Abstract

We improve the theorem of Beck giving a lower bound on the number of \(k\)-flats spanned by a set of points in real space, and improve the bound of Elekes and Tóth on the number of incidences between points and \(k\)-flats in real space.

1 Introduction

Let \(P\) be a set of \(n\) points in real affine space. A \(k\)-flat (\(k\)-dimensional affine subspace) \(\Gamma\) is \(r\)-rich if it contains \(r\) points of \(P\), and \(\Gamma\) is spanned by \(P\) if \(\Gamma\) contains \(k+1\) points of \(P\) that are not contained in a \((k-1)\)-flat. This paper gives new results on two well-studied questions:

1. How many \(r\)-rich \(k\)-flats can be determined by \(P\)?
2. How few \(k\)-flats can be spanned by \(P\)?

A fundamental result in combinatorial geometry is the Szemerédi-Trotter theorem, which gives an upper bound on the number of \(r\)-rich lines determined by a set of points.

**Theorem 1** (Szemerédi, Trotter). The number of \(r\)-rich lines determined by \(P\) is bounded above by \(O(n^2r^{-3} + nr^{-1})\).

Note that there is an equivalent formulation of Theorem 1 in terms of the maximum possible number of incidences between a fixed set of points and lines, where an incidence is a pair of a point and a line such that the point is contained in the line, and bounds of the this type are often called incidence bounds.

Szemerédi-Trotter type incidence bounds play a major role in combinatorial geometry, and have numerous applications in other areas of mathematics and computer science. The surveys and books of Dvir, Guth, Tao, and Vu, Elekes, and Matoušek, are all good resources on Szemerédi-Trotter type bounds and their applications.

The first result of this paper is an upper bound on \(r\)-rich \(k\)-flats, for \(k > 1\). In order to prove a nontrivial bound in this context, we need to place some restriction on the points or the flats. To illustrate this point, let \(L\) be a set of
planes that each contain a fixed line, and $P$ a set of $n$ points contained in the same line. Then, each plane of $L$ is $n$-rich.

Several point-flat incidence bounds have been proved, using several different nondegeneracy conditions. Initial work on point-plane incidences was by Edelsbrunner, Guibas, and Sharir [7], who considered point sets with no 3 collinear, and also incidences between planes and vertices of their arrangement. Agarwal and Aronov [1] gave a tight bound on the number of incidences between vertices and flats of an arrangement of $k$-flats; this bound was generalized by the author, Purdy, and Smith [13] to incidences between vertices of an arrangement of flats and a subset of the flats of the arrangement. Brass and Knauer [4], as well as Apfelbaum and Sharir [2], considered incidences between points and flats such that the intersection of $t$ of the flats does not contain $s$ of the points. Sharir and Solomon [16] obtain a stronger bound than that of Apfelbound and Sharir for point-plane incidences by adding the condition that the points are contained in an algebraic variety of bounded degree.

In this paper, we use the nondegeneracy assumption introduced by Elekes and Tóth [9]. A $k$-flat $\Gamma$ is $\alpha$-degenerate if at most $\alpha |P \cap \Gamma|$ points of $P$ lie on any $(k-1)$-flat contained in $\Gamma$. Using this definition, Elekes and Tóth proved the following Szemerédi-Trotter type theorem for points and planes.

**Theorem 2** (Elekes, Tóth). For any $\alpha < 1$, the number of $\alpha$-degenerate, $r$-rich planes is bounded above by $O(\alpha(n^3r^{-4} + n^2r^{-2}))$.

The subscript in the $O$-notation indicates that the implied constant depends on those parameters listed in the subscript.

Elekes and Tóth generalized Theorem 2 to higher dimensions in the following, weaker form.

**Theorem 3** (Elekes, Tóth). For each $k > 2$ there is a constant $\beta_k$ such that, for any $\alpha < \beta_k$, the number of $\alpha$-degenerate, $r$-rich $k$-flats is bounded above by $O_{\alpha,k}(n^{k+1}r^{-k-2} + n^k r^{-k})$.

Elekes and Tóth remarked that their argument can’t be improved to replace the constants $\beta_k$ with 1 for $k > 2$ in Theorem 3. The contribution of this paper is the following strong generalization of Theorem 2 which removes this limitation of Theorem 3.

**Theorem 4.** For each $k > 2$ and any $\alpha < 1$, the number of $\alpha$-degenerate, $r$-rich $k$-flats is bounded above by $O_{\alpha,k}(n^{k+1}r^{-k-2} + n^k r^{-k})$.

One well-known application of an incidence bound between points and lines is Beck’s theorem [3]. Proving a conjecture of Erdős [10], Beck used a slightly weaker incidence bound than Theorem 4 to show that, if $P$ is a set of points such that no more than $s$ points of $P$ lie on any single line, then the number of lines spanned by $P$ is $\Omega(n(n-s))$. In the same paper, Beck gave the following bound for flats of higher dimensions.

**Theorem 5** (Beck). For each $k \geq 1$, there is a constant $c_k$ such that either $c_k n$ points of $P$ are contained in a single $k$-flat, or $P$ spans $\Omega_k(n^{k+1})$ $k$-flats.
How large can the constant $c_k$ be in Theorem 5? For $k = 1$, Beck showed that Theorem 5 holds for any $c_1 < 1$. For $k = 2$, if $P$ is a set of $n$ points of which $n/2$ lie on each of two non-intersecting lines, then $P$ spans $n$ planes, but no plane contains more than $n/2 + 1$ points of $P$. Hence, Theorem 5 does not hold for $c_2 = 1/2$. In general, Beck’s proof implies a bound on $c_k$ that depends exponentially on $k$; Do [5] improved this by showing that Theorem 5 holds for any value of $c_k < 1/k$ and $n$ sufficiently large.

Here, we give the following improvement to Theorem 5.

**Theorem 6.** For each $k \geq 1$, either $(\frac{1}{2} - o(1))n$ points of $P$ are contained in a single $(k - 1)$-flat, or $P$ spans $\Omega_k(n^{k+1})$ $k$-flats, or $k$ is odd and $P$ lies in the union of $k$ lines.

Note that this shows that we can take any value strictly less than $1/2$ for each $c_k$ in Theorem 5. Indeed, if $k$ is odd and $P$ lies in the union of $k$ lines, then any subset of $(k + 1)/2$ of the lines is contained in a $k$-flat. Hence, there must be some $k$-flat that contains at least $(k + 1)n/(2k) > n/2$ points of $P$.

As noted above, Theorem 5 does not hold for $c_2 = 1/2$. Indeed, Theorem 5 does not hold for $c_k = 1/2$ for any $k > 1$. This is shown by taking $P$ to be contained in the union of a $(k - 1)$-flat $\Gamma$ and a line $\ell$, with each of $\Gamma$ and $\ell$ containing $n/2$ points. In this case, any $k$-flat spanned by $P$ contains either $\Gamma$ or $\ell$, and so $P$ spans at most $n/2 + \binom{n/2}{k-1} = O(n^{k-1})$ $k$-flats. Hence, from Theorem 6, we get a tight bound on $c_k$ for all $k$.

We remark that all of the new results in this paper hold for point sets in complex space, using the generalization of the Szemerédi-Trotter bound to complex space proved by Tóth [20] and Zahl [21].

2 Projective geometry and essential dimension

In this section, we fix notation and review some basic facts of projective geometry, as well as results and definitions we need from [12]. Note that it is sufficient to work in projective space rather than affine space, since we can always embed a set of points in affine space into projective space without changing the number of $r$-rich flats, or the number of flats spaned by the points.

The span of a set $X$ is the smallest flat that contains $X$, and is denoted $\overline{X}$. We denote by $\Lambda, \Gamma$ the span of $\Lambda \cup \Gamma$. It is a basic fact of projective geometry that, for any flats $\Lambda, \Gamma$,

$$\dim(\overline{\Lambda, \Gamma}) + \dim(\Lambda \cap \Gamma) = \dim(\Lambda) + \dim(\Gamma).$$

(1)

More generally, using the fact that $\dim(\Lambda \cap \Gamma) \geq -1$, we have for any set $\mathcal{H}$ of flats that

$$\dim \mathcal{H} \leq |\mathcal{H}| - 1 + \sum_{\Lambda \in \mathcal{H}} \dim(\Lambda).$$

(2)

For a $k$-flat $\Lambda$, the projection from $\Lambda$ is the map

$$\pi_\Lambda : \mathbb{P}^d \setminus \Lambda \to \mathbb{P}^{d-k-1}$$
that sends a point \( p \) to the intersection of the \((k+1)\)-flat \( \overline{p, \Lambda} \) with an arbitrary \((d-k-1)\)-flat disjoint from \( \Lambda \).

Defined in [12], the essential dimension \( K = K(P) \) of \( P \) is the minimum \( t \) such that there exists a set \( \mathcal{G} \) of flats such that

1. \( P \) is contained in the union of the flats of \( \mathcal{G} \),
2. each flat \( \Lambda \in \mathcal{G} \) has dimension \( \dim(\Lambda) \geq 1 \), and
3. \( \sum_{\Lambda \in \mathcal{G}} \dim(\Lambda) = t \).

We denote by \( f_k \) the number of \( k \)-flats spanned by \( P \). Here is the main result on point sets with bounded essential dimension that we need from [12].

**Theorem 7.** For each \( k \), there is a constant \( c_k \) such that, if \( K > k \) and \( n - g_k > c_k \), then

\[
    f_k = \Theta \left( \prod_{i=0}^{k} (n - g_i) \right).
\]

If \( k \geq K \), then

\[
    f_k = O \left( \prod_{i=0}^{2(k-1)-k} (n - g_i) \right),
\]

and either \( f_{k-1} = f_k = 0 \) or \( f_{k-1} > f_k \).

### 3 Proof of Theorem 4

Recall from the introduction that a \( k \)-flat \( \Lambda \) is \( \alpha \)-degenerate if at most \( \alpha |P \cap \Lambda| \) points of \( P \) lie on any \((k-1)\)-flat contained in \( \Lambda \). We further say that \( \Lambda \) is essentially-\( \alpha \)-degenerate if for each \( P' \subseteq P \cap \Lambda \) such that the essential dimension of \( P' \) is at most \( k - 1 \), we have \( |P'| \leq \alpha |P \cap \Lambda| \). Note that an essentially-\( \alpha \)-degenerate flat is also \( \alpha \)-degenerate, but not necessarily the other way around.

The following bound on essentially-\( \alpha \)-degenerate flats was proved by Do [5].

**Theorem 8 (Do).** For any \( k \) and any \( \alpha < 1 \), the number of essentially-\( \alpha \)-degenerate, \( r \)-rich \( k \)-flats is bounded above by \( O(n^{k+1-r-k-2} + n^k r^{-k}) \).

We remark that Theorem 8 is also an immediate consequence of Theorem 7 together with the following theorem of Elekes and Tóth [9]. A \( k \)-flat \( \Lambda \) is \( \gamma \)-saturated if \( \Lambda \cap P \) spans at least \( \gamma |\Lambda \cap P|^k \) different \((k-1)\)-flats.

**Theorem 9 (Elekes, Tóth).** The number of \( r \)-rich \( \gamma \)-saturated \( k \)-flats is at most \( O(n^{k+1-r-k-2} + n^k r^{-k}) \).

In the remainder of this section, we deduce Theorem 4 from Theorem 8.

#### 3.1 A simple proof for the case \( k = 3 \)

The case \( k = 3 \) admits a simpler proof than the general theorem, which we give first. The proof for arbitrary \( k \) does not depend on this special case, but is built around a similar idea.
Theorem 10. For any $\alpha < 1$, the number of $\alpha$-degenerate, $r$-rich 3-flats is bounded above by $O(n^{4r-5} + n^3r^{-3})$.

Proof. By Theorem 8 the number of essentially-$\alpha^{1/2}$-degenerate $r$-rich 3-flats is bounded above by $O(n^{4r-5} + n^3r^{-3})$. If an $r$-rich 3-flat $\Lambda$ is $\alpha$-degenerate but not essentially-$\alpha^{1/2}$-degenerate, then at least $\alpha^{1/2}|P \cap \Lambda| \geq \alpha^{1/2}r$ points of $P$ are contained in the union of two skew lines, neither of which contains more than $\alpha|P \cap \Lambda|$ points of $P$; hence, each of these lines contains at least $(\alpha^{1/2} - \alpha)r$ points. By the Szemerédi-Trotter theorem, the maximum number of pairs of $((\alpha^{1/2} - \alpha)r)$-rich lines is bounded above by $O(n^{4r-6} + n^2r^{-2})$, which implies the conclusion of the theorem.

3.2 Proof of general case

The proof of Theorem 10 given above does not generalize to higher dimensions, but the basic approach of bounding the number of $r$-rich $\alpha$-degenerate flats that are not also essentially-$\alpha'$-degenerate does still work in higher dimensions. This idea is captured by the following lemma.

Lemma 11. Let $\mathcal{F} = \mathcal{F}_{\alpha,r}$ be a set of $k$-flats satisfying the following property. If $\Lambda \in \mathcal{F}$, then $\Lambda$ contains a set $\mathcal{G}$ of flats so that

1. $\sum_{\Gamma \in \mathcal{G}} \dim(\Gamma) < k$,
2. $\mathcal{G} = \Lambda$,
3. each flat of $\mathcal{G}$ is $r$-rich and $\alpha$-degenerate.

Then, $|\mathcal{F}| = O(n^{k+1}r^{-k-2} + n^k r^{-k})$.

We remark that Lemma 11 is not tight, in general; for example, a stronger bound of $O(n^{4r-6} + n^2r^{-2})$ was given for the case $k = 3$ in the proof of Theorem 10 above.

The proof of Lemma 11 is by induction on $k$; in order to prove Lemma 11 for $k$-flats, we use Theorem 4 for $k'$-flats, for each $k' < k$. Before giving the proof of Lemma 11 we show that it implies Theorem 4.

Proof of Theorem 4. Let $\alpha'$ so that $(k + \alpha)(k + 1)^{-1} < \alpha' < 1$; note that $\alpha' > \alpha$. The required bound on the number of $r$-rich, essentially-$\alpha'$-degenerate $k$-flats is given by Theorem 8 so it only remains to bound the number of $r$-rich, $\alpha$-degenerate $k$-flats that are not also essentially-$\alpha'$-degenerate.

Let $\Lambda$ be an $r$-rich, $\alpha$-degenerate $k$-flat that is not essentially-$\alpha'$-degenerate (assuming that such a flat exists). From the definition of essentially-$\alpha'$-degenerate, there is a collection $\mathcal{G}'$ of sub-flats of $\Lambda$ with $\sum_{\Gamma' \in \mathcal{G}'} \dim(\Gamma') < k$ such that $|\bigcup_{\Gamma' \in \mathcal{G}'} \Gamma' \cap P| > \alpha'|P \cap \Lambda|$. We obtain a set $\mathcal{G}$ satisfying the conditions of Lemma 11 from $\mathcal{G}'$ as follows. If $\Gamma \in \mathcal{G}'$ is not $\alpha'$-degenerate, then replace $\Gamma$ with the smallest flat $\Gamma' \subset \Gamma$ that contains at least $(\alpha')^{\dim(\Gamma') - \dim(\Gamma')'} |P \cap \Gamma'|$ points. Note that, since any flat $\Gamma'' \subset \Gamma'$ with $\dim(\Gamma'') = \dim(\Gamma') - 1$ contains fewer than $(\alpha')^{\dim(\Gamma') - \dim(\Gamma')'} |P \cap \Gamma'| \leq \alpha'|P \cap \Gamma'|$ points of $P$, it follows that $\Gamma'$ is $\alpha'$-degenerate. Furthermore, fewer than $(\dim(\Gamma) - \dim(\Gamma'))(1 - \alpha') |P \cap \Lambda|$ points lie in $\Gamma'$ and not in $\Gamma'$. 

5
Hence, it will suffice to show that, for each \( \Lambda \in \mathcal{F} \), the size of each \( \mathcal{G}_1 \) is at most \( \sum_{\Gamma \in \mathcal{G}'} \dim(\Gamma)(1 - \alpha')(|P \cap \Lambda| - k(1 - \alpha')|P \cap \Lambda| < (\alpha' - \alpha)|P \cap \Lambda| \) points; hence, \( \bigcup_{\Gamma \in \mathcal{G}} \Gamma \cap P > \alpha|P \cap \Lambda| \). If \( \dim(\mathcal{G}) < k \), then \( \Lambda \) is \( \alpha \)-degenerate, contrary to our assumption. Hence, \( \dim(\mathcal{G}) = k \), and hence \( \Lambda \) belongs to the set \( \mathcal{F} \) of Lemma 11. Now Lemma 11 (using \( \alpha' \) in the hypothesis of Lemma 11) implies the required bound on the number of possible choices for \( \Lambda \), which completes the proof.

Next, we prove Lemma 11.

**Proof of Lemma 11.** We proceed by induction on \( k \); the induction uses Theorem 4 and (for \( k = 1 \)) the Szemerédi-Trotter theorem.

We partition \( \mathcal{F} \) into subsets \( \mathcal{F}_b \), for \( 1 \leq b < k \), and separately bound the size of each \( \mathcal{F}_b \), as follows.

For each \( \Lambda \in \mathcal{F} \), let \( \mathcal{G}_\Lambda \) be the set of flats given by the hypothesis of Lemma 11. We may suppose that \( \mathcal{G}_\Lambda \) is minimal, so that \( \mathcal{G}_\Lambda^* = \Lambda \) but \( \mathcal{G}_\Lambda^* \cap \Gamma \subseteq \Lambda \) for each \( \Gamma \in \mathcal{G}_\Lambda \). Let \( \Gamma_\Lambda \) be an arbitrary flat in \( \mathcal{G}_\Lambda \), let \( b_\Lambda = \dim(\mathcal{G}_\Lambda^* \cap \Gamma_\Lambda) \), and let \( \mathcal{G}_\Lambda' = \mathcal{G}_\Lambda \setminus \Gamma_\Lambda \). Assign \( \Lambda \) to \( \mathcal{F}_{b_\Lambda} \).

For each \( b \), the inductive hypothesis (on Theorem 4) implies that

\[
|\{ \mathcal{G}_\Lambda^* : \Lambda \in \mathcal{F}_b \}| = O(n^{b+1}r^{-b-2} + n^b r^{-b}).
\]

Hence, it will suffice to show that, for each \( \Lambda \in \mathcal{F}_b \), we have \( |\{ \Lambda' \in \mathcal{F}_b : \mathcal{G}_{\Lambda'} = \mathcal{G}_\Lambda^* \}| = O(n^{k-b}r^{-b-1+k}). \)

Let \( \mathcal{R} \in \{ \mathcal{G}_\Lambda^* : \Lambda \in \mathcal{F}_b \} \). Let \( \pi_\mathcal{R} : \mathbb{R}^d \to \mathbb{R}^{d-k+b} \) be projection from \( \mathcal{R} \). We consider the projection \( \pi_\mathcal{R}(P) \) of \( P \) to be a multiset of points, with the multiplicity of a point \( p \in \mathcal{R}(P) \) equal to the number of points \( p' \in P \) such that \( \pi_\mathcal{R}(p') = p \).

For each \( \Lambda \in \mathcal{F}_b \) such that \( \mathcal{G}_\Lambda^* = \mathcal{R} \), there is an \( \alpha \)-degenerate, \( r \)-rich flat \( \Gamma \) such that \( \Gamma, R = \Lambda \). Since \( \Gamma, R = \Lambda \) and \( \pi_\mathcal{R}(\Gamma) \) is disjoint from \( \mathcal{R} \), we have that \( \dim \pi_\mathcal{R}(\Gamma) = k - 1 - b \). We claim that \( \pi_\mathcal{R}(\Gamma) \) is \( (1 - \alpha)r \)-rich and \( \alpha \)-degenerate (with regard to \( \pi_\mathcal{R}(P) \), counting all points with multiplicity). First, note that \( |\pi_\mathcal{R}(\Gamma) \cap \pi_\mathcal{R}(P)| = |\Gamma \cap P| - |\Gamma \cap R \cap P| \). Since \( \Gamma \) is \( \alpha \)-degenerate, \( |\Gamma \cap R \cap P| < \alpha|\Gamma \cap P| \), so \( \pi_\mathcal{R}(\Gamma) \) is \( (1 - \alpha)r \)-rich. Let \( \Gamma' \) be a subflat of \( \pi_\mathcal{R}(\Gamma) \), and let \( \pi^{-1}(\Gamma') \subseteq \Gamma \) be the preimage of \( \Gamma' \) in \( \Gamma \). Note that \( \dim \pi^{-1}(\Gamma') \leq \dim \Gamma \), hence \( |\Gamma' \cap \pi_\mathcal{R}(P)| \leq \alpha|\Gamma \cap P| - |\Gamma \cap R \cap P| \leq \alpha|\pi_\mathcal{R}(\Gamma) \cap \pi_\mathcal{R}(P)| \). Hence, \( \pi_\mathcal{R}(\Gamma) \) is \( \alpha \)-degenerate.

To complete the proof, we will use the following lemma, proved below.

**Lemma 12.** Let \( M \) be a multiset of points with total multiplicity \( n \). The number of \( r \)-rich, \( \alpha \)-degenerate \( k \)-flats spanned by \( M \) is bounded above by \( (1 - \alpha)^{-k} n^{k+1} r^{-k-1} \).
From Lemma 12, we get the required bound of $O(n^{k-b}r^{k-b})$ on the number of $(1 - \alpha)r$-rich, $\alpha$-degenerate $(k - 1 - b)$-flats spanned by $\pi_\mathcal{R}(P)$, and this completes the proof of Lemma 11.

**Proof of Lemma 12.** There are $n^{k+1}$ ordered lists of $k + 1$ points in $M$ (with repetitions allowed). We show below that for any $r$-rich, $\alpha$-degenerate $k$-flat $\Lambda$, the are at least $(1 - \alpha)r^{k+1}$ distinct lists of $k + 1$ points such that all of the points are contained in $\Lambda$, and the points are affinely independent. Since $k + 1$ affinely independent points are contained in exactly one $k$-flat, this implies the conclusion of the lemma.

Let $\Lambda$ be an $r$-rich, $\alpha$-degenerate $k$-flat. We will show, by induction, that, for each $0 \leq k' \leq k$, $\Lambda$ contains $(1 - \alpha)r^{k'+1}$ distinct ordered lists of $k'+1$ affinely independent points. The base case of $k' = 0$ is immediate from the fact that $\Lambda$ is $r$-rich.

Choose uniformly at random a pair $(v, p)$, where $v$ is an ordered list of $k'$ affinely independent points contained in $\Lambda$, and $p$ is a point of $P$ contained in $\Lambda$. By the inductive hypothesis, we know that there are $(1 - \alpha)r^{k'-1}$ choices for $v$, and there are clearly $|P \cap \Lambda| \geq r$ choices for $p$. If the probability that $p$ is affinely dependent on the points of $v$ is more than $\alpha$, then there is some $v$ for which the number of points in $P \cap \Lambda$ that are affinely dependent on $v$ is more than $\alpha|P \cap \Lambda|$. Since these points must all be contained in the $k' - 1$-dimensional span of $v$, this contradicts the hypothesis that $\Lambda$ is $\alpha$-degenerate. Hence, the number of choices of $(v, p)$ such that $p$ is affinely independent of $v$ is at least $(1 - \alpha)r^{k'+1}$, which is what was to be proved.

4 Proof of Theorem 6

In this section, we show that Theorem 6 follows easily from Theorem 7.

**Proof.** Suppose that $f_k = o(n^{k+1})$. By Theorem 7, there is a set $\mathcal{G}$ of flats such that $\sum_{\Gamma \in \mathcal{G}} \dim(\Gamma) \leq k$, at least $(1 - o(1))n$ points of $P$ lie in some flat of $\mathcal{G}$, and each flat of $\mathcal{G}$ has dimension at least 1. We show below that, unless $k$ is odd and $\mathcal{G}$ is the union of $k$ lines, we can partition $\mathcal{G}$ into $\mathcal{G}_1, \mathcal{G}_2$ such that $\dim(\mathcal{G}_1) \leq k - 1$. Since either $|G_1 \cap P| \geq 1/2 |G \cap P|$ or $|G_2 \cap P| \geq 1/2 |G \cap P|$, this is enough to prove the theorem.

Let $\mathcal{G} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$, with $\dim(\Gamma_1) \leq \dim(\Gamma_2) \leq \ldots \leq \dim(\Gamma_m)$. Let $\mathcal{G}_1 = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_{m_1}\}$, with $m_1$ chosen as large as possible under the constraint $\dim(\mathcal{G}_1) \leq k - 1$.

Let $s = \dim(\Gamma_{m_1} + 1)$. Note that $\dim(\mathcal{G}_1) + s \geq k - 1$, since otherwise $\Gamma_{m_1} + 1$ would be included in $\mathcal{G}$. Next, since each flat in $\mathcal{G}$ has dimension at most $s$, by
we have
\[ \sum_{\Gamma \in \mathcal{G}_1} \dim \Gamma \geq \dim \mathcal{G}_1 + 1 - |\mathcal{G}|, \]
\[ \geq k - s - \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_1} \dim \Gamma. \]
Hence,
\[ k - \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma \geq \sum_{\Gamma \in \mathcal{G}_1} \dim \Gamma \geq \frac{k - s}{1 + s^{-1}}. \]
and so
\[ \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma \leq \frac{s + ks^{-1}}{1 + s^{-1}}. \]
Since each flat in \( \mathcal{G}_2 \) has dimension at least \( s \), we have \(|\mathcal{G}_2| \leq \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma\). Applying (2), we have
\[ \dim \mathcal{G}_2 \leq |\mathcal{G}_2| - 1 + \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma, \]
\[ \leq (1 + s^{-1}) \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma - 1, \]
\[ \leq s + ks^{-1} - 1, \]
\[ \leq k, \]
with equality only if \( s = 1 \) and \(|\mathcal{G}_2| = \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_2} \dim \Gamma\); this occurs only if \( \mathcal{G} \) is a set of lines. If \( \mathcal{G} \) is a set of lines and \( k \) is even, then \( \mathcal{G}_2 \) consists of at most \( k/2 \) lines, which span at most a \((k - 1)\)-flat. Hence, if \( \dim \mathcal{G}_2 = k \), then \( \mathcal{G} \) must be a set of lines, and \( k \) must be odd. Since \( \dim \mathcal{G}_1 \leq k - 1 \) by construction, this completes the proof.

\[ \square \]

References

[1] Pankaj K. Agarwal and Boris Aronov. Counting facets and incidences. *Discrete & Computational Geometry*, 7(1):359–369, 1992.

[2] Roel Apfelbaum and Micha Sharir. Large complete bipartite subgraphs in incidence graphs of points and hyperplanes. *SIAM Journal on Discrete Mathematics*, 21(3):707–725, 2007.

[3] József Beck. On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*, 3(3-4):281–297, 1983.

[4] Peter Brass and Christian Knauer. On counting point-hyperplane incidences. *Computational Geometry*, 25(1-2):13–20, 2003.
[5] Thao Do. Extending Erdős-Beck’s theorem to higher dimensions. arXiv:1607.00048, 2016.

[6] Zeev Dvir. Incidence theorems and their applications. Foundations and Trends in Theoretical Computer Science, 6(4):257–393, 2012.

[7] Herbert Edelsbrunner, Leonidas Guibas, and Micha Sharir. The complexity of many cells in arrangements of planes and related problems. Discrete & Computational Geometry, 5(1):197–216, 1990.

[8] György Elekes. Sums versus products in number theory, algebra and Erdős geometry. Paul Erdos and his Mathematics II, 11:241–290, 2001.

[9] György Elekes and Csaba D Tóth. Incidences of not-too-degenerate hyperplanes. In Proceedings of the twenty-first annual symposium on Computational geometry, pages 16–21. ACM, 2005.

[10] Paul Erdős. On some problems of elementary and combinatorial geometry. Annali di Matematica pura ed applicata, 103(1):99–108, 1975.

[11] Larry Guth. Polynomial methods in combinatorics, volume 64. American Mathematical Soc., 2016.

[12] Ben Lund. Essential dimension and the flats spanned by a point set. arXiv:1602.08002, 2016.

[13] Ben D Lund, George B Purdy, and Justin W Smith. A bichromatic incidence bound and an application. Discrete & Computational Geometry, 46(4):611–625, 2011.

[14] Jirí Matoušek. Lectures on discrete geometry, volume 212. Springer Science & Business Media, 2002.

[15] János Pach and Micha Sharir. Geometric incidences. Contemporary Mathematics, 342:185–224, 2004.

[16] Micha Sharir and Noam Solomon. Incidences between points on a variety and planes in $\mathbb{R}^3$. arXiv:1603.04823, 2016.

[17] Endre Szemerédi and William T Trotter Jr. Extremal problems in discrete geometry. Combinatorica, 3(3-4):381–392, 1983.

[18] Terence Tao. Algebraic combinatorial geometry: the polynomial method in arithmetic combinatorics, incidence combinatorics, and number theory. arXiv:1310.6482, 2013.

[19] Terence Tao and Van H Vu. Additive combinatorics, volume 105. Cambridge University Press, 2006.

[20] Csaba D Tóth. The Szemerédi-Trotter theorem in the complex plane. Combinatorica, 35(1):95–126, 2015.
[21] Joshua Zahl. A Szemerédi-Trotter type theorem in $\mathbb{R}^4$. *Discrete & Computational Geometry*, 54(3):513–572, 2015.