Delayed feedback control of fractional-order chaotic systems

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Abstract
We study the possibility to stabilize unstable steady states and unstable periodic orbits in chaotic fractional-order dynamical systems by the time-delayed feedback method. By performing a linear stability analysis, we establish the parameter ranges for successful stabilization of unstable equilibria in the plane parameterized by the feedback gain and the time delay. An insight into the control mechanism is gained by analyzing the characteristic equation of the controlled system, showing that the control scheme fails to control unstable equilibria having an odd number of positive real eigenvalues. We demonstrate that the method can also stabilize unstable periodic orbits for a suitable choice of the feedback gain, providing that the time delay is chosen to coincide with the period of the target orbit. In addition, it is shown numerically that delayed feedback control with a sinusoidally modulated time delay significantly enlarges the stability region of steady states in comparison to the classical time-delayed feedback scheme with a constant delay.

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1. Introduction
The fractional-order dynamical systems have attracted remarkable attention in the last decade. Many authors have studied the chaotic and hyperchaotic dynamics of various fractional-order systems, such as those of Duffing, Lorenz, Rössler, Chua, Lü, Chen, etc., which are introduced by changing the time derivative in the corresponding ODE systems, usually with the fractional derivative in the Caputo or Riemann–Liouville sense of order $0 < \alpha < 1$ \cite{1–5}. One interesting problem is to analyze the lowest value of parameter $\alpha$ under which fractional-order dynamical systems show chaotic or hyperchaotic behaviors. Stability analysis, synchronization and control of fractional-order systems by using different techniques are also widely investigated \cite{6–14} and are of great interest due to their application in control theory, signal processing, complex networks, etc \cite{15–20}.
In this paper we investigate the possibility to control unstable equilibria and unstable periodic orbits in fractional-order chaotic systems by a time-delayed feedback method. Pyragas introduced the time-delayed feedback control (TDFC) in 1992 by constructing a control force in the form of a continuous feedback proportional to the difference between the present and an earlier value of an appropriate system variable \( \eta \), i.e. \( K [\eta(t - T) - \eta(t)] \), with \( K \) and \( T \) being the constant control parameters denoting the feedback gain and the time delay, respectively \([21–23]\). To stabilize unstable periodic orbits, the delay \( T \) is chosen to match the period of the unstable orbit. In the case of controlling unstable equilibria, the optimal delay is related to the intrinsic characteristic time scale given by the eigenfrequencies of the uncontrolled system. In both cases, the control force vanishes when the target state is reached, rendering the noninvasive method. In this sense, the unstable states of the uncontrolled system are not changed, since the control force acts only if the system deviates from the state to be stabilized.

The main advantage of the Pyragas method over the other control methods is that it does not require the knowledge of system’s equations and the positions of the unstable states whose control is desirable. The method has been used in many concrete applications \([24–33]\), and also rigorously investigated analytically \([34–50]\). In recent works \([51, 52]\), it has been shown that the efficiency of the method is greatly improved by deterministic or stochastic modulation of the time delay \( T \). This variable-delay feedback control (VDFC) has been shown successful in stabilization of unstable equilibria in systems described by both ordinary and delay differential equations.

This paper is organized as follows. In section 2 we present the TDFC for stabilization of unstable equilibria in fractional-order dynamical systems. We first look at a general non-diagonal case of such a control scheme and derive the stability conditions for the equilibrium points in the presence and absence of control. To illustrate the control method, we choose a fractional-order Rössler system in a chaotic regime controlled via a single state variable with a Pyragas-type feedback force. By performing a linear stability analysis of the controlled system, we calculate the domain of successful control of the unstable equilibria in the plane parameterized by the feedback gain and the delay time. It is shown, both numerically and analytically that the unstable equilibrium point with an odd number of positive real eigenvalues cannot be controlled by the time-delayed feedback method for any values of the control parameters, thus extending the validity of the odd-number limitation theorem \([46, 47]\) to the case of fractional-order systems. In section 3 we perform a numerical calculation of the control domain by the Pyragas-delayed feedback with a modulated time delay in a form of a sine-wave. In section 4 we give a numerical evidence that a TDFC can successfully stabilize unstable periodic orbits, and estimate the corresponding feedback gain intervals for which such a control is possible. A summary of the obtained results is given in section 5.

2. Delayed feedback control of unstable equilibria

2.1. Stability analysis

We consider a general \( n \)-dimensional nonlinear fractional-order dynamical system under a non-diagonal form of delayed feedback control in the sense of Pyragas:

\[
\begin{align*}
D_\alpha^\omega x_1(t) &= f_1(x(t)) + F_1(t), \\
D_\alpha^\omega x_2(t) &= f_2(x(t)) + F_2(t), \\
&\vdots \\
D_\alpha^\omega x_n(t) &= f_n(x(t)) + F_n(t),
\end{align*}
\] (1)

This page appears to contain a mathematical equation, which is not properly formatted for readability. The equation seems to be related to fractional-order dynamical systems with delayed feedback control.
where

$$F_i(t) = \sum_{j=1}^{n} K_{ij} [x_j(t - T) - x_j(t)]$$

(2)

is the delayed feedback force applied to the $i$th component of the system, consisting of contributions of all the system components, $K_{ij}$ are the gain factors of the feedback terms, $T$ is the constant time delay, $x = (x_1, x_2, \ldots, x_n)$ is the state vector, and $f = (f_1, f_2, \ldots, f_n)$ is the nonlinear vector field that determines the dynamics of the unperturbed system. The notation $D_x^\alpha$ is the time fractional derivative in the Caputo sense defined as [53]

$$D_x^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t f^{(m)}(\tau) (t - \tau)^{\alpha - m} \, d\tau, & m - 1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t), & \alpha = m, \quad m \in \mathbb{N}, \end{cases}$$

(3)

which is related to the famous Riemann–Liouville fractional integral [54]

$$J_x^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) (t - \tau)^{\alpha - 1} \, d\tau,$$

(4)

i.e.

$$D_x^\alpha f(t) = J_x^{m-\alpha} \frac{d^m}{dt^m} f(t),$$

(5)

where $m = \lfloor \alpha \rfloor$, i.e. $m$ is the first integer which is not less than $\alpha$. In the following, we consider the fractional-orders $\alpha_i$ to be in interval $(0, 1)$.

In the case $K_{ij} = K \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta, the generalized control scheme (1) reduces to TDFC with a diagonal coupling, and when the control force is applied only to a single system component and consists only of contributions of the same component, it yields the original TDFC control scheme introduced by Pyragas.

Let $P = (x_1^*, x_2^*, \ldots, x_n^*)$ be an arbitrary equilibrium point of the system (1) in the absence of control ($K_{ij} = 0$) being a solution to the nonlinear algebraic system:

$$\begin{align*}
    f_1(x_1, x_2, \ldots, x_n) &= 0, \\
    f_2(x_1, x_2, \ldots, x_n) &= 0, \\
    \vdots \\
    f_n(x_1, x_2, \ldots, x_n) &= 0.
\end{align*}$$

(6)

Assuming that $P$ is an unstable equilibrium point of the uncontrolled system, we wish to find the domain in the parameter space of the feedback gains $K_{ij}$ and the time-delay $T$ for which $P$ becomes locally asymptotically stable under TDFC force (2). The stability of $P$ under a non-diagonal feedback control (1) and (2) can be determined by linearizing (1) around $P$, which leads to the linear autonomous system:

$$\begin{pmatrix}
    D_x^\alpha \tilde{x}_1(t) \\
    D_x^\alpha \tilde{x}_2(t) \\
    \vdots \\
    D_x^\alpha \tilde{x}_n(t)
\end{pmatrix} = \tilde{\Lambda} \cdot \begin{pmatrix}
    \tilde{x}_1(t) \\
    \tilde{x}_2(t) \\
    \vdots \\
    \tilde{x}_n(t)
\end{pmatrix} + \begin{pmatrix}
    \tilde{F}_1(t) \\
    \tilde{F}_2(t) \\
    \vdots \\
    \tilde{F}_n(t)
\end{pmatrix},$$

(7)

where

$$\tilde{\Lambda} = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}$$

(8)
Theorem 1. The equilibrium point $P$ of the system (1) and (2) is locally asymptotically stable if and only if all the roots $s$ of the characteristic equation:

$$
\det[\Delta(s)] = 0,
$$

is the Jacobian matrix of the free-running system, with $a_{ij} = (\partial f_i/\partial x_j)$ calculated at $P$, $\tilde{x}_i(t) = x_i(t) - x_i^*$ are the transformed coordinates in which the equilibrium point is at the origin, and $\tilde{F}_i(t)$ are the components of the feedback control force in the new coordinates, i.e.

$$
\tilde{F}_i(t) = \sum_{j=1}^{n} K_{ij}[\tilde{x}_j(t - T) - \tilde{x}_j(t)].
$$

By applying the Laplace transform to equations (7) and by using the formula for the Laplace transform of the fractional derivative in the Caputo sense [56]:

$$
\mathcal{L}[D^\alpha t^\alpha \tilde{x}_i(t)] = s^\alpha X_i(s) - \sum_{k=0}^{m-1} \tilde{x}_i^{(k)}(0+)s^{\alpha-1-k},
$$

where $X_i(s) = \mathcal{L}[\tilde{x}_i(t)]$ are the Laplace images and $\tilde{x}_i^{(k)}(0)$ are the initial conditions, we obtain

$$
\Delta(s) \cdot X(s) = B(s),
$$

where

$$
\Delta(s) = \begin{pmatrix}
n - a_{11} + K_{11}(1 - e^{-sT}) & -a_{12} + K_{12}(1 - e^{-sT}) & \ldots & -a_{1n} + K_{1n}(1 - e^{-sT}) \\
-a_{21} + K_{21}(1 - e^{-sT}) & n - a_{22} + K_{22}(1 - e^{-sT}) & \ldots & -a_{2n} + K_{2n}(1 - e^{-sT}) \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} + K_{n1}(1 - e^{-sT}) & -a_{n2} + K_{n2}(1 - e^{-sT}) & \ldots & n - a_{nn} + K_{nn}(1 - e^{-sT})
\end{pmatrix}
$$

represents a characteristic matrix of the system (7), $X(s) = \text{col}(X_1(s), X_2(s), \ldots, X_n(s))$ is the column vector of the Laplace images, and

$$
B(s) = \begin{pmatrix}
\tilde{x}_1(0)s^{a_1-1} + e^{-sT} \sum_{j=1}^{n} K_{1j} \int_{0}^{T} \tilde{x}_j(t) e^{-st} \, dt \\
\tilde{x}_2(0)s^{a_2-1} + e^{-sT} \sum_{j=1}^{n} K_{2j} \int_{0}^{T} \tilde{x}_j(t) e^{-st} \, dt \\
\vdots \\
\tilde{x}_n(0)s^{a_n-1} + e^{-sT} \sum_{j=1}^{n} K_{nj} \int_{0}^{T} \tilde{x}_j(t) e^{-st} \, dt
\end{pmatrix}.
$$

For the sake of the argument, let us assume that all the roots $s$ of $\det[\Delta(s)] = 0$ are positioned in the left complex $s$-plane ($\text{Re}(s) < 0$). The assumption is equivalent to the claim that $\det[\Delta(s)] \neq 0$ for $\text{Re}(s) \geq 0$, meaning that $\Delta(s)$ is the invertible matrix in the right complex $s$-plane, and thus $X(s) = \Delta(s)^{-1} B(s)$ has a unique solution $X(s)$ for $\text{Re}(s) \geq 0$. Furthermore, it is easy to check that

$$
\lim_{s \to 0} sX(s) = \lim_{s \to 0} [\Delta(s)^{-1} \cdot sB(s)] = 0,
$$

from which we have

$$
\lim_{t \to +\infty} \tilde{x}(t) = \lim_{s \to 0} sX(s) = 0
$$

by using the final-value theorem for the Laplace transform. Thus, we prove the following theorem.

**Theorem 1.** The equilibrium point $P$ of the system (1) and (2) is locally asymptotically stable if and only if all the roots $s$ of the characteristic equation:

$$
\det[\Delta(s)] = 0,
$$

4
have negative real parts, i.e.

\[ |\arg(s)| > \pi/2. \]  

(17)

The matrix \( \Delta(s) \) is given by equation (12), and its components \( a_{ij} = (\partial f_i/\partial x_j) \) are evaluated at the equilibrium point \( P \).

**Remark 1.** For the case when there is no control \( (K_{ij} = 0) \), the conditions for stability of a particular equilibrium point is still provided by theorem 1. In this case, the characteristic matrix of the uncontrolled system simplifies to

\[ \Delta(s) = S \cdot \hat{I} - \hat{A}, \]  

(18)

where \( S = \text{col}(s^{\alpha_1}, s^{\alpha_2}, \ldots, s^{\alpha_n}) \), \( \hat{I} \) is an \( n \times n \) identity matrix and \( \hat{A} \) is the Jacobian matrix of the unperturbed system evaluated at the equilibrium point, and given by equation (8). In the special case \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha \), the characteristic equation (18) can be recast into the form

\[ \det(\lambda \hat{I} - \hat{A}) = 0, \]  

(19)

which is a polynomial of degree \( n \) in \( \lambda \), where \( \lambda = s^{\alpha} \). In this case, the stability condition (17) can be rewritten in the form of Matignon [6]

\[ |\arg(\lambda)| > \alpha \pi/2, \]  

(20)

meaning that the stability region is bounded by a cone, with vertex at the origin, extending into the right half of the complex \( \lambda \)-plane such that it encloses an angle of \( \pm \alpha \pi/2 \) with the positive real axis. Therefore, the equilibrium point of the unperturbed system is stable if and only if all the roots of the characteristic polynomial (19) are placed outside this cone.

To derive another important result related to the limitation of the time-delayed feedback method (1) and (2), we consider the function \( \det(\Delta(s)) \), with \( \Delta(s) \) given by equation (12), and \( s \in \mathbb{R}^+ \). We take \( P \) to be an unstable equilibrium of (1) in the absence of external perturbation \( (K_{ij} = 0) \), and \( \hat{A} \) the Jacobian matrix of \( P \). One can easily deduce that

\[ \lim_{s \to +\infty} \det(\Delta(s)) = +\infty, \]  

(21)

and

\[ \lim_{s \to 0^+} \det(\Delta(s)) = \det(-\hat{A}) = \prod_{i=1}^{n} (-e_i), \]  

(22)

where \( e_i \) are the eigenvalues of \( \hat{A} \). Evidently, if \( \hat{A} \) has an odd number of positive real eigenvalues, then \( \lim_{s \to 0^+} \det(\Delta(s)) < 0 \). In this case, the sign of \( \det(\Delta(s)) \) is changed from negative to positive when \( s \) sweeps the real interval \( [0, +\infty) \). Since \( \det(\Delta(s)) \) is a smooth function in \( s \), there exists at least one positive real root of the characteristic equation \( \det(\Delta(s)) = 0 \), meaning that the equilibrium point \( P \) cannot be stabilized by the TDFC (2).

The result is summarized in the following theorem.

**Theorem 2 (odd-number limitation).** Let \( P \) be an unstable equilibrium point of the fractional-order system (1) in the absence of control \( (K_{ij} = 0) \), and \( \hat{A} \) the corresponding Jacobian matrix at \( P \). If \( \hat{A} \) has an odd number of positive real eigenvalues, then the TDFC (2) cannot stabilize the unstable equilibrium \( P \) for any values of the control parameters \( K_{ij} \) and \( T \).

The result is an extension of the odd-number limitation theorem [46, 47] to fractional-order systems with respect to unstable fixed points. We note that the odd-number limitation has recently been refuted by Fiedler et al [48, 49] for the case of unstable periodic orbits in systems described by ordinary differential equations.
Figure 1. The phase plots of the chaotic attractor in the free-running fractional-order Rössler system. The parameters are $a = 0.4$, $b = 0.2$, $c = 10$ and $\alpha = 0.9$.

2.2. Numerical example

To illustrate the TDFC in fractional-order chaotic systems, we consider a fractional-order Rössler system in the form

\[
\begin{align*}
D_\alpha^x x(t) &= -y(t) - z(t), \\
D_\alpha^y y(t) &= x(t) + ay(t) + F(t), \\
D_\alpha^z z(t) &= z(t) [x(t) - c] + b, \\
\end{align*}
\]

where

\[F(t) = K [y(t - T) - y(t)]\]

is the Pyragas feedback controller applied through a single component (y-channel), and $a$, $b$ and $c$ are the parameters of the free-running system. In the following, we take $a = 0.4$, $b = 0.2$, $c = 10$ and $\alpha = 0.9$, for which the uncontrolled system ($K = 0$) has a chaotic attractor (see figure 1) [3].

The unperturbed Rössler system has two equilibrium points $P_1 = (x_1^*, y_1^*, z_1^*)$ and $P_2 = (x_2^*, y_2^*, z_2^*)$, where

\[
\begin{align*}
x_\pm^* &= \frac{c}{2} \left( 1 \pm \sqrt{1 - \frac{4ab}{c^2}} \right), \\
y_\pm^* &= -\frac{c}{2a} \left( 1 \pm \sqrt{1 - \frac{4ab}{c^2}} \right), \\
z_\pm^* &= \frac{c}{2a} \left( 1 \pm \sqrt{1 - \frac{4ab}{c^2}} \right).
\end{align*}
\]

Linearization around the equilibrium points leads to the following linear autonomous system:

\[
\begin{pmatrix}
D_\alpha^x \tilde{x}(t) \\
D_\alpha^y \tilde{y}(t) \\
D_\alpha^z \tilde{z}(t)
\end{pmatrix} = \tilde{\Lambda} \cdot \begin{pmatrix}
\tilde{x}(t) \\
\tilde{y}(t) \\
\tilde{z}(t)
\end{pmatrix},
\]

where $\tilde{\Lambda}$ is the matrix of linearization. The eigenvalues of $\tilde{\Lambda}$ determine the stability of the equilibrium points.
where
\[
\hat{\mathbf{A}} = \begin{pmatrix}
0 & -1 & -1 \\
1 & a & 0 \\
z_+^* & 0 & x_+^* - c
\end{pmatrix}
\]
(29)
is the Jacobian matrix, and \(\hat{x}(t) = x(t) - x_+^*, \hat{y}(t) = y(t) - y_+^*, \hat{z}(t) = z(t) - z_+^*\) are the transformed coordinates in which the corresponding fixed point is at the origin. According to equation (20), the equilibrium point of the linearized system (28) is asymptotically stable if and only if \(|\arg(\lambda)| > \alpha \pi/2\) for all the eigenvalues \(\lambda\) of the Jacobian matrix \(\hat{\mathbf{A}}\). The equilibrium point \(P_1 = (x_+^*, y_+^*, z_+^*) = (9.9919, -24.98, 24.98)\) has eigenvalues \(\lambda_1 = 0.3844\) and \(\lambda_{2,3} = 0.0038 \pm 5.0964i\). It is an unstable saddle point of index 1 since \(\lambda_1 > 0\), \(|\arg(\lambda_{2,3})| = 1.5701 > \alpha \pi/2\). The equilibrium point \(P_2 = (x_+^*, y_+^*, z_+^*) = (0.008, -0.02, 0.02)\) has eigenvalues \(\lambda_1 = -9.9900\) and \(\lambda_{2,3} = 0.1990 \pm 0.9797i\), and it is an unstable saddle point of index 2 since \(\lambda_1 < 0\), \(|\arg(\lambda_{2,3})| = 1.3704 < \alpha \pi/2\) [55].

In the presence of TDFC, the linearized version of the system (23) around the equilibrium points states:
\[
\begin{pmatrix}
D^0_\tau \hat{x}(t) \\
D^0_\tau \hat{y}(t) \\
D^0_\tau \hat{z}(t)
\end{pmatrix} = \hat{\mathbf{A}} \cdot \begin{pmatrix}
\hat{x}(t) \\
\hat{y}(t) \\
\hat{z}(t)
\end{pmatrix} + K \begin{pmatrix}
0 \\
\hat{y}(t - T) - \hat{y}(t) \\
0
\end{pmatrix}.
\]
(30)
According to theorem 1, the zero solution of the system (30) is asymptotically stable if and only if all the roots \(s\) of the characteristic equation:
\[
\det[\Delta(s)] = 0
\]
(31)
have negative real parts, i.e. \(|\arg(s)| > \pi/2\), where the characteristic matrix \(\Delta(s)\) is given by
\[
\Delta(s) = \begin{pmatrix}
s^\alpha & 1 & 1 \\
-1 & s^\alpha - a + K(1 - e^{-sT}) & 0 \\
-z_+^* & 0 & s^\alpha - (x_+^* - c)
\end{pmatrix}.
\]
(32)
The characteristic equation (31) can be numerically analyzed to obtain the domains of control for the unstable steady states \(P_{1,2}\) in the plane parameterized by the feedback gain \(K\) and the time delay \(T\).

In the absence of control, the equilibrium point \(P_1\) has an odd number (one) of positive real eigenvalues, and according to theorem 2, it cannot be stabilized by the TDFC method. This result has been confirmed by a numerical analysis of the characteristic equation (31), showing absence of stability domain in the \((K, T)\) parameter plane. This observation is further confirmed by a numerical simulation of the system (23) under TDFC (24). On the other hand, the fixed point \(P_2\) can be controlled by TDFC, and the resulting stability domain is shown in figure 2. The stability islands (shaded areas) denote the values of the control parameters \(K\) and \(T\) for which all the eigenvalues \(s\) of the characteristic equation (31) are lying on the left complex \(s\)-plane, thus satisfying the stability condition (17). For these values of the control parameters, the control of the fixed point \(P_2\) is successful. As a verification, we performed a computer simulation of TDFC by numerically integrating the system (23) and (24). The resulting diagrams are shown in figure 3. The simulations were done by using a predictor-corrector Adams–Bashford–Moulton numerical scheme for solving fractional-order differential equations [58]. Panels (a), (b) and (c) depict the dynamics of the state variables \(x(t), y(t)\) and \(z(t)\), respectively, and panel (d) shows the corresponding time series of the control signal \(F(t)\). In the simulations, the control parameters were \(K = 2\) and \(T = 3\),

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Figure 2. The stability domain in the \((K, T)\) parametric plane of the unstable equilibrium \(P_2\) in the fractional-order Rössler system (23) under TDFC (24). The shaded areas correspond to the control parameter values for which stabilization of the fixed point \(P_2\) is achievable. The parameters of the free-running system are \(a = 0.4\), \(b = 0.2\), \(c = 10\) and \(\alpha = 0.9\).

belonging to the domain of successful TDFC control depicted in figure 2. As expected, the simulation confirms a successful stabilization of the unstable equilibrium \(P_2\). Moreover, as indicated from panel (d) in figure 3, the control signal \(F(t)\) vanishes when the control is achieved, meaning that the control scheme is noninvasive.

We note that the above analysis has been repeated for different parameter values of the free-running system. In each case, the resulting stability domains computed from equations (31)–(32) are in agreement with the numerical simulation of the TDFC method. Specifically, for \(a = 0.4\), \(b = 0.2\), \(c = 10\) and variable \(\alpha\), we observed a decrease in the stability region as \(\alpha\) is increased from \(\alpha = 0.9\) to \(\alpha = 1\). On the other hand, as \(\alpha\) becomes smaller than 0.9, the complex-conjugate eigenvalues of the equilibrium point \(P_2\) eventually escape the instability region described by the Matignon formula (20), resulting in a stable equilibrium \(P_2\) even without control. The critical value \(\alpha = \alpha_c\) that corresponds to this eigenvalue-crossing of the conic surface between the different stability regions can be calculated from equation (20). In this case, \(\alpha_c = 0.8724\).

3. Variable-delay feedback control of unstable equilibria

In recent papers [51, 52], it has been demonstrated, both numerically and analytically, that the original Pyragas TDFC scheme can be improved significantly by modulating the time delay in an \(\varepsilon\) interval around some nominal delay value \(T_0\). In both deterministic and stochastic variants of such a delay variation, the stability domain was considerably changed, resulting in an extension of the stability area in the control parameter space if appropriate modulation is chosen. In the following, we will demonstrate numerically the successfulness of this VDFC in the case of fractional-order chaotic Rössler system (23) with \(a = 0.4\), \(b = 0.2\), \(c = 10\) and \(\alpha = 0.9\). In this case, the feedback force \(F(t)\) is given by

\[
F(t) = K[y(t - T(t)) - y(t)],
\]

where we choose a time-varying delay \(T(t)\) in a form

\[
T(t) = T_0 + \varepsilon \sin(\omega t),
\]
modulated around a nominal delay value $T_0$ with a sine-wave modulation of amplitude $\varepsilon$ and frequency $\omega$. Obviously, if $\varepsilon = 0$, then $T(t) = T_0 = \text{const}$, and the VDFC is reduced to the classical Pyragas TDFC scheme. With this choice of the feedback force $F(t)$, the control parameters of the proposed variable-delay scheme are $K$, $T_0$, $\varepsilon$ and $\omega$, and thus, the control parameter space is four-dimensional. For visualization purposes, we may fix two of the control parameters and investigate the stability domains in the parametric plane spanned by the remaining two control parameters. To demonstrate the superiority of VDFC over TDFC, we fix the modulation amplitude $\varepsilon$ and the frequency $\omega$ and investigate the control domain in $(K, T_0)$ parameter space. Numerical simulations show that the stability area is gradually increasing as $\varepsilon$ is increased from zero. Figure 4 shows such a control domain for $\varepsilon = 1$ and $\omega = 10$. The gray region indicates those values of the control parameters $K$ and $T_0$ for which the control of the unstable equilibrium $P_2$ is successful. The stability domain is obtained by numerically integrating the linearized system

$$
\begin{pmatrix}
D^\alpha x(t) \\
D^\alpha y(t) \\
D^\alpha z(t)
\end{pmatrix} = \tilde{A} \cdot \begin{pmatrix} x(t) \\
\dot{x}(t) \\
\ddot{x}(t)
\end{pmatrix} + K \begin{pmatrix} 0 \\
\tilde{y}(t - T(t)) - \tilde{y}(t) \\
0
\end{pmatrix},
$$

(35)

with the Jacobian matrix $\tilde{A}$ given by equation (29). It is evident that the control domain is significantly enlarged in comparison to the one in TDFC in figure 2. We note that numerical
integration of the system (35) for the unstable equilibrium $P_1$ shows failure of the variable-delay control scheme for any values of $K$ and $T_0$, suggesting validity of the odd-number limitation theorem also in the case of a time-varying delay.

As a demonstration of the VDFC in the fractional-order Rössler system, in panels (a)–(c) of figure 5 we show the dynamics of the state variables $x(t)$, $y(t)$ and $z(t)$ for $K = 3$ and $T_0 = 7$, fixing the modulation amplitude $\varepsilon = 2$ and frequency $\omega = 10$. The time series indicate a successful stabilization of the unstable equilibrium $P_2$. The method of control is again noninvasive, as indicated by the vanishing feedback force $F(t)$ in panel (d) of figure 5. We note that for these parameter values, the control via TDFC ($\varepsilon = 0$) is unsuccessful, as can be perceived from the stability domain in the TDFC case depicted in figure 2.

4. Delayed feedback control of unstable periodic orbits

The Pyragas-delayed feedback control method was originally aimed to stabilize unstable periodic orbits embedded into the chaotic attractor of the free-running system [22]. For this purpose, the time delay in the feedback loop was chosen to coincide with the period of the target orbit. By tuning the feedback gain to an appropriate value, the stabilization is achieved and the controller perturbation vanishes, leaving the target orbit and its period unaltered.

In this section, we will give a brief demonstration of the Pyragas method to control unstable periodic orbits in the fractional-order Rössler system (23) and (24). As in the previous discussion, we use $a = 0.4$, $b = 0.2$, $c = 10$ and $\alpha = 0.9$, for which the system is chaotic in the absence of external perturbation.

In order to estimate the periods of the unstable orbits which are typically not known a priori, we use the fact that the signal difference $\mathcal{F}(t) = y(t - \tau) - y(t)$ at a successful control asymptotically tends to zero if the delay $\tau$ of the controller is adjusted to match the period
Figure 5. A simulation of the VDFC in the chaotic fractional-order Rössler system ($a = 0.4$, $b = 0.2$, $c = 10$, $\alpha = 0.9$). The delay modulation is in a form of a sine-wave with amplitude $\epsilon = 2$ and frequency $\omega = 10$. The time series of the variables $x(t)$, $y(t)$ and $z(t)$ depicted in panels (a)–(c) indicate a successful control of the unstable equilibrium $P_2$. The vanishing feedback force $F(t)$ in panel (d) shows the noninvasiveness of the control method. The control parameters were: $K = 3$, $T_0 = 7$. The control was activated at $t = 70$. The total time span shown in each panel is 200 time units.

$T$ of the target orbit. The method consists of calculating the dispersion $\langle F^2 \rangle$ of the control signal at a fixed value of the feedback gain $K$ for a given range of values of the delay $\tau$, excluding the transient period [22, 57]. The resulting logarithmic plots of the dependence of the dispersion $\langle F^2 \rangle$ on the delay $\tau$ may contain several segments of finite $\tau$-intervals for which $\langle F^2 \rangle$ is practically zero, and a sequence of isolated resonance peaks with a very deep minima. The former correspond to the stability domain of the fixed point $P_2$, and the latter are the points at which $\tau$ coincides with some accuracy to the periods of the unstable periodic orbits in the original system. The estimated values of the periods $T$ can be made more accurate if one repeats this ‘spectroscopy’ procedure for a larger sampling resolution of the $\tau$ interval encompassing the resonance peaks. In this way, we have obtained the periods of the unstable period-one, period-two and period-three orbits: $T_1 \approx 6.2$, $T_2 \approx 12.49$ and $T_3 \approx 18.89$. The plot of the dispersion $\langle F^2 \rangle$ versus the delay $\tau$ for $K = 0.2$ is shown in panel (a) of figure 6.

The same approach could be used to calculate the intervals of the feedback gain $K$ for which the corresponding orbits can be stabilized with the Pyragas controller. In panel (b) of figure 6 we depict the dependence of the dispersion $\langle F^2 \rangle$ on the feedback gain $K$ when the delay time $\tau$ coincides with the period of the first unstable periodic orbit $\tau = T_1 = 6.2$. In this case, the interval of the parameter $K$ for which the orbit is stabilized is estimated to be $K = [0.14, 0.65]$. A similar analysis yields $K = [0.1, 0.22]$ for a period-two, and
Figure 6. Example of the ‘spectroscopic’ procedure to determine the periods of unstable periodic orbits and their control domains under TDFC in the fractional-order Rössler system ($a = 0.4$, $b = 0.2$, $c = 10$, $\alpha = 0.9$). (a) The dependence of the dispersion of the control signal $\mathcal{F}(t) = y(t - \tau) - y(t)$ upon the time delay $\tau$ for $K = 0.2$ reveals segments of finite length related to a fixed point stabilization (compare with the $T$-intervals in figure 2 at $K = 0.2$), and a resonance peaks at those values of $\tau$ corresponding to the periods of unstable periodic orbits. The period-three resonance point becomes more pronounced for smaller values of $K$. (b) Calculation of the stability domain for the period-one orbit. Note the peak at $K \approx 0.195$, for which the control of the orbit is the most robust.

$K = [0.06, 0.15]$ for a period-three orbit. It is observed that the control interval of the feedback gain $K$ becomes narrower as the period of the target orbit is increased.

In figure 7 we show the results of the stabilization of period-one orbit ($T = 6.2$) for $K = 0.25$. Panels (a) and (b) show the projection of the system trajectory in $xy$ and $xz$ planes, respectively, after the control of the target period-one orbit has been established. The time-series of the state variables are given in panels (c)–(e), and panel (f) shows the feedback force that vanish after the controller is switched-on, warranting a noninvasiveness of the control procedure. Analogous results related to the stabilization of period-two and period-three orbits are given in figures 8 and 9.

A detailed bifurcation analysis of the chaotic Rössler system described by ordinary differential equations and subjected to a TDFC has been performed recently [57], revealing multistability and a large variety of different attractors that are not present in the free-running system. A similar analysis in the case of fractional-order chaotic systems is left for future studies.

5. Summary and conclusions

We have shown that the TDFC can be used to stabilize unstable steady states and unstable periodic orbits in fractional-order chaotic systems. Although the control method was illustrated
specifically for the fractional-order Rössler system, it has also successfully been applied to stabilize unstable equilibria and unstable periodic orbits in various other fractional-order dynamical systems. In all the cases, delayed feedback control with a variable time-delay significantly enlarges the stability region of the steady states in comparison to the classical Pyragas TDFC scheme with a constant delay.

We find that equilibrium points that have an odd number of positive real eigenvalues cannot be stabilized by TDFC for any values of the feedback control parameters. The result is known as the odd-number limitation theorem, which extends to the case of fractional-order systems, as purported by theorem 2. The odd-number limitation is also confirmed numerically in the case of a VDFC.
Figure 8. Results of stabilization of period-two orbit in the chaotic fractional-order Rössler system. The control parameters are $K = 0.12$ and $T = 12.49$. The parameters of the unperturbed system are as in figure 7. The control was activated at $t = 50$. The total time span shown in each panel is 250 time units. Note that one of the period-two peaks in panel (e) is barely visible.

An analytical treatment of delayed feedback control of unstable periodic orbits in fractional-order systems is still lacking, and constitutes a promising subject for a future research. Applying the extended versions of the delayed feedback controller [37, 39] to fractional-order systems is another interesting topic not tackled in this paper. This is especially important regarding the observations for the system used in this paper that the control domains are becoming smaller for higher orbits, such that the periodic orbits of periods higher than three practically cannot be stabilized by the original controller.

A detailed analysis of the VDFC in fractional-order systems including a theoretical understanding of the method and numerical computation of the stability domains in different parameter planes and for different types of delay modulations are also left for future studies.
Figure 9. Results of stabilization of period-three orbit in the chaotic fractional-order Rössler system. The control parameters are $K = 0.08$ and $T = 18.89$. The parameters of the unperturbed system are as in figure 7. The control was activated at $t = 50$. The total time span shown in each panel is 300 time units. Note that one of the period-three peaks in panel (e) is too small to be seen on the scale of this figure.

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