MIXED RESOLUTIONS AND SIMPLICIAL SECTIONS

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Abstract. We introduce the notions of mixed resolutions and simplicial sections, and prove a theorem relating them. This result is used (in another paper) to study deformation quantization in algebraic geometry.

0. Introduction

Let $K$ be a field of characteristic 0. In this paper we present several technical results about the geometry of $K$-schemes. These results were discovered in the course of work on deformation quantization in algebraic geometry, and they play a crucial role in [Ye3]. This role will be explained at the end of the introduction. The idea behind the constructions in this paper can be traced back to old work of Bott [Bo, HY].

Let $\pi: Z \to X$ be a morphism of $K$-schemes, and let $U = \{ U_{(0)}, \ldots, U_{(m)} \}$ be an open covering of $X$. A simplicial section $\sigma$ of $\pi$, based on the covering $U$, consists of a family of morphisms $\sigma_i: \Delta^q_K \times U_i \to Z$, where $i = (i_0, \ldots, i_q)$ is a multi-index; $\Delta^q_K$ is the $q$-dimensional geometric simplex; and $U_i := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$. The morphisms $\sigma_i$ are required to be compatible with $\pi$ and to satisfy simplicial relations. See Definition 5.1 for details. An important example of a simplicial section is mentioned at the end of the introduction.

Another notion we introduce is that of mixed resolution. Here we assume the $K$-scheme $X$ is smooth and separated, and each of the open sets $U_{(i)}$ in the covering $U$ is affine. Given a quasi-coherent $O_X$-module $M$ we define its mixed resolution $\text{Mix}_U(M)$. This is a complex of sheaves on $X$, concentrated in non-negative degrees. As the name suggests, this resolution mixes two distinct types of resolutions: a de Rham type resolution which is related to the sheaf $P_X$ of principal parts of $X$ and its Grothendieck connection, and a simplicial-Cech type resolution which is related to the covering $U$. The precise definition is too complicated to state here – see Section 4.

Let $C^+(\text{QCoh} O_X)$ denote the abelian category of bounded below complexes of quasi-coherent $O_X$-modules. For any $M \in C^+(\text{QCoh} O_X)$ the mixed resolution $\text{Mix}_U(M)$ is defined by totalizing the double complex $\bigoplus_{p,q} \text{Mix}^q_U(M^p)$. The derived category of $K$-modules is denoted by $D(\text{Mod} K)$.

Theorem 0.1. Let $X$ be a smooth separated $K$-scheme, and let $U = \{ U_{(0)}, \ldots, U_{(m)} \}$ be an affine open covering of $X$.

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(1) There is a functorial quasi-isomorphism \( \mathcal{M} \to \text{Mix}_U(\mathcal{M}) \) for \( \mathcal{M} \in \text{C}^+(\text{QCoh}\mathcal{O}_X) \).

(2) Given \( \mathcal{M} \in \text{C}^+(\text{QCoh}\mathcal{O}_X) \), the canonical morphism \( \Gamma(X, \text{Mix}_U(\mathcal{M})) \to \text{R}\Gamma(X, \text{Mix}_U(\mathcal{M})) \) in \( \text{D}(\text{Mod} K) \) is an isomorphism.

(3) The quasi-isomorphism in part (1) induces a functorial isomorphism \( \Gamma(X, \text{Mix}_U(\mathcal{M})) \cong \text{R}\Gamma(X, \mathcal{M}) \) in \( \text{D}(\text{Mod} K) \).

This is repeated as Theorem 4.15 in the body of the paper. Note that part (3) is a formal consequence of parts (1) and (2).

A useful corollary of the theorem is the following (see Corollary 4.16). Suppose \( \mathcal{M} \) and \( \mathcal{N} \) are two complexes in \( \text{C}^+(\text{QCoh}\mathcal{O}_X) \), and \( \phi : \text{Mix}_U(\mathcal{M}) \to \text{Mix}_U(\mathcal{N}) \) is a \( K \)-linear quasi-isomorphism. Then

\[
\Gamma(X, \phi) : \Gamma(X, \text{Mix}_U(\mathcal{M})) \to \Gamma(X, \text{Mix}_U(\mathcal{N}))
\]

is a quasi-isomorphism.

Here is the connection between simplicial sections and mixed resolutions.

**Theorem 0.2.** Let \( X \) be a smooth separated \( K \)-scheme, let \( \pi : Z \to X \) be a morphism of schemes, and let \( U \) be an affine open covering of \( X \). Suppose \( \sigma \) is a simplicial section of \( \pi \) based on \( U \). Let \( \mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N} \) be quasi-coherent \( \mathcal{O}_X \)-modules, and let

\[
\phi : \prod_{i=1}^r \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})
\]

be a continuous \( \mathcal{O}_Z \)-multilinear sheaf morphism on \( Z \). Then there is an induced \( K \)-multilinear sheaf morphism

\[
\sigma^*(\phi) : \prod_{i=1}^r \text{Mix}_U(\mathcal{M}_i) \to \text{Mix}_U(\mathcal{N})
\]

on \( X \).

In the theorem, the continuity and the complete pullback \( \pi^* \) refer to the dir-inv structures on these sheaves, which are explained in Section 1. A more detailed statement is Theorem 4.22 in the body of the paper.

Let us explain, in vague terms, how Theorem 0.2 or rather Theorem 5.2 is used in the paper [Ye3]. Let \( X \) be a smooth separated \( n \)-dimensional \( K \)-scheme. As we know from the work of Kontsevich [Ko], there are two important sheaves of DG Lie algebras on \( X \), namely the sheaf \( \mathcal{T}_{\text{poly}, X} \) of poly derivations, and the sheaf \( \mathcal{D}_{\text{poly}, X} \) of poly differential operators. Suppose \( U \) is some affine open covering of \( X \). The inclusions \( \mathcal{T}_{\text{poly}, X} \to \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) \) and \( \mathcal{D}_{\text{poly}, X} \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X}) \) are then quasi-isomorphisms of sheaves of DG Lie algebras (cf. Theorem 4.1). The goal is to find an \( L_\infty \) quasi-isomorphism

\[
\Psi : \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X})
\]

between these sheaves of DG Lie algebras. Having such an \( L_\infty \) quasi-isomorphism pretty much implies the solution of the deformation quantization problem for \( X \).

Let \( \text{Coor} X \) denote the coordinate bundle of \( X \). This is an infinite dimensional bundle over \( X \), endowed with an action of the group \( \text{GL}_n(K) \). Let \( \text{LCC} X \) be the quotient bundle \( \text{Coor} X / \text{GL}_n(K) \). In [Ye4] we proved that if the covering \( U \) is fine
enough (the condition is that each open set \( U_{i,j} \) admits an étale morphism to \( \mathbb{A}^n_k \)), then the projection \( \pi : \text{LCC} X \to X \) admits a simplicial section \( \sigma \).

Now the universal deformation formula of Kontsevich [Ko] gives rise to a continuous \( L_\infty \) quasi-isomorphism
\[
\mathcal{U} : \hat{\pi}^\ast (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \to \hat{\pi}^\ast (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X})
\]
on \text{LCC} X. This means that there is a sequence of continuous \( \mathcal{O}_{\text{LCC} X} \)-multilinear sheaf morphisms
\[
\mathcal{U}_r : \prod^r \hat{\pi}^\ast (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \to \hat{\pi}^\ast (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}),
\]
r \geq 1, satisfying very complicated identities. Using Theorem 5.2 we obtain a sequence of multilinear sheaf morphisms
\[
\sigma^\ast (\mathcal{U}_r) : \prod^r \text{Mix}_{\mathcal{U}_r} (\mathcal{T}_{\text{poly},X}) \to \text{Mix}_{\mathcal{U}_r} (\mathcal{D}_{\text{poly},X})
\]
on \( X \). After twisting these morphisms suitably (this is needed due to the presence of the Grothendieck connection; cf. [Ye2]) we obtain the desired \( L_\infty \) quasi-isomorphism \( \Psi \).

We believe that mixed resolutions, and the results of this paper, shall have additional applications in algebraic geometry (e.g. algebro-geometric versions of results on index theorems in differential geometry, cf. [NT]; or a proof of Kontsevich’s famous yet unproved claim on Hochschild cohomology of a scheme [Ko Claim 8.4]).

1. Review of Dir-Inv Modules

We begin the paper with a review of the concept of dir-inv structure, which was introduced in [Ye2]. A dir-inv structure is a generalization of adic topology.

Let \( C \) be a commutative ring. We denote by \( \text{Mod} C \) the category of \( C \)-modules.

**Definition 1.1.**

1. Let \( M \in \text{Mod} C \). An inv module structure on \( M \) is an inverse system \( \{F^i M\}_{i \in \mathbb{N}} \) of \( C \)-submodules of \( M \). The pair \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) is called an inv \( C \)-module.

2. Let \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) and \( (N, \{F^i N\}_{i \in \mathbb{N}}) \) be two inv \( C \)-modules. A function \( \phi : M \to N \) is said to be continuous if for every \( i \in \mathbb{N} \) there exists \( i' \in \mathbb{N} \) such that \( \phi(F^{i'} M) \subset F^i N \).

3. Define \( \text{Inv Mod} C \) to be the category whose objects are the inv \( C \)-modules, and whose morphisms are the continuous \( C \)-linear homomorphisms.

There is a full and faithful embedding of categories \( \text{Mod} C \hookrightarrow \text{Inv Mod} C \), \( M \mapsto (M, \{\ldots, 0, 0\}) \).

Recall that a directed set is a partially ordered set \( J \) with the property that for any \( j_1, j_2 \in J \) there exists \( j_3 \in J \) such that \( j_1, j_2 \leq j_3 \).

**Definition 1.2.**

1. Let \( M \in \text{Mod} C \). A dir-inv module structure on \( M \) is a direct system \( \{F_j M\}_{j \in J} \) of \( C \)-submodules of \( M \), indexed by a nonempty directed set \( J \), together with an inv module structure on each \( F_j M \), such that for every \( j_1 \leq j_2 \) the inclusion \( F_{j_1} M \hookrightarrow F_{j_2} M \) is continuous. The pair \( (M, \{F_j M\}_{j \in J}) \) is called a dir-inv \( C \)-module.

2. Let \( (M, \{F_j M\}_{j \in J}) \) and \( (N, \{F_k N\}_{k \in K}) \) be two dir-inv \( C \)-modules. A function \( \phi : M \to N \) is said to be continuous if for every \( j \in J \) there exists \( k \in K \) such that \( \phi(F_j M) \subset F_k N \), and \( \phi : F_j M \to F_k N \) is a continuous homomorphism between these two inv \( C \)-modules.
(3) Define $\text{Dir Inv Mod } C$ to be the category whose objects are the dir-inv $C$-modules, and whose morphisms are the continuous $C$-linear homomorphisms.

An inv $C$-module $M$ can be endowed with a dir-inv module structure $\{F_j M\}_{j \in J}$, where $J := \{0\}$ and $F_0 M := M$. Thus we get a full and faithful embedding $\text{Inv Mod } C \rightrightarrows \text{Dir Inv Mod } C$.

Inv modules and dir-inv modules come in a few “flavors”: trivial, discrete and complete. A discrete inv module is one which is isomorphic, in $\text{Inv Mod } C$, to an object of $\text{Mod } C$ (via the canonical embedding above). A complete inv module is an inv module $(M, \{F^i M\}_{i \in \mathbb{N}})$ such that the canonical map $M \to \varprojlim_{i \to \infty} F^i M$ is bijective. A discrete (resp. complete) dir-inv module is one which is isomorphic, in $\text{Dir Inv Mod } C$, to a dir-inv module $(M, \{F_j M\}_{j \in J})$, where all the inv modules $F_j M$ are discrete (resp. complete), and the canonical map $\varprojlim_j F_j M \to M$ in $\text{Mod } C$ is bijective. A trivial dir-inv module is one which is isomorphic to an object of $\text{Mod } C$. Discrete dir-inv modules are complete, but there are also other complete modules, as the next example shows.

Example 1.3. Assume $C$ is noetherian and $c$-adically complete for some ideal $c$. Let $M$ be a finitely generated $C$-module, and define $F^i M := c^{i+1} M$. Then $\{F^i M\}_{i \in \mathbb{N}}$ is called the $c$-adic inv structure, and of course $(M, \{F^i M\}_{i \in \mathbb{N}})$ is a complete inv module. Next consider an arbitrary $C$-module $M$. We take $\{F_j M\}_{j \in J}$ to be the collection of finitely generated $C$-submodules of $M$. This dir-inv module structure on $M$ is called the $c$-adic dir-inv structure. Again $(M, \{F_j M\}_{j \in J})$ is a complete dir-inv $C$-module. Note that a finitely generated $C$-module $M$ is discrete as inv module iff $c^i M = 0$ for $i \gg 0$; and a $C$-module is discrete as dir-inv module iff it is a direct limit of discrete finitely generated modules.

The category $\text{Dir Inv Mod } C$ is additive. Given a collection $\{M_k\}_{k \in K}$ of dir-inv modules, the direct sum $\bigoplus_{k \in K} M_k$ has a structure of dir-inv module, making it into the coproduct of $\{M_k\}_{k \in K}$ in the category $\text{Dir Inv Mod } C$. Note that if the index set $K$ is finite and each $M_k$ is a nonzero discrete inv module, then $\bigoplus_{k \in K} M_k$ is a discrete dir-inv module which is not trivial. The tensor product $M \otimes_C N$ of two dir-inv modules is again a dir-inv module. There is a completion functor $M \mapsto \hat{M}$. (Warning: if $M$ is complete then $\hat{M} = M$, but it is not known if $\hat{M}$ is complete for arbitrary $M$.) The completed tensor product is $M \hat{\otimes}_C N := M \hat{\otimes}_C N$. Completion commutes with direct sums: if $M \cong \bigoplus_{k \in K} M_k$ then $\hat{M} \cong \bigoplus_{k \in K} \hat{M}_k$. See [Y22] for full details.

A graded dir-inv module (or graded object in $\text{Dir Inv Mod } C$) is a direct sum $M = \bigoplus_{k \in \mathbb{Z}} M_k$, where each $M_k$ is a dir-inv module. A DG algebra in $\text{Dir Inv Mod } C$ is a graded dir-inv module $A = \bigoplus_{k \in \mathbb{Z}} A^k$, together with continuous $C$-(bi)linear functions $\mu : A \times A \to A$ and $d : A \to A$, which make $A$ into a DG $C$-algebra. If $A$ is a super-commutative associative unital DG algebra in $\text{Dir Inv Mod } C$, and $\mathfrak{g}$ is a DG Lie algebra in $\text{Dir Inv Mod } C$, then $A \hat{\otimes}_C \mathfrak{g}$ is a DG Lie Algebra in $\text{Dir Inv Mod } C$.

Let $A$ be a super-commutative associative unital DG algebra in $\text{Dir Inv Mod } C$. A DG $A$-module in $\text{Dir Inv Mod } C$ is a graded object $M$ in $\text{Dir Inv Mod } C$, together with continuous $C$-(bi)linear functions $\mu : A \times M \to M$ and $d : M \to M$, which make $M$ into a DG $A$-module in the usual sense. A DG $A$-module Lie algebra in $\text{Dir Inv Mod } C$ is a DG Lie algebra $\mathfrak{g}$ in $\text{Dir Inv Mod } C$, together with a continuous $C$-bilinear function $\mu : A \times \mathfrak{g} \to \mathfrak{g}$, such that such that $\mathfrak{g}$ becomes a DG $A$-module,
and
\[ [a_1\gamma_1, a_2\gamma_2] = (-1)^{i_2j_2} a_1a_2 [\gamma_1, \gamma_2] \]
for all \( a_k \in A_{i_k} \) and \( \gamma_k \in g_{i_k} \).

All the constructions above can be geometrized. Let \((Y, \mathcal{O})\) be a commutative ringed space over \(K\), i.e. \(Y\) is a topological space, and \(\mathcal{O}\) is a sheaf of commutative \(K\)-algebras on \(Y\). We denote by \(\text{Mod}\mathcal{O}\) the category of \(\mathcal{O}\)-modules on \(Y\).

**Example 1.4.** Geometrizing Example 1.3, let \(X\) be a noetherian formal scheme, with defining ideal \(I\). Then any coherent \(O_X\)-module \(M\) is an inv \(O_X\)-module, with system of submodules \(\{I^{i+1}M\}_{i\in\mathbb{N}}\), and \(M \cong \hat{M}\); cf. [EGA I]. We call an \(O_X\)-module dir-coherent if it is the direct limit of coherent \(O_X\)-modules. Any dir-coherent module is quasi-coherent, but it is not known if the converse is true. At any rate, a dir-coherent \(O_X\)-module \(M\) is a dir-inv \(O_X\)-module, where we take \(\{F_jM\}_{j\in J}\) to be the collection of coherent submodules of \(M\). Any dir-coherent \(O_X\)-module is then a complete dir-inv module. This dir-inv module structure on \(M\) is called the \(I\)-adic dir-inv structure. Note that a coherent \(O_X\)-module \(M\) is discrete as inv module iff \(I^iM = 0\) for \(i \gg 0\); and a dir-coherent \(O_X\)-module is discrete as dir-inv module iff it is a direct limit of discrete coherent modules.

If \(f : (Y', \mathcal{O}') \to (Y, \mathcal{O})\) is a morphism of ringed spaces and \(M \in \text{Dir Inv Mod} \mathcal{O}\), then there is an obvious structure of dir-inv \(\mathcal{O}'\)-module on \(f^*M\), and we define \(f^*M := \hat{f}^*\hat{M}\). If \(M\) is a graded object in \(\text{Dir Inv Mod} \mathcal{O}\), then the inverse images \(f^{-1}\mathcal{O}\) and \(f^{-1}\mathcal{O}'\) are graded objects in \(\text{Dir Inv Mod} \mathcal{O}'\). If \(\mathcal{G}\) is a sheaf of topological \(R\)-modules, then \(f^{-1}\mathcal{G}\) is a sheaf of topological \(f^{-1}R\)-modules.

**Example 1.5.** Let \((Y, \mathcal{O})\) be a ringed space and \(V \subset Y\) an open set. For a dir-inv \(\mathcal{O}\)-module \(M\) there is an obvious way to make \(\Gamma(V, M)\) a dir-inv \(\mathcal{O}\)-module. If \(M\) is a complete inv \(\mathcal{O}\)-module then \(\Gamma(V, M)\) is a complete inv \(\mathcal{O}\)-module. If \(V\) is quasi-compact and \(M\) is a complete dir-inv \(\mathcal{O}\)-module, then \(\Gamma(V, M)\) is a complete dir-inv \(\Gamma(V, \mathcal{O})\)-module.

## 2. Complete Thom-Sullivan Cochains

From here on in \(K\) is a field of characteristic 0. Let us begin with some abstract notions about cosimplicial modules and their normalizations, following [HS] and [HY]. We use the notation \(\text{Mod}K\) and \(\text{DGMod}K\) for the categories of \(K\)-modules and DG (differential graded) \(K\)-modules respectively. Let \(\Delta\) denote the category with objects the ordered sets \([q] := \{0, 1, \ldots, q\}\), \(q \in \mathbb{N}\). The morphisms \([p] \to [q]\) are the order preserving functions, and we write \(\Delta_p^q := \text{Hom}_\Delta([p], [q])\). The \(i\)-th co-face map \(\partial^i : [p] \to [p+1]\) is the injective function that does not take the value \(i\); and the \(i\)-th co-degeneracy map \(s^i : [p] \to [p - 1]\) is the surjective function that takes the value \(i\) twice. All morphisms in \(\Delta\) are compositions of various \(\partial^i\) and \(s^i\).

An element of \(\Delta_p^q\) may be thought of as a sequence \(i = (i_0, \ldots, i_p)\) of integers with \(0 \leq i_0 \leq \cdots \leq i_p \leq q\). Given \(i \in \Delta_p^m, j \in \Delta_p^m\), and \(\alpha \in \Delta_p^q\), we sometimes write \(\alpha(i) := i \circ \alpha \in \Delta_p^m\) and \(\alpha*(j) := \alpha \circ j \in \Delta_p^q\).

Let \(C\) be some category. A cosimplicial object in \(C\) is a functor \(C : \Delta \to C\). We shall usually refer to the cosimplicial object as \(C = \{C^p\}_{p\in\mathbb{N}}\), and for any \(\alpha \in \Delta_p^q\),
the corresponding morphism in $C$ will be denoted by $\alpha^*: C^p \to C^q$. A simplicial object in $C$ is a functor $C: \Delta \op \to C$. The notation for a simplicial object will be $C = \{C_p\}_{p \in \mathbb{N}}$ and $\alpha^*: C_q \to C_p$.

Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial $K$-module. The standard normalization of $M$ is the DG module $N M$ defined as follows: $N^q M := \bigcap_{i=0}^{q-1} \text{Ker}(s^i: M^q \to M^{q-1})$. The differential is $\partial := \sum_{i=0}^{q-1} (-1)^i \partial^i: N^q M \to N^{q+1} M$. We get a functor $N: \Delta \text{Mod } K \to \text{DGMod } K$.

For any $q$ let $\Delta^q_C$ be the geometric $q$-dimensional simplex $\Delta^q_C := \text{Spec } K[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1)$.

The $i$-th vertex of $\Delta^q_C$ is the $K$-rational point $x$ such that $t_i(x) = 1$ and $t_j(x) = 0$ for all $j \neq i$. We identify the vertices of $\Delta^q_C$ with the ordered set $[q] = \{0, 1, \ldots, q\}$.

For any $\alpha: [p] \to [q]$ in $\Delta$ there is a unique linear morphism $\alpha: \Delta^q_C \to \Delta^p_C$ extending it, and in this way $\{\Delta^q_C\}_{q \in \mathbb{N}}$ is a cosimplicial scheme.

For a $K$-scheme $X$ we write $\Omega^p(X) := \Gamma(X, \Omega^p_{X/K})$. Taking $X := \Delta^q_C$ we have a super-commutative associative unital DG $K$-algebra $\Omega(\Delta^q_C) = \bigoplus_{p \in \mathbb{N}} \Omega^p(\Delta^q_C)$, that is generated as $K$-algebra by the elements $t_0, \ldots, t_q, dt_0, \ldots, dt_q$. The collection $\{\Omega(\Delta^q_C)\}_{q \in \mathbb{N}}$ is a simplicial DG algebra, namely a functor from $\Delta \op$ to the category of DG $K$-algebras.

In [HY] we made use of the Thom-Sullivan normalization $\hat{N} M$ of a cosimplicial $K$-module $M$. For some applications (specifically [Ye3]) a complete version of this construction is needed. Recall that for $M, N \in \text{Dir Inv Mod } K$, we can define the complete tensor product $N \hat{\otimes} M$. The $K$-modules $\Omega^p(\Delta^q_C)$ are always considered as discrete inv modules, so $\Omega(\Delta^q_C)$ is a discrete dir-inv DG $K$-algebra.

**Definition 2.1.** Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial dir-inv $K$-module, namely each $M^q \in \text{Dir Inv Mod } K$, and the morphisms $\alpha^*: M^p \to M^q$, for $\alpha \in \Delta^q_C$, are continuous $K$-linear homomorphisms. Let

$$\hat{N}^q M = \prod_{l=0}^{\infty} \left( \Omega^q(\Delta^l_C) \hat{\otimes} M^l \right)$$

be the submodule consisting of all sequences $(u_0, u_1, \ldots)$, with $u_l \in \Omega^q(\Delta^l_C) \hat{\otimes} M^l$, such that

$$(1 \otimes \alpha^*)(u_k) = (\alpha_\ast \otimes 1)(u_l) \in \Omega^q(\Delta^l_C) \hat{\otimes} M^l$$

for all $k, l \in \mathbb{N}$ and all $\alpha \in \Delta^l_C$. Define a coboundary operator $\partial: \hat{N}^q M \to \hat{N}^{q+1} M$ using the exterior derivative $d: \Omega^q(\Delta^l_C) \to \Omega^{q+1}(\Delta^l_C)$. The resulting DG $K$-module $(\hat{N} M, \partial)$ is called the complete Thom-Sullivan normalization of $M$.

The $K$-module $\hat{N} M = \bigoplus_{q \in \mathbb{N}} \hat{N}^q M$ is viewed as an abstract module. We obtain a functor

$$\hat{N}: \Delta \text{ Dir Inv Mod } K \to \text{DGMod } K.$$

**Remark 2.4.** In case each $M^l$ is a discrete dir-inv module one has $\Omega^q(\Delta^l_C) \hat{\otimes} M^l = \Omega^q(\Delta^l_C) \otimes M^l$, and therefore $\hat{N} M = \hat{N} M$.

The standard normalization $N M$ also makes sense here, via the forgetful functor $\Delta \text{ Dir Inv Mod } K \to \Delta \text{ Mod } K$. The two normalizations $\hat{N}$ and $N$ are related as follows. Let $\int_{\Delta^l}: \Omega(\Delta^l_C) \to K$ be the $K$-linear map of degree $-l$ defined by
integration on the compact real $l$-dimensional simplex, namely $\int_{\Delta^l} \frac{dt_1 \wedge \cdots \wedge dt_l}{t_l} = \frac{1}{l!}$ etc. Suppose each dir-inv module $M^q$ is complete, so that using \cite[Proposition 1.5]{Ye2} we get a functorial $\mathbb{K}$-linear homomorphism

$$\int: \Omega(\Delta^l_k) \hat{\otimes} M^l \to \mathbb{K} \hat{\otimes} M^l \cong M^l.$$ 

**Proposition 2.5.** Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial dir-inv $\mathbb{K}$-module, with all dir-inv modules $M^q$ complete. Then the homomorphisms $\int_{\Delta^l}$ induce a quasi-isomorphism

$$\int_{\Delta^l}: \hat{\tilde{N}}M \to NM$$

in $\text{DGMod} \mathbb{K}$.

**Proof.** This is a complete version of \cite[Theorem 1.12]{HY}. Let $\Delta^l$ be the simplicial set $\Delta^l := \text{Hom}_\Delta([-],[l])$; so its set of $p$-simplices is $\Delta^l_p$. Define $C_l$ to be the algebra of normalized cochains on $\Delta^l$, namely

$$C_l := \text{NHom}_\text{Sets}(\Delta^l, \mathbb{K}) \cong \text{Hom}_\text{Sets}(\Delta^l, \mathbb{K}).$$

Here $\Delta^l_{\text{nd}}$ is the (finite) set of nondegenerate simplices, i.e. those sequences $i = (i_0, \ldots, i_p)$ satisfying $0 \leq i_0 < \cdots < i_p \leq l$. As explained in \cite[Appendix A]{HY} we have simplicial DG algebras $C = \{C_l\}_{l \in \mathbb{N}}$ and $\Omega(\Delta^l_k) = \{\Omega(\Delta^l_k)\}_{l \in \mathbb{N}}$, and a homomorphism of simplicial DG modules $\rho: \Omega(\Delta^l_k) \to C$.

It turns out (this is work of Bousfield-Gugenheim) that $\rho$ is a homotopy equivalence in $\Delta^l \text{op} \text{DGMod} \mathbb{K}$, i.e. there are simplicial homomorphisms $\phi: C \to \Omega(\Delta^l_k)$, $h: C \to C$ and $h': \Omega(\Delta^l_k) \to \Omega(\Delta^l_k)$ such that $1 - \rho \circ \phi = h \circ d + d \circ h$ and $1 - \phi \circ \rho = h' \circ d + d \circ h'$.

Now for $M = \{M^q\} \in \Delta \text{Dir Inv Mod} \mathbb{K}$ and $N = \{N_q\} \in \Delta^l \text{op} \text{Mod} \mathbb{K}$ let $N \hat{\odot}_{\text{\Lambda}^l} M$ be the complete version of \cite[formula (A.1)]{HY}, so that in particular $\Omega(\Delta^l_k) \hat{\odot}_{\text{\Lambda}^l} M \cong \hat{\tilde{N}}M$ and $C \hat{\odot}_{\text{\Lambda}^l} M \cong NM$. Moreover

$$\rho \hat{\odot}_{\text{\Lambda}^l} 1_M = \int_{\Delta^l}: \hat{\tilde{N}}M \to NM.$$ 

It follows that $\int_{\Delta}$ is a homotopy equivalence in $\text{DGMod} \mathbb{K}$. \hfill \Box

Suppose $A = \{A^q\}_{q \in \mathbb{N}}$ is a cosimplicial DG algebra in $\text{Dir Inv Mod} \mathbb{K}$ (not necessarily associative nor commutative). This is a pretty complicated object: for every $q$ we have a DG algebra $A^q = \bigoplus_{i \in \mathbb{Z}} A^q_i$ in $\text{Dir Inv Mod} \mathbb{K}$. For every $\alpha \in \Delta^l_p$ there is a continuous DG algebra homomorphism $\alpha^* : A^\ell \to A^p$, and the $\alpha^*$ have to satisfy the simplicial relations.

Anyhow, both $\tilde{N}A$ and $NA$ are DG algebras. For $\tilde{N}A$ the DG algebra structure comes from that of the DG algebras $\Omega(\Delta^l_k) \otimes A^l$, via the embeddings \cite{LZ}. In case each $A^l$ is an associative super-commutative unital DG $\mathbb{K}$-algebra, then so is $\tilde{N}A$. Likewise for DG Lie algebras. (The algebra $NA$, with its Alexander-Whitney product, is very noncommutative.)

Assume that each $A^q_i$ is complete, so that the integral $\int_{\Delta^l}: \tilde{N}A \to NA$ is defined. This is not a DG algebra homomorphism. However:
Proposition 2.6. Suppose $A = \{A^q\}_{q \in \mathbb{N}}$ is a cosimplicial DG algebra in $\text{Dir Inv Mod} \mathbb{K}$, with all $A^q$ complete. Then the homomorphisms $\int_{\Delta}$ induce an isomorphism of graded algebras

$$H(\int_{\Delta}) : H^\wedge \tilde{N}A \xrightarrow{\sim} HNA.$$ 

Proof. This is a complete variant of [HY, Theorem 1.13]. The proof is identical, after replacing "$\otimes$" with "$\hat{\otimes}$" where needed; cf. proof of previous proposition. \[ \Box \]

Remark 2.7. If $A$ is associative then presumably $\int_{\Delta}$ extends to an $A_\infty$ quasi-isomorphism $\tilde{N}A \to NA$.

3. Commutative Čech Resolutions

In this section $\mathbb{K}$ is a field of characteristic 0 and $X$ is a noetherian topological space. We denote by $\mathbb{K}_X$ the constant sheaf $\mathbb{K}$ on $X$. We will be interested in the category $\text{Dir Inv Mod} \mathbb{K}_X$, whose objects are sheaves of $\mathbb{K}$-modules on $X$ with dir-inv structures. Note that any open set $V \subset X$ is quasi-compact.

Let $X = \bigcup_{i=0}^{m} U(i)$ be an open covering, which we denote by $U$. For any $i = (i_0, \ldots, i_q) \in \Delta^m_q$ define $U_i := U(i_0) \cap \cdots \cap U(i_q)$, and let $g_i : U_i \to X$ be the inclusion. Given a dir-inv $\mathbb{K}_X$-module $\mathcal{M}$ and natural number $q$ we define a sheaf $C^q(U, \mathcal{M}) := \prod_{i \in \Delta^m_q} g_i^* g_i^{-1} \mathcal{M}$.

This is a finite product. For an open set $V \subset X$ we then have

$$\Gamma(V, C^q(U, \mathcal{M})) = \prod_{i \in \Delta^m_q} \Gamma(V \cap U_i, \mathcal{M}).$$

For any $i$ the $\mathbb{K}$-module $\Gamma(V \cap U_i, \mathcal{M})$ has a dir-inv structure. Hence $\Gamma(V, C^q(U, \mathcal{M}))$ is a dir-inv $\mathbb{K}$-module. If $\mathcal{M}$ happens to be a complete dir-inv $\mathbb{K}_X$-module then $\Gamma(V, C^q(U, \mathcal{M}))$ is a complete dir-inv $\mathbb{K}$-module, since each $V \cap U_i$ is quasi-compact.

Keeping $V$ fixed we get a cosimplicial dir-inv $\mathbb{K}$-module $\{\Gamma(V, C^q(U, \mathcal{M}))\}_{q \in \mathbb{N}}$.

Applying the functors $N^q$ and $\tilde{N}^q$ we obtain $\mathbb{K}$-modules $N^q \Gamma(V, C(U, \mathcal{M}))$ and $\tilde{N}^q \Gamma(V, C(U, \mathcal{M}))$. As we vary $V$ these become presheaves of $\mathbb{K}$-modules, which we denote by $N^q \mathbb{C}(U, \mathcal{M})$ and $\tilde{N}^q \mathbb{C}(U, \mathcal{M})$.

Recall that a simplex $i = (i_0, \ldots, i_q)$ is nondegenerate if $i_0 < \cdots < i_q$. Let $\Delta^m_{\text{nd}}$ be the set of non-degenerate simplices inside $\Delta^m_q$.

Lemma 3.1. For every $q$ the presheaves

$$N^q \mathbb{C}(U, \mathcal{M}) : V \mapsto N^q \Gamma(V, C(U, \mathcal{M}))$$

and

$$\tilde{N}^q \mathbb{C}(U, \mathcal{M}) : V \mapsto \tilde{N}^q \Gamma(V, C(U, \mathcal{M}))$$

are sheaves. There is a functorial isomorphism of sheaves

$$N^q \mathbb{C}(U, \mathcal{M}) \cong \prod_{i \in \Delta^m_{\text{nd}}} g_i^* g_i^{-1} \mathcal{M},$$

(3.2)
and functorial embeddings of sheaves

\begin{equation}
\hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M}) \hookrightarrow \prod_{l \in \mathbb{N}} \prod_{i \in \Delta^n} g_{i*} g_{i}^{-1} (\Omega^q(\Delta^n_k) \otimes \mathcal{M})
\end{equation}

and

\begin{equation}
\mathcal{M} \hookrightarrow \hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M}).
\end{equation}

\textbf{Proof.} Since \{\mathcal{C}^q(U, \mathcal{M})\}_{q \in \mathbb{N}} is a cosimplicial sheaf we get the isomorphism \cite{32}.

As for \(\hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M})\), consider the sheaf \(\Omega^q(\Delta^n_k) \otimes \mathcal{M}\) on \(X\). Take any open set \(V \subset X\) and \(i \in \Delta^n_m\). Since \(V \cap U_i\) is quasi-compact we have

\[\Omega^q(\Delta^n_k) \otimes \Gamma(V \cap U_i, \mathcal{M}) \cong \Gamma(V \cap U_i, \Omega^q(\Delta^n_k) \otimes \mathcal{M}) = \Gamma(V, g_{i*} g_{i}^{-1} (\Omega^q(\Delta^n_k) \otimes \mathcal{M})).\]

By Definition \ref{def:1} there is an exact sequence of presheaves on \(X\):

\[
0 \to \hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M}) \to \prod_{l \in \mathbb{N}} \prod_{i \in \Delta^n} g_{i*} g_{i}^{-1} (\Omega^q(\Delta^n_k) \otimes \mathcal{M}) \\
\xrightarrow{1 \otimes \alpha - \alpha \otimes 1} \prod_{k,l \in \mathbb{N}} \prod_{i \in \Delta^n} g_{i*} g_{i}^{-1} (\Omega^q(\Delta^n_k) \otimes \mathcal{M}).
\]

Since the presheaves in the middle and on the right are actually sheaves, it follows that \(\hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M})\) is also a sheaf.

Finally the embedding \cite{34} comes from the embeddings \(\mathcal{M} \hookrightarrow \Omega^q(\Delta^n_k) \otimes \mathcal{M}\), \(w \mapsto 1 \otimes w\).

Thus we have complexes of sheaves \(\text{NC}(U, \mathcal{M})\) and \(\hat{\mathcal{N}} \mathcal{C}(U, \mathcal{M})\). There are functorial homomorphisms \(\mathcal{M} \to \text{NC}(U, \mathcal{M})\) and \(\mathcal{M} \to \hat{\mathcal{N}} \mathcal{C}(U, \mathcal{M})\). Note that the complex \(\Gamma(X, \text{NC}(U, \mathcal{M}))\) is nothing but the usual global Čech complex of \(\mathcal{M}\) for the covering \(U\).

\textbf{Definition 3.5.} The complex \(\hat{\mathcal{N}} \mathcal{C}(U, \mathcal{M})\) is called the \textit{commutative Čech resolution} of \(\mathcal{M}\).

The reason for the name is that \(\hat{\mathcal{N}} \mathcal{C}(U, \mathcal{O}_X)\) is a sheaf of super-commutative DG algebras, as can be seen from the next lemma.

\textbf{Lemma 3.6.} Suppose \(\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}\) are dir-inv \(\mathbb{K}_X\)-modules, and \(q_1, \ldots, q_r \in \mathbb{N}\). Let \(q := q_1 + \cdots + q_r\). Suppose that for every \(l \in \mathbb{N}\) and \(i \in \Delta^n\) we are given \(\mathbb{K}\)-multilinear sheaf maps

\[
\phi_{q_1, \ldots, q_r, i} : (\Omega^{q_1}(\Delta^n_k) \otimes (\mathcal{M}_1|_{U_i})) \times \cdots \times (\Omega^{q_r}(\Delta^n_k) \otimes (\mathcal{M}_r|_{U_i})) \to \Omega^q(\Delta^n_k) \otimes (\mathcal{N}|_{U_i})
\]

that are continuous (for the dir-inv module structures), and are compatible with the simplicial structure as in Definition \ref{def:1}. Then there are unique \(\mathbb{K}\)-multilinear sheaf maps

\[
\phi_{q_1, \ldots, q_r} : \hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M}_1) \times \cdots \times \hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{M}_r) \to \hat{\mathcal{N}}^q \mathcal{C}(U, \mathcal{N})
\]

that commute with the embeddings \(\hat{\mathcal{N}}^q\).

\textbf{Proof.} Direct verification. \(\square\)
Lemma 3.7. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be dir-inv $\mathbb{K}_X$-modules, and $\phi : \prod M_i \to \mathcal{N}$ a continuous $\mathbb{K}$-multilinear sheaf homomorphism. Then there is an induced homomorphism of complexes of sheaves

$$\phi : \hat{\mathcal{N}}C(U, \mathcal{M}_1) \otimes \cdots \otimes \hat{\mathcal{N}}C(U, \mathcal{M}_r) \to \hat{\mathcal{N}}C(U, \mathcal{N}).$$

Proof. Use Lemma 3.6. \qed

In particular, if $\mathcal{M}$ is a dir-inv $\mathcal{O}_X$-module then $\hat{\mathcal{N}}C(U, \mathcal{M})$ is a DG $\hat{\mathcal{N}}C(U, \mathcal{O}_X)$-module.

If $\mathcal{M} = \bigoplus_p \mathcal{M}^p$ is a graded dir-inv $\mathbb{K}_X$-module then we define

$$\hat{\mathcal{N}}C(U, \mathcal{M})^i := \bigoplus_{p+q=i} \hat{\mathcal{N}}^qC(U, \mathcal{M}^p)$$

and

$$\hat{\mathcal{N}}C(U, \mathcal{M}) := \bigoplus_i \hat{\mathcal{N}}C(U, \mathcal{M})^i.$$  

Due to Lemma 3.7 if $\mathcal{M}$ is a complex in $\text{Dir Inv Mod} \mathbb{K}_X$, then $\hat{\mathcal{N}}C(U, \mathcal{M})$ is also a complex (in $\text{Mod} \mathbb{K}_X$), and there is a functorial homomorphism of complexes $\mathcal{M} \to \hat{\mathcal{N}}C(U, \mathcal{M})$.

Theorem 3.8. Let $X$ be a noetherian topological space, with open covering $U = \{U_i\}_{i=0}^m$. Let $\mathcal{M}$ be a bounded below complex in $\text{Dir Inv Mod} \mathbb{K}_X$, and assume each $\mathcal{M}^p$ is a complete dir-inv $\mathbb{K}_X$-module. Then:

1. For any open set $V \subset X$ the homomorphism

$$\Gamma(V, \int_{\Delta} : \Gamma(V, \hat{\mathcal{N}}C(U, \mathcal{M})) \to \Gamma(V, \mathcal{N}C(U, \mathcal{M}))$$

is a quasi-isomorphism of complexes of $\mathbb{K}$-modules.

2. There are functorial quasi-isomorphism of complexes of $\mathbb{K}_X$-modules

$$\mathcal{M} \to \hat{\mathcal{N}}C(U, \mathcal{M}) \overset{\Delta}{\longrightarrow} \mathcal{N}C(U, \mathcal{M}).$$

Proof. (1) Lemma 3.1 and Proposition 2.5 imply that for any $p$ the homomorphism of complexes

$$\Gamma(V, \int_{\Delta} : \Gamma(V, \hat{\mathcal{N}}C(U, \mathcal{M}^p)) \to \Gamma(V, \math{N}C(U, \mathcal{M}^p))$$

is a quasi-isomorphism. Now use the standard filtration argument (the complexes in question are all bounded below).

(2) From (1) we deduce that

$$\Gamma(V, \int_{\Delta} : \Gamma(V, \hat{\mathcal{N}}C(U, \mathcal{M})) \to \Gamma(V, \math{N}C(U, \mathcal{M}))$$

is a quasi-isomorphism. Hence

$$\int_{\Delta} : \hat{\mathcal{N}}C(U, \mathcal{M}) \to \mathcal{N}C(U, \mathcal{M})$$

is a quasi-isomorphism of complexes of sheaves.

It is a known fact that $\mathcal{M}^p \to \mathcal{N}C(U, \mathcal{M}^p)$ is a quasi-isomorphism of sheaves (see [Ha] Lemma 4.2). Again this implies that $\mathcal{M} \to \mathcal{N}C(U, \mathcal{M})$ is a quasi-isomorphism. And therefore the homomorphism $\mathcal{M} \to \hat{\mathcal{N}}C(U, \mathcal{M})$ coming from (3.4) is also a quasi-isomorphism. \qed
Now let us look at a separated noetherian formal scheme \( \mathfrak{X} \). Let \( \mathcal{I} \) be some defining ideal of \( \mathfrak{X} \), and let \( X \) be the scheme with structure sheaf \( \mathcal{O}_X := \mathcal{O}_{\mathfrak{X}}/\mathcal{I} \). So \( \mathfrak{X} \) and \( X \) have the same underlying topological space. Recall that a \( \mathcal{I} \)-coherent \( \mathcal{O}_X \)-module is a quasi-coherent \( \mathcal{O}_X \)-module which is the union of its coherent sub-modules.

**Corollary 3.10.** Let \( \mathfrak{X} \) be a noetherian separated formal scheme over \( \mathbb{K} \), with defining ideal \( \mathcal{I} \) and underlying topological space \( X \). Let \( U = \{ U(i) \}_{i=0}^m \) be an affine open covering of \( X \). Let \( \mathcal{M} \) be a bounded below complex of sheaves of \( \mathbb{K} \)-modules on \( X \). Assume each \( \mathcal{M}^p \) is a \( \mathcal{I} \)-coherent \( \mathcal{O}_X \)-module, and the coboundary operators \( \mathcal{M}^p \to \mathcal{M}^{p+1} \) are continuous for the \( \mathcal{I} \)-adic dir-inv structures (but not necessarily \( \mathcal{O}_X \)-linear). Then:

1. The canonical morphism
   \[
   \Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M})) \to R\Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M}))
   \]
   in \( D(\text{Mod} \ \mathbb{K}) \) is an isomorphism.
2. There is a functorial isomorphism
   \[
   \Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M})) \cong R\Gamma(X, \mathcal{M})
   \]
   in \( D(\text{Mod} \ \mathbb{K}) \).

**Proof.** (1) Consider the commutative diagram

\[
\begin{array}{ccc}
\Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M})) & \xrightarrow{\Gamma(X, \delta)} & \Gamma(X, \mathcal{NC}(U, \mathcal{M})) \\
\downarrow & & \downarrow \\
R\Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M})) & \xrightarrow{R\Gamma(X, \delta)} & R\Gamma(X, \mathcal{NC}(U, \mathcal{M}))
\end{array}
\]

in \( D(\text{Mod} \ \mathbb{K}) \), in which the vertical arrows are the canonical morphisms. By part (1) of the theorem (with \( V = X \)) the top arrow is a quasi-isomorphism. And by part (2) the bottom arrow is an isomorphism. Hence it is enough to prove that the right vertical arrow is an isomorphism.

Using a filtration argument we may assume that \( \mathcal{M} \) is a single \( \mathcal{I} \)-coherent \( \mathcal{O}_X \)-module. Now \( \Gamma(X, \mathcal{NC}(U, \mathcal{M})) \) is the usual Čech resolution of the sheaf \( \mathcal{M} \) with respect to the covering \( U \) (cf. equation 3.2). So it suffices to prove that for all \( q \) and \( i \in \Delta^0 \) the sheaves \( g_i, g_i^{-1} \mathcal{M} \) are \( (X, -) \)-acyclic.

First let’s assume \( \mathcal{M} \) is a coherent \( \mathcal{O}_X \)-module. Let \( U_i \) be the open formal subscheme of \( \mathfrak{X} \) supported on \( U_i \). Then \( g_i^{-1} \mathcal{M} \) is a coherent \( \mathcal{O}_{U_i} \)-module, and both \( g_i : U_i \to \mathfrak{X} \) and \( U_i \to \text{Spec} \ \mathbb{K} \) are affine morphisms. By [EGA I, Theorem 10.10.2] it follows that \( g_i, g_i^{-1} \mathcal{M} = Rg_i, g_i^{-1} \mathcal{M} \), and also

\[
\Gamma(U_i, g_i^{-1} \mathcal{M}) = R\Gamma(U_i, g_i^{-1} \mathcal{M}) \cong R\Gamma(X, Rg_i, g_i^{-1} \mathcal{M}) \cong R\Gamma(X, g_i, g_i^{-1} \mathcal{M}).
\]

We conclude that \( H^j(X, g_i, g_i^{-1} \mathcal{M}) = 0 \) for all \( j > 0 \).

In the general case when \( \mathcal{M} \) is a direct limit of coherent \( \mathcal{O}_X \)-modules we still get \( H^j(X, g_i, g_i^{-1} \mathcal{M}) = 0 \) for all \( j > 0 \).

(2) By part (2) of the theorem we get a functorial isomorphism \( R\Gamma(X, \mathcal{M}) \cong R\Gamma(X, \tilde{\mathcal{N}}C(U, \mathcal{M})) \). Now use part (1) above. \( \square \)
4. Mixed Resolutions

In this section $\mathbb{K}$ is a field of characteristic 0 and $X$ is a finite type $\mathbb{K}$-scheme. Let us begin by recalling the definition of the sheaf of principal parts $\mathcal{P}_X$ from 

Recall that a connection $\nabla$ on an $\mathcal{O}_X$-module $M$ is a $\mathcal{O}_X$-linear homomorphism $\nabla : M \to \Omega^1 \otimes_{\mathcal{O}_X} M$ satisfying the Leibniz rule $\nabla(fm) = d(f) \otimes m + f \nabla(m)$ for local sections $f \in \mathcal{O}_X$ and $m \in M$.

**Definition 4.1.** Consider the de Rham differential $d_{X^2/X} : \mathcal{O}_{X^2} \to \Omega^1_{X^2/X}$ relative to the morphism $p_2 : X^2 \to X$. Since $\Omega^1_{X^2/X} \cong p_1^* \Omega^1_X = p_1^{-1} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X^2}$ we obtain a $\mathbb{K}$-linear homomorphism $d_{X^2/X} : \mathcal{O}_{X^2} \to p_1^* \Omega^1_X$. Passing to the completion along the diagonal $\Delta(X)$ we get a connection of $\mathcal{O}_X$-modules

$$\nabla_{p_2} : \mathcal{P}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

called the Grothendieck connection.

Note that the connection $\nabla_{p_2}$ is $p_2^{-1} \mathcal{O}_X$-linear. It will be useful to describe $\nabla_{p_2}$ on the level of rings. Let $U = \text{Spec } C \subset X$ be an affine open set. Then

$$\Gamma(U, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega^1_C \otimes C \cong \Omega^1_C \otimes \mathcal{O}_X,$$

the $I$-adic completion, where $I := \ker(C \otimes C \to C)$. And $\nabla_{p_2} : \mathcal{C} \otimes C \to \Omega^1_C \otimes C$ is the completion of $d \otimes 1 : C \otimes C \to \Omega^1_C \otimes C$.

As usual the connection $\nabla_{p_2}$ of (4.2) induces differential operators of left $\mathcal{O}_X$-modules

$$\nabla_{p_2} : \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \to \Omega^{i+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

for all $i \geq 0$, by the rule

$$\nabla_{p_2}(\alpha \otimes b) = d(\alpha) \otimes b + (-1)^i \alpha \wedge \nabla_{p_2}(b).$$

**Theorem 4.4.** Assume $X$ is a smooth $n$-dimensional $\mathbb{K}$-scheme. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then the sequence of sheaves on $X$

$$0 \to \mathcal{M} \xrightarrow{\text{mult} \otimes 1} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla_{p_2} \otimes 1 \mathcal{M}} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\ldots} \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to 0$$

is exact.

**Proof.** The proof is similar to that of \[Theorem 4.5\]. We may restrict to an affine open set $U = \text{Spec } B \subset X$ that admits an étale coordinate system $s = (s_1, \ldots, s_n)$, i.e. $\mathbb{K}[s] \to B$ is an étale ring homomorphism. It will be convenient to have another copy of $B$, which we call $C$; so that $\Gamma(U, \mathcal{P}_X) = B \otimes C$, the $I$-adic completion, where $I := \ker(B \otimes C \to B)$. We shall identify $B$ and $C$ with their images inside $B \otimes C$, and denote the copy of the element $s_i$ in $C$ by $r_i$. Letting $t_i := r_i - s_i \in B \otimes C$ we then have $t_i = s_i - s_i \otimes 1$ in our earlier notation. Note that $\Omega^i_{\mathbb{K}[s]} \subset \Omega^i_B$ is a sub DG algebra, and $B \otimes_{\mathbb{K}[s]} \Omega^i_{\mathbb{K}[s]} \to \Omega^i_B$ is a bijection.

By definition

$$\Gamma(U, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega^1_B \otimes_B (B \otimes C) \cong \Omega^1_B \otimes C.$$
The differential $\nabla_p$ on the left goes to the differential $d_B \otimes 1_C$ on the right. Consider the sub DG algebra $\Omega_{K[s]} \otimes C \subset \Omega_B \otimes C$. We know that $K \to \Omega_{K[s]}$ is a quasi-isomorphism; therefore so is $C \to \Omega_{K[s]} \otimes C$.

Because $t_i + s_i = r_i \in C$ we see that $C[s] = C[t] \subset B \otimes C$. Therefore we obtain $C$-linear isomorphisms

$$\Omega^p_{K[s]} \otimes C \cong \Omega^p_{K[s]} \otimes_{K[s]} C[s] = \Omega^p_{K[s]} \otimes_{K[s]} C[t].$$

So there is a commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
C \\
\downarrow \\
\Omega^1_{K[s]} \otimes_{K[s]} C[t] \\
\downarrow \\
\cdots \\
\downarrow \\
\Omega^n_{K[s]} \otimes_{K[s]} C[t] \\
\downarrow \\
0
\end{array}
$$

in which the top tow is continuously $C$-linearly split, and the vertical arrow are bijections. Hence the bottom row is split exact. Comparing this to (4.6) we conclude that for any $\mathcal{O}_X$-module $\mathcal{M}$ the sequence (4.6) is transformed to the commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
C \\
\downarrow \\
\Omega^1_{K[s]} \otimes_{K[s]} C[t] \\
\downarrow \\
\cdots \\
\downarrow \\
\Omega^n_{K[s]} \otimes_{K[s]} C[t] \\
\downarrow \\
0
\end{array}
$$

in which the top tow is continuously $C$-linearly split, and the vertical arrow are bijections. Hence the bottom row is split exact. Comparing this to (4.6) we conclude that the sequence of right $\mathcal{O}_U$-modules

$$
0 \to \mathcal{O}_U \xrightarrow{\mathcal{P}_X|_U} \mathcal{P}_X|_U \xrightarrow{\nabla_p} (\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X)|_U \xrightarrow{\nabla_p} \cdots (\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X)|_U \to 0
$$

is split exact.

Therefore it follows that for any $\mathcal{O}_X$-module $\mathcal{M}$ the sequence (4.6) is split exact. □

Let us now fix an affine open covering $U = \{U_{(0)}, \ldots, U_{(m)}\}$ of $X$.

Let $\mathcal{I}_X = \text{Ker}(\mathcal{P}_X \to \mathcal{O}_X)$. This is a defining ideal of the noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_X) := (X, \mathcal{P}_X)$. So $\mathcal{P}_X$ is an inv module over itself with the $\mathcal{I}_X$-adic inv structure. Given quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, the tensor product $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a dir-coherent $\mathcal{P}_X$-module, and so it has the $\mathcal{I}_X$-adic dir-inv structure. See Example 1.24. In particular

$$
\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus_{p \geq 0} \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}
$$

becomes a dir-inv $K[X]$-module.
Lemma 4.8. $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a DG $\Omega_X$-module in Dir Inv Mod $\mathbb{K}_X$, with differential $\nabla_{\mathcal{P}} \otimes 1_{\mathcal{M}}$.

Proof. Because $\nabla_{\mathcal{P}} \otimes 1_{\mathcal{M}}$ is a differential operator of $\mathcal{P}_X$-modules, it is continuous for the $I_X$-adic dir-inv structure. See [Ye2, Proposition 2.3]. □

Henceforth we will write $\nabla_{\mathcal{P}}$ instead of $\nabla_{\mathcal{P}} \otimes 1_{\mathcal{M}}$.

Definition 4.9. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. For any $p, q \in \mathbb{N}$ define

$$\operatorname{Mix}^{p,q}_U(\mathcal{M}) := \hat{\mathbb{N}}^qC(U, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

The Grothendieck connection $\nabla_{\mathcal{P}} : \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ induces a homomorphism of sheaves

$$\nabla_{\mathcal{P}} : \operatorname{Mix}^{p,q}_U(\mathcal{M}) \to \operatorname{Mix}^{p+1,q}_U(\mathcal{M}).$$

We also have $\partial : \operatorname{Mix}^{p,q}_U(\mathcal{M}) \to \operatorname{Mix}^{p,q+1}_U(\mathcal{M})$. Define

$$\operatorname{Mix}^i_U(\mathcal{M}) := \bigoplus_{p+q=i} \operatorname{Mix}^{p,q}_U(\mathcal{M}),$$

$$\operatorname{Mix}_U(\mathcal{M}) := \bigoplus_i \operatorname{Mix}^i_U(\mathcal{M})$$

and

$$(4.10) \quad d_{\text{mix}} := \partial + (-1)^q\nabla_{\mathcal{P}} : \operatorname{Mix}^{p,q}_U(\mathcal{M}) \to \operatorname{Mix}^{p+1,q}_U(\mathcal{M}) \oplus \operatorname{Mix}^{p,q+1}_U(\mathcal{M}).$$

The complex $(\operatorname{Mix}_U(\mathcal{M}), d_{\text{mix}})$ is called the mixed resolution of $\mathcal{M}$.

There are functorial embeddings of sheaves

$$(4.11) \quad \mathcal{M} \subset \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \subset \hat{\mathbb{N}}^0C(U, \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \operatorname{Mix}^{0,0}_U(\mathcal{M})$$

and

$$(4.12) \quad \operatorname{Mix}^{p,q}_U(\mathcal{M}) \subset \prod_{l \in \mathbb{N}} \prod_{i \in \Delta^p} g_{i*}^{-1}(\Omega^q(\Delta^i_k) \otimes (\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}));$$

see Lemma 3.1.

Proposition 4.13. (1) $\operatorname{Mix}_U(\mathcal{O}_X)$ is a sheaf of super-commutative associative unital DG $\mathbb{K}$-algebras. There are two $\mathbb{K}$-algebra homomorphisms $p^*_1, p^*_2 : \mathcal{O}_X \to \operatorname{Mix}^0_U(\mathcal{O}_X)$.

(2) Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\operatorname{Mix}_U(\mathcal{M})$ is a left DG $\operatorname{Mix}_U(\mathcal{O}_X)$-module.

(3) If $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of finite rank then the multiplication map

$$\operatorname{Mix}_U(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \to \operatorname{Mix}_U(\mathcal{M})$$

is an isomorphism.

Proof. By by Lemmas 3.1 and 3.7 □

Note that $d_{\text{mix}} \circ p^*_2 : \mathcal{O}_X \to \operatorname{Mix}_U(\mathcal{O}_X)$ is zero, but $d_{\text{mix}} \circ p^*_1 \neq 0$. 
Proposition 4.14. Let $M_1,\ldots,M_r,N$ be quasi-coherent $O_X$-modules. Suppose
\[
\phi : \prod_{i=1}^r (\Omega_X \otimes_{O_X} P_X \otimes_{O_X} M_i) \to \Omega_X \otimes_{O_X} P_X \otimes_{O_X} N
\]
is a continuous $\Omega_X$-multilinear sheaf morphism of degree $d$. Then there is a unique $K$-multilinear sheaf morphism of degree $d$
\[
\tilde{NC}(U,\phi) : \text{Mix}_U(M_1) \times \cdots \times \text{Mix}_U(M_r) \to \text{Mix}_U(N)
\]
which is compatible with $\phi$ via the embedding \([4.12]\).

Proof. This is an immediate consequence of Lemma 3.7. \qed

Suppose we are given $M \in C^+(\text{QCoh} O_X)$. Define
\[
\text{Mix}_U(M) := \bigoplus_{p+q=i} \text{Mix}_U^q(M^p)
\]
with differential
\[
d_{\text{mix}} + (-1)^q d_M : \text{Mix}_U^q(M^p) \to \text{Mix}_U^{q+1}(M^p) \oplus \text{Mix}_U^q(M^{p+1}).
\]

Theorem 4.15. Let $X$ be a smooth separated $K$-scheme, and let $U = \{U(0),\ldots,U(m)\}$ be an affine open covering of $X$.

1. There is a functorial quasi-isomorphism $M \to \text{Mix}_U(M)$ for $M \in C^+(\text{QCoh} O_X)$.
2. Given $M \in C^+(\text{QCoh} O_X)$, the canonical morphism $\Gamma(X,\text{Mix}_U(M)) \to R\Gamma(X,\text{Mix}_U(M))$ in $D(\text{Mod} K)$ is an isomorphism.
3. The quasi-isomorphism in part (1) induces a functorial isomorphism $\Gamma(X,\text{Mix}_U(M)) \cong \Gamma(X,M)$ in $D(\text{Mod} K)$.

Proof. (1) Write $N := \Omega_X \otimes_{O_X} P_X \otimes_{O_X} M$. A filtration argument and Theorem 4.4 show that the inclusion $M \to N$ is a quasi-isomorphism. Next we view $N$ as a bounded below complex in $\text{Dir Inv Mod} K_X$. By Theorem 3.8(2) we have a quasi-isomorphism $N \to \tilde{NC}(U,N) = \text{Mix}_U(M)$.

(2) This is due to Corollary 3.10(1), applied to the formal scheme $(X,P_X)$ and the complex $N$ of dir-coherent $P_X$-modules defined above.

(3) This assertion is an immediate consequence of parts (1) and (2). \qed

Corollary 4.16. In the situation of the theorem, suppose $M,N \in C^+(\text{QCoh} O_X)$ and $\phi : \text{Mix}_U(M) \to \text{Mix}_U(N)$ is a $K$-linear quasi-isomorphism. Then
\[
\Gamma(X,\phi) : \Gamma(X,\text{Mix}_U(M)) \to \Gamma(X,\text{Mix}_U(N))
\]
is a quasi-isomorphism.

Proof. Consider the commutative diagram
\[
\begin{array}{ccc}
\Gamma(X,\text{Mix}_U(M)) & \xrightarrow{\Gamma(X,\phi)} & \Gamma(X,\text{Mix}_U(N)) \\
\downarrow & & \downarrow \\
R\Gamma(X,\text{Mix}_U(M)) & \xrightarrow{R\Gamma(X,\phi)} & R\Gamma(X,\text{Mix}_U(N))
\end{array}
\]
in $\mathcal{D}(\text{Mod } \mathbb{K})$. By part (2) of the theorem the vertical arrows are isomorphisms. Since $\phi$ is an isomorphism in $\mathcal{D}(\text{Mod } \mathbb{K}_X)$ it follows that the bottom arrow is an isomorphism. \hfill \Box

Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and an integer $i$ define
$$G^i \text{Mix}_U(\mathcal{M}) := \bigoplus_{q \geq i} \text{Mix}_U^q(\mathcal{M}).$$
Then $\{G^i \text{Mix}_U(\mathcal{M})\}_{i \in \mathbb{Z}}$ is a descending filtration of $\text{Mix}_U(\mathcal{M})$ by subcomplexes, satisfying $G^i \text{Mix}_U(\mathcal{M}) = \text{Mix}_U(\mathcal{M})$ for $i \ll 0$ and $\bigcap_i G^i \text{Mix}_U(\mathcal{M}) = 0$. For any $i$
$$\text{gr}_G^i \text{Mix}_U(\mathcal{M}) := G^i \text{Mix}_U(\mathcal{M}) / G^{i+1} \text{Mix}_U(\mathcal{M}).$$
The functor $\text{gr}_G^i \text{Mix}_U : \text{QCoh} \mathcal{O}_X \to \text{Mod } \mathbb{K}_X$
is additive, but we do not know whether it is exact. The next theorem asserts this in a very special case.

Consider the sheaves of DG Lie algebras $\mathcal{T}_{\text{poly},X}$ and $\mathcal{D}_{\text{poly},X}$ as complexes of quasi-coherent $\mathcal{O}_X$-modules (cf. [Ye3, Proposition 3.18]). According to [Ye1, Theorem 0.4] there is a quasi-isomorphism
$$\mathcal{U}_1 : \mathcal{T}_{\text{poly},X} \to \mathcal{D}_{\text{poly},X}.$$  

**Theorem 4.17.** For any $i$ the homomorphism of complexes
$$\text{gr}_G^i \text{Mix}_U(\mathcal{U}_1) : \text{gr}_G^i \text{Mix}_U(\mathcal{T}_{\text{poly},X}) \to \text{gr}_G^i \text{Mix}_U(\mathcal{D}_{\text{poly},X})$$
is a quasi-isomorphism.

**Proof.** Given a point $x \in X$ choose an affine open neighborhood $V$ of $x$ which admits an étale morphism $V \to \mathbb{A}^n_k$. By [Ye2, Theorem 4.11] the map of complexes
$$\mathcal{U}_1|_V : \mathcal{T}_{\text{poly},X}|_V \to \mathcal{D}_{\text{poly},X}|_V$$
is a homotopy equivalence in $C^+(\text{QCoh } \mathcal{O}_V)$. Since $\text{gr}_G^i \text{Mix}_U$ is an additive functor we see that $\text{gr}_G^i \text{Mix}_U(\mathcal{U}_1)|_V$ is a quasi-isomorphism. \hfill \Box

**Remark 4.18.** We know very little about the structure of the sheaves $\hat{\mathcal{N}}^q C(U, \mathcal{M})$, even when $\mathcal{M} = \mathcal{O}_X$. Cf. [HS].

5. Simplicial Sections

Let $X$ be a $\mathbb{K}$-scheme, and let $X = \bigcup_{i=0}^m U(i) \to X$. We denote this covering by $\mathcal{U}$. For any multi-index $i = (i_0, \ldots, i_q) \in \Delta^n_k$ we write $U_i := \bigcap_{j=0}^i U(j)$, and we define the scheme $U_q := \prod_{i \in \Delta_q^n} U_i$. Given $\alpha \in \Delta^n_q$ and $i \in \Delta_q^n$ there is an inclusion of open sets $\alpha_\ast : U_i \to U_{\alpha_\ast(i)}$. These patch to a morphism of schemes $\alpha_\ast : U_q \to U_p$, making $\{U_q\}_{q \in \mathbb{N}}$ into a simplicial scheme. The inclusions $g(i) : U(i) \to X$ induce inclusions $g_q : U_q \to X$ and morphisms $g_q : U_q \to X$; and one has the relations $g_p \circ \alpha_\ast = g_q$ for any $\alpha \in \Delta^n_q$.

**Definition 5.1.** Let $\pi : Z \to X$ be a morphism of $\mathbb{K}$-schemes. A simplicial section of $\pi$ based on the covering $\mathcal{U}$ is a sequence of morphisms
$$\sigma = \{\sigma_q : \Delta^n_q \times U_q \to Z\}_{q \in \mathbb{N}}$$
satisfying the following conditions.
Figure 1. An illustration of a simplicial section $\sigma$ based on an open covering $U = \{U_i\}$. On the left we see two components of $\sigma$ in dimension $q = 0$; and on the right we see one component in dimension $q = 1$.

(i) For any $q$ the diagram

$$
\begin{array}{ccc}
\Delta^q_K \times U_q & \xrightarrow{\sigma_q} & Z \\
p_2 \downarrow & & \downarrow \pi \\
U_q & \xrightarrow{g_q} & X
\end{array}
$$

is commutative.

(ii) For any $\alpha \in \Delta^p_K$ the diagram

$$
\begin{array}{ccc}
\Delta^p_K \times U_p & \xrightarrow{\sigma_p} & Z \\
1 \times \alpha^* \downarrow & & \downarrow \sigma_q \\
\Delta^q_K \times U_q & \xrightarrow{\alpha \times 1} & \Delta^q_K \times U_q
\end{array}
$$

is commutative.

Given a multi-index $i \in \Delta^m_q$ we denote by $\sigma_i$ the restriction of $\sigma_q$ to $\Delta^q_K \times U_i$. See Figure 1 for an illustration.

As explained in the introduction, simplicial sections arise naturally in several contexts, including deformation quantization.

Let $A$ be an associative unital super-commutative DG $K$-algebra. Consider homogeneous $A$-multilinear functions $\phi : M_1 \times \cdots \times M_r \to N$, where $M_1, \ldots, M_r, N$ are DG $A$-modules. There is an operation of composition for such functions: given
functions $\psi_i : \prod_i L_{i,j} \to M_i$ the composition is $\phi \circ (\psi_1 \times \cdots \times \psi_r) : \prod_i L_{i,j} \to N$. There is also a summation operation: if $\phi_j : \prod_i M_i \to N$ are homogeneous of equal degree then so is their sum $\sum_j \phi_j$. Finally, let $d : \prod_i M_i \to \prod_i M_i$ be the function

$$d(m_1, \ldots, m_r) := \sum_{i=1}^r \pm (m_1, \ldots, d(m_i), \ldots, m_r)$$

with Koszul signs. All the above can of course be sheafified, i.e. $A$ is a sheaf of DG algebras on a scheme $Z$ etc.

As before let $\pi : Z \to X$ be a morphism if $\mathbb{K}$-schemes, and let $U = \{U_{(i)}\}$ be an open covering of $X$. Suppose $\sigma$ is a simplicial section of $\pi$ based on $U$. We consider $\Omega^n_X$ as a discrete inv $\mathbb{K}_X$-module, and $\Omega_X = \bigoplus_{p \geq 0} \Omega^p_X$ has the $\bigoplus$ dir-inv structure. Likewise for $\Omega_Z = \bigoplus_{p \geq 0} \Omega^p_Z$.

Suppose $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Then, as explained in Section 4, $\Omega_Z \otimes_{\mathcal{O}_X} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a DG $\Omega_Z$-module on $Z$, with the Grothendieck connection $\nabla_p$. And $\text{Mix}_U(\mathcal{M})$ is a DG $\text{Mix}_U(\mathcal{O}_X)$-module on $X$, with differential $d_{\text{mix}}$.

**Theorem 5.2.** Let $\pi : Z \to X$ be a morphism of schemes, and suppose $\sigma$ is a simplicial section of $\pi$ based on an open covering $U$ of $X$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules, and let

$$\phi : \prod_{i=1}^r (\Omega_Z \otimes_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)) \to \Omega_Z \otimes_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous $\Omega_Z$-multilinear sheaf morphism on $Z$ of degree $k$. Then there is an induced $\text{Mix}_U(\mathcal{O}_X)$-multilinear sheaf morphism of degree $k$

$$\sigma^*(\phi) : \text{Mix}_U(\mathcal{M}_1) \times \cdots \times \text{Mix}_U(\mathcal{M}_r) \to \text{Mix}_U(\mathcal{N})$$

on $X$ with the following properties:

(i) The assignment $\phi \mapsto \sigma^*(\phi)$ respects the operations of composition and summation.

(ii) If $\phi = \pi^*(\phi_0)$ for some continuous $\Omega_X$-multilinear morphism

$$\phi_0 : \prod_{i=1}^r (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

then $\sigma^*(\phi) = \tilde{\text{NC}}(U, \phi_0)$.

(iii) Assume that

$$\nabla_p \circ \phi - (-1)^k \phi \circ \nabla_p = \psi$$

for some continuous $\Omega_Z$-multilinear sheaf morphism

$$\psi : \prod_{i=1}^r (\Omega_Z \otimes_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)) \to \Omega_Z \otimes_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

of degree $k + 1$. Then

$$d_{\text{mix}} \circ \sigma^*(\phi) - (-1)^k \sigma^*(\phi) \circ d_{\text{mix}} = \sigma^*(\psi).$$

Before the proof we need an auxiliary result.
Lemma 5.3. Let $A$ and $B$ be complete DG algebras in $\text{Dir Inv Mod} \mathbb{K}$, and let $f^* : A \to B$ be a continuous DG algebra homomorphism. To any DG $A$-module $M$ in $\text{Dir Inv Mod} \mathbb{K}$ we assign the DG $B$-module $f^*M := B \hat{\otimes}_A M$. Then to any continuous $A$-multilinear function $\phi : \prod_i M_i \to N$ we can assign a continuous $B$-multilinear function $f^*(\phi) : \prod_i f^*(M_i) \to f^*(N)$. This assignment is functorial in $f^*$, and respects the operations of composition and summation. If $\phi$ and $\psi$ are such continuous $A$-multilinear functions, homogeneous of degrees $k$ and $k+1$ respectively and satisfying
\[ d \circ \phi - (-1)^k \phi \circ d = \psi, \]
then
\[ d \circ f^*(\phi) - (-1)^k f^*(\phi) \circ d = f^*(\psi). \]

Proof. This is all straightforward, except perhaps the last assertion. For that we make the calculations. By continuity and multilinearity it suffices to show that
\[ (d \circ f^*(\phi))(\beta) - (-1)^k (f^*(\phi) \circ d)(\beta) = f^*(\psi)(\beta) \]
for $\beta = (\beta_1, \ldots, \beta_r)$, with $\beta_i = b_i \otimes m_i$, $b_i \in B^{p_i}$ and $m_i \in M^{q_i}$. Then
\[ (d \circ f^*(\phi))(\beta) = d(\pm b_1 \cdots b_r \cdot \phi(m_1, \ldots, m_r)) \]
\[ = \pm d(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot d(\phi(m_1, \ldots, m_r)) \]
with Koszul signs. Since
\[ d(\beta_i) = d(b_i) \otimes m_i \pm b_i \otimes d(m_i) \]
we also have
\[ (f^*(\phi) \circ d)(\beta) = \sum_i \pm f^*(\phi)(\beta_1, \ldots, d(\beta_i), \ldots, \beta_r) \]
\[ = \sum_i \left( \pm b_1 \cdots b_i \cdots b_r \cdot \phi(m_1, \ldots, m_r) \right) \]
\[ \pm b_1 \cdots b_r \cdot \phi(m_1, \ldots, d(m_i) \cdots m_r) \]
\[ = \pm d(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot \phi(d(m_1, \ldots, m_r)). \]
Finally
\[ f^*(\psi)(\beta) = \pm b_1 \cdots b_r \cdot \psi(m_1, \ldots, m_r), \]
and the signs all match up.

Proof of the theorem. For a sequence of indices $i = (i_0, \ldots, i_l) \in \Delta_+^n$ let us introduce the abbreviation $Y_i := \Delta^n_{i_0} \times U_i$, and let $p_2 : Y_i \to U_i$ be the projection. The simplicial section $\sigma$ restricts to a morphism $\sigma_i : Y_i \to Z$.

By Lemma 5.3 applied with respect to the DG algebra homomorphism $\sigma_i^* : \Omega^{-1} Y_i \to \Omega Y_i$, there is an induced continuous $\Omega Y_i$-multilinear morphism
\[ \sigma_i^* (\phi) : \prod_j \left( \Omega X \hat{\otimes}_{\sigma_i^{-1} \Omega Y_i} \sigma_i^{-1} (\Omega Z \hat{\otimes} \mathcal{O} \pi^\omega \mathcal{M} j) \right) \]
\[ \to \Omega Y_i \hat{\otimes}_{\sigma_i^{-1} \Omega Y_i} \sigma_i^{-1} (\Omega Z \hat{\otimes} \mathcal{O} \pi^\omega \mathcal{N} j) \]

Now for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we have an isomorphism of dir-inv DG $\Omega Y_i$-modules
\[ \Omega Y_i \hat{\otimes}_{\sigma_i^{-1} \Omega Y_i} \sigma_i^{-1} (\Omega Z \hat{\otimes} \mathcal{O} \pi^\omega \mathcal{M} j) \cong \Omega Y_i \hat{\otimes} \mathcal{O}_{Y_i} P^I_2 (\mathcal{P} \hat{\otimes} \mathcal{O}_X \mathcal{M}). \]
Under the DG algebra isomorphism $p_2^*\Omega_Y \cong \Omega(\Delta_k^t) \otimes \Omega_U$, there is a dir-inv DG module isomorphism

$$p_2^*\left(\Omega_Y \otimes_{\mathcal{O}_Y} p_2^*(P_X \otimes_{\mathcal{O}_X} \mathcal{M})\right) \cong \Omega(\Delta_k^t) \otimes (\Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{M})|_{U_i}.$$

Thus we obtain a family of morphisms

$$\sigma_i^*(\phi) : \prod_{j=1}^r \left(\Omega(\Delta_k^t) \otimes (\Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{M}_j)|_{U_i}\right) \rightarrow \Omega(\Delta_k^t) \otimes (\Omega_X \otimes_{\mathcal{O}_X} P_X \otimes_{\mathcal{O}_X} \mathcal{N})|_{U_i}$$

indexed by $i$ and satisfying the simplicial relations. Now use Lemma 3.6 to obtain $\sigma^*(\phi)$. Properties (i-iii) follow from Lemma 5.7.

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