Optimization of perturbative similarity renormalization group for Hamiltonians with asymptotic freedom and bound states

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A model Hamiltonian that exhibits asymptotic freedom and a bound state, is used to show on example that similarity renormalization group procedure can be tuned to improve convergence of perturbative derivation of effective Hamiltonians, through adjustment of the generator of the similarity transformation. The improvement is measured by comparing the eigenvalues of perturbatively calculated renormalized Hamiltonians that couple only a relatively small number of effective basis states, with the exact bound state energy in the model. The improved perturbative calculus leads to a few-percent accuracy in a systematic expansion.

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I. INTRODUCTION

The matrix of canonical QCD Hamiltonian is too large for a direct diagonalization using computers. One cause of the forbidding size is the dynamical coupling among bare degrees of freedom across an infinite range of energy scales. This difficulty is manifest in all local quantum field theories of physical interest through ultraviolet divergences. One possible method to attack the divergence problem in the case of asymptotically free theories is to first evaluate an effective Hamiltonian and then attempt diagonalization of the corresponding effective matrix. For some well adjusted effective degrees of freedom may interact over only a limited range of energy scales, whereby a definition of a finite effective eigenvalue sub-problem becomes conceivable and its subsequent solution using computers is not immediately excluded. To derive the effective Hamiltonians one can use similarity renormalization group (RG) procedure [1]. This article discusses the accuracy that the similarity procedure can achieve in evaluating the effective Hamiltonians in perturbation theory. The latter is the only systematic method that exists for handling the initial canonical QCD Hamiltonian.

Having a perturbatively calculated effective Hamiltonian, $\mathcal{H}(\lambda)$, that strongly couples effective basis states only if their free energies, i.e. eigenvalues of certain $\mathcal{H}_0(\lambda)$, do not differ by more than a finite width parameter $\lambda$, one can cut out of the matrix elements in the effective basis. The energy range of the sub-matrix $W_\lambda$ can be limited to few or several widths $\lambda$. Such $W_\lambda$ contains only a fraction of the matrix elements of $\mathcal{H}(\lambda)$, which is why $W_\lambda$ is called a window. One can diagonalize the window numerically and obtain the part of the spectrum of $\mathcal{H}(\lambda)$ that is matched by the selected window. For example, the energy range of the window matrix $W_\lambda$ can be limited to the region where a bound state is formed. The question is then how accurate the perturbative RG derivation of $\mathcal{H}(\lambda)$ can be for the matrix elements in such $W_\lambda$.

The issue was studied previously [2] using a particularly elegant version of the similarity transformation generator, which was taken from Wegner’s flow equation for Hamiltonians in solid state physics [3]. In the model studied in Ref. [2], the Hamiltonian $\mathcal{H}(\lambda)$ could be calculated numerically with arbitrary accuracy. One could also calculate the same Hamiltonian in perturbation theory. Then, one could compare results of numerical diagonalization of the resulting windows cut out from the exact and from the perturbative results for $\mathcal{H}(\lambda)$.

The model study [2] showed that Wegner’s generator produced quite useful first three terms in an effective coupling constant expansion for the window Hamiltonian, $W_\lambda = W_0 + g_\lambda W_{\lambda 1} + g_\lambda^2 W_{\lambda 2}$. Diagonalization of this $W_\lambda$ produced surprisingly good accuracy on the order of 10%. The bound state eigenvalue of $W_\lambda$ matched the exact one with so small error even when $\lambda$ was reduced down to the range that corresponded to 1 GeV in QCD. Moreover, only a handful of basis states were sufficient in the numerical diagonalization. However, $g_\lambda$ grew in the asymptotically free model when $\lambda$ was made small, which is a generic feature for asymptotic freedom, and $g_\lambda$ became comparable with 1 for $\lambda \sim 1$ GeV. At the same time, the coefficients of the perturbative expansion for matrix elements of $W_\lambda$ beyond the first three terms turned out to form an alternating series with growing coefficients and, eventually, for $\lambda$ still much larger than 1 GeV, results including terms proportional to $g_\lambda^3, g_\lambda^4$, etc. turned out to be useless. Higher-than-second order terms led to erratic behavior of $W_\lambda$ and its eigenvalues. This result required a closer inspection to decide if one could improve the perturbative part of the similarity approach by using other generators than Wegner’s, and whether any serious investment in a vastly more complex QCD calculations should ever be made using this method. The problem was not only quantitative, with details depending on the model chosen for study, but also qualitative. Namely, one needed to know if any significant improvement was possible in principle, because the growth of $g_\lambda$ for decreasing $\lambda$ in asymptotically free theories is in basic conflict with the goal of reducing the width $\lambda$ down to the binding scale using perturbation theory. Since there are not many exactly soluble models with asymptotic freedom and bound states, it was important to find out if any improvement could be obtained in the model of Ref. [2].

The optimization task may be attempted in similarity RG procedure by taking advantage of the fact that the generator of

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the similarity transformation can be chosen in infinitely many different ways. It is plausible that the convergence problems emerged in the Wegner case because interactions with large energy changes were very quickly eliminated and such rapid elimination could produce large feedback in the low-energy dynamics, too large for easy reproduction through a perturbative expansion. Therefore, this work is focused on the question if varying the similarity generator, especially the rate at which the off-diagonal terms are removed, can improve convergence of at least the first five terms in the expansion, i.e. all terms up to and including order $g_4^\lambda$. The number 4 is distinguished by the fact that one needs at least four orders of perturbation theory to simultaneously account for effective masses and running coupling in quark-gluon dynamics in QCD. The answer found in the model studied here is positive. Namely, it is shown below that one can obtain a systematic expansion for $W_\lambda$ into powers of $g_\lambda$, and the sum of first 5 terms in the expansion produces at least three times better accuracy than in Ref. [2]. When an optimized version of the similarity generator is identified, one can include higher order terms than 4th. For example, it will be shown here that 6th order improves the accuracy to 10 times better than achieved previously using Wegner’s generator in the model. Sensitivity of the effective theory to the finite bare cutoff in numerical calculations is also reduced by at least an order of magnitude (this will be explained later) and control over numerical instabilities that emerge in the RG evolution is also enhanced (this aspect is only mentioned). The main result reported here, however, is not that a class of similarity generators leads to improvement in the model chosen for this study, but that the flexibility available in the similarity renormalization group procedure does lead to considerable options for improved convergence pattern.

Since the similarity approach to QCD has already evolved in the effective particle version to third order terms [4] and systematic studies of bound states will become possible once the fourth order terms are brought under numerical control [5], the options for improving convergence need to be further explored in more complex models. Also, the development of Wegner’s flow equations in solid state physics and field theory [6, 7, 8, 9, 10], has already demonstrated that new options for optimization may have much broader range of applicability than the QCD bound state problem could suggest by itself. Besides, if the window Hamiltonians of interest could be reliably evaluated using perturbation theory, the standard quantum mechanical calculus could then be employed to describe with them the time evolution of selected states with efficiency probably hard to match by any other method.

This paper is organized as follows. Section II describes the model. Section III discusses results concerning perturbation theory and optimization of the similarity generator, which is followed by concluding remarks in Section IV.

II. MODEL

Matrix elements of the model studied here are (see Ref. [2] for introduction)

$$H_{mn} = E_m\delta_{mn} - g\sqrt{E_mE_n}, \quad (2.1)$$

where $E_n = b^n, b > 1$, and $n$ is an integer, with a convention that energy equal 1 corresponds to 1 GeV. The model is cut off in infrared and ultraviolet regimes by limiting the subscripts, $M \leq m,n \leq N$, $M$ being negative and $N$ positive, both of much larger magnitude than 1. The ultraviolet renormalizability of the model, its asymptotic freedom, and its lack of sensitivity to the infrared cutoff, were discussed in [2]. The bare coupling constant, $g = 0.060600631$, is adjusted to obtain a bound state with eigenvalue -1 GeV with 8 digits of accuracy for $b = 2, M = -21$ and $N = 16$, in exactly the same way as in Ref. [2]. With these choices, the Hamiltonian $H$ is a $38 \times 38$ matrix with 37 positive eigenvalues, and one negative. The largest energy scale reaches 65 TeV, and the smallest one is 0.5 KeV. This energy range saturates the needs of contemporary theories where asymptotic freedom and bound states are of interest.

The differential similarity RG procedure is based on a class of equations that can be written as

$$\frac{dH}{d\lambda} = [F_\lambda \{H\}, H], \quad (2.2)$$

with the initial condition set at $\lambda = \infty$, forcing the RG trajectory to start from the initial Hamiltonian, $H(\infty) = H$, of the bare theory with counterterms. The generator of the similarity transformation can be written as ($D_m = H_{mn}$)

$$[F_\lambda \{H\}]_{mn} = f_{mn}(D_m - D_n)H_{mn}, \quad (2.3)$$

where different choices of $f_{mn}$ lead to different results for the trajectory of effective Hamiltonians. $f_{mn} = \phi_{mn} ds/d\lambda$, where $s = 1/\sqrt{\lambda}$ and $\phi_{mn} \equiv 1$, gives Wegner’s equation, which has a considerable record of applications in solid state physics [3, 6, 7, 8, 9], independently of the similarity renormalization group studies of asymptotically free theories. Note also, that multiplication of $\phi_{mn}$ by a constant, say $a$, is equivalent to change of variables from $\lambda$ to $\lambda' = \lambda/\sqrt{a}$, and corresponds to mere shift on logarithmic scale.

Behavior of matrix elements $H_{mn}(\lambda)$ as functions of $\lambda$ has already been extensively discussed in the case of Wegner’s generator in Ref. [2]. For $\phi_{mn}$ that is independent of $\lambda$, it can be qualitatively represented as

$$H_{mn}(\lambda) = E_m\delta_{mn} - g_\lambda \sqrt{E_mE_n} \times \exp \left[ -\phi_{mn}(E_m - E_n)^2/\lambda^2 \right], \quad (2.4)$$

where $g_\lambda$ is the effective coupling constant. The Gaussian-like fall-off function is not exact, but it is not far from the actual form factor that makes the interaction Hamiltonian matrix narrow on the energy scale, i.e., of width $\lambda$. It is clear that in the case of Eq. (2.4) the effective coupling constant could be evaluated from the following formula,

$$g_\lambda = 1 - \frac{H_{MM}(\lambda)}{E_M}. \quad (2.5)$$
The same definition is applied in the case of exact solution. The resulting effective coupling $g_{\lambda}$ can be computed numerically in all cases considered here and its dependence on $\lambda$ is known, see Fig. 1. For $\lambda \gg 1$, one has $g_{\lambda} \sim 1/\log \lambda$, and the growth of $g_{\lambda}$ toward smaller $\lambda$ continues in case of Wegner’s generator until the coupling constant slightly exceeds 1, which happens when the effective Hamiltonian width becomes comparable to the momentum-space width of the bound state wave function. For comparison, the optimized generator result is labeled $c = 1$. It will be discussed later.

Near $\lambda = 1$ GeV the bound state eigenvalue begins to build up on the diagonal of $H$, with participation of a limited number of neighboring matrix elements. The limitation comes from the size of $\lambda$ and the actual wave function width. Thus, when one extracts a sub-matrix from $H(\lambda \sim 1)$, which forms a window, $W_\lambda$, that embraces all matrix elements that count in the bound state dynamics, the diagonalization of that window alone can produce the right value of the bound state energy. This is shown in Figure 2 in case of Wegner’s equation for three different choices of the window matrix, i.e. for three different choices of $M$ and $N$ that limit $W_\lambda$ in the same fashion as $M$ and $N$ limit $H$ in Eq. (2.1).

Figure 2 demonstrates that an exactly calculated small window may provide correct results when $\lambda$ is appropriately small, i.e. below about 4 GeV. The only problem is that the calculation of a window Hamiltonian with finite $\lambda$ in realistic theories is not feasible beyond perturbation theory. Thus, one is led to the question how well the same small windows can be calculated perturbatively. The question is of key importance to QCD because the region of $\lambda \leq 4$ GeV is where a significant raise of the effective coupling constant $g_{\lambda}$ is expected. But the question may also be relevant in other areas of physics than strong interactions.

III. PERTURBATION THEORY

This Section describes difficulties that similarity approach encounters with convergence of perturbation theory, treating the case with similarity transformation generator taken from Wegner’s flow equation as a benchmark, and shows how one can optimize the convergence by changing the generator. The quality of the perturbative expansions is established in the model by comparing the negative eigenvalue of perturbatively calculated windows $W_\lambda$ with the exact value of $-1$ GeV.

Perturbation theory is constructed in the following way. First one writes a series for $H(\lambda)$ in the form

$$H(\lambda) = \sum_{n=0}^{\infty} g^n H^{(n)}(\lambda),$$

(3.1)

where $g$ is the bare coupling constant from $H$ of Eq. (2.1). For example, in second order expansion the solution of Eq. (2.2) with $\phi_{mn}$ independent of $\lambda$, reads

$$H(\lambda) = E_m \delta_{mn} - g \sqrt{E_m E_n} \exp \left[-\phi_{mn}(E_m - E_n)^2/\lambda^2\right] + g^2 \sqrt{E_m E_n} \exp \left[-\phi_{mn}(E_m - E_n)^2/\lambda^2\right] \times \sum_{k=M}^{N} E_k \frac{\phi_{mk}(E_m - E_k) - \phi_{kn}(E_k - E_n)}{A_{mnk}} \times \left[1 - \exp \left(-\frac{A_{mnk}}{\lambda^2}\right)\right] - 2\phi_{mn} \frac{(E_m - E_n)^2}{\lambda^2} + \text{corrections},$$

(3.2)

where

$$A_{mnk} = \phi_{mk}(E_m - E_k)^2 + \phi_{nk}(E_n - E_k)^2$$
and corrections are of higher order in $g$ than second. The latter terms are also calculable analytically in orders 3rd, 4th, and higher, but they are too complex to display here and hardly useful in the analytic form for numerical calculations in the present study, because they involve combinations of mutually canceling terms that contain ratios of functions that approach zero in numerator and denominator and require careful evaluation. Instead of such analytic expressions, which are discussed here only to demonstrate what the required expansion consists in, it is more practical in the model to apply a numerical algorithm and generate matrices $H^{(n)}(\lambda)$ order by order through Runge-Kutta integration of $(N-M+1)(N-M)/2$ coupled non-linear differential equations.

Once the expansion of Eq. (3.1) is calculated to some order, one uses Eq. (2.5) to express $g_{\lambda}$ as a series in powers of $g$ to the same order and one inverts that series to obtain $g$ in the form of a new series in powers of $g_{\lambda}$ up to that same order. The latter series is then inserted into Eq. (3.1), which provides the desired effective coupling constant expansion for the whole matrix $H(\lambda)$, i.e. the expansion in terms of powers of $g_{\lambda}$. Note that this procedure can be carried out without detailed analytic knowledge of how the matrices $H^{(n)}(\lambda)$ depend on $\lambda$, although such knowledge and selection of relevant, marginal, and irrelevant matrices may be of great help in calculations with more than one coupling constant if one can reliably identify correlations among the couplings. In fact, the accuracy achieved here in plain perturbation theory can most probably be further enhanced using techniques similar to the ones employed in Ref. [10].

The accuracy of the perturbative procedure for evaluating effective Hamiltonians in case of Wegner’s equation, i.e. with $\phi_{mn} \equiv 1$, is shown in Fig. 3, in terms of the negative eigenvalues of a selected window $W_{\lambda}$. While the window matrix elements are evaluated in perturbation theory, the eigenvalues of the window are obtained from non-perturbative diagonalization. Note that the vertical axis has logarithmic scale. While the second order result produces about 10% accuracy after diagonalization, the next orders in the expansion are wrong by huge factors and completely unacceptable, including the fact that they render spurious negative eigenvalues. Since it is not known how to select the right negative eigenvalue of the windows obtained in third and fourth order calculation, all negative eigenvalues of the windows are displayed, labeled with the number that indicates the order of the expansion for $W_{\lambda}$.

The window $W_{\lambda}$ used in Fig. 3 was selected on the basis of results analogous to those shown in Fig. 2. An arbitrarily set requirement was adopted, that a good window should produce the negative eigenvalue with accuracy around 0.5 percent for $4 \geq \lambda \geq 1$ when calculated exactly. The 0.5% may be considered a large error but it is introduced taking into account that low order perturbation theory for $W_{\lambda}$ is expected to lead to much larger errors. The bound state energies calculated by diagonalization of exact windows with different values of $\tilde{N}$ and one value of $M = -8$ are shown in Figure 2. It is clear that one should choose $\tilde{N} = 2$ to make sure that the correct result is contained in the window of choice when $\lambda \leq 4$. For $\tilde{N}$ kept equal 2 at $\lambda = 2$, one obtains the eigenvalue $-9.9599$ with $M = -8$, $-9.9199$ with $M = -7$, and $-9.8398$ with $M = -6$. Therefore, $M = -8$ and $\tilde{N} = 2$ were selected in Fig. 3.

![FIG. 3: Negative eigenvalues obtained from diagonalization of the window $W_{\lambda}$ with $M = -8$ and $\tilde{N} = 2$, obtained with Wegner’s generator in first four orders of perturbation theory. The orders 3 and 4 produce spurious negative eigenvalues (see the text for details) with no sign of convergence.](image)

The summary of results obtained for $W_{\lambda}$ from the perturbative expansion of Wegner’s equation is that the expansion produces accuracy on the order of 10% in second order but completely fails in higher orders. Therefore, it is important to verify what happens when one considers $f_{mn}$ with $\phi_{mn} \neq 1$ in Eq. (2.3). Two basic options can be considered. One is to make $\phi_{mn} > 1$, which accelerates changes in the matrix elements $H_{mn}(\lambda)$ as functions of $\lambda$, and the other is to make $\phi_{mn} < 1$, which slows down the changes.

Numerical calculations show that the acceleration with a factor of the type $\phi_{mn} = 1 + c |m-n|$ with $c$ ranging between 0.01 and 100 does not change the erratic behavior of the window to any better. In contrast, slowing down the evolution by the factor of the type

$$\phi_{mn} = \frac{1}{1 + c |m-n|},$$

produces significant improvements. The type of results one obtains from the slowed down similarity transformation is illustrated by an example in Fig. 4, using Eq. (3.4) with $c = 1$. In this case, one obtains only one negative eigenvalue in the range $[-2, 0]$ (no spurious negative eigenvalues appear in that range). Moreover, the accuracy of the eigenvalue calculated by diagonalization of the fourth order window reaches 3%, which is unattainable using Wegner’s generator.

The particular choice of $c = 1$ follows from the observation that 4th order results are less accurate than the 3rd order for $c \lesssim 0.5$, while for $c \gtrsim 2$ no further improvement is obtained.
in the accuracy of the 4th order terms. Figure 4 shows also results of including orders 5th and 6th for \( c = 1 \). Inclusion of 6th order contribution achieves the accuracy of 1\% for the bound state eigenvalue, which is 10 times better than with the Wegner generator. Since the contribution of 6th order is larger than 5th, one may consider larger \( c \), around 2, to obtain a more systematic inclusion of successive orders as required. This issue will not be further discussed here, since already the 4th order inclusion of successive orders as required. This issue will not be further discussed here, since already the 4th order terms as required. This issue will not be further discussed here, since already the 4th order terms are sufficient for effective window Hamiltonians in QCD is currently not known.

Finally, one has to address the issue of renormalizability since for \( c \to \infty \) the similarity transformation generator approaches zero within the bounds set by the ultraviolet cutoff \( N \) and no RG evolution occurs. The question is then how strongly the effective window Hamiltonians \( W_\lambda \) for \( c \sim 1 \) and \( \lambda \lesssim 4 \) GeV, depend on \( N \). This can be studied by inspection of examples and the result is that the generator with \( c = 1 \) produces considerably more efficient approach to renormalized effective theory than Wegner’s. The measure of how quickly the renormalized effective theory is achieved is found by evaluating 64 matrix elements of the exact window \( W_\lambda \) with \( \tilde{M} = -4 \) and \( \tilde{N} = 3 \) for \( N = 16 \), and for \( N = 20 \). These matrix elements are most important in the formation of bound-state in the effective dynamics. In both cases the bare coupling constant is adjusted to produce the same bound state eigenvalue of -1 (one obtains \( g = 0.04878048667 \) in the case \( N = 20 \)). Then, one evaluates

\[
r^2(\lambda) = \frac{1}{(\tilde{N} - \tilde{M})^2} \sum_{m,n=\tilde{M}}^{\tilde{N}} \left[ \frac{W_{\lambda mn}(N = 16)}{W_{\lambda mn}(N = 20)} - 1 \right]^2,
\]

(3.5)
in the range of \( \lambda \) between 4 and 1 GeV where, as indicated by Figs. 3 and 4, the chosen window is most suitable for diagonalization. One cannot rely on a comparison at one value of \( \lambda \) because, as it is visible in Fig. 1, the formation of bound states occurs in Wegner’s case (\( c = 0 \)) and for \( c \neq 0 \) at different values of \( \lambda \), being shifted toward smaller values of \( \lambda \) for \( c = 1 \). It turns out that the resulting measure \( r(\lambda) \) for the Wegner generator varies between \( 10^{-5} \) and \( 10^{-4} \), with numerical accuracy being of significance, while for \( c = 1 \) one obtains a numerically stable result, approximately equal \( 10^{-6} \) in the whole range between 4 and 1 GeV.

The utility of logarithmic (i.e. depending on exponents \( m \) and \( n \) of the base \( b \)) slowing down of RG evolution of the off-diagonal matrix elements originates from two sources. One is the smaller size of the effective coupling constant in the region where the bound state is formed, as seen in Fig. 1. For example, at \( \lambda = 2 \) GeV, in Wegner’s case \( g_\lambda = 0.72373722 \), and the slowed down case with \( c = 1 \) gives \( g_\lambda = 0.28518167 \), which is about 2.5 times smaller and produces about 40 times smaller result for \( g_\lambda^2 \). The other source is that coefficients of the perturbative series in powers of \( g_\lambda \) for matrix elements \( \mathcal{H}(\lambda) \) derived with slower generator are not growing with the order as they do in Wegner’s case. This is illustrated in Tab. I on two generic examples. Namely, the matrix elements of coefficient matrices \( a_i \) with \( i = 0, 1, 2, 3, \) and 4, in

\[
\mathcal{H}(\lambda) = a_0 + a_1 g_\lambda + a_2 g_\lambda^2 + a_3 g_\lambda^3 + a_4 g_\lambda^4 + \ldots,
\]

(3.6)
are tabulated for \((m, n)\) equal \((-1, -1)\) and \((2, 1)\) at \( \lambda = 2 \) GeV, in Wegner’s case and for \( c = 1 \), as indicated. It is visible that the ratios \( a_4/a_3 \) are about an order of magnitude smaller in size for \( c = 1 \) than in Wegner’s case.

| \( \lambda = 2 \) | \( F(\mathcal{H}) \) | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) |
|-------------------|-----------------|--------|--------|--------|--------|--------|
| \( \mathcal{H}_{-1, -1} \) Wegner | 0.5 | -0.498 | 0.044 | 0.253 | -0.902 |
| \( \mathcal{H}_{2, 1} \) Wegner | 0 | -1.050 | 0.270 | 6.116 | -24.788 |
| \( c = 1 \) | 0 | -1.724 | -2.685 | -2.741 | 0.960 |

TABLE I: Numerical values of perturbative coefficients of successive powers of \( g_\lambda \) in Eq. (3.6) at \( \lambda = 2 \) GeV for selected matrix elements in case of Wegner’s generator and for \( c = 1 \). See the text for details.

IV. CONCLUSION

Similarity renormalization group approach to solving asymptotically free theories could employ various generators of the similarity transformation to calculate effective Hamiltonians. The calculation can be carried out in perturbation theory and then the Hamiltonians can be diagonalized using computers with a reasonable chance for obtaining accurate answers, if the perturbative procedure for evaluating effective Hamiltonians is sufficiently precise. It was shown here, using Wegner’s generator as a benchmark, that in order to obtain stable 3\% accuracy in the bound state spectrum one needs to slow down the rate of elimination of the off-diagonal matrix elements. In the model study, the rate of change of bare
Hamiltonian matrix elements with largest energy changes was slowed down 38 times. The stabilizing effect of slowing down the RG flow is evident from comparison of Figs. 3 and 4. Perturbation theory including sixth order terms with $c = 1$ in Eq. (3.4) achieved 1% accuracy in evaluating effective dynamics, as measured by the bound state eigenvalue.

However, there is a stiff price to pay. Namely, the second order perturbative expansions with slower generators produce only about 30% accuracy, in comparison to 10% in Wegner’s case, and to take advantage of the improved convergence one has to go to higher orders, which requires considerably more effort. On the other hand, orders higher than second are not attainable using Wegner’s generator, and the outstanding accuracy of about 10%, achievable already in the second order, cannot be improved by including higher orders in a plain expansion in one running coupling constant. Moreover, the slower generator with $c = 1$ leads also to less sensitivity of the numerically evaluated effective theory to the bare cutoff, when the latter is kept large but finite.

Detailed studies will disclose how much work is actually required to achieve the few-percent accuracy with realistic Hamiltonians. However, the model study described here shows already that optimization of the similarity generator may in principle lead to a desired convergence pattern in the case of bound states in asymptotically free theories. Perhaps, the optimization found here may turn out to be useful also in other applications of similarity than QCD.

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