Quintessence Model With Double Exponential Potential

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(May 30, 2018)

We have reinvestigated the quintessence model with minimally coupled scalar field in the context of recent Supernova observation at $z = 1.7$. By assuming the form of the scale factor which gives both the early time deceleration and late time acceleration, consistent with the observations, we show that one needs a double exponential potential. We have also shown that the equation of state and the behaviour of dark energy density are reasonably consistent with earlier constraints obtained by different authors. This work shows again the importance of double exponential potential for a quintessence field.

Over the first few years we are experiencing some of the most interesting cosmological observations. Data from the luminosity distance-redshift observations of the type Ia Supernova (SNIa) collected by two survey teams, The Supernova Cosmology Project and the High-z Supernova Search team, [1,2] predict that the universe is currently going through an accelerating expansion phase. Although there are two different interpretation of the Supernova observations — intergalactic dust and SN luminosity evolution [3], the recent observations of SN 1997ff, at $z \sim 1.7$ [4] put the accelerating Universe hypothesis on a firm footing. This also provides the first evidence for an early epoch of decelerating universe. On the hand, the recent observations of the acoustic peaks of the Cosmic Microwave Background (CMB) temperature fluctuations [5] favour a spatially flat universe, as predicted by the inflationary models.

If the results of these two observations are put together, one immediate conclusion is that the

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energy density of the universe is currently dominated by a form of matter having negative pressure, commonly referred as "dark energy". This component is qualitatively different from the standard dark matter in the sense that it has large negative pressure and it is approximately homogeneous, not clustering with matter on scales of clusters of galaxies. The first and obvious choice for this dark energy component is the cosmological constant $\Lambda$ which represents the energy of a quantum vacuum. However the problem of $\Lambda$ being the dominant component of the total energy density stems from the fact that the energy scale involved is lower than the normal energy scale of most particle physics model by a factor $\sim 10^{-120}$.

So to find some alternative candidate for this acceleration a dynamical $\Lambda$ [6] in the form of a scalar field with some self interacting potential [7] is considered whose slowly varying energy density mimics an effective cosmological constant. The idea of this candidate, called *quintessence* [6], is borrowed from the inflationary phase of the early universe, with the difference that it evolves at a much lower energy scale. The energy density of this field, though dominant at present epoch, must remain subdominant at very early stages and has to evolve in such a way that it becomes comparable with the matter density $\Omega_m$ now. This type of specific evolution, needs several constraints on the initial conditions and fine tuning of parameters for the potential. A new form of quintessence field called “*tracker field*” [9] has been proposed to solve this problem. It has an equation of motion with an attractor like solution in a sense that for a wide range of initial conditions the equation of motion converges to the same solution.

There are a number of quintessence models which have been put forward in recent years. They involve a scalar field rolling down its potential [10]- [20], an axion field [21], scalar tensor theories of gravity [22]- [37], dilaton in context of string theory [38], and also fields arising from compactifications of the multidimensional Einstein-Yang-Mills system [39]. In a very recent work, Zimdahl and Pavon [40] have shown that a suitable coupling between a minimally coupled quintessence field and the pressureless cold dark matter gives a constant ratio of the energy densities of both components which is compatible with the late time acceleration of the universe. They have termed it "*interacting quintessence*". In another recent work, Tocchini-Valentini and Amendola have investigated the cosmological models when this coupling between the quintessence field
and the perfect fluid dark matter is linear \([41]\).

Although all of these have their own merits in explaining the dark energy of the universe, there are number of difficulties with these models. One of them is to smoothly match the current accelerating universe with matter or radiation dominated decelerated universe. The current accelerated expansion is obviously a recent phenomena as one needs a sufficiently long matter dominated decelerated phase which should last until a recent past for the observed structure to develop from the density inhomogeneities. Further the success of big bang nucleosynthesis gives us a strong evidence of the radiation dominated decelerated phase when the universe is few seconds old. Although this required feature of the decelerated expansion is by far observationally untested, the recent observation of the SN1997ff at \(z = 1.7\) \([4]\) confirms this essential feature of the history of the universe. In a recent analysis, using a new technique which is independent of the content of the universe, Turner and Riess \([42]\) have shown that supernova data favour past deceleration \((z > 0.5)\) and a recent acceleration \((z < 0.5)\).

Also there are a number of investigations in order to constrain the equation of state of the dark energy component taking both the supernova data and data for the cmb measurements into account \([46]\). In one of the recent analysis, Corassaniti and Copeland \([47]\) have shown that most of the potentials used so far, including the inverse power law one as well as the Supergravity inspired potential, are not satisfactory as far as these constraints on the equation of state are concerned.

In this work, we have investigated these issues of the dark energy in a different way. We have taken the dark energy to be a minimally coupled scalar field rolling down its potential. Instead of assuming the form of its potential, we have assumed the form of the scale factor (which in turn gives the form of the Hubble parameter) keeping in mind that although the universe is presently accelerating but it was decelerating in recent past. In their work, Turner and Riess have emphasised \([42]\) the importance of assumption about \(H(z)\) in order to use the SNe data to probe the history of the universe. This method of finding exact solutions for scalar field cosmology was first used by Ellis and Madsen \([43]\) for inflationary models. They showed that one can determine the potential which gives the best behaviour in terms of its implications for cosmology. Later Uggla et.al have discussed this method for a more generalised situation \([44]\). Assuming some specific form of the
scale factor which gives both the decelerating universe in the past as well as the accelerating one at present, we have tried to fit our model with the SNe data including the recent data at $z = 1.7$. For particular value of the parameter in our model for which our model fits reasonably well with the observational data, we have found that the potential turns out to be a double exponential one. This sort of potential has been considered earlier by different authors for quintessence models \cite{13,20}. We have also shown that energy density for the scalar field remains sufficiently below that of the matter field for higher redshifts and starts dominating in the recent past.

Let us consider a spatially flat, homogeneous, isotropic universe, with a pressureless (dust) matter fluid and a scalar field $\phi$ with potential $V(\phi)$ minimally coupled with gravity. The equations of motion are given by

$$3H^2 = 8\pi G [\rho_m + \frac{1}{2} \dot{\phi}^2 + V(\phi)]$$

(1)

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

(2)

$$\dot{\rho}_m + 3H \rho_m = 0$$

(3)

where $\rho_m$ is the energy density for the matter fluid and $H = \dot{R}(t)/R(t)$ is the Hubble parameter and $R(t)$ is the scale factor. Here overdot and prime mean differentiations with respect to time and scalar field $\phi$ respectively. In this system of equations, one has three independent equations and four unknowns, which demands one assumption about the unknowns to solve the system. In most of the previous works with minimally coupled scalar field, the form of the potential has been assumed in order to solve the system. Although the assumption of the form of the potential from the particle physics viewpoint is a reasonable way, many a times it leads to complicated equations for the scale factor to solve and also to study different observable quantities. In this work, we proceed in a different way. The Supernova observations, made by two teams (Perlmutter et.al and Riess et.al), favour a present day accelerating universe. Also the recent observation of SN 1997ff at $z = 1.7$ provides the evidence of decelerating universe around that redshift. In a recent paper, Turner and Riess have shown that SN data favour a recent acceleration ($z < 0.5$) and a past deceleration ($z > 0.5$). Keeping this in mind, we assume scale factor $R(t)$ of the form...
\[ R(t) = \frac{R_0}{\alpha} \left[ \sinh\left(\frac{t}{t_0}\right) \right]^\beta \]  

where \( t_0 \) is the present time, \( R_0 = R(t = t_0) \) is the present day scale factor and \( \alpha = [\sinh(1)]^\beta \), \( \beta \) being a constant. The interesting feature of this scalar factor is that for \( \beta < 1 \), the universe is decelerating for \( t << t_0 \), and exponentially accelerating for \( t >> t_0 \). As an example, for \( \beta = 2/3 \),

\[ R(t) \propto t^{2/3} \quad \text{for} \quad t << t_0 \]  

\[ R(t) \propto \exp(t) \quad \text{for} \quad t >> t_0 \]  

In terms of redshift, the expression for the Hubble parameter is given by

\[ H(z) = \frac{\beta}{t_0} \left[ 1 + \left(\frac{y}{\alpha}\right)^{2/\beta} \right]^{1/2} \]  

where \( y = 1 + z \). One can also calculate the deceleration parameter \( q(z) \) in our model which is given by

\[ q(z) = \frac{1}{\beta} \left[ \frac{(y/\alpha)^{2/\beta}}{(y/\alpha)^{2/\beta} + 1} \right] - 1 \]  

In fig 1 we have plotted the deceleration parameter \( q(z) \) for different values of \( \beta \). The figure shows

![Graph of deceleration parameter](image)

\[ \text{FIG. 1. The deceleration parameter } q \text{ vs redshift } z \]

that although the universe is accelerating at present, it was decelerating in recent past. For all the
different values of $\beta$ we have used in the figure, the universe is decelerating at $z = 1.7$ which is in agreement with the recent observation.

Astronomers measure luminosities in logarithmic units, called magnitudes defined by

$$m_B(z) = M + 5 \log_{10}(D_l)$$

(9)

where $M = M - 5 \log_{10}(H_0)$ and $D_l = H_0 d_l$ with $M$ is the absolute luminosity of the object and $d_l$ is the luminosity distance defined by

$$d_l = R(t_0)(1 + z)r_1$$

(10)

for an event at $r = r_1$ and at time $t = t_1$. One can show that for nearby sources (in the low redshift limit) the equation (9) can be written as

$$m_B(z) = M + 5 \log_{10} z$$

(11)

which can be used to measure the $M$ by using low-redshift supernovae-measurements. In fig 2 we have plotted the $m_B(z)$ for different values of $\beta$.

![Graph showing effective magnitude $m_B$ vs redshift $z$.](image)

**FIG. 2.** The effective magnitude $m_B$ vs redshift $z$

We now obtain the best-fit value of $\beta$ by comparing our model predictions with the SN1a data. We use the high-z data of the Supernova Cosmology Project (SCP) by Permutter et al. [1] and the low-z data from Calan-Tolodo survey [48] for our study. Of the 60 data point points, we use 54
data points for our analysis (Fit C–D of the SCP data; for details of the excluded data points see Perlmutter et al. 1998). In addition we use the $z \simeq 1.7$ datum reported by Riess [4] in 2001. The best-fit value and 1$\sigma$ errors from the SN1a data are $\beta = 0.81^{+0.18}_{-0.16}$, the best-fit $\chi^2/dof = 1.08$. In this analysis we marginalize over $H_0$.

In Fig 3, we show the $\chi^2/dof$ as a function of the parameter $\beta$. But as the parameter $\beta$ effectively determines the turnover point from acceleration to deceleration, given in equation (8) by $q(z) = 0$, one can interpret this as a likelihood analysis of the turnover redshift. Fig 3 shows that the value of $\chi^2$ is not very sensitive to the value of $\beta$ for $0.5 \simeq \beta \simeq 1$. This can also be seen in Fig 2 where we have plotted the effective magnitude $m_B$ with respect to the redshift $z$ for different choice of $\beta$. There also, it is very difficult to distinguish models with different values of $\beta$ up to redshift $z \sim 1$. This means, from Fig 1, that the epoch at which the universe passes from the accelerating to the decelerating phase is not very well determined. We can compare our conclusions with the results of Turner and Riess [42] who claim that the universe is accelerating for $z \leq 0.5$ and is decelerating for higher redshifts. Though their result is consistent with our conclusions ($\beta \simeq 0.65$ implies (Fig 1) the results of Turner and Riess [42] and it is within 1$\sigma$ of our best fit value), we cannot conclude it. On the other hand, our results are in greater accord with their assertion that the deceleration parameter is increasing as the redshift increases; it is evident from the range of allowed $\beta$ values in our analysis (Fig 1).

In this paper, we use only $\beta = 2/3$; this value is within 1$\sigma$ of the best fit value. It is important to use this value of $\beta$ in order to have a early time matter dominated decelerated universe. The age of the universe with this choice of parameter turns out to be approximately 14 GYrs with $H_0 = 0.6 \times 10^{-10}$ per yr.

Now using equations (1)-(4) one can write

\[ 2\ddot{H} + 3H^2 = 8\pi G \left[ -\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] = \frac{4}{3t_0^2} \]  

(12)

which gives

\[ V(\phi) = \frac{1}{2} \dot{\phi}^2 + \frac{1}{6\pi G t_0^2} \]  

(13)
From the above equation, one can write

\[ V'(\phi) = \ddot{\phi} \]  

(14)

Using this relation in the scalar field wave equation (2) one can solve \( \phi \) as

\[ \phi = A t_0 \log_e \left[ \tanh \left( \frac{t}{2t_0} \right) \right] \]  

(15)

where \( A \) is a constant of integration. Using the expression \( \phi \) and equation (13) one can get the form of the potential which is given by

\[ V(\phi) = \frac{A^2}{8} \left( e^{2a\phi} + e^{-2a\phi} \right) + V_0 \]  

(16)

where \( a = \frac{1}{At_0} \) and \( V_0 = \frac{1}{6\pi G t_0} - \frac{A^2}{4} \). This type of potential has been earlier used by Barreiro et al. [13] and Rubano et al. [20]. In a recent paper [47], Corasaniti and Copeland have used the Supernova data (excluding the recent data at \( z = 1.7 \)) and measurements of the position of the acoustic peaks of the CMBR spectra to constrain a general class of potentials including the inverse power law models and the recently proposed Supergravity inspired potential. They have argued that in order to have the equation of state parameter \( \omega_{\phi} \sim -1 \), the quintessence field has to undergo damped oscillations around the minimum of the potential or has to evolve in a very flat region of the potential. And in this respect double exponential potential can be a good choice although they have not included it in their analysis.
The expression for the energy density $\rho_\phi$ for the scalar field and the equation of state $\omega_\phi$ for the scalar field are given by

$$\rho_\phi = A^2 \sinh^{-2}(t/t_0) + \frac{1}{6\pi G t_0^2}$$

(17)

$$\omega_\phi = -\frac{1}{6\pi G t_0^2 A^2 \sinh^{-2}(t/t_0) + 1}$$

(18)

Using the expression for $\rho_\phi$, $H$ and $\rho_m = \rho_{m0}(1 + z)^3$ in equation (1), one can get

$$A^2 = \frac{1}{6\pi G t_0^2} - \rho_{m0} \alpha^3$$

(19)

Also if one has to the link the potential given in (16) with that of Rubano et.al [20] one has to set $A^2 t_0^2 = \frac{1}{3\pi G}$ which is not possible from equation (19).

Baccigalupi et.al [45] have shown in recent work that the position of the first doppler peak prefers a quintessence model with $\omega_{\phi0} \sim -0.8$ for the prior $\Omega_\phi = 0.7$. If we use this value of $\Omega_\phi$ in our model the constraint on the constant $A$ turns out to be $6\pi G t_0^2 A^2 = 0.286$ and the equation of state for the scalar field at present $\omega_\phi(z = 0)$ comes out to be $\sim -0.83$ which is reasonably consistent with the earlier bound on $\omega_\phi(z = 0)$ obtained by different authors [45,46].

In fig 4 we have plotted the ratio of the two energy densities $\rho_\phi/\rho_m$. One can look at this figure to see that the ratio is approximately constant and also much less than one for the higher redshift.
It starts dominating in the recent time (around $z = 0.5$). It shows that the scalar field scales below
the matter energy density in the early universe. This is feature is quite similar to that of the tracker
field discussed earlier.

In conclusion, we have re-investigated the role of double exponential potential for the
quintessence field in the context of the recent observation of SN1997ff at $z = 1.7$ but in a dif-
ferent manner. We have assumed the form of the scale factor for which the universe interpolates
between the early time decelerated expansion and the late time accelerated expansion. We then
tried to fit our model with the SN1a observation including the recent observation at $z = 1.7$. There
are two free parameters in our model: $H_0$ and $\beta$. We have marginalised over $H_0$ in our analysis.
We then have obtained that the best fit value of the parameter $\beta$ appearing in the form of the scale
factor. Constancy of $\chi^2$ over a wide values of $\beta$ does suggest that it is difficult to pinpoint the
exact turn around from acceleration to deceleration. It is largely owing the quality of the data. It
is not possible to say if the universe is decelerating or accelerating taking points up to $z = 0.83$
(Milne universe with $a(t) \propto t$ is not such a bad fit to the data up to $z = 0.83$, as shown in the
original paper of Perlmutter et al.). However inclusion of the new data point at $z = 1.7$ suggests
that the deceleration parameter is increasing as the redshift increases (this also happens to be the
strongest claim of Turner and Riess). Given the insensitivity of $\chi^2$ on $\beta$, this is also one of our
main conclusion of our analysis.

To have a early matter dominated decelerated universe we have chosen $\beta = 2/3$ for which our
model fits reasonably with observation. We have then showed that the potential one needs is a dou-
ble exponential potential. We have also investigated the different relevant parameters, such as the
equation of state and the ratio of the two energy densities and showed that they behave reasonably
well as far as the consistency of the model is concerned. Although this method of solving the field
equations is completely ad hoc as it does not result from a known particle physics model, how-
ever it does results potential which gives the right behaviour for the expanding universe. Also the
potential, we obtain, has earlier been considered by different authors for quintessence fields. In
the context of recent SN1a observation we have shown in this paper that this may really be a good
choice. It will be worthwhile to study the extrapolation of this model to radiation dominant era to
check its consistency with the big bang nucleosynthesis. Also one should check the consistency of this model with recent CMB observation. These issues will be addressed later.

**Acknowledgments:** We would like to thank Claudio Rubano, Bob Jantzen and Diego Pavon for their valuable comments and suggestions and also for giving references of some important papers. We are also thankful to the anonymous referee for his comments which helped us to improve the clarity of the paper.

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