Confidence Decision Trees via Online and Active Learning for Streaming (BIG) Data

Rocco De Rosa
derosa@dis.uniroma1.it

Abstract—Decision tree classifiers are a widely used tool in data stream mining. The use of confidence intervals to estimate the gain associated with each split leads to very effective methods, like the popular Hoeffding tree algorithm. From a statistical viewpoint, the analysis of decision tree classifiers in a streaming setting requires knowing when enough new information has been collected to justify splitting a leaf. Although some of the issues in the statistical analysis of Hoeffding trees have been already clarified, a general and rigorous study of confidence intervals for splitting criteria is missing. We fill this gap by deriving accurate confidence intervals to estimate the splitting gain in decision tree learning with respect to three criteria: entropy, Gini index, and a third index proposed by Kears and Mansour. Our confidence intervals depend in a more detailed way on the tree parameters. We also extend our confidence analysis to a selective sampling setting, in which the decision tree learner adaptively decides which labels to query in the stream. We furnish theoretical guarantee bounding the probability that the classification is non-optimal learning the decision tree via our selective sampling strategy. Experiments on real and synthetic data in a streaming setting show that our trees are indeed more accurate than trees learned incrementally is close to the one learned via standard batch learning. A similar yet stronger guarantee is achieved by the Hoeffding tree algorithm, which is at the core of the state-of-the-art VFDT system. Alternative approaches, such as NIP-H and NIP-N, use Gaussian approximations instead of Hoeffding bounds in order to compute confidence intervals. Several extensions of VFDT have been proposed, also taking into account non-stationary data sources —see, e.g., [10], [9], [2], [35], [27], [13], [19], [21], [11], [34], [20], [29], [8]. All these methods are based on the classical Hoeffding bound [14]:

\[
\epsilon_{\text{Hoeff}}(m, \delta) = R \sqrt{\frac{1}{2m} \ln \frac{1}{\delta}}.
\]

The problem of computing the confidence interval for the split gain estimate can be phrased as follows: we are given a set of unknown numbers \( G \) (i.e., the true gains for the available splits) and want to find the largest of them. We do that by designing a sample-based estimator \( \hat{G} \) of each \( G \), and then use an appropriate version of the Hoeffding bound to control the probability that \( |\hat{G} - G| > \varepsilon \) for any given \( \varepsilon > 0 \). It is easy to see that this allows to pick the best split at any given node: assume that \( \hat{G}_F \) is the highest empirical gain (achieved by the split function \( F \)) and \( \hat{G}_{F_2} \) is the second-best (achieved by the split function \( F_2 \)). If \( \hat{G}_F - \hat{G}_{F_2} > 2\varepsilon \) then with probability at least \( 1 - \delta \) the split function \( F \) is optimal —see Figure 1

![Confident Split](image)

Fig. 1. The condition \( \hat{G}_F - \hat{G}_{F_2} > 2\varepsilon \) guarantees that the confidence intervals for the true gains \( G_F \) and \( G_{F_2} \) are non-overlapping.

Hoeffding bound [14] to compute the confidence intervals for

\[\text{papers build on the idea of [24], which advocates the use of measures to evaluate the confidence in choosing a split. These works include Sequential ID3 [13], VFDT [7], NIP-H and NIP-N [17]. Sequential ID3 uses a sequential probability ratio test in order to minimize the number of examples sufficient to choose a good split. This approach guarantees that the tree learned incrementally is close to the one learned via standard batch learning. A similar yet stronger guarantee is achieved by the Hoeffding tree algorithm, which is at the core of the state-of-the-art VFDT system. Alternative approaches, such as NIP-H e NIP-N, use Gaussian approximations instead of Hoeffding bounds in order to compute confidence intervals. Several extensions of VFDT have been proposed, also taking into account non-stationary data sources —see, e.g., [10], [9], [2], [35], [27], [13], [19], [21], [11], [34], [20], [29], [8]. All these methods are based on the classical Hoeffding bound [14]: after \( m \) independent observations of a random variable taking values in a real interval of size \( R \), with probability at least \( 1 - \delta \) the true mean does not differ from the sample mean by more than\]
The work [22] directly uses the classification error as splitting the authors do not consider other types of split functions. However, this still ignores the bias of the estimate and, besides, components, and apply the Hoeffding bound to each one of them, derive the entropy gain calculation in three components. The goal is to find a function $f(\mathbf{X})$ assigning the correct category $Y \in \{0,1\}$ to a new instance $\mathbf{X}$. We consider binary decision trees based on a class $\mathcal{F}$ of binary split

The problem of computing the confidence interval for the splitting gain estimate has two main source of difficulties: First, splitting criteria —like entropy or Gini index— are nonlinear functions of the distribution at each node. Hence, appropriate large deviation bounds (such as the McDiarmid bound [23]) must be used. As the McDiarmid bound controls the deviations $|\hat{G} - \mathbb{E}G|$, further work is needed to control the bias $|\mathbb{E}\hat{G} - G|$. The first problem was solved (for entropy and Gini) in [28]. The authors used McDiarmid bound to derive estimates of confidence intervals for various split measures. For instance, in a problem with $K$ classes, the bound on the confidence interval for the entropy criterion is

$$
\varepsilon_{md}(m, \delta) = C(K, m) \sqrt{\frac{1}{2m} \ln \frac{1}{\delta}}
$$

where $C(K, m) = 6(K \log_2 e + \log_2 2m) + 2 \log_2 K$. The authors proposed to replace Hoeffding bound (1) by McDiarmid bound (2) in the VFDT algorithm and its successors. However, although this allows to control the deviations, the bias of the estimate is ignored. More recently, in [8] the same authors apply the Hoeffding bound to the entropy splitting criterion, focusing on binary trees and binary classification. They decompose the entropy gain calculation in three components, and apply the Hoeffding bound to each one of them, obtaining a confidence interval estimate for the splitting gains. However, this still ignores the bias of the estimate and, besides, the authors do not consider other types of split functions. The work [22] directly uses the classification error as splitting criterion rather than a concave approximation of it (like the entropy or the Gini index). Though this splitting criterion can be easily analyzed via the Hoeffding bound, its empirical performance is generally not very good —see Section [11] for more discussion on this.

In this work, we significantly simplify the approach of [28] and extend it to a third splitting criterion. Moreover, we also solve the bias problem, controlling the deviations of $\hat{G}$ from the real quantity of interest (i.e., $G$ rather than $\mathbb{E}\hat{G}$). Moreover, unlike [22] and [8], our bounds apply to the standard splitting criteria. Our analysis shows that the confidence intervals associated with the choice of a suboptimal split not only depend on the number of leaf examples $m$ —as in bounds (1) and (2)— but also on other problem dependent parameters, as the dimension of the feature space, the depth of the leaves, and the overall number of examples seen so far by the algorithm. As revealed by the experiments in Section VII-A this allows a more cautious and accurate splitting in complex problems. Furthermore, we point out that our technique can be easily applied to all extensions of VFDT (see Section I) yielding similar improvements, as these extensions all share the same Hoeffding-based confidence analysis as the Hoeffding tree algorithm.

Standard decision tree learning approaches assume that all training instances are labeled and available beforehand. In a true incremental learning setting, instead, in which the classifier is asked to predict the label of each incoming sample, active learning techniques allows us to model the interaction between the learning system and the labeler agent (typically, a human annotator) [30]. More specifically, such techniques help the learner select a small number of instances for which the annotator should be invoked in order to obtain the true label. The overall goal is to maximize predictive accuracy (measured on the entire set of predicted samples, irrespective to whether the label was queried or not) at any given percentage of queried labels. Recently, in the work [36] is showed a general active learning framework which is applied to Hoeffding trees. They present different strategies to annotate the samples considering the output leaf class probabilities. These techniques rely on the class probability estimates at the leaves level without considering confidence-based techniques, that is they do not consider if the estimates are supported by a small or large number of sample labels. In Section VI we develop a selective sampling version of our algorithm using the new confidence bounds. The approach we propose is based on using the purity of a leaf to decide (at some confidence level) whether its classification is optimal. In Section VIII we compare our labeling strategy with a recent baseline. Section IX concludes the paper.

II. RELATED WORK

The rest of the paper is organized as follows. Section [11] discusses related work. In Section [11] we state the basic decision tree learning concepts and introduce the notation used in the rest of the paper. In Section [11] we derive the new bounds for the splitting criteria. In Section [11] we apply the confidence bounds to the incremental learning of a decision tree and derive a formal guarantee (Theorem [1]) on the probability that examples in the stream are classified using suboptimal splits based on any of the three splitting criteria. These theoretical guidelines are empirically tested in Section VII where we show that our more refined bounds deliver better splits that the splits performed by the other techniques. In Section VI we develop a selective sampling version of our algorithm using the new confidence bounds. In this setting, the learner has the possibility of adaptively subsampling the labels of examples in the stream. In other words, everytime a new example arrives, the learner may decide to save the cost of obtaining the label. Note that the learner’s predictive performance is nevertheless evaluated on the entire stream, including the examples whose label remains unknown. The approach we propose is based on using the purity of a leaf to decide (at some confidence level) whether its classification is optimal. In Section VIII we compare our labeling strategy with a recent baseline. Section IX concludes the paper.
functions $F : \mathbb{R}^d \rightarrow \{0, 1\}$, that is, the test functions through which the feature space is partitioned\[^4\]. Training examples $(X_1, Y_1), (X_2, Y_2), \ldots \in \mathbb{R}^d \times \{0, 1\}$ are i.i.d. draws from a fixed but unknown probability distribution. Decision tree classifiers are typically constructed in an incremental way, starting from a tree consisting of a single node. The tree grows through a sequence of splitting operations applied to its leaves. If a split is decided for a leaf $i$, then the leaf is assigned some split function $F \in \mathcal{F}$ and two nodes $i_0$ and $i_1$ are added to the tree as children of the split node. Examples are recursively routed through the tree starting from the root as follows: when an example $(X_i, Y_i)$ reaches an internal node $i$ with split function $F$, then it is routed to child $i_0$ if $F(X_i) = 0$ and to child $i_1$ otherwise. A decision tree $T$ induces a classifier $f_T : \mathbb{R}^d \rightarrow \{0, 1\}$. The prediction $f_T(X)$ of this classifier on an example $X$ is computed by routing the example $X$ through the tree until a leaf is reached. We use $X \rightarrow i$ to indicate that $X$ is routed to the leaf $i$. Then $f_T(X)$ is set to the most frequent label $y = \arg \max_y P(Y = y | X \rightarrow i)$ among the labels of all observed examples that reach that leaf. The goal of the learning process is to control the binary classification risk $P(f_T(X) \neq Y)$ of $f_T$. For any leaf $i$, let $Y_i = P(Y | X \rightarrow i)$ be the random variable denoting the label of a random instance $X$ given that $X \rightarrow i$. Let $\mathcal{L}(T)$ be the leaves of $T$. The risk of $f_T$ can then be upper bounded, with the standard bias-variance decomposition, as follows

$$
P(f_T(X) \neq Y) \leq \sum_{i \in \mathcal{L}(T)} P(Y \neq y^*_i | X \rightarrow i) P(X \rightarrow i)$$

$$+ \sum_{i \in \mathcal{L}(T)} P(f_T(X) \neq y^*_i | X \rightarrow i) P(X \rightarrow i)$$

where $y^*_i = I\{P(Y = 1 | X \rightarrow i) \geq \frac{1}{2}\}$ is the optimal label for leaf $i$ and $I\{}$ is the indicator function of the event at argument. The variance terms are the easiest to control: $f_T(X)$ is determined by the most frequent label of the leaf $i$ such that $X \rightarrow i$. Hence, conditioned on $X \rightarrow i$, the event $f_T(X) = y^*_i$ holds with high probability whenever the confidence interval for the estimate of $y^*_i$ does not cross the $\frac{1}{2}$ boundary.\[^3\] The bias terms compute the Bayes error at each leaf. The error vanishes quickly when good splits for expanding the leaves are available. However, due to the large number of available split functions $F$, the confidence intervals for choosing such good splits shrink slower than the confidence interval associated with the bias error. Our Theorem\[^3\] accurately quantifies the dependence of the split confidence on the various problem parameters. Let $\Psi(Y)$ be a shorthand for $\min\{P(Y = 0), P(Y = 1)\}$. Every time a leaf $i$ is split using $F$, the term $\Psi(Y_i)$ gets replaced by

$$P(F = 0 | X \rightarrow i) \Psi(Y_i | F = 0)$$

$$+ P(F = 1 | X \rightarrow i) \Psi(Y_i | F = 1)$$

corresponding to the newly added leaves (here and in what follows, $F$ also stands for the random variable $F(X)$). The concavity of $\min$ ensures that no split of a leaf can ever make that sum bigger. Of course, we seek the split maximizing the risk decrease (or “gain”),

$$\Psi(Y_i) - P(F = 0 | X \rightarrow i) \Psi(Y_i | F = 0)$$

$$- P(F = 1 | X \rightarrow i) \Psi(Y_i | F = 1).$$

In practice, splits are chosen so to approximately maximize a gain functional defined in terms of a concave and symmetric function $\Phi$, which bounds from the above the min function $\Psi$ (used in\[^2\] as splitting criterion). The curvature of $\Phi$ helps when comparing different splits, as opposed to $\Psi$ which is piecewise linear. Indeed $\Psi$ gives nonzero gain only to splits generating leaves with disagreeing majority labels — see, e.g.,\[^6\] for a more detailed explanation. Let $Z$ be a Bernoulli random variable with parameter $p$. Three gain functions used in practice are: the scaled binary entropy $H_1/2(Z) = -\frac{1}{2} \ln p - \frac{1}{2} \ln (1 - p)$ used in C4.5; the Gini index $J(Z) = 2p(1 - p)$ used in CART, and the function $Q(Z) = \sqrt{p(1 - p)}$ introduced by Kearns and Mansour in\[^13\] and empirically tested in\[^6\]. Clearly, the binary classification risk can be upper bounded in terms of any upper bound $\Phi$ on the min function $\Psi$,

$$P(f_T(X) \neq Y) \leq \sum_{i \in \mathcal{L}(T)} \Phi(Y_i) P(X \rightarrow i)$$

$$+ \sum_{i \in \mathcal{L}(T)} P(f_T(X) \neq y^*_i | X \rightarrow i) P(X \rightarrow i).$$

The gain for a split $F$ at node $i$, written in terms of a generic entropy-like function $\Phi$, takes the form

$$G_{i,F} = \Phi(Y_i) - \Phi(Y_i | F)$$

$$= \Phi(Y_i) - P(F = 0 | X \rightarrow i) \Phi(Y_i | F = 0)$$

$$- P(F = 1 | X \rightarrow i) \Phi(Y_i | F = 1).$$

Now, in order to choose splits with a high gain (implying a significant reduction of risk), we must show that $G_{i,F}$ (for the different choices of $\Phi$) can be reliably estimated from the training examples. In this work we focus on estimates for choosing the best split $F$ at any given leaf $i$. Since the term $\Phi(Y_i)$ in $G_{i,F}$ is invariant with respect to this choice, we may just ignore it when estimating the gain\[^6\].

**IV. CONFIDENCE BOUND FOR SPLIT FUNCTIONS**

In this section we compute estimates $\hat{\Phi}_{i,F}$ of $\Phi(Y_i | F)$ for different choices of $\Phi$, and compute confidence intervals for these estimates. As mentioned in Section\[^4\] we actually bound the deviations of $\Phi_{i,F}$ from $\Phi(Y_i | F)$, which is the

\[\text{Note that, for all functions } \Phi \text{ considered in this paper, the problem of estimating } \Phi(Y_i) \text{ can be solved by applying essentially the same techniques as the ones we used to estimate } \Phi(Y_i | F).\]
real quantity of interest here. Due to the nonlinearity of $\Phi$, this problem is generally harder than controlling the deviations of $\hat{\Phi}_{i|F}$ from its expectation $E\hat{\Phi}_{i|F}$—see, e.g., [23] for weaker results along these lines. In the rest of this section, for each node $i$ and split $F$ we write $p_k = P(Y = 1, F = k)$ and $q_k = 1 - p_k$ for $k \in \{0, 1\}$; moreover, we use $\hat{p}_k, \hat{q}_k$ to denote the empirical estimates of $p_k, q_k$.

A. Bound for the entropy

Let $\Phi(Z)$ be the (scaled) binary entropy $H_{1/2}(Z) = -\frac{1}{2} \ln p - \frac{1}{2} \ln (1 - p)$ for $Z$ Bernoulli of parameter $p$. In the next result, we decompose the conditional entropy as a difference between entropies of the joint and the marginal distribution. Then, we apply standard results for plug-in estimates of entropy.

Theorem 1: Pick a node $i$ and route $m$ i.i.d. examples $(X_t, Y_t)$ to $i$. For any $F \in \mathcal{F}$, let

$$\hat{\Phi}_{i|F} = \hat{H}_{1/2}(Y_t | F) - \hat{H}_{1/2}(F)$$

where $\hat{H}_{1/2}$ denotes the empirical scaled entropy (i.e., the scaled entropy of the empirical measure defined by the i.i.d. sample). Then, for all $\delta > 0$,

$$\left| \hat{\Phi}_{i|F} - H_{1/2}(Y_t | F) \right| \leq \varepsilon_{\text{ent}}(m, \delta)$$

where

$$\varepsilon_{\text{ent}}(m, \delta) = (\ln m) \sqrt{\frac{2}{m} \ln \frac{4}{\delta} + \frac{2}{m}}$$

with probability at least $1 - \delta$ over the random draw of the $m$ examples.

Proof: In appendix A

B. Bound for the Gini index

In the Bernoulli case, the Gini index takes the simple form $J(Z) = 2p(1 - p)$ for $Z$ Bernoulli of parameter $p$. First we observe that $J(Y_t | F)$ is the sum of harmonic averages, then we use the McDiarmid inequality to control the variance of the plug-in estimate for these averages.

Theorem 2: Pick a node $i$ and route $m$ i.i.d. examples $(X_t, Y_t)$ to $i$. For any $F \in \mathcal{F}$, let

$$\hat{\Phi}_{i|F} = \text{HM}(\hat{p}_t, \hat{q}_t) + \text{HM}(\hat{p}_0, \hat{q}_0)$$

where $\text{HM}$ denotes the harmonic mean $\text{HM}(p, q) = \frac{pq}{p + q}$. Then, for all $\delta > 0$,

$$\left| \hat{\Phi}_{i|F} - J(Y_t | F) \right| \leq \varepsilon_{\text{Gini}}(m, \delta)$$

where

$$\varepsilon_{\text{Gini}}(m, \delta) = \sqrt{\frac{8}{m} \ln \frac{2}{\delta} + 4 \sqrt{\frac{1}{m}}}$$

with probability at least $1 - \delta$ over the random draw of the $m$ examples.

Proof: In appendix B

C. Bound for the Kernels-Mansour index

The third entropy-like function we analyze is $Q(Z) = \sqrt{p(1 - p)}$ for $Z$ Bernoulli of parameter $p$. The use of this function was motivated in [13] by a theoretical analysis of decision tree learning as a boosting procedure. See also [31] for a simplified analysis and some extensions.

In this case McDiarmid inequality is not applicable and we control $Q(Y_t | F)$ using a direct argument based on classical large deviation results.

Theorem 3: Pick a node $i$ and route $m$ i.i.d. examples $(X_t, Y_t)$ to $i$. For any $F \in \mathcal{F}$, let

$$\hat{\Phi}_{i|F} = \sqrt{\hat{p}_t \hat{q}_t} + \sqrt{\hat{p}_0 \hat{q}_0}$$

Then, for all $\delta > 0$,

$$\left| \hat{\Phi}_{i|F} - Q(Y_t | F) \right| \leq \varepsilon_{\text{KM}}(m, \delta)$$

where

$$\varepsilon_{\text{KM}}(m, \delta) = 4 \sqrt{\frac{1}{m} \ln \frac{8}{\delta}}$$

with probability at least $1 - \delta$ over the random draw of the $m$ examples.

Proof: In appendix C

V. CONFIDENCE DECISION TREE ALGORITHM

A setting in which confidence intervals for splits are extremely useful is online or stream-based learning. In this setting, examples are received incrementally, and a confidence interval can be used to decide how much data should be collected at a certain leaf before a good split $F$ can be safely identified. A well-known example of this approach are the so-called Hoeffding trees [7]. In this section, we show how our confidence interval analysis can be used to extend and refine the current approaches to stream-based decision tree learning.

For $t = 1, 2, \ldots$ we assume the training example $(X_t, Y_t)$ is received at time $t$. C-Tree (Algorithm 1) describes the online decision tree learning approach. A stream of examples is fed to the algorithm, which initially uses a 1-node decision tree. At time $t$, example $(X_t, Y_t)$ is routed to a leaf $\ell_t$. If the leaf is not pure (both positive and negative examples have been

Algorithm 1 C-Tree

Input: Threshold $\tau > 0$
1: Build a 1-node tree $T$
2: for $t = 1, 2, \ldots$ do
3: Route example $(X_t, Y_t)$ through $T$ until a leaf $\ell_t$ is reached
4: if $\ell_t$ is not pure then
5: Let $\hat{F} = \arg\max_{F \in \mathcal{F}} \hat{\Phi}_{\ell_t, F}$ and $F_2 = \arg\max_{F \in \mathcal{F}, F \neq \hat{F}} \hat{\Phi}_{\ell_t, F}$
6: if $\hat{\Phi}_{\ell_t, \hat{F}} - \hat{\Phi}_{\ell_t, F_2} < 2\varepsilon_t$ or $\varepsilon_t \leq \tau$ then
7: Let $F_t = \hat{F}$ and expand $\ell_t$ using split $F_t$
8: end if
9: end if
10: end for
Figure 2– An example of not \( \delta \)-consistent leaf. The class probability confidence interval overlaps \( 0.5 \). In this case we are not sure, at the desired confidence level, if the prediction made by the leaf is the same of that of the Bayesian Optimal classifier in the correspondent sub-region.

Definition 1: Recall that \( \ell_t \) is the leaf to which example \( (X_i, Y_i) \) is routed, and that \( m_{\ell_t,t} \) is the number of queried data points routed to leaf \( \ell_t \) in the first \( t-1 \) time steps. Let \( \tilde{Y}_{\ell_t,t} \) be the fraction of positive examples among those points, so that \( f_T(X_i) = \mathbb{I}\{\tilde{Y}_{\ell_t,t} \geq \frac{1}{2}\} \). We say that \( \ell_t \) is \( \delta \)-consistent if

\[
\frac{\tilde{Y}_{\ell_t,t} - \frac{1}{2}}{\delta} > \varepsilon_{tc}(m_{\ell_t,t}, t, \delta)
\]

where \( \varepsilon_{tc}(m, t, \delta) = \sqrt{\frac{1}{2m} \ln \frac{2t}{\delta}} \).

Let \( p_i = \mathbb{P}(Y_i = 1) = \mathbb{P}(Y = 1 \mid X \rightarrow i) \). If \( \ell_t \) is \( \delta \)-consistent but \( f_T(X_i) \neq y_i^* \), then it must be that \( |\tilde{Y}_{\ell_t,t} - p_{\ell_t}| \geq \sqrt{\frac{1}{2m_{\ell_t,t}} \ln \frac{2}{\delta}} \). Hence, when a leaf becomes \( \delta \)-consistent we are confident that its classification is optimal at a certain confidence level—see Figure 2. On the contrary, when the leaf is not \( \delta \)-consistent we need to let the leaf seeing more data to reach the consistency. Even in the case of consistency, the leaves need further data in order to discover possible new good splits. In this direction our active method asks a small portion of the input data in order to let exploration even if the leaf is consistent. In this case, as we will describe later with more details, we set the query rate depending on some statistics observed at the leaf level. In particular, we require more exploration for weakly consistent leaf (i.e., a leaf supported by a small portion of data and not pure class distribution) respect to the very confident ones (i.e., a leaf supported by a huge amount of data and with pure class distribution). On the other hand, when the leaf is not consistent, all the labels have to be requested in order to reach the consistency.

As labels are generally obtained via queries to human annotators, any practical active learning system for streaming settings should impose a bound on the query rate. The active framework we propose is taken from [36]—see Algorithm 3 (ACTIVE setting). Whenever a new sample is presented to the model, the system makes a prediction and then invokes the active learning module in order to determine whether the label should be requested. If this is the case, then a query is issued to the annotator unless the query rate budget is violated. When the label is not requested, the model is not updated. Our labeling strategy is described in Algorithm 2. In summary, if the incoming sample falls into a not \( \delta \)-consistent leaf—see Figure 2—the annotation is requested. Contrary, on \( \delta \)-consistent
leaf, the annotations are moderated with random sampling to guarantee exploration, that is a controlled growth of the tree. In this case the sampling probability, which gives the priority on the labeling requests, depends on the budget rate $B$ (more budget more probability), the leaf confidence error $\varepsilon_{le}$ (less samples support the probability estimates more priority) and the class distribution purity $|Y_{t,t} - \frac{1}{2}|$ (distribution toward uniformity more priority).

**Theorem 5:** The probability that the classification of a $\delta$-consistent leaf is non-optimal is at most $\delta$:

$$P(f_T(X_t) \neq y_t^*, \ell_t \text{ is } \delta\text{-consistent}) \leq \delta.$$ 

**Proof:** In appendix E. Similarly to Theorem 4, choosing $\delta = \frac{1}{2}$ and applying the union bound allows to conclude that at most a logarithmic number of examples in the stream are misclassified by $\delta$-consistent leaves. We use this setting in all the experiments. In the work [36] are presented different labeling strategies respect to Algorithm 2 These techniques confide only in the leaf class distribution $Y_{t}$ and do not take into account the confidence information. The resulting performances are less robust respect that ones achieved by our approaches. This is empirically verified in the experiments showed in Section VIII.

VII. FULL SAMPLING EXPERIMENTS

We ran experiments on synthetic datasets and popular benchmarks, comparing our C-Tree (Algorithm 1) against two baselines: H-Tree (VDFT algorithm [27]) and CorrH-Tree (the method from [8] using the classification error as splitting criterion). The bounds of [28] are not considered because of their conservativeness. In fact, these bounds generate 1-node trees in all the experiments, even when the confidence is set to a very low value.

The three methods (ours and the two baselines) share the same core, i.e., the HoeffdingTree (H-Tree) algorithm implemented in MOA. In order to implement C-tree the baseline CorrH-Tree, we directly modified the H-Tree code in MOA. The grace period parameter $\delta$ was set to 100. In contrast to the typical experimental settings in the literature, we did not consider the tie-break parameter because in the experiments we observed that it caused the majority of the splits. Based on Theorem 4 and Remark 1, we used the following version of our confidence bounds $\varepsilon_{KM}$ and $\varepsilon_{Gini}$ (the bound for $\varepsilon_{ent}$ contains an extra $\ln m$ factor),

$$\varepsilon_{KM} = \varepsilon_{Gini} = c \sqrt{\frac{1}{m} \ln(m^2h^2td)} \quad (10)$$

where the parameter $c$ is used to control the number of splits.

In a preliminary round of experiments, we found that the Gini index delivered a performance comparable to that of entropy and Kearns-Mansour, but —on average— produced trees that were more compact for all three algorithms (ours and the two baselines). Hence, we ran all remaining experiments using the Gini index.

In all experiments we measured the online performance. This is the average performance (either accuracy or F-measure) when each new example in the stream is predicted using the tree trained only over the past examples in the stream (“Interleaved Test-Then-Train” validation in MOA) —see Algorithm 3 (FULL setting).

Algorithm 3 Online Stream Validation Protocol

**Input:** labeling budget $B$, Active Strategy, examples stream $(x_1,y_1),(x_2,y_2),...$

1: Initialize online accuracy $M_0 = 0$
2: for $i = 1,2,...$ do
3: Receive sample $x_i$
4: Predict $\hat{y}_i$
5: Update $M_i = (1 - \frac{1}{i})M_{i-1} + \frac{1}{i}\mathbb{1}\{\hat{y}_i = y_i\}$
6: if FULL setting then
7: Receive true label $y_i$
8: Update model using new example $(x_i,y_i)$
9: else
10: // ACTIVE setting
11: if budget $B$ not exceeded then
12: if Strategy$(x_i,y_i)$ then
13: Request true label $y_i$
14: Update query rate
15: Update model using new example $(x_i,y_i)$
16: end if
17: end if
18: end if
19: end for
the number of its leaves is the same as the one of the tree generating the stream. The random binary trees were generated according to Algorithm 2 with fixed class distributions in each leaf. The random binary trees are constructed through recursive random splits. More precisely, we start at the root with a budget of \( n \) leaves. Then we assign to the left and right sub-trees \( \lfloor nX \rfloor \) and \( n - \lfloor nX \rfloor \) leaves respectively, where \( X \) is uniformly distributed in the unit interval. This splitting continues with i.i.d. draws of \( X \) until the left and right sub-trees are left with one leaf each. Whenever a split is generated, we assign it a uniformly random attribute and a random threshold value. In the experiment, we generated 1000 random binary trees with \( n = 50 \) leaves. The random splits are performed choosing among \( d = 5 \) attributes. For simplicity, we only considered numerical attributes and thresholds in the \([0, 1]\) interval. A random binary tree is then used to generate a stream as follows: for each leaf of the tree, 10,000 examples are uniformly drawn from the subregion of \([0, 1]^d\) defined by the leaf, obtaining 500,000 examples. Each of these examples is given label 1 with probability 0.7 for a left leaf and with probability 0.3 for a right leaf. In Figure 3 we show online performances averaged over 1000 streams, each generated using a different random binary tree. In order to span a wide range of tree sizes, we used a grid of 200 different values for the algorithms’ parameters controlling the growth of the trees. Namely, the parameter \( \delta \) available in MOA implementation of H-Tree and CorrH-Tree, and the parameter \( c \) of (10) for C-Tree (in C-Tree \( \delta \) is set to \( \frac{1}{2} \) according to Remark 1). The plots are obtained as follows: for each dataset and algorithm we logged the running average of the online performance \(-M_i\) in Algorithm 3 and the total number of leaves in the tree as the stream was being fed to the algorithm.

B. Experiments on real-world data

We constructed ten different streams from each dataset listed below here by taking a random permutation of the examples in it. A9A, COD-RNA and COVERTYPE are from the LIBSVM binary classification repository[^1] and ELECTRICITY are from the MOA collection[^2]. On the unbalanced datasets (marked with a star in Table I) we used the F-measure on the smallest class to measure performance whereas accuracy was used for the remaining datasets. The parameters \( \delta \) (H-Tree and CorrH-Tree) and \( c \) (C-Tree) were individually tuned on each dataset using a grid of 200 values, hence plots show the online performance of each algorithm when it is close to be optimally tuned. Even if the datasets are not particularly large, the plots show that trees generated by our algorithm compare favourably with respect to the baselines especially in the first learning phases.

### VIII. Selective Sampling Experiments

The validation protocol of the active strategies experiment is described in Algorithm 4 (ACTIVE setting), where we used a different labeling strategy (line 12) for each compared approach. The labeled instances are stored and used to update the model. The query rate is upper bounded by an input budget parameter \( B \in [0, 1] \). In these experiments, we calculated the query rate as the fraction of instances for which a label was

[^1]: www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html
[^2]: moa.cms.waikato.ac.nz/datasets/
Fig. 4. Online performance (accuracy or F-measure) against number of leaves for each bin and dataset achieved by C-Tree (cross and red line), H-Tree (circle and blue line) and CorrH-Tree (star and green line).

requested among the ones observed so far —see [36]. We compared our active strategy of Section VI against a baseline of five techniques proposed in [36]:

**Rnd**. The Random Strategy (Algorithm 5) is a naive method that queries the labels of incoming instances with probability equal to the query rate budget \( B \) without considering the actual incoming instance \( x_t \).

**Algorithm 5 Random Strategy**

**Input**: labeling budget \( B \)

**Output**: labeling \( \in \{true, false\} \) indicates whether to request the true label \( y_t \) for \( x_t \)

1: Generate a uniform random variable \( \text{rand} \in [0,1] \);
2: return \( \mathbb{I}\{\text{rand} \leq B\} \) //where \( \mathbb{I}\{\cdot\} \) is the Indicator Fun.

**VarUn**. We used Variable Uncertainty Strategy described in Algorithm 6 to decide for which instances manual annotation is requested. The confidence threshold \( \Theta \) which determines requests for new labels, is continuously updated. If the classifier’s confidence was above the current threshold \( \Theta \) over the time interval associated with the last instance, the latter is increased by a fraction \( s \) in order to query only the most uncertain instances. In the opposite case the threshold is reduced, with the goal of acquiring more labels in regions where the estimator is less confident. As explained in [36] the parameter \( s \) can be easily set to a default value 0.01. We performed all the experiments with this setting.

**Algorithm 6 Variable Uncertainty Strategy**

**Input**: incoming sample \( x_t \), decision tree \( T \), threshold adjustment step \( s \in (0,1] \)

**Output**: labeling \( \in \{true, false\} \)

1: **Initialize**: confidence threshold \( \Theta = 1 \) and store the latest value during operation
2: Route instance \( x_t \) through \( T \) until a leaf \( \ell_t \) is reached
3: if \( \max\{p_{\ell_t}, 1 - p_{\ell_t}\} < \Theta \) then
4: // confidence below the threshold
5: decrease the confidence threshold \( \Theta = (1 - s)\Theta \)
6: return true
7: else // confidence above the threshold
8: // confidence above the threshold
9: increase the confidence threshold \( \Theta = (1 + s)\Theta \)
10: return false
11: end if

**RndVar**. This method is essentially the same as VarUn described above. VarUn always labels the instances that are close to the decision boundary. However, in data streams changes may happen anywhere in the instance space. When concept drift [33] happens in labels, the classifier will not notice it without the true labels. In order not to miss concept drift, this technique randomize the labeling threshold by multiplying by a normally distributed random variable that follows \( \mathcal{N}(1, \delta = 1) \). This way, the instances that are close to the decision boundary are labeled more often, but occasionally
Fig. 5. Online performance (accuracy or F-measure) against the labeling budget.

also some distant instances are annotated—see [36] for more details.

**Sel-Samp.** The Selective Sampling method is based on [4], and uses a variable labeling threshold $B + \frac{1}{2} \delta \ell_t$ similar to our random sampling mechanism for $\delta$-consistent leaves. The threshold is based on certainty expectations, and the labels are queried at random.

**Split.** Many adaptive learning methods use change-detection mechanisms that monitor streaming error. Change detectors (e.g., DDM [12]) are built with an implicit assumption that the errors are distributed uniformly over time unless a change has happened. The uncertainty strategy asks for labels based on a prediction. Since the predictive model adapts over time, the stream of labeled data is not distributed identically to the stream of unlabeled data. Thus, change detectors may have problems to distinguish a change in distribution due to active labeling from a change in distribution due to concept drift. To overcome that problem, this Split Strategy (Algorithm 7) splits a stream at random into two streams. One of the new streams is labeled according to the Variable Uncertainty Strategy, while the other is labeled according to the Random Strategy. Both streams are used to train a classifier. But only the random stream is used for change detection. In the experiments we set the parameter $\nu = 0.2$.

We compared against the above baseline our Confidence Tree Strategy (Algorithm 3) that we define as ConfTree. We also coupled the Split Strategy (Algorithm 7) with our approach substituting the VarUn procedure with ConfTree method, we define this approach SplitConfTree. All our experiments was performed using the MOA data stream software suite. We added change detection to the base classifier to improve its performance. We chose DDM [12] as in [36]. All the tested ACTIVE strategies used C-Tree (Algorithm 1) as the base learner with the same parameters setting of Section VII.

The algorithms had to predict the label of each new incoming sample. After each prediction, if the active learning system requested the true label, the sample together with its label were fed to it as a new training example—see Algorithm 3.

We ran all the competing algorithms with the same range

**Algorithm 7 Split Strategy**

**Input:** incoming sample $x_t$, decision tree $T$, threshold adjustment step $s \in (0,1]$, proportion of random labeling $\nu \in (0,1]$

**Output:** labeling $\in \{true, false\}$

```plaintext
1: Initialize: confidence threshold $\Theta = 1$ and store the latest value during operation
2: Route instance $X_t$ through $T$ until a leaf $\ell_t$ is reached
3: Generate a uniform random variable $\text{rand} \in [0,1]$
4: if $\text{rand} \leq \nu$ then
5: Change Detection Method()
6: return $\text{Rnd}(B)$
7: else
8: return $\text{VarUn}(x_t, T, s)$
9: end if
```
Let $H$ be the standard (unscaled) entropy. Using the standard identity $H(Y_{i} \mid F) = H(Y_{i}, F) - H(F)$, we have that $\hat{H}_{1/2}(Y_{i} \mid F) = \hat{H}_{1/2}(Y_{i}, F) - \hat{H}_{1/2}(F)$. We now use part (iii) of the remark following Corollary 1, we have that

$$
\left| \hat{H}_{1/2}(F) - E \hat{H}_{1/2}(F) \right| \leq \frac{\ln m}{2} \sqrt{\frac{2}{m} \ln \frac{4}{\delta}}
$$

and

$$
\left| \hat{H}_{1/2}(Y_{i}, F) - E \hat{H}_{1/2}(Y_{i}, F) \right| \leq \frac{\ln m}{2} \sqrt{\frac{2}{m} \ln \frac{4}{\delta}}
$$

Note that the budget is only an upper limit to the actual query rate – algorithms generally ask for a smaller number of annotations.

Next, we apply Proposition 1, which states that

$$
-\ln \left( 1 + \frac{N - 1}{m} \right) \leq \mathbb{E} \hat{H}(Z) - H(Z) \leq 0
$$

for any random variable $Z$ which takes $N$ distinct values. In our case, $N = 2$ for $F = 0$ and $N = 4$ for $F = (Y_{i}, F)$. Hence, using $-a \leq -\ln(1 + a)$ for all $a$, we get

$$
\left| \hat{H}_{1/2}(F) - E \hat{H}_{1/2}(F) \right| \leq \frac{1}{2m}
$$

$$
\left| \hat{H}_{1/2}(Y_{i}, F) - E \hat{H}_{1/2}(Y_{i}, F) \right| \leq \frac{3}{2m}.
$$

Putting everything together gives the desired result.

**APPENDIX B**

**Proof Theorem 2**

**Lemma 2 (McDiarmid’s inequality):** Let $G$ be a real function of $m$ independent random variables $X_1, \ldots, X_m$ such that

$$
G(x_1, \ldots, x_i, \ldots, x_m) - G(x_1, \ldots, x'_i, \ldots, x_m) \leq c \quad (11)
$$

for some constant $c \in \mathbb{R}$ and for all realizations $x_1, \ldots, x_i, x'_i, \ldots, x_m$. Then

$$
P \left( |G - \mathbb{E} G| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2\epsilon^2}{mc^2} \right).
$$

If we set the right-hand side equal to $\delta$, then

$$
|G - \mathbb{E} G| \leq c \sqrt{\frac{m}{2\ln \frac{2}{\delta}}}
$$

is true with probability at least $1 - \delta$.

Note the following fact

$$
J(Y_{i} \mid F)
$$

$$
= \mathbb{P}(F = 1) \frac{\mathbb{P}(Y_{i} = 1, F = 1)}{\mathbb{P}(F = 1)} \frac{\mathbb{P}(Y_{i} = 0, F = 1)}{\mathbb{P}(F = 1)}
$$

$$
+ \mathbb{P}(F = 0) \frac{\mathbb{P}(Y_{i} = 1, F = 0)}{\mathbb{P}(F = 0)} \frac{\mathbb{P}(Y_{i} = 0, F = 0)}{\mathbb{P}(F = 0)}
$$

$$
= 2 \mathbb{P}(Y_{i} = 1, F = 1) \frac{\mathbb{P}(Y_{i} = 0, F = 1)}{\mathbb{P}(F = 1)}
$$

$$
+ 2 \mathbb{P}(Y_{i} = 1, F = 0) \frac{\mathbb{P}(Y_{i} = 0, F = 0)}{\mathbb{P}(F = 0)}
$$

$$
= \overline{HM}(p_1, q_1) + \overline{HM}(p_0, q_0). \quad (12)
$$

In view of applying McDiarmid inequality, let $\tilde{p}_k = \frac{r}{m}$ and $\tilde{q}_k = \frac{r^t}{m}$. We can write the left-hand side of condition (11) in Lemma 2 for each term of (12) as

$$
\frac{2}{m} \left[ \frac{rs}{r + s} - \frac{r's'}{r' + s'} \right]
$$

where $r, s = 1, \ldots, m$ and $r', s'$ may take the following forms: $(r + 1, s - 1)$ — when a label of an example in the current leaf is flipped; $(r + 1, s)$ — when an example is moved from the sibling leaf to the current leaf, and $(r - 1, s)$ — when an example is moved from the current leaf to the sibling leaf.
Since the harmonic mean is symmetric in \( r \) and \( s \), we can ignore the cases \( (r - 1, s + 1) \), \( (r, s + 1) \), and \( (r, s - 1) \). A tedious but simple calculation shows that

\[
\left| \frac{rs}{r + s} - \frac{r's'}{r' + s'} \right| \leq 1.
\]

Therefore, we may apply Lemma 2 with \( c = \frac{4}{m} \) and obtain that

\[
\left| \Phi_{i,F} - \mathbb{E} \Phi_{i,F} \right| \leq \sqrt{\frac{8}{m} \ln \frac{2}{\delta}}
\]

holds with probability at least \( 1 - \delta \).

Next, we control the bias of \( \text{HM}(\hat{p}_k, \hat{q}_k) \) as follows,

\[
0 \leq \text{HM}(p_k, q_k) - \mathbb{E} \left[ \text{HM}(\hat{p}_k, \hat{q}_k) \right]
= 2 \frac{p_k q_k - \hat{p}_k \hat{q}_k}{p_k + q_k} - 2 \mathbb{E} \left[ \frac{\hat{p}_k \hat{q}_k}{\hat{p}_k + \hat{q}_k} \right]
= 2 \mathbb{E} \left[ \frac{p_k \hat{p}_k q_k - \hat{q}_k}{(p_k + q_k)(\hat{p}_k + \hat{q}_k)} \right]
\geq 2 \mathbb{E} \left[ (q_k - \hat{q}_k) \right]
\leq 2 \sqrt{\mathbb{E} \left[ (q_k - \hat{q}_k)^2 \right]} + 2 \mathbb{E} \left[ (p_k - \hat{p}_k)^2 \right]
\leq \frac{2}{\sqrt{m}}
\]

(15)

where the first inequality is due to the concavity of \( \text{HM} \). Combining (13) and (15) concludes the proof.

**APPENDIX C**

**PROOF THEOREM 3**

**Lemma 3:** Let \( B \) binomially distributed with parameters \((m, p)\). Then

\[
\left| \sqrt{\frac{B}{m}} - \sqrt{p} \right| \leq \sqrt{\frac{1}{m} \ln \frac{2}{\delta}}
\]

is true with probability at least \( 1 - \delta \).

**Proof:** The result is an immediate consequence of the bounds in [15] —see also [3, Exercise 2.13]. In particular,

\[
\mathbb{P} \left( \left| \frac{B}{m} - p \right| \geq \varepsilon \right) \leq e^{-mD(p+\varepsilon||p)} + e^{-mD(p-\varepsilon||p)}
\]

where \( D(q||p) = q \ln \frac{q}{p} + (1-q) \ln \frac{1-q}{1-p} \) is the KL divergence, and

\[
D(p + \varepsilon||p) \geq \left( \sqrt{p + \varepsilon} - \sqrt{p} \right)^2
\]

\[
D(p - \varepsilon||p) \geq 2 \left( \sqrt{p - \varepsilon} - \sqrt{p} \right)^2.
\]

Simple algebraic manipulation concludes the proof.

**APPENDIX D**

**PROOF THEOREM 4**

Fix some arbitrary tree \( T \) of depth \( H \) and let \( D_h \) be the set of internal nodes at depth \( h \). Clearly,

\[
\sum_{i \in D_h} \mathbb{P}(X \rightarrow i) \leq 1.
\]
Now, for any internal node \( i \) of \( T \), let \( F_i \) be the split used at that node. We have

\[
\mathbb{P}(X \text{ routed via a } \tau \text{-suboptimal split}) = \mathbb{P}(\exists i : X \to i, G_{i,F} + \tau \leq \max_{F \in \mathcal{F}} G_{i,F}) \\
\leq \sum_i \sum_{h=0}^{H} \mathbb{P}(G_{i,F} + \tau \leq \max_{F \in \mathcal{F}} G_{i,F} \mid X \to i) \mathbb{P}(X \to i) \\
= \sum_{h=0}^{H} \sum_{i \in \mathcal{D}_h} \mathbb{P}(\Phi_{i,F} > \Phi_{i,F}) \\
- 2\varepsilon \left( m_{i,t} \frac{\delta (h+1)}{t dm_{i,t}} \right) \mathbb{P}(X \to i) \\
\leq \sum_{h=0}^{H} \sum_{i \in \mathcal{D}_h} \sum_{s=0}^{t-1} \mathbb{P}(\Phi_{i,F} > \Phi_{i,F}) \\
- 2\varepsilon \left( s \frac{\delta (h+1)}{t ds} \right) \mathbb{P}(X \to i) \\
\leq \sum_{h=0}^{H} \sum_{i \in \mathcal{D}_h} \sum_{s=0}^{t-1} \delta (h+1)(h+2)^{s} \mathbb{P}(X \to i) \\
\leq \sum_{h=0}^{H} \sum_{i \in \mathcal{D}_h} \frac{\delta (h+1)(h+2)^{t}}{t} \leq \delta .
\]

**APPENDIX E**

**PROOF THEOREM 5**

\[
\mathbb{P}(f_T(X_t) \neq y_{t,i}, \ell_t \text{ is } \delta \text{-consistent}) \\
\leq \mathbb{P}(\left| Y_{t,i} - p_{t,i} \right| \geq \sqrt{\frac{1}{2m_{t,i}} \ln \frac{2t}{\delta}}) \\
\leq \sum_{s=0}^{t-1} \mathbb{P}(\left| Y_{t,i} - p_{t,i} \right| \geq \sqrt{\frac{1}{2s} \ln \frac{2t}{\delta}}) \\
\leq \sum_{s=0}^{t-1} \sum_{i \in \mathcal{T}} \mathbb{P}(\left| Y_{t,i} - p_{t,i} \right| \geq \sqrt{\frac{1}{2s} \ln \frac{2t}{\delta}}) = \mathbb{P}(X_t \to i) \\
\leq \sum_{s=0}^{t-1} \sum_{i \in \mathcal{T}} \frac{\delta}{t} \mathbb{P}(X_t \to i) = \delta
\]

where we used the standard Chernoff bound in the last step.

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Rocco De Rosa received the Ph.D. degree from University of Milan, Milan, Italy, in 2014. He is currently a Researcher in Machine Learning Theory and Application at University of Milan. His current research interests include mining evolving streaming data, change detection, multivariate time series classification, online learning, stream mining, adaptive learning, and predictive analytics applications.