TYKHONOV WELL-POSEDNESS OF A VISCOPLASTIC CONTACT PROBLEM

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ABSTRACT. We consider an initial and boundary value problem \( P \) which describes the frictionless contact of a viscoplastic body with an obstacle made of a rigid body covered by a layer of elastic material. The process is quasistatic and the time of interest is \( \mathbb{R}_+ = [0, +\infty) \). We list the assumptions on the data and derive a variational formulation \( P_V \) of the problem, in a form of a system coupling an implicit differential equation with a time-dependent variational-hemivariational inequality, which has a unique solution. We introduce the concept of Tykhonov triple \( T = (I, \Omega, C) \) where \( I \) is set of parameters, \( \Omega \) represents a family of approximating sets and \( C \) is a set of sequences, then we define the well-posedness of Problem \( P_V \) with respect to \( T \). Our main result is Theorem 3.4, which provides sufficient conditions guaranteeing the well-posedness of \( P_V \) with respect to a specific Tykhonov triple. We use this theorem in order to provide the continuous dependence of the solution with respect to the data. Finally, we state and prove additional convergence results which show that the weak solution to problem \( P \) can be approached by the weak solutions of different contact problems. Moreover, we provide the mechanical interpretation of these convergence results.

1. Introduction. The concept of Tykhonov well-posedness for a mathematical problem is based on two main ingredients: the existence and uniqueness of solution to the problem and the convergence to it of any approximating sequence. It was introduced in [33] for a minimization problem and then it has been generalized for different optimization problems. References in the field include [15, 36], [16] and [4], where the concepts of extended well-posedness, Levitin-Polyak well-posedness

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†This paper is dedicated to Professor Meir Shillor on the occasion of his 70th birthday.

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and generic well-posedness for minimization problems have been introduced, respectively. For more details on well-posedness of optimization problems, we refer the readers to the monographs [5, 20]. There, a comprehensive mathematical theory of such problems is presented in an unified way.

The well-posedness in the sense of Tykhonov (well-posedness, for short) has been extended in recent years to various mathematical problems like inequalities, inclusions, fixed point and saddle point problems. The well-posedness for variational inequalities was studied for the first time in [18, 19] and the study of well-posedness for hemivariational inequalities was initiated in [9]. A reference in the field is [35] where some metric characterizations of well-posed hemivariational inequalities were provided and the equivalence of the well-posedness with the corresponding inclusions problems were proved. A general framework which unifies the view on well-posedness for abstract problems in metric spaces was recently considered in [32]. There, the well-posedness concept has been introduced by using approximating sequences which are defined by a family of subsets \( \{\Omega(\theta)\} \) indexed upon a positive parameter \( \theta > 0 \). For more references on well-posedness for variational inequalities, hemivariational inequalities and other related problems, we refer readers to [8, 14, 25, 31, 34].

Process of contact between deformable bodies arise in industry and daily life. Because of their importance in various real word applications, a large effort has been put into their modeling, analysis and numerical simulations. The mathematical literature on this field includes [6, 7, 22, 23, 24, 26] and, more recently, [1, 21, 28, 30, 29]. There, the unique weak solvability of various models of contact, expressed in terms of elliptic, time-dependent or evolutionary nonlinear boundary value problems, were provided. The mathematical tools used to solve these problems are based on arguments of differential inclusions, variational and hemivariational inequalities, convex and nonsmooth analysis, among others. References on numerical analysis of contact problems with elastic, viscoelastic and viscoplastic materials, within the framework of linearized strain theory, include [10, 11, 12, 13, 17].

In this paper we consider a relevant mathematical model which describes the contact between a viscoplastic body and a rigid foundation covered by a layer of elastic material. In variational formulation, the model leads to a system which couples an implicit differential equation for the stress field and a variational-hemivariational inequality for the displacement field, both defined on the unbounded interval of time \( \mathbb{R}_+ = [0, +\infty) \). The structure of this system suggests to introduce a new concept of well-posedness, based on the notion of Tykhonov triple \( T = (I, \Omega, C) \) where \( I \) is set of parameters, \( \Omega \) represents a family approximating sets and \( C \) is a set which defines a criterion of convergence. More precisely, \( I \) will represent either the set of positive real numbers or the set of sequences of positive real numbers, \( \Omega \) is a multivalued function defined on \( I \) with values in the product of two Fréchet spaces of continuous functions associated to the displacement and the stress field, respectively, and \( C \) represents a set of sequences of elements of \( I \).

Our aim in this paper is twofold. The first one is to study the well-posedness of the above-mentioned system with respect to various Tykhonov triples. This enlarges the functional framework considered in [32] and allows us to obtain a new result, Theorem 3.4. Our second aim is to illustrate the consequence of Theorem 3.4 in the variational analysis of a viscoplastic contact model. Thus, we start by proving that Theorem 3.4 allows us to obtain the continuous dependence of the solution with respect to the data. We also show that the choice of the Tykhonov
triple is crucial in order to obtain this result. Moreover, we use Theorem 3.4 to show that the weak solution of the viscoplastic contact problem can be approached by of a sequence of weak solutions of various contact problems, constructed with interface laws which are different from those used in the original contact problem. We conclude from above that the analysis of the contact model we consider in this paper leads to a new concept of well-posedness and, in turn, this new concept leads to interesting convergence results and mechanical interpretations. In this way we illustrate the cross fertilization between the models and applications, in one hand, and the nonlinear functional analysis, on the other hand, which represents the main feature of the Mathematical Theory of Contact Mechanics.

The rest of the paper is structured as follows. In Section 2 we introduce the viscoplastic contact problem, list the assumptions on the data and derive its variational formulation. Then, we state and prove an existence and uniqueness result, Theorem 2.1. In Section 3 we introduce the concept of well-posedness for the contact problem, associated to a given Tykhonov triple, and provide several examples of such triples. Then, we state and prove our main result, Theorem 3.4. It provides sufficient conditions for the well-posedness of the problem. In Section 4 we state and prove a convergence result, which represents a first consequence of Theorem 3.4. Finally, in Section 5 we presents additional convergence results and provide their mechanical interpretations.

We end this section with some notation which are needed in the rest of the paper. Let j : \mathbb{R} \to \mathbb{R} be a locally Lipschitz function. The generalized (Clarke) directional derivative of j at x ∈ \mathbb{R} in the direction v ∈ \mathbb{R} is defined by

\[ j^0(x; v) := \limsup_{y \to x, \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}. \]

The generalized (Clarke) subdifferential of j at x is a subset of \mathbb{R} given by

\[ \partial j(x) := \{ \zeta \in \mathbb{R} | j^0(x; v) \geq \zeta v, \forall v \in \mathbb{R} \}. \]

More details on the generalized subdifferential for locally Lipschitz functions defined on a normed space can be found in [2, 3].

Everywhere in this paper we use \mathbb{N} to represent the set of positive integers, i.e. \mathbb{N} = \{1, 2, 3, \ldots\}. For a real Banach space X we use \| \cdot \|_X and 0_X to represent its norm and zero element, respectively, and we denote by C(\mathbb{R}^+; X) the space of continuous functions on \mathbb{R}^+ with values in X. It is well known that C(\mathbb{R}^+; X) can be organized in a canonical way as a Fréchet space, i.e., it is a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The convergence of a sequence \{v_n\} to an element v, in the space C(\mathbb{R}^+; X), can be described as follows:

\[
\begin{align*}
\begin{cases}
v_n \to v \text{ in } C(\mathbb{R}^+; X) & \text{as } n \to \infty \text{ if and only if } \\
\max_{t \in [0,m]} \|v_n(t) - v(t)\|_X \to 0 & \text{as } n \to \infty, \forall m \in \mathbb{N}.
\end{cases}
\end{align*}
\]

Finally, for a subset K ⊂ X, we use the notation C(\mathbb{R}^+; K) for the set of functions defined on \mathbb{R}^+ with values in K.

2. The contact model. The physical setting is as follows. A viscoplastic body occupies, in the reference configuration, the domain D ⊂ \mathbb{R}^d (d = 2, 3) with smooth boundary Γ divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that meas(Γ_1) > 0. The body is fixed on Γ_1, is acted upon by time-dependent body
forces and surface tractions on $\Gamma_2$, and is in frictionless contact on $\Gamma_3$ with an obstacle, the so-called foundation. As a result its mechanical state evolves and the time interval of interest is $\mathbb{R}_+ = [0, +\infty)$. To describe this evolution we denote by $u$ the displacement field, by $\sigma$ the stress field and by $\varepsilon(u)$ the linearized strain tensor. Also, we use $\nu$ for the unit outward normal to $\Gamma$ and the indices $\nu$ and $\tau$ for the normal and tangential components of vectors and tensors, respectively. Finally, we denote by $S^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use the dot above to indicate the derivative with respect to the time variable. With these preliminaries, the mathematical model we consider in the paper is the following.

**Problem $\mathcal{P}$.** Find a displacement field $u: \mathbb{D} \times \mathbb{R}_+ \to \mathbb{R}^d$, a stress field $\sigma: \mathbb{D} \times \mathbb{R}_+ \to S^d$ and an interface function $\xi_\nu: \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ such that

\[
\mathbf{\dot{\sigma}} = \mathcal{E}_\varepsilon(u) + \mathcal{G}(\sigma, \varepsilon(u)) \quad \text{in } \mathbb{D} \times \mathbb{R}_+, \quad (2)
\]

\[
\text{Div } \sigma + f_0 = 0 \quad \text{in } \mathbb{D} \times \mathbb{R}_+, \quad (3)
\]

\[
u \quad \mathbf{\sigma} = f_2 \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \quad (5)
\]

\[
\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \quad (4)
\]

\[
\mathbf{u}_\nu \leq g, \quad \mathbf{\sigma} + \xi_\nu \leq 0, \quad (\mathbf{u}_\nu - g)(\mathbf{\sigma} + \xi_\nu) = 0, \quad \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (6)
\]

\[
\mathbf{\sigma}_\tau = 0 \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (7)
\]

\[
\mathbf{\sigma}(0) = \mathbf{\sigma}_0, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \mathbb{D}. \quad (8)
\]

A brief description of the equations, boundary and initial conditions in Problem $\mathcal{P}$ is the following. First, equation (2) represents the viscoplastic constitutive law of the material in which $\mathcal{E}$ is the fourth order elasticity tensor and $\mathcal{G}$ is a given nonlinear constitutive function. Example of viscoplastic laws of this form can be found in [6, 10, 28, 30], for instance. They have been used to model the properties of real materials like metals, rubbers, polymers and rocks, among others. Equation (3) is the equation of equilibrium in which $f_0$ denotes the density of body forces. We use it here since we assume that the process is quasistatic and, therefore, we neglect the inertial term in the equation of motion. Conditions (4), (5) represent the displacement and traction boundary conditions, respectively, in which $f_2$ is the density of surface tractions. Next, condition (6) represents the contact condition in which $g > 0$ and $\partial j_\nu$ denotes the Clarke subdifferential of a given function $j_\nu$ described below. This condition models the contact with a rigid foundation covered by a layer of elastic material. The bound $g$ represents the thickness of the layer and $j_\nu$ is the so-called normal compliance function describing the reaction it exerts towards the viscoplastic body. Details can be found in [30, p. 224] and [11, p. 188] and, therefore, we do not provide them. Condition (7) represents the frictionless condition on the contact surface and, finally, condition (8) represents the initial conditions in which $\mathbf{\sigma}_0$ and $\mathbf{u}_0$ are the initial stress and the initial displacement, respectively.

To derive the variational formulation of Problem $\mathcal{P}$ we need to introduce additional notation and preliminary material. First, the inner product, the norm and the zero element of the spaces $\mathbb{R}^d$ and $S^d$ will be denoted by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ and $0$, respectively. Next, besides the standard Lebesgue and Sobolev spaces associated to
\(D\) and \(\Gamma\), we use the space \(G\) for the displacement field and the space \(V\) for the stress field. These are real Hilbert spaces endowed with the canonical inner products given by

\[
(u, v)_V = \int_D \varepsilon(u) : \varepsilon(v) \, dx, \quad (\sigma, \tau)_Q = \int_D \sigma \cdot \tau \, dx
\]

and the associated norms \(\|\cdot\|_V\) and \(\|\cdot\|_Q\), respectively. Here, \(\varepsilon : V \to Q\) represents the linearized deformation operator, that is

\[
\varepsilon(v) = \frac{1}{2} \left( \nabla v + \nabla^T v \right) \quad \forall v \in V.
\]

For an element \(v \in V\) we still write \(v\) for the trace of \(v\) to \(\Gamma\). Moreover, the normal and tangential components of \(v\) on \(\Gamma\) are given by \(v_n = v \cdot \nu\) and \(v_\tau = v - v_n \nu\), respectively. In addition, we denote by \(\|\gamma_2\|\) and \(\|\gamma_3\|\) the norms of the trace operators \(\gamma_2 : V \to L^2(\Gamma_2)^d\) and \(\gamma_3 : V \to L^2(\Gamma_3)^d\), and recall that the following inequalities hold:

\[
\|\gamma_2 v\|_{L^2(\Gamma_2)^d} \leq \|\gamma_2\||v||_V \quad \forall v \in V, \quad (10)
\]

\[
\|\gamma_3 v\|_{L^2(\Gamma_3)^d} \leq \|\gamma_3\||v||_V \quad \forall v \in V. \quad (11)
\]

Finally, for a regular function \(\sigma : D \to \mathbb{S}^d\) we have \(\sigma_v = (\sigma \nu) \cdot \nu, \sigma_\tau = \sigma \nu - \sigma_v \nu\) and, moreover, the following Green’s formula holds:

\[
\int_P \sigma \cdot \varepsilon(v) \, dx + \int_P \text{Div} \sigma \cdot v \, dx = \int_\Gamma \sigma \nu \cdot v \, d\Gamma \quad \forall v \in H^1(D)^d. \quad (12)
\]

In the study of Problem \(\mathcal{P}\) we assume that the elasticity tensor \(E\) and the viscoplastic function \(G\) satisfy the following conditions.

\[
\begin{align*}
\mathcal{E} = (\mathcal{E}_{ijkl}) : D \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\
(13) & \mathcal{E}_{ijkl} = \mathcal{E}_{klji} = \mathcal{E}_{jikl} \in L^\infty(D), \ 1 \leq i, j, k, l \leq d, \\
(13a) & \text{there exists } \alpha_E > 0 \text{ such that} \\
& \mathcal{E}(x) \tau \cdot \tau \geq \alpha_E \|\tau\|^2 \text{ for all } \tau \in \mathbb{S}^d, \ \text{a.e. } x \in D.
\end{align*}
\]

\[
\begin{align*}
\mathcal{G} : D \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\
(14a) & \text{there exists } L_G > 0 \text{ such that} \\
& \|\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)\| \\
& \leq L_G (\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\|) \\
& \text{for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{a.e. } x \in D, \\
(14b) & \text{the mapping } x \mapsto \mathcal{G}(x, \sigma, \varepsilon) \text{ is measurable on } D, \\
(14c) & \text{for any } \sigma, \varepsilon \in \mathbb{S}^d, \\
& \text{the mapping } x \mapsto \mathcal{G}(x, \sigma, \varepsilon) \text{ belongs to } Q.
\end{align*}
\]
Moreover, the normal compliance function $j_\nu$ and the densities of body forces and tractions satisfy the conditions

$$j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$$

such that

1. $j_\nu(\cdot, r)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$ and there exists $\bar{\epsilon}_0 \in L^2(\Gamma_3)$ such that $j_\nu(\cdot, \bar{\epsilon}(\cdot)) \in L^1(\Gamma_3)$,

2. $j_\nu(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Gamma_3$,

3. $|\partial j_\nu(x, r)| \leq \bar{c}_0 + \bar{c}_1 |r|$ for a.e. $x \in \Gamma_3$ and for all $r \in \mathbb{R}$ with $\bar{c}_0, \bar{c}_1 \geq 0$,

4. $j_\nu^0(x, r_1; r_2 - r_1) + j_\nu^0(x, r_2; r_2 - r_1) \leq \alpha_{j_\nu} |r_1 - r_2|^2$ for a.e. $x \in \Gamma_3$ and for all $r_1, r_2 \in \mathbb{R}$ with $\alpha_{j_\nu} \geq 0$.

Examples of functions $j_\nu$ which satisfy condition (15) can be found in [11, 30], for instance. We also recall that $g > 0$, and, finally, we assume that the initial data have the regularity

$$u_0 \in V, \quad \sigma_0 \in Q$$

and the following smallness condition holds:

$$\alpha_{j_\nu} \|\gamma_\beta\|^2 < \alpha_\varepsilon.$$  

Note that assumption (13) implies that there exists $L_\mathcal{E} > 0$ such that

$$\|\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})\|_Q \leq L_\mathcal{E}\|\mathbf{u} - \mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V$$

and, moreover,

$$\langle \mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v}), \mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v}) \rangle_Q \geq \alpha_\mathcal{E}\|\mathbf{u} - \mathbf{v}\|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V.$$  

To derive the variational formulation of Problem $\mathcal{P}$ we first integrate equation (8) to obtain

$$\sigma(t) = \mathcal{E}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\sigma(s), \mathcal{E}(\mathbf{u}(s))) \, ds + \sigma_0 - \mathcal{E}(\mathbf{u}_0) \quad \forall t \in \mathbb{R}_+.$$  

Next, we introduce the set of admissible displacement fields $K$, the function $f : \mathbb{R}_+ \to V$ and the subset $Y$ of the product space $C(\mathbb{R}_+, V) \times C(\mathbb{R}_+, Q)$ defined by

$$K = \{ \mathbf{v} \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3 \},$$

$$f(t), v)_V = \int_0^t \int_{\Gamma_3} f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot \mathbf{v} \, d\Gamma \quad \forall \mathbf{v} \in V, \quad t \in \mathbb{R}_+,$$

$$Y = C(\mathbb{R}_+, K) \times C(\mathbb{R}_+, Q).$$

Following a standard approach based on the Green formula (12) and the definition of the Clarke subdifferential of the function $j_\nu$, it follows that if $(\mathbf{u}, \sigma)$ represents a regular solution of Problem $\mathcal{P}$ then

$$\int_0^t \int_{\Gamma_3} \sigma(t) \cdot (\mathcal{E}(\mathbf{v}) - \mathcal{E}(\mathbf{u}(t))) \, dx + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma$$

$$\geq \int_0^t \int_{\Gamma_3} f_0(t) \cdot (\mathbf{v} - \mathbf{u}(t)) \, dx + \int_{\Gamma_2} f_2(t) \cdot (\mathbf{v} - \mathbf{u}(t)) \, d\Gamma$$

and

$$\int_0^t \int_{\Gamma_3} \sigma(t) \cdot (\mathcal{E}(\mathbf{v}) - \mathcal{E}(\mathbf{u}(t))) \, dx + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma$$

$$\geq \int_0^t \int_{\Gamma_3} f_0(t) \cdot (\mathbf{v} - \mathbf{u}(t)) \, dx + \int_{\Gamma_2} f_2(t) \cdot (\mathbf{v} - \mathbf{u}(t)) \, d\Gamma$$
for all $v \in K$ and $t \in \mathbb{R}^+$. We now combine equation (22) with regularity $u(t) \in K$, inequality (26) and definition (24) to obtain the following variational formulation of Problem $P_V$.

**Problem $P_V$.** Find a displacement field $u \in C(\mathbb{R}^+, K)$ and a stress field $\sigma : C(\mathbb{R}^+, Q)$ such that

$$
\sigma(t) = \mathcal{E}(u(t)) + \int_0^t \mathcal{G}(\sigma(s), \mathcal{E}(u(s))) \, ds + \sigma_0 - \mathcal{E}(u_0),
$$

(27)

$$
(\sigma(t), \mathcal{E}(v) - \mathcal{E}(u(t)))_Q + \int_{\Gamma_3} j_{\nu}^0(u_{\nu}(t); v_{\nu} - u_{\nu}(t)) \, d\Gamma \geq (f(t), v - u(t))_V
$$

(28)

for all $v \in K$ and $t \in \mathbb{R}^+$.

The unique solvability of Problem $P_V$ is given by the following existence and uniqueness result.

**Theorem 2.1.** Assume that (13)–(19) hold. Then Problem $P_V$ has a unique solution $(u, \sigma) \in Y$.

**Proof.** We use arguments similar to those used in the proof of Theorem 114 in [30] and, for this reason, we skip the details. We restrict ourselves to mention that the proof is structured in four steps, as follows.

(i) First, we prove that there exists an operator $\mathcal{R} : C(\mathbb{R}^+; V) \to C(\mathbb{R}^+; Q)$ such that, for all functions $u \in C(\mathbb{R}^+; V)$ and $\sigma \in C(\mathbb{R}^+; Q)$, equality (27) holds for all $t \in \mathbb{R}^+$, if and only if

$$
\sigma(t) = \mathcal{E}(u(t)) + \mathcal{R}(u(t))
$$

(29)

for all $t \in \mathbb{R}^+$. Moreover, the operator $\mathcal{R} : C(\mathbb{R}^+; V) \to C(\mathbb{R}^+; Q)$ is a history-dependent operator, i.e.,

$$
\left\{
\begin{array}{l}
\text{for every } m \in \mathbb{N} \text{ there exists } d_m > 0 \text{ such that }\\
\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_V \leq d_m \int_0^t \|u_1(s) - u_2(s)\|_V \, ds
\end{array}
\right.
$$

(30)

$\forall u_1, u_2 \in C(\mathbb{R}^+; V), \forall t \in [0, m]$. 

(ii) Next, we consider the auxiliary problem of finding a function $u \in C(\mathbb{R}^+; K)$ such that

$$
(\mathcal{E}(u(t)), \mathcal{E}(v) - \mathcal{E}(u(t)))_Q + (\mathcal{R}(u(t)) : (\mathcal{E}(v) - \mathcal{E}(u(t)))_Q
$$

$$
+ \int_{\Gamma_3} j_{\nu}^0(u_{\nu}(t); v_{\nu} - u_{\nu}(t)) \, d\Gamma \geq (f(t), v - u(t))_V
$$

(31)

for all $v \in K$ and $t \in \mathbb{R}^+$. This problem is obtained by substituting the stress field given by equality (29) in the variational-hemivariational inequality (28). Then it is easy to see that the pair $(u, \sigma)$ is a solution to Problem $P_V$ with regularity $(u, \sigma) \in Y$ if and only if both equality (29) and inequality (31) hold, for all $v \in K$ and $t \in \mathbb{R}^+$.

(iii) Note that $K$ is a nonempty closed convex subset of the space $V$. Therefore, the property (30) of the operator $\mathcal{R}$ combined with assumptions (13), (15), (19), among others, allows us to apply Theorem 93 in [30] in order to deduce that inequality (31) has a unique solution $u \in C(\mathbb{R}^+; K)$. 

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(iv) Let $\sigma$ be the function defined by equality (29). Then, it is easy to see that $\sigma \in C(\mathbb{R}_+; Q)$ and, moreover, using step (ii), it follows that the pair $(u, \sigma)$ is the unique solution of Problem $P_V$ with the regularity $(u, \sigma) \in Y$. \hfill $\square$

We end this section by recalling that a pair of functions $(u, \sigma) \in Y$ which satisfies (27) and (28) is called a weak solution to Problem $P$. Note that the problem of finding the interface function $\xi_{\nu} : \mathbb{R}_{+} \times \Gamma_3 \to \mathbb{R}$ is left open.

3. Tykhonov well-posedness. Everywhere in this section we assume that (13)–(19) hold, even if we do not mention it explicitly. As already mentioned in the Introduction, the concept of well-posedness for Problem $P_V$ is associated to the so-called Tykhonov triple which is defined as follows.

Definition 3.1. A Tykhonov triple is a mathematical object of the form $T = (I, \Omega, C)$ where $I$ is a given nonempty set, $\Omega : I \to 2^Y - \{\emptyset\}$ and $C \subset S(I)$ is a nonempty set, where $S(I)$ denotes the set of sequences whose elements belong to $I$ and $2^Y$ is the power set of $Y$.

Below in this paper we shall use the following notation and terminology: a) an element of $I$ will be denoted by $\theta$ when $I \subset \mathbb{R}$, and by $\theta = \{\theta^m\}_m$ when $I \subset S(\mathbb{R})$; b) an element of $S(I)$ will be denoted by $\theta = \{\theta^m\}_m$ when $I \subset \mathbb{R}$, and by $\theta = \{\theta^m\}_n$ with $\theta^m = \{\theta^m_n\}_n$ when $I \subset S(\mathbb{R})$; c) for any $\theta$ or $\theta^m \in I$, we refer to the sets $\Omega(\theta)$ or $\Omega(\theta^m) \subset Y$, respectively, as approximating sets.

Next, inspired by our previous work [32], we consider the following definitions.

Definition 3.2. Given a Tykhonov triple $T = (I, \Omega, C)$, a sequence $\{(u_n, \sigma_n)\}_n \subset Y$ is called a $T$-approximating sequence if there exists a sequence $\{\theta_n\}_n \subset C$, such that $(u_n, \sigma_n) \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

Note that approximating sequences always exist, since, by assumption, $C \neq \emptyset$ and, moreover, for any sequence $\{\theta_n\}_n \subset C$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty.

Definition 3.3. The Problem $P_V$ is said to be well-posed with respect to the Tykhonov triple $T = (I, \Omega, C)$ if it has a unique solution and every $T$-approximating sequence for Problem $P_V$ converges in $C(\mathbb{R}_+, V) \times C(\mathbb{R}_+, Q)$ to the solution.

In other words, Problem $P_V$ is well-posed with respect to $T$ if there exists a unique couple of functions $(u, \sigma) \in Y$ which satisfies (27) and (28) for all $v \in K$ and $t \in \mathbb{R}_+$ and, moreover, for any $T$-approximating sequence $\{(u_n, \sigma_n)\}_n$, we have

$$u_n \to u \quad \text{in} \quad C(\mathbb{R}_+, V) \quad \text{as} \quad n \to \infty, \quad (32)$$

$$\sigma_n \to \sigma \quad \text{in} \quad C(\mathbb{R}_+, Q) \quad \text{as} \quad n \to \infty. \quad (33)$$

Note that the concept of well-posedness defined above depends on the Tykhonov triple $T$ and, as explained below in this paper, the choice of $T$ is crucial for the analysis of the well-posedness for Problem $P$. In what follows we construct a relevant example of such triple.

Example 1. Keep the assumption in Theorem 2.1 and take $T = (I, \Omega, C)$ where

$$I = \{ \theta = \{\theta^m\}_m : \theta^m \in \mathbb{R}, \theta^m > 0 \quad \forall m \in \mathbb{N} \}, \quad (34)$$

$$C = \{ \{\theta^n\}_n : \theta_n = \{\theta^m_n\}_m \in I \quad \forall n \in \mathbb{N},$$

$$\theta^m_n \to 0 \quad \text{as} \quad n \to \infty, \quad \forall m \in \mathbb{N} \} \quad (35)$$
and, for each $\theta = \{\theta^m\}_m \in I$, the set $\Omega(\theta)$ is defined as follows: $(u, \sigma) \in \Omega(\theta)$ if and only if the following conditions hold:

$$ (u, \sigma) \in Y, $$

$$ \left\| \sigma(t) - E\varepsilon(u(t)) - \int_0^t G(\sigma(s), \varepsilon(u(s))) \, ds - \sigma_0 + E\varepsilon(u_0) \right\|_Q \leq \theta^m $$

\(\forall t \in [0, m], \ m \in \mathbb{N},\)

$$ (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + \int_{\Gamma_3} j^0_{\nu}(u_{\nu}(t); v_{\nu} - u_{\nu}(t)) \, d\Gamma $$

$$ + \theta^m \|v - u(t)\|_V \geq (f(t), v - u(t))_V \quad \forall v \in K, \ t \in [0, m], \ m \in \mathbb{N}. $$

Note that for each $\theta \in I$ the solution $(u, \sigma)$ obtained in Theorem 2.1 belongs to $\Omega(\theta)$ and, therefore, $\Omega(\theta) \neq \emptyset$. Moreover, in Sections 4–5 we show that our choice above, based on the conditions (36), (37) and (38), gives rise to a well-posedness concept with interesting mechanical interpretations.

Our main result in this section is the following.

**Theorem 3.4.** Assume that (13)–(19) hold. Then Problem $P_V$ is well-posed with respect to the Tykhonov triple in Example 1.

**Proof.** We start by recalling that the existence of a unique solution $(u, \sigma) \in Y$ to Problem $P_V$ was provided in Theorem 2.1. To proceed, we consider a $T$-approximating sequence for the Problem $P_V$, denoted by $\{(u_n, \sigma_n)\}_n$. Then, according to Definition 3.2 it follows that there exists a sequence $\{\theta_n\}_n \in \mathbb{C}$ with $\theta_n = \{\theta^m_n\}_m \in I$ such that

$$ \theta^m_n \to 0 \quad as \quad n \to \infty, \quad \text{for each } m \in \mathbb{N} $$

and, moreover, for each $n \in \mathbb{N}$, the following properties hold.

$$ (u_n, \sigma_n) \in Y, $$

$$ \left\| \sigma_n(t) - E\varepsilon(u_n(t)) - \int_0^t G(\sigma_n(s), \varepsilon(u_n(s))) \, ds - \sigma_0 + E\varepsilon(u_0) \right\|_Q \leq \theta^m_n $$

\(\forall t \in [0, m], \ m \in \mathbb{N},\)

$$ (\sigma(t), \varepsilon(v) - \varepsilon(u_n(t)))_Q + \int_{\Gamma_3} j^0_{\nu}(u_{\nu}(t); v_{\nu} - u_{\nu}(t)) \, d\Gamma $$

$$ + \theta^m_n \|v - u_n(t)\|_V \geq (f(t), v - u_n(t))_V \quad \forall v \in K, \ t \in [0, m], \ m \in \mathbb{N}. $$

Let $n \in \mathbb{N}$ be fixed. We introduce the functions $\eta_n : \mathbb{R}_+ \to Q$ and $\eta : \mathbb{R}_+ \to Q$ defined by

$$ \eta_n(t) = \int_0^t G(\sigma(s), \varepsilon(u_n(s))) \, ds + \sigma_0 - E\varepsilon(u_0), $$

$$ \eta(t) = \int_0^t G(\sigma(s), \varepsilon(u(s))) \, ds + \sigma_0 - E\varepsilon(u_0), $$

for all $t \in \mathbb{R}_+$. Consider now $m \in \mathbb{N}$ and let $t \in [0, m]$. Then, using (41) and (22) it follows that

$$ \left\| \sigma_n(t) - E\varepsilon(u_n(t)) - \eta_n(t) \right\|_Q \leq \theta^m_n, $$

$$ \sigma(t) = E\varepsilon(u(t)) + \eta(t). $$
Moreover, using (46) it follows that
\[ \|\sigma_n(t) - \sigma(t)\|_Q \leq \|\sigma_n(t) - \mathcal{E}\varepsilon(u_n(t)) - \eta_n(t)\|_Q + \|\mathcal{E}\varepsilon(u_n(t)) - \mathcal{E}\varepsilon(u(t))\|_Q + \|\eta_n(t) - \eta(t)\|_Q. \]

We now use inequalities (45) and (20) to find that
\[ \|\sigma_n(t) - \sigma(t)\|_Q \leq \theta_n^m + L_G\|u_n(t) - u(t)\|_V + \|\eta_n(t) - \eta(t)\|_Q. \] (47)

On the other hand, we test in (42) with \( v = u(t) \), then in (28) with \( v = u_n(t) \), both in \( K \), and add the resulting inequalities to obtain
\[(\sigma_n(t) - \sigma(t), \varepsilon(u_n(t)) - \varepsilon(u(t)))_Q \leq \int_{\Gamma_3} (j_\nu^0(u_{nt}(t); u_{nt} - u_{nt}(t))) d\Gamma + \theta_n^m\|u(t) - u_n(t)\|_V. \] (48)

We now use (46) to write
\[ \sigma_n(t) - \sigma(t) = \sigma_n(t) - \mathcal{E}\varepsilon(u_n(t)) - \eta_n(t) + \mathcal{E}\varepsilon(u_n(t)) + \eta_n(t) - \mathcal{E}\varepsilon(u(t)) - \eta(t), \]
then we substitute this equality in (48) and use (21), (15)(d) to find that
\[ \alpha\mathcal{E}\|u_n(t) - u(t)\|_V \leq (\sigma_n(t) - \mathcal{E}\varepsilon(u_n(t)) - \eta_n(t), \varepsilon(u(t)) - \varepsilon(u_n(t)))_Q + (\eta_n(t) - \eta(t), \varepsilon(u(t)) - \varepsilon(u_n(t)))_Q + \alpha j_\nu \int_{\Gamma_3} |u_{nt}(t) - u_{nt}(t)|^2 d\Gamma + \theta_n^m\|u(t) - u_n(t)\|_V. \]

Next, exploiting inequalities (45) and (11) we deduce that
\[ (\alpha\mathcal{E} - \alpha j_\nu, \|\gamma_3\|)^2\|u_n(t) - u(t)\|_V \leq 2\theta_n^m + \|\eta_n(t) - \eta(t)\|_Q \]
and, using the smallness assumption (19), we find that
\[ \|u_n(t) - u(t)\|_V \leq \frac{2}{\alpha\mathcal{E} - \alpha j_\nu, \|\gamma_3\|} \theta_n^m + \frac{1}{\alpha\mathcal{E} - \alpha j_\nu, \|\gamma_3\|} \|\eta_n(t) - \eta(t)\|_Q. \] (49)

To proceed, we use definitions (43) and (44) together with property (14)(a) of the function \( \mathcal{G} \) to deduce that
\[ \|\eta_n(t) - \eta(t)\|_Q \leq L_G \int_0^t \|\sigma_n(s) - \sigma(s)\|_Q ds + L_G \int_0^t \|u_n(s) - u(s)\|_V ds. \] (50)

We now combine inequalities (47), (49) and (50), use inequality \( t \leq m \) and, after some algebra, find that
\[ \|\eta_n(t) - \eta(t)\|_Q \leq mL_G (1 + \frac{2(L_G + 1)}{\alpha\mathcal{E} - \alpha j_\nu, \|\gamma_3\|} \theta_n^m \]
\[ + L_G (\frac{L_G + 1}{\alpha\mathcal{E} - \alpha j_\nu, \|\gamma_3\|} + 1) \int_0^t \|\eta_n(s) - \eta(s)\|_Q ds. \]

We deduce from here that there exists two positive constants \( c_0 > 0 \) and \( c_1 > 0 \) which depend on \( \mathcal{E}, \mathcal{G}, j \) and \( \|\gamma_3\| \) but are independent of \( t, m \) and \( n \) such that
\[ \|\eta_n(t) - \eta(t)\|_Q \leq c_0 m \theta_n^m + c_1 \int_0^t \|\eta_n(s) - \eta(s)\|_Q ds. \]

Therefore, using the Gronwall inequality we deduce that
\[ \|\eta_n(t) - \eta(t)\|_Q \leq c_0 m e^{c_1 t} \theta_n^m \]
and, moreover,
\[
\max_{t \in [0, m]} \| \eta_n(t) - \eta(t) \|_Q \leq c_0 m e^{c_1 m} \theta_n^m. \tag{51}
\]

We now use inequalities (49), (51) and the convergence (39) to find that
\[
\max_{t \in [0, m]} \| u_n(t) - u(t) \|_V \to 0 \quad \text{as} \quad n \to \infty. \tag{52}
\]

Finally, inequalities (47), (51) and convergences (39), (52) guarantee that
\[
\max_{t \in [0, m]} \| \sigma_n(t) - \sigma(t) \|_Q \to 0 \quad \text{as} \quad n \to \infty. \tag{53}
\]

Now, it follows from (1) and convergences (52), (53) that (32) and (33) hold, which concludes the proof. \qed

We now consider a second example of Tykhonov triple with whom Problem \( \mathcal{P}_V \) is also well-posed.

**Example 2.** Keep the assumption in Theorem 2.1 and take \( \mathcal{T} = (I, \Omega, C) \) where
\[
I = \mathbb{R}^*_+ = \mathbb{R}^+ \setminus \{0\}, \tag{54}
\]
\[
C = \{ \{ \theta_n \} : \theta_n \in I \; \forall n \in \mathbb{N}, \; \theta_n \to 0 \; \text{as} \; n \to \infty \} \tag{55}
\]

and, for each \( \theta \in I \), the set \( \Omega(\theta) \) is defined as follows: \((u, \sigma) \in \Omega(\theta)\) if and only if the following conditions hold.
\[
(u, \sigma) \in Y, \tag{56}
\]
\[
\left\| \sigma(t) - \mathcal{E} \varepsilon(u(t)) - \int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s))) \, ds - \sigma_0 + \mathcal{E} \varepsilon(u_0) \right\|_Q \leq \theta \quad \forall t \in \mathbb{R}^+_+, \tag{57}
\]
\[
(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + \int_{\Gamma_3} J^0_\nu(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma \tag{58}
\]
\[
+ \theta \| v - u(t) \|_V \geq (f(t), v - u(t))_V \quad \forall v \in K, \; t \in \mathbb{R}^+_+.
\]

Note that, again, using Theorem 2.1 it follows that \( \Omega(\theta) \neq \emptyset \), for each \( \theta \in I \).

We have the following well-posednes result.

**Theorem 3.5.** Assume that (13)–(19) hold. Then Problem \( \mathcal{P}_V \) is well-posed with respect to the Tykhonov triple \( \mathcal{T} \) in Example 2.

The proof of this theorem is similar to those of Theorem 3.4 and, therefore, we skip it. Note that in this case some estimates are simpler since they do not depend on \( m \).

We end this section with an example of Tykhonov triple with whom Problem \( \mathcal{P}_V \) is not well-posed in the sense of Definition 3.3.

**Example 3.** Keep the assumption in Theorem 2.1 and take \( \mathcal{T} = (I, \Omega, C) \) where the sets \( I \) and \( C \) is defined by (34) and (35), respectively and, for each \( \theta = \{ \theta^m \} \in I \), the set \( \Omega(\theta) \) is defined as follows: \((u, \sigma) \in \Omega(\theta)\) if and only if
\[
(u, \sigma) \in Y, \tag{59}
\]
\[
\left\| \sigma(t) - \mathcal{E} \varepsilon(u(t)) - \int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s))) \, ds - \sigma_0 + \mathcal{E} \varepsilon(u_0) \right\|_Q \leq \theta^m + a \quad \forall t \in [0, m], \; m \in \mathbb{N}, \tag{60}
\]
(\sigma(t), \varepsilon(v) - \varepsilon(u(t))) + \int_{\Gamma_3} j_0^b(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma \\
+ (\theta^m + a) \|v - u(t)\|_V \geq (f(t), v - u(t))_V \quad \forall v \in K, t \in [0, m], \ m \in \mathbb{N};

Here \( a > 0 \) is a positive constant which will be defined below. Consider the case when the constitutive function \( \mathcal{G} \) does not depend on \( \sigma \) and let \( \sigma^* \in Q \) be given with \( \sigma^* \neq 0_Q \). For each \( n \in \mathbb{N} \), let \( u_n = u, \sigma_n = \sigma + \sigma^* \) where \( (u, \sigma) \) denotes the solution of Problem \( \mathcal{P}_V \) obtained in Theorem 2.1 and let \( \theta_n = \{\theta^m_n\}_m \) where \( \theta^m_n = \frac{1}{n} \) for each \( m \in \mathbb{N} \). Then it is easy to see that, for each \( n \in \mathbb{N} \), the couple of functions \((u_n, \sigma_n)\) satisfies conditions (59)–(61) with \( a = \|\sigma^*\|_Q > 0 \) and, therefore, the sequence \( \{(u_n, \sigma_n)\} \) is a \( T \)-approximating sequence. Nevertheless, condition (33) is not satisfied. We conclude from Definition 3.3 that Problem \( \mathcal{P}_V \) is not well-posed with the Tykhonov triple \( T \) above defined.

4. A continuous dependence result. The solution of Problem \( \mathcal{P}_V \) depends on the data \( f_0, f_2, u_0 \) and \( \sigma_0 \). Its continuous dependence with respect these data is provided by the following convergence result.

**Theorem 4.1.** Assume that (13)–(19) hold and denote by \((u, \sigma)\) the solution of Problem \( \mathcal{P}_V \). Moreover, for each \( n \in \mathbb{N} \), denote by \((u_n, \sigma_n)\) the solution of Problem \( \mathcal{P}_V \) for the data \( f_{0n}, f_{2n}, u_{0n} \) and \( \sigma_{0n} \) which satisfy

\[
\begin{align*}
f_{0n} &\in C(\mathbb{R}_+; L^2(D)^d), \quad f_{2n} \in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \\
u_{0n} &\in V, \quad \sigma_{0n} \in Q.
\end{align*}
\]

In addition, assume that

\[
\begin{align*}
f_{0n} &\to f_0 \text{ in } C(\mathbb{R}_+; L^2(D)^d), \quad f_{2n} \to f_2 \text{ in } C(\mathbb{R}_+; L^2(\Gamma_2)^d), \\
u_{0n} &\to u_0 \text{ in } V, \quad \sigma_{0n} \to \sigma_0 \text{ in } Q
\end{align*}
\]

as \( n \to \infty \). Then,

\[
\begin{align*}
u_n &\to u \text{ in } C(\mathbb{R}_+; V), \quad \sigma_n \to \sigma \text{ in } C(\mathbb{R}_+; Q)
\end{align*}
\]

as \( n \to \infty \).

**Proof.** Let \( n, m \in \mathbb{N}, t \in [0, m] \) and \( v \in K \). Then, using the statement of Problem \( \mathcal{P}_V \) it follows that

\[
\sigma_n(t) = \mathcal{E}\varepsilon(u_n(t)) + \int_0^t \mathcal{G}(\sigma_n(s), \varepsilon(u_n(s))) \, ds + \sigma_{0n} - \mathcal{E}\varepsilon(u_{0n}),
\]

\[
(\sigma_n(t), \varepsilon(v) - \varepsilon(u_n(t))) + \int_{\Gamma_3} j_0^b(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma \\
\geq (f_n(t), v - u(t))_V
\]

where

\[
(f_n(t), v)_V = \int_D f_{0n}(t) \cdot v \, dx + \int_{\Gamma_2} f_{2n}(t) \cdot v \, d\Gamma \quad \text{for all } v \in V, t \in \mathbb{R}_+.
\]

We use equality (67) and inequality (20) to deduce that

\[
\|\sigma_n(t) - \mathcal{E}\varepsilon(u_n(t)) - \int_0^t \mathcal{G}(\sigma_n(s), \varepsilon(u_n(s))) \, ds - \sigma_0 + \mathcal{E}\varepsilon(u_0)\|_Q \leq (70)
\]

\[
\leq \|\sigma_{0n} - \sigma_0\|_Q + L_\varepsilon \|u_{0n} - u_0\|_V.
\]
On the other hand, inequality (68) implies that
\[
(\sigma_n(t), \varepsilon(v) - \varepsilon(u_n(t)))_Q + \int_{\Gamma_3} j^{(0)}_\nu(u_{n\nu}(t); v_v - u_{n\nu}(t)) \, d\Gamma \\
+ (f(t) - f_n(t), v - u_n(t))_V \geq (f(t), v - u(t))_V,
\]
which, combined with (24), (69) and (10), yields
\[
(\sigma_n(t), \varepsilon(v) - \varepsilon(u_n(t)))_Q + \int_{\Gamma_3} j^{(0)}_\nu(u_{n\nu}(t); v_v - u_{n\nu}(t)) \, d\Gamma \\
+ \left(\|f_{0n}(t) - f_0(t)\|_{L_2(\nu)} + \|\gamma_1\| \|f_{2n}(t) - f_2(t)\|_{L_2(\Gamma_3)}\right) \|v - u_n(t)\|_V \\
\geq (f(t), v - u(t))_V.
\]
Consider now the sequence \( \theta = \{\theta^n\}_m \in S(I) \) defined by
\[
\theta^n_m = \max_{t \in [0, m]} \left\{ \|\sigma_{0n} - \sigma_0\|_Q + L_\varepsilon \|u_{0n} - u_0\|_V, \right. \\
\left. \max_{t \in [0, m]} \left\{ \|f_{0n}(t) - f_0(t)\|_{L_2(\nu)} + \|\gamma_1\| \|f_{2n}(t) - f_2(t)\|_{L_2(\Gamma_3)}\right\} \right\}.
\]
Then, (70)–(72) show that the inequalities (41) and (42) hold which implies that \( (u_n, \sigma_n) \in \Omega(\theta_n) \). On the other hand, notation (72) and assumptions (64), (65) show that \( \theta^n_m \to 0 \) as \( n \to \infty \). We conclude from Definition 3.2 that \( \{u_n, \sigma_n\}_m \) is an approximating sequence for Problem \( P_V \). Therefore, Theorem 3.4 and Definition 3.3 guarantee that the convergences (66) hold, which concludes the proof.

Example 4. Consider Problem \( P_V \) and the Tykhonov triple \( T \) in Example 2 in the particular case when \( \Gamma_3 = \emptyset \). Note that in this particular case \( K = V \) and, moreover, inequalities (28) and (58) become
\[
(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q = (f(t), v - u(t))_V,
\]
\[
(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + \theta \|v - u(t)\|_V \geq (f(t), v - u(t))_V
\]
for all \( v, t \in \mathbb{R}_+ \).

Let \( f_0 \in L^2(D)^d, f_2 \in L^2(\Gamma_2)^d \) and note that in this case the function \( f \) defined by (24) does not depend on \( t \). Moreover, assume that \( f_0 \neq 0_{L^2(D)} \) which implies that \( f \neq 0_V \) and, for each \( n \in \mathbb{N} \), consider the functions \( f_{0n} \) and \( f_{2n} \) defined by
\[
f_{0n}(t) = f_0 + \frac{t}{n} f_0, \quad f_{2n}(t) = f_2 + \frac{t}{n} f_2 \quad \forall t \in \mathbb{R}_+.
\]
Then, it is easy to see that conditions (62) and (64) are satisfied. Denote in what follows by \( (u_n, \sigma_n) \) the solution of Problem \( P_V \) for the data \( f_{0n}, f_{2n}, u_0, \sigma_0 \). Then, using (73) we deduce that
\[
(\sigma_n(t), \varepsilon(v) - \varepsilon(u_n(t)))_Q = (f_n(t), v - u(t))_V
\]
for each \( v \in V, t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \) where, recall, \( f_n \) is the function defined by (69).

We claim that the sequence \( \{\mathbf{u}_n, \sigma_n\}\) is not a \( T \)-approximating sequence for Problem \( PV \). Indeed, arguing by contradiction, assume that \( \{\mathbf{u}_n, \sigma_n\}\) is a \( T \)-approximating sequence. Then, using (74) we deduce that there exists a sequence \( \{\theta_n\} \subset \mathbb{R}_+^* \) such that \( \theta_n \to 0 \) and, for each \( n \in \mathbb{N} \), the couple \( (\mathbf{u}_n, \sigma_n) \) satisfies the inequality

\[
(\sigma_n(t), \varepsilon(v) - \varepsilon(\mathbf{u}_n(t)))_Q + \theta_n \|v - \mathbf{u}_n(t)\|_V \geq (f, v - \mathbf{u}_n(t))_V
\]

for each \( v \in V \) and \( t \in \mathbb{R}_+ \). Then using (76) and (77) yields

\[
(f - f_n(t), v)_V \leq \theta_n \|v\|_V
\]

for each \( v \in V, t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). On the other hand, using (24), (69) and (75) we find that

\[
(f - f_n(t), v)_V = -\frac{t}{n}(f, v)_V
\]

for each \( v \in V, t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). We now combine (78) and (79) then we test with \( v = -f \) in the resulting inequality to deduce that

\[
t \leq \frac{n \theta_n}{\|f\|_V}
\]

for all \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). We now fix \( n \), then we pass to the limit when \( t \to \infty \) and obtain a contradiction. We conclude from above that the sequence \( \{\mathbf{u}_n, \sigma_n\}\) is not a \( T \)-approximating sequence for Problem \( PV \) and, therefore, Theorem 3.5 cannot be used to obtain the convergences (66).

On the other hand, since condition (64) holds, it follows from Theorem 3.4 that the convergences (66) hold. We conclude from here that there exist sequences \( \{(\mathbf{u}_n, \sigma_n)\}\subset Y \) which are not \( T \)-approximating sequences but converge to the solution \( (\mathbf{u}, \sigma) \) of Problem \( P_V \). This shows that the choice of the Tykhonov triple plays a crucial role to deduce convergence results in the study of Problem \( P_V \).

5. Convergence results and mechanical interpretations. In this section we present two additional convergence results which represent a direct consequence of Theorem 3.5. To this end we start by introducing two additional contact problems. The first one describes the frictional contact of a viscoplastic material with a foundation made by a rigid body covered by a layer of elastic material. Its statement is as follows.

**Problem Q1.** Find a displacement field \( \mathbf{u} : D \times \mathbb{R}_+ \to \mathbb{R}^d \), a stress field \( \sigma : D \times \mathbb{R}_+ \to S^d \) and an interface function \( \xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R} \) such that

\[
\begin{align*}
\mathbf{u} &= \mathcal{E} \varepsilon(\mathbf{u}) + \mathcal{G}(\sigma, \varepsilon(\mathbf{u})) & \text{in } D \times \mathbb{R}_+, \\
\text{Div} \sigma + f_0 &= 0 & \text{in } D \times \mathbb{R}_+, \\
\mathbf{u} &= 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\
\sigma \nu &= f_2 & \text{on } \Gamma_2 \times \mathbb{R}_+, \\
u_\nu &\leq g, & \sigma_\nu + \xi_\nu &\leq 0, \\
(u_\nu - g)(\sigma_\nu + \xi_\nu) &= 0, & \xi_\nu &\in \partial f_\nu(u_\nu) & \text{on } \Gamma_3 \times \mathbb{R}_+
\end{align*}
\]
\[ \|\boldsymbol{\sigma}\| \leq F, \quad -\boldsymbol{\sigma} = F_0 \frac{\dot{\mathbf{u}}}{\|\dot{\mathbf{u}}\|} \quad \text{if } \dot{\mathbf{u}} \neq 0 \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (86) \]

\[ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } D. \quad (87) \]

Note that Problem \( P_1 \) is obtained from Problem \( P \) by replacing the frictionless condition (7) with the Coulomb’s law of dry friction (86). Here \( F \) represents a positive function, the friction bound, which can depend on the spatial variable and other process variables assumed to take values in a set \( Z \). Thus, \( F = F(\mathbf{x}, r) \) where \( \mathbf{x} \in \Gamma_3 \) and \( r \in Z \), where \( r \) denotes the corresponding process variables. For instance, a contact model in which \( F = F(\mathbf{x}, u_\nu, \|\mathbf{u}_\tau\|) \) was considered in [30] and in this case \( Z = \mathbb{R} \). A dependence of the form \( F = F(\mathbf{x}, u_\nu, \|\mathbf{u}_\tau\|) \) can also be considered and, in this case, \( Z = \mathbb{R} \times \mathbb{R}_+ \).

The second contact problem we consider in this section describes the frictionless contact with a rigid body covered by a layer of elastic-plastic material and can be formulated as follows.

\textbf{Problem} \( P_2 \). \textit{Find a displacement field} \( \mathbf{u}: D \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \), a stress field \( \boldsymbol{\sigma}: D \times \mathbb{R}_+ \rightarrow \mathbb{S}^d \) and two interface functions \( \eta_\nu: \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R}, \xi_\nu: \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R} \) such that

\[ \dot{\sigma} = \varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\sigma, \varepsilon(\mathbf{u})) \quad \text{in } D \times \mathbb{R}_+, \quad (88) \]

\[ \text{Div } \boldsymbol{\sigma} + f_0 = 0 \quad \text{in } D \times \mathbb{R}_+, \quad (89) \]

\[ \mathbf{u} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \quad (90) \]

\[ \sigma_\nu = f_2 \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \quad (91) \]

\[ u_\nu \leq g, \quad \sigma_\nu + \eta_\nu + \xi_\nu \leq 0, \quad (u_\nu - g)(\sigma_\nu + \eta_\nu + \xi_\nu) = 0, \quad (92) \]

\[ \eta_\nu = \begin{cases} 0 & \text{if } u_\nu < 0, \\ F & \text{if } u_\nu > 0, \end{cases} \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (92) \]

\[ \xi_\nu \in \partial j_\nu(u_\nu) \]

\[ \sigma_\tau = 0 \quad \text{on } \Gamma_3 \times \mathbb{R}_+, \quad (93) \]

\[ \sigma(0) = \sigma_0, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } D. \quad (94) \]

Note that Problem \( P_2 \) is obtained from Problem \( P \) by replacing the contact condition (6) by the contact condition (92). Here \( F \) represents a positive bound which can be interpreted as the yield limit of the elastic-plastic layer. As in the case of Problem \( P_1 \), we assume that \( F \) can depend on the spatial variable and other process variables. For instance, the case when \( F \) depends on the history of the penetration, defined by

\[ z(\mathbf{x}, t) = \int_0^t u_\nu(\mathbf{x}, s) \, ds \quad \forall t \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3, \]

was considered in [30]. There, more details and mechanical interpretations on the boundary contact conditions (86) and (92) can be found.

In the study of Problem \( P_1 \) and \( P_2 \) we assume that (13)–(19) hold and, in addition, we assume that there exists \( \omega > 0 \) such that

\[ 0 \leq F(\mathbf{x}, r) \leq \omega \quad \forall r \in Z, \text{ a.e. } \mathbf{x} \in \Gamma_3. \quad (95) \]

Under these assumptions we consider the following variational problem.
Problem $Q_V$. Find a displacement field $u \in C(\mathbb{R}_+, K)$ and a stress field $\sigma : C(\mathbb{R}_+, Q)$ such that

$$\sigma(t) = \mathcal{E}\varepsilon(u(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s))) \, ds + \sigma_0 - \mathcal{E}(u_0),$$

(96)

and

$$(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma$$

(97)

$$+ \omega \|\gamma_3\| \|v - u(t)\|_V \geq (f(t), v - u(t))_V$$

for all $v \in K$ and $t \in \mathbb{R}_+$.

We have the following result.

**Theorem 5.1.** Under assumptions (13)–(19) and (95), the following statements hold.

(a) If $(u, \sigma)$ is a regular solution to Problem $Q_1$ then $(u, \sigma)$ is a solution to Problem $Q_V$.

(b) If $(u, \sigma)$ is a regular solution to Problem $Q_2$ then $(u, \sigma)$ is a solution to Problem $Q_V$.

(c) Problem $Q_V$ has at least one solution, for each $\omega > 0$. Moreover, any sequence of solutions of Problem $Q_V$ converge to the solution of Problem $P_V$ as $\omega \to 0$.

**Proof.** (a) Assume that $(u, \sigma)$ is a regular solution to Problem $Q_1$ and let $v \in K$ and $t \in \mathbb{R}_+$. Then, using the bound $\|\sigma_f\| \leq F$ in (86) we have

$$\int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - u_\tau(t)) \, d\Gamma \geq - \int_{\Gamma_3} \|\sigma_f\| \|v_\tau - u_\tau(t)\| \, d\Gamma \geq - \int_{\Gamma_3} F \|v - u(t)\| \, d\Gamma.$$

We combine this inequality with assumption (95) and use the trace inequality (11) to deduce that

$$\int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - u_\tau(t)) \, d\Gamma \geq - \omega \|\gamma_3\| \|v - u(t)\|_V.$$

We now use this inequality, the Green formula and standard arguments to deduce that (97) holds. Finally, note that (96) follows by integrating the constitutive law (81) with the initial conditions (87).

(b) The proof of this part of the theorem is similar to the proof of part (a). The difference arises in the fact that now we use the inequality

$$\int_{\Gamma_3} \sigma_\nu \cdot (v_\nu - u_\nu(t)) \, d\Gamma \geq - \int_{\Gamma_3} F \|v - u(t)\| \, d\Gamma - \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) \, d\Gamma$$

for any $v \in K$ and $t \in \mathbb{R}_+$, which is a direct consequence of the boundary conditions (92).

(c) Let $(u, \sigma)$ be the solution to Problem $P_V$ obtained in Theorem 2.1. Then it is easy to see that $(u, \sigma)$ is a solution to Problem $Q_V$ for any $\omega > 0$, which proves the first part of the statement. In order to prove the second part we consider the Tykhonov triple $T = (I, \Omega, \mathcal{C})$ in Example 2 as well as a sequence $\{\omega_n\}_n \in S(\mathbb{R}_+)$ such that $\omega_n \to 0$ as $n \to \infty$. Let $\{(u_n, \sigma_n)\}_n$ be a sequence of couples such that $(u_n, \sigma_n)$ is a solution to Problem $Q_V$ with $\omega = \omega_n$, for each $n \in \mathbb{N}$, and let $\theta = \{\theta_n\}_n$ be the sequence defined by $\theta_n = \|\gamma_3\|\omega_n$, for each $n \in \mathbb{N}$. Then, using (96) and (97) it follows that (40)–(42) are satisfied and, therefore $(u_n, \sigma_n) \in \Omega(\theta_n)$, for each $n \in \mathbb{N}$. Now, since $\omega_n \to 0$ as $n \to \infty$ we deduce that $\theta_n \to 0$ as $n \to \infty$, for each $n \in \mathbb{N}$.
which implies that \( \{\theta_n\}_n \in \mathcal{C} \) and, therefore, the sequence \( \{(u_n, \sigma_n)\}_n \) is a \( \mathcal{T} \)-approximating sequence of Problem \( P_V \). Its convergence to the unique solution of Problem \( P_V \) is a direct consequence of Theorem 3.5 and Definition 3.3. \( \square \)

We end this section with the following comments and mechanical interpretation of Theorem 5.1.

(i) First, Theorem 5.1 (a), (b) allows us to consider Problem \( Q_V \) as a variational formulation of both Problems \( Q_1 \) and \( Q_2 \). Consequently, any solution of Problem \( Q_V \) can be considered as a weak solution of Problems \( Q_1 \) and \( Q_2 \). Nevertheless, this variational formulation is very weak, since it takes into account the boundary conditions of the corresponding contact models only partially. For instance, to obtain the variational-hemivariational inequality (97), in the case of Problem \( Q_1 \) it is enough to use the bound in (86). The rest of the conditions in the Coulomb’s law of dry friction (86) are not used and a similar comment can be made concerning Problem \( Q_2 \). This explain why the two contact problems give rise to the same variational formulation.

(ii) Theorem 5.1 (c) guarantees the weak solvability of both contact problems \( Q_1 \) and \( Q_2 \). Nevertheless, it does not guarantee their unique weak solvability. The reason is that the variational formulation considered, \( Q_V \), is very weak, as explained above.

(iii) The mechanical interpretation of the convergence result in Theorem 5.1 (c) is the following: any weak solution of the frictional contact problem \( Q_1 \) converges to the weak solution of the frictionless contact problem \( \mathcal{P} \) as the friction bound \( F_b \) converges to zero. So does any weak solution of the frictionless contact problem \( Q_2 \) as the yield limit \( F \) converges to zero. Using now the parts (a) and (b) of the theorem we deduce similar convergence results for the regular solutions of Problems \( Q_1 \) and \( Q_2 \), if such solutions exists. These results are interesting from the mechanical point of view since the nature of problem \( \mathcal{P} \), in one hand, and Problems \( Q_1 \) and \( Q_2 \), on the other hand, is different. This shows a stability property of the weak solution to Problem \( \mathcal{P} \), since it can be approached by the weak solutions of other contact problems, constructed with different interface laws.

We conclude from here the importance of the concept in the sense of Tykhonov in the study of mathematical models of contact. Indeed, besides the unique weak solvability and the continuous dependence of the weak solution with respect to the data, it provides a functional framework in which different convergence results can be established, illustrating in this way the link between different models of contact.

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