Anisotropic cosmic inflation with nonminimally coupled 
Einstein-Maxwell gravity

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Abstract

By using Bianchi I type of homogenous and anisotropic background metric having cylindrical symmetry in \( x \) direction of a local cartesian coordinate systems, we solve Einstein Maxwell metric field equations for three different cases. We assume in each case just one component of the four vector potential is non zero. This important condition reaches to some compatible metric solutions. There are obtained several exponentially inflationary expanding metric solutions. This encourages us to investigate their stability nature via dynamical system approach. We obtain unstable source nature in phase space for case \( A_t(t) \neq 0 \) while there are stable sink nature in phase space for \( A_x(t) \neq 0 \) with \( A_y = A_z \neq 0 \). For \( A_x(t) \neq 0 \) signature of the metric is Lorentzian \((-,+,+,+))\) while for \( A_y = A_z \neq 0 \) it is Euclidean \((-,-,-,-))\). Furthermore our mathematical calculations show that the anisotropy property plays critical behavior in expansion of the universe.

1 Introduction

Isotropy property of our universe on the large scales is known as one of fundamental assumption in the standard cosmological model. The well known Bianchi cosmological solutions [1, 2] of the Einstein metric equations are used usually to break this assumption. These models are investigated also via observational date from Plank probe. For instance, Saadeh et al [3] considered all degrees of freedom in the Bianchi solutions for the first time, to conduct a general test of isotropy using cosmic microwave background temperature and polarization data from Planck probe. By considering the vector mode associated with vorticity, they obtained a limit on the anisotropic

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expansion of the universe which is an order of magnitude tighter than previous Planck results that used cosmic microwave background temperature only. They also place upper limits on other modes of anisotropic expansion, with the weakest limit arising from the regular tensor mode. By including all degrees of freedom simultaneously for the first time they obtained a statement where anisotropic expansion of the Universe may be strongly disfavored. But from the point of view of the theoretical physics the anisotropy and other assumptions in the standard cosmology can be still considered as open problem. Applying the dynamical system approach the Aluri et al [4] studied a Bianchi I universe in presence of the anisotropic sources and obtained some stable critical points in the extended phase space. They also checked the obtained solutions with the observational date, correspondence between analytical solutions with numerical solutions, and the de Sitter phase. They obtained also that the CMB anisotropy maps due to shear are also generated in this scenario, assuming that the universe contains anisotropic matter along with the usual (dark) matter and vacuum (dark) energy since decoupling. Their solutions have also contributions dominantly to the CMB quadrupole and possible any cosmic preferred axis present in the data. We know now that the observational data from the Plank probe predicts an anisotropy axis close to the mirror symmetry axis seen in the cosmic microwave background. Sharif and Waheed [5] are considered the Brans Dicke scalar tensor gravity with self-interacting potential by using perfect, anisotropic and magnetized anisotropic fluids model to study a Bianchi I type cosmology. They assumed that the expansion scalar is proportional to the shear scalar and also take a power law ansatz for the scalar field and concluded that contrary to the universe model, the anisotropic fluid approaches isotropy at later times in all cases, which is consistent with observational data. Shamir [6] recently used Gauss-Bonnet topological invariant together in presence of the trace of the energy-momentum tensor as an alternative \( f(F,T) \) gravity to study anisotropic universe and so concluded that presence of term \( T \) in the bivariate function \( f(G,T) \) may give many cosmologically important solutions of the field equations. In general one can seek in the literature to obtain numerous gravity models where the anisotropy property of the universe is investigated to reach to great achievements (see for instance some of recent works as [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and references therein). Today, We understood this inevitable fact where the magnetic fields are present throughout the Universe and play an important role in a multitude of astrophysical situations. For instance the solar winds.
are effect on shape of the Earth and other planets magnetosphere in the solar system of our galaxy. Many other spiral galaxies are endowed with coherent magnetic fields. They are also affect on dynamics of the pulsars, white dwarfs and even black holes. Theoretically the Einstein-Maxwell gravity is well known to study cosmological systems where the both of electromagnetic and gravity have high intensity. In usual way the latter model is obtained which come from the 5 dimensional Kaluza-Klein gravity induced in 4 dimension (for a good review one can see [22]). According to the work [23] we apply an alternative non-minimal coupling of the Einstein-Maxwell gravity to study inflationary phase of a Bianchi I type of anisotropic cosmology via dynamical system approach. Non minimal coupling lagrangian counter terms in this model are made by contraction of the electromagnetic four vector potential and Recci tensor and Recci scalar. We should point that at the minimal coupling regime the Einstein-Maxwell gravity is gauge invariance while at the non-minimal coupling model under consideration this property is broken reaching to violate the charge conservation. Because these additional terms break conformal invariance property of the electromagnetic fields which is more important for amplification of the weak cosmic magnetic fields. Authors of the work [23] showed importance of these non-minimal coupling terms where an inflationary expansion of the isotropic and homogenous FRW cosmology can be produced just by including high intensity cosmic magnetic fields. We should point that the magnetic fields produced are uninterestingly small unless the conformal invariance of the electromagnetic fields is broken. However, due to the physical importance of this model both in particle physics and in large-scale physics, we analyze it in this paper for Bianchi I Cosmology. Layout of this work is as follows.

In section 2, we define the gravity model under consideration and calculate general form of the field equations. In section 3, we define general form of the background metric for the Bianchi I type of cosmology and generate metric field equations for this background. In section 4, we apply to obtain metric solutions and whose stabilizations for the Bianchi I space time via the dynamical system approach. Section 5 denotes to concluding remark.
2 The Model

Let us we start with the nonminimally coupled Einstein-Maxwell gravity \[23\]

\[
I = - \int dx^4 \sqrt{g} \left\{ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{\alpha}{2} R A^2 + \frac{\beta}{2} R_{\mu \nu} A^\mu A^\nu \right\} \tag{2.1}
\]

where \( F_{\mu \nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) is the Maxwell tensor field, \( A_\mu \) is four vector potential with norm \( A^2 = g_{\mu \nu} A^\mu A^\nu \). \( R \) and \( R_{\mu \nu} \) are the Recci scalar and the Recci tensor respectively. \( g \) is absolute value of determinant of the background metric \( g_{\mu \nu} \). \( \alpha \) and \( \beta \) are nonminimally coupling constants of the metric field \( g_{\mu \nu} \) and the vector gauge field \( A_\mu \). \( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \) is vacuum sector of the Maxwell field Lagrangian density while other two latter terms are nonminimally interacting parts. They come usually from QED in curved space times. They break explicitly the conformal invariance of \( U(1) \) group through the gravitational coupling. Varying (2.1) with respect to the four vector potential \( A^\mu \) we obtain equation of motion for the Maxwell tensor field as follows.

\[
\nabla_\mu F^{\mu \nu} - \alpha R A^\nu - \beta A^\mu R^{\mu \nu} = 0 \tag{2.2}
\]
in which

\[
\nabla_\mu F^{\mu \nu} = \frac{\partial_\mu (\sqrt{g} F^{\mu \nu})}{\sqrt{g}} \tag{2.3}
\]

and for antisymmetric Maxwell tensor field \( F_{\mu \nu} \) we can use the identities

\[
\partial_\mu F_{\lambda \kappa} + \partial_\kappa F_{\mu \lambda} + \partial_\lambda F_{\kappa \mu} = 0. \tag{2.4}
\]

Varying the action functional (2.1) with respect to the metric field \( g^{\mu \nu} \) we obtain the metric field equation such that

\[
\alpha A^2 G_{\mu \nu} = - T^{(EM)}_{\mu \nu} - \alpha T^{(a)}_{\mu \nu} - \frac{\beta}{2} T^{(b)}_{\mu \nu} \tag{2.5}
\]

where we defined

\[
T^{(EM)}_{\mu \nu} = F^{\lambda}_{\mu} F_{\nu \lambda} - \frac{1}{4} g_{\mu \nu} F^2, \quad F^2 = F^\epsilon \epsilon, T^{(a)}_{\mu \nu} = R A_\mu A_\nu + \frac{\nabla_\mu \nabla_\nu (\sqrt{g} A^2)}{\sqrt{g}} - g_{\mu \nu} \Box A^2, \quad \Box = \frac{\partial_\epsilon (g^\epsilon \epsilon \sqrt{g} \partial_\eta)}{\sqrt{g}} \tag{2.6}
\]

\[
T^{(b)}_{\mu \nu} = \frac{\nabla_\mu \nabla_\nu (\sqrt{g} A^2)}{\sqrt{g}} - g_{\mu \nu} \Box A^2. \tag{2.7}
\]
and
\[ T^{(\beta)}_{\mu\nu} = \frac{\nabla_\lambda \nabla_\nu (A^\lambda A_\mu \sqrt{g})}{\sqrt{g}} + \frac{\nabla_\lambda \nabla_\mu (A^\lambda A_\nu \sqrt{g})}{\sqrt{g}} - g_{\mu\nu} \frac{\nabla_\lambda \nabla_\eta (A^\lambda A^\eta \sqrt{g})}{\sqrt{g}} - g_{\mu\nu} R^a_{\lambda \eta} A^\lambda A^\eta - \Box(A_\mu A_\nu). \] (2.8)

Regarding the Lorentz gauge condition \( \nabla_\mu A^\mu = 0 \) and substituting the definition\( F_{\mu\nu} = \nabla_\mu A^\nu - \nabla_\nu A^\mu \) and identity \( (\nabla_a \nabla_b - \nabla_b \nabla_a) A^c = R^c_{\ abd} A^d \) for which \( \nabla_\mu \nabla^\nu A^\mu = \nabla^\nu \nabla_\mu A^\mu - R^\nu_{\ \lambda \mu} A^\lambda \) into the equation (2.2) one can show that the vector potential \( A_\mu \) satisfies at the wave equation
\[ \Box A^\nu = \alpha R A^\nu + (\beta - 1) A^\mu R^\mu_{\ \nu}. \] (2.9)

In the next section we want to study this model for particular anisotropic homogenous cosmological space time which is called as the Bianchi I model.

## 3 Bianchi I type cosmology

To redefine the covariant form of the Maxwell equations (2.2) and (2.4) versus the electric and magnetic vector fields \( \vec{E} \) and \( \vec{B} \) we should fix the background metric where we will use time-dependent homogenous and anisotropic Bianchi I model as follows.
\[ ds^2 = -dt^2 + e^{2a(t)} \{ e^{-4b(t)} dx^2 + e^{2b(t)} (dy^2 + dz^2) \} \] (3.1)

which has cylindrical symmetry in \( x \)-direction and all dynamical fields propagating in this metric are just time dependent as \( A_\mu(t), \vec{E}(t) \) and \( \vec{B}(t) \). Applying (3.1) one can infer the electromagnetic antisymmetric tensor field can be written as
\[
F_{\mu\nu} = \begin{pmatrix}
0 & -e^{a-b} E_x & -e^{a+b} E_y & -e^{a+b} E_z \\
e^{a-2b} E_x & 0 & e^{2a-b} B_z & -e^{2a-b} B_y \\
e^{a+b} E_y & -e^{a-2b} B_z & 0 & e^{2a+2b} B_x \\
e^{a+b} E_z & e^{2a-b} B_y & -e^{2a+2b} B_x & 0
\end{pmatrix} \] (3.2)

which is obtained by following steps. At first step we choose a local coordinate transformation
\[ t' = t, \quad x' = e^{a-2b} x, \quad y' = e^{a+b} y, \quad z' = e^{a+b} z \] (3.3)
in which the chart \((t', x', y', z')\) is characterized by a flat Minkowski space
time \(ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2\) for which

\[
F_{\mu'\nu'} = \begin{pmatrix}
0 & -E_{x'} & -E_{y'} & -E_{z'} \\
E_{x'} & 0 & B_{z'} & -B_{y'} \\
E_{y'} & -B_{z'} & 0 & B_{x'} \\
E_{z'} & B_{y'} & -B_{x'} & 0
\end{pmatrix}
\]

is well known but the chart \((t, x, y, z)\) corresponds to the background met-
ric \((3.1)\). At the second step we apply \((3.3)\) to transform \((3.4)\) as

\[
F_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} F_{\mu'\nu'}
\]

which reduces to the equation \((3.2)\) where \(E_{x,y,z}\) and \(B_{x,y,z}\) are
Cartesian components of the electric and magnetic vector fields re spectively.
Substituting \((3.1)\) the equation \((2.2)\) reads

\[
\ddot{A}_t + 3\dot{a}\dot{A}_t + [3(\beta - 4\alpha)\dot{a}^2 + 6(\beta - \alpha)\dot{b}^2 + 3(\beta - 2\alpha)\ddot{a}]A_t = 0, \tag{3.5}
\]

\[
\ddot{A}_x + (8\dot{b} - \dot{a})\dot{A}_x + [(4 + 2\beta)\dot{b} - (2 + 6\alpha + \beta)\dot{a} - (2 + 12\alpha + 3\beta)\dot{a}^2
+ (16 - 6\beta - 4\dot{a}\dot{b})A_x = 0, \tag{3.6}
\]

\[
\ddot{A}_y - (\dot{a} + 4\dot{b})\dot{A}_y + [(4 - 6\alpha)\dot{b}^2 - (2 + 12\alpha + 3\beta)\dot{a}^2 + (2 - 3\beta)\dot{a}\dot{b}
- (2 + \beta + 6\alpha)\ddot{a} - (2 + \beta)\ddot{b}]A_y = 0 \tag{3.7}
\]

and

\[
\ddot{A}_z - (\dot{a} + 4\dot{b})\dot{A}_z + [(4 - 6\alpha)\dot{b}^2 - (2 + 12\alpha + 3\beta)\dot{a}^2 + (2 - 3\beta)\dot{a}\dot{b}
- (2 + \beta + 6\alpha)\ddot{a} - (2 + \beta)\ddot{b}]A_z = 0. \tag{3.8}
\]

Applying the metric equation \((3.1)\) one can calculate simply non-vanishing
Einstein tensor components \(G_{\mu\nu}\) as

\[
G_{tt} = 3(\dot{b}^2 - \dot{a}^2), \tag{3.9}
\]

\[
G_{xx} = e^{2a - 4b}[3\dot{a}^2 + 6\dot{a}\dot{b} + 3\dot{b}^2 + 2\ddot{a} + 2\ddot{b}], \tag{3.10}
\]

\[
G_{yy} = G_{zz} = e^{2a + 2b}[3\dot{a}^2 - 3\dot{a}\dot{b} + 3\dot{b}^2 + 2\ddot{a} - \ddot{b}] \tag{3.11}
\]

and for the stress tensors \((2.6), (2.7)\) and \((2)\) respectively as follows.

\[
T^{(EM)}_{tt} = \frac{1}{2}\{e^{-2a + 4b}\dot{A}_x^2 + e^{-2a - 2b}(\dot{A}_y^2 + \dot{A}_z^2)\}, \tag{3.12}
\]

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\[ T_{xx}^{(EM)} = \frac{1}{2} \{-\dot{A}_x^2 + e^{-6\beta}(\dot{A}_y^2 + \dot{A}_z^2)\}, \quad (3.13) \]
\[ T_{yy}^{(EM)} = \frac{1}{2} \{e^{6\beta}\dot{A}_x^2 - \dot{A}_y^2 + \dot{A}_z^2\}, \quad (3.14) \]
\[ T_{zz}^{(EM)} = \frac{1}{2} \{e^{6\beta}\dot{A}_x^2 + \dot{A}_y^2 - \dot{A}_z^2\}, \quad (3.15) \]
\[ T_{tt}^{(\alpha)} = -(12\dot{a}^2 + 6\dot{b}^2 + 6\dot{a})A_t^2 + 3\dot{a}\ddot{\Sigma} + 3\dot{a}\Sigma, \quad (3.16) \]
\[ T_{xx}^{(\alpha)} = -(12\dot{a}^2 + 6\dot{b}^2 + 6\dot{a})A_x^2 + e^{2a-4b}[3\dot{a}\ddot{\Sigma} + 3\dot{a}\Sigma], \quad (3.17) \]
\[ T_{yy}^{(\alpha)} = -(12\dot{a}^2 + 6\dot{b}^2 + 6\dot{a})A_y^2 + e^{2a+2b}[3\dot{a}\ddot{\Sigma} + 3\dot{a}\Sigma], \quad (3.18) \]
\[ T_{zz}^{(\alpha)} = -(12\dot{a}^2 + 6\dot{b}^2 + 6\dot{a})A_z^2 + e^{2a+2b}[3\dot{a}\ddot{\Sigma} + 3\dot{a}\Sigma], \quad (3.19) \]

for which
\[ \Sigma(t) = A_{\mu}A^{\mu} = -A_t^2 + e^{-2a+4b}A_x^2 + e^{-2a-2b}(A_y^2 + A_z^2) \quad (3.20) \]

and
\[ T_{tt}^{(\beta)} = -2A_t\ddot{A}_t + 15\dot{a}A_t\dot{A}_t + (2\dot{a}^2 + 4\dot{b}^2 + 4\dot{a}\dot{b} - 6\dot{a})A_t^2 \]
\[ + e^{-2a+4b}[13\dot{a}^2 + 16\dot{b}^2 - 34\dot{a}\dot{b} + \ddot{a} - 2\dot{b}]A_x^2 \]
\[ + e^{-2a-2b}[13\dot{a}^2 + 2\dot{b}^2 + 17\dot{a}\dot{b} + \ddot{a} + \ddot{b}](A_y^2 + A_z^2), \quad (3.21) \]
\[ T_{xx}^{(\beta)} = (\ddot{a} - 2\dot{b} + 7\dot{a}^2 - 22\dot{a}\dot{b} + 16\dot{b}^2)A_x^2 + 6\dot{a}A_x\dot{A}_x \]
\[ - 2A_x\ddot{A}_x - 2A_x^2 - e^{2a-4b}((14\dot{a} + 8\dot{b})A_t\dot{A}_t + 2\dot{A}_t^2 \]
\[ + (22\dot{a}^2 + 32\dot{a}\dot{b} - 6\dot{b}^2)A_t^2 \} + e^{-6b}\{2(\ddot{a} + \ddot{b})(A_y\dot{A}_y + A_z\dot{A}_z) \]
\[ + (\ddot{a} + \ddot{b} + 7\dot{a}^2 + 5\dot{a}\dot{b} - 4\dot{b}^2)(A_y^2 + A_z^2)\}, \quad (3.22) \]
\[ T_{yy}^{(\beta)} = (\ddot{a} + \ddot{b} + 7\dot{a}^2 + 11\dot{a}\dot{b} + 6\dot{b}^2)A_y^2 - (\ddot{a} + \ddot{b} + 7\dot{a}^2 + 5\dot{a}\dot{b} - 4\dot{b}^2)A_z^2 \]
\[ + (10\dot{a} - 2\dot{b})A_y\dot{A}_y - 2(\ddot{a} + \ddot{b})A_x\dot{A}_x + 4\dot{A}_y^2 + 4A_y\ddot{A}_y \]
\[ + e^{2a+2b}\{(\ddot{a} + \ddot{b} + 8\dot{a}\dot{b} - 29\dot{a}^2 - \dot{b}^2)A_y^2 + (2\dot{b} - 16\dot{a})A_t\dot{A}_t - 2\dot{A}_t^2 \}
\[ - e^{6b}\{(\ddot{a} - 2\dot{b} + 7\dot{a}^2 - 10\dot{a}\dot{b} - 8\dot{b}^2)A_x^2 + 2(\ddot{a} - 2\dot{b})A_x\dot{A}_x\}, \quad (3.23) \]
\[ T_{zz}^{(\beta)} = (\ddot{a} + \ddot{b} + 7\dot{a}^2 + 11\dot{a}\dot{b} + 6\dot{b}^2)A_z^2 - (\ddot{a} + \ddot{b} + 7\dot{a}^2 + 5\dot{a}\dot{b} - 4\dot{b}^2)A_y^2 \]
\[ + (10\dot{a} - 2\dot{b})A_x\dot{A}_x - 2(\ddot{a} + \ddot{b})A_y\dot{A}_y + 4\dot{A}_x^2 + 4A_x\ddot{A}_x \]
\[ + e^{2a+2b}\{(\ddot{a} + \ddot{b} + 8\dot{a}\dot{b} - 29\dot{a}^2 - \dot{b}^2)A_x^2 + (2\dot{b} - 16\dot{a})A_t\dot{A}_t - 2\dot{A}_t^2 \} \]
We should point the non-diagonal components of the stress tensor (2.6) vanish trivially but not for $T^{(\alpha)}_{\mu \nu}$ and $T^{(\beta)}_{\mu \nu}$. Non-diagonal components for two latter stress tensors can be vanish just for three different assumptions which we will consider here as $A^\mu = (A^t(t), 0, 0, 0)$, $A^\mu = (0, A^z(t), 0, 0)$ or $A^\mu = (0, 0, A^\nu(t), 0)$. The choice $A^\mu = (0, 0, 0, A^z(t))$ is similar to $A^\mu = (0, 0, A^\nu(t), 0)$ because of the cylindrical symmetry of the space time in $x$ axis direction. Hence we solve dynamical field equations (2.2) and (2.5) separately for all three different kinds of the above assumptions as follows.

### 4 Metric solutions analysis

To solve the metric field equations we use dynamical system approach because they become nonlinear differential equations which possibly having chaotic solutions. To do so we substitute

$$X = \dot{a}, \quad Y = \dot{b}, \quad Z = \frac{\dot{A}_t}{A_t}, \quad U = \frac{\dot{A}_x}{A_x}, \quad V = \frac{\dot{A}_y}{A_y}$$

into the relations (3.5), (3.3), (3.3), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), (3.3), (3) and produce exact form of the equations (2.2) and (2.5) separately for all three case ($A_t \neq 0, A_{x,y,z} = 0$), ($A_x \neq 0, A_{t,y,z} = 0$) and ($A_y \neq 0, A_{t,x,z} = 0$) respectively as follows.

#### 4.1 The Case $A_t \neq 0, A_x = A_y = A_z = 0$

In this case the equation (3.5) reads

$$3(\beta - 2\alpha) \dot{X} + \dot{Z} = 3(4\alpha - \beta)X^2 + 6(\alpha - \beta)Y^2 - 3XZ - Z^2$$

and the equations (3.3), (3.3), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), (3.3), (3) and produce non vanishing components of the metric equation (2.5) become

$$6(3\alpha + \beta) \dot{X} + 2\beta \dot{Z} = 2(\beta - 15\alpha)X^2 + 2(2\beta - 3\alpha)Y^2 + 4\beta XY$$

$$+(15\beta - 12\alpha)XZ - 2\beta Z^2,$$

$$- \alpha \dot{X} + 2\alpha \dot{Y} = (11\beta - 3\alpha)X^2 + (16\beta - 6\alpha)XY - 3(\alpha + \beta)Y^2$$
\begin{align*}
(7\beta + 6\alpha)XZ + \beta Z^2 + 4\beta YZ
\end{align*}

and

\begin{align*}
(\beta - 2\alpha)(\dot{X} + \dot{Y}) &= 4(3\alpha + 4\beta)XZ - 2\beta YZ + 2(3\alpha - 4\beta)XY \\
+ 2\beta Z^2 + (29\beta - 6\alpha)X^2 + (\beta - 6\alpha)Y^2.
\end{align*}

One of simple solutions which one can check easily is logarithmic time dependent solutions for the functions \(X(t), Y(t)\) and \(Z(t)\). These logarithmic solutions reach to power-law time dependent forms for the scale factors \(e^{2a(t)}\) and \(e^{2b(t)}\) and the vector potential \(A_t(t)\). Their integral constants can be set by e-folding parameter which with observational date is estimated as \(N = \int_{t_i}^{t_f} \dot{X} dt > 100\) where \(i\) and \(f\) denote to the initial and finial time of the inflation. Observations predict that the Universe reaches to an inflationary phase after the possible Big Bang phase at \(t_i \approx 10^{-33}\text{sec}\) and stays for durations where the finishing time is approximately \(t_f = 10^{-32}\text{sec}\). What is theoretically important for our model is to find the stability conditions of the solutions near their critical points. To do so we should first obtain critical points of the above equations via dynamical system approach. We assume a three dimensional phase space defined by a vector field \(\vec{\xi}(t) = \{X(t), Y(t), Z(t)\}\) which is a constant vector field at the critical points. Thus one can infer that the critical points in the phase space are determined by solving the equations \(\vec{\xi} = 0\). Near the critical points one can obtain time evolutions of the vector field \(\vec{\xi}(t)\) by the linear equations \(\dot{\xi}_i = \sum_{j=1}^{N} J_{ij} \xi_j\) for \(N\) dimensional phase space where \(J_{ij} = \frac{\partial \dot{\xi}_i}{\partial \xi_j}|_{\vec{\xi}_c}\) is Jacobian matrix of the dynamical field equations at the critical points \(\vec{\xi}_c\). By solving the Jacobian secular equation one can obtain eigen values where negative (positive) real eigen values show stable (unstable) nature of the obtained solutions. If the obtained eigen values have imaginary part then the nature of the solutions near the critical points will be spiral stable (unstable) if their real part become negative (positive). Usually in the dynamical system approach stable (unstable) state called as sink (source) for absolutely real eigen values and spiral stable (unstable) for eigen values with complex valued (see [24] and references therein). Nevertheless we investigate now to obtain critical points of the field equations \((4.2), (4.1), (4.1)\) and \((4.1)\) as \((X_c, Y_c, Z_c)\) which are determined via \(\dot{X} = 0 = \dot{Y} = \dot{Z}\) as follows.

\[\alpha = -Z_c(6X_c^3 + 12X_c^2Y_c + 47X_c^2Z_c + 12X_cY_c^2 + 4X_cY_cZ_c)\]
\[
+9X_cZ_c^2 + 4Y_c^2Z_c - 2Z_c^3)/(11X_c^4 - 8X_c^3Y_c - 24X_c^2Z_c + 23X_cY_c^2 \\
+4X_c^2Z_c^2 - 4X_cY_c^3 - 3X_cY_c^2Z_c + 2Y_c^4 + 2Y_c^2Z_c^3)6 
\]

(4.6)

\[
\beta = -Z_c(15X_c^3 + X_c^2Z_c + 3X_cY_c^2 + 2X_cZ_c^2 + Y_c^2Z_c)/(11X_c^4 \\
-8X_c^3Y_c - 24X_c^2Z_c + 23X_c^2Y_c^2 + 4X_c^2Z_c^2 - 4X_cY_c^3 \\
-3X_cY_c^2Z_c + 2Y_c^4 + 2Y_c^2Z_c^2) 
\]

(4.7)

\[
Y_c = -4X_c(36717728X_c^8 + 156881576X_c^7Z_c + 201052172X_c^6Z_c^2 \\
+145258054X_c^5Z_c^3 + 55503731X_c^4Z_c^4 + 14371036X_c^3Z_c^5 + 1922364X_c^2Z_c^6 \\
+149920X_cZ_c^7 - 3904Z_c^8)/(617549744X_c^8 + 666183168X_c^7Z_c + 427921544X_c^6Z_c^2 \\
+16592284X_c^5Z_c^3 + 47818923X_c^4Z_c^4 + 12400394X_c^3Z_c^5 \\
+4813320X_c^2Z_c^6 + 1600864X_cZ_c^7 + 101504Z_c^8) 
\]

(4.8)

and for \((X_c, Z_c) \neq (0, 0)\) all possible values for \((X_c, Z_c)\) are obtained by the following equation.

\[
(3 + T)(1 + 2.8475T + 4.1278T^2 + 3.739T^3 + 2.2831T^4 + 0.95010T^5 \\
+0.25688T^6 + 0.038653T^7 + 0.0018811T^8) = 0 
\]

(4.9)

with real roots

\[
T_1 = -3, \quad T_2 = -2.021048944, \quad T_3 = -12.12347764 
\]

(4.10)

where we defined

\[
T = \frac{Z_c}{X_c}. 
\]

(4.11)

Substituting (4.11) into the equation (4.1) we obtain

\[
\frac{Y_c}{X_c} = 4K(T) 
\]

(4.12)

in which numerical values of the constant \(K(T)\) are obtained for (4.10) respectively as follows.

\[
K_1 = -1.0401, \quad K_2 = 0.65253, \quad K_3 = -3.0061 
\]

(4.13)
Substituting (4.12) and (4.13) into the relations (4.1) and (4.1) we obtain
\[
\alpha = \frac{2K^4 - 4K^2T^2 - 9K^3 - 4K^2T - 12KT^2 - 47K^2 - 12KT - 6K}{12K^2T^2 + 12T^4 - 18KT^2 - 24T^3 + 24K^2 + 138T^2 - 144K - 48T + 66}
\] (4.14)
and
\[
\beta = \frac{-K^2T^2 - 2K^3 - 3KT^2 - 11K^2 - 15K}{2K^2T^2 + 2T^4 - 3KT^2 - 4T^3 + 4K^2 + 23T^2 - 24K - 8T + 11}.
\] (4.15)

which for numerical valued (4.10) and (4.13) read respectively
\[
\alpha_1 = 0.0048, \quad \alpha_2 = -0.0451, \quad \alpha_3 = -0.000025
\] (4.16)
and
\[
\beta_1 = 0.0413, \quad \beta_2 = -0.14765, \quad \beta_3 = -0.000046
\] (4.17)

For these numerical values we plot arrow diagrams of the dynamical equations \(\dot{Z}\) and \(\dot{X}\) obtained from (4.2), (4.1), (4.1) and (4.1) on the \((Z, X)\) plane of the phase space in figure 1. In these diagrams we should fix an arbitrary numerical value for \(X_c\) and apply the obtained solutions (4.13), to fix numerical values for \(Y_c\) and \(Z_c\). For an accelerating expanding universe we must be choose positive values for \(X_c\) and so we set \(X_c = 1\) to plot the diagrams in figure 1. Looking to these diagrams one can infer that the case \(A_t \neq 0, A_{x,y,z} = 0\) shows a source (unstable) nature for these critical points regretfully and so we should investigate other choices as \(A_t = 0, A_{x,y,z} \neq 0\) in the subsequent parts. We will obtain some stable solutions which show an inflationary expanding anisotropic universe. Substituting (4.10) and (4.13) into the relations (4.11) and (4.12) we obtain three critical points by setting \(X_c = 1\) for this case as follows.

\[
C_1: (X_c^{(1)}, Y_c^{(1)}, Z_c^{(1)}) = (1, -4.16, -3)
\] (4.18)
\[
C_2: (X_c^{(2)}, Y_c^{(2)}, Z_c^{(2)}) = (1, 2.61, -2.02)
\] (4.19)
\[
C_3: (X_c^{(3)}, Y_c^{(3)}, Z_c^{(3)}) = (1, -12.02, -12.12).
\] (4.20)

We calculate Jacobi matrix for each of the above critical points as follows.

\[
J_1 = \begin{pmatrix}
-9.891929742 & -15.54567026 & 2.481118586 \\
-286.51058482 & 177.9204982 & 10.54264262 \\
9.808122519 & 2.769501583 & 2.764045622
\end{pmatrix}
\] (4.21)
with eigen values
\[ \lambda_1 \approx -32.34, \quad \lambda_2 \approx 3.85, \quad \lambda_3 \approx 199.29, \quad (4.22) \]

\[ J_2 = \begin{pmatrix}
-3.251751662 & 6.820857322 & 1.463718051 \\
62.70799335 & -6.863019235 & 10.57713618 \\
5.303060602 & 4.070863669 & 1.292271806 \\
\end{pmatrix} \quad (4.23) \]

with eigen values
\[ \lambda_1 \approx -26.15, \quad \lambda_2 \approx -0.89, \quad \lambda_3 \approx 18.22 \quad (4.24) \]

and
\[ J_3 = \begin{pmatrix}
-15.97739370 & -19.15122988 & 0.9173567669 \\
-246.0358968 & 104.5547850 & -10.40212162 \\
36.35986773 & -0.002849292095 & 21.23998899 \end{pmatrix} \quad (4.25) \]

with eigen values
\[ \lambda_1 \approx -46.88, \quad \lambda_2 \approx 20.67, \quad \lambda_3 \approx 136.03 \quad (4.26) \]

where the eigen values are obtained from the secular equation \( \text{det}\{J_{ij} - \delta_{ij}\lambda\} = 0 \) of each Jacobi matrix. In general, stable nature of the system is just for case where all of eigen values have negative valued. Hence one can infer unstable or quasi stable nature of the system by comparing these obtained eigen values with diagrams at figure 1. We obtain more physical

Figure 1: Diagrams of the critical points for \( A_t \neq 0, A_{x,y,z} = 0 \)

statement for the obtained solutions if we investigate more about the source of this solutions. Here we assume the stress tensor which supports this inflation
given by right side of the equation (2.5) treats as anisotropic perfect fluid meaning that they have no viscosity or heat flow. In this case we write
\[
\left\{ T_\nu^{(EM)} + \alpha T_\nu^{(\alpha)} + \frac{\beta}{2} T_\nu^{(\beta)} \right\} / \alpha A^2 = \text{diag}\{ \rho, -p_x, -p_y, -p_z \} \tag{4.27}
\]
where \( \rho \) is the mass density and \( p_{x,y,z} \) are the directional pressures for metric signature \((-++,++)\). Substituting (4.1), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), (3), (3), (3), into the identity (4.27) we obtain
\[
\rho = \left( \frac{\beta}{\alpha} - 12 \right) X^2 + \left( \frac{2\beta}{\alpha} - 6 \right) Y^2 + 6Z + \frac{15\beta}{2\alpha} XZ + \frac{2\beta}{\alpha} XY - \frac{\beta}{\alpha} Z - 3 \left( 1 + \frac{\beta}{\alpha} \right) \dot{X},
\]
\[
p_x = \frac{11\beta}{\alpha} X^2 - \frac{3\beta}{\alpha} Y^2 - \frac{\beta}{\alpha} Z^2 + \frac{16\beta}{\alpha} XY + \frac{4\beta}{\alpha} YZ + \left( \frac{7\beta}{\alpha} - 6 \right) XZ - 3\dot{X},
\]
\[
p_y = p_z = -\frac{29\beta}{2\alpha} X^2 - \frac{\beta}{2\alpha} Y^2 - \frac{\beta}{2\alpha} Z^2 + \frac{4\beta}{\alpha} XY + \frac{\beta}{\alpha} YZ - 2 \left( 3 + \frac{4\beta}{\alpha} \right) XZ
\]
\[+ \left( \frac{\beta}{2\alpha} - 3 \right) \dot{X} + \frac{\beta}{2\alpha} \dot{Y}, \tag{4.30}
\]
which at the critical points (4.18), (4.19) and (4.20) read
\[
\rho(t) = (92.60931666, -83.80340642, -629.5641280) \tag{4.31}
\]
\[
p_x = (-580.4783001, -119.3804948, 127.9619520) \tag{4.32}
\]
and
\[
p_y = p_z = (-212.7017999, -71.34399144, -25.8528480) \tag{4.33}
\]
where we substitute (4.16) and (4.17). This stress tensor dose not describe physical system because the mass density has negative valued for these critical points. Usually such this un-physical matters are called as tachyon. However we does not consider this solution to be physical and so we are encouraged to explore other possibilities of the solutions by considering \( A_t = 0 \) and \( A_{x,y,z} \neq 0 \) in the subsequent sections.
4.2 The case $A_x \neq 0, A_t = A_y = A_z = 0$

In this case the Maxwell equations (3.5), (3), (3) vanish trivially and a non vanishing equation is just (3) which by substituting (4.1) become

$$- (2 + 6\alpha + \beta)\dot{X} + (4 + 2\beta)\dot{Y} + \ddot{U} = -U^2 + XU$$

$$-8UY + (2 + 12\alpha + 3\beta)X^2 + (6\alpha - 16)Y^2 + (4 - 6\beta)XY$$

(4.34)

and for the Einstein field equation (2.5) we obtain

$$(6\alpha + \beta)\dot{X} - 2\beta\dot{Y} = -U^2 - 12\alpha UX + 2(17\beta - 12\alpha)XY$$

$$+ [6\alpha(2 - e^{2a-4b}) - 13\beta]X^2$$

(4.35)

for $G_{tt}$ component

$$[2\alpha(2e^{2a-4b} - 3) + \beta]\dot{X} + 2[2\alpha e^{2a-4b} - \beta]\dot{Y} - 2\beta\ddot{U} =$$

$$[6\alpha(6 - e^{2a-4b}) - 7\beta]X^2 + [22\beta - 12\alpha(2 + e^{2a-4b})]XY$$

$$+ [6\alpha(2 - e^{2a-4b}) - 16\beta]Y^2 + (1 + 4\beta)\dot{U}^2 - 6(\beta + 2\alpha)\dot{X}U$$

(4.36)

for $G_{xx}$ component and

$$[\alpha(2e^{2a-4b} + 3) - \beta]\dot{X} + [2\beta - \alpha e^{2a-4b}]\dot{Y} =$$

$$[7\beta + 3\alpha(2 + e^{2a-4b})]X^2 + [3\alpha e^{2a-4b} - 8\beta]Y^2 + (2\beta - 6\alpha)\dot{X}U$$

$$- 4\beta\dot{Y}U - [10\beta + 3\alpha(4 + e^{2a-4b})]XY - \frac{\dot{U}^2}{2}$$

(4.37)

for $G_{yy}$ and $G_{zz}$ components. To obtain critical points $(X_c, Y_c, U_c)$, we solve $\dot{X} = 0 = \dot{Y} = \ddot{U}$ to reach

$$\alpha = (1/6)(544H^5 - 1144H^4K + 616H^3K^2 + 34H^2K^3 - 58HK^4 - 2K^5 + 272H^4$$

$$- 666H^3K + 375HK^2 - 19HK^3 - 9K^4 + 40H^3 - 136H^2K + 82HK^2 - 9K^3 - 8HK$$

$$+ 4K^2)/(34H^5 - 87H^4K + 143H^3K^2 - 203H^2K^3 + 108HK^4 - 10K^5 - 20H^3K$$

$$+ 22H^2K^2 - 46HK^3 + 20K^4)$$

(4.38)

$$\beta = (64H^5 - 112H^4K^2 + 176H^3K^3 - 92H^2K^4 - 4HK^5 + 8K^6 + 64H^4K - 92H^3K^2$$

$$+ 154HK^3 - 54HK^4 - 4K^5 + H^4 + 19H^3K - 20HK^2 + 42HK^3 - 4K^4 + 2H^2K$$

$$- 70HK^2 - 8K^3 - 18HK + 18K + 2HK)/(34H^5 - 87H^4K + 143H^3K^2 - 203H^2K^3 + 108HK^4 - 10K^5 - 20H^3K$$

$$+ 22H^2K^2 - 46HK^3 + 20K^4)$$
\[ e^{2a_c - 4b_c} = (1024H^5K + 2688H^4K^2 - 4320H^3K^3 + 1360H^2K^4 + 216HK^5 - 108K^6 + 1536H^4K + 2368H^3K^2 - 3080H^2K^3 + 680HK^4 + 38K^5 + 16H^4 + 886H^3K + 541H^2K^2 - 552HK^3 + 62K^4 + 8H^3 + 188H^2K + 12HK^2 - 46K^3 + 16HK - 8K^2)/K(544H^5 - 1144H^4K + 616H^3K^2 + 34H^2K^3 - 58HK^4 - 2K^5 + 272H^4 - 666H^3K + 375H^2K^2 - 19HK^3 - 9K^4 + 40H^3 - 136H^2K + 82HK^2 - 9K^3 - 8HK + 4K^2) \]  

where we defined

\[ K = \frac{X_c}{U_c}, \quad H = \frac{Y_c}{U_c} \]  

in which \((H, K)\) satisfy the identity \(\Omega(H, K)\Pi(H, K) = 0\). In the latter equation \(\Omega\) and \(\Pi\) have long length and they are given in the appendix. To obtain all possible real values for \(K\) and \(H\) we plot diagram \(\Omega(H, K)\Pi(H, K) = 0\) in figure 2. When we solve the equation \(\Omega(H, K)\Pi(H, K) = 0\) versus \(K\) we can obtain parametric solutions which one of them is real as follows.

\[ H(K) = \frac{43K}{204} - \frac{1}{6} + \frac{\Upsilon}{408} + \frac{408}{\Upsilon^3} \left( \frac{2257}{41616}K^2 - \frac{35K}{2448} - \frac{1}{306} \right) \]  

where we defined

\[ \Upsilon(K) = 101728 + 229704K - 670956K^2 - 776440K^3 + 612\times \{27200 + 119136K - 2696976K^2 - 1467992K^3 - 515163K^4 + 1228080K^5 - 355008K^6\}^{1/2} \]  

and its diagram is given in figure 2. Substituting (4.42) into the relations (4.12), (4.12) and (4.12) we plot their diagrams versus \(K\) in figure 2. They show that we can obtain more critical points by fixing the \(H\) and \(K\) parameters. Hence we study stability nature of the solutions for particular critical point obtained from \((\Pi, \Omega) = (0, 0)\) as

\[ H = 0.01779112114, \quad K = -1.139414992 \]  

(4.44)
which satisfy the relation (4.42) and so trivially $\Omega(K, H)\Pi(K, H) = 0$. Substituting these solutions the equations (4.2), (4.2) and (4.2) read

$$\alpha = 0.02375564657, \quad \beta = -0.1490821715, \quad e^{2ac-4b} = 12.56623097. \quad (4.45)$$

Applying the above obtained numerical values for the critical parameters we investigate stability nature of the system just by plotting arrow diagrams of the dynamical equations $\dot{X}$ and $\dot{Y}$ on the $XY$ plane of the phase space. They are given in figure 2 showing nature of stable sink (middle) and spiral sink (left side) and unstable (right side) for the cosmic system. Choosing other initial conditions from the line diagrams in figure 2 for the critical points one can check stabilization of our cosmic system in other points in phase space. Usually in the dynamical system approach we can investigate stability situations of the solutions of the dynamical equations via two method as analytically by determining sign of the eigen values of the secular equation of the Jacobi matrix or by plotting arrow diagrams. Full stable state of the system is happened when we obtain negative real valued for all eigen values in N dimensional phase space. If at least one of them takes positive value then the system will be quasi stable. For complex valued eigen values stable state is happened when their real part to be negative. In the latter case we call usually spiral sink. In the previous section we study both of analytical approach via calculation of the eigen values of the Jacobi matrix and plotting arrow diagrams of the dynamical field equations but in this and next sections we investigate stabilization of our chaotic system just by plotting the arrow diagram. Usually in the stable nature directions of the arrows in the diagram show stable attraction point. For instance in the figure 2, the attraction point in the arrow diagram is $(X_a \approx -12.5, Y_a \approx -1.1)$ for top-middle and $(X_a \approx -1.25, Y_a \approx 0)$ for top-left. For top-right in figure 2, the arrow diagram does not show an attractor point in phase space. Top-middle diagram is plotted for $U_c = -1$ for which $X_c = KU_c = 1.139414992$ takes positive value showing an exponentially accelerating expanding universe. Applying this result one can integrate easily (4.1) to obtain $a(t) = X_c t$ while top-left is plotted for $U_c = 1$ for which $X_c = -1.139414992$ describing an exponentially accelerating collapsing universe. One important statement which one can obtain from the top-left arrow diagram in figure 2 is this: Vanishing anisotropy property of the space time for which the attractor point has $Y_a = 0$ dose not describe an expanding universe but collapsing one. Thus we can result anisotropy property of the universe may to be an important factor in production of expansion in dynamics of the universe. This is a
A promising result for our solutions where a non-minimal interacting Einstein-Maxwell gravity can produce an anisotropic exponentially inflationary stable universe without to use the additional cosmological constant or dark sector of the cosmic matter. This can be checked easily by studying behavior of the barotropic index of the source used in this section as follows. To study beh-

Figure 2: Diagrams of the critical points for case $A_x \neq 0, A_{t,y,z} = 0$.

avior of the stress tensor near the critical points (4.14) we substitute (1.1), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), (3), (3), (3), (3), into the
identity (4.27) and we obtain

\[
\rho = 6XU + \left(12 + \frac{17\beta}{\alpha}\right)XY - \left(6 + \frac{13\beta}{2\alpha}\right)X^2 - \frac{U^2}{2\alpha} - \frac{8\beta}{\alpha}Y^2 + \left(3 - \frac{\beta}{2\alpha}\right)\dot{X} + \frac{\beta}{\alpha}\dot{Y}
\]  

(4.46)

\[
p_x = \left(6 - \frac{7\beta}{2\alpha}\right)X^2 + \left(\frac{1}{2\alpha} + \frac{2\beta}{\alpha}\right)U^2 - \frac{8\beta}{\alpha}Y^2 - \left(6 + \frac{3\beta}{\alpha}\right)XY
\]

\[+ \left(\frac{11\beta}{\alpha} - 12\right)XY - \frac{\beta}{2\alpha} \dot{X} + \frac{\beta}{\alpha} \dot{Y} + \frac{\beta}{\alpha} \dot{U}
\]  

(4.47)

\[
p_y = p_z = \left(6 - \frac{7\beta}{2\alpha}\right)X^2 - \left(6 + \frac{3\beta}{\alpha}\right)XU - \left(12 - \frac{5\beta}{\alpha}\right)XY + \frac{4\beta}{\alpha}Y^2
\]

\[+ \frac{2\beta}{\alpha} YU - \frac{U^2}{\alpha} + \frac{\beta}{\alpha} \dot{Y} - \left(3 + \frac{\beta}{2\alpha}\right) \dot{X},
\]  

(4.48)

which at the critical point

\[C : (X_c, Y_c, U_c) = \{1.1394414992, -0.01779112114, -1\}
\]  

(4.49)

obtained from (4.44) for \(U_c = -1\) read

\[\{\rho, p_x, p_y = p_z\} = (10.30695919, 36.12576551, 0.30639358)
\]  

(4.50)

This stress tensor can be featured by massive real particles of the perfect fluid because of positive valued mass density for which one can obtain directional barotropic indexes as

\[
\gamma_x \approx \frac{p_x}{\rho} = 3.5, \quad \gamma_y = \frac{p_y}{\rho} \cong 0.03
\]  

(4.51)

which means this anisotropic perfect fluid behaves as supper-sonic fluid in \(x\)-direction but not in the \(y\)-direction. What is physically important is the positive value of the barotropic indexes where the exponentially expansion of the anisotropic universe is supported by cosmic EM visible field coming from the Big Bang remains instead of unknown dark matter/energy for which anisotropy property plays a critical role in the expansion.
4.3 The case \( A_y \neq 0, \quad A_t = A_x = A_z = 0 \)

In this case we substitute (4.1) into the equation (3) to obtain

\[
\dot{\mathbf{V}} - (2 + \beta + 6\alpha)\dot{X} - (2 + \beta)\dot{Y} = XV + 4YV + (3\beta - 2)XY + (6\alpha - 4)Y^2 \\
+(2 + 12\alpha + 3\beta)X^2 - V^2 \\
\]

and so the Einstein equation (2.5) reads

\[
(\beta + 6\alpha)\dot{X} + \beta\dot{Y} = [6\alpha(2 + e^{2a+2b}) - 13\beta]X^2 - V^2 - 12\alpha XV \\
+(12\alpha - 17\beta)XY - 2(\beta + 3\alpha e^{2a+2b})Y^2 \\
\]

for \( G_{tt} \) component,

\[
[\beta + 2\alpha(3 + 2e^{2a+2b})]\dot{X} + [\beta + 4\alpha e^{2a+2b}]\dot{Y} = [6\alpha(2 - e^{2a+2b}) - 7\beta]X^2 \\
+2[2\beta - 3\alpha e^{2a+2b}]Y^2 + [12\alpha(1 - e^{2a+2b}) - 5\beta]XY - 2(\beta + 6\alpha)XV \\
-2\beta YV - V^2 \\
\]

for \( G_{xx} \) component,

\[
[\beta + 6\alpha(1 + e^{2a+2b})]\dot{X} + [\beta - 2\alpha e^{2a+2b}]\dot{Y} + 4\beta\dot{V} = [6\alpha(6 + e^{2a+2b}) - 7\beta]X^2 \\
+[6\alpha(2 + e^{2a+2b}) - 6\beta]Y^2 - (1 + 8\beta)V^2 + [6\alpha(2 - e^{2a+2b}) - 11\beta]XY \\
-(12\alpha + 10\beta)XV + 2\beta YV \\
\]

for \( G_{yy} \) component, and

\[
[6\alpha(e^{2a+2b} + 1) - \beta]\dot{X} - [2\alpha e^{2a+2b} + \beta]\dot{Y} = [7\beta + 6\alpha(2 - e^{2a+2b})]X^2 \\
-2[2\beta + 3\alpha e^{2a+2b}]Y^2 + [5\beta + 6\alpha(2 + e^{2a+2b})]XY + 2(\beta - 6\alpha)XV + 2\beta YV - V^2. \\
\]

for \( G_{zz} \) component. To solve the above dynamical equations near the critical points \((X_c, Y_c, V_c)\) we should first solve the equations \( \dot{X} = 0 = \dot{Y} = \dot{V} \) to obtain

\[
e^{2a_c+2b_c} = 2(32J^5S + 8J^4S^2) - 116J^3S^3 - 100J^2S^4 + 20JS^5 + 28S^6 \\
-80J^4S - 16J^3S^2 + 166J^2S^3 + 96JS^4 - 6S^5 - 4J^4 + 77J^3S + 25J^2S^2 \\
-56JS^3 - 18S^4 + 2J^3 - 26J^2S - 14JS^2 + 10S^3 - 4JS \\
\]

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\[
\frac{+4S^2}{(32J^6 - 8J^5S - 72J^4S^2 - 78J^3S^3 - 68J^2S^4 + 6JS^5 + 28S^6 - 48J^5}
- 16J^4S + 22J^3S^2 + 63J^2S^3 + 77JS^4 + 22S^5 + 24J^4 + 22J^3S + 28J^2S^2
+ 14JS^3 - 10S^4 - 4J^3 - 6J^2S - 6JS^2 - 4S^3),
\]
\[
\alpha = (1/6)(32J^6 - 8J^5S - 72J^4S^2 - 78J^3S^3 - 68J^2S^4 + 6JS^5 + 28S^6 - 48J^5
- 16J^4S + 22J^3S^2 + 63J^2S^3 + 77JS^4 + 22S^5 + 24J^4 + 22J^3S + 28J^2S^2 + 14JS^3
- 10S^4 - 4J^3 - 6J^2S - 6JS^2 - 4S^3)/(8J^6 - 6J^5S + 5J^4S^2 - 11J^3S^3 - 16JS^5 - 28S^6
- 4J^5 - 6J^4S - 14J^3S^2 - 34J^2S^3 - 30JS^4 - 8S^5)
\]
\[
\beta = 3SJ(8J^3S + 12J^2S^2 - 4S^4 - 16J^2S - 14JS^2 + 2S^3 - J^2 + 10JS + 2S^2
- 2S)/(8J^6 - 6J^5S + 5J^4S^2 - 11J^3S^3 - 16JS^5 - 28S^6 - 4J^5 - 6J^4S
- 14J^3S^2 - 34J^2S^3 - 30JS^4 - 8S^5)
\]
where we defined
\[
S = \frac{X_e}{V_e}, \quad J = \frac{Y_e}{V_e}
\]
in which the parameters \(S, J\) are obtained by the following equations.
\[
(S + J)(8J^3S + 12J^2S^2 - 4S^4 - 16J^2S - 14JS^2 + 2S^3 - J^2 + 10JS + 2S^2 - 2S)
\times (16J^4 - 12J^3S - 46J^2S^2 - 4JS^3 + 14S^4 - 24J^3 + 4J^2S + 33JS^2 + 11S^3
+ 12J^2 + 8JS - 2S^2 - 2J - 2S)(J^2 + 19JS + 14S^2 + J + 4S) = 0
\]
and
\[
(16J^4 - 12J^3S - 46J^2S^2 - 4JS^3 + 14S^4 - 24J^3 + 4J^2S + 33JS^2 + 11S^3 + 12J^2 + 8JS
- 2S^2 - 2J - 2S)(32J^8 + 56J^7S - 80J^6S^2 - 506J^5S^3 - 620J^4S^4 - 382J^3S^5 - 196J^2S^6
+ 80JS^7 + 112S^8 - 48J^7 - 176J^6S + 102J^5S^2 + 517J^4S^3 + 399J^3S^4 + 522J^2S^5
+ 388JS^6 + 32S^7 + 16J^6 + 187JS^5 - 138J^4S^2 - 31J^3S^3 + 166J^2S^4 - 78JS^5 - 56S^6
- 62J^4S + 230J^3S^2 + 16J^2S^3 - 78JS^4 + 125S^5 + 20J^3S - 120J^2S^2
+ 12JS^3 + 8S^4 + 24JS^2) = 0
\]
Solving the equations (4.3) and (4.1) as synchronously we obtain
\[
\{J = 0.3333333333, \quad S = -0.3333333333\}
\]
and substituting these numerical values into the relations (4.3), (4.3) and (4.3) we obtain

\[ e^{2a \epsilon + 2b \epsilon} = -1.481481489 \times 10^{10}, \quad \alpha = 6.750000011, \quad \beta = 4.500000021. \]

We plot the arrow diagrams of the equations \( \dot{X} \) and \( \dot{Y} \) on the XY plane in the phase space for \( \dot{V}_c = -1 \) in figure 3. It shows a linear attractor sink (stable) for \( (X, Y) > 0 \) while for \( (X, Y) < 0 \) it behaves as linear source (unstable) nature of the system. Physically this means stable exponentially accelerating expanding anisotropic universe. Here we set \( \dot{V}_c = -1 \) to obtain an exponentially expanding scale factor as \( a(t) = X_c t = SV_c t = 0.33 t \) for the model under consideration. Other statement which one can infer from (4.64) is change of the metric signature in the case \( A_y = A_z \neq 0 \) with respect to the cases \( A_x \neq 0, A_t \neq 0 \). In the latter case signature of the metric become Euclidian \((-,-,-,-)\) while in the two former cases is Lorentzian \((-,+,+,-)\). A fixed value for \( \dot{V}_c \) is important to determine e-folding param-
eter of the cosmic inflation defined by

\[ N = 2 \int_{t_b}^{t_f} \dot{a}(t) dt = -0.66V_c(t_f - t_b) \] (4.65)

in which \( t_b \) and \( t_f \) are the beginning time of the inflation and the finishing time of it. Observations predict that the universe reaches to an inflationary phase after the possible Big Bang phase at \( t_b \approx 10^{-33} \text{sec} \) and stays for durations where the finishing time is approximately \( t_f = 10^{-32} \text{sec} \). In these times the e-folding number should be more than \( N > 100 \). For these experimental values the equation (4.65) reads

\[ 0 > V_c > -1.68 \times 10^{34}. \] (4.66)

Substituting (4.1), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), (3), (3), (3), into the identity (4.27) we obtain

\[
\rho = \left(6 - \frac{13\beta}{2\alpha}\right)X^2 - \frac{2\beta}{\alpha}Y^2 - \frac{V^2}{2\alpha} + \left(6 - \frac{17\beta}{2\alpha}\right)XY - 6XV - \left(3 + \frac{\beta}{2\alpha}\right)\dot{X} - \frac{\beta}{2\alpha}\dot{Y} \tag{4.67}
\]

\[
p_x = \left(6 - \frac{7\beta}{2\alpha}\right)X^2 + \frac{2\beta}{\alpha}Y^2 - \frac{V^2}{2\alpha} + \left(6 - \frac{5\beta}{2\alpha}\right)XY - \left(6 + \frac{\beta}{\alpha}\right)XV - \frac{\beta}{2\alpha}YV - \left(3 + \frac{\beta}{2\alpha}\right)\dot{X} - \frac{\beta}{2\alpha}\dot{Y} \tag{4.68}
\]

\[
p_y = \left(\frac{1}{2\alpha} - \frac{4\beta}{\alpha}\right)V^2 + \left(6 - \frac{\beta}{\alpha}\right)Y^2 + \left(18 - \frac{7\beta}{2\alpha}\right)X^2 + \left(6 - \frac{11\beta}{2\alpha}\right)XY - \left(6 + \frac{5\beta}{2\alpha}\right)XV + \frac{\beta}{\alpha}YV - \frac{\beta}{2\alpha}X - \frac{\beta}{2\alpha}Y - \frac{2\beta}{\alpha}V, \tag{4.69}
\]

and

\[
p_z = -\frac{V^2}{\alpha} + \left(6 + \frac{7\beta}{2\alpha}\right)X^2 - \frac{2\beta}{\alpha}Y^2 + \left(\frac{\beta}{\alpha} - 6\right)XV + \left(6 + \frac{5\beta}{2\alpha}\right)XY + \frac{\beta}{\alpha}YV + \frac{\beta}{2\alpha}\dot{Y} + \left(\frac{\beta}{2\alpha} - 3\right)\dot{X} \tag{4.70}
\]

which by substituting the critical point

\[(X_c, Y_c, V_c) = \{0.3333333333, \ -0.3333333333, \ -1\}\] (4.71)
obtained from (4.63) for \( V_c = -1 \) read

\[
\rho(t) = (2, 2, 2.814814810, 1.777777778) \tag{4.72}
\]

with directional barotropic indexes

\[
\gamma_x = 1, \quad \gamma_y = 1.407407405, \quad \gamma_z = 0.8888888890 \tag{4.73}
\]

which in \( x \) and \( y \) directions treats as super-sonic fluid but not in the \( z \) direction. This case is also similar to the previous case shows that the matter source behaves as visible baryonic matter which support the anisotropic exponentially expansion of the Bianchi I model defined by the modified Einstein Maxwell gravity under consideration.

## 5 Concluding Remarks

In this work we used non-minimally coupling Einstein-Maxwell gravity to study anisotropic Bianchi I cosmology with a cylindrical symmetry property. To do so we solved metric field equations for three situations for each of these cases just one component of the electromagnetic four potential is non zero. When the time component of this potential is just non-zero we have not a physical metric solution because matter density has negative value. When the vector potential is alignment in direction of the cylindrical symmetry of the space time namely \( x \) direction there is obtained an exponentially accelerating expanding scale factor of the space time with anisotropy and stable nature. In this case matter density of the system takes positive values describing an visible baryonic matter. In this case the signature of the space time is still Lorentzian. At last we solved dynamical field equations for case where the vector potential is perpendicular to axis of the cylindrical symmetry of the space time and obtained exponentially cosmic inflation with anisotropy property which its nature is linear stable in the phase space. In the latter case source matter density takes positive non-zero value which show visible matter which is support the inflation instead of the unknown dark matter/energy. In the latter case signature of the space time changes to the Euclidean form. In short, the outlook of this paper is that anisotropy is a necessary condition for accelerated exponential expansion. As a future work we like to investigate quantum cosmological approach of this work.
Appendix

We defined
\[ \Omega(H, K) = 960H^7K + 6000H^6K^2 - 1264H^5K^3 - 1460H^4K^4 - 1636H^3K^5 
+ 824H^2K^6 + 136HK^7 - 40K^8 + 2016H^6K + 6452H^5K^2 + 122H^4K^3 - 186H^3K^4 
- 2272H^2K^5 + 404HK^6 + 108K^7 + 32H^6 + 830H^5K + 3493H^4K^2 - 1018H^3K^3 
+ 1778H^2K^4 - 1028HK^5 + 24K^6 + 8H^5 - 14H^4K + 918H^3K^2 - 782H^2K^3 
+ 560HK^4 - 88K^5 - 4H^4 - 60H^3K + 116H^2K^2 - 196HK^3 + 40K^4 
- 8H^2K + 8HK^2 - 16K^3 \] (5.1)

and
\[ \Pi(H, K) = 544H^3 - 344H^2K - 16HK^2 + 26K^3 + 272H^2 - 138HK 
+ 13K^2 + 40H - 16K. \] (5.2)

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