CHARACTERS OF REPRESENTATIONS OF AFFINE KAC-MOODY LIE ALGEBRAS AT THE CRITICAL LEVEL

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1. Introduction and Main Results

1.1. Let $\bar{g}$ be a complex simple Lie algebra of rank $l$, $g$ non-twisted affine Kac-Moody Lie algebra associated with $\bar{g}$:

$$ g = \bar{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D. $$

(1)

The commutation relations of $g$ are given by the following.

$$ [X(m), Y(n)] = [X, Y](m+n) + m\delta_{m+n,0}(X|Y)K, $$

$$ [D, X(m)] = mX(m), $$

$$ [K, g] = 0 $$

for $X, Y \in \bar{g}$, $m, n \in \mathbb{Z}$, where $X(m) = X \otimes t^m$ with $X \in \bar{g}$ and $m \in \mathbb{Z}$ and $(\cdot | \cdot)$ is the normalized invariant inner product of $\bar{g}$. We identify $\bar{g}$ with $\bar{g} \otimes \mathbb{C} \subset g$. Fix the triangular decomposition $\bar{g} = \bar{n}^- \oplus \bar{h} \oplus \bar{n}^+$, and the Cartan subalgebra of $g$ as $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$. We have $\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\Lambda_0$ and $\delta$ are elements dual to $K$ and $D$, respectively.

Let $L(\lambda)$ be the irreducible highest weight representation of $g$ of highest weight $\lambda \in \mathfrak{h}^*$ with respect to the standard triangular decomposition $g = n^- \oplus \mathfrak{h} \oplus n^+$, where $n^- = \bar{n}^- \oplus \bar{g} \otimes \mathbb{C}[t^{-1}]t^{-1}$, $n^+ = \bar{n}^+ \oplus \bar{g} \otimes \mathbb{C}[t]t$.

The central element $K$ acts on $L(\lambda)$ as the multiplication by the constant $\langle \lambda, K \rangle$, which is called the level of $L(\lambda)$. The level $\langle \lambda, K \rangle = -h^\vee$ is called critical, where $h^\vee$ is the dual Coxeter number of $\bar{g}$.

1.2. Let $\text{ch} L(\lambda)$ be the formal character of $L(\lambda)$:

$$ \text{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} e^{\mu} \dim \mathbb{C} L(\lambda)^\mu, $$

where $L(\lambda)^\mu$ is the weight space of $L(\lambda)$ of weight $\mu$.

The Weyl-Kac character formula [Kac74] gives an explicit formula of $\text{ch} L(\lambda)$ in the case that $L(\lambda)$ is an integrable representations of $g$. It is known that Kac-Wakimoto admissible representations [KW88, KW89] also have the Weyl-Kac type character formulas. The celebrated Kazhdan-Lusztig conjecture [KL79] (proved by [BB81, BK81]) has been generalized to $g$ by Kashiwara-Tanisaki [KT93, KT96] [KT98, KT00] and Casian [Cas96]. As a result the character $\text{ch} L(\lambda)$ is known for any $L(\lambda)$ provided that its level is not critical (see [KT00] for the most general formula).

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1.3. On the contrary not much is known about the characters of $L(\lambda)$ at the critical level. It seems that only the generic case is known, that is, the case that $\lambda$ satisfies the condition that

$$\langle \lambda + \rho, \alpha \rangle \not\in \mathbb{N} \quad \text{for all } \alpha \in \Delta^+_r,$$

where $\Delta^+_r$ is the set of positive real roots of $\mathfrak{g}$, $\rho = \bar{\rho} + h^\vee \Lambda_0$, $\bar{\rho} = 1/2 \sum_{\bar{\alpha} \in \bar{\Delta}_+} \bar{\alpha}$ and $\bar{\Delta}_+ \subset \Delta^+_r$ is the set of positive roots of $\bar{\mathfrak{g}}$. In this case the Kac-Kazhdan conjecture [KK79] (which is a theorem proved by [Hay88, GW89, FF88, Ku89]) gives the following character formula of $L(\lambda)$:

$$\text{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta^+_r} (1 - e^{-\alpha}).$$

By the existence of the Wakimoto modules at the critical level [Wak86, FF90b, Pre05], it follows that the irreducible representation $L(\lambda)$ at the critical level in general has a character equal to or smaller than the right hand side of (3).

1.4. In this paper we study the irreducible highest weight representations of $\mathfrak{g}$ at the critical level which are integrable over $\bar{\mathfrak{g}}$. Denote by $\bar{\lambda}$ the restriction of $\lambda \in \mathfrak{h}^*$ to $\bar{\mathfrak{h}}$. Set

$$P^+_{\text{crit}} = \{ \lambda \in \mathfrak{h}^*; \bar{\lambda} \in \bar{P}^+, \langle \lambda, K \rangle = -h^\vee \},$$

where $\bar{P}^+$ the set of integral dominant weights of $\bar{\mathfrak{g}}$:

$$\bar{P}^+ = \{ \bar{\lambda} \in \bar{\mathfrak{h}}^*; \langle \bar{\lambda}, \alpha \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \bar{\Delta}_+ \}.$$

The $L(\lambda)$ at the critical level is integrable over $\bar{\mathfrak{g}}$ if and only if $\lambda$ belongs to $P^+_{\text{crit}}$.

We have the following result.

**Theorem 1.** Let $\lambda \in P^+_{\text{crit}}$. The character of $L(\lambda)$ is given by

$$\text{ch} L(\lambda) = \sum_{w \in \bar{W}} (-1)^{\ell(w)} e^{w \circ \lambda} \prod_{\alpha \in \bar{\Delta}_+} \left( 1 - q^{-\langle \lambda + \rho, \alpha \rangle} \right) \prod_{\alpha \in \Delta^+_r} (1 - e^{-\alpha}),$$

where $q = e^\delta$, $\bar{W}$ is the Weyl group of $\bar{\mathfrak{g}}$, $\ell(w)$ is the length of $w$ and $w \circ \lambda = w(\lambda + \rho) - \rho$.

1.5. Recently the representations of $\mathfrak{g}$ at the critical level have been studied in detail by Frenkel and Gaitsgory [FG04, FG05, FG06, FG07] in the view point of the geometric Langlands program. Our original motivation was to confirm Conjecture 5 of [Pre06] in the case that the opers are “graded” (see Theorem 10) by applying the method of the quantum Drinfeld-Sokolov reduction [FF90a, FF92, FBZ04] (cf. [FKW92, KRW03, KW04, Ara04, Ara05, Ara06, Ara07a]). Theorem 1 has been obtained as a byproduct of the proof.

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1. See also [Mat03, Pre05, Ara06].
2. After finishing this paper we were notified that Frenkel and Gaitsgory have recently proved Conjecture 5 of [Pre06] and that Theorem 1 was known to E. Frenkel.
2. **Endomorphism rings, duality and the character level**

2.1. **The category $O^{KL}_{\text{crit}}$.** Denote by $O^{KL}_{\text{crit}}$ the full subcategory of the category of $\mathfrak{g}$-modules consisting of objects $M$ such that the following hold: (1) $K$ acts on $M$ as the multiplication by $-h^\vee$, (2) $D$ acts on $M$ semisimply: $M = \bigoplus_{d \in \mathbb{C}} M_d$, where $M_d = \{ m \in M; Dm = dm \}$, (3) dim $M_d < \infty$ for all $d$, (4) there exists a finite subset $\{ d_1, \ldots, d_r \}$ of $\mathbb{C}$ such that $M_d = 0$ unless $d \in \bigcup_{i=1}^r d_i - Z_{\geq 0}$.

Any object $M$ of $O^{KL}_{\text{crit}}$ admits a weight space decomposition: $M = \bigoplus_{\lambda \in \mathbb{C}} M^\lambda$, $M^\lambda = \{ m \in M; hm = \langle \lambda, h \rangle m \ \forall h \in \mathfrak{h} \}$. We set $\text{ch} M = \sum_{\lambda \in \mathbb{C}} e^\lambda \dim M^\lambda$.

The Weyl module

\begin{equation}
V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t] \oplus CK)} E(\lambda)
\end{equation}

with $\lambda \in P^+_{\text{crit}}$ belongs to $O^{KL}_{\text{crit}}$. Here $E(\lambda)$ is the irreducible finite-dimensional representation of $\mathfrak{g}$ of highest weight $\lambda$, considered as a $\mathfrak{g} \otimes \mathbb{C}[t] \oplus CK \oplus CD$-module on which $\mathfrak{g} \otimes \mathbb{C}[t]$ acts by zero, and $K$ and $D$ act as the multiplication by $-h^\vee$ and $\langle \lambda, D \rangle$, respectively. By the Weyl character formula one has

\begin{equation}
\text{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^{f(w)} e^w \omega_\lambda}{\prod_{i \geq 1} (1-q^{-i})^\ell \prod_{i \in \Delta^+} (1-e^{-\alpha})}.
\end{equation}

The $V(\lambda)$ has $L(\lambda)$ as its unique simple quotient.

2.2. **The derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$.** Let $\mathfrak{g}'$ be the derived algebra of $\mathfrak{g}$:

\begin{equation}
\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus CK.
\end{equation}

One sees that each $L(\lambda)$ remains irreducible over $\mathfrak{g}'$.

2.3. **The vertex algebra associated with $\bar{\mathfrak{g}}$ at the critical level.** The vacuum Weyl module

\begin{equation}
\mathcal{V}_{\mathfrak{g}, \text{crit}} := V(-h^\vee \Lambda_0)
\end{equation}

has the natural structure of vertex algebras (see eg. [Kac98] [FBZ04]). The $\mathcal{V}_{\mathfrak{g}, \text{crit}}$ is called the universal affine vertex algebra associated with $\bar{\mathfrak{g}}$ at the critical level. Each object of $O^{KL}_{\text{crit}}$ can be regarded as a $\mathcal{V}_{\mathfrak{g}, \text{crit}}$-module.

For a vertex algebra $V$ in general we denote by

\begin{equation}
Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in (\text{End} V)[[z, z^{-1}]])
\end{equation}

the quantum field corresponding to $a \in V$. Also, we write $\text{Zh}(V)$ for the Zhu algebra [FZ92] [Zhu96] (see also [NT05]) of $V$. One knows by [FZ92] that there is a natural isomorphism

\begin{equation}
\text{Zh}(\mathcal{V}_{\mathfrak{g}, \text{crit}}) \cong U(\bar{\mathfrak{g}}).
\end{equation}

2.4. **Feigin-Frenkel’s theorem.** The vertex algebra $\mathcal{V}_{\mathfrak{g}, \text{crit}}$ has a large center [Hay88] [FF92], which we denote by $\mathfrak{Z}(\mathcal{V}_{\mathfrak{g}, \text{crit}})$:

$\mathfrak{Z}(\mathcal{V}_{\mathfrak{g}, \text{crit}}) = \{ a \in \mathcal{V}_{\mathfrak{g}, \text{crit}}; a_{(n)} v = 0 \text{ for all } n \in \mathbb{Z}_{\geq 0}, v \in \mathcal{V}_{\mathfrak{g}, \text{crit}} \}$.

Then

$\mathfrak{Z}(\mathcal{V}_{\mathfrak{g}, \text{crit}}) = \{ a \in \mathcal{V}_{\mathfrak{g}, \text{crit}}; [a_{(m)}, v_{(n)}] = 0 \text{ for all } m, n \in \mathbb{Z}, v \in \mathcal{V}_{\mathfrak{g}, \text{crit}} \}$.

Let $a \in \mathfrak{Z}(\mathcal{V}_{\mathfrak{g}, \text{crit}})$ and $n \in \mathbb{Z}$. The action of $a_{(n)}$ on $M \in O^{KL}_{\text{crit}}$ commutes with the action of $\mathfrak{g}'$. Therefore each $a_{(n)}$ acts as the multiplication by a constant on $L(\lambda)$. 
Let $I = \{1, 2, \ldots, l\}$ ($l = \text{rank } \mathfrak{g}$), $\{d_i; i \in I\}$ the exponents of $\mathfrak{g}$, $\mathcal{Z}(\mathfrak{g})$ the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

There is a remarkable realization of $3(V_{\mathfrak{g}, \text{crit}})$ due to Feigin and Frenkel \[FF92\] as a chiralization of Kostant’s Whittaker model \[Kos78\] of $\mathcal{Z}(\mathfrak{g})$ (for details, see \[FBZ04\]). As a result one has the following description of $3(V_{\mathfrak{g}, \text{crit}})$.

**Theorem 2** (E. Frenkel and B. Feigin \[FF92\]). There exist homogeneous vectors $p^i \in 3(V_{\mathfrak{g}, \text{crit}}) \cap (V_{\mathfrak{g}, \text{crit}})_{-d_i - 1}$ with $i \in I$ that generate a PBW basis of $3(V_{\mathfrak{g}, \text{crit}})$: that is, there is a linear isomorphism

$$\mathbb{C}[p^i_{(-n)}; i \in I, n \in \mathbb{Z}_{\geq 1}] \xrightarrow{\sim} 3(V_{\mathfrak{g}, \text{crit}}),$$

where $|0\rangle$ is the highest weight vector of $V_{\mathfrak{g}, \text{crit}}$.

2.5. **The linkage principle for $O_{\text{KL}}^{\text{crit}}$**. Let $p^i$ with $i \in I$ be generators of $3(V_{\mathfrak{g}, \text{crit}})$ as in Theorem 2. (We have $p^i = p^i_{(-1)}|0\rangle$.) Write

$$Y(p^i, z) = \sum_{n \in \mathbb{Z}} p^i_n z^{-n-d_i-1},$$

so that

$$[D, p^i_n] = np^i_n$$
on $M \in O_{\text{KL}}^{\text{crit}}$. Set

$$R_Z = \mathbb{C}[p^i_n; i \in I, n \in \mathbb{Z}].$$

An object $M$ of $O_{\text{KL}}^{\text{crit}}$ is regarded as a $R_Z$-module naturally.

There is a natural map

$$\mathbb{C}[p^i_0; i \in I] \to \text{Zh}(3(V_{\mathfrak{g}, \text{crit}})),$$

which is actually an isomorphism. To be precise we have the following assertion.

**Theorem 3.** The natural map $\text{Zh}(3(V_{\mathfrak{g}, \text{crit}})) \to \text{Zh}(V_{\mathfrak{g}, \text{crit}}) = U(\mathfrak{g})$ is injective and its image coincides with $\mathcal{Z}(\mathfrak{g})$.

See \[Ara07a\] for a proof of Theorem 3. We shall identify $\text{Zh}(3(V_{\mathfrak{g}, \text{crit}}))$ with $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})$ through Theorem 3.

Let $o(p^i)$ be the image of $p^i_0$ in $\text{Zh}(3(V_{\mathfrak{g}, \text{crit}}))$. If $v_\lambda \in M \in O_{\text{KL}}^{\text{crit}}$ is annihilated by $n_+$, then one has

$$p^i_0 v = o(p^i)v = \tilde{\chi}_\lambda(o(p^i))v,$$

where

$$\tilde{\chi}_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

is the evaluation of $\mathcal{Z}(\mathfrak{g})$ at the Verma module of $\mathfrak{g}$ of highest weight $\tilde{\lambda}$.

Because $p^i_0$ commutes with the action of $\mathfrak{g}'$ on $M \in O_{\text{KL}}^{\text{crit}}$, the following assertion follows immediately.

**Proposition 4.** If $L(\mu)$ appears in the local composition factor of $V(\lambda)$ then $\mu = \lambda - n\delta$ for some $n \in \mathbb{Z}_{\geq 0}$.
2.6. The conjecture of Frenkel and Gaitsgory for graded opers. Let $\bar{\lambda} \in \bar{P}^+$. The character $\bar{\chi}_\lambda$ naturally extends to the graded central character of $\mathfrak{g}_{\text{crit}}$, that is, to the ring homomorphism

\begin{equation}
\chi_{\bar{\lambda}} : R_{\mathfrak{g}} \to \mathbb{C}
\end{equation}

defined by

\begin{equation}
\chi_{\bar{\lambda}}(p^i_n) = \begin{cases} 
\bar{\chi}_\lambda(a(p^i_n)) & \text{if } n = 0, \\
0 & \text{if } n \neq 0.
\end{cases}
\end{equation}

The $\ker \chi_{\bar{\lambda}} \cdot V(\lambda)$ is a submodule of $V(\lambda)$. One has

\begin{equation}
\ker \chi_{\bar{\lambda}} \cdot V(\lambda) = \sum_{n>0, i \in I} U(n_-)p^i_{-n}|\lambda|,
\end{equation}

where $|\lambda|$ is the highest weight vector of $V(\lambda)$. Thus $\ker \chi_{\bar{\lambda}} \cdot V(\lambda)$ is a proper submodule of $V(\lambda)$ which lies in $O^{KL}_{\text{crit}}$. Hence there is a following exact sequence in $O^{KL}_{\text{crit}}$:

\begin{equation}
V(\lambda)/\ker \chi_{\bar{\lambda}} \cdot V(\lambda) \to L(\lambda) \to 0.
\end{equation}

The following assertion is clear.

**Proposition 5.** Any vector of $L(\lambda)$ is annihilated by $\ker \chi_{\bar{\lambda}}$.

Let $\mathcal{M}^\lambda_0$ be the full subcategory $O^{KL}_{\text{crit}}$ consisting of objects $M$ such that $\ker \chi_{\bar{\lambda}} \cdot M = 0$. Any simple object of $\mathcal{M}^\lambda_0$ is isomorphic to $L(\mu)$ with $\mu \in P^+_{\text{crit}}$ such that $\bar{\mu} = \bar{\lambda}$ (thus all the simple modules are mutually isomorphic as $\mathfrak{g}'$-modules).

The following striking assertion was conjectured by Frenkel and Gaitsgory (announced in [Fre06]).

**Conjecture 1** (E. Frenkel and D. Gaitsgory).

(i) The category $\mathcal{M}^\lambda_0$ is semisimple for any $\bar{\lambda} \in \bar{P}^+$.

(ii) For each $\lambda \in P^+_{\text{crit}}$ there is an isomorphism $V(\lambda)/\ker \chi_{\bar{\lambda}} \cdot V(\lambda) \cong L(\lambda)$.

**Remark 6.**

(i) By the “Langlands duality” [FF92], the $\chi_{\bar{\lambda}}$ can be considered as an element of the $\mathfrak{g}$-oper $\text{Op}_{\mathfrak{g}}(D^\infty)$ [BD04] on the punctured disk $D^\infty$, which is “graded”. The original conjecture (Conjecture 5 of [Fre06]) of Frenkel and Gaitsgory is more general and applies to any (not necessarily graded) central character $\chi$ (i.e. to any element of $\text{Op}_{\mathfrak{g}}^\lambda$, see Remark 5 and [Fre06]).

(ii) In the case that $\lambda = 0$, Conjecture 1 follows from Theorem 6.3 of [FG04] (applied to the graded oper $\chi$).

2.7. Endomorphism rings of Weyl modules. Let $\lambda \in P^+_{\text{crit}}$. Recall that $|\lambda|$ denotes the highest weight vector of $V(\lambda)$. Define

\begin{equation}
R^\lambda_{\mathfrak{g}} = R_{\mathfrak{g}}/\text{Ann}_{R_{\mathfrak{g}}} |\lambda|.
\end{equation}

Note that $R^\lambda_{\mathfrak{g}}$ is naturally graded by $D$:

\begin{equation}
R^\lambda_{\mathfrak{g}} = \bigoplus_{d \in -Z_{\geq 0}} (R^\lambda_{\mathfrak{g}})_d, \quad (R^\lambda_{\mathfrak{g}})_d = \{a \in R^\lambda_{\mathfrak{g}}; [D_a] = da\}.
\end{equation}

There is a natural algebra homomorphism

\begin{equation}
R^\lambda_{\mathfrak{g}} \to \text{End}_{U(\mathfrak{g})}(V(\lambda)).
\end{equation}
If $\bar{\lambda} = 0$, then $R^0_{\bar{\lambda}} \cong \mathcal{Z}(\mathcal{V}_{\text{crit}})$. In this case it is known by [FF92] that (22) gives an isomorphism

$$\mathcal{Z}(\mathcal{V}_{\text{crit}}) \cong \text{End}_{U(\mathfrak{g}')} (\mathcal{V}_{\text{crit}}).$$

This is true for any $\lambda \in P^+_\text{crit}$.

**Theorem 7.** Let $\lambda \in P^+_{\text{crit}}$.

(i) The map (22) gives the isomorphism $R^\lambda_{\bar{\lambda}} \cong \text{End}_{U(\mathfrak{g}')} (V(\lambda))$.

(ii) Set $\text{ch} R^\lambda_{\bar{\lambda}} = \bigoplus_{d \in \mathbb{C}} q^d \dim (R^\lambda_{\bar{\lambda}})_d$. Then

$$\text{ch} R^\lambda_{\bar{\lambda}} = \prod_{\alpha \in \Delta^+} (1 - q^{-(\lambda + p, \alpha')}).$$

Theorem 7 was obtained earlier in [FG07]. In [Ara07b] we give an independent proof of Theorem 7 by the method of quantum Drinfeld-Sokolov reduction.

**Remark 8.** According to Frenkel and Gaitsgory [FG07], one has

$$\text{Spec } R^\lambda_{\bar{\lambda}} \cong \text{Op}^\lambda_G,$$

where $\text{Op}^\lambda_G$ is a certain sub-pro-variety of $\text{Op}_G(D^\infty)$ described in [Fre04] (cf. (7.17) of [Fre94]).

### 2.8. An equivalence of categories.

Let $\mathcal{M}^{\bar{\lambda}}$ be the full subcategory of $\mathcal{O}^{K_{\text{crit}}}_{\text{crit}}$ consisting of objects $M$ that are annihilated by

$$p_i^j - \chi_{\bar{\lambda}}(p_n^i) \text{ for all } i \in I, n \geq 0.$$

Then $V(\lambda), L(\lambda) \in \mathcal{M}^{\bar{\lambda}}$. Also, $\mathcal{M}_0^{\bar{\lambda}}$ is a full subcategory of $\mathcal{M}^{\bar{\lambda}}$.

Let $R^\lambda_{\bar{\lambda}}$-gmod be the full subcategory of the category of graded $R^\lambda_{\bar{\lambda}}$-modules consisting of objects $X = \bigoplus_{d \in \mathbb{C}} X_d$ such that (1) $\dim X_d < \infty$ for all $d \in \mathbb{C}$; (2) there exists a finite subset $d_1, \ldots, d_r \subset \mathbb{C}$ such that $X_{d_i} = 0$ unless $d \in \bigcup d_i - \mathbb{Z}_{>0}$. Then any simple object of $R^\lambda_{\bar{\lambda}}$-gmod is isomorphic to

$$R^\lambda_{\bar{\lambda}} / \ker \chi_{\bar{\lambda}} : R^\lambda_{\bar{\lambda}}$$

as $R^\lambda_{\bar{\lambda}}$-modules.

Set

$$F(M) = \text{Hom}_{U(\mathfrak{g}')} (V(\lambda), M)$$

for $M \in \mathcal{M}^{\bar{\lambda}}$. Then $F(M) \cong M^{n_+}$. The $F(M)$ is naturally a graded $R^\lambda_{\bar{\lambda}}$-module:

$$F(M) = \bigoplus_{d \in \mathbb{Z}} F(M)_d$$

where $F(M)_d = M^{n_+} \cap M_d$. Thus $F$ defines a functor from $\mathcal{M}^{\bar{\lambda}}$ to $R^\lambda_{\bar{\lambda}}$-gmod.

Next let

$$G(X) = V(\lambda) \otimes_{R^\lambda_{\bar{\lambda}}} X$$

for $X \in R^\lambda_{\bar{\lambda}}$-gmod (See 1) of Theorem 7). Then $G(X)$ is an object of $\mathcal{M}^{\bar{\lambda}}$. Here the action of $D$ on $G(X)$ is defined in an obvious way.

**Theorem 9.** Let $\lambda \in P^+_{\text{crit}}$.

(i) The $V(\lambda)$ is a free $R^\lambda_{\bar{\lambda}}$-module.
(ii) (cf. Theorem 6.3 of [FG04]) The functor $F$ gives an equivalence of categories $M^\lambda \cong R^\lambda_{Z}\mathsf{grmod}$. The inverse functor is given by $G$.

The proof of Theorem (ii) is given in [Ara07b].

The following assertion follows immediately from Theorem 9.

Theorem 10. Conjecture 7 holds.

Because $L(\lambda) = G(C_{\chi}^\lambda)$, by (ii) of Theorem 9, one has

$$\text{ch} V(\lambda) = \text{ch} R^\lambda_Z \cdot \text{ch} L(\lambda).$$

Therefore (i) and (ii) of Theorem 10 give Theorem 1.

Remark 11. Using Theorem 9, one can show the irreducibility of the $g'$-module $V(\lambda)/\ker \chi \cdot V(\lambda)$ for any $\chi \in \text{Op}_{\mu}^\lambda L^{G}$. This confirms the original conjecture of Frenkel and Gaitsgory (Conjecture 5 of [Pre06], see Remarks 5 and 8) partially.

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