COBORDISM MAPS IN EMBEDDED CONTACT HOMOLOGY

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ABSTRACT. Given an exact symplectic cobordism \((X, \lambda)\) between contact 3-manifolds \((Y_+, \lambda_+)\) and \((Y_-, \lambda_-)\) with no elliptic Reeb orbits up to a certain action, we define a chain map \(\Phi\) from the embedded contact homology (ECH) chain complex of \((Y_+, \lambda_+)\) to that of \((Y_-, \lambda_-)\), both taken with coefficients in \(\mathbb{Z}/2\mathbb{Z}\). The map is defined by counting punctured holomorphic curves with ECH index 0 in the completion of the cobordism and new objects that we call ECH buildings.

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1. Introduction

The goal of this paper is to define, given contact 3-manifolds \((Y_+, \lambda_+)\) and \((Y_-, \lambda_-)\) with no elliptic Reeb orbits up to a certain action \(L\) and an exact symplectic cobordism \((X, \lambda)\) from \((Y_+, \lambda_+)\) to \((Y_-, \lambda_-)\), a chain map

\[
\Phi_{X, \lambda, L, c} : ECC^L(Y_+, \lambda_+, J_+) \to ECC^L(Y_-, \lambda_-, J_-)
\]
by counting $J$-holomorphic curves in the completion $\hat{X}$ and new objects that we call **index 0 ECH buildings**. Here, $J_\pm$ is a generic almost complex structure on the symplectization $\mathbb{R} \times Y_\pm$, we choose a generic almost complex structure $J$ on $\hat{X}$ that is compatible with $J_+$ at the positive end and with $J_-$ at the negative end, and $c$ is a choice of auxiliary data that is explained in Definition 1.4.3. The definition of $\Phi_{X,\lambda,J,c}$ relies on some new developments for holomorphic curves in the $L$-supersimple setting of Bao-Honda [BH1, BH2] and Colin-Ghiggini-Honda [CGH1, CGH2, CGH3], and we restrict our attention to that setting throughout the paper.

ECH is isomorphic to both Heegaard Floer homology and Seiberg-Witten Floer (co)homology (see [KLT1, KLT2, KLT3, KLT4, KLT5, CGH1, CGH2, CGH3]), and the latter isomorphism was used by Hutchings-Taubes in [HT3] to define maps induced by exact symplectic cobordisms between contact 3-manifolds. However, a definition of such maps that involves counting $J$-holomorphic curves has proved elusive. Chris Gerig has given a construction in a special case [G], and Hutchings has given an example where one must take into account multi-level SFT buildings [H3, Section 5].

In Sections 2 and 3, we give appropriate background information for ECH and the evaluation map defined by Bao-Honda. In Sections 4 to 7, we discuss the details of these new developments. In Section 8, we prove the main result of this paper, namely, that $\Phi_{X,\lambda,J,c}$ is a chain map. The remainder of this section is an outline of the paper, culminating in the definition of $\Phi_{X,\lambda,J,c}$; see Theorem 1.5.3 and Definition 1.5.2 which depend on some auxiliary definitions in this section.

This paper is a heavily revised version of the author’s doctoral thesis [R], from which portions of this work have been excerpted.

### 1.1. The $L$-supersimple setting and filtered ECH

We begin with a discussion of the $L$-supersimple setting. Recall that the **action** of a Reeb orbit $\alpha$ on the contact manifold $(Y, \lambda)$ is the integral $A(\alpha) = \int_\alpha \lambda$, while the **total action** of an orbit set $\alpha$ is the sum $A(\alpha) = \sum_{\alpha \in \alpha} \int_\alpha \lambda$.

**Definition 1.1.1.** A contact form $\lambda$ on a smooth 3-manifold $Y$ is called $L$-supersimple if every Reeb orbit with action less than $L$ is non-degenerate, hyperbolic, and satisfies the conclusions of Theorem 2.5.1.

Our chain map is defined on the level of filtered ECH, defined as follows. Let $(Y, \lambda)$ be a non-degenerate contact 3-manifold, and let $J$ by a generic, compatible almost complex structure on $\mathbb{R} \times Y$. Let $L > 0$ and consider the
subgroup $ECC^L(Y, \lambda, J) \subset ECC(Y, \lambda, J)$ generated by orbit sets $\alpha$ with total action $A(\alpha) < L$. Every non-degenerate contact form can be made into an $L$-supersimple form by a small perturbation. That is, for any $L > 0$ and $\varepsilon > 0$, there is a positive smooth function $f$ on $Y$ that is $C^1$-close to 1 such that $f\lambda$ is $L$-supersimple. Furthermore, if $f_i\lambda$ is $L_i$-supersimple for $i = 1, 2$ and $L_1 < L_2$, we can ensure that the set of Reeb orbits of $f_2\lambda$ with action less than $L_1$ coincides with the corresponding set of Reeb orbits for $f_1\lambda$, i.e., that there is a natural inclusion map

$$ECC^{L_1}(Y, f_1\lambda, J) \hookrightarrow ECC^{L_2}(Y, f_2\lambda, J).$$

See [BH1, Theorem 2.0.2] and [CGH1, Theorem 2.5.2] for details.

We can reconstruct $ECH(Y, \lambda, J)$ from these filtered groups in the following way, as described in [CGH0, Theorem 3.2.1]. Let $\{f_i\}_{i=1}^\infty$ be a sequence of positive smooth functions on $Y$ with $1 \geq f_1 \geq f_2 \geq \cdots$ and such that $f_i\lambda$ is $L_i$-supersimple for some sequence $\{L_i\}_{i=1}^\infty$ of positive real numbers with $\lim_{i \to \infty} L_i = \infty$. Then there is a canonical isomorphism

$$ECH(Y, \lambda, J) \simeq \lim_{i \to \infty} ECH^{L_i}(Y, f_i\lambda, J).$$

Thus, it suffices to define the chain map $\Phi_{X,\lambda,J,c}$ on each level of the filtration $ECC^{L_i}(Y, f_i\lambda, J)$, where there are no elliptic Reeb orbits. There is also not much loss of generality in assuming that the contact forms on $Y_\pm$ are $L$-supersimple aside from the need to assume invariance results of Hutchings-Taubes [HT3].

1.2. The ECH index inequality. The first of our developments is an improvement to the ECH index inequality in the $L$-supersimple setting. On one-dimensional moduli spaces, the inequality is in fact an equality and gives information about the topology of punctured $J$-holomorphic curves that violate the ECH partition conditions. One can also show that the improved equality is an equality for generic curves with higher Fredholm index using the evaluation map from Section 3. The inequality is implicit in the work of Hutchings [H2]. Gardiner-Hind-McDuff give a similar improvement in [CGHD], and Gardiner-Hutchings-Zhang recently showed that the improved inequality is an equality for generic curves [CGHZ]. The advantages of the $L$-supersimple setting are that (1) the extra term in the improved inequality is given by a simple formula that involves only the multiplicities of the ends of the curve, and (2) the analysis required to prove generic equality is greatly simplified.
The starting point for our improved inequality is Hutchings’ ECH index inequality from [H2]: if $u$ is a somewhere injective $J$-holomorphic curve in a symplectization $\mathbb{R} \times Y$, then

\[(1.2.1) \quad I(u) \geq \text{ind}(u) + 2\delta(u),\]

where $\delta(u)$ is a non-negative count of singularities of $u$.

**Definition 1.2.1.** Let $u: \tilde{\Sigma} \to \mathbb{R} \times Y$ be a punctured $J$-holomorphic curve asymptotic to an orbit set $\alpha$ at the positive ends and to an orbit set $\beta$ at the negative ends. We say that $\alpha$ is the positive orbit set of $u$ and that $\beta$ is the negative orbit set of $u$.

**Definition 1.2.2.** Let $\Gamma^+(u)$ denote the set of embedded Reeb orbits in the positive orbit set of $u$ (i.e., forgetting their multiplicities), and let $\Gamma^-(u)$ denote the set of embedded Reeb orbits in the negative orbit set of $u$.

**Definition 1.2.3.** The ECH deficit of $u$ at an orbit $\gamma \in \Gamma^+(u)$ is defined as follows. If $\gamma$ is negative hyperbolic, suppose $u$ has ends at (covers of) $\gamma$ of multiplicities $q_1, \ldots, q_n$, ordered so that the first $k$ ends have odd multiplicity and the last $n - k$ ends have even multiplicity. Then

\[\Delta(u, \gamma) = \sum_{i=1}^{k} \left( \frac{q_i - 1}{2} + i - 1 \right) + \sum_{i=k+1}^{n} \left( \frac{q_i}{2} - 1 \right)\]

If $\gamma$ is positive hyperbolic and $u$ has ends at (covers of) $\gamma$ of multiplicities $q_1, \ldots, q_n$, then

\[\Delta(u, \gamma) = \sum_{i=1}^{n} (q_i - 1).\]

The ECH deficit $\Delta(u, \gamma)$ for $\gamma \in \Gamma^-(u)$ is defined similarly.

**Definition 1.2.4.** The ECH deficit of $u$ is

\[\Delta(u) = \sum_{\gamma \in \Gamma^+(u)} \Delta(u, \gamma) + \sum_{\gamma \in \Gamma^-(u)} \Delta(u, \gamma).\]

**Theorem 1.2.5.** If $J$ is generic and $u$ is a somewhere injective and connected $J$-holomorphic curve in a symplectization, then

\[(1.2.2) \quad I(u) \geq \text{ind}(u) + 2\delta(u) + \Delta(u).\]

Equality holds if $\mathcal{A}(\alpha) < L$ and $\text{ind}(u) = 1$. 
1.3. Degenerations of one-dimensional families in cobordisms. The next development is an analysis of possible degenerations of one-dimensional families of punctured holomorphic curves in exact symplectic cobordisms, which we discuss in Section 5.

Let $(Y_{\pm}, \lambda_{\pm})$ be $L$-supersimple contact 3-manifolds and let $(X, \lambda)$ be an exact symplectic cobordism from $(Y_{+}, \lambda_{+})$ to $(Y_{-}, \lambda_{-})$. Let $J$ be a generic, $L$-simple, admissible almost complex structure on the completion $(\hat{X}, \hat{\lambda})$ that restricts to $L$-simple, admissible almost complex structures $J_{+}$ and $J_{-}$ on the ends $[0, \infty) \times Y_{+}$ and $(-\infty, 0] \times Y_{-}$, respectively, of $\hat{X}$. Let $\alpha$ be a generator of $ECC^{L}(Y_{+}, \lambda_{+}, J_{+})$ and let $\beta$ be a generator of $ECC^{L}(Y_{-}, \lambda_{-}, J_{-})$. Let $M = M^{1}_{J}(\alpha, \beta)$ denote the moduli space of punctured $J$-holomorphic curves in $\hat{X}$ with $\text{ind}(u) = I(u) = 1$ and asymptotic to $\alpha$ at the positive ends and to $\beta$ at the negative ends. Let $\overline{M}$ denote the SFT compactification of $M$ as described in [BEHWZ]. We denote an SFT building in $\partial M$ by $v_{0} \cup v_{1} \cup \cdots \cup v_{n}$, where the levels go from bottom to top as we read from left to right.

**Theorem 1.3.1.** The points in $\partial \overline{M}$ are two-level buildings $u_{-} \cup u_{+}$, where one of the levels is in $\hat{X}$ and has Fredholm index 0, and the other level is in $\mathbb{R} \times Y_{\pm}$ and has Fredholm index 1.

Let $\gamma$ denote the negative orbit set of $u_{+}$. When $\gamma$ is a generator of the ECH chain complex $ECC^{L}(Y_{\pm}, \lambda_{\pm}, J_{\pm})$, the cobordism level has $I = 0$, the symplectization level has $I = 1$, and both levels are somewhere injective. When $\gamma$ is not a generator of $ECC^{L}(Y_{\pm}, \lambda_{\pm}, J_{\pm})$, the buildings occur in pairs unless the following conditions hold:

1. $u_{+}$ is in $\mathbb{R} \times Y_{+}$ and is somewhere injective;
2. $u_{-}$ is in $\hat{X}$ and is multiply covered;
3. $I(u_{+}) > 1$ and $I(u_{-}) < 0$;
4. each Reeb orbit in $\gamma$ has multiplicity 1 except for finitely many negative hyperbolic orbits $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{k}$ with multiplicities $n_{1}, n_{2}, \ldots, n_{k}$, respectively;
5. $u_{+}$ has $n_{i}$ negative ends at $\gamma_{i}$, each with multiplicity 1;
6. each multiply covered component of $u_{-}$ is an unbranched, disconnected multiple cover of an embedded holomorphic plane with its positive end at a negative hyperbolic orbit.

1.4. The prototypical gluing problem. The last development is an obstruction bundle gluing calculation for certain branched covers of trivial cylinders with high Fredholm index, which we discuss in Section 6. Here, a trivial
Figure 1. A schematic drawing of an endpoint of $\partial \mathcal{M}$ where
$\gamma$ is not a generator and the building is not part of a canceling
pair.

cylinder is a cylinder $\mathbb{R} \times \alpha \subset \mathbb{R} \times Y$, where $\alpha$ is an embedded Reeb orbit in
$Y$. We use the notation of Hutchings-Taubes from [HTT] for moduli spaces of
such branched covers.

Definition 1.4.1. Let $\alpha$ be a Reeb orbit in $Y$. Let
$\mathcal{M}(a_1, a_2, \ldots, a_k \mid a_{-1}, a_{-2}, \ldots, a_{-\ell})$
denote the moduli space of genus 0 branched covers $\hat{\Sigma} \to \mathbb{R} \times \alpha$ with ends
labeled and asymptotically marked and such that the $i^{th}$ end is asymptotic to
an $a_i$-fold cover of $\alpha$.

Definition 1.4.2. For each $n \geq 3$, let $\mathcal{M}_n = \mathcal{M}(1, 1, \ldots, 1 \mid 1, 1, \ldots, 1, 3)$,
where there are $n$ positive ends of multiplicity 1, $n - 3$ negative ends of mul-
tiplicity 1, and one negative end of multiplicity 3.

The prototypical gluing problem considered in this paper is the following.
Let $(Y, \lambda)$ be a smooth 3-manifold with an $L$-supersimple contact form and let
$u_+: \hat{\Sigma} \to \mathbb{R} \times Y$ be a holomorphic embedding with $\text{ind}(u) = 1$ such that
(1) the positive ends of $u_+$ are asymptotic to an ECH generator $\alpha$ with
$A(\alpha) < L$;
(2) the negative ends of $u_+$ are asymptotic to an orbit set $\beta$ in which each
Reeb orbit has multiplicity 1 except for a single negative hyperbolic
orbit $\beta_0$;
(3) the curve $u_+$ has $n$ negative ends at $\beta_0$, each with multiplicity 1;
(4) $I(u_+) = 1 + \binom{n}{2}$.

We wish to glue branched covers in $\mathcal{M}_n$ to the curve $u_+$.

The main source of trouble in the above gluing problem is that the moduli
spaces $\mathcal{M}_n$ are not transversely cut out. However, by standard techniques,
there should be an obstruction bundle

$$\mathcal{O} \to [R, \infty) \times (\mathcal{M}_n/\mathbb{R}),$$

for $R \gg 0$ sufficiently large, with fiber

$$\mathcal{O}_{(T,u)} = \text{Hom} \left( \text{Coker} \, D^N_u, \mathbb{R} \right),$$

where $D^N_u$ is the normal part of the linearized $\bar{\partial}$-operator for $u$.

In analogy with [HT2, Definition 5.9], there should also be an obstruction section $s$ for $\mathcal{O}$ whose zero set is the set of branched covers that glue to $u_+$. Such glued curves lie in a moduli space, which we denote by $\widetilde{\mathcal{M}}_n$. Note that $\dim_{\mathbb{R}} \widetilde{\mathcal{M}}_n = 2n - 3$.

**Definition 1.4.3.** Let $u: \hat{\Sigma} \to \mathbb{R} \times Y$ be a punctured $J$-holomorphic curve in a moduli space $\mathcal{M}_J(\alpha, \beta)$. Choose a subset of the negative ends of $u$ at negative hyperbolic orbits with odd multiplicity and label them by elements of $I_- = \{-1, \ldots, -n\}$. The curve $u$ satisfies the asymptotic restrictions $c \in \mathbb{C}^n$ if $ev_{I_-}^1(u) = c$, where the evaluation map $ev_{I_-}^1$ maps $u$ to the leading complex coefficient in the asymptotic expansion of $u$ at the negative ends labeled by $I_-$. See Definition 3.2.3 for the full definition of the evaluation map.

**Definition 1.4.4.** We say that $c \in (\mathbb{C}^\ast)^n$ is an admissible asymptotic restriction if it is not in the big diagonal of $(\mathbb{C}^\ast)^n$.

**Theorem 1.4.5.** In the prototypical gluing problem, $s^{-1}(0)$ is non-empty. For any choice of admissible asymptotic restriction $c \in (\mathbb{C}^\ast)^{n-2}$ and any $T \geq R$, the mod 2 count of curves in $s^{-1}(0)$ with gluing parameter $T$ and that satisfy the asymptotic restriction is 1.

1.5. **Definition of the chain map.** As described above, there are two contributions to the curve count in the definition of $\Phi_{X,\lambda,J,c}$. Suppose that we have ECH generators $\alpha \in ECC(Y_+, \lambda_+, J_+)$ and $\beta \in ECC(Y_-, \lambda_-, J_-)$, that we write

$$\Phi_{X,\lambda,J,c}(\alpha) = \sum_{A(\beta) < A(\alpha)} \langle \Phi_{X,\lambda,J,c}(\alpha), \beta \rangle \cdot \beta,$$

and that we want to define the coefficient $\langle \Phi_{X,\lambda,J,c}(\alpha), \beta \rangle$. The first contribution is the mod 2 count $\#_2(\mathcal{M}_I^0(\alpha, \beta))$ of $J$-holomorphic curves in $\tilde{X}$ with ECH index $I = 0$. The second contribution is the mod 2 count of new objects that we call ECH buildings satisfying certain admissible asymptotic restrictions.
Definition 1.5.1. Assume the setup described above. An index 0 ECH building from $\alpha$ to $\beta$ satisfying the admissible asymptotic restriction $c$ is a pair $(u_, u_0)$ satisfying the following conditions:

1. the curve $u_0$ is in $\mathbb{R} \times Y$ and the curve $u_-$ is in $\hat{X}$;
2. the curve $u_0$ has positive orbit set $\alpha$ and the curve $u_-$ has negative orbit set $\beta$;
3. the negative orbit set $\gamma$ of $u_0$ coincides with the positive orbit set of $u_-$;
4. the partition of the negative ends of $u_0$ coincides with the partition of the positive ends of $u_-$, except possibly for some negative hyperbolic Reeb orbits $\gamma_1, \ldots, \gamma_\ell$ in $\gamma$ of multiplicities $m_1, \ldots, m_\ell$ where the partition for the negative ends of $u_0$ at each $\gamma_i$ is $(3, 1, \ldots, 1)$ and the partition for the positive ends of $u_-$ at each $\gamma_i$ is $(1, 1, \ldots, 1)$;
5. we have $\text{ind}(u_-) = 0$ and $I(u_-) = -\sum_{j=1}^\ell \binom{m_j}{2}$;
6. we have $\text{ind}(u_0) = \sum_{j=1}^\ell (2m_j - 4)$ and $I(u_0) = -I(u_-)$; and
7. the curve $u_0$ satisfies the asymptotic restriction $c$, where we use all of the negative ends at the orbits $\gamma_1, \ldots, \gamma_\ell$ for the evaluation map.

We denote the set of index 0 ECH buildings from $\alpha$ to $\beta$ satisfying the admissible asymptotic restriction $c$ by $B^0_J(\alpha, \beta; c)$.

Definition 1.5.2. Let $(Y_\pm, \lambda_\pm)$ be $L$-supersimple contact 3-manifolds and let $(X, \lambda)$ be an exact symplectic cobordism from $(Y_+, \lambda_+)$ to $(Y_-, \lambda-)$. Let $J$ be a generic, $L$-simple, admissible almost complex structure on the completion $(\hat{X}, \hat{\lambda})$ that restricts to $L$-simple, admissible almost complex structures $J_+$ and $J_-$ on the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, respectively, of $\hat{X}$. Let $c$ be a choice of admissible asymptotic restriction. The map

$$\Phi_{X,\lambda,J,c}: \text{ECC}^L(Y_+, \lambda_+, J_+) \to \text{ECC}^L(Y_-, \lambda_-, J_-)$$

induced by $(X, \lambda)$ is defined by

$$\Phi_{X,\lambda,J,c}(\alpha) = \sum_{A(\beta) < A(\alpha)} \left[ \#_2 \left( M^0_J(\alpha, \beta) \right) + \#_2 \left( B^0_J(\alpha, \beta; c) \right) \right] \cdot \beta.$$
Acknowledgements. First and foremost, the author thanks Ko Honda for his generous support and endless patience. The author also thanks Michael Hutchings, Katrin Wehrheim, and Erkao Bao for helpful conversations during the development of the ideas in this paper.

2. Background

In this section, we establish some notation, briefly review the definition of embedded contact homology, and recall some basic facts about the $L$-supersimple setting of Bao-Honda.

2.1. Basic definitions. Let $Y$ be a smooth 3-manifold, let $\lambda$ be a non-degenerate contact form on $Y$, let $\xi = \text{Ker}(\lambda)$ be the associated contact structure, and let $R_\lambda$ be the Reeb vector field of $\lambda$, defined as the unique vector field on $Y$ satisfying $\lambda(R_\lambda) = 1$ and $d\lambda(R_\lambda, \cdot) = 0$.

**Definition 2.1.1.** An almost complex structure $J$ on $\mathbb{R} \times Y$ is admissible if it satisfies the following properties:

1. $J$ is invariant under $\mathbb{R}$-translation;
2. $J(\partial_s) = R_\lambda$, where $s$ is the $\mathbb{R}$-coordinate of $\mathbb{R} \times Y$;
3. $J$ restricts to an orientation-preserving isomorphism of the contact structure $\xi$.

We will always assume that the almost complex structures on symplectizations are admissible.

Let $\alpha$ be a Reeb orbit in $(Y, \lambda)$ and let $\tau$ be a trivialization of $\xi$ over $\alpha$. We denote the **Conley-Zehnder** index of $\alpha$ in the trivialization $\tau$ by $CZ_\tau(\alpha)$. We recall here some simple expressions for the Conley-Zehnder index in dimension 3: (1) if $\alpha$ is elliptic, then there is some irrational number $\theta \in (0, 1)$ such that $CZ_{\tau}(\alpha^k) = 2\lfloor k\theta \rfloor + 1$; (2) if $\alpha$ is hyperbolic, then $CZ_{\tau}(\alpha^k) = kn$ for some integer $n$. In the latter case, we say that $\alpha$ is positive hyperbolic if $n$ is even and negative hyperbolic if $n$ is odd.

**Definition 2.1.2.** An orbit set is a tuple of ordered pairs

$$\alpha = ((\alpha_1, m_1), (\alpha_2, m_2), \ldots, (\alpha_k, m_k))$$

such that each $\alpha_i$ is an embedded Reeb orbit in $Y$ and each $m_i$ is a positive integer.
2.2. Punctured holomorphic curves. Let $(\Sigma, j)$ be a closed Riemann surface with complex structure $j$. Let $P \subset \Sigma$ be a finite set of points, called punctures, which are partitioned into subsets $P^+$ and $P^-$ of positive and negative punctures, respectively. Define $\hat{\Sigma} = \Sigma \setminus P$; we refer to $\hat{\Sigma}$ as a punctured Riemann surface. If $J$ is an admissible almost complex structure on $\mathbb{R} \times Y$, a punctured holomorphic curve is a smooth map $u: \hat{\Sigma} \to \mathbb{R} \times Y$

such that

$$du + J \circ du \circ j = 0.$$

A $J$-holomorphic curve $u: \hat{\Sigma} \to \mathbb{R} \times Y$ is said to be multiply covered if it factors through a (possibly branched) cover $\phi: \Sigma' \to \hat{\Sigma}$ for some punctured Riemann surface $\Sigma'$. A connected curve is said to be simply connected if it is not multiply covered. We will also refer to such curves as simple in this paper.

2.3. Moduli spaces. We distinguish between two types of moduli spaces of $J$-holomorphic curves in this paper: marked and unmarked. We use both types in this paper. Marked moduli spaces are used in Section 6 for obstruction bundle gluing problems, and ECH is defined using unmarked moduli spaces.

Let $u: \hat{\Sigma} \to \mathbb{R} \times Y$ be $J$-holomorphic, and assume that $u$ is asymptotic to Reeb orbits $\alpha_1, \alpha_2, \ldots, \alpha_n$ at the positive punctures and to $\beta_1, \beta_2, \ldots, \beta_m$ at the negative punctures. For each such Reeb orbit, let $(\alpha_i)_e$ (resp. $(\beta_j)_e$) denote the underlying embedded Reeb orbit for $\alpha_i$ (resp. $\beta_j$). Choose a point $\zeta_i$ on each $(\alpha_i)_e$ (resp. $\eta_j$ on $(\beta_j)_e$), and for each $z_i \in P^+$ (resp. $w_j \in P^-$), choose an element $r_i \in (T_{z_i} \Sigma \setminus \{0\})/\mathbb{R}$ (resp. $r_j \in (T_{w_j} \Sigma \setminus \{0\})/\mathbb{R}$) that maps to $\zeta_i$ under the map $\alpha_i \to (\alpha_i)_e$ (resp. $\eta_j$ under the map $\beta_j \to (\beta_j)_e$). We refer to each such choice as an asymptotic marker at the relevant puncture; we refer to markers at positive punctures as positive markers and to markers at negative punctures as negative markers. Let $r$ denote the set of markers we have chosen.

Given orbit sets $\alpha$ and $\beta$, the moduli space of marked, punctured holomorphic curves from $\alpha$ to $\beta$ in $\mathbb{R} \times Y$ is the space of pairs $(u, r)$, where $u$ is asymptotic to $\alpha$ at the positive punctures, $u$ is asymptotic to $\beta$ at the negative punctures, and $r$ is a set of asymptotic markers for $u$, modulo biholomorphisms of domains that send positive punctures to positive punctures, negative punctures to negative punctures, positive markers to positive markers, and negative markers to negative markers. We denote the moduli space of such curves by...
In the compactified moduli spaces, we have such moduli spaces can be compactified using SFT buildings; see [BEHWZ] for details.

Unmarked moduli spaces are defined similarly to marked moduli spaces, except we do not choose asymptotic markers at each puncture. Consequently, we identify two such maps if they are related by a biholomorphism of the domains that maps positive punctures to positive punctures and negative punctures to negative punctures. ECH uses unmarked moduli spaces and identifies two maps if they represent the same current in $\mathbb{R} \times Y$. We denote unmarked moduli spaces of curves by $\mathcal{M}_J(\alpha, \beta)$.

A curve $u \in \mathcal{M}_J(\alpha, \beta)$ has a Fredholm index given by

$$\text{ind}(u) = -\chi(\hat{\Sigma}) + CZ_\tau(\alpha) - CZ_\tau(\beta) + 2c_1(u^*\xi, \tau),$$

where $c_1(u^*\xi, \tau)$ is the relative first Chern class of $\xi$ over $u$ in the trivialization $\tau$. See [H2, Section 2] for the definition of the relative first Chern class. If $\mathcal{M}_J(\alpha, \beta)$ is transversely cut out, then the (real) dimension of a neighborhood of $u \in \mathcal{M}_J(\alpha, \beta)$ is precisely $\text{ind}(u)$ by results of Dragnev [D].

**2.4. The ECH chain complex.** We now define the ECH chain complex with $\mathbb{Z}/2\mathbb{Z}$ coefficients. (It is possible to define ECH with $\mathbb{Z}$ coefficients, but we do not treat that case here.) Let $\Gamma \in H_1(Y)$ and let $J$ be a generic, admissible almost complex structure on $\mathbb{R} \times Y$. The groups $ECC(Y, \lambda, \Gamma, J)$ are generated by orbits sets $\alpha = ((\alpha_1, m_1), (\alpha_2, m_2), \ldots, (\alpha_k, m_k))$ such that $m_i = 1$ if $\alpha_i$ is hyperbolic and such that

$$\sum_{i=1}^{k} m_i[\alpha_i] = \Gamma.$$

Hutchings defines an ECH index $I$ for $J$-holomorphic currents $\mathcal{C}$ in $\mathbb{R} \times Y$. More specifically, he defines a relative self-intersection number $Q_\tau(\mathcal{C})$ and sets

$$I(\mathcal{C}) = c_1(\xi|_{\mathcal{C}}, \tau) + Q_\tau(\mathcal{C}) + CZ_I^\tau(\alpha, \beta),$$

where

$$CZ_I^\tau(\alpha, \beta) = \sum_{i}^{m_i} \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_{j}^{n_j} \sum_{k=1}^{n_j} CZ_\tau(\beta_j^k).$$

The differential $\partial$ counts punctured $J$-holomorphic curves with ECH index 1 in $\mathbb{R} \times Y$ going from $\alpha$ to $\beta$. More precisely, consider the moduli space $\mathcal{M}_J^{I=1}(\alpha, \beta)$ of $J$-holomorphic currents $\mathcal{C}$ with $I(\mathcal{C}) = 1$ that are asymptotic to $\alpha$ at the positive ends and asymptotic to $\beta$ at the negative ends. There is
an $\mathbb{R}$-action on each $\mathcal{M}_{j}^{I=\mathbb{R}}(\alpha, \beta)$ induced by translation in the $\mathbb{R}$-direction of $\mathbb{R} \times Y$.

**Lemma 2.4.1.** If $\mathcal{M}_{j}^{I=\mathbb{R}}(\alpha, \beta)$ is non-empty, then $\mathcal{A}(\beta) < \mathcal{A}(\alpha)$.

**Proof.** See [H3, Section 5]. □

**Lemma 2.4.2.** [H3, Lemma 5.10] If $J$ is generic and $\alpha$ and $\beta$ are orbit sets, then $\mathcal{M}_{j}^{I=\mathbb{R}}(\alpha, \beta)/\mathbb{R}$ is finite.

The differential $\partial$ on the chain complex $ECC(Y, \lambda, \Gamma, J)$ is defined by

$$\partial(\alpha) = \sum_{\mathcal{A}(\beta) < \mathcal{A}(\alpha)} \#(\mathcal{M}_{j}^{I=\mathbb{R}}(\alpha, \beta)/\mathbb{R}) \cdot \beta.$$  

**Remark 2.4.3.** The proof in [HT1, HT2] that $\partial^2 = 0$ is quite difficult and uses obstruction bundle gluing to describe the boundary of 2-dimensional moduli spaces of punctured $J$-holomorphic curves in symplectizations. All of the difficulty comes from the presence of elliptic orbits, and proof in the $L$-supersimple setting is trivial.

Curves counted by the differential $\partial$ satisfy a rigid requirement on the multiplicities of their positive and negative ends. This requirement is crucial in [HT1, HT2] to show that $\partial^2 = 0$ and is leveraged extensively in this paper.

**Definition 2.4.4.** Let $\alpha$ be an embedded hyperbolic Reeb orbit in $Y$. Let $u$ be a $J$-holomorphic curve $\mathbb{R} \times Y$ with positive ends of multiplicities $m_1, m_2, \ldots, m_k$ and negative ends of multiplicities $n_1, n_2, \ldots, n_l$ at covers of $\alpha$. We say that $u$ satisfies the **ECH partition conditions** at $\alpha$ if the multiplicities $m_i$ and $n_j$ are as in Table 1.

| $\gamma$ | $n$ even | $n$ odd |
|---------|---------|--------|
| positive hyperbolic | $(1, \ldots, 1)$ | $(1, \ldots, 1)$ |
| negative hyperbolic | $(2, \ldots, 2)$ | $(2, \ldots, 2, 1)$ |

**Table 1.** The partition conditions for hyperbolic Reeb orbits.

**Remark 2.4.5.** We do not concern ourselves with the partition conditions for elliptic Reeb orbits in this paper, as we work completely in the $L$-supersimple setting. Interested readers can consult [H3] for details.
2.5. The $L$-supersimple setting. We now review the relevant background for the $L$-supersimple setting of Bao-Honda. As stated in Section 1, every non-degenerate contact form can be made into an $L$-supersimple form by a small perturbation. The precise statement of this result, which we take from [BH1], is as follows.

**Theorem 2.5.1.** [BH1, Theorem 2.0.2] Let $\lambda$ be a non-degenerate contact form for $(Y, \xi)$. Then, for any $L > 0$ and $\epsilon > 0$, there exists a smooth function $\phi : Y \to \mathbb{R}_+$ such that

1. $\phi$ is $\epsilon$-close to 1 with respect to a fixed $C^1$-norm;
2. all the orbits of $R_{\phi \lambda}$ of $\phi \lambda$-action less than $L$ are hyperbolic.

Moreover, we may assume that

1. each positive hyperbolic orbit $\alpha$ has a neighborhood $(\mathbb{R}/\mathbb{Z}) \times D^2_{\delta_0}$ with coordinates $(t, x, y)$ such that
   - $D^2_{\delta_0} = \{x^2 + y^2 \leq \delta_0\}$, where $\delta_0 > 0$ is small;
   - $\phi \lambda = H \, dt + \eta$;
   - $H = c(\alpha) - \epsilon xy$, with $c(\alpha), \epsilon > 0$ and $c(\alpha) \gg \epsilon$;
   - $\eta = 2x \, dy + y \, dx$;
   - $\alpha = \{ x = y = 0 \}$.
2. each negative hyperbolic orbit $\alpha$ has a neighborhood $([0, 1] \times D^2_{\delta_0})/\sim$ with coordinates $(t, x, y)$, where $\sim$ identifies $1, x, y$ with $(0, -x, -y)$ and the conditions (a) through (e) above hold.

One major advantage of working in the $L$-supersimple setting is that the Fredholm index is well-behaved under taking multiple covers.

**Lemma 2.5.2.** [BH1, Lemma 3.3.2] Let $Y$ be a smooth 3-manifold, let $\lambda$ be an $L$-supersimple contact form on $Y$, and let $\alpha$ and $\beta$ be orbit sets of total action less than $L$. If $u \in \mathcal{M}_J(\alpha', \beta')$ is a degree $k$ branched cover of a somewhere injective curve $v \in \mathcal{M}_J(\alpha, \beta)$, and if $b$ is the total branching order of $u$, then

$$\text{ind}(u) = k \text{ind}(v) + b.$$ 

In particular, $\text{ind}(u) \geq 0$ for all $u \in \mathcal{M}_J(\alpha, \beta)$.

Another major advantage of the $L$-supersimple setting is that, by choosing the almost complex structure $J$ appropriately, we can ensure that the $\overline{\partial}$-equation is linear for curves that are close to and graphical over trivial cylinders. The set of $J$ for which this assertion is true is described in the following definitions.
Definition 2.5.3. [BH2, Definition 3.1.2] Let $\lambda$ be a contact form on $Y$. An almost complex structure $J$ on $\mathbb{R} \times Y$ is $\lambda$-tame if the following three conditions hold:

1. $J$ is $\mathbb{R}$-invariant;
2. $J(\partial_s) = gR_\lambda$ for some positive function $g$ on $Y$; and
3. there exists a 2-plane field $\xi'$ on $Y$ such that $J$ preserves $\xi'$, $d\lambda$ is a symplectic form on $\xi'$, and $J$ restricts to an orientation-preserving isomorphism on $\xi'$.

Definition 2.5.4. [BH2, Definition 3.1.3] Let $L > 0$, let $\lambda$ be an $L$-supersimple contact form, and let $\alpha$ be an embedded Reeb orbit of $\lambda$. A $\lambda$-tame almost complex structure $J$ is $L$-simple for $\lambda$ if, inside the neighborhood of $\alpha$ given by Theorem 2.5.1, the following conditions hold:

1. $\xi' = \text{Span} (\partial_x, \partial_y)$;
2. $J(\partial_x) = \partial_y$; and
3. the function $g$ in Definition 2.5.3 satisfies $gR_\lambda = \partial_t + X_H$, where $X_H$ is the Hamiltonian vector field of the function $H$ from Theorem 2.5.1 with respect to the symplectic form $dx \wedge dy$.

We can now state the second advantage precisely.

Proposition 2.5.5. [BH2] Let $\lambda$ be an $L$-supersimple contact form on $Y$ and let $J$ be an $L$-simple almost complex structure for $\lambda$. If $u: [R, \infty) \times S^1 \to \mathbb{R} \times Y$ is a $J$-holomorphic half-cylinder asymptotic to a Reeb orbit $\alpha$, and if we write $u(s, t) = (s, t, \eta(s, t))$, then the function $\eta$ satisfies

$$
\partial_s \eta + j_0 \partial_t \eta + S \eta = 0,
$$

where $j_0$ is the standard complex structure on $\mathbb{R}^2$ and

$$
S = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}.
$$

Proof. See the discussion in [BH2] between Definition 3.1.3 and Convention 3.1.4.

3. The Evaluation Map

In this section, we review the Bao-Honda evaluation map from [BH1, BH2]. It is used in Section 7 to carefully cut out 1-dimensional families of holomorphic curves in high-dimensional moduli spaces.
Throughout this section, let $Y$ be a smooth 3-manifold, let $\lambda$ be a non-degenerate, $L$-supersimple contact form on $Y$, and let $R_{\lambda}$ be the Reeb vector field of $\lambda$ on $Y$. All Reeb orbits and orbit sets under consideration in this section are implicitly assumed to have (total) action less than $L$.

3.1. The asymptotic operator. Let $\gamma$ be a Reeb orbit of $\lambda$ with period $2\pi a$, where $a \in \mathbb{Z}_+$. Recall from [BH1] that there is an asymptotic operator $A_{\gamma}: W^{1,2}(\mathbb{R}/2\pi a\mathbb{Z}, \mathbb{R}^2) \to L^2(\mathbb{R}/2\pi a\mathbb{Z}, \mathbb{R}^2)$ defined by

$$A_{\gamma} = -j_0 \frac{\partial}{\partial t} - S(t).$$

Here, $j_0$ is the standard complex structure on $\mathbb{R}^2$ and $S(t)$ is a loop of $2 \times 2$ symmetric matrices. Recall from [BH1] that the eigenspaces of $A_{\gamma}$ have dimension at most 2. If $\gamma$ is negative hyperbolic, then every eigenspace of $A_{\gamma}$ has real dimension 2, and if we label the eigenvalues so that

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

then we can choose the corresponding eigenfunctions $\{f_i(t)\}_{i \in \mathbb{Z}\setminus\{0\}}$ so that they form an orthonormal basis for $L^2(\mathbb{R}/2\pi a\mathbb{Z}, \mathbb{R}^2)$. If $\gamma$ is positive hyperbolic, then the eigenvalues can be labeled so that

$$\cdots \leq \lambda_{-3} \leq \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

the eigenspaces for $\lambda_{\pm 1}$ have real dimension 1, and all other eigenspaces have real dimension 2. The corresponding eigenfunctions can again be chosen to be an orthonormal basis for $L^2(\mathbb{R}/2\pi a\mathbb{Z}, \mathbb{R}^2)$.

Next, we recall from [BH1] some properties of the above-mentioned eigenfunctions. Let $u$ be a punctured holomorphic curve in $\mathbb{R} \times Y$ and suppose that $u$ has a negative end at $\gamma$. If we choose trivialization $\tau$ of the contact structure $\xi = \text{Ker}(\lambda)$ over $\gamma$, then the negative end of $u$ in question can be written in cylindrical coordinates $(s, t) \in (-\infty, -R] \times (\mathbb{R}/2\pi a\mathbb{Z})$, $R \gg 0$, as the graph of a function $\eta(s, t)$, i.e., we have

$$u(s, t) = (s, t, \eta(s, t)).$$

In the $L$-supersimple setting, the function $\eta$ admits a Fourier-type expansion

$$\eta(s, t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i s} f_i(t),$$
where the $c_i$ are real constants. Similarly, a positive end of $u$ asymptotic to $\gamma$ can be written as the graph of a function that has a Fourier-type expansion in negative-indexed eigenfunctions of $A_\gamma$.

Let $\text{wind}_u(f_i)$ denote the winding number of the eigenfunction $f_i$ of $A_\gamma$. Recall the following facts from [H1, Lemma 6.4].

**Fact 3.1.1.**

1. If $i \leq j$, then $\text{wind}_u(f_i) \leq \text{wind}_u(f_j)$.
2. We have $\text{wind}_u(f_1) = \left\lceil CZ_\tau(\gamma) \right\rceil$ and $\text{wind}_u(f_{-1}) = \left\lfloor CZ_\tau(\gamma) \right\rfloor$.

### 3.2. The evaluation map

We now recall the definition of the evaluation map in [BH1] and review some of the map’s properties. Throughout, we use $\mathcal{M}_J(\alpha, \beta)$ to denote a transversely cut out moduli space of $J$-holomorphic curves in $\mathbb{R} \times Y$ with positive orbit set $\alpha$ and negative orbit set $\beta$.

**Definition 3.2.1.** Let $u: (\mathbb{R} / 2\pi a\mathbb{Z}) \times [0, \infty) \to \mathbb{R} \times Y$ be a $J$-holomorphic half-cylinder, and assume that $u$ is asymptotic to a Reeb orbit $\gamma$ at the negative end. Write $u$ in cylindrical coordinates as the graph of a function $\eta(s, t)$, and write the Fourier-type expansion of $\eta$ as

$$\eta(s, t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i s} f_i(t).$$

Then the **order $k$ evaluation map** on $u$ is defined as

$$\text{ev}_u^k(u) = (c_1, c_2, \ldots, c_k).$$

We can define a similar map for half-cylinders with positive ends.

**Definition 3.2.2.** Let $u \in \mathcal{M}_J(\alpha, \beta)$. Label the positive ends of $u$ by $1, 2, \ldots, n$ and the negative ends by $-1, -2, \ldots, -m$. The **order $k$ evaluation map** at the end of $u$ labeled $i$ is defined as

$$\text{ev}_i^k: \mathcal{M}_J(\alpha, \beta) \to \mathbb{R}^k$$

$$u \mapsto (c_1, c_2, \ldots, c_k),$$

where we have identified $u$ with a half-cylinder near the $i^{th}$ end and used the evaluation map from Definition 3.2.1.
Definition 3.2.3. Assume the setup in Definition 3.2.2. Let $I_+ = \{i_1, \ldots, i_p\}$ and $I_- = \{i_{-1}, \ldots, i_{-q}\}$ be subsets of $\{1, \ldots, n\}$ and $\{-1, \ldots, -m\}$ respectively. At the $l$th end of $u$, write the Fourier-type series as

$$\sum_{i \in \mathbb{Z} \setminus \{0\}} c_{l,i} e^{\lambda_i s} f_i(t),$$

where

$$c_{l,i} = 0 \quad \text{for} \quad \begin{cases} i > 0 & \text{at a positive end} \\ i < 0 & \text{at a negative end} \end{cases}.$$ 

Choose a positive integer $k_i$ for each $i \in I_+ \cup I_-$ and set

$$k = (k_{i_1}, \ldots, k_{i_p}, k_{-i_{-1}}, \ldots, k_{-i_{-q}}).$$

The order $k$ evaluation map at the ends specified by $I_+$ and $I_-$ is defined as

$$\text{ev}^k_{I_+ \cup I_-} : M_J(\alpha, \beta) \to \prod_{i \in I_+ \cup I_-} \mathbb{R}^{k_i}$$

$$u \mapsto (\text{ev}^k_i(u))_{i \in I_+ \cup I_-}.$$ 

We set

$$|k| = \sum_{i \in I_+ \cup I_-} k_i$$

and call $|k|$ the total order of the map $\text{ev}^k_{I_+ \cup I_-}$.

If all of the ends labeled by $I_+$ and $I_-$ have odd multiplicity, the asymptotic eigenspaces at those ends all have multiplicity 2. If the relevant asymptotic operators are also complex-linear, we can view the eigenspaces as complex vector spaces and take complex coefficients with the evaluation map. This modification is used extensively in Section 7.

Fact 3.2.4. The above evaluation maps are all smooth.

3.3. Transversality for the evaluation map. One of the key advantages of the $L$-supersimple setting exploited in [BH1, BH2] is the abundance of transversality for the evaluation map at ends of punctured holomorphic curves. We now briefly justify why similar transversality results hold for evaluation maps on multiple ends, beginning with a mild generalization of [BH1, Theorem 6.0.4].

Theorem 3.3.1. Let $J$ be generic, and let $M_J(\alpha, \beta)$ be a transversely cut out moduli space of curves in $\mathbb{R} \times Y$ with Fredholm index $k$. Let $K \subset M_J(\alpha, \beta)$ be compact and let $Z \subset \mathbb{R}^{k-1}$ be a submanifold. Then there exists a generic
\( J' \), arbitrarily close to \( J \), and a compact subset \( K' \subset \mathcal{M}_f(\alpha, \beta) \), arbitrarily close to \( K \), such that the evaluation map \( \text{ev}_{I_+ \cup I_-}^k \) on \( K' \) is transverse to \( Z \).

**Proof.** Let \( u \in \mathcal{M}_f(\alpha, \beta) \). The perturbation constructed in the proof of [BH1, Theorem 6.0.4] is supported over a single end, so we can repeat the construction over the relevant ends of \( u \) separately. \( \square \)

**Proposition 3.3.2.** Let \( J \) be generic, let \( \mathcal{M} \) be a transversely cut out moduli space of punctured holomorphic curves in \( \mathbb{R} \times Y \), and consider the evaluation map \( \text{ev}_{I_+ \cup I_-}^k \) on \( \mathcal{M} \). The set of \( u \in \mathcal{M} \) such that \( \text{ev}_{I_+ \cup I_-}^k(u) \) intersects a coordinate hyperplane \( \{x_i = 0\} \) in \( \mathbb{R}^{1|k} \) has codimension 1 in \( \mathcal{M} \).

**Proof.** By Theorem 3.3.1, we can make \( \text{ev}_{I_+ \cup I_-}^k \) transverse to the coordinate plane \( \{x_i = 0\} \) if \( J \) is generic. \( \square \)

4. **Index Calculations**

We prove Theorem 1.2.5 in this section. We will use the inequality in Section 5 to classify degenerations of 1-dimensional families of \( J \)-holomorphic curves in the \( L \)-supersimple setting. The proof involves strengthening the various inequalities involved in Hutchings’ proof of the inequality (1.2.1). The relationship between \( \Delta(u) \) and the ECH partition conditions is partially expressed in the following result, whose proof follows easily from the derivation of the formulas for \( \Delta(u, \gamma) \) in Section 4.3. The subsequent corollary is a crucial ingredient to our arguments in Section 8.

**Proposition 4.0.1.** If \( J \) is generic, \( u \) is a somewhere injective and connected \( J \)-holomorphic curve in a symplectization, and \( \Delta(u) = 0 \), then \( u \) satisfies the ECH partition conditions.

**Corollary 4.0.2.** If \( J \) is generic and \( u \) is a somewhere injective and connected \( J \)-holomorphic curve in a symplectization with \( I(u) = \text{ind}(u) \), then \( u \) is embedded and satisfies the ECH partition conditions.

4.1. **Ingredients in the proof of Hutchings’ inequality.** We begin by fixing notation and collating the results used in Hutchings’ proof of the inequality (1.2.1). Our notation closely, but not exactly, matches that used in [H1]. We only analyze negative ends in this discussion; the analysis for positive ends is similar.

**Notation 4.1.1.** Let \( u \) be a somewhere injective \( J \)-holomorphic curve in a symplectization. Let \( \beta \) be the negative orbit set of \( u \) and fix an embedded
Reeb orbit $\beta \in \Gamma^-(u)$. Let $m$ be the multiplicity of $\beta$ in the orbit set $\beta$, let $n$ be the number of negative ends of $u$ that are asymptotic to (covers of) $\beta$, and let $q_1, q_2, \ldots, q_n$ be the multiplicities of these negative ends. Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the braids determined by these negative ends, and let $\zeta$ denote the union of all braids determined by negative ends of $u$ at (covers of) $\beta$. Let $\tau$ denote a trivialization of $\xi$ over $\beta$. Let $\mu_\tau(\beta^k)$ denote the Conley-Zehnder index of the $k$-fold cover of $\beta$. For each braid $\zeta_i$, let $\text{wind}_\tau(\zeta_i)$ denote the winding number of $\zeta_i$ around $\beta$ with respect to $\tau$, let $\text{w}_\tau(\zeta_i)$ the asymptotic writhe of $\zeta_i$ with respect to $\tau$, and let $\ell_\tau(\zeta_i, \zeta_j)$ denote the linking number of the braids $\zeta_i$ and $\zeta_j$ with respect to $\tau$.

With the above notation, the five ingredients in the proof of Hutchings’ inequality are the following.

\begin{align}
(4.1.1) \quad & \text{wind}_\tau(\zeta_i) \geq \left\lceil \frac{\mu_\tau(\beta^q_i)}{2} \right\rceil \\
(4.1.2) \quad & \text{w}_\tau(\zeta_i) \geq (q_i - 1) \text{wind}_\tau(\zeta_i) \\
(4.1.3) \quad & \ell_\tau(\zeta_i, \zeta_j) \geq \min(q_i \text{wind}_\tau(\zeta_j), q_j \text{wind}_\tau(\zeta_i)) \\
(4.1.4) \quad & \text{w}_\tau(\zeta_\beta) \geq \sum_{i=1}^{n} \text{wind}_\tau(\zeta_i)(q_i - 1) + \sum_{i \neq j} \min(q_i \text{wind}_\tau(\zeta_j), q_j \text{wind}_\tau(\zeta_i)) \\
(4.1.5) \quad & \sum_{i=1}^{n} \text{wind}_\tau(\zeta_i)(q_i - 1) + \sum_{i \neq j} \min(q_i \text{wind}_\tau(\zeta_j), q_j \text{wind}_\tau(\zeta_i)) \geq \sum_{k=1}^{m} \mu_\tau(\beta^k) - \sum_{i=1}^{n} \mu_\tau(\gamma^{q_i})
\end{align}

4.2. The writhe bound. We first use Proposition 3.3.2 to improve (4.1.1) slightly. Our proof of the next Lemma closely follows the one given for [H1, Lemma 6.6]

\begin{lemma}
Let $u$ be a somewhere injective curve in a symplectization $\mathbb{R} \times Y$. Assume that the contact form on $Y$ is $L$-supersimple and that $\beta$ is an orbit in the negative orbit set $\beta$ of $u$. If $J$ is generic, then

\begin{align}
(4.2.1) \quad & \text{wind}_\tau(\zeta_i) \geq \left\lceil \frac{\mu_\tau(\beta^{q_i})}{2} \right\rceil.
\end{align}
\end{lemma}
Equality holds if $A(\beta) < L$ and $\text{ind}(u) = 1$.

Proof. The inequality is proved in [H1, Lemma 6.6]. So assume that $A(\beta) < L$ and $\text{ind}(u) = 1$. Let $(s, t)$ be cylindrical coordinates over the relevant negative end of $u$ and take an asymptotic expansion

$$u(s, t) = \left(s, t, \sum_{i=1}^{\infty} c_i e^{-\lambda_i s} f_i(t)\right)$$

of $u$ for $s \ll 0$, as in Section 3. Since $u$ has Fredholm index 1 and $J$ is generic, Proposition 3.3.2 implies that $c_1 \neq 0$. Thus, $\text{wind}_r(\zeta_i)$ equals the winding number of $f_i$ around $\beta$ in the trivialization $\tau$. By computations in [HWZ, Section 3], said winding number is precisely \[\left\lceil \frac{\mu_+(\gamma^q_i)}{2} \right\rceil.\] □

We now turn our attention to (4.1.2). Our proof of the next Lemma closely follows the one given for [H1, Lemma 6.7].

Lemma 4.2.2. Let $u$ be a somewhere injective curve in a symplectization $\mathbb{R} \times Y$. Assume that the contact form on $Y$ is $L$-supersimple and that $\beta$ is an orbit in the negative orbit set $\beta$ of $u$. If $J$ is generic, then

(4.2.2) \[w_r(\zeta_i) \geq \text{wind}_r(\zeta_i)(q_i - 1) + (d_i - 1),\]

where $d_i = \gcd(q_i, \text{wind}_r(\zeta_i))$. Equality holds if $A(\beta) < L$ and $\text{ind}(u) = 1$.

Proof. For simplicity of notation, we write $\rho_i = \text{wind}_r(\zeta_i)$ in this proof. As a preliminary step, direct calculation shows that equality holds when $\rho_i = q_i$. We proceed by complete induction on $\rho_i$. First assume that $d_i = 1$. (This is true when $\rho_i = 1$, but the more general result is useful in the inductive step.) The proof of [H1, Lemma 6.7] shows that $w_r(\zeta_i) = \rho_i(q_i - 1)$. Now assume $\rho_i > 1$ and $d_i > 1$. The same proof shows that $\zeta_i$ is the cabling of a braid $\zeta_i'$ with $q_i/d_i$ strands and winding number $\rho_i/d_i$ by a braid $\zeta_i''$ with $d_i$ strands and winding number $\rho_i' \geq \rho_i$. Write $\rho_i' = \rho_i + k$ and $d_i' = \gcd(\rho_i', d_i)$. We know inductively that

$$w_r(\zeta_i') = \frac{\rho_i}{d_i} \left(\frac{q_i}{d_i} - 1\right) \quad \text{and} \quad w_r(\zeta_i'') \geq (\rho_i + k)(d_i - 1) + (d_i' - 1),$$

and thus

$$w_r(\zeta_i) = d_i^2 w_r(\zeta_i') + w_r(\zeta_i'') \geq \rho_i(q_i - d_i) + (\rho_i + k)(d_i - 1) + (d_i' - 1) = \rho_i(q_i - 1) + k(d_i - 1) + (d_i' - 1).$$
If $k = 0$, then $d'_i = d_i$ and
\[
\rho_i(q_i - 1) + k(d_i - 1) + (d'_i - 1) = \rho_i(q_i - 1) + (d_i - 1).
\]
If $k > 0$, then
\[
\rho_i(q_i - 1) + k(d_i - 1) + (d'_i - 1) = \rho_i(q_i - 1) + (d_i - 1) + (k - 1)(d_i - 1) + (d'_i - 1)
\geq \rho_i(q_i - 1) + (d_i - 1).
\]

Now assume that $A(\beta) < L$ and $\text{ind}(u) = 1$. Equality is also proved by induction on $\rho_i$, and the case $\rho_i = 1$ is handled in the same way (i.e., by proving the result when $d_i = 1$). So assume $\rho_i > 1$ and $d_i > 1$. By Proposition 3.3.2, either $\rho'_i = \rho_i$ or $\rho'_i = \rho_i + 1$. The former is the case $k = 0$ above, where $d'_i = d_i$. Here, we know inductively that $w_\tau(\zeta''_i) = \rho_i(d_i - 1) + (d_i - 1) = (\rho_i + 1)(d_i - 1)$, so
\[
w_\tau(\zeta_i) = d_i^2 w_\tau(\zeta'_i) + w_\tau(\zeta''_i) = \rho_i(q_i - d_i) + (\rho_i + 1)(d_i - 1) = \rho_i(q_i - 1) + (d_i - 1).
\]
The latter is the case $k = 1$ above, where $d'_i = 1$. Here, we again know inductively that $w_\tau(\zeta''_i) = (\rho_i + 1)(d_i - 1)$, and equality follows as in the previous case. □

4.3. Linking numbers. Now we turn out attention to (4.1.3). By [HT2, Proposition 3.9], if $J$ is a generic almost complex structure on $\mathbb{R} \times Y$, any Fredholm index 1, connected, simple curve $u$ in $\mathbb{R} \times Y$ has no overlapping ends. In particular, the proof of [H1, Lemma 6.9] implies the following strengthened result.

**Lemma 4.3.1.** Let $u$ be a somewhere injective curve in a symplectization $\mathbb{R} \times Y$. Assume that the contact form on $Y$ is $L$-supersimple and that $\beta$ is an orbit in the negative orbit set $\beta$ of $u$. If $J$ is generic, then
\[
\ell_\tau(\zeta_i, \zeta_j) \geq \min(q_i \rho_j, q_j \rho_i).
\]
Equality holds if $A(\beta) < L$ and $\text{ind}(u) = 1$.

Now we put the preceding lemmas together to derive stronger versions of (4.1.4) and (4.1.5) in the $L$-supersimple setting; these new inequalities are implicit in work of Hutchings [H2].
**Lemma 4.3.2.** Let $u$ be a somewhere injective curve in a symplectization $\mathbb{R} \times Y$. Assume that the contact form on $Y$ is $L$-supersimple and that $\beta$ is a negative hyperbolic orbit in the negative orbit set $\beta$ of $u$. As in Notation 4.1.1, suppose that $u$ has negative ends of multiplicity $q_1, \ldots, q_n$ at $\beta$. In addition, order the ends of $u$ at $\beta$ so that $q_1 \geq q_2 \geq \cdots \geq q_k$ are the ends with odd multiplicity, ordered such that $q_1 \geq q_2 \geq \cdots \geq q_k$, and so that $q_{k+1}, q_{k+2}, \ldots, q_n$ are the ends with even multiplicity. Then

\begin{equation}
(w_{\tau}(\zeta)) \geq \sum_{i=1}^{m} \mu_{\tau}(\beta^i) - \sum_{i=1}^{n} \mu_{\tau}(\beta^{q_i}) + k \left( \frac{q_i - 1}{2} + 1 - 1 \right) + \sum_{i=k+1}^{n} (\rho_i - 1). \tag{4.3.2}
\end{equation}

Equality holds if $A(\beta) < L$ and $\text{ind}(u) = 1$.

**Proof.** Choose the trivialization $\tau$ so that $\mu_{\tau}(\beta) = 1$. Then note that

\[ \rho_i = \begin{cases} 
\frac{q_i - 1}{2}, & i = 1, 2, \ldots, k \\
\frac{q_i}{2}, & i = k + 1, k + 2, \ldots, n
\end{cases} \]

and

\[ d_i = \begin{cases} 
1, & i = 1, 2, \ldots, k \\
\rho_i, & i = k + 1, k + 2, \ldots, n
\end{cases} \]

Hence, with this choice of $\tau$, the inequalities (4.2.2) and (4.3.1) imply that

\begin{align*}
w_{\tau}(\zeta) &= \sum_{i=1}^{n} w_{\tau}(\zeta_i) + \sum_{i \neq j} \ell_{\tau}(q_i \rho_j, q_j \rho_i) \\
&\geq \sum_{i=1}^{n} [\rho_i(q_i - 1) + (d_i - 1)] + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i) \\
&= \sum_{i=k+1}^{n} (\rho_i - 1) + \sum_{i=1}^{n} \rho_i(q_1 - 1) + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i).
\end{align*}

By a computation in the proof of [H2, Lemma 4.19], we have

\[ \sum_{i=1}^{n} \rho_i(q_i - 1) + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i) = \sum_{i=1}^{m} \mu_{\tau}(\beta^i) - \sum_{i=1}^{n} \mu_{\tau}(\beta^{q_i}) + \sum_{i=1}^{k} \left( \frac{q_i - 1}{2} + i - 1 \right), \]

and the result follows. \qed

**Lemma 4.3.3.** Let $u$ be a somewhere injective curve in a symplectization $\mathbb{R} \times Y$. Assume that the contact form on $Y$ is $L$-supersimple and that $\beta$ is a
positive hyperbolic orbit in the negative orbit set $\beta$ of $u$. As in Notation 4.1.1, suppose that $u$ has negative ends of multiplicity $q_1, \ldots, q_n$ at $\beta$. Then

$$w_\tau(\zeta) \geq \sum_{i=1}^{m} \mu_\tau(\beta^i) - \sum_{i=1}^{n} \mu_\tau(\beta^q_i) + \sum_{i=1}^{n} (q_i - 1).$$

Equality holds if $\mathcal{A}(\beta) < L$ and $\text{ind}(u) = 1$.

**Proof.** Choose the trivialization $\tau$ so that $\mu_\tau(\beta_i) = 0$. Then note that $\rho_i = 0$ for $i = 1, 2, \ldots, n$, so that $d_i = \rho_i$. Hence, with this choice of $\tau$, the inequalities (4.2.2) and (4.3.1) imply that

$$w_\tau(\zeta) = \sum_{i=1}^{n} w_\tau(\zeta_i) + \sum_{i \neq j} \ell_\tau(q_i \rho_j, q_j \rho_i)$$

$$\geq \sum_{i=1}^{n} [\rho_i (q_i - 1) + (d_i - 1)] + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i)$$

$$= \sum_{i=1}^{n} (q_i - 1) + \sum_{i=1}^{n} \rho_i (q_i - 1) + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i).$$

With our choice of $\tau$, it is easy to see that

$$\sum_{i=1}^{n} \rho_i (q_i - 1) + \sum_{i \neq j} \min(q_i \rho_j, q_j \rho_i) = \sum_{i=1}^{m} \mu_\tau(\beta^i) - \sum_{i=1}^{n} \mu_\tau(\beta^q_i),$$

and the result follows. \qed

4.4. **Proof of the inequality.** The proofs of Lemmas 4.3.2 and 4.3.3 show that we can write

$$w_\tau(\zeta) = \sum_{i=1}^{m} \mu_\tau(\beta^i) - \sum_{i=1}^{n} \mu_\tau(\beta^q_i) + \Delta(u, \beta).$$

If $u$ has a positive end at $\alpha$, computations similar to those in Lemmas 4.3.2 and 4.3.3 show that

$$w_\tau(\zeta) \leq \sum_{i=1}^{m} \mu_\tau(\alpha^i) - \sum_{i=1}^{n} \mu_\tau(\alpha^q_i) - \Delta(u, \alpha)$$

and that equality holds if $\mathcal{A}(\alpha) < L$ and $\text{ind}(u) = 1$. Thus, if we set

$$w_\tau(u) = \sum_{\text{positive ends}} w_\tau(\zeta) - \sum_{\text{negative ends}} w_\tau(\zeta),$$

we have

$$w_\tau(u) \leq \mu^I_\tau(\alpha, \beta) - \mu_\tau(\alpha, \beta) - \Delta(u),$$

where $\mu^I_\tau(\alpha, \beta)$ is the obstruction term.

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and equality holds if $A(\alpha) < L$ and $\text{ind}(u) = 1$. Thus, $\Delta(u)$ measures how much the curve $u$ violates the ECH partition conditions at its ends.

\textbf{Proof of Theorem 1.2.5.} \ Recall the relative adjunction formula for somewhere injective curves:

$$c_1(u^*\xi, \tau) = \chi(\hat{\Sigma}) + Q_\tau(u) + w_\tau(u) - 2\delta(u).$$

By the above formula for the asymptotic writhe, we have

$$I(u) = c_1(u^*\xi, \tau) + Q_\tau(u) + \mu_\tau^L(\alpha, \beta)$$

$$= -\chi(\hat{\Sigma}) + 2c_1(u^*\xi, \tau) - w_\tau(u) + 2\delta(u) + \mu_\tau^L(\alpha, \beta)$$

$$\geq -\chi(\hat{\Sigma}) + 2c_1(u^*\xi, \tau) + \mu_\tau(\alpha) - \mu_\tau(\beta) + 2\delta(u) + \Delta(u)$$

$$= \text{ind}(u) + 2\delta(u) + \Delta(u).$$

Equality clearly holds if $A(\alpha) < L$ and $\text{ind}(u) = 1$. \hfill \Box

5. \textbf{Degenerations in Cobordisms}

In this section, we prove Theorem 1.3.1. We first recall the setup for that theorem. Let $(Y_\pm, \lambda_\pm)$ be $L$-supersimple contact 3-manifolds and let $(X, \lambda)$ be an exact symplectic cobordism from $(Y_+, \lambda_+)$ to $(Y_-, \lambda_-)$. Let $J$ be a generic, $L$-simple, admissible almost complex structure on the completion $(\hat{X}, \hat{\lambda})$ that restricts to $L$-simple, admissible almost complex structures $J_+$ and $J_-$ on the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, respectively, of $\hat{X}$. Let $\alpha$ be a generator of $ECC^L(Y_+, \lambda_+, J_+)$ and let $\beta$ be a generator of $ECC^L(Y_-, \lambda_-, J_-)$. Let $M = M^1_J(\alpha, \beta)$ denote the moduli space of punctured $J$-holomorphic curves in $\hat{X}$ with $\text{ind}(u) = I(u) = 1$ and asymptotic to $\alpha$ at the positive ends and to $\beta$ at the negative ends. Let $\overline{M}$ denote the SFT compactification of $M$ as described in \cite{BEHWZ}. We denote an SFT building in $\partial \overline{M}$ by $v_{-m} \cup v_{-m+1} \cup \ldots \cup v_0 \cup \ldots \cup v_{n-1} \cup v_n$, where $n$ and $m$ are positive integers and the levels go from bottom to top as we read from left to right.

Let $v_{-m} \cup v_{-m+1} \cup \ldots \cup v_0 \cup \ldots \cup v_{n-1} \cup v_n$ be an SFT building in $\partial \overline{M}$, where the levels with negative indices are in $\mathbb{R} \times Y_-$, the level $v_0$ is in $\hat{X}$, and the levels with positive indices are in $\mathbb{R} \times Y_+$. By Lemma 2.5.2, each level of the building has non-negative Fredholm index, and the symplectization levels have positive Fredholm index. Since $\text{ind}$ is additive and the total Fredholm index of the building is 1, there must be only one symplectization level, which has Fredholm index 1. Consequently, the cobordism level $v_0$ has Fredholm index 0. In the $L$-supersimple setting, we can say even more about these buildings.
5.1. Multiply covered curves. We begin a classification of multiply covered curves in $\hat{X}$ with non-positive ECH index.

**Lemma 5.1.1.** A $J$-holomorphic curve $u$ in $\hat{X}$ with $\text{ind}(u) = 0$ has negative ECH index if and only if it is an unbranched, disconnected cover of a $J$-holomorphic plane, in which case

$$I(u) = -\frac{d(d-1)}{2},$$

where $d$ is the degree of the covering.

**Proof.** Suppose that $I(u) < 0$. By the ECH index inequality (1.2.1), somewhere injective curves in cobordisms have non-negative ECH index, so $u$ must be a $d$-fold multiple cover of a somewhere injective curve $v: \hat{\Sigma}' \rightarrow \hat{X}$ with $\text{ind}(v) \geq 0$ and $d \geq 2$. Recall the index inequality

$$I(u) \geq d \cdot I(v) + \left(\frac{d^2 - d}{2}\right) \left(2g(\hat{\Sigma}') - 2 + \text{ind}(v) + h(v)\right)$$

from [H2], where $h(v)$ is the number of ends of $v$ at hyperbolic orbits. Since $h(v) \geq 1$ and $\text{ind}(v) \geq 0$, the only way for $I(u)$ to be negative is if $g(\hat{\Sigma}') = 0$ and $\text{ind}(v) + h(v) = 1$. Since $\text{ind}(u) = 0$, Lemma 2.5.2 implies that $u$ is an unbranched cover of $v$ and $\text{ind}(v) = 0$. Hence $h(v) = 1$. It follows that $u$ is an unbranched, disconnected cover of a plane $v$.

Suppose there is a component $\hat{\Sigma}$ of the domain of $u$ such that $\hat{\Sigma} \rightarrow \hat{\Sigma}'$ is an $m$-fold (unbranched) covering with $m \geq 2$. Then $m = \chi(\hat{\Sigma}) = 2 - 2g(\hat{\Sigma}) - m$, so $g(\hat{\Sigma}) = 1 - m < 0$ which is impossible. It follows that every component of the domain of $u$ maps diffeomorphically onto $\hat{\Sigma}$.

Conversely, suppose that $u: \hat{\Sigma} \rightarrow \hat{X}$ is such a cover of a plane $v: \hat{\Sigma}' \rightarrow \hat{X}$ with a positive end at a hyperbolic orbit $\gamma$. If we choose the trivialization $\tau$ of $\gamma^*\xi$ such that $c_1(v^*\xi, \tau) = 0$ we see that

$$0 = \text{ind}(v)$$

$$= -\chi(\hat{\Sigma}') + 2c_1(v^*\xi, \tau) + \mu_\tau(\gamma)$$

$$= \mu_\tau(\gamma) - 1,$$

so $\mu_\tau(\gamma) = 1$. Thus,

$$0 = I(v)$$

$$= c_1(v^*\xi, \tau) + Q_\tau(v) + \mu_\tau(\gamma)$$

$$= Q_\tau(v) + 1,$$
so \( Q_\tau(v) = -1 \). The relative self-intersection number \( Q_\tau \) is quadratic under taking multiple covers (see the discussion in [H2 Section 3.5]), so \( Q_\tau(u) = -d^2 \) and

\[
I(u) = c_1(u^*\xi, \tau) + Q_\tau(u) + \mu_\tau^I(\gamma)
\]

\[
= -d^2 + \sum_{i=1}^{d} i
\]

\[
= -\frac{d(d - 1)}{2},
\]

as desired. \( \square \)

**Lemma 5.1.2.** Let \( \alpha \) and \( \beta \) be orbit sets such that \( \alpha \) satisfies the ECH partition conditions and \( \beta \) is a generator of the ECH chain complex. Then a curve \( u \in \mathcal{M}_0^0(\alpha, \beta) \) with \( \text{ind}(u) = 0 \) is a multiple cover if and only if the underlying somewhere injective curve is a \( J \)-holomorphic cylinder with ECH index 0 and no negative ends. In this case, the map \( u \) is an immersion.

**Proof.** Assume first that \( u \) is a \( d \)-fold cover, \( d \geq 2 \), of a somewhere injective curve \( v: \hat{\Sigma}' \to \hat{X} \), which necessarily satisfies \( I(v) = 0 \). Since \( \text{ind}(u) = 0 \), Lemma [2.5.2] implies that \( u \) is necessarily an unbranched cover of \( v \). Hence \( u \) is an immersion. Since \( \beta \) is an ECH generator, it follows immediately that \( u \) has no negative ends. Since \( I(u) = 0 \), the inequality (5.1.1) implies that \( 2g(\hat{\Sigma}') - 2 + h(v) \leq 0 \). Thus, \( h(v) = 1 \) or \( 2 \). If \( h(v) = 1 \), then \( v \) is a plane and, by the arguments in the proof of Lemma [5.1.1], \( I(u) < 0 \). It follows that \( h(v) = 2 \) and \( v \) is a cylinder. Clearly \( v \) has no negative ends.

Now assume that \( u \) is a \( d \)-fold multiple cover, \( d \geq 2 \), of a \( J \)-holomorphic cylinder \( v: \hat{\Sigma}' \to \hat{X} \) with \( I(v) = 0 \) and no negative ends. Let \( \gamma_+ \) be the positive orbit set of \( v \). Note that the orbits in \( \gamma_+ \) must be positive hyperbolic \( u \) will not satisfy the partition conditions at the positive ends. Choose \( \tau \) such that \( \mu_\tau(\gamma) = 0 \) for all \( \gamma \in \gamma_+ \). Then we have

\[
0 = \text{ind}(v) = 2c_1(v^*\xi, \tau),
\]

and hence

\[
0 = I(v) = Q_\tau(v).
\]

It follows that \( c_1(u^*\xi, \tau) = Q_\tau(u) = 0 \), so \( I(u) = 0 \). \( \square \)

### 5.2. Canceling degenerations

Now we prove a sequence of lemmas that serve to eliminate various cases in our analysis of \( \partial\mathcal{M} \) by showing that certain types of buildings occur in canceling pairs.
Lemma 5.2.1. If $v_- \cup v_+$ is a two-level building in $\partial \overline{M}$, then the symplectization level is somewhere injective.

Proof. Without loss of generality, assume that $v_+$ is the symplectization level. If it is multiply covered, Lemma 2.5.2 implies it is a branched cover of a trivial cylinder in $\mathbb{R} \times Y_+$, contradicting the assumption that its positive orbit set $\alpha$ is a generator of the ECH chain complex for $(Y_+, \lambda_+)$.

Lemma 5.2.2. If $v_- \cup v_+$ is a two-level building in $\partial \overline{M}$, then $I(v_+) \geq 0$.

Proof. First assume that $v_+$ is the symplectization level, so that $\text{ind}(v_+) = 1$. Then $v_+$ is somewhere injective by Lemma 5.2.1 so $I(v_+) \geq 1$ by Hutchings’ index inequality (1.2.1).

Now assume that $v_+$ is the cobordism level, so that $\text{ind}(v_+) = 0$. If $I(v_+) < 0$, then $v_+$ must contain an unbranched, disconnected cover of a plane by Lemma 5.1.1. The underlying embedded plane cannot have a negative end, by the maximum principle, so $\alpha$ must contain a Reeb orbit with multiplicity greater than 1, which contradicts the assumption that it is a generator of $ECC(Y_+, \lambda_+, J_+)$. □

Lemma 5.2.3. The count of buildings $v_- \cup v_+$ in $\partial \overline{M}$ with $I(v_-) \geq 0$ and such that $\gamma$ has at least one orbit of multiplicity greater than 1 is even.

Proof. First assume that $v_+$ is the symplectization level. Then it is somewhere injective by Lemma 5.2.1 and $I(v_+) \geq 1$ by the proof of Lemma 5.2.2. Since $I(v_-) \geq 0$ and $I(v_-) + I(v_+) = 1$, we see that in fact $I(v_-) = 0$. By Corollary 4.0.2, $v_+$ satisfies the ECH partition conditions. Hence so does $v_-$, since its negative orbit set $\beta$ is a generator of the ECH chain complex for $(Y_-, \lambda_-)$. But then the multiply covered components of $v_-$ are unbranched covers of cylinders with no negative ends by Lemma 5.1.2 and the count of such buildings is even.

Now assume that $v_+$ is the cobordism level. Then $v_-$ is somewhere injective by Lemma 5.2.1, so by Hutchings’ index inequality (1.2.1), $I(v_-) \geq \text{ind}(v_-) = 1$. By the same argument as above, we conclude that $I(v_-) = \text{ind}(v_-) = 1$ and $I(v_+) = 0$. By Corollary 4.0.2, $v_-$ satisfies the ECH partition conditions. If $v_+$ is multiply covered, then its multiply covered components are unbranched covers of cylinders with no negative ends by Lemma 5.1.2. This gives a contradiction, as $\alpha$ cannot be a generator of $ECC(Y_+\!, \lambda_+\!, J_+)$. Hence $v_+$ is also somewhere injective. Since $\gamma$ has a hyperbolic orbit with multiplicity greater than 1, $v_+$ must either have multiple negative ends with multiplicity 1 asymptotic to the same positive hyperbolic orbit or at least one negative end with
multiplicity 2 asymptotic to a double cover of a negative hyperbolic orbit. In either case, the count of such buildings is even. □

5.3. The proof of Theorem 1.3.1. Note that $v_-$ must be the cobordism level. Let $\gamma$ denote the negative orbit set of $v_+$, and let $n_\gamma$ denote the multiplicity of the orbit $\gamma$ in $\gamma$. By Lemma 5.1.1 $v_-$ must be multiply covered and must contain at least one unbranched cover of a plane. Let $\Gamma^+(v_-)$ denote the subset of orbits $\gamma$ such that $v_-$ contains an unbranched cover of a plane with its positive end at $\gamma$. For each $\gamma \in \Gamma^+(v_-)$, let $m_\gamma$ denote the multiplicity of the covering of the plane with its positive end at $\gamma$. By Theorem 1.2.5.

\begin{equation}
I(v_+) \geq 1 + \sum_{\gamma \in \Gamma^+(v_-)} \left( \frac{m_\gamma}{2} \right)
\end{equation}

and

\begin{equation}
I(v_-) \geq - \sum_{\gamma \in \Gamma^+(v_-)} \left( \frac{m_\gamma}{2} \right).
\end{equation}

Since $I(v_-) + I(v_+) = 1$, both inequalities must in fact be equalities.

We claim that buildings where $v_-$ contains other multiply covered components occur in canceling pairs. Any multiple covers besides the planes have non-negative ECH index. Covers with ECH index 0 are unbranched covers of cylinders with no negative ends. No such connected covers have multiplicity greater than 2, as then the inequality (5.3.1) is strict and $I(v_-) + I(v_+) > 1$. Buildings where $v_-$ has connected covers of such cylinders with multiplicity 2 or disconnected covers with multiplicity at least 2 clearly occur in canceling pairs. There are no multiply covered components of $v_-$ with positive ECH index, as then the inequality (5.3.2) is again strict and $I(v_-) + I(v_+) > 1$.

We now claim that buildings where there exists a $\gamma \in \Gamma^+(v_-)$ with $m_\gamma < n_\gamma$ occur in canceling pairs. So assume that such a $\gamma$ exists. If $v_-$ has a non-planar component with a positive end asymptotic to $\gamma^k$ for $k$ odd, then (5.3.1) is a strict inequality and $I(v_-) + I(v_+) > 1$. If $v_-$ has a non-planar component with a positive end asymptotic to $\gamma^k$ for $k \geq 4$ even, then again $I(v_-) + I(v_+) > 1$. The buildings where $v_-$ has a non-planar component with a positive end asymptotic to $\gamma^2$ occur in canceling pairs.

Finally, we claim that every Reeb orbit $\gamma$ in $\Gamma^+(v_-) \setminus \Gamma^+(v_-)$ has multiplicity 1. So suppose not. By Theorem 1.2.5 the negative ends of $v_+$ at covers of $\gamma$ satisfy the ECH partition conditions. If $\gamma$ is positive hyperbolic, there are at
least two negative ends of \( v^+ \) of multiplicity 1 at \( \gamma \), and such buildings occur in canceling pairs. If \( \gamma \) is negative hyperbolic, there is at least one negative end of \( v^+ \) at \( \gamma ^2 \), and such buildings again occur in canceling pairs.

6. Obstruction Bundle Gluing

In this section, we set up the gluing machinery in preparation for the proof of Theorem 1.4.5 in Section 7. We first review the prototypical gluing problem from Section 1. Recall that \((Y, \lambda)\) is a smooth 3-manifold with an \( L \)-supersimple contact form, and \( u^+ : \Sigma \to \mathbb{R} \times Y \) is a holomorphic embedding with \( \text{ind}(u^+) = 1 \) such that

1. the positive ends of \( u^+ \) are asymptotic to an ECH generator \( \alpha \) with \( A(\alpha) < L \);
2. the negative ends of \( u^+ \) are asymptotic to an orbit set \( \beta \) in which each Reeb orbit has multiplicity 1 except for a single negative hyperbolic orbit \( \beta_0 \);
3. the curve \( u^+ \) has \( n \) negative ends at \( \beta_0 \), each with multiplicity 1;
4. \( I(u^+) = 1 + \binom{n}{2} \).

Recall that, for each \( n \geq 3 \), we set \( M_n = M(1,1,\ldots,1|1,1,\ldots,1,3) \), where there are \( n \) positive ends of multiplicity 1, \( n - 3 \) negative ends of multiplicity 1, and one negative end of multiplicity 3. We wish to glue branched covers in \( M_n \) to the curve \( u^+ \) above. Note that each branched cover in \( M_n \) has total branching index \( 2n - 4 \).

**Proposition 6.0.1.** Let \((Y, \lambda)\) be a non-degenerate contact 3-manifold, and let \( J \) be a generic \( \mathbb{R} \)-invariant almost complex structure on \( \mathbb{R} \times Y \). Let \( \alpha \) be a negative hyperbolic Reeb orbit of \( \lambda \), and let \( u \) be a branched cover of the trivial cylinder \( \mathbb{R} \times \alpha \) in \( \mathbb{R} \times Y \). If \( u \) has \( k \) branch points, counted with multiplicity, then \( \text{ind}(u) = k \) and \( \dim \text{Coker} D_u^N = k \). In particular, the obstruction bundle \( \mathcal{O} \to [R, \infty) \times (M_n/\mathbb{R}) \) has rank \( 2n - 4 \).

**Proof.** The computation of \( \text{ind}(u) \) follows immediately from Lemma 2.5.2. From [W Theorem 3], we know that \( \dim \ker D_u^N = \dim \ker D_{\nabla J}^N - 2k = 0 \). From the computation immediately preceding that theorem, we also know that \( \text{ind}(D_u^N) = \text{ind}(u) - 2k = -k \), so \( \dim \text{coker} D_u^N = k \), as desired. \( \square \)
Notation 6.0.2. For any two disjoint subsets \( \{p_1, \ldots, p_n\} \) and \( \{q_1, \ldots, q_{n-2}\} \) of \( \mathbb{C} \), where the \( p_i \) and \( q_j \) are pairwise distinct, we set

\[
A(z) = \prod_{i=1}^{n} (z - p_i), \quad A_k(z) = \prod_{i=1, i \neq k}^{n} (z - p_i),
\]

\[
A_k(z) = \prod_{i=1}^{n-2} (z - p_i), \quad A_k(z) = \prod_{i=1, i \neq k}^{n} (z - p_i),
\]

\[
B(z) = \prod_{i=2}^{n-2} (z - q_i), \quad B_k(z) = \prod_{i=2, i \neq k}^{n-2} (z - q_i).
\]

Note that we suppress the dependence on \( n \) for the functions considered above.

6.1. Parametrization of the moduli space. We parametrize the reduced moduli space \( \mathcal{M}_n/\mathbb{R} \) by choosing a smooth section of the bundle \( \mathcal{M}_n \to \mathcal{M}_n/\mathbb{R} \) in the following way. Curves in \( \mathcal{M}_n \) have genus 0, so the domain for each map is a punctured Riemann sphere \( \hat{\mathbb{C}} \setminus (P^+ \cup P^-) \), where \( P^+ = \{p_1, p_2, \ldots, p_n\} \) and \( P^- = \{q_1, q_2, \ldots, q_{n-2}\} \) are the (disjoint) sets of positive and negative punctures and \( q_1 \) is the multiplicity 3 negative puncture. View \( \hat{\mathbb{C}} \) as \( \mathbb{C} \cup \{\infty\} \), fix the positive punctures \( p_{n-1} \) and \( p_n \) in \( \mathbb{C} \), and fix the negative puncture \( q_1 \) to be the point at infinity. The other punctures \( p_1, \ldots, p_{n-2}, q_2, \ldots, q_{n-2} \) are free to move in \( \mathbb{C} \). Then the data consisting of the punctures \( p_1, \ldots, p_{n-2}, q_2, \ldots, q_{n-2} \in \mathbb{C} \) and \( \theta \in \mathbb{R}/6\pi\mathbb{Z} \) are sent to the map

\[
u : \hat{\mathbb{C}} \setminus (P^+ \cup P^-) \to \mathbb{C}^*
\]

\[
z \mapsto e^{i\theta} \frac{B(z)}{A(z)}.
\]

Roughly speaking, changing the parameter \( \theta \) simultaneously rotates the branch points of \( u \) in the \( S^1 \)-factor of the image cylinder.

The asymptotic marker \( \tau \in S^1 \) at each puncture is determined as follows. For a positive puncture \( p_i \), there is an \( \epsilon > 0 \) and a complex-valued function \( f(t), 0 < t < \epsilon \), such that \( \lim_{t \to 0^+} f(t) = 0 \) and \( u(p_i + f(t)) = e^{1/t} \). Then

\[
(6.1.1) \quad \tau_i = \lim_{t \to 0^+} \left| \frac{f(t)}{|f(t)|} \right| \left| \frac{B(p_i)}{A_i(p_i)} \right|^n \left| \frac{B(p_i)}{A_i(p_i)} \right|^{-1}.
\]

For a negative puncture \( q_j, j = 2, \ldots, n-2 \), there is an \( \epsilon > 0 \) and a complex-valued function \( f(t), 0 < t < \epsilon \), such that \( \lim_{t \to 0^+} f(t) = 0 \) and \( u(q_j + f(t)) = e^{-1/t} \).
Then
\[ \tau_{-j} = \lim_{t \to 0^+} \frac{f(t)}{|f(t)|} = e^{-i\theta} \frac{A(q_j)}{B_j(q_j)} \left| \frac{A(q_j)}{B_j(q_j)} \right|^{-1}. \]

For \( q_1 \), there is an \( \epsilon > 0 \) and a complex-valued function \( f(t), 0 < t < \epsilon \), such that \( \lim_{t \to 0^+} f(t) = 0 \) and \( u(\frac{1}{f(t)}) = e^{-1/t} \). Then
\[ \tau_{3-1} = \lim_{t \to 0^+} \frac{f(t)}{|f(t)|} = e^{-i\theta}. \]

6.2. The obstruction sections. Recall from [HT1, HT2] that there is a linearized obstruction section \( s_1 \) homotopic to the full obstruction section \( s \) and defined as follows. Assume that the positive ends of \( u \) and the negative ends of \( u_+ \) are labeled so that the \( i \)th positive end of \([u] \in \mathcal{M}_n/\mathbb{R}\) matches up with the \( i \)th negative end of \( u_+ \). We first restrict our attention to the \( i \)th positive end of \( u \). Consider the asymptotic expansion of \( u_+ \) over its \( i \)th negative end, written in cylindrical coordinates, and let \( \alpha_i \) denote its projection onto the leading eigenspace of the asymptotic operator \( A_{\beta_0} \) from Section 3.1. Let \( \sigma \in \text{Coker}(D_u^N) \) and let \( \sigma_i \) denote the restriction of \( \sigma \) to the \( i \)th positive end of \( u \), written in cylindrical coordinates. Then
\[ s_1(T, u)(\sigma) = \sum_{i=1}^n \langle \alpha_i, \sigma_i(T, \cdot) \rangle. \]

For each \( m > 1 \), we define a section \( s_m \) of \( \mathcal{O} \) that is similar to \( s_1 \) except that, on the \( i \)th positive end of \( u \in \mathcal{M}_n \), we project the asymptotic expansion of the \( i \)th negative end of \( u_+ \) to the direct sum of the leading \( m \) eigenspaces of \( A_{\beta_0,1} \) before taking the inner product with \( \sigma_i \). Thus, if the projection of the \( i \)th negative end of \( u_+ \) is
\[ \Pi_{i,m}u_+ = \alpha_{i,1}e^{-\lambda_1s}f_1(t) + \alpha_{i,2}e^{-\lambda_2s}f_2(t) + \cdots + \alpha_{i,m}e^{-\lambda_ms}f_m(t), \]
then
\[ s_m(T, u)(\sigma) = \sum_{i=1}^n \langle \Pi_{i,m}u_+, \sigma_i(T, \cdot) \rangle. \]

Notation 6.2.1. We denote the zero set \( s_1^{-1}(0) \) by \( \mathcal{Z}_m \). We denote the zero set \( s^{-1}(0) \) of the full obstruction section by \( \mathcal{Z} \).

---

1 Hutchings-Taubes use the notation \( s_0 \) for the linearized obstruction section. However, we use a \( \mathbb{Z}_+ \)-indexed family of sections homotopic to \( s \) in this paper, and it makes more sense for the linearized section to be written as \( s_1 \).
6.3. **A basis for the cokernel.** We now choose a convenient basis for the space \( \text{Coker}(D_N^u) \), which we identify with \( \text{Ker}(D_N^u)^* \). If \( \sigma \in \text{Coker}(D_N^u) \) and \( \tau \) is a trivialization of \( \xi \) over \( \beta_0 \), let \( \text{wind}_\tau(\sigma_i) \) denote the **asymptotic winding number** of \( \sigma \) restricted to the \( i^{th} \) positive end of \( u \) in the trivialization \( \tau \), defined as follows. On the \( i^{th} \) positive end, write \( \sigma = \sigma_i \otimes (ds - idt) \) in cylindrical coordinates. Then \( \text{wind}_\tau(\sigma_i) \) is defined as the winding number of the leading asymptotic eigenfunction in the series expansion of \( \sigma_i \). Recall from [HWZ, Section 3] that, for each positive end of \( u \), we have \( 2 \text{wind}_\tau(\sigma_i) \geq \mu_\tau(\beta_0) \) and for each negative end, we have \( 2 \text{wind}_\tau(\sigma_i) \leq \mu_\tau(\beta_0) \).

**Lemma 6.3.1.** If \( u \in \mathcal{M}_n \), where \( n \geq 3 \), and \( \sigma \in \text{Coker}(D_N^u) \), then \( |\#\sigma^{-1}(0)| \leq n - 3 \), where the zeros of \( \sigma \) are counted with multiplicities.

**Proof.** Note that every zero of \( \sigma \) has negative multiplicity. We first compute 
\[
\chi(\hat{\Sigma}) = 4 - 2n.
\]
On the ends of \( u \), write 
\[
\text{wind}_\tau(\sigma_i) = \left\lfloor \frac{\mu_\tau(\beta_0)}{2} \right\rfloor + k_i
\]
for \( i > 0 \),
\[
\text{wind}_\tau(\sigma_j) = \left\lfloor \frac{\mu_\tau(\beta_0)}{2} \right\rfloor - k_j
\]
for \( j = -2, \ldots, -(n - 2) \), and
\[
\text{wind}_\tau(\sigma_{-1}) = \left\lfloor \frac{\mu_\tau(\beta_0^3)}{2} \right\rfloor - k_{-1}.
\]
Then, choosing \( \tau \) so that \( \mu_\tau(\beta_0) = 1 \), we have
\[
0 \geq |\#\sigma^{-1}(0)|
\]
\[
= \chi(\hat{\Sigma}) + \sum_{i=1}^{n} \text{wind}_\tau(\sigma_i) - \sum_{j=1}^{n-2} \text{wind}_\tau(\sigma_{-j})
\]
\[
= 4 - 2n + \sum_{i=1}^{n} \left\lfloor \frac{\mu_\tau(\beta_0)}{2} \right\rfloor - \sum_{j=2}^{n-2} \left\lfloor \frac{\mu_\tau(\beta_0)}{2} \right\rfloor - \left\lfloor \frac{\mu_\tau(\beta_0^3)}{2} \right\rfloor + \sum_{i=1}^{n} k_i + \sum_{j=1}^{n-2} k_{-j}
\]
\[
= 3 - n + \sum_{i=1}^{n} k_i + \sum_{j=1}^{n-2} k_{-j}
\]
\[
\geq 3 - n,
\]
as claimed. \( \square \)
Remark 6.3.2. The proof of Lemma 6.3.1 also shows that a cokernel element $\sigma$ cannot be too degenerate at the ends. More precisely, we have

\begin{equation}
0 \leq \#\sigma^{-1}(0) + \sum_{i=1}^{n} k_i + \sum_{j=1}^{n-2} k_{-j} \leq n - 3.
\end{equation}

**Proposition 6.3.3.** There exists a basis $\sigma^1, \sigma^2, \ldots, \sigma^{2n-5}, \sigma^{2n-4}$ for $\text{Ker}(D_u^N)^*$ such that, for each $i = 1, 2, \ldots, n-2$, the projection of $\{\sigma^{2i-1}, \sigma^{2i}\}$ to the leading eigenspace on the $j$th positive end of $u$ is a basis for that eigenspace if $j = i$, $n - 1$, or $n$ and vanishes otherwise.

**Proof.** Let $\sigma^1, \sigma^2, \ldots, \sigma^{2n-5}, \sigma^{2n-4}$ be a basis for $\text{Ker}(D_u^N)^*$. We give an algorithm to converting this basis into one with the desired properties.

First, note that there must be a pair of basis elements whose projections to the leading eigenspace on the positive end labeled 1 are linearly independent. For if not, then row reduction yields a cokernel element $\sigma$ with $k_1 \geq n - 2$, which contradicts Remark 6.3.2. After possibly relabeling the elements of the basis, we may assume that $\sigma^1$ and $\sigma^2$ are the above basis elements. By subtracting appropriate multiples of $\sigma^1$ and $\sigma^2$ from the other basis elements, we may assume that $k_1 \geq 1$ for each $\sigma^i$ with $i \neq 1, 2$.

Assume that the elements $\sigma^1, \sigma^2, \ldots, \sigma^{2\ell-1}, \sigma^{2\ell}$ are such that, for each $i = 1, 2, \ldots, \ell$, the projection of $\{\sigma^{2i-1}, \sigma^{2i}\}$ to the leading eigenspace on the $j$th positive end of $u$, is a basis for that eigenspace if $j = i$ and vanishes if $1 \leq j \leq \ell$ and $j \neq i$. Assume also that, for each $\sigma^i$ with $i = 2\ell+1, 2\ell+2, \ldots, 2n-5, 2n-4$, we have $k_j \geq 1$ for $j = 1, 2, \ldots, \ell$. There must be a pair of vectors among $\sigma^{2\ell+1}, \sigma^{2\ell+2}, \ldots, \sigma^{2n-5}, \sigma^{2n-4}$ whose projections to the leading eigenspace on the positive end labeled $\ell + 1$ are linearly independent. For if not, row reduction yields a cokernel element $\sigma$ with $\sum_{j=1}^{\ell+1} k_j \geq n - 2$, which contradicts Remark 6.3.2. After possibly relabeling $\sigma^{2\ell+1}, \sigma^{2\ell+2}, \ldots, \sigma^{2n-5}, \sigma^{2n-4}$, we may assume that $\sigma^{2\ell+1}$ and $\sigma^{2\ell+2}$ are the above basis elements. By subtracting appropriate multiples of $\sigma^{2\ell+1}$ and $\sigma^{2\ell+2}$ from $\sigma^{2\ell+3}, \sigma^{2\ell+4}, \ldots, \sigma^{2n-5}, \sigma^{2n-4}$, we may assume that $k_{\ell+1} \geq 1$ for each $\sigma^i$ with $i \neq 2\ell + 1, 2\ell + 2$.

After step $n-2$ of this algorithm, we arrive at our desired basis. \hfill \Box

### 6.4. Deformation of the asymptotic operator

To make our calculations easier, we now replace the elements of $\text{Coker}(D_u^n)$, which we identify with $\text{Ker}(D_u^N)^*$, with meromorphic (0, 1)-forms by perturbing the asymptotic operator $A_{\beta_0}$ for covers of $\beta_0$. First, define a homotopy of the asymptotic operator...
by
\[ A_{\beta_0^k,\nu} = -j_0 \frac{\partial}{\partial t} - \left( \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \right) - (1 - \nu) \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \]
where \( \nu \in [0,1] \). When \( k \) is odd, the operator is non-degenerate throughout the homotopy. However, when \( k \) is even, the operator is non-degenerate when \( 0 \leq \nu < 1 \) and singular when \( \nu = 1 \). More specifically, let \( \lambda_{+,\nu} \) denote the smallest positive eigenvalue of \( A_{\beta_0^k,\nu} \) and \( \lambda_{-\nu} \) the largest negative eigenvalue. Then both \( \lambda_{+,\nu} \) and \( \lambda_{-\nu} \) monotonically converge to 0 as \( \nu \to 1 \).

We correct for the degeneration by putting some asymptotic weights \( \delta_\nu = (\delta_\nu, \ldots, \delta_\nu) \) on our Sobolev spaces for \((D_N^u)^*\), where \( \delta_\nu = (1 - \nu)\lambda_{-,0} + \nu \delta \) and \( \delta \) is a sufficiently small positive real number. When \( \nu = 1 \), the operator \((D_N^u)^*\) is complex-linear, and the elements of \( \text{Ker}(D_N^u)^* \) are, in cylindrical coordinates near the positive ends, equal to \( \sigma(s,t) \otimes (ds - idt) \), where \( \sigma(s,t) \) satisfies the equation
\[ (\sigma_i)_s - i(\sigma_i)_t + \frac{1}{2} \sigma_i = 0. \]

If we set \( \eta_i(s,t) = e^{-s/2} \sigma_i(s,t) \) over such an end, we see that \( \eta_i \) is antimeromorphic in the usual sense. Finally, we single out the real 1-dimensional subspace of the 0-eigenspace of \( A_{\beta_0^k,1} \) that corresponds to the \( \lambda_{+,0} \)-eigenspace of \( A_{\beta_0^k,0} \) by requiring that the leading eigenfunction in the asymptotic expansion of \( \eta \) near an even-multiplicity end be a real scalar multiple of the vector in \( \mathbb{C} \) representing the stable direction of \( \beta_0^k \). We say that the leading eigenfunction follows the stable direction of \( \beta_0^k \).

**Definition 6.4.1.** A meromorphic \((0,1)\)-form \( \eta \) is a replacement for \( \sigma \in \text{Ker}(D_N^u)^* \) if, in cylindrical coordinates \((s,t)\) near each puncture, we have \( \eta(s,t) = e^{-s/2} \sigma(s,t) \).

**Remark 6.4.2.** The point of using replacements instead of using elements of \( \text{Ker}(D_N^u)^* \) directly is that we can write down explicit expressions for replacements, and hence explicit equations for the zero sets \( \mathcal{Z}_m \) and \( \mathcal{Z} \).

**6.5. The gluing problem.** We now write down a collection of meromorphic \((0,1)\)-forms on \( \hat{\Sigma} \) that are replacements, in the sense described above, for the basis from Proposition [6.3.3]

**Notation 6.5.1.** Set
\[ Q_k(z) = \frac{A_k(z)}{B(z)}. \]
for \( k = 1, \ldots, n - 2 \). For future notational convenience, set

\[ r_i = \frac{B(p_i)}{A_i(p_i)}. \]

(6.5.1)

for \( i = 1, \ldots, n \), set

\[ r_{-1} = 1, \]

and set

\[ r_{-j} = \frac{A(q_j)}{B_j(q_j)} \]

for \( j = 2, \ldots, n - 2 \).

**Proposition 6.5.2.** The meromorphic \((0,1)\)-forms

\[ \eta_k(z) = \overline{Q_k(z)}d\bar{z} \]

are replacements for a basis of \( \text{Ker}(D^N_u)^* \) as constructed in Proposition 6.3.3.

**Proof.** Near \( p_i \), we have \( z = p_i + \tau_i e^{-s-it} \), where \((s, t) \in [R, \infty) \times (\mathbb{R}/2\pi\mathbb{Z})\). Hence

\[ u(z) = e^{i\theta} e^{s+it} \frac{B(p_i + e^{-s-it})}{A_i(p_i + e^{-s-it})}, \]

so

\[ \log |u(z)| = s + \log \left| \frac{B(p_i + e^{-s-it})}{A_i(p_i + e^{-s-it})} \right|. \]

Thus, we must change our \( s \)-coordinate to

\[ \tilde{s} = s + \log \left| \frac{B(p_i + e^{-s-it})}{A_i(p_i + e^{-s-it})} \right|. \]

If \( s \gg 0 \), we have

\[ \tilde{s} \approx s + \log \left| \frac{B(p_i)}{A_i(p_i)} \right| = s + \log |r_i|. \]

A similar change must be made in cylindrical coordinates around the negative punctures \( q_j, j = 2, \ldots, n - 2 \).

Now fix a value of \( k \). We claim that each \( \eta_k \) has winding number 1 at \( p_k \), \( p_{n-1} \), and \( p_n \), has winding number 2 at all other \( p_i \), has winding number 1 at \( q_1 \), and has winding number 0 at all other \( q_j \).

If we change to cylindrical coordinates around \( p_k \), we can write \( z = p_k + \tau_k e^{-s-it}, (s, t) \in [R, \infty) \times (\mathbb{R}/2\pi\mathbb{Z}) \). Then the first term in the asymptotic expansion of \( \eta_k \) in these coordinates is approximately

\[ -\overline{\tau_k r_k Q_k(p_k)} e^{-\tilde{s}+it} \otimes (d\tilde{s} - idt), \]
which has winding number 1. Similarly, the winding number of \( \eta_k \) in cylindrical coordinates on the positive ends at \( p_{n-1} \) and \( p_n \) is also 1. The first term in the asymptotic expansion of \( \eta_k \) vanishes in cylindrical coordinates around \( p_i, i \neq k, n-1, n \), and the winding number at each of those ends is 2.

If we change to cylindrical coordinates around \( q_j, j = 2, \ldots, n-2 \), we can write

\[
\eta_k(z) = \left[ \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k(p_i)}{dz^\ell} (p_i)(z-p_i)^\ell \right] \frac{d}{dz}
\]

which has winding number 0. For \( q_1 \), if we change coordinates to \( \zeta = \frac{1}{z} \), we see that

\[
\eta_k = -\frac{1}{\zeta} \frac{A_k(\zeta)}{B(\zeta)} d\zeta.
\]

Hence, if we change to cylindrical coordinates around \( \zeta = 0 \), we can write

\[
\eta_k(z) = \frac{1}{\tau_{-1}|r_{-1}|} e^{-(\bar{s}+it)/\tau} \approx -\frac{1}{\tau_{-1}|r_{-1}|} \left( d\bar{s} - idt \right),
\]

which has winding number 1.

\[\Box\]

**Corollary 6.5.3.** If \( R \gg 0 \), the section \( s_m \) on \([R, \infty) \times (M_n/\mathbb{R})\) is close to a section whose zero set is defined by

\[(6.5.2) \quad 0 = \sum_{i=1}^{n} \sum_{\ell=1}^{m} \frac{1}{(\ell-1)!} \frac{d^{\ell-1}Q_k(p_i)}{dz^{\ell-1}} \frac{B(p_i)^\ell}{A_i(p_i)^{\ell+1}} e^{-\epsilon r} e^{i\theta_\ell} \alpha_{i,\ell},\]

where \( k = 1, 2, \ldots, n-2 \).

**Proof.** We make the same change to the s-coordinate near a positive puncture as in Proposition 6.5.2. Near \( p_i \), we have

\[
\eta_k(z) = \left[ \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k(p_i)}{dz^\ell} (p_i)(z-p_i)^\ell \right] \frac{d}{dz}
\]

near \( p_i \), we have

\[
\eta_k(z) = -\left[ \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k(p_i)}{dz^\ell} (p_i)e^{-\ell s} \right] \tau_{i} e^{-s-\ell t} \approx -\left( d\bar{s} + idt \right).
\]

Hence, if we change to cylindrical coordinates around \( \zeta = 0 \), we can write
\[
- \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k}{dz^\ell} (p_i) r_i^{\ell+1} |r_i|^{\ell+1} e^{-\ell(\tilde{s} + it)} e^{\tilde{s} - it} \otimes (d\tilde{s} + idt)
\]

\[
- \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k}{dz^\ell} (p_i) e^{i(\ell+1)\theta} r_i^{\ell+1} e^{-(\ell+1)(\tilde{s} + it)} \otimes (d\tilde{s} + idt)
\]

Since the obstruction bundle $O$ is complex in this case, we can, following Hutchings-Taubes [HT1], identify the section $s_m$ with a section $s_m^C$ defined by

\[
s_m^C(T, u)(\sigma) = s_m(T, u)(\sigma) + is_m(T, u)(-i\sigma).
\]

As noted in [HT1], the definition of $s_m^C$ is equivalent to the replacing the real inner products in the original definition with complex inner products. For the rest of this paper, we implicitly identify $s$ and $s_m$ with their complexified versions $s^C$ and $s_m^C$. Making this identification, we have

\[
s_m(T, [u])(\eta_k) = \sum_{i=1}^{n} \left( \sum_{\ell=0}^{m-1} \alpha_{i,\ell+1} e^{i(\ell+1)\eta_k} - \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell Q_k}{dz^\ell} (p_i) r_i^{\ell+1} e^{-i(\ell+1)\theta} e^{-(\ell+1)T} e^{(\ell+1)it} \right)
\]

\[
= - \sum_{i=1}^{n} \sum_{\ell=0}^{m-1} \frac{1}{(\ell-1)!} \frac{d^{\ell-1} Q_k}{dz^{\ell-1}} (p_i) r_i^\ell e^{i\theta} e^{-\ell T} \alpha_{i,\ell+1}
\]

and the result follows. \(\square\)

Remark 6.5.4. We can recast the system of equations given by (6.5.2), $k = 1, \ldots, n - 2$, as a matrix equation, which will be useful in the proof of Theorem 7.1.2

\[
\sum_{\ell=1}^{m} \frac{1}{(\ell-1)!} \begin{pmatrix}
\frac{d^{\ell-1} Q_1}{dz^{\ell-1}} (p_1) & \cdots & \frac{d^{\ell-1} Q_1}{dz^{\ell-1}} (p_n) \\
\vdots & \ddots & \vdots \\
\frac{d^{\ell-1} Q_{n-1}}{dz^{\ell-1}} (p_1) & \cdots & \frac{d^{\ell-1} Q_{n-1}}{dz^{\ell-1}} (p_n)
\end{pmatrix}
\begin{pmatrix}
r_1^\ell e^{i\theta} e^{-\ell T} \alpha_{1,\ell} \\
\vdots \\
r_n^\ell e^{i\theta} e^{-\ell T} \alpha_{n,\ell}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

For future use, set

\[
B_\ell = \frac{1}{(\ell-1)!} \begin{pmatrix}
\frac{d^{\ell-1} Q_1}{dz^{\ell-1}} (p_1) & \cdots & \frac{d^{\ell-1} Q_1}{dz^{\ell-1}} (p_n) \\
\vdots & \ddots & \vdots \\
\frac{d^{\ell-1} Q_{n-1}}{dz^{\ell-1}} (p_1) & \cdots & \frac{d^{\ell-1} Q_{n-1}}{dz^{\ell-1}} (p_n)
\end{pmatrix}
\]

(6.5.3)
It will also be useful in Section 7 to note that the partial fraction decomposition of \( Q_i(z) \) is

\[
Q_i(z) = 1 - \sum_{k=2}^{n-2} \frac{A_i(q_k)}{B_k(q_k) q_k - z}.
\]

**Corollary 6.5.5.** The equations

\[
(p_{n-1} - p_n)\alpha_{k,1} - (p_k - p_n)\alpha_{n-1,1} + (p_k - p_{n-1})\alpha_{n,1} = 0,
\]

\( k = 1, 2, \ldots, n-2 \), determine \( Z_1 \). Moreover, if \( (T, [u]) \in Z_1 \), then \( p_1, p_2, \ldots, p_{n-2} \) are determined by \( p_{n-1}, p_n \), and the coefficients \( \alpha_{1,1}, \alpha_{2,1}, \ldots, \alpha_{n,1} \).

**Proof.** We use the notation \( r_i \) from Notation 6.5.1. Note that \( Q_k(p_i) = 0 \) when \( i \neq k, n - 1, \) or \( n \). Note also that, when \( m = 1 \), (6.5.2) has an overall factor of \( e^{-Te^{i\theta}} \). Thus, (6.5.2) reduces to

\[
0 = \sum_{i=1}^{n} Q_k(p_i) r_i \alpha_{i,1}
\]

\[
= Q_k(p_k) r_k \alpha_{k,1} + Q_k(p_{n-1}) r_{n-1} \alpha_{n-1,1} + Q_k(p_n) r_n \alpha_{n,1}
\]

\[
= \frac{\alpha_{k,1}}{(p_k - p_{n-1})(p_k - p_n)} + \frac{\alpha_{n-1,1}}{(p_{n-1} - p_k)(p_{n-1} - p_n)} + \frac{\alpha_{n,1}}{(p_n - p_k)(p_n - p_{n-1})},
\]

\( k = 1, \ldots, n-2 \), which is equivalent to (6.5.5). \( \square \)

### 6.6. The auxiliary gluing problem.

There is one case in Theorem 1.3.1 that is not addressed by the prototypical gluing problem: the case where the curve in \( \hat{X} \) has a double cover of a plane, where we must glue a branched cover with two multiplicity 1 positive ends and one multiplicity 2 negative end. Accordingly, we now calculate the zero set of the obstruction section for the moduli space \( \mathcal{M}(1, 1|2) \). This calculation is slightly different, due to the presence of the multiplicity 2 negative end. To begin, let \( \mathcal{M} \) denote the moduli space \( \mathcal{M}(1, 1|2) \). Any \( u \in \mathcal{M} \) has one simple branch point, and we can and do make the identification \( \mathcal{M} = \mathbb{R}/4\pi\mathbb{Z} \). Note that \( \text{ind}(u) = \dim \text{Coker}(D_u^N) = 1 \).

The obstruction bundle \( \mathcal{O} \) has rank 1. Choose a trivialization \( \tau \) of \( \xi \) over \( \beta_0 \) so that \( \mu_\tau(\beta_0) = 1 \), as before. Any element \( \sigma \in \text{Ker}(D_u^N) \) satisfies

\[
0 \geq \# \sigma^{-1}(0)
\]

\[
= \chi(\hat{\Sigma}) + \text{wind}_\tau(\sigma_1) + \text{wind}_\tau(\sigma_2) - \text{wind}_\tau(\sigma_{-1})
\]

\[
\geq 0.
\]

Thus, every non-zero element of \( \text{Ker}(D_u^N) \) is non-vanishing.
We parametrize $\mathcal{M}$ in the following way. Fix the positive puncture $p = p_1$, set $p_2 = -p_1$, and set the negative puncture $q_1 = 0$. Then send $\theta \in \mathbb{R}/4\pi \mathbb{Z}$ to

$$u_\theta: \mathbb{C} \setminus \{\pm p, 0\} \to \mathbb{C}^*$$

$$z \to e^{i\theta} \frac{z^2}{z^2 - p^2}.$$

The markers at the positive ends are given by

$$\tau_1 = e^{i\theta} \frac{p}{|p|} \quad \text{and} \quad \tau_2 = -e^{i\theta} \frac{p}{|p|} = -\tau_1,$$

while the marker at the negative end is determined by

$$\tau_{-1} = -e^{-i\theta} \frac{p^2}{|p|^2}.$$

We also have

$$r_1 = r_2 = p \quad \text{and} \quad r_{-1}^2 = p^2.$$

The meromorphic $(0,1)$-form

$$\eta_\theta(z) = 2i \bar{p} e^{i\theta/2} \frac{d\bar{z}}{\bar{z}^2}$$

is a replacement for a spanning element of $\text{Ker}(D_{u_\theta}^N)^*$; in particular, it follows the stable direction at the negative end.

We now compute the zero set $Z_1$. Note that, up to a real scalar multiple, we have

$$s_1(u_\theta)(\eta_\theta) = \langle \alpha_1, -2ie^{-i\theta/2} \rangle + \langle \alpha_2, 2ie^{-i\theta/2} \rangle$$

$$= \langle \alpha_1 - \alpha_2, -2ie^{-i\theta/2} \rangle.$$

Thus, there are two values of $\theta \in \mathbb{R}/4\pi \mathbb{Z}$ that such that $s_1(u_\theta)(\eta_\theta) = 0$.

The branched covers corresponding to these two values of $\theta$ differ only in the choice of asymptotic marker at the negative end. Thus, the two curves we obtain by gluing also differ only in the choice of asymptotic marker at the multiplicity 2 negative end in question. Moduli spaces for ECH consist of holomorphic currents that are not asymptotically marked, so we have over-counted by a factor of 2.

7. **Gluing Models and Evaluation Map Calculations**

In this section, we construct models for the curves obtained after performing the gluing procedure from Section 6. We then use those models to compute the degree of certain evaluation maps and prove Theorem 1.4.5.
7.1. Gluing models. Assume that the point \((T, [u]) \in [R, \infty) \times (\mathcal{M}_n/R)\) glues to \(u_+\). Denote by \(u \# u_+\) the curve obtained by gluing. Part of the domain of \(u \# u_+\) can be identified with the Riemann surface \(\Sigma_1\) obtained from the domain \(\hat{\Sigma}\) of the branched cover \(u\) by truncating the positive ends at height \(T\). Over \(\Sigma_1\), we can write \(u \# u_+\) as the graph of a section \(\nu\) of the pullback of the normal bundle of the trivial cylinder \(R \times \beta_0\) in \(R \times Y\). Since the normal bundle is a trivial complex holomorphic line bundle, we can view \(\nu\) as a complex-valued function on \(\Sigma_1\). After using the similarity principle, we may assume that \(\nu\) is genuinely holomorphic away from the punctures. We view \(\Sigma_1\) as the extended complex plane \(\hat{\mathbb{C}}\) with a finite number of disks removed. Using the analysis in the proof of [BH1, Proposition 8.7.2], we can write \(\nu\) in cylindrical coordinates in an annulus around a positive puncture \(p_i\) as

\[
\nu(s, t) = \left( s, t, \sum_{\ell=1}^{\infty} (\alpha_{i,\ell} e^{-\ell T} + d_{i,\ell}(T)) e^{\lambda_i s} f_i(t) + \text{(lower-order terms)} \right),
\]

where the \(d_{i,\ell}(T)\) are constants, depending on \(T\), coming from the perturbation in the obstruction bundle gluing construction and satisfy \(|\alpha_{i,\ell}| \gg |e^T d_{i,\ell}(T)|\).

Definition 7.1.1. An order \(m\) model associated to \((T, [u]) \in [R, \infty) \times (\mathcal{M}_n/R)\) is a meromorphic function with a pole of order \(m\) at each positive puncture \(p_i\), a zero of order 2 at infinity, and simple zeros at \(q_2, \ldots, q_{n-2}\), such that the asymptotic expansion in cylindrical coordinates of \(u\) near \(p_i\) is

\[
\alpha_{i,1} e^{-T r_i e^{s + it}} + \alpha_{i,2} e^{-2T r_i^2 e^{2(s + it)}} + \cdots + \alpha_{i,m} e^{-mT r_i^m e^{m(s + it)}}
\]

plus lower-order terms, where the \(r_i\) are as in Notation 6.5.1.

Note that an order \(m\) model for \((T, [u]) \in [R, \infty) \times \mathcal{M}_n\), if one exists, is unique and is given by

\[
g(z) = \sum_{i=1}^{n} \sum_{\ell=1}^{m} \frac{\alpha_{i,\ell} e^{-\ell T r_i^\ell |r_i|^{\ell}}}{(z - p_i)^\ell}.
\]

As the name suggests, the models defined above are related to curves obtained by gluing branched covers of trivial cylinders. An order \(m\) model associated to \(u\) is the approximation to the function \(\nu\) on \(\Sigma_1\) obtained by truncating the principal part at each singularity to the leading \(m\) terms.

The following theorem describes when a branched cover has an associated order \(m\) model. The proof is given in Appendix A.

Theorem 7.1.2. A point \((T, [u]) \in [R, \infty) \times \mathcal{M}_n/R\) has an associated order \(m\) model if and only if \((T, [u]) \in \mathcal{Z}_m\).
Definition 7.1.3. A full model associated to \((T, [u]) \in [R, \infty) \times (\mathcal{M}_n/R)\) is complex-valued function \(g\) that is holomorphic except for essential singularities at \(p_1, \ldots, p_n\), that has a zero of order 2 at infinity, that has simple zeros at \(q_2, \ldots, q_{n-2}\), and such that the principal part of \(g\) at \(p_i\) in cylindrical coordinates is

\[
(\alpha_{i,1}e^{-T} + d_{i,1}(T)r_i e^{\tau_1} + (\alpha_{i,2}e^{-2T} + d_{i,2}(T)r_i^2 e^{2\tau_1} + \cdots).
\]

Note that a full model associated to \((T, [u])\), if one exists, is unique and is given by

\[
(7.1.1) \quad g(z) = \sum_{i=1}^{n} \sum_{\ell=1}^{\infty} \tau_i^\ell r_i^\ell \frac{(\alpha_{i,\ell}e^{-\ell T} + d_{i,\ell}(T))}{(z - p_i)^\ell},
\]

An examination of the proof of Theorem 7.1.2 yields the following corollary.

Corollary 7.1.4. A point \((T, [u]) \in [R, \infty) \times \mathcal{M}_n/R\) has an associated full model if and only if \((T, [u]) \in \mathcal{Z}\).

7.2. The evaluation map in the model. We define an evaluation map that is suitable for use in an order \(m\) model, allowing for translations of the glued curve in the \(R\)-direction of \(R \times Y\), and compute the count of gluings in the case \(m = 1\).

Definition 7.2.1. The evaluation map of order \(k\) associated to the model of order \(m\) on \(\mathcal{Z}_1 \times \mathbb{R}\) is defined as follows. Let \((T, [u], s) \in \mathcal{Z}_1 \times \mathbb{R}\) and let \(g\) be the model of order \(m\) associated to \((T, [u])\). We define the evaluation map of order \(k\) associated to the model at the negative puncture \(q_j\), \(j = 2, 3, \ldots, n - 2\), by

\[
\text{mev}^k_{q_j}(T, [u], s) = \left( e^{-s}\frac{g'(q_j)}{\tau_{-j}} r_{-j}, \ldots, e^{-ks\frac{g^{(k)}(q_j)}{k!}} \frac{\tau^{(k)}_{-j} r_{-j}^{(k)}}{k!} \right).
\]

Let \(h(z) = g(z^{-1})\). We define the evaluation map of order \(k\) associated to the model at the negative puncture \(q_1\) by

\[
\text{mev}^k_{q_1}(T, [u], s) = \left( e^{-s/3} \frac{h''(q_1)}{2!} \frac{\tau^2_{-1} r_{-1}^2}{\tau_{-1}} ^2, \ldots, e^{-(2k+1)s/3} \frac{h^{(k+1)}(q_1)}{(k+1)!} \frac{\tau^{(k+1)}_{-1} r_{-1}^{(k+1)}}{k!} \right).
\]

If \(I_\subseteq\) is a subset of \(\{1, 2, \ldots, n - 2\}\), we choose a positive integer \(k_j\) for each \(j \in I_\subseteq\), and we set \(k = (k_j)_{j \in I_\subseteq}\), then we define the evaluation map of order \(k\) associated to the model at the negative punctures \(q_j\), \(j \in I_\subseteq\), by

\[
\text{mev}^k_{I_\subseteq}(T, [u], s) = \left( \text{mev}^k_{q_j}(T, [u], s) \right)_{j \in I_\subseteq}.
\]
We now compute the degree of \(\text{mev}_{1}^{(1,1,\ldots,1)}\) on \((\mathcal{Z}_1 \cap \{T\} \times (\mathcal{M}_n / \mathbb{R}))\) when \(T\) is fixed and sufficiently large. We begin with the puncture \(q_1\). For simplicity of notation, set

\[
H_i = \frac{B(p_i)}{A_i(p_i)} \alpha_{i,1},
\]

so that

\[
g(z) = e^{i\theta} e^{-T} \sum_{i=1}^{n} \frac{B(p_i)}{A_i(p_i)} \alpha_{i,1} \left( z - p_i \right) = e^{i\theta} e^{-T} \sum_{i=1}^{n} \frac{H_i}{z - p_i}.
\]

**Lemma 7.2.2.** The leading coefficient in the Taylor expansion of the order 1 model at the puncture \(q_1\) is

\[
e^{i\theta/3} e^{-T} \sum_{i=1}^{n} \frac{p_i B(p_i)}{A_i(p_i)} \alpha_{i,1}.
\]

**Proof.** Recall that

\[
\tau_3 \sim r^{-1} \sim e^{-i\theta}.
\]

and that \(h(z) = g(z^{-1})\). Thus, we want to compute the leading coefficient in the Taylor expansion of \(h\) at \(z = 0\). Since \(h'(0) = 0\), we have

\[
\sum_{i=1}^{n} H_i = 0,
\]

and hence

\[
h(z) = e^{i\theta/3} e^{-T} z \sum_{i=1}^{n} \frac{H_i}{1 - p_i z} \left( z - p_i z \right)
\]

\[
= e^{i\theta/3} e^{-T} z \sum_{i=1}^{n} \left[ \frac{H_i}{1 - p_i z} - H_i \right]
\]

\[
= e^{i\theta/3} e^{-T} z^2 \sum_{i=1}^{n} \frac{p_i H_i}{1 - p_i z}.
\]

The result follows. \(\square\)

Now we compute the leading coefficient in the Taylor expansion of the order 1 model at the punctures \(q_2, \ldots, q_{n-2}\). It is advantageous to first simplify the expression for the order 1 model.

**Lemma 7.2.3.** If \((T, [u]) \in \mathcal{Z}_1\), we have

\[
\sum_{i=1}^{n} \frac{H_i}{(q_{k_i} - p_i) \cdots (q_{k_j} - p_i)} = 0,
\]
where \( j = 1, 2, \ldots, n - 3 \) and \( k_1, \ldots, k_j \) are distinct elements of \( \{2, \ldots, n - 2\} \).

**Proof.** The proof is by induction on \( j \leq n - 3 \). Since \( g(q_k) = 0 \) for \( k = 2, 3, \ldots, n - 2 \), we have

\[
0 = \sum_{i=1}^{n} \frac{H_i}{q_k - p_i}
\]

for \( k = 2, 3, \ldots, n - 2 \). Since \( q_k \neq 0 \), we have

\[
0 = \sum_{i=1}^{n} \frac{p_i H_i}{q_k - p_i}
\]

for \( k = 2, 3, \ldots, n - 2 \), which establishes the case \( j = 1 \).

Now assume the result for some \( j \leq n - 4 \). If \( k_1, \ldots, k_{j+1} \) are distinct elements of \( \{2, \ldots, n - 2\} \), we see that

\[
0 = \sum_{i=1}^{n} \frac{H_i}{(q_{k_1} - p_i) \cdots (q_{k_{j+1}} - p_i)} - \sum_{i=1}^{n} \frac{H_i}{(q_{k_1} - p_i) \cdots (q_{k_{j+1}} - p_i)(q_{k_{j+1}} - p_i)}
\]

\[
= \sum_{i=1}^{n} \frac{H_i}{(q_{k_1} - p_i) \cdots (q_{k_{j+1}} - p_i)} \left[ \frac{1}{q_{k_j} - p_i} - \frac{1}{q_{k_{j+1}} - p_i} \right]
\]

\[
= (q_{k_{j+1}} - q_{k_j}) \sum_{i=1}^{n} \frac{H_i}{(q_{k_1} - p_i) \cdots (q_{k_{j+1}} - p_i)}.
\]

Since \( q_{k_{j+1}} - q_{k_j} \neq 0 \), the inductive step follows, and we are done. \( \square \)

**Lemma 7.2.4.** If \((T, [u]) \in Z_1\), we can write

\[
g(z) = e^{i \theta} e^{-T} B(z) \sum_{i=1}^{n} \frac{a_{i,1}}{A_i(p_i)} \frac{1}{z - p_i}.
\]

**Proof.** Use Lemma 7.2.3 repeatedly to write

\[
g_C(z) = e^{i \theta} e^{-T} \sum_{i=1}^{n} \frac{H_i}{z - p_i}
\]

\[
= e^{i \theta} e^{-T} \left[ \sum_{i=1}^{n} \frac{H_i}{z - p_i} - \sum_{i=1}^{n} \frac{H_i}{q_2 - p_i} \right]
\]

\[
= e^{i \theta} e^{-T} (q_2 - z) \sum_{i=1}^{n} \frac{H_i}{(q_2 - p_i)(z - p_i)}
\]

\[
= e^{i \theta} e^{-T} (q_2 - z) \left[ \sum_{i=1}^{n} \frac{H_i}{(q_2 - p_i)(z - p_i)} - \sum_{i=1}^{n} \frac{H_i}{(q_2 - p_i)(q_3 - p_i)} \right]
\]
\[
= e^{i\theta} e^{-T}(q_2 - z)(q_3 - z) \sum_{i=1}^{n} \frac{H_i}{(q_2 - p_i)(q_3 - p_i)(z - p_i)}
\]
\[
\vdots
\]
\[
= e^{i\theta} e^{-T} \left( \prod_{k=2}^{n-2} (q_k - z) \right) \sum_{i=1}^{n} \frac{H_i}{\prod_{k=2}^{n-2} (q_k - p_i) (z - p_i)} \frac{1}{z - p_i}
\]
\[
= e^{i\theta} e^{-T} B(z) \sum_{i=1}^{n} \frac{H_i}{B(p_i)} \frac{1}{z - p_i}
\]
\[
= e^{i\theta} e^{-T} B(z) \sum_{i=1}^{n} \frac{\alpha_{i,1}}{A_i(p_i)} \frac{1}{z - p_i},
\]
as desired. \qed

**Lemma 7.2.5.** The leading coefficient in the Taylor expansion of the order 1 model at the puncture \( q_k, k = 2, \ldots, n - 2 \), is
\[
(-1)^n e^{-T} (\alpha_{n-1} - \alpha_n) q_k - (p_n \alpha_{n-1} - p_{n-1} \alpha_n) p_{n-1} / p_n.
\]

**Proof.** Recall that
\[
\tau_{-k} |r_{-k}| = e^{-i\theta} \frac{A(q_k)}{B_k(q_k)}
\]
for \( k = 2, \ldots, n - 2 \). Set
\[
\Delta = \prod_{1 \leq i < j \leq n} (p_i - p_j)
\]
and
\[
\Delta_i = \prod_{1 \leq j < k \leq n} (p_j - p_k).
\]
Let \( E_\ell \) be the \( \ell \)th elementary symmetric polynomial and set
\[
E_{\ell,i} = E_\ell(p_1, \ldots, \hat{p_i}, \ldots, p_n).
\]
The leading coefficient at the puncture \( q_k \) is
\[
\tau_{-k} |r_{-k}| e^{i\theta} e^{-T} B_k(q_k) \sum_{i=1}^{n} \frac{\alpha_{i,1}}{A_i(p_i)} \frac{1}{q_k - p_i}
\]
\[ e^{-T}A(q_k)\sum_{i=1}^{n} \frac{\alpha_{i,1}}{A_i(p_i) q_k - p_i} \]

\[ = e^{-T} \sum_{i=1}^{n} \frac{A_i(q_k)}{A_i(p_i)} p_i \alpha_{i,1} \]

\[ = e^{-T} \sum_{i=1}^{n} (-1)^{i-1} A_i(q_k) \alpha_{i,1} \Delta_i \]

\[ = e^{-T} \sum_{i=1}^{n} (-1)^{i-1} \alpha_{i,1} \Delta_i \left( \sum_{\ell=0}^{n-1} (-1)^{\ell} E_{\ell,i} q_k^{n-1-\ell} \right) \]

\[ = e^{-T} \sum_{\ell=0}^{n-1} (-1)^{\ell} q_k^{n-1-\ell} \left( \sum_{i=1}^{n} (-1)^{i-1} E_{\ell,i} \alpha_{i,1} \Delta_i \right) \]

\[ = e^{-T} \sum_{\ell=0}^{n-1} (-1)^{\ell} q_k^{n-1-\ell} \det M_\ell, \]

where

\[
M_\ell = \begin{pmatrix}
E_{\ell,1} \alpha_{1,1} & \cdots & E_{\ell,n} \alpha_{n,1} \\
p_1^{n-2} & \cdots & p_n^{n-2} \\
\vdots & \ddots & \vdots \\
p_i & \cdots & p_n \\
1 & \cdots & 1
\end{pmatrix}.
\]

**Claim 7.2.6.** We have

\[
(-1)^n \frac{\det M_\ell}{\Delta} = \begin{cases}
\frac{p_n \alpha_{n-1} - p_{n-1} \alpha_n}{p_n - p_{n-1}} & \ell = n-1 \\
\frac{\alpha_{n-1} - \alpha_n}{p_n - p_{n-1}} & \ell = n-2 \\
0 & \ell < n-2
\end{cases}.
\]

The lemma follows from Claim 7.2.6. The proof of the claim is an exercise in careful row-reduction and is given in Appendix B.

**Proposition 7.2.7.** Let \( I_- = \{-1, -2, \ldots, -(n-2)\} \) and \( k = (1, 1, \ldots, 1) \). Suppose that \( R \gg 0 \) is sufficiently large in the prototypical gluing problem. For any fixed \( T \geq R \) and any admissible asymptotic restriction \( c \in (\mathbb{C}^*)^{n-2} \), the degree of the restriction of \( \text{mev}_k^l \) to \((Z_1 \cap (\{T\} \times (\mathcal{M}_n/\mathbb{R}))) \times \mathbb{R}\) is 1 (mod 2).
Thus, there are an odd number of values of $s$ than $\theta$ of the first equation are equal. Since $7.3$. □ that solves the first equation.

Choose an admissible asymptotic restriction $c = (c_1, \ldots, c_{n-2}) \in (\mathbb{C}^*)^{n-2}$. We must count solutions of the equations

$$e^{-s/3}e^{i\theta/3}e^{-T/2} \sum_{i=1}^{n} \frac{p_i B(p_i)}{A_i(p_i)} \alpha_{i,1} = c_1,$$

$$(-1)^n e^{-s}e^{-T} \frac{(\alpha_{n-1} - \alpha_n) q_2 - (p_n \alpha_{n-1} - p_{n-1} \alpha_n)}{p_{n-1} - p_n} = c_2,$$

$$\vdots$$

$$(-1)^n e^{-s}e^{-T} \frac{(\alpha_{n-1} - \alpha_n) q_{n-2} - (p_n \alpha_{n-1} - p_{n-1} \alpha_n)}{p_{n-1} - p_n} = c_{n-2},$$

where $T$ is fixed by our assumptions, the $p_i$ are fixed, distinct points in $\mathbb{C}$ by Corollary 6.5.5 $q_2, \ldots, q_{n-2}$ are allowed to vary in $\mathbb{C} \setminus \{p_1, \ldots, p_n\}$, $s$ is allowed to vary in $\mathbb{R}$, and $\theta$ is allowed to vary in $(\mathbb{R}/6\pi \mathbb{Z})$. The last $n - 3$ equations have solutions

$$q_k = \frac{(-1)^n e^{s}e^{T}(p_{n-1} - p_n)c_k + (p_n \alpha_{n-1} - p_{n-1} \alpha_n)}{\alpha_{n-1} - \alpha_n},$$

$k = 2, \ldots, n - 2$. Now substitute back into the first equation and consider the norm of the left-hand side. If $s$ is very large and positive, the norm is larger than $|c_1|$, while if $s$ is very large and negative, the norm is smaller than $|c_1|$. Thus, there are an odd number of values of $s$ for which the norms of both sides of the first equation are equal. Since $\theta \in \mathbb{R}/6\pi \mathbb{Z}$, there is a unique choice of $\theta$ that solves the first equation. □

### 7.3. Reduction to the first-order model

We now show that the full evaluation map and the evaluation map in the order 1 model have the same degree.

**Proposition 7.3.1.** Let $I_\pi = \{-1, -2, \ldots, -(n - 2)\}$ and $k = (1, 1, \ldots, 1)$. Suppose that $R \gg 0$ is sufficiently large in the prototypical gluing problem. For any fixed $T \geq R$ and any admissible asymptotic restriction $c \in (\mathbb{C}^*)^{n-2}$, the mod 2 degree of the restriction of $\text{mev}_I^k$ to $(Z_1 \cap \{T\} \times (\mathcal{M}_n/\mathbb{R})) \times \mathbb{R}$ is equal to the mod 2 degree of the restriction of $\text{ev}_I^k$ to $(Z \cap \{T\} \times (\mathcal{M}_n/\mathbb{R})) \times \mathbb{R}$.

The following corollary is an analogue of Lemmas 7.2.2 and 7.2.5 for full models. It follows directly from 7.1.1.

**Corollary 7.3.2.** The leading coefficient in the full model at the puncture $q_1$ is

$$e^{i\theta/3}e^{-T} \sum_{i=1}^{n} \frac{p_i B(p_i)}{A_i(p_i)} \alpha_{i,1} + e^{4i\theta/3}e^{-2T} \sum_{i=1}^{n} \frac{B(p_i)^2}{A_i(p_i)^2} \alpha_{i,2}$$

for which the norms of both sides
The leading coefficient at the puncture \( q_k, \ k = 2, \ldots, n - 2 \), is
\[
\frac{1}{B_k(q_k)} \sum_{i=1}^{n} \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{A_i(q_k) B_k(p_i)^\ell}{A_i(p_i)^\ell} \alpha_i \ell e^{(\ell-1)i\theta} e^{-\ell T}
\]

**Proof of Proposition 7.3.1.** Let \( c \) be an admissible asymptotic restriction and let \( T \gg R \). Define a map \( F: (\mathcal{M}_n/\mathbb{R}) \times \mathbb{R} \to \mathbb{C}^{n-2} \times \mathbb{C}^{n-2} \) by
\[
F([u], s) = \begin{pmatrix}
\mathfrak{s}(T, [u])(\eta_1) \\
\vdots \\
\mathfrak{s}(T, [u])(\eta_{n-2}) \\
\operatorname{ev}^k_{\lambda, \nu}(T, [u], s)
\end{pmatrix}.
\]
Thus, the set of all \((T, [u]) \in \{T\} \times (\mathcal{M}_n/\mathbb{R})\) that glue to \( u_+ \) and satisfy the admissible asymptotic restrictions \( c \) when translated in the \( \mathbb{R} \)-direction by \( s \) is
\[
F^{-1}(\{0\} \times \{c\})
\]

Define a homotopy \( F_\nu \) of \( F \) in the following way. For \( \nu \in [0, \frac{1}{2}] \), we can define a homotopy \( \mathfrak{s}_\nu \) of the obstruction section \( \mathfrak{s} \) such that \( \mathfrak{s}_0 = \mathfrak{s} \) and \( \mathfrak{s}_{\frac{1}{2}} \) is the linear portion of \( \mathfrak{s} \). See, e.g., [BHI] Section 8.7] for a similar construction. Let \( \zeta_k(p_1, \ldots, p_n, q_2, \ldots, q_{n-2}, s) \) denote the leading asymptotic coefficient at the puncture \( q_k, k = 1, 2, \ldots, n - 2 \), multiplied by \( e^{-s/3} \) if \( k = 1 \) or \( e^{-s} \) if \( k > 1 \). Define a homotopy of each \( \zeta_k \) for \( \nu \in [0, \frac{1}{2}] \) by replacing each \( \alpha_i \ell e^{-\ell T} + d_i \ell(T) \) with \( \alpha_i \ell e^{-\ell T} + (1 - 2\nu)d_i \ell(T) \), and set
\[
\operatorname{ev}^k_{\lambda, \nu}(T, [u], s) = (\zeta_{k, \nu})^{n-2}_{j=1}.
\]
Then for \( \nu \in [0, \frac{1}{2}] \), we define
\[
F_\nu([u], s) = \begin{pmatrix}
\mathfrak{s}_\nu(T, [u])(\eta_1) \\
\vdots \\
\mathfrak{s}_\nu(T, [u])(\eta_{n-2}) \\
\operatorname{ev}^k_{\lambda, \nu}(T, [u], s)
\end{pmatrix}.
\]

For \( \nu \in [\frac{1}{2}, 1] \), let \( \eta_{k, \nu} \) denote the linear interpolation from \( \eta_k \) to \( \Pi \eta_k \), the projection of \( \eta_k \) onto the leading eigenspace at each positive end. Let \( \zeta_{j, \nu}, \nu \in [\frac{1}{2}, 1] \), be the linear interpolation that kills all terms with \( \ell \geq 2 \) and set
\[
\operatorname{ev}^k_{\lambda, \nu} = (\zeta_{j, \nu})^{n-2}_{j=1}.
\]
Then for \( \nu \in [\frac{1}{2}, 1] \), we define
\[
F_\nu([u], s) = \begin{pmatrix}
\mathfrak{s}_{\frac{1}{2}}(T, [u])(\eta_{1, \nu}) \\
\vdots \\
\mathfrak{s}_{\frac{1}{2}}(T, [u])(\eta_{n-2, \nu}) \\
\operatorname{ev}^k_{\lambda, \nu}(T, [u], s)
\end{pmatrix}.
\]
Note that $F_0 = F$ and
\[ F_1([u], s) = \begin{pmatrix} s_1(T, [u]) (\eta_1) \\ \vdots \\ s_1(T, [u]) (\eta_{n-2}) \\ \text{mev}^k_{m-1}(T, [u], s) \end{pmatrix}. \]

**Claim 7.3.3.** Let $K \subset (\mathcal{M}_n/\mathbb{R}) \times \mathbb{R}$ be a compact set such that $F_{1}^{-1}(\{0, c\})$ is contained in the interior of $\{T\} \times K$. If $T \gg 0$ is sufficiently large, then $F_{\nu}^{-1}(\{0, c\}) \cap (\{T\} \times \partial K) = \emptyset$ for all $\nu \in [0, 1]$.

**Proof of Claim 7.3.3.** Suppose that the claim is false. Then there is a sequence $\{T_k, \nu_k\}$ with $T_k \to \infty$ such that $F_{\nu_k}^{-1}(\{0, c\}) \cap (\{T_k\} \times \partial K) \neq \emptyset$ for all $k$. We will arrive at a contradiction by showing that the homotopy $F_{\nu}$ is very small on $\{T\} \times K$ if $T$ is sufficiently large.

Let $\text{ev}^{-1}_{k, \nu}$ denote the projection of $\text{ev}^{-1}_{k, \nu}$ to the $k^{th}$ coordinate. Since $K$ is compact, there is a large positive constant $C$ such that
\[ |p_i - p_j| > C^{-1} \text{ and } |q_i - q_j| > C^{-1} \]
for all $i \neq j$. In addition, we may assume that
\[ C^{-1} < s < C. \]

We may also assume that each of the punctures $p_1, \ldots, p_n, q_2, \ldots, q_{n-2}$, except possibly for one of the positive punctures $p_i$, is contained in the disk of diameter $C$ centered at the origin in $\mathbb{C}$. If all of the punctures in question are contained in the disk, then
\[ |p_i - q_j| < C \]
for all $i \neq j$. In this case, we have, for $\nu \in [\frac{1}{2}, 1]$ and $j = 2, \ldots, n - 2$,
\[ |\text{ev}^{-1}_{k, \nu}(T, [u], s) - \text{mev}^{-1}_{k}(T, [u], s)| \leq \frac{e^C}{|B_k(q_k)|} \sum_{i=1}^{n} \sum_{\ell=2}^{\infty} \ell \left| \frac{A_i(q_k) B_k(q_k)^{\ell} \alpha_{i, \ell}}{A_i(p_i)^{\ell}} \right| e^{-\ell T} \]
\[ \leq e^C \sum_{i=1}^{n} \sum_{\ell=2}^{\infty} \ell C^{(\ell+1)(2n-5)} |\alpha_{i, \ell}| e^{-\ell T}, \]
and the right-hand side can be made as small as we like by taking $T$ to be sufficiently large.

Now assume that one of the positive punctures, say $p_j$, is outside of the above-mentioned disk. There is a constant $D > C$, depending only on $n$ and
By (6.5.4), we have
\[ \left| \frac{B_k(p_j)^\ell}{A_j(p_j)^\ell} \right| < 1 \quad \text{and} \quad \left| \frac{A_i(q_k)}{A_i(p_i)^\ell} \right| < 1 \]
for \( i \neq j \) and \( \ell \geq 2 \). Moreover, when \( C \leq |p_j| \leq D \), we have
\[ \left| \frac{B_k(p_j)^\ell}{A_j(p_j)^\ell} \right| \leq C^{(n-1)\ell}(C + D)^{(n-3)\ell} \quad \text{and} \quad \left| \frac{A_i(q_k)}{A_i(p_i)^\ell} \right| \leq C^{(n-1)\ell+n-2}(C + D) \]
for all \( i \neq j \). It follows that, in this case, we can make
\[ |\text{ev}^1_{-k,\nu}(T, [u], s) - \text{mev}^1_{-k}(T, [u], s)| \]
as small as we like by taking \( T \) to be sufficiently large.

Since \( |d_{i,\ell}(T)| \ll |\alpha_{i,\ell}|e^{-\ell T} \), a similar estimate shows that, for \( \nu \in [0, \frac{1}{2}] \),
\[ |\text{ev}^1_{-j,\nu}(T, [u], s) - \text{ev}^1_{-j,0}(T, [u], s)| \]
can be made as small as we like by taking \( T \) to be sufficiently large. Another similar estimate shows the same result for the leading coefficient at \( q_1 \).

Now we prove a similar result for the homotopy of the obstruction section. By (6.5.4), we have
\[ \frac{1}{(\ell - 1)!} \frac{d^{\ell-1}Q_i}{dz^{\ell-1}}(z) = -n \frac{\sum_{k=2}^{n-2} A_i(q_k)}{\sum_{k=2}^{n-2} B_k(q_k)} \frac{1}{(q_k - z)^\ell} \]
for \( i = 1, \ldots, n \) and \( \ell \geq 1 \). If each of \( p_1, \ldots, p_n, q_2, \ldots, q_{n-2} \) is in the disk of diameter \( C \) centered at the origin, then, for \( \nu \in [\frac{1}{2}, 1] \),
\[ |\mathcal{S}_{\frac{1}{2}}(T, [u])(\eta_{k,\nu}) - \mathcal{S}_1(T, [u])(\eta_1)| \leq \sum_{i=1}^{n} \sum_{\ell=2}^{n} \sum_{k=2}^{n-2} e^{-\ell T} |\alpha_{i,\ell}| \frac{|A_i(q_k)B_k(p_i)^\ell|}{|B_k(q_k)A_i(p_i)^\ell|} \]
\[ \leq C^{2n-6} \sum_{i=1}^{n} \sum_{\ell=2}^{n} \sum_{k=2}^{n-2} e^{-\ell T} |\alpha_{i,\ell}| C^{(2n-4)\ell} \]
\[ = n(n - 3)C^{2n-6} \sum_{\ell=2}^{n} \sum_{k=2}^{n-2} e^{-\ell T} |\alpha_{i,\ell}| C^{(2n-4)\ell}, \]
which can be made as small as we like by taking \( T \) to be sufficiently large. When some positive puncture, say \( p_j \), is outside of the disk of diameter \( C \), the same argument used for the evaluation map shows that we can make
\[ |\mathcal{S}_{\frac{1}{2}}(T, [u])(\eta_{k,\nu}) - \mathcal{S}_1(T, [u])(\eta_1)| \]
as small as we like by taking \( T \) to be sufficiently large. A similar estimate shows that, for \( \nu \in [0, \frac{1}{2}] \),
\[
|s_{\nu}(T, [u])(\eta_k) - s_{\frac{1}{2}}(T, [u])(\eta_k)|
\]
can be made as small as we like by taking \( T \) to be sufficiently large.

We now finish the proof of the claim. Since \( K \) is compact and \( F_1(\{\{0\}\} \times \partial K) \) does not intersect \( \{0, c\} \), the distance between \( F_1(\{\{T\}\} \times \partial K) \) and \( \{0, c\} \) is bounded below by a positive constant. By our above estimates, the homotopy \( F_\nu \) can be made as small as we like on \( \{\{T\}\} \times K \) by taking \( T \) to be sufficiently large. Thus, the distance between \( F_\nu(\{\{T\}\} \times \partial K) \) and \( \{0, c\} \) is bounded below by a (possibly smaller) positive constant for all \( \nu \in [0, 1] \), which contradicts our assumption that \( T_k \to \infty \) in the sequence \( \{T_k, \nu_k\} \).

The claim implies Proposition 7.3.1 if the number of points in \( F_0^{-1}(\{0, c\}) \) is even, there is a large compact subset \( K \subset (\mathcal{M}_n/\mathbb{R}) \times \mathbb{R} \) containing \( F_0^{-1}(\{0, c\}) \) in its interior such that \( F_\nu^{-1}(\{0, c\}) \cap (\{T\} \times \partial K) \neq \emptyset \) for some \( \nu \in (0, 1) \). □

**Proof of Theorem 1.4.5.** Combine Proposition 7.2.7 and Proposition 7.3.1. □

### 8. The Cobordism Map

In this section, we complete the proof of Theorem 1.5.3 using the evaluation map discussed in Section 3 together with the degree calculations in Section 7. Let \( \widetilde{\mathcal{M}}_n \) denote the moduli space in which the curves obtained by gluing branched covers in \( \mathcal{M}_n/\mathbb{R} \) to \( u_0 + u_1 \) live. Let \( I_\pm = \{-1, -2, \ldots, -(n-2)\} \), and \( k = (1, \ldots, 1) \).

**Proposition 8.0.1.** Let \( c \in (\mathbb{C}^*)^{n-2} \) be a choice of admissible asymptotic restriction. The pre-image of \( \{c\} \) under \( ev^k_{I_+ \cup I_-} \) is a real 1-dimensional sub-manifold of \( \widetilde{\mathcal{M}}_n \) whose intersection with the end corresponding to \( Z \times \mathbb{R} \) has an odd number of components.

**Proof.** The pre-image is a submanifold if \( J \) is generic by Theorem 3.3.1. The number of components is odd by Proposition 7.2.7 and Proposition 7.3.1. □

**Proposition 8.0.2.** Let \( c \in (\mathbb{C}^*)^{n-2} \) be a choice of admissible asymptotic restriction. Each endpoint of the pre-image of \( \{c\} \) under \( ev^k_{I_+ \cup I_-} \) is a two-level building \( u_0 \cup u_1 \) where \( \text{ind}(u_1) = 1 \) and either (1) both levels are regular or (2) \( u_0 \) is a branched cover in \( \mathcal{M}_n/\mathbb{R} \).

**Proof.** An endpoint of the pre-image is a multi-level SFT building \( u_0 \cup u_1 \cup \cdots \cup u_k \). First assume that the bottom level \( u_0 \) is regular. If \( J \) is generic, the
evaluation map is a submersion on such boundary strata and \( u_0 \) has Fredholm index \( 2n - 4 \). It follows that \( k = 1 \) and \( \text{ind}(u_1) = 1 \).

Now assume that \( u_0 \) is not regular. Since \( c \) is admissible, all components of \( u_0 \) must be simple, except possibly for branched covers of trivial cylinders.

**Lemma 8.0.3.** If \( u \) is a branched cover of the trivial cylinder \( \mathbb{R} \times \beta_0 \) that glues to a somewhere injective curve \( u_1 \) and the glued curve is also somewhere injective, then the partition of the negative ends of \( u \) cannot be \((1, \ldots, 1)\). If the partition is \((3, 1, \ldots, 1)\), then the partition of the positive ends of \( u \) must be \((1, \ldots, 1)\).

**Proof of Lemma 8.0.3.** We may assume that \( u_1 \) is embedded. (If not, there is a generic nearby curve \( u'_1 \) with only nodal singularities, and the branched cover \( u \) still glues to a small, generic resolution of \( u'_1 \).) First assume that the partition of the negative ends of \( u \) is \((1, \ldots, 1)\). If \( u \# u_1 \) is the curve obtained by gluing \( u \) to the relevant negative ends of \( u_1 \), then \( \text{ind}(u \# u_1) > \text{ind}(u_1) \), \( I(u \# u_1) = I(u_1) \), and \( \Delta(u \# u_1) \geq \Delta(u_1) \). Since both \( u_1 \) and \( u \# u_1 \) are somewhere injective, the inequality (1.2.2) yields a contradiction.

Now assume that the partition of the negative ends of \( u \) is \((3, 1, \ldots, 1)\). If the partition of the positive ends of \( u \) is not \((1, \ldots, 1)\), then \( \text{ind}(u \# u_1) > \text{ind}(u_1) \), \( I(u \# u_1) = I(u_1) \), and \( \Delta(u \# u_1) \geq \Delta(u_1) \), as in the first case. The inequality (1.2.2) again yields a contradiction. \( \square \)

Let \( v_1, \ldots, v_m \) denote the components of \( u_0 \) that are branched covers of trivial cylinders, and let \( w_1, \ldots, w_\ell \) denote the simple components. Note that \( \text{ind}(w_i) > 0 \) for all \( i \), or else each \( w_i \) is an unbranched cover of a trivial cylinder, and we are in case (1). Note also that \( v_1, \ldots, v_m \) are all branched covers of \( \mathbb{R} \times \beta_0 \), as all other orbits in the negative orbit set \( \beta \) of \( u_0 \) other than \( \beta_0 \) have multiplicity 1. By Lemma 8.0.3, we must have \( m = 1 \), as only one of the branched covers can have a multiplicity 3 negative end. Finally, note that the partition of the positive ends of \( v_1 \) must be \((1, \ldots, 1)\).

Now consider the curve \( u = (w_1 \sqcup \cdots \sqcup w_\ell) \# u_1 \# \cdots \# u_k \) obtained by gluing the simple components of \( u_0 \). The partition of the negative ends of \( u \) at \( \beta_0 \) is \((1, \ldots, 1)\), and the negative ends of \( u \) have total multiplicity \( n \) at that orbit. Thus,

\[
\Delta(u) = \binom{n}{2}.
\]
But \( \text{ind}(u) > 1 \), so we have
\[
1 + \binom{n}{2} = I(u) \geq \text{ind}(u) + 2\delta(u) + \Delta(u) > 1 + \binom{n}{2},
\]
which is a contradiction. Thus, if \( u_0 \) is not regular, it must have only one non-trivial component, which is necessarily a branched cover of \( \mathbb{R} \times \beta_0 \).

Assume that \( u_0 \) is such a curve. The partition of its positive ends at \( \beta_0 \) must be \((1, \ldots, 1)\), with \( m \) positive ends, and the negative partition must be \((3, 1, \ldots, 1)\), with \( m - 2 \) negative ends, for some \( m \) with \( 1 \leq m \leq n \). We claim that \( m = n \), which will complete the proof. For if \( m < n \), the non-trivial component of \( u_0 \) must have positive genus. Truncate the curve \( u_1 \# \cdots \# u_k \) to a collection of \( m \) \( J \)-holomorphic cylinders close to and graphical over \( \beta_0 \) and then glue the non-trivial component of \( u_0 \) to get a \( J \)-holomorphic curve \( \tilde{u} \). The existence of \( \tilde{u} \) contradicts the genus-minimizing property of \( J \)-holomorphic curves, as there is a topological curve with the same profile and smaller genus, namely, the curve we get by gluing a branched cover in \( \mathcal{M}_m \) to the truncated half-cylinders. \( \square \)

**Lemma 8.0.4.** In case (2) of Proposition 8.0.2, we have \( I(u_1) = 1 \) and \( I(u_0) = \binom{n}{2} \).

*Proof.* The partition of the negative ends of \( u_0 \) at \( \beta_0 \) is \((3, 1, \ldots, 1)\), so \( \Delta(u_0) \geq 1 + \binom{n - 2}{2} \). Thus,
\[
I(u_0) \geq \text{ind}(u_0) + 2\delta(u_0) + \Delta(u_0) \\
\geq (2n - 4) + 2 + \binom{n - 2}{2} \\
= \binom{n}{2}.
\]
Since \( I(u_1) \geq 1 \), \( I(u_0) \geq \binom{n}{2} \), and \( I(u_1) + I(u_0) = 1 + \binom{n}{2} \), both inequalities must in fact be equalities. \( \square \)

By Proposition 8.0.1, an odd number of endpoints \( u_0 \cup u_1 \) of the pre-image of \( \{c\} \) are such that \( u_0 \in \mathcal{M}_n \). Thus, by Proposition 8.0.2, an odd number of endpoints \( u_0 \cup u_1 \) of the pre-image of \( \{c\} \) are regular. Let \( \gamma \) denote the negative orbit set of \( u_1 \). Since \( \text{ind}(u_1) = I(u_1) = 1 \) by Lemma 8.0.4, the negative ends of \( u_1 \) satisfy the partition conditions. Therefore, the count of such endpoints where \( \gamma \) is not a generator of the ECH complex is even. We conclude that the count of such endpoints where \( \gamma \) is an ECH generator, so that \( u_1 \) is counted
by the ECH differential, is odd. The proof of Theorem 1.5.3 is now complete when the gluing problem is in in the case \( n \geq 3 \).

All that remains is the case \( n = 2 \). When we glue branched covers in \( \mathcal{M}(1,1|2) \) to \( u_+ \), the result lives in a connected moduli space \( \widetilde{\mathcal{M}} \) of curves with Fredholm index 2. Then \( \widetilde{\mathcal{M}}/\mathbb{R} \) has dimension 1, and the other endpoint must be a two-level SFT building \( u_0 \cup u_1 \) with \( \text{ind}(u_1) = \text{ind}(u_0) = 1 \) and \( I(u_0) = I(u_1) = 1 \).

### Appendix A. Existence of Models

In this appendix, we prove Theorem 7.1.2. Recall that our branched covers have multiplicity 1 positive punctures \( p_1, \ldots, p_n \), a multiplicity 3 negative puncture \( q_1 \), and multiplicity 1 negative punctures \( q_2, \ldots, q_{n-2} \). An order \( m \) model, if it exists, is necessarily given by

\[
g(z) = \sum_{i=1}^{n} \sum_{\ell=1}^{m} \alpha_{i,\ell} e^{-\ell T \tau_i |r_i|^\ell} (z - p_i)^\ell.
\]

As before, set \( h(z) = g(z^{-1}) \) and note that \( h \) has a removable singularity at \( z = 0 \) and vanishes there. The function \( g \) is an order \( m \) associated to a branched cover \( u \) if and only if

\[
(A.1) \quad h'(0) = 0 \quad \text{and} \quad g(q_2) = \cdots = g(q_{n-2}) = 0.
\]

By (7.2.1), (7.2.2), (6.1.1), and (6.5.1), the equation \( h'(0) = 0 \) is equivalent to

\[
\sum_{i=1}^{n} \tau_i |r_i| \alpha_{i,1} e^{-i\theta} = 0,
\]

which is equivalent to

\[
\sum_{i=1}^{n} \tau_i |r_i| \alpha_{i,1} e^{-T} = 0.
\]

The equations (A.1) are therefore equivalent to the equations

\[
\sum_{i=1}^{n} \tau_i |r_i| \alpha_{i,1} e^{-T} = 0,
\]

\[
\sum_{i=1}^{n} \sum_{\ell=1}^{m} \tau_{i,\ell} |r_i|^\ell \alpha_{i,\ell} e^{-\ell T} \frac{(q_2 - p_i)^\ell}{(q_3 - p_i)^\ell} = 0,
\]

\[
\sum_{i=1}^{n} \sum_{\ell=1}^{m} \tau_{i,\ell} |r_i|^\ell \alpha_{i,\ell} e^{-\ell T} \frac{(q_3 - p_i)^\ell}{(q_3 - p_i)^\ell} = 0.
\]
\[
\sum_{i=1}^{n} \sum_{\ell=1}^{m} \frac{\tau_{\ell} |r_i|}{{(q_{n-2} - p_i)}^\ell} \alpha_{i,\ell} e^{-\ell T} = 0
\]

For each \( \ell = 1, \ldots, m \), define an \((n - 2) \times n\) matrix \( A_{\ell} \) whose \( i \)th row, \( i = 2, \ldots, n - 2 \), is
\[
(A_{\ell})_i = \left( \frac{1}{(q_{n-2} - p_1)^\ell} \quad \frac{1}{(q_{n-2} - p_2)^\ell} \quad \cdots \quad \frac{1}{(q_{n-2} - p_n)^\ell} \right)
\]
and whose first row is
\[
(A_{1})_1 = (1 \quad 1 \quad \cdots \quad 1)
\]
when \( \ell = 1 \) and
\[
(A_{\ell})_1 = (0 \quad 0 \quad \cdots \quad 0)
\]
when \( \ell > 1 \). The above equations hold if and only if the column vector
\[
\alpha_{\ell} = (\tau_1 |r_1| |\alpha_{1,\ell} e^{-\ell T} \quad \tau_2 |r_2| |\alpha_{2,\ell} e^{-\ell T} \quad \cdots \quad \tau_n |r_n| |\alpha_{n,\ell} e^{-\ell T} )^T
\]
is in the nullspace of \( A_{\ell} \) for all \( \ell = 1, \ldots, m \). This condition holds if and only if the \( nm \times 1 \) column vector
\[
\alpha = (\alpha_1^T \quad \alpha_2^T \quad \cdots \quad \alpha_m^T)^T
\]
is in the nullspace of the \((n - 2) \times nm \) block matrix
\[
A = \begin{pmatrix}
A_1 & A_2 & \cdots & A_m
\end{pmatrix}.
\]

We relate the above equations to the equations (6.5.2) defining \( Z_m \) by performing row operations on the matrix \( A \) to put it in echelon form, at which point the block \( A_{\ell} \) has been converted to the matrix \( B_{\ell} \) from (6.5.3). We first describe the row-reduction algorithm as a sequence of steps. Fix \( \ell \) and write our starting matrix \( A_{\ell} \) as a matrix of row vectors:
\[
A_{\ell} = \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_{n-2}
\end{pmatrix}.
\]

At every step of the row-reduction process, we will refer to the matrix obtained at that step by \( \tilde{A} \) and the rows of \( \tilde{A} \) by \( \tilde{A}_1, \ldots, \tilde{A}_{n-2} \). The rows of the original matrix \( A_{\ell} \) will always be denoted \( A_1, \ldots, A_{n-2} \).
The \( j \)th step of our row-reduction algorithm, \( j = 1, \ldots, n - 2 \), is as follows. If \( j > 1 \), then for \( i = 1, \ldots, j - 1 \), replace the row \( \tilde{A}_i \) with
\[
\tilde{A}_i + \frac{q_j - p_j}{p_j - p_i} \tilde{A}_j
\]
and multiply the resulting row by \( \frac{p_j - p_i}{q_j - p_i} \). If \( j < n - 2 \), then for \( i = j + 1, \ldots, n - 2 \), replace the row \( \tilde{A}_i \) with
\[
\tilde{A}_i - \frac{q_j - p_j}{q_i - p_j} \tilde{A}_j
\]
and multiply the resulting row by \( \frac{q_i - p_j}{q_j - q_i} \). Finally, multiply rows 1, \ldots, \( j \) by \(-1\).

**Notation A.1.** If \( I \subset \{1, \ldots, n - 2\} \) is a set of indices, define
\[
A_I(z) = \prod_{i=1 \atop i \notin I}^{n-2} (z - p_i)
\]
and
\[
B_I(z) = \prod_{i=2 \atop i \notin I}^{n-2} (z - q_i),
\]
where an empty product is defined to be 1. Note that
\[
A_{\{i\}}(z) = A_i(z) \quad \text{and} \quad B_{\{i\}}(z) = B_i(z)
\]
in the notation from Notation 6.0.2. For any \( j \in \{1, \ldots, n - 2\} \), define
\[
I_j = \{j, \ldots, n - 2\}.
\]

**Claim A.2.** After step \( j \) of the row reduction, the \( i \)th row \( \tilde{A}_i \) of the resulting matrix \( \tilde{A} \) is as follows. For \( 1 \leq i \leq j \),
\[
(-1)^j \left[ A_1 - \sum_{k=2}^{j} \frac{A_{\{i\} \cup I_{j+1}}(q_k)}{B_{\{k\} \cup I_{j+1}}(q_k)} A_k \right].
\]

For \( j + 1 \leq i \leq n - 2 \),
\[
(-1)^j \left[ A_1 - \frac{A_{I_{j+1}}(q_i)}{B_{I_{j+1}}(q_i)} A_i - \sum_{k=2}^{j} \frac{A_{I_{j+1}}(q_k)}{(q_k - q_i)B_{\{k\} \cup I_{j+1}}(q_k)} A_k \right].
\]
In both cases, an empty sum is defined to be the zero row vector.
Proof of Claim A.2. We proceed by induction on $j$. Note that, after step 1, the matrix $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} -A_1 \\ (q_2 - p_1)A_2 - A_1 \\ \vdots \\ (q_{n-2} - p_1)A_{n-2} - A_1 \end{pmatrix}.$$ 

Now assume that the claim holds at step $j$. Then the $(j+1)^{st}$ row of $\tilde{A}$ is already in the correct form for step $j+1$, and after that step, the other rows of $\tilde{A}$ are as follows. For $i \leq j$, the $i^{th}$ row is

$$(-1)^{j+1} \left[ A_1 - \sum_{k=2}^{j} \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{B_{(k) \cup I_{j+1}}(q_k)} A_k \right. \right.$$

$$+ \frac{q_{j+1} - p_{j+1}}{p_{j+1} - p_i} \left( A_1 - \sum_{k=2}^{j+1} \frac{\mathbb{A}_{I_{j+1}}(q_k)}{B_{(k) \cup I_{j+1}}(q_k)} A_k \right) \left. \frac{p_{j+1} - p_i}{q_{j+1} - p_i} \right]$$

$$= (-1)^{j+1} \left[ A_1 - \sum_{k=2}^{j} \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{B_{I_{j+1}}(q_{j+1})} A_{j+1} \right. \right.$$

$$- \sum_{k=2}^{j} \left[ \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{(q_{j+1} - p_i)B_{(k) \cup I_{j+1}}(q_k)} (q_{j+1} - p_i)(q_k - p_{j+1}) A_k \right]$$

$$= (-1)^{j+1} \left[ A_1 - \sum_{k=2}^{j} \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{B_{I_{j+1}}(q_{j+1})} A_{j+1} \right. \right.$$

$$- \sum_{k=2}^{j+1} \left[ \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{(q_{j+1} - p_i)B_{(k) \cup I_{j+1}}(q_{k+1})} (q_{j+1} - p_i)(q_k - p_{j+1}) A_k \right]$$

$$= (-1)^{j+1} \left[ A_1 - \sum_{k=2}^{j+1} \frac{\mathbb{A}_{\{i\} \cup I_{j+1}}(q_k)}{B_{(k+1) \cup I_{j+3}}(q_{k+1})} A_k \right].$$

For $j + 2 \leq i \leq n - 2$, the $i^{th}$ row is

$$(-1)^j \left[ \left( A_1 - \frac{\mathbb{A}_{I_{j+1}}(q_i)}{B_{I_{j+1}}(q_i)} A_i \right) - \sum_{k=2}^{j} \frac{\mathbb{A}_{I_{j+1}}(q_k)}{(q_k - q_i)B_{(k) \cup I_{j+1}}(q_k)} A_k \right].$$
\[-q_{j+1} - p_{j+1} \begin{pmatrix} A_1 - \sum_{k=2}^{j+1} \frac{\hat{A}_{I_{j+2}}(q_k)}{B_{(k) \cup I_{j+2}}(q_k)} A_k \end{pmatrix} \begin{pmatrix} q_i - p_{j+1} \\ q_{j+1} - q_i \end{pmatrix} \]

\[= (-1)^j \left[ -A_1 + \frac{\hat{A}_{I_{j+2}}(q_i)}{B_{I_{j+2}}(q_i)} A_i + \frac{\hat{A}_{I_{j+2}}(q_{j+1})}{(q_{j+1} - q_i) B_{I_{j+2}}(q_{j+1})} A_{j+1} \right. \]

\[ - \sum_{k=1}^j \left( \frac{\hat{A}_{I_{j+1}}(q_k)}{(q_{j+1} - q_i) (q_k - q_i) B_{(k) \cup I_{j+2}}(q_k)} \right) \]

\[ \cdot \left( (q_{j+1} - p_{j+1})(q_k - q_i) - (q_i - p_{j+1})(q_k - q_{j+1}) \right) A_k \right] \]

\[= (-1)^{j+1} \left[ A_1 - \frac{\hat{A}_{I_{j+2}}(q_i)}{B_{I_{j+2}}(q_i)} A_i - \frac{\hat{A}_{I_{j+2}}(q_{j+1})}{(q_{j+1} - q_i) B_{I_{j+2}}(q_{j+1})} A_{j+1} \right. \]

\[- \sum_{k=1}^j \frac{\hat{A}_{I_{j+1}}(q_k)}{(q_{j+1} - q_i) (q_k - q_i) B_{(k) \cup I_{j+2}}(q_k)} (q_{j+1} - q_i) (q_k - p_{j+1}) A_k \right] \]

\[= (-1)^{j+1} \left[ A_1 - \frac{\hat{A}_{I_{j+2}}(q_i)}{B_{I_{j+2}}(q_i)} A_i - \frac{\hat{A}_{I_{j+2}}(q_{j+1})}{(q_k - p_i) B_{(k) \cup I_{j+2}}(q_k)} A_{j+1} \right. \]

The claim follows by induction. \(\square\)

After step \(n - 2\) of the row reduction, multiply the matrix \(\tilde{A}\) by \((-1)^{n-2}\).

By Claim \(A.2\), the \(i\)th row of the resulting matrix is

\[
A_1 - \sum_{k=2}^{n-2} \frac{\hat{A}_i(q_k)}{B_k(q_k)} A_k.
\]

The \(j\)th entry of this row is

\[
1 - \sum_{k=2}^{n-2} \frac{\hat{A}_i(q_k)}{B_k(q_k)} \frac{1}{q_k - p_j}
\]

when \(\ell = 1\) and

\[
- \sum_{k=2}^{n-2} \frac{\hat{A}_i(q_k)}{B_k(q_k)} \frac{1}{(q_k - p_j)\ell}
\]

when \(\ell > 1\). In either case, the result is equal to

\[
\frac{1}{(\ell - 1)!} \frac{d^{\ell-1}}{dz^{\ell-1}} (p_j)
\]

by (6.5.4), and we are done.
Appendix B. Determinant Calculations

In this appendix, we prove Claim 7.2.6. For notational simplicity, we abbreviate

\[ E_{\ell,i} = E_\ell(p_1, \ldots, \hat{p}_i, \ldots, p_n) \]

and

\[ E_{\ell,(i,j)} = E_\ell(p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n). \]

For compactness of notation, we also abbreviate

\[ \alpha_i = \alpha_{i,1}. \]

To reduce bookkeeping with signs, we will instead compute

\[
\begin{vmatrix}
1 & \cdots & 1 \\
p_1 & \cdots & p_n \\
p_1^2 & \cdots & p_n^2 \\
\vdots & \ddots & \vdots \\
p_1^{n-2} & \cdots & p_n^{n-2} \\
\alpha_1 E_{\ell,1} & \cdots & \alpha_n E_{\ell,n}
\end{vmatrix},
\]

which differs from our desired determinant by a factor of \((-1)^{\binom{n}{2}}\).

Lemma B.1. If \(n \geq 4\) and \(0 \leq \ell \leq n - 1\), we have

\[
\begin{vmatrix}
1 & \cdots & 1 \\
p_1 & \cdots & p_n \\
p_1^2 & \cdots & p_n^2 \\
\vdots & \ddots & \vdots \\
p_1^{n-2} & \cdots & p_n^{n-2} \\
E_{\ell,1} & \cdots & E_{\ell,n}
\end{vmatrix} = \begin{cases} 
0, & 0 \leq \ell \leq n - 2 \\
(-1)^{\binom{n-1}{2}} \Delta, & \ell = n - 1
\end{cases}.
\]

Proof. The proof is by induction on \(n\). Our base case is \(n = 4\). When \(\ell = 0\), the result is trivial, as the last row is equal to the first row. So assume \(\ell > 0\).

We have

\[
\begin{vmatrix}
p_1 & p_2 & p_3 & p_4 \\
p_1^2 & p_2^2 & p_3^2 & p_4^2 \\
E_{\ell,1} & E_{\ell,2} & E_{\ell,3} & E_{\ell,4}
\end{vmatrix}
= \begin{vmatrix}
p_2 - p_1 & p_3 - p_1 & p_4 - p_1 \\
p_2(p_2 - p_1) & p_3(p_3 - p_1) & p_4(p_4 - p_1) \\
E_{\ell,2} - E_{\ell,1} & E_{\ell,3} - E_{\ell,1} & E_{\ell,4} - E_{\ell,1}
\end{vmatrix}
\]
When \( \ell = 1 \), we are done, as the last row is equal to the first row. When \( \ell = 2 \), the last determinant is
\[
\begin{vmatrix}
1 & 1 & 1 \\
p_2 & p_3 & p_4 \\
p_3 + p_4 & p_2 + p_4 & p_2 + p_3
\end{vmatrix} = \begin{vmatrix}
1 & 1 & 1 \\
p_2 & p_3 & p_4 \\
p_3 p_4 & p_2 p_4 & p_2 p_3
\end{vmatrix} = 0
\]
since the second and third rows are parallel. When \( \ell = 3 \), we have
\[
\begin{vmatrix}
1 & 1 & 1 \\
p_2 & p_3 & p_4 \\
p_3 p_4 & p_2 p_4 & p_2 p_3
\end{vmatrix} = \begin{vmatrix}
1 & 1 & 1 \\
p_2 & p_3 & p_4 \\
p_3 p_4 & p_2 p_4 & p_2 p_3
\end{vmatrix} = - (p_2 - p_3)(p_2 - p_4)(p_3 - p_4),
\]
and the result follows.

Now assume the result for \( n - 1 \) and all \( \ell = 0, \ldots, n - 2 \). We prove it for \( n \) and all \( \ell = 0, \ldots, n - 1 \). We have
\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
p_1 & p_2 & \cdots & p_n \\
p_1^2 & p_2^2 & \cdots & p_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
p_1^{n-2} & p_2^{n-2} & \cdots & p_n^{n-2} \\
E_{\ell,1} & E_{\ell,2} & \cdots & E_{\ell,n}
\end{vmatrix} = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
0 & p_2 - p_1 & \cdots & p_n - p_1 \\
0 & p_2(p_2 - p_1) & \cdots & p_n(p_n - p_1) \\
0 & \vdots & \ddots & \vdots \\
0 & p_2^{n-3}(p_2 - p_1) & \cdots & p_n^{n-3}(p_n - p_1) \\
0 & E_{\ell,2} - E_{\ell,1} & \cdots & E_{\ell,n} - E_{\ell,1}
\end{vmatrix}
\]
\[
= \begin{vmatrix}
p_2 - p_1 & \cdots & p_n - p_1 \\
p_2(p_2 - p_1) & \cdots & p_n(p_n - p_1) \\
\vdots & \ddots & \vdots \\
p_2^{n-3}(p_2 - p_1) & \cdots & p_n^{n-3}(p_n - p_1) \\
(p_1 - p_2)E_{\ell-1,1} & \cdots & (p_1 - p_n)E_{\ell-1,n}
\end{vmatrix}
\]
where

\[ k \] show that its determinant vanishes:

with

\[ \square \]

\[ \alpha \]

\[ (B.1) \]

We now prove Claim 7.2.6 when

\[ 0 \leq \ell \leq n - 2, \]

\[ (1) \]

\[ (-1)^nA_1(p_1)(-1)^{(\ell-2)/2}\Delta_1, \quad \ell = n - 1 \]

\[ = \begin{cases} 
0, & 0 \leq \ell \leq n - 2, \\
(-1)^nA_1(p_1)(-1)^{(\ell-2)/2}\Delta, & \ell = n - 1 
\end{cases} \]

\[ = \begin{cases} 
0, & 0 \leq \ell \leq n - 2, \\
(-1)^{(\ell-1)/2}\Delta, & \ell = n - 1 
\end{cases} \]

by the induction hypothesis.

We start with the case \( \ell \leq 0 \) and \( \ell = 1 \). The case \( \ell = n - 1 \) is done later as a separate calculation.

We start with the case \( \ell = 0 \). First, note that the equations (6.5.5) imply that

\[ \alpha_k - \alpha_1 = \frac{(p_k - p_1)(\alpha_{n-1} - \alpha_n)}{p_{n-1} - p_n}, \]

where \( k = 1, \ldots, n \). With (B.1), we can row-reduce the matrix in question to show that its determinant vanishes:

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
p_1 & p_2 & \cdots & p_n \\
p_1^2 & p_2^2 & \cdots & p_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
p_1^{n-2} & p_2^{n-2} & \cdots & p_n^{n-2} \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\end{vmatrix}
= \begin{vmatrix}
1 & 1 & \cdots & 1 \\
0 & p_2 - p_1 & \cdots & p_n - p_1 \\
0 & p_2(p_2 - p_1) & \cdots & p_n(p_n - p_1) \\
\vdots & \vdots & \ddots & \vdots \\
0 & p_2^{n-2}(p_2 - p_1) & \cdots & p_n^{n-2}(p_n - p_1) \\
0 & \alpha_2 - \alpha_1 & \cdots & \alpha_n - \alpha_1 \\
\end{vmatrix}
= \begin{vmatrix}
p_2 - p_1 & \cdots & p_n - p_1 \\
p_2(p_2 - p_1) & \cdots & p_n(p_n - p_1) \\
\vdots & \ddots & \vdots \\
p_2^{n-3}(p_2 - p_1) & \cdots & p_n^{n-3}(p_n - p_1) \\
(p_2 - p_1)^{\alpha_{n-1} - \alpha_n} & \cdots & (p_n - p_1)^{\alpha_{n-1} - \alpha_n} \\
\end{vmatrix}
= 0.
\]
Now we treat the inductive step. Assume the result is true for \( \ell - 1 \leq n - 3 \). We prove it for \( \ell \). First, note that
\[
p_1\alpha_k - p_k\alpha_1 = (p_k - p_1) \frac{p_n\alpha_{n-1} - p_{n-1}\alpha_n}{p_{n-1} - p_n}
\]
by \((6.5.5)\), so that
\[
E_{\ell,k}\alpha_k - E_{\ell,1}\alpha_1 = E_{\ell-1,(1,k)}(p_1\alpha_k - p_k\alpha_1) + E_{\ell,(1,2)}(\alpha_k - \alpha_1)
= (p_k - p_1) \left[ E_{\ell-1,(1,k)} \frac{p_n\alpha_{n-1} - p_{n-1}\alpha_n}{p_{n-1} - p_n} + E_{\ell,(1,k)} \frac{\alpha_{n-1} - \alpha_n}{p_{n-1} - p_n} \right].
\]

Hence,
\[
\begin{align*}
\det & \begin{pmatrix}
1 & 1 & \cdots & 1 \\
p_1 & p_2 & \cdots & p_n \\
p_1^2 & p_2^2 & \cdots & p_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
p_1^{n-2} & p_2^{n-2} & \cdots & p_n^{n-2} \\
E_{\ell,1}\alpha_1 & E_{\ell,2}\alpha_2 & \cdots & E_{\ell,n}\alpha_n
\end{pmatrix} \\
& = \det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & p_2 - p_1 & \cdots & p_n - p_1 \\
0 & p_2(p_2 - p_1) & \cdots & p_n(p_n - p_1) \\
\vdots & \vdots & \ddots & \vdots \\
0 & p_2^{n-3}(p_2 - p_1) & \cdots & p_n^{n-3}(p_n - p_1) \\
0 & E_{\ell,2}\alpha_2 - E_{\ell,1}\alpha_1 & \cdots & E_{\ell,n}\alpha_n - E_{\ell,1}\alpha_1
\end{pmatrix} \\
& = (-1)^{n-1}A_1(p_1) \frac{p_{n}\alpha_{n-1} - p_{n-1}\alpha_n}{p_{n-1} - p_n} \det \begin{pmatrix}
1 & \cdots & 1 \\
p_2 & \cdots & p_n \\
\vdots & \ddots & \vdots \\
p_2^{n-3} & \cdots & p_n^{n-3} \\
E_{\ell-1,(1,2)} & \cdots & E_{\ell-1,(1,n)}
\end{pmatrix} \\
& + (-1)^{n-1}A_1(p_1) \frac{\alpha_{n-1} - \alpha_n}{p_{n-1} - p_n} \det \begin{pmatrix}
1 & \cdots & 1 \\
p_2 & \cdots & p_n \\
\vdots & \ddots & \vdots \\
p_2^{n-3} & \cdots & p_n^{n-3} \\
E_{\ell,(1,2)} & \cdots & E_{\ell,(1,n)}
\end{pmatrix}
\]

When \( 0 \leq \ell \leq n - 3 \), both determinants vanish by Lemma \([B.1]\). When \( \ell = n - 2 \), Lemma \([B.1]\) tells us that the first determinant vanishes and that the
second is

\[(−1)^{(n−1)}A_1(p_1)\frac{α_{n−1} − α_n}{p_{n−1} − p_n}Δ_1 = (−1)^{(n−1)}A_1\frac{α_{n−1} − α_n}{p_{n−1} − p_n}\].

Thus, we have

\[\det\begin{pmatrix} α_1 E_ℓ,1 & \cdots & α_n E_ℓ,n \\
p_1^{n−2} & \cdots & p_n^{n−2} \\
\vdots & \ddots & \vdots \\
p_1^2 & \cdots & p_n^2 \\
p_1 & \cdots & p_n \\
1 & \cdots & 1 \end{pmatrix} = (−1)^n \frac{α_{n−1} − α_n}{p_{n−1} − p_n}\]

as desired.

We now prove the claim when \(\ell = n − 1\). In this case, we have

\[E_{n−1,k}α_k − E_{n−1,1}α_1 = (p_k − p_1)E_{n−2,(1,k)}\frac{p_1α_{n−1} − p_{n−1}α_n}{p_{n−1} − p_n},\]

so, by Lemma [B.1]
\[ = (-1)^{n-1} A_1(p_1) \frac{p_n \alpha_{n-1} - p_{n-1} \alpha_n}{p_{n-1} - p_n} (-1)^{\binom{n-2}{2}} \Delta_1 \]
\[ = (-1)^{\binom{n-1}{2} - 1} \frac{p_n \alpha_{n-1} - p_{n-1} \alpha_n}{p_{n-1} - p_n} \Delta. \]

Thus, we have

\[
\det \begin{pmatrix}
\alpha_1 E_{n-1,1} & \ldots & \alpha_n E_{n-1,n} \\
p_1^{n-2} & \ldots & p_n^{n-2} \\
\vdots & \ddots & \vdots \\
p_1^2 & \ldots & p_n^2 \\
p_1 & \ldots & p_n \\
1 & \ldots & 1
\end{pmatrix} = (-1)^{\binom{n}{2}} (-1)^{\binom{n-1}{2} - 1} \Delta \frac{p_n \alpha_{n-1} - p_{n-1} \alpha_n}{p_{n-1} - p_n}
\]

\[ = (-1)^n \Delta \frac{p_n \alpha_{n-1} - p_{n-1} \alpha_n}{p_{n-1} - p_n}, \]

as desired.

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