STATIC PROPERTIES OF QUARK SOLITONS

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Abstract

It has been conjectured that at distances smaller than the confinement scale but large enough to allow for nonperturbative effects, QCD is described by an effective $SU(N_c \times N_f)_L \times SU(N_c \times N_f)_R$ chiral Lagrangian. The soliton solutions of such a Lagrangian are extended objects with spin $\frac{1}{2}$. For $N_c=3$, $N_f=3$ they are triplets of color and flavor and have baryon number $\frac{1}{3}$, to be identified as constituent quarks. We investigate in detail the static properties of such constituent-quark solitons for the simplest case $N_f=1$, $N_c=3$. The mass of these objects comes from the energy of the static soliton and from quantum effects, described semiclassically by rotation of collective coordinates around the classical solution. The quantum corrections tend to be large, but can be controlled by exploring the Lagrangian’s parameter space so as to maximize the inertia tensor. We comment on the acceptable parameter space and discuss the model’s further predictive power.
1 Introduction

One of the outstanding questions in particle physics continues to be the relationship between the phenomenologically successful non-relativistic constituent-quark model of hadrons, and QCD’s fundamental degrees of freedom[1]-[5]. The u, d, s valence quarks appearing in hadron spectroscopy have the same quantum numbers as the QCD current-quark fields, yet are much heavier and in some sense embody the strongly interacting QCD dynamics. The experimental results[6] on spin structure of the proton indicate that the constituent quarks ought to be thought of as composite objects with internal structure. Finding a theoretical description of this structure is an important and interesting challenge.

Kaplan has put forward a model[7] combining some of the features of the chiral quark[1] and skyrmion[8, 9, 10] approaches. In this model it is postulated that at distances smaller than the confinement scale but large enough to allow for nonperturbative phenomena the effective dynamics of QCD is described by a chiral Lagrangian whose target space is $SU(N_c \times N_f) \times SU(N_c \times N_f)_R$, resulting from misalignment of the $\bar{q}q$ condensate in the color space. One then makes the additional assumption that the solitons of this effective theory are stable and may be quantized semiclassically. Remarkably enough, it turns out that such solitons have the same quantum numbers as the constituent quarks in the quark model. Thus the constituent quarks in this model are “skyrmions” in color space.

This is a very attractive idea, but some crucial elements of the puzzle are still missing, most importantly a derivation of the effective dynamics from QCD. However, recent work on QCD in 1+1 dimensions[11] (QCD$_2$) provides evidence in support of this picture. Exact non-abelian bosonization allows the Lagrangian of QCD$_2$ to be re-expressed in terms of a chiral field $u(x) \in U(N_c \times N_f)$ and of the usual gauge field. This bosonized Lagrangian has topologically nontrivial static solutions that have the quantum numbers of baryons and mesons, constructed out of constituent quark and anti-quark solitons. The identification of these solitons with constituent, rather than current quarks, goes beyond the usual fermion-boson duality in two dimensions. It is based on the fact that their mass is generated dynamically and depends on the coupling constant.

When $N_c$ solitons of this type are combined, a static, finite-energy, color singlet solution is formed, corresponding to a baryon. Similarly, static meson solutions are formed out of a soliton and an anti-soliton of different flavors. The stability of the mesons against annihilation is ensured by flavor conservation. These results can be viewed as a derivation of the constituent quark model in QCD$_2$. Thus the idea of constituent quarks as solitons of a Lagrangian with colored chiral fields becomes exact in $D = 1+1$.

Pending a more formal justification of the effective dynamics in $D = 3+1$, it is interesting to see whether the quark solitons or “qualitons” have phenomenologically acceptable static properties, beyond their quantum numbers. The model discussed in Ref. [7] is the simplest one with the required properties. It is the analogue of the Skyrme model in color space.

It is quite probable that the actual effective Lagrangian in four dimensions is considerably more complicated, but retains the same transformation properties under flavor and color. Past experience with skyrmion-like models shows that their qualitative predictions are remarkably independent of specific details of the Lagrangian[12, 13]. The group theoretical
analysis of collective-coordinate quantization is thus likely to be correct, regardless of the specific dynamics. So until the correct approximate dynamics is found, it is worthwhile to investigate in some detail the simplest model possessing the right symmetries.

2 Model and Solution

We start from the Lagrangian

\[ \mathcal{L} = -\frac{1}{4g_s^2} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{4} \mu^2 \chi f^2 \left\{ -\text{Tr} \hat{R}_\mu \hat{R}^\mu + \frac{1}{4} c_4 \text{Tr} \left[ \hat{R}_\mu, \hat{R}_\nu \right]^2 + c_\alpha \frac{\alpha_s}{4\pi} \text{Tr} T_a \Sigma T_a \Sigma^\dagger + \cdots \right\} \]  

(1)

plus a Wess-Zumino term, where \( G^a_{\mu\nu} \) is the gluon field strength, \( \mu \chi \) is the chiral symmetry breaking scale, \( f \) is analogous to the pion decay constant \( f_\pi \), \( T_a \) is a color SU(3) generator normalized by \( \text{Tr} T_a T_b = \frac{1}{2} \delta_{ab} \), \( \hat{R}_\mu = \left( \Sigma^\dagger D_\mu \Sigma \right) / \mu \chi \), \( \Sigma = \exp(2i \Pi^a T_a / f) \), \( D_\mu = \partial_\mu + i A^a_\mu T_a \) is the color covariant derivative, \( \alpha_s = g_s^2 / 4\pi \) is the effective color fine structure constant and the dots (\( \cdots \)) refer to other operators with four or more derivatives and higher powers of \( \alpha_s \).

Of the five parameters in (1), we have some experimental information only on \( f \) and \( \alpha_s \), and even that is quite limited. The two-flavor model’s axial flavor current suggests that to leading order in \( \alpha_s \), \( f \approx f_\pi / \sqrt{3} \approx 53.7 \text{ MeV} \), but higher derivative operators ignored in this treatment might renormalize \( f \) away from that value, just as in the usual Skyrme model. Here \( \alpha_s \) is the strength of the effective color interaction between the color-carrying condensate and the gluon field, in principle derivable by nonperturbatively matching the effective and fundamental theories. As no such calculation is available at present, in practice we can take as a rough estimate the value \( \alpha_s \sim 0.28 \) provided by the non-relativistic quark model.

For the other parameters, dimensional analysis suggests that \( \mu_\chi \sim 4\pi f \) and \( c_4 \sim 1 \sim c_\alpha \).

Given a parameter set \( \{ \alpha_s, c_4, c_\alpha, f, \mu_\chi \} \), we seek classical static solutions for both \( \Sigma \) and gauge fields, adopting a radially-symmetric skyrmion ansatz

\[ \Sigma = e^{iF(r)\vec{r} \cdot \vec{\sigma}_c}, \quad F(0) = \pi, \quad F(\infty) = 0, \]  

(2)

where \( \vec{\sigma}_c \) acts on a two-dimensional subspace of SU(3)_c. To obtain the classical gluon configuration we note that it should have the same symmetry as \( \Sigma \) under combined spin and color isospin rotations. We work in \( A_0 = 0 \) gauge and eliminate the remaining gauge freedom, imposing the Gauss law constraint for unit winding number. We also note that energy minimization will remove the pure Yang-Mills terms, resulting in a configuration whose nonvanishing components are

\[ A^i_m = \frac{\gamma(r)}{r} \epsilon_{imj} \hat{r}_j, \quad m = 1, 2, 3, \]

\[ \tilde{A}_8 = -t \frac{g_s^2}{\sqrt{12} 8\pi^2 r^2} \left[ 2(\pi - F) + (1 + \gamma)^2 \sin 2F \right]. \]  

(3)

Then the radial energy density of a qualiton (relative to the classical vacuum) is

\[ \rho(r) = \frac{\alpha_s}{96\pi^2 r^2} \left[ 2(\pi - F) + (1 + \gamma)^2 \sin (2F) \right]^2. \]
\[ + \frac{1}{2r^2 \alpha_s} \left[ 2(r \gamma')^2 + \gamma^2(2 + \gamma)^2 \right] \]
\[ + 2 \pi f^2 \left[ (r F')^2 + 2(1 + \gamma)^2 \sin^2 F \right] \]
\[ + \frac{c_4 4 \pi f^2}{r^2 \mu_\chi^2} \sin^2 F (1 + \gamma)^2 \left[ 2(r F')^2 + (1 + \gamma)^2 \sin^2 F \right] \]
\[ + c_\alpha \frac{\alpha_s}{2 \mu_\chi^2} f^2 r^2 \left[ 2 - \cos F - \cos^2 F \right] , \]

and the classical contribution to the rest mass is
\[ M_{cl} = \int_0^\infty dr \rho(r) . \]

The five terms represent the respective contributions of the color electric field, the color magnetic field, the kinetic term, the four-derivative term, and the symmetry-breaking term. Minimizing the classical mass yields the variational equations for \( F(r) \) and \( \gamma(r) \):

\[ 4 \pi f^2 \left[ r^2 + \frac{4 c_4}{\mu_\chi^2} \sin^2 F(\gamma+1)^2 \right] F'' - 4 \pi f^2 \left[ (\gamma+1)^2 \sin 2F - 2r F' \right] \]
\[ - \frac{\alpha_s}{24 \pi^2 r^2} \left[ 2(\pi - F) + (\gamma+1)^2 \sin 2F \right] \left[ (\gamma+1)^2 \cos 2F - 1 \right] \]
\[ - \frac{8 c_4 \pi f^2}{\mu_\chi^2} \left[ \frac{1}{r^2} \sin 2F \sin^2 F(\gamma+1)^4 - \sin 2F(\gamma+1)^2(F')^2 - 4 \sin^2 F(\gamma + 1) \gamma' F' \right] \]
\[ - \frac{1}{2} c_\alpha \frac{\alpha_s}{\mu_\chi^2} f^2 r^2 (\sin F + \sin 2F) = 0 ; \]

subject to the boundary conditions (2) and \( \gamma(0) = 0 = \gamma(\infty) \). The equations were solved with the COLSYS package[14].

Given a particular solution \( \{\Sigma, A\} \), minimizing the classical mass (5), the symmetry generates a family of new solutions with the same energy:

\[ \Sigma \rightarrow \Omega \Sigma \Omega^\dagger , \quad A \rightarrow \Omega A \Omega^\dagger + i \Omega \nabla \Omega^\dagger , \]

where \( \Omega(\vec{r}, t) \in SU(3) \) is a collective coordinate. Quantizing these zero modes in the gauged case[7] is more subtle than in the global-symmetry Skyrme model because the quantum states have to satisfy the Gauss law constraint. In the external background \( \{\Sigma, A\} \) corresponding to a given parameter set this constraint leads to differential equations for \( \Omega \). We solve these equations numerically, obtaining the two moments of inertia \( I_1 \) and \( I_2 \), which govern the
Clearly, an equally valid alternative is to take a larger minimum mass becomes
where \( C \) and \( j \) are respectively the \( SU(3) \) Casimir and the spin in the appropriate representation. The ground state solitons are color triplets with \( j = \frac{1}{2} \), and mass

\[
M = M_{cl} + \frac{3}{8I_1} + \frac{1}{4I_2} \equiv M_{cl} + M_1 + M_2 \,.
\]

In the analogous expression in the usual Skyrme model the quantum contribution, \( M_1 + M_2 \), is suppressed by \( 1/N_c^2 \) relative to the classical mass. This justifies the semiclassical approach in the Skyrme model. In the present case, however, the large-\( N_c \) expansion does not help, because both the classical mass and the moments of inertia scale like \( \sim N_c^0 \); parametrically the classical and quantum terms are of the same order. For the semiclassical approach to make sense, the parameters of the model have to be chosen so as to make the quantum contribution numerically small, while at the same time giving phenomenologically acceptable results.

Starting with the “natural” guess discussed earlier, \( \{ \alpha_s = 0.28, c_4 = 1, c_\alpha = 1, f = 53.7 \text{ MeV}, \mu_\chi = 4\pi f \} \), we obtain \( M = M_{cl} + M_1 + M_2 = (361+605+443) = 1409 \text{ MeV} \). This is doubly unacceptable: \( M \) is four times the phenomenological value of 350 MeV, and the ratio of quantum corrections to classical contribution is large, \( \eta \equiv (M_1+M_2)/M_{cl} = 2.2 \). We can also compute the “half-height radius” \( r_{\frac{1}{2}} \) defined by \( F(r_{\frac{1}{2}}) = \pi/2 \), and obtain \( r_{\frac{1}{2}} = 0.354 \text{ fm} \), which does look reasonable compared to the proton charge radius \( \approx 0.86 \text{ fm} \). This situation is rather generic: it is easy to find parameters yielding a smaller mass with a bigger radius or vice versa, but one cannot keep both of them small, while decreasing \( \eta \): moments of inertia scale like \( M_{cl} r^2 \) and so from (11), \( \eta \sim 1/(M_{cl} r)^2 \).

We can learn more by systematically investigating the parameter space. This task can be simplified by expressing all dimensionful parameters in terms of \( f \). Then the SSB scale \( \mu_\chi \) appears only in the dimensionless combinations \( \tilde{c}_4 \equiv (4\pi f/\mu_\chi)^2 c_4 \) and \( \tilde{c}_\alpha \equiv (\mu_\chi/4\pi f)^2 c_\alpha \). Together with \( \alpha_s \) they form a three dimensional parameter space, which determines the physics up to an overall scale factor. A variational estimate then suggests that \( M \) has a local minimum, \( \sim 20f \), near \( \tilde{c}_4 \sim 3 \). In the \( \tilde{c}_\alpha \) direction the minimum occurs only at the boundary, \( \tilde{c}_\alpha = 0 \). Near the minimum the mass changes slowly as a function of \( \tilde{c}_4 \), allowing partial minimization of either \( \eta \sim 2/(\tilde{c}_4)^2 \), or of \( r_{\frac{1}{2}} f \sim 0.1\sqrt{\tilde{c}_4} \).

Numerical results confirm these expectations. With \( \alpha_s = 0.28 \) and \( \tilde{c}_\alpha = 1 \), a local minimum \( M = 18.3f \) occurs at \( \tilde{c}_4 = 2.72 \), yielding \( \eta = 0.377 \). When \( c_\alpha \) is fine-tuned small, the minimum mass becomes \( M \approx 16f \), falling to \( M \approx 13f \) for \( \alpha_s = 1 \). Thus for any reasonable values of \( c_\alpha \) and \( \alpha_s \), \( M \) is well above the colored pseudo-Goldstone mass \( m_{\Pi} = (\frac{2}{3} \tilde{c}_4 4\pi \alpha_s)^{1/2} f \), and comparable to \( \mu_\chi \sim 4\pi f \). Furthermore, a 350 MeV constituent quark requires \( f \) to be considerably below 53.7 MeV, implying a large value for \( r_{\frac{1}{2}} \approx 0.35\sqrt{\tilde{c}_4} (53.7 \text{ MeV}/f) \) fm. Clearly, an equally valid alternative is to take a larger \( f \), so as to get a smaller \( r_{\frac{1}{2}} \), at the expense of increasing \( M \) above 350 MeV. For example, \( f = 36 \text{ MeV} \) with \( c_\alpha \) small yields \( r_{\frac{1}{2}} = 0.86 \text{ fm} \), \( M = 570 \text{ MeV} \), with \( \eta \) unchanged.
At first it therefore seems that to obtain realistic properties of constituent quarks one must modify the model— which, as noted earlier, is only the simplest of a large family of Lagrangians possessing the right symmetries. The variational argument indicates that additional four-derivative terms are unlikely to make a qualitative difference, but it also indicates possibilities for quantitative improvements.

We would like to point out, however, that there is another, somewhat unusual possibility: the radius of a constituent quark might actually be larger than the radius of the proton. We are used to composite objects being larger than the building blocks out of which they are constructed, but there are some exceptions to this rule when the interactions among the constituents are very strong. Clearly, when something like that happens, there must be a very substantial deformation of the original constituents. An example is provided by the fusion of two Deuterium nuclei[15] to form a nucleus of Helium-4. The r.m.s. charge radius of $^2\text{H}$ is 2.12 fm, with binding energy of 2.2 MeV, while the corresponding figures for $^4\text{He}$ are 1.67 fm and 28.3 MeV. The confining interaction between qualitons is stronger than the interactions that shape the qualiton. It is therefore possible that a similar phenomenon occurs when qualitons bind together to form hadrons. We do not know whether Nature realizes this scenario, or whether the effective Lagrangian then remains a useful guide, but if it does, we can find a broad region of the parameter space corresponding to a phenomenologically acceptable mass.

To this effect, we first choose a somewhat small $\tilde{c}_\alpha$, to minimize the dimensionless mass. This will allow $M = 350$ MeV to correspond to a slightly larger $f$. Next we decrease $\tilde{c}_4$ in the neighborhood of the minimum, so as to decrease the radius, while constraining $\eta < 0.33$ to protect the validity of the semi-classical approximation. With $\alpha_s = 0.28$, $c_\alpha = 0.1$, and $\mu_\chi = 4\pi f$, we thus find parameters $c_4 = 2.46$, $f = 21.4$ MeV, giving $r_\frac{1}{2} = 1.43$ fm and $\eta = 0.33$. Fig. 1 plots the resulting shape functions.

To test the sensitivity of this choice we note that taking instead $c_\alpha = 0.3$ decreases $f$ only slightly to 20.6 MeV, and increases $r_\frac{1}{2}$ to 1.50 fm, with a range $1.44 \text{ fm} < r_\frac{1}{2} < 1.56$ fm corresponding to $0.1 < \alpha_s < 1$. Even taking $c_\alpha = 1, \alpha_s = 1$ raises $r_\frac{1}{2}$ only to 1.67 fm. Thus in the phenomenologically relevant regions of parameter space the results are rather stable and do not require fine-tuning.

3 Summary and outlook

We have studied the static properties of constituent quark solitons for the simplest case $N_f=1$, $N_c=3$. Self-consistency of the semiclassical calculation indicates that in order to obtain phenomenologically acceptable constituent quark masses, such objects are likely to have rather large radii, before binding into hadrons. A more detailed investigation of this possibility requires extending the present work to multi-soliton configurations. Such an extension would also make it possible to compute modifications of constituent quark color magnetic moments in bound states, giving baryon hyperfine mass splittings and thus fixing the effective $\alpha_s$.

It would also be interesting to extend the calculation to an $N_f\geq2$ model, since for $N_f=1$ there is no chiral flavor symmetry and hence no pion. This would allow a computation of
the constituent quark axial-vector coupling $g_A^q$. The framework can also be fruitfully applied in the study of the spin and strangeness content of the proton, or in a computation of the Isgur-Wise function[16] representing the light-quark degrees of freedom in the formalism of the heavy quark symmetry.

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Figure Caption

The profiles $F(r)$, $\gamma(r)$, and the classical mass density $\rho(r)$ in MeV/fm, with $\alpha_s = 0.28$, $c_4 = 2.46$, $c_\alpha = 0.1$, $f = 21.4$ MeV, and $\mu_\chi = 4\pi f$. $M = 350$ MeV, $\eta \equiv M_{\text{quantum}}/M_{\text{cl}} = 0.33$. 
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