The Cobordism Hypothesis in Dimension 1

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Abstract

In [Lur1] Lurie published an expository article outlining a proof for a higher version of the cobordism hypothesis conjectured by Baez and Dolan in [BaDo]. In this note we give a proof for the 1-dimensional case of this conjecture. The proof follows most of the outline given in [Lur1], but differs in a few crucial details. In particular, the proof makes use of the theory of quasi-unital ∞-categories as developed by the author in [Har].

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1 Introduction

Let \( \mathcal{B}_1^{or} \) denote the 1-dimensional oriented cobordism ∞-category, i.e. the symmetric monoidal ∞-category whose objects are oriented 0-dimensional closed manifolds and whose morphisms are oriented 1-dimensional cobordisms between them.

Let \( \mathcal{D} \) be a symmetric monoidal ∞-category with duals. The 1-dimensional cobordism hypothesis concerns the ∞-category

\[
\text{Fun}^\otimes(\mathcal{B}_1^{or}, \mathcal{D})
\]

of symmetric monoidal functors \( \varphi : \mathcal{B}_1^{or} \rightarrow \mathcal{D} \). If \( X_+ \in \mathcal{B}_1^{or} \) is the object corresponding to a point with positive orientation then the evaluation map \( Z \mapsto Z(X_+) \) induces a functor

\[
\text{Fun}^\otimes(\mathcal{B}_1^{or}, \mathcal{D}) \rightarrow \mathcal{D}
\]

It is not hard to show that since \( \mathcal{B}_1^{or} \) has duals the ∞-category \( \text{Fun}^\otimes(\mathcal{B}_1^{or}, \mathcal{D}) \) is in fact an ∞-groupoid, i.e. every natural transformation between two functors
\( F, G : B_1^{\text{or}} \to D \) is a natural equivalence. This means that the evaluation map \( Z \mapsto Z(X_+) \) actually factors through a map

\[
\text{Fun}^\otimes(B_1^{\text{or}}, D) \to \tilde{D}
\]

where \( \tilde{D} \) is the maximal \( \infty \)-groupoid of \( D \). The cobordism hypothesis then states

**Theorem 1.1.** The evaluation map

\[
\text{Fun}^\otimes(B_1^{\text{or}}, D) \to \tilde{D}
\]

is an equivalence of \( \infty \)-categories.

**Remark 1.2.** From the consideration above we see that we could have written the cobordism hypothesis as an equivalence

\[
\text{Fun}^\otimes(B_1^{\text{or}}, D) \cong \tilde{D}
\]

where \( \text{Fun}^\otimes(B_1^{\text{or}}, D) \) is the maximal \( \infty \)-groupoid of \( \text{Fun}^\otimes(B_1^{\text{or}}, D) \) (which in this case happens to coincide with \( \text{Fun}^\otimes(B_1^{\text{or}}, D) \)). This \( \infty \)-groupoid is the fundamental groupoid of the space of maps from \( B_1^{\text{or}} \) to \( D \) in the \( \infty \)-category \( \text{Cat}^\otimes \) of symmetric monoidal \( \infty \)-categories.

In his paper [Lur1] Lurie gives an elaborate sketch of proof for a higher dimensional generalization of the 1-dimensional cobordism hypothesis. For this one needs to generalize the notion of \( \infty \)-categories to \( (\infty, n) \)-categories. The strategy of proof described in [Lur1] is inductive in nature. In particular in order to understand the \( n = 1 \) case, one should start by considering the \( n = 0 \) case.

Let \( B_0^{\text{un}} \) be the 0-dimensional unoriented cobordism category, i.e. the objects of \( B_0^{\text{un}} \) are 0-dimensional closed manifolds (or equivalently, finite sets) and the morphisms are diffeomorphisms (or equivalently, isomorphisms of finite sets). Note that \( B_0^{\text{un}} \) is a (discrete) \( \infty \)-groupoid.

Let \( X \in B_0^{\text{un}} \) be the object corresponding to one point. Then the 0-dimensional cobordism hypothesis states that \( B_0^{\text{un}} \) is in fact the free \( \infty \)-groupoid (or \( (\infty, 0) \)-category) on one object, i.e. if \( \mathcal{G} \) is any other \( \infty \)-groupoid then the evaluation map \( Z \mapsto Z(X) \) induces an equivalence of \( \infty \)-groupoids

\[
\text{Fun}^\otimes(B_0^{\text{un}}, \mathcal{G}) \cong \mathcal{G}
\]

**Remark 1.3.** At this point one can wonder what is the justification for considering non-oriented manifolds in the \( n = 0 \) case oriented ones in the \( n = 1 \) case. As is explained in [Lur1] the desired notion when working in the \( n \)-dimensional cobordism \( (\infty, n) \)-category is that of \( n \)-framed manifolds. One then observes that 0-framed 0-manifolds are unoriented manifolds, while taking 1-framed 1-manifolds (and 1-framed 0-manifolds) is equivalent to taking the respective manifolds with orientation.
Now the 0-dimensional cobordism hypothesis is not hard to verify. In fact, it holds in a slightly more general context - we do not have to assume that $\mathcal{G}$ is an $\infty$-groupoid. In fact, if $\mathcal{G}$ is any symmetric monoidal $\infty$-category then the evaluation map induces an equivalence of $\infty$-categories

$$\text{Fun}^\otimes(\mathcal{B}_0^\text{un}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

and hence also an equivalence of $\infty$-groupoids

$$\tilde{\text{Fun}}^\otimes(\mathcal{B}_0^\text{un}, \mathcal{G}) \xrightarrow{\simeq} \tilde{\mathcal{G}}$$

Now consider the under-category $\text{Cat}^\otimes_{\mathcal{B}_0^\text{un}}/\mathcal{D}$ equipped with a functor $\mathcal{B}_0^\text{un} \rightarrow \mathcal{D}$. Since $\mathcal{B}_0^\text{un}$ is free on one generator this category can be identified with the $\infty$-category of pointed symmetric monoidal $\infty$-categories, i.e. symmetric monoidal $\infty$-categories with a chosen object. We will often not distinguish between these two notions.

Now the point of positive orientation $X_+ \in \mathcal{B}_1^\text{or}$ determines a functor $\mathcal{B}_0^\text{un} \rightarrow \mathcal{B}_1^\text{or}$, i.e. an object in $\text{Cat}^\otimes_{\mathcal{B}_0^\text{un}}/\mathcal{D}$, which we shall denote by $\mathcal{B}_1^+$. The 1-dimensional cobordism hypothesis is then equivalent to the following statement:

**Theorem 1.4. [Cobordism Hypothesis 0-to-1]** Let $\mathcal{D} \in \text{Cat}^\otimes_{\mathcal{B}_0^\text{un}}$ be a pointed symmetric monoidal $\infty$-category with duals. Then the $\infty$-groupoid

$$\tilde{\text{Fun}}^\otimes_{\mathcal{B}_0^\text{un}}/(\mathcal{B}_1^+, \mathcal{D})$$

is contractible.

Theorem 1.4 can be considered as the inductive step from the 0-dimensional cobordism hypothesis to the 1-dimensional one. Now the strategy outlined in [Lur1] proceeds to bridge the gap between $\mathcal{B}_0^\text{un}$ to $\mathcal{B}_1^\text{or}$ by considering an intermediate $\infty$-category

$$\mathcal{B}_0^\text{un} \hookrightarrow \mathcal{B}_1^\text{cv} \hookrightarrow \mathcal{B}_1^\text{or}$$

This intermediate $\infty$-category is defined in [Lur1] in terms of framed functions and index restriction. However in the 1-dimensional case one can describe it without going into the theory of framed functors. In particular we will use the following definition:

**Definition 1.5.** Let $\iota : \mathcal{B}_1^\text{cv} \hookrightarrow \mathcal{B}_1^\text{or}$ be the subcategory containing all objects and only the cobordisms $M$ in which every connected component $M_0 \subseteq M$ is either an identity segment or an evaluation segment.

Let us now describe how to bridge the gap between $\mathcal{B}_0^\text{un}$ and $\mathcal{B}_1^\text{cv}$. Let $\mathcal{D}$ be an $\infty$-category with duals and let

$$\varphi : \mathcal{B}_1^\text{cv} \rightarrow \mathcal{D}$$

be a symmetric monoidal functor. We will say that $\varphi$ is non-degenerate if for each $X \in \mathcal{B}_1^\text{cv}$ the map

$$\varphi(\text{ev}_X) : \varphi(X) \otimes \varphi(\bar{X}) \simeq \varphi(X \otimes \bar{X}) \rightarrow \varphi(1) \simeq 1$$
is **non-degenerate**, i.e. identifies $\varphi(\check{X})$ with a dual of $\varphi(X)$. We will denote by

$$\text{Cat}^\text{nd}_{B^1_+} \subseteq \text{Cat}^\otimes_{B^1_+}$$

the full subcategory spanned by objects $\varphi : B^1_+ \to D$ such that $D$ has duals and $\varphi$ is non-degenerate.

Let $X_+ \in B^1_+$ be the point with positive orientation. Then $X_+$ determines a functor

$$B^\text{un}_0 \to B^1_+$$

The restriction map $\varphi \mapsto \varphi|_{B^\text{un}_0}$ then induces a functor

$$\text{Cat}^\text{nd}_{B^1_+} \to \text{Cat}^\otimes_{B^\text{un}_0}$$

Now the gap between $B^1_+$ and $B^\text{un}_0$ can be climbed using the following lemma (see [Lur1]):

**Lemma 1.6.** The functor

$$\text{Cat}^\text{nd}_{B^1_+} \to \text{Cat}^\otimes_{B^\text{un}_0}$$

is fully faithful.

**Proof.** First note that if $F : D \to D'$ is a symmetric monoidal functor where $D, D'$ have duals and $\varphi : B^1_+ \to D$ is non-degenerate then $f \circ \varphi$ will be non-degenerate as well. Hence it will be enough to show that if $D$ has duals then the restriction map induces an equivalence between the $\infty$-groupoid of non-degenerate symmetric monoidal functors

$$B^1_+ \to D$$

and the $\infty$-groupoid of symmetric monoidal functors

$$B^\text{un}_0 \to D$$

Now specifying a non-degenerate functor

$$B^1_+ \to D$$

is equivalent to specifying a pair of objects $D_+, D_- \in D$ (the images of $X_+, X_-$ respectively) and a non-degenerate morphism

$$e : D_+ \otimes D_- \to 1$$

which is the image of $\text{ev}_{X_+}$. Since $D$ has duals the $\infty$-groupoid of triples $(D_+, D_-, e)$ in which $e$ is non-degenerate is equivalent to the $\infty$-groupoid of triples $(D_+, D_-, f)$ where $f : D_+ \to D_-$ is an equivalence. Hence the forgetful map $(D_+, D_-, e) \mapsto D_+$ is an equivalence. \(\square\)
Now consider the natural inclusion \( \iota : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{B}_{1}^{\text{or}} \) as an object in \( \text{Cat}_{\mathcal{B}_{1}^{\text{ev}}}^{\text{nd}} \). Then by Lemma 1.6 we see that the 1-dimensional cobordism hypothesis will be established once we make the following last step:

**Theorem 1.7 (Cobordism Hypothesis - Last Step).** Let \( \mathcal{D} \) be a symmetric monoidal \( \infty \)-category with duals and let \( \varphi : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{D} \) be a non-degenerate functor. Then the \( \infty \)-groupoid

\[
\tilde{\text{Fun}}_{\mathcal{B}_{1}^{\text{ev}}/(\mathcal{B}_{1}^{\text{or}}, \mathcal{D})}
\]

is contractible.

Note that since \( \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{B}_{1}^{\text{or}} \) is essentially surjective all the functors in

\[
\tilde{\text{Fun}}_{\mathcal{B}_{1}^{\text{ev}}/(\mathcal{B}_{1}^{\text{or}}, \mathcal{D})}
\]

will have the same essential image of \( \varphi \). Hence it will be enough to prove for the claim for the case where \( \varphi : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{D} \) is essentially surjective. We will denote by

\[
\text{Cat}_{\mathcal{B}_{1}^{\text{ev}}}^{\text{sur}} \subseteq \text{Cat}_{\mathcal{B}_{1}^{\text{ev}}}^{\text{nd}}
\]

the full subcategory spanned by essentially surjective functors \( \varphi : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{D} \). Hence we can phrase Theorem 1.7 as follows:

**Theorem 1.8 (Cobordism Hypothesis - Last Step 2).** Let \( \mathcal{D} \) be a symmetric monoidal \( \infty \)-category with duals and let \( \varphi : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{D} \) be an essentially surjective non-degenerate functor. Then the space of maps

\[
\text{Map}_{\text{Cat}_{\mathcal{B}_{1}^{\text{ev}}}^{\text{sur}}/(\iota, \varphi)}
\]

is contractible.

The purpose of this paper is to provide a formal proof for this last step. This paper is constructed as follows. In §2 we prove a variant of Theorem 1.8 which we call the quasi-unital cobordism hypothesis (Theorem 2.6). Then in §3 we explain how to deduce Theorem 1.8 from Theorem 2.6. Section §2 relies on the notion of quasi-unital \( \infty \)-categories which is developed rigorously in [Har] (however §2 is completely independent of [Har]).

## 2 The Quasi-Unital Cobordism Hypothesis

Let \( \varphi : \mathcal{B}_{1}^{\text{ev}} \rightarrow \mathcal{D} \) be a non-degenerate functor and let \( \text{Grp}_{\infty} \) denote the \( \infty \)-category of \( \infty \)-groupoids. We can define a lax symmetric functor \( M_{\varphi} : \mathcal{B}_{1}^{\text{ev}} \rightarrow \text{Grp}_{\infty} \) by setting

\[
M_{\varphi}(X) = \text{Map}_{\mathcal{D}}(1, \varphi(X))
\]

We will refer to \( M_{\varphi} \) as the fiber functor of \( \varphi \). Now if \( \mathcal{D} \) has duals and \( \varphi \) is non-degenerate, then one can expect this to be reflected in \( M_{\varphi} \) somehow. More precisely, we have the following notion:
**Definition 2.1.** Let $M : \mathcal{B}_1^{ev} \to \text{Grp}_\infty$ be a lax symmetric monoidal functor. An object $Z \in M(X \otimes \hat{X})$ is called **non-degenerate** if for each object $Y \in \mathcal{B}_1^{ev}$ the natural map

$$M(Y \otimes \hat{X}) \xrightarrow{Id \times \hat{F}} M(Y \otimes \hat{X}) \times M(X \otimes \hat{X}) \to M(Y \otimes \hat{X} \otimes X \otimes \hat{X}) \xrightarrow{M(1 \otimes \overline{\text{ev}} \otimes 1)} M(Y \otimes \hat{X})$$

is an equivalence of $\infty$-groupoids.

**Remark 2.2.** If a non-degenerate element $Z \in M(X \otimes \hat{X})$ exists then it is unique up to a (non-canonical) equivalence.

**Example 1.** Let $M : \mathcal{B}_1^{ev} \to \text{Grp}_\infty$ be a lax symmetric monoidal functor. The lax symmetric structure of $M$ includes a structure map $1_{\text{Grp}_\infty} \to M(1)$ which can be described by choosing an object $Z_1 \in M(1)$. The axioms of lax monoidality then ensure that $Z_1$ is non-degenerate.

**Definition 2.3.** A lax symmetric monoidal functor $M : \mathcal{B}_1^{ev} \to \text{Grp}_\infty$ will be called **non-degenerate** if for each object $X \in \mathcal{B}_1^{ev}$ there exists a non-degenerate object $Z \in M(X \otimes \hat{X})$.

**Definition 2.4.** Let $M_1, M_2 : \mathcal{B}_1^{ev} \to \text{Grp}_\infty$ be two non-degenerate lax symmetric monoidal functors. A lax symmetric natural transformation $T : M_1 \to M_2$ will be called **non-degenerate** if for each object $X \in \text{Bord}^{ev}$ and each non-degenerate object $Z \in M(X \otimes \hat{X})$ the objects $T(Z) \in M_2(X \otimes \hat{X})$ is non-degenerate.

**Remark 2.5.** From remark 2.2 we see that if $T(Z) \in M_2(X \otimes \hat{X})$ is non-degenerate for at least one non-degenerate $Z \in M_1(X \otimes \hat{X})$ then it will be true for all non-degenerate $Z \in M_1(X \otimes \hat{X})$.

Now we claim that if $\mathcal{D}$ has duals and $\varphi : \mathcal{B}_1^{ev} \to \mathcal{D}$ is non-degenerate then the fiber functor $M_\varphi$ will be non-degenerate: for each object $X \in \mathcal{B}_1^{ev}$ there exists a coevaluation morphism

$$\text{coev}_{\varphi(X)} : 1 \to \varphi(X) \otimes \varphi(\hat{X}) \simeq \varphi(X \otimes \hat{X})$$

which determines an element in $Z_X \in M_\varphi(X \otimes \hat{X})$. It is not hard to see that this element is non-degenerate.

Let $\text{Fun}_{lax}^{\infty}(\mathcal{B}_1^{ev}, \text{Grp}_\infty)$ denote the $\infty$-category of lax symmetric monoidal functors $\mathcal{B}_1^{ev} \to \text{Grp}_\infty$ and by

$$\text{Fun}_{lax}^{\infty}(\mathcal{B}_1^{ev}, \text{Grp}_\infty) \subseteq \text{Fun}_{lax}^{\infty}(\mathcal{B}_1^{ev}, \text{Grp}_\infty)$$

the subcategory spanned by non-degenerate functors and non-degenerate natural transformations. Now the construction $\varphi \mapsto M_\varphi$ determines a functor

$$\text{Cat}_{\mathcal{B}_1^{ev}}^{\text{nd}} \to \text{Fun}_{lax}^{\infty}(\mathcal{B}_1^{ev}, \text{Grp}_\infty)$$

In particular if $\varphi : \mathcal{B}_1^{ev} \to \mathcal{C}$ and $\psi : \mathcal{B}_1^{ev} \to \mathcal{D}$ are non-degenerate then any functor $T : \mathcal{C} \to \mathcal{D}$ under $\mathcal{B}_1^{ev}$ will induce a non-degenerate natural transformation

$$T_* : M_\varphi \to M_\psi$$

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The rest of this section is devoted to proving the following result, which we call the "quasi-unital cobordism hypothesis":

**Theorem 2.6 (Cobordism Hypothesis - Quasi-Unital).** Let $\mathcal{D}$ be a symmetric monoidal $\infty$-category with duals, let $\varphi : \mathcal{B}^{\text{ev}}_{1} \to \mathcal{D}$ be a non-degenerate functor and let $\iota : \mathcal{B}^{\text{ev}}_{1} \to \mathcal{B}^{\text{or}}_{1}$ be the natural inclusion. Let $M_{\iota}, M_{\varphi} \in \text{Fun}^{\text{lax}}_{\text{nil}}$ be the corresponding fiber functors. Then the space of maps

$$\text{Map}_{\text{Fun}^{\text{lax}}_{\text{nil}}}(M_{\iota}, M_{\varphi})$$

is contractible.

**Proof.** We start by transforming the lax symmetric monoidal functors $M_{\iota}, M_{\varphi}$ to left fibrations over $\mathcal{B}^{\text{ev}}_{1}$ using the symmetric monoidal analogue of Grothendieck’s construction, as described in [Lur1], page 67–68.

Let $M : \mathcal{B} \to \text{Grp}_{\infty}$ be a lax symmetric monoidal functor. We can construct a symmetric monoidal $\infty$-category $\text{Groth}(\mathcal{B}, M)$ as follows:

1. The objects of $\text{Groth}(\mathcal{B}, M)$ are pairs $(X, \eta)$ where $X \in \mathcal{B}$ is an object and $\eta$ is an object of $M(X)$.
2. The space of maps from $(X, \eta)$ to $(X', \eta')$ in $\text{Groth}(\mathcal{B}, M)$ is defined to be the classifying space of the $\infty$-groupoid of pairs $(f, \alpha)$ where $f : X \to X'$ is a morphism in $\mathcal{B}$ and $\alpha : f_{*}\eta \to \eta$ is a morphism in $M(X')$. Composition is defined in a straightforward way.
3. The symmetric monoidal structure on $\text{Groth}(\mathcal{B}, M)$ is obtained by defining

$$(X, \eta) \otimes (X', \eta') = (X \otimes X', \beta_{X,Y}(\eta \otimes \eta'))$$

where $\beta_{X,Y} : M(X) \times M(Y) \to M(X \otimes Y)$ is given by the lax symmetric structure of $M$.

The forgetful functor $(X, \eta) \mapsto X$ induces a left fibration

$$\text{Groth}(\mathcal{B}, M) \to \mathcal{B}$$

**Theorem 2.7.** The association $M \mapsto \text{Groth}(\mathcal{B}, M)$ induces an equivalence between the $\infty$-category of lax-symmetric monoidal functors $\mathcal{B} \to \text{Grp}_{\infty}$ and the full subcategory of the over $\infty$-category $\text{Cat}^{\otimes}_{\mathcal{B}}$ spanned by left fibrations.

**Proof.** This follows from the more general statement given in [Lur1] Proposition 3.3.26. Note that any map of left fibrations over $\mathcal{B}$ is in particular a map of coCartesian fibrations because if $p : \mathcal{C} \to \mathcal{B}$ is a left fibration then any edge in $\mathcal{C}$ is $p$-coCartesian.

**Remark 2.8.** Note that if $\mathcal{C} \to \mathcal{B}$ is a left fibration of symmetric monoidal $\infty$-categories and $\mathcal{A} \to \mathcal{B}$ is a symmetric monoidal functor then the $\infty$-category

$$\text{Fun}^{\otimes}_{\mathcal{B}}(\mathcal{A}, \mathcal{C})$$
is actually an $\infty$-groupoid, and by Theorem 2.7 is equivalent to the $\infty$-groupoid of lax-monoidal natural transformations between the corresponding lax monoidal functors from $B$ to $\text{Grp}_\infty$.

Now set
\[ F_\iota \overset{\text{def}}{=} \text{Groth}(B_1^{\text{ev}}, M_\iota) \]
\[ F_\varphi \overset{\text{def}}{=} \text{Groth}(B_1^{\text{ev}}, M_\varphi) \]

Let
\[ \text{Fun}_/^\text{nd} B_1^{\text{ev}} (F_\iota, F_\varphi) \subseteq \text{Fun}_/^\otimes B_1^{\text{ev}} (F_\iota, F_\varphi) \]
denote the full sub $\infty$-groupoid of functors which correspond to non-degenerate natural transformations $M_\iota \rightarrow M_\varphi$ under the Grothendieck construction. Note that $\text{Fun}_/^\text{nd} B_1^{\text{ev}} (F_\iota, F_\varphi)$ is a union of connected components of the $\infty$-groupoid $\text{Fun}_/^\otimes B_1^{\text{ev}} (F_\iota, F_\varphi)$.

We now need to show that the $\infty$-groupoid $\text{Fun}_/^\text{nd} B_1^{\text{ev}} (F_\iota, F_\varphi)$ is contractible.

Unwinding the definitions we see that the objects of $F_\iota$ are pairs $(X, M)$ where $X \in B_1^{\text{ev}}$ is a 0-manifold and $M \in \text{Map}_{B_1^{\text{ev}}} (\emptyset, X)$ is a cobordism from $\emptyset$ to $X$. A morphism in $\varphi$ from $(X, M)$ to $(X', M')$ consists of a morphism in $B_1^{\text{ev}}$
\[ N : X \rightarrow X' \]
and a diffeomorphism
\[ T : M \bigsqcup_X N \cong M' \]
respecting $X'$. Note that for each $(X, M) \in F_\iota$ we have an identification $X \cong \partial M$. Further more the space of morphisms from $(\partial M, M)$ to $(\partial M', M')$ is homotopy equivalent to the space of orientation-preserving $\pi_0$-surjective embeddings of $M$ in $M'$ (which are not required to respect the boundaries in any way).

Now in order to analyze the symmetric monoidal $\infty$-category $F_\iota$ we are going to use the theory of $\infty$-operads, as developed in [Lur2]. Recall that the category $\text{Cat}^\otimes$ of symmetric monoidal $\infty$-categories admits a forgetful functor
\[ \text{Cat}^\otimes \rightarrow \text{Op}^\infty \]
to the $\infty$-category of $\infty$-operads. This functor has a left adjoint
\[ \text{Env} : \text{Op}^\infty \rightarrow \text{Cat}^\otimes \]
called the monoidal envelope functor (see [Lur2] §2.2.4). In particular, if $\mathcal{C}^\otimes$ is an $\infty$-operad and $\mathcal{D}$ is a symmetric monoidal $\infty$-category with corresponding $\infty$-operad $\mathcal{D}^\otimes \rightarrow N(\Gamma_*)$ then there is an equivalence of $\infty$-categories
\[ \text{Fun}^\otimes (\text{Env}(\mathcal{C}^\otimes), \mathcal{D}) \simeq \text{Alg}_{\mathcal{C}} (\mathcal{D}^\otimes) \]
Where $\text{Alg}_C (\mathcal{D}^\otimes) \subseteq \text{Fun}_{/N(\Gamma_\ast)} (\mathcal{E}^\otimes, \mathcal{D}^\otimes)$ denotes the full subcategory spanned by $\infty$-operad maps (see Proposition 2.2.4.9 of [Lur2]).

Now observing the definition of monoidal envelop (see Remark 2.2.4.3 in [Lur2]) we see that $\mathcal{F}_i$ is equivalent to the monoidal envelope of a certain simple $\infty$-operad

$$F_i \simeq \text{Env} (\mathcal{O}_i^\otimes)$$

which can be described as follows: the underlying $\infty$-category $\mathcal{O}_i$ of $\mathcal{O}_i^\otimes$ is the $\infty$-category of connected 1-manifolds (i.e. either the segment or the circle) and the morphisms are orientation-preserving embeddings between them. The (active) $n$-to-1 operations of $\mathcal{O}_i$ (for $n \geq 1$) from $(M_1, \ldots, M_n)$ to $M$ are the orientation-preserving embeddings

$$M_1 \coprod \cdots \coprod M_n \to M$$

and there are no 0-to-1 operations.

Now observe that the induced map $\mathcal{O}_i^\otimes \to (\mathcal{B}_1^{\text{ev}})^\infty$ is a fibration of $\infty$-operads. We claim that $\mathcal{F}_i$ is not only the enveloping symmetric monoidal $\infty$-category of $\mathcal{O}_i^\otimes$, but that $\mathcal{F}_i \to \mathcal{B}_1^{\text{ev}}$ is the enveloping left fibration of $\mathcal{O}_i \to \mathcal{B}_1^{\text{ev}}$. More precisely we claim that for any left fibration $\mathcal{D} \to \mathcal{B}_1^{\text{ev}}$ of symmetric monoidal $\infty$-categories the natural map

$$\text{Fun}_{/\mathcal{B}_1^{\text{ev}}} (\mathcal{F}_i, \mathcal{D}) \to \text{Alg}_{\mathcal{O}_i/\mathcal{B}_1^{\text{ev}}} (\mathcal{D}^\otimes)$$

is an equivalence of $\infty$-groupoids (where both terms denote mapping objects in the respective over-categories). This is in fact not a special property of $F_i$:

**Lemma 2.9.** Let $\mathcal{O}$ be a symmetric monoidal $\infty$-category with corresponding $\infty$-operad $\mathcal{O}^\otimes \to N(\Gamma_\ast)$ and let $p : \mathcal{E}^\otimes \to \mathcal{O}^\otimes$ be a fibration of $\infty$-operads such that the induced map

$$p : \text{Env} (\mathcal{E}^\otimes) \to \mathcal{O}$$

is a left fibration. Let $\mathcal{D} \to \mathcal{O}$ be some other left fibration of symmetric monoidal categories. Then the natural map

$$\text{Fun}_{/\mathcal{O}} (\text{Env} (\mathcal{E}^\otimes), \mathcal{D}) \to \text{Alg}_{\mathcal{E}/\mathcal{O}} (\mathcal{D}^\otimes)$$

is an equivalence of $\infty$-categories. Further more both sides are in fact $\infty$-groupoids.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
\text{Fun}^\otimes (\text{Env} (\mathcal{E}^\otimes), \mathcal{D}) & \to & \text{Alg}_C (\mathcal{D}^\otimes) \\
\downarrow & & \downarrow \\
\text{Fun}^\otimes (\text{Env} (\mathcal{E}^\otimes), \mathcal{O}) & \to & \text{Alg}_C (\mathcal{O}^\otimes)
\end{array}
$$

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Now the vertical maps are left fibrations and by adjunction the horizontal maps are equivalences. By [Lur3] Proposition 3.3.1.5 we get that the induced map on the fibers of $p$ and $\mathcal{F}$ respectively

$$\text{Fun}_{/\mathcal{O}}(\text{Env}(\mathcal{C}^{\otimes}), \mathcal{D}) \to \text{Alg}_{\mathcal{O}/\mathcal{D}}$$

is a weak equivalence of $\infty$-groupoids. \qed

Remark 2.10. In [Lur2] a relative variant $\text{Env}_{\mathcal{B}_1^{cv}}$ of Env is introduced which sends a fibration of $\infty$-operads $\mathcal{C}^{\otimes} \to (\mathcal{B}_1^{cv})^{\otimes}$ to its enveloping coCartesian fibration $\text{Env}_{\mathcal{O}}(\mathcal{C}^{\otimes}) \to \mathcal{B}_1^{cv}$. Note that in our case the map $\mathcal{F}_i \to \mathcal{B}_1^{cv}$ is not the enveloping coCartesian fibration of $\mathcal{O}\mathcal{F}^{\otimes} \to (\mathcal{B}_1^{cv})^{\otimes}$. However from Lemma 2.9 it follows that the map

$$\mathcal{F}_i \to \mathcal{B}_1^{cv}$$

is a covariant equivalence over $\mathcal{B}_1^{cv}$, i.e. induces a weak equivalence of simplicial sets on the fibers (where the fibers on the left are $\infty$-groupoids and the fibers on the right are $\infty$-categories). This claim can also be verified directly by unwinding the definition of $\text{Env}_{\mathcal{B}_1^{cv}}(\mathcal{O}\mathcal{F}^{\otimes})$.

Summing up the discussion so far we observe that we have a weak equivalence of $\infty$-groupoids

$$\text{Fun}_{/\mathcal{B}_1^{cv}}(\mathcal{F}_i, \mathcal{F}_\varphi) \simeq \text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{cv}}(\mathcal{F}^{\otimes}_\varphi)$$

Let

$$\text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{cv}}^{nd}(\mathcal{F}^{\otimes}_\varphi) \subseteq \text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{cv}}(\mathcal{F}^{\otimes}_\varphi)$$

denote the full sub $\infty$-groupoid corresponding to

$$\text{Fun}_{/\mathcal{B}_1^{cv}}(\mathcal{F}_i, \mathcal{F}_\varphi) \subseteq \text{Fun}_{/\mathcal{B}_1^{cv}}(\mathcal{F}_i, \mathcal{F}_\varphi)$$

under the adjunction. We are now reduced to prove that the $\infty$-groupoid

$$\text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{cv}}^{nd}(\mathcal{F}^{\otimes}_\varphi)$$

is contractible.

Let $\mathcal{O}\mathcal{F} \subseteq \mathcal{O}\mathcal{F}^{\otimes}$ be the full sub $\infty$-operad of $\mathcal{O}\mathcal{F}^{\otimes}$ spanned by connected 1-manifolds which are diffeomorphic to the segment (and all n-to-1 operations between them). In particular we see that $\mathcal{O}\mathcal{F}$ is equivalent to the non-unital associative $\infty$-operad. We begin with the following theorem which reduces the handling of $\mathcal{O}\mathcal{F}^{\otimes}$ to $\mathcal{O}\mathcal{F}$.
Theorem 2.11. Let $q: \mathcal{C} \to \mathcal{O}$ be a left fibration of $\infty$-operads. Then the restriction map

$$\text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) \to \text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{O})$$

is a weak equivalence.

Proof. We will base our claim on the following general lemma:

Lemma 2.12. Let $A \to B$ be a map of $\infty$-groupoids and let $q: \mathcal{C} \to \mathcal{O}$ be left fibration of $\infty$-operads. Suppose that for every object $B \in B$, the category

$$\mathcal{F}_B = A_{\text{act}} \times_{B_{\text{act}}} B_{/B}$$

is weakly contractible (see [Lur2] for the terminology). Then the natural restriction map

$$\text{Alg}_{A/\mathcal{O}}(\mathcal{C}) \to \text{Alg}_{B/\mathcal{O}}(\mathcal{C})$$

is a weak equivalence.

Proof. In [Lur2] §3.1.3 it is explained how under certain conditions the forgetful functor (i.e. restriction map)

$$\text{Alg}_{A/\mathcal{O}}(\mathcal{C}) \to \text{Alg}_{B/\mathcal{O}}(\mathcal{C})$$

admits a left adjoint, called the free algebra functor. Since $\mathcal{C} \to \mathcal{O}$ is a left fibration both these $\infty$-categories are $\infty$-groupoids, and so any adjunction between them will be an equivalence. Hence it will suffice to show that the conditions for existence of left adjoint are satisfies in this case.

Since $q: \mathcal{C} \to \mathcal{O}$ is a left fibration $q$ is compatible with colimits indexed by weakly contractible diagrams in the sense of [Lur2] Definition 3.1.1.18 (because weakly contractible colimits exists in every $\infty$-groupoid and are preserved by any functor between $\infty$-groupoids). Combining Corollary 3.1.3.4 and Proposition 3.1.1.20 of [Lur2] we see that the desired free algebra functor exists. 

In view of Lemma 2.12 it will be enough to check that for every object $M \in \mathcal{O}$ (i.e. every connected 1-manifolds) the $\infty$-category

$$\mathcal{F}_M \overset{\text{def}}{=} \mathcal{O}_{\text{act}} \times_{\mathcal{O}\text{act}} (\mathcal{O}_\text{act})_{/M}$$

is weakly contractible.

Unwinding the definitions we see that the objects of $\mathcal{F}_M$ are tuples of 1-manifolds $(M_1, ..., M_n)$ ($n \geq 1$), such that each $M_i$ is diffeomorphic to a segment, together with an orientation preserving embedding

$$f : M_1 \coprod ... \coprod M_n \to M$$

A morphisms in $\mathcal{F}_M$ from

$$f : M_1 \coprod ... \coprod M_n \to M$$

are given by

$$f' : M_1' \coprod ... \coprod M_n' \to M$$

such that $f'$ is obtained from $f$ by adjusting the embeddings $M_i \to M$. It is sufficient to show that the space of such $f'$ is weakly contractible.

To see this we choose a surjective map $X \to \mathcal{O} \text{act}$ such that the colimit of $X$ is the one-object groupoid represented by $M$. This can be done e.g. by choosing $X$ to be the groupoid obtained by freely adding a morphism $X \to M$.

This map $X \to \mathcal{O} \text{act}$ gives a map $F : \mathcal{F}_M \to X$ which is a surjective submersion. It is well known (see [Lur2] §3.1.3) that a map of $\infty$-groupoids $F : \mathcal{G} \to X$ is a surjective submersion if and only if the map of $\infty$-categories $F : \mathcal{G} \to \text{Ind}(X)$ is a surjective submersion. In our case, $\mathcal{F}_M$ is a $\infty$-category, and $\text{Ind}(X)$ is a $\infty$-groupoid.

Since $X$ is a $\infty$-groupoid, it follows that $\text{Ind}(X)$ is an $\infty$-groupoid. Therefore, $F : \mathcal{F}_M \to \text{Ind}(X)$ is a surjective submersion. It follows that $\mathcal{F}_M$ is weakly contractible.

Therefore, the natural restriction map

$$\text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) \to \text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{O})$$

is a weak equivalence.
to
\[ g : M'_1 \coprod \ldots \coprod M'_m \hookrightarrow M \]
is a \( \pi_0 \)-surjective orientation-preserving embedding
\[ T : M_1 \coprod \ldots \coprod M_n \rightarrow M'_1 \coprod \ldots \coprod M'_m \]
together with an isotopy \( g \circ T \sim f \).

Now when \( M \) is the segment then \( F_M \) contains a terminal object and so is weakly contractible. Hence we only need to take care of the case of the circle \( M = S^1 \).

It is not hard to verify that the category \( F_{S^1} \) is in fact discrete - the space of self isotopies of any embedding \( f : M_1 \coprod \ldots \coprod M_n \hookrightarrow M \) is equivalent to the loop space of \( S^1 \) and hence discrete. In fact one can even describe \( F_{S^1} \) in completely combinatorial terms. In order to do that we will need some terminology.

**Definition 2.13.** Let \( \Lambda_{\infty} \) be the category whose objects correspond to the natural numbers \( 1, 2, 3, \ldots \) and the morphisms from \( n \) to \( m \) are (weak) order preserving maps \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) such that \( f(x+n) = f(x) + m \).

The category \( \Lambda_{\infty} \) is a model for the the universal fibration over the cyclic category, i.e., there is a left fibration \( \Lambda_{\infty} \rightarrow \Lambda \) (where \( \Lambda \) is connes' cyclic category) such that the fibers are connected groupoids with a single object having automorphism group \( \mathbb{Z} \) (or in other words circles). In particular the category \( \Lambda_{\infty} \) is known to be weakly contractible. See [Kal] for a detailed introduction and proof (Lemma 4.8).

Let \( \Lambda_{\infty}^{\text{sur}} \) be the subcategory of \( \Lambda_{\infty} \) which contains all the objects and only surjective maps between. It is not hard to verify explicitly that the map \( \Lambda_{\infty}^{\text{sur}} \rightarrow \Lambda_{\infty} \) is cofinal and so \( \Lambda_{\infty}^{\text{sur}} \) is contractible as well. Now we claim that \( F_{S^1} \) is in fact equivalent to \( \Lambda_{\infty}^{\text{sur}} \).

Let \( \Lambda_{\text{big}}^{\text{sur}} \) be the category whose objects are linearly ordered sets \( S \) with an order preserving automorphisms \( \sigma : S \rightarrow S \) and whose morphisms are surjective order preserving maps which commute with the respective automorphisms. Then \( \Lambda_{\infty}^{\text{sur}} \) can be considered as a full subcategory of \( \Lambda_{\text{big}}^{\text{sur}} \) such that \( n \) corresponds to the object \((\mathbb{Z}, \sigma_n)\) where \( \sigma_n : \mathbb{Z} \rightarrow \mathbb{Z} \) is the automorphism \( x \mapsto x + n \).

Now let \( p : \mathbb{R} \rightarrow S^1 \) be the universal covering. We construct a functor \( F_{S^1} \rightarrow \Lambda_{\text{big}}^{\text{sur}} \) as follows: given an object
\[ f : M_1 \coprod \ldots \coprod M_n \hookrightarrow S^1 \]
of \( F_{S^1} \) consider the fiber product
\[ P = \left[ M_1 \coprod \ldots \coprod M_n \right] \times_{S^1} \mathbb{R} \]
note that \( P \) is homeomorphic to an infinite union of segments and the projection
\[ P \rightarrow \mathbb{R} \]
is injective (because $f$ is injective) giving us a well defined linear order on $P$. The automorphism $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ of $\mathbb{R}$ over $S^1$ given by $x \mapsto x + 1$ gives an order preserving automorphism $\tilde{\sigma} : P \rightarrow P$.

Now suppose that $((M_1, \ldots, M_n), f)$ and $((M'_1, \ldots, M'_m), g)$ are two objects and we have a morphism between them, i.e. an embedding

$$T : M_1 \coprod \ldots \coprod M_n \rightarrow M'_1 \coprod \ldots \coprod M'_m$$

and an isotopy $\psi : g \circ T \sim f$. Then we see that the pair $(T, \psi)$ determine a well defined order preserving map

$$\left[ M_1 \coprod \ldots \coprod M_n \right] \times_{S^1} \mathbb{R} \rightarrow \left[ M'_1 \coprod \ldots \coprod M'_m \right] \times_{S^1} \mathbb{R}$$

which commutes with the respective automorphisms. Clearly we obtain in this way a functor $u : F_{S^1} \rightarrow \Lambda^\text{sur}_{\infty}$ whose essential image is the same as the essential image of $\Lambda^\text{sur}_{\infty}$. It is also not hard to see that $u$ is fully faithful. Hence $F_{S^1}$ is equivalent to $\Lambda^\text{sur}_{\infty}$ which is weakly contractible. This finishes the proof of the theorem.

\[ \square \]

Let $\text{Alg}_{\text{O} \text{I} / \text{B} \text{ev}}^\text{nd} (\mathcal{F}^\otimes_\varphi) \subseteq \text{Alg}_{\text{O} \text{I} / \text{B} \text{ev}} (\mathcal{F}^\otimes_\varphi)$ denote the full sub $\infty$-groupoid corresponding to the full sub $\infty$-groupoid

$$\text{Alg}_{\text{O} \text{I} / \text{B} \text{ev}}^\text{nd} (\mathcal{F}^\otimes_\varphi) \subseteq \text{Alg}_{\text{O} \text{I} / \text{B} \text{ev}} (\mathcal{F}^\otimes_\varphi)$$

under the equivalence of Theorem [2.11].

Now the last step of the cobordism hypothesis will be complete once we show the following:

**Lemma 2.14.** The $\infty$-groupoid

$$\text{Alg}_{\text{O} \text{I} / \text{B} \text{ev}}^\text{nd} (\mathcal{F}^\otimes_\varphi)$$

is contractible.

**Proof.** Let

$$q : p^*\mathcal{F}_\varphi \rightarrow \mathcal{O}^\otimes$$

be the pullback of left fibration $\mathcal{F}_\varphi \rightarrow B^\otimes_1$ via the map $p : \mathcal{O}^\otimes \rightarrow B^\otimes_1$, so that $q$ is a left fibration as well. In particular, since $\mathcal{O}^\otimes$ is the non-unital associative $\infty$-operad, we see that $q$ classifies an $\infty$-groupoid $q^{-1}(\mathcal{O})$ with a non-unital monoidal structure. Unwinding the definitions one sees that this $\infty$-groupoid is the fundamental groupoid of the space

$$\text{Map}_c(1, \varphi(X_+) \otimes \varphi(X_-))$$
where \( X_+, X_- \in \mathcal{B}^{ev} \) are the points with positive and negative orientations respectively. The monoidal structure sends a pair of maps

\[
f, f' : 1 \to \varphi(X_+) \otimes \varphi(X_-)
\]

to the composition

\[
1 \overset{f \otimes f'}{\longrightarrow} [\varphi(X_+) \otimes \varphi(X_-)] \otimes [\varphi(X_+) \otimes \varphi(X_-)] \overset{\cong}{\longrightarrow} \\
\varphi(X_+) \otimes [\varphi(X_-) \otimes \varphi(X_+)] \otimes \varphi(X_-) \overset{\text{Id \otimes \varphi(ev) \otimes \text{Id}}}{\longrightarrow} \\
\varphi(X_+) \otimes \varphi(X_-)
\]

Since \( \mathcal{C} \) has duals we see that this monoidal \( \infty \)-groupoid is equivalent to the fundamental \( \infty \)-groupoid of the space

\[
\text{Map}_\mathcal{C}(\varphi(X_+), \varphi(X_+))
\]

with the monoidal product coming from \textbf{composition}.

Now

\[
\text{Alg}_{O\mathcal{J}/B_1^\ast}(\mathcal{F}_\varphi) \simeq \text{Alg}_{O\mathcal{J}/O\mathcal{J}}(p^*\mathcal{F}_\varphi)
\]

classifies \( O\mathcal{J}^\circ \)-algebra objects in \( p^*\mathcal{F}_\varphi \), i.e. non-unital algebra objects in

\[
\text{Map}_\mathcal{C}(\varphi(X_+), \varphi(X_+))
\]

with respect to composition. The full sub \( \infty \)-groupoid

\[
\text{Alg}_{nd/O\mathcal{J}/B_1^\ast}(\mathcal{F}_\varphi) \subseteq \text{Alg}_{O\mathcal{J}/B_1^\ast}(\mathcal{F}_\varphi)
\]

will then classify non-unital algebra objects \( A \) which correspond to \textbf{self equivalences}

\[
\varphi(X_+) \longrightarrow \varphi(X_+)
\]

It is left to prove the following lemma:

**Lemma 2.15.** Let \( \mathcal{C} \) be an \( \infty \)-category. Let \( X \in \mathcal{C} \) be an object and let \( \mathcal{E}_X \) denote the \( \infty \)-groupoid of self equivalences \( u : X \to X \) with the monoidal product induced from composition. Then the \( \infty \)-groupoid of non-unital algebra objects in \( \mathcal{E}_X \) is contractible.

**Proof.** Let \( \text{Ass}_{nu} \) denote the non-unital associative \( \infty \)-operad. The identity map \( \text{Ass}_{nu} \to \text{Ass}_{nu} \) which is in particular a left fibration of \( \infty \)-operads classifies the terminal non-unital monoidal \( \infty \)-groupoid \( A \) which consists of single automorphismless idempotent object \( a \in A \). The non-unital algebra objects in \( \mathcal{E}_X \) are then classified by non-unital lax monoidal functors

\[
A \longrightarrow \mathcal{E}_X
\]

Since \( \mathcal{E}_X \) is an \( \infty \)-groupoid this is same as non-unital monoidal functors (without the lax)

\[
A \longrightarrow \mathcal{E}_X
\]
Now the forgetful functor from unital to non-unital monoidal ∞-groupoids has a left adjoint. Applying this left adjoint to \( \mathcal{A} \) we obtain the ∞-groupoid \( \mathcal{U} \mathcal{A} \) with two automorphismless objects

\[
\mathcal{U} \mathcal{A} = \{1, a\}
\]

such that 1 is the unit of the monoidal structure and \( a \) is an idempotent object.

Hence we need to show that the ∞-groupoids of monoidal functors

\[
\mathcal{U} \mathcal{A} \rightarrow \mathcal{E}_X
\]

is contractible. Now given a monoidal ∞-groupoid \( \mathcal{G} \) we can form the ∞-category \( \mathcal{B}(\mathcal{G}) \) having a single object with endomorphism space \( \mathcal{G} \) (the monoidal structure on \( \mathcal{G} \) will then give the composition structure). This construction determines a fully faithful functor from the ∞-category of monoidal ∞-groupoids and the ∞-category of pointed ∞-categories (see [Lur1] Remark 4.4.6 for a much more general statement). In particular it will be enough to show that the ∞-groupoid of pointed functors

\[
\mathcal{B}(\mathcal{U} \mathcal{A}) \rightarrow \mathcal{B}(\mathcal{E}_X)
\]

is contractible. Since \( \mathcal{B}(\mathcal{E}_X) \) is an ∞-groupoid it will be enough to show that \( \mathcal{B}(\mathcal{U} \mathcal{A}) \) is weakly contractible.

Now the nerve \( \mathcal{N}(\mathcal{B}(\mathcal{U} \mathcal{A})) \) of \( \mathcal{B}(\mathcal{U} \mathcal{A}) \) is the simplicial set in which for each \( n \) there exists a single non-degenerate \( n \)-simplex \( \sigma_n \in \mathcal{N}(\mathcal{B}(\mathcal{U} \mathcal{A}))_n \) such that \( d_i(\sigma_n) = \sigma_{n-1} \) for all \( i = 0, ..., n \). By Van-Kampen it follows that \( \mathcal{N}(\mathcal{B}(\mathcal{U} \mathcal{A})) \) is simply connected and by direct computation all the homology groups vanish.

This finishes the proof of Lemma 2.14.

This finishes the proof of Theorem 2.6.

3 From Quasi-Unital to Unital Cobordism Hypothesis

In this section we will show how the quasi-unital cobordism hypothesis (Theorem 2.6) implies the last step in the proof of the 1-dimensional cobordism hypothesis (Theorem 1.8).

Let \( M : \mathcal{B}_1^\nu \rightarrow \text{Grp}_\infty \) be a non-degenerate lax symmetric monoidal functor. We can construct a pointed non-unital symmetric monoidal ∞-category \( \mathcal{C}_M \) as follows:

1. The objects of \( \mathcal{C}_M \) are the objects of \( \mathcal{B}_1^\nu \). The marked point is the object \( X_+ \).
2. Given a pair of objects \( X, Y \in \mathcal{C}_M \) we define

\[
\text{Map}_{\mathcal{C}_M}(X, Y) = M(X \otimes Y)
\]
Given a triple of objects $X,Y,Z \in C_M$ the composition law

$$\text{Map}_{C_M}(\tilde{X}, Y) \times \text{Map}_{C_M}(\tilde{Y}, Z) \to \text{Map}_{C_M}(\tilde{X}, Z)$$

is given by the composition

$$M(\tilde{X} \otimes Y) \times M(\tilde{Y} \otimes Z) \to M(\tilde{X} \otimes Y \otimes \tilde{Y} \otimes Z) \to M(\tilde{X} \otimes Z)$$

where the first map is given by the lax symmetric monoidal structure on the functor $M$ and the second is induced by the evaluation map

$$ev_Y : \tilde{Y} \otimes Y \to 1$$

in $B_1^{ev}$.

3. The symmetric monoidal structure is defined in a straight forward way using the lax monoidal structure of $M$.

It is not hard to see that if $M$ is non-degenerate then $C_M$ is quasi-unital, i.e. each object contains a morphism which behaves like an identity map (see [Har]). This construction determines a functor

$$G : \text{Fun}^{lax}_\text{nd}(B_1^{ev}, \text{Grp}_\infty) \to \text{Cat}^{\text{qu}, \otimes}_{B_0^{un}}$$

where $\text{Cat}^{\text{qu}, \otimes}$ is the $\infty$-category of symmetric monoidal quasi-unital categories (i.e. commutative algebra objects in the $\infty$-category $\text{Cat}^{\text{qu}}$ of quasi-unital $\infty$-categories). In [Har] it is proved that the forgetful functor

$$S : \text{Cat} \to \text{Cat}^{\text{qu}}$$

From $\infty$-categories to quasi-unital $\infty$-categories is an equivalence and so the forgetful functor

$$S^\otimes : \text{Cat}^\otimes \to \text{Cat}^{\text{qu}, \otimes}$$

is an equivalence as well.

Now recall that

$$\text{Cat}^{\text{sur}}_{B_1^{ev}} \subseteq \text{Cat}^{\text{nd}}_{B_1^{ev}}$$

is the full subcategory spanned by essentially surjective functors $\varphi : B_1^{ev} \to C$. The fiber functor construction $\varphi \mapsto M_\varphi$ induces a functor

$$F : \text{Cat}^{\text{sur}}_{B_1^{ev}} \to \text{Fun}^{\text{lax}}_{\text{nd}}(B_1^{ev}, \text{Grp}_\infty)$$

The composition $G \circ F$ gives a functor

$$\text{Cat}^{\text{sur}}_{B_1^{ev}} \to \text{Cat}^{\text{qu}, \otimes}_{B_0^{un}}$$

We claim that $G \circ F$ is in fact equivalent to the composition

$$\text{Cat}^{\text{sur}}_{B_1^{ev}} \xrightarrow{T} \text{Cat}^{\text{sur}}_{B_0^{un}} \xrightarrow{S} \text{Cat}^{\text{qu}, \otimes}_{B_0^{un}}$$

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where $T$ is given by the restriction along $X_+ : \mathcal{B}^\text{un}_0 \hookrightarrow \mathcal{B}^\text{cv}_1$ and $S$ is the forgetful functor.

Explicitly, we will construct a natural transformation

$$N : G \circ F \xrightarrow{\sim} S \circ T$$

In order to construct $N$ we need to construct for each non-degenerate functor $\varphi : \mathcal{B}^\text{cv}_1 \rightarrow \mathcal{D}$ a natural pointed functor

$$N_\varphi : \mathcal{M}_\varphi \longrightarrow \mathcal{D}$$

The functor $N_\varphi$ will map the objects of $\mathcal{M}_\varphi$ (which are the objects of $\mathcal{B}^\text{cv}_1$) to $\mathcal{D}$ via $\varphi$. Then for each $X, Y \in \mathcal{B}^\text{cv}_1$ we can map the morphisms

$$\text{Map}_{\mathcal{M}_\varphi}(X, Y) = \text{Map}_{\mathcal{D}}(1, \check{X} \otimes Y) \longrightarrow \text{Map}_{\mathcal{D}}(X, Y)$$

via the duality structure - to a morphism $f : 1 \rightarrow \check{X} \otimes Y$ one associates the morphism $\hat{f} : X \longrightarrow Y$ given as the composition

$$X \xrightarrow{id \otimes f} X \otimes \check{X} \otimes Y \xrightarrow{\varphi(\text{ev}_X) \otimes Y} Y$$

Since $\mathcal{D}$ has duals we get that $N_\varphi$ is fully faithful and since we have restricted to essentially surjective $\varphi$ we get that $N_\varphi$ is essentially surjective. Hence $N_\varphi$ is an equivalence of quasi-unital symmetric monoidal $\infty$-categories and $N$ is a natural equivalence of functors.

In particular we have a homotopy commutative diagram:

$$\begin{array}{ccc}
\text{Cat}^\text{sur}_{\mathcal{B}^\text{cv}_1} & \xrightarrow{F} & \text{Fun}_{\text{lax}}^\text{lax}(\mathcal{B}^\text{cv}_1, \text{Grp}_\infty) \\
\downarrow T & & \downarrow \text{S} \\
\text{Cat}^\otimes_{\mathcal{B}^\text{un}_0} & \xleftarrow{G} & \text{Cat}^\otimes_{\mathcal{B}^\text{un}_0} \\
\end{array}$$

Now from Lemma 1.6 we see that $T$ is fully faithful. Since $S$ is an equivalence of $\infty$-categories we get

**Corollary 3.1.** The functor $G \circ F$ is fully faithful.

We are now ready to complete the proof of 1.8. Let $\mathcal{D}$ be a symmetric monoidal $\infty$-category with duals and let $\varphi : \mathcal{B} \longrightarrow \mathcal{D}$ be a non-degenerate functor. We wish to show that the space of maps

$$\text{Map}_{\text{Cat}^\text{sur}_{\mathcal{B}^\text{cv}_1}}(\iota, \varphi)$$

is contractible. Consider the sequence
\[ \text{Map}_{\text{Cat}_{\text{sur}}^{\text{H}}}(t, \varphi) \rightarrow \text{Map}_{\text{Fun}_{\text{H}}^{\text{H}}}(\mathcal{B}_1^\varphi, \text{Grp}_{\infty})(M_t, M_\varphi) \rightarrow \text{Map}_{\text{Cat}_{\text{qH}}^{\text{H}}}(\mathcal{B}_1^\varphi, \mathcal{D}) \]

By Theorem 2.6 the middle space is contractible and by lemma 3.1 the composition

\[ \text{Map}_{\text{Cat}_{\text{sur}}^{\text{H}}}(t, \varphi) \rightarrow \text{Map}_{\text{Cat}_{\text{qH}}^{\text{H}}}(\mathcal{B}_1^\varphi, \mathcal{D}) \]

is a weak equivalence. Hence we get that

\[ \text{Map}_{\text{Cat}_{\text{sur}}^{\text{H}}}(t, \varphi) \]

is contractible. This completes the proof of Theorem 1.8.

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