RICCI ITERATIONS AND CANONICAL KÄHLER-EINSTEIN CURRENTS ON LOG CANONICAL PAIRS

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Abstract

In this article we construct a canonical Kähler-Einstein current on a LC (log canonical) pairs of log general type as the limit of a sequence of canonical Kähler-Einstein currents on KLT (Kawamata log terminal) pairs of log general type. We call the volume form associated with the canonical Kähler-Einstein current the canonical measure of the LC pair. We prove that the relative canonical measure on a projective family of LC pairs of log general type defines a singular hermitian metric on the relative log canonical bundle and the metric has semipositive curvature in the sense of current. This is the first semipositivity result for relative log canonical bundles of a family of LC pairs in general dimension.

Our proof depends on certain Ricci iterations and dynamical systems of Bergman kernels. MSC: 53C25(32G07 53C55 58E11)

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1 Introduction

In [T9] I have constructed canonical measures in terms of the dynamical systems of Bergman kernels. This dynamical construction is considered to be a polynomial approximation of canonical measures and naturally leads us to the logarithmic pluri-subharmonicity of relative canonical measure on an algebraic fiber space with nonnegative relative Kodaira dimension ([T9]).

In algebraic geometry, it is natural and useful to consider KLT pairs instead of projective varieties themselves. For example, the finite generation of canonical rings has been proven not only for smooth projective varieties but also KLT pairs ([B-C-H-M]). Moreover, the proof of the finite generation in [B-C-H-M] also depends on the generalization to KLT pairs. Another example is the canonical bundle formula which reduces the canonical rings of projective varieties of arbitrary Kodaira dimension to the log canonical rings of KLT pairs of log general type ([F-M]).

In this way generalizations to KLT pairs are quite essential in algebraic geometry. Roughly speaking in differential geometric context, to consider KLT pairs corresponds to consider orbifold structures on the projective varieties.

But so far most of the results for KLT pairs have not yet been generalized to the case of LC pairs. For example, the finite generation of log canonical rings of LC pairs has not yet been proven. The main reason is that the usual tools to study KLT pairs, such as the branched covering trick, variation of Hodge structures and the Kawamata-Viehweg vanishing theorem are no longer effective in the case of LC pairs. At present there are big obstacles to go beyond KLT pairs.

The purpose of this article is to extend the results in [T7, T9] to the case of LC pairs of log general type. More precisely, first we extend the result in [T7, T9] to the case of KLT pairs and then passing to the limits, we extend the results to the case of LC pairs.

The new feature here is the use of Ricci iterations. In fact, in this case we cannot construct the Kähler-Einstein current in terms of a single dynamical

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1 Throughout this article, an algebraic fiber space \( f : X \to Y \) means that \( X, Y \) are smooth projective and \( f \) is a dominant morphism with connected fibers.
system of Bergman kernels and the construction requires another dynamical system, namely Ricci iterations.

The main motivation to construct canonical Kähler-Einstein currents in terms of dynamical systems of Bergman kernels is to prove the semipositivity of relative (log) canonical bundles or the direct images of the pluri(log)canonical systems in terms of the recent breakthrough due to Bo Berndtsson which proves the pluri-subharmonic variation properties of Bergman kernels ([B1], [B2], [B3]).

We note that the corresponding result has already been obtained in the context of algebraic geometry much earlier (almost 30 years ago) by Y. Kawamata in terms of variation of Hodge structures as follows. Let \( f : X \rightarrow Y \) be an algebraic fiber space over a projective curve \( Y \). In [Ka1], Y. Kawamata proved that for every positive integer \( m \), \( f_* \mathcal{O}_X(K_X^{\otimes m}/Y) \) is semipositive in the sense that any quotient of it has semipositive degree on \( Y \). Since then, this result has been used extensively in algebraic geometry. Later Kawamata generalized his result to the case of KLT pairs ([Ka2], [Ka3], p.175, Theorem 1.2). In particular he proved the subadjunction theorem ([Ka2]). We note that the KLT condition is quite essential in his argument because he applied the theory of variation of Hodge structures ([G], [S], [Sch]) after taking a suitable branched covering of the original variety. Hence in this sense LC pairs are out of reach of the Hodge theory.

The advantage of the differential geometric method in this article is two folds. First, it gives a canonical metric with semipositive curvature in the sense of current on the relative log canonical bundle of a projective family of LC pairs. The canonicity of the metric is quite essential to consider the moduli problems. See [T3] for other canonical metrics. Second, it gives a general method to consider LC pairs as a limit of KLT pairs. Although LC pairs are out of reach of the Hodge theory, nevertheless intuitively, a LC pair is a limit of a sequence of KLT pairs. Hence it is natural to expect that if one obtains a theorem for KLT pairs, then the corresponding result for LC pairs would be obtained by taking a limit in a suitable sense. In fact we implement this philosophy in terms of canonical Kähler-Einstein currents on KLT pairs and obtain the semipositivity of the relative log canonical bundle for a family of LC pairs (Theorem 1.10). The prototypes of this kind of argument were already observed in [T1], [T2], (cf. Example 2.9), where I considered a quasiprojective manifold as a limit of orbifolds. I hope that we may justify the philosophy in much broader context in future and construct the theory of LC pairs such as finite generation of log canonical rings of LC pairs. The new feature here is that the semipositivity is not on the direct image, but on the relative log canonical bundle itself. Unlike the KLT case, this is an essential difference, i.e., it seems to be difficult to obtain the semipositivity of the direct images of relative pluri log canonical systems for a projective family of LC pairs in general.

The organization of this article is as follows. In Section 1, I explain the results in this article. In Section 2, I prove the existence of canonical Kähler-Einstein currents on LC pairs of log general type. In Section 3, I construct a Ricci iterations which converges to the canonical Kähler-Einstein current on LC pairs of log general type. In Section 4, I decompose the Ricci iteration into a series of dynamical systems of Bergman kernels. In Section 5, I prove the logarithmic pluri-subharmonicity of canonical Kähler-Einstein volume forms on a projective variety.

\footnote{Maybe we may use mixed Hodge theory. But it is not clear now.}
family of LC pairs. In Section 6, I apply the pluri-subharmonic variation property of relative canonical measures to deduce the weak semistability of the direct images of relative pluri-log-canonical systems for a projective family of KLT pairs. This is a direct application of the result in Section 5 and Viehweg’s ingenious ideas in [V].

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I would like to express my sincere thanks to Professor Y. Rubinstein, who sent me his preprint [Ru] and drew my attention to Ricci iterations. In fact the Ricci iterations used in this article are variants of the Ricci iterations considered in [Ru]. I also would like to express my sincere thanks to Dr. V. Tosatti who pointed out a serious incompleteness in the previous version.

Notations

- For a real number $a$, $\lceil a \rceil$ denotes the minimal integer greater than or equal to $a$ and $\lfloor a \rfloor$ denotes the maximal integer smaller than or equal to $a$.
- Let $X$ be a projective variety and let $D$ be a Weil divisor on $X$. Let $D = \sum d_i D_i$ be the irreducible decomposition. We set
  \[
  [D] := \sum \lceil d_i \rceil D_i, \quad \lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i.
  \]
- Let $f : X \to Y$ be an algebraic fiber space and let $D$ be a $\mathbb{Q}$-divisor on $X$. Let
  \[
  D = D^h + D^v
  \]
  be the decomposition such that an irreducible component of $\text{Supp} \, D$ is contained in $\text{Supp} \, D^h$ if and only if it is mapped onto $Y$. $D^h$ is the horizontal part of $D$ and $D^v$ is the vertical part of $D$.
- Let $(X, D)$ be a pair of a normal variety and a $\mathbb{Q}$-divisor on $X$. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution. Then we have the formula:
  \[
  K_Y = f^*(K_X + D) + \sum a_i E_i,
  \]
  where $E_i$ is a prime divisor and $a_i \in \mathbb{Q}$. The pair $(X, D)$ is said to be subKLT (resp. subLC, if $a_i > -1$ (resp. $a_i \geq -1$) holds for every $i$. $(X, D)$ is said to be KLT (resp. LC), if $(X, D)$ is subKLT (resp. subLC) and $D$ is effective.
- Let $X$ be a projective variety and let $\mathcal{L}$ be an invertible sheaf on $X$. $\mathcal{L}$ is said to be semiample, if there exists a positive integer $m$ such that $|\mathcal{L}^\otimes m|$ is base point free.
- $f : X \to Y$ be a morphism between projective varieties. Let $\mathcal{L}$ be an invertible sheaf on $X$. $\mathcal{L}$ is said to be $f$-semiample, if for every $y \in Y$, $\mathcal{L}|f^{-1}(y)$ is semiample.
Let $D, D' \in \text{Div}(X) \otimes \mathbb{Q}$ be $\mathbb{Q}$-Cartier divisors on a normal projective variety $X$. We denote
\[ D \preceq D', \]
if $D' - D$ is effective.

Let $L$ be a $\mathbb{Q}$-line bundle on a compact complex manifold $X$, i.e., $L$ is a formal fractional power of a genuine line bundle on $X$. A singular hermitian metric $h$ on $L$ is given by
\[ h = e^{-\varphi} \cdot h_0, \]
where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(X)$ is an arbitrary function on $X$. We call $\varphi$ the weight function of $h$ with respect to $h_0$. We note that $h$ makes sense, since a hermitian metric is a real object.

The curvature current $\Theta_h$ of the singular hermitian $\mathbb{Q}$-line bundle $(L, h)$ is defined by
\[ \Theta_h := \Theta_{h_0} + \partial \bar{\partial} \varphi, \]
where $\partial \bar{\partial} \varphi$ is taken in the sense of current. We define the multiplier ideal sheaf $I(h)$ of $(L, h)$ by
\[ I(h)(U) := \{ f \in \mathcal{O}_X(U); |f|^2 e^{-\varphi} \in L^1_{\text{loc}}(U) \}, \]
where $U$ runs open subset of $X$.

For a closed positive $(1, 1)$ current $T$, $T_{abc}$ denotes the absolutely continuous part of $T$.

For a Cartier divisor $D$, we denote the corresponding line bundle by the same notation. Let $D$ be an effective $\mathbb{Q}$-divisor on a smooth projective variety $X$. Let $a$ be a positive integer such that $aD$ is Cartier. We identify $D$ with a formal $a$-th root of the line bundle $aD$. We say that $\sigma$ is a multivalued global holomorphic section of $D$ with divisor $D$, if $\sigma$ is a formal $a$-th root of a global holomorphic section of $aD$ with divisor $aD$. And $1/|\sigma|^2$ denotes the singular hermitian metric on $D$ defined by
\[ \frac{1}{|\sigma|^2} := \frac{h_D}{h_D(\sigma, \sigma)}, \]
where $h_D$ is an arbitrary $C^\infty$ hermitian metric on $D$.

For a singular hermitian line bundle $(F, h_F)$ on a compact complex manifold $X$ of dimension $n$, $K(X, K_X + F, h_F)$ denotes (the diagonal part of) the Bergman kernel of $H^0(X, \mathcal{O}_X(K_X + F) \otimes I(h_F))$ with respect to the $L^2$-inner product:
\[ (\sigma, \sigma') := (\sqrt{-1})^n \int_X h_F \cdot \sigma \wedge \bar{\sigma}', \]
i.e.,
\[ K(X, K_X + F, h_F) = \sum_{i=0}^N |\sigma_i|^2, \]
where $\{\sigma_0, \cdots, \sigma_N\}$ is a complete orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + F) \otimes I(h_F))$. It is clear that $K(X, K_X + F, h_F)$ is independent of the choice of the complete orthonormal basis.
1.1 Twisted Kähler-Einstein equations for ample hermitian adjoint bundles

First let us consider an easy case. Let $X$ be a smooth projective $n$-fold and let $(L, h_L)$ be a $\mathbb{Q}$-line bundle on $X$ with a $C^\infty$ hermitian metric $h_L$. Here a $\mathbb{Q}$-line bundle means a formal fractional power of a genuine line bundle on $X$. We shall assume that $K_X + L$ is ample. We consider the following equation:

$$-\text{Ric}_\omega + \sqrt{-1} \Theta_{h_L} = \omega,$$

where $\omega$ is a $C^\infty$ Kähler form such that the Kähler class $[\omega]$ is equal to $2\pi c_1(K_X + L)$ in $H^2(X, \mathbb{R})$ and $\Theta_{h_L}$ denotes the curvature form of $h_L$. We call (1.5) the twisted Kähler-Einstein equation associated with $(L, h_L)$.

The equation (1.5) is reduced to a complex Monge-Ampère equation as follows. Let $\omega_0$ be a Kähler form on $X$ with $[\omega_0] = 2\pi c_1(K_X + L)$. Let $\Omega$ be a $C^\infty$ volume form on $X$ such that

$$\omega_0 = -\text{Ric}_\Omega + \sqrt{-1}\Theta_{h_L}$$

holds. Then there exists a $C^\infty$ function $u$ on $X$ such that

$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u$$

and the equation (1.5) is equivalent to the Monge-Ampère equation:

$$\log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} = u.$$

By the solution of Calabi’s conjecture ([A, Y1]) the equation (1.8) has a unique $C^\infty$ solution $u$. Hence (1.5) has a $C^\infty$ solution $\omega$.

**Definition 1.1** Let $X$ be a smooth projective variety and let $(L, h_L)$ be a $C^\infty$ hermitian (holomorphic) line bundle on $X$ such that $K_X + L$ is ample. The $C^\infty$ Kähler form $\omega$ satisfying the equation (1.5) is said to be the twisted Kähler-Einstein form associated with $(X, (L, h_L))$.

We call $(L, h_L)$ the boundary of the pair $(X, (L, h_L))$. Of course twisted Kähler-Einstein forms are not essentially new. But this new terminology gives us freedom to think about Kähler-Einstein-like forms on a smooth projective variety whose first Chern class is not necessarily definite.

For example, in this terminology the canonical Kähler current on the base of an Iitaka fibration is considered to be the twisted Kähler-Einstein form associated with the metrized Hodge $\mathbb{Q}$-line bundle (cf. [T9]) as the boundary.

1.2 Ricci iterations

Let $X$ be a compact Kähler manifold and let $L$ be a $\mathbb{Q}$-line bundle on $X$. Let $h_L$ be a (possibly singular) hermitian metric on $L$. Let $\omega_0$ be a $C^\infty$ Kähler form on $X$ and let $t$ be a fixed positive number in $(0, 1)$. In this article, we shall consider the Ricci iteration of the form:

$$-\text{Ric}_{\omega_m} + t\omega_{m-1} + \sqrt{-1}\Theta_{h_L} = \omega_m$$

for $m \geq 1$. We call the term $\sqrt{-1}\Theta_{h_L}$ the drift term of the Ricci iteration.
Suppose that \( h_L \) is \( C^\infty \). In this case if the iteration (1.9) has \( C^\infty \) solution \( \{\omega_m\} \) for \( m = 0, \cdots, \ell \), then
\[
(1.10) \quad [\omega_m] = \left( \frac{1 - t^m}{1 - t} \right) 2\pi c_1(K_X + L) + t^m[\omega_0]
\]
holds for every \( m = 0, \cdots, \ell \), where \([\omega_m]\) denotes the de Rham cohomology class of \( \omega_m \). In particular \( \{[\omega_m]\} \) moves on the segment connecting \([\omega_0]\) and \( 2\pi c_1(K_X + D) \).

Conversely by the solution of Calabi’s conjecture, if
\[
\left( \frac{1 - t^\ell}{1 - t} \right) 2\pi c_1(K_X + L) + t^\ell[\omega_0]
\]
is in the Kähler cone of \( X \) for some positive integer \( \ell \), then we see that the sequence of \( C^\infty \) Kähler forms \( \{\omega_m\}_{m=0}^\ell \) satisfying (1.9) exists. Moreover if \( c_1(K_X + L) \) is in the Kähler cone and \( \sqrt{-1}\Theta_{h_L} \geq 0 \), then one can prove easily that \( \{\omega_m\}_{m=0}^\ell \) converges to a \( C^\infty \) solution \( \omega_\infty \) of
\[
(1.11) \quad - \text{Ric}_{\omega_\infty} + \sqrt{-1}\Theta_{h_L} = \frac{1}{1 - t} \omega_\infty
\]
as \( m \) tends to infinity (cf. Theorem 3.1).

In this article, we shall consider the case that \( K_X + L \) is big and \( h_L \) is a singular hermitian metric of the form:
\[
h_L = \frac{1}{|\sigma|^2},
\]
where \( \sigma \) is a multivalued global holomorphic section of \( L \) such that \((X,(\sigma))\) is KLT, where \((\sigma)\) denotes the effective \( \mathbb{Q} \)-divisor associated with \( \sigma \) on \( X \). We note that in this case the solution is Kähler-Einstein on the locus (contained in \( X\setminus \text{Supp}(\sigma) \)), where \( \omega_\infty \) is a \( C^\infty \) Kähler form.

### 1.3 Dynamical systems of Bergman kernels

Here we shall review the construction in [T9]. Let \( X \) be a smooth projective variety and let \((L, h_L)\) be a \( C^\infty \) hermitian line bundle on \( X \) such that \( K_X + L \) is ample. Suppose that \( \sqrt{-1}\Theta_{h_L} \) is semipositive on \( X \). Then as in [T9], we may construct the twisted Kähler-Einstein form \( \omega \) satisfying (1.5) in terms of the dynamical system of Bergman kernels.

Let \( X \) be a smooth projective variety and let \((L, h_L)\) be a \( C^\infty \) hermitian line bundle on \( X \) with semipositive curvature. We assume that \( K_X + L \) is ample.

Let \( A \) be an ample line bundle such that for every pseudo-effective singular hermitian line bundle \((F, h_F), O_X(A + F) \otimes I(h_F)\) is globally generated. Such \( A \) exists by Nadel’s vanishing theorem [N, p.561].

We shall construct a sequence of Bergman kernels \( \{K_m\}_{m=1}^\infty \) and a sequence of singular hermitian metrics \( \{h_m\}_{m=1}^\infty \) as follows.

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3Theorem 3.1 is weaker than this fact. But one may easily generalize it.
We set

\[ K_1 := \begin{cases} 
  K(X, K_X + A, h_A), & \text{if } a > 1 \\
  K(X, K_X + L + A, h_L \cdot h_A), & \text{if } a = 1
\end{cases} \]

(1.12)

where \( K(X, K_X + A, h_A) \) and \( K(X, K_X + L + A, h_L \cdot h_A) \) are the Bergman kernels defined as (1.3). Then we set

\[ h_1 := (K_1)^{-1}. \]

(1.13)

We continue this process. Suppose that we have constructed \( K_m \) and the singular hermitian metric \( h_m \). Let \( h_m \) be the sequence of hermitian metrics as above and let \( n \) denote \( \dim X \). Then

\[ K_{m+1} := \begin{cases} 
  K(X, (m+1)K_X + \lfloor \frac{m+1}{a} \rfloor aL + A, h_m), & \text{if } m+1 \not\equiv 0 \mod a \\
  K(X, (m+1)(K_X + L) + A, h_L^a \otimes h_m), & \text{if } m+1 \equiv 0 \mod a
\end{cases} \]

and

\[ h_{m+1} := (K_{m+1})^{-1}. \]

(1.14)

Thus inductively we construct the sequences \( \{h_m\}_{m \geq 1} \) and \( \{K_m\}_{m \geq 1} \). This inductive construction is essentially the same one originated by the author in [T6]. The following theorem asserts that the above dynamical system yields the twisted Kähler-Einstein metric on \((X, (L, h_L))\).

**Theorem 1.2** Let \( X \) be a smooth projective \( n \)-fold and let \((L, h_L)\) be a \( C^\infty \) hermitian \( \mathbb{Q} \)-line bundle on \( X \) with semipositive curvature \( \sqrt{-1} \Theta h_L \) such that \( K_X + L \) is ample. Let \( \{h_m\}_{m \geq 1} \) be the sequence of hermitian metrics as above and let \( n \) denote \( \dim X \). Then

\[ h_\infty := \lim inf_{m \to \infty} \frac{n!}{(m!)^n} h_m \]

is a \( C^\infty \) hermitian metric on \( K_X + L \) such that

\[ \omega = \sqrt{-1} \Theta h_\infty \]

(1.16)

is the twisted Kähler-Einstein form (Definition 1.7).

In particular \( \omega = \sqrt{-1} \Theta h_\infty \) (in fact \( h_\infty \)) is unique and is independent of the choice of \( A \) and \( h_A \).

The proof of Theorem 1.2 is completely the same as the one of [T9 Theorem 1.8]. Hence we omit it. Theorem 1.2 naturally leads us to the following semipositivity theorem.

**Theorem 1.3** Let \( f : X \to Y \) be an algebraic fiber space and let \((L, h_L)\) be a hermitian \( \mathbb{Q} \)-line bundle on \( X \) with \( C^\infty \) hermitian metric \( h_L \). Suppose that
\[ -\text{Ric}_{\omega_y} + \sqrt{-1} \Theta_{h_L} |X_y = \omega_y. \]

Let \( n \) be the relative dimension of \( f : X \to Y \). Then the singular hermitian metric \( h^\omega \) on \( K_{X/Y} + L|f^{-1}(Y_0) \) defined by
\[(1.19) h^\omega |X_y := \left( \frac{1}{n!} \omega_y^n \right)^{-1} \cdot h_L \quad (y \in Y_0) \]
extends to a singular hermitian metric \( h \) on \( K_{X/Y} + L \) and has semipositive curvature in the sense of current everywhere on \( X \).

**Remark 1.4** \( \omega \) is \( C^\infty \) over the complement of the discriminant locus of \( f \). This is an easy consequence of the implicit function theorem. \( \square \)

The proof of Theorem \( 1.3 \) is completely the same as the one of [T9, Theorem 4.1]. Hence we omit it. In fact Theorem \( 1.2 \) and the logarithmic pluri-subharmonicity of Bergman kernels ([B3, B-P, T7]) implies Theorem \( 1.3 \).

### 1.4 Canonical Kähler-Einstein currents on KLT pairs

Now we shall consider the twisted Kähler-Einstein form for a singular \((L,h_L)\).

Let \( X \) be a smooth projective \( n \)-fold and let let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) is KLT. We assume that \((X, D)\) is of log general type, i.e., \( K_X + D \) is big. Let us consider the equation:
\[(1.20) -\text{Ric}_{\omega_K} + 2\pi [D] = \omega_K, \]
where \([D]\) denotes the closed positive current associated with the effective \( \mathbb{Q} \)-divisor \( D \). Let \( \sigma_D \) be a multivalued global holomorphic section of \( D \) with divisor \( D \) and let \( h_{\sigma_D} \) be the singular hermitian metric on \( D \) defined by
\[(1.21) h_{\sigma_D} := \frac{1}{|\sigma_D|^2}. \]

We construct a solution \( \omega_K \) of \( (1.20) \) which satisfies the followings:

1. \( \omega_K \) is a closed positive current on \( X \).
2. There exists a nonempty Zariski open subset \( U \) of \( X \) such that \( \omega_K|U \) is \( C^\infty \) Kähler form.
3. We define the singular hermitian metric \( h_K \) on \( K_X + D \) by
\[ h_K := \left( \frac{1}{n!} \omega_K^n \right)^{-1} \cdot h_{\sigma_D}. \]

Then \( h_K \) is an AZD (cf. [T4, T5]) of \( K_X + D \), i.e.,

1. \( \sqrt{-1} \Theta_{h_K} \) is a closed semipositive current on \( X \),
(b) For every positive integer \(m\) such that \(m(K_X + D)\) is Cartier

\[
H^0(X, \mathcal{O}_X(m(K_X + D))) \otimes \mathcal{T}(h_K^m)) \simeq H^0(X, \mathcal{O}_X(m(K_X + D)))
\]

holds.

We note that if such \(\omega_K\) exists, then \(\omega_K\) is Kähler-Einstein on \(U\), i.e., \(-\text{Ric}_{\omega_K} = \omega_K\) holds on \(U\).

**Definition 1.5** Let \((X, D)\) be a KLT pair of log general type. A closed positive current \(\omega_K\) satisfying the above properties: (1),(2),(3) is said to be the canonical Kähler-Einstein current on \((X, D)\). □

The following theorem asserts that the canonical Kähler-Einstein current always exists on a KLT pair \((X, D)\) of log general type.

**Theorem 1.6** Let \(X\) be a smooth projective variety and let \(D\) be an effective divisor such that \((X, D)\) is a KLT pair of log general type. Then there exists a unique canonical Kähler-Einstein current \(\omega_K\) on \((X, D)\). □

Later we shall prove a stronger uniqueness of the canonical Kähler-Einstein currents on KLT pairs (Theorem 1.9). The proof of Theorem 1.6 here is very similar to the case that \(D = 0\) in [19]. The proof is done by solving a Monge-Ampère equation as the solution of Calabi’s conjecture in [A, Y1]. The Monge-Ampère equation is quite similar to the one in [Y1, p.409, Theorem 8]. But to prove Theorem 1.6 we cannot apply [Y1, p.409, Theorem 8] directly, since the coefficients of \(D\) may exceed \(1/n(n = \dim X)\). We overcome this difficulty by introducing an orbifold structure on \((X, D)\), i.e., we resolve the singularity in terms of local cyclic branched coverings.

### 1.5 Canonical measures on KLT pairs of nonnegative Kodaira dimension

Similar twisted Kähler-Einstein currents naturally appear in algebraic geometry of KLT pairs.

Let \((X, D)\) be a KLT pair of nonnegative Kodaira dimension, i.e., \(|m!(K_X + D)| \neq \emptyset\) for every sufficiently large \(m\).

Let \(f : X \to Y\) be the Iitaka fibration associated with the log canonical divisor \(K_X + D\). By replacing \(X\) and \(Y\) by suitable modifications, we may assume the followings:

1. \(X, Y\) are smooth and \(f\) is a morphism with connected fibers.
2. \(\text{Supp } D\) is a divisor with normal crossings.
3. There exists an effective divisor \(\Sigma\) on \(Y\) such that \(f\) is smooth over \(Y - \Sigma\), \(\text{Supp } D^h\) is relatively normal crossings over \(Y - \Sigma\) and \(f(D^v) \subset \Sigma\), where \(D^h, D^v\) denote the horizontal and the vertical component of \(D\) respectively (cf. [L2]).
4. There exists a positive integer \(m_0\) such that for every \(m \geq m_0\), \(m!(K_X + D)\) is Cartier and \(f_!\mathcal{O}_X(m!(K_X + D))^{**}\) is a line bundle on \(Y\), where ** denotes the double dual.
We note that adding effective exceptional $\mathbb{Q}$-divisors does not change the log canonical ring. Such a modification exists by [F-M, p.169, Proposition 2.2]. We define the $\mathbb{Q}$-line bundle $L_{X/Y,D}$ on $Y$ by

\begin{equation}
L_{X/Y,D} = \frac{1}{m_0!} f_* \mathcal{O}_X (m_0! (K_X + D))^*.
\end{equation}

$L_{X/Y}$ is independent of the choice of $m_0$. Similarly as before we may define the singular hermitian metric $h_{L_{X/Y,D}}$ on $L_{X/Y,D}$ by

\begin{equation}
h_{L_{X/Y,D}}^{m!} (\sigma, \sigma)(y) := \left( \int_{X_y} |\sigma|^2 \right)^{m!},
\end{equation}

where $y \in Y - \Sigma$ and $X_y := f^{-1}(y)$. We call the singular hermitian $\mathbb{Q}$-line bundle $(L_{X/Y,D}, h_{L_{X/Y,D}})$ the **Hodge $\mathbb{Q}$-line bundle** of the Iitaka fibration $f : X \to Y$ associated with the KLT pair $(X, D)$. We note that since $(X, D)$ is KLT, $h_{L_{X/Y,D}}$ is well defined. By the same strategy as in the proof of Theorem [1.6] and [T9, Theorem 1.6], we have the following theorem:

**Theorem 1.7** In the above notations, there exists a unique singular hermitian metric on $h_K$ on $K_Y + L_{X/Y,D}$ and a nonempty Zariski open subset $U$ of $Y$ such that

1. $h_K$ is an AZD of $K_Y + L_{X/Y,D}$.
2. $f^* h_K$ is an AZD of $K_X + D$.
3. $h_K$ is $C^\infty$ on $U$.
4. $\omega_Y = \sqrt{-1} \Theta_{h_K}$ is a Kähler form on $U$.
5. $-\text{Ric}_{\omega_Y} + \sqrt{-1} \Theta_{L_{X/Y,D}} = \omega_Y$ holds on $U$. □

We may construct $\omega_Y$ in Theorem [1.7] in terms of a family of dynamical systems of Bergman kernels as Theorem[1.8] below and we obtain the same semipositivity. We define the **canonical measure** $d\mu_{\text{can}}$ of $(X, D)$ by

\begin{equation}
d\mu_{\text{can}} := \frac{1}{n!} f^* \omega^n_{Y,abc} \cdot h_{L_{X/Y,D}}^{n-1},
\end{equation}

where $n = \dim Y$. Then $d\mu_{\text{can}}$ is considered to be a singular volume form on $X$.

### 1.6 Variation of canonical Kähler-Einstein currents

One of the important properties of canonical Kähler-Einstein currents constructed in Theorem [1.6] is the following pluri-subharmonic variation property.

**Theorem 1.8** Let $f : X \to Y$ be an algebraic fiber space and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose that there exists a nonempty Zariski open subset $Y_0$ of $Y$ such that

1. $f$ is smooth over $Y_0$.
(2) \((X_y, D_y)(X_y := f^{-1}(y), D_y := D \cap X_y)\) is a KLT pair of log general type.

Let \(\omega_{K,y}\) be the canonical Kähler-Einstein current on \((X_y, D_y)(y \in Y_0)\) constructed as in Theorem 1.6. Let \(n\) be the relative dimension of \(f : X \to Y\).

Then the singular hermitian metric \(h_K\) on \(K_{X/Y} + D\mid f^{-1}(Y_0)\) defined by

\[
h_K|_{X_y} := \left(\frac{1}{n!} (\omega_{K,y})^n_{abc} \right)^{-1} \cdot h_{\sigma_D}|_{X_y} \quad (y \in Y_0)
\]

extends to a singular hermitian metric \(h_K\) on \(K_{X/Y} + D\) and has semipositive curvature in the sense of current everywhere on \(X\).

This theorem is a natural generalization of the result in [T7], where we deal with the case that \(D = 0\). We note that Theorem 1.8 is closely related to the semipositivity of the direct image of a relative pluri-log-canonical systems of a family of KLT pairs [Ka3, p.175, Theorem 1.2]. In fact Theorem 1.8 is considered to be a generalization of it. More generally we have the following theorem.

**Theorem 1.9** Let \(f : X \to Y\) be an algebraic fiber space and let \(D\) be an effective \(\mathbb{Q}\)-divisor on \(X\). Suppose that there exists a nonempty Zariski open subset \(Y_0\) of \(Y\) such that

1. \(f\) is smooth over \(Y_0\).
2. \((X_y, D_y)(X_y := f^{-1}(y), D_y := D \cap X_y)\) is a KLT pair of nonnegative Kodaira dimension.

Let \(\mu_{\text{can},y}\) be the canonical measure on \((X_y, D_y)(y \in Y_0)\) as in Section 1.5. Then the singular hermitian metric \(h_K^2|_{X_y}\) on \(K_{X/Y} + D\mid f^{-1}(Y_0)\) defined by

\[
h_K^2|_{X_y} := \mu_{\text{can},y}^{-1} \cdot h_{\sigma_D}|_{X_y} \quad (y \in Y_0)
\]

extends to a singular hermitian metric on \(K_{X/Y} + D\) and has semipositive curvature in the sense of current everywhere on \(X\).

This is a natural generalization of [T9, Theorem 4.1].

To prove Theorem 1.8, we use the dynamical construction of canonical Kähler-Einstein currents as in [T9]. But there is a major difference between the dynamical construction here and the one in [T9].

The most naive way to construct canonical Kähler current by dynamical system of Bergman kernels is to replace the Hodge \(\mathbb{Q}\)-line bundle \((L, h_L)\) in [T9] by \((D, |\sigma_D|^{-2})\), where \(\sigma_D\) is a multivalued holomorphic section of the \(\mathbb{Q}\)-line bundle \(D\) with divisor \(D\). Let \(a\) denote the minimal positive integer such that \(aD\) is Cartier. Then in the above naive dynamical construction (cf. Section 1.3) we tensorize \((\mathcal{O}_X(ad), |\sigma_D|^{-2a})\) in every \(a\)-steps. But unfortunately this construction does not yield the desired canonical Kähler-Einstein current, since \(|\sigma_D|^{-2a}\) has too large singularities in general. Moreover the dynamical system itself is not well defined. Actually this construction works only when \(D\) is \(\mathbb{Q}\)-linear equivalent to a Cartier divisor essentially.

To overcome this difficulty we consider a Ricci iteration and a family of dynamical systems of Bergman kernels associated with the Ricci iteration instead of a single dynamical system.
1.7 Semipositivity of the relative log canonical bundles of a family of LC pairs

Let $X$ be a smooth projective variety and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is a LC pair. Then for every rational number $t \in [0, 1)$, we see that $(X, tD)$ is a KLT pair. Suppose that $(X, D)$ is of log general type. Then for a sufficiently small rational number $0 < \epsilon << 1$, $(X, (1 - \epsilon)D)$ is a KLT pair of log general type. Let $\omega_{\epsilon}$ be the canonical Kähler-Einstein current on $(X, (1 - \epsilon)D)$ for $0 < \epsilon << 1$ (cf. Theorem 1.6). Then the Kähler-Einstein volume form $dV_{\epsilon}$ of $\omega_{\epsilon}$ is monotone increasing as $\epsilon$ tends to 0 and converges to a singular volume form on $X$ (see Lemma 2.4 below). Using this simple fact, Theorem 1.8 can be generalized to the case of LC pairs.

Theorem 1.10 Let $f : X \rightarrow Y$ be an algebraic fiber space and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose that there exists a nonempty Zariski open subset $Y_0$ of $Y$ such that

(1) $f$ is smooth over $Y_0$.

(2) For every $y \in Y_0$, $(X_y, D_y)(X_s := f^{-1}(y), D_y := D \cap X_y)$ is a LC pair of log general type.

Let $\omega_{K,y}$ be the canonical Kähler-Einstein current on $(X_y, D_y)(y \in Y_0)$ constructed as in Theorem 2.5 below. Let $n$ be the relative dimension of $f : X \rightarrow Y$.

Then the singular hermitian metric $h_K$ on $K_{X/Y} + D|f^{-1}(Y_0)$ defined by

$$h_K^n|_{X_y} := \left( \frac{1}{n!} \langle \omega_{K,y} \rangle_{abc} \right)^{-1} \cdot h_{\sigma_D}|_{X_y} \quad (y \in Y_0)$$

extends to a singular hermitian metric $h_K$ on $K_{X/Y} + D$ and has semipositive curvature in the sense of current everywhere on $X$. 

For families of LC pairs not necessarily of log general type, we have the following corollary (see 

Corollary 1.11 Let $f : X \rightarrow Y$ be an algebraic fiber space and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose that there exists a nonempty Zariski open subset $Y_0$ of $Y$ such that

(1) $f$ is smooth over $Y_0$,

(2) For every $y \in Y_0$, $(X_y, D_y)(X_s := f^{-1}(y), D_y := D \cap X_y)$ is a LC pair such that $K_{X_y} + D_y$ is pseudo-effective.

Then $K_{X/Y} + D$ is pseudo-effective. 

Proof of Corollary 1.11 Let $A$ be an ample effective divisor on $X$. Then for every positive rational number $\epsilon$, by Theorem 1.10 we see that $K_{X/Y} + D + \epsilon A$ is pseudo-effective. Letting $\epsilon \downarrow 0$, we complete the proof of Corollary 1.11.

The following corollary is a typical special case of Theorem 1.10.
Corollary 1.12 Let $f : X \to Y$ be an algebraic fiber space and let $D$ be an effective divisor on $X$ with normal crossings. Let $n$ denote the relative dimension of $f : X \to Y$. Suppose that there exists a nonempty Zariski open subset $Y_0$ of $Y$ such that

1. $K_{X/Y} + D$ is relatively ample over $Y_0$,
2. $f$ is smooth over $Y_0$,
3. For every $y \in Y_0$, $D_y = X_y \cap D$ (the scheme theoretic intersection) is a divisor with normal crossings in $X_y := f^{-1}(y)$.

Let $\omega_{E,y}$ be the complete Kähler-Einstein form on $X_y \setminus D_y$ such that $-\text{Ric}_{\omega_{E,y}} = \omega_{E,y}$ constructed as in [Ko]. We define the metric $h_K^0$ on $K_{X/Y} + D | f^{-1}(Y_0)$ by

$$h_K^0|_{X_y} := \left( \frac{1}{n!} \omega_{E,y}^n \right)^{-1} \cdot h_{\sigma_D}|_{X_y}.$$

Then $h_K^0$ extends to a singular hermitian metric $h_K$ on $K_{X/Y} + D$ with semipositive curvature in the sense of current.

Remark 1.13 In Corollary 1.12, by the implicit function theorem, we see that $h_K$ is $C^\infty$ on a Zariski open subset of $X$. □

For a family of KLT pairs of not necessarily of log general type, we have the following useful theorem.

Theorem 1.14 Let $f : X \to Y$ be an algebraic fiber space and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose that there exists a nonempty Zariski open subset $Y_0$ of $Y$ such that

1. $f$ is smooth over $Y_0$,
2. For every $y \in Y_0$, $(X_y, D_y)(X_s := f^{-1}(y), D_s := D \cap X_y)$ is a KLT pair of nonnegative Kodaira dimension.

Let $d\mu_{\text{can},X/Y}$ be the relative canonical measure defined by

$$d\mu_{\text{can},X/Y}|_{X_y} := d\mu_{\text{can},y} \quad (y \in Y_0)$$

where $d\mu_{\text{can},y}$ denotes the canonical measure on $(X_y, D_y)(y \in Y_0)$ constructed as in Theorem 1.7. Then the singular hermitian metric

$$h_K^0|_{X_y} := d\mu_{\text{can},y}^{-1} \cdot h_{\sigma_D}|_{X_y} \quad (y \in Y_0)$$

on $K_{X/Y} + D | f^{-1}(Y_0)$ extends to a singular hermitian metric $h_K$ on $K_{X/Y} + D$ and has semipositive curvature in the sense of current everywhere on $X$. □

There are numerous applications of the results in this article. They will be discussed in the subsequent papers ([T10, T11]).

2 Existence of canonical Kähler-Einstein currents

In this section we shall prove Theorem 1.6. The proof is not essentially new except the use of orbifold structures. This simple technique overcomes the difficulty arising from the poles of the right-hand side of the Monge-Ampère equation: (2.9) below, i.e., we can eliminate the poles of the Monge-Ampère equation in terms of local cyclic coverings.
2.1 Existence of canonical Kähler-Einstein currents on KLT pairs of log general type

In this subsection, we shall prove Theorem 1.6. Let $X$ be a smooth projective $n$-fold and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is KLT and $K_X + D$ is big. Then by [B-C-H-M] the canonical ring:

\[(2.1) \quad R(X, K_X + D) = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{O}_X(\lfloor m(K_X + D) \rfloor))\]

is finitely generated. Hence there exists a log resolution

\[(2.2) \quad \mu : Y \rightarrow X\]

of $(X, D)$ which satisfies the followings:

1. If we write $K_Y = \mu^*(K_X + D) + \sum a_i E_i$, then $a_i > -1$ holds for every $i$, where $\{E_i\}$ are prime divisors.

2. There exists a decomposition

\[(2.3) \quad \mu^*(K_X + D) = P + N (P, N \in \text{Div}(Y) \otimes \mathbb{Q})\]

into the $\mathbb{Q}$-divisors $P$ and $N$ such that $P$ is semiample, $N$ is effective and

\[(2.4) \quad H^0(Y, \mathcal{O}_Y(\mu^*(a m P))) \simeq H^0(Y, \mathcal{O}(a m \mu^*(K_X + D)))\]

holds for every $m \geq 0$, where $a$ is the minimal positive integer such that $aD, aP \in \text{Div}(X)$ hold.

We call the decomposition (2.3) a Zariski decomposition of $\mu^*(K_X + D)$.

We note that adding effective exceptional $\mathbb{Q}$-divisors does not change the log canonical ring. We set $I := \{i | a_i < 0\}$. Then replacing $X$ by $Y$ and $D$ by

\[(2.5) \quad D_Y := \sum_{i \in I} \left( -a_i \right) E_i,\]

we obtain a new KLT pair $(Y, D_Y)$ such that

(a) $R(Y, K_Y + D_Y) \simeq R(X, K_X + D)$,

(b) There exists a Zariski decomposition: $K_Y + D_Y = P + (N + \sum_{i \notin I} a_i E_i)$.

Hence we may assume that Supp $D$ and Supp $N$ are divisors with normal crossings and the decomposition $K_X + D = P + N$ holds on $X$ from the beginning. In fact if we construct a canonical Kähler-Einstein current $\omega_K$ on $Y$, then the push forward $f_* \omega_K$ is a canonical Kähler-Einstein current on $(X, D)$.

Let $\sigma_D$ be a multivalued holomorphic section of the $\mathbb{Q}$-line bundle $D$ with divisor $D$. Then $|\sigma_D|^{-2}$ is an singular hermitian metric on $L$. Similarly we consider the singular hermitian metric $|\sigma_N|^{-2}$ on the $\mathbb{Q}$-line bundle $N$. Let $h_D, h_N$ be $C^\infty$ hermitian metric on $D$ and $N$. Then since $P$ is semiample, there exists a $C^\infty$ volume form $\Omega$ on $X$ such that

\[(2.6) \quad h_P := \Omega^{-1} \cdot h_D \cdot h_N^{-1}\]

is a $C^\infty$ hermitian metric such that the curvature $\sqrt{-1} \Theta_{h_P}$ is the positive constant times the pull back of the Fubini-Study form by the base point free linear
system $|m_0|P$ for $m_0 \gg 1$. Let $D = \sum d_iD_i$ be the irreducible decomposition of $D$ and let $\sigma_i$ be a nontrivial global section of $\mathcal{O}_X(D_i)$ with divisor $D_i$ and let $\|\sigma_i\|$ be the hermitian norm of $\sigma_i$ with respect to a $C^\infty$ hermitian metric on $\mathcal{O}_X(D_i)$. By multiplying a small positive constant to $\sigma_i$, we may and do assume that $\|\sigma_i\| < 1$ holds for every $i$ on $X$. Let $b$ be a positive integer such that

$$d_i < \frac{b - 1}{b} \tag{2.7}$$

holds for every $i$. We set

$$\omega_P := \sqrt{-1} \Theta_{h_P}. \tag{2.8}$$

We consider the Monge-Ampère equation

$$\left(\omega_P + \sqrt{-1} \partial \bar{\partial} u\right)^n = \frac{\|\sigma_N\|^2_{h_N}}{\|\sigma_D\|^2_{h_D}} \cdot \Omega \cdot e^u \tag{2.9}$$

on $X$. To solve (2.9), we consider the perturbation of the equation (2.9) and construct the solution as the limit of the solutions of the perturbed equations. We may assume that there exists an effective exceptional $\mathbb{Q}$-divisor $E$ with respect to the morphism $\Phi_{|m|P} : X \to \mathbb{P}^\nu$ for a sufficiently large $m$ such that

1. $P - \delta E$ is ample for every $\delta \in (0, 1]$,

2. If we define

$$U := \{x \in X | \langle m | P \rangle \text{ defines an embedding for every } m \gg 1 \text{ on a neighbourhood of } x\}, \tag{2.10}$$

then

$$X \setminus \text{Supp } E = U \tag{2.11}$$

holds.

The existence of such $E$ follows from the definition (2.10) of $U$ and the trivial fact that for any successive blowing up

$$\varpi : \tilde{\mathbb{P}}^\nu \to \mathbb{P}^\nu$$

of $\mathbb{P}^\nu$ with smooth centres, there exists an effective $\mathbb{Q}$-divisor $B$ supported on the exceptional divisors of $\varpi$ such that $\varpi^* \mathcal{O}(1) - B$ is ample. Hence by taking a suitable successive blowing up with smooth centres over $X \setminus U$, we may assume the existence of such an effective $\mathbb{Q}$-divisor $E$. Then there exists a $C^\infty$ hermitian metric $h_E$ on $E$ such that $h_P \cdot h_E^{-1}$ is a metric with strictly positive curvature on $X$. For every $0 < \delta \leq 1$, we define the orbifold Kähler form $\omega_{P, \delta}$ on $X$ by

$$\omega_{P, \delta} := (1 - \delta)\omega_P + \delta \left(\omega_P - \sqrt{-1} \Theta_{h_E} + \varepsilon \sum_i \sqrt{-1} \partial \bar{\partial} \log (1 - \|\sigma_i\|^2)\right), \tag{2.12}$$

where $\varepsilon$ is a fixed sufficiently small positive number so that $\omega_{P, \delta}$ is an orbifold Kähler form on $X$ branching along $D$ with order $b$ for every $\delta \in (0, 1]$.

\footnote{Since $(X, D)$ is KLT, $d_i < 1$ holds for every $i$.}
More precisely, $\omega_{P,\delta}(\delta \in (0,1])$ is an orbifold Kähler form in the following sense. Let $(V, (z_1, \cdots, z_n))$ be a local coordinate on $X$ such that $V$ is biholomorphic to the unit open polydisk $\Delta^n$ in $\mathbb{C}^n$ with centre $O$ and

$$V \cap D = \{p \in V | z_1(p) \cdots z_k(p) = 0\}$$

hold for some $k$. Let

$$\pi_V : \Delta^n \rightarrow V$$

be the morphism defined by

$$\pi_V(t_1, \cdots, t_n) = (t^b_1, \cdots, t^b_k, t_{k+1}, \cdots, t_n).$$

If we take a positive number $\varepsilon$ sufficiently small, then $\pi^*_V(\omega_{P,\delta}|V)(\delta \in (0,1])$ is a $C^\infty$ Kähler form on $\Delta^n$. In this sense $\omega_{P,\delta}(\delta \in (0,1])$ is an orbifold Kähler form on $(X, D)$.

Now we consider the perturbed equation:

$$\omega_{P,\delta} + \sqrt{-1}\partial\bar{\partial}u_\delta^n = \pi^*_V \left( \frac{\| \sigma_N \|_{h_N}^2 \left( (\prod_i (1 - \| \sigma_i \|_{h_E}^{-\varepsilon}) \| \sigma_E \|_{h_E}^2 \right)^\delta}{\| \sigma_D \|_{h_D}^2} \cdot \Omega \cdot e^{u_\delta} \right)$$

for $\delta \in (0,1]$. Then for $\pi_V : \Delta^n \rightarrow V$ as above, pulling back (2.15) by $\pi_V$, we obtain

$$\pi^*_V(\omega_{P,\delta} + \sqrt{-1}\partial\bar{\partial}u_\delta^n) = \pi^*_V \left( \frac{\| \sigma_N \|_{h_N}^2 \left( (\prod_i (1 - \| \sigma_i \|_{h_E}^{-\varepsilon}) \| \sigma_E \|_{h_E}^2 \right)^\delta}{\| \sigma_D \|_{h_D}^2} \cdot \Omega \right) e^{\pi^*_V u_\delta}.$$ 

In (2.16),

$$\pi^*_V \left( \frac{\| \sigma_N \|_{h_N}^2 \left( (\prod_i (1 - \| \sigma_i \|_{h_E}^{-\varepsilon}) \| \sigma_E \|_{h_E}^2 \right)^\delta}{\| \sigma_D \|_{h_D}^2} \cdot \Omega \right)$$

degenerates along $\{(t_1, \cdots, t_n) \in \Delta^n | t_1 \cdots t_k = 0\}$ by the choice of $b$ (cf. (2.7)). Hence the equation (2.15) is considered to be a complex Monge-Ampère equation with degeneracy along $\text{Supp} \ D$ as an equation on the orbifold branching along $\text{Supp} \ D$ with order $b$. Hence by [1, p.387, Theorem 6], there exists a solution $u_\delta$ of (2.15) on $X$ such that

1. $u_\delta$ is $C^\infty$ on the complement of $\text{Supp} \ D \cup \text{Supp} \ N$,
2. $\sup |u_\delta| < +\infty$,
3. $|\Delta_{\omega_{P,\delta}} u_\delta|$ is bounded, where $\Delta_{\omega_{P,\delta}}$ denotes the Laplacian with respect to the orbifold Kähler form $\omega_{P,\delta}$,
4. The solution $u_\delta$ satisfying the above properties is unique.

\footnote{We have assumed that $\text{Supp} \ D$ is a divisor with normal crossings on $X$}
We set

\[ (2.17) \quad \tilde{\omega}_{P,\delta} := \omega_{P,\delta} + \sqrt{-1} \partial \bar{\partial} u_{\delta}. \]

The following lemma asserts that \( \{\omega_{P,\delta}^n\} \) is monotone decreasing with respect to \( \delta \).

**Lemma 2.1** For \( 0 < \delta < \delta' < 1 \), we have that

\[ \tilde{\omega}_{P,\delta}^n \geq \tilde{\omega}_{P,\delta'}^n \]

holds on \( U = X \setminus \text{Supp } E \).

**Proof.** Let \( \delta < \delta' \) be positive numbers as above. By (2.15),

\[ (2.18) \quad \frac{(\omega_{P,\delta} + \sqrt{-1} \partial \bar{\partial} u_{\delta})^n}{(\omega_{P,\delta'} + \sqrt{-1} \partial \bar{\partial} u_{\delta'})^n} = \left( \prod_i (1 - ||\sigma_i||^2b) \right)^{\delta - \delta'} \cdot e^{u_{\delta} - u_{\delta'}.} \]

holds on \( X \). We set

\[ (2.19) \quad w_{\delta,\delta'} := u_{\delta} - u_{\delta'} + (\delta - \delta') \log \left( \left( \prod_i (1 - ||\sigma_i||^2b) \right)^{-\epsilon ||\sigma_E||^2_{h_E}} \right). \]

Then by (2.18) we see that

\[ (2.20) \quad \int_0^1 \Delta_t w_{\delta,\delta'} \, dt = w_{\delta,\delta'} \]

holds, where \( \Delta_t \) denotes trace\((1-t)\tilde{\omega}_{\delta} + t \omega_{\delta'}\sqrt{-1} \partial \bar{\partial} \). Since

\[ (\delta - \delta') \log \left( \left( \prod_i (1 - ||\sigma_i||^2b) \right)^{-\epsilon ||\sigma_E||^2_{h_E}} \right) \to +\infty \quad \text{as } x \to \text{Supp } E, \]

by the boundedness of \( u_{\delta}, u_{\delta'} \) and the definition of \( w_{\delta,\delta'} \) (cf. (2.19)), there exists a point \( p_0 \) where \( w_{\delta,\delta'} \) takes its minimum. Then by (2.20), we see that

\[ w_{\delta,\delta'}(p_0) \geq 0 \]

holds. Hence \( w_{\delta,\delta'} \geq 0 \) holds on \( U \). Hence by (2.18) and (2.19), we see that

\[ \tilde{\omega}_{\delta}^n \geq \tilde{\omega}_{\delta'}^n \]

holds on \( U \). This completes the proof of Lemma 2.1.

To obtain a uniform estimate of \( u_{\delta} \) with respect to \( \delta \), we set

\[ (2.21) \quad v_{\delta} := u_{\delta} - (1 - \delta) \cdot \log \left( \prod_i (1 - ||\sigma_i||^2b) \right)^{-\epsilon ||\sigma_E||^2_{h_E}} \]

and estimate \( v_{\delta} \). By (2.15) \( v_{\delta} \) satisfies the equation:

\[ (2.22) \quad (\omega_{P,1} + \sqrt{-1} \partial \bar{\partial} v_{\delta})^n = \frac{||\sigma_N||^2_{h_N} \left( \prod_i (1 - ||\sigma_i||^2b) \right)^{-\epsilon ||\sigma_E||^2_{h_E}}}{||\sigma_D||^2_{h_D}} \cdot e^{v_{\delta}}. \]
Since $u_\delta$ is bounded on $X$, we see that
\[
\log \frac{(\omega_{P,1} + \sqrt{-1} \partial \bar{\partial} u_\delta)^n}{\|\sigma_N\|_{h_N}^2 \left( \prod_i (1 - \|\sigma_i\|_F^2)^{-\varepsilon} \|\sigma_E\|_{h_E}^2 \right)} = v_\delta
\]
is bounded from below and blows up along $E$. Hence we see that there exists a point $q_0$ on $X \setminus \text{Supp } E$ where $v_\delta$ takes minimum. Then by the maximum (minimum) principle, we see that
\[
(2.23) \quad \tilde{\omega}_\delta(q_0) \geq \omega_{P,1}(q_0)
\]
holds. Hence if we set
\[
(2.24) \quad C_- := \inf \frac{\omega_{P,1}^n}{\|\sigma_N\|_{h_N}^2 \left( \prod_i (1 - \|\sigma_i\|_F^2)^{-\varepsilon} \|\sigma_E\|_{h_E}^2 \right)} > 0,
\]
then
\[
(2.25) \quad v_\delta > C_-
\]
holds. We note that $C_-$ is independent of $\delta \in (0, 1]$. Let us fix $0 < \delta_0 < 1$. We set
\[
(2.26) \quad v_{\delta, \delta_0} := u_\delta - (\delta_0 - \delta) \cdot \log \left( \prod_i (1 - \|\sigma_i\|_F^2) \|\sigma_E\|_{h_E}^2 \right).
\]
Then by the same argument, we see that there exists a constant $C_-(\delta_0)$ such that for every $\delta \in (0, \delta_0)$
\[
(2.27) \quad v_{\delta, \delta_0} \geq C_-(\delta_0)
\]
holds on $X$. Hence by (2.21) for every $\delta \in (0, \delta_0)$
\[
(2.28) \quad v_\delta \geq C_-(\delta_0) - (1 - \delta_0) \cdot \log \left( \prod_i (1 - \|\sigma_i\|_F^2) \|\sigma_E\|_{h_E}^2 \right).
\]
holds on $X$.

Next we shall estimate $v_\delta$ from above. We note that by (2.16)
\[
\int_X \frac{\|\sigma_N\|_{h_N}^2 \left( \prod_i (1 - \|\sigma_i\|_F^2)^{-\varepsilon} \|\sigma_E\|_{h_E}^2 \right)^{\delta}}{\|\sigma_D\|_{h_D}^2} \cdot \Omega \cdot e^{u_\delta} = \int_X (\omega_{P,\delta} + \sqrt{-1} \partial \bar{\partial} u_\delta)^n = (2\pi)^n (P - \delta E)^n
\]
hold. Then by the concavity of logarithm,
\[
(2.28) \quad \int_X \left( u_\delta + \log \frac{\|\sigma_N\|_{h_N}^2 \left( \prod_i (1 - \|\sigma_i\|_F^2)^{-\varepsilon} \|\sigma_E\|_{h_E}^2 \right)^{\delta}}{\|\sigma_D\|_{h_D}^2} \right) \Omega \leq (\log(2\pi)^n (P - \delta E)^n) \int_X \Omega
\]
holds. Hence there exists a positive constant $C$ independent of $\delta \in (0, 1]$ such that

\[ \int_X u_\delta \Omega < C \]

holds for every $\delta \in (0, 1]$. Then since $\omega_{P, \delta} + \sqrt{-1} \partial \overline{\partial} u_\delta$ is a closed positive current on $X$ and $\omega_{P, \delta}$ is an orbifold Kähler form on $X$, we see that $u_\delta$ is almost plurisubharmonic function on $X$ in the sense of orbifold. Hence by the submeanvalue inequality for subharmonic functions, by (2.29) there exists a positive constant $C_+$ independent of $\delta \in (0, \delta]$ such that

\[ u_\delta \leq C_+ \]

holds on $X$. Hence by (2.21), for every $\delta \in (0, 1]$

\[ v_\delta \leq C_+ - (1 - \delta) \cdot \log \left( \prod_i (1 - \| \sigma_i \|_b^2) \right)^{\frac{\varepsilon}{2}} \| \sigma E \|_{h_E}^2 \]

holds on $X$.

Now we shall estimate the $C^2$-norm of $v_\delta$. First we note that $\omega_{P, 1}$ is a $C^\infty$ orbifold Kähler form on $X$. In particular, the bisectional curvature $R_{\alpha \bar{\alpha} \beta \bar{\beta}}$ of $\omega_{P, 1}$ is bounded on $X$. To get the $C^2$-estimate, we shall estimate $e^{-Cv_\delta (n + \Delta_{P, 1} v_\delta)}$, where $C$ is a positive constant which will be specified later and and $\Delta_{P, 1}$ denotes the Laplacian with respect to the orbifold Kähler form $\omega_{P, 1}$.

**Lemma 2.2** ([T3, p. 127, Lemma 2.2]) We set

\[ f := \log \frac{\omega_{P, 1}^n}{\Omega}. \]

Let $C$ be a positive number such that

\[ C + \inf_{\alpha \neq \beta} R_{\alpha \bar{\alpha} \beta \bar{\beta}} > 1 \]

holds on $Y$, where $R_{\alpha \bar{\alpha} \beta \bar{\beta}}$ denotes the bisectional curvature of $\omega_{P, 1}$.

Then

\[ e^{Cv_\delta} \Delta_\delta (e^{-Cv_\delta (n + \Delta_{P, 1} v_\delta)}) \geq (n + \Delta_{P, 1} v_\delta) \]

\[ + \Delta_{P, 1} \left( f + \log \frac{\| \sigma_N \|_{h_N}^2 \left( \prod_i (1 - \| \sigma_i \|_b^2) \right)^{-\varepsilon} \| \sigma E \|_{h_E}^2}{\| \sigma_D \|_{h_D}^2} \right) \]

\[ (n + \Delta_{P, 1} v_\delta) \exp \left( \frac{-1}{n - 1} (v_\delta + f) \right) \]

holds, where $\Delta_{P, 1}$ denotes the Laplacian with respect to $\omega_{P, 1}$ (i.e., $\Delta_{P, 1} = \text{trace}_{\omega_{P, 1}} \sqrt{-1} \partial \overline{\partial}$) and $\Delta_\delta$ denotes the Laplacian with respect to $\tilde{\omega}_\delta$. \hfill \Box
Since $u_\delta$ is bounded, by the definition of $v_\delta$ (cf. (2.21)) and the boundedness of $\Delta_{1,\delta}u_\delta$, we see that there exists a point $x_0 \in X \setminus \text{Supp } E$ such that $e^{-C v_\delta(n + \Delta P_1 v_\delta)}$ takes its maximum at $x_0$. Then by Lemma 2.2 and the lower estimate (2.27) (taking $\delta_0 = 2/3$ for example), if we take $C$ sufficiently large, we see that there exists a positive constant $C_2$ such that for every $\delta \in (0, 1/2)$

$$e^{-C v_\delta(x_0)}(n + \Delta_{1,\delta}v_\delta)(x_0) \leq C_2 \quad (\delta \in (0, 1/2))$$

holds. This implies that for every $\delta \in (0, 1/2)$

$$n + \Delta_{P,1} v_\delta \leq C_2 e^{C v_\delta} \leq C_2 \cdot e^{CC_+ \left( \prod_i (1 - \| \sigma_i \|_{b_i})^{-\tau} \| \sigma_E \|_{h_E}^2 \right)^{-C(1-\delta)}}$$

holds on $X$. Hence we see that $\{v_\delta \mid \delta \in (0, 1/2)\}$ is uniformly $C^2$-bounded on every compact subset of $X \setminus \text{Supp } E$. Then by the general theory of fully nonlinear equations of 2nd order ([T r]), we see that $\{v_\delta \mid \delta \in (0, 1/2)\}$ is uniformly $C^{2,\alpha}$ bounded for some $\alpha \in (0, 1)$ on every compact subset of $X \setminus \text{Supp } E$. Then by the standard theory for linear elliptic partial differential equation of 2nd order, we see that for every $k \geq 0$, $\{v_\delta \mid \delta \in (0, 1/2)\}$ is uniformly $C^k$-bounded on every compact subset of $X \setminus \text{Supp } E$. Hence there exists a sequence $\{\delta_k\}, \delta_k \downarrow 0$ such that

$$v := \lim_{k \to \infty} v_{\delta_k}$$

exists in $C^\infty$-topology on every compact subset of $X \setminus \text{Supp } E$. Then by (2.21) and (2.15), we see that

$$u := v + \log \left( \prod_i (1 - \| \sigma_i \|_{b_i})^{-\tau} \| \sigma_E \|_{h_E}^2 \right)$$

is $C^\infty$ on $X \setminus \text{Supp } E$ and satisfies the Monge-Ampère equation (2.9) and $\omega_K := \omega_P + \sqrt{-1}\partial\bar\partial u$ is a well defined current by the estimates (2.30) and (2.25). Moreover by Lemma 2.1, $u$ is independent of the choice of the subsequence $\{\delta_k\}$. We set

$$\omega_K := \omega_P + \sqrt{-1}\partial\bar\partial u.$$

To prove that $\omega_K$ is the canonical Kähler-Einstein current on $(X, D)$, we need to show that

$$h_K := \left( \frac{1}{n!} \omega_K^n \right)^{-1} \cdot h_{\sigma_D}$$

is an AZD of $K_X + D$. By (2.21), (2.24) and (2.30), we obtain the following **almost boundedness** of $u_\delta$ and $u$.

**Lemma 2.3** For every $\delta_0 \in (0, 1]$, there exists a constant $C_-(\delta_0)$ such that for every $\delta \in (0, \delta_0)$

$$C_-(\delta_0) + (\delta_0 - \delta) \log \left( \prod_i (1 - \| \sigma_i \|_{b_i})^{-\tau} \| \sigma_E \|_{h_E}^2 \right) \leq u_\delta \leq C_+$$

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holds. And in particular
\[
C_-(\delta_0) + \delta_0 \log \left( \prod_i \left( 1 - \| \sigma_i \|_2 \right)^{-\epsilon} \| \sigma_E \|_{h_E}^2 \right) \leq u \leq C_+
\]
holds. \(\square\)

Then by (2.9) and Lemma 2.3, letting \(\delta_0\) tend to 0, we see that \(h_K\) is an AZD of \(K_X + D\).

The uniqueness is the direct consequence of the dynamical construction (cf. Theorem 4.9 below). This completes the proof of Theorem 1.6. \(\square\)

2.2 Monotonicity Lemma and Kähler-Einstein currents on LC pairs

Using Theorem 1.6, we shall construct a canonical Kähler-Einstein current on a LC pair of log general type.

Let \(X\) be a smooth projective variety and let \(D\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, D)\) is a LC pair. Then for every rational number \(t \in [0, 1]\), we see that \((X, tD)\) is a KLT pair. Suppose that \((X, D)\) is of log general type. Then for a sufficiently small rational number \(0 < \epsilon << 1\), \((X, (1 - \epsilon)D)\) is a KLT pair of log general type. Let \(\epsilon_0\) be a positive number such that for every \(t \in (1 - \epsilon_0, 1)\), \((X, tD)\) is of log general type. For \(t \in (1 - \epsilon_0, 1)\), let \(\omega_t\) be the canonical Kähler-Einstein current on \((X, tD)\) as in Theorem 1.6. We set
\[
d\mu_{\text{can}, t} := \frac{1}{n!} \omega_{t, abc}^n
\]
and call it the canonical Kähler-Einstein volume form on \((X, tD)\). \(d\mu_{\text{can}, t}\) is nothing but the canonical measure on \((X, tD)\). The following monotonicity lemma is essential for our purpose.

**Lemma 2.4** (Monotonicity Lemma) \(d\mu_{\text{can}, t}\) is (weakly) monotone increasing with respect to \(t \in (1 - \epsilon_0, 1) \cap \mathbb{Q}\). \(\square\)

**Proof of Lemma 2.4** Let \(t < t'\) be positive numbers in \((1 - \epsilon_0, 1)\). Taking a suitable modification of \(X\), we may assume that \(\text{Supp} D\) is a divisor with normal crossings and there exist Zariski decompositions:
\[
K_X + tD = P_t + N_t
\]
and
\[
K_X + t'D = P_{t'} + N_{t'}.
\]
We note that adding exceptional divisor does not affect the log canonical ring. It is clear that
\[
N_{t'} \leq N_t
\]
holds. Let \(h_{P_t}\) be a \(C^\infty\) hermitian metric on \(P_t\) induced by the Fubini-Study metric by the morphism \(\Phi_{|\nu!P_t|}\) associated with the base point free linear system \(|\nu!P_t|\) from \(X\) into a projective space for some sufficiently large \(\nu\). We set
\[
\omega_{P_t} := \sqrt{-1} \Theta_{h_{P_t}}.
\]
Let $h_{tD}$ be a $C^\infty$ hermitian metric on $tD$ and let $h_{N_t}$ be a $C^\infty$ hermitian metric on $N_t$. Let $\sigma_{tD}$ be a multivalued global holomorphic section on $tD$ with divisor $tD$ and let $\sigma_{N_t}$ be a multivalued global holomorphic section of $N_t$ with divisor $N_t$. Let $\Omega$ be a $C^\infty$ volume form on $X$ such that

$$h_P = \Omega^{-1} \cdot h_{tD} \cdot h_{N_t}^{-1}$$

holds. We consider

$$(\omega_{P_t} + \sqrt{-1} \partial \bar{\partial} u_t)^n = \frac{\| \sigma_{N_t} \|^2_{h_{N_t}}}{\| \sigma_{tD} \|^2_{h_{tD}}} \cdot \Omega \cdot e^{u_t}$$

on $X$ as (2.3) in Section 2.1 such that

$$\omega_t = \omega_{P_t} + \sqrt{-1} \partial \bar{\partial} u_t + 2\pi t[D]$$

is the canonical Kähler-Einstein current on $X$. As in Section 2.1, let $E$ be an effective $\mathbb{Q}$-divisor on $X$ such that $P_t - E$ is ample and $X \setminus \text{Supp } E$ is contained in

$$(2.45) \quad W := \{ x \in X | d\mu_{can,t}(x), d\mu_{can,t'}(x) > 0 \} \setminus \text{Supp } D$$

By the proof of Theorem 1.6, $W$ is a nonempty Zariski open subset of $X$ and $\omega_t, \omega_t'$ are $C^\infty$ on $W$. Let $D = \sum d_i D_i$ be the irreducible decomposition of $D$ and let $\sigma_i$ be a nontrivial global section of $O_X(D_i)$ with divisor $D_i$ and let $\| \sigma_i \|$ be the hermitian norm of $\sigma_i$ with respect to a $C^\infty$ hermitian metric on $O_X(D_i)$. We may and do assume that $\| \sigma_i \| < 1$ holds for every $i$ on $X$. Let $b$ be a positive integer such that

$$d_i < \frac{b - 1}{b}$$

holds for every $i$ as (2.7). Let $\sigma_E$ be a global multivalued holomorphic section of $E$ with divisor $E$ and let $h_E$ be a $C^\infty$ hermitian metric on $E$ such that $h_P \cdot h_E^{-1}$ has strictly positive curvature on $X$. For every $0 < \delta \leq 1$, we define the orbifold Kähler form $\omega_{P,\delta}$ on $X$ by

$$(2.46) \quad \omega_{P,\delta} := (1 - \delta) \omega_{P_t} + \delta \left( \omega_{P_t} - \sqrt{-1} \Theta_{h_E} + \varepsilon \sum \sqrt{-1} \partial \bar{\partial} \log(1 - \| \sigma_i \|^2_{h_E}) \right),$$

where $\varepsilon$ is a fixed sufficiently small positive number so that $\omega_{P,\delta}$ is an orbifold Kähler form on $X$ branching along $D$ with order $m$ for every $\delta \in (0, 1]$.

For $\delta > 0$ we consider the perturbed equation:

$$(2.47) \quad (\omega_{P,\delta} + \sqrt{-1} \partial \bar{\partial} u_{t,\delta})^n = \frac{\| \sigma_{N_t} \|^2_{h_{N_t}} \left( \prod (1 - \| \sigma_i \|^2_{h_E}) \right)^{-\varepsilon} \| \sigma_{tD} \|^2_{h_{tD}}}{\| \sigma_{tD} \|^2_{h_{tD}}} \cdot \Omega \cdot e^{u_{t,\delta}}$$

We see that (2.47) has a unique bounded solution $u_{t,\delta}$ whose $C^2$-norm with respect to $\omega_{P,\delta}$ is bounded by [11, p.387, Theorem 6]. By the uniqueness of the solution of (2.43) (cf. Theorem 4.9) and the uniform weighted $C^2$-estimate of
\{u_{t,\delta}|t \in (0,1)\} parallel to the one of the solution of (2.15) in Section 2.1, we see that

\begin{equation}
\tag{2.48}
u_t := \lim_{\delta \downarrow 0} u_{t,\delta}
\end{equation}

exists in $C^\infty$-topology on $X \setminus \text{Supp } E$ and is the unique solution of (2.43) under the condition that $u_t$ is almost bounded on $X$ (cf. Lemma 2.3) or equivalently $\omega_{P_t} + \sqrt{-1}\partial\bar{\partial}u_t$ is the canonical Kähler-Einstein current on $(X, tD)$. We set

\begin{equation}
\tag{2.49}
\omega_{t,\delta} := \omega_{P_t,\delta} + \sqrt{-1}\partial\bar{\partial}u_{t,\delta}.
\end{equation}

Then by the equation (2.47), we see that

\begin{equation}
\tag{2.50}
- \text{Ric}_{\omega_{t,\delta}} = \omega_{t,\delta}
\end{equation}

holds on $X \setminus \text{Supp } E$.

Now we shall compare $\omega^n_{t,\delta}(t \in (0,1))$ and $\omega^n_t(t < t')$. We note that

\begin{equation}
\tag{2.51}
F_\delta(x) := \frac{\omega^n_{t',\delta}(x)}{\omega^n_{t,\delta}(x)}
\end{equation}

tends to $+\infty$ as $x$ tends to Supp $E$ by (2.41), (2.47) and the equation (2.43) replacing $t$ by $t'$. Hence there exists a point $x_0 \in X \setminus \text{Supp } E$ where $F(x)$ takes its minimum. Then by the maximum (minimum) principle and the Kähler-Einstein condition, we see that

\begin{equation}
\omega^n_{t'}(x_0) \geq \omega^n_{t,\delta}(x_0)
\end{equation}

holds. Hence we see that $F_\delta(x) \leq 1$ holds on $X$ and

\begin{equation}
\tag{2.52}
\omega^n_{t,\delta} \leq \omega^n_t
\end{equation}

holds on $X$. Hence letting $\delta$ tend to 0, by the convergence (2.48), we see that

\begin{equation}
\tag{2.53}
\omega^n_t \leq \omega^n_{t'}
\end{equation}

holds on $X$. This completes the proof of Lemma 2.4.

Now we shall construct an AZD of $K_X + D$ by using Lemma 2.4.

**Theorem 2.5** In the above notations

\begin{equation}
\tag{2.54}d\mu_{\text{can}} := \lim_{t \uparrow 1} d\mu_{\text{can},t}
\end{equation}

exists. And $d\mu_{\text{can}}^{-1} \cdot h_{\sigma_D}$ is an AZD of $K_X + D$. □

**Proof.** First we shall prove the convergence. By Lemma 2.4, it is enough to prove that \{d\mu_{\text{can},t}|t \in (1 - \epsilon_0,1)\} is locally uniformly bounded on some nonempty Zariski open subset of $X$. Taking modification of $X$, we may assume that Supp $E$ is a divisor with normal crossings. Let $H$ be an ample divisor such that Supp $E + H$ is a divisor with normal crossings on $X$. We set

\begin{equation}
\tag{2.55}
U_0 := X \setminus (\text{Supp } E \cup H)
\end{equation}

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Then by \([Ko]\), there exists a complete Kähler-Einstein form \(\omega_H\) on \(U_0\) such that

\[-\text{Ric}_H = \omega_H\]

and it extends to a closed positive current on \(X\). Then by Yau’s Schwarz lemma ([Y2]), we have that

\[(2.56) \quad d\mu_{\text{can}, t} \leq \frac{1}{n!} \omega_H^n\]

holds on \(U_0\). Then by the monotonicity of \(\{d\mu_{\text{can}, t}\}\) (Lemma 2.4), this completes the proof of the convergence of \(d\mu_{\text{can}, t}\) as \(t \uparrow 1\). Henceforth we shall identify \(d\mu_{\text{can}, 1}^{-1}\) with the singular hermitian metric \(d\mu_{\text{can}, 1}^{-1} \cdot h_{\sigma_0}^t\) on \(K_X + tD\).

Next we shall prove that \(d\mu_{\text{can}}^{-1}\) is an AZD of \(K_X + D\). Let us fix an arbitrary \(t_0 \in (1 - \varepsilon_0, 1) \cap \mathbb{Q}\). Then \(K_X + t_0D\) is big. Let \(t_0\) be a sufficiently large positive integer such that \(B := t_0(K_X + t_0D)\) is an integral divisor and \(|B| \neq \emptyset\). Let \(m\) be an arbitrary positive integer such that \(m(K_X + D)\) is Cartier. We note that \(d\mu_{\text{can}, t}\) is an AZD of \(K_X + tD\) for every \(t \in (1 - \varepsilon_0, 1) \cap \mathbb{Q}\). Then for any positive integer \(\ell\), if we set

\[(2.57) \quad s := \frac{t_0 \ell_0 + \ell m}{t_0 + \ell m},\]

\[(2.58) \quad H^0(X, \mathcal{O}_X(m\ell(K_X + D) + B)) \subseteq H^0(X, \mathcal{O}_X(m\ell + t_0)(K_X + D)) \otimes \mathcal{I}(d\mu_{\text{can}, s}^{-2(m + t_0)}))\]

holds by the definition of \(s\). We note that \(d\mu_{\text{can}, s} \leq d\mu_{\text{can}}\) holds on \(X\) by Lemma 2.4. Hence we see that for every nonzero global section \(\tau \in H^0(X, \mathcal{O}_X(B))\) and arbitrary section \(\sigma \in H^0(X, \mathcal{O}_X(m(K_X + D))))\),

\[(2.59) \quad \sigma^t \otimes \tau \in H^0(X, \mathcal{O}_X(m\ell + t_0)(K_X + D)) \otimes \mathcal{I}(d\mu_{\text{can}}^{-2(m + t_0)}))\]

holds. Let us fix a \(C^\infty\) volume form \(dV\) on \(X\). Let us take a positive integer \(\ell\) sufficiently large so that

\[(2.60) \quad \int_X (|\tau|^2 \cdot d\mu_{\text{can}}^{-t_0})^{-\frac{1}{2}} dV < \infty\]

holds. Then by Hölder’s inequality, we see that

\[(2.61) \quad \int_X |\sigma|^2 \cdot (d\mu_{\text{can}})^{-m} dV \leq \left(\int_X |\sigma|^{2} \cdot (d\mu_{\text{can}})^{-(t_0 + m)} dV\right)^{\frac{1}{2}} \cdot \left(\int_X (|\tau|^2 \cdot d\mu_{\text{can}})^{-\frac{1}{2}} dV\right)^{\frac{1}{2}} < +\infty\]

hold. Since \(\sigma\) is arbitrary, this means that \(d\mu_{\text{can}}^{-1} \cdot h_{\sigma_D}\) is an AZD of \(K_X + D\). This completes the proof of Lemma 2.4.

**Remark 2.6** There are many other way to approximate the LC pair \((X, D)\) by a sequence KLT pairs \(\{(X, D_k)|D_k \leq D\}\) such that \(D_k \leq D_{k+1}\) and \(D_k \uparrow D\) as \(k\) tends to infinity. We may easily generalize Lemma 2.4 including all such approximations. Hence we may generalize Theorem 2.5 including such approximations.

**Definition 2.7** Let \((X, D)\) be a LC pair of log general type. Then \(d\mu_{\text{can}}\) constructed as above is said to be the canonical measure on \((X, D)\).
I believe that $d\mu_{\text{can}}$ constructed is $C^\infty$ on a nonempty Zariski open subset of $X$. But at present it is not clear. In the following examples, the canonical measures are generically $C^\infty$.

**Example 2.8** Let $X$ be a smooth projective variety and let $D$ be a divisor with normal crossings on $X$ such that $K_X + D$ is ample. Then $(X, D)$ is a LC pair. Then by [Ka], there exists a complete Kähler-Einstein form $\omega_E$ on $X\setminus D$ such that $-\text{Ric}_{\omega_E} = \omega_E$ holds on $X\setminus D$. Then $\omega_E$ extends to be a closed positive current on $X$ cohomologous to $2\pi c_1(K_X + D)$ and is a canonical Kähler-Einstein current on $X$.

**Example 2.9** Let $X$ be a smooth projective $n$-fold and let $D$ be a divisor with normal crossings on $X$ such that for every sufficiently small positive rational number $\varepsilon$, $K_X + (1 - \varepsilon)D$ is ample. Then as in [T1], for every sufficiently large positive integer $m$, there exists an orbifold Kähler-Einstein form $\omega_m$ on the KLT pair $(X, m^{-1}D)$. Then as in [T2], we see that $\omega_\infty := \lim_{m \to \infty} \omega_m$ exists on $X$ as a closed positive current and in $C^\infty$-topology on $X\setminus D$. Then $(n!)^{-1}\omega_\infty$ is the canonical measure on the LC pair $(X, D)$.

### 3 Ricci iterations

In this section, we shall construct solutions of (1.8) and (2.9) as the limit of dynamical systems of Kähler forms or closed semipositive currents. The dynamical systems are defined by Ricci iterations. This construction will be used to define family of dynamical systems of Bergman kernels which converge to the canonical Kähler-Einstein current on a KLT pair of log general type. More precisely we use the Ricci iteration to eliminate the effect of singularities of singular hermitian metrics on the $\mathbb{Q}$-line bundle. In this section, we shall use the notations in the previous section, if without fear of confusion.

#### 3.1 The case of twisted Kähler-Einstein forms

Let $X$ be a smooth projective $n$-fold and let $(L, h_L)$ be a $C^\infty$ hermitian $\mathbb{Q}$-line bundle on $X$ such that $\sqrt{-1}\Theta_{h_L} \geq 0$ on $X$. We shall assume that $K_X + L$ is ample. Let $a$ be a positive integer such that $aL$ is a genuine line bundle on $X$. Let $\omega_0$ be a $C^\infty$ Kähler form representing $2\pi ac_1(K_X + L)$. For $m \geq 0$, we shall define a sequence of Kähler forms $\{\omega_m\}_{m=0}^\infty$ inductively by:

$$
-\text{Ric}_{\omega_m} + \frac{a-1}{a}\omega_{m-1} + \sqrt{-1}\Theta_{h_L} = \omega_m \quad (m = 1, 2, \cdots).
$$

Let us reduce (3.1) to the sequence of Monge-Ampère equation as follows. Let $\Omega$ be a $C^\infty$ volume form on $X$ such that

$$
\omega_0 = a \left(-\text{Ric} \Omega + \sqrt{-1}\Theta_{h_L} \right)
$$

holds. We define the sequence of $C^\infty$-functions $\{u_m\}_{m=0}^\infty$ by $u_0 = 0$ and for $m \geq 1$

$$
\log \left( \frac{\omega_0 + \sqrt{-1}\partial\bar{\partial}u_m}{\Omega} \right)^n = u_m - \frac{a-1}{a}u_{m-1}.
$$
The existence of \( \{u_m\}_{m=0}^{\infty} \) follows from the solution of Calabi’s conjecture (\[A, Y1\]). Then we see that the sequence of Kähler forms:

\[
\omega_m := \omega_0 + \sqrt{-1} \partial \bar{\partial} u_m \quad (m \geq 0)
\]

satisfy the successive equations (3.1). Now we shall state the result.

**Theorem 3.1** Let \( \{\omega_m\}_{m=0}^{\infty} \) be the sequence of Kähler forms defined inductively by (3.1) as above. Then

\[
\omega := \lim_{m \to \infty} \omega_m
\]

exists on \( X \) in \( C^\infty \)-topology and \( \omega \) is a \( C^\infty \) Kähler form on \( X \). And \( \omega \) satisfies

\[
- \operatorname{Ric}_\omega + \sqrt{-1} \Theta_{h_L} = \omega
\]
on \( X \), i.e., \( \omega \) is the twisted Kähler-Einstein form associated with \( (X, (L, h_L)) \) (cf. Definition \[LJ\]).

**Proof of Theorem 3.1** By (3.3), we have that

\[
\log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} u_m}{\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{m-1}} \right)^n = u_m - u_{m-1} - \frac{a-1}{a} (u_{m-1} - u_{m-2})
\]

holds for every \( m \geq 2 \). We note that

\[
\log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} u_m}{\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{m-1}} \right)^n = \int_0^1 \Delta_t (u_m - u_{m-1}) dt
\]

holds, where \( \Delta_t (t \in [0, 1]) \) denotes the Laplacian with respect to the Kähler form \((1-t)\omega_{m-1} + t \omega_m\). Then by the maximum principle and (3.7), we see that

\[
\sup_X (u_m - u_{m-1}) \leq \frac{a-1}{a} \sup_X (u_{m-1} - u_{m-2})
\]

and

\[
\inf_X (u_m - u_{m-1}) \geq \frac{a-1}{a} \inf_X (u_{m-1} - u_{m-2})
\]

hold for every \( m \geq 2 \). Hence by (3.9) and (3.10),

\[
\lim_{m \to \infty} (u_m - u_{m-1}) = 0
\]

holds and there exists a positive constant \( C_0 \) independent of \( m \) such that

\[
|u_m| \leq C_0
\]

holds on \( X \).

Now we shall consider the uniform \( C^2 \)-estimate of \( \{u_m\} \). We follow the argument in \[Ru, Theorem 3.3\] modeled after \[B-K\]. Let \( f : (M, g) \to (N, h) \) be a holomorphic map between Kähler manifolds. The Chern-Lu inequality in the context of Yau’s Schwarz lemma (\[Lu, Y2\]) gives:

\[
\Delta_g \log |\partial f|^2 \geq \frac{\operatorname{Ric}_g (\partial f, \bar{\partial} f)}{|\partial f|^2} - \frac{\operatorname{Bise}_{h} (\partial f, \bar{\partial} f, \partial f, \bar{\partial} f)}{|\partial f|^2},
\]

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where $\Delta_g$ denotes the Laplacian with respect to $g$. Let $f$ be the identity map $1_X : (X, \omega_m) \to (X, \omega_0)$. By rewriting the equation (3.1) as:

$$\text{Ric}_{\omega_m} = \frac{a-1}{a} \omega_m - \sqrt{-1} \Theta_{h_L} - \omega_m,$$

we see that Ricci curvature of $(X, \omega_m)$ is uniformly bounded from below along the Ricci iteration by our assumption that $\sqrt{-1} \Theta_{h_L}$ is semipositive. Since

$$|\partial f|^2 = \text{tr}_{\omega_m} \omega_0 = n - \Delta_{\omega_m} u_m,$$

we see that

(3.14) $\Delta_{\omega_m} \log(n - \Delta_{\omega_m} u_m) \geq -C(1 + n - \Delta_{\omega_m} u_m)$

holds for some positive constant $C$ independent of $m$. Hence

(3.15) $\Delta_{\omega_m} (\log(n - \Delta_{\omega_m} u_m) - (C+1) u_m) \geq -n - C(1+n) + (n - \Delta_{\omega_m} u_m)$

holds. Hence by the maximum principle and the uniform estimate (3.12) we see that there exists a positive constant $C_2$ such that

(3.16) $\|u_m\|_{C^2,\alpha} \leq C_2$

holds, where $\|\|_{C^2}$ denotes the $C^2$-norm with respect to $\omega_0$. Then by the general theory of nonlinear elliptic equations of 2nd order ([11]), we see that there exist a positive number $\alpha < 1$ and a positive constant $C_{2,\alpha}$ such that

(3.17) $\|u_m\|_{C^{2,\alpha}} \leq C_{2,\alpha}$

holds, where $\|\|_{C^{2,\alpha}}$ denotes the $C^{2,\alpha}$-norm with respect to $\omega_0$. Hence there exists a subsequence $\{m_k\}$ of $N$ such that

(3.18) $u := \lim_{k \to \infty} u_{m_k}$

exists in $C^{2,\alpha}$-topology. Then by (3.3) and (3.11), we see that $u$ satisfies the equation:

(3.19) $\log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} u}{\Omega} \right)^n = \frac{1}{a} u$

on $X$. We note that (3.19) has a unique solution as in [Y1]. Hence by (3.9) and (3.10) the limit $u$ exists in $C^{2,\alpha}$-topology without taking the subsequence $\{m_k\}$. Hence by taking $\sqrt{-1} \partial \bar{\partial}$ of both sides of (3.19), by the definition of $\Omega$ (cf. (3.20)), we see that if we define $\omega$ by

(3.20) $\omega := \frac{1}{a} (\omega_0 + \sqrt{-1} \partial \bar{\partial} u),$

then

(3.21) $-\text{Ric}_\omega + \sqrt{-1} \Theta_{h_L} = \omega$

holds. This completes the proof of Theorem 3.1. $\square$
3.2 The case of canonical Kähler-Einstein currents

Next we shall construct the canonical Kähler-Einstein current in Theorem 1.6 in terms of Ricci iterations. The major difference is that the $C^\infty$ hermitian $\mathbb{Q}$-line bundle $(L, h_L)$ is replaced by a singular hermitian $\mathbb{Q}$-line bundle $(D, h_{\sigma_D})$ (cf. (1.21)).

Let $X$ be a smooth projective $n$-fold and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is a KLT pair. We assume that $\text{Supp} D$ is a divisor with normal crossings and $K_X + D$ is big. We assume that there exists a Zariski decomposition:

$$K_X + D = P + N \quad (P, N \in \text{Div}(X) \otimes \mathbb{Q})$$

i.e., $P$ is semisample, $N$ is effective and

$$H^0(X, \mathcal{O}_X(maP)) \simeq H^0(X, \mathcal{O}(ma(K_X + D)))$$

holds for every $m \geq 0$, where $a$ is the minimal positive integer such that $aD, aP, aN \in \text{Div}(X)$. We may assume that $(X, D)$ satisfies the above conditions without loss of generality as was observed in Section 2.1.

Moreover since adding exceptional effective $\mathbb{Q}$-divisors does not change the log canonical ring and the canonical Kähler-Einstein current depends only on the log canonical ring essentially as is seen in Theorem 4.9 below.

Let $h_P$ be a $C^\infty$ hermitian metric on $P$ with semipositive curvature. We set

$$U := \{ x \in X \mid |\nu!P| \text{ is very ample on a neighbourhood of } x \text{ for every } \nu \gg 1 \}.$$ 

Let $a$ be a positive integer such that $aD \in \text{Div}(X)$. For $m \geq 0$, we shall define inductively a sequence of closed positive current $\{\omega_m\}_{m=0}^\infty$ satisfying the following conditions:

(P1) $\omega_0 = a (\sqrt{-1} \Theta_{h_P} + 2\pi [N])$, where $[N]$ denotes the current of integration over $N$.

(P2) The cohomology class of $[\omega_m]$ of $\omega_m$ is equal to $2\pi a \cdot c_1(K_X + D)$.

(P3) $\omega_m$ is $C^\infty$ on $U$.

(P4) $\{\omega_m\}_{m=0}^\infty$ satisfies the equation:

$$- \text{Ric}_{\omega_m} + a \cdot \frac{1}{a} \omega_{m-1} + 2\pi [D] = \omega_m$$

on $U$ for every $m \geq 1$.

(P5) We define the singular hermitian metric $h_m$ by

$$h_m := n! (\omega^n_{m,abc} \cdot (\omega_{m-1,abc})^{a-1})^{-\frac{1}{a}}$$

on $K_X + D$. Then $h_m$ is an AZD of $K_X + D$ for every $m \geq 1$.

Now we shall state the main result in this subsection.
Theorem 3.2  The dynamical system $\{\omega_m\}_{m=0}^{\infty}$ satisfying (P1)-(P5) above exists and unique and the limit
\begin{equation}
\omega_K := \frac{1}{a} \lim_{m \to \infty} \omega_m
\end{equation}
exists in $C^\infty$-topology on every compact subset of $U$ (cf. (3.24)) and also in the sense of current on $X$. The closed positive current $\omega_K$ satisfies the equation:
\[-\text{Ric}_{\omega_K} + 2\pi [D] = \omega_K\]
on $X$ and $\omega_K$ is the canonical Kähler-Einstein current on $(X, D)$ (cf. Definition 1.5).

Let us reduce (3.25) to successive Monge-Ampère equations. Let $\sigma_N$ be a multi-valued holomorphic section of $N$ with divisor $N$. Let $h_D, h_N$ are $C^\infty$ hermitian metric on $D$ and $N$ respectively. Let $\Omega$ be a $C^\infty$ volume form on $X$ such that
\begin{equation}
h_P := \Omega^{-1} \cdot h_D \cdot h_N^{-1}.
\end{equation}
holds. Then
\begin{equation}
\omega_0 = a (-\text{Ric} \Omega + 2\pi [N])
\end{equation}
holds. We set
\begin{equation}
\omega_P := \sqrt{-1} \Theta_{h_P}.
\end{equation}
As in Section 2.1 let $E$ be an effective $\mathbb{Q}$-divisor such that $P - \delta E$ is ample for every $\delta \in (0, 1)$. As before we may and do assume that $X \setminus \text{Supp} E = U$ holds (cf. (2.11) in Section 2.1). Then there exists a $C^\infty$ hermitian metric $h_E$ on $E$ such that $h_P \cdot h_E^{-1}$ is a metric with strictly positive curvature on $X$. We define the sequence of functions $\{u_m\}_{m=0}^{\infty}$ by $u_0 = 0$ and for $m \geq 1$ as follows.

(Q1) $\{u_m\}$ are almost bounded in the sense of Lemma 2.3 in Section 2.1, i.e., for every $\delta_0 \in (0, 1]$ there exists a constant $C_- (\delta_0)$ depending only on $\delta_0 \in (0, 1]$ and a constant $C_+$ independent of $m$ such that for every $\delta \in (0, \delta_0)$,
\begin{equation}
C_- (\delta_0) + \delta_0 \log \left( \prod_i \left( 1 - \| \sigma_i \|_{h_E}^2 \right)^{-\epsilon} \| \sigma_N \|_{h_N}^2 \| \sigma_D \|_{h_D} \right) \leq u_m \leq C_+
\end{equation}
holds, where we have used the notations in Section 2.1 (cf. (2.7), (2.46) etc.).

(Q2) $u_m \in C^\infty (X \setminus \text{Supp} E)$.

(Q3) $\{u_m\}$ satisfy the successive equations:
\begin{equation}
\log \left( \frac{\omega_P + \sqrt{-1} \partial \bar{\partial} u_m}{\Omega} \right)^n = \log \frac{\sigma_N}{\sigma_D} \frac{\sigma_N}{h_N} + \left( u_m - \frac{a - 1}{a} u_{m-1} \right)
\end{equation}
for $m \geq 1$. Then as in the Section 2, (2.9), (3.31) is equivalent to (3.25), i.e., the sequence of closed positive currents $\{\omega_m\}$ defined by
\begin{equation}
\omega_m := a (\omega_P + 2\pi [N]) + \sqrt{-1} \partial \bar{\partial} u_m
\end{equation}
satisfies the conditions (P1) to (P5) above. For example the almost boundedness (Q1) implies the condition (P5) and (Q2) implies (P3) etc.

The construction of \( \{u_m\}_{m=0}^{\infty} \) is very much similar to the one of Theorem 1.6 in Section 2. The only difference is that we need to estimate inductively. To construct \( \{u_m\}_{m=0}^{\infty} \) we consider the perturbed equation as follows. For a fixed \( \delta \in (0,1] \), we set

\[
\omega_{m,\delta,0} := \omega_P + \delta \left( \frac{a-1}{a} \right)^m \left( \sqrt{-1} \Theta_{hE} + \varepsilon \sqrt{-1} \bar{\partial} \partial \sum_i \log(1 - \| \sigma_i \|^2 \right),
\]

where \( \varepsilon \) is a sufficiently small positive number such that \( \omega_{m,\delta,0} \) is an orbifold Kähler form on \( X \) for every \( \delta \in (0,1] \) as in Secion 2.1. We consider the successive equations:

\[
\log \left( a \omega_{m,\delta,0} + \sqrt{-1} \bar{\partial} \partial u_{m,\delta} \right)^n = \log \frac{\| \sigma_N \|^2_{hE} + \left( u_{m,\delta} - \frac{a-1}{a} u_{m-1,\delta} \right)}{\| \sigma_D \|^2_{hD}} \quad \text{for } m \geq 1
\]

and we set

\[
\omega_{m,\delta} := a \omega_{m,\delta,0} + \sqrt{-1} \bar{\partial} \partial u_{m,\delta}.
\]

We set \( u_{0,\delta} = 0 \). Then inductively by using the same strategy as in the proof of Theorem 1.6 in Section 2, by [Y1] p.387,Theorem 6], we see that (3.34) has a sequence of bounded solutions \( \{u_{m,\delta}\}_{m=0}^{\infty} \) such that for every \( m \), \( u_{m,\delta} \) has a bounded \( C^2 \)-norm on \( X \) with respect to the orbifold Kähler form \( \omega_{1,1,0} \).

**Lemma 3.3** For every \( \epsilon > 0 \) there exists a positive constant \( C_{0,\epsilon} \) such that

\[
u_{1,\delta} \geq -C_{0,\epsilon} + \epsilon \log \| \sigma_E \|_{hE}^2
\]

holds for every \( \delta \in (0,1] \).

**Proof.** We set

\[
u_{1,\delta,\epsilon} := u_{1,\delta} - \epsilon \left( \log \| \sigma_E \|_{hE} - \varepsilon \sum_i \log(1 - \| \sigma_i \|^2) \right)
\]

and

\[
\omega_{1,\delta,0,\epsilon} := \omega_{1,\delta,0} + \epsilon \left( -\Theta_{hE} - \varepsilon \partial \bar{\partial} \sum_i \log(1 - \| \sigma_i \|^2) \right).
\]

Then \( \nu_{1,\delta,\epsilon} \) satisfies the equation:

\[
\log \left( a \omega_{1,\delta,0,\epsilon} + \sqrt{-1} \bar{\partial} \partial u_{1,\delta,\epsilon} \right)^n = \log \frac{\| \sigma_N \|^2_{hE} \cdot \| \sigma_E \|^2_{hE} \cdot \Omega}{\| \sigma_D \|^2_{hD} \prod_i (1 - \| \sigma_i \|^2) \epsilon \cdot \omega_{1,\delta,0,\epsilon}^{n} + u_{1,\delta,\epsilon}}.
\]

We note that since \( \omega_{1,\delta,0,\epsilon} \) is an orbifold Kähler form, by the choice of \( b \) (cf. [2,1]), there exists a positive constant \( C_{0,\epsilon} \) depending only on \( \epsilon \) such that

\[
\log \frac{\| \sigma_N \|^2_{hE} \cdot \| \sigma_E \|^2_{hE} \cdot \Omega}{\| \sigma_D \|^2_{hD} \prod_i (1 - \| \sigma_i \|^2) \epsilon \cdot \omega_{1,\delta,0,\epsilon}^{n}} \leq C_{0,\epsilon}
\]

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holds on $X$. We note that since $u_{1,\delta}$ is bounded on $X$, $u_{1,\delta,\epsilon}(x)$ tends to $+\infty$ as $x$ tends to $\text{Supp } E$. Hence by applying the maximum principle, we obtain that

$$u_{1,\delta,\epsilon} \geq -C_{0,\epsilon}.$$  

holds on $X$. By (3.36), we obtain the lemma. □

As (3.37) we have that

$$\log \frac{(a\omega_{m,\delta},0 + \sqrt{-1}\partial u_{m,\delta})n}{(a\omega_{m-1,\delta},0 + \sqrt{-1}\partial u_{m-1,\delta})n} = (u_{m,\delta} - u_{m-1,\delta}) - \frac{a-1}{a}(u_{m-1,\delta} - u_{m-2,\delta})$$

for $m \geq 2$. We note that on $X \setminus \text{Supp } E$,

$$\log \frac{(a\omega_{m,\delta},0 + \sqrt{-1}\partial u_{m,\delta})n}{(a\omega_{m-1,\delta},0 + \sqrt{-1}\partial u_{m-1,\delta})n} = \log \frac{(a\omega_{m-1,\delta},0 + \sqrt{-1}\partial (u_{m,\delta} - \alpha_m \delta \log \| \sigma_E \| _{h_E}^2))n}{(a\omega_{m-1,\delta},0 + \sqrt{-1}\partial u_{m-1,\delta})n}$$

holds, where

$$\alpha_m := \frac{1}{a} \left( \frac{a-1}{a} \right)^{m-1}.$$  

The definition of $\{\alpha_m\}$ follows from the successive equations in cohomology classes:

$$2\pi[P] + \frac{a-1}{a} [\omega_{m-1,\delta}] = [\omega_{m,\delta}] \quad (m \geq 1)$$

arising from (3.25).

We note that the equality:

$$(u_{m,\delta} - u_{m-1,\delta}) - \frac{a-1}{a}(u_{m-1,\delta} - u_{m-2,\delta}) =$$

$$(u_{m,\delta} - u_{m-1,\delta} - \alpha_m \delta \log \| \sigma_E \| _{h_E}^2) - \frac{a-1}{a}(u_{m-1,\delta} - u_{m-2,\delta} - \alpha_{m-1} \delta \log \| \sigma_E \| _{h_E}^2)$$

holds. We also note that by the boundedness of $|u_{m,\delta}|$ and $|u_{m-1,\delta}|$ on $X$,

$$(u_{m,\delta} - u_{m-1,\delta} - \alpha_m \delta \log \| \sigma_E \| _{h_E}^2)(x) \rightarrow +\infty \quad \text{as } x \rightarrow \text{Supp } E$$

holds. Hence applying the minimum principle to (3.37), by using the similar formula as (3.38),

$$\inf_X (u_{m,\delta} - u_{m-1,\delta} - \alpha_m \delta \log \| \sigma_E \| _{h_E}^2 \geq \frac{a-1}{a} \inf_X (u_{m-1,\delta} - u_{m-2,\delta} - \alpha_{m-1} \delta \log \| \sigma_E \| _{h_E}^2$$

holds for every $m \geq 2$. Hence $\{u_{m,\delta}\}$ is an almost monotone increasing sequence in this sense and by Lemma 3.3 (taking $\epsilon > 0$ sufficiently small), there exists a positive constant $C_0(\delta)$ depending $\delta$ such that

$$u_{m,\delta} - u_{m-1,\delta} - \alpha_m \delta \log \| \sigma_E \| _{h_E}^2 \geq -C_0(\delta) \left( \frac{a-1}{a} \right)^{m-1}$$
holds for every $m \geq 1$. By (3.39) and (3.44), we see that
\begin{equation}
(3.45) \quad u_{m,\delta} \geq -C_0(\delta) \cdot a + \delta \log \| \sigma_E \|_{h_{E}}^2
\end{equation}
holds. But this estimate depends on $\delta$.

To get the uniform lower estimate with respect to $\delta$, we set
\begin{equation}
(3.46) \quad u_{m,\delta}(\epsilon) := u_{m,\delta} - \epsilon \left( \frac{a-1}{a} \right)^m \log \| \sigma_E \|_{h_{E}}^2
\end{equation}
for $m \geq 0$. Then by Lemma 3.3, the similar argument as above replacing $u_{m,\delta}$ by $u_{m,\delta}(\epsilon)$ and $\omega_{m,\delta,0}$ by
\begin{equation}
(3.47) \quad \omega_{m,\delta,0}(\epsilon) := \omega_{m,\delta,0} - \epsilon \sqrt{-1} \left( \frac{a-1}{a} \right)^m \Theta_{h_{E}}
\end{equation}
we see that for every $\epsilon > 0$, there exists a positive constant $C_{0,\epsilon}$ (same as in Lemma 3.3) independent of $\delta$ such that
\begin{equation}
(3.48) \quad u_{m,\delta}(\epsilon) - u_{m-1,\delta}(\epsilon) - \alpha_m(\delta + \epsilon) \log \| \sigma_E \|_{h_{E}}^2 \geq -C_{0,\epsilon} \left( \frac{a-1}{a} \right)^{m-1}
\end{equation}
holds. Hence we have the uniform lower estimate (with respect to $\delta$):
\begin{equation}
(3.49) \quad u_{m,\delta} \geq -C_{0,\epsilon} \cdot a + \left\{ \left( \frac{a-1}{a} \right)^m \epsilon + (\delta + \epsilon) \right\} \log \| \sigma_E \|_{h_{E}}^2.
\end{equation}

On the other hand, since
\[
\int_X \exp \left( u_{m,\delta} - \frac{a-1}{a} u_{m-1,\delta} \right) \Omega = (2\pi)^n \left( P - \left( \frac{a-1}{a} \right)^m \delta E \right)^n
\]
holds. By the concavity of logarithm as (2.28) and (2.29), we see that there exists a positive constant $C_0$ independent of $m$ and $\delta$ such that
\begin{equation}
(3.50) \quad \int_X \left( u_{m,\delta} - \frac{a-1}{a} u_{m-1,\delta} \right) \Omega \leq C_0
\end{equation}
holds. Hence by the almost monotonicity and the almost pluri-subharmonicity:
\[
\omega_{m,\delta} = \omega_{m,\delta,0} + \sqrt{-1} \partial \bar{\partial} u_{m,\delta} \geq 0,
\]
we see that there exists a positive constant $C_+$ independent of $m$ and $\delta$ such that
\begin{equation}
(3.51) \quad u_{m,\delta} \leq C_+
\end{equation}
holds. Hence by the almost monotonicity of $\{ u_{m,\delta} \}_{m=0}^\infty$ we see that
\begin{equation}
(3.52) \quad u_\delta := \lim_{m \to \infty} u_{m,\delta}
\end{equation}
and
\begin{equation}
(3.53) \quad u := \lim_{m \to \infty} u_m
\end{equation}
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exist. Let \( \omega_{P,\delta}(\delta \in (0,1]) \) be the orbifold Kähler form defined as (2.46) in Section 2, i.e.,

\[
\omega_{P,\delta} := \omega_P + \delta \left( -\sqrt{-1} \Theta_E + \varepsilon \sum_i \sqrt{-1} \partial \overline{\partial} \log(1- \| \sigma_i \| \frac{2}{a}) \right).
\]

As in Section 2 we consider the perturbed equation:

\[
(a \omega_{P,1} + \sqrt{-1} \partial \overline{\partial} v_{m,\delta})^n = \frac{\| \sigma_N \|^2_{h_N} \left( \prod_i (1- \| \sigma_i \| \frac{2}{a}) \right)^{-\varepsilon} \| \sigma_E \|^2_{h_E}}{\| \sigma_D \|^2_{h_D}} \cdot \exp \left( v_{m,\delta} - \frac{a-1}{a} v_{m-1,\delta} \right) \cdot \Omega,
\]

where

\[
v_{m,\delta} := u_{m,\delta} - (1 - \alpha_m \delta) \cdot \log \left( \prod_i (1- \| \sigma_i \| \frac{2}{a}) \right)^{-\varepsilon} \| \sigma_E \|^2_{h_E}.
\]

This equation is equivalent to

\[
- \text{Ric}_{\omega_{m,\delta}} + \frac{a-1}{a} \omega_{m-1,\delta} + 2\pi [D] = \omega_{m,\delta}
\]
as before, where

\[
\omega_{m,\delta} := a(\omega_{P,1} + 2\pi [N]) + \sqrt{-1} \partial \overline{\partial} v_{m,\delta}.
\]

Then by the weighted inductive \( C^2 \)-estimate as in the proof of Theorem 1.6 (or the uniform weighted \( C^2 \)-estimate below as the proof of Theorem 3.1 using Schwarz type lemma) letting \( \delta \downarrow 0 \), we see that \( \{u_{m,\delta}\} \) exists for every \( m \geq 0 \) and \( \delta \in (0,1] \). Here we note that \( \Delta_{P,1} v_{m,\delta} \) is bounded on \( X \) for every \( m \) and \( \delta > 0 \), hence, we may take the constant \( C \) in Lemma 2.2 independent of \( m \) and \( \delta \), since \( e^{-C v_{m,\delta}(n + \Delta_{P,1} v_{m,\delta})} \) is bounded on \( X \). Then as in the proof of Theorem 1.6 we have the following lemma.

**Lemma 3.4**

\[
u_m := \lim_{\delta \downarrow 0} u_{m,\delta}
\]

exists on \( X \setminus \text{Supp} E \) compact uniformly in \( C^\infty \)-topology for every \( m \) and \( \{u_m\} \) satisfies the family of equations (3.22). And \( \{u_m\} \) satisfies the properties (Q1),(Q2) and (Q3) above. \( \square \)

Then by Theorem 1.9 below, we see that \( \{u_m\}_{m=1}^\infty \) is unique and gives a sequence of closed positive currents \( \{\omega_m\} \) satisfying (P1)-(P5) above. But we should note that the above inductive estimate may not be uniform with respect to \( m \). A priori the estimate may get worse as \( m \) tends to infinity, because to estimate the \( C^2 \)-norm of \( u_{m,\delta} \) by using Lemma 2.2 the \( (\text{weighted}) C^2 \)-norm of \( u_{m-1,\delta} \) comes into the estimate. The following estimate gives a better \( C^2 \)-estimate of \( \{u_m\} \) (or \( \{u_m,\delta\} \)).

To prove the convergence, we proceed just as the proof of Theorem 3.1. But here we need to handle the incompleteness of \( X \setminus \text{Supp} E \) with respect to the orbifold Kähler form \( \omega_{P,1} \) (cf. (2.40)). First we note that by (3.51), \( \text{Ric}_{\omega_{m,\delta}} \)
is uniformly bounded from below along the Ricci iteration just as before. We note that the bisectional curvature of the Kähler orbifold \((X, a\omega_{P,1})\) is uniformly bounded. Then applying the Schwarz type lemma: \((3.13)\) to the identity map \(1_X : (X, \omega_{m,\delta}) \rightarrow (X, a\omega_{P,1})\), as \((3.14)\), we see that there exists a positive constant \(C_1\) independent of \(m\) and \(\delta\) such that
\[
\Delta_{\omega_{m,\delta}} \left( \log(n - \Delta_{\omega_{m,\delta}} v_{m,\delta}) - (C_1 + 1)v_{m,\delta} \right) \geq n - C_1(1 + n) + (n - \Delta_{\omega_{m,\delta}} v_{m,\delta}),
\]
holds on \(U\). We note that \(v_{m,\delta}\) blows up (to \(+\infty\)) toward \(\text{Supp} E\) by the definition \((3.56)\) and the \(C^0\)-estimate \((3.49)\) (taking \(\epsilon\) sufficiently small). Then by the inductive \(C^2\)-estimate of \(v_{m,\delta}\) using Lemma 2.2 as in Section 2 (cf. \((2.36)\)), if we take \(C_1\) (independent of \(m\) and \(\delta\)) sufficiently large, then
\[
\log(n - \Delta_{\omega_{m,\delta}} v_{m,\delta}) - (C_1 + 1)v_{m,\delta} \to -\infty \quad \text{as} \quad x \to \text{Supp} E
\]
holds and there exists a point \(x_{m,\delta}\) on \(X\setminus\text{Supp} E\), where \(\log(n - \Delta_{\omega_{m,\delta}} v_{m,\delta}) - (C_1 + 1)v_{m,\delta}\) takes it maximum. Now we are ready to apply the maximum principle to \((3.60)\) and we see that
\[
\log(n - \Delta_{\omega_{m,\delta}} v_{m,\delta})(x_{m,\delta}) \leq C_1(1 + n) - n
\]
holds. Since \(v_{m,\delta}\) is uniformly bounded from below by the definition of \(v_{m,\delta}\) (cf. \((3.56)\)) and the uniform lower estimate \((3.49)\), there exists a positive constant \(C_2\) independent of \(m\) such that
\[
\log(n - \Delta_{\omega_{m,\delta}} v_{m,\delta}) - (C_1 + 1)v_{m,\delta} \leq C_2
\]
holds on \(U\). Letting \(\delta\) tend to 0, this gives a weighted \(C^2\)-estimate of \(u_m\) on every compact subset. Hence using \([T]\) and the almost monotonicity of \(\{u_m\}\), we conclude that
\[
u = \lim_{m \to \infty} u_m
\]
exists on \(X\setminus\text{Supp} E\) compact uniformly in \(C^\infty\)-topology without taking a subsequence and \(u\) is almost bounded by \((3.49)\) and \((3.51)\).

Hence by \((3.31)\), \(u\) satisfies the Monge-Ampère equation:
\[
\log \frac{(a\omega_P + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} = \log \frac{\|\sigma_N\|_{h_N}}{\|\sigma_D\|_{h_D}} + \frac{1}{a} u.
\]
Then it is clear that
\[
\omega_K := \frac{1}{a} (a\omega_P + \sqrt{-1}\partial\bar{\partial}u) + 2\pi [N]
\]
satisfies the equation:
\[-\text{Ric}_{\omega_K} + 2\pi [D] = \omega_K.
\]
Now we define the singular hermitian metric:
\[
h_K := \left( \frac{1}{n!} \omega^n_{K,abc} \right)^{-1} \cdot h_D
\]
on \(K_X + D\). Then by the equation \((3.64)\) and the uniform \(C^0\)-estimates \((3.49)\) from below, we see that \(h_K\) is an AZD of \(K_X + D\). Hence \(\omega_K\) is nothing but the canonical Kähler-Einstein current on \((X, D)\). This completes the proof of Theorem 3.2. \(\square\)
4 Dynamical systems of Bergman kernels

In this section, we shall construct twisted Kähler-Einstein forms and canonical Kähler-Einstein currents (on KLT pairs of log general type) in terms of dynamical systems of Bergman kernels. The proof is essentially the same as in \[T7, T9\]. But here the dynamical construction cannot be applied directly to the twisted Kähler-Einstein forms or canonical Kähler-Einstein currents. In fact we apply the dynamical construction as in \[T7, T9\] to the solutions of (3.1) or (3.25). Hence the dynamical constructions here have two parameters.

4.1 Dynamical construction of twisted Kähler-Einstein forms

Let \( X \) be a smooth projective \( n \)-fold and let \((L, h_L)\) is a \( C^\infty \) hermitian \( \mathbb{Q} \)-line bundle on \( X \) such that

1. \( \sqrt{-1} \Theta_{h_L} \) is semipositive,
2. \( K_X + L \) is ample.

Let \( \alpha \) be a positive number such that \( \alpha L \) is a genuine line bundle. Let \( h_0 \) be a \( C^\infty \) hermitian metric on \( \alpha (K_X + L) \) such that

\[
\omega_0 := \sqrt{-1} \Theta_{h_0}
\]

is a \( C^\infty \) Kähler form on \( X \). Let us consider the dynamical system of Bergman kernel \( \{ K_\ell, 1 \}_{\ell=1}^\infty \) as follows. Let \( A \) be a sufficiently ample line bundle on \( X \) such that for every pseudo-effective singular hermitian line bundle \((F, h_F)\) on \( X \), \( \mathcal{O}_X(A + F) \otimes I(h_F) \) and \( \mathcal{O}_X(K_X + A + F) \otimes I(h_F) \) are globally generated. Such \( A \) exists by Nadel’s vanishing theorem (cf. \[N, p.561\]). Let \( h_A \) be a \( C^\infty \) hermitian metric on \( A \) with strictly positive curvature. We set

\[
K_{1, 1} := K(X, A + \alpha (K_X + L), h_A \cdot \frac{h_0}{h_0} \cdot h_L),
\]

where \( K(X, A + \alpha (K_X + L), h_A \cdot \frac{h_0}{h_0} \cdot h_L) \) denotes the Bergman kernel defined as \[1.4\]. By the choice of \( A \), \( \mathcal{O}_X(A + \alpha (K_X + L)) \) is globally generated. Hence we may define the \( C^\infty \) hermitian metric \( h_{1, 1} \) on \( A + \alpha (K_X + L) \) by

\[
h_{1, 1} := \frac{1}{K_{1, 1}}
\]

where \( K(X, A + \alpha (K_X + L), h_A \cdot \frac{h_0}{h_0} \cdot h_L) \)

For \( \ell \geq 2 \), we define \( K_{\ell, 1} \) and the \( C^\infty \) hermitian metric \( h_{\ell, 1} \) on \( A + \ell \alpha (K_X + L) \) by

\[
K_{\ell, 1} := K(X, A + \ell \alpha (K_X + L), h_{\ell-1, 1} \cdot \frac{h_0}{h_0} \cdot h_L)
\]

and define a \( C^\infty \) hermitian metric on \( A + \ell \alpha (K_X + L) \) by

\[
h_{\ell, 1} := \frac{1}{K_{\ell, 1}}
\]

Then as in \[T9\], we have the following lemma.
Lemma 4.1

\[(4.6) \quad K_1 := \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell)!^{-n} K_{\ell,1}} \]

is bounded and

\[(4.7) \quad h_1 := \frac{n!}{(2\pi)^n} K_1^{-1} \]

is a $C^\infty$ hermitian metric on $a(K_X + L)$. Here the constant $(2\pi)^{-n}n!$ is not so important anyway. The reason why we put such a constant will become clear in the next subsection. If we set

\[(4.8) \quad \omega_1 := \sqrt{-1} \Theta h_1, \]

then $\omega_1$ is a Kähler form on $X$ and satisfies the equation:

\[(4.9) \quad -\text{Ric}_{\omega_1} + \frac{a-1}{a} \omega_0 + \sqrt{-1} \Theta h_L = \omega_1. \]

Then replacing $h_0, \omega_0$ by $h_1, \omega_1$ respectively. We obtain the dynamical system of Bergman kernels $\{K_{\ell,2}\}_{\ell=1}^\infty$. Again

\[(4.10) \quad K_2 := \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell)!^{-n} K_{\ell,2}} \]

exists and $h_2 := \frac{n!}{(2\pi)^n} K_2^{-1}$ is a $C^\infty$ hermitian metric on $a(K_X + L)$. If we set

\[(4.11) \quad \omega_2 := \sqrt{-1} \Theta h_2, \]

then $\omega_2$ is a Kähler form on $X$ and satisfies the equation:

\[(4.12) \quad -\text{Ric}_{\omega_2} + \frac{a-1}{a} \omega_1 + \sqrt{-1} \Theta h_L = \omega_2. \]

Inductively, for every positive integers $m$, we define the dynamical system of Bergman kernels $\{K_{\ell,m}\}_{\ell=1}^\infty$ and a $C^\infty$ hermitian metric $h_m$ on $a(K_X + L)$. Then by Theorem 3.1 we have the following theorem.

**Theorem 4.2** Let $\{K_{\ell,m}\}_{\ell=1}^\infty$ be the system of Bergman kernels defined as above. If we define

\[(4.13) \quad K_m := \text{the upper-semi-continuous envelope of } \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell)!^{-n} K_{\ell,m}}, \]

then

\[(4.14) \quad h_m := \frac{n!}{(2\pi)^n} K_m^{-1} \]

is a $C^\infty$ hermitian metric on $a(K_X + L)$ with strictly positive curvature.

\[\text{7} \quad \text{The upper-semi-continuous envelope of the supremum of a family of pluri-subharmonic functions is again pluri-subharmonic by \cite[Theorem 5]{Le}, if the family is locally bounded from above. This operation is often used in this article. But anyway the adjustment occurs only on the set of measure 0.} \]

\[37 \]


then $\{\omega_m\}_{m=1}^{\infty}$ satisfy the equations
\begin{equation}
-\text{Ric}_{\omega_m} + \frac{a-1}{a} \omega_{m-1} + \sqrt{-1} \Theta_{h_L} = \omega_m
\end{equation}
on X for every $m \geq 1$ as (3.1). And
\[
\omega := \frac{1}{a} \lim_{m \to \infty} \omega_m
\]
exists on $X$ in $C^\infty$-topology and $\omega$ is the unique $C^\infty$-solution of the equation:
\[
-\text{Ric}_\omega + \sqrt{-1} \Theta_{h_L} = \omega.
\]
on $X$, i.e., $\omega$ is the twisted Kähler-Einstein form associated with $(X, (L, h_L))$.

\[\square\]

### 4.2 Dynamical construction of canonical Kähler-Einstein currents

In this subsection, we shall construct canonical Kähler-Einstein currents in terms of dynamical systems of Bergman kernels.

Let $X$ be a smooth projective variety of dimension $n$ and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is a KLT pair. We assume that $\text{Supp} D$ is a divisor with normal crossings and $K_X + D$ is big. We also assume that there exists a Zariski decomposition:
\begin{equation}
K_X + D = P + N \quad (P, N \in \text{Div}(X) \otimes \mathbb{Q}),
\end{equation}
i.e., $P$ is semiample, $N$ is effective and
\begin{equation}
H^0(X, \mathcal{O}_X(\nu a P)) \simeq H^0(X, \mathcal{O}(ma(K_X + D)))
\end{equation}
holds for every $m \geq 0$, where $a$ is the minimal positive integer such that $aD, aP, aN \in \text{Div}(X)$ hold. These assumptions are not actual restrictions as is stated in the beginning of Section 2.1.

Let $h_P$ be a $C^\infty$ hermitian metric on $P$ with semipositive curvature. We set
\begin{equation}
U := \{x \in X \mid |\nu P| \text{ is very ample on a neighbourhood of } x \text{ for every } \nu \gg 1\}.
\end{equation}
Let $\sigma_D$ be a multivalued holomorphic section of $D$ with divisor $D$. And we define the singular hermitian metric $h_{\sigma_D}$ by
\begin{equation}
h_{\sigma_D} := \frac{1}{|\sigma_D|^2}.
\end{equation}
Let $h_P$ be a $C^\infty$ hermitian metric on $P$ such that
\begin{equation}
\omega_P := a\sqrt{-1} \Theta_{h_P}
\end{equation}
is a $C^\infty$ Kähler form on $X$. Let $\sigma_N$ be a global multivalued holomorphic section of $N$ with divisor $N$. Let $h_{\sigma_N}$ be the singular hermitian metric on $N$ defined by
\begin{equation}
h_{\sigma_N} := \frac{1}{|\sigma_N|^2}.
\end{equation}
Let \( a \) be the minimal positive integer such that \( aD, aP, aN \in \text{Div}(X) \) hold. Let us consider the dynamical system of Bergman kernel \( \{K_{\ell,1}\}_{\ell=1}^\infty \) as follows. As before, let \( A \) be a sufficiently ample line bundle on \( X \) such that for every pseudo-effective singular hermitian line bundle \( (F, h_F) \) on \( X, \) \( \mathcal{O}_X(K_X + A + F) \otimes \mathcal{I}(h_F) \) and \( \mathcal{O}_X(A + F) \otimes \mathcal{I}(h_F) \) are globally generated.

The following lemma is trivial. But it explains why we can handle KLT pairs in the dynamical system of Bergman kernels.

**Lemma 4.3**

\[
(4.23) \quad \otimes \sigma^a_{\sigma_D} : \mathcal{O}_X(\ell aP) \hookrightarrow \mathcal{O}_X(\ell a(K_X + D)) \otimes \mathcal{I}((h_P \cdot h_{\sigma_N})^{(a-1) \cdot h_{\sigma_D}})
\]

is a well defined injection for every \( \ell \geq 0. \)

**Proof of Lemma 4.3** Let \( \sigma \) be an arbitrary holomorphic section of \( \ell aP \) on an open subset \( V \) in \( X. \) Then \( \sigma^{1/\ell a} \cdot \sigma_N \) is a multivalued holomorphic section of \( K_X + D \) on \( V. \) Hence we see that

\[
\frac{\sigma^a \cdot \sigma_N}{\sigma_D}
\]

is locally \( L^2 \)-integrable multivalued meromorphic form on \( V, \) because \( (X, D) \) is KLT. We note that the equality:

\[
((h_P \cdot h_{\sigma_N})^{(a-1) \cdot h_{\sigma_D}}) \sigma \cdot \sigma^a_{\sigma_N} = \left| \frac{\sigma^a \cdot \sigma_N}{\sigma_D} \right|^2 \cdot (h_P \cdot h_N)^{\ell a-1} \left| \sigma^{1-\frac{1}{\ell a}} \cdot \sigma^a_{\sigma_N} \right|^2
\]

holds. Since \( (h_P(h_N)^{a-1} \left| \sigma^{1-\frac{1}{\ell a}} \cdot \sigma^a_{\sigma_N} \right|^2 = h_P^{\ell a-1} \left| \sigma \right|^{2(1-1/\ell a)} \) is locally bounded, we see that (4.23) is a well defined injection.

Let \( h_A \) be a \( C^\infty \) hermitian metric on \( A \) with strictly positive curvature. We set

\[
(4.24) \quad K_{1,1} := K(X, A + a(K_X + D), h_A \cdot (h_P \cdot h_{\sigma_N})^{(a-1) \cdot h_{\sigma_D}})
\]

and define a singular hermitian metric \( h_{1,1} \) on \( A + a(K_X + D) \) by

\[
(4.25) \quad h_{1,1} := \frac{1}{K_{1,1}}.
\]

By Lemma 4.3 we have the natural injection:

\[
(4.26) \quad H^0(X, \mathcal{O}(A + aP)) \hookrightarrow H^0(X, \mathcal{O}_X(A + a(K_X + D)) \otimes \mathcal{I}(h_A \cdot (h_P \cdot h_{\sigma_N})^{(a-1) \cdot h_{\sigma_D}})).
\]

We note that by the choice of \( A, \) \( \mathcal{O}(A + aP) \) is globally generated. Hence we have that

\[
(4.27) \quad h_{1,1} = O(h_A \cdot h_{P}^a \cdot h_{\sigma_N}^a)
\]

holds. Suppose that we have already defined \( K_{1,\ell-1} \) and \( h_{1,\ell-1} \) such that

\[
(4.28) \quad h_{\ell-1,1} = O(h_A \cdot h_{P}^{(\ell-1)a} \cdot h_{\sigma_N}^{(\ell-1)a}).
\]
Then we define $K_{\ell,1}$ and the singular hermitian metric $h_{\ell,1}$ on $A + \ell a(K_X + D)$ by

\[
K_{\ell,1} := K(X, A + \ell a(K_X + D), h_{\ell-1,1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D})
\]

and define the singular hermitian metric on $A + \ell a(K_X + L)$ by

\[
h_{\ell,1} := \frac{1}{K_{\ell,1}}
\]

By the choice of $A$, we see that $\mathcal{O}_X(A + \ell aP)$ is globally generated and by (4.28) and Lemma 4.3, there is a natural injection:

\[
H^0(X, \mathcal{O}_X(A + \ell aP)) \hookrightarrow H^0(X, \mathcal{O}_X(A + \ell a(K_X + D)), \mathcal{I}(h_{\ell-1,1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D})).
\]

Hence $h_{\ell,1} = O(h_A \cdot h_P^a \cdot h_{\sigma_N}^a)$ holds. In this way by the induction on $\ell$, we see that $K_{\ell,1}, h_{\ell,1}$ are well defined for every $\ell \geq 1$ and

\[
h_{\ell,1} = O(h_A \cdot h_P^a \cdot h_{\sigma_N}^a)
\]

holds for every $\ell \geq 1$.

Let us make (4.31) quantitative. Let $\{\omega_m\}_{m=1}^{\infty}$ be the Ricci iteration constructed as in Theorem 3.2 in Section 3. We set

\[
h_1 := \left(\frac{1}{n!} \omega_n^{abc} \right)^{-1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D}.
\]

Then by the equation (3.25), we see that

\[
\omega_1 = \sqrt{-1} \Theta h_1 = -\text{Ric}_{\omega_1} + \frac{a-1}{a} \omega_0
\]

hold on $X$, where $\omega_0 = a(\sqrt{-1} \Theta h_P + 2\pi[N])$ as in Theorem 3.2. Let $\ell \geq 2$ and suppose that there exists a positive constant $C_{\ell-1}$ such that

\[
K_{\ell-1,1} \geq C_{\ell-1} \cdot h_1^{-(\ell-1)} \cdot h_A^{-1}
\]

holds. We note that by the extremal property of Bergman kernels,

\[
K_{\ell,1}(x) := \sup \{||\sigma||^2(x) : \int_X ||\sigma||^2 \cdot \left(h_{\ell-1,1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D} \right) = 1 \}
\]

holds for every $x \in X$, where $\sigma$ runs all the elements in

\[
H^0 \left(X, \mathcal{O}_X(A + \ell a(K_X + D)) \otimes \mathcal{I}(h_{\ell-1,1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D}) \right).
\]

And this vector space contains $H^0(X, \mathcal{O}_X(A + \ell aP))$ by Lemma 4.3 and (4.31).

Let us estimate $K_{\ell,1}$ from below by using the $L^2$-estimate for $\bar{\partial}$-operators and (4.35) above. Let $p \in U(= X \setminus \text{Supp } E)$ be an arbitrary point. Then since $\omega_1$ is strictly positive on a neighbourhood of $p$, by (4.32), there exists a holomorphic normal coordinate $(V, (z_1, \cdots, z_n))$ of $(X, \omega_1)$ around $p$ and a multivalued holomorphic functions $f_D, f_N$ defining effective $\mathbb{Q}$-divisors $D, N$ respectively on $V$ such that

\[
h_1 = \left\{\prod_{i=1}^{n} (1 - |z_i|^2) + O(||z||^3) \right\} \cdot \frac{|f_D|^{2a}}{2^{-2an} |dz_1 \wedge \cdots \wedge dz_n|^{2a} \cdot |f_N|^{2a}}
\]

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holds on $V$. We may assume that $V$ is biholomorphic to the polydisk $\Delta^n(r)$ of radius $r$ with center $O$ in $\mathbb{C}^n$ for some $0 < r < 1$ via $(z_1, \ldots, z_n)$. Taking $r < 1$ sufficiently small we may assume that there exists a $C^\infty$-function $\rho$ on $X$ such that

1. $\rho$ is identically 1 on $\Delta^n(r/3)$.
2. $0 \leq \rho \leq 1$.
3. $\text{Supp } \rho \subset V$.
4. $|d\rho| < 3/r$, where $|| \cdot ||$ denotes the pointwise norm with respect to $\omega_1$.
5. There exists a local holomorphic frame $e_A$ of $A$ on $V$ such that $h_A(e_A, e_A)(p) = 1$.

We set

$$\tau_\ell := e_A \otimes \frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes \ell} \cdot f_{\sigma}^{\ell}}{f_D^{\ell}}.$$  

(4.37)

Then the $L^2$-norm $\|\rho \cdot \tau_\ell\|$ of the cut off $\rho \cdot \tau_\ell$ with respect to $h_1^\ell \cdot h_A$ and $\omega_1$ concentrates around $p$ as $\ell$ tends to infinity. More precisely, noting the equality:

$$\int_{\Delta} (1 - |t|^2)^{\ell} \, dt \, dt = \frac{2\pi}{\ell + 1}$$

(4.38)

where $\Delta := \{ t \in \mathbb{C}; |t| < 1 \}$, we see that

$$\|\rho \cdot \tau_\ell\|^2 \sim 2^{2n\ell} \left(\frac{2\pi}{\ell}\right)^n$$

(4.39)

as $\ell$ tends to infinity, where $\sim$ means that the ratio of the both sides converges to $1$ as $\ell$ tends to infinity. We set

$$\phi := n\rho \log \sum_{i=1}^n |z_i|^2.$$  

(4.40)

We may and do assume that $\ell$ is sufficiently large so that

$$(\ell - 1)\omega_1 + \sqrt{-1} \Theta h_A + \sqrt{-1} \Theta h_P + \sqrt{-1} \partial \bar{\partial} \phi > 0$$

holds on $X$. We note that $\bar{\partial}(\rho \cdot \tau_\ell)$ vanishes on the polydisc of radius $r/3$ with center $p$ as above. Then by (4.39), the $L^2$-norm

$$\|\bar{\partial}(\rho \cdot \tau_\ell)\|_\phi$$

(4.42)

of $\bar{\partial}(\rho \cdot \tau_\ell)$ with respect to $e^{-\phi} \cdot h_A \cdot h_1^{\ell-1}$ and $\omega_1$ satisfies the inequality:

$$\|\bar{\partial}(\rho \cdot \tau_\ell)\|^2 \leq C_0 \cdot \left(\frac{3}{\ell}\right)^{2n+2} 2^{2n} \left(\frac{2\pi}{\ell}\right)^n$$

(4.43)

for every $\ell$, where $C_0$ is a positive constant independent of $\ell$. By Hörmander’s $L^2$-estimate applied to the adjoint line bundle of the hermitian line bundle:

$$\{ (\ell - 1)a + (a - 1) \}(K_X + D) + D, e^{-\phi} \cdot h_A \cdot (h_P \cdot h_{\sigma^N})^{a-1} \cdot h_1^{\ell-1} \cdot h_{\sigma_D}),$$

(4.44)
we see that for every sufficiently large $\ell$, there exists a $C^\infty$-solution $u$ of the equation
\begin{equation}
\bar{\partial}u = \bar{\partial}(\rho \cdot \tau_\ell)
\end{equation}
such that
\begin{equation}
  u(p) = 0
\end{equation}
and
\begin{equation}
  \|u\|_{L^2} \leq \frac{2}{\ell} \|\bar{\partial}(\rho \cdot \tau_\ell)\|_{L^2}
\end{equation}
hold, where $\|\cdot\|$'s denote the $L^2$-norms with respect to $e^{-\phi} \cdot h_A \cdot h_1^{\ell-1} \cdot h_{\sigma_D}$ and $\omega_1$ respectively. Then $\rho \cdot \tau_\ell - u$ is a holomorphic section of $\ell a(K_X + D) + A$ such that
\begin{equation}
  (\rho \cdot \tau_\ell - u)(p) = \tau_\ell(p)
\end{equation}
and
\begin{equation}
  \|\rho \cdot \tau_\ell - u\|^2 \leq \left( 1 + C_0 \cdot \left( \frac{3}{\ell} \right)^{2n+2} \sqrt{\frac{2}{\ell}} \cdot 2^{2an\ell} \cdot \left( \frac{2\pi}{\ell} \right)^n \right).
\end{equation}
Hence by the inductive assumption \eqref{inductive_assumption_1} and \eqref{inductive_assumption_2}, this implies that there exists a positive constant $C$ independent of $\ell$ such that
\begin{equation}
  K_{\ell,1}(p) \geq \left( 1 - \frac{C}{\sqrt{\ell}} \right) \cdot \ell^n \cdot (2\pi)^{-n} \cdot C_{\ell-1} \cdot (h_A^{-1} \cdot h_1^{-\ell})(p)
\end{equation}
holds, since the point norm of $\tau_\ell$ at $p$ (with respect to $h_A \cdot h_1^{\ell}$) is asymptotically equal to $2^n$. Then by induction on $\ell$, using \eqref{inductive_assumption_3} and \eqref{inductive_assumption_4}, we see that there exists a positive constant $C'$ and a positive integer $\ell_1$ such that $C/\sqrt{\ell_1} < 1$ and for every $\ell > \ell_1$
\begin{equation}
  K_{\ell,1}(p) \geq C' \cdot \left( \prod_{k=\ell_1}^\ell \left( 1 - \frac{C}{\sqrt{k}} \right) \right) \cdot (\ell!)^n \cdot (2\pi)^{-\ell n} \cdot h_A^{-1} \cdot h_1^{-\ell}(p)
\end{equation}
holds. Since $p \in U$ is arbitrary, we have the following lower estimate.

**Lemma 4.4**
\begin{equation}
\limsup_{\ell \to \infty} \sqrt{\ell} K_{\ell,1} \geq (2\pi)^{-n} h_1^{-1}.
\end{equation}
holds on $X$. 

Next we shall estimate $K_{\ell,1}$ from above as in \eqref{inductive_assumption_5}. We set
\begin{equation}
\mu(X, K_X + D) = n! \cdot a^{-n} \limsup_{\ell \to \infty} \dim H^0(X, O_X(\ell a(K_X + D))) / \ell^n.
\end{equation}
We call $\mu(X, K_X + D)$ the **volume** of $X$ with respect to $K_X + D$. 
We set for every $\ell \geq 1$

\[(4.54)\quad dV_\ell := K_{\ell,1}^* \cdot (h_P \cdot h_{\sigma_X})^{a-1} \cdot h_{\sigma_D} \cdot h_A^* \cdot h_{\ell-1} \cdot (h_P \cdot h_{\sigma_X})^{a-1} \cdot h_{\sigma_D} \cdot h_A \]

Then $dV_\ell$ is a singular volume form on $X$. But by the construction, $\pi_*^* dV_\ell$ is a locally bounded volume form on $V$ for any $\pi : \Delta^n \to V$ as (2.13) and (2.14). By using Hölder’s inequality, we have the following lemma.

**Lemma 4.5**

\[(4.55)\quad \limsup_{\ell \to \infty} (\ell!)^{-\frac{n}{\ell}} \int_X dV_\ell \leq a^n \mu(X, K_X + D)\]

holds.

**Proof.** The following proof is essentially the same as [T9, Lemma 3.2]. By Hölder’s inequality we have that

\[(4.56)\quad \int_X dV_\ell \leq \left( \int_X \left( \frac{dV_\ell}{dV_{\ell-1}} \right)^{\ell} \right)^{\frac{1}{\ell}} \cdot \left( \int_X dV_{\ell-1} \right)^{\frac{1}{\ell-1}}\]

holds for every $m \geq 2$. By direct calculation, we see that

\[(4.57)\quad \left( \frac{dV_\ell}{dV_{\ell-1}} \right)^{\ell} dV_{\ell-1} = K_{\ell,1} \cdot h_{\ell-1} \cdot (h_P \cdot h_{\sigma_X})^{a-1} \cdot h_{\sigma_D} \cdot h_A\]

holds. Hence by the definition of the Bergman kernel $K_{\ell,1}$ and (4.57) we have that

\[(4.58)\quad \int_X \left( \frac{dV_\ell}{dV_{\ell-1}} \right)^{\ell} dV_{\ell-1} \leq \dim |\ell a(K_X + D) + A| + 1\]

holds. Combining (4.56) and (4.58), we have that

\[(4.59)\quad \int_X dV_\ell \leq \left( \dim |\ell a(K_X + D) + A| + 1 \right)^{\frac{1}{\ell}} \cdot \left( \int_X dV_{\ell-1} \right)^{\frac{1}{\ell-1}}\]

holds. Repeating the same estimate (if $m \geq 3$),

\[(4.60)\quad \int_X dV \leq (\dim |\ell a(K_X + D) + A| + 1)^{\frac{1}{\ell}} \cdot \left( \dim |(\ell - 1) a(K_X + D) + A| + 1 \right) \cdot \left( \int_X dV_{\ell-2} \right)^{\frac{1}{\ell-2}}\]

holds. Continuing this process, we obtain the inequality:

\[(4.61)\quad \int_X dV_\ell \leq \left( \prod_{k=1}^m \left( \dim |A + ka(K_X + D)| + 1 \right) \right)^{\frac{1}{\ell}}\]

Then by (4.61) and (4.53),

\[(4.62)\quad \limsup_{\ell \to \infty} (\ell!)^{-\frac{n}{\ell}} \int_X dV_\ell \leq a^n \mu(X, K_X + D)\]
holds. This completes the proof of Lemma 4.5.

By the assumption that $K_X + D$ is big, we see that $\mu(X, K_X + D)$ is positive. Moreover by the existence of the Zariski decomposition (4.17), we see that

\[(4.63) \quad \mu(X, K_X + D) = P^n\]

holds.

On the other hand, since $h_1$ is an AZD of $a(K_X + D)$, we see that the absolutely continuous part $\omega_{1,abc}$ of $\omega_1$ is cohomologous to $2a\pi c_1(P)$ and we have that

\[(4.64) \quad \int_X \omega_{1,abc}^n = (2\pi)^n a^n P^n\]

holds. Hence by (4.63) and (4.64), we have that

\[(4.65) \quad \int_X \omega_{1,abc}^n = (2\pi)^n a^n \mu(X, K_X + D)\]

holds. Then by (4.62) and (4.65), we see that

\[(4.66) \quad \limsup_{\ell \to \infty} (\ell!)^{-\frac{1}{n}} \int_X dV_\ell \leq \frac{1}{(2\pi)^n} \int_X \omega_{1,abc}^n\]

holds. By Lemma 1.4 and (4.32), we see that

\[(4.67) \quad \limsup_{\ell \to \infty} (\ell!)^{-\frac{1}{n}} dV_\ell \geq (2\pi)^{-n} h_1^{-1} \cdot (h_P \cdot h_{\sigma N})^{-(a-1)} \cdot h_{\sigma_D}^{-1} = (2\pi)^{-n} \omega_1^n\]

hold. We note that $(\ell!)^{-\frac{1}{n}} K_{1,1}^{\ell}$ is the $\ell$-th root of sum of absolute value squares of holomorphic sections and is considered to be pluri-subharmonic with respect to a local holomorphic trivialization of $A+\ell a(K_X + D)$. Then by the definition of $dV_\ell$ (cf. (4.54)) and (4.66), we see that $(\ell!)^{-\frac{1}{n}} \pi_v^* dV_\ell$ is locally uniformly bounded and semipositive volume form on $\Delta^n$ for any local cyclic branched covering $\pi_v: \Delta^n \to V$ as (2.13) and (2.14) in Section 2. Hence by Lebesgue’s bounded convergence theorem,

\[(4.68) \quad \int_X \limsup_{\ell \to \infty} (\ell!)^{-\frac{1}{n}} dV_\ell = \limsup_{\ell \to \infty} (\ell!)^{-\frac{1}{n}} \int_X dV_\ell\]

holds. Hence by (4.66) and (4.67), we have

\[(4.69) \quad \limsup_{\ell \to \infty} (\ell!)^{-\frac{1}{n}} dV_\ell = \frac{1}{(2\pi)^n} \omega_{1,abc}^n\]

hold almost everywhere on $X$. Hence by (4.69) and (4.54)

\[(4.70) \quad \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell!)^{-n} K_{1,1}} = (\omega_{1,abc}^n) \cdot (h_P \cdot h_{\sigma N})^{-(a-1)} \cdot h_{\sigma_D}^{-1} = \frac{n!}{(2\pi)^n} h_1^{-1}\]

hold almost everywhere on $X$. Then by (4.32) imply the following lemma.
Lemma 4.6

(4.71) \( K_1 := \text{the uppersemicontinuous envelope of } \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell!)}^{-n} K_{\ell,1} \)

exists and

\[ h_1 := \frac{(2\pi^n)^n}{n!} K_1^{-1} \]

is an AZD of \( a(K_X + D) \). If we set

(4.72) \( \omega_1 := \sqrt{-1} \Theta h_1 \),

then \( \omega_1 \) is a closed positive current on \( X \) and satisfies the equation:

(4.73) \[- \text{Ric}_{\omega_1} + \frac{a-1}{a} \omega_0 + 2\pi[D] = \omega_1.\]

on \( X \) and

\[ h_1 = \left( \frac{\omega_1^{1,abc}}{n!} \right)^{-1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D} \]

holds on \( X \). \( \square \)

Next replacing \( h_P, \omega_0 \) by \( h_1, \omega_1 \) respectively. We obtain the dynamical system of Bergman kernels \( \{K_{\ell,2}\}_{\ell=1}^\infty \) and

(4.74) \[ K_2 := \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell!)}^{-n} K_{\ell,2} \]

exists and \( h_2 := K_2^{-1} \) is a \( C^\infty \) hermitian metric on \( a(K_X + D) \) on \( U \) and is a singular hermitian metric on \( a(K_X + D) \). If we set

(4.75) \[ \omega_2 := \sqrt{-1} \Theta h_2, \]

then \( \omega_2 \) is a Kähler form on \( X \) and satisfies the equation:

(4.76) \[- \text{Ric}_{\omega_2} + \frac{a-1}{a} \omega_1 + 2\pi[D] = \omega_2.\]

Inductively, for every positive integer \( m \), we define the dynamical system of Bergman kernels \( \{K_{\ell,m}\}_{\ell=1}^\infty \). Continuing this process, we obtain the following theorem.

Theorem 4.7 Let \( \{K_{\ell,m}\}_{\ell=1}^\infty \) be the system of Bergman kernels defined as above. If we define

(4.77) \[ K_m := \text{the upper-semi-continuous envelope of } \limsup_{\ell \to \infty} \sqrt[\ell]{(\ell!)}^{-n} K_{\ell,m}, \]

then

(4.78) \[ h_m := \frac{n!}{(2\pi)^n} K_m^{-1} \]

is a singular hermitian metric on \( a(K_X + D) \) with strictly positive curvature on \( U \) and is an AZD of \( a(K_X + D) \). And if we define the closed positive current \( \omega_m \) on \( X \) by

(4.79) \[ \omega_m := \sqrt{-1} \Theta h_m, \]

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then \( \{ \omega_m \}_{m=1}^{\infty} \) satisfy the equations

\[
(4.80) \quad - \text{Ric}_{\omega_m} + \frac{a^{-1}}{a} \omega_{m-1} + 2\pi [D] = \omega_m
\]
on \( U \) for every \( m \geq 1 \) as (6.25). Hence by Theorem 6.2,

\[
\omega_K := \frac{1}{a} \lim_{m \to \infty} \omega_m
\]
exists in the sense of current on \( X \) and in \( C^\infty \)-topology on \( U \). And \( \omega_K \) is the unique solution of the equation:

\[
- \text{Ric}_{\omega_K} + 2\pi [D] = \omega_K,
\]
on \( U \) such that

\[
h_K := \left( \frac{1}{n!} \omega_K^{n,0} \right)^{-1} \cdot h_{\sigma_D}
\]
is an AZD of \( K_X + D \), i.e. \( \omega_K \) is the canonical Kähler-Einstein current on \((X, D)\).

Now we shall consider the uniqueness of canonical Kähler-Einstein currents on KLT pairs. To state the result we shall introduce an equivalence relation between KLT pairs.

**Definition 4.8** Let \((X_1, D_1), (X_2, D_2)\) be KLT (resp. LC) pairs. We say that \((X_i, D_i)(i = 1, 2)\) are birationally equivalent, if there exists a KLT (resp. LC) pair \((Y, D_Y)\) and compositions of blowing ups with smooth centers: \( \mu_i : Y \to X_i(i = 1, 2) \)

\[
(4.81)
\]
such that

\[
K_Y + D_Y - \mu_i^*(K_X + D_i) \quad (i = 1, 2)
\]
are effective exceptional divisors respectively. Here \( \mu_2 \circ \mu_1^{-1} \) is a birational rational map.

Using Theorem 4.7, we obtain the following uniqueness of canonical Kähler-Einstein currents on KLT pairs.

**Theorem 4.9** Let \( X \) be a smooth projective variety and let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) is KLT pair of log general type. Then canonical Kähler-Einstein current \( \omega_K \) on the KLT pair \((X, D)\) is unique.

Moreover if two such KLT pairs \((X_i, D_i)(i = 1, 2)\) are equivalent in the sense of Definition 4.8, then for any KLT pair \((Y, D_Y)\) and the morphisms \( \mu_i : Y \to X_i(i = 1, 2) \) as in Definition 4.8, we have that for the canonical Kähler-Einstein current \( \omega_K \) on \((Y, D_Y)\), \((\mu_i)_* \omega_K(i = 1, 2)\) are unique canonical Kähler-Einstein currents on \((X_i, D_i)(i = 1, 2)\) respectively.

\[\text{Here } h_{\sigma_D} \text{ appears just to identify a multivalued holomorphic section of } K_X + D \text{ with a } (n,0) \text{ form with pole along Supp } D\]
Proof. As in Section 2.1 we take a log resolution \( \mu : Y \to X \) of \((X, D)\) which satisfies the followings:

1. If we write \( K_Y = \mu^*(K_X + D) + \sum a_i E_i \), where \( \{ E_i \} \) are prime divisors, then \( a_i > -1 \) holds for every \( i \).

2. There exists a Zariski decomposition: of \( \mu^*(K_X + D) = P + N \) of \( \mu^*(K_X + D) \) as \((2.3)\).

We set \( I := \{ i | a_i < 0 \} \). Then replacing \( X \) by \( Y \) and \( D \) by \( (4.82) \)

\[
D_Y := \sum_{i \in I} (-a_i) E_i,
\]

we obtain a new KLT pair \((Y, D_Y)\) such that

(a) \( R(Y, K_Y + D_Y) \simeq R(X, K_X + D) \),

(b) There exists a Zariski decomposition: \( K_Y + D_Y = P + (N + \sum_{i \notin I} a_i E_i) \).

Suppose that Theorem 4.9 holds for \((Y, D_Y)\). Then if there exist canonical Kähler-Einstein currents \( \omega_{K,1}, \omega_{K,2} \) on \((X, D)\), we see that \( \mu^* \omega_{K,1} = \mu^* \omega_{K,2} \) holds. Hence \( \omega_{K,1} = \omega_{K,2} \) holds on \( X \).

Hence to prove Theorem 4.9, we may assume that \( \text{Supp} D \) and \( \text{Supp} N \) are divisors with normal crossings and the Zariski decomposition \( K_X + D = P + N \) exists on \( X \) from the beginning.

We set \( n := \dim X \). By the construction of \( \{ K_{\ell,m} \} \) as above, we see that

\[
K_1 := \limsup_{\ell \to \infty} \sqrt{\left( \frac{\ell!}{(\ell-n)!} \right)^n K_{\ell,1}}
\]

does not depend on \((A, h_A)\) and if we set

\[
h_1 := \text{the lower-semi-continuous envelope of} \ K_1^{-1}
\]
is an AZD of \( a(K_X + D) \). On the other hand let \( \omega'_1 \) be a solution of

\[
-\text{Ric} \omega'_1 + \frac{a-1}{a} \omega_0 + 2\pi [D] = \omega'_1,
\]
such that

\[
h'_1 := \left( \frac{1}{n!} (\omega'_{1,abc})^n \right)^{-1} \cdot (h_P \cdot h_{\sigma_N})^{a-1} \cdot h_{\sigma_D}
\]
is an AZD of \( a(K_X + D) \). Then by the proof of Theorem 4.7 above, we see that

\[
(4.83) \quad \omega_1 = \sqrt{-1} \Theta_{h_1} = \sqrt{-1} \Theta_{h'_1}
\]

holds. Hence \((4.73)\) has a unique solution. If \( a = 1 \), then we have nothing to prove. If \( a > 1 \), then by the equation \((3.26)\), we have that \( \omega_K \) is unique up to the choice of \( h_P \).

Now we shall prove that \( \omega_K \) does not depend on the choice of \( h_P \). This follows from the contraction property of the Ricci iterations. Let us recall the equation \((3.34)\). Let \( \{ u_{m,\delta} \}_{m=0}^{\infty}, \{ u'_{m,\delta} \}_{m=0}^{\infty} \) be the systems of solutions
corresponding to the metrics $h_P$ and $h'_P$ respectively. Then by (3.34), we have that
\begin{equation}
\log \left( \frac{(a\omega_{m,\delta,0} + \sqrt{-1} \partial \bar{\partial} u_{m,\delta})^n}{(a\omega_{m-1,\delta,0} + \sqrt{-1} \partial \bar{\partial} u'_{m-1,\delta})^n} \right) = \left( u_{m,\delta} - u'_{m-1,\delta} \right) - \frac{1}{a} \left( u_{m-1,\delta} - u'_{m-2,\delta} \right)
\end{equation}
hold for $m \geq 1$. Let $\epsilon$ be a sufficiently small positive number and we define $u_{m,\delta}(\epsilon), u'_{m,\delta}(\epsilon)$ as (3.46). Then repeating the same argument as in Section 3.2 (cf. (3.48)) by the maximum principle, we see that there exists a positive constants $C(-\cdot)$ independent of $m$ and $\delta$ such that
\begin{equation}
\lim_{m \rightarrow \infty} \left( u_{m,\delta}(\epsilon) - u'_{m-1,\delta}(\epsilon) \right) = 0
\end{equation}
holds, where $\alpha_m$ is as (3.39). hold for every $m \geq 1$. Switching $u_{m,\delta}(\epsilon)$ and $u'_{m,\delta}(\epsilon)$, we see that there exists a positive constant $C_+$ independent of $m$ and $\delta$ such that
\begin{equation}
\lim_{m \rightarrow \infty} \left( u_{m-1,\delta}(\epsilon) - u'_{m,\delta}(\epsilon) + \alpha_m(\delta + \epsilon) \log \| \sigma_E \|_{h_E}^2 \right) \leq C_+ \left( \frac{1}{a} \right)^{m-1}
\end{equation}
holds. In Section 3.2, we have already observed that \{ $u_{m,\delta}$\}$_{m=0}^{\infty}$ and \{ $u'_{m,\delta}$\}$_{m=0}^{\infty}$ converge. Hence by the estimates (4.85) and (4.86)
\begin{equation}
\lim_{m \rightarrow \infty} \left( u_{m,\delta} - u'_{m,\delta} \right) = 0
\end{equation}
holds on $U$. Now we note that the estimates (4.85) and (4.86) are independent of $\delta$. Then letting $\delta$ tend to 0, by (3.32) we have that $\omega_K = \lim_{\delta \rightarrow 0} \omega_m$ is independent of the choice of $h_P$. This completes the proof of Theorem 4.9. 

5 Variation of canonical measures

In this section we shall prove Theorems 1.8, 1.9, 1.10 and 1.14. In Section 4, we have decomposed the Ricci iteration in Section 3 into iterations of the dynamical systems of Bergman kernels. Using this decomposition, by the logarithmic plurisubharmonicity of Bergman kernels we shall prove Theorem 1.8. The strategy of the proof is similar to the proof of the logarithmic plurisubharmonicity of the relative canonical measure in [T9]. But here we need to use the double induction.

Let $f : X \rightarrow Y$ be an algebraic fiber space and let let $D$ be an effective $\mathbb{Q}$-divisor on $X$ and such that $K_{X/Y} + D$ is $f$-big. We set
\begin{equation}
Y_0 := \{ y \in Y | f \text{ is smooth over } y \text{ and } (X_y, D_y) \text{ is a KLT pair} \}.
\end{equation}
Let $n = \dim X - \dim Y$ be the relative dimension of $f$. Let $d\mu_{can,(X,D)/Y}$ be the relative canonical measure $(X,D)$, i.e., for every $y \in Y_0$,
\begin{equation}
d\mu_{can,(X,D)/Y}|X_y := \frac{1}{n!} (\omega_{K,y})^n_{abc},
\end{equation}
where $\omega_{K,y}$ is the canonical Kähler-Einstein current on $(X_y, D_y)$. Let $(W, (t_1, \cdots, t_s))$ be a local coordinate on $Y$ which is biholomorphic to the unit open polydisk
Let $\ell_0$ be a sufficiently large positive integer and $f, O_X(\ell_0((K_X/Y + D))|W$ is generated by global sections $\{\tau_1, \cdots, \tau_M\}$ on $W$. We set

$$h^*_p := \left(\sum_{i=1}^{M} |\tau_i|^2\right)^{-\frac{1}{2}}.$$  

Then $h^*_p$ is a singular hermitian metric on $K_{X/Y} + D|f^{-1}(W)$ with semipositive curvature current. We note that $h^*_p$ corresponds $h_p \cdot h_N$ in (4.2). We set

$$W_0 := \{y \in W| h^*_p \text{ is globally generated whenever } f, h$$

Taking $\ell_0$ sufficiently large by the finite generation of relative log canonical rings ([B-C-H-M]), we may assume that $W_0$ is a nonempty Zariski open subset of $W$. Let $A$ be a sufficiently ample line bundle on $X$ such that for any pseudo-effective singular hermitian line bundle $(F, h_F)$ on $X$, $O_X(A + F) \otimes I(h_F)$ is globally generated on $X$ and also for every $y \in Y_0$ and $O_{X_y}(A + F|X_y) \otimes I(h_F|X_y)$ is globally generated whenever $h_F|X_y$ is well defined. Again such $A$ exists by Nadel’s vanishing theorem ([N, p.561]). Then for every $y \in W_0$, we consider the family of dynamical systems of Bergman kernels as in Section 4. Let us fix an arbitrary point $y \in W_0$. We set

$$K_{1, y} := K(X_y, A + a(K_X + D_y), h_A \cdot (h^*_p)^{a-1} : h_{\sigma_D}|X_y)$$

and define a singular hermitian metric $h_{1, y}$ on $A + a(K_X + D_y)$ by

$$h_{1, y} := \frac{1}{K_{1, y}}.$$  

Suppose that we have already defined $K_{\ell, y}$ and $h_{1, y}$. Then we define $K_{\ell, y}$ and the singular hermitian metric $h_{\ell, y}$ on $A + \ell a(K_X + D)$ by

$$K_{\ell, y} := K(X, A + \ell a(K_X + D_y), h_{\ell-1, y} \cdot (h^*_p)^{a-1} : (h_{\sigma_D})|X_y)$$

and define the singular hermitian metric on $A + \ell a(K_X + L)$ by

$$h_{\ell, y} := \frac{1}{K_{\ell, y}}.$$  

As in the proof of Theorem 4.7 we construct families of dynamical system of Bergman kernels $\{K_{\ell, y}\}_{\ell=1}^{\infty}$ for every $y \in W_0$ and $m \geq 1$. And by Theorem 4.7 we see that if we define

$$K_{m, y} := \text{the upper-semi-continuous envelope of } \limsup_{\ell \to \infty} \sqrt{\ell!}^{-n} K_{\ell, m, y},$$

then

$$h_{m, y} := K_{m, y}^{-1}$$

is a singular hermitian metric on $a(K_X + D)$ with semipositive curvature in the sense of current. And if we define the closed positive current $\omega_{m, y}$ on $X_y$ by

$$\omega_{m, y} := \sqrt{-1} \Theta_{h_{m, y}},$$
then \( \{ \omega_{m,y} \}_{m=1}^\infty \) satisfy the successive equations:

\[
(5.11) \quad - \text{Ric}_{\omega_{m,y}} + \frac{a - 1}{a} \omega_{m-1,y} + 2\pi [D] |X_y = \omega_{m,y}
\]
on a nonempty Zariski open subset \( U_y \) of \( X_y \) defined as \([3.24]\) for \( m \geq 1 \) by Theorem 3.2.

\[
\omega_{K,y} := \frac{1}{a} \lim_{m \to \infty} \omega_{m,y}
\]
e exists in \( C^\infty \)-topology compact uniformly on \( U_y \) and also in the sense of current on \( X_y \). Then \( \omega_{K,y} \) is the unique solution of the equation:

\[
- \text{Ric}_{\omega_{K,y}} + 2\pi [D] |X_y = \omega_{K,y}
\]
such that \( n! ((\omega_{K,y})^{abc})^{-1} \) is an AZD of \( K X_y + D_y \). i.e., \( \omega_{K,y} \) is the canonical Kähler-Einstein current on \((X_y, D_y)\).

Now we quote the following theorem mainly due to B. Berndtsson.

**Theorem 5.1** \([\text{B2, B3, B-P, T7}]\) Let \( f : X \to S \) be a projective family of projective varieties over a complex manifold \( S \). Let \( S^0 \) be the maximal nonempty Zariski open subset such that \( f \) is smooth over \( S^0 \).

Let \( (L, h_L) \) be a pseudo-effective singular hermitian line bundle on \( X \).

Let \( K_s := K(X_s, K_X + L|_{X_s}, h|_{X_s}) \) be the Bergman kernel of \( K_{X_s} + (L|_s) \) with respect to \( h|_s \) for \( s \in S^0 \). Then the singular hermitian metric \( h \) of \( K_{X/S} + L| f^{-1}(S^0) \) defined by

\[
h|_s := K_{s}^{-1}(s \in S^0)
\]
has semipositive curvature on \( f^{-1}(S^0) \) and extends on \( X \) as a singular hermitian metric on \( K_{X/S} + L \) with semipositive curvature in the sense current. \( \square \)

Now we define the family of Bergman kernels by

\[
K_{\ell,m}|X_y = K_{\ell,m,y}
\]
over \( W_0 \) and set

\[
K^*_{\ell,m} := \text{the upper-semi-continuous envelope of } K_{\ell,m}.
\]

And we define the singular hermitian metric \( h_{\ell,m} \) on \( A + \ell a(K_{X/Y} + D)|f^{-1}(W_0) \) by

\[
h_{\ell,m} := (K^*_{\ell,m})^{-1}.
\]

Now by Theorem 5.1 and \([5.12]\), we see that \( h_{1,1} \) has semipositive curvature in the sense of current on \( f^{-1}(W_0) \) and extends to a singular hermitian metric of semipositive curvature current on \( f^{-1}(W) \). Then by the induction on \( \ell \), using Theorem 5.1 and \([5.6]\), we see that \( h_{\ell,1} \) has has semipositive curvature in the sense of current for every \( \ell \) and extends to \( f^{-1}(W) \) as a singular hermitian metric on \( A + \ell a(K_{X/Y} + D)|f^{-1}(W) \) with semipositive curvature in the sense of current. We define \( K_1 \) on \( f^{-1}(W_0) \) by

\[
K_1|X_y = K_{1,y} \quad (y \in W_0)
\]

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and set
\[(5.16) \quad K^*_1 := \text{the upper-semi-continuous envelope of } K_1.\]

Then by Theorems 4.7 and 5.1 we see that the singular hermitian metric:
\[(5.17) \quad h^*_1 := (K^*_1)^{-1}\]
on a(K_{X/Y} + D)|f^{-1}(W_0) extends to a singular hermitian metric $h_1$ on $a(K_{X/Y} + D)|f^{-1}(W)$ and the extended metric has semipositive curvature in the sense of current. We denote the extended metric again by $h_1$.

Now we begin the second induction. Repeating the same argument replacing $h^*_P$ by $h_1$, we define the singular hermitian metrics $\{h_{\ell,2}\}_{\ell=1}^\infty$ and $h_2$ similarly as above and conclude that all of these metrics have semipositive curvature in the sense of current on $f^{-1}(W)$. Thus inductively, we obtain the sequences of singular hermitian metrics $\{h_{\ell,m}\}_{\ell=1}^\infty$ on $A + \ell a(K_{X/Y} + D)|f^{-1}(W)$ and $\{h_m\}_{m=1}^\infty$ on $a(K_{X/Y} + D)|f^{-1}(W)$. And all these metrics have semipositive curvature in the sense of current. Hence by Theorem 4.7 we see that
\[(5.18) \quad h_K = \text{the lower-semi-continuous envelope of } \liminf_{m \to \infty} h_m\]
is a well defined singular hermitian metric on $a(K_{X/Y} + D)|f^{-1}(W)$ and has semipositive curvature in the sense of current. The relative canonical measure $d\mu_{\text{can.}(X,D)/Y}$ of $(X,D)$ with respect to $f : X \to Y$ is related to $h_K$ as
\[(5.19) \quad d\mu_{\text{can.}(X,D)/Y} = h_K^{-1} \cdot h_\sigma D\]
almost everywhere on $f^{-1}(W)$. Again by Theorem 4.7 and Theorem 4.9 we see that $h_K$ does not depend on the choice of $W$ and the metric $h^*_P$ (cf. (5.2)) above. Hence we conclude that the logarithm of the relative canonical measure $d\mu_{\text{can.}(X,D)/Y}$ is pluri-subharmonic on $X$, more precisely $h_K$ is naturally defined on $X$ as a singular hermitian metric on $a(K_{X/Y} + D)$ and has semipositive curvature in the sense of current. This completes the proof of Theorem 1.8.

The proof of Theorem 1.9 is similar. The only difference is that since the Hodge metric is singular, we need to consider the smoothing as in (1.9).

The proof of Theorem 1.10 is similar.

Theorem 1.14 is obtained by Theorem 1.8 by passing to the limit by using Theorem 2.5 \( \Box \)

6 Weak semistability of relative log canonical bundles

In this section, we shall prove the weak semistability of relative pluri-log-canonical bundles as an application of the logarithmic pluri-subharmonic variation property of canonical measures for KLT pairs. The proof depends heavily on Viehweg’s idea. But we use the logarithmic pluri-subharmonicity of canonical measures (Theorem 6.5) instead of variation of Hodge structures and the branched covering trick.
6.1 Weak semistability

First we shall recall several definitions in [V]. To measure the positivity of coherent sheaves, we shall introduce the following notion.

**Definition 6.1** Let \( Y \) be a quasi-projective reduced scheme, \( Y_0 \subseteq Y \) an open dense subscheme and let \( \mathcal{G} \) be locally free sheaf on \( Y \), of finite constant rank. Then \( \mathcal{G} \) is **weakly positive** over \( Y_0 \), if for an ample invertible sheaf \( \mathcal{H} \) on \( Y \) and for a given number \( \alpha > 0 \) there exists some \( \beta > 0 \) such that \( S^{\alpha \beta}(\mathcal{G}) \otimes \mathcal{H}^\beta \) is globally generated over \( Y_0 \).

The notion of weak positivity is a natural generalization of the notion of nefness of line bundles. Roughly speaking, the weak semipositivity of \( \mathcal{G} \) over \( Y_0 \) means that \( \mathcal{G} \otimes \mathcal{H}^\varepsilon \) is \( \mathbb{Q} \)-globally generated over \( Y_0 \) for every \( \varepsilon > 0 \).

**Definition 6.2** Let \( \mathcal{F} \) be a locally free sheaf and let \( \mathcal{A} \) be an invertible sheaf, both on a quasi-projective reduced scheme \( Y \). We denote

\[
\mathcal{F} \geq \frac{b}{a} \mathcal{A},
\]

if \( S^a(\mathcal{F}) \otimes \mathcal{A}^{-b} \) is weakly positive over \( Y \), where \( a, b \) are positive integers.

Let \( X \) be a normal variety. We define the canonical sheaf \( \omega_X \) of \( X \) by

\[
\omega_X := i_* \mathcal{O}_{X_{\text{reg}}}(K_{X_{\text{reg}}}),
\]

where \( i : X_{\text{reg}} \hookrightarrow X \) is the canonical injection. The following notion introduced by Viehweg is closely related to the notion of logcanonical thresholds.

**Definition 6.3** Let \( (X, \Gamma) \) be a pair of normal variety \( X \) and an effective Cartier divisor \( \Gamma \). Let \( \pi : X' \to X \) be a log resolution of \( (X, \Gamma) \) and let \( \Gamma' := \pi^* \Gamma \). For a positive integer \( N \) we define

\[
\omega_X \left\{ \frac{-\Gamma}{N} \right\} = \pi_* \left( \omega_{X'} \left( -\frac{\Gamma'}{N} \right) \right)
\]

and

\[
\mathcal{C}_X(\Gamma, N) = \text{Coker} \left\{ \omega_X \left\{ \frac{-\Gamma}{N} \right\} \to \omega_X \right\}.
\]

If \( X \) has at most rational singularities, one defines:

\[
e(\Gamma) = \min \{ N > 0 \mid \mathcal{C}_X(\Gamma, N) = 0 \}.
\]

If \( \mathcal{L} \) is an invertible sheaf, \( X \) is proper with at most rational singularities and \( H^0(X, \mathcal{L}) \neq 0 \), then one defines

\[
e(\mathcal{L}) = \sup \{ e(\Gamma) \mid \Gamma : \text{effective Cartier divisor with } \mathcal{O}_X(\Gamma) \simeq \mathcal{L} \}.
\]

Now we state the result of E. Viehweg.
Theorem 6.4 ([(1), p.191, Theorem 6.22]) Let \( f : X \to Y \) be a flat surjective projective Gorenstein morphism of reduced connected quasi-projective schemes. Assume that the relative canonical sheaf \( \omega_{X/Y} : \omega_X \otimes f^* \omega_Y^{-1} \) is \( f \)-semi-ample and that the fibers \( X_y = f^{-1}(y) \) are reduced normal varieties with at most rational singularities. Then one has:

1. **Functoriality**: For \( m > 0 \) the sheaf \( f^* \omega^m_{X/Y} \) is locally free of rank \( r(m) \) and it commutes with arbitrary base change.

2. **Weak semipositivity**: For \( m > 0 \) the sheaf \( f^* \omega^m_{X/Y} \) is weakly positive over \( Y \).

3. **Weak semistability**: Let \( m > 1, e > 0 \) and \( \nu > 0 \) be chosen so that \( f^* \omega^m_{X/Y} \neq 0 \) and

\[
(6.7) \quad e \geq \sup \left\{ \frac{k}{m-1}, e(\omega^k_{X_y}) : y \in Y \right\}
\]

hold. Then

\[
(6.8) \quad f^* \omega^m_{X/Y} \succeq \frac{1}{e \cdot r(k)} \det(f^* \omega^k_{X/Y})
\]

holds. \( \square \)

The weak semistability of \( f^* \omega^m_{X/Y} \) is very important in the application of the weak semipositivity. For example, if \( \dim Y = 1 \) and \( \deg \det(f^* \omega^m_{X/Y}) > 0 \) hold, then for a sufficiently large \( r \) \( S^r(f^* \omega^m_{X/Y}) \) is globally generated. Moreover by the finite generation of canonical rings ([B-C-H-M]), for every sufficiently large \( m \), \( f^* \omega^m_{X/Y} \) is globally generated on \( Y \).

### 6.2 Weak semistability of the direct image of the relative log canonical bundle of a family of KLT pairs

Now we state our generalization of Theorem 6.4.

**Theorem 6.5** Let \( f : X \to Y \) be an algebraic fiber space and let \( D \) be an effective \( \mathbb{Q} \) divisor on \( X \) such that \((X, D)\) is KLT. Let \( Y^\circ \) denote the complement of the discriminant locus of \( f \). We set

\[
(6.9) \quad Y_0 := \{ y \in Y | y \in Y^\circ, (X_y, D_y) \text{ is a KLT pair} \}
\]

1. **Weak semistability 1**: Let \( r \) denote rank \( f_* \mathcal{O}_X(\lfloor m(K_{X/Y} + D) \rfloor) \). Let \( X^r := X \times_Y X \times_Y \cdots \times_Y X \) be the \( r \)-times fiber product over \( Y \) and let \( f^r : X^r \to Y \) be the natural morphism. And let \( D^r \) denote the divisor on \( X^r \) defined by \( D^r = \sum_{i=1}^r \pi_i^* D \), where \( \pi_i : X^r \to X \) denotes the projection: \( X^r \ni (x_1, \cdots, x_n) \mapsto x_i \in X \).

There exists a canonically defined effective divisor \( \Gamma \) (depending on \( m \)) on \( X^r \) which does not contain any fiber \( X^r_y(y \in Y^\circ) \) such that if we define the number \( \varepsilon_0 \) by

\[
(6.10) \quad \varepsilon_0 := \sup \{ \varepsilon | (X^r_y, D^r + \varepsilon \Gamma_y) \text{ is KLT for all } y \in Y^\circ \},
\]
then for every $0 < \varepsilon < \varepsilon_0$, there exists a singular hermitian metric $h_{m,\varepsilon}$ on
\[
(6.11) \quad \left(\frac{1}{m} + \varepsilon\right) r(|m(K_{X/Y} + D)|) - \varepsilon \cdot f^* \det f_* O_X(|m(K_{X/Y} + D)|)^{**}
\]
such that
(a) $\sqrt{-1} \Theta_{h_{m,\varepsilon}} \geq 0$ holds on $X$ in the sense of current.
(b) For every $y \in Y_0$, $h_{m,\varepsilon}|_{X_y}$ is well defined and is an AZD of
\[
\left(\frac{1}{m} + \varepsilon\right) r(|m(K_{X/Y} + D)|) - \varepsilon \cdot f^* \det f_* O_X(|m(K_{X/Y} + D)|)^{**}|_{X_y}.
\]

(2) **Weak semistability 2:** In addition, suppose that $K_{X/Y} + D$ is $\mathbb{Q}$-linear equivalent to a Cartier divisor $G$ on $X$. Then for every $m \geq 1$ and a rational number $0 < \varepsilon < \varepsilon_0$, there exists an ample line bunel $A$ on $Y$ such that for every sufficiently large positive integer $\ell$ with $\ell \varepsilon \in \mathbb{Z} > 0$,
\[
f_* O_X(\ell mG) \otimes (\det f_* O_X(mG))^{-\ell \varepsilon} \otimes O_Y(A)
\]
is globally generated over $Y_0$. And for every sufficiently large $\ell$
\[
(6.12) \quad f_* O_X(\ell mG) \geq \frac{\ell m \varepsilon}{1 + (1 + m \varepsilon) \ell r} \det f_* O_X(mG)^{\otimes \ell}
\]
holds over $Y_0$.

The main advantage of Theorem 6.5 to Theorem 6.4 is that we do not assume the $f$-semiampleness of the relative log canonical bundles.

### 6.3 Proof of Theorem 6.5

Let us prove Theorem 6.5. The proof follows closely the one of Theorem in [V]. But we replace the use of branched coverings in [V] by the use of Theorem 1.14. This enables us to get rid of the assumption that $K_{X/Y}$ is $f$-semiampl.

Let us start the proof. Let $f : X \to Y$ be an algebraic fiber space.

Let $Y_0$ be the Zariski open subset of $Y$ defined as (6.9) above. For $r = \text{rank } f_* O_X(|m(K_{X/Y} + D)|)$ we set $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the $r$-times fiber product over $Y$ and let $f^r : X^r \to Y$ be the natural morphism. Then we have the natural morphism
\[
(6.13) \quad \det f_* O_X(|m(K_{X/Y} + D)|) \to \otimes^r f_* O_X(|m(K_{X/Y} + D)|) = f_*^r O_{X^r}(|m(K_{X^r/Y} + D^r)|).
\]

Hence we have the canonical global section:
\[
(6.14) \quad \gamma \in H^0\left(X^r, f^{**}(\det f_* O_X(|m(K_{X/Y} + D)|))^{-1} \otimes O_{X^r}(|m(K_{X^r/Y} + D^r)|)\right).
\]

Let $\Gamma$ denote the zero divisor of $\gamma$. It is clear the $\Gamma$ does not contain any fiber over $Y_0$. Now we define $\delta_0$ as (6.10). Let us take a positive rational number $\varepsilon < \delta_0$. We set
\[
(6.15) \quad \Theta := \frac{1}{m}[mD^r] + \varepsilon \Gamma.
\]

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Then there exists the relative canonical measure $d\mu_{can,(X^r,\Theta)}$ on $f : (X^r,\Theta) \to Y$ as in Theorems [1.7] and [1.14]. By the logarithmic pluri-subharmonicity of the relative canonical measure (Theorem [1.14]), we see that

$$(6.16) \quad \sqrt{-1}\partial\bar{\partial}\log d\mu_{can,(X^r,\Theta)/Y} \geq 0$$

holds on $X$ in the sense of current. We set

$$(6.17) \quad H_{m,\epsilon} := d\mu_{can,(X^r,\Theta)/Y}^{-1}$$

Then $H_{m,\epsilon}$ is a singular hermitian metric on

$$(6.18) \quad \left(\frac{1}{m} + \epsilon\right) \left(\lfloor m(K_{X^r/Y} + D^r)\rfloor \right) - \epsilon \cdot f^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)$$

with semipositive curvature current by Theorem [1.14] and $H_{m,\epsilon}|_{X_y^r}$ is an AZD of

$$(6.19) \quad \left(\frac{1}{m} + \epsilon\right) \left(\lfloor m(K_{X^r/Y} + D^r)\rfloor \right) \lfloor X^r - \epsilon \cdot f^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)\rfloor |_{X_y^r}$$

for every $y \in Y_0$. Let $h_{m,\epsilon}$ be the restriction of $H_{m,\epsilon}$ to the diagonal $\Delta(X^r)$ of $X^r$. Then since the restriction of (6.18) to $\Delta(X^r) \subset X$ is isomorphic to

$$(6.20) \quad L := \left(\frac{1}{m} + \epsilon\right) r(\lfloor m(K_{X/Y} + D)\rfloor) - \epsilon \cdot f^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor),$$

this implies that $h_{m,\epsilon}$ is considered to be a singular hermitian metric on $L$ with semipositive curvature in the sense of current and for every $y \in Y_0$, $h_{m,\epsilon}|_{X_y}$ is an AZD of the restriction of $L|_{X_y}$.

Now we shall prove the assertion (2) in Theorem [6.5]. Let $G$ be a Cartier divisor on $X$ which is $\mathbb{Q}$-linear equivalent to $K_{X/Y} + D$. For a positive rational number $t$ we consider

$$(6.21) \quad \Theta_t := (1 + t)D^r + \epsilon r$$

instead of $\Theta$ in [6.15] above, where we have considered $D$ (resp. $D^r$) with the Cartier divisor $G - K_{X/Y}$ (resp. $G^r - K_{X^r/Y}$). Then we obtain the $\mathbb{Q}$-line bundle:

$$(6.22) \quad L(t) := rK_{X/Y} + (1 + t)rD + \epsilon rmg - \epsilon \cdot f^*\det f_*\mathcal{O}_X(mG)$$

and a singular hermitian metric $h_{m,\epsilon}(t)$ on $L(t)$ with semipositive curvature in the sense of current and for every $y \in Y_0$, $h_{m,\epsilon}(t)|_{X_y}$ is an AZD of $L(t)|_{X_y}$. Let $\ell$ be a positive integer such that $\ell \epsilon$ is a positive integer. And we set $t = 1/\ell r$. We note that if we have taken $\ell$ sufficiently large, then $t$ is enough small so that $(X^r,\Theta_t)$ is still a KLT pair. Then

$$(6.23) \quad K_{X/Y} + \ell L(t) = ((1 + \ell r) + \ell \epsilon m)G - \ell \epsilon \cdot f^*\det f_*\mathcal{O}_X(mG)$$

holds. Let us fix a $C^\infty$ volume form $dV_Y$ on $Y$.

Let $(A,h_A)$ be a sufficiently ample hermitian line bundle on $Y$ such that for every point $y \in Y$, the followings hold:
(1) There exists a local coordinate \((U, t)\) with biholomorphic to an open unit polydisk \(\Delta^n\) in \(\mathbb{C}^n\) with center \(O\) and \(t(y) = 0\),

(2) For \(y \in Y\) and \((U, t)\), there exists a \(\rho \in C^\infty(Y)\) such that \(\text{Supp} \rho \subset \subset t^{-1}(\Delta^n)\) and \(\rho \equiv 1\) on a neighbourhood of \(y\).

(3) \(\text{Ric } dv_Y + (2 \text{dim } Y) \sqrt{-1} \partial \bar{\partial} (\rho \log |t|) + \sqrt{-1} \Theta_A\) dominates a Kähler form on \(Y\), where \(|t| = \sqrt{\sum_{j=1}^k |t_j|^2}, t = (t_1, \cdots, t_k)(k = \text{dim } Y)\).

Such a hermitian line bundle \((A, h_A)\) certainly exists. Then since \(h_{m, \varepsilon}(t)\) has semipositive curvature in the sense of current, by the choice of \((A, h_A)\) and the \(L^2\)-extension theorem \([O, O-T]\), we see that for every \(y \in Y_0\) and any element of

\[H^0(X_y, \mathcal{O}_{X_y}((1 + (1 + m\varepsilon)\ell r) G) \otimes \mathcal{I}(h_{m, \varepsilon}(t)^f |X_y))\]

extends to an element of

\[H^0(X, \mathcal{O}_X((1 + (1 + m\varepsilon)\ell r) G + f^* A - \ell \varepsilon f^* \det f_* \mathcal{O}_X(mG)) \otimes \mathcal{I}(h_{m, \varepsilon}(t)^f).\]

We note that \(L(t)|X_y - (1 + m\varepsilon)\ell r) G|X_y = D|X_y\) is effective. Then since \(h_{m, \varepsilon}(t)|X_y\) is an AZD of \(L(t)|X_y\),

\[(6.24) \quad H^0(X_y, \mathcal{O}_{X_y}((1+(1+m\varepsilon)\ell r) G)) = H^0(X_y, \mathcal{O}_{X_y}((1+(1+m\varepsilon)\ell r) G) \otimes \mathcal{I}(h_{m, \varepsilon}(t)^f |X_y))\]

holds. Hence we see that for every sufficiently large \(\ell\) with \(\ell \varepsilon \in \mathbb{Z}_{>0}\),

\[f_* \mathcal{O}(\ell mG) \otimes (\det f_* \mathcal{O}_X(mG))^{-\ell \varepsilon} \otimes \mathcal{O}_Y(A)\]

is globally generated over \(Y_0\). Then \((6.12)\) follows from the finite generation of the relative log canonical bundles \([B-C-H-M]\). This completes the proof of Theorem 6.5.

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