On the Application of Gluon to Heavy Quarkonium Fragmentation Functions

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Abstract

We analyze the uncertainties induced by different definitions of the momentum fraction $z$ in the application of gluon to heavy quarkonium fragmentation function. We numerically calculate the initial $g \rightarrow J/\psi$ fragmentation functions by using the non-covariant definitions of $z$ with finite gluon momentum and find that these fragmentation functions have strong dependence on the gluon momentum $\vec{k}$. As $|\vec{k}| \rightarrow \infty$, these fragmentation functions approach to the fragmentation function in the light-cone definition. Our numerical results show that large uncertainties remains while the non-covariant definitions of $z$ are employed in the application of the fragmentation functions. We present for the first time the polarized gluon to $J/\psi$ fragmentation functions, which are fitted by the scheme exploited in this work.

PACS number(s): 13.87.Fh, 14.70.Dj
1 Introduction

Fragmentation refers to the process of a parton which carries large transverse momentum and subsequently forms a jet containing the expected hadron [1]. At sufficiently large transverse momentum of the heavy quarkonium production, the direct leading order production scheme is normally suppressed while the fragmentation scheme becomes dominant, though it is formally of higher order in the strong coupling constant $\alpha_s$ [2, 3, 4, 5].

Generally, the fragmentation processes of heavy quarkonium $H$ production can be expressed as [6]

$$d\sigma[A + B \rightarrow H(p_T) + X] = \sum_{a,b,c} \int_0^1 dx_a \int_0^1 dx_b \int_0^1 dz f_{a/A}(x_a, Q) \times f_{b/B}(x_b, Q) d\hat{\sigma}(a + b \rightarrow c + X) D_{c \rightarrow H}(z, Q),$$

where $a$ and $b$ are incident partons in the colliding hadrons $A$ and $B$ respectively; $f_{a/A}$ and $f_{b/B}$ are the parton distribution functions at the scale $Q^2$ of the partonic subprocess $a + b \rightarrow c + X$; $c$ is the fragmenting parton (either a gluon or a quark) and the sum runs over all possible parton contributions; $D_{c \rightarrow H}(z, Q)$ is the fragmentation function with respect to the scale $Q^2$ which can be obtained by evolving from the initial fragmentation function $D_{g \rightarrow H}(z, Q_0)$ using Altarelli-Parisi equations [7]

$$Q \frac{\partial}{\partial Q} D_{i \rightarrow H}(z, Q) = \sum_j \int_{y}^{1} \frac{d y}{y} P_{ij}(z/y, Q) D_{j \rightarrow H}(y, Q),$$

where $P_{ij}$ are the splitting functions of a parton $i$ into a parton $j$. The initial gluon fragmentation function $D_{g \rightarrow H}(z, Q_0)$ is a universal function defined by factorization in the infinite momentum frame of the fragmenting gluon. It can also be obtained by calculating a specific physical process in perturbative QCD in the finite momentum frame of fragmenting gluon [1, 8], where the $z$ is defined in the Lorentz boost invariant form (e.g. Eq. (1) or (2)). The resulting initial fragmentation function, independent of the momentum of the parent gluon, is equivalent to the one derived in the infinite momentum frame.

$$z = \frac{E_H + p_T^H}{E_g + p_T^g},$$

(1)

$$z = \frac{p_g \cdot p_H}{(p_g)^2}.$$  

(2)

In above equations we take the $Z$ axis along the momentum of fragmenting gluon. Then, $E^H, p_T^H, E^g$ and $p_T^g$ are the energies and $Z$-components of the four-momenta of the fragmenting gluon and the produced heavy quarkonium $H$, respectively. In Eq. (1) $z$ is defined as the usual light-cone form.
The Eqs. (1) and (2) are hard to be employed in the application of the gluon fragmentation functions, because they involve the transverse momentum of the resulting heavy quarkonium. Instead, usually the non-covariant definitions as follows are used approximately:

\[ z = \frac{\sqrt{M_H^2 + (p_z^H)^2}}{E^g}, \]  

(3)

\[ z = \frac{\sqrt{M_H^2 + (p_z^H)^2 + p_z^H}}{E^g + p_z^g}, \]  

(4)

\[ z = \frac{p_z^H}{p_z^g}. \]  

(5)

When the fragmenting gluon momentum \( |\vec{k}| \to \infty \), these non-covariant definitions are equivalent to the light-cone definition in Eq. (1). However when the momentum of the fragmenting gluon remains finite, these non-covariant definitions of \( z \) may induce large uncertainties in the application of the fragmentation functions. The Eq. (3) can be re-expressed as

\[ |p_z^H| = \sqrt{(zE^g)^2 - M_H^2}, \]

which shows that at certain values of \( z \), there are two possibilities for momentum directions of quarkonium \( H \). Similarly, the definition of Eq. (4) does not give unique direction of \( H \) for certain \( z \), which does not comply with the original idea of fragmentation. Thus, the definition of Eq. (5) is better and widely used. About a decade ago, people noticed that the parton to heavy quarkonium fragmentation functions can be obtained analytically [4, 8, 9, 10]. However, in fact the initial scale gluon fragmentation function can also be obtained numerically and works equally well as the analytic one in the application.

In section 2, by using the perturbative QCD we numerically calculate the initial \( g \to J/\psi \) fragmentation functions \( D_{g\to J/\psi}(z, M_{J/\psi}) \) with the non-covariant definitions given above at finite momentum of fragmenting gluon. We compare them with the fragmentation functions in the light-cone definition Eq. (1). In section 3, We give out numerically the polarized gluon to \( J/\psi \) fragmentation functions with light-cone definition of \( z \), and transform it into analytic ones by fitting. We also present the \( \chi^2 \)-square fitted functions for both polarized and unpolarized fragmentation functions in the light-cone definition of \( z \). In section 4, we give a brief summary of our results.

2 Unpolarized fragmentation functions w.r.t. different definitions of \( z \) and the resulting uncertainties

The Feynman diagram of the process \( qg \to qg^* \to J/\psi gg \) is shown in Fig. 1 and the corresponding matrix element is composed of two parts: (1) the production of a virtual
gluon $g^*$ with momentum $k$ and invariant mass $\sqrt{s}$; (2) the virtual gluon decaying into a $J/\psi$ and a gluon pair, i.e.,

$$\mathcal{M} = M_1^\mu \left( \frac{g_{\mu \nu}}{k^2} \right) M_2^\nu(\kappa),$$

$$= -\sum_\lambda M_1^\mu(\lambda) \frac{1}{k^2} M_2^\nu(\kappa) \varepsilon^*_\nu(\lambda),$$

where $M_1^\mu(\lambda)$ and $M_2^\nu(s)\varepsilon^*_\nu(\lambda)$ represent the virtual gluon production and decay sectors, respectively. The $\lambda$ and $\kappa$ denote the virtual gluon and $J/\psi$ polarization indices. In deriving out the above second equation, the following condition on the summation of gluon polarization vector is used:

$$g_{\mu \nu} = -\sum_\lambda \varepsilon_{\mu}(\lambda) \varepsilon^*_\nu(\lambda).$$

Then, the differential cross-section of process $g + q \rightarrow q + g^*; g^* \rightarrow J/\psi + g + g$, as shown in Fig. 1, can be expressed as:

$$d\sigma = |\mathcal{M}|^2 [d\phi]$$

$$= \int d^3k \frac{d\hat{\sigma}(qg \rightarrow qg^*)}{d^3k} \int_{M_{J/\psi}^2 \pi s^3/2}^{\infty} \frac{ds}{\pi s^3/2} \sum_\kappa d\Gamma_{\kappa}(g^* \rightarrow J/\psi gg),$$

where the sum runs over the $J/\psi$ polarization for the unpolarized production, and

$$d\hat{\sigma}(qg \rightarrow qg^*) = \delta^4(p_1 + p_2 - k - p_6) \sum_\lambda |M_1|^2 \lambda \lambda [d\phi_1] \frac{d^3k}{(2\pi)^3 2E_6},$$

$$d\Gamma_{\kappa}(g^* \rightarrow J/\psi gg) = \frac{(2\pi)^4}{2\sqrt{s}} \delta^4(k - p_3 - p_4 - p_5) \left( \frac{1}{2} \sum_\lambda |M_2(\kappa)|^2 \lambda \lambda [d\phi_2] \right).$$

Hence, the initial scale gluon fragmentation function corresponding to the finite gluon momentum satisfies

$$\int_0^1 dz D_{g \rightarrow J/\psi}(z, M_{J/\psi}) = \int dp_{4z} \int_{M_{J/\psi}^2 / \pi s^3/2}^{\infty} \frac{ds}{\pi s^3/2} \sum_\kappa \int d\Gamma_{\kappa}(g^* \rightarrow J/\psi gg) dp_{4z}.$$
Therefore, from Eq. (12) one can extract the unpolarized initial gluon fragmentation functions with either boost-invariant or non-invariant definitions of $z$. We numerically calculate the fragmentation functions with various definitions of $z$ with a specific gluon momentum and compare the differences among them.

Fig. 2 is the fragmentation functions corresponding to the boost-invariant definitions of $z$ given in Eq. (1) and (2), which are numerically calculated using the FDC (Feynman Diagram Calculation) program [11] and they agree with those given in Refs. [4] and [12], respectively. Given different virtual gluon momentums in the numerical calculation, we obtain the same fragmentation function distribution versus $z$ as shown in Fig. 2, which indicates explicitly the independence of the fragmentation functions on virtual gluon momentum, as they should be from the definitions of $z$. 

Figs. 3 and 4 present the fragmentation function distributions over variable $z$ in non-covariant definition of Eqs. (3) and (4). It shows that the fragmentation function distributions rely on the momentum $|\vec{k}|$ of the virtual gluon. When $|\vec{k}| \to \infty$, the corresponding fragmentation functions approach to the one in the light-cone definition of Eq. (1). However, when $|\vec{k}|$ is finite, these fragmentation functions distribute distinctively from the one in the light-cone definition, which will bring certain uncertainties in the application of the fragmentation functions. With the non-covariant definition in Eqs. (3) and (4), there is the possibility that the $J/\psi$ momentum is opposite to the virtual gluon momentum $\vec{k}$, and this possibility becomes larger as $|\vec{k}|$ decreases. Therefore, the large peaks at small $z$ in Figs. 3 and 4 correspond to the lower momentum $|\vec{k}|$.

Figure 5 gives the distribution of the fragmentation functions versus $z$ in the definition of (5) at various virtual gluon momentums. Since it is possible that the $J/\psi$ may possess momentum opposite to that of the virtual gluon, in the definition of (5), the condition of
Figure 3: Fragmentation functions w.r.t. Eq. (3)

Figure 4: Fragmentation functions w.r.t. Eq. (4)
$z < 0$ exists. And because of the non-zero invariant mass of the virtual gluon, even $z > 1$ happens. Thus, in Fig. 5 the fragmentation functions distribute in the scope of $-1 \leq z \leq 2$. As shown in the figure, when the virtual gluon momentum $|\vec{k}| \to \infty$, fragmentation functions approach to the one in the light-cone definition. When $|\vec{k}| = 5, 10, 20, 80$ and 1280GeV, the ratios of fragmentation probabilities in the definition of (5), that is the integrations of the corresponding fragmentation functions over $z$, to the one in light-cone definition are 0.49, 0.65, 0.77, 0.91 and 0.97 respectively in the range of $0 \leq z \leq 1$. This indicates that the smaller the virtual gluon momentum is, the more possible the produced $J/\psi$ will be opposite to the gluon. Therefore, uncertainties occur in the application of the fragmentation function in the definition of (5) when gluon momentum $|\vec{k}|$ is not big enough. While the virtual gluon energy is greater than 80GeV, the uncertainties induced by the definition of (5) will drop to less than 10%.

Figure 6 presents the differential cross-sections of processes $pp(\bar{p}) \to gg$ versus gluon energy at the colliding energies of Fermilab Tevatron and LHC respectively. In our numerical calculation, the parton distribution function of CTEQ6L [13] is employed. As shown in the figure, the cross-section decreases quite fast as the gluon momentum increases. Whereas, from the analysis above, we know that at some low energies of the fragmenting gluon there are large discrepancies between the fragmentation functions with non-covariant definitions and boost-invariant definitions of $z$. Therefore, it is possible that the non-covariant definitions of $z$ may induce large errors in the application of fragmentation functions.

Figure 5: Fragmentation function w.r.t. Eq. (5)
3 Polarized fragmentation functions w.r.t. light-cone definition of \( z \)

The polarized fragmentation functions of gluon fragmenting into P-wave charmonium \( \chi_{cJ} \) exist in the literature [14], however, there has been no relevant work on polarized fragmentation functions of gluon to S-wave vector Quarkonium states. With the FDC program and procedure described in above, in fact it is straightforward for us to obtain the gluon to polarized \( J/\psi \) fragmentation functions numerically. And then get the analytic ones by fitting, which are equivalent in use as the ones obtained directly from the analytic calculation, at least to large extent.

With the numerical method described in Sec. 2, we can also obtain the polarized fragmentation functions which satisfy

\[
\int_0^1 dz D^T_{g \rightarrow J/\psi}(z, M_{J/\psi}) = \int dp_{4z} \int_{M_{J/\psi}^2}^{\infty} \frac{ds}{\pi s^{3/2}} \sum_{\kappa=1,2} \int d\Gamma_{\kappa}(g^* \rightarrow J/\psi gg) \frac{d\Gamma_{\kappa}(g^* \rightarrow J/\psi gg)}{dp_{4z}},
\]

(13)

\[
\int_0^1 dz D^L_{g \rightarrow J/\psi}(z, M_{J/\psi}) = \int dp_{4z} \int_{M_{J/\psi}^2}^{\infty} \frac{ds}{\pi s^{3/2}} \int d\Gamma_3(g^* \rightarrow J/\psi gg) \frac{d\Gamma_3(g^* \rightarrow J/\psi gg)}{dp_{4z}},
\]

(14)

where \( D^T_{g \rightarrow J/\psi} \) and \( D^L_{g \rightarrow J/\psi} \) are fragmentation functions of gluon to \( J/\psi \) in transverse and longitudinal polarizations, respectively. The polarized fragmentation functions can then be extracted from Eqs. (13) and (14) in different definitions of \( z \). Fig. 7 shows the polarized fragmentation functions corresponding to the light-cone definition of Eq. (1). Explicit numerical calculation indicates that these polarized fragmentation functions in light-cone definition are independent of the virtual gluon momentum, as they should be.
In Ref. [8], the initial unpolarized fragmentation function of gluon to \( J/\psi \) is given in a form of two-dimensional integrals, which in practice can only be evaluated numerically and thus is not very convenient for the application of the Altarelli-Parisi evolution equation [7]. In use of the obtained initial gluon to unpolarized \( J/\psi \) fragmentation function [8], one can integrate out the integrals numerically [15] and fit the \( z \) distribution into a combination of functions of \( z \). Then the application of the gluon to \( J/\psi \) fragmentation function will be more easier for the phenomenological use. To our knowledge the polarized fragmentation functions of gluon to \( J/\psi \) are still absent so far in literatures. However, they may be necessary for future careful analysis about the polarization situation of Quarkonium production.

We proceed the \( \chi \)-square fitting of gluon to polarized and unpolarized \( J/\psi \) fragmentation functions into polynomials and exponents in the light-cone definition of \( z \), which tells

\[
D_{g \to J/\psi}(z, M_{J/\psi}) = \alpha_s^3(M_{J/\psi}) \frac{|R(0)|^2}{(2M_{J/\psi})^3} e^{a_0z(1-z)} (1 - z) \sum_{n=1,9} a_n z^n ,
\]

(15)

\[
D_{T g \to J/\psi}(z, M_{J/\psi}) = \alpha_s^3(M_{J/\psi}) \frac{|R(0)|^2}{(2M_{J/\psi})^3} e^{a_0^T z(1-z)} (1 - z) \sum_{n=1,9} a_n^T z^n .
\]

(16)

Here, \( \alpha_s(M_{J/\psi}) \), \( M_{J/\psi} \) and \( R(0) \) are strong coupling, the \( J/\psi \) mass and radial wave function at the origin, respectively. The fitted coefficients of \( a_n \) and \( a_n^T \) are presented in Table 1.

| \( a_0 \)       | \( a_1 \)       | \( a_1^T \)      | \( a_2 \)       | \( a_2^T \)      | \( a_3 \)       | \( a_3^T \)      | \( a_4 \)       | \( a_4^T \)      | \( a_5 \)       | \( a_5^T \)      | \( a_6 \)       | \( a_6^T \)      | \( a_7 \)       | \( a_7^T \)      | \( a_8 \)       | \( a_8^T \)      | \( a_9 \)       | \( a_9^T \)      |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| -17.0402262     | 5.1820321 \times 10^{-7} | 3.3809309 \times 10^{-7} | -7.5744212 \times 10^{-6} | -5.0184819 \times 10^{-6} | 1.2310370 \times 10^{-4} | 8.2006550 \times 10^{-5} | -7.6536059 \times 10^{-4} | -5.1990241 \times 10^{-4} | 3.1398595 \times 10^{-3} | 2.1421231 \times 10^{-3} | -7.3523074 \times 10^{-3} | -4.9941798 \times 10^{-3} | 9.2419267 \times 10^{-3} | 6.2434863 \times 10^{-3} | -5.8527342 \times 10^{-3} | -3.9354343 \times 10^{-3} | 1.4728577 \times 10^{-3} | 9.8679226 \times 10^{-4} |

Table 1: Coefficients of fitted functions for the unpolarized and transversely polarized initial gluon to \( J/\psi \) fragmentation functions.
4 Summary

In this work, we show that although the fragmentation functions corresponding to the boost invariant definitions of (1) and (2) are equivalent to the ones in the infinite momentum frame of gluon and thus independent of fragmenting gluon momentum, they are inconvenient in use, because they involve the transverse momentum of the parent gluon. Instead, the non-covariant definitions, such as in Eqs. (3), (4) and (5), are used as an approximation. They are equivalent to the definition of the light-cone coordinate form when the fragmenting gluon momentum $|\vec{k}| \to \infty$. However, as shown in our calculation, these non-covariant definitions may induce large errors while $|\vec{k}|$ is finite, especially small. In practice, the definition of (3) and (4) are somewhat inconvenient because they possess, in the range of $0 \leq z \leq 1$, the possibility of $J/\psi$ moving in the opposite direction of the fragmenting gluon. The definition of (5) excludes this situation, but it may bring larger errors in the colliding energies of Tevatron and LHC. This should be taken into account in the careful phenomenological analysis in the application of the gluon fragmentation functions.

In this paper we give out the initial fragmentation functions of gluon into the polarized $J/\psi$ for the first time. We use a different method in getting them from the normal analytic calculation in the literature. The $\chi^2$-square fittings for both polarized and unpolarized fragmentation functions of gluon into $J/\psi$ are done and presented, which are suitable for future phenomenological use. Finally, although the polarized fragmentation functions obtained in this work are schematically for charmonium, the $J/\psi$, in fact they can be directly applied to the $\Upsilon$ system with some simple replacements.
Acknowledgments

The work of C. F. Q. was supported in part by the Natural Science Foundation of China and by the Scientific Research Fund of GUCAS (No. 055101BM03); the work of J. X. W. and W. Q. was supported by the Natural Science Foundation of China (No. 10475083).

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