ABSTRACT: The scalar potential of the Minimal Supersymmetric Standard Model (MSSM) is nearly flat along many directions in field space. We provide a catalog of the flat directions of the renormalizable and supersymmetry-preserving part of the scalar potential of the MSSM, using the correspondence between flat directions and gauge-invariant polynomials of chiral superfields. We then study how these flat directions are lifted by non-renormalizable terms in the superpotential, with special attention given to the subtleties associated with the family index structure. Several flat directions are lifted only by supersymmetry-breaking effects and by supersymmetric terms in the scalar potential of surprisingly high dimensionality.
1. Introduction

Supersymmetric gauge theories often possess a remarkable vacuum degeneracy at the classical level. The renormalizable scalar potential in supersymmetry is a sum of squares of $F$-terms and $D$-terms, and so may vanish identically along certain “flat directions” in field space. The space of all such flat directions is called the moduli space, and the massless chiral superfields whose expectation values parameterize the flat directions are known as moduli. The properties of the space of flat directions of a supersymmetric model are crucial considerations for cosmology and whenever the behavior of the theory at large field strengths is an issue.

In realistic models such as the Minimal Supersymmetric Standard Model [1] (MSSM), the “flat” directions are only approximately flat; the vacuum degeneracy of the scalar potential is lifted by soft supersymmetry-breaking terms, and by non-renormalizable terms in the superpotential. The soft terms contribute terms to the scalar potential which are schematically of the form

$$V_{\text{soft}} = m^2 |\phi|^2$$  \hspace{1cm} (1.1)

where $\phi$ represents the scalar component of the moduli fields. Now, if supersymmetry is to provide a successful explanation for the hierarchy problem associated with the mass of the Higgs scalar boson, $m$ must be of the order of the electroweak scale. The terms in (1.1) can never be forbidden by any symmetry (unlike soft terms of the form $\phi^2$ and $\phi^3$), and so we expect that all flat directions will be lifted weakly in this way.

The question of which non-renormalizable terms in the superpotential also lift a given flat direction is more complicated. It is useful to think of the non-renormalizable superpotential as an expansion in inverse powers of some large mass scale $M$ (presumably the Planck scale or some other physical cutoff); schematically

$$W = W_{\text{renorm}} + \sum_{n>3} \frac{\lambda}{M^{n-3}} \Phi^n.$$  \hspace{1cm} (1.2)

Each flat direction may be labeled by an order parameter modulus $\phi$ which can take on values with $|\phi| < M$. Therefore it is sufficient to consider separately the contributions to the superpotential first from renormalizable terms $W_{\text{renorm}}$ and then for each value of $n > 3$ in turn. Renormalizable flat directions are those for which all $F$-terms following from $W_{\text{renorm}}$ and all $D$-terms vanish. Of these renormalizable flat directions, some are lifted when $F$-terms from the $n = 4$ superpotential are included; some may survive until
\( n = 5 \) superpotential terms are included, etc. The lowest order contribution to the scalar potential for a modulus \( \phi \) corresponding to a flat direction lifted at level \( n \) is

\[
V \supset |F|^2 \sim \frac{|\lambda|^2}{M^{2n-6}} |\phi|^{2n-2}
\]

(1.3)

so that (nearly) flat directions which survive to higher \( n \) are at least formally “flatter”. We may therefore describe the relative flatness of a renormalizable flat direction at large field strength by specifying the level \( n \) at which it is first lifted by non-renormalizable terms in the superpotential. The non-renormalizable terms (1.3) dominate over the soft supersymmetry breaking terms (1.1) in lifting flat directions for field strengths

\[
\left( \frac{mM^{n-3}}{|\lambda|} \right)^{1/(n-2)} < |\phi| < M.
\]

(1.4)

The scalar potential should also include supersymmetry-breaking terms of the form

\[
V \supset \frac{m}{M^{s-3}} \phi^s + \text{c.c.}
\]

(1.5)

where again \( m \) must be of the order of the electroweak scale if supersymmetry is to solve the hierarchy problem. A phase rotation on \( \phi \) can always make terms of this type negative. However, these terms cannot dominate over both (1.1) and (1.3) for renormalizable flat directions except when \( s < n \) (and even then only for a limited range of \(|\phi|\)). In the MSSM, \( s < n \) does not occur for any of the renormalizable flat directions assuming “generic” values for all couplings, as we shall see. In an inflationary epoch, the \( m \) appearing in (1.1) and (1.5) may be identified [2] with the (much larger) Hubble expansion parameter. In most of our remaining discussion, we will concentrate on the moduli space of the supersymmetric part of the scalar potential.

Consider a model with \( N \) chiral superfields \( X_I \) transforming under the gauge group \( G \) as a (in general reducible) representation in which the generators of the Lie gauge algebra are matrices \( T^A \). In principle, one could attempt to find all flat directions in the scalar potential \( V = (g_A^2/2)D^AD^A + \sum |F_{X_I}|^2 \) by solving the simultaneous nonlinear equations

\[
D^A \equiv X^\dagger T^AX = 0, \quad (1.6)
\]

\[
F_{X_I} \equiv \partial W/\partial X_I = 0 \quad (1.7)
\]

for the scalar components \( X_I \). However, a more useful and elegant way of characterizing the moduli space relies on the correspondence [3,4] between flat directions and gauge-invariant, holomorphic polynomials of the chiral superfields \( X_I \). In particular, the moduli space of
$D$-flat directions is parameterized by a finite set of gauge-invariant monomials of the chiral superfields which obey a finite set of redundancy relations. The smaller moduli space of $D$- and $F$-flat directions is parameterized by the same basis of monomials and redundancy constraints subject to additional constraints (some linear and some non-linear) following from $F_{X_I} = 0$ (see [5] and references therein). This result is the foundation for most of our discussion in this paper. Intuitively, the order parameter which describes motion along a given flat direction is the scalar component of the corresponding gauge invariant polynomial of chiral superfields. Examples of this correspondence between flat directions and the gauge invariant polynomials abound in the literature (see for example [2-11]).

In the absence of exact global symmetries which would make the non-renormalizable superpotential non-generic, one expects that all flat directions will be lifted at some finite $n$. The number of $F$-constraints (1.7) is formally equal to the number of degrees of freedom in the chiral superfields $X_I$. In addition, one has dim$(G)$ real gauge constraints (although not all of these constraints are always independent), and generally even more degrees of freedom can be absorbed in gauge-fixing. Therefore the system ought to be overconstrained, typically leaving only the trivial solution $X_I = 0$ as the minimum of the supersymmetric part of the potential. Of course, this counting presupposes that the $F = 0$ constraints are all non-trivial and independent, an assumption which clearly does not hold at the renormalizable level in many theories. Furthermore, the requirement of gauge invariance (and, in realistic models, matter parity or equivalently R-parity invariance) of the superpotential severely limits the non-renormalizable superpotential terms for smaller values of $n$, and so may not allow flat directions to be lifted except for some (perhaps large) $n$. It is thus a non-trivial problem for a given supersymmetric model to identify the flat directions, and the level $n$ of the non-renormalizable superpotential terms required to lift each flat direction.

In this paper we will study the (nearly) flat directions of the MSSM. The obvious relevance of understanding the structure of the scalar potential of this model (which may well describe nature up to a scale $M_U \sim 2 \times 10^{16}$ GeV at which the gauge couplings appear [12] to unify) has recently been highlighted by Dine, Randall and Thomas [2] in the context of baryogenesis[13]. Here we will provide a complete catalog of MSSM flat directions, and the level at which they are expected to be lifted by non-renormalizable terms in the superpotential. We will assume that all non-renormalizable couplings are “generic”, i.e., not subject to any constraints other than gauge invariance and $R$-parity. The rest of this paper is organized as follows. In section 2 we will describe our notation for the MSSM.
and provide a basis of gauge-invariant monomials which parameterize all $D$-flat directions. These monomials are subject to redundancy relations which are easily understood in terms of identities obtained by antisymmetrizing over $SU(3)_C$ and $SU(2)_L$ indices. We will then identify a smaller basis of monomials which parameterize all renormalizable $F$-flat and $D$-flat directions. In section 3 we will study how each of the renormalizable flat directions associated with the monomials identified in section 2 are lifted by non-renormalizable terms in the superpotential. Most of the flat directions are lifted already at the $n = 4$ level by non-renormalizable terms in the superpotential. However, we will show that there exists a unique flat direction (which carries $B - L = 1$) which is not lifted by non-renormalizable operators until the $n = 9$ level, and one other flat direction (which carries $B - L = -3$) which is not lifted until the $n = 7$ level. We will also identify other flat directions which survive until the $n = 5$ and $n = 6$ levels. Section 4 contains some concluding remarks.

2. Renormalizable flat directions of the MSSM

Let us begin by specifying our notation and assumptions regarding the MSSM. The chiral superfields consist of three families of $SU(2)_L$-doublet quarks $Q$ and leptons $L$ and $SU(2)_L$-singlet quarks and leptons $u, d, e$, and two Higgs superfields $H_u$ and $H_d$ which couple respectively to up- and down-type quark superfields. We will use the same symbol for chiral superfields and for their scalar components. We will often be able to suppress gauge and family indices in the following, but when necessary, Greek letters $\alpha, \beta, \gamma, \ldots$ will be used to refer to $SU(2)_L$ indices; Latin letters $a, b, c, \ldots$ to refer to $SU(3)_C$ indices; and Latin letters $i, j, k, \ldots = 1, 2, 3$ to refer to family indices. All interactions (including soft and non-renormalizable ones) are assumed to be invariant under an exact $Z_2$ matter parity which is trivially related to $R$-parity by a minus sign for fermions and which is defined by $(-1)^{3(B-L)}$. [This assumption follows most naturally [14] in models in which $B - L$ is gauged at very high energies and is broken by order parameter(s) with only even values of $3(B - L)$.] The renormalizable superpotential is given by

$$W_{\text{renorm}} = \mu H_u H_d + y_u^{ij} H_u Q_i u_j + y_d^{ij} H_d Q_i d_j + y_e^{ij} H_d L_i e_j . (2.1)$$

We will assume in the following that the $3 \times 3$ Yukawa matrices $y_u^{ij}, y_d^{ij}, y_e^{ij}$ are each non-degenerate (have rank 3), although it is worth noting that this assumption is perhaps not inevitable.

The configuration space of the scalar fields of the MSSM has 49 complex dimensions (18 for $Q_i$; 9 each for $u_i$ and $d_i$; 6 for $L_i$; 3 for $e_i$; and 2 each for $H_u$ and $H_d$). The
subspace of $D$-flat directions on which the 12 real $D$-term constraints [8 for $SU(3)_C$; 3 for $SU(2)_L$; and 1 for $U(1)_Y$] are satisfied is therefore 37 complex dimensional [2], after 12 corresponding phase degrees of freedom are gauge-fixed. We wish to find a basis $B$ of gauge-invariant monomials with the property that any gauge-invariant polynomial in $(Q, L, u, d, e, H_u, H_d)$ can be written as a polynomial in the elements of $B$. The number of distinct monomials in $B$ is necessarily much greater than 37, because they will be subject to many non-linear redundancy constraints. The space of $D$-flat directions of the MSSM will then be parameterized by the elements of the basis $B$, subject to this finite set of constraints. Our strategy for constructing $B$ is as follows. First we will construct a basis $B_3$ of $SU(3)_C$-singlet monomials which transform under $SU(2)_L$ as singlets, doublets, and a single spin-3/2 representation. Using the elements of $B_3$ as building blocks, we can then construct a basis $B_{32}$ of monomials which generate all $SU(3)_C \times SU(2)_L$ invariant polynomials. Finally we can combine the elements of $B_{32}$ into weak-hypercharge singlets to find the basis $B$ of monomials invariant under the full gauge group.

Under $SU(3)_C$, the chiral superfields of the MSSM transform as 13 singlets ($e_i, L_i, H_u, H_d$), six 3’s ($Q_i$), and six $\overline{3}$’s ($u_i, d_i$). It is useful to adopt temporarily a generic notation $q_I$ for 3’s and $\overline{q}_I$ for $\overline{3}$’s of $SU(3)_C$, with $I = 1 \ldots 6$. Then all $SU(3)_C$-invariant polynomials in the $q_I$ and $\overline{q}_I$ are generated by the 76 monomials

\[
(q_I \overline{q}_J) \equiv q^a_I \overline{q}_a J \\
(q_I q_J q_K) \equiv q^a_I q^b_J q^c_K \epsilon_{abc} \\
(\overline{q}_I \overline{q}_J \overline{q}_K) \equiv \overline{q}_a I \overline{q}_b J \overline{q}_c K \epsilon^{abc}.
\]

These monomials are not all independent, but are subject to the constraints following from

\[
q^a_I (q_K q_L q_M) = q^a_K (q_I q_L q_M) + q^a_L (q_K q_I q_M) + q^a_M (q_K q_L q_I)
\]

and the analogous relation for $\overline{q}$’s, and by

\[
(q_I q_J q_K) (\overline{q}_L \overline{q}_M \overline{q}_N) = (q_I \overline{q}_L) (q_J \overline{q}_M) (q_K \overline{q}_N) \pm (\text{permutations})
\]

In (2.2), the free color index $a$ may be contracted with those from additional superfields to form either a $(q_I q_J)$ or a $(qqq)$. The identity (2.3) shows that any $SU(3)_C$-invariant polynomial can be written in a form in which no term contains both a $(qqq)$ and a $(q_I q_J)$.

Now, a color-singlet product of three $Q$’s can transform under $SU(2)_L$ as either a 2 or a 4 of $SU(2)_L$, since $2 \times 2 \times 2 = 2 + 2 + 4_S$. If $(QQQ)$ transforms as a 4 of $SU(2)_L$, it must be totally antisymmetric on its family indices, since it is antisymmetric on color indices
and symmetric on $SU(2)_L$ indices. There is therefore a unique $SU(3)_C$-singlet monomial made out of three $Q$’s which is a 4 of $SU(2)_L$, namely
\[(QQQ)^{(\alpha\beta\gamma)}_4 \equiv Q^{\alpha a}_i Q^{\beta b}_j Q^{\gamma c}_k \epsilon_{abc} \epsilon^{ijk}. \quad (2.4)\]

The remaining combinations of $(QQQ)$ which are $SU(3)_C$-invariant are $SU(2)_L$ doublets and can be written in the form
\[ (Q_i Q_j Q_k)^{\alpha} \equiv Q^{\beta a}_i Q^{\gamma b}_j Q^{\alpha c}_k \epsilon_{abc} \epsilon^{\beta\gamma} \quad (2.5) \]
subject to the constraints that not all three of the family indices may be the same. The basis $B_3$ need not include terms of the form (2.5) with $i = k$ or $j = k$, since it is easy to show that such monomials can be written as linear combinations of terms of the same type with $i = j$. In fact, there are only 8 linearly independent $SU(2)_L$-doublet $(QQQ)^{\alpha}$ monomials, since $(Q_i Q_j Q_k)^{\alpha}$ is symmetric under interchange of $i$ and $j$, and $(Q_3 Q_2 Q_1)^{\alpha}$ is a linear combination of $(Q_1 Q_2 Q_3)^{\alpha}$ and $(Q_1 Q_3 Q_2)^{\alpha}$. The family indices on monomials of the form $(uuu)$, $(uud)$, etc. are subject to the obvious antisymmetrization constraints that follow from their definitions.

We therefore find that a basis $B_3$ of monomials which generate all $SU(3)_C$-singlet polynomials of chiral superfields is as given in Table 1. The gauge index structure is now unambiguous and so is suppressed. The family indices are suppressed, but take on values 1,2,3 on each chiral superfield (in any convenient basis), except in the case of $(QQQ)_4$ and with other restrictions as just discussed. The $SU(2)_L \times U(1)_Y$ quantum numbers for the monomials and the number of linearly independent monomials of each type (taking family structure into account) are also listed in Table 1. Note that the basis of $SU(3)_C$-invariant monomials contains 23 1’s, 31 2’s and one 4 of $SU(2)_L$.

We can now construct a monomial basis $B_{32}$ for the $SU(3)_C \times SU(2)_L$-singlet polynomials of chiral superfields, by combining the elements of Table 1 into $SU(2)_L$-singlet combinations. This is made much easier with the realization that any term in which an $SU(2)_L$ index from a $(QQQ)_4$ is contracted with another $Q$ can always be rewritten as a polynomial in terms of $SU(2)_L$-doublet $(QQQ)$’s instead. This fact can be proved directly by examining each possible such term that can arise; it is interesting to note that if there were 4 or more families, explicit calculation shows that some terms of this form cannot be rewritten in the appropriate way. Therefore, our monomial basis only includes terms in which the $SU(2)_L$ indices of $(QQQ)_4$ are contracted with color singlet chiral superfields $H_u$, $H_d$, or $L$. Besides such terms, the $SU(2)_L$-singlet monomials include terms of the
form

\[(\varphi_i \varphi_j) \equiv \varphi_i^\alpha \varphi_j^\beta \epsilon_{\alpha \beta}\]

where the \(\varphi_i\) are any of the \(SU(2)_L\) doublets in Table 1. These monomials are clearly antisymmetric under \(I \leftrightarrow J\) and subject to the constraint relations

\[(\varphi_i \varphi_j) (\varphi_k \varphi_L) = (\varphi_i \varphi_k) (\varphi_j \varphi_L) + (\varphi_i \varphi_L) (\varphi_k \varphi_j).\]  \hspace{1cm} (2.6)

The complete basis \(B_{32}\) of monomials which generate all \(SU(3)_C \times SU(2)_L\)-singlet polynomials is given in Table 2, along with their \(U(1)_Y\) quantum numbers, which take on the values \(-2, -1, 0, 1, 2\). In Table 2 we have adopted the following notational conventions. Each consecutive triplet \(QQQ\) reading left-to-right in each monomial is assumed to form a color singlet, as are consecutive pairs \(Qu\) and \(Qd\) and consecutive triplets \(uuu, uud,\) etc. If a \(QQQ\) is not enclosed with parentheses and a subscript 4, it is assumed to form a 2 of \(SU(2)_L\), with the \(SU(2)_L\) indices contracted as in (2.5). The contractions of the remaining \(SU(2)_L\) indices within each monomial in Table 2 are then uniquely determined. [For example, in \((QQQ)_4LLL\), the three \(L\)'s must form a 4 of \(SU(2)_L\) in order for the monomial to be an \(SU(2)_L\) singlet.] The suppressed family indices may take on values 1, 2, 3 (in any convenient family basis) with constraints as discussed before. In many cases, the number of linearly independent monomials in the basis can be reduced by using identities (2.2), (2.3), and (2.6). The remaining basis elements are subject to further non-linear constraints following from the same identities. In the case of monomials involving \(Q\), one often must use these identities several times in order to obtain a non-linear redundancy constraint which is written explicitly in terms of monomials in the basis \(B_{32}\). These redundancy constraints are again themselves highly redundant.

Now we are ready to find the basis \(B\) of \(SU(3)_C \times SU(2)_L \times U(1)_Y\)-invariant monomials, by combining elements of Table 2 into \(U(1)_Y\)-singlets. Referring to the elements of \(B_{32}\) in the generic form \(\chi_{-2}, \chi_{-1}, \chi_0, \chi_1, \chi_2\) with the subscript indicating the weak hypercharge, it is clear that \(B\) consists of monomials of the form \(\chi_0, \chi_1 \chi_{-1}, \chi_2 \chi_{-2}, \chi_2 \chi_{-1} \chi_{-1}\); and \(\chi_{-2} \chi_1 \chi_1\). (Note that there is exactly one \(\chi_2\) and one \(\chi_{-2}\).) However, using the relations (2.2), (2.3), and (2.6), one can show that many of the terms formed in this way are actually polynomials of other monomials in the basis. After eliminating as many redundancies as possible in this way, we find the list of monomials in \(B\) given in Table 3. The conventions for contracting the suppressed gauge indices are as discussed above for Table 2. This basis is quite overcomplete, in the sense that there are many non-linear constraints relating the
basis elements following from (2.2), (2.3), and (2.6), as well as

\[(\chi_1^I \chi_1^J)(\chi_1^K \chi_1^L) = (\chi_1^I \chi_1^K)(\chi_1^J \chi_1^L)\; ; \]
\[(\chi_1^I \chi_1^J)^{-1}(\chi_1^K \chi_1^L)^{-1} = (\chi_1^I \chi_1^K)^{-1}(\chi_1^J \chi_1^L)^{-1}; \]

etc. It is a trivial, if tedious, exercise to write out all of these redundancy constraints explicitly in terms of the monomials appearing in Table 3. It should be kept in mind that these redundancy constraints are themselves highly redundant.

Each polynomial formed out of the elements of Table 3 corresponds to a \(D\)-flat direction of the MSSM. Though the correspondence between flat directions and gauge-invariant operators has been discussed extensively in the literature in simpler models, a short example may be in order. Consider the flat directions associated with the leptonic sector of the MSSM alone. In the absence of a superpotential or soft breaking terms, the scalar potential in this sector is:

\[V = \frac{g^2}{2} \left(D_1^2 + D_2^2 + D_3^2\right) + \frac{g'^2}{2} D_Y^2\]
\[D_1 = \frac{1}{2} \sum_i (L_i^\dagger L_i + L_i^\dagger L_i^\dagger); \quad D_2 = \frac{i}{2} \sum_i (L_i^\dagger L_i^\dagger - L_i^\dagger L_i^\dagger); \]
\[D_3 = \frac{1}{2} \sum_i (|L_i^\dagger|^2 - |L_i^\dagger|^2); \quad D_Y = \frac{1}{2} \sum_i (2|e_i|^2 - |L_i^\dagger|^2 - |L_i^\dagger|^2).\]

It is easy to see that \(D_Y = D_1 = D_2 = D_3 = 0\) and thus \(V = 0\) for a class of flat directions of the form

\[L_i = \begin{pmatrix} \phi \\ 0 \end{pmatrix}; \quad L_j = \begin{pmatrix} 0 \\ \phi \end{pmatrix}; \quad e_k = \phi \]

with \(i \neq j\), where \(\phi\) is the sliding VEV along the flat direction. Each such flat direction is labeled by a gauge-invariant monomial \(L_i L_j e_k\). This notation is useful for two reasons. First, the correspondence between \(D\)-flat directions and gauge-invariant monomials conveniently obviates the necessity of directly solving the non-linear equations (1.6). Secondly, the \(F\)-flatness conditions (1.7) can be directly imposed as constraints on the gauge-invariant operators.

The renormalizable flat directions of the MSSM correspond to the gauge invariant monomials in Table 3, subject to the additional constraints

\[F_{H_u}^\alpha = \mu H_{d}^\alpha + y_{u}^{ij} Q_{i}^\alpha u_j = 0 \quad (2.8)\]
\[F_{H_d}^\alpha = -\mu H_{u}^\alpha + y_{d}^{ij} Q_{d}^\alpha d_j + y_{e}^{ij} L_i^\alpha e_j = 0 \quad (2.9)\]
Using the assumed invertibility of the Yukawa matrices $y_u$, $y_d$ and $y_e$, one finds from the $F_u = 0$, $F_d = 0$ and $F_e = 0$ constraints that any monomial containing an $SU(2)_L$ contraction of $H_d$ with $L$ or $Q$, or of $H_u$ with $Q$, is immediately constrained to vanish. (A flat direction is lifted when the corresponding gauge-invariant operator is constrained to vanish.) The $F_L = 0$ constraint shows that an $H_d$ and $e$ cannot coexist in any flat direction, while $F_Q = 0$ yields a more complicated constraint on terms containing $uH_u$ and $dH_d$. Note that at the renormalizable level, only the four complex constraints following from $F_{H_u}^\alpha = 0$ and $F_{H_d}^\alpha = 0$ can lift flat directions which do not contain a Higgs field.

In the last column of Table 3, we have indicated with a check mark the monomial flat directions which are always lifted by renormalizable $F$-term constraints [namely (2.11)-(2.14)] regardless of family index structure. The only remaining monomials in the basis which involve the Higgs fields are $LH_u$ and $HuH_d$. The moduli space of renormalizable flat directions of the MSSM is now parameterized by the gauge-invariant monomials in Table 3 without check marks, subject to the redundancy constraints implied by (2.2), (2.3), (2.6), (2.7), and the additional constraints obtained by contracting (2.8)-(2.14) with additional chiral superfields to form gauge singlets. For most applications, including cosmological ones, one may neglect the contribution to (2.8) and (2.9) from the $\mu$ term, since for phenomenological reasons $\mu$ should be of order the electroweak scale $m_W$ and thus its contributions to the scalar potential at large field strength $|\phi| \gg m_W$ are suppressed by $m_W/|\phi|$ compared to those from the dimensionless couplings.

3. Lifting of the flat directions by the non-renormalizable superpotential

The MSSM is presumably an effective theory, valid only at scales below the physical cutoff $M$ which might be associated with a GUT ($M \sim 10^{16}$ GeV) or perhaps a supergravity or superstring model ($M \sim 10^{19}$ GeV). In any case, the new physics associated with the scale $M$ will give rise to non-renormalizable terms suppressed by powers of $M$, as indicated schematically by (1.2). Throughout this section, we will assume that all terms
consistent with the gauge symmetries of the MSSM and matter parity will be generated by short distance effects, with generic couplings of order 1. This is tantamount to rejecting the possibility of (nearly) exact global symmetries (including continuous $R$-symmetries) or additional gauged symmetries left unbroken below the scale $M$.

The problem of characterizing all of the non-renormalizable operators which can appear in the superpotential is nearly equivalent to the problem of finding $D$-flat directions, solved in section 2. All such non-renormalizable operators can be generated using the monomials appearing in Table 3, with the additional proviso due to matter parity conservation that each superpotential term contains an even number of odd matter parity fields ($Q, L, u, d, e$). In this section we will consider how the renormalizable flat directions are lifted by such non-renormalizable superpotential interactions. Since the validity of the expansion (1.2) presumes that the field strengths are less than $M$, it is sufficient at least formally to consider the various $F = 0$ constraints separately level-by-level in $n$. At each level $n$, the surviving moduli space of flat directions is described by the basis of monomials as before, subject to additional constraints obtained by contracting $F = 0$ with additional fields to form polynomials in the basis monomials. Some of the additional constraints obtained in this way will be linear in the basis monomials, but in general one finds that some of the $F = 0$ constraints can only be realized non-linearly on the basis monomials.

It is convenient to start by considering the flat directions associated with the monomials $H_u H_d$ and $L H_u$ which contain Higgs fields. (All flat directions involving Higgs fields are lifted already considering only the renormalizable $\mu$ term, but as we remarked at the end of the previous section it is appropriate in many applications to neglect $\mu \approx m_W$.) The $n = 4$ superpotential includes terms

$$W_4 \supset \frac{\lambda}{M} (H_u H_d)^2 + \frac{\lambda^{ij}}{M} (L_i H_u) (L_j H_u)$$  \hspace{1cm} (3.1)

The $F_{H_d} = 0$ constraint following from (3.1) is $\lambda H_u^a (H_u H_d) = 0$, which upon contraction with $\epsilon_{\alpha \beta} H_d^\beta$ immediately implies $H_u H_d = 0$ if $\lambda \neq 0$. The constraint $F_{H_u} = 0$ then similarly implies, after multiplying by $H_u$, that $\lambda^{ij} (L_i H_u) (L_j H_u) = 0$, which in turn requires $L_i H_u = 0$ for all $i$ as long as $\det \lambda^{ij} \neq 0$. Since the terms in (3.1) are the only ones allowed in the $n = 4$ superpotential which involve $H_u$ and $H_d$, it is clear that there can be no interference between the $F$-constraints on the monomials $H_u H_d$ and $L_i H_u$ and those on the remaining monomials in the renormalizable flat basis. Thus we find that with generic couplings $\lambda \neq 0$ and $\det \lambda^{ij} \neq 0$, all flat directions involving $H_u$ or $H_d$ are lifted by $n = 4$ terms in the superpotential, i.e., dimension 6 terms in the scalar potential.
The remaining renormalizable flat directions which do not involve Higgs fields are described by gauge-invariant monomials

$$
\begin{align*}
LLe; & \quad udd; \\
udd; & \quad QdL; \\
QdQd; & \quad QQL; \\
QdQd; & \quad QuQd; \\
QQQQu; & \quad (QQQ)_4 LLe; \\
(QQQ)_4 LLe; & \quad uuuee; \\
uuuee; & \quad QuQe; \\
uudQQd. & 
\end{align*}
$$

With all monomials containing Higgs fields already constrained to vanish, the only superpotential terms which can play a role in lifting these are ones which contain neither $H_u$ nor $H_d$ (if $n$ is even), or exactly one of $H_u$ or $H_d$ (if $n$ is odd), because of matter parity conservation. For example, at $n = 4$ the part of the superpotential which is pertinent for flat directions not involving Higgs fields is just a sum over the 213 linearly independent monomials of the correct dimensionality:

$$
\begin{align*}
W_4 &= W_4^{QQQL} + W_4^{QuQd} + W_4^{QuLe} + W_4^{uude}, \\
W_4^{QQQL} &= \sum_{I=1}^{24} \frac{\alpha_I}{M} (QQQL)_I; \\
W_4^{QuQd} &= \sum_{I=1}^{81} \frac{\beta_I}{M} (QuQd)_I; \\
W_4^{QuLe} &= \sum_{I=1}^{81} \frac{\gamma_I}{M} (QuLe)_I; \\
W_4^{uude} &= \sum_{I=1}^{27} \frac{\delta_I}{M} (uude)_I.
\end{align*}
$$

For odd $n$, only $F_{H_u} = 0$ or $F_{H_d} = 0$ constraints can help to lift the remaining flat directions, while for even $n$, only $F_Q = 0$, $F_L = 0$, $F_u = 0$, $F_d = 0$, and $F_e = 0$ impose constraints on the remaining polynomials.

The existence of flat directions can be viewed as an “accidental” consequence of the limitations imposed on the superpotential at a given level $n$ by the requirements of gauge invariance and matter parity. As discussed in the Introduction, the number of $F$-constraints always formally exceeds the number of gauge-invariant degrees of freedom, so that flat directions only occur because in certain exceptional directions in field space, the $F$-constraints are trivially satisfied. It might seem intuitively clear, then, that the flat directions which have the greatest possibility of surviving unlifted to large $n$ are those which involve only a few different superfields, since in such special directions many of the $F$-terms vanish automatically. This intuition works well in most cases, but as we shall see, it suffers one major exception in the MSSM. In the following discussion, we study first the cases of flat directions which involve only two different types of chiral superfields.

- **$L, e$ flat directions.** Flat directions involving only the chiral superfields $L$ and $e$ are parameterized by gauge-invariant operators $LLe$. There are 9 linearly independent monomials of this type, since there are 3 ways to assign family indices to $LL$ and 3 $e$’s. However, not all
of these are functionally independent, because of the redundancy constraints (2.7), which in this case take the form

$$(L_i L_j e_k) (L_i' L_j' e_{k'}) = (L_i L_j e_{k'}) (L_i' L_j' e_k). \quad (3.3)$$

These constraints allow us to solve for the 9 $LL e$ monomials in terms of just 5 of them, which may be taken to be $L_1 L_2 e_i$, $L_1 L_3 e_1$, $L_2 L_3 e_1$ in an arbitrary basis, as long as $L_1 L_2 e_1 \neq 0$. [Alternatively, one may note that there are 6 complex degrees of freedom in $L$ and 3 in $e$, subject to 4 real gauge constraints from $SU(2)_L \times U(1)_Y$ D-flatness, leaving 5 complex degrees of freedom after 4 additional phases are gauge-fixed.] There are 2 independent constraints on these monomials due to the $F_{H_d}^0 = 0$ condition following from the renormalizable superpotential. Therefore the moduli subspace of renormalizable flat directions of the type $LL e$ has 3 complex dimensions. None of the $n = 4$ superpotential terms can lift the remaining flat directions, since they all involve at least two fields other than $L$ and $e$, so that the $F$-terms resulting from them must involve at least one field other than $L$ and $e$. At the $n = 5$ level the only relevant superpotential terms are of the form

$$W_5 \supset \frac{1}{M^2} H_u LLL e. \quad (3.4)$$

Following from this are two independent constraints $F_{H_u}^\alpha = 0$ which after multiplying by $\epsilon_{\alpha \beta} L^\beta e$ are realized non-linearly on the monomials $LL e$. So the moduli subspace of flat directions which remain unlifted at $n = 5$ has one complex dimension. Finally at the $n = 6$ level the 9 independent constraints $F_L = 0$ and $F_e = 0$ obtained from

$$W_6 \supset \frac{1}{M^3} LLeLL e \quad (3.5)$$

provide an overconstraining set of requirements on the monomials $LL e$. Therefore we find that all flat direction of the type $LL e$ are lifted by the $n = 6$ non-renormalizable superpotential.

- u,d flat directions. Flat directions involving only $u$ and $d$ fields are labeled by the gauge-invariant monomials $udd$. There are 9 linearly (and functionally) independent such monomials in the basis $B$. Clearly, none of the renormalizable ($n = 3$) superpotential terms can yield $F$-constraints which affect these monomials. At the $n = 4$ level, the relevant part of the superpotential is $W_4^{udd}$ of (3.2). The resulting $F_e = 0$ constraints are realized non-linearly on the $udd$ monomials, after multiplying by $ddd$ and using (2.2), schematically:

$$ddd \frac{\partial}{\partial e} W_4^{udd} = \sum (udd)(udd) = 0. \quad (3.6)$$
Since there are only 3 such constraints that are functionally independent, (one for each of the \( e_i \) with respect to which the derivatives are taken), the moduli subspace of directions of the \( udd \) type which remain flat at the \( n = 4 \) level has 6 complex dimensions. At the \( n = 5 \) level of the superpotential, there are no available \( F \)-terms which can constrain the monomials \( udd \) by themselves. At the \( n = 6 \) level, one finds terms of the form

\[
W_6 \supset \frac{1}{M^3} uddudd . \tag{3.7}
\]

The 18 independent constraints \( F_u = 0 \) and \( F_d = 0 \) following from this will clearly over-constrain the remaining 6 complex degrees of freedom, so that all monomials \( udd \) must vanish. So we find that all flat directions involving \( u, d \) are lifted at the level \( n = 6 \).

- \( Q,L \) flat directions. Flat directions involving only fields \( Q \) and \( L \) are associated with gauge-invariant monomials \( QQQL \). There are 24 linearly independent monomials of the type \( QQQL \). [Recall that there are 8 linearly independent doublets \( QQQ \) and three \( L \)s, and that the results of the previous section show that monomials involving \( (QQQ)_4 \) can be eliminated except in terms which involve Higgs fields or \( e \), which are not in question here.]

Using the identities (2.2) and (2.6), one can show that only 12 of these monomials are actually functionally independent. [This can alternatively be understood by the following counting: there are 18 independent \( Q \)’s and 6 independent \( L \)’s, subject to 12 real gauge constraints from \( SU(3)_C \), \( SU(2)_L \), and \( U(1)_Y \), and 12 phase gauge-fixings.] None of the \( F \)-constraints from \( n = 3 \) superpotential terms can restrict these monomials. At the \( n = 4 \) level, the relevant superpotential terms are just those which have the same form as the monomials we are trying to lift, namely \( W_4^{QQQL} \) of (3.2). The 24 functionally independent \( F \)-constraints

\[
\frac{\partial}{\partial Q_i^a} W_4^{QQQL} = 0; \quad \frac{\partial}{\partial L_i^a} W_4^{QQQL} = 0 \tag{3.8}
\]

ought to overconstrain the 12 functionally independent variables \( QQQL \). Let us demonstrate the validity of this sort of counting argument by proving explicitly that all of the flat directions \( QQQL \) are indeed lifted. From (3.8) one obtains constraints linear in the monomials \( QQQL \):

\[
Q_j \frac{\partial}{\partial Q_i} W_4^{QQQL} = \sum (QQQL) = 0; \quad L_j \frac{\partial}{\partial L_i} W_4^{QQQL} = \sum (QQQL) = 0 . \tag{3.9}
\]

Of these 18 equations, 17 are linearly independent. In practice, one can choose random numerical values for the 24 couplings \( \alpha_J \), and use (3.9) to solve for the 24 \( QQQL \) monomials in terms of 7 unknowns \( x_1 \ldots x_7 \). [The last constraint encountered in (3.9) will always
amount to 0 = 0.] Then the non-linear redundancy constraints (2.2) can be rewritten using (2.6) in the form
\[ \sum (QQQL)(QQQL) = 0 \] (3.10)
which become homogeneous quadratic equations in the 7 unknowns \( x_1 \ldots x_7 \).

Now, there do exist mathematical algorithms well-known in algebraic geometry (e.g. the methods of resultant polynomials [15] or Gröbner bases [16]) which can in principle be used to decide whether such systems of simultaneous non-linear equations have non-trivial solutions. Unfortunately, these methods are of no practical use at the level of complexity encountered here or in the examples below; the required number of algebraic operations is demonstrably finite, but quite astronomical.

In the present example, fortunately, it is still possible to prove rigorously and explicitly using a more brutish method that there is generically no non-trivial solution to the constraints. One may simply treat the 28 unknowns \( z_{ij} \equiv x_i x_j \) as independent variables. Then one can show that 20 of the equations (3.10) are linearly independent in the \( z_{ij} \). To complete the proof, one may note that we have so far only used 17 of the 24 functionally independent equations (3.8). Additional constraints are obtained from
\[ LLQQQ^a \frac{\partial}{\partial L^a} W_4^{QQQ} = \sum (QQQL)(QQQL) = 0 \] (3.11)
These constraints are quadratic in the monomials \( QQQL \), hence linear in the \( z_{ij} \). One can check that indeed exactly \( 24 - 17 = 7 \) of these constraints are linearly independent of each other and the previous constraints, in terms of the \( z_{ij} \). Thus one can solve for all of the \( z_{ij} \) in terms of, say, \( z_{11} \). Finally one can compute \( z_{11} z_{22} - z_{12}^2 \), which is constrained to be 0 by construction. Written in terms of the last remaining unknown \( z_{11} \), it is easy to check that only the trivial solution \( z_{11} = 0 \) exists. Hence, all of the \( QQQL \) monomials are constrained to be zero, so that all of the flat directions associated with them are indeed lifted by the \( n = 4 \) superpotential.

• \( d,L \) flat directions. Flat directions involving just the fields \( d,L \) are labeled by the gauge-invariant monomials \( dddLL \). These flat directions are a particularly exceptional case, because few terms in the superpotential involve only \( d \) and \( L \) and just one other chiral superfield. (Superpotential terms containing two or more fields other than \( d \) and \( L \) will result in \( F \)-terms which cannot constrain the \( dddLL \) monomials, except in combinations with other monomials.) The number of linearly independent monomials \( dddLL \) is 3, because there is only one color-singlet combination \( ddd \), and 3 possible family index assignments
for $LL$. All three monomials $dddLL$ are also functionally independent, in the sense that none of them can be eliminated by using the non-linear redundancy identities. It is easy to see that the lowest dimension term in the superpotential which can lift flat directions of the type $dddLL$ is the $n = 5$ term

$$W_5 \supset \frac{\lambda^i}{M^2} dddH_dL_i$$

(3.12)

The requirement $F_{H_d}^\alpha = 0$ following from this is just

$$\lambda^i dddL_i^\alpha = 0$$

(3.13)

which yields only 2 independent constraints on the 3 monomials $dddLL$. Therefore, if $\lambda^i \neq 0$, exactly one of the flat directions of the type $dddLL$ is not lifted at the level $n = 5$. To make this transparent, one may choose a basis in family space such that $\lambda^i \propto \delta^i_1$. Then clearly (3.13) forces the monomials $dddL_1L_2$ and $dddL_1L_3$ to vanish, but can never constrain the monomial $dddL_2L_3$. There is no possibility of $F$-constraints on $dddLL$ flat directions coming from $n = 6$ terms in the superpotential, since these all involve at least two fields other than $d$ and $L$, so that at least one field other than $d$ and $L$ remains after taking the derivative. At the $n = 7$ level, one finds superpotential terms of the form

$$W_7 \supset \frac{1}{M^4} H_uLLLddd$$

(3.14)

The two independent $F_{H_u}^\alpha = 0$ constraints following from this term are realized non-linearly on the monomials $dddLL$ (after multiplying by $dddL$) and finally lift the last remaining $dddLL$ flat direction. Therefore we find that exactly one of the $dddLL$ flat directions is not lifted until the $n = 7$ level.

- $u,e$ flat directions. The flat directions involving only the fields $u$ and $e$ correspond to gauge-invariant monomials $uuuee$. There are 6 independent monomials of the type $uuuee$, since there is only one color-singlet combination $uuu$, and 6 independent products $e_i e_j$. Of these, only three combinations are functionally independent, because of identities of the type (2.7), e.g., $(uuue_1e_2)^2 = (uuue_1e_1)(uuue_2e_2)$. The only superpotential terms with $n < 9$ which can yield $F$-terms constraining these monomials alone is $W_4^{uude}$ of (3.2). It is not difficult to show that the 9 independent constraints

$$\frac{\partial}{\partial d^i} W_4^{uude} = 0$$

(3.15)

are sufficient to require all $uuuee$ monomials to be zero. [One can simply consider the linear equations obtained by contracting (3.15) with $u_j^\theta e_k$, with random numerical values
for the couplings $\delta_I$ in (3.2).] Therefore we find that all flat directions associated with the monomials $uuuee$ are lifted by the non-renormalizable superpotential with $n = 4$.

- **Q,u flat directions.** These are parameterized by the monomials $QQQQu$ in the basis $B$ of Table 3. One can show, using (2.2) and (2.6) that there are 54 linearly independent monomials $QQQQu$. Of these, 15 are functionally independent in the sense that they cannot be eliminated using non-linear constraints also following from (2.2) and (2.6). At the renormalizable ($n = 3$) level, there are 2 functionally independent $F^\alpha_{H_u}$ constraints, so that the moduli subspace of renormalizable flat directions involving $Q$ and $u$ has 13 complex dimensions. At the $n = 4$ level, the pertinent terms in the superpotential are $W^u_{QuQd}^4$ and $W^QQQQL^4$ in (3.2). We obtain 9 functionally independent constraints on the monomials $QQQQu$ from $F_d$ of $W^u_{QuQd}$, and 6 more from $F_L$ of $QQQL$. Thus there are a total of 17 independent constraints from the $n = 3$ and $n = 4$ superpotentials, which should therefore overconstrain the 15 functionally independent degrees of freedom, so that all of the flat directions are lifted at $n = 4$.

This concludes the discussion of the flat directions which involve only two different types of chiral superfields. Note that there can be no flat directions of the type $Q,d$ or $Q,e$ or $d,e$ or $L,u$, since these clearly cannot satisfy the $D_Y = 0$ constraint. Next we consider the flat directions which involve three different types of chiral superfields.

- **L,d,e flat directions.** The flat directions involving only $L$, $d$, and $e$ chiral superfields are associated with the 12 linearly independent monomials $LLe$ and $dddLL$. Of these, only 6 are functionally independent when the non-linear constraints (2.7) are taken into account. Of course, these flat directions contain as two exceptional subspaces the flat directions $L,e$ (with $ddd = 0$) and $d,L$ (with all $e_i = 0$), which have already been discussed, so here we can assume that $ddd \neq 0$ and some $e_i \neq 0$ (and of course some $L_iL_j \neq 0$), so that these can be divided by in reducing the number of independent complex degrees of freedom to 6. Now at the level of the renormalizable ($n = 3$) superpotential, there are 2 independent $F^\alpha_{H_d}$ = 0 constraints. None of the $n = 4$ superpotential terms produces a relevant $F$-constraint. At the level $n = 5$, one finds 2 constraints $F^\alpha_{H_u}$ = 0 from the superpotential term (3.4) and 2 independent constraints $F^\alpha_{H_d}$ = 0 from the superpotential term (3.12). Thus other than the degenerate cases discussed above in which either $ddd$ or all $e_i$ vanish, the $L,d,e$ flat directions are lifted at the $n = 5$ level.

- **L,d,u flat directions.** These flat directions are labeled by the 12 linearly (and functionally) independent monomials $udd$ and $LLddd$. None of the renormalizable superpotential terms help to lift these flat directions. At the $n = 4$ level, one obtains 3 constraints $F_e = 0$
from the superpotential term $W_{uude}^4$ of (3.2). At the $n = 5$ level, one finds 2 constraints $F_{H_u} = 0$ from the superpotential term

$$W_5 \supset \frac{1}{M^2} uddH_uL$$

(3.16)

and 2 constraints $F_{H_d} = 0$ from (3.12). At the $n = 6$ level one obtains 18 constraints $F_Q = 0$ from the superpotential terms

$$W_6 \supset \frac{1}{M^3} uddQdL$$

(3.17)

so that the system is overconstrained. Therefore, other than the exceptional case discussed above with all $u_i = 0$, the $L, d, u$ flat directions are lifted at $n = 6$.

• **L,u,e flat directions.** The flat directions involving $L, u, e$ are parameterized by the 15 linearly independent monomials $LLe$ and $uuuee$, subject to non-linear constraints of the type (2.7) which reduce the number of functionally independent monomials to 6. At the renormalizable level one has only 2 constraints on these 6 degrees of freedom from $F_{H_d} = 0$. At the $n = 4$ level, the 18 $F_Q = 0$ constraints from $W_4^{Qul}e$ and the 9 $F_d = 0$ constraints from $W_4^{uude}$ clearly overconstrain the system, lifting all of these flat directions at $n = 4$, except the ones treated above with $u_i = 0$.

• **u,d,e flat directions.** These are associated with the 42 linearly independent monomials $udd$, $uuuee$ and $uude$. Of these, only 12 are functionally independent after taking into account the non-linear identities of the form (2.2) and (2.7), besides the exceptional cases discussed above in which all $e_i$ vanish or all $d_i$ vanish. The renormalizable superpotential does not lift any of these terms. At the $n = 4$ level, the 21 independent constraints $F_u = 0$, $F_d = 0$ and $F_e = 0$ following from the superpotential term $W_4^{uude}$ are enough to lift all flat directions of this type, other than in the special case treated above that all $e_i$ vanish.

• **Q,u,e flat directions.** This is a quite exceptional case, for a somewhat subtle reason. These flat directions are associated with gauge-invariant monomials $uuuee$, $QQQQu$, and $QuQue$. Using the non-linear redundancy constraints, one finds that 18 of these monomials are functionally independent. Now, at the renormalizable level, there are 2 independent constraints $F_{H_u}$, so that the moduli subspace of renormalizable flat directions involving $Q, u, e$ has 16 complex dimensions. At the $n = 4$ level, one has 6 independent constraints $F_L$ and 9 independent constraints $F_d$, which schematically take the form

$$0 = F_L = \sum (QQQ) + \sum (Que);$$

(3.18)

$$0 = F_d = \sum (QuQ) + \sum (uue)$$

(3.19)
respectively. These constraints are realized linearly on the monomials after multiplying (3.18) by $Qu$ and (3.19) by $QQ$ and $ue$, but only 15 such constraints are functionally independent. Therefore, one complex degree of freedom must remain unlifted at the $n = 4$ level! One can easily check that no superpotential terms with $5 \leq n \leq 8$ can affect the $Q, u, e$ flat directions, since they all contain at least two superfields other than $Q, u, e$. At the $n = 9$ level there are many terms which can lift the last remaining flat direction, for example

$$W_9 \supset \frac{1}{M^6} QuQuQuH_{ded} .$$

The $F_{H_d}$ constraints coming from this term will clearly lift the last remaining $Q, u, e$ flat direction. The unique $Q, u, e$ flat direction which survives to $n = 9$ is at least formally the “flattest” of the flat directions of the MSSM, being lifted only by terms of dimension 16 in the supersymmetry-preserving part of the scalar potential. This result is perhaps somewhat surprising, as we found above that all of the $Q, u$ flat directions and $u, e$ flat directions, which are special cases of $Q, u, e$ flat directions, are lifted already at $n = 4$. The subtlety is that at the $n = 4$ level, the $F$-terms which were responsible for lifting the $Q, u$ and $u, e$ flat directions are the same, and cannot be counted separately in the more general case $Q, u, e$. The specific identification of the flat direction which survives to $n = 9$ depends in an extremely complicated non-linear way on the unknown $n = 4$ couplings.

- **$Q,L,d$ flat directions.** These flat directions are associated with the 54 linearly independent monomials $QQQL$, $ddLL$, and $QdL$. Only 21 of these monomials are functionally independent after taking into account the non-linear redundancy constraints. At the renormalizable level, there are just 2 constraints $F_{H_d} = 0$ on these 21 degrees of freedom. At the $n = 4$ level, there are 24 constraints $F_Q = 0$ and $F_L = 0$ from $W_{4}^{QQQL}$ of (3.2) and 9 more constraints $F_u = 0$ from $W_{4}^{QuQu}$. Hence all $Q, L, d$ flat directions are lifted at $n = 4$, other than the exceptional case of $L, d$ flat directions as discussed above.

- **$Q,u,d$ flat directions.** Here the monomials which label the $D$-flat directions are $udd$, $QQQQu$, $QuQd$, and $uddQdQd$. Of these, 24 can be shown to be functionally independent using the non-linear redundancy constraints. At the renormalizable level, there are 4 independent constraints from $F_{H_u} = 0$ and $F_{H_d} = 0$. At the $n = 4$ level, one has 36 constraints $F_Q = 0$, $F_u = 0$ and $F_d = 0$ from the superpotential term $W_{4}^{QuQd}$ of (3.2) and 3 more constraints $F_e = 0$ from $W_{4}^{uude}$. Thus all $Q, u, d$ flat directions are lifted at $n = 4$ except in the case treated above in which the monomials dependent on $Q$ vanish.

- **$Q,L,e; Q,L,u; Q,L,u,e; Q,L,d,e; Q,L,u,d; Q,u,d,e; L,u,d,e; Q,L,u,d,e$ flat directions.** Using methods which should be clear by now, one can show that each of these flat directions
is lifted by the $n = 4$ superpotential, except in degenerate cases already covered in the above discussion. Note that there can be no flat directions of the type $Q, d, e$, since these clearly cannot satisfy the $D_Y = 0$ constraint.

Some of the essential features of the preceding discussion are summarized in Tables 4 and 5. In Table 4, for each type of flat direction we put a check mark for each superpotential term which produces a pertinent $F$-constraint. (In “overkill” cases where all flat directions of a given type are already lifted by superpotential terms with smaller $n$, no entry is made.) In Table 5, we show the complex dimension of the moduli space associated with each type of flat direction which remains unlifted at each level $n$. A check mark indicates the minimum level $n$ at which all flat directions of a given type are lifted.

4. Discussion

In this paper we have provided a catalog of the (nearly) flat directions of the MSSM. We found that there exists a unique direction, involving the fields $(Q, u, e)$ in a non-trivial way, which is formally “flattest”. It is lifted only by soft supersymmetry-breaking effects and by $n = 9$ terms in the superpotential, which correspond to dimension 16 terms in the scalar potential. The next flattest direction involves the fields $(L, d)$ and is lifted at the level $n = 7$, corresponding to dimension 12 supersymmetric terms in the scalar potential. There are several flat directions with sliding VEVs for fields $(L, e)$, for $(u, d)$, and for $(L, u, d)$ which are lifted at the $n = 6$ level corresponding to dimension 10 terms in the scalar potential, and some flat directions involving $(L, d, e)$ are lifted at the $n = 5$ level corresponding to dimension 8 terms in the scalar potential. All other flat directions are lifted already at the $n = 4$ level, corresponding to dimension 6 terms in the supersymmetric part of the scalar potential. Our results seem to differ slightly with those found in the second reference in [2], although this fact is probably quite peripheral to the main points of that paper. It should be noted that the notion of “generic” couplings may have to allow for dimensionless numbers which are several orders of magnitude less than unity; after all, the Yukawa coupling for the electron in the standard model is less than $10^{-5}$. Therefore the formal distinction between flat directions lifted at $n = 6$ and at higher $n$ may be quite moot in practice, in view of our lack of knowledge of the non-renormalizable couplings. This situation could change in the context of model extensions of the MSSM which make more specific predictions for the non-renormalizable couplings.

One possible application of our results is to the question of the existence of unbounded
from below directions or non-trivial global minima of the scalar potential along nearly flat directions. It is possible that the running (mass)\(^2\) parameters of the scalars of the MSSM could become negative at very high energy scales. Even within the framework of unified supersymmetric models, it is conceivable [17] that the common scalar (mass)\(^2\) parameter \(m_0^2\) could be negative at the unification scale. This does not necessarily lead to VEVs for squarks or sleptons. The reason is that the prediction of such VEVs is only trustworthy if the running parameters are evaluated at the scale of the would-be VEVs. The important terms in the scalar potential in this regard are (1.1) and (1.3). Thus the running (mass)\(^2\) parameter \(m^2(Q)\) corresponding to a renormalizable flat direction lifted at level \(n\) must remain positive for, roughly,

\[
Q < \left( \frac{m_WM^{n-3}}{|\lambda|} \right)^{1/(n-2)}
\]

so that equations for a putative VEV can have no solution. Such a constraint has been examined very recently in [18] for one particular flat direction of the type described by the monomials \(udd\).

Terms in the scalar potential of the form (1.5) are also potentially dangerous with regard to unwanted global minima of the scalar potential at large field strength if \(s < n\). However, it is easy to check that for the seemingly most-dangerous \(Q, u, e\) and \(L, d\) flat directions, one has \(s \geq 10\) because of the requirement of matter parity invariance. Similarly, for the flat directions with smaller \(n\) one finds that the maximum \(n\) never exceeds the corresponding \(s\).

We must emphasize that our results depend on certain controvvertible assumptions about physics at very high energies. For example, if there exist at an intermediate scale below \(M\) some chiral superfields with vector-like quantum numbers, these could have non-trivial superpotential couplings to the chiral superfields of the MSSM, lifting some of the flat directions at smaller \(n\) than found here. As an example, one could imagine introducing gauge-singlet neutrino chiral superfields \(\nu\) so that \(W_{\text{renorm}}\) would include couplings schematically of the form \(\mu_I\nu\nu + y_\nu H_u L\nu\), implementing a supersymmetric seesaw mechanism [19] for neutrino masses. At the \(n = 4\) level, one would then expect to have matter parity invariant couplings

\[
W_4 \supset \frac{1}{M}(LLe\nu + udd\nu + QdL\nu),
\]

and the \(F_\nu = 0\) constraints could help to lift some of the flat directions discussed above which survived beyond \(n = 4\) in the MSSM alone. New symmetries can also clearly
affect our conclusions. In the case of global symmetries, some of the couplings assumed
to be generic here could actually be zero or otherwise restricted, so that flat directions
could remain unlifted to higher $n$. Conversely, if there are additional unbroken gauged
symmetries above some intermediate scale, the corresponding $D$-terms would lift many of
the flat directions. Clearly for any extension of the MSSM, the status of flat directions at
very large field strength must be reexamined. The methods used here may also serve as a
helpful guide in such cases.

Acknowledgments: We are grateful to Michael Dine, Gordy Kane, John March-Russell
and Lisa Randall for helpful discussions. The work of TG and SPM was supported in
part by the U.S. Department of Energy. The work of CK was supported in part by the
U.S. Department of Energy under contract #DE-FG02-90ER40542 and by the Monell
Foundation.

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### Table 1: The basis $B_3$ of $SU(3)_C$-invariant monomials

|  | SU(2)$_L \times$ U(1)$_Y$ | No. |
|---|---|---|
| e | (1,1) | 3 |
| L | (2,−1/2) | 3 |
| H$_u$ | (2,1/2) | 1 |
| H$_d$ | (2,−1/2) | 1 |
| uuu | (1,−2) | 1 |
| uud | (1,−1) | 9 |
| udd | (1,0) | 9 |
| ddd | (1,1) | 1 |
| Qu | (2,−1/2) | 9 |
| Qd | (2,1/2) | 9 |
| QQQ | (2,1/2) | 8 |
| (QQQ)$_4$ | (4,1/2) | 1 |

### Table 2: The basis $B_{32}$ of $SU(3)_C \times SU(2)_L$-invariant monomials

|  | U(1)$_Y$ |  | U(1)$_Y$ |  | U(1)$_Y$ |  |
|---|---|---|---|---|---|---|
| e | 1 | QuH$_u$ | 0 | (QQQ)$_4$LLL | −1 |  |
| LH$_u$ | 0 | QdH$_u$ | 0 | (QQQ)$_4$LLH$_d$ | −1 |  |
| H$_u$H$_d$ | 0 | QuH$_d$ | −1 | (QQQ)$_4$LLH$_u$ | 0 |  |
| LH$_d$ | −1 | QQH$_u$ | 1 | (QQQ)$_4$LH$_u$H$_d$ | 0 |  |
| LL | −1 | QQH$_d$ | 0 | (QQQ)$_4$H$_u$H$_d$H$_d$ | 0 |  |
| uuu | −2 | QQQL | 0 | (QQQ)$_4$LLH$_d$H$_d$ | −1 |  |
| uud | −1 | QdQd | 1 | (QQQ)$_4$LH$_u$H$_u$ | 1 |  |
| udd | 0 | QuQd | 0 | (QQQ)$_4$LH$_u$H$_d$ | 1 |  |
| ddd | 1 | QuQu | −1 | (QQQ)$_4$H$_u$H$_d$H$_d$ | 2 |  |
| QdL | 0 | QQQQu | 0 | (QQQ)$_4$H$_u$H$_u$H$_d$ | −1 |  |
| QuL | −1 | QQQQQu | 1 |  |  |  |
Table 3: The basis $B$ of gauge-invariant monomials

|                            | $B - L$ | Always lifted by $W_{\text{renorm.}}$? |
|-----------------------------|---------|---------------------------------------|
| $LHu$                       | -1      |                                       |
| $HuHd$                      | 0       |                                       |
| $uud$                       | -1      |                                       |
| $LLe$                       | -1      |                                       |
| $QdL$                       | -1      |                                       |
| $QuHu$                      | 0       | $\sqrt{}$                            |
| $QdHd$                      | 0       | $\sqrt{}$                            |
| $LH_{a\varepsilon}$        | 0       | $\sqrt{}$                            |
| $QQQL$                      | 0       |                                       |
| $QuQd$                      | 0       |                                       |
| $QuLe$                      | 0       |                                       |
| $uude$                      | 0       |                                       |
| $QQQH_d$                    | 1       | $\sqrt{}$                            |
| $QuH_d\varepsilon$         | 1       | $\sqrt{}$                            |
| $dddLL$                     | -3      |                                       |
| $uuuee$                     | 1       |                                       |
| $QuQue$                     | 1       |                                       |
| $QQQQu$                     | 1       |                                       |
| $dddLH_d$                   | -2      | $\sqrt{}$                            |
| $uudQdH_u$                  | -1      | $\sqrt{}$                            |
| $(QQQ)_4LLHu$               | -1      | $\sqrt{}$                            |
| $(QQQ)_4LHuH_d$             | 0       | $\sqrt{}$                            |
| $(QQQ)_4H_uH_dH_d$          | 1       | $\sqrt{}$                            |
| $(QQQ)_4LLe$                | -1      |                                       |
| $uudQdQd$                   | -1      |                                       |
| $(QQQ)_4LLH_{a\varepsilon}$ | 0       | $\sqrt{}$                            |
| $(QQQ)_4LH_dH_{a\varepsilon}$ | 1   | $\sqrt{}$                            |
| $(QQQ)_4H_dH_dH_{a\varepsilon}$ | 2 | $\sqrt{}$ |
Table 4: Flat directions (not involving Higgs fields) and the superpotential terms which lift them

| Flat directions | Superpotential Terms | n=3 | n=4 | n=5 | n=6 | n=7 | n=8 | n=9 |
|-----------------|----------------------|-----|-----|-----|-----|-----|-----|-----|
| L,d             |                      |     |     |     |     |     |     |     |
| L,e             | √                    |     |     |     |     |     |     |     |
| u,d             | √                    |     |     |     |     |     |     |     |
| u,e             | √                    |     |     |     |     |     |     |     |
| Q,L             | √                    |     |     |     |     |     |     |     |
| Q,u             | √                    |     |     |     |     |     |     |     |
| Q,u,e           | √                    |     |     |     |     |     |     |     |
| L,u,d           | √                    |     |     |     |     |     |     |     |
| L,d,e           | √                    |     |     |     |     |     |     |     |
| L,u,e           | √                    |     |     |     |     |     |     |     |
| u,d,e           | √                    |     |     |     |     |     |     |     |
| Q,L,e           | √                    |     |     |     |     |     |     |     |
| Q,L,d           | √                    |     |     |     |     |     |     |     |
| Q,L,u           | √                    |     |     |     |     |     |     |     |
| Q,u,d           | √                    |     |     |     |     |     |     |     |
| L,u,d,e         | √                    |     |     |     |     |     |     |     |
| Q,L,d,e         | √                    |     |     |     |     |     |     |     |
| Q,L,u,e         | √                    |     |     |     |     |     |     |     |
| Q,u,d,e         | √                    |     |     |     |     |     |     |     |
| Q,L,u,d         | √                    |     |     |     |     |     |     |     |
| Q,L,u,d,e       | √                    |     |     |     |     |     |     |     |
## Table 5: Complex dimensionalities of moduli subspaces in the MSSM

| Flat directions | D-flat | Superpotential Terms | n=3 | n=4 | n=5 | n=6 | n=7 | n=8 | n=9 |
|-----------------|--------|----------------------|-----|-----|-----|-----|-----|-----|-----|
| L, d            | 3      | 3                    | 3   | 3   | 1   | 1   | √   |     |     |
| L, e            | 5      | 3                    | 3   | 3   | 1   |     |     |     | √   |
| u, d            | 9      | 9                    | 6   | 6   |     |     |     |     |     |
| u, e            | 3      | 3                    |     |     |     |     |     |     |     |
| Q, L            | 12     | 12                   |     |     |     |     |     |     |     |
| Q, u            | 15     | 13                   |     |     |     |     |     |     |     |
| Q, u, e         | 18     | 16                   | 1   | 1   | 1   | 1   | 1   |     | √   |
| L, u, d         | 12     | 12                   | 9   | 5   |     |     |     |     |     |
| L, d, e         | 6      | 4                    | 4   |     |     |     |     |     |     |
| L, u, e         | 6      | 4                    |     |     |     |     |     |     |     |
| u, d, e         | 12     | 12                   |     |     |     |     |     |     |     |
| Q, L, e         | 15     | 13                   |     |     |     |     |     |     |     |
| Q, L, d         | 21     | 19                   |     |     |     |     |     |     |     |
| Q, L, u         | 21     | 19                   |     |     |     |     |     |     |     |
| Q, u, d         | 24     | 20                   |     |     |     |     |     |     |     |
| L, u, d, e      | 15     | 13                   |     |     |     |     |     |     |     |
| Q, L, d, e      | 24     | 22                   |     |     |     |     |     |     |     |
| Q, L, u, e      | 24     | 20                   |     |     |     |     |     |     |     |
| Q, u, d, e      | 27     | 23                   |     |     |     |     |     |     |     |
| Q, L, u, d      | 30     | 26                   |     |     |     |     |     |     |     |
| Q, L, u, d, e   | 33     | 29                   |     |     |     |     |     |     |     |