WEAK CONVERGENCE OF ORTHOGONAL POLYNOMIALS

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Abstract. The weak convergence of orthogonal polynomials is given under conditions on the asymptotic behaviour of the coefficients in the three-term recurrence relation. The results generalize known results and are applied to several systems of orthogonal polynomials, including orthogonal polynomials on a finite set of points.

1. Introduction

Let \( p_n(x) \) be a system of orthonormal polynomials on the real line, with orthogonality measure \( \mu \), i.e., \( \mu \) is a probability measure for which all the moments exist and

\[
\int p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n}, \quad m, n \geq 0.
\]

When the support of \( \mu \) consists of a finite number of points \( x_1, x_2, \ldots, x_N \), then we only consider the polynomials up to degree \( N \) and \( p_N(x) \) has its zeros at the support \( \{x_1, x_2, \ldots, x_N\} \). It is well known that orthonormal polynomials satisfy a three-term recurrence relation

\[
xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0,
\]

with initial values \( p_0(x) = 1 \) and \( p_{-1}(x) = 0 \). Here

\[
a_{n+1} = \int xp_n(x)p_{n+1}(x)\,d\mu(x), \quad b_n = \int xp_n^2(x)\,d\mu(x) \in \mathbb{R}, \quad n \geq 0.
\]

Usually the orthonormal polynomials are chosen in such a way that the leading coefficient \( \gamma_n = (a_1a_2\cdots a_n)^{1/n} \) is positive, and then \( a_n = \gamma_n/\gamma_{n-1} > 0 \) for every \( n \geq 1 \).

We are interested in the weak asymptotic behaviour of the orthonormal polynomials \( p_n(x) \). This means that we want to investigate the behaviour for \( n \to \infty \) of integrals of the form

\[
\int f(x)p_n^2(x)\,d\mu(x), \quad f \in C_b,
\]

where \( C_b \) is the linear space of bounded and continuous functions on \( \mathbb{R} \).

We will consider a one-parameter orthogonality measure \( \mu_k \) \( (k \in \mathbb{N}) \), the parameter being discrete. This implies that the recurrence coefficients and the orthogonal polynomials all depend on this discrete parameter and we write

\[
xp_n(x; \mu_k) = a_{n+1,k}p_{n+1}(x; \mu_k) + b_{n,k}p_n(x; \mu_k) + a_{n,k}p_{n-1}(x; \mu_k), \quad n \geq 0.
\]

Our main result will be the limit as \( n \to \infty \) of integrals of the form
The special case \(k = l = 0\) then gives the desired weak convergence, but the general case with \(k, l \in \mathbb{Z}\) also has useful applications: these integrals are related to the transition probabilities of birth and death processes and random walks.

Such one-parameter families do occur frequently in various applications and limiting procedures. For example the rescaling of orthogonal polynomials

\[ p_n(c_k x; \mu) = p_n(x; \mu_k) \]

gives the one-parameter family of measures \(\mu_k\) with distribution functions satisfying

\[ \mu_k(t) = \mu(c_k t), \quad t \in \mathbb{R}. \]

Other examples include orthogonal polynomials in which some of the parameters are allowed to tend to infinity together with the degree, e.g., if \(P_n^{(\alpha, \beta)}(x)\) is the Jacobi polynomial of degree \(n\), with weight function

\[ w(x) = (1 - x)^\alpha (1 + x)^\beta, \quad -1 < x < 1, \]

then \(P_n^{(k\alpha + \gamma, k\beta + \delta)}(x)\) is an orthogonal polynomial of degree \(n\) with weight function

\[ w_k(x) = w^k(x)(1 - x)^\gamma (1 + x)^\delta, \quad -1 < x < 1. \]

The main result will be in terms of a doubly infinite Jacobi matrix

\[
\mathcal{J} = \begin{pmatrix}
\ddots & \ddots & & \\
& \ddots & \ddots & \\
&& \ddots & b_{-2}^0 a_{-1}^0 \\
&&& b_{-1}^0 a_{-1}^0 b_{-1}^0 a_{0}^0 \\
&&&& b_{0}^0 a_{0}^0 b_{0}^0 a_{1}^0 \\
&&&&& b_{1}^0 a_{1}^0 b_{1}^0 a_{2}^0 \\
&&&&&& b_{2}^0 a_{2}^0 b_{2}^0 a_{3}^0 \\
&&&&&&& \ddots \ddots \ddots \\
\end{pmatrix},
\]

and is the following

**Theorem.** Suppose that the recurrence coefficients in (1.2) satisfy

\[
\lim_{n \to \infty} a_{n+k,n} = a_k^0 > 0, \quad \lim_{n \to \infty} b_{n+k,n} = b_k^0 \in \mathbb{R}
\]

for every \(k \in \mathbb{Z}\). Then

\[
\lim_{n \to \infty} \int f(x) p_{n+k}(x; \mu_n) p_{n+1}(x; \mu_n) \, d\mu_n(x)
= \int f(x) \begin{pmatrix} A_k(x) \\ B_k(x) \end{pmatrix} d\mu(x) \begin{pmatrix} A_l(x) \\ B_l(x) \end{pmatrix},
\]

for every polynomial \(f\). Here \(\mu\) is the spectral matrix of measures for the doubly infinite Jacobi matrix \(\mathcal{J}\) with entries \(a_n^0, b_n^0 (n \in \mathbb{Z})\) and

\[
\begin{pmatrix} A_n(x) \\ B_n(x) \end{pmatrix} = \begin{cases}
\begin{pmatrix} a_0^0 & p_n(1) \\ a_1^0 & p_n(x) \\ \vdots & \vdots \\
q_{n-1}(x) & q_{n-1}(x) \\ \vdots & \vdots \\
-\frac{a_{n-1}^0}{a_{n-2}^0} & q_{n-2}(1) \\
\end{pmatrix} & \text{for } n \geq 0, \\
\begin{pmatrix} \frac{a_{-n}^0}{a_{-n-1}^0} & p_{-n}(1) \\
p_{-n}(x) & q_{-n}(x) \\
\vdots & \vdots \\
q_{-n-1}(x) & q_{-n-1}(x) \\
-\frac{a_{-n-2}^0}{a_{-n-3}^0} & q_{-n-3}(1) \\
\end{pmatrix} & \text{for } n < 0,
\end{cases}
\]

with \(p_n(x)\) and \(q_n(x)\) the orthonormal polynomials with recurrence coefficients respectively \(a_n^0, b_n^0 (n \in \mathbb{Z})\).
2. Spectral theory for Jacobi operators

If we put the coefficients $a_n > 0$ ($n = 1, 2, \ldots$) and $b_n \in \mathbb{R}$ ($n = 0, 1, 2, \ldots$) of the recurrence relation (1.1) in a tridiagonal matrix

\begin{equation}
J = \begin{pmatrix}
    b_0 & a_1 & 0 & 0 & \cdots \\
    a_1 & b_1 & a_2 & 0 & \cdots \\
    0 & a_2 & b_2 & a_3 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{equation}

then $J$ is the Jacobi matrix associated with these orthogonal polynomials. If $\mu$ is supported on $N$ points, then $a_n = 0 = b_n$ for $n \geq N$ and $J$ is a $N \times N$ matrix with eigenvalues at the support $\{x_1, x_2, \ldots, x_N\}$.

Note that

\begin{equation}
J = \left( \int x p_i(x)p_j(x) \, d\mu(x) \right)_{i,j=0,1,\ldots},
\end{equation}

and by induction we find

\begin{equation}
J^k = \left( \int x^k p_i(x)p_j(x) \, d\mu(x) \right)_{i,j=0,1,\ldots}.
\end{equation}

In fact, $J : \ell_2(\mathbb{N}, \mathbb{C}) \to \ell_2(\mathbb{N}, \mathbb{C})$ acts as a linear operator on the Hilbert space $\ell_2(\mathbb{N}, \mathbb{C}) = \{\psi : \psi_i \in \mathbb{C} \text{ and } \sum_{i=0}^{\infty} |\psi_i|^2 < \infty\}$. The operator is symmetric on the initial domain consisting of finite linear combinations of the basic vectors $\{e_n^+ = (0, 0, 0, \ldots, 0, 1, 0, \ldots) : n \geq 0\}$ and by imposing appropriate conditions on the recurrence coefficients $a_{n+1}, b_n$ (e.g., boundedness) this operator can be extended in a unique way to a self-adjoint operator on the maximal domain $\{\psi \in \ell_2(\mathbb{N}, \mathbb{C}) : J\psi \in \ell_2(\mathbb{N}, \mathbb{C})\}$. This operator has a cyclic vector, i.e., if we take $e_0^+ = (1, 0, 0, 0, \ldots)$, then the linear span of $\{J^k e_0^+ : k = 0, 1, \ldots\}$ is dense in $\ell_2(\mathbb{N}, \mathbb{C})$. The spectral theorem (see, e.g., Akhiezer and Glazman [2] or Stone [19]) then implies the existence of a measure $\mu$ and a linear mapping $\Lambda : \ell_2(\mathbb{N}, \mathbb{C}) \to L_2(\mu)$ with $\Lambda e_0^+ = 1$ and $\Lambda J\psi = M\Lambda\psi$ for every $\psi \in \ell_2(\mathbb{N}, \mathbb{C})$, where $M$ is the multiplication operator

\[(Mf)(t) = tf(t).\]

The mapping $\Lambda$ is unitary, meaning $\langle \Lambda\psi, \Lambda\phi \rangle = \langle \psi, \phi \rangle$. The mapping $\Lambda$ thus maps $e_0^+$ to the constant function $t \mapsto 1$, $Je_0^+$ to the identity $t \mapsto t$, and in general $\Lambda$ maps $J^n e_0^+$ to the monomial $t \mapsto t^n$. Hence the fact that $e_0^+$ is a cyclic vector is equivalent with the density of polynomials in $L_2(\mu)$. By induction and by using the recurrence relation (1.1) we see that $\Lambda$ maps the basic vector $e_n^+$ to the polynomial $p_n$. The unitarity thus implies that

\[\int p_n(t)p_m(t) \, d\mu(t) = \langle e_n^+, e_m^+ \rangle = \delta_{m,n},\]

which shows that the spectral measure $\mu$ for the operator $J$ is the orthogonality measure for the orthogonal polynomials $p_n(x)$ ($n = 0, 1, 2, \ldots$). These elements from spectral theory, applied to the semi-infinite Jacobi matrix $J$, are well-known (see e.g., Akhiezer [1], Dombrowski [6], Sarason [18], Stone [19]).

We will also need to use doubly infinite Jacobi matrices of the form

\[
\begin{pmatrix}
    \cdots & \cdots & \cdots \\
    \cdots & \cdots & \cdots \\
    \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
with \(a_k > 0, b_k \in \mathbb{R}\) for \(k \in \mathbb{Z}\). Again these are operators, now acting on the Hilbert space \(\ell_2(\mathbb{Z}, \mathbb{C}) = \{\psi \in \mathbb{C}^\mathbb{Z} : \sum_{i=-\infty}^{\infty} |\psi_i|^2 < \infty\}\). The spectral theory of such matrices is less known, but has also been developed (see, e.g., Berezanski˘ı [4], Nikishin [17], Masson and Repka [13]). We will briefly recall some of the elements, which we have taken from Nikishin [17] (see also Berezanski˘ı [4, Chapter VII, §3]). We will first consider bounded matrices \(\mathcal{J}\), i.e., we assume that \(a_n\) and \(b_n\) \((n \in \mathbb{Z})\) are bounded sequences. It turns out that it is convenient to introduce the \(2 \times 2\) matrices

\[
B_0 = \begin{pmatrix} b_{-1} & a_0 \\ a_0 & b_0 \end{pmatrix}, \quad B_n = \begin{pmatrix} b_{n-1} & 0 \\ 0 & b_n \end{pmatrix}, \quad n = 1, 2, \ldots
\]

\[
A_n = \begin{pmatrix} a_{-n} & 0 \\ 0 & a_n \end{pmatrix}, \quad n = 1, 2, \ldots.
\]

We can now study the semi-infinite Jacobi block matrix

\[
\mathcal{J} = \begin{pmatrix} B_0 & A_1 & A_2 & A_3 & \cdots \\ A_1 & B_1 & A_2 & A_3 & \cdots \\ A_2 & B_2 & A_3 & \cdots & \cdots \\ A_3 & B_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

(2.5)

which contains \(2 \times 2\) matrices. As an operator it acts on \(\ell_2(\mathbb{N}, \mathbb{C}^2)\) in the sense that \(\mathcal{J}\psi\) for \(\psi \in \ell_2(\mathbb{Z}, \mathbb{C})\) corresponds to \(\mathcal{J}\Psi\) with \(\Psi \in \ell_2(\mathbb{N}, \mathbb{C}^2)\) given by

\[
\Psi_n = \begin{pmatrix} \psi_{-n-1} \\ \psi_n \end{pmatrix}, \quad n = 0, 1, 2, \ldots.
\]

In this way we have transformed the study of the doubly-infinite Jacobi matrix to the study of a semi-infinite Jacobi block matrix. Such semi-infinite block matrices are closely connected to orthogonal matrix polynomials, in the same way as ordinary Jacobi matrices are connected to scalar orthogonal polynomials (see, e.g., Aptekarev and Nikishin [3]).

Consider the standard basis \(e_n\) \((n \in \mathbb{Z})\) in \(\ell_2(\mathbb{Z}, \mathbb{C})\), i.e.,

\[
(e_n)_i = \delta_{i,n}, \quad i, n \in \mathbb{Z},
\]

then the linear span of \(\{\mathcal{J}^k e_1, \mathcal{J}^l e_0, k, l \in \mathbb{N}\}\) is dense in \(\ell_2(\mathbb{Z}, \mathbb{C})\). By the spectral theorem there exists a matrix-measure

\[
\mu = \begin{pmatrix} \mu_{1,1} & \mu_{1,2} \\ \mu_{1,2} & \mu_{2,2} \end{pmatrix}
\]

and a unitary linear mapping \(\Lambda : \ell_2(\mathbb{Z}, \mathbb{C}) \rightarrow L_2(\mu)\) with

\[
\Lambda e_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Lambda e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

such that

\[
\Lambda \mathcal{J} \psi = M \Lambda \psi,
\]

where now \(M\) is the multiplication operator in the space \(L_2(\mu)\) of vector valued functions. The inner product in the space \(L_2(\mu)\) is given by
whereas $J^n e_0$ is mapped to
\[ t \mapsto \begin{pmatrix} 0 \\ t^n \end{pmatrix}. \]

Let $J^+$ be the semi-infinite Jacobi matrix
\[
J^+ = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & a_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix},
\]
and similarly $J^-$ be the semi-infinite Jacobi matrix
\[
J^- = \begin{pmatrix} b_{-1} & a_{-1} & & \\ a_{-1} & b_{-2} & a_{-2} & \\ & a_{-2} & b_{-3} & a_{-3} \\ & & \ddots & \ddots & \ddots \end{pmatrix}.
\]

We denote by $p_n(x)$ the orthonormal polynomials corresponding to the Jacobi-matrix $J^+$ satisfying the recurrence relation
\[
(2.6) \quad xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),
\]
with initial values $p_{-1}(x) = 0, p_0(x) = 1$, and by $q_n(x)$ the orthonormal polynomials for $J^-$ satisfying
\[
(2.7) \quad xq_n(x) = a_{-n-1} q_{n+1}(x) + b_{-n-1} q_n(x) + a_{-n} q_{n-1}(x),
\]
with initial values $q_{-1}(x) = 0, q_0(x) = 1$. We will show, by induction, that for $n \in \mathbb{N}$ the mapping $\Lambda$ maps the basis vector $e_n$ to $\Lambda e_n$ which is the vector function
\[
(2.8) \quad t \mapsto \begin{pmatrix} -\frac{a_0}{a_1} p_n^{(1)}(t) \\ p_n(t) \end{pmatrix}
\]
and the basisvector $e_{-n}$ to $\Lambda e_{-n}$ given by
\[
(2.9) \quad t \mapsto \begin{pmatrix} -\frac{a_0}{a_{-1}} q_{n-1}^{(1)}(t) \\ -q_n(t) \end{pmatrix}.
\]

Here $p_n^{(1)}(x)$ and $q_n^{(1)}(x)$ are the associated polynomials, i.e., the orthonormal polynomials corresponding to the Jacobi matrices $J^+$ and $J^-$, with the first row and column deleted. This is clear for $e_{-1}$ and $e_0$. Assume that this is true for $0 \leq n \leq k$, then from
\[
J e_k = a_{k+1} e_{k+1} + b_k e_k + a_k e_{k-1}
\]
it follows that
\[
\Lambda J e_k = a_{k+1} \Lambda e_{k+1} + b_k \Lambda e_k + a_k \Lambda e_{k-1},
\]
and since $\Lambda J e_k = M \Lambda e_k$ this gives
and $\Lambda \mathcal{J} e_{-k} = M \Lambda e_{-k}$, this gives

$$(\Lambda e_{-k-1})(t) = \frac{1}{a_{-k}} \left( \begin{array}{cc} (t - b_{-k})q_{k-1}(t) - a_{-k+1}q_{k-2}(t) \\ -\frac{a_{0}}{a_{-1}}(t - b_{-k})q_{k-2}(t) - a_{-k+1}q_{k-3}(t) \end{array} \right) = \left( \begin{array}{c} q_{k}(t) \\ -\frac{a_{0}}{a_{-1}}q_{k-1}(t) \end{array} \right).$$

From the unitarity we find for $m, n \in \mathbb{Z}$

$$\langle \Lambda e_{n}, \Lambda e_{m} \rangle = \delta_{m,n},$$

Hence the matrix polynomials

$$P_n(t) = \left( \begin{array}{cc} q_n(t) \\ -\frac{a_{0}}{a_{-1}}q_{n-1}(t) \end{array} \right)$$

satisfy

$$\int P_n(t)d\mu(t)P_m(t)^* = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \delta_{m,n}.$$

Therefore these matrix polynomials are orthonormal with respect to the matrix-measure $\mu$. Note that these matrix polynomials satisfy

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_nP_{n-1}(t),$$

$$P_0(t) = \left( \begin{array}{cc} 1 \\ 0 \end{array} \right),$$

so that they are the orthonormal polynomials corresponding to the block Jacobi matrix $\mathcal{J}$ given in (2.5).

The polynomials $p_n(x) \ (n = 0, 1, 2, \ldots)$ are orthonormal with respect to some probability measure $\mu^+$. and the polynomials $q_n(x) \ (n = 0, 1, 2, \ldots)$ are orthonormal with respect to some probability measure $\mu^-$. In order to find a relation between the matrix-measure $\mu$ and the measures $\mu^+$ and $\mu^-$, we observe that the unitarity of $\Lambda$ implies

$$\langle (z - \mathcal{J})^{-1}e_n, e_m \rangle = \int \frac{1}{z - t}(\Lambda e_n)^*d\mu(t)\Lambda e_m,$$

where $(z - \mathcal{J})^{-1}$ is the resolvent of $\mathcal{J}$, which is well-defined for every $z$ outside the spectrum of $\mathcal{J}$. Since $\mathcal{J}$ is symmetric and bounded, it follows that $\mathcal{J}$ is self-adjoint so that its spectrum is a subset of the real line. The Stieltjes transform of the matrix-measure $\mu$ is determined by

$$\langle (z - \mathcal{J})^{-1}e_{-1}, e_{-1} \rangle = \int \frac{1}{z - t}d\mu_{1,1}(t), \quad \langle (z - \mathcal{J})^{-1}e_{-1}, e_0 \rangle = \int \frac{1}{z - t}d\mu_{1,2}(t),$$

$$\langle (z - \mathcal{J})^{-1}e_0, e_0 \rangle = \int \frac{1}{z - t}d\mu_{2,2}(t).$$

On the other hand, if we write

$$(z - \mathcal{J})^{-1}e_0 = r = (\ldots, r_{-2}, r_{-1}, r_0, r_1, r_2, \ldots),$$

then $(z - \mathcal{J})r = e_0$, which gives the infinite system of equations
combination of the orthogonal polynomials \( p_n(z) \) (respectively \( q_n(z) \)) and the functions of the second kind \( \tilde{p}_n(z) \) (respectively \( \tilde{q}_n(z) \)) given by

\[
\tilde{p}_n(z) = \int \frac{p_n(x)}{z - x} \, d\mu^+(x), \quad \tilde{q}_n(z) = \int \frac{q_n(x)}{z - x} \, d\mu^-(x),
\]

with \( a_0\tilde{p}_{-1}(z) = 1 = a_0\tilde{q}_{-1}(z) \). These functions of the second kind have the property that they are a minimal solution of the recurrence relation, and for \( z \in \mathbb{C} \setminus \mathbb{R} \) they satisfy \( \lim_{n \to \infty} \tilde{p}_n(z) = \lim_{n \to \infty} \tilde{q}_n(z) = 0 \). The fact that \( r \in \ell_2(\mathbb{Z}, \mathbb{C}) \) thus implies that \( r_n \) and \( r_{-n} \) are (up to a constant factor) given by the functions of the second kind \( \tilde{p}_n(z) \) and \( \tilde{q}_{n-1}(z) \) respectively. The constant multiple is determined by setting \( n = 0 \), giving

\[
r_n = r_0 \frac{\tilde{p}_n(z)}{\tilde{p}_0(z)}, \quad r_{-n} = a_0 r_0 \tilde{q}_{n-1}(z), \quad n \geq 0.
\]

In particular \( r_1 = r_0 \tilde{p}_1(z)/\tilde{p}_0(z) \) and \( r_{-1} = a_0 r_0 \tilde{q}_0(z) \). Inserting in (2.10) gives

\[
r_0 = \frac{1}{z - a_0^2 \tilde{q}_0(z) - b_0 - a_1 \tilde{p}_1(z)/\tilde{p}_0(z)},
\]

which by using \( a_1 \tilde{p}_1(z) = (z - b_0)\tilde{p}_0(z) - 1 \) gives

\[
r_0 = \frac{\tilde{p}_0(z)}{1 - a_0^2 \tilde{p}_0(z)\tilde{q}_0(z)}.
\]

In a similar way we may investigate

\[
(z - \mathcal{J})^{-1} e_{-1} = s = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots),
\]

which gives the infinite system of linear equations

\[
zs_k = a_k s_{k-1} + b_k s_k + a_{k+1} s_{k+1}, \quad k \geq 0,
zs_{-k} = a_{-k} s_{-k-1} + b_{-k} s_{-k} + a_{-k+1} s_{-k+1}, \quad k \geq 2,
zs_{-1} = a_{-1} s_{-2} + b_{-1} s_{-1} + a_0 s_0.
\]

Now we find

\[
s_n = a_0 s_{-1} \tilde{p}_n(z), \quad s_{-n} = s_{-1} \frac{\tilde{q}_{n-1}(z)}{\tilde{q}_0(z)},
\]

and inserting this in (2.11) gives

\[
s_{-1} = \frac{1}{z - a_{-1} \tilde{q}_1(z)/\tilde{q}_0(z) - b_{-1} - a_0^2 \tilde{p}_0(z)},
\]

which by using \( a_{-1} \tilde{q}_1(z) = (z - b_{-1})\tilde{q}_0(z) - 1 \) becomes

\[
s_{-1} = \frac{\tilde{q}_0(z)}{1 - a_0^2 \tilde{p}_0(z)\tilde{q}_0(z)}.
\]

The Stieltjes transform of the matrix of measures \( \mu \) is thus given by

\[
\int \frac{1}{z - t} \, d\mu_{1,1}(t) = \frac{\tilde{q}_0(z)}{1 - a_0^2 \tilde{p}_0(z)\tilde{q}_0(z)}, \quad \int \frac{1}{z - t} \, d\mu_{2,2}(t) = \frac{\tilde{p}_0(z)}{1 - a_0^2 \tilde{p}_0(z)\tilde{q}_0(z)}.
\]
3. PROOF OF THE THEOREM

The first important observation is that for \( f(x) = x^m \) one has

\[
\int x^m p_{n+k}(x; \mu_n) p_{n+l}(x; \mu_n) \, d\mu_n(x) = \langle J_n^m e_{n+k}, e_{n+l}^+ \rangle,
\]

where \( J_n \) is the semi-infinite Jacobi matrix with the recurrence coefficients \( a_{k,n}, b_{k,n}, \) \( (k = 0, 1, 2, \ldots) \). Consider the semi-infinite operator \( J_n \) as a doubly infinite Jacobi matrix by taking \( (J_n)_{i,j} = 0 \) whenever \( i < 0 \) or \( j < 0 \). If \( S \) is the shift operator on \( \ell_2(\mathbb{Z}, \mathbb{C}) \) acting as \( Se_k = e_{k+1} \), then

\[
\int x^m p_{n+k}(x; \mu_n) p_{n+l}(x; \mu_n) \, d\mu_n(x) = \langle (S^*)^n J_n^m S^n e_k, e_l \rangle.
\]

One easily verifies that this expression is given by

\[
\langle (S^*)^n J_n^m S^n e_k, e_l \rangle = \sum_{i_1, i_2, \ldots, i_{m-1} \in \{-1, 0, 1\}} (J_n)_{n+k,n+k+i_1} (J_n)_{n+k+i_1,n+k+i_1+i_2} \cdots (J_n)_{n+k+i_1+i_2, n+k+i_2+i_{m-1}, n+l}.
\]

This is a finite sum, containing the matrix entries \( (J_n)_{n+r,n+s} \) where \( r \) and \( s \) remain bounded. The hypothesis \((1.3)\) implies that \((S^*)^n J_n S^n\) converges entrywise to \( J \), where \( J \) is the doubly infinite Jacobi matrix containing the coefficients \( a_k^0, b_k^0, \) \( (n \in \mathbb{Z}) \). The hypothesis \((1.3)\) thus implies that

\[
\lim_{n \to \infty} \int x^m p_{n+k}(x; \mu_n) p_{n+l}(x; \mu_n) \, d\mu_n(x) = \sum_{i_1, i_2, \ldots, i_{m-1} \in \{-1, 0, 1\}} J_{k,k+i_1} J_{k+i_1,k+i_1+i_2} \cdots J_{k+i_1+i_2, \ldots, i_{m-1}, l},
\]

and the latter expression is equal to

\[
\langle J^m e_k, e_l \rangle.
\]

Therefore we find that \((S^*)^n J_n^m S^n\) converges entrywise to \( J^m \). By the unitarity of the mapping \( \Lambda : \ell_2(\mathbb{Z}, \mathbb{C}) \to L_2(\mu) \), transforming the action of \( J \) on \( \ell_2(\mathbb{Z}, \mathbb{C}) \) to the action of the multiplication operator \( M \) on \( L_2(\mu) \), we have

\[
\langle J^m e_k, e_l \rangle = \int t^m(\Lambda e_k)^* d\mu(t) \Lambda e_l,
\]

and thus the theorem for \( f(x) = x^m \) (and hence for every polynomial \( f \)) follows from \((2.8)\) and \((2.9)\).

In order to prove the theorem for every \( f \in C_b \), we consider the linear operator \( H_n = (S^*)^n J_n S^n \). We would like to prove that \( f(H_n) = (S^*)^n f(J_n) S^n \) converges weakly to \( f(J) \), since then

\[
\lim_{n \to \infty} \langle (S^*)^n f(J_n) S^n e_k, e_l \rangle = \langle f(J) e_k, e_l \rangle,
\]

and this is precisely the weak convergence stated in our theorem. Observe that \( H_n e_k = a_{k+n,n} e_{k-1} + b_{k+n,n} e_k + a_{k+n+1,n} e_{k+1} \) whenever \( k + n > 0 \), hence the condition \((1.3)\) implies the convergence of \( H_n e_k \) to \( J e_k = a_k^0 e_{k-1} + b_k^0 e_k + a_{k+1}^0 e_{k+1} \) for every \( k \in \mathbb{Z} \). All finite linear combinations of the basis elements \( e_k \) converge in \( L_2(\mu) \) to \( J e_k \), hence the theorem follows.
4. Examples

The class $M(a, b)$. The class $M(a, b)$ consists of all orthogonal polynomials $p_n(x; \mu)$ (or all probability measures $\mu$) with recurrence coefficients that satisfy

$$
\lim_{n \to \infty} a_n = a/2, \quad \lim_{n \to \infty} b_n = b.
$$

We can apply the theorem with the family of measures $\mu_k \equiv \mu$, i.e., with all the orthogonality measures the same. If $a > 0$ then we can, without loss of generality, only consider $M(1, 0)$. The doubly infinite Jacobi matrix $J$ then consists of 0 on the diagonal and $1/2$ on the subdiagonals. The semi-infinite Jacobi matrices $J^+$ and $J^-$ are the same and the corresponding orthogonal polynomials are the Chebyshev polynomials of the second kind, which are orthogonal with respect to the measure $(2/\pi)\sqrt{1 - x^2} \, dx$ on the interval $[-1, 1]$. The Stieltjes transform of this measure is

$$
\tilde{p}_0(z) = \tilde{q}_0(z) = 2[z - \sqrt{z^2 - 1}].
$$

Therefore the Stieltjes transform of the spectral matrix of measures for the Jacobi matrix $J$ is given by

$$
\int \frac{1}{z - x} \, d\mu_{1,1}(x) = \int \frac{1}{\sqrt{z^2 - 1}} = \int \frac{1}{z - x} \, d\mu_{2,2}(x),
$$

$$
\int \frac{1}{z - x} \, d\mu_{1,2}(x) = \frac{z - \sqrt{z^2 - 1}}{\sqrt{z^2 - 1}},
$$

from which one easily finds that the spectrum of $J$ is $[-1, 1]$ and

$$
d\mu_{1,1}(x) = d\mu_{2,2}(x) = \frac{1}{\pi} \frac{dx}{\sqrt{1 - x^2}}, \quad d\mu_{1,2}(x) = \frac{1}{\pi} \frac{x \, dx}{\sqrt{1 - x^2}}.
$$

From our theorem we thus find

$$
\lim_{n \to \infty} \int f(x) p_n^2(x; \mu) \, d\mu(x) = \int f(x) \, d\mu_{2,2}(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx,
$$

and in general

$$
\lim_{n \to \infty} \int f(x) p_n(x; \mu) p_{n+k}(x; \mu) \, d\mu(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x) U_k(x) - xU_{k-1}(x)}{\sqrt{1 - x^2}} \, dx
$$

$$
= \frac{1}{\pi} \int_{-1}^{1} f(x) \frac{T_k(x)}{\sqrt{1 - x^2}} \, dx.
$$

Here we have used the identity

$$
U_k(x) - xU_{k-1}(x) = T_k(x),
$$

where $T_k(x)$ is the Chebyshev polynomial of the first kind. This result is well-known and can already be found in [15, Theorem 13 on p. 45]. See also [20, Theorem 2 on p. 438].

Unbounded recurrence coefficients. Suppose that we have a sequence of orthogonal polynomials $p_n(x; \mu)$ satisfying the three-term recurrence relation (1.1). If we rescale the variable by a positive and increasing sequence $c_k$ and consider the one-parameter family of polynomials $p_n(c_k x; \mu)$, then these polynomials satisfy a recurrence relation of the form (1.2) with
orthogonal on the geometric sequence

\{ q^n, n = 1, 2, 3, \ldots \}

have recurrence coefficients

\[ a_n(b, q) = q^n \sqrt{b(1-q^n)(1-bq^{n-1})}, \quad b_n(b, q) = q^n[b + q - (1+q) bq^n]. \]

If we consider the Wall polynomials \( w_n(x; b, c^{1/k}) \), where \( 0 < c < 1 \), then we have a one-parameter family of orthogonal polynomials with recurrence coefficients

\[ a_{n,k} = a_n(b, c^{1/k}), \quad b_{n,k} = b_n(b, c^{1/k}), \]

and one easily verifies that

\[ \lim_{n \to \infty} a_{n+k,n} = c \sqrt{b(1-c)(1-bc)} = A/2, \quad \lim_{n \to \infty} b_{n+k,n} = (b + 1 - 2bc)c = B. \]

Hence we can apply the theorem, where \( J \) again is a doubly infinite Jacobi matrix with constant entries \( B \) on the diagonal and \( A/2 \) on the subdiagonal. Therefore again we have the asymptotic behaviour in terms of Chebyshev polynomials (first and second kind) as in the previous two examples. These three examples are all covered by Theorem 2 in [21, p. 307] which covers the special case of our theorem with a doubly infinite Jacobi matrix with constant entries. In particular this asymptotic behaviour for Wall polynomials was used in [21] to show that the product formulas for Legendre polynomials are a limiting case of the product formulas for little \( q \)-Legendre polynomials as \( q \to 1 \).

**Jacobi polynomials** \( P_n^{(\alpha, \beta)}(x) \). The Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) are orthogonal on \([-1, 1]\) with the weight function \((1-x)^\alpha(1+x)^\beta\). The orthonormal Jacobi polynomials

\[ p_n^{(\alpha, \beta)}(x) = \sqrt{2n + \alpha + \beta + 1 \over n![(\alpha+\beta+2)_n] n!} P_n^{(\alpha, \beta)}(x) \]
If we consider the Jacobi polynomials $p_n^{(ak+\alpha, bk+\beta)}(x)$ with $a > 0, b > 0$, then the recurrence coefficients are $a_{n,k} = a_n(ak + \alpha, bk + \beta)$ and $b_{n,k} = b_n(ak + \alpha, bk + \beta)$, and one easily finds

$$\lim_{n \to \infty} a_{n+k,n} = \frac{2 \sqrt{(a+1)(b+1)(a+b+1)}}{(a+b+2)^2}, \quad \lim_{n \to \infty} b_{n+k,n} = \frac{b^2 - a^2}{(a+b+2)^2}.$$ 

Hence our theorem applies again with a constant doubly infinite Jacobi matrix $J$. This result has not been given in the literature, but complements the known results concerning strong asymptotics and zero behaviour given in [5] [9] [14].

**Laguerre polynomials $L_n^{\alpha+n}(nx)$**. The Laguerre polynomials $L_n^{\alpha}$ are orthogonal on $[0, \infty)$ with weight function $x^\alpha e^{-x}$. The orthonormal Laguerre polynomials

$$p_n^{\alpha}(x) = (-1)^n \binom{n + \alpha}{n}^{-1/2} L_n^{\alpha}(x)$$

have recurrence coefficients

$$a_n(\alpha) = \sqrt{n(n+\alpha)}, \quad b_n(\alpha) = 2n + \alpha + 1.$$ 

If we consider the polynomials $p_n^{ak+\alpha}(kx)$, then we have $a_{n,k} = a_n(ak + \alpha)/k$ and $b_{n,k} = b_n(ak + \alpha)/k$, hence one easily finds

$$\lim_{n \to \infty} a_{n+k,n} = a + 1, \quad \lim_{n \to \infty} b_{n+k,n} = a + 2,$$

so that once more our theorem can be applied with a constant Jacobi matrix $J$. This complements the asymptotic behaviour for such Laguerre polynomials given in [5] and [8].

**Dual Hahn polynomials**. The dual Hahn polynomials $R_n(x) = R_n(x; \alpha, \beta, N)$ are given by the recurrence relation

$$-xR_k(x) = D_kR_{k-1}(x) - (D_k + B_k)R_k(x) + B_kR_{k+1}(x),$$

with initial condition $R_0(x) = 1$ and $R_{-1}(x) = 0$ [11]. Here

$$B_k = (N - 1 - k)(\alpha + 1 + k), \quad D_k = k(N + \beta - k),$$

and the polynomials are orthogonal on the quadratic lattice \{x_k = k(k+\alpha+\beta+1) : k = 0, 1, 2, \ldots, N-1\} with weights

$$\pi_k = \pi_k(\alpha, \beta, N) = \frac{\Gamma(\beta+N)}{\Gamma(N+\alpha+\beta+k+1)} \frac{\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(\alpha+1)} (2k+\alpha+\beta+1)$$

at these points $x_k$, so that these polynomials are only defined up to degree $N$. The orthonormal polynomials are

$$p_n(x; \alpha, \beta, N) = \frac{(N+\alpha+\beta)^{1/2}}{(\alpha+n)^{1/2}(\beta+N-1-n)^{1/2}} \frac{1}{R_{n-1}(x; \alpha, \beta, N)},$$

with recurrence coefficients $a_n^2 = D_nB_{n-1}$ and $b_n = D_n + B_n$. These polynomials are useful in the description of a genetic model of Moran, as is worked out in [11] [12] and [7]. Consider the polynomials
One easily finds
\[
\lim_{n \to \infty} a_{n+k,n}^2 = \begin{cases} 
0 & \text{for } k \geq 0, \\
-k(\beta - k) & \text{for } k < 0,
\end{cases}
\lim_{n \to \infty} b_{n+k,n} = \begin{cases} 
0 & \text{for } k \geq 0, \\
-2k + \beta - 1 & \text{for } k < 0.
\end{cases}
\]

The doubly infinite Jacobi matrix \( J \) corresponding to these asymptotic formulas is therefore only a semi-infinite Jacobi matrix which coincides with the Jacobi matrix \( J^- \), for which the corresponding orthogonal polynomials are the Laguerre polynomials \( L_\alpha^\beta(x) \). The spectral matrix of measures of \( J \) thus reduces to the Laplace model in genetics \([7]\). Weak limits of this kind are useful when studying fluctuation theory in the Moran (or Bernoulli-Laplace) model in genetics \([7]\).

In general we have for \( k, l \geq 1 \)
\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j/n)p_{n-k}(x_j; \alpha, \beta, n)\pi_{j,n} = \frac{(-1)^{k+l}}{\sqrt{h_k h_{l-1}}} \int_{-\infty}^{\infty} f(x) L_{k-1}^\beta(x) L_{l-1}^\beta(x) \frac{x^\beta e^{-x}}{\Gamma(\beta + 1)} \, dx,
\]
where \( x_j = j(\alpha + \beta + 1) \) and \( \pi_{j,n} = \pi_j(\alpha, \beta, n) \). In general we have for \( k, l \geq 1 \)
\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j/n)p_{n-k}(x_j; \alpha, \beta, n)p_{n-l}(x_j; \alpha, \beta, n)\pi_{j,n}
= \frac{(-1)^{k+l}}{\sqrt{h_k h_{l-1}}} \int_{-\infty}^{\infty} f(x) L_{k-1}^\beta(x) L_{l-1}^\beta(x) \frac{x^\beta e^{-x}}{\Gamma(\beta + 1)} \, dx,
\]
where \( h_k = \binom{k + \beta}{k} \) is the norm of \( L_k^\beta(x) \).

If we consider the polynomials \( p_n(k^3/2x + k^2/2; \alpha, k/2, k) \), then the recurrence coefficients are zero for \( n \geq k \) and for \( n < k \) we have
\[
a_{n,k}^2 = \frac{n(k - n)(3k/2 - n)(\alpha + n)}{k^3},
\]
\[
b_{n,k} = \frac{(k - n - 1)(\alpha + n + 1) + n(3k/2 - n) - k^2/2}{k^3/2}.
\]
Now one easily finds
\[
\lim_{n \to \infty} a_{n+k,n}^2 = \begin{cases} 
0 & \text{for } k \geq 0, \\
-k/2 & \text{for } k < 0,
\end{cases}
\lim_{n \to \infty} b_{n+k,n} = 0 \quad \text{for } k \in \mathbb{Z}.
\]

The doubly infinite Jacobi matrix \( J \) again coincides with \( J^- \), which is now the semi-infinite Jacobi matrix for the Hermite polynomials \( H_n(x) \). We thus find
\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} f \left( \frac{x_j - n^2/2}{n^{3/2}} \right) p_{n-1}(x_j; \alpha, n/2, n)\pi_{j,n} = \int_{-\infty}^{\infty} f(x) e^{-x^2} \sqrt{\pi} \, dx,
\]
where now \( \pi_{j,n} = \pi_j(\alpha, n/2, n) \), and in general
\[
\lim \sum_{j=0}^{n-1} f \left( \frac{x_j - n^2/2}{n^{3/2}} \right) p_{n-1,j}(x_j; \alpha, n/2, n)p_{n,k}(x_k; \alpha, n/2, n)\pi_{j,n}.
\]
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