Convergence rates in precise asymptotics for a kind of complete moment convergence

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Abstract

In Liu and Lin (Statist. Probab. Letters, 2006), they introduced a kind of complete moment convergence which includes complete convergence as a special case. Inspired by the study of complete convergence, in this paper, we study the convergence rates of the precise asymptotics for this kind of complete moment convergence and get the corresponding convergence rates.

Keywords: Convergence rate; precise asymptotics; complete moment convergence

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1. Introduction

Let \( \{X, X_n, n \in \mathbb{N}\} \) be a sequence of i.i.d. random variables. Hsu and Robbins [16] introduced the complete convergence, and proved that if \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[X^2] < \infty \), then

\[
\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) < \infty, \ \epsilon > 0,
\]

where

\[
S_n = \sum_{k=1}^{n} X_k, n \in \mathbb{N}.
\]

Since then, it has attracted a lot of interest. For example, Erdös [7, 8] proved the necessity. For more information, we refer to Baum and Katz [1], Davis [5, 6], Lai [19] and Gut [9].

Here we point out that the sum in (1.1) tends to infinity as \( \epsilon \searrow 0 \). Due to this fact, it is interesting to study the precise asymptotic problem, that is, finding an elementary function \( f(\epsilon) \) such that \( f(\epsilon) \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) \) has a non-degenerate limit as \( \epsilon \searrow 0 \). Many results about this topic have been established. For example, Heyde [15] showed that

\[
\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) = \mathbb{E}[X^2],
\]

whenever \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[X^2] < \infty \). For more results on precise asymptotic problems, we refer to Chen [4], Gut and Spătaru [10, 11], Li and Spătaru [20], Spătaru [22, 23] and the references therein.

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Noting the Heyde’s result (1.2), it is natural to consider the rate of the convergence for the precise asymptotic problem. This topic has been studied extensively. For example, Klesov [17] proved that if \( E[X] = 0 \), \( E[X^2] > 0 \) and \( E[|X|^3] < \infty \), then
\[
\epsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) - E[X^2] = o(\epsilon^{1.2}). \tag{1.3}
\]
Furthermore, by replacing \( E[|X|^3] < \infty \) in Klesov [17] with \( E[|X|^q] < \infty \) for some \( q \in (2, 3] \), He and Xie [14] got a much faster rate for Heyde’s result. Recently, Gut and Steinebach [12, 13], and Kong [18] also extended the Klesov’s [17] result, respectively, and got some new results.

As an extension of the complete convergence, Liu and Lin [21] introduced a new kind of complete moment convergence, and got the precise asymptotic results for this kind of complete moment convergence. These results read as follows.

**Theorem 1.1** Let \( \{X, X_n, n \in \mathbb{N}\} \) be a sequence of i.i.d. random variables with partial sums \( \{S_n, n \in \mathbb{N}\} \). For any \( \epsilon > 0 \), set
\[
\lambda_1(\epsilon, p) = \sum_{n=1}^{\infty} \frac{1}{n^p} E[|S_n|^p I\{|S_n| \geq \epsilon n\}], \quad \text{for any } p \in [0, 2],
\]
\[
\lambda_2(\epsilon, \delta) = \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} E[S_n^2 I\{|S_n| \geq \epsilon \sqrt{n \log n}\}], \quad \text{for any } \delta \in (0, 1].
\]

(a) For any \( p \in [0, 2) \), we have
\[
\lim_{\epsilon \searrow 0} \epsilon^{2-p} \lambda_1(\epsilon, p) = \frac{2\sigma^2}{2-p} \tag{1.6}
\]
if and only if \( E[X] = 0 \) and \( E[X^2] = \sigma^2 < \infty \).

(b) For any \( \delta \in (0, 1] \), we have
\[
\lim_{\epsilon \searrow 0} \epsilon^{2\delta} \lambda_2(\epsilon, \delta) = \frac{\sigma^{2\delta+2}}{\delta} E[|N|^{2\delta+2}] \tag{1.7}
\]
if and only if \( E[X] = 0 \), \( E[X^2] = \sigma^2 \) and \( E[X^2(\log^+ |X|)^\delta] < \infty \), where \( N \) denotes the standard normal random variable and \( \log^+ x = \log(x \lor e) \).

Similar to the study of the rate of convergence in the complete convergence, it is also interesting to consider the convergence rates in precise asymptotics for Liu and Lin’s type complete moment convergence, which is the motivation of our work. In this paper, we will carry out this work.

The rest of this paper is organized as follows. In Section 2, we state the main results of this paper. Section 3 is devoted to presenting the detailed proofs of these results.

We end this section with some notations. We use \( C \) to denote a positive constant whose value may vary from place to place. Let \( N \) denote the standard normal random variable and \( \Phi(x) = \mathbb{P}(|N| \geq x) \) for any \( x \geq 0 \).
2. Main Results

In this section, we state the main results of this paper. Before we do it, we first introduce some notations. Define

\[ B_{n,\theta} = \sum_{j=1}^{n} j^{\theta} - \frac{n^{\theta+1}}{\theta+1}, \quad \text{for any } -1 < \theta < 0; \tag{2.1} \]

\[ C_{n,\delta} = \sum_{j=2}^{n} \frac{(\log j)^{\delta-1}}{j} - \frac{(\log n)^{\delta}}{\delta}, \quad \text{for any } 0 < \delta \leq 1. \tag{2.2} \]

We use \( B_{\theta} \) and \( C_{\delta} \) to denote the limits of the sequences \( \{B_{n,\theta}, n \in \mathbb{N}\} \) and \( \{C_{n,\delta}, n \geq 2\} \), respectively.

Remark 2.1 Gut and Steinebach [13] proved the convergence of the sequence \( \{B_{n,\theta}; n \in \mathbb{N}\} \). For the convergence of the sequence \( \{C_{n,\delta}; n \geq 2\} \), we will give the proof in Section 3 below.

Now, we state our main results. We have

Theorem 2.1 Let \( \{X, X_n, n \in \mathbb{N}\} \) be a sequence of i.i.d. random variables with partial sums \( \{S_n, n \in \mathbb{N}\} \). Suppose that

\[ \mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = \sigma^2 > 0, \quad \text{and } \mathbb{E}[|X|^q] < \infty \text{ for some } q \in (2, 3). \tag{2.3} \]

(a) For any \( p \in (0, 2) \), we have

\[ \lim_{\epsilon \to 0} \epsilon^{(\gamma-1)(2-p)} \left[ \epsilon^{2-p} \lambda_{1}(\epsilon, p) - \frac{2\sigma^2}{2-p} \right] = 0, \tag{2.4} \]

where \( \gamma = \frac{q-p}{2q-2p} \) and \( \lambda_{1}(\epsilon, p) \) is given by [2.4].

(b) For any \( 0 < \delta \leq 1 \), we have

\[ \lim_{\epsilon \to 0} (\log \epsilon)^{-\delta} \epsilon^{-2\delta} \left[ \epsilon^{2\delta} \lambda_{2}(\epsilon, \delta) - \frac{\sigma^{2\delta+2}}{\delta} \mathbb{E}[|N|^{2\delta+2}] \right] = 0, \tag{2.5} \]

where \( \lambda_{2}(\epsilon, \delta) \) is defined by [1.5].

Remark 2.2 Under the condition of Theorem 2.1, Theorem 2.1 includes Theorem 1.1 as a special case.

Remark 2.3 It is obvious that \( 0 < \gamma < 1 \), since \( p \in (0, 2) \), \( q \in (2, 3) \).

Remark 2.4 [2.4] obtains the convergence rate for \( 0 < p < 2 \) in [1.6]. When \( p = 0 \), then

\[ \lambda_{1}(\epsilon, 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{P}(|S_n| \geq \epsilon n). \]

Under the assumptions of Theorem 2.1 He and Xie [14] got that

\[ \epsilon^{2} \lambda_{1}(\epsilon, 0) - \sigma^2 = o(\epsilon^{q-2}). \tag{2.6} \]

Moreover, Liu and Lin [21] also got the precise asymptotics for the case \( p = 2 \). However, the method we used in this paper seems unable to deal with the case \( p = 2 \) and we will study this case in the future work.
The proof of Theorem 2.1 is based on the following two propositions, one of which is concerned with the Gaussian case (Proposition 2.1) and the other is related to two Berry-Esseen type remainder terms (Proposition 2.2).

**Proposition 2.1** Suppose that \( \{X, X_n, n \in \mathbb{N}\} \) is a sequence of i.i.d. normal random variables with mean 0 and variance \( \sigma^2 > 0 \). For any \(-1 < \theta < 0 \) and \( 0 < \delta \leq 1 \), \( B_\theta \) and \( C_\delta \) denote the limits of the sequences \( \{B_{n, \theta}, n \in \mathbb{N}\} \) and \( \{C_{n, \delta}; n \geq 2\} \), respectively.

(a) For any \( p \in (0, 2) \), we have, as \( \epsilon \to 0 \),

\[
\lambda_1(\epsilon, p) = \frac{2}{2-p} \epsilon^{p-2} \sigma^2 + B_\theta \mathbb{E}[|N|^p] \sigma^p + O(\epsilon^\gamma \log \frac{1}{\epsilon}).
\]  
(2.7)

(b) For any \( \delta \in (0, 1] \), we have, as \( \epsilon \to 0 \),

\[
\lambda_2(\epsilon, \delta) = \frac{1}{\delta} \mathbb{E}[|N|^{2\delta+2}] \epsilon^{-2\delta} \sigma^{2\delta+2} + C_\delta \mathbb{E}[N^2] \sigma^2 + O(\epsilon^2).
\]  
(2.8)

**Proposition 2.2** Let \( \{X, X_n; n \in \mathbb{N}\} \) be a sequence of i.i.d. random variables with \( \mathbb{E}[X] = 0 \), \( \mathbb{E}[X^2] = \sigma^2 > 0 \), \( \mathbb{E}[|X|^q] < \infty \) for some \( q \in (2, 3] \), and partial sums \( S_n = \sum_{k=1}^{n} X_k, \ n \in \mathbb{N} \).

(a) For any \( p \in (0, 2) \), we have,

\[
\lim_{\epsilon \to 0} \epsilon^{\gamma(2-p)} \sum_{n=1}^{\infty} \frac{1}{np} \int_{\epsilon n}^{\infty} px^{p-1} \mathbb{P}(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\ dx = 0, \]  
(2.9)

where \( \gamma = \frac{q-p}{2q-2-p} \).

(b) For any \( \delta \in (0, 1] \), we have,

\[
\lim_{\epsilon \to 0} (\log \frac{1}{\epsilon})^{-\delta} \sum_{n=2}^{\infty} \frac{\log n}{n^2} \int_{\epsilon \sqrt{n} \log n}^{\infty} 2x \mathbb{P}(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\ dx = 0. \]  
(2.10)

**Remark 2.5** By Propositions 4.2, 4.3, 5.2 and 5.3 in Liu and Lin [21], it’s easy to get the following results

\[
\lim_{\epsilon \to 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{np} \int_{\epsilon n}^{\infty} px^{p-1} \mathbb{P}(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\ dx = 0,
\]

\[
\lim_{\epsilon \to 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{n} \log n}^{\infty} 2x \mathbb{P}(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\ dx = 0.
\]

We improve the above results by (2.9) and (2.10) with \( 0 < \gamma < 1 \).
3. Proofs

In this section, we give the detailed proofs of Theorem 2.1 and Propositions 2.1 and 2.2. Without loss of generality, we assume \( \sigma = 1 \) in this section. In order to reach our aim, we first introduce some technical lemmas. The first one comes from Billingsley [2].

**Lemma 3.1** For \( x \) large enough, We have 
\[
\Phi(x) \sim \frac{2}{\sqrt{2\pi x}} e^{-\frac{x^2}{2}}.
\]

The following lemma comes from Gut and Steinebach [13] which shows the convergence of the sequence \( \{B_{n,\theta}, n \in \mathbb{N}\} \) for any \( \theta \in (-1, 0) \).

**Lemma 3.2** We have, as \( n \to \infty \), 
\[
B_{n,\theta} = B_\theta + O(n^\theta), \quad \text{for } -1 < \theta < 0,
\]
where \( B_{n,\theta} \) is defined by (2.1) and \( B_\theta \) is a constant with 
\[
-\frac{1}{\theta+1} \leq B_\theta < \frac{\theta}{\theta+1} < 0.
\]

Similar to Lemma 3.2, the following lemma shows the convergence of the sequence \( \{C_{n,\delta}, n \geq 2\} \) for any \( \delta \in (0, 1] \).

**Lemma 3.3** For any \( 0 < \delta \leq 1 \), we have, as \( n \to \infty \), 
\[
C_{n,\delta} = C_\delta + O\left(\frac{(\log n)^{\delta-1}}{n}\right),
\]
where \( C_{n,\delta} \) is defined by (2.2) and \( C_\delta \) is a constant with 
\[
-\frac{\log 2}{\delta} \leq C_\delta \leq 0.
\]

**Proof:** It follows from (2.2) and the mean value theorem that 
\[
C_{n+1,\delta} - C_{n,\delta} = \frac{[\log(n+1)]^{\delta-1}}{n+1} - \frac{[\log(n+1)]^{\delta}}{\delta}
\]
\[
= \frac{[\log(n+1)]^{\delta-1}}{n+1} - \frac{(\log \xi_n)^{\delta-1}}{\xi_n}
\]
for some \( \xi_n \in (n, n+1) \).

On the other hand, noting 
\[
C_{n,\delta} = \sum_{j=2}^{n} \int_{j-1}^{j} \left[ \frac{(\log j)^{\delta-1}}{j} - \frac{(\log x)^{\delta-1}}{x} \right] dx,
\]
we get that 
\[
0 \geq C_{n,\delta} \geq \sum_{j=3}^{n} \left[ \frac{(\log j)^{\delta-1}}{j} - \frac{(\log (j-1))^{\delta-1}}{j-1} \right] + \frac{(\log 2)^{\delta-1}}{2} - \int_{1}^{2} \frac{(\log x)^{\delta-1}}{x} dx
\]
\[
= \frac{(\log n)^{\delta-1}}{n} - \frac{(\log 2)^{\delta}}{\delta} \geq -\frac{(\log 2)^{\delta}}{\delta},
\]
since \( f(x) = \frac{(\log x)^{\delta-1}}{x} \) is a decreasing function.
Since the function $f(x) = \frac{(\log x)^{\delta-1}}{x}$ is decreasing with $\delta \in (0, 1]$, we obtain that the sequence $\{C_{n,\delta}, n \geq 2\}$ is also decreasing. By using this fact, we get from the monotone bounded theorem that $\lim_{n \to \infty} C_{n,\delta}$ exists, since (3.4) implies that $\{C_{n,\delta}, n \geq 2\}$ is a bounded sequence.

For any $\delta \in (0, 1]$, let $C_{\delta}$ denote the limit of the sequence $\{C_{n,\delta}, n \geq 2\}$. Moreover, from (3.4) we get that $-\frac{(\log 2)^{\delta}}{\delta} \leq C_{\delta} \leq 0$.

Given $m > n$, using the mean value theorem again, we have

$$0 > C_{m,\delta} - C_{n,\delta} = \sum_{j=n+1}^{m} \left[ \frac{(\log j)^{\delta-1}}{j} - \frac{(\log j)^{\delta} - (\log (j - 1))^\delta}{\delta} \right]$$

$$= \sum_{j=n+1}^{m} \left[ \frac{(\log j)^{\delta-1}}{j} - \frac{(\log j)^{\delta}}{\xi_j} \right]$$

for some $\xi_j \in (j - 1, j)$

$$> \sum_{j=n+1}^{m} \left[ \frac{(\log j)^{\delta-1}}{j} - \frac{(\log (j - 1))^\delta}{j - 1} \right]$$

$$= \frac{(\log m)^{\delta-1}}{m} - \frac{(\log n)^{\delta-1}}{n}.$$ (3.5)

Letting $m \to \infty$ in (3.5), we finish the proof of Lemma 3.3. □

Now we come to a point where we can prove the Propositions 2.1 and 2.2. Next, we first prove the Proposition 2.1.

Proof of Proposition 2.1. Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. normal random variables with mean 0 and variance $\sigma^2 > 0$. Since we assume $\sigma = 1$, we indeed deal with the standard normal random variable $N$.

We first prove the part (a) of Proposition 2.1. Note that, for $0 < p < 2$,

$$\lambda_1(\epsilon, p) = \epsilon^p \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) + \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon n}^{\infty} px^{p-1} \mathbb{P}(|S_n| \geq x) dx.$$ (3.6)

On the other hand, from Klesov [17], we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \epsilon n) = \frac{\sigma^2}{\epsilon^2} - \frac{1}{2} + o(1).$$ (3.7)

Thus, in order to prove (3.6), we only need to consider the second term in (3.6).

The change of variable $t = \frac{x}{\epsilon n}$ yields that

$$\frac{1}{n^p} \int_{\epsilon n}^{\infty} px^{p-1} \mathbb{P}(|S_n| \geq x) dx = \frac{1}{n^p} \int_{\epsilon \sqrt{n}}^{\infty} pt^{p-1} \Phi(t) dt$$

$$= \frac{1}{n^p} \sum_{j=n}^{\infty} \int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} pt^{p-1} \Phi(t) dt,$$ (3.8)

since

$$\frac{S_n}{\sqrt{n}} \overset{d}{=} N$$

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and

\[ \Phi(t) = \mathbb{P}(|N| \geq t). \]

It follows from the Fubini's theorem, (3.8) and Lemma 3.2 that

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon_n}^{\infty} px^{p-1} \mathbb{P}(|S_n| \geq x) dx \\
= \sum_{j=1}^{\infty} \left( \sum_{n=1}^{j} \frac{1}{n^p} \right) \int_{\epsilon \sqrt{j+1}}^{\epsilon \sqrt{j+1}} pt^{p-1} \Phi(t) dt \\
= I_{31} + I_{32} + I_{33},
\]

where

\[
I_{31} = \frac{2p}{2-p} \sum_{j=1}^{\infty} j^{-\frac{p}{2}+1} \int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} t^{p-1} \Phi(t) dt,
\]

\[
I_{32} = B_{-\frac{p}{2}} \sum_{j=1}^{\infty} \int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} pt^{p-1} \Phi(t) dt,
\]

\[
I_{33} = \sum_{j=1}^{\infty} O(j^{-\frac{p}{2}}) \sum_{j=1}^{\infty} \int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} pt^{p-1} \Phi(t) dt.
\]

Now, we deal with \( I_{31} \). The integer mean theorem implies that for some \( \xi \in (j, j+1) \)

\[
\int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} t^{p-1} \Phi(t) dt = e^p \xi^{\frac{p-1}{2}} \Phi(\epsilon \sqrt{\xi})(\sqrt{j+1} - \sqrt{j}).
\]

Furthermore, by the Taylor expansion, we have

\[
\xi^{\frac{p-1}{2}} = j^{\frac{p-1}{2}} + O(j^{\frac{p-3}{2}}),
\]

\[
\Phi(\epsilon \sqrt{\xi}) = \Phi(\epsilon \sqrt{j}) + e^{-\epsilon^2 j} O(j^{-\frac{3}{2}}),
\]

\[
\sqrt{j+1} - \sqrt{j} = \frac{1}{2} j^{-\frac{1}{2}} + O(j^{-\frac{3}{2}}).
\]

From the above argument, we have

\[
I_{31} = \frac{pe^p}{2-p} \sum_{j=1}^{\infty} \left[ \Phi(\epsilon \sqrt{j}) + e^{-\epsilon^2 j} O(j^{-\frac{3}{2}}) + \Phi(\epsilon \sqrt{j}) O(j^{-1}) + O(j^{-\frac{3}{2}}) \right].
\]

(3.10)

Letting \( y = \epsilon \sqrt{t} \), we have

\[
\frac{pe^p}{2-p} \sum_{j=1}^{\infty} \Phi(\epsilon \sqrt{j}) = \frac{pe^p}{2-p} \int_{\epsilon}^{\infty} \Phi(\epsilon \sqrt{t}) dt + O(e^p)
\]

\[
= \frac{pe^{p-2}}{2-p} \int_{\epsilon}^{\infty} \Phi(y) dy + O(e^p)
\]

\[
= \frac{pe^{p-2} \mathbb{E}[N^2]}{2-p} - \frac{pe^{p-2}}{2-p} \int_{0}^{\epsilon} 2t \Phi(y) dy + O(e^p)
\]

\[
= \frac{pe^{p-2} \mathbb{E}[N^2]}{2-p} + O(e^p).
\]

(3.11)
Similar to (3.11), we can obtain that
\[
\frac{pe^{p+1}}{2-p} \sum_{j=1}^{\infty} O\left(j^{-\frac{p}{2}}\right) e^{-\frac{j^2}{2}} = O(e^p),
\]
(3.12)

\[
\frac{pe^p}{2-p} \sum_{j=1}^{\infty} O\left(j^{-1}\right) \Phi\left(\epsilon \sqrt{j}\right) = O(e^p \log \frac{1}{\epsilon}),
\]
(3.13)

and
\[
\frac{pe^p}{2-p} \sum_{j=1}^{\infty} O\left(j^{-\frac{p}{2}}\right) = O(e^p).
\]
(3.14)

From (3.11) to (3.14), we have
\[
I_{31} = \frac{pe^{p-2} \mathbb{E}[N^2]}{2-p} + O(e^p \log \frac{1}{\epsilon}).
\]
(3.15)

For \(I_{32}\), we have
\[
I_{32} = B_{-\frac{p}{2}} \int_{\epsilon}^{\infty} pt^{p-1} \Phi(t) dt
= B_{-\frac{p}{2}} \mathbb{E}[|N|^p] - B_{-\frac{p}{2}} \int_{0}^{\epsilon} pt^{p-1} \Phi(t) dt
= B_{-\frac{p}{2}} \mathbb{E}[|N|^p] + O(e^p).
\]
(3.16)

Next, we deal with \(I_{33}\). Noting that
\[(j+1)^{\frac{p}{2}} - j^{\frac{p}{2}} = O(j^{\frac{p}{2}-1}), \text{ as } j \to \infty,
\]
we have
\[
\int_{\epsilon \sqrt{j}}^{\epsilon \sqrt{j+1}} pt^{p-1} \Phi(t) dt \leq e^p \Phi(\epsilon \sqrt{j})(j+1)^{\frac{p}{2}} - j^{\frac{p}{2}} = e^p \Phi(\epsilon \sqrt{j})O(j^{\frac{p}{2}-1}),
\]
(3.17)
since \(\Phi(t)\) is a decreasing function.

It follows from (3.17) that
\[
I_{33} \leq C e^p \sum_{j=1}^{\infty} j^{-1} \Phi(\epsilon \sqrt{j})
= C e^p \int_{1}^{\infty} \frac{\Phi(\epsilon \sqrt{x})}{x} dx + O(e^p)
= C e^p \int_{\epsilon}^{\infty} \frac{\Phi(t)}{t} dt + O(e^p)
\leq C e^p \int_{\epsilon}^{1} t^{-1} dt + C e^p \int_{1}^{\infty} \Phi(t) dt + O(e^p) = O(e^p \log \frac{1}{\epsilon}),
\]
(3.18)

where the second equation in (3.18) follows from the change of variable \(t = \epsilon \sqrt{x}\) again. (3.15), (3.16) and (3.19) imply that
\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon n}^{\infty} p x^{p-1} \mathbb{P}(|S_n| \geq x) dx = \frac{pe^{p-2} \mathbb{E}[N^2]}{2-p} + B_{-\frac{p}{2}} \mathbb{E}[|N|^p] + O(e^p \log \frac{1}{\epsilon}).
\]
(3.20)
From (3.6), (3.7) and (3.20), we get the part (a) in Proposition 2.1. Below, we prove the part (b). The proof of this part is similar to that of the part (a). However, some modifications are needed to characterize the lower bound $\epsilon \sqrt{n \log n}$. Note that, for any $\delta \in (0, 1)$,

$$
\lambda_2(\epsilon, \delta) = \epsilon^2 \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \mathbb{P}(\lfloor S_n \rfloor \geq \epsilon \sqrt{n \log n}) + \\
\sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{n \log n}}^{\infty} 2x \mathbb{P}(\lfloor S_n \rfloor \geq x)dx.
$$

(3.21)

It follows from Kong [18] that

$$
\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \mathbb{P}(\lfloor S_n \rfloor \geq \epsilon \sqrt{n \log n}) = \frac{\epsilon^{-2\delta-2} \mathbb{E}[|N|^{2\delta+2}]}{\delta + 1} + O(1).
$$

(3.22)

Hence, we only need to compute the second term in (3.21). In fact, by the Fubini’s theorem and Lemma 3.3 we have

$$
\sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{n \log n}}^{\infty} 2x \mathbb{P}(\lfloor S_n \rfloor \geq x)dx \\
= \sum_{j=2}^{\infty} \left( \sum_{n=2}^{j} \frac{(\log n)^{\delta-1}}{n} \right) \int_{\epsilon \sqrt{\log j}}^{\epsilon \sqrt{\log(j+1)}} 2t \Phi(t)dt \\
=: I_{34} + I_{35} + I_{36},
$$

(3.23)

where

$$
I_{34} = \frac{1}{\delta} \sum_{j=2}^{\infty} (\log j)^{\delta} \int_{\epsilon \sqrt{\log j}}^{\epsilon \sqrt{\log(j+1)}} 2t \Phi(t)dt, \\
I_{35} = C\delta \sum_{j=2}^{\infty} \int_{\epsilon \sqrt{\log j}}^{\epsilon \sqrt{\log(j+1)}} 2t \Phi(t)dt, \\
I_{36} = \sum_{j=2}^{\infty} O\left( \frac{(\log j)^{\delta-1}}{j} \right) \int_{\epsilon \sqrt{\log j}}^{\epsilon \sqrt{\log(j+1)}} 2t \Phi(t)dt.
$$

We first consider $I_{34}$. The integer mean theorem shows that

$$
\int_{\epsilon \sqrt{\log j}}^{\epsilon \sqrt{\log(j+1)}} 2t \Phi(t)dt = \epsilon^2 \Phi(\epsilon \sqrt{\log \xi})(\log(j+1) - \log j) \text{ for some } \xi \in (j, j+1).
$$

(3.24)

On the other hand, it follows from the Taylor expansion that

$$
\Phi(\epsilon \sqrt{\log \xi}) = \Phi(\epsilon \sqrt{\log j}) + \epsilon O(j^{-1} - \xi^{-1} (\log j)^{-\frac{1}{2}}), \\
\log(j+1) - \log j = j^{-1} + O(j^{-2}).
$$
By the above argument, we have

$$I_{34} = \frac{C}{\delta} \sum_{j=2}^{\infty} \frac{(\log j)^{\delta}}{j} \Phi(\sqrt{\log j}) + O(\epsilon^2).$$

Putting $t = \epsilon \sqrt{\log x}$, we have

$$\int_{x_0}^{\infty} \frac{(\log x)^{\delta}}{x} \Phi(\epsilon \sqrt{\log x})dx = \frac{2 \epsilon^{-2\delta}}{\delta} \int_{\epsilon \sqrt{\log x}}^{\infty} t^{2\delta+1} \Phi(t) dt = \frac{\epsilon^{-2\delta} \E[N^{2\delta+2}]}{\delta(\delta+1)} + O(\epsilon^2), \quad (3.25)$$

since

$$\int_{0}^{\epsilon \sqrt{\log x}} 2t^{2\delta+1} \Phi(t) dt \leq \int_{0}^{\epsilon \sqrt{\log x}} 2t^{2\delta+1} dt = O(\epsilon^{2\delta+2}).$$

It follows from (3.25) that

$$I_{34} \leq \frac{\epsilon^{-2\delta} \E[N^{2\delta+2}]}{\delta(\delta+1)} + O(\epsilon^2). \quad (3.26)$$

For $I_{35}$, we have

$$I_{35} = C_\delta \int_{0}^{\infty} 2t \Phi(t) dt - C_\delta \int_{\epsilon \sqrt{\log x}}^{\infty} 2t \Phi(t) dt = C_\delta \E[N^2] + O(\epsilon^2). \quad (3.27)$$

Finally, we look at $I_{36}$. By (3.24), we have

$$I_{36} \leq C \epsilon^2 \sum_{j=2}^{\infty} \frac{(\log j)^{\delta-1}}{j^2} \Phi(\sqrt{\log j}) + O(\epsilon^2) = O(\epsilon^2), \quad (3.28)$$

where

$$0 < \delta \leq 1, \text{ and } \Phi(\sqrt{\log j}) \leq 1 \text{ for any } j \geq 2.$$

Hence, from (3.26) to (3.28), we obtain

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{\log n}}^{\infty} 2x \P(|S_n| \geq x) dx = \frac{\epsilon^{-2\delta} \E[N^{2\delta+2}]}{\delta(\delta+1)} + C_\delta \E[N^2] + O(\epsilon^2). \quad (3.29)$$

Therefore, we get from (3.21), (3.22) and (3.29) that the part (b) holds. The proof of this proposition is finished.

□

Next, we focus on the proof of Proposition 2.2.

Proof of Proposition 2.2: Let \( \{X, X_n; n \geq 1\} \) be a sequence of i.i.d. random variables with mean zero, \( \E[X^2] = 1 \) and \( \E[|X|^q] < \infty \) for some \( q \in (2, 3) \). Moreover, for any \( n \in \N \), set

$$\Delta_n := \sup_t \left| \P(|S_n| \geq \sqrt{n}t) - \Phi(t) \right|. \quad (3.30)$$
We first prove the part (a). For any $M \geq 1$, let

$$H_1(\epsilon) = M \epsilon^{-2\gamma},$$

where $\gamma = \frac{q-p}{2q-2p} > 0$ with $0 < p < 2$ and $2 < q \leq 3$.

By the change of variable $t = \frac{x}{\sqrt{n}}$, we have

$$\sum_{n=1}^{\infty} n^{-p} \int_{\epsilon n}^{\infty} px^{p-1} \left| \mathbb{P}(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| dx = \sum_{n=1}^{\infty} n^{-\frac{p}{2}} \int_{\epsilon \sqrt{n}}^{\infty} \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt. \quad (3.31)$$

Based on $H_1(\epsilon)$, we split the right hand side of (3.31) into two parts. We first observe the first part, that is,

$$\sum_{n \leq H_1(\epsilon)} n^{-\frac{p}{2}} \int_{\epsilon \sqrt{n}}^{\infty} \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt \leq \sum_{n \leq H_1(\epsilon)} n^{-\frac{p}{2}} (J_1 + J_2), \quad (3.32)$$

where

$$J_1 = \int_{0}^{\Delta_n^{-\frac{1}{2p}}} pt^{p-1} \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt$$

and

$$J_2 = \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} pt^{p-1} \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt.$$

It follows from

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N$$

that $\Delta_n \to 0$ as $n \to \infty$. Thus,

$$J_1 \leq \Delta_n \int_{0}^{\Delta_n^{-\frac{1}{2p}}} pt^{p-1} dt = \Delta_n^{-\frac{1}{2p}} \xrightarrow{n \to \infty} 0 \quad (n \to \infty). \quad (3.33)$$

On the other hand, it follows from $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N$ and Lemma 3.1 that for large enough $t$,

$$\left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| \leq Ce^{-\frac{t^2}{2}} t^{-1} \leq Ce^{-t} t^{1-p}. \quad (3.34)$$

Hence, as $n$ goes to $\infty$,

$$J_2 \leq C \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} e^{-t} dt \to 0. \quad (3.35)$$

By the Toeplitz lemma, (3.33) and (3.35), we have that for any $M \geq 1$,

$$\lim_{\epsilon \downarrow 0} \epsilon^{(2-p)} \sum_{n \leq H_1(\epsilon)} n^{-\frac{p}{2}} (J_1 + J_2) = \lim_{\epsilon \downarrow 0} \frac{M^{1-\frac{p}{2}}}{(H_1(\epsilon))^{1-\frac{p}{2}}} \sum_{n \leq H_1(\epsilon)} n^{-\frac{p}{2}} (J_1 + J_2) = 0. \quad (3.36)$$
Next, we look at the second part. Recall that Bikjalis [3] got the following non-uniform large deviation estimate, for any $x > 0$,
\[
\left| \Pr(S_n > \sqrt{nx}) - \Pr(N > x) \right| \leq \frac{C \mathbb{E}|X|^q}{n^{q-1}(1 + x^q)}, \quad 2 < q \leq 3.
\] (3.37)

Hence
\[
\sum_{n \geq H_1(\epsilon)} n^{-\frac{p}{2}} \int_{\epsilon \sqrt{n}}^\infty p t^{p-1} \left[ \Pr(|S_n| \geq \sqrt{nt}) - \Phi(t) \right] dt \leq C \sum_{n \geq H_1(\epsilon)} \frac{n^{-\frac{p}{2}} \int_{\epsilon \sqrt{n}}^\infty C t^{p-1} dt}{n^{q-1}(1 + t^q)}
\]
\[
\leq Cn^{1-\frac{p}{2} - \frac{q}{2}} \int_{\epsilon \sqrt{n}}^\infty t^{p-1} dt = C \epsilon^{p-q} \sum_{n \geq H_1(\epsilon)} n^{1-q}.
\]

Thus,
\[
\limsup_{\epsilon \searrow 0} \epsilon^{2-2p} \sum_{n \geq H_1(\epsilon)} n^{-\frac{p}{2}} \int_{\epsilon \sqrt{n}}^\infty p t^{p-1} \left[ \Pr(|S_n| \geq \sqrt{nt}) - \Phi(t) \right] dt \leq C \limsup_{\epsilon \searrow 0} \epsilon^{2-2p+(p-q)} (H_1(\epsilon))^{2-q}
\]
\[
= CM^{2-q} \searrow 0, \quad \text{as } M \nearrow \infty,
\] (3.38)
since $q > 2$.

Finally, we get from (3.36) and (3.38) that the part (a).

Below, we prove the part (b). In fact, the proof of this part is similar to that of part (a) and we only write the modifications in the following.

For any $M \geq 1$, define
\[
H_2(\epsilon) = M \epsilon^{-2} (\log \frac{1}{\epsilon})^{\frac{2}{2-q}}.
\]

It is obvious that as $\epsilon \searrow 0$,
\[
H_2(\epsilon) \nearrow \infty.
\]

The change of variable $t = \frac{x}{\sqrt{n}}$ yields that
\[
\sum_{n=2}^\infty \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{n} \log n}^\infty 2x \left| \Pr(|S_n| \geq x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| dx
\]
\[
= \sum_{n=2}^\infty \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon \sqrt{n} \log n}^\infty 2t \left| \Pr(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt.
\] (3.39)

According to $H_2(\epsilon)$, we split the sum in (3.39) into two part. For the first part,
\[
\sum_{n=2}^\infty \frac{(\log n)^{\delta-1}}{n} \int_{\epsilon \sqrt{n} \log n}^\infty 2t \left| \Pr(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt \leq \sum_{n=2}^\infty \frac{(\log n)^{\delta-1}}{n} (J_3 + J_4),
\]
where
\[ J_3 = \int_0^{\Delta_n^{-\frac{1}{4}}} 2t \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt \]
and
\[ J_4 = \int_{\Delta_n^{-\frac{1}{4}}}^{\infty} 2t \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt \]
with \( \Delta_n \) being defined by (3.30). Similar to (3.33) and (3.35), we obtain that \( J_3 \) and \( J_4 \) go to 0 as \( n \to \infty \), respectively. Thus, by using the Toeplitz lemma, we have, for any \( M \geq 1 \),
\[
\lim (\log \frac{1}{\epsilon})^{-\delta} \sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-1} n (J_3 + J_4) = \lim (\log \frac{1}{\epsilon})^{-\delta} \sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-1} n (J_3 + J_4) = 0. \tag{3.40}
\]
Furthermore, by using the large deviation (3.37), we have
\[
\sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-1} \int_{\epsilon \sqrt{\log n}}^{\infty} 2t \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt 
\leq \sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-1} \int_{\epsilon \sqrt{\log n}}^{\infty} \frac{Ct \mathbb{E}[|X|^q]}{1 + t^q} dt 
= C\epsilon^{2-q} \sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-\frac{q}{2}} n^{-\frac{q}{2}}. \tag{3.41}
\]
Note that
\[ 0 < \delta \leq 1 \text{ and } 2 < q \leq 3. \]
Hence, we have \( \delta - \frac{q}{2} < 0 \) and
\[
\limsup (\log \frac{1}{\epsilon})^{-\delta} \sum_{n \geq H_2(\epsilon)} (\log n)^{\delta-1} \int_{\epsilon \sqrt{\log n}}^{\infty} 2t \left| \mathbb{P}(|S_n| \geq \sqrt{nt}) - \Phi(t) \right| dt 
\leq C \limsup (\log \frac{1}{\epsilon})^{-\delta} \epsilon^{2-q} \sum_{n \geq H_2(\epsilon)} n^{-\frac{q}{2}} 
\leq C \limsup (\log \frac{1}{\epsilon})^{-\delta} \epsilon^{2-q} (H_2(\epsilon))^{1-\frac{q}{2}} 
\leq CM^{1-\frac{q}{2}} \epsilon \to 0, \text{ as } M \to \infty.
\]
Thus, we get from (3.40) and (3.41) the part (b) of Proposition 2.2.

Finally, we prove the main result of this paper.

**Proof of Theorem 2.1:** In order to prove it, we first should point out that (3.6) and (3.21) also hold under the conditions of Theorem 2.1. We prove this theorem by two steps.
We first prove the part (a) of this theorem. Recall that He and Xie \cite{14} obtained that
\[ \sum_{n=1}^{\infty} \mathbb{P}(\mid S_n \mid \geq \epsilon n) = \epsilon^{-2}(\sigma^2 + o(\epsilon^{q-2})). \] (3.42)

Combing (3.6) and (3.42), we get that for any \( 0 < p < 2 \)
\[ \lambda_1(\epsilon, p) = \sigma^2 \epsilon^{p-2} + \epsilon^p q^4 o(1) + \sum_{n=1}^{\infty} n^{-p} \int_{\epsilon n}^{\infty} px^{p-1} \mathbb{P}(\mid S_n \mid \geq x) \, dx. \]
Thus, by (3.20), we have
\[ \lambda_1(\epsilon, p) = \frac{2\sigma^2 \epsilon^{p-2}}{2-p} + B \frac{q}{2q-2-p} \sigma^p \mathbb{E}[\mid N \mid^p] + \epsilon^p q^4 o(1) + O(\epsilon^p \log \frac{1}{\epsilon}) \]
\[ + \sum_{n=1}^{\infty} n^{-p} \int_{\epsilon n}^{\infty} px^{p-1} \left[ \mathbb{P}(\mid S_n \mid \geq x) - \Phi \left( \frac{x}{\sqrt{n}} \right) \right] \, dx. \] (3.43)

Hence, combing (3.43) and part (a) of Proposition 2.2 together, we get that
\[ \lim_{\epsilon \to 0} \epsilon^{\gamma(2-p)} \left[ \lambda_1(\epsilon, p) - \frac{2\sigma^2 \epsilon^{p-2}}{2-p} \right] = 0, \]
where
\[ 0 < p < 2, \quad 0 < \gamma = \frac{q-p}{2q-2-p} < 1, \]
and
\[ \gamma(2-p) + p + q - 4 = \frac{2(q-2)^2}{2q-2-p} > 0. \]
Hence the part (a) holds.

Next, we prove the part (b). The arguments for the part (b) are similar to those for the part (a). Kong \cite{18} got that (3.22) also holds for all random variables satisfying (2.3). Thus, by (3.29) and part (b) of Proposition 2.2 we get that
\[ \lim_{\epsilon \to 0} (\log \frac{1}{\epsilon})^{-\delta} \left[ \lambda_2(\epsilon, \delta) - \frac{\sigma^{2\delta+2}}{\delta} \mathbb{E}[(\mid N \mid)^{2\delta+2}] \epsilon^{-2\delta} \right] = 0. \]
Therefore, we get part (b). The proof of Theorem 2.1 is finished. \( \square \)

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