Quasi-Coregular Modules

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Abstract: In this paper, we introduce the concept of quasi – copure submodules which is a generalization of a copure submodules. We used this concept to define the class of quasi – coregular module, where an R-module M is called quasi – coregular module if every submodule of M is quasi-co-pure. Many results about this concept are proved.

1. Introduction
Throughout this note, R is commutative ring with identity and M be a unitary R-module. A submodule N of an R-module M is called pure in M if IN = N ∩ IM for every ideal I of R [1]. An R-module M is regular module if every submodule of M is pure [2]. Ansari and F. Darshadifar in [3] introduced the concept of copure submodules, where a submodule N of M is called copure if [N : I] = N + [0 ; M] for each ideal I of R. First recall that a submodule N of an R-module M is called a quasi – pure if for each x ∈ M and x /∈ N, there exists a pure submodule L of M such that N ⊆ L and x /∈ L. This paper is structured in two sections. In section one we give new results about quasi-copure submodules. In section two, we study the concept of quasi – coregular mmodules. We give some relationships between quasi – coregular modules (rings) and quasi – regular modules (rings).

2. Quasi -Copure Submodules:
In this section we introduce the concept of quasi -copure submodule. We investigate the basic properties of these type of submodules are analogous to the properties of copure submodules.

Definition (2.1): Let M be an R-module. A submodule N of M is called a quasi – copure submodule of M if for each x ∈ M and x /∈ N, there exists a copure submodule L of M such that N ⊆ L and x /∈ L.

Remarks and Examples (2.2):
(1) Every copure submodule is quasi – copure. But the converse is not true in general. We have no example.
(2) Let M = Z₁ ⊕ Z₂ as a Z-module, and N = Z (2, 1), then N = {(0, 0), (2, 1)} is quasi – pure submodule of M. Since N is pure [4]. But is not a quasi – copure submodule of M, since there exists no copure submodule contain N = Z (2, 1).
(3) {0, 2} in Z₄ as a Z-module is not quasi – copure, since there is no copure submodule contain {0, 2}. And not quasi – pure [4].
(4) If M is semisimple R-module, then every submodule of M is quasi-copure.
(5) If R is a principal ideal ring, then every submodule of a coregular R-module is quasi-copure.
(6) Let $M = Z_2 \bigoplus Z_2$ as a $Z$-module, and $\text{add} N = \{(0, 0), (2, 0)\} = 2Z_2 \bigoplus 0 \cong Z_2$. It is easy to check that $N$ is quasi-copure submodule of $M$, since $N$ is a submodule of $M$.

(7) In any R-module $M$, the submodules $N$ and $M$ are always quasi-copure submodules.

Recall that an $RR$-module $M$ is called copure simple if $M$ and $< 0 >$ are the only copure submodule of $M$.

(8) Every copure simple $R$-module $M$ does not contain quasi-copure submodule except $< 0 >$ and $M$, because the copure simple module has non-zero proper copure submodule. For example, the $Z$-modules $Q, Z_p$.

**Proposition (2.3):** If $A$ and $B$ are quasi-copure submodules of an $R$-module $M$, then $A \cap B$ is quasi-copure submodule of $M$.

**Proof:** Let $x \in M$ and $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$. Assume that $x \notin A$. Since $A$ is quasi-copure in $M$, then there exists a copure submodule $D$ of $M$ such that $A \subseteq D$ and $x \notin D$. This implies that $A \cap B \subseteq D$ and $x \notin D$. That is $A \cap B$ is quasi-copure submodule of $M$.

**Proposition (2.4):** Let $M$ and $M'$ be two $R$-modules. If $f: M \longrightarrow M'$ be an epimorphism and $N$ is a quasi-copure submodule of $M$ such that $f(N)$ is a copure submodule in $M'$. If $y \in M$ and $y \notin f(N)$, then $y \notin f(N)$.

**Proof:** Let $y \in M'$ and $y \notin f(N)$, since $f$ is epimorphism, then there exists $x \in M$ such that $y = f(x)$ and $x \notin N$. Since $N$ is a quasi-copure submodule of $M$, then there exists a copure submodule $L$ of $M$ such that $N \subseteq L$ and $x \notin L$. Since $L$ is a copure submodule of $M$, hence $f(L)$ is a copure submodule of $M'$.

**Corollary (2.5):** Let $M$ be a $R$-module and $N$ be a submodule of an $R$-module $M$. If $N$ is quasi-copure in $M$, then $N \cap B$ is copure in $B$.

**Proof:** It follows directly by proposition (2.4), by taking the natural epimorphism $\pi: M \longrightarrow M/N$.

**Proposition (2.6):** Let $N$ be a submodule of an $R$-module $M$. Then $N$ is quasi-copure in $M$ if and only if $N = \bigcap_{\alpha} L_{\alpha}$, where $L_{\alpha}$ are copure submodules of $M$ containing $N$.

**Proof:** Assume that $N$ is a quasi-copure submodule of $M$. It clear that $N \subseteq \bigcap_{\alpha} L_{\alpha}$. We have to show that $\bigcap_{\alpha} L_{\alpha} \subseteq N$. Let $y \in \bigcap_{\alpha} L_{\alpha}$, then $y \notin L_{\alpha}$, for some $\alpha$. Suppose $y \notin N$. Since $N$ is quasi-copure, hence $y$ is not contained in any copure submodule that contains $N$, which is a contradiction, thus $y \in N$. Then $\bigcap_{\alpha} L_{\alpha} \subseteq N$ and $N = \bigcap_{\alpha} L_{\alpha}$.

Conversely, assume that $N = \bigcap_{\alpha} L_{\alpha}$, where $L_{\alpha}$ are copure submodules of $M$, for each $\alpha$, and containing $N$. Let $x \in M$ and $x \notin N$. Since $N = \bigcap_{\alpha} L_{\alpha}$, then $x \notin L_{\alpha}$, for some $\alpha$. Thus $N \subseteq L_{\alpha}$ and $x \notin L_{\alpha}$, for some $\alpha$. Hence $N$ is quasi-copure.

**Proposition (2.7):** Let $N_1 \subseteq N_2 \subseteq \ldots$ be a ascending chain of quasi-copure submodules of an $R$-module $M$. Then $\bigcup_{i=1}^{\infty} N_i$ is quasi-copure in $M$.

**Proof:** Let $x \in M$ and $x \notin \bigcup_{i=1}^{\infty} N_i$, then $x \notin N_i$, for each $i$. Since $N_i$ is quasi-copure, for each $i$, then there exists copure submodule $L_i$, such that $N_i \subseteq L_i$ and $x \notin L_i$, for each $i$. Then $\bigcup_{i=1}^{\infty} N_i \subseteq \bigcup_{i=1}^{\infty} L_i$, and $\bigcup_{i=1}^{\infty} L_i$ is copure submodule in $M$.

**Remark (2.8):** Every direct summand of an $R$-module $M$ is quasi-copure.

**Proof:** Since every direct summand of an $R$-module $M$ is copure, and every copure is quasi-copure, hence is quasi-copure.

**Proposition (2.9):** Let $M = M_1 \bigoplus M_2$ be an $R$-module and $N_1, N_2$ be submodules of $M_1, M_2$ respectively. Then $N = N_1 \oplus N_2$ is quasi-copure submodule of $M$ if and only if $N_1$ is quasi-copure submodule of $M_1$, for each $i = 1, 2$.
Submodule of $H$ hence $N$ is quasi-copure in $M$. 

For the converse, let $x \in M$ and $x \notin N$ such that $x \notin N = N_1 \oplus N_2$, $x = (x_1, x_2)$ then neither $x_1 \notin N_1$ or $x_2 \notin N_2$. Assume that $x_1 \notin N_1$, since $N_1$ is quasi-copure in $M_1$, so there exists a copure submodule $L_1$ of $M_1$ such that $L_1$ containing $N_1$ and $x_1 \notin L_1$. 

Similarly, if $x_2 \notin N_2$, then there exists a copure submodule $L_2$ of $M_2$ such that $L_2$ containing $N_2$ and $x_2 \notin L_2$. Since $L_1$ and $L_2$ are copure in $M_1$ and $M_2$ respectively. Then $L = L_1 \oplus L_2$ is a copure submodule of $M$. 

Theorem (2.10): Let $M$ be a direct sum of $R$-modules $M_1, M_2, \ldots, M_n$. If $N_i \subseteq M_i$ for each $i = 1, \ldots, n$. Then $N = \bigoplus N_i$ is a copure submodule if and only if $N_i$ is quasi-copure in $M_i$, for each $i = 1, n$.

Proposition (2.11): Let $N$ be a submodule of an $R$-module $M$. If $N$ is quasi-copure in $M$, then $N_\mathfrak{p}$ is quasi-copure in $M_\mathfrak{p}$ as $R_\mathfrak{p}$-module for every maximal ideal $\mathfrak{p}$ of $R$.

Theorem (2.12): Let $M$ be a faithful finitely generated multiplication $R$-module, and $N$ be a submodule of $M$. Then $N$ is quasi-copure in $M$ if and only if $[N : R]$ is a quasi-copure ideal of $R$.

Proposition (2.13): Let $M$ be a faithful generated multiplication $R$-module, let $N$ be a submodule of $M$. The following statements are equivalent:

1. $N$ is quasi-copure submodule in $M$.
2. $[N : R]$ is a quasi-copure ideal of $R$.
3. $N = IM$ for some quasi-copure ideal $I$ in $R$.

Proof: (1) $\iff$ (2) follows by Theorem (2.12).

(2) $\implies$ (3) It is clear.

(3) $\implies$ (2) Suppose $N = IM$ and $I$ is a quasi-copure ideal of $R$. Since $M$ is multiplication, then $N = [N : R]$ is a quasi-copure ideal of $R$. Thus $[N : R]$ is a quasi-copure ideal of $R$.

Proposition (2.14): Let $M$ be a multiplication $R$-module with $\text{ann}_R(M)$ is a pure ideal in $R$. If $N$ is a multiplication quasi-copure submodule of $M$, then $N$ is contained a pure submodule of $M$.

Proof: It is clear that $N = M[N : R]$ is a pure submodule of $N$. Since $N$ is quasi-copure in $M$, then for each $x \in M$ and $x \notin N$ there exists a copure submodule $L$ of $M$, such that $N \subseteq L$. $[L : R]$ is a copure ideal in $R$ and $N \subseteq L$, $x \notin L$. Then $N = \bigoplus N_i$ is a copure submodule of $M$, for each $i = 1, n$. Hence $N$ is quasi-copure in $M$.
and hence \( L = \{ L R M \} L \), since \( M \) is multiplication. Then by [9, Theorem 1.1 (1)⇒ (2)]. Therefor \( L \) is pure submodule in \( M \), but \( N \subseteq L \) and \( N \) quasi – co-pure in \( M \).

**Proposition (2.15):** Let \( M_1 \) and \( M_2 \) be \( R \)-module, and let \( A \) be submodule in \( M_1 \) and \( B \) be a submodule in \( M_2 \) such that \( \text{ann}_R(M_1) + \text{ann}_R(M_2) = R \). If \( A \oplus B \) is quasi – copure submodule in \( M = M_1 \oplus M_2 \), then \( A \) is quasi – copure in \( M_1 \) and \( B \) is quasi – copure submodule in \( M_2 \).

**Proof:** To show that \( A \) is quasi –copure in \( M_1 \). Let \( m \in M_1 \) and \( m \notin A \). Then \((m, 0) \notin A \oplus B \). Since \( A \oplus B \) is quasi – copure submodule in \( M \). So there exists a co-pure submodule \( D \) in \( M \) such that \( D \subseteq \text{ann}_R(M_1) \oplus \text{ann}_R(M_2) \). Let \( (m, 0) \notin D \) since \( \text{ann}_R(M_1) + \text{ann}_R(M_2) = R \). Then by apart of the proof of [10, Proposition (4.2), CH.1], any submodule of \( M = M_1 \oplus M_2 \) can be written as direct sum of two submodule of \( M_1 \) and \( M_2 \). Thus \( D = N \oplus K \) for some submodules \( N \) and \( K \) of \( M_1 \) and \( M_2 \) respectively. It follows by [5] that \( N \) is copure submodule in \( M_1 \) and \( K \) is copure submodule in \( M_2 \). Since \( A \oplus B \subseteq N \oplus K \), so \( A \subseteq N \) and \( B \subseteq K \). But \((m, 0) \notin D = N \oplus K \), then \( m \notin N \). Therefore \( A \) is quasi – copure submodule in \( M_1 \). Similarly, \( B \) is quasi – copure submodule in \( M_2 \).

**Remark (2.16):** The condition \( \text{ann}_R(M_1) + \text{ann}_R(M_2) = R \) is necessary in proposition (2.15). For example, the module \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \) as \( \mathbb{Z} \)-module. Clearly that \( \text{ann}_R(\mathbb{Z}_4) + \text{ann}_R(\mathbb{Z}_2) = 2\mathbb{Z} \neq \mathbb{Z} \). As we have seen in Remark and Examples (2.1), the submodule \(< 2, 0 > = < 2 > \oplus < 0 > \) is quasi – copure submodule in \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \). But \( 2\mathbb{Z}_4 \) is not quasi – copure submodule in \( \mathbb{Z}_4 \).

3. **Basic Results for Quasi –co-regular modules**

In this section, we introduce and study the class of quasi –co-regular modules. However, we give some basic results about this concept. Beside these we study the direct summand of quasi-co-regular modules and direct sum of quasi-co-regular modules.

Recall that an \( R \)-module \( M \) is called coregular if every submodule is co-pure and a ring \( R \) is co-regular if every ideal of \( R \) is co-pure [5]. An \( R \)-module \( M \) is quasi-regular if every submodule of \( M \) is quasi-pure [4].

**Definition (3.1):** An \( R \)-module \( M \) is called quasi – coregular if every submodule of \( M \) is quasi – copure.

**Remark and Example (3.2):**

1. Clearly that every coregular \( R \)-module is quasi – coregular. For example, the \( \mathbb{Z} \)-module \( \mathbb{Z}_6 \) is quasi-coregular. Since every semisimple module is coregular, hence \( \mathbb{Z}_6 \) is quasi – coregular. But the converse is not true in general. We have no example.

2. Every semisimple \( R \)-module is coregular, hence is quasi – coregular. But the converse is not true in general. We have no example.

3. If \( M \) is copure simple, then \( M \) is not quasi-coregular. For example each of the \( \mathbb{Z} \)-module \( \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \) are not quasi – coregular.

4. \( \mathbb{Z}_4 \) as \( \mathbb{Z} \)-module is not quasi – coregular since not coregular [4].

**Proposition (3.3):** Let \( M \) be a quasi- co-regular \( R \)-module and \( N \) be submodule of \( M \), then \( \frac{M}{N} \) is quasi – coregular \( R \)-module.

**Proof:** Let \( \frac{K}{N} \) be submodule of \( \frac{M}{N} \), where \( K \) is a submodule of \( M \). To show that \( \frac{K}{N} \) is quasi – coregular in \( \frac{M}{N} \), let \( \bar{x} \in \frac{M}{N} \) and \( \bar{x} \notin \frac{K}{N} \), then \( x + N \notin \frac{K}{N} \), hence \( x \notin K \). Since \( K \) is quasi – co-pure in \( M \), then there exists a co-pure submodule \( L \) of \( M \) such that \( K \subseteq L \) and \( x \notin L \), hence \( x \notin \frac{L}{N} \), since \( L \) is copure, then by [5] \( \frac{L}{N} \) is copure. Also \( \frac{K}{N} \subseteq \frac{L}{N} \). So that \( \frac{M}{N} \) is quasi – coregular.

**Corollary (3.4):** If \( M_1, M_2 \) are isomorphic \( R \)-modules, then \( M_1 \) is quasi – coregular if and only if \( M_2 \) is quasi – coregular.
Proof: Since $M_1$ is isomorphic to $M_2$, then there exists $f: M_1 \longrightarrow M_2$ an isomorphism. If $M_1$ is quasi-co-regular, let $K$ be submodule of $M_2$, then $K = f^{-1}(K)$, but $f^{-1}(K)$ is quasi - copure submodule in $M_1$, hence by proposition (2.4), $K$ is quasi – co-pure in $M_2$.

Similarly if $M_2$ is quasi – coregular R-module, then $M_1$ is quasi – coregular.

Corollary (3.5): Let $f: M_1 \longrightarrow M_2$ be an epimorphism. If $M_1$ is quasi – coregular R-module, then $M_2$ is quasi – coregular.

Proof: By the first fundamental theorem. $M_1 \cong M_2$, but $M_1 \cong M_2$ is quasi – coregular by Proposition (3.3). Hence $M_2$ is quasi – coregular by Corollary (3.4).

Corollary (3.6): If $M = M_1 \oplus M_2$ and $M$ is quasi – coregular R-module, then $M_1$ and $M_2$ are quasi – coregular.

Proof: Since $M_1 \cong M_2$ and $M_2 \cong M_2$ hence by Proposition (3.3), Corollary (3.4), $M_1$ and $M_2$ are quasi – coregular.

Corollary (3.7): A direct summand of quasi – coregular R-module $M$ is quasi – coregular.

Proof: Let $N$ be a directs sum and of $M$. Then $M = N \oplus K$ forsome $K$ is submodule of $M$, hence $M = N \oplus K$. But $M = N \oplus K$ is quasi – coregular by Proposition (3.3). Hence $N$ is quasi – coregular by Corollary (3.4).

Lemma (3.8): Let $M$ be afinitely generated faithful multiplication R-module. If $J$ is copure ideal in $R$, then $M$ is copure submodule in $M$.

Proof: Let $I$ be an ideal of $R$. To prove $[JM:M] = JM + [0:M]$, since $J$ is copure ideal in $R$, then $[JR] = J + [0]$, and hence $[JR:M] = JM + [0:M]$, since $M$ is multiplication, wehave $[JM:M] \subseteq [JR:M]$, Let $x \in [JM:M]$ and $x \in [JR:M]$, then $x = \sum a_i m_i$, $a_i \in [JM:M]$, $m_i \in M$. Since $M$ is finitely generated faithful R-module, then $a_i M \subseteq JM$. The reverse inclusion isclear, Therefor $[JM:M] = JM + [0:M]$ that is $JM$ is copure.

Definition (3.9): Let $R$ be a ring, $R$ is called a quasi – coregular ring if every ideal in $R$ is quasi – copure.

Remark and Example (3.10):

(1) It is clear that every coregular ring is quasi-coregular and we have no example of quasi-coregular, which isnot coregular.

(2) Every coregular ring is quasi – regular.

(3) If $R$ is Noetherian regular ring, then $R$ is quasi – coregular.

Proof: Since every Noetherian regular ring is coregular by [5], hence is quasi – coregular by remark (1).

(4) $Z_{10}$, $Z_{p^q}$; $p$, $q$ are prime numbers are quasi – coregular.

It is well – known thataan ideal $I$ of a ring $R$ is called an annihilator ideal if $I = ann_R ann_R I$, $(I, e) = [0_R: [0_R: I]], [3]$.

(5) Let $R$ be a regular ring with every ideal is an annihilator ideal. Then $R$ is quasi – coregular.

Proof: Let $I$ be an ideal of a ring $R$, since $R$ is regular ring and every ideal of $R$ is annihilator ideal, then by [5], $R$ is coregular and $I$ is a copure ideal in $R$. Hence $R$ is quasi – coregular.

Proposition (3.11): Every quasi – coregular ring $R$ is quasi – regular.

Proof: Let $I$ be an ideal of a ring $R$ and $r \in R$. Let $r \notin I$ we have to show thereexists a pure ideal $J$ of $R$ such that $r \notin J$ and $I \subseteq J$. Since $I$ is a quasi – coregular ideal of $R$, then thereexists a copure ideal $J$
of $R$ such that $r \in J$ and $I \subseteq J$. Since $J$ is a copure ideal of $R$, then $[J : R K] = J + [0 : R K]$, for each ideal $K$ of $R$, hence $[J : R J] = J + [0 : R J]$, so $R = J + [0 : R J]$. This implies that $I = b + c$ where $b \in J$ and $c \in [0 : R J]$. To show that $J$ is pure, we have to show that $J \cap K = K J$ for all ideal $K$ of $R$, let $a \in J \cap K$. So $a = ab + ac$. But $ac = 0$. Hence $a = a, b \in K J$. This show that $I$ is pure because the reverse inclusion is clear. Thus $I$ is quasi – pure and $R$ is quasi – regular.

**Proposition (3.12):** If $R$ is a quasi – regular ring and $J(R) = \{0\}$, with every ideal is an annihilator ideal, then $R$ is quasi – regular.

**Proof:** Since $R$ is quasi – regular and $J(R) = \{0\}$, then by [4], $R$ is regular. Since every ideal of $R$ is an annihilator ideal, then $R$ is coregular, hence by remark and examples (3.10) (1) $R$ is quasi – coregular.

**Remark (3.13):** If $R$ is a co-regular ring and $M$ is a finitely generated faithful multiplication $R$-module. If $J$ is an ideal in $R$, then $J M$ is copure in $M$.

**Proof:** By Lemma (3.8).

**Proposition (3.14):** Let $R$ be a quasi – co-regular ring and $M$ be a faithfully multiplication $R$-module. Then $M$ is quasi – coregular.

**Proof:** Let $N$ be a submodule of $M$. Since $M$ is multiplication, then $N = IM$ for some ideal $I$ of $R$. Since $R$ is quasi – co-regular, then by [11], $I = \cap_{\alpha \in A} I_{\alpha}$, where $I_{\alpha}$ is copure ideal of $R$ containing $I$ for each $\alpha \in A$. Then $N = (\cap_{\alpha \in A} I_{\alpha}) M$. Since $M$ is faithful multiplication, then $(\cap_{\alpha \in A} I_{\alpha}) M = \cap_{\alpha \in A} (J M)_{\alpha}$ [12]. By remark (3.13), $IM$ is copure containing $N$. Hence $M$ is quasi – coregular.

**Proposition (3.15):** Let $R$ be a ring and $M$ be a finitely generated faithful multiplication quasi – coregular $R$-module. Then $R$ is quasi – coregular.

**Proof:** Let $I$ be an ideal of $R$. We have to show that $I$ is quasi – copure. $IM$ is a submodule of $M$. Since $M$ is multiplication, then $IM = \cap_{\alpha \in A} I_{\alpha} M$, where $I_{\alpha}$ is copure in $M$ containing $IM$ for each $\alpha \in A$. Put $L_{\alpha} = I_{\alpha} M$. Thus $IM = \cap_{\alpha \in A} L_{\alpha} = \cap_{\alpha \in A} I_{\alpha} M$, since $M$ is finitely generated faithful, then by $\frac{1}{2}$ cancellation property [13]. $I = \cap_{\alpha \in A} I_{\alpha}$. Claim $I_{\alpha}$ is copure in $R$ containing $I$. Also since $I_{\alpha} M = IM \subseteq L_{\alpha}$. Thus $I \subseteq L_{\alpha}$. So that $R$ is quasi – coregular.

**Theorem (3.16):** Let $M$ be a faithful finite $I$ generated multiplication $R$-module. The following statements are equivalent:

1. $M$ is quasi – coregular $R$-module.
2. $R$ is quasi – coregular ring.

**Proof:**

1. $\implies$ 2. By Proposition (3.15).

2. $\implies$ 1. By Proposition (3.14).

**Theorem (3.17):** Let $R$ be a ring with every ideal of $R$ is an annihilator ideal, $J(R) = \{0\}$ and $M$ be a finitely generated faithful multiplication. Consider the following statements:

1. $M$ is a quasi – coregular $R$-module.
2. $R$ is a quasi – coregular ring.
3. $R$ is a quasi – regular ring and $J(R) = \{0\}$.

Then (1) $\iff$ (2) and (3) $\implies$ (2) if $J(R) = \{0\}$ and every ideal in $R$ is an annihilator ideal (i.e.) the statements are equivalent.

**Proof:**

1. $\implies$ 2. By Proposition (3.15).

2. $\iff$ 1. By Proposition (3.14).

3. $\implies$ (3) It is clear by Proposition (3.11).

3. $\implies$ (2) Suppose $R$ is quasi-regular ring. Since every ideal of $R$ is annihilator ideal and then by proposition (3.2.12) $R$ is a quasi-coregular ring.
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