Nonlinearity in Quantum Theory and Closed Timelike Curves

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Abstract

We examine consequences of the density matrix approach to quantum theory in the context of a model spacetime containing closed timelike curves. We find that in general, an initially pure state will evolve in a nonlinear way to a mixed quantum state. CPT invariance and the implications of this nonlinearity for the statistical interpretation of quantum theory are discussed.
1. Introduction

Recently there has been a lot of interest in the question of whether one can formulate a well defined quantum field theory on spacetimes containing closed timelike curves (CTCs) and if so, how the theory will behave. Studies of a class of spacetimes which have well behaved initial and final chronal regions and a compact region of CTCs in the middle have yielded some interesting results. If an S matrix is defined relating the initial and final quantum states in the asymptotic regions, then it was found [1] that free field scattering is unitary but for the case of interacting fields, the Feynman propagator fails to satisfy the identities (analogous to the flat space Cutkosky rules) which would establish perturbative unitarity of the S matrix. This paper is concerned with the various attempts that have been made to try and recover a Copenhagen interpretation, consistent with observations made both before and after the region of CTCs, for interacting fields on these spacetimes.

In the context of generalised quantum mechanics, Hartle [2] suggests that a consistent probabilistic interpretation can be recovered by normalising the amplitudes with a factor which depends on the initial state of the system. However, in doing so, he introduces a fundamental nonlinearity into quantum theory. The path integral approach will be reviewed briefly in Section 2.

Deutsch [4] looked at the problem from the point of view of quantum computation by considering the world lines of a finite number of particles, whose quantum states are described by density matrices. In his approach, a density matrix corresponding to a pure initial state can evolve into a density matrix corresponding to a mixed quantum state. This shows that the evolution cannot be given by $SS^\dagger$, where $S$ is a unitary matrix. In Section 3, we compare the evolution of density matrices (given by Deutsch’s rules) in a spacetime containing a compact region of CTCs with the standard density matrix analysis in a black hole geometry. For the CTC spacetime, one might guess that the evolution would be described by a linear superscattering operator, $\$, as in the black hole case but one of the results of this paper is to show that the density matrix evolves nonlinearly under
the rules that Deutsch proposes. This nonlinearity is a much worse feature than loss of quantum coherence and calls into question the statistical interpretation of quantum theory.

In a recent paper, Politzer [5] applied Deutsch’s general method to a specific example, which has received much attention due to its simplicity. In this spacetime, two flat, space-like, 3 dimensional disks of radius \( r \) are located in Minkowski space at \( t = 0 \) and \( t = T \), with the same spatial coordinates. To introduce CTCs, one simply identifies the upper surface of the bottom disk with the lower surface of the top disk. Also, the lower surface of the bottom disk is identified with the upper surface of the top disk, in order to avoid free surfaces. In his analysis, Politzer considers the world lines of only two interacting fermions, one inside and one outside the radius \( r \). This model spacetime is reviewed in Section 4.

In Section 5, we examine the evolution of density matrices in the Deutsch/Politzer example and find that, when the initial density matrix is pure, the evolution is nonlinear and hence superposition is lost. CPT invariance of the time machine and the implications of this nonlinearity for the evolution of initially mixed states are discussed in Section 6.

2. The Sum over Histories Approach

This section presents a review of Hartle’s approach [2] to spacetimes containing CTCs. In generalised quantum mechanics, developed for closed systems by Gell–Mann and Hartle [see, for example, 3], there is still a time ordering of field operators so in the case of spacetimes containing CTCs, Hartle tries to circumvent this problem by use of the sum over histories formalism.

As in Feynman’s sum over histories prescription, one considers amplitudes for particular 4 dimensional field configurations \( \phi(x) \), given by:

\[
\text{Amplitude} \propto e^{iS[\phi(x)]/\hbar},
\]

where \( S \) is the action functional. Sets of these configurations represent coarse grained histories and a decoherence functional \( D(\alpha; \alpha') \) measures the amount of quantum interference.
between two histories \( \alpha, \alpha' \in A \), where \( A \) is an exhaustive set of histories for the closed system.

If \( D(\alpha; \alpha') \) is negligibly small for all pairs of different \( \alpha, \alpha' \in A \), then the set is said to decohere and probabilities can be assigned to the individual histories, given by \( p(\alpha) = D(\alpha; \alpha) \). However, if \( D(\alpha; \alpha') \neq 0 \) for any \( \alpha, \alpha' \in A \), then that means there is nonnegligible quantum interference between \( \alpha \) and \( \alpha' \), so there is no probabilistic interpretation for that particular set of coarse grained histories.

Familiar Hamiltonian quantum mechanics can be cast in this form. The dynamics is described by a characteristic decoherence functional, \( D(\alpha; \alpha') \) and all the fundamental principles of quantum mechanics (e.g. superposition, hermiticity, positivity) can be related to certain properties of \( D(\alpha; \alpha') \). Other decoherence functionals are acceptable in this framework as long as these basic properties are respected. Therefore, by suitably changing the form of \( D(\alpha; \alpha') \), one can generalise Hamiltonian quantum mechanics.

This is exactly what Hartle does in a recent paper [2] to try and recover a reasonable probabilistic interpretation of quantum theory in the presence of nonunitary evolution. He considers a single scalar field in a general spacetime containing a compact region of CTCs. The chronal regions can still be foliated with spacelike hypersurfaces and on each surface, a Hilbert space of states can be defined. Transition amplitudes between field configurations on spacelike surfaces before and after the nonchronal region are given by a typical path integral over all 4–dimensional field configurations which match the initial and final conditions. Therefore, an S matrix defining scattering through the region of CTCs can be defined and calculated. However, the fact that the spacetime is not globally hyperbolic implies that the S matrix is nonunitary (the propagator no longer has the form of half advanced minus retarded Green functions and hence does not satisfy the unitarity identities [1]).

Hartle’s proposal for a consistent probabilistic interpretation is to introduce a decoherence functional which reduces to that of ordinary Hamiltonian quantum mechanics.
when there are no CTCs, but provides a generalised theory otherwise. This is given by:

\[ D(\beta', \alpha'; \beta, \alpha) = \frac{\text{tr}(C_{\beta'}XC_{\alpha'}\rho C_{\alpha}^\dagger X^\dagger C_{\beta}^\dagger)}{\text{tr}(X\rho X^\dagger)}. \]

In the above equation, \( X \) is an operator describing the nonunitary evolution through the nonchronal region and \( \rho \) is the density matrix for the initial state of the system.

The \( C_{\alpha,\beta} \) represent chains of unitary evolution operators and projection operators in the choral regions before/after the region of CTCs. The initial state evolves unitarily between the spacelike surfaces and the projections are on to relevant subspaces of the Hilbert spaces defined on these surfaces. This functional does lead to consistent assignment of probabilities but unfortunately the normalisation factor in the denominator renders the theory nonlinear in the initial density matrix \( \rho \).

If one takes a quantum cosmological point of view, where the unique initial state might be given by the Hartle-Hawking no boundary state, for example, then this is not such a problem but even so it is still an undesirable feature of the theory. For example, if an advanced civilisation were to create a wormhole sometime in the future, one can see (in principle at least) that evolved information from nonlinear effects detected now might prevent the civilisation from ever building the wormhole in the first place!

Anderson [6] also proposed a rule for quantum evolution in the presence of CTCs. The essence of this proposal involves a ‘renormalisation’ of the measure density for the Hilbert space of states defined after the acausal region. This simple idea provides a quantum mechanics which is causal (in the sense that probabilities in the present are not affected by achronal regions of spacetime in the future) and also linear in the initial density matrix.

However, Fewster and Wells [7] found a problem with this procedure on a physical level. In normal quantum mechanics, we think of a physical observable as an operator which is self-adjoint with respect to the inner product of the Hilbert space. In the context of Anderson’s proposal, one would expect that observables must be represented by operators which are self-adjoint with respect to both inner products. However, it is shown that as
a result of this requirement, allowed operators must commute with the nonunitary part of the evolution, and their expectation values will only evolve according to the unitary part. This is a severe restriction: for example, one can argue that momentum is unlikely to be in the class of allowed observables in this proposal.

Fewster and Wells suggest an alternative way to restore unitarity based on the theory of unitary dilations. They replace the nonunitary evolution operator \( X \) by a unitary dilation of \( X \), which maps between enlarged (and possibly indefinite) inner product spaces. In essence, the extra dimensions which are introduced provide a space of states for the particles in the CTC region which is inaccessible to outside observers and so overall, global unitarity is maintained. However, when observers reduce the state on the enlarged Hilbert space to the part they can measure, there will be loss of quantum coherence.

3. Mappings between tensor product Hilbert spaces

Consider a quantum system composed of two subsystems labelled 1 and 2. If there are no interactions between the subsystems, the total Hilbert space of states can be written as a tensor product:

\[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \]

In this paper, we shall be interested in mappings between Hilbert spaces of this form. If an observer can only measure one part of the system, then all possibilities for the second subsystem must be summed over and in general, the resulting state relevant for the observer will be mixed.

The natural context in which to describe mixed states is given by the density matrix formalism. A density matrix \( \rho \) describing a quantum system can be written as:

\[ \rho = \rho^A_B |A\rangle \langle B|, \]

where summation over \( A \) and \( B \) is implied, \( \{|A\rangle\} \) is a basis for \( \mathcal{H} \) and \( \langle B| \) is a basis for the complex conjugate Hilbert space \( \overline{\mathcal{H}} \). The \( \rho^A_B \) can be thought of as the components of
a \((\frac{1}{2})\) tensor on Hilbert space or alternatively as a positive semi-definite matrix with unit trace. In a basis in which \(\rho^A_B\) is diagonal, we can write:

\[
\rho^A_B = \delta^A_B P_A \quad \text{(no implicit summation)}
\]

where \(P_A\) is the probability for the system to be in the pure state \(|A\rangle\). If any \(P_A = 1\) with all the others zero, then the system is said to be in a pure state. Otherwise, \(\rho\) describes a mixed state (i.e. a sum of pure states weighted with probabilities \(P_A\) such that \(\sum_A P_A = 1\)).

In the usual situation where a system is described by density matrices \(\rho_-\) and \(\rho_+\) in asymptotic past and future regions, the superscattering operator \(\$ : \rho_- \rightarrow \rho_+\) provides a unique linear mapping

\[
\rho_+^A_B = \$^{A}_{BC}D \rho_-^{-C_D}
\]

between density matrices. \(\$^{A}_{BC}D\) is Hermitian in each of its pairs of indices (AB,CD), probability is conserved, i.e.

\[
\$^A_{AC}D = \delta^D_C
\]

and when there is no loss of quantum coherence (i.e. a pure state evolves to a pure state), it can be factorised into a product of a standard unitary \(S\) operator and its adjoint,

\[
\$^{A}_{BC}D = S^{A}_{C}S^{D}_{B}
\]

One can see how, in this approach, nonunitarity of the \(S\) matrix is reinterpreted by saying that the system loses quantum coherence in the evolution and transition probabilities are given by the superscattering operator, \(\$\), instead of \(|S|^2\). A more detailed analysis of \(\$\) in the context of the Euclidean approach to quantum gravity is given in [8]. We note here that one can really only use \(\$\) if the evolution of density matrices is linear. In this paper, nonlinear evolution operators will be denoted by \(\Omega : \rho_- \rightarrow \rho_+\).

Locally, density matrices evolve according to the Schrödinger equation:

\[
U \rho_- U^\dagger = \rho_+
\]
where $U$ is a standard unitary evolution operator. We can write this equation in a component notation which will be useful for discussing density matrices on tensor product Hilbert spaces:

$$U^{EF}_{AB} (\rho_-)^{AB}_{CD} U^\dagger_{GH} = (\rho_+)^{EF}_{GH}.$$ 

If $\rho_-$ is a density operator on $\mathcal{H}^-$, and the system can be decomposed as $\mathcal{H}^- = \mathcal{H}_1^- \otimes \mathcal{H}_2^-$, then we can write:

$$\rho_- = \rho_1^- \otimes \rho_2^- \quad \text{or} \quad (\rho_-)^{AB}_{CD} = (\rho_1^-)^A_C (\rho_2^-)^B_D.$$ 

Even if an operator cannot be decomposed in this form (like, for example, the operator $U$ above), it is still useful to think of the first (second) index in a pair as referring to subsystem 1 (2). A pair of indices can be thought of as a single index referring to the system as a whole.

As an example, let us consider the case of a black hole spacetime. In this example, we can write both the initial and final Hilbert spaces as tensor products:

$$\mathcal{H}^- = \mathcal{H}_1^- \otimes \mathcal{H}_2^- \quad \text{and} \quad \mathcal{H}^+ = \mathcal{H}_1^+ \otimes \mathcal{H}_2^+.$$ 

After the formation of the black hole, $\mathcal{H}_2^+$ is interpreted as the Hilbert space of states for the particles hidden behind the event horizon. Before the black hole forms, we can still define $\mathcal{H}_2^-$, but the density operator on this space is assumed to be in the vacuum state, ie $\rho_2^- = |O_2^-\rangle\langle O_2^-|$. The total initial density matrix will be given by

$$\rho_- = |A_1^-\rangle |O_2^-\rangle \langle B_1^-| \langle O_2^-|,$$

and in component form, the total evolution equation is

$$(\rho_+)^{CD}_{EF} = U^{CD}_{AO} (\rho_-)^{AO}_{BO} U^\dagger_{BO}^{BO} U E F.$$
The particle states behind the event horizon are unobservable, so we must sum over all possibilities to obtain the reduced superscattering operator relevant for subsystem 1, the region outside the horizon:

$$S_{EA}^{C} = U_{AO}^{CD} U_{ED}^{BO}.$$  

To emphasize the point made earlier, the mapping provided by this $S_{EA}^{C}$ is only defined for subsystem 1 and the final reduced density operator obtained, $\rho_1^+$, will in general correspond to a mixed state, even though the full density operator describes a pure state on $\mathcal{H}^+$.

We now apply this formalism to the evolution of density matrices on a spacetime containing a compact region of CTCs. Deutsch’s proposed rules for the evolution consist of:

(a) an assumption that if the initial quantum state outside the time machine, $\rho_1^-$, is pure at $t = 0$, then the total quantum state can be written as a tensor product $\rho_1 = \rho_1^- \otimes \rho_2^-$. The final quantum state at $t = T$ is not, in general, a tensor product but we can still obtain the density matrices relevant for the individual subsystems by tracing over the degrees of freedom inside/outside the time machine,

(b) an imposed boundary condition to obtain consistency around the CTC:

$$(\rho_+)_{EH}^{EF} \equiv (\rho_2^+)_{F H}^{AB} U_{CD}^{AB} U_{EH}^{CD} = (\rho_2^-)_{F H}^{E F}.$$  

We wish to obtain an expression for the operator which maps initial to final density matrices, just as in the black hole case. Generically, the final external state $\rho_1^+$ will correspond to a mixed state and so there is a loss of quantum coherence as in the black hole case. However, the imposed consistency condition creates an essential difference. When this boundary condition is solved, one finds that the components $(\rho_2^-)_{AB}$ are, in general, a nonlinear function of $(\rho_1^-)_{C D}$ so this implies that the evolution operator obtained on the basis of the above rules is nonlinear.
The final state outside the region of CTCs is given by:

\[
(\rho_1^+)_G^E = U_{AB}^{EF} (\rho_-)_C^{AB} U_{CD}^{\dagger GF} (\rho_-)_D^{AB}
\]

Hence, the reduced evolution operator relevant for the state outside the time machine is

\[
\Omega^E_{GA} = U_{AB}^{EF} (\rho_2^-)_C^D U_{CD}^{\dagger GF}.
\]

By inspection, one can see that \(\Omega\) depends on \((\rho_2^-)_B^A\) which implies that the overall scattering will also be nonlinear.

So far, the discussion has been fairly general. In a recent paper [5], Politzer applies Deutsch’s density matrix proposal to a specific spacetime containing CTCs. In his example, pure initial states generically evolve to mixed states. The evolution of initially mixed states is treated by employing the standard statistical interpretation, \textit{i.e.} by decomposing the state into its constituent weighted pure states, evolving each pure state separately and then recombining them in the appropriate way at the end. In Section 5, we focus on a specific example of the nonlinear evolution discussed above in the context of Politzer’s spacetime and consider the implications for the evolution of initially mixed states in Section 6. First of all, though, we shall briefly review the Deutsch method applied to Politzer’s model spacetime.
4. The Deutsch/Politzer example

The example consists of a fermion with fixed spatial coordinate and world line defined for all time \( t \), labelled subsystem 1. For \( t \) between 0 and capital \( T \), where \( T > 0 \), the fermion can interact with an identical particle at a different fixed spatial location, subsystem 2. For simplicity, we have assumed that system 2 does not exist for \( t > T \) or \( t < 0 \).

In this region of interaction, the Hilbert space for the system is 4-dimensional, a suitable basis of states is given by the \( \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \) basis and the state of the whole system at time \( t \) can be described by a 4x4 density matrix \( \rho(t) \). The great advantage of considering fermions in this example is that all calculations are reduced to 4x4 matrix manipulations. So for example, by taking partial traces, we can obtain the 2x2 matrices describing the state of each individual particle. For the external particle, this is given by \( \rho_1(t) \) or \( \text{tr}_2\rho(t) \), and similarly the CTC particle is described by \( \rho_2(t) \).

Certain matching conditions need to be imposed. The boundary condition on \( \rho_2 \) that brings the CTC into the system is given by

\[
\rho_2(T^-) = \rho_2(0^+) \\
\]

where \( 0^+ \equiv 0 + \epsilon \) and \( T^- \equiv T - \epsilon \) (where \( \epsilon \) is small). This is just rule (b) of Section 3 which enforces consistency of information around the CTC. The other matching conditions are given by

\[
\rho_1(0^-) = \rho_1(0^+) ,
\]

\[
\rho_2(0^-) = \rho_2(T^+) \quad \text{and}
\]

\[
\rho_1(T^+) = \rho_1(T^-) .
\]

Evolution of the density matrices is given by the Schrödinger equation

\[
i\dot{\rho} = [H, \rho] \quad \text{or alternatively} \quad \rho(t) = U(t)\rho(0)U(t)^{-1}.
\]

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The evolution matrix $U(t) = e^{-iHt}$ is unitary, the evolution is linear in $\rho$, and in the above basis, the Hamiltonian can be written in matrix form as:

$$H = \begin{pmatrix}
2\omega_0 + \lambda & \gamma_1 & \gamma_2 & 0 \\
\gamma_1 & \omega_0 & \omega_1 & 0 \\
\gamma_2 & \omega_1 & \omega_0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

where $\omega_0$ is like a mass parameter, $\omega_1$ corresponds to a kinetic parameter, $\gamma_1$ and $\gamma_2$ represent Yukawa 3–particle coupling parameters, and $\lambda$ corresponds to a 4–fermion coupling parameter.

### 5. Evolution of the Density Matrices

We now look at what happens when the external particle evolves towards $t = 0$ and beyond in a pure quantum state, according to Deutsch’s prescription. So we wish to use the boundary condition and the Schrödinger evolution to determine $\rho_1(T^+)$ for a given $\rho_1(0^-)$. According to rule (a) of Section 3, if $\rho_1(0^-)$ is pure (in other words it can be written as $\rho_1(0^-) = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is some quantum state), then at $t = 0$ the most general 4x4 density matrix describing the system can be written in this tensor product form:

$$\rho(0^-) = \rho_1(0^-) \otimes \begin{pmatrix} a & b \\ b^* & (1 - a) \end{pmatrix}.$$

The 2x2 matrix is just $\rho_2(0^+)$ and $a$ and $b$ are undetermined parameters.

The final output will be given by

$$\rho_1(T^+) = \text{tr}_2 \left[ U \left[ \rho_1(0^-) \otimes \begin{pmatrix} a & b \\ b^* & (1 - a) \end{pmatrix} \right] U^\dagger \right]$$

and the parameters $a$ and $b$ are determined by the constraint equations which are just another way of writing the boundary condition:

$$\text{tr}_1 \left[ U \left[ \rho_1(0^-) \otimes \begin{pmatrix} a & b \\ b^* & (1 - a) \end{pmatrix} \right] U^\dagger \right] = \begin{pmatrix} a & b \\ b^* & (1 - a) \end{pmatrix}.$$

In this case when the initial state is pure, the boundary condition provides enough constraints to determine $\rho_1(T^+)$ uniquely.
As Deutsch noted, the boundary condition can be written as

\[ \rho_2(0^+) = \rho_2(0^+), \]

where

\[ * = \text{tr}_1 \left[ U \left[ \rho_1(0^-) \otimes * \right] U^\dagger \right] \]

(* denotes the position of the operand).

In other words, \( \rho_2 \) is a fixed point of \$, where \$ is given by the above and is a linear superscattering operator on the space of density matrices describing subsystem 2. In his paper, he proves that every operator of this form has a fixed point so there will always be a solution to this boundary condition and hence in his words - “...closed timelike lines place no retrospective constraint on the state of a quantum system.”

We now give a specific example to show that the map from initial to final density matrices (for subsystem 1) is in fact nonlinear as described in Section 3. We assume the tensor product form:

\[ \rho(0^+) = \rho_1(0^+) \otimes \rho_2(0^+) \]

where \( \rho_1 \) describes the initial (pure) quantum state of the external particle and \( \rho_2 \) describes the CTC particle. The final 4x4 density matrix, and hence \( \rho_1(T^+) \), are determined by both \( \rho_1 \) and \( \rho_2 \). However, as was discussed in Section 3, \( \rho_2(0^+) \) depends nonlinearly on \( \rho_1(0^+) \) and hence the output \( \rho_1(T^+) \) will also be a nonlinear function of the initial state.

For example, suppose we write

\[ \rho(0^+) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \otimes \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \]

where \( \gamma = 1 - \alpha \), \( c = 1 - a \), \( \alpha \) and \( \beta \) are fixed parameters describing the initial state and \( a \) and \( b \) are initially undetermined. We evolve \( \rho \) using the evolution matrix

\[ U = \begin{pmatrix} 1 & \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
and calculate the following constraint equations:

$$\text{tr}_1 \left[ U \left[ \left( \begin{array}{cc} \alpha & \beta \\ \beta^* & \gamma \end{array} \right) \otimes \left( \begin{array}{cc} a & b \\ b^* & c \end{array} \right) \right] U^\dagger \right] = \left( \begin{array}{cc} a & b \\ b^* & c \end{array} \right).$$

It turns out that the parameters $a$ and $b$ are given by:

$$a = \frac{(\alpha - \alpha \cos \theta + 2 |\beta|^2 \cos \theta)}{(1 - \cos \theta + 4 |\beta|^2 \cos \theta)} \quad \text{and} \quad b = \frac{\beta \sin \theta (2a - 1)}{(1 - \cos \theta)}$$

i.e. there is a nonlinear dependence of $a, b$ on $\alpha$ and $\beta$ so the final state $\rho_1(T^\pm)$ will be a nonlinear function of the initial state and superposition is lost.

6. Conclusions

In this paper, we have examined a proposal for evolution of density matrices (describing initially pure states) in a spacetime containing closed timelike curves. The proposal assumes a tensor product form for the initial quantum state and a boundary condition imposing consistency of information around the CTC. We found that the evolution of the reduced density matrix describing the state of particles outside (but interacting with) the CTC region is described by a nonlinear operator $\Omega$. This result poses serious problems for the statistical interpretation of quantum theory. In the context of Politzer’s example, this is basically the question of what to take as the initial 4x4 density matrix when the external particle is in a mixed quantum state at $t=0$.

Recall that in the case when the external fermion was in a pure state, the single tensor product form was the most general total density matrix such that $\text{tr}_2 \rho = \rho_1$. If $\rho_1$ is mixed, then now the most general total density matrix $\rho$ compatible with this condition will not be of this simple form and it will have a greater number of free parameters. As a consequence, the boundary condition will not provide enough constraint equations to determine the final output uniquely.
One could (and Deutsch does) argue that a product form is still justified because of the lack of correlation between the 2 spins at \( t=0 \). Before \( t=0 \), the particles cannot interact so why should they be correlated at \( t=0 \)? However, Politzer noted that from the point of view of particle number 2 on the CTC, it has been interacting with particle 1 in its past, even if those interactions were not in the other particle’s past. Correlations between the 2 systems need to be taken into consideration.

So how could we perform the evolution if we did not assume this simple product form? The standard interpretation of a mixed density matrix \( \rho \) is to decompose it as a statistical ensemble of pure states weighted with probabilities ie \( \rho = \sum c_i \rho_i \), where \( \rho_i \) are pure density matrices and the probabilities \( c_i \) obey \( \sum c_i = 1, 0 < c_i < 1 \). So if this interpretation were valid, the evolved mixed state could be unambiguously determined by summing the results for the evolved pure states which have been weighted by the initial probabilities \( c_i \). This interpretation cannot be used here because the initial to final map is nonlinear. Politzer [5] proposed a dynamics based on a similar decomposition which retains correlations between the particles at \( t = 0 \). One of his results is that a pure state evolves to a mixed state even for external particles which do not interact with the CTC region. In this example it is difficult to see where quantum coherence is being lost as there is no interaction to carry the information. If there was a closed timelike curve on the other side of the galaxy, it shouldn’t affect what happens here. The explanation for this counterintuitive result is precisely what we have found here - the nonlinear evolution ruins any sort of statistical interpretation.

So where does the proposal go wrong? Solving the boundary condition creates the nonlinear dependence but all the condition really enforces is consistency of information around the CTC, which seems reasonable. It seems more likely that the assumption of a tensor product form as the initial state is where the problem lies. The tensor product rules out the possibility of correlations between the 2 systems, but it seems likely that correlations are indeed present. A more realistic proposal may be given based on, for
example, the direct sum [7] of Hilbert spaces, in which the systems are not assumed independent.

Finally, we consider whether the Deutsch/Politzer time machine is time symmetric (or CPT invariant). If strong CPT was obeyed, then one would expect the final total density matrix $\rho(T^-)$ to be a tensor product when $\rho_2$ is the fixed point of $$. However, this is not the case as is readily verified in simple examples.

Weak CPT (or detailed balance) does seem to be obeyed – that is, the probability of transition from A to B is the same as the probability of evolving from the CPT reverse of B to the CPT reverse of A. This can be seen in the context of the 2 spin example quite easily. Under the CPT operation, a spin up (down) particle becomes a spin up (down) antiparticle. The inner product used to calculate transition amplitudes for density matrices is essentially just the trace of the dot product of the matrices so weak CPT follows from the cyclic property of the trace.

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