AN ELEGANT 3-BASIS FOR INVERSE SEMIGROUPS

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Abstract. It is well known that in every inverse semigroup the binary operation and the unary operation of inversion satisfy the following three identities:

\[ x = (xx')x \quad (xx')(y'y) = (y'y)(xx') \quad (xy)z = x(yz'). \]

The goal of this note is to prove the converse, that is, we prove that an algebra of type \(\langle 2, 1 \rangle\) satisfying these three identities is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups.

1. Introduction

In the language of a binary operation \(\cdot\) and a unary operation \('\), a set of \(n\) independent identities is an \(n\)-basis for inverse semigroups, if those identities define the variety of inverse semigroups considered as algebras \((S, \cdot, ')\) of type \(\langle 2, 1 \rangle\), where the unary operation coincides with the natural inversion. Denoting by \(x'\) the inverse of an element \(x\) in an inverse semigroup, we then have \(x = (xx')x\) (as inverse semigroups are regular semigroups) and \((xx')(y'y) = (y'y)(xx')\) (as both \(xx'\) and \(y'y\) are idempotents, and idempotents commute in inverse semigroups). Thus we might be tempted to think that the following identities provide a 3-basis for inverse semigroups:

\[ x = (xx')x, \quad (xx')(y'y) = (y'y)(xx') \quad \text{and} \quad (xy)z = x(yz'). \]  

(1.1)

However, for \(S = \{0, 1\}\) with \(xy = 0\), except for \(11 = 1\), and defining \(x' = 1\), we have the previous identities satisfied, but \(0' \neq 00'\) and hence \('\) does not coincide with the natural inversion in \((S, \cdot)\).

B.M. Schein [4] repaired the defect of \((1.1)\) by adjoining two additional identities: \(x'' = x\) and \((xy)' = y'x'\). The resulting set of five identities indeed provides a 4-basis for inverse semigroups. (The identity \((xy)' = y'x'\) is dependent upon the others, and hence can be discarded. However it is worth observing that in the same paper Schein also provided a 5-basis using \(xx'x'x = x'xxx'\) instead of \(xx'y'y = y'yx'x'\); see [4, Theorem 1.6] and [2] p. 15, Ex. 20(b)). Therefore the natural question to ask would be: is it possible to find a 3-basis for inverse semigroups? This question was first answered in the affirmative in [1], but the 3-basis given there requires an extremely complicated proof (it is still an open problem to provide a reasonable proof for that result).

The aim of this note is to repair \((1.1)\) by providing an easy, transparent and elegant 3-basis for inverse semigroups.

Theorem. Let \((S, \cdot', )\) be an algebra of type \(\langle 2, 1 \rangle\). Then this algebra is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups if and only if

\[ (E_1) \quad x = (xx')x, \quad (E_2) \quad (xx')(y'y) = (y'y)(xx'), \quad (E_3) \quad (xy)z = x(yz'). \]
2. Proof of the Theorem

In this section we prove that the identities \((E_1)-(E_3)\) imply Schein’s 4-basis for inverse semigroups. As the converse is obvious, the equivalence of the two bases will follow.

Throughout this section let \((S, \cdot, \cdot)\) be an algebra of type \((2, 1)\) satisfying \((E_1)-(E_3)\). We start by proving a few handy identities.

Lemma 1. The following identities hold.

\[
\begin{align*}
    x'x'' &= x'x & (2.1) \\
    (xy')y &= x(y'y) & (2.2) \\
    x &= x(x'x) & (2.3) \\
    x'' &= (x'x')x = x''(x'x) & (2.4) \\
    x'''x &= x'''x^{(4)} & (2.5)
\end{align*}
\]

Proof. Firstly, for (2.1), we have

\[
x'x'' \overset{(E_1)}{=} x'[x''x''']x'' = x'[x''x''']x = x''(x'x) \overset{(E_1)}{=} x'x.
\]

Next, for (2.2), we compute \((xy')y \overset{(E_3)}{=} x(y'y') \overset{(2.1)}{=} x(y'y).

Regarding (2.3), we have \(x(x'x) \overset{(2.3)}{=} x(x'x) = x.

Then for (2.4), we compute \(x'' = x''(x'x') \overset{(E_1)}{=} x''x''x'x = x''x'x.

Finally, for (2.5), we have

\[
x''x = x'''x'''x'' = x'''x''x'' = x''x''x'' = x''x''x'' = x''x''x''x''.
\]

The next two lemmas are the key tools in the proof that the identities \((E_1)-(E_3)\) imply \(x'' = x\).

Lemma 2. \((x'x)x''' = x''\).

Proof. We start with two observations. Firstly, as

\[
[x(y'y)]y' \overset{(E_3)}{=} x[(y''y')y'] \overset{(2.6)}{=} x[(y''y'')y''] \overset{(E_1)}{=} xy'',
\]

we have

\[
(x(y'y))y' = xy''.
\]

Secondly,

\[
(x'x')(x'''x) \overset{(2.6)}{=} (x'x')(x'''x) = (x'''x)(x'x) \overset{(2.4)}{=} [(x'''x)x']x = x''x,
\]

so that

\[
(x'x)(x'''x) = x''x.
\]

Now we have all we need to prove the lemma.

\[
x'' \overset{(2.6)}{=} (x''x)' \overset{(2.6)}{=} (x''x)x' \overset{(2.4)}{=} [(x'x)(x''x)]x' \overset{(2.6)}{=} (x'x)x''.
\]

\[
\Box
\]
Lemma 3. \((xy)z' = x(yz')\).

Proof. We start by proving that
\[
x'' = x'.
\] (2.8)
In fact we have \(xx' \equiv [x(x')x'] \equiv (E_2) x[(x'x)x''] = x x''\), using Lemma 2 in the last equality. Thus
\[
xx'' = xx'.
\] (2.9)
Now, by Lemma 2
\[
x'' = (x'x)x'' \equiv (x'x'')(E_3) = x'(x''x'(5)) \equiv x'(x''x'(E_2)) = (E_3) x'(E_1) x' = x'.
\]
Replacing \(z\) by \(z'\) in \((E_3)\), we get
\[
(xy)z' = x(yz'') = x(yz'),
\]
where the last equality follows from (2.8). The lemma is proved. \(\square\)

We have everything we need to prove our main result.

Theorem 1. The identities \((E_1)-(E_3)\) imply \(x'' = x\) and the associative law.

Proof. First, we have
\[
x''x' \equiv [(x''x)x'] = (x''x')(x'') \equiv (E_2) (xx')(x''x') = (E_3) x' = xx',
\]
where we have used Lemma 3 in the unlabeled equalities. Thus
\[
x''x' = xx'.
\] (2.10)
Now \(x'' \equiv (x''x)x \equiv (xx')x \equiv x\), as claimed.

Associativity now follows easily: \((xy)z \equiv (E_1) x(yz'') = x(yz)\). \(\square\)

3. Other Sets of Axioms

It is natural to ask how sensitive the axioms \((E_1)-(E_3)\) are to certain modifications, such as shifting the parentheses in \((E_1)\) or changing the placement of the double inverse in \((E_3)\).

If, for instance, we leave \((E_2)\) intact, replace \((E_1)\) with \(x(x') = x\) and replace \((E_3)\) with \((x''y)z = x(yz)\), then we obtain a set of identities which are dual to \((E_1)-(E_3)\). By an argument dual to that in \((3)\) this set of identities is another 3-basis for inverse semigroups.

Thus to dispense with these sorts of obvious dualities, we will assume that both \((E_1)\) and \((E_2)\) are left intact, and consider only alternative placement of the double inverse in \((E_3)\). Using \textsc{Prover9}, we found that each of the following identities can substitute for \((E_3)\) to give another 3-basis for inverse semigroups:

\[
(xy)z = x''(yz) \quad (xy)z = x(y''z) \\
x(yz) = (xy'')z \quad x(yz) = (xy)z''.
\]

The remaining possibility, \(x(yz) = (x''y)z\), does not work. Using \textsc{Mace4}, we found the counterexample given by the following tables. It satisfies \((E_1)\), \((E_2)\) and \(x(yz) = (x''y)z\), but the binary operation is not associative \(((0\cdot0)\cdot0 = 1\cdot0 = 7 \neq 6 = 0\cdot1 = 0\cdot(0\cdot0))\), and the unary operation clearly fails to satisfy \(x'' = x\).
Does there exist a 2-basis for inverse semigroups?

We guess that the answer is no.

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