On $p$-Adic Convergence of Perturbative Invariants of Some Rational Homology Spheres.

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Abstract

R. Lawrence has conjectured that for rational homology spheres, the series of Ohtsuki’s invariants converges $p$-adically to the $SO(3)$ Witten-Reshetikhin-Turaev invariant. We prove this conjecture for Seifert rational homology spheres. We also derive it for manifolds constructed by a surgery on a knot in $S^3$. Our derivation is based on a conjecture about the colored Jones polynomial that we have formulated in our previous paper. We also present numerical examples of $p$-adic convergence for some simple manifolds.
1 Introduction

The cyclotomic properties of the $SO(3)$ Witten-Reshetikhin-Turaev (WRT) invariant $Z'(M; K)$ of 3d manifolds $M$ have attracted a lot of attention recently. This invariant was defined by R. Kirby and P. Melvin [7] by modifying the Reshetikhin-Turaev surgery formula of [18]. H. Murakami showed that if $M$ is a rational homology sphere (RHS) and $K$ is an odd prime number, then

$$Z'(M; K) \in \mathbb{Z}[	ilde{q}], \quad \tilde{q} = e^{\frac{2\pi i}{K}},$$

(1.1)

here $\mathbb{Z}[	ilde{q}]$ is a cyclotomic ring:

$$\frac{\tilde{q}^K - 1}{\tilde{q} - 1} = 0 \quad (1.2)$$

(alternative proofs of (1.1) were presented in [12] and [23]).

As an element of $\mathbb{Z}[	ilde{q}]$, $Z'(M; K)$ can be presented as a polynomial in $\tilde{q} - 1$:

$$Z'(M; K) = \sum_{n=0}^{K-2} a_n(M; K) h^n, \quad h = \tilde{q} - 1. \quad (1.3)$$

The numbers $a_n(M; K) \in \mathbb{Z}$ depend on both the RHS $M$ and the ‘level’ $K$. T. Ohtsuki showed in [16] and [17] how to reprocess $a_n(M; K)$ into the invariants of $M$ which are independent of $K$.

We introduce the following notations. Since our prime number is $K$, we use the term ‘$K$-adic’ instead of the usual term ‘$p$-adic’. For $q \in \mathbb{Z}$, $q \neq 0 \pmod{K}$ let $q^*$ denote $K$-adic inverse of $q$. In other words, $q^*$ is a formal series in positive powers of $K$

$$q^* = \sum_{n=0}^{\infty} q^*_{(n)} K^n, \quad 0 \leq q^*_{(n)} \leq K - 1, \quad (1.4)$$

such that for any $N \geq 0$

$$q \sum_{n=0}^{N} q^*_{(n)} K^n = 1 \pmod{K^{N+1}}. \quad (1.5)$$

We denote by $\vee$ an operation that converts rational numbers whose denominators are not divisible by $K$, into $K$-adic numbers:

$$\left(\frac{p}{q}\right) \vee = pq^*, \quad p, q \in \mathbb{Z}, \quad q \neq 0 \pmod{K}. \quad (1.6)$$
Theorem 1.1 (Ohtsuki [16], [17]) Let $M$ be a RHS. Then there exists an infinite sequence of rational invariants $\lambda_n(M)$, $n \geq 0$ such that if $K$ is an odd prime and $K > |H_1(M, \mathbb{Z})|$, $|H_1(M, \mathbb{Z})|$ being the order of first homology, then

$$\{ |H_1(M, \mathbb{Z})| \}_K |H_1(M, \mathbb{Z})| a_n(M; K) = \lambda_n^\vee(M) \pmod{K} \quad \text{for } n \leq \frac{K-3}{2},$$

(1.7)

where $\{ |H_1(M, \mathbb{Z})| \}_K$ is the Legendre symbol: for $p \in \mathbb{Z}$, $\{ p \}_K = 1$ if there exists some $q \in \mathbb{Z}$ such that $p = q^2 \pmod{K}$, $\{ p \}_K = -1$ otherwise.

R. Lawrence suggested that this theorem can be strengthened:

Conjecture 1.1 (Lawrence [9]) If $M$ is an integer homology sphere then $\lambda_n(M) \in \mathbb{Z}$ and the cyclotomic series $\sum_{n=0}^\infty \lambda_n(M)h^n$ converges $K$-adically:

$$\sum_{n=0}^\infty \lambda_n(M)h^n = Z'(M; K).$$

(1.8)

If $M$ is a rational homology sphere then

$$\lambda_n(M) \in \mathbb{Z} \left[ \frac{1}{2}, \frac{1}{|H_1(M, \mathbb{Z})|} \right]$$

(1.9)

and if $|H_1(M, \mathbb{Z})| \neq 0 \pmod{K}$ then the cyclotomic series $\sum_{n=0}^\infty \lambda_n^\vee(M)h^n$ converges $K$-adically:

$$\sum_{n=0}^\infty \lambda_n^\vee(M)h^n = \{ |H_1(M, \mathbb{Z})| \}_K |H_1(M, \mathbb{Z})| Z'(M; K).$$

(1.10)

R. Lawrence proved her conjecture for the subclass

$$X \left( \frac{-p}{r}, \frac{q}{s}, l + pq \right), \quad l, p, q, r, s \in \mathbb{Z}, \quad q = 2, \quad ps - qr = 1$$

(1.11)

of 3-fibered Seifert rational homology spheres. The manifolds (1.11) were constructed by Dehn’s surgery on a $(p, q)$ torus knot with $q = 2$ and framing $l$.

Let us comment briefly on the notion of $K$-adic limit in the cyclotomic ring. A cyclotomic relation (1.2) tells us that

$$h^{K-1} = -K \sum_{n=0}^{K-2} \frac{1}{K} \binom{K}{n+1} h^n.$$
If K is prime then \( \frac{1}{K}(\frac{K}{n+1}) \in \mathbb{Z} \) for \( 0 \leq n \leq K - 2 \). Therefore eq. (1.12) allows us to reduce any cyclotomic polynomial of h to the 'fundamental' powers \( h^n, 0 \leq n \leq K - 2 \). We denote this reduction by♠. Since the coefficients of \( [h^{n(K-1)+m}]\), \( m, n \geq 0 \), are divisible by \( K^n \), we see that the map♠ converts the 'analytic' smallness of high powers of h into K-adic smallness of the fundamental coefficients. As a result, eq. (1.10) means that for any \( N_1 > 0 \) there exists \( N_2 \) such that for any \( N > N_2 \), the coefficients of the polynomial

\[
\left[ |H_1(M, \mathbb{Z})| K |H_1(M, \mathbb{Z})| Z'(M; K) - \sum_{n=0}^{N} \lambda_n^\vee(M) h^n \right]^{\star}
\]

are all divisible by \( K^{N_1} \).

A certain amount of information about the first Ohtsuki invariants is already known. H. Murakami showed [14], [15] that \( \lambda_0(M) = 1 \) and \( \lambda_1(M) = 3 \lambda_{\text{CW}}(M) \), here \( \lambda_{\text{CW}}(M) \) is the Casson-Walker invariant. X-S. Lin and Z. Wang proved [11] that for any integer homology sphere \( M, \lambda_2(M) \in 3 \mathbb{Z} \).

The Casson-Walker invariant is known to satisfy the following properties [1], [28]:

\[
\lambda_{\text{CW}}(M) \in 2 \mathbb{Z} \quad \text{if} \quad |H_1(M, \mathbb{Z})| = 1,
\]

\[
6|H_1(M, \mathbb{Z})| \lambda_{\text{CW}} \in \mathbb{Z} \quad \text{if} \quad |H_1(M, \mathbb{Z})| > 1.
\]

We conjecture that the fraction \( \frac{1}{2} \) in (1.9) is due completely to (1.15), so that both parts of Conjecture [11] can be united in one statement:

**Conjecture 1.2** For a RHS \( M \) there exists a sequence of rational invariants \( \tilde{\lambda}_n(M) \) such that

\[
\tilde{\lambda}_n(M) \in \mathbb{Z} \left[ \frac{1}{|H_1(M, \mathbb{Z})|} \right],
\]

and if \( |H_1(M, \mathbb{Z})| \not\equiv 0 \pmod{K} \) then the cyclotomic series \( \sum_{n=0}^{\infty} \tilde{\lambda}_n^\vee(M) h^n \) converges K-adically:

\[
q^{(3\lambda_{\text{CW}}(M))^\vee} \sum_{n=0}^{\infty} \tilde{\lambda}_n^\vee(M) h^n = \{ |H_1(M, \mathbb{Z})| \}_{K} |H_1(M, \mathbb{Z})| Z'(M; K).
\]
Conjecture 1.1 follows from this conjecture because of the properties (1.14) and (1.15) of the Casson-Walker invariant.

In our previous paper [23] we showed that the formal power series $\sum_{n=0}^{\infty} \lambda_n(M) h^n$ coincides (up to a normalization) with the ‘perturbative invariant’ $Z^{(tr)}(M; K)$ which is a generating function of invariants $\Delta_n(M)$ (see [24] and references therein for the definition of $Z^{(tr)}(M; K)$ and $\Delta_n(M)$). In other words, $\lambda_n(M)$ appear as coefficients in the re-expansion of the formal power series $\sum_{n=0}^{\infty} \Delta_n(M) K^{-n}$ in powers of $h = \frac{2n i}{K} + O(K^{-2})$ rather than $K^{-1}$. More precisely, in the notations of [24],

$$
\frac{|H_1(M, \mathbb{Z})|^2}{\sqrt{2 K} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(M; K) = \left( \frac{\pi}{K} \right) \sum_{n=0}^{\infty} \Delta_n(M) K^{-n} = \sum_{n=0}^{\infty} \lambda_n(M) h^n
$$

(1.18)

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$$

(1.18)

The relation (1.18) tells us that we can calculate Ohtsuki’s invariants $\lambda_n(M)$, $\tilde{\lambda}_n(M)$ with the help of the same surgery formula that we used for $\Delta_n(M)$. If $M$ is constructed by a surgery on a link $L \subset S^3$, then $\lambda_n(M)$ and $\tilde{\lambda}_n(M)$ can be expressed explicitly [23] in terms of the derivatives of the colored Jones polynomial of $L$. These expressions lead us in Section 2 to the proof of Conjecture 1.2 for Seifert rational homology spheres. They also help us in Section 4 to derive Conjecture 1.2 for RHS constructed by a surgery on a knot $K \subset S^3$ from the ‘weak conjecture’ of [25]. In Section 4 we present numerical examples of Conjecture 1.1 for integral homology spheres. In Section 5 we briefly discuss this conjecture as a relation between the path integral and number theoretical approaches to Witten-Reshetikhin-Turaev invariant.

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2The proof of [23] relies on the Reshetikhin formula which is a special integral representation of the colored Jones polynomial of a link. A ‘path integral proof’ of this formula was presented in [22]. We will give a mathematically rigorous proof in [26]. It is exactly the same proof as in [23] except that we use Kontsevich’s integral formula [8] for the $1/K$ expansion of the Jones polynomial rather than a Bar-Natan style [2] perturbation theory (see Appendix of [24] for the sketch of this argument).
2 Seifert Rational Homology Spheres

In this section we will prove Conjecture 1.2 for Seifert rational homology spheres. A Seifert RHS $X\left(\frac{p_1}{q_1}, \ldots, \frac{p_N}{q_N}\right)$ can be constructed by a surgery on a framed $(N + 1)$-component link $L \subset S^3$. This link consists of the ‘primary’ unknot and $N$ ‘secondary’ unknots simply connected to the primary unknot in a Hopf link style. The primary unknot has zero framing while the secondary unknots have rational framings $\frac{p_j}{q_j}$, $0 \leq j \leq N$ (this means that their parallels go $p_j$ times around the unknots and $q_j$ times along the unknots).

We introduce a notation

$$P = \prod_{j=1}^{N} p_j, \quad H = P \sum_{j=1}^{N} \frac{q_j}{p_j}. \quad (2.1)$$

$H$ determines the order of the first homology of the Seifert manifold $X$:

$$|H_1(X, \mathbb{Z})| = |H|. \quad (2.2)$$

Let $K$ be an odd prime number. In what follows we require in view of eq. (2.2) that

$$H \neq 0 \pmod{K}. \quad (2.3)$$

This condition implies together with eq. (2.1) that there could be at most one $p_j$ divisible by $K$. Therefore we assume that

$$p_j \neq 0 \pmod{K}, \quad 2 \leq j \leq N. \quad (2.4)$$

In fact, we will assume for simplicity that $p_j, q_j \neq 0 \pmod{K}, 1 \leq j \leq N$. This will allow us to use the numbers $p_j^*, q_j^*$ freely in the intermediate equations. We claim however that the only essential assumption is (2.3).

The formula for the $SO(3)$ invariant of the Seifert RHS $X$ was derived in [23] (eq.(4.38)):

$$Z'(X; K) = -\frac{i}{2\sqrt{K}}e^{\frac{\pi}{4}K \text{sign}(H/\pi)}e^{\frac{\pi}{4}H} e^{\frac{1}{2}\text{sign}(H/\pi)} e^{\frac{1}{2}K} x^{\text{sign}(H/\pi)}$$

$$\times \left\{ \sum_{0 \leq \beta < 2K} q^{4^* p_H H^* H + 3} \sum_{j=1}^{N} s(y_{j^*}) - \frac{1}{q^{2^*} - q^{2^*} \beta} \right\}$$

$$\times \sum_{\beta \in \mathbb{Z}} q^{-4^* p_H H^* H} \prod_{j=1}^{N} \left( q^{-2^* \beta - q^{2^*} \beta^*} \right). \quad (2.5)$$
here \( s(p, q) \) is a Dedekind sum

\[
s(p, q) = \frac{1}{4q} \sum_{j=1}^{q-1} \cot \left( \frac{\pi j}{q} \right) \cot \left( \frac{\pi p j}{q} \right)
\]  

(2.6)

and

\[
\kappa = \begin{cases} 
1 & \text{if } K = 1 \mod 4 \\
-1 & \text{if } K = -1 \mod 4.
\end{cases}
\]  

(2.7)

In [23] * meant only inverse mod \( K \): \( pp^* = 1 \mod K \), while in this paper * means \( K \)-adic inverse. This does not make any difference in the context of eq. (2.5) because all ‘asterisk’ numbers are in the exponent of \( \tilde{q} \), so only their value mod \( K \) is important.

To prove Conjecture 1.2 for the Seifert RHS \( X \) it is better to use a slightly different formula for its \( SO(3) \) invariant.

**Lemma 2.1** The \( SO(3) \) invariant \( Z'(M; K) \) can be presented as a \( K \)-adicly converging series:

\[
Z'(X; K) = -\{ |H| \}_K \text{sign} (H) \sum_{\mu_1 = \pm 1} \mu_1 \times \tilde{q}^{(3\lambda_{\text{CW}})^2 + 4^* (N-1)(N-2) H^* P - 2^* (N-2) H^* P \sum_{j=1}^{N} \mu_j p_j^* + 2^* H^* P \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} p_j^* p_{j'}^* + 2^*} \\
\times \frac{1}{1 - \tilde{q}} \sum_{m \geq 0} \sum_{l=0}^{2m} \sum_{k=0}^{m} C_{l,k;m} \left( \frac{H^* P}{l} \right) \frac{1}{k!} \partial^{(k)} \sum_{j=3}^{N} \left( 1 - (1 - T^* (\epsilon)) p_j^* \right) \left| T^{N-2}_* (\epsilon) \right|_{\epsilon=0},
\]  

(2.8)

where

\[
\mu_j = \begin{cases} 
\pm 1 & \text{for } j = 1 \\
1 & \text{for } 2 \leq j \leq N,
\end{cases}
\]  

(2.9)

and \( \lambda_{\text{CW}} \) is the Casson-Walker invariant of \( X \) as calculated by C. Lescop [10]:

\[
\lambda_{\text{CW}} = \frac{1}{12} \frac{P}{H} \left( 2 - N + \sum_{j=1}^{N} \frac{1}{p_j^2} \right) - \frac{1}{4} \text{sign} \left( \frac{H}{P} \right) + \frac{1}{12} \frac{H}{P} - \sum_{j=1}^{N} s(q_j, p_j).
\]  

(2.10)

The coefficients \( C_{l,k;m} \in \mathbb{Z} \) are defined by eq. (2.19) and

\[
T^* (\epsilon) = 1 - (1 + \epsilon) \tilde{q}^{-H^* P \left( \sum_{j=1}^{N} \mu_j p_j^* - N + 2 \right)}.
\]  

(2.11)
Proof of Lemma 2.1. We derive eq. (2.8) from eq. (2.5). First of all, we want to remove the factor $\frac{1}{2}$ from the r.h.s. of eq. (2.5). All the factors in the sum over $\beta$ have definite parity. The range of summation could also be made symmetric ($-K \leq \beta \leq K$, $\beta \in 2\mathbb{Z} + 1$) due to the periodicity of the summand under the shift $\beta \to \beta + K$. Therefore we can place $2\tilde{q}^{-2}p_j^3\beta$ instead of $\tilde{q}^{-2}p_j^3\beta - \tilde{q}^{2}p_j^3\beta$ without changing the sum.

Next, we are going to transform the summand of eq. (2.5) by 'completing the square' in the exponent:

$$
\tilde{q}^{-4}P^*H\beta^2\tilde{q}^{-2}p_j^3\beta = \tilde{q}^{-4}P^*H(\beta\mp H^*Pp_j^3)^2 + 4H^*Pp_j^2.
$$

We introduce a new variable $\beta'$:

$$
\beta = \beta' - H^*P \left( \sum_{j=1}^{N} \mu_j p_j^* - N + 2 \right) \tag{2.13}
$$

($\mu_j$ are defined by eq. (2.9)) and rewrite eq. (2.5) as

$$
Z'(X; K) = -\frac{i}{2\sqrt{K}} e^{\frac{i}{4} \kappa \text{sign}(\beta)} e^{\frac{i}{4} \frac{K}{2\mu} \text{sign}(\beta)}
\times \{ |P| \}_K \text{sign}(P) \tilde{q}^{4P^*H-3\sum_{j=1}^{N} s_j(q_j,p_j)+4H^*P \left( \sum_{j=1}^{N} \mu_j p_j^* - N+2 \right)^2}
\times \frac{1}{\tilde{q}^{2^*} - \tilde{q}^2} \sum_{0 \leq \beta' \leq 2K \atop \beta \in 2\mathbb{Z} + 1} \tilde{q}^{-4P^*H\beta^2} \prod_{j=3}^{N} \left( 1 - \tilde{q}^{p_j^3\beta} \right) \left( 1 - \tilde{q}\beta \right)^{-N-2}.
$$

We kept the variable $\beta$ in the last factor of this equation meaning that it is a function (2.14) of $\beta'$.

To calculate the sum over $\beta'$ we present the last fraction factor of the r.h.s. of eq. (2.14) as an explicit polynomial in $\tilde{q}^n\beta$:

$$
\prod_{j=3}^{N} \left( 1 - \tilde{q}^{p_j^3\beta} \right) \left( 1 - \tilde{q}\beta \right)^{-N-2} = \sum_{0 \leq n \leq \sum_{j=3}^{N} p_j^3 - N+2} C_n \tilde{q}^n\beta, \quad C_n \in \mathbb{Z}. \tag{2.15}
$$

The coefficients $C_n$ depend, of course, on the numbers $p_j^3$ and $N$. The sum over $\beta'$ for individual monomials $\tilde{q}^n\beta$ can be calculated by completing the square:

$$
\sum_{0 \leq \beta' \leq 2K \atop \beta \in 2\mathbb{Z} + 1} \tilde{q}^{-4P^*H\beta^2} \tilde{q}^n\beta = \sum_{0 \leq \beta' \leq 2K \atop \beta \in 2\mathbb{Z} + 1} \tilde{q}^{-4P^*H(\beta' - 2H^*Pn)^2 + 4H^*Pn^2 - H^*P \left( \sum_{j=1}^{N} \mu_j p_j^* - N+2 \right)^2} \tilde{q}^{H^*P \left( \sum_{j=1}^{N} \mu_j p_j^* - N+2 \right)^n} \tag{2.16}
$$

$$
= \sqrt{Ke^{\frac{i}{4} \kappa (\beta - 1)}} \left( P^*H \right)_K \tilde{q}^{H^*Pn^2} \tilde{q}^{-H^*P \left( \sum_{j=1}^{N} \mu_j p_j^* - N+2 \right)^n}.
$$
Let us substitute $\tilde{q} = 1 + h$ in $\tilde{q}^{H^* P n^2}$ and present the latter as a $K$-adicly convergent series

$$\tilde{q}^{H^* P n^2} = (1 + h)^{H^* P n^2} = \sum_{m \geq 0} \binom{H^* P n^2}{m} h^m,$$  \hfill (2.17)

here by definition

$$\binom{x}{m} = \frac{1}{m!} \prod_{l=0}^{m-1} (x-l), \quad x \in \mathbb{R}, \ m \in \mathbb{Z}, \ m \geq 0. \hfill (2.18)$$

The polynomial $P(x, n) = \binom{x^2}{m}$ takes integer values when $x, n \in \mathbb{Z}$. Therefore it can be presented as an integer linear combination of the product of elementary binomial polynomials of $x$ and $n$:

$$\binom{x^2}{m} = \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k,m} \binom{x}{l} \binom{n}{k}, \quad C_{l,k,m} \in \mathbb{Z}. \hfill (2.19)$$

After presenting a binomial polynomial $\binom{n}{k}$ as a derivative

$$\binom{n}{k} = \frac{1}{k!} \partial_{\epsilon}^k (1 + \epsilon)^n \bigg|_{\epsilon=0}, \hfill (2.20)$$

we finally re-express $\tilde{q}^{H^* P n^2}$ as

$$\tilde{q}^{H^* P n^2} = \sum_{m \geq 0} h^m \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k,m} \binom{H^* P}{l} \frac{1}{k!} \partial_{\epsilon}^k (1 + \epsilon)^n \bigg|_{\epsilon=0}. \hfill (2.21)$$

Now we combine eqs. (2.15), (2.16) and (2.21) in order to calculate the sum over $\beta'$:

$$\sum_{0 \leq \beta' < 2K} \tilde{q}^{-H^* P H \beta'^2} \tilde{q}^{n\beta} = \sqrt{K} e^{i\pi (\kappa - 1)} \{ P^* H \}_K \sum_{0 \leq n \leq \sum_{j=3}^{N} \mu_j p_j^* - N + 2} C_n \tilde{q}^{-H^* P \left( \sum_{j=1}^{N} \mu_j p_j^* - N + 2 \right)} n \hfill (2.22)$$

$$\times \sum_{m \geq 0} \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k,m} \binom{H^* P}{l} \frac{1}{k!} \partial_{\epsilon}^k (1 + \epsilon)^n \bigg|_{\epsilon=0}$$

$$= \sqrt{K} e^{i\pi (\kappa - 1)} \{ P^* H \}_K \sum_{m \geq 0} h^m \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k,m} \binom{H^* P}{l}$$

$$\times \frac{1}{k!} \partial_{\epsilon}^k \left( \prod_{j=3}^{N} \frac{1 - (1 - T_n(\epsilon)) p_j^*}{T_n^{N-2}(\epsilon)} \right) \bigg|_{\epsilon=0},$$

the function $T_n(\epsilon)$ was defined in eq. (2.11). We used eq. (2.15) ‘backwards’ in order to get rid of the sum over $n$. If we substitute the formula (2.22) in eq. (2.14) and use eq. (2.10)
together with simple relations

\[ 4^* = \frac{1 - \kappa K}{4} \pmod{K}, \quad (2.23) \]

\[ e^{\frac{i\pi}{4}(\kappa - 1)} \{|P|\}_K \{P^* H\}_K = e^{\frac{i\pi}{4}(\kappa - 1) \text{sign}(H)} \{|H|\}_K, \quad (2.24) \]

\[ i \text{sign}(P) e^{-\frac{i\pi}{4} \text{sign}(H)} = \text{sign}(H), \quad (2.25) \]

in order to simplify the exponentials in front of the sum over \(\beta'\), then we obtain eq. (2.8). \(\square\)

Note that the requirement \(p_1 \neq 0 \pmod{K}\) looks artificial indeed, because whenever \(p_1^*\) appears in eq. (2.8), it is always canceled by a factor of \(P\).

Now we turn to the perturbative invariant \(Z^{(tr)}(X; K)\) which generates Ohtsuki’s invariants \(\tilde{\lambda}_n(X)\) via eq. (1.18). The formula for \(Z^{(tr)}(X; K)\) was derived in [19], [21]. It has obvious similarities with eq. (2.5):

\[ Z^{(tr)}(X; K) = -\frac{1}{4K \sqrt{|P|}} e^{\frac{i\pi}{4} \text{sign}(H)} \text{sign}(P) q^{\frac{1}{4} \left( H^2 - 3 \text{sign}(H) - 12 \sum_{j=1}^{N} s(q_j, p_j) \right)} \quad (2.26) \]

\[ \times \int_{-\infty}^{+\infty} d\beta \bar{q}^{-\frac{1}{8} H^2} \beta^2 \prod_{j=1}^{N} \left( \bar{q}^{-\frac{1}{2} p_j} - \bar{q}^{\frac{1}{2} p_j} \right) \left( \bar{q}^{-\frac{1}{2}} - \bar{q}^{\frac{1}{2}} \right)^{N-2}. \]

The symbol \(\int_{-\infty}^{+\infty} \int_{[\beta = 0]}\) means that we take only the contribution of the stationary phase point \(\beta = 0\) to the integral. In other words, we have to expand the factor

\[ \prod_{j=1}^{N} \left( \bar{q}^{-\frac{1}{2} p_j} - \bar{q}^{\frac{1}{2} p_j} \right) \left( \bar{q}^{-\frac{1}{2}} - \bar{q}^{\frac{1}{2}} \right)^{N-2} \quad (2.27) \]

in Taylor series in \(\beta\) at \(\beta = 0\) and then integrate the series together with the gaussian exponential \(\bar{q}^{-\frac{1}{8} H^2} \beta^2\) term by term in order to produce the expansion (1.18). The result would be the series (2.30) (cf. eq.(1.8) of [19]).

A more effective formula for the perturbative invariant follows from eqs. (3.21), (3.23) of [20] specialized to the trivial connection. It can be obtained from eq. (2.26) by rotating the integration contour in the complex plane by \(\frac{\pi}{4}\) and rescaling the integration variable.
Proposition 2.1  The generating function \( \frac{|H_1(X, Z)|^\frac{3}{2}}{\sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(X; K) \) is an analytic function of \( K \) in the area where \( iK\frac{H}{P} \) is not a positive real number:

\[
\frac{|H_1(X, Z)|^\frac{3}{2}}{\sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(X; K) = -\frac{i}{4} \frac{H}{\sin \left( \frac{\pi}{K} \right)} e^{\frac{3\pi i}{4} \left\{ \frac{\pi}{K} - 3 \text{sign} \left( \frac{\pi}{K} \right) - 12 \sum_{j=1}^{N} s(q_j, p_j) \right\} - \frac{1}{2} \sum_{j=1}^{N} n_i} \times \int_{-\infty}^{\infty} dx \ e^{-\pi x^2} F \left( e^{-\frac{\pi i}{4} \text{sign} \left( \frac{\pi}{K} \right) \sqrt{\frac{2}{K}} P X}, \frac{P}{\sqrt{K}} \right),
\]

\[
F(b) = \frac{\prod_{j=1}^{N} \left( e^{-i\pi b_j} - e^{i\pi b_j} \right)}{(e^{-i\pi b} - e^{i\pi b})^{N-2}}.
\]

In the limit of large \( K \) the integral can be approximated by expanding the pre-exponential factor \( F \) in powers of \( x^2/K \) and integrating each term of the expansion separately with the gaussian factor. This procedure results in the asymptotic series in \( 1/K \):

\[
\frac{|H_1(X, Z)|^\frac{3}{2}}{\sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(X; K) = -\frac{i}{4} \frac{H}{\sin \left( \frac{\pi}{K} \right)} e^{\frac{3\pi i}{4} \left\{ \frac{\pi}{K} - 3 \text{sign} \left( \frac{\pi}{K} \right) - 12 \sum_{j=1}^{N} s(q_j, p_j) \right\} - \frac{1}{2} \sum_{j=1}^{N} n_i} \times \sum_{m \geq 1} \frac{1}{m!} \left( \frac{P}{H} \right)^m \frac{1}{(2\pi i K)^m} \partial_b^{2m} \left. \frac{\prod_{j=1}^{N} \left( e^{-i\pi b_j} - e^{i\pi b_j} \right)}{(e^{-i\pi b} - e^{i\pi b})^{N-2}} \right|_{b=0}.
\]

The expansion (2.30) contains too many fractions with ‘large’ denominators, so we are going to construct a slightly different expansion which would look similar to eq. (2.8).

Lemma 2.2  The perturbative invariant \( \frac{|H_1(X, Z)|^\frac{3}{2}}{\sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(X; K) \) which generates Ohtsuki’s invariants \( \tilde{\lambda}_n(X) \), can be presented as an asymptotic series in \( h \):

\[
\frac{|H_1(X, Z)|^\frac{3}{2}}{\sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right)} Z^{(tr)}(X; K) = \hat{q}^{3\lambda_{cw}(X)} \sum_{n=0}^{\infty} \tilde{\lambda}_n(X) h^n
\]

\[
= -\text{sign} (H) \sum_{\mu_1 = \pm 1} \mu_1 \hat{q}^{3\lambda_{cw}(X) + \frac{1}{2N} \left( N(N-1) - \frac{1}{2} \right) \sum_{j=1}^{N} \frac{n_j}{p_j} + \sum_{1 \leq j < j' \leq N} \frac{n_j p_j}{p_j} + \frac{1}{2}} \times \frac{1}{1 - \hat{q}} \sum_{m \geq 0} h^m \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k;m} \left( \frac{P}{H}, l \right) \frac{1}{k!} \partial_{(k)}^{\epsilon} \left. \prod_{j=3}^{N} \left( 1 - (1 - T(\epsilon)) \frac{1}{p_j} \right) \right|_{\epsilon=0}^{T(\epsilon) = 1 - (1 + \epsilon) \hat{q}^{-\frac{P}{H} \left( \sum_{j=1}^{N} \frac{n_j + 1}{p_j} N + 2 \right)}}
\]

here

\[
T(\epsilon) = 1 - (1 + \epsilon) \hat{q}^{-\frac{P}{H} \left( \sum_{j=1}^{N} \frac{n_j + 1}{p_j} N + 2 \right)}
\]
Proof of Lemma 2.2. The proof is absolutely similar to that of Lemma 2.1. We place $2\tilde{q}^{-\frac{1}{4}H}$ instead of $\tilde{q}^{-\frac{1}{4}H} - \tilde{q}^{\frac{1}{4}H}$ and complete the square by shifting the integration variable:

$$\beta = \beta' - \frac{P}{H} \left( \sum_{j=1}^{N} \frac{\mu_j}{p_j} - N + 2 \right), \quad (2.33)$$

so that eq. (2.26) becomes

$$Z^{(tr)}(X; K) = -\frac{1}{2K} \sqrt{|P|} e^{i\pi \frac{3}{4} \text{sgn}(\#) \text{sgn}(P)} \sum_{\mu_1=\pm 1} \mu_1 \times \tilde{q}^{-3 \text{sgn}(\#)-12 \sum_{j=1}^{N} s(q_{j,p_j}) + \frac{P}{H} \left( \sum_{j=1}^{N} \frac{\mu_j}{p_j} - N + 2 \right)^2} \times \int_{[\beta'=0]}^{+\infty} d\beta' \tilde{q}^{-\frac{1}{4}H} \beta'^{\frac{N}{2}} \prod_{j=3}^{N} \frac{1 - \tilde{q}^{\beta_j}}{(1 - \tilde{q}^\beta)^{N-2}}. \quad (2.34)$$

In the last factor of the r.h.s. of this equation $\beta$ means the function (2.33) of $\beta'$.

We present the last factor of the r.h.s. of eq. (2.34) as a ‘generalized’ geometric series in powers of $\tilde{q}^\beta$:

$$\prod_{j=3}^{N} \frac{1 - \tilde{q}^{\beta_j}}{(1 - \tilde{q}^\beta)^{N-2}} = \lim_{x \to 1} \sum_{n \in \Lambda} C_n x^n q^{n\beta}, \quad C_n \in \mathbb{Z}, \quad (2.35)$$

here $\Lambda$ is a certain set of non-negative rational numbers and $C_n$ are multiplicities with which the powers $n$ appear in the expansion. The individual powers $q^{n\beta}$ are easy to integrate:

$$\int_{-\infty}^{+\infty} d\beta' \tilde{q}^{-\frac{1}{4}H} \beta'^{2n} q^{n\beta} = (2K)^{-\frac{1}{2}} P \frac{1}{H} e^{-\frac{1}{4}H \text{sgn}(\#) \text{sgn}(P)} q^{n\beta} \frac{\frac{P}{H}}{q^{n\beta} \frac{N}{2} \sum_{j=1}^{N} \frac{\mu_j}{p_j} - N + 2}. \quad (2.36)$$

We transform the quadratic exponential further:

$$\tilde{q}^{\frac{P}{H} n^2} = (1 + h)\frac{P}{H} n^2 = \sum_{m \geq 0} h^m \sum_{l=0}^{m} \sum_{k=0}^{2m} C_{l,k;m} \left( \frac{n}{l} \right) \left( \frac{n}{k} \right), \quad (2.37)$$

eq \text{eq. (2.20) defines a binomial coefficient } \left( \frac{x}{i} \right) \text{ for rational } x.$$

Combining eqs. (2.35), (2.36) and (2.37) we obtain the following formula for the integral over $\beta'$:

$$\int_{-\infty}^{+\infty} d\beta' \tilde{q}^{-\frac{1}{4}H} \beta'^{2n} q^{n\beta} = (2K)^{-\frac{1}{2}} P \frac{1}{H} \prod_{j=3}^{N} \frac{1 - \tilde{q}^{\beta_j}}{(1 - \tilde{q}^\beta)^{N-2}}. \quad (2.38)$$
\[ e^{-\frac{ie}{4}\text{sign}(\frac{H}{P})} \sum_{n \in \Lambda} C_n q^{-n} \left( \sum_{j=1}^{N} \frac{\mu_j - N + 2}{p_j} \right) \sum_{m \geq 0} h^m \sum_{l=0}^{2m} C_{l,k; m} \left( \frac{p_H}{l} \right) \frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k} (1 + \epsilon)^n \bigg|_{\epsilon = 0} \]

\[ = \sqrt{2K} \left( \frac{P}{H} \right)^{\frac{1}{2}} e^{-\frac{ie}{4}\text{sign}(\frac{H}{P})} \sum_{m \geq 0} h^m \sum_{l=0}^{2m} C_{l,k; m} \left( \frac{p_H}{l} \right) \frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k} \prod_{j=3}^{N} \left( 1 - (1 - T(\epsilon))^{p_j} \right) \right]_{\epsilon = 0} \]

The function \( T(\epsilon) \) is defined by eq. (2.32). We used eq. (2.35) backwards to get rid of the sum over \( n \) in the middle expression of eq. (2.38). After substituting the expression (2.38) into eq. (2.34) and using eq. (2.10) we arrive at eq. (2.31). \( \Box \)

Now we are going to prove Conjecture 1.2 with the help of two lemmas:

**Lemma 2.3** For \( q, m \in \mathbb{Z} \),

\[ \left( \frac{q}{p} \right) \in \mathbb{Z} \left[ \frac{1}{p} \right] \]  
(2.39)

and hence

\[ \left( \frac{q}{p} \right)^\vee = \left( \frac{p\ast q}{m} \right) \quad \text{for} \quad p \in \mathbb{Z}, \quad p \neq 0 \pmod{K}, \]  
(2.40)

\[ \hat{q}^q = \sum_{m \geq 0} \left( \frac{q}{p} \right) h^m \in \mathbb{Z} \left[ \frac{1}{p} \right] [h]. \]  
(2.41)

**Lemma 2.4**

\[ P \left( \frac{(N - 1)(N - 2)}{2} + (N - 2) \sum_{j=1}^{N} \frac{\mu_j}{p_j} + \sum_{1 \leq j < j' \leq N} \frac{\mu_j \mu_j'}{p_j p_j'} \right) + \frac{1}{2} \in \mathbb{Z} \left[ \frac{1}{H} \right]. \]  
(2.42)

**Proof of Lemma 2.4.** It is obvious that

\[ P \left( \frac{(N - 1)(N - 2)}{2} + (N - 2) \sum_{j=1}^{N} \frac{\mu_j}{p_j} + \sum_{1 \leq j < j' \leq N} \frac{\mu_j \mu_j'}{p_j p_j'} \right) + \frac{1}{2} \in \mathbb{Z} \left[ \frac{1}{H}, \frac{1}{2} \right], \]  
(2.43)

so (2.42) is true if \( H \) is even.

We assume now that \( H \) is odd. It is sufficient to prove that

\[ P \left( \frac{(N - 1)(N - 2)}{2} + (N - 2) \sum_{j=1}^{N} \frac{\mu_j}{p_j} + \sum_{1 \leq j < j' \leq N} \frac{\mu_j \mu_j'}{p_j p_j'} + H \right) \in 2\mathbb{Z}. \]  
(2.44)
The parity of all three terms in this expression is determined by the parity of \(N\) and individual numbers \(p_j\). Since \(H\) is odd, there can be at most one even number among \(p_j\). Therefore (2.44) can be checked directly for all possible combinations. \(\square\)

**Proof of Conjecture 1.2.** First, we prove eq. (1.10). We compare the formulas (2.8) and (2.31) with the help of eq. (2.41). It is obvious that

\[
\left[ \frac{\tilde{q}^*}{P} \left( \frac{(N-1)(N-2)}{2} + (N-2) \sum_{j=1}^{N} \frac{\mu_j}{p_j} + \sum_{1 \leq j < j' \leq N} \frac{\mu_j \mu_j'}{p_j p_j'} \right) + \frac{1}{2} \right]^\vee
\]

\[
= q^* H^* P \left( 2^*(N-1)(N-2) - (N-2) \sum_{j=1}^{N} \mu_j p_j^* + \sum_{1 \leq j < j' \leq N} \mu_j \mu_j' p_j^* p_j'^* \right) + 2^*
\]

Also,

\[
\left[ \frac{p}{\ell} \left( \sum_{j=1}^{N} \frac{\mu_j}{p_j} - N + 2 \right) \right]^\vee = q^{-H^* P \left( \sum_{j=1}^{N} \mu_j p_j^* - N + 2 \right)},
\]

hence

\[
[T(\varepsilon)]^\vee = T_*(\varepsilon).
\]

Then we see that for \(3 \leq j \leq N\)

\[
\left[ \frac{1 - (1 - T(\varepsilon))^\frac{1}{p_j}}{T(\varepsilon)} \right]^\vee = \left[ - \sum_{m \geq 0} (-1)^m \left( \frac{1}{m+1} \right) T^m(\varepsilon) \right]^\vee
\]

\[
= \sum_{m \geq 0} (-1)^m \left( \frac{p_j^*}{m+1} \right) T_*^m(\varepsilon) = \frac{1 - (1 - T_*(\varepsilon)) p_j^*}{T_*(\varepsilon)}.
\]

Finally,

\[
\left( \frac{p}{\ell} \right)^\vee = \left( H^* P \right)^\vee.
\]

Combining eqs. (2.45), (2.48) and (2.49) we arrive at eq. (1.17) for the Seifert RHS \(X\).

Now we turn to condition (1.16). The invariants \(\tilde{\lambda}_n(X)\) appear as coefficients in the expansion of the r.h.s. of eq. (2.31) (multiplied by \(\tilde{q}^{-3\lambda_{CW}(X)}\)) in powers of \(h\). Lemma 2.3 tells us that these coefficients are rational and their denominators contain only the divisors of \(H\) and \(p_j\) for \(3 \leq j \leq N\):

\[
\tilde{\lambda}_n(X) \in \mathbb{Z} \left[ \frac{1}{H}, \frac{1}{p_3}, \ldots, \frac{1}{p_N} \right]
\]
In our derivation of eq. (2.31) we made an arbitrary selection of two factors $\tilde{q}^{-\frac{3}{2j}} - \tilde{q}^{-\frac{1}{2j}}$, $j = 1, 2$ to be absorbed by the gaussian exponential. We could select any other pair of numbers $j$ and obtain a modified condition (2.50). The formula (2.1) for $H$ indicates that if a number is a divisor of at least one of any $N - 2$ numbers among all $p_j$, then that number divides $H$. This means that the intersection of rings (2.50) corresponding to all possible choices of two missing numbers $p_j$, is exactly $\mathbb{Z} \lbrack \frac{1}{H} \rbrack$. This proves (1.16).\[\square\]

3 Surgery on a Knot

In this section we will study rational homology spheres constructed by a rational surgery on a knot $K \subset S^3$. We will derive Conjecture 1.2 for these manifolds from Conjecture 1.1 of [25].

Let $M$ be a RHS constructed by a rational $(p, q)$ surgery on a knot $K \subset S^3$. Since

$$|H_1(M, \mathbb{Z})| = |p|,$$  \hspace{1cm} (3.1)

we assume that

$$p \neq 0 \pmod{K}. \hspace{1cm} (3.2)$$

We also assume for simplicity that $q \neq 0 \pmod{K}$, but we claim that the latter assumption is not essential.

The formula for the $SO(3)$ invariant of $M$ was derived in [23] along the lines of [6] from the Kirby-Melvin [7] modification of the Reshetikhin-Turaev [18] surgery formula:

$$Z'(M; K) = -\frac{i}{2\sqrt{K}} \{|q|\}_K \text{sign}(q) e^{-\frac{i\pi K \text{sign}(\frac{q}{k})}{4}} e^{-i\frac{\pi K^2 \text{sign}(\frac{q}{k})}{4}}$$

$$\times \tilde{q}^{3s'(p,q)-4s'p} \sum_{0 \leq a \leq 2K \atop a \in \mathbb{Z}+1} \tilde{q}^{4s'p\alpha^2} \left( \tilde{q}^{-2s'\alpha} - \tilde{q}^{2s'\alpha} \right) J_\alpha(K; \tilde{q}),$$  \hspace{1cm} (3.3)

here $J_\alpha(K; t)$ is the ‘unframed’ (i.e. reduced to zero self-linking number) colored Jones polynomial of the knot $K$ (we use the complex variable $t$ in order to distinguish it from the
cyclotomic variable $\tilde{q} = e^{\frac{2\pi i}{K}}$. The Jones polynomial is normalized in such a way that it is multiplicative under a disconnected sum and

$$J_\alpha(\text{unknot}; t) = \frac{t^{-\frac{\tilde{q}}{2}} - t^{\frac{\tilde{q}}{2}}}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}.$$  \hspace{1cm} (3.4)

The Casson-Walker invariant of $M$ is calculated by the surgery formula of [28]:

$$\lambda_{\text{cw}}(M) = -s(q, p) + \frac{q}{p} \Delta''_A(\mathcal{K}),$$  \hspace{1cm} (3.5)

here $\Delta''_A(\mathcal{K})$ is the second derivative of the Alexander polynomial of $\mathcal{K}$ at $z = 0$, $\Delta''_A(\mathcal{K}) \in 2\mathbb{Z}$.

It is convenient to use the Jones polynomial in a slightly different normalization:

$$V_\alpha(\mathcal{K}; t) = \frac{J_\alpha(\mathcal{K}; t)}{J_\alpha(\text{unknot}; t)}, \quad V_\alpha(\mathcal{K}; t) \in \mathbb{Z}[t, t^{-1}].$$  \hspace{1cm} (3.6)

After substituting $V_\alpha(\mathcal{K}; \tilde{q})$ instead of $J_\alpha(\mathcal{K}; \tilde{q})$ in eq. (3.1) and shifting the summation variable $\alpha$ in order to complete the square in the exponent, we come to the ‘semi-analog’ of Lemma 2.1.

**Lemma 3.1** The SO(3) invariant of $M$ is expressed in terms of $V_\alpha(\mathcal{K}; \tilde{q})$ as a sum

$$Z'(M; K) = \frac{i}{\sqrt{K}} \{|q|\} \kappa \text{sign}(q) e^{\frac{2\pi i}{K} (\kappa + 1)} \text{sign}(\frac{\tilde{q}}{q}) \tilde{q}^{(3\lambda_{\text{cw}}(M))'} \sum_{\mu \in \pm 1} \mu q^{-3\rho^* q \Delta''_A(\mathcal{K}) - 2\rho^* p^* + 2^*} \tilde{q}^{4\rho^* p^* \alpha_2} V_\alpha(\mathcal{K}; \tilde{q}).$$  \hspace{1cm} (3.7)

In this equation $\alpha$ denotes a function of $\alpha'$:

$$\alpha = \alpha' + p^* (1 + \mu q)$$  \hspace{1cm} (3.8)

**Proof of Lemma 3.1.** First, we use the symmetries of the summand and summation range in eq. (3.3) in order to substitute $2\tilde{q}^{-2^* q^* \alpha}$ instead of $\tilde{q}^{-2^* q^* \alpha} - \tilde{q}^{2^* q^* \alpha}$. Since $\alpha$ is kept odd in the sum of eq. (3.3) we can also change $\tilde{q}^\alpha_2$ into $-\tilde{q}^{2^* \alpha}$ in the formula for $J_\alpha(\text{unknot}; \tilde{q})$, because

$$2^* = \frac{1 - \kappa K}{2}. $$  \hspace{1cm} (3.9)
As a result,

\[ J_\alpha(K; \tilde{q}) = \frac{\tilde{q}^{-2\alpha} - \tilde{q}^{2\alpha}}{\tilde{q}^{-2} - \tilde{q}^2} V_\alpha(K; \tilde{q}). \]  

(3.10)

The change of variables (3.8) allows us to absorb the factors \( \tilde{q}^{-2\alpha} \) and \( \tilde{q}^{2\alpha} \) into the gaussian exponential \( \tilde{q}^{4q_\alpha^2} \). Finally, the relation

\[ s(p, q) + s(q, p) = \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4} \text{sign}(pq) \]  

(3.11)

leads us to eq. (3.7). □

Now we turn to the perturbative invariant \( Z^{(tr)}(M; K) \) which is a generating function of Ohtsuki’s invariants \( \lambda_n(M) \) according to eq. (1.18). The formula for \( Z^{(tr)}(M; K) \) was derived in [21] (see also [24] and references therein):

\[ Z^{(tr)}(M; K) = i \frac{\sin \left( \frac{\pi}{K} \right)}{2K \sqrt{|q|}} \text{sign}(q) e^{-i\pi \frac{3}{4} \frac{K-2}{K} \text{sign}(q)} \tilde{q}^{3s(p, q) - \frac{3}{4} \eta} \]  

\[ \times \int_{\alpha=0}^{+\infty} d\alpha \tilde{q}^{\frac{1}{2} \alpha^2} \left( \tilde{q}^{-\frac{1}{2} \alpha} - \tilde{q}^{\frac{1}{2} \alpha} \right) J_\alpha(K; \tilde{q}). \]  

(3.12)

Here the symbol \( \int_{\alpha=0}^{+\infty} \) means that we take only the contribution of the stationary phase point \( \alpha = 0 \) to the integral. This contribution is calculated in the form of formal power series in \( K^{-1} \) (or in \( h = \tilde{q} - 1 \)) of eq. (1.18) by expanding the colored Jones polynomial in Taylor series in \( K^{-1} \) and \( \alpha \) and then integrating monomials of \( \alpha \) individually with the gaussian factor \( \tilde{q}^{\frac{1}{2} \alpha^2} \). Note that since we assume the polynomial \( J_\alpha(K; \tilde{q}) \) to be expanded in powers of \( \alpha \), it does make sense to evaluate it not only for \( \alpha \in \mathbb{Z} \) but also for \( \alpha \in \mathbb{R} \).

By using \( V_\alpha(K; \tilde{q}) \) instead of \( J_\alpha(K; \tilde{q}) \), substituting

\[ \alpha = \alpha' + \frac{1 + \mu q}{p} \]  

(3.13)

and using eq. (3.11) we obtain a modified expression for the perturbative invariant.
Lemma 3.2 The generating function \( \frac{|H_1(M, \mathbb{Z})|^3}{\sqrt{2\pi \sin \left( \frac{\pi}{K} \right)}} Z^{(tr)}(M; K) \) can be presented as a stationary phase integral:

\[
\frac{|H_1(M, \mathbb{Z})|^3}{\sqrt{2\pi \sin \left( \frac{\pi}{K} \right)}} Z^{(tr)}(M; K) = \frac{i}{\sqrt{2K}} \left| \frac{p}{q} \right|^{\frac{1}{2}} \text{sign}(q) e^{-i\pi \frac{4}{9} \text{sign}(\frac{q}{\bar{q}}) \frac{3}{4} \lambda_{\text{CW}}(M)} \sum_{\mu=\pm 1} \mu \\
\times \bar{q}^{-3\frac{2}{9} \Delta'_{\alpha}(\mathcal{K}) - \frac{4}{9} \frac{1}{p} + \frac{1}{q}} \frac{1}{1 - \bar{q}} \int_{-\infty}^{+\infty} d\alpha' \bar{q}^{\frac{1}{2} \text{sign}(\bar{p})} V_\alpha(\mathcal{K}; \bar{q}),
\]

\( \alpha \) in the r.h.s. of this equation denotes the function (3.13).

The proof of this lemma is completely similar to the proof of Lemma 3.1 and we drop it.

A simple form of the coefficients of Taylor expansion of \( V_\alpha(\mathcal{K}; t) \) in powers of \( t - 1 \) allows us to calculate the sum of eq. (3.7) and the integral of eq. (3.14) explicitly. We will prove Conjecture 1.2 for \( M \) by comparing these expressions.

For a fixed value of \( \alpha, V_\alpha(\mathcal{K}; t) \in \mathbb{Z}[t, t^{-1}] \). Therefore we can expand \( V_\alpha(\mathcal{K}; t) \) in Taylor series in powers of \( t - 1 \) at \( t = 1 \). According to the Melvin-Morton conjecture [13] which was proven by D. Bar-Natan and S. Garoufalidis [3], the coefficients of this expansion are polynomials in \( \alpha \) of a limited degree:

\[
V_\alpha(\mathcal{K}; t) = \sum_{n \geq 0} \sum_{0 \leq m \leq \frac{n}{2}} D_{m,n}(\mathcal{K}) \alpha^{2m}(t - 1)^n,
\]

(3.15)

here \( D_{m,n}(\mathcal{K}) \in \mathbb{Q} \) are Vassiliev invariants of the knot \( \mathcal{K} \) of degree \( n \). We can rearrange eq. (3.15) as an expansion in powers of \( \alpha(t - 1) \) and \( t - 1 \):

\[
V_\alpha(\mathcal{K}; t) = \sum_{n \geq 0} \sum_{m \geq 0} D_{m,n+2m}(\mathcal{K}) [\alpha(t - 1)]^{2m}(t - 1)^n.
\]

(3.16)

We go one step further and use a variable

\[
t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}} = \alpha(t - 1) + \cdots
\]

(3.17)

instead of \( \alpha(t - 1) \):

\[
V_\alpha(\mathcal{K}; t) = \sum_{n \geq 0} \sum_{m \geq 0} d_{m,n}(\mathcal{K}) \left(t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}}\right)^{2m}(t - 1)^n.
\]

(3.18)

We made the following ‘weak’ conjecture in [25]:
Conjecture 3.1  All coefficients \(d_{m,n}(K)\) in the expansion (3.18) are integer:

\[
d_{m,n}(K) \in \mathbb{Z}. \tag{3.19}
\]

Consider an ‘approximate’ Jones polynomial

\[
V^{(N)}_\alpha(K; t) = \sum_{m+n \leq N} d_{m,n}(K) \left(t^\frac{\alpha}{2} - t^{-\frac{\alpha}{2}}\right)^{2m} (t - 1)^n. \tag{3.20}
\]

It is easy to see that for a fixed \(\alpha\), the coefficients of its first \(N + 1\) terms in Taylor expansion in powers of \(t - 1\) coincide with those of the ‘exact’ polynomial \(V_\alpha(K; t)\). The following simple lemma is an analog of Lemma 2.3 of [14].

Lemma 3.3  If \(f(t) \in \mathbb{Z}[t, t^{-1}]\) and \(\partial_t^n f(t) \big|_{t=0} = 0\) for \(0 \leq n \leq N\), then there exists \(g(t) \in \mathbb{Z}[t, t^{-1}]\) such that \(f(t) = (t - 1)^{N+1}g(t)\).

It follows from this lemma that there exists a function \(g_\alpha(t)\) such that \(g_\alpha(t) \in \mathbb{Z}[t, t^{-1}]\) for \(\alpha \in \mathbb{Z}\) and

\[
V_\alpha(K; t) = V^{(N)}_\alpha(K; t) + (t - 1)^{N+1}g_\alpha(t). \tag{3.21}
\]

Denote by \(Z'_\lambda(N)(M; K)\) the l.h.s. of eq. (3.7) if we substitute \(V^{(N)}_\alpha(K; t)\) instead of \(V_\alpha(K; t)\) in its r.h.s..

Lemma 3.4  The ‘approximate invariant’ \(Z'_\lambda(N)(M; K)\) belongs to the cyclotomic ring \(\mathbb{Z}[\bar{q}]\)

\[
Z'_\lambda(N)(M; K) \in \mathbb{Z}[\bar{q}] \tag{3.22}
\]

and converges \(K\)-adically to the ‘exact’ invariant \(Z'(M; K)\):

\[
\lim_{N \to \infty} Z'_\lambda(N)(M; K) = Z'(M; K). \tag{3.23}
\]

Proof of Lemma 3.4.  To prove (3.22) we calculate explicitly the sum of eq. (3.7) for an individual monomial \(d_{m,n} \left(\bar{q}^\frac{\alpha}{2} - \bar{q}^{-\frac{\alpha}{2}}\right)^{2m} (\bar{q} - 1)^n:\n\]

\[
\frac{1}{\sqrt{K}} \sum_{0 \leq \alpha' \leq 2K} \sum_{\alpha \in 2 \mathbb{Z} + 1} \bar{q}^{4^*q^*p_\alpha t^2} \left(\bar{q}^\frac{\alpha}{2} - \bar{q}^{-\frac{\alpha}{2}}\right)^{2m} (\bar{q} - 1)^n = \frac{1}{\sqrt{K}} \sum_{0 \leq \alpha' \leq 2K} \sum_{\alpha \in 2 \mathbb{Z} + 1} \bar{q}^{4^*q^*p_\alpha t^2} \sum_{k=0}^{2m} \binom{2m}{k} \bar{q}^{\alpha(m-k)} \tag{3.24}
\]
\[
\frac{1}{\sqrt{K}} \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) \bar{q}^{-p^*q(m-k)^2+p^*(1+\mu q)(m-k)} \sum_{0 \leq \alpha' < 2K} \bar{q}^{4^*p^*\alpha'^2} \\
= e^{\frac{i\pi}{4}(1-\kappa)} \left\{ q^*p \right\}_K \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) \bar{q}^{-p^*q(m-k)^2+p^*(1+\mu q)(m-k)}. 
\]

Since

\[ -i \{|q|\}_K \text{sign} (q) e^{\frac{i\pi}{4}(\kappa+1)\text{sign}(\frac{q}{q})} e^{\frac{i\pi}{4}(1-\kappa)} \{ q^*p \}_K = \text{sign} (p) \{|p|\}_K, \quad (3.25) \]

we conclude that

\[
Z'_{(N)}(M; K) = \frac{|p|}_K \text{sign} (p) q^{3\lambda_{CW}(M)} \sum_{\mu = \pm 1} \mu \bar{q}^{-3p^*q\Delta^\alpha_q(K)-2^*p^*+2^*} \\
\times \sum_{m+n \leq N} d_{m,n}(K) h^n \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) \bar{q}^{-p^*q(m-k)^2+p^*(1+\mu q)(m-k)}.
\]

The divisibility by \( h = 1-\bar{q} \) in the r.h.s. of this equation is easy to see. Indeed, the coefficient at \( h^0 \) in the expansion of the summand in the sum over \( \mu \) can be obtained by substituting there \( \bar{q} = 1 \). The resulting expression is independent of \( \mu \) except for the overall prefactor of \( \mu \), hence it is eliminated by the sum over \( \mu \). Thus eq. (3.26) demonstrates (3.22).

To prove the \( K \)-adic limit (3.23) we recall that

\[
\sum_{0 \leq \alpha < K} \bar{q}^{-\alpha^2} = \sqrt{K} e^{\frac{i\pi}{4}(\kappa-1)} 
\]

and

\[
\sum_{0 \leq \alpha < K} \bar{q}^{-\alpha^2} = h^\frac{K-1}{2} w, \quad w, w^{-1} \in \mathbb{Z}[\bar{q}], \quad (3.28)
\]

hence

\[
\frac{i}{\sqrt{K}} e^{\frac{i\pi}{4}(\kappa+1)\text{sign}(\frac{q}{q})} = e^{\frac{i\pi}{4}(\kappa+1)(\text{sign}(\frac{q}{q})+1)} \frac{w^{-1}}{h^{\frac{K-1}{2}}} = \frac{w'}{h^{\frac{K-1}{2}}}, \quad w' \in \mathbb{Z}[\bar{q}].
\]

Therefore if we substitute the ‘error term’ \((\bar{q} - 1)^{N+1}g_\alpha(\bar{q})\) of eq. (3.21) into the r.h.s. of eq. (3.7) instead of \( V_\alpha(K; \bar{q}) \), then we find that

\[
Z'(M; K) - Z'_{(N)}(M; K) = h^{N-\frac{K-1}{2}} w'', \quad w'' \in \mathbb{Z}[\bar{q}].
\]

(3.30)
This relation implies $K$-adic limit (3.23). □

There is an ‘asymptotic’ counterpart of Lemma 3.4. Denote by $Z^{(\text{tr})}_{(N)}(M; K)$ an ‘approximate perturbative invariant’ which results from substituting $V_\alpha^{(N)}(K; \bar{q})$ instead of $V_\alpha(K; \bar{q})$ in the r.h.s. of eq. (3.14).

**Lemma 3.5** The approximate perturbative invariant $Z^{(\text{tr})}_{(N)}(M; K)$ converges ‘formally’ to the exact invariant $Z^{(\text{tr})}(M; K)$:

$$\lim_{N \to \infty} Z^{(\text{tr})}_{(N)}(M; K) = Z^{(\text{tr})}(M; K).$$

(3.31)

The ‘formal limit’ (3.31) means that for any $N' > 0$ there exists $N_0$ such that for any $N \geq N_0$ the first $N'$ approximate invariants $\lambda^{(N)}_n(M)$, $0 \leq n < N'$, extracted from $Z^{(\text{tr})}_{(N)}(M; K)$ via the analog of eq. (1.18), coincide with the exact invariants $\lambda_n(M)$.

**Proof of Lemma 3.5.** An easy estimate

$$\frac{1}{\sqrt{K}} \int_{-\infty}^{+\infty} d\alpha' \bar{q}^{\frac{1}{p} \alpha'^2} \left( \bar{q}^{\frac{\alpha}{2}} - \bar{q}^{-\frac{\alpha}{2}} \right)^{2m} = O(h^m)$$

(3.32)

demonstrates that an individual term $d_{m,n} \left( \bar{q}^{\frac{\alpha}{2}} - \bar{q}^{-\frac{\alpha}{2}} \right)^{2m} h^n$ of the sum (3.20) contributes only to $\lambda^{(N')}_n(M)$, $n' \geq n + m - 1$. Therefore the terms with $d_{m,n}$, $m + n \geq N + 1$ do not contribute to $\lambda_n(M)$, $n \leq N - 1$. As a result,

$$\lambda^{(N)}_n(M) = \lambda_n(M) \quad \text{for } n \leq N - 1.$$ 

(3.33)

This proves the Lemma. □

Our final lemma is

**Lemma 3.6** Approximate invariants $Z^{(\text{tr})}_{(N)}(M; K)$ and $Z^{(\text{tr})}_{(N)}(M; K)$ satisfy Conjecture 1.2. Namely, the ‘invariants’ $\bar{\lambda}^{(N)}_n(M)$ extracted from $Z^{(\text{tr})}_{(N)}(M; K)$ via the analog of eq. (1.18) have only ‘simple’ denominators

$$\bar{\lambda}^{(N)}_n(M) \in \mathbb{Z} \left[ \frac{1}{p} \right],$$ 

(3.34)
and there is a $K$-adic relation

$$\left[ \frac{|H_1(M, \mathbb{Z})|_{(N)}}{\sqrt{2} \sin \left( \frac{\pi}{K} \right)} \right]^{\frac{3}{2}} Z_{(N)}^{(tr)}(M; K) = \left\{ \frac{|H_1(M, \mathbb{Z})||H_1(M, \mathbb{Z})|Z'(M; K)}{K} \right\}.$$  (3.35)

**Proof of Lemma 3.6.** An integral of eq. (3.14) can be easily evaluated for an individual term of the sum (3.20):

$$\int_{-\infty}^{+\infty} d\alpha' \frac{q^{1/2} m^2}{\alpha'} \left( q^\frac{\alpha}{\alpha'} - q^{-\frac{\alpha}{\alpha'}} \right)^{2m} \int_{-\infty}^{+\infty} d\alpha' \frac{q^{1/2} m^2}{\alpha'} \sum_{k=0}^{2m} \left( \frac{2m}{k} \right) q^{\alpha(m-k)}$$

$$= \sqrt{2K} \left| \frac{q}{p} \right|^{1/2} e^{\frac{2\pi i}{p} \text{sign}(\frac{q}{p})} \sum_{k=0}^{2m} \left( \frac{2m}{k} \right) q^{\frac{1+\mu k}{p}(m-k)} q^{-\frac{\mu k}{p}(m-k)^2}$$

Since

$$i \text{sign}(q) e^{-\frac{2\pi i}{p} \text{sign}(\frac{q}{p})} = \text{sign}(p),$$

we find that

$$\frac{|H_1(M, \mathbb{Z})|_{(N)}}{\sqrt{2} \sin \left( \frac{\pi}{K} \right)} Z_{(N)}^{(tr)}(M; K) = \frac{\text{sign}(q)}{1-q} q^{3\lambda_{CW}(M)} \sum_{\mu=\pm 1} \mu q^{-3\#\Delta'(K)-\frac{1}{2}+\frac{i}{2}}$$

$$\times \sum_{m+n \leq N} d_{m,n}(K)h^n \sum_{k=0}^{2m} \left( \frac{2m}{k} \right) q^{\frac{1+\mu k}{p}(m-k)} q^{-\frac{\mu k}{p}(m-k)^2}.$$  (3.38)

We analyze the coefficients in the expansion of this formula in powers of $h$ with the help of Lemma 2.3. The factor $\frac{1}{1-q} = -\frac{1}{h}$ does not produce negative powers of $h$ because the term at $h^0$ in the summand of $\sum_{\mu=\pm 1}$ is killed by the sum over $\mu$ the same way as it happened in eq. (3.26). All the exponents of $q$ belong explicitly to $\mathbb{Z} \left[ \frac{1}{p} \right]$ except the term

$$-\frac{\mu}{2p} + \frac{1}{2} = \frac{1}{p} \cdot \frac{p - \mu}{2}.$$  (3.39)

If $p$ is even then $\frac{1}{2} \in \mathbb{Z} \left[ \frac{1}{p} \right]$. If $p$ is odd then $p - \mu \in 2\mathbb{Z}$. Thus we proved (3.34).

Equation (3.35) comes from comparing the expansions of the r.h.s. of eq. (3.26) and (3.38) in powers of $h$ and applying eq. (2.40). $\square$

The Conjecture 1.2 for $M$ is a direct consequence of Lemmas 3.4, 3.5 and 3.6.
4 Numerical Examples

In her paper [10], R. Lawrence has already presented some numerical evidence in support of Conjecture 1.1 for the manifolds constructed by a (1, 1) surgery on torus knots. The proof of Conjecture 1.2 in Section 3 for manifolds constructed by a rational surgery on a knot, was based on another conjecture and therefore deserves a numerical check. In fact, we verified the underlying Conjecture 3.1 in [25] numerically with certain precision for some simple knots. This result translates through the proofs of Section 3 into numerical verification of Conjecture 1.3. Here we present the examples of Ohtsuki and $SO(3)$ invariants of some simple integer homology spheres for mainly illustrative purposes.

Let $\chi_q(K)$ denote an integer homology sphere constructed by a $(1, q)$ surgery on a knot $K \subset S^3$. We calculated $SO(3)$ invariants at $K = 5$ for six manifolds $\chi_{1,2,3}(4_1)$ and $\chi_{1,2,3}(6_1)$ ($4_1$ is known as ‘8-knot’, see e.g. [4] for the pictures of knots; $\chi_{1}(4_1)$ coincides with the manifold $M_{3, -4}$ of [9]). The invariants are presented as polynomials (1.3). Their coefficients $a_n$ are listed in Table 1. The Ohtsuki invariants $\lambda_n(M)$, $0 \leq n \leq 11$ of the same manifolds were calculated with the help of eqs. (1.18) and (3.14). They are collected in Table 2. The Table 3 contains the coefficients $\tilde{a}_n$ defined by the relation

$$\sum_{n=0}^3 \tilde{a}_n(M) h^n = \left[ \sum_{n=0}^{11} \lambda_n(M) h^n \right]^{\bullet}. \quad (4.1)$$

It is easy to check that

$$a_n = \tilde{a}_n \pmod{5^3} \quad (4.2)$$

in full agreement with $K$-adic limit (1.8) of Conjecture 1.1.

5 Discussion

E. Witten [27] has originally defined the $SU(2)$ WRT invariant of a 3d manifold $M$ as a path integral over the $SU(2)$ connections $A$ on $M$:

$$Z(M; K) = \int D A e^{\frac{i K - 2}{4\pi} \int_M (A, dA + \frac{1}{4}[A, A])}, \quad (5.1)$$
\[
\begin{array}{cccccccc}
\chi_1(4_1) & \chi_2(4_1) & \chi_3(4_1) & \chi_1(6_1) & \chi_2(6_1) & \chi_3(6_1) \\
\hline
a_0 & 1 & 6 & 1 & -4 & 1 & 1 \\
a_1 & 4 & 8 & 2 & -7 & 1 & -1 \\
a_2 & 4 & 5 & 3 & -4 & 1 & 0 \\
a_3 & 1 & 1 & 1 & -1 & 0 & 0 \\
\end{array}
\]

Table 1: The coefficients \(a_n(\chi_{1,2,3}(4_1))\) and \(a_n(\chi_{1,2,3}(6_1))\).

| \lambda_0 | \lambda_1 | \lambda_2 | \lambda_3 | \lambda_4 | \lambda_5 | \lambda_6 | \lambda_7 | \lambda_8 |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \chi_1(4_1) | 1 | -6 | 69 | -1064 | 20770 | -492052 | 13724452 | -440706098 | 16015171303 |
| \chi_2(4_1) | 1 | -12 | 270 | -8284 | 324109 | -15440692 | 867600594 | -56182136200 | 4119997542641 |
| \chi_3(4_1) | 1 | -18 | 603 | -27684 | 1624005 | -116107654 | 9795346273 | -952637958170 | 104938749980019 |
| \chi_1(6_1) | 1 | -12 | 246 | -6916 | 248171 | -10848488 | 559466999 | -33256127501 | 223888918356 |
| \chi_2(6_1) | 1 | -24 | 996 | -57200 | 4207360 | -377589690 | 40010000129 | -4889051681203 | 676832345117565 |
| \chi_3(6_1) | 1 | -36 | 2250 | -195060 | 21677895 | -2940578892 | 471099854664 | -87035716226366 | 1821988681495290 |

Table 2: Ohtsuki invariants \(\lambda_n(\chi_{1,2,3}(4_1))\) and \(\lambda_n(\chi_{1,2,3}(6_1))\).
Table 3: The coefficients $\tilde{a}_n(\chi_{1,2,3}(4_1))$ and $a_n(\chi_{1,2,3}(6_1))$.

here $\langle , \rangle$ is an appropriately normalized Killing form on $su(2)$. The methods of quantum field theory allow us to calculate this integral by the stationary phase approximation for large values of $K$. The stationary points of the exponent (which is proportional to the Chern-Simons invariant) are flat connections. Therefore the invariant $Z(M; K)$ is presented in the large $K$ limit as a sum over the contributions $Z^{(c)}(M; K)$ coming from connected components of the moduli space of flat $SU(2)$ connections on $M$ ($c$ indexes these components):

$$Z(M; K) = \sum_c Z^{(c)}(M; K). \quad (5.2)$$

The individual contributions are expressed generally as a product of a ‘classical exponential’, a certain power of $K$ and an asymptotically convergent series:

$$Z^{(c)}(M; K) = e^{\frac{2\pi i}{4\pi} S_{CS}^{(c)} K^{\dim H_1^{(c)} - \dim H_0^{(c)}}} \sum_{n=0}^{\infty} \Delta_n^{(c)}(M) K^{-n}. \quad (5.3)$$

In this formula $S_{CS}^{(c)}$ is the Chern-Simons invariant of connections of $c$, $H_0^{(c)}$ are the cohomologies of $d$ twisted by a connection of $c$ and the coefficients $\Delta_n^{(c)}(M)$ are called ‘$(n + 1)$-loop corrections’.

The asymptotic form (5.1), (5.2) of the WRT invariant was tested numerically and analytically by D. Freed, R. Gompf [5], L. Jeffrey [6] as well as in [19] for lens spaces and Seifert manifolds.
If $M$ is a rational homology sphere, then the trivial connection is a separate point in the moduli space of flat connections on $M$. Therefore the trivial connection should produce a distinct contribution to the sum (5.2). We checked [19] that the ‘perturbative invariant’ (2.26) represents such a contribution for Seifert rational homology spheres. We have reasons to believe (see path integral arguments in Section 3 of [21]) that this is a general case: perturbative invariant $Z^{(\text{tr})}(M; K)$ which is defined through the surgery formula without any reference to path integrals (see [24] and references therein), is equal to the trivial connection contribution to the ‘total’ invariant $Z(M; K)$. Therefore Conjecture 1.1 leads to the following speculation:

\[ \text{Up to a normalization, the same power series } \sum_{n \geq 0} \lambda_n(M) h^n \text{ converges asymptotically to the trivial connection contribution into the SU(2) WRT invariant and converges $K$-adicly to the SO(3) WRT invariant.} \]

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