The variety generated by all the ordinal sums of perfect MV-chains

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“To my friend Erika, and to her invaluable talent in finding surprisingly deep connections among poetry, art, philosophy and logic.”

Abstract

We present the logic $\text{BL}_{\text{Chang}}$, an axiomatic extension of BL (see [Haj98]) whose corresponding algebras form the smallest variety containing all the ordinal sums of perfect MV-chains. We will analyze this logic and the corresponding algebraic semantics in the propositional and in the first-order case. As we will see, moreover, the variety of $\text{BL}_{\text{Chang}}$-algebras will be strictly connected to the one generated by Chang’s MV-algebra (that is, the variety generated by all the perfect MV-algebras): we will also give some new results concerning these last structures and their logic.

1 Introduction

MV-algebras were introduced in [Cha58] as the algebraic counterpart of Łukasiewicz (infinite-valued) logic. During the years these structures have been intensively studied (for a historical overview, see [Cig07]): the book [CDM99] is a reference monograph on this topic.

Perfect MV-algebras were firstly studied in [BDL93] as a refinement of the notion of local MV-algebras: this analysis was expanded in [DL94], where it was also shown that the class of perfect MV-algebras $\text{Perf}(MV)$ does not form a variety, and the variety generated by $\text{Perf}(MV)$ is also generated by Chang’s MV-algebra (see Section 2.2 for the definition). Further studies, about this variety and the associated logic have been done in [BDG07a, BDG07b].

On the other side, Basic Logic BL and its correspondent variety, BL-algebras, were introduced in [Haj98]: Łukasiewicz logic results to be one of the axiomatic extensions of BL and MV-algebras can also be defined as a subclass of BL-algebras. Moreover, the connection between MV-algebras and BL-algebras is even stronger: in fact, as shown in [AM03], every ordinal sum of MV-chains is a BL-chain.

For these reasons one can ask if there is a variety of BL-algebras whose chains are (isomorphic to) ordinal sums of perfect MV-chains: even if the answer to this question
is negative, we will present the smallest variety (whose correspondent logic is called $\text{BL}_\text{Chang}$) containing this class of BL-chains.

As we have anticipated in the abstract, there is a connection between the variety of $\text{BL}_\text{Chang}$-algebras and the one generated by Chang’s MV-algebra. In fact the first-one is axiomatized (over the variety of BL-algebras) with an equation that, over MV-algebras, is equivalent to the one that axiomatize the variety generated by Chang MV-algebras: however, the two equations are not equivalent, over BL.

The paper is structured as follows: in Section 2 we introduce the necessary logical and algebraic background: moreover some basic results about perfect MV-algebras and other structures will be listed. In Section 3 we introduce the main theme of the article: the variety of $\text{BL}_\text{Chang}$ and the associated logic. The analysis will be done in the propositional case: completeness results, algebraic and logical properties and also some results about the variety generated by Chang’s MV-algebra. We conclude with Section 4, where we will analyze the first-order versions of $\text{BL}_\text{Chang}$ and $\text{Ł}_\text{Chang}$: for the first-one the completeness results will be much more negative.

To conclude, we list the main results.

- $\text{BL}_\text{Chang}$ enjoys the finite strong completeness (but not the strong one) w.r.t. $\omega \forall$, where $\omega \forall$ represents the ordinal sum of $\omega$ copies of the disconnected rotation of the standard cancellative hoop.

- $\text{Ł}_\text{Chang}$ (the logic associated to the variety generated by Chang’s MV-algebra) enjoys the finite strong completeness (but not the strong one) w.r.t. $\forall$, $\forall$ being the disconnected rotation of the standard cancellative hoop.

- There are two BL-chains $\mathcal{A}, \mathcal{B}$ that are strongly complete w.r.t., respectively $\text{Ł}_\text{Chang}$ and $\text{BL}_\text{Chang}$.

- Every $\text{Ł}_\text{Chang}$-chain that is strongly complete w.r.t. $\text{Ł}_\text{Chang}$ is also strongly complete w.r.t $\text{Ł}_\text{Chang} \forall$.

- There is no $\text{BL}_\text{Chang}$-chain to be complete w.r.t. $\text{BL}_\text{Chang} \forall$.

2 Preliminaries

2.1 Basic concepts

Basic Logic BL was introduced by P. Hájek in [Haj98]. It is based over the connectives $\&$, $\rightarrow$, $\bot$ and a denumerable set of variables $\text{VAR}$. The formulas are defined inductively, as usual (see [Haj98] for details).

Other derived connectives are the following.

- negation: $\neg \varphi := \varphi \rightarrow \bot$; verum or top: $\top := \neg \bot$; meet: $\varphi \land \psi := \varphi \land (\varphi \rightarrow \psi)$; join: $\varphi \lor \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi)$. 

BL is axiomatized as follows.

\begin{align*}
(\phi \to \psi) & \to ((\psi \to \chi) \to (\phi \to \chi)) \\ (\phi \& \psi) & \to \phi \\ (\phi \& \psi) & \to (\psi \& \phi) \\ (\phi \& (\phi \to \psi)) & \to (\psi \& (\psi \to \phi)) \\ (\phi \to (\psi \to \chi)) & \to ((\phi \& \psi) \to \chi) \\ ((\phi \& \psi) \to \chi) & \to (\phi \to (\psi \to \chi)) \\ ((\phi \to \psi) \to \chi) & \to (((\psi \to \phi) \to \chi) \to \chi) \\ \bot & \to \phi
\end{align*}

(A1) \quad (A2) \quad (A3) \quad (A4) \quad (A5a) \quad (A5b) \quad (A6) \quad (A7)

*Modus ponens* is the only inference rule:

\[
\frac{\phi \quad \phi \to \psi}{\psi}. \quad (MP)
\]

Among the extensions of BL (logics obtained from it by adding other axioms) there is the well known Łukasiewicz (infinitely-valued) logic \(\mathbb{L}\), that is, BL plus

\[\neg\neg\phi \to \phi. \quad (INV)\]

On Łukasiewicz logic we can also define a strong disjunction connective (in the following sections, we will introduce a strong disjunction connective, for BL, that will be proved to be equivalent to the following, over \(\mathbb{L}\))

\[\phi \mathbin{\triangledown} \psi := \neg(\neg\phi \& \neg\psi).\]

The notations \(\phi^n\) and \(n\phi\) will indicate \(\phi \& \ldots \& \phi\) and \(\phi \mathbin{\triangledown} \cdots \mathbin{\triangledown} \phi\).

Given an axiomatic extension \(L\) of BL, a formula \(\phi\) and a theory \(T\) (a set of formulas), the notation \(T \vdash_{L} \phi\) indicates that there is a proof of \(\phi\) from the axioms of \(L\) and the ones of \(T\). The notion of proof is defined like in classical case (see [Haj98]).

We now move to the semantics: for all the unexplained notions of universal algebra, we refer to [BS81, Gra08].

**Definition 2.1.** A BL-algebra is an algebraic structure of the form \(A = (A, *, \Rightarrow, \sqcap, \sqcup, 0, 1)\) such that

- \((A, \sqcap, \sqcup, 0, 1)\) is a bounded lattice, where 0 is the bottom and 1 the top element.
- \((A, *, 1)\) is a commutative monoid.
- \((*, \Rightarrow)\) forms a residuated pair, i.e.

\[
z * x \leq y \quad \text{iff} \quad z \leq x \Rightarrow y, \quad (\text{res})
\]

it can be shown that the only operation that satisfies \((\text{res})\) is \(x \Rightarrow y = \max\{z : z * x \leq y\}\).
• \( \mathcal{A} \) satisfies the following equations

\[ (x \Rightarrow y) \sqcup (y \Rightarrow x) = 1 \]  
\[ x \sqcap y = x \ast (x \Rightarrow y). \]  

(pl) \hspace{1cm} (div)

Two important types of BL-algebras are the followings.

• A BL-chain is a totally ordered BL-algebra.

• A standard BL-algebra is a BL-algebra whose support is \([0, 1]\).

Notation: in the following, with \( \sim x \) we will indicate \( x \Rightarrow 0 \).

Definition 2.2. An MV-algebra is a BL-algebra satisfying

\[ x = \sim \sim x. \]  

(inv)

A well known example of MV-algebra is the standard MV-algebra \([0, 1]_L = \langle [0, 1], \ast, \Rightarrow, \min, \max, 0, 1 \rangle \), where \( x \ast y = \max(0, x + y - 1) \) and \( x \Rightarrow y = \min(1, 1 - x + y) \).

In every MV-algebra we define the algebraic equivalent of \( \gamma \), that is

\[ x \oplus y := \sim (\sim x \ast \sim y). \]

The notations (where \( x \) is an element of some BL-algebra) \( x^n \) and \( nx \) will indicate \( \underbrace{x \ast \cdots \ast x}_{n \text{ times}} \) and \( \underbrace{x \oplus \cdots \oplus x}_{n \text{ times}} \).

Given a BL-algebra \( \mathcal{A} \), the notion of \( \mathcal{A} \)-evaluation is defined in a truth-functional way (starting from a map \( v : \text{VAR} \rightarrow \mathcal{A} \), and extending it to formulas), for details see [Haj98].

Consider a BL-algebra \( \mathcal{A} \), a theory \( T \) and a formula \( \varphi \). With \( \mathcal{A} \models \varphi \) (\( \mathcal{A} \) is a model of \( \varphi \)) we indicate that \( v(\varphi) = 1 \), for every \( \mathcal{A} \)-evaluation \( v \); \( \mathcal{A} \models T \) denotes that \( \mathcal{A} \models \psi \), for every \( \psi \in T \). Finally, the notation \( T \models_{\mathcal{A}} \varphi \) means that if \( \mathcal{A} \models T \), then \( \mathcal{A} \models \varphi \).

A BL-algebra \( \mathcal{A} \) is called L-algebra, where L is an axiomatic extension of BL, whenever \( \mathcal{A} \) is a model for all the axioms of L.

Definition 2.3. Let \( L \) be an axiomatic extension of BL and \( K \) a class of L-algebras. We say that \( L \) is strongly complete (respectively: finitely strongly complete, complete) with respect to \( K \) if for every set \( T \) of formulas (respectively, for every finite set \( T \) of formulas, for \( T = \emptyset \)) and for every formula \( \varphi \) we have

\[ T \models_L \varphi \iff T \models_K \varphi. \]
2.2 Perfect MV-algebras, hoops and disconnected rotations

We recall that Chang’s MV-algebra ([Cha58]) is a BL-algebra of the form

\[ C = \langle \{ a_n : n \in \mathbb{N} \} \cup \{ b_n : n \in \mathbb{N} \}, \ast, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle. \]

Where for each \( n, m \in \mathbb{N} \), it holds that \( b_n < a_m \), and, if \( n < m \), then \( a_m < a_n \), \( b_n < b_m \); moreover \( a_0 = 1 \), \( b_0 = 0 \) (the top and the bottom element).

The operation \( \ast \) is defined as follows, for each \( n, m \in \mathbb{N} \):

\[ b_n \ast b_m = b_0, \quad b_n \ast a_m = b_{\max(0,n-m)}, \quad a_n \ast a_m = a_{n+m}. \]

**Definition 2.4 ([BDL93]).** Let \( \mathcal{A} \) be an MV-algebra and let \( x \in \mathcal{A} \): with \( \text{ord}(x) \) we mean the least (positive) natural \( n \) such that \( x^n = 0 \). If there is no such \( n \), then we set \( \text{ord}(x) = \infty \).

- An MV-algebra is called local\(^1\) if for every element \( x \) it holds that \( \text{ord}(x) < \infty \) or \( \text{ord}(\sim x) < \infty \).
- An MV-algebra is called perfect if for every element \( x \) it holds that \( \text{ord}(x) < \infty \) iff \( \text{ord}(\sim x) = \infty \).

An easy consequence of this definition is that every perfect MV-algebra cannot have a negation fixpoint.

With \( \text{Perfect}(\text{MV}) \) and \( \text{Local}(\text{MV}) \) we will indicate the class of perfect and local MV-algebras. Moreover, given a BL-algebra \( \mathcal{A} \), with \( \text{V}(\mathcal{A}) \) we will denote the variety generated by \( \mathcal{A} \).

**Theorem 2.1 ([BDL93]).** Every MV-chain is local.

Clearly there are local MV-algebras that are not perfect: \([0,1]_L\) is an example.

Now, in [DL94] it is shown that

**Theorem 2.2.**

- \( \text{V}(C) = \text{V} \left( \text{Perfect}(\text{MV}) \right) \),
- \( \text{Perfect}(\text{MV}) = \text{Local}(\text{MV}) \cap \text{V}(C) \).

It follows that the class of chains in \( \text{V}(C) \) coincides with the one of perfect MV-chains. Moreover

**Theorem 2.3 ([DL94]).** An MV-algebra is in the variety \( \text{V}(C) \) iff it satisfies the equation \( (2x)^2 = 2(x^2) \).

As shown in [BDG07a], the logic correspondent to this variety is axiomatized as \( \mathcal{L} \) plus \( (2\varphi)^2 \leftrightarrow 2(\varphi^2) \): we will call it \( \mathcal{L}_{\text{Chang}} \).

\(^1\)Usually, the local MV-algebras are defined as MV-algebras having a unique (proper) maximal ideal. In [BDL93], however, it is shown that the two definitions are equivalent. We have preferred the other definition since it shows in a more transparent way that perfect MV-algebras are particular cases of local MV-algebras.
We now recall some results about hoops

**Definition 2.5** ([Fer92] [BF00]). A hoop is a structure \( \mathcal{A} = \langle A, \ast, \Rightarrow, 1 \rangle \) such that 
\( \langle A, \ast, 1 \rangle \) is a commutative monoid, and \( \Rightarrow \) is a binary operation such that 
\[
x \Rightarrow x = 1, \quad x \Rightarrow (y \Rightarrow z) = (x \ast y) \Rightarrow z \quad \text{and} \quad x \ast (x \Rightarrow y) = y \ast (y \Rightarrow x).
\]

In any hoop, the operation \( \Rightarrow \) induces a partial order \( \leq \) defined by 
\[
x \leq y \iff x \Rightarrow y = 1.
\]
Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (pocrims) in which the meet operation \( \sqcap \) is definable by 
\[
x \sqcap y = x \ast (x \Rightarrow y).
\]
Finally, hoops satisfy the following divisibility condition:

\[\text{If } x \leq y, \text{ then there is an element } z \text{ such that } z \ast y = x. \quad \text{(div)}\]

We recall a useful result.

**Definition 2.6.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two algebras of the same language. Then we say that

- \( \mathcal{A} \) is a partial subalgebra of \( \mathcal{B} \) if \( A \subseteq B \) and the operations of \( \mathcal{A} \) are the ones of \( \mathcal{A} \) restricted to \( A \). Note that \( A \) could not be closed under these operations (in this case these last ones will be undefined over some elements of \( A \)): in this sense \( \mathcal{A} \) is a partial subalgebra.

- \( \mathcal{A} \) is partially embeddable into \( \mathcal{B} \) when every finite partial subalgebra of \( \mathcal{A} \) is embeddable into \( \mathcal{B} \). Generalizing this notion to classes of algebras, we say that a class \( K \) of algebras is partially embeddable into a class \( M \) if every finite partial subalgebra of a member of \( K \) is embeddable into a member of \( M \).

**Definition 2.7.** A bounded hoop is a hoop with a minimum element; conversely, an unbounded hoop is a hoop without minimum.

Let \( \mathcal{A} \) be a bounded hoop with minimum \( a \): with \( \mathcal{A}^+ \) we mean the (partial) subalgebra of \( \mathcal{A} \) defined over the universe \( A^+ = \{ x \in A : x > x \Rightarrow a \} \).

A hoop is Wajsberg iff it satisfies the equation 
\[
(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x.
\]
A hoop is cancellative iff it satisfies the equation 
\[
x \ast (y \Rightarrow x) = y \ast (x \Rightarrow y).
\]

**Proposition 2.1** ([Fer92] [BF00] [AFM07]). Every cancellative hoop is Wajsberg. Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the 0-free reducts of) MV-algebras.

We now recall a construction introduced in [Jen03] (and also used in [EGHM03] [NEG05]), called disconnected rotation.

**Definition 2.8.** Let \( \mathcal{A} \) be a cancellative hoop. We define an algebra, \( \mathcal{A}^* \), called the disconnected rotation of \( \mathcal{A} \), as follows. Let \( \mathcal{A} \times \{0\} \) be a disjoint copy of \( A \). For every \( a \in A \) we write \( a' \) instead of \( \langle a, 0 \rangle \). Consider \( \langle A' = \{ a' : a \in A \} \,, \leq \rangle \) with the inverse order and let \( A^* := A \cup A' \). We extend these orderings to an order in \( A^* \) by putting
$a' < b$ for every $a, b \in A$. Finally, we take the following operations in $A^*$: $1 := 1_{A'}$, $0 := 1'$, $\cap_{A'}, \cup_{A'}$ as the meet and the join with respect to the order over $A^*$. Moreover,

\[
\sim_{A'} a := \begin{cases} 
    a' & \text{if } a \in A \\
    b & \text{if } a = b' \in A'
\end{cases}
\]

\[
a_{A'} b := \begin{cases} 
    a_{A'} b & \text{if } a, b \in A \\
    \sim_{A'} (a \Rightarrow_{A'} \sim_{A'} b) & \text{if } a \in A, b \in A' \\
    \sim_{A'} (b \Rightarrow_{A'} \sim_{A'} a) & \text{if } a \in A', b \in A \\
    0 & \text{if } a, b \in A'
\end{cases}
\]

\[
a \Rightarrow_{A'} b := \begin{cases} 
    a_{A'} b & \text{if } a, b \in A \\
    \sim_{A'} (a \Rightarrow_{A'} \sim_{A'} b) & \text{if } a \in A, b \in A' \\
    1 & \text{if } a \in A', b \in A \\
    \sim_{A'} (b) \Rightarrow_{A'} (\sim_{A'} a) & \text{if } a, b \in A'.
\end{cases}
\]

**Theorem 2.4 ([NEG05] theorem 9).** Let $\mathcal{A}$ be an MV-algebra. The followings are equivalent:

- $A$ is a perfect MV-algebra.
- $A$ is isomorphic to the disconnected rotation of a cancellative hoop.

To conclude the section, we present the definition of ordinal sum.

**Definition 2.9 ([AM03]).** Let $(I, \leq)$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded. Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the ordinal sum of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

\[
x \Rightarrow y = \begin{cases} 
    x \Rightarrow_{A_i} y & \text{if } x, y \in A_i \\
    y & \text{if } \exists i > j(x \in A_i \text{ and } y \in A_j) \\
    1 & \text{if } \exists i < j(x \in A_i \setminus \{1\} \text{ and } y \in A_j)
\end{cases}
\]

\[
x \ast y = \begin{cases} 
    x \ast_{A_i} y & \text{if } x, y \in A_i \\
    x & \text{if } \exists i > j(x \in A_i \setminus \{1\}, y \in A_j) \\
    y & \text{if } \exists i < j(y \in A_i \setminus \{1\}, x \in A_j)
\end{cases}
\]

When defining the ordinal sum $\bigoplus_{i \in I} \mathcal{A}_i$ we will tacitly assume that whenever the condition $A_i \cap A_j = \{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the $\mathcal{A}_i$ by isomorphic copies satisfying such condition. Moreover if all $\mathcal{A}_i$'s are isomorphic to some $\mathcal{A}'$, then we will write $I_{\mathcal{A}}$, instead of $\bigoplus_{i \in I} \mathcal{A}_i$. Finally, the ordinal sum of two hoops $\mathcal{A}$ and $\mathcal{B}$ will be denoted by $\mathcal{A} \oplus \mathcal{B}$.

Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MV-algebra, then the previous definition also works with these structures.
Theorem 2.5 ([AM03 theorem 3.7]). Every BL-chain is isomorphic to an ordinal sum whose first component is an MV-chain and the others are totally ordered Wajsberg hoops.

Note that in [Bus04] it is presented an alternative and simpler proof of this result.

3 The variety of BL\textsubscript{Chang}-algebras

Consider the following connective

\[
\varphi \downarrow \psi := ((\varphi \rightarrow (\varphi \& \psi)) \rightarrow \psi) \land ((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)
\]

Call \(\vartriangledown\) the algebraic operation, over a BL-algebra, corresponding to \(\downarrow\); we have that

Lemma 3.1. In every MV-algebra the following equation holds

\[
x \vartriangledown y = x \oplus y.
\]

Proof. It is easy to check that \(x \vartriangledown y = x \oplus y\), over [0, 1]\textsubscript{MV}, for every \(x, y \in [0, 1]\).

We now analyze this connective in the context of Wajsberg hoops.

Proposition 3.1. Let \(\mathcal{A}\) be a linearly ordered Wajsberg hoop. Then

- If \(\mathcal{A}\) is unbounded (i.e. a cancellative hoop), then \(x \vartriangledown y = 1\), for every \(x, y \in \mathcal{A}\).

- If \(\mathcal{A}\) is bounded, let \(a\) be its minimum. Then, by defining \(\sim x := x \Rightarrow a\) and \(x \oplus y = \sim (\sim x \ast \sim y)\) we have that \(x \oplus y = x \vartriangledown y\), for every \(x, y \in \mathcal{A}\).

Proof. An easy check.

Now, since the variety of cancellative hoops is generated by its linearly ordered members (see [EGHM03]), then we have that

Corollary 3.1. The equation \(x \vartriangledown y = 1\) holds in every cancellative hoop.

We now characterize the behavior of \(\vartriangledown\) for the case of BL-chains.

Proposition 3.2. Let \(\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i\) be a BL-chain. Then

\[
x \vartriangledown y = \begin{cases} x \oplus y, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is bounded} \\ 1, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is unbounded} \\ \max(x, y), & \text{otherwise.} \end{cases}
\]

for every \(x, y \in \mathcal{A}\).

Proof. If \(x, y\) belong to the same component of \(\mathcal{A}\), then the result follows from Lemma 3.1 and Proposition 3.1. For the case in which \(x\) and \(y\) belong to different components of \(\mathcal{A}\), this is a direct computation.
Remark 3.1. From the previous proposition we can argue that $\cup$ is a good approximation, for BL, of what that $\oplus$ represents for MV-algebras. Note that a similar operation was introduced in [ABM09]: the main difference with respect to $\cup$ is that, when $x$ and $y$ belong to different components of a BL-chain, then the operation introduced in [ABM09] holds.

In the following, for every element $x$ of a BL-algebra, with the notation $\pi x$ we will denote $x \cup \cdots \cup x$; analogously $\pi \varphi$ means $\varphi \lor \cdots \lor \varphi$.

Definition 3.1. We define $\text{BL}_{\text{Chang}}$ as the axiomatic extension of BL, obtained by adding

$$((\exists \varphi)^2 \leftrightarrow \exists (\varphi^2)).$$

That is, writing it in extended form

$$(\varphi^2 \rightarrow (\varphi^2 \& \varphi^2) \rightarrow \varphi^2) \leftrightarrow ((\varphi \rightarrow \varphi^2) \rightarrow \varphi)^2.$$  

Clearly the variety corresponding to $\text{BL}_{\text{Chang}}$ is given by the class of BL-algebras satisfying the equation $(\exists x)^2 = \exists (x^2)$.

Moreover,

Definition 3.2. We will call pseudo-perfect Wajsberg hoops those Wajsberg hoops satisfying the equation $(\exists x)^2 = \exists (x^2)$.

Remark 3.2. Thanks to Lemma 3.1 we have that

$$\vdash_L ((\exists \varphi)^2 \leftrightarrow \exists (\varphi^2)) \leftrightarrow ((\exists \varphi)^2 \leftrightarrow \exists (\varphi^2)),$$

that is, if we add $(\exists \varphi)^2 \leftrightarrow \exists (\varphi^2)$ or $(2 \varphi)^2 \leftrightarrow 2 (\varphi^2)$ to $L$, then we obtain the same logic $L_{\text{Chang}}$.

These formulas, however are not equivalent over $BL$: see Remark 3.3 for details.

Theorem 3.1. Every totally ordered pseudo-perfect Wajsberg hoop is a totally ordered cancellative hoop or (the $0$-free reduct of) a perfect MV-chain.

More in general, the variety of pseudo-perfect Wajsberg hoops coincides with the class of the $0$-free subreducts of members of $\text{V}(C)$.

Proof. In [EGHM03] it is shown that the variety of Wajsberg hoops coincides with the class of the $0$-free subreducts of MV-algebras. The results easily follow from this fact and from Proposition 2.1, Theorem 2.3 and Definition 3.2.

As a consequence, we have

Theorem 3.2. Let $WH, CH, psWH$ be, respectively, the varieties of Wajsberg hoops, cancellative hoops, pseudo-perfect Wajsberg hoops. Then we have that

$$CH \subset psWH \subset WH.$$
Proof. An easy consequence of Theorem 3.1.

The first inclusion follows from the fact that $\text{psWH}$ contains all the totally ordered cancellative hoops and hence the variety generated by them. For the second inclusion note that, for example, the 0-free reduct of $[0, 1]_L$ belongs to $\text{WH} \setminus \text{psWH}$. □

We now describe the structure of BL-Chang-chains, with an analogous of the Theorem 2.5 for BL-chains.

Theorem 3.3. Every BL-Chang-chain is isomorphic to an ordinal sum whose first component is a perfect MV-chain and the others are totally ordered pseudo-perfect Wajsberg hoops.

It follows that every ordinal sum of perfect MV-chains is a BL-Chang-chain.

Proof. Thanks to Theorems 2.2 and 2.3, Remark 3.2 and Definition 3.2, we have that every MV-chain (Wajsberg hoop) satisfying the equation $(\neg x)^2 = \neg(x^2)$ is perfect (pseudo-perfect): using these facts and Proposition 3.2 we have that a BL-chain satisfies the equation $(\neg x)^2 = \neg(x^2)$ if it holds true in all the components of its ordinal sum. From these facts and Theorem 2.5 we get the result. □

As a consequence, we obtain the following corollaries.

Corollary 3.2. The variety of BL-Chang-algebras contains the ones of product-algebras and Gödel-algebras: however it does not contains the variety of MV-algebras.

Proof. From the previous theorem it is easy to see that the variety of BL-Chang-algebras contains $[0, 1]_\Pi$ and $[0, 1]_G$, but not $[0, 1]_L$. □

Corollary 3.3. Every finite BL-Chang-chain is an ordinal sum of a finite number of copies of the two elements boolean algebra. Hence the class of finite BL-Chang-chains coincides with the one of finite Gödel chains.

For this reason it is immediate to see that the finite model property does not hold for BL-Chang.

We conclude with the following remark.

Remark 3.3. • One can ask if it is possible to axiomatize the class $\text{BL}_{\text{perf}}$ of BL-algebras, whose chains are the BL-algebras that are ordinal sum of perfect MV-chains: the answer, however, is negative. In fact, the class of bounded Wajsberg hoops does not form a variety: for example, it is easy to check that for every bounded pseudo-perfect Wajsberg hoop $A$, its subalgebra $A^+$ (see Definition 2.7) forms a cancellative hoop. Hence $\text{BL}_{\text{perf}}$ cannot be a variety.

However, as we will see in Section 3.2, the variety of BL-Chang-algebras is the “best approximation” of $\text{BL}_{\text{perf}}$, in the sense that it is the smallest variety to contain $\text{BL}_{\text{perf}}$.

• In [DSE+02] (see also [CT06]) it is studied the variety, called $P_0$, generated by all the perfect BL-algebras (a BL-algebra $A$ is perfect if, by calling $\text{MV}(A)$...
the biggest subalgebra of \( \mathcal{A} \) to be an MV-algebra, then \( MV(\mathcal{A}) \) is a perfect MV-algebra. \( P_0 \) is axiomatized with the equation
\[
\neg (\neg (x^2))^2 = (\neg (\neg x))^2.
\] \( (p_0) \)

One can ask which is the relation between \( P_0 \) and the variety of BL-Chang-algebras. The answer is that the variety of BL-Chang-algebras is strictly contained in \( P_0 \). In fact, an easy check shows that a BL-chain is perfect if and only if the first component of its ordinal sum is a perfect MV-chain. Hence we have:

- Every BL-Chang-chain is a perfect BL-chain.
- There are perfect BL-chains that are not BL-Chang-chains: an example is given by \( C \oplus [0,1]_L \).

Now, since the variety of BL-Chang-algebras is generated by its chains (like any variety of BL-algebras, see [Haj98]), then we get the result.

Finally note that \( [p_0] \) is equivalent to \( 2(x^2) = (2x)^2 \); hence, differently to what happens over \( L \) (see Remark 3.2), the equations \( 2(x^2) = (2x)^2 \) and \( \neg \neg (x^2) = (\neg \neg x)^2 \) are not equivalent, over BL.

### 3.1 Subdirectly irreducible and simple algebras

We begin with a general result about Wajsberg hoops.

**Theorem 3.4** ([Fer92, Corollary 3.11]). Every subdirectly irreducible Wajsberg hoop is totally ordered.

As a consequence, we have:

**Corollary 3.4.** Every subdirectly irreducible pseudo-perfect Wajsberg hoop is totally ordered.

We now move to simple algebras.

It is shown in [Tur99, Theorem 1] that the simple BL-algebras coincide with the simple MV-algebras, that is, with the subalgebras of \( [0,1]_L \) (see [CDM99, Theorem 3.5.1]). Therefore we have:

**Theorem 3.5.** The only simple BL-Chang-algebra is the two elements boolean algebra \( 2 \).

An easy consequence of this fact is that the only simple \( L_{\text{Chang}} \)-algebra is \( 2 \).

### 3.2 Completeness

We begin with a result about pseudo-perfect Wajsberg hoops.

**Theorem 3.6.** The class \( p\text{MV} \) of 0-free reducts of perfect MV-chains generates \( \text{ps}_{\text{WHL}} \).
Proof. From Theorems 2.4 and 3.1 it is easy to check that the variety generated by $pMV$ contains all the totally ordered pseudo-perfect Wajsberg hoops.

From these facts and Corollary 3.4, we have that $pMV$ must be generic for $p\text{S\hbar\bar{W}}$.

\begin{theorem}[\text{CEG'09}] Let $L$ be an axiomatic extension of $BL$, then $L$ enjoys the finite strong completeness w.r.t a class $K$ of $L$-algebras iff every countable $L$-chain is partially embeddable into $K$.
\end{theorem}

As shown in [Haj98] product logic enjoys the finite strong completeness w.r.t $[0,1]_\Pi$ and hence every countable product chain is partially embeddable into $[0,1]_\Pi \simeq 2 \oplus (0,1)_C$, with $(0,1)_C$ being the standard cancellative hoop (i.e. the 0-free reduct of $[0,1]_\Pi \setminus \{0\}$). Since every totally ordered product chain is of the form $2 \oplus \mathcal{A}$, where $\mathcal{A}$ is a cancellative hoop (see [EGHM03]), it follows that:

\begin{proposition}
Every countable totally ordered cancellative hoop partially embeds into $(0,1)_C$.
\end{proposition}

\begin{theorem}
Every countable perfect MV-chain partially embeds into $\mathcal{V} = (0,1)_C^*$ (i.e. the disconnected rotation of $(0,1)_C$).
\end{theorem}

\begin{proof}
Immediate from Proposition 3.3 and Theorem 2.4.
\end{proof}

\begin{corollary}
The logic $L\text{Chang}$ is finitely strongly complete w.r.t. $\mathcal{V}$.
\end{corollary}

\begin{theorem}
$BL\text{Chang}$ enjoys the finite strong completeness w.r.t. $\omega \mathcal{V}$. As a consequence, the variety of $BL\text{Chang}$-algebras is generated by the class of all ordinal sums of perfect MV-chains and hence is the smallest variety to contain this class of algebras.
\end{theorem}

\begin{proof}
Thanks to Theorem 3.7 it is enough to show that every countable $BL\text{Chang}$-chain partially embeds into $\omega \mathcal{V}$ (i.e. the ordinal sum of “$\omega$ copies” of $\mathcal{V}$). This fact, however, follows immediately from Proposition 3.3 and Theorems 3.3 and 3.8.
\end{proof}

But we cannot obtain a stronger result: in fact

\begin{theorem}
$BL\text{Chang}$ is not strongly complete w.r.t. $\omega \mathcal{V}$.
\end{theorem}

\begin{proof}
Suppose not: from the results of [CEG'09, Theorem 3.5] this is equivalent to claim that every countable $BL\text{Chang}$-chain embeds into $\omega \mathcal{V}$. But, this would imply that every countable totally ordered cancellative hoop embeds into $(0,1)_C$: this means that every countable product-chain embeds into $[0,1]_\Pi$, that is product logic is strongly complete w.r.t $[0,1]_\Pi$. As it is well known (see [Haj98, Corollary 4.1.18]), this is false.
\end{proof}

With an analogous proof we obtain

\begin{theorem}
$L\text{Chang}$ is not strongly complete w.r.t. $\mathcal{V}$
\end{theorem}

However, thanks to [Mon11, Theorem 3] we can claim

\begin{theorem}
There exist a $L\text{Chang}$-chain $\mathcal{A}$ and a $BL\text{Chang}$-chain $\mathcal{B}$ such that $L\text{Chang}$ is strongly complete w.r.t. $\mathcal{A}$ and $BL\text{Chang}$ is strongly complete w.r.t. $\mathcal{B}$.
\end{theorem}

\begin{problem}
Which can be some concrete examples of such $\mathcal{A}$ and $\mathcal{B}$?
\end{problem}
4 First-order logics

We assume that the reader is acquainted with the formalization of first-order logics, as developed in [Haj98, CH10].

Briefly, we work with (first-order) languages without equality, containing only predicate and constant symbols; as quantifiers we have ∀ and ∃. The notions of terms and formulas are defined inductively like in classical case.

As regards to semantics, given an axiomatic extension L of BL we restrict to L- chains: the first-order version of L is called L ∀ (see [Haj98, CH10] for an axiomatization). A first-order A-interpretation (A being an L-chain) is a structure M = ⟨M, {rP}P∈P, {mc}c∈C⟩, where M is a non-empty set, every rP is a fuzzy arity(P)-ary relation, over M, in which we interpretate the predicate P, and every mc is an element of M, in which we map the constant c.

Given a map v: VAR → M, the interpretation of ∥ϕ∥A M, v in this semantics is defined in a Tarskian way: in particular the universally quantified formulas are defined as the infimum (over A) of truth values, whereas those existentially quantified are evaluated as the supremum. Note that these inf and sup could not exist in A: an A-model M is called safe if ∥ϕ∥A M, v is defined for every ϕ and v.

A model is called witnessed if the universally (existentially) quantified formulas are evaluated by taking the minimum (maximum) of truth values in place of the infimum (supremum): see [Haj07, CH06, CH10] for details.

The notions of soundness and completeness are defined by restricting to safe models (even if in some cases it is possible to enlarge the class of models: see [BM09]): see [Haj98, CH10, CH06] for details.

We begin with a positive result about ŁChang ∀.

Definition 4.1. Let L be an axiomatic extension of BL. With L ∀w we define the extension of L ∀ with the following axioms

(∃y)(φ(y) → (∀x)φ(x)) (CV ∀)

(∃y)((∃x)φ(x) → φ(y)). (C∃ ∀)

Theorem 4.1 ([CH06, Proposition 6]). L ∀ coincides with L ∀w, that is L ∀ ⊢ [(CV ∀)(C∃ ∀)].

An immediate consequence is:

Corollary 4.1. Let L be an axiomatic extension of L. Then L ∀ coincides with L ∀w.

Theorem 4.2 ([CH06, Theorem 8]). Let L be an axiomatic extension of BL. Then L ∀w enjoys the strong witnessed completeness with respect to the class K of L-chains, i.e.

T ⊢L ∀w ϕ iff ∥ϕ∥M = 1,

for every theory T, formula ϕ, algebra A ∈ K and witnessed A-model M such that ∥ψ∥M = 1 for every ψ ∈ T.
Lemma 4.1 ([Mon11] Lemma 1). Let $L$ be an axiomatic extension of $BL$, let $\mathcal{A}$ be an $L$-chain, let $\mathcal{B}$ be an $L$-chain such that $A \subseteq B$ and let $M$ be a witnessed $\mathcal{A}$-structure. Then for every formula $\varphi$ and evaluation $v$, we have $\|\varphi\|_M = \|\varphi\|_B$.

Theorem 4.3. There is a $L_{\text{Chang}}$-chain that is strongly complete w.r.t. $L_{\text{Chang}}$. More in general, every $L_{\text{Chang}}$-chain that is strongly complete w.r.t. $L_{\text{Chang}}$ is also strongly complete w.r.t. $L_{\text{Chang}}$.

Proof. An adaptation of the proof for the analogous result, given in [Mon11] Theorem 16], for $L_{\mathcal{V}}$.

From Theorem 3.12 we know that there is a $L_{\text{Chang}}$-chain $\mathcal{A}$ strongly complete w.r.t. $L_{\text{Chang}}$: from [CEG+09] Theorem 3.5] this is equivalent to claim that every countable $L_{\text{Chang}}$-chain embeds into $\mathcal{A}$. We show that $\mathcal{A}$ is also strongly complete w.r.t. $L_{\text{Chang}}$.

Suppose that $T \not\models_{L_{\text{Chang}}} \varphi$. Thanks to Corollary 4.1 and Theorem 4.2 there is a countable $L_{\text{Chang}}$-chain $\mathcal{G}$ and a witnessed $\mathcal{G}$-model $M$ such that $\|\varphi\|_M = 1$, for every $\psi \in T$, but $\|\varphi\|_M < 1$. Finally, from Lemma 4.1 we have that $\|\psi\|_M = 1$, for every $\psi \in T$ and $\|\varphi\|_M = \|\varphi\|_M < 1$: this completes the proof.

For $BL_{\text{Chang}}$, however, the situation is not so good.

Theorem 4.4. $BL_{\text{Chang}}$ cannot enjoy the completeness w.r.t. a single $BL_{\text{Chang}}$-chain.

Proof. The proof is an adaptation of the analogous result given in [Mon11] Theorem 17 for $BL_{\mathcal{V}}$.

Let $\mathcal{A}$ be a $BL_{\text{Chang}}$-chain: call $\mathcal{A}_0$ its first component. We have three cases

- $\mathcal{A}_0$ is finite: from Theorem 3.3 we have that $\mathcal{A}_0 = 2$ and hence $\mathcal{A} \models (\neg \neg x) \rightarrow (\neg \neg x)^2$. However $\mathcal{V} \not\models (\neg \neg x) \rightarrow (\neg \neg x)^2$, where $\mathcal{V}$ is the chain introduced in Section 3.2 and hence $\mathcal{A}$ cannot be complete w.r.t. $BL_{\text{Chang}}$.

- $\mathcal{A}_0$ is infinite and dense. As shown in [Mon11] Theorem 17 the formula $(\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x)$ is a tautology in every $BL$-chain whose first component is infinite and densely ordered: hence we have that $\mathcal{A} \models (\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x)$.

- $\mathcal{A}_0$ is infinite and not dense. As shown in [Mon11] Theorem 17 the formula $(\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x) \vee \neg (\forall x)P(x) \rightarrow (\forall x)P(x)$ is a tautology in every $BL$-chain whose first component is infinite and not densely ordered: hence we have that $\mathcal{A} \models (\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x) \vee \neg (\forall x)P(x) \rightarrow (\forall x)P(x)^2$. Also in this case, however, this formula fails in $[0, 1]_G$, using the same model $M$ of the previous case.
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