1. Introduction

Oscillating integrals are integrals of the form

$$\int e^{itf(x)} \varphi(x)\,dx.$$  

They frequently occur in applied mathematics and mathematical physics. In this article, we investigate their asymptotic behaviour when the parameter $t$ tends to infinity, in terms of the geometry of the Newton polyhedron of the phase $f$. It is well-known that the greatest contributions to this asymptotic behaviour arise from the critical points of $f$: if $f$ is regular on the support of $\varphi$, subsequent oscillations will more or less cancel each other as $t$ grows bigger and the integrand starts oscillating faster, so that the integral tends to zero more rapidly than any power of $t$. This phenomenon is called the principle of the stationary phase. When $f$ has only non-degenerate critical points, we can apply Morse’s lemma to give a description of the asymptotic expansion of the integral; see [1]. In the present paper, we will consider a much larger class of phase functions $f$: real analytic functions which are non-degenerate with respect to their Newton polyhedron.

The key result is Theorem 1 in section 5, which yields, together with formula (1) in section 3, an expression of $\mu(\varphi)$ in terms of principal value integrals, where $\mu(\varphi)$ is the coefficient of the expected leading term in the asymptotic expansion of our oscillating integral. A similar - but more complicated - expression was given in [3] in terms of a different kind of principal value integrals. A direct consequence of Theorem 1 is Corollary 1, which states that the coefficients $\mu(\varphi)$ for $f$ and $f_{\tau_0}$ differ only by an easy nonzero factor. The much simpler polynomial $f_{\tau_0}$, as defined in the next section, is obtained by omitting all monomials of $f$ whose exponents do not lie on the face $\tau_0$. Here $\tau_0$ is the smallest face of the Newton polyhedron of $f$ intersecting the diagonal.

As a first application, we give in Section 6 a more transparent proof of the fact that $\mu(\varphi) = 0$ whenever $\tau_0$ is unstable. This result, conjectured by Denef and Sargos, was first proven in [3]. As a second application, we give a very explicit formula for $\mu(\varphi)$ in terms of gamma functions, assuming that $\tau_0$ is a simplex of codimension 1, whose vertices are the only integral points on $\tau_0$ corresponding to monomials of $f$. This is done in Section 7.

In Section 8 we develop an analogous residue formula for the complex local zeta function. This allows us to give, in section 9 a partial proof of the stability conjecture of Denef and Sargos, using a theorem of Loeser on the relation between the spectrum of a complex polynomial, and the poles of the associated complex local zeta function.

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We conclude this introduction with an overview of the structure of the paper. Section 2 contains some preliminaries concerning Newton polyhedra, while section 3 deals with some known results about the asymptotic behaviour of oscillating integrals. In section 4, we summarize the analytic construction of toric varieties, which will be used to desingularize \( f \) at the origin. Section 5 establishes the real residue formula. In section 6, we prove that \( \mu(x) \) vanishes if \( \tau_0 \) is unstable. The explicit formula for \( \mu(x) \) is deduced in section 7. We use the standard notation \( \mathbb{R}_+ \) for the set of positive real numbers, and \( \mathbb{R}_0, \mathbb{C}_0, \text{resp. } \mathbb{N}_0, \) for \( \mathbb{R} \setminus \{0\}, \mathbb{C} \setminus \{0\}, \text{resp. } \mathbb{N} \setminus \{0\} \). To avoid confusion: \( \mathbb{R}_+^n \) means \((\mathbb{R}_0)^n\), and \( \mathbb{C}_0^n \) means \((\mathbb{C}_0)^n\).

2. Newton polyhedra: some terminology

Let \( f \) be a nonconstant real analytical function in \( n \) variables \( x = (x_1, \ldots, x_n) \) on an neighbourhood of \( 0 \in \mathbb{R}^n \), which has a critical point at the origin and satisfies \( f(0) = 0 \). Let \( f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \) be the Taylor series of \( f \) about the origin. We define the support \( \text{supp}(f) \) of \( f \) to be the set of exponents \( \alpha \in \mathbb{N}^n \) for which \( a_\alpha \neq 0 \). The Newton polyhedron \( \Gamma(f) \) of \( f \) is the convex hull of \( \text{supp}(f) + \mathbb{R}_+^n \). For each subset \( \gamma \) of \( \mathbb{N}^n_+ \) we note by \( f_\gamma \) the function \( f_\gamma(x) = \sum_{\alpha \in \gamma \cap \mathbb{N}^n} a_\alpha x^\alpha \). When \( \Gamma_c \) is the union of the compact faces of \( \Gamma(f) \), we call \( f_{\Gamma_c} \) the principal part of \( f \). Let \( (t_0, \ldots, t_0) \) be the intersection point of the union of faces of \( \Gamma(f) \) with the diagonal \( x_1 = \cdots = x_n \), and \( \tau_0 \) the smallest face of \( \Gamma(f) \) containing this point. Then \( \tau_0 \) denotes the value \(-1/t_0\) and \( \rho \) denotes the codimension of \( \tau_0 \) in \( \mathbb{R}^n \). We will sometimes write \( \tau_0(f) \) and \( s_0(f) \) instead of \( \tau_0 \) and \( s_0 \), to make \( f \) explicit.

We say that \( f \) is non degenerate over \( \mathbb{R} \) with respect to \( \Gamma(f) \) if the following holds: for each compact face \( \gamma \) of \( \Gamma(f) \) the polynomial \( f_\gamma \) has no critical points in \( \mathbb{R}_0^n \). "Almost all" phase functions \( f \) are non degenerate with respect to \( \Gamma(f) \). To specify this assertion a little further: given a fixed Newton polyhedron \( \Gamma \), the principal parts of functions \( f \) that are non degenerate with respect to \( \Gamma = \Gamma(f) \) form a semi-algebraic subset of the space of principal parts with Newton polyhedron \( \Gamma \), and its complement is everywhere dense. From now on, we will always assume that \( f \) is non degenerate with respect to its Newton polyhedron \( \Gamma(f) \).

To this Newton polyhedron we can associate a fan of rational cones subdividing \( \mathbb{R}_+^n \). The trace function \( l_{\Gamma} \) maps a vector \( a \) in \( \mathbb{R}_+^n \) to the value \( \min_{k \in \Gamma} \langle a, k \rangle \), where \( \langle, \rangle \) denotes the inner product. We define the trace of \( a \) to be the compact face

\[
\tau_a = \{k \in \Gamma_c \mid \langle a, k \rangle = l_{\Gamma}(a)\}.
\]

In this way each \( k \)-dimensional compact face \( \tau \) corresponds to a \((n-k)\)-dimensional cone \( \t \) consisting of all the vectors with trace \( \tau \). Geometrically, this is the cone spanned by the normal vectors of the facets of \( \Gamma(f) \) containing \( \tau \). It is clear that these cones form a fan \( \Gamma \). We say that a fan is subordinate to \( \Gamma(f) \) if each cone of the fan is contained in a cone of \( \Gamma \).

An important role in this article is fulfilled by the notion of instability. A face \( \tau \) of \( \Gamma(f) \) is unstable over \( \mathbb{R} \) with respect to the variable \( x_j \), if \( \tau \) is contained in the region \( \{y \in \mathbb{R}_0^n \mid 0 \leq y_j \leq 1\} \), but not entirely in the hyperplane defined by \( x_j = 0 \), and if furthermore, each compact face \( \gamma \) of \( \Gamma(f) \) that is contained in the hyperplane \( x_j = 1 \), is subject to the condition that \( f_\gamma \) has no zero in \( \mathbb{R}_0^n \).
These notions can also be defined in the complex case. Let \( g \) be a non-constant complex analytic function on a neighbourhood of the origin in \( \mathbb{C}^n \), which has a critical point at the origin and satisfies \( g(0) = 0 \). The Newton polyhedron \( \Gamma(g) \) of \( g \) at 0 is defined as in the real case. We say that \( g \) is non degenerate over \( \mathbb{C} \) with respect to \( \Gamma(g) \) if the following holds: for each compact face \( \gamma \) of \( \Gamma(g) \) the polynomial \( g_\gamma \) has no critical points in \( \mathbb{C}^n_0 \), where \( g_\gamma \) is defined in the same way as before. Again, “almost all” complex phase functions \( g \) are non degenerate with respect to \( \Gamma(g) \).

We conclude this section with some additional notation: given a vector \( \xi_i \) we mean by \( \nu_i \) the sum of its coordinates, and \( N_i \) is short for \( \nu_i \Gamma(\xi_i) \). If the vector these notations are relating to is not explicitly indicated it should be clear from the context which vector is meant.

### 3. Oscillating Integrals

Let \( \varphi \) be a \( C^\infty \)-function on \( \mathbb{R}^n \) with support in a sufficiently small neighbourhood of 0.

It is well-known that the oscillating integral

\[
I(t) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx
\]

has for \( t \to \infty \) an asymptotic expansion

\[
\sum_p \sum_{i=0}^{n-1} a_{p,i}(\varphi) t^p (\ln t)^i \quad (*)
\]

where \( p \) runs through a finite number of arithmetic progressions, not depending on the amplitude \( \varphi \), that consist of negative rational numbers. Since our objective is to study the asymptotic behaviour of \( I \), our primary interest goes out to the largest \( p \) occurring in this expansion. Let \( S \) be the set of tuples \((p, i)\) such that for each neighbourhood of 0 in \( \mathbb{R}^n \) there exists a \( C^\infty \)-function \( \varphi \) with support in this neighbourhood for which \( a_{p,i}(\varphi) \neq 0 \). We define the oscillating index \( \beta \) of \( f \) to be the maximum of values \( p \) for which we can find an \( i \) so that \((p, i)\) belongs to \( S \); the maximum of these \( i \) is called the multiplicity \( \kappa \) of \( \beta \). The index \( \beta \) contains information about the nature of the singularity 0 of \( f \).

In this paper, we will derive information about this asymptotic expansion from the geometry of the Newton polyhedron \( \Gamma(f) \) of the phase function \( f \), always assuming that \( f \) is non-degenerate with respect to \( \Gamma(f) \). Using the notation introduced in the previous section, the expansion (*) can be written as

\[
\mu(\varphi)t^{s_0}(\log t)^{\rho-1} + O(t^{s_0}(\log t)^{\rho-2}).
\]

This is the main result in Varchenko’s paper [8]. Moreover, it is known (as one can verify in [1]) that \( \beta = s_0 \) and \( \kappa = \rho - 1 \) if at least one of the following additional conditions is satisfied:

\[
\begin{cases}
  a) & s_0 > -1 \\
  b) & f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \\
  c) & \tau_0 \text{ is compact, } s_0 \text{ is not an odd integer, and } f_{\tau_0} \text{ does not vanish in } \mathbb{R}^n_0.
\end{cases}
\]

On \( D = \{ s \in \mathbb{C} \mid \Re(s) > 0 \} \) we can define the functions

\[
Z_\pm(s) = \int_{\mathbb{R}^n} f_\pm(x)^s \varphi(x) dx.
\]
Here \( f_+ = \max(f, 0) \) and \( f_- = \max(-f, 0) \). It is well known that these functions allow a meromorphic continuation to the whole of \( \mathbb{C} \), which we denote again by \( Z_\pm(s) \). The study of the asymptotic behaviour of \( I(t) \) can be reduced to an investigation of the poles of \( Z_\pm(s) \): the terms in the development (*) are related to the singular part of the Laurent expansion about the poles of \( Z_\pm(s) \) (cf. [1]).

For a negative integer pole of \( Z_\pm(s) \), it occurs that terms in the singular part of the Laurent expansion at this pole do not correspond to anything in the asymptotic expansion of \( I(t) \). This is why while studying the distribution \( f_\pm^s \) we will always assume that \( s_0 \notin \mathbb{Z} \); the other case is dealt with in the last paragraph of this section.

When \( s_0 \notin \mathbb{Z} \), \( Z_\pm(s) \) can have a pole of order at most \( \rho \) at \( s = s_0 \). We define \( \mu_+(\varphi) \) and \( \mu_-(\varphi) \) by requiring that

\[
Z_\pm(s) = \frac{\mu_+(\varphi)}{(s - s_0)^\rho} + O\left(\frac{1}{(s - s_0)^{\rho - 1}}\right)
\]

for \( s \to s_0 \). Using material in [1], one obtains the following expression for \( \mu(\varphi) \) in terms of \( \mu_+(\varphi) \) and \( \mu_-(\varphi) \):

\[
\mu(\varphi) = \frac{1}{(\rho - 1)!} \Gamma(-s_0)[\mu_+(\varphi)e^{-\frac{i\pi s_0}{2}} + \mu_-(\varphi)e^{\frac{i\pi s_0}{2}}] .
\]

Note that, since \( s_0 \notin \mathbb{Z} \), this equality implies that \( \mu(\varphi) = 0 \) iff \( \mu_+(\varphi) = \mu_-(\varphi) = 0 \).

This formula enables us to reduce the study of \( \mu(\varphi) \) to the study of the candidate pole \( s_0 \) of \( Z_\pm(s) \). By the relationship

\[
Z_\pm(s) = \sum_{\theta \in \mathbb{Z}^n} \int_{\mathbb{R}^n_+} f_\pm(\theta_1 x_1, \ldots, \theta_n x_n)^s \varphi(\theta x) \, dx ,
\]

it even suffices to investigate the properties of

\[
\int_{\mathbb{R}^n_+} f_\pm(x)^s \varphi(x) \, dx .
\]

The question remains what happens when \( s_0 \in \mathbb{Z} \). However, we can reduce this problem to the case \( s_0 \notin \mathbb{Z} \) by introducing an additional variable \( y \): if we define a function \( f^* \) on \( \mathbb{R}^n \) by \( f^*(x_1, \ldots, x_n, y) = f(x_1, \ldots, x_n) + y^2 \) and put \( \varphi^*(x_1, \ldots, x_n, y) = \varphi(x_1, \ldots, x_n) \psi(y) \), where \( \psi \) is a test function on \( \mathbb{R} \) satisfying \( \psi(0) = 1 \), and we define \( \tau_0^* \), \( \rho^* \), \( s^*_0 \) and \( \mu^*(\varphi^*) \) in the obvious way, we obtain the following properties:

- \( f^* \) is non-degenerate over \( \mathbb{R} \) with respect to its Newton polyhedron
- \( s^*_0 = s_0 - \frac{1}{2} \)
- \( \rho^* = \rho \)
- \( \tau_0^* \) is the convex hull of \( \tau_0 \) and the point \( (0, \ldots, 0, 2) \) in \( \mathbb{R}^{n+1} \)
- \( \tau_0^* \) is stable iff \( \tau \) is stable.

Moreover, \( I^*(t) = I(t) \int_{\mathbb{R}} e^{it y^2} \psi(y) \, dy \). Since the asymptotic expansion of this last factor equals

\[
e^{\frac{\pi i}{4}\sqrt{\frac{1}{t}} + O(t^{-\frac{3}{2}})} ,
\]

we conclude that

\[
\mu(\varphi) = e^{-\frac{\pi i}{4}} \sqrt{\frac{\pi}{t}} \mu^*(\varphi^*) .
\]
4. Toric varieties

Toric varieties form an important topic in algebraic geometry because the geometric properties of this large class of varieties are related to the combinatorial properties of the fans used to construct them \( \mathbb{Q} \). Here we will introduce the analytical counterpart of this construction, using an atlas with monomial transition functions, following the approach in \( \mathbb{I} \).

Let \( L \subset \mathbb{R}^n \) be a lattice, e.g. \( L = \mathbb{Z}^n \). A cone in \( \mathbb{R}^n \) is called rational if it can be generated by vectors in \( L \). We say the cone is simplicial if it can be generated by a free set of vectors in \( L \), and simple if this set can be extended to a basis of \( L \). Starting from a fan \( F \) of \( L \)-simple cones in \( \mathbb{R}^n \), we will construct a real analytic manifold \( X_{L,F} \); this is the toric manifold associated to \( L \) and \( F \).

We number once and for all the 1-dimensional cones in our fan \( F \); this will allow us to speak of an ordered basis of generators of a \( n \)-dimensional cone. The analytical structure of \( X_{L,F} \) is defined by giving an atlas for \( X_{L,F} \), or more specifically by giving a number of copies of \( \mathbb{R}^n \) and the transition functions between the parts of these copies that will overlap once we identify these copies with open parts of \( X_{L,F} \).

The charts \( U_\tau \) in our atlas correspond to the \( n \)-dimensional cones \( \tau \) in our fan \( F \), and an ordered basis of generators of this cone provides standard coordinates on the corresponding chart. Now we explain how you can travel from one chart to another. Given two charts \( U_{\tau_1} \) and \( U_{\tau_2} \), we consider the matrix \( A = [a_{i,j}] \) the \( j \)-th column of which contains the coordinates of the \( j \)-th base vector \( \xi_j^{\tau_1} \) of the first cone expressed in terms of the ordered \( L \)-basis generating the second cone. The matrix \( A \) is an element of \( GL_n(\mathbb{Z}) \). The associated monomial mapping \( h_A \) is defined by

\[
y_j \circ h_A : D \to \mathbb{R} : x \mapsto \prod_{i=1}^n x_i^{a_{j,i}},
\]

where the domain \( D \) consists of \( \mathbb{R}^n \) minus the coordinate hyperplanes on which \( h_A \) is ill-defined: these are the hyperplanes corresponding to the variables \( x_k \) for which not all entries \( a_{i,k} \) are positive.

It is clear that \( h_{A,B} = h_A \circ h_B \) in points where both sides are defined. We show that \( h_{A^{-1}} \) is defined on the image of \( h_A \). Suppose that \( x \) belongs to the domain of \( h_A \), \( y = h_A(x) \) and \( y_k = 0 \). We have to prove that all entries in the \( k \)-th column of \( A^{-1} \) are positive, or equivalently, that \( \xi_k^{\tau_1} \) belongs to \( \tau_1 \). The fact that \( y_k = 0 \) implies the existence of an index \( i \) such that \( a_{k,i} > 0 \) and \( a_{j,i} \geq 0 \) for all \( j \); this means that \( \xi_i^{\tau_1} \) belongs to \( \tau_2 \). Since \( F \) is a fan \( \xi_i^{\tau_1} \) has to be contained in a common face of \( \tau_1 \) and \( \tau_2 \), so \( a_{j,i} = \delta_{j,k} \). Thus \( \xi_1^{\tau_1} \) and \( \xi_k^{\tau_1} \) are one and the same.

The preceding shows that we have constructed a well-defined atlas for \( X_{L,F} \). Furthermore, the transition functions map points with positive coordinates in one chart to points with positive coordinates in another, so the positive part \( X_{L,F}(\mathbb{R}_+) \) of \( X_{L,F} \) is well-defined. When we work with two fans \( F \) and \( F' \) we say that \( F' \) is finer than \( F \) (notation: \( F' < F \)) if each cone of \( F' \) is contained in a cone of \( F \). In this case there exists a natural mapping from \( X_{L,F'} \) to \( X_{L,F} \); on a standard chart of \( X_{L,F'} \) associated to a \( n \)-dimensional cone \( \tau' \) of \( F' \) it is defined as the monomial mapping associated to the couple of ordered \( L \)-bases formed by generators of \( \tau' \) and generators of the unique cone \( \tau \) of \( F \) containing \( \tau' \). Note that the inclusion of \( \tau' \) in \( \tau \) implies that the domain of this mapping coincides with the whole chart. From the nature of this definition it is clear that all this mappings are compatible as \( \tau' \) ranges over the \( n \)-dimensional cones of \( F' \), so they glue together to a well-defined
analytical mapping $\pi : X_{L,F'} \to X_{L,F}$. In the special case where $L = \mathbb{Z}^n$ and $F$ is the positive orthant $\mathbb{R}_n^+$, we get a mapping $\pi : X_{L,F'} \to \mathbb{R}^n$.

This construction can be generalized by considering two lattices $L, L'$, an $L$-simple fan $F$ and an $L'$-simple fan $F'$, where $F' \subset F$. Let $\tau'$ be a $n$-dimensional cone of $F'$ and $\tau$ a cone of $F$ containing $\tau'$. Expressing the generators of $\tau'$ in the $L$-basis consisting of the ordered set of generators for $\tau$ yields a monomial map with nonnegative real exponents, and by gluing we obtain a map $\pi : X_{L,F'}(\mathbb{R}_+) \to X_{L,F}(\mathbb{R}_+)$.

The geometric properties of toric varieties are reflected in the characteristics of the fans used to define them. The mapping $\pi$ will be proper if and only if the union of the cones in $F$ coincides with the union of those in $F'$. When $L = \mathbb{Z}^n$ and $F$ is a fan subdividing $\mathbb{R}^n_+$ and subordinate to the Newton polyhedron of $f$, the associated mapping $\pi : X_{\mathbb{Z}^n,F} \to \mathbb{R}^n$ has a very nice property: it desingularizes $f$ at the origin of $\mathbb{R}^n$. As always, we assume $f$ to be non-degenerate with respect to its Newton polyhedron.

It is clear that the construction of $X_{L,F}$ can be copied verbatim to the complex case, simply by extending the transition functions in our atlas to $\mathbb{C}^n$, to obtain a complex analytic variety.

5. A residue formula

Before proceeding, we have to state some conventions. For every facet $\phi$ of $\Gamma(f)$, let $\xi_{\phi}$ be the primitive vector (i.e. with components relatively prime in $\mathbb{N}$) orthogonal to $\phi$. Let $\phi_1, \ldots, \phi_r$ be the facets that contain $\tau_0$ and let $\tilde{\tau}_b$ be the $r$-dimensional subspace of $\mathbb{R}^n$ spanned by these $\xi_{\phi_i}$. Permutating the coordinates of $\mathbb{R}^n$ if necessary, we may assume that $\mathbb{R}^n = \tilde{\tau}_0 \oplus \sum_{j=r+1}^n \mathbb{R}e_j$ and $\tau_0$ is parallel with $e_{m+1}, \ldots, e_n (m \geq r)$ and with none of the other $e_i$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. Putting $N_i = l_{\Gamma}(\xi_{\phi_i})$ we define $C$ to be the convex hull of $\{0, \tilde{\xi}_{\phi_1}, \ldots, \tilde{\xi}_{\phi_r}, e_{r+1}, \ldots, e_n\}$. If it is not clear from the context which polynomial $C$ is associated to, we write it explicitly as $C(f)$.

For $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ we define a function $J^{(\lambda)}_\pm$ on $D = \{s \in \mathbb{C} | \Re(s) > 0\}$ by

$$J^{(\lambda)}_\pm(s) = \int_{\mathbb{R}^n_+} f_\pm(x)^s x^{-\lambda} \varphi(x) \, dx,$$

where $x^{-\lambda} = x_1^{\lambda-1} \ldots x_n^{\lambda-1}$. It is known that $J_\pm$, considered as a function in $\lambda$ and $s$, has a meromorphic continuation to the whole of $\mathbb{C}^2$. This essentially comes down to the observation that the assertion holds when $f$ is a monomial and a reduction to this particular case via a resolution of singularities. If we fix $\lambda$ in $\mathbb{R}_+ \setminus 0$ such that $\lambda s_0 \notin \mathbb{Z}$, then the non-integral poles of $J^{(\lambda)}_\pm$ are not greater than $\lambda s_0$, and the polar multiplicity at $\lambda s_0$ is at most $r$. For let $F$ be a fan, subdividing the positive orthant, and subordinate to the Newton polyhedron of $f$, and let $\pi : X_{\mathbb{Z}^n,F} \to \mathbb{R}^n$ be the corresponding proper toric morphism. We know that $\pi$ desingularizes $f$ at the origin. Using a partition of unity, it suffices to investigate the poles of the meromorphic continuation of the integral

$$J^{(\lambda)}_\pm(s) = \int_{\mathbb{R}^n_+} w^s \prod_j y_j^{M_j s + \kappa_j \lambda - 1} \varphi(y) \, dy,$$
The principal value integral
\[ \text{PV} \int \varphi(0) dy, \]
where
\[ \varphi \] is sufficiently small, then
\[ \bar{\mu}_\pm(\varphi) = n! \text{Vol}(C) \varphi(0) PV \int_{\mathbb{R}_+^n} f_{\tau_0}(1, \ldots, 1, y_{p+1}, \ldots, y_n)^{s_0} dy, \]
where
\[ \bar{\mu}_\pm = \lim_{s \to s_0} (s - s_0)^\rho \int_{\mathbb{R}_+^n} f_\pm^\rho \varphi dx. \]
The principal value integral \( \text{PV} \int \) is defined as the value of the analytic continuation at \( \lambda = 1 \) of the function
\[ K_\pm(\lambda) = \int_{\mathbb{R}_+^n} f_{\tau_0}(1, \ldots, 1, y_{p+1}, \ldots, y_n)^{s_0\lambda} y^{\lambda-1} dy, \]
where \( K_\pm(\lambda) \) is defined for \( \lambda \in \mathbb{R}_+ \setminus 0 \), and \( s_0\lambda > -1 \). Here \( y = \prod_{i=p+1}^n y_i \) and \( dy = dy_{p+1} \wedge \ldots \wedge dy_n \).

Some explanation:
we will show that \( n! \text{Vol}(C) \varphi(0) K_\pm(\lambda) \) equals \( \lim_{s \to s_0} (s - s_0)^\rho J_\pm^{(\lambda)}(s) \) on its domain of definition mentioned above, and that, in particular, the integral \( K_\pm \) converges on this domain. This shows that \( K_\pm \) has indeed an analytic continuation at \( \lambda = 1 \) - which is necessarily unique - since we will show, using resolution of singularities, that, whenever \( \Re(\lambda) > 0 \) and \( s_0 \lambda \notin \mathbb{Z} \), \( (s - s_0)^\rho J_\pm^{(\lambda)}(s) \) is analytic on a neighbourhood of \( (s_0\lambda, \lambda) \). Details can be found in the proof.

In particular, we see that a possible dropping of the polar multiplicity of \( Z_\pm(s) \) in \( s_0 \) only depends on \( f_{\tau_0} \).

**Proof.** We may assume that \( \tau_0 \) is simple, for the general case is obtained by subdividing \( \tau_0 \) into simple cones. Let \( L_1 = \mathbb{Z}^n \) and let \( F_1 \) be a \( L_1 \)-simple fan, subordinate to the Newton polyhedron \( \Gamma(f) \) of \( f \) at 0, and containing the cone \( \tau_0 \). The natural map \( \pi_1 : X_{L_1,F_1} \to \mathbb{R}^n \) is an embedded resolution of singularities of \( f \) in a neighbourhood of the origin in \( \mathbb{R}^n \). Next, we define the closed submanifold \( Y \) of \( X_{L_1,F_1} \), by requiring for every \( n \)-dimensional cone \( \Delta \) in \( F_1 \) that
\[ U_{L_1,F_1,\tau_0} \cap Y = \emptyset \text{ if } \tau_0 \not\subseteq \Delta, \]
\[ U_{L_1,F_1,\tau_0} \cap Y = \text{ locus } \{ y_1 = y_2 = \ldots = y_p = 0 \} \text{ if } \tau_0 \subseteq \Delta, \]
where \( (y_1, \ldots, y_n) \) are the standard coordinates in the chart \( U_{L_1,F_1,\tau_0} \), associated to an ordered basis \( \{\xi_1, \ldots, \xi_n\} \) of \( \Delta \) with \( \xi_1, \ldots, \xi_p \in \tau_0 \). One can easily verify that \( Y = X_{L_2,F_2} \) where \( F_2 \) is obtained by projecting the cones in \( F_1 \) containing \( \tau_0 \) onto \( \mathbb{R}e_{p+1} + \ldots + \mathbb{R}e_n = \mathbb{R}^{n-p} \), parallel to \( \tau_0 \), and the lattice \( L_2 \) is the image of \( L_1 \) under the same projection. Note that the cones of \( F_2 \) are \( L_2 \)-simple.

Put \( L_3 = \mathbb{Z}e_{p+1} + \ldots + \mathbb{Z}e_n \subset \mathbb{R}^{n-p} \) and let \( F_3 \) be the fan in \( \mathbb{R}^{n-p} \) induced by all orthants. Then \( X_{L_3,F_3} = (\mathbb{P}^1_{\mathbb{R}})^{n-p} \), where \( \mathbb{P}^1_{\mathbb{R}} \) denotes the real projective line.
By refining the fan $F_1$ we may suppose that $F_2 < F_3$. As a consequence of this there exists a natural map

$$\pi_2 : Y(\mathbb{R}_+) = X_{L_2,F_2}(\mathbb{R}_+) \rightarrow X_{L_3,F_3}(\mathbb{R}_+) .$$

Now the idea is to pull back the integral defining $K_\pm(\lambda)$ along the mapping $\pi_2$, in order to compare $K_\pm$ with $\lim_{s \to \lambda \delta_0}(s - \lambda s_0)^\rho J_\pm^\lambda(s).$ Let $\gamma$ on $(\mathbb{R}^n_{\mathbb{R}})^{n-r}$ be given by

$$\gamma = f_{\gamma}(1, \ldots, 1, z_{\rho+1}, \ldots, z_n)^{n_\lambda} \prod_{i=\rho+1}^n z_i^{\lambda-1} dz_{\rho+1} \wedge \ldots \wedge dz_n ,$$

where $z_{\rho+1}, \ldots, z_n$ are standard affine coordinates on $\mathbb{R}^{n-r}$. With this notation,

$$K_\pm(\lambda) = \int_{\mathbb{R}^{n-r}^+} \gamma = \int_{Y(\mathbb{R}^+)} \pi_2^*(\gamma).$$

Let $\Delta$ be a $n$-dimensional cone of $F_1$, generated by $\xi_1, \ldots, \xi_n$, with $\xi_1, \ldots, \xi_\rho \in \pi_0$. On $Y(\mathbb{R}_+) \cap U_\Delta$, where $U_\Delta$ is the coordinate neighbourhood in $X_{L_1,F_1}$ corresponding to $\Delta$, we have

$$(n! Vol(C) \phi(0) \prod_{i=1}^\rho N_i \pi_2^*(\gamma)) = \frac{\prod_{i=1}^\rho y_i \pi_1^* (\phi f_\gamma^{N_{\lambda}} x_\rho^{\lambda-1} dx) dy_1 \wedge \ldots \wedge dy_\rho}{dy_1 \wedge \ldots \wedge dy_\rho} \bigg|_{y_1 = \ldots = y_\rho = 0}$$

where $(y_1, \ldots, y_n)$ are the standard coordinates associated to $(\xi_1, \ldots, \xi_n)$. This equality is straightforward but crucial for what follows.

Now we can exploit the special properties of the map $\pi_1$. By Theorem page 202, there exists a neighbourhood of $\pi_1^{-1}(0)$ in $X_{L_1,F_1}$, so that at each point $P$ in this neighbourhood and belonging to $Y(\mathbb{R}_+) \cap U_\Delta$, we can find a system of local coordinates $y'_1, \ldots, y'_n$ satisfying

- $y_i = y'_i$ if $y_i(P) = 0$; in particular this holds for $i \in \{\rho, 1, \ldots, \rho\}$,
- $\pi_1^*(f_\gamma x_\rho^{\lambda-1} dx) = v_1^i v_2^i v_3 \prod_{i=1}^\rho y_i^{N_{i}'} dx'$, where $v_1, v_2, v_3$ are positive nonvanishing analytic functions, $(N_i', \nu_i') = (N_i, \nu_i)$ whenever $y_i(P) = 0$, and $(N_i', \nu_i')$ equals either $(0, 1)$ or $(1, 1)$ if $y_i(P) \neq 0$,
- the points where $y_i \geq 0$ for each $i$ and $f_\pm > 0$ are exactly the points where $y_j' > 0$ for each $j$ satisfying $N_j' \neq 0$, and $y_j' \geq 0$ for each $j$ satisfying $y_j(P) = 0$.

We briefly recall the construction of this new system of local coordinates $(y'_1, \ldots, y'_n)$. Let us suppose that $y_1(P) = \ldots = y_\rho(P) = 0$, and that $y_j(P) \neq 0$ when $j > \rho$. Let $\gamma$ be the common trace of $\xi_1, \ldots, \xi_\rho$. By the definition of $\pi_1$, we can write $f \circ \pi_1$ as

$$y_1^{N_1} \cdots y_n^{N_s} (f_0(y_{s+1}, \ldots, y_n) + O(y_1, \ldots, y_s)) .$$

Now there are two possibilities. If $f_0(y_{s+1}, \ldots, y_n)$ is nonzero at $P$, the factor between brackets is a unit in the local ring at $P$. If $f_0(y_{s+1}, \ldots, y_n)$ vanishes, we can use the factor between brackets as a new coordinate $y_{s+1}'$, since

$$f_\gamma \circ \pi_1 = y_1^{N_1} \cdots y_{s+1}^{N_s} f_0 ,$$

$\pi_1$ induces a local diffeomorphism $\mathbb{R}_0^n \rightarrow \mathbb{R}_0^n$, and $f$ is non-degenerate with respect to its Newton polyhedron. If $P$ were a critical point of $f_0$, $(1, \ldots, 1, y_{s+1}(P), \ldots, y_n(P))$ would be a critical point of $f_\gamma$.

It will be important for our purposes, in particular for the remark following the corollary, that the case $(N_i', \nu_i') = (1, 1)$ only occurs when $f_0$ is zero at $P$. 
The choice of local coordinates implies that on $Y(\mathbb{R}^+) \cap U_{\Delta}$, the function

$$
(n! \text{Vol}(C) \varphi(0) \prod_{i=1}^{\rho} N_i) \pi_2^*(\gamma)
$$
equals

$$
\prod_{i=\rho+1}^{n} \int_{\mathbb{R}_+^n} y_i^{iN_i' \lambda + \nu_i'^{-1}} (v_1 v_2 v_3 (\varphi \circ \pi_1)) y_i \ldots y_n = 0 \ dy_{\rho+1}' \land \ldots \land dy_n'.
$$

Now observe that in the expression

$$
\lim_{s \to \lambda s_0} (s - \lambda s_0)^{\rho} \int_{\mathbb{R}^n} \prod_{i=1}^{\rho} n_i^{iN_i'(s + \nu_i') - 1} \int_{\mathbb{R}^{\rho-n}} v_1 v_2 v_3 \theta \prod_{i=\rho+1}^{n} y_i^{iN_i' \lambda + \nu_i'^{-1}} dy'
$$

where $\theta$ is a Schwarz function on $\mathbb{R}^n$, i.e. a $C^\infty$-function with compact support, the inner integral converges for $\lambda > 0$ sufficiently small and $s$ near $s_0 \lambda$, since the exponents $N_i' s_0 \lambda + \nu_i' - 1$, for $i = \rho + 1, \ldots, n$, are either 0, $\lambda s_0$, or $(N_i s_0 + \nu_i) \lambda - 1 > -1$. Hence we can apply the formula

$$
\lim_{t \to t_0} (t - t_0)^{\rho} \int_{[0, a]^n} \prod_{i=1}^{\rho} z_i^{N_i (t - t_0) - 1} \psi(t, z) \ dz = \frac{\psi(t_0, 0)}{\prod_{i=1}^{\rho} N_i},
$$

which holds for every continuous mapping $\psi$ and any $a \in \mathbb{R}^{n+}$. Since $N_i s_0 + \nu_i = 0$ for $i = 1, \ldots, \rho$, this formula yields that

$$
\lim_{s \to \lambda s_0} (s - \lambda s_0)^{\rho} \int_{X_{L_1, F_1} (\mathbb{R}^+)} \theta \pi_1^* (f_1 x^\lambda) dx
$$
is equal to

$$
n! \text{Vol}(C) \int_{\mathbb{R}^{\rho-n}} \theta(0, \ldots, 0, y_{\rho+1}', \ldots, y_n') \pi_2^*(\gamma),
$$

provided that the support of $\theta$ is contained in a sufficiently small neighbourhood of a point of $Y(\mathbb{R}^+) \cap U_{\Delta}$.

To conclude the proof of the theorem, one only has to observe that the expression

$$
\varphi(0) \prod_{i=1}^{\rho} N_i \varphi
$$
vanishes when $\theta$ is a Schwarz function with compact support disjoint with $Y$ (simply apply formula (3) again), and invoke a suitable partition of unity for $X_{L_1, F_1}$. A similar construction shows that $(s - s_0 \lambda)^{\rho} J_{-1}^\lambda (s)$ is analytic in a neighbourhood of $\{(s_0 \lambda, \lambda) \in \mathbb{C}^2 \mid \Re(\lambda) > 0, \lambda s_0 \notin \mathbb{Z}\}$: simply apply integration by parts to the integral in (4) with respect to $y_{\rho+1}'$, $\ldots$, $y_n'$, and with respect to the $y_j'$, $j > \rho$, with $N_j' \neq 0$, in order to increase their exponent until its real part becomes greater than $\lambda$.

Note that the compactness of $\tau_0$ implies that for each $i \in \{\rho + 1, \ldots, n\}$ there exists an index $j \in \{1, \ldots, \rho\}$ for which $(\xi_j) \neq 0$, so setting $y_1, \ldots, y_{\rho}$ equal to zero indeed reduces $\pi_1^* (\varphi)$ to $\varphi(0)$. If $\tau_0$ fails to be compact, the factor $\varphi(0)$ has to be replaced by a factor $\varphi(0, \ldots, 0, y_{\rho+1}, \ldots, y_n)$ in the integrand of the principal value integral. In particular, the following immediate consequence of Theorem 1 will still be valid:

**Corollary 1.** The coefficients $\mu_{+1}(\varphi)$ for $f$ and $f_{\tau_0}$ differ only by a nonzero factor, which depends only on the Newton polyhedron of $f$. 
Remark: If $f$ satisfies one of the conditions $\sharp$, then the principal value integral actually converges for $\lambda = 1$, and thus $\mu(\varphi)$ is nonzero, so we recover the result mentioned in section 3. The fact that condition (a) is sufficient is obvious. As for condition (b), observe that in formula (2) in the proof of the theorem, $f_0$ will not vanish at $P$ if $f \geq 0$ on $\mathbb{R}^n$, since this would mean that $f_0$ has a critical point at $P$. As a consequence, the exponent $\lambda s_0$ does not occur in $\mathbb{R}^n$. To conclude, condition (c) implies that $f_0$ will not vanish at $P$ in this case either: since

$$f_0 \circ \pi_1 = y_1^{N_1} \cdots y_s^{N_s} (f_0(y_{s+1}, \ldots, y_n) + O(y_1, \ldots, y_s)),$$

the equality $f_0(P) = 0$ would induce a zero of $f_0$ in $\mathbb{R}^n$.

6. A NEW PROOF OF THE CONJECTURE

The objective of this section is to prove the following conjecture formulated by Denef and Sargos [5]:

Conjecture 1. If $\tau_0$ is unstable, then $\mu(\varphi) = 0$ for any $C^\infty$-function $\varphi$ with support in a sufficiently small neighbourhood of 0 in $\mathbb{R}^n$.

From the discussion in the last paragraph of section 3, it follows that we may confine ourselves to the case $s_0 \notin \mathbb{Z}$. Moreover, the material in that section shows that it suffices to prove that $\mu_{\pm}(\varphi) = 0$; this is the assertion stated in Theorem 2. Another - still open - conjecture of Denef and Sargos claims that the reverse is also true: the vanishing of $\mu(\varphi)$ whenever the support of $\varphi$ is small enough implies the instability of $\tau_0$.

Theorem 2. If $\tau_0$ is unstable with respect to a variable $x_j$, then the polar multiplicity of $Z_{\pm}(s)$ in $s_0$ is strictly less than $\rho$.

Proof. Because of corollary 1, we may suppose that $f = f_{\tau_0}$. We will proceed by constructing an appropriate resolution of singularities of $f$. To simplify notation we suppose that $j = n$.

Let $F$ be a fan subdividing of the positive orthant $\mathbb{R}^n_+$ into $\mathbb{Z}^n$-simple cones such that $F$ is subordinate to $\Gamma(f)$. Let $F'$ be the fan consisting of the simple cones $\text{conv}(\Delta \cap H_n, e_n)$ with $\Delta \in F$, where $H_n$ is the hyperplane in $\mathbb{R}^n$ defined by $x_n = 0$. Let $Y$ be the toric manifold associated to $(\mathbb{Z}^n, F')$, and $\pi: Y \to \mathbb{R}^n$ the natural map.

We know that $Y$ is nonsingular and $\pi$ is proper. Furthermore, $\pi$ is an isomorphism on the complement of the coordinate hyperplanes in $\mathbb{R}^n$. In order to prove that $\pi$ is a resolution for $f$ we need to show that locally $f \circ \pi$ and the Jacobian $J_\pi$ of $\pi$ can be written as the product of a monomial with a unit.

Let $\Delta$ be an $n$-dimensional cone of $F'$ spanned by an ordered basis $\{\xi_1, \ldots, \xi_{n-1}, e_n\}$ and $U$ be the associated open part of $Y$. Choosing a point $w$ on $U$ we may suppose that $1, \ldots, s$ are the indices $i \neq n$ for which $w_i = 0$.

On $U$ the Jacobian of $\pi$ is a scalar multiple of $\prod_{i=1}^{n-1} y_i^{y_i^{-1}}$ and $f \circ \pi$ can be written as

$$(\prod_{i=1}^s y_i^{N_i})(g(y_{s+1}, \ldots, y_{n-1}) + h(y_{s+1}, \ldots, y_n) + O(y_1, \ldots, y_s)).$$

When the $g + h y_n$ part differs from zero in $w$ we have found our unit; when it equals zero but $\frac{\partial}{\partial y_n}(g + h y_n)(w) \neq 0$ we can introduce the whole second factor...
between brackets as a new variable. So it suffices to show that \( \frac{\partial}{\partial y_n}(g + h y_n) = 0 \) has no solutions in \( \mathbb{R}^{n-1} \times \mathbb{R} \). But if it has, this means that \( h \) has a zero in \( \mathbb{R}_0^{n-1} \), contradicting the definition of unstableness because \( (\prod_{i=1}^s y_i^{\gamma_i}) h = f_\gamma \circ \pi \), with \( \gamma \) denoting the intersection of the common trace of \( \xi_1, \ldots, \xi_s \) with the hyperplane defined by \( x_n = 1 \).

A resolution of singularities for \( f \) determines a set of candidate poles of \( Z_+(s) \) containing the actual poles, and provides an upper bound for their polar multiplicities (cf. \cite{1} and \cite{12}). This is why we constructed a resolution that takes into account the instability of \( \tau_0 \); this piece of extra information yields a sharper upper bound for the polar multiplicity at \( s_0 \). To make things concrete: under the assumption that \( s_0 \) is not an integer, this polar multiplicity is not greater than the maximal number of vectors \( \xi_i \) occurring as generators of the same cone of the fan \( F \) for which \( N_i \neq 0 \) and the value \( v_i/N_i \) equals \( -s_0 \). As is easily seen, this condition is equivalent to the property that the traces of the \( \xi_i \) contain \( \tau_0 \). Since these \( \xi_i \) have to be linearly independent and they will automatically be contained in the hyperplane \( x_n = 0 \) (recall that \( s_0 \notin \mathbb{Z} \)), their number can never be greater than \( \rho - 1 \). Now it becomes clear why we chose this specific form for our resolution: when we consider a fan subordinate to \( \Gamma(f) \), the vectors \( \xi_i \) no longer have to be contained in \( x_n = 0 \) and their number can rise up to \( \rho \). \( \square \)

7. An explicit formula

In this section, we give an explicit formula for the residue \( \mu(\varphi) \), in the case where \( \tau_0 \) is a simplex of codimension 1, such that the only lattice points in the intersection of \( \tau_0 \) with the support of \( f \), are its vertices. We still suppose that \( s_0 \notin \mathbb{Z} \), the other case can be dealt with by introducing a new variable, as was done at the end of Section 3. Special cases of our explicit formula were obtained already in the Ph.D. thesis of A. Laeremans \cite{10} under the direction of the first author. We will use the technique of decoupages, developed in \cite{5}.

Let \( f = f_{\gamma_0} \) be the polynomial \( \sum_{i=1}^n \varepsilon_i x^{a_i} \), with \( x = (x_1, \ldots, x_n) \), with \( \varepsilon_i \in \mathbb{R}_0 \), and with \( a_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{N}^n \) for \( i = 1, \ldots, n \), such that the set of vectors \( \{a_i\}_i \) linearly independent over \( \mathbb{Q} \). Here \( x^{a_i} \) is short for \( \prod_j x_j^{a_{i,j}} \). We define an \( n \)-tuple \( \gamma \) of positive real numbers \( \gamma_i \), by the expression

\[
\tilde{\gamma} = (1, \ldots, 1) = \sum_{i=1}^n \gamma_i a_i .
\]

Note that, since \( s_0 \tilde{\gamma} \) belongs to \( \tau_0 \), the \( \gamma_i \) satisfy \( \sum_i \gamma_i = -s_0 \). We denote the real matrix of order \( n \), with the vector \( a_i \) as \( i \)-th column, by \( A \). For any matrix \( X \), we will write \( X' \) for its transpose. We will denote the \( i \)-th column of \( A' \) by \( a^i \).

Now let \( \varphi \) be, as before, a Schwarz function on \( \mathbb{R}^n \), i.e. a \( C^\infty \)-function with compact support \( \text{Support}(\varphi) \). We assume that \( \text{Support}(\varphi) \cap \mathbb{R}^n_+ \subset [0,1]^n \). We denote the linear polynomial \( \sum_{i=1}^n \varepsilon_i y_i \) by \( g(y) \), and we define \( \tilde{Z}_+(s) \) by the integral

\[
\tilde{Z}_+(s) = \int_{[0,1]^n} g(x)(y)^s y^{-1} \varphi(y) dy ,
\]

for \( s \in \mathbb{C} \), with \( \Re(s) > 0 \), where \( y^{-1} \) means \( \prod_{i=1}^n y_i^{-\gamma_i} \). By \cite{5}, Lemme 3.1, the function \( \tilde{Z}_+(s) \) has a meromorphic continuation to the whole complex plane, which we will denote again by \( \tilde{Z}_+(s) \).
Let \( \Sigma \) be a simplicial fan subdividing the positive orthant \( \mathbb{R}^n_+ \), such that the rays of \( \Sigma \) are generated by elements of the set \( \{a^1, \ldots, a^n, e_1, \ldots, e_n\} \), where \( (e_i)_{i=1}^n \) is the ordered standard basis for \( \mathbb{R}^n \). We denote the cones of \( \Sigma \) of dimension \( n \) by \( \Delta_0, \ldots, \Delta_m \), and we assume that \( \Delta_0 \) is generated by the vectors \( a^i \). Such a fan \( \Sigma \) always exists, by [4], Lemme 2.3.

We will follow the terminology in [4]. We define a function \( L \) by
\[
L : [0, 1]^n \to \mathbb{R}^+_n : (x_1, \ldots, x_n) \mapsto (-\log x_1, \ldots, -\log x_n).
\]
The cones \( \Delta_j \) induce a decoupage \( \{L^{-1}(\Delta_j)\}_{j=1}^m \) of \([0, 1]^n \). It is obvious that \( g \) is compatible with this decoupage (in the sense of [4]). For each \( j = 1, \ldots, m \), we consider the integral
\[
\int_{L^{-1}(\Delta_j)} g_\pm(y)^s y^{\gamma-1} \varphi(y) dy,
\]
for \( s \in \mathbb{C}, \Re(s) > 0 \), and we denote by \( Z_\pm^{(j)}(s) \) its meromorphic continuation to the whole of \( \mathbb{C} \), which exists by [4], Lemme 5.4.

Now let \( \omega \) be the monomial transformation associated to \( \Delta_0 \), i.e.
\[
\omega : [0, 1]^n \to [0, 1]^n : (x_i) \mapsto (\prod_{i=1}^n x_i^{a_{i,k}})_k.
\]
If we pull back the integral defining \( Z_\pm^{(0)} \) along \( \omega \), we get
\[
Z_\pm^{(0)}(s) = |\det A| \int_{[0, 1]^n} f_\pm(x)^s \varphi \circ \omega(x) dx.
\]

**Lemma 1.** The function \( Z_\pm^{(0)}(s) \) is the only term in the sum \( \bar{Z}_\pm(s) = \sum_{j=0}^m Z_\pm^{(j)}(s) \) with a pole at \( s = s_0 \).

**Remark.** Our explicit computations below, will show that \( \bar{Z}_\pm(s) \) has, indeed, a pole at \( s = s_0 \).

**Proof.** First, we show that the vector \( \bar{1} \) is contained in the interior of \( \Delta_0 \). We define an \( n \)-tuple of real numbers \( \zeta = (\zeta_i)_i \) by \( \zeta A' = \bar{1} \). This means that \( A' \zeta = \bar{1} \), hence \( \sum \zeta_i z_i = 0 \) is the equation of \( \tau_0 \), which implies that \( \zeta_i > 0 \) for every \( i \).

We subdivide \( \Delta_0 \) into \( n \) simplicial subcones \( \Delta_{0,l} \), where \( \Delta_{0,l} \) is generated by \( \bar{1} \), and \( \{a^1, \ldots, a^l, \ldots, a^n\} \). Let \( \Sigma' \) be the induced subdivision of the fan \( \Sigma \). We say that a ray of \( \Sigma' \), generated by some vector \( \xi \), contributes to the pole \( s_0 \) of \( \bar{Z}_\pm \), if there exists a positive integer \( \alpha \), such that
\[
s_0 = -\frac{<\xi, \gamma> + \alpha}{l_{\Gamma(g)}(\xi)},
\]
where \( l_{\Gamma(g)} \) is, as in Section 2, the trace map associated to the Newton polyhedron \( \Gamma(g) \) of \( g \). If \( l_{\Gamma(g)}(\xi) \) is zero, we define the right part of the equation to be equal to \( -\infty \). It is clear that the vector \( \bar{1} \) contributes to \( s_0 \), and we will show it is the only ray of \( \Sigma' \) that does. By [4], Lemme 5.4, this concludes the proof.
For a tuple $\xi = (\xi_i)$ of real numbers, we will write $\min(\xi)$ for $\min_i\{\xi_i\}$. Let $T$ be the torus $\mathbb{R}^n_0$. For $\xi$ in $T(\mathbb{R}_+^n)$, and $\xi \notin \overline{I}$, we see that

$$\frac{\langle \xi, \gamma \rangle}{l_{T(\mathbb{R}^n_0)}(\xi)} = \frac{\langle \xi, \gamma \rangle}{\min(\xi)} > \frac{\min(\xi)I, \gamma >}{\min(\xi)} = -s_0$$

This shows that $\overline{I}$ is the only vector contributing to $s_0$.

Hence, in order to know the residue of $Z^{(0)}_\pm(s)$ at $s = s_0$, it suffices to study the residue of $\bar{Z}_\pm(s)$ at $s_0$.

The function $Z^{(0)}_\pm(s)$ is more or less the meromorphic function whose residue in $s_0$ we want to investigate. The problem is that the function $(\varphi \circ \omega)\chi_{[0,1]^n}$ is not $C^\infty$, and that the support of $\varphi \circ \omega$ cannot be chosen in an arbitrarily small neighbourhood of 0, since it contains $\omega^{-1}(0)$ as soon as $\varphi(0) \neq 0$. The following lemma deals with these difficulties.

**Lemma 2.** Let $\{\psi_\alpha\}$ be a partition of unity for $\mathbb{R}^n$, with $\psi_\alpha \equiv 1$ on a neighbourhood of 0. If the supports of $\varphi$ and $\psi_\alpha$ are sufficiently small,

$$\lim_{s \to s_0} (s - s_0)^\rho Z^{(0)}_\pm(s) = |\det A| \lim_{s \to s_0} (s - s_0)^\rho Z_{\alpha_0}(s),$$

where we write $Z_{\alpha_0}$ for the meromorphic continuation of

$$\int_{\mathbb{R}^n_+} f_\pm(x)^s \varphi \circ \omega(x)\psi_{\alpha_0}(x)dx.$$

**Proof.** Our proof is similar to the proof of the previous lemma. We can construct a subcone $\Delta'$ of $\Delta_0$, containing $\overline{I}$ in its interior, such that

$$\omega^{-1}(\mathcal{L}^{-1}(\Delta') \cap \text{Support}(\varphi)) \subset \psi^{-1}_{\alpha_0}(1),$$

provided the support of $\varphi$ is sufficiently small. If we extend $\{\Delta'\}$ to a subdivision of $\Delta_0$, the only cone contributing to the residue of $\bar{Z}_\pm(s)$ at $s = s_0$ will be $\Delta'$ itself.

We denote by $Z^{(0)}_\pm$ the meromorphic continuation of

$$\int_{\mathbb{R}^n_+} f_\pm(x)^s \varphi \circ \omega(x)dx.$$

We introduce a new function $J^{(\lambda)}_\Delta(s)$, which is defined, for $\Re(\lambda) > 0$ and $\Re(s) > 0$, by

$$J^{(\lambda)}_\Delta(s) = \int_{\mathcal{L}^{-1}(\Delta')} g_\pm(z)^s y^{\lambda - 1} \varphi(y)dy.$$

Pulling back the integral via $\omega$ yields

$$\int_{\mathcal{L}(\omega)^{-1}(\Delta')} f_\pm(x)^s x^{\lambda - 1} \varphi \circ \omega(x)dx.$$

Slightly adapting the proofs of [4], Lemme 3.1 and Lemme 5.4, we see that this function has a meromorphic continuation to $\mathbb{C}^2$, which we will again denote by $J^{(\lambda)}_\Delta(s)$. If we fix $\lambda$ in $\mathbb{R}_+ \setminus 0$ such that $\lambda s_0 \notin \mathbb{Z}$, then the non-integral poles of $J^{(\lambda)}_\Delta$
are not greater than $\lambda s_0$, and the polar multiplicity at $\lambda s_0$ is at most $\rho$. Similarly, we define a function $J_{\alpha_0}^{(\lambda)}(s)$ as the meromorphic continuation of

$$\int_{\mathbb{R}_+^n} f_{\pm}(x)^s x^{\lambda} \varphi \circ \omega(x) \psi_{\alpha_0}(x) dx.$$ 

Now observe that formula 4 in our proof of Theorem 1 holds, as soon as $\psi$ is continuous in a neighbourhood of $(t_0, 0)$. This implies that, when the support of $\psi_{\alpha_0}$ is sufficiently small, and $\lambda > 0$ is sufficiently small,

$$\lim_{s \to \lambda s_0} (s - \lambda s_0)^\rho J_{\alpha_0}^{(\lambda)}(s) = |\det A| \lim_{s \to \lambda s_0} (s - \lambda s_0)^\rho J_{\alpha_0}^{(\lambda)}(s).$$

The function $(s - \lambda s_0)^\rho J_{\alpha_0}^{(\lambda)}(s)$ is meromorphic on $\mathbb{C}^2$, and by (6), the plane $s = \lambda s_0$ is not contained in its polar locus (this also follows from the results in [4]). Hence, its restriction to this plane is a meromorphic function in $\lambda$, and by the identity principle, it coincides with the restriction of $|\det A|(s - \lambda s_0)^\rho J_{\alpha_0}^{(\lambda)}(s)$. Taking values in $\lambda = 1$ yields

$$\lim_{s \to s_0} (s - s_0)^\rho Z_{\pm}^{(0)}(s) = |\det A| \lim_{s \to s_0} (s - s_0)^\rho Z_{\alpha_0}^{(0)}(s),$$

since both sides are equal to the residue of $\bar{Z}_{\pm}(s)$ at $s_0$. □

It follows from Theorem 1 that

$$\lim_{s \to s_0} (s - s_0)^\rho Z_{\pm}(s) = |\det A| \lim_{s \to s_0} (s - s_0)^\rho Z_{\alpha_0}(s),$$

with $\bar{Z}$ the meromorphic continuation of

$$\int_{[0,1]^n} f_{\pm}(x)^s \varphi(x) dx,$$

whenever $\phi$ is a Schwarz function with sufficiently small support, and $\phi(0) = \varphi(0)$.

We define $\tilde{I}(t)$ as the oscillating integral

$$\tilde{I}(t) = \int_{[0,1]^n} e^{it\sigma(y)} y^{\gamma-1} \varphi(y) dy .$$

As the coefficient of $t^{\gamma_0}$ in its asymptotic expansion only depends on $\varphi(0)$, provided the support of $\varphi$ is sufficiently small, we may assume that $\varphi$ is of the form $\prod_{i=1}^n \theta_i(y_i)$, where $\theta_i$ is a Schwarz function on $\mathbb{R}$. Then $\tilde{I}(t)$ splits into a product of integrals of the type

$$\mathcal{I}(t; \varepsilon, \eta) = \int_0^1 e^{it\varepsilon z} z^{\eta-1} \theta(z) dz,$$

where $\varepsilon$ and $\eta$ are non-zero real numbers, $\eta > 0$. By [1], 7.2.3 (11), we get, as $t \to +\infty$,

$$\mathcal{I}(t; \varepsilon, \eta) \sim \theta(0) \frac{\Gamma(\eta)}{(-i\varepsilon t)^\eta}.$$
where \( \arg(\pm it) = \pm \pi/2 \), and \( \Gamma \) is the Gamma function. Bringing these factors together, we see that the coefficient of \( t^{s_0} \) in the asymptotic expansion of \( \tilde{I}(t) \) is equal to

\[
\varphi(0) \prod_{j=1}^{n} \Gamma(\gamma_j) |\varepsilon_j|^{-\gamma_j} e^{\text{sign}(\varepsilon_j) \frac{\pi i}{2} \gamma_j}.
\]

**Theorem 3.** Suppose that \( f \) is non-degenerate with respect to its Newton polyhedron, and \( \tau_0 \) is a simplex of codimension \( \rho = 1 \), such that the only lattice points in the intersection of \( \tau_0 \) with the support of \( f \), are its vertices. We suppose as well that \( s_0 \notin \mathbb{Z} \). If \( \varphi \) is a Schwarz function with sufficiently small support,

\[
\mu(\varphi) = |\det A|^{-1} \varphi(0) \prod_{j=1}^{n} \Gamma(\gamma_j) |\varepsilon_j|^{-\gamma_j} \sum_{\beta \in \{-1,1\}^n} \prod_{j=1}^{n} e^{\text{sign}(\varepsilon_j \beta_j) \frac{\pi i}{2} \gamma_j}.
\]

**Proof.** This follows immediately from our residue formula, and the arguments above. \( \square \)

Conjecture 4 of \([5]\) predicts the following remarkable combinatorial assertion.

**Conjecture 2.** With the notation and hypotheses of Theorem 3, if \( \tau_0 \) is stable, then

\[
\sum_{\beta \in \{-1,1\}^n} \prod_{j=1}^{n} e^{\text{sign}(\varepsilon_j \beta_j) \frac{\pi i}{2} \gamma_j} \neq 0.
\]

8. A complex residue formula

In this section, we will establish the complex analogue of our residue formula in Theorem 1. The complex local zeta function \( Z(s) \), associated to a complex analytic germ \( f \) at \((\mathbb{C}^n, 0)\), is defined for \( s \in \mathbb{C}, \Re(s) > 0 \) as

\[
\int_{\mathbb{C}^n} |f|^2 s |x|^2 \varphi(x) dx
\]

where \( \varphi \) is a positive \( \mathcal{C}^\infty \)-function with sufficiently small support, and the integration is conducted with respect to the Haar-measure on \( \mathbb{C}^n \) (this is just the Lebesgue measure on \( \mathbb{C}^n = \mathbb{R}^{2n} \)). In the definition, we use \( |f|^2 s \), rather than \( |f|^s \), for reasons of uniformity: \(|.|^2\) is the modulus associated to the Haar-measure on the local field \( \mathbb{C} \), as are \(|.| \) in the real, and \(|.|_p \) in the \( p \)-adic case \([7]\). It is known that \( Z \) has a meromorphic continuation to the whole of \( \mathbb{C} \), which we will denote again by \( Z \).

We again assume that \( f \) vanishes at the origin, and that \( f \) is non-degenerate with respect to its Newton polyhedron. We define \( \tau_0, s_0, \) and \( \rho \), as in the real case. We will use the same notation and conventions as in Section 5.

As before, a toric embedded resolution establishes \( s_0 \) as the largest non-trivial candidate pole of \( Z(s) \), where we say that a pole is trivial if it is integer and has multiplicity 1. We will assume that \( s_0 \notin \mathbb{Z} \). In this case, the polar multiplicity of \( s_0 \) is at most \( \rho \). We want to determine the residue

\[
|\mu|(|\varphi|) = \lim_{s \to s_0} (s - s_0)^\rho Z(s).
\]

For \( \lambda \in \mathbb{C}, \Re(\lambda) > 0 \) we define a function \( J^{(\lambda)} \) on \( D = \{ s \in \mathbb{C} | \Re(s) > 0 \} \) by

\[
J^{(\lambda)}(s) = \int_{\mathbb{C}^n} |f(x)|^{2s} |x|^{2\lambda - 2} \varphi(x) dx.
\]
where \( |x|^{2\lambda - 2} = |x_1|^{2\lambda - 2} \cdots |x_n|^{2\lambda - 2} \). As in the real case, \( J \), considered as a function in \( \lambda \) and \( s \), has a meromorphic continuation to the whole of \( \mathbb{C}^2 \), and if we fix \( \lambda \) in \( \mathbb{R}_+ \setminus 0 \) such that \( \lambda s_0 \notin \mathbb{Z} \), then the non-integral poles of \( J^{(\lambda)}(s) \) are not greater than \( \lambda s_0 \), and the polar multiplicity at \( \lambda s_0 \) is at most \( \rho \).

The following theorem is the complex counterpart of Theorem 4.

**Theorem 4.** Assume, as always, that \( f \) is non-degenerate with respect to its Newton polyhedron, and furthermore that \( \tau_0 \) is compact; the latter condition is included only to simplify formulae. When the support of \( \varphi \) is sufficiently small, then

\[
|\mu(\varphi)| = \pi^\rho n! \text{Vol}(C) \varphi(0) \text{PV} \int_{\mathbb{C}^n-\rho} |f_{\tau_0}(1, \ldots, 1, z_{\rho+1}, \ldots, z_n)|^{2s_0} \, dz,
\]

where the principal value integral is defined as the value of the meromorphic continuation at \( \lambda = 1 \) of the function

\[
K(\lambda) = \int_{\mathbb{C}^n-\rho} |f_{\tau_0}(1, \ldots, 1, z_{\rho+1}, \ldots, z_n)|^{2s_0 \lambda} |z|^{2\lambda-2} \, dz,
\]

where \( K(\lambda) \) is defined for \( \lambda \in \mathbb{R}_+ \setminus 0 \) and \( \lambda s_0 > -1 \). Here \( z = \prod_{i=\rho+1}^n z_i \), and \( dz \) is the Haar measure on \( \mathbb{C}^n-\rho \).

Some explanation: we will show, again, that \( \pi^\rho n! \text{Vol}(C) \varphi(0) K(\lambda) \) equals \( \lim_{s \to \lambda s_0} (s - \lambda s_0)^\rho J^{(\lambda)}(s) \) on its domain of definition mentioned above, and that, in particular, the integral \( K \) converges on this domain. This shows that \( K \) has indeed an analytic continuation at \( \lambda = 1 \) - which is necessarily unique - since we will show, using resolution of singularities, that, whenever \( \Re(\lambda) > 0 \) and \( \lambda s_0 \notin \mathbb{Z} \), \( (s - \lambda s_0)^\rho J^{(\lambda)}(s) \) is analytic on a neighbourhood of \( (s_0, \lambda, \lambda) \). Details can be found in the proof.

In particular, we see that the dropping of the polar multiplicity of \( Z(s) \) in \( s_0 \) only depends on \( f_{\tau_0} \).

**Proof.** The proof is almost the same as in the real case. We define lattices \( L_i \) and fans \( F_i, i = 1, \ldots, 3 \), as before, as well as the resolution morphism \( \pi_1 \), and the submanifold \( Y \) of \( X_{L_i, F_i} \). Refining the fan \( F_1 \), we may suppose that \( F_2 < F_3 \). We define \( L_3 \) as \( \pi L_3 \), with \( N \in \mathbb{N}_0 \). The toric morphism

\[
\pi' : X_{L_3', F_3} \to X_{L_3, F_3}
\]

is an \( N(n-\rho) \)-fold cover of \( (\mathbb{P}^1_{\mathbb{C}})^{n-\rho} \), ramified over the orbits of codimension one. For an appropriate choice of \( N \), a set of base vectors of \( L_2 \) has integer coordinates with respect to \( L_3 \). As a consequence of this, there exist natural maps

\[
\pi_2 : X_{L_3', F_2} \to X_{L_3, F_3} \quad \text{and} \quad \pi_3 : X_{L_3', F_2} \to X_{L_2, F_2} = Y.
\]

Let \( \gamma \) on \( (\mathbb{P}^1_{\mathbb{C}})^{n-\rho} \) be given by

\[
\gamma = |f_{\tau_0}(1, \ldots, 1, w_{\rho+1}, \ldots, w_n)|^{2s_0 \lambda} \prod_{i=\rho+1}^n |w_i|^{2\lambda-2} dw_{\rho+1} \wedge \ldots \wedge dw_n,
\]

where \( w_\rho, \ldots, w_n \) are standard affine coordinates on \( \mathbb{C}^{n-\rho} \). With this notation,

\[
K(\lambda) = \int_{\mathbb{C}^{n-\rho}} \gamma = \int_{X_{L_3, F_2}} \pi_2^*(\gamma).
\]
Let $\Delta$ be a $n$-dimensional cone of $F_1$, generated by $\xi_1, \ldots, \xi_n$, with $\xi_1, \ldots, \xi_\rho \in \tau_0$. On $Y \cap U_\Delta$, where $U_\Delta$ is the coordinate neighbourhood in $X_{L_1,F_1}$ corresponding to $\Delta$, we have

$$\left< n! \text{Vol}(C) \varphi(0) \mathbb{P}(i=1)^\rho N_i \mathbb{P}(\gamma) = \mathbb{P}(i=1)^\rho \left| y_i^2 \right| \mathbb{P}(i=1)^\rho \left| f^{2s_0} z |^{2\lambda - 2} dz \right| y_1 = \ldots, y_n = 0 \right>$$

where $(y_1, \ldots, y_n)$ are the standard coordinates associated to $(\xi_1, \ldots, \xi_n)$.

We can again find local coordinates $y'_i$, around each point $P$ of $Y \cap U_\Delta$ in a neighbourhood of $\pi_1^{-1}(0)$, such that

- $y_i = y'_i$ if $y_i(P) = 0$; in particular this holds for $i \in \{1, \ldots, \rho\}$,
- $\pi_1^v(\int [f^{2s}|z|^{2\lambda - 2} dz) = \left| v_1 \right|^v | v_2 |^v | v_3 | \prod_{i=1}^{\rho} y_i^{2Ns_0 + 2\nu_i - 2} dy'$, where $v_1, v_2, v_3$ are nonvanishing analytic functions, $(N_i, \nu_i)$ for $y_i(P) = 0$, and $(N_i', \nu_i')$ equals either $(0, 1)$ or $(1, 1)$ if $y_i(P) \neq 0$.

In the expression

$$(s - \lambda s_0)^\rho \int_{\mathbb{C}^\rho} \prod_{i=1}^{\rho} y_i^{2Ns_0 + 2\nu_i - 2} \int_{\mathbb{C}^n, \rho} | v_1 |^v | v_2 |^v | v_3 | \theta \prod_{i=\rho+1}^{n} y_i^{2Ns_0 + 2\nu_i - 2} dy'$$

where $\theta$ is a Schwarz function on $\mathbb{C}^n = \mathbb{R}^{2n}$, the inner integral converges for $\lambda$ sufficiently small and $s > s_0 \lambda$, since the exponents $N_i s_0 \lambda + \nu_i - 1$, for $i = \rho + 1, \ldots, n$, are either 0, $s_0$, or $(N_i s_0 + \nu_i) \lambda - 1 > -1$. Hence we can apply the formula

$$\lim_{t \to t_0} t \theta^\rho \int \prod_{i=1}^{\rho} z_i^{2N_i (t-t_0) - 2} \psi(t, z) dz = \frac{\pi^\rho \psi(t_0, 0)}{\prod_{i=1}^{\rho} N_i},$$

which holds for every continuous mapping $\psi$ on $\mathbb{R} \times \mathbb{C}^\rho$.

To conclude the proof of the theorem, one only has to observe that

$$\lim_{s \to \lambda s_0} (s - \lambda s_0)^\rho \int_{X_{L_1,F_1}} \theta \pi_1^v (\int [f^{2s}|z|^{2\lambda - 2} dz) = 0$$

when $\theta$ is a Schwarz function with compact support disjoint with $Y$, and invoke a suitable partition of unity for $X_{L_1,F_1}$. A similar construction shows that the function $(s - \lambda s_0)^\rho J(\lambda)(s)$ is analytic in a neighbourhood of

$$\{(s_0, \lambda) \in \mathbb{C}^2 \mid \Re(\lambda) > 0, \lambda s_0 \notin \mathbb{Z} \}.$$

If $\tau_0$ fails to be compact, the factor $\varphi(0)$ has to be replaced by a factor $\varphi(0, \ldots, 0, z_{m+1}, \ldots, z_n)$ in the integrand of the principal value integral ($m$ is defined as in the real case). In particular, the following immediate consequence of Theorem 1 will still be valid:

**Corollary 2.** The coefficients $|\mu|(<\varphi>$ for $f$ and $f_{\tau_0}$ differ only by a nonzero factor, which depends only on the Newton polyhedron of $f$.

If $s_0 > -1$, it follows from the proof that the principal value integral actually converges for $\lambda = 1$, and hence that $|\mu|(<\varphi>$ is nonzero.
9. THE STABILITY CONJECTURE

Denef and Sargos stated their conjecture in the complex case, as well [3]. As always, we suppose that \( s_0 \notin \mathbb{Z} \).

**Conjecture 3** (Denef-Sargos). Let \( f \) be a singular complex analytic germ at \( 0 \in \mathbb{C}^n \), with \( f(0) = 0 \), and let \( \varphi \) be a Schwarz function on \( \mathbb{C}^n \) with sufficiently small support, such that \( \varphi(0) \neq 0 \). We denote by \( Z(s) \) the complex local zeta function associated to \( f \) and \( \varphi \), and we suppose \( s_0 \notin \mathbb{Z} \). The polar multiplicity of \( Z(s) \) at \( s_0 \) equals \( \rho \) if and only if \( \tau_0 \) is a stable face.

We can copy the proof of Theorem 2 verbatim to obtain a proof of its complex analogue.

**Theorem 5.** If \( \tau_0 \) is unstable with respect to a variable \( z_j \), then the polar multiplicity of \( Z(s) \) in \( s_0 \) is strictly less than \( \rho \).

However, in the complex case, we can say more. We start by proving the following statement.

**Proposition 1.** If \( \tau_0 \) is compact, and \( \rho = 1 \), the complex local zeta function \( Z(s) \) has a pole of order \( \rho \) at \( s = s_0 \).

**Proof.** First, we reduce to the case where \( f \) has an isolated singularity at the origin. If not, we can always modify \( f \) without changing \( f_{\tau_0(f)} \), which is the only part of \( f \) that really matters as far as the polar properties of \( s_0 \) are concerned, by adding, for each \( i = 1, \ldots, n \), a monomial \( b_i x_i^{\alpha_i} \) to \( f \), with \( \alpha_i \) sufficiently large. The new polynomial \( f \), obtained in this way, will still be non-degenerate with respect to its Newton polyhedron, and furthermore, it will have an isolated singularity at the origin, provided we make suitable choices for \( a_i, b_i \) (cf. [2]).

Loeser showed in [11], that the fact that \( f \) has an isolated singularity at the origin, implies that the set \( \{ - (\alpha + m) | \alpha \in Sp(f), m \in \mathbb{N}_0 \} \) is contained in the set of poles of \( Z(s) \), where \( Sp(f) \) is the spectrum of Steenbrink-Varchenko associated to \( f \). Since \( -s_0 - 1 \) is the smallest value in \( Sp(f) \) (cf. [9]) we can conclude that \( s_0 \) is indeed a pole of \( Z \).

This Proposition does not contradict the Conjecture, since in the case \( \rho = 1 \), \( \tau_0 \) will automatically be stable. Using this Proposition, and our complex residue formula, we can derive a combinatorial sufficient condition that implies the Conjecture.

**Lemma 3.** Suppose that \( \tau_0(f) \) is compact, and that we can find a complex analytic function \( g(z_{\rho+1}, \ldots, z_n) \) on a neighbourhood of the origin in \( \mathbb{C}^{n-\rho+1} \), such that

- \( g(0) = 0 \) and \( g \) has a singularity at the origin
- \( g \) is non-degenerate with respect to its Newton polyhedron
- \( \tau_0(g) \) is compact, and has codimension 1
- \( s_0(g) = s_0(f) \)
- \( \tau_0(g) \oplus \sum_{j=\rho+1}^{n} \mathbb{R} e_j = \mathbb{R}^{n-\rho+1} \)
- \( g_{\tau_0(g)}(1, z_{\rho+1}, \ldots, z_n) = f_{\tau_0(f)}(1, \ldots, 1, z_{\rho+1}, \ldots, z_n) \)

Then \( Z(s) \) has a pole at \( s = s_0 \) of order \( \rho \).

**Proof.** We will denote the complex local zeta function, associated to a complex analytic germ \( h \), and a Schwarz function \( \phi \), by \( Z_{h, \phi}(s) \). We deduce from Proposition
and base in Repeating this argument, shows that $\tau$ is unstable with respect to subspace $V$ has codimension 1. Our assumption implies that $\tau$ is stable. The stability of $g$ for each $\sigma$ is satisfied by $\pi$, where $\tilde{\tau}$ and our residue formula that
$$\lim_{s \to s_0} (s - s_0)^\rho Z_{f,\varphi}(s) = c \lim_{s \to s_0} (s - s_0) Z_{g,\varphi}(s) \neq 0$$
where $\varphi(x_\rho, \ldots, x_n) = \varphi(0, \ldots, 0, x_\rho, \ldots, x_n)$ and $c$ is a nonzero constant. So $s_0$ is a pole of order $\rho$ of $Z(s)$. \hfill \square

The natural choice for the function $g$ would be
$$g(z_\rho, \ldots, z_n) = f_{\tau_0}(1, \ldots, 1, z_\rho, \ldots, z_n),$$
maybe after permutating the first $\rho$ variables $z_i$. However, this does not work in general, because it can happen that $g \neq g_{\tau_0(g)}$, as is illustrated by the following example.

**Example 1.** Consider the polynomial function
$$f(z_1, z_2, z_3, z_4) = z_1^2 z_2 z_3 z_4 + z_1^2 z_3 z_4^2 + z_1^2 z_3 z_4^2 z_4.$$ It is easy to see that $\tau_0(f) = \text{supp}(f)$, that $\rho = 2$, and that $\tau_0$ is stable. The faces of $\Gamma(f)$ that contain $\tau_0$ are contained in the hyperplanes $z_2 + z_4 = 3$, and $z_1 + z_3 = 3$, which means that any coordinate could figure as $z_1$. However, no matter how we permute the coordinates, $g(z_2, z_3, z_4) = f(1, z_2, z_3, z_4)$ will never satisfy $g_{\tau_0(g)} = g$.

However, this choice does work under some additional assumptions. We denote by $\tau_j$ the projection of $\mathbb{R}^n$ onto $\sum_{k=j}^n \mathbb{R}e_j$, with $1 \leq j \leq n$.

**Theorem 6.** Suppose that $\tau_0$ is stable and compact, and suppose that, maybe after permutating $(z_1, \ldots, z_\rho)$, the function $g$ defined by $g(z_\rho, \ldots, z_n) = f_{\tau_0}(1, \ldots, 1, z_\rho, \ldots, z_n)$ satisfies $\tau_0(g) = \tau_\rho(\tau_0(f))$. Then $Z(s)$ has a pole of order $\rho$ at $s = s_0$.

**Proof.** It suffices to check the conditions of Lemma 3. By assumption, $g_{\tau_0(g)} = g$, and $\tau_\rho(\tau_0(g))$ is compact.

Let us first show that the dimension of $\tau_0(g)$ is still equal to $n - \rho$, hence $\tau_0(g)$ has codimension 1. Our assumption implies that
$$\pi_m(\tau_0(f)) = \tau_0(f(1, \ldots, 1, z_m, \ldots, z_n)),$$
for each $m = 1, \ldots, \rho$. Choose vertices $v_0, \ldots, v_{n - \rho}$ of $\tau_0(f)$, spanning the affine subspace $V$ of $\mathbb{R}^n$ generated by $\tau_0(f)$. Since $\tau_0(f)$ is compact, $\dim \pi_2(V) = \dim V$, because $V$ can not contain vectors parallel to one of the coordinate axes. Repeating this argument, shows that $\tau_0(g)$ has codimension 1. It is clear that
$$\tau_0(g) \oplus \sum_{j=\rho+1}^n \mathbb{R}e_j = \mathbb{R}^{n-\rho+1}.$$ The stability of $\tau_0(g)$ follows from the stability of $\tau_0(f)$. For suppose that $\tau_0(g)$ is unstable with respect to $z_k$. This means that $\tau_0$ is a pyramid with top in $z_k = 1$ and base in $z_k = 0$. Then the same holds for the face $\tau_\rho(\tau_0(g))$, since it cannot be parallel to one of the projection axes. This is impossible, as we assumed that $\tau_0(f)$ is stable. The stability of $\tau_0(g)$ implies, in particular, that the origin is a singularity for $g$.

To conclude, we show that $g$ is non-degenerate with respect to its Newton polyhedron. Let $\sigma$ be a face of $\tau_0(g)$, and suppose that $(\alpha_\rho, \ldots, \alpha_n)$ is a singular point of $g_\sigma$. The face $\sigma$ corresponds to a face $\sigma'$ of $\tau_0(f)$. Our aim is to prove that one of the
\( \alpha_i \) is zero, and we proceed by showing that \( \alpha = (1, \ldots, 1, \alpha_{\rho}, \ldots, \alpha_n) \) is a singular point of \( f_{\sigma'} \). The quasihomogeneity of \( f_{\sigma'} \) implies that \( N_j f_{\sigma'} = \sum_{i=1}^n (\xi_{\phi_i})_i z_i \frac{\partial f_{\sigma'}}{\partial z_i} \) for \( j = 1, \ldots, r \). Here, the \( \phi_j \) are the facets of \( \Gamma(f) \) containing \( \tau, \xi_{\phi_j} \) is the primitive normal vector on \( \phi_j \), and \( N_j \) equals \( l_{\Gamma(f)}(\xi_{\phi_j}) \). Applying both sides of the expression to \( \alpha \), yields a linear system of equations of rank \( \rho - 1 \) in the first \( \rho - 1 \) partial derivatives of \( f \) in \( \alpha \), obliging all of these to become zero as well. \( \square \)

In spite of the additional assumptions, the proof of Theorem 3 gives some intuition about the reasons behind the stability condition: it guarantees that \( g \) still has a singularity at the origin.

**Corollary 3.** If \( \tau_0 \) is a stable, compact face of dimension 1, then \( Z(s) \) has a pole of order \( \rho \) at \( s = s_0 \).

**Proof.** By Proposition 1 we may assume \( n \geq 3 \). Let \( v_0, v_1 \) be the vertices of \( \tau_0 \), and suppose that we can find, for each \( i = 1, \ldots, n \), an index \( j(i) \in \{0, 1, \ldots, n\} \), such that

\[
\bar{\pi}_i(v_{j(i)}) \in \bar{\pi}_i(v_{1-j(i)}) + \mathbb{R}^n_+,
\]

where \( \bar{\pi}_i \) is the projection

\[
\bar{\pi}_i : \oplus_{k=1}^n \mathbb{R}e_k \to \oplus_{k \neq i} \mathbb{R}e_k.
\]

Then there would be two indices \( i_1, i_2 \) such that \( j(i_1) = j(i_2) \), which would imply that already \( v_{j(i_1)} \in v_{1-j(i_1)} + \mathbb{R}^n_+ \). This is impossible. Hence, we may suppose that \( \bar{\pi}_2(v_{j}) \notin \bar{\pi}_2(v_{1-j}) + \mathbb{R}^n_+ \), for \( j = 0, 1 \). By induction, we may assume that \( \bar{\pi}_{n-1}(v_{j}) \notin \bar{\pi}_{n-1}(v_{1-j}) + \mathbb{R}^2_+ \), for \( j = 0, 1 \).

This implies that \( g(z_{n-1}, z_n) := f_{\tau_0}(1, \ldots, 1, z_{n-1}, z_n) \) satisfies \( g = g_{\tau_0(g)} \). By Theorem 3 the only thing left to prove, is that we can find an index \( i \in \{n-1, n\} \), such that \( \bar{\tau}_0(f) + \mathbb{R}e_i = \mathbb{R}^n \). But if this were not the case, \( \bar{\tau}_0(f) \) would contain \( \mathbb{R}e_{n-1} + \mathbb{R}e_n \), and this contradicts that \( \bar{\pi}_{n-1}(v_{j}) \notin \bar{\pi}_{n-1}(v_{1-j}) + \mathbb{R}^2_+ \), for \( j = 0, 1 \). \( \square \)

To conclude, we give a proof of the Conjecture for \( n = 3 \).

**Proposition 2.** Let \( f \) be a complex analytic singular germ at \( 0 \in \mathbb{C}^3 \), let \( \varphi \) be a Schwarz function on \( \mathbb{C}^3 \) with sufficiently small support, nonzero at \( 0 \). We denote by \( Z(s) \) the complex local zeta function associated to \( f \) and \( \varphi \), and we suppose that \( s_0 \notin \mathbb{Z} \). The polar multiplicity of \( Z(s) \) at \( s_0 \) equals \( \rho \) if and only if \( \tau_0 \) is a stable face.

**Proof.** The case \( \rho = 3 \) is trivial, so we may assume \( \rho < 3 \). By Theorem 3 Proposition 1 and Corollary 3 we may furthermore suppose that \( \tau_0 \) is stable, and not compact. As is easily seen, this implies that \( t_0 \geq 1 \). \( \square \)

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