Improved robust $H_\infty$ exponential mean square stabilization for uncertain Markov jump delay systems based on memory-state feedback control

Jie Wang  |  Guangming Zhuang  |  Jianwei Xia  |  Huasheng Zhang  |  Wei Sun

School of Mathematical Sciences, Liaocheng University, Liaocheng, P. R. China

Correspondence
Guangming Zhuang, School of Mathematical Sciences, Liaocheng University, Liaocheng 252000, P. R. China.
Email: zgmtsg@126.com

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Abstract
This paper investigates the problem of robust $H_\infty$ exponential mean square stabilisation for uncertain Markov jump systems with time-varying and mode-dependent delays based on memory-state feedback control. Attention is focused on the design of memory-state feedback controller for uncertain Markov jump systems with time-varying and mode-dependent delays such that the closed-loop uncertain Markov jump systems satisfies robust $H_\infty$ exponential mean square stability. The improved conditions for the solvability of the memory-state feedback control problem are obtained via designing mode-dependent and delay-dependent L-K functional. The desired memory-state feedback controller is given by using linear matrix inequalities. Two simulation examples including a numerical example and a practical example of the industrial non-isothermal continuous stirred tank reactor are used to demonstrate the effectiveness and usefulness of the delayed feedback control technique that this paper proposes.

1 | INTRODUCTION
In the past many years, the study of Markov jump theory has attracted more and more attention due to the wide range of applications of Markov jump systems (MJSs) in the fields of chemical plants, robots, airplanes, and so on [1–5]. MJSs are a kind of hybrid system, which combine continuous and discrete values and are widely used in many physics and engineering such as network control systems and fault-tolerant control systems, where the environmental changes, system failures and maintenance can be reflected effectively [6–10]. With the rapid development of MJSs, stability analysis and control synthesis have been extensively researched in the past years, and some important results in feedback control and filtering have been obtained [11–17].

As an inevitable natural phenomenon, time delays often appear in various dynamic systems, which can easily lead to system performance degradation even instability [9, 18–24]. Recently, the mode-dependent and time-varying delays have been accorded great and prompt attention [25, 26]. Moreover, in order to obtain delay-dependent results, the free-weighting matrix technique and the Jensen’s integral inequality were applied in [27–30] and references therein. Note that uncertain parameters often occur in many dynamic systems, for instance, interconnected power systems, hydraulic systems and aerospace systems etc., where some parameters change linearly with time, while the other parameters change randomly within a certain range and satisfy a certain relation [1, 31–33].

Recently, the $H_\infty$ control theory has attracted extensive attention and made significant progress [34–37]. Practically speaking, [31, 38] considered robust $H_\infty$ control problems for uncertain stochastic systems. What’s more, a sufficient condition for solvability of the robust $H_\infty$ control problem for uncertain time-delay systems was proposed in [18]. And then, the $H_\infty$ filter
design problem for MJSs with time delays was solved in [39], where delay-independent results were presented in the form of linear matrix inequalities (LMIs).

It is worth noting that exponential stability, mean square stability and mean square asymptotic stability for plenty of dynamic systems have been investigated in [31, 40, 41]. As the combination of exponential stability and mean square stability, exponential mean square stability (EMSS) has been paid much attention due to its faster convergence rate [1, 42]. However, the EMSS has not been fully studied for uncertain delayed MJSs. On the other hand, feedback control of various dynamic systems can effectively guarantee system stability and improve the system performance [43–49]. However, in practice, the efficiency of the closed-loop system can be reduced due to time delays. How to overcome time delays has been the focus of attention over the past few decades. Recently, many famous scholars have conducted in-depth research on memory-state feedback control, where the delayed state information was considered in the controller design [50, 51]. Yet, as far as we know, robust $H_\infty$ EMSS problem and memory-state feedback control for MJSs have not been adequately studied, which are full of challenge due to EMSS index and time-varying delay characteristics in feedback controller.

The intervention of exponential function for EMSS and mode-dependent time-varying delay makes the solution to the stabilisation problem will be the focus of this paper, which makes the solution even more complicated [50, 51]. How to deal with robust $H_\infty$ EMSS for delayed MJS based on memory-state feedback control deserves further study. This is our research motivation in this work.

In this work, robust $H_\infty$ exponential mean square stabilisation for UMJSs with mode-dependent and time-varying delays based on memory-state feedback control will be investigated. The design of the memory-state feedback controller and the robust $H_\infty$ stabilisation problem will be the focus of this paper, and the closed-loop system will be EMSS under all allowable uncertainties, mode-dependent time-varying delays. Apart from the above requirements, $H_\infty$ performance level will be satisfied. Furthermore, the improved conditions of solvable problems can be obtained by using LMIs. In the end, a numerical example and the industrial non-isothermal continuous stirred tank reactor (INCTSTR) [52] will be employed to demonstrate the effectiveness and usefulness of the method that this work brings up. The main contributions of this paper are summarised as follows:

1. The mode-independent and mode-dependent time-varying delays are considered both in the memory-state feedback controller and in the UMJS;
2. On the premise of achieving EMSS of UMJS, robust $H_\infty$ exponential mean-square stabilisation for closed-loop UMJS is achieved based on a memory-state feedback controller;
3. The mode-dependent memory-state feedback control can be extended to asynchronous feedback control and can be reduced to the special memoryless-state feedback control via the proposed method in this work.

Notation: $L_2[0, \infty)$ represents the space of square-integrable vector functions over $[0, \infty)$, $C([-\tau, 0]; \mathbb{R}^n)$ denotes the space of continuous function $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$ with norm $\|\phi\| = \sup_{t \in [0, \infty)} |\phi(t)|$.

2 SYSTEM DESCRIPTION

Given a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the following UMJS with time-varying delays:

$$
\begin{align*}
\dot{x}(t) &= A(t, r_t) x(t) + A_a(t, r_t) x(t - \tau_{i-1}(t)) + A_d(t, r_t) x(t - \tau_i(t)) + B(t, r_t) U(t) + D(t, r_t) \omega(t), \\
\zeta(t) &= E(t, r_t) x(t) + D(t, r_t) \omega(t), \\
\xi(t) &= \phi(t), \quad \forall t \in [-\tau, 0],
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the state; $U(t) \in \mathbb{R}^m$ is the control input; $\omega(t) \in \mathbb{R}^l$ is the noise signal in $L_2[0, \infty)$; $\zeta(t) \in \mathbb{R}^n$ is the estimated signal. $\{r_t\}$ is Markovian jump process, where $r_t \in S = \{1, 2, \ldots, N\}$ and transfer rate matrix $\Pi = [\pi_{ij}]$ meets the general condition in [1–3]. Let $b = \max_{r \in S} |\pi_{ij}|$.

In the system (1), for $r_t = i \in S$, the time-varying delays $\tau(t)$ and $\tau_i(t)$ satisfying

$$
0 < \tau_i(t) \leq \tau < \infty, \quad \tau_i(t) \leq s_i < 1,
$$

where $\tau$, $s_i$ and $s$ are given scalars. $A(r_t), A_a(r_t), A_d(r_t), B(r_t), D(r_t), E(r_t)$ are known real constant matrices with compatible dimensions. $A(t, r_t), A_a(t, r_t), A_d(t, r_t), B(t, r_t), D(t, r_t)$ are rewritten as

$$
\begin{align*}
A(t, r_t) &= A(r_t) + \Delta A(t, r_t), \\
A_a(t, r_t) &= A_a(r_t) + \Delta A_a(t, r_t), \\
A_d(t, r_t) &= A_d(r_t) + \Delta A_d(t, r_t), \\
B(t, r_t) &= B(r_t) + \Delta B(t, r_t), \\
D(t, r_t) &= D(r_t) + \Delta D(t, r_t).
\end{align*}
$$

$\Delta A(t, r_t), \Delta A_a(t, r_t), \Delta A_d(t, r_t), \Delta B(t, r_t), \Delta D(t, r_t)$ represent unknown matrices with time-varying parameter uncertainty, where

$$
\begin{align*}
&\begin{bmatrix} \Delta A(t, r_t) & \Delta A_a(t, r_t) & \Delta A_d(t, r_t) & \Delta B(t, r_t) & \Delta D(t, r_t) \end{bmatrix} \\
&= G(r_t) F(t, r_t) M_1(r_t) M_2(r_t) M_3(r_t) M_4(r_t) M_5(r_t)
\end{align*}
$$
then $G(\tau_1), M_1(\tau_1), M_2(\tau_1), M_3(\tau_1), M_4(\tau_1), M_5(\tau_1)$ are known real constant matrices for all $\tau_1 \in \mathcal{S}$, and $F(t, \tau_1)$ are the uncertain time-varying matrices satisfying

$$F(t, \tau_1)^TF(t, \tau_1) \leq I, \quad \forall \tau_1 \in \mathcal{S}.$$ 

Now, the memory-state (time delays) feedback controller is designed as follows:

$$U(t) = K_1(\tau_1)x(t) + K_2(\tau_1)x(t - \tau_1(\tau_1)) + K_3(\tau_1)x(t - \tau(\tau_1)), \quad (3)$$

where $K_1(\tau_1), K_2(\tau_1), K_3(\tau_1) \in \mathbb{R}^{n \times n}$ are the parameters of the memory-state feedback controller with the appropriate dimension to be determined. In order to facilitate the following research, for each $\tau_1 = i \in \mathcal{S}$, $A(t, \tau_1)$ is rewritten as $A_i(t)$ etc.

**Remark 1.** Feedback control of various dynamic systems can effectively guarantee system stability and improve the system performance [42–49]. However, in practice, due to time delays, the efficiency of the closed-loop system can be reduced. How to overcome the delays has been paid much attention during the past decades. The state feedback control can be a better and more effective control strategy to make the system work steadily and normally. Recently, memory-state feedback control has received much attention, where the delayed state information is considered in the controller design [50, 51]. In this work, the mode-independent and mode-dependent time-varying delays are considered both in the memory-state feedback controller and in the UMJS.

In order to effectively design the mode-dependent memory-state feedback controller, the following definitions are given firstly.

**Definition 1** ([1, 17]). The unforced UMJS (1) achieves robust EMSS, if for any finite $\phi(t) \in \mathbb{R}^p$ defined on $[-\tau, 0]$, and initial mode $r_0 \in \mathcal{S}$, there exist constant scalars $b > 0$ and $n > 0$ such that

$$E_2[|x(t, \phi(\tau_0))|^2] \leq b \sup_{-\tau_0 \leq \tau < 0} |\phi(\tau)|^2e^{-\alpha\tau}. \quad (4)$$

**Definition 2** ([1, 17]). For non-zero $\omega \in L_2^\infty[0, \infty)$ and zero initial condition, the UMJS (1) satisfies $H_{\infty}$ performance level $\gamma(\gamma > 0)$ if the following condition holds:

$$\|Z\|_{E_2} < \gamma \|\omega\|_2. \quad (5)$$

### 3 MAIN RESULTS

First of all, we consider the following UMJS with mode-independent and mode-dependent time-varying delays as follows:

$$\begin{align*}
    \dot{x}(t) &= A(t, \tau_1)x(t) + A_0(t, \tau_1)x(t - \tau(\tau_1)) + A_3(t, \tau_1)x(t - \tau(\tau_1)) + D(t, \tau_1)\omega(t), \\
    \tau(t) &= E(t, \tau_1)x(t) + D(t, \tau_1)\omega(t), \\
    x(t) &= \phi(t), \quad \forall t \in [-\tau, 0], \quad (6)
\end{align*}$$

**Theorem 1.** The unforced MJS (6) achieves EMSS and satisfies robust $H_{\infty}$ disturbance attenuation $\gamma$ if there exist matrices $P_1 > 0, P_2 > 0, \ldots, P_{N'} > 0, Q > 0$ and $\omega > 0$ such that the following matrix inequalities hold for $\ell = 1, 2, \ldots, N'$:

$$\Gamma_{\ell} = \begin{bmatrix}
    \Psi_{\ell} & P_\ell A_{\ell}(t) & \Delta P_\ell(t) + E_{\ell}D_{\ell}(t) \\
    * & (\ell - 1)Q & * \\
    * & * & -\gamma^2I + D_{\ell}^T(t)D_{\ell}(t)
\end{bmatrix} < 0, \quad (7)$$

where

$$\begin{align*}
    \Psi_{\ell} = \sum_{j=1}^{N'} \pi_{ij}Q_j < Q, \quad (8)
    \Psi_{\ell} &= \sum_{j=1}^{N'} \pi_{ij} P_j + \Delta P_j(t) + A^T(t) + (1 + \alpha(\alpha)Q + \alpha(\alpha))Q + \alpha(\alpha)E_{\ell}, \quad (9)
\end{align*}$$

**Proof.** First, set a new process $\{x(t, \tau_1), t \geq 0\}$ by

$$x(t, \tau_1) = x(t + \delta), \quad -\tau \leq \delta \leq 0.$$

Define a L-K functional candidate for the unforced MJS (6) as:

$$V'(t, \tau_1) = V_1'(t, \tau_1) + V_2'(t, \tau_1) + V_3'(t, \tau_1) + V_4'(t, \tau_1), \quad (10)$$

where

$$\begin{align*}
    V_1'(t, \tau_1) &= x^T(t)P(t_1)x(t), \\
    V_2'(t, \tau_1) &= \int_{-\tau(t)}^{t} x^T(t)Q(t)x(t)dt, \\
    V_3'(t, \tau_1) &= (1 + \beta)\int_{-\tau(t)}^{t} x^T(t)Q(t)x(t)dt, \\
    V_4'(t, \tau_1) &= \int_{-\tau(t)}^{t} x^T(t)Q(t)x(t)dt.
\end{align*}$$

Set $\xi$ be the weak infinitesimal generator of the stochastic process $\{x(t, \tau_1)\}$. Then, for each $\tau_1 = i \in \mathcal{S}$ and any scalar $u > 0,$
we have

\[ E[e^{\mu t}V(x_t, \delta)] = e^{\mu t} x^T(t) \left[ \sum_{j=1}^{N} \pi_{ij} P_j + P_e A_i(t) \right. \]

\[ + \ A_e^T(t) P_j + (1 + \tau + h\tau) Q + Q \left] x(t) \right] \]

\[ + 2e^{\mu t} x^T(t) P_j A_{sl}(t) x(t - \tau(t)) \]

\[ + 2e^{\mu t} x^T(t) P_j A_{sl}(t) x(t - \tau(t)) \]

\[ + e^{\mu t} \sum_{j=1}^{N} \pi_{ij} \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ - e^{\mu t} (1 - \beta(t)) x^T(t - \tau(t)) Qx(t - \tau(t)) \]

\[ - e^{\mu t} (1 + h) \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ + e^{\mu t} \int_{t-\tau(t)}^{t} x^T(\alpha) \left( \sum_{j=1}^{N} \pi_{ij} Q_j \right) x(\alpha) d\alpha \]

\[ - (1 - \beta(t)) e^{\mu t} x^T(t - \tau(t)) Qx(t - \tau(t)) \]

\[ + n e^{\mu t} V(x_t, \delta). \]

Noting that \( \pi_{ij} \geq 0 \), for each \( j \neq i \) and \( \pi_{ii} \leq 0 \), we have

\[ \sum_{j=1}^{N} \pi_{ij} \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ \leq \sum_{j=1}^{N} \pi_{ij} \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ = -\pi_{ii} \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ \leq 0. \]

Noting that

\[ \int_{t-\tau(t)}^{t} x^T(\alpha) \left( \sum_{j=1}^{N} \pi_{ij} Q_j \right) x(\alpha) d\alpha \]

\[ \leq \int_{t-\tau(t)}^{t} x^T(\alpha) \left( \sum_{j=1}^{N} \pi_{ij} Q_j \right) x(\alpha) d\alpha \]

\[ \leq \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha. \]

Applying the method in [49], we have

\[ 2x^T(t) P_e A_{sl}(t) x(t - \tau(t)) \]

\[ - (1 - \beta(t)) x^T(t - \tau(t)) Qx(t - \tau(t)) \]

\[ \leq (1 - \beta(t)) x^T(t) P_e A_{sl}(t) Q^{-1} A_e^T(t) P_e x(t), \]

and we can get

\[ E[e^{\mu t}V(x_t, \delta)] \leq e^{\mu t} x^T(t) \left[ \sum_{j=1}^{N} \pi_{ij} P_j + P_e A_i(t) \right. \]

\[ + A_e^T(t) P_j + (1 + \tau + h\tau) Q + Q \left] x(t) \right] \]

\[ + 2e^{\mu t} x^T(t) P_j A_{sl}(t) x(t - \tau(t)) \]

\[ - (1 + h) \int_{t-\tau(t)}^{t} x^T(\alpha) Qx(\alpha) d\alpha \]

\[ + 2e^{\mu t} x^T(t) P_j A_{sl}(t) x(t - \tau(t)) \]

\[ + 2e^{\mu t} \int_{t-\tau(t)}^{t} x^T(\alpha) \left( \sum_{j=1}^{N} \pi_{ij} Q_j \right) x(\alpha) d\alpha \]

\[ - (1 - \beta(t)) e^{\mu t} x^T(t - \tau(t)) Qx(t - \tau(t)) \]

\[ + n e^{\mu t} V(x_t, \delta). \]

Using the Schur complement formula to (17), we have

\[ Y_i = \begin{bmatrix} \Psi_i & P_e A_{sl}(t) \\ * & - (1 - \beta(t)) \end{bmatrix} \begin{bmatrix} P_e A_{sl}(t) \\ * \end{bmatrix} \begin{bmatrix} P_e A_{sl}(t) \\ * \end{bmatrix} Q \begin{bmatrix} 0 \\ - (1 - \beta(t)) \end{bmatrix} \leq 0, \]

where

\[ \Psi_i = \sum_{j=1}^{N} \pi_{ij} P_j + P_e A_i(t) + A_e^T(t) P_j + (1 + \tau + h\tau) Q + Q. \]
Then, there exists constant $\theta > 0$ such that
\[
\sum_{k=1}^{n} \pi_{k} P_{k} + P_{k} A_{w}(t) + A_{w}^{T}(t) P_{k} + (1 + \tau + k\tau) Q
\]
\[
+ (1 - k^{-1}) P_{k} A_{w}(t) Q_{r}^{-1} A_{w}^{T}(t) P_{k} + Q
\]
\[
+ (1 - k^{-1}) P_{k} A_{w}(t) Q_{r}^{-1} A_{w}^{T}(t) P_{k} + \theta I < 0.
\]

From the above, we can get
\[
\mathcal{E}[\varphi(t) V(x(t), r)] < -\theta \varphi(t^*) |x(t)|^2 + \varphi(t) V(x(t), r).
\]

Set
\[
\eta = \{ \lambda_{\max}(P), \lambda_{\max}(Q), \lambda_{\max}(Q) \},
\]

then
\[
\int_{-\tau}^{t} x^{T}(\alpha) Q x(\alpha) d\alpha \\
\leq \tau \int_{-\tau}^{t} x^{T}(\alpha) Q x(\alpha) d\alpha
\]
\[
\leq \eta \tau \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha,
\]

and
\[
\mathcal{E}[\varphi(t) V(x(t), r)] < -\theta \varphi(t^*) |x(t)|^2 \\
+ \eta \nu(\beta + \tau) \varphi(t) \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha \\
+ 2\eta \nu \varphi(t) \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha + \eta \nu \varphi(t) |x(t)|^2 \\
= (\eta \nu - \theta) \varphi(t^*) |x(t)|^2 \\
+ (2\eta \nu + b \eta \tau) \varphi(t) \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha.
\]

By using Dynkin's formula, for any $T > 0$, $\beta > 0$, we have
\[
\mathbb{E}[\varphi(T) V(x_{T}, r)] - V(x_{0}, r_{0}) \\
\leq (2\eta + b \eta \tau) \int_{0}^{T} \varphi(t) \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha dt \\
+ (\eta \nu - \theta) \int_{0}^{T} \varphi(t) |x(t)|^2 dt.
\]

And noting that
\[
\int_{0}^{T} \varphi(t) \int_{-\tau}^{t} |x(\alpha)|^2 d\alpha dt \\
\leq \int_{-\tau}^{T} \varphi(t) |x(\alpha)|^2 d\alpha
\]
\[
+ \int_{0}^{T} \varphi(t) |x(\alpha)|^2 d\alpha
\]
\[
= \tau \int_{-\tau}^{T} \varphi(t) |x(\alpha)|^2 d\alpha,
\]

therefore, for any scalar $\beta > 0$, we have
\[
\mathbb{E}[\varphi(T) V(x_{T}, r)] < V(x_{0}, r_{0}) + [\eta \nu - \theta] \\
+ (\eta \nu + b \eta \tau) \int_{0}^{T} \varphi(t) |x(t)|^2 dt
\]
\[
+ (\eta \nu + b \eta \tau) \varphi(T) |x(t)|^2 dt.
\]

Choose a scalar $\eta > 0$ such that
\[
\eta \nu - \theta + (\eta \nu + b \eta \tau) \varphi(t) = 0,
\]

then
\[
\mathbb{E}[\varphi(T) V(x_{T}, r)] < V(x_{0}, r_{0}) \\
+ (\eta \nu + b \eta \tau) \int_{0}^{T} \varphi(t) |x(t)|^2 dt.
\]

Noting that
\[
\mathbb{E}[\varphi(T) V(x_{T}, r)] > \eta \nu |x(t)|^2 + \varphi(t) V(x(t), r),
\]

we have
\[
\mathbb{E}[|x(t)|^2] < \frac{1}{\eta \nu} \sup_{t_{2} < t_{1} < t_{2} < t_{1}} |x(\sigma)|^2 e^{-\nu T},
\]

where
\[
\sup_{t_{2} < t_{1} < t_{2} < t_{1}} |x(\sigma)|^2 = V(x_{0}, r_{0}) \\
+ (\eta \nu + b \eta \tau) \int_{0}^{T} \varphi(t) |x(t)|^2 dt.
\]

Hence, the unforced MJ 6 (6) achieves EMSS.
Then, for \( \omega(t) \neq 0 \in L_2[0, \infty) \) and the zero initial condition, we have

\[
J(t) = E \left\{ \int_0^t [\xi^T(\alpha)\xi(\alpha) - \gamma^2 \omega^T(\alpha)\omega(\alpha)]d\alpha \right\} 
= E \left\{ \int_0^t [\xi^T(\alpha)\xi(\alpha) - \gamma^2 \omega^T(\alpha)\omega(\alpha)]d\alpha \right\} 
+ \epsilon \left\{ V(\chi, t) \right\}
\]

where

\[
\psi^T(t) = \begin{bmatrix} x^T(t) \\ x^T(t - \tau_i(t)) \\ x^T(t - \tau(t)) \\ \omega^T(t) \end{bmatrix},
\]

then

\[
\int_0^t [\xi^T(\alpha)\xi(\alpha) - \gamma^2 \omega^T(\alpha)\omega(\alpha)]d\alpha \leq \int_0^t \psi^T(\alpha) \Gamma \psi(\alpha) d\alpha.
\]

Observing (10) and by calculating, we can get

\[
\epsilon \left\{ V(\chi, t) \right\}
\]

for \( \epsilon > 0 \), we can find that the MJS (6) with mode-dependent time-varying delays and memory-state feedback controller (3) achieves EMSS as follows:

\[
\dot{x}(t) = \begin{bmatrix} A(\tau_i) + \Delta A(\tau_i) \\
+ (B(\tau_i) + \Delta B(\tau_i))K_1 \end{bmatrix} x(t)
+ \begin{bmatrix} A_d(\tau_i) + \Delta A_d(\tau_i) \\
+ (B(\tau_i) + \Delta B(\tau_i))K_2 \end{bmatrix} x(t - \tau_i(t))
+ \begin{bmatrix} A_d(\tau_i) + \Delta A_d(\tau_i) \\
+ (B(\tau_i) + \Delta B(\tau_i))K_3 \end{bmatrix} x(t - \tau(t))
+ \begin{bmatrix} D(\tau_i) + \Delta D(\tau_i) \end{bmatrix} \omega(t),
\]

where

\[
\psi_1 = \sum_{j=1}^N \pi_j P_j + P_{\omega}(A_i + \Delta A_i) + (A_i + \Delta A_i)P^T \psi_4,
\]

\[
\psi_2 = P_{\omega}(A_i + \Delta A_i) + (A_i + \Delta A_i)P^T \psi_3,
\]

\[
\psi_3 = P_{\omega}(A_i + \Delta A_i) + (A_i + \Delta A_i)P^T \psi_4,
\]

\[
\psi_4 = \gamma I + (D_i + \Delta D_i)P^T(D_i + \Delta D_i),
\]

\[
\psi_5 = \gamma I + (D_i + \Delta D_i)P^T(D_i + \Delta D_i).
\]
Proof. Using the same L-K functional as Theorem 1 and the similar proof method of Theorem 1, we can get

\[
\mathcal{L}[\varphi^T V(x_i, r_i)] < c^T e^T \sum_{j=1}^{N_i} \pi_{ij} P_j + P_i (A_i + \Delta A_i(t))
\]

\[
+ (A_i^T + \Delta A_i^T(t)) P_i + P_i (B_i + \Delta B_i(t)) K_i
\]

\[
+ K_i^T (B_i^T + (1 + \tau + h\tau) Q + Q_i + B_i^T(t)) P_i
\]

\[
+ (1 - \sigma_i)^{-1} P_i (A_{ji} + \Delta A_{ji}(t)) Q_i^{-1} (A_{ji} + \Delta A_{ji}(t))^T P_i
\]

\[
+ (1 - \sigma_i)^{-1} P_i (B_i + \Delta B_i(t)) K_{ji} Q_i^{-1} K_{ji}^T (B_i + \Delta B_i(t))^T P_i
\]

\[
+ (1 - \sigma_i)^{-1} P_i (A_{di} + \Delta A_{di}(t)) Q_i^{-1} (A_{di} + \Delta A_{di}(t))^T P_i
\]

\[
+ (1 - \sigma_i)^{-1} P_i (B_i + \Delta B_i(t)) K_{di} Q_i^{-1} K_{di}^T (B_i + \Delta B_i(t))^T P_i
\]

\[
+ n e^T V(x_i, \tau_i)
\]

Thus,

\[
E[\left| x(T) \right|^2] < \frac{1}{\eta} \sup_{-\tau_i < t < 0} \left| x(\sigma) \right|^2 e^{-\eta T},
\]

where

\[
\sup_{-\tau_i < t < 0} \left| x(\sigma) \right|^2 = V(x_0, n_0)
\]

\[
+ (\sigma \eta T + \alpha M^T) e^T \int_0^T |x(t)|^2 dt.
\]

Hence, closed-loop UMJS (37) with mode-dependent time-varying delays and memory-state feedback controller (3) satisfies robust exponential mean square stabilisation.

For \( \omega(t) \neq 0 \in L_2(0, \infty) \) and zero initial condition, we have

\[
\mathcal{L}[V(x_i, r_i)] \leq \sum_{j=1}^{N_i} \pi_{ij} P_j + P_i (A_i + \Delta A_i(t))
\]

\[
+ (A_i^T + \Delta A_i^T(t)) P_i + P_i (B_i + \Delta B_i(t)) K_i
\]

\[
+ K_i^T (B_i^T + \Delta B_i^T(t)) P_i + (1 + \tau + h\tau) Q + Q_i + E_i^T E_i
\]

\[
\times (t)
\]

\[
- (1 - \tau_i(t)) e^T (t - \tau_i(t)) Q x(t - \tau_i(t))
\]

\[
- (1 - \tau_i(t)) x^T (t - \tau_i(t)) Q x(t - \tau_i(t))
\]

\[
+ 2 x^T (t) \left[ P_i (A_{ji} + \Delta A_{ji}(t))
\]

\[
+ P_i (B_i + \Delta B_i(t)) K_{ji} \right] x(t - \tau_i(t))
\]

\[
+ 2 x^T (t) \left[ P_i (A_{di} + \Delta A_{di}(t))
\]

\[
+ P_i (B_i + \Delta B_i(t)) K_{di} \right] x(t - \tau_i(t))
\]

\[
+ 2 x^T (t) \left[ P_i (A_{di} + \Delta A_{di}(t))
\]

\[
+ P_i (B_i + \Delta B_i(t)) K_{di} \right] x(t - \tau_i(t))
\]

\[
= \rho^T (t) \tilde{y},\rho (t).
\]

Noting that \( \Gamma_i < 0 \), we have that the closed-loop UMJS (37) with mode-dependent time-varying delays achieves \( H_\infty \) disturbance attenuation \( \gamma \).

Now, we design the memory/delayed state feedback controller for closed-loop UMJS (37).

Theorem 3. The closed-loop UMJS (37) with mode-dependent time-varying delays and memory-state feedback controller (3) achieves
EMSS and $H_{\infty}$ performance $\gamma$ if there exist matrices $N_i > 0, N_i > 0, \cdots, N_N > 0, Q_i > 0, Q_i > 0$ and scalars $\epsilon > 0, \epsilon_k > 0$ such that (8) and the following LMIs hold for $i = 1, 2 \ldots, N$:

$$
\Xi_j = \begin{bmatrix}
\hat{\Psi}_j + \hat{\Omega}_{1j} & A_{ii}N_i + B_iY_i & * & -(1 - s)I & * & * & * & K_j & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0, \quad (48)
$$

where

$$
\hat{\Psi}_j = \begin{bmatrix}
\Psi_j + \hat{\Omega}_{1j} & A_{ii}N_i + B_iY_i & * & -(1 - s)I & * & * & * & K_j & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Then, the appropriate memory-state feedback controller can be realised by

$$
K_{1j} = X_iN_i^{-1}, \quad K_{2j} = Y_iN_i^{-1}, \quad K_{3j} = Z_iN_i^{-1}. \quad (49)
$$

Proof. Noting that $\Delta A(t, r), \Delta A_i(t, r), \Delta A_j(t, r), \Delta B(t, r), \Delta D(t, r)$ are unknown matrices representing time-varying parameter uncertainties, there exist $\epsilon > 0$ and $\epsilon_k > 0$, such that

$$
2x^T(t)P_3G_iF_i(t)[M_{1i} M_{2i} M_{3i} M_{3j}M_{3j}M_{3j}] \begin{bmatrix} x(t) \\
x(t - \tau(t)) \\
x(t - \tau(t)) \omega(t) \end{bmatrix}
$$

$$
\leq \epsilon_k \begin{bmatrix} s \tau(t) \\
x^T(t - \tau(t)) \\
x^T(t - \tau(t)) \omega^T(t) \end{bmatrix}
$$

and

$$
2x^T(t)P_3G_iF_i(t)[K_{1j} K_{2j} K_{3j}] \begin{bmatrix} x(t) \\
x(t - \tau(t)) \\
x(t - \tau(t)) \omega(t) \end{bmatrix}
$$

$$
\leq \epsilon_k \begin{bmatrix} s \tau(t) \\
x^T(t - \tau(t)) \\
x^T(t - \tau(t)) \omega^T(t) \end{bmatrix}
$$

then, (38) in Theorem 2 can be rewritten as

$$
\hat{\Gamma}_j = \begin{bmatrix}
\Psi_j + \hat{\Omega}_{1i} & P_i A_{ii} + P_i B_i K_j & P_i A_{ii} + P_i B_i K_j & 0 \\
* & * & * & * \\
* & * & * & * \\
\end{bmatrix}
$$

$$
\hat{\Gamma}_j = \begin{bmatrix}
\Psi_j + \hat{\Omega}_{1i} & P_i A_{ii} + P_i B_i K_j & P_i A_{ii} + P_i B_i K_j & 0 \\
* & * & * & * \\
* & * & * & * \\
\end{bmatrix}
$$
\[
P_{i} = \text{diag}\{ -P_{i}^{-1} \}, \quad \pi_{i} = \begin{bmatrix} \sqrt{\pi_{i1}} I & \cdots & \sqrt{\pi_{i(N-1)}} I & \sqrt{\pi_{iN}} I \end{bmatrix}
\]

where

\[
\Xi_{i} = \begin{bmatrix} \gamma_{i} \chi_{i} \end{bmatrix} < 0,
\]

and applying Schur complement formula to (53), we can acquire \( \Xi_{i} < 0 \), then Theorem 3 is satisfied based on Theorem 2. The desired memory-delayed feedback controller is given.
Theorem 3 are sufficiently guaranteed to be LMIs, which by:
\[ K_1 N_i = X_i, \quad K_2 N_i = Y_i, \quad K_3 N_i = Z_i. \] (55)

Remark 2. In order to obtain the expected memory-state feedback controller, we have used the method of pre- and post-multiplying (53) by \( \text{diag}(P_i^{-1}, P_i^{-1}, P_i^{-1}, I, I, I \cdots I) \) and \( \text{diag}(P_i^T, P_i^T, P_i^T, I, I, I \cdots I) \), respectively. Next, (53) is not a linear matrix inequality, and we need to use the inequality relation in (54) to convert (53) to (48).

Based on the inequality relation in (54), the conditions of Theorem 3 are sufficiently guaranteed to be LMIs, which enable us to obtain the expected memory-state feedback controller.

Remark 3. Robust \( \mathcal{H}_\infty \) EMSS of open-loop MJSs with time-varying delays is acquired firstly. Then, on the premise of achieving EMSS, robust \( \mathcal{H}_\infty \) exponential mean-square stabilisation for closed-loop UMJS (37) is achieved based on memory-state feedback controller.

It should be pointed out that the computation is complicated in the proposed approach. When the size of LMI gets bigger, the calculation process will be more complicated and the solvability of the LMIs will be affected. Therefore, the complicated positive definite matrices are mentioned in this work, and the strict LMIs have been employed to improve the accuracy of numerical calculations.

Remark 4. Theorem 1 illustrates that the unforced MJS achieves EMSS and satisfies robust \( \mathcal{H}_\infty \) disturbance attenuation \( \gamma \) via linear matrix inequality. Theorem 2 shows that the closed-loop UMJSs with mode-dependent time-varying delays and memory-state feedback controller achieves EMSS and robust \( \mathcal{H}_\infty \) performance \( \gamma \) via matrix inequalities; Theorem 3 states that the closed-loop UMJSs with mode-dependent time-varying delays and memory-state feedback controller achieves EMSS and \( \mathcal{H}_\infty \) performance \( \gamma \) via linear matrix inequality. Furthermore, we construct and implement the expected memory-state feedback controller in Theorem 3.

Note the controller (3) can be simplified as
\[ \mathcal{U}_1 (t) = K_1 (r_t) x(t), \] (56)
\[ \mathcal{U}_2 (t) = K_1 (r_t) x(t) + K_2 (r_t) x(t - \tau (t)), \] (57)
\[ \mathcal{U}_3 (t) = K_1 (r_t) x(t) + K_3 (r_t) x(t - \tau (t)). \] (58)

Here, we give the first two forms of state feedback controller based on Theorem 3. The corresponding corollaries are as follows:

Corollary 1. The closed-loop UMJS (37) with mode-dependent time-varying delays and memory-state feedback controller (56) achieves exponential mean square stabilisation and \( \mathcal{H}_\infty \) performance \( \gamma \) if there exist matrices \( N_1 > 0, N_2 > 0, \ldots, N_N > 0, Q > 0, \Omega > 0 \) and scalars \( \epsilon > 0, \epsilon_k > 0 \) such that (8) and the following LMIs hold for \( i = 1, 2, \ldots, N \):
\[ \mathcal{E}_1 = \begin{bmatrix} \tilde{\mathcal{G}}_i & \mathcal{M}_i \mathcal{N}_i & \tilde{h}_i \\ \ast & \tilde{\mathcal{P}}_i & 0 & 0 \\ \ast & \ast & \tilde{\mathcal{X}}_k & 0 \\ \ast & \ast & \ast & \mathcal{R} \end{bmatrix} < 0, \] (59)
where
\[ \mathcal{E}_1 = \begin{bmatrix} \Psi_i + \Omega_i & A_{\mu} N_i & A_{\mu} N_i \\ \ast & -(1 - \kappa) I & 0 \\ \ast & \ast & -(1 - \kappa) I \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} \]
\[ \mathcal{M}_1^T = \begin{bmatrix} X_i^T M_{4i}^T & 0 & 0 & 0 \end{bmatrix}, \]
\[ \Psi_i = \pi_{\mu} N_i + A_{\mu} N_i + N_i A_{\mu}^T + B_{\mu} X_i + X_i B_{\mu}^T, \] (60)
\[ \mathcal{R}_k = \text{diag}\{-\epsilon_k^{-1} I\}, \]
In this case, the appropriate memory-state feedback controller can be realised by
\[ K_1 = X_i N_i^{-1}. \] (61)

Corollary 2. The closed-loop UMJS (37) with mode-dependent time-varying delays and memory-state feedback controller (37) achieves exponential mean square stabilisation and \( \mathcal{H}_\infty \) performance \( \gamma \) if there exist matrices \( N_1 > 0, N_2 > 0, \ldots, N_N > 0, Q > 0, \Omega > 0 \) and scalars \( \epsilon > 0 \) and scalar \( \epsilon_k > 0 \) such that (8) and the following LMIs hold for \( i = 1, 2, \ldots, N \):
\[ \mathcal{E}_1 = \begin{bmatrix} \tilde{\mathcal{G}}_i & \mathcal{M}_i \mathcal{N}_i & \tilde{h}_i \\ \ast & \tilde{\mathcal{P}}_i & 0 & 0 \\ \ast & \ast & \tilde{\mathcal{X}}_k & 0 \\ \ast & \ast & \ast & \mathcal{R} \end{bmatrix} < 0, \] (62)
where

\[
\mathbf{M}_i = \begin{bmatrix}
X^T_i M^T_i N^{-1}_i \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

\[
\hat{\mathbf{R}}_k = \text{diag}\{-\varepsilon_k^{-1}I, -\varepsilon_k^{-1}I\},
\]

In this case, the appropriate memory-state feedback controller can be realised by

\[
K_{1i} = X_i N^{-1}_i, \quad K_{2i} = Y_i N^{-1}_i.
\] (63)

Remark 5. Corollary 1 and Corollary 2 are special cases of Theorem 3. The purpose of Corollary 1 and Corollary 2 is to compare the control effectiveness of the memory-state feedback controller with that of Theorem 3.

Remark 6. The controller in this work can not only ensure the stability of the closed-loop system, but also reduce the interference of time-varying delays in the system. The memory-state feedback controller can be extended to the asynchronous memory-state feedback controller and non-fragile memory-state feedback controller for delayed MFSIs based on sampled-data and event-triggered mechanism, and can be simplified to the memoryless-state feedback controller.

4 | SIMULATION EXAMPLES

Example 1. Consider delayed UMFS (37) with three modes and the following parameters:

\[
A_1 = \begin{bmatrix} 5 & 2 \\ 2 & 9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix},
\]

\[
A_{d2} = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix},
\]

\[
D_1 = \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & -0.3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 60 & 50 \\ -30 & 50 \end{bmatrix},
\]

\[
M_{11} = \begin{bmatrix} 0.012 & 0.021 \\ 0.021 & 0.012 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0.011 & 0.023 \\ 0.025 & 0.013 \end{bmatrix},
\]

\[
M_{31} = \begin{bmatrix} 0.021 & 0.015 \\ 0.014 & 0.025 \end{bmatrix}, \quad M_{41} = \begin{bmatrix} 0.021 & 0.032 \\ 0.015 & 0.042 \end{bmatrix},
\]

\[
M_{51} = \begin{bmatrix} 0.012 & 0.015 \\ 0.015 & 0.022 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.10 \\ 0.15 \end{bmatrix},
\]

and the time-varying delay \(\tau_1(t)\) meets (2) with \(\tau_1 = 3.5 \sin(0.5t), s_1 = 0.2\).

\[
A_2 = \begin{bmatrix} 5 & 8 \\ 6 & 8 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix},
\]

\[
A_{d2} = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 2 \\ 4 & -4 \end{bmatrix},
\]

\[
D_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 70 & 80 \\ 40 & 50 \end{bmatrix},
\]

\[
M_{12} = \begin{bmatrix} 0.013 & 0.032 \\ 0.013 & 0.020 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0.026 & 0.021 \\ 0.012 & 0.014 \end{bmatrix},
\]

\[
M_{32} = \begin{bmatrix} 0.014 & 0.025 \\ 0.025 & 0.016 \end{bmatrix}, \quad M_{42} = \begin{bmatrix} 0.035 & 0.013 \\ 0.012 & 0.024 \end{bmatrix},
\]

\[
M_{52} = \begin{bmatrix} 0.016 & 0.023 \\ 0.016 & 0.015 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix},
\]

and the time-varying delay \(\tau_2(t)\) meets (2) with \(\tau_2 = 2.2 \sin(0.6t), s_2 = 0.2\).

\[
A_3 = \begin{bmatrix} 8 & 2 \\ 3 & 5 \end{bmatrix}, \quad A_{d3} = \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.4 \end{bmatrix},
\]

\[
A_{d3} = \begin{bmatrix} 0.2 & 0.5 \\ 0.3 & 0.6 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 5 & -4 \\ 2 & 1 \end{bmatrix},
\]

\[
D_3 = \begin{bmatrix} 0.5 & -0.2 \\ -0.3 & 0.8 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 80 & 40 \\ 50 & 50 \end{bmatrix},
\]

\[
M_{13} = \begin{bmatrix} 0.008 & 0.011 \\ 0.005 & 0.012 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} 0.025 & 0.012 \\ 0.015 & 0.016 \end{bmatrix},
\]

\[
M_{33} = \begin{bmatrix} 0.004 & 0.013 \\ 0.006 & 0.004 \end{bmatrix}, \quad M_{43} = \begin{bmatrix} 0.008 & 0.015 \\ 0.006 & 0.004 \end{bmatrix},
\]

\[
M_{53} = \begin{bmatrix} 0.014 & 0.025 \\ 0.021 & 0.013 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},
\]

and the time-varying delay \(\tau_3(t)\) satisfies (4) with \(\tau_3 = 2.1 \sin(0.5t), s_3 = 0.3\). And the transition rate matrix is described as:

\[
\Pi = \begin{bmatrix} -3.6 & 1.8 & 1.8 \\ 2 & -2.6 & 0.6 \\ 1 & 2 & -3 \end{bmatrix}.
\]
Let $h = 3.6$, $\tau = 1$ and $s = 0.4$. Solving LMIs (48) and (59), we get $H_\infty$ disturbance attenuation $\gamma = 6.8747$ and the following parameters:

\[
\begin{align*}
K_{11} &= \begin{bmatrix} -12.8342 & -121.7117 \\ -28.0314 & -97.7131 \end{bmatrix}, \\
K_{12} &= \begin{bmatrix} -89.2874 & -104.2496 \\ -79.3108 & -91.6749 \end{bmatrix}, \\
K_{13} &= \begin{bmatrix} -21.1702 & -18.1535 \\ -6.9784 & -12.9754 \end{bmatrix}, \\
K_{21} &= \begin{bmatrix} -0.0089 & -0.0087 \\ -0.0069 & -0.0077 \end{bmatrix}, \\
K_{22} &= \begin{bmatrix} -0.0187 & -0.0210 \\ -0.0155 & -0.0181 \end{bmatrix}, \\
K_{23} &= \begin{bmatrix} -0.0150 & -0.0116 \\ -0.0121 & -0.0098 \end{bmatrix}, \\
K_{31} &= \begin{bmatrix} -0.0063 & 0.0021 \\ 0.0039 & -0.0157 \end{bmatrix}, \\
K_{32} &= \begin{bmatrix} -0.0052 & -0.0003 \\ -0.0110 & -0.0256 \end{bmatrix}, \\
K_{33} &= \begin{bmatrix} -0.0071 & -0.0013 \\ -0.0020 & -0.0114 \end{bmatrix}.
\end{align*}
\]

\[\begin{array}{l}
\text{TABLE 1 Different values of } \gamma \text{ in Example 1} \\
\hline
\text{Case} & \tau = 1 & \tau = 1.2 & \tau = 1.5 \\
\hline
\gamma \text{ by Theorem 3 } (s = 0.4) & 6.8747 & 6.8806 & 6.8846 \\
\gamma \text{ by Corollary 1 } (s = 0.4) & 9.9780 & 9.9793 & 14.8335 \\
\gamma \text{ by Theorem 3 } (s = 0.5) & 9.1029 & 9.1774 & 9.1184 \\
\gamma \text{ by Corollary 1 } (s = 0.5) & 10.2851 & 10.2903 & 10.2950 \\
\gamma \text{ by Theorem 3 } (s = 0.6) & 9.1717 & 9.2503 & 9.1709 \\
\gamma \text{ by Corollary 1 } (s = 0.6) & 10.2801 & 10.3235 & 10.3835 \\
\hline
\end{array}\]

Let

\[
\begin{align*}
x(0) &= \begin{bmatrix} 2.5 \\ -3.6 \end{bmatrix}, \\
\omega(t) &= \begin{bmatrix} e^{-0.1t} \sin(0.3t) \\ \text{uni frnd}[-0.1, 0.1] \end{bmatrix},
\end{align*}
\]

where uni frnd$[-0.1, 0.1]$ means uniformly distributed noise between the upper and lower bounds. Figure 1 shows closed-loop UMJS state $x(t)$ with memory-state feedback controller containing jump modes, Figure 2 describes closed-loop UMJS state $x(t)$ with memoryless-state feedback controller containing jump modes and Figure 3 depicts estimated output $z(t)$ containing jump modes.

$H_\infty$ performance indices of memory-state feedback control (Theorem 3) are compared with that of memoryless-state feedback control (Corollary 1) in Table 1:

**Example 2.** Consider the INCSTR [52], where the reactor parameters will change because of the internal and external environment, so the reactor can be considered as an UMJS. Let $x_1(t)$ and $x_2(t)$ represent the reactant and reaction temperature, respectively. External interference
\[
\omega(t) = [\omega_1(t) \quad \omega_2(t)]^T \text{ represents the inlet temperature, } U(t) \text{ represents the cooling medium, and the INCSTR can be represented by UMJS (1) with three modes and the following parameters:}
\]

\[
A_1 = \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 \\ 4 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & -0.2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 80 & 60 \\ -60 & 50 \end{bmatrix}.
\]
and the time-varying delay \( \tau(t) \) meet (2) with \( \tau_3 = 1.2 \sin(t), \ s_3 = 0.5. \) And the transition rate matrix is described as

\[
\Pi = \begin{bmatrix} -3 & 1.5 & 1.5 \\ 1.8 & -2 & 0.2 \\ 1.2 & 2.2 & -3.4 \end{bmatrix},
\]

Let \( h = 3.4, \tau = 1 \) and \( s = 0.6. \) Solving LMI s (48) and (59), we obtain \( H_{\infty} \) disturbance attenuation \( \gamma = 5.3848 \) and the following parameters:

\[
K_{11} = \begin{bmatrix} 5.6722 & -38.6180 \\ -12.3518 & -35.2153 \end{bmatrix},
\]

\[
K_{12} = \begin{bmatrix} -39.0724 & -47.1014 \\ -32.7492 & -35.9425 \end{bmatrix},
\]

\[
K_{13} = \begin{bmatrix} -21.1101 & -15.8221 \\ -9.7317 & -18.3632 \end{bmatrix},
\]

\[
K_{21} = \begin{bmatrix} -0.0350 & -0.0291 \\ -0.0244 & -0.0275 \end{bmatrix},
\]

\[
K_{22} = \begin{bmatrix} -0.0193 & -0.0267 \\ -0.0142 & -0.0239 \end{bmatrix},
\]

\[
K_{23} = \begin{bmatrix} -0.0234 & -0.0185 \\ -0.0155 & -0.0183 \end{bmatrix},
\]

\[
K_{31} = \begin{bmatrix} -0.0865 & -0.1061 \\ -0.0678 & -0.1111 \end{bmatrix},
\]

\[
K_{32} = \begin{bmatrix} -0.0514 & -0.0912 \\ -0.0434 & -0.0920 \end{bmatrix},
\]

\[
K_{33} = \begin{bmatrix} -0.0536 & -0.0717 \\ -0.0386 & -0.0774 \end{bmatrix}.
\]
FIGURE 4  Closed-loop system state $x(t)$ containing jump modes in Example 2

FIGURE 5  Closed-loop system state $x(t)$ containing jump modes in Example 2
Let

\[ x(0) = \begin{bmatrix} 2.8 \\ -3.9 \end{bmatrix}, \quad \omega(t) = \begin{bmatrix} 2e^{-0.1't} \sin(0.8't) \\ \text{uni frnd}[-0.15, 0.15] \end{bmatrix}, \]

where \( \text{uni frnd}[-0.15, 0.15] \) means uniformly distributed noise between the upper and lower bounds. Figure 4 depicts closed-loop UMJS state \( x(t) \) with memory-state feedback controller containing jump modes. Figure 5 describes closed-loop UMJS state \( x(t) \) with memoryless-state feedback controller containing jump modes and Figure 6 shows estimated output \( z(t) \) containing jump modes.

\( H_\infty \) performance indices of memory-state feedback control (Theorem 3) are compared with that of memoryless-state feedback control (Corollary 1) in Table 2:

6 FIGURE 6 Estimated output \( z(t) \) containing jump modes in Example 2

Remark 7. In the simulation study, simulation examples are obtained by using Theorem 3 and Corollary 1. Figures 1 and 4 are simulation results of Theorem 3; Figures 2 and 5 are simulation results of Corollary 1. As can be seen from the enlarged figures, the memory-state feedback controller makes the convergence and stability effectiveness of state \( x(t) \) better than the memoryless-state feedback controller.

5 CONCLUSION

This paper has investigated the problem of robust \( H_\infty \) EMSS for UMJSs with time-varying and mode-dependent delays based on memory-state feedback control. Attention has been focused on the design of memory-state feedback controller for UMJSs with time-varying and mode-dependent delays such that the closed-loop UMJS satisfies robust \( H_\infty \) EMSS. The improved conditions for the solvability of the memory-state feedback control problem has been obtained via designing mode-dependent and delay-dependent L-K functional. The desired memory-state feedback controller has been given by using LMIs. Two simulation examples including the INCSTR have been used to demonstrate the effectiveness and usefulness of the delayed feedback control technique that this work proposes.

In the future work, we will deeply research the asynchronous tracking and non-fragile memory-state feedback control for delayed UMJSs based on sampled-data and event-triggered mechanisms. The aforesaid recommendations leave a good prospect for future research.

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