GK-DIMENSION OF THE LIE ALGEBRA OF GENERIC $2 \times 2$ MATRICES

VESSELIN DRENSKY, PLAMEN KOSHLUKOV, GUSTAVO GRINGS MACHADO

Abstract. Recently Machado and Koshlukov have computed the Gelfand-Kirillov dimension of the relatively free algebra $F_m = F_m(\text{var}(sl_2(K)))$ of rank $m$ in the variety of algebras generated by the three-dimensional simple Lie algebra $sl_2(K)$ over an infinite field $K$ of characteristic different from 2. They have shown that $\text{GKdim}(F_m) = 3(m - 1)$. The algebra $F_m$ is isomorphic to the Lie algebra generated by $m$ generic $2 \times 2$ matrices. Now we give a new proof for $\text{GKdim}(F_m)$ using classical results of Procesi and Razmyslov combined with the observation that the commutator ideal of $F_m$ is a module of the center of the associative algebra generated by $m$ generic traceless $2 \times 2$ matrices.

1. Introduction

Let $R$ be a (not necessarily associative) algebra generated by $m$ elements $r_1, \ldots, r_m$ over a field $K$ and let $V_n$ be the vector subspace of $R$ spanned by all products $r_{i_1} \cdots r_{i_k}$, $k \leq n$. The growth function of $R$ with respect to the given system of generators is

$$g_R(n) = \dim(V_n), \quad n \geq 0.$$ 

The Gelfand-Kirillov dimension of $R$ is defined as

$$\text{GKdim}(R) = \limsup_{n \to \infty} \log_n(g_R(n)).$$

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It does not depend on the choice of the generators of \( R \). See the book \[9\] for a background on GKdim. If the algebra \( R \) is graded,
\[
R = \bigoplus_{n \geq 0} R^{(n)},
\]
where \( R^{(n)} \) is the homogeneous component of degree \( n \) of \( R \), then the Hilbert series of \( R \) is the formal power series
\[
H(R, t) = \sum_{n \geq 0} \dim(R^{(n)}) t^n.
\]
If \( R \) is generated by its homogeneous elements of first degree, then its growth function is
\[
g_R(n) = \sum_{l=0}^{n} \dim(R^{(l)}).
\]
In the general case, if \( R \) is a graded algebra generated by a finite system of (homogeneous) elements of arbitrary degree, its Gelfand-Kirillov dimension can be expressed using again its Hilbert series as
\[
\text{GKdim}(R) = \limsup_{n \to \infty} \log_n \left( \sum_{l=0}^{n} \dim(R^{(l)}) \right).
\]
When studying varieties of \( K \)-algebras \( \mathfrak{V} \), all information for the \( m \)-generated algebras in \( \mathfrak{V} \) is carried by the relatively free algebra \( F_m(\mathfrak{V}) \) of rank \( m \) in \( \mathfrak{V} \). When the base field \( K \) is of characteristic 0, a lot is known for the Gelfand-Kirillov dimension of relatively free associative algebras, see the book \[9\], the survey article \[4\], or the paper \[11\]. In particular, \( \text{GKdim}(F_m(\mathfrak{V})) \) is an integer for all proper varieties of associative algebras. Almost nothing is known for relatively free Lie algebras. Using the bases of free nilpotent-by-abelian Lie algebras given by Shmelkin \[17\], it is easy to see that
\[
\text{GKdim}(F_m(\mathfrak{V}, \mathfrak{A})) = \text{GKdim}(L_m/(L_m^{(c+1)}) = mc,
\]
where \( m > 1 \) and \( L_m \) is the free \( m \)-generated Lie algebra. Together with free nilpotent Lie algebras where the Gelfand-Kirillov dimension is equal to 0, these are the only free polynilpotent Lie algebras of finite Gelfand-Kirillov dimension, see Petrogradsky \[12\].

Recently Machado and Koshlukov \[11\] have computed the Gelfand-Kirillov dimension of the relatively free algebra \( F_m(\text{var}(sl_2(K))) \) of rank \( m > 2 \) in the variety of algebras generated by the three-dimensional simple Lie algebra \( sl_2(K) \) over an infinite field \( K \) of characteristic different from 2. They have shown that \( \text{GKdim}(F_m) = 3(m-1) \). Their proof is based on a careful analysis of the explicit expression of
the Hilbert series of $F_m$ obtained by Drensky [3]. The case $m = 2$ was handled before by Bahturin [2] who showed that GKdim($F_2$) = 3. The algebra $F_m$ is isomorphic to the Lie algebra generated by $m$ generic traceless $2 \times 2$ matrices. The purpose of our paper is to give a new proof for GKdim($F_m$) using classical results of Procesi [13, 14] on Gelfand-Kirillov dimension of the algebra of generic matrices and Razmyslov [16] on the weak polynomial identities of matrices, combined with the observation that the commutator ideal of $F_m$ is a module over the center of the associative algebra generated by $m$ generic traceless $2 \times 2$ matrices. We believe that the present approach is more adequate for generalizations for other finite dimensional simple Lie algebras than the approach in [11].

2. The proof

The following statement and its corollary are folklorely known. We include the proof for self-completeness of the exposition and also because we were not able to find an explicit reference.

**Lemma 1.** Let $R$ be a finitely generated graded algebra with Hilbert series of the form

$$H(R, t) = h(t) \prod_{i=1}^{s} \frac{1}{(1 - t^{d_i})},$$

where $h(t) \in \mathbb{C}[t]$ is a polynomial and the $d_i$’s are positive integers. Then the Gelfand-Kirillov dimension of $R$ is equal to the multiplicity of 1 as a pole of $H(R, t)$.

**Proof.** It is sufficient to consider the case when $R$ is not finite dimensional and hence its Hilbert series has a nontrivial denominator. Let $d$ be the least common multiple of the degrees $d_i$. Then

$$H(R, t) = \sum_{n \geq 0} a_n t^n = f(t) + \sum_{p=1}^{k} \sum_{q=0}^{d-1} \frac{\alpha_{pq}}{(1 - \omega_d t)^p}$$

$$= f(t) + \sum_{n \geq 0} \left( \sum_{p=1}^{k} \binom{n+p-1}{p-1} \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n \right) t^n,$$

where $f(t) \in \mathbb{C}[t]$, $\alpha_{pq} \in \mathbb{C}$, $\omega_0 = 1, \omega_1, \ldots, \omega_{d-1}$ are the $d$-th roots of 1, and at least one of the coefficients $\alpha_{kq}$ is different from zero. Since $\omega_q^d = 1$, the sequences

$$\beta_{pn} = \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n, \quad p = 1, \ldots, k,$$
are periodic with period $d$ and for $n$ large enough the coefficients $a_n$ of the Hilbert series $H(R, t)$ are bounded by polynomials of degree $k - 1$ in $n$. Hence the sequence
\[ \sum_{l=0}^{n} a_l = \sum_{l=0}^{n} \dim(R^{[l]}) \]
needed for the definition of the Gelfand-Kirillov dimension of $R$ is bounded by a polynomial of degree $k$ in $n$ and
\[ \text{GKdim}(R) \leq k. \]
The asymptotics of the coefficients $a_n$ of
\[ H(R, t) = f(t) + \sum_{n \geq 0} \left( \sum_{p=1}^{k} \left( \frac{n + p - 1}{p - 1} \right) \beta_{pn} \right) t^n, \]
is determined by $\beta_{kn}$. Since $a_n$ are positive integers, we derive that the periodic sequence $\beta_{kn}$, $n = 0, 1, 2, \ldots$, consists of nonnegative reals and at least one of them is positive. Since $\omega_q^d = 1$, if $\omega_q \neq 1$, then $1 + \omega_q + \omega_q^2 + \cdots + \omega_q^{d-1} = 0$. Hence
\[ 0 < \sum_{l=0}^{d-1} \beta_{k,dn+l} = \sum_{l=0}^{d-1} \sum_{q=0}^{d-1} \alpha_{kq} \omega_q^{dn+l} \]
\[ = \sum_{q=0}^{d-1} \alpha_{kq} \sum_{l=0}^{d-1} \omega_q^l = d\alpha_{k0}. \]
Therefore $\alpha_{k0} > 0$. We consider the partial sum $p_{dn} = a_0 + a_1 + \cdots + a_{dn}$ of the coefficients of the Hilbert series $H(R, t)$. Its asymptotics is determined by
\[ \tilde{p}_{dn} = \sum_{c=0}^{dn} (c + k - 1) \frac{1}{(k-1)!} \sum_{c=0}^{dn} c^{k-1} \beta_{kc} \approx \frac{1}{(k-1)!} \sum_{c=0}^{dn} (ed)^{k-1} \beta_{k,ed+t} \]
\[ \approx \frac{1}{(k-1)!} \sum_{e=0}^{n} (ed)^{k-1} \sum_{l=0}^{d-1} \beta_{k,ed+l} = \frac{d\alpha_{k0}}{(k-1)!} \sum_{e=0}^{n} (ed)^{k-1} \]
and this is a polynomial of degree $k$ in $n$. Hence
\[ \text{GKdim}(R) = \limsup_{n \to \infty} \log_n \left( \sum_{l=0}^{n} a_l \right) \geq \limsup_{n \to \infty} \log_n \left( \sum_{c=0}^{dn} a_c \right) \]
\[ = \limsup_{n \to \infty} \log_n (p_{dn}) = \limsup_{n \to \infty} \log_n (\tilde{p}_{dn}) = k \]
which, together with the opposite inequality $\text{GKdim}(R) \leq k$, completes the proof. $\square$
Corollary 2. Let $R$ be a finitely generated graded algebra and let $C$ be a finitely generated graded subalgebra of the center of $R$ such that $R$ is a finitely generated $C$-module. Then the Gelfand-Kirillov dimension of $R$ is equal to the multiplicity of 1 as a pole of $H(R, t)$.

Proof. By the Hilbert-Serre theorem (see e.g., [1]), the Hilbert series of any finitely generated graded module $M$ over a finitely generated graded commutative algebra $C$ is of the form

$$H(M, t) = h(t) \prod_{i=1}^{k} \frac{1}{(1 - t^{d_i})}, \quad h(t) \in \mathbb{C}[t], d_i > 0.$$ 

Hence the proof follows immediately from Lemma 1. □

In the sequel we assume that the base field $K$ is of characteristic 0. Let $\Omega_{km} = K[Y_{km}] = K[y_{pq}^{(i)} | p, q = 1, \ldots, k, i = 1, \ldots, m]$ be the algebra of polynomials in $k^2m$ commuting variables and let $y_{i} = (y_{pq}^{(i)}), \quad i = 1, \ldots, m,$ be $m$ generic $k \times k$ matrices. We consider the following algebras:

$R_{km}$ – the generic matrix algebra. This is the subalgebra generated by $y_{1}, \ldots, y_{m}$ of the associative $k \times k$ matrix algebra $M_{k}(\Omega_{km})$ with entries from $\Omega_{km}$.

$C_{km}$ – the pure trace algebra. This is the subalgebra of $\Omega_{km}$ generated by the traces of the products, $\text{tr}(y_{i_{1}} \cdots y_{i_{l}})$. We embed $C_{km}$ in $M_{k}(\Omega_{km})$ by $f(Y_{km}) \rightarrow f(Y_{km})I_{k}$, where $I_{k}$ is the identity matrix.

$T_{km}$ – the mixed trace algebra. This is the subalgebra of $M_{k}(\Omega_{km})$ generated by $R_{km}$ and $C_{km}$.

For a background on generic matrices see e.g., [14] or [7]. Below we summarize the results we need.

Proposition 3. Let $k, m \geq 2$. Then:

(i) The mixed trace algebra $T_{km}$ has no zero divisors;

(ii) The pure trace algebra $C_{km}$ coincides with the center of $T_{km}$. It is finitely generated and $T_{km}$ is a finitely generated $C_{km}$-module;

(iii) $\text{GKdim}(T_{km}) = \text{GKdim}(C_{km}) = \text{GKdim}(R_{km}) = k^{2}(m - 1) + 1$.

Further, we consider the generic traceless $k \times k$ matrices

$$z_{i} = (z_{pq}^{(i)}) = y_{i} - \frac{1}{k}\text{tr}(y_{i})I_{k}, \quad i = 1, \ldots, m,$$

and the subalgebra $W_{km}$ of $T_{km}$ generated by $z_{1}, \ldots, z_{m}$, the subalgebra $C_{km}^{(0)}$ of $C_{km}$ generated by the traces of the products, $\text{tr}(z_{i_{1}} \cdots z_{i_{l}})$, and
the subalgebra $T_{km}^{(0)}$ of $T_{km}$ generated by $W_{km}$ and $C_{km}^{(0)}$. Finally, let $L_{km}$ be the Lie subalgebra of $W_{km}$ generated by $z_1, \ldots, z_m$.

**Proposition 4.** Let $k, m \geq 2$. Then

(i) (Procesi [15])

$$T_{km} \cong K[\text{tr}(y_1), \ldots, \text{tr}(y_m)] \otimes K T_{km}^{(0)},$$

$$C_{km} \cong K[\text{tr}(y_1), \ldots, \text{tr}(y_m)] \otimes K C_{km}^{(0)};$$

(ii) (Razmyslov [16])

$$W_{km} \cong K\langle x_1, \ldots, x_m \rangle / \text{Id}(M_k(K), \text{sl}_k(K))$$

where $\text{Id}(M_k(K), \text{sl}_k(K))$ is the ideal of all weak polynomial identities in $m$ variables for the pair $(M_k(K), \text{sl}_k(K))$, i.e., the polynomials in the free associative algebra $K\langle x_1, \ldots, x_m \rangle$ which vanish when evaluated on $\text{sl}_k(K)$ considered as a subspace in $M_k(K)$.

(iii) (Razmyslov [16]) The Lie algebra $L_{km}$ is isomorphic to the relatively free algebra $F_m(\text{var}(\text{sl}_k(K))$ in the variety of Lie algebras generated by $\text{sl}_k(K)$.

**Corollary 5.** For $k, m \geq 2$

$$\text{GKdim}(T_{km}^{(0)}) = \text{GKdim}(C_{km}^{(0)}) = (k^2 - 1)(m - 1).$$

**Proof.** The algebras $T_{km}$ and $C_{km}$ satisfy the conditions of Corollary [2]. Hence the multiplicity of 1 as a pole of the Hilbert series of $T_{km}$ and $C_{km}$ is equal to their Gelfand-Kirillov dimension $k^2(m - 1) + 1$ from Proposition [3] (iii). Proposition [4] (i) gives that

$$H(T_{km}, t) = H(K[\text{tr}(y_1), \ldots, \text{tr}(y_m)], t)H(T_{km}^{(0)}, t) = \frac{1}{(1 - t)^m}H(T_{km}^{(0)}, t),$$

$$H(C_{km}, t) = \frac{1}{(1 - t)^m}H(C_{km}^{(0)}, t).$$

Hence the multiplicity of 1 as a pole of $H(T_{km}^{(0)}, t)$ and $H(C_{km}^{(0)}, t)$ is equal to $(k^2(m - 1) + 1) - m = (k^2 - 1)(m - 1)$. Both algebras $T_{km}^{(0)}$ and $C_{km}^{(0)}$ are finitely generated and graded. Hence the proof follows from Corollary [2].

Now we shall summarize the information for $2 \times 2$ generic matrices.

**Proposition 6.** Let $k = 2$ and $m \geq 2$. Then:

(i) (Sibirskii [18]) The trace polynomials

$$\text{tr}(y_i), \ i = 1, \ldots, m, \ \text{tr}(y_iy_j), \ 1 \leq i \leq j \leq m,$$

$$\text{tr}(y_{i_1}y_{i_2}y_{i_3}), \ 1 \leq i_1 < i_2 < i_3 \leq m,$$
form a minimal system of generators of $C_{2m}$.

(ii) (Procesi [15]) The algebras $T_{2m}^{(0)}$ and $W_{2m}$ coincide. The algebra $C_{2m}^{(0)}$ is generated by

$$\text{tr}(z_iz_j), 1 \leq i \leq j \leq m, \quad \text{tr}(z_iz_iz_3), 1 \leq i_1 < i_2 < i_3 \leq m,$$

which belong to $W_{2m}$.

(iii) (Drensky [5]) The algebra $C_{2m}^{(0)}$ is generated by

$$z_i^2, i = 1, \ldots, m, \quad z_iz_j + z_jz_i, 1 \leq i \leq j \leq m,$$

$$s_3(z_{i_1}, z_{i_2}, z_{i_3}), 1 \leq i_1 < i_2 < i_3 \leq m,$$

where

$$s_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

is the standard polynomial of degree 3.

Proof. We shall present the proof of (ii) and (iii) as a consequence of (i). Clearly $C_{2m}^{(0)}$ is generated by $\text{tr}(z_iz_j), 1 \leq i \leq j \leq m,$ and $\text{tr}(z_iz_iz_3), 1 \leq i_1 < i_2 < i_3 \leq m.$ Now the proof of (ii) and (iii) follows immediately from the equalities in $T_{2m}^{(0)}$

$$\text{tr}(z_1^2) = 2z_1^2, \quad \text{tr}(z_1z_2) = z_1z_2 + z_2z_1,$$

$$\text{tr}(z_1z_2z_3) = \frac{1}{3}s_3(z_1, z_2, z_3)$$

which may be checked by direct verification. □

Lemma 7. The commutator ideal $L'_{2m}$ is a $C_{2m}^{(0)}$-module.

Proof. The following equalities which can be verified directly hold in $W_{2m}$:

$$[z_1, z_2]z_3^2 = \frac{1}{4}([z_1, z_2, z_3, z_3] - [[z_1, z_3], [z_2, z_3]],$$

$$[z_1, z_2](z_3z_4 + z_4z_3) = \frac{1}{4}([z_1, z_2, z_3, z_4] + [z_1, z_2, z_4, z_3]$$

$$-[[z_1, z_3], [z_2, z_4]] - [[z_1, z_4], [z_2, z_3]],$$

$$z_4s_3(z_1, z_2, z_3) = \frac{3}{8} \sum_{\sigma \in S_3} \text{sign}(\sigma)[z_4, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}].$$

The elements of the commutator ideal are linear combinations of (left normed) commutators $u_i = [z_{i_1}, z_{i_2}, \ldots, z_{i_n}]$. If $v$ is a generator of $C_{2m}^{(0)}$, then

$$u_iv = [z_{i_1}, z_{i_2}, \ldots, z_{i_n}]v = [[z_{i_1}, z_{i_2}]v, \ldots, z_{i_n}]$$

and the above equalities guarantee that $u_iv$ is a linear combination of commutators, i.e., belongs to $L'_{2m}$ again. Hence $L'_{2m}C_{2m}^{(0)} \subset L'_{2m}$. □
Remark 8. It is known that $W_{2m}$ is a $C_{2m}^{(0)}$-module generated by $1$, $z_i$, $i = 1, \ldots, m$, and $[z_i, z_j]$, $1 \leq i < j \leq m$. Using the equality

$$[z_1, z_2, z_3] = 2(z_1 z_2 z_3 + z_3 z_2) - z_2(z_1 z_3 + z_3 z_1),$$

as in the proof of Lemma 7 we can show that $L'_{2m}$ is a $C_{2m}^{(0)}$-module generated by all commutators $[z_i, z_j]$ and $[z_i, z_i, z_j]$. For $m = 2$ the commutator ideal $L'_{22}$ is a free $C_{22}^{(0)}$-module generated by $[z_1, z_2]$, $[z_1, z_2, z_1]$, $[z_1, z_2, z_2]$, see [6].

The proof of the following theorem established in [11] is the main result of our paper.

**Theorem 9.** Let $K$ be a field of characteristic 0 and let $L_{2m}$ be the Lie algebra generated by $m$ generic traceless $2 \times 2$ matrices, $m \geq 2$. Then

$$\text{GKdim}(L_{2m}) = \text{GKdim}(F_m(\text{var}(sl_2(K)))) = 3(m - 1).$$

**Proof.** Let

$$H(C_{2m}^{(0)}, t) = \sum_{n \geq 0} c_n t^n, \quad H(L_{2m}, t) = \sum_{n \geq 1} l_n t^n, \quad H(W_{2m}, t) = \sum_{n \geq 1} w_n t^n$$

be the Hilbert series of $C_{2m}^{(0)}$, $L_{2m}$, and $W_{2m}$, respectively. Since the algebra $L_{2m}$ is finitely generated, its Gelfand-Kirillov dimension is

$$\text{GKdim}(L_{2m}) = \limsup_{n \to \infty} \log_n \left( \sum_{k=1}^{n} l_k \right).$$

The algebra $W_{2m}$ has no zero divisors and hence $[z_1, z_2]C_{2m}^{(0)} \subset L'_{2m} \subset L_{2m}$ is a free $C_{2m}^{(0)}$-module. Therefore

$$\sum_{k=0}^{n-2} c_k \leq \sum_{k=1}^{n} l_k \leq \sum_{k=0}^{n} w_k,$$

which implies that

$$3(m - 1) = \text{GKdim}(C_{2m}^{(0)}) \leq \text{GKdim}(L_{2m}) \leq \text{GKdim}(W_{2m}) = 3(m - 1).$$

□

**Remark 10.** As in [11] the formula for the Gelfand-Kirillov dimension of $F_m(\text{var}(sl_2(K)))$ obtained in characteristic 0 holds also for any infinite field $K$ of characteristic different from 2.

**Remark 11.** In characteristic 2 the algebra $sl_2(K)$ is nilpotent of class 2 and hence $F_m(\text{var}(sl_2(K)))$ is isomorphic to the free nilpotent of class 2 Lie algebra $F_m(N_2)$ which is finite dimensional. Therefore
GK-dim(F_m(var(\text{sl}_2(K)))) = 0. When K is an infinite field of characteristic 2, a much more interesting object is the relatively free algebra F_m(var(M_2(K)^(-))) of the variety generated by the 2\times 2 matrix algebra M_2(K) considered as a Lie algebra. Vaughan-Lee [19] showed that the algebra M_2(K)^(-) does not have a finite basis of its polynomial identities. (It is easy to see that the four-dimensional Lie algebra constructed in [19] is isomorphic to M_2(K)^(-).) The algebra M_2(K)^(-) satisfies the center-by-metabelian polynomial identity 

\[[[x_1, x_2], [x_3, x_4]], x_5] = 0.

It is well known that the free center-by-metabelian Lie algebra F_m([\mathfrak{A}^2, \mathfrak{E}]) over any field K is spanned by

\[[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}], [[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}], [x_{i_{n+1}}, x_{i_{n+2}}]],

where \(i_1 > i_2 \leq i_3 \leq \cdots \leq i_n\) and the commutators are left normed, e.g., \([x_1, x_2, x_3] = [[x_1, x_2], x_3]\). (A basis of F_m([\mathfrak{A}^2, \mathfrak{E}]) is given by Kuzmin [10].) Since the commutators \([x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]\) form a basis of the free metabelian Lie algebra F_m(\mathfrak{A}^2) and are linearly independent in F_m(var(M_2(K)^(-))), we obtain immediately that

\[\text{GKdim}(F_m(var(M_2(K)^(-)))) = m, \quad m > 1.\]

In characteristic 2 there is another three-dimensional simple Lie algebra which is an analogue of the Lie algebra of the three-dimensional real vector space with the vector multiplication. It is interesting to see whether this algebra has a finite basis of its polynomial identities (probably not) and, when the field is infinite, to compute the Gelfand-Kirillov dimension of the corresponding relatively free algebras.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria

E-mail address: drensky@math.bas.bg

Department of Mathematics, IMECC, UNICAMP, Sérgio Buarque de Holanda 651, Campinas, SP 13083-859, Brazil

E-mail address: plamen@ime.unicamp.br

Department of Mathematics, CCNE, UFSM, Faixa de Camobi, Campus UFSM, Santa Maria, RS 97105-900, Brazil

E-mail address: grings@smail.ufsm.br