Quantum chains with a Catalan tree pattern
of conserved charges:
the $\Delta = -1$ XXZ model and the isotropic octonionic chain

M. P. Grabowski$^a$ and P. Mathieu$^b$

$^a$Département de physique, Université Laval, Québec, Canada G1K 7P4
$^b$e-mail: pmathieu@phy.ulaval.ca

Abstract

A class of quantum chains possessing a family of local conserved charges with a Catalan tree pattern is studied. Recently, we have identified such a structure in the integrable $SU(N)$-invariant chains. In the present work we find sufficient conditions for the existence of a family of charges with this structure in terms of the underlying algebra. Two additional systems with a Catalan tree structure of conserved charges are found. One is the spin 1/2 XXZ model with $\Delta = -1$. The other is a new octonionic isotropic chain, generalizing the Heisenberg model. This system provides an interesting example of an infinite family of noncommuting local conserved quantities.

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1. Introduction

A remarkably simple explicit expression for all the local conserved charges of the periodic or infinite XXX model (spin 1/2 quantum chain) has been found recently [1,2] (see also [3] for an earlier presentation of an equivalent result for the infinite chain). In an appropriate basis, each charge is described in terms a Catalan tree pattern. A direct argument, based on three simple identities, has been devised to prove that these integrals commute with the hamiltonian and among themselves. This provides thus an alternative and new proof of the integrability of the XXX model. But this in itself is not very interesting given that the integrability of this model, which is probably the most intensively studied quantum integrable system, has already been proved in diverse ways. The interest, of course, lies in the possibility of generalizing the argument in view of discovering new integrable systems or even unraveling the structure of the conserved charges of other known integrable systems.

Such an extension has already been found in [2], where it is shown that the XXX pattern is in fact common to all isotropic $su(N)$ quantum models formulated in their fundamental representation. The proof in that case simply boils down to showing that the three identities alluded to above hold true in this case, when the Pauli matrices are replaced by the Gell-Mann matrices and the $su(2)$ structure constants by their $su(N)$ counterparts. The fundamental representation is singled out from the requirement that the anticommutator of the generators should close in the algebra, a necessary condition for the validity of two of the identities.

In view of enlarging the potential applications of this direct method, we consider here two sorts of generalization of the three identities at the core of the algebraic direct method. As a first step, we reformulate the model in terms of an unspecified algebra, and introduce as many free parameters as possible in the three identities. When the normalization of the generators is fixed, two such free parameters can be introduced. Once the algebra will be specified, these parameters will automatically be determined from the expression for the contraction of two (resp. three) indices in the product of two (resp. three) structure constants. Notice that the hamiltonian is the sum of the nearest-neighbor bilinear in the algebra generators and it does not contain any parameter, i.e. it is isotropic. One then constructs the generalization of the XXX charges, with $su(2)$ quantities replaced by their analogues in the yet unspecified algebra, and see under which conditions these charges commute with the hamiltonian. These conditions turn out to be simply a linear relation.
between our two free parameters. Thus we end up with a one parameter set of identities, sufficient to guarantee the existence of an infinite number of conservation laws for an infinite chain. By fixing the algebra, which amounts to fixing the two parameters, we can thus readily check whether the linear relation between the two now-determined parameters is satisfied. If it is, then not only the existence of an infinite number of conserved charges is automatically proved, but as a bonus, we have an explicit expression for them. By construction, these charges have a Catalan tree pattern. Moreover, for a particular value of the free parameter, our construction ensures automatically mutual commutativity of all the charges, hence demonstrating the integrability of the model.

Here we present a new quantum chain, for which the existence of an infinite family of conservation laws can be proved exactly in this way. This is the isotropic octonionic chain. Since the $su(2)$ XXX model is actually a quaternionic chain, this generalization looks, in retrospect, rather natural.

Another direction where a generalization can be contemplated is for anisotropic models. Considering for concreteness the XYZ model, we can determine by a systematic procedure how the building blocks of the XXX charges (i.e. the Catalan tree vertices) would have to be modified to take into account the anisotropy. Unfortunately, for the generic anisotropic model, the three identities cannot be satisfied. However, once we have identified the point where the proof breaks down in the general case, we can look for special anisotropic models that would still make the argument go through. In this way, we find that the XXZ model

$$H = \sum_{j \in \Lambda} [\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} + \Delta \sigma^z_j \sigma^z_{j+1}], \quad (1.1)$$

with $\Delta = -1$ satisfies all the conditions. Its conservation laws have then a Catalan tree pattern. This solution appears not to have a $su(N)$ extension.

2. Algebraic framework

In this section we formulate quantum “spin” chains in a general algebraic framework and define the sequence of charges with Catalan tree pattern. Recall first a few basic definitions [4]. An algebra over a field $\mathbf{K}$ is a vector space $A$ over $\mathbf{K}$ equipped with a product (a mapping $A \times A \to A$) satisfying the properties

$$ (\alpha + \beta)x = \alpha x + \beta x, $$

$$ \alpha(x + y) = \alpha x + \alpha y, \quad (2.1) $$

$$ \alpha(xy) = (\alpha x)y = x(\alpha y) $$
for all $\alpha, \beta$ in $K$ and $x, y$ in $A$. An algebra is associative if the product is associative, i.e.

$$(xy)z = x(yz)\quad (2.2)$$

for all $x, y, z$ in $A$. An algebra is alternative if

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2\quad (2.3)$$

for all $x, y \in A$. It follows immediately that any associative algebra is alternative.

We will consider models defined on a lattice $\Lambda$ which may be either infinite ($\Lambda = \mathbb{Z}$) or finite; in the latter case, periodic boundary conditions are assumed ($\Lambda = \{1, \ldots, N\}$, with $N + 1 \equiv 1$). Let $A$ be a finite-dimensional (possibly nonassociative) algebra with unity over a field $K$. We denote by $\mathcal{A}$ the tensor product algebra

$$\mathcal{A} = \bigotimes_{i \in \Lambda} A. \quad (2.4)$$

Let us recall now some definitions from [2]. Let $\mathcal{B}^{(n)}(\Lambda)$ be the set of all $n$-element sequences of points in $\Lambda$. We define $n$-clusters as ordered elements of $\mathcal{B}^{(n)}$, i.e. an $n$-cluster is a sequence of $n$ lattice sites $C = \{i_1, \ldots, i_n\}$, with $i_1 < i_2 < \ldots < i_n$. We denote the set of $n$-clusters as $C^{(n)}(\Lambda)$. Let

$$\mathcal{B}(\Lambda) = \bigcup_{n=1}^{\left|\Lambda\right|} \mathcal{B}^{(n)}(\Lambda), \quad C(\Lambda) = \bigcup_{n=1}^{\left|\Lambda\right|} C^{(n)}(\Lambda),$$

where $\left|\Lambda\right|$ is the number of sites in $\Lambda$. The sequences in $\mathcal{B}(\Lambda)$ which are not in $C(\Lambda)$ will be called disordered clusters. For any $n$-cluster we define the number of its holes $n_h$, i.e. the number of sites in between $i_1$ to $i_n$ that are not included in $C$: $n_h(C) = i_n - i_1 + 1 - n$. Let

$$C^{(n,k)}(\Lambda) = \{C \in C^{(n)}|n_h(C) = k\} \quad (2.5)$$

be the set of all $n$-clusters of $\Lambda$ with $k$ holes.

Let $h : \mathcal{B}(\Lambda) \rightarrow \mathcal{A}$ be some function and let $H = \sum_{C \in \mathcal{B}(\Lambda)} h(C)$. A triple $(A, \Lambda, H)$ defines then a general quantum chain with a hamiltonian $H$ on a lattice $\Lambda$.

We also recall the recurrence relation defining generalized Catalan numbers $\alpha_{k, \ell}$:

$$\alpha_{k, \ell} = \alpha_{k-1, \ell-1} + \alpha_{k, \ell+1}, \quad (2.6)$$
with the understanding that $\alpha_{k,\ell} = 0$ if $\ell > k$.

Consider now an arbitrary function $\tilde{f} : \mathcal{B}(\Lambda) \to \mathcal{A}$. We will use the notation $\tilde{f}_n$ to denote a restriction of $\tilde{f}$ to $n$-clusters: $\tilde{f}_n = \tilde{f}|_{\mathcal{C}(n)(\Lambda)}$. We next define an elementary operation (which is a derivation if $\mathcal{A}$ is associative) $\delta_{ij}$ by

$$\delta_{ij}X \equiv [\tilde{f}_2(i,j), X] \quad (2.7)$$

for $X$ in $\mathcal{A}$. To any such function $\tilde{f}$ we can associate a sequence $\{H_n(\tilde{f})\}$ $(n = 2, \ldots, |\Lambda|)$ of elements of $\mathcal{A}$, defined by

$$H_n = F_{n,0} + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \alpha_{k,\ell} F_{n-2k,\ell}, \quad (2.8)$$

where the square bracket indicates the integer part and

$$F_{n,k} = \sum_{\mathcal{C} \in \mathcal{C}(n,k)} \tilde{f}(\mathcal{C}), \quad (2.9)$$

We will call (2.8) a Catalan tree sequence corresponding to the function $\tilde{f}$, since each element of this sequence can be represented in terms of a simple tree, known as Catalan tree [1-2]. We also define

$$\delta_{H_2}X = \sum_{j \in \Lambda} \delta_{jj+1}X = [H_2, X] \quad (2.10)$$

for $X$ in $\mathcal{A}$. Note that the elements of the Catalan tree sequence are given by local expressions, in the sense that terms involving a certain set of sites vanish when the distances between the sites become sufficiently large. A Catalan tree sequence will be called conserved if

$$[H_2, H_n] = \sum_{j \in \Lambda} \delta_{jj+1}H_n = 0. \quad (2.11)$$

---

2 If the characteristic of $K$ is not zero, (2.8) should be understood as

$$H_n = F_{n,0} + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \sum_{m=1}^{\alpha_{k,\ell}} F_{n-2k,\ell}$$
3. Sufficient conditions for the conservation of the family \( \{H_n\} \)

We first formulate the following technical:

Lemma

Suppose that there exist a function \( \tilde{f} : \mathcal{B}(\Lambda) \to A \) such that the operation \( \delta_{ij} \) has the following properties:

(i) for any \( n \)-cluster \( C = \{i_1, \ldots, i_n\} \)

\[
\delta_{jj+1} \tilde{f}(C) = -\kappa_1 \tilde{f}(i_1-1, i_1, \ldots, i_n) \quad \text{if} \quad j = i_1 - 1,
\]

\[
= \kappa_1 \tilde{f}(i_1, \ldots, i_n, i_n + 1) \quad \text{if} \quad j = i_n,
\]

\[
= \kappa_1 \tilde{f}(i_1, i_1 + 1, i_2, \ldots, i_n) \quad \text{if} \quad j = i_1 \neq i_2 - 1,
\]

\[
= -\kappa_1 \tilde{f}(i_1, \ldots, i_n - 1, i_n) \quad \text{if} \quad j + 1 = i_n \neq i_{n-1} + 1,
\]

\[
= 2\kappa_2[\tilde{f}(i_2, i_3, \ldots, i_n) - \tilde{f}(i_1, i_3, \ldots, i_n)] \quad \text{if} \quad j = i_1 = i_2 - 1,
\]

\[
= 2\kappa_2[\tilde{f}(i_1, \ldots, i_{n-2}, i_n) - \tilde{f}(i_1, \ldots, i_{n-2}, i_{n-1})] \quad \text{if} \quad j = i_{n-1} = i_n - 1,
\]

\[
= \kappa_3[\tilde{f}(i_1, \ldots, i_{k-1}, i_k + 1, \ldots, i_n) - \tilde{f}(i_1, \ldots, i_k, i_{k+2}, \ldots, i_n)] \quad \text{if} \quad j = i_k = i_{k+1} - 1, i_k \neq 1, n - 1.
\]  

(3.1)

(ii) if \( j = i_k \) and \( i_{k+1} \neq i_k + 1 \)

\[
\delta_{jj+1}[\tilde{f}(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_n) + \tilde{f}(i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_n)] = 0 \quad (3.2)
\]

Then we have:

(a) The Catalan tree sequence \( H_n \) corresponding to \( \tilde{f} \) is conserved

\[
[H_2(\tilde{f}), H_n(\tilde{f})] = 0 \quad (3.3)
\]

iff

\[
\kappa_1 + \kappa_3 = 2\kappa_2. \quad (3.4)
\]

(b) Moreover, if \( A \) is alternative and if \( \kappa_1 = \kappa_2 = \kappa_3 \), the Catalan tree sequence \( \{H_n(\tilde{f})\} \) forms a mutually commuting associative family:

\[
[H_n(\tilde{f}), H_m(\tilde{f})] = 0 \quad \text{for all} \quad n, m \geq 2. \quad (3.5)
\]

Proof: Part (a) of the lemma can be proved by a direct calculation, which is essentially a straightforward extension of the proof presented in [1-2] for the case when \( A = M_2(\mathbb{C}) \)
(the algebra of complex $2 \times 2$ matrices), and which will not be repeated here. The proof of part (b) makes use of the properties of the operation:

$$
\delta_B = \sum_{j \in \Lambda} j \delta_{jj+1} = [B, ],
$$

where

$$
B = \sum_{j \in \Lambda} j \tilde{f}(j, j + 1).
$$

Suppose first that the chain is infinite $|\Lambda| = \infty$. If $\kappa_1 = \kappa_2 = \kappa_3$ then $\delta_B$ generates recursively all the $H_{n>2}$ starting from $H_2$. This can be seen from the fact that $\delta_B H_n$ is a linear combination of elements $H_{n+1-2k}$, $k = 0, \ldots, [n/2]$. More exactly, we have:

$$
H_{n+1} = \frac{1}{(n-1)}[B, H_n] + R_n,
$$

where $R_n$ is a linear combination of the charges $H_{m<n}$. In the associative case, we can use (3.8) in an inductive argument to prove (3.5). We outline this argument below (cf. [2], section 4.2.(i)). Assuming that $[H_n, H_m] = 0$ for all $n, m < n_0$, we prove that $[H_{n_0+1}, H_k] = 0$, for $k < n_0$. For $k = 2$ this holds by construction. For $k = 3$, the commutativity of $H_{n_0+1}$ and $H_3$ can be established using the Jacobi identity and the fact that $[H_{n_0+2}, H_2] = 0$. Similarly, one may successively show that $[H_{n_0+1}, H_{k>3}] = 0$ using the Jacobi identity and the relations $[H_{n_0+\ell}, H_2] = 0$ for $\ell = 1, \ldots, k-1$.

If the algebra $A$ (and hence $\mathcal{A}$) is nonassociative, the Jacobi identity does not hold. However, when $A$ is alternative we have [4]

$$
[[x, y], z] + [[y, z], x] + [[z, x], y] = 6(x, y, z),
$$

for all $x, y, z \in \mathcal{A}$, where

$$
(x, y, z) = (xy)z - x(yz)
$$

is the associator. Moreover, the Artin’s theorem [4] shows that the subalgebra of an alternative algebra generated by any two elements is associative. Since $H_{n\geq 2}$ belong to the subalgebra generated by $B$ and $H_2$, we have then

$$
(B, H_n, H_m) = (H_n, H_m, H_k) = 0
$$

for all $n, m, k \geq 2$. In consequence, the Jacobi identity holds for the commutators of the type $[B, [H_n, H_m]]$ and the inductive proof of (3.5) goes through. Although strictly
speaking the action of $\delta_B$ is incompatible with periodic boundary conditions, the formula (3.8) remains valid for finite chains, if coefficients in (3.6) are understood modulo $|\Lambda|$. In consequence, (3.5) holds also for finite chains. ♦

In physical terms, part (a) of the above lemma can be interpreted as giving a sufficient condition for the existence of a family of conservation laws in the chain with the hamiltonian $H_2$. Similarly, part (b) gives a sufficient condition for the integrability of such chain. However, the lemma does not indicate how to construct a function $\tilde{f}$ with the desired properties. In the following, this lemma will be used as a tool for verifying explicit constructions of $\tilde{f}$.

Next we present a theorem identifying sufficient conditions for the existence of $\tilde{f}$ with the properties (i)-(ii), when the multiplication in $A$ satisfies certain constraints. Clearly any such function can be determined from its restrictions to $C(n)(\Lambda)$ (denoted $\tilde{f}_n$) for each $n$, which we will construct below.

Let $u^\alpha (\alpha = 0, \ldots, d-1)$ be a basis in $A$. The product in $A$ is completely specified by the multiplication table $a_{\gamma}^{\alpha \beta}$, defined by

$$u^\alpha u^\beta = a_{\gamma}^{\alpha \beta} u^\gamma. \quad (3.12)$$

Identifying $u^0$ with the identity, $u^0 \equiv 1$, we have then

$$a_{0 \gamma}^{\alpha} = a_{\gamma}^{0 \alpha} = \delta_{\gamma}, \quad a_{0}^{00} = 1. \quad (3.13)$$

For an algebra with unity we have then $(d-1)^3$ elements of the field $K$ uniquely determining the algebra. We decompose the structure constants $a_{\gamma}^{\alpha \beta}$ into a symmetric and an antisymmetric part:

$$[u^a, u^b] = c_{\gamma}^{abg} u^g + \eta^{ab} u^0,$$

$$\{u^a, u^b\} = d_{\gamma}^{abg} u^g + \theta^{ab} u^0, \quad (3.14)$$

(the Latin indices go from 1 to $d-1$, the Greek ones from 0 to $d-1$). The antisymmetric part $c_{\gamma}^{abg}$ can be used to define a “vector product”

$$(u_i \times u_j)^g = \beta^{-1} c_{\gamma}^{abg} u_i^a u_j^b, \quad (3.15)$$

where the vector $u_i$ is defined in terms of its components $u_i^a$, and $\beta$ is a nonzero constant. We also define dot product in a natural way, i.e.

$$u_i \cdot u_j = \delta^{ab} u_i^a u_j^b \quad (3.16)$$
Assigning on each site of a cluster $C = \{i_1, ..., i_n\}$ a basis element, we then construct $n$-linear polynomials

$$f_n(C) = v_{n-1} \cdot u_{i_n},$$

(3.17)

where the vector $v_{n-1}$ is obtained from the nested vector product of the basis vectors $u^a$ at the first $n - 1$ sites of the cluster:

$$v_1 = u_{i_1},$$
$$v_2 = (u_{i_1} \times u_{i_2}),$$
$$v_3 = ((u_{i_1} \times u_{i_2}) \times u_{i_3}),$$
$$...$$
$$v_m = (v_{m-1} \times u_{i_m}).$$

(3.18)

In other words,

$$f_n(C) = (\ldots (u_{i_1} \times u_{i_2}) \ldots) \times u_{i_{n-1}} \cdot u_{i_n}.$$

(3.19)

If the tensor $c^{abc}$ is cyclic\(^3\), the polynomials $f_n$ have the property that the dot product can be placed at an arbitrary position, provided that parentheses to its left (right) are nested toward the left (right), e.g:

$$f_n(C) = (u_{i_1} \cdot (u_{i_2} \times (u_{i_3} \ldots \times (u_{i_{n-1}} \times u_{i_n}) \ldots)).$$

(3.20)

We form next the sequence $H_n(f)$. The first few elements are

$$H_2 = \sum_{j \in \Lambda} u_j \cdot u_{j+1},$$
$$H_3 = \sum_{j \in \Lambda} (u_j \times u_{j+1}) \cdot u_{j+2},$$
$$H_4 = \sum_{j \in \Lambda} [((u_j \times u_{j+1}) \times u_{j+2}) \cdot u_{j+3} + u_j \cdot u_{j+2}],$$
$$H_5 = \sum_{j \in \Lambda} [(((u_j \times u_{j+1}) \times u_{j+2}) \times u_{j+3}) \cdot u_{j+4}$$
$$+ (u_j \times u_{j+2}) \cdot u_{j+4} + (u_j \times u_{j+3}) \cdot u_{j+4}].$$

(3.21)

We have then the following

\(^3\) Note that the requirement of cyclicity for $c^{abc}$ implies that it is a completely antisymmetric tensor.
Theorem

Suppose that the structure constants of an algebra with unity $A$ satisfy the following relations:

(a) $c$ is cyclic:
\[ c_{abg} = c_bga = c_gab \]  
\[ (3.22) \]

(b) there exist constants $b_2$, $b_3$ such that
\[ c_{abc}(\theta^sa c_{sb} + \eta^sa d_{sb}) = b_2\delta^{lc}, \]
\[ c_{lap} c_{pbr}(\theta^sa c_{sb} + \eta^sa d_{sb}) = b_3c^{lmr}, \]  
\[ (3.23) \]

(c) for all $c, l, r, m$:
\[ c_{abc}(\eta^sa \theta^sb + \theta^sa \eta^sb) = c_{abc}(c_{sal} d_{sbr} + d_{sal} c_{sbr}) = 0, \]
\[ c_{lap} c_{pbr}(\eta^sa \theta^sb + \theta^sa \eta^sb) = c_{lap} c_{pbr}(c_{sam} d_{sbc} + d_{sam} c_{sbc}) = 0. \]  
\[ (3.24) \]

Then if
\[ 2\beta^2 + b_3 = b_2 \]  
\[ (3.25) \]
the Catalan tree sequence $\{H_n(f)\}$ is conserved:
\[ [H_n(f), H_2(f)] = 0. \]  
\[ (3.26) \]

**Proof:** The theorem is proven by a direct calculation verifying that when the constraints (a)-(c) hold, the function $f$ defined in (3.17) satisfies the conditions (i) and (ii) of the preceding lemma with
\[ \kappa_1 = \beta, \quad 2\kappa_2 = b_2/(2\beta), \quad \kappa_3 = b_3/(2\beta) \]  
\[ (3.27) \]
(recall that $\beta$ is the constant in the definition (3.15)). ♦

The assumptions of the theorem may not seem to be very transparent. However, the conditions (a)-(c) above may be equivalently rewritten in terms of the following three simple identities. Let $\mathbf{L}$ and $\mathbf{R}$ be vectors build out of the algebra generators, but involving only sites on the left of $i$ and on the right of $i + 1$ respectively. The identities are:
\[ [\mathbf{u}_i \cdot \mathbf{u}_{i+1}, \mathbf{u}_{i+1} \cdot \mathbf{R}] = -\kappa_1(\mathbf{u}_i \times \mathbf{u}_{i+1}) \cdot \mathbf{R}, \]  
\[ (3.28) \]
\[ [\mathbf{u}_i \cdot \mathbf{u}_{i+1}, (\mathbf{u}_i \times \mathbf{u}_{i+1}) \cdot \mathbf{R}] = 2\kappa_2\{(\mathbf{u}_{i+1} \cdot \mathbf{R}) - (\mathbf{u}_i \cdot \mathbf{R})\}, \]  
\[ (3.29) \]
\[ [\mathbf{u}_i \cdot \mathbf{u}_{i+1}, ((\mathbf{L} \times \mathbf{u}_i) \times \mathbf{u}_{i+1}) \cdot \mathbf{R}] = \kappa_3 \{(\mathbf{L} \times \mathbf{u}_{i+1}) \cdot \mathbf{R} - (\mathbf{L} \times \mathbf{u}_i) \cdot \mathbf{R}\}. \] (3.30)

These identities are equivalent to the conditions (a)-(c) above if \( \kappa_1 = \beta, \ 2\kappa_2 = b_2/(2\beta), \) and \( \kappa_3 = b_3/(2\beta) \).

Moreover, a simple calculation shows that if the assumptions of the theorem hold,

\[ H_1^a = \sum_{j \in \Lambda} u_j^a \] (3.31)

commutes with \( H_2 \).

Note that the construction of the Catalan tree sequence makes use of an explicit choice of basis in the algebra \( A \). In particular, the assumptions of the Theorem in section 2 requires cyclic symmetry of \( c^{abc} \); this property is basis dependent.

The Catalan tree sequence of charges for the \( SU(N) \)-invariant models corresponds to \( A = M_N(\mathbf{C}) \). A convenient basis for \( M_N(\mathbf{C}) \) is provided by the unit matrix \( \mathbf{I} \) and the set of \( su(N) \) Gell-Mann matrices \( t^a, (a = 1, \ldots, N^2 - 1) \) satisfying

\[ [t^a, t^b] = 2i f^{abc} t^c, \]
\[ t^a t^b + t^b t^a = 4(\delta_{ab}/M) \mathbf{I} + 2 \hat{d}^{abc} t^c, \] (3.32)

where \( f^{abc} \) are the structure constants of \( su(N) \), and \( \hat{d}^{abc} \) is a completely symmetric tensor, nontrivial for all \( N > 2 \). Choosing the \( u^a \) basis as \( u_0 = \mathbf{I}, \) and \( u^a = t^a, (a = 1, \ldots, N^2 - 1) \) we have then

\[ c^{abg} = 2i f^{abg}, \quad \theta^{ab} = (4/N) \delta^{ab}, \quad d^{abc} = 2i \hat{d}^{abc}. \] (3.33)

With the choice of \( \beta = 2i \) for the constant in the vector product, we obtain the identities (3.28-3.30) with \( \kappa_1 = \kappa_2 = \kappa_3 = 2i. \)

In the subsequent sections, we will present two other examples of systems with a Catalan tree sequence of charges.

4. The isotropic octonionic chain

In this section we consider the algebra of octonions \( \mathbf{O} \), a nonassociative alternative division algebra over \( \mathbf{R} \). Let \( \{e^0 = 1, e^a\} (a = 1, \ldots, 7) \) be a basis in \( \mathbf{O} \), with the octonionic multiplication defined by

\[ e^a e^b = -\delta^{ab} + f^{abc} e^c, \] (4.1)
where \( f^{abc} \) are the Cayley structure constants. There are 480 different possible choices for the multiplication table; one aesthetically appealing choice, which we will adopt here, is: \( f^{abc} = 1 \) for the cycles

\[
(abc) = (123), (246), (435), (367), (571), (714).
\]  

(4.2)

The Cayley structure constants obey the identity [5]:

\[
f^{abc} f^{gde} = \delta^{ag} \delta^{be} - \delta^{ae} \delta^{bg} + \phi^{abge},
\]

(4.3)

where \( \phi^{abge} \) is a completely antisymmetric tensor defining the octonionic associator:

\[
(e^a, e^b, e^g) = (e^a e^b) e^g - e^a (e^b e^g) = 2 \phi^{abge} e^e
\]

(4.4)

(with the understanding that \( a, b, g \) above are all different). Tensors \( \phi \) and \( f \) are dual to each-other:

\[
\phi^{abcd} = -\frac{1}{3!} \epsilon^{abcd} f_{egh}.
\]

(4.5)

The identity (4.3) implies that

\[
f^{abc} f^{gbc} = 6 \delta^{ag}.
\]

(4.6)

We have also

\[
f^{abc} f^{cge} f^{epa} = 3 f^{bap}.
\]

(4.7)

In the basis \( u^a = e^a, u^0 = e^0 \) we have then

\[
c^{abc} = 2 f^{abc}, \quad \theta^{ab} = -2 \delta^{ab}, \quad \eta^{ab} = d^{abc} = 0.
\]

(4.8)

We define the vector product as

\[
(e_j \times e_k)^c = \gamma^{-1} f^{abc} e_j^a e_k^b,
\]

(4.9)

where \( \gamma \) is a constant. Using the identities for the products of the Cayley structure constants, we find that the identities (3.28-3.30) are satisfied when

\[
\kappa_1 = 2\gamma, \quad 2\kappa_2 = -12/\gamma, \quad \kappa_3 = 6/\gamma.
\]

(4.10)

Therefore the condition \( \kappa_1 + \kappa_3 = 2\kappa_2 \) is satisfied for the choice of \( \gamma = \pm 3i \). The value of this constant turns out to be complex, which may be accounted for by extending the octonionic algebra to

\[
A = \mathbb{C} \otimes \mathbb{O}
\]

(4.11)
This complexified octonionic algebra is still an alternative algebra, but no longer a division algebra, as it has nontrivial zero divisors.

By the theorem in section 3 the sequence $H_n(f)$ (defined by (2.8) with $f$ defined in (3.19)) is conserved for $\gamma = \pm 3i$. In other words, the octonionic model with the hamiltonian

$$H_2 = \sum_{j \in \Lambda} e_j^a e_{j+1}^a \quad (4.12)$$

has $|\Lambda| - 2$ conservation laws $H_{n>2}$. The first few are:

$$H_3 = \sum_{j \in \Lambda} \gamma^{-1} f^{abc} e_j^a e_{j+1}^b e_{j+2}^c,$$

$$H_4 = \sum_{j \in \Lambda} \left[ \gamma^{-2} f^{ab} f^{scd} e_j^a e_{j+1}^b e_{j+2}^c e_{j+3}^d + e_j^a e_{j+2}^a \right], \quad (4.13)$$

$$H_5 = \sum_{j \in \Lambda} \gamma^{-1} \left[ \gamma^{-2} f^{ab} f^{scm} f^{mdh} e_j^a e_{j+1}^b e_{j+2}^c e_{j+3}^d e_{j+4}^h \right.$$

$$+ f^{abc} (e_j^a e_{j+2}^b e_{j+3}^c + e_j^a e_{j+1}^a e_{j+3}^c)].$$

This model appears as the octonionic generalization of the Heisenberg (XXX) model

$$H = \sum_{j \in \Lambda} \sigma_j^a \sigma_{j+1}^a, \quad (4.14)$$

where $\sigma^a$, $a = x, y, z$ are the Pauli sigma matrices, or equivalently imaginary unit quaternions.

We can see that both choices for sign of $\gamma$ lead to equivalent results (under a change $\gamma \to -\gamma$ the charges $H_n$ of even order are invariant, while the charges of odd orders change signs). Observe also that a trivial redefinition $H_{2k+1} \to \gamma H_{2k+1}$ gives a system of conserved charges with real coefficients, thus yielding conservation laws in $\otimes_{j \in \Lambda} O$ (without the complexification (4.11)).

Note that for $\gamma = \pm 3i$ the condition $\kappa_1 = \kappa_2 = \kappa_3$ is not satisfied, and hence part (b) of the lemma in section 3 cannot be used. Mutual commutativity of the $H_n$’s (which is needed to classify the octonionic chain as an integrable system) must be checked independently. Surprisingly, higher order charges do not commute. In particular, for $|\Lambda| \geq 6$, $[H_3, H_4]$ does not vanish.

Note that (up to a constant) $[B, H_2] = H_3$, but $[B, H_3]$ is not a conserved quantity. Thus there is no boost construction for the charges in this case.

For $|\Lambda| = 4, 5$ $[H_3, H_4] = 0$. 

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4 Note that (up to a constant) $[B, H_2] = H_3$, but $[B, H_3]$ is not a conserved quantity. Thus there is no boost construction for the charges in this case.

5 For $|\Lambda| = 4, 5$ $[H_3, H_4] = 0$. 

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12
5. The XXZ model with $\Delta = -1$

Can the pattern of conserved charges in an anisotropic chain be described in terms of a Catalan tree? To investigate this problem, we concentrate on the XYZ model, and analyze possible modifications of the three identities (3.28)-(3.30) to take the anisotropy into account. We will see that this leads in a natural way to a construction of a function $\tilde{f}$ satisfying the requirements of the lemma in section 3, which is a kind of deformation of the polynomials $f$.

The XYZ hamiltonian to be considered is

$$H = \sum_{j \in \Lambda} [\lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y + \lambda_z \sigma_j^z \sigma_{j+1}^z],$$

(5.1)

where $\lambda_x, \lambda_y, \lambda_z$ are constants, and $\sigma_j^a$’s are the Pauli matrices. We rewrite it in the form

$$H = \sum_{j \in \Lambda} \hat{\sigma}_j^a \hat{\sigma}_{j+1}^a,$$

(5.2)

where

$$\hat{\sigma}_j^a = \sqrt{\lambda_a} \sigma_j^a.$$

(5.3)

At first sight, a natural guess for the modified form of the $f$ polynomials is to replace all spin factors by their hat versions (which ensures that $H_2$ coincides with the XYZ hamiltonian). To see whether this is appropriate, we first consider the commutator of $\hat{\sigma}_i \cdot \hat{\sigma}_{i+1}$ with $\hat{\sigma}_{i+1} \cdot R$, where $R$ involves only spin variables at sites to the right of $i + 1$:

$$[\hat{\sigma}_i^a \hat{\sigma}_{i+1}^a, \hat{\sigma}_{i+1}^b R^b] = 2i \sqrt{\frac{\lambda_a \lambda_b}{\lambda_c}} \epsilon_{abc} \hat{\sigma}_i^a \hat{\sigma}_{i+1}^c R^b$$

(5.4)

The goal is to express commutators of $\hat{\sigma}_i \cdot \hat{\sigma}_{i+1}$ with the polynomials $f_n$ in terms of higher or lower order but similar polynomials. Due the to extra factor $\sqrt{\lambda_a \lambda_b/\lambda_c}$, the right hand side of the above equation cannot be rewritten in the form $(\hat{\sigma}_i \times \hat{\sigma}_{i+1}) \cdot R$. However, we can rescale the spin variable at site $i + 1$ by

$$\hat{\sigma}_{i+1}^c = \sqrt{\frac{\lambda_x \lambda_y \lambda_z}{\lambda_c}} \sigma_{i+1}^c$$

(5.5)

In the rest of this section summation over indices repeated two or more times is implied, unless indicated otherwise.
In this way, we find

\[ [\hat{\sigma}_i \cdot \hat{\sigma}_{i+1}, \hat{\sigma}_{i+1} \cdot \mathbf{R}] = -2i(\hat{\sigma}_i \times \hat{\sigma}_{i+1}) \cdot \mathbf{R} \]  

(5.6)

This suggests to replace in the expression for the \( f \)'s, the spins at the border of the cluster by their hat versions and all the other ones by their tilde versions. But this is not enough. As the next calculation will show, an extra multiplying factor, depending on the position of the holes, has to be introduced. Consider:

\[ [\hat{\sigma}_i \cdot \hat{\sigma}_{i+1}, (\hat{\sigma}_i \times \hat{\sigma}_{i+1}) \cdot \mathbf{R}] = \lambda_a \lambda_c \sqrt{\lambda_e} \epsilon_{cde} [\sigma_i^a \sigma_{i+1}^a, \sigma_i^c \sigma_{i+1}^d] R^e. \]  

(5.7)

Using

\[ 2[\sigma_i^a \sigma_{i+1}^a, \sigma_i^c \sigma_{i+1}^d] = [\sigma_i^a, \sigma_i^c] [\sigma_{i+1}^a, \sigma_{i+1}^d] + [\sigma_i^a, \sigma_i^d] [\sigma_{i+1}^a, \sigma_{i+1}^c], \]  

(5.8)

(5.7) is found to be

\[ 2i \lambda_a \lambda_c \sqrt{\lambda_e} \epsilon_{cde} (\delta_{ad}\epsilon_{acf} \sigma_i^f + \delta_{ac}\epsilon_{adf} \sigma_{i+1}^f) R^e. \]  

(5.9)

To investigate the effect of holes, we concentrate on the first term. We cannot use directly the identity

\[ \epsilon_{cae} \epsilon_{caf} = 2\delta_{ef}, \]  

(5.10)

to simplify this term, because the prefactor depends on the indices \( a \) and \( c \), over which we want to sum. But the cure is simple: one just notices that

\[ \lambda_a \lambda_c \epsilon_{cae} = \frac{\lambda_x \lambda_y \lambda_z}{\lambda_e} \epsilon_{cae} \]  

(5.11)

(no summation). With this transformation, the troublesome prefactor is eliminated and the first term in (5.9) becomes

\[ -4i \frac{\lambda_x \lambda_y \lambda_z}{\lambda_e} \sigma_i^e R^e. \]  

(5.12)

This shows that to account for the presence of a hole, here at site \( i+1 \), an extra factor \( \lambda_x \lambda_y \lambda_z / \lambda_e \) has to be introduced, where the index \( e \) stands for the overall component of the vector product to the right (or, equivalently to the left) of the hole. Recall that (as a consequence of the cyclicity of \( \epsilon^{abg} \)) in the \( f \)-type polynomials, the dot (scalar product) can be placed anywhere, provided that all vector products at its left are nested toward the left and all those at its right are nested toward the right. In the XYZ context, it is nothing but a retranscription of the familiar vector identity

\[ (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \]  

(5.13)
The component of the vector product say to the left of a hole is easily read off when the dot product is placed at the left of the hole site. We now complete the evaluation of (5.9), whose second term reads

\[ 2i\lambda_a^2 \sqrt{\lambda_e} \epsilon_{ade} \epsilon_{adf} \sigma_{i+1}^f R^e. \]  

(5.14)

But it presents a serious difficulty: a prefactor depending upon the index \( a \) (or the index \( d \), after a transformation of the type (5.11)) cannot be avoided, so that the result cannot be written in terms of a scalar product. The only way to proceed is to assume that all \( \lambda_a^2 \) are the same. In that case, without loss of generality, we can let \( \lambda_a^2 = 1 \). For this second term, we get then

\[ 4i \hat{\sigma}_{i+1} \cdot \mathbf{R}. \]  

(5.15)

Let us summarize the situation at this point in the anisotropic case. We have found that for the general XYZ model, we cannot prove the second of the three identities, unless \( \lambda_a^2 \) is independent of \( a \). With \( \lambda_a^2 = 1 \), this gives two independent solutions (up to trivial relabelings of the space directions), namely the XXX isotropic model, where \( \lambda_x = \lambda_y = \lambda_z = 1 \), and the XXZ model with \( \lambda_x = \lambda_y = 1 \) and \( \lambda_z \equiv \Delta = -1 \). This second solution, denoted XXZ\(_{-1}\) from now on, is a new candidate for a model with a Catalan tree pattern. Moreover, by studying these first two identities, we have seen how the polynomials \( f \) need to be modified in the anisotropic case:

\[ f_n \rightarrow \tilde{f}_n = g_{\{\text{holes}\}}((\cdots (\hat{\sigma}_i \times \tilde{\sigma}_{i+1}) \times \hat{\sigma}_{i+n-1}) \times \tilde{\sigma}_{i+n}) \cdot \hat{\sigma}_{i+n}, \]  

(5.16)

where \( g \) is the product of the factors \( \lambda_x \lambda_y \lambda_z / \lambda_{e_j} \), one for each hole \( (j = 1, \ldots, k) \), where \( e_j \) is the overall group index at the left of the hole at site \( j \). When \( \lambda_a^2 = 1 \), \( g \) is just a sign. (Note that we have just rederived, in a different way, the rules for constructing these anisotropic polynomials obtained in [2] from the boost construction of the charges.)

Consider now the third identity in the context of the XXZ\(_{-1}\) model. A simple calculation yields

\[ [\hat{\sigma}_i \cdot \hat{\sigma}_{i+1}, ((\mathbf{L} \times \hat{\sigma}_i) \times \tilde{\sigma}_{i+1}) \cdot \mathbf{R}] = 2i \{g_{\{i\}}(\mathbf{L} \times \hat{\sigma}_{i+1}) \cdot \mathbf{R} - g_{\{i+1\}}(\mathbf{L} \times \tilde{\sigma}_i) \cdot \mathbf{R}\}. \]  

(5.17)

Hence, with appropriate correction factors and rescalings, this identity is also satisfied.

In the isotropic case, the three identities ensure the validity of all the steps in the proof of the commutativity of \( H_n \) with the hamiltonian: in other words the function \( f \) satisfies
both the conditions (i) and (ii) of the lemma in section 3. But when these identities are modified as above, only part (i) follows automatically. A key step of the argument, part (ii), must be reconsidered entirely. In the language of [2] this amounts to establishing the cancellation of the terms corresponding to disordered clusters. Such terms have the form

\[(L \times (\tilde{\sigma}_i \times \tilde{\sigma}_{i+1})) \cdot R \neq ((L \times \tilde{\sigma}_i) \times \tilde{\sigma}_{i+1}) \cdot R,\]

(5.18)

and they arise in the commutator of $\tilde{\sigma}_i \cdot \tilde{\sigma}_{i+1}$ with $\tilde{f}_n(C)$ for clusters $C$ having a hole either at $i$ or $i+1$. In the appendix, we use the Jacobi identity to show that such terms can be rewritten as a sum of fully nested terms plus terms that cannot be nested. When we sum over clusters in (2.8), the latter part is shown to cancel for the general XYZ case. Hence, even though there is no Catalan tree pattern for the generic XYZ model, the conserved charges can all be expressed in terms of sums over the polynomials $\tilde{f}_n$, as proved in [2]. For the particular XXZ$_{-1}$ case, the first part also adds up to zero when summed over appropriate clusters. This ensures that the condition (ii) of the lemma holds and it is enough to guarantee the Catalan tree pattern of conserved charges.

To summarize: we have constructed a function $\tilde{f}$ satisfying the requirements of the lemma in section 3 with $\kappa_1 = \kappa_2 = \kappa_3$. The elements of the Catalan tree sequence corresponding to $\tilde{f}$ are linear combinations of the charges obtained from the logarithmic derivatives of the transfer matrix [2], with $H_2$ coinciding with the XXZ$_{-1}$ Hamiltonian. By part (b) of the lemma, the elements of the Catalan tree sequence $H_n(\tilde{f})$ mutually commute (this of course is also an immediate consequence of the transfer matrix formalism).

We present an example of a polynomial $\tilde{f}$. For $C = \{1, 2, 5\}$, we have

\[\tilde{f}_3(1, 2, 5) = g_{\{3\}} g_{\{4\}} (\tilde{\sigma}_1 \times \tilde{\sigma}_2) \cdot \tilde{\sigma}_5,\]

\[= \left(\frac{\lambda_x \lambda_y \lambda_z}{\lambda_c}\right)^2 \epsilon_{abc} \hat{\sigma}_1^a \hat{\sigma}_2^b \hat{\sigma}_5^c,\]

\[= \left(\frac{\lambda_x \lambda_y \lambda_z}{\lambda_c}\right)^2 \sqrt{\lambda_a} \sqrt{\frac{\lambda_x \lambda_y \lambda_z}{\lambda_b}} \sqrt{\lambda_c} \epsilon_{abc} \sigma_1^a \sigma_2^b \sigma_5^c.\]

(5.19)

Specializing to the XXZ$_{-1}$ model, we set

\[\hat{\sigma}^x = \sigma^x, \quad \hat{\sigma}^y = \sigma^y, \quad \hat{\sigma}^z = \sqrt{-1} \sigma^z,\]

\[= \sqrt{-1} \sigma^x, \quad \hat{\sigma}^y = \sqrt{-1} \sigma^y, \quad \hat{\sigma}^z = \sigma^z.\]

(5.20)

We give below the first few nontrivial charges of the XXZ$_{-1}$ model read off from (2.8). The first nontrivial charge beyond the Hamiltonian is:

\[H_3 = F_{3,0} = \sum_{j \in \Lambda} (\hat{\sigma}_j \times \hat{\sigma}_{j+1}) \cdot \hat{\sigma}_{j+2}.\]

(5.21)
The four-spin charge is:

\[ H_4 = F_{4,0} + F_{2,1}, \quad (5.22) \]

where

\[ F_{4,0} = \sum_{j \in \Lambda} ((\hat{\sigma}_j \times \hat{\sigma}_{j+1}) \times \hat{\sigma}_{j+2}) \cdot \hat{\sigma}_{j+3}, \]
\[ F_{2,1} = -\sum_{j \in \Lambda} \lambda_a \hat{\sigma}_j^a \hat{\sigma}_{j+2}^a, \quad (5.23) \]

The five-spin charge is:

\[ H_5 = F_{5,0} + F_{3,1}, \quad (5.24) \]

with

\[ F_{5,0} = \sum_{j \in \Lambda} ((\hat{\sigma}_j \times \hat{\sigma}_{j+1}) \times \hat{\sigma}_{j+2}) \times \hat{\sigma}_{j+3} \cdot \hat{\sigma}_{j+4}, \]
\[ F_{3,1} = -\sum_{j \in \Lambda} \lambda_a \epsilon^{abc} (\hat{\sigma}_j^a \hat{\sigma}_{j+2}^b \hat{\sigma}_{j+3}^c + \hat{\sigma}_j^b \hat{\sigma}_{j+1}^c \hat{\sigma}_{j+3}^a). \quad (5.25) \]

In deriving the correction factors for the polynomials \( f \), we have used in a crucial way the fact that there are only three generators (in (5.11) in particular). This hints that the present analysis is most probably not applicable to algebras other than \( M_2(\mathbb{C}) \). Indeed, we have not found anisotropic \( M_{N>2}(\mathbb{C}) \) models which would satisfy the generalized version of the three identities.

6. Conclusions

In this work we have established sufficient conditions for the existence of a system of conserved charges with a Catalan tree pattern. These conditions are formulated in the lemma and theorem of section 3. The lemma is rather technical, and does not indicate how to construct the sequence of charges from the underlying algebraic structure. The theorem gives such a construction, which is possible if the algebra obeys a number of constraints. These conditions can be viewed as being equivalent to the three equalities (3.28)-(3.30), which are parametrized by three coefficients \( \kappa_1, \kappa_2, \kappa_3 \). Our construction gives a sequence of elements \( \{H_n(f)\} \) such that \([H_2(f), H_n(f)] = 0\), provided that the coefficients \( \kappa_1, \kappa_2, \kappa_3 \) are related by the condition \( \kappa_1 + \kappa_3 = 2\kappa_2 \).

Because an overall rescaling of a function \( \tilde{f} \) is irrelevant, and due to the constraint \( \kappa_1 + \kappa_3 = 2\kappa_2 \) needed to ensure the conservation of the Catalan tree sequence, this construction

\[ \text{We have used the fact that } \lambda_x \lambda_y \lambda_z / \lambda_a = -\lambda_a \text{ for the XXZ}_{-1} \text{ model.} \]
gives a one-parameter family of systems. For example, we can choose for the free parameter the ratio \( \kappa = \kappa_2/\kappa_3 \). For \( A = M_N(C) \) and for the XXZ\(_{-1}\) case, this ratio is \( \kappa = 1 \); for the octonionic system described in section 4, \( \kappa = -1 \). Each model in this family can be regarded as a quantum chain with a hamiltonian \( H_2 \) and a sequence of \(|\Lambda| - 2\) conserved charges \( \{H_{n>2}\} \) (we can also add \( H_1^a \) defined in (3.31)). Since these systems possess an infinity of conserved charges (when \(|\Lambda| = \infty\)), they may be expected to be integrable. However, the standard definition of quantum integrability requires not only the existence of a family conserved charges, but their mutual commutativity as well. For \( \kappa_1 = \kappa_2 = \kappa_3 \), the commutativity of the family \( \{H_n\} \) (if the algebra is alternative) can be established by means of a recursive argument based on the properties of the boost operator \( B \). This argument uses the Jacobi identity, or its generalization (3.9), and it breaks down for a nonalternative algebra. In the general case \((\kappa_1 : \kappa_2 : \kappa_3 \neq 1 \) or if \( A \) is nonalternative\) mutual commutativity of the conserved charges cannot be taken for granted, as the example of the octonionic system shows.

A number of interesting problems remains to be studied. Are there other models with the Catalan tree pattern of conserved charges? In particular, is there such a model for every value of \( \kappa \)? When do the resulting charges mutually commute? How does the presence of such a special pattern of charges reflect itself in the Yang-Baxter equations, the usual hallmark of integrability? Finally, the octonionic chain is interesting in its own right and deserves further study.

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Appendix A. Cancellation of disordered clusters in the XXZ\(_{-1}\) model

We show in this appendix that in the summation over all clusters appearing in (2.8), terms in \([\vec{\sigma}_i \cdot \vec{\sigma}_{i+1}, H_n]\) corresponding to disordered clusters cancel two by two. Such terms originate from the polynomials of the form \( g_{\{j\}} (L \times \vec{\sigma}_j) \cdot R \) with \( j \) either \( i \) or \( i+1 \). As usual, \( L \) and \( R \) are assumed to commute with \( \sigma_i \) and \( \sigma_{i+1} \). We will prove that the non-nested parts of

\[
[\vec{\sigma}_i \cdot \vec{\sigma}_{i+1}, g_{\{i+1\}} (L \times \vec{\sigma}_i) \cdot R + g_{\{i\}} (L \times \vec{\sigma}_{i+1}) \cdot R]
\]  

(A.1)
vanish. This result holds true for the general XYZ model\textsuperscript{8}. The evaluation of the first commutator yields\textsuperscript{9}

\begin{equation}
2i \frac{\lambda_s (\lambda_x \lambda_y \lambda_z)^{3/2}}{\lambda_c \sqrt{\lambda_b}} \epsilon_{abc} \epsilon_{sbn} L^a \sigma^n_i \sigma^{s}_{i+1} R^c. \tag{A.2}
\end{equation}

Using (5.11) and the Jacobi identity

\begin{equation}
\epsilon_{acb} \epsilon_{sbn} = \epsilon_{sab} \epsilon_{cbn} + \epsilon_{csb} \epsilon_{abn} \tag{A.3}
\end{equation}

(no summation) we get:

\begin{equation}
-2i \frac{\lambda_x \lambda_y \lambda_z \lambda_s \sqrt{\lambda_s \lambda_n}}{\lambda_c} (\epsilon_{abc} \epsilon_{cbn} + \epsilon_{csb} \epsilon_{abn}) L^a \sigma^n_i \sigma^{s}_{i+1} R^c. \tag{A.4}
\end{equation}

The first part corresponds to a disordered term: $L$ is first multiplied by the spin at site $i+1$ and the result is multiplied by the one at site $i$. The second part is ordered and properly nested. The second commutator in (A.1) is evaluated in the same way, with the result

\begin{equation}
-2i \frac{\lambda_x \lambda_y \lambda_z \lambda_s \sqrt{\lambda_s \lambda_n}}{\lambda_a} (\epsilon_{abc} \epsilon_{cbn} + \epsilon_{csb} \epsilon_{abn}) L^a \sigma^n_i \sigma^{s}_{i+1} R^c. \tag{A.5}
\end{equation}

Here the second term is not ordered. The addition of the first term of (A.4) to the second term of (A.5) gives the total contribution of the unwanted terms, corresponding to disordered clusters:

\begin{equation}
2i \lambda_x \lambda_y \lambda_z \sqrt{\lambda_s \lambda_n} \epsilon_{asb} \epsilon_{ncb} L^a \sigma^n_i \sigma^{s}_{i+1} R^c \left( \frac{\lambda_s}{\lambda_c} - \frac{\lambda_n}{\lambda_a} \right). \tag{A.6}
\end{equation}

Due to the presence of the antisymmetric tensors, we may write

\begin{equation}
\frac{\lambda_s}{\lambda_c} = \frac{\lambda_x \lambda_y \lambda_z}{\lambda_a \lambda_b} \left( \frac{\lambda_n}{\lambda_x \lambda_y \lambda_z} \right) = \frac{\lambda_n}{\lambda_a} \tag{A.7}
\end{equation}

(no summation) which shows that the unwanted terms cancel. The sum of the remaining terms of (A.4) and (A.5) is

\begin{equation}
2i \frac{\lambda_s \lambda_n \lambda_x \lambda_y \lambda_z}{\lambda_b} \epsilon_{asb} \epsilon_{ncb} L^a \sigma^n_i \sigma^{s}_{i+1} R^c \left( \frac{1}{\lambda_a^2} - \frac{1}{\lambda_c^2} \right). \tag{A.8}
\end{equation}

\textsuperscript{8} This was stated in [2] but without a detailed proof.

\textsuperscript{9} As in section 5, summation over all indices that appear twice or more is implied, unless indicated otherwise.
This vanishes only if all $\lambda_i^2$ are identical, that is for the XXX or the XXZ$^{-1}$ model. In consequence, for these models (A.1) is zero, and thus the condition (ii) of the lemma in section 3 is satisfied. In contrast, for the general XYZ model this condition does not hold.

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