Free divisors and duality for $\mathcal{D}$-modules *

F.J. Castro-Jiménez and J.M. Ucha

March 14, 2001

Abstract

The relationship between $\mathcal{D}$-modules and free divisors has been studied in a general setting by L. Narváez and F.J. Calderón. Using the ideas of these works we prove in this article a duality formula between two $\mathcal{D}$-modules associated to a class of free divisors on $\mathbb{C}^n$ and we give some applications.

Keywords: $\mathcal{D}$-modules, Differential Operators, Gröbner Bases, Logarithmic Comparison Theorem.

Math. Classification: 32C38, 13N10, 14F40, 13P10.

1 Free divisors

Here we summarize some results of K. Saito [14].

Let us denote $X = \mathbb{C}^n$. Denote by $\mathcal{O} = \mathcal{O}_X$ the sheaf of holomorphic functions on $X$. Let $D \subset X$ be a divisor and $x \in D$. Denote by $\text{Der}(\mathcal{O}_x)$ the $\mathcal{O}_x$-module of $\mathbb{C}$-derivations of $\mathcal{O}_x$ (the elements in $\text{Der}(\mathcal{O}_x)$ are called vector fields).

A vector field $\delta \in \text{Der}(\mathcal{O}_x)$ is said to be logarithmic w.r.t. $D$ if $\delta(f) = af$ for some $a \in \mathcal{O}_x$, where $f$ is a local (reduced) equation of the germ $(D, x) \subset (\mathbb{C}^n, x)$. The $\mathcal{O}_x$-module of logarithmic vector fields (or logarithmic derivations) is denoted by $\text{Der}(\log D)_x$. This yields a $\mathcal{O}$-module sheaf denoted by $\text{Der}(\log D)$.

The divisor $D$ is said to be free at the point $x \in D$ if the $\mathcal{O}_x$-module $\text{Der}(\log D)_x$ is free (and, in this case, of rank $n$). The divisor $D$ is called free if it is free at each point $x \in D$.

Smooth divisors are free. A normal crossing divisor $D \equiv (x_1 \cdots x_t = 0) \subset \mathbb{C}^n$ is free because we have $\text{Der}(\log D) = \bigoplus_{i=1}^t \mathcal{O}_{\mathbb{C}^n} x_i \partial_i \oplus \bigoplus_{j=t+1}^n \mathcal{O}_{\mathbb{C}^n} \partial_j$. By [14] any reduced germ of plane curve $D \subset \mathbb{C}^2$ is a free divisor.

Saito’s criterium to test the freedom of a divisor $D$ at a point $p$ is:

Lemma 1.1.1. ([14, (1.9)]) Let $\delta_i = \sum_{j=1}^n a_{ij} \partial_j$, $i = 1, \ldots, n$ be a system of holomorphic vector fields at $p \in \mathbb{C}^n$, such that:

i) $[\delta_i, \delta_j] \in \sum_{k=1}^n \mathcal{O}_p \delta_k$, for $i, j = 1, \ldots, n$.

ii) $\det(a_{ij}) = h$ defines a reduced hypersurface $D$.

Then, for $D \equiv (h = 0)$, $\delta_1, \ldots, \delta_n$ belong to $\text{Der}(\log D)_p$ and hence $\{\delta_1, \ldots, \delta_n\}$ is a free basis of $\text{Der}(\log D)_p$.

*Partially supported by DGESIC-97-0723 and HF-1998-0105. Second author partially supported by MSRI (Berkeley)
2 The logarithmic comparison theorem

Let $X$ be a complex manifold and $D \subset X$ a divisor. We have a canonical inclusion

$$i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\ast D)$$

where $\Omega^\bullet(\ast D)$ is the meromorphic de Rham complex and $\Omega^\bullet(\log D)$ is the de Rham logarithmic complex, both w.r.t $D$. A meromorphic form $\omega \in \Omega^p(\ast D)$ is said to be logarithmic if $fw \in \Omega^p$ and $df \wedge \omega \in \Omega^{p+1}$ for each local equation $f$ of $D$.

A classical natural problem is to find the class of divisors $D \subset X$ for which $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\ast D)$ is a quasi-isomorphism (i.e. $i_D$ induces an isomorphism on cohomology).

By Grothendieck’s comparison theorem we know that the complexes $\Omega^\bullet(\ast D)$ and $Rj_*(\mathcal{C})$ are naturally quasi-isomorphic, where $j : U = X \setminus D \to X$ is the natural inclusion. So, if $i_D$ is a quasi-isomorphism we say that the logarithmic comparison theorem holds for $D$ (or simply LCT holds for $D$).

**Definition 2.1.2.** ([8]) A divisor $D \subset X$ is locally quasi-homogeneous if for all $q \in D$ there exist local coordinates $(V; x_1, \ldots, x_n)$ centered at $q$ such that $D \cap V$ has a weighted homogeneous defining equation w.r.t. $(x_1, \ldots, x_n)$.

Smooth divisors and normal crossing divisors are locally quasi-homogeneous. A weighted homogeneous polynomial $f \in \mathbb{C}[x,y]$ defines a locally quasi-homogeneous divisor $D \equiv (f = 0) \subset \mathbb{C}^2$.

Suppose $D \subset X$ is a locally quasi-homogeneous free divisor. The main result of [8] is that LCT holds for $D$, i.e.

$$i_D : \Omega^\bullet(\log D) \to Rj_*(\mathcal{C})$$

is a quasi-isomorphism.

3 Logarithmic $\mathcal{D}$-modules

Let us denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf (of rings) of linear differential operators with holomorphic coefficients on $X$.

A local section $P$ of $\mathcal{D}$ is a finite sum

$$P = \sum_{\alpha} a_{\alpha} \partial^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $a_{\alpha}$ is a local section of $\mathcal{O}$ on some chart $(U; x_1, \ldots, x_n)$ and $\partial = (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

The sheaf $\mathcal{D}$ is naturally filtered by the order of the differential operators. The associated graded ring $\text{gr}(\mathcal{D})$ is commutative. In fact, we can identify $\text{gr}(\mathcal{D})$ with the sheaf $\mathcal{O}[\xi_1, \ldots, \xi_n]$ of polynomials in the variables $\xi = (\xi_1, \ldots, \xi_n)$ and with coefficients in $\mathcal{O}$.

Assume the operator $P = \sum a_{\alpha} \partial^\alpha$ has order $d$ (i.e. $d = \max\{|\alpha| = \alpha_1 + \cdots + \alpha_n | a_{\alpha} \neq 0\}$) then the principal symbol of $P$ is

$$\sigma(P) = \sum_{|\alpha| = d} a_{\alpha} \xi^\alpha \in \mathcal{O}[\xi].$$

For each left ideal $I$ in $\mathcal{D}$ the graded ideal associated to $I$ is the ideal of $\text{gr}(\mathcal{D})$ generated by the set of principal symbol $\sigma(P)$ for $P \in I$. This ideal is denoted by $\text{gr}(I)$.

The characteristic variety of the $\mathcal{D}$-module $M = \mathcal{F}$ is, by definition, the analytic sub-variety of the cotangent bundle $T^*X$ defined by $\mathcal{O}_{T^*X}\text{gr}(I)$. This characteristic variety if denoted by $\text{Ch}(M)$. The cycle defined in $T^*X$ by the ideal $\mathcal{O}_{T^*X}\text{gr}(I)$ is denoted by $CC\text{Ch}(M)$ and it is called the characteristic cycle of the $\mathcal{D}$-module $M$. 

2
For any divisor $D \subset \mathbb{C}^n$ the sheaf $\mathcal{O}[\star D]$ of meromorphic functions with poles along $D$ is naturally a left coherent $\mathcal{D}$-module (that follows from the results of Bernstein-Björk on the existence of the $b$-function for each local equation $f$ of $D$, [4, 3]). Even more, Kashiwara proved that the dimension of $\text{Ch}(\mathcal{O}[\star D])$ is equal to $n$ (i.e. $\mathcal{O}[\star D]$ is holonomic, [11]).

In [3] and [4] the author considers the (left) ideal $I^{\log}D \subset \mathcal{D}$ generated by the logarithmic vector fields $\text{Der}(\log D)$ (see [4]). We will denote simply $I^{\log} = I^{\log}D$ and $\mathcal{M}^{\log}$ the quotient $\mathcal{D}/I^{\log}$ if no confusion is possible.

### 3.1 Koszul free divisors

Let us give the main result of F.J. Calderón, [3] (see also [4]). Let $D \subset X$ be a divisor and $x \in D$.

**Definition 3.1.1.** ([4, Def. 4.1.1]) The divisor $D$ is said to be Koszul free at the point $x \in D$ if it is free at $x$ and there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $\text{Der}(\log D)_x$ such that the sequence $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ of principal symbols is a regular sequence in the ring $\text{gr}^F(\mathcal{D})$. The divisor $D$ is Koszul free if it is Koszul free at any point of $D$.

By [14] and [4, 4.2.2] any plane curve $D \subset \mathbb{C}^2$ is a Koszul free divisor. By [4, Prop. 4.1.2] if $D$ is a Koszul free divisor then $\mathcal{M}^{\log}$ is holonomic and

**Theorem 3.1.2.** ([4, Th. 4.2.1]) If $D$ is a Koszul free divisor then $\Omega^*(\log D)$ and $\mathbb{R}\text{Hom}_{\mathcal{D}}(\mathcal{M}^{\log}, \mathcal{O})$ are naturally quasi-isomorphic.

### 3.2 $\widetilde{\mathcal{M}}^{\log}$

In [17] (see also [3]) L. Narváez suggested the study of the $\mathcal{D}$-module $\widetilde{\mathcal{M}}^{\log}$ defined as follows: Let us denote by $I^{\log}$ the left ideal of $\mathcal{D}$ generated by the set $\{\delta + a | \delta \in I^{\log} \text{ and } \delta(f) = af\}$. Let us write $\widetilde{\mathcal{M}}^{\log} = \mathcal{D}/I^{\log}$. There exists a natural morphism $\phi_D : \widetilde{\mathcal{M}}^{\log} \to \mathcal{O}[\star D]$ defined by $\phi_D(\mathcal{P}) = P(1/f)$ where $\mathcal{P}$ denotes the class of the operator $P \in \mathcal{D}$ modulo $I^{\log}$. The image of $\phi_D$ is $\mathcal{D}_f^+$. As a natural question we ask for the class of $D$ such that the morphism $\phi_D$ is an isomorphism (see [5, 2]).

### 4 The duality theorem

Suppose here that the divisor $D \subset X$ is free, and let $f \in \mathcal{O}$ be a local equation of $D$ and let $\{\delta_1, \ldots, \delta_n\}$ be a basis of the logarithmic derivations. We will use the following notation:

- $\delta_i(f) = m_i f$ for some $m_i \in \mathcal{O}$.
- $\delta_i = \sum_{k=1}^n a_{ik} \partial_k$ for some $a_{ik} \in \mathcal{O}$.
- $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$
- $[\delta_i, \delta_j] = \sum_{k=1}^n a_{ij}^k \delta_k$ for some $a_{ij}^k \in \mathcal{O}$.

**Lemma 4.1.1.** For any $i = 1, \ldots, n$ we have

\[
\delta_i([A]) = \sum_{k=1}^n (\delta_i(a_{k1}), \ldots, \delta_i(a_{kn})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}
\]

where $A_{kj}$ is the adjoint matrix of $a_{kj}$.
Proof. From the very definition of the determinant developed from the $k$-th row. \[\square\]

The lemma above is true in fact for any derivation, not only for elements in the basis.

Lemma 4.1.2. We have
\[
f(\alpha_{ij}^1, \ldots, \alpha_{ij}^n) = (\delta_i(a_{j1}) - \delta_j(a_{i1}), \ldots, \delta_i(a_{jn}) - \delta_j(a_{in})) Adj(A)^t.
\]

Proof. It is only necessary to consider that
\[
[\delta_i, \delta_j] = (\alpha_{ij}^1, \ldots, \alpha_{ij}^n) \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} =
\]
\[
= (\delta_i(a_{j1}) - \delta_j(a_{i1}), \ldots, \delta_i(a_{jn}) - \delta_j(a_{in})) \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.
\]

\[\square\]

We consider the augmented Spencer logarithmic complex as in [4, page 712]. We have
\[
\mathcal{D} \otimes \mathcal{O} \wedge^\bullet \text{Der}(\log \mathcal{D}) \to M^\log \to 0.
\]

We say that a free divisor $D$ is of Spencer type if this complex is a (locally) free resolution of $M^\log$ and this last $\mathcal{D}$-module is holonomic. By [4, Prop. 4.1.3 ] if $D$ is Koszul free (in particular if $D$ is a plane curve) then it is of Spencer type but the converse is not true, see [4, Remark 4.2.4] and section 5.3.

The following proposition is a consequence of [4, Th. 4.2.1].

Proposition 4.1.3. If $D$ is of Spencer type then $\text{Sol}(M^\log)$ is naturally quasi-isomorphic to $\Omega^\bullet(\log \mathcal{D})$.

Theorem 4.1.4. Suppose $D$ is of Spencer type. Then $(M^\log)^* \simeq \tilde{M}^\log$.

Proof. Using the Spencer logarithmic free resolution of the holonomic $\mathcal{D}$-module $M^\log$, we first compute a presentation of the right $\mathcal{D}$-module $E := \text{Ext}_D^n(M^\log, \mathcal{D})$ and then we prove that left $\mathcal{D}$-module associated to $E$ is $\tilde{M}^\log$.

The matrix of the $n$-th morphism in the resolution of $M^\log$ (see [4, page 712]) has components of the form
\[
(-1)^{i-1}\delta_i + (-1)^i \sum_{l \neq i} \alpha_{il}^j,
\]
so it is enough to prove that
\[
(-\delta_i + \sum_{k \neq i} \alpha_{ik}^k)^* = \delta_i + \sum_{k=1}^n \partial_k(a_{ik}) + \sum_{k \neq i} \alpha_{ik}^k = \delta_i + m_i.
\]

In order to prove the last equality, we will show that
\[
m_i f = \delta_i(f) = \delta_1(|A|) =
\]
\[
= \sum_{k=1}^n f \partial_k(a_{ik}) + \sum_{k \neq i} f \alpha_{ik}^k.
\]
Using 4.1.2, we obtain

\[
\sum_{k \neq i} f \alpha_k^i = \sum_{k \neq i} (\delta_i(a_{k1}) - \delta_k(a_{i1}), ..., \delta_i(a_{kn}) - \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} = \\
= \sum_{k=1}^{n} (\delta_i(a_{k1}), ..., \delta_i(a_{kn})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} - \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}
\]

So we have collected in the first sum precisely (see 4.1.1) \( \delta_i(\|A\|)\). It remains to check that

\[
\sum_{k=1}^{n} f \partial_k(a_{ik}) = \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}.
\]

As \( f = (a_{k1}, ..., a_{kn})(A_{k1}, ..., A_{kn})^t \), we have

\[
= \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} \partial_j(a_{i1}), ..., \sum_{j=1}^{n} a_{kj} \partial_j(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix} = \\
= \sum_{k=1}^{n} (\delta_k(a_{i1}), ..., \delta_k(a_{in})) \begin{pmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{pmatrix}.
\]

\]

5 Some applications

5.1 LCT in dimension 2. Regularity of \( M^{\log} \) and \( \tilde{M}^{\log} \)

Let \( D \subset \mathbb{C}^2 \) be a plane curve.

**Theorem 5.1.1.** (\( \mathbb{F} \), Theorem 3.9) The morphism \( i_D : \Omega^* (\log D) \to \Omega^*(\star D) \) is a quasi-isomorphism if and only if \( D \) is locally quasi-homogeneous.

**Proof.** We show here how to read the original (topological) proof of \( \mathbb{F} \) to give a differential proof of “only if” part. Part “if” is a consequence of \( \mathbb{F} \) because any plane curve is a free divisor \( \mathbb{F} \).

The problem is local. Suppose the local equation \( f \) of \( D \) is defined in a small open neighbourhood such that the only singular point of \( f = 0 \) is the origin. Denote \( \mathcal{O}[1/f] = \mathcal{O}[\star D] \).

Let us consider (see 3.2) the natural surjective morphism

\[
\phi_D : \tilde{M}^{\log} \to \mathcal{D} \frac{1}{f} \simeq \mathcal{O}[\star D]
\]
where the last isomorphism follows by a result of Varchenko (i.e. the local b-function $b_f(s)$ of f verifies $b_f(-k) \neq 0$ for any integer $k \geq 2$, [18]). The kernel $K$ of $\phi_D$ is supported by the origin (because f is smooth outside $(0,0)$) and $CCh(M^{\log}) = CCh(K) + CCh(\mathcal{O}[*D])$. In particular $\widetilde{M}^{\log}$ and $M^{\log} = (\widetilde{M}^{\log})^*$ are regular holonomic (cf. [12]) because as we said before $D$ satisfies the hypothesis of theorem 1.1.4.

Let us denote $Sol(M^{\log}) = \mathbf{R}\mathcal{H}om_D(M^{\log}, \mathcal{O})$ the solution complex of $M^{\log}$.

Assume LCT holds for $D$. Then we have

$$DR(\mathcal{O}[*D]) \simeq \Omega^{\bullet}(D) \simeq \Omega^{\bullet}((log D) \simeq Sol(M^{\log}) \simeq DR((M^{\log})^*) \simeq DR(\widetilde{M}^{\log}).$$

Then both $\mathcal{D}$-modules $\mathcal{O}[*D]$ and $\widetilde{M}^{\log}$ have the same de Rham complex and then the same characteristic cycle. In this case $K = 0$ and $\widetilde{M}^{\log} \simeq \mathcal{O}[*D]$. Finally, by [16], page 88 (or by [14], 2.2.6), see also [3]) $f$ is weighted homogeneous in suitable coordinates. That proves the “only if” part of the theorem.

5.2 On the comparison of $\widetilde{M}^{\log}$ and $\mathcal{O}[*D]$

In the previous section we proved (in dimension 2) that if $\widetilde{M}^{\log} \simeq \mathcal{O}[*D]$ then $f$ is weighted homogeneous and the converse is also true (cf. [3]). So, $\widetilde{M}^{\log} \simeq \mathcal{O}[*D]$ if and only if $f$ is weighted homogeneous if and only if LCT holds for $D$.

Now we return to dimension $n$.

**Theorem 5.2.1.** If $D \subset \mathbb{C}^n$ is a free, locally quasi-homogeneous (l.q-h.) divisor then $M^{\log}$ and $\widetilde{M}^{\log}$ are regular holonomic. Moreover $\widetilde{M}^{\log,D}$ and $\mathcal{O}[*D]$ are naturally isomorphic.

**Proof.** By [3] $D$ is Koszul free and then $M^{\log}$ is holonomic and the dual of $M^{\log}$ is $\widetilde{M}^{\log}$ (see 1.1.4). So, $M^{\log}$ is also holonomic. It is enough to prove that $\widetilde{M}^{\log}$ is regular.

To avoid confusion we will denote $\widetilde{M}^{\log,D}$ to emphasize the divisor $D$. In fact we will prove, by induction on $n$, that the natural morphism

$$\phi_D : \widetilde{M}^{\log,D} \rightarrow \mathcal{O}[*D]$$

is an isomorphism. We follows here the argument of [3] 4.3]. There is nothing to prove in the case $n = 1$. We note that in dimension 2 the result is proved in [17] (see 5.1). Suppose the result is true for any free, l.q-h. divisor in dimension $\leq n - 1$. Let $D \subset \mathbb{C}^n$ be a free, l.q-h. For any $x \in D$ there exists an open neighbourhood $U$ of $x$ such that for any $y \in U \cap D \setminus \{x\}$ the germ $(\mathbb{C}^n, D, y)$ is isomorphic to $(\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0,0))$, where $D'$ is a free, l.q-h. divisor in $\mathbb{C}^{n-1}$ (see [3] prop. 2.4, lemma 2.2). So, by induction hypothesis the morphism $\phi_D : \widetilde{M}^{\log,D'} \rightarrow \mathcal{O}[*D']$ is an isomorphism. Then, by applying the functor $\phi^*$ (where $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ is the projection), we have that for any $y \in U \cap D$, $y \neq x$, the morphism $\phi_{D,y} : \widetilde{M}_y^{\log,D'} \rightarrow \mathcal{O}[*D']_y$. We owe this argument to L. Narváez. So, the kernel of $\phi_D : \widetilde{M}^{\log,D} \rightarrow N^{\log,D}$ is concentrated on a discrete set and it is regular holonomic (here $N^{\log,D}$ is the $\mathcal{D}$-module $D_{\frac{1}{n}}(k)$, where $k$ is a local equation of $D$). As $N^{\log,D} \subset \mathcal{O}[*D]$ is regular holonomic we deduce the regularity of $\widetilde{M}^{\log,D}$. On the other hand, by [3] the logarithmic comparison theorem holds for $D$. So, by using duality 1.1.4 and the natural quasi-isomorphism $Sol(M^{\log,D}) \rightarrow \Omega^{\bullet}(log D)$ (3.1.3), we deduce (as in 5.1) that $DR(\widetilde{M}^{\log,D})$ and $DR(\mathcal{O}[*D])$ are naturally quasi-isomorphic and therefore, by Riemann-Hilbert correspondence, $\widetilde{M}^{\log,D}$ and $\mathcal{O}[*D]$ are naturally isomorphic, i.e. $\phi_D$ is an isomorphism. Thus we have concluded the induction. \qed
5.3 An example in dimension 3

In [10] the authors give an example of a non Koszul free divisor in dimension 3 for which LCT holds. We will treat here, following the same lines as in [10], the case of the surface \( D \subset \mathbb{C}^3 \) defined by \( f = y(x^2 + y)(x^2 z + y) = 0 \).

The surface is free because computing the syzygies among \( f, f_x, f_y, f_z \) we obtain

\[
(-3, \frac{1}{2} x, y, 0) \\
(-x^2, 0, 0, x^2 z + y) \\
(-xz - x, \frac{1}{2} x^2 + \frac{1}{2} y, 0, x z^2 - xz),
\]

which produce the logarithmic vector fields

\[
\delta_1 = \frac{1}{2} x \partial_x + y \partial_y \\
\delta_2 = (x^2 z + y) \partial_z \\
\delta_3 = (\frac{1}{2} x^2 + \frac{1}{2} y) \partial_x + (x z^2 - xz) \partial_z,
\]

whose coefficients have a determinant equal to \( 1/2f \).

This surface is not Koszul-free because the set of the symbols (with respect to the total order) \( \sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3) \) do not form a regular sequence. If we write \( \sigma(\partial_x) = \xi, \sigma(\partial_y) = \eta, \sigma(\partial_z) = \zeta \). We have \( y z \eta^2 \zeta + \frac{1}{4} \xi^2 \zeta \notin (\sigma(\delta_1), \sigma(\delta_2)) \) but \( y z \eta^2 \zeta + \frac{1}{4} \xi^2 \zeta \sigma(\delta_3) \in (\sigma(\delta_1), \sigma(\delta_2)) \).

We compute a free resolution of \( M_{\log} = D/\langle \partial(\delta_1, \delta_2, \delta_3) \rangle \) using Gröbner basis. We obtain that the module of syzygies \( Syz(\delta_1, \delta_2, \delta_3) \) is generated by \( s_{12}, s_{13}, s_{23} \) deduced from the commutators \( [\delta_i, \delta_j] \) where:

- \( [\delta_1, \delta_2] = \delta_2 \)
- \( [\delta_1, \delta_3] = \frac{1}{2} \delta_3 \)
- \( [\delta_1, \delta_2] = (xz - x) \delta_2 \).

The second module of syzygies (among the \( s_{ij} \)) it is generated by only one element \( t \), so we have finished the resolution. This element is

\[
t = (t_1, t_2, t_3) = (-x^2 z \partial_z + x z \partial_z - \frac{1}{2} x^2 \partial_x - \frac{1}{2} y \partial_x - xz + x, x^2 z \partial_z + y \partial_z, -y \partial_y - \frac{1}{2} x \partial_x + \frac{3}{2}),
\]

precisely the one required in the Spencer logarithmic complex for \( M_{\log} \) in dimension 3, which is in fact a free resolution of \( M_{\log} \). We can check that in this case \( M_{\log} \) is holonomic (use for example [12] or [13], so \( D \) is of Spencer type and we apply:

a) The theorem 4.1.4 to obtain that \( (M_{\log})^* \simeq \tilde{M}_{\log} \).

b) Proposition 4.1.3 to obtain \( Sol(M_{\log}) \simeq \Omega^*(\log D) \).

Besides, the global \( b \)-function of \( f \) is

\[
(6s + 5)(3s + 2)(2s + 1)(3s + 4)(6s + 7)(s + 1)^3,
\]

so we can assure that \( O(\frac{1}{f}) \simeq \tilde{M}_{\log} \), because \( \tilde{M}_{\log} = Ann_D(1/f) \). We have used [10] and [13] again to compute the \( b \)-function and the annihilating ideal of \( 1/f \), that is to say, the algorithms of [13].

Finally, we have the following chain of quasi-isomorphisms

\[
DR(O[\ast D]) \simeq \Omega^*(\ast D) \simeq \Omega^*(\log D) \simeq Sol(M_{\log}) \simeq DR((M_{\log})^*) \simeq DR(\tilde{M}_{\log}),
\]

and the LCT holds for \( D \).
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