Abstract

A general theorem is rigorously proved for the case, when an observable is a sum of linearly independent terms: The dispersion of a global observable is normal if and only if all partial dispersions of its terms are normal, and it is anomalous if and only if at least one of the partial dispersions is anomalous. This theorem, in particular, rules out the possibility that in a stable system with Bose-Einstein condensate some fluctuations of either condensed or non-condensed particles could be anomalous. The conclusion is valid for arbitrary systems, whether uniform or nonuniform, interacting weakly or strongly. The origin of fictitious fluctuation anomalies, arising in some calculations, is elucidated.

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The problem of fluctuations of observable quantities is among the most important questions in statistical mechanics, being related to the fundamentals of the latter. A great revival, in recent years, of interest to this problem is caused by intensive experimental and theoretical studies of Bose-Einstein condensation in dilute atomic gases (see e.g. reviews [1–3]). The fact that the ideal uniform Bose gas possesses anomalously large number-of-particle fluctuations has been known long ago [4,5], which has not been of much surprise, since such an ideal gas is an unrealistic and unstable system. However, as has been recently suggested in many papers, similar anomalous fluctuations could appear in real interacting Bose systems. A number of recent publications has addressed the problem of fluctuations in Bose gas, proclaiming controversial statements of either the existence or absence of anomalous fluctuations (see discussion in review [6]). So that the issue has not been finally resolved. In the present paper, the problem of fluctuations is considered from the general point of view, independent of particular models or calculational methods. A general theorem is rigorously proved, from which it follows that there are no anomalous fluctuations in any stable equilibrium systems.

It is worth stressing that no phase transitions are considered in this paper. As can be easily inferred from any textbook on thermodynamics or statistical mechanics, the points of phase transitions are, by definition, the points of instability. A phase transition occurs exactly because one phase becomes unstable and has to change to another stable phase. It is well known that at the points of second-order phase transitions fluctuations do become anomalous, yielding divergent susceptibilities, as it should be at the points of instability. After a phase transition has occurred, the system, as is also well known, becomes stable and susceptibilities go finite. However, in many papers on Bose systems, the claims are made that fluctuations remain anomalous far below the condensation point, in the whole region of the Bose-condensed system. As is shown below, these claims are incorrect, since such a system with anomalous fluctuations possesses a divergent compressibility, thus, being unstable.

Observable quantities are represented by Hermitian operators from the algebra of observables. Let $\hat{A}$ be an operator from this algebra. Fluctuations of the related observable quantity are quantified by the dispersion

$$\Delta^2(\hat{A}) \equiv <\hat{A}^2> - <\hat{A}>^2,$$

where $<\ldots>$ implies equilibrium statistical averaging. The dispersion itself can be treated as an observable quantity, which is the average of an operator $(\hat{A} - <\hat{A}>)^2$, since each dispersion is directly linked to a measurable quantity. For instance, the dispersion for the number-of-particle operator $\hat{N}$ defines the isothermal compressibility

$$\kappa_T \equiv -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{\Delta^2(\hat{N})}{N\rho k_B T},$$

in which $P$ is pressure, $\rho \equiv N/V$ is density, $N = <\hat{N}>$, $V$ is volume, and $T$ temperature. The dispersion $\Delta^2(\hat{N})$ is also connected with the sound velocity $s$ through the equation

$$s^2 \equiv \frac{1}{m} \left( \frac{\partial P}{\partial \rho} \right)_T = \frac{1}{m\rho \kappa_T} = \frac{N k_B T}{m \Delta^2(\hat{N})},$$
where $m$ is particle mass, $P$ pressure, and with the central value

$$
S(0) = \rho k_B T \kappa T = \frac{k_B T}{m s^2} = \frac{\Delta^2(\hat{N})}{N}
$$

(4)
of the structural factor

$$
S(k) = 1 + \rho \int [g(r) - 1] e^{-ik \cdot r} dr,
$$

where $g(r)$ is the pair correlation function. The fluctuations of the Hamiltonian $\hat{H}$ characterize the specific heat

$$
C_V \equiv \frac{1}{N} \left( \frac{\partial E}{\partial T} \right)_V = \frac{\Delta^2(\hat{H})}{N k_B T^2},
$$

(5)

where $E \equiv < \hat{H} >$ is internal energy. In magnetic systems, with the Zeeman interaction $-\mu_0 \sum_i \mathbf{B} \cdot \mathbf{S}_i$, the longitudinal susceptibility

$$
\chi_{\alpha\alpha} \equiv \frac{1}{N} \left( \frac{\partial M_\alpha}{\partial B_\alpha} \right) = \frac{\Delta^2(\hat{M}_\alpha)}{N k_B T},
$$

(6)
is related to the fluctuations of the magnetization $M_\alpha \equiv < \hat{M}_\alpha >$, where $\hat{M}_\alpha \equiv \mu_0 \sum_{i=1}^{N} S^\alpha_i$.

All relations (2) to (6) are exact and hold true for any equilibrium system. The stability conditions for such systems require that at any finite temperature, except the points of phase transitions, quantities (2) to (6) be positive and finite for all $N$, including the thermodynamic limit, when $N \to \infty$. At the phase transition points, these quantities can of course be divergent, since, as is well known, the phase transition points are the points of instability. Summarizing this, we may write the general form of the necessary stability condition as

$$
0 < \frac{\Delta^2(\hat{A})}{N} < \infty,
$$

(7)

which must hold for any stable equilibrium systems at finite temperature and for all $N$, including the limit $N \to \infty$. The value $\Delta^2(\hat{A})/N$ can become zero only at $T = 0$. Condition (7) is nothing but a representation of the well known fact that the susceptibilities in stable systems are positive and finite.

The stability condition (7) shows that the dispersion $\Delta^2(\hat{A})$ has to be of order $N$. When $\Delta^2(\hat{A}) \sim N$, one says that the dispersion is normal and the fluctuations of an observable $\hat{A}$ are normal, since then the stability condition (7) is preserved. But when $\Delta^2(\hat{A}) \sim N^\alpha$, with $\alpha > 1$, then such a dispersion is called anomalous and the fluctuations of $\hat{A}$ are anomalous, since then $\Delta^2(\hat{A})/N \sim N^{\alpha-1} \to \infty$ as $N \to \infty$, hence the stability condition (7) becomes broken. A system with anomalous fluctuations is unstable. For example, an ideal uniform Bose gas, with $\Delta^2(\hat{N}) \sim N^2$, is unstable [4–6].

Thus, in any stable equilibrium system, the fluctuations of global observables must be normal, $\Delta^2(\hat{A}) \sim N$. The situation becomes more involved, if the operator of an observable is represented by a sum

$$
\hat{A} = \sum_i \hat{A}_i,
$$

(8)
of Hermitian terms \( \hat{A}_i \), and one is interested in the fluctuations of the latter. Then the intriguing question is: Could some partial dispersions \( \Delta^2(\hat{A}_i) \) be anomalous, while the total dispersion \( \Delta^2(\hat{A}) \) remaining normal? Exactly such a situation concerns the systems with Bose condensate. Then the total number of particles \( N = N_0 + N_1 \) is a sum of the numbers of condensed, \( N_0 \), and noncondensed, \( N_1 \), particles. We know that for a stable system \( \Delta^2(\hat{N}) \) must be normal. But could it happen that at the same time either \( \Delta^2(\hat{N}_0) \) or \( \Delta^2(\hat{N}_1) \), or both would be anomalous, as is claimed by many authors?

Considering the sum (8), it is meaningful to keep in mind a nontrivial case, when all \( \hat{A}_i \) are linearly independent. In the opposite case of linearly dependent terms, one could simply express one of them through the others and reduce the number of terms in sum (8). The consideration also trivializes if some of \( \hat{A}_i \) are \( c \)-numbers, since then \( \Delta^2(c) = 0 \).

The dispersion of operator (8) reads as

\[
\Delta^2(\hat{A}) = \sum_i \Delta^2(\hat{A}_i) + 2 \sum_{i<j} \text{cov}(\hat{A}_i, \hat{A}_j),
\]

where the covariance

\[
\text{cov}(\hat{A}_i, \hat{A}_j) \equiv \frac{1}{2} < \hat{A}_i \hat{A}_j + \hat{A}_j \hat{A}_i > - < \hat{A}_i > < \hat{A}_j >
\]

is introduced. The latter is symmetric, \( \text{cov}(\hat{A}_i, \hat{A}_j) = \text{cov}(\hat{A}_j, \hat{A}_i) \). The dispersions are, by definition, non-negative, but the covariances can be positive as well as negative. One might think that an anomalous partial dispersion \( \Delta^2(\hat{A}_i) \) could be compensated by some covariances, so that the total dispersion \( \Delta^2(\hat{A}) \) would remain normal. This is just the way of thinking when one finds an anomalous dispersion of condensed, \( \Delta^2(\hat{N}_0) \), or noncondensed, \( \Delta^2(\hat{N}_1) \), particles, presuming that the system as a whole could remain stable, with the normal total dispersion \( \Delta^2(\hat{N}) \). However, the following theorem rules out such hopes.

**Theorem.** The total dispersion (9) of an operator (8), composed of linearly independent Hermitian operators, is anomalous if and only if at least one of the partial dispersions is anomalous, with the power of the total dispersion defined by that of its largest partial dispersion. Conversely, the total dispersion is normal if and only if all partial dispersions \( \Delta^2(\hat{A}_i) \) are normal.

**Proof.** First of all, we notice that it is sufficient to prove the theorem for the sum of two operators, for which

\[
\Delta^2(\hat{A}_i + \hat{A}_j) = \Delta^2(\hat{A}_i) + \Delta^2(\hat{A}_j) + 2 \text{cov}(\hat{A}_i, \hat{A}_j),
\]

where \( i \neq j \). This is because any sum of terms more than two can always be represented as a sum of two new terms. Also, we assume that both operators in Eq. (10) are really operators but not \( c \)-numbers, since if at least one of them, say \( \hat{A}_j = c \), is a \( c \)-number, then Eq. (10) reduces to a simple equality \( \Delta^2(\hat{A}_i + c) = \Delta^2(\hat{A}_i) \) of positive (or semipositive) quantities, both of which simultaneously are either normal or anomalous.

Introduce the notation

\[
\sigma_{ij} \equiv \text{cov}(\hat{A}_i, \hat{A}_j).
\]

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As is evident, \( \sigma_{ii} = \Delta^2(\hat{A}_i) \geq 0 \) and \( \sigma_{ij} = \sigma_{ji} \). The set of elements \( \sigma_{ij} \) forms the covariance matrix \([\sigma_{ij}]\), which is a symmetric matrix. For a set of arbitrary real-valued numbers, \( x_i \), with \( i = 1, 2, \ldots, n \), where \( n \) is an integer, one has

\[
< \sum_{i=1}^{n} \left( \hat{A}_i - <\hat{A}_i> \right) x_i >^2 = \sum_{ij} \sigma_{ij} x_i x_j \geq 0 .
\]

(12)

The right-hand side of equality (12) is a semipositive quadratic form. From the theory of quadratic forms [7] one knows that a quadratic form is semipositive if and only if all principal minors of its coefficient matrix are non-negative. Thus, the sequential principal minors of the covariance matrix \([\sigma_{ij}]\), with \( i, j = 1, 2, \ldots, n \), are all non-negative. In particular, \( \sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji} \geq 0 \). This, owing to the symmetry \( \sigma_{ij} = \sigma_{ji} \), transforms to the inequality \( \sigma_{ij}^2 \leq \sigma_{ii} \sigma_{jj} \). Then the correlation coefficient

\[
\lambda_{ij} \equiv \sigma_{ij} / \sqrt{\sigma_{ii} \sigma_{jj}}
\]

possesses the property \( \lambda_{ij}^2 \leq 1 \).

The equality \( \lambda_{ij}^2 = 1 \) holds true if and only if \( \hat{A}_i \) and \( \hat{A}_j \) are linearly dependent. The sufficient condition is straightforward, since if \( \hat{A}_j = a + b \hat{A}_i \), where \( a \) and \( b \) are any real numbers, then \( \sigma_{ij} = b \sigma_{ii} \) and \( \sigma_{jj} = b^2 \sigma_{ii} \), hence \( \lambda_{ij} = b / |b| \), from where \( \lambda_{ij}^2 = 1 \). To prove the necessary condition, assume that \( \lambda_{ij}^2 = 1 \). This implies that \( \lambda_{ij} = \pm 1 \). Consider the dispersion

\[
\Delta^2 \left( \frac{\hat{A}_i}{\sqrt{\sigma_{ii}}} \pm \frac{\hat{A}_j}{\sqrt{\sigma_{jj}}} \right) = 2(1 \pm \lambda_{ij}) \geq 0 .
\]

The value \( \lambda_{ij} = 1 \) is possible then and only then, when

\[
\Delta^2 \left( \frac{\hat{A}_i}{\sqrt{\sigma_{ii}}} - \frac{\hat{A}_j}{\sqrt{\sigma_{jj}}} \right) = 0 .
\]

The dispersion can be zero if and only if

\[
\frac{\hat{A}_i}{\sqrt{\sigma_{ii}}} - \frac{\hat{A}_j}{\sqrt{\sigma_{jj}}} = \text{const} ,
\]

that is, the operators \( \hat{A}_i \) and \( \hat{A}_j \) are linearly dependent. Similarly, the value \( \lambda_{ij} = -1 \) is possible if and only if

\[
\frac{\hat{A}_i}{\sqrt{\sigma_{ii}}} + \frac{\hat{A}_j}{\sqrt{\sigma_{jj}}} = \text{const} ,
\]

which again means the linear dependence of the operators \( \hat{A}_i \) and \( \hat{A}_j \). As far as these operators are assumed to be linearly independent, one has \( \lambda_{ij}^2 < 1 \). The latter inequality is equivalent to \( \sigma_{ij}^2 \leq \sigma_{ii} \sigma_{jj} \), which, in agreement with notation (11), gives

\[
|\text{cov}(\hat{A}_i, \hat{A}_j)|^2 < \Delta^2(\hat{A}_i) \Delta^2(\hat{A}_j) .
\]

(14)

The main relation (10) can be written as
\[ \Delta^2(\hat{A}_i + \hat{A}_j) = \sigma_{ii} + \sigma_{jj} + 2\lambda_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}, \]  

(15)

where, as is shown above, \(|\lambda_{ij}| < 1\). Altogether there can occur no more than four following cases. First, when both partial dispersions \(\sigma_{ii} = \Delta^2(\hat{A}_i)\) and \(\sigma_{jj} = \Delta^2(\hat{A}_j)\) are normal, so that \(\sigma_{ii} \sim N\) and \(\sigma_{jj} \sim N\). Then from Eq. (15) it is evident that the total dispersion \(\Delta^2(\hat{A}_i + \hat{A}_j) \sim N\) is also normal. Second, one of the partial dispersions, say \(\sigma_{ii} \sim N\), is normal, but another is anomalous, \(\sigma_{jj} \sim N_\alpha\), with \(\alpha > 1\). From Eq. (15), because of \((1 + \alpha)/2 < \alpha\), one has \(\Delta^2(\hat{A}_i + \hat{A}_j) \sim N_\alpha\), so that the total dispersion is anomalous, having the same power \(\alpha\) as \(\sigma_{jj}\). Third, both partial dispersions are anomalous, \(\sigma_{ii} \sim N^{\alpha_1}\) and \(\sigma_{jj} \sim N^{\alpha_2}\), with different powers, say \(1 < \alpha_1 < \alpha_2\). From Eq. (15), taking into account that \((\alpha_1 + \alpha_2)/2 < \alpha_2\), we get \(\Delta^2(\hat{A}_i + \hat{A}_j) \sim N^{\alpha_2}\), that is, the total dispersion is also anomalous, with the same power \(\alpha_2\) as the largest partial dispersion \(\sigma_{jj}\). Fourth, both partial dispersions are anomalous, \(\sigma_{ii} = c_i^2 N^\alpha\) and \(\sigma_{jj} = c_j^2 N^\alpha\), where \(c_i > 0\) and \(c_j > 0\), with the same power \(\alpha\). Then Eq. (15) yields

\[
\Delta^2(\hat{A}_i + \hat{A}_j) = c_{ij} N^\alpha
\]

with

\[
c_{ij} = (c_i - c_j)^2 + 2c_i c_j (1 + \lambda_{ij}) > 0,
\]

which is strictly positive in view of the inequality \(|\lambda_{ij}| < 1\). Hence, the total dispersion is anomalous, with the same power \(\alpha\) as both partial dispersions. After listing all admissible cases, we see that the total dispersion \(\Delta^2(\hat{A}_i + \hat{A}_j)\) is anomalous if and only if at least one of the partial dispersions is anomalous, with the power of \(N\) of the total dispersion being equal to the largest power of partial dispersions. Oppositely, the total dispersion is normal if and only if all partial dispersions are normal. This concludes the proof of the theorem.

As an example, let us consider a Bose system with Bose-Einstein condensate, whose total number-of-particle operator \(\hat{N} = \hat{N}_0 + \hat{N}_1\) consists of two terms, corresponding to condensed, \(\hat{N}_0\), and noncondensed, \(\hat{N}_1\), particles. Since for a stable system the dispersion \(\Delta^2(\hat{N})\) is normal, then from the above theorem it follows that both dispersions \(\Delta^2(\hat{N}_0)\) as well as \(\Delta^2(\hat{N}_1)\) must be normal. No anomalous fluctuations can exist in a stable system, neither for condensed nor for noncondensed particles. This concerns any type of stable systems, either uniform or nonuniform. And this result does not depend on the method of calculations, provided the latter are correct.

How then could one explain the appearance of numerous papers claiming the existence of anomalous fluctuations in Bose-condensed systems in the whole region far below the critical point? If these anomalous fluctuations would really exist, then the compressibility would be divergent everywhere below the critical point. A system, whose compressibility is divergent everywhere in its region of existence, as is known from any textbook on statistical mechanics, is unstable. Such anomalous fluctuations are usually obtained as follows. One considers a low-temperature dilute Bose gas, at \(T \ll T_c\), when the Bogolubov theory [8] is applicable. In the frame of this theory, one calculates the dispersion \(\Delta^2(\hat{N}_1)\) of noncondensed particles, where \(\hat{N}_1 = \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k\). To find \(< \hat{N}_1^2 >\), one needs to work out the four-operator expression \(< \hat{a}^\dagger_k \hat{a}_k \hat{a}^\dagger_q \hat{a}_q >\), or after employing the Bogolubov canonical transformation \(\hat{a}_k = u_k \hat{b}_k + v_k \hat{b}^\dagger_{-k}\), to consider \(< \hat{b}^\dagger_k \hat{b}_q \hat{b}^\dagger_q \hat{b}_q >\). Such four-operator expressions are treated by invoking the Wick decoupling. Then one finds that the dispersion \(\Delta^2(\hat{N}_1)\) diverges as \(N \int dk/k^2\).
Discretizing the phonon spectrum, one gets $\Delta^2(\hat{N}_1) \sim N^{4/3}$. Another way [6] could be by limiting the integration by the minimal $k_{\text{min}} = 1/L$, where $L \sim N^{1/3}$ is the system length. Then again $\Delta^2(\hat{N}_1) \sim N^{4/3}$. In any case, one obtains anomalous fluctuations of noncondensed particles. This result holds true for both canonical and grand canonical ensembles, which is a direct consequence of the Bogolubov theory [8]. But the anomalous behaviour of $\Delta^2(\hat{N}_1)$, according to the theorem proved above, immediately leads to the same anomalous behaviour of the total dispersion $\Delta^2(\hat{N})$, which would imply the instability of the system, since then compressibility (2) and structural factor (4) become divergent. As far as the system is assumed to be stable, there should be something wrong in such calculations.

The drawback of these calculations is in the following. One of the basic points of the Bogolubov theory is in omitting in the Hamiltonian all terms of orders higher than two with respect to the operators $a_k$ of noncondensed particles. This is a second-order theory with respect to $a_k$. It means that the same procedure of keeping only the terms of second order, but ignoring all higher-order terms, must be done in calculating any physical quantities. In considering $<\hat{N}_1^2>$, one meets the fourth-order terms with respect to $a_k$. Such fourth-order terms are not defined in the Bogolubov theory. The calculation of the fourth-order products in the second-order theory is not self-consistent. This inconsistency leads to incorrect results.

A correct calculations of $\Delta^2(\hat{N})$ in the frame of the Bogolubov theory can be done in the following way [9]. Writing down the pair correlation function, one should retain there only the terms not higher than of the second order with respect to $a_k$, omitting all higher-order terms. Then for a uniform system one gets

$$g(r) = 1 + \frac{2}{\rho} \int \left( <a_k^+ a_k> + <a_k a_{-k}> \right) e^{i k \cdot r} \frac{d k}{(2\pi)^3}.$$ 

This gives us the structural factor $S(0) = k_B T/m c^2$, $c \equiv \sqrt{(\rho/m) \Phi_0}$, $\Phi_0 = \int \Phi(r) d r$, where $\Phi(r)$ is an interaction potential. Because of the exact relation $\Delta^2(\hat{N}) = N S(0)$, we find the dispersion $\Delta^2(\hat{N}) = N k_B T/m c^2$, which is, of course, normal, as it should be for a stable system. And since the total dispersion $\Delta^2(\hat{N})$ is normal, the theorem tells us that both the partial dispersions, $\Delta^2(\hat{N}_0)$ as well as $\Delta^2(\hat{N}_1)$, must also be normal. Note that if in defining the pair correlation function $g(r)$ one would retain the higher-order terms, one would again get the anomalous total dispersion $\Delta^2(\hat{N})$.

It is easy to show that the same type of fictitious anomalous fluctuations appear for arbitrary systems, if one uses the second-order approximation for the Hamiltonian but intends to calculate fourth-order expressions. This is immediately evident from the analysis of susceptibilities for arbitrary systems with continuous symmetry, as is done by Patashinsky and Pokrovsky [10] (Chapter IV), when the system Hamiltonian is restricted to hydrodynamic approximation. Following Ref. [10], one may consider an operator $\hat{A} = \hat{A}(\varphi)$ being a functional of a field $\varphi$. Let this operator be represented as a sum $\hat{A} = \hat{A}_0 + \hat{A}_1$, in which the first term is quadratic in the field $\varphi$, so that $\hat{A}_0 \sim \varphi^+ \varphi$, while the second term depends on the field fluctuations $\delta \varphi$ as $\hat{A}_1 \sim \delta \varphi^+ \delta \varphi$. Keeping in the Hamiltonian only the second-order field fluctuations is equivalent to the hydrodynamic approximation. The dispersion $\Delta^2(\hat{A}) \sim N \chi$ is proportional to a longitudinal susceptibility $\chi$. The latter is given by the integral $\int C(r) d r$ over the correlation function $C(r) = g(r) - 1$, where $g(r)$ is a pair correlation function. Calculating $\Delta^2(\hat{A})$, one meets the fourth-order term $<\delta \varphi^+ \delta \varphi \delta \varphi^+ \delta \varphi>$. If this is treated by invoking the Wick decoupling and the quadratic hydrodynamic Hamiltonian, one gets
\[ C(\mathbf{r}) \sim 1/r^{2(d-2)} \] for any dimensionality \( d > 2 \). Consequently, \( \chi \sim \int C(\mathbf{r}) d\mathbf{r} \sim N^{(d-2)/3} \) for \( 2 < d < 4 \), and the dispersion \( \Delta^2(\hat{A}) \sim N\chi \sim N^{(d+1)/3} \). For \( d = 3 \), this results in the anomalous dispersion \( \Delta^2(\hat{A}) \sim N^{4/3} \). If this would be correct, this would mean, according to the necessary stability condition (7), that the system is unstable. That is, there could not exist any stable systems with continuous symmetry, such as magnetic systems or liquid helium. Of course, we know that such systems perfectly exist, but the above contradiction has arisen solely due to an inconsistent calculational procedure, when the fourth-order term \( < \delta \varphi^+ \delta \varphi \delta \varphi^+ \delta \varphi > \) was treated in the frame of the hydrodynamic approximation, which is a second-order theory with respect to \( \delta \varphi \).

Concluding, there are no anomalous fluctuations of any physical quantities in arbitrary stable equilibrium systems. Fictitious anomalous fluctuations in realistic systems far outside any points of phase transitions might appear only due to drawbacks in a calculational procedure. The absence of anomalous fluctuations follows from the general theorem, rigorously proved in this paper.

It is important to stress that not only the anomalous fluctuations as such signify the occurrence of instability, though they are the explicit signals of the latter. But also one should not forget that the dispersions for the operators of observables are directly related to the corresponding susceptibilities, as in Eqs. (2) to (6). It is the wrong behaviour of these susceptibilities, which manifests the instability.

Thus, the anomalous number-of-particle fluctuations are characterized by the anomalous dispersion \( \Delta^2(\hat{N}) \). The latter is connected, through the general and exact relation (2), with the isothermal compressibility \( \kappa_T \). The anomalous dispersion \( \Delta^2(\hat{N}) \) implies the divergence of this compressibility. From the definition of the compressibility \( \kappa_T \equiv -(1/V)(\partial V/\partial P) \), it is obvious that its divergence means the following: An infinitesimally small positive fluctuation of pressure abruptly squeezes the system volume to a point. Respectively, an infinitesimally small negative fluctuation of pressure suddenly expands the system volume to infinity. It is more than evident that a system which is unstable with respect to infinitesimally small fluctuations of pressure, immediately collapsing or blowing up, has to be termed unstable.

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