Polynomial running times
for polynomial-time oracle machines

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Abstract
This paper introduces a more restrictive notion of feasibility of functionals on Baire space than the established one from second-order complexity theory. Thereby making it possible to consider functions on the natural numbers as running times of oracle Turing machines and avoiding second-order polynomials, which are notoriously difficult to handle. Furthermore, all machines that witness this stronger kind of feasibility can be clocked and the different traditions of treating partial operators from computable analysis and second-order complexity theory are equated in a precise sense. The new notion is named 'strong polynomial-time computability', and proven to be a strictly stronger requirement than polynomial-time computability. It is proven that within the framework for complexity of operators from analysis introduced by Kawamura and Cook the classes of strongly polynomial-time computable operators and polynomial-time computable operators coincide.

Contents
1 Introduction 2
2 Second-order complexity and relativization 4
  2.1 Descriptions of second-order polynomials 6
  2.2 Polynomial majorants 7
  2.3 Relativization 8
  2.4 Incompatibility with relativization and a partial recovery 10
3 Query dependent step restrictions 13
  3.1 Step-counts 13
  3.2 Finite length-revision 15
  3.3 Strong polynomial-time computability 17
  3.4 Compatibility with relativization 18
  3.5 Comparison to polynomial-time on $\Sigma^{**}$ 18
4 Conclusion 20
1 Introduction

Modern applications of second-order complexity theory almost exclusively use time-restricted oracle Turing machines to define and argue about the class of polynomial-time computable functionals \[\text{Lam}06, \text{Kaw}11, \text{FHI}13, \text{FGH}14, \text{FZ}15, \text{KSZ}16b, \text{SS}17, \text{etc.}\]. The acceptance of this model of computation goes back to a result by Kapron and Cook \[\text{KC}96\] that characterizes the class of basic feasible functionals introduced by Mehlhorn \[\text{Meh}76\].

There are several reasons for the popularity of this model of computation. Firstly, it intuitively reflects what a programmer would require of an efficient program if oracle Turing machines are interpreted as programs with subroutine calls. I.e. the time taken to evaluate the subroutine is not counted towards the time consumption (the oracle query takes one time step) and if the result is complicated the machine is given more time for further operations. Secondly, it is superficially quite close to classical polynomial-time computability: There is a type of functions that take sizes of the inputs and return an allowed number of steps. A subclass of these functions are considered polynomial, or ‘fast’ running times.

On closer inspection, however, this generalization introduces a whole bunch of new difficulties: Running times of oracle Turing machines, and also the functions that are considered polynomial running times, are functions of type \(\omega^\omega \times \omega \rightarrow \omega\). These so-called second-order polynomials are a lot less well-behaved than their first-order counterparts. There are no normal-form theorems, structural induction turns out to be complicated, there is no established notion of degree and so on \[\text{KP}14, \text{KSZ}16a\]. Even worse: Second-order polynomials turn out not to be time-constructible \[\text{SS}17\].

The framework introduced by Kawamura and Cook \[\text{KC}12\] addresses this problem by restricting to length-monotone string functions, thereby forcing time-constructibility of second-order polynomials. However, it has been argued that the restriction to length-monotone string functions seems to be an unnatural one in practice \[\text{SS}17\] and that it is too restrictive to reflect some situations from practice \[\text{BS}17\]. Thus, this paper investigates different solutions to the above problems.

The content of this paper

The first part of the paper investigates the boundaries of the polynomial-time framework. Descriptions of second-order polynomials are introduced as a replacement of a normal-form theorem which currently seems to be out of reach. They turn out to be a very useful tool to prove existential claims about second order polynomials. In particular it allows us to prove the existence of polynomial majorants: For any second-order polynomial there is a polynomial and a number such that the values of the second-order polynomial can be bounded by an easy formula only involving these. The polynomial majorants come in handy in several of the later parts of the paper.

Then complexity of partial functionals is investigated. The traditions of how
to handle partial operators differ a lot between computable analysis and second-order complexity theory. The two corresponding notions are introduced and the new tool-set for second-order polynomials is put to a use to prove that there really is a difference. This can be considered to be a very strong version of the statement that second-order polynomials are not time-constructible. Finally, it is proven that in the most important example of use of an intermediate of the two conventions, namely the framework for complexity of operators in analysis as introduced by Kawamura and Cook, could have equivalently used the convention from second-order complexity theory.

The second part of the paper presents a restriction on the behavior of oracle Turing machines such that use of running time of higher type is not necessary anymore. It proves that the corresponding class of functionals, which are named ‘strongly polynomial-time computable functionals’, is a subclass of the class of polynomial-time computable functionals as defined in second-order complexity theory. It provides an example of an operator that is polynomial-time computable but not strongly polynomial-time computable. The example is not a natural example, but it is pointed out without a proof that there is a candidate for a more natural example. Finally it presents some evidence that strong polynomial-time computability is more compatible with partial operators and proves that within the framework for complexity of operators in analysis introduced by Kawamura and Cook, it is equivalent to polynomial-time computability.

Conventions

Fix the finite alphabet $\Sigma := \{0, 1\}$ and let $\Sigma^*$ denote the set of finite binary strings. Elements of $\Sigma^*$ are denoted by $a, b, \ldots$. The set of non-negative integers is denoted by $\mathbb{N}$ or by $\omega$ if they are used as size measurements (making this difference stems from complexity theory where elements of $\mathbb{N}$ are usually encoded in binary and elements of $\omega$ in unary). Elements of both sets are denoted by $n, m, \ldots$. We identify the Baire space with the set $\mathcal{B} := (\Sigma^*)^{\Sigma^*}$ of string functions. Elements of $\mathcal{B}$ are denoted as $\phi, \psi, \ldots$. We assume the reader to be familiar with the notions of computability and complexity theory for elements of the Baire space introduced via Turing machines.

To compute functions on the Baire space, this paper uses oracle Turing machines: An oracle Turing machine $M^\varphi$ is a Turing machine that has an additional oracle query tape and an oracle query state. The oracle slot may be occupied by any string function $\varphi \in \mathcal{B}$. If the computation of $M^\varphi$ with oracle $\varphi$ and input $a$, also referred to as the run of $M^\varphi$ on $a$, enters the query state, the content of the oracle query tape, say $a$, is replaced with the value $\varphi(a)$. For measuring time consumption, overwriting $a$ with $\varphi(a)$ is considered to be done in one time step and the reading/writing head is not moved. The outcome of the run of $M^\varphi$ with oracle $\varphi$ on input $a$ is denoted by $M^\varphi(a)$. For $A \subseteq \mathcal{B}$ an functional $F : A \rightarrow \mathcal{B}$ reps. a functional $\tilde{F} : A \times \Sigma^* \rightarrow \Sigma^*$ if for all $\varphi \in A$ and $a \in \Sigma^*$ it holds that $F(\varphi)(a) = M^\varphi(a)$ resp. $\tilde{F}(\varphi, a) = M^\varphi(a)$.
2 Second-order complexity and relativization

Oracle Turing machines may be understood to compute functionals of type \( B \times \Sigma^* \rightarrow \Sigma^* \). For complexity considerations, computations on ‘big’ oracles \( \varphi \) should be granted more time. The size of an oracle is not a natural number anymore, but a function on the natural numbers:

**Definition 2.1** For a string function \( \varphi \in B \), define its size function \(|\varphi| : \omega \rightarrow \omega \) by

\[
|\varphi| (n) := \max\{|\varphi(a)| \mid |a| \leq n\}.
\]

Thus, running times are objects of the type \( T : \omega^\omega \times \omega \rightarrow \omega \): \( T \) is a running time for an oracle Turing machine \( M^\varphi \) if for any oracle \( \varphi \) and string \( a \), the run of \( M^\varphi \) on input \( a \) terminates within \( T(|\varphi|, |a|) \) steps. Note that it is not a priori clear what running times should be considered polynomial. The class of second-order polynomials is the smallest class of functions \( P : \omega^\omega \times \omega \rightarrow \omega \) such that:

- All of the functions \((l,n) \mapsto p(n)\) are contained, where \( p \) is a polynomial with natural numbers as coefficients.

And which is closed under the following operations:

- Whenever \( P \) and \( Q \) are contained, then so is their point-wise sum \( P + Q \).
- Whenever \( P \) and \( Q \) are contained, then so is their point-wise product \( P \cdot Q \).
- Whenever \( P \) is contained then so is the function \( P^+ \) defined by
  \[
P^+(l,n) := l(P(l,n)).
\]

**Definition 2.2** A functional on Baire space is called *polynomial-time computable* if there is an oracle Turing machine that computes it and has a second-order polynomial as running time.

The above definition is based on a characterization by Kapron and Cook of the class of basic feasible functionals originally introduced by Mehlhorn. Note that, while being obvious from Mehlhorn’s original definition, it is not obvious from the above definition that the class of polynomial-time computable functionals is closed under composition. We need a generalization of this statement later and thus give a proof relying on the above definition.

**Proposition 2.3** Let \( F \) and \( G \) be functionals on Baire space that can be computed within polynomial-times \( P \) and \( Q \). Then \( F \circ G \) can be computed in time

\[
(l,n) \mapsto C \cdot (Q(l,\cdot), n) + Q(l, P(Q(l,\cdot), n)) \cdot P(Q(l,\cdot), n))
\]

for some \( C \in \omega \).
Proof Let $M^2$ and $N^2$ be machines that compute the operators $F$ and $G$ in polynomial time. The machine to compute $F \circ G$ proceeds on oracle $\varphi$ and input $a$ by first following the steps that $M^1$ takes on input $a$ but replacing the oracle query tape with a memory tape. On oracle call it switches to carrying out the computation of $N^\varphi$ with the content of the memory tape as input and replacing the output tape by a memory tape. When the computation has terminated it returns to carry out the computation of $M^2$ but instead of reading form the oracle answer tape reads from the memory tape.

To see that the machine finishes within the specified time, note that, since $N^2$ computes $G$, the computation of $M^1$ is runs as if it was given $G(\varphi)$ as oracle. Now, from the running time restriction of $G$, it follows that $|G(\varphi)| (n) \leq Q(|\varphi|, n)$. Thus, at most $P(Q(|\varphi|, \cdot), |a|)$ steps are spent carrying out the operations of the machine $M^2$. This is also a bound for the number of oracle queries made and the size of the queries. Finally, due to the running time restriction of $F$ and these bounds, the number of steps carried out imitating the machine $N^\varphi$ is bounded by

$$P(Q(|\varphi|, \cdot), |a|) \cdot Q(|\varphi|, P(Q(|\varphi|, \cdot), |a|))$$

Adding these, and accounting for the additional memory tapes which have to be simulated, leads to the time bound from the statement.

The proof that the above running time is indeed a second-order polynomial is quite troublesome when carried out by induction on the term-structure, however, an easy argument is given in Proposition 2.8 in the next section.

Lemma 2.4 The polynomial-time computable operators are closed under composition.

Another property of polynomial time computable functionals that should be mentioned is that they preserve the class of polynomial-time computable functions. This can easily be proven by combining the program of a polynomial-time machine computing the function with the program of a polynomial-time oracle Turing machine computing the functional.

Second-order polynomials were introduced as functions of type $\omega^\omega \times \omega \rightarrow \omega$. This is natural since they are considered running times. However, it also regularly leads to difficulties: It is not clear how to decide equality of two second-order polynomials from the construction procedures. The reader may for instance try to prove that the inequality $P \neq Q$ of two second-order polynomials as functions implies that also $P^+ \neq Q^+$. While a proof for the general case is not known to the authors, it is possible to prove this in the case where $P \neq Q$ is realized by a strictly monotone function argument. Note, that while it is not an unreasonable idea to restrict the domain of the second order polynomials, it should at least contain all (not necessarily strictly) monotone functions, as these show up as length functions of string functions. Just like for the general case, a proof of the above if the inequality is realized by a monotone function is not known to the authors. This leads to problems when trying to recursively define functions on the second-order polynomials.
This paper handles these difficulties by using descriptions of how to construct second-order polynomials instead.

2.1 Descriptions of second-order polynomials

This section presents some technical results about second-order polynomials that are needed several times throughout the paper. On first reading it may be skipped and rolled back to when the results are needed. Note that all uses of the closure under addition and multiplication rules that happen between two applications of the function argument application rule can be bundled together to applying a multivariate polynomial. This paper always assumes that second order polynomials are specified in the following way:

**Definition 2.5** A polynomial tree is a finite tree $T$ whose nodes are elements of $\mathbb{N}[X_0, \ldots, X_k]$ where $k$ coincides with the number of children the node has and there is a specified linear order on the children of each node.

Recursively assign to each node of a polynomial tree a second-order polynomial: To a leaf $t$ assign the second order polynomial $(l, n) \mapsto t(n)$. Now assume that second-order polynomials $P_1, \ldots, P_k$ were assigned to each of the children $t_1, \ldots, t_k$ of a node $t$. Assign to $t$ the second-order polynomial

$$(l, n) \mapsto t(n, l(P_1(l, n)), \ldots, l(P_k(l, n))) = t(n, P_1^+(l, n), \ldots, P_k^+(l, n)).$$

**Definition 2.6** A polynomial tree is called a description of a second-order polynomial $P$ if $P$ is assigned to the root of the tree by the above procedure.

Note that there may be many polynomial trees such that the same second-order polynomial is assigned. For instance both of the polynomial trees on the right hand side get the second-order polynomial $(l, n) \mapsto 2l(n)$ assigned to the root. Whether or not these ambiguities can completely be avoided depends on whether or not the operation $P \mapsto P^+$ is injective and is briefly discussed at the end of the section.

An easy structural induction proves:

**Lemma 2.7** Every second-order polynomial has a description.

**Proof** For the base case note that the a description consisting of a single node $p \in \mathbb{N}[X_0]$ is a description of the second-order polynomial $(l, n) \mapsto p(n)$.

To obtain a description of the point-wise sum $P + Q$ from descriptions of $P$ and of $Q$, let $t_P \in \mathbb{N}[X_0, \ldots, X_k]$ be the polynomial at the root of $P$’s description and $t_Q \in \mathbb{N}[X_0, \ldots, X_m]$ the polynomial at the root of $Q$’s description. A description of $P + Q$ is given by merging the root of the two descriptions to a node labeled with the polynomial

$$i(X_0, \ldots, X_{k+m+1}) := t_P(X_0, \ldots, X_k) + t_Q(X_{k+1}, \ldots, X_{k+m+1}).$$

For the point-wise product replace $t_P \cdot t_Q$ in the above procedure by $t_P \cdot t_Q$. 

6
Finally note that if \( P \) is a second order polynomial and \( T \) a description of \( P \) then adding a single node containing the polynomial \( X_1 \) above the root of \( T \) is a description of \( P^+ \).

This enables us to close a gap in the previous section:

**Proposition 2.8** Whenever \( P \) and \( Q \) are second-order polynomials, then so are \( (l, n) \mapsto P(Q(l, \cdot), n) \) and \( (l, n) \mapsto P(l, Q(l, n)) \).

**Proof** A description of the latter can be specified by replacing each leaf \( p \) of a description of \( P \) with a description of \( Q \) where the root \( t \) of \( Q \) is replaced by \( p \circ t \). For the former one each edge of in a description of \( P \) has to be replaced with a description of \( Q \) (where a copy of the part of the description of \( P \) below the edge is appended to each leaf of the description of \( Q \) and there are compositions again in the roots and the leaves).

The extent of ambiguity in descriptions is closely connected to injectivity of the mapping \( P \mapsto P^+ \). Which is, as mentioned before, an open question to the knowledge of the authors. If injectivity holds, it is likely that it is possible to use descriptions to construct a normal form for second-order polynomials. Such a normal form theorem is highly desirable, since it would allow to define functions on the second-order polynomials by defining them on the normal forms. Some authors go as far as restricting to strictly monotone functions to make this possible [KP14].

### 2.2 Polynomial majorants

As an example of a quantity that is well-defined on descriptions and of use later in the paper consider the following:

**Definition 2.9** A pair \((N, p)\) is called a **polynomial majorant** of a second order polynomial \( P \) if \( p(n) \geq n \) and there exists a description \( T \) of \( P \) such that

- \( N \) is the height of the tree \( T \).
- For each integer \( n \) and each node \( t \) of the tree \( T \) it holds that \( p(n) \geq t(n, \ldots, n) \).

The following is the reason for the name ‘majorant’:

**Lemma 2.10** Let \((N, p)\) be a polynomial majorant of a second-order Polynomial \( P \). Define a sequence of functions \( p_i : \omega^\omega \times \omega \rightarrow \omega \) recursively by

\[
p_0(l, n) := p(n) \quad \text{and} \quad p_{i+1}(l, n) := p(\max\{n, l(p_i(n))\}) \}
\]

Whenever \( l : \omega \rightarrow \omega \) is monotone and \( n \in \omega \) is arbitrary it holds that

\[
P(l, n) \leq p_N(l, n).
\]

7
The proof proceeds by induction over the height of the description witnessing that \((N, p)\) is a polynomial majorant.

For height 0 the second order polynomial is of the form \((l, n) \mapsto q(n)\) for some polynomial \(q\). By the assumption that \((N, p)\) is a polynomial majorant of \(P\) it follows that

\[
P(l, n) = q(n) \leq p(n) = p_0(l, n).
\]

Next assume that the statement has been proven for all descriptions of height \(n < N\). Note that each of the \(k\) children of the root can be regarded as a root of a description \(T_k\) of a second-order polynomial \(Q_k\). Each \(T_k\) is a proper subtree of \(T\), thus its height \(n_k\) is strictly smaller than \(Q_k\). From the induction hypothesis it follows that for all \(l : \omega \rightarrow \omega\) and \(n \in \omega\)

\[
Q_k(l, n) \leq p_{n_k}(l, n).
\]

Let \(q\) be the polynomial at the root of \(T\). Thus,

\[
P(l, n) = q(n, l(Q_1(l, n)), \ldots, l(Q_k(l, n)))
\]

First note that for all monotone \(l\) it holds that \(p_i(l, n) \leq p_{i+1}(l, n)\). Since \((N, p)\) is a polynomial majorant of \(P\) it holds that \(p(m) \geq q(m, \ldots, m)\). Therefore, under the assumption that \(l\) is monotone, it holds that

\[
P(l, n) \leq q(n, l(p_{N-1}(l, n)), \ldots, l(p_{N-1}(l, n)))
\]

\[
\leq p(\max\{n, l(p_{N-1}(l, n))\})
\]

\[
= p_N(l, n).
\]

This proves the assertion. \(\blacksquare\)

Each second-order polynomial has a description. From a description a polynomial majorant can be constructed by taking \(N\) to be the height of the tree and \(p\) to have as coefficients the maximum of the coefficients of the polynomials that arise from the nodes of the description by setting each of the variables to \(n\).

This proves:

**Lemma 2.11** Any second-order polynomial has a polynomial majorant.

### 2.3 Relativization

Second-order complexity theory usually only considers total functionals. However, the application we are most interested in is real complexity theory, which stems from computable analysis. In computable analysis, computations on continuous structures are carried out by encoding the objects by string functions and operating on these. In this process, partial functionals are used. To see how, we recall the most basic notions.

**Definition 2.12** A representation \(\xi\) of a space \(X\) is a partial surjective mapping \(\xi : B \rightarrow X\).
An element of $\xi^{-1}(x)$ is called a $\xi$-name of $x$ or just a name, if the representation is clear from the context. A pair $X = (X, \xi_X)$ of a set and a representation of that set is called a represented space.

Computations on represented spaces can be carried out by operating on names:

**Definition 2.13** Let $f : X \to Y$ be a function between represented spaces. A partial functional $F : \subseteq B \to B$ is called a realizer of $f$ if it translates $\xi_X$-names of $x$ to $\xi_Y$-names of $f(x)$, that is if

$$\forall \varphi \in \text{dom}(\xi_X) : \xi_Y(F(\varphi)) = f(\xi_X(\varphi)).$$

A function is then called computable if it has a computable realizer. Here it is tradition not to make any assumptions about the behavior of the realizer outside of the domain of $\xi_X$. This poses a difficulty when going to complexity as one would like to use second-order complexity theory where traditionally only total functionals are considered. However, the characterization polynomial-time computability of a functional by Kapron and Cook can straightforwardly be relaxed in an appropriate way.

**Definition 2.14** Let $A \subseteq B$. We say that an oracle Turing machine $M^\varphi$ runs in $A$-restricted polynomial-time if there exists a second-order polynomial $P$ such that for each oracle $\varphi$ from $A$ and every string $a$ the run of $M^\varphi$ on input $a$ terminates after at most $P(|\varphi|, |a|)$ steps. We denote the set of all operators $F : A \to B$ such that there exists a machine that computes $F$ in $A$-restricted polynomial time by $P(A)$.

There are two main examples of this definition covertly showing up in literature:

**Example 2.15 (relativization)** Oracle machines are used in classical complexity theory to talk about polynomial-time computability of a string function $\varphi : \Sigma^* \to \Sigma^*$ relative to some oracle $\psi : \Sigma^* \to \{0, 1\}$ interpreted as a subset of the strings. Under the assumption that $\psi$ only returns 0 or 1, one can check that the following are equivalent:

- $\varphi$ is polynomial-time computable relative to $\psi$.
- The constant functional returning $\varphi$ is $\{\psi\}$-restricted polynomial-time computable.

This is the reason for the name of this chapter and remains true as long as $\psi$ has at most polynomial length.

The second example is Kawamura and Cook’s framework for complexity for operators in analysis. Recall that Kawamura and Cook introduce the following subclass of Baire space:

**Definition 2.16 ([KC12])** A string function $\varphi \in B$ is called length-monotone if for all strings $a$ and $b$ it holds that

$$|a| \leq |b| \Rightarrow |\varphi(a)| \leq |\varphi(b)|.$$

The set of all length-monotone string functions is denoted by $\Sigma^{**}$. 


Polynomial-time computability of functionals from $\Sigma^{**}$ to $\Sigma^{**}$ is then defined as $\Sigma^{**}$-restricted polynomial-time computability. (Of course it is not referred to by this name, but the definitions are identical.)

The tradition in second-order complexity theory is to impose the running time requirement independently of the domain of the operator.

**Definition 2.17** For $A \subseteq B$ denote the class of all functionals $F: A \to B$ that have a polynomial-time computable extension to all of Baire space by $P|_A$.

For an operator $F: A \to B$ there are now two approaches to define polynomial-time computability. One could require that $F$ is $A$-restricted polynomial-time computable, i.e., $F \in P(A)$, or one could use the more restrictive definition that $F$ has a total polynomial-time computable extension, i.e., $F \in P|_A$. The first definition fits into the tradition of computable analysis, where usually no assumptions about a realizer are made outside of the domain of the operator it realizes. The second definition is in the tradition of second-order complexity theory, where one usually only considers polynomial-time computability of total operators.

### 2.4 Incompatibility with relativization and a partial recovery

Of course, the above distinction only makes sense if the classes $P(A)$ and $P|_A$ differ in general. Note that by definition $P(A) \supseteq P|_A$. The basic idea towards proving that this inclusion is proper is to take a functional which is known to not be polynomial-time computable, for instance the length function, and restrict it to inputs of exponential length. However, one should note that restricting the domain increases the number of machines that are considered to compute the functional. And indeed the restriction of the length function to oracles of exponential length can be computed as follows: Do a brute-force search as long as you do not have to ask more than twice as many oracle queries than the length of the biggest return value you have found so far. If you need more steps, terminate on output $\varepsilon$. This machine does indeed compute the restriction of the length function on the exponentially growing functions while running in polynomial time for all inputs and returning something that differs from the length on the shorter functions (this is allowed since they are not in the domain). Thus, the argument has to be more elaborate.

To prove that the other inclusion fails is our first application of the polynomial majorants:

**Theorem 2.18 (in general $P|_A \subsetneq P(A)$)** There exist a set $E \subseteq B$ and an operator $F: E \to B$ such that $F$ is $E$-restricted polynomial-time computable but has no total polynomial-time computable extension.

**Proof** Consider the set

$$E := \{ \varphi \in B \mid \forall n \in \omega : |\varphi| (2^{2^n}) \geq 2^{2^{2n}} \}$$
and the operator on $E$ defined by
\[ F : E \to B, \quad F(\varphi)(a) := 0^{|\varphi|(|a|)}. \]

$F$ is $E$-restricted polynomial-time computable. To see that this is true first note that $3(n + 2) \geq 2^{2^{\lceil \log(3(n+2)^2) \rceil} - 1} \in \mathbb{N}$ and $\log(n^2) \leq 3(n + 2)$ (this is implied by the inequality $\ln(x) \leq \frac{e-1}{\sqrt{x}}$). Thus, for $\varphi \in E$ it holds that
\[
|\varphi|(|\varphi|(3(n + 2))) \geq |\varphi|(|\varphi|(2^{2^{\lceil \log(3(n+2)^2) \rceil} - 1})) \geq |\varphi|(2^{2^{\lceil \log(3(n+2)^2) \rceil} - 1}) \\
\geq 2^{2^{2^{\lceil \log(3(n+2)^2) \rceil} - 1}} \geq 2^{2^{\sqrt{3(n+2)}}} \geq 2^n
\]

This means that a second-order polynomial provides sufficient time to find the value of $|\varphi|(|a|)$ in $E$-restricted polynomial time using a brute-force search.

However, $F$ does not have a total polynomial-time computable extension, as can be seen as follows: Towards a contradiction assume that there is an oracle Turing machine $M'$ that computes $F$ in time bounded by some second-order polynomial $P$. For each $n \in \omega$ define an oracle $\varphi_n \in E$ as follows: First define a sequence of functions $\varphi_{n,k}$ as follows: Let $\varphi_{n,0}$ be the constant function returning $\varepsilon$ and define $\varphi_{n,k+1}$ to return the same values as $\varphi_{n,k}$ unless the computation of $M'$ with oracle $\varphi_{n,k}$ and input $0^n$ does an oracle query $a$ of length $2^{2m}$ for some integer $m$, has done all queries of this length before and $\varphi_{n,k}$ returns $\varepsilon$ for all these queries. If this is the case, then let $\varphi_{n,k+1}$ return a string of length $2^{2^{2m}}$ zeros. Since the machine terminates, the sequence is eventually constant. Let $k_0$ big enough such that the sequence does not change anymore afterwards.

Let $\varphi_n$ be the function that is identical to $\varphi_{n,k_0}$ unless $\varphi_{n,k_0}$ vanishes on all inputs of some length $2^{2^m}$. In this case the machine did not ask all the queries of length $2^{2^m}$. Pick one query of this length that was not asked and let $\varphi_n$ return the string of $2^{2^{2^m}}$ zeros on this string. This guarantees that $\varphi_n \in E$.

Let $m(n)$ be the least integer such that for $k(n) := 2^{2^{m(n)}}$ the number of steps of $M^\varphi(0^n)$ is less than $2^{k(n)}$. Since the machine can not ask all queries of length $k(n)$, the oracle $\varphi_n$ can be replaced with an oracle $\psi_n$ such that $|\psi_n|$ is bounded by $2^{\sqrt{k(n)}}$ (due to the double exponential, this is an integer) and the runs $M^{\varphi_n}(0^n)$ and $M^{\psi_n}(0^n)$ are identical. It follows that
\[ 2^{k(n)} \leq P(|\psi_n|, n). \]

Now let $(N, p)$ be a polynomial majorant that exists by Lemma 2.11. Since being a polynomial majorant is preserved under increasing the polynomial we may assume that $p(n) \geq n$. An easy induction together with Lemma 2.10 proves that whenever $l$ is monotone and bounded by $k$, i.e. $l(m) \leq k$ for all $m \in \omega$, then
\[ P(l, m) \leq \max\{p^N(m), p^{N-1}(k)\}. \]
Therefore,
\[ 2^{k(n)} \leq P(|\psi_n|, |a|) \leq \max\{p^N(n), C2^{d\sqrt{k(n)}}\} \]

11
holds for all \( n \) and appropriate \( C, d \in \omega \). It can be easily checked that the maximum on the right hand side has to assume the value \( p^N(n) \) whenever \( k(n) > (2\text{lb}(C) + d^2 + d\sqrt{2\text{lb}(C) + d^2})/2 \). Thus \( 2^{k(n)} \) is majorized by a polynomial \( q \). Choose some \( N \) such that \( q(N) < 2^N \). Since \( 2^{k(N)} \leq q(N) < 2^N \) is a bound on the number of steps of the computation of \( M' \) on oracle \( \varphi_N \) and input \( 0^N \), the machine \( M^{\varphi_N} \) can on input \( 0^N \) not query \( \varphi_N \) in all strings of length \( N \). Force an invalid output by replacing \( \varphi_N \) with an oracle that only differs in its value on one of the unasked queries and is huge there.

Therefore, for an arbitrary set \( A \subseteq \mathcal{B} \), it can not be expected that every \( A \)-restricted polynomial-time computable operator has a total polynomial-time computable extension. However, \( \Sigma^{**} \) is far from being an arbitrary set. Recall the following notion:

**Definition 2.19** Let \( A \) be a subset of \( \mathcal{B} \). A mapping \( R: \mathcal{B} \rightarrow A \) is called a retraction of \( \mathcal{B} \) onto \( A \), if for all \( \varphi \in A \) it holds that \( R(\varphi) = \varphi \).

A property of \( \Sigma^{**} \) that guarantees the existence of total polynomial-time computable extensions is the following:

**Lemma 2.20** There is a polynomial-time computable retraction from \( \mathcal{B} \) onto \( \Sigma^{**} \).

**Proof** For a string \( a \) let \( \hat{a} \leq n \) denote its initial segment of length \( n \) (or the string itself if it has less than \( n \) bits). Consider the mapping

\[
R(\varphi)(a) := \varphi(\hat{a}) \leq |\varphi(0^n)|q_{\text{max}}\{|\varphi(0^n)|-|\varphi(a)|,0\}.
\]

This mapping is a polynomial-time computable retraction from \( \mathcal{B} \) onto \( \Sigma^{**} \).

**Theorem 2.21** Whenever there is a polynomial-time computable retraction from \( \mathcal{B} \) onto \( A \), then any \( A \)-restricted polynomial-time computable operator has a total polynomial-time computable extension.

**Proof** Note that the proof that the composition of two polynomial-time computable operators is polynomial-time computable remains valid if the assumptions are weakened to \( F \) being \( G(\mathcal{B}) \)-restricted polynomial-time computable. Thus, the composition of the \( A \)-restricted polynomial-time computable operator with the retraction is polynomial-time computable.

The previous two results directly entail the following:

**Corollary 2.22** (\( P(\Sigma^{**}) = P|_{\Sigma^{**}} \)) An operator \( F : \Sigma^{**} \rightarrow \mathcal{B} \) is polynomial-time computable in the sense of Kawamura and Cook if and only if it has a total polynomial-time computable extension.
3 Query dependent step restrictions

Recall that for a regular Turing machine the time function \( \text{time}_M : \Sigma^* \rightarrow \omega \) is defined to return on input \( a \) the number of steps that it takes until the machine terminates on input \( a \). A running time of the machine is then defined to be a function \( t : \omega \rightarrow \omega \) such that

\[
\forall n \in \omega, \forall a \in \Sigma^* : |a| \leq n \Rightarrow \text{time}_M(a) \leq t(n). \tag{rt}
\]

For an oracle Turing machine, each of the time functions \( \text{time}_{M^{\varphi}} \) may be different. Thus, the above definition has to be replaced. The most common one is that of second-order complexity theory and is discussed in the first part of the paper. Therefore it requires a new definition of what it means for a running-time to be ‘polynomial’.

However, there exist other approaches of how to do this in literature. Some of them stay with functions of type \( \omega \rightarrow \omega \) for running times. So does the notion this part of the paper introduces. To distinguish these objects from the time function and the running times from second-order complexity theory, we refer to such objects as ‘step-counts’ instead of ‘running times’. One example of a definition in this vein is from Stephen Cook [Coo91]. He bounds the steps an oracle Turing machine may take by modifying Equation (rt) as follows: He replaces \( M \) by \( M^{\varphi} \), \( |a| \) by the maximum of \( |a| \) and the biggest length of any return value of the oracle in the run of \( M^{\varphi} \) and additionally universally quantifies over \( \varphi \in \mathcal{B} \). He refers to the class of functionals that can be computed by a machine that has a polynomial bound in the above sense as \( \text{OPT} \) (for ‘oracle polynomial time’).

3.1 Step-counts

We use a slightly more complicated definition that turns out to be considerably more well-behaved.

**Definition 3.1** Let \( M^? \) be an oracle Turing machine. For a given oracle \( \varphi \) and a given input \( a \) denote the content of the oracle answer tape in the \( k \)-th step of the computation by \( b_k \). Define the length revision function \( o_{\varphi,a} : \omega \rightarrow \omega \) recursively as follows:

\[
o_{\varphi,a}(0) := |a| \quad \text{and} \quad o_{\varphi,a}(n+1) := \max\{o_{\varphi,a}(n), |b_{n+1}|\}.
\]

Note that \( o_{\varphi,a}(k+1) > o_{\varphi,a}(k) \) means that in the \( k \)-th step of the computation, the machine asked an oracle query and the answer was bigger than both the input \( a \) and any of the answers the oracle has given earlier in the computation. We call this a length revision as it means that it became apparent to the machine that its input is bigger than what the previous evidence indicated.

For an oracle Turing machine \( M^? \) with a fixed oracle \( \varphi \in \mathcal{B} \) let \( \text{time}_{M^{\varphi}}(a) \in \omega \cup \{\infty\} \) be the number of steps that the computation of \( M^{\varphi} \) takes on input \( a \). I.e. the machine is explicitly allowed to diverge on some inputs.
Figure 1: Verifying that \( \varphi \) and \( a \) are not a counterexample of \( t \) being a step-count. Under the assumption that \( t \) is invertible on the set \([t(0), \infty)\).

**Definition 3.2 (compare fig. 1)** A function \( t: \omega \rightarrow \omega \) is a step-count for an oracle Turing machine \( M \) if

\[ \forall \varphi \in \mathcal{B}, \forall n \in \omega, \forall a \in \Sigma^* : n \leq \text{time}_{M\varphi}(a) \Rightarrow n \leq t(o_{\varphi,a}(n)). \]

Denote the set of all operators on the Baire space that can be computed by an oracle Turing machine that has a polynomial step-count by PSC.

Note that in contrast to Equation (rt), the above is not void if the machine diverges on some inputs. The relationship between termination of a machine and the existence of a step-count is quite involved. For instance: If a machine has a step-count and diverges, then the machine queries the oracle an infinite number of times. Furthermore, if there is an integer bound on the length of all return values of an oracle, then every machine that has a step-count terminates when given that oracle and an arbitrary input.

The notion introduced by Cook (discussed before these definitions and which lead to the class OPT) can be reproduced by instead of requiring the above for all \( n \in \omega \) only considering the case \( n = \text{time}_{M\varphi}(a) \). In upcoming proofs it is used that it is possible to clock a machine while basically maintaining the same step-count by checking in each step, that the requirement above is fulfilled. Note that this is not possible for the machines used by Cook without increasing the step-count considerably, as his framework allows to retroactively justify high time-consumption early in the computation by a big oracle answer late in the computation.

The very example that Cook used to disregard the class OPT as a candidate for the class of polynomial-time functionals can be used to also disregard
the class PSC of functionals that are computed by a machine that allows a polynomial step-count:

**Example 3.3 (PSC \( \not\subseteq P \))** The total functional \( F : B \to B \) defined by

\[
F(\varphi)(a) := \varphi^{\left|a\right|}(0)
\]

can be computed by an oracle Turing machine that has a polynomial step-count but does not carry polynomial-time computable input to polynomial-time computable output.

To see that this machine has a polynomial step-count, note that it can be computed by the machine that proceeds as follows: It copies the input to the memory tape and writes 0 to the oracle query tape. Then as long as the memory tape is not empty it repeats the following steps: First copies the content of the oracle answer band to the oracle query band. Then it removes the content of the last non-empty cell from the memory band. Finally it enters the oracle query state. When the memory tape is empty it copies the content of the oracle answer band to the output tape and enters the termination state.

Copying a string of length \( n \) takes \( O(n) \) steps. The length of the string that has to be copied is always bounded by the previous oracle answers. The loop is carried out exactly \( |a| \) times. Therefore, there is some step-count in \( O(n^2) \).

To verify that the functional does not preserve the class of polynomial-time computable functionals consider the polynomial-time computable functional \( \psi(a) := aa \). Note that

\[
F(\psi)(a) = \psi^{\left|a\right|}(0) = 0^{2^{\left|a\right|}}.
\]

Therefore, writing \( F(\psi)(a) \) takes at least \( 2^{\left|a\right|} \) steps and thus \( F(\psi) \) can not be polynomial-time computable.

This means that further restrictions are necessary. The most popular way to introduce such is to use the accepted framework for second-order complexity instead. (This is for instance what [Coo91] does at this point). However, this paper presents a different set of restrictions that can be used.

### 3.2 Finite length-revision

Let \( M^? \) be an oracle Turing machine that always terminates. Then for any oracle \( \varphi \) and any string \( a \) the computation of \( M^\varphi \) on \( a \) is finite and only queries the oracle a finite number of times. This implies that the following statement about the length revision functions of \( M^? \) holds true in this case:

\[
\forall \varphi \in B, \forall a \in \Sigma^* : \exists N \in \mathbb{N} : \#o_{\varphi,a}(\omega) \leq N.
\]

Of course, in general the \( N \) depends on the choice of the oracle and the string. Our restriction on the behavior of the machine is basically that there is an \( N \) that works independently of the choice of the oracle and the input.
Definition 3.4 We say that an oracle Turing machine $M^?$ has finite length-revision if there is an integer $N$ such that no matter what the oracle and the input are, no more than $N$ length revisions happen. That is, if its length revision functions $o_{\varphi,a}$ fulfill

$$\exists N \in \mathbb{N} : \forall \varphi \in B, \forall a \in \Sigma^* : \#o_{\varphi,a}(\omega) \leq N.$$ 

We denote the set of all functionals on the Baire space that can be computed by machines with finite length-revision by $\text{FLR}$.

Finite length revision does a priori neither restrict the number of oracle questions nor the length of the oracle answers: The restriction is that there is a finite number of length revisions, that is, only a finite number of times it happens that a query is asked such that the answer is strictly bigger than the input and any earlier oracle answer.

Example 3.5 ($\text{P} \not\subseteq \text{FLR}$) Consider the operator

$$F : B \rightarrow B, \quad F(\varphi)(a) := 0^\max(|\varphi(0^n)|, n \leq |a|).$$

The straightforward implementation asks $n$ queries, compares their lengths and returns the maximum. This can be done in time $P(l, n) = C(n + n \cdot l(n)) + C$ for some $C$. However, since $|\varphi(0^n)|$ may be strictly increasing when $n$ increases, this machine does not have finite length revision.

Indeed, no machine with finite length revision can compute $F$, as can be proven via contradiction as follows: Assume that there was such a machine $M^?$. Let $N$ be a bound on the length-revisions $M^?$ does. Define an oracle such that the output of $M^?(0^{N+1})$ is incorrect as follows: Let $a_1$ be the first oracle query that is asked in the run of the machine $M^?(0^N)$. Set $\varphi(a_1) := 0^{N+1}$. Thus, a length-revision happens. Let $a_2$ be the next oracle query that the machine poses. Set $\varphi(a_2) := 0^{N+2}$. This means that another length revision happens. Carry on in that way until $\varphi(a_N)$ is set to $0^{2N}$. After asking the query $a_N$, the machine can not ask another query as we may as well set the return value to be bigger again and no further length revision is allowed.

Note that the run of the machine on $0^N$ is identical for any oracle that fulfills $\psi(a_i) = 0^{N+i}$. Let $M$ be the number of steps the machine $M^N$ takes for any of these oracles to terminate. There are $N + 1$ strings of the form $0^n$ for $n \leq N$. Thus, at least one of these strings is not contained within $a_1, \ldots, a_N$. Let $0^m$ be this string. Let $\varphi$ be the string function defined as follows:

$$\varphi(b) = \begin{cases} 
0^{N+i} & \text{if } b = a_i \\
0^{M+1} & \text{if } b = 0^m \\
\varepsilon & \text{otherwise.}
\end{cases}$$

Obviously, the run of $M^\varphi$ on $0^N$ coincides with the one described above. Therefore the return value can have at most $M$ bits. Since $m \leq N$ it holds that $|F(\varphi)(0^N)| \geq |\varphi(0^m)| \geq M + 1$. Thus $M^\varphi$ can on input $0^N$ not produce the right return value.
3.3 Strong polynomial-time computability

While neither finite length revision nor having a step-count implies termination of the machine, the combination does: We mentioned that a machine that has a step-count may only diverge with oracle $\varphi$ if there is no bound on the oracle answers. This, however, is forbidden by finite length revision. Therefore, if $M^2$ is a machine that has finite length-revision and a step-count, then the computation of $M^2$ with any oracle and on any input terminates.

**Definition 3.6** Call an operator $F : B \to B$ strongly polynomial-time computable if there is an oracle Turing machine computing $F$ that has both finite length-revision and a polynomial step-count (see Definition 3.2).

As the name suggests, strong polynomial-time computability implies polynomial-time computability.

**Lemma 3.7 (SP ⊆ P)** Any strongly polynomial-time computable operator is polynomial-time computable.

**Proof** Let $M^2$ be the Turing machine that verifies that the total operator is strongly polynomial time computable, $p$ a polynomial step-count of the machine and $N$ a bound of the number of length revisions it does. To see that the machine runs in polynomial time fix some arbitrary oracle $\varphi$ and a string $a$. By the definition of being a step count, the first oracle query in the run of $M^2$ on input $a$ has at most $p(|a|)$ bits. Thus the return value of the oracle has at most length $|\varphi| (p(|a|))$. Therefore, again since $p$ is a step count, the next oracle query that leads to a length revision can not have more than $p(|\varphi| (p(|a|)))$ bits. Repeating the above argument $N$ times and using that $N$ is a bound of the number of length revisions proves that the computation terminates within at most $(p \circ |\varphi|)^N (p(n))$ steps. That is, that the second order polynomial

$$P(l, n) := (p \circ l)^N (p(n))$$

is a running time of $M^2$. 

On the other hand, it is a strictly stronger notion.

**Lemma 3.8 (SP ⊊ P)** There exists a polynomial-time computable operator that is not computable with finite length-revision. In particular, this operator is not strongly polynomial-time computable.

**Proof** An operator that is polynomial-time computable but not computable with finite length revision was discussed in detail in Example 3.5. Since SP ⊊ FLR, this indeed proves that the inclusion SP ⊊ P from the previous result is strict.

A candidate for a more natural example of an operator that is not strongly polynomial-time computable is the evaluation operator of the minimal representation constructed in [BS17].

17
3.4 Compatibility with relativization

For strong polynomial-time computability, relativized notions can be introduced analogously to Section 2.3. Let \( A \subseteq B \). A machine \( M^\varphi \) is said to run in \( A \)-restricted strongly polynomial time if the number of length revisions \( M^\varphi \) does on oracles from \( A \) is bounded by a number and there is a polynomial step-count that is valid whenever the oracle is from \( A \). That is if the formulas from Definition 3.2 and Definition 3.4 are fulfilled if ‘\( \forall \varphi \in B \)’ is replaced by ‘\( \forall \varphi \in A \)’. Again we denote the set of all operators whose domain is \( A \) and that can be computed by an \( A \)-restricted strongly polynomial-time machine by \( \text{SP}(A) \) and the set of all operators whose domain is contained in \( A \) and that have a total strongly polynomial-time computable extension by \( \text{SP}|_A \). For strong polynomial-time computability these classes coincide. This may be interpreted as strong polynomial-time computability being more well behaved with respect to partial operators.

Lemma 3.9 \((\text{SP}(A) = \text{SP}|_A)\) An operator that is \( A \)-restricted strongly polynomial-time computable has a total strongly polynomial-time computable extension.

**Proof** Let \( F : A \rightarrow B \) be an \( A \)-restricted strongly polynomial-time computable operator and let \( M^\varphi \) be a machine that witnesses the strong polynomial-time computability of the operator. Let \( N \) be maximum number of length revisions \( M^\varphi \) does on any oracle from \( A \) and let \( p \) be a polynomial step-count valid for input from \( A \). Define a new machine \( \tilde{M}^\varphi \) as follows: \( \tilde{M}^\varphi \) starts by initializing a counter with \( N \) written on it. Furthermore it saves the length of the input string and produces the coefficients of \( p \) on the memory tape. It applies the polynomial \( p \) to the length of the input and initializes a second counter holding this value. Now it follows the exact same steps \( M^\varphi \) does as long as no oracle query is done and meanwhile counts down the second counter. If the second counter hits zero, it terminates and returns \( \varepsilon \). If before that happens, an oracle call is done, it decreases the first counter. If the counter was already zero, it terminates and returns \( \varepsilon \). If it was not, it writes the maximum of the previous content and the length of the return value to where it originally noted the length of the input. It applies the polynomial to this new value and adds the difference to the previous value to the second counter. Then it continues as before.

It is clear that the machine described above runs with length revision \( N + 1 \), that it has a polynomial step-count (that depends only on \( p \) and \( N \)) and that whenever the oracle is from \( A \), none of the counters will hit zero and \( \tilde{M}^\varphi \) and \( M^\varphi \) produce the same values in the end. Thus \( \tilde{M}^\varphi \) computes a total strongly polynomial-time computable extension of \( F \).

3.5 Comparison to polynomial-time on \( \Sigma^{**} \)

Recall that originally polynomial-time computability was only defined for machines that compute total functions. Kawamura and Cook’s framework for complexity of operators in analysis, however, does not require an operator to
have a total polynomial-time computable extension, but instead gives a new
definition of what polynomial time computability of an operator explained on
$\Sigma^{**}$ means. Earlier, this notion of complexity was called being $\Sigma^{**}$-restricted
polynomial-time computable and the class of these operators was denoted by
$P(\Sigma^{**})$.

This section proves that, for an operator whose domain is contained in $\Sigma^{**}$,
the four notions of having a total extension from $P$ or $SP$, or having an extension
to all of $\Sigma^{**}$ that is from $SP(\Sigma^{**})$ or $P(\Sigma^{**})$ all coincide. Part of this was already
proven in Lemma 3.9 which implies that having a total strongly polynomial-
time computable extension is equivalent to being from $SP(\Sigma^{**})$. Note that
this implies, that the domain of the operator considered in Example 3.5 was
necessarily not contained in $\Sigma^{**}$.

Finally we prove that strong polynomial-time computability is equivalent to
polynomial-time computability for operators whose domain is contained in $\Sigma^{**}$.

**Theorem 3.10 (SP(Σ**) = P(Σ**))** An operator is $\Sigma^{**}$-restricted polynomial-
time computable if and only if it is strongly polynomial-time computable.

**Proof (That SP(Σ**) ⊆ P(Σ**))** This direction follows from previous results:
Let $F : A \to B$ be an $\Sigma^{**}$-restricted polynomial-time computable operator.
By Lemma 3.9 $F$ has a total strongly polynomial-time computable extension.
By Lemma 3.7 this total extension is polynomial-time computable, and since
$\Sigma^{**}$-restricted polynomial-time computability is a weaker requirement, it is also
contained in $P(\Sigma^{**})$.

The other direction of the proof heavily relies on the notions discussed in
Section 2.2

**Proof (That P(Σ**) ⊆ SP(Σ**))** Let $F : \Sigma^{**} \to B$ be computable in $\Sigma^{**}$-
restricted polynomial time. By Lemma 3.9 this operator has a total polynomial
time computable extension. Let $M^?$ be a machine that computes this extension
in time bounded by a second-order polynomial $P$. From Lemma 2.11 it fol-
lows that there exists a polynomial majorant $(N, p)$ of $P$. Define a new oracle
Machine $\tilde{M}^?$ as follows: When given $\varphi$ as oracle and a string $a$ as input, the
machine computes $p(m)$ with $m := |a|$. It then poses the oracle query $\varphi(0^p(m))$
and takes the maximum of the length of the return value and $m$. It repeats this
procedure with $m$ set to be this maximum. The above is repeated $N$ times. It
writes the result into a counter, does a final query of the oracle of this length
and then carries out the computations $M^?$ does on oracle $\varphi$ and input $a$ while
counting this counter down. If the counter runs empty it terminates and returns
$\varepsilon$. If $M^?$ terminates before this happens, it returns $M^?(a)$.

Whenever the oracle $\varphi$ is length monotone, the above procedure is easily
checked to first produce a value of the function $p_N(l, n)$ from Lemma 2.10
thereby doing at most $N+1$ length revisions and within a polynomial step-count.
Then it simulates the machine $M^?$, which by Lemma 2.10 can not ask queries
big enough to lead to another length revision and comes to an end before the
timer is empty. Thus, the machine $\tilde{M}^?$ runs in $\Sigma^{**}$-restricted strong polynomial
time. Finally note that by Lemma 3.9 $\Sigma^{**}$-restricted strong polynomial-time computability and strong polynomial-time computability coincide for operators whose domain is contained in $\Sigma^{**}$.

4 Conclusion

Many of the results of this paper are very tightly connected to questions of whether or not it is possible to add clocks to certain machines. Clocking is a standard procedure to increase the domain of machines while maintaining its behavior on a set of "important" oracles and inputs. For regular Turing machines, clocking allows to make any machine that runs in polynomial time on the inputs the user cares about into a machine that actually runs in polynomial time: Just take the polynomial that bounds the running time on the important inputs and in each step check if this number of steps was exceeded. That a machine that does this and runs in about the same time as the original machine can be found is due to the time constructibility of polynomials. When moving to oracle Turing machines, the polynomials have to be replaced by second-order polynomials and unfortunately, as mentioned before, these turn out not to be time constructible. Thus, for oracle Turing machines the above procedure does not extend in a straight forward manner.

One of the main motivations Kawamura and Cook had when they restricted to length-monotone functions was to force clockability of polynomial time machines [KC12]. And indeed, in this framework the second-order polynomials can be proven to be time-constructible [SSI17]. The notion of strong polynomial-time computability introduced in this paper tackles the same problem from another angle: It introduces a subclass of the polynomial-time operators such that clocking is possible on any domain. However, strong polynomial-time computability is a strictly stronger condition than polynomial-time computability. While it is improbable that the strongly polynomial-time computable operators are the biggest class of operators such that clocking is possible, the proof that polynomial-time does not behave well with respect to relativization (Theorem 2.18) implies that it is impossible to clock a general polynomial-time machine without making further assumptions. This can be understood as a very strong version of the failure of time-constructibility of second-order polynomials. Indeed: Any notion of time-constructibility should imply relativization.

We hope that strong polynomial-time computability turns out to be a useful concept. We think it has potential for usefulness and that it is a further step towards the the expectations of programmers what programs with subroutine calls should be considered fast as it removes the dependence of the running time on information that can not be read from the oracle in a fast way.
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