A quasi-two dimensional superconductor Sr$_2$RuO$_4$ is an unconventional superconductor, and the pairing symmetry is suggested to be a chiral $p$-wave pairing with the basic form $p_\pm = p_x \pm ip_y$ and in-plane equal-spin pairing.$^{12}$ The spin-triplet pairing is supported by the fact that the Knight shift does not change below the superconducting transition temperature $T_c$. Since the spontaneous moment appears below $T_c$ in the observation of the muon spin relaxation ($\mu$SR), time-reversal symmetry of the pairing function is broken.$^8$ The internal field distribution of the vortex state, which is observed by a small angle neutron scattering (SANS), is consistent to the chiral $p$-wave pairing state.$^8$ While the pairing function of Sr$_2$RuO$_4$ may have an additional horizontal node, such as $(p_x \pm ip_y) \cos p\hat{z}$, it is intrinsic in our study that the pairing function has a factor of the chiral component $p_x \pm ip_y$.

In the chiral $p$-wave superconductor, a $p_+$ state and a $p_-$ state are degenerate in free energy. Therefore, the multi-domain structure may realize, i.e., some regions in a sample are $p_+$ domains and others are $p_-$ domains. Between the $p_+$ and $p_-$ domains, domain walls appear as topological defects, which are not easily destroyed.$^2$ In the vortex state when a magnetic field $H$ is applied to $\hat{z}$ direction, the degeneracy of the $p_+$ and $p_-$ states is removed. When $H \parallel \hat{z}$ and charge $e > 0$ (or equivalently when $H \parallel -\hat{z}$ and $e < 0$), the free energy of the $p_-$ state is lower than that of the $p_+$ state$^2$ because the vortex structures are different between the $p_+$ and $p_-$ domains. In the $p_-$ ($p_+$) state, the opposite $p_+$ ($p_-$) component induced around the vortex core has large (small) amplitude.$^{2,9,14}$ Since the $p_-$ state is stable in the vortex state, it is interesting to see how the multi-domain structure at a zero field changes to the single $p_-$ domain, when a magnetic field is applied. We note that the chirality dependence is defined relative to the magnetic field direction, the free energy of the $p_+$ state becomes smaller when the magnetic field is applied to the reverse direction $-\hat{z}$. Study of vortices trapped at the domain wall is also important since they are considered to have a strange structure called a “vortex sheet”$^8$ where half flux-quantum vortices are aligned along the domain wall.$^{8,13,14,15,16}$

The purpose of this paper is to study the magnetization process in order to understand how the multi-domain structure changes to the single $p_-$ domain by applying magnetic fields in the chiral $p$-wave superconductor. We also investigate the roles and properties of the vortex sheet structure at the domain wall appearing in the magnetization process. The static properties of the vortex or domain wall structure were studied also by the two-component Ginzburg-Landau (GL) theory$^{2,8,10,17,18}$ For the study of the magnetization process, we use the time-dependent Ginzburg-Landau (TDGL) theory. In this paper, we use the GL free energy for the chiral $p$-wave superconductor, and study the magnetization process.

To obtain the two-component GL equation in the chiral $p$-wave superconductor, the pair potential is decomposed as $\Delta(r, p) = \eta_+(r)\phi_+(p) + \eta_-(r)\phi_-(p)$ with the order parameter $\eta_\pm(r)$, where $r$ is the center-of-mass coordinate of the Cooper pair. The pairing functions $\phi_\pm(p)$, depending on the relative momentum $p$ of the pair, are given by the chiral $p$-wave type such as $p_x \pm ip_y$. The GL free energy density is written as

$$
\tilde{f} = -\left(1 - \frac{T}{T_c}\right) \left(\eta_+^2 + \eta_-^2\right) + \frac{1}{2} |\eta_+|^4 + \frac{1}{2} |\eta_-|^4 + 2|\eta_+|^2|\eta_-|^2 + C_1 (\eta_+^2 \eta_-^2 + \eta_+ \eta_-^2) + \eta_+ \eta_- (q_z^2 + q_y^2) \eta_+ \eta_- + C_2 (\eta_+^2 q_y^2 \eta_-^2 + \eta_-^2 q_y^2 \eta_+) + C_3 (\eta_+^2 q_y^2 \eta_- + \eta_-^2 q_y^2 \eta_+)$$

in the dimensionless form$^{13,14}$ where $q_z = q_y = 0$, $q = (h/i) \nabla - 2\pi(2e/hc)A$ with vector potential $A$, $hc/2|e| = \phi_0$ is a flux-quantum. The coefficients are related to the pairing function and the Fermi surface structure as

\begin{align}
C_1 & = \frac{\langle \phi_+^2 \phi_-^2 \rangle}{2\langle |\phi_+|^4 \rangle}, & C_2 & = \frac{\langle v_+^2 \phi_+^2 \phi_- \rangle}{2\langle v_+ v_- |\phi_+|^2 \rangle}, \\
C_3 & = \frac{\langle v_+^2 \phi_+^2 \phi_- \rangle}{2\langle v_+ v_- |\phi_+|^2 \rangle},
\end{align}

where $v_\pm = (v_x \pm iv_y)/2$ with a Fermi velocity $(v_x, v_y)$, and $\langle \cdot \cdot \cdot \rangle$ indicates the average on $p$ along the Fermi sur-
face. $C_1$ and $C_3$ are anisotropy parameters which are finite when the pairing functions or the Fermi surface have fourfold symmetric structure. For the isotropic case, $C_1 = C_3 = 0$ due to the vanishing Fermi surface average in Eq. (2). As the detailed forms of $\phi_i$ and the Fermi surface structure have not been established yet, we treat the coefficients in Eq. (2) as arbitrary parameters. In this study we set $C_1 = C_3 = 0$ for simplicity to exclude the additional anisotropy effect. We show results calculated for $C_2 = 0.3$ and $T = 0.5T_c$.

In our simulations, we use the TDGL equation coupled with Maxwell equations\textsuperscript{19,20}:

$$\frac{\partial}{\partial t} \eta_1 = -\frac{1}{12} \frac{\partial f}{\partial \eta_1}, \quad \frac{\partial}{\partial t} \eta_2 = -\frac{1}{12} \frac{\partial f}{\partial \eta_2}, \quad \frac{\partial}{\partial t} A = \tilde{j}_s - \kappa^2 \nabla \times B, \quad B = \nabla \times A. \quad (3)$$

The supercurrent $\tilde{j}_s = (\tilde{j}_{s,x}, \tilde{j}_{s,y}) \propto (\partial \tilde{\phi} / \partial A_x, \partial \tilde{\phi} / \partial A_y)$ is given by $\tilde{j}_{s,x} = \text{Re} [\eta_1^* (q_x \eta_+ + q^*_- (q_- \eta_-))] + C_2 (\eta_{+}^* (q_x \eta_-) + \eta_{-}^* (q_- \eta_+)) + C_3 (\eta_{+}^* (q_+ \eta_-) + \eta_{-}^* (q_- \eta_+))]$, $\tilde{j}_{s,y} = \text{Re} [q_x \eta_1^* (q_x \eta_+) + q^*_- (q_- \eta_-) - i C_2 (\eta_{+}^* (q_- \eta_-) - \eta_{-}^* (q_+ \eta_+)) + i C_3 (\eta_{+}^* (q_- \eta_+)) + i C_3 (\eta_{+}^* (q_+ \eta_-) - \eta_{-}^* (q_- \eta_+))]$. The length, field, and time are, respectively, scaled by the coherence length $\xi_0$, $H_{c2,0} = \phi_0 / 2 \pi \xi_0^2$, and $t_0 = 4 \pi \xi_0^2 \kappa^2 / c^2$ with the normal state conductivity $\sigma$.\textsuperscript{19,20} However, we here scale $\eta_\pm$ by $\eta_0$ instead of $\eta_0 (T) = \eta_0 (1 - T/T_c)^{1/2}$. $\eta_0$ is a uniform solution of $\eta_\pm$ when $\eta_\pm = 0$ and $T = 0$. The calculations are performed in a two-dimensional rectangular area with a size $200 \xi_0 \times 100 \xi_0$. Outside the open boundary, we set $\eta_\pm = 0$ and $B(r) = 0$ with an applied field $H$. We set the GL parameter $\kappa = 2.7$.

Our calculation of the magnetization process for the multi-domain state is shown in Fig. 1. At a zero field, we prepare the state where the right-hand side half region is a $p_+\_x$ state ($\eta_+ \sim 1$, $\eta_- = 0$), and the left-hand side half region is a $p_-\_x$ state ($\eta_+ \sim 1$, $\eta_- = 0$). A straight domain wall appears at the center between the $p_+\_x$ and $p_-\_x$ domains. The relative phase of $\eta_\pm$ is $\pi$, which minimizes the free energy in our case. We increase $H$ gradually with a slow rate $\delta H / \delta t = 5 \times 10^{-6}$. The left panels in Fig. 1 show the color-density plot of $|\eta_+(r)|$ and $|\eta_-(r)|$. A green (red) region indicates the $p_+\_x$ ($p_-\_x$) domain. The center panels show $|\eta_x(r)|$ and $|\eta_y(r)|$, when we define the $x$ component $\eta_x$ and the $y$ component $\eta_y$ as $\Delta (r, \phi) = \eta_x (r) \phi_x (p) + \eta_y (r) \phi_y (p)$ with $\phi_x = (\phi_+ + \phi_-) / 2 \sim p_x$ and $\phi_y = (\phi_+ - \phi_-) / 2i \sim p_y$. Yellow region indicates that $|\eta_x (r)| \sim |\eta_y (r)| \sim 1$ in the $p_+\_x$ or $p_-\_x$ domains. The right panels represent the internal field distribution $B(r)$, which can be observed directly.

At low fields in the Meissner state [Fig. 1(a)], magnetic fields penetrate inside through the domain wall, forming the vortex sheet structure. A straight domain wall at $H = 0$ begins to meander by the penetration of vortices at finite fields. To see the vortex sheet structure, we show the spatial structure in Fig. 2 magnifying the enclosed area in Fig. 1(a). In the amplitude $|\eta_- (r)|$ and $|\eta_+ (r)|$ in Figs. 2(a) and 2(b), we do not see the singularity of the vortex center. It is because the singularity points with the phase winding $2\pi$ of vortices are at the opposite region across the domain wall, where the amplitude of the order parameter is well suppressed. That is, the winding center of $\eta_+ (\eta_-)$ is located in the $p_-\_x$ ($p_+\_x$) domain, as shown by a solid circle in Fig. 2(b) (Fig. 2(a)). Therefore, vortices at the domain wall are core-less vortices.

When we consider $\eta_\pm$ and $\eta_\mp$ instead of $\eta_\pm$ and $\eta_\mp$, as shown in the center panels in Fig. 1 the core-less vortices of the vortex sheet are seen as different structures. At a zero field, the domain wall is presented as a green line in the color density plot, since $\eta_\mp \sim 1$ and $\eta_\pm \sim 0$ due to the sign change of $\eta_\pm$ at the domain wall. When the magnetic field penetrates, as shown in the center panel of Fig. 1(a) or in Figs. 2(c) and 2(d), vortices of $\eta_\mp$ (red circle) enter along the domain wall from the boundary. These $\eta_\mp$ vortices are located slightly at the $p_+\_x$ domain side in the domain wall region. The green line between the $\eta_\mp$ vortices changes to the vortex of $\eta_x$ (green circle), when an inter-vortices distance of the $\eta_\mp$ vortices becomes short with increasing $H$. That is, the order parameters $\eta_\pm$ and $\eta_\mp$ have vortex cores with a winding $2\pi$ at different positions. This vortex sheet structure with the $\eta_\pm$ and $\eta_\mp$ vortices alternately aligning along the domain wall is the same vortex sheet structure reported in our previous work\textsuperscript{13,14}. The $B(r)$ distribution has a ridge along the domain wall, and has a peak at the $\eta_\mp$ vortices, as shown in the right panel of Fig. 1(a) or in Fig. 2(c).

With increasing $H$, first, vortices penetrate inside only in the $p_-\_x$ domain [Fig. 1(b)]. Later at higher field, vortices penetrate also in the $p_+\_x$ domain [Fig. 1(c)]. These indicate that the lower critical field $H_{c1}$ in the $p_-\_x$ domain is lower than that in the $p_+\_x$ domain. This corresponds to the fact that the upper critical field $H_{c2}$ in the $p_-\_x$ state is higher than that of the $p_+\_x$ state.\textsuperscript{24} The amplitude of the induced $p_+\_x$ component around vortices in the $p_-\_x$ state is larger than that of the induced $p_-\_x$ component in the $p_+\_x$ state.\textsuperscript{24} Since $H_{c1}$ is related to the creation energy of the vortex core, therefore, the $p_-\_x$ state with smaller creation energy of the vortex core has smaller $H_{c1}$ compared to the $p_+\_x$ state.\textsuperscript{24} In Figs. 1(b) and 1(c), we see vortices at the boundary region. This is because the penetrating magnetic field decreases towards inside of the superconductor in the length scale of the penetration depth. Near $H_{c1}$ when $H$ increases, vortices first appear at the boundary region where locally $B(r) > H_{c1}$.

With increasing the number of vortices along the domain wall, the domain wall line moves meanderingly. With further increasing $H$, the domain wall moves so that the area of the $p_+\_x$ domain shrinks as is seen in Fig. 1(d), where the vortex sheet structure still appears along the meandering domain wall. Finally at enough high fields, the $p_+\_x$ domain vanishes, and the single $p_-\_x$ domain state is realized.

Around the vortex cores in the $p_+\_x$ domain and the $p_-\_x$ domain, the opposite chiral component is induced as discussed in previous works.\textsuperscript{9,10,11} We also observe some double-winding $4\pi$ vortex near the boundary regions in
FIG. 1: (Color) Magnetization process for the multi-domain state in a $200\xi_0 \times 100\xi_0$ area with the open boundary. We start from the zero-field state where the left (right) hand side half region is a $p_+$ ($p_-$) domain. The applied field $H$ at the boundary is gradually increased. (a) $H/H_{c2}^0 = 0.08$, (b) 0.11, (c) 0.13 and (d) 0.32. The left panels show the color density plots of $|\eta_+(r)|$ and $|\eta_-(r)|$. The center panels are for $|\eta_y(r)|$ and $|\eta_0(r)|$. The right panels present the internal field distribution $B(r)$.

Finally, the $p_+$ domain shrinks to vanish also in this case, realizing the single $p_-$ domain state.

Lastly, we compare the magnetization curves in Fig. 4 for three cases: (i) the multi-domain case when the $p_+$ domain and the $p_-$ domain coexist at low fields as shown in Fig. 1 (ii) the case of the single $p_+$ domain at low fields as shown in Fig. 3 and (iii) the case of the single $p_-$ domain. In the single domain case, $B = 0$ in the Meissner state at $H < H_{c1}$, and $B$ appears in the mixed state at $H > H_{c1}$ due to the penetration of vortices. We find that $H_{c1}(p_-\text{ state}) < H_{c1}(p_+\text{ state})$ also in Fig. 4. In the multi-domain case (thin solid line in Fig. 4), $B$ is small but finite even in the Meissner state, because the magnetic field penetrates along the domain wall. In this case, the slope of $B$ curve changes both at $H_{c1}(p_+\text{ state})$ and at $H_{c1}(p_-\text{ state})$.

In summary, we performed the simulation of the magnetization process in a chiral $p$-wave superconductor, using the TDGL theory with two components of the $p_+$ and $p_-$ states. In the multi-domain case of the $p_+$ and $p_-$ domains, the magnetic field penetrates as core-less vortices along the domain wall, forming the vortex sheet.
structure, even in the Meissner state. With increasing external fields, the domain wall meanderingly moves, so that the area of the $p_+$ domain shrinks. Then, the unstable $p_+$ domain vanishes at a high field, and the single domain of the stable $p_-$ state is realized. Even in the case of the single $p_+$ domain at a zero field, the $p_+$ domain changes to the single $p_-$ domain at a high field as in a similar way, after small $p_-$ domains are created at the boundary.

Recently, anomalous internal field distribution due to the vortex coalescence was observed. These may be related to the intrinsic character of the chiral $p$-wave superconductor, such as a domain structure.

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