THE FRACTIONAL S-TRANSFORM ON BMO AND HARDY SPACES

BABY KALITA and SUNIL KUMAR SINGH

Abstract. In this paper, the authors studied the fractional S-transform on BMO and Hardy spaces and generalized the results given in [15]. In introduction section, the definitions of the S-transform, fractional Fourier transform and fractional S-transform are given. In Section 2 continuity and boundedness results for the fractional S-transform in BMO and Hardy spaces are obtained. Furthermore, in Section 3 the fractional S-transform is studied on the weighted BMO and weighted Hardy spaces associated with a tempered weight function.

Key words: S-transform, Fractional S-transform, BMO space, Hardy space

2010 AMS Subject Classification: 65R10; 32A37; 30H10

1. Introduction

The S-transform is a new time-frequency analysis method, which is deduced from short-time Fourier transform and an extension of wavelet transform. It was first introduced by Stockwell et al. [17] in 1996 for analyzing geophysics data and since then has been applied in several discipline, such as geophysics, oceanography, atmospheric physics, medicine, hydrogeology. The continuous S-transform of a function \( f \) with respect to the window function \( \omega \) is defined as [18]

\[
(S_\omega f)(\tau, \xi) = \int_{\mathbb{R}} f(t) \omega(\tau - t, \xi) e^{-i 2\pi \xi t} dt, \quad \text{for } \tau, \xi \in \mathbb{R},
\]

where the window \( \omega \) is assumed to satisfy the following:

\[
\int_{\mathbb{R}} \omega(t, \xi) dt = 1 \text{ for all } \xi \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}.
\]

The most usual window \( \omega \) is a Gaussian one

\[
\omega(t, \xi) = \frac{|\xi|}{k\sqrt{2\pi}} e^{-\xi^2 t^2/2k^2}, \quad k > 0,
\]

where \( \xi \) is the frequency, \( t \) is the time variable, and \( k \) is a scaling factor that controls the number of oscillations in the window.

Equation (1.1) can be rewritten as a convolution

\[
(S_\omega f)(\tau, \xi) = \left( f(\cdot) e^{-i 2\pi \xi \cdot} \ast \omega(\cdot, \xi) \right)(\tau).
\]

Applying the convolution property for the Fourier transform in (1.4), we can obtain a direct relation between S-transform and Fourier transform as follows:

\[
(S_\omega f)(\tau, \xi) = \mathcal{F}^{-1} \left\{ \hat{f}(\cdot + \xi) \hat{\omega}(\cdot, \xi) \right\}(\tau).
\]
where $\hat{f}(\eta) = (\mathcal{F}f)(\eta) = \int_{\mathbb{R}} f(t) e^{-2\pi i \eta t} dt$ is the Fourier transform of $f$. Certain examples and basic properties of the S-transform can be found in [7, 8, 9, 10, 11, 12, 13, 14].

1.1. The fractional Fourier transform. The fractional Fourier transform was introduced by Almeida [1] as a generalization of classical Fourier transform. It has several known applications in the signal processing and many other scientific fields. The $a^{th}$ order FRFT (fractional Fourier transform) of a signal $f$ is defined as

$$F_a^f(\xi) = \int_{\mathbb{R}} f(t) K_a(t, \xi) dt,$$

where the transform kernel is given by

$$K_a(t, \xi) = \begin{cases} A_\theta e^{\pi i (\xi^2 \cot \theta - 2t \csc \theta + t^2 \cot \theta)}, & \text{if } \theta \neq j\pi \\ \delta(t - \xi), & \text{if } \theta = 2j\pi \\ \delta(t + \xi), & \text{if } \theta + \pi = 2j\pi, \end{cases}$$

where $A_\theta = \sqrt{1 - i \cot \theta}, \theta = a\frac{\pi}{2}, a \in [0, 4], i$ is the complex unit, $j$ is an integer and $\xi$ is the fractional Fourier frequency. The inverse FRFT of equation (1.6) is

$$f(t) = \int_{\mathbb{R}} F_a^f(\xi) K_a(t, \xi) d\xi.$$

1.2. The fractional S-transform. The fractional S-transform is used by Xu et.al. [19] in 2012 as a generalization of the S-transform. The $a^{th}$ order continuous fractional S-transform (FRST) of a signal $f(t)$ is defined as

$$FRST_a^f(\tau, \xi) = \int_{\mathbb{R}} f(t) g(\tau - t, \xi) K_a(t, \xi) dt,$$

where the window function $g$ is given by

$$g(t, \xi) = \frac{\lvert \xi \csc \theta \rvert^p}{k\sqrt{2\pi}} e^{-2(\xi \csc \theta)^2/2k^2}; k, p > 0,$$

which satisfies the condition:

$$\int_{\mathbb{R}} g(t, \xi) dt = 1, \text{ for all } \xi \in \mathbb{R}_0. \quad (1.10)$$

Inverse fractional S-transform is defined as

$$f(t) = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} FRST_a^f(\tau, \xi) d\tau \right] K_a(t, \xi) d\xi.$$

The fractional S-transform depends on a parameter $\theta$ and can be interpreted as a rotation by an angle $\theta$ in the time-frequency plane. The parameter $p$ and $k$ can be used to adjust the window function space. The fractional S-transform has been studied on distribution spaces by Singh [9, 10, 11].
2. The fractional S-transform on BMO and Hardy spaces

The bounded mean oscillation space \( BMO(\mathbb{R}) \) also known as John-Nirenberg space was first introduced by F. John and L. Nirenberg in 1961 [5]. It is the dual space of the real Hardy space \( H^1 \) and serves in many ways as a substitute space for \( L^\infty \). The \( BMO(\mathbb{R}) \) spaces play an important role in various areas of mathematics, for example many operators which are bounded on \( L^p, 1 < p < \infty \), but not on \( L^\infty \), are bounded when considered as operators on \( BMO \).

Now, we recall the definitions of the BMO and Hardy spaces.

**Definition 2.1.** The bounded mean oscillation space \( BMO(\mathbb{R}) \) is defined in [16, p. 140] as the space of all Lebesgue integrable (locally) functions defined on \( \mathbb{R} \) such that

\[
\| f \|_{BMO} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty, \quad (2.1)
\]

here the supremum is taken over all intervals \( I \) in \( \mathbb{R} \) of measure \( |I| \) and \( f_I \) stands for the mean of \( f \) on \( I \), namely

\[
f_I := \frac{1}{|I|} \int_I f(x) \, dx \leq \frac{1}{|I|} \int_I |f(x)| \, dx \leq m < \infty. \quad (2.2)
\]

**Definition 2.2.** The Hardy space is defined in [16, pp. 90-91] as the space of all functions \( f \in L^1(\mathbb{R}) \) such that

\[
\| f \|_{H^1} = \int_{\mathbb{R}} \sup_{t > 0} |(f * \phi_t)(x)| \, dx < \infty, \quad (2.3)
\]

where \( \phi \) is any test function with \( \int_{\mathbb{R}} \phi(x) \, dx \neq 0 \) and \( \phi_t(x) = t^{-1} \phi(x/t); \, t > 0, x \in \mathbb{R} \).

Our main results in this section are as follows.

**Lemma 2.1.** If \( K_\theta(x, \xi) \) is the transform kernel defined in (1.7) and \( \theta \neq j\pi \), then for any Lebesgue integrable (locally) function defined on \( \mathbb{R} \) we have

\[
\| f(\cdot)K_\theta(\cdot, \xi) \|_{BMO} \leq |A_\theta| (\| f \|_{BMO} + 2m)
\]

where \( m \) is a constant given in equation (2.2).
Proof. By using the inequality $|K_a(x, \xi)| \leq |A_\theta|$, we have

$$
\|f(\cdot)K_a(\cdot, \xi)\|_{BMO}
= \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left| f(x)K_a(x, \xi) - \frac{1}{|I|} \int_I f(t)K_a(t, \xi)dt \right| dx
= \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left| f(x)K_a(x, \xi) - \frac{K_a(x, \xi)}{|I|} \int_I f(t)dt + \frac{K_a(x, \xi)}{|I|} \int_I f(t)dt \right| dx
\leq \frac{1}{|I|} \int_I f(t)K_a(t, \xi)dt \right| dx
+ \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left| K_a(x, \xi) \left( f(x) - \frac{1}{|I|} \int_I f(t)dt \right) \right| dx
\leq |A_\theta| \|f\|_{BMO} + |A_\theta| \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f| \, dx + |A_\theta| \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left| f(t)K_a(t, \xi) \right| dt dx
\leq |A_\theta| \left( \|f\|_{BMO} + \frac{1}{|I|} m|I| + \frac{1}{|I|} \int_I m \, dx \right)
\leq |A_\theta| \left( \|f\|_{BMO} + \frac{1}{|I|} m|I| + \frac{1}{|I|} \int_I m \, dx \right)
= |A_\theta| \left( \|f\|_{BMO} + 2m \right).
\]

\[\square\]

Theorem 2.2. For any fixed $\xi \in \mathbb{R}_0$ and $\theta \neq j\pi$, the operator $\text{FRST}^\theta : BMO(\mathbb{R}) \to BMO(\mathbb{R})$ is continuous and

$$
\| (\text{FRST}^\theta)(\cdot, \xi) \|_{BMO} \leq |A_\theta| (\|f\|_{BMO} + 2m).$$

Proof. Since

$$
| (\text{FRST}^\theta)(\tau, \xi) - (\text{FRST}^\theta)_I(\tau, \xi) |
= \int_\mathbb{R} f(\tau - x)g(x, \xi)K_a(\tau - x, \xi) dx
- \frac{1}{|I|} \int_I \int_\mathbb{R} f(\alpha - x)g(x, \xi)K_a(\alpha - x, \xi) dx \, d\alpha
= \int_\mathbb{R} f(\tau - x)g(x, \xi)K_a(\tau - x, \xi) dx
- \int_\mathbb{R} g(x, \xi) \left( \frac{1}{|I|} \int_I f(\alpha - x)K_a(\alpha - x, \xi) d\alpha \right) dx
= \int_\mathbb{R} f(\tau - x)K_a(\tau - x, \xi) - \frac{1}{|I|} \int_I f(\alpha - x)K_a(\alpha - x, \xi) d\alpha \, dx
\leq \int_\mathbb{R} g(x, \xi) \left| f(\tau - x)K_a(\tau - x, \xi) - \frac{1}{|I|} \int_I f(\alpha - x)K_a(\alpha - x, \xi) d\alpha \right| dx.
$$
Therefore,

\[
\| (FRST_f^g)(\cdot, \xi) \|_{BMO} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left| (FRST_f^g)(\tau, \xi) - (FRST_f^g)_I(\tau, \xi) \right| d\tau \\
\leq \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \int_{\mathbb{R}} |g(x, \xi)| \left| f(\tau - x)K_a(\tau - x, \xi) \right| - \frac{1}{|I|} \int_{\mathbb{R}} f(\alpha - x)K_a(\alpha - x, \xi) d\alpha \right| dx d\tau \\
= \int_{\mathbb{R}} |g(x, \xi)| \left( \sup_{J \subset \mathbb{R}} \frac{1}{|J|} \int_J \left| f(\tau - x)K_a(\tau - x, \xi) \right| - \frac{1}{|J|} \int_J f(y)K_a(y, \xi) dy \right| dt \right) dx \\
(\text{here } J = I - x \text{ for } x \in \mathbb{R}) \\
= \| f(\cdot)K_a(\cdot, \xi) \|_{BMO} \int_{\mathbb{R}} |g(x, \xi)| dx \\
= \| f(\cdot)K_a(\cdot, \xi) \|_{BMO}.
\]

Now by using above lemma we get

\[
\| (FRST_f^g)(\cdot, \xi) \|_{BMO} \leq |A_\theta| (\| f \|_{BMO} + 2m).
\]

\[\square\]

**Theorem 2.3.** Let \( f \in L^1(\mathbb{R}) \) such that

\[
\sup_{t > 0} \left| \int_{\mathbb{R}} f(x - y)\phi_t(y) dy \right| = \sup_{t > 0} \int_{\mathbb{R}} |f(x - y)\phi_t(y)| dy < \infty. \tag{2.4}
\]

Then for any fixed \( \xi \in \mathbb{R}_0 \), the operator \( FRST_f^g : H^1(\mathbb{R}) \to H^1(\mathbb{R}) \) is continuous and

\[
\| (FRST)(\cdot, \xi) \|_{H^1} \leq |A_\theta| \| f \|_{H^1}.
\]
Proof. We have
\[
\|\text{FRST}_p^\theta(\cdot, \xi)\|_{H^1}
= \int_{\mathbb{R}} \sup_{t > 0} \left| \int f(\cdot - x)g(x, \xi)K_a(\cdot - x, \xi)dx \right| \phi_t(\cdot)(\tau)d\tau
= \int_{\mathbb{R}} \sup_{t > 0} \left| \int f(\tau - x - y)K_a(\tau - x - y, \xi)\phi_t(y)dy \right| dx\, d\tau
\leq \int_{\mathbb{R}} |g(x, \xi)| \left( \int_{\mathbb{R}} \sup_{t > 0} \left| \int f(\tau - x - y)K_a(\tau - x - y, \xi)\phi_t(y)dy \right| d\tau \right) dx
= |A_\theta| \int_{\mathbb{R}} |g(x, \xi)| \left( \int_{\mathbb{R}} \sup_{t > 0} \left| \int f(\tau - x - z)\phi_t(\tau - x - z)dz\, d\tau \right| dx \right)
\]
(by using equation (2.4) and then definition of convolution)
\[
= |A_\theta| \|f\|_{H^1} \int_{\mathbb{R}} |g(x, \xi)| dx
= |A_\theta| \|f\|_{H^1}.
\]

\[\Box\]

3. The fractional S-transform on weighted function spaces

Let us recall the definitions of relevant weighted function spaces.

Definition 3.1. A positive function \(\kappa\) defined on \(\mathbb{R}\) is called a tempered weight function if there exist positive constants \(C\) and \(N\) such that
\[
\kappa(x + \eta) \leq (1 + C|\xi|)^N \kappa(\eta) \quad \text{for all} \ \xi, \eta \in \mathbb{R}, \quad (3.1)
\]
and the set of all such functions \(\kappa\) is denoted by \(\mathcal{K}\). Certain examples and basic properties of weight function \(\kappa\) can be found in [4].

Definition 3.2. For \(1 \leq p < \infty\), the weighted Lebesgue space \(L^p_\kappa(\mathbb{R})\) is defined as the space of all measurable functions \(f\) on \(\mathbb{R}\) such that
\[
\|f\|_{L^p_\kappa} = \left( \int_{\mathbb{R}} |f(x)|^p \kappa(x)dx \right)^{1/p} < \infty.
\]

Definition 3.3. The weighted bounded mean oscillation space \(\text{BMO}_\kappa(\mathbb{R})\) is defined as the space of all weighted Lebesgue integrable (locally) functions defined on \(\mathbb{R}\) such that
\[
\|f\|_{\text{BMO}_\kappa} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|_\kappa} \int_I |f(x) - f_I| \kappa(x)dx < \infty,
\]
where the supremum is taken over all intervals \(I\) in \(\mathbb{R}\) and \(|I|_\kappa = \int_I \kappa(x)dx\).
Definition 3.4. The weighted Hardy space is defined as the space of all functions \( f \in L^1_\kappa(\mathbb{R}) \) such that

\[
\| f \|_{H^1_\kappa} = \int_{\mathbb{R}} \sup_{t > 0} |(f * \phi_t)(x)| \kappa(x) dx < \infty.
\]

where \( \phi \) is any test function with \( \int_{\mathbb{R}} \phi(x) dx \neq 0 \) and \( \phi_t(x) = t^{-1} \phi(x/t); \ t > 0, \ x \in \mathbb{R} \).

Our main results on weighted function spaces are as follows.

Lemma 3.1. If \( g(x, \xi) \) is defined by (1.9) and \( \theta \neq j\pi \), then for any fixed \( \xi \in \mathbb{R}_0 \) we have the following estimate

\[
\int_{\mathbb{R}} |g(x, \xi)|(1 + C|x|)^N dx \leq A_{\xi,N},
\]

where \( N \) is positive constant, and \( A_{\xi,N} \) is a constant depend on \( \xi \) and \( N \).

Proof. Using the well-known inequality, \(|x + y|^n \leq 2^{n-1}(|x|^n + |y|^n), \ \forall x, y \in \mathbb{R}, \ \forall n \in \mathbb{N}\), we have

\[
\int_{\mathbb{R}} |g(x, \xi)|(1 + C|x|)^N dx \\
\leq \int_{\mathbb{R}} |g(x, \xi)| 2^N \left( 1 + |C_\xi|^{[N]+1} \right) dx \\
= 2^N \left( 1 + C^{[N]+1} \int_{\mathbb{R}} |g(x, \xi)||x|^{[N]+1} dx \right) \\
= 2^N \left( 1 + 2C^{[N]+1} \int_{0}^{\infty} x^{[N]+1} g(x, \xi) dx \right) \\
= 2^N \left( 1 + 2C^{[N]+1} \frac{2^{([N]+1)/2} \xi^{[N]+1}}{2\sqrt{\pi} \mid \xi \mid \csc \theta |^{[N]+1}} \Gamma([N]/2 + 1) \right) \\
\leq A_{\xi,N}.
\]

\[ \square \]

Lemma 3.2. If \( K_a(x, \xi) \) is the transform kernel defined in (1.7) and \( \theta \neq j\pi \), then for any locally Lebesgue integrable function defined on \( \mathbb{R} \) we have

\[
\| f(\cdot)K_a(\cdot, \xi) \|_{BMO_\kappa} \leq |A_\theta| \left( \| f \|_{BMO_\kappa} + 2m \right)
\]

where \( m \) is a constant given in equation (2.2).
Proof. We have

$$\| f(\cdot)K_a(\cdot, \xi) \|_{BMO_\kappa}$$

$$= \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I f(x)K_a(x, \xi) - \frac{1}{|I|} \int_I f(t)K_a(t, \xi) dt \kappa(x) dx$$

$$= \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I f(x)K_a(x, \xi) - \frac{K_a(x, \xi)}{|I|} \int_I f(t) dt + \frac{K_a(x, \xi)}{|I|} \int_I f(t) dt$$

$$- \frac{1}{|I|} \int_I f(t)K_a(t, \xi) dt \kappa(x) dx$$

$$= \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I K_a(x, \xi) \left( f(x) - \frac{1}{|I|} \int_I f(t) dt \right) + \frac{K_a(x, \xi)}{|I|} \int_I f(t) dt$$

$$- \frac{1}{|I|} \int_I f(t)K_a(t, \xi) dt \kappa(x) dx$$

$$\leq \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I K_a(x, \xi) \left( f(x) - \frac{1}{|I|} \int_I f(t) dt \right) \kappa(x) dx$$

$$+ \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \frac{K_a(x, \xi)}{|I|} \int_I f(t) dt \kappa(x) dx$$

$$+ \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \frac{1}{|I|} \int_I f(t)K_a(t, \xi) dt \kappa(x) dx$$

$$\leq |A_\phi| \| f \|_{BMO_\kappa} + |A_\phi| \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f|_I \kappa(x) dx$$

$$+ |A_\phi| \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I \left( \frac{1}{|I|} \int_I |f(t)| dt \right) \kappa(x) dx$$

$$\leq |A_\phi| \left( \| f \|_{BMO_\kappa} + \frac{1}{|I|} m |I| + \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I m \kappa(x) dx \right)$$

$$\leq |A_\phi| \left( \| f \|_{BMO_\kappa} + \frac{1}{|I|} m |I| + \frac{1}{|I|} m |I| \right)$$

$$= |A_\phi| \left( \| f \|_{BMO_\kappa} + 2m \right).$$

\[ \square \]

**Theorem 3.3.** For any fixed $\xi \in \mathbb{R}_0$, the operator $FRST_f^a : BMO_\kappa(\mathbb{R}) \rightarrow BMO_\kappa(\mathbb{R})$ is continuous. Furthermore, we have

$$\| (FRST_f^a)(\cdot, \xi) \|_{BMO_\kappa} \leq A_{\xi,N} |A_\phi| \left( \| f \|_{BMO_\kappa} + 2m \right)$$

where $m$ is a constant given in equation (2.2).
Proof.

\[
\| (\text{FRST}_f^\tau)(\cdot, \xi) \|_{BMO_{\kappa}}
= \sup_{I \subset \mathbb{R}} \frac{1}{|I|^{\kappa}} \int_I \left| (\text{FRST}_f^\tau)(\tau, \xi) - (\text{FRST}_f^\tau)_I(\tau, \xi) \right| \kappa(\tau) d\tau
\]

\[
= \sup_{I \subset \mathbb{R}} \frac{1}{|I|^{\kappa}} \int_I \left| \int_{\mathbb{R}} g(x, \xi) f(\tau - x) K_{\alpha}(\tau - x, \xi) d\alpha \right| \kappa(\tau) d\tau
\]

\[
- \frac{1}{|I|} \int_I f(\alpha - x) K_{\alpha}(\alpha - x, \xi) d\alpha \right| \kappa(\tau) d\tau
\]

\[
\leq \int_{\mathbb{R}} |g(x, \xi)| \left( \sup_{J \subset \mathbb{R}} \frac{1}{|J|^{\kappa}} \int_J \left| f(\tau - x) K_{\alpha}(\tau - x, \xi) \right| \kappa(\tau) d\tau \right) dx
\]

\[
- \frac{1}{|J|^{\kappa}} \int_J f(\alpha - x) K_{\alpha}(\alpha - x, \xi) d\alpha \left| \kappa(\tau) d\tau \right| dx
\]

\[
= \int_{\mathbb{R}} |g(x, \xi)| \left( \sup_{J \subset \mathbb{R}} \frac{1}{|J|^{\kappa}} \int_J \left| f(y) K_{\alpha}(y, \xi) - \frac{1}{|J|^{\kappa}} \int_J f(t) K_{\alpha}(t, \xi) dt \right| \kappa(x + y) dy \right) dx
\]

(here \( J = I - x \) for \( x \in \mathbb{R} \))

\[
\leq \int_{\mathbb{R}} |g(x, \xi)| \left( \sup_{J \subset \mathbb{R}} \frac{1}{|J|^{\kappa}} \int_J \left| f(y) K_{\alpha}(y, \xi) \right| \kappa(x + y) dy \right) dx
\]

\[
- \frac{1}{|J|^{\kappa}} \int_J f(t) K_{\alpha}(t, \xi) dt (1 + C|x|)^N \kappa(y) dy \right) dx
\]

\[
= \| f(\cdot) K_{\alpha}(\cdot, \xi) \|_{BMO_{\kappa}} \int_{\mathbb{R}} |g(x, \xi)| (1 + C|x|)^N dx.
\]

So by using lemma 3.1 and 3.2, we get

\[
\| (\text{FRST}_f^\tau)(\cdot, \xi) \|_{BMO_{\kappa}} \leq A_{\xi, N} |A_{\theta}| (\| f \|_{BMO_{\kappa}} + 2m).
\]

\[\Box\]

**Theorem 3.4.** Let \( f \in L^1(\mathbb{R}) \) and satisfies the condition (2.4). Then, for any fixed \( \xi \in \mathbb{R}_0 \), the operator \( \text{FRST}_f^\tau : H^1_\kappa(\mathbb{R}) \rightarrow H^1_\kappa(\mathbb{R}) \) is continuous. Furthermore, we have

\[
\| \text{FRST}_f^\tau(\cdot, \xi) \|_{H^1_\kappa} \leq A_{\xi, N} |A_{\theta}| \| f \|_{H^1_\kappa}.
\]
Proof. We have
\[ \| \text{FRST}^\theta_f (\cdot, \xi) \|_{L^1_+} \]
\[ = \int_{\mathbb{R}} \sup_{t > 0} \left| \left( \text{FRST}^\theta_f (\cdot, \xi) * \phi_t (\cdot) \right)(\tau) \right| \kappa(\tau) d\tau \]
\[ = \int_{\mathbb{R}} \sup_{t > 0} \left| \int_{\mathbb{R}} g(x, \xi) \left( \int_{\mathbb{R}} f(\tau - x - y) K_\alpha(\tau - x - y, \xi) \phi_t(y) dy \right) dx \right| \kappa(\tau) d\tau \]
\[ \leq \int_{\mathbb{R}} \left| g(x, \xi) \right| \left( \int_{\mathbb{R}} \sup_{t > 0} \left| \int_{\mathbb{R}} f(\tau - x - y) K_\alpha(\tau - x - y, \xi) \phi_t(y) \right| \kappa(\tau) d\tau \right) dx \]
\[ = \int_{\mathbb{R}} \left| g(x, \xi) \right| \left( \int_{\mathbb{R}} \sup_{t > 0} \left| f(\tau - x - y) K_\alpha(\tau - x - y, \xi) \phi_t(y) \right| \kappa(\tau) d\tau \right) dx \]
\[ \leq |A_\theta| \int_{\mathbb{R}} \left| g(x, \xi) \right| \left( \int_{\mathbb{R}} \sup_{t > 0} \left| f(\tau - x - y) \phi_t(y) \right| (1 + C|x|)^N \kappa(\tau) d\tau \right) dx \]
\[ = |A_\theta| \| f \|_{L^1_+} \int_{\mathbb{R}} \left| g(x, \xi) \right| (1 + C|x|)^N dx. \]

Now by using lemma 3.1 we get
\[ \| \text{FRST}^\theta_f (\cdot, \xi) \|_{L^1_+} \leq A_{\xi, N} |A_\theta| \| f \|_{L^1_+}. \]

\[ \square \]

References

[1] L. B. Almeida : The fractional Fourier transform and time-frequency representations. IEEE Trans. Signal Process. 42(11) (1994), 3084-3091.
[2] N. M. Chuong, D. V. Duong : Boundedness of the wavelet integral operator on weighted function spaces. Russ. J. Math. Phys. 20(3) (2013), 268-275.
[3] C. Fefferman, E. M. Stein : $H^p$ spaces of several variables, Acta Math. 129(1) (1972), 137-193.
[4] L. Hörmander : The Analysis of Linear Partial Differential Operators II. Springer, Berlin (1983).
[5] F. John, L. Nirenberg : On functions of bounded mean oscillation. Commun. Pure Appl. Math. 14 (1961), 415-426.
[6] R. S. Pathak, S. K. Singh : Boundedness of the wavelet transform in certain function spaces. J. Inequal. Pure Appl. Math. 8(1) (2007), Article 23
[7] S. K. Singh : The S-transform on spaces of type S. Integral Transforms Spec. Funct. 23(7) (2012), 481-494.
[8] S. K. Singh : The S-transform on spaces of type W. Integral Transforms Spec. Funct. 23(12) (2012), 891-899.
[9] S. K. Singh : The fractional S-transform of tempered ultradistributions. Investig. Math. Sci. 2(2) (2012), 315-325.
[10] S. K. Singh : The fractional S-transform on spaces of type S. J. Math. 2013. doi: 10.1155/2013/105848. Article ID 105848.
[11] S. K. Singh : The fractional S-transform on spaces of type W. J. Pseudo-Differ. Oper. Appl. 4(2) (2013), 251-265.
[12] S. K. Singh : A new integral transform: Theory part. Investig. Math. Sci. 3(1) (2013), 135-139.
[13] S. K. Singh : The S-Transform of distributions. Sci. World J. 2014. doi: 10.1155/2014/623294. Article ID 623294.
[14] S. K. Singh: Besov norms in terms of the S-transform. Afr. Mat. (2015). doi: 10.1007/s13370-015-0365-0.
[15] S. K. Singh, B. Kalita: The S-transform on Hardy spaces and its duals. International J. Anal. Appl. 7(2) (2015), 171-178.
[16] E. M. Stein: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993).
[17] R. G. Stockwell, L. Mansinha, R. P. Lowe: Localization of the complex spectrum: The S transform. IEEE Trans. Signal Process. 44(4) (1996), 998-1001.
[18] S. Ventosa, C. Simon, M. Schimmel, J. Dañobeitia, A. Manuel: The S-transform from a wavelet point of view. IEEE Trans. Signal Process. 56(07) (2008), 2771-2780.
[19] D. P. Xu, K. Guo: Fractional S transform -Part 1:Theory. Applied Geophysics. 9(1) (2012), 73-79.

Department of Mathematics, Rajiv Gandhi University
Doimukh-791112, Arunachal Pradesh, India
1 Email: kalitababy478@gmail.com
2 Email: sks_math@yahoo.com
* Corresponding author