SYMmetric UNION DIAGRAMS AND ALEXANDER IDEALS

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ABSTRACT. Eisermann and Lamm introduced a notion of symmetric equivalence among symmetric union diagrams and studied it using a refined form of the Jones polynomial. We introduced invariants of symmetric equivalence via refined versions of topological spin models and provided a partial answer to a question left open by Eisermann and Lamm. In this paper we adopt a new approach to the symmetric equivalence problem and give a complete answer to the original question left open by Eisermann and Lamm.

1. INTRODUCTION

Eisermann and Lamm introduced a notion of symmetric equivalence among symmetric union diagrams and defined a Laurent polynomial invariant under symmetric equivalence [2]. The authors of the present paper tackled the problem of symmetric equivalence, by considering a stronger version of symmetric equivalence and using topological spin models to define invariants for both types of equivalence [1]. Here we introduce a different approach to study symmetric equivalence and, as an application, we resolve a question left open in both [2] and [1]. In Subsections 1.1, 1.2 and 1.3 we collect the necessary background material and in Subsection 1.4 we state our results.

1.1. Symmetric diagrams and symmetric equivalences. The involution of the real two-plane \( \rho : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \rho(x, y) = (-x, y) \) fixes the axis \( \ell = \{0\} \times \mathbb{R} \subset \mathbb{R}^2 \) pointwise. We declare two diagrams \( D, D' \subset \mathbb{R}^2 \) to be identical if one is sent to the other by an orientation-preserving diffeomorphism \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( h \circ \rho = \rho \circ h \) and \( h(D) = D' \). An oriented link diagram \( D \subset \mathbb{R}^2 \) is symmetric if \( \rho(D) \) is obtained from \( D \) by changing the orientation and switching all the crossings on the axis. A symmetric diagram \( D \) is a symmetric union if \( \rho \) sends each component \( \hat{D} \) of \( D \) to itself in an orientation-reversing fashion, implying that \( \hat{D} \) crosses the axis perpendicularly in exactly two non–crossing points. Figure 4 illustrates the symmetric union diagrams \( D_4 \) and \( D'_4 \), first considered by Eisermann and Lamm [2]. Following [2], we define

![Figure 1. The symmetric union diagrams \( D_4 \) (left) and \( D'_4 \) (right).](image)

a symmetric Reidemeister move off the axis as an ordinary Reidemeister move on a symmetric diagram carried out, away from the axis \( \ell \), together with its mirror-symmetric counterpart with respect to \( \ell \). A symmetric Reidemeister move on the axis is one of the moves illustrated in

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Eisermann and Lamm consider also two extra moves \( S_1(\pm) \) and \( S_2(v) \), some of which are illustrated in Figure 3. It is understood all of these moves admit variants obtained by turning

\[
\begin{align*}
S_3(o-) & \quad S_2(h) \\
S_2(v) & \quad S_4(--) \\
S_3(o-) & \quad S_2(v)
\end{align*}
\]

**Figure 2. Symmetric Reidemeister moves on the axis**

**Figure 3. Moves \( S_1(-) \) and \( S_2(v) \)**

the corresponding pictures upside down, mirroring or rotating them around the axis (cf. [2, §2.3]).

**Definitions 1.1.** Two oriented, symmetric diagrams which can be obtained from each other via a finite sequence of symmetric Reidemeister moves on and off the axis (sR-moves) and \( S_1 \)-moves will be called **symmetrically equivalent**. If they can be obtained from each other using sR-moves, \( S_1 \)- and \( S_2(v) \)-moves, we will say that the diagrams are **weakly symmetrically equivalent**.

1.2. **Eisermann and Lamm’s results.** Eisermann and Lamm [2] showed that there exists an infinite family of pairs \((D_n, D_n')\) of symmetric union 2-bridge knot diagrams such that \( D_n \) and \( D_n' \) are Reidemeister equivalent but not weakly symmetrically equivalent for \( n = 3 \) and \( n \geq 5 \). The diagrams \( D_{2k} \) and \( D_{2k}' \), \( k \geq 1 \), are illustrated in Figure 4. For \( k = 2 \) one obtains the special cases of Figure 1. Eisermann and Lamm established their result using an invariant of weak symmetric equivalence defined as follows. Let \( \mathcal{G} \) denote the set of oriented planar link diagrams \( D \subset \mathbb{R}^2 \) transverse to the axis \( \ell = \{0\} \times \mathbb{R} \). Let \( \mathbb{Z}(s^{1/2}, t^{1/2}) \) be the quotient field of the ring of Laurent polynomials with integer coefficients in the variables \( s^{1/2} \) and \( t^{1/2} \). Eisermann and Lamm [2] define a map \( W : \mathcal{G} \to \mathbb{Z}(s^{1/2}, t^{1/2}) \) such that \( W(D) = W(D') \) if \( D \) and \( D' \) are weakly symmetrically equivalent. By [2, Proposition 5.6], if \( D \in \mathcal{G} \) represents a link \( L \) and has no crossings on the axis then

\[
W(D) = \left( \frac{s^{1/2} + s^{-1/2}}{t^{1/2} + t^{-1/2}} \right)^{n-1} V_L(t),
\]

where \( V_L(t) \) is the Jones-polynomial of the link \( L \), normalized so that on the \( n \)-component unlink it takes the value \( (-t^{1/2} - t^{-1/2})^{n-1} \). Moreover, if \( D \) has crossings on the axis, then the following skein-like recursion formulas hold:

\[
\begin{align*}
W\left( \begin{array}{c}
\includegraphics{figure1}
\end{array} \right) &= -s^{-1/2} W\left( \begin{array}{c}
\includegraphics{figure2}
\end{array} \right) - s^{-1} W\left( \begin{array}{c}
\includegraphics{figure3}
\end{array} \right), \\
W\left( \begin{array}{c}
\includegraphics{figure4}
\end{array} \right) &= -s^{1/2} W\left( \begin{array}{c}
\includegraphics{figure5}
\end{array} \right) - s W\left( \begin{array}{c}
\includegraphics{figure6}
\end{array} \right),
\end{align*}
\]
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FIGURE 4. The symmetric union diagrams $D_{2k}$ (left) and $D'_{2k}$ (right)

It turns out [2, Proposition 1.8] that when $D$ is a symmetric union knot diagram, then $W(D)$ is an honest Laurent polynomial that we shall call the refined Jones polynomial. The diagrams $D_4$ and $D'_4$ of Figure 1 have the same refined Jones polynomial, so the question of their weak symmetric equivalence was left unanswered in [2].

1.3. Invariants from topological spin models. The theory of topological spin models for links in $S^3$ was introduced in [3]. Here we follow the reformulation used in [1], to which we refer the reader for further details. Fix an integer $n \geq 2$, denote by $\text{Mat}_n(\mathbb{C})$ the space of square $n \times n$ complex matrices, and let $d \in \{\pm \sqrt{n}\}$. Given a symmetric, complex matrix $W^+ \in \text{Mat}_n(\mathbb{C})$ with non-zero entries, let $W^- \in \text{Mat}_n(\mathbb{C})$ be the matrix uniquely determined by the equation

$$W^+ \circ W^- = J,$$

where $\circ$ is the Hadamard, i.e. entry-wise, product and $J$ is the all-1 matrix. Define, for each matrix $X \in \text{Mat}_n(\mathbb{C})$ with non-zero entries and $a, b \in \{1, ..., n\}$, the vector $Y_{ab}^X \in \mathbb{C}$ by setting

$$Y_{ab}^X(x) := \frac{X(x, a)}{X(x, b)} \in \mathbb{C}, \quad x \in \{1, ..., n\}.$$ 

Then, the pair $M = (W^+, d)$ is a spin model if the following equations hold:

$$W^+ Y_{ab}^{W^+} = d W^- (a, b) Y_{ab}^{W^-} \quad \text{for every } a, b \in \{1, ..., n\}.$$ 

The following definition was introduced in [1, Remark 1.3].

Definition 1.2. A Potts-refined spin model is a triple $\widehat{M} = (W^+, V^+, d)$ such that:

- $M = (W^+, d)$ is a spin model;
- $V^+ = (\xi^{-3} - \xi) I + \xi (J - I)$, where $\xi$ is one of the four complex numbers such that $d = -\xi^2 - \xi^{-2}$.

Let $\widehat{M} = (W^+, V^+, d)$ be a a Potts-refined spin model, $D$ a symmetric union diagram and $c$ a chequerboard colouring of $\mathbb{R}^2 \setminus D$. Let $\Gamma_D$ be the planar, signed medial graph associated to the black regions of $c$. Let $\Gamma_D^0, \Gamma_D^1$ be the sets of vertices, respectively edges of $\Gamma_D$ and let $N = |\Gamma_D^0|$. Given $e \in \Gamma_D^1$, we denote by $v_e$ and $w_e$ (in any order) the vertices of $e$. The set $\Gamma_D^1$ contains the
set $\Gamma_h^1$ of edges corresponding to crossings on the axis. Let $V^- = (-\xi^3)I + \xi^{-1}(J - I)$, and define the partition function $Z_{h}\tilde{\Gamma}(D, c)$ by the formula

$$Z_{h}\tilde{\Gamma}(D, c) := d^{-N} \sum_{\sigma : \Gamma_{h}^1 \rightarrow \{1, \ldots, n\}} \prod_{e \in \Gamma_{h}^1} V^{s(e)}(\sigma(v_e), \sigma(w_e)) \prod_{e \in \Gamma_{h}^1 \setminus \Gamma_{h}^1} W^{s(e)}(\sigma(v_e), \sigma(w_e)),$$

where $s(e) \in \{+, -\}$ is the sign of the edge $e$, and the normalized partition function $I_{h}\tilde{\Gamma}(D, c)$ by

$$I_{h}\tilde{\Gamma}(D, c) := (-\xi)^{p_t(D) - n_t(D)} Z_{h}\tilde{\Gamma}(D, c),$$

where $p_t(D)$ and $n_t(D)$ denote, respectively, the numbers of positive and negative crossings on the axis. When $D$ is not connected $Z_{h}\tilde{\Gamma}(D, c)$ and $I_{h}\tilde{\Gamma}(D, c)$ are defined as the product of the values of $Z_{h}\tilde{\Gamma}$ and, respectively, $I_{h}\tilde{\Gamma}$ on its connected components with the induced colourings. It turns out [1] that the complex number $I_{h}\tilde{\Gamma}(D, c)$ is independent of the choice of $c$, so we can write more simply $I_{h}\tilde{\Gamma}(D)$. Moreover, by a special case of [1] Theorem 1.3], if $D$ and $D'$ are oriented, weakly symmetrically equivalent symmetric union diagrams, then

$$I_{h}\tilde{\Gamma}(D) = I_{h}\tilde{\Gamma}(D').$$

In [1] Subsection 4.2] we showed that, for a suitable choice of $M$, the invariant $I_{h}\tilde{\Gamma}$ defined above can distinguish, up to weak symmetric equivalence, infinitely many Reidemeister equivalent symmetric union diagrams. We also showed [1] Subsection 4.2] that more general invariants can distinguish the diagrams $D_4$ and $D'_4$ up to symmetric equivalence, but we were unable to use invariants coming from spin models to rule out that the diagrams $D_4$ and $D'_4$ of Figure [1] are weakly symmetrically equivalent.

### 1.4. Statements of results.

Given a symmetric union diagram $D$ and an integer $h \in \mathbb{Z}$, define a new symmetric union diagram $D(h)$ by replacing each crossing on the axis with $|h|$ consecutive crossings, having the same or opposite type depending on the sign of $h$. The precise convention is specified in Figure [5] where a number $m = \pm h$ inside a box denotes a sequence of $|m|$ consecutive half-twists on the axis, each of sign equal to $\text{sgn}(m)$, the sign of $m$.

![Figure 5. Definition of $D(h), h \in \mathbb{Z}$](image)

The following theorem is established in Section [2]

**Theorem 1.3.** Let $D$ and $D'$ be two symmetric diagrams. If $D$ and $D'$ are (weakly) symmetrically equivalent, then $D(h)$ and $D'(h)$ are (weakly) symmetrically equivalent for each $h \in \mathbb{Z}$.

Clearly, if for any integer $h \in \mathbb{Z}$ the diagrams $D(h)$ and $D'(h)$ can be shown to be (weakly) symmetrically inequivalent, it follows from Theorem [1.3] that $D$ and $D'$ cannot be (weakly) symmetrically equivalent. It is therefore natural to ask whether the weak symmetric equivalence of $D_4$ and $D'_4$ could be decided by showing that $D_4(h)$ and $D'_4(h)$ have different refined Jones polynomials or different Potts-refined spin model invariants. It turns out that this is impossible: in Section [3] we show that, for any $h \in \mathbb{Z}$, the diagrams $D_4(h)$ and $D'_4(h)$ have the same refined Jones polynomial and Potts-refined spin model invariants. Nevertheless, in Section [4] we use Theorem [1.3] to prove the following.

**Theorem 1.4.** Let $s \geq 1$. Then, the Reidemeister equivalent symmetric union diagrams $D_{4s}$ and $D'_{4s}$ are not weakly symmetrically equivalent. In particular, $D_4$ and $D'_4$ are not weakly symmetrically equivalent.
Notice that Theorem 1.4 resolves the question left open in [2, 1] about the weak symmetric equivalence of the diagrams $D_4$ and $D'_4$ of Figure 1. The proof of Theorem 1.4 exploits the following simple remark. If two symmetric union diagrams $D$ and $D'$ are symmetrically equivalent then they are, in particular, Reidemeister equivalent and therefore represent the same link in $S^3$. We establish Theorem 1.4 by showing that the knots $K_s$ and $K'_s$, represented respectively by the diagrams $D_{4s}(2)$ and $D'_{4s}(2)$, are distinct because they have different second Alexander ideals for $s \geq 1$.

The paper is organized as follows. In Section 2 we prove Theorem 1.3. In Section 3 we show that $D_4(h)$ and $D'_4(h)$ have the same refined Jones polynomial and Potts-refined spin model invariants. In Section 4 we prove Theorem 1.4. The Appendix contains the proof of a lemma used in Section 4.

2. Proof of Theorem 1.3

The following Lemmas 2.1 and 2.2 deal with generalizations of, respectively, the $S_4$-move and the $S_2$-move. The lemmas play a key rôle in the proof of Theorem 1.3.

**Lemma 2.1.** Suppose that the symmetric diagrams $D$ and $D'$ differ by the $S_4(m, n)$-move defined in Figure 6. Then, $D$ and $D'$ are connected by a sequence of symmetric Reidemeister moves off the axis and $S_4$-moves. In particular, $D$ and $D'$ are symmetrically equivalent.

**Proof.** Since when $n = 0$ or $m = 0$ an $S_4(m,n)$-move reduces to a symmetric pair of second Reidemeister moves off the axis, we may assume without loss of generality that $mn \neq 0$. We suppose first that $m$ and $n$ are both positive and we establish the statement by induction on $m$ and $n$. The basis of the induction holds because an $S_4(1,1)$-move is just an ordinary $S_4$-move. Assume that the statement holds for $S_4(h,k)$-moves with $1 \leq h < m$ and $1 \leq k < n$. The inductive step is established by proving the statement for $S_4(m,k)$-moves and $S_4(h,n)$-moves. Figure 7 shows that an $S_4(m,k)$-move can be decomposed into a sequence of symmetric Reidemeister moves and $S_4(h,k)$-moves with $1 \leq h < m$. More precisely, to go from Figure 7A to Figure 7B we use two symmetric second Reidemeister moves off the axis, to go from Figure 7B to Figure 7C one $S_4(1,k)$-move and to go from Figure 7C to Figure 7D one $S_4(m - 1, k)$-move. A similar sequence of moves can be used to prove the inductive step for an $S_4(h,n)$-move.

For the other choices of signs of $m$ and $n$ the argument is essentially the same, except that one needs to perform the double induction on $|m|$ and $|n|$ and modify accordingly Figure 7 and its analogue for the $S_4(h,n)$-move. The obvious details are left to the reader. □

**Lemma 2.2.** Suppose that the symmetric diagrams $D$ and $D'$ differ by the $S_2(\pm, n)$-move defined in Figure 8. Then, $D$ and $D'$ are connected by a sequence of symmetric Reidemeister moves off the axis, $S_4$-moves and $|n|$ $S_2$-moves. In particular, $D$ and $D'$ are symmetrically equivalent.
Proof. Note that the statement is obvious for $n = 0, 1, -1$. We describe the proof for the $S(-, n)$-move with $n < 0$ because the other cases can be proved similarly. We are going to argue by induction on $n$, so we start assuming that the statement is true for $S(-, k)$-moves with $n < k \leq -1$. Performing a symmetric $R2$-move on the left-hand side tangle of Figure 8 we obtain the tangle of Figure 9A. After an $S4(-1, n + 1)$-move and some symmetric Reidemeister moves off the axis, the tangle of Figure 9A can be modified into the tangle in Figure 9B. By Lemma 2.1 this means that the tangles of Figures 9A and 9B are obtained from each other via a sequence of $S4$-moves and Reidemeister moves off the axis. By a third Reidemeister move off the axis followed by a second Reidemeister move off the axis, the tangle of Figure 9B can be turned into the tangle in Figure 9C. Now we make use of the inductive hypothesis and perform an $S(-, n + 1)$-move to get the tangle of Figure 9D. Finally, a single $S2$-move leads us from Figure 9D to the right-hand side of Figure 8, concluding the proof. \[\square\]

Proof of Theorem 1.3. We will argue that, whenever the diagrams $D$ and $D'$ differ by a symmetric Reidemeister move $M$, then $D(n)$ and $D'(n)$ are weakly symmetrically equivalent if $M$ is of type $S2(v)$, and symmetrically equivalent otherwise. This is clear for symmetric Reidemeister
moves off the axis and $S_2(h)$-moves because they do not involve crossings on the axis. Suppose that $D'$ is obtained from $D$ by applying an $S_1$-move. Then, it is immediate that $D'(n)$ is obtained from $D(n)$ by applying $|n|$ $S_1$-moves. A similar reasoning applies to $S_2(v)$-moves and $S_3$-moves, and since all the verifications are very simple we leave them to the reader. If $D'$ is obtained from $D$ by an $S_4(\epsilon_1, \epsilon_2)$-move with $\epsilon_i \in \{\pm\}$, then $D'(n)$ is obtained from $D(n)$ via an $S_4(\epsilon_1 n, \epsilon_2 n)$-move and by Lemma 2.1 $D(n)$ and $D'(n)$ are symmetrically equivalent. Finally, if $D'$ is obtained from $D$ by an $S_2(\pm)$-move, then $D'(n)$ is obtained from $D(n)$ via a $S_2(\pm, n)$-move and by Lemma 2.2 $D(n)$ and $D'(n)$ are symmetrically equivalent.

3. Negative results for $W$ and the Potts-refined spin model invariants

Our aim in this section is to show that the diagrams $D_4(h)$ and $D'_4(h)$ cannot be distinguished up to any symmetric equivalence using neither Eisermann and Lamm’s refined Jones polynomial nor any invariant coming from a Potts-refined topological spin model. This will be established in Corollary 3.2. We start with the following Proposition 3.1 which will be used in the proof of Corollary 3.2. Recall that $\mathcal{D}$ denotes the set of oriented planar link diagrams $D \subset \mathbb{R}^2$ transverse to the axis $\ell = \{0\} \times \mathbb{R}$. Let $U^m$, for $m \geq 1$, denote any crossingless, symmetric union diagram of the $m$-component unlink.

**Proposition 3.1.** Let $R$ be a ring and $\Psi : \mathcal{D} \rightarrow R$ a map such that

1. $\Psi(D) = \Psi(D')$ if $D$ and $D'$ are weakly symmetrically equivalent;
2. $\Psi \left( \begin{array}{c} \ \ \ \ \ \\ x \\
 \end{array} \right) = a_+ \Psi \left( \begin{array}{c} \ \ \ \ \ \\ x \\
 \end{array} \right) + b_+ \Psi \left( \begin{array}{c} \ \ \ \ \ \\ \end{array} \right)$ and $\Psi \left( \begin{array}{c} \ \ \ \ \ \\ \end{array} \right) = a_- \Psi \left( \begin{array}{c} \ \ \ \ \ \\ x \\
 \end{array} \right) + b_- \Psi \left( \begin{array}{c} \ \ \ \ \ \\ \end{array} \right)$, with
3. $b_+ b_- = 1$ and $(a_+ b_- + a_- b_+) \Psi(U^2) + a_+ a_- \Psi(U^0) = 0$.

Then, $\Psi(D_4(h)) = \Psi(D'_4(h))$ for each $h \in \mathbb{Z}$.

**Proof.** Clearly $D_4(0) = D'_4(0)$. We only prove the statement for $h > 0$ because the proof for $h < 0$ is essentially the same. Let $D_4(t, h, b), t, b \geq 0, h \geq 1$, be the diagram obtained
from $D_4$ by replacing the top (respectively bottom) crossing on the axis with $t$ (respectively $b$) consecutive crossings on the axis of the same sign, and each of the other crossings on the axis with $h$ consecutive crossings on the axis of the same sign. Observe that $D_4(h, h, h) = D_4(h)$ and $D_4(0, h, 0) = D_4'(h)$ for each $h \geq 1$. Therefore, it suffices to prove that $\Psi(D_4(k, h, k)) = \Psi(D_4(h))$ for each $h \geq k \geq 0$ with $h \geq 1$. This follows by an easy downward induction on $k$ starting from $k = h \geq 1$ once we show that the equality $\Psi(D_4(k, h, k)) = \Psi(D_4(k - 1, h, k - 1))$ holds. It will be convenient to use the following terminology and notation. We call a horizontal resolution of a crossing on the axis a 0-resolution and a vertical resolution of such a crossing a 1-resolution. We denote by $D_{xy}$, with $x, y \in \{0, 1\}$, any symmetric union diagram obtained from $D_4(k, h, k)$ by an $x$-resolution of any of its $k$ top crossings on the axis and a $y$-resolution of any of its $h$ bottom crossings on the axis. It is easy to check that $D_{01}$ and $D_{10}$ are weakly symmetrically equivalent to $U^2$, $D_{00}$ is weakly symmetrically equivalent to $U^3$ and $D_{11} = D_4(k - 1, h, k - 1)$.

A simple calculation using Assumption (2) yields

$$\Psi(D_4(k, h, k)) = b_+b_-\Psi(D_4(k - 1, h, k - 1)) + (a_+b_- + a_-b_+)\Psi(U^2) + a_+a_-\Psi(U^3),$$

which by (3) gives the claimed equality $\Psi(D_4(k, h, k)) = \Psi(D_4(k - 1, h, k - 1))$. 

**Corollary 3.2.** For any $h \in \mathbb{Z}$, the refined Jones polynomial and any Potts-refined spin model invariant take the same values on $D_4(h)$ and $D_4'(h)$.

**Proof.** Let $W$ be the refined Jones polynomial from [2]. We recalled in Subsection 1.2 that $W$ satisfies Assumption 1 of Proposition 5.1. By Equation (1.1), since $V_{U^m}(t) = (-t^{1/2} - t^{-1/2})^{m-1}$ we obtain $W(U^m) = (-s^{1/2} - s^{-1/2})^{m-1}$. Together with Equations (1.2) and (1.3) this immediately implies that $W$ satisfies Assumptions (2) and (3) of Proposition 5.1 and therefore $W(D_4(h)) = W(D_4'(h))$ for every $h \in \mathbb{Z}$.

Let $\hat{M} = (W^+, V^+, d)$ be a Potts-refined spin model. By Equation (1.6) we know that $I_{\hat{M}}$ satisfies Assumption 1 of Proposition 5.1 and it follows immediately from the definition that $I_{\hat{M}}(U^m) = d^m$. Using that $V^+ = (-\xi^3 - \xi)I + \xi J = d\xi^{-1}I + \xi J$, $V^- = -\xi^3I + \xi^{-1}(J - I) = (-\xi^3 - \xi^{-1})I + \xi^{-1}J = d\xi I + \xi^{-1}J$ and $d = -\xi^2 - \xi^2$, it is straightforward to check that

$$I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right) = -\xi^{-2}I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right) - \xi^{-4}I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right) \quad \text{and} \quad I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right) = -\xi^2I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right) - \xi^4I_{\hat{M}}\left(\begin{array}{c}
\circ
\end{array}\right).$$

Thus, $I_{\hat{M}}$ satisfies Assumptions (2) and (3) of Proposition 5.1 and $I_{\hat{M}}(D_4(h)) = I_{\hat{M}}(D_4'(h))$ for every $h \in \mathbb{Z}$. 

**4. PROOF OF THEOREM 1.4**

In this section we show that the diagrams $D_{4s}(2)$ and $D_{4s}'(2)$ represent knots $K_s$ and $K'_s$ having different second Alexander ideals for every $s \geq 1$. Since this implies that $D_{4s}(2)$ and $D_{4s}'(2)$ are not Reidemeister equivalent, combining this fact with Theorem 1.3 yields Theorem 1.4.

Recall [4] Chapter 6) that the second Alexander ideal of a knot $K \subset S^3$ is the second elementary ideal $E_2$ of the Alexander module $M_K$, and that $E_2$ is generated by the minors of order $m - 1$ of any $m \times m$ presentation matrix of $M_K$. A square presentation matrix of $M_K$ as a $\mathbb{Z}[t, t^{-1}]$-module is given by the Alexander matrix $tV - V'$, where $V$ is a Seifert matrix of $K$. Therefore, we start by computing Seifert matrices for the knots represented by the diagrams $D_{2k}(2)$ and $D_{2k}'(2)$, for $k = 2s$. In the following Lemma 4.1 we will use the computations of the Seifert matrices to obtain convenient presentation matrices for the two Alexander modules.

Consider, for any $k \geq 1$, the Seifert surface $\Sigma_k$ for $D_{2k}(2)$ and the basis for its first homology group shown in Figure 10. The generators are divided into two groups, each of which is shown separately in Figure 10 to maximize readability. Now assume that $k = 2s$. Then, it is a tedious but straightforward exercise to check that the entries of the associated Seifert matrix

$$V = (v_{i,j})_{1 \leq i, j \leq 8s+2},$$

following the convention of [4, chapter 6], are:

$$v_{4i,r} = 1 \quad \text{for } r = 4i \pm 1 \text{ and } 1 \leq i \leq 2s, \quad v_{8s+2,r} = 1 \quad \text{for } r = 4i - 1 \text{ and } 1 \leq i \leq 2s,$$
The remaining entries of $V$ are trivial.

Next, we consider a Seifert surface $\Sigma'_s$ for $D'_4s(2)$ and the basis for $H_1(\Sigma'_s, \mathbb{Z})$ illustrated in Figure 11. As before, the generators are divided into two groups, which are shown separately. It
is easy to check that the corresponding Seifert matrix $V'$ is given by

$$V' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & s \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ s & 0 & 1 & 0 & s - 1 & 0 & 1 & 2s \end{pmatrix}$$

Using the Seifert matrices $V$ and $V'$ we can now construct Alexander matrices and then, by applying suitable elementary row and column operations, find convenient presentation matrices for the corresponding Alexander modules. The end result is the following Lemma 4.1, the proof of which is deferred to the Appendix for the sake of presentation.

**Lemma 4.1.** For every $s \geq 1$:

1. the Alexander module of the knot $K_s$ underlying the oriented diagram $D_{4s}(2)$ is presented by the matrix $\begin{pmatrix} 0 & -p_1(t) \\ p_1(t)/t & -p_2(t)/t \end{pmatrix}$, where $p_1(t) = 2st^2 - (4s + 1)t + 2s$ and $p_2(t) = s(6s + 1)(t - 1)^3$.

   In particular, the Alexander polynomial of $K_s$ is $p_1(t)^2$ (up to units) and the second Alexander ideal is $(p_1(t), p_2(t)) \subset \mathbb{Z}[t, t^{-1}]$.

2. the Alexander module of the knot $K'_s$ underlying the oriented diagram $D'_{4s}(2)$ is presented by the matrix $\begin{pmatrix} 0 & q_1(t)/t \\ -q_1(t) & -q_2(t)/t \end{pmatrix}$, where $q_1(t) = 2st^2 - (4s + 1)t + 2s$ and $q_2(t) = (t - 1)(s(4s - 1)t^2 - (8s^2 - 1)t + s(4s - 1))$.

   In particular, the Alexander polynomial of $K'_s$ is $q_1(t)^2$ (up to units) and the second Alexander ideal is $(q_1(t), q_2(t)) \subset \mathbb{Z}[t, t^{-1}]$.

**Proof.** See the Appendix.

We are now ready to apply Lemma 4.1 to show that the knots represented by the diagrams $D_{4s}(2)$ and $D'_{4s}(2)$ have distinct second Alexander ideals.

**Lemma 4.2.** Let $s \geq 1$ and let $K_s$, respectively $K'_s$ be the knot represented by $D_{4s}(2)$, respectively $D'_{4s}(2)$. Then, $K_s$ and $K'_s$ have distinct second Alexander ideals. In particular, $D_{4s}(2)$ and $D'_{4s}(2)$ are not Reidemeister equivalent.

**Proof.** Let $J = (p_1(t), p_2(t))$ and $J' = (q_1(t), q_2(t))$ be the second Alexander ideals of $K_s$ and $K'_s$, given by Lemma 4.1. Consider the ring homomorphism $\Phi : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}/(6s + 1)[t, t^{-1}]$ defined by

$$\Phi \left( \sum_{i=-n}^{m} \alpha_i t^i \right) = \sum_{i=-n}^{m} [\alpha_i] t^i.$$

Clearly $\Phi(p_2(t)) = [0]$. Moreover, it follows from $[-4s - 1] = [2s]$ and $[-3][2s] = [1]$ that

$$\Phi(J) = (\Phi(p_1(t))) = (t^2 + t + [1]).$$

Let $R$ be the quotient ring $\mathbb{Z}/(6s + 1)[t, t^{-1}]/(t^2 + t + [1])$. 


and \( \pi : \mathbb{Z}/(6s + 1)[t, t^{-1}] \to R \) the natural projection. To prove \( J \neq J' \) it suffices to show that \( \pi(\Phi(J')) = R \). In fact, it is an easy exercise to check that \( R \) is non-trivial, and clearly \( \pi(\Phi(J)) = (0) \). Since \( q_2(t) = (2s - 1)p_1(t)(t - 1) + s(t - 1)^3 \), we have
\[
\pi(\Phi(J')) = (\pi(\Phi(q_2(t))) = (\pi([s]\Phi((t - 1)^3))) = (\pi(\Phi((t - 1)^3))),(n)
\]
where the last equality holds because \( \pi([s]) \) is invertible in \( R \). Moreover,
\[
p_1(t) = -1 + (t - 1)(2s(t - 1) - 1),
\]
therefore \( \pi(\Phi(t - 1)) \) is invertible as well, and it follows that \( \pi(\Phi(J')) = R \).

**Proof of Theorem 1.4** The statement is an immediate consequence of Lemma 4.2 and Theorem 1.3

**APPENDIX**

In this appendix we prove Lemma 4.1. Let \( V \) be the Seifert matrix associated to the oriented diagram \( D_{4s} \) given in Section 4. It is well-known that the Alexander matrix \( A = tV - V^t \) presents the Alexander module of the knot represented by the diagram \( D_{4s}(2) \). The entries of the \((8s + 2) \times (8s + 2)\) matrix \( A = (a_{i,j}) \) are as follows:

\[
a_{4i+2,r} = \begin{cases}
1 & \text{if } r = 4i + 1, 0 \leq i < 2s, \\
1 & \text{if } r = 4i + 1, 0 \leq i < 2s
\end{cases}
\]

\[
a_{8s+2,r} = \begin{cases}
t & \text{if } r = 4i + 3, 0 \leq i < 2s \\
1 & \text{if } r = 4i + 1, 0 \leq i < 2s \\
-1 & \text{if } r = 8s + 1 \\
t & \text{if } r = 8s - 1, i = 2s
\end{cases}
\]

\[
a_{4i+1,r} = \begin{cases}
t & \text{if } r = 4i + 2, 0 \leq i < 2s \\
-1 & \text{if } r = 4i, 1 \leq i < 2s
\end{cases}
\]

\[
a_{4i+3,r} = \begin{cases}
t & \text{if } r = 4i + 4, 0 \leq i < 2s \\
-1 & \text{if } r = 8s + 2, 0 \leq i < 2s
\end{cases}
\]

and the remaining entries are zeros. In particular, for \( s = 1 \) we have

\[
A = tV - V^t = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -t \\
-1 & t & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & t & t-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & t & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & t & 0 & 0 & -t \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
1 & 0 & t & 0 & 0 & 0 & 0 & t & t-1 & 0 & 2t-2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & t & t-1 & 0 & 2t-2
\end{pmatrix}
\]

We are going to perform a sequence of row and column operations on \( A \) to obtain a new presentation matrix of the Alexander module. The \( i \)-th row (column, respectively) of a matrix \( M \) will be denoted \( M_i \) (\( M^i \), respectively). We start by swapping the \((2i - 1)\)-th with the \(2i\)-th row for every \( i = 1, \ldots, 4s \), calling the resulting matrix \( B \). For \( s = 1 \) we obtain

\[
B = \begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t & 0 & 0 & 0 & 0 & 0 & 0 & -t \\
0 & 0 & t & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\
1 & 0 & t & 0 & 0 & 0 & 0 & -1 & -1 & -1
\end{pmatrix}
\]
Observe that the first $8s$ diagonal entries of $B = (b_{i,j})$ are as follows:

$$b_{ii} = \begin{cases} -1 & \text{if } i \equiv 0 \text{ or } 3 \text{ mod } 4 \\ t & \text{if } i \equiv 1 \text{ or } 2 \text{ mod } 4, \end{cases} \quad i = 1, \ldots, 8s.$$ 

Note that these diagonal entries are all units of $\mathbb{Z}[t, t^{-1}]$. For $i = 1, \ldots, 8s$ we now divide the $i$-th row of $B$ by $b_{ii}$. The first $8s$ diagonal entries of the resulting matrix $C$ are now all 1's, and

$$c_{2i+2,2i} = \begin{cases} -t & \text{for } i \equiv 1 \text{ mod } 2 \\ -1/t & \text{for } i \equiv 0 \text{ mod } 2, \end{cases} \quad i = 1, \ldots, 4s - 1,$$

For $s = 1$, we have that

$$C = \begin{pmatrix} 1 & t-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1-1/t & 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t-1 & 0 & 0 \\ 0 & 0 & 0 & -1/t & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2-2/t \\ 1 & t & 0 & 1 & 0 & 0 & -1 & 2t-2 \\ \end{pmatrix}$$

We now perform the following row operations on $C$:

$$C_{2i+2} \mapsto C_{2i+2} - c_{2i+2,2i} \cdot C_{2i}, \quad i = 1, \ldots, 4s - 1.$$ 

For $s = 1$ we obtain the matrix $E = (e_{ij})$ which is as follows:

$$E = \begin{pmatrix} 1 & t-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1-1/t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2-2/t \\ 0 & 0 & 0 & 0 & 0 & 0 & t & t-1 \\ 1 & 0 & t & 0 & 1 & 0 & 0 & -1 \\ \end{pmatrix}$$

We remark that, for any $s$, the entry $e_{i,j} = 0$ if $j \leq 8s - 2$ and $j < i < 8s + 2$, while

$$e_{i,i+1} = \begin{cases} 0 & \text{if } i \text{ is even}, \\ t-1 & \text{if } i \equiv 1 \text{ mod } 4, \quad i = 1, \ldots, 8s - 2, \\ 1-1/t & \text{if } i \equiv 3 \text{ mod } 4, \end{cases}$$

Moreover, it is easy to check that the first $8s$ entries of the last column of $E$ are given by

$$e_{i+1,8s+2} = \begin{cases} 0 & \text{if } i \text{ is even}, \\ ((i-1)-(i+3)/4)t & \text{if } i \equiv 1 \text{ mod } 4, \quad \text{for } i = 0, \ldots, 8s - 1, \\ (i+1)(1-t)/4 & \text{if } i \equiv 3 \text{ mod } 4 \end{cases}$$

We perform the following row and column operations on $E$:

$$E_{2i-1} \mapsto E_{2i-1} - e_{2i-1,2i} \cdot E_{2i}, \quad i = 1, \ldots, 4s - 1,$$

$$E^{8s+2} \mapsto E^{8s+2} - c_{2i,8s+2} \cdot E^{2i}, \quad i = 1, \ldots, 4s - 1.$$ 

The resulting matrix $F = (f_{i,j})$ is as follows for $s = 1$:

$$F = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+t \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0+t \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/t-2+t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/t-3+2t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & t & 0 & 1 & 0 & t & 0 & -1 & 2t-2 \\ \end{pmatrix}$$
Note that
\begin{align*}
\begin{cases}
0 & \text{if } i \text{ is even}, \\
(1-t)(i-1 -(i+3)t)/4t & \text{if } i \equiv 1 \text{ mod } 4, \quad i = 1, \ldots, 8s - 2, \\
(i+1)(1-t)^2/4t & \text{if } i \equiv 3 \text{ mod } 4,
\end{cases}
\end{align*}
\tag{4.1}

Performing the following column operations on $F$
\begin{align*}
F^{2i+1} \mapsto F^{2i+1} - F^{2i-1}, \quad i = 1, \ldots, 4s - 1
\end{align*}
we obtain the matrix $G$, which for $s = 1$ is:
\begin{align*}
G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1+t & 0 & 2-t & 0 & 2t-2 & 0 & -1 & 2t-2
\end{pmatrix}
\end{align*}

For general $s$, the submatrix of $G$ identified by the first $8s - 1$ rows and columns is the identity matrix and the square submatrix of $G$ specified by the last four rows and columns is given by
\begin{align*}
\begin{pmatrix}
1 & 0 & 2-2/t & 1-1/t & 0 \\
0 & 1 & 1 & 0 & t \\
0 & 1 & 0 & 1 & -1 \\
2st-2s & t-1 & 0 & -1 & t
\end{pmatrix}
\end{align*}

Moreover, the last column of $G$ coincides with the last column of $F$, while an easy computation shows that the first $8s - 2$ entries of the last row of $G$ are given by
\begin{align*}
g_{8s+2,j} = \begin{cases}
0 & \text{if } j \text{ is even}, \\
(j+3 -(j-1)t)/4 & \text{if } j \equiv 1 \text{ mod } 4, \quad j = 1, \ldots, 8s - 2, \\
(j+1)(t-1)/4 & \text{if } j \equiv 3 \text{ mod } 4,
\end{cases}
\tag{4.2}
\end{align*}

Finally, we apply to $G$ the following column operations:
\begin{align*}
G^{8s+2} \mapsto G^{8s+2} - g_{2t-1,8s+2} \cdot G^{2t-1}, \quad i = 1, \ldots, 8s - 1.
\end{align*}
The result is a matrix $H$ of the form $(\begin{smallmatrix} I & 0 \\ 0 & L \end{smallmatrix})$, where $I$ is the square identity matrix of size $8s - 2$ and $L$ is given by
\begin{align*}
L = \begin{pmatrix}
1 & 2-2/t & 1-1/t & 0 \\
0 & 1 & 1 & 0 & t \\
0 & 1 & 0 & 1 & -1 \\
2st-2s & t-1 & 0 & -1 & t
\end{pmatrix}
\end{align*}
for some $p(t) \in \mathbb{Z}[t, t^{-1}]$. Taking into account the last column operations performed on $G$ and the fact that the last column of $G$ coincides with the last column of $F$, using (4.1) and (4.2) we get
\begin{align*}
p(t) = 2st - 2s - \sum_{i=1}^{4s-1} g_{8s+2,2t-1} f_{2t-1,8s+2} = s(2s-1)(t-1) \left(1 - \frac{2t-2}{t}\right).
\end{align*}

Applying to $L = (l_{i,j})$ the row operations
\begin{align*}
L_i \mapsto L_i - l_{i1} L_1, \quad i = 3, 4, \\
L_i \mapsto L_i - l_{i2} L_2, \quad i = 3, 4
\end{align*}
we obtain a matrix of the form $(\begin{smallmatrix} T & 0 \\ 0 & M \end{smallmatrix})$, where $T = \begin{pmatrix} 1 & 2-2/t & 1-1/t \\ 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 2st-2s & t-1 & 0 & -1 & t
\end{pmatrix}$,
\begin{align*}
M = \begin{pmatrix}
p_1(t)/t & -p_1(t) \\
p_2(t)/t & p_2(t)
\end{pmatrix}
\end{align*}
and
\begin{align*}
p_1(t) &= 2st^2 - (4s+1)t + 2s, \\
p_2(t) &= -s(6s+1)(t-1)^3/t.
\end{align*}
This concludes the proof of Part (1) of Lemma 4.1. Now we can proceed with the proof of Part (2). A Seifert matrix $V'$ associated to the oriented diagram $D_{4s}^t(2)$ is given in Section 4. The corresponding Alexander matrix is

$$A' = tV' - (V')^t = \begin{pmatrix} 0 & t & 0 & 0 & 0 & 0 & st-s \\ -1 & t-1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & t & 1-t & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & t & 1-s+st \\ 0 & 0 & 0 & 0 & 1 & -1+t & 0 & 0 \\ st-s & 0 & t & 0 & (s-1)t-s & 0 & t & 2st-2s \end{pmatrix}.$$  

We leave to the reader the simple verification that via a sequence of row and column operations analogous, but simpler, to the one used in the proof of Part (1), one obtains a matrix of the form $\begin{pmatrix} I & * \\ 0 & B' \end{pmatrix}$, where $I$ is the identity matrix of size 5 and

$$B' = \begin{pmatrix} 1 & -1 & 2s/t-2s \\ t-1 & 1-t & -s/t+s+1-(s+1)t+st^2 \\ -2s+4st-2st^2 & t & -s/t+s+1-(s+1)t+st^2 \end{pmatrix}.$$  

By further row operations we get to

$$C' = \begin{pmatrix} 1 & -1 & 2s/t-2s \\ t-1 & 1-t & \frac{q_1(t)}{t} \\ 0 & 0 & \frac{q_2(t)}{t} \end{pmatrix},$$  

where $q_1(t) = 2st^2 - (4s + 1)t + 2s$ and $q_2(t) = (t - 1)(s(4s - 1)t^2 - (8s^2 - 1)t + s(4s - 1))$.

This clearly suffices to prove Part (2) of Lemma 4.1.

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