An integral sliding-mode parallel control approach for general nonlinear systems via piecewise affine linear models

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Abstract
The fundamental problem of stabilizing a general nonaffine continuous-time nonlinear system is investigated via piecewise affine linear models (PALMs) in this article. A novel integral sliding-mode parallel control (ISMPC) approach is developed, where an uncertain piecewise affine system (PWA) is constructed to model a nonaffine continuous-time nonlinear system equivalently on a compact region containing the origin. A piecewise sliding-mode parallel controller is designed to globally stabilize the PALM and, consequently, to semiglobally stabilize the original nonlinear system. The proposed scheme enjoys three favorable features: (i) some restrictions on the system input channel are eliminated, thus the developed method is more relaxed compared with the published approaches; (ii) it is convenient to be used to deal with both matched and unmatched uncertainties of the system; and (iii) the proposed piecewise parallel controller generates smooth control signals even around the boundaries between different subspaces, which makes the developed control strategy more implementable and reliable. Moreover, we provide discussions about the universality analysis of the developed control strategy for two kinds of typical nonlinear systems. Simulation results from two numerical examples further demonstrate the performance of the developed control approach.

Keywords
nonlinear systems, integral sliding-mode parallel control, piecewise affine linear models, universality

1 | INTRODUCTION

PALMs possess the convenience of control design and simplicity of structure, which have resulted in its extensive employment in analysis and control of diverse industrial systems with nonlinearity.¹,² By dividing the premise state space into a series of adjacent subspaces, PALMs model a nonlinear system equivalently by an affine linear system with norm-bounded uncertainties that can be made small enough via appropriate design in each subspace.³ Based on the powerful linear system theory and a quadratic Lyapunov function, this comparatively simple framework promotes systematic analysis and
As another research frontier in the robust control theory, sliding-mode control (SMC) has drawn growing research interests and has been used in many industrial applications. The SMC strategy holds various favorable characteristics, like unique robustness against disturbances and uncertainties, distinguished transient performance and fast response. The core idea of the SMC approach is to construct the closed-loop control system such that the system trajectories are first driven onto a well-designed linear sliding surface covering the equilibrium, and are forced to maintain on the surface with preferred convergent characteristic towards the equilibrium. The special dynamics the closed-loop control system behaves while its trajectories are moving on the sliding surface is called the sliding motion. An alternative approach is the integral sliding-mode control (ISMC) scheme where an integral form sliding surface is used instead of a linear one. Different from the common SMC scheme, in the ISMC approach, the system trajectories maintain on the sliding surface during the whole time interval. Consequently, the reaching phase can be removed from the dynamics w.r.t. the controlled system, which demonstrates stronger robustness of the ISMC approach than the SMC approach.

By now, there have been few published results on ISMC design for PALMs. Considering the equivalence between a T-S fuzzy system and an uncertain PALM, a convenient extension might be made such that the fuzzy ISMC strategy in Reference can be applied to PALMs. However, the fuzzy ISMC strategy in Reference suffers from a restrictive assumption that each submodel of the fuzzy system must hold an identical constant input matrix, thus it is confronted with significant conservativeness when being applied to general nonlinear systems. Various attempts have been made to weaken or remove this restrictive assumption. The approach in Reference replaced this assumption by a less conservative assumption in SMC design for fuzzy systems. A piecewise ISMC approach allowing different local input matrices can be found in Reference where the region of interest was split into a series of subspaces and unique integral sliding surface was constructed w.r.t. all subspaces. However, the high complexity of the approach in Reference obstructs its wide implementation in practice.

Compared with the T-S fuzzy model based approach in Reference, a PALM models a general nonlinear system using linearization method, which yields fewer plant rules in general and thus less conservative controller design. Nevertheless, the affine terms appearing in the local models of the PALM, on the other hand, lead to more complicated analysis and synthesis. Moreover, the commonly used piecewise static feedback controllers for PALMs suffer from abrupt changes around the boundaries between different subspaces due to the switching behaviors of the systems. How to avoid this undesired chattering phenomenon needs further investigation.

In this article, motivated by the previous fuzzy-model-based result in Reference, an appropriate ISMC design for general nonaffine continuous-time nonlinear systems through PALMs is developed to eliminate the restriction on system matrices and to avoid the chatter phenomenon around boundaries between subregions. Specifically, a PALM is constructed on a compact region to express a controlled nonlinear system first and then an ISMPC method is proposed for global asymptotic stabilization of the PALM and, consequently, for ensuring the semi-global asymptotic stability of the original general nonlinear system. In particular, the constructed integral type sliding surface function depends on the system state and control signal. Compared with the commonly designed static state feedback controller, the ISMPC strategy utilizes a new parallel control law and, in each partition of the whole system space, the resultant sliding-mode controller has a dynamical parallel compensator form. The corresponding control gains are obtained by calculating a series of linear matrix inequalities (LMIs) with the aid of a common quadratic Lyapunov function. This ISMPC strategy holds three favorable features:

(i) Different input matrices w.r.t. the PALMs are allowed, thus the ISMPC scheme is applicable for general nonaffine nonlinear systems;
(ii) The uncertainties arising during the approximation procedure, either “matched” or “unmatched”, are eliminated in the control channel w.r.t. the resultant controlled system, which introduces stronger robustness; and
(iii) The proposed piecewise sliding-mode parallel controller admits a time-integral form of solution and naturally generates smooth control signals, even around the boundaries of the partitioned subspaces where the controller gains switch abruptly. This helps reduce chattering phenomenon and makes the control law more implementable and reliable in real applications.

The universality discussion w.r.t. the developed ISMPC scheme in this article is another key contribution. The key concern is that, for any given stabilizable general nonlinear system with a smooth system function, can one always construct a piecewise integral sliding surface and a corresponding piecewise integral sliding-mode parallel controller such that the
resultant closed-loop control system behaves a stable sliding motion since initially? This universality characteristic of the proposed ISMPC strategy is analyzed for two classes of typical continuous-time nonlinear systems, that is, globally asymptotically/exponentially stabilizable (GAS/GES) nonlinear systems, respectively. It is believed that these discussions may provide confidence in applying the developed approach to wider industrial practice.

The rest of this article is structured as: Section 2 formulates the PLAM and problems. In Section 3, an ISMPC strategy is developed to globally robustly stabilize a PLAM and, correspondingly, to semi-globally stabilize the corresponding original nonlinear system on the predefined system space, then, the universality discussion of the developed ISMPC scheme for GAS/GES nonaffine nonlinear systems are presented respectively. The numerical simulation is implemented in Section 4. Conclusion lies in Section 5.

Notations: The notation $\star$ in a matrix expresses the entries induced by matrix symmetry. Given a vector or a matrix $Q$, then $Q^T$ and $\|Q\|$ denote its transpose and induced norm respectively, and $Q > 0$ indicates a matrix $Q$ is positive definite. Let $\alpha : [0, a) \to [0, \infty)$ be a continuous function, then $\alpha$ belongs to the class $\mathcal{K}$, if $\alpha(0) = 0$ and it is strictly increasing; $\alpha$ belongs to the class $\mathcal{K}_\infty$, if $a \to \infty$ as $t \to \infty$ and it belongs to the class $\mathcal{K}_0$.

\section{SYSTEM DESCRIPTION AND PRELIMINARIES}

This article will principally concentrate on the following general nonaffine continuous-time nonlinear system defined on a compact region $X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$:

$$\dot{x}(t) = f(x(t), u(t)), \quad (1)$$

where $x := [x_1, \ldots, x_n]^T \in X$, $u := [u_1, \ldots, u_m]^T \in U$. The following assumption is necessary:

\textbf{Assumption 1.}

1. $X \times U$ contains the origin.
2. The system function $f$ is continuously differentiable on $X \times U$ and has the origin as its equilibrium.

In view of the universal approximation capability of PALMs,

the following PALM can be constructed to express the controlled system in (1) equivalently on $X \times U$:

$$\bar{x}(t) = (A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t) + C_i + \Delta C_i(t), \quad (2)$$

where

$$C_0 = \Delta C_0(t) \equiv 0, \quad \|\Delta C_i\| \leq \varepsilon_{\delta}, \quad \|\Delta A_0(t),\Delta B_0(t)\| \leq \varepsilon_{f_0}, \quad \|\Delta A_j(t),\Delta B_j(t)\| \leq \varepsilon_{f,j} \quad j \in \Phi := \{1, 2, …, l\}, \quad (3)$$

$$\bar{x}(t) = [x^T(t), u^T(t)]^T,$$  

the norm-bounds of the approximation error $\varepsilon_{f_0}, \varepsilon_f$, and $\varepsilon_{\delta}$ can be made arbitrarily small,

$X_i$ and $U_i$ are the partitions of $X$ and $U$, respectively, $\overline{X}_i = X_i \times U_i$ are the adjacent partitions of $X \times U$, $l + 1$ denotes the number of partitions, and $\overline{X}_0$ contains the origin.

Note that the partitions w.r.t. the PALM in (2) are inherently polyhedral regions on the compact region. To outer approximate the polyhedral regions $P_i$, an ellipsoid $F_i$ is used. Assume that the matrices $Q_i$ and $f_i$ can be designed to satisfy

$$P_i \subseteq F_i, \quad F_i = \{\bar{x}|\|Q_i\bar{x} + f_i\| \leq 1\} \quad (4)$$

Suppose the polyhedral regions $P_i$ are slabs, which are appropriately described as

$$P_i = \{\bar{x}|\sigma_{i_1} \leq \theta^T \bar{x} \leq \sigma_{i_2}\}, \quad (5)$$

where $\sigma_{i_1} \in \mathbb{R}$, $\sigma_{i_2} \in \mathbb{R}$, and $\theta = \mathbb{R}^{nx1}$. Then each region can be precisely illustrated as a degenerate ellipsoid in (4) with

$$Q_i = \frac{\sigma_{i_3}}{\sigma_{i_1} - \sigma_{i_2}} f_i = -\frac{\sigma_{i_3}}{\sigma_{i_2} - \sigma_{i_1}} f_i, \quad (6)$$

new $Q_i$ and $f_i$. Then each region can be precisely illustrated as a degenerate ellipsoid in (4) with

\begin{align*}
Q_i &= \frac{\sigma_{i_3}}{\sigma_{i_1} - \sigma_{i_2}} f_i = -\frac{\sigma_{i_3}}{\sigma_{i_2} - \sigma_{i_1}} f_i, \\
& \quad \text{where} \quad \sigma_{i_1} \in \mathbb{R}, \sigma_{i_2} \in \mathbb{R}, \text{and} \quad \theta = \mathbb{R}^{nx1}.
\end{align*}
Based on (4), the state $\bar{x}(t)$ within an ellipsoid region $P_i$ satisfies

$$
\begin{bmatrix}
\bar{x}(t) \\
1
\end{bmatrix}^T
\begin{bmatrix}
Q_i^T Q_i & Q_i^T f_i \\
* & f_i^T f_i - 1
\end{bmatrix}
\begin{bmatrix}
\bar{x}(t) \\
1
\end{bmatrix} \leq 0.
$$

(7)

It is noted that the PALM in (2) approximates a smooth nonlinear system via linearization at multi-operating points in both the system state space and the control space.\(^{31}\) However, some prior knowledge of the system behavior, which is often very difficult to obtain for complicated systems, is essential in this modeling process. One can refer to Reference 24 for an approach to identifying these operating points via clustering algorithms. With the linearization points determined, the system space can be partitioned into a series of slab subspaces in (5), each one of which envelops an operating point,\(^1\) and can be formulated as in (7).

By regarding the approximation errors arising during the modeling procedure as norm-bounded uncertainty terms, one can conclude that an ISMC scheme robustly stabilizing the PALM in (2) can stabilize the original nonlinear system in (1) simultaneously. Considering the equivalence between a T-S fuzzy system and an uncertain PALM,\(^24\) one may extend the fuzzy ISMC approach in References 26-28 to our case. However, the approaches in References 26-28 are useful only when the nonlinear system in (1) has a constant and linear input channel. This motivates us to develop a new ISMC scheme that stabilizes the general nonaffine nonlinear system in (1) based on its corresponding PALM in (2) and to remove these restrictions.

3 INTEGRAL SLIDING-MODE PARALLEL CONTROL

A new ISMPC strategy will be presented in this section to robustly stabilize the PALM in (2) and, correspondingly, to stabilize the nonlinear system in (1).

3.1 An integral sliding-mode parallel controller

Considering the PALM in (2), or equivalently (1), we propose a novel piecewise integral sliding surface as

$$
\dot{s}(t) = S_x [x(t) - x(0)] - \int_0^t S_x (A_i x(s) + B_i u(s) + C_i) \, ds + S_u [u(t) - u(0)] - \int_0^t S_u (F_i x(s) + G_i u(s) + D_i) \, ds,
$$

(8)

where $S_x \in \mathbb{R}^{m \times n}$ and $S_u \in \mathbb{R}^{m \times m}$ represent the sliding surface matrices to be designed and $S_u$ is required to be nonsingular. The matrices $F_i \in \mathbb{R}^{m \times n}$, $G_i \in \mathbb{R}^{m \times m}$ and $D_i \in \mathbb{R}^{m \times 1}$ will be determined later and here we set $D_0 \equiv 0$.

The following theorem provides an appropriate design of the sliding-mode control law to guarantee that the integral sliding surface in (8) can be maintained from the beginning of evolution.

**Theorem 1.** For the PALM in (2), or correspondingly, the controlled nonlinear system in (1), by insulting a piecewise sliding-mode parallel controller as

$$
\dot{x}(t) = F_i x(t) + G_i u(t) + D_i - (\gamma + \alpha_i + \nu_i(t)) S_u^{-1} \text{sgn}(s(t))
$$

(9)

with $u(0) = 0$,

$$
\alpha_0 = 0, \alpha_j = \varepsilon_g \|S_x\|, \nu_0(t) = \varepsilon_f \|S_x\| \|x^T(t) \cdot u^T(t)\|, \nu_j(t) = \varepsilon_f \|S_x\| \|x^T(t) \cdot u^T(t)\|^j, j \in \Phi,
$$

(10)

where $F_i$ and $G_i$ are defined in (8), $\gamma > 0$ is a scalar, the norm-bounds of the approximation error $\varepsilon_f$, $\varepsilon_f$ and $\varepsilon_g$ denote in (3), then the piecewise integral sliding surface in (8) is reached and maintained since initially in potential.
Remark 1. Notice that the equivalence between the PALM in (2) and the continuous-time nonlinear system in (1) is ensured only within $X \times U$. Put another way, the closed-loop control system consisting of (1) and (9), which is named the practical closed-loop control system in this article, behaves the sliding motion since initially only when its trajectories keep moving within $X \times U$ during the time interval of interest. This is, however, often not the case in practice even for a stable closed-loop control system. When the initial states are very close to the boundary of $X \times U$, the system trajectories are highly possible to move out $X \times U$. In this case, the PALM in (2) and the piecewise integral sliding surface in (8) are both undefined, thus the developed ISMPC approach no longer works. Therefore, it is stated that by using the proposed approach, the ideal sliding mode can be only realized “in potential”. This will be illustrated by Figure 3 in the simulation section. Practically, designing a compact region where the PLAM is constructed big enough can improve this situation.

Remark 2. The designed sliding-mode parallel control law in (9) is in form of a dynamical parallel compensator, which distinguishes the proposed scheme from the published feasible solutions in References 26-28. One can observe an important advantage of this ISMPC strategy is that it can be applied to the PALM in (2) without requiring that each local model holds an identical input channel, while in this general case, the methods in References 26-28 cannot be directly used.

Proof. The Lyapunov function candidate of the piecewise integral sliding surface in (8) can be constructed to be

$$A(t) = s^T(t) s(t).$$

(11)

For the sake of simplicity, the case that $\bar{x}(t) \subseteq \bar{X}_i$, $i \in \Phi$ is considered exclusively in this proof. This proof can be extended to the case that $\bar{x}(t) \subseteq \bar{X}_0$ similarly.

Then, from (8) and (9), we have

$$\dot{s}(t) = S_x \dot{x}(t) - S_c (A_c x(t) + B_c u(t) + C_c) + S_u u(t) - S_d (F_d x(t) + G_d u(t) + D_d)$$

$$= S_x [\Delta A_i x(t) + \Delta B_i u(t) + \Delta C_i] - (\gamma + \alpha_i + \nu_i(t)) \text{sgn}(s(t)).$$

(12)

Substituting (11) into (12) yields

$$\dot{A}(t) = 2s^T(t) \dot{s}(t) = 2s^T(t) \left\{ S_x [\Delta A_i x(t) + \Delta B_i u(t) + \Delta C_i] - (\gamma + \alpha_i + \nu_i(t)) \text{sgn}(s(t)) \right\}. $$

(13)

Since we have

$$2s^T(t) S_x [\Delta A_i x(t) + \Delta B_i u(t) + \Delta C_i]$$

$$\leq 2 \| s(t) \| \| S_x \| \| \Delta A_i x(t) + \Delta B_i u(t) + \Delta C_i \|$$

$$\leq 2 \| s(t) \| \| S_x \| (\| \Delta A_i x(t) + \Delta B_i u(t) \| + \| \Delta C_i \|)$$

$$\leq 2 \| s(t) \| \| S_x \| (\varepsilon_f \left\| [x^T(t), u^T(t)] \right\| + \varepsilon_g)$$

$$= 2 \varepsilon_f \| s(t) \| \| S_x \| \left\| [x^T(t), u^T(t)] \right\| + 2 \varepsilon_g \| s(t) \| \| S_x \|$$

(14)

and

$$2 (\gamma + \alpha_i + \nu_i(t)) \| s(t) \| \leq 2s^T(t) (\gamma + \alpha_i + \nu_i(t)) \text{sgn}(s(t)), $$

(15)

then

$$\dot{A}(t) \leq 2 \varepsilon_f \| s(t) \| \| S_x \| \left\| [x^T(t), u^T(t)] \right\| + 2 \varepsilon_g \| s(t) \| \| S_x \| - 2 (\gamma + \alpha_i + \nu_i(t)) \| s(t) \|. $$

(16)

Combining (10)-(16), one can conclude that

$$\dot{A}(t) \leq -2 \gamma \| s(t) \| = -2 \gamma \sqrt{A(t)}, $$

(17)

which implies that in finite time, $s(t)$ can converge to zero. Since initially $s(0) = 0$ and, consequently, the piecewise integral sliding surface in (8) is maintained subsequently.
Remark 3. In this article, we only consider the nominal general nonlinear systems as in (1). Nevertheless, practical nonlinear plants often face issues such as input saturation, undirectional input constraints, dead-zone, and unmodeled dynamics. There have been several control strategies in the literature addressing these issues, such as those in References 27, 28, 33-35, where neural networks or robust integral terms are included in the control law. It is worth pointing out that (i) sliding-mode control design under control constraints has been a tough research topic and more efforts must be made to solve this problem; (ii) the proposed approach in this article still works when the dead-zone function can be described by a smooth function, and the unmodeled dynamics have norm-bounds as in (3); and (iii) the existing approaches, like those in Reference 27 and 28 cannot be extended to our case in (1) because the control input gain cannot be represented by a constant matrix. However, it would be an interesting research topic to investigate the constraint control design problem by integrating the ideas in References 33-35 and the proposed ISMPC approach.

Denote \( \bar{A}_i = [A_i, B_i] \), \( \bar{C}_i = [C_i^T, D_i^T]^T \), \( \bar{K}_i = [F_i, G_i] \), \( \bar{S} = [S_x, S_u] \), \( R_1 = [I_n, 0_{nxm}]^T \), \( R_2 = [0_{mxn}, I_m]^T \). Then, one has a more compact form of (8) as

\[
\text{for } \bar{x}(t) \subseteq \bar{X}, i \in \varphi
\]

\[
s(t) = \bar{S} \left\{ \bar{x}(t) - \bar{x}(0) - \int_0^t \left[ (R_1 \bar{A}_i + R_2 \bar{K}_i) \bar{x}(r) + \bar{C}_i \right] dr \right\} = 0. \tag{18}
\]

In the literature, the PWA

\[
\text{for } \bar{x}(t) \subseteq \bar{X}, i \in \varphi
\]

\[
\dot{\bar{x}}(t) = (R_1 \bar{A}_i + R_2 \bar{K}_i) \bar{x}(t) + \bar{C}_i \tag{19}
\]

is usually named the “nominal closed-loop control system”. One observes that the integral sliding surface variable \( s(t) \) in (18) is in fact the real time difference between the trajectories of the practical closed-loop control system defined in Remark 1 and those of (19), multiplying by a weight matrix \( \bar{S} \). This represents the core idea of the proposed ISMPC approach, that is, to achieve a sliding motion that is as close to (19) as possible. In other words, one can design (19) according to desired control criteria to force the practical closed-loop control system to behave desirable control performance.

The following lemma provides a constructive procedure for designing (19):

**Lemma 1.** (32). Given a series of matrices \( D_i, i \in \Phi \) of (19), which is generated by Algorithm 1, the PWA in (19) is asymptotically stable, if the following LMIs are feasible w.r.t. a set of scalars \( \lambda_i > 0 \), a matrix \( W \in \mathbb{R}^{(m+n)(m+n)} > 0 \), and matrices \( H_j \in \mathbb{R}^{m \times (m+n)}, j \in \varphi \), and furthermore, the control gains are given as \( \bar{K}_i = H_i W^{-1} \):

\[
R_1 \bar{A}_i W + R_2 H_0 + (R_1 \bar{A}_i W + R_2 H_0)^T < 0,
\]

\[
\begin{bmatrix}
\Omega_i - \lambda_i \bar{C}_i \bar{C}_i^T & W Q_i - \lambda_i \bar{C}_i \bar{C}_i^T \\
* & \lambda_i (I - f_i f_i^T)
\end{bmatrix} < 0, \tag{20}
\]

where \( \Omega_i = R_1 \bar{A}_i W + R_2 H_1 + (R_1 \bar{A}_i W + R_2 H_1)^T \) and \( Q_i \) and \( f_i \) are defined in (4).

**Proof.** Lemma 1 derives directly from lemma 4.1 in Reference 32 and the proof is omitted here.

The detailed design procedure to calculate the control matrices \( \bar{K}_i \) and \( D_i \) in (19) is summarized as follows:

**Algorithm 1.** (sample method 32). Step 1: Choose a grid for the domain of the vector \([D_1 \ D_2 \ \cdots \ D_l]\) and sample its value at \( N \) points.

Step 2: Solve Lemma 20 for each point in the grid. If a feasible solution is given, stop;

Step 3: Increase the sampling density of the grid and return to Step 1.


3.2 The sliding motion

How to obtain the sliding surface matrix \( \bar{S} \) through stability analysis of the sliding motion will be further shown in this subsection. For the sliding motion w.r.t. (8), two conditions must be fulfilled simultaneously:
\[ s(t) = 0, \]
\[ \dot{s}(t) = S_s \dot{x}(t) - S_x (A_t x(t) + B_t u(t) + C_t) + S_u \dot{u}(t) - S_u (F_t x(t) + G_t u(t) + D_t) = 0. \] (21)

Since \( S_u \) is nonsingular, (21) yields

\[ \text{for } \bar{x}(t) \subseteq \bar{X}_i, i \in \varphi \]
\[ \dot{u}(t) = F_t x(t) + G_t u(t) + D_t - S_u^{-1} S_x [\Delta A_t (t) x(t) + \Delta B_t (t) u(t) + \Delta C_t (t)]. \] (22)

which is usually named the equivalent sliding-mode control law. The corresponding sliding motion is then referred to the dynamical system consisting of (2) and (22):

\[ \text{for } \bar{x}(t) \subseteq \bar{X}_i, i \in \varphi \]
\[ \begin{cases} \dot{x}(t) = A_t x(t) + B_t u(t) + C_t + \Delta A_t (t) x(t) + \Delta B_t (t) u(t) + \Delta C_t (t) \\ \dot{u}(t) = F_t x(t) + G_t u(t) + D_t - S_u^{-1} S_x [\Delta A_t (t) x(t) + \Delta B_t (t) u(t) + \Delta C_t (t)] \end{cases} \] (23)

or equivalently

\[ \bar{x}(t) = \left( R_1 \bar{A}_i + R_2 \bar{K}_i + (R_1 - R_2 S_u^{-1} S_x) \Delta \bar{A}_i (t) \right) \bar{x}(t) + \left( R_1 - R_2 S_u^{-1} S_x \right) \Delta C_i (t). \] (24)

One has the following theorem:

**Theorem 2.** The asymptotic stability of the sliding motion in (24) can be ensured, when the LMIs in (25) are feasible w.r.t. a series of positive scalars \( \eta_0, \eta_i, \eta_1, i \in \Phi \) and a matrix \( P \in \mathcal{R}^{(m+n)(m+n)} > 0 \). Additionally, the integral sliding surface matrix is \( \bar{S} = R_2^T P. \)

\[
\begin{bmatrix}
A_0 + \eta_0 \varepsilon_i^2 I_{m+n} & PR_1 & PR_2 \\
\ast & R_1^T PR_1 - \eta_i I_n & 0 \\
\ast & \ast & -R_2^T PR_2
\end{bmatrix} < 0, \]
\[
\begin{bmatrix}
A_i + \eta_1 \varepsilon_i^2 I_{m+n} - \eta_i Q_i^T Q_i & PR_1 & PR_1 & PR_2 \\
\ast & R_1^T PR_1 - \eta_i I_n & 0 & 0 \\
\ast & \ast & R_1^T PR_1 - \eta_i I_n & 0 \\
\ast & \ast & \ast & \eta_i \varepsilon_i^2 - \eta_i (f_i^T f_i - 1) & 0 \\
\ast & \ast & \ast & \ast & -\frac{1}{2} R_2^T PR_2
\end{bmatrix} < 0, \ i \in \Phi. \] (25)

where \( A_i = P \left( R_1 \bar{A}_i + R_2 \bar{K}_i \right) + \left( R_1 \bar{A}_i + R_2 \bar{K}_i \right)^T P, i \in \varphi. \)

**Proof.** The following Lyapunov function is used for stability analysis:

\[ V(\bar{x}(t)) = \bar{x}^T(t) \bar{P} \bar{x}(t). \] (26)

Along the trajectories of the sliding motion in (24), the derivative of (26) can be obtained as

\[
\dot{V}(\bar{x}(t)) = \bar{x}^T(t) \dot{\bar{P}} \bar{x}(t) + \bar{x}^T(t) \dot{\bar{x}}(t) \\
= \bar{x}^T(t) \left( P \left( R_1 \bar{A}_i + R_2 \bar{K}_i \right) + \left( R_1 \bar{A}_i + R_2 \bar{K}_i \right)^T P + \Delta \bar{A}_i^T(t) \left( PR_1 - PR_2 S_u^{-1} S_x \right)^T + \left( PR_1 - PR_2 S_u^{-1} S_x \right) \Delta \bar{A}_i(t) \right) \bar{x}(t) \\
+ \bar{x}^T(t) P \bar{C}_i + \bar{C}_i^T \bar{x}(t) PR_1 \Delta C_i(t) + \Delta C_i^T(t) R_1^T P \bar{x}(t) \\
- \bar{x}^T(t) PR_2 S_u^{-1} S_x \Delta C_i(t) - \left( \bar{x}^T(t) \bar{P} R_2 S_u^{-1} S_x \Delta C_i(t) \right)^T. \] (27)
Let $M$ be a positive definite matrix satisfying $M = \sqrt{P}$. Then, it follows from the fact $\bar{S} = R_x^T P$ that

$$-PR_x S_u^{-1} S_x \Delta \bar{A}_i (t) - (PR_x S_u^{-1} S_x \Delta \bar{A}_i (t))^T$$

$$= -PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x \Delta \bar{A}_i (t) - (PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x \Delta \bar{A}_i (t))^T$$

$$\leq PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x (R_x^T PR_x)^{-1} R_x^T P + \Delta \bar{A}_i^T (t) R_x^T M M R_x \Delta \bar{A}_i (t)$$

$$= PR_x (R_x^T PR_x)^{-1} R_x^T P + \Delta \bar{A}_i^T (t) R_x^T P R_x \Delta \bar{A}_i (t)$$

(28)

and

$$-\bar{x}^T (t) PR_x S_u^{-1} S_x \Delta C_i (t) - (\bar{x}^T (t) PR_x S_u^{-1} S_x \Delta C_i (t))^T$$

$$= -\bar{x}^T (t) PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x \Delta C_i (t) - (\bar{x}^T (t) PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x \Delta C_i (t))^T$$

$$\leq \bar{x}^T (t) PR_x (R_x^T PR_x)^{-1} R_x^T M M R_x (R_x^T PR_x)^{-1} R_x^T P \bar{x} (t) + \Delta C_i^T (t) R_x^T P R_x \Delta C_i (t)$$

$$= \bar{x}^T (t) PR_x (R_x^T PR_x)^{-1} R_x^T P \bar{x} (t) + \Delta C_i^T (t) R_x^T P R_x \Delta C_i (t)$$

(29)

Then, combining (28) and (29), we can obtain

$$\dot{V} (\bar{x} (t)) \leq \bar{x}^T (t) \{ A_i + PR_x \Delta \bar{A}_i (t) + (PR_x \Delta \bar{A}_i (t))^T + \Delta \bar{A}_i^T R_x^T P R_x \Delta \bar{A}_i + 2PR_x (R_x^T PR_x)^{-1} R_x^T P \} \bar{x} (t)$$

$$+ \bar{x}^T (t) PC_i + C_i^T P \bar{x} (t) + \Delta C_i^T (t) R_x^T P R_x \Delta C_i (t) + \bar{x}^T (t) PR_x \Delta C_i (t) + \Delta C_i^T (t) R_x^T P \bar{x} (t).$$

(30)

It follows from (30) that $\dot{V} (\bar{x} (t)) < 0$ if

$$\bar{x}^T (t) \{ 2PR_x (R_x^T PR_x)^{-1} R_x^T P + A_i + \Delta \bar{A}_i^T (t) R_x^T P R_x \Delta \bar{A}_i (t) + PR_x \Delta \bar{A}_i (t) + (PR_x \Delta \bar{A}_i (t))^T \} \bar{x} (t)$$

$$+ \Delta C_i^T (t) R_x^T P R_x \Delta C_i (t) + \bar{x}^T (t) PC_i + C_i^T P \bar{x} (t) + \bar{x}^T (t) PR_x \Delta C_i (t) + \Delta C_i^T (t) R_x^T P \bar{x} (t) < 0.$$

(31)

Besides, (31) is equivalent to

$$\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix}^T
\begin{bmatrix}
A_i + \Pi_0 & PR_x & PR_x & PC_i \\
\ast & R_x^T PR_x & 0 & 0 \\
\ast & \ast & R_x^T PR_x & 0 \\
1 & \ast & \ast & \ast & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix} < 0,$$

(32)

where $\Pi_0 = 2PR_x (R_x^T PR_x)^{-1} R_x^T P$.

Furthermore, the norm-bound of approximation errors defined in (3) satisfies

$$\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix}^T
\begin{bmatrix}
-\epsilon^2 I_m n & 0 & 0 & 0 \\
\ast & I_n & 0 & 0 \\
\ast & \ast & 0 & 0 \\
1 & \ast & \ast & \ast & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix} \leq 0,$$

(33)

and

$$\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\ast & 0 & 0 & 0 \\
\ast & \ast & -\epsilon^2 I_n & 0 \\
1 & \ast & \ast & 1
\end{bmatrix}
\begin{bmatrix}
\bar{x} (t) \\
\Delta \bar{A}_i (t) \bar{x} (t) \\
\Delta C_i (t) \\
1
\end{bmatrix} \leq 0.$$

(34)
For the partitions \( \mathcal{X}_i, i \in \Phi \), by defining \( \xi(t) = \left[ \bar{x}^T(t) \quad (\Delta A_i(t) \bar{x}(t))^T \quad \Delta C_i^T \right]^T \), (7) is rewritten as

\[
\xi^T(t) \begin{bmatrix}
Q_i^T Q_i & 0 & Q_i^T f_i \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & f_i^T f_i - 1
\end{bmatrix} \xi(t) \leq 0.
\] (35)

By applying Lemma 2 in Appendix A and combining the LMIs in (32)–(35), one can conclude that (32) is fulfilled if there exists a series of positive scalars \( \eta_1, \eta_2, \eta_3, i \in \Phi \) such that

\[
\begin{bmatrix}
A_i + P_0 + \eta_i \varepsilon^2 I_{m+n} - \eta_i Q_i^T Q_i & PR_1 & PR_1 \\
\ast & R_1^T PR_1 - \eta_i I_n & 0 \\
\ast & \ast & R_1^T PR_1 - \eta_i I_n \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
P \bar{C}_i - \eta_i Q_i^T f_i \\
0 \\
0 \\
\eta_i \varepsilon^2 - \eta_i (f_i^T f_i - 1)
\end{bmatrix} < 0.
\] (36)

Based on the Schur’s complement in Reference 36, the LMIs in (36) are proved to be equivalent to those in (25). One can obtain that the derivative of \( V(\bar{x}(t)) \) is strictly negative if the LMIs in Theorem 2 can hold simultaneously. Therefore, the sliding motion (24) is asymptotically stable if the LMIs in (25) can be fulfilled.

\[\blacksquare\]

It’s worth noting that the sliding surface matrix \( S_u \) in (8) is required to be nonsingular because its inverse is essential in constructing the sliding mode controller in (9). By designing \( \bar{S} = R_2^T P \), one has \( S_u = R_2^T PR_2 \) with \( R_2 = [0_{m \times n}, I_m]^T \). One can then conclude that \( S_u \) is positive definite and thus invertible since \( P \) is positive definite.

Remark 4. The common quadratic Lyapunov function used as in (26) tends to be conservative, especially when dealing with more complicated nonlinear systems. In order to achieve less conservative control synthesis, the more relaxing piecewise/fuzzy Lyapunov functions in References 37–39 could be potentially employed. However, preliminary research along this direction has shown that the control design would tend to be extreme complex in practice. 29

Corollary 1. If the conditions in Theorem 2 are fulfilled simultaneously, the practical closed-loop control system consisting of (1) and (9) behaves a semi-globally asymptotically stable sliding motion since initially.

Proof. Corollary 1 can be concluded based on the fact that only within \( X \times U \), the PALM in (2) is equivalent to the original nonlinear system in (1). The proof will be thus omitted here. \[\blacksquare\]

A systematic algorithm for implementing the ISMPC approach concerning the nonlinear systems in (1) can be summarized as follows:

**Algorithm 2.** For any controlled nonlinear system in (1), a piecewise sliding-mode parallel controller in (9) can be constructed such that a semi-globally asymptotically stable sliding motion can be achieved since initially, by conducting the subsequent procedure:

1. **Step 1:** To obtain the PALM via the linearization approach of the controlled nonlinear system \( X \times U \).
2. **Step 2:** To obtain the control matrices \( \bar{K}_i = [F_i, G_i] \) and \( D_i \) via Algorithm 1.
3. **Step 3:** To obtain the sliding surface matrix \( \bar{S} \) based on Theorem 2.
4. **Step 4:** To choose a suitable parameter \( \gamma > 0 \) to ensure that the system trajectories move within \( X \times U \). In case Algorithm 2 cannot return a feasible solution, raise the amount of partitions \( l \) and repeat the loop until \( l \) exceeds the pre-chosen threshold.

Remark 5. The computational complexity of Algorithm 2 consists of implementing Algorithm 1 and solving the LMIs (25). Following the result, 40 given the maximum amount of sampled points in the grid \( M \), the complexity of Algorithm 1 could be calculated as \( \mathcal{O} \left( M l^2 n^2 \right) \); similarity, the complexity of solving the LMIs (25) is \( \mathcal{O} \left( l^2 n^2 \right) \). Therefore, the computational complexity of Algorithm 2 is \( \mathcal{O} \left( M l^2 n^2 \right) \).
3.3  |  Universality discussion

In Sections 3.1 and 3.2, we have developed an ISMPC scheme to stabilize a general nonlinear system as in (1) through PALMs. It is shown that the scheme works if a series of LMIs is fulfilled simultaneously. Therefore, the universality of such a scheme is questionable, which motivates the study in the remaining of this section. To be specific, we will answer the subsequent question: for any given stabilizable nonlinear system in (1), can one always design a piecewise sliding-mode parallel controller as in (9) such that the resultant closed-loop control system behaves a stable sliding motion since initially?

In the remaining of this article, we say that a general nonaffine continuous-time nonlinear system in (1) is GAS/GES, if there exists a control law in the form of

\[ \dot{x}(t) = f(x(t), u(t)) \]

is globally asymptotically/exponentially stable. For brevity, we rewrite (37) as

\[ \tilde{x}(t) = F(\tilde{x}(t)) = R_1 f(x(t), u(t)) + R_2 g(x(t), u(t)), \]

where \( R_1 \) and \( R_2 \) are defined in Section 3.1 below (17).

The subsequent two theorems (Theorems 3 and 4) summarize the main results of this subsection.

**Theorem 3.** For a GES nonlinear system in (1), one can always design a piecewise sliding-mode parallel controller as in (9) such that the resultant closed-loop control system behaves a semi-globally exponentially stable sliding motion from the beginning of evolution.

**Proof.** See Appendix B.

Before proceeding with the more general case, a preliminary result from the Lyapunov converse theorem in References 41 is presented.

Suppose the nonlinear system in (38) is globally asymptotically stable, then along the trajectories of (38), there exist two functions \( \alpha_1(\cdot) \) and \( \alpha_2(\cdot) \) belonging to the class \( \mathcal{K}_{\infty} \), a function \( \alpha_3(\cdot) \) belonging to the class \( \mathcal{K} \), a scalar \( h > 0 \), and a Lyapunov function \( V(\tilde{x}) \) satisfying

\[ \alpha_1(\|\tilde{x}\|) \leq V(\tilde{x}) \leq \alpha_2(\|\tilde{x}\|), \]

\[ \frac{\partial V(\tilde{x})}{\partial \tilde{x}} F(\tilde{x}) \leq -\alpha_3(\|\tilde{x}\|), \]

\[ \left\| \frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right\| \leq h, \]

where \( F(\tilde{x}) \) is defined in (38).

**Theorem 4.** For a GAS nonlinear system in (1), one can always design a piecewise sliding-mode parallel controller as in (9) such that the resultant closed-loop control system behaves a semi-globally asymptotically stable sliding motion from the beginning of evolution, if for the function \( \alpha_3(\cdot) \) defined in (40), the condition

\[ \inf_{\tilde{x} \in \mathbb{X} \times \mathbb{U}} \alpha_3(\|\tilde{x}\|) \geq \rho \|\tilde{x}\| + \alpha_4(\|\tilde{x}\|) \]

holds for a \( \mathcal{K} \) function \( \alpha_4(\cdot) \) and a scalar \( \rho > 0 \).

**Proof.** See Appendix C.

**Remark 6.** As shown in Appendix B and Appendix C, the construction of the piecewise sliding-mode parallel control law depends on the norm-bounds of the uncertainties. Therefore, for a given GES nonlinear system or a GAS nonlinear
system satisfying the condition in (42), one can always design a corresponding parallel control law to stabilize the original system in (1) by decreasing the norm-bounds of uncertainties until the LMIs in Theorem 2 are fulfilled. This fact brings us great confidence in applying the easy-checking control design approach in Theorem 2 to industrial practice.

4 | SIMULATION STUDIES

Two different types of numerical examples are given to demonstrate the effectiveness and advantages of the developed ISMPC approach.

4.1 | Nonlinear Chua’s Circuit

The famous Chua’s circuit has the following dynamical equation:42

\[
\begin{align*}
C_1 \frac{dx_1}{dt} &= \frac{x_2 - x_1}{R} - g(x_1) - u, \\
C_2 \frac{dx_2}{dt} &= \frac{x_1 - x_2}{R} - x_3, \\
L \frac{dx_3}{dt} &= x_2 - v_d,
\end{align*}
\]

where \(x_1\) and \(x_2\) are the voltages across the capacitors \(C_1\) and \(C_2\), respectively, \(x_3\) is the current passing the inductor, \(u\) is the control current used to stabilize the nonlinear circuit and \(v_d\) is the voltage loss \(R_0x_3\) or the external disturbance. The nonlinear function \(g(x_1)\) describes the nonlinearity of the resistor is given by

\[g(x_1) = ax_1 + cx_1^3, a < 0, c > 0.\]

Here, we use the approximation model built in Reference 43, where the system parameters and subspaces can be referred to, for the control design. Note that the dynamical equation of the Chua’s circuit could be reformulated as \(\dot{x}(t) = f(x(t)) + Bu(t)\), where \(f(x(t)) = \left[ \begin{array}{cc} -\frac{g(x_1)}{C_1} & 0 \\ \frac{1}{C_1R} & -\frac{1}{C_2R} \end{array} \right] \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + \left[ \begin{array}{c} 0 \\ -1 \\ \frac{1}{L} \end{array} \right] \left( \begin{array}{c} x_3 \\ \dot{x}_3 \end{array} \right)\)

\(x(t)\) and \(B = \left[ \begin{array}{cc} 0 \\ \frac{1}{C_1} \end{array} \right]^T\).

In practice, it is very difficult to calculate the approximation error bounds precisely. In this experiment, following the approach as in Reference 30, the norm-bounds of uncertainties are determined by

\[\varepsilon_{f_0} = \max \left\{ \frac{\|f(x(t)) - A_0x(t)\|}{\|x(t)\|} \right\}, \varepsilon_f = \frac{1}{2} \max \left\{ \frac{\|A_0x(t) + C_0 - f(x(t))\|}{\|x(t)\|} \right\}, \varepsilon_g = \frac{1}{2} \max \|A_1x(t) + C_1 - f(x(t))\|, i = \{1, 2\}\]

(43)

at a series of vertex points, which could be sampled uniformly or randomly, within the operating region \((x_1, x_2, x_3) \in [-5.5] \times [-5.5] \times [-5.5]\), which yields \(\varepsilon_{f_0} = 0.007, \varepsilon_f = 0.003, \varepsilon_g = 0.005\) by numerically calculating (43) on these sampled vertex points. It is noticed that only finite points could be implemented. However, one could sample more vertex points within the operating region to enhance the precision of the obtained norm-bounds of uncertainties.

Based on Algorithm 2, one can obtain the controller matrices and the integral sliding surface matrix as

\[
K_0 = \begin{bmatrix} 6.0518 & 49.6777 & 20.8074 & -2.5596 \end{bmatrix}, K_1 = \begin{bmatrix} 6.1412 & 49.5742 & 20.8064 & -2.5411 \end{bmatrix},
\]

\[
K_2 = \begin{bmatrix} 5.7869 & 48.5033 & 20.3217 & -2.5140 \end{bmatrix}, D_1 = -D_2 = 0.200,
\]

\[
S = \begin{bmatrix} -0.3318 & -4.8582 & -1.6437 & 0.4322 \end{bmatrix}.
\]

In particular, the PALM of the Chua’s circuit possesses an identical constant input matrix in each subspace. It can be observed that the Chua’s circuit behaves desirable control performance.
In order to avoid singular problem and reduce chattering phenomenon, $\text{sgn}(s(t))$ in (9) is replaced by its approximation function:

$$s(t) \frac{s(t)}{||s(t)|| + \delta}$$

with $\delta = 0.001$.

This simulation sets $[4 \quad 0 \quad 1 \quad 0]$ as the initial state $\bar{x}(0)$. The voltages across the capacitors $C_1$ and $C_2$ are shown in Figure 1A, respectively, and the currents in the circuit are presented in Figure 1B. One can observe that the control input is pretty smooth during the simulation time.

It is also observed from Figure 1C that the integral sliding surface can be reached and maintained since the time $t = 0.59s$, which is prior to the time when the system trajectories converge to zero (which is about $t = 25s$ according to Figure 1A,B). This coincides with the theoretical analysis in this article that the controlled system should enter the sliding mode first and then behave stable sliding motion. Note that theoretically, the practical closed-loop control system should enter and keep the sliding mode from the beginning of this simulation. The approximation signum function utilized in the constructed controller in (9) results in this deviation from the ideal sliding mode. Note that the approximation
signum function is used to avoid critical chattering phenomenon during the sliding motion. Better approximation can be achieved if the positive constant \( \delta \) is chosen to be smaller, which yields better control performance (closer to the ideal case). However, more evident chattering phenomenon would be caused as a result, which is undesirable in practice. How to determine the value of \( \delta \) is a tradeoff in practice and depends on specific applications.

4.2 Inverted pendulum

Stabilization of the inverted pendulum is always used to demonstrate the advantages and effectiveness of various control methods. The inverted pendulum system in Reference 30 is chosen. The inverted pendulum has the following dynamics:

\[
\begin{align*}
\dot{x}_1 (t) &= x_2 (t) \\
\dot{x}_2 (t) &= \frac{g \sin(x_1) - a m l x_2^2 \sin(2x_1)/2 - a \cos(x_1) u(t)}{4l/3 - a m l \cos^2 (x_1)},
\end{align*}
\]

where \( x_1 \) is the angle of pendulum from the vertical, \( x_2 \) denotes the angular velocity, and \( u(t) \) is the input signal. \( g = 9.8 \text{ m/s}^2 \) is called as the gravity constant, \( M = 4.0 \text{ kg} \) is the mass of the cart, the mass and length of the pendulum is \( m = 2.0 \text{ kg} \) and \( l = 0.5 \text{ m} \) respectively, \( a = \frac{1}{M+m} \).

\( X \times U \) is selected as \([-\pi/2, 5\pi/6] \times [-3, 3] \times [-300, 300] \). To obtain the corresponding PLAM of the inverted pendulum by linearization around the operating points \((0; 0; 0), (\pm \pi; 0; 0)\) and \((\pm 13\pi/30; 0; 0)\). And the subspaces are selected as

\[
\begin{align*}
P_0 &= \left\{ x \mid -\frac{\pi}{3} \leq x_1 \leq \frac{\pi}{3} \right\}, P_1 = \left\{ x \mid \frac{\pi}{3} \leq x_1 \leq \frac{5\pi}{12} \right\}, P_2 = \left\{ x \mid \frac{5\pi}{12} \leq x_1 \leq \frac{\pi}{2} \right\}, \\
P_3 &= \left\{ x \mid -\frac{5\pi}{12} \leq x_1 \leq -\frac{\pi}{3} \right\}, P_4 = \left\{ x \mid -\frac{\pi}{2} \leq x_1 \leq -\frac{5\pi}{12} \right\}.
\end{align*}
\]

One obtains

\[
\begin{align*}
\tilde{A}_0 &= \begin{bmatrix} 0 & 1 & 0 \\ 19.6000 & 0 & -0.6667 \end{bmatrix}, \tilde{A}_1 = \tilde{A}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 4.7040 & 0 & -0.2667 \end{bmatrix}, \tilde{A}_2 = \tilde{A}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1.5955 & 0 & -0.1585 \end{bmatrix}, \\
C_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_1 = -C_3 = \begin{bmatrix} 0 \\ 8.6533 \end{bmatrix}, C_2 = -C_4 = \begin{bmatrix} 0 \\ 12.3638 \end{bmatrix}.
\end{align*}
\]

Notice that the dynamics equation of the inverted pendulum can be expressed as an affine nonlinear form by \( \dot{x}(t) = f(x(t)) + g(x(t))u(t) \), where \( f(x(t)) = \frac{g \sin x_1 - a m l x_2^2 \sin(2x_1)/2}{4l/3 - a m l \cos^2 x_1} \) and \( g(x(t)) = \frac{-a \cos x_1}{4l/3 - a m l \cos^2 x_1} \).

Similarly, the approximation error bounds are difficult to be obtained precisely. Thus, in this experiment, based on the method in References 30, the norm-bounds of uncertainties are calculated as

\[
\epsilon_{f_0} = \max \left\{ \frac{\| A_0 x(t) - f(x(t)) \|}{\| x(t) \|}, \frac{\| g(x(t)) - B_0 \|}{\| x(t) \|} \right\}, \epsilon_f = \max \left\{ \frac{\| A_1 x(t) + C_1 - f(x(t)) \|}{\| x(t) \|}, \frac{\| g(x(t)) - B_1 \|}{\| x(t) \|} \right\}, \epsilon_\delta = \frac{1}{2} \max \| A_i x(t) + C_i - f(x(t)) \|, i \in \{1, 2, 3, 4\}
\]

at a series of vertex points within the operating region \((x_1, x_2) \in [-\pi/2, 5\pi/6] \times [-3, 3] \). One can calculate that \( \epsilon_{f_0} = 0.02, \epsilon_f = 0.01, \) and \( \epsilon_\delta = 0.35 \) through the numerical calculation of (44) on these chosen vertex points.

Base on Algorithm 2, one can obtain the controller matrices and the sliding surface matrix as follows:

\[
\begin{bmatrix} D_1 & D_3 \\ D_2 & D_4 \end{bmatrix} = \begin{bmatrix} 3.00 & -3.00 \\ 5.00 & -5.00 \end{bmatrix}, \quad \tilde{K}_0 = \begin{bmatrix} 46381.5662 & 13843.0990 & -437.2131 \end{bmatrix}
\]
FIGURE 2 The numerical experiment.

\[
K_1 = K_3 = \begin{bmatrix} 13997.0179 & 4213.0535 & -133.2537 \end{bmatrix}
\]

\[
K_2 = K_4 = \begin{bmatrix} 8287.5168 & 1620.6117 & -51.5002 \end{bmatrix}
\]

\[
\bar{S} = \begin{bmatrix} -0.1269 & -0.0501 & 0.00066 \end{bmatrix}
\]

It is noted that the constructed PALM has different input matrices in each partition. Therefore, the method in References 26-28 cannot be extended to our case trivially.

Similarly, the function \( \text{sgn}(s(t)) \) is approximated by the function

\[
\frac{s(t)}{||s(t)|| + 0.020}
\]

In the simulation, we set \( \bar{x}(0) \) as \([82^\circ \ 0 \ 0]\). Figure 2A,B shows the inverted pendulum trajectories. One can observe that both \( x(t) \) and \( u(t) \) are moving within the chosen compact region and behave the asymptotic stable sliding motion. In view of highly nonlinearities w.r.t. the inverted pendulum, a large control input is required to restrict system dynamics on the proposed integral sliding manifold and stabilize nonlinear system behaviors.
The inverted pendulum trajectories move out $X \times U$ (A) and the integral sliding surface is not maintained (B) in Figure 3, which shows an undefined case.

It is also observed from Figure 2C that the sliding mode is achieved prior to $t = 0.63s$ while the system trajectories converge to the origin after $t = 3s$. That is, the controlled inverted pendulum enters the sliding mode before the system trajectories are stabilized to the origin and behaves the ideal system dynamics afterwards.

It can be observed from the numerical results and Figure 2D that the inverted pendulum trajectories converge to the equilibrium with satisfied performance and no obvious chattering phenomenon appears. Note that the considered system plant has no constant input matrices. The successful application shown in the numerical results demonstrates the advantages of the developed ISMPC scheme.

In order to demonstrate the statement we made in Remark 1 that the sliding mode can be only achieved “in potential”, we further consider the case that the system trajectories are initially placed at $[88^\circ \; 0 \; 0]$, which is very close to the boundary of the region of interest, and the control design results are shown in Figure 3. One could observe that the system trajectories move out $X \times U$ and the ideal stable sliding motion defined in this article cannot be realized. This is because outside $X \times U$, the approximation PALM in (2) and the piecewise integral sliding surface in (8) are both undefined.

One can also observe from the simulation results of the two examples that first, the developed stabilization strategy is implementable to both nonlinear systems in (1) with different input matrices or identical input channel; and second, the proposed ISMPC generates smooth control signals even around the boundary between different subspaces, while the closed-loop control system behaves stable sliding motion.

5 CONCLUSIONS

A new ISMPC scheme has been proposed to stabilize the general continuous-time nonaffine nonlinear systems through PALMs. The proposed control strategy removes a restrictive assumption that is required in relevant research and shows significant convenience in coping with general nonlinear systems by constructing a piecewise integral sliding surface and a corresponding piecewise sliding-mode parallel controller. Moreover, results on the universality of the proposed ISMPC scheme have been provided, which further demonstrates its usefulness. Future research topics include conservatism reduction, universality analysis of different types of controllers and practical applications of the developed method.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.
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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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**APPENDIX A. A USEFUL LEMMA**

**Lemma 2** (S-procedure\textsuperscript{36}). Given some symmetric matrices $S_0, \ldots, S_q \in \mathbb{R}^{n\times n}$, the following conditions on $S_0, \ldots, S_q$, $\xi^T S_0 \xi > 0$, for $\forall \xi \neq 0$.

$\xi^T S_i \xi \geq 0$, for $i = 1, \ldots, q$, are fulfilled when there exists a series of scalars $\tau_1 \geq 0, \ldots, \tau_q \geq 0$, such that

$$S_0 - \sum_{i=1}^{q} \tau_i S_i > 0.$$  

**APPENDIX B. PROOF OF THEOREM 3**

For a GES nonlinear system in (1), a globally exponentially stable system as in (37) or (38) can be constructed. In view of the universal approximation capability of the PALM,\textsuperscript{4} for any positive scalars $\epsilon_f, \epsilon_g, \epsilon_h$ and $\epsilon_i$, one can build the following approximation system w.r.t. (37) on $X \times U$:

$$\begin{cases} \dot{x}(t) = \hat{f}(x,u) = f(x,u) + \epsilon_f(x,u) + \epsilon_g(x,u) \\ \dot{u}(t) = \hat{g}(x,u) = g(x,u) + \epsilon_h(x,u) + \epsilon_i(x,u) \end{cases} \quad (B1)$$
such that

\[
\begin{align*}
\|e_f(x, u)\| &\leq \|x^T, u^T\|, \\
\|e_h(x, u)\| &\leq \|x^T, u^T\|,
\end{align*}
\]

(B2)

where \(f(x, u)\) and \(g(x, u)\) are defined in (37).

In particular, for the partition \(X_0\), we have \(\max \{e_f, e_h\} \equiv 0\).

The system in (B1) is rewritten as

\[
\dot{x}(t) = \dot{F}(\bar{x}(t)) = F(\bar{x}(t)) + \bar{e}_f(\bar{x}(t)) + \bar{e}_g(\bar{x}(t)),
\]

(B3)

where

\[
\begin{align*}
\dot{F}(\bar{x}(t)) &= \left[\dot{f}(x, u)^T, \dot{g}(x, u)^T\right]^T, \\
\dot{e}_f(\bar{x}(t)) &= \left[e_f(x, u)^T, e_h(x, u)^T\right]^T, \\
\dot{e}_g(\bar{x}(t)) &= \left[e_g(x, u)^T, e_i(x, u)^T\right]^T.
\end{align*}
\]

(B4)

Then, the piecewise integral sliding surface is designed as

\[
s(t) = \bar{S}\left\{\bar{x}(t) - \bar{x}(0) - \int_0^t \dot{F}(\bar{x}(\theta)) d\theta\right\} = 0,
\]

(B5)

while the piecewise sliding-mode parallel controller is constructed as

\[
\begin{align*}
\text{for } \bar{x}(t) &\subseteq \bar{X}_i, i \in \varphi \\
\dot{u}(t) &= \hat{g}(\bar{x}(t)) - (\gamma + a_i + v_i(t)) S_u^{-1} \text{sgn}(s(t)).
\end{align*}
\]

(B6)

It can be concluded from the procedure of Theorem 1 that (B5) is reached and maintained from the beginning of evolution and the resultant sliding motion is

\[
\dot{x}(t) = \dot{F}(\bar{x}(t)) - (R_1 - R_2 S_u^{-1} S_x) e_f(x(t), u(t)) - (R_1 - R_2 S_u^{-1} S_x) e_g(x(t), u(t)).
\]

(B7)

It follows from the Lyapunov converse theorem in Reference 41 that, along the trajectories of (38), there exist a Lyapunov function \(V(\bar{x})\) and four positive scalars \(b_1, b_2, b_3,\) and \(b_4\) satisfying

\[
b_1\|\bar{x}\|^2 \leq V(\bar{x}) \leq b_2\|\bar{x}\|^2.
\]

(B8)

\[
\frac{\partial V(\bar{x})}{\partial \bar{x}} F(\bar{x}) \leq -b_3\|\bar{x}\|^2.
\]

(B9)

\[
\left\|\frac{\partial V(\bar{x})}{\partial \bar{x}}\right\| \leq b_4\|\bar{x}\|.
\]

(B10)

where \(F(\bar{x})\) is defined in (38).

On the other hand, the derivative of \(V(\bar{x})\) along the trajectories of (B3) satisfies

\[
V(\bar{x}(t)) = \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} \dot{F}(\bar{x}(t)).
\]
\[ \frac{dV}{dt}(\tilde{x}(t)) = \frac{dV}{d\tilde{x}}(\tilde{x}(t)) + \frac{dV}{d\tilde{x}}(\tilde{x}(t)) + \tilde{e}_g(\tilde{x}(t)) \]
\[ \leq -b_3 \|\tilde{x}(t)\|^2 + b_4 \max \{\epsilon_f, \epsilon_h\} \|\tilde{x}(t)\|^2 + b_4 \max \{\epsilon_f, \epsilon_i\} \|\tilde{x}(t)\|. \]  

**(Case i):** For the trajectories that move within the partition \(X_0\), one has that \(\max \{\epsilon_g, \epsilon_i\} \equiv 0\). It then follows from (B11) that
\[ V(\tilde{x}(t)) \leq -(b_3 - b_4 \max \{\epsilon_f, \epsilon_h\}) \|\tilde{x}(t)\|^2 \] 
by choosing the appropriate approximation error bounds \(\epsilon_f, \epsilon_h\) such that
\[ \max \{\epsilon_f, \epsilon_h\} < \frac{b_3}{b_4}. \]  

**(Case ii):** For trajectories that move within the partitions \(X_i, i \in \Phi\), one can conclude that the norm \(\|\tilde{x}(t)\|\) is lower bounded.

Given a scalar \(0 < \lambda < 1/2\), one can always choose the approximation error bounds \(\epsilon_g, \epsilon_h, \epsilon_f, \epsilon_i\), and \(\epsilon_h\) such that
\[ \max \{\epsilon_g, \epsilon_i\} < \lambda \frac{b_3}{b_4} \]
and
\[ \max \{\epsilon_f, \epsilon_h\} < (1 - \lambda) \frac{b_3}{b_4} \]
which means \(b_3 - b_4 \max \{\epsilon_f, \epsilon_h\} - \lambda b_3 > 0\).

By submitting (B14) and (B15) into (B11), we have
\[ V(\tilde{x}(t)) \leq -(b_3 - b_4 \max \{\epsilon_f, \epsilon_h\} + \lambda b_3) \|\tilde{x}(t)\|^2. \]  

From (B14)-(B16), one has that along the trajectories of (B1) moving within \(X_i, i \in \Phi\),
\[ V(\tilde{x}(t)) \leq -\tilde{b} \|\tilde{x}(t)\|^2, \]  
where
\[ \tilde{b} = b_3 - (b_4 \max \{\epsilon_f, \epsilon_h\} + c_3 \lambda) > 0. \]  

Therefore, the semiglobal exponential stability of (B3) is concluded based on the Lyapunov stability theory and the results in (B12) and (B17).

We are now ready to analyze the stability of (B7) that can be treated as a perturbation system of (B3). By following the similar procedure of (B8)-(B18) and based on the subsequent facts:
\[ \| (R_1 - R_2 S_u^{-1} S_x) \epsilon_f (x(t), u(t)) \| \leq (1 + \|R_2 S_u^{-1}\|\|S_u\|) \|\epsilon_f\|\|\tilde{x}(t)\|, \]  
\[ \| (R_1 - R_2 S_u^{-1} S_x) \epsilon_g (x(t), u(t)) \| \leq (1 + \|R_2 S_u^{-1}\|\|S_u\|) \|\epsilon_g\|, \]
\[ \max \{\epsilon_g\} < \lambda \frac{b_3}{b_4} \min \|\tilde{x}(t)\| \leq \frac{b_3}{b_4} \|\tilde{x}(t)\|, \]

the semi-global exponential stability of (B7) is guaranteed, by appropriately designing the approximation error bounds \(\epsilon_f, \epsilon_h, \epsilon_g, \text{and} \epsilon_i\) such that
\[ (1 + \|R_2 S_u^{-1}\|\|S_u\|) \epsilon_f + \max \{\epsilon_f, \epsilon_g\} < \left[1 - (2 + \|R_2 S_u^{-1}\|\|S_u\|) \lambda\right] \frac{b_3}{b_4}. \]
where
\[ 0 < \lambda < \frac{1}{2 + \|R_2S_u^{-1}\|\|S_x\|}. \]

and are defined in (B9) and (B10) respectively.

**APPENDIX C. PROOF OF THEOREM 4**

By using the piecewise sliding-mode parallel controller in (B6), we have shown in Appendix B that the piecewise integral sliding surface in (B5) can be reached and maintained from the beginning of evolution. So now our main goal is to show that, for a GAS nonlinear system in (1), its corresponding sliding motion in (B7) will be controlled to behave the semi-global asymptotic stability. This can be done with the aid of \( V(\bar{x}(t)) \) defined in (39)-(41).

The derivative of \( V(\bar{x}(t)) \) defined in (39)-(41) along the trajectories of (B3) satisfies
\[
\dot{V}(\bar{x}(t)) = \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} \dot{\bar{x}}(t) = \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} F(\bar{x}(t)) + \frac{\partial V(\bar{x}(t))}{\partial \bar{x}} \left( \bar{e}_f(\bar{x}(t)) + \bar{e}_h(\bar{x}(t)) \right) \\
\leq -\alpha_3 (|\bar{x}(t)|) + h \max \{\epsilon_f, \epsilon_h\} |\bar{x}(t)| + h \max \{\epsilon_g, \epsilon_i\}. \tag{C1}
\]

**Case i:** For the trajectories that move within the partition \( \bar{X}_0 \), one has
\[
\dot{V}(\bar{x}(t)) \leq -\alpha_3 (|\bar{x}(t)|) + \rho |\bar{x}(t)| \leq -\alpha_4 (|\bar{x}(t)|), \tag{C2}
\]
if (42) holds and the norm-bounds of the approximation error \( \epsilon_f \) and \( \epsilon_h \) satisfy
\[
\max \{\epsilon_f, \epsilon_h\} < \frac{\rho}{h}. \tag{C3}
\]

**Case ii:** For trajectories that move within the partitions \( \bar{X}_i, i \in \Phi \), the norm \( |\bar{x}(t)| \) is lower bounded.

Given a scalar \( 0 < \mu < 1/2 \), one can always choose the approximation error bounds \( \epsilon_g, \epsilon_i, \epsilon_f \), and \( \epsilon_h \) such that
\[
\max \{\epsilon_g, \epsilon_i\} \leq \mu \frac{\rho}{h} \min |\bar{x}(t)| \leq \mu \frac{\rho}{h} |\bar{x}(t)| \leq \mu \frac{\rho}{h} |\bar{x}(t)|. \tag{C4}
\]
and
\[
\max \{\epsilon_f, \epsilon_h\} < (1 - \mu) \frac{\rho}{h}. \tag{C5}
\]
which means
\[
h \max \{\epsilon_f, \epsilon_h\} + \mu \rho < \rho. \tag{C6}
\]

By submitting (C4)-(C6) into (C1), one has
\[
\dot{V}(\bar{x}(t)) \leq -\alpha_3 (|\bar{x}(t)|) + (h \max \{\epsilon_f, \epsilon_h\} + \mu \rho) |\bar{x}(t)| \\
\leq -\alpha_3 (|\bar{x}(t)|) + \rho |\bar{x}(t)| \\
\leq -\alpha_4 (|\bar{x}(t)|). \tag{C7}
\]
if (42) holds. The semi-global asymptotic stability of (B1) is then concluded by following from both (C2) and (C7).
The sliding motion in (B7) can be treated as a perturbation system of (B3). Similar with the procedure of (C1)-(C7), by choosing the norm-bounds of the approximation error $\varepsilon_f, \varepsilon_h, \varepsilon_g,$ and $\varepsilon_i$ such that

$$(1 + \| R_2 S_u^{-1} \| \| S_x \| ) \varepsilon_f + \max \{ \varepsilon_f, \varepsilon_g \} < [1 - (2 + \| R_2 S_u^{-1} \| \| S_x \| ) \mu] \frac{\rho}{R},$$

where

$$0 < \mu < \frac{1}{2 + \| R_2 S_u^{-1} \| \| S_x \| },$$

$\rho$ and $h$ are defined in (41) and (42), respectively, one can conclude that the (B7) is semi-globally asymptotically stable.