On the Calculation of the Fe K-alpha Line Emissivity of Black Hole Accretion Disks

H. Krawczynski and B. Beheshtipour
Physics Department and McDonnell Center for the Space Sciences, Washington University in St. Louis, 1 Brookings Drive, CB 1105, St. Louis, MO 63130, USA; krawcz@wustl.edu

Received 2017 August 14; revised 2017 September 18; accepted 2017 September 18; published 2017 October 31

Abstract

Observations of the fluorescent Fe Kα emission line from the inner accretion flows of stellar mass black holes in X-ray binaries and supermassive black holes in active galactic nuclei have become an important tool to study the magnitude and inclination of the black hole spin, and the structure of the accretion flow close to the event horizon of the black hole. Modeling spectral, timing, and soon also X-ray polarimetric observations of the Fe Kα emission requires the calculation of the specific intensity in the rest frame of the emitting plasma. We revisit the derivation of the equation used for calculating the illumination of the accretion disk by the corona. We present an alternative derivation leading to a simpler equation, and discuss the relation to previously published results.

Key words: accretion, accretion disks – black hole physics – line: formation – quasars: emission lines – X-rays: binaries

1. Introduction

The Fe Kα line in the X-ray energy spectra of stellar mass black holes in X-ray binaries and supermassive black holes in active galactic nuclei (AGNs) is thought to originate as fluorescent emission when the disk is illuminated by the hard X-rays from a hot corona (see Reynolds 2014 and Miller & Miller 2015 for recent reviews). The shape and origin of the corona are still a matter of debate although spectral, timing, and—in the case of gravitationally lensed quasars—the amplitude of microlensing amplification favor extremely compact coronas close to the event horizons of the black holes (e.g., Reis & Miller 2013). X-ray energy spectra are commonly modeled by folding the specific emissivity $I$ with Cunningham’s transfer function $f$ (Cunningham 1975; Speith et al. 1995). For a geometrically thin equatorial accretion disk of a black hole described by a stationary axis symmetric, and asymptotically flat spacetime, the luminosity at energy $E$ per unit energy and steradian is given by

$$
\frac{d^2L_E}{dE \, d\Omega} = \int_{r_1}^{\infty} \pi r^2 g^* \frac{I(E/g, r, \mu)}{\sqrt{g^*(1 - g^*)}} f(g^*, r, \vartheta, \theta) \, dg^* \, dr,
$$

where $r_1$ is the inner edge of the accretion disk, $r$ is the radial coordinate, $g$ is the ratio between the observed and emitted energy of the photon, $g^*$ is the same ratio rescaled so that the minimum and maximum values for the emission from the considered accretion disk ring are 0 and 1, respectively, $\mu$ is the cosine of the polar angle of the emitted emission with respect to the disk normal in the rest frame of the accretion disk plasma, and $\vartheta$ is the inclination of the observer measured from the rotation axes of the black hole and the accretion disk (assumed to have co-aligned spin axes). The $g$ and $g^*$ terms in the integral reduce the $g$-dependence of $f$. The parameter $g^*$ is double-valued for each ring, and the integral has to be performed twice.

Assuming axial symmetry, the emitted intensity $I$ of the Fe Kα emission follows from convolving the hard X-ray flux impinging on the accretion disk $F_X$ with a reflection function $R$ of

$$
I(E, r, \mu) = \int \int dE' \, d\mu' \, F_X(E', r, \mu') R(E, \mu; E', r, \mu').
$$

Here, $F_X(E', r, \mu')$ is the energy flux of the X-ray radiation per unit energy at energy $E'$, incident at radial coordinate $r$ from a direction with direction cosine $\mu'$ per unit time and area (all quantities in the rest frame of the accreting plasma) averaged over the azimuthal incident angles. $R$ (defined here in a loose analogy to Chandrasekhar’s scattering function $S$; Chandrasekhar 1960, Chapter 13) is the specific intensity of the reflected emission with energy $E$ emitted into the directions with direction cosine $\mu$. We have chosen the most general form of $R$ consistent with Cunningham’s transfer function approach, and the $r$ dependence could arise, for example, from the disk ionization changing with $r$. The double integral in Equation (2) is commonly replaced by the product of an incident flux times an emission coefficient. A notable exception is García et al. (2014) who account for the $\mu$-dependence of the reprocessed emission. If the corona emission follows a power-law distribution with a photon index $\Gamma$ ($dN/dE \propto E^{-\Gamma}$) or equivalently, with a spectral index $\alpha$ ($E \, dN/dE \propto E^{-\alpha}$ with $\alpha = \Gamma - 1$), the relevant flux is the photon flux $N(> E_{thr})$ above the threshold energy $E_{thr}$ required for the emission of an Fe Kα fluorescent photon. The integral photon flux above $E_{thr}$ scales with $E_{thr}^{-\alpha}$.

As we acquire more and more accurate Fe Kα spectral and timing observations, there is a growing interest in pinning down the physical properties (shape, location, and energy spectra) of the corona by comparing the detailed predictions for specific corona geometries with the observations. For example, Fukumura & Kazanas (2007) study the disk illumination by isotropically and an-isotropically emitting point-like lamppost coronae. Wilkins & Fabian (2012) analyze lamppost coronae on the symmetry axis as well as laterally offset coronae orbiting the symmetry axis, disk-shaped coronae, and lamppost corona moving along the symmetry axis. Dauser et al. (2013) discuss the illumination pattern created by static point-like and radially...
elargated static and accelerating coronas (see also Gonzalez et al. 2017).

All of these studies start with a determination of the flux \(F_X\) impinging on the accretion disk. In Section 2 of this paper, we review the standard argument of how to convert the radial distribution of the corona photons (derived from integrating the geodesic equations) into the photon flux per proper time and proper area. We present a much simpler derivation, obtain a much simpler equation, and clarify the relation between our results and previous results. We close with a brief summary and outlook in Section 3. Although we limit the following discussion to general relativity’s Kerr metric in Boyer–Lindquist (BL; Boyer & Lindquist 1967) coordinates, the results can easily be adapted to work for other stationary, axisymmetric, and asymptotically flat black hole spacetimes (see Johannsen 2016 and Bambi 2017, for recent reviews describing observational constraints on alternative black hole spacetimes).

We use geometric units \((G = c = 1)\) and express all distances in units of the gravitational radius \(r_g = GM/c^2\) with \(M\) being the mass of the black hole. \(a\in[-1,1]\) denotes the black hole angular momentum per unit mass.

### 2. The Coronal Photon Flux Impinging on the Accretion Disk Per Proper Time and Proper Area

In terms of the BL coordinates \(x^\mu = (t, r, \theta, \phi)\), the Kerr metric is given by

\[
d\lambda^2 = g_{00} dt^2 + 2g_{03} dt d\phi + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2
\]

with \(g_{00} = -(1-2r/\Sigma),\ g_{03} = -2ar \sin^2\theta/\Sigma,\ g_{11} = \Sigma/\Delta,\ g_{22} = \Sigma,\ \text{and}\ g_{33} = (r^2 + a^2 + 2a^2 r \sin^2\theta/\Sigma) \sin^2\theta,\ \Sigma = r^2 + a^2,\ \Delta = r^2 - 2r + a^2\).

We introduce two observers, the Zero Angular Momentum Observer (ZAMO, subscript \(Z\)) and the Keplerian observer (KO, subscript \(K\)) orbiting the black hole with the angular frequencies of \(\omega = -g_{03}/g_{33} = 2a/(r^3 + a^2 r + 2a^2)\) and \(\Omega = \pm(3^{3/2} + a)^{-1}\), respectively. The upper sign refers to direct orbits, and the lower sign refers to retrograde orbits. In the following, we limit the discussion to direct orbits. As in standard thin disk theory, we assume that the accretion disk plasma orbits the black hole on Keplerian orbits, and that the KO frame is thus the rest frame of the accretion disk plasma.

The four velocities of the two observers are given by \(u_Z\) and \(u_K\) with the contravariant components (Bardeen et al. 1972) of

\[
u^\mu_Z = u_0^Z (1, 0, 0, \omega)
\]

\[
u^\mu_K = u_0^K (1, 0, 0, \Omega).
\]

Normalizing the four velocities to \(-1\) gives the zero components

\[
u^0_Z = \sqrt{(r^2 + a^2 + 2a^2 r/2r)}/(r^2 - 2r + a^2)\]

\[
u^0_K = (a + r^{3/2})/\sqrt{r^3 - 2r^2 + 2a^{3/2}}.
\]

For an observer moving with four velocity \(u\), we associate a tetrad \((a\) system of orthogonal and normalized basis vectors of the tangent vector space (Chandrasekhar 1983, Chapter 7)) by defining a time-like basis vector \(e^\mu_{(0)} = u\) and three space-like basis vectors \(e^\mu_{(i)} = A \partial_i, e^\mu_{(2)} = B \partial_2,\) and \(e^\mu_{(3)} = C \partial_3 + D \partial_s,\) The orthonormality conditions \(e^\mu_{(i)} \cdot e^\mu_{(j)} = \delta^\mu_{j} (i, j = 1 ... 3)\) together with \(e^0_{(0)} \cdot e^\mu_{(i)} = 0\) determine the values of \(A, B, C,\) and \(D\) (see the Appendix for the ZAMO and KO values). Given the tetrad for a point \(P\) and the coordinates of an event in an infinitesimal neighborhood of \(P\) in tetrad coordinates \(x^\nu\), the BL coordinates \(x^\mu\) are given by

\[
x^\nu(x^\mu) = x^\nu(P) + x^\nu(\mu')e_{(\mu')},
\]

where we sum over \(\mu'\). The relation gives the transformation matrix for tangent vectors \(\Lambda^\mu_{\nu'} = e^\nu_{(\mu')}\) and we obtain the inverse transformation by matrix inversion.

#### 2.1. Traditional Derivation

In the following, we refer to the photon flux hitting the accretion disk per proper time and proper area as \(f\) and the corresponding energy flux as \(F\). The standard way of determining \(f\) or \(F\) consists of integrating the geodesics of photons leaving the corona to determine \(N(r, dr)\), the number of photon trajectories intersecting the equatorial plane with a radial coordinate between \(r\) and \(r + dr\) per considered time interval, and calculating \(f\) with an equation of the form (e.g., Wilkins & Fabian 2012; Dauser et al. 2013; Gonzalez et al. 2017)

\[
f(r) = \frac{8\pi K}{\gamma_{ZK} A_Z(r, dr)} N(r, dr).
\]

Here, \(A_Z(r, dr)\) is the proper area of an accretion disk ring extending from \(r\) to \(r + dr\) as measured by a ZAMO.

\[
A_Z(r, dr) = \left(\int_0^{2\pi} \left| \frac{\partial(x^\nu, x^3)}{\partial(r, \phi)} \right| d\phi \right) dr
\]

\[
= 2\pi \sqrt{g_{11} g_{33}} dr = 2\pi \sqrt{r^2 + a^2 r^2 + 2a^2 r} dr.
\]

The factor \(\gamma_{ZK}\) is the Lorentz factor of the KO relative to the ZAMO, and describes how much larger the ring area is for the (faster moving) KO than for the ZAMO. We obtain the \(\phi\)-component of the relative velocity and the corresponding Lorentz factor with the equations

\[
v_0 = \frac{u_K \cdot e_{(3)}}{-u_K \cdot e_{(0)}}
\]

\[
= \frac{r^{3/2} - 2ar + a^2 \sqrt{r}}{(r^{3/2} + a) \sqrt{r^3 - 2r^2 + a^{3/2}}}.
\]
and 1 over the global BL
Figure 3. of the area¢
depend on the coordinates and the constants of motion, and
The energy
symmetry axis. As
being the photon
with
E
parameterized as follows
Equation
. The factor transforms the
consider photons emanating from the lamppost corona for a
time
in BL coordinate time, and consider that the accretion
disk ring from
r to
r +
dr is made of infinitesimally small area elements. For each area element the exposure
E of the area
element to corona photons equals the product of the proper
time
dx0 and the proper area
dx1dx2.
The dashed coordinates refer to the coordinates of a KO corotating with the accretion
disk plasma.

Figure 1 shows
νφ and γZK for a rapidly spinning black hole.
Finally, gLK is the relative change of the photon energy between its arrival at the accretion disk (measured by a
corotating KO) and its emission (measured in the reference
frame of the lamppost corona). The factor transforms the
photon emission rate in the lamppost reference frame to the
photon emission rate observed by a KO. The ratio can be calculated by
projecting the photon’s four velocity onto the four velocity of the
disk plasma and lamppost corona, respectively, as

\[ g_{LK} = \frac{u_K \cdot p_{\gamma,K}}{u_L \cdot p_{\gamma,L}}, \]  

where \( u_K \) is the four velocity of the KO from Equation (5), and

\[ u_L^\mu = ( (g_{00})^{-1/2}, 0, 0, 0) \]

is the four velocity of the lamppost corona with the subscript L

denoting the evaluation at the lamppost position. The four
momentum of a null geodesic of the Kerr spacetime can be parameterized as follows (Chandrasekhar 1983, Section 62, Equation (187))

\[ p_{K,\mu} = (E, \sqrt{R}/\Delta, \sqrt{\Theta}, -L_\Sigma), \]

with \( E \) being the photon energy, \( R \) and \( \Theta \) being functions that depend on the coordinates and the constants of motion, and \( L_\Sigma \) being the photon’s angular momentum with respect to the
symmetry axis. As \( L_\Sigma \) vanishes for photons from a lamppost
corona, \( g_{LK} \) is given by

\[ g_{LK} = \frac{u_K^0}{u_L^0}. \]

Figure 2 shows \( g_{LK} \) for a rapidly spinning black hole.
Combining the results and simplifying considerably, we
obtain

\[ f(r) = \frac{1}{2\pi r} \sqrt{1 - \frac{2h}{a^2 + h^2}} \frac{dN(r, dr)}{dr} . \]

The energy flux per proper time and proper area is obtained by
multiplying \( f(r) \) with the photon energy in the lamppost frame

\[ F(r) = g_{LK} E_L f(r). \]

2.2. Alternative Derivation

We refer to Figure 3 for an alternative derivation. We consider photons emanating from the lamppost corona for a
time \( dt \) in BL coordinate time, and consider that the accretion
disk ring from \( r \) to \( r + dr \) is made of infinitesimally small area
elements. For each area element the exposure \( E \) of the area
element to corona photons equals the product of the proper
time \( dx^0 \) and the proper area \( dx^1dx^2 \). The dashed coordinates refer to the coordinates of a KO corotating with the accretion
disk plasma.

We find the exposure of all area elements of the ring by
integrating the locally defined exposure \( E \) over the global BL
coordinate \( \phi \) using the Jacobian for the transformation from
\((x^0, x^1, x^3)\) to \((t, r, \phi)\) to express \( dx^0dx^1x^3 \) in terms of \( dt, dr \), and \( d\phi \) as

\[ E = \left( \int_0^{2\pi} \frac{\partial(x^0, x^1, x^3)}{\partial(t, r, \phi)} d\phi \right) dr dt \]

\[ = \left( \int_0^{2\pi} \sqrt{-g_{00}} d\phi \right) dr dt = 2\pi r dr dt. \]

The second line follows from the general result that the
Jacobian is given by the square root of the ratio of the metric
determinants \( \sqrt{g_{00}} / g_{013} \) with \( g_{00} = -r^2 \) being the determinant
of the \( t, r \), and \( \phi \) part of the metric in BL coordinates, and
\( g_{013} = -1 \) being the determinant of the 0, 1, and 3 part of the
metric in the KO coordinates (e.g., Poisson 2004, Section 1.7).

Dividing the number of photons \( N(r, dr) \) impinging on the
accretion disk ring from \( r \) to \( r + dr \) in the coordinate time
interval \( dt \) by the exposure, we obtain the photon flux per
proper time and proper area as

\[ f_2(r) = \frac{1}{2\pi r} \frac{dN(r, dr)}{dr}. \]

Equation (18) is considerably simpler than Equation (7). Somewhat amusingly, discussing Equation (7) Wilkins & Fabian (2012, p. 1287) noted that “the general relativistic
effects on the area of orbiting annuli in the accretion disc are
cancelled exactly by one factor of the redshift.” Indeed, the \( g_{LK} \)
in the numerator of Equation (7) approximately cancels the $\gamma_{ZK}$ in the denominator. This cannot be an exact cancellation as $g_{LK}$ depends on the height of the lamppost while $\gamma_{ZK}$ does not.

What is the relation between Equations (7) and (18)? The attentive reader will anticipate the answer: whereas the latter equation converts the number of photons $N(r, dr)$ impinging on the ring per unit coordinate time into the photon flux per unit proper time and unit proper area, the former converts the number of photons $N(r, dr)$ impinging on the ring per unit lamppost time into the photon flux per unit proper time and unit proper area. Indeed, when converting $f_2$ from per coordinate time into per lamppost time, we recover $f$ as

$$f_2(r) \frac{dt}{dl_c} = f_2(r) \frac{1}{u_L} = f_2(r) \sqrt[2]{-(g_{00})_L}.$$  

$$= f_2(r) \sqrt{1 - \frac{2h}{a^2 + b^2}} = f(r).$$ (19)

The proportionality factor $\sqrt[2]{-(g_{00})_L}$ is shown in Figure 4. $f$ is lower than $f_2$ as a certain photon emission rate in the lamppost frame corresponds to a lower rate measured in the Boyer–Lindquist coordinate time.

3. Discussion

In this paper, we discuss equations for the determination of the photon flux per proper time and proper area impinging on the accretion disk of a black hole in the Kerr metric. Our discussion clarifies the assumptions underlying the derivation of the standard equations used in the literature. In particular, the quantity $N(r, dr)$ in Equation (7) refers to the rate of photons impinging on an accretion disk ring extending from $r$ to $r + dr$ per unit lamppost time. We present a conceptually simpler derivation that does not require the transformation into the auxiliary ZAMO reference frame and relies on the concept of a relativistically invariant proper exposure (product of exposure time and area).

In the near term, *Chandra*, *XMM-Newton*, *Swift*, and *NuSTAR* will continue to give new spectroscopy and timing Fe Kα data. The results derived here can be used to calculate the Fe Kα emissivity via Cunningham’s transfer function. Around 2020, NASA’s and ESA’s *Imaging X-ray Polarimetry Explorer* (Weisskopf et al. 2016) will add qualitatively new information. Interestingly, this will require modelers to update their existing codes, as Cunningham’s transfer function does not account for the change of the polarization direction along the photon geodesics.

Our current work focuses on deriving the emissivity profiles for a number of 3D corona models, and compares the predictions with a large body of observational timing and spectral results (B. Beheshtipour et al. 2017, in preparation).

We thank NASA (grant #NNX14AD19G) for financial support, and the anonymous referee for excellent comments.

Appendix

The constants appearing in the definition of the ZAMO tetrad are given by

$$A = g_{11}^{-1/2}, \quad B = g_{22}^{-1/2}$$

$$C = 0, \quad D = (r^2 + a^2 + 2a^2/r)^{-1/2}.$$  

For the Keplerian observer the constants read as

$$A = g_{11}^{-1/2}, \quad B = g_{22}^{-1/2}$$

$$C = \sqrt{r^2 - 2\sqrt{r}a + a^2 - a^2/2}$$

$$D = \frac{r^{3/2} - 2\sqrt{r}a + a}{\sqrt{r^3 - 3r^2 + 2ar^{3/2}(r^2 - 2r + a^2)}}.$$  

ORCID iDs

H. Krawczynski @ https://orcid.org/0000-0002-1084-6507  
B. Beheshtipour @ https://orcid.org/0000-0002-8524-1537

References

Bambi, C. 2017, *RvMP*, 89, 025001  
Bardeen, J. M., Press, W. H., & Teukolsky, S. A. 1972, *ApJ*, 178, 347  
Boyer, R. H., & Lindquist, R. W. 1967, *JMP*, 8, 265  
Chandrasekhar, S. 1960, *Radiative Transfer* (New York: Dover)  
Chandrasekhar, S. 1983, *The Mathematical Theory of Black Holes* (New York: Oxford Univ. Press)  
Cunningham, C. T. 1975, *ApJ*, 202, 788  
Dauser, T., Garcia, J., Wilms, J., et al. 2013, *MNRAS*, 430, 1694  
Fukumura, K., & Kazanas, D. 2007, *ApJ*, 664, 14  
García, J., Dauser, T., Lohfink, A., et al. 2014, *ApJ*, 782, 76  
Gonzalez, A. G., Wilkins, D. R., & Gallo, L. C. 2017, *MNRAS*, 472, 1932  
Johannsen, T. 2016, *CQGra*, 33, 124001  
Miller, M. C., & Miller, J. M. 2015, *P	extsc{r}	extsc{i}	extsc{r}	extsc{r}	extsc{i}	extsc{r}	extsc{i}, 548, 1  
Poisson, E. 2004, *A Relativist’s Toolkit* (Cambridge: Cambridge Univ. Press)  
Reis, R. C., & Miller, J. M. 2013, *ApJL*, 769, L7  
Reynolds, C. S. 2014, *SSRv*, 183, 277R  
Speith, R., Riffert, H., & Ruder, H. 1995, *CoPhC*, 88, 109  
Thorne, K. S. 1974, *ApJ*, 191, 507  
Weisskopf, M. C., Ramsey, B., O’Dell, S., et al. 2016, *Proc. SPIE*, 9905, 990517  
Wilkins, D. R., & Fabian, A. C. 2012, *MNRAS*, 424, 1284