Cosmological Parameter Estimation using $H(z)$ Measurements for Non-flat $\Lambda$ Cold Dark Matter Universe

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Abstract

Accurate estimations of cosmological density parameters ($\Omega_{\text{m}}h^2$, $\Omega_{\text{k}}h^2$, $\Omega_{\Lambda}h^2$) and Hubble constant ($h_0$) provide detailed understanding of our universe. In this article, we propose a new procedure to estimate these parameters for non-flat $\Lambda$ cold dark matter ($\Lambda$CDM) universe in the Friedmann-Robertson-Walker (FRW) background. We utilize the Hubble parameter ($H(z)$) measurements, which are available in the redshift range $0.07 \leq z \leq 2.36$, in our analysis. These $H(z)$ are measured using two techniques (e.g., $31 H(z)$ using differential age (DA) technique in the redshift range $0.07 \leq z \leq 1.965$ and 24 $H(z)$ using baryon acoustic oscillation (BAO) technique in the redshift range $0.24 \leq z \leq 2.36$). We generalize the two-point diagnostic method first proposed by Sahni et al. (2008) to the case of non-flat spatial curvature of the universe. Using three independent $H(z)$ measurements at a time, we solve for three fundamental cosmological density parameters ($\Omega_{\text{m}}h^2$, $\Omega_{\text{k}}h^2$, $\Omega_{\Lambda}h^2$) and repeat the procedure for all possible combinations of three measurements of Hubble parameters. We divide the $H(z)$ data into three groups comprising all, 31 DA only and 24 BAO only measurements. We perform our analysis separately on each group. In our method, we use weighted-mean and median statistics to estimate the values of density parameters. Using these estimated values of density parameters, we also find the values of $h_0$ for each of these three sets corresponding to each of two statistics. We conclude that median statistic is more reliable in our analysis, since the sample specific uncertainties (corresponding to density parameters) are non-Gaussian in nature. Finally, we achieve the reliable results $\Omega_{\text{m}}h^2 = 0.1485^{+0.0041}_{-0.0051}$, $\Omega_{\text{k}}h^2 = -0.0137^{+0.0125}_{-0.0125}$, $\Omega_{\Lambda}h^2 = 0.3126^{+0.0131}_{-0.009}$ and $h_0 = 0.6689^{+0.0141}_{-0.0121}$ using median statistic in our analysis corresponding to the set of total 53 $H(z)$. These estimated median values of cosmological parameters show an excellent agreement with Planck Collaboration VI. (2020) results.

Keywords: Non-flat $\Lambda$CDM universe - Cosmological density parameters - Hubble parameter measurements - Hubble constant

1 Introduction

The cosmological principle states that our universe is homogeneous and isotropic on a large enough scale ($\gtrsim 300$ Mpc). The recent observations of cosmic microwave background (CMB) radiation and its interpretations establish the explanations for the accelerated expansion, dark energy domination and flatness of the universe (Planck Collaboration VI., 2020). $\Lambda$CDM model is known as the standard model of Big Bang cosmology. This is the simplest model of our universe considering three fundamental density components, such as cosmological constant ($\Lambda$), cold dark matter and visible matter density.

Dark matter and dark energy are the mysterious components of the universe. We are familiar with some properties regarding these components. Dark matter has zero pressure, same as ordinary matter, and it interacts with nothing except gravitation. Dark energy, which perhaps causes the accelerated expansion of the universe (Riess et al., 1998; Perlmutter et al., 1999), has negative pressure. Cosmological constant ($\Lambda$), first envisioned by Albert Einstein almost a century ago (Einstein, 1917), is treated as the simplest form of dark energy which has the pressure exactly equal to the density with a negative sign. Recent observations show that our universe contains around 4.6% visible baryonic matter, 24% dark matter and 71.4% cosmological constant.

CMB data show an excellent agreement with $\Lambda$CDM model of the universe. However, many research projects show serious disagreement to accept the $\Lambda$CDM model as a final interpretable model of the universe. Zunckel & Clarkson (2008) formulated a 'litmus test' to verify the acceptance of $\Lambda$CDM model. They showed a significant deviation of the equation of state of dark energy for $\Lambda$CDM model as well as other dark energy models in their analysis. In these literature (Macaulay et al., 2013; Raveri, 2016), Canada-France-Hawaii-Telescope Lensing Survey measurements also showed a tension with the results of Planck.

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Collaboration VI. (2020). Testing of the Copernican principle (Uzan et al., 2008; Valkenburg et al., 2014) allows us to understand the existence as well as the evolution of dark energy. These types of concerns encourage the researchers to test the reliability of ΛCDM model.

Another two interesting powerful procedures are $Om$ and $Omh^2$ diagnostics for a null test of the ΛCDM model. Sahni et al. (2008) developed $Om(z)$ diagnostic to express the density components of the universe in terms of redshift and corresponding Hubble parameter for flat ΛCDM universe. $Om(z)$ is the matter density parameter ($\Omega_0$) at a particular redshift. Sahni et al. (2008) also defined two-point diagnostic $Om(z_i, z_j) = Om(z_i) - Om(z_j)$ for null test of flat ΛCDM model of the universe. Shafieloo et al. (2012) modified the two-point diagnostic defining by $Om h^2(z_i, z_j) = Omh^2(z_i) - Omh^2(z_j)$, where $\Omega_0 h_0^2$ is denoted by $Omh^2(z)$ and $h_0$ is the Hubble constant in $100 km Mpc^{-1} sec^{-1}$ unit. $Om h^2(z_i, z_j)$ is also a powerful probe for null test of the flat ΛCDM universe (Shafieloo et al., 2012; Sahni et al., 2014). Sahni et al. (2014) utilized the two-point diagnostic $Om h^2(z_i, z_j)$ for the null test using three Hubble parameter measurements ($H(z)$) estimated by baryon acoustic oscillation (BAO) technique. These three $H(z)$ were measured by Riess et al. (2011) and Planck Collaboration XVI. (2014), the $H(z = 0.57)$ from Sloan Digital Sky Survey Data Release 9 (SDSS DR9) (Samushia et al., 2013), and the $H(z = 2.34)$ from Lyα forest in SDSS DR11 (Delubac et al., 2015). Using these three $H(z)$, Sahni et al. (2014) showed that their null test experiment gives a strong tension with the value of $\Omega_0 h_0^2$ from Planck Collaboration XVI. (2014). $Om(z_i, z_j)$ and $Om h^2(z_i, z_j)$ diagnostics were also applied by Zheng et al. (2016) for three different models (ΛCDM, wCDM and Chevalier-Polarski-Linder(CPL; Chevalier & Polarski (2001); Linder (2003))) of the flat universe. They also found significant tension with Planck Collaboration XVI. (2014) results, for each of these three models.

In our work, considering non-flat ΛCDM universe we use three-point diagnostics, in a similar fashion of $Om h^2(z_i, z_j)$ diagnostic (Shafieloo et al., 2012; Sahni et al., 2014), to estimate the sample specific values of matter ($\Omega_0 h_0^2$), curvature ($\Omega_k h_0^2$) and cosmological constant ($\Omega_0 \Lambda^0 h_0^2$) density parameter without assuming the value of Hubble constant ($h_0$). Thus, our method is completely parameter independent. We utilize weighted-mean and median statistics in our analysis. After finding the weighted-mean and median values of these three density parameters, we also calculate the value of Hubble constant using these values of the density parameters. We show the non-Gaussian nature of the sample specific uncertainties corresponding to density parameters. Since the uncertainties are non-Gaussian, median statistic is more reliable in our analysis. Interestingly, we find the values of density parameters ($\Omega_0 h_0^2$, $\Omega_k h_0^2$, $\Omega_0 \Lambda^0 h_0^2$) and Hubble constant ($h_0$), which gives excellent agreement with the results of Planck Collaboration VI. (2020), using median statistic for the set of total 53 Hubble parameter measurements.

Rest of our paper is arranged as follows. In section 2.1 we describe the methodology of three-point diagnostics. We discuss about weighted-mean and median statistics in section 2.2 and 2.3. In section 3, we show the Hubble parameter measurements used in our analysis. In section 3.1 and section 3.2, we give a little overview about DA and BAO techniques. In section 4.1, we discuss the three-point statistics for processing of sample specific values of density parameters. In section 4.2, we present our estimated results and corresponding uncertainty ranges for cosmological parameters. In section 4.2, we also show the Hubble parameter curves using our estimated results as well as the results of Planck Collaboration VI. (2020). We discuss the non-Gaussian nature of uncertainties corresponding to sample specific values of density parameters in section 4.3. Finally, in section 5, we conclude our analysis.

2 Formalism

2.1 Three-point diagnostics

The well known Einstein’s field equation is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $R_{\mu\nu}$ is Ricci curvature tensor, $R$ is Ricci curvature scalar, $g_{\mu\nu}$ is metric tensor, $T_{\mu\nu}$ is energy-momentum tensor, $G$ is universal gravitational constant and $c$ is the velocity of light in vacuum.

The Friedmann-Robertson-Walker (FRW) line element, in spherical coordinate system, can be expressed as

$$ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right]$$

where $dt^2 = dv^2 + r^2 d\phi^2$. In Eqn. 2, $r, \theta, \phi$ are comoving co-ordinates. Scale factor of the universe is defined by $a(t)$ and the curvature constant is denoted by $k$. Zero value of $k$ defines the spatially flat universe. Positive value of $k$ represents that universe is closed and negative value of $k$ signifies that universe is open.
Using Eqn. 1 and Eqn. 2, we find two Friedmann equations which can be written as

\[
\frac{\dot{a}^2}{a^2} + \frac{k c^2}{a^2} = \frac{8\pi G}{3} \rho(t) \tag{3}
\]

\[
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k c^2}{a^2} = \frac{8\pi G}{c^2} P(t) \tag{4}
\]

where \(\rho(t)\) is the density and \(P(t)\) is the gravitational pressure of the universe. In Eqn. 3 and Eqn. 4, \(\dot{a}\) represents the first order time derivative of scale factor and \(\ddot{a}\) defines the second order time derivative of scale factor.

From these two Friedmann equations (Eqn. 3 and Eqn. 4) for \(\Lambda\)CDM model (neglecting radiation density term), the Hubble parameter as a function of redshift is defined by

\[
H^2(z) = H_0^2 \left[ \Omega_{\text{om}}(1 + z)^3 + \Omega_{\text{ok}}(1 + z)^2 + \Omega_{\text{m}} \right]. \tag{5}
\]

The density parameter (\(\Omega\)) is specified by the ratio between density (\(\rho\)) and critical density (\(\rho_c\)). The critical density (\(\rho_c\)) is expressed as \(3H^2/8\pi G\). In Eqn. 5, \(\Omega_{\text{om}}\) is the matter density parameter, \(\Omega_{\text{ok}}\) (defined as \(-k c^2/a^2H^2\)) is the curvature density parameter and \(\Omega_{\text{m}}\) is the cosmological constant density parameter.

We can also explain the spatial curvature of the universe by looking into the value of \(\Omega_{\text{ok}}\). If \(\Omega_{\text{ok}} = 0\), then universe is spatially flat. Positive and negative values of \(\Omega_{\text{ok}}\) indicate that universe is open and closed respectively.

Defining \(h(z) = H(z)/100 \text{ km} \text{Mpc}^{-1} \text{s}^{-1}\) and \(h_0 = H_0/100 \text{ km} \text{Mpc}^{-1} \text{s}^{-1}\), the Hubble parameter expression at \(\alpha\)-th redshift can be written as

\[
h^2(z_\alpha) = \Omega_{\text{om}} h_0^2 (1 + z_\alpha)^3 + \Omega_{\text{ok}} h_0^2 (1 + z_\alpha)^2 + \Omega_{\text{m}} h_0^2. \tag{6}
\]

Similarly, the Hubble parameter expressions, at \(\beta\)-th and \(\gamma\)-th redshifts, are given by

\[
h^2(z_\beta) = \Omega_{\text{om}} h_0^2 (1 + z_\beta)^3 + \Omega_{\text{ok}} h_0^2 (1 + z_\beta)^2 + \Omega_{\text{m}} h_0^2 \tag{7}
\]

and

\[
h^2(z_\gamma) = \Omega_{\text{om}} h_0^2 (1 + z_\gamma)^3 + \Omega_{\text{ok}} h_0^2 (1 + z_\gamma)^2 + \Omega_{\text{m}} h_0^2. \tag{8}
\]

Inevitably, we have three equations (Eqn. 6, Eqn. 7 and Eqn. 8) now. Three unknown coefficients corresponding to these equations are \(\Omega_{\text{om}} h_0^2\), \(\Omega_{\text{ok}} h_0^2\) and \(\Omega_{\text{m}} h_0^2\). So these unknown coefficients can be easily resolved from these three equations Eqn. 6, Eqn. 7 and Eqn. 8.

The matter density parameter (\(\Omega_{\text{om}} h_0^2\)), using Eqn. 6, Eqn. 7 and Eqn. 8, can be expressed by

\[
\Omega_{\text{om}} h_0^2 = \frac{h^2(z_\alpha) \left[ (1 + z_\beta)^2 - (1 + z_\gamma)^2 \right]}{-A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\beta) \left[ (1 + z_\gamma)^2 - (1 + z_\alpha)^2 \right]}{-A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\gamma) \left[ (1 + z_\alpha)^2 - (1 + z_\beta)^2 \right]}{-A(z_\alpha, z_\beta, z_\gamma)}. \tag{9}
\]

The curvature density parameter (\(\Omega_{\text{ok}} h_0^2\)), using Eqn. 6, Eqn. 7 and Eqn. 8, is given by

\[
\Omega_{\text{ok}} h_0^2 = \frac{h^2(z_\alpha) \left[ (1 + z_\beta)^3 - (1 + z_\gamma)^3 \right]}{A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\beta) \left[ (1 + z_\gamma)^3 - (1 + z_\alpha)^3 \right]}{A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\gamma) \left[ (1 + z_\alpha)^3 - (1 + z_\beta)^3 \right]}{A(z_\alpha, z_\beta, z_\gamma)}. \tag{10}
\]

The cosmological constant density parameter (\(\Omega_{\text{m}} h_0^2\)), using Eqn. 6, Eqn. 7 and Eqn. 8, can be written as

\[
\Omega_{\text{m}} h_0^2 = \frac{h^2(z_\alpha)(1 + z_\beta)^2(1 + z_\gamma)^2(z_\gamma - z_\beta)}{A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\beta)(1 + z_\alpha)^2(1 + z_\gamma)^2(z_\alpha - z_\gamma)}{A(z_\alpha, z_\beta, z_\gamma)} + \frac{h^2(z_\gamma)(1 + z_\alpha)^2(1 + z_\beta)^2(z_\beta - z_\alpha)}{A(z_\alpha, z_\beta, z_\gamma)}. \tag{11}
\]
where \( A(z_0, z_β, z_γ) = (1 + z_0)^2 (1 + z_β)^2 (z_β - z_0) + (1 + z_β)^2 (1 + z_γ)^2 (z_γ - z_β) + (1 + z_γ)^2 (1 + z_0)^2 (z_0 - z_γ) \).

Using error propagation formula in Eqn. 9, Eqn. 10 and Eqn. 11, the corresponding uncertainties of \( Ω_{0m} h_0^2 \), \( Ω_k h_0^2 \) and \( Ω_{0Λ} h_0^2 \) are given by

\[
\sigma^2_{Ω_{0m} h_0^2} = \frac{4h^2(z_0)\sigma^2_{h(z_0)} [(1 + z_β)^2 - (1 + z_γ)^2]^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_β)\sigma^2_{h(z_β)} [(1 + z_γ)^2 - (1 + z_0)^2]^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_γ)\sigma^2_{h(z_γ)} [(1 + z_0)^2 - (1 + z_β)^2]^2}{A^2(z_0, z_β, z_γ)}
\]

(12)

\[
\sigma^2_{Ω_k h_0^2} = \frac{4h^2(z_0)\sigma^2_{h(z_0)} [(1 + z_β)^3 - (1 + z_γ)^3]^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_β)\sigma^2_{h(z_β)} [(1 + z_γ)^3 - (1 + z_0)^3]^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_γ)\sigma^2_{h(z_γ)} [(1 + z_0)^3 - (1 + z_β)^3]^2}{A^2(z_0, z_β, z_γ)}
\]

(13)

and

\[
\sigma^2_{Ω_{0Λ} h_0^2} = \frac{4h^2(z_0)\sigma^2_{h(z_0)} (1 + z_β)^4 (1 + z_γ)^4 (z_γ - z_β)^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_β)\sigma^2_{h(z_β)} (1 + z_γ)^4 (1 + z_0)^4 (z_0 - z_γ)^2}{A^2(z_0, z_β, z_γ)} + \frac{4h^2(z_γ)\sigma^2_{h(z_γ)} (1 + z_0)^4 (1 + z_β)^4 (z_β - z_0)^2}{A^2(z_0, z_β, z_γ)}
\]

(14)

The Hubble constant \( (h_0) \), Hubble parameter at redshift \( z = 0 \), can be expressed by

\[
h_0^2 = Ω_{0m} h_0^2 + Ω_k h_0^2 + Ω_{0Λ} h_0^2.
\]

(15)

Corresponding uncertainty of the Hubble constant \( (h_0) \), using error propagation formula in Eqn. 15, is given by

\[
\sigma^2_{h_0} = \frac{\sigma^2_{Ω_{0m} h_0^2}}{h_0^2} + \frac{\sigma^2_{Ω_k h_0^2}}{h_0^2} + \frac{\sigma^2_{Ω_{0Λ} h_0^2}}{h_0^2}.
\]

(16)

So we can easily calculate the value of \( h_0 \) and \( σ_{h_0} \), from Eqn. 15 and Eqn. 16, using the calculated values of \( Ω_{0m} h_0^2 \), \( Ω_k h_0^2 \) and \( Ω_{0Λ} h_0^2 \).

For \( n \) number of samples of redshift and corresponding \( H(z) \) measurement, we can generate \( ^nC_3 = n(n-1)(n-2)/6 \) number of sample specific values for each of the coefficients \( Ω_{0m} h_0^2 \), \( Ω_k h_0^2 \) and \( Ω_{0Λ} h_0^2 \) (Zheng et al., 2016). We analyse these sample specific values for each of three density parameters using weighted-mean and median statistics. Theoretically, we should obtain the same values in every calculation for each of the coefficients. Since we use the observed \( H(z) \) values and corresponding standard deviations in our analysis, we can’t obtain the same results in each calculation of numerical analysis.

### 2.2 Weighted-Mean Statistic

Weighted-mean statistic (Zheng et al., 2016) is a popular way to analyse a collection of measurements, where the weightage factor also comes into consideration for each measurement. In this statistic, we need to assume that the uncertainties corresponding to the measurements are Gaussian in nature. We consider the inverses of the corresponding variances of the measurements as the weightage factors in our weighted-mean calculations. The weighted-mean formula for three-point diagnostic \( Ω_{0x} h_0^2(z_α, z_β, z_γ) \), where ‘\( x \)’ stands for matter \((m)\), curvature \((k)\), or cosmological constant \((Λ)\), is given by

\[
Ω_{0x} h_0^2_{(w.m.)} = \frac{\sum_{i=1}^{n-2} \sum_{j=γ+1}^{n} \sum_{α=β+1}^{n} Ω_{0x} h_0^2(z_α, z_β, z_γ)}{\sum_{i=γ+1}^{n-1} \sum_{j=β+1}^{n} \sum_{α=β+1}^{n} \sigma^2_{Ω_{0x} h_0^2(z_α, z_β, z_γ)}}.
\]

(17)
The corresponding weighted-mean uncertainty is expressed by

$$\sigma^2_{\text{DA}, k^2_{(w,m)}} = \left( \sum_{\gamma=1}^{n-2} \sum_{\beta=\gamma+1}^{n-1} \sum_{\alpha=\beta+1}^{n} \frac{1}{\sigma^2_{\text{DA}, k^2_{(z_\alpha, z_\beta, z_\gamma)}}} \right)^{-1}. \quad (18)$$

Using Eqn. 17 and Eqn. 18 we can calculate the weighted-mean value as well as the corresponding uncertainty for each of three density parameters.

### 2.3 Median Statistic

Median statistic (Gott et al., 2001; Zheng et al., 2016) is also an excellent approach for analysing a large number of samples without assuming the Gaussian nature of the uncertainties corresponding to samples. In this statistic, we will consider that all data points are statistically independent and they have no systematic errors. Each statistically independent measurement has 50% probability to be above or below the true median. The probability of n-th observation (from total N measurements), higher than true median is given by the binomial distribution $P = 2^{-N}N!/[n!(N-n)!]$. This binomial property gives us the liberty to calculate any required confidence intervals of the median.

Let us assume that we have total $(N+1)$ number of measurements $(M_i)$, where $i = 0, 1, ..., N$, arranged in ascending order. If $N$ is even, the median value will be the $\frac{N}{2}$-th measurement. If $N$ is odd then the median value will be the simple average of the $\frac{N+1}{2}$-th and $\frac{N+1}{2}$-th measurements. Gott et al. (2001) introduced a nice method to calculate the range of the median value. Defining $r = i/N$, we can write $M(r) = M_i$ and index variable becomes $r$ showing the range converted to $0 < r < 1$. The average of $r$, for true median, is $\langle r \rangle = 0.5$ and the standard deviation is given by $\langle r^2 - \langle r \rangle^2 \rangle^{1/2} = \frac{1}{\sqrt{4N}}$. So the range of the median value can be written as

$$\left(0.5 - \frac{1}{\sqrt{4N}}\right) N \text{-th value} < \text{median} < \left(0.5 + \frac{1}{\sqrt{4N}}\right) N \text{-th value} \quad (19)$$

where Eqn. 19 represents 68% confidence range (1σ error) of the median value.

### 3 Data

In our analysis, we use a total 55 $H(z)$ measurements (Sharov & Vasiliev, 2018) estimated by two different techniques (DA & BAO). In Table 1, We show these Hubble parameter data in $km Mpc^{-1} sec^{-1}$ unit.

#### 3.1 DA technique

One of the methods is differential dating of cosmic chronometers which is suggested by Jimenez & Loeb (2002). This process utilizes the differential relation between Hubble parameter and redshift, which is given by

$$H(z) = -\frac{1}{(1+z)} \frac{dz}{dt} \quad (20)$$

where $dz/dt$ is the time derivative of redshift.

Cosmic chronometers are the selected galaxies showing similar metallicities and low star formation rates. The best cosmic chronometers are those galaxies which evolve passively on a longer time scale than age differences between them. These selected galaxies are usually called passively evolving galaxies. Age difference ($\Delta t$) between two passively evolving galaxies can be measured by observing D4000 break at 4000 Å in galaxy spectra (Moresco et al., 2016). Measuring the age difference ($\Delta t$) between these types of galaxies, separated by a small redshift interval ($\Delta z$), one can infer the derivative $dz/dt$ from the ratio $\Delta z / \Delta t$. Then, substituting the value of $dz/dt$ in Eqn. 20, we can estimate the Hubble parameter ($H(z)$) at an effective redshift ($z$). This technique measured the values of $H(z)$ in the redshift range 0.07 $\leq z \leq 1.965$. Some of these DA $H(z)$ contain large standard deviations.

#### 3.2 BAO technique

The most recent method is the observation of peak in matter correlation function due to baryon acoustic oscillations in the pre-recombination epoch (Delubac et al., 2015). Angular separation of BAO peak, at a redshift $z$, is given by $\Delta \theta = r_d / (1 + z) D_A$, where $r_d$ is the sound horizon at drag epoch and $D_A$ is the angular diameter distance. The redshift separation of BAO peak, at a particular redshift $z$, can be expressed as $\Delta z = r_d / D_H$, where $D_H$ is Hubble distance ($c/H$). Measuring the BAO peak position at any redshift $z$, we can obtain $H(z)$ from the determination of $D_H / r_d$ and $D_A / r_d$. BAO $H(z)$ were measured in the redshift range 0.24 $\leq z \leq 2.36$. The standard deviations of these BAO $H(z)$ are smaller than DA technique.
Table 1: The available Hubble parameter \( (H(z)) \) and corresponding uncertainties \( (\sigma_H(z)) \), measured by DA and BAO techniques in \( km \text{Mpc}^{-1} \text{sec}^{-1} \) unit, are shown at different redshifts \( (z) \).

| \( z \)  | \( H(z) \) | \( \sigma_H(z) \) | Method & Refs.                          |
|--------|-----------|-----------------|----------------------------------------|
| 0.07   | 69        | 19.6            | DA (Zhang et al., 2014)                |
| 0.09   | 69        | 12              | DA (Jimenez et al., 2003)              |
| 0.12   | 68.6      | 26.2            | DA (Zhang et al., 2014)                |
| 0.17   | 83        | 8               | DA (Simon et al., 2005)                |
| 0.1791 | 75        | 4               | DA (Moresco et al., 2012)              |
| 0.1993 | 75        | 5               | DA (Moresco et al., 2012)              |
| 2      | 72.9      | 29.6            | DA (Zhang et al., 2014)                |
| 0.24   | 79.69     | 2.99            | BAO (Gaztañaga et al., 2009)           |
| 0.27   | 77        | 14              | DA (Simon et al., 2005)                |
| 0.28   | 88.8      | 36.64           | DA (Zhang et al., 2014)                |
| 0.30   | 81.7      | 6.22            | BAO (Oka et al., 2014)                 |
| 0.31   | 78.18     | 4.74            | BAO (Wang et al., 2017)                |
| 0.34   | 83.8      | 3.66            | BAO (Gaztañaga et al., 2009)           |
| 0.35   | 82.7      | 8.4             | BAO (Chuang & Wang, 2013)              |
| 0.3519 | 83        | 14              | DA (Moresco et al., 2012)              |
| 0.36   | 79.94     | 3.38            | BAO (Wang et al., 2017)                |
| 0.38   | 81.5      | 1.9             | BAO (Alam et al., 2017)                |
| 0.3802 | 83        | 13.5            | DA (Moresco et al., 2016)              |
| 0.4    | 95        | 17              | DA (Simon et al., 2005)                |
| 0.4    | 82.04     | 2.03            | BAO (Wang et al., 2017)                |
| 0.4004 | 77        | 10.2            | DA (Moresco et al., 2016)              |
| 0.4247 | 87.1      | 11.2            | DA (Moresco et al., 2016)              |
| 0.43   | 86.45     | 3.97            | BAO (Gaztañaga et al., 2009)           |
| 0.44   | 84.81     | 1.83            | BAO (Wang et al., 2017)                |
| 0.4497 | 92.8      | 12.9            | DA (Moresco et al., 2016)              |
| 0.47   | 89        | 34              | DA (Ratsimbazafy et al., 2017)         |
| 0.4783 | 80.9      | 9               | DA (Moresco et al., 2016)              |
| 0.48   | 97        | 62              | DA (Stern et al., 2010)                |
| 0.48   | 87.79     | 2.03            | BAO (Wang et al., 2017)                |
| 0.51   | 90.4      | 1.9             | BAO (Alam et al., 2017)                |
| 0.52   | 94.35     | 2.64            | BAO (Wang et al., 2017)                |
| 0.56   | 93.34     | 2.3             | BAO (Wang et al., 2017)                |
| 0.57   | 96.8      | 3.4             | BAO (Anderson et al., 2014)            |
| 0.59   | 98.48     | 3.18            | BAO (Wang et al., 2017)                |
| 0.5929 | 104       | 13              | DA (Moresco et al., 2012)              |
| 0.6    | 87.9      | 6.1             | BAO (Blake et al., 2012)               |
| 0.61   | 97.3      | 2.1             | BAO (Alam et al., 2017)                |
| 0.64   | 98.82     | 2.98            | BAO (Wang et al., 2017)                |
| 0.6797 | 92        | 8               | DA (Moresco et al., 2012)              |
| 0.73   | 97.3      | 7               | BAO (Blake et al., 2012)               |
| 0.7812 | 105       | 12              | DA (Moresco et al., 2012)              |
| 0.8754 | 125       | 17              | DA (Moresco et al., 2012)              |
| 0.88   | 10         | 40              | DA (Stern et al., 2010)                |
| 0.9    | 117       | 23              | DA (Simon et al., 2005)                |
| 1.037  | 154       | 20              | DA (Moresco et al., 2012)              |
| 1.3    | 168       | 17              | DA (Simon et al., 2005)                |
| 1.363  | 160       | 33.6            | DA (Moresco, 2015)                     |
| 1.43   | 177       | 18              | DA (Simon et al., 2005)                |
| 1.53   | 140       | 14              | DA (Simon et al., 2005)                |
| 1.75   | 202       | 40              | DA (Simon et al., 2005)                |
| 1.965  | 186.5     | 50.4            | DA (Moresco, 2015)                     |
| 2.3    | 224       | 8               | BAO (Busca et al., 2013)               |
| 2.33   | 224       | 8               | BAO (Bautista et al., 2017)            |
| 2.34   | 222       | 7               | BAO (Delubac et al., 2015)             |
| 2.36   | 226       | 8               | BAO (Font-Ribera et al., 2014)         |
4 Analysis and Results

4.1 Three-point statistics

We employ our cosmological parameters estimation procedure (three-point diagnostics) for three different sets of $H(z)$ measurements. In the first case, we use total 53 $H(z)$ excluding two DA data corresponding to the redshift points 0.4 and 0.48. We choose these two same $H(z)$ data with low standard deviations measured by BAO technique. For the second case, we work with 31 $H(z)$ measured by DA technique. In the last case, we perform our analysis using only 24 BAO $H(z)$ measurements. We generate the sample specific values (using Eqn. 9, Eqn. 10 & Eqn. 11) of three-point diagnostics $\Omega_{0m}h_0^2(z_\alpha, z_\beta, z_\gamma)$, $\Omega_{0k}h_0^2(z_\alpha, z_\beta, z_\gamma)$ and $\Omega_{0\Lambda}h_0^2(z_\alpha, z_\beta, z_\gamma)$ as well as the uncertainties (using Eqn. 12, Eqn. 13 & Eqn. 14) corresponding to these sample specific values for each of these $H(z)$ sets. Here, $z_\alpha$, $z_\beta$ and $z_\gamma$ are three different redshift points (e.g., see section 2.1). We show the representations of these sample specific values (with errorbars), with respect to the absolute values of redshift differences $|z_\gamma - z_\alpha|$, in individual figures for each of the density parameters and each of these three sets of $H(z)$.

![Figure 1](image1.png)

Figure 1: Each sub-figure shows the representation of the sample specific values (red) of matter density parameter ($\Omega_{0m}h_0^2$) with corresponding uncertainties (green) for a particular set of $H(z)$ measurements. Left sub-figure is corresponding to total (BAO & DA) 53 $H(z)$, middle sub-figure represents the values corresponding to only 31 DA $H(z)$ and the right sub-figure shows the distribution of $\Omega_{0m}h_0^2$ for only 24 BAO $H(z)$. Horizontal axis of each sub-figure defines the absolute values of the differences between two redshifts.

![Figure 2](image2.png)

Figure 2: Left sub-figure shows the representation of the sample specific values (red) of curvature density parameter ($\Omega_{0k}h_0^2$) with corresponding uncertainties (green) for the set of total (BAO & DA) 53 $H(z)$ measurements. Similarly, middle and right sub-figures represent the sample specific values (red) of $\Omega_{0k}h_0^2$ for only 31 DA and only 24 BAO $H(z)$ sets. In each sub-figure, horizontal axis represents the absolute values of the differences between two redshifts.

Fig. 1 represents three separate distributions of the matter density parameter ($\Omega_{0m}h_0^2$) with errorbars for three different sets of $H(z)$. For total 53 $H(z)$ measurements, we obtain 23426 number of sample specific values of $\Omega_{0m}h_0^2$. For only 31 DA $H(z)$ measurements, we find 4495 number of values of this density parameter. We obtain 2024 number of values for only 24 BAO $H(z)$ measurements. We find that the range of the matter density parameter is in between $-4715$ and $3982$ for the set of total 53 $H(z)$. For only 31 DA $H(z)$, the range is about $-18543$ to $15008$. We notice that there is a large difference in the ranges of the matter density parameter between these two sets of $H(z)$ measurements, since we exclude two $H(z)$ of DA technique at redshifts 0.4 and 0.48 for the set of total (DA & BAO) 53 $H(z)$. When we consider only 31 DA $H(z)$, it shows the large range for including the redshift points 0.4 and 0.48. The range of the sample specific values of the matter density parameter, for the set of 24 BAO $H(z)$, lies within $-298$ to $774$. We also produce similar representations, shown in Fig. 2, for sample specific values of the curvature density parameter ($\Omega_{0k}h_0^2$) for same three sets of $H(z)$. For the set of total 53 $H(z)$, the range of the values of $\Omega_{0k}h_0^2$ lies within $-8484$ to $9832$. The range of the sample specific values of $\Omega_{0k}h_0^2$, for the set of 31 DA $H(z)$, is about $-31797$ to $38669$. The same thing happens here that the difference between the ranges of
Using weighted-mean and median statistics, we estimate the values of three cosmological density parameters ($\Omega_{\text{m}}h^2_0$, $\Omega_{\text{b}}h^2_0$, $\Omega_{\text{c}}h^2_0$) from the sample specific values of these density parameters and also calculate the uncertainties corresponding to these estimated values for each of the sets of $H(z)$ measurements. Thereafter, using the values and uncertainties of these density parameters, we obtain the value of Hubble constant ($h_0$) as well as the corresponding uncertainty in each of the cases. In Table 2, we show the weighted-mean and median values of these cosmological parameters corresponding to three $H(z)$ sets (total 53, only 31 DA and only 24 BAO). We obtain 1σ uncertainties (using Eqn. 18) for weighted-mean statistic and 68% (1σ) confidence intervals (using Eqn. 19) for median statistic. Usually, median statistic shows larger uncertainties, corresponding to estimated values, than weighted-mean uncertainties.

For total 53 Hubble data, we get smaller value of the matter density ($\Omega_{\text{m}}h^2_0$) as well as the corresponding uncertainty in case of median statistic. The values of the cosmological constant density ($\Omega_{\text{c}}h^2_0$) are almost same in each case of both statistics. Weighted-mean statistic gives positive value of the curvature density ($\Omega_{\text{k}}h^2_0$) and median statistic shows negative value of this parameter corresponding to this set of $H(z)$. Comparing the estimated values of $\Omega_{\text{k}}h^2_0$ with corresponding uncertainties for each of these two statistics, we can conclude both statistics showing nearly flatness of the universe. Interestingly, we can notice that our estimated median values of these cosmological parameters agree with the results of Planck Collaboration VI. (2020) excellently for this set of total 53 $H(z)$ data. In Table 3, we show the values of cosmological density parameter and Hubble constant obtained by Planck Collaboration VI. (2020). The uncertainties corresponding to our estimated median values (which agree with Planck Collaboration VI. (2020) results) are greater than the uncertainties obtained by Planck Collaboration VI. (2020), since the Hubble data utilized by us are available in a small number than CMB data.

Figure 3: Each sub-figure represents the sample specific values (red) of cosmological constant density parameter ($\Omega_{\text{c}}h^2_0$) with corresponding uncertainties (green) for a particular set of $H(z)$ measurements. Three sub-figures (left, middle and right) are corresponding to total (BAO & DA) 53, only 31 DA and only 24 BAO $H(z)$ measurements. The absolute values of the differences between two redshifts are presented on the horizontal axis of each sub-figure.
Table 2: Estimated weighted-mean and median values of three cosmological density parameters ($\Omega_{\text{om}}h_0^2$, $\Omega_{\text{ok}}h_0^2$, $\Omega_{\text{OA}}h_0^2$) and Hubble constant ($h_0$) corresponding to total 53, only 31 DA, and only 24 BAO Hubble parameter measurements. We estimate 1σ uncertainty range for the cases of former statistic and 68% confidence interval for the cases of later statistic. Detailed descriptions of the results are given in section 4.2.

| Parameter | Total 53 Hubble parameter data (0.07 ≤ z ≤ 2.36) | Only 31 DA Hubble parameter data (0.07 ≤ z ≤ 1.965) | Only 24 BAO Hubble parameter data (0.24 ≤ z ≤ 2.36) |
|-----------|---------------------------------------------|---------------------------------|---------------------------------|
|           | w.m. value(±1σ) | median value(68% range) | w.m. value(±1σ) | median value(68% range) | w.m. value(±1σ) | median value(68% range) |
| $\Omega_{\text{om}}h_0^2$ | 0.1072 ± 0.0022 | 0.1485−0.0051 | 0.1651 ± 0.0136 | 0.167−0.0145+0.0022 | 0.0805 ± 0.0004 | 0.0887−0.0088+0.0074 |
| $\Omega_{\text{ok}}h_0^2$ | 0.0523 ± 0.0075 | −0.0137−0.0129+0.0313 | −0.0782 ± 0.0373 | 0.001−0.0301 | 0.1702 ± 0.0143 | 0.1437−0.0027+0.0267 |
| $\Omega_{\text{OA}}h_0^2$ | 0.3232 ± 0.0082 | 0.3126−0.0093+0.0311 | 0.3865 ± 0.0325 | 0.2668−0.0261 | 0.1603 ± 0.0179 | 0.1941−0.0209+0.0257 |
| $h_0$ | 0.6947 ± 0.0082 | 0.6689−0.0121+0.0141 | 0.6880 ± 0.0373 | 0.6594−0.0321+0.0379 | 0.6411 ± 0.0182 | 0.6531−0.0297+0.0289 |

Table 3: Table shows the cosmological parameters obtained by Planck Collaboration VI. (2020) from standard ΛCDM model. $\Omega_{\text{om}}$, $\Omega_{\text{ok}}$, $\Omega_{\text{OA}}$ are matter, curvature, and cosmological density parameters. $h_0$ is today’s Hubble parameter in 100 km Mpc$^{-1}$ sec$^{-1}$ unit.

| Parameter | Planck's Value |
|-----------|----------------|
| $\Omega_{\text{om}}h_0^2$ | 0.1430 ± 0.0011 |
| $\Omega_{\text{ok}}$ | 0.001 ± 0.002 |
| $\Omega_{\text{OA}}$ | 0.6847 ± 0.0073 |
| $h_0$ | 0.6736 ± 0.0054 |

Combining the value of $\Omega_{\text{ok}}$ and $\Omega_{\text{OA}}$ with the value of $h_0$

| Parameter | Value |
|-----------|-------|
| $\Omega_{\text{ok}}h_0^2$ | $\Omega_{\text{OA}}h_0^2$ |
| 0.0005 | 0.3107 |

In case of only 24 BAO $H(z)$, the values of matter density corresponding to both statistics are almost equal. The weighted-mean value of Hubble constant is also equivalent to the median value of this parameter. We find that the value of the curvature density is large positive for each of both statistics. The value of the cosmological constant density is smaller for weighted-mean statistic compared to median statistic in this case. Observing the uncertainties of $\Omega_{\text{ok}}h_0^2$ and $\Omega_{\text{OA}}h_0^2$ for this set of $H(z)$, we can notice that the weighted-mean values are comparable with the median values of these two density parameters. Between three sets of $H(z)$, our estimated values of cosmological parameters corresponding to only 24 BAO data deviate the most from Planck Collaboration VI. (2020) results. Though BAO technique shows lower standard deviations in $H(z)$ measurements than DA technique, the available $H(z)$ measured by former technique are less number (even lower than DA $H(z)$ measurements). Since the less number of the available BAO $H(z)$ data and non-availability of the values of BAO $H(z)$ in some redshift regions (at $z < 0.24$ and $0.73 < z < 2.3$), the analysis corresponding to 24 BAO data set also does not show reliable estimations of cosmological parameters.

We also analyse the entire $H(z)$ measurements excluding four high redshift data ($z = 2.3, 2.33, 2.34, 2.36$). We show the estimated results corresponding to these 49 $H(z)$ in Table 4. In this case, the estimated values of density parameters and Hubble constant, corresponding to each of both statistics, are strongly deviated from the results of Planck Collaboration VI. (2020). For this case, the estimated uncertainties corresponding...
to values of parameters are also higher than the uncertainties corresponding to total 53 \( H(z) \) data. We find spatially closed universe, which gives a tension in fundamental cosmology, for this case of 49 data in each of both statistics. So we conclude that the cosideration of four high redshift data in the analysis is more crucial to estimate the values (consistent with Planck Collaboration VI. (2020) results) of cosmological parameters.

Table 4: Estimated weighted-mean and median values of three cosmological density parameters (\( \Omega_{0\text{m}}h^2 \), \( \Omega_{0k}h^2 \), \( \Omega_{0\Lambda}h^2 \)) and Hubble constant (\( h_0 \)) corresponding to 49 Hubble parameter measurements. We estimate 1\( \sigma \) uncertainty range for the cases of former statistic and 68\% confidence interval for the cases of later statistic. Detailed descriptions of the results are given in section 4.2.

| parameter        | 49 Hubble parameter data (0.07 \( \leq z \leq 1.965 \)) |
|------------------|-------------------------------------------------------|
|                  | w.m. value(\( \pm \sigma \)) | median value(68\% range) |
| \( \Omega_{0\text{m}}h^2 \) | 0.1895 \( \pm \) 0.0055 | 0.237 \( ^{+0.0082}_{-0.0062} \) |
| \( \Omega_{0k}h^2 \)    | \(-0.1384 \pm 0.0143\) | \(-0.1856 \pm 0.0162\) |
| \( \Omega_{0\Lambda}h^2 \) | 0.4468 \( \pm \) 0.0124 | 0.4124 \( ^{+0.0117}_{-0.0099} \) |
| \( h_0 \)              | 0.7056 \( \pm \) 0.0139 | 0.6811 \( ^{+0.0159}_{-0.0148} \) |

We show the Hubble parameter (\( H(z) \)) curves in Fig. 4 using the values of cosmological parameters obtained from our analysis. Horizontal axis of this figure represents the values of redshift. In the same figure, we represent \( H(z) \) curve using the density parameter and Hubble constant values from Planck Collaboration VI. (2020). Moreover, in this figure, we also show the available 31 DA and 24 BAO \( H(z) \) data points with corresponding uncertainties. \( H(z) \) curve using our median values (corresponding to 53 Hubble data) of cosmological parameters shows better agreement with the \( H(z) \) curve obtained by using the results of Planck Collaboration VI. (2020). Further, \( H(z) \) curve obtained by using our weighted-mean results (corresponding to 31 DA Hubble data) is also very nearly equal to the same curve obtained by using the results of Planck Collaboration VI. (2020). Except these two cases, rest of \( H(z) \) curves of Fig. 4 indicate significant tension with Planck Collaboration VI. (2020) results.

Figure 4: Figure shows Hubble parameter (\( H(z) \)) curves using our estimated values of cosmological parameters as well as using Planck Collaboration VI. (2020) results. Horizontal axis of the figure defines the values of redshift. Moreover, available 55 \( H(z) \) measurements (31 DA (yellow) & 24 BAO (red)) are also shown in the same figure. The Hubble curve corresponding to 49 data represents the curve corresponding to all Hubble data excluding four high redshift (\( z = 2.3, 2.33, 2.34, 2.36 \)) points. Detailed discussions about these \( H(z) \) curves are given in section 4.2.

### 4.3 Non-Gaussianity of uncertainties

We calculate the weighted-mean values of density parameters assuming the Gaussian nature of sample specific uncertainties corresponding to the density parameters. However, these uncertainties do not show the Gaussian behaviour. In case of median statistic, these non-Gaussian uncertainties do not effect the estimated
values, since we do not need to use sample specific uncertainties for median analysis. Chen et al. (2003); Crandall & Ratra (2014); Crandall et al. (2015) developed a procedure to measure the non-Gaussianity of sample specific uncertainties. They defined the number of standard deviations ($N_\sigma$) which is a measurement of deviation from the central estimation of an observable for a particular statistic. Using the distribution of this number of standard deviation, we can find that the nature of uncertainties is Gaussian or not. For instance, the number of standard deviation, corresponding to weighted-mean (w.m.) statistic for a particular measurement, is defined as

$$N_{\sigma, \alpha\beta\gamma} = \frac{\Omega_{m}h^2_0(z_\alpha, z_\beta, z_\gamma) - \Omega_{k}h^2_0(\omega_m)}{\sigma_{\Omega_{m}h^2_0(z_\alpha, z_\beta, z_\gamma)}}$$  \hspace{0.5cm} (21)$$

where 'x' is anyone of matter ($m$), curvature ($k$) and cosmological constant ($\Lambda$). We can also use the Eqn. 21 for the same purpose corresponding to median (md) statistic. The percentage of the measurements $|N_\sigma| < 1$ helps us to find the deviation from the Gaussian nature of the uncertainties. If the uncertainty distribution is Gaussian, then the percentage should be equal to approximately 68.3%. In Table 5 and Table 6, we
represent our results for the percentage of the measurements | \( N_\sigma \) | < 1 corresponding to weighted-mean and median statistic. We find that the deviation from the 1σ uncertainty is larger for both sets of total 53 and only 31 DA \( H(z) \) data compared to the set of only 24 BAO \( H(z) \) corresponding to both statistics. We can also understand the deviation of uncertainty looking into the probability density of \( N_\sigma \). In Fig. 5 and Fig. 6, we show the probability density (normalized histogram) of \( N_\sigma \) for cosmological density parameters corresponding to three sets of \( H(z) \) in case of both weighted-mean and median statistic. In these figures, horizontal axis of each sub-figure defines the values of \( N_\sigma \).

Table 5: Percentage of the measurements | \( N_\sigma \) | < 1 for three cosmological density parameters (\( \Omega_{0m}h^2_0 \), \( \Omega_{0k}h^2_0 \), \( \Omega_{0\Lambda}h^2_0 \)) corresponding to three sets of \( H(z) \) in case of weighted-mean (w.m.) statistic.

| Parameter | Total data(53) | DA(31) | BAO(24) |
|-----------|----------------|--------|---------|
| \( \Omega_{0m}h^2_0 \) | 86.28% | 92.50% | 81.03% |
| \( \Omega_{0k}h^2_0 \) | 86.71% | 92.77% | 81.03% |
| \( \Omega_{0\Lambda}h^2_0 \) | 87.84% | 93.73% | 81.42% |

Table 6: Percentage of the measurements | \( N_\sigma \) | < 1 for three cosmological density parameters (\( \Omega_{0m}h^2_0 \), \( \Omega_{0k}h^2_0 \), \( \Omega_{0\Lambda}h^2_0 \)) corresponding to three sets of \( H(z) \) in case of median (md) statistic.

| Parameter | Total data(53) | DA(31) | BAO(24) |
|-----------|----------------|--------|---------|
| \( \Omega_{0m}h^2_0 \) | 86.83% | 92.52% | 81.03% |
| \( \Omega_{0k}h^2_0 \) | 87.15% | 92.48% | 81.32% |
| \( \Omega_{0\Lambda}h^2_0 \) | 87.79% | 93.24% | 81.92% |

5 Discussions and Conclusions

In this article, we use a new three-point diagnostics method to analyze the available Hubble parameter measurements (DA & BAO). We use Hubble parameter and redshift relation (Eqn. 5) and then using the three-point diagnostics we find the three important cosmological density parameters \( \Omega_{0m}h^2_0 \), \( \Omega_{0k}h^2_0 \), \( \Omega_{0\Lambda}h^2_0 \) in terms of values of redshifts and the corresponding Hubble parameter (e.g., Eqn. 9, Eqn. 10 and Eqn. 11). We also find the uncertainties in the measured values of the density parameters following Eqn. 12, Eqn. 13 and Eqn. 14. We use all possible three-point combinations of the available data points to obtain a set of different values of the density parameters using all or different subsets of the \( H(z) \) data. We employ weighted-mean statistic as well as median statistic to estimate the density parameters and the corresponding uncertainties. Using the values of density parameters we derive the today’s Hubble parameter value and the corresponding uncertainties using Eqn. 15 and Eqn. 16.

Our analysis shows that the uncertainty distributions of the density parameters are highly non-Gaussian. Since a fundamental assumption in using the weighted-mean statistic reliably is the validity of the Gaussian nature of the error distributions, we conclude that results of the weighted-mean statistics are difficult for interpretation in absence of suitable error estimates. The median statistic on the other hand is more reliable since one does not need to make any assumption about the specific distributions of the error distributions of the Hubble data observations. Using the median statistic we obtain excellent agreement of the estimates of the density and Hubble parameters of this work with those reported by Planck Collaboration VI. (2020) (e.g., see Table 2 and Table 3). In our analysis, the estimated uncertainties are larger than the uncertainties of parameters calculated by Planck Collaboration VI. (2020), since the number of available Hubble data (used by us) is significantly smaller than CMB data.

Though, the weighted-mean value of curvature density parameter signifies almost nearly equal to spatially flat curvature of universe in each of the cases of total 53 \( H(z) \) and only 31 BAO data, the weighted-mean values of other parameters do not show reliability in respect of median results corresponding to total 53 \( H(z) \) data. The median value of curvature density parameter also shows excellently spatial flatness of universe in case of 31 DA \( H(z) \), but median values of other parameters do not show any consistency in this case. In case of only 24 BAO data, both statistics show high tension with Planck Collaboration VI. (2020) for the values of all estimated parameters.

Interestingly although we do not assume a spatially flat universe to begin with, the spatial curvature estimated by us following median statistic using all 53 Hubble data becomes \(-0.0137^{+0.0129}_{-0.0125}\) indicating a spatially flat universe compatible with Planck Collaboration VI. (2020) results. Moreover, the value of today’s Hubble parameter estimated by us \( h_0 = 0.6689^{+0.0141}_{-0.0121} \) is in close agreement with the corresponding
value obtained by the Planck Collaboration VI. (2020). From Supernovae type Ia measurements, the values of $h_0$ is 0.74 ± 0.014. Our analysis using Hubble data measured from relatively local universe therefore is consistent global measurement of the same from the CMB data avoiding the so-called 'Hubble tension'.

We summarize the major conclusions of our analysis as follows.

(i) We note that the intrinsic errors in Hubble parameter measurements based upon the DA approach are larger than the errors using the BAO measurements. This leads to larger error on estimated cosmological parameters for both weighted mean and median statistics using the DA data alone.

(ii) In case of 31 DA and 24 BAO $H(z)$ data, the sample specific values of density parameters are smaller than the same corresponding to total 53 $H(z)$ measurements since the large number ($n$) of data generates the larger number ($\binom{n}{3}$) of sample specific values. Interestingly, applying median statistic on these sample specific values (for entire 53 $H(z)$ data), we find the values of fundamental cosmological parameters which agree with Planck Collaboration VI. (2020) results excellently.

(iii) Excluding four high redshift points ($z = 2.3, 2.33, 2.34, 2.36$) in the analysis leads to the inconsistent values of the cosmological parameters. Including these four redshift points, we increase the number of sample specific values and also reduce the estimated uncertainties corresponding to the values of parameters. Thus, we find the results (consistent with Planck Collaboration VI. (2020) results) of the fundamental cosmological parameters for the set of total 53 $H(z)$ using our three-point diagnostics.

(iv) Because of the limitations of the application of weighted-mean statistic (specifically non-validating assumption of Gaussian nature of uncertainty distribution), median statistic is more reliable in our analysis.

(v) We estimate the values of fundamental cosmological density parameters and Hubble constant without assuming any prior value of any cosmological parameters. We analyse these parameters using our three-point diagnostics procedure for only $\Lambda$CDM model of the universe. In future, we will apply our three-point diagnostics method in various cosmological model (e.g. $w$CDM, CPL (Chevalier & Polarski, 2001; Linder, 2003) models) to estimate the fundamental cosmological parameters.

References

Alam, S. et al., 2017, MNRAS, 470, 2617
Anderson, L., Aubourg, E., Bailey, S. et al., 2014, MNRAS, 439, 83
Bautista, J. E., Busca, N. G. et al., 2017, A & A, 603, A12
Blake, C., Brough, S., Colless, M. et al., 2012, MNRAS, 425, 405
Busca, N. G., Delubac, T., Rich, J., Bailey, S., Font-Ribera, A. et al., 2013, A & A, 552, A96
Chen, G., Gott, J. R., III, & Ratra, B., 2003, PASP, 115, 1269
Chevalier, M. & Polarski, D., 2001, IJMPD, 10, 213
Chuang, C. H., & Wang, Y., 2013, MNRAS, 435, 255
Crandall, S., & Ratra, B., 2014, PHYS. LETT. B, 732, 330
Crandall, S., Houston, S., & Ratra, B., 2015, MOD. PHYS. LETT. A, 30, 1550123
Delubac, T., Bautista, J. E., Busca, N. G. et al., 2015, A & A, 574, A59
Einstein, A., Sitz. Preuss. Akad. d. Wiss. Phys.-Math 142 (1917).
Font-Ribera, A., Kirkby, D., Busca, N.G. et al., 2014, JCAP, 05, 027
Gaztañaga, E., Cabré, A. & Hui, L., 2009, MNRAS, 399, 1663
Gott, J. R., III, Vogeley, M. S., Podariu, S., & Ratra, B., 2001, ApJ, 549, 1
Jimenez, R., & Loeb, A., 2002, ApJ, 573, 37
Jimenez, R., Verde, L., Treu, T., & Stern, D., 2003, ApJ, 593, 622
Linder E. V., 2003, Phys. Rev. Lett., 90, 091301
Macauley, E., Wehus, I. K. and Eriksen, H. K., 2013, Phys. Rev. Lett., 111, 161301
Moresco, M., Cimatti, A., Jimenez, R. et al., 2012, JCAP, 08, 006
Moresco, M., 2015, MNRAS, 450, L16
Moresco, M., Pozzetti, L., Cimatti, A. et al., 2016, JCAP, 05, 014
