Analysis and Optimization of Cache-Enabled Networks with Random DTX Scheme

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Abstract

In this paper, we focus on the meta distribution for the cache-enabled networks where the locations of base stations (BSs) are modeled as Poisson point process (PPP). Under the random caching framework, we derive the moments of the conditional successful transmission probability (STP), the exact meta distribution and its beta approximation by utilizing stochastic geometry. The closed-form expression of the mean local delay is also derived. We consider the maximization of the STP and the minimization of the mean local delay by optimizing the caching probability and the BS active probability, respectively. For the former, a convex optimization problem is formulated and the optimal caching probability and BS active probability are achieved. Moreover, most popular caching (MPC) is proved to optimal under the constraint that the mean local delay is finite. For the latter, a non-convex optimization problem is formulated and an iterative algorithm is proposed to obtain the optimal solution. The backhaul delay has a significant influence on the caching strategy. MPC is proved to be optimal when the backhaul delay is relatively low and the uniform caching (UC) is the optimal caching strategy when the backhaul delay is very large. Finally, the numerical results reveal the effect of the key network parameters on the cache-enabled networks in terms of STP, variance, meta distribution and mean local delay.

Index Terms

Caching strategy, meta distribution, mean local delay, random DTX scheme.

I. INTRODUCTION

Due to the rapid proliferation of various multi-media applications and smart mobile devices, the mobile data traffic has witnessed an unprecedented growth and imposed heavy burden on the backhaul links. Moreover, 5%-10% files are required by the majority of the user equipment [1]. Stimulated by these facts, equipping base stations (BSs) with caches to pre-fetch the popular contents in the off-peak time is proposed as a promising approach to deal with the potential bottleneck issue of backhauling.

The caching strategy in the cellular networks has been studied by utilizing stochastic geometry as the analyzing tool. In [2], the analytical expressions for the average delivery rate and outage probability of the cache-enabled networks where the locations of BSs were modeled as Poisson point process (PPP) were investigated. In [3], the closed-form expression of the cache hit probability for the cache-enabled heterogeneous cellular networks was derived. In addition, an sequential computation approach was proposed to obtain the optimal caching probability under the uniform signal-to-interference-ratio (SIR) threshold and an algorithm was proposed to achieve the sub-optimal solution under the non-uniform SIR threshold. The authors in [4] proposed an optimal caching strategy to maximize the successful transmission probability (STP) and the area spectral efficiency (ASE) of the cache-enabled networks. In addition, the relationship between the optimal caching probability and the network parameters are revealed. In [5], a cluster-centric cellular network was proposed and a cooperative transmission strategy was designed to strike the balance between the content diversity and the transmission reliability.

Most existing works utilize the STP as a critical performance metric, which is a function of the SIR at the typical user and is achieved by averaging over the point process and the channel fading. The STP only provides limited information on the performance of the cache-enabled networks, i.e., the mean of a random variable and cannot reflect the performance variation among each individual link. In order to overcome this limitation and obtain a fine-grained analysis on the network performance, the meta distribution was proposed in [7]. Different from the STP which only answers the question “what fraction of users experience successful transmission?”, the meta distribution characterizes the distribution of the STP and answers the question “What fraction of users can achieve x% successful transmission probability?”. The meta distribution of the cellular networks has been investigated extensively in previous works. Note that the foundation of the concept, i.e., the meta distribution is laid by [7], where the moments of the conditional success probability, the exact expression, bounds and approximation of the meta distribution for the cellular networks and bipolar networks are derived, respectively. The exact analytical expressions and the beta approximations of the meta distribution were obtained in various scenarios, including the heterogeneous networks [8], non-Poisson networks [9], even with D2D communication [10], coordinated multipoint transmission scheme [11], non-orthogonal multi-access [12] and fractional power control [13]. Note that the mean local delay was also derived in [8–13].

Delay is another important performance metric reflecting the service quality and network reliability [6]. In general, two kinds of delays exist in the wireless networks: the queuing delay, i.e., the waiting time in the service queues, and the transmission delay, i.e., the time spent in the successful transmission of the data. The local delay is a basic form of the transmission delay and a lower bound of the system delay can be provided by analyzing the local delay. The local delay of the wireless networks has been investigated in the previous works. In [14], a mathematical framework was proposed for the derivation of the local...
delay by utilizing stochastic geometry. The work in [15] achieved the analytical expression of the local delay in the mobile Poisson networks. In [16], the optimal power control policies were provided to minimize the local delay for different fading statistics. The authors in [17] adopted two multi-access-control protocols, i.e., ALOHA and frequency-hop multiple access, to reduce the interference correlation and corresponding parameters were optimized to obtain the minimization of the local delay under both protocols. Note that the same set of interfering BSs may be seen by a user in different time slots, which introduces the interference correlation. The local delay may be degraded by the interference correlation and the discontinuous transmission (DTX) scheme is proposed as an effective technology to manage such correlation by increasing randomness of the interfering BSs. In [18], the local delay and the energy efficiency were analyzed in the wireless networks with random DTX scheme.

While the meta distribution in the cellular networks has been investigated extensively, the meta distribution in the cache-enabled networks needs to be further investigated. Compared with the traditional cellular networks, the meta distribution in the cache-enabled networks is also affected by the caching probability. For the cache-enabled networks, the main challenge is the cache capacity limitation of the BSs affecting the availability of the files with relatively low popularity. In order to improve the network performance and enhance the file diversity, the random caching strategy can be adopted. In general, the STP is employed for the performance evaluation of the cache-enabled networks under the random caching framework. However, the STP only provides the overall SIR distribution of the networks. In order to capture the statistics of the cache-enabled networks, we derive the meta distribution and provide a fine-grained analysis on the network performance.

In the cache-enabled networks, the STP is adopted as a critical performance metric and the maximization of the STP is obtained by optimizing the caching probability without considering the mean local delay. However, a phase transition may occur where the mean local delay changes from the finite regime to the infinite regime. When the mean local delay approaches infinity, the file cannot be transmitted effectively. Therefore, the mean local delay is an important performance metric to measure the effectiveness of the file transmission and adopted as an optimization objective in this paper.

We consider a downlink cache-enabled network where the random caching and random DTX scheme are applied to reduce the mean local delay. Our goal is to obtain the maximization of the STP in the finite mean local delay regime or the minimization of the mean local delay for the cache-enabled networks by joint optimization of the random DTX scheme and random caching strategy. The main contributions are summarised as follows

1) We derive the \( b \)-th moment of the conditional STP given a realization of the spatial location of the BSs and the exact expression of the meta distribution by utilizing stochastic geometry. The beta approximation of the meta distribution is also derived. Moreover, we derive the expression of the mean local delay in the cache-enabled networks. The simplified expression of the local delay is also provided in the high and low data rate regime. From the derived numerical results, the critical value of the caching probability and the BS active probability is obtained and the relationship between the caching probability and the BS active probability is revealed.

2) We consider the maximization of the STP and the minimization of the mean local delay, respectively. For the former, a convex optimization problem is formulated to maximize the STP under the constraint that the mean local delay is finite. For the latter, a non-convex optimization problem is formulated to minimize the mean local delay by optimizing the BS active probability and the caching probability. By exploring the optimality property, the problem is converted to a convex one and an iterative algorithm is proposed to achieve the optimal BS active probability and the caching probability.

3) By numerical results, we reveal the effect of the key network parameters, i.e., the BS active probability, the caching probability and the SIR threshold, on the meta distribution and the mean local delay.

II. SYSTEM MODEL

We consider a downlink cache-enabled network. The BSs are assumed to follow the PPP \( \Phi \) with density \( \lambda \). The BSs transmit with power \( P \) and a path loss function, i.e., \( \ell(x) = x^{-\alpha} \) is adopted to model the attenuation of the transmitted signal, where \( \alpha \) denotes the path loss exponent and \( x \) the distance between a user and its serving BS. The small-scale fading is assumed to be Rayleigh, i.e., \( h \sim CN(0,1) \). Without loss of generality, according to Slivnyak’s theorem, we study the performance of the typical user \( u_0 \) located at the origin. The network is considered to be static, i.e., the locations of the BSs and users follow the fixed, but arbitrary, realization of the point processes.

Let \( \mathcal{F} \triangleq \{1, 2, \ldots, F\} \) denote a set of \( F \) files in the network. For analytical tractability, all files are assumed to be of the same size\(^1\) and the file popularity distribution is identical among all users. Let \( p_f \) denote the probability that File \( f \) is requested by a user. In other words, the popularity of File \( f \) is \( p_f \), where \( \sum_{f=1}^{F} p_f = 1 \). In addition, we can always assume that \( p_1 \geq p_2 \geq \cdots \geq p_F \). Hence, the file popularity distribution can be expressed as \( p \triangleq \{p_1, p_2, \ldots, p_F\} \), which is assumed to be known a priori. Note that the popularity of different files evolves at a relatively slow timescale and can be estimated in practice (e.g. by the machine learning [20]). Each BS is equipped with a cache of size \( C \leq F \) to store \( C \) different files from \( \mathcal{F} \).

\(^1\)Note that the results can easily be extended to the case where the contents have different file sizes (e.g. by combining multiple files of different sizes to form files of equal size or splitting files of different sizes into segments of equal size) [21].
A. Random Caching

To improve the system performance and provide the spatial diversity, the random caching strategy [21] is adopted. Let \( q_f \) denotes the probability that File \( f \) is cached at a BS. Then, we have [21]

\[
0 \leq q_f \leq 1
\]

and second moments. An alternative method is adopted where the beta distribution is utilized to approximate the meta distribution by matching the first probability

\[
\sum_{f=1}^{F} q_f = C,
\]

which indicates that the sum of all the caching probabilities of the files cached at a BS is limited by the BS’s storage capacity.

Consider \( u_0 \) requesting File \( f \). If File \( f \) is not stored in any tier, \( u_0 \) will download the corresponding file from the core network through the nearest BS. Otherwise, \( u_0 \) is associated with the BS which provides the maximum biased received signal power among all BSs caching File \( f \), referred to as the serving BS.

B. Random DTX scheme

We assume that the time is divided into slots with equal duration and a transmission attempt requires a single time slot. In the random DTX scheme, the DTX mode in each time slot is modeled as a Bernoulli trial with an parameter \( \beta \) called BS active probability. That is, a BS in each time slot is temporally independently active with probability \( \beta \) and is muted with probability \( 1 - \beta \). Let \( \Phi_f \) and \( \Phi_{-f} \) denote the set of active BSs with/without caching File \( f \) in time slot \( t \), respectively. The interference from the BSs with/without caching File \( f \) at time slot \( t \) is given by

\[
I_{t,f} = \sum_{x \in \Phi_f \setminus x_0} 1(x \in \Phi_f(t)) P h_x x^{-\alpha},
\]

\[
I_{t,-f} = \sum_{x \in \Phi_{-f}} 1(x \in \Phi_{-f}(t)) P h_x x^{-\alpha},
\]

where \( 1 \) denotes the indicator function, \( \Phi_f(t) \) and \( \Phi_{-f}(t) \) denotes the set of BSs with/without caching File \( f \) at time slot \( t \), respectively. We consider the interference-limited scenario and the noise is neglected. Therefore, the SIR at \( u_0 \) is given by

\[
\text{SIR} = \frac{P h_{x_0} x_0^{-\alpha}}{I_f + I_{-f}}.
\]

C. Performance Metric

In this paper, we consider two performance metrics, i.e., the meta distribution and the local delay. Note that a retransmission occurs if a transmission in time slot \( t \) fails. Therefore, local delay is defined as the number of retransmissions needed until a successful transmission occurs. Due to the appliance of the random DTX scheme, the transmission is considered to be successful if the serving BS is in the active mode and the SIR at \( u_0 \) exceeds a pre-defined threshold \( \theta \), i.e.,

\[
\mathcal{P}(\theta|\Phi) = \beta \mathbb{P} \left[ \text{SIR} > \theta | \Phi \right].
\]

The STP is the complementary cumulative distribution function (CCDF) of the SIR. Accordingly, the meta distribution can be defined as the CCDF of the conditional STP, which is given by

\[
\bar{F}_P(x) \triangleq \mathbb{P}(\mathcal{P}(\theta) > x), \ x \in [0, 1].
\]

Due to the ergodicity of the point processes, the meta distribution can be interpreted as the fraction of active links with the conditional success probability greater than \( x \).

To obtain the analytical expression of the meta distribution, it is necessary to obtain the \( b \)-th moment of the conditional STP. When \( u_0 \) requests File \( f \), we denote by \( M_{b,f} \) the \( b \)-th moment of \( \mathcal{P} \). The expression of meta distribution for \( u_0 \) requesting File \( f \) can be derived by utilizing the Gil-Pelaez theorem. Since the exact form of the meta distribution is complex, a simpler alternative method is adopted where the beta distribution is utilized to approximate the meta distribution by matching the first and second moments.

We denote the mean local delay by \( D \). Conditioned on \( \Phi \), the local delay is geometrically distributed with parameter \( \mathcal{P}(\theta|\Phi) \). Letting \( D_\Phi = (D|\Phi) \), we have

\[
\mathbb{P}(D_\Phi = d) = (1 - \mathcal{P}(\theta|\Phi))^{d-1} \mathcal{P}(\theta|\Phi).
\]

Accordingly, the mean of the geometrically distributed random variable is \( \frac{1}{\mathcal{P}(\theta|\Phi)} \). The local delay can be obtained by calculating the expectation with respect to \( \Phi \) as follows

\[
D = \mathbb{E}_\Phi \left[ \mathbb{E}[D_\Phi] \right] = \mathbb{E}_\Phi \left[ \frac{1}{\mathcal{P}(\theta|\Phi)} \right].
\]

From [2], it can be observed that the mean local delay is the \(-1\)-st moment of the conditional STP, i.e., \( M_{-1,f} \). In this paper, our goal is to obtain the maximization of the STP or the minimization of the mean local delay by optimizing the caching probability and the BS active probability.
III. Analytical Results

In this section, we first analyze the meta distribution, then derive the mean local delay and obtain some useful insight.

A. Meta Distribution

In this subsection, we obtain the provide the b-th moment of the conditional STP when File f is requested by u₀, then provide the exact expression of the meta distribution. In addition, the beta approximation of the meta distribution is derived.

**Theorem 1:** When u₀ requests File f, the b-th moment of the meta distribution is given by

\[
M_{b,f} = q_f \left( q_f + \sum_{n=1}^{\infty} \frac{b}{n} (-1)^{n+1} (\delta(1-q_f)^{n+1} B(\delta, n-\delta)) \right)^{-1} \tag{10}
\]

**Proof:** See the Appendix A.

When u₀ requests File f, the meta distribution of the SIR can be obtained by applying the Gil-Pelaez theorem, which is given by

\[
\hat{F}_{\delta f} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{J(e^{-jt\log z} M_{f,\delta})}{t} \, dt,
\]

where J(z) is the imaginary part of z ∈ c. Since the numerical evaluation of (11) is cumbersome and it is difficult to obtain further insight, we resort to a beta distribution to approximate the meta distribution by matching the first and second moments, which can be easily obtained from the result in (10):

\[
M_{1,f} = q_f \left( q_f + \frac{q_f \beta \delta}{1-\delta} 2F_1(1,1-\delta;2-\delta;\theta) \right)^{-1} \tag{12}
\]

\[
M_{2,f} = q_f \left( q_f + \frac{2q_f \beta \delta}{1-\delta} 2F_1(1,1-\delta;2-\delta;\theta) \right)^{-1} \tag{13}
\]

By matching the variance and mean of the beta distribution, i.e., \(M_{2,f} - M_{1,f}^2\) and \(M_{1,f}\), the approximated meta distribution of the SIR can be given by

\[
\hat{F}_{\delta f} \approx 1 - I_x \left( \frac{M_{1,f}}{1-M_{1,f}} \chi, \, x \in [0,1], \right) \tag{14}
\]

where

\[
\chi = \frac{(M_{1,f} - M_{2,f})(1-M_{1,f})}{M_{2,f} - M_{1,f}^2} \tag{15}
\]

and \(I_x(a, b)\) is the regularized incomplete beta function

\[
I_x(a, b) \triangleq \int_0^x t^{a-1}(1-t)^{b-1} \, dt. \tag{16}
\]

Figs. 1 and 2 plot the statistical information of the distribution of the SIR, i.e., mean \(M_{1,f}\) and variance. From Fig. 1 we can observe that the simulation results match the numerical results obtained from (12) well, verifying the correctness of the numerical analysis. From Fig. 2 we can observe that the variances of the STP first increase with the SIR threshold. After reaching its maximum point, it starts to decrease. Note that the maximum value of the variance increases with the caching probability. Fig. 3 illustrates the meta distribution as a function of the STP under different caching probabilities. It can be observed that the meta distribution decreases rapidly at start, then the variation of the meta distribution tends to be gentle. When the STP approaches 1, the decreasing rate of the meta distribution becomes larger. This phenomenon indicates that the STPs of all links between the users and serving BSs vary in a wide range and the STPs of a considerable portion of users is either very large or small.
Fig. 1. $M_{1,f}$ (STP) vs SIR threshold $\theta$. $F=10$, $C=2$, $q_1=0.9$, $q_2=0.8$, $q_3=0.3$

Fig. 2. Variance vs SIR threshold $\theta$. $F=10$, $C=2$, $q_1=0.9$, $q_2=0.8$, $q_3=0.3$

Fig. 3. Meta distribution vs SIR threshold $\theta$. $F=10$, $C=2$, $q_1=0.9$, $q_2=0.8$, $q_3=0.3$
B. Mean Local Delay

In this subsection, we derive the closed-form expression of the mean local delay, then provide a simplified expression of the local delay under the special case where the data rate is sufficiently large or small. While File \( f \) is transmitted by the serving BS, the interfering BSs are divided into two groups: one group is formed by the active BSs caching the requested file, the other group is formed by the active BSs not caching the requested file. By carefully handling the interference from two groups, the expression of the service delay can be obtained in the following theorem.

**Theorem 2:** The mean local delay is given by

\[
M_{-1,f} = \frac{q_f}{\beta(C_3q_f - C_1)}
\]

where \( \delta = \frac{\beta}{\alpha} \).

\[
C_1 = \delta(1 - \beta)^{\delta-1} B(\delta, 1 - \delta)
\]

and

\[
C_2 = \frac{\delta\beta}{1 - \delta} F_1(1, 1 - \delta; 2 - \delta; -(1 - \beta)\theta)
\]

\[
C_3 = 1 + C_1 - C_2;
\]

(17)

(18)

(19)

(20)

**Proof:** See the Appendix B.

From Theorem 2, we can see that the mean local delay \( M_{-1} \) is affected by the file-related parameters, i.e., the popularity and the caching probability, and the network parameters, i.e., the active probability \( \beta \) and the path loss exponent \( \alpha \). The impact of the file-related parameters and the network parameters are coupled. In order to obtain more insight, we also derive the simplified expression of the mean local delay \( M_{-1} \) in the low or high SIR threshold \( \theta \) regime, i.e., \( \theta \to 0 \) or \( \theta \to \infty \), in the following lemma.

**Lemma 1:** When \( \theta \to \infty \), the local delay for File \( f \) is given by

\[
M_{-1,f,\infty} = \frac{q_f}{\beta(C_3,\infty q_f - C_1)}.
\]

(21)

where

\[
C_{3,\infty} = 1 - \frac{\beta}{(1 - \beta)^2\theta}.
\]

When \( \theta \to 0 \), the local delay for File \( f \) is given by

\[
M_{-1,f,0} = \frac{q_f}{\beta(C_{3,0} q_f - C_1)},
\]

(23)

where

\[
C_{3,0} = \frac{1}{1 - \beta} - \frac{\beta\delta}{(1 - \beta)^2(1 + \delta)\theta}.
\]

(24)

**Proof:** See the Appendix B.

Note that the service delay is independent of the BS density \( \lambda \) and the transmit power \( P \). The reason is given as follows. When a larger number of BSs are added into the networks, the distance between \( u_0 \) and its serving BS becomes smaller, resulting in a stronger signal received by \( u_0 \). Meanwhile, the interference received by \( u_0 \) becomes stronger. Overall, the statistics of SIR remains unchanged. In addition, from (17), we can observe that the local delay is an increasing function of the SIR threshold \( \theta \). The reason can be stated as follows. When the SIR threshold \( \theta \) increases, the probability that File \( f \) is successfully transmitted by the BS in a certain time slot decreases. Therefore, a larger number of retransmissions are needed until the transmission succeeds. Since the local delay reaches infinity when the SIR threshold \( \theta \) exceeds a certain value. We say that a phase transition occurs when the mean local delay changes from finite to infinity. The corresponding value of \( \theta \) is called the critical value, denoted by \( \theta_c \). When the caching probability \( q_f \), the active probability \( \beta \) are fixed, the critical value of the SIR threshold can be obtained by solving the following equation:

\[
q_f - \delta(1 - q_f)(1 - \beta)\delta B(\delta, 1 - \delta)
\]

\[
- \frac{q_f\delta\beta\theta_c}{1 - \delta} F_1(1, 1 - \delta; 2 - \delta; -(1 - \beta)\theta_c) = 0.
\]

(25)

The equation indicates that the local delay for File \( f \) \( M_{-1,f} \to \infty \) when \( \theta > \theta_c \) given the fixed caching probability \( q_f \) and active probability \( \beta \).

Next, We investigate the effect of the caching probability and the active probability on the mean local delay \( M_{-1} \). Fig. 4 plots the mean local delay as a function of active probability under different caching probabilities. From Fig. 4, it can be observed that the mean local delay decreases with the caching probability. The minimum value of local delay can be achieved when the caching probability is 1. This can be explained as follows. When the caching probability becomes larger, the distance between \( u_0 \) and its serving BS is smaller, leading to the stronger signal received by \( u_0 \). Hence, the SIR increases and the mean
local delay decreases. It can also observed that the mean local delay approaches infinity when the caching probability is below a certain value. The critical caching probability $q_c$ can be obtained by letting the denominator of (17) equal to 0. Therefore, the critical caching probability can be expressed as follows

$$q_c \sim \frac{C_1}{C_3}. \quad (26)$$

Since the mean local delay when $\theta \to \infty$ is derived in Lemma 1, the critical caching probability in the high SIR threshold regime, i.e., $q_{c,\infty}$, can be expressed as follows

$$q_{c,\infty} \sim \frac{C_1}{C_{3,\infty}}. \quad (27)$$

Note that With $q_{f,c,\infty}$, there is no phase transition for any given SIR threshold $\theta$.

From Fig. 4, it is also noteworthy that the critical caching probability $q_c$ increases with the BS active probability. This phenomenon indicates that the caching probability needs to keep large to make the local delay finite in the large active probability regime.

Fig. 5 illustrates the effect of the active probability on the mean local delay under different caching probabilities. From Fig. 5 we can observe that the mean local delay approaches infinity when the active probability $\beta$ reaches zero or a certain value. This value is defined as the critical active probability $\beta_c$ and it can be obtained following the similar methodology with the derivation of $q_c$. Note that there exists an optimal value of active probability $\beta^*$ within $(0, \beta_c)$. The mean local delay increases with the gap $|\beta - \beta^*|$. The reason can be stated as follows. When $\beta < \beta^*$, the active probability for the serving BS becomes smaller, leading to the increasing number of retransmission needed until the transmission succeeds. When $\beta > \beta^*$, the interference received by $u_0$ becomes larger, leading to the decrease of SIR. In addition, $u_0$ is more likely to be interfered by the common set of BSs in different time slots. The interference temporal correlation together with the increasing interference will reduce the probability that the file is transmitted by the BS successfully.

IV. OPTIMIZATION OF STP AND SERVICE DELAY

In this section, we focus on the optimization of the STP and service delay, respectively.

A. Optimization of STP

In this subsection, we focus on the maximization of the STP by optimizing the caching probability and the BS active probability. Traditionally, only the cache size limitation is considered while searching for the optimal solution. Here, we also consider the constraint where the mean local delay is finite. Then the optimization problem can be formulated as follows

Problem 1 (Optimization of Service Delay):

$$\max_{q,\beta} M_1(q, \beta) \quad (28)$$

s.t. 1, 2

$$M_{-1,f} < \infty \quad (29)$$

$$M_{-1,f} < \infty \quad (30)$$
In order to satisfy constraint (30), the caching probability need to exceed the critical value $q_c$ given a fixed BS active probability $\beta$. It can be easily verified that $M_1(q, \beta)$ is a monotonically increasing function of the BS active probability $\beta$. Given a fixed $\beta$, $\min \left( \left\lfloor \frac{F}{q_c} \right\rfloor - 1, F \right)$ files can be possibly cached in the BSs. Since the critical caching probability $q_c$ increases with the BS active probability $\beta$, the largest number of files that can be possibly cached in the BSs decreases with $\beta$, as shown in Fig. 6. From Fig. 6 we can also observe that a slightly performance gain can be obtained with the increasing number of cached files. However, the reduction in the signal strength caused by the decreasing $\beta$ cannot be compensated by the interference reduction and the enhancement of the file diversity. Therefore, it can be concluded that the maximum STP can be achieved when $\beta = \beta_c$ and $q_1 = \cdots = q_C = 1$, $q_{C+1} = \cdots = q_F = 0$. In other words, the optimal caching strategy is MPC.

B. Optimization of Service Delay

In this subsection, the service delay is first introduced. The service delays are different for the file cached and not cached in the BSs. When $u_0$ requests a file cached in the BS, $u_0$ can directly obtain the corresponding file from the BS and only the local delay, denoted by $T_{bh}$, is considered. However, when the requested file is not cached in the BS, $u_0$ needs to retrieve the corresponding file from the core network through the backhaul. Hence, the service delay includes the mean local delay and the backhaul delay, denoted by $T_{bh}$. Since the DTX scheme is employed, only the interference from the active BSs will be received by $u_0$. Then the expression of the service delay can be obtained as follows

$$D = \sum_{f=1}^{F} p_f \left( 1(q_f > 0)M_{-1,f} + 1(q_f = 0)T_{bh} \right),$$

(31)
where \(1(\cdot)\) denotes the indicator function, the mean local delay is given by (17) and the backhaul delay is given by

\[
T_{bh,f} = \frac{1}{\beta(1 - C_1)} + \xi
\]

Next, we obtain the minimization of the service delay by optimizing the caching probability \(q\) and the active probability \(\beta\). The optimization problem is formulated as follows

\[
\begin{align*}
\min_{q,\beta} & \quad D(q, \beta) \\
\text{s.t.} & \quad (1), (2)
\end{align*}
\]

where (1) is the capacity constraint and (2) is the probability constraint. In order to facilitate the optimization process, we explore the optimality property of this problem as follows

1. The network parameters affect the service delay of all files and the popularity only affects the service delay of the corresponding file.
2. Given a fixed active probability \(\beta\), there exists \(F^*_c \in [C, \min\left(\frac{C}{q_c} - 1, F\right)]\) such that the caching probabilities of the files stored in the BSs exceed the critical caching probability, i.e., \(q_f > q_c, f \in [1, F^*_c]\) and \(q_f = 0, f \in (F^*_c, F]\), where \(q_c\) is determined by (26). Given a fixed \(q\), there exists an optimal \(\beta^* \in (0, \beta_c)\) such that the minimum service delay is achieved.
3. When \(F^*_c = C\), we have \(q_f > q_c, f \in [1, C]\) and \(q_f = 0, f \in (C, F]\), indicating that the optimal caching strategy degrades to Most popular content (MPC) (2).

Due to the existence of the indicator function, it is difficult to obtain the derivative of the objective function. The gradient projection method [21] or interior point method [22] cannot be directly applied to obtain the optimal solution. Therefore, an equivalent problem is constructed by utilizing the optimality property of Problem 1. Since \(F^*_c\) files are cached in the BSs, an auxiliary variable \(F_c\) is introduced and the objective function of Problem 1 can be rewritten as follows

\[
D(q, \beta) = \sum_{f=0}^{F_c} M_{-1,f} + \sum_{f=F_c+1}^{F} T_{bh}.
\]

For notation simplicity, we denote \(\sum_{f=0}^{F_c} M_{-1,f}\) and \(\sum_{f=F_c+1}^{F} T_{bh}\) by \(D_1(q, F_c)\) and \(D_2(F_c)\), respectively. We can see that \(D(q, \beta)\) is differentiable with respect to \(q\) and \(\beta\). The equivalent problem is formulated as follows

\[
\begin{align*}
\min_{q,\beta,F_c} & \quad D(q, \beta, F_c) \\
\text{s.t.} & \quad C < F_c < \frac{C}{q_c} \\
& \quad \beta_c < \beta < 1 \\
& \quad q_c < q_f < 1 \\
& \quad \sum_{f=1}^{F_c} q = C
\end{align*}
\]

where constraint (37) limits the search space of \(F_c\). The probability constraint (39) and capacity constraint (40) are refined from (1) and (2) by utilizing the optimality property. From (36), we can observe that two kinds of variables are to be determined. One is the continuous variables \(q\) and the other is the discrete variables \(\beta\) and \(F_c\). \(\beta\) and \(q\) can be updated iteratively until a stationary point is reached. Specifically, two sub-problems are formulated to obtain the optimal \(\beta\) and \(q\), respectively.

**Sub-Problem 1 (Optimization of Caching Probability for Given Active Probability):**

\[
D^*(q, \beta, F_c) \equiv \min_{F_c} D_1^*(F_c) + D_2(F_c)
\]

\[
\begin{align*}
\text{s.t.} & \quad (37)
\end{align*}
\]

where

\[
D_1(F_c)^* \equiv \min_{q} D_1(q, F_c).
\]

\[
\begin{align*}
\text{s.t.} & \quad (39), (40)
\end{align*}
\]

Note that \(F_c\) can be determined by exhaustive search of complexity \(O(F)\). For a given \(F_c\), the constraint (39) and (40) are linear and the second-order derivative of \(D(q, \beta, F_c)\) with respect to \(q\) is positive within the feasible region. Therefore, the
problem is a convex and satisfies Slater’s condition, indicating that the strong duality holds. Then we can obtain the optimal caching probability in the following lemma.

**Lemma 2:** The optimal caching probability can be obtained by utilizing the Karush-Kuhn-Tucker (KKT) conditions as follows

\[
q^*_f(t) = \min \left\{ \max \left\{ \frac{1}{C_3} \sqrt{-\frac{C_1}{\eta^* \beta} + \frac{C_1}{C_3}}, q_c \right\}, 1 \right\},
\]

where \( \eta^* \) satisfies that

\[
\min \left\{ \max \left\{ \frac{1}{C_3} \sqrt{-\frac{C_1}{\eta^* \beta} + \frac{C_1}{C_3}}, q_c \right\}, 1 \right\} = C.
\]

When \( \theta \to \infty \), the optimal caching probability is given by

\[
q^*_{f, \infty}(t) = \min \left\{ \max \left\{ \frac{1}{C_{3, \infty}} \sqrt{-\frac{C_1}{\eta^* \beta} + \frac{C_1}{C_{3, \infty}}}, q_c \right\}, 1 \right\},
\]

When \( \theta \to 0 \), the optimal caching probability is given by

\[
q^*_{f, 0}(t) = \min \left\{ \max \left\{ \frac{1}{C_{3, 0}} \sqrt{-\frac{C_1}{\eta^* \beta} + \frac{C_1}{C_{3, 0}}}, q_c \right\}, 1 \right\},
\]

**Sub-Problem 2 (Optimization of BS Active Probability for Given Caching Probability):**

\[
\min_{\beta} D(q, \beta, F_c)
\]

s.t. \( \text{[48]} \)

It is difficult to determine the convexity of the problem. Since the derivative of the objective function is achieved, we therefore resort to the gradient projection method [21] to obtain the locally optimal BS active probability. The derivative of the objective function is given by

\[
\frac{\partial D(q, \beta, F_c)}{\partial \beta} = q_f \left( 2\Omega_2 \beta - \Omega_1 \sum_{n=0}^{\delta-1} \left( \frac{\delta - 1}{n} \right) (n + 2)(-\beta)^{n+1} \right)
\]

\[-q_f \left( \Omega_1 (1 - \beta)^{\delta-1} \beta - \Omega_2 \beta + q_f \right)^2
\]

where

\[
\Omega_1 = \frac{q_f \delta \theta}{1 - \delta^2} \left( F_1(1, 1 - \delta; 2 - \delta; -(1 - \beta)\theta) \right)
\]

\[
\Omega_2 = \frac{q_f \delta^2 \theta}{1 - \delta^2} \left( F_1(1, 1 - \delta; 2 - \delta; -(1 - \beta)\theta) \right)
\]

The overall process of the two-stage method is summarised in Algorithm 2 below.

**Algorithm 1 Optimal Solution of Problem 2**

**Input:** Number of Files \( F \), Cache size \( C \).

**Output:** Optimal caching probability \( q^* \).

**Initialize:** set \( \chi = 0 \).

1: repeat
2: Obtain the critical caching probability \( q_c \).
3: for \( F_c = C \) to \( q \) do
4: Obtain \( F^*_c \) and \( q^* \) by solving the optimization in [43] using the KKT conditions.
5: if \( D^*(q, \beta, F_c) < D^*_f(F_c) + D_2(F_c) \) then
6: \( D^*(q, \beta, F_c) \leftarrow D_f(F_c) + D_2(F_c) \) and \( q^* \leftarrow q^*(F^*_c) \)
7: end if
8: end for
9: Obtain the critical BS active probability \( \beta^* \).
10: Obtain \( \beta^* \) by solving the optimization in [49] using the gradient projection method.
11: \( \chi \rightarrow \chi + 1 \)
12: until convergence criterion is satisfied.
13: return the optimal caching probability \( q^* \).
In this section, we first illustrate the effect of the critical network parameters, i.e., the active probability and the SIR threshold, on the meta distribution. Then the effect of network parameters on the mean local delay is also presented. Unless otherwise stated, the parameters are set as follows. The BS transmit power is $P = 23$ dBm, the BS density is $\lambda = 10^{-4}/m^2$, the user density is $\lambda_u = 3 \times 10^{-4}/m^2$, and the path loss exponent $\alpha = 3$.

Figs. 7 and 8 plot the statistical information of the distribution of the SIR. From Fig. 7, we can observe that STP increases with the BS active probability and the caching probability. From Fig. 8, it can be observed that there exists a maximum variance when the caching probability $q$ is relatively large, i.e., $q = 0.9$ or $0.8$. Given $q = 0.5, 0.6$ or $0.7$, the variance increases with $\beta$.

This phenomenon indicates that random DTX scheme has different impact on the files with different caching probabilities. For the files with relatively low caching probabilities, the random DTX scheme improves the stability of the network performance. For the files with relatively large caching probabilities, the random DTX scheme harms the stability of the network performance.

Fig. 10 shows the effect of the caching probability on the variance of the distribution of SIR. When the BS active probability is large, the variance increases at start and reaches its maximum point, then decreases with the caching probability. Here we provide an intuitive explanation. The stability of the network performance is mainly dependent on two components: the variance of the signal power and the interference strength from the dominant interfering BS, i.e., the nearest BS not caching File $f$. When File $f$ is requested, the distance between the user and its serving BS is shortened with the increasing caching probability of File $f$. Then the performance fluctuation caused by the channel fading becomes larger. When the caching probability further increases, the distance between the user and its dominant interfering BS becomes larger, then the performance fluctuation...
becomes smaller.

Fig. 11 illustrates the service delay as a function of the cache size under different backhaul delays. We can observe that the optimal service delay decreases with the cache size. This is because the increasing cache size can not only enhance the file diversity but also shorten the distance between the user and the serving BS, leading to the increase of the SIR and the decrease of the mean local delay. Moreover, the service delay increases with the backhaul delay and the performance gap between the service delays under different backhaul delays become smaller with the increasing cache size. There are two reasons for this phenomenon. First, compared to the cache-enabled networks with small backhaul delay, a larger number of files tend to be cached in the BSs and the optimal caching probability for each file turns out to be smaller in the large backhaul delay regime. In order to keep the mean local delay finite, the BS active probability needs to be smaller, resulting in the increasing mean local delay. Second, when the cache size increases, a larger number of files are cached in the BSs and fewer files needs to be retrieved from the core network through the backhaul link. Therefore, the dominant factor that affects the service delay is the mean local delay rather than the backhaul delay. Specifically, the service delay are identical when the cache size is 30 since all the files are cached in the BSs and no file needs to be retrieved from the core network.

Fig. 12 plots the effect of the backhaul delay on the number of cached files under different cache sizes. From Fig. 12 it can be observed that the the optimal number of cached files increases with the backhaul delay. When the backhaul delay $\xi=0$, 5 or 10, the optimal number of cached files is equal to the cache size, indicating that the optimal caching strategy is MPC. When the backhaul delay increases, the optimal number of cached files tends to be larger than the cache size and the optimal number can be determined by the proposed algorithm. The reason is that the users are less likely to suffer from the severe backhaul delay when more files can be directly obtained from the BSs. When the backhaul delay further increases, the optimal number of cached files is equal to $F$, indicating that the optimal caching strategy is UC. Note that the optimal caching strategy
VI. Conclusion

We provide a fine-grained analysis on the cache-enabled networks. The moments of the conditional successful transmission probability (STP), the exact meta distribution and its beta approximation were derived by utilizing stochastic geometry. The closed-form expression of the mean local delay was also derived. We considered the maximization of the STP and the minimization of the mean local delay by optimizing the caching probability and the BS active probability, respectively. For the former, a convex optimization problem was formulated and the optimal caching probability and BS active probability were achieved. For the latter, a non-convex optimization problem was formulated and an iterative algorithm was proposed to obtain the optimal solution. We demonstrated the impact of the backhaul delay on the caching strategy. That is, MPC is proved to be optimal when the backhaul delay is relatively low and the uniform caching (UC) is the optimal caching strategy when the backhaul delay is very large. The structure of the optimal caching strategy provides useful insights for designing the cache-enabled networks.
A. Proof of Theorem 1

Assuming that $u_0$ requests File $f$, the STP conditioned on the realization $\Phi$ can be derived as follows

$$
P(\theta|\Phi) \overset{(a)}{=} \beta \mathbb{P}(\text{SIR}_{k,t} > \theta|\Phi)
= \beta \mathbb{P}(P_{h0}r^{-\alpha} > \theta(I_{t,f} + I_{t,-f})|\Phi)
\overset{(b)}{=} \beta \mathbb{E}_f \left[ \exp \left( -\frac{\theta r^\alpha}{P} I_{t,f} \right) \exp \left( -\frac{\theta r^\alpha}{P} I_{t,-f} \right) \right]
\overset{(c)}{=} \beta \mathbb{E}_\Phi \left[ L_{I_{t,f}} \left( \frac{\theta r^\alpha}{P} \Phi \right) L_{I_{t,-f}} \left( \frac{\theta r^\alpha}{P} \Phi \right) \right],
$$

(54)

where (a) follows from the definition of $P(\theta)$, (b) follows from that $h \sim \exp(1)$. In the last step, $L_{I_{t,f}} \left( \frac{\theta r^\alpha}{P} \Phi \right)$ and $L_{I_{t,-f}} \left( \frac{\theta r^\alpha}{P} \Phi \right)$ denote the Laplace transforms of the interferences of the BSs with/without caching File $f$, respectively. Assuming $s = \frac{\theta r^\alpha}{P}$, the Laplace transform can be derived as follows

$$
L_{I_{t,f}}(s|\Phi) = \mathbb{E}_{I_{t,f}}[\exp(-sI_{t,f})]
= \mathbb{E}_{h_i} \left[ \exp \left( -s \sum_{x \in \Phi_{t,f}\setminus x_0} P h_i x^{-\alpha} \right) \right]
= \prod_{x \in \Phi_{t,f}\setminus x_0} \mathbb{E}_{h_i} \left[ \beta \exp \left( -s P h_i x^{-\alpha} \right) + 1 - \beta \right]
= \prod_{x \in \Phi_{t,f}\setminus x_0} \left( \frac{\beta}{1 + s P x^{-\alpha}} + 1 - \beta \right)
$$

(55)

Similarly, the Laplace transform of the interferences from the BSs not caching File $f$ can be evaluated as

$$
L_{I_{t,-f}}(s|\Phi) = \prod_{x \in \Phi_{t,-f}} \left( \frac{\beta}{1 + s P x^{-\alpha}} + 1 - \beta \right)
$$

(56)

Accordingly, the $b$-th moment of the conditional STP can be derived as follows

$$
M_{b,f} = \mathbb{E}_\Phi \left[ \mathbb{P}(\text{SIR} > \theta|\Phi)^b \right]
= \beta \mathbb{E}_\Phi \left[ L_{I_{t,f}} \left( \frac{\theta r^\alpha}{P} \Phi \right)^b L_{I_{t,-f}} \left( \frac{\theta r^\alpha}{P} \Phi \right)^b \right].
$$

(57)

Assuming $s = \frac{\theta r^\alpha}{P}$, $L_{I_{t,f}}(s|\Phi)^b$ can be derived as follows

$$
L_{I_{t,f}}(s|\Phi)^b = \prod_{x \in \Phi_{t,f}\setminus x_0} \left( \frac{\beta}{1 + s P x^{-\alpha}} + 1 - \beta \right)^b
\overset{(a)}{=} \exp \left( -2\pi \lambda q_f \int_r^\infty \left( 1 - \left( \frac{\beta}{1 + \theta r^\alpha x^{-\alpha}} + 1 - \beta \right)^b \right) dx \right)
= \exp \left( -2\pi \lambda q_f \int_r^\infty \sum_{n=0}^\infty \frac{b^n}{n!} \left( \frac{\beta \theta r^\alpha x^{-\alpha}}{1 + \theta r^\alpha x^{-\alpha}} \right)^n dx \right)
\overset{(a)}{=} \exp \left( -\pi \delta l q_f \sum_{n=1}^\infty \frac{b^n}{n!} \left( -1 \right)^{n+1} \frac{(\beta \theta)^n r^{2n}}{(n-\delta)} \right)
$$

(58)

$$
2F_1 \left( n, n - \delta; n - \delta + 1, -\theta \right).
$$
Similarly, \( L_{t_{r,f}}(s|\Phi)^b \) can be derived as follows

\[
L_{t_{r,f}}(s|\Phi)^b = \prod_{x \in \Phi_{t_{r,f}} \setminus x_0} \left( \frac{\beta}{1 + sP} x^\alpha + 1 - \beta \right)^b
\]

\[

(a) \quad \exp \left( -2\pi \lambda (1 - q_f) \int_0^\infty \left( 1 - \frac{\beta}{1 + \theta r^\alpha x^{-\alpha} + 1 - \beta} \right)^b x \, dx \right)
\]

\[

(b) \quad \exp \left( 2\pi \lambda (1 - q_f) \int_0^\infty \frac{\beta \theta r^\alpha x^{-\alpha}}{1 + \theta r^\alpha x^{-\alpha}} x \, dx \right)
\]

\[

= \exp \left( -2\pi \lambda (1 - q_f) \int_0^\infty \sum_{n=1}^\infty \frac{b}{n} (-1)^{n+1} \left( \frac{\beta \theta r^\alpha x^{-\alpha}}{1 + \theta r^\alpha x^{-\alpha}} \right)^n x \, dx \right)
\]

\[

= \exp \left( -\pi \delta \lambda (1 - q_f) \sum_{n=1}^\infty \frac{b}{n} (-1)^{n+1} \beta^n \theta^2 \text{B}(\delta, n - \delta) \right).
\]

According to [23], the probability density function (PDF) of the distance between \( u_0 \) and the serving BS conditioned on that File \( f \) is requested is given by

\[
f_{R}(r) = 2\pi \lambda q_f r \exp(-\pi \lambda q_f r^2)
\]

(60)

The \( b \)-th moment of the conditional STP can be derived as follows

\[
M_{b,f} = \int_0^\infty 2\pi \lambda q_f r \exp(-\pi \lambda q_f r^2)
\]

\[
L_{t_{r,f}} \left( \frac{\theta r^\alpha}{P} \mid \Phi \right)^b L_{t_{r,f}} \left( \frac{\theta r^\alpha}{P} \mid \Phi \right) \, dr
\]

\[
= q_f \left( q_f + \delta (1 - q_f) \sum_{n=1}^\infty \frac{b}{n} (-1)^{n+1} \beta^n \theta^2 \text{B}(n, n - \delta) \right)
\]

\[
= \delta q_f \sum_{n=1}^\infty \frac{b}{n} (-1)^{n+1} \frac{(\beta \theta)^n}{n - \delta} \text{F}_1(n, n - \delta; n - \delta + 1; -\theta)^{-1}
\]

(61)

B. Proof of Theorem 2

The mean local delay can be derived as follows

\[
M_{-1,c} = \mathbb{E}_\Phi \left[ \mathbb{P} \left( \text{SIR} > \theta \mid \Theta \right)^{-1} \right]
\]

\[
= \beta \mathbb{P} \left( P_{\text{hor}} r^{-\alpha} > \theta (I_{t_{r,f}} + I_{t_{r,f}}) \mid \Phi \right)^{-1}
\]

\[
= \beta \mathbb{E}_\Phi \left[ \frac{1}{L_{t_{r,f}} \left( \frac{\theta r^\alpha}{P} \mid \Phi \right)} \frac{1}{L_{t_{r,f}} \left( \frac{\theta r^\alpha}{P} \mid \Phi \right)} \right],
\]

we derive the expression of \( \frac{1}{L_{t_{r,f}}(s)} \) as follows

\[
\left( \frac{1}{L_{t_{r,f}}(s)} \right)^b
\]

\[
(a) \quad \exp \left( -2\pi \lambda \int_r^\infty \left( 1 - \frac{\beta}{1 + sP} x^\alpha + 1 - \beta \right)^{-1} x \, dx \right)
\]

\[
(b) \quad \exp \left( 2\pi \lambda \int_r^\infty \frac{\beta \theta r^\alpha}{1 + \beta \theta r^\alpha x^{-\alpha}} x \, dx \right)
\]

\[
= \exp \left( \pi \delta q_f \beta \theta r^\alpha \int_{r^\alpha}^\infty \frac{u^{\delta-1}}{u + (1 - \beta) \theta r^\alpha} \, du \right)
\]

\[
= \exp \left( \pi \delta q_f \frac{\beta \theta r^2}{1 - \delta} \text{F}_1(1, 1 - \delta; 2 - \delta; -(1 - \beta) \theta) \right),
\]

(63)
where (a) is obtained by utilizing the probability generating functional (PGFL) of the PPP, (b) follows from the binomial theorem.

Similarly, the derivation of $\frac{1}{L_{t_{-f}}(s)}$ is provided as follows

$$
\frac{1}{L_{t_{-f}}(s)} = \exp \left( \pi \delta \lambda (1 - q_f) \beta \theta r^\alpha \int_0^\infty \frac{u^{\delta - 1}}{u + (1 - \beta \theta r^\alpha)} du \right)
= \exp \left( \pi \delta \lambda (1 - q_f) \beta \theta r^2 B(\delta, 1 - \delta) \right).
$$

(64)

The local delay for File $f$ can be derived by averaging over $r$ as follows

$$
M_{-1,f} = \int_0^\infty \frac{2 \pi \lambda q_f r}{L_{t_{-f}}(\frac{q_f r^2}{2}) L_{t_{-f}}(\frac{q_f r^2}{2})} \exp \left( -\pi \lambda q_f r^2 \right) dr
= q_f \beta (q_f - \delta(1 - q_f)(1 - \beta)\delta^{-1} \beta \theta \delta B(\delta, 1 - \delta)
- \frac{q_f \beta}{1 - \delta} F_1(1, 1 - \delta; 2 - \delta, -(1 - \beta)\theta) \right)^{-1},
$$

(65)

where (a) follows from the fact that $\int_0^\infty 2re^{-Ar^2} = 1/A$. Note that the backhaul delay can be obtained by substituting $q_f = 1$ into (17). Then the proof is completed.

C. Proof of Lemma 7

When $\theta \to \infty$, the local delay for File $f$ $M_{-1,f}$ can be rewritten as follows

$$
M_{-1,f} = q_f \beta (q_f - \delta(1 - q_f)(1 - \beta)\delta^{-1} \beta \theta \delta B(\delta, 1 - \delta)
- \frac{q_f \beta}{1 - \delta} F_1(1, 1 - \delta; 2 - \delta, -(1 - \beta)\theta) \right)^{-1},
$$

(66)

where (a) is from the fact that

$$
F_1(1, 1 - \delta; 2 - \delta, -(1 - \beta)\theta) = \frac{1 - \delta}{(1 - \beta)\theta} \delta B(\delta, 1 - \delta)
- \frac{1 - \delta}{(1 - \beta)\theta} F_1(1, 1 - \delta; 1 + \delta, \frac{1}{1 - \beta}) \right),
$$

(67)

and (b) follows from $\lim_{\theta \to \infty} F_1(1, 1; 1 + \delta, \frac{1}{1 - \beta}) = 1$.

When $\theta \to 0$, the local delay can be derived as follows

$$
M_{-1,f} = q_f \beta (q_f - \delta(1 - \beta)\delta^{-1} \beta \theta \delta B(\delta, 1 - \delta)
+ \frac{q_f \beta}{1 - \delta} F_1(1, 1 + \delta; \frac{1}{1 - \beta}) \right)^{-1},
$$

(68)

where (a) follows from $F_1(a, b; c; x) = 1 + \frac{ab}{c}x + O(x^2)$ for $x \to 0$. 
D. Proof of Lemma 2

By establishing the Lagrange function of Problem 1, we obtain

\[
L(q, \rho, v, \eta) = D + \sum_{f=1}^{F} \rho_f (q_f - q_c) + \sum_{f=1}^{F} v_f (1 - q_f) + \eta \left( C - \sum_{f=1}^{F} q_f \right),
\]

where \( \rho_f, v_f \) and \( \eta \) are the Lagrange multipliers. \( \rho_f, v_f \) is associated with (1), \( \eta \) is associated with (2). Note that \( \rho \triangleq (\rho_f)_{f \in \mathcal{F}} \) and \( v \triangleq (v_f)_{f \in \mathcal{F}} \). Thus, we have

\[
\frac{\partial L(q, \rho, v, \eta)}{\partial q_f} = -\frac{C_1}{\beta (C_3 q_f - C_1)^2} + \rho_f - v_f - \eta,
\]

The KKT conditions can be written as follows

\[
\frac{\partial L(q^*, \rho, v, \eta)}{\partial q_f} = 0, \forall f \in \mathcal{F},
\]

\[
\rho_f (q_f^* - q_c) = 0, v_f (1 - q_f^*) = 0, \forall f \in \mathcal{F},
\]

\[
\eta (C - \sum_{f=1}^{F} q_f^*) = 0,
\]

\[
\sum_{f=1}^{F} q_f^* = C, 0 \leq q_f^* \leq 1, \forall f \in \mathcal{F},
\]

\[
\rho \geq 0, v \geq 0, \forall f \in \mathcal{F},
\]

where (72) is the complementary slackness and (72) is the dual constraint.

According to (70) and (71), we have

\[
\eta = \frac{-C_1}{\beta (C_3 q_f - C_1)^2} + \rho_f - v_f, \forall f \in \mathcal{F}.
\]

Next, we analyze the optimal solution by considering three cases as follows

1) If \( q_f^* = q_c \), then \( \rho_f \geq 0, v_f = 0 \) and \( \eta = \frac{-C_1}{\beta (C_3 q_f - C_1)^2} + \rho_f \), implying that \( \eta \geq \frac{-C_1}{\beta (C_3 q_f - C_1)^2} \).

2) If \( q_f^* = 1 \), then \( \rho_f = 0, v_f \geq 0 \) and \( \eta = \frac{-C_1}{\beta (C_3 q_f - C_1)^2} - v_f \), implying that \( \eta \leq \frac{-C_1}{\beta (C_3 q_f - C_1)^2} \).

3) If \( 0 < q_f^* < 1 \), then \( \rho_f = 0, v_f = 0 \) and \( \eta = \frac{-C_1}{\beta (C_3 q_f - C_1)^2} \).

Therefore, the proof is completed.

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