THE FRANK TENSOR AS A BOUNDARY CONDITION IN INTRINSIC LINEARIZED ELASTICITY

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Abstract. The Frank tensor plays a crucial role in linear elasticity, and in particular in the presence of dislocation lines, since its curl is exactly the elastic strain incompatibility. Further, the Frank tensor also appears in Cesaro decomposition, and in Volterra theory of dislocations and disclinations, since its jump is the Frank vector around the defect line. The purpose of this paper is to show to which functional space the compatible strain $\epsilon$ belongs in order to imply a homogeneous boundary conditions for the induced displacement field on a portion $\Gamma_0$ of the boundary. This will allow one to define the homogeneous, or even the mixed problem of linearized elasticity in a variational setting involving the strain $\epsilon$ in place of displacement $u$. With other purposes, this problem was originally treated by Ph. Ciarlet and C. Mardare, and termed the intrinsic formulation. In this paper we propose alternative conditions on $\epsilon$ expressed in terms of $\epsilon$ and the Frank tensor $\text{Curl}^t \epsilon$ only, yielding a clear physical understanding and showing as equivalent to Ciarlet-Mardare boundary condition.

1. Introduction.

1.1. Overview and description of the main results. In the mathematical treatment of linearized elasticity, the basic model variable is most often the displacement field $u : \Omega \to \mathbb{R}^3$ (where $\mathbb{R}^3$ is the body manifold), with respect to which elastic problems are stated and solved. The linearized strain is then the symmetric gradient of displacement field, $\epsilon = \epsilon(u) := \nabla^S u$. However, in many computations and experiments, the strain is most naturally observable, hence becoming the main model variable. Sometimes, because of defects or other incompatibilities, the very notion of a displacement field does not make sense. Therefore, one would like to state the linear-elastic problem in terms of the strain $\epsilon$ (not necessarily taken as a symmetric gradient) as the basic unknown instead of the displacement $u$. This choice was made in [5], where the resulting model was called the intrinsic formulation of elasticity. The main difficulty is the treatment of the boundary conditions, since a condition such as $u|_{\Gamma_0} = 0$ for some $\Gamma_0 \subset \partial \Omega$ is not easily translated to a boundary condition on $\epsilon$.

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The main contribution in this paper at hand is giving the boundary conditions on $\epsilon$ a clearer geometric meaning, and also a definition of applicable strain spaces (defined at the beginning of Section 4). This is done by a Cartan-like approach of working with a moving orthonormal frame instead of an arbitrary coordinate system. Since this approach respects the geometry of the problem (the geometry of $\partial \Omega$ in this case), it gives rise to much simpler conditions.

1.2. The intrinsic formulation of elasticity. On the one hand, Physicists and mechanical Engineers mostly consider strain and stress as their basic model variables in Elasticity, both for theoretical and computational reasons. Indeed, given the stress tensor, the strain is well defined as soon as a constitutive law is provided, here a linear homogeneous, isothermal and isotropic law: the strain-stress constitutive law reads $\epsilon = C \sigma$, with $C$ the compliance tensor, i.e., fourth-rank (inverse) tensor of elasticity. On the other hand, Mathematicians prefer the displacement as basic model variable, from which the strain is defined by the kinematic relation $\epsilon = \nabla^S u$, whence the stress by a constitutive law. This choice presumably comes from the study of elliptic boundary-value problems, where the elasticity system is seen as a vector-valued extension of the elliptic equations in divergence form. Moreover, weak and variational formulations are most easily derived by means of the displacement, and show a convenient and elegant way of solving problems in Elasticity.

There are profound theoretical reasons to refrain from taking the displacement as main model variable. For instance, its possible multivaluedness, which is not to avoid from a Physical standpoint, since multivaluedness may have a meaning, but which must be addressed in an adequate manner in the chosen mathematical formalism. Another example is the reference configuration, from which the displacement is defined and which by definition is arbitrary: while natural in finite elasticity, it becomes somehow artificial in linearized elasticity, since Eulerian and Lagrangian field representations often coincide. Let us also mention plasticity or elastic bodies with defects, where stress and defect-free reference configurations might not exist (simultaneously), not to mention the possible use of intermediate configurations, which induce a plastic and an elastic deformation, whose Physical meaning is far from clear. In fact, what is a plastic distortion (i.e., the gradient of a plastic displacement) if no constitutive law exist for the rotations (i.e., its skewsymmetric part)? Further, we should also mention the fact that any rigourous model should in principle be proven independent of the choice the reference configuration. Lastly, in the presence of crystal defects like dislocations the very notion of displacement or velocity is not clearly defined at any scale. For instance, at the atomic scale, bonds can move while atoms remain fixed.

For these reasons, the intrinsic approach in linearized elasticity by Ph. Ciarlet and C. Mardare [5] constitutes a major breakthrough in Mathematical elasticity, which was able to reconcile in an elegant manner the two aforementioned approaches and Scientific communities. In their presentation, the strain is the main model variable, while variational formulations are sought. The displacement only appears in a second step if the Riemannian curvature tensor associated to the elastic metric vanishes (for the use of differential geometry concepts in Elasticity with defects, see e.g. [9, 10, 18]). These strains are said compatible, being immediately remarked [14] that the above geometric condition is equivalent (to the first order) to requiring that the strain has vanishing incompatibility. In Ciarlet and Mardare approach, a Differential Geometry setting is chosen (see also [3]), where the boundary under
analysis is defined by means of smooth enough immersions, from which the curvilinear basis, the metric, the symmetric connexion and the curvature tensors are derived. It should be emphasized that such derived curvilinear basis are indeed defined in the body itself as well as on its boundary, but are not mutually orthogonal, and have no particular physical meaning.

In contrast, while also based on differential geometry concepts, our wish is to define orthonormal basis in the body, which have a physical meaning, that is, are in some sense intrinsic to the body. Only in a second step, the appropriate functional spaces with hilbertian structure are introduced and proved well suited to study the elasticity problem in intrinsic terms. Further, being our curvilinear basis orthonormal, the associated metric (in the body and on its surface) reduces to the identity tensor. In particular, the covariant and contravariant components of a vector/tensor are coincident notions.

Our model variables will be the strain and the Frank tensor, defined as the transpose of the strain curl, and also bears a clear physical meaning. Our goal is thus to determine a well-posed variational problem in terms of the strain only and with prescribed conditions of these two quantities on a connected subset \( \Gamma_0 \) of the boundary. In our approach, the displacement appears only in a second step, as one of a two-field strain decomposition, and is to be considered by this procedure, as a mere mathematical object, whose interpretation as displacement is made for convenience, and obviously coincide with the classical displacement field in the special case of compatible strains, that is in defect-free bodies.

1.3. Notations and conventions. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \), i.e., the body \( \Omega \) is embedded in a Euclidean manifold. By smooth we mean \( C^\infty \), but this assumption could be considerably weakened. Let \( \mathcal{M}^3 \) denote the space of square 3-matrices, and \( \mathbb{S}^3 \) of symmetric 3-matrices. Curl, incompatibility and cross product with second-rank tensors are defined componentwise as follows with the summation convention on repeated indices. Let the Cartesian base be denoted by \( \{ e^i \} \) and the associated Cartesian coordinates by \( x^i \). In the following definition, \( E \) represents a second-rank tensor, \( N \) is a unit vector (which will be extended from the boundary to the domain), and \( \epsilon \) is the Levi-Civita symbol. Symbol \( \partial_x \) stands for partial derivative with respect to \( x_i \) (further in the paper \( \partial_i \) will mean a curvilinear derivative). In the Cartesian basis, one has:

\[
\begin{align*}
(\text{Curl } E)_{ij} &:= (\nabla \times E)_{ij} = \epsilon_{jkm} \partial_{x_k} E_{im}, \\
(\text{inc } E)_{ij} &:= (\text{Curl} \text{Curl}^t E)_{ij} = \epsilon_{ikm} \epsilon_{jln} \partial_{x_k} \partial_{x_l} E_{mn}, \\
(N \times E)_{ij} &:= -(E \times N)_{ij} = \epsilon_{jkm} N_k E_{im}.
\end{align*}
\]

Note that the expression of the incompatibility in a general curvilinear basis is a difficult issue addressed in [19]. With \( \epsilon \) the elastic strain, \( \text{Curl}^t \epsilon \) will be called the Frank tensor, and \( \text{inc } \epsilon \), the strain incompatibility.

1.4. Origin of the approach: Frank tensor and Cesaro-Volterra identities. As a first step, let us recall the problem of reconstructing a displacement from a given symmetric tensor (see, e.g., [13] for this classical topic). In linearized elasticity, if all functions involved are smooth enough, we prove that the displacement field \( u \) is completely defined in terms of the linearized strain tensor \( e \) and by recursive integral formulae (see (8)), which are computed explicitly.

Let \( e \in \mathcal{C}^\infty(\Omega, \mathcal{M}^3) \) be a symmetric tensor field such that \( \text{inc } e = 0 \) on \( \Omega \). Let us fix \( x_0, x \in \Omega \), and let \( \gamma \in \mathcal{C}^1([0,1], \Omega) \) be a curve in \( \Omega \) such that \( \gamma(0) = x_0 \) and
γ(1) = x. We define the following quantities:

\[
\begin{align*}
    w_i(x; \gamma) & := w_i(x_0) + \int_\gamma \epsilon_{ipn} \partial_{p} \epsilon_{mn}(y) dy, \\
    u_i(x; \gamma) & := u_i(x_0) + \int_\gamma (\epsilon_{iil} - \epsilon_{iit} w_l(y)) dy.
\end{align*}
\]

(1) (2)

Let us now prove that the quantities \(w(x)\) and \(u(x)\) defined in (1) and (2) do not depend on the choice of the path from \(x_0\) to \(x\). We will show that this is a consequence of the fact that \(\text{inc } e = 0\). In such case, the quantities \(w\) and \(u\) define two \(C^\infty\)-functions in \(\Omega\) that will be called the rotation and the displacement vectors associated to the strain \(e\), respectively. In order to prove this fact, we compute the jump of \(w\) and \(u\) between two arbitrary curves with the same endpoints, and observe that this quantity is zero if and only if the incompatibility tensor vanishes. These are exactly the well known Saint-Venant compatibility relations. Then rotation and displacement jumps are defined as

\[
\begin{align*}
    [w_i](x; x_0) & := w_i(x; \gamma) - w_i(x; \tilde{\gamma}), \\
    [u_i](x; x_0) & := u_i(x; \gamma) - u_i(x; \tilde{\gamma}),
\end{align*}
\]

(3) (4)

respectively. Here, \(\gamma\) and \(\tilde{\gamma}\) are two distinct curves with start- and endpoints \(x_0\) and \(x\). By Stokes theorem, the jumps will be non-vanishing as soon as \(\gamma - \tilde{\gamma}\) encloses at least one dislocation line (see [14]). In particular \([w_i]\) defines the Frank vector, associated to the Frank tensor by (1), while \([u_i]\) defines the Burgers vector, associated to the strain and to the Frank tensor by (2).

**Theorem 1.1** (Rotation and displacement jumps [14]). Let \(\Omega \subseteq \mathbb{R}^3\) be a simply-connected domain, let \(x_0 \in \Omega\) be prescribed, and let \(w, u \in C^\infty(\Omega, \mathbb{R}^3)\) be the functions defined in (1) and (2), respectively. Then the following formulae hold:

\[
\begin{align*}
    [w_i](x; x_0) & = \int_{S_{\gamma-\tilde{\gamma}}} (\text{inc } e(y))_{im} dS_m(y), \\
    [u_i](x; x_0) & = \int_{S_{\gamma-\tilde{\gamma}}} (y_m - x_m) \epsilon_{imk} (\text{inc } e(y))_{jk} dS_k(y),
\end{align*}
\]

(5) (6)

for all \(x \in \Omega\), and where \(S_{\gamma-\tilde{\gamma}}\) is a surface enclosed by the the closed path \(\gamma - \tilde{\gamma}\). In particular, \([w_i], [u_i] = 0\) for each couple of curves \(\gamma, \tilde{\gamma}\) iff \(\text{inc } e = 0\).

**Remark 1.** As a consequence of \(\text{inc } e = 0\), (1) and (2) do not depend on the choice of the curve \(\gamma \in C^1([0, 1], \Omega)\) connecting \(x_0\) to \(x\). In particular, the vector fields \(w \in C^\infty(\Omega, \mathbb{R}^3)\) and \(u \in C^\infty(\Omega, \mathbb{R}^3)\) are univoquely defined. Thus, in (1) and (2), one can use the notation \(\int_\gamma = \int_{x_0}^x\).

It is straightforward to prove the following result:

**Corollary 1** (Saint-Venant compatibility conditions in \(C^\infty; [14]\)). Let \(\Omega\) be a simply-connected and bounded open set in \(\mathbb{R}^3\) and let \(e \in C^\infty(\Omega, M^3)\) be a symmetric tensor field. There exists \(u \in C^\infty(\Omega, \mathbb{R}^3)\) (given by (2)) such that \(e = \nabla^* u\), if and only if

\[
\text{inc } e = 0.
\]

(7)

Now, the following classical quantities can be introduced:
Definition 1.2 (Strain and Frank tensor). Let \( u : \Omega \to \mathbb{R}^3 \) be a smooth displacement field, i.e., it writes in the Cartesian base \( \{ e^i \} \) as \( u = \hat{u}_i e^i \). Let us introduce the following quantities:

(i) the linear elastic strain \( e_{ij} := (\nabla u) \cdot e^i \otimes e^j = \frac{1}{2} (\partial_{x_j} \hat{u}_i + \partial_{x_i} \hat{u}_j) \);

(ii) the Frank tensor \( \epsilon_{ipk} \partial_p e_{jk} = \text{(Curl of)} e_{ij} \),

where \( x_i \) stands for the \( i \)th Cartesian coordinate.

The term Frank tensor comes from the fact that its integration along a closed curve encircling a dislocation, yields by (1) the jump of the rotation, denoted as Frank vector \([11, 12]\).

Remark 2. Let \( x \in \Omega \) and \( \gamma_x \) be a smooth curve joining \( x_0 \) to \( x \). Let \( y \in \gamma_x \) and \( \gamma_y \) be a smooth curve joining \( x_0 \) to \( y \). By Eqs (1) and (2), the displacement writes as a recursive line integral involving the strain and the Frank tensor, i.e.,

\[
\begin{align*}
u_i(x; \gamma_x) &= \nu_i(x_0) + \int_{\gamma_x} \left( e_{ik}(y) - \epsilon_{ilk}(\int_{\gamma_y} (\text{Curl } e)_{im}(\xi) d\xi_m)(y) \right) dy_l.
\end{align*}
\]

1.5. Motivation and goals of the paper. By (8), the assumed-smooth displacement in linearized elasticity can be expressed by means of the strain, here a general symmetric tensor \( e \), in practice a tensor derived from the stress tensor by a constitutive law, and of the Frank tensor. On the one hand, this classical formula is at the basis of the intrinsic, strain-based formulation of elasticity, on the other, it directly provided the Frank and Burgers vectors in terms of the strain and its curl, thence endowing the Frank tensor an important role in elasticity with dislocations.

Based on these considerations, the first goal of this work is to elucidate to which functional space \( e \) belongs so as to provide a homogeneous boundary conditions in terms of the displacement field on a connected subset \( \Gamma_0 \) of \( \partial \Omega \). This will allow one to define the homogeneous, or even the mixed problem of elasticity in a variational setting involving \( e \) in place of \( u \). This problem was originally treated by Ph. Ciarlet and C. Mardare, and termed the intrinsic approach \([5]\), where they determined in differential geometric terms which conditions \( e \) should satisfy on the boundary. In this paper, we present an alternative conditions expressed in terms of the strain \( e \) and the associated Frank tensor, thereby providing a Physical understanding of the boundary condition. We also show that it is equivalent to Ciarlet-Mardare condition. Moreover, we believe that this intrinsic approach is mandatory when dislocation lines are present, since the displacement, being multiple-valued by (2), is an uncomfortable model variable (a rigorous manner of introducing the displacement as a main variable, though, is for instance to consider torus-valued maps as done in \([15]\)). In the presence of dislocations, the Frank tensor is a model variable, beside the elastic strain, and its curl is precisely the strain incompatibility as related to the dislocation density. For these reasons, we find worthwhile to model linearized elasticity in terms of these two tensors. Of course, the classical elasticity system is recovered as soon as the dislocation density vanishes (and thus the strain incompatibility). In brief, in this work we address the relation between strain/Frank tensor-based and displacement-based boundary conditions in Elasticity.

2. Extension and differentiation of the normal and tangent vectors to a surface. We would like to construct a curvilinear basis on the boundary which should be smooth and orthonormal, starting from the vector \( N_{\partial \Omega} \) normal to the
boundary and defining two tangent vectors perpendicular to \( N_{\partial \Omega} \). This basis is then extended to the whole body. The natural moving frame sought is close in spirit to the Darboux frame of surfaces, though in principle the latter may only be defined at non-umbilical points. As a matter of fact, in order to achieve a certain level of generality, we will not consider principal lines of curvature with their associated principal curvatures, and hence the gradient of the normal vector will be given by a symmetric matrix with possibly non-zero off-diagonal entries.

2.1. Signed distance function and extended unit normal. We denote by \( N_{\partial \Omega} \) the outward unit normal to \( \partial \Omega \), and by \( b \) the signed distance to \( \partial \Omega \), i.e.,

\[
b(x) = \begin{cases} 
\text{dist}(x, \partial \Omega) & \text{if } x \notin \Omega, \\
-\text{dist}(x, \partial \Omega) & \text{if } x \in \Omega.
\end{cases}
\]

We recall the following result.

**Theorem 2.1** ([7], Chap. 5, Thms 3.1 and 4.3). There exists an open neighborhood \( W \) of \( \partial \Omega \) such that

1. \( b \) is smooth in \( W \);
2. every \( x \in W \) admits a unique projection \( p_{\partial \Omega}(x) \) onto \( \partial \Omega \);  
3. this projection satisfies \( p_{\partial \Omega}(x) = x - \frac{1}{2} \nabla b^2(x) \), \( x \in W \);
4. it holds \( \nabla b(x) = N_{\partial \Omega}(p_{\partial \Omega}(x)) \), \( x \in \overline{W} \).

In particular, this latter property shows that \( \nabla b(x) = N_{\partial \Omega}(x) \) for all \( x \in \partial \Omega \) and \( |\nabla b(x)| = 1 \) for all \( x \in W \). Therefore, we define the extended unit normal by

\[
N(x) := \nabla b(x) = N_{\partial \Omega}(p_{\partial \Omega}(x)), \quad x \in W. 
\]  

(9)

2.2. Tangent vectors and orthonormal frame on \( \partial \Omega \). For all \( x \in \partial \Omega \), we denote by \( T_{\partial \Omega}(x) \) the tangent plane to \( \partial \Omega \) at \( x \), that is, the orthogonal complement of \( N_{\partial \Omega}(x) \). Being \( \partial \Omega \) smooth, there exists a covering of \( \partial \Omega \) by open balls \( B_1, ..., B_M \) of \( \mathbb{R}^3 \) such that, for each index \( k \), two smooth vector fields \( \tau^A_{\partial \Omega}, \tau^B_{\partial \Omega} \) can be constructed on \( \partial \Omega \cap B_k \) where, for all \( x \in \partial \Omega \cap B_k \), \( (\tau^A_{\partial \Omega}(x), \tau^B_{\partial \Omega}(x)) \) is an orthonormal basis of \( T_{\partial \Omega}(x) \). In all the sequel, the index \( k \) will be implicitly considered as fixed and the restriction to \( B_k \) will be omitted. In fact, for our needs, global properties and constructions will be easily obtained from local ones through a partition of unity subordinate to the covering.

By symmetry of the Jacobian matrix \( DN(x) = D^2 b(x) \) of \( N(x) \), differentiating the equality \( |N(x)|^2 = 1 \) yields \( \partial_N N(x) = DN(x)N(x) = 0 \), \( x \in W \). In other words, \( N(x) \) is an eigenvector of \( DN(x) \) for the eigenvalue 0. For all \( x \in \partial \Omega \), the system \( (\tau^A_{\partial \Omega}(x), \tau^B_{\partial \Omega}(x), N_{\partial \Omega}(x)) \) is an orthonormal basis of \( \mathbb{R}^3 \), where the gradient of \( N \) takes the form

\[
DN(x) = \begin{pmatrix} \kappa^A_{\partial \Omega}(x) & \xi_{\partial \Omega}(x) & 0 \\ \xi_{\partial \Omega}(x) & \kappa^B_{\partial \Omega}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x \in \partial \Omega, 
\]

(10)

with \( \kappa^A_{\partial \Omega}, \kappa^B_{\partial \Omega} \) and \( \xi \) smooth scalar fields defined on \( \partial \Omega \). If \( R \in \{A, B\} \), we denote by \( R^* \) the complementary index of \( R \), i.e., \( R^* = B \) if \( R = A \) and \( R^* = A \) if \( R = B \).

2.3. Extended tangent vectors and the parallel curvinormal frame. Let \( d \) be defined in \( W \) by

\[
d := (1 + b \kappa^A_{\partial \Omega} \circ p_{\partial \Omega}) \left(1 + b \kappa^B_{\partial \Omega} \circ p_{\partial \Omega}\right) - \left(b \xi_{\partial \Omega} \circ p_{\partial \Omega}\right)^2.
\]
Possibly adjusting \( W \) so that \( d(x) > 0 \) for all \( x \in W \), we define in \( W \):

\[
\tau^R = \tau^R_{\partial}\circ p_{\partial\Omega}, \quad \kappa^R = d^{-1} \left( (1 + b \kappa^R_{\partial}\circ p_{\partial\Omega})\left(\kappa^R_{\partial}\circ p_{\partial\Omega}\right) - b \left( \xi_{\partial\Omega}\circ p_{\partial\Omega}\right)^2 \right), \quad \xi = d^{-1} \xi_{\partial\Omega}\circ p_{\partial\Omega}, \quad \kappa = \kappa^A + \kappa^B, \quad \gamma^R = \text{div} \, \tau^R.
\]

Obviously, for each \( x \in W \), the triple \((\tau^A(x), \tau^B(x), N(x))\) forms an orthonormal basis of \( \mathbb{R}^3 \), which we call the the curvinormal (parallel) frame. Next, we compute the normal and tangential derivatives of these vectors. We denote the tangential derivative \( \partial_{\tau, n} \) by \( \partial_R \) for simplicity, i.e., \( \partial_R u := (Du)\tau^R \), where \( Du \) stands for the differential of \( u \), and \( \partial_R u \) its value in the direction \( \tau^R \).

**Theorem 2.2** ([1]). The following holds in \( W \):

\[
\begin{align*}
\partial_N \tau^R &= 0, \quad \partial_R N = \kappa^R \tau^R + \xi \tau^R, \quad \partial_R \tau^R = -\kappa^R N - \gamma^R \tau^R, \\
\partial_R \cdot \tau^R &= \gamma^R \tau^R - \xi N, \\
\text{div} \, N &= \text{tr} \, DN = \Delta b = \kappa. 
\end{align*}
\]

In this paper we make use of a orthonormal frame parallel to the boundary. The induced coordinates are non-holonomic in the following sense.

**Corollary 2** (Non-holonomic curvilinear frame [1]). If \( f \) is twice differentiable in \( \Omega \) it holds

\[
\partial_R \partial_N f - \partial_N \partial_R f = \kappa^R \partial_R f + \xi \partial_R f.
\]

3. **Differential geometry on the boundary with curvinormal basis.** At each point \( x \in \partial\Omega \) the curvilinear basis \((g^i(x))_{i=A,B,N} := (\tau^A(x), \tau^B(x), N_{\partial\Omega}(x))\) is orthonormal and differentiable by Theorem 2.2. Remark that indices \( P, Q, R \) will stand for \( A \) or \( B \), and denote one of the two orthogonal tangent vectors on the boundary, whereas index \( N \) will always be associated to the normal \( N_{\partial\Omega} \). In some sense, the chosen curvilinear basis is a generalization to general surfaces of the spherical or cylindrical frames. We recall that \( \partial_1 \) means the differential in the direction \( g^1 \). For the scalar \( u \), \( \partial_1 u = \partial_1 u = \tau^R \cdot \nabla u \) for \( R = A, B \), or \( \partial_1 u = N \cdot \nabla u \) for \( i = N \), where \( \nabla = e^i \partial_{e^i} \) is the Cartesian gradient, and \( e^i \) stands for the \( i \)th Cartesian basis vector. Recall that partial curvilinear derivatives do not commute, as shown in Corollary 2. For instance, the gradient in spherical coordinates reads \( \nabla u = \partial_\theta u e_\theta + \frac{1}{\tau} \partial_\phi u e_\phi + \frac{1}{\tau \sin \phi} \partial_\theta u e_\phi \) and hence \( \partial_A = \frac{1}{\tau} \partial_\phi \) and \( \partial_B = \frac{1}{\tau \sin \phi} \partial_\theta \).

Letting \( x = x_i e^i \) denote the position vector, and \( g^i = g^i_k e^k \) be the \( i \)th curvilinear basis vector, then by definition, \( \partial_i x = g^i_k \partial_{e^k} x = g^i_k e^k = g^i \).

3.1. **Christoffel symbols and Riemannian curvature.** The displacement field will write as \( u = u_i g^i \) with \( u_i \) its covariant components. Moreover, the extrinsic metric is Euclidean, since \( g^{ij} := g^i \cdot g^j = \delta^{ij} \). Let \( g_i := g^{ij} g^j \) be the dual of the basis vector. The Christoffel symbol of second kind \( \Gamma_{ij}^p \) is defined as the linear operator such that [3]

\[
\partial_j g^p = -\Gamma_{ij}^p g^i, \quad \text{called the Levi-Civita connection. In other words,}
\]

\[
\Gamma_{ij}^p := -g_i \cdot \partial_j g^p.
\]

Remark that the body manifold in this paper is Euclidean and hence the associated connection is symmetric. However, the associated Christoffel symbols are not symmetric due to the choice of a non-holonomic frame, since \( \Gamma_{ij} := -g_i \cdot \partial_j g^p = \)
As an example, in a spherical coordinates/components system, it holds
\[ u_i = \partial_j(u_ig^j) = (\partial_j u_i - \Gamma^p_{ij} u_p)g^j = u_{ij}g^j, \]
where the \( j \)th covariant derivative of the covariant component of \( u \) reads
\[ u_{ij} := \partial_j u_i - \Gamma^p_{ij} u_p. \]

By Theorem 2.2, it is easily deduced by identification with (15) that the only nonvanishing components of \( \Gamma^p_{ij} \) read (with no sum on repeated indices)
\[ \Gamma^N_{RR} = -\xi, \quad \Gamma^N_{RR} = -\kappa R, \quad \Gamma^R_{RR} = \xi, \quad \Gamma^R_{RR} = \gamma R^*, \quad \Gamma^R_{RR} = \kappa R, \quad \Gamma^R_{RR} = -\gamma R. \]

As an example, in a spherical coordinates/components system, it holds\(^1\) \( i, j \in \{ \phi, \theta \} \), \( \kappa = \frac{1}{r} \), \( \gamma = \frac{1}{\tan \phi} \), \( \gamma = 0 \), and hence
\[
\begin{align*}
\Gamma^r_{ij} &= \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & -\frac{1}{r} & 0 \\
0 & 0 & -\frac{1}{r} 
\end{array} \right), \\
\Gamma^\phi_{ij} &= \left( \begin{array}{ccc}
0 & \frac{1}{r} & 0 \\
0 & 0 & -\frac{1}{r \tan \phi} \\
0 & 0 & -\frac{1}{r \tan \phi} 
\end{array} \right), \\
\Gamma^\theta_{ij} &= \left( \begin{array}{ccc}
0 & 0 & \frac{1}{r} \\
0 & 0 & \frac{1}{r \tan \phi} \\
0 & 0 & 0 
\end{array} \right). 
\end{align*}
\]

Moreover, it is observed that \( \Gamma^p_{ij} \) is not symmetric, i.e., \( \Gamma^p_{ij} \neq \Gamma^p_{ji} \). This Euclidean metric is therefore associated with a nonvanishing “anholonomicity torsion”,
\[ T^p_{ij} := \Gamma^p_{ij} - \Gamma^p_{ji}. \]

(as opposed to the connection torsion, which here vanishes). In the curvilinear basis, it is easily computed that the only nonvanishing components of \( T^p_{ij} \) are
\[ T^r_{ij} = \kappa R \delta_{ij} + \xi \delta_{ij} - (\gamma R^* - \gamma R) \delta_{ij}. \]

In particular in spherical coordinates one has
\[ T^r_{ij} = 0, \quad T^\phi_{ij} = \left( \begin{array}{ccc}
0 & \frac{1}{r} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{array} \right), \quad T^\theta_{ij} = \left( \begin{array}{ccc}
0 & 0 & \frac{1}{r} \\
0 & 0 & \frac{1}{r \tan \phi} \\
0 & 0 & 0 
\end{array} \right). \]

The Riemann curvature tensor is defined as (see [8])
\[ \text{Riem}^q_{ijk} := \partial_k \Gamma^q_{ij} - \partial_j \Gamma^q_{ik} + \Gamma^q_{ik} \Gamma^p_{jp} - \Gamma^q_{jp} \Gamma^p_{ik}. \]

Accordingly, the Ricci curvature tensor is defined as its \( jq \)-trace, viz.,
\[ \text{Ric}^q_{ik} := \text{Riem}^q_{ikq} := \partial_k \Gamma^q_{ij} - \partial_j \Gamma^q_{ik} + \Gamma^q_{ik} \Gamma^q_{jk} - \Gamma^q_{jk} \Gamma^q_{ik}. \]

Lastly, the scalar curvature is the trace of Ric, i.e.,
\[ R := \text{Ric}^q_{kk} := \partial_k \Gamma^q_{kq} - \partial_q \Gamma^q_{kk} + \Gamma^q_{kk} \Gamma^q_{qq} - \Gamma^q_{qq} \Gamma^q_{kk}. \]

In the curvilinear basis, it is easily computed from (23) and (18) that,
\[ R = 2\partial_N^2 + 2\partial_A \gamma^A + 2(\gamma^2 + (\gamma^A)^2) \]
(with sum on \( A \)), which for a spherical surface or radius \( r \) (for which \( \xi = 0, \gamma = \frac{1}{r} \), \( \kappa = 1/r \)), yields \( R = 4/r^2 - 2/(2r^2 \sin^2 \phi) + 2/(2r^2 \tan^2 \phi) = 2/r^2 \), recognized as twice the Gaussian curvature, as expected.

\(^1\)Here, \( \phi \) denotes the polar, and \( \theta \) the azimuthal coordinate, respectively.
Remark 3. Let us emphasize that Eq. (24) yields a relation between the scalar curvature and the normal derivative of the mean curvature $H = \kappa / 2$.

3.2. Some identities in the curvilinear frame. Recall that if $u$ is a vector, $u = u_i g^i = \hat{u}_j e_j$, then $(\nabla u)_{mn} = \partial_{x_n} \hat{u}_m$, and we write

$$\text{grad} u := (\nabla u)_{mn} e^m \otimes e^n = u_{ij} g^i \otimes g^j = \partial_i u \otimes g^i. \quad (25)$$

Let $q_R$ be the curvilinear coordinate associated to $g^R$ in the sense that $g^R = \frac{\partial}{\partial q_R} x$, with $h_R := \| \partial q_R x \|$, and where $x$ stands for the position vector of a point. Otherwise said, $q_R$ is the curvilinear abscissa of the curve with tangent vector $\tau^R$. Indeed,

$$\partial_{q_R} u = \partial_{x_i} u \frac{\partial x_i}{\partial q_R} = h_R g^i \partial_{x_i} u = h_R \partial_{R} u. \quad (26)$$

By Lemma 2, remark that the $\partial_{R}$-derivatives do not commute, contrarily to $\partial_{q_R}$, because of the factors $h_R$. This object of anholonomicity (see [17, 18]) reflects itself in the non-symmetry of the Christoffel symbols. As an example, consider the spherical frame, where $h_N = h_r = 1, h_A = h_\phi = \frac{1}{r}, h_B = h_\theta = \frac{1}{r \tan \phi}$, and $q_A = \phi$ (polar angle), $q_B = \theta$ (azimuthal angle). One has

$$\left( \partial_A \partial_r - \partial_r \partial_A \right) = \frac{1}{r^2} \partial_\phi, \quad \left( \partial_B \partial_r - \partial_r \partial_B \right) = \frac{1}{r^2 \sin \phi} \partial_\theta, \quad (27)$$

Partial derivatives in curvilinear frames might be defined in a weak sense by means of surface integrals and (26) as follows:

$$\int_{\omega^A \times \omega^B} \partial_R u \cdot \varphi J dq_A dq_B := - \int_{\omega^A \times \omega^B} u \cdot h_R \partial_R \left( \frac{1}{h_R} \varphi J \right) dq_A dq_B, \quad (28)$$

with $J$ the surface Jacobian, $\varphi$, a test function with compact support, and where $\omega^R$ stands for a subset of the domain of $q_R$.

Setting

$$e_{ij} = e_{ij}(u) := \frac{1}{2}(u_{i||j} + u_{j||i}), \quad (29)$$

one has by (16) that

$$e_{ij} = \frac{1}{2}(\partial_k u \cdot g_j + \partial_j u \cdot g_i), \quad (30)$$

and in particular,

$$e_{NN} = u_{N||N} = (\partial_N u) \cdot N. \quad (31)$$

In general, for the covariant components of a second-rank tensor $A$ it holds (see [8])

$$A_{i||j} = \partial_k A_{ij} - \Gamma^l_{ik} A_{lj} - \Gamma^l_{jk} A_{il}. \quad (32)$$

Let $A_{ij} = u_{i||j}$. Then by (32) one has $u_{i||j} := (u_{i||j})_{||k}$ and hence

$$u_{i||jk} g^i = \partial_k u_{i||j} g^i - (\Gamma^l_{ik} u_{i||j} + \Gamma^l_{jk} u_{i||l}) g^i = \partial_k (u_{i||j} g^i) + u_{i||j} \Gamma^l_{ik} g^i - (\Gamma^l_{ik} u_{i||j} + \Gamma^l_{jk} u_{i||l}) g^i = \partial_k (u_{i||j} g^i) - \Gamma^l_{jk} u_{i||l} g^i,$$

where (17) and a change of dumb indices have been used. Therefore,

$$\partial_k (\partial_j u) = (u_{i||jk} + \Gamma^l_{jk} u_{i||l}) g^i. \quad (33)$$
In particular, it follows by (16) that
\[ \partial_k (\partial_j u) \cdot N = u_N \| jk + (\Gamma^j_{ik} \partial_i u) \cdot N. \] (34)

Moreover, one has
\[ \partial_k (\partial_j u) \cdot R^R = \partial_k (\partial_j u \cdot R^R) - \partial_j u \cdot \partial_k R^R = \partial_k (\partial_j u_R) - \partial_j u \cdot \partial_k R^R = u_R \| jk + (\Gamma^j_{ik} \partial_i u) \cdot R^R. \] (35)

Remark also that
\[ u_{ij} \| jk - u_{ij} \| k = (\partial_k \partial_j - \partial_j \partial_k) u \cdot g^j - T^j_{jk} u_{ij} \| i, \] (36)
where we recall that \( T \) stands for the anholonomicity torsion. In a spherical frame, one has by (20) and (27) (and with a slight abuse of notations, since one writes \( q_R \) instead of \( R \) as covariant differentiation indice)
\[ u_{ij} \| r \phi - u_{ij} \| \phi r = u_{ij} \| r \theta - u_{ij} \| \theta r - u_{ij} \| \phi \theta = 0, \] (37)
that is, the second covariant derivatives do commute. Further, \( \epsilon_{ijk} u_{ij} \| jk = 0. \)

Some identities are most easily derived for the contravariant components, and only expressed in covariant components in a second step. Let \( u' \) be the contravariant coordinate of the vector \( u = u' g_i. \) It is well known that the covariant derivative of vector \( u \) and tensor \( A \) are expressed as (see [8]),
\[ u'_{ij} := \partial_j u^i + \Gamma^i_{jk} u^j \text{ and } (A')_{jk} := \partial_k A_j^i + \Gamma^i_{pk} A_j^p - \Gamma^p_{jk} A_i^p. \]
Moreover, \( u'_{ij} \| k = (u'_{ij}) \| k, \) and the Ricci identity for contravariant components \( u' \) then reads (with sum on \( q \) and \( p \))
\[ u'_{ij} \| jk - u'_{ij} \| k = \text{Riem}^i_{qjk} u^q + T^p_{jk} u^i \| p, \]
which is an alternative formula to (36) which shows only the value of \( u \) and of its derivative in the RHS. Therefore,
\[ u^Q_{\| NR} - u^Q_{\| RN} = \text{Riem}^Q_{qNR} u^q + T^p_{NR} u^i \| p, \]
which is written in the curvilinear frame (with sum on \( q \) but not on \( R \)) as
\[ u^Q_{\| NR} - u^Q_{\| RN} = \partial_N \Gamma^Q_{qR} u^q + T^p_{NR} u^Q_{\| p} = \partial_N \Gamma^Q_{qR} u^q + \kappa u^Q_{\| R} + \xi u^Q_{\| R}, \]
\[ = (\partial_N \Gamma^Q_{qR} + \kappa \Gamma^Q_{qR} + \xi \Gamma^Q_{qR}) u^q + \kappa u^R_{\| q} + \xi u^R_{\| R} \]
\[ = \ell_{NR} u^q + \kappa u^R_{\| q} + \xi u^R_{\| R}, \]
(38)
where \( \ell \) stands for a short notation for the expression inside the parenthesis in the RHS of the second line. Thus, (38) is rewritten as
\[ \delta u^Q_{\| NR} := u^Q_{\| NR} - u^Q_{\| RN} = \ell^Q_{NR} u^q + \hat{\ell}_{NR} u^Q + \hat{\ell}_{NR} \partial_R u^q, \]
where \( \hat{\ell} \) depend on \( \kappa \) and \( \xi. \) It is obvious that the covariant differentiation of the curvilinear metric \( g_{ij\| k} \) vanishes, since the metric is the identity tensor. Thus, lowering indices and covariant differentiation mutually commute [8] and the counterpart of (39) for the covariant components reads
\[ \delta u_{m} \| jR := u_{m} \| jR - u_{m} \| Rj = \delta Q_m \ell_{NR} u^q + \hat{\ell}_{NR} \partial_R u_m + \hat{\ell}_{NR} \partial_R u_m. \]
(40)
where \( \ell^Q_{mNR} := \delta Q_m \ell^Q_{NR} \) \( q.\) Now, it is easily deduced from (29) and (39) that
\[ e_{QN} + e_{RN} - e_{QR} = e_{NQ} + \frac{1}{2} (\delta u_{QR}^N + \delta u_{RN}^Q), \]
which by (17) and (34) yields

$$
e_{QN||R} + e_{RN||Q} - e_{QR||N} = u_{N||QR} + \frac{1}{2}(\delta u_{Q||NR} + \delta u_{R||NQ})$$

$$= (\partial_R \partial_Q u - \Gamma^I_{QR} \partial_I u) \cdot N + \frac{1}{2}(\delta u_{Q||NR} + \delta u_{R||NQ}).$$

Hence, by (31),

$$e_{QN||R} + e_{RN||Q} - e_{QR||N} + \Gamma^N_{QR} e_{NN} = (\partial_R \partial_Q u - \Gamma^P_{QR} \partial_P u) \cdot N$$

$$+ \frac{1}{2}(\delta u_{Q||NR} + \delta u_{R||NQ}).$$

3.3. **Restatement of Ciarlet-Mardare results in [5].** Basically, the differential geometry on the boundary by Ph. Ciarlet and C. Mardare [3, 5] is defined by means of a holonomic non-orthogonal frame, whereas our choice of basis vectors on the boundary, suitably extended in the domain, are orthonormal, but non-holonomic, thereby inducing a non-vanishing anholonomy (non-symmetry of the Christoffels symbols).

Let \( \Gamma_0 \) be a connected subset of \( \partial \Omega \). Let \( u \in \mathcal{C}^2(\Omega) \) and let \( u_{\Gamma_0} \) denote the boundary trace of \( u \) on \( \Gamma_0 \). Set

$$e = e(u) := \text{grad}^S u = (\nabla^S \delta u)_{mn} e^m \otimes e^n = \frac{1}{2}(u_{i||j} + u_{j||i}) g^i \otimes g^j$$

$$= \frac{1}{2}(\delta_i u \cdot g^j + \delta_j u \cdot g^i) g^i \otimes g^j,$$  

and let \( e_{\Gamma_0} \) be the boundary trace of \( e \) on \( \Gamma_0 \). Let us define the operator \( \gamma : \mathcal{C}^2(\Gamma_0, \mathbb{R}^3) \rightarrow \mathcal{C}^1(\Gamma_0, \mathbb{R}^3), \)

$$\gamma(u_{\Gamma_0}) := \gamma_{QR}(u_{\Gamma_0}) g^Q \otimes g^R \in \mathcal{C}^1(\Gamma_0, \mathbb{R}^3),$$

called the **linearized change of metric** induced by \( u_{\Gamma_0} \) [5], whose components read

$$\gamma_{QR}(u_{\Gamma_0}) := \frac{1}{2}(\partial_Q u_{\Gamma_0} \cdot g^R + \partial_R u_{\Gamma_0} \cdot g^Q).$$

Further, we introduce the operator \( \gamma^\sharp : \mathcal{C}^1(\bar{\Omega}) \rightarrow \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^3) \)

$$\gamma^\sharp(e) := \gamma^\sharp_{QR}(e) g^Q \otimes g^R \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^3),$$

where

$$\gamma^\sharp_{QR}(e) := (e_{\Gamma_0})_{QR}. \quad (46)$$

It is immediate from (42) and the curvinormal frame approach that

$$\gamma^\sharp_{QR}(e) = \gamma_{QR}(u_{\Gamma_0}). \quad (47)$$

**Definition 3.1.** Let the linearized change of curvature induced by \( u_{\Gamma_0} \) be given by

$$\rho(u_{\Gamma_0}) := \rho_{QR}(u_{\Gamma_0}) g^Q \otimes g^R \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^3),$$

with the components given by

$$\rho_{QR}(u_{\Gamma_0}) := (\partial_R \partial_Q u_{\Gamma_0} - \Gamma^P_{QR} \partial_P u_{\Gamma_0}) \cdot N. \quad (49)$$

Let

$$\rho^\sharp(e) := \rho^\sharp_{QR}(e) g^Q \otimes g^R \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^3),$$

with

$$\rho^\sharp_{QR}(e) := (e_{QN||R} + e_{RN||Q} - e_{QR||N} + \Gamma^N_{QR} e_{NN})_{\Gamma_0}. \quad (51)$$
The main preliminary results of [5] are now restated in the curvinormal frame:

**Theorem 3.2** (Ciarlet-Mardare [5]). One has

\[ \rho^e = \rho(u_{\Gamma_0}) + \delta \rho(u_{\Gamma_0}) \quad \text{on} \quad \Gamma_0, \]  

(52)

where

\[ (\delta \rho)_{QR}(u) := \frac{1}{2} (\delta u_Q||NR + \delta u_R||NQ). \]  

(53)

Moreover, there exist positive constants \(C_1, C_2, C_3\) and \(C_4\) such that

\[ \| \gamma^e \|_{H^{-1}(\Gamma_0)} + \| \rho^e \|_{H^{-2}(\Gamma_0)} \leq C_1 \inf_{v \in \mathcal{R}(\Omega)} \|(u + r)_{\Gamma_0}\|_{L^2(\Gamma_0)} \leq C_2 \|e\|_{L^2(\Omega)}, \]  

(54)

with \(\mathcal{R}(\Omega)\) the set of rigid displacements in \(\Omega\), and

\[ \inf_{v \in \mathcal{R}(\Gamma_0)} \|(u + r)_{\Gamma_0}\|_{L^2(\Gamma_0)} \]

\[ \leq C_3 (\| \gamma(u_{\Gamma_0}) \|_{H^{-1}(\Gamma_0)} + \| \rho(u_{\Gamma_0}) \|_{H^{-2}(\Gamma_0)}) \]  

(55)

\[ \leq C_4 (\| \gamma(u_{\Gamma_0}) \|_{H^{-1}(\Gamma_0)} + \| (\rho + \delta \rho)(u_{\Gamma_0}) \|_{H^{-2}(\Gamma_0)}) \]  

(56)

\[ \leq C_4 (\| \gamma^e \|_{H^{-1}(\Gamma_0)} + \| \rho^e \|_{H^{-2}(\Gamma_0)}), \]  

(57)

with \(\mathcal{R}(\Gamma_0)\) the set of rigid displacement in \(\Gamma_0\).

Note that (57) stems from (56) by Eqs. (47) and (52). Note also that in Ciarlet-Mardare original formulation, \(\delta \rho = 0\) because their connection is symmetric (i.e., their frame is holonomic). The proof of (52) is basically Eq. (41), here proved in the curvinormal basis. By inspecting the original proof in [5], Eq. (54) is also easily demonstrated, since by formulae (39) and (28) one has

\[ \left| \int_{\Gamma_0} (\delta \rho)_{QR}(u) \cdot \varphi dS(x) \right| \leq \int_{\omega^0} u \cdot \Psi dq^A dq^B \| \leq C_1 \|u_{\Gamma_0}\|_{L^2(\Gamma_0)}, \]

for some positive constant \(C_1\), some vector \(\Psi\) with compact support independent of \(u\), and with \(\omega^0\) the domain of \((q^A, q^B)\) associated to \(\Gamma_0\). Lastly, to prove (55), scrutinizing again the original proof in [5], it suffices to bound the terms \(\partial R \partial Q(u \rho)_{\Gamma_0}\) in \(H^{-2}(\omega^0)\) in terms of \(\| \frac{1}{2} (\partial R(u_{\Omega_0}) \partial Q + \partial Q(u_{\Omega_0}) \partial R) \|_{H^{-1}(\omega^0)}\) and \(\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}\), then use an argument of Nečas [2], noting that the extra terms appearing due to the nonsymmetric property of the connexion are also bounded by \(\| \frac{1}{2} (\partial R(u_{\Omega_0}) \partial Q + \partial Q(u_{\Omega_0}) \partial R) \|_{H^{-1}(\omega^0)}\) and \(\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}\), from formulae (35) and (39). Note that the bound (56) is obtained by writing \(\|\rho(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)} \leq \| (\rho + \delta \rho)(u_{\Gamma_0}) \|_{H^{-2}(\Gamma_0)} + \| \delta \rho(u_{\Gamma_0}) \|_{H^{-2}(\Gamma_0)}\) and using the fact that by (53) and (38), one bounds \(\| \delta \rho(u_{\Gamma_0}) \|_{H^{-2}(\Gamma_0)}\) by means of \(\| \frac{1}{2} (\partial R(u_{\Omega_0}) \partial Q + \partial Q(u_{\Omega_0}) \partial R) \|_{H^{-1}(\omega^0)}\) and \(\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}\).

The main result in [5] is now stated in the curvinormal basis. Its proof basically follows from (52)-(55) and standard extension operators.

**Theorem 3.3** (Ciarlet-Mardare Main result [5]). Let \(u \in H^1(\Omega)\) and let \(e = e_{ij}(u) g^i \otimes g^j\) with

\[ e_{ij}(u) = \frac{1}{2} (\partial_i u \cdot g^j + \partial_j u \cdot g^i). \]  

(58)

Then the following conditions satisfy (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii):

(i) \(u_{\Gamma_0} = 0\)

(ii) \(\gamma^e = \tilde{\rho} \gamma^e = 0\)

(iii) \(u_{\Gamma_0} \in \mathcal{R}(\Gamma_0)\).

where \(\gamma^e\) and \(\tilde{\rho} \gamma^e\) are suitable extensions of \(\gamma^e\) and \(\rho^e\).
4. Function spaces for the incompatibility operator.

4.1. Definitions and basic properties. Let \( \Gamma_0 \) be a connected subset of \( \partial \Omega \). Define

\[
\mathcal{H}_{\text{comp}}^2(\Omega) := \{ E \in H^2(\Omega, \mathbb{S}^3) : \text{inc } E = 0 \text{ in } \Omega \}
\]

\[
\mathcal{H}_{\Gamma_0,\text{comp}}^2(\Omega) := \{ E \in \mathcal{H}_{\text{comp}}^2(\Omega) : E = \text{Curl}^E E \times N = 0 \text{ on } \Gamma_0 \}
\]

\[
\mathcal{H}_{\text{inc}}(\Omega) := \{ E \in L^2(\Omega, \mathbb{S}^3) : \text{inc } E \in L^2(\Omega, \mathbb{S}^3) \}
\]

\[
\mathcal{H}_{\text{comp}}(\Omega) := \{ E \in L^2(\Omega, \mathbb{S}^3) : \text{inc } E = 0 \text{ in } \Omega \}
\]

\[
\mathcal{H}_{\Gamma_0,\text{comp}}(\Omega) := \{ E \in \mathcal{H}_{\text{inc}}(\Omega) : \text{inc } E = 0 \text{ in } \Omega, E = \text{Curl}^E E \times N = 0 \text{ on } \Gamma_0 \}.
\]

The spaces \( \mathcal{H}(\Omega), \mathcal{H}_0(\Omega) \) and the above affine spaces are naturally endowed with the Hilbertian structure of \( H^2(\Omega, \mathbb{S}^3) \). Note that in order to define \( \mathcal{H}_{\text{inc}}(\Omega) \) we should precise that \( \text{inc } E \in H^{-2}(\Omega, \mathbb{S}^3) \) and that the boundary traces in the space \( \mathcal{H}_{\text{comp}}(\Omega) \) are defined in a weak sense, whose exact meaning is the object of Corollary 3 (see Eqs. (76) and (77)).

**Lemma 4.1** (Amstutz-Van Goethem [1]). For all \( E \in H^2(\Omega, \mathbb{S}^3) \) it holds an open neighborhood \( W \) of \( \partial \Omega \):

\[
\text{Curl}^E E \times N = - (\partial_N E \times N)^t \times N + \left( \sum_R \tau^R \times \partial_R E \right)^t \times N \quad \text{on } \partial \Omega.
\]

In particular, if \( (\partial_N E \times N)^t \times N = 0 = E \text{ on } \partial \Omega \), then

\[
\text{Curl}^E E \times N = 0 \quad \text{on } \partial \Omega.
\]

**Lemma 4.2.** Let \( E \in H^2(\Omega, \mathbb{S}^3) \). If \( \text{Curl}^E E \times N = 0 \) on \( \Gamma_0 \), then (i) \( \partial_R E_{RR} = \partial_N E_{RR}, (ii) \partial_R E_{R'N} = \partial_N E_{R'R'}, \) and (iii) \( \partial_R E_{R'R'} - \partial_R E_{RR'} = 0 \) on \( \Gamma_0 \) and for \( R = A \) or \( B \). In particular, if \( E = 0 \) on \( \Gamma_0 \) then \( \partial_N E_{RR} = \partial_N E_{RR'} = 0 \) on \( \Gamma_0 \).

**Proof.** Let us compute \( \text{Curl}^E E \times N \) in the local basis \( (\tau^A, \tau^B, N) \). From

\[
(\text{Curl}^E E) \times N)_{im} = \epsilon_{mij} \epsilon_{ikl} \partial_k E_{jl} N_p,
\]

one has \( N_p = \delta_{pN} \) and statement (i) follows from (take \( R = A \) or \( B \))

\[
(\text{Curl}^E E) \times N)_{RR'} = \epsilon_{R'j} \epsilon_{R'R} \partial_k E_{jl} = (\delta_{Rk} \delta_{Nl} - \delta_{Rl} \delta_{Nk}) \partial_k E_{rl} = \partial_R E_{RR'} - \partial_R E_{RR'},
\]

Moreover, statement (ii) follows from

\[
(\text{Curl}^E E) \times N)_{RR'} = \epsilon_{R'j} \epsilon_{R'R} \partial_k E_{jl} = \epsilon_{R'R} \epsilon_{R'} \partial_k E_{jl} = \partial_N E_{RR'} - \partial_N E_{RR'},
\]

and (iii) by

\[
(\text{Curl}^E E) \times N)_{NR} = \epsilon_{Rj} \epsilon_{Nkl} \partial_k E_{jl} = \epsilon_{Nkl} \partial_k E_{jl} = \partial_R E_{R'N} - \partial_R E_{R'R},
\]

with \( R = A, B \), proving the result, since \( \partial_R E = E = 0 \) on \( \Gamma_0 \) if \( E = 0 \) on \( \Gamma_0 \).

The central contribution of the present work is the following Theorem, whose proof follows from the preceding discussion.

**Theorem 4.3.** Let \( \epsilon \in C^1(\bar{\Omega}) \cap \mathcal{H}_{\Gamma_0,\text{comp}}^2(\Omega) \). The following conditions are equivalent:

1. \( \epsilon = \text{Curl}^E E \times N = 0 \text{ on } \Gamma_0 \)
2. \( \gamma^E(\epsilon) = \rho^E(\epsilon) = 0 \text{ on } \Gamma_0 \).

**Proof.** This is a direct consequence of Eqs. (47) and (51), Eq. (32), and items (i) and (ii) of Lemma 4.2. 

\[ \square \]
4.2. **Green formula and weak trace operators.** Denote $A^S = (A + A^T)/2$ the symmetric part of a tensor $A$.

**Theorem 4.4** (Green formula for the incompatibility[1]). Suppose that $E \in C^2(\Omega, S^3)$ and $\eta \in H^2(\Omega, S^3)$. Then

\[
\int_{\Omega} E \cdot \text{inc} \, \eta \, dx = \int_{\Omega} \text{inc} \, E \cdot \eta \, dx + \int_{\partial \Omega} T_1(E) \cdot \eta \, dS(x) + \int_{\partial \Omega} T_0(E) \cdot \partial N \eta \, dS(x),
\]

with the trace operators defined as

\[
T_0(E) := (E \times N)^t \times N, \quad (60)
\]

\[
T_1(E) := (\text{Curl}^t (E \times N))^S + ((\partial_N + \kappa)E \times N)^t \times N + (\text{Curl}^t E \times N)^S, \quad (61)
\]

where we recall that $\kappa$ stands for twice the mean curvature (see Section 2.3).

It is crucial to note that by (60), only the tangential components of $\partial_N \eta$ are to be considered in the right-hand side of (59).

As we have seen, the two Dirichlet boundary conditions imposed in the space $H_{F,\text{comp}}^1(\Omega)$ read $\mathcal{D}_1(E) = E_{\Gamma_0}$ (i.e., the trace of $E$ on $\Gamma_0$) and $\mathcal{D}_2(E) = (\text{Curl}^t E \times N)_{\Gamma_0}$ (i.e., the trace of $\text{Curl}^t E \times N$ on $\Gamma_0$, where $N$ is a suitable extension of the normal). In particular, let

\[
\mathcal{E}_{\text{comp}}(\Omega) := \{ \nabla^S u : u \in C^2(\Omega, \mathbb{R}^3) \} \subset C^1(\Omega, S^3).
\]

One has

\[
\mathcal{D}_1 : C^1(\Omega, S^3) \rightarrow C^1(\Gamma_0), \quad E \in \mathcal{E}_{\text{comp}}(\Omega) \rightarrow \mathcal{D}_1(E) = E_{\Gamma_0} \in C^1(\Gamma_0), \quad (62)
\]

and

\[
\mathcal{D}_2 : C^1(\Omega, S^3) \rightarrow C(\Gamma_0), \quad E \in \mathcal{E}_{\text{comp}}(\Omega) \rightarrow \mathcal{D}_2(E) = (\text{Curl}^t E \times N)_{\Gamma_0} \in C(\Gamma_0). \quad (63)
\]

In the following result, trace operators will be introduced in a weak form. While $\mathcal{T}_0$ and $\mathcal{T}_1$ are extensions of (60) and (61), respectively, the operators $\mathcal{D}_2$ and $\mathcal{D}_1$ stand for the traces of $\text{Curl}^t E \times N$ and $E$ on $\partial \Omega$, respectively.

**Corollary 3** (Weak trace operators). Let $\Phi \in H^2(\Omega, S^3)$, with $\gamma_0(\Phi) \in H^{3/2}(\partial \Omega, S^3)$ the boundary trace of $\Phi$. Let $\gamma_i(\Phi) \in H^{1/2}(\partial \Omega, S^3)$ $(i = 1, 2)$ the boundary traces of $\nabla \Phi \cdot N$ and $\text{Curl}^t \Phi \times N$ for $i = 1$ or $i = 2$, respectively. Let $E \in H_{\text{inc}}(\Omega)$.

Then there exists $^2 \mathcal{T}_0(E) \in H^{-1/2}(\partial \Omega) := (H^{1/2}(\partial \Omega, S^3))^t$ such that

\[
\langle \mathcal{T}_0(E), \gamma_1(\Phi) \rangle := \langle E, \text{inc} \, \Phi \rangle - \langle \text{inc} \, E, \Phi \rangle, \quad (66)
\]

for all $\Phi$ such that $\gamma_0(\Phi) = 0$. Moreover, there exists $\mathcal{T}_1(E) \in H^{-3/2}(\partial \Omega) := (H^{3/2}(\partial \Omega, S^3))^t$ such that

\[
\langle \mathcal{T}_1(E), \gamma_0(\Phi) \rangle := \langle E, \text{inc} \, \Phi \rangle - \langle \text{inc} \, E, \Phi \rangle, \quad (67)
\]

for all $\Phi$ such that $\gamma_1(\Phi) = 0$.

---

$^2$There is no need to write $H^{3/2}(\partial \Omega, S^3)$ since the boundary is a closed surface and this space is defined by density of smooth functions with compact support.
Furthermore, there exists $\tilde{\mathcal{T}}_2(E) \in H^{-3/2}(\partial\Omega)$ such that
\begin{equation}
\langle \tilde{\mathcal{T}}_2(E), \gamma_0(\Phi) \rangle := \langle E, \text{inc } \Phi \rangle - \langle \text{inc } E, \Phi \rangle, \tag{68}
\end{equation}
for all $\Phi$ such that $\gamma_2(\Phi) = 0$. Moreover, there exists $\tilde{\mathcal{T}}_1(E) \in H^{-1/2}(\partial\Omega)$ such that
\begin{equation}
\langle \tilde{\mathcal{T}}_1(E), \gamma_2(\Phi) \rangle := \langle \text{inc } E, \Phi \rangle - \langle E, \text{inc } \Phi \rangle, \tag{69}
\end{equation}
for all $\Phi$ such that $\gamma_0(\Phi) = 0$, respectively.

Here symbol $\langle \cdot \rangle$ stands for the duality operator in appropriate spaces.

**Proof.** By Green formula (59), let us define the linear functional on $H^{1/2}(\partial\Omega)$ by
\begin{equation}
\langle T_0(E), \gamma_1 \rangle := \langle E, \text{inc } \Phi \rangle - \langle \text{inc } E, \Phi \rangle,
\end{equation}
where $\Phi \in H^2(\Omega, S^3)$ satisfies $\gamma_0(\Phi) = 0$ and $\gamma_1(\Phi) = \gamma_1$ for a given $\gamma_1 \in H^{3/2}(\partial\Omega)$; define also the linear functional on $H^{3/2}(\partial\Omega)$ by
\begin{equation}
\langle \mathcal{T}_1(E), \gamma_0 \rangle := \langle E, \text{inc } \Phi \rangle - \langle \text{inc } E, \Phi \rangle,
\end{equation}
where $\Phi \in H^2(\Omega, S^3)$ satisfies $\gamma_1(\Phi) = 0$ and $\gamma_0(\Phi) = \gamma_0$ for a given $\gamma_0 \in H^{1/2}(\partial\Omega)$. First observe that $\mathcal{T}_i(E)$ ($i = 0, 1$) does not depend on the chosen extension. If $\Phi_1, \Phi_2$ are two such extensions, then their difference has zero trace and
\begin{equation}
0 = \langle E, \text{inc } (\Phi_1 - \Phi_2) \rangle - \langle \text{inc } E, (\Phi_1 - \Phi_2) \rangle,
\end{equation}
by (59), since $\gamma_0(\Phi_1 - \Phi_2) = \gamma_1(\Phi_1 - \Phi_2) = 0$. It has been proved in [1] that a lifting operator $L_{\partial\Omega} : H^{1/2}(\partial\Omega, S^3) \times H^{3/2}(\partial\Omega, S^3) \to H^2(\Omega, S^3)$ exists and can be chosen so that by its linearity and continuity (note that in Lemma 3.11. of [1] such a lifting can be taken as solenoidal on the boundary), it holds (for $i = 0, 1$)
\begin{equation}
|\langle \mathcal{T}_i(E), \gamma_i \rangle| \leq C \left( \|E\|_{L^2} + \|\text{inc } E\|_{L^2} \right) \left\| L_{\partial\Omega}(\gamma_i) \right\|_{H^2(\Omega)}
\leq C \left( \|E\|_{L^2} + \|\text{inc } E\|_{L^2} \right) \left\| \gamma_i \right\|_{H^{3/2} - 1(\partial\Omega)},
\end{equation}
achieving the proof of the first two statements.

As for the last one, take $E \in C^2(\Omega, S^3)$ and $\eta \in H^2(\Omega, S^3)$ such that $\gamma_2(\eta) = 0$ and compute by a series of integrations by parts,
\begin{equation}
\int_{\Omega} E \cdot \text{inc } \eta dx = \int_{\Omega} \text{Curl } E \cdot \text{Curl}^\dagger \eta dx = \int_{\Omega} \text{Curl}^\dagger E \cdot \text{Curl } \eta dx
= \int_{\Omega} \text{inc } E \cdot \eta dx + \int_{\partial\Omega} \text{Curl}^\dagger E \times N \cdot \eta dS(x). \tag{70}
\end{equation}

By (65), one has
\begin{equation}
\langle \mathcal{D}_2(E), \gamma_0 \rangle = \langle E, \text{inc } \Phi \rangle - \langle \text{inc } E, \Phi \rangle, \tag{71}
\end{equation}
where $\Phi \in H^2(\Omega, S^3)$ satisfies $\gamma_0(\Phi) = \gamma_0$ and $\gamma_2(\Phi) = 0$. Now, let $E \in \mathcal{H}_{\text{inc}}(\Omega)$ and define
\begin{equation}
\langle \mathcal{D}_2(E), \gamma_0 \rangle := \langle E, \text{inc } \Phi \rangle - \langle \text{inc } E, \Phi \rangle \tag{72}
\end{equation}
where $\Phi \in H^2(\Omega, S^3)$ satisfies $\gamma_0(\Phi) = \gamma_0$ and $\gamma_2(\Phi) = 0$. By the above lifting operator $L_{\partial\Omega}$ and provided Lemma 4.1 (which states that given the curl transpose and the value of $E$ on the boundary yields the tangential components of $\partial_N E$), one obtains
\begin{equation}
|\langle \mathcal{D}_2(E), \gamma_0 \rangle| \leq C \left( \|E\|_{L^2} + \|\text{inc } E\|_{L^2} \right) \left\| \gamma_0 \right\|_{H^{3/2}(\partial\Omega)}, \tag{73}
\end{equation}
whence linearity and continuity of $\mathcal{D}_2$ in $H^{-3/2}(\partial\Omega)$. 

\[\]
Inverting the roles of $E$ and $\eta$ in (70), and defining
\[ \langle \mathcal{P}_1(E), \gamma_2 \rangle := \langle \text{inc } E, \Phi \rangle - \langle E, \text{inc } \Phi \rangle, \]
where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_0(\Phi) = 0$ and $\gamma_2(\Phi) = \gamma_2$, also yields linearity and continuity of $\mathcal{P}_1$ in $H^{-1/2}(\partial \Omega)$, achieving the proof.

Obviously (75) holds for any $\gamma_0 \in C_0^\infty(\Gamma_0)$ and hence
\[ ||\langle \mathcal{P}_2(E), \gamma_0 \rangle|| \leq C (\|E\|_{L^2} + \|\text{inc } E\|_{L^2}) \|\gamma_0\|_{H^{1/2}(\Gamma_0)}, \]
for any $\gamma_0 \in H_0^{3/2}(\Gamma_0)$. A similar reasoning can be made for $\mathcal{P}_1$ and hence (63) and (65) can be extended as follows:
\[ \mathcal{P}_{1,0} : \mathcal{H}_{\text{inc}}(\Omega) \rightarrow H^{-1/2}(\Gamma_0) \]
\[ \mathcal{P}_{2,0} : \mathcal{H}_{\text{inc}}(\Omega) \rightarrow H^{-3/2}(\Gamma_0). \]

4.3. Saint-Venant conditions and Beltrami decomposition. The following result is given for the sake of generality in $L^p(\Omega)$ with $1 < p < \infty$ but should here be considered for $p = 2$.

**Theorem 4.5** (Saint-Venant compatibility conditions in $L^p$ [14]). Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain, let $1 < p < +\infty$, and let $E \in L^p(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then
\[ \text{inc } E = 0 \text{ in } W^{-2,p}(\Omega, \mathbb{S}^3) \iff E = \nabla^S u \]
for some $u \in W^{1,p}(\Omega, \mathbb{R}^3)$. Moreover, $u$ is unique up to rigid displacements.

The following result is again given for the sake of generality in $L^p(\Omega)$ with $1 < p < \infty$ but should here be considered for $p = 2$.

**Theorem 4.6** (Beltrami decomposition in $L^p$ [14]). Assume that $\Omega$ is simply-connected. Let $p \in (1, +\infty)$ be a real number and let $E \in L^p(\Omega, \mathbb{S}^3)$. Then, for any $u_0 \in W^{1,p}(\partial \Omega, \mathbb{R}^3)$, there exists a unique $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $u = u_0$ on $\Gamma_0 \subset \partial \Omega$ and a unique $F \in L^p(\Omega, \mathbb{S}^3)$ with $\text{Curl } F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, $\text{inc } F \in L^p(\Omega, \mathbb{S}^3)$, $\text{div } F = 0$ and $FN = 0$ on $\partial \Omega$ such that
\[ E = \nabla^S u + \text{inc } F. \]
We call $\nabla^S u$ the compatible part and $\text{inc } F$ the (solenoidal) incompatible part of the Beltrami decomposition.

Observe that if $\text{inc } E = 0$ in $\Omega$ and $u_0 = 0$ on $\Gamma_0$ then $u$ is uniquely determined such that
\[ E = \nabla^S u, \]
since $F = 0$ is the unique solution of the decomposition. Indeed, $u$ is the unique solution of
\[ -\text{div } (\nabla^S u) = -\text{div } E \in \Omega, \]
with as boundary conditions, $u = 0$ on $\Gamma_0$ and $(\nabla^S u)N = EN$ on $\partial \Omega \setminus \Gamma_0$. Now, if there were two solutions $F_1$ and $F_2$ satisfying (78), then $\text{inc } (F_1 - F_2) = 0$, which by Theorem 4.5 implies that $F_1 - F_2 = \nabla^S v$ for some $v$, whence $F_1 = F_2$, since $\text{div } (\nabla^S v) = 0$ in $\Omega$ with $(\nabla^S v)N = 0$ on $\partial \Omega$. 
5. The intrinsic approach with an anholonomic curvilinear frame: Statement of the main results.

**Definition 5.1** (Equivalence class). We write \( u = u_0 \) on \( \Gamma_0 \) to mean that there exists a rigid displacement \( r \in \mathcal{R}(\Gamma_0) \) in \( \Gamma_0 \), such that \( u - r = u_0 \).

**Lemma 5.2.** The following conditions are equivalent:

(i) \( e \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\text{comp}}(\Omega) \),

(ii) there exists a unique \( u \in \mathcal{C}^2(\bar{\Omega}) \cap H^3(\Omega, \mathbb{R}^3) \) such that \( e = \nabla^S u \) and \( u = 0 \) on \( \Gamma_0 \). It also holds \( u \equiv 0 \) on \( \Gamma_0 \) \iff \( e = 0 = (\text{Curl}^k e) \times N \) on \( \Gamma_0 \).

**Proof.** If \( e \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\text{comp}}(\Omega) \), then inc \( e = 0 \) and by Theorems 4.5 and 4.6, there exists a unique \( u \in \mathcal{C}^2(\bar{\Omega}) \cap H^3(\Omega, \mathbb{R}^3) \) such that \( e = \nabla^S u \) and \( u = 0 \) on \( \Gamma_0 \). Moreover, \( e = 0 = \text{Curl}^k e \times N \) on \( \Gamma_0 \) and hence Theorem 4.3 entails that \( \gamma^i(e) = \rho^i(e) = 0 \), and Theorem 3.3 yields \( u \equiv 0 \) on \( \Gamma_0 \). Now, \( u \in \mathcal{R}(\Gamma_0) \) implies by (54) and Theorem 4.3 that \( e = 0 = (\text{Curl}^k e) \times N \) on \( \Gamma_0 \).

Now, for \( e \in \mathcal{H}^1_{\Gamma_0;\text{comp}}(\Omega) \), Theorem 4.5 yields \( e = \nabla^S u \), where uniqueness of \( u \) follows as soon as \( \mathcal{H}^1(\Gamma_0) > 0 \) (see [4]). Moreover, Corollary 3 provides an extension meaning to the traces, since from (74) it follows that \( e \in H^{-1/2}(\partial \Omega) \) and from (75) that \( \text{Curl}^k e \in H^{-3/2}(\partial \Omega) \). In particular, for any \( e \in \mathcal{H}^1_{\Gamma_0;\text{comp}}(\Omega) \), there exists a sequence \( e_n \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\text{comp}}(\Omega) \) such that \( e_n \to e \) strongly in \( L^2(\Omega, \mathbb{R}^3) \). Of course by Korn’s inequality, it also holds \( u_n \to u \) strongly in \( H^1(\Omega, \mathbb{R}^3) \).

By the above considerations (in particular the weak traces of Section 4.2) and classical density arguments, we are now in position to state and prove a general form of Lemma 5.2. The following theorem is a restatement of Theorem 3.3 without appealing to the change of metric and curvature tensors as in the original version of [5], rather by letting \( e \) belong to a specific function space.

**Theorem 5.3** (Intrinsic version of the homogeneous condition). The following conditions are equivalent:

(i) \( e \in \mathcal{H}^1(\bar{\Omega}) \),

(ii) there exists a unique \( u \in H^1(\Omega) \) such that \( e = \nabla^S u \) and \( u = 0 \) on \( \Gamma_0 \). Further,

\[
u \equiv 0 \text{ on } \Gamma_0 \iff \mathcal{D}_{1,0}(e) = \mathcal{D}_{2,0}(e) = 0,
\]

where \( \mathcal{D}_i \) are the boundary operators given by (76) and (77) for \( i = 1 \) and \( i = 2 \), respectively.

Let us remark that the non-homogeneous problem could be considered in this setting, too. In contrast with Theorem 5.4 it is not clear how the nonhomogeneous boundary condition may be handled by means of the change of metric and curvature tensors as in the original version of [5]. Within our formalism, it is immediate, as stated in the following result.

**Theorem 5.4** (Intrinsic version of the non-homogeneous condition). Let \( u_0 \in H^{3/2}(\Gamma_0, \mathbb{R}^3) \). If \( e \in \mathcal{H}^1(\bar{\Omega}) \), there exists a unique \( u \in H^1(\Omega) \) such that \( e = \nabla^S u \) and \( u = u_0 \) on \( \Gamma_0 \). It also holds

\[
u \equiv u_0 \text{ on } \Gamma_0 \iff \mathcal{D}_{1,0}(e - \nabla^S \hat{u}) = \mathcal{D}_{2,0}(e - \nabla^S \hat{u}) = 0,
\]

where \( \hat{u} = \mathcal{L}(u_0) \) is a \( H^1 \)-boundary lifting of \( u_0 \).
**Proof.** The first part of the statement follows from Saint-Venant compatibility condition and Beltrami decomposition in $L^2$ (i.e., Theorems 4.5 and 4.6), where the rigid displacement is fixed by the condition $u = u_0$ on $\Gamma_0$. The second part is an obvious consequence of Theorem 5.3. \qed

**Definition 5.5.** We introduce the following quotient space with respect to the equivalence relation of Definition 5.1:

$$\hat{H}^1_{\Gamma_0}(\Omega, \mathbb{R}^3) = H^1_{\Gamma_0}(\Omega, \mathbb{R}^3) / \mathcal{R}(\Gamma_0) = \{ u \in H^1(\Omega, \mathbb{R}^3) : u = 0 \text{ on } \Gamma_0 \}.$$

**Corollary 4.** The map

$$F^* : \mathcal{H}_{\Gamma_0;\text{comp}}(\Omega) \to \hat{H}^1_{\Gamma_0}(\Omega, \mathbb{R}^3) : e \mapsto F^*(e) = u \text{ s.t. } e = \nabla^S u$$  \hspace{1cm} (80)

is well defined, linear and continuous with respect to the $L^2$-norm. \qed

**Proof.** Well-definedness and linearity follow from Theorem 5.3, and continuity from Korn inequality [14].

It is now obvious that the strong form of linearized elasticity may be rewritten as a variational problem in terms of the compatible strain $e = e(u) = \nabla^S u$.

**Theorem 5.6 (Intrinsic version of linearized elasticity).** The variational problem

$$\inf_{e \in \mathcal{H}_{\Gamma_0;\text{comp}}(\Omega)} \mathcal{E}(e) = \frac{1}{2} \int_{\Omega} C^{-1} e \cdot edx - \int_{\Omega} f \cdot F^*(e) dx - \int_{\partial \Omega \setminus \Gamma_0} g \cdot F^*(e) dS(x),$$

achieves its minimum $e^*$, which satisfies the strong form

$$\begin{cases}
- \text{div} (C^{-1} e^*) = f & \text{in } \Omega \\
e^* = (\text{Curl}^e e^*) \times N = 0 & \text{on } \Gamma_0 \\
(C^{-1} e^*) N = g & \text{on } \partial \Omega \setminus \Gamma_0
\end{cases},$$

where the traces are intended in a weak sense. Furthermore $u^* := F^*(e^*)$ satisfies

$$\begin{cases}
- \text{div} (C^{-1} \nabla^S u^*) = f & \text{in } \Omega \\
u^* = 0 & \text{on } \Gamma_0 \\
(C^{-1} \nabla^S u^*) N = g & \text{on } \partial \Omega \setminus \Gamma_0
\end{cases}.$$

**Proof.** By Korn inequality, it is easily computed that $0 > \mathcal{E}(e) > C\|e\|_{L^2} - \beta$ for some $C > 0$ and $\beta \geq 0$. Existence follows by the direct method, since $e \mapsto \mathcal{E}(e)$ is continuous by Corollary 4. \qed

Remark that the non-homogeneous case can also be written as a variational problem in $e$, by a simple boundary lifting, and change of variables.

Theorem 5.6 is the counterpart of Theorems 7.1 and 7.2. in [5]. Instead of a condition on the variations of metric curvature, which is not easy to give a clear physical meaning, here we give a condition on the strain and on the Frank tensor on $\Gamma_0$, which both bear a precise physical meaning. Moreover, our formalism also allows one to write the problem in a variational form, since the boundary condition on the compatible strain is included in the function space $\mathcal{H}_{\Gamma_0;\text{comp}}(\Omega)$. 

6. Discussion and concluding remarks.

6.1. Application to multiscale analysis of dislocations. In a crystal $\Omega$, a mesoscopic dislocation $L$ is a loop (or a curve ending at the crystal boundary) which renders the strain field $e$ (and the displacement $u$) singular, because it generates a strain of the order $O\left(1/d(\cdot,L)\right)$, with $d(x,L)$ the distance from $x \in \Omega$ to $L$. This is due to the constrain $\oint_{C_L} \nabla u dH^1 = B$, where the Burgers vector $B$ represents the jump of the displacement vector, and where $C_L$ stands for a circuit around the line $L$. Note that in the latter formula, $\nabla u$ stands for the absolutely continuous part of the distributional derivative $Du = \nabla u_L - B \otimes \tau H^1_L$ with $S_L$ a surface enclosed by $C_L$ (see [16]) with unit normal $N$. By Stokes theorem, it is deduced that $\text{Curl} \nabla u = \Lambda_T L = B \otimes \tau H^1_L$ which implies (as proved in [20]) that

$$\text{inc } e = \text{Curl } (\Lambda_L - \frac{I_2}{2} \text{tr } \Lambda_L),$$

where $e := \nabla S u \in L^1(\Omega)$ is the incompatible strain (wheras $D S u$ is obviously compatible). As a consequence of the strain behaviour in $O\left(1/d(\cdot,L)\right)$, the potential energy $E(e) := \frac{1}{2} \int_{\Omega} C^{-1} e \cdot e dx - \int_{\Omega} f \cdot F^*(e) dx - \int_{\partial \Omega \setminus \Gamma_0} g \cdot F^*(e) dS(x)$ is unbounded, due to the unboundedness of the quadratic stored elastic energy $\mathcal{W}(e) := \frac{1}{2} \int_{\Omega} C^{-1} e \cdot e dx.$

As easily seen, for a single dislocation line, the singularity of $\mathcal{W}(e)$ is of the order $O\left(\log \left( d(\cdot,L) \right) \right)$. Note also that by Theorem 5.6, the boundary condition on $u$ may be non-homogeneous.

Let $\varepsilon$ be a small parameter and $L_\varepsilon$ be a finite family of dilute dislocations, whose number is bounded by $N_\varepsilon$. Let $e_\varepsilon$ be the (symmetric) strain at scale $\varepsilon$ satisfying $\text{inc } e_\varepsilon = \text{Curl } (\Lambda_{L_\varepsilon} - \frac{I_2}{2} \text{tr } \Lambda_{L_\varepsilon})$ in $\Omega$. Now, let $\hat{e}_\varepsilon$ be the associated cut-off, i.e. $\hat{e}_\varepsilon = 0$ in a tubular neigbourhood of $L_\varepsilon$ of radius depending $\varepsilon$, while $\hat{e}_\varepsilon = e_\varepsilon$ in the remaining of the domain. By such a procedure, the modified strain $\hat{e}_\varepsilon$ is compatible in the punctured domain and the intrinsic approach to elasticity with dislocations applies for any $\varepsilon > 0$. Let us consider the rescaled stored elastic energy

$$\mathcal{W}_\varepsilon(\hat{e}_\varepsilon; \Lambda_{L_\varepsilon}) = \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{\Omega} C^{-1} \hat{e}_\varepsilon \cdot \hat{e}_\varepsilon dx.$$

The problem we address is whether the functional $\mathcal{W}_\varepsilon$ converges as $\varepsilon \to 0$ in an appropriate sense, as for instance $\Gamma$-convergence (see [6]). A crucial step when dealing with $\Gamma$-convergence, is to determinate the topology involved, and hence to know the functional spaces inherent to the problem. Here, $\Lambda_{L_\varepsilon}$ is a bounded Radon measure, and the strains $\hat{e}_\varepsilon$ belong to $L^2(\Omega, S^3)$. Thanks to our formalism, we can consider the case in which the displacement is prescribed on a portion $\Gamma_0$ of the boundary, as in Theorem 5.6, that is, we take $\hat{e}_\varepsilon \in \mathcal{H}_{\Gamma_0,\text{comp}}(\Omega) \subset L^2(\Omega, S^3)$. Therefore, the problem is to compute

$$\Gamma - \lim_{\varepsilon \to 0} \mathcal{W}_\varepsilon(\hat{e}_\varepsilon; \Lambda_{L_\varepsilon}).$$
There are good reasons to believe (but is hard to prove, and will be the goal for future works) that this limit writes as

\[ W(\hat{\epsilon}) + \int_{\Omega} \varphi \left( \frac{d\mu}{d|\mu|} \right) d|\mu|, \]

where \( \frac{\hat{\epsilon}}{\sqrt{N \log \epsilon}} \to \hat{\epsilon} \in H_{\Gamma_0; \text{comp}}(\Omega) \) weakly in \( L^2 \), and \( \Lambda_{\epsilon} \to \mu \) in the Radon-measure sense, and for some \( \varphi \) to be determined. Now, let \( \Lambda_{\epsilon} \) be fixed and set

\[ \mathcal{E}_\epsilon(\hat{\epsilon}_x; \Lambda_{\epsilon}) := W(\hat{\epsilon}_x; \Lambda_{\epsilon}) - \int_{\Omega} f \cdot F^*(\hat{\epsilon}_x) dx - \int_{\partial \Omega \setminus \Gamma_0} g \cdot F^*(\hat{\epsilon}_x) dS(x). \]

Then the above postulated \( \Gamma \)-convergence result together with Corollary 4 and Theorem 5.6 yield

\[
\Gamma - \lim_{\epsilon \to 0} \left\{ \inf_{\hat{\epsilon}_x \in H_{\Gamma_0; \text{comp}}(\Omega)} \mathcal{E}_\epsilon(\hat{\epsilon}_x; \Lambda_{\epsilon}) \right\} = \mathcal{E}(\hat{\epsilon}) + \int_{\mathcal{L}} \varphi(B \otimes \tau) dH^1,
\]

where \( \varphi \) is interpreted as a line-tension functional accounting for the core regularization of the dislocations, i.e., related to the self energy of the dislocation network.

Summarizing, our formalism of the intrinsic approach will allow us address the aforementioned homogenization problem, where one passes from a singular elasticity problem at the mesoscopic scale, to a regularized macroscopic elasto-plastic model, where the macroscopic plasticity is represented by means of the limit measure \( \mu \).

Note that it is important in dislocation modeling to be able to consider a complete boundary-value problem of mixed type, since typical crystals, in particular in industrial crystal growth processes, show force-free portion of their boundary together with interfaces subjected to an imposed displacement (as for instance the melting temperature front).

6.2. General conclusion. The motivation for this work was the study of dislocations where the displacement must be replaced by the strain as a model variable. This work represents the first step towards a systematic use of the Frank tensor in various contexts, and in particular in the study of dislocations, where it most naturally appears in the form of its curl as the incompatibility tensor, that is, by Kröner’s formula [12], as a measure of the dislocation density in the body:

\[ \text{inc } \epsilon = \text{Curl } \kappa, \]

with \( \kappa \) the contortion tensor as related to the dislocation density \( \Lambda \) by \( \kappa := \Lambda - \frac{1}{2} \text{tr } \Lambda \), and \( \epsilon \) the elastic strain, related to the Cauchy stress \( \sigma \) by the constitutive law \( \epsilon = C \sigma \). For this reason, this work will permit various contributions in dislocation modeling, in particular dealing with homogenization, as briefly exposed in Section 6.1.

The original intrinsic formulation by Ph. Ciarlet and C. Mardare is at the origin of the present work. The authors have in mind various applications in shell and plate theories, and the definitive impact of such a novel presentation will certainly appear clear in near future. Therefore, also the role of the Frank tensor, as related to the rotation gradient,

\[ \text{Curl}^\dagger \epsilon = \nabla w, \]

will presumably play a role in variational formulations of low-dimensional, membrane theories.
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