a family \( \Phi = (\varphi_x)_{x \in X} \) with \( \varphi_x \in K \) for all \( x \in X \) is a continuous frame in \( K \) if

\[
\exists 0 < A \leq B : \forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2
\]
A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if $\Sigma$ is the discrete $\sigma$-algebra and $\mu$ is the counting measure.

3 A spectral criterion for checking the frame condition for sequences of the type $(f(T)(e_n))_{n \in \mathbb{N}}$

There may be situations where, in a separable Hilbert space $K$, we are interested in sequences of the form $(f(T)(e_n))_{n \in \mathbb{N}}$ where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis, $T$ is a bounded operator in $K$ and $f$ is a holomorphic function. In the present section, we will see a simple criterion allowing us to check that such sequences are frames whenever they are Riesz bases. Since checking the Riesz basis condition is easy to do in this situation using the holomorphic spectral theorem, our main result is very helpful for asserting or rejecting the frame property. We will prove for instance, that $(e_n + e_{n+1})_{n \in \mathbb{N}}$ is not a frame when $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of a separable Hilbert space $K$, an example that appeared in 5.4.6 p. 132 of [2] (see example 3.1 of this article).

We recall that given a complete orthonormal basis $(e_i)_{i \in I}$ in a Hilbert space $K$ of dimension $|I|$ and a bounded operator $T$ in $K$, $(T(e_i))_{i \in I}$ is a frame if and only if $T$ is surjective (see theorem 5.5.4 p. 138 of [2]), a Riesz basis if and only if $T$ is invertible, and a complete orthonormal basis if and only if $T$ is unitary.

The following theorem is the main result of this section.

**Theorem 3.1.** Let $H$ be a Hilbert space and $T \in B(K)$. Then we have

$$\sigma_{ap}(T^*) = \sigma(T^*) \iff \left( \forall f : \Omega \to \mathbb{C} \text{ holomorphic} \right) f(T) \text{ surjective } \iff f(T) \text{ invertible },$$

where $\Omega$ is any open subset of $\mathbb{C}$ containing $\sigma(T)$.

This theorem gives indeed a simple solution to the problem since, by the holomorphic spectral mapping theorem (see theorem VII.3.11 p. 569 of [6]), the invertibility of $f(T)$ amounts to checking that $0 \notin f(\sigma(T))$.

**Remark 3.1.** Ignoring the holomorphic functional calculus, it is easily seen that

$$0 \in \sigma(T^*) \Rightarrow 0 \in \sigma_{ap}(T^*) \iff (T \text{ surjective } \Rightarrow T \text{ invertible}).$$

The theorem says that requiring the more restrictive condition $\sigma_{ap}(T^*) = \sigma(T^*)$ makes it possible for the second part of the equivalence to be true not only for $T$ but for the whole unital commutative algebra $\{f(T)/f : \Omega \to \mathbb{C} \text{ holomorphic}\}$.

**Theorem 3.1** (see proof 3) follows easily from three lemmas, two of them are already present in the literature.

It is interesting to note that normal and compact operators satisfy both parts of the equivalence in the proposition and that they make up the most basic type of operators $T$ which can be used to generate frames or non-frames using this spectral criterion. This is because for a normal operator $T$: 3
1. $T^*$ is normal;

2. $\sigma_{ap}(T) = \sigma(T)$ (see propositions XI.1.1 p. 347 and XI.1.4 p. 349 of [3]);

3. $f(T)$ is normal for every holomorphic function $\forall f : \Omega \to \mathbb{C}$ (see theorem VII.3.10 p. 568 of [6]);

4. $T$ surjective $\Rightarrow T$ invertible (because $Ker(T) = Ker(T^*) = Ran(T)^\perp$),

Similarly, compact operators satify properties 1-4 with 'normal' replaced with 'compact' and the inconsequential exception that $f(T)$ is invertible when $f$ is a holomorphic function satisfying $f(0) \neq 0$ (see [10]). The reasons in this case are that $\sigma(T) \backslash \{0\} = \sigma_p(T)$ and $0 \in cl(\sigma_p(T))$ which imply $\sigma(T) = cl(\sigma_p(T)) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$, and a compact operator in infinite-dimensional space is non-surjective.

However, there are other types of operators satisfying $\sigma_{ap}(T^*) = \sigma(T^*)$, which makes the proposition even more interesting.

**Example 3.1.** Consider the unilateral left shift operator $S^*$ on $\ell^2(\mathbb{N})$ defined by $S^*((u_n)_{n \in \mathbb{N}}) = (u_{n+1})_{n \in \mathbb{N}}$ for all $(u_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$. It is equivalently defined by its values on the canonical basis $(\delta_n)_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$ : $S^*(\delta_0) = 0$ and $S^*(\delta_n) = \delta_{n-1}$ for all $n \geq 1$. The adjoint of $S^*$ is the unilateral right shift operator $S$ of $\ell^2(\mathbb{N})$ and is defined by $S((u_n)_{n \in \mathbb{N}}) = (0, u_0, u_1, u_2, \cdots)$ for all $(u_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$, or equivalently $S(\delta_n) = \delta_{n+1}$ for all $n \in \mathbb{N}$. Notice that while $S^*S = Id$ ($S$ is an isometry), $SS^* = 7\overline{d}$ is false. So $S$ is not normal. Moreover, it is easy to show that $\sigma_p(S^*) = \{z \in \mathbb{C} : |z| < 1\}$ and $\sigma(S^*) = \{z \in \mathbb{C} : |z| \leq 1\} = cl(\mathbb{D})$ which tells us that $S^*$ (nor $S$) is not compact and that $\sigma_{ap}(S^*) = \sigma(S^*) = cl(\mathbb{D})$. Hence, theorem 3.1 can be applied. So a Bessel sequence of the form $(f(S)(e_n))_{n \in \mathbb{N}}$ is a frame iff it is a Riesz basis iff $0 \notin f(\sigma(S)) = f(cl(\mathbb{D}))$.

In particular, taking $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = 1 + z$ for all $z \in \mathbb{C}$, we recover the well-known result that $(e_n + e_{n+1})_{n \in \mathbb{N}}$ is not a frame (see example 5.4.6 p. 132 of [2]), since $0 = f(-1) \notin f(cl(\mathbb{D}))$. More generally, $(e_n + e_{n+1} + \cdots + e_{n+k})_{n \in \mathbb{N}}$ is not a frame for any $k \in \mathbb{N}^*$ since $f(z) = 1 + z + \cdots + z^k$ admits a root in $cl(\mathbb{D})$.

The proof of theorem 3.1 (see proof 3) builds on three lemmas.

**Lemma 3.1.** Let $K$ be a Hilbert space, $T \in B(K)$ and $\lambda \in \mathbb{C}$. Then we have

$$\overline{\lambda} \notin \sigma_{ap}(T^*) \Leftrightarrow T - \lambda.Id \text{ surjective}$$

For a reference, see proposition XI.1.1 p. 347 of [3]. Here is the proof for convenience.

**Proof.** ($\Rightarrow$) Suppose that $\overline{\lambda} \notin \sigma_{ap}(T^*)$. By definition, this means that $\inf_{\|x\|=1}\|T^*x - \overline{\lambda}x\| > 0$. Therefore there exists a constant $C > 0$ such that $\forall x \in K : \|T^*x - \overline{\lambda}x\| \geq C\|x\|$. This implies clearly that $Ker(T^* - \overline{\lambda}) = \{0\}$. It also implies that $R(T^* - \overline{\lambda})$ is closed ($R(S)$ denotes the range of the operator $S$): suppose that we have a sequence $y_n = (T^* - \overline{\lambda})(x_n)$ such that $y_n \to y \in K$. Then $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence which implies that $(x_n)_{n \in \mathbb{N}}$ is too since $\|x_n - x_m\| \leq \frac{1}{C}\|y_n - y_m\|$. Since $K$ is a complete metric space, $(x_n)_{n \in \mathbb{N}}$ converges to some $x \in K$. By continuity of $T$, we then have $y = \lim_n (T^* - \overline{\lambda})(x_n) = (T^* - \overline{\lambda})(x)$,
which proves that \( R(T^* - \lambda) \) is closed. By the closed range theorem, \( R(T - \lambda) \) is also closed. Moreover, \( cl(R(T - \lambda)) = Ker(T^* - \lambda)^{\perp} = K \). Hence \( T - \lambda \) is surjective.

\( \left(\Rightarrow\right) \) Suppose that \( T - \lambda \) is surjective. Then, introducing \( B \), the Moore-Penrose pseudo-inverse operator of \( T - \lambda \), we have \( B(T - \lambda) = Id \). This implies that \( \forall x \in K \) s.t. \( \|x\| = 1 \). Suppose that \( \| (T^* - \lambda) B * x \| = 1 \), and so \( \lambda \notin \sigma_{ap}(T^*) \).

\[ \square \]

**Lemma 3.2.** Let \( K \) be a Hilbert space and \( T \in B(K) \). Then we have

\[ \sigma_{ap}(T^*) = \sigma(T^*) \Leftrightarrow \left( \forall \lambda \in \mathbb{C} \right) \ T - \lambda \cdot Id \text{ surjective} \Rightarrow T - \lambda \cdot Id \text{ invertible} \]

\[ \text{Proof.} \ (\Rightarrow) \ \text{Suppose that} \ \sigma_{ap}(T^*) = \sigma(T^*) \ 	ext{Let} \ \lambda \in \mathbb{C} \ \text{Suppose that} \ T - \lambda \cdot Id \text{ surjective. Then} \ \lambda \notin \sigma_{ap}(T^*) \ 	ext{by lemma 3.1. Since} \ \sigma_{ap}(T^*) = \sigma(T^*) \ 	ext{this implies that} \ \lambda \notin \sigma(T^*). \ This \ \text{last set is the image of} \ \sigma(T) \ \text{by the conjugation map, so} \ \lambda \notin \sigma(T), \ \text{which means that} \ T - \lambda \cdot Id \text{ is invertible.} \ (\Leftarrow) \ \text{Suppose that} \ \left( \forall \lambda \in \mathbb{C} \right) \ T - \lambda \cdot Id \text{ surjective} \Rightarrow \ T - \lambda \cdot Id \text{ invertible}. \ \text{Let} \ \lambda \in \sigma(T^*) \setminus \sigma_{ap}(T^*). \ \text{So} \ T - \lambda \cdot Id \text{ is surjective, and so it is invertible by the hypothesis. This means} \ \lambda \notin \sigma(T), \ \text{and so} \ \lambda \notin \sigma(T^*), \ \text{which is a contradiction.} \ \square \]

**Lemma 3.3.** Let \( K \) be a Hilbert space and \( T \in B(K) \). Then we have

1. \( cl(\sigma_p(T)) = \sigma(T) \Rightarrow (\forall f : \Omega \rightarrow \mathbb{C} \text{ holomorphic}) \ cl(\sigma_p(f(T))) = \sigma(f(T)) \);
2. \( \sigma_{ap}(T) = \sigma(T) \Rightarrow (\forall f : \Omega \rightarrow \mathbb{C} \text{ holomorphic}) \ \sigma_{ap}(f(T)) = \sigma(f(T)), \)

where \( \Omega \) is any open subset of \( \mathbb{C} \) containing \( \sigma(T) \).

This lemma appears without proofs as exercises XI.1.3 and XI.1.4 p. 349 of [3]. In particular, this means that the set of bounded operators \( T \) satisfying \( \sigma_{ap}(T) = \sigma(T) \) is stable by the holomorphic functional calculus. Here is the proof for convenience.

\[ \text{Proof.} \ 1. \ \text{Suppose that} \ cl(\sigma_p(T)) = \sigma(T). \ \text{Let} \ f : \Omega \rightarrow \mathbb{C} \ \text{a holomorphic function, where} \ \Omega \ \text{is an open subset of} \ \mathbb{C} \ \text{containing} \ \sigma(T). \ \text{Since} \ \sigma_p(f(T)) \subseteq \sigma(f(T)) \ \text{and} \ \sigma(f(T)) \ \text{is closed, we have} \ cl(\sigma_p(f(T))) \subseteq \sigma(f(T)). \ \text{Conversely, let} \ \lambda \in \sigma(f(T)) = f(\sigma(T)) \ \text{holomorphic spectral mapping theorem). Hence, there exists} \ s \in \sigma(T) \ \text{such that} \ \lambda = f(s). \ \text{Since} \ \sigma(T) = cl(\sigma_p(T)), \ \text{there exists a sequence} \ (s_n)_{n \in \mathbb{N}} \ \text{such that} \ s_n \rightarrow s \ \text{and} \ s_n \in \sigma_p(T) \ \text{for all} \ n \in \mathbb{N}. \ \text{Since} \ s_n \in \sigma_p(T), \ \text{there exists} \ v_n \in K \setminus \{0\} \ \text{such that} \ T(v_n) = s_n v_n. \ \text{We then have that} \ f(T)(v_n) = f(s_n) v_n, \ \text{which means that} \ f(s_n) \in \sigma_p(f(T)) \ \text{for all} \ n \in \mathbb{N}. \ \text{Since} \ f(s_n) \rightarrow f(s) = \lambda \ \text{by continuity of} \ f, \ \text{we deduce that} \ \lambda \in cl(\sigma_p(f(T))) \ \text{and we conclude that the two sets are equal.}

2. \ \text{Suppose that} \ \sigma_{ap}(T) = \sigma(T). \ \text{Let} \ f : \Omega \rightarrow \mathbb{C} \ \text{a holomorphic function, where} \ \Omega \ \text{is an open subset of} \ \mathbb{C} \ \text{containing} \ \sigma(T). \ \text{We have} \ \sigma_{ap}(f(T)) \subseteq \sigma(f(T)). \ \text{Conversely, let} \ \lambda \in \sigma(f(T)) = f(\sigma(T)) \ \text{Hence, there exists} \ s \in \sigma(T) \ \text{such that} \ \lambda = f(s). \ \text{Since} \ \sigma(T) = \sigma_{ap}(T), \ \text{there exists a sequence} \ (x_n)_{n \in \mathbb{N}} \ \text{with} \ x_n \in H \ \text{and} \ \|x_n\| = 1 \ \text{for all} \ n \in \mathbb{N}, \ \text{and} \ (T - s \cdot Id)(x_n) \rightarrow 0. \ \text{Consider the function} \ g : \Omega \rightarrow \mathbb{C} \ \text{defined by}\ g(z) = f(z) - f(s) \ \text{for all} \ z \in \Omega. \ \text{Since} \ g \ \text{is holomorphic and}\ g(s) = 0, \ \text{there exists} \]
a holomorphic function $h : \Omega \rightarrow \mathbb{C}$ such that \( g(z) = h(z)(z - s) \) for all \( z \in \Omega \). We have then

\[
(f(T) - \lambda)(x_n) = h(T)(T - sId)(x_n) \rightarrow 0,
\]

by continuity of \( h(T) \). Hence \( \lambda \in \sigma_{ap}(f(T)) \) and we conclude that the two sets are equal.

We are now ready to prove theorem 3.1.

Proof. (of theorem 3.1)

(\( \Rightarrow \)) Suppose that \( \sigma_{ap}(T^*) = \sigma(T^*) \). Let \( f : \Omega \rightarrow \mathbb{C} \) be a holomorphic function, where \( \Omega \) is an open subset of \( \mathbb{C} \) containing \( \sigma(T) \). Then \( \sigma_{ap}(f(T)^*) = \sigma(f(T)^*) \) by lemma 3.3 part 2. By lemma 3.2 for \( \lambda = 0 \), we have \( [f(T) \text{ surjective } \Rightarrow f(T) \text{ invertible}] \).

(\( \Leftarrow \)) Suppose that

\[
(\forall f : \Omega \rightarrow \mathbb{C} \text{ holomorphic}) \ f(T) \text{ surjective } \Rightarrow f(T) \text{ invertible},
\]

where \( \Omega \) is any open subset of \( \mathbb{C} \) containing \( \sigma(T) \).

Let \( \lambda \in \mathbb{C} \). Taking \( f : \Omega \rightarrow \mathbb{C} \) to be the function given by \( f(z) = z - \lambda \) for all \( z \in \Omega \), we have

\[
T - \lambda.Id \text{ surjective } \Rightarrow T - \lambda.Id \text{ invertible}.
\]

Using the equivalence of lemma 3.2, we conclude that \( \sigma_{ap}(T^*) = \sigma(T^*) \).

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References

[1] P. G. Casazza. The art of frame theory. Taiwanese Journal of Mathematics, vol. 4, no. 2, June 2000, pp. 129-201.

[2] O. Christensen. An Introduction to Frames and Riesz Bases. 2nd edition, Birkhäuser/Springer, Switzerland, 2016.

[3] J. B. Conway. A course in functional analysis. Second edition. Springer-Verlag, New York, 1990.

[4] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. J. Math. Phys. vol. 27, 1986, pp. 1271-1283.

[5] R. Duffin and A. Schaffer. A class of non-harmonic Fourier series. Trans. Amer. Math. Soc. vol. 72, 1952, pp. 341-366.
[6] N. Dunford and J. T. Schwartz. *Linear operators. I: General theory*. Pure and Applied Mathematics 6, John Wiley, 1958.

[7] D. Gabor. *Theory of communications*. Jour. Inst. Elec. Eng. (London), vol. 93, 1946, pp. 429-457.

[8] P. R. Halmos. *Shifts on Hilbert spaces*. J. Reine Angew. Math., no. 208, 1961, pp. 102-112.

[9] https://math.stackexchange.com/questions/1771203/approximate-point-spectrum-of-a-normal-operator

[10] https://math.stackexchange.com/questions/3746484/is-a-function-of-a-compact-operator-a-compact-operator

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