On Genera of Curves from High-loop Generalized Unitarity Cuts

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\textbf{Abstract:} Generalized unitarity cut of a Feynman diagram generates an algebraic system of polynomial equations. At high-loop levels, these equations may define a complex curve or a (hyper-)surface with complicated topology. We study the curve cases, i.e., a 4-dimensional $L$-loop diagram with $(4L - 1)$ cuts. The topology of a complex curve is classified by its genus. Hence in this paper, we use computational algebraic geometry to calculate the genera of curves from two and three-loop unitarity cuts. The global structure of degenerate on-shell equations under some specific kinematic configurations is also sketched. The genus information can also be used to judge if a unitary cut solution could be rationally parameterized.

\textbf{Keywords:} Generalized unitarity cut, Loop amplitudes, Computational algebraic geometry
1 Introduction

Systematic approach to the multi-loop scattering amplitude study is now on the road, based on the fruitful progresses of tree and one-loop amplitude computations in the past a few years. After struggling with complicated calculations from Feynman diagrams for decades, the changes started with a new way of computing tree amplitudes. By complexifying the amplitude through certain momentum shifting on the complex plane, the amplitude becomes an analytic function of single complex variable $A(z)$ with simple poles. The physical amplitude, defined as the amplitude at $z = 0$ of the complex plane, is then calculated by the residues of $A(z)$. The locus of residues are the phase space points where propagators are on-shell. Thus, the amplitude can be obtained by considering only on-shell diagrams with lower-point tree amplitudes as the input via Britto-Cachazo-Feng-Witten(BCFW) recursion relation [1, 2]. This point of view of scattering amplitude not only provides an efficient calculation method, but also greatly deepens our understanding of gauge field theories [6].

The breakthrough in tree amplitude calculations also inspires progresses in loop amplitude calculations. Decades ago, the unitarity cut method [3] has already been used to...
compute one-loop amplitude from tree amplitude [4, 5]. After the finding of BCFW recursion relation, this method is applicable to practical calculations [7, 8]. The difficulty is resolved because of the simple and compact tree amplitudes produced by the new method, so the focus of unitarity cut method is switched to the study of unitarity cut information.

Take an one-loop amplitude as example, schematically it can be expanded on some one-loop integral basis [9],

$$A^{1\text{-loop}} = \sum_{i \in \text{basis}} c_i I_i + \mathcal{R}. \quad (1.1)$$

The basis $I_i$’s are scalar integrals, i.e., integrals whose numerators of the integrand are 1. They can be used universally for any one-loop amplitudes in renormalizable theories. Expansion coefficients $c_i$ and the remainder $\mathcal{R}$ are rational functions of external momenta. By cutting both sides of 1.1, i.e., computing the branch-cut discontinuities across various kinematical channels, we can get equalities between expressions in both sides, which are products of tree amplitudes. The coefficients can be determined by comparing expressions in both sides. Now the one-loop amplitude calculation is simple, because there are only finite number of integrals in the basis. For example, in 4-dimensional theory, the integral basis $I_i$ can only be box, triangle, bubble or tadpole topologies, and the numerators are always constants. The solution space defined by the unitarity cut of these diagrams is simple: it is just the solution of one quadratic equation. In this sense, we can easily find a parametrization for the loop momentum, and work out the expansion coefficients $c_i$ systematically from quadruple, triple, double and single cuts. Another way of extracting coefficients $c_i$ is the generalized unitarity cut method [10, 11], which uses the information of cuts and also contour integration.

The main concept of loop-amplitude calculations by unitarity-cut method is to expand the amplitude onto some known integral basis and then find the coefficients. So first of all, we need a set of integral basis. The basis should be large enough so that it is complete for expanding every loop amplitude, yet it should be as small as possible to simplify the calculation. It is somehow difficult to find the integral basis for multi-loop higher-point amplitudes. A conventional way is using Integrate-By-Parts(IBP) method [12] to find relations among different integrals. Sometimes the IBP calculation can be heavy, so instead we can define a set of integrand basis before integration. The denominator of the integrand is a product of propagators. The numerator is a polynomial of Lorentz Invariant Scalar Product (ISP). Ossola-Papadopoulos-Pittau(OPP) decomposition method [13] defines the integrand basis and the corresponding reduction scheme in the one-loop level. This method has been successfully applied to one-loop amplitude calculations, where the amplitude is decomposed into some Master Integrals(MIs) plus the rational terms [14–19].

It still takes some time to generalize above methods to multi-loop amplitude calculations beyond the one-loop level calculation. The difficulty is obvious: the basis of multi-loop amplitudes is far more complicated than those of one-loop amplitudes. Some works have been done on the integral basis of diagrams such as two-loop double-box using IBP method [20] or generalized unitarity cut method [21–26]. By the latter method, the master contours, which produce the coefficients of double-box integrals in the basis decomposition,
are uniquely defined [23]. For more complicated diagrams, the study of integral basis is very complicated, thus we can consider the integrand basis first. After works in [27, 28], systematic study of integrand basis begins with the introduction of computational algebraic geometry method, for example, Gröbner basis, to multi-loop amplitude calculations [29, 30]. This method can determine a small enough yet sufficient set of the integrand basis, and also analyze the solution space of unitarity-cut equations systematically for two-loop [31–33] and also three-loop amplitudes [34].

The main objects we get, after applying the cuts on loop amplitudes, are unitarity-cut equations. These equations come from setting corresponding propagators on-shell, i.e., $D_i = \ell_i^2 - m_i^2 = 0$, where $i$’s ranges from all propagators being cut, $\ell_i$ is the loop momentum and $m_i$ is the mass of propagator. In this paper, we consider only massless theories, so $m_i = 0$.

The studies on unitarity-cut equations are crucial for the application of unitarity methods. For example, the unitarity-cut equations for $4D$ massless four-point double-box diagram contain six irreducible components, and all the six components should be parameterized to find the constraints for contours [21]. Furthermore, Simon and Kasper in [23] study the two-loop double-box diagram with six massive external legs, and find that the unitarity cut equations define an elliptic curve, i.e. a genus-one curve. This solution space can be parameterized by elliptic functions, but not rational functions. The structure of unitarity cuts for all other kinematics of $4D$ double-box can be easily illustrated via the degeneracy of the elliptic curve [23].

Unitarity cut equations may be simple, if the order of loop is low and the number of external legs is small. In these cases, one-variable complex analysis is sufficient for the analysis of unitarity cut equations. However, for high-loop diagrams or diagrams with many legs, the analysis would be complicated. Since the cut equations are all polynomial equations with complex coefficients, the natural tool to study them is algebraic geometry.

The generalized unitarity cut of $L$-loop diagrams may define discrete points, complex curves or (hyper-)surfaces. All of them can be studied by algebraic geometry methods. In this paper, we consider the simplest non-trivial cases: the curves. More explicitly, we consider the cut equations of $L$-loop diagrams with $4L-1$ propagators in 4-dimensional theory. Since the topology of a complex curve is completely characterized by its genus, so we study the genera explicitly for the curves defined by generalized unitarity cut from one-loop to three-loop level, i.e. $L = 1, 2, 3$. The same mathematical approach should work for even higher-loop diagrams. However, the computation would be more involved.

There are many ways to analyze genus and describe the global structure of multi-loop unitarity cuts, by algebraic geometry. In this paper, we mainly use the following methods,

- The relation between arithmetic genus and geometric genus. In practice, we birationally project the curve to a plane curve. For the latter one, the arithmetic genus can be easily found, and then we use the relation between arithmetic genus and geometric genus to find the genus of the original curve.
- Riemann-Hurwitz formula. For a projection of a curve onto another one, this formula gives a relation between the genera of the two curves.
The genus of a curve is a birational invariant, i.e., invariant under rational re-parametrization. Furthermore, if equations of unitarity cuts generate a genus-zero curve, then we can find a rational parametrization for the solution space. However, if the genus is larger than zero, the solution space cannot be described by rational parameters no matter how we choose the coordinates. So information of genus is the first judgement for the difficulty of an unitarity cut computation.

Given a diagram, we first consider the kinematics with the maximal number of massive external legs. We call this case the prime case. Its unitarity cut sometimes gives high-genus curves. Then we consider the degenerate cases, i.e., the same diagram with fewer external legs or with several massless external legs. Then usually we get a reducible curve, which is the union of several irreducible curves. Each curve’s genus is lower than the one of the prime case. For the double-box case in [23], we show again that the global structure of unitarity cut is given by the prime case, i.e., if we find the intersection of curves from the degenerate case and “sew” them together, we get the topology of the prime case. In this sense, the genus information is the guideline for the global structure of unitarity cut solutions for all degenerate kinematics.

Throughout this paper, we find that the genus of high-loop unitarity cuts can be higher than 1. For example, for a generic non-planar two-loop crossed-box diagram with 6 massive momenta, the on-shell equations of unitarity cuts generate a complex curve with genus 3.

The paper is organized as follows. In section 2, we present some mathematical background, where basic concepts of algebraic curves, such as genus and singular points, are introduced. In section 3,4 and 5, we study the genera of algebraic curves from one, two and three-loop diagrams respectively. Detailed analysis is given for two-loop diagrams, and some typical diagrams of three-loop are also carefully studied. Discussions and comments are presented in the conclusion.

### 2 Mathematical preliminaries

In this section, we review the definition and computation of the genus of a complex algebraic curve. For detailed mathematical proofs, please refer to [35] [36].

There are two types of genera for complex curves, arithmetic genus and geometric genus. The topology of a complex curve is characterized by its geometric genus. Roughly speaking, the geometric genus is the number of handles of a complex curve. (The rigorous definition of the two genera will be given in this section.) Throughout the paper, if not specified, “genus” means geometric genus.

We calculate the geometric genus of a complex curve directly by computational algebraic geometry. The strategies are,

- Calculate the arithmetic genus of a complex curve. This is easily done by polynomial computations, which is automated by the computational algebraic geometry softwares, for example, 'Macaulay2' [37]. If the curve is non-singular, then the arithmetic genus equals the geometric genus and we are done. Otherwise, we project the curve onto to a plane curve and re-calculate the arithmetic genus. For the plane
curve, the geometric genus is simply related to the arithmetic genus by singular-point counting and the blow-up process.

- Alternatively, project the curve onto a known curve with smaller genus. This projection needs not to be birational. Then we use Riemann-Hurwitz formula to determine the genus of the first curve.

2.1 Projective algebraic curves and the arithmetic genus

Consider a complex curve $C$ in $\mathbb{C}^n$, defined by polynomial equations

$$f_1(z_1, \ldots z_n) = \ldots = f_k(z_1, \ldots z_n) = 0 .$$

The polynomials $f_1, \ldots f_k$ generate an ideal $J = \langle f_1, \ldots f_k \rangle$. In the formal language, $C$ is the zero locus of $J$.

To discuss its topology, we need to extend $C$ to a projective curve $C$ in $\mathbb{P}^n$. Define homogenous coordinates by $z_i = Z_i/Z_0$, $C$ is defined by homogenous polynomials

$$Z_0^{\deg f} \cdot f(Z_1/Z_0, \ldots Z_n/Z_0) \equiv F(Z_0, \ldots, Z_n) = 0 , \ \forall f \in J .$$

All such $F$’s generate a homogenous ideal $I$. In practice, to get $I$, we do not need to find the homogenous form of all polynomials in $J$. Instead, it is sufficient to consider the homogenous form of polynomials in the Gröbner basis of $J$, in a graded monomial order.

Furthermore, we assume that $C$ is irreducible, i.e., cannot be written as the union of two different curves. Equivalently, it means that $I$ is a prime ideal. For the cases of reducible curves, we consider the genus of each irreducible component. Irreducible components can be found by primary decomposition,

$$I = \bigcap_{i} I_i , \ i = 1, \ldots, k ,$$

where all $I_i$’s are prime ideals. Primary decomposition technique has been used in high-loop unitarity cuts.

The arithmetic genus can be defined by the Hilbert polynomial of the quotient ring $S/I$, where $S = \mathbb{C}[Z_0, Z_1, \ldots Z_k]$. $S/I$ is a graded $S$-module,

$$S/I = \bigoplus_{i=0}^{\infty} S_i .$$

There exists a unique polynomial $P(x)$ and a positive integer $N$ such that, for $n > N$, $n \in \mathbb{Z},$

$$P(n) = \dim_{\mathbb{C}} S_n .$$

This polynomial is called Hilbert polynomial (of the projective curve $C$). Then the arithmetic genus of $C$ is defined to be

$$g_A(C) \equiv -P(0) + 1.$$
In practice, the Hilbert polynomial and arithmetic genus can be easily calculated by Gröbner basis method. In particular, for a plane curve, i.e. a curve in $\mathbb{CP}^2$, defined by one homogenous polynomial with degree $d$, the arithmetic genus is simply
\[ g_A = \frac{1}{2}(d - 1)(d - 2) \quad (2.7) \]

### 2.2 Singular points and the geometric genus

The arithmetic genus is not directly related to the topological properties of a curve, since intuitively it counts not only the number of handles but also the singular points. A singular point on $C$ is a point $(a_0, a_1, ..., a_n)$ such that the rank of the Jacobian matrix,
\[
\left| \frac{\partial F_i}{\partial Z_j}(a_0, ..., a_n) \right|, \quad 1 \leq i \leq k, \quad 0 \leq j \leq n, \quad (2.8)
\]
is less than $n - 1$ [35], where $F_i$, $1 \leq i \leq k$ are the generators of equations for the curve. A singular point on $C$ is normal if all tangent lines on the singular point are distinct.

For a smooth curve, i.e., an irreducible projective curve without a singular point, we define the geometric genus to be its arithmetic genus [35]. For an irreducible projective curve $C$ with singular points, there exists a smooth irreducible projective curve $\tilde{C}$, which is the normalization of $C$ [35],
\[
\pi : \tilde{C} \to C. \quad (2.9)
\]

We define the geometric genus of $C$ to be the arithmetic genus of $\tilde{C}$. This definition is consistent with the topological definition of the genus. In particular, for an irreducible plane curve with normal singular points only, the geometric genus is related to the arithmetic genus as
\[
g_G = g_A - \sum_{P \in \text{Sing}(C)} \frac{1}{2} \mu_P(\mu_P - 1) \quad (2.10)
\]
\[
= \frac{1}{2}(d - 1)(d - 2) - \sum_{P \in \text{Sing}(C)} \frac{1}{2} \mu_P(\mu_P - 1), \quad (2.11)
\]
where $\text{Sing}(C)$ is the collection of all singular points $P$ on $C$. $\mu_P$ is the multiplicity of $P$, i.e., the number of tangent lines at $P$. In practice, $\text{Sing}(C)$ can be found by computational algebraic geometry softwares, like Macaulay2 [37].

We may frequently re-parametrize a curve, and intuitively, the topological information should be invariant after re-parametrization. To realize the re-parametrization process rigorously, we introduce the concept of birational map: A rational map $r$ from the irreducible curves, $C_1$ and $C_2$, is a map from a non-empty open set of $C_1$ to $C_2$, such that, in terms of the coordinates, all components of $f$ are rational functions. A birational map $f$ between two irreducible curves is a rational map $f$ from $C_1$ to $C_2$ with an inverse rational map $g$ from $C_2$ to $C_1$, such that $gf$ is the identity map on an open set of $C_1$ and $fg$ is the identity map on an open set of $C_2$. The geometric genus is a birational invariant [35].

In particular, since $\mathbb{CP}^1$ has the genus zero, a project curve with $g_G > 0$ is not birational to $\mathbb{CP}^1$. It implies that there is no rational parametrization for such a curve.

We present some simple examples on the genera of algebraic curves:
1. \( y^2 = x^3 + 1 \). This is a cubic plane curve. The corresponding projective curve is \( Y^2Z = X^3 + Z^3 \). There is no singular point on the projective curve. The arithmetic genus is \( (3 - 2)(3 - 1)/2 = 1 \) and the geometric genus is also 1. So this is an elliptic curve.

2. \( y^2 = x^3 + x^2 \). This is also a cubic plane curve. The corresponding projective curve is \( Y^2 Z = X^3 + X^2 Z \). There is a singular point at \( X = 0, Y = 0, Z = 1 \) and there are two distinct tangent lines at this point: \( x - y = 0 \) and \( x + y = 0 \). So this is an ordinary singular point with the multiplicity 2. The arithmetic genus is still 1 but the geometric genus is \( 1 - 2(2 - 1)/2 = 0 \). So this is not an elliptic curve. We can blow up the singular point at \( (x, y) = (0, 0) \) as follows: define a new variable \( t \) via \( y = xt \), and the equation becomes \( -x^2(1 - t^2 + x) = 0 \). Factorize it and keep the curve part, we get a new curve \( 1 - t^2 + x = 0 \). This new curve is birational to the original curve, and is smooth. Since the new curve is conic, so we show that the original curve is birationally equivalent to a conic and can be rational parameterized.

3. The projective curve,

\[
 XT - YZ = 0, \quad X^2Z + Y^3 + YZT = 0, \quad XZ^2 + Y^2T + ZT^2 = 0
\]  

(2.12)

is non-singular [36]. From Macaulay2, we find that the arithmetic genus is 2, so its geometric genus is also 2.

There are many different ways to calculate the genus of an algebraic curve. In this paper, for the algebraic curves from unitarity cuts in two-loop and three-loop orders, we mainly use the relation between arithmetic genus and geometric genus for the computation. Explicitly, if the curve is non-singular, then we simply calculate its arithmetic genus and by the definition, it is the geometric genus. Otherwise, we birationally project the curve onto a plane, and then use (2.10) to determine the genus if all singular points are normal. If some singular points are not normal, we use the blow up process to resolve them, as described in [36].

Another useful formula for curve genus computation is Riemann-Hurwitz formula [35]. Let \( f : C_1 \to C_2 \) be a covering map between two irreducible complex curves \( C_1 \) and \( C_2 \), and \( n \) is the degree of \( f \). Then \( g(C_1) \) and \( g(C_2) \), the genera of \( C_1 \) and \( C_2 \) are related by

\[
 2g(C_1) - 2 = n \cdot (2g(C_2) - 2) + \sum_{P \in C_1} (e_P - 1) ,
\]

(2.13)

where \( e_P \) is the ramification index of \( P \). For all but finite points on \( C_1 \), the ramification index equals one so it is just a finite sum. Therefore if \( g(C_2) \) is known, we can get \( g(C_1) \) by counting the ramified points on \( C_1 \).

3 Genera of algebraic curves from one-loop diagrams

For one-loop diagrams, there is only one loop momentum, and it has 4 components. Diagrams with 3 propagators after maximal unitarity cut define algebraic curves. The only
diagram we should consider is the triangle diagram. The propagators are given by

\[ D_0 = \ell^2, \quad D_1 = (\ell - p_1)^2, \quad D_2 = (\ell - p_1 - p_2)^2, \quad (3.1) \]

where external momenta at each vertices are \( p_1, p_2, p_3 \), and we assume they are all massive. We can get 2 polynomials \( L_1 = D_0 - D_1 \), \( L_2 = D_0 - D_2 \), which are linear in \( \ell \). If we use 4 variables \( (x_1, x_2, x_3, x_4) \) to expand the loop momentum, then \( L_1 = L_2 = 0 \) are 2 linear equations in them. By solving these equations and substituting solutions back to \( D_0 \), we get a quadratic polynomials of 2 variables. The resulting equation \( D_0 = 0 \) is then a conic section, and topologically equivalent to a genus-0 Riemann sphere.

If some of the external momenta are massless, the on-shell equations become degenerate. The topological picture can be given by combing another Riemann sphere from the original one, and the result is that two Riemann spheres are connected at a single point. This agrees with the analysis of intersection pattern in [31].

4 Genera of algebraic curves from two-loop diagrams

In this section, we calculate the genera of all curves from the 4D two-loop unitarity cuts, for massless theories.

For 4-dimensional two-loop diagrams, there are 8 components from two loop momenta. The 8 variables associated to the 8 components of loop momenta could be defined as the expansion coefficients of loop momenta on a chosen momentum basis. They can also be defined by Lorentz invariant scalar product of loop momentum and external momentum. The resulting propagators, which expressed as polynomials of these 8 variables, will be different according to the definitions of momentum basis and variables. But in general, they are always quadratic polynomials and only differ by a linear transformation.

If we apply the maximal unitarity cut on the diagram, all propagators become on-shell, then we get a polynomial equation system. These equations define an algebraic set, i.e. the solution set of cut equations in the space spanned by 8 variables. When considering diagrams with 7 cuts, the solution set is described by 7 equations with respect to 8 variables. It is an algebraic curve.

It is not easy to analyze this curve directly from its defining polynomial equation system. So the first step is to eliminate as many variables as possible from the linear equations which can be constructed from 7 on-shell equations. In practice, it is always possible to obtain 4 linear equations. By solving them, we can express 4 variables as linear combinations of the remaining 4 variables. The resulting 3 quadratic polynomial equations of 4 variables is isomorphic to the original algebraic set. Furthermore, by coordinate transformations, we can rewrite the 3 equations as a single meromorphic function of 2 variables. It defines a plane curve, which is birationally equivalent to the original algebraic set. Though the explicit definitions of curves in above three descriptions are different, the topological invariant objects remain the same. So we can get a unique result for the genus.
Figure 1. Two-loop diagrams with 7 propagators: (a) planar pentagon-triangle diagram, (b) planar double-box diagram, (c) non-planar crossed-box diagram. All external momenta are out-going and massive. The loop momenta are denoted by $\ell_1, \ell_2$.

4.1 Two-loop planar pentagon-triangle diagram, genus 0

As a warm-up, we first analyze this simple diagram, in Figure 1.a). The 7 propagators are given by

\[
D_0 = \ell_2^2, \quad D_1 = (\ell_1 - p_1)^2, \quad D_2 = (\ell_1 - p_1 - p_2)^2, \quad D_3 = (\ell_1 - p_1 - p_2 - p_3)^2, \\
\tilde{D}_0 = \ell_2^2, \quad \tilde{D}_1 = (\ell_2 - p_4)^2, \quad \hat{D}_0 = (\ell_1 + \ell_2 + p_5)^2. 
\] (4.1)

There are 4 propagators containing $\ell_1$ only, and from the on-shell equations $D_i = 0$, we can completely fix the variables in $\ell_1$. These equations $D_i = 0, i = 1, 2, 3, 4$ have two solutions. For each solution, since $\ell_1$ has already been completely fixed, we can treat it as a four-vector similar to an external momentum. Thus the on-shell propagators highlighted by dashed lines in Figure 1.a) can be effectively treated as massless external momenta. Then for each solution, the result of two-loop planar pentagon-triangle diagram is the same as the result of the one-loop triangle diagram. It is associated with a genus-0 Riemann sphere. In some specific kinematic configurations where degeneracy exists, the topological picture becomes that of two Riemann spheres connected at a single point.

4.2 Two-loop planar double-box diagram, genus up to 1

It is already known in [23] that on-shell equations of the two-loop double-box diagram, with six massive legs, define an one-dimensional elliptic curve which is associated with a genus-1 torus.

We can reproduce this result by directly computing the arithmetic genus and singular points of the curve. Taking the convention shown in Figure 1.b), the 7 propagators can be written as,

\[
D_0 = \ell_2^2, \quad D_1 = (\ell_1 - p_1)^2, \quad D_2 = (\ell_1 - p_1 - p_2)^2, \\
\tilde{D}_0 = \ell_2^2, \quad \tilde{D}_1 = (\ell_2 - p_4)^2, \quad \hat{D}_0 = (\ell_1 + \ell_2 + p_5)^2, 
\] (4.2)

In order to compute the arithmetic genus and singular points, we first rewrite above equations by expanding all momenta on a momentum basis. A simple analysis is as follows.
Figure 2. Topological pictures of on-shell equations from the two-loop double-box diagram under specific kinematic configurations. The pictures should be understood as complex curves, or two-dimensional real surfaces. (a) the curve is irreducible and the solution set is a torus (b) the curve has 2 irreducible branches, (c) the curve has 4 irreducible branches, (d) the curve has 6 irreducible branches. For general kinematics the curve is genus 1. In degenerate limit, tubes shrink to points along dashed lines. The resulting Riemann surfaces for each branch can only be a sphere. This explains why we get Riemann spheres connected by points and linked in a chain in degenerate limits.

We have variables \((x_1, x_2, x_3, x_4)\) for 4 components of \(\ell_1\) and \((y_1, y_2, y_3, y_4)\) for 4 components of \(\ell_2\). Using on-shell equations \(D_i = 0, \tilde{D}_i = 0\), we can eliminate 2 variables of \(x_i\) and 2 of \(y_i\), and get 3 quadratic polynomials \(Q_1 = Q_1(x_1, x_2), Q_2 = Q_2(y_1, y_2), Q_3 = Q_3(x_1, x_2, y_1, y_2)\). This can be done by Yang’s BasisDet package\[29\]. We can further eliminate 2 variables and 2 equations via Gröbner basis method and get a plane curve. This plane curve has degree 8, so the arithmetic genus is

\[
ge_A = \frac{(d-1)(d-2)}{2} = 21 .
\] (4.3)

There are 10 singular points, of which 8 have the multiplicity \(\mu_P = 2\) and 2 have the multiplicity \(\mu_P = 4\), so the geometric genus is

\[
ge_G = g_A - \sum_{P \in \text{Sing}(C)} \frac{1}{2} \mu_P(\mu_P - 1) = 21 - 8 \times 1 - 2 \times 6 = 1 .
\] (4.4)

This result is consistent with that in \[23\].

Another explicit way of getting an equivalent plane curve can be taken as follows. Through coordinate transformation it is easy to rewrite \(Q_1 \rightarrow Q_1' = x_1' x_2' - c_1, Q_2 \rightarrow Q_2' = y_1' y_2' - c_2\), so we can do the following substitution \(x_1' = c_1/x_2', y_1' = c_2/y_2'\) in \(Q_3\). The resulting \(Q_3' = n(x_2', y_2')/d(x_2', y_2')\) is a meromorphic function which is equivalent to original equations, and the numerator \(n(x_2', y_2')\) defines a plane curve. This plane curve \(n(x_2', y_2')\) has the degree 4, so we have \(g_A = 3\). There are 2 singular points of the multiplicity \(\mu_P = 2\), thus we get the geometric genus \(g_G = 3 - 2 = 1\). This result again agrees with that given in \[23\].

The topological pictures of genus of degenerate algebraic curves under specific kinematic configurations can be easily found, because in degenerate limit the tube shrinks to point, and there is only one handle for the torus. The contraction will break the handle, and
we should get genus-0 Riemann spheres connected at points. Equations of each irreducible branch can be obtained by Macaulay2 [37] via primary decomposition. By computing the arithmetic genus and singular points, we found that their geometric genera are indeed zero. The 3 kinds of contraction shown in Figure (2) describe topological pictures of degenerate algebraic curves under all possible specific kinematic configurations. There are 2, 4 or 6 Riemann spheres connected by points along dashed lines where tubes shrink, and linked adjacently into a chain. This is exactly the picture described in [23].

4.3 Two-loop non-planar crossed-box diagram, genus up to 3

For the two-loop non-planar crossed-box diagram, the explicit form of propagators for generic kinematics is very complicated. Abstractly, we write the 7 propagators as,

\[
\begin{align*}
D_0 &= \ell_1^2, \quad D_1 = (\ell_1 - p_1)^2, \quad D_2 = (\ell_1 - p_1 - p_2)^2, \quad \tilde{D}_0 = \ell_2^2, \\
\tilde{D}_1 &= (\ell_2 - p_3)^2, \quad \tilde{D}_0 = (\ell_1 + \ell_2 + p_3)^2, \quad \tilde{D}_1 = (\ell_1 + \ell_2 + p_4 + p_5)^2.
\end{align*}
\] (4.5)

From above propagators, we can at most get 4 polynomials which are linear in \(\ell\),

\[
\begin{align*}
L_1 &= D_0 - D_1 = 2\ell_1 p_1 - p_1^2, \quad L_2 = D_0 - D_2 = 2\ell_1 (p_1 + p_2) - (p_1 + p_2)^2, \\
L_3 &= \tilde{D}_0 - \tilde{D}_1 = 2\ell_2 p_3 - p_3^2, \quad L_4 = \tilde{D}_0 - \tilde{D}_1 = -2(\ell_1 + \ell_2 + p_3) p_4 - p_4^2,
\end{align*}
\]

and 3 polynomials which are quadratic in \(\ell\),

\[
\begin{align*}
Q_1 &= D_0 - \ell_1^2, \quad Q_2 = \tilde{D}_0 = \ell_2^2, \quad Q_3 = \tilde{D}_0 - \tilde{D}_0 = 2\ell_1 \ell_2 + 2(\ell_1 + \ell_2) p_5 + p_5^2.
\end{align*}
\]

If we use momentum basis \(e_1, e_2, e_3, e_4\) to expand loop momenta as,

\[
\ell_1 = x_2 e_1 + x_1 e_2 + x_4 e_3 + x_3 e_4, \quad \ell_2 = y_2 e_1 + y_1 e_2 + y_4 e_3 + y_3 e_4,
\] (4.6)

and the 8 variables are taken as the expansion coefficients \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\), then there will be quadratic terms from products of two loop momenta \(\ell_i \ell_j\), and linear terms from products like \(\ell_i p_j\). Systematically we can generate the Gröbner basis by 7 on-shell equations \(D_i = \tilde{D}_i = 0\), and find all linear terms in the Gröbner basis. We can remove as many variables as possible by solving these linear terms, and express other terms with remaining variables. In this case, we get an equivalent but simplified polynomial equation system. A more explicit way is to solve linear equations \(L_i = 0\), and express 4 variables as linear functions of remaining 4 variables. Substituting solutions back to quadratic polynomial equations \(Q_i = 0\), we get an alternate algebraic set. It is possible to analyze the three quadratic equations by algebraic geometry program such as Macaulay2 [37].

Let us consider the two-loop non-planar crossed-box diagram drawn in Figure (1.c). All external momenta are general, and we construct momentum basis \(e_i\) from \(p_1, p_3\). External

\footnote{Momentum basis \(e_i\) have the following orthogonal and normalization relations \(e_1 e_2 = e_3 e_4 = 1\), otherwise 0. The explicit form can be found by Gram-Schmidt process.}
momenta are expanded as
\[ p_1 = \alpha_1 e_1 + \alpha_1 e_2, \quad p_1 + p_2 = \sum_{i=1}^{4} \alpha_i e_i, \quad p_3 = \beta_1 e_1 + \beta_1 e_2, \]
\[ p_5 = \sum_{i=1}^{4} \gamma_i e_i, \quad p_4 + p_5 = \sum_{i=1}^{4} \gamma_i e_i, \quad p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5), \quad (4.7) \]
where \( \alpha, \beta, \gamma \) are the projection coefficients of momenta on the corresponding momentum basis, for example \( \alpha_{11} = p_1 e_2, \alpha_{12} = p_1 e_1 \). Using this expansion, the four linear polynomials \( L_i \) become
\[ L_1 = 2(\alpha_{11} x_1 + \alpha_{12} x_2 - \alpha_{11} \alpha_{12}), \]
\[ L_2 = 2(\alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3 + \alpha_{24} x_4 - \alpha_{21} \alpha_{22} - \alpha_{23} \alpha_{24}), \]
\[ L_3 = 2(\beta_{11} y_1 + \beta_{12} y_2 - \beta_{11} \beta_{12}), \]
\[ L_4 = -2 \sum_{i=1}^{4} (x_i + y_i)(\gamma_{2i} - \gamma_{1i}) - 2(\gamma_{21} \gamma_{22} + \gamma_{23} \gamma_{24} - \gamma_{11} \gamma_{12} - \gamma_{13} \gamma_{14}), \quad (4.8) \]
and the three quadratic polynomials \( Q_i \) become
\[ Q_1 = 2(x_1 x_2 + x_3 x_4), \quad Q_2 = 2(y_1 y_2 + y_3 y_4), \]
\[ Q_3 = 2(x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3) + 2 \sum_{i=1}^{4} (x_i + y_i) \gamma_{1i} + 2(\gamma_{11} \gamma_{12} + \gamma_{13} \gamma_{14}). \quad (4.9) \]
The equations until now are still simple, but after solving \( L_i = 0 \) and substituting solutions back to \( Q_i \), they will become quadratic polynomials with complicated coefficients of \( \alpha, \beta, \gamma \). Anyway, \( Q_i \)'s are still 3 quadratic polynomials in 4 variables, and they define a one-dimensional curve.

As we have mentioned, different definitions of momentum basis and variables, and the choices of choosing remaining variables after solving linear equations, will describe different curves. But they are all birationally equivalent. Birational invariants will stay the same in each description. So we project the three quadratic equations onto a plane curve. By a coordinate transformation, we can eliminate 2 equations as well as 2 variables, and the resulting equation defines a plane curve describing an equivalent algebraic set. This equation has the degree \( d = 8 \), so the arithmetic genus is,
\[ g_A = \frac{1}{2}(d - 1)(d - 2) = 21. \quad (4.10) \]
The singular points can be obtained directly from the definition, i.e., for projective plane curve defined by homogeneous polynomial \( P(x, y, z) \) of degree \( d \), the singular points are solutions of equations
\[ P(x, y, z) = P_x'(x, y, z) = P_y'(x, y, z) = P_z'(x, y, z) = 0. \quad (4.11) \]
For this plane curve, there are 18 normal singular points of the multiplicity $\mu_p = 2$, so the geometric genus is again

$$ g_G = g_A - \sum_{P \in \text{Sing}(C)} \frac{1}{2} \mu_P (\mu_P - 1) = 21 - 18 = 3. \quad (4.12) $$

Another way of getting a plane curve from original on-shell equations is to use the elimination process via \textit{Gröbner} basis method. We can use \textit{BasisDet} package to generate on-shell equations, where the variables are defined as independent Lorentz Invariant Scalar Products (ISPs) of loop momentum and external momentum. Using these cut equations and \textit{Gröbner} basis associated with them, we get a plane curve. The degree of the plane curve is 8, so we have $g_A = 21$. There are 18 normal singular points of the multiplicity 2, so again we get the geometric genus $g_G = 21 - 18 = 3$.

To summarize, the maximal unitarity cut of two-loop non-planar crossed-box diagram can fix 7 components of loop momenta, and the solution set of on-shell equations is described by a free parameter corresponding to the remaining one degree of freedom. The variety is associated with a genus-3 Riemann surface.

### 4.3.1 Degeneracy under specific kinematics

In the prime case, external momenta $p_i$ are massive and the algebraic set defined by on-shell equations is irreducible. Under some specific kinematic configurations, the variety will be decomposed to many irreducible ideals after primary decomposition. For example, if $p_1$ is massless in Figure (1.c), the algebraic set decomposes into 2 irreducible ideals, i.e., it has two branches. Each irreducible branch is associated with a Riemann surface, and it should have connection with the original genus-3 Riemann surface.

We can compute the geometric genus for each irreducible branch using the same method in previous section. Let us start from the kinematic configuration where at least one momentum of $p_1, p_2$ is massless. Simple calculation shows that for $p_1^2 = 0$, we have $Q_1 = x_3 x_4$, so $Q_1 = 0$ implies that either $x_3 = 0$ or $x_4 = 0$. For $p_2^2 = 0$, we have $Q_1 = f_1(x_3, x_4) f_2(x_3, x_4)$, where $f_1, f_2$ are linear polynomials of $x_3, x_4$, and $Q_1 = 0$ implies that either $f_1 = 0$ or $f_2 = 0$. Combined with $Q_2 = 0, Q_3 = 0$, there are two branches. For each branch, we can transfer the 3 equations to a plane curve by coordinate transformation. This plane curve has the degree 4, so the arithmetic genus is $g_A = 3$. There are 2 normal singular points of the multiplicity $\mu_p = 2$. Finally the geometric genus is $g_G = 3 - 2 = 1$ for each branch, so they are associated with a genus-1 torus. The two tori intersect at 2 points [31]. From these it is easy to obtain that, the topological picture under this kinematic configuration is given by contracting two tubes to points along dashed lines, as shown in Figure (3.b). The up and down parts become tori, and they are connected by 2 points, which are originally 2 tubes.

If we consider kinematic configurations where only one of $p_3, p_4$ is massless, or only one of $p_5, p_6$ is absent (which means that only one vertex of those attached to $p_3, p_4, p_5, p_6$ is a three-vertex with chirality), the algebraic set is reducible, and it has two irreducible branches. For $p_3^2 = 0$, we have $Q_2 = y_3 y_4$, and we can get two branches directly from $Q_2 = 0$. For other kinematic configurations, it is not easy to see the degeneracy directly.
Figure 3. Topological pictures of degenerate on-shell equations from non-planar two-loop crossing-box diagram under specific kinematic configurations. (a) the curve is irreducible, the geometric genus is 3, (b) the curve has 2 irreducible branches, each branch is genus 1, and they are connected by two points, (c) the curve has 2 irreducible branches, each branch is genus 0, and they are connected by four points, (d) no kinematic configuration corresponds to this topological picture.

From on-shell equations, since none of $Q_1, Q_2, Q_3$ can be obviously factorized into two parts. Then we generate an ideal from $Q_1, Q_2, Q_3$, get two irreducible ideals using primary decomposition method of Macaulay2 \[37\]. For each irreducible ideal, we can obtain a plane curve after coordinate transformations. Each plane curve has the degree 4, so the arithmetic genus is $g_A = 3$. There are 3 normal singular points of the multiplicity $\mu_P = 2$, so we have $g_G = 3 - 3 = 0$. It means that for all kinematic configurations considered in this paragraph, each irreducible branch is associated with a genus-0 Riemann sphere. Another way of analysis is to get a plane curve directly from $Q_1 = Q_2 = Q_3 = 0$, and the resulting equation has the degree 8. This equation can be factorized to 2 polynomials of the degree 4, and each factor is equivalent to one irreducible branch. We can again get geometric genus $g_G = 3 - 3 = 0$. These two Riemann spheres intersect at 4 points\[31\], so the possible topological picture can be given by contracting 4 tubes to points along dashed lines as shown in Figure (3.c). The left and right parts become Riemann spheres, and they intersect at 4 points.

Above discussion involves all kinematic configurations where the algebraic set has two irreducible branches. From a genus-3 Riemann surface, there is another possibility of contracting two tubes to get two Riemann surfaces as shown in Figure (3.d), and the resulting topological picture is: a genus-0 Riemann sphere and genus-2 Riemann surface intersect at two single points. But no kinematic configuration is found to guarantee the on-shell equations having such degeneracy. Intuitively, they can be understood as the consequence of symmetry between two branches. Usually, the two branches are related by the parity symmetry and they would have the same genus.

If we denote kinematic condition as,

- $K_1$: at least one of $p_1, p_2$ is massless,
- $K_2$: $p_3$ is massless,
• $K^b_2$: $p_4$ is massless,
• $K^a_3$: $p_5$ is absent,
• $K^b_3$: $p_6$ is absent,

then the general rules for degenerate topological picture can be given as follows: condition $K_1$ will contract the tubes at dashed lines in Figure (3.b), and condition $K^a_2$, $K^b_2$, $K^a_3$ or $K^b_3$ will contract the tubes at dashed lines in Figure (3.c). Topological picture of cases with more than two irreducible branches can be determined by combining these conditions.

Explicitly, if we combine one condition $K_1$ with one of $K^a_2$, $K^b_2$, $K^a_3$, $K^b_3$, the topological picture is given in Figure (4.a), which is the overlap of (3.b) and (3.c). This gives 4 Riemann spheres, which means that there are 4 irreducible branches. The on-shell equations of this kinematic configuration indeed have 4 irreducible branches, and by the explicit computation, each one is a genus-0 curve, which agrees with the topological picture.

If we combine one condition of $K^a_2$, $K^b_2$ with one of $K^a_3$, $K^b_3$, the topological picture is given in Figure (4.b). This gives 4 Riemann spheres. It is not hard to find that Figure (4.c) is given by further combining $K_1$ with above configuration, and the resulting picture gives 6 Riemann spheres. For kinematic configurations with both $K^a_2$, $K^b_2$ or both $K^a_3$, $K^b_3$, the picture is given by overlapping the double copy of Figure (3.c), as shown in Figure (4.d). This also gives 6 Riemann spheres.

Furthermore, combined with $K_1$, we get the topological picture (4.e), which has 8 Riemann spheres. (This case was studied in [28].) We computed the geometric genera of all irreducible branches in this paragraph, and found that they are all zero. Intersection points computation from on-shell equations of irreducible branches, also agrees with this analysis. This verifies the result in [31].

5 Genera of algebraic curves from three-loop diagrams

We also consider algebraic curves defined by equations of maximal unitarity cut from 4-dimensional three-loop diagrams. There are $4 \times 3 = 12$ components in loop momenta, so diagrams with 11 propagators would generate algebraic curves. We study the genera of several three-loop curve examples. Although the algebraic system is more complicated in these cases, the genus computation process is similar to that of two-loop cases. We will project curves to plane curves, and use the knowledge of arithmetic genus and singular points to compute the geometric genera.

5.1 Three-loop planar pentagon-box-box diagram, genus up to 1

We consider the generic pentagon-box-box diagram with 9 massive external legs, as shown in Figure (5.a). The 11 propagators are given by

\[
D_0 = \ell_1^2, \quad D_1 = (\ell_1 - p_1)^2, \quad D_2 = (\ell_1 - p_1 - p_2)^2, \quad D_3 = (\ell_1 - p_1 - p_2 - p_3)^2, \\
\tilde{D}_0 = \ell_2^2, \quad \tilde{D}_1 = (\ell_2 - p_5)^2, \quad \tilde{D}_2 = (\ell_2 - p_4 - p_5)^2, \quad D_0 = \ell_3^2, \\
\tilde{D}_1 = (\ell_3 - p_1 - p_2 - p_3 - p_6 - p_7)^2, \quad \tilde{D}_0 = (\ell_3 - \ell_1 - p_6)^2, \quad \tilde{D}_1 = (\ell_3 + \ell_2 + p_8)^2.
\]
Figure 4. Topological pictures of degenerate on-shell equations from non-planar two-loop crossing-box diagram under specific kinematic configurations, where the solution set has more than 2 branches. These 5 pictures includes the degeneracies under all possible kinematic configurations. Each irreducible branch is genus 0 sphere, and they are connected by points along dashed lines where tubes have been contracted.

Figure 5. Planar three-loop diagrams with 11 propagators: (a) pentagon-box-box diagram, (b) box-pentagon-box diagram. All external momenta are out-going and massive. The loop momenta are denoted by $\ell_1, \ell_2, \ell_3$.

The 11 on-shell equations $D_i = \tilde{D}_i = \bar{D}_i = \hat{D}_i = 0$ of maximal unitarity cut define an algebraic curve, and its topology is actually simple. To see this, note that 4 on-shell equations $D_i = 0$ contain only $\ell_1$, and we can solve them first. Since $\ell_1$ has 4 components, these 4 equations will completely fix $\ell_1$. Explicitly, after solving 3 linear equations from $D_i = 0$, the remaining one becomes a quadratic equation of a single variable. This equation will give two solutions for $\ell_1$, and on each solution, $\ell_1$ is completely fixed. Hence $\ell_1$ can be
treated as a 4-vector similar to external momenta. The remaining $11 - 4 = 7$ cut equations for $\ell_2$ and $\ell_3$ are the same as the two-loop double-box case.

More explicitly, for each solution of $\ell_1$, the on-shell propagators highlighted by dashed lines in Figure (5.a) can be treated as massless external momenta. So the discussion reduces to the two-loop double-box diagram with general kinematics if $p_6, p_7$ are not absent. As we know it is a genus-1 torus. In some specific kinematic configurations of $(p_4, p_5, p_6, p_7, p_8, p_9)$, the algebraic curves are degenerate, and the topological pictures are given by genus-0 Riemann spheres connected at points and linked in a chain, just as the two-loop double-box diagram. The two solutions of $\ell_1$ are separated.

### 5.2 Three-loop planar box-pentagon-box diagram, genus up to 5

On-shell equations of the three-loop box-pentagon-box diagram are much more complicated. Naive elimination of linear equations will generate 5 quadratic equations in 6 variables, and they define a one-dimensional curve. More explicitly, we consider the generic box-pentagon-box diagram with 9 massive external momenta as shown in Figure (5.b). Among the 11 propagators, there are 3 containing $\ell_1$ only

$$D_0 = \ell_1^2, \ D_1 = (\ell_1 - p_1)^2, \ D_2 = (\ell_1 - p_1 - p_2)^2, \quad (5.1)$$

3 containing $\ell_2$ only

$$\tilde{D}_0 = \ell_2^2, \ \tilde{D}_1 = (\ell_2 - p_5)^2, \ \tilde{D}_2 = (\ell_2 - p_4 - p_5)^2, \quad (5.2)$$

and 3 containing $\ell_3$ only

$$D_0 = \ell_3^2, \ D_1 = (\ell_3 - p_1 - p_2 - p_6 - p_7)^2, \ D_2 = (\ell_3 - p_1 - p_2 - p_3 - p_6 - p_7)^2. \quad (5.3)$$

The remaining two propagators contain terms of mixed loop momenta, and they are given by,

$$\tilde{D}_0 = (\ell_3 - \ell_1 - p_6)^2, \ \tilde{D}_1 = (\ell_2 + \ell_3 + p_8)^2. \quad (5.4)$$

We have 4 variables for each loop momentum, denoted as $(x_1, x_2, x_3, x_4)$ for $\ell_1$, $(y_1, y_2, y_3, y_4)$ for $\ell_2$, and $(z_1, z_2, z_3, z_4)$ for $\ell_3$. As usual, for each loop momentum we have 2 linear equations, and by solving them, we can eliminate 2 variables and 2 equations. So finally we get 5 quadratic polynomials,

$$Q_1(x_3, x_4) = D_0, \ Q_2(y_3, y_4) = \tilde{D}_0, \ Q_3(z_3, z_4) = D_0,$$

$$Q_4(x_3, x_4, z_3, z_4) = \tilde{D}_0 - D_0 - D_0, \ Q_5(y_3, y_4, z_3, z_4) = \tilde{D}_1 - \tilde{D}_0 - \tilde{D}_0. \quad (5.5)$$

Notice that $Q_1(x_3, x_4) = 0$, $Q_2(y_3, y_4) = 0$ and $Q_3(z_3, z_4) = 0$ are conics. So it is always possible to find rational parametrization for each loop momentum, and express them by one free parameter as $\ell_1 = \ell_1(x)$, $\ell_2 = \ell_2(y)$ and $\ell_3 = \ell_3(z)$. The remaining two equations

\footnote{By coordinate transformation, we can rewrite a non-degenerate conic $Q(x, y) = 0$ in the form $Q'(x', y') = x'y' - c = 0$, and express one parameter as a rational function of the other as $x' = c/y'$. So the loop momentum can be expanded as a rational function of a single variable.}
\[
\hat{D}_0 \text{ and } \hat{D}_1 \text{ are meromorphic functions after substituting } \ell_i \text{ back, and the numerators define an algebraic plane curve as,}
\]
\[
a_2(x)z^2 + a_1(x)z + a_0(x) = 0 ,
b_2(y)z^2 + b_1(y)z + b_0(y) = 0 .
\tag{5.6}
\]

For generic massive external momenta, \(a_i\) and \(b_i\) are quadratic polynomials. We can eliminate \(z\) by computing the resultant of these two equations to get a plane curve \(F(x, y) = 0\).

This plane curve is birationally equivalent to (5.6) via the inverse map,
\[
z = \frac{-a_2(x)b_0(y) + a_0(x)b_2(y)}{a_2(x)b_1(y) - a_1(x)b_2(y)} ,
\tag{5.7}
\]
so they have the same geometric genus. For generic kinematic configurations, the plane curve \(F(x, y)\) has the degree 8, so the arithmetic genus is
\[
g_A = \frac{(d-1)(d-2)}{2} = 21 .
\tag{5.8}
\]

There are 4 singular points of the multiplicity \(\mu_P = 2\), and 2 singular points of the multiplicity \(\mu_P = 4\), so the geometric genus is given by
\[
g_G = g_A - \sum_{P \in \text{Sing}(P)} \frac{1}{2} \mu_P(\mu_P - 1) = 21 - 4 \times 1 - 2 \times 6 = 5 .
\tag{5.9}
\]

So this is a genus 5 curve.

Alternatively, we can also find the genus of (5.6) by Riemann-Hurwitz formula (2.13)\(^3\). Let \(C\) be the complex curve defined by (5.6). The projection \(f : (x, y, z) \mapsto z\) is a covering map from \(C\) to the complex plane of \(z\). For all but finite points, this is a four-fold covering so the degree of the map is 4. We determine the ramified points as follows:

1. Consider the first equation in (5.6) as a quadratic equation for \(x\) and the discriminant is \(\Delta_1(z)\). For a fixed value of \(z\), there are two corresponding \(x\) unless \(\Delta_1(z)\) vanishes. We can checked that \(\Delta_1(z) = 0\) have four distinct roots on the \(z\) plane, namely \(z_1, z_2, z_3\) and \(z_4\).

2. Consider the second equation in (5.6) as a quadratic equation for \(y\) and the discriminant is \(\Delta_2(z)\). We can checked that \(\Delta_2(z) = 0\) have four distinct roots on the \(z\) plane, namely \(z_5, z_6, z_7\) and \(z_8\).

3. We checked \(z_i\)'s are all distinct, \(i = 1, \ldots, 8\). Hence for each \(z_i, i = 1, \ldots, 4\), there are two corresponding points on \(C\), namely \((x_i, y_i^{(1)}, z_i)\) and \((x_i, y_i^{(2)}, z_i)\). For each \(z_i, i = 5, \ldots, 8\), there are two corresponding points on \(C\), namely \((x_i^{(1)}, y_i, z_i)\) and \((x_i^{(2)}, y_i, z_i)\). So there are \(2 \times 4 + 2 \times 4 = 16\) ramified points on \(C\) and each point has the ramification index 2.

\(^3\)We thank Simon Caron-Huot for the discussion of using this formula.
Then we can use Riemann-Hurwitz formula (2.13) and the fact that the complex plane has the genus zero,

$$2g(C) - 2 = 4 \times (2 \times 0 - 2) + 16 \cdot (2 - 1) ,$$

(5.10)

Again we get the same conclusion that the curve has the genus 5.

Finally, we can also find the genus with the help of two-loop-diagram genus information. If we forget the cut equations involves $l_2$ for a while, the rest equations form the hepta-cut equations of the two-loop massive double box diagram. Intuitively, it is clear that if all propagators involving $l_2$ are pinched, the resulting diagram is the massive double box. Explicitly, neglect the second equation in (5.6) and we find that the remaining equation,

$$a_2(x)z^2 + a_1(x)z + a_0(x) = 0 ,$$

(5.11)

which corresponds to the massive double box topology, defines an elliptic curve with the genus 1. Then we consider the projection of the variety (5.6) to this elliptic curve, via

$$(x, y, z) \mapsto (x, z)$$

(5.12)

This is a ramified double covering. The ramified point are determined by the discriminant $\Delta(z)$ for the equation,

$$b_2(y)z^2 + b_1(y)z + b_0(y) = 0 ,$$

(5.13)

in $y$. The simultaneous equations $\Delta(z) = 0$ and $a_2(x)z^2 + a_1(x)z + a_0(x) = 0$ has 8 distinct solutions. Then we can use Riemann-Hurwitz formula (2.13) and the fact that the complex plane has the genus zero,

$$2g(C) - 2 = 2 \times (2 \times 1 - 2) + 8 \cdot (2 - 1) ,$$

(5.14)

As expected, we get the same conclusion that the curve has the genus 5. Note that here we explicitly used the genus information of the massive double-box topology. In this sense, Riemann-Hurwitz formula provides an induction relation between the three-loop box-penta-box diagram and two-loop double box diagram.

### 5.2.1 Degeneracy under specific kinematics

The degeneracy pattern of the genus-5 Riemann surface is more complicated than that of genus-3 Riemann surface of two-loop example. Here we will discuss some degenerate cases.

If at least one of $p_1, p_2$ is massless, $Q_1(x_3, x_4)$ is factorized and there are two irreducible branches. Similarly, if at least one of $p_4, p_5$ is massless, $Q_2(y_3, y_4)$ has two factors. Also, if $p_3$ is massless, $Q_3(z_3, z_4)$ has two factors. When one of $p_6, p_7, p_8, p_9$ is absent, the algebraic set defined by $Q_i, i = 1, \ldots, 5$ is also reducible, but the factorization is not obvious. Again, we can get two irreducible branches from primary decomposition method by Macaulay2 [37].

Let us assume $p_1^2 = 0$, and the quadratic polynomial of $\ell_1$ has two factors $Q_1(x_3, x_4) = f_1(x_3, x_4)f_2(x_3, x_4)$, where $f_1, f_2$ are both linear in $x_3, x_4$. The two irreducible branches are defined by $I_1 = \langle f_1, Q_2, Q_3, Q_4, Q_5 \rangle$ and $I_2 = \langle f_2, Q_2, Q_3, Q_4, Q_5 \rangle$. By the elimination
Figure 6. Topological pictures of degenerate on-shell equations from planar three-loop box-pentagon-box diagram under specific kinematic configurations. (a) the curve is irreducible, and the genus is 5, (b) the curve has 2 irreducible branches, each branch has genus 1. They are connected by 4 points, (c) the curve has 4 irreducible branches, each branch has genus 0. They are linked in chain with two adjacent branches connected by 2 points.

method via Gröbner basis, we can get a plane curve of degree 8 for each branch, so the arithmetic genus is $g_A = 21$. There are 8 singular points of the multiplicity $\mu_P = 2$, and 2 singular points of the multiplicity $\mu_P = 4$, so the geometric genus is $g_G = 21 - 8 \times 1 - 2 \times 6 = 1$. We can also count the intersection points between two curves, and there are 4 single points. So the topological picture of this degeneracy can be given by contracting tubes to points along dashed lines, as shown in Figure (6.b).

We consider another kinematic configuration where $p_1$ and $p_5$ are massless. In this case both $Q_1(x_3, x_4)$ and $Q_2(y_3, y_4)$ are degenerate, and we can get 4 irreducible branches. From Macaulay2, we can also get 4 irreducible ideals. For each branch, we can get a plane curve of degree 4 via Gröbner basis method, so the arithmetic genus is $g_A = 3$. There are 3 singular points of the multiplicity $\mu_P = 2$, so the geometric genus is $g_G = 3 - 3 \times 1 = 0$. Counting intersecting points, we found that there are 8 points in total: the 4 irreducible branches are linked in chain, and the adjacent two branches intersect at 2 points. The topological picture can be given by contracting tubes to points around dashed lines, as shown in Figure (6.c). Each branch is a genus-0 Riemann sphere.

5.3 Three-loop non-planar box-crossed-pentagon diagram, genus up to 9

In this subsection, we discuss the three-loop box-crossed-pentagon diagram as shown in Figure (7.a). This is a non-planar three-loop diagram with 9 general massive external momenta. As usual, the maximal unitarity cut of this diagram gives 11 on-shell equations of 12 variables, and they define an algebraic curve. Again, there are 3 propagators containing only $\ell_1$,

$$D_0 = \ell_1^2 , \quad D_1 = (\ell_1 - p_1)^2 , \quad D_2 = (\ell_1 - p_1 - p_2)^2 .$$  

(5.15)
From on-shell equations $D_i = 0$ we can parameterize $\ell_1$ rationally by one free parameter $x$. Similarly, there are 3 propagators containing only $\ell_3$,

$$\tilde{D}_0 = \ell_3^2, \quad \tilde{D}_1 = (\ell_3 - p_4)^2, \quad \tilde{D} = (\ell_3 - p_1 - p_2 - p_3 - p_4 - p_9)^2.$$  \hfill (5.16)

Using on-shell equations $\tilde{D}_i = 0$, $\ell_3$ is rationally parameterized by one free parameter $w$. However, only 2 propagators

$$\tilde{D}_0 = \ell_2^2, \quad \tilde{D}_1 = (\ell_2 - p_6)^2$$  \hfill (5.17)

containing single loop momentum $\ell_2$, so $\ell_2$ is rationally parameterized by 2 free parameters, namely, $y$ and $z$ from on-shell equations $\tilde{D}_i = 0$.

The remaining 3 propagators contain terms of mixed loop momenta,

$$\tilde{D}_0 = (\ell_2 - \ell_3 - p_5 - p_6)^2, \quad \tilde{D}_1 = (\ell_2 - \ell_3 - p_5 - p_6 - p_7)^2, \quad \tilde{D}_2 = (\ell_1 + \ell_3 - p_1 - p_2 - p_3 - p_4)^2.$$  \hfill (5.18)

After substituting $\ell_1(x), \ell_2(y, z), \ell_3(w)$ back, they become meromorphic functions. The numerators $f_1, f_2, f_3$ of these 3 meromorphic functions are polynomials in $(x, y, z, w)$, and equations

$$f_1(y, z, w) = f_2(y, z, w) = f_3(x, w) = 0$$  \hfill (5.19)

define the algebraic curve. They are not necessary quadratic.

Our strategy is to eliminate $y$ and $w$ from the equations and then get a plane curve in $x$ and $z$. This can be done automatically by Gröbner basis method. However, it is helpful to eliminate $y$ and $w$ step by step, so that we can explicitly show this projection process is birational. Furthermore, we can see the induction relation from the two-loop non-planer diagram to this diagram.
First, note that if we combine the external legs $p_1$, $p_2$, $p_3$ and $p_9$ to one external leg and neglect the four cut equations involving $l_1$, then we get the prime case of two-loop non-planar crossed-box diagram. In other words, $f_1(y, z, w) = f_2(y, z, w) = 0$ defines a genus-3 complex curve. Hence we can eliminate $y$ like the case (5.6) and it is birational via an inverse transformation like (5.7). The resulting reduced algebraic system is

$$a_8 w^8 + a_7(z)w^7 + a_6(z)w^6 + a_5(z)w^5 + a_4(z)w^4 + a_3(z)w^3 + a_2(z)w^2 + a_1(z)w + a_0(z) = 0,$$
$$b_2(x)w^2 + b_1(x)w + b_0(x) = 0,$$  \hspace{1cm}(5.20)

where for generic massive external momenta, $a_i, i = 0, \ldots, 6$ and $b_i, i = 0, \ldots, 2$ are quadratic polynomials while $a_7(z)$ is linear and $a_8$ is a constant. Again, we use the resultant to eliminate $w$ and get a plane curve $F(x, z) = 0$. This step is also birational, and the inverse map is

$$w = \frac{p(x, z)}{q(x, z)},$$  \hspace{1cm}(5.21)

where the explicit expressions for $p(x, z)$ and $q(x, z)$ can be found by Gröbner basis method.

Finally, the plane curve $F(x, z) = 0$ has the degree 20, so the arithmetic genus is $g_A = 171$. There are 32 normal singular points of the multiplicity $\mu_P = 2$, one normal singular point of the multiplicity $\mu_P = 4$, and one 16-fold point. The 16-fold point has 8 ordinary tangent lines with the multiplicity 1 and 4 tangent lines with the multiplicity 2. Since this 16-fold point is not a normal singular point, we need to perform a blow up. The new blow-up curve resolves the 16-fold point into 8 smooth points and 4 normal double points. In summary, the genus is

$$g_G = \frac{1}{2}(20 - 1)(20 - 2) - 32 \cdot 1 - \frac{1}{2} \cdot 4 \cdot (4 - 1) - \frac{1}{2} \cdot 16 \cdot (15 - 1) - 4 = 9.$$  \hspace{1cm}(5.22)

So the maximal unitarity cut of the non-planar three-loop box-crossed-pentagon diagram, with all massive legs, defines a one-dimensional algebraic curve of the genus 9.

Alternatively, we can use Riemann-Hurwitz formula to get the genus of the curve (5.20). The first equation of (5.20) generates 8 ramified points on the $w$ plane while the second equation generates 4 ramified points on the $w$ plane. All these points are distinct, so there are $8 \cdot 2 + 4 \cdot 2 = 24$ ramified points on the curve (5.20). So Riemann-Hurwitz formula reads

$$2g - 2 = 4(0 - 2) + 24,$$  \hspace{1cm}(5.23)

which again implies that (5.20) has the genus 9.

5.3.1 Degeneracy under specific kinematics

Before going to the discussion of degenerate cases, let us first verify an observation of on-shell equations\footnote{We thank Simon Caron-Huot for this observation.}. If we forget the 4 equations that containing $\ell_1$, i.e., $D_0 = D_1 = D_2 = \tilde{D}_2 = 0$, then the remaining 7 equations describe the prime case of two-loop non-planar crossed-box diagram, so it should have the genus 3. To see this, we could first eliminate 4 variables from 4 linear equations $\tilde{D}_0 - D_1 = \tilde{D}_0 - \tilde{D}_2 = 0, \tilde{D}_0 - \tilde{D}_1 = 0$ and
Figure 8. Topological pictures of on-shell equations from the non-planar three-loop box-crossed-pentagon diagram under specific kinematic configurations. (a) the curve has 2 irreducible branches, each branch has the genus 3 and they are connected by 4 points, (b) the curve has 2 irreducible branches, each branch has the genus 1, and they are connected by 8 single points, (c) the curve has 4 irreducible branches, each branch has the genus 0, and they are linked in chain, (d) the curve has 8 irreducible branches, and each branch has the genus 0.

\[ \hat{D}_0 - \hat{D}_0 - \tilde{D}_0 = 0. \]  

Then the remaining 3 equations of 4 variables define an equivalent curve, and we can birationally project it to a plane curve. The resulting plane curve has the degree 8, while there are 12 normal singular points of the multiplicity \( m = 2 \), 1 normal singular point of the multiplicity \( m = 4 \), so the genus is

\[ g_G = \frac{1}{2}(8 - 1)(8 - 2) - 12 \cdot 1 - \frac{1}{2}4(4 - 1) = 3, \]  

which agrees with the observation.

The genus-9 curve of generic kinematics is complicated. However, as previous examples, we can still consider some kinematic limits where on-shell equations are degenerated. In these cases the solution space has several branches, and each branch defines a curve with the lower genus. Here we will show some simple degenerate cases.

After solving linear equations, we get 5 quadratic equations,

\[
Q_1(x_1, x_2) = 0, \quad Q_2(y_1, y_2, z_1, z_2) = 0, \quad Q_3(z_1, z_2) = 0,
\]

\[
Q_4(x_1, x_2, z_1, z_2) = 0, \quad Q_5(y_1, y_2, z_1, z_2) = 0,
\]  

where \( Q_1, Q_2, Q_3 \) come from \( \ell_1^2, \ell_2^2, \ell_3^2 \) and \( Q_4, Q_5 \) come from the mixed terms \( \ell_1 \ell_3, \ell_2 \ell_3 \). If \( p_1 \) or \( p_2 \) is massless, then \( Q_1 \) has two factors, while if \( p_4 \) is massless, then \( Q_3 \) has two factors. In these kinematic configurations, there are two irreducible branches. As usual, we can project each branch onto a plane curve. The resulting plane curve has the degree 12, so the arithmetic genus is \( g_A = 55 \). There are 16 normal singular points of the multiplicity \( m = 2 \), 1 normal singular point of the multiplicity \( m = 4 \), one 8-fold singular point. This 8-fold point is not normal, and after blowing up, we get 2 normal singular points of the multiplicity \( m = 2 \). So finally we get,

\[ g_G = 55 - 16 \cdot 1 - \frac{1}{2}4(4 - 1) - \frac{1}{2}8(8 - 1) - 2 \cdot 1 = 3. \]  

(5.24)
Therefore there are two genus-3 curves. We can also calculate the intersecting points between these two curves, and find that there are 4 points. The topological picture can be reproduced from the genus-9 picture by contracting tubes to points along dashed lines as shown in Figure (8.a).

If \( p_6 \) or \( p_7 \) is massless, \( Q_2 \) has two factors. In this case, again, there are two irreducible branches. Similarly, after birationally projecting each branch onto a plane curve, we get an equation of the degree 12. There are 20 normal singular points of the multiplicity \( m_P = 2 \), one normal singular point of the multiplicity \( m_P = 4 \) and one normal singular point of the multiplicity \( m_P = 8 \), so finally we get genus

\[
g_G = \frac{1}{2}(12 - 1)(12 - 2) - 20 \cdot 1 - \frac{1}{2}4(4 - 1) - \frac{1}{2}8(8 - 1) = 1 . \tag{5.27}
\]

There are 8 intersection points between the two irreducible branches. So the topological picture can be reproduced from the genus-9 picture by contracting tubes to points along dashed lines, as shown in Figure (8.b).

In fact, if we know the genus for generic kinematics, together with the number of irreducible branches for specific kinematic configuration and the intersecting points between different branches, it is possible to predict the genus of each branch by directly studying the topological picture. Take the kinematic configuration in previous paragraph as an example: intuitively, these two branches are symmetric, and they should have the same genus. A simple calculation determined that there are 8 intersecting points. Then it is clear that (8.b) is the only possible topological picture which satisfies these requirements. So we can conclude that each branch is genus-1 without doing any further calculation.

If both \( p_1 \) and \( p_6 \) are massless, \( Q_1 \) and \( Q_2 \) are factorized, and each of them has two factors. Thus, the variety has 4 irreducible branches. The on-shell equation system of each branch is quite simple, and the corresponding plane curve for each branch has the degree 4. There are 3 normal singular points of the multiplicity \( m_P = 2 \), so the genus is \( g_G = (4 - 1)(4 - 2)/2 - 3 = 0 \). The topological picture is shown in Figure (8.c).

If we further consider the case with massless \( p_1, p_4, p_6 \), there are 8 irreducible branches. The plane curve of each branch has the degree 3, and there is only one normal singular point of the multiplicity \( m_P = 2 \), so the genus \( g_G = (3-1)(3-2)/2-1 = 0 \). After obtaining the intersecting points between branches, we draw the topological picture in Figure (8.d).

5.4 Three-loop non-planar crossed-crossed-pentagon diagram, genus up to 13

The same methods can be used to study the non-planar three-loop crossed-crossed-pentagon diagram, as shown in Figure (7.b), although the computation is more complicated. Nevertheless we should start from the on-shell equations. There are 2 propagators containing only \( \ell_1 \)

\[
D_0 = \ell_1^2 , \quad D_1 = (\ell_1 - p_1)^2 , \tag{5.28}
\]

2 propagators containing only \( \ell_2 \)

\[
\bar{D}_0 = \ell_2^2 , \quad \bar{D}_1 = (\ell_2 - p_3)^2 , \tag{5.29}
\]
and 3 propagators containing only $\ell_3$

$$
\tilde{D}_0 = \ell_3^2, \quad \tilde{D}_1 = (\ell_3 - p_1 - p_2 - p_7 - p_9)^2, \quad \tilde{D}_2 = (\ell_3 + p_4 + p_5 + p_6 + p_8)^2.
$$

(5.30)

From maximal unitarity cut, we can rationally parameterize $\ell_1$ by two parameters $x, y$, $\ell_2$ by $z, w$, and $\ell_3$ by $\tau$. The remaining 4 on-shell equations $\hat{D}_0 = \hat{D}_1 = \hat{D}_2 = \hat{D}_3 = 0$, where

$$
\hat{D}_0 = (\ell_3 - \ell_1 - p_7)^2, \quad \hat{D}_1 = (\ell_3 - \ell_1 - p_7 - p_9)^2, \\
\hat{D}_2 = (\ell_3 + \ell_2 + p_6)^2, \quad \hat{D}_3 = (\ell_3 + \ell_2 + p_6 + p_8)^2,
$$

(5.31)

become

$$
f_1(x, y, \tau) = f_2(x, y, \tau) = f_3(z, w, \tau) = f_4(z, w, \tau) = 0,
$$

(5.32)

which are polynomial equations. Again, it is clear that this diagram contains two copies of non-planar two-loop diagrams. In other words, $f_1 = f_2 = 0$ defines a genus-3 curve and $f_3 = f_4 = 0$ defines another genus-3 curve. Following the previous analysis, we can birationally eliminate $y$ and $w$ and get,

$$
g_1(x, \tau) = 0, \quad g_2(z, \tau) = 0,
$$

(5.33)

where $g_1$ is quadratic in $x$ and $g_2$ is quadratic in $z$.

We can further eliminate $\tau$ to get a plane curve $F(x, z) = 0$ in $z$. The plane curve has the degree 24, so the arithmetic genus is $g_A = 253$. There are 184 normal singular points of the multiplicity $\mu_P = 2$, two normal singular point of the multiplicity $\mu_P = 8$. In summary, the genus is

$$
g_G = \frac{1}{2}(24 - 1)(24 - 2) - 184 \cdot 1 - \frac{1}{2} \cdot 8 \cdot (8 - 1) \cdot 2 = 13.
$$

(5.34)

Alternatively, each of $g_1$ and $g_2$ generates 8 ramified points on the $\tau$ plane. All these points are distinct, so there $8 \cdot 2 + 8 \cdot 2 = 32$ ramified points on the curve (5.33). So Riemann-Hurwitz formula reads

$$
2g - 2 = 4(0 - 2) + 32,
$$

(5.35)

which again implies that the curve from crossed-crossed-pentagon diagram, with all massive legs, has the genus 13.

5.4.1 Degeneracy under specific kinematics

Again, if we forget the 4 equations containing $\ell_1$ (or $\ell_2$), as in the box-crossed-pentagon example, the remaining 7 equations also describe the generic two-loop non-planar crossed-box diagram, thus it should be genus 3. This can be verified by the same method in the previous subsection.

We discuss some kinematic configurations, where the on-shell equations are degenerated. Again, after solving linear equations $D_0 - D_1 = 0$ and $\tilde{D}_0 - \tilde{D}_0 - D_0 = 0$, we can formally write the other two quadratic equations as

$$
Q_1(x_1, x_2, z_1, z_2) = 0, \quad Q_4(x_1, x_2, z_1, z_2) = 0.
$$

(5.36)
If $p_1$ is massless, $Q_1$ has two factors $Q_1 = f_1 f_2$, where $f_1, f_2$ are linear functions. Thus, there are two irreducible branches, from ideals $I_1 = \langle f_1, Q_2, Q_3, Q_4, Q_5 \rangle$ and $I_2 = \langle f_2, Q_2, Q_3, Q_4, Q_5 \rangle$. These two branches intersect at 8 points, so we can guess that the topological picture should be Figure (9.a) and each branch gives a curve with the genus 3. To verify this, as usual, we can project the curve onto a plane curve, and the resulting equation has the degree 12. There are 40 normal singular points of the multiplicity $m_P = 2$, 2 normal singular points of the multiplicity $m_P = 4$, so the genus is

$$g_G = \frac{1}{2}(12 - 1)(12 - 2) - 40 \cdot 1 - 2 \cdot \frac{1}{2} 4(4 - 1) = 3 ,$$

which is exactly the result shown in Figure (9.a).

If $p_3$ is massless, $Q_3$ has two factors. There are two irreducible branches. The corresponding plane curve of each branch has the degree 12. And there are 38 normal singular points of the multiplicity $m_P = 2$, 2 normal singular points of the multiplicity $m_P = 4$, so the genus is given by

$$g_G = \frac{1}{2}(12 - 1)(12 - 2) - 38 \cdot 1 - 2 \cdot \frac{1}{2} 4(4 - 1) = 5 .$$

The number of intersecting points between two branches is 4, so the topological picture is given by contracting tubes to points from the genus-13 picture, as shown in Figure (9.b).

It is not hard to predict that if both $p_1, p_3$ are massless, the topological picture should be given by overlapping of Figure (9.a) and (9.b). In other words, there are 4 genus-1 tori, which are linked in a chain. In fact, in this kinematic configuration, both $Q_1$ and $Q_3$ are factorized, and there are indeed 4 branches. The plane curve of each branch has the degree 6, so the arithmetic genus is $g_A = (6 - 1)(6 - 2)/2 = 10$. There are 9 normal singular
Figure 10. Non-planar three-loop "Mercedes" logo diagram. All external momenta are out-going and massive. The loop momenta are denoted by $\ell_1, \ell_2, \ell_3$.

points of the multiplicity $m_P = 2$, so the genus from explicit calculation is

$$g_G = 10 - 9 \cdot 1 = 1,$$

(5.39)
as expected.

If both $p_1, p_5$ are massless, $Q_1, Q_2$ are factorized. So there are 4 irreducible branches. The plane curve of each branch has the degree 6, while there are 10 normal singular points of the multiplicity $m_P = 2$, so the genus is 0. If $p_1, p_3, p_5$ are massless, we further get 8 branches. The plane curve of each branch has the degree 3, and there is only one singular point of the multiplicity 1, so the genus is again 0. After obtaining the intersecting points between branches, we can sketch the topological pictures of both kinematic configurations from the genus-13 picture in Figure (9.c) and (9.d).

5.5 Three-loop Mercedes-logo diagram, genus up to 9

So far, we only consider the three-loop diagrams with "ladder" type. The same mathematical method also works for three-loop diagrams with "Mercedes-logo" topology.

In this subsection, we explicitly calculate a three-loop "Mercedes-logo" diagram with 11 propagators and 9 massive external legs as shown in Figure (10). Different from the "ladder" diagrams, all possible terms of loop momenta $\ell_1 \ell_2, \ell_1 \ell_3, \ell_2 \ell_3$ will appear. This will somehow complicate the cut equations. There are two propagators containing only $\ell_1$

$$D_0 = \ell_1^2, \quad D_1 = (\ell_1 - p_1)^2,$$

(5.40)

and three propagators containing only $\ell_2$

$$\bar{D}_0 = \ell_2^2, \quad \bar{D}_1 = (\ell_2 - p_6)^2, \quad \bar{D}_2 = (\ell_2 + p_7)^2,$$

(5.41)

and three propagators containing only $\ell_3$

$$\tilde{D}_0 = \ell_3^2, \quad \tilde{D}_1 = (\ell_3 - p_3)^2, \quad \tilde{D}_2 = (\ell_3 + p_4)^2.$$

(5.42)
The remaining three propagators contain mixed terms as

\[
\hat{D}_0 = (\ell_2 - \ell_1 + p_7 + p_1 + p_8)^2, \quad \hat{D}_1 = (\ell_3 - \ell_2 + p_6 + p_5)^2, \quad \hat{D}_2 = (\ell_1 - \ell_3 + p_3 + p_2)^2.
\]

(5.43)

(5.44)

It is not possible to construct linear equations directly from equations of mixed propagators, so we can only get 5 linear equations from \(D_i = \bar{D}_i = \tilde{D}_i = 0\). The one-dimensional curve is described by 6 quadratic equations of 7 variables.

As before, we can parameterize \(\ell_1\) by two variables \(x\) and \(y\), \(\ell_2\) by one variable \(z\), and \(\ell_3\) by one variables \(w\). The rest equations have the following form

\[
f(x, y, z) = 0, \quad g(w, z) = 0, \quad h(x, y, w) = 0.
\]

(5.45)

We birationally project the variety on a plane curve \(F(y, w) = 0\). This plane curve has the degree 20, 36 normal double point, one normal 16-fold point and one normal 4-fold point. So the geometric genus is,

\[
g_G = \frac{(20 - 1)(20 - 2)}{2} - 36 - \frac{16(16 - 1)}{2} - \frac{4(4 - 1)}{2} = 9.
\]

(5.46)

Interestingly, genus of Mercedes logo diagrams equals to the genus of box-crossed-pentagon diagrams, they both have genus 9. We examine some simple degenerate equations under specific kinematic configurations, and find that the topological pictures are the same as box-crossed-pentagon. If \(p_4\) is massless, there are two irreducible branches. After birationally projecting each branch onto plane curve, we get an equation of degree 12. There are 18 normal singular points of \(m_P = 2\), 1 normal singular point of \(m_P = 4\) and 1 normal singular point of \(m_P = 8\), so the genus is 3. These two branches intersect at 4 points, and the topological picture is the same as shown in Figure (8.a). If \(p_1\) is massless, cut equations also degenerate to two irreducible branches. The plane curve of each branch has degree 12. There are 20 normal singular points of \(m_P = 2\), 1 singular point of \(m_P = 4\) and 1 singular point of \(m_P = 8\), so the genus is 1. There are 8 intersecting points between two branches, and the picture is the same as Figure (8.b). If both \(p_1, p_4\) are massless, there are 4 irreducible branches. The plane curve after birationally projecting each branch has degree 6 and 10 normal singular points, so each of them is genus-0 curve. The topical picture is the same as Figure (8.c). Further considering \(p_1, p_4, p_6\) massless, we get 8 irreducible branches. Each branch can be projected onto a plane curve of degree 3 and 1 singular point, so they are again genus-0 curve. After getting the intersecting points among branches, the topological picture can be found to be the same as the one shown in Figure (8.d).

6 Conclusion

In this paper, we study the global structure of on-shell equations from generalized unitarity cut of loop diagrams. We focus on the \(L\)-loop diagram with \(4L - 1\) unitarity cuts in four-dimensional theories. There is one degree of freedom for the solution space, which is a
complex algebraic curve or union of complex algebraic curves. Since the topology of a complex algebraic curve is completely determined by the genus, in order to study the global structure, we compute the genera of algebraic curves. This is systematically done by computational algebraic geometry methods.

In this paper, we used two algebraic geometry methods to determine the genera of curves from unitarity cuts: (1) the relation between arithmetic genus and geometric genus (2) Riemann-Hurwitz formula. These methods clearly work for all algebraic curves, and are quite efficient for those from two and three-loop unitarity cuts.

The only algebraic curve for the one-loop case is from the one-loop triangle diagram with 3 propagators. The defining equation is simply a conic section, and it is equivalent to a genus-0 Riemann sphere, so we can always find the rational parametrization for the solution space.

Two-loop diagrams with 7 propagators also define algebraic curves. We find that the genus could be as high as 3, in the two-loop non-planar crossed-box diagram with the maximal number of massive external momenta. In the degenerate limit, i.e., some external momenta being massless or absent, the solution space degenerates to the union of several complex curves. These curves have lower genera, 1 or 0. As the global structure analysis in [23], from intersection structure of these curves, we can show that the union of these curves exactly comes from a degenerate genus-3 curve.

The topology of algebraic curves from three-loop diagrams with 11 propagators is more complicated. We consider four examples. The pentagon-box-box diagram is, in some sense, similar to the two-loop double box diagram. The curve from the planar box-pentagon-box diagram with generic kinematics has the genus 5. For the non-planar box-crossed-pentagon diagram with generic kinematics, the curve can have the genus 9. And for the non-planar crossed-crossed-pentagon diagrams, the curve can have the genus 13. Again, for diagrams with massless external legs, we show that the global structure of solutions comes from a degenerate high-genus curve.

The information of genus and the topological picture under degenerate limit is important for calculating multi-loop amplitude via unitarity cut method:

- The genus is the criterion for rational parametrization: if the curve has the genus zero, then it can be rationally parameterized. Otherwise, the rational parametrization does not exist. This property is important, since for unitarity computation, we may parameterize solution space before tree amplitude computations. So it is useful to know the difficulty of parametrization before unitarity computation.

- The topological picture under degenerate limit is important for calculating expansion coefficients of integrand basis. Usually, all branches should be studied to get these coefficients. Knowing the genus for generic kinematics, it is possible to predict the topology for branches under degenerate limits from the knowledge of intersection pattern. Especially, we can use the genus information to check if all the cut solutions are identified or not.
We comment that in all cases considered in this paper, the genera of curves, if not zero, are always odd. We expect that this feature can be understood by the loop-by-loop induction relation, in the future work.

Unitarity cuts with fewer propagators are more complicated, since in these cases, the on-shell equations define algebraic (hyper-)surfaces instead of curves. The topological structures of surfaces are much more complicated than those of curves. However, we expect that the techniques of computational algebraic geometry would still be important for obtaining the topological information of unitarity cuts. Analysis of the global structure of generalized unitarity cuts is just the beginning of multi-loop amplitude calculation, and more works, such as parametrization or the branch-by-branch polynomial fitting [34], can be done based by this information.

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