GROUP BUNDLE DUALITY

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Abstract. This paper introduces a generalization of Pontryagin duality for locally compact Hausdorff Abelian groups to locally compact Hausdorff Abelian group bundles.

First, recall that a group bundle is just a groupoid where the range and source maps coincide. An Abelian group bundle is a bundle where each fibre is an Abelian group. When working with a group bundle $G$, we will use $X$ to denote the unit space of $G$ and $p : G \to X$ to denote the combined range and source maps. Furthermore, we will use $G_x$ to denote the fibre over $x$. Group bundles, like general groupoids, may not have a Haar system but when they do the Haar system has a special form. If $G$ is a locally compact Hausdorff group bundle with Haar system, denoted by $\{\beta_x\}$ throughout the paper, then $\beta_x$ is Haar measure on the fibre $G_x$ for all $x \in X$. At this point, it is convenient to make the standing assumption that all of the locally compact spaces in this paper are Hausdorff.

Now suppose $G$ is an Abelian, second countable, locally compact group bundle with Haar system $\{\beta_x\}$. Then $C^*(G, \beta)$ is a separable Abelian $C^*$-algebra and in particular $\hat{G} = C^*(G, \beta)^\wedge$ is a second countable locally compact Hausdorff space [1, Theorem 1.1.1]. We cite [2, Section 3] to see that each element of $\hat{G}$ is of the form $(\omega, x)$ with $x \in X$ and $\omega$ a character in the Pontryagin dual of $G_x$, denoted $(G_x)^\wedge$. The action of $(\omega, x)$ on $C_c(G)$ is given by

$$ (\omega, x)(f) = \int_G f(s) \omega(s) d\beta(s). $$

Since every element in $\hat{G}$ is a character on a fibre of $G$, we are justified in thinking of $\hat{G}$ as a bundle over $X$ with fibres $\hat{G}_x = (G_x)^\wedge$ and action on $C^*(G, \beta)$ given by (1). We will use $\hat{p}$ to denote the projection from $\hat{G}$ to $X$ and $\omega$ to denote the element $(\omega, \hat{p}(\omega))$ in $\hat{G}$.

At this point, it is clear that $\hat{G}$ is algebraically a group bundle. In order for it to be a topological groupoid, we must show that the groupoid operations are continuous with respect to the Gelfand topology on $\hat{G}$. To this end, we reference the following characterization of the topology on $\hat{G}$.

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Lemma 1 ([2, Proposition 3.3]). Let $G$ be a second countable locally compact abelian group bundle with Haar system. Then a sequence $\{\omega_n\}$ in $\hat{G}$ converges to $\omega_0$ in $\hat{G}$ if and only if

(a) $\hat{p}(\omega_n)$ converges to $\hat{p}(\omega_0)$ in $X$, and

(b) if $s_n \in G_{\hat{p}(\omega_n)}$ for all $n \geq 0$ and $s_n \to s_0$ in $G$, then $\omega_n(s_n) \to \omega_0(s_0)$.

The first thing we can conclude from this lemma is that the restriction of the topology on $\hat{G}$ to $\hat{G}_x$ is the same as the topology on $\hat{G}_x$ as the dual group of $G_x$. The second thing we conclude is that the topology on $\hat{G}$ is independent of the Haar system $\beta$. Furthermore, recall that the groupoid operations on $\hat{G}$ are those coming from the dual operations on $\hat{G}_x$. In other words, the operations are pointwise multiplication and conjugation of characters, and it follows from Lemma 1 that these operations are continuous. Therefore, we have proven the lemma.

Lemma 2 ([2, Corollary 3.4]). Let $G$ be a second countable locally compact Abelian group bundle with Haar system. Then $\hat{G}$, equipped with the Gelfand topology, is a second countable locally compact Abelian group bundle with fibres $\hat{G}_x = (G_x)^\wedge$.

Now we can make our first definition.

Definition 3. If $G$ is a second countable locally compact Abelian group bundle with Haar system, then we define the dual bundle to be $\hat{G} = C^*(G)^\wedge$ equipped with the groupoid operations coming from the identification of $\hat{G}_x$ as the dual of $G_x$. We will use $\hat{p}$ to denote the projection on this bundle.

This definition gives rise to the notion of a duality theorem for group bundles. The main result of this paper is to prove the following theorem, stated without proof in [3, Proposition 1.3.7].

Theorem 4. If $G$ is a second countable locally compact (Hausdorff) Abelian group bundle with Haar system then the dual $\hat{G}$ has a dual group bundle, denoted $\hat{\hat{G}}$. Furthermore, the map $\Phi : G \to \hat{\hat{G}}$ such that

$$\Phi(s)(\omega) = \hat{s}(\omega) := \omega(s)$$

is a (topological) group bundle isomorphism between $G$ and $\hat{\hat{G}}$.

Before we continue, it will be useful to see that the group bundle notion of duality is a natural extension of the usual Pontryagin dual, as illustrated by the following proposition.

Proposition 5. Let $G$ be a second countable locally compact Abelian group bundle with Haar system. Then $C^*(G) \cong C_0(\hat{G})$ via the Gelfand transform. Furthermore, if $f \in C_c(G)$ then the Gelfand transform of $f$ restricted to $\hat{G}_x$ is the Fourier transform of $f |_{G_x}$.
Proof. The first statement follows from the fact that we defined $\hat{G}$ to be the spectrum of the Abelian $C^*$-algebra $C^*(G)$. Next, let $\hat{f}$ be the Gelfand transform of $f$. Then for $\omega \in \hat{G}$ we see from (1) that
\[
\hat{f}(\omega) = \omega(f) = \int_{G_{\hat{p}(\omega)}} f(s)\omega(s)d\beta(\omega)(s).
\]
This of course implies that $\hat{f}$ is the usual Fourier transform on $\hat{G}$.

We can now begin the process of proving Theorem 4. The first step is to show that $\hat{\hat{G}}$ has a dual bundle. We have already verified that $\hat{G}$ is a second countable locally compact Abelian group bundle. The only remaining requirement is that $\hat{\hat{G}}$ has a Haar system. Recall that given a locally compact Abelian group $H$ and Haar measure $\lambda$ the Plancharel theorem guarantees the existence of a dual Haar measure $\hat{\lambda}$ such that $L^2(H, \lambda) \cong L^2(\hat{H}, \hat{\lambda})$. The existence of a dual Haar system is then taken care of by the following lemma.

Lemma 6 ([2, Proposition 3.6]). If $G$ is an Abelian second countable locally compact group bundle with Haar system $\{\beta_x\}$, then the collection of dual Haar measures $\{\hat{\beta}_x\}$ is a Haar system for $\hat{\hat{G}}$.

Now that $\hat{\hat{G}}$ is well defined, we must show that $\Phi$ is a group bundle isomorphism. In some sense, the following proposition gets us most of the way there.

Proposition 7. The map $\Phi : G \to \hat{\hat{G}} : s \mapsto \hat{s}$ is a continuous bijective groupoid homomorphism.

Proof. It follows from Lemma [2] that $\hat{\hat{G}}_x$ is the double dual of $G_x$. Furthermore, classical Pontryagin duality says that $s \to \hat{s}$ is an isomorphism from $G_x$ onto $\hat{\hat{G}}_x$ [4, Theorem 1.7.2]. Since $\Phi$ is formed by gluing all of these fibre isomorphisms together it is clear that $\Phi$ is a bijective groupoid homomorphism. Next, we need to see that it is continuous. Suppose $s_i \to s_0$ in $G$. We know from Lemma [1] that it will suffice to show that
\begin{itemize}
  \item[(a)] $\hat{p}(\Phi(s_i)) \to \hat{p}(\Phi(s_0))$, and
  \item[(b)] given $\omega_i \in \hat{G_{\hat{p}(\Phi(s_i))}}$ such that $\omega_i \to \omega_0$ in $\hat{G}$ then $\Phi(s_i)(\omega_i) \to \Phi(s_0)(\omega_0)$.
\end{itemize}

First, let $x_i = p(s_i) = \hat{p}(\Phi(s_i))$. Since $p$ is continuous, it is clear that $x_i \to x_0$ and that the first condition is satisfied. Now suppose $\omega_i \in \hat{G}_{x_i}$ for all $i \geq 0$ such that $\omega_i \to \omega_0$. All we have to do is cite Lemma [1] again to see that
\[
\Phi(s_i)(\omega_i) = \omega_i(s_i) \to \omega_0(s_0) = \Phi(s_0)(\omega_0).
\]
If we were working with groups, we would be done since continuous bi-
jections between second countable locally compact groups are automatically
homeomorphisms [3, Theorem D.3], [1, Corollary 2, p. 72]. However, there
currently no automatic continuity results for the inverse of a continuous bi-
jective group bundle homomorphism. Regardless, we can still show that in
this case \( \Phi \) is a homeomorphism.

Proof of Theorem 4. Given Proposition 7, all we need to do to prove that \( \Phi \) is
a homeomorphism is show that if \( \hat{s}_i \to \hat{s}_0 \) in \( \hat{G} \) then \( s_i \to s_0 \) in \( G \). First, we let
\( x_i = p(\hat{s}_i) \) for all \( i \). Recall that \( \hat{G} \) has the Gelfand topology as the spectrum
of \( \mathbb{C}^*(\hat{G}, \hat{\beta}) \). Therefore, for all \( \phi \in \mathbb{C}_c(\hat{G}) \) we have \( \hat{s}_i(\phi) \to \hat{s}_0(\phi) \). When we
remember that characters in \( \hat{G} \) act on functions in \( \mathbb{C}_c(\hat{G}) \) via equation (1) we
see that this says, for all \( \phi \in \mathbb{C}_c(\hat{G}) \),

\[
(2) \quad \int_{\hat{G}} \phi(\omega)\omega(s_i)d\hat{\beta}x_i(\omega) \to \int_{\hat{G}} \phi(\omega)\omega(s_0)d\hat{\beta}x_0(\omega).
\]

Now suppose we have a relatively compact open neighborhood \( V \) of \( x_0 \) in \( G \).
Then using the continuity of multiplication, there exists a relatively compact
open neighborhood \( U \) of \( x_0 \) in \( G \) such that \( U^2 \subseteq V \). Choose \( h \in \mathbb{C}_c(G) \) such
that \( h(x_0) = 1 \) and \( \text{supp}(h) \subseteq U \). Let \( f = h^* \ast h \). Then \( f \in \mathbb{C}_c(G) \) and a
simple calculation shows that \( \text{supp}(f) \subseteq V \). From now on, let \( f^x \) denote the
restriction of \( f \) to \( G_x \). It is clear from the definition of \( f \) and [4, Section 1.4.2]
that it is a positive definite function on each fibre and therefore satisfies the
conditions of Bochner’s theorem and the inversion theorem on each fibre. In
particular, it can be shown using [4, Section 1.4.3] that for each \( x \) there exists
a finite positive measure \( \mu^x \) on \( \hat{G}_x \) (extended to \( \hat{G} \) by giving everything else
measure zero) such that

\[
f(s) = \int_{\hat{G}} \omega(s)d\mu^x(\omega).
\]

Furthermore, it is easy to prove using [4, Section 1.4.1] that \( \mu^x(\hat{G}) = \mu^x(\hat{G}_x) = ||f^x||_\infty \leq ||f||_\infty \) for all \( x \in X \) so that \( \{\mu^x\} \) is a bounded collection of finite
measures. Additionally, it is shown in the proof of [4, Section 1.5.1] that, as
measures on \( \hat{G}_x \),

\[
f^x d\hat{\beta}^x = d\mu^x.
\]

Proposition 5 states that given \( f \in \mathbb{C}_c(G) \) the Gelfand transform of \( f \) restricts
to the usual Fourier transform fibrewise. Therefore, since everything outside
\( \hat{G}_x \) has measure zero, we may as well write

\[
(3) \quad \hat{f} d\hat{\beta}^x = d\mu^x.
\]
Now, if $\phi \in C_c(\hat{G})$ then $\hat{\phi}f$ is compactly supported. It follows from (2) that

\[
\int_{\hat{G}} \phi(\omega) \hat{f}(\omega) \omega(s_i) d\hat{\beta}^{x_i}(\omega) \to \int_{\hat{G}} \phi(\omega) \hat{f}(\omega) \omega(s_0) d\hat{\beta}^{x_0}(\omega).
\]

Using (3), we can rewrite (4) as

\[
\int_{\hat{G}} \phi(\omega) \omega(s_i) d\mu^{x_i}(\omega) \to \int_{\hat{G}} \phi(\omega) \omega(s_0) d\mu^{x_0}(\omega).
\]

We can extend (5) to functions $\phi \in C_c(\hat{G})$ by noting that $C_c(\hat{G})$ is uniformly dense in $C_0(\hat{G})$ and doing a straightforward approximation argument using the fact that the $\{\mu^{x_i}\}$ are uniformly bounded.

Let $g \in C_c(G)$. Observe that

\[
\hat{g}^{x_i}(\omega) \omega(s_i) = \int_{G_{x_i}} \hat{g}^{x_i}(s) \omega(s) \omega(s_i) d\beta^{x_i}(s)
\]

\[
= \int_{G_{x_i}} \hat{g}^{x_i}(s) \omega(s^{-1} s_i) d\beta^{x_i}(s)
\]

\[
= \int_{G_{x_i}} \hat{g}^{x_i}(s_i s) \omega(s^{-1}) d\beta^{x_i}(s)
\]

\[
= (\text{It}_{s_i^{-1}} g^{x_i})^{\wedge}(\omega).
\]

Therefore, for all $i$, we have

\[
\int_{\hat{G}} \hat{g}(\omega) \omega(s_i) d\mu^{x_i}(\omega) = \int_{\hat{G}} \hat{g}(\omega) \hat{f}(\omega) \omega(s_i) d\hat{\beta}^{x_i}(\omega)
\]

\[
= \int_{G_{x_i}} \hat{g}^{x_i}(\omega) \hat{f}(\omega) \omega(s_i) d\hat{\beta}^{x_i}(\omega)
\]

\[
= \int_{G_{x_i}} (\text{It}_{s_i^{-1}} g^{x_i})^{\wedge} \hat{f}^{x_i} d\hat{\beta}^{x_i}
\]

\[
= \int_{G_{x_i}} (\text{It}_{s_i^{-1}} g^{x_i})^{\wedge} f^{x_i} d\beta^{x_i},
\]

where the last equality follows from the Plancherel theorem [4, Theorem 1.6.1]. Since $\hat{g} \in C_0(\hat{G})$, it follows from (5) that

\[
\int_{G_{x_i}} \text{It}_{s_i^{-1}} g^{x_i} f^{x_i} d\beta^{x_i} \to \int_{G_{x_0}} \text{It}_{s_0^{-1}} g^{x_0} f^{x_0} d\beta^{x_0}.
\]

We are now ready to attack the convergence of the $s_i$. Choose an open neighborhood $O$ of $s_0$. Using the continuity of multiplication, we can find relatively compact open neighborhoods $V$ and $W$ in $G$ such that $x_0 \in V$, $s_0 \in W$ and $V W \subseteq O$. Furthermore, by intersecting $V$ and $V^{-1}$ we can assume that $V^{-1} = V$. Construct $f$ for $V$ as in the beginning of the proof.
Now choose $g \in C(G)$ so that $0 \leq g \leq 1$, $g(s_0) = 1$, and $g$ is zero off $W$. Then $g \in C_c(G)$ and $\mathcal{F} = g$ so that by equation (6) we have

$$\int_{G_{x_i}} g(s_i t) f(t) d\beta^{x_i}(t) \to \int_{G_{x_0}} g(s_0 t) f(t) d\beta^{x_0}(t).$$

It turns out that $\int g(s_i t) f(t) d\beta^{x_i}(t) = 0$ unless $s_i \in WV^{-1} = WV \subseteq O$. Furthermore, both $g(s_0 x_0)$ and $f(x_0)$ are nonzero by construction, and since both functions are continuous, this implies

$$\int_{G_{x_0}} g(s_0 t) f(t) d\beta^{x_0}(t) \neq 0.$$

It follows from (7) that eventually $s_i \in O$. This of course implies that $s_i \to s_0$ and we are done. \hfill \Box

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