RANK-2 ATTRACTORS AND FERMAT TYPE CY n-FOLDS

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Abstract. The Fermat type Calabi-Yau n-fold, denoted by \( F_n \), is the hypersurface of \( \mathbb{P}^{n+1} \) defined by \( \sum_{i=0}^{n+1} x_i^{n+2} = 0 \), which is the smooth fiber over the Fermat point \( \psi = 0 \) of the Fermat pencil

\[
\sum_{i=0}^{n+1} x_i^{n+2} - (n+2) \psi \prod_{i=0}^{n+1} x_i = 0.
\]

The nowhere vanishing holomorphic \( n \)-form on \( F_n \) defines an \( n+1 \) dimensional sub-Hodge structure of \( (H^n(F_n, \mathbb{Q}), F_p) \). In this paper, we will formulate a conjecture which says that this \( n+1 \) dimensional sub-Hodge structure splits completely into the direct sum of pure Hodge structures with dimensions \( \leq 2 \), among which is a direct summand \( H_{a,1}^n \) whose Hodge decomposition is

\[
H_{a,1}^n = H^{n,0}(F_n) \oplus H^{0,n}(F_n).
\]

Using numerical methods, we are able to explicitly construct such a split for the cases where \( n = 3, 4, 6 \), while we also construct a partial split for the cases where \( n = 8, 10 \).

For \( n = 3, 4, 6, 8, 10 \), we have numerically found that the value of the mirror map \( t \) for the Fermat pencil at the Fermat point \( \psi = 0 \) is of the form

\[
t|_{\psi=0} = \frac{1}{2} + \xi i,
\]

where \( \xi \) is a real algebraic number that intuitively depends on the integer \( n+2 \). Furthermore, we have also numerically found that the quotient \( c^+(H_{a,1}^n)/c^-(H_{a,1}^n) \) of the Deligne’s periods of \( H_{a,1}^n \) is an algebraic number for the cases where \( n = 3, 4, 6, 8, 10 \), and in fact we will formulate a stronger conjecture generalizing this observation.

Keywords: Calabi-Yau, attractor, Fermat point, Hodge structure, periods.

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1. Introduction

The Fermat type CY (Calabi-Yau) $n$-fold $\mathcal{F}_n$ is by definition the hypersurface of $\mathbb{P}^{n+1}$ defined by the degree-$(n + 2)$ equation

$$\mathcal{F}_n: x_0^{n+2} + x_1^{n+2} + \cdots + x_{n+1}^{n+2} = 0,$$

(1.1)

which is a CY $n$-fold rationally defined over $\mathbb{Q}$. When $n = 1$, $\mathcal{F}_1$ is a CM elliptic curve with $j$-invariant $0$. The pure Hodge structure on $H^1(\mathcal{F}_1, \mathbb{Q})$ is two dimensional, whose Hodge decomposition only has $(1, 0)$ and $(0, 1)$ components. When $n = 2$, $\mathcal{F}_2$ is usually called the Fermat quartic, which is a singular K3 surface. The pure Hodge structure on $H^2(\mathcal{F}_2, \mathbb{Q})$ has a two dimensional sub-Hodge structure $H^2_a$ with Hodge decomposition [15]

$$H^2_a = H^{2,0}(\mathcal{F}_2) \oplus H^{0,2}(\mathcal{F}_2).$$

(1.2)

Hence one might wonder whether this property of $\mathcal{F}_1$ and $\mathcal{F}_2$ admits any generalizations for $n \geq 3$?

The natural generalization that comes to our mind is: **Does there exist a real number field $k$ over which the pure Hodge structure on $H^n(\mathcal{F}_n, \mathbb{Q})$ has a sub-Hodge structure $H^n_a$ with Hodge decomposition**

$$H^n_a = H^{n,0}(\mathcal{F}_n) \oplus H^{0,n}(\mathcal{F}_n)?$$

(1.3)

For a CY threefold whose middle Hodge structure has such a two dimensional direct summand with $k = \mathbb{Q}$, it is called a rank-2 attractor, where the terminology comes from string theory [8]. Rank-2 attractors have very interesting applications in the constructions of BPS black holes in type IIB string theory [8]. In this paper, we will follow this terminology and call $\mathcal{F}_n$ a rank-2 attractor if the answer to the previous question is yes.

Now recall that the Fermat pencil of CY $n$-folds is defined by the equation

$$\mathcal{F}_\psi: \sum_{i=0}^{n+1} x_i^{n+2} - (n + 2) \psi \prod_{i=0}^{n+1} x_i = 0,$$

(1.4)

which is smooth if $\psi^{n+2} \neq 1$ and $\psi \neq \infty$. Notice that the smooth fiber of this pencil over the Fermat point $\psi = 0$ is just $\mathcal{F}_n$. There is a canonical way to construct a nowhere vanishing holomorphic $n$-form $\Omega_\psi$ on a smooth fiber $\mathcal{F}_\psi$, which is defined over $\mathbb{Q}$ if $\psi \in \mathbb{Q} - \{1\}$ [9, 12]. Suppose the underlying differentiable manifold of a smooth fiber $\mathcal{F}_\psi$ is denoted by $X$. The holomorphic $n$-form $\Omega_\psi$ defines an $n + 1$ dimensional sub-Hodge structure $(H^{n,a}(X, \mathbb{Q}), F^{p,a}_\psi)$ of the pure Hodge structure $(H^n(X, \mathbb{Q}), F^{p}_\psi)$ on $\mathcal{F}_\psi$. In this paper, we will use the Picard-Fuchs equation of $\Omega_\psi$ to numerically compute the periods of the $n$-form $\Omega_0$ (at $\psi = 0$) when $n = 3, 4, 6, 8, 10$. Based on these numerical results, we will show that the Fermat type CY $n$-fold $\mathcal{F}_n$ is indeed a rank-2 attractor when $n = 3, 4, 6, 8, 10$. In fact, we have discovered something much stronger!

Here we summarize our results:

1. When $n = 3$, i.e. the Fermat quintic $\mathcal{F}_3$, the four dimensional sub-Hodge structure $(H^{3,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_3$ splits into the following direct sum over $\mathbb{Q}(\sqrt{5})$

$$\left( H^{3,a}(X, \mathbb{Q}), F^{p,a}_0 \right)_2 = H^3_{a,1} \oplus H^3_{a,2}. $$

(1.5)
Here the Hodge decomposition of the direct summand $H^3_{a,1}$ is given by
\[ H^3_{a,1} = H^{3,0}(\mathcal{F}_3) \oplus H^{0,3}(\mathcal{F}_3), \] (1.6)
and the Hodge type of $H^3_{a,2}$ is $(2, 1) + (1, 2)$. More concretely, using numerical methods we have found two charges $\rho_1, \rho_2 \in H^3(X, \mathbb{Q})$ whose Hodge decompositions only have $(3, 0)$ and $(0, 3)$ components. Moreover, the numerical value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ agrees with
\[ t|_{\psi=0} = \frac{1}{2} + \left( \frac{1}{4} + \sqrt{5} \frac{1}{10} \right)^{1/2} i. \] (1.7)

(2) When $n = 4$, i.e. the Fermat sextic $\mathcal{F}_4$, the five dimensional sub-Hodge structure $(H^{4,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_4$ splits into the following direct sum over $\mathbb{Q}$
\[ (H^{4,a}(X, \mathbb{Q}), F^{p,a}_0) = H^4_{a,1} \oplus H^4_{a,2} \oplus H^4_{a,3}. \] (1.8)
Here the Hodge decomposition of the direct summand $H^4_{a,1}$ is given by
\[ H^4_{a,1} = H^{4,0}(\mathcal{F}_4) \oplus H^{0,4}(\mathcal{F}_4). \] (1.9)
While the Hodge type of the two dimensional summand $H^4_{a,2}$ is $(3, 1) + (1, 3)$, and that of the one dimensional summand $H^4_{a,3}$ is $(2, 2)$. Moreover, the numerical value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ agrees with
\[ t|_{\psi=0} = \frac{1}{2} + \frac{i}{2} \sqrt{3.} \] (1.10)

(3) When $n = 6$, i.e. the Fermat octic $\mathcal{F}_6$, the seven dimensional sub-Hodge structure $(H^{6,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_6$ splits into the following direct sum over $\mathbb{Q}(\sqrt{2})$
\[ (H^{6,a}(X, \mathbb{Q}), F^{p,a}_0) = H^6_{a,1} \oplus H^6_{a,2} \oplus H^6_{a,3} \oplus H^6_{a,4}. \] (1.11)
Here the Hodge decomposition of the direct summand $H^6_{a,1}$ is given by
\[ H^6_{a,1} = H^{6,0}(\mathcal{F}_6) \oplus H^{0,6}(\mathcal{F}_6). \] (1.12)
While the Hodge type of the two dimensional summand $H^6_{a,2}$ is $(5, 1) + (1, 5)$, and that of the two dimensional summand $H^6_{a,3}$ is $(4, 2) + (2, 4)$; and that of the one dimensional summand $H^6_{a,4}$ is $(3, 3)$. Moreover, the numerical value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ agrees with
\[ t|_{\psi=0} = \frac{1}{2} + \frac{1}{2} \left( 1 + \sqrt{2} \right) i. \] (1.13)

(4) When $n = 8$, i.e. the Fermat decic $\mathcal{F}_8$, the nine dimensional sub-Hodge structure $(H^{8,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_8$ splits into the following direct sum over $\mathbb{Q}(\sqrt{5})$
\[ (H^{8,a}(X, \mathbb{Q}), F^{p,a}_0) = H^8_{a,1} \oplus H^8_{a,2} \oplus H^8_{a,3} \oplus H^8_{a,4}. \] (1.14)
Here the Hodge decomposition of the direct summand $H^8_{a,1}$ is given by
\[ H^8_{a,1} = H^{8,0}(\mathcal{F}_8) \oplus H^{0,8}(\mathcal{F}_8). \] (1.15)
While the Hodge type of the two dimensional summand $H^8_{a,2}$ is $(7, 1) + (1, 7)$, and that of the two dimensional summand $H^8_{a,3}$ is $(6, 2) + (2, 6)$; and that of the three
dimensional summand $H^8_{4,4}$ is $(5, 3) + (4, 4) + (3, 5)$. Moreover, the numerical value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ agrees with
\[
t_{\psi=0} = \frac{1}{2} + \frac{i}{2} \sqrt{5 + 2\sqrt{5}}.
\] (1.16)

(5) When $n = 10$, i.e. the Fermat dodecic $\mathcal{F}_{10}$, the eleven dimensional sub-Hodge structure $(H^{10,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_{10}$ splits into the following direct sum over $\mathbb{Q}(\sqrt{3})$
\[
(H^{10,a}(X, \mathbb{Q}), F^{p,a}_0) = H^{10}_{a,1} \oplus H^{10}_{a,2} \oplus H^{10}_{a,3} \oplus H^{10}_{a,4}.
\] (1.17)

Here the Hodge decomposition of the direct summand $H^{10}_{a,1}$ is given by
\[
H^{10}_{a,1} = H^{10,0}(\mathcal{F}_{10}) \oplus H^{0,10}(\mathcal{F}_{10}).
\] (1.18)
While the Hodge type of the two dimensional summand $H^{10}_{a,2}$ is $(9,1) + (1,9)$, and that of the two dimensional summand $H^{10}_{a,3}$ is $(8,2) + (2,8)$; and that of the five dimensional summand $H^{10}_{a,4}$ is $(7,3) + (6,4) + (5,5) + (4,6) + (3,7)$. Moreover, the numerical value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ agrees with
\[
 t_{\psi=0} = \frac{1}{2} + \left(1 + \frac{\sqrt{3}}{2}\right)i.
\] (1.19)

Furthermore, for every two dimensional Hodge structure $H^n_{a,j}$ with $n = 3, 4, 6, 8, 10$ listed previously, we have numerically computed their Deligne’s periods $c^\pm(H^n_{a,j})$. Our numerical results have shown that the quotient $c^+(H^n_{a,j})/c^-(H^n_{a,j})$ is always an algebraic number. For example, when $n = 3$, we have the split 1.5, and the quotient of their Deligne’s periods (with respect to a special Betti cohomology basis) satisfies
\[
\begin{align*}
\frac{c^+(H^3_{a,1})}{c^-(H^3_{a,1})} &= i\sqrt{5 - 2\sqrt{5}}, \\
\frac{c^+(H^3_{a,2})}{c^-(H^3_{a,2})} &= i\sqrt{5 + 2\sqrt{5}};
\end{align*}
\] (1.20)
which are of course up to multiplications by nonzero elements of $\mathbb{Q}(\sqrt{5})$. More results will be provided later in this paper.

Our numerical results have prompted us to formulate three conjectures about the Fermat type CY $n$-fold $\mathcal{F}_n$ 1.1. The first is about the value of the mirror map at the Fermat point.

**Conjecture 1.1.** For every positive integer $n$, the value of the mirror map for the Fermat pencil 1.4 at the Fermat point $\psi = 0$ is of the form
\[
t_{\psi=0} = \frac{1}{2} + \xi i,
\] (1.21)
where $\xi$ is a real algebraic number.

Intuitively, this real algebraic number $\xi$ depends on the integer $n+2$. In the paper [8], Moore has formulated a similar conjecture for the attractors which are CY threefolds. Our second conjecture is about the split of the $n+1$ dimensional sub-Hodge structure $(H^{n,a}(X, \mathbb{Q}), F^{p,a}_0)$ on $\mathcal{F}_n$ 1.1.
Conjecture 1.2. There exists a real algebraic number field \( k \) such that the pure Hodge structure \((H^n, \mathcal{F}_0, F^{p.a})\) on \( \mathcal{F}_n \), which is an \( n+1 \) dimensional sub-Hodge structure of \((H^n, \mathcal{F}_0, F^p)\), splits completely into the direct sum

\[
(H^n, \mathcal{F}_0, F^{p,a}) = \begin{cases}
H^n_{a,1} \oplus H^n_{a,2} \oplus \cdots \oplus H^n_{a,(n+1)/2}, & \text{if } n \text{ is odd} \\
H^n_{a,1} \oplus H^n_{a,2} \oplus \cdots \oplus H^n_{a,n/2} \oplus \mathbb{Q}(-n/2), & \text{if } n \text{ is even}
\end{cases}
\]  

(1.22)

Here the Hodge decomposition of \( H^n_{a,1} \) is

\[
H^n_{a,1} = H^{n,0}(\mathcal{F}_n) \oplus H^{0,n}(\mathcal{F}_n);
\]

and the Hodge type of \( H^n_{a,j} \) is \((n-j+1, j-1) + (j-1, n-j+1)\). If such a field \( k \) exists, we will always assume it is smallest among all possible choices. If assuming Conjecture 1.1, then \( k \) is a subfield of \( \mathbb{Q}(\xi) \).

Recall that \( \mathbb{Q}(j), j \in \mathbb{Z} \) is the one dimensional pure Hodge structure with Hodge type \((-j, -j)\), which is also called the Hodge-Tate object \([10]\). Our third conjecture is about the Deligne’s periods of the direct summand \( H^n_{a,j} \) in the split 1.22 \([4, 13]\).

Conjecture 1.3. Assuming Conjecture 1.2, then the Deligne’s periods \( c^\pm(H^n_{a,j}) \) of the two dimensional direct summand \( H^n_{a,j} \) in the split 1.22 are well defined up to multiplications by nonzero elements of the field \( k \). Their quotient is of the form

\[
\frac{c^+(H^n_{a,j})}{c^-(H^n_{a,j})} = \sigma i,
\]

(1.24)

where \( \sigma \) is a real algebraic number. If further assuming Conjecture 1.1, then \( \sigma \) is in the real field \( \mathbb{Q}(\xi) \).

Notice that these three conjectures are true when \( n = 1, 2 \) \([15]\).

The outline of this paper is as follows. Section 2 is an overview of the Fermat pencil of CY \( n \)-folds, which includes the Picard-Fuchs equation of its holomorphic \( n \)-form and the canonical periods. Section 3 discusses the variation of Hodge structures of the Fermat pencil, which shows how the holomorphic \( n \)-form defines an \( n+1 \) dimensional sub-Hodge structure. It also introduces the charge equations for the splitting of this sub-Hodge structure. From Section 4 to Section 8, we will numerically show that the Fermat type CY \( n \)-fold \( \mathcal{F}_n \) for \( n = 3, 4, 6, 8, 10 \) does satisfy the predictions of Conjectures 1.1, 1.2 and 1.3. In the appendix, we will provide the numerical data needed in this paper.

2. The Fermat pencil of Calabi-Yau \( n \)-folds

In this section, we will introduce the Picard-Fuchs equation of the Fermat pencil of Calabi-Yau \( n \)-folds and its canonical solutions. We will follow the paper \([12]\) closely.

2.1. The Fermat pencil and its holomorphic forms. The Fermat pencil of Calabi-Yau \( n \)-folds, denoted by \( \mathcal{F}_\psi \), is a one-parameter family of \( n \)-dimensional hypersurfaces in the projective space \( \mathbb{P}^{n+1} \)

\[
\mathcal{F}_\psi : \{ f_\psi = 0 \} \subset \mathbb{P}^{n+1}, \quad \text{with} \quad f_\psi = \sum_{i=0}^{n+1} x_i^{n+2} - (n+2) \psi \prod_{i=0}^{n+1} x_i.
\]

(2.1)
Here \((x_0, x_1, \cdots, x_{n+1})\) is the projective coordinate of \(\mathbb{P}^{n+1}\). In a more formal language, the polynomial equation in formula 2.1 defines a rational fibration over \(\mathbb{P}^1\)

\[
\pi : \mathcal{X} \to \mathbb{P}^1,
\]

whose singular fibers are over the points

\[
\{\psi^{n+2} = 1\} \cup \{\psi = \infty\}.
\]

Moreover, if \(\psi\) is a rational number, \(\mathcal{X}_\psi\) is a variety defined over \(\mathbb{Q}\). The point \(\psi = 0\) is called the Fermat point, and the smooth fiber over it is also denoted by

\[
\mathcal{F}_n : \sum_{i=0}^{n+1} x_i^{n+2} = 0,
\]

which is called the Fermat type Calabi-Yau \(n\)-fold.

The adjunction formula tells us that a smooth fiber \(\mathcal{X}_\psi\) is a Calabi-Yau manifold. In fact, there is a canonical way to construct a holomorphic \(n\)-form on \(\mathcal{X}_\psi\) \([3, 5, 9]\). On \(\mathbb{P}^{n+1}\), there is a meromorphic \((n+1)\)-form \(\Theta\)

\[
\Theta := \sum_{i=0}^{n+1} \left(\frac{-1}{f_\psi}\right)^i \left(x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}\right),
\]

which is well-defined on the open subvariety \(\mathbb{P}^{n+1} - \mathcal{X}_\psi\). The residue of \(\Theta\) on the hypersurface \(\mathcal{X}_\psi\) gives us a holomorphic \(n\)-form \(\Omega\) on \(\mathcal{X}_\psi\)

\[
\Omega := \text{Res}_{\mathcal{X}_\psi}(\Theta).
\]

Computations in the open affine subvarieties of \(\mathcal{X}_\psi\) have shown that \(\Omega\) is in fact nowhere vanishing \([5, 9]\). For example, over the affine open subset of \(\mathcal{X}_\psi\) defined by \(x_{n+1} = 1\), the residue of \(\Theta\) is

\[
\Omega = \frac{1}{\partial f_\psi/\partial x_n} \left. dx_0 \wedge \cdots \wedge dx_{n-1}\right|_{\mathcal{X}_\psi},
\]

which does not vanish. Moreover, if \(\psi\) is rational, then \(\Omega\) is also defined over \(\mathbb{Q}\). In particular, when \(\psi = 0\), we obtain a nowhere vanishing \(n\)-form \(\Omega_0\) on \(\mathcal{F}_n\).

2.2. The Picard-Fuchs equation. When discussing the Picard-Fuchs equation of the \(n\)-form \(\Omega\), it is more convenient to define a new parameter \(\varphi\) by

\[
\varphi = \psi^{-(n+2)}.
\]

The Picard-Fuchs equation satisfied by the \(n\)-form \(\psi\Omega\) is well-known \([9, 12]\)

\[
\left(\psi^{n+1} - \varphi \prod_{k=1}^{n+1} \left(\vartheta + \frac{k}{n + 2}\right)\right)(\psi \Omega) = 0, \quad \vartheta = \varphi \frac{d}{d\varphi}.
\]

For simplicity, let us denote the Picard-Fuchs operator in the formula 2.9 by

\[
D_n = \varphi^{n+1} - \varphi \prod_{k=1}^{n+1} \left(\vartheta + \frac{k}{n + 2}\right), \quad \vartheta = \varphi \frac{d}{d\varphi};
\]
which can be solved by the Frobenius method [12]. It has \( n + 1 \) canonical solutions of the form
\[
\varpi_j(\varphi) = \frac{1}{(2\pi i)^j} \sum_{k=0}^{j} \binom{j}{k} h_k(\varphi) \log^{j-k} ((n+2)^{(n+2)}\varphi), \quad j = 0, 1, \cdots, n; \tag{2.11}
\]
where \( h_k(\varphi) \) is a power series in \( \varphi \). If we impose the following boundary conditions
\[
h_0(0) = 1, \quad h_1(0) = \cdots = h_n(0) = 0, \tag{2.12}
\]
then the power series \( h_k(\varphi) \) becomes unique [12]. The Picard-Fuchs operator \( D_n \) has three regular singularities \( \{0, 1, \infty\} \), thus the power series \( h_k(\varphi) \) converges on the unit disc
\[
\Delta = \{ |\varphi| < 1 \}. \tag{2.13}
\]
The canonical periods \( \{\varpi_j\}_{j=0}^{n} \) are linearly independent and form a basis for the solution space of \( D_n \) [12].

The monodromy of the canonical periods \( \varpi_j \) at \( \varphi = 0 \) is induced by the analytic continuation \( \log \varphi \rightarrow \log \varphi + 2\pi i \), under which \( \varpi_j \) transforms in the way
\[
T_0 : \varpi_j(\varphi) \mapsto \sum_{k=0}^{j} \binom{j}{k} \varpi_k(\varphi). \tag{2.14}
\]
If we define the canonical period vector \( \varpi \) to be the column vector
\[
\varpi = (\varpi_0, \varpi_1, \cdots, \varpi_n)^T, \tag{2.15}
\]
then the monodromy action can be expressed as
\[
\varpi \rightarrow T_0 \varpi, \tag{2.16}
\]
where \( T_0 \) is an \( (n+1) \times (n+1) \) matrix
\[
T_0 = \begin{pmatrix}
1, & 0, & 0, & 0, & \cdots & 0, \\
1, & 1, & 0, & 0, & \cdots & 0, \\
1, & 2, & 1, & 0, & \cdots & 0, \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1, & \binom{n}{1}, & \binom{n}{2}, & \binom{n}{3}, & \cdots & \binom{n}{n}
\end{pmatrix}. \tag{2.17}
\]
This matrix \( T_0 \) satisfies the equation
\[
(T_0 - \text{Id})^{n+1} = 0, \tag{2.18}
\]
therefore the monodromy of \( \varpi \) at 0 is maximally unipotent [3, 5]. We will call \( \varphi = 0 \) (i.e. \( \psi = \infty \)) the large complex structure limit of the Picard-Fuchs operator \( D_n \).

3. The middle pure Hodge structure at the Fermat point

In this section, we will discuss the middle pure Hodge structure of the Fermat type \( n \)-fold \( \mathcal{F}_n \) and the conditions for it to split. First, let us show how the holomorphic \( n \)-form of the Fermat pencil induces an \( n + 1 \) dimensional sub-Hodge structure [12].
3.1. The expansions of the holomorphic $n$-form and its derivatives. For simplicity, the underlying differentiable manifold of a smooth fiber of the Fermat pencil 2.2 will be denoted by $X$. A period of the $n$-form $\Omega_\psi$ is by definition an integral of the form $\int_C \Omega_\psi$, where $C$ is a homological cycle of $H_n(X, \mathbb{C})$. Let $H^n_b(X, \mathbb{Q})$ be the subspace of $H_n(X, \mathbb{Q})$ defined by the condition

$$C \in H^n_b(X, \mathbb{Q}) \iff \int_C \Omega_\psi \equiv 0.$$  \hspace{1cm} (3.1)

The Poincaré duality induces a non-degenerate bilinear form on $H_n(X, \mathbb{Q})$, and let $H^n_a(X, \mathbb{Q})$ be the orthogonal complement of $H^n_a(X, \mathbb{Q})$ with respect to this form, i.e.

$$H_n(X, \mathbb{Q}) = H^n_a(X, \mathbb{Q}) \oplus H^n_b(X, \mathbb{Q}).$$  \hspace{1cm} (3.2)

Let the dual of $H^n_a(X, \mathbb{Q})$ (resp. $H^n_b(X, \mathbb{Q})$) be $H^n_a(X, \mathbb{Q})$ (resp. $H^n_b(X, \mathbb{Q})$), then the cohomology group $H^n(X, \mathbb{Q})$ splits into the direct sum

$$H^n(X, \mathbb{Q}) = H^n_a(X, \mathbb{Q}) \oplus H^n_b(X, \mathbb{Q}).$$  \hspace{1cm} (3.3)

The (nontrivial) periods of $\Omega_\psi$ are given by its integration over the cycles in $H^n_a(X, \mathbb{Q}) = H^n_a(X, \mathbb{Q}) \otimes \mathbb{C}$.

From the Picard-Fuchs equation 2.9, there exist homological cycles $C_j \in H^n_a(X, \mathbb{C})$ such that

$$\psi^{-1} \omega_j(\varphi) = \int_{C_j} \Omega_\psi.$$  \hspace{1cm} (3.4)

Since $\{\omega_j(\varphi)\}_{j=0}^n$ are linearly independent and form a solution basis of $\mathcal{D}_n$, we deduce that the dimension of $H^n_a(X, \mathbb{C})$ is $n + 1$ and $\{C_j\}_{j=0}^n$ form a basis for it. Now let the dual of the basis $\{C_j\}_{j=0}^n$ be $\{\gamma_j\}_{j=0}^n$, i.e. they satisfy the pairing relations

$$\gamma_j(C_k) = \delta_{jk}.$$  \hspace{1cm} (3.5)

Then $\{\gamma_j\}_{j=0}^n$ form a basis of $H^n_a(X, \mathbb{C}) = H^n_a(X, \mathbb{Q}) \otimes \mathbb{C}$. Under the comparison isomorphism between Betti and algebraic de Rham cohomology, the $n$-form $\Omega_\psi$ admits an expansion

$$\Omega_\psi = \sum_{j=0}^n \gamma_j \int_{C_j} \Omega_\psi = \sum_{j=0}^n \gamma_j \psi^{-1} \omega_j(\varphi).$$  \hspace{1cm} (3.6)

Similarly, the derivative $\Omega_\psi^{(k)} = d^k \Omega_\psi/d\psi^k$ admits an expansion

$$\Omega_\psi^{(k)} = \sum_{j=0}^n \gamma_j \int_{C_j} d^k \Omega_\psi/d\psi^k = \sum_{j=0}^n \gamma_j d^k \left( \psi^{-1} \omega_j(\varphi) \right) /d\psi^k.$$  \hspace{1cm} (3.7)

For every $\psi$ such that $\mathcal{X}_\psi$ is smooth, the forms

$$\Omega_\psi, \, \Omega_\psi^{(1)}, \, \cdots, \, \Omega_\psi^{(n)}$$

are linearly independent, therefore they also form a basis of $H^n_a(X, \mathbb{C})$ [12].
3.2. Variations of Hodge structures and the period matrix. Given a smooth fiber $X_{\psi}$ of the Fermat pencil $2.1$, from Hodge theory, there exists a Hodge decomposition

$$H^n(X, \mathbb{Q}) \otimes \mathbb{C} = H^{n,0}(X_{\psi}) \oplus H^{n-1,1}(X_{\psi}) \oplus \cdots \oplus H^{1,n-1}(X_{\psi}) \oplus H^{0,n}(X_{\psi}).$$

It defines a weight-$n$ pure Hodge structure $(H^n(X, \mathbb{Q}), F^p)$ with the Hodge filtration

$$F^p = \oplus_{k \geq p} H^{k,n-k}(X_{\psi}),$$

which varies holomorphically with respect to $\psi$. From Griffiths transversality [10], we have

$$\Omega^{(k)}(\psi) \in F^{n-k}_{\psi}, \quad k = 0, 1, \cdots, n.$$ (3.12)

Together with the linear independence of $\Omega^{(k)}(\psi)$, it shows there is a pure Hodge structure on $H^{n,a}(X, \mathbb{Q})$ with Hodge filtration

$$F_{p,a}^p := \oplus_{k=0}^{n-p} \mathbb{C} \Omega^{(k)}(\psi).$$ (3.13)

Therefore the pure Hodge structure on $H^n(X, \mathbb{Q})$ splits into the direct sum

$$(H^n(X, \mathbb{Q}), F^p_{\psi}) = (H^{n,a}(X, \mathbb{Q}), F^p_{\psi,a}) \oplus (H^{n,b}(X, \mathbb{Q}), F^p_{\psi,b}),$$ (3.14)

where the pure Hodge structure $(H^{n,b}(X, \mathbb{Q}), F^p_{\psi,b})$ is induced by $(H^n(X, \mathbb{Q}), F^p_{\psi})$ [10, 12]. Furthermore, the Hodge numbers of the sub-Hodge structure $(H^{n,a}(X, \mathbb{Q}), F^p_{\psi,a})$ satisfy

$$h^{n,0} = h^{n-1,1} = \cdots = h^{1,n-1} = h^{0,n} = 1.$$ (3.15)

However, in order to obtain more information about $(H^{n,a}(X, \mathbb{Q}), F^p_{\psi,a})$, we will need to know the transformation matrix between the canonical basis

$$\gamma = (\gamma_0, \cdots \gamma_n)$$ (3.16)

and a rational vector space $\alpha$ of $H^{n,a}(X, \mathbb{Q})$

$$\alpha = (\alpha_0, \cdots, \alpha_n).$$ (3.17)

Denote this transformation matrix by $P$, i.e.

$$\gamma_j = \sum_{j=0}^{n} \alpha_j \cdot P_{ji}.$$ (3.18)

When $n = 3$, the period matrix $P$ is determined by mirror symmetry [1, 7], while when $n = 4, 5, 6, 7, 8, 9, 10, 11, 12$, $P$ has been numerically computed in the paper [12]. Suppose the dual of the basis $\alpha$ is

$$A = (A_0, \cdots, A_n),$$ (3.19)

which forms a basis of $H^{a}_n(X, \mathbb{Q})$. The rational periods $\Pi_j(\psi)$ are defined by

$$\Pi_j(\psi) = \int_{A_j} \Omega_{\psi}, \quad j = 0, 1, \cdots, n.$$ (3.20)

Then under the comparison isomorphism between Betti and algebraic de Rham cohomology, $\Omega_{\psi}$ also has an expansion

$$\Omega_{\psi} = \sum_{j=0}^{n} \alpha_j \Pi_j(\psi),$$ (3.21)
hence from formulas 3.7 and 3.18, we deduce
\[
\Pi_j(\psi) = \sum_{k=0}^{n} P_{jk} \psi^{-1} \omega_k(\phi).
\] (3.22)

In order to study the pure Hodge structure \((H^{n,a}(X, Q), F_0^{p,a})\) on \(\mathcal{F}_n\) at the Fermat point \(\psi = 0\), we will need to compute the values of \(\Pi_i(\psi)\) and its derivatives at \(\psi = 0\). In this paper, we will numerically compute these values to a very high precision, which allows us to obtain essential properties of the sub-Hodge structure \((H^{n,a}(X, Q), F_0^{p,a})\) when \(n = 3, 4, 6, 8, 10\).

3.3. The charge equations for the Hodge structures to split. Suppose we want to show the sub-Hodge structure \((H^{n,a}(X, Q), F_0^{p,a})\) on the Fermat type CY \(n\)-fold \(\mathcal{F}_n\) has a two dimensional direct summand \(H^n_{a,1}\) over a real number field \(k\) with Hodge decomposition
\[
H^n_{a,1} = H^{n,0}(\mathcal{F}_n) \oplus H^{0,n}(\mathcal{F}_n).
\] (3.23)

This equation is equivalent to the existences of two linearly independent charges \(\gamma_1, \gamma_2\) in the space \(H^n(\mathcal{F}_n, Q) \otimes_k k\) with Hodge decomposition
\[
\rho_1 = \rho_1^{n,0} + \rho_1^{0,n}, \quad \rho_2 = \rho_2^{n,0} + \rho_2^{0,n}.
\] (3.24)

Here the terminology ‘charge’ comes from string theory, which means the charges of BPS black holes \([6,8]\). From formula 3.21, the \(n\)-form \(\Omega_0\) on \(\mathcal{F}_n\) admits an expansion
\[
\Omega_0 = \sum_{j=0}^{n} \alpha_j \Pi_j(0).
\] (3.25)

The one dimensional vector space \(H^{n,0}(\mathcal{F}_n)\) is spanned by the \(n\)-form \(\Omega_0\), hence formula 3.24 is equivalent to the existence of two nonzero constants \(c_1, c_2 \in \mathbb{C}\) such that
\[
\rho_1 = \sum_{i=0}^{n} \left( c_1 \Pi_i(0) + c_1 \Pi_i(0) \right) \alpha_i, \quad \rho_2 = \sum_{i=0}^{n} \left( c_2 \Pi_i(0) + c_2 \Pi_i(0) \right) \alpha_i.
\] (3.26)

Similarly, the existence of other two dimensional sub-Hodge structures of \((H^{n,a}(X, Q), F_0^{p,a})\) is equivalent to equations similar to formula 3.26, but involving the values of the derivatives of \(\Pi_i(\psi)\) at the Fermat point \(\psi = 0\).

In general, it is very difficult, if not entirely impossible, to compute the value of \(\Pi_i(0)\) analytically. Moreover, it is certainly much more difficult to show whether there exist the two elements \(\rho_1\) and \(\rho_2\) that satisfy the condition 3.26 for some real number field \(k\). Hence in this paper, we will resort to numerical methods to evaluate the periods \(\Pi_i(0)\) using Mathematica programs. Then we will numerically search constants \(C_1\) and \(C_2\) that satisfy the condition 3.26. Our search is successful for the cases
\[
n = 3, 4, 6, 8, 10,
\] (3.27)

which has provided very strong evidences to Conjecture 1.2. At the same time, our numerical results also provide strong evidences to Conjecture 1.1 and 1.3. Now let us first look at the case where \(n = 3\), i.e. Fermat quintic CY threefold.
4. The Fermat quintic CY threefold

The Fermat quintic CY threefold \( \mathcal{F}_3 \) is by definition
\[ \{ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \} \subset \mathbb{P}^4. \] (4.1)

From the terminologies in previous sections, its underlying differentiable manifold will be denoted by \( X \). Recall from Section 3 that the pure Hodge structure \((H^3(X, \mathbb{Q}), F^0_0)\) on \( \mathcal{F}_3 \) has a four dimensional direct summand \((H^{3,a}(X, \mathbb{Q}), F^{0,a}_0)\) that is induced by the holomorphic threeform \( \Omega_\psi \) of the Fermat quintic pencil 2.1. In this section, we will explicitly construct the following split over the quadratic field \( \mathbb{Q}(\sqrt{5}) \)
\[ (H^{3,a}(X, \mathbb{Q}), F^{0,a}_0) = H^3_{a,1} \oplus H^3_{a,2}, \] (4.2)
where the Hodge decomposition of \( H^3_{a,1} \) is given by
\[ H^3_{a,1} = H^{3,0}(\mathcal{F}_3) \oplus H^{0,3}(\mathcal{F}_3). \] (4.3)

More concretely, we will use numerical method to find two charges \( \rho_1, \rho_2 \in H^3(X, \mathbb{Q}) \) whose Hodge decomposition only have \((3,0)\) and \((0,3)\) components.

4.1. The period matrix for the Fermat quintic pencil. From Section 2, when \( n = 3 \), we have \( \varphi = \psi^{-5} \) by formula 2.8. The Picard-Fuchs operator
\[ D_3 = \vartheta^4 - \varphi \prod_{k=1}^4 \left( \vartheta + \frac{k}{5} \right), \quad \vartheta = \varphi \frac{d}{d\varphi} \] (4.4)
for the Fermat quintic pencil 2.1 has four canonical solutions of the form
\[ \varpi_j(\varphi) = \frac{1}{(2\pi i)^j} \sum_{k=0}^j \binom{j}{k} h_k(\varphi) \log^{j-k} \left( 5^{-5} \varphi \right), \quad j = 0, 1, 2, 3; \] (4.5)
which have played a crucial role in the mirror symmetry of the quintic CY threefolds [1, 5]. The first several terms of the power series \( h_i(\varphi) \) are
\[ h_0 = 1 + \frac{24}{625} \varphi + \frac{4536}{390625} \varphi^2 + \frac{1345344}{244140625} \varphi^3 + \frac{488864376}{152587890625} \varphi^4 + \cdots, \]
\[ h_1 = \frac{154}{625} \varphi + \frac{32409}{390625} \varphi^2 + \frac{29965432}{244140625} \varphi^3 + \frac{12207031250}{152587890625} \varphi^4 + \cdots, \]
\[ h_2 = \frac{125}{625} \varphi^2 + \frac{168327}{781250} \varphi^3 + \frac{271432352}{2197265625} \varphi^4 + \cdots, \]
\[ h_3 = \frac{276}{125} \varphi^2 - \frac{79161}{156250} \varphi^3 - \frac{3732972559}{3430531250} \varphi^4 + \cdots. \] (4.6)

From Section 3.1, there exist homological cycles \( C_j \in H^3_3(X, \mathbb{C}) \) such that
\[ \psi^{-1} \varpi_j(\varphi) = \int_{C_j} \Omega_\psi, \quad j = 0, 1, 2, 3. \] (4.7)
The dual of \( \{C_j\}_{j=0}^3 \), denoted by \( \{\gamma_j\}_{j=0}^3 \), forms a basis of \( H^{3,a}(X, \mathbb{C}) \). The threeform \( \Omega_\psi \) admits an expansion
\[
\Omega_\psi = \sum_{j=0}^3 \gamma_j \psi^{-1} \varpi_j(\varphi). \tag{4.8}
\]
Similarly, the form \( \Omega_\psi^{(k)} = d^k\Omega_\psi / d\psi^k \) admits an expansion \([5, 7]\)
\[
\Omega_\psi^{(k)} = \sum_{i=0}^3 \gamma_i d^k \left( \psi^{-1} \varpi_i(\varphi) \right) / d\psi^k. \tag{4.9}
\]

From the mirror symmetry of quintic CY threefolds, there exist a symplectic basis of the rational vector space \( H^{3,a}(X, \mathbb{Q}) \) \([1, 3, 5, 7]\)
\[
\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \tag{4.10}
\]
such that the corresponding cup product pairing matrix is given by
\[
(\int_X \alpha_i \cup \alpha_j) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}. \tag{4.11}
\]
The period matrix \( P \) between the two basis \( \gamma \) and \( \alpha \), i.e. \( \gamma = \alpha \cdot P \) is determined by the perturbative part of the prepotential for the quintic mirror pair, which is carefully discussed in the papers \([1, 7]\). Here we give the matrix \( P \) without further details
\[
P = l_3(2\pi i)^3 \begin{pmatrix}
-25i\zeta(3)/\pi^3 & 25/12 & 0 & 5/6 \\
25/12 & -11/2 & -5/2 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \; l_3 \in \mathbb{Q}^\times, \tag{4.12}
\]
where \( l_3 \) is a nonzero rational number. From formula \( 3.22 \), the integral period \( \Pi_i(\psi) \) is given by
\[
\Pi_i(\psi) = \sum_{k=0}^3 P_{jk} \psi^{-1} \varpi_k(\varphi), \; j = 0, 1, 2, 3. \tag{4.13}
\]

With respect to the rational basis \( \alpha \), \( \Omega_\psi \) has an expansion
\[
\Omega_\psi = \alpha \cdot \Pi(\psi) = \sum_{j=0}^3 \alpha_j \Pi_j(\psi). \tag{4.14}
\]
From the period matrix \( P \) \(4.12\), \( l_3(2\pi i)^3 \varpi_0 \) and \( l_3(2\pi i)^3 \varpi_1 \) are the integrals of the threeform \( \Omega_\psi \) over rational homological cycles of \( H^3_a(X, \mathbb{Q}) \), and their quotient is by definition the mirror map \( t \) \([1, 3, 5, 7]\)
\[
t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}. \tag{4.15}
\]
As \( \mathcal{F}_3 \) is a variety defined over \( \mathbb{Q} \), the complex conjugation acts on its complex points, which induces an involution \( F_\infty \) on the cohomology group \( H^{3,a}(X, \mathbb{Q}) \) \([4, 13]\)
\[
F_\infty : H^{3,a}(X, \mathbb{Q}) \to H^{3,a}(X, \mathbb{Q}). \tag{4.16}
\]
The matrix of $F_\infty$ with respect to the basis $\alpha$ has been computed in the paper [13]

$$
F_\infty = \begin{pmatrix}
1 & 1 & -5 & 8 \\
0 & -1 & 8 & -16 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}.
$$

(4.17)

4.2. The numerical evaluations of periods and their derivatives at $\psi = 0$. In order to study the pure Hodge structure $(H^{3,\alpha}(X, \mathbb{Q}), F_0^{p,\alpha})$ on $\mathcal{F}_3$ at the Fermat point $\psi = 0$, we will need to compute the values of $\Pi_i^{(k)}(\psi)$ at $\psi = 0$. But the power series $h_i(\varphi)$ in the formula 4.6 only converges in the disc $|\varphi| < 1$, i.e. $\psi > 1$, while the Fermat point $\psi = 0$ is not in this convergence region. Hence we will resort to numerical methods in this paper.

The Fermat point $\psi = 0$ is a smooth point for the Fermat quintic pencil 2.1. From formula 2.9, the threeform $\Omega_\psi$ satisfies the following Picard-Fuchs equation

$$
(1 - \psi^5)\frac{d^4\Omega_\psi}{d\psi^4} - 10\psi^4\frac{d^3\Omega_\psi}{d\psi^3} - 25\psi^3\frac{d^2\Omega_\psi}{d\psi^2} - 15\psi^2\frac{d\Omega_\psi}{d\psi} - \psi\Omega_\psi = 0,
$$

(4.18)

from which we can also see the Fermat point $\psi = 0$ is in fact a smooth point. Now choose a point $\psi_0 = -3$, i.e. $\varphi_0 = -3^{-5}$. The power series $h_i(\varphi)$ in the formula 4.6 converges very fast at the point $\varphi_0$, which allows us to compute the numerical values of $\psi_0^{-1}\varpi_i(\varphi_0)$ to a very high precision. Similarly, we can also compute the values of the derivatives of $\psi^{-1}\varpi_i$ at $\psi_0$ to a very high precision. With them as boundary conditions, we can numerically solve the Picard-Fuchs equation 4.18 over the closed interval $\psi \in [-3, 0]$ and obtain the value of $\psi^{-1}\varpi_i$ at $\psi = 0$ to a very high precision. Here we list the first dozens digits of them

$$
\begin{align*}
\psi^{-1}\varpi_0|_{\psi=0} &= -2.498836213357162381371483738493768932484871519\cdots; \\
\psi^{-1}\varpi_1|_{\psi=0} &= -1.24941810667858141593657041869246868446624243579\cdots \\
&\quad - i\cdot1.719676493141723462166954291797891220476917996\cdots; \\
\psi^{-1}\varpi_2|_{\psi=0} &= 0.28456812308856920225085785787763144205108426532\cdots \\
&\quad - i\cdot1.719676493141723462166954291797891220476917996\cdots; \\
\psi^{-1}\varpi_3|_{\psi=0} &= 1.0515612379721445113445719961626815052407838576780\cdots \\
&\quad - i\cdot1.4589100573179453148308494931916829184851124375\cdots.
\end{align*}

(4.19)

From them, we immediately find that the value of the mirror map $t$ 4.15 at the Fermat point $\psi = 0$ agrees with an algebraic number

$$
t|_{\psi=0} = \lim_{\psi \to 0} \varpi_1/\varpi_0 = \frac{1}{2} + i\sqrt{\frac{1}{4} + \frac{\sqrt{5}}{10}}.
$$

(4.20)

To obtain further information about the pure Hodge structure $(H^{3,\alpha}(X, \mathbb{Q}), F_0^{p,\alpha})$, we will also need the values of the derivatives of $\psi^{-1}\varpi_i$ at $\psi = 0$. However the ODE satisfied by $\Omega_\psi'$ is of the form

$$
\psi(1 - \psi^5)\frac{d^4\Omega_\psi'}{d\psi^4} - (1 + 14\psi^5)\frac{d^3\Omega_\psi'}{d\psi^3} - 55\psi^4\frac{d^2\Omega_\psi'}{d\psi^2} - 65\psi^3\frac{d\Omega_\psi'}{d\psi} - 16\psi^2\Omega_\psi' = 0,
$$

(4.21)
which has a singularity at $\psi = 0$. But through extrapolation, Mathematica still can compute the values of $(\psi^{-1} \omega_i)'$ at $\psi = 0$ to a very high precision. Here we list the first dozens digits of them

$$
\begin{align*}
(\psi^{-1} \omega_0)'|_{\psi=0} &= -2.2550683836960622558295812512914672499779372160315 \cdots \\
(\psi^{-1} \omega_1)'|_{\psi=0} &= -1.12753419184803112791479062564573362498896860801576 \cdots \\
&\quad - i 0.36635806710747798519065311128934465216399570358 \cdots \\
(\psi^{-1} \omega_2)'|_{\psi=0} &= -1.76018670120301548276885157450397059554092744317944 \cdots \\
&\quad - i 0.36635806710747798519065311128934465216399570358 \cdots \\
(\psi^{-1} \omega_3)'|_{\psi=0} &= -2.0765129558050766019588204893308908081690686076128 \cdots \\
&\quad - i 3.2974468423280359976607145327721968031946480597382 \cdots 
\end{align*}
$$

Then the numerical values of $\Pi_i(0)$ (resp. $\Pi_i'(0)$) are obtained from that of $(\psi^{-1} \omega_i)|_{\psi=0}$ (resp. $(\psi^{-1} \omega_i)'|_{\psi=0}$) and the period matrix $P_{4.12}$.

### 4.3. The charges for the split at the Fermat point.

In order to find a split 4.2 over a real number field $k$, we will need to find two charges $\rho_1$ and $\rho_2$ in the vector space $H^{3,a}(X, Q) \otimes_Q k$ whose Hodge decompositions only have $(3, 0)$ and $(0, 3)$ components. Namely there exist two nonzero constants $c_1, c_2 \in \mathbb{C}$ for $\rho_1$ and $\rho_2$ such that

$$
\rho_1 = \sum_{i=0}^{3} \left( c_1 \Pi_i(0) + c_1 \Pi_i(0) \right) \alpha_i, \quad \rho_2 = \sum_{i=0}^{3} \left( c_2 \Pi_i(0) + c_2 \Pi_i(0) \right) \alpha_i.
$$

(4.23)

After extensive searching, we have found two such charges that satisfy the condition 4.23 over the quadratic field $\mathbb{Q}(\sqrt{5})$

$$
\begin{align*}
\rho_1 &= \alpha \cdot \left( \frac{1}{2} \left( 5 - \sqrt{5} \right), -8, 0, 1 \right)^\top, \\
\rho_2 &= \alpha \cdot \left( \frac{1}{2} \left( 5 - \sqrt{5} \right), -3 + \sqrt{5}, 2, 1 \right)^\top.
\end{align*}
$$

(4.24)

Therefore there does exist a split 4.2 with $k = \mathbb{Q}(\sqrt{5})$, and the Fermat quintic CY threefold $\mathcal{F}_3$ 4.1 is a rank-2 attractor. In particular, the underlying vector space of $H^{3,a}_{4,1}$ is spanned by the two charges $\rho_1$ and $\rho_2$ 4.24 over $\mathbb{Q}(\sqrt{5})$.

The orthogonal complement of $H^{3,a}(X, Q) \otimes_Q \mathbb{Q}(\sqrt{5})$ with respect to the cup product pairing 4.11 is spanned by the charges

$$
\begin{align*}
\rho_3 &= \alpha \cdot \left( \frac{1}{2} \left( 5 + \sqrt{5} \right), -8, 0, 1 \right)^\top, \\
\rho_4 &= \alpha \cdot \left( \frac{1}{2} \left( 5 + \sqrt{5} \right), -3 - \sqrt{5}, 2, 1 \right)^\top.
\end{align*}
$$

(4.25)

It is very interesting to notice that under the involution $\iota$ of $\text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$ which sends $\sqrt{5}$ to $-\sqrt{5}$, we have

$$
\iota(\rho_1) = \rho_3, \quad \iota(\rho_2) = \rho_4.
$$

(4.26)
The cup product pairings between $\rho_3$, $\rho_4$ and $\Omega_0$ vanish
\[ \int_X \rho_3 \sim \Omega_0 = \int_X \rho_4 \sim \Omega_0 = 0, \quad (4.27) \]
and the cup product pairings between $\rho_1$, $\rho_2$ and $\Omega'_0$ also vanish
\[ \int_X \rho_1 \sim \Omega'_0 = \int_X \rho_2 \sim \Omega'_0 = 0. \quad (4.28) \]
Hence the underlying vector space of the direct summand $H^3_{a,2}$ is spanned by $\rho_3$ and $\rho_4$. The Fermat quintic CY threefold $\mathcal{F}_3$ also forms a supersymmetric flux vacuum in IIB string theory, the interesting physics of which can be found in the paper [6]. It is very interesting to explore the physics of the elliptic curve associated to the pure Hodge structure $H^3_{a,2}$ [6, 11].

4.4. **Deligne’s periods for the Fermat quintic.** Now we are ready to compute the Deligne’s periods for $H^3_{a,1}$ and $H^3_{a,2}$ [4, 13]. First, as $H^3_{a,1}$ and $H^3_{a,2}$ are defined over the quadratic field $\mathbb{Q}(\sqrt{5})$, their Deligne’s periods are only well defined up to multiplications by nonzero elements of $\mathbb{Q}(\sqrt{5})$. Let us first look at the Deligne’s periods for $H^3_{a,1}$! The charge $\rho_1$ (resp. $\rho_2$) is an eigenvector of the involution $F_\infty$ 4.17 with eigenvalue 1 (resp. $-1$), i.e.
\[ F_\infty(\rho_1) = \rho_1, \quad F_\infty(\rho_2) = -\rho_2. \quad (4.29) \]
From [4, 13], the Deligne’s periods $c^+(H^3_{a,1})$ are given by the pairings
\[ c^+(H^3_{a,1}) = \frac{1}{(2\pi i)^3} \int_X \rho_1 \sim \Omega_0, \quad c^-(H^3_{a,1}) = \frac{1}{(2\pi i)^3} \int_X \rho_2 \sim \Omega_0, \quad (4.30) \]
which can be evaluated immediately using formula 4.14. The numerical value of $c^+(H^3_{a,1})$ is
\[ c^+(H^3_{a,1}) = -l_3 \times 5.587567637704784064376190685029719491579683585192 \cdots, \quad (4.31) \]
where $l_3$ is the nonzero rational constant appears in the period matrix 4.12. Furthermore, we have the following very interesting quotient
\[ \frac{c^+(H^3_{a,1})}{c^-(H^3_{a,1})} = \frac{\int_X \rho_1 \sim \Omega_0}{\int_X \rho_2 \sim \Omega_0} = i \sqrt{5 - 2\sqrt{5}}. \quad (4.32) \]
Similarly, the charge $\rho_3$ (resp. $\rho_4$) is an eigenvector of the involution $F_\infty$ 4.17 with eigenvalue 1 (resp. $-1$), i.e.
\[ F_\infty(\rho_3) = \rho_3, \quad F_\infty(\rho_4) = -\rho_4. \quad (4.33) \]
Similarly from [4, 13], the Deligne’s periods $c^+(H^3_{a,2})$ are given by the pairings
\[ c^+(H^3_{a,2}) = \frac{1}{(2\pi i)^3} \int_X \rho_3 \sim \Omega'_0, \quad c^-(H^3_{a,2}) = \frac{1}{(2\pi i)^3} \int_X \rho_4 \sim \Omega'_0, \quad (4.34) \]
which can be evaluated immediately. The numerical value of $c^+(H^3_{a,2})$ is
\[ c^+(H^3_{a,2}) = l_3 \times 5.0424861998549736541282891203674075610741855844668 \cdots. \quad (4.35) \]
Similarly, we have the following very interesting quotient
\[ \frac{c^+(H^3_{a,2})}{c^-(H^3_{a,2})} = \frac{\int_X \rho_3 \sim \Omega'_0}{\int_X \rho_4 \sim \Omega'_0} = i \sqrt{5 + 2\sqrt{5}}. \quad (4.36) \]
5. The Fermat sextic CY fourfold

The Fermat sextic CY fourfold $\mathcal{F}_4$ is by definition

$$\{x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0\} \subset \mathbb{P}^5. \quad (5.1)$$

From the terminologies in previous sections, its underlying differentiable manifold will be denoted by $X$. Recall from Section 3 that the pure Hodge structure $(H^4(X, \mathbb{Q}), F^0_p)$ on $\mathcal{F}_4$ has a five dimensional direct summand $(H^{4,a}(X, \mathbb{Q}), F^0_p)$ that is induced by the holomorphic fourform $\Omega_\psi$ of the Fermat sextic pencil 2.1. In this section, we will explicitly construct the following split over $\mathbb{Q}$

$$(H^{4,a}(X, \mathbb{Q}), F^0_p) = H^4_{a,1} \oplus H^4_{a,2} \oplus H^4_{a,3}, \quad (5.2)$$

where the Hodge decomposition of $H^4_{a,1}$ is given by

$$H^4_{a,1} = H^{4,0}(\mathcal{F}_4) \oplus H^{0,4}(\mathcal{F}_4). \quad (5.3)$$

While the Hodge type of the two dimensional summand $H^4_{a,2}$ is $(3, 1) + (1, 3)$, and that of the one dimensional summand $H^4_{a,3}$ is $(2, 2)$. More concretely, we will use numerical methods to find two charges $\rho_1, \rho_2 \in H^{4,a}(X, \mathbb{Q})$ (resp. $\rho_3, \rho_4 \in H^{4,a}(X, \mathbb{Q})$) whose Hodge decompositions only have $(4, 0)$ and $(0, 4)$ (resp. $(3, 1)$ and $(1, 3)$) components.

5.1. The period matrix for the Fermat sextic pencil. From Section 2, when $n = 4$, we have $\varphi = \psi^{-6}$ by formula 2.8. The Picard-Fuchs operator

$$D_4 := \varphi^5 - \varphi \prod_{k=1}^5 \left( \vartheta + \frac{k}{6} \right), \quad \vartheta = \varphi \frac{d}{d\varphi} \quad (5.4)$$

for the Fermat sextic pencil 2.1 has five canonical solutions of the form

$$\varomega_j = \frac{1}{(2\pi i)^j} \sum_{k=0}^j \binom{j}{k} h_k(\varphi) \log^{i-k} (6^{-6}\varphi), \; j = 0, 1, 2, 3, 4, \quad (5.5)$$

where $h_j(\varphi)$ is a power series in $\varphi$. From Section 3.1, there exist homological cycles $C_j \in H^4_a(X, \mathbb{C})$ such that $[7]$

$$\psi^{-1} \varomega_j(\varphi) = \int_{C_j} \Omega_\psi, \; j = 0, 1, 2, 3, 4. \quad (5.6)$$

The dual of $\{C_j\}_{j=0}^4$, denoted by $\{\gamma_j\}_{j=0}^4$, forms a basis of $H^{4,a}(X, \mathbb{C})$. The fourform $\Omega_\psi$ admits an expansion

$$\Omega_\psi = \sum_{i=0}^4 \gamma_i \psi^{-1} \varomega_i(\varphi). \quad (5.7)$$

Similarly, the form $\Omega^{(k)}_\psi$ admits an expansion

$$\Omega^{(k)}_\psi = \sum_{i=0}^4 \gamma_i d^k (\psi^{-1} \varomega_i(\varphi)) / d\psi^k. \quad (5.8)$$
The cup product pairing on $H^{4,a}(X, \mathbb{C})$ can be computed by the equations
\[
\int_X \Omega_\psi \wedge \Omega_\psi = 0, \quad \int_X (\Omega_\psi \wedge \Omega^{(1)}_\psi) = 0, \tag{5.9}
\]
and with respect to the canonical basis $\{\gamma_i\}_{i=0}^4$, the cup product pairing matrix is
\[
(\int_X \gamma_i \smile \gamma_j) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -4 & 0 \\
0 & 0 & 6 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{5.10}
\]
From the paper [12], $H^{4,a}(X, \mathbb{Q})$ has a rational basis
\[
\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \tag{5.11}
\]
with respect to which the period matrix $P$ between the basis $\gamma$ and $\alpha$, i.e. $\gamma = \alpha \cdot P$, is given by
\[
P = l_4(2\pi i)^4 \begin{pmatrix}
1, & 0, & 0, & 0, & 0, \\
0, & 1, & 0, & 0, & 0, \\
0, & 0, & 1, & 0, & 0, \\
-420 \zeta(3)/(2\pi i)^3, & 0, & 0, & 1, & 0, \\
0, & -1680 \zeta(3)/(2\pi i)^3, & 0, & 0, & 1,
\end{pmatrix}, \tag{5.12}
\]
where $l_4$ is a nonzero rational number. From formula 3.22, the integral period $\Pi_i(\psi)$ is given by
\[
\Pi_i(\psi) = \sum_{j=0}^4 P_{ij} \psi^{-1} \varpi_j(\phi). \tag{5.13}
\]
With respect to the rational basis $\alpha$, $\Omega_\psi$ has an expansion
\[
\Omega_\psi = \alpha \cdot \Pi(\psi) = \sum_{i=0}^4 \alpha_i \Pi_i(\psi). \tag{5.14}
\]
From the period matrix 5.12, $l_4(2\pi i)^4 \varpi_0$ and $l_4(2\pi i)^4 \varpi_1$ are the integrals of the fourform $\Omega_\psi$ over rational homological cycles of $H^4_a(X, \mathbb{Q})$, and their quotient is by definition the mirror map $t$ [7]
\[
t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}. \tag{5.15}
\]
As $\mathcal{F}_4$ is a variety defined over $\mathbb{Q}$, the complex conjugation acts on its complex points, which induces an involution $F_\infty$ on the cohomology group $H^{4,a}(X, \mathbb{Q})$ [4, 13]
\[
F_\infty : H^{4,a}(X, \mathbb{Q}) \rightarrow H^{4,a}(X, \mathbb{Q}). \tag{5.16}
\]
The matrix of $F_\infty$ with respect to the basis $\alpha$ has been computed by the method developed in the paper [13], which is given by

$$F_\infty = \begin{pmatrix}
\frac{75}{64} & 0 & -\frac{15}{8} & 0 & -\frac{1}{4} \\
0 & -1 & 0 & 0 & 0 \\
\frac{55}{256} & 0 & -\frac{43}{32} & 0 & -\frac{5}{16} \\
0 & 0 & 0 & -1 & 0 \\
-\frac{121}{1024} & 0 & \frac{165}{128} & 0 & \frac{75}{64}
\end{pmatrix}. \quad (5.17)$$

5.2. The charges for the split at the Fermat point. The numerical values of $\psi^{-1} \varpi_j(\phi)$, $(\psi^{-1} \varpi_j(\phi))'$ and $(\psi^{-1} \varpi_j(\phi))''$ at the Fermat point $\psi = 0$ have been computed using the method introduced in Section 4.2, which are listed in Appendix A. Together with the period matrix $P$ 4.12, we obtain the numerical values of $\Pi_i(0)$, $\Pi'_i(0)$ and $\Pi''_i(0)$. From these numerical results, we immediately learn that the value of the mirror map $t$ 5.15 at the Fermat point $\psi = 0$ agrees with the following algebraic number

$$t|_{\psi=0} = \lim_{\psi \to 0} \frac{\varpi_1}{\varpi_0} = \frac{1}{2} + \frac{i}{2} \sqrt{3}. \quad (5.18)$$

In order to construct the split 4.2 over $\mathbb{Q}$, we will need to find five charges $\rho_i, i = 1, \ldots, 5$ in the rational vector space $H^{1,\alpha}(X, \mathbb{Q})$ such that

1. The Hodge decompositions of $\rho_1$ and $\rho_2$ only have $(4, 0)$ and $(0, 4)$ components;
2. The Hodge decompositions of $\rho_3$ and $\rho_4$ only have $(3, 1)$ and $(1, 3)$ components;
3. The Hodge decomposition of $\rho_5$ only has $(2, 2)$ components.

Numerically, we have found the following two charges

$$\rho_1 = \alpha \cdot \begin{pmatrix}
1, 0, -\frac{3}{4}, 0, \frac{101}{16}
\end{pmatrix}^\top, \quad \rho_2 = \alpha \cdot \begin{pmatrix}
1, 2, \frac{5}{4}, -\frac{5}{2}, -\frac{11}{16}
\end{pmatrix}^\top \quad (5.19)$$

that satisfy the charge equations

$$\rho_1 = \sum_{i=0}^4 \left( c_1 \Pi_i(0) + c_1 \Pi_i(0) \right) \alpha_i, \quad \rho_2 = \sum_{i=0}^4 \left( c_2 \Pi_i(0) + c_2 \Pi_i(0) \right) \alpha_i \quad (5.20)$$

for nonzero constants $c_1, c_2 \in \mathbb{C}$. Hence their Hodge decompositions only have $(4, 0)$ and $(0, 4)$ components. Moreover, the cup product pairings between $\rho_1$ ($\rho_2$) and $\Omega'_0$, $\Omega''_0$ vanish, i.e.

$$\int_X \rho_1 \cup \Omega'_0 = \int_X \rho_2 \cup \Omega'_0 = \int_X \rho_1 \cup \Omega''_0 = \int_X \rho_2 \cup \Omega''_0 = 0. \quad (5.21)$$

Therefore the underlying vector space of the direct summand $H^4_{\alpha,1}$ in the formula 5.2 is spanned by the charges $\rho_1$ and $\rho_2$ 5.19.
Similarly, we have found another two linearly independent charges

\[ \rho_3 = \alpha \cdot \left( 1, 0, \frac{7}{12}, 0, -\frac{59}{16} \right)^\top, \]
\[ \rho_4 = \alpha \cdot \left( \frac{3}{2}, 1, \frac{15}{8}, \frac{11}{4}, -\frac{33}{32} \right)^\top \]

that satisfy the charge equations

\[ \rho_3 = \sum_{i=0}^{4} \left( c_3 \Pi_i'(0) + c_3 \Pi'_i(0) \right) \alpha_i, \]
\[ \rho_4 = \sum_{i=0}^{4} \left( c_4 \Pi_i'(0) + c_4 \Pi'_i(0) \right) \alpha_i \]

for nonzero constants \( c_3, c_4 \in \mathbb{C} \). Moreover, the cup product pairings between \( \rho_3, \rho_4 \) and \( \Omega_0, \Omega_0'' \) vanish, i.e.

\[ \int_X \rho_3 \smile \Omega_0 = \int_X \rho_4 \smile \Omega_0 = \int_X \rho_3 \smile \Omega_0'' = \int_X \rho_4 \smile \Omega_0'' = 0. \]

(5.24)

Hence the underlying vector space of the direct summand \( H_{a,1}^4 \) in the formula 5.2 is spanned by \( \rho_3, \rho_4 \).

We also have found a fifth charge

\[ \rho_5 = \alpha \cdot \left( 1, \frac{1}{2}, \frac{5}{4}, \frac{13}{8}, -\frac{11}{16} \right), \]

that satisfies the charge equation

\[ \rho_5 = \sum_{i=0}^{4} \left( c_5 \Pi_i''(0) + \overline{c_5 \Pi'_i(0)} \right) \alpha_i \]

for a nonzero constant \( c_5 \in \mathbb{C} \). The cup product pairings between \( \rho_5 \) and \( \Omega_0, \Omega_0'' \) vanish, i.e.

\[ \int_X \rho_5 \smile \Omega_0 = \int_X \rho_5 \smile \Omega_0'' = 0, \]

(5.27)

hence the underlying vector space of the direct summand \( H_{a,2}^4 \) in the formula 5.2 is spanned by \( \rho_5 \). The upshot is that we have explicitly constructed the split 5.2 numerically.

5.3. Deligne’s periods for Fermat sextic. Now we are ready to compute the Deligne’s periods for \( H_{a,1}^4 \) and \( H_{a,2}^4 \) [4, 13]. First, as \( H_{a,1}^4 \) and \( H_{a,2}^4 \) are defined over \( \mathbb{Q} \), their Deligne’s periods are only well defined up to nonzero rational multiples. Let us first look at the Deligne’s periods for \( H_{a,1}^4 \). The charge \( \rho_1 \) (resp. \( \rho_2 \)) is an eigenvector of the involution \( F_\infty \) with eigenvalue 1 (resp. \( -1 \)), i.e.

\[ F_\infty(\rho_1) = \rho_1, \quad F_\infty(\rho_2) = -\rho_2. \]

(5.28)

From [4, 13], the Deligne’s periods \( c^+(H_{a,1}^4) \) are given by

\[ c^+(H_{a,1}^4) = \frac{1}{(2\pi i)^4} \int_X \rho_1 \smile \Omega_0, \quad c^-(H_{a,1}^4) = \frac{1}{(2\pi i)^4} \int_X \rho_2 \smile \Omega_0. \]

(5.29)

From the numerical results in Appendix A, the numerical value of \( c^+(H_{a,1}^4) \) is

\[ c^+(H_{a,1}^4) = -l_4 \times 42.088012626742807536142059740344624777125095306 \ldots, \]

(5.30)
where \( l_4 \) is the nonzero rational constant appears in the period matrix 5.12. We have also found an interesting quotient between Deligne’s periods

\[
\frac{c^+(H_{a,1}^4)}{c^-(H_{a,1}^4)} = \frac{\int_X \rho_1 \sim \Omega_0}{\int_X \rho_2 \sim \Omega_0} = -\frac{\sqrt{3}}{3}i
\]  

(5.31)

Similarly, the charge \( \rho_3 \) (resp. \( \rho_4 \)) 5.22 is an eigenvector of the involution \( F_\infty \) 5.17 with eigenvalue 1 (resp. -1), i.e.

\[
F_\infty(\rho_3) = \rho_3, \quad F_\infty(\rho_4) = -\rho_4.
\]  

(5.32)

From [4, 13], the Deligne’s periods \( c^+(H_{a,2}^4) \) are given by

\[
c^+(H_{a,2}^4) = \frac{1}{(2\pi i)^4} \int_X \rho_3 \sim \Omega'_0, \quad c^-(H_{a,2}^4) = \frac{1}{(2\pi i)^4} \int_X \rho_4 \sim \Omega'_0.
\]  

(5.33)

From the numerical results in Appendix A, the numerical value of \( c^+(H_{a,2}^4) \) is

\[
c^+(H_{a,2}^4) = l_4 \times 9.41456533191957346749114895059375683751750691454905533 \cdots,
\]  

(5.34)

and we also have an interesting quotient

\[
\frac{c^+(H_{a,2}^4)}{c^-(H_{a,2}^4)} = \frac{\int_X \rho_3 \sim \Omega'_0}{\int_X \rho_4 \sim \Omega'_0} = -\frac{2\sqrt{3}}{3}i.
\]  

(5.35)

The charge \( \rho_5 \) 5.25 is an eigenvector of \( F_\infty \) with eigenvalue -1, i.e.

\[
F_\infty(\rho_5) = -\rho_5.
\]  

(5.36)

Numerically we have

\[
c^-(H_{a,3}^4) = \frac{1}{(2\pi i)^4} \int_X \rho_5 \sim \Omega_0 = l_4 \times \frac{63i}{(2\pi i)^2},
\]  

(5.37)

which agrees with the fact that the one dimensional sub-Hodge structure \( H_{a,3}^4 \) in the formula 5.2 is isomorphic to the Hodge-Tate structure \( \mathbb{Q}(-2) \).

6. The Fermat Octic CY Sixfold

The Fermat octic CY sixfold \( \mathcal{S}_6 \) is by definition

\[
\{x_0^8 + x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^8 + x_6^8 = 0\} \subset \mathbb{P}^7.
\]  

(6.1)

From the terminologies in previous sections, its underlying differentiable manifold will be denoted by \( X \). Recall from Section 3 that the pure Hodge structure \( (H^6(X, \mathbb{Q}), F^p_0) \) on \( \mathcal{S}_6 \) has a seven dimensional direct summand \( (H^{6,a}(X, \mathbb{Q}), F^{p,a}_0) \) that is induced by the holomorphic sixform \( \Omega_\psi \) of the Fermat octic pencil 2.1. In this section, we will explicitly construct the following split over \( \mathbb{Q}(\sqrt{2}) \)

\[
(H^{6,a}(X, \mathbb{Q}), F^{p,a}_0) = H_{a,1}^6 \oplus H_{a,2}^6 \oplus H_{a,3}^6 \oplus H_{a,4}^6,
\]  

(6.2)

where the Hodge decomposition of \( H_{a,1}^6 \) is

\[
H_{a,1}^6 = H^{6,0}(\mathcal{S}_6) \oplus H^{0,6}(\mathcal{S}_6).
\]  

(6.3)

While the Hodge type of the two dimensional summand \( H_{a,2}^6 \) is \((5, 1) + (1, 5)\), and that of the two dimensional summand \( H_{a,3}^6 \) is \((4, 2) + (2, 4)\); and that of the one dimensional summand
$H_{6,4}^6$ is $(3, 3)$. More concretely, we will use numerical methods to find two charges $\rho_1$ and $\rho_2$ in $H^{6,a}(X, \mathbb{Q})$ whose Hodge decompositions only have $(6, 0)$ and $(0, 6)$ components, etc.

6.1. The period matrix for the Fermat octic pencil. From Section 2, when $n = 6$, we have $\varphi = \psi^{-8}$ by formula 2.8. The Picard-Fuchs operator

$$D_6 := \vartheta^7 - \varphi \prod_{k=1}^{7} \left( \vartheta + \frac{k}{8} \right), \quad \vartheta = \varphi \frac{d}{d\varphi}$$

has seven canonical solutions of the form

$$\omega_j = \frac{1}{(2\pi i)^j} \sum_{k=0}^{j} \binom{j}{k} h_k(\varphi) \log^{i-k} \left( 8^{-8} \varphi \right), \quad j = 0, 1, \cdots, 6,$$

where $h_j(\varphi)$ is a power series in $\varphi$. From Section 3.1, there exist homological cycles $C_j \in H_6^a(X, \mathbb{C})$ such that

$$\psi^{-1} \omega_j(\varphi) = \int_{C_j} \Omega_\psi, \quad j = 0, 1, \cdots, 6.$$  

The dual of $\{C_j\}_{j=0}^6$, denoted by $\{\gamma_j\}_{j=0}^6$, forms a basis of $H^{6,a}(X, \mathbb{C})$. The sixform $\Omega_\psi$ admits an expansion

$$\Omega_\psi = \sum_{i=0}^{6} \gamma_i \psi^{-1} \omega_i(\varphi).$$

Similarly, the form $\Omega_\psi^{(k)}$ admits an expansion

$$\Omega_\psi^{(k)} = \sum_{i=0}^{6} \gamma_i d^k \left( \psi^{-1} \omega_i(\varphi) \right) / d\psi^k.$$

The cup product pairing on $H^{6,a}(X, \mathbb{C})$ can be computed by the equations

$$\int_X \Omega_\psi \wedge \Omega_\psi = 0, \quad \int_X \Omega_\psi \wedge \Omega_\psi^{(1)} = 0,$$

and with respect to the canonical basis $\{\gamma_i\}_{i=0}^6$, the cup product pairing matrix is

$$\left( \int_X \gamma_i \sim \gamma_j \right) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -6 & 0 \\
0 & 0 & 0 & 15 & 0 & 0 \\
0 & 0 & 0 & 0 & -20 & 0 & 0 \\
0 & 0 & 15 & 0 & 0 & 0 & 0 \\
0 & -6 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}.$$  

From the paper [12], $H^{6,a}(X, \mathbb{Q})$ has a basis

$$\alpha = (\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),$$

with respect to which the period matrix $P$ between the basis $\gamma$ and $\alpha$, i.e. $\gamma = \alpha \cdot P$, has been numerically computed. The matrix $P$ is of the form

$$P = l_6 \left( 2\pi i \right)^6 \cdot P_\zeta, \quad l_6 \in \mathbb{Q}^\times,$$
where the entries of the $7 \times 7$ matrix $P_\zeta$ satisfy
\[
(P_\zeta)_{i,i} = 1; \quad (P_\zeta)_{i,j} = 0, \; \forall j > i; \quad (P_\zeta)_{i,j} = \binom{i}{j} (P_\zeta)_{i-j,0}, \; \forall j < i.
\] (6.13)

Now let $\tau_{6,3}$ and $\tau_{6,5}$ be
\[
\tau_{6,3} = -168 \zeta(3)/(2\pi i)^3, \quad \tau_{6,5} = -6552 \zeta(5)/(2\pi i)^5,
\] (6.14)
then from the paper [12], we have
\[
(P_\zeta)_{1,0} = (P_\zeta)_{2,0} = (P_\zeta)_{4,0} = 0, \quad (P_\zeta)_{3,0} = 3! \tau_{6,3}, \quad (P_\zeta)_{5,0} = 5! \tau_{6,5}, \quad (P_\zeta)_{6,0} = 6! \left(\frac{1}{2!} \tau_{6,3}^2\right).
\] (6.15)

From formula 3.22, the integral period $\Pi_i(\psi)$ is given by
\[
\Pi_i(\psi) = \sum_{j=0}^{6} P_{ij} \psi^{-1} \varpi_j(\phi),
\] (6.16)
and with respect to the rational basis $\alpha$, $\Omega_\psi$ has an expansion
\[
\Omega_\psi = \alpha \cdot \Pi(\psi) = \sum_{i=0}^{6} \alpha_i \Pi_i(\psi).
\] (6.17)

From the period matrix 5.12, $l_6(2\pi i)^6 \varpi_0$ and $l_6(2\pi i)^6 \varpi_1$ are the integrals of the sixform $\Omega_\psi$ over rational homological cycles of $H^6_a(X, \mathbb{Q})$, and their quotient is by definition the mirror map $t$
\[
t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}.
\] (6.18)

6.2. The charges for the split at the Fermat point. The numerical values of $\psi^{-1} \varpi_j(\phi)$, $(\psi^{-1} \varpi_j(\phi))'$ and $(\psi^{-1} \varpi_j(\phi))''$ at the Fermat point $\psi = 0$ have been computed using the method introduced in Section 4.2, which are listed in Appendix B. However, the singularity at the Fermat point $\psi = 0$ is too severe for the ODE satisfied by $\Omega_\psi^{(k)}, k \geq 3$. As a result, we can not obtain the numerical values of $(\psi^{-1} \varpi_j(\phi))^{(k)}$ at $\psi = 0$ when $k \geq 3$. Together with the period matrix $P$ 6.12, we obtain the numerical values of $\Pi_i(0), \Pi_i'(0)$ and $\Pi_i''(0)$. From these numerical results, we immediately learn that the value of the mirror map $t$ 6.18 at the Fermat point $\psi = 0$ agrees with an algebraic number
\[
t|_{\psi=0} = \lim_{\psi \to 0} \frac{\varpi_1}{\varpi_0} = \frac{1}{2} + \frac{1}{2} \left(1 + \sqrt{2}\right) i.
\] (6.19)

In order to construct the split 6.2 over the field $\mathbb{Q}(\sqrt{2})$, we will need to find six charges $\rho_i$ with $i = 1, \cdots, 6$ in the vector space $H^{6,a}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ such that

(1) The Hodge decompositions of $\rho_1$ and $\rho_2$ only have $(6, 0)$ and $(0, 6)$ components;
(2) The Hodge decompositions of $\rho_3$ and $\rho_4$ only have $(5, 1)$ and $(1, 5)$ components;
(3) The Hodge decompositions of $\rho_5$ and $\rho_6$ only have $(4, 2)$ and $(2, 4)$ components.
Numerically, we have found the following two charges
\[ \rho_1 = \alpha \cdot \left( 1, 0, \frac{1}{3} - \sqrt{2}, 0, \frac{53}{10} + 9\sqrt{2}, 0, -\frac{593}{2} - \frac{651}{2}\sqrt{2} \right)^\top, \]
\[ \rho_2 = \alpha \cdot \left( 1, 2 + \sqrt{2}, \frac{7}{3}, -2 - 4\sqrt{2}, -\frac{7}{10}, \frac{395}{3} + \frac{605}{6}\sqrt{2}, \frac{229}{2} \right)^\top, \]
that satisfy the charge equations
\[ \rho_1 = \sum_{i=0}^{6} \left( c_i \Pi_i(0) + c_i \Pi_i(0) \right) \alpha_i, \quad \rho_2 = \sum_{i=0}^{6} \left( c_i \Pi_i(0) + c_i \Pi_i(0) \right) \alpha_i \]  
for nonzero constants \( c_1, c_2 \in \mathbb{C} \). Moreover, the cup product pairings between \( \rho_1 (\rho_2) \) and \( \Omega'_0, \Omega''_0 \) vanish, i.e.
\[ \int_X \rho_1 \smile \Omega'_0 = \int_X \rho_2 \smile \Omega'_0 = \int_X \rho_1 \smile \Omega''_0 = \int_X \rho_2 \smile \Omega''_0 = 0. \]  
Thus the Hodge decomposition of \( \rho_1 \) (resp. \( \rho_2 \)) \ref{6.20} only has \((6,0)\) and \((0,6)\) components.

The underlying vector space of the direct summand \( \mathbf{H}^6_{a,1} \) in the formula \ref{6.2} is spanned by \( \rho_1 \) and \( \rho_2 \ref{6.20} \).

We have also found two charges
\[ \rho_3 = \alpha \cdot \left( 1, 0, \frac{4}{3}, 0, -\frac{97}{10}, 0, 239 \right)^\top, \]
\[ \rho_4 = \alpha \cdot \left( 1, 1, \frac{7}{3}, 5, -\frac{7}{10}, -\frac{205}{6}, \frac{229}{2} \right)^\top, \]
that satisfy the equation
\[ \rho_3 = \sum_{i=0}^{6} \left( c_i \Pi'_i(0) + c_i \Pi'_i(0) \right) \alpha_i, \quad \rho_4 = \sum_{i=0}^{6} \left( c_i \Pi'_i(0) + c_i \Pi'_i(0) \right) \alpha_i \]  
for nonzero constants \( c_3, c_4 \in \mathbb{C} \). Moreover, the cup product pairings between \( \rho_3 (\rho_4) \) \ref{6.23} and \( \Omega_0, \Omega''_0 \) vanish, i.e.
\[ \int_X \rho_3 \smile \Omega_0 = \int_X \rho_4 \smile \Omega_0 = \int_X \rho_3 \smile \Omega''_0 = \int_X \rho_4 \smile \Omega''_0 = 0. \]  
Thus the Hodge decomposition of \( \rho_3 \) (resp. \( \rho_4 \)) only has \((5,1)\) and \((1,5)\) components, and the underlying vector space of the direct summand \( \mathbf{H}^6_{a,2} \) in the formula \ref{6.2} is spanned by \( \rho_3 \) and \( \rho_4 \ref{6.23} \). Notice that \( \rho_3 \) and \( \rho_4 \) are defined over \( \mathbb{Q} \), i.e. their components are rational, hence the direct summand \( \mathbf{H}^6_{a,2} \) is a sub-Hodge structure defined over \( \mathbb{Q} \).

We have also found another two charges
\[ \rho_5 = \alpha \cdot \left( 1, 0, \frac{1}{3} + \sqrt{2}, 0, \frac{53}{10} - 9\sqrt{2}, 0, -\frac{593}{2} + \frac{651}{2}\sqrt{2} \right)^\top, \]
\[ \rho_6 = \alpha \cdot \left( 1, 2 - \sqrt{2}, \frac{7}{3}, -2 + 4\sqrt{2}, -\frac{7}{10}, \frac{395}{3} - \frac{605}{6}\sqrt{2}, \frac{229}{2} \right)^\top, \]
that satisfy the charge equations

$$\rho_5 = \sum_{i=0}^{6} \left( c_5 \Pi''_i(0) + c_5 \Pi''_i(0) \right) \alpha_i, \quad \rho_6 = \sum_{i=0}^{6} \left( c_6 \Pi''_i(0) + c_6 \Pi''_i(0) \right) \alpha_i \tag{6.27}$$

for nonzero constants $c_5, c_6 \in \mathbb{C}$. The cup product pairings between $\rho_5$ ($\rho_6$) and $\Omega_0$, $\Omega'_0$ vanish, i.e.

$$\int_X \rho_5 \sim \Omega_0 = \int_X \rho_6 \sim \Omega_0 = \int_X \rho_5 \sim \Omega'_0 = \int_X \rho_6 \sim \Omega'_0 = 0. \tag{6.28}$$

Thus the Hodge decomposition of $\rho_5$ (resp. $\rho_6$) only has $(4, 2)$ and $(2, 4)$ components, and the underlying vector space of the direct summand $H^6_{a,3}$ in the formula 6.2 is spanned by $\rho_5$ and $\rho_6$ 6.26.

The orthogonal complement of the six charges $\rho_1, \ldots, \rho_6$ with respect to the cup product pairing 6.10 is spanned by the charge

$$\rho_7 = \begin{pmatrix} 1, 1, 7, 13, -7, 85, 229 \end{pmatrix}. \tag{6.29}$$

The cup product pairings between $\rho_7$ and $\Omega_0$, $\Omega'_0$ and $\Omega''_0$ vanish, i.e.

$$\int_X \rho_7 \sim \Omega_0 = \int_X \rho_7 \sim \Omega'_0 = \int_X \rho_7 \sim \Omega''_0 = 0. \tag{6.30}$$

Hence the one dimensional summand $H^6_{a,4}$ in the formula 6.2 is spanned by $\rho_7$ 6.29. Moreover, the charge $\rho_7$ is defined over $\mathbb{Q}$, i.e. its components are rational. So $H^6_{a,4}$ is a sub-Hodge structure defined over $\mathbb{Q}$ and it is isomorphic to $\mathbb{Q}(-3)$.

6.3. Deligne’s periods for Fermat octic. Now we are ready to compute the Deligne’s periods for $H^6_{a,1}$, $H^6_{a,2}$ and $H^6_{a,3}$ [4, 13]. As $\mathcal{F}_6$ is a variety defined over $\mathbb{Q}$, the complex conjugation acts on its complex points, which induces an involution $F_{\infty}$ on the cohomology group $H^6(X, \mathbb{Q})$

$$F_{\infty} : H^6(X, \mathbb{Q}) \to H^6(X, \mathbb{Q}). \tag{6.31}$$

The matrix of $F_{\infty}$ with respect to the basis $\alpha$ can be computed by the method developed in the paper [13], however, in this section we will use a property of the Deligne’s periods to determine the action of $F_{\infty}$. Namely, the Deligne’s period $c^+(H^6_{a,j})$, $j = 1, 2, 3$ is a real number, and the Deligne’s period $c^-(H^6_{a,j})$, $j = 1, 2, 3$ is a purely imaginary number. From this property, we find that the charge $\rho_1$ (resp. $\rho_2$) 6.20 is an eigenvector of $F_{\infty}$ with eigenvalue 1 (resp. $-1$), i.e.

$$F_{\infty}(\rho_1) = \rho_1, \quad F_{\infty}(\rho_2) = -\rho_2. \tag{6.32}$$

From [4, 13], the Deligne’s periods $c^\pm(H^6_{a,1})$ are given by

$$c^+(H^6_{a,1}) = \frac{1}{(2\pi i)^6} \int_X \rho_1 \sim \Omega_0, \quad c^-(H^6_{a,1}) = \frac{1}{(2\pi i)^6} \int_X \rho_2 \sim \Omega_0. \tag{6.33}$$

From the numerical results in Appendix B, the the numerical value of $c^+(H^6_{a,1})$ is

$$c^+(H^6_{a,1}) = l_0 \times 8007.10875567897668453754447710594661081111628358109 \cdots, \tag{6.34}$$
where $l_6$ is the nonzero rational constant appears in the period matrix 6.12. We also have the following interesting quotient

$$\frac{c^+(H_{a,1}^6)}{c^-(H_{a,1}^6)} = \frac{\int_X \rho_1 \sim \Omega_0}{\int_X \rho_2 \sim \Omega_0} = \left(1 - \sqrt{2}\right)i. \quad (6.35)$$

Similarly, the charge $\rho_3$ (resp. $\rho_4$) 6.23 is an eigenvector of $F_\infty$ with eigenvalue 1 (resp. -1), i.e.

$$F_\infty(\rho_3) = \rho_3, \quad F_\infty(\rho_4) = -\rho_4. \quad (6.36)$$

From [4, 13], the Deligne’s periods $c^+(H_{a,2}^6)$ are given by

$$c^+(H_{a,2}^6) = \frac{1}{(2\pi i)^6} \int_X \rho_3 \sim \Omega'_0, \quad c^-(H_{a,2}^6) = \frac{1}{(2\pi i)^6} \int_X \rho_4 \sim \Omega'_0. \quad (6.37)$$

From the numerical results in Appendix B, the numerical value of $c^+(H_{a,2}^6)$ is

$$c^+(H_{a,2}^6) = -l_6 \times 444.843717266932339518041219005289087906852670921\ldots. \quad (6.38)$$

We also have the following interesting quotient

$$\frac{c^+(H_{a,2}^6)}{c^-(H_{a,2}^6)} = \frac{\int_X \rho_3 \sim \Omega'_0}{\int_X \rho_4 \sim \Omega'_0} = -i. \quad (6.39)$$

Similarly, the charges $\rho_5$ (resp. $\rho_6$) 6.26 is an eigenvector of $F_\infty$ with eigenvalue 1 (resp. -1), i.e.

$$F_\infty(\rho_5) = \rho_5, \quad F_\infty(\rho_6) = -\rho_6. \quad (6.40)$$

From [4, 13], the Deligne’s periods $c^+(H_{a,3}^6)$ are given by

$$c^+(H_{a,3}^6) = \frac{1}{(2\pi i)^6} \int_X \rho_5 \sim \Omega''_0, \quad c^-(H_{a,3}^6) = \frac{1}{(2\pi i)^6} \int_X \rho_6 \sim \Omega''_0. \quad (6.41)$$

From the numerical results in Appendix B, the numerical value of $c^+(H_{a,3}^6)$ is

$$c^+(H_{a,3}^6) = l_6 \times 286.85024228542971686694718641015790360409402\ldots, \quad (6.42)$$

and again we have an interesting quotient

$$\frac{c^+(H_{a,3}^6)}{c^-(H_{a,3}^6)} = \frac{\int_X \rho_5 \sim \Omega''_0}{\int_X \rho_6 \sim \Omega''_0} = -\left(1 + \sqrt{2}\right)i. \quad (6.43)$$

### 7. The Fermat decic CY eightfold

The Fermat decic CY eightfold $\mathcal{F}_8$ is by definition

$$\{x_0^{10} + x_1^{10} + x_2^{10} + x_3^{10} + x_4^{10} + x_5^{10} + x_6^{10} + x_7^{10} + x_8^{10} + x_9^{10} = 0\} \subset \mathbb{P}^9. \quad (7.1)$$

From the terminologies in previous sections, its underlying differentiable manifold will be denoted by $X$. Recall from Section 3 that the pure Hodge structure $(H^8(X, \mathbb{Q}), F^p_0)$ on $\mathcal{F}_8$ has a nine dimensional direct summand $(H^{8,a}(X, \mathbb{Q}), F^{p,a}_0)$ that is induced by the holomorphic eightform $\Omega_\psi$ of the Fermat decic pencil 2.1. In this section, we will explicitly construct the following split over $\mathbb{Q}(\sqrt{5})$

$$(H^{8,a}(X, \mathbb{Q}), F^{p,a}_0) = H^8_{25} \oplus H^8_{a,2} \oplus H^8_{a,3} \oplus H^8_{a,4}, \quad (7.2)$$
where the Hodge decomposition of $H_{a,1}^8$ is

$$H_{a,1}^8 = H^{8,0}(\mathcal{F}_8) \oplus H^{0,8}(\mathcal{F}_8). \quad (7.3)$$

While the Hodge type of the two dimensional summand $H_{a,2}^8$ is $(7, 1) + (1, 7)$, and that of the two dimensional summand $H_{a,3}^8$ is $(6, 2) + (2, 6)$; and that of the three dimensional summand $H_{a,4}^8$ is $(5, 3) + (4, 4) + (3, 5)$. More concretely, we will use numerical methods to find two charges $\rho_1$ and $\rho_2$ in $H^{8,a}(X, \mathbb{Q})$ whose Hodge decompositions only have $(8, 0)$ and $(0, 8)$ components, etc.

### 7.1. The period matrix for the Fermat decic pencil.

From Section 2, when $n = 8$, we have $\varphi = \psi^{-10}$ by formula 2.8. The Picard-Fuchs operator

$$D_8 := \varphi^9 - \varphi \prod_{k=1}^{9} \left( \vartheta + \frac{k}{10} \right), \quad \vartheta = \varphi \frac{d}{d\varphi} \quad (7.4)$$

has nine canonical solutions of the form

$$\omega_j = \frac{1}{(2\pi i)^j} \sum_{k=0}^{j} \binom{j}{k} h_k(\varphi) \log^{j-k}(10^{-10}\varphi), \quad j = 0, 1, \ldots, 8, \quad (7.5)$$

where $h_j(\varphi)$ is a power series in $\varphi$. From Section 3.1, there exist homological cycles $C_j \in H^8(X, \mathbb{C})$ such that

$$\psi^{-1} \omega_j(\varphi) = \int_{C_j} \Omega_\psi, \quad j = 0, 1, \ldots, 8. \quad (7.6)$$

The dual of $\{C_j\}_{j=0}^8$, denoted by $\{\gamma_j\}_{j=0}^8$, forms a basis of $H^{8,a}(X, \mathbb{C})$. The eightform $\Omega_\psi$ admits an expansion

$$\Omega_\psi = \sum_{i=0}^{8} \gamma_i \psi^{-1} \omega_i(\varphi). \quad (7.7)$$

Similarly, the form $\Omega^{(k)}_\psi$ admits an expansion

$$\Omega^{(k)}_\psi = \sum_{i=0}^{8} \gamma_i d^k \left( \psi^{-1} \omega_i(\varphi) \right) / d\psi^k. \quad (7.8)$$

Similarly, the cup product pairing on $H^{8,a}(X, \mathbb{C})$ can be computed by the equations

$$\int_X \Omega_\psi \wedge \Omega_\psi = 0, \quad \int_X \Omega_\psi \wedge \Omega^{(1)}_\psi = 0; \quad (7.9)$$
and with respect to the canonical basis \( \{ \gamma_i \}_{i=0}^8 \), its matrix is

\[
\left( \int_X \gamma_i \sim \gamma_j \right) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 28 & 0 & 0 \\
0 & 0 & 0 & -56 & 0 & 0 & 0 \\
0 & 0 & 70 & 0 & 0 & 0 & 0 \\
0 & -56 & 0 & 0 & 0 & 0 & 0 \\
0 & 28 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(7.10)

From the paper \cite{12}, \( H^{8,a}(X, \mathbb{Q}) \) has a basis

\[
\alpha = (\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8),
\]

(7.11)

with respect to which the period matrix \( P \) between the basis \( \gamma \) and \( \alpha \), i.e. \( \gamma = \alpha \cdot P \), has been numerically computed. The matrix \( P \) is of the form

\[
P = l_8(2\pi i)^8 \cdot P_\zeta,
\]

(7.12)

where the entries of the \( 9 \times 9 \) matrix \( P_\zeta \) satisfy

\[
(P_\zeta)_{i,i} = 1; \quad (P_\zeta)_{i,j} = 0, \quad \forall j > i; \quad (P_\zeta)_{i,j} = \binom{i}{j} (P_\zeta)_{i-j,0}, \quad \forall j < i.
\]

(7.13)

Now let \( \tau_{8,3}, \tau_{8,5} \) and \( \tau_{8,7} \) be

\[
\tau_{8,3} = -330 \zeta(3)/(2\pi i)^3, \quad \tau_{8,5} = -19998 \zeta(5)/(2\pi i)^5, \quad \tau_{8,7} = -1428570 \zeta(7)/(2\pi i)^7;
\]

(7.14)

and we have \cite{12}

\[
(P_\zeta)_{1,0} = (P_\zeta)_{2,0} = (P_\zeta)_{4,0} = 0, \quad (P_\zeta)_{3,0} = 3! \tau_{8,3}, \quad (P_\zeta)_{5,0} = 5! \tau_{8,5}, \quad (P_\zeta)_{6,0} = 6! \left( \frac{1}{2!} \tau_{8,3}^2 \right), \quad (P_\zeta)_{7,0} = 7! \tau_{8,7}, \quad (P_\zeta)_{8,0} = 8! \tau_{8,3} \tau_{8,5}.
\]

(7.15)

From formula 3.22, the integral period \( \Pi_i(\psi) \) is given by

\[
\Pi_i(\psi) = \sum_{j=0}^8 P_{ij} \psi^{-1} \omega_j(\phi),
\]

(7.16)

and with respect to the rational basis \( \alpha \), \( \Omega_\psi \) has an expansion

\[
\Omega_\psi = \alpha \cdot \Pi(\psi) = \sum_{i=0}^8 \alpha_i \Pi_i(\psi).
\]

(7.17)

From the period matrix 7.12, \( l_8(2\pi i)^8 \omega_0 \) and \( l_8(2\pi i)^8 \omega_1 \) are the integrals of the eightform \( \Omega_\psi \) over rational homological cycles of \( H^a_8(X, \mathbb{Q}) \), and their quotient is by definition the mirror map \( t \)

\[
t = \frac{\omega_1(\varphi)}{\omega_0(\varphi)^{27}}.
\]

(7.18)
7.2. **The charges for the split at the Fermat point.** The numerical values of \( \psi^{-1} \omega_j(\phi) \), \((\psi^{-1} \omega_j(\phi))^'\) and \((\psi^{-1} \omega_j(\phi))^''\) at the Fermat point \( \psi = 0 \) have been computed using the method introduced in Section 4.2, which are listed in Appendix C. However, the singularity at the Fermat point \( \psi = 0 \) is too severe for the ODE satisfied by \( \Omega_k^{(k)} \), \( k \geq 3 \). As a result, we can not obtain the numerical values of \((\psi^{-1} \omega_j(\phi))^{(k)}\) at \( \psi = 0 \) when \( k \geq 3 \). Together with the period matrix \( P \), we obtain the numerical values of \( \Pi_i(0) \), \( \Pi'_i(0) \) and \( \Pi''_i(0) \). From these numerical results, we immediately learn that the value of the mirror map \( t \) at the Fermat point \( \psi = 0 \) agrees with an algebraic number

\[
t|_{\psi=0} = \lim_{\psi \to 0} \frac{\omega_1}{\omega_0} = \frac{1}{2} + \frac{i}{2} \sqrt{5 + 2\sqrt{5}}.
\]  

(7.19)

In order to find a split 7.2 over the field \( \mathbb{Q}(\sqrt{5}) \), we will need to find six charges \( \rho_i \) with \( i = 1, \ldots, 6 \) in the vector space \( H^{8,a}(X, \mathbb{Q}) \otimes \mathbb{Q}(\sqrt{5}) \) such that

1. The Hodge decompositions of \( \rho_1 \) and \( \rho_2 \) only have \((8, 0)\) and \((0, 8)\) components;
2. The Hodge decompositions of \( \rho_3 \) and \( \rho_4 \) only have \((7, 1)\) and \((1, 7)\) components;
3. The Hodge decompositions of \( \rho_5 \) and \( \rho_6 \) only have \((6, 2)\) and \((2, 6)\) components.

Numerically, we have found the following two charges

\[
\begin{align*}
\rho_1 &= \alpha \cdot \left( 1, 0, \frac{3}{4} - \sqrt{5}, 0, \frac{225}{16}, \frac{25}{2} \sqrt{5}, 0, -\frac{492059}{448}, -\frac{11851}{16} \sqrt{5}, 0, \right. \\
&\quad \left. \frac{37679073}{16} + \frac{1430785}{\sqrt{5}} \right)^\top, \\
\rho_2 &= \alpha \cdot \left( 1, 3 + \sqrt{5}, \frac{15}{4}, -\frac{21}{4}, \frac{23}{4} \sqrt{5}, \frac{9}{16}, \frac{7983}{16}, \frac{3781}{16} \sqrt{5}, \frac{7983}{448}, \frac{194305}{16}, \right. \\
&\quad \left. -\frac{2904321}{64} - \frac{1375003}{\sqrt{5}}, -\frac{10628943}{256} \right)^\top
\end{align*}
\]  

(7.20)

that satisfy the charge equations

\[
\begin{align*}
\rho_1 &= \sum_{i=0}^{8} \left( c_1 \Pi_i(0) + c_1 \Pi_i(0) \right) \alpha_i, \\
\rho_2 &= \sum_{i=0}^{8} \left( c_2 \Pi_i(0) + c_2 \Pi_i(0) \right) \alpha_i
\end{align*}
\]  

(7.21)

for nonzero constants \( c_1, c_2 \in \mathbb{C} \). Moreover, the cup product pairings between \( \rho_1 (\rho_2) \) 7.20 and \( \Omega'_0, \Omega''_0 \) vanish, i.e.

\[
\int_X \rho_1 \sim \Omega'_0 = \int_X \rho_2 \sim \Omega'_0 = \int_X \rho_1 \sim \Omega''_0 = \int_X \rho_2 \sim \Omega''_0 = 0.
\]  

(7.22)

Thus the Hodge decomposition of \( \rho_1 \) (resp. \( \rho_2 \) 7.20) only has \((8, 0)\) and \((0, 8)\) components, and the underlying vector space of the direct summand \( H^8_{a,1} \) in the formula 7.2 is spanned by \( \rho_1 \) and \( \rho_2 \) 7.20.
Similarly, we have also found the following two charges

\[
\rho_3 = \alpha \cdot \left( 1, 0, \frac{11}{4} - \frac{1}{5} \sqrt{5}, 0, -\frac{1259}{80} - \frac{23}{10} \sqrt{5}, 0, \frac{297933}{448} + \frac{6101}{80} \sqrt{5}, 0, -\frac{15637823}{16} - \frac{87811}{16} \sqrt{5} \right)^\top,
\]

\[
\rho_4 = \alpha \cdot \left( 1, 1 + \frac{1}{5} \sqrt{5}, \frac{15}{4}, \frac{173}{20}, \frac{5}{4} \sqrt{5}, \frac{9}{16}, \frac{1019}{16} - \frac{80}{16} \sqrt{5}, \frac{194305}{448}, \frac{1475929}{320} + \frac{82033}{64} \sqrt{5}, \frac{267}{16} - \frac{334}{16} \sqrt{5} \right)^\top,
\]

that satisfy the charge equations

\[
\rho_3 = \sum_{i=0}^{8} \left( c_3 \Pi_i'(0) + \overline{c_3 \Pi_i'(0)} \right) \alpha_i, \quad \rho_4 = \sum_{i=0}^{8} \left( c_4 \Pi_i'(0) + \overline{c_4 \Pi_i'(0)} \right) \alpha_i
\]

for nonzero constants \(c_3, c_4 \in \mathbb{C}\). Moreover, the cup product pairings between \(\rho_3\) (resp. \(\rho_4\)) \(7.23\) and \(\Omega_0, \Omega_0''\) vanish, i.e.

\[
\int_X \rho_3 \smile \Omega_0 = \int_X \rho_4 \smile \Omega_0 = \int_X \rho_3 \smile \Omega_0'' = \int_X \rho_4 \smile \Omega_0'' = 0.
\]

Thus the Hodge decomposition of \(\rho_3\) (resp. \(\rho_4\)) \(7.23\) only has \((7, 1)\) and \((1, 7)\) components, and the underlying vector space of the direct summand \(H^8_{a,1}\) in the formula \(7.2\) is spanned by \(\rho_3\) and \(\rho_4\) \(7.23\).

Similarly, we have found another two charges

\[
\rho_5 = \alpha \cdot \left( 1, 0, \frac{3}{4} + \sqrt{5}, 0, \frac{225}{16}, -\frac{25}{2} \sqrt{5}, 0, -\frac{492059}{448} + \frac{11851}{16} \sqrt{5}, 0, -\frac{37670973}{16} - \frac{1430785}{16} \sqrt{5} \right)^\top,
\]

\[
\rho_6 = \alpha \cdot \left( 1, 3 - \sqrt{5}, \frac{15}{4}, -\frac{21}{4}, \frac{23}{4} \sqrt{5}, \frac{9}{16}, \frac{7983}{16} - \frac{3781}{16} \sqrt{5}, \frac{194305}{448}, \frac{2904321}{64} + \frac{1375003}{64} \sqrt{5}, \frac{10628943}{256} \right)^\top
\]

that satisfy the charge equations

\[
\rho_5 = \sum_{i=0}^{8} \left( c_5 \Pi_i''(0) + \overline{c_5 \Pi_i''(0)} \right) \alpha_i, \quad \rho_6 = \sum_{i=0}^{8} \left( c_6 \Pi_i''(0) + \overline{c_6 \Pi_i''(0)} \right) \alpha_i
\]

for nonzero constants \(c_5, c_6 \in \mathbb{C}\). The cup product pairings between \(\rho_5\) (resp. \(\rho_6\)) \(7.26\) and \(\Omega_0, \Omega_0''\) vanish, i.e.

\[
\int_X \rho_5 \smile \Omega_0 = \int_X \rho_6 \smile \Omega_0 = \int_X \rho_5 \smile \Omega_0'' = \int_X \rho_6 \smile \Omega_0'' = 0.
\]

Thus the Hodge decomposition of \(\rho_5\) (resp. \(\rho_6\)) \(7.26\) only has \((6, 2)\) and \((2, 6)\) components, and the underlying vector space of the direct summand \(H^8_{a,2}\) in the formula \(7.2\) is spanned by \(\rho_5\) and \(\rho_6\) \(7.26\).
7.3. Deligne’s periods for Fermat decic. Now we are ready to compute the Deligne’s periods for \( \mathbf{H}_{a,1}^8, \mathbf{H}_{a,2}^8 \) and \( \mathbf{H}_{a,3}^8 \) \cite{[4, 13]}. As \( \mathcal{F}_8 \) is a variety defined over \( \mathbb{Q} \), the complex conjugation acts on its complex points, which induces an involution \( F_\infty \) on the cohomology group \( H^{8,\alpha}(X, \mathbb{Q}) \)

\[
F_\infty : H^{8,\alpha}(X, \mathbb{Q}) \to H^{8,\alpha}(X, \mathbb{Q}).
\] (7.29)

The matrix of \( F_\infty \) with respect to the basis \( \alpha \) can be computed by the method developed in the paper \cite{[13]}, however, in this section we will use a property of the Deligne’s periods to determine \( F_\infty \). Namely, the Deligne’s period \( c^+(\mathbf{H}_{a,j}^8), j = 1, 2, 3 \) is a real number, and the Deligne’s period \( c^-(\mathbf{H}_{a,j}^8), j = 1, 2, 3 \) is a purely imaginary number. From this property, we find that the charge \( \rho_1 \) (resp. \( \rho_2 \)) 7.20 is an eigenvector of \( F_\infty \) with eigenvalue 1 (resp. \(-1\)), i.e.

\[
F_\infty(\rho_1) = \rho_1, \quad F_\infty(\rho_2) = -\rho_2.
\] (7.30)

From \cite{[4, 13]}, the Deligne’s periods \( c^+(\mathbf{H}_{a,1}^8) \) are given by

\[
c^+(\mathbf{H}_{a,1}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_1 \sim \Omega_0, \quad c^-(\mathbf{H}_{a,1}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_2 \sim \Omega_0.
\] (7.31)

From the numerical results in Appendix C, the numerical value of \( c^+(\mathbf{H}_{a,1}^8) \) is

\[
c^+(\mathbf{H}_{a,1}^8) = -l_8 \times 5212961.1265694976222689791525301848232107600478095 \cdots,
\] (7.32)

where \( l_8 \) is the nonzero rational constant appears in the period matrix 7.12. We have the following interesting quotient

\[
\frac{c^+(\mathbf{H}_{a,1}^8)}{c^-(\mathbf{H}_{a,1}^8)} = \frac{\int_X \rho_1 \sim \Omega_0}{\int_X \rho_2 \sim \Omega_0} = -\frac{i}{\sqrt{5 + 2\sqrt{5}}},
\] (7.33)

Similarly, the charge \( \rho_3 \) (resp. \( \rho_4 \)) 7.23 is an eigenvector of \( F_\infty \) with eigenvalue 1 (resp. \(-1\)), i.e.

\[
F_\infty(\rho_3) = \rho_3, \quad F_\infty(\rho_4) = -\rho_4.
\] (7.34)

From \cite{[4, 13]}, the Deligne’s periods \( c^+(\mathbf{H}_{a,2}^8) \) are given by

\[
c^+(\mathbf{H}_{a,2}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_3 \sim \Omega'_0, \quad c^-(\mathbf{H}_{a,2}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_4 \sim \Omega'_0.
\] (7.35)

From the numerical results in Appendix C, the numerical value of \( c^+(\mathbf{H}_{a,2}^8) \) is

\[
c^+(\mathbf{H}_{a,2}^8) = l_8 \times 79815.6087659784105899046934127518572733437994818904 \cdots
\] (7.36)

We have also found an interesting quotient

\[
\frac{c^+(\mathbf{H}_{a,2}^8)}{c^-(\mathbf{H}_{a,2}^8)} = \frac{\int_X \rho_3 \sim \Omega'_0}{\int_X \rho_4 \sim \Omega'_0} = -i\sqrt{5 - 2\sqrt{5}}.
\] (7.37)

The charges \( \rho_5 \) (resp. \( \rho_6 \)) 7.26 is an eigenvector of \( F_\infty \) with eigenvalue 1 (resp. \(-1\)), i.e.

\[
F_\infty(\rho_5) = \rho_5, \quad F_\infty(\rho_6) = -\rho_6.
\] (7.38)

From \cite{[4, 13]}, the Deligne’s periods \( c^+(\mathbf{H}_{a,3}^8) \) are given by

\[
c^+(\mathbf{H}_{a,3}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_5 \sim \Omega''_0, \quad c^-(\mathbf{H}_{a,3}^8) = \frac{1}{(2\pi i)^8} \int_X \rho_6 \sim \Omega''_0.
\] (7.39)
From the numerical results in Appendix C, the numerical value of $c^+(H_{a,3}^8)$ is
\[ c^+(H_{a,3}^8) = -l_8 \times 20875.2118612791484236100896801533250654143203133 \cdots, \] (7.40)
and again we have an interesting quotient
\[ \frac{c^+(H_{a,3}^8)}{c^-(H_{a,3}^8)} = \int X \rho_5 \sim \Omega''_0 = -\frac{i}{\sqrt{5} - 2\sqrt{5}}. \] (7.41)

8. The Fermat dodecic CY tenfold

The Fermat dodecic CY tenfold $\mathcal{F}_{10}$ is by definition
\[ \{ x_0^{12} + x_1^{12} + x_2^{12} + x_3^{12} + x_4^{12} + x_5^{12} + x_6^{12} + x_8^{12} + x_9^{12} + x_{10}^{12} + x_{11}^{12} = 0 \} \subset \mathbb{P}^{11}. \] (8.1)

From the terminologies in previous sections, its underlying differentiable manifold will be denoted by $X$. Recall from Section 3 that the pure Hodge structure $(H^{10}(X, \mathbb{Q}), F^p_0)$ on $\mathcal{F}_{10}$ has an eleven dimensional direct summand $(H^{10,a}(X, \mathbb{Q}), F^{p,a}_0)$ that is induced by the holomorphic tenform $\Omega_\psi$ of the Fermat dodecic pencil 2.1. In this section, we will explicitly construct the following split over $\mathbb{Q}(\sqrt{3})$
\[ (H^{10,a}(X, \mathbb{Q}), F^{p,a}_0) = H^{10}_{a,1} \oplus H^{10}_{a,2} \oplus H^{10}_{a,3} \oplus H^{10}_{a,4}, \] (8.2)
where the Hodge decomposition of the direct summand $H^{10}_{a,1}$ is given by
\[ H^{10}_{a,1} = H^{10,0}(\mathcal{F}_{10}) \oplus H^{0,10}(\mathcal{F}_{10}). \] (8.3)

While the Hodge type of the two dimensional summand $H^{10}_{a,2}$ is $(9, 1) + (1, 9)$, and that of the two dimensional summand $H^{10}_{a,3}$ is $(8, 2) + (2, 8)$; and that of the five dimensional summand $H^{10}_{a,4}$ is $(7, 3) + (6, 4) + (5, 5) + (4, 6) + (3, 7)$. More concretely, we will use numerical methods to find two charges $\rho_1$ and $\rho_2$ in $H^{10,a}(X, \mathbb{Q})$ whose Hodge decompositions only have $(10, 0)$ and $(0, 10)$ components, etc.

8.1. The period matrix for the Fermat dodecic pencil. From Section 2, when $n = 10$, we have $\varphi = \psi^{-12}$ by formula 2.8. The Picard-Fuchs operator
\[ D_{10} := \vartheta^{11} - \varphi \prod_{k=1}^{11} \left( \vartheta + \frac{k}{12} \right), \varrho = \varphi \frac{d}{d\varphi} \] (8.4)
has eleven canonical solutions of the form
\[ \varpi_j = \frac{1}{(2\pi i)^j} \sum_{k=0}^{j} \binom{j}{k} h_k(\varphi) \log^{j-k} \left( 12^{-12} \varphi \right), j = 0, 1, \cdots, 10, \] (8.5)
where $h_j(\varphi)$ is a power series in $\varphi$. From Section 3.1, there exist homological cycles $C_j \in H^{a}_{10}(X, \mathbb{C})$ such that
\[ \psi^{-1} \varpi_j(\varphi) = \int_{C_j} \Omega_\psi, j = 0, 1, \cdots, 10. \] (8.6)
The dual of \( \{C_i\}_{i=0}^{10} \), denoted by \( \{\gamma_i\}_{i=0}^{10} \), forms a basis of \( H^{10,a}(X,\mathbb{C}) \). The tenform \( \Omega_\psi \) admits an expansion of the form

\[
\Omega_\psi = \sum_{i=0}^{10} \gamma_i \psi^{-1} \varpi_i(\varphi) .
\]  

(8.7)

Similarly, the form \( \Omega_\psi^{(k)} \) admits an expansion

\[
\Omega_\psi^{(k)} = \sum_{i=0}^{10} \gamma_i d^k (\psi^{-1} \varpi_i(\varphi)) / d\psi^k .
\]  

(8.8)

The cup product pairing on \( H^{10,a}(X,\mathbb{C}) \) can be computed by the equations

\[
\int_X \Omega_\psi \wedge \Omega_\psi = 0, \quad \int_X \Omega_\psi \wedge \Omega_\psi^{(1)} = 0,
\]  

(8.9)

and with respect to the canonical basis \( \{\gamma_i\}_{i=0}^{10} \), its matrix is

\[
\left( \int_X \gamma_i \smile \gamma_j \right) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -120 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 210 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -252 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 210 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} .
\]  

(8.10)

From the paper [12], \( H^{10,a}(X,\mathbb{Q}) \) has a basis

\[
\alpha = (\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}),
\]  

(8.11)

with respect to which the period matrix \( P \) between the basis \( \gamma \) and \( \alpha \), i.e. \( \gamma = \alpha \cdot P \), has been explicitly computed. The matrix \( P \) is of the form

\[
P = l_{10}(2\pi i)^{10} \cdot P_\zeta ,
\]  

(8.12)

where the entries of the \( 11 \times 11 \) matrix \( P_\zeta \) satisfy

\[
(P_\zeta)_{i,i} = 1; \quad (P_\zeta)_{i,j} = 0, \quad \forall j > i; \quad (P_\zeta)_{i,j} = \binom{i}{j} (P_\zeta)_{i-j,0}, \quad \forall j < i;
\]  

(8.13)

Now let \( \tau_{10,3}, \tau_{10,5}, \tau_{10,7} \) and \( \tau_{10,9} \) be

\[
\tau_{10,3} = -572 \zeta(3)/(2\pi i)^3, \quad \tau_{10,5} = -49 764 \zeta(5)/(2\pi i)^5 ,
\]

\[
\tau_{10,7} = -5 \times 118 828 \zeta(7)/(2\pi i)^7, \quad \tau_{10,9} = -\frac{1 719 926 780}{32} \zeta(9)/(2\pi i)^9 ,
\]

(8.14)
then we have [12]
\[
(P_ζ)_{1,0} = (P_ζ)_{2,0} = (P_ζ)_{4,0} = 0, \quad (P_ζ)_{3,0} = 3! \tau_{10,3}, \quad (P_ζ)_{5,0} = 5! \tau_{10,5},
\]
\[
(P_ζ)_{6,0} = 6! \left( \frac{1}{2!} \tau_{10,3}^2 \right), \quad (P_ζ)_{7,0} = 7! \tau_{10,7}, \quad (P_ζ)_{8,0} = 8! \tau_{10,5} \tau_{10,5},
\]
\[
(P_ζ)_{9,0} = 9! \left( \tau_{10,9} + \frac{1}{3!} \tau_{10,3}^3 \right), \quad (P_ζ)_{10,0} = 10! \left( \frac{1}{2!} \tau_{10,5}^2 + \tau_{10,5} \tau_{10,7} \right).
\]

From formula 3.22, the integral period \( \Pi_i(\psi) \) is given by
\[
\Pi_i(\psi) = \sum_{j=0}^{10} P_{ij} \psi^{-1} \varpi_j(\phi), \quad (8.16)
\]
and with respect to the rational basis \( \alpha \), \( \Omega_\psi \) has an expansion
\[
\Omega_\psi = \alpha \cdot \Pi(\psi) = \sum_{i=0}^{10} \alpha_i \Pi_i(\psi). \quad (8.17)
\]

From the period matrix \( P_{8.12} \), \( l_{10}(2\pi i)^{10} \varpi_0 \) and \( l_{10}(2\pi i)^{10} \varpi_1 \) are the integrals of the tenform \( \Omega_\psi \) over rational homological cycles of \( H_{10}^a(X, \mathbb{Q}) \), and their quotient is by definition the mirror map \( t \)
\[
t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}.
\]

8.2. The charges for the split at the Fermat point. The numerical values of \( \psi^{-1} \varpi_j(\phi) \), \( (\psi^{-1} \varpi_j(\phi))^' \) and \( (\psi^{-1} \varpi_j(\phi))^'' \) at the Fermat point \( \psi = 0 \) have been computed using the method introduced in Section 4.2, which are listed in Appendix D. However, the singularity at the Fermat point \( \psi = 0 \) is too severe for the ODE satisfied by \( \Omega_\psi^{(k)} \), \( k \geq 3 \). As a result, we can not obtain the numerical values of \( (\psi^{-1} \varpi_j(\phi))^{(k)} \) at \( \psi = 0 \) when \( k \geq 3 \). Together with the period matrix \( P_{8.12} \), we obtain the numerical values of \( \Pi_i(0) \), \( \Pi'_i(0) \) and \( \Pi''_i(0) \). From these numerical results, we immediately learn that the value of the mirror map \( t \) 8.18 at the Fermat point \( \psi = 0 \) agrees with an algebraic number
\[
t|_{\psi=0} = \lim_{\psi \to 0} \frac{\varpi_1}{\varpi_0} = \frac{1}{2} + \left( 1 + \frac{\sqrt{3}}{2} \right) i.
\]

In order to construct the split 8.2 over the field \( \mathbb{Q}(\sqrt{3}) \), we will need to find six charges \( \rho_i \) with \( i = 1, \cdots, 6 \) in the vector space \( H^{10,a}(X, \mathbb{Q}) \otimes \mathbb{Q}(\sqrt{3}) \) such that

1. The Hodge decompositions of \( \rho_1 \) and \( \rho_2 \) only have \((10, 0)\) and \((0, 10)\) components;
2. The Hodge decompositions of \( \rho_3 \) and \( \rho_4 \) only have \((9, 1)\) and \((1, 9)\) components;
3. The Hodge decompositions of \( \rho_5 \) and \( \rho_6 \) only have \((8, 2)\) and \((2, 8)\) components.
After extensive numerical search, we have found the following two charges
\[
\rho_1 = \alpha \cdot \left( 1, 0, \frac{3}{2} - 2\sqrt{3}, 0, \frac{182}{5} + 28\sqrt{3}, 0, -\frac{279023}{84}, -2699\sqrt{3}, 0, \right. \\
\left. \frac{6287743}{10} + \frac{1426516}{3} \sqrt{3}, 0, -\frac{772191889}{4}, -138556957\sqrt{3} \right)^\top,
\]
\[
\rho_2 = \alpha \cdot \left( 1, 4 + 2\sqrt{3}, \frac{11}{2}, -14 - 13\sqrt{3}, \frac{22}{5}, 1512 + 876\sqrt{3}, \frac{108889}{84}, \\ - \frac{586535}{3} - \frac{694265}{6} \sqrt{3}, -\frac{5247011}{30}, -\frac{229380382}{5} + \frac{134971631}{5} \sqrt{3}, \frac{169569847}{4} \right)^\top,
\]
that satisfy the charge equations
\[
\rho_1 = \sum_{i=0}^{10} \left( c_1 \Pi_i(0) + c_1 \Pi_i(0) \right) \alpha_i, \quad \rho_2 = \sum_{i=0}^{10} \left( c_2 \Pi_i(0) + c_2 \Pi_i(0) \right) \alpha_i
\]
for nonzero constants \(c_1, c_2 \in \mathbb{C}\). Moreover, the cup product pairing between \(\rho_1 (\rho_2)\) 8.20 and \(\Omega_0', \Omega_0''\) vanish, i.e.
\[
\int_X \rho_1 \sim \Omega_0' = \int_X \rho_2 \sim \Omega_0' = \int_X \rho_1 \sim \Omega_0'' = \int_X \rho_2 \sim \Omega_0'' = 0.
\]
Thus the Hodge decomposition of \(\rho_1 (\rho_2)\) 8.20 only has \((10, 0)\) and \((0, 10)\) components, and the underlying vector space of the direct summand \(H_{4,1}^{10}\) in the formula 8.2 is spanned by \(\rho_1\) and \(\rho_2\) 8.20.

Similarly, we have also found two charges
\[
\rho_3 = \alpha \cdot \left( 1, 0, \frac{7}{2}, 0, -\frac{198}{5}, 0, \frac{199693}{84}, 0, -\frac{9080491}{30}, 0, \frac{274607139}{4} \right)^\top,
\]
\[
\rho_4 = \alpha \cdot \left( 1, 2, \frac{11}{2}, 23, \frac{22}{5}, -324, \frac{108889}{84}, \frac{183955}{6}, -\frac{5247011}{30}, -\frac{25293169}{5}, \frac{169569847}{4} \right)^\top
\]
that satisfy the charge equations
\[
\rho_3 = \sum_{i=0}^{10} \left( c_3 \Pi_i'(0) + c_3 \Pi_i'(0) \right) \alpha_i, \quad \rho_4 = \sum_{i=0}^{10} \left( c_4 \Pi_i'(0) + c_4 \Pi_i'(0) \right) \alpha_i
\]
for nonzero constants \(c_3, c_4 \in \mathbb{C}\). Moreover, the cup product pairings between \(\rho_3 (\rho_4)\) 8.23 and \(\Omega_0, \Omega_0''\) vanish, i.e.
\[
\int_X \rho_3 \sim \Omega_0 = \int_X \rho_4 \sim \Omega_0 = \int_X \rho_3 \sim \Omega_0'' = \int_X \rho_4 \sim \Omega_0'' = 0.
\]
Thus the Hodge decomposition of \(\rho_3 (\rho_4)\) 8.23 only has \((9, 1)\) and \((1, 9)\) components, and the underlying vector space of the direct summand \(H_{a,2}^{10}\) 8.2 is spanned by \(\rho_3\) and \(\rho_4\) 8.23. Notice that the two charges \(\rho_3\) and \(\rho_4\) 8.23 have rational components, hence \(H_{a,2}^{10}\) is sub-Hodge structure over \(\mathbb{Q}\).
We have also found another two charges

$$\rho_5 = \alpha \cdot \left(1, 0, \frac{9}{2}, 0, -\frac{118}{5}, 0, \frac{132871}{84}, 0, -\frac{2176657}{10}, 0, \frac{207439973}{4}\right)^\top,$$

$$\rho_6 = \alpha \cdot \left(1, 1, \frac{11}{2}, \frac{29}{2}, \frac{22}{5}, -72, \frac{108889}{84}, \frac{124105}{12}, -\frac{5247011}{30}, -\frac{18289309}{10}, \frac{169569847}{4}\right)^\top. \quad (8.26)$$

that satisfy the charge equations

$$\rho_5 = \sum_{i=0}^{8} \left(c_5 \Pi_i^0(0) + c_5 \Pi_i^0(0)\right) \alpha_i, \quad \rho_6 = \sum_{i=0}^{8} \left(c_6 \Pi_i^0(0) + c_6 \Pi_i^0(0)\right) \alpha_i \quad (8.27)$$

for nonzero constants $c_5, c_6 \in \mathbb{C}$. The cup product pairings between $\rho_5$ (resp. $\rho_6$) 8.26 and $\Omega_0, \Omega_0'$ vanish, i.e.

$$\int_X \rho_5 \smile \Omega_0 = \int_X \rho_6 \smile \Omega_0 = \int_X \rho_5 \smile \Omega_0' = \int_X \rho_6 \smile \Omega_0' = 0. \quad (8.28)$$

Thus the Hodge decomposition of $\rho_5$ (resp. $\rho_6$) 8.26 only has (8, 2) and (2, 8) components, and the underlying vector space of the direct summand $H_{a,3}^{10}$ in the formula 8.2 is spanned by $\rho_5$ and $\rho_6$ 8.26. Notice that the two charges $\rho_5$ and $\rho_6$ 8.26 have rational components, hence $H_{a,3}^{10}$ is sub-Hodge structure over $\mathbb{Q}$.

8.3. Deligne’s periods for Fermat dodecic. Now we are ready to compute the Deligne’s periods for $H_{a,1}^{10}, H_{a,2}^{10}$ and $H_{a,3}^{10}$ [4, 13]. As $\mathcal{F}_{10}$ is a variety defined over $\mathbb{Q}$, the complex conjugation acts on its complex points, which induces an involution $F_{\infty}$ on the cohomology groups $H^{10,a}(X, \mathbb{Q})$

$$F_{\infty} : H^{10,a}(X, \mathbb{Q}) \to H^{10,a}(X, \mathbb{Q}). \quad (8.29)$$

The matrix of $F_{\infty}$ with respect to the basis $\alpha$ can be computed by the method developed in the paper [13], however, in this section we will use a property of the Deligne’s periods to determine $F_{\infty}$. Namely, the Deligne’s period $c^+(H_{a,j}^{10}), j = 1, 2, 3$ is a real number, and the Deligne’s period $c^-(H_{a,j}^{10}), j = 1, 2, 3$ is a purely imaginary number. From this property, we find that the charge $\rho_1$ (resp. $\rho_2$) 8.20 is an eigenvector of $F_{\infty}$ with eigenvalue 1 (resp. $-1$), i.e.

$$F_{\infty}(\rho_1) = \rho_1, \quad F_{\infty}(\rho_2) = -\rho_2. \quad (8.30)$$

From [4, 13], the Deligne’s periods $c^+(H_{a,1}^{10})$ are given by

$$c^+(H_{a,1}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_1 \smile \Omega_0, \quad c^-(H_{a,1}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_2 \smile \Omega_0. \quad (8.31)$$

From the numerical results in Appendix D, the numerical value of $c^+(H_{a,1}^{10})$ is

$$c^+(H_{a,1}^{10}) = l_{10} \times 8628829314.63181296956648940152332863328728264485086 \cdots, \quad (8.32)$$

where $l_{10}$ is the nonzero rational constant appears in the period matrix 8.12. We have an interesting quotient

$$\frac{c^+(H_{a,1}^{10})}{c^-(H_{a,1}^{10})} = \frac{\int_X \rho_1 \smile \Omega_0}{\int_X \rho_2 \smile \Omega_0} = \left(-2 + \sqrt{3}\right)i. \quad (8.33)$$
Similarly, the charge $\rho_3$ (resp. $\rho_4$) is an eigenvector of $F_\infty$ with eigenvalue 1 (resp. $-1$), i.e.

$$F_\infty(\rho_3) = \rho_3, \quad F_\infty(\rho_4) = -\rho_4. \quad (8.34)$$

From [4, 13], the Deligne’s periods $c^+(H_{a,2}^{10})$ are given by

$$c^+(H_{a,2}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_3 \sim \Omega'_0, \quad c^-(H_{a,2}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_4 \sim \Omega'_0. \quad (8.35)$$

From the numerical results in Appendix D, the numerical value of $c^+(H_{a,2}^{10})$ is

$$c^+(H_{a,2}^{10}) = -l_{10} \times 36916404.2175706170751471487392255869214751161125761 \ldots, \quad (8.36)$$

and we also have an interesting quotient

$$\frac{c^+(H_{a,2}^{10})}{c^-(H_{a,2}^{10})} = \int_X \rho_3 \sim \Omega'_0 = -i. \quad (8.37)$$

The charges $\rho_5$ (resp. $\rho_6$ is an eigenvector of $F_\infty$ with eigenvalue 1 (resp. $-1$), i.e.

$$F_\infty(\rho_5) = \rho_5, \quad F_\infty(\rho_6) = -\rho_6. \quad (8.38)$$

From [4, 13], the Deligne’s periods $c^+(H_{a,3}^{10})$ are given by

$$c^+(H_{a,3}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_5 \sim \Omega''_0, \quad c^-(H_{a,3}^{10}) = \frac{1}{(2\pi i)^{10}} \int_X \rho_6 \sim \Omega''_0. \quad (8.39)$$

From the numerical results in Appendix D, the numerical value of $c^+(H_{a,3}^{10})$ is

$$c^+(H_{a,3}^{10}) = l_{10} \times 4474246.1550369742331223061834922036711622476664701339 \ldots. \quad (8.40)$$

Again we have an interesting quotient

$$\frac{c^+(H_{a,3}^{10})}{c^-(H_{a,3}^{10})} = \int_X \rho_5 \sim \Omega''_0 = -i. \quad (8.41)$$

9. Conclusions and further prospects

In this paper, we have formulated three conjectures, Conjectures 1.1, 1.2 and 1.3, about the properties of the Fermat type CY $n$-fold $\mathcal{F}_n$

$$\mathcal{F}_n : \{ \sum_{i=0}^{n+1} x_i^{n+2} = 0 \} \subset \mathbb{P}^{n+1}. \quad (9.1)$$

When $n = 1, 2$, these conjectures have been shown to be true [9, 15]. Using numerical methods, we have explicitly shown that these three conjectures are satisfied for the cases where $n = 3, 4, 6$; while we have also provided partial results for the cases where $n = 8, 10$. Hence the numerical results in this paper have provided strong and enlightening evidences to Conjectures 1.1, 1.2 and 1.3.

There are many interesting open questions about the Fermat type CY $n$-fold $\mathcal{F}_n$. The first and most important one is of course to prove Conjectures 1.1, 1.2 and 1.3. Then it is also very interesting to see whether the split 1.22 is motivic and the two dimensional sub-objects in this split are modular! For example, when $n = 4$, we can ask whether the Galois representation $H^4_{\text{ét}}(\mathcal{F}_4, \mathbb{Q}_\ell)$ has a two dimensional sub-representation that corresponds
to $H_{a,1}^4$ in the split 1.8. If so, whether this two dimensional sub-representation is modular, and associated to it is a weight-5 newform? It is also very interesting to see whether $H_{a,1}^4$ satisfies the predictions of Deligne’s conjecture on the special values of $L$-functions at critical integers [13, 14, 15]. There are also other interesting open questions related to string theory, for example the Fermat quintic $f_3$ is a flux vacua, do the results in this paper have any string theoretic interpretations and vice versa [2, 6, 11]!
APPENDIX A. THE NUMERICAL DATA FOR THE FERMAT SEXTIC CY FOURFOLD

In this section, we will provide the numerical values of the canonical periods of Fermat sextic pencil 2.1 and its derivatives at the Fermat point $\psi = 0$. When $n = 4$, the fourform $\Omega_\psi$ of the Fermat sextic pencil 2.1 satisfies the following Picard-Fuchs equation

\[(1 - \psi^6)\frac{d^5\Omega_\psi}{d\psi^5} - 15\psi^5\Omega_\psi - 65\psi^4d^3\Omega_\psi - 90\psi^3d^2\Omega_\psi - 31\psi^2d\Omega_\psi - \psi\Omega_\psi = 0. \quad (A.1)\]

Notice that the Fermat point $\psi = 0$ is a smooth point of this ODE. Using the method introduced in Section 4.2, we have computed the numerical values of $\psi^{-1}\varpi_i$ at $\psi = 0$ to a very high precision

\[
\begin{align*}
\psi^{-1}\varpi_0|_{\psi=0} &= -2.63050078917142547210088787337715390487570318456678 \cdots \\
&\quad - i \times 1.51872033873164550661945035986597266836741161215 \cdots ; \\
\psi^{-1}\varpi_1|_{\psi=0} &= -i \times 3.03744067746329101323890071973079453367348232430 \cdots \\
\psi^{-1}\varpi_2|_{\psi=0} &= 1.9728755918785691040756659050328654286427738842508 \cdots \\
&\quad - i \times 1.898400423414556883274312949831746583354926451519 \cdots ; \\
\psi^{-1}\varpi_3|_{\psi=0} &= 3.09109620007393123171034111212622607065482265809874 \cdots \\
&\quad - i \times 1.557134822782026723313217753331204179071266412835 \cdots ; \\
\psi^{-1}\varpi_4|_{\psi=0} &= 8.12373336894682656097384724694048016279757991221218 \cdots \\
&\quad + i \times 1.044120232878006285800872122407460620845029548335 \cdots .
\end{align*}
\]

The first derivative of the fourform $\Omega_\psi$, $\Omega^{(1)}_\psi = d\Omega_\psi/d\psi$, satisfies the following ODE

\[
\psi(1 - \psi^6)\frac{d^5\Omega_\psi}{d\psi^5} - (1 + 20\psi^6)\frac{d^4\Omega_\psi}{d\psi^4} - 125\psi^5\frac{d^3\Omega_\psi}{d\psi^3} \\
- 285\psi^4\frac{d^2\Omega_\psi}{d\psi^2} - 211\psi^3\frac{d\Omega_\psi}{d\psi} - 32\psi^2\Omega_\psi = 0, \quad (A.2)
\]

which however has $\psi = 0$ as a singularity. But using the method introduced in Section 4.2, Mathematica can still compute the values of $(\psi^{-1}\varpi_i)'$ at $\psi = 0$ to a very high precision

\[
\begin{align*}
(\psi^{-1}\varpi_0)'|_{\psi=0} &= -1.765230999734920251545904282363294070345325464779 \cdots \\
&\quad - i \times 3.0574697786364848492294183051855823667429225187366 \cdots ; \\
(\psi^{-1}\varpi_1)'|_{\psi=0} &= -i \times 2.038313185757565661529455367903882444952816791578 \cdots \\
(\psi^{-1}\varpi_2)'|_{\psi=0} &= -0.0297180831787033480068444164711921541034773187788 \cdots \\
&\quad - i \times 3.8218372232956060615367728814819779584286531484208 \cdots ; \\
(\psi^{-1}\varpi_3)'|_{\psi=0} &= 6.2229582192051493989229936831634050808663578555865 \cdots \\
&\quad - i \times 9.1981878638474427383992464553611084815861408077043 \cdots ; \\
(\psi^{-1}\varpi_4)'|_{\psi=0} &= 23.103844562736249323218868692557211570750126380035 \cdots \\
&\quad + i \times 2.10201047281258333845225084815087877135759231631 \cdots .
\end{align*}
\]
The second derivative of the fourform $\Omega$, $\Omega^{(2)}_\psi = d^2 \Omega_\psi / d\psi^2$, satisfies the following ODE
\[
\phi^2(1 - \phi^6) \frac{d^5 \Omega_\psi}{d\phi^5} - \phi(2 + 25 \phi^6) \frac{d^4 \Omega_\psi}{d\phi^4} + (2 - 205 \phi^6) \frac{d^3 \Omega_\psi}{d\phi^3}
- 660 \phi^5 \frac{d^2 \Omega_\psi}{d\phi^2} - 781 \phi^4 \frac{d \Omega_\psi}{d\phi} - 243 \phi^3 \Omega_\psi = 0,
\] (A.3)
which also has $\psi = 0$ as a singularity. Again using the method introduced in Section 4.2, Mathematica can still compute the values of $\left(\psi^{-1} \varpi_i\right)'\psi$ at $\psi = 0$ to a very high precision
\[
\left(\psi^{-1} \varpi_0\right)'|_{\psi=0} = -i \times 3.64756261112415977197966067550195005569158882009 \ldots;
\left(\psi^{-1} \varpi_1\right)'|_{\psi=0} = -i \times 1.823781305562079885989830337775097500278457941004 \ldots;
\left(\psi^{-1} \varpi_2\right)'|_{\psi=0} = -i \times 4.55945326390519971497457584443774375069661448602511 \ldots;
\left(\psi^{-1} \varpi_3\right)'|_{\psi=0} = 7.4239915270996435118023941654188079356976001723687 \ldots
- i \times 5.9272892430767596294669485977690668759049883183264 \ldots;
\left(\psi^{-1} \varpi_4\right)'|_{\psi=0} = 14.8479830541992870236047888330837615871395200344737 \ldots
+ i \times 2.5076992951478598432360167144075906288279673138 \ldots.
\]

However, this numerical method does not work for the case $\Omega^{(k)}_\psi$ where $k \geq 3$, as the singularity $\psi = 0$ of its ODE becomes too bad. Hence we are not able to obtain the numerical values of $\left(\psi^{-1} \varpi_i\right)^{(k)}$ at $\psi = 0$ when $k \geq 3$. The previous analysis for the Fermat sextic also works for the cases where $n = 6, 8, 10$.

**Appendix B. The numerical data for the Fermat octic CY sixfold**

When $n = 6$, the sixform $\Omega$, of the Fermat pencil 2.1 satisfies the following Picard-Fuchs equation
\[
(1 - \phi^8) \frac{d^7 \Omega_\psi}{d\phi^7} - 28 \phi^7 \frac{d^6 \Omega_\psi}{d\phi^6} - 266 \phi^6 \frac{d^5 \Omega_\psi}{d\phi^5} - 1050 \phi^5 \frac{d^4 \Omega_\psi}{d\phi^4}
- 1701 \phi^4 \frac{d^3 \Omega_\psi}{d\phi^3} - 966 \phi^3 \frac{d^2 \Omega_\psi}{d\phi^2} - 127 \phi^2 \frac{d \Omega_\psi}{d\phi} - \psi \Omega_\psi = 0.
\] (B.1)
The numerical values of $\psi^{-1} \varpi_i$, at $\psi = 0$ are
\[
\psi^{-1} \varpi_0|_{\psi=0} = -3.8161185324494391627280485350027433391930427001819 \ldots
- i \times 1.580668051763869710593134880695845278143943643408 \ldots;
\psi^{-1} \varpi_1|_{\psi=0} = -i \times 5.39680658421330873211834156985886173369863435898 \ldots
\psi^{-1} \varpi_2|_{\psi=0} = 4.1247670733968291524118339040310075042726387768625 \ldots
- i \times 3.6882721207823626580506480549569723156692018346186 \ldots;
\psi^{-1} \varpi_3|_{\psi=0} = 7.721330184157759717829449534390813156394109936795 \ldots
- i \times 6.5378420238843938904675174363369161662427061596022 \ldots;
\]
\[ \psi^{-1} \omega_4|_{\psi=0} = 36.652393457346875929742257420586211238782395049501 \ldots \\
+ i \times 1.106481636234708797415194416487091694700760550386 \ldots ; \\
\psi^{-1} \omega_5|_{\psi=0} = 48.56601528580657980260958172737059824866887266236 \ldots \\
+ i \times 85.662936954758874887607628716480310364817126748075 \ldots ; \\
\psi^{-1} \omega_6|_{\psi=0} = -79.54076015393166770421214663319403245834356423335 \ldots \\
+ i \times 196.18203141762876382343979311137939091471723541690 \ldots . \]

The first derivative of the sixform \( \Omega_\psi \), \( \Omega^{(1)}_\psi = d\Omega_\psi/d\psi \), satisfies the following ODE

\[
\begin{align*}
\psi(1 - \psi^8) \frac{d^7 \Omega_\psi}{d\psi^7} - (1 + 35\psi^8) \frac{d^6 \Omega_\psi}{d\psi^6} - 434\psi^7 \frac{d^5 \Omega_\psi}{d\psi^5} - 2380\psi^6 \frac{d^4 \Omega_\psi}{d\psi^4} \\
-5901\psi^5 \frac{d^3 \Omega_\psi}{d\psi^3} - 6069\psi^4 \frac{d^2 \Omega_\psi}{d\psi^2} - 2059\psi^3 \frac{d\Omega_\psi}{d\psi} - 128\psi^2 \Omega_\psi = 0.
\end{align*}
\]

(B.2)

The numerical values of \( (\psi^{-1} \omega_i)' \) at \( \psi = 0 \) are

\[
(\psi^{-1} \omega_0)'|_{\psi=0} = -4.942759685521470439464490243339210097674280745468 \ldots \\
- i \times 4.942759685521470439464490243339210097674280745468 \ldots ; \\
(\psi^{-1} \omega_1)'|_{\psi=0} = -i \times 4.942759685521470439464490243339210097674280745468 \ldots \\
(\psi^{-1} \omega_2)'|_{\psi=0} = -6.590346247808627252519320324452280130232374327291 \ldots \\
- i \times 11.5331059365500976920838105677914902279066507259 \ldots ; \\
(\psi^{-1} \omega_3)'|_{\psi=0} = 24.144346199631754134766810249282126273984872457973 \ldots \\
- i \times 48.8581446288924893544990553709173132278601283473 \ldots ; \\
(\psi^{-1} \omega_4)'|_{\psi=0} = 144.52215375129284265347796533167573889053954216206 \ldots \\
+ i \times 3.4599317800965029307625143170337474068371996218 \ldots ; \\
(\psi^{-1} \omega_5)'|_{\psi=0} = 151.86433666610878607797681930356237316793770042894 \ldots \\
+ i \times 258.45674793015711260452829152066785740527148338213 \ldots ; \\
(\psi^{-1} \omega_6)'|_{\psi=0} = -56.50663208075046042412449309954795381401563406213 \ldots \\
+ i \times 613.45477679643925947001990457993602385235785703484 \ldots . 
\]

The second derivative of the sixform \( \Omega_\psi \), \( \Omega^{(2)}_\psi = d^2 \Omega_\psi/d\psi^2 \), satisfies the following ODE

\[
\begin{align*}
\psi^2(1 - \psi^8) \frac{d^7 \Omega_\psi}{d\psi^7} - 2\psi(1 + 21\psi^8) \frac{d^6 \Omega_\psi}{d\psi^6} + (2 - 644\psi^8) \frac{d^5 \Omega_\psi}{d\psi^5} - 4550\psi^7 \frac{d^4 \Omega_\psi}{d\psi^4} \\
-15421\psi^6 \frac{d^3 \Omega_\psi}{d\psi^3} - 23772\psi^5 \frac{d^2 \Omega_\psi}{d\psi^2} - 14197\psi^4 \frac{d\Omega_\psi}{d\psi} - 2187\psi^3 \Omega_\psi = 0.
\end{align*}
\]

(B.3)

The numerical values of \( (\psi^{-1} \omega_i)'' \) at \( \psi = 0 \) are

\[
(\psi^{-1} \omega_0)''|_{\psi=0} = -4.6441270357678197835318681937020736973496995008407 \ldots 
\]
Again we are not able to obtain the numerical values of \((\psi^{-1}w_i)^{(k)}\) at \(\psi = 0\) when \(k \geq 3\).

**Appendix C. The numerical data for the Fermat decic CY eightfold**

When \(n = 8\), the eightform \(\Omega_\psi\) of the Fermat pencil 2.1 satisfies the following Picard-Fuchs equation

\[
(1 - \psi^{10}) \frac{d^9 \Omega_\psi}{d\psi^9} - 45 \psi^9 \frac{d^8 \Omega_\psi}{d\psi^8} - 750 \psi^8 \frac{d^7 \Omega_\psi}{d\psi^7} - 5880 \psi^7 \frac{d^6 \Omega_\psi}{d\psi^6} - 22827 \psi^6 \frac{d^5 \Omega_\psi}{d\psi^5} - 42525 \psi^5 \frac{d^4 \Omega_\psi}{d\psi^4} - 34105 \psi^4 \frac{d^3 \Omega_\psi}{d\psi^3} - 9330 \psi^3 \frac{d^2 \Omega_\psi}{d\psi^2} - 511 \psi^2 \frac{d \Omega_\psi}{d\psi} - \psi \Omega_\psi = 0.
\]

(C.1)

The numerical values of \(\psi^{-1}w_i\) at \(\psi = 0\) are

\[
\begin{align*}
\psi^{-1}w_0|_{\psi=0} &= -4.97857997204969305057747921898461986154403080395360 \cdots \\
&- i \times 1.617638692189617533723143530317410597717480219659793 \cdots ; \\
\psi^{-1}w_1|_{\psi=0} &= -i \times 8.470066155338955310286234546684833703778139605822 \cdots \\
\psi^{-1}w_2|_{\psi=0} &= 7.3985082698848468682153621311358138742741758915494 \cdots \\
&- i \times 6.066145095711065751428823869028974140550823724223 \cdots ; \\
\psi^{-1}w_3|_{\psi=0} &= 15.521444569480047815142982675953666859605588432469 \cdots \\
&- i \times 18.4788783377448928854035659145879861256677111260147 \cdots ; \\
\psi^{-1}w_4|_{\psi=0} &= 115.91853403069468516781873205098538275483924168831 \cdots \\
&- i \times 0.9099921764356659862714323580354346121608262355863 \cdots ; \\
\psi^{-1}w_5|_{\psi=0} &= 171.005356744857492814073395549799982886575099854 \cdots \\
&+ i \times 313.1107005243444300728156503099488804899979649409 \cdots ; \\
\psi^{-1}w_6|_{\psi=0} &= -237.42106810694690857324251245335547964181161368426 \cdots \\
&- i \times 11.2119144751342304076255771697966315994796737217232 \cdots .
\end{align*}
\]
\[
\psi^{-1} \omega_7|_{\psi=0} = -1686.4530516055867838288932723145591297712408139489 \ldots
\]
\[
+ i \times 2487.5611102581987861695160697051963159470628728096 \ldots;
\]
\[
\psi^{-1} \omega_8|_{\psi=0} = -16471.87499563888078145875111564803979149214263967 \ldots
\]
\[
+ i \times 2669.737705830431680695313456131602195111703562215 \ldots;
\]

The first derivative of the eightform \( \Omega_\psi \), \( \Omega^{(1)}_\psi = d\Omega_\psi/d\psi \), satisfies the following ODE

\[
\psi (1 - \psi^{10}) \frac{d^2 \Omega_\psi}{d\psi^2} - (1 + 54 \psi^{10}) \frac{d^6 \Omega_\psi}{d\psi^6} - 1110 \psi^6 \frac{d^4 \Omega_\psi}{d\psi^4} - 11130 \psi^8 \frac{d^6 \Omega_\psi}{d\psi^6}
\]
\[
- 58107 \psi^7 \frac{d^5 \Omega_\psi}{d\psi^5} - 156660 \psi^6 \frac{d^4 \Omega_\psi}{d\psi^4} - 204205 \psi^5 \frac{d^5 \Omega_\psi}{d\psi^5} = 0.
\]

The numerical values of \((\psi^{-1} \omega_i)'\) at \( \psi = 0 \) are

\[
(\psi^{-1} \omega_0)'|_{\psi=0} = -9.4515283097221295111620761073770856827308224012874 \ldots
\]
\[
- i \times 6.8669372716597515173892931197897419902508025689766 \ldots;
\]
\[
(\psi^{-1} \omega_1)'|_{\psi=0} = -i \times 9.937924978991380417804297825896214304821128435529 \ldots
\]
\[
(\psi^{-1} \omega_2)'|_{\psi=0} = -21.764850893375382549160726113885873784005422767293 \ldots
\]
\[
- i \times 25.75101476872406819020984919921153246344050963662 \ldots;
\]
\[
(\psi^{-1} \omega_3)'|_{\psi=0} = 65.8891177237430062893580413788045538161303 \ldots
\]
\[
- i \times 169.28127096143616856337721687187304104334984144693 \ldots;
\]
\[
(\psi^{-1} \omega_4)'|_{\psi=0} = 578.774732156589941682626391512097869643789354217 \ldots
\]
\[
- i \times 3.86265221530861022853147737988172986951607644505 \ldots;
\]
\[
(\psi^{-1} \omega_5)'|_{\psi=0} = 725.90372471074018452000092887153842873212578027030 \ldots
\]
\[
+ i \times 1112.0917803031733410226082215910015228622649834693 \ldots;
\]
\[
(\psi^{-1} \omega_6)'|_{\psi=0} = 734.732425832101156603191002563328821589323505706 \ldots
\]
\[
+ i \times 3343.8387239270019981135415695286231521929827003302 \ldots;
\]
\[
(\psi^{-1} \omega_7)'|_{\psi=0} = -7159.056823312710140677901536796030146715697968528 \ldots
\]
\[
+ i \times 17617.206368197243150988346082275778437584920753486 \ldots;
\]
\[
(\psi^{-1} \omega_8)'|_{\psi=0} = -90626.647351920809172619095575434756523108044279743 \ldots
\]
\[
+ i \times 11333.13727363132785251570150320925114590343898229 \ldots;
\]

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The second derivative of the eightform $\Omega_\psi$, $\Omega^{(2)}_\psi = d^2\Omega_\psi/d\psi^2$, satisfies the following ODE

$$
\psi^2(1 - \psi^{10}) \frac{d^3\Omega_\psi}{d\psi^3} - \psi \left( 2 + 63\psi^{10} \right) \frac{d^5\Omega_\psi}{d\psi^5} + \left( 2 - 1542\psi^{10} \right) \frac{d^7\Omega_\psi}{d\psi^7} - 18900\psi^9 \frac{d^6\Omega_\psi}{d\psi^6} \\
-124887\psi^8 \frac{d^5\Omega_\psi}{d\psi^5} - 447195\psi^7 \frac{d^4\Omega_\psi}{d\psi^4} - 830845\psi^6 \frac{d^3\Omega_\psi}{d\psi^3} \quad (C.3)
$$

The numerical values of $(\psi^{-1}w_i)^{''}$ at $\psi = 0$ are

$$
(\psi^{-1}w_0)^{''} |_{\psi=0} = -16.806561019422849312269832352009342977876907609673 \ldots \\
- i \times 23.132246732429185448909462521434209901405921690322 \ldots.
$$

$$
(\psi^{-1}w_1)^{''} |_{\psi=0} = -i \times 17.6714640312785087772564686649298706358119787922072 \ldots.
$$

$$
(\psi^{-1}w_2)^{''} |_{\psi=0} = -50.185533671994791383045017753800144421329074951460 \ldots \\
- i \times 86.745925246609445343410484455378287130272206338706 \ldots.
$$

$$
(\psi^{-1}w_3)^{''} |_{\psi=0} = 221.95678624559547983862578412450033693316841867100 \ldots \\
- i \times 337.23708719611925373678195407163657311580200977125 \ldots.
$$

$$
(\psi^{-1}w_4)^{''} |_{\psi=0} = 911.65498350419032128827888645530659780751810786407 \ldots \\
- i \times 13.01188887699144681501157266830674306954083095081 \ldots.
$$

$$
(\psi^{-1}w_5)^{''} |_{\psi=0} = 2445.3090802647727023962766701289574672992521571192 \ldots \\
+ i \times 137.0876884056578473166187857839123664517133498014 \ldots.
$$

$$
(\psi^{-1}w_6)^{''} |_{\psi=0} = 12924.689781352231768392200114509752353568594909033 \ldots \\
+ i \times 11264.19236630580269289219839958196122134298556442 \ldots.
$$

$$
(\psi^{-1}w_7)^{''} |_{\psi=0} = -24116.292643564989239122644020586952813081558966530 \ldots \\
+ i \times 82912.02356972969069056143745831002306604299659552 \ldots.
$$

$$
(\psi^{-1}w_8)^{''} |_{\psi=0} = -329667.12654737598673990727297340604105653962245347 \ldots \\
+ i \times 38177.271364933958485078828891496332434294946359 \ldots;
$$

Again we are not able to obtain the numerical values of $(\psi^{-1}w_i)^{(k)}$ at $\psi = 0$ when $k \geq 3.$
APPENDIX D. THE NUMERICAL DATA FOR THE FERMAT DUODECIC CY TENFOLD

When \( n = 10 \), the tenform \( \Omega_\psi \) of the Fermat pencil 2.1 satisfies the following Picard-Fuchs equation

\[
(1 - \psi^{12}) \frac{d^{11}\Omega_\psi}{d\psi^{11}} - 66\psi^{11} \frac{d^{10}\Omega_\psi}{d\psi^{10}} - 1705\psi^{10} \frac{d^9\Omega_\psi}{d\psi^9} - 22275\psi^9 \frac{d^8\Omega_\psi}{d\psi^8} - 159027\psi^8 \frac{d^7\Omega_\psi}{d\psi^7} \\
- 62739\psi^7 \frac{d^6\Omega_\psi}{d\psi^6} - 1323652\psi^6 \frac{d^5\Omega_\psi}{d\psi^5} - 1379400\psi^5 \frac{d^4\Omega_\psi}{d\psi^4} - 611501\psi^4 \frac{d^3\Omega_\psi}{d\psi^3} \\
- 86526\psi^3 \frac{d^2\Omega_\psi}{d\psi^2} - 2047\psi^2 \frac{d\Omega_\psi}{d\psi} - \psi \Omega_\psi = 0.
\] (D.1)

The numerical values of \( \psi^{-1}w_i \) at \( \psi = 0 \) are

\[
\psi^{-1}w_0|_{\psi=0} = -6.12870418485103129685249777725011929437602147015724 \cdots \\
- i \times 1.6421813369800760117009813252928577078246336026536 \cdots ;
\]

\[
\psi^{-1}w_1|_{\psi=0} = -i \times 12.25740836970206259370499554500238585875204294031447 \cdots 
\]

\[
\psi^{-1}w_2|_{\psi=0} = 12.037397788167426218729281780667266943751269548624 \cdots \\
- i \times 9.0319973539041806435539728911071739303548148195 \cdots ;
\]

\[
\psi^{-1}w_3|_{\psi=0} = 27.31201945440609550685735436482159271220735573794 \cdots \\
- i \times 41.962865901850661431406798885595895225150432471364 \cdots ;
\]

\[
\psi^{-1}w_4|_{\psi=0} = 295.12756484443208569907663512586667581674686532870 \cdots \\
- i \times 7.225597882712334451484317831288573914122838785168 \cdots ;
\]

\[
\psi^{-1}w_5|_{\psi=0} = 463.75600668745516042030737797516188949327333240609 \cdots \\
+ i \times 902.76482747701071847012187563965326292891991245097 \cdots ;
\]

\[
\psi^{-1}w_6|_{\psi=0} = -490.69921111341616727780720190148335902338846167065 \cdots \\
+ i \times 2413.6561548381486952814854354510879904491955029020 \cdots ;
\]

\[
\psi^{-1}w_7|_{\psi=0} = -5226.7262548673167712243263674925206960014866170004 \cdots \\
+ i \times 12399.268179993229192355329354337360924991254110172 \cdots ;
\]

\[
\psi^{-1}w_8|_{\psi=0} = -87758.9303047344623994680541395928760375671329147 \cdots \\
+ i \times 19613.60676985268655805354950322264728882778074506 \cdots ;
\]

\[
\psi^{-1}w_9|_{\psi=0} = -267243.0664103435491475517276367782480469039201236 \cdots \\
- i \times 334637.57216828511139424612305925841814677484655499 \cdots ;
\]

\[
\psi^{-1}w_{10}|_{\psi=0} = 408550.8332803312580661833410910600778638248660215 \cdots \\
- i \times 1510914.1272862746388791419893869308111718569019075 \cdots ;
\]
The first derivative of the tenform $\Omega_\psi$, $\Omega_\psi^{(1)} = d\Omega_\psi/d\psi$, satisfies the following ODE

\[
\begin{align*}
\psi(1 - \psi^{12}) & \frac{d^{11}\Omega_\psi}{d\psi^{11}} - (1 + 77\psi^{12}) \frac{d^{10}\Omega_\psi}{d\psi^{10}} - 2365\psi^{11} \frac{d^9\Omega_\psi}{d\psi^9} - 37620\psi^{10} \frac{d^8\Omega_\psi}{d\psi^8} - 337227\psi^9 \frac{d^7\Omega_\psi}{d\psi^7} \\
& - 1740585\psi^8 \frac{d^6\Omega_\psi}{d\psi^6} - 5088028\psi^7 \frac{d^5\Omega_\psi}{d\psi^5} - 7997660\psi^6 \frac{d^4\Omega_\psi}{d\psi^4} - 6129101\psi^5 \frac{d^3\Omega_\psi}{d\psi^3} \\
& - 1921029\psi^4 \frac{d^2\Omega_\psi}{d\psi^2} - 175099\psi^3 \frac{d\Omega_\psi}{d\psi} - 2048\psi^2 \Omega_\psi = 0.
\end{align*}
\]

(D.2)
The second derivative of the tenform $\Omega_\psi$, $\Omega^{(2)}_\psi = d^2\Omega_\psi/d\psi^2$, satisfies the following ODE

$$
\psi^2(1 - \psi^{12})d^{11}\Omega_\psi/d\psi^{11} - 2\psi(1 + 4\psi^{12})d^{10}\Omega_\psi/d\psi^{10} + (2 - 3135\psi^{12})d^9\Omega_\psi/d\psi^9 - 58905\psi^{11}d^8\Omega_\psi/d\psi^8
$$

$$
- 638187\psi^{10}d^7\Omega_\psi/d\psi^7 - 4101174\psi^9d^6\Omega_\psi/d\psi^6 - 15531538\psi^8d^5\Omega_\psi/d\psi^5 - 33437800\psi^7d^4\Omega_\psi/d\psi^4
$$

$$
- 38119741\psi^6d^3\Omega_\psi/d\psi^3 - 20308332\psi^5d^2\Omega_\psi/d\psi^2 - 4017157\psi^4d\Omega_\psi/d\psi - 177147\psi^3\Omega_\psi = 0. \quad (D.3)
$$

The numerical values of $(\psi^{-1}\varpi_i)^n$ at $\psi = 0$ are

$$(\psi^{-1}\varpi_0)^n|_{\psi=0} = -39.455433465934517046933916962012378052577139915962 \cdots$$

$$- i \times 39.455433465934517046933916962012378052577139915962 \cdots$$

$$(\psi^{-1}\varpi_1)^n|_{\psi=0} = -i \times 39.455433465934517046933916962012378052577139915962 \cdots$$

$$(\psi^{-1}\varpi_2)^n|_{\psi=0} = -177.54945059670532671120262632905570123659712962183 \cdots$$

$$- i \times 217.004884062636983475813654329106807928917426953779 \cdots$$

$$(\psi^{-1}\varpi_3)^n|_{\psi=0} = 656.20497693943580860604696213301138327999534626 \cdots$$

$$- i \times 1228.30876221999407804122585251131249314564852412771 \cdots$$

$$(\psi^{-1}\varpi_4)^n|_{\psi=0} = 3555.9681376518289257503766665520241675739404834017 \cdots$$

$$- i \times 173.6039072501114750065902436328544634313394156302 \cdots$$

$$(\psi^{-1}\varpi_5)^n|_{\psi=0} = 11142.3104466241837998456923217759224035990926436254 \cdots$$

$$- i \times 1739.4694674374627638596097348897010699717386162135 \cdots$$

$$(\psi^{-1}\varpi_6)^n|_{\psi=0} = 87332.205496291727701229573587282838816862475017034 \cdots$$

$$+ i \times 57991.06814962687253482415211767158784663892415139 \cdots$$

$$(\psi^{-1}\varpi_7)^n|_{\psi=0} = -125578.5493049191434099884430834486477924660112436 \cdots$$

$$+ i \times 60639.18512793261821499297171690837438454398094743 \cdots$$

$$(\psi^{-1}\varpi_8)^n|_{\psi=0} = -2561879.1036671212504084858709923461980238260507624 \cdots$$

$$+ i \times 471241.110532919480251858416136699223561285729958 \cdots$$

$$(\psi^{-1}\varpi_9)^n|_{\psi=0} = -6420844.512443746442103936142557274055334472129359 \cdots$$

$$- i \times 5796765.9685458185237645260271137091523284624676218 \cdots$$

$$(\psi^{-1}\varpi_{10})^n|_{\psi=0} = -16486768.5665125675483377222302137737111510111756 \cdots$$

$$- i \times 36301576.737872400519539269895777255152616367270300 \cdots$$

Again we are not able to obtain the numerical values of $(\psi^{-1}\varpi_i)^{(k)}$ at $\psi = 0$ when $k \geq 3$. 

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