WHEN THE THEORIES MEET: KOHANOV HOMOLOGY
AS HOCHSCHILD HOMOLOGY OF LINKS

JOZEF H. PRZYTYCKI

Version of October 6, 2005

ABSTRACT. We show that Khovanov homology and Hochschild homology theories share common structure. In fact they overlap: Khovanov homology of a $(2, n)$-torus link can be interpreted as a Hochschild homology of the algebra underlining the Khovanov homology. In the classical case of Khovanov homology we prove the concrete connection. In the general case of Khovanov-Rozansky, $sl(n)$, homology and their deformations we conjecture the connection. The best framework to explore our ideas is to use a comultiplication-free version of Khovanov homology for graphs developed by L. Helme-Guizon and Y. Rong and extended here to $M$-reduced case, and in the case of a polygon to noncommutative algebras. In this framework we prove that for any unital algebra $A$ the Hochschild homology of $A$ is isomorphic to graph homology over $A$ of a polygon. We expect that this paper will encourage a flow of ideas in both directions between Hochschild/cyclic homology and Khovanov homology theories.

1. HÖCHSCHILD HOMOLOGY AND CYCLIC HOMOLOGY

We recall in this section definition of Hochschild homology and cyclic homology and we sketch two classical calculations for tensor algebras and symmetric tensor algebras. More calculations are reviewed in Section 4 in which we use our main result, Theorem 3.1, to obtain new results in Khovanov homology, in particular solving some conjectures from [H-P-R]. We follow [Lo] in our exposition of Hochschild homology.

Let $k$ be a commutative ring and $A$ a $k$-algebra (not necessarily commutative). Let $M$ be a bimodule over $A$ that is a $k$-module on which $A$ operates linearly on the left and on the right in such a way that $(am)a' = a(ma')$ for $a, a' \in A$ and $m \in M$. The actions of $A$ and $k$ are always compatible (e.g. $m(\lambda a) = (m\lambda)a = \lambda(ma)$). When $A$ has a unit element $1$ we always assume that $1m = m1 = m$ for all $m \in M$. Under this unital hypothesis, the bimodule $M$ is equivalent to a right $A \otimes A^{op}$-module via $m(a' \otimes a) = ama'$. Here $A^{op}$ denotes the opposite algebra of $A$ that is $A$ and $A^{op}$ are the same as sets but the product $a \cdot b$ in $A^{op}$ is the product $ba$ in $A$. The product map
of \( A \) is usually denoted \( \mu : A \otimes A \to A, \mu(a, b) = ab. \)

In this paper we work only with unital algebras. We also assume, unless otherwise stated, that \( A \) is a free \( k \)-module, however in most cases, it suffices to assume that \( A \) is \( k \)-projective, or less restrictively, that \( A \) is flat over \( k \). Throughout the paper the tensor product \( A \otimes B \) denotes the tensor product over \( k \), that is \( A \otimes_k B \).

**Definition 1.1 ([Hoch](#) [Lo](#)).** The Hochschild chain complex \( C_\ast(A, \mathbb{M}) \) is defined as:

\[
\ldots \xrightarrow{b} \mathbb{M} \otimes A^{\otimes n} \xrightarrow{b} \mathbb{M} \otimes A^{\otimes n-1} \xrightarrow{b} \ldots \mathbb{M} \otimes A \xrightarrow{b} \mathbb{M}
\]

where \( C_\ast(A, \mathbb{M}) = \mathbb{M} \otimes A^{\otimes n} \) and the Hochschild boundary is the \( k \)-linear map \( b : \mathbb{M} \otimes A^{\otimes n} \to \mathbb{M} \otimes A^{\otimes n-1} \) given by the formula \( b = \sum_{i=0}^{n}(-1)^i d_i \), where the face maps \( d_i \) are given by

\[
d_0(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n),
\]

\[
d_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_{i+1}, \ldots, a_n) \quad \text{for} \quad 1 \leq i \leq n - 1,
\]

\[
d_n(m, a_1, \ldots, a_n) = (a_n m, a_1, \ldots, a_{n-1}).
\]

In the case when \( \mathbb{M} = A \) the Hochschild complex is called the cyclic bar complex.

By definition, the \( n \)th Hochschild homology group of the unital \( k \)-algebra \( A \) with coefficients in the \( A \)-bimodule \( \mathbb{M} \) is the \( n \)th homology group of the Hochschild chain complex denoted by \( HH_n(A, \mathbb{M}) \). In the particular case \( \mathbb{M} = A \) we write \( HH_n(A) \) instead of \( C_\ast(A, A) \) and \( HH_\ast(A) \) instead of \( H_\ast(A, A) \).

The algebra \( A \) acts on \( C_\ast(A, \mathbb{M}) \) by \( a \cdot (m, a_1, \ldots, a_n) = (am, a_1, \ldots, a_n) \). If \( A \) is a commutative algebra then the action commutes with boundary map \( b \), therefore \( HH_n(A) \) is an \( A \)-module.

If \( A \) is a graded algebra and \( \mathbb{M} \) a coherently graded \( A \)-bimodule, and the boundary maps are grading preserving, then the Hochschild chain complex is a bigraded chain complex with \( (b : C_{i,j}(A, \mathbb{M}) \to C_{i-1,j}(A, \mathbb{M})) \), and \( HH_\ast(A, \mathbb{M}) \) is a bigraded \( k \)-module. In the case of abelian \( A \) and \( A \)-symmetric \( \mathbb{M} \) (i.e. \( am = ma \)), \( HH_\ast(A, \mathbb{M}) \) is bigraded \( A \)-module. The main examples coming from the knot theory are \( A_m = \mathbb{Z}[x]/(x^m) \) and \( \mathbb{M} \) the ideal in \( A_m \) generated by \( x^{m-1} \).

We complete this survey by describing, after [Lo](#), two classical results in Hochschild homology – the computation of Hochschild homology for a tensor algebra and for a symmetric tensor algebra.

**Theorem 1.2.**

Let \( V \) be any \( k \)-module and let \( A = T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \ldots \) be its tensor algebra. We denote by \( \tau_n : V^{\otimes n} \to V^{\otimes n} \) the cyclic permutation, \( \tau_n(v_1, \ldots, v_{n-1}, v_n) = (v_n, v_1, \ldots, v_{n-1}) \). Then the Hochschild homology of \( A = T(V) \) is:

\[
HH_0(A) = \oplus_{i \geq 0} V^{\otimes i}/(1 - \tau_i),
\]

\[
HH_1(A) = \oplus_{i \geq 1} (V^{\otimes i})^{\tau_i}, \text{ where } (V^{\otimes i})^{\tau_i} \text{ is the space of invariants, that is the}
\]
kernel of $1 - \tau_i$.
$HH_n(A) = 0$ for $n \geq 2$.

The main idea of the proof is to show that the Hochschild chain complex of $T(V)$ is quasi-isomorphic\(^1\) to the “small” chain complex:

$$C_{\text{small}}(T(V)) : \ldots \rightarrow 0 \rightarrow \mathcal{A} \otimes V \xrightarrow{\hat{b}} \mathcal{A}$$

where the module $\mathcal{A}$ is in degree 0 and where the map $\hat{b}$ is given by $\hat{b}(av - va)$. Therefore $\hat{b}$ restricted to $V^{\otimes n-1} \otimes V$ is precisely $(1 - \tau_n) : V^{\otimes n} \rightarrow V^{\otimes n}$ and Theorem 1.2 follows. See Proposition 3.1.2 and Theorem 3.1.4 of [Lo].

**Theorem 1.3.** Symmetric and polynomial algebras.

Let $V$ be a module over $k$ and let $S(V)$ be the symmetric tensor algebra over $V$; $S(V) = k \oplus V \oplus S^2(V) \oplus \ldots$. If $V$ is free of dimension $n$ generated by $x_1, \ldots, x_n$ then $S(V)$ is the polynomial algebra $k[x_1, \ldots, x_n]$. Assume that $V$ is a flat $k$-module (e.g. a free module). Then there is an isomorphism

$$S(V) \otimes \Lambda^n V \cong HH_n(S(V))$$

where $\Lambda^n V$ is the exterior algebra of $V$. If $V$ is free of dimension $n$ generated by $x_1, \ldots, x_n$ then $\Lambda^n V$ is a free $\left(\begin{array}{c} n \\ m \end{array}\right)$ $k$-module with a basis $v_1 \wedge v_2 \wedge \ldots \wedge v_m$, $i_1 < i_2 < \ldots < i_m$.

The above theorem is a special case of Hochschild-Konstant-Rosenberg theorem about Hochschild homology of smooth algebras [HKR], which we discuss in section 4. Here we stress, after Loday, that the isomorphism $\varepsilon : S(V) \otimes \Lambda^* V \rightarrow HH_n(S(V))$ is induced by a chain map, that is not true in general for smooth algebras. $S(V) \otimes \Lambda^* V$ is a chain complex with the zero boundary maps. The chain map $\varepsilon : S(V) \otimes \Lambda^* V \rightarrow C_n(S(V), S(V))$ is given by $\varepsilon(a_0 \otimes a_1 \wedge \ldots \wedge a_n) = \varepsilon_n(a_0, a_1, \ldots, a_n)$ where $\varepsilon_n$ is the antisymmetrization map given as the sum $\sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma((a_0, a_1, \ldots, a_n))$ and the permutation $\sigma \in S_n$ acts by $\sigma((a_0, a_1, \ldots, a_n)) = (a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$.

In the second section we describe Khovanov homology of links and its (“comultiplication-free”) version for graphs introduced in [I-R-2]. In order to compare them with Hochschild homology we offer various generalizations, relaxing a condition that underlying algebra needs to be commutative (in the case of a polygon or a line graph), and allowing an $M$-reduced case. Another innovation in our presentation is that we start with cohomology of a functor from the category of subsets of a fixed set to the category of modules. In this setting we describe graph homology introduced in [I-R-2] and its generalizations to $M$-reduced cohomology and to cohomology of a polygon or a line graphs with a noncommutative underlying algebra. A set

\(^1\)Two chain complexes $C$ and $C'$ are called quasi-isomorphic if there is a chain map $f : C \rightarrow C'$ which induces an isomorphism on homology, $f_* : H_n(C) \rightarrow H_n(C')$, for all $n$. The map $f$ is called a quasi-isomorphism, chain equivalence, or homologism.
in this case is the set of edges of a graph. Finally, we describe a generalization to “supersets” which allows homology for signed graphs, link diagrams and, in some cases, links (with the classical Khovanov homology as the main example). In examples of functors from “supersets” we utilize comultiplication in the underlying algebra \( A \), after Khovanov \cite{Kh-1}, we assume that \( A \) is a Frobenius algebra \( \text{Abr} [\text{Kock}] \).

In the third section we prove our main result relating Hochschild homology \( HH_* (A) \) to graph cohomology and Khovanov homology of links, and the homology \( H_n (A, M) \) to reduced (\( M \)-reduced, more precisely) cohomology of graphs and links.

In the fourth section we use our main result to describe graph cohomology of polygons for various algebras solving, in particular, several problems from \cite{H-P-R}.

We envision the connection between Connes cyclic homology and Khovanov type homology, possibly when analyzing symmetry of graphs and links. For the idea of cyclic homology it is the best to quote from J-L. Loday \cite{Lo}:

"...in his search for a non-commutative analogue of de Rham homology theory, A. Connes discovered in 1981 the following striking phenomenon:
- the Hochschild boundary map \( b \) is still well defined when one factors out the module \( A \otimes A^{\otimes n} = A^{\otimes n+1} \) by the action of the (signed) cyclic permutation of order \( n + 1 \).
Hence a new complex was born whose homology is now called (at least in characteristic zero) cyclic homology.”

2. Khovanov homology

We start this section with a very abstract definition based on Khovanov construction but, initially, devoid of topological or geometric context. In this setting we recall and generalize the concept of graph cohomology \cite{H-R-2} and of classical Khovanov homology for unoriented framed links. We follow, in part, the exposition in \cite{H-R-1, H-R-2, Vi}. We review, after \cite{H-P-R}, the connection between Khovanov homology of links and graph cohomology of associated Tait graphs. We define homology of link diagrams related to graph cohomology for any commutative algebra\(^2\).

2.1. Cohomology of a functor on sets.

Let \( k \) be a commutative ring and \( E \) a finite (or countable) set.

**Definition 2.1.** Let \( \Phi \) be a functor from the category of subsets of \( E \) (i.e. subsets of \( E \) are objects and inclusions are morphisms) to the category of \( k \)-modules. We define the “Khovanov cohomology” of the functor, \( H^i(\Phi) \), as follows. We start from the graded \( k \)-module \( \{ C^i(\Phi) \} \), where \( C^i(\Phi) \) is the direct sum of \( \Phi(s) \) over all \( s \in E \) of \( i \) elements \( (|s| = i) \). To define

\(^2\)We can extend the definition to noncommutative algebras in the case of (2, \( n \))-torus link diagrams.
It suffices to define face maps \(d_e(s) = \Phi(s \cup s \cup e)(s)\) for \(e \in E - s\) (notice that \(s \subset s \cup e\) is the unique morphism in \(\text{Mor}(s, s \cup e)\)). Now, as usually, \(d(s) = \sum_{e \in s}(-1)^{t(s,e)}d_e(s)\), where \(t(s,e)\) requires ordering of elements of \(E\) and is equal the number of elements of \(s\) smaller then \(e\). Because \(\Phi\) is a functor, therefore we have \(d_{e_2}d_{e_1}(s) = d_{e_1}d_{e_2}(s)\) for any \(e_1, e_2 \notin s\). The sign convention guarantees now that \(d^2 = 0\) and \(\{C^{i}\}, d\) is a cochain complex. Now we define, in a standard way, cohomology as \(H^i(\Phi) = \ker(d(C^i(\Phi) \to C^{i+1}(\Phi))/d(C^{i-1}(\Phi)).\) The standard argument shows that \(H^i(\Phi)\) is independent on ordering of \(E\).

In the case when \(E\) are edges of a graph \(G\) we can define specific functors in various ways taking into account a structure of \(G\). We construct below our main example: a generalization of a graph cohomology, defined in [H-R-2], to \(\mathbb{M}\)-reduced case and its translation to homology of alternating diagrams. In the case of the algebra \(A_2 = \mathbb{Z}[x]/(x^2)\), this homology agrees partially with the classical Khovanov homology (see Theorem 2.7). To deal with all link diagrams we will later expand Definition 2.1 to “supersets” (Definition 2.4) in a construction which can incorporate multiplication and comultiplication in \(A\) (Example 2.5).

**Definition 2.2** (Cohomology introduced in [H-R-2] and extended to \(\mathbb{M}\)-reduced case).

1. We define here \(\mathbb{M}\)-reduced cohomology denoted by \(H^*_\mathbb{A}\mathbb{M}(G,v_1)\).
   If we assume \(\mathbb{M} = \mathbb{A}\) we obtain (comultiplication-free) cohomology of graphs, \(H^*_\mathbb{A}(G)\), defined in [H-R-2].

Let \(G\) be a graph with an edge set \(E = E(G)\), and a chosen base vertex \(v_1\). Fix a commutative \(k\)-algebra \(\mathbb{A}\) and a \(\mathbb{A}\)-module \(\mathbb{M}\). We define a functor \(\Phi\) on a category of subsets of \(E\) as follows:

**(Objects)** To define the functor \(\Phi\) on objects \(s \subset E\) we define it more generally on any subgraph \(H \subset G\), starting from a connected \(H\), to be \(\Phi(H) = \mathbb{M}\) if \(v_1 \in H\), and \(\Phi(H) = \mathbb{A}\) if \(v_1 \notin H\). If \(H\) has connected components \(H_1, ..., H_k\) then we define \(\Phi(H) = \Phi(H_1) \otimes ... \otimes \Phi(H_k)\). Finally, \(\Phi(s) = \Phi([G : s])\) where \([G : s]\) is a subgraph of \(G\) containing all vertices of \(G\) and edges \(s\).

In what follows \(k(s)\) is the number of components of \([G : s]\).

**(Morphisms)** It suffices to define \(\Phi(s \subset s \cup e)\) where \(e \notin s\). The definition depends now on the role of \(e\) in \([G : s]\) as follows:

(i) Assume that \(e\) connects different components of \([G : s]\):

(i') If \(e\) connects components \(u_i\) and \(u_{i+1}\) not containing \(v_1\) then we define \(\Phi(s \subset s \cup e)(m, a_1, ..., a_i, a_{i+1}, ..., a_{k(s)-1}) = (m, a_1, ..., a_i a_{i+1}, ..., a_{k(s)-1})\).

(i'') If \(e\) connects a component of \([G : s]\) containing \(v_1\) with another component of \([G : s]\), say \(u_1\), then we put \(\Phi(s \subset s \cup e)(m, a_1, ..., a_{k(s)-1}) = (ma_1, ..., a_{k(s)-1})\).
When the theories meet

(ii) Assume that \( e \) connects vertices of the same component of \( [G : s] \), then \( \Phi(s \subset s \cup e) = \Phi(s \cup e) = M \otimes A^{k(s)-1} \)

In the proof that \( \Phi \) is a functor commutativity of \( A \) is important (compare (3)).

2. If we modify the functor \( \Phi \) from (1) to a new functor, \( \hat{\Phi} \) which differs from \( \Phi \) only in the rule (1)(ii) that \( \hat{\Phi}(s \subset s \cup e) \) is a zero map if \( e \) connects vertices of the same component of \( [G : s] \). We denote the cohomology yielded by the functor \( \hat{\Phi} \) by \( \hat{H}^i_{A,M}(G,v_1) \). For \( M = A \) this cohomology, \( \hat{H}^i_A(G) \), is introduced in [H-R-2].

3. For a polygon or a line graph the cohomology \( \hat{H}^i_{A,M}(G,v_1) \) is defined also for a noncommutative algebra \( A \) and any \( A \)-bimodule \( M \). We consider a polygon or a line graph as a directed graph: from left to right (in the case of a line graph, Fig. 3.1) and in the anti-clockwise orientation (in the case of a polygon). In the formula for the morphism, \( \hat{\Phi}(s \subset s \cup e) \), of Definition 2.2(2) we use the product \( xy \) if \( x \) is the weight of the initial point of the directed edge \( e \) connecting different components of \( [G : s] \). We use this graph cohomology of the directed polygon when comparing graph cohomology with Hochschild homology.

Notice that cohomology described in (1) and (2) coincide to certain degree. Namely \( H^i_{A,M}(G,v_1) = \hat{H}^i_{A,M}(G,v_1) \) for all \( i < \ell - 1 \) where \( \ell \) is the length of the shortest cycle in \( G \).

Furthermore, if \( k \) is a principal ideal domain (e.g. \( k = Z \)) and \( A \) and \( M \) are free \( k \)-modules then \( Tor(H^i_{A,M}(G,v_1)) = Tor(\hat{H}^i_{A,M}(G,v_1)) \) for \( i = \ell - 1 \) (compare Theorem 2.7).

**Remark 2.3.**

(i) One can generalize\(^3\) construction in Definition 2.2(1) and (2) by choosing the sequence of elements \( f_1, f_2, f_3, ..., f_{|E|} \) in \( A \) and modifying functors \( \Phi \) and \( \hat{\Phi} \) on morphisms to get the functors \( \Phi' \) and \( \hat{\Phi}' \). We put \( \Phi'(s \subset s \cup e) = f_{|s|+1} \Phi(s \subset s \cup e) \) and, similarly, \( \hat{\Phi}'(s \subset s \cup e) = f_{|s|+1} \hat{\Phi}(s \subset s \cup e) \).

(ii) If \( A \) is not commutative and we work with cohomology of a line graph or a polygon (as in Definition 2.2(3)) we have to assume, in order to have \( d^2 = 0 \), that \( f_i \)'s are in the center of \( A \). We define \( f \cdot (m,a_1, ..., a_{k(s)-1})(s) = (fm,a_1, ..., a_{k(s)-1})(s) \).

We can define Khovanov cohomology on an alternating link diagram, \( D \) by considering associated plane graph, \( G(D) \) (Tait graph; compare Fig. 2.1) and its cohomology described in Definition 2.2 and Remark 2.3.

In order to define Khovanov cohomology on any link diagram (and take both multiplication and comultiplication into account) we have to define

---

\(^3\)We are motivated here by [Sto].
cohomology on any signed planar graph. We can start, as in Definition 2.1, from the very general setting (again cohomology of a functor) and to produce specific examples of a cohomology of signed planar graphs using a coherent algebra and coalgebra structures (Frobenius algebra).

**Definition 2.4.** Let $k$ be a commutative ring and $E = E_+ \cup E_-$ a finite set divided into two disjoint subsets (positive and negative sets). We consider the category of subsets of $E$ ($E \supset s = s_+ \cup s_-$ where $s_+ = s \cap E_+$). The set $\text{Mor}(s, s')$ is either empty or has one element if $s_- \subset s'_-$ and $s_+ \supset s'_+$. Objects are graded by $\sigma(s) = |s_-| - |s_+|$. Let us call this category the superset category (as the set $E$ is initially $Z_2$-graded). We define “Khovanov cohomology” for every functor, $\Phi$, from the superset category to the category of $k$-modules. We define cohomology of $\Phi$ in the similar way as for a functor from the category of sets (which corresponds to the case $E = E_-$). The cochain complex corresponding to $\Phi$ is defined to be $\{C^i(\Phi)\}$ where $C^i(\Phi)$ is the direct sum of $\Phi(s)$ over all $s \in E$ with $\sigma(s) = i$. To define $d : C^i(\Phi) \to C^{i+1}(\Phi)$ we first define face maps $d_e(s)$ where $e = e_+ \notin s_-$ ($e_- \in E_-$) or $e = e_+ \in s_+$. In such a case $d_e(s) = \Phi(s \subset s_+ \cup s_-)(s)$ and $d_e(s) = \Phi(s \supset s - e_+)$. We define $d(s) = \sum_{e \notin s} (-1)^{t(s, e)} d_e(s)$, where $t(s, e)$ requires ordering of elements of $E$ and is equal the number of elements of $s_-$ smaller than $e$ plus the number of elements of $s_+$ bigger than $e$. We obtain the cochain complex whose cohomology does not depend on ordering of $E$.

**Example 2.5.** Let $G$ be a signed plane graph with an edge set $E = E_+ \cup E_-$ were $E_+$ is the set of positive edges and $E_-$ is the set of negative edges. We define the functor from the superset category $E$ using the fact that $G$ is the (signed) Tait graph of a link diagram $D(G)$ (with white infinite region). See Figure 2.1 for conventions).

![Figure 2.1](image)

**Figure 2.1**

To define the functor $\Phi$ we fix a Frobenius algebra $A$ with multiplication $\mu$ and comultiplication $\Delta$ (the main example used by Khovanov is the algebra of truncated polynomials $A_m = Z[x]/(x^m)$ with a coproduct $\Delta(x^k) = \sum_{i+j=m-1+k} x^i \otimes x^j$). To get our functor on objects, $\Phi(s)$, we consider the Kauffman state defined by $s$ (so also denoted by $s$) and we associate $A$ to every circle of $D_G$ obtained from $D(G)$ by smoothing every crossing according to $s$ and then taking tensor product of these copies of $A$ (compare [4]). To get $\Phi$ on morphisms, without going into details, we say succinctly that if $\text{Mor}(s, s') \neq \emptyset$ and $\sigma(s') = \sigma(s) + 1$ then $\Phi(\text{Mor}(s, s'))$ is defined using product or coproduct depending on whether $D_G$ has more or less circles than $D_{s'}$. For $A_2$ we obtain the classical Khovanov homology.
Remark 2.6. One can also extend Example 2.5 to include the concept of $M$-reduced cohomology. We can consider, for example $M$ to be an ideal in $A$ with $\Delta(M) \subset M \otimes M$. An example, considered by Khovanov, is $A_m$ with $M$ generated by $x^{m-1}$ (in which case $\Delta(x^{m-1}) = x^{m-1} \otimes x^{m-1}$).

One can build more delicate (co)homology theory for ribbon graphs (flat vertex graph) using the fact that they embed uniquely into the closed surface. For $A = A_2$ it can be achieved using the approach presented in [APS], while for more general (Frobenius) algebras it is not yet done (most likely one should not use Frobenius algebra alone but its proper enhancement like in $A_2$ case).

In [H-P-R] we proved the following relation between graph cohomology and classical Khovanov homology of alternating links.

Theorem 2.7. Let $D$ be the diagram of an unoriented framed alternating link and let $G$ be its Tait graph. Let $\ell$ be the length of the shortest cycle in $G$. For all $i < \ell - 1$, we have

$$H^i_{A_2}(G) \cong H_{a,b}(D)$$

with

\[
\begin{align*}
  a &= E(G) - 2i, \\
  b &= E(G) - 2V(G) + 4j.
\end{align*}
\]

where $H_{a,b}(D)$ are the Khovanov homology groups of the unoriented framed link defined by $D$, as explained in [Vi].

Furthermore, $\text{Tor}(H^i_{A_2}(G)) = \text{Tor}(H_{a,b}(D))$ for $i = \ell - 1$.

We also speculate that for other $sl(m)$ Khovanov-Rozansky homology [Kh-R-1] [Kh-R-2] the graph cohomology of a polygon with $A = A_m = \mathbb{Z}[x]/(x^m)$ keeps essential information on $sl(m)$-homology of a torus link.

3. Relation between Hochschild homology and Khovanov homology

The main goal of our paper is to demonstrate relation between Khovanov homology and Hochschild homology. Initially I observed this connection for a commutative algebra $A$ by showing that for every commutative unital algebra $A$ the graph cohomology of $(n + 1)$-gon, $H^i_A(P_{n+1})$, is isomorphic to Hochschild homology of $A$, $H_{n-i}(A)$; $0 < i < n$. From this, via Theorem 2.7, relation between classical Khovanov homology of $(2, n + 1)$ torus link and Hochschild homology of $A_2 = \mathbb{Z}[x]/(x^2)$, follows. This relation was also observed independently by Magnus Jacobson [Jac].

In this paper we prove more general result. In order to formulate it we use an extended version of (Khovanov type) graph cohomology (working with noncommutative algebras and $M$-reduced cohomology):

(i) We can work with a noncommutative algebra, because, as mentioned in Section 2 (Definition 2.2(3)), for a polygon the graph cohomology is defined also for noncommutative algebras.
(ii) We can fix an \( \mathcal{A} \)-bialgebra \( \mathbb{M} \) and compare \( \mathbb{M} \)-reduced Khovanov type graph cohomology with the Hochschild homology of \( \mathcal{A} \) with coefficients in \( \mathcal{A} \)-bimodule \( \mathbb{M} \).

One can observe that we generalize notion of graph (co)homology while we keep the original definition of Hochschild homology. Our point is that the graph (co)homology is the proper generalization of Hochschild homology: from a polygon to any graph. We have this interpretation only for a commutative \( \mathcal{A} \). It seems to be, that if one work with general graphs and not necessary commutative algebras then these algebras should satisfy some "multiface" properties. Very likely that planar algebras or operads provide the proper framework\(^4\).

The interpretation of Hochschild homology as a homology of \( \mathcal{A} \) treated as an algebra over \( \mathcal{A} \otimes \mathcal{A}^{op} \) allows us to use the standard tool of homological algebra, that is we find appropriate (partial) free resolution of \( \mathcal{A} \otimes \mathcal{A}^{op} \) module \( \mathcal{A} \) using graph cochain complex of a line graph (Figure 3.1). The graph cohomology of the polygon is the cohomology obtained from this resolution. In Theorem 3.1 we use cohomology \( \hat{H}^i_{\mathcal{A},\mathbb{M}}(P_{n+1}) \) because we do not assume that \( \mathcal{A} \) is commutative.

**Theorem 3.1.** Let \( \mathcal{A} \) be a unital algebra\(^5\), \( \mathbb{M} \) an \( \mathcal{A} \)-bimodule and \( P_{n+1} \) – the \((n+1)\) -gon. Then for \( 0 < i \leq n \) we have:

\[
\hat{H}^i_{\mathcal{A},\mathbb{M}}(P_{n+1}) = H_{n-i}(\mathcal{A}, \mathbb{M}).
\]

Furthermore, if \( \mathcal{A} \) is a graded algebra and \( \mathbb{M} \) a coherently graded module then \( \hat{H}^i_{\mathcal{A},\mathbb{M}}(P_{n+1}) = H_{n-i}(\mathcal{A}, \mathbb{M}) \), for \( 0 < i \leq n \) and every \( j \).

**Corollary 3.2.** \( \hat{H}^{i,j}_{\mathcal{A}}(P_{n+1}) = HH_{n-i,j}(\mathcal{A}) \), for \( 0 < i \leq n \) and every \( j \).

Furthermore, for a commutative \( \mathcal{A} \), \( \hat{H}^{i,j}_{\mathcal{A}}(P_{n+1}) = \hat{H}^{i,j}_{\mathcal{A}}(P_{n+1}) \), for \( 0 < i < n \) and \( H^{n,j}_*(P_{n+1}) = 0 \), \( \hat{H}^{n,j}_*(P_{n+1}) = HH_{0,*}(\mathcal{A}) = \mathcal{A} \). For a general \( \mathcal{A} \), \( \hat{H}^{n,j}_*(P_{n+1}) = HH_{0,*}(\mathcal{A}) = \mathcal{A}/(ab - ba) \).

**Proof of Theorem 3.1**

We consider graph cohomology for a unital, possibly noncommutative algebra \( \mathcal{A} \) and any \( \mathcal{A} \)-bimodule \( \mathbb{M} \). There is no difference in the proof between commutative and noncommutative case except that we have to prove some property of cohomology given in [H-R-2] for a commutative \( \mathcal{A} \) (Lemma 3.3). The main idea of our proof is to interpret the graph cochain complex of

---

\(^4\)The author’s idea of working with directed graphs (quivers) seems to work, as observed by Y. Rong, only for line graphs and polygons.

\(^5\)We assume in this paper that \( \mathcal{A} \) is a free \( k \)-module, but we could relax the condition to have \( \mathcal{A} \) to be projective or, more generally, flat over a commutative ring with identity \( k \); compare [L]. We require \( \mathcal{A} \) to be a unital algebra in order to have an isomorphism \( \mathbb{M} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n+2} = \mathbb{M} \otimes \mathcal{A}^{\otimes n} \); the isomorphism is given by \( \mathbb{M} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n+2} \ni (m, a_0, a_1, ..., a_n, a_{n+1}) \rightarrow (a_{n+1} ma_0, a_1, ..., a_n, 1) \). We can write succinctly as \( (a_{n+1} ma_0, a_1, ..., a_n) \in \mathbb{M} \otimes \mathcal{A}^{\otimes n} \). We should stress that in \( \mathbb{M} \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n+2} \) the tensor product is taken over \( \mathcal{A} = \mathcal{A} \otimes \mathcal{A}^{op} \) while in \( \mathbb{M} \otimes \mathcal{A}^{\otimes n} \) the tensor product is taken over \( k \).
a line graph as a (partial) resolution of $A$. It was proved in \cite{H-R-2} that $H_i^A(\text{line graph}) = 0$, $i > 0$ for a commutative algebra $A$. We give here the proof for any unital algebra $A$. Let $L_n$ be the (directed) line graph of $n + 1$ vertices $(v_0, \ldots, v_n)$ and $n$ edges $(e_1, \ldots, e_n)$; Fig. 3.1.

![Graph](image)

**Figure 3.1**

**Lemma 3.3.** The graph cochain complex of $L_n$:

$C^*_{A}(L_n) : C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n$, is acyclic, except for the first term. That is, $H_i^A(L_n) = 0$ for $i > 0$ and $H_0^A(L_n)$ is usually nontrivial\(^6\).

For a line graph $\hat{H}_A = H_A$ so we will use $H_A$ to simplify notation. We prove Lemma 3.3 by induction on $n$. For $n = 0$, $L_0$ is the one vertex graph, thus $H^*_A(L_0) = A$ and the Lemma 3.3 holds. Assume that the lemma holds for $L_k$ with $k < n$. In order to perform inductive step we construct the long exact sequence of cohomology of line graphs. For a commutative algebra $A$ it is a special case\(^7\) of the exact sequence of graph cohomology in \cite{H-R-2} which in turn resemble the skein exact sequence of Khovanov homology \cite{V1}:

$$0 \to H^0_A(L_n) \to H^0_A(L_n) \otimes A \xrightarrow{\partial} H^0_A(L_{n-1}) \to H^1_A(L_n) \to \cdots$$

$$\to H^{i-1}_A(L_{n-1}) \to H^i_A(L_n) \to H^i_A(L_n) \otimes A \to \cdots$$

such that $\partial : H^0_A(L_n) \otimes A \to H^0_A(L_{n-1})$ is an epimorphism. From this exact sequence the inductive step follows.

To construct the above exact sequence we consider the short exact sequence of chain complexes (for the notation see Figure 3.1):

$$0 \to C^i_{A}(L_n/e_n) \xrightarrow{\alpha} C^i(L_n) \xrightarrow{\beta} C^i(L_n - e_n) \to 0.$$

This exact sequence is constructed in the same way as in the case of commutative algebra $A$, that is $\alpha(a_0 \otimes a_1 \otimes \cdots a_{n-i})(e_{j_1}, \ldots, e_{j_{i-1}}) = (a_0 \otimes a_1 \otimes \cdots a_{n-i})(e_{j_1}, \ldots, e_{j_{i-1}}, e_n)$. Further, $\beta$ is defined in such a way that if $e_n \in s$ then $\beta(S) = 0$, and if $e$ is not in $s$ then $\beta$ is the identity map (up to $(-1)^{|s|}$).

\(^6\)From the fact that chromatic polynomial of $L_n$ is equal to $\lambda(\lambda - 1)^n$ follows that $\text{rank}(H^0_A(L_n)) = \text{rank}(A)(\text{rank}(A) - 1)^n$, we assume here that $k$ is a principal ideal domain. It was proven in \cite{H-R-2} that for a commutative algebra $A$ decomposable into $k1 \oplus A/k$ one has that $H^0_A(L_n) = H^0_A(L_0) = A \otimes (A/k)^{\otimes n}$.

\(^7\)One can construct an exact sequence of functor cohomology imitating deleting-contracting exact sequence. One have to define properly two functors on $E \cup e$, one “covariant” and one “contravariant” and the exact sequence will be based on a functor on subsets of $E$ and these two additional functors. We will discuss this idea in a sequel paper.
Exactness of the sequence follows from the definition. This exact sequence leads to the long exact sequence of cohomology:

$$0 \to H^i_A(L_n) \to H^i_A(L_n - e_n) \xrightarrow{\partial} H^i_A(L_n/e_n) \to$$

$$... \to H^{i-1}_A(L_n/e_n) \to H^i_A(L_n) \to H^i_A(L_n - e_n) \to ...$$

Now $L_{n-1} = L_n/e_n$ and $L_n - e_n$ is $L_{n-1}$ with an additional isolated vertex, therefore by a Künneth formula (see for example 1.0.16 of [Lo]) we have $H^i_A(L_n - e_n) = H^i_A(L_n) \otimes A$. From this we get the exact sequence used in the proof of Lemma 3.3. To see the epimorphism of $\partial$ notice that the map $H^0_A(L_n - e_n) \to H^0_A(L_{n-1})$ is an epimorphism almost by the definition (we can think of decorating the last vertex of $L_n$ by 1, to see the epimorphism. There is chain map epimorphism $\partial_c : C^*_A(L_n - e_n) \to C^*_A(L_{n-1})$ which is obtained by multiplying the weight of component containing $v_{n-1}$ by the weight of $v_n$ which descends to $\partial$. This map has a chain map section $\partial^{-1}_c : C^*_A(L_{n-1}) \to C^*_A(L_n - e_n)$ such that in the image $v_n$ has always weight 1. Because $\partial_c \partial^{-1}_c = Id$ therefore, on the cohomology level $\partial$ is an epimorphism.

We can continue now with the proof of Theorem 3.1. The (partially) acyclic chain complex of Lemma 3.3 is the chain complex of $A^e = A \otimes A^{op}$ modules. It is a (partial) free resolution of the $A^e$-module $A$. Upon tensoring this resolution with $M$ considered as a right module over $A^e$ we obtain the cochain complex,

$$\{M \otimes A^e C^i\}_{i=0}^{n-1} : M \otimes A^e C^0 \xrightarrow{\partial^0} M \otimes A^e C^1 \xrightarrow{\partial^1} ... \to M \otimes A^e C^{n-2} \xrightarrow{\partial^{n-1}} M \otimes A^e C^{n-1} \to 0$$

whose cohomology (except possibly $H^0$) are the Hochschild homology of $A$ with coefficients in $M$ (compare for example [Wei]). Having in mind relation between indexing we get that $H^i = H_{n-i}(A, M)$ for $i > 0$. To get exactly the chain complex of the $M$-reduced (directed) graph homology of $P_n$, $H^*_{\Lambda, M}(P_n)$, we extend this chain complex to

$$\{M \otimes A^e C^i\}_{i=0}^{n} : M \otimes A^e C^0 \xrightarrow{\partial^0} M \otimes A^e C^1 \xrightarrow{\partial^1} ... \to M \otimes A^e C^{n-1} \xrightarrow{\partial^{n-1}} M \otimes A^e C^n = M \to 0$$

where the homomorphism $\partial^{n-1}$ is the zero map.

To complete the proof of Theorem 3.1 we show that this complex is exactly the same as the cochain complex of the $M$-reduced (directed) graph cohomology of $P_n$. We consider carefully the map $M \otimes A^e C^j \xrightarrow{\partial^j} M \otimes A^e C^{j+1}$. In the calculation we follow the proof of Proposition 1.1.13 of [Lo]. The idea is to “bend” the line graph $L_n$ to the polygon $P_n$ and show that it corresponds to tensoring, over $A^e$ with $M$; compare Figure 3.2.
Let us order components of \([G : s]\) (\(G\) is equal to \(P_n\) or \(L_n\)) in the anticlockwise orientation, starting from the component containing \(v_0\) (decorated by an element of \(M\) if \(G = P_n\)). The element of \(C_{\mathcal{A}}^j(L_n)\) will be denoted by \((a_{i_0}, a_{i_1}, ..., a_{i_{n-j-1}}, a_{i_{n-j}})(s)\) and of \(C_{\mathcal{A},M}^j(P_n)\) by \((m, a_{i_0}, ..., a_{i_{n-j-1}}, a_{i_{n-j}})(s)\). In the isomorphism \(M \otimes_{\mathcal{A}^e} C_{\mathcal{A}}^j(L_n) \rightarrow C_{\mathcal{A},M}^j(P_n)\) \((j < n)\) the element \((m, a_{i_0}, a_{i_1}, ..., a_{i_{n-j-1}}, a_{i_{n-j}})(s)\) is sent to \((a_{i_{n-j}} ma_{i_0}, a_{i_1}, ..., a_{i_{n-j-1}})(s)\). One can easily check that it yields a cochain map (compare \([L_0]\)) so it induces the isomorphism on cohomology. Note that \((m, a_{i_0}, a_{i_1}, ..., a_{i_{n-j-1}}, a_{i_{n-j}}) = (a_{i_{n-j}} ma_{i_0}, 1, a_{i_1}, ..., a_{i_{n-j-1}}, 1)\) in \(M \otimes_{\mathcal{A}^e} C_{\mathcal{A}}^j(L_n)\). The proof of Theorem 3.1 is completed.

4. **Calculations and speculations**

There is an extensive literature on Hochschild homology and a lot of ingenious methods of computing them (e.g. \([L_0, W_0, R_0, K_{0n}]\)). Our main result, Theorem 3.1, allows us to use these methods to compute graph cohomology for polygons and, to some extend, to other graphs (using for example an observation that some properties of a cohomology of a polygon propagate to graphs containing it (compare \([A-P, H-P-R]\)). Properties of Hochschild homology (and, equally well, cyclic homology) should eventually shed light on Khovanov type homology of links.

We start from adapting Theorem 1.3 about Hochschild homology of symmetric tensor algebra. The simplest case of one variable polynomials \(\mathcal{A} = \mathcal{A}_\infty = Z[x]\) allows us to extend Theorem 27 of \([H-P-R]\) from the triangle to any polygon. \(\mathcal{A}_\infty\) is a graded algebra with \(x^i\) being of degree \(i\). Consequently Hochschild homology of \(\mathcal{A}_\infty\) is a bigraded module. We treat \(\mathcal{A}_\infty\) as a \(Z\)-module (an abelian group) and to simplify description of homology we use the Poincaré polynomial of \(HH_{ss}(\mathcal{A}_\infty)\) to describe the free part of homology. Recall that the Poincaré polynomial (or series) of bigraded finitely generated \(Z\)-modules \(H_{ss}\) is \(PP(t, q) = PP(H_{ss})(t, p) = \sum_{i,j} a_{i,j} t^i q^j\) where \(a_{i,j}\) is the rank of the group \(H_{i,j}\).
Corollary 4.1. For an n-gon $P_n$ the graph cohomology groups $H_{\mathcal{A}_{\infty}}^{i,j}(P_n)$ are free abelian with Poincaré polynomial $(q+q^2+q^3+...)^3 + t^{n-2}(q+q^2+q^3+...) = \left(\frac{q}{1-q}\right)^3 + t^{n-2}\frac{q}{1-q}$.

Proof. From Theorem 3.1 we obtain Corollary 4.1. For an $q$ observed in [H-R-2] that $H_{\mathcal{A}_{\infty}}^{i,j}(P_n) = 0$ for $i \geq n-1$. To find $H_{\mathcal{A}_{\infty}}^{0,i}(P_n)$ we use the fact that the chromatic polynomial of $P_n$ is equal to $(\lambda-1)^n(\lambda-1)$ and the graph cohomology categorify the chromatic polynomial. That is, if we substitute $t = -1$ and $1+q+q^2+q^3+... = \lambda$ in the Poincaré polynomial we obtain the chromatic polynomial [H-R-2].

Another illustration of the power of our connection is for the algebra $A = A_{p(x)} = Z[x]/(p(x))$ where $p(x)$ is a polynomial in $Z[x]$. We discuss the general case later, here let us notice that two special cases of $p(x) = x^m$ and $p(x) = x^m - 1$ are of great interest in knot theory (in Khovanov-Rozansky homology [Kh-1,1] and its deformations [Gor]). Let us apply first the knowledge of Hochschild homology for $A_m = Z[x]/(x^m)$ (compare [Lo]) and solving Conjectures 30 and 31 of [HP-R].

Theorem 4.2. (Free) The Poincaré polynomial of $HH_{\ast\ast}(A_m)$ is equal to:

$$(1+q+...+q^{m-1}) + t(q+q^2+...+q^{m-1}) + (t^2+t^3)(q+q^2+...+q^{m-1})q^m + (t^4+t^5)(q+q^2+...+q^{m-1})q^{2m} + ... + (t^{2i}+t^{2i+1})(q+q^2+...+q^{m-1})q^{im} + ...$$

(Torsion) $\text{Tor}(H_{\ast\ast}(A_m)) = \bigoplus_{i=1}^{\infty} H_{2i-1,im}(A_m)$ where each summand is isomorphic to $Z_m$.

We solve Conjectures 30 and 31 of [HP-R] by applying Theorems 4.2 and 3.1.

Corollary 4.3. (Odd) For $n = 2g+1$ we have:

$\text{Tor}(H_{\mathcal{A}_m}^{i,i}(P_{2g+1})) = H_{\mathcal{A}_m}^{2,2m}(P_{2g+1}) \oplus H_{\mathcal{A}_m}^{4,2m}(P_{2g+1}) \oplus ... \oplus H_{\mathcal{A}_m}^{1,2m}(P_{2g+1})$

with each summand isomorphic to $Z_m$.

The Poincaré polynomial of $H_{\mathcal{A}_m}^{i,i}(P_{2g+1})$ is equal to

$(q + ... + q^{m-1})^{n-1} + (q + ... + q^{m-1})(t^{n-2} + t^{n-3}q + (t^2 + t)q^{m-1})$.

(Even) For $n = 2g+2$ we have:

$\text{Tor}(H_{\mathcal{A}_m}^{i,i}(P_{2g+2})) = H_{\mathcal{A}_m}^{2,2m}(P_{2g+2}) \oplus H_{\mathcal{A}_m}^{4,2m}(P_{2g+2}) \oplus ... \oplus H_{\mathcal{A}_m}^{2,2m}(P_{2g+2})$

with each summand isomorphic to $Z_m$.

The Poincaré polynomial of $H_{\mathcal{A}_m}^{i,i}(P_{2g+2})$ is equal to

$(q + ... + q^{m-1})^{n-2} + q^{m(n/2)-1}(q + ... q^{m-1}) + (q + ... + q^{m-1})(t^{n-2} + t^{n-3}q + (t^2 + t)q^{m-1} + t^2q^{m-1})$.

Assume that $m = 2$ in Corollary 4.3. Then, using Theorem 2.7 we can recover Khovanov computation of homology of the torus link $T_{2,n}$ [Kh-1]. In particular we get:

Corollary 4.4. [Kh-2] Let $T_{2,-n}$ be a left-handed torus link of type $(2,-n)$, $n > 2$. Then the torsion part of the Khovanov homology of $T_{2,-n}$ is given
by (in the description of homology we use notation of \( \psi \) treating \( T_{2,-n} \) as a framed link):

(Odd) For \( n \) odd, all the torsion of \( H_{**}(T_{2,-n}) \) is supported by
\[
H_{n-2,3n-4}(T_{2,-n}) = H_{n-4,3n-8}(T_{2,-n}) = \ldots = H_{-n+4,-n+8}(T_{2,-n}) = Z_2.
\]

(Even) For \( n \) even, all the torsion of \( H_{**}(T_{2,-n}) \) is supported by
\[
H_{n-4,3n-8}(T_{2,-n}) = H_{n-6,3n-12}(T_{2,-n}) = \ldots = H_{-n+4,-n+8}(T_{2,-n}) = Z_2.
\]

For a right-handed torus link of type \((2,n)\), \( n > 2 \), we can use the formula for the mirror image (Khovanov duality theorem; see for example A-P APS: \( H_{-i,-j}(D) = H_{ij}(D)/\text{Tor}(H_{ij}(D)) \oplus \text{Tor}(H_{i-2,j}(D)) \)).

The result on Hochschild homology of symmetric algebras has a major generalization to the large class of algebras called smooth algebras.

**Theorem 4.5.** \([\text{Lo}] \text{HKR}\) For any smooth algebra \( A \) over \( k \), the antisymmetrization map \( \varepsilon_\ast : \Lambda_{A[k]}^n \rightarrow HH_n(A) \) is an isomorphism of graded algebras. Here \( \Omega_{A[k]}^n = \Lambda^n \Omega_{A[k]}^1 \) is an \( A \)-module of differential \( n \)-forms.

We refer to \([\text{Lo}]\) for a precise definition of a smooth algebra, here we only recall that the following are examples of smooth algebras:

(i) Any finite extension of a perfect field \( k \) (e.g. a field of characteristic zero).

(ii) The ring of algebraic functions on a nonsingular variety over an algebraically closed field \( k \), e.g. \( k[x], k[x_1, \ldots, x_n], k[x, y, z, t]/(xt - yz - 1) \) \([\text{Lo}]\).

Not every quotient of a polynomial algebra is a smooth algebra. For example, \( C[x, y](x^2y^3) \) or \( Z[x]/(x^n) \) are not smooth. The broadest, to my knowledge, treatment of Hochschild homology of algebras \( C[x_1, \ldots, x_n]/(\text{Ideal}) \) is given by Kontsevich in \([\text{Kon}]\). For us the motivation came from one variable polynomials, Theorem 40 of \([\text{HHR}]\). In particular we generalize Theorem 40(i) from a triangle to any polygon that is we compute the graph cohomology of a polygon for truncated polynomial algebras and their deformations. Thus, possibly, we can approximate Khovanov-Rozansky \( sl(n) \) homology and their deformations.

**Theorem 4.6.**

(i) \( HH_1(A_{p(x)}) = Z[x]/(p(x), p'(x)) \) for \( i \) odd and \( HH_i(A_{p(x)}) = \{ [q(x)] \in Z[x]/(p(x)) | q(x)p'(x) \text{ is divisible by } p(x) \} \), for \( i \) even \( i \geq 2 \). In both cases the \( Z \) rank of the group is equal to the degree of \( \gcd(p(x), p'(x)) \).

(ii) In particular, for \( p(x) = x^{m+1} \), we obtain homology of the ring of truncated polynomials, \( A_{m+1} = Z[x]/x^{m+1} \) for which: \( HH_i(A_{m+1}) = Z_{m+1} \oplus Z^m \) for odd \( i \) and \( HH_i(A_{m+1}) = Z^m \) for even \( i \geq 2 \).

(iii) The graph cohomology of a polygon \( P_n, H^i_{A_{p(x)}}(P_n) \), is given by:
\[
H^{n-2i}_{A_{p(x)}}(P_n) = A_{p(x)}/(p'(x)) \quad \text{for } 1 \leq i \leq \frac{v^2}{2}, \text{ and}
\]
\[
H^{n-2i-1}_{A_{p(x)}}(P_n) = \ker(A_{p(x)} \xrightarrow{p'(x)} A_{p(x)}) \quad \text{for } 1 \leq i \leq \frac{v-2}{2}.
\]
Furthermore, $H^k_{A_p(x)}(P_n) = 0$ for $k \geq n - 1$ and $H^0_{A_p(x)}(P_n)$ is a free abelian group of rank $(d - 1)^n + (-1)^n(d - 1)$ for $n$ even (where $d$ denotes the degree of $p(x)$) and it is of rank $(d - 1)^n + (-1)^n(d - 1) - \text{rank}(H^i_{A_p(x)}(P_n))$ if $n$ is odd (notice that $(d - 1)^n + (-1)^n(d - 1)$ is the Euler characteristic of $\{H^i_{A_p(x)}(P_n)\}$).

Proof. Theorem 4.6(i) is proven by considering a resolution of $A_p(x)$ as an $A_p(x)\otimes A_p(x)$ module:
\[
\cdots \to A_p(x) \otimes A_p(x) \xrightarrow{u} A_p(x) \otimes A_p(x) \xrightarrow{v} A_p(x) \otimes A_p(x) \xrightarrow{u} \cdots \to A_p(x)
\]
where $u = x \otimes 1 - 1 \otimes x$ and $v = \Delta(p(x))$ is a coproduct given by $\Delta(x^{i+1}) = x^i \otimes 1 + x^{i-1} \otimes x + \cdots + x \otimes x \otimes 1 + x^i$.

Curious but not accidental observation is that by choosing coproduct $\Delta(1) = v$ we define a Frobenius algebra structure on $A$. In Frobenius algebra $(x \otimes 1)\Delta(1) = (1 \otimes x)\Delta(1)$ which makes $uv = vu = 0$ in our resolution. Furthermore the distinguished element of the Frobenius algebra $\mu \Delta(1) = p'(x)$.

References

[Abr] L. Abrams, Two-dimensional topological quantum field theories and Frobenius algebras, Journal of the Knot Theory and Its Ramifications, 5(5), 1995, 569-587.

[A-P] M. M. Asaeda, J. H. Przytycki, Khovanov homology: torsion and thickness, Advances in Topological Quantum Field Theory, in Advances in topological quantum field theory, 135–166, Kluwer Acad. Publ., Dordrecht, 2004; e-print: http://www.arxiv.org/math.GT/0402402

[APS] M. M. Asaeda, J. H. Przytycki, A. S. Sikora, Categorification of the Kauffman bracket skein module of $I$-bundles over surfaces, Algebraic & Geometric Topology (AGT), 4, 2004, 1177-1210. e-print: http://front.math.ucdavis.edu/math.QA/0403527

[C-G-V] G. Cortinas, J. Guccione, O. Villamayor, Cyclic homology of $K[Z/pZ]$, $K$-theory, 2, 1989, 603-616.

[D-G-R] N. M. Dunfield, S. Gukov, J.Rasmussen, The Superpolynomial for Knot Homologies, e-print: http://front.math.ucdavis.edu/math.GT/0505662

[Gor] B. Gornik, Note on Khovanov link cohomology, e-print: http://front.math.ucdavis.edu/math.QA/0402266

[Gu-Sch-Va] S. Gukov, A. Schwarz, C. Vafa, Khovanov-Rozansky Homology and Topological Strings; e-print: http://arxiv.org/abs/hep-th/0412243

[H-P-R] L. Helme-Guizon, J. H. Przytycki, Y. Rong, Torsion in Graph Homology, for Fundamenta Mathematicae, preprint (July 2005). e-print: http://arxiv.org/abs/math.GT/0507245
When the theories meet

[H-R-1] L. Helme-Guizon, Y. Rong, A Categorification for the Chromatic Polynomial, to appear in Algebraic and Geometric Topology (AGT), e-print: http://front.math.ucdavis.edu/math.CO/0412264

[H-R-2] L. Helme-Guizon, Y. Rong, Graph Cohomologies from Arbitrary Algebras, e-print: http://front.math.ucdavis.edu/math.QA/0506023

[Hoch] G. Hochschild, On the cohomology groups of an associative algebra, Annals of Math., 46, 1945, 58-67.

[HKR] G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102, 1962, 383-408.

[Jac] M. Jacobsson, Personal communication at AMS-IMS-SIAM Joint Summer Research Conference; Quantum Topology–Contemporary Issues and Perspectives, Snowbird, Utah, June 5-9, 2005.

[Kh-1] M. Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000), no. 3, 359–426, http://xxx.lanl.gov/abs/math.QA/9908171

[Kh-2] M. Khovanov, Patterns in knot cohomology I, Experiment. Math. 12(3), 2003, 365-374, http://arxiv.org/abs/math/0201306

[Kh-R-1] M. Khovanov, L. Rozansky, Matrix factorizations and link homology, e-print: http://www.arxiv.org/math.QA/0401268

[Kh-R-2] M. Khovanov, L. Rozansky, Matrix factorizations and link homology II, e-print: http://www.arxiv.org/math.QA/0505056

[Kock] J. Kock, Frobenius algebras and 2D topological quantum field theories, in London Mathematical Society Student Texts, 59, Cambridge University Press, 2003.

[Kon] M. Kontsevich, Hochschild and Harrison cohomology of complete intersections, Appendix to the paper Quantization on Curves by C. Fronsdal, e-print: http://front.math.ucdavis.edu/math-ph/0507021

[Kon-2] M. Kontsevich, Operads and Motives in Deformation Quantization, e-print: http://front.math.ucdavis.edu/math.QA/9904055

[Lo] J-L. Loday, Cyclic Homology, Grund. Math. Wissen. Band 301, Springer-Verlag, Berlin, 1992 (second edition, 1998).

[Pr] J. H. Przytycki, KNOTS: From combinatorics of knot diagrams to the combinatorial topology based on knots, Cambridge University Press, accepted for publication, to appear 2007, pp. 600.

[Ros] J. Rosenberg, Algebraic K-theory and its applications, Graduate Texts in Mathematics, 147, 1st ed 1994. Corr. 2nd printing, 1996, X, 392 p.

[Shu] A. Shumakovitch, Torsion of the Khovanov Homology, Geometry and Topology (GT), to appear. e-print: http://arxiv.org/abs/math.GT/0405474

[Sto] M. Stosic, Categorification of the Dichromatic Polynomial for Graphs, e-print: http://arxiv.org/abs/math.GT/0504239

[Vi] O. Viro, Remarks on definition of Khovanov homology, e-print: http://front.math.ucdavis.edu/math.GT/0202199

[Wei] C. A. Weibel, An introduction to homological algebra, Cambridge studies in advanced mathematics, 38, Cambridge University Press, 1995.

Dept. of Mathematics, Old Main Bldg., 1922 F St. NW The George Washington University, Washington, DC 20052

e-mail: przytyck@gwu.edu