Removability of singularity for nonlinear elliptic equations with $p(x)$-growth*

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Abstract

Using Moser’s iteration method, we investigate the problem of removable isolated singularities for elliptic equations with $p(x)$-type nonstandard growth. We give a sufficient condition for removability of singularity for the equations in the framework of variable exponent Sobolev spaces.

Keywords: variable exponent space; isolated singularity; removable singularity.

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1 Introduction

In recent years, the research of elliptic equations with variable exponent growth conditions has been an interesting topic. These problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator $-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is inhomogeneous, and these problems have many important applications, see [1, 2, 3]. Since Kováčik and Rákosník first studied the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [4], many results have been obtained concerning these kinds of variable exponent spaces, see examples in [5−12].

In this paper, we study solutions to nonlinear elliptic equations with nonstandard growth in the divergence form

$$-\text{div}A(x,u,\nabla u) + g(x,u) = 0.$$  \hspace{1cm} (1.1)

in a punctured domain $\Omega \setminus \{0\}$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Throughout the paper we suppose that the functions $A(\cdot,\xi,\eta) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $g(\cdot,\xi) : \Omega \times \mathbb{R} \to \mathbb{R}^N$ are measurable for all $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^N$, and $A(x,\cdot,\cdot)$, $g(x,\cdot)$ are continuous for almost all $x \in \Omega$. We also assume that the following structure conditions

$$A(x,\xi,\eta)\eta \geq \mu_1|\eta|^{p(x)},$$  \hspace{1cm} (1.2)

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\[ |A(x, \xi, \eta)| \leq \mu_2|\eta|^{p(x)-1}, \quad (1.3) \]
\[ A(x, \xi, -\eta) = -A(x, \xi, \eta) \quad (1.4) \]
\[ |x|^{-\alpha}|\eta|^q \leq g(x, \xi) \text{sgn} \xi \leq C|x|^{-\alpha}|\eta|^q \quad (1.5) \]
are fulfilled for almost all \( x \in \bar{\Omega}, \xi \in \mathbb{R}, \eta \in \mathbb{R}^N \), where \( \mu_1, \mu_2 > 0, \alpha < N, C > 1 \) are constants, \( p, q \in C(\bar{\Omega}) \), \( 1 < p^- \leq p(x) \leq p^+ < N \), and \( q(x) \gg p(x) - 1 \).

Here we denote
\[
p^- = \inf_{x \in \bar{\Omega}} p(x), \quad p^+ = \sup_{x \in \bar{\Omega}} p(x),
\]
and denote by \( q(x) \gg p(x) - 1 \) the fact that \( \inf_{x \in \bar{\Omega}} (q(x) - p(x) + 1) > 0 \).

For the Laplace’s equation, a set of capacity zero constitutes a removable singularity for a bounded harmonic function, while, a single point \( x_0 \) is removable if the solution is \( o(\log|x-x_0|) \) or \( o(|x-x_0|^{2-N}) \).

Serrin [13] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of \( g(x, u) \equiv 0 \), it is shown that at an isolated singularity a positive solution has precisely the order of growth \( |x-x_0|^{\frac{p-N}{p-1}} \) if \( 1 < p < N \), or \( \log \frac{1}{|x-x_0|} \) if \( p = N \).

Brezis and Veron [14] studied the equation of form (1.1) with a Laplace operator in the principal part. They proved the removability of isolated singularities for solutions under condition \( g(x, \xi) \text{sgn} \xi \geq |\xi|^q \) and \( q \geq \frac{N}{N-2}, N \geq 3 \).

For the equation of the form:
\[-\text{div} A(x, u, \nabla u) + a_0(x, u, \nabla u) = 0\]
Serrin [13, 15] considered the conditions of removability of an isolated singular point \( x_0 \), the condition has the form
\[ u(x) = o \left( |x-x_0|^{\frac{p-N}{p-1}}^{\frac{p-N}{p-1}+\tau} \right), \quad 1 < p < N, \]
with positive number \( \tau \). Nicolosi et al. [16] obtained a precise condition for the removability of singularities, it has the form
\[ u(x) = o \left( |x-x_0|^{\frac{p-N}{p-1}} \right), \quad 1 < p < N. \]

For equations with weighted functions \( v, w \), Mamedov and Harman [17] proved that an isolated singular point \( x_0 \) is removable for solutions of equation (1.1) if the condition of weighted functions
\[ v(B(x_0, \varepsilon)) \left( \frac{w(B(x_0, \varepsilon))}{\varepsilon^pv(B(x_0, \varepsilon))} \right)^{\frac{q}{q-p+1}} = o(1), \quad \varepsilon \to 0, \]
and \( p > 1, q > p - 1 \) are fulfilled. For the removability of singularities for solutions of elliptic equations with absorption term (see [18, 19]).

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Recently, there have been a few papers on the study of the removability of singularities for the equations with nonstandard growth. Lukkari [20] investigated the removability of a compact set for the equation \(-\text{div} \left( |Du|^{p(x)-2} Du \right) = 0\). For the anisotropic elliptic equation, the removability of a compact set was proved by Cianci [21]. Cataldo and Cianci [22] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of \(g(x, u) = |u|^{q-2} u\).

In this paper, following Moser’s method [23], we establish the condition

\[
1 < \frac{(p(x) - \alpha)q(x)}{q(x) - p(x) + 1} + \alpha \ll N \quad \text{a.e. on } \overline{\Omega}
\] (1.6)

to ensure the removability of singularities.

2 Preliminaries

We first recall some facts on spaces \(L^{p(x)}(\Omega)\) and \(W^{k,p(x)}(\Omega)\). For the details see [4, 8].

Let \(P(\Omega)\) be the set of all Lebesgue measurable functions \(p: \Omega \to [1, \infty]\), we denote

\[\rho_p(u) = \int_{\Omega, \Omega_\infty} |u|^{p(x)} \, dx + \sup_{x \in \Omega_\infty} |u(x)|,\]

where \(\Omega_\infty = \{x \in \Omega: p(x) = \infty\}\).

The variable exponent Lebesgue space \(L^{p(x)}(\Omega)\) is the class of all functions \(u\) such that \(\rho_{p(x)}(tu) < \infty\), for some \(t > 0\). \(L^{p(x)}(\Omega)\) is a Banach space equipped with the norm

\[\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0: \rho_{p(x)} \left( \frac{u}{\lambda} \right) \leq 1\}.\]

For any \(p \in P(\Omega)\), we define the conjugate function \(p'(x)\) as

\[p'(x) = \begin{cases} 
\infty, & x \in \Omega_1 = \{x \in \Omega: p(x) = 1\}, \\
1, & x \in \Omega_\infty, \\
\frac{p(x)}{p(x) - 1}, & x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty).
\end{cases}\]

**Theorem 2.1** Let \(p \in P(\Omega)\). For any \(u \in L^{p(x)}(\Omega)\) and \(v \in L^{p'(x)}(\Omega)\),

\[\int_{\Omega} |uv| \, dx \leq 2\|u\|_{L^{p(x)}} \|v\|_{L^{p'(x)}}.\]

**Theorem 2.2** Let \(p \in P(\Omega)\) with \(p^+ < \infty\). For any \(u \in L^{p(x)}(\Omega)\), we have

1) if \(\|u\|_{L^{p(x)}} \geq 1\), then \(\|u\|_{L^{p^-(x)}} \leq \int_{\Omega} |u|^{p^-(x)} \, dx \leq \|u\|_{L^{p^+(x)}});\]
(2) if \( \|u\|_{L^p(x)} < 1 \), then \( \|u\|_{L^p(x)}^+ \leq \int_\Omega |u|^{p(x)} \, dx \leq \|u\|_{L^p(x)}^- \).

The variable exponent Sobolev space \( W^{1,p(x)}(\Omega) \) is the class of all functions \( u \in L^{p(x)}(\Omega) \) such that \( |\nabla u| \in L^{p'(x)}(\Omega) \). \( W^{1,p(x)}(\Omega) \) is a Banach space equipped with the norm

\[
\|u\|_{W^{1,p(x)}} = \|u\|_{L^p(x)} + \|\nabla u\|_{L^{p(x)}}.
\]

We say that the function \( u(x) \) belongs to the space \( W^{1,p(x)}_{loc}(\Omega) \) if \( u(x) \) belongs to \( W^{1,p(x)}(G) \) in any subdomain \( G, \overline{G} \subset \Omega \).

**Theorem 2.3** For any \( u \in W^{1,p(x)}(\Omega) \), we have

(1) if \( \|u\|_{W^{1,p(x)}} \geq 1 \), then \( \|u\|_{W^{1,p(x)}}^+ \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \leq \|u\|_{W^{1,p(x)}}^- \);

(2) if \( \|u\|_{W^{1,p(x)}} < 1 \), then \( \|u\|_{W^{1,p(x)}}^+ \leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \leq \|u\|_{W^{1,p(x)}}^- \).

From Zhikov [5, 6], we know smooth functions are not dense in \( W^{1,p(x)}(\Omega) \) without additional assumptions on the exponent \( p(x) \). To study the Lavrentiev phenomenon, he considered the following log-Hölder continuous condition

\[
|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)} \tag{2.1}
\]

for all \( x, y \in \Omega \) such that \( |x - y| \leq \frac{1}{2} \). If the log-Hölder continuous condition holds, then smooth functions are dense in \( W^{1,p(x)}(\Omega) \) and we can define the Sobolev spaces with zero boundary values \( W^{1,p(x)}_0(\Omega) \), as the closure of \( C_0^{\infty}(\Omega) \) with the norm of \( \| \cdot \|_{W^{1,p(x)}(\Omega)} \).

**Theorem 2.4** If \( u \in W^{1,p}_0(B_R(a)) \), \( 1 \leq p < N \), then for any \( 1 \leq q \leq p^* \), the inequality

\[
\left( \int_{B_R(a)} |u|^q \, dx \right)^{\frac{1}{q}} \leq C(N, p)R^{1+\frac{N}{p} - \frac{N}{q}} \left( \int_{B_R(0)} |Du|^p \, dx \right)^{\frac{1}{p}} \tag{2.2}
\]

is valid, where \( B_R(a) \) is the ball of radius \( R \) with centre \( a \).

We define \( p_\delta^+ = \sup_{y \in B_\delta(0) \cap \Omega} p(y), \quad p_\delta^- = \inf_{y \in B_\delta(0) \cap \Omega} p(y), \quad q_\delta^+ = \sup_{y \in B_\delta(0) \cap \Omega} q(y), \quad q_\delta^- = \inf_{y \in B_\delta(0) \cap \Omega} q(y), \)

where \( \delta > 0 \) is a constant.

**Lemma 2.1** Since \( q(x) \gg p(x) - 1 \), then the set \( S = \{ \delta : p_\delta^+ - 1 < q_\delta^- \} \) is nonempty, bounded above and \( \delta_0 = \sup \{ \delta : p_\delta^+ - 1 < q_\delta^- \} < +\infty \).

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\textbf{Proof.} As \( q(x) \), \( p(x) \) are continuous on \( \overline{\Omega} \), for \( \varepsilon_1 \in (0, 1) \) and \( 0 \in \Omega \), there exists \( \delta > 0 \) such that \(|q(0) - q(y)| < \varepsilon_1 \) and \(|p(0) - p(y)| < \varepsilon_1 \) whenever \(|y| < \delta \). For any \( y \in B_\delta(0) \cap \overline{\Omega} \), we have
\[
p(y) - 1 < p(0) - 1 + \varepsilon_1,
\]
and
\[
q(y) > q(0) - \varepsilon_1.
\]
As \( q(x) \gg p(x) - 1 \), take \( \varepsilon_1 = \frac{1}{4} \inf_{x \in \Omega} (q(x) - p(x) + 1) \),
\[
q(0) - \varepsilon_1 - (p(0) - 1 + \varepsilon_1) \geq \frac{1}{2} \inf_{x \in \Omega} (q(0) - p(0) + 1) > 0,
\]
then
\[
p(y) - 1 < p(0) - 1 + \varepsilon_1 < q(0) - \varepsilon_1 < q(y),
\]
and further
\[
p^+_\delta - 1 = \sup_{y \in B_\delta(0) \cap \Omega} (p(y) - 1) < q^-_\delta = \inf_{y \in B_\delta(0) \cap \Omega} q(y).
\]
So the set \( S = \{ \delta : p^+_\delta - 1 < q^-_\delta \} \) is nonempty. From the definition of the \( q(x) \gg p(x) - 1 \), we know the set \( S \) is bounded above. By the Continuum Property, it has a smallest upper bound \( \delta_0 \). This smallest upper bound \( \delta_0 \) is called the supremum of the set \( S \). We write \( \delta_0 = \sup S = \sup \{ \delta : p^+_\delta - 1 < q^-_\delta \} \).

Consider a solution \( u(x) \) of equation (1.1) with an isolated singularity. Assume that \( 0 \in \Omega \) and zero is a singular point of the solution \( u(x) \). We say that \( u(x) \) is a solution in \( \Omega \setminus \{0\} \) if \( u \in W^{1,p(x)}(\Omega \setminus \{0\}) \) and for any test function \( \varphi \in W^{1,p(x)}_0(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\}) \) in \( \Omega \setminus \{0\} \), the following equality is true:
\[
\int_\Omega (A(x, u, \nabla u) \varphi + g(x, u) \varphi) \, dx = 0.
\]
(2.3)

We say that the solution \( u(x) \) of equation (1.1) has a removable singularity at the point \( 0 \) if the function \( u(x) \) is a solution in \( \Omega \setminus \{0\} \) and \( u \in W^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\}) \) implies that it belongs to the space \( W^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \) and satisfies (2.3) for any test function \( \varphi \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega) \).

\section{Proof of theorems}

In this section we state and prove the following theorems.

In the sequel by \( C \) we denote a constant, the value of which may vary from line to line.
Theorem 3.1 Let \( u \in W^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\}) \) be a solution of equation (1.1) in \( \Omega \setminus \{0\} \). Assume that conditions (1.2) – (1.5), (2.1) are satisfied. Then for any \( 0 < |x| \leq R < \min\{\text{dist}(0, \partial \Omega), \delta_0, 1\} \), the estimate

\[
|u(x)| \leq C|x|^{-Q},
\]

holds almost everywhere, where \( Q = Q(N, \alpha, p_R, p_R^+, q_R) \) and \( C = C(N, m, n, \mu_2, p_R^+, q_R, q_R^+, R) \).

Proof. For \( \rho < R \) we define a smooth cut-off function \( \varphi_1(x) \) satisfying conditions: \( \varphi_1(x) = 1 \) for \( \frac{\rho}{2} < |x| < \frac{3\rho}{4} \), \( \varphi_1(x) = 0 \) outside the set for \( \frac{\rho}{4} \leq |x| \leq \rho \), \( |\nabla \varphi_1(x)| \leq \frac{C}{\rho} \) and \( 0 \leq \varphi_1(x) \leq 1 \).

Take the test function

\[
\psi(x) = (1 + |u(x)|)^m u(x) \varphi_1(x)^{n+p_R^+} \in W_0^{1,p(x)}(B_R(0) \setminus \{0\}),
\]

\( m, n \geq 0 \) are nonnegative numbers to be determined later, and then

\[
\nabla \psi(x) = m(1 + |u(x)|)^{m-1} \nabla u(x) |u(x)| \varphi_1(x)^{n+p_R^+} + (1 + |u(x)|)^m \nabla u(x) \varphi_1(x)^{n+p_R^+} \\
+ (1 + |u(x)|)^m u(x)(n + p_R^+) \varphi_1(x)^{n+p_R^+ - 1} \nabla \varphi_1(x).
\]

We substitute the test function \( \psi(x) \) into the integral identity (2.3), we obtain

\[
\int_{B_R(0)} mA(x, u, \nabla u)(1 + |u(x)|)^{m-1} \nabla u(x) |u(x)| \varphi_1(x)^{n+p_R^+} dx \\
+ \int_{B_R(0)} A(x, u, \nabla u)(1 + |u(x)|)^m \nabla u(x) \varphi_1(x)^{n+p_R^+} dx \\
+ \int_{B_R(0)} g(x, u)(1 + |u(x)|)^m u(x) \varphi_1(x)^{n+p_R^+} dx \\
+ \int_{B_R(0)} A(x, u, \nabla u)(1 + |u(x)|)^m u(x)(n + p_R^+) \varphi_1(x)^{n+p_R^+ - 1} \nabla \varphi_1(x) dx = 0.
\]

By virtue of the conditions (1.2) – (1.5),

\[
\int_{B_R(0)} \mu_1 m |\nabla u(x)|^{p(x)}(1 + |u(x)|)^{m-1} |u(x)| \varphi_1(x)^{n+p_R^+} dx \\
+ \int_{B_R(0)} \mu_1 |\nabla u(x)|^{p(x)}(1 + |u(x)|)^m \varphi_1(x)^{n+p_R^+} dx \\
+ \int_{B_R(0)} |x|^{-\alpha}|u(x)|^{q(x)+1}(1 + |u(x)|)^m \varphi_1(x)^{n+p_R^+} dx \\
\leq \int_{B_R(0)} \mu_2 (n + p_R^+) |\nabla u(x)|^{p(x)-1}(1 + |u(x)|)^{m+1} \varphi_1(x)^{n+p_R^+ - 1} |\nabla \varphi_1(x)| dx,
\]

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and using Young’s inequality, we have

\[
\int_{B_R(0)} \mu_1 |\nabla u(x)|^{p(x)} (1 + |u(x)|)^m \varphi_1(x)^{n+p_R} dx + \int_{B_R(0)} |x|^{-\alpha} |u(x)|^{q(x)+m+1} \varphi_1(x)^{n+p_R} dx
\]

\[
\leq \mu_2 \int_{B_R(0)} (1 + |u(x)|)^m \varphi_1(x)^{n+p_R} \left[ |\nabla u(x)|^{p(x)-1} \right] \left[ (n + p_R^+(1 + |u(x)|) \varphi_1(x)^{-1} |\nabla \varphi_1(x)| \right] dx
\]

\[
\leq \mu_2 C(\varepsilon_2) \int_{B_R(0)} (n + p_R^+)^{p(x)} (1 + |u(x)|)^{p(x)+m} \varphi_1(x)^{n+p_R^--p(x)} |\nabla \varphi_1(x)|^{p(x)} dx
\]

\[
+ \mu_2 \varepsilon_2 \int_{B_R(0)} (1 + |u(x)|)^m \varphi_1(x)^{n+p_R^--p(x)} |\nabla u(x)|^{p(x)} dx
\]

Take \( \varepsilon_2 = \frac{\mu_1}{2\mu_2} \), we have

\[
\frac{\mu_1}{2} \int_{B_R(0)} |\nabla u(x)|^{p(x)} (1 + |u(x)|)^m \varphi_1(x)^{n+p_R^+} dx + \int_{B_R(0)} |x|^{-\alpha} |u(x)|^{q(x)+m+1} \varphi_1(x)^{n+p_R^+} dx
\]

\[
\leq C(\mu_1, \mu_2) \int_{B_R(0)} (n + p_R^+) \frac{1}{\rho^{p(x)}} (1 + |u(x)|)^{p(x)+m} \varphi_1(x)^{n+p_R^--p(x)} dx.
\]

(3.2)

Denote \( p_R^* = \frac{N p_R^-}{N - p_R^-} = k p_R^- \). Since \( u(x) \in W^{1,p(x)}(B_R(0) \setminus \{0\}) \), then \( u(x) \in W^{1,p_R^-}(B_R(0) \setminus \{0\}) \) and \( \phi(x) = \left[ (1 + |u(x)|)^{t+p_R^-} \varphi_1(x)^{s+p_R^-} \right]^{\frac{1}{\rho^{p_R^-}}} \in W_0^{1,p_R^-}(B_R(0)) \), where \( t + p_R^+ > k p_R^- \), \( s + p_R^+ > k p_R^- \).

As \( 1 < p_R^- < N \), applying (2.2) to the function \( \phi(x) \), we have

\[
\int_{B_R(0)} (1 + |u(x)|)^{t+p_R^+} \varphi_1(x)^{s+p_R^+} dx
\]

\[
\leq C(N, p_R^-) \left( \int_{B_R(0)} |\nabla \phi(x)|^{p_R^-} dx \right)^k
\]

\[
= C(N, p_R^-) \left\{ \int_{B_R(0)} \left[ \left( \frac{t + p_R^+}{k p_R^-} \right)^{p_R^-} (1 + |u(x)|)^{\frac{t+p_R^+}{k} - p_R^-} |\nabla u(x)|^{p_R^-} \varphi_1^{\frac{s+p_R^+}{k} - p_R^-} \right] dx \right\}^k
\]

\[
+ \left( \frac{s + p_R^+}{k p_R^-} \right)^{p_R^-} (1 + |u(x)|)^{\frac{s+p_R^+}{k} - p_R^-} |\nabla \varphi_1|^{p_R^-} dx \right\}^k
\]

\[
\leq C(N, p_R^-) \left( \frac{t + s + p_R^+}{k p_R^-} \right)^{p_R^-} \left\{ \int_{B_R(0)} \left[ (1 + |u(x)|)^{\frac{t+p_R^+}{k} - p_R^-} |\nabla u(x)|^{p_R^-} \varphi_1^{\frac{s+p_R^+}{k} - p_R^-} \right] dx \right\}^k
\]

\[
+ \left( \frac{1}{\rho} \right)^{p_R^-} (1 + |u(x)|)^{\frac{t+p_R^+}{k} - p_R^-} \varphi_1^{\frac{s+p_R^+}{k} - p_R^-} dx \right\}^k
\]

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Taking \( m = \frac{t+p_R^-}{k} - p_R^- \), \( n + p_R^+ = \frac{s+p_R^-}{k} \) in (3.2) and using Young’s inequality, we have

\[
\int_{B_R(0)} (1 + |u(x)|) \frac{t+p_R^-}{x} - p_R^- |\nabla u(x)|^{p_R^-} \varphi_1^{s+p_R^-} \, dx \\
\leq \int_{B_R(0)} (1 + |u(x)|) \frac{t+p_R^-}{x} - p_R^- |\nabla u(x)|^{p(x)} \varphi_1^{s+p_R^-} \, dx + \int_{B_R(0)} (1 + |u(x)|) \frac{t+p_R^-}{x} - p_R^- \varphi_1^{s+p_R^-} \, dx
\]

(3.4)

\[
\leq C(\mu_1, \mu_2) \left( s + p_R^+ \right) \frac{1}{\rho p_R^-} \int_{B_R(0)} (1 + |u(x)|) \frac{t+p_R^-}{x} - p_R^- + p(x) \, dx.
\]

From (3.3) and (3.4) we get

\[
\int_{B_R(0)} (1 + |u(x)|)^{t+p_R^+} \varphi_1(x)^{s+p_R^-} \, dx \\
\leq C(s + p_R^+)^{kp_R^-} (t + s + p_R^+) \frac{1}{\rho p_R^-} \left[ \int_{B_R(0)} (1 + |u(x)|)^{t+p_R^-} \varphi_1^{s+p_R^-} \, dx \right]^k,
\]

(3.5)

where \( C = C(N, \mu_1, \mu_2, p_R^+, p_R^-) \).

Denote

\[
I_i = \int_{B_R(0)} (1 + |u(x)|)^{t_i+p_R^-} \varphi_1(x)^{s_i+p_R^-} \, dx,
\]

\[
t_i = (q_R^- + kp_R^-) k^i - p_R^+ + \frac{(p_R^- - p_R^-) N}{p_R^-},
\]

\[
s_i = \left( s_0 + p_R^+ + \frac{N p_R^+}{p_R^-} \right) k^i - p_R^+ - \frac{N p_R^+}{p_R^-},
\]

where

\[
s_0 = \frac{p_R^+ \left( q_R^- + kp_R^- + \frac{(p_R^- - p_R^-) N}{p_R^- + 1} \right)}{k} - p_R^+ + 1.
\]

From (3.5), we get

\[
I_i \leq C(N, \mu_1, \mu_2, p_R^+, p_R^-) \left( t_i + s_i + p_R^+ \right) \frac{2kp_R^-}{\rho p_R^-} I_{i-1}^k.
\]

(3.6)

Since

\[
t_i + s_i + p_R^+ \leq \left( q_R^- + kp_R^- \right) k^i + \frac{(p_R^- - p_R^-) N}{p_R^-} + \left( s_0 + p_R^+ + \frac{N p_R^+}{p_R^-} \right) k^i - \frac{N p_R^+}{p_R^-}
\]

\[
\leq \left( q_R^- + kp_R^- + s_0 + p_R^+ + \frac{N p_R^+}{p_R^-} \right) k^i.
\]
iterate (3.6), then we have
\[
I_i \leq C \left( q_R^- + k p_R^- + s_0 + p_R^+ + \frac{N p_R^+}{p_R} \right)^{2 k p_R^+} \left( \frac{1}{\rho} k^{2 k p_R^+} t_R^{k^{2 k p_R^+}} \right) I_0 k^i,
\]
then
\[
\left[ \int_{B_R(0)} |u(x)| \left( q_R^- + k p_R^- + s_0 + p_R^+ + \frac{N p_R^+}{p_R} \right)^{2 \sum_{j=1}^{i} k^{j-1} p_R^- + 2 \sum_{j=1}^{i} (i+1-j) k^{i-j} p_R^-} \varphi_1(x)^{s_0 + p_R^+} dx \right]^{1/\sigma} \quad \leq \quad C \left( q_R^- + k p_R^- + s_0 + p_R^+ + \frac{N p_R^+}{p_R} \right)^{2 \sum_{j=1}^{i} k^{j-1} p_R^- + 2 \sum_{j=1}^{i} (i+1-j) k^{i-j} p_R^-} \varphi_1(x)^{s_0 + p_R^+} dx \right]^{1/\sigma},
\]
(3.7)

and passing to the limit as \( i \to \infty \), we obtain
\[
\left\| u(x) \right\|_{L^\infty(q < |x| < \frac{2 \rho}{\sigma R})} \leq \left\| 1 + |u(x)| \right\|_{L^\infty(q < |x| < \frac{2 \rho}{\sigma R})} \left[ \int_{B_R(0)} |u(x)| \left( q_R^- + k p_R^- + s_0 + p_R^+ + \frac{N p_R^+}{p_R} \right)^{2 \sum_{j=1}^{i} k^{j-1} p_R^- + 2 \sum_{j=1}^{i} (i+1-j) k^{i-j} p_R^-} \varphi_1(x)^{s_0 + p_R^+} dx \right]^{1/\sigma},
\]
(3.8)

where \( C = C(N, \mu_1, \mu_2, p_R^+, p_R^-) \).

Taking \( m = k p_R^- + \left( \frac{p_R^+ - p_R^-}{p_R} \right)^N \), \( s_0 \) in (3.2), we have
\[
\left[ \int_{B_R(0)} |x|^{-\alpha} |u(x)| \left( q(x) + k p_R^- + \frac{\left( p_R^+ - p_R^- \right)^N}{p_R} \right)^{p_R^+ + 1} \varphi_1(x)^{s_0 + p_R^+} dx \right]^{1/\rho} \leq \quad C(N, \mu_1, \mu_2, p_R^+, p_R^-) \left[ \int_{B_R(0)} |x|^{-\alpha} |u(x)| \left( p(x) + k p_R^- + \frac{\left( p_R^+ - p_R^- \right)^N}{p_R} \right)^{p_R^+ + 1} \varphi_1(x)^{s_0 + p_R^+ - p(x)} dx \right],
\]
(3.10)

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and further by (3.10), we get

\[
\int_{B_R(0)} (1 + |u(x)|)^{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N + 1} \varphi_1(x)^{s_0 + p^+_R} dx
\]

\[
\leq C(N, p^+_R, p^-_R, q^-_R) \int_{B_R(0)} (1 + |u(x)|)^{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N + 1} \varphi_1(x)^{s_0 + p^+_R} dx
\]

\[
\leq C + C \int_{B_R(0)} \rho^{\alpha - p^+_R} (1 + |u(x)|)^{p(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N} \varphi_1(x)^{s_0 + p^+_R} dx
\]

\[
\leq C + C \varepsilon \int_{B_R(0)} (1 + |u(x)|)^{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N} \varphi_1(x)^{s_0 + p^+_R} dx +
\]

\[
C(\varepsilon) \int_{B_R(0)} \rho^{\alpha - p^+_R} \frac{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N}{q(x) - p(x) + 1} \varphi_1(x)^{s_0 + p^+_R} dx.
\]

Take \(\varepsilon = \frac{1}{2C}\), we have

\[
\int_{B_R(0)} (1 + |u(x)|)^{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N} \varphi_1(x)^{s_0 + p^+_R} dx
\]

\[
\leq C \left( 1 + \int_{B_R(0)} \rho^{\alpha - p^+_R} \frac{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N}{q(x) - p(x) + 1} dx \right),
\]

where \(C = C(N, \mu_1, \mu_2, p^+_R, p^-_R, q^-_R, R)\).

From (3.9), we have

\[
||u(x)||_{L^\infty(\frac{\rho}{2} < |x| < \frac{\rho}{4})} \leq C \left( \rho^{-\frac{k^+_R}{q^-_R}} + \rho^{-\frac{k^+_R}{q^-_R}} \int_{B_R(0)} \rho^{\alpha - p^+_R} \frac{q(x) + k^+_R + \left(\frac{p^+_R - p^+_R}{p^+_R}\right)^N}{q(x) - p(x) + 1} dx \right). \quad (3.11)
\]

If \(p^+_R \leq \alpha < N\), we have

\[
||u(x)||_{L^\infty(\frac{\rho}{2} < |x| < \frac{\rho}{4})} \leq C \rho^{-\frac{k^+_R}{q^-_R}},
\]

and

\[
|u(x)| \leq C |x|^{-\frac{k^+_R}{(\alpha-1)q^-_R}}, \quad \text{a.e.}
\]

where \(C = C\left(N, \mu_1, \mu_2, p^+_R, p^-_R, q^-_R, q^-_R, R\right)\).
If \( \alpha < p_R^+ \), we have
\[
||u(x)||_{L^\infty}^{q_R^-} \left( \frac{q_R^- \kappa}{p_R^-} \right) \leq C \rho \left( \frac{q_R^- \kappa p_R^-}{p_R^-} \right)^{q_R^- - 1},
\]
and
\[
|u(x)| \leq C |x| \left\{ \frac{(p_R^- - \alpha) \left( q_R^- + \kappa p_R^- \right)^{q_R^- - 1}}{q_R^-} + \frac{k p_R^+}{k - 1} \right\}
\]
where \( C = C \left( N, \mu_1, \mu_2, p_R^+, p_R^-, q_R^+, q_R^-, R \right) \).

The following is the main theorem in this paper.

**Theorem 3.2** Let conditions (1.2) – (1.6), (2.1) be fulfilled. If \( u \) is a solution of equation (1.1) in \( \Omega \setminus \{0\} \), then the singularity of \( u(x) \) at the point 0 is removable.

**Proof.** For \( 0 < r < R < \min \{ \text{dist}(0, \partial \Omega), \delta_0, 1 \} \), we denote \( m(r) = \sup \{ |u(x)| : r \leq |x| \leq R \} \). For sufficiently small \( r \leq \min \{ \frac{1}{e^2}, R^2 \} \), we define the function \( \psi_r(x) \) as follows:
\[
\psi_r(x) \equiv 0 \quad \text{for} \quad |x| < r,
\]
\[
\psi_r(x) \equiv 1 \quad \text{for} \quad |x| > \sqrt{r},
\]
\[
\psi_r(x) = \frac{2}{\ln \frac{r}{\sqrt{r}}} \ln \frac{|x|}{r} \quad \text{for} \quad r \leq |x| \leq \sqrt{r}.
\]

We take the following test function
\[
\varphi(x) = \psi_r^\gamma(x) \left[ \ln \frac{u}{m(\varrho)} \right]_+^{(3.12)}
\]
for any \( x \in \Omega_\varrho \), where \( 0 < \varrho < R \), \( \Omega_\varrho = \{ x \in B_R(0) : u(x) > m(\varrho) \} \), \( \gamma = \sup_{x \in \Pi} \frac{p(x) q(x)}{q(x) - p(x) + 1} \) is a constant and \( \varphi(x) \equiv 0 \) for \( x \notin \Omega_\varrho \).

For some \( 0 < \varrho < R \), let the domain \( \Omega_\varrho \) be nonempty. Since \( \varphi(x) \in W_0^{1,p(x)}(\Omega \setminus \{0\}) \cap L^\infty(\Omega \setminus \{0\}) \), testing the equality (2.3) by \( \varphi \), we have
\[
\int_{\Omega_\varrho} A(x, u, \nabla u) \nabla u \frac{\psi_r^\gamma}{u} + g(x, u) \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx
\]
\[
+ \int_{\Omega_\varrho} A(x, u, \nabla u) \gamma \psi_r^{\gamma - 1}(x) \nabla \psi_r \ln \frac{u}{m(\varrho)} dx = 0.
\]

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By virtue of the conditions (1.2) – (1.4), we have

\[
\int_{\Omega_e} \mu_1 \frac{|\nabla u|^p(x)}{u} \psi_r^\gamma(x) dx + \int_{\Omega_e} |x|^{-\alpha} u^{g(x)} \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx \\
\leq \mu_2 \gamma \int_{\Omega_e} |\nabla u|^{p(x)-1} |\nabla \psi_r| \psi_r^{\gamma-1}(x) \ln \frac{u}{m(\varrho)} dx.
\]

By Young’s inequality,

\[
\begin{align*}
\mu_2 \gamma & \int_{\Omega_e} |\nabla u|^{p(x)-1} |\nabla \psi_r| \psi_r^{\gamma-1}(x) \ln \frac{u}{m(\varrho)} dx \\
& \leq C(\varepsilon_4) \int_{\Omega_e} u^{p(x)-1} \psi_r^{\gamma-p(x)} |\nabla \psi_r|^{p(x)} \left( \ln \frac{u}{m(\varrho)} \right)^{p(x)} dx + \mu_2 \gamma \varepsilon_4 \int_{\Omega_e} \psi_r^{\gamma-1} |\nabla u|^{p(x)} dx, \end{align*}
\]

take \( \varepsilon_4 = \frac{\mu_1}{2\mu_2 \gamma} \), then

\[
\begin{align*}
\frac{\mu_1}{2} & \int_{\Omega_e} \frac{|\nabla u|^p(x)}{u} \psi_r^\gamma(x) dx + \int_{\Omega_e} |x|^{-\alpha} u^{g(x)} \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx \\
& \leq C(\mu_1, \mu_2, \gamma) \int_{\Omega_e} u^{p(x)-1} \psi_r^{\gamma-p(x)} |\nabla \psi_r|^{p(x)} \left( \ln \frac{u}{m(\varrho)} \right)^{p(x)} dx.
\end{align*}
\]

Further,

\[
\begin{align*}
\int_{\Omega_e} u^{p(x)-1} \psi_r^{\gamma-p(x)} |\nabla \psi_r|^{p(x)} \left( \ln \frac{u}{m(\varrho)} \right)^{p(x)} dx & \\
& \leq C(\varepsilon_5) \int_{\Omega_e} |x|^{\frac{\alpha g(x)}{p(x)-1}+1-\alpha} \left( \ln \frac{u}{m(\varrho)} \right)^{1+\frac{g(x)-1)(q(x))}{p(x)-1}} |\nabla \psi_r|^{p(x) q(x)} dx \\
& \quad + \varepsilon_5 \int_{\Omega_e} |x|^{-\alpha} \ln \frac{u}{m(\varrho)} u^{q(x)} \psi_r^{(\gamma-p(x))q(x)}^{p(x)-1} dx.
\end{align*}
\]

Take \( \varepsilon_5 = \frac{1}{2C(\mu_1, \mu_2, \gamma)} \). Since \( \frac{\gamma-p(x)q(x)}{p(x)-1} > \gamma, \psi_r(x) \leq 1 \), we have

\[
\begin{align*}
\frac{\mu_1}{2} & \int_{\Omega_e} \frac{|\nabla u|^p(x)}{u} \psi_r^\gamma(x) dx + \frac{1}{2} \int_{\Omega_e} |x|^{-\alpha} u^{q(x)} \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx \\
& \leq C(\mu_1, \mu_2, \gamma) \int_{\Omega_e} |x|^{\frac{\alpha g(x)}{\gamma-p(x)+1}-\alpha} \left( \ln \frac{u}{m(\varrho)} \right)^{1+\frac{g(x)-1)(q(x))}{\gamma-p(x)} |\nabla \psi_r|^{p(x) q(x)} dx. \tag{3.13}
\end{align*}
\]
By Lemma 2.1, we get \(0 < 1 + \frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} < \infty\). Denote \( \lambda = \sup_{x \in \Omega} \left( \frac{p(x) - \alpha}{q(x) - p(x) + 1} + \alpha \right) \), and from Theorem 3.1 and (3.13), we have

\[
\frac{\mu_1}{2} \int_{\Omega} \frac{|\nabla u|^p(x)}{u} \psi_r^\gamma(x) dx + \frac{1}{2} \int_{\Omega} |x|^{-\alpha} u^{q(x)} \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx \\
\leq C \int_{\Omega \cap \{|x| \leq \sqrt{r}\}} |x|^{\alpha q(x)} \frac{1}{q(x) - p(x) + 1} - \alpha \left( \ln |x|^{-Q} + C \right)^{\frac{p(x) - 1}{q(x) - p(x) + 1}} \left( \frac{2}{|x| \ln \frac{1}{|x|}} \right)^{\frac{p(x)q(x)}{q(x) - p(x) + 1}} dx
\]

\[
\leq C \left( \ln \frac{1}{r} \right)^{-\frac{q R_R p R_R}{q R_R - p R_R + 1}} \int_{\Omega \cap \{|x| \leq \sqrt{r}\}} |x|^{\alpha q(x)} \frac{1}{q(x) - p(x) + 1} - \alpha \left( \ln \frac{1}{|x|} \right)^{1+\frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} \left( \frac{1}{|x|} \right)^{\lambda}} dx
\]

\[
\leq C \left( \ln \frac{1}{r} \right)^{-\frac{q R_R p R_R}{q R_R - p R_R + 1}} \int_{\sqrt{r}}^{\sqrt{r}} \left( \frac{1}{t} \right)^{\lambda} \left( \ln \frac{1}{t} \right)^{1+\frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} t^{N-1} dt}
\]

where \( C = C \left( N, \mu_1, \mu_2, \gamma, p R_R, p R_R, q R_R, q R_R, R \right) \).

Further, by (1.6), we get \( \lambda < N \), then

\[
\left( \ln \frac{1}{r} \right)^{-\frac{q R_R p R_R}{q R_R - p R_R + 1}} \int_{\sqrt{r}}^{\sqrt{r}} \left( \frac{1}{t} \right)^{\lambda} \left( \ln \frac{1}{t} \right)^{1+\frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} t^{N-1} dt}
\]

\[
\leq \left( \ln \frac{1}{r} \right)^{-\frac{q R_R p R_R}{q R_R - p R_R + 1}} \left( \ln \frac{1}{r} \right)^{1+\frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} \int_{\sqrt{r}}^{\sqrt{r}} t^{N-1} dt}
\]

\[
= \left( \ln \frac{1}{r} \right)^{-\frac{q R_R p R_R}{q R_R - p R_R + 1}} \left( \ln \frac{1}{r} \right)^{1+\frac{(p R_R - 1)q R_R}{q R_R - p R_R + 1} \frac{1}{N - \lambda} r^{\frac{1}{2}(N-\lambda)} \left( 1 - r^{\frac{1}{2}(N-\lambda)} \right)}
\]

\[
\rightarrow 0,
\]

as \( r \rightarrow 0 \). Therefore, we obtain

\[
\lim_{r \rightarrow 0} \frac{\mu_1}{2} \int_{\Omega} \frac{|\nabla u|^p(x)}{u} \psi_r^\gamma(x) dx + \frac{1}{2} \int_{\Omega} |x|^{-\alpha} u^{q(x)} \psi_r^\gamma(x) \ln \frac{u}{m(\varrho)} dx \leq 0,
\]

then

\[
\mu_1 \int_{\Omega} \frac{\nabla u|^p(x)}{u} dx + \int_{\Omega} |x|^{-\alpha} u^{q(x)} \ln \frac{u}{m(\varrho)} dx = 0.
\]

Hence \( u(x) = m(\varrho) \) almost everywhere in \( \Omega \) and the Lebesgue measure of \( \Omega \) equals to zero. Considering further the function \(-u(x)\) instead of \(u(x)\), we obtain the boundedness of \(-u(x)\) in a neighborhood of the point 0. Thus we have proved that \( u \in L^\infty(\Omega) \).
Next, we take the test function
\[
\tilde{\varphi} = \psi^p u,
\]
where \( \psi \equiv 1 \) in \( B_{2\rho}(0) \setminus B_{\rho}(0) \), \( \psi \equiv 0 \) outside \( B_{2\rho}(0) \setminus B_{\rho}(0) \), \( 0 \leq \psi(x) \leq 1 \), \( |\nabla \psi| \leq \frac{C}{\rho} \) and \( 0 < \rho \leq 1 \). Testing the equality (2.3) by \( \tilde{\varphi} \), we have
\[
\int_{\Omega} A(x, u, \nabla u) \left( p^+ \psi^{p^+ - 1} u \nabla \psi + \psi^{p^+} \nabla u \right) + g(x, u) \psi^p u \, dx = 0.
\]
By virtue of the conditions (1.2) – (1.5), we have
\[
\int_{B_{5\rho}^\pm(0)} \mu_1 |\nabla u|^{p(x)\psi^p} \left| |x|^{-\alpha} |u|^{p(x)\psi^p} \nabla \psi \right| dx
\]
\[
\leq p^+ \mu_2 \int_{B_{5\rho}^\pm(0)} |\nabla u|^{p(x)\psi^p} - 1 |\nabla \psi| |u| dx
\]
\[
= p^+ \mu_2 \int_{B_{5\rho}^\pm(0)} \left[ |\nabla \psi| |u|^{\psi^p - 1 - p^+ \frac{\rho^+}{p(x)}} \right] \left[ |\nabla u|^{p(x)\psi^p - 1} \psi^p \right] dx
\]
\[
\leq C(\mu_2, p^+, \varepsilon_6) \int_{B_{5\rho}^\pm(0)} |\nabla \psi| |u|^{p(x)\psi^p - p(x)} dx + p^+ \mu_2 \varepsilon_6 \int_{B_{5\rho}^\pm(0)} |\nabla u|^{p(x)\psi^p} dx.
\]
Take \( \varepsilon_6 = \frac{\mu_1}{2p^\pm \mu_2} \), we have
\[
\int_{B_{5\rho}^\pm(0)} |\nabla u|^{p(x)\psi^p} dx \leq C(\mu_1, \mu_2, p^+) \int_{B_{5\rho}^\pm(0)} |\nabla \psi|^{p(x)\psi^p - p(x)} dx
\]
\[
\leq C \frac{1}{\rho^{p^+}} \max \left\{ ||u||_p^+, ||u||_\infty^p \right\} \left| B_{5\rho}^\pm(0) \right|
\]
\[
\leq C \frac{1}{\rho^{p^+} \omega_n} \left( \frac{5\rho}{2} \right)^N
\]
\[
= C(\mu_1, \mu_2, p^+) \rho^{N - p^+},
\]
where \( \omega_n \) is the volume of the unit ball, \( |B_{5\rho}^\pm(0)| \) is the volume of the ball \( B_{5\rho}^\pm(0) \).

Further,
\[
\int_{B_{2\rho}(0) \setminus B_{\rho}(0)} |\nabla u|^{p(x)} dx \leq C(\mu_1, \mu_2, p^+) \rho^{N - p^+}, \tag{3.14}
\]
then we obtain

\[
\int_{B_\rho(0)} |\nabla u|^p(x) dx = \sum_{j=1}^\infty \int_{B_{2^{-j}\rho}(0)\setminus B_{2^{-j-1}\rho}(0)} |\nabla u|^p(x) dx \\
\leq C \sum_{j=1}^\infty (2^{-j}\rho)^{N-p^+} \\
\leq C(\mu_1, \mu_2, p^+) \rho^{N-p^+} \\
\to 0,
\]

as \( \rho \to 0 \). So \(|\nabla u| \in L^p(x)(\Omega)\).

Thus, we have proved that \( u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \).

Next, we will show that \( u(x) \) is a solution of equation (1.1) in the domain \( \Omega \). Pick \( \eta_\rho \in C_0^\infty(\mathbb{R}^N) \) be the cutoff function for the ball \( B_\rho(0) \), \( \eta_\rho \equiv 1 \) in \( B_\rho(0) \), \( \eta_\rho \equiv 0 \) outside the ball \( B_{2\rho}(0) \), \( |\nabla \eta_\rho| \leq \frac{C}{\rho} \) and \( 0 < \rho \leq 1 \). Let \( \varphi \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega) \). Testing the equation (2.3) by the test function \((1 - \eta_\rho)\varphi\), we have

\[
\int_{\Omega} A(x, u, \nabla u)\nabla[(1 - \eta_\rho)\varphi] dx + \int_{\Omega} g(x, u)(1 - \eta_\rho)\varphi dx = 0,
\]

that is,

\[
\int_{\Omega} A(x, u, \nabla u)(1 - \eta_\rho)\nabla \varphi dx - \int_{\Omega} A(x, u, \nabla u)\varphi \nabla \eta_\rho dx + \int_{\Omega} g(x, u)(1 - \eta_\rho)\varphi dx = 0.
\]

Indeed,

\[
|A(x, u, \nabla u)(1 - \eta_\rho)\nabla \varphi| \leq \mu_2 |\nabla u|^{p(x)-1}|\nabla \varphi| \\
\leq \mu_2 \left( \frac{p(x)-1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{p(x)} |\nabla \varphi|^{p(x)} \right) \\
\in L^1(\Omega),
\]

therefore, by Lebesgue’s Dominated Convergence Theorem, we have

\[
\lim_{\rho \to 0} \int_{\Omega} A(x, u, \nabla u)(1 - \eta_\rho)\nabla \varphi dx = \int_{\Omega} A(x, u, \nabla u)\nabla \varphi dx.
\]

In the same way,

\[
\lim_{\rho \to 0} \int_{\Omega} g(x, u)(1 - \eta_\rho)\varphi dx = \int_{\Omega} g(x, u)\varphi dx.
\]
Meanwhile, by (3.14), we have

\[
\left| \int_{\Omega} A(x, u, \nabla u) \varphi \nabla \eta_\rho dx \right| \leq \frac{C \mu_2}{\rho} \int_{B_{2\rho}(0) \setminus B_{\rho}(0)} |\nabla u|^{p(x)-1} dx
\]

\[
\leq \frac{C(\mu_2)}{\rho} \| |\nabla u|^{p(x)-1} \|_{L^{p(x)}(B_{2\rho}(0) \setminus B_{\rho}(0))} 1 \|_{L^{p(x)}(B_{2\rho}(0) \setminus B_{\rho}(0))} \leq C(\mu_1, \mu_2, p^+) \rho \left( \frac{\rho^{-1}(N-p^+)}{\rho^+} \right)^{\frac{1}{p^+}}
\]

\[
= C(\mu_1, \mu_2, p^+) \rho \left( \frac{\rho^{-1}(N-p^+)}{\rho^+} \right)^{\frac{1}{p^+}} \rightarrow 0
\]

as \( \rho \rightarrow 0 \).

So we have obtained that equality (2.3) is fulfilled for any test function.
Therefore, the isolated singular point 0 is removable for solutions of equation (1.1).

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