I. INTRODUCTION

The energy momentum tensor plays a crucial role in quantum field theory since it arises as a Noether current of the Poincare group. It is conserved in all relativistic local theories even when there are no other conserved charges. According to the standard Noether construction (see e.g. Refs. [1, 2] for recent reviews on the classical theory), the energy momentum tensor can be computed as a functional derivative of the action with respect to an external space-time-dependent metric, $g_{\mu\nu}(x)$, around the flat space-time metric $\eta_{\mu\nu}$ (we take the signature $(-,-)$).

$$\frac{1}{2} \theta^{\mu\nu}(x) = \frac{\delta S}{\delta g_{\mu\nu}(x)}|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

where

$$S = \int d^4x \sqrt{-g} \mathcal{L}(x).$$

The chiral Lagrangian $\mathcal{L}$ contains only metric contributions, and it is obtained by promoting the flat space-time metric $\eta_{\mu\nu}$ to the curved one $g_{\mu\nu}$. It takes the form given in Refs. [3, 4]. In QCD, the energy momentum tensor operator probes the interaction of quarks and gluons to gravitons. At the quantum level the high energy behavior of $\theta_{\mu\nu}$ is improved if suitable additional transverse corrections are implemented [4], and within such a set up there is a trace anomaly relating $\theta^{\mu\nu}$ to the divergence of the dilatation current [4, 8, 8], signaling the anomalous breaking of scale invariance. The non-vanishing vacuum expectation value, $\langle 0 | \theta^{\mu\nu}(0) | 0 \rangle$, is related to the existence of a non-vanishing gluon condensate generating a bag constant [4] and scale Ward identities [9, 10]. Deep inelastic scattering provides also some information on the momentum fraction carried by quarks and gluons in a hadron at a given scale [10]. Direct experimental determination on the basis of one graviton exchange is out of question due to the smallness of the gravitational constant as compared to weak and strong processes. The gravitational pion form factor can be used to determine the width of a light Higgs boson into two pions [13], and also the decay of a scalar glueball into two pions [14]. There have also been few attempts in the past to define $\theta^{\mu\nu}$ on the lattice [15], but so far there are no results of practical interest concerning matrix elements between hadron states carrying different momentum.

At low energies, the spontaneous breaking of chiral symmetry dominates and in the meson sector any operator can be described in terms of a non-linearly transforming Pseudoscalar Goldstone boson field $U$ with an infinite number of low energy constants (LEC’s) [2, 3, 16, 17, 18] (for review see, e.g., Refs. [15, 20]). In a chiral expansion, the most general structure of $\theta_{\mu\nu}$ up to and including fourth order corrections reads [18]

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)} + \theta_{\mu\nu}^{(2)} + \theta_{\mu\nu}^{(4)} + \ldots$$

with

$$\theta_{\mu\nu}^{(0)} = -\eta_{\mu\nu} \mathcal{L}^{(0)},$$

$$\theta_{\mu\nu}^{(2)} = \frac{f^2}{4} \langle D_\mu U^\dagger D_\nu U \rangle - \eta_{\mu\nu} \mathcal{L}^{(2)},$$

$$\theta_{\mu\nu}^{(4)} = -\eta_{\mu\nu} \mathcal{L}^{(4)} + 2L_4 \langle D_\mu U^\dagger D_\nu U \rangle \langle \chi^\dagger U + U^\dagger \chi \rangle$$

where

$$\langle A \rangle = \text{tr} A \text{ means trace in flavor space and the expansion for the Lagrangian reads}$$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

The chiral expansion corresponds to an expansion in powers of external momenta. The upper-script means the
order of the contribution. The zeroth order vacuum contribution reads

$$\mathcal{L}^{(0)} = B,$$

(8)

where $B$ is the vacuum energy density constant. The second order contributions reads

$$\mathcal{L}^{(2)} = \frac{f^2}{4} (\partial_\mu U^\dagger \partial^\mu U + (\chi^\dagger U + U^\dagger \chi)), $$

(9)

and the fourth order contribution is

$$\mathcal{L}^{(4)} = L_1 \langle D_\mu U^\dagger D^\mu U \rangle + L_2 \langle D_\mu U^\dagger D_\nu U \rangle^2 + L_3 \langle (D_\mu U^\dagger D^\mu U) \rangle + L_4 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_5 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_6 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_7 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_8 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_9 \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + L_{10} \langle (D_\mu U^\dagger D^\mu U) \rangle^2 + H_1 \langle (F_{\mu\nu}^R)^2 \rangle + H_2 \langle \chi^\dagger \chi \rangle. $$

(10)

Here, we have introduced the standard chiral covariant derivatives and gauge field strength tensors,

$$D_\mu U = D_\mu^L U - U D_\mu^R = \partial_\mu U - i A_\mu^L U + i U A_\mu^R, $$

$$F_{\mu\nu} = i [D_\mu^L, D_\nu^R] = \partial_\mu A_\nu^L - \partial_\nu A_\mu^L - i [A_\mu^L, A_\nu^L],$$

with $r = L, R, f$ is the pion weak decay constant in the chiral limit and $U = e^{i\sqrt{2}n/f}$ a unitary matrix and $\Pi$ a non-linear transforming field under the chiral group representing the octet of pseudoscalar mesons given by

\[
\Pi = \begin{pmatrix}
\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta \\
\pi^+ \\
\pi^- \\
K^0 \\
K^+ \\
K^- \\
\end{pmatrix}.
\]

(12)

Finally, $\chi = 2B_0 \text{diag}(m_s, m_d, m_u)$ is the current quark mass matrix with $B_0 = \langle \bar{q}q \rangle/f^2$ where $\langle \bar{q}q \rangle$ is the quark condensate for one flavor.

Note that the coefficients $L_1 - L_{10}$ appear in $\mathcal{L}^{(4)}$ given by Eq. (10). As one sees, the terms containing $L_{11} - L_{13}$ cannot be obtained by computing the energy momentum tensor from the chiral effective Lagrangian in flat-space time and from this viewpoint are genuine new contributions to $\theta^{\mu\nu}$. The gravitational low energy contributions arise at the hadronic level due to quantum effects and corresponds to the inclusion of curvature terms in the Lagrangian which generate purely transverse terms in the energy momentum tensor. Actually, from a calculational viewpoint it is more advantageous to couple the system to gravity and compute the chiral Lagrangian in curved space-time.

Although the LEC’s should be deduced from QCD itself solely in terms of $\Lambda_{\text{QCD}}$ and the quark masses, their numerical values are obtained from a fit to low energy data, and systematic calculations of the corresponding non-gravitational low-energy constants (LEC’s) $L_{1-10}$ have been carried out in the recent past up to two loop accuracy [21, 22, 23, 24], or by using the Roy equations [25] (see also [26, 27]). For strong and electroweak processes involving pseudoscalar mesons the bulk of the LEC’s is saturated in terms of resonance exchanges [28, 29, 30, 31], which can be justified in the large-$N_c$ limit in a certain low-energy approximation [32] by imposing relevant QCD short-distance constraints. In the case of gravitational processes similar ideas apply [15], although less model independent information is known. (A one loop calculation has been carried out in Ref. [33]).

In a more model dependent set up the LEC’s can also be understood from a more microscopic viewpoint by using chiral quark models [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] for the non-gravitational coefficients $L_1 - L_{10}$. The calculation of $L_{11}, L_{12}$ and $L_{13}$, encoding the coupling to gravitational sources, has only been done for a chiral quark model in Ref. [48] and more recently in the spectral quark model [49]. These values could be used to predict the experimentally unmeasured gravitational form factors of pseudoscalar mesons (see Ref. [32] for a calculation within ChPT).

In this paper we provide the values of the gravitational LEC’s $L_{11,12,13}$, as well as $H_{0,1,2,3}$, in several chiral quark models. Namely, we use in Sect. [10] the Nambu-Jona-Lasinio (NJL) model [50, 51] and the Georgi-Manohar (GM) model [52] in the presence of gravity. Although these models look very different in appearance they do incorporate chiral symmetry breaking and quarks have a constituent mass $M \sim 300 \text{MeV}$. The main difference between them is that while in the NJL model this mass is dynamically generated through interactions, the GM already assumes that this breaking has taken place. Moreover, while in the GM model there are only pseudoscalar Goldstone bosons as dynamical degrees of freedom in the NJL there are additional $qq$ scalar fields. Finally, in the NJL model quarks appear to have an axial coupling constant $g_A$ for the quarks equal to one, while in the GM model one starts already with $g_A \neq 1$. A more detailed analysis [42] shows that if in NJL $G_V \neq 0$ then $g_A \neq 1$ (see also below). More specifically, we consider the generalized SU(3) NJL with scalar, pseudoscalar, vector and axial couplings extending the original calculation of Ref. [42] (see also Ref. [45] where also an estimate on the gluonic effects of the LEC’s was done). To this end we apply, after bosonization and Pauli-Villars regularization, a coordinate and frame invariant derivative expansion by means of the heat kernel technique in curved space-time (Sect. [111]). After that, the scalars, vectors and axial vectors are eliminated at the mean field level (Sect. [113] and the full results are presented in Sect. [114]).

For notation details, we rely heavily on previous work [42, 44, 52]. Nevertheless, the inclusion of gravity is a bit messy, so we present the minimal amount of formulas to make the paper more comprehensive and also
to point out some differences with previous calculations non-gravitational LEC’s by one of us (ERA) \cite{42} and other authors \cite{45} as well as our previous work with Brønnowski on the gravitational LEC’s in the spectral quark model \cite{49}.

II. CHIRAL QUARK MODELS IN CURVED SPACE TIME

The coupling of gravity\(^1\) to chiral quark models has been described at length in Ref. \cite{49} where more details on the notation and this specific model can be found in the context of the Spectral Quark Model (SQM) \cite{52,53,54}. In this paper we discuss several constituent chiral quark models which have the common feature of incorporating dynamical chiral symmetry breaking at the one quark loop level: The Nambu–Jona-Lasinio (NJL) model \cite{55,56} and the Georgi-Manohar (GM) model \cite{52}. In these models quarks have a constituent mass \(M\) model \cite{52}. The main difference between them has to do with the presence or not of dynamical \(\bar{q}q\) scalar fields respectively. In addition, while the NJL model dynamically generates spontaneous chiral symmetry breaking, the GM model already starts in a chirally broken phase. To avoid unnecessary duplication we will discuss explicitly the NJL model with scalar and vector couplings from which the relevant implications for the GM model can be more easily deduced.

A. Nambu–Jona-Lasinio model

The Nambu–Jona-Lasinio (NJL) action in curved Minkowski space-time with metric tensor \(g_{\mu\nu}(x)\) reads

\[
S = \int d^4x \sqrt{-g} \mathcal{L}_{NJL} \hspace{1cm} (13)
\]

where \(g = \text{Det}(g_{\mu\nu})\) and the Lagrangian is given by

\[
\mathcal{L}_{NJL} = \bar{q}(i\slashed{D} + \mu - \tilde{M}_0)q + \frac{G_S}{2} \sum_{a=0}^{N_f-1} ((\bar{q}\lambda_a q)^2 + (\bar{q}\lambda_a i\gamma_5 q)^2) - \frac{G_V}{2} \sum_{a=0}^{N_f-1} ((\bar{q}\lambda_a \gamma_\mu q)^2 + (\bar{q}\lambda_a \gamma_\mu \gamma_5 q)^2) \hspace{1cm} (14)
\]

where \(q = (u, d, s, \ldots)\) represents a quark spinor with \(N_c\) colors and \(N_f\) flavors. The \(\lambda\)'s represent the Gell-Mann matrices of the \(U(N_f)\) group and \(\tilde{M}_0 = \text{diag}(m_u, m_d, m_s, \ldots)\) stands for the current quark mass matrix. In the limiting case of vanishing current quark masses the classical NJL-action is invariant under the global \(U(N_f)_R \otimes U(N_f)_L\) group of transformations. The derivative \(\partial_\mu - i\omega_\mu\) is frame (local Lorentz) and general-coordinate covariant and it includes the spin connection,

\[
\omega_\mu(x) = \frac{i}{8} \langle \gamma^\nu(x), \gamma_{\nu\mu}(x) \rangle \hspace{1cm} (15)
\]

where the frame covariant derivative is defined as usual

\[
\gamma_{\nu\mu} = \partial_\mu \gamma_\nu(x) - \Gamma^\lambda_{\mu\nu}(x)\gamma_\lambda(x) \hspace{1cm} (16)
\]

and \(\Gamma^\lambda_{\mu\nu}\) are the Christoffel symbols which for a Riemannian space read,

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\sigma\tau} \{ \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu} \} \hspace{1cm} (17)
\]

The space-time dependent Dirac matrices, \(\gamma_\mu(x)\), fulfill the standard anticommutation rules \(\{ \gamma^\mu(x), \gamma^\nu(x) \} = 2g^{\mu\nu}(x)\). The \(\gamma_5\) matrix is, on the contrary, \(x\)-independent (we use the conventions of Ref. \cite{58}). The Dirac slash is constructed as \(i\slashed{D}(x) = i\gamma_\mu(x)\partial_\mu q(x)\).

The vacuum to vacuum transition amplitude in the presence of external scalar, pseudoscalar, vector, axial and metric bosonic fields \((s, p, v, a, g)\) Lagrangian can be written as a path integral in the form

\[
Z[s, p, v, a, g] = \langle 0 | T \exp \left\{ i \int d^4x \sqrt{-g} \mathcal{L}_{\text{ext}}(x) \right\} | 0 \rangle \hspace{1cm} (18)
\]

where the external field Lagrangian is given by

\[
\mathcal{L}_{\text{ext}}(x) = \bar{q} \left( \not\!D + \not\!q \gamma_5 - (s + i\gamma_5 p) \right) q \hspace{1cm} (19)
\]

Following the standard procedure \cite{54} it is convenient to introduce auxiliary bosonic fields \((S, P, V, A, s, p, v, a, g)\) so that after formally integrating the quarks out one gets the equivalent generating functional

\[
Z[s, p, v, a, g] = \int DSDPDVDAd\tau[S, P, V, A; s, p, v, a, g] \hspace{1cm} (20)
\]

\(^2\) Note that in the curved case \(\partial^\mu \varphi(x) = g^{\mu\nu}(x)\partial_\nu \varphi(x)\) and the self-adjoint \(D\)’Alambertian on a spin-0 field is given by \(\partial^\mu \varphi(x) = 1/\sqrt{-g} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi(x))\) so that

\[
\int d^4x \sqrt{-g} \varphi_1(x) \partial^2 \varphi_2(x) = \int d^4x \sqrt{-g} \varphi_1(x) \partial^2 \varphi_1(x).
\]

Likewise, for a Dirac spinor the form \(\int d^4x \sqrt{-g} \bar{\psi}_1 (i\slashed{D} + \not\!\psi) \psi_2\) is Hermitian.

---

\(^1\) We consider only Einstein gravity, i.e., we use the Riemann connection, uniquely defined by being torsionless and metric preserving (metricity). Extension to torsion gravity \cite{52} is possible but will not be studied here.
where the effective action
\[ \Gamma[S, P, V, A; s, p, v, a, g] = -iN_c \text{Tr} \log (iD) \]
\[ + \int d^4x \sqrt{-g} \mathcal{L}_m \]  
(21)
have been introduced. Here, the Dirac operator is given by
\[ iD = i\bar{\psi} + \psi - \bar{M}_0 + (\bar{V} + A\gamma_5 - \bar{S} - i\gamma^5\bar{P}) \]
where we introduce the additive combinations
\[ \bar{S} = s + S, \quad \bar{P} = p + P, \quad \bar{V} = v + V, \quad \bar{A} = a + A \]
(22)
and the mass term
\[ \mathcal{L}_m = -\frac{1}{4G_S} (S^2 + P^2) + \frac{1}{4G_V} (V^\mu V^\mu + A_\mu A^\mu) \]  
(23)
has been defined. Here, \((S, P, V, A)\) are dynamical scalar, pseudoscalar, vector and axial meson fields respectively, whereas \((s, p, v, a, g)\) represent external sources. With the exception of the metric tensor \(g_{\mu\nu}(x)\), all fields are expanded in terms of the \(\lambda\) flavor matrices. Notice also that for the path integral in the bosonic fields to be well defined in Minkowski space we must use the prescription \(G_S^{-1} \rightarrow G_S^{-1} - i\varepsilon\) and \(G_V^{-1} \rightarrow G_V^{-1} - i\varepsilon\).

The additive contribution to the effective action can be separated into a \(\gamma_5\)-even and \(\gamma_5\)-odd parts corresponding to normal and abnormal parity processes respectively. Introducing the operator
\[ \mathcal{D}_5[S, P, V, A] = \gamma_5 \mathcal{D}[S, -\bar{P}, \bar{V}, -\bar{A} \gamma_5, \]  
(24)
the \(\gamma_5\)-even contribution is quadratically divergent and can be regularized in a chiral gauge invariant manner by means of the Pauli-Villars scheme \[60\]
\[ \Gamma_{\text{even}} = -iN_c \text{Tr} \sum_i c_i \log (DD_5 + \Lambda^2 + i\varepsilon) \]
\[ - \frac{1}{4G_S} (S^2 + P^2) + \frac{1}{4G_V} (V^\mu V^\mu + A_\mu A^\mu) \]  
(25)
where the Pauli-Villars regulators fulfill \(c_0 = 1, \Lambda_0 = 0\) and the conditions \(\sum_i c_i = 0, \sum_i c_i \Lambda_i^2 = 0\) which render finite the logarithmic and quadratic divergences respectively. In practice, we take two cut-offs in the coincidence limit \(\Lambda_1 \rightarrow \Lambda_2 = \Lambda\) and hence \(\sum_i c_i f(\Lambda_i^2) = f(0) - f(\Lambda^2) + \Lambda^2 f'(\Lambda^2)\). In the vacuum \((s = p = v = a = 0)\) one has spontaneous chiral symmetry breaking, since by minimizing \(\Gamma_{\text{even}}\) one gets \(S = M\) with \(M\) the constituent quark mass and \(P = V = A = 0\), and the gap equation
\[ \frac{1}{G_S} = -iN_c \sum_i c_i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2 - \Lambda_i^2} \]  
(26)
where the Pauli-Villars regularization has been used. Thus, according to Goldstone's theorem we will write the decomposition
\[ \mathcal{M} = \bar{S} + i\bar{P} = MU + \overline{\mathcal{M}} \]
(27)
with \(U = e^{i\sqrt{2}H/f}\) and \(\Pi\) is given by Eq. (12).

### B. Georgi-Manohar model

In the presence of gravity, the Georgi-Manohar model Lagrangian \[52\] reads
\[ \mathcal{L}_{\text{GM}} = \bar{q} \left( iD + \frac{1}{2}(1 - g_A^2)U^5 \partial_\mu U^5 \right) q + \frac{1}{2} (1 - g_A) U_{\mu\nu} \]  
(28)
where \(U_5 = e^{i\sqrt{2}H(f)}\), and \(\Pi\) is given by Eq. (12), \(M\) the constituent quark mass, and \(g_A\) the axial coupling of the quarks, which will be assumed as suggested in Ref. [52] to be different from one. The generating functional now reads,
\[ Z[s, p, v, a, g] = \int DU e^{i\Gamma[U; s, p, v, a, g]} \]  
(29)
with the effective action given by
\[ \Gamma[U; s, p, v, a, g] = -iN_c \text{Tr} \log (iD) \]  
(30)

Direct comparison with the one quark loop effective action of the NJL, Eq. (21) shows that it corresponds to a similar model without mass terms \(\mathcal{L}_m\) and with a Dirac operator as in Eq. (22) with a specific choice of the spin-1 dynamical field operators,
\[ V_\mu = \frac{1}{4}(1 - g_A) \left[ U^\dagger \partial_\mu U - \partial_\mu U^\dagger U \right], \]  
(31)
\[ A_\mu = \frac{1}{4}(1 - g_A) \left[ U^\dagger \partial_\mu U + \partial_\mu U^\dagger U \right]. \]
(32)

In Eq. (30) we understand that the same Pauli-Villars regularization as in the NJL model has been implemented.
III. HEAT KERNEL EXPANSION

In the chiral expansion of the action, Eq. 21, the counting is the standard one, the pseudoscalar field $U$ and the curved space-time metric $g_{\mu\nu}$ are zeroth order, the vector and axial fields $v_{\mu}$ and $a_{\mu}$ are first order, and any derivative $\partial_{\mu}$ is taken to be first order. The external scalar and pseudoscalar fields $s$ and $p$ and the current mass matrix $\hat{m}_0$ are taken to be second order. At the one quark loop level this chiral expansion corresponds to a derivative expansion which should be gauge, frame and coordinate invariant.

In order to carry out the low energy expansion of the action via a heat kernel method it proves useful to use the proper-time integral representation of the logarithm

$$\sum_i c_i \text{Tr} \log (D_\phi + \Lambda_i^2) = -\text{Tr} \int_0^\infty \frac{dt}{t} e^{-t D_\phi} \phi(\tau)$$

(33)

where $\phi(\tau)$ is the proper time representation of the Pauli-Villars regularization,

$$\phi(\tau) = \sum_i c_i e^{-i\tau\Lambda_i^2}$$

(34)

with the conditions

$$\sum_i c_i = 0 \quad \sum_i c_i \Lambda_i^2 = 0$$

(35)

which fulfills the conditions $\phi(0) = 0$ and $\phi'(0) = 0$, thus killing the quadratic and logarithmic divergencies respectively. The operator inside the logarithm is of Klein-Gordon type in curved space-time with some spinorial structure,

$$D_\phi D = \frac{1}{\sqrt{-g}} \left[ \mathcal{D}_\mu \left( \sqrt{-g} g^{\mu\nu} \mathcal{D}_\nu \right) \right] + \mathcal{V},$$

(36)

with

$$\mathcal{V} = \mathcal{V}_R P_R + \mathcal{V}_L P_L$$

(37)

and

$$\mathcal{V}_R = -\frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu}^R + \frac{1}{4} R - i \gamma^\mu \nabla_\mu \mathcal{M} + \mathcal{M}^\dagger \mathcal{M},$$

$$\mathcal{V}_L = -\frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu}^L + \frac{1}{4} R - i \gamma^\mu \nabla_\mu \mathcal{M}^\dagger + \mathcal{M} \mathcal{M}^\dagger.$$ (38)

As we saw in the previous section, the one quark loop contribution to the effective action depends additively on the external and dynamical fields. Although this complicates matters one can borrow from previous results taking into account some minor modifications regarding this fact and the regularization.

The form of the operator $D_\phi D$ in Eq. 38 is suitable to make a heat kernel expansion in curved space-time as the one of Ref. 62. For a review see e.g. 63 and references therein. In our particular case, before undertaking the heat kernel expansion we separate a constituent quark mass squared, $M^2$, contribution from the operator $D_\phi D$ which we treat exactly,

$$\langle x | e^{-i\tau D_\phi D} | x \rangle = e^{-i\tau M^2} \langle x | e^{-i(D_\phi D - M^2)} | x \rangle$$

(39)

$$= \frac{i}{(4\pi\tau)^2} e^{-i\tau M^2} \sum_{n=0}^\infty a_n(x) (i\tau)^n.$$

The $\tau$ integrals appearing in the effective action are of the form

$$\mathcal{I}_2 \tau := M^2 \int_0^\infty d\tau \phi(\tau) (i\tau)^n e^{-i\tau M^2}.$$ (40)

For completeness, we list here the particular values,

$$M^4 \mathcal{I}_{-4} = -\frac{1}{2} \sum_i c_i (\lambda_i^2 + M^2)^2 \log(\lambda_i^2 + M^2),$$ (41)

$$M^2 \mathcal{I}_{-2} = \sum_i c_i (\lambda_i^2 + M^2) \log(\lambda_i^2 + M^2),$$ (42)

$$\mathcal{I}_0 = -\sum_i c_i \log(\lambda_i^2 + M^2),$$ (43)

$$\mathcal{I}_{2n} = \Gamma(n) \sum_i c_i \left( \frac{M^2}{\lambda_i^2 + M^2} \right)^n, \text{Re}(n) > 0,$$ (44)

After evaluation of the Dirac traces, the second order Lagrangian is given by

$$\mathcal{L}^{(2)}_\mathfrak{q} = \frac{N_c}{(4\pi)^2} \left\{ M^2 \mathcal{I}_0(\nabla_\mu U^\dagger \nabla^\mu U) + 2 M^2 \mathcal{I}_{-2}(\overline{m} U + U^\dagger \overline{m}) + \frac{M^2}{6} \mathcal{I}_{-2}(R) \right\},$$ (45)

whereas the fourth order becomes

$$\mathcal{L}^{(4)}_\mathfrak{q} = \frac{N_c}{(4\pi)^2} \left\{ -\frac{1}{6} \mathcal{I}_0(\overline{m} \overline{m})^2 + (\overline{m} \overline{m})^2 + \mathcal{I}_0 \left( \frac{1}{144} R^2 + \frac{1}{90} R_{\mu\nu} R_{\mu\nu} \right) - \frac{i}{2} \mathcal{I}_2 \left[ (\overline{D}^R U)^\dagger \overline{D}^R U + (\overline{D}^L U)^\dagger \overline{D}^L U \right] + \frac{1}{12} \mathcal{I}_4 \left( (\overline{\nabla}_\mu U)^\dagger \overline{\nabla}_\mu U \right)^2 - \frac{1}{6} \mathcal{I}_4 \left( (\overline{\nabla}_\mu U)^\dagger \overline{\nabla}_\mu U \right)^2 + \frac{1}{6} \mathcal{I}_4 \left( (\overline{\nabla}_\mu U)^\dagger \overline{\nabla}_\mu U \right)^2 + 2 M^2 \mathcal{I}_{-2}(\overline{m} U + U^\dagger \overline{m})^2 - M^2 \mathcal{I}_2 (\overline{m} U + U^\dagger \overline{m}) + M^2 \mathcal{I}_0 (\overline{m} U + U^\dagger \overline{m}) + M^2 \mathcal{I}_0 (\overline{m} U + U^\dagger \overline{m}) - \frac{M^2}{6} \mathcal{I}_0 (U^\dagger \overline{m} + \overline{m} U) - \frac{1}{12} \mathcal{I}_2 R (\overline{\nabla}_\mu U)^\dagger \overline{\nabla}_\mu U \right\}.$$ (46)

---

5 This additivity works only for the normal parity contribution of the action, but must be modified in the abnormal parity sector in order to reproduce the QCD anomaly. See e.g. Ref. 63, 64.
Here $R^\lambda_{\mu
u}, R_{\mu
u}$ and $R$ are the Riemann curvature tensor, the Ricci tensor, and the curvature scalar, respectively\(^6\),

\[- R^\lambda_{\sigma\mu\nu} = \partial_\nu \Gamma^\lambda_{\sigma\mu} - \partial_\mu \Gamma^\lambda_{\sigma\nu} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho},
R_{\mu
u} = R^\lambda_{\mu\lambda\nu}; \quad R = g^{\rho\sigma} R_{\rho\sigma} .
\] (47)

The Riemann connection is given by the Christoffel symbols, $\Gamma^\lambda_{\mu\nu}$ defined in Eq. (17).

We have introduced the gauge and frame covariant derivatives and the field strength tensor containing the external and internal (bosonized) degrees of freedom,

\[\nabla_\mu U = \nabla_\mu U - iV_\mu^L U + iU V_\mu^R,\]
\[F_\mu^\nu = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],\]

with $r = L, R$ and the spin 0 additive combination

\[m = (S + iP - MU) + m .\] (49)

The form of Eq. (46) is not yet ready for comparison with the result of Ref. [4, 18]. To do that we have to eliminate all degrees of freedom other than the on-shell pions. We proceed in three steps: Firstly we integrate out the vector and axial degrees of freedom, afterwards we eliminate the scalars and finally we exploit the classical equations of motion for the pseudoscalars.

**IV. MEAN FIELD EQUATIONS OF MOTION**

**A. Elimination of Vector and Axial Vectors**

To eliminate the $V_\mu$ and $A_\mu$ fields at the mean field level we minimize the corresponding Lagrangian with respect to those fields. To the order in the chiral expansion we are working it is enough to deal with the terms containing vector mesons with two Lorentz indices, i.e. the mass term and the second order term stemming from the quark determinant,

\[L^{(2)}_{A,V} = \frac{N_c}{(4\pi)^2} M^2 I_0 \langle \nabla_\mu U U^\dagger \nabla^\mu U \rangle + \frac{1}{4G_V} (V_\mu V^\mu + A_\mu A^\mu) .\] (50)

The result looks like the GM model Eqs. [31, 32] as noted in Ref. [42],

\[\nabla_\mu^R = \nabla_\mu^L + \frac{i}{2} (1 - g_A) U U^\dagger \nabla_\mu U ,\]
\[\nabla_\mu^I = \nabla_\mu^L + \frac{i}{2} (1 - g_A) U U^\dagger \nabla_\mu U ,\]

which looks like a chiral gauge transformation of the additive fields $\nabla$ and $A$. Then it is straightforward to obtain the following relations,

\[F_\mu^R = \frac{1}{2} (1 + g_A) F_\mu^R + \frac{1}{2} (1 - g_A) U U^\dagger F_\mu^L U ,\]
\[- \frac{i}{4} (1 - g_A^2) \left( \nabla_\mu U U^\dagger \nabla_\mu U - \nabla_\mu U \nabla^\mu U \right) ,\] (53)
\[F_\mu^I = \frac{1}{2} (1 - g_A) U F_\mu^R U^\dagger + \frac{1}{2} (1 + g_A) F_\mu^L U^\dagger ,\]
\[- \frac{i}{4} (1 - g_A^2) \left( \nabla_\mu U \nabla^\mu U - \nabla_\mu U \nabla^\mu U \right) ,\] (54)
\[\nabla_\mu U = g_A \nabla_\mu U ,\]
\[\nabla^2 U = g_A \nabla^2 U + ig_A (1 - g_A) U \nabla_\mu U \nabla^\mu U .\] (56)

**B. Elimination of Scalars**

The elimination of scalars proceeds along similar lines as in the vector and axial case. If we do a chiral rotation

\[S + iP = \sqrt{U} \Sigma \sqrt{U} ,\] (57)

with $\Sigma^\dagger = \Sigma$, and using that $\Sigma = M + \Phi$ with $\Phi$ the fluctuation around the vacuum value we have

\[m = \sqrt{U} \Phi \sqrt{U} + \frac{1}{2B_0} \chi .\] (58)

The mass term becomes

\[L_m = - \frac{1}{4G_S} \langle M^2 + 2M\Phi + \Phi^2 \rangle .\] (59)

Using the gap equation, Eq. (20), linear terms in $\Phi$ not containing external fields vanish. As a consequence, the part of the Lagrangian comprising the scalar field is given by

\[L_\Phi(x) = - \frac{N_c}{(4\pi)^2} M^2 I_0 \Phi^2 + \frac{1}{3} M I_0 R \Phi + M I_0 \sqrt{U} \Phi \sqrt{U} \left(U \nabla^2 U + \nabla^\mu U \nabla_\mu U \right) + 2M^2 \left(2I_0 - I_{-2} \right) \sqrt{U} \Phi \sqrt{U} \left(U m^\dagger + U m \right) + 2M I_2 \sqrt{U} \Phi \sqrt{U} \nabla_\mu U \nabla^\mu U \right) ,\] (60)

from which the well known mass formula $M_S = 2M$ follows. Minimizing with respect to the field $\Phi$ the classical equation of motion follow,

\[\sqrt{U} \Phi \sqrt{U} \left(U \nabla^2 U + \nabla^\mu U \nabla_\mu U \right) - \frac{1}{2} \left(1 - \frac{I_{-2}}{2I_0} \right) \left(U m^\dagger + m U \right) .\] (61)

Substituting this equation into the Lagrangian $L_\Phi$ we obtain the contribution to the effective Lagrangian stemming from the integration of scalars.

\(^6\) Note the opposite sign of our definition for the Riemann tensor as compared to Ref. [15]. We follow Ref. [53].
C. Mean field equations for pseudoscalars

The relevant equations of motion for the non-linear $U$ field are obtained by minimizing $\mathcal{L}^{(2)}$. One obtains a set of relations which are valid in the presence of curvature,

$$
\langle \nabla^2 U \nabla^2 U \rangle = \left( \langle \nabla_\mu U \nabla^\mu U \rangle \right)^2 - \frac{1}{4} \left( \langle \chi^I U - U^I \chi \rangle \right)^2 \\
+ \frac{1}{12} \left( \langle \chi^I U - U^I \chi \rangle \right)^2
$$

and

$$
\langle \chi^I \nabla^2 U + \nabla^2 U \chi \rangle = 2 \langle \chi^I \chi \rangle - \frac{1}{2} \left( \langle \chi^I U + U^I \chi \rangle \right)^2 \\
- \langle \chi^I U + U^I \chi \rangle \nabla^\mu U \nabla_\mu \rangle \\
+ \frac{1}{6} \left( \langle \chi^I U + U^I \chi \rangle \right)^2.
$$

In the case of the $U(3)$ group one has $\text{Det} \ U = e^{im_0/I}$ not necessarily equal to unity and the last two terms involving $\langle \chi^I U \pm U^I \chi \rangle^2$ in Eqs. (62) and (63) should be dropped. There is another integral identity which proves very useful

$$
\int d^4x \sqrt{-g} \left( \langle \nabla_\mu \nabla_\nu U \nabla^\nu \nabla^\mu U \rangle - \frac{1}{2} \langle \nabla^2 U \nabla^2 U \rangle \right) \\
i \int d^4x \sqrt{-g} \left( \langle F_{\mu\nu}^R \nabla^\nu U \nabla^\mu U \rangle + \langle F_{\mu\nu}^L \nabla^\nu U \nabla^\mu U \rangle \right) \\
- \frac{1}{2} \int d^4x \sqrt{-g} \left( \langle F_{\mu\nu}^R \nabla^\nu U \rangle^2 + \langle F_{\mu\nu}^L \nabla^\nu U \rangle^2 \right) \\
+ \int d^4x \sqrt{-g} \langle R_{\mu\nu} \nabla_\mu U \nabla_\nu U \rangle.
$$

Finally, we also have the SU(3) identity

$$
\langle \langle \nabla_\mu U \nabla^\mu U \rangle^2 \rangle = -2 \langle \langle \nabla_\mu U \nabla^\mu U \rangle \rangle + \langle \nabla_\mu U \nabla^\mu U \rangle \langle \nabla_\mu U \nabla^\mu U \rangle + \frac{1}{2} \langle \nabla_\mu U \nabla^\mu U \rangle^2.
$$

V. RESULTS FOR THE GASSER-LEUTWYLER–DONOGHUE COEFFICIENTS

Following the steps of Ref. [139] one gets the chiral effective Lagrangian in the presence of gravity in the form of Gasser, Leutwyler and Donoghue form [144,148]

$$
\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2,g)} + \mathcal{L}^{(2,R)} + \mathcal{L}^{(4,g)} + \mathcal{L}^{(4,R)} + \ldots
$$

The zeroth order vacuum contribution reads

$$
\mathcal{L}^{(0)} = B = -\frac{2N_f N_c}{(4\pi)^2} M^4 T_{-4},
$$

where $B$ is the vacuum energy density constant. The metric contributions read

$$
\mathcal{L}^{(2,g)} = \frac{f^2}{4} \langle \nabla_\mu U \nabla^\mu U \rangle + \langle \chi^I U - U^I \chi \rangle, \quad (68)
$$

and

$$
\mathcal{L}^{(4,g)} = L_1 \langle \nabla_\mu U \nabla^\mu U \rangle^2 + L_2 \langle \nabla_\mu U \nabla^\mu U \rangle^2 + L_3 \langle \nabla_\mu U \nabla^\mu U \rangle^2 + L_4 \langle \nabla_\mu U \nabla^\mu U \rangle^2 + L_5 \langle \nabla_\mu U \nabla^\mu U \rangle^2 + L_6 \langle \nabla_\mu U \nabla^\mu U \rangle^2
$$

The curvature contributions to the chiral Lagrangian can be written in the form proposed in Ref. [148] and are given by

$$
\mathcal{L}^{(2,R)} = -H_0 R
$$

and

$$
\mathcal{L}^{(4,R)} = -L_{11} R \langle \nabla_\mu U \nabla^\mu U \rangle + L_{12} R^{\mu\nu} \langle \nabla_\mu U \nabla^\mu U \rangle + L_{13} R \langle \chi^I U + U^I \chi \rangle + H_3 R^2 + H_4 R_{\mu\nu} R^{\mu\nu}
$$

The curvature terms reflect the composite nature of the pseudoscalar fields, since in the considered model they correspond to the coupling of the gravitational external field at the quark level.

Note that the pseudoscalar matrix $U$ is a coordinate and frame scalar. So, once and only once the identities (63) and (65) have been used one can substitute coordinate-frame covariant derivative by the covariant derivative, i.e., $\nabla_\mu U = D_\mu U$. In this way Eqs. (67) and (10) of this manuscript, and Eq. (10) of Ref. [139] are deduced.

A. GM model

The Gasser-Leutwyler-Donoghue coefficients for the GM model do not contain any contributions from the scalars, i.e. the spin 0 field. So, the only contribution stems from the quark loop. For this model, the pion weak decay constant is

$$
f^2 = \frac{N_c}{4\pi^2} g_A^2 M^2 T_0.
$$

The normalization factor for the field $\chi$ is

$$
B_0 = M \frac{T_{-2}}{T_0},
$$

With

$$
\rho \equiv \frac{M}{B_0} = M \frac{f^2}{\langle \langle qq \rangle \rangle} = g_A^2 \frac{T_0}{T_{-2}}
$$

(74)
the result we find for the GLD coefficients reads

\[ L_1 = \frac{N_c}{48(4\pi)^2} \left[ (1 - g_A^2) I_0 + 4g_A^2(1 - g_A^2) I_2 + 2g_A^4 I_4 \right], \]

\[ L_2 = 2L_1, \]

\[ L_3 = -\frac{N_c}{24(4\pi)^2} \left[ 3(1 - g_A^2) I_0 + 8g_A^4 I_4 + 4g_A^2(3 - 4g_A^2) I_2 \right], \]

\[ L_4 = 0, \]

\[ L_5 = \frac{N_c}{2(4\pi)^2} \rho g_A^2 [I_0 - I_2], \]

\[ L_6 = 0, \]

\[ L_7 = -\frac{N_c}{24(4\pi)^2} N_f g_A [6\rho I_0 - 6 I_2], \]

\[ L_8 = -\frac{N_c}{6(4\pi)^2} \left[ 6(1 - g_A^2) I_0 + 2g_A^2 I_2 \right], \]

\[ L_9 = \frac{N_c}{6(4\pi)^2} \left[ (1 - g_A^2) I_0 + 2g_A^2 I_2 \right], \]

\[ L_{10} = -\frac{N_c}{6(4\pi)^2} \left[ (1 - g_A^2) I_0 + g_A^2 I_2 \right], \]

\[ L_{11} = \frac{N_c}{12(4\pi)^2} g_A^2 I_2, \]

\[ L_{12} = -\frac{N_c}{6(4\pi)^2} g_A^2 I_2, \]

\[ L_{13} = \frac{N_c}{12(4\pi)^2} \rho I_0 = \rho \frac{f^2}{48M^2} \frac{g_A}{g_A^2}, \]

\[ H_0 = -\frac{N_c N_f}{6(4\pi)^2} M^2 I_{-2} = -\frac{N_f f^2}{24} \frac{g_A^2}{\rho}, \]

\[ H_1 = \frac{N_c}{12(4\pi)^2} \left[ -(1 + g_A^2) I_0 + g_A^2 I_2 \right], \]

\[ H_2 = \frac{N_c N_f}{12(4\pi)^2} \left[ 6\rho I_{-2} - 6\rho(\rho + g_A) I_0 + g_A^2 I_2 \right], \]

\[ H_3 = -\frac{N_c N_f}{144(4\pi)^2} I_0 = \frac{N_f f^2}{576M^2} \frac{g_A^2}{g_A^2}, \]

\[ H_4 = \frac{N_c N_f}{90(4\pi)^2} I_0 = \frac{N_f f^2}{360M^2} \frac{g_A^2}{g_A^2}, \]

\[ H_5 = \frac{7N_c N_f}{720(4\pi)^2} I_0 = \frac{7N_f f^2}{2880M^2} \frac{g_A^2}{g_A^2}. \]

We take \( M = 300 \text{ MeV} \) and \( g_A = 0.75 \). For given values of \( M \) and \( g_A \), the cutoff \( \Lambda \) is adjusted to reproduce \( f_L = 93.2 \text{ MeV} \). This yields

\[ \Lambda = 1470 \text{ MeV}, \quad B_0 = 4913 \text{ MeV}, \]

\[ I_{-2} = 20.8, \quad I_0 = 2.26, \quad I_2 = 0.922, \]

\[ I_4 = 0.995. \]

The constituent chiral quark model (QC) corresponds to take \( g_A = 1 \) in the previous coefficients. Using the same value for \( M \), we have for this model

\[ \Lambda = 828 \text{ MeV}, \quad B_0 = 1299 \text{ MeV}, \]

\[ I_{-2} = 5.50, \quad I_0 = 1.27, \quad I_2 = 0.781, \]

\[ I_4 = 0.963. \]

The numerical values for the GLD coefficients are displayed in Table 4.

**B. NJL model**

The GLD coefficients for this model will have two different contributions: one coming from the quark loop and subsequent integration of spin 1 fields, and another coming from the integration of spin 0 fields. For the quark loop contribution we have the same expressions as Eq. (75)-(93). The pion weak decay constant is

\[ f^2 = \frac{N_c}{4\pi} g_A M^2 I_0. \]

Note the power \( g_A \) as opposed to the power \( g_A^2 \) in the GM model, Eq. (72). The difference has to do with the absence of a mass term like \( L_m \) in the GM model. We use the notation

\[ B_0 = \frac{M}{2Gf^2} = \frac{M I_{-2}}{g_A I_0}, \quad g_A = 1 - 2Gf^2. \]

With

\[ \rho \equiv \frac{M}{B_0} = g_A \frac{I_0}{I_{-2}}, \]

the spin 0 contribution becomes

\[ L_3^S = \frac{N_c}{4(4\pi)^2} \frac{g_A^4}{g_A^2} [I_0 - I_4]^2, \]

\[ L_5^S = \frac{N_c}{4(4\pi)^2} \frac{g_A^2}{g_A^2} (g_A - 2\rho) [I_0 - I_2], \]

\[ L_8^S = \frac{N_c}{16(4\pi)^2} (g_A - 2\rho)^2 I_0, \]

\[ L_{11}^S = \frac{N_c}{12(4\pi)^2} g_A^2 [I_0 - I_2], \]

\[ L_{13}^S = \frac{N_c}{144(4\pi)^2} g_A^2 (g_A - 2\rho) I_0, \]

\[ H_3^S = \frac{N_c}{24(4\pi)^2} g_A^2 I_0, \]

\[ H_4^S = \frac{N_c}{144(4\pi)^2} g_A^2 I_0 = \frac{N_f f^2}{576M^2} \frac{g_A^2}{g_A^2}. \]

The remaining \( L_1^S, H_5^S \) vanish identically. The sum of both contributions will give the GLD coefficients for the NJL model, which read

\[ L_1 = \frac{N_c}{48(4\pi)^2} \left[ (1 - g_A^2) I_0 + 4g_A^2(1 - g_A^2) I_2 + 2g_A^4 I_4 \right], \]
\( L_2 = 2L_1, \quad L_3 = -\frac{N_c}{24(4\pi)^2} \left[ 3(1 - 2g_A^2 - g_A^4)I_0 + 8g_A^4I_4 + 2g_A^2 \left( 2(3 - g_A^2 - 3g_A^2I_2) \frac{I_0}{I_2} \right) \right], \quad (107) \)

\( L_4 = 0, \quad L_5 = \frac{N_c}{4(4\pi)^2} g_A^3 [I_0 - I_2], \quad (109) \)

\( L_6 = 0, \quad L_7 = -\frac{N_c}{24(4\pi)^2} N_f \left[ 6\rho I_0 - g_A I_2 \right], \quad (112) \)

\( L_8 = \frac{N_c}{48(4\pi)^2} [3I_0 - 2I_2], \quad (113) \)

\( L_9 = \frac{N_c}{6(4\pi)^2} \left[ (1 - g_A^2)I_0 + 2g_A^2I_2 \right], \quad (114) \)

\( L_{10} = \frac{N_c}{6(4\pi)^2} \left[ (1 - g_A^2)I_0 + g_A^2I_2 \right], \quad (115) \)

\( L_{11} = \frac{N_c}{12(4\pi)^2} g_A^2I_0 = \frac{g_A f^2}{48M^2}, \quad (116) \)

\( L_{12} = -\frac{N_c}{6(4\pi)^2} g_A^2I_2, \quad (117) \)

\( L_{13} = \frac{N_c}{24(4\pi)^2} g_A I_0 = \frac{f^2}{96M^2}, \quad (118) \)

\( H_0 = -\frac{N_c N_f}{6(4\pi)^2} M^2 \rho I_2 = -\frac{N_f f^2}{24 \rho}, \quad (119) \)

\( H_1 = \frac{N_c}{12(4\pi)^2} \left[ -1 + g_A^2 \frac{I_0}{I_2} + g_A^2 I_2 \right], \quad (120) \)

\( H_2 = \frac{N_c}{24(4\pi)^2} \left[ 12\rho^2 I_2 - 3g_A(g_A - 8\rho)I_0 + 2g_A^2 I_2 \right], \quad (121) \)

\( H_3 = 0, \quad (122) \)

\( H_4 = \frac{N_c N_f}{90(4\pi)^2} I_0 = \frac{N_f f^2}{360M^2 g_A}, \quad (123) \)

\( H_5 = \frac{7N_c N_f}{720(4\pi)^2} I_0 = \frac{7N_f f^2}{2880M^2 g_A}, \quad (124) \)

The coefficients \( L_1, L_2, L_4, L_6, L_7, L_9, L_{10}, L_{12}, H_0, H_1, H_2 \) and \( H_5 \) in the GM model coincide with those of the NJL model. However, we prefer to display them explicitly for easier reference and because the expressions for \( f^2 \) (Eqs. 122 and 123) do not coincide.

Note that this model reproduces the relation \( L_3 = -6L_1 \), provided terms \( O(N_c g_A^4) \) are neglected. We note several differences with previous works. The values \( L_1, L_2, L_3, L_4, L_5, L_6, L_9, L_{10}, H_1 \) and \( H_2 \) coincide with Ref. 45. \( L_8 \) differs in two powers of \( g_A \) in the term proportional to \( I_2 \). (We reproduce their results for every separate pieces: quark loop contribution and spin 0 contribution.)

The value of \( L_7 \) is non-zero, if one imposes the condition \( \text{Det}(U) = 1 \) within \( SU(N_f) \) flavor symmetry. Both in Ref. 42 and 45 this term is not obtained, in spite of the fact that in those papers it is explicitly mentioned that one works in \( SU(N_f) \) flavor symmetry. The situation can be mended if one considers \( U(N_f) \) flavor symmetry instead where \( L_7 = 0 \).

The coefficients \( L_{1-10} \) where given in Ref. 42. The present values of \( L_4, L_5, L_6, L_8, L_9 \) and \( L_{10} \) coincide with those 42 where an extra erroneous term in \( L_4 \) appears. \( L_3 \) differs from Ref. 42 in all factors except for the one in \( I_4 \). We correct here these errors (see also Ref. 45). \( H_1 \) and \( H_2 \) did not appear in that reference.

The coefficients \( L_{11}, L_{12} \) and \( L_{13} \), as well as \( H_{0,3-5} \), are new and are the main result of this work. They have been also evaluated some time ago 48 in a quark model without scalars and vectors and more recently by us 49 in the spectral quark model.

The numerical values for these coefficients Eq. (106)-(124) are displayed in Table I in two different cases: one for the generalized \( SU(3) \) NJL model, and another one in which integration of spin 1 fields is not considered, i.e. \( g_A = 1 \). For the first case we take \( g_A = 0.606 \) as a reasonable value. Using \( M = 300 \text{ MeV} \), this yields

\[ \Lambda = 1344 \text{ MeV}, \quad B_0 = 4015 \text{ MeV}, \quad I_2 = 17.0, \quad I_0 = 2.10, \quad I_2 = 0.907, \quad I_4 = 0.993. \quad (125) \]

On the other hand, for \( g_A = 1 \), the numerical values of \( \Lambda, B_0, \rho \) and \( I_2n \) are identical to those for the QC model, Eq. (25). The LEC’s in the NJL model with \( g_A = 1 \) and in the QC model differ due to the scalar contributions \( L_{3,5,8,11,13} \) and \( H_{2,3} \) which are not present in the QC case.

VI. SUMMARY AND CONCLUSIONS

In the present work we have calculated the low energy constants of the energy momentum tensor. Some of these LEC’s are directly determined by the standard Gasser-Leutwyler coefficients, whereas other, \( L_{11-13} \) and \( H_{0,3-5} \), are new as are driven by operators not present in the (flat space) chiral Lagrangian. Technically, the best way to proceed is to consider QCD in a curved space-time because these allows us to work with a single scalar, the low energy Lagrangian, rather than its variation, the energy momentum tensor, making easier both the computation and the imposition of symmetry restrictions. In addition the coupling to gravity is also obtained as a byproduct. The curved space chiral Lagrangian, contains two kinds of contributions. On the one hand, those obtained by “minimal coupling” from the flat space case, \( \mathcal{L}^{(S)} \), on the other, non-minimal pieces containing the Riemann curvature tensor \( \mathcal{L}^{(R)} \). In the spirit of not introducing new fields other than the metric tensor, we have considered only Einstein’s gravity. Further new pieces would appear by using connections with torsion or violating metricity. As happens with gauge couplings (e.g. the magnetic moment) the non-minimal gravitational pieces are not fixed.
TABLE I: The dimensionless low energy constants and $H_0$ compared with some reference values and other models. The values quoted for $L_{1-10}, H_{1-5}$ are to be multiplied by $10^{-2}$. The value quoted for $H_0$ is to be multiplied by $10^3$ MeV$^2$.

|       | ChPT$^a$ | NJL | NJL ($g_A = 1$) | QC | GM | SQM$^b$ (MDM) | Large $N_c^c$ | Dual$^b$ |
|-------|----------|-----|----------------|----|----|--------------|---------------|---------|
| $L_1$ | 0.53 ± 0.25  | 0.77 | 0.76           | 0.76 | 0.78 | 0.79         | 0.9           | 0.79    |
| $L_2$ | 0.71 ± 0.27  | 1.54 | 1.52           | 1.52 | 1.56 | 1.58         | 1.8           | 1.58    |
| $L_3$ | −2.72 ± 1.12 | −4.02 | −2.73         | −3.62 | −4.25 | −3.17       | −4.3          | −3.17   |
| $L_4$ | 0         | 0    | 0              | 0    | 0   | 0            | 0             | 0       |
| $L_5$ | 0.91 ± 0.15  | 1.26 | 2.32           | 1.08 | 0.44 | 2.0 ± 0.1    | 2.1           | 3.17    |
| $L_6$ | 0         | 0    | 0              | 0    | 0   | 0            | 0             | 0       |
| $L_7$ | −0.32 ± 0.15 | −0.06 | −0.26          | −0.26 | −0.03 | −0.07 ± 0.01 | −0.3          |         |
| $L_8$ | 0.62 ± 0.20  | 0.65 | 0.89           | 0.46 | 0.04 | 0.08 ± 0.04  | 0.8           | 1.18    |
| $L_9$ | 5.93 ± 0.43  | 6.31 | 4.95           | 4.95 | 6.41 | 6.33         | 7.1           | 6.33    |
| $L_{10}$ | −4.40 ± 0.70$^d$ | −5.25 | −2.47        | −2.47 | −4.77 | −3.17       | −5.4          | −4.75   |
| $L_{11}$ | 1.85 ± 0.90$^e$ | 1.22 | 2.01          | 1.24 | 0.82 | 1.58         | 1.6$^e$       |         |
| $L_{12}$ | −2.7$^e$ | −1.06 | −2.47        | −2.47 | −1.64 | −3.17       | −2.7$^e$      |         |
| $L_{13}$ | 1.7 ± 0.80$^e$ | 1.01 | 1.01          | 0.47 | 0.22 | 0.33 ± 0.01  | 1.1$^e$       |         |
| $H_0$ |          | −14.6 | −4.67        | −4.67 | −17.7  | 1.09        |               |         |
| $H_1$ |          | −4.01 | −2.78        | −2.78 | −4.76  |     |               |         |
| $H_2$ |          | 1.46  | 1.45          | 0.59 | 0.49   | −1.0 ± 0.2  |               |         |
| $H_3$ |          | 0     | 0             | −0.50 | −0.89  |     |               |         |
| $H_4$ |          | 1.33  | 0.80          | 0.80 | 1.43   |     |               |         |
| $H_5$ |          | 1.16  | 0.70          | 0.70 | 1.25   |     |               |         |

$^a$The two-loop calculation of Ref. 22.
$^b$Ref. 17.
$^c$Ref. 28.
$^d$Ref. 24, 65.
$^e$Ref. 18.

(nor demanded) by general covariance of the chiral Lagrangian and to obtain them requires to couple gravity directly to QCD quarks and gluons previously to their integration out to yield the low energy Lagrangian.

The standard LEC’s, which are expected to follow from QCD, have also been computed in QCD-like models containing chiral quarks, with some degree of success. Here we apply the same approach for the non-minimal pieces. We do so for various models, namely, the constituent quark model, the NJL model with and without vector mesons, and the Georgi-Manohar model. This extends our previous calculation carried out within the spectral quark model 14 where also a dual large $N_c$ model was introduced. The results quoted in Table I include standard and non minimal coefficients in these models as well as ChPT calculations and the single resonance Large $N_c$ model. The results for the LEC’s look roughly quite similar. As a rule all models and fits give the same sign for all coefficients, with the exception of $H_0$ and $H_2$ in the SQM. For the standard Gasser-Leutwyler coefficients $L_{1-10}$ the best overall agreement with the two-loop ChPT calculation of 22 is achieved by the NJL model with vector mesons, for which the value of the reduced chi-square $\chi^2/N$, $(N = 10)$, is 2.2, although the QC and GM models give results of near quality: 2.5 and 3.6 respectively. For the new coefficients there are no widely accepted values. The closest agreement with the resonance saturation and large $N_c$ estimates of 18 for $L_{11-13}$ is provided by the NJL model without vectors mesons, for which $\chi^2/N = 0.29$, but this is, of course, not conclusive. We should also mention the remarkable agreement between the prediction of the SQM for these three coefficients and that of the chiral bosonization model of 48.

APPENDIX: A SPECTRAL QUARK MODEL WITH $g_A \neq 1$?

One of the common features of both the NJL model with $G_V \neq 0$ and the GM model analyzed in this paper is the presence of an axial coupling constant for the quarks $g_A \neq 1$. In this appendix we analyze the possibility of extending the Spectral Quark Model to have a coupling similar to the GM model. Let us remind that the SQM 53, 54 is characterized by a generalized Lehmann representation for the quark propagator in terms of a spectral function $\rho(\omega)$ defined on a suitable contour on the complex mass plane, which according to the rules devised in those works should satisfy the spec-
tral conditions for the positive moments

$$\rho_n = \int d\omega \rho(\omega) \omega^n = \delta_{n0}, \quad n \geq 0. \quad (A.1)$$

The vanishing of even moments Eq. (A.1), remove the dependence on the scale \( \mu \) in Eq. (A.2) and thus prevents the occurrence of dimensional transmutation for all moments except \( \rho_0 \). Assuming that the quark model is defined on a very definite renormalization scale, any low energy hadronic observable should not depend on any additional arbitrary scale, so we expect that \( \rho_0 \) does not determine hadronic properties, and in particular low energy constants corresponding to process involving pseudoscalars mesons, i.e. \( L_2 = L_{11} \). Actually, in our low energy analysis of the SQM the zeroth log-moment, \( \rho_0 \), only contributed to non hadronic LEC’s corresponding to vacuum polarization diagrams, like the photon and graviton wave functions and the quadratic current quark

mass corrections to the vacuum energy density. As such, this scale dependence triggered by dimensional transmutation in the SQM is innocuous since QED and quantum gravity also renormalize the photon and graviton wave functions respectively.

If we assume a Spectral Quark Model a la Georgi-Manohar and consider a low energy expansion along the lines explained in the paper we would generate the scale dependent term

$$\mathcal{L} = \frac{N_c}{(4\pi)^2} \frac{1}{6} \rho_0 \left[ (1 - g_A^2) \langle F_{\mu\nu} U F_{\mu\nu}^R U^\dagger \rangle - i(1 - g_A^2) \langle F_{\mu L} D\mu U D\nu U^\dagger + F_{\mu R} D\mu U^\dagger D\nu U \rangle \right. $$

$$\left. - \frac{1}{4} (1 - g_A^2) \langle (D_\mu U^\dagger D_\nu U)^2 \rangle + \frac{1}{4} (1 - g_A^2) \langle (D_\mu U^\dagger D^\mu U)^2 \rangle \right]. \quad (A.3)$$

This terms arises from the photon wave function contribution and because of the additivity of the \( (1 - g_A) \) terms and the external spin one fields. This yields contributions to the LEC’s involving pseudoscalars which depend on the arbitrary scale generated by dimensional transmutation. It is not clear whether this deficiency can be attributed to the \( g_A \neq 1 \) assumption or the spectral quark model construction.

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