Multifractality in stock indexes: Fact or fiction?

Zhi-Qiang Jiang \textsuperscript{a,b}, Wei-Xing Zhou \textsuperscript{a,b,c,*}

\textsuperscript{a}School of Business, East China University of Science and Technology, Shanghai 200237, China
\textsuperscript{b}School of Science, East China University of Science and Technology, Shanghai 200237, China
\textsuperscript{c}Research Center of Systems Engineering, East China University of Science and Technology, Shanghai 200237, China

Abstract

Multifractal analysis and extensive statistical tests are performed upon intraday minutely data within individual trading days for four stock market indexes (including HSI, SZSC, S&P500, and NASDAQ) to check whether the indexes (instead of the returns) possess multifractality. We find that the mass exponent $\tau(q)$ is linear and the singularity $\alpha(q)$ is close to 1 for all trading days and all indexes. Furthermore, we find strong evidence showing that the scaling behaviors of the original data sets cannot be distinguished from those of the shuffled time series. Hence, the so-called multifractality in the intraday stock market indexes is merely an illusion.

Key words: Econophysics, Multifractal analysis, Bootstrapping, Stock markets

1 Introduction

Econophysics is an emerging interdisciplinary field applying concepts, theories, and tools borrowed from statistical physics, nonlinear sciences, applied mathematics, and complexity sciences to understand the complex self-organizing behaviors of financial markets \cite{1,2,3,4}. This field has become to flourish since the pioneering work of Mantegna and Stanley on the scaling behavior in the dynamics of the Standard & Poor’s 500 index \cite{5}, which is closely related to the Pareto-Lévy law.

* Corresponding author. Address: 130 Meilong Road, School of Business, P.O. Box 114, East China University of Science and Technology, Shanghai 200237, China, Phone: +86 21 64253634, Fax: +86 21 64253152.
Email address: wxzhou@ecust.edu.cn (Wei-Xing Zhou).
proposed by Mandelbrot in the description of cotton price fluctuations [6]. Econophysi- 
cists have uncovered remarkable similarities between financial markets and 
turbulent flows [1, 4]. Such analogues include (but not limited to) the evolution of 
probability densities of financial returns [7] based on the variational theory in turbu-
lence [8, 9, 10, 11], inverse statistics in stock markets [12, 13, 14] motivated by the 
inverse structure function analysis of velocity [15, 16, 17, 18, 19], scale-invariant 
distribution of multipliers defined from volatility of equities [20] and from dissi-
pating energy [21, 22, 23, 24], and intermittency and multifractality of asset returns 
[7, 25].

Indeed, the multifractal nature of equity returns is one of the most important styl-
ized facts. A small part of this literature contains the studies on the foreign exchange 
rate [7, 25, 26, 27, 28, 29, 30, 31], gold price [28], commodity price [32], returns 
of stock price or indexes [32, 33, 34, 35, 36, 37, 38, 39, 40, 41], and so on. We 
note that the quantity price (or its logarithm) in financial markets is the analogue 
of velocity in turbulence. Similarly, the counterpart of velocity difference in fluid 
mechanics is the asset return. In this framework, it is natural that numerous multi-
fractal analyses have been carried out on the returns for financial equities similar to 
the velocity differences for turbulent flows.

However, there are exceptions, where analysis is performed on several indexes di-
rectly rather than their variations (the returns) and the presence of multifractality in 
the several indexes is claimed [42, 43, 44]. Specifically, they performed multifractal 
analysis on the intraday high-frequency data of Hang Seng Index (HSI), Shanghai 
Stock Exchange Composite Index (SSEC), and Shenzhen Stock Exchange Com-
posite Index (SZEC) within individual trading days. The extracted “multifractal” 
spectra \( f(\alpha) \) were then utilized to predict abnormal price movements and serve as 
a risk measure in risk management. It seems to us that a careful scrutiny on the 
obtained multifractality should be undertaken based on the extremely narrow spec-
tra of the singularity \( \alpha \). Two problems arise, casting doubts on the aforementioned 
analysis [45].

Firstly, based on the multifractal theory, there exists a constant \( \alpha(t) \) for each mo-
ment \( t \) such that the investigated measure \( \mu \) on the neighbor \( B(t, l) \) of \( x \) scale with 
\( l \) when the scale \( l \to 0 \),

\[
\mu \left( B(t, l) \right) \sim l^{\alpha(t)}. \tag{1}
\]

The measure \( \mu \) is singular at arbitrary moment \( t \) with the singularity strength being 
\( \alpha(t) \). When \( \mu \) is defined as the sum of index prices within a given time interval, 
\( \mu \left( B(t, l) \right) \) is approximately proportional to \( l \), that is, \( \alpha(t) \approx 1 \) for all \( t \). This sug-
gests that the measure \( \mu \) does not possess multifractal nature. This inference is 
further supported by the fact that the span of singularity strength \( \Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}} \approx 0 \) in the real data [42, 43, 44].

Secondly, in the analysis of multifractality in turbulence or high-frequency financial 
data, the moment order \( q \) should not be greater than 8 in order to make the partition
function converge. Specifically, it is shown that the size of a time series should be no less than one million to ensure the estimate of its eighth order partition function statistically significant [19, 46]. The situation is similar for high-frequency financial data [20]. Hence, it is of little significance to compute partition function for higher orders. In the analysis of minutely (or five-minute) data within a time period of one day [42, 43, 44], the size of the intraday high-frequency data is no more than 240 while the moment order is taken to be \(-120 \leq q \leq 120\). This usually broad interval of \(q\) casts further doubts on the reported multifractality in the indexes.

Despite of the specific considerations discussed above, it is worthwhile to put further comments in general on the investigation of multifractality in financial data. The multifractal features in financial series have attracted great interests, however, the origin and significance of the extracted “multifractality” is less concerned. On one hand, it has been shown that an exact monofractal financial model can lead to an artificial multifractal behavior [47]. On the other hand, a time series of the price fluctuations possessing multifractal nature usually has either fat tails in the distribution or long-range temporal correlation or both [48]. However, possessing long memory is not sufficient for the presence of multifractality and one has to have a nonlinear process with long-memory in order to have multifractality [49]. In many cases, the null hypothesis that the reported multifractal nature is stemmed from the large fluctuations of prices cannot be rejected [50].

In this work, we focus on the presence of multifractal feature in stock market indexes and testing whether the obtained empirical multifractality stems from random fluctuations. To address these issues, we adopt the bootstrap approach by shuffling the intraday index series and perform multifractal analysis on them. The results are compared with that from original data. This paper is organized as follows. In Sec. 2 we describe the data sets we investigate. The basic multifractal method is explained in detail in Sec. 3. Multifractal analysis of the data sets is presented in Sec. 4. Statistical bootstrapping tests are conducted in Sec. 5. Finally, Sec. 6 concludes.

2 Data sets

To gain a more profound insight into the multifractality in intraday stock market indexes, we investigate four important indexes, i.e., the Hang Seng Index (HSI), Shenzhen Stock Exchange Composite Index (SZSC), Standard & Poor’s 500 Index (S&P 500), and the National Association of Securities Dealers Automated Quotation (NASDAQ). HSI and SZSC are selected since they were used in the original work of this topic [42, 43, 44]. Both the Hongkong Stock Exchange and Shenzhen Stock Exchange are emerging markets. The S&P 500 and NASDAQ that are representative of mature stock markets are chosen for comparison.

The data have been recorded at each minute in trading days. The HSI index covers
from Jan. 2, 1997 to May 28, 1997, the SZSC index is from Nov. 12, 2001 to Aug. 17, 2006, the S&P 500 index is recorded from Jan. 2, 1997 to Feb. 26, 1999, and the NASDAQ index ranges from Aug. 18, 2000 to Oct. 30, 2000. Eliminating the weekend, holidays and, the days having recording errors, there are 101 days for the HSI data, 1149 days for the SZSC data, 448 days for the S&P 500 data, and 45 days for the NASDAQ data, respectively.

3 Method

We use the box counting method following the work of [42, 43, 44] to investigate the multifractal nature of the index series of each trading day. Denote the intraday index series as \( \{ I(t) : t = 1, 2, \ldots, T \} \), where \( T = 240 \) for HSI and SZSC, \( T = 405 \) for S&P 500, and \( T = 390 \) for NASDAQ, respectively. For a given box size \( l \), we obtain \( N = T/l \) boxes and construct a measure \( \mu \) on each box as follows,

\[
\mu(n; l) = \mu([((n-1)l+1, nl]]) = \sum_{i=1}^{l} I[((n-1)l+i], \tag{2}
\]

where \( [((n-1)l+1, nl] \) is the \( n \)-th box and \( l \in [1, 2, 3, 4, 6, 10, 15, 20, 30, 40, 60, 80, 120, 240] \) for HSI and SZSC, \( l \in [1, 3, 5, 9, 15, 27, 45, 81, 135, 405] \) for S&P 500, and \( l \in [1, 2, 3, 5, 10, 13, 15, 26, 30, 39, 78, 130, 195, 390] \) for NASDAQ, respectively. The sizes \( l \) of the boxes are chosen such that the number of boxes of each size is an integer to cover the whole time series.

We then construct the partition function \( \chi_q \) as

\[
\chi_q(l) = \sum_{n=1}^{N} \left[ \frac{\mu(n; l)}{\sum_{m=1}^{N} \mu(m; l)} \right]^{q}, \tag{3}
\]

and expect it to scale as

\[
\chi_q(l) \sim l^{\tau(q)}, \tag{4}
\]

which defines the exponent \( \tau(q) \). The local singularity exponent \( \alpha \) of the measure \( \mu \) and its spectrum \( f(\alpha) \) are related to \( \tau(q) \) through a Legendre transformation [51]

\[
\left\{ \begin{array}{l}
\alpha = d\tau(q)/dq \\
f(\alpha) = q\alpha - \tau(q)
\end{array} \right. \tag{5}
\]

In order to keep the comparability of our results with those in [44], we also pose \(-120 \leq q \leq 120\).

When \( \mu(n; l) / \sum \mu(m; l) \ll 1 \) and \( q \gg 1 \), the estimate of the partition function \( \chi \) will be very difficult since the value is so small that it is out of the memory. To overcome this problem, we can calculate the logarithm of the partition function
\[ \ln \chi_q(l) = \ln \sum_{n=1}^{N} \left[ \frac{\mu(n; l)}{\max_m \{ \mu(m; l) \}} \right]^q + q \ln \left[ \frac{\max_m \{ \mu(m; l) \}}{\sum \mu(m; l)} \right], \] 

where \( \max_m \{ \mu(m; l) \} \) is the maximum of \( \mu(m; l) \) for \( m = 1, 2, \cdots, N \).

4 Multifractal analysis

Four dates (Jan. 8, 1997 for HSI, Nov. 26, 2001 for SZSC, Feb. 10, 1997 for S&P 500, and Aug. 22, 2000 for NASDAQ) are taken as examples to show the results of multifractal analysis. Figure 1 shows the dependence of the partition function \( \chi_q(l) \) on the box size \( l \) for different values of \( q \) in log-log coordinates. Excellent power-law scaling of \( \chi_q(l) \) with respect to \( l \) has been observed and the scaling range covers all the selected values of \( l \). The solid lines are the best linear fits to the data.

Fig. 1. Plots of \( \chi_q(l) \) as a function of the box size \( l \) for different values of \( q \) in log-log coordinates. The solid lines are the least-squares fits to the data using linear regression (in log-log coordinates) corresponding to power laws. (a) HSI, (b) SZSC, (c) S&P 500, and (d) NASDAQ.

The scaling exponents \( \tau(q) \) are given by the slopes of the linear fits to \( \ln \chi_q(l) \) with respect to \( \ln l \) for different values of \( q \). Figure 2 plots the dependence of
the mass exponents $\tau(q)$ as a function of the moment order $q$. One observes that there is an evident linear relationship between $\tau(q)$ and $q$ for all the four examples. The solid lines are the least-squares fits to the data. The slopes of the lines are respectively $\bar{\alpha} = 1.000 \pm 0.001$ for HSI, $\bar{\alpha} = 1.00000 \pm 0.00003$ for SZSC, $\bar{\alpha} = 1.00000 \pm 0.00001$ for S&P 500, and $\bar{\alpha} = 1.0001 \pm 0.0001$ for NASDAQ, respectively. All the corresponding correlation coefficients of the linear fits are equal to 1.0000. Furthermore, the linear relationships are also hold for other trading days. Therefore, there is no evidence of nonlinearity in the functions $\tau$ and the intraday stock market index do not exhibit multifractal nature. Since $\alpha(q) = d\tau(q)/dq$, we expect that $\alpha(q) \approx 1$ for all $q$, as expected in our discussion in Sec. I.

Fig. 2. Dependence of the scaling exponent $\tau(q)$ on the order $q$. The solid lines are the least-squares fits to the data. (a) HSI, (b) SZSC, (c) S&P 500, and (d) NASDAQ.

Figure 3 presents the multifractal singularity spectra $f(\alpha)$ obtained through Legendre transformation of $\tau(q)$ defined by Eq. (5). The curves in Fig. 3 have the geometrical features of the conservable multifractal spectra [4, 5], which makes them look as if there is sound evidence for the presence of multifractality. However, when looking at the dispenseness of the singularity strength $\Delta \alpha \triangleq \alpha_{\text{max}} - \alpha_{\text{min}}$, we find that $\Delta \alpha$ is very close to zero. It is well-known that $\Delta \alpha$ is an important parameter qualifying the width of the extracted multifractal spectrum. The larger is the $\Delta \alpha$, the stronger is the multifractality. According to Fig. 3 even in the case of $-120 \leq q \leq 120$, $\Delta \alpha < 0.002$ for NASDAQ. One can see that the values of $\Delta \alpha$ for other indexes are much smaller than that of NASDAQ. This observation indicates that there is no multifractality in stock market indexes.
5 Statistical tests for multifractality

We access further the statistical significance of the empirical multifractality in the spirit of bootstrapping. For a given intraday time series, we reshuffle the series to remove any potential temporal correlation and carry out the same multifractal analysis as for the original data. For the four examples discussed in Sec. 4, we compute the multifractal spectra of ten reshuffled time series for each index. The results are illustrated in Fig. 4 where the solid lines are associated with real stock market
indexes, while the dotted lines are obtained from the shuffled data of the corresponding indexes. We find that the multifractal spectra of the real indexes $f(\alpha)$ and that of the shuffled data $f_{\text{rnd}}(\alpha_{\text{rnd}})$ are almost overlapping together in Fig.4(b) and (c). Although the solid lines and the dotted line can be distinguished clearly in Fig.4 (a) and (d), the differences between $\alpha$ and $\alpha_{\text{rnd}}$ are ignorable. In other words, the multifractal nature in the real indexes is insignificant in these examples.

For each intraday time series, we shuffle the data for 1000 times. The associated multifractal spectra are obtained. For each singularity spectrum, we calculate two characteristic quantity, $\Delta \alpha$ and $F \triangleq [f(\alpha_{\text{min}}) + f(\alpha_{\text{max}})]/2$. Figure 5 shows the scatter plots of $F_{\text{rnd}}$ for the shuffled data versus the corresponding $\Delta \alpha_{\text{rnd}}$ for the four example trading days. Clear linear relationship between $F_{\text{rnd}}$ and $\Delta \alpha_{\text{rnd}}$ for each case is observed and we have

$$F_{\text{rnd}} = k \Delta \alpha_{\text{rnd}} + b,$$

where $k = -30.31$ and $b = 1.00$ for HSI, $k = -30.10$ and $b = 1.00$ for SZSC, $k = -30.05$ and $b = 1.05$ for S&P 500, and $k = -30.31$ and $b = 1.06$ for NASDAQ, respectively. The open circle in each plot of Fig.5 presents the values of $F$ and $\Delta \alpha$ for the real data.

![Fig. 5. Scatter plots of the dependence of the shuffled $F_{\text{rnd}}$ and the corresponding $\Delta \alpha_{\text{rnd}}$. (a) HSI, (b) SZSC, (c) S&P 500, and (d) NASDAQ.](image)

Two striking facts emerge from Fig. 5. First, the 4000 points of $(\Delta \alpha_{\text{rnd}}, F_{\text{rnd}})$ collapse on a same linear line since the values of $k$ and $b$ are identical for the four plots. Second, the four points of $(\Delta \alpha, F)$ for the four real data sets also locate on
the same line. For other trading days, we have observed similar phenomena, which put further evidence on our conclusion that the real and reshuffled time series have undistinguishable scaling behaviors.

The values of $\Delta \alpha$ and $F$ for each original time series are compared with the averages $\langle \Delta \alpha_{\text{rnd}} \rangle$ and $\langle F_{\text{rnd}} \rangle$ of the 1000 corresponding shuffled data sets. The results for the four indexes are illustrated in Fig. 6. The solid line is the main diagonal $y = x$. We find that $\Delta \alpha \approx \langle \Delta \alpha_{\text{rnd}} \rangle$ and $F \approx \langle F_{\text{rnd}} \rangle$ for all cases, which implied that the multifractal spectra of the shuffled data are very close to that of the real data and the $f(\alpha)$ curves of real index data can be completely interpreted by the random fluctuations of the original data sets. We stress that there are no extreme values in the intraday index prices so that one can not attribute the observed multifractality to tail fatness that is absent in the present case. Hence, the multifractal property in high-frequency stock market indexes obtained by partition function method is not statistically significant. It is just an illusion.

Fig. 6. Comparison of $F$ and $\Delta \alpha$ obtained from the shuffled data and the real data. (a) HSI, (b) SZSC, (c) S&P 500, and (d) NASDAQ.

In the presence case, to test the presence of multifractality amounts to testing whether the local singularity exponent $\alpha \neq 1$, or $\Delta \alpha \neq 0$. As a last step, we impose a very strict null hypothesis to investigate whether the $f(\alpha)$ spectrum is wider than those produced by chance. The null hypothesis is the following:

$$H_0 : \Delta \alpha \leq \Delta \alpha_{\text{rnd}}.$$  \hspace{1cm} (8)

We can compute the $p$-value, which is the probability that the null hypothesis is
true. The smaller the \( p \)-value, the stronger the evidence against the null hypothesis and favors the alternative hypothesis that the presence of multifractality is statistically significant. The false probability is estimated by

\[
p_1 = \Pr(\Delta \alpha \leq \Delta \alpha_{\text{rnd}}).
\]  

(9)

Under the conventional significance level of 0.05, the multifractal phenomenon is statistically significant if and only if \( p_1 \leq 0.05 \). While \( p_1 > 0.05 \), the null hypothesis cannot be rejected. A similar null hypothesis can be described as follows:

\[
H_0 : F \geq F_{\text{rnd}},
\]

(10)

where the false probability is

\[
p_2 = \Pr(F \geq F_{\text{rnd}}).
\]

(11)

Using the conventional significance level of 0.05, the multifractal phenomenon is statistically significant if and only if \( p_2 \leq 0.05 \).

For the four examples shown in Fig. 5, we find that \( p_1 = 1 \) and \( p_2 = 1 \) for HSI, \( p_1 = 0.108 \) and \( p_2 = 0.109 \) for SZSC, \( p_1 = 0.452 \) and \( p_2 = 0.456 \) for S&P 500, and \( p_1 = 0 \) and \( p_2 = 0 \) for NASDAQ. Obviously, we can not distinguish the real data from the shuffled data beside NASDAQ for the chosen trading days. We also find that \( p_1 \approx p_2 \) for all the trading days. More generally, Table 1 shows the statistical tests for the all the each trading days. About half of the trading days can not pass the statistical inference, indicating that multifractality is absent in the those trading series.

Table 1
Statistical tests for the presence of multifractal nature in the four indexes investigated.

| Indexes       | HSI   | SZSC  | S&P 500 | NASDAQ |
|---------------|-------|-------|---------|--------|
| Percentage of \( p_1 \leq 0.05 \) | 54.6% | 56.1% | 54.4%   | 53.9%  |
| Percentage of \( p_2 \leq 0.05 \) | 54.4% | 55.8% | 53.6%   | 53.6%  |

6 Conclusion

We have investigated the multifractal features in intraday minutely high-frequency stock market indexes (including HSI, SZSC, S&P 500, and NASDAQ) for individual trading days. The resultant scaling functions \( \tau(q) \) have been confirmed to be linear and the singularities \( \alpha \) are close to 1 so that \( \Delta \alpha \) is close to 0. This analysis implies that there is no multifractality in the indexes. Further evidence based on bootstrapping technique shows that the scaling behavior of the shuffled data is undistinguishable from that of the raw data. Specifically, we find that,
almost all points \((\Delta \alpha, F)\) of the raw data sets locate on the same straight line
\[ F_{\text{rnd}} = -30 \Delta \alpha_{\text{rnd}} + 1 \]
extracted from the points \((\Delta \alpha_{\text{rnd}}, F_{\text{rnd}})\) of the shuffled data;
(2) for each time series, \(\Delta \alpha \approx \langle \Delta \alpha_{\text{rnd}} \rangle\) and \(F \approx \langle F_{\text{rnd}} \rangle\); and (3) the two rather
strict null hypotheses cannot be rejected for about half of the time series. There is
thus no doubt that the reported multifractal nature in the indexes of HSI and SZSC
is not a fact but a fiction. This conclusion is further verified by two
indexes (S&P 500 and NASDAQ) in a developed stock market. We believe that our
analysis and conclusion apply for other market indexes or common stock prices
when one concerns intraday stock prices or indexes rather than their returns.

In addition, we cast doubts on the efforts to use this illusionary multifractal feature
to forecast the stock market [43] and to define a risk index for risk management
[44]. However, to be more conservative, we do not deny the potential usefulness
of those techniques proposed based on some nonexistent properties. The idea to
use multifractal nature to predict or to manage risks in stock markets should be
investigated based on the returns or other alternative financial quantities. After all,
one cannot build a palace on a sand beach.

Acknowledgments:

We are indebted to Prof. Bing-Hong Wang for providing the HSI data and fruitful
discussion. This work was partly supported by the National Natural Science Foun-
dation of China (Grant No. 70501011), the Fok Ying Tong Education Foundation
(Grant No. 101086), and the Shanghai Rising-Star Program (No. 06QA14015).

References

[1] R. N. Mantegna, H. E. Stanley, An Introduction to Econophysics: Correlations
and Complexity in Finance, Cambridge University Press, Cambridge, 2000.
[2] J.-P. Bouchaud, M. Potters, Theory of Financial Risks: From Statistical
Physics to Risk Management, Cambridge University Press, Cambridge, 2000.
[3] D. Sornette, Why Stock Markets Crash: Critical Events in Complex Financial
Systems, Princeton University Press, Princeton, 2003.
[4] W.-X. Zhou, Guide to Econophysics (in Chinese), Shanghai University of Fi-
nance and Economics Press, Shanghai, 2007.
[5] R. N. Mantegna, H. E. Stanley, Scaling behaviour in the dynamics of an eco-
nomic index, Nature 376 (1995) 46–49.
[6] B. B. Mandelbrot, The variation of certain speculative prices, J. Business 36
(1963) 394–419.
[7] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, Y. Dodge, Turbulent cas-
cades in foreign exchange markets, Nature 381 (1996) 767–770.
[8] B. Castaing, Y. Gagne, E. J. Hopfinger, Velocity probability density functions
of high Reynolds number turbulence, Physica D 46 (1990) 177–200.
[9] B. Castaing, Y. Gagne, M. Marchand, Log-similarity for turbulent flows?, Physica D 68 (1993) 387–400.
[10] B. Castaing, B. Chabaud, B. Hébral, A. Naert, J. Peinke, Velocity probability density functions in developed turbulence: A finite Reynolds theory, Physica B 194-196 (1994) 695–696.
[11] B. Castaing, Scalar intermittency in the variational theory of turbulence, Physica D 73 (1994) 31–37.
[12] I. Simonsen, M. H. Jensen, A. Johansen, Optimal investment horizons, Eur. Phys. J. B 27 (2002) 583–586.
[13] W.-X. Zhou, W.-K. Yuan, Inverse statistics in stock markets: Universality and idiosyncracy, Physica A 353 (2005) 433–444.
[14] K. Karpio, M. A. Zaluska-Kotur, A. Orłowski, Gain-loss asymmetry for emerging stock markets, Physica A 375 (2007) 599–604.
[15] M. H. Jensen, Multiscaling and structure functions in turbulence: An alternative approach, Phys. Rev. Lett. 83 (1999) 76–79.
[16] L. Biferale, M. Cencini, D. Vergni, A. Vulpiani, Exit time of turbulent signals: A way to detect the intermediate dissipative range, Phys. Rev. E 60 (1999) R6295–R6298.
[17] F. Schmitt, Explicit predictability and dispersion scaling exponents in fully developed turbulence, Phys. Lett. A 342 (2005) 448–458.
[18] B. R. Pearson, W. van de Water, Inverse structure functions, Phys. Rev. E 71 (2005) 036303.
[19] W.-X. Zhou, D. Sornette, W.-K. Yuan, Inverse statistics and multifractality of exit distances in 3D fully developed turbulence, Physica D 214 (2006) 55–62.
[20] Z.-Q. Jiang, W.-X. Zhou, Scale invariant distribution and multifractality of volatility multiplier in stock markets, Physica A 381 (2007) 343–350.
[21] A. B. Chhabra, K. R. Sreenivasan, Negative dimensions: Theory, computation and experiment, Phys. Rev. A 43 (1991) 1114–1117.
[22] A. B. Chhabra, K. R. Sreenivasan, Scale-invariant multiplier distribution in turbulence, Phys. Rev. Lett. 68 (1992) 2762–2765.
[23] B. Jouault, P. Lipa, M. Greiner, Multiplier phenomenology in random multiplicative cascade processes, Phys. Rev. E 59 (1999) 2451–2454.
[24] B. Jouault, M. Greiner, P. Lipa, Fix-point multiplier distributions in discrete turbulent cascade models, Physica D 136 (2000) 125–144.
[25] R. N. Mantegna, H. E. Stanley, Turbulence and financial markets, Nature 383 (1996) 587–588.
[26] N. Vandewalle, M. Ausloos, Sparseness and roughness of foreign exchange rates, Int. J. Modern Phys. C 9 (1998) 711–719.
[27] F. Schmitt, D. Schertzer, S. Lovejoy, Multifractal analysis of foreign exchange data, Appl. Stoch. Models Data Analysis 15 (1999) 29–53.
[28] K. Ivanova, M. Ausloos, Low q-moment multifractal analysis of Gold price, Dow Jones Industrial Average and BGL-USD exchange rate, Eur. Phys. J. B 8 (1999) 665–669.
[29] R. Baviera, M. Pasquini, M. Serva, D. Vergni, A. Vulpiani, Correlations and multi-affinity in high frequency financial datasets, Physica A 300 (2001) 551–
S. V. Muniandy, S. C. Lim, R. Murugan, Inhomogeneous scaling behaviors in Malaysian foreign currency exchange rates, Physica A 301 (2001) 407–428.

Z.-X. Xu, R. Gençay, Scaling, self-similarity and multifractality in FX markets, Physica A 323 (2003) 578–590.

K. Matia, Y. Ashkenazy, H. E. Stanley, Multifractal properties of price fluctuations of stock and commodities, Europhys. Lett. 61 (2003) 422–428.

A. Turiel, C. J. Pérez-Vicente, Multifractal geometry in stock market time series, Physica A 322 (2003) 629–649.

P. Oświęcimka, J. Kwapien, S. Drożdż, R. Rak, Investigating multifractality of stock market fluctuations using wavelet and detrending fluctuation methods, Acta Physica Polonica B 36 (2005) 2447–2457.

L. Olsen, Multifractal geometry, Progress in Probability 46 (2000) 3–37.

A. Turiel, C. J. Pérez-Vicente, Role of multifractal sources in the analysis of stock market time series, Physica A 355 (2005) 475–496.

P. Norouzzadeh, G. R. Jafari, Application of multifractal measures to Tehran price index, Physica A 356 (2005) 609–627.

A. Bershadskii, Multifractal diffusion in NASDAQ, J. Phys. A 34 (2001) L127–130.

I. Andreadis, A. Serletis, Evidence of a random multifractal turbulent structure in the Dow Jones Industrial Average, Chaos Solitons & Fractals 13 (2002) 1309–1315.

A. Z. Górski, S. Drożdż, J. Speth, Financial multifractality and its subtleties: An example of DAX, Physica A 316 (2002) 496–510.

M. Balcilar, Multifractality of the Istanbul and Moscow stock market returns, Emerging Markets Fin. Trade 39 (2) (2003) 5–46.

X. Sun, H.-P. Chen, Z.-Q. Wu, Y.-Z. Yuan, Multifractal analysis of Hang Seng index in Hong Kong stock market, Physica A 291 (2001) 553–562.

X. Sun, H.-P. Chen, Y.-Z. Yuan, Z.-Q. Wu, Predictability of multifractal analysis of Hang Seng stock index in Hong Kong, Physica A 301 (2001) 473–482.

Y. Wei, D.-S. Huang, Multifractal analysis of SSEC in Chinese stock market: A different empirical result from Heng Seng, Physica A 355 (2005) 497–508.

W.-X. Zhou, Illusionary multifractality in high-frequency data of Shanghai Stock Exchange Composite Index (in Chinese), preprint (2007).

V. S. L’vov, E. Podivilov, A. Pomyalov, I. Procaccia, D. Vandembroucq, Improved shell model of turbulence, Phys. Rev. E 58 (1998) 1811–1822.

J.-P. Bouchaud, M. Potters, M. Meyer, Apparent multifractality in financial time series, Eur. Phys. J. B 13 (2000) 595–599.

J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, H. E. Stanley, Multifractal detrended fluctuation analysis of non-stationary time series, Physica A 316 (2002) 87–114.

A. Saichev, D. Sornette, Generic multifractality in exponentials of long memory processes, Phys. Rev. E 74 (2006) 011111.

T. Lux, Detecting multifractal properties in asset returns: The failure of the “scaling estimator”, Int. J. Modern Phys. C 15 (2004) 481–491.
[51] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, B. I. Shraiman, Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A 33 (1986) 1141–1151.

[52] W.-X. Zhou, Fractals and Applied Multifractals with Application to Gas-Liquid Two-Phase Turbulent Jets (in Chinese), Ph.D. thesis, ECUST, Shanghai (March 2001).