The translational oscillations of a cylindrical bubble in a bounded volume of a liquid with free deformable interface

A A Alabuzhev¹² and M I Kaysina²
¹Institute of Continuous Media Mechanics UB RAS, Perm 614013, Russia
²Perm State University, Perm 614990, Russia

E-mail: alabuzhev@mail.ru

Abstract. We consider the eigen and forced translation oscillations of a cylindrical gas bubble surrounded by an incompressible fluid with free deformable interface. The bubble has an equilibrium cylindrical shape and is bounded axially by two parallel solid surfaces. Dynamics of contact lines is taken into account by an effective boundary condition: velocity of the contact line is assumed to be proportional to contact angle deviation from the equilibrium value. The equilibrium contact angle is right. Eigen frequency decreases with liquid outer free surface radius decreasing and increases with the radius-to-height ratio increasing. We found that the main translational frequency of eigen oscillation vanishes at a certain value of the Hocking parameter (the so-called wetting parameter). The eigen frequencies of the bubble are higher than the eigen frequencies of fluid volume.

1. Introduction

Problems with the contact-line dynamics taken into account have been examined in various formulations [1-4]. In the study of the dynamic contact angle Hocking condition is widely used (by virtue of its simplicity), and it is the one employed in a study [5] of the damping of standing waves between two vertical walls. This condition assumes a linear relationship between the velocity of the contact line motion and the contact angle (for the case of right equilibrium contact angle)

\[ \frac{\partial \zeta^*}{\partial t} = \Lambda \cdot \vec{\nabla} \zeta^*, \]  

(1)

where \( \zeta^* \) is the deviation of the interface from the equilibrium position, \( \vec{k} \) is the external normal to the solid surface, \( \Lambda \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter) having the dimension of the velocity. The special cases of the boundary condition (1) are the requirement of the fixed contact line (\( \zeta^* = 0 \), the so-called pinned-end edge condition) and the constant contact angle (\( \vec{k} \cdot \vec{\nabla} \zeta^* = 0 \)). Article [5] also presents a qualitative comparison with the experimental studies of the other authors. The equation (1) was used in the study of oscillations of the sessile drop and bubble [6-8], cylindrical drop and bubble [9,10], capillary bridge [11] and “sandwiched” drop [12]. The particular case of pinned-end edge was used for capillary bridge in [13,14] and sessile drop [15]; the case of constant contact angle was used for cylindrical drop in [16] and for sessile drop in [17].

The more difficult condition was proposed in [18]. This condition involves an ambiguous dependence of the contact angle on the contact line velocity (for simplicity, the equilibrium contact angle is generally assumed to be a right angle)

¹ To whom any correspondence should be addressed.
where $\gamma = \partial z^*/\partial r^*$ is the contact angle deviation from the equilibrium value, $z^*$ is the coordinate normal to the solid wall and extending deep into the fluid. Results of physical experiments [19,20] are described well by formula (2) for small deviations of the contact angle. The equation (2) was used in the study of oscillations of the sessile drop [21], hemispherical bubble [22] on solid plate and cylindrical drop [23].

In the present study, we consider the translational oscillations of cylindrical bubble which surrounded by a liquid with free deformable interface. We apply condition (1) and continue the study of the axisymmetric oscillations [10,24,25]. It is important to note that the external liquid has the free nondeformable interface in [10,24] and the rigid wall in [25].

2. Problem formulation

Consider a gaseous bubble surrounded by a liquid with a free external surface (see figure 1). The system is bounded by two parallel solid surfaces separated by a distance $h^*$. The bubble and liquid volume are assumed to be cylindrical, with a radius $r_0^*$ and $R_0^*$ in equilibrium, respectively, implying that the every equilibrium contact angle is $90^\circ$. The system is acted upon by an external vibration force with amplitude $A^*$ and oscillation frequency $\omega^*$. The force directed parallel to the solid surfaces.

The characteristic amplitude of oscillations $A^*$ is small compared to the equilibrium radius $r_0^*$. We assume that, on the one hand, the fundamental oscillation frequency $\omega^*$ is large enough for the viscosity could be ignored, and, on the other hand, the oscillation frequency is small enough, so that we can use the incompressibility conditions $\delta^2 = \sqrt{\nu/\omega^*} < r_0^*$ and $\omega^* r_0^* < c$ ( $\delta$ is the boundary-layer thickness, $c$ is the sound velocity and $\nu$ is the kinematic viscosity of liquid).

Since the problem is symmetric, it is convenient to introduce cylindrical coordinates $(r^* , \alpha , z^*)$. The azimuthal angle $\alpha$ is reckoned from the $x$ axis. Let the lateral surface of the bubble and the external surface of liquid volume be described by the following equations, respectively

$$r^* = r_0^* + \zeta^* (\alpha , z^*, t^*), \quad r^* = R_0^* + \xi^* (\alpha , z^*, t^*).$$

Following [10,24], we use $\sqrt{\rho_o r_0^3/\sigma^*}$, $r_0^*$, $h^*$, $A^*$, $A^* r_0^3/\sigma^*$, $A^* \sqrt{\sigma^*/\rho_o r_0^3}$ as the scales for the time, length, height, deviation of bubble surface and free surface from its equilibrium position, pressure, and velocity potential, respectively. Here, $\sigma^*$ is the surface tension and $\rho_o^*$ is the liquid density. Thus, the dimensionless boundary value problem is determined by (intermediate steps can be found in [10,24])

$$p_e = -\varphi (\varphi + \omega^* r \cos (\alpha) e^{i\omega^*}) = 0, \quad p_i = -2n_p^0 P_{\varphi \varphi}^* r_0^3/\sigma^* \langle \zeta^* \rangle \equiv -P_0 \langle \zeta^* \rangle,$$

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + \frac{b^2}{\partial z^2} \zeta^*,$$

\[ (2) \]
In view of translation algebra:

\[ \mathbf{r} = 1: \ \xi = \varphi_1, \ [p] = \xi + \varphi_2^2 + b^2 \xi_2, \quad (4) \]

\[ \mathbf{r} = R_0: \ \xi = \varphi_1, \ [p] = \xi + \varphi_2^2 + b^2 \xi_2, \quad (5) \]

\[ z = \pm \frac{1}{2} \ \varphi_2 = 0, \quad (6) \]

\[ \mathbf{r} = 1, \ z = \pm \frac{1}{2} \ \xi = \pm \lambda \xi_z, \quad (7) \]

\[ \mathbf{r} = R_0, \ z = \pm \frac{1}{2} \ \xi = \pm \lambda \xi_z, \quad (8) \]

where \( \varphi_2 \) is the liquid pressure, \( \varphi_1 \) is potential of liquid velocity, \( p_1 \) is the gas pressure in the bubble, \( n_2 \) is polytropic (e.g., adiabatic) exponent, \( P_0 \) is dimension gas pressure in the bubble. The boundary-value problem (3)–(7) involves five parameters: the aspect ratio, the radius of free surface, the wetting parameter, the frequency and amplitude

\[ b = r_0^3 / h^3, \quad R_0 = R_0^* / r_0^*, \quad \lambda = \kappa / \sqrt{\rho_0^* / \rho}, \ \omega = \omega^* \sqrt{\rho_0^* / \rho}, \ \varepsilon = A^* / r_0^*. \]

### 3. Natural translational oscillations of the bubble

In order to investigate the problem, it is convenient to begin with a consideration of the natural oscillations of a cylindrical bubble. By the evenness of the natural oscillation modes is meant the evenness of the functions under a change of sign of the axial coordinate \( z \). In view of translational symmetry, the solution of the boundary value problem (3)–(8) without external force is written as

\[ \varphi(r, z, t, \alpha) = \sum_{m=0}^{\infty} \left( a_m(R_0) + b_m(R_0) \right) \sin \left( \left( 2n+1 \right) \pi z \right) + \sum_{m=0}^{\infty} \left( a_m(R_0) + b_m(R_0) \right) \cos \left( \left( 2n+1 \right) \pi z \right) \right) e^{\omega t} e^{\omega t}, \quad (9) \]

\[ \xi(\alpha, z, t) = d_z + d_z z^2 + \sum_{m=0}^{\infty} h_m \sin \left( \left( 2n+1 \right) \pi z \right) + \sum_{m=0}^{\infty} c_m \cos \left( \left( 2n+1 \right) \pi z \right) \right) e^{\omega t} e^{\omega t}, \quad (10) \]

\[ \xi(\alpha, z, t) = D_z + D_z z^2 + \sum_{m=0}^{\infty} H_m \sin \left( \left( 2n+1 \right) \pi z \right) + \sum_{m=0}^{\infty} C_m \cos \left( \left( 2n+1 \right) \pi z \right) \right) e^{\omega t} e^{\omega t}, \quad (11) \]

where \( \Omega \) is eigen frequency, \( R_{2n}^i(r) = 1, \left( (2n+1) \pi b r \right), \ R_{2n}^i(r) = K_2, \left( (2n+1) \pi b r \right), \ R_{2n}^i(r) = r^2, \ R_{2n}^i(r) = 1, \left( 2n \pi b r \right), \ R_{2n}^i(r) = r^2, \ R_{2n}^i(r) = 1, \left( 2n \pi b r \right), \ R_{2n}^i(r) = K_1, \left( 2n \pi b r \right), \ I_1 \) and \( K_1 \) are modified Bessel functions of the first order.

Substituting solutions (9)–(11) into (3)–(8), we obtain a spectral-amplitude problem which eigenvalues are the values of the natural oscillation frequency \( \Omega \). From the solution of this problem it follows that the eigenvalues are found from the equations:

\[ \sum_{m=0}^{\infty} \frac{1}{\pi^2 n^2 \omega^2 - \Omega^2(R_0)} = -\frac{\lambda}{i \omega}, \quad (12) \]

\[ \sum_{m=0}^{\infty} \frac{\Omega^2(R_0)}{\omega^2 - \Omega^2(R_0)} S_k + S_k(S_1 - S_k) = \frac{S_2}{S_4} \quad (13) \]

\[ \Omega^2_{2n}(1) = 4\pi^2 n^2 b^2 R_{n_1}^i(R_0) R_{n_2}^i(R_0) - R_{n_2}^i(R_0) R_{n_1}^i(R_0), \quad \Omega^2_{2n}(1) = 4\pi^2 n^2 b^2 R_{n_2}^i(R_0) R_{n_1}^i(R_0) - R_{n_1}^i(R_0) R_{n_2}^i(R_0), \quad S_2 = \Omega^2_{1n}(1) + \Omega^2_{11}(R_0) \quad (14) \]

\[ S_3 = \Omega^2_{11}(1), S_4 = \Omega^2_{1n}(1) \cdot \Omega^2_{11}(R_0), \quad S_5 = -\frac{4b^2 R_0^2}{(R_0 - 1)^2}, \quad S_6 = \frac{R_0 + 1}{2 R_0^2}, \quad S_7 = \frac{2b^2 (R_0^2 + 1)}{(R_0 - 1)^2}. \]
Odd modes

\[ \sum_{n=0}^{4} \frac{\Omega_n^2}{\pi^2 (2n+1)^2 \Omega_n^2(1) - \Omega^2} + \frac{1}{2} = \frac{-\lambda}{i \Omega}, \quad (14) \]

\[ \sum_{n=0}^{4} \frac{\Omega_n^2(R_0)}{\pi^2 (2n+1)^2 \Omega_n^2(R_0) - \Omega^2} = \frac{-\lambda}{i \Omega}, \quad (15) \]

\[ \Omega_n^2(1) = (2n + 1)^2 \pi^2 b^2 \frac{R_n^{\epsilon c}(1)R_n^{\epsilon c}(R_0) - R_n^{\epsilon c}(R_0)R_n^{\epsilon c}(1)}{R_n^{\epsilon c}(1)R_n^{\epsilon c}(R_0) - R_n^{\epsilon c}(R_0)R_n^{\epsilon c}(1)}, \]

\[ \Omega_n^2(R_0) = \frac{(2n + 1)^2 \pi^2 b^2 (R_n^{\epsilon c}(1)R_n^{\epsilon c}(R_0) - R_n^{\epsilon c}(R_0)R_n^{\epsilon c}(1))}{R_n^{\epsilon c}(1)R_n^{\epsilon c}(R_0) - R_n^{\epsilon c}(R_0)R_n^{\epsilon c}(1)}. \]

Here, \( \Omega_n^2(1) \), \( \Omega_n^2(R_0) \), \( \Omega_n^2(1) \), \( \Omega_n^2(R_0) \) are the natural oscillation frequencies of the freely moving contact line. The complex algebraic equations (12)–(15) have complex solutions, this leads to oscillation damping due to the dissipation on the contact line.

**Figure 2.** Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter \( \lambda \) for \( \Omega_0 \) (\( R_0 = 5 \), \( P_0 = 5 \)).

\( b = 1 \) – line 1, \( b = 2 \) – line 2, liquid – solid line, bubble – dashed line.

**Figure 3.** Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter \( \lambda \) for \( \Omega_2 \) (\( R_0 = 5 \), \( P_0 = 5 \), \( b = 1 \)).

liquid – solid line, bubble – dashed line.

Equations (12)–(15) were solved numerically with the usage of the two-dimensional secant method. Figures 2 and 3 show the real part of \( \text{Re}(\Omega) \) (oscillation frequency) and imaginary part
Im(Ω) (damping ratio) of the complex natural frequency Ω for the oscillation even modes Ω_0 (i.e., k = 0 is the wavenumber) and Ω_2. Here and below for the frequency indices, consecutive numbering of the wavenumber is used: even values of k correspond to the even modes (the solution of Eqs. (12), (13)) and odd values of k correspond to the odd modes (the solution of Eqs. (14), (15)). The zero translational mode k = 0 describes the displacement of the liquid volume as a whole. In the considered case (with the contact line dynamics taken into account) the displacement is larger in the central part of the column than near the ends. Elastic forces cause the liquid volume to take the original shape, resulting in the return motion of its center of mass. As the capillary parameter increases, the shift between the center and periphery of the free surface decreases. For a certain value of the capillary parameter, the difference of the value of the shift disappears and the zero eigen frequency vanishes (figure 2a). For large values of λ, the damping ratio takes two values (figure 2b).

From figure 3 it follows that as λ increases, the frequency decreases monotonically, the damping ratio has a maximum for a finite value of the capillary parameter and tends to zero as λ → 0 and λ → ∞. Note, that the eigen frequencies of the bubble are higher than the eigen frequencies of the fluid volume. It is the result of the influence of the surface tension force on external interface.

Figure 4 shows the real part of Re(Ω) (oscillation frequency) and imaginary part Im(Ω) (damping ratio) of the complex natural frequency Ω for the oscillation odd mode Ω_1. From figures 2 and 4 it follows that the natural oscillation frequency increases as the parameter b increases (i.e., as the bubble equilibrium radius increases or as the bubble height decreases).

**Figure 4.** Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter λ for Ω_2 (R_0 = 5, P_0 = 5).

b = 1 – line 1, b = 2 – line 2, liquid – solid line, bubble – dashed line.

4. **Forced translation oscillations of a bubble**

Here we consider the problem of forced oscillations of a cylindrical bubble. The fields of velocity potential and surfaces deflections are defined by

\[ \varphi(r, z, t, \alpha) = \text{Re} \left\{ \sum_{n=0}^{\infty} \left( a_n R_n(r) + b_n R_n(r) \right) \cos(2\pi nz) \cos(\alpha) e^{i\omega t} \right\}, \]

\[ \zeta(\alpha, z, t) = \text{Re} \left[ z^2 d + \sum_{n=0}^{\infty} c_n \cos(2\pi nz) \cos(\alpha) e^{i\omega t} \right], \]

\[ \xi(\alpha, z, t) = \text{Re} \left[ z^2 D + \sum_{n=0}^{\infty} C_n \cos(2\pi nz) \cos(\alpha) e^{i\omega t} \right]. \]

Substituting solutions (16)–(18) into (3)–(8), we obtain expressions for the unknown amplitudes a_n, b_n, c_n, C_n, d and D. These expressions are not given due to their cumbersome forms.
5. Conclusions

Eigen frequency decreases with the decrease of the liquid outer free surface radius and increase with the bubble radius-to-height ratio increasing. We found that the main translational frequency of the natural oscillations vanishes at a certain value of the Hocking parameter. The eigen frequencies of the bubble are higher than the eigen frequencies of the fluid volume. It is the result of the influence of the surface tension force on external interface. Also note that the frequencies of the translational modes are independent of the gas pressure inside the bubble.

Well-marked resonance effects are found in the study of the forced oscillations. Thus, one can choose the bubble radius-to-height ratio such that the characteristic frequency of any mode is equal to zero, in order to ultimately, to determine the wetting parameter. There are not the «anti-resonant» frequencies under the translational vibrations, i.e. the vibration frequencies at which the amplitude of contact line deviation is zero [7,10].

Acknowledgments

This work was supported by the Russian Foundation for Basic Research (project 14-01-96017-r-ural).

6. References

[1] De Gennes P G 1985 Rev. Mod. Phys. 57 827
[2] Voinov O V 1976 Fluid Dyn 11 714–721
[3] Bonn D, Eggers J, Indekeu J, Meunier J and Rolley E 2009 Rev. Mod. Phys. 81 739–805
[4] Zhang L and Thiessen D B 2013 J. Fluid Mech. 719 295–313
[5] Hocking L M 1987 J. Fluid Mech. 179 253-66
[6] Lyubimov D V, Lyubimova T P and Shklyaev S V 2004 Fluid Dynamics 39 851–62
[7] Lyubimov D V, Lyubimova T P and Shklyaev S V 2006 Phys. Fluids 18 012101
[8] Shklyaev S and Straube A V 2008 Phys. Fluids 20 052102
[9] Alabuzhev A A and Lyubimov D V 2007 J. Appl. Mech. Tech. Phys. 48 686–93
[10] Alabuzhev A A 2014 Computational Continuum Mechanics 7 151–61 (in Russian)
[11] Borkar A and Tsamopoulos J 1991 Phys. Fluids A 3 2866–74
[12] Alabuzhev A A and Lyubimov D V 2012 J. Appl. Mech. Tech. Phys. 53 9–19
[13] Tsamopoulos J, Chen T and Borkar A 1992 J. Fluid Mech. 235 579–609
[14] Demin V A 2008 Fluid Dyn 43 524–532
[15] Kartavih N N and Shklyaev S V 2007 Bulletin of Perm University. Series: Physics no 1(6) 23–28 (in Russian)
[16] Alabuzhev A A and Lyubimov D V 2005 Fluid Dyn 40 183-192
[17] Ivantsov A O 2012 Bulletin of Perm University. Series: Physics no 3(21) 16–23 (in Russian)
[18] Hocking L M 1987 J. Fluid Mech. 179 267
[19] Ablett R 1923 Philos. Mag. 46 244
[20] Dussan V E B 1979 Annu. Rev. Fluid Mech. 11 371
[21] Fayzrakhmanova I S and Straube A V 2009 J. Fluid Mech. 21 072104
[22] Fayzrakhmanova I S, Straube A V and Shklyaev S 2011 J. Fluid Mech. 23 102105
[23] Alabuzhev A A 2012 Bulletin of Perm University. Series: Physics no 1(29) 35-41 (in Russian)
[24] Alabuzhev A A and Kaysina M I 2015 Bulletin of Perm University. Series: Physics no 1(29) 35-41 (in Russian)
[25] Alabuzhev A A and Kaysina M I 2015 Bulletin of Perm University. Series: Physics no 2(30) 56-68 (in Russian)