Quasisymmetric rigidity of Sierpiński carpets $F_{n,p}$

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Abstract. We study a new class of square Sierpiński carpets $F_{n,p}$ ($5 \leq n$, $1 \leq p < (n/2) - 1$) on $S^2$, which are not quasisymmetrically equivalent to the standard Sierpiński carpets. We prove that the group of quasisymmetric self-maps of each $F_{n,p}$ is the Euclidean isometry group of $F_{n,p}$. We also establish that $F_{n,p}$ and $F'_{n',p'}$ are quasisymmetrically equivalent if and only if $(n, p) = (n', p')$.

1. Introduction

The quasisymmetric geometry of Sierpiński carpets is related to the study of Julia sets in complex dynamics and boundaries of Gromov hyperbolic groups. For background and research progress, we recommend the survey of Bonk [3].

Let $S^2$ be the unit sphere in $\mathbb{R}^3$. Let $S = S^2 \setminus \bigcup_{i \in \mathbb{N}} D_i$ be the complement in $S^2$ of countably many pairwise disjoint open Jordan regions $D_i \subset S^2$. We call $S$ a (Sierpiński) carpet if $S$ has empty interior, diam$(D_i) \to 0$ as $i \to \infty$, and $\partial D_i \cap \partial D_j = \emptyset$ for all $i \neq j$. The boundary of $D_i$, denoted by $C_i$, is called a peripheral circle of $S$.

Topologically all carpets are the same [12]. A much richer structure arises if we consider quasisymmetric geometry of metric carpets.

The concept of a quasisymmetric map between metric spaces was defined by Tukia and Väisälä [11]. Let $f : X \to Y$ be a homeomorphism between two metric spaces $(X, d_X)$ and $(Y, d_Y)$. Then $f$ is quasisymmetric if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right) \quad \text{for all } x, y, z \in X, x \neq z.$$

It follows from the definition that the set of quasisymmetric self-maps of $X$ form a group QS($X$).

A round carpet is a carpet on $S^2$ such that all of its peripheral circles are geometric circles. Typical examples of round carpets arise from limit sets of convex co-compact Kleinian groups. The famous conjecture of Kapovich–Kleiner [9] predicts that if $G$...
Figure 1. The standard Sierpiński carpet $S_3$.

is a hyperbolic group with boundary $\partial_{\infty}G$ homeomorphic to a carpet, then $G$ acts geometrically (the action is isometrical, properly discontinuous and co-compact) on a convex subset of $\mathbb{H}^3$ with non-empty totally geodesic boundary. The Kapovich–Kleiner conjecture is equivalent to the conjecture that $\partial_{\infty}G$ (endowed with a ‘visual’ metric) is quasisymmetrically equivalent to a round carpet on $S^2$. The conjecture is known to be true if $\partial_{\infty}G$ can be quasisymmetrically embedded into $S^2$ [4].

1.1. Motivation. In the work of Bonk and Merenkov [6], it was proved that:

1. every quasisymmetric self-homeomorphism of the standard 1/3-Sierpiński carpet $S_3$ (see Figure 1) is a Euclidean isometry [6, Theorem 1.1];

2. for the standard 1/p-Sierpiński carpets $S_p$, $p \geq 3$ odd, the groups $\text{QS}(S_p)$ of quasisymmetric self-maps are finite dihedral [6, Theorem 1.2]. It was conjectured that $\text{QS}(S_p) = \text{Isom}(S_p)$;

3. the carpets $S_p$ and $S_q$ are quasisymmetrically equivalent if and only if $p = q$ [6, Theorem 1.3].

The main tool in their proof is carpet modulus, which is a certain discrete modulus of a path family and is preserved under quasisymmetric maps of carpets.

The aim of this paper is to extend their results to a new class of Sierpiński carpets $F_{n,p}$ ($5 \leq n$, $1 \leq p < (n/2) - 1$). We will show that $\text{QS}(F_{n,p}) = \text{Isom}(F_{n,p})$. This is a further application of the works [4–6].

1.2. Main results. Unless otherwise indicated, we will equip a carpet $S = S^2 \setminus \bigcup_{i \in \mathbb{N}} D_i$ with the spherical metric. Note that when a carpet is contained in a compact set $K$ of $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} \cong S^2$, the Euclidean and the spherical metrics are bi-Lipschitz equivalent on $K$. 
Let $5 \leq n$, $1 \leq p < (n/2) - 1$ be integers. Let $Q^{(0)}_{n,p} = [0, 1] \times [0, 1]$ be the closed unit square in $\mathbb{R}^2$. We first subdivide $Q^{(0)}_{n,p}$ into $n^2$ subsquares with equal side-length $1/n$ and remove the interior of four subsquares; each has side-length $1/n$ and is of distance $\sqrt{2}p/n$ to one of the four corner points of $Q^{(0)}_{n,p}$.

The resulting set $Q^{(1)}_{n,p}$ consists of $(n^2 - 4)$ squares of side-length $1/n$. Inductively, $Q^{(k+1)}_{n,p}$, $k \geq 1$, is obtained from $Q^{(k)}_{n,p}$ by subdividing each of the remaining squares in the subdivision of $Q^{(k)}_{n,p}$ into $n^2$ subsquares of equal side-length $1/n^{k+1}$ and removing the interior of four subsquares as we have done before.

The Sierpiński carpet $F_{n,p}$ is the intersection of all the sets $Q^{(k)}_{n,p}$, i.e.,

$$F_{n,p} = \bigcap_{k=0}^{+\infty} Q^{(k)}_{n,p}.$$ 

See Figure 2.

The following theorem will be proved in §4. It shows that, from the point of view of quasiconformal geometry, the carpets $F_{n,p}$ are different to the standard Sierpiński carpets $S_m$, $m \geq 3$ odd (note that the standard Sierpiński carpets $S_m$ are constructed from a similar process, by removing the interior of the middle square in each step).

**Theorem 1.** Let $5 \leq n$, $1 \leq p < (n/2) - 1$ be integers. The carpet $F_{n,p}$ is not quasisymmetrically equivalent to any standard Sierpiński carpet $S_m$, $m \geq 3$ odd.

In §6, we will show the following.

**Theorem 2.** Let $f$ be a quasisymmetric self-map of $F_{n,p}$. Then $f$ is a Euclidean isometry.
Note that the Euclidean isometric group of $F_{n,p}$ (and $S_m$), consisting of eight elements, is the group generated by the reflections in the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$ and the vertical line $\{(x, y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$.

We will also prove the following theorem.

**Theorem 3.** Two Sierpiński carpets $F_{n,p}$ and $F_{n',p'}$ are quasisymmetrically equivalent if and only if $(n, p) = (n', p').$

1.3. **Idea of proofs.** The main tools to prove our theorems are the so-called carpet modulus and weak tangent, both of which were investigated in [6].

Our arguments follow the same outline as [6]. One of the most important observations in [6] is that a quasisymmetric self-map $f$ of $S_3$ should preserve the pair $\{M, O\}$, where $M$ and $O$ are the inner and outer peripheral circles of $S_3$, respectively. By counting the orbits of points under the action of $\text{QS}(S_3)$, Bonk and Merenkov [6] then showed that $f$ maps distinguished points (points of $S_3$ on the corner or on the middle of peripheral circles) to distinguished points. Moreover, $f$ induces a ‘tangent map’ between weak tangents of distinguished points, which are also quasisymmetric maps. The study of carpet modulus with respect to the normalized quasisymmetry group of weak tangent shows that $f$ must map $M$ to $M$ and $O$ to $O$.

We will first concentrate on the carpet modulus of the families of curves connecting the boundary of the annulus domains bounded by pairs of distinct peripheral circles of $F_{n,p}$. The extremal mass distribution of such a carpet modulus exists and is unique (Proposition 3.6). This, together with some auxiliary results proved by [6] (see §3), allows us to show that (see §4) any quasisymmetric self-map $f$ of $F_{n,p}$ should preserve the set $\{O, M_1, M_2, M_3, M_4\}$, where $O$ is the boundary of the unit square and $M_1, M_2, M_3, M_4$ are the boundary of the first four squares removed from the unit square.

It is more difficult to see that $f$ should map $O$ to $O$. To show this, we first study the weak tangents of the carpets (this is our main work in §5). In §6, we prove that $f(O) = O$ by counting the orbit of a corner of $O$ or $M_1$ under the group $\text{QS}(F_{n,p})$. The proofs of Theorems 2 and 3 are given at the end of §6.

In §7, we give a remark that our results can be generalized to a more general class of Sierpiński carpets.

It is worthwhile to note that our proofs (and the proofs of Bonk and Merenkov [6]) rely on the assumption that the carpets are symmetric with respect to the reflections in the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$ and the vertical line $\{(x, y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$. It seems that the methods here cannot be applied to a wider class of carpets. For example, let $S$ be a carpet defined by the same iterative procedure as that of $S_p$, $p \geq 5$ odd, removing some subsquare not on the center of the square instead. Then it is unknown whether a quasisymmetric self-map $f$ of $S$ preserves the pair of inner and outer peripheral circles of $S$ or not.

2. **Carpet modulus**

An important tool in the study of quasisymmetric maps is the conformal modulus of a given family of paths. The notion of conformal modulus (or extremal length) was first
introduced by Beurling and Ahlfors [2]. It has a lot of applications in complex analysis and metric geometry [7, 10].

In this section, we shall recall the definitions of conformal modulus and carpet modulus. The latter was introduced by Bonk and Merenkov [6]. We will collect several important properties of carpet modulus that will be used in the rest of our paper. In many cases, we will neglect the proof and refer to [4–6] instead.

2.1. Conformal modulus. A path $\gamma$ in a metric space $X$ is a continuous map $\gamma : I \to X$ of a finite interval $I$. Without causing confusion, we shall identify the map with its image $\gamma(I)$ and denote a path by $\gamma$. We say that $\gamma$ is open if $I = (a, b)$. The limits $\lim_{t \to a} \gamma(t)$ and $\lim_{t \to b} \gamma(t)$, if they exist, are called the end points of $\gamma$. If $A, B \subseteq X$, then we say that $\gamma$ connects $A$ and $B$ if $\gamma$ has endpoints such that one of them lies in $A$ and the other lies in $B$. If $I = [a, b]$ is a closed interval, then the length of $\gamma : I \to X$ is defined by

$$\text{length}(\gamma) := \sup \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|$$

where the supremum is taken over all finite sequences $a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b$. If $I$ is not closed, then we set

$$\text{length}(\gamma) := \sup_{J} \text{length}(\gamma|J),$$

where $J$ is taken over all closed subintervals of $I$ and $\gamma|J$ denotes the restriction of $\gamma$ on $J$. We call $\gamma$ rectifiable if its length is finite. Similarly, a path $\gamma : I \to X$ is locally rectifiable if its restriction to each closed subinterval is rectifiable. Any rectifiable path $\gamma : I \to X$ has a unique extension $\overline{\gamma}$ to the closure $\overline{I}$ of $I$.

Let $\Gamma$ be a family of paths in $S^2$. Let $\sigma$ be the spherical measure and $ds$ be the spherical line element on $S^2$ induced by the spherical metric. A non-negative Borel function $\rho : S^2 \to [0, \infty]$ is called admissible if

$$\int_{\gamma} \rho \, ds \geq 1$$

for all locally rectifiable paths $\gamma \in \Gamma$. The conformal modulus of $\Gamma$ is defined as

$$\text{mod}(\Gamma) := \inf_{\rho} \int_{S^2} \rho^2 \, d\sigma,$$

where the infimum is taken over all admissible functions $\rho$.

Conformal modulus is monotone and subadditive (see [1]).

- If $\Gamma_1$ and $\Gamma_2$ are two families of paths such that each path $\gamma$ in $\Gamma_1$ contains a subpath $\gamma' \in \Gamma_2$, then

$$\text{mod}(\Gamma_1) \leq \text{mod}(\Gamma_2). \tag{2.1}$$

- For any countable path families $\{\Gamma_i\}_{i=1}^{\infty}$, we have

$$\text{mod} \left( \bigcup_{i=1}^{\infty} \Gamma_i \right) \leq \sum_{i=1}^{\infty} \text{mod}(\Gamma_i). \tag{2.2}$$
In this paper, we shall adopt the metric definition of quasiconformal maps \[8, \text{ Definition 1.2}\] and allow them to be orientation-reversing. Suppose that \( f : X \to Y \) is a homeomorphism between two metric spaces \( X \) and \( Y \). Then \( f \) is quasiconformal if there is a constant \( H \geq 1 \) (the distortion of \( f \)), such that for all \( x \in X, \)

\[
\lim_{r \to 0^+} \sup \max \left\{ \frac{d(f(x), f(y))}{d(x, y) \leq r} : d(x, y) \leq r \right\} \leq H.
\]

Let \( \Omega \) and \( \Omega' \) be regions in \( \mathbb{S}^2 \) and let \( \Gamma \) be a family of paths in \( \Omega \). Denote by \( f(\Gamma) = \{ f \circ \gamma \mid \gamma \in \Omega \} \). Suppose that \( f : \Omega \to \Omega' \) is a quasiconformal map. Then

\[
\frac{1}{K} \text{mod}(\Gamma) \leq \text{mod}(f(\Gamma)) \leq K \text{mod}(\Gamma),
\]

where \( K \geq 1 \) is a constant depending on the distortion of \( f \).

Conformal modulus is a conformal invariant, that is, \( \text{mod}(\Gamma) = \text{mod}(f(\Gamma)) \) if \( f : \Omega \to \Omega' \) is a conformal map. From (2.3), a quasiconformal map preserves the modulus of a path family up to a fixed multiplicative constant. In particular, if \( \Gamma_0 \subseteq \Gamma \) and \( \text{mod}(\Gamma_0) = 0 \), then \( \text{mod}(f(\Gamma_0)) = 0 \).

2.2. Carpet modulus. If a certain property for paths in \( \Gamma \) holds for all paths outside an exceptional family \( \Gamma_0 \subseteq \Gamma \) with \( \text{mod}(\Gamma_0) = 0 \), then we say that it holds for almost every path in \( \Gamma \).

Let \( S = \mathbb{S}^2 \setminus \bigcup_{i=1}^{\infty} D_i \) be a carpet with \( C_i = \partial D_i \), and let \( \Gamma \) be a family of paths in \( \mathbb{S}^2 \). A mass distribution \( \rho \) is a function that assigns to each \( C_i \) a non-negative number \( \rho(C_i) \). A mass distribution \( \rho \) that satisfies

\[
\sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i) \geq 1
\]

for almost every path in \( \Gamma \) is called admissible.

The carpet modulus of \( \Gamma \) with respect to \( S \) is defined as

\[
\text{mod}_S(\Gamma) = \inf_{\rho} \sum_i \rho(C_i)^2,
\]

where the infimum is taken over all mass distributions \( \rho \).

It is straightforward to check that the carpet modulus is monotone and countably subadditive, the same properties as conformal modulus in (2.1) and (2.2). An crucial property of carpet modulus is its invariance under quasiconformal maps.

**Lemma 2.1.** [6, Lemma 2.1] Let \( D, \tilde{D} \subseteq \mathbb{S}^2 \) be regions and \( f : D \to \tilde{D} \) be a quasiconformal map. Let \( S \subseteq D \) be a carpet and \( \Gamma \) be a family of paths such that \( \gamma \subset D \) for each \( \gamma \in \Gamma \). Then

\[
\text{mod}_{f(S)}(f(\Gamma)) = \text{mod}_S(\Gamma).
\]

Let \( S = \mathbb{S}^2 \setminus \{D_i\}, C_i = \partial D_i \) be a carpet and \( \Gamma \) be a family of paths on \( \mathbb{S}^2 \). An admissible mass distribution \( \rho \) for a carpet modulus \( \text{mod}_S(\Gamma) \) is called extremal if \( \text{mod}_S(\Gamma) \) is obtained by \( \rho \),

\[
\text{mass}(\rho) = \sum_i \rho(C_i)^2 = \text{mod}_S(\Gamma).
\]
The peripheral circles \( \{C_i\} \) are \textit{uniform quasicircles} if there exists a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that every \( C_i \) is the image of an \( \eta \)-quasisymmetric map of the unit circle.

**Proposition 2.2.** [6, Proposition 2.4] Let \( S \) be a carpet in \( S^2 \) whose peripheral circles are uniform quasicircles, and let \( \Gamma \) be an arbitrary path family in \( S^2 \) with \( \text{mod}_S(\Gamma) < +\infty \). Then the extremal mass distribution for \( \text{mod}_S(\Gamma) \) exists and is unique.

We also need the notion of \textit{carpet modulus with respect to a group}.

Let \( S = S^2 \setminus \bigcup_{i \in \mathbb{N}} D_i \) be a carpet and \( C_i = \partial D_i \). Let \( G \) be a group of homeomorphisms of \( S \). If \( g \in G \) and \( C \subseteq S \) is a peripheral circle of \( S \), then \( g(C) \) is also a peripheral circle of \( S \). Let \( \mathcal{O} = \{g(C) : g \in G\} \) be the orbit of \( C \) under the action of \( G \).

Let \( \Gamma \) be a family of paths in \( S^2 \). A admissible \( G \)-invariant mass distribution \( \rho : \{C_i\} \to [0, +\infty] \) is a mass distribution such that:

1. \( \rho(g(C)) = \rho(C) \) for all \( g \in G \) and all peripheral circles \( C \) of \( S \);
2. almost every path \( \gamma \) in \( \Gamma \) satisfies
   \[ \sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i) \geq 1. \]

The \textit{carpet modulus} \( \text{mod}_{S/G}(\Gamma) \) with respect to the action of \( G \) on \( S \) is defined as
\[
\text{mod}_{S/G}(\Gamma) := \inf_{\rho} \sum_{\mathcal{O}} \rho(\mathcal{O})^2,
\]
where the infimum is taken over all admissible \( G \)-invariant mass distributions. In the above definition, \( \rho(\mathcal{O}) \) is defined by \( \rho(C) \) for any \( C \in \mathcal{O} \). Since \( \rho \) is \( G \)-invariant, \( \rho(\mathcal{O}) \) is well defined. Note that each orbit contributes exactly one term to the sum \( \sum_{\mathcal{O}} \rho(\mathcal{O})^2 \).

**Lemma 2.3.** [6, Lemma 3.1] Let \( D \) be a region in \( S^2 \) and \( S \) be a carpet contained in \( D \). Let \( G \) be a group of homeomorphisms on \( S \). Suppose that \( \Gamma \) is a family of paths with \( \gamma \subseteq D \) for each \( \gamma \in \Gamma \) and \( f : D \to \tilde{D} \) a quasiconformal map onto another region \( \tilde{D} \subseteq S^2 \). We denote \( \tilde{S} = f(S), \tilde{\Gamma} = f(\Gamma) \) and \( \tilde{G} = (f|_S) \circ G \circ (f|_S)^{-1} \), then
\[
\text{mod}_{S/G}(\Gamma) = \text{mod}_{\tilde{S}/\tilde{G}}(\tilde{\Gamma}).
\]

**Lemma 2.4.** [6, Lemma 3.3] Let \( S \) be a carpet in \( S^2 \) and \( \Psi : S^2 \to S^2 \) be a quasiconformal map with \( \Psi(S) = S, \Psi := \Psi|_S \). Assume that \( \Gamma \) is a \( \Psi \)-invariant path family in \( S^2 \) such that for every peripheral circle \( C \) of \( S \) that meets some path in \( \Gamma \) we have \( \Psi^n(C) \neq C \) for all \( n \in \mathbb{Z} \). Then \( \text{mod}_{S/(\Psi^k)}(\Gamma) = k \text{mod}_{S/(\Psi)}(\Gamma) \) for every \( k \in \mathbb{N} \).

In Lemma 2.4, \( \langle \psi \rangle \) denotes the cyclic group of homeomorphisms on \( S \) generated by \( \psi \), and \( \Gamma \) is called \( \Psi \)-\textit{invariant} if \( \Psi(\Gamma) = \Gamma \). This lemma gives a precise relationship between the carpet modulus with respect to a cyclic group and its subgroups.

There is a criterion for the existence of an extremal mass distribution for carpet modulus with respect to a group (see [6, Proposition 3.2] and the proof of Lemma 5.4 below).
3. Auxiliary results

3.1. Uniformization and rigidity. Let \( S = \mathbb{S}^2 \backslash \{ D_i \}, C_i = \partial D_i \) be a carpet. The peripheral circles \( \{ C_i \} \) of \( S \) are called uniformly relatively separated if the pairwise distances are uniformly bounded away from zero. i.e., there exists \( \delta > 0 \) such that

\[
\Delta(C_i, C_j) = \frac{\text{dist}(C_i, C_j)}{\min\{\text{diam}(C_i), \text{diam}(C_j)\}} \geq \delta
\]

for any two distinct \( i \) and \( j \). This property is preserved under quasisymmetric maps [4, Corollary 4.6].

**Proposition 3.1.** [4, Proposition 5.1] Let \( S \) be a carpet in \( \mathbb{S}^2 \) whose peripheral circles are uniform quasicircles and let \( f \) be a quasisymmetric map of \( S \) onto another carpet \( \tilde{S} \subseteq \mathbb{S}^2 \). Then there exists a self-quasiconformal map \( F \) on \( \mathbb{S}^2 \) whose restriction to \( S \) is \( f \).

A carpet \( S = \mathbb{S}^2 \backslash \bigcup D_i \) is called round if each \( D_i \) is an open spherical disk.

**Theorem 3.2.** [4, Corollary 1.2] Let \( S \) be a carpet in \( \mathbb{S}^2 \) whose peripheral circles are uniformly relatively separated uniform quasicircles, then there exists a quasisymmetric map of \( S \) onto a round carpet.

A Möbius transformation is a fractional linear transformation on \( \mathbb{S}^2 \cong \hat{\mathbb{C}} \) or the complex-conjugate of such a map. We allow a Möbius transformation to be orientation-reversing.

**Theorem 3.3.** [5, Theorem 1.2] Let \( S \) be a round carpet in \( \mathbb{S}^2 \) of measure zero. Then every quasisymmetric map of \( S \) onto any other round carpet is the restriction of a Möbius transformation.

Let \( S \subseteq \mathbb{S}^2 \) be a carpet. A homeomorphism embedding \( f : S \to \mathbb{S}^2 \) is called orientation-preserving if some homeomorphic extension \( F : \mathbb{S}^2 \to \mathbb{S}^2 \) of \( f \) is orientation-preserving on \( \mathbb{S}^2 \) (such an extension exists and the definition is independent of the choice of extension, see the proof of [4, Lemma 5.3]).

**Theorem 3.4.** [6] Let \( S \) be a carpet in \( \mathbb{S}^2 \) of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles and \( C_i, i = 1, 2, 3 \), be three distinct peripheral circles of \( S \). Let \( f \) and \( g \) be two orientation-preserving quasisymmetric self-maps of \( S \). Then we have the following rigidity results.

1. Assume that \( f(C_i) = g(C_i) \) for \( i = 1, 2, 3 \). Then \( f = g \).
2. Assume that \( f(C_i) = g(C_i) \) for \( i = 1, 2 \) and \( f(p) = g(p) \) for a given point \( p \in S \). Then \( f = g \).
3. Assume that \( G \) is the group of all orientation-preserving quasisymmetric self-maps of \( S \) that fix \( C_1, C_2 \). Then \( G \) is a finite cyclic group.
4. Assume that \( G \) is the group of all orientation-preserving quasisymmetric self-maps of \( S \) that fix \( C_1 \) and fix a given point \( q \in C_1 \). Then \( G \) is an infinite cyclic group.

3.2. Square carpets. A \( \mathbb{C}^* \)-cylinder \( A \) is a set of the form

\[
A = \{ z \in \mathbb{C}; r \leq |z| \leq R \}
\]
with \(0 < r < R < +\infty\). The metric on \(A\) is induced by the length element \(|dz|/|z|\) which is the flat metric. Equipped with this metric, \(A\) is isometric to a finite cylinder of circumference \(2\pi\) and length \(\log(R/r)\). The boundary components \(\{z \in \mathbb{C} : |z| = r\}\) and \(\{z \in \mathbb{C} : |z| = R\}\) are called the inner and outer boundary components of \(A\), respectively.

A \(\mathbb{C}^*\)-square \(Q\) is a Jordan region of the form
\[
Q = \{\rho e^{i\theta} : a < \rho < b, \alpha < \theta < \beta\}
\]
with \(0 < \log(b/a) = \beta - \alpha < 2\pi\). We call the quantity
\[
l_{\mathbb{C}^*}(Q) = \log(b/a) = \beta - \alpha
\]
its side-length. Clearly, two opposite sides of \(Q\) are parallel to the boundaries of \(A\), while the other two are perpendicular to the boundaries of \(A\).

A square carpet \(T\) in a \(\mathbb{C}^*\)-cylinder \(A\) is a carpet that can be written as
\[
T = A \bigcup_i Q_i,
\]
where the sets \(Q_i, i \in I\), are \(\mathbb{C}^*\)-squares whose closures are pairwise disjoint and contained in the interior of \(A\).

**Theorem 3.5.** [4, Theorem 1.6] Let \(S\) be a carpet of measure zero in \(\mathbb{S}^2\) whose peripheral circles are uniformly relatively separated uniform quasicircles, and \(C_1\) and \(C_2\) two distinct peripheral circles of \(S\). Then there exists a quasisymmetric map \(f\) from \(S\) onto a square carpet \(T\) in a \(\mathbb{C}^*\)-cylinder \(A\) such that \(f(C_1)\) is the inner boundary component of \(A\) and \(f(C_2)\) is the outer one.

Let \(S\) be a carpet in \(\mathbb{S}^2\) and \(C_1, C_2\) be two distinct peripheral circles of \(S\). Suppose that the Jordan regions \(D_1\) and \(D_2\) are the complementary components of \(S\) bounded by \(C_1\) and \(C_2\) respectively. We let \(\Gamma(C_1, C_2)\) be the family of all open paths in \(\mathbb{S}^2 \setminus \overline{D}_1 \cup \overline{D}_2\) that connect \(\overline{D}_1\) and \(\overline{D}_2\).

**Proposition 3.6.** [4, Corollary 12.2] Let \(S\) be a carpet of measure zero in \(\mathbb{S}^2\) whose peripheral circles are uniformly relatively separated uniform quasicircles, and \(C_1\) and \(C_2\) two distinct peripheral circles of \(S\). Then:

1. \(\text{mod}_S(\Gamma(C_1, C_2))\) has finite and positive total mass;
2. let \(f\) be a quasisymmetric map of \(S\) onto a square carpet \(T\) in a \(\mathbb{C}^*\)-cylinder \(A = \{z \in \mathbb{C} : r \leq |z| \leq R\}\) such that \(C_1\) corresponds to the inner and \(C_2\) to the outer boundary components of \(A\). Then the extremal mass distribution of \(\text{mod}_S(\Gamma(C_1, C_2))\) is given as
\[
\rho(C_1) = \rho(C_2) = 0, \quad \rho(C) = \frac{l_{\mathbb{C}^*}(f(C))}{\log(R/r)}
\]
with the peripheral circles \(C \neq C_1, C_2\) of \(S\).

Let \(S\) be a carpet in a closed Jordan region \(D \subset \hat{\mathbb{C}}\); \(S\) is called a square carpet if \(\partial D\) is a peripheral circle of \(S\) and all other peripheral circles are squares with sides parallel to the coordinate axes.
Theorem 3.7. [6, Theorem 1.4] Let $S$ and $\tilde{S}$ be square carpets of measure zero in rectangles $K = [0, a] \times [0, 1] \subseteq \mathbb{R}^2$ and $\tilde{K} = [0, \tilde{a}] \times [0, 1] \subseteq \mathbb{R}^2$, respectively, where $a$, $\tilde{a} > 0$. If $f$ is an orientation-preserving quasisymmetric homeomorphism from $S$ onto $\tilde{S}$ that takes the corners of $K$ to the corners of $\tilde{K}$ with $f(0) = 0$, then $a = \tilde{a}$, $S = \tilde{S}$, and $f$ is the identity on $S$.

4. Distinguished peripheral circles

Let $n \geq 5$, $1 \leq p < (n/2) - 1$ be integers. Let $F_{n,p}$ be the Sierpiński carpet that was defined in §1.2. We shall endow $F_{n,p}$ with the Euclidean metric in $\mathbb{R}^2$. Since $F_{n,p}$ is a subset of $[0, 1] \times [0, 1]$, the Euclidean metric (measure) is comparable with the spherical metric (measure).

If $Q$ is a peripheral circle of $F_{n,p}$, then we denote by $\ell_Q$ the Euclidean side-length of $Q$. Denote by $Q_0$ the unit square $[0, 1] \times [0, 1]$.

Proposition 4.1. The carpet $F_{n,p}$ is of measure zero. The peripheral circles of $F_{n,p}$ are uniform quasicircles and uniformly relatively separated.

Proof. It follows from the construction that $F_{n,p}$ is a carpet of Hausdorff dimension

$$\log(n^2 - 4)/\log n < 2.$$ 

So the measure of $F_{n,p}$ is equal to zero.

Since each peripheral circle of $F_{n,p}$ can be mapped to the boundary of $Q_0$ by a Euclidean similarity, the peripheral circles of $F_{n,p}$ are uniform quasicircles.

Finally, the peripheral circles of $F_{n,p}$ are uniformly relatively separated in the Euclidean metric. Indeed, consider any two distinct peripheral circles $C_1$, $C_2$ of $F_{n,p}$. The Euclidean distance between $C_1$ and $C_2$ satisfies

$$\text{dist}(C_1, C_2) \geq \min\{\ell(C_1), \ell(C_2)\} = \frac{1}{\sqrt{2}} \min\{\text{diam}(C_1), \text{diam}(C_2)\}. \quad \square$$

By Proposition 4.1, all the results in §3 apply to $F_{n,p}$.

4.1. Distinguished pairs of non-adjacent peripheral circles. We denote by $O$ the boundary of the unit square $Q_0$. In the first step of the inductive construction of $F_{n,p}$, there are four squares $Q_1$, $Q_2$, $Q_3$, $Q_4$ of side-length $1/n$, i.e., the lower-left, lower-right, upper-right and upper-left squares respectively, removed from $Q_0$. We denote by $M_i, i = 1, \ldots, 4$ the boundary of $Q_i, i = 1, \ldots, 4$, respectively.

In the following discussions, we call $O$ the outer circle of $F_{n,p}$ and $M_i, i = 1, \ldots, 4$ the inner circles of $F_{n,p}$. We say that two disjoint peripheral circles $C$, $C'$ are adjacent if there exists a copy $F$ of $F_{n,p}$ (here $F \subset F_{n,p}$ is considered as a carpet scaled from $F_{n,p}$ by some factor $1/n^k$) such that $C$, $C'$ are inner circles of $F$. For example, any two distinct inner circles $M_i$ and $M_j$ are adjacent. Two disjoint peripheral circles $C$, $C'$ which are not adjacent are called non-adjacent.
**Figure 3.** Every path in $\Gamma(C, C')$ must intersect with $C_0$.

**Lemma 4.2.** Let $\{C, C'\}$ be any pair of non-adjacent distinct peripheral circles of $F_{n,p}$. Then

$$\text{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \text{mod}_{F_{n,p}}(\Gamma(O, M)),$$

where $M$ denotes any inner circle of $F_{n,p}$. Moreover, the equality holds if and only if $\{C, C'\} = \{O, M\}$ for some inner circle $M$.

**Proof.** Assume that $\{C, C'\} \neq \{O, M\}$ for any inner circle $M$. By Propositions 4.1 and 3.6, $\text{mod}_{F_{p,q}}(\Gamma(C, C'))$ is a finite and positive number. Without loss of generality we may assume that $\ell(C) = 1/n^m \leq \ell(C')$. Note that there exists a copy $F \subset F_{n,p}$, rescaled from $F_{n,p}$ by a factor $1/n^m$, so that $C$ corresponds to some inner circle, say, $M_1$ of $F_{n,p}$.

Denote the outer circle of $F$ by $C_0$. Since $C$ and $C'$ are disjoint and $\ell(C) \leq \ell(C')$, $C'$ is disjoint with the interior region of $C_0$. Hence every path in $\Gamma(C, C')$ must intersect with $C_0$ and then contains a subpath in $\Gamma(C, C_0)$. See Figure 3. Therefore, by monotonicity of carpet modulus,

$$\text{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \text{mod}_{F_{n,p}}(\Gamma(C, C_0)). \quad (4.1)$$

On the other hand, since every path in $\Gamma(C, C_0)$ meets exactly the same peripheral circles of $F$ and $F_{n,p}$, we have

$$\text{mod}_{F_{n,p}}(\Gamma(C, C_0)) = \text{mod}_F(\Gamma(C, C_0)).$$

Moreover, by the similarity of $F_{n,p}$ and $F$,

$$\text{mod}_F(\Gamma(C, C_0)) = \text{mod}_{F_{n,p}}(\Gamma(M, O)). \quad (4.2)$$

Combining (4.2) with (4.1), we have

$$\text{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \text{mod}_{F_{n,p}}(\Gamma(M_1, O)).$$

We next show that the equality case in (4.1) cannot happen. We argue by contradiction. Assume that

$$\text{mod}_{F_{n,p}}(\Gamma(C, C')) = \text{mod}_{F_{n,p}}(\Gamma(C, C_0)).$$
By Proposition 3.6, the carpet modulus we considered above is finite. There exist unique extremal mass distributions, say \( \rho \) and \( \rho' \), for \( \text{mod}_{F_{n,p}}(\Gamma(C, C')) \) and \( \text{mod}_{F_{n,p}}(\Gamma(O, M_1)) \), respectively.

Let \( C \) be the set of all peripheral circles of \( F_{n,p} \). According to the description in Proposition 3.6, \( \rho \) and \( \rho' \) are supported on \( C \setminus \{C, C'\} \) and \( C \setminus \{O, M_1\} \), respectively.

By transplanting \( \rho' \) to the carpet \( F \) using a suitable Euclidean similarity between \( F \) and \( F_{n,p} \), we get an admissible mass distribution \( \tilde{\rho} \) for \( F \) supported only on the set of peripheral circles of \( F \) except \( C \) and \( C_0 \). Note that the total mass of \( \tilde{\rho} \) is the same as mass(\( \rho' \)).

We extend \( C \to \tilde{\rho}(C) \) by zero if \( C \) belonging to \( C \) does not intersect the interior region of \( C_0 \). Then \( \tilde{\rho} \) is an admissible mass distribution for \( \text{mod}_{F_{n,p}}(\Gamma(C, C_0)) \), thus for \( \text{mod}_{F_{n,p}}(\Gamma(C, C')) \) as well. However, \( \tilde{\rho} \neq \rho \) and mass(\( \tilde{\rho} \)) = mass(\( \rho' \)), so we arrive at a contradiction by Proposition 2.2.

In summary, we get the following crucial inequality:

\[
\text{mod}_{F_{n,p}}(\Gamma(C, C')) < \text{mod}_{F_{n,p}}(\Gamma(O, M_1)) \tag{4.3}
\]

where \( \{C, C'\} \neq \{O, M_i\}, i = 1, 2, 3, 4 \) and are non-adjacent.

**Corollary 4.3.** Let \( f \) be a quasisymmetric self-map of \( F_{n,p} \). Then

\[
f((O, M_1, M_2, M_3, M_4)) = \{O, M_1, M_2, M_3, M_4\}.
\]

**Proof.** We argue by contradiction. Assume that \( f \) maps \( \{O, M_1\} \) to some pair of peripheral circles \( \{C, C'\} \not\subseteq \{O, M_1, M_2, M_3, M_4\} \) and \( f(O) = C \). By Proposition 3.1, \( f \) extends to a quasiconformal homeomorphism on \( \mathbb{S}^2 \). In particular, \( \Gamma(C, C') = f(\Gamma(O, M_1)) \). Then Proposition 2.2 implies

\[
\text{mod}_{F_{n,p}}(\Gamma(O, M_1)) = \text{mod}_{F_{n,p}}(\Gamma(C, C')).
\]

We distinguish the argument into two cases depending on the type of the squares \( C \) and \( C' \), i.e., whether they are adjacent or not.

**Case 1:** \( C, C' \) are non-adjacent. This is only possible if

\[
\{C, C'\} \subseteq \{O, M_1, M_2, M_3, M_4\}
\]

by Lemma 4.2. Then we get a contradiction.

**Case 2:** \( C, C' \) are adjacent. Suppose \( C, C' \) are inner circles of some copy \( F \subset F_{n,p} \). Consider \( f(M_i), i = 2, 3, 4 \). They must be inner circles of \( F \) as well. Otherwise, for example, suppose that \( f(M_2) \) is not an inner circle of \( F \). Since \( C \) and \( f(M_2) \) are non-adjacent, we can apply Lemma 4.2 to show that

\[
\text{mod}_{F_{n,p}}(\Gamma(C, f(M_2))) < \text{mod}_{F_{n,p}}(\Gamma(O, M_1)),
\]

which is contradicted with the fact that

\[
\text{mod}_{F_{n,p}}(\Gamma(C, f(M_2))) = \text{mod}_{F_{n,p}}(\Gamma(O, M_2)) = \text{mod}_{F_{n,p}}(\Gamma(O, M_1)).
\]

As a result, \( \{f(O), f(M_1), f(M_2), f(M_3), f(M_4)\} \) are pairwise adjacent and all of them are inner circles of \( F \). However, \( F \) contains exactly four inner circles. So Case 2 cannot happen.

By the same argument to pairs \( O \) and \( M_i, i = 2, 3, 4 \), the corollary follows. \( \square \)
4.2. **QS**\((F_{n,p})\) is finite. Let \(H\) denote the Euclidean isometry group which consists of eight elements: four of them rotate around the center by \(\pi/2, \pi, 3\pi/2,\) and \(2\pi,\) respectively; the others are orientation-preserving and reflecting by lines \(x = 0, \ x = y, \ y = 0\) and \(x + y = 0,\) respectively. It is obvious that \(H\) is contained in \(QS(F_{n,p})\).

**COROLLARY 4.4.** Let \(5 \leq n, \ 1 \leq p < (n/2) - 1\) be integers. Then the group \(QS(F_{n,p})\) of quasisymmetric self-maps of \(F_{n,p}\) is finite.

**Proof.** According to Corollary 4.3, \(\{O, M_1, M_2, M_3, M_4\}\) are preserved under every quasisymmetric self-map of \(F_{n,p}\). The group \(G\) of all orientation-preserving quasisymmetric self-maps of \(F_{n,p}\) is finite by the proof of case (1) in Theorem 3.4. Since \(G\) is a subgroup of \(QS(F_{p,q})\) with index two, \(QS(F_{p,q})\) is finite. \(\square\)

4.3. **Proof of Theorem 1.** Recall that the standard carpet \(S_m, m \geq 3\) odd, is obtained by subdividing \([0, 1] \times [0, 1]\) into \(m^2\) subsquares of equal size, removing the interior of the middle square, and repeating these operations to every subsquare, inductively.

**Proof of Theorem 1.** Let \(\mathcal{M}, \mathcal{O}\) be the inner circle and outer circle of \(S_m\) respectively. The same proof as Lemma 4.2 (see also [6, Lemma 5.1]) shows that \(\text{mod}_{S_m}(\Gamma(O, \mathcal{M}))\) is strictly larger than the carpet modulus of any other path family \(\Gamma(C, C')\) with respect to \(S_m,\) where \(C\) and \(C'\) are peripheral circles of \(S_m.\)

On the other hand, according to the symmetry, the carpet \(F_{n,p},\) has at least two pairs of peripheral circles the maximum of \(\{\text{mod}_{F_{n,p}}\Gamma(C_1, C_2) : C_1, C_2 \in \mathcal{C}\}\). Since any quasisymmetric map from \(F_{n,p}\) to \(S_m\) must preserve such a maximum property, there is no such quasisymmetric map. \(\square\)

5. **Weak tangent spaces**

The results in this section generalize the discussion in [6, §7].

At first, we explain the definition of weak tangent of a carpet. Then we show that a quasisymmetric map between two carpets \(F_{n,p}\) induces a quasisymmetric map between weak tangents.

5.1. **Weak tangents.** In general, the **weak tangents** of a metric space \(M\) at a point \(p \in M\) can be defined as the Gromov–Hausdorff limits of the pointed metric spaces

\[
\lim_{\lambda \to \infty} (\lambda M, \ p)
\]

where \(\lambda M\) is the same set of points with \(M\) equipped with the original metric multiplied by \(\lambda.\) If the limit is unique up to multiplication by positive constants, then the weak tangent is usually called the **tangent cone** of \(M\) at \(p.\)

In the following, as in [6], we will use a suitable definition of weak tangents for subsets of \(\mathbb{S}^2\) equipped with the spherical metric.

Suppose that \(a, b \in \mathbb{C}, \ a \neq 0\) and \(M \subseteq \hat{\mathbb{C}}.\) We denote

\[
a M + b := \{az + b : z \in M\}.
\]
Let $A$ be a subset of $\hat{\mathbb{C}}$ with a distinguished point $z_0 \in A, z_0 \neq \infty$. We say that a closed set $W_A(z_0) \subseteq \hat{\mathbb{C}}$ is a weak tangent of $A$ if there exists a sequence $(\lambda_n)$ with $\lambda_n \to \infty$ such that the sets $A_n := \lambda_n(A - z_0)$ converge to $W_A(z_0)$ as $n \to \infty$ in the sense of Hausdorff convergence on $\hat{\mathbb{C}}$ equipped with the spherical metric. In this case, we use the notation

$$W_A(z_0) = \lim_{n \to \infty} (A, z_0, \lambda_n).$$

Since for every sequence $(\lambda_n)$ with $\lambda_n \to \infty$, there is a subsequence $(\lambda_{n_k})$ such that the sequence of the sets $A_{n_k} = \lambda_{n_k}(A - z_0)$ converges as $k \to \infty$, $A$ has weak tangents at each point $z_0 \in A \setminus \{\infty\}$. In general, weak tangents at a point are not unique. In particular, $\lambda W_A(z_0)$ is also a weak tangent.

Now we apply the notion to our carpets $F_{n,p}$. In fact, the following arguments work for a general class of carpets, such as the standard Sierpiński carpet $S_m$ and carpets which satisfy some self-similarity property.

A weak tangent of a point $z_0 \in F_{n,p}$ is a closed set $W_{F_{n,p}}(z_0) \subseteq \hat{\mathbb{C}}$ such that

$$W_{F_{n,p}}(z_0) = \lim_{j \to \infty} (F_{n,p}, z_0, n^k),$$

where $k_j \geq 1$ and $k_j \to \infty$ as $j \to \infty$.

At the point 0 the carpet $F_{n,p}$ has the unique weak tangent

$$W_{F_{n,p}}(0) = \lim_{j \to \infty} (F_{n,p}, 0, n^j) = \{\infty\} \cup \bigcup_{j \in \mathbb{N}_0} n^j F_{n,p}. \quad (5.1)$$

This follows from the inclusions $n^j F_{n,p} \subseteq n^{j+1} F_{n,p}$.

Similarly, at each corner of $O$ there exists a unique weak tangent of $F_{n,p}$ obtained by a suitable rotation of the set $W_{F_{n,p}}(0)$ around 0.

Let $c = p/n + ip/n$ be the lower-left corner of $M_1$. Then at $c$ the carpet $F_{n,p}$ has unique weak tangent

$$W_{F_{n,p}}(c) = \lim_{j \to \infty} (F_{n,p}, c, n^j) = \{\infty\} \cup \bigcup_{j \in \mathbb{N}_0} n^j (i F_{n,p} \cup (-i) F_{n,p} \cup (-1) F_{n,p}).$$

Note that $W_{F_{n,p}}(c)$ can be obtained by pasting together three copies of $W_{F_{n,p}}$. If $z_0$ is a corner of a peripheral circle $C \neq O$ of $F_{n,p}$, then $F_{n,p}$ has a unique weak tangent at $z_0$ obtained by a suitable rotation of the set $W_{F_{n,p}}(c)$ around 0.

**Lemma 5.1.** Let $z_0$ be a corner of a peripheral circle of $F_{n,p}$. Then the weak tangent $W_{F_{n,p}}(z_0)$ is a carpet of measure zero. If $W_{F_{n,p}}(z_0)$ is equipped with the spherical metric, then the family of peripheral circles of $W_{F_{n,p}}(z_0)$ are uniform quasicircles and uniformly relatively separated.

**Proof.** We can assume that $z_0$ equals 0. The proof works for other cases.

First note that (5.1) implies that $W_{F_{n,p}}(0)$ is a carpet of measure zero, since $W_{F_{n,p}}(0)$ is the union of countably many sets of measure zero.

Let $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0\}$. Then $\partial \Omega$ is a peripheral circle of $W_{F_{n,p}}(0)$. It is easy to construct a bi-Lipschitz map between $\partial \Omega$ and the unit circle (both equipped with the spherical metric). Hence $\partial \Omega$ is a quasicircle. Note that all other peripheral
circles of $W_{F_{n,p}(0)}$ are squares. As a result, the peripheral circles of $W_{F_{n,p}(0)}$ are uniform quasicircles.

To show that the peripheral circles are uniformly relatively separated, we only need to check the inequality
\[
\text{dist}(C_1, C_2) \geq \min\{\ell(C_1), \ell(C_2)\}
\]
for any peripheral circles $C_1, C_2 \neq \partial \Omega$. Here dist(\cdot, \cdot) and $\ell(\cdot)$ denote the Euclidean distance and Euclidean side-length.

The inequality implies that the peripheral circles are uniformly relatively separated with respect to the Euclidean metric. To see that they are uniformly relatively separated properly with respect to the spherical metric, we can apply an argument of [6, Lemma 7.1].

5.2. Quasisymmetric maps between weak tangents. We are interested in quasisymmetric maps $g : W \to W'$ between weak tangents $W$ of $F_{n,p}$ and weak tangents $W'$ of $F_{n,p}$. Note that $0, \infty \in W, W'$. We call $g$ normalized if $g(0) = 0$ and $g(\infty) = \infty$.

**Lemma 5.2.** Let $z_0$ be a corner of a peripheral circle of $F_{n_1,p_1}$ and let $w_0$ be a corner of a peripheral circle of $F_{n_2,p_2}$. Suppose that $f : F_{n_1,p_1} \to F_{n_2,p_2}$ be a quasisymmetric map with $f(z_0) = w_0$. Then $f$ induces a normalized quasisymmetric map $g$ between the weak tangent $W_{F_{n_1,p_1}(z_0)}$ and $W_{F_{n_2,p_2}(w_0)}$.

**Proof.** By Proposition 3.1 we can extend $f$ to a quasiconformal self-homeomorphism $F$ of $\hat{\mathbb{C}}$. There exists a relative neighborhood $N_1$ of $z_0$ in $F_{n_1,p_1}$ and a relative neighborhood $N_2$ of $w_0$ in $F_{n_2,p_2}$ with $F(N_1) = N_2$ such that
\[
W_{F_{n_1,p_1}(0) \setminus \{\infty\}} = \bigcup_{j \in \mathbb{N}_0} n_j^{\prime}(N_1 - z_0)
\]
and
\[ W_{F_{n_2,p_2}}(0) \setminus \{ \infty \} = \bigcup_{j \in \mathbb{N}_0} n_2^j (N_2 - w_0). \]

Pick a point \( u_0 \in N - z_0, u_0 \neq 0 \). Then for each \( j \in \mathbb{N}_0 \) we have \( F(z_0 + n_1^{-j} u_0) \neq u_0, \infty \) in \( F_{n_2,p_2} \).

We consider the following quasiconformal self-map \( F_j \) of \( \hat{\mathbb{C}} \) with \( F_j(n_1^j (N_1 - z_0)) = n_2^j (N_2 - w_0) \)

\[ F_j : u \mapsto n_2^j (F(z_0 + n_1^{-j} u) - w_0) \]

for \( u \in \hat{\mathbb{C}} \), where \( k(j) \) is the unique integer such that \( 1 \leq |F_j(u_0)| < n_2 \).

Note that \( k(j) \to \infty \) as \( j \to \infty \) and \( F(\infty) \neq w_0 \). This implies that \( F_j(\infty) \to \infty \) as \( j \to \infty \). We also have \( F_j(0) = 0 \). So the images of \( 0, \infty \) and \( u_0 \) under \( F_j \) have mutual spherical distance uniformly bounded from below independent of \( j \). Moreover, \( F_j \) is obtained from \( F \) by post-composing and pre-composing Möbius transformations. Hence the sequence \( (F_j) \) is uniformly quasiconformal, and it follows that we can find a subsequence of \( (F_j) \) that converges uniformly on \( \hat{\mathbb{C}} \) to a quasiconformal map \( F_\infty \). Without loss of generality, we assume that \( (F_j) \) converges uniformly to \( F_\infty \).

Note that \( F_\infty(0) = 0 \) and \( F_\infty(\infty) = \infty \). To prove the statement of the lemma, it suffices to show that \( F_\infty(W_{F_{n_1,p_1}}(z_0)) = W_{F_{n_2,p_2}}(w_0) \), because then \( g := F_\infty|W_{F_{n_1,p_1}}(z_0) \) is an induced normalized quasisymmetric map between \( W_{F_{n_1,p_1}}(z_0) \) and \( W_{F_{n_2,p_2}}(w_0) \), as desired.

Let \( u \) be an arbitrary point in \( W_{F_{n_1,p_1}}(z_0) \). There exists a sequence \( (u_j) \) with \( u_j \in n_1^j (N_1 - z_0) \) converging to \( u \). We have \( F_j(u_j) \in n_2^j (N_2 - w_0) \) and a subsequence of \( (F_j(u_j)) \) converging to some point \( v \) in \( W_{F_{n_2,p_2}}(w_0) \). By the definition of \( F_\infty \), we have \( F_\infty(u) = v \). Hence \( F_\infty(W_{F_{n_1,p_1}}(z_0)) \subseteq W_{F_{n_2,p_2}}(w_0) \).

For every point \( v \) in \( W_{F_{n_2,p_2}}(w_0) \), there exists a sequence \( (u_j) \) with \( u_j \in n_1^j (N_1 - z_0) \) such that \( (F_j(u_j)) \) converges to \( v \). Then we can choose a subsequence of \( (u_j) \) converging to some point \( u \) in \( W_{F_{n_1,p_1}}(z_0) \) and so \( F_\infty(u) = v \).

It follows that \( F_\infty(W_{F_{n_1,p_1}}(z_0)) = W_{F_{n_2,p_2}}(w_0) \) and we are done. \( \square \)

By Corollary 4.3, a quasisymmetric self-map \( f \) of \( F_{n,p} \) maps \( \{ O,M_1,M_2,M_3,M_4 \} \) to \( \{ O,M_1,M_2,M_3,M_4 \} \). In the remaining part of this section, we will show that there is no quasisymmetric self-map \( f \) of \( F_{n,p} \) with \( f(0) = c \), where \( c \) is a corner of an inner circle. By Lemma 5.2, if such an \( f \) exists, then it would induce a normalized quasisymmetric map from \( W_{F_{n,p}}(0) \) to \( W_{F_{n,p}}(c) \). However, the following proposition contradicts this statement.

**Proposition 5.3.** There is no normalized quasisymmetric map from \( W_{F_{n,p}}(0) \) to \( W_{F_{n,p}}(c) \).

To prove the proposition, we need two lemmas.

Let \( G \) and \( \tilde{G} \) be the group of normalized orientation-preserving quasisymmetric self-maps of \( W_{F_{n,p}}(0) \) and \( W_{F_{n,p}}(c) \), respectively. By Theorem 3.4, \( G \) and \( \tilde{G} \) are infinite cyclic groups. Note that the map \( \mu(z) := nz \) is contained in \( G \cap \tilde{G} \). We assume that \( G = \langle \phi \rangle \) and \( \mu = \phi^s \) for some \( s \in \mathbb{Z}_+ \). Since the peripheral circles of \( W_{F_{n,p}}(0) \) are uniformly
quasicircles and uniformly relatively separated, there exists a quasiconformal extension \( \Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of \( \phi \). Let \( H \) be the group generated by the reflection in the real and in the imaginary axes. We may assume that \( \Phi \) is equivalent under the action of \( H \) (see [6, p. 42] for the discussion).

Let \( \Omega = \{ z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0 \} \). Then \( C_0 := \partial \Omega \) is a peripheral circle of \( W_{F_n,p}(0) \). Since \( \Phi(C_0) = C_0 \) and \( \Phi \) is orientation-preserving, \( \Phi(\Omega) = \Omega \).

Let \( \Gamma \) be the family of all open paths in \( \Omega \) that connect the positive real and positive imaginary axes. Since the paths in \( \Omega \) are open, they don’t intersect with \( C_0 \). For any peripheral circle \( C \) of \( W_{F_n,p}(0) \) that meets some path in \( \Gamma \), note that \( \phi^k(C) \neq C \) for all \( k \in \mathbb{Z} \setminus \{0\} \) (otherwise, \( \phi \) would be of finite order, contradicted with the fact that \( \phi \) is the generator of the infinite cyclic group \( G \)). So we can apply Lemma 2.4 to conclude that

\[
\text{mod}_{W_{F_n,p}(0)/(\mu)}(\Gamma) = \text{mod}_{W_{F_n,p}(0)/(\phi^k)}(\Gamma) = s \text{ mod}_{W_{F_n,p}(0)/G}(\Gamma).
\]

Note that without the action of the group \( G \), the carpet modulus \( \text{mod}_{W_{F_n,p}(0)}(\Gamma) \) is equal to infinity.

**Lemma 5.4.** We have \( 0 < \text{mod}_{W_{F_n,p}(0)/G}(\Gamma) < \infty \).

**Proof.** Let us first show that \( \text{mod}_{W_{F_n,p}(0)/(\mu)}(\Gamma) < \infty \) by constructing an admissible mass distribution of finite mass.

Let \( \text{pr} : \mathbb{C} \setminus \{0\} \to S^1 \) be the projection \( z \mapsto (z/|z|) \). If \( C \neq C_0 \) is a peripheral circle of \( W_{F_n,p}(0) \), we let \( \theta(C) \) be the arc length of \( \text{pr}(C) \). We set

\[
\rho(C) := \begin{cases} 
0 & \text{if } C = C_0, \\
\frac{2}{\pi} \theta(C) & \text{if } C \neq C_0.
\end{cases}
\]

Note that \( \rho \) is \( (\mu) \)-invariant.

Let \( \Gamma_0 \) be the family of paths \( \gamma \in \Gamma \) that are not locally rectifiable or for which \( \gamma \cap W_{F_n,p}(0) \) has positive length. Since \( W_{F_n,p}(0) \) is a set of measure zero, we have \( \text{mod}(\Gamma_0) = 0 \), i.e., \( \Gamma_0 \) is an exceptional subfamily of \( \Gamma \).

For any \( \gamma \in \Gamma \setminus \Gamma_0 \), note that

\[
\sum_{\gamma \cap C \neq \emptyset} \rho(C) = \frac{2}{\pi} \sum_{\gamma \cap C \neq \emptyset} \theta(C) \geq 1.
\]

As a result, \( \rho \) is admissible.

Let \( Q_0 = [0, 1] \times [0, 1] \). Note that every \( (\mu) \)-orbit of a peripheral circle \( C \neq C_0 \) has a unique element contained in the set \( F = \mu(Q_0) \setminus Q_0 \). There is a constant \( K > 0 \) such that

\[
\theta(C) \leq K \ell(C)
\]

for all peripheral circles \( C \subset F \). It follows that

\[
\frac{4}{\pi^2} \sum_{C \subset F} \theta(C)^2 \leq \sum_{C \subset F} \ell(C)^2 = \text{Area}(F) = n^2 - 1.
\]

Hence \( \rho \) is a finite admissible mass distribution for \( \text{mod}_{W_{F_n,p}(0)/(\mu)}(\Gamma) \).
To show that \( \text{mod}_{W_{F_{n,p}}(0)/\langle \mu \rangle} (\Gamma) > 0 \), we only need to show that the carpet satisfies the assumptions in [6, Proposition 3.2]. Then the extremal mass distribution for \( \text{mod}_{W_{F_{n,p}}(0)/\langle \mu \rangle} (\Gamma) \) exists and this is only possible if \( \Gamma \) itself is an exceptional family, that is, \( \text{mod}(\Gamma) = 0 \).

In fact, for \( k \in \mathbb{N} \) we let \( C_k \) be the set of all peripheral circles \( C \) of \( W_{F_{n,p}}(0) \) with \( C \subset F_k = \mu^k(Q_0) \setminus \mu^{-k}(Q_0) \). Then:

1. every \( \langle \mu \rangle \)-orbit of a peripheral circle \( C \neq C_0 \) has exactly \( 2k \) elements in \( C_k \);
2. let \( \Gamma_k \) be the family of paths in \( \Gamma \) that only meet peripheral circles in \( C_k \). Then \( \Gamma = \bigcup_k \Gamma_k \).

As a result, the assumptions in [6, Proposition 3.2] are satisfied. \( \square \)

Let \( \bar{\Omega} = \mathbb{C} \setminus \bar{\Omega} \). The closure of \( \bar{\Omega} \) contains \( W_{F_{n,p}}(c) \) and \( C_0 = \partial \Omega = \partial \bar{\Omega} \) is a peripheral circle of \( W_{F_{n,p}}(c) \). Denote \( \psi = \Phi|_{W_{F_{n,p}}(c)} \). Then we have \( \psi \in \tilde{G} \). Let \( \tilde{\Gamma} \) be the family of all open paths in \( \bar{\Omega} \) that join the positive real and the positive imaginary axes.

**Lemma 5.5.** We have \( \text{mod}_{W_{F_{n,p}}(c)/\langle \psi \rangle} (\tilde{\Gamma}) \leq \frac{1}{3} \text{mod}_{W_{F_{n,p}}(0)}/G (\Gamma) \).

**Proof.** Let \( \rho \) be an arbitrary admissible invariant mass distribution for \( \text{mod}_{W_{F_{n,p}}(0)}/G (\Gamma) \), with exceptional family \( \Gamma_0 \subset \Gamma \). We set

\[
\tilde{\rho}(\tilde{C}) := \begin{cases} 0 & \text{if } \tilde{C} = C_0, \\ \frac{1}{3} \rho(\alpha(\tilde{C})) & \text{otherwise,} \end{cases}
\]

if there is an \( \alpha \in H \) such that \( \alpha(\tilde{C}) \) is a peripheral circle of \( W_{F_{n,p}}(0) \) (such an \( \alpha \) exists and is unique).

Since \( \Phi \) is \( H \)-equivalent and \( \rho \) is \( G \)-invariant, \( \tilde{\rho} \) is \( \langle \psi \rangle \)-invariant.

Let \( \tilde{\Gamma}_0 \) be the family of paths in \( \bar{\Omega} \) that have a subpath that can be mapped to a path in \( \Gamma_0 \) by an element of \( \alpha \in H \). Then \( \text{mod}(\tilde{\Gamma}_0) = 0 \).

Let \( \gamma \in \tilde{\Gamma} \). Note that \( \gamma \) has three disjoint open subpaths: one for each quarter-plane of \( \bar{\Omega} \) and by suitable elements in \( H \), the three subpaths are mapped to paths in \( \Gamma \). Denote the images by \( \gamma_1, \gamma_2, \gamma_3 \). If \( \gamma \in \tilde{\Gamma} \setminus \tilde{\Gamma}_0 \), then \( \gamma_i \in \Gamma \setminus \Gamma_0, i = 1, 2, 3 \) and

\[
\sum_{\gamma \cap \tilde{C} \neq \emptyset} \tilde{\rho}(\tilde{C}) \geq \frac{1}{3} \sum_{i=1}^{3} \sum_{\gamma_i \cap C \neq \emptyset} \rho(C) \geq 1.
\]

Hence \( \tilde{\rho} \) is admissible for \( \text{mod}_{W_{F_{n,p}}(c)/\langle \psi \rangle} (\tilde{\Gamma}) \) and

\[
\text{mod}_{W_{F_{n,p}}(c)/\langle \psi \rangle} (\tilde{\Gamma}) \leq \text{mass}_{W_{F_{n,p}}(c)/\langle \psi \rangle} (\tilde{\rho}) \leq \frac{1}{3} \text{mass}_{W_{F_{n,p}}(0)/G} (\rho).
\]

Since \( \rho \) is an arbitrary mass distribution for \( \frac{1}{3} \text{mod}_{W_{F_{n,p}}(0)/G} (\Gamma) \), the statement follows. \( \square \)

**Proof of Proposition 5.3.** Suppose that the proposition is false, so there exists a normalized quasisymmetric map \( f : W_{F_{n,p}}(0) \rightarrow W_{F_{n,p}}(c) \). Pre-composing \( f \) by the reflection in the diagonal line \( \{x = y\} \) if necessary, we may assume that \( f \) is orientation-preserving. Then \( \tilde{G} = f \circ G \circ f^{-1} \) and \( \tilde{\phi} = f \circ \phi \circ f^{-1} \) is a generator for \( \tilde{G} \).
Let $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal extension of $f$. Then $\tilde{\Gamma} = F(\Gamma)$. By quasisymmetric invariance of carpets modulus,
\[
\mod_{W_{F_n,p}(c)/\tilde{G}}(\tilde{\Gamma}) = \mod_{W_{F_n,p}(0)/G}(\Gamma).
\]
Assume that $\psi = \tilde{\phi}^m$. Then similarly to our discussion before Lemma 5.4, we have
\[
\mod_{W_{F_n,p}(c)/\langle \psi \rangle}(\tilde{\Gamma}) = \frac{1}{|m|} \mod_{W_{F_n,p}(c)/\tilde{G}}(\tilde{\Gamma}).
\]
Hence by Lemma 5.5 we have
\[
\mod_{W_{F_n,p}(0)/G}(\Gamma) = \mod_{W_{F_n,p}(c)/\tilde{G}}(\tilde{\Gamma}) = \frac{1}{|m|} \mod_{W_{F_n,p}(c)/\langle \psi \rangle}(\tilde{\Gamma}) \leq \frac{1}{3|m|} \mod_{W_{F_n,p}(0)/G}(\Gamma).
\]
This is possible only if $\mod_{W_{F_n,p}(0)/G}(\Gamma)$ is equal to 0 or $\infty$. But this is a contradiction to Lemma 5.4. □

6. Quasisymmetric rigidity

Let $D$ be the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$ and $V$ be the vertical line $\{(x, y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$. We denote the reflections in $D$ and $V$ by $R_D$ and $R_V$, respectively. The Euclidean isometry group of $F_{n,p}$ is generated by $R_D$ and $R_V$.

Let $QS(F_{n,p})$ be the group of quasisymmetric self-maps of $F_{n,p}$. By Corollary 4.4, $QS(F_{n,p})$ is a finite group.

**Proposition 6.1.** Let $f$ be a quasisymmetric self-map of $F_{n,p}$. Then $f(\{O\}) = \{O\}$ and $f(\{M_1, M_2, M_3, M_4\}) = \{M_1, M_2, M_3, M_4\}$.

**Proof.** From Corollary 4.3, we argue by contradiction and assume that there exists a quasisymmetric self-map $f$ of $F_{n,p}$ and some $i \in \{1, 2, 3, 4\}$ such that $f(\{O\}) = \{M_i\}$. By pre-composing and post-composing suitable elements in the Euclidean isometry group, we can suppose that $f$ is orientation-preserving and $f(\{O\}) = \{M_1\}$.

Let $G$ be the subgroup of $QS(F_{n,p})$,
\[
G = \{g \in QS(F_{n,p}) \mid g(O) = O, g(M_1) = M_1\}.
\]
Then $G$ has a subgroup $G'$ with index two consisting of orientation-preserving elements. Then
\[
G = G' \bigsqcup G' \circ R_D.
\]
We denote by
\[
\mathcal{O}_G(z) = \{g(z) : g \in G\}
\]
the orbit of $z$ under the action of $G$ for arbitrary $z \in F_{n,p}$. Let $c = (p/n, p/n)$ and $c' = ((p + 1)/n, (p + 1)/n)$ be the lower-left and upper-right corners of $M_1$, respectively.

Now we consider the map
\[
\Phi_0 : G' \to \mathcal{O}_G(0)
\]
\[
g \mapsto g(0).
\]
Note that $\Phi_0$ is an isomorphism. In fact, for any $g(0) \in O_G(0)$, if $g$ is orientation-preserving, then $\Phi_0(g) = g(0)$; otherwise, $\Phi_0(g \circ R_D) = g(0)$. So $\Phi_0$ is a surjection. On the other hand, if $\Phi_0(g_1) = \Phi_0(g_2)$ for any $g_1, g_2 \in G'$, then case (2) of Theorem 3.4 gives $g_1 = g_2$. So $\Phi_0$ is a injection.

Similarly, we can also define the isomorphism
\[
\Phi_c : G' \longrightarrow O_G(c)
\]
\[
g \longmapsto g(c).
\]
These isomorphisms $\Phi_0$ and $\Phi_c$ imply that
\[
\# O_G(0) = # G' = # O_G(c).
\]
(6.1)

On the other hand, $f$ induces the following isomorphism
\[
f_* : G \longrightarrow G
\]
\[
g \longmapsto f \circ g \circ f^{-1}.
\]
We denote $m = f(0)$. Then
\[
O_G(m) = \{g(m) : g \in G\} = \{f \circ g \circ f^{-1}(m) : g \in G\}
\]
\[
= \{f \circ g(0) : g \in G\} = f(O_G(0)).
\]
Hence
\[
\# O_G(m) = # G' = # O_G(0)
\]
and so the orbits $O_G(m)$ and $O_G(c)$ have the same number of elements.

If $G' \neq \{id\}$, we claim that $G'$ is a cyclic group of order three. Indeed, for any $g \neq id$ in $G'$, $g(M_3) \neq M_3$, otherwise case (1) of Theorem 3.4 implies $g = id$. By Corollary 4.3, either $g(M_3) = M_4$, $g(M_4) = (M_2)$ or $g(M_3) = M_2$, $g(M_2) = (M_4)$. In both cases, $g$ is of order three, i.e., $g^3 = id$. Using Corollary 4.3 again we know that $G'$ is generated by $g$. So the claim follows.

Hence, we have $\# O_G(m) = # G' = 1$ or 3. There must be some $h \in G$ with $h(m) = c$ or $c'$. Otherwise, $O_G(m)$ does not contain $c$, $c'$. For any point $p \in O_G(m)$, the point $R_D(p) \in O_G(m) \text{ and } R_D(p) \neq p$. Then $\# O_G(m)$ is even, which is impossible.

By Lemma 5.2, $h \circ f$ induces a normalized quasisymmetric map between the weak tangent $W_{F_n,p}(0)$ and $W_{F_n,p}(c)$ or $W_{F_n,p}(c')$. This contradicts Proposition 5.3. So we have proved the proposition.

6.1. Proof of main theorems.

Proof of Theorem 2. We adopt the notation as used previously. The proof of Proposition 6.1 implies that $G'$ is a cyclic group of order three or a trivial group. To prove the theorem, it suffices to show that the former case cannot happen. We argue by contradiction and assume that $G'$ is a cyclic group of order three.

By Theorem 3.2, there exists a quasisymmetric map $f$ from $F_{n,p}$ onto some round carpet $S$. After post-composing suitable fraction linear transformation, we can assume
that \( f(O) \) is the unit disc \( \mathbb{D} \) and \( f(M_1) \) lies in \( \mathbb{D} \) with center \((0, 0)\). Then \( f \) induces the isomorphism

\[
 f_* : QS(F_{n,p}) \longrightarrow QS(S) \\
g \longmapsto f \circ g \circ f^{-1}.
\]

Combined with Theorem 3.3, \( f_*(G') \) is a cyclic group of order three consisting of Möbius transformations. Moreover, elements in \( f_*(G') \) preserve \( \partial \mathbb{D} \) and the circle \( O_1 = f(M_1) \). Hence we have

\[
 f_*(G') = \{ \text{id}, z \mapsto e^{2\pi i/3}z, \ z \mapsto e^{4\pi i/3}z \}.
\]

**Claim.** We claim that \( O_2 = f(M_2), O_3 = f(M_3), O_4 = f(M_4) \) are round circles with the same diameter and equidistributed clockwise in the annuli bounded by \( \partial \mathbb{D} \) and \( O_1 \).

**Proof of the claim.** In fact, by the proof of Proposition 6.1, we may assume that \( G' = \langle g \rangle \), where \( g(M_3) = M_4, g(M_4) = M_2 \) and \( g(M_2) = M_3 \). Note that

\[
 O_3 = f(M_3) = f \circ g(M_2) = f \circ g \circ f^{-1}(O_2),
\]

where \( f \circ g \circ f^{-1} \) is equal to the rotation \( z \mapsto e^{2\pi i/3}z \). Similarly, one can show that \( O_4 = f \circ g \circ f^{-1}(O_3) \). As a result, \( O_3 \) is obtained from \( O_2 \) by a rotation of angle \( 2\pi/3 \) and \( O_4 \) is obtained from \( O_2 \) by a rotation of angle \( 4\pi/3 \). The claim follows.

Let \( R \) be the rotation in the isometry group of \( F_{n,p} \) with \( R(M_1) = M_2, R(M_2) = M_3, R(M_3) = M_4, \) and \( R(M_4) = M_1 \). By Theorem 3.3, the composition

\[
h = f \circ R \circ f^{-1} : S \rightarrow S
\]

is also a Möbius transformation which maps \( \partial \mathbb{D} \to \partial \mathbb{D}, O_2 \to O_3, O_3 \to O_4 \). Such a Möbius transformation must be \( \varphi = z \mapsto e^{2\pi i/3}z \). If not, let \( \varphi' \) be another Möbius transformation that satisfies the conditions. Then \( \varphi' \circ \varphi^{-1} \) fixes three non-concentric circles \( \partial \mathbb{D}, O_2 \) and \( O_3 \) and so \( \varphi' \circ \varphi^{-1} = \text{id} \). Hence \( \varphi' = \varphi \). But \( h(O_1) = O_2 \), which is impossible. So the theorem follows.

**Proof of Theorem 3.** Suppose there exists a quasisymmetric map \( f : F_{n,p} \to F'_{n',p'} \).

Firstly, we claim that \( f(O) = O' \), \( f([M_1, M_2, M_3, M_4]) = [M'_1, M'_2, M'_3, M'_4] \). Indeed, from Theorem 2, we know that every quasisymmetric self-map of \( F_{n,p} \) and \( F'_{n',p'} \) is an isometry and so preserves the peripheral circle \( O \) and \( O' \). For any \( g \) in \( QS(F_{n,p}) \), \( f \circ g \circ f^{-1} \) is a quasisymmetric self-map of \( F'_{n',p'} \) and \( f \circ g \circ f^{-1}(f(O)) = f(O) \). So \( f(O) \) is fixed by any element in \( QS(F'_{n',p'}) \). Hence we have \( f(O) = O' \). If for some inner circles \( M_i \), say \( M_1 \), of \( F_{n,p} \), \( f(M_1) \) is not an inner circle of \( F'_{n',p'} \), then by Proposition 3.1, \( f \) extends to a quasiconformal self-map of \( S^2 \). We have

\[
 \text{mod}_{F_{n,p}}(\Gamma(M_1, O)) = \text{mod}_{F'_{n',p'}}(\Gamma(f(M_1), O'))
\]

and

\[
 \text{mod}_{F'_{n',p'}}(\Gamma(M'_1, O')) = \text{mod}_{F_{n,p}}(\Gamma(f^{-1}(M'_1), O)).
\]

While Lemma 4.2 implies

\[
 \text{mod}_{F'_{n',p'}}(\Gamma(f(M_1), O)) < \text{mod}_{F'_{n',p'}}(\Gamma(M'_1, O))
\]
and
\[ \text{mod}_{F_{n,p}} \Gamma(f^{-1}(M'_1), O) \leq \text{mod}_{F_{n,p}} (\Gamma(M_1, O)). \]

Hence \( \text{mod}_{F_{n,p}} (\Gamma(M_1, O)) < \text{mod}_{F_{n,p}} (\Gamma(M_1, O)) \) and we get a contradiction.

Secondly, by pre-composing and post-composing with Euclidean isometries, we can assume that \( f \) is orientation-preserving and \( f(M_1) = M'_1 \). We claim that \( f((0,0)) = (0,0) \) and \( f((1,1)) = (1,1) \) or interchanges them and \( f(M_3) = M'_3 \). In fact, the orientation-preserving quasisymmetric map
\[ f^{-1} \circ R_D \circ f \circ R_D : F_{n,p} \to F_{n,p}, \]
fixes peripheral circles \( O \) and \( M_1 \). Then, by Theorem 2, \( f^{-1} \circ R_D \circ f \circ R_D \) is a Euclidean isometry and so it is the identity on \( F_{n,p} \). This implies \( f \circ R_D = R_D \circ f \). Hence the claim follows.

We now distinguish two cases to analyze.

Case 1: \( f((0,0)) = (0,0) \) and \( f((1,1)) = (1,1) \). We denote the reflection in the line \( \{ (x, y) \in \mathbb{R}^2 : x + y = 1 \} \) by \( R'_D \). Then the map \( f^{-1} \circ R'_D \circ f \circ R_D \) is an orientation-preserving quasisymmetric map in \( QS(F_{n,p}) \), which fixes peripheral circles \( O, M_1 \), and the point \( (0,0) \). Hence this map is the identity on \( F_{n,p} \) and so \( f \circ R'_D = R'_D \circ f \). It follows that \( f \) fixes \( (1,0) \) and \( (0,1) \) or interchanges them. Since \( f \) is orientation-preserving, the latter cannot happen. By Theorem 3.7 the map \( f \) must be the identity. Hence \( (n, p) = (n', p') \).

Case 2: \( f((0,0)) = (1,1) \) and \( f((1,1)) = (0,0) \). The map \( g = R_D \circ f \circ R'_D : F_{n,p} \to F_{n',p'} \) is an orientation-preserving quasisymmetry which fixes points \( (0,0) \) and \( (1,1) \) and peripheral circle \( O \) and maps \( M_1 \) to \( M'_1 \). Similarly to Case 1, \( g^{-1} \circ R'_D \circ g \circ R_D \) is an orientation-preserving isometry map fixing \( (0,0), (1,1) \) and \( O \) and so is the identity. Then \( g \) fixes \( (1,0) \) and \( (0,1) \) or interchanges them. The orientation-preserving of \( g \) implies the latter case is impossible. By Theorem 3.7 the map \( g \) is the identity, which contradicts with \( g(M_1) = M'_1 \). So Case 2 cannot happen.

7. Remark

Our arguments in this paper apply to a more general class of Sierpiński carpets \( F_{n,p,r} \), \( r \geq 1, p \geq 1, n \geq 5, 1 \leq p + r < (n/2) \). Let \( Q_{n,p,r}^{(0)} = [0, 1] \times [0, 1] \). Subdivide \( Q_{n,p,r}^{(0)} \) into \( n^2 \) subsquares and remove the interior of four bigger subsquares with side-length \( r/n \) and of distance \( \sqrt{2}p/n \) to one of the four corner points of \( Q_{n,p,r}^{(0)} \). So the resulting set \( Q_{n,p,r}^{(1)} \) has \( (n^2 - 4r^2) \) subsquares with side-length \( 1/n \). Repeating the operation to the subsquares, we obtain \( Q_{n,p,r}^{(2)} \). Inductively, we have \( Q_{n,p,r}^{(k)} \). Then the carpet \( F_{n,p,r} = \bigcap_{k \geq 0} Q_{n,p,r}^{(k)} \). See Figure 5. Note that \( F_{n,p} = F_{n,p,1} \).

Similarly, \( F_{n,p,r} \) is not quasisymmetrically equivalent to \( S_m, m \geq 3 \) odd and \( QS(F_{n,p,r}) \) is the isometric group. Moreover, \( F_{n,p,r} \) and \( F_{n',p',r} \) are quasisymmetrically equivalent if and only if \( (n, p, r) = (n', p', r') \). Since the proof of the above conclusions is of no essential difference from that of \( F_{n,p} \), we shall omit it.
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