The stability of persistent homology of hypergraphs

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Abstract

Hypergraph is the most general model for complex networks involving group interactions. Taking the ideas of path homology from Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau [18–22], Stephane Bressan, Jingyan Li and the authors of this article introduced embedded homology of hypergraphs [6] in 2019, which has led to successful applications in protein-ligand binding network [24, 25] in 2021. A fundamental question arising from practical applications is about the stability of the persistent embedded homology of hypergraphs. In this paper, we prove the stability of the persistent embedded homology as well as the persistent homology of the associated simplicial complex with respect to perturbations of the filtration on a hypergraph. We apply the persistent homology methods to morphisms of hypergraphs and prove the stability with respect to perturbations of the filtrations. We prove the constancy of the persistent Betti numbers under some conditions on the simple-homotopy types of hypergraphs.

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1 Introduction

Let $V$ be a finite set equipped with a total order. Let $2^V$ denote the power-set of $V$. Let $\emptyset$ denote the empty set. A hypergraph $\mathcal{H}$ on $V$ is a subset of $2^V \setminus \{\emptyset\}$ (cf. [4, 27]). We call $V$ the vertex-set and call an element of $V$ a vertex. We call an element in $\mathcal{H}$ a hyperedge. From the view of algebraic topology, a hypergraph may be considered as an incomplete simplicial complex with missing some faces. More precisely, if a hypergraph satisfies the condition: for any $\sigma \in \mathcal{H}$ and any nonempty subset $\tau \subseteq \sigma$, the set $\tau$ must be a hyperedge in $\mathcal{H}$, then it is an (abstract) simplicial complex. A hyperedge of a simplicial complex is a simplex. Let $\mathcal{H}$ be a hypergraph on $V$ and let $\mathcal{H}'$ be a hypergraph on $V'$. A morphism $\varphi : \mathcal{H} \rightarrow \mathcal{H}'$ is a map $\varphi : V \rightarrow V'$ such that each hyperedge of $\mathcal{H}$ is sent to a hyperedge of $\mathcal{H}'$. A morphism of hypergraphs between two simplicial complexes is a simplicial map.

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People have been used networks having group interactions in various disciplines. Pair-wise representations are not necessarily appropriate when the fundamental interactions involve more than two entities at the same time, such as bionetworks on protein-protein interactions, protein-ligand interactions and microbiome interactions as well as many social networks. Hypergraph is the most general mathematical model without constraint conditions for complex networks involving group interactions, which has been extensively used by the recent researchers \[1, 28, 32\], see the review articles \[2, 3, 29\]. This demands mathematical explorations on hypergraphs for introducing (more) mathematical features that can detect the structures of hypergraphs.

Since 2010s, Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau started to study the topological features of graphs and digraphs, with introducing path homology theory for graphs and digraphs in their series work \[18–22\], which breaks through some frames of the classical algebraic topology in the sense that their homology theory is able to apply to incomplete complexes with missing faces. Taking the ideas of path homology, Stephane Bressan, Jingyan Li and the authors of this article introduced embedded homology of hypergraphs \[6\] in 2019, which seems one of the most canonical extensions of simplicial homology to hypergraphs in the sense that the representation of a hypergraph in a chain complex induces infimum complex and supremum complex having the same chain homotopy type that is used for defining the embedded homology, see Subsection 2.2 for details.

The newly introduced embedded homology of hypergraphs has leaded to successful applications in protein-ligand binding network \[24, 25\] in 2021, where the persistent method for hypergraphs plays an important role for the experimental results. A fundamental question arising from practical applications is then whether the barcodes derived from persistent embedded homology is robust. In terms of mathematics, the question is about the stability of the persistent embedded homology of hypergraphs. The purpose of this article is to give the stability theorem on persistent embedded homology of hypergraphs.

Our methodology for studying the stability of the persistent embedded homology for hypergraphs is similar to that on the stability of persistent simplicial homology \[9, 10\], while some technical difficulties have to be overcome in the case of hypergraphs. Since the embedded homology of hypergraphs is a canonical extension of simplicial homology, our stability theorem on persistent embedded homology of hypergraphs can be viewed as a generalization of the stability theorem on persistent simplicial homology that gives a robust topological approach to data modeled by hypergraph.

Now let us give the detailed discussions. We consider real-valued functions on hypergraphs and use the persistent of the embedded homology groups as well as the homology of the associated simplicial complexes to study their qualitative and quantitative behavior. Specifically, we study the topological features of hypergraphs as well as morphisms between hypergraphs. By applying the persistent of the embedded homology as well as the homology of the associated simplicial complexes to the induced pull-back filtrations as well as the induced push-forward filtrations, we use persistent diagrams to encode the topological features of the morphisms and prove the stability of this encoding. The stability that will be proved in this paper gives the theoretical foundation for the applications of the persistent embedded homology for hypergraphs, which says that the out-put of the persistent homology is stable under small perturbations of the observation quantity.

The first main result proves that the persistent diagram of the persistent homology for a hypergraph is stable with respect to perturbations of a filtration of the hypergraph. Let \(\mathcal{H}\) be a hypergraph. Let \(f\) be a real function on \(\mathcal{H}\). Let \(\mathcal{H}^f\) be the filtration of \(\mathcal{H}\) with re-
pect to $f$, i.e. $\mathcal{H}^f$ is a family of hypergraphs $\mathcal{H}_t^f$, $t \in \mathcal{R}$, where $\mathcal{H}_t^f = f^{-1}(-\infty, t]$. We have the persistent homology $H_n(\Delta(\mathcal{H}_t^f))$ of the filtered associated simplicial complex $\Delta(\mathcal{H}_t^f)$ whose persistent diagram is denoted by $D(\Delta \mathcal{H}, f)$, the persistent homology $H_n(\delta(\mathcal{H}_t^f))$ of the lower-associated simplicial complex $\delta(\mathcal{H}_t^f)$ whose persistent diagram is denoted by $D(\delta \mathcal{H}, f)$, and the persistent of the embedded homology $H_n(\mathcal{H}_t^f)$ whose persistent diagram is denoted by $D(\mathcal{H}, f)$. Let $D \mathcal{H}^f$ be the triple $(D(\Delta \mathcal{H}, f), D(\delta \mathcal{H}, f), D(\mathcal{H}, f))$. Then the stability of the persistent diagram makes sense for $D \mathcal{H}^f$.

**Theorem 1.1** (Main Result I). Let $\mathcal{H}$ be a hypergraph. Let $f$ and $g$ be real-valued functions on $\mathcal{H}$. Then the bottleneck distance $d^\infty_B(D \mathcal{H}^f, D \mathcal{H}^g)$ between the persistent diagrams induced from $\mathcal{H}^f$ and $\mathcal{H}^g$ is bounded above by the $L^\infty$-distance $||f - g||_\infty$ between $f$ and $g$.

The second main result proves that the persistent diagram of a morphism between two hypergraphs is stable with respect to perturbations of the filtrations on both the source hypergraph and the target hypergraph. Let $\varphi : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of hypergraphs. Let $f$ be a real function on $\mathcal{H}$ and let $f'$ be a real function on $\mathcal{H}'$. Then we have a filtration $\mathcal{H}_t^f$ of $\mathcal{H}$ and a filtration $\mathcal{H}_t'^f$ of $\mathcal{H}'$. We will define in Subsection 4.2 an induced filtration of $\mathcal{H}$ by pulling back $\mathcal{H}_t'^f$ via $\varphi$ and define twenty-one induced homomorphisms of persistent homology groups, denoted as $\Phi_f$, in the commutative diagram in Subsection 4.2 (A). We will also define in Subsection 4.2 an induced filtration of $\mathcal{H}'$ by pushing forward $\mathcal{H}_t^f$ via $\varphi$ and define twenty-one induced homomorphisms of persistent homology groups, denoted as $\Phi'_g$, in the commutative diagram in Subsection 4.2 (B). We define the persistent diagram of a homomorphism between two persistent homology groups to be the triple consisting of the persistent diagram of the kernel, the persistent diagram of the image, and the persistent diagram of the cokernel. Then the stability of the persistent diagram makes sense for $\Phi_f$ and $\Phi'_g$.

**Theorem 1.2** (Main Result II). Let $\epsilon > 0$. Suppose the persistent homology is with field coefficients.

(i). If $||f - g||_\infty \leq \epsilon$, then each of the persistent linear map among the twenty-one persistent linear maps in the commutative diagram in (B) (cf. Subsection 4.2), denoted as $\Phi_f$ and $\Phi_g$, satisfies $d_B^\infty(D(\Phi_f), D(\Phi_g)) \leq \epsilon$;

(ii). If $||f' - g'||_\infty \leq \epsilon$, then each of the persistent linear map among the twenty-one persistent linear maps in the commutative diagram in (A) (cf. Subsection 4.2), denoted as $\Phi'_f$ and $\Phi'_g$, satisfies $d_B^\infty(D(\Phi'_f), D(\Phi'_g)) \leq \epsilon$.

Thanks to the stability theorem of the persistent homology of simplicial complexes, Theorem 1.1 and Theorem 1.2 can be proved. The notion of persistent diagram was introduced in [15]. The stability of persistent diagrams was proved in [10]. An interleaving condition was discovered in [8], which provides a way to measure the distance between persistent modules and ensure the stability. Homomorphisms of persistent modules and the stability was studied in [11]. The methodology for the proofs of Theorem 1.1 and Theorem 1.2 is originated from [8, 12, 14, 15] with some technical details overcome.

For any real numbers $t \leq s$, the persistent Betti number $\beta_n^{t,s}$ is the dimension of the persistent homology $H_n^{t,s}$ (cf. [33]). We define the simple-homotopy types of hypergraphs in Subsection 3.1. For any hypergraph $\mathcal{H}$ and any real function $f$ on $\mathcal{H}$, the third main result proves the constancy of the persistent Betti numbers under some conditions on the simple-homotopy types of the filtered hypergraphs.
Theorem 1.3 (Main Result III). Let $\mathcal{H}$ be a hypergraph and let $f : \mathcal{H} \rightarrow \mathbb{R}$. For any $t \leq s$,

(1) if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have all the vertices as 0-hyperedges and have the same strong simple-homotopy type, then the persistent Betti number $\beta_n^{t,s}$ of the embedded homology is constant;

(2) if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have the same weak simple-homotopy type, then the persistent Betti number $\beta_{\Delta,n}^{t,s}$ of the homology of the associated simplicial complexes is constant;

(3) if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have the same weak simple-homotopy type via a sequence of weak simplicial collapses and weak simplicial expansions such that in each weak elementary simplicial collapse or weak elementary simplicial expansion, $\eta$ is a free face of $\xi$ and one of (a), (b) and (c) in Lemma 3.5 is satisfied, then the persistent Betti number $\beta_{\delta,n}^{t,s}$ of the homology of the lower-associated simplicial complexes is constant.

2 The homology of hypergraphs

2.1 Hypergraphs and simplicial complexes

Let $V$ be a finite set with a total order $\prec$. A hyperedge on $V$ is $\sigma = \{v_0, v_1, \ldots, v_n\}$ where $v_0, v_1, \ldots, v_n \in V$ and $v_0 \prec v_1 \prec \cdots \prec v_n$. We call $n$ the dimension of $\sigma$ and call $\sigma$ an $n$-hyperedge. A hypergraph $\mathcal{H}$ on $V$ is a collection of hyperedges on $V$. An (abstract) simplicial complex $\mathcal{K}$ on $V$ is a hypergraph on $V$ such that $\tau \in \mathcal{K}$ for any $\sigma \in \mathcal{K}$ and any nonempty subset $\tau \subseteq \sigma$. A hyperedge of a simplicial complex is called a simplex.

For a hyperedge $\sigma = \{v_0, v_1, \ldots, v_n\}$ on $V$, the associated simplicial complex $\Delta^\sigma$ of $\sigma$ is the collection of all the nonempty subsets of $\sigma$

$$\Delta^\sigma = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n, 0 \leq k \leq n\}.$$ 

Let $\sigma = \{v_0, v_1, \ldots, v_n\}$ and $\tau = \{u_0, u_1, \ldots, u_m\}$ be two hyperedges on $V$. Suppose $\sigma \cap \tau = \emptyset$. The join $\sigma \ast \tau$ is an $(n + m + 1)$-hyperedge

$$\sigma \ast \tau = \{v_0, v_1, \ldots, v_n, u_0, u_1, \ldots, u_m\},$$

where the right-hand side is up to a rearrangement of the vertices $v_0, v_1, \ldots, v_n, u_0, u_1, \ldots, u_m$ with respect to the total order $\prec$. It is direct that $\Delta(\sigma \ast \tau) = (\Delta\sigma) \ast (\Delta\tau)$. Let $\mathcal{H}$ be a hypergraph on $V$. The associated simplicial complex

$$\Delta\mathcal{H} = \{\tau \in \Delta^\sigma \mid \sigma \in \mathcal{H}\}$$

is the smallest simplicial complex that $\mathcal{H}$ can be embedded in (cf. [27]) and the lower-associated simplicial complex

$$\delta\mathcal{H} = \{\sigma \in \mathcal{H} \mid \Delta^\sigma \subseteq \mathcal{H}\} = \{\tau \in \Delta^\sigma \mid \Delta^\sigma \subseteq \mathcal{H}\}$$

is the largest simplicial complex that can be embedded in $\mathcal{H}$.

Let $V$ and $V'$ be two totally-ordered finite sets. Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs on $V$ and $V'$ respectively. A morphism of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$ is a map $\varphi$ from $V$ to $V'$ such that for any $k \geq 0$, whenever $\sigma = \{v_0, \ldots, v_k\}$ is a hyperedge of $\mathcal{H}$, its image $\varphi(\sigma) = \{\varphi(v_0), \ldots, \varphi(v_k)\}$ is a hyperedge of $\mathcal{H}'$. Here $v_0, \ldots, v_k$ are distinct in $V$ while
\(\varphi(v_0), \ldots, \varphi(v_k)\) may not be distinct in \(V'\). Such a morphism \(\varphi : \mathcal{H} \to \mathcal{H}'\) induces simplicial maps \(\delta \varphi : \mathcal{H} \to \mathcal{H}'\) and \(\Delta \varphi : \mathcal{H} \to \mathcal{H}'\) such that \(\varphi = (\Delta \varphi) |_{\mathcal{H}}\) and \(\delta \varphi = \varphi |_{\delta \mathcal{H}}\). Suppose in addition that \(V \cap V' = \emptyset\). Let \(V \cup V'\) be the disjoint union. We define the join \(\mathcal{H} \ast \mathcal{H}'\) to be a hypergraph on \(V \cup V'\) by

\[\mathcal{H} \ast \mathcal{H}' = \{\sigma \ast \sigma' \mid \sigma \in \mathcal{H} \text{ and } \sigma' \in \mathcal{H}'\} \cup \mathcal{H} \cup \mathcal{H}'.\]

### 2.2 The homology of hypergraphs

For each \(n \geq 0\), let \((\Delta \mathcal{H})_n\) be the set of all the \(n\)-simplices in \(\Delta \mathcal{H}\). For each \(0 \leq i \leq n\), consider the face map \(d_i : (\Delta \mathcal{H})_n \to (\Delta \mathcal{H})_{n-1}\) given by \(d_i\{v_0, v_1, \ldots, v_n\} = \{v_0, \ldots, \hat{v}_i, \ldots, v_n\}\). Let \(R\) be a commutative ring with unit. Let \(C_n(\Delta \mathcal{H}; R)\) be the free \(R\)-module generated by all the elements in \((\Delta \mathcal{H})_n\). Let \(\partial_n = \sum_{i=0}^n (-1)^i d_i\), which extends linearly over \(R\). We have a chain complex

\[\cdots \to C_n(\Delta \mathcal{H}; R) \xrightarrow{\partial_n} C_{n-1}(\Delta \mathcal{H}; R) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(\Delta \mathcal{H}; R) \xrightarrow{\partial_0} 0,\]

denoted as \(C_*(\Delta \mathcal{H}; R)\). Let \(R(\mathcal{H})_n\) be the collection of all the formal linear combinations of the \(n\)-hyperedges in \(\mathcal{H}\) with coefficients in \(R\). Then \(R(\mathcal{H})_*\) is a graded sub-\(R\)-module of \(C_*(\Delta \mathcal{H}; R)\). By [6, Section 2], the infimum chain complex

\[\text{Inf}_n(R(\mathcal{H})_*) = R(\mathcal{H})_n \cap \partial_n^{-1}(R(\mathcal{H})_{n-1}), \quad n \geq 0\]

is the largest sub-chain complex of \(C_*(\Delta \mathcal{H}; R)\) contained in \(R(\mathcal{H})_*\) and the supremum chain complex

\[\text{Sup}_n(R(\mathcal{H})_*) = R(\mathcal{H})_n + \partial_{n+1}(R(\mathcal{H})_{n+1}), \quad n \geq 0\]

is the smallest sub-chain complex of \(C_*(\Delta \mathcal{H}; R)\) containing \(R(\mathcal{H})_*\). The canonical inclusion \(i : \text{Inf}_n(R(\mathcal{H})_*) \to \text{Sup}_n(R(\mathcal{H})_*)\) induces an isomorphism

\[i_* : H_n(\text{Inf}(R(\mathcal{H})_*)) \cong H_n(\text{Sup}(R(\mathcal{H})_*)). \quad (2.1)\]

This homology group is the \(n\)-th embedded homology of \(\mathcal{H}\) and is denoted as \(H_n(\mathcal{H}; R)\) (cf. [6, Subsection 3.2]).

Let \(\varphi : \mathcal{H} \to \mathcal{H}'\) be a morphism of hypergraphs. We have a commutative diagram of chain complexes

\[
\begin{array}{ccccccccc}
\text{Ker}((\Delta \varphi)_#) & \to & C_*((\Delta \mathcal{H}; R) & \xrightarrow{(\Delta \varphi)_#} & C_*((\Delta \mathcal{H}'; R) & \to & \text{Coker}((\Delta \varphi)_#) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Ker}(\text{Sup}(\varphi)) & \to & \text{Sup}_*(\mathcal{H}) & \xrightarrow{\text{Sup}(\varphi)} & \text{Sup}_*(\mathcal{H}') & \to & \text{Coker}(\text{Sup}(\varphi)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Ker}(\text{Inf}(\varphi)) & \to & \text{Inf}_*(\mathcal{H}) & \xrightarrow{\text{Inf}(\varphi)} & \text{Inf}_*(\mathcal{H}') & \to & \text{Coker}(\text{Inf}(\varphi)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Ker}((\delta \varphi)_#) & \to & C_*((\delta \mathcal{H}; R) & \xrightarrow{(\delta \varphi)_#} & C_*((\delta \mathcal{H}'; R) & \to & \text{Coker}((\delta \varphi)_#) \\
\end{array}
\]
where the vertical arrows in the first three columns are inclusions and the vertical arrows in the last column are homomorphisms of chain complexes. This induces a commutative diagram of homology groups

\[
\begin{array}{cccc}
H_\ast(\text{Ker}((\Delta \varphi)_\#)) & \xrightarrow{(\Delta \varphi)_\ast} & H_\ast(\Delta H; R) & \xrightarrow{(\varphi)_\ast} & H_\ast(\Delta H'; R) & \xrightarrow{(\varphi)_\ast} & H_\ast(\text{Coker}((\Delta \varphi)_\#)). \\
H_\ast(\text{Ker}(\text{Sup}(\varphi))) & & & & & & \\
H_\ast(\text{Ker}(\text{Inf}(\varphi))) & & & & & & \\
H_\ast(\text{Ker}((\delta \varphi)_\#)) & \xrightarrow{(\delta \varphi)_\ast} & H_\ast(\delta H; R) & \xrightarrow{(\delta \varphi)_\ast} & H_\ast(\delta H'; R) & \xrightarrow{(\delta \varphi)_\ast} & H_\ast(\text{Coker}((\delta \varphi)_\#)). \\
\end{array}
\]

The next proposition follows from [6, Theorem 3.10].

**Proposition 2.1** (Mayer-Vietoris sequences). Let \( \mathcal{H} \) and \( \mathcal{H}' \) be two hypergraphs satisfying the hypothesis\((P)\). for any \( \sigma \in \mathcal{H} \) and any \( \sigma' \in \mathcal{H}' \), either \( \sigma \cap \sigma' \) is the emptyset or \( \sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}' \).

Then we have a commutative diagram

\[
\begin{array}{cccc}
\cdots & \xrightarrow{\delta} & H_n(\delta \mathcal{H} \cap \delta \mathcal{H}') & \xrightarrow{\delta} & H_n(\delta \mathcal{H}) \oplus H_n(\delta \mathcal{H}') & \xrightarrow{\delta} & \cdots \\
\cdots & & \xrightarrow{\delta} & H_n(\mathcal{H} \cap \mathcal{H}') & \xrightarrow{\delta} & H_n(\mathcal{H}) \oplus H_n(\mathcal{H}') & \xrightarrow{\delta} & \cdots \\
\cdots & & & \xrightarrow{\delta} & H_n(\Delta \mathcal{H} \cap \Delta \mathcal{H}') & \xrightarrow{\delta} & H_n(\Delta \mathcal{H}) \oplus H_n(\Delta \mathcal{H}') & \xrightarrow{\delta} & \cdots \\
\end{array}
\]

such that each row is a long exact sequence and each vertical map is a homomorphism induced by the canonical inclusions of hypergraphs.

The next proposition follows from [30, Theorem 1.1].
Proposition 2.2 (Künneth-type formulae). Let $R$ be a principal ideal domain with unit 1. We have a commutative diagram

\[
\begin{array}{c}
\bigoplus_{p+q+1=n} H_{p+1}(\delta H) \otimes H_{q+1}(\delta H') \\
\downarrow \\
\bigoplus_{p+q+1=n} H_{p+1}(H) \otimes H_{q+1}(H') \\
\downarrow \\
\bigoplus_{p+q+1=n} H_{p+1}(\Delta H) \otimes H_{q+1}(\Delta H') \\
\downarrow \\
\bigoplus_{t+s+1=n} \text{Tor}_R(H_{p+1}(\Delta H), H_{q}(\Delta H')) \\
\end{array}
\]

such that each row is a short exact sequence and each vertical map is a homomorphism induced by the canonical inclusions of hypergraphs.

3 Simple-homotopy types of hypergraphs and homology

3.1 Collapses of hypergraphs

Let $\mathcal{H}$ be a hypergraph. Let $\eta, \xi \in \mathcal{H}$. If $\eta \subseteq \xi$ and $\eta$ is not a proper face of any other hyperedges in $\mathcal{H}$, then we say that $\eta$ is a free face of $\xi$ in $\mathcal{H}$. Let $\mathcal{A} \subseteq \mathcal{H}$ be a sub-hypergraph of $\mathcal{H}$. We call $(\mathcal{H}, \mathcal{A})$ a pair of hypergraphs.

Definition 1. (1). We say that there exists a weak elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$ if there exist $\eta, \xi \in \mathcal{H}$ such that $\eta$ is a free face of $\xi$ in $\mathcal{H}$ and $\mathcal{A} = \mathcal{H} \setminus \{\eta, \xi\}$ is not the emptyset;

(2). We say that there exists a strong elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$ if there exist a weak elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$ such that (i) $\mathcal{H} = \mathcal{A} \cup \Delta(\{v\} * \eta)$ and (ii) $\mathcal{A} \cap \Delta(\{v\} * \eta) = \{v\} * \text{bd}(\eta)$ for some $v \in V$ and some $\eta \in \mathcal{H}$. Equivalently, there exists a strong elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$ if both (i) and (ii) are satisfied such that $\eta$ is a free face of $\{v\} * \eta$ in $\mathcal{H}$.

Definition 2. We say that $\mathcal{H}$ (weakly, strongly) collapses simplicially to $\mathcal{A}$ or $\mathcal{A}$ (weakly, strongly) expands simplicially to $\mathcal{H}$ if there is a finite sequence of (weak, strong) elementary simplicial collapses $\mathcal{H} = \mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_q = \mathcal{A}$.

Definition 3. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs. We say that $\mathcal{H}$ and $\mathcal{H}'$ have the same (weak, strong) simple-homotopy type if there is a finite sequence $\mathcal{H} = \mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_q = \mathcal{H}'$, where each arrow represents a (weak, strong) simplicial expansion or a (weak, strong) simplicial collapse.
Example 1. Let $\mathcal{H} = \{\{v_0, v_1\}, \{v_0, v_1, v_2\}, \{v_0, v_1, v_3\}, \{v_0, v_1, v_2, v_4, v_5\}\}$. Then (i). \{v_0, v_1\} is not a free face of \{v_0, v_1, v_2\} in $\mathcal{H}$, (ii). \{v_0, v_1, v_2\} is a free face of \{v_0, v_1, v_2, v_4, v_5\} in $\mathcal{H}$, (iii). \{v_0, v_1\} is not a free face of \{v_0, v_1, v_2, v_4, v_5\} in $\mathcal{H}$.

Example 2. Let $(\mathcal{H}, \mathcal{A})$ by $\mathcal{H} = \{\{v_0, v_1\}, \{v_2, v_3\}, \{v_0, v_1, v_2, v_3\}\}$ and $\mathcal{A} = \{\{v_2, v_3\}\}$. Let $\eta = \{v_0, v_1\}$ and $\xi = \{v_0, v_1, v_2, v_3\}$. Then $\eta$ is a free face of $\xi$ in $\mathcal{H}$ and $\mathcal{A} = \mathcal{H}\{\eta, \xi\}$. Thus there is a weak elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$. However, there does not exist any strong elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$.

Example 3. Let $(\mathcal{H}, \mathcal{A})$ by

\[ \mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}\}, \]

\[ \mathcal{A} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}, \{v_0, v_1, v_2, v_3\}\}. \]

Let $\eta = \{v_0, v_1\}$ and $v = v_2$. Then

\[ \Delta(\{v\} * \eta) = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}\}, \]

\[ \{v\} * \text{bd}(\eta) = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_2\}, \{v_1, v_2\}\}. \]

Since both $\eta$ and $\{v\} * \eta$ are proper faces of $\{v_0, v_1, v_2, v_3\}$, $\eta$ is not a free face of $\{v\} * \eta$. Thus there does not exist any weak elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$.

Let $(\mathcal{K}, \mathcal{A})$ be a pair of simplicial complexes. By [7, p. 3] and [16, p. 106], the weak simple-homotopy type as well as the strong simple-homotopy type is the same as the usual simple-homotopy type (cf. [7, pp. 3-4] and [16, p. 106]).

3.2 Collapses and the embedded homology

Lemma 3.1. Let $(\mathcal{H}, \mathcal{A})$ be a pair of hypergraphs such that all the vertices are 0-hyperedges. If there is a strong elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$, then the canonical inclusion of $\mathcal{A}$ into $\mathcal{H}$ induces an isomorphism from $H_*(\mathcal{A})$ to $H_*(\mathcal{H})$.

Proof. Suppose there is a strong elementary simplicial collapse from $\mathcal{H}$ to $\mathcal{A}$. Then there exist a vertex $v$ of $\mathcal{H}$ and a hyperedge $\eta \in \mathcal{H}$ such that (1). $\mathcal{H} = \mathcal{A} \cup \Delta(\{v\} * \eta)$, (2). $\mathcal{A} \cap \Delta(\{v\} * \eta) = \{v\} * \text{bd}(\eta)$, and (3). $\eta$ is a free face of $\{v\} * \eta$ in $\mathcal{H}$. It follows from (1) and (2) that $\mathcal{A} = \mathcal{H}\{\eta, \{v\} * \eta\}$. It follows from (3) that $\{v\} * \eta$ is a maximal hyperedge in $\mathcal{H}$. In (3), we substitute $\mathcal{H}$ with $\mathcal{A}$ and substitute $\mathcal{H}'$ in with $\Delta(\{v\} * \eta)$. It follows that for any $\tau \in \mathcal{A}$ and any $\tau' \in \Delta(\{v\} * \eta)$, either $\tau \cap \tau'$ is the emptyset or $\tau \cap \tau'$ is a proper subset of $\{v\} * \eta$ such that $\tau \cap \tau'$ is not equal to $\eta$. Thus either $\tau \cap \tau'$ is the emptyset or $\tau \cap \tau'$ is a simplex of $\mathcal{A} \cap \Delta(\{v\} * \eta)$. The Mayer-Vietoris sequence gives a long exact sequence

\[
\cdots \rightarrow H_n(\mathcal{A} \cap \Delta(\{v\} * \eta)) \rightarrow H_n(\mathcal{A}) \oplus H_n(\Delta(\{v\} * \eta)) \rightarrow H_n(\mathcal{H}) \rightarrow H_{n-1}(\mathcal{A} \cap \Delta(\{v\} * \eta)) \rightarrow \cdots \tag{3.1}
\]

Since both $\mathcal{A} \cap \Delta(\{v\} * \eta)$ and $\Delta(\{v\} * \eta)$ are simplicial complexes such that

$$|\mathcal{A} \cap \Delta(\{v\} * \eta)| = |\{v\} * \text{bd}(\eta)| \simeq |\Delta(\{v\} * \eta)| \simeq |v|,$$
Then we have strong elementary simplicial collapses $H \to H$. 

Consider the hypergraphs $\mathcal{H}$.

Example 4. Let $\mathcal{H}$ be a hypergraph. Proposition 3.2.

There is a finite sequence $\mathcal{H} = \mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_q = \mathcal{H}'$ of hypergraphs where each arrow represents a strong simplicial expansion or a strong simplicial collapse. By Proposition 3.1, for each $i = 1, \ldots, q$, the embedded homology groups $H_s(\mathcal{H}_i)$ and $H_s(\mathcal{H}_{i-1})$ are isomorphic. Thus the embedded homology groups $H_s(\mathcal{H})$ and $H_s(\mathcal{H}')$ are isomorphic.

Proposition 3.2. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs with all the vertices as 0-hyperedges and of the same strong simple-homotopy type. Then their embedded homology groups $H_s(\mathcal{H})$ and $H_s(\mathcal{H}')$ are isomorphic.

Proof. There is a finite sequence $\mathcal{H} = \mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_q = \mathcal{H}'$ of hypergraphs where each arrow represents a strong simplicial expansion or a strong simplicial collapse. By Proposition 3.1, for each $i = 1, \ldots, q$, the embedded homology groups $H_s(\mathcal{H}_i)$ and $H_s(\mathcal{H}_{i-1})$ are isomorphic. Thus the embedded homology groups $H_s(\mathcal{H})$ and $H_s(\mathcal{H}')$ are isomorphic.

Example 4. Consider the hypergraphs

$\mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}\}.$

$\mathcal{H}' = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2, v_3\}\}.$

Then we have strong elementary simplicial collapses $\mathcal{H} \to \mathcal{H}_1 \to \mathcal{H}'$, where

$\mathcal{H}_1 = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}\}.$

Thus $\mathcal{H}$ and $\mathcal{H}'$ have the same strong simple-homotopy type. We have

$H_n(\mathcal{H}) \cong H_n(\mathcal{H}') = \begin{cases} R \oplus R, & n = 0, \\ 0, & n \geq 1. \end{cases}$
3.3 Collapses and the homology of the associated simplicial complexes

We have a long exact sequence

\[ \cdots \rightarrow H_n(\Delta(H \setminus A)) \rightarrow H_n(\Delta H) \rightarrow \tilde{H}_n(\Delta H/|\Delta(H \setminus A)|) \rightarrow H_{n-1}(\Delta(H \setminus A)) \rightarrow \cdots, \]  

(3.3)

where

\[ \tilde{H}_n(\Delta H/|\Delta(H \setminus A)|) = H_n(C_*(\Delta H)/C_*(\Delta(H \setminus A))) \]

is the \( n \)-th reduced homology group of the quotient topological space.

Lemma 3.3. Let \((H, A)\) be a pair of hypergraphs. If there is a weak elementary simplicial collapse from \(H\) to \(A\), then the canonical inclusion of \(\Delta A\) into \(\Delta H\) induces an isomorphism from \(H_\ast(\Delta A)\) to \(H_\ast(\Delta H)\).

Proof. Suppose \(A = H \setminus \{\eta, \xi\}\), where \(\eta\) is a free face of \(\xi\) in \(H\). Let \(\max(H)\) be the collection of all the maximal hyperedges in \(H\). Then

\[ \Delta(H \setminus \{\eta, \xi\}) = \begin{cases} \Delta H \setminus \{\eta, \xi\}, & \text{if } \xi \in \max(H), \\ \Delta H, & \text{if } \xi \notin \max(H). \end{cases} \]

It follows that \(|\Delta H/(\Delta H \setminus \{\eta, \xi\})| \simeq *\), which implies

\[ \tilde{H}_n(\Delta H/\Delta(H \setminus \{\eta, \xi\})) = 0 \]  

(3.4)

for each \(n \geq 0\). In (3.3), substitute \(A\) with \(\{\eta, \xi\}\). With the help of (3.4),

\[ H_n(\Delta A) = H_n(\Delta(H \setminus \{\eta, \xi\})) \cong H_n(\Delta H), \]

where the isomorphism is induced by the canonical inclusion of \(\Delta A\) into \(\Delta H\).

Proposition 3.4. Let \(H\) and \(H'\) be two hypergraphs of the same weak simple-homotopy type, then the homology groups \(H_\ast(\Delta H)\) and \(H_\ast(\Delta H')\) are isomorphic.

Proof. There is a finite sequence \(H = H_0 \rightarrow H_1 \rightarrow \cdots \rightarrow H_q = H'\) of hypergraphs where each arrow represents a weak simplicial expansion or a weak simplicial collapse. By Lemma 3.3 for each \(i = 1, \ldots, q\), \(H_\ast(\Delta H_i)\) and \(H_\ast(\Delta H_{i-1})\) are isomorphic. Thus \(H_\ast(\Delta H)\) and \(H_\ast(\Delta H')\) are isomorphic.

Example 5. Consider the hypergraphs

\[ H = \{\{v_0\}, \{v_2\}, \{v_0, v_1, v_2\}, \{v_2, v_3, v_4\}, \{v_0, v_1, v_2, v_3, v_4\}\}, \]

\[ H' = \{\{v_0\}\}. \]

Then we have weak elementary simplicial collapses \(H \rightarrow H_1 \rightarrow H'\), where

\[ H_1 = \{\{v_0\}, \{v_2\}, \{v_0, v_1, v_2\}\}. \]

Thus \(H\) and \(H'\) have the same weak simple-homotopy type. We have \(\Delta H = \Delta[v_0, v_1, v_2, v_3, v_4]\), \(\Delta H' = \{\{v_0\}\}\) and

\[ H_n(\Delta H) \cong H_n(\Delta H') = \begin{cases} G, & n = 0, \\ 0, & n \geq 1. \end{cases} \]
3.4 Collapses and the homology of The lower-associated simplicial complexes

We have a long exact sequence

\[ \cdots \longrightarrow H_n(\delta(H \setminus A)) \longrightarrow H_n(\delta H) \longrightarrow \tilde{H}_n(|\delta H|/|\delta(H \setminus A)|) \]
\[ \longrightarrow H_{n-1}(\delta(H \setminus A)) \longrightarrow \cdots . \] (3.5)

Lemma 3.5. Let \((H, A)\) be a pair of hypergraphs. Suppose there is a weak elementary simplicial collapse from \(H\) to \(A\), say \(A = H \setminus \{\eta, \xi\}\) where \(\eta, \xi \in H\) such that \(\eta\) is a free face of \(\xi\).

(i). If one of the followings is satisfied:

(a). \(\dim \eta = 0\),
(b). \(\sigma \in H\) for any nonempty subset \(\sigma \subseteq \xi\),
(c). there exists a nonempty subset \(\tau \subseteq \eta\) such that \(\tau \notin H\),

then the canonical inclusion of \(\delta A\) into \(\delta H\) induces an isomorphism \(H_n(\delta A) \cong H_n(\delta H)\) for any \(n \geq 0\);

(ii). If all the followings are satisfied:

(a)' . \(\dim \eta \geq 1\),
(b)' . there exists a nonempty subset \(\sigma \subseteq \xi\) such that \(\sigma \notin H\),
(c)' . \(\tau \in H\) for any nonempty subset \(\tau \subseteq \eta\),

then

(1). the canonical inclusion of \(\delta A\) into \(\delta H\) induces an isomorphism \(H_n(\delta A) \cong H_n(\delta H)\) for any \(n \neq \dim \eta, \dim \eta - 1\),
(2). there is an injection from \(H_{\dim \eta}(\delta A)\) into \(H_{\dim \eta}(\delta H)\) whose cokernel is a subgroup of \(R\),
(3). there is a surjection from \(H_{\dim \eta-1}(\delta A)\) into \(H_{\dim \eta-1}(\delta H)\) whose kernel is a quotient group of \(R\).

Proof. Suppose \(A = H \setminus \{\eta, \xi\}\), where \(\eta\) is a free face of \(\xi\) in \(H\). In (3.5), substitute \(A\) with \(\{\eta, \xi\}\). We obtain a long exact sequence

\[ \cdots \longrightarrow H_n(\delta(H \setminus \{\eta, \xi\})) \longrightarrow H_n(\delta H) \longrightarrow \tilde{H}_n(\delta H/\delta(H \setminus \{\eta, \xi\})) \]
\[ \longrightarrow H_{n-1}(\delta(H \setminus \{\eta, \xi\})) \longrightarrow \cdots . \] (3.6)

Case 1. \(\sigma \in H\) for any nonempty subset \(\sigma \subseteq \xi\).

Then \(\xi \in \delta H\). Since \(\eta\) is not a face of any hyperedge \(\kappa \neq \xi\) in \(H\), it follows that \(\eta\) is not a face of any simplex \(\kappa \neq \xi\) in \(\delta H\). Thus for any \(\kappa \in \delta H\) such that \(\kappa \neq \eta, \xi\) we have \(\kappa \in \delta(H \setminus \{\eta, \xi\})\). Hence \(\delta(H \setminus \{\eta, \xi\}) = \delta H \setminus \{\eta, \xi\}\). Therefore,

\[ |\delta H/(\delta H \setminus \{\eta, \xi\})| \simeq *. \]

Case 2. there exists a nonempty subset \(\sigma \subseteq \xi\) such that \(\sigma \notin H\).

Then \(\xi \notin \delta H\).
Subcase 2.1. $\tau \in \mathcal{H}$ for any nonempty subset $\tau \subseteq \eta$.

Then $\eta \in \delta \mathcal{H}$. Since $\xi \notin \delta \mathcal{H}$ and $\eta$ is not a proper face of any hyperedge $\tau \neq \xi$, it follows that $\eta$ is a maximal simplex of $\delta \mathcal{H}$. Similar with the argument of Case 1 we have $\delta(\mathcal{H} \setminus \{\eta, \xi\}) = \delta \mathcal{H} \setminus \{\eta\}$. Therefore,

$$|\delta \mathcal{H}/(\delta \mathcal{H} \setminus \{\eta, \xi\})| \simeq \begin{cases} \mathbb{S}^{\dim \eta}, & \text{if } \dim \eta \geq 1, \\ \ast, & \text{if } \dim \eta = 0. \end{cases} \quad (3.7)$$

Subcase 2.2. there exists a nonempty subset $\tau \subseteq \eta$ such that $\tau \notin \mathcal{H}$.

Then $\eta \notin \delta \mathcal{H}$. Similar with the argument of Case 1 we have $\delta(\mathcal{H} \setminus \{\eta, \xi\}) = \delta \mathcal{H} \setminus \{\eta\}$. Therefore,

$$|\delta \mathcal{H}/(\delta \mathcal{H} \setminus \{\eta, \xi\})| = \ast.$$

(i). Suppose one of (a), (b) and (c) is satisfied. By summarizing the above cases,

$$\tilde{H}_n(\delta \mathcal{H}/\delta(\mathcal{H} \setminus \{\eta, \xi\})) = 0 \quad (3.8)$$

for each $n \geq 0$. By (3.6) and (3.8) we obtain

$$H_n(\delta(\mathcal{H} \setminus \{\eta, \xi\})) \simeq H_n(\delta \mathcal{H}). \quad (3.9)$$

(ii). Suppose all of (a)', (b)' and (c)' are satisfied. By (3.7),

$$|\delta \mathcal{H}/(\delta \mathcal{H} \setminus \{\eta, \xi\})| \simeq \mathbb{S}^{\dim \eta}.$$

With the help of (3.6),

$$H_n(\delta \mathcal{H}) \cong H_n(\delta A) \quad (3.10)$$

for any $n \neq \dim \eta, \dim \eta - 1$ and an exact sequence

$$0 \to H_{\dim \eta}(\delta(\mathcal{H} \setminus \{\eta, \xi\})) \to H_{\dim \eta}(\delta \mathcal{H}) \to R \to H_{\dim \eta - 1}(\delta(\mathcal{H} \setminus \{\eta, \xi\})) \to H_{\dim \eta - 1}(\delta \mathcal{H}) \to 0. \quad (3.10)$$

Therefore, (1) follows from (3.9), (2) follows from the second arrow of (3.10), and (3) follows from the fifth arrow of (3.10).

Proposition 3.6. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs of the same weak simple-homotopy type via a sequence of weak simplicial collapses and weak simplicial expansions such that in each weak elementary simplicial collapse or weak elementary simplicial expansion, $\eta$ is a free face of $\xi$ and one of (a), (b) and (c) in Lemma 3.5 is satisfied. Then the homology groups $H_*(\delta \mathcal{H})$ and $H_*(\delta \mathcal{H}')$ are isomorphic.

Proof. There is a finite sequence $\mathcal{H} = \mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_q = \mathcal{H}'$ of hypergraphs where each arrow represents a weak simplicial expansion or a weak simplicial collapse. In each weak elementary simplicial collapse or weak elementary simplicial expansion, $\eta$ is a free face of $\xi$ and one of (a), (b) and (c) in Lemma 3.5 is satisfied. Thus by Lemma 3.5(i), for each $i = 1, \ldots, q$, $H_n(\delta \mathcal{H}_i)$ and $H_n(\delta \mathcal{H}_{i-1})$ are isomorphic. Thus $H_n(\delta \mathcal{H})$ and $H_n(\delta \mathcal{H}')$ are isomorphic.
Example 6. Consider the hypergraphs

\[ \mathcal{H} = \{ \{v_0\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_0, v_3\}, \{v_0, v_4\}, \{v_3, v_4\}, \{v_0, v_1, v_2\}, \{v_0, v_3, v_4\}, \{v_0, v_2, v_5\}, \{v_0, v_2, v_5, v_6, v_7\} \}, \]

\[ \mathcal{H}' = \{ \{v_0\} \}. \]

Then we have weak elementary simplicial collapses \( \mathcal{H} \to \mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{H}_3 \to \mathcal{H}_4 \to \mathcal{H}' \),
where

\[ \mathcal{H}_1 = \mathcal{H} \setminus \{ \{v_0, v_2, v_5\}, \{v_0, v_2, v_5, v_6, v_7\} \} \]

with \( \{v_0, v_2, v_5\} \) a free face of \( \{v_0, v_2, v_5, v_6, v_7\} \) satisfying Lemma 3.5 (c),
\[ \mathcal{H}_2 = \mathcal{H}_1 \setminus \{ \{v_3, v_4\}, \{v_0, v_3, v_4\} \} \]

with \( \{v_3, v_4\} \) a free face of \( \{v_0, v_3, v_4\} \) satisfying Lemma 3.5 (b),
\[ \mathcal{H}_3 = \mathcal{H}_2 \setminus \{ \{v_1\}, \{v_0, v_1, v_2\} \} \]

with \( \{v_1\} \) a free face of \( \{v_0, v_1, v_2\} \) satisfying Lemma 3.5 (a), and
\[ \mathcal{H}_4 = \mathcal{H}_3 \setminus \{ \{v_3\}, \{v_0, v_3\} \}, \quad \mathcal{H}' = \mathcal{H}_4 \setminus \{ \{v_4\}, \{v_0, v_4\} \} \]

are elementary simplicial collapses of simplicial complexes. We have \( \delta \mathcal{H} = \Delta[v_0, v_3, v_4], \)
\[ \delta \mathcal{H}' = \{ \{v_0\} \} \]

and

\[ H_n(\delta \mathcal{H}) \cong H_n(\delta \mathcal{H}') = \begin{cases} G, & n = 0, \\ 0, & n \geq 1. \end{cases} \]

4 The stability of persistent homology of hypergraphs

4.1 Filtrations of hypergraphs and interleavings of the persistent homology

Let \( \mathcal{H} \) be a hypergraph. A filtration \( \{\mathcal{H}_t\}_{t \in \mathbb{R}} \) of \( \mathcal{H} \) is a family \( \mathcal{H}_t, t \in \mathbb{R} \), of sub-hypergraphs of \( \mathcal{H} \) such that for any \( t \leq s \), there is an inclusion \( i_{t,s} : \mathcal{H}_t \hookrightarrow \mathcal{H}_s \) such that \( i_{s,r} \circ i_{t,s} = i_{t,r} \).

Let \( f \) be a function on \( \mathcal{H} \) assigning a real number \( f(\sigma) \) to each \( \sigma \in \mathcal{H} \). For each \( t \in \mathbb{R} \), let the level hypergraph be

\[ \mathcal{H}_t^f := f^{-1}((-\infty, t]) = \{ \sigma \in \mathcal{H} \mid f(\sigma) \leq t \}. \tag{4.1} \]

Let \( t \leq s \). Let \( i_t^s(f) : \mathcal{H}_t^f \to \mathcal{H}_s^f \) be the canonical inclusion of hypergraphs, which gives a filtration \( \mathcal{H}^f := \{ \mathcal{H}_t^f \}_{t \in \mathbb{R}} \) of \( \mathcal{H} \). We have the induced inclusions \( \delta(i_t^s(f)) : \delta(\mathcal{H}_t^f) \to \delta(\mathcal{H}_s^f) \) and \( \Delta(i_t^s(f)) : \Delta(\mathcal{H}_t^f) \to \Delta(\mathcal{H}_s^f) \) of simplicial complexes, which give the filtration \( \{ \Delta(\mathcal{H}_t^f) \}_{t \in \mathbb{R}} \) of \( \Delta \mathcal{H} \) and the filtration \( \{ \delta(\mathcal{H}_t^f) \}_{t \in \mathbb{R}} \) of \( \delta \mathcal{H} \) respectively. Let \( \mathbb{F} \) be a field. We have three persistent vector spaces

1. \( \{ H_n(\mathcal{H}_t^f) \}_{t \in \mathbb{R}} := \{ H_n(\mathcal{H}_t^f ; \mathbb{F}), (i_t^s(f))_* \mid r, t, s \in \mathbb{R}, t \leq s \} ; \)
2. \( \{ H_n(\delta(\mathcal{H}_t^f)) \}_{t \in \mathbb{R}} := \{ H_n(\delta(\mathcal{H}_t^f) ; \mathbb{F}), (\delta(i_t^s(f)))_* \mid r, t, s \in \mathbb{R}, t \leq s \} ; \)
3. \( \{ H_n(\Delta(\mathcal{H}_t^f)) \}_{t \in \mathbb{R}} := \{ H_n(\Delta(\mathcal{H}_t^f) ; \mathbb{F}), (\Delta(i_t^s(f)))_* \mid r, t, s \in \mathbb{R}, t \leq s \} ; \)

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Moreover, by Definition 4.2, we obtain (i). The proofs for (ii) and (iii) are similar.

Taking the embedded homology, we have that for any $t, s \in \mathbb{R}, t \leq s$

whose persistent diagrams are respectively

1. $D(\mathcal{H}, f) := D(\{H_n(\mathcal{H}_t^f)\}_{t \in \mathbb{R}})$;
2. $D(\delta \mathcal{H}, f) := D(\{H_n(\delta(\mathcal{H}_t^f))\}_{t \in \mathbb{R}})$;
3. $D(\Delta \mathcal{H}, f) := D(\{H_n(\Delta(\mathcal{H}_t^f))\}_{t \in \mathbb{R}})$.

Let $f, g : \mathcal{H} \to \mathbb{R}$ be two real valued functions on $\mathcal{H}$. The $L^\infty$-distance between $f$ and $g$ is defined by

$$||f - g||_\infty = \sup_\sigma |f(\sigma) - g(\sigma)|$$

where $\sigma$ ranges over all hyperedges of $\mathcal{H}$. Let $d_B^\infty$ be the bottleneck distance (cf. § 10).

**Theorem 4.1.** Let $\mathcal{H}$ be a hypergraph. Let $f$ and $g$ be real-valued functions on $\mathcal{H}$. Then $d_B^\infty(D\mathcal{H}^f, D\mathcal{H}^g) \leq ||f - g||_\infty$. Here

$$d_B^\infty(D\mathcal{H}^f, D\mathcal{H}^g) = \max\{d_B^\infty(D(\mathcal{H}, f), D(\mathcal{H}, g)),
$$

$$d_B^\infty(D(\delta \mathcal{H}, f), D(\delta \mathcal{H}, g)),
$$

$$d_B^\infty(D(\Delta \mathcal{H}, f), D(\Delta \mathcal{H}, g))\}.
$$

**4.1.1 Proof of Theorem 4.1**

**Lemma 4.2.** Let $n \geq 0$. Suppose

$$||f - g||_\infty \leq \epsilon \quad (4.2)$$

for some $\epsilon > 0$. Then the following persistent vector spaces are strongly $\epsilon$-interleaved:

1. the persistent embedded homology $\{H_n(\mathcal{H}_t^f)\}_{t \in \mathbb{R}}$ and $\{H_n(\mathcal{H}_t^g)\}_{t \in \mathbb{R}}$;
2. the persistent homology $\{H_n(\delta(\mathcal{H}_t^f))\}_{t \in \mathbb{R}}$ and $\{H_n(\delta(\mathcal{H}_t^g))\}_{t \in \mathbb{R}}$ of the associated simplicial complexes;
3. the persistent homology $\{H_n(\Delta(\mathcal{H}_t^f))\}_{t \in \mathbb{R}}$ and $\{H_n(\Delta(\mathcal{H}_t^g))\}_{t \in \mathbb{R}}$ of the lower-associated simplicial complexes.

**Proof.** For any $t \in \mathbb{R}$ there are inclusions

$$\varphi_t : \mathcal{H}_t^f \to \mathcal{H}_t^g, \quad \psi_t : \mathcal{H}_t^g \to \mathcal{H}_t^f$$

of hypergraphs. By a diagram chasing,

$$\psi_s \circ i_t^s(g) \circ \varphi_{t-\epsilon} = i_t^{s+\epsilon}(f), \quad i_t^{s+\epsilon}(f) \circ \psi_t = \psi_s \circ i_t^s(g),$$

$$\varphi_s \circ i_t^s(f) \circ \psi_{t-\epsilon} = i_t^{s+\epsilon}(g), \quad i_t^{s+\epsilon}(g) \circ \varphi_t = \varphi_s \circ i_t^s(f).$$

Taking the embedded homology, we have that for any $t \in \mathbb{R}$ there are homomorphisms

$$(\varphi_t)_* : H_*(\mathcal{H}_t^f; \mathbb{F}) \to H_*(\mathcal{H}_t^g; \mathbb{F}), \quad (\psi_t)_* : H_*(\mathcal{H}_t^g; \mathbb{F}) \to H_*(\mathcal{H}_t^f; \mathbb{F}).$$

Moreover,

$$(\psi_s)_* \circ (i_t^s(g))_* \circ (\varphi_{t-\epsilon})_* = (i_t^{s+\epsilon}(f))_*, \quad (i_t^{s+\epsilon}(f))_* \circ (\psi_t)_* = (\psi_s)_* \circ (i_t^s(g))_*,$$

$$(\varphi_s)_* \circ (i_t^s(f))_* \circ (\psi_{t-\epsilon})_* = (i_t^{s+\epsilon}(g))_*, \quad (i_t^{s+\epsilon}(g))_* \circ (\varphi_t)_* = (\varphi_s)_* \circ (i_t^s(f))_*.$$
Proof of Theorem 4.4: By [8, Theorem 4.4] and Lemma 4.2 (a), $d_B^H(D(H, f), D(H, g)) \leq \|f - g\|_{\infty}$. By [8, Theorem 4.4] and Lemma 4.2 (b), $d_B^H(D(\delta H, f), D(\delta H, g)) \leq \|f - g\|_{\infty}$. By [8, Theorem 4.4] and Lemma 4.2 (c), $d_B^H(D(\Delta H, f), D(\Delta H, g)) \leq \|f - g\|_{\infty}$.

4.2 Persistent homology for morphisms between hypergraphs and the stability

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs on $V$ and $V'$ respectively. Let $\varphi : \mathcal{H} \to \mathcal{H}'$ be a morphism of hypergraphs.

(A). Suppose $\{\mathcal{H}'_t\}_{t \in \mathbb{R}}$ is a filtration of $\mathcal{H}'$. We have a pull-back filtration $\{\varphi^*\mathcal{H}'_t\}_{t \in \mathbb{R}}$ of $\mathcal{H}$ induced from $\varphi$, where for each $t \in \mathbb{R}$, 

$$\varphi^*\mathcal{H}'_t = \{\sigma \in \mathcal{H} \mid \varphi(\sigma) \in \mathcal{H}'_t\}.$$ 

we have morphisms of hypergraphs

$$\varphi^*_t : \varphi^*\mathcal{H}'_t \to \mathcal{H}'_t, \ t \in \mathbb{R},$$

where for each $t \in \mathbb{R}$, $\varphi^*_t$ is the restriction of $\varphi$ to $\varphi^*\mathcal{H}'_t$. Consequently, we have a commutative diagram of persistent homology:

$$\begin{array}{ccc}
H_*(\text{Ker}(\Delta(\varphi^*_t)#)) & \rightarrow & H_*(\text{Ker}(\text{Sup}(\varphi^*_t))) \\
\uparrow & & \downarrow \\
H_*(\varphi^*\mathcal{H}'_t) & \rightarrow & H_* \mathcal{H}'_t \\
\downarrow & & \downarrow \\
H_*(\text{Ker}(\delta(\varphi^*_t)#)) & \rightarrow & H_*(\text{Ker}(\text{Inf}(\varphi^*_t)))
\end{array}$$

(B). Suppose $\{\mathcal{H}_t\}_{t \in \mathbb{R}}$ is a filtration of $\mathcal{H}$. We have a push-forward filtration $\{\varphi_*\mathcal{H}_t\}_{t \in \mathbb{R}}$ of $\mathcal{H}'$ induced from $\varphi$, where for each $t \in \mathbb{R}$,

$$\varphi_*\mathcal{H}_t = \{\varphi(\sigma) \mid \sigma \in \mathcal{H}_t\}.$$ 

we have morphisms of hypergraphs

$$\varphi_* : \mathcal{H}_t \to \varphi_*\mathcal{H}_t, \ t \in \mathbb{R},$$

where for each $t \in \mathbb{R}$, $\varphi_*$ is the restriction of $\varphi$ to $\mathcal{H}_t$. Consequently, we have a commu-
4.2.1 Proof of Theorem 4.3

Let $f, g : \mathcal{H} \rightarrow \mathbb{R}$ and let $f', g' : \mathcal{H}' \rightarrow \mathbb{R}$. We have filtrations $\mathcal{H}_t^f = f^{-1}((\infty, t])$ and $\mathcal{H}_t^g = g^{-1}((\infty, t])$ of $\mathcal{H}$ as well as filtrations $\mathcal{H}_t^{f'} = f'^{-1}((\infty, t])$ and $\mathcal{H}_t^{g'} = g'^{-1}((\infty, t])$, $t \in \mathbb{R}$, of $\mathcal{H}'$. Let $\varphi : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of hypergraphs. We have the pull-back filtrations $\varphi^*(\mathcal{H}_t^{f'})$ and $\varphi^*(\mathcal{H}_t^{g'})$ of $\mathcal{H}$ as well as the push-forward filtrations $\varphi_*(\mathcal{H}_t^f)$ and $\varphi_*(\mathcal{H}_t^g)$ of $\mathcal{H}'$.

**Theorem 4.3.** Let $\epsilon > 0$. Let $f$ and $g$ be real-valued functions on $\mathcal{H}$. Let $f'$ and $g'$ be real-valued functions on $\mathcal{H}'$.

(i). If $\|f - g\|_{\infty} \leq \epsilon$, then each of the persistent linear map among the twenty-one persistent linear maps in the commutative diagram in (B), denoted as $\Phi_f$ and $\Phi_g$, satisfies

$$d_B^\infty(D(\Phi_f), D(\Phi_g)) \leq \epsilon.$$ 

Here

$$d_B^\infty(D(\Phi_f), D(\Phi_g)) = \max\{d_B^\infty(D(\Ker(\Phi_f)), D(\Ker(\Phi_g))),$$

$$d_B^\infty(D(\Im(\Phi_f)), D(\Im(\Phi_g))),$$

$$d_B^\infty(D(\Coker(\Phi_f)), D(\Coker(\Phi_g))))\};$$

(ii). If $\|f' - g'\|_{\infty} \leq \epsilon$, then each of the persistent linear map among the twenty-one persistent linear maps in the commutative diagram in (A), denoted as $\Phi_f'$ and $\Phi_g'$, satisfies

$$d_B^\infty(D(\Phi_f'), D(\Phi_g')) \leq \epsilon.$$
\( \mathcal{U}, \mathcal{V}' \) and \( \mathcal{U}' \) respectively. Let \( \Phi : \mathcal{V} \to \mathcal{U} \) and \( \Phi' : \mathcal{V}' \to \mathcal{U}' \) be homomorphisms of persistent modules (cf. [Section 1.3]). We have a persistent sub-R-module \( \text{Ker}(\Phi) = \{ \text{Ker}(\Phi_t) \}_{t \in R} \) of \( \mathcal{V} \) and a persistent sub-R-module \( \text{Im}(\Phi) = \{ \text{Im}(\Phi_t) \}_{t \in R} \) of \( \mathcal{U} \). We also have a persistent quotient R-module \( \text{Coker}(\Phi) = \{ \text{Coker}(\Phi_t) \}_{t \in R} \) of \( \mathcal{U} \). As persistent R-modules, \( \mathcal{V}/\text{Ker}(\Phi) \cong \text{Im}(\Phi) \) and \( \mathcal{U}/\text{Im}(\Phi) = \text{Coker}(\Phi) \). Let \( \epsilon > 0 \).

**Definition 4.** We say that \( \Phi \) and \( \Phi' \) are strongly \( \epsilon \)-interleaved if all of the followings are satisfied:

(a). there exist two families of homomorphisms \( \{ \alpha_t : V_t \to V_{t+\epsilon} \}_{t \in R} \) and \( \{ \alpha'_t : V'_t \to V'_{t+\epsilon} \}_{t \in R} \) such that \( \mathcal{V} \) and \( \mathcal{V}' \) are strongly \( \epsilon \)-interleaved via these two families of homomorphisms;

(b). there exist two families of homomorphisms \( \{ \beta_t : U_t \to U_{t+\epsilon} \}_{t \in R} \) and \( \{ \beta'_t : U'_t \to U'_{t+\epsilon} \}_{t \in R} \) such that \( \mathcal{U} \) and \( \mathcal{U}' \) are strongly \( \epsilon \)-interleaved via these two families of homomorphisms;

(c). for any \( t \leq s \), the following four diagrams commute

\[
\begin{array}{c}
V_t \xrightarrow{\phi_t} V_s & V'_t \xrightarrow{\phi'_t} V'_s & V_t \xrightarrow{\phi_t} V_{t+\epsilon} & V'_t \xrightarrow{\phi'_t} V'_{t+\epsilon} \\
\downarrow \mu^t & \downarrow \mu'^t & \downarrow \alpha_t & \downarrow \alpha'_t \\
U_t & U'_s & U_t & U'_t \\
\end{array}
\]

**Lemma 4.4.** Suppose two homomorphisms \( \Phi : \mathcal{V} \to \mathcal{U} \) and \( \Phi' : \mathcal{V}' \to \mathcal{U}' \) are strongly \( \epsilon \)-interleaved. Then the following homomorphisms are also strongly \( \epsilon \)-interleaved:

(i). \( \Phi : \text{Ker}(\Phi) \to 0 \) and \( \Phi' : \text{Ker}(\Phi') \to 0 \);

(ii). \( \Phi/\text{Ker} : \mathcal{V}/\text{Ker}(\Phi) \to \text{Im}(\Phi) \) and \( \Phi'/\text{Ker} : \mathcal{V}'/\text{Ker}(\Phi') \to \text{Im}(\Phi') \);

(iii). \( \Phi/\text{Im} : 0 \to \text{Coker}(\Phi) \) and \( \Phi'/\text{Im} : 0 \to \text{Coker}(\Phi') \).

**Proof.** Let \( t \in R \). Both \( \beta_t \) and \( \beta'_t \) send \( 0 \) to \( 0 \). By the third and the forth commutative diagrams in Definition 4 (c) respectively, \( \alpha_t \) sends \( \text{Ker}(\Phi_t) \) to \( \text{Ker}(\Phi'_{t+\epsilon}) \) and \( \alpha'_t \) sends \( \text{Ker}(\Phi'_t) \) to \( \text{Ker}(\Phi_{t+\epsilon}) \). Hence \( \Phi : \text{Ker}(\Phi) \to 0 \) and \( \Phi' : \text{Ker}(\Phi') \to 0 \), which are the restrictions of \( \Phi \) and \( \Phi' \) to \( \text{Ker}(\Phi) \) and \( \text{Ker}(\Phi') \) respectively, are well-defined. By the strong \( \epsilon \)-interleaving property of \( \Phi \) and \( \Phi' \), \( \Phi \) and \( \Phi' \) satisfy (a), (b) and (c) in Definition 4. Hence \( \Phi : \text{Ker}(\Phi) \to 0 \) and \( \Phi' : \text{Ker}(\Phi') \to 0 \) satisfy (a), (b) and (c) in Definition 4 as well. Thus (i) follows.

The following two families

\[
\{ \alpha_t/\text{Ker} : V_t/\text{Ker}(\Phi_t) \to V'_{t+\epsilon}/\text{Ker}(\Phi'_{t+\epsilon}) \}_{t \in R}, \quad (4.3)
\]

\[
\{ \alpha'_t/\text{Ker} : V'_t/\text{Ker}(\Phi'_t) \to V'_{t+\epsilon}/\text{Ker}(\Phi'_{t+\epsilon}) \}_{t \in R}, \quad (4.4)
\]

of homomorphisms are well-defined. Via the two families (4.3) and (4.4) of homomorphisms, the persistent modules \( \mathcal{V}/\text{Ker}(\Phi) \) and \( \mathcal{V}'/\text{Ker}(\Phi') \) are strongly \( \epsilon \)-interleaved. By substituting \( \mathcal{V} \) and \( \mathcal{V}' \) in Definition 4 with \( \mathcal{V}/\text{Ker}(\Phi) \) and \( \mathcal{V}'/\text{Ker}(\Phi') \) respectively and substituting \( \{ \alpha_t \}_{t \in R} \) and \( \{ \alpha'_t \}_{t \in R} \) in Definition 4 with (4.3) and (4.4) respectively, the four diagrams in (c) still commute. This implies that \( \Phi/\text{Ker} \) and \( \Phi'/\text{Ker} \) are strongly
\(\epsilon\)-interleaved via the persistent modules \(\mathcal{V}/\text{Ker}(\Phi), \mathcal{V}'/\text{Ker}(\Phi'), \mathcal{U}\) and \(\mathcal{U}'\) and the families \(\{\alpha_t/\text{Ker}\}_{t \in \mathbb{R}}\) in (4.3), \(\{\alpha'_t/\text{Ker}\}_{t \in \mathbb{R}}\) in (4.4), \(\{\beta_t\}_{t \in \mathbb{R}}\) and \(\{\beta'_t\}_{t \in \mathbb{R}}\) of homomorphisms. Thus (ii) follows.

Similar with the proof of (i), the homomorphisms \(\Phi/\text{Im} : 0 \rightarrow \text{Coker}(\Phi)\) and \(\Phi'/\text{Im} : 0 \rightarrow \text{Coker}(\Phi')\) as well as the families

\[
\begin{align*}
\{\beta_t/\text{Im} : U_t/\text{Im}(\Phi_t) & \rightarrow U'_{t+\epsilon}/\text{Im}(\Phi'_{t+\epsilon})\}_{t \in \mathbb{R}}, \\
\{\beta'_t/\text{Im} : U'_t/\text{Im}(\Phi'_t) & \rightarrow U'_{t+\epsilon}/\text{Im}(\Phi'_{t+\epsilon})\}_{t \in \mathbb{R}}
\end{align*}
\]

(4.5) and (4.6) of homomorphisms are well-defined. the persistent modules \(\mathcal{U}/\text{Im}(\Phi)\) and \(\mathcal{U}'/\text{Im}(\Phi')\) are strongly \(\epsilon\)-interleaved via (4.5) and (4.6). By substituting \(\mathcal{U}\) and \(\mathcal{U}'\) in Definition 4.4 with \(\mathcal{U}/\text{Im}(\Phi)\) and \(\mathcal{U}'/\text{Im}(\Phi')\) respectively and substituting \(\{\beta_t\}_{t \in \mathbb{R}}\) and \(\{\beta'_t\}_{t \in \mathbb{R}}\) in Definition 4.4 with (4.5) and (4.6) respectively, the four diagrams in (c) commute. This implies that \(\Phi/\text{Im}\) and \(\Phi'/\text{Im}\) are strongly \(\epsilon\)-interleaved via the persistent modules \(\mathcal{V}, \mathcal{V}', \mathcal{U}/\text{Im}(\Phi)\) and \(\mathcal{U}'/\text{Im}(\Phi')\) and the families \(\{\alpha_t\}_{t \in \mathbb{R}}, \{\alpha'_t\}_{t \in \mathbb{R}}, \{\beta_t/\text{Im}\}_{t \in \mathbb{R}}\) in (4.5) and \(\{\beta'_t/\text{Im}\}_{t \in \mathbb{R}}\) in (4.6) of homomorphisms. Thus (iii) follows.

The next corollary (i), (ii) and (iii) follow from Lemma 4.4 (i), (ii) and (iii) respectively.

**Corollary 4.5.** Suppose two homomorphisms \(\Phi : \mathcal{V} \rightarrow \mathcal{U}\) and \(\Phi' : \mathcal{V}' \rightarrow \mathcal{U}'\) are strongly \(\epsilon\)-interleaved. Then the following persistent \(\mathcal{R}\)-modules are also strongly \(\epsilon\)-interleaved:

(i). \(\text{Ker}(\Phi)\) and \(\text{Ker}(\Phi')\);

(ii). \(\mathcal{V}/\text{Ker}(\Phi)\) and \(\mathcal{V}'/\text{Ker}(\Phi')\), or equivalently, \(\text{Im}(\Phi)\) and \(\text{Im}(\Phi')\);

(iii). \(\text{Coker}(\Phi)\) and \(\text{Coker}(\Phi')\).

Let \(\mathcal{R}\) be a field \(\mathbb{F}\). Choose a persistent basis \(b(\text{Ker}(\Phi))\) for \(\text{Ker}(\Phi)\). By \[12\] Theorem 1.1, we may extend \(b(\text{Ker}(\Phi))\) to a persistent basis \(b(\text{Ker}(\Phi)) \sqcup b(\Phi)\) of \(\mathcal{V}\), where \(b(\Phi)\) is a persistent basis for \(\mathcal{V}/\text{Ker}(\Phi)\). The persistent linear map \(\Phi\) sends \(b(\Phi)\) bijectively to a persistent basis \(\Phi(b(\Phi))\) of \(\text{Im}(\Phi)\). By \[12\] Theorem 1.1, we may extend \(\Phi(b(\Phi))\) to a persistent basis \(\Phi(b(\Phi)) \sqcup b(\text{Coker}(\Phi))\) of \(\mathcal{U}\), where \(b(\text{Coker}(\Phi))\) is a persistent basis for \(\text{Coker}(\Phi)\). Similarly, we have a persistent basis \(b(\text{Ker}(\Phi'))\) for \(\text{Ker}(\Phi')\), a persistent basis \(b(\Phi')\) for \(\mathcal{V}'/\text{Ker}(\Phi')\), a persistent basis \(\Phi'(b(\Phi'))\) for \(\text{Im}(\Phi')\), and a persistent basis \(b(\text{Coker}(\Phi'))\) for \(\text{Coker}(\Phi')\). By taking the birth-times and the death-times of the elements in the persistent bases, we have the corresponding persistent diagrams.

**Lemma 4.6.** Let \(\mathcal{R}\) be a field. Suppose the persistent \(\mathcal{F}\)-linear maps \(\Phi : \mathcal{V} \rightarrow \mathcal{U}\) and \(\Phi' : \mathcal{V}' \rightarrow \mathcal{U}'\) are strongly \(\epsilon\)-interleaved. Then

(i). \(d_\mathcal{B}(\Phi(\text{Ker}(\Phi)), \Phi(\text{Ker}(\Phi'))) \leq \epsilon\);

(ii). \(d_\mathcal{B}((\Phi(\text{Im}(\Phi))), (\Phi(\text{Im}(\Phi'))) \leq \epsilon\);

(iii). \(d_\mathcal{B}(\Phi(\text{Coker}(\Phi)), \Phi(\text{Coker}(\Phi'))) \leq \epsilon\).

**Proof.** By applying \[8\] Theorem 4.4 to Corollary 4.5 (i), (ii) and (iii) respectively, we obtain (i), (ii) and (iii) in Lemma 4.6.

Define the persistent diagram of a persistent linear map \(\Phi\) as the triple

\[D(\Phi) = (\Phi(\text{Ker}(\Phi)), \Phi(\text{Im}(\Phi)), \Phi(\text{Coker}(\Phi))).\]
Define the $L^\infty$-bottleneck distance between two persistent linear maps $\Phi$ and $\Phi'$ as
\[
d_B^\infty(D(\Phi), D(\Phi')) = \max\{d_B^\infty(D(\text{Ker}(\Phi)), D(\text{Ker}(\Phi'))),
\]
\[
d_B^\infty(D(\text{Im}(\Phi)), D(\text{Im}(\Phi'))),
\]
\[
d_B^\infty(D(\text{Coker}(\Phi)), D(\text{Coker}(\Phi')))\}.
\]

The next corollary follows from Lemma 4.6 directly.

**Corollary 4.7.** Suppose two persistent linear maps $\Phi$ and $\Phi'$ are strongly $\epsilon$-interleaved. Then
\[
d_B^\infty(D(\Phi), D(\Phi')) \leq \varepsilon. \quad \square
\]

**Proof of Theorem 4.5.** (i). Let $\Phi_f$ and $\Phi_g$ be the persistent linear maps. Suppose $||f - g||_\infty \leq \varepsilon$. Then Definition 3 (a), (b) and (c) hold. Thus $\Phi_f$ and $\Phi_g$ are strongly $\epsilon$-interleaved. By Corollary 4.7 we have (i).

(ii). Let $\Phi'_f$, and $\Phi'_g'$ be the persistent linear maps. Suppose $||f' - g'||_\infty \leq \varepsilon$. Similar with (i), $\Phi'_f$, and $\Phi'_g'$ are strongly $\epsilon$-interleaved. By Corollary 4.7 we have (ii). \quad \square

### 4.3 The stability of the Mayer-Vietoris sequences

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs. Let $\{\mathcal{H}_t\}_{t \in \mathbb{R}}$ be a filtration of $\mathcal{H}$ and let $\{\mathcal{H}'_t\}_{t \in \mathbb{R}}$ be a filtration of $\mathcal{H}'$ such that for each $t \in \mathbb{R}$, $\mathcal{H}_t$ and $\mathcal{H}'_t$ satisfy the hypothesis (P) in Proposition 2.1. Then we have the following commutative diagram of persistent homology

\[
\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_n(\delta(\mathcal{H}_t) \cap \delta(\mathcal{H}'_t)) & \overset{\alpha_n}{\longrightarrow} & H_n(\delta(\mathcal{H}_t) \oplus H_n(\delta(\mathcal{H}'_t)) & \overset{\beta_n}{\longrightarrow} & \\
\cdots & \longrightarrow & H_n(\mathcal{H}_t \cap \mathcal{H}'_t) & \overset{\beta_n}{\longrightarrow} & H_n(\mathcal{H}_t) \oplus H_n(\mathcal{H}'_t) & \overset{\beta'_n}{\longrightarrow} & \\
\cdots & \longrightarrow & H_n(\Delta(\mathcal{H}_t) \cap \Delta(\mathcal{H}'_t)) & \overset{\alpha_n}{\longrightarrow} & H_n(\Delta(\mathcal{H}_t)) \oplus H_n(\Delta(\mathcal{H}'_t)) & \overset{\beta_n}{\longrightarrow} & \\
\end{array}
\]

where each row is a long exact sequence of persistent modules and each vertical map is a homomorphism of persistent modules induced by the canonical inclusions of hypergraphs. We have
\[
\begin{align*}
\text{Ker}(\alpha_n) &= \text{Im}(\gamma_{n+1}), & \text{Ker}(\beta_n) &= \text{Im}(\alpha_n), & \text{Ker}(\gamma_n) &= \text{Im}(\beta_n), \\
\text{Coker}(\alpha_n) &\cong \text{Im}(\beta_n), & \text{Coker}(\beta_n) &\cong \text{Im}(\gamma_n), & \text{Coker}(\gamma_n) &\cong \text{Im}(\alpha_{n-1}).
\end{align*}
\]
The persistent diagram of the persistent linear map $\alpha_n$ is
\[
D(\alpha_n) = (D(\text{Ker}(\alpha_n)), D(\text{Im}(\alpha_n)), D(\text{Coker}(\alpha_n)))
\]
\[
= (D(\text{Ker}(\beta_{n+1})), D(\text{Ker}(\beta_n)), D(\text{Im}(\beta_n)))
\]
\[
= (D(\text{Im}(\gamma_{n+1})), D(\text{Coker}(\gamma_{n+1})), D(\text{Ker}(\gamma_n))).
\]
(4.7)

Similarly, we can give the explicit expressions for $D(\beta_n)$ and $D(\gamma_n)$. Let $f, g : \mathcal{H} \rightarrow \mathbb{R}$ and $f', g' : \mathcal{H}' \rightarrow \mathbb{R}$ such that for each $t \in \mathbb{R}$, both the pair $(\mathcal{H}_t^g, \mathcal{H}_t^{g'})$ and the pair $(\mathcal{H}_t^g, \mathcal{H}_t^{g'})$ satisfy the hypothesis (P) in Proposition 2.1. Denote the persistent linear maps $\alpha_n$, $\beta_n$ and $\gamma_n$ induced by $f$ and $g$ as $\alpha_{n,f,g}$, $\beta_{n,f,g}$ and $\gamma_{n,f,g}$ respectively. The maximum among the three distances
\[
d_B^\infty(\alpha_{n,f,g}, \alpha_{n,f',g'}), \quad d_B^\infty(\beta_{n,f,g}, \beta_{n,f',g'}), \quad d_B^\infty(\gamma_{n,f,g}, \gamma_{n,f',g'})
\]
equals to the maximum among the five distances
\[
d_B^\infty(\text{Ker}(\alpha_{n,f,g}), \text{Ker}(\alpha_{n,f',g'})), \quad d_B^\infty(\text{Ker}(\alpha_{n-1,f,g}), \text{Ker}(\alpha_{n-1,f',g'})),
\]
\[
d_B^\infty(\text{Im}(\alpha_{n,f,g}), \text{Im}(\alpha_{n,f',g'})), \quad d_B^\infty(\text{Im}(\alpha_{n-1,f,g}), \text{Im}(\alpha_{n-1,f',g'})),
\]
\[
d_B^\infty(\text{Coker}(\alpha_{n,f,g}), \text{Coker}(\alpha_{n,f',g'})),
\]
which equals to the maximum among the five distances
\[
d_B^\infty(\text{Ker}(\beta_{n,f,g}), \text{Ker}(\beta_{n,f',g'})), \quad d_B^\infty(\text{Ker}(\beta_{n-1,f,g}), \text{Ker}(\beta_{n-1,f',g'})),
\]
\[
d_B^\infty(\text{Coker}(\beta_{n,f,g}), \text{Coker}(\beta_{n,f',g'})), \quad d_B^\infty(\text{Coker}(\beta_{n+1,f,g}), \text{Coker}(\beta_{n+1,f',g'})),
\]
\[
d_B^\infty(\text{Im}(\beta_{n,f,g}), \text{Im}(\beta_{n,f',g'})),
\]
as well as the maximum among the five distances
\[
d_B^\infty(\text{Im}(\gamma_{n,f,g}), \text{Im}(\gamma_{n,f',g'})), \quad d_B^\infty(\text{Im}(\gamma_{n+1,f,g}), \text{Im}(\gamma_{n+1,f',g'})),
\]
\[
d_B^\infty(\text{Coker}(\gamma_{n,f,g}), \text{Coker}(\gamma_{n,f',g'})), \quad d_B^\infty(\text{Coker}(\gamma_{n+1,f,g}), \text{Coker}(\gamma_{n+1,f',g'})),
\]
\[
d_B^\infty(\text{Ker}(\gamma_{n,f,g}), \text{Ker}(\gamma_{n,f',g'})).
\]

Analogous assertions hold for the persistent linear maps $\alpha_{\delta,*}$, $\beta_{\delta,*}$ and $\gamma_{\delta,*}$ as well as the persistent linear maps $\alpha_{\Delta,*}$, $\beta_{\Delta,*}$ and $\gamma_{\Delta,*}$.

Choose a persistent module in the last diagram. Denote the persistent module induced by $f$ and $g$ as $M_{f,g}$ and denote the persistent module induced by $f'$ and $g'$ as $M_{f',g'}$. Choose an arrow in the last diagram. Denote the arrow induced by $f$ and $g$ as $\Phi_{f,g}$ and denote the arrow induced by $f'$ and $g'$ as $\Phi_{f',g'}$.

Theorem 4.8. If $||f - f'||_\infty \leq \epsilon$ and $||g - g'||_\infty \leq \epsilon$, then $d_B^\infty(D(M_{f,g}), D(M_{f',g'})) \leq \epsilon$ and $d_B^\infty(D(\Phi_{f,g}), D(\Phi_{f',g'})) \leq \epsilon$.

Proof. Suppose $||f - f'||_\infty \leq \epsilon$ and $||g - g'||_\infty \leq \epsilon$. Then $M_{f,g}$ and $M_{f',g'}$ are strongly $\epsilon$-interleaved. Thus $d_B^\infty(D(M_{f,g}), D(M_{f',g'})) \leq \epsilon$. Moreover, $\Phi_{f,g}$ and $\Phi_{f',g'}$ are strongly $\epsilon$-interleaved. Thus $d_B^\infty(D(\Phi_{f,g}), D(\Phi_{f',g'})) \leq \epsilon$.

4.4 The stability of the Künneth-type formulae

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs. Let $\{\mathcal{H}_t\}_{t \in \mathbb{R}}$ be a filtration of $\mathcal{H}$ and let $\{\mathcal{H}'_t\}_{t \in \mathbb{R}}$ be a filtration of $\mathcal{H}'$. Let $R$ be a principal ideal domain with unit 1. By Proposition 2.2 we
have a commutative diagram of persistent modules

\[
\begin{align*}
\bigoplus_{p+q+1=n} H_{p+1}(\delta(H_t)) \otimes H_{q+1}(\delta(H'_t)) & \rightarrow H_{n+1}(\delta(H_t) \ast \delta(H'_t)) \\
\bigoplus_{p+q+1=n} H_{p+1}(H_t) \otimes H_{q+1}(H'_t) & \rightarrow H_{n+1}(H_t \ast H'_t) \\
\bigoplus_{p+q+1=n} H_{p+1}(\Delta(H_t)) \otimes H_{q+1}(\Delta(H'_t)) & \rightarrow H_{n+1}(\Delta(H_t) \ast \Delta(H'_t))
\end{align*}
\]

\[
\begin{align*}
\rightarrow \bigoplus_{p+q+1=n} \text{Tor}_R(H_{p+1}(\delta(H_t)), H_q(\delta(H'_t))) \\
\rightarrow \bigoplus_{p+q+1=n} \text{Tor}_R(H_{p+1}(H_t), H_q(H'_t)) \\
\rightarrow \bigoplus_{p+q+1=n} \text{Tor}_R(H_{p+1}(\Delta(H_t)), H_q(\Delta(H'_t)))
\end{align*}
\]

such that each row is a short exact sequence of persistent modules and each vertical map is a homomorphism of persistent modules induced by canonical inclusions of hypergraphs. Let \( R = \mathbb{F} \) be a field. Then all the torsions in the last diagram are trivial.

Let \( D = \{(b_i, d_i) \mid i = 1, \ldots, l \} \cup \{(t, t) \mid t \in \mathbb{R} \} \) and \( D' = \{(b'_j, d'_j) \mid j = 1, \ldots, m \} \cup \{(t, t) \mid t \in \mathbb{R} \} \), where \(-\infty \leq b_i < d_i \leq +\infty \) for each \( 1 \leq i \leq l \) and \(-\infty \leq b'_j < d'_j \leq +\infty \) for each \( 1 \leq j \leq m \), be two persistent diagrams. We define the product of \( D \) and \( D' \) as

\[
DD' = \{(\max(b_i, b'_j), \min(d_i, d'_j)) \mid 1 \leq i \leq l, 1 \leq j \leq m \} \cup \{(t, t) \mid t \in \mathbb{R} \}.
\]

The multiplicity of \((\max(b_i, b'_j), \min(d_i, d'_j))\) is the product of the multiplicity of \((b_i, d_i)\) in \( D \) and the multiplicity of \((b'_j, d'_j)\) in \( D' \). We define the sum of \( D \) and \( D' \) as

\[
D + D' = \{(b_i, d_i) \mid 1 \leq i \leq l \} \cup \{(b'_j, d'_j) \mid 1 \leq j \leq m \} \cup \{(t, t) \mid t \in \mathbb{R} \}.
\]

For any \((b, d) \in D + D'\), if \((b, d) = (b_i, d_i) = (b'_j, d'_j)\) for some \( 1 \leq i \leq l \) and some \( 1 \leq j \leq m \), then the multiplicity of \((b, d)\) is the sum of the multiplicity of \((b_i, d_i)\) in \( D \) and the multiplicity of \((b'_j, d'_j)\) in \( D' \).

**Proposition 4.9.** We have

\[
D(H_{n+1}(\delta(H_t) \ast \delta(H'_t)))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(\delta(H_t)))_{t \in \mathbb{R}}
\]

\[
D(H_{n+1}(H_t \ast H'_t))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(H_t))_{t \in \mathbb{R}}
\]

\[
D(H_{n+1}(\Delta(H_t) \ast \Delta(H'_t)))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(\Delta(H_t)))_{t \in \mathbb{R}}
\]

\[
D(H_{q+1}(\delta(H'_t)))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(\delta(H'_t)))_{t \in \mathbb{R}}, \quad (4.8)
\]

\[
D(H_{q+1}(H'_t))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(H'_t))_{t \in \mathbb{R}}, \quad (4.9)
\]

\[
D(H_{q+1}(\Delta(H'_t)))_{t \in \mathbb{R}} = \sum_{p+q+1=n} D(H_{p+1}(\Delta(H'_t)))_{t \in \mathbb{R}}.
\] (4.10)
Proof. Take the persistent diagram of each persistent module in the last diagram. The equations (4.8), (4.9) and (4.10) follow from the first row, the second row and the third row in the last diagram respectively.

Let $f, g : \mathcal{H} \to \mathbb{R}$ and let $f', g' : \mathcal{H}' \to \mathbb{R}$. Choose a persistent module in the last diagram. Denote the persistent module induced by $f$ and $g$ as $M_{f,g}$ and denote the persistent module induced by $f'$ and $g'$ as $M_{f',g'}$. Choose an arrow in the last diagram. Denote the arrow induced by $f$ and $g$ by $\Phi_{f,g}$ and denote the arrow induced by $f'$ and $g'$ by $\Phi_{f',g'}$.

**Theorem 4.10.** If $\|f - f'\|_{\infty} \leq \epsilon$ and $\|g - g'\|_{\infty} \leq \epsilon$, then $d_{B}^{\infty}(D(M_{f,g}), D(M_{f',g'})) \leq \epsilon$ and $d_{B}^{\infty}(D(\Phi_{f,g}), D(\Phi_{f',g'})) \leq \epsilon$.

Proof. The proof is an analog of Theorem 4.8.

5 The constancy of persistent Betti numbers of hypergraphs

Given a filtration $\{\mathcal{H}_{t}\}_{t \in \mathbb{R}}$ of a hypergraph $\mathcal{H}$, consider the persistent embedded homology

$$H_{n}^{t,s}(\{\mathcal{H}_{t}\}_{t \in \mathbb{R}}) = \text{Im}((\iota_{t})_{s})$$

$$= \frac{\text{Ker}(\partial_{n}(\mathcal{H}_{t}))}{\text{Ker}(\partial_{n}(\mathcal{H}_{t})) \cap \text{Im}(\partial_{n+1}(\mathcal{H}_{s}))},$$

where $t \leq s$, $\partial_{n}(\mathcal{H}_{t})$ is the restriction of the boundary map $\partial_{n}$ of $\Delta \mathcal{H}$ to $\text{Inf}_{n}(\mathcal{H}_{t})$, $\iota_{t}$ is the canonical inclusion of $\mathcal{H}_{t}$ into $\mathcal{H}_{s}$ and $((\iota_{t})_{s})$ is the induced homomorphism of the embedded homology groups. Similarly, consider the persistent homology

$$H_{n}^{t,s}(\{\Delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}) = \text{Im}(((\iota_{\Delta})_{t})_{s})$$

$$= \frac{\text{Ker}(\partial_{n}(\Delta(\mathcal{H}_{t})))}{\text{Ker}(\partial_{n}(\Delta(\mathcal{H}_{t})) \cap \text{Im}(\partial_{n+1}(\Delta(\mathcal{H}_{s}))))},$$

$$H_{n}^{t,s}(\{\delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}) = \text{Im}(((\iota_{\delta})_{t})_{s})$$

$$= \frac{\text{Ker}(\partial_{n}(\delta(\mathcal{H}_{t})))}{\text{Ker}(\partial_{n}(\delta(\mathcal{H}_{t})) \cap \text{Im}(\partial_{n+1}(\delta(\mathcal{H}_{s}))))},$$

where $\partial_{n}(\Delta(\mathcal{H}_{t}))$ (resp. $\partial_{n}(\delta(\mathcal{H}_{t}))$) is the restriction of the boundary map $\partial_{n}$ of $\Delta \mathcal{H}$ (resp. $\delta \mathcal{H}$) to $C_{n}(\Delta(\mathcal{H}_{t}); R)$ (resp. $C_{n}(\delta(\mathcal{H}_{t}); R)$) and $((\iota_{\Delta})_{t})$ (resp. $((\iota_{\delta})_{t})$) is the canonical inclusion of $\Delta(\mathcal{H}_{t})$ (resp. $\delta(\mathcal{H}_{t})$) into $\Delta(\mathcal{H}_{s})$ (resp. $\delta(\mathcal{H}_{s})$). Take the coefficients of the persistent homology (5.1), (5.2) and (5.3) in a field $F$. The persistent Betti numbers of the filtrations $\{\mathcal{H}_{t}\}_{t \in \mathbb{R}}$, $\{\Delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}$ and $\{\delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}$ are respectively

$$\beta_{n}^{t,s} = \dim_{F} H_{n}^{t,s}(\{\mathcal{H}_{t}\}_{t \in \mathbb{R}}),$$

$$\beta_{t,s}^{\Delta,n} = \dim_{F} H_{n}^{t,s}(\{\Delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}),$$

$$\beta_{t,s}^{\delta,n} = \dim_{F} H_{n}^{t,s}(\{\delta(\mathcal{H}_{t})\}_{t \in \mathbb{R}}).$$

With the help of [13, Section 2.3], $\beta_{n}^{t,s}$ is the sum of the multiplicities $m_{b,d}$ of the points $(b, d)$ in the persistent diagram $D(\{H_{n}(\mathcal{H}_{t})\}_{t \in \mathbb{R}})$ such that $b \leq t$ and $d > s$, $\beta_{t,s}^{\Delta,n}$ is the sum of the multiplicities $m_{b,d}$ of the points $(b, d)$ in the persistent diagram $D(\{H_{n}(\Delta(\mathcal{H}_{t}))\}_{t \in \mathbb{R}})$ such that $b \leq t$ and $d > s$, and $\beta_{t,s}^{\delta,n}$ is the sum of the multiplicities $m_{b,d}$ of the points $(b, d)$ in the persistent diagram $D(\{H_{n}(\delta(\mathcal{H}_{t}))\}_{t \in \mathbb{R}})$ such that $b \leq t$ and $d > s$. 

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Theorem 5.1. Let $\mathcal{H}$ be a hypergraph and let $f : \mathcal{H} \to \mathbb{R}$. For any $t \leq s$,

1. if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have all the vertices as 0-hyperedges and have the same strong simple-homotopy type, then $\beta_{n,t}^s = \beta_n(\mathcal{H}_t^f) = \beta_n(\mathcal{H}_s^f)$, where $\beta_n$ is the Betti number of the embedded homology group;

2. if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have the same weak simple-homotopy type, then $\beta_{n,t}^s = \beta_n(\Delta(\mathcal{H}_t^f)) = \beta_n(\Delta(\mathcal{H}_s^f))$;

3. if $\mathcal{H}_t^f$ and $\mathcal{H}_s^f$ have the same weak simple-homotopy type via a sequence of weak simplicial collapses and weak elementary simplicial expansion, $\eta$ is a free face of $\xi$ and one of (a), (b) and (c) in Lemma 3.5 is satisfied, then $\beta_{n,t}^s = \beta_n(\delta(\mathcal{H}_t^f)) = \beta_n(\delta(\mathcal{H}_s^f))$.

Proof. For any $t \leq s$, we have $\mathcal{H}_t^f \subseteq \mathcal{H}_s^f$, $1 \leq i \leq k$.

1. By our assumption, there is a sequence $\mathcal{H}_s^f = \mathcal{H}_t^f \to \mathcal{H}^1 \to \cdots \to \mathcal{H}^n = \mathcal{H}_s^f$ of hypergraphs such that each arrow is a strong elementary simplicial collapse and for each $0 \leq j \leq n$, all the vertices of $\mathcal{H}_j^f$ are 0-hyperedges. By Lemma 3.1 for each $1 \leq j \leq n$, the canonical inclusion of $\mathcal{H}_j^f$ into $\mathcal{H}_j^{j-1}$ induces an isomorphism from $H_* (\mathcal{H}_j^f)$ to $H_* (\mathcal{H}_j^{j-1})$. Thus the inclusion of $\mathcal{H}_t^f$ into $\mathcal{H}_s^f$ induces an isomorphism from $H_* (\mathcal{H}_t^f)$ to $H_* (\mathcal{H}_s^f)$. Therefore, $\beta_{n,t}^s = \beta_n(\mathcal{H}_t^f) = \beta_n(\mathcal{H}_s^f)$.

2. By our assumption, there is a sequence $\mathcal{H}_s^f = \mathcal{H}_t^f \to \mathcal{H}^1 \to \cdots \to \mathcal{H}^n = \mathcal{H}_s^f$ of hypergraphs such that each arrow is a weak elementary simplicial collapse. By Lemma 3.3 for each $1 \leq j \leq n$, the canonical inclusion of $\mathcal{H}_j^f$ into $\mathcal{H}_j^{j-1}$ induces an isomorphism from $H_* (\Delta \mathcal{H}_j^f)$ to $H_* (\Delta \mathcal{H}_j^{j-1})$. Thus the inclusion of $\mathcal{H}_t^f$ into $\mathcal{H}_s^f$ induces an isomorphism from $H_* (\Delta(\mathcal{H}_t^f))$ to $H_* (\Delta(\mathcal{H}_s^f))$. Therefore, $\beta_{n,t}^s = \beta_n(\Delta(\mathcal{H}_t^f)) = \beta_n(\Delta(\mathcal{H}_s^f))$.

3. By our assumption, there is a sequence $\mathcal{H}_s^f = \mathcal{H}_t^f \to \mathcal{H}^1 \to \cdots \to \mathcal{H}^n = \mathcal{H}_s^f$ of hypergraphs such that each arrow is a weak elementary simplicial collapse satisfying the hypothesis in (3). By Lemma 3.5 for each $1 \leq j \leq n$, the canonical inclusion of $\mathcal{H}_j^f$ into $\mathcal{H}_j^{j-1}$ induces an isomorphism from $H_* (\delta \mathcal{H}_j^f)$ to $H_* (\delta \mathcal{H}_j^{j-1})$. Thus the inclusion of $\mathcal{H}_t^f$ into $\mathcal{H}_s^f$ induces an isomorphism from $H_* (\delta(\mathcal{H}_t^f))$ to $H_* (\delta(\mathcal{H}_s^f))$. Therefore, $\beta_{n,t}^s = \beta_n(\delta(\mathcal{H}_t^f)) = \beta_n(\delta(\mathcal{H}_s^f))$. $\square$

References

[1] Timothy G. Barraclough, How do species interactions affect evolutionary dynamics across whole communities? Annu. Rev. Ecol. Evol. Syst. 46 (2015), 25-48.

[2] Federico Battiston, Giulia Cencetti, Iacopo Iacopini, Vito Latora, Maxime Lucas, Alice Patania, Jean-Gabriel Young and Giovanni Petri, Networks beyond pairwise interactions: Structure and dynamics. Phys. Rep. 874(25) (2020), 1-92.

[3] Federico Battiston, Enrico Amico, Alain Barrat, Ginestra Bianconi, Guilherme Ferraz de Arruda, Benedetta Franceschiello, Iacopo Iacopini, Sonia Kéfi, Vito Latora, Yamir Moreno, Micah M. Murray, Tiago P. Peixoto, Francesco Vaccarino and Giovanni Petri, The physics of higher-order interactions in complex systems. Nat. Phys. 17(10) (2021), 1093-1098.

[4] Claude Berge, Graphs and hypergraphs. North-Holland Mathematical Library, Amsterdam, 1973.
[5] G. E. Bredon, *Sheaf theory*. Graduate Texts in Mathematics (GTM vol. 170), Springer, New York, 1997.

[6] Stephane Bressan, Jingyan Li, Shiquan Ren and Jie Wu, *The embedded homology of hypergraphs and applications*. Asian J. Math. 23(3) (2019), 479–500.

[7] Marshall M. Cohen, *A Course in simple-homotopy theory*. Springer-Verlag New York Inc., 1973.

[8] Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J. Guibas and Steve Y. Oudot, *Proximity of persistence modules and their diagrams*. Proceedings of the 25-th annual symposium on computational geometry, ACM, 2009, 237–246.

[9] Frédéric Chazal, Vin de Silva, Marc Glisse and Steve Oudot, *The structure and stability of persistence modules*. Springer Cham, 2016.

[10] David Cohen-Steiner, Herbert Edelsbrunner and John Harer, *Stability of persistence diagrams*. Discrete Comput. Geom. 37 (2007), 103–120.

[11] David Cohen-Steiner, Herbert Edelsbrunner, John Harer and Dmitriy Morozov, *Persistent homology for kernels, images and cokernels*. Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (2009), 1011–1020.

[12] William Crawley-Boevey, *Decomposition of point-wise finite dimensional persistence modules*. J. Algebra Appl. 14(5) (2015), 1550066.

[13] Trinh Khanh Duy, Yasuaki Hiraoka and Tomoyuki Shirai, *Limit theorems for persistence diagrams*. Ann. Appl. Probab. 28(5) (2018), 2740-2780.

[14] Herbert Edelsbrunner and John Harer, *Computational topology: an introduction*. The American Mathematical Society, 2010.

[15] Herbert Edelsbrunner, David Letscher and Afra Zomorodian, *Topological persistence and simplification*. Discrete Comput. Geom. 28 (2002), 511–533.

[16] Robin Forman, *Morse theory for cell complexes*. Adv. Math. 134 (1) (1998), 90-145.

[17] Jelena Grbić, Jie Wu, Kelin Xia and Guo-Wei Wei, *Aspects of topological approaches for data science*. Foundations of Data Science, American Institute of Mathematical Sciences 4(2) (2022), 165-216.

[18] Alexander Grigor’yan, Yong Lin and Shing-Tung Yau, *Torsion of digraphs and path complexes*. arXiv: 2012.07302v1, 2020.

[19] Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau, *Homologies of path complexes and digraphs*. arXiv: 1207.2834, 2013.

[20] Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau, *Homotopy theory for digraphs*. Pure Appl. Math. Q. 10(4) (2014), 619–674.

[21] Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau, *Cohomology of digraphs and (undirected) graphs*. Asian J. Math. 15(5) (2015), 887-932.

[22] Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau, *Path complexes and their homologies*. J. Math. Sci. 248(5) (2020), 564-599.

[23] Allen Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2001.
[24] Xiang Liu, Xiangjun Wang, Jie Wu and Kelin Xia, *Hypergraph-based persistent cohomology (HPC) for molecular representations in drug design*. Brief. Bioinform. **22**(5) (2021), DOI: 10.1093/bib/bbaa411.

[25] Xiang Liu, Huitao Feng, Jie Wu and Kelin Xia, *Persistent spectral hypergraph based machine learning (PSH-ML) for protein-ligand binding affinity prediction*. Brief. Bioinform. **22**(5) (2021), no. bbab127.

[26] James R. Munkres, *Elements of algebraic topology*. Addison-Wesley Publishing Company, California, 1984.

[27] A. D. Parks and S. L. Lipscomb, *Homology and hypergraph acyclicity: a combinatorial invariant for hypergraphs*. Naval Surface Warfare Center, 1991.

[28] John J. Stachowicz, *Mutualism, facilitation, and the structure of ecological communities: positive interactions play a critical, but underappreciated, role in ecological communities by reducing physical or biotic stresses in existing habitats and by creating new habitats on which many species depend*. Bioscience **51** (2001), 235-246.

[29] Jean-Gabriel Young, Giovanni Petri and Tiago P. Peixoto, *Hypergraph reconstruction from network data*. Commun. Phys. **135** (2021), https://doi.org/10.1038/s42005-021-00637-w.

[30] Chong Wang, Shiquan Ren and Jian Liu, *A Küneth formula for finite sets*. Chin. Ann. of Math. Ser. B **42**(6) (2021), 801-812.

[31] Chengyuan Wu, Shiquan Ren, Jie Wu and Kelin Xia, *Discrete Morse theory for weighted simplicial complexes*. Topol. Appl. **270** (2020), Article 107038.

[32] Shuang Wu, Libo Jiang, Xiaqing He, Yi Jin, Christopher H. Griffin and Rongling Wu, *A quantitative decision theory of animal conflict*. Heliyon **7** (2021) e07621.

[33] A. Zomorodian and G. Carlsson, *Computing persistent homology*. Discrete Comput. Geom. **33**(2) (2005), 249-274.

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