SCALING LIMITS FOR BEAM WAVE PROPAGATION IN ATMOSPHERIC TURBULENCE

ALBERT C. FANNNJIANG∗ KNUT SOLNA†

ABSTRACT. We prove the convergence of the solutions of the parabolic wave equation to that of the Gaussian white-noise model widely used in the physical literature. The random medium is isotropic and is assumed to have integrable correlation coefficient in the propagation direction. We discuss the limits of vanishing inner scale and divergent outer scale of the turbulent medium.

1. Introduction

The small-scale refractive index variations, called the refractive turbulence, in the atmosphere is the result of small scale fluctuations of temperature, pressure and humidity caused by the turbulence of air velocities. For optical propagation in the atmosphere the influence of the temperature variations on the refractive index field is dominant whereas in the microwave range, the effect of the humidity variations is more important. The refractive turbulence results in the phenomena of beam wander, beam broadening and intensity fluctuation (scintillation). It is important to note that these effects depend on the length scales of the waves as well as the refractive turbulence [19].

The refractive turbulence is modeled on the basis of Kolmogorov theory of turbulence which introduces the notion of the inertial range bounded by the outer scale \( L_0 \) (of the order of \( 100 \text{m} - 1 \text{km} \)) and the inner scale \( \ell_0 \) (of the order of \( 1 - 10 \text{mm} \)). Other features of the refractive turbulence in the open clear atmosphere include [22]: (i) small changes (typical value of \( 3 \times 10^{-4} \) at sea level) in the refractive index related to small variations in temperature (on the order of \( 0.1 - 1^\circ \text{C} \)), (ii) small scattering angle which is of the order \( \lambda/\ell_0 \) and has the typical value \( 3 \times 10^{-4} \text{rad} \) for \( \lambda = 0.6 \text{mm} \) and \( \ell_0 = 2 \text{mm} \). Perturbation methods for solving the Maxwell equations are adequate provided that the propagation distance is less than, say, \( 100 \text{m} \), a severe limitation on their applicability to imaging or communication problems. Our motivation is mainly from laser or microwave beams but our consideration and results apply equally well to ultrasound waves in atmospheric turbulence. The results are also relevant in the context of ultrasound waves penetrating through complicated multiscale fluctuating (interface) zones in for instance human tissue.

Under the condition \( \lambda = O(\ell_0) \) (including the millimeter and the sub-millimeter range) the depolarization term in the Helmholtz equation for the electric field is negligible [22] and one can use the (scalar) Helmholtz equation

\[
\nabla^2 E + k^2 \tilde{n}^2 (1 + \hat{n})^2 E = 0
\]

with appropriate boundary conditions where \( k \) is the wavenumber, \( \tilde{n} \) is the mean refractive index field and \( \hat{n} \) is the normalized fluctuation of the refractive index.

1.1. The rescaled parabolic approximation. The well-known parabolic approximation to equation (1) is applicable in a regime where the variations of the index of refraction are small on the
scale of the wavelength so that backscattering is negligible \cite{22}. This is almost always valid for laser beam in the atmosphere.

In this paper we study the initial value problem for the parabolic wave equation

\[ \nabla^2 \Psi(z, x) + 2i \frac{\partial \Psi(z, x)}{\partial z} = -2k^2 \tilde{n}(z, x) \Psi(z, x), \quad \Psi(0, x) = F_0 \left( \frac{x}{a} \right) \in L^2(\mathbb{R}^2) \]

where \( z \) is the longitudinal coordinate in the direction of the propagation, \( x = (x_1, x_2) \) is the transverse coordinates, \( \nabla_\perp \) is the transverse gradient and \( \Psi \) is related to the scalar wave field \( E \) by \( E = \Psi(z, x) \exp(ik\tilde{n}z) \). The initial condition has a typical width \( a \) which is the aperture. Below we will drop the perp in denoting the derivatives in the transverse directions.

The difficulty in solving equation (2) lies in the random multiscale nature of \( \tilde{n}(z, x) \). First we non-dimensionalize eq. (2) as follows. Let \( L_z \) be the propagation distance in the longitudinal direction. Let \( \lambda_0 \) be the characteristic wavelength. The corresponding central wavenumber is \( k_0 = 2\pi/\lambda_0 \). The Fresnel length \( L_f \) is then given by

\[ L_f = \sqrt{L_z/k_0}. \]

We introduce dimensionless wave number and coordinates

\[ \tilde{k} = k/k_0, \quad \tilde{x} = x/L_f, \quad \tilde{z} = z/L_z \]

and rewrite the equation in the form

\[ 2i\tilde{k} \frac{\partial \Psi}{\partial \tilde{z}} + \Delta \Psi + 2\tilde{k}^2 k_0 L_z \tilde{n}(L_z \tilde{x}, \tilde{L}_f) \Psi = 0, \quad \Psi(0, \tilde{x}) = F_0(\gamma^{1/2} \tilde{x}) \in L^2(\mathbb{R}^2) \]

after dropping the tilde in the coordinate variables where

\[ \gamma = \left( \frac{L_L}{a} \right)^2 \]

is assumed to be \( O(1) \), thus the source is supported on the scale determined by the Fresnel length.

1.2. Model spectra. A widely used model for the structure function of the refractive index field of the atmosphere is based on the Kolmogorov theory of turbulence and has the following modified Von K’arm’an spectral density

\[ \Phi_n(\tilde{k}) = 0.033C_n^2(\tilde{k}^2 + K_0^2)^{-11/6} \exp(-|\tilde{k}|^2/K_m^2) \]

where \( \tilde{k} = (\xi, \mathbf{k}) \), with \( \xi \in \mathbb{R}, \mathbf{k} \in \mathbb{R}^2 \) the Fourier variables conjugate to the longitudinal and transversal coordinates, respectively. Here \( K_0 = 2\pi/L_0, K_m = 5.92/\ell_0 \). This spectrum has the correct behavior only in the inertial subrange, i.e.

\[ \Phi_n(\tilde{k}) \sim |\tilde{k}|^{-11/3}, \quad |\tilde{k}| \in (2\pi L_0^{-1}, 2\pi \ell_0^{-1}). \]

Outside of this range, particularly for \( |\tilde{k}| \ll 2\pi L_0^{-1} \) there is no physical basis for their behavior; they are just mathematically convenient expressions of the cutoffs. In particular, if the wave statistics strongly depend on \( \ell_0 \) or \( L_0 \), then the problem probably requires more accurate information on the refractive index field outside of the inertial range \cite{6}, \cite{12}, \cite{13}. Note that the ratio \( L_0/\ell_0 \) grows like \( \text{Re}^{3/4} \) as the Reynolds number \( \text{Re} \) tends to infinity.

There are several variants of (4) arising from modeling more detailed features of the refractive index field. One of them is the Hill spectrum \cite{2,15} to account for the “bump” at high wave numbers which is known to occur near the inner scale

\[ 6\Phi_n(\tilde{k}) = 0.033C_n^2 \left[ 1 + 1.802|\tilde{k}|/K_m - 0.254(|\tilde{k}|/K_m)^{7/6} \right] (|\tilde{k}|^2 + K_0^2)^{-11/6} \exp(-|\tilde{k}|^2/K_m^2) \]

where \( K_m = 3.3/\ell_0 \). The coefficient \( C_n^2 \) is itself a random variable that depends on time as well as the altitude. Note that in atmospheric turbulence the inner and outer scales and the exponent in
the power law may also have to be modeled as stochastic processes \[21\]. The temporal dependence is irrelevant for optical propagation; the altitude dependence has a rather permanent, non-universal structure with length scales much greater than the outer scale \(L_0\) \[19\].

We will consider a class of spectra satisfying the upper bound

\[\Phi(H, \mathbf{k}) \leq K(L_0^{-2} + |\mathbf{k}|^2)^{-H-3/2}(1 + \ell_0^2|\mathbf{k}|^2)^{-2}, \quad \mathbf{k} = (\xi, \mathbf{k}) \in \mathbb{R}^3, H \in (0, 1)\]

with some constant \(K < \infty\) as the ratio \(L_0/\ell_0 \to \infty\) in the high Reynolds number limit. The details of the spectrum are not pertinent to our results, only the exponent \(H\) is. In particular, \(H = 1/3\) for the modified Von K’arm’an spectrum \[4\].

1.3. **White noise scaling.** Let us introduce the non-dimensional parameters that are pertinent to our scaling:

\[\varepsilon = \sqrt{\frac{L_f}{L_z}}, \quad \eta = \frac{L_f}{\ell_0}, \quad \rho = \frac{L_f}{\ell_0}\]

In terms of the parameters and the power-law spectrum in \[17\] we rewrite \[3\] as

\[2i\mathbf{k} \frac{\partial \Psi^\varepsilon}{\partial z} + \Delta \Psi^\varepsilon + \frac{\mathbf{k}^2}{\varepsilon} \sigma_H \mathcal{V}(\frac{z}{\varepsilon^2}, \mathbf{x}) \Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, \mathbf{x}) = F_0(\gamma^{1/2} \mathbf{x}) \in L^2(\mathbb{R}^2)\]

with

\[\sigma_H = \frac{L_f^H}{\varepsilon^3} \mu\]

where \(\mu\) is the standard deviation of the refractive index field corresponding to \(\Phi(H, \mathbf{k})\). The spectrum for the (normalized) process \(\mathcal{V}\) is given by

\[\Phi_{\eta, \rho}(\mathbf{k}) \leq K(\eta^2 + |\mathbf{k}|^2)^{-H-3/2}(1 + \rho^{-2}|\mathbf{k}|^2)^{-2}, \quad \mathbf{k} = (\xi, \mathbf{k}) \in \mathbb{R}^3, H \in (0, 1)\]

which is rescaled version of \[17\]. For high Reynolds number one has \(L_0/\ell_0 = \rho/\eta \gg 1\) which is always the case in our study.

In the beam approximation one has \(\varepsilon \ll 1\). The beam approximation is well within the range of validity of the parabolic approximation. The white-noise scaling then corresponds to \(\sigma_H = \mathcal{O}(1)\). We set it to unity by absorbing the constant into \(\mathcal{V}\). This implies relatively weak fluctuations of the index field, i.e.

\[C_n \sim L_f^{3/2-H} L_z^{-3/2} \ll 1, \quad \text{as } L_z \to \infty\]

in view of the fact that \(H \in (0, 1)\) and \(\varepsilon \ll 1\).

In the present paper we first study the white-noise scaling with \(\rho < \infty\) and \(\eta > 0\) fixed as \(\varepsilon \to 0\). We then discuss the resulting white-noise model with \(\rho \to \infty\) and \(\eta \to 0\). For the proof, we adopt the approach of \[10\] where the turbulent transport of passive scalars is studied. In \[11\] the white noise limit is studied via the so called Wigner distribution.

2. **Formulation and main results**

2.1. **Martingale formulation.** We consider the weak formulation of the equation:

\[i\mathbf{k} \langle (\Psi^\varepsilon_s, \theta) - (\Psi^\varepsilon_0, \theta) \rangle = -\int_0^z \frac{1}{2} \langle \Psi^\varepsilon_s, \Delta \theta \rangle \, ds - \int_0^z \frac{\mathbf{k}^2}{\varepsilon} \int_0^s \langle \Psi^\varepsilon_s, \mathcal{V}(\frac{s}{\varepsilon^2} \cdot) \cdot \theta \rangle \, ds\]

for any test function \(\theta \in C_c^\infty(\mathbb{R}^2)\), the space of smooth functions with compact support. The tightness result (Section 4.1) implies that for \(L^2\) initial data the limiting measure \(\mathbb{P}\) is supported in the Skorohod space \(D([0, \infty]; L_w^2(\mathbb{R}^2))\). Here and below \(L_w^2(\mathbb{R}^2)\) denotes the standard \(L^2\)-function space with the weak topology.
For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^{\varepsilon}$ will play an important role. Let $\mathcal{V}_z^\varepsilon \equiv \mathcal{V}(z/\varepsilon^2, \cdot)$, $\mathcal{F}_z^\varepsilon$ the $\sigma$-algebras generated by $\mathcal{V}_z^\varepsilon$, $s \leq z$ and $\mathbb{E}^\varepsilon_z$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Let $\mathcal{M}^{\varepsilon}$ be the space of measurable functions adapted to $\{\mathcal{F}_z^\varepsilon, \forall z\}$ such that $\sup_{z < s_0} \mathbb{E}[f(z)] < \infty$. We say $f(\cdot) \in \mathcal{D}(\mathcal{A}^{\varepsilon})$, the domain of $\mathcal{A}^{\varepsilon}$, and $\mathcal{A}^{\varepsilon} f = g$ if $f, g \in \mathcal{M}^{\varepsilon}$ and for $f^{\delta}(z) \equiv \delta^{-1} [\mathbb{E}^\varepsilon_z f(z + \delta) - f(z)]$ we have

$$\sup_{z, \delta} \mathbb{E}|f^{\delta}(z)| < \infty$$

$$\lim_{\delta \to 0} \mathbb{E}|f^{\delta}(z) - g(z)| = 0, \ \forall z.$$  

Consider the special class of admissible functions $f(z) = \phi(\langle \Psi^\varepsilon_z, \theta \rangle)$, $f'(z) = \phi'(\langle \Psi^\varepsilon_z, \theta \rangle), \forall \phi \in C^\infty(\mathbb{R})$, then we have the following expression from (11) and the chain rule

$$\mathcal{A}^{\varepsilon} f(z) = i f'(z) \left[ \frac{1}{2k} \langle \Psi^\varepsilon_z, \Delta \theta \rangle + \frac{k}{\varepsilon} \langle \Psi^\varepsilon_z, \mathcal{V}^\varepsilon_z \theta \rangle \right].$$  

A main property of $\mathcal{A}^{\varepsilon}$ is that

$$f(z) - \int_0^z \mathcal{A}^{\varepsilon} f(s) ds \ \text{is a} \ \mathcal{F}^\varepsilon_z \text{-martingale, } \forall f \in \mathcal{D}(\mathcal{A}^{\varepsilon}).$$  

Also,

$$\mathbb{E}^\varepsilon_s f(z) - f(s) = \int_s^z \mathbb{E}^\varepsilon_s \mathcal{A}^{\varepsilon} f(\tau) d\tau \ \forall s < z \ \text{a.s.}$$  

(see [17]). We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $\mathcal{V}_z(\cdot) = \mathcal{V}(z, \cdot)$.

Define

$$\Gamma^{(1)}(x, y) = \int \int_0^\infty \cos ((x - y) \cdot p) \cos (s\xi) \Phi_{(\eta, \rho)}(\xi, p) ds \, d\xi \, dp$$

$$= \pi \int \cos ((x - y) \cdot p) \Phi_{(\eta, \rho)}(0, p) \, dp$$

$$\Gamma^{(1)}_0(x) = \Gamma^{(1)}(x, x)$$

where we have written the wavevector $\tilde{k} \in \mathbb{R}^3$ as $\tilde{k} = (\xi, p)$ with $p \in \mathbb{R}^2$.

Now we formulate the solutions for the Gaussian Markovian model as the solutions to the corresponding martingale problem: Find a measure $\mathbb{P}$ (of $\Psi_z(\cdot)$) on the subspace of $\mathcal{D}([0, \infty); L^2_w(\mathbb{R}^2))$ whose elements have the initial condition $F_0(\gamma^{1/2} x)$ such that

$$f(\langle \Psi_z, \theta \rangle) - \int_0^z \left\{ f'(\langle \Psi_s, \theta \rangle) \left[ \frac{i}{2k} \langle \Psi_s, \Delta \theta \rangle - \tilde{k}^2 \langle \Psi_s, \Gamma^{(1)}_0(\theta) \rangle \right] - \tilde{k}^2 f''(\langle \Psi_s, \theta \rangle) \langle \theta, \mathcal{K}_{\Psi, \theta} \rangle \right\} ds$$

is a martingale w.r.t. the filtration of a cylindrical Wiener process, for each $f \in C^\infty(\mathbb{R})$

where

$$\mathcal{K}_{\Psi, \theta} = \int \Psi_s(x)\Psi_s(y) \Gamma^{(1)}(x, y) \theta(y) dy.$$  

The Gaussian Markovian model has been extensively studied for beam wander, broadening and scintillation effects in the literature (see, e.g. [5], [14]). It can also been written as the Itô's
$$d\Psi_z = \left( \frac{i}{2k} \Delta - \tilde{k}^2 \Gamma_0^{(1)} \right) \Psi_z \, dz + i\tilde{k} \left( K\Psi_z \right)^{1/2} \, dW_z$$

$$= \left( \frac{i}{2k} \Delta - \tilde{k}^2 \Gamma_0^{(1)} \right) \Psi_z \, dz + i\tilde{k} \Psi_z \, d\tilde{W}_z$$

(18)

where $\circ$ stands for the Stratonovich integral, and $W_z(x)$ and $\tilde{W}_z(x)$ are the Brownian fields with the spatial covariance $\delta(x - y)$ and $\Gamma^{(1)}(x, y)$, respectively.

The existence and uniqueness for the Schrödinger-Itô eq. (18) with $\rho < \infty$ and $\eta > 0$ has been studied in [8] by using the Wiener chaos expansion. Note that the following limit exists

$$\bar{\Gamma}(x, y) = \lim_{\rho \to \infty} \pi \int \cos \left( (x - y) \cdot p \right) \Phi_{(\eta, \rho)}(0, p) \, dp$$

(19)

By the well-posedness result of [8] and a standard weak-compactness argument one can prove the existence of weak solution in $D([0, \infty); L^2_w(\mathbb{R}^2))$ for $\rho = \infty, H \in (0, 1)$.

Next we consider the limiting case $\eta = 0$. This would induce uncontrollable large scale fluctuation the Gaussian, Markovian model which should be factored out first. Thus we consider the solution of the form

$$\Psi(z, x) = \Psi'(z, x) \exp \left( i\tilde{k} \int_0^z \tilde{W}_s(0) \, ds \right)$$

and the resulting equation

$$d\Psi_z = \frac{i}{2k} \Delta \Psi_z \, dz + i\tilde{k} \Psi_z \, d\tilde{W}_z', \quad \Psi_0(x) = F_0(\gamma^{1/2}x)$$

(20)

where $\tilde{W}_z'$ is given by

$$\tilde{W}_z'(x) = \tilde{W}_z(x) - \tilde{W}_z(0)$$

(21)

with the covariance function

$$\bar{\Gamma}^\prime(x, y) = \pi \int (e^{ixp} - 1)(e^{-iyp} - 1) \Phi_{(0, \infty)}(0, p) \, dp.$$ 

Note that the above integral is convergent only if

$$H < 1/2;$$

in particular, the limit exists for the modified Von K’arm’an spectrum $H = 1/3$. Since $H < 1/2$, the limiting model is only Hölder continuous in the transverse coordinates.

Again by the well-posedness result of [8] and a standard weak-compactness argument one can prove the existence of weak solution in $D([0, \infty); L^2_w(\mathbb{R}^2))$ for $\rho = \infty, \eta = 0, H \in (0, 1/2)$.

2.2. Uniqueness. Because of the non-smoothness (when $\rho = \infty$) and the non-homogeneity (when $\eta = 0$) of the white-noise potential in the transverse coordinates the uniqueness argument of [8] does not apply here.

Taking the function $f(r) = r^n$ in the martingale formulation, we arrive after some algebra at the following equation

$$\frac{\partial F_z^{(n)}}{\partial z} = \mathcal{C}_1 F_z^{(n)} + \mathcal{C}_2 F_z^{(n)}$$

(22)
for the \( n \)-point correlation function

\[
F_z^{(n)}(x_1, \ldots, x_n) \equiv \mathbb{E} [\Psi_z(x_1) \cdots \Psi_z(x_n)]
\]

where

\[
C_1 = \frac{i}{2k} \sum_{j=1}^{n} \Delta x_j
\]

\[
C_2 = -k^2 \sum_{j,k=1}^{n} \Gamma(x_j, x_k),
\]

or

\[
C_2 = -k^2 \sum_{j,k=1}^{n} \Gamma'(x_j, x_k)
\]

We will now establish the uniqueness for eq. (22) with the initial data

\[
F_0^{(n)}(x_1, \ldots, x_n) = \mathbb{E} [\Psi_0(x_1) \cdots \Psi_0(x_n)], \quad \Psi_0 \in L^2(\mathbb{R}^2).
\]

In the former case (24) \( C_2 \) is a bounded, Hölder continuous function and we rewrite eq. (22) in the mild formulation

\[
F_z^{(n)} = \exp (zC_1)F_0^{(n)} + \int_0^z \exp [(z-s)C_1]C_2F_s^{(n)} \, ds
\]

whose local existence and uniqueness can be easily established by straightforward application of the contraction mapping principle. By linearity, local well-posedness can be extended to global well-posedness.

In the latter case (25) \( C_2 \) is unbounded, Hölder continuous function with sub-Lipschitz growth. We first note that \( C_2 \) is non-positive everywhere since

\[
\sum_{j,k=1}^{n} \Gamma'(x_j, x_k) = \pi \int \sum_{j} (e^{i x_j \cdot p} - 1) \sum_{k} (e^{i x_k \cdot p} - 1) \Phi_{(0, \infty)}(p) dp \geq 0.
\]

Hence both \( C_1 \) and \( C_2 \) are generators of one-parameter contraction semigroups on \( L^2(\mathbb{R}^{2n}) \), thus by the product formula (Theorem 3.30, [7]) we have

\[
\lim_{m \to \infty} \left[ \exp \left( \frac{z}{m} C_1 \right) \exp \left( \frac{z}{m} C_2 \right) \right]^m F = \exp \left[ z(C_1 + C_2) \right] F
\]

for all \( F \in L^2(\mathbb{R}^{2n}) \), which then gives rise to a unique semigroup on \( L^2(\mathbb{R}^{2n}) \).

2.3. Main assumptions and theorem. Let \( V_z \) be a \( z \)-stationary, \( x \)-homogeneous square-integrable process whose spectral density satisfies the upper bound (10).

Let \( F_z \) and \( F_z^+ \) be the sigma-algebras generated by \( \{V_s : \forall s \leq z\} \) and \( \{V_s : \forall s \geq z\} \), respectively. Define the correlation coefficient

\[
\rho(t) = \sup_{h \in F_z} \sup_{\mathbb{E}[h]=0, \mathbb{E}[h^2]=1} \mathbb{E}[hg].
\]

**Assumption 1.** The correlation coefficient \( \rho(t) \) is integrable

When \( V_z \) is a Gaussian process, the correlation coefficient \( \rho(t) \) equals the linear correlation coefficient \( r(t) \) which has the following useful expression

\[
r(t) = \sup_{g_1, g_2} \int R(t - \tau_1 - \tau_2, k) g_1(\tau_1, k) g_2(\tau_2, k) dk d\tau_1 d\tau_2
\]
where
\[ R(t, k) = \int e^{it\xi} \Phi(\xi, k) d\xi \]
and the supremum is taken over all \( g_1, g_2 \in L^2(\mathbb{R}^{d+1}) \) which are supported on \((−∞, 0] \times \mathbb{R}^d\) and satisfy the constraint
\[
\int R(t - t', k) g_1(t, k) \bar{g}_1(t', k) dt dt' \, dk = 1.
\]
(28)

There are various criteria for the decay rate (e.g., exponential decay) of the linear correlation coefficients, see \cite{16}.

**Lemma 1.** Assumption 1 implies that the random field
\[
\tilde{V}_z(x) = \int_0^\infty \mathbb{E}_z[V_t(x)] dt
\]
is \(x\)-homogeneous and has a finite second moment which satisfies the upper bound:
\[
\mathbb{E}[\tilde{V}_z^2] \leq \int_z^\infty \int_z^\infty |\mathbb{E}[\mathbb{E}_z[V_s][\mathbb{E}_z[V_t]]]| ds dt \leq \mathbb{E}[V_z^2] \left( \int_0^\infty \rho(t) dt \right)^2.
\]

**Proof.** Consider in the definition of the correlation coefficient
\[
h_1 = \mathbb{E}_z(V_s) \in L^2(\Omega, P, \mathcal{F}_z)
\]
\[
h_2 = V_t \in L^2(\Omega, P, \mathcal{F}_t^+).
\]
We then have
\[
|\mathbb{E}[\mathbb{E}_z[V_s(x)][\mathbb{E}_z[V_t(x)]]]| = |\mathbb{E}[\mathbb{E}_z[V_s(x)][V_t(x)]]| \leq \rho(t - z) \mathbb{E}[V_z^2] \mathbb{E}[V_t^2].
\]
Hence by setting \( s = t \) first and the Cauchy-Schwartz inequality we have
\[
\mathbb{E}[\mathbb{E}_z[V_s]] \leq \rho^2(s - z) \mathbb{E}[V_z^2]
\]
\[
|\mathbb{E}[\mathbb{E}_z[V_s(x)][\mathbb{E}_z[V_t(x)]]]| \leq \rho(t - z) \rho(s - z) \mathbb{E}[V_z^2], \quad s, t \geq z.
\]
Therefore
\[
\mathbb{E}[\tilde{V}_z^2] \leq \int_z^\infty \int_z^\infty |\mathbb{E}[\mathbb{E}_z[V_s][\mathbb{E}_z[V_t]]]| ds dt \leq \mathbb{E}[V_z^2] \left( \int_0^\infty \rho(t) dt \right)^2
\]
(21)
which, together with the integrability of \( \rho(t) \), implies a finite second order moment of \( \tilde{V}_z \).

\[ \square \]

**Corollary 1.** For each \( L, z_0 < \infty \) and \( \rho < \infty, \eta > 0 \) there exists a constant \( \tilde{C} \) such that
\[
\sup_{z < z_0 \atop |x| \leq L} \mathbb{E} \left[ \Delta \tilde{V}_{\lambda z} \right]^2 \leq \tilde{C}
\]
for all \( H \in (0, 1), \lambda \geq 1 \).
Proof. Analogous to (21) we have
\[
\mathbb{E}[\Delta \tilde{V}_z]^2 \leq \mathbb{E}[\Delta V_z]^2 \left( \int_0^\infty \rho(t)dt \right)^2.
\]
A straightforward spectral calculation shows that
\[
\mathbb{E}[\Delta V_z(x)]^2 = O(\rho^{4-2H}), \quad \forall x, z.
\]
It is easy to see that
\[
A \tilde{V}_z = -V_z
\]
and that
\[
\Gamma^{(1)}(x, y) = \mathbb{E} [\tilde{V}_z(x)V_z(y)]
\]
where \( \Gamma^{(1)} \) is given by (15).

Next we assume the following quasi-Gaussian property:

**Assumption 2.**
\[
(21) \quad \sup_{|y| \leq L} \mathbb{E} [V^\varepsilon_z(y)]^4 \leq C_1 \sup_{|y| \leq L} \mathbb{E}^2 [V^\varepsilon_z]^2 (y)
\]
\[
(22) \quad \sup_{|y| \leq L} \mathbb{E} \left[ \tilde{V}^\varepsilon_z (y) \right]^4 \leq C_2 \sup_{|y| \leq L} \mathbb{E}^2 \left[ \tilde{V}^\varepsilon_z (y) \right]^2
\]
\[
(23) \quad \sup_{|y| \leq L} \mathbb{E} \left[ V^\varepsilon_z (y) \tilde{V}^\varepsilon_z (y) \right]^4 \leq C_3 \left\{ \left( \sup_{|y| \leq L} \mathbb{E} [V^\varepsilon_z (y)]^2 \right) \left( \sup_{|y| \leq L} \mathbb{E}^2 \left[ \tilde{V}^\varepsilon_z (y) \right]^2 \right) \right. \\
\left. + \left( \sup_{|y| \leq L} \mathbb{E}^2 \left[ V^\varepsilon_z (y) \tilde{V}^\varepsilon_z (y) \right] \right) \left( \sup_{|y| \leq L} \mathbb{E} \left[ \tilde{V}^\varepsilon_z (y) \right]^2 \right) \right\}
\]
for all \( L < \infty \) where the constants \( C_1, C_2, \) and \( C_3 \) are independent of \( \varepsilon, \eta, \rho, \gamma \).

**Assumption 3.** For any fixed \( \eta > 0 \) and every \( \theta \in C_c^\infty (\mathbb{R}^2) \)
\[
(23) \quad \sup_{z < z_0} \| \theta \tilde{V} \|_2 = O \left( \frac{1}{\varepsilon} \right), \quad \forall \varepsilon \leq 1 \leq \rho
\]
with a random constant of finite moments independent of \( \rho \) and \( \varepsilon \).

When \( \mathcal{V} \) is Gaussian, \( \tilde{\mathcal{V}} \) is also Gaussian and condition (23) is always satisfied
\[
(24) \quad \sup_{z < z_0} \| \theta \tilde{V} \|_2 \leq C \log \left( \frac{z_0}{\varepsilon^2} \right)
\]
where the random constant \( C \) has a Gaussian-like tail by a simple application of Borell’s inequality \[1\].

**Theorem 1.** Let \( \mathcal{V} \) satisfy Assumptions 1, 2 and 3. Let \( \eta > 0 \) and \( \rho < \infty \) be fixed as \( \varepsilon \to 0 \). Then the weak solution \( \Psi^\varepsilon \) of (11) converges in the space of \( D([0, \infty); L^2_w(\mathbb{R}^2)) \) to that of the Gaussian white-noise model with the covariance functions \( \Gamma^{(1)} \) and \( \Gamma^{(1)}_0 \).

Note that in the limiting model with \( \rho = \infty \) the white-noise velocity field has transverse regularity of Hölder exponent \( H + 1/2 \).

The convergence of the white-noise limit has been established in \[3\] and \[4\] for a refractive index field that is a function of \( z \) only.
3. Proof of Theorem 1

3.1. Tightness. In the sequel we will adopt the following notation

\[ f(z) \equiv f(\langle \Psi^\varepsilon, \theta \rangle), \quad f'(z) \equiv f'(\langle \Psi^\varepsilon, \theta \rangle), \quad f''(z) \equiv f''(\langle \Psi^\varepsilon, \theta \rangle), \quad \forall f \in C^\infty(\mathbb{R}). \]

Namely, the prime stands for the differentiation w.r.t. the original argument (not \( z \)).

A family of processes \( \{ \Psi^\varepsilon, 0 < \varepsilon < 1 \} \subset D([0, \infty); L^2_w(\mathbb{R}^2)) \) is tight if and only if the family of processes \( \{ \langle \Psi^\varepsilon, \theta \rangle, 0 < \varepsilon < 1 \} \subset D([0, \infty); L^2_w(\mathbb{R}^2)) \) is tight for all \( \theta \in C^\infty_c(\mathbb{R}^2) \). We use the tightness criterion of [18] (Chap. 3, Theorem 4), namely, we will prove: Firstly, for each \( \varepsilon \)

\[ \lim_{N \to \infty} \limsup_{\varepsilon \to 0} \mathbb{P}\{ \sup_{z < z_0} | \langle \Psi^\varepsilon, \theta \rangle | \geq N \} = 0, \quad \forall z_0 < \infty. \]  

Secondly, for each \( f \in C^\infty(\mathbb{R}) \) there is a sequence \( f^\varepsilon(z) \in D(A^\varepsilon) \) such that for each \( z_0 < \infty \)

\[ \{ A^\varepsilon f^\varepsilon(z), 0 < \varepsilon < 1, 0 < z < z_0 \} \text{ is uniformly integrable and} \]

\[ \lim_{\varepsilon \to 0} \mathbb{P}\{ \sup_{z < z_0} | f^\varepsilon(z) - f(\langle \Psi^\varepsilon, \theta \rangle) | \geq \delta \} = 0, \quad \forall \delta > 0. \]

Then it follows that the laws of \( \{ \langle \Psi^\varepsilon, \theta \rangle, 0 < \varepsilon < 1 \} \) are tight in the space of \( D([0, \infty); L^2_w(\mathbb{R}^2)) \).

Condition (25) is satisfied because the \( L^2 \)-norm is preserved. Let

\[ f_1^\varepsilon(z) = \frac{i\kappa}{\varepsilon} \int_z^\infty \mathbb{E}_z^\varepsilon f'(z) \langle \Psi^\varepsilon, V^\varepsilon \theta \rangle \, ds \]

be the 1-st perturbation of \( f(z) \). Let

\[ \tilde{V}^\varepsilon = \frac{1}{\varepsilon^2} \int_z^\infty \mathbb{E}_z^\varepsilon V^\varepsilon \, ds. \]

We obtain

\[ f_1^\varepsilon(z) = i\kappa \varepsilon f'(z) \langle \Psi^\varepsilon, \tilde{V}^\varepsilon \theta \rangle. \]

Proposition 1.

\[ \limsup_{\varepsilon \to 0} \mathbb{E}|f_1^\varepsilon(z)| = 0, \quad \limsup_{\varepsilon \to 0} |f_1^\varepsilon(z)| = 0 \text{ in probability}. \]

Proof. We have

\[ \mathbb{E}[|f_1^\varepsilon(z)|] \leq \varepsilon \|

\[ f'(z) \|_\infty \|

\[ \Psi_0 \|_2 \|

\[ \tilde{V}^\varepsilon \|_2 \]

\]

and

\[ \sup_{z < z_0} |f_1^\varepsilon(z)| \leq \varepsilon \|

\[ f'(z) \|_\infty \|

\[ \Psi_0 \|_2 \sup_{z < z_0} \|

\[ \tilde{V}^\varepsilon \|_2. \]

The right side of (27) is \( O(\varepsilon) \) by Lemma 11 while that of (28) is \( o(1) \) in probability by Assumption 3.

\[ \square \]

Set \( f^\varepsilon(z) = f(z) + f_1^\varepsilon(z) \). A straightforward calculation yields

\[ A^\varepsilon f_1^\varepsilon = -\varepsilon f''(z) \left[ \langle \Psi^\varepsilon, \Delta \theta \rangle + \frac{k^2}{\varepsilon} \langle \Psi^\varepsilon, V^\varepsilon \theta \rangle \right] \langle \Psi^\varepsilon, \tilde{V}^\varepsilon \theta \rangle \]

\[ -\varepsilon f'(z) \left[ \frac{1}{2} \langle \Psi^\varepsilon, \Delta(\tilde{V}^\varepsilon \theta) \rangle + \frac{k^2}{\varepsilon} \langle \Psi^\varepsilon, V^\varepsilon \tilde{V}^\varepsilon \theta \rangle \right] - \frac{i\kappa}{\varepsilon} f'(z) \langle \Psi^\varepsilon, V^\varepsilon \theta \rangle \]
and, hence
\begin{equation}
A^\varepsilon f^\varepsilon(z) = \frac{i}{2k} f'(z) \langle \Psi^\varepsilon_z, \Delta \theta \rangle - \tilde{k}^2 f'(z) \langle \Psi^\varepsilon_z, \nabla \bar{V}^\varepsilon \bar{Z}^\varepsilon \theta \rangle - \tilde{k}^2 f''(z) \langle \Psi^\varepsilon_z, \nabla \bar{V}^\varepsilon \bar{Z}^\varepsilon \theta \rangle \\
- \frac{\varepsilon}{2} \left[ f'(z) \langle \Psi^\varepsilon_z, \Delta (\bar{V}^\varepsilon \theta) \rangle + f''(z) \langle \Psi^\varepsilon_z, \Delta \theta \rangle \langle \Psi^\varepsilon_z, \bar{V}^\varepsilon \theta \rangle \right] = A^\varepsilon_1(z) + A^\varepsilon_2(z) + A^\varepsilon_3(z) + A^\varepsilon_4(z)
\end{equation}

where $A^\varepsilon_2(z)$ and $A^\varepsilon_3(z)$ are the $O(1)$ statistical coupling terms.

For the tightness criterion stated in the beginnings of the section, it remains to show

**Proposition 2.** \{$A^\varepsilon f^\varepsilon\}$ are uniformly integrable and

$$
\lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E} |A^\varepsilon_4(z)| = 0.
$$

**Proof.** We show that \{$A^\varepsilon_i\}, i = 1, 2, 3, 4$ are uniformly integrable. To see this, we have the following estimates.

$$
|A^\varepsilon_1(z)| \leq \frac{1}{2k} \|f''\|_\infty \|\Psi_0\|_2 \|\Delta \theta\|_2
$$

$$
|A^\varepsilon_2(z)| \leq \tilde{k}^2 \|f''\|_\infty \|\Psi_0\|_2 \|\nabla \bar{V} \bar{Z} \theta\|_2
$$

$$
|A^\varepsilon_3(z)| \leq \tilde{k}^2 \|f''\|_\infty \|\Psi_0\|_2 \|\nabla \bar{V} \bar{Z} \theta\|_2

For fixed $\eta$, the second moments of the right hand side of the above expressions are uniformly bounded as $\varepsilon \to 0$ and hence $A^\varepsilon_1(z), A^\varepsilon_2(z), A^\varepsilon_3(z)$ are uniformly integrable.

$$
|A^\varepsilon_4| \leq \frac{\varepsilon}{2} \left[ \|f''\|_\infty \|\Psi_0\|_2 \|\Delta \theta\|_2 \|\bar{V} \bar{Z} \theta\|_2 + \|f''\|_\infty \|\bar{V} \bar{Z} \theta\|_2 \|\Delta \bar{V} \bar{Z} \theta\|_2 \right].
$$

By Lemma 1 and Corollary 1, $A^\varepsilon_4$ is uniformly integrable. Finally, it is clear that

$$
\lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E} |A^\varepsilon_4(z)| = 0.
$$

\[\square\]

### 3.2. Identification of the limit.

Once the tightness is established we can use another result in \cite{JS} (Chapter 3, Theorem 2) to identify the limit. The setting there is finite-dimensional but the argument is entirely applicable to the infinite-dimensional setting here (cf. \cite{I}).

Let $\bar{A}$ be a diffusion or jump diffusion operator such that there is a unique solution $\omega_z$ in the space $D([0, \infty); L^2_w(\mathbb{R}^2))$ such that

\begin{equation}
f(\omega_z) - \int_0^z \bar{A} f(\omega_s) \, ds
\end{equation}

is a martingale. We shall show that for each $f \in C^\infty(\mathbb{R})$ there exists $f^\varepsilon \in \mathcal{D}(\bar{A}^\varepsilon)$ such that

\begin{equation}
\sup_{z < z_0, \varepsilon} \mathbb{E} |f^\varepsilon(z) - f(\langle \Psi^\varepsilon_z, \theta \rangle)| < \infty
\end{equation}

\begin{equation}
\lim_{\varepsilon \to 0} \mathbb{E} |f^\varepsilon(z) - f(\langle \Psi^\varepsilon_z, \theta \rangle)| = 0, \quad \forall z < z_0
\end{equation}

\begin{equation}
\sup_{z < z_0, \varepsilon} \mathbb{E} |A^\varepsilon f^\varepsilon(z) - \bar{A} f(\langle \Psi^\varepsilon_z, \theta \rangle)| < \infty
\end{equation}

\begin{equation}
\lim_{\varepsilon \to 0} \mathbb{E} |A^\varepsilon f^\varepsilon(z) - \bar{A} f(\langle \Psi^\varepsilon_z, \theta \rangle)| = 0, \quad \forall z < z_0.
\end{equation}

Then it follows that any tight processes $\langle \Psi^\varepsilon_z, \theta \rangle$ converges in law to the unique process generated by $\bar{A}$. As before we adopt the notation $f(z) = f(\langle \Psi^\varepsilon_z, \theta \rangle)$.
For this purpose, we introduce the next perturbations $f_2^\varepsilon, f_3^\varepsilon$. Let

\begin{align}
A_2^{(1)}(\phi) &\equiv \int \int \theta(x)\phi(x)\Gamma^{(1)}(x,y)\phi(y)\theta(y) \, dx \, dy \\
A_3^{(1)}(\phi) &\equiv \int \Gamma^{(1)}(x,x)\phi(x)\theta(x) \, dx
\end{align}

where

\begin{equation}
\Gamma^{(1)}(x,y) \equiv \mathbb{E} \left[ V_x^\varepsilon(x)\tilde{V}_y^\varepsilon(y) \right].
\end{equation}

It is easy to see that

\begin{equation}
A_2^{(1)}(\phi) = \mathbb{E} \left[ \langle \phi, V_x^\varepsilon \theta \rangle \left\langle \phi, \tilde{V}_x^\varepsilon \theta \right\rangle \right].
\end{equation}

Define

\begin{align}
f_2^\varepsilon(z) &\equiv \tilde{k}^2 f''(z) \int_{\mathbb{R}}^\infty \mathbb{E}^z \left[ \langle \Psi_x^\varepsilon, V_x^\varepsilon \theta \rangle \left\langle \Psi_x^\varepsilon, \tilde{V}_x^\varepsilon \theta \right\rangle - A_2^{(1)}(\Psi_x^\varepsilon) \right] \, ds \\
f_3^\varepsilon(z) &\equiv \tilde{k}^2 f'(z) \int_{\mathbb{R}}^\infty \mathbb{E}^z \left[ \left\langle \Psi_x^\varepsilon, \tilde{V}_x^\varepsilon \Psi_x^\varepsilon \theta \right\rangle - A_3^{(1)}(\Psi_x^\varepsilon) \right] \, ds.
\end{align}

Let

\begin{equation}
\Gamma^{(2)}(x,y) \equiv \mathbb{E} \left[ \tilde{V}_x^\varepsilon(x)\tilde{V}_y^\varepsilon(y) \right],
\end{equation}

and

\begin{align}
A_2^{(2)}(\phi) &\equiv \int \int \theta(x)\phi(x)\Gamma^{(2)}(x,y)\phi(y)\theta(y) \, dx \, dy \\
A_3^{(2)}(\phi) &\equiv \int \Gamma^{(2)}(x,x)\phi(x)\theta(x) \, dx,
\end{align}

we then have

\begin{align}
f_2^\varepsilon(z) &= \frac{\varepsilon^2 \tilde{k}^2}{2} \int f''(z) \left[ \left\langle \Psi_x^\varepsilon, \tilde{V}_x^\varepsilon \theta \right\rangle \right]^2 - A_2^{(2)}(\Psi_x^\varepsilon) \right] \\
f_3^\varepsilon(z) &= \frac{\varepsilon^2 \tilde{k}^2}{2} \int f'(z) \left[ \left\langle \Psi_x^\varepsilon, \tilde{V}_x^\varepsilon \tilde{V}_x^\varepsilon \theta \right\rangle - A_3^{(2)}(\Psi_x^\varepsilon) \right].
\end{align}

**Proposition 3.**

\begin{equation*}
\lim_{\varepsilon \to 0} \sup_{t < z_0} \mathbb{E} |f_2^\varepsilon(t)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{t < z_0} \mathbb{E} |f_3^\varepsilon(t)| = 0.
\end{equation*}

**Proof.** We have the bounds

\begin{align}
\sup_{z < z_0} \mathbb{E} |f_2^\varepsilon(z)| &\leq \sup_{z < z_0} \varepsilon^2 \tilde{k}^2 \| f'' \|_{\infty} \left[ \| \Psi_0 \|_2^2 \mathbb{E} \| \tilde{V}_x^\varepsilon \theta \|_2^2 + \mathbb{E} \left[ A_2^{(2)}(\Psi_x^\varepsilon) \right] \right] \\
\sup_{z < z_0} \mathbb{E} |f_3^\varepsilon(z)| &\leq \sup_{z < z_0} \varepsilon^2 \tilde{k}^2 \| f' \|_{\infty} \left[ \| \Psi_0 \|_2 \mathbb{E} \| \tilde{V}_x^\varepsilon \tilde{V}_x^\varepsilon \theta \|_2 + \mathbb{E} \left[ A_3^{(2)}(\Psi_x^\varepsilon) \right] \right];
\end{align}

both of them tend to zero. \hfill \Box

We have

\begin{align}
\mathcal{A}^\varepsilon f_2^\varepsilon(z) &= \tilde{k}^2 f''(z) \left[ - \left\langle \Psi_x^\varepsilon, V_x^\varepsilon \theta \right\rangle \left\langle \Psi_x^\varepsilon, \tilde{V}_x^\varepsilon \theta \right\rangle + A_2^{(1)}(\Psi_x^\varepsilon) \right] + R_2^\varepsilon(z) \\
\mathcal{A}^\varepsilon f_3^\varepsilon(z) &= \tilde{k}^2 f'(z) \left[ - \left\langle \Psi_x^\varepsilon, V_x^\varepsilon \tilde{V}_x^\varepsilon \theta \right\rangle + A_3^{(1)}(\Psi_x^\varepsilon) \right] + R_3^\varepsilon(z)
\end{align}
with

\[
R_2^\epsilon(z) = i\epsilon^2 \bar{k} f'''(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta \theta \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \theta \rangle \right] \left[ \langle \Psi_\epsilon^z, \nabla \hat{\psi}_\epsilon \rangle^2 - A_2^2(\Psi_\epsilon^z) \right] + i\epsilon^2 \bar{k} f''(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta(\nabla \hat{\psi}_\epsilon \theta) \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \hat{\psi}_\epsilon \theta \rangle \right]
\]

(29)

\[
- i\epsilon^2 \bar{k} f''(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta(\nabla \hat{G}_\epsilon^\theta \Psi_\epsilon^z) \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \hat{G}_\epsilon^\theta \Psi_\epsilon^z \rangle \right]
\]

where \( \hat{G}_\epsilon^\theta \) denotes the operator

\[
\hat{G}_\epsilon^\theta (\psi) = \int \theta(x) \hat{G}_\epsilon(\psi) \psi_0(y) \, dy.
\]

Similarly

\[
R_3^\epsilon(z) = i\epsilon^2 \bar{k} f'(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta(\gamma z \hat{\psi}_\epsilon \theta) \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \hat{\psi}_\epsilon \theta \rangle \right]
\]

\[
+ i\epsilon^2 \bar{k} f''(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta(\nabla \hat{\psi}_\epsilon \theta) \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \hat{\psi}_\epsilon \theta \rangle \right] \left[ \langle \Psi_\epsilon^z, \nabla \hat{\psi}_\epsilon \rangle^2 - A_3^2(\Psi_\epsilon^z) \right]
\]

\[
- i\epsilon^2 \bar{k} f'(z) \left[ \frac{1}{2} \langle \Psi_\epsilon^z, \Delta(\nabla \hat{G}_\epsilon^\theta \Psi_\epsilon^z) \rangle + \frac{\bar{k}^2}{\epsilon} \langle \Psi_\epsilon^z, \gamma z \hat{G}_\epsilon^\theta \Psi_\epsilon^z \rangle \right]
\]

where

\[
\Gamma_0^2(\psi) \equiv \Gamma^2(\psi, \psi).
\]

**Proposition 4.**

\[
\lim_{\epsilon \to 0} \sup_{z < z_0} E|R_2^\epsilon(z)| = 0, \quad \lim_{\epsilon \to 0} \sup_{z < z_0} E|R_3^\epsilon(z)| = 0.
\]

The argument is entirely analogous to that for Proposition 3. The most severe factors involve \( \Delta(\nabla \hat{\psi}_\epsilon \theta) \) and \( \Delta(\nabla \hat{\psi}_\epsilon \theta) \), both of which have uniformly bounded second moments by Assumption 2 and Corollary 1. Therefore the corresponding terms are \( O(\epsilon^2) \).

Consider the test function \( f^\epsilon(z) = f(z) + f_1^\epsilon(z) - f_2^\epsilon(z) - f_3^\epsilon(z) \). We have

(27) \[ A^\epsilon f^\epsilon(z) = i \frac{f'(z) \langle \Psi_\epsilon^z, \Delta \theta \rangle + \bar{k}^2 f''(z) A_2^1(\Psi_\epsilon^z) - \bar{k}^2 f' A_3^1(\Psi_\epsilon^z) + R_2^\epsilon(z) - R_3^\epsilon(z) + A_4^\epsilon(z). \]

Set

(27) \[ R^\epsilon(z) = R_1^\epsilon(z) - R_2^\epsilon(z) - R_3^\epsilon(z), \quad \text{with} \quad R_1^\epsilon(z) = A_4^\epsilon(z). \]

It follows from Propositions 3 and 5 that

\[
\lim_{\epsilon \to 0} \sup_{z < z_0} E|R^\epsilon(z)| = 0.
\]
Recall that
\[
M_z^ε(θ) = f^ε(z) - \int_0^z \mathcal{A}^ε f^ε(s) \, ds
\]
\[
= f(z) + f_1'(z) - f_2^ε(z) - f_3^ε(z) - \int_0^z \frac{i}{2k} f'(z) \langle \Psi^ε z, θ \rangle \, ds
\]
\[
+ \int_0^z k^2 \left[ f''(s) A_2(1)(\Psi^ε s) + f'(s) A_3(1)(\Psi^ε s) \right] \, ds - \int_0^z R^ε(s) \, ds
\]
is a martingale. Now that (23)-(26) are satisfied we can identify the limiting martingale to be
\[
M_z(θ) = f(z) - \int_0^z \left\{ f'(s) \left[ \frac{i}{2k} \langle \Psi_s, θ \rangle - k^2 A_3(1)(\Psi_s) \right] - k^2 f''(s) A_2(1)(\Psi_s) \right\} ds.
\]
Since \( \langle \Psi^ε z, θ \rangle \) is uniformly bounded
\[
||\langle \Psi^ε z, θ \rangle|| \leq ||\Psi_0||_2||θ||_2
\]
we have the convergence of the second moment
\[
\lim_{ε \to 0} \mathbb{E} \{ \langle \Psi^ε z, θ \rangle^2 \} = \mathbb{E} \{ \langle \Psi z, θ \rangle^2 \}.
\]
Use \( f(r) = r \) and \( r^2 \) in (32)
\[
M_z^{(1)}(θ) = \langle \Psi z, θ \rangle - \int_0^z \left[ \frac{i}{2k} \langle \Psi_s, θ \rangle - k^2 A_3(1)(\Psi_s) \right] \, ds
\]
is a martingale with the quadratic variation
\[
\left[ M^{(1)}(θ), M^{(1)}(θ) \right]_z = -k^2 \int_0^z A_2(1)(\Psi_s) \, ds = -k^2 \int_0^z \langle θ, \mathcal{K}(Ψ_s) \theta \rangle \, ds
\]
where
\[
\mathcal{K}(Ψ_s) = \int \Psi_{s}(x) \Gamma^{(1)}(x, y) Ψ_{s}(y) \theta(y) \, dy.
\]
Therefore,
\[
M_z^{(1)} = i k \int_0^z \sqrt{\mathcal{K}(Ψ_s)} dW_s
\]
where \( W_s \) is a real-valued, cylindrical Wiener process (i.e. \( dW_s(x) \) is a space-time white noise field) and \( \sqrt{\mathcal{K}(Ψ_s)} \) is the square-root of the positive-definite operator given in (17).

References
[1] R.J. Adler: An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes, Institute of Mathematical Statistics, Hayward, California, 1990.
[2] L.C. Andrew: An analytical model for the refractive index power spectrum and its application to optical scintillations in the atmosphere. J. Mod. Opt. 39, 1849-1853 (1992).
[3] F. Bailly, J.P. Clouet and J.-P. Fouque: Parabolic and Gaussian white noise approximation for wave propagation in random media. SIAM J. Appl. Math. 56:5, 1445-1470 (1996).
[4] F. Bailly and J.-P. Fouque: High frequency wave propagation in random media, preprint, 1997.
[5] A.V. Balakrishnan: Spectral density of laser beam scintillation in wind turbulence. I. Theory. Comput. Appl. Math. 17:2, 173-199 (1998).
[6] R.S. Cole, K.L. Ho and N.D. Mavrokeoukoulakis: The effect of the outer scale of turbulence and wavelength on scintillation fading at millimeter wavelengths. IEEE Trans. Antennas Propag.26(5), 712-715 (1978).
[7] E.B. Davies: One-Parameter Semigroups. Academic Press, London, 1980.
[8] D. Dawson and G. Papapicoulaou: A random wave process. Appl. Math. Optim.12, 97-114 (1984).
[9] S.N. Ethier and T.G. Kurtz: Markov Processes - Characterization and Convergence. John Wiley and Sons, New York, 1986.
[10] A. Fannjiang: Invariance principle for inertial-scale behavior of scalar fields in Kolmogorov-type turbulence, Phys. D 179: 3-4(2003), 161-182.
[11] A. Fannjiang: White-noise and geometrical optical limits of Wigner-Moyal equation for wave beams in turbulent media, e-print, arxiv: math-ph/0304024

[12] R.L. Fante: Electromagnetic beam propagation in turbulent media: an update. Proc. IEEE 68(11), 1424-1443 (1980).

[13] R.L. Fante: Inner-scale size effect on the scintillations of light in the turbulent atmosphere. Jour. Optical Soc. Amer. 73(3) (1983).

[14] J.P. Fouque, G. Papanicolaou and Y. Samueldes: Forward and Markov approximation: The strong intensity fluctuations regime revisited., preprint, 1997.

[15] R.J. Hill: Models of the scalar spectrum for turbulent advection. J. Fluid Mech. 88, 541-562 (1978).

[16] I.A. Ibragimov and Y.A. Rozanov: Gaussian Random Processes. Springer-Verlag, New York, 1978.

[17] T. G. Kurtz: Semigroups of conditional shifts and approximations of Markov processes, Ann. Prob. 3: 4, 618-642 (1975).

[18] H. J. Kushner: Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory, The MIT Press, Cambridge, Massachusetts (1984).

[19] R.M. Manning: Stochastic Electromagnetic Image Propagation. McGraw-Hill, Inc., New York, 1993.

[20] G. Papanicolaou, L. Ryzhik and K. Solna: Statistical stability in time reversal, preprint, 2002.

[21] G. Papanicolaou and K. Solna: Wavelet based estimation of Kolmogorov turbulence, In Long-range Dependence: Theory and Applications, P. Doukhan, G. Oppenmeim and M. S. Taqu, editors, Birkhauser, 473-505, (2002).

[22] J.W. Strohbehn: Laser Beam Propagation in the Atmosphere. Springer-Verlag, Berlin, 1978.

[23] V.I. Tartarski, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 4, 551 (1960).

[24] V.I. Tartarski: The Effects of the Turbulent Atmosphere on Wave Propagation. U.S. Department of Commerce, Springfield, Va., 1971.