A metric result for special subsequences of the Halton sequences

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August 4, 2018

Abstract

In this paper we investigate the special subsequence of the Halton sequence indexed by $\lfloor n\beta \rfloor$ with $\beta \in \mathbb{R}$ and prove a metric almost low-discrepancy result.

KEYWORDS: Halton sequences, Subsequences, Discrepancy.

MSC2010: 11K31, 11K38.

1 Introduction

The discrepancy of the first $N$ terms of any sequence $(z_n)_{n \geq 0}$ of points in $[0, 1)^s$ is defined by

$$D_N(z_n) = \sup J \left| \frac{A_N(J)}{N} - \lambda_s(J) \right|,$$

where the supremum is extended over all half-open subintervals $J$ of $[0, 1)^s$, $\lambda_s$ denotes the $s$-dimensional Lebesgue measure, and the counting function $A_N(J)$ is given by

$$A_N(J) = \# \{0 \leq n \leq N - 1 : z_n \in J \}.$$

For the sake of simplicity, we will sometimes write $D_N$ instead of $D_N(z_n)$. If the supremum is extended over all half-open subintervals $J$ of $[0, 1)^s$ with the lower left point in the origin then we arrive at the notion of the star-discrepancy $D^*_N$. It is not so hard to see that $D^*_N \leq D_N \leq 2^s D^*_N$. A sequence $(z_n)_{n \geq 0}$ of points in $[0, 1)^s$ is called uniformly distributed if $\lim_{N \to \infty} D^*_N = 0$.

It is frequently conjectured in the theory of irregularities of distribution, that for every sequence $(z_n)_{n \geq 0}$ in $[0, 1)^s$ we have

$$D_N \geq c_s \log^s N$$

*supported by the Austrian Science Fund (FWF): Project F5505-N26, which is a part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”
for a constant $c_ε > 0$ and for infinitely many $N$. Therefore sequences whose discrepancy satisfies $D_N ≤ C \log^s N/N$ for all $N$ with a constant $C > 0$ that is independent of $N$ (or $D_N = O(\log^s N/N)$), are called low-discrepancy sequences. Well-known examples of low-discrepancy sequences are, for example, the $s$-dimensional Halton sequences and one-dimensional Kronecker sequences $(\{nα\})_{n \geq 0}$ with $α$ irrational and having bounded continued fraction coefficients.

For an integer $b ≥ 2$, let $Z_b = \{0, 1, \ldots, b−1\}$ denote the least residue system modulo $b$. Let $n = \sum_{j=1}^{∞} a_j(n) b^{j−1}$ with all $a_j(n) ∈ Z_b$ and $a_j(n) = 0$ for all sufficiently large $j$ be the unique digit expansion of the integer $n ≥ 0$ in base $b$. The radical-inverse function $φ_b$ in base $b$ is defined by

$$φ_b(n) = \sum_{j=1}^{∞} a_j(n) b^{−j}.$$  

The Halton sequence $(y_n)_{n ≥ 0}$ (in the bases $b_1, \ldots, b_s$) is given by

$$y_n = (φ_{b_1}(n), \ldots, φ_{b_s}(n)) ∈ [0, 1)^s, \quad n = 0, 1, 2, \ldots.$$  

It is well known (see [7, Theorem 3.6]) that the discrepancy of the Halton sequence in pairwise coprime bases $b_1, \ldots, b_s ≥ 2$ satisfies a low-discrepancy bound, i.e.,

$$D_N(y_n) = O_{b_1, \ldots, b_s}( (\log N)^s/N ) \quad \text{for all } N ≥ 2. \quad (1)$$  

The one-dimensional Kronecker sequences $((\{nα\}))_{n ≥ 0}$ related to the real number $α$ is uniformly distributed if $α$ is irrational. This sequence is a low-discrepancy sequence if the continued fraction coefficients of $α$ are bounded. Furthermore, the discrepancy $D_N$ satisfies $N D_N ≤ C(α, ε) \log^{1+ε} N$ for all $N$ and all $ε > 0$ for almost all $α ∈ \mathbb{R}$ in the sense of Lebesgue measure. Here the positive constant $C(α, ε)$ may depend on $α$ and $ε > 0$ but is independent of $N$.

Subsequences of the one-dimensional Kronecker sequence and Halton sequences are frequently studied objects (see for instance [1], [2], [3], [4]). Here we mention a result of Hofer and Ramare [5] who studied the sequence $(\{nα\})_{n ≥ 0}$. It is known to be u.d. mod 1 when $α$ is a nonzero rational real if and only if $β$ is irrational. When $α = 0$ the sequence is for no real $β$ u.d. mod 1. And when $α$ is irrational the uniform distribution modulo one of $(\{nα\})_{n ≥ 0}$ is equivalent to the condition, $1, α, αβ$ being linearly independent over the rationals. (For a proof see [5, Chapter 5, Theorem 1.8].) In [3] many results on the discrepancy of this sequence were proved. Here we mention just one result, i.e., for almost all pairs of real numbers $(α, β)$ in the sense of Lebesgue measure we have for every $ε > 0$ that $D_N([nα]/β) ≪_{α, β, ε} N^{−1+ε}$.

In this paper we investigate the subsequence $(x_n)_{n ≥ 0}$ of the Halton sequence of this particular form

$$x_n := (φ_{b_1}([nβ]), \ldots, φ_{b_s}([nβ])) ∈ [0, 1)^s, \quad n = 0, 1, 2, \ldots \quad (2)$$

with nonzero $β ∈ \mathbb{R}$ and integers $b_1, \ldots, b_s ≥ 2$ pairwise coprime. See Figure 1 for examples.
In [2, Example 5.3] the uniform distribution of (2) was studied. A sufficient criterion for the one-dimensional sequence is, for example, the linear independence of 1, $\beta$ over $\mathbb{Q}$. This criterion is not a necessary one, as the choice $\beta = 1/d$ with $d \in \mathbb{N}$ also yields a uniformly distributed sequence (2). If $\beta = d \in \mathbb{N}$ such that $(d, b_i) = 1$ for $i = 1, \ldots, s$ then the sequence (2) is even a low-discrepancy sequence (see [4]).

In this paper we prove that the sequence (2) is for almost all $\beta \in \mathbb{R}$ an almost low-discrepancy sequence by the following theorem.

**Theorem 1.** For almost all $\beta \in \mathbb{R}$ in the sense of Lebesgue measure the dis-
crepancy of \((x_n)_{n \geq 0}\) satisfies for all \(\epsilon > 0\)
\[
ND_N = O_{\beta, \epsilon, b_1, \ldots, b_s, s}(\log^{s+1+\epsilon} N).
\]

A proof of this theorem will be outsourced in the next section.

## 2 Proof of Theorem \[1\]

We start with the case \(\beta \in (0, 1)\). We write \(\alpha := 1/\beta\). Note that \(\alpha > 1\). Then Theorem \[1\] immediately follows from the following three auxiliary results.

**Proposition 1.** The star-discrepancy of the first \(N\) points of the sequence (2) satisfies
\[
ND_N^*(x_n) \leq b_1 \cdots b_s \sum_{j_1=0}^{f_1} \cdots \sum_{j_s=0}^{f_s} N\Delta_N^{(j_1, \ldots, j_s)}(\{n/(\alpha b_1^{j_1} \cdots b_s^{j_s})\}) + s,
\]
where
\[
f_i := \left\lfloor \frac{1}{\log b_i} \log N \right\rfloor \text{ for } 1 \leq i \leq s \tag{3}
\]
and
\[
N\Delta_N^{(j_1, \ldots, j_s)}(\{n/(\alpha b_1^{j_1} \cdots b_s^{j_s})\}) := \sup_{0 \leq R < b_1^{j_1} \cdots b_s^{j_s}} |A_N(\left[\frac{R}{b_1^{j_1} \cdots b_s^{j_s}}, \frac{R+1}{b_1^{j_1} \cdots b_s^{j_s}}\right]) - \frac{N}{b_1^{j_1} \cdots b_s^{j_s}}|.
\]

**Proof.** The first aim in the proof is to compute or estimate the function \(A_N(J) - N\lambda_s(J)\) relative to the points \(x_n\) in (2), where \(J \subseteq [0, 1)^s\) is an interval of the form
\[
J = \prod_{i=1}^s [0, v_i b_i^{-f_i})
\]
with \(v_1, \ldots, v_s \in \mathbb{Z}, 1 \leq v_i < b_i^{f_i}\) for \(1 \leq i \leq s\). We extend \(v_i\) in base \(b_i\), i.e.,
\[
v_i = u_{f_i}^{(i)} + u_{f_i-1}^{(i)} b_i + \cdots + u_{1}^{(i)} b_i^{f_i-1}.
\]
We can write \(J\) as a union of \(\prod_{i=1}^s (\sum_{j=1}^{f_i} u_j^{(i)} \leq (b_i - 1) \cdots (b_i - 1) b_1 \cdots b_s)\) elementary intervals of the form
\[
\prod_{i=1}^s \left( \bigcup_{j=1}^{f_i} \bigcup_{k=1}^{u_j^{(i)}} \sum_{l=1}^{i-1} \frac{u_l^{(i)} - 1}{b_l^{f_i}} + \frac{k}{b_l^{f_i}} + \sum_{l=1}^{i-1} \frac{u_l^{(i)}}{b_l^{f_i}} \right)
\]

The crucial step is to estimate \(A(I_e, N) - N\lambda_s(I_e)\) for an elementary interval \(I_e\) by exploiting special properties of the Halton sequence. Using the definition of the sequence we obtain for any integer \(n \geq 0\) that the following three assertions are equivalent.
\[(\phi_n([n/\alpha]), \ldots, \phi_n([n/\alpha])) \in \prod_{i=1}^{s} \left( \sum_{l=1}^{j_i-1} \frac{u_{l}^{(i)}}{b_{l}^{(i)}} + \frac{k_i - 1}{b_{j_i}^{(i)}} \sum_{l=1}^{j_i-1} \frac{u_{l}^{(i)}}{b_{l}^{(i)}} + k_i \right), \]

where \(1 \leq j_i \leq f_i\) and \(1 \leq k_i \leq u_{j_i}^{(i)}\) for \(i = 1, \ldots, s\).

\[|n/\alpha| \equiv a_1 \pmod{b_1^{(i)}}, \ldots, |n/\alpha| \equiv a_s \pmod{b_s^{(i)}} \quad (5)\]

with \(a_i = (k_i - 1)b_i^{j_i-1} + \sum_{l=1}^{j_i-1} u_{l}^{(i)}b_i^{l-1}\).

\[|n/\alpha| \equiv R \pmod{b_1^{(i)} \cdots b_s^{(i)}}\]

where the \(R\) is uniquely determined by the \(a_1, \ldots, a_s\).

By observing the fact that

\[|n/\alpha| \equiv |n/\alpha - b_1^{(i)} \cdots b_s^{(i)}| \equiv R \pmod{b_1^{(i)} \cdots b_s^{(i)}}\]

we easily obtain the equivalence of the above assertion to

\[\left\{ \frac{n}{\alpha b_1^{(i)} \cdots b_s^{(i)}} \right\} \in \left[ \frac{R}{b_1^{(i)} \cdots b_s^{(i)}}, \frac{R+1}{b_1^{(i)} \cdots b_s^{(i)}} \right].\]

Hence,

\[|A(J, N) - N\lambda_s(J)| \leq (b_1 - 1) \cdots (b_s - 1) \sum_{j_1=1}^{f_1} \cdots \sum_{j_s=1}^{f_s} N\Delta_N^{(j_1, \ldots, j_s)}(\{n/(\alpha b_1^{(i)} \cdots b_s^{(i)})\})\]

An arbitrary interval \(I \subseteq [0, 1)^s\) of the form

\[I = \prod_{i=1}^{s} [0, w_i] \quad (6)\]

with \(0 < w_i \leq 1\) for \(1 \leq i \leq s\) can be approximated from below by an interval \(J\) of the form \([\bar{b}, \underline{b}]\), by taking the nearest fraction to the left of \(w_i\) of the form \(v_i b_i^{f_i}\) with \(v_i \in \mathbb{Z}\). We easily get

\[|A(I, N) - N\lambda_s(I)| \leq |A(J, N) - N\lambda_s(J)| + \max(A(I \setminus J, N), N\lambda_s(I \setminus J)).\]

The definition of the \(f_i\) in \(\square\) yields \(f_i \geq \log_{b_i} N\). Hence \(N\lambda_s(I \setminus J) \leq N \sum_{i=1}^{s} b_i^{f_i} \leq s\). It remains to estimate \(A(I \setminus J, N)\): We trivially have \(A(I \setminus J, N) \leq A(S, N) \leq |A(S, N) - N\lambda_s(S)| + N\lambda_s(S)\) with \(S\) a union of elementary intervals \(I_e\) of the form

\[S = \bigcup_{k=1}^{s} \left[ \prod_{i=1}^{k-1} [0, 1) \times \left[ \sum_{l=1}^{j_i-1} \frac{u_{l}^{(i)}}{b_{l}^{(i)}} + \frac{u_{j_i}^{(i)}}{b_{j_i}^{(i)}} \sum_{l=1}^{j_i-1} \frac{u_{l}^{(i)}}{b_{l}^{(i)}} + u_{j_i}^{(i)} + 1 \right] \times \prod_{i=k+1}^{s} [0, 1) \right].\]
Now $N\lambda_s(I_e) \leq 1$ by the definition of $f_k$ and

$$|A(I_e, N) - N\lambda_s(I_e)| \leq N\Delta_N^{(0, \ldots, 0, f_k, 0, \ldots, 0)}(\{n/(\alpha b_k^s)\})$$

and the result follows.

The discrepancy of a one-dimensional Kronecker sequence $(\{n\beta\})_{n \geq 0}$ can be related to the continued fraction expansion of $\beta$

$$\beta = [\beta; a_1(\beta), a_2(\beta), a_3\beta, \ldots] = [\beta] + \frac{1}{a_1(\beta) + \frac{1}{a_2(\beta) + \frac{1}{a_3(\beta) + \cdots}}}$$

This expansion is obtained by setting $\{\beta\} = x_1$ and defining inductively

$$a_k(\beta) = \lfloor 1/x_k \rfloor \text{ and } x_{k+1} = \{1/x_k\}.$$

The convergents $p_k/q_k$ to $\beta$ are defined by $p_k/q_k = [\beta; a_1(\beta), a_2(\beta), \ldots, a_k(\beta)]$.

The denominators $q_k$ obey the following recursion

$$q_k = a_k(\beta)q_{k-1} + q_{k-2} \text{ for } k \geq 1 \text{ and with } q_0 = 1, q_{-1} = 0.$$

**Lemma 1.** We have

$$N\Delta_N^{(j_1, \ldots, j_s)}(\{n/(\alpha b_1^{i_1} \cdots b_s^{i_s})\}) \leq \alpha + 1 + 2 \left( \sum_{k=1}^{[\log_3(N)]} a_k((\alpha b_1^{i_1} \cdots b_s^{i_s})) \right).$$

**Proof.** We follow the idea of [5] page 125, proof of Theorem 3.4. Now we use the fact that $\beta \in (0, 1)$ and hence $\alpha > 1$. Then $\alpha b_1^{i_1} \cdots b_s^{i_s} > 1$, and the first coefficient $a_1(1/(\alpha b_1^{i_1} \cdots b_s^{i_s})) = [\alpha b_1^{i_1} \cdots b_s^{i_s}]$. We compute the Ostrowski expansion using the denominators $1 = q_0 = q_1 < q_2 < q_3 < \cdots$ of the convergents to $1/(\alpha b_1^{i_1} \cdots b_s^{i_s})$:

$$N = N_0 + N_1 q_1 + N_2 q_2 + \cdots + N_r q_r$$

with $q_r \leq N < q_{r+1}$ and $N_i \leq a_{i+1}(1/\alpha b_1^{i_1} \cdots b_s^{i_s})$.

We split

$$N\Delta_N^{(j_1, \ldots, j_s)}(\{\frac{n}{\alpha b_1^{i_1} \cdots b_s^{i_s}}\}) \leq N_0 \Delta_N^{(j_1, \ldots, j_s)}(\{\frac{n}{\alpha b_1^{i_1} \cdots b_s^{i_s}}\}) + (N-N_0) \Delta_N^{(j_1, \ldots, j_s)}(\{\frac{n+N_0}{\alpha b_1^{i_1} \cdots b_s^{i_s}}\})$$

and proceed for the second term as in [5] page 125, proof of Theorem 3.4] and obtain

$$(N - N_0) \Delta_N^{(j_1, \ldots, j_s)}(\{\frac{n+N_0}{\alpha b_1^{i_1} \cdots b_s^{i_s}}\}) \leq 2 \sum_{k=2}^{[\log_3(N)]+1} a_k(1/(\alpha b_1^{i_1} \cdots b_s^{i_s}))$$

and

$$(N - N_0) \Delta_N^{(j_1, \ldots, j_s)}(\{\frac{n+N_0}{\alpha b_1^{i_1} \cdots b_s^{i_s}}\}) \leq 2 \sum_{k=1}^{[\log_3(N)]} a_k(1/(\alpha b_1^{i_1} \cdots b_s^{i_s})).$$
Finally, it is not so hard to see that $N_0 \Delta_{N_0}^{(j_1, \ldots, j_s)} \{n/(ab_1^{j_1} \cdots b_s^{j_s})\} \leq \alpha + 1$ with $N_0 \leq [ab_1^{j_1} \cdots b_s^{j_s}]$. Note the fact that $[R/b_1^{j_1} \cdots b_s^{j_s})$, $(R + 1)/b_1^{j_1} \cdots b_s^{j_s})$ either remains empty or contains at most $\lceil \alpha \rceil + 1$ points, and $N_0 1/b_1^{j_1} \cdots b_s^{j_s} \leq \lfloor \alpha \rfloor + 1$.

\[\square\]

**Lemma 2.** Let $\alpha \in \mathbb{R}$. For any $L \in \mathbb{N}$ let

$$S_L(\alpha) := \sum_{j_1=0}^{L} \cdots \sum_{j_s=0}^{L} \sum_{k=1}^{L} a_k (\alpha b_1^{j_1} \cdots b_s^{j_s}).$$

Then for almost all $\alpha \in \mathbb{R}$ we have for every $\epsilon > 0$

$$S_L(\alpha) = O_{\alpha, \epsilon, b_1, \ldots, b_s}(L^{s+1+\epsilon}).$$

**Proof.** See e.g. [6, Lemma 2].

Finally we consider $\beta > 1$ and set $\alpha := \beta/(\beta + 1)$. Then $1/2 < \alpha < 1$, $\lfloor n \alpha \rfloor = n - \lceil n 1/\beta + 1 \rceil$ and hence

$$\{ \lfloor n \alpha \rfloor ; n \in \mathbb{N} \} = \{ \lfloor n \beta \rfloor ; n \in \mathbb{N} \} \cup \mathbb{N}_0.$$

Let

$$B := \left\{ \alpha \in (1/2, 1) : D_N(\phi_{b_1}(\lfloor n \alpha \rfloor), \ldots, \phi_{b_s}(\lfloor n \alpha \rfloor)) = O_{\alpha, c, b_1, \ldots, b_s} \left( \frac{(\log N)^{s+1+\epsilon}}{N} \right) \text{ for all } \epsilon > 0 \right\}.$$

From the above case we know $\lambda(B) = 1/2$. Let $A := \{ \beta \in (1, \infty) : \beta/(\beta + 1) \in B \}$. Then $\lambda((1, \infty) \setminus A) = 0$.

Now let $\beta \in A$. Then $\alpha \in B$. Since

$$D_N(\phi_{b_1}(\lfloor n \beta \rfloor), \ldots, \phi_{b_s}(\lfloor n \beta \rfloor)) = O_{\beta, c, b_1, \ldots, b_s} \left( \frac{(\log N)^{s+1+\epsilon}}{N} \right) \text{ for all } \epsilon > 0$$

we have

$$D_N(\phi_{b_1}(\lfloor n \beta \rfloor), \ldots, \phi_{b_s}(\lfloor n \beta \rfloor)) = O_{\beta, \epsilon, b_1, \ldots, b_s} \left( \frac{(\log N)^{s+1+\epsilon}}{N} \right) \text{ for all } \epsilon > 0$$

and the proof is complete.

**Acknowledgments**

The author would like to thank Gerhard Larcher for valuable comments and fruitful discussions.
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