The classical Hankel transform in the Kirillov model of the discrete series

Ehud Moshe Baruch*

Department of Mathematics, Technion, Haifa 32000, Israel

(Received 28 April 2012; final version received 2 May 2012)

In this paper, we give a new and simple proof of the Hankel inversion formula for the classical Hankel transform of index \( \nu \) which holds for \( \text{Re}(\nu) > -1 \). Using the proof of this formula, we obtain the full description of the Kirillov model for discrete series representations of \( SL(2, \mathbb{R}) \) and \( GL(2, \mathbb{R}) \).

Keywords: Bessel functions; Hankel transform; Kirillov model; discrete series

1991 Mathematics Subject Classification: Primary: 22E50; Secondary: 22E35,11S37

1. Introduction

It has been noted by Cogdell and Piatetski-Shapiro [3] (see also [2,7,15,17]) that the action of the Weyl element in the Kirillov model [11] of an irreducible unitary representation of \( GL(2, \mathbb{R}) \) is given by a certain integral transform and that in the case of the discrete series this is a classical Hankel transform of integer order. It has been proved in [5] that the classical Hankel transform of real order \( \nu \) with \( \nu > -1 \) is an isomorphism of order 2 of a certain ‘Schwartz’ space. In this paper, we give a simple and elementary proof of this fact using ‘representation theoretic’ ideas and extend the result to complex \( \nu \) with \( \text{Re}(\nu) > -1 \). Using these methods for the Hankel transform of integer order, we determine the smooth space of the discrete series in the Kirillov model, thus giving for the first time an explicit model which is suitable for various applications.

The inversion formula for the Hankel transform which states that the Hankel transform is self-reciprocal was studied by many authors starting with Hankel [10]. This formula is classically stated as follows [20, p. 453]. Let \( J_{\nu}(z) \) be the classical J-Bessel function. Let \( f \) be a complex-valued function defined on the positive real line. Then, under certain assumptions on \( f \) and \( \nu \) (see [20] or Theorem 3.2), we have

\[
f(z) = \int_0^\infty \int_0^\infty f(x)J_{\nu}(xy)J_{\nu}(yz)xy \, dx \, dy. \tag{1.1}
\]

In a more modern notation, we define the Hankel transform of order \( \nu \) of \( f \) to be

\[
h_{\nu}(f)(y) = \int_0^\infty f(x)J_{\nu}(xy) \, dx.
\]

*Email: emb@math.technion.ac.il

© 2013 Taylor & Francis
Then, under certain assumptions on \( f \) and \( \nu \), the Hankel transform is self-reciprocal, that is, \( h^2 = Id \). A general discussion of the history of this result and various attempts at a proof can be found in [20, p. 454]. It is mentioned in [20] that Hankel was the first to give the formula in 1869 and that Weyl [21, p. 324] was the first to give a complete proof when \( f \) is in a certain space of twice differential functions and \( \nu \) is real, \( \nu > -1 \). Watson gave a complete proof when \( \nu \) is real, \( \nu > -1/2 \) and \( \phi \in L^1((0, \infty), \sqrt{x} \, dx) \), and is of bounded variation in an interval around the point \( z \) in Equation (1.1). MacRobert [14] proved the result for \( \Re(\nu) > -1 \) under the additional assumption that \( \phi \) is analytic. In more recent studies, it has been proved by Zemanian [22] that the Hankel transform preserves a certain ‘Schwartz’ space. Duran [5] gave a very elegant proof of the inversion formula on the ‘Schwartz’ space when \( \nu > -1 \). In this paper, we extend the inversion formula on a ‘Schwartz’ space to complex \( \nu \) such that \( \Re(\nu) > -1 \). Our method is to ‘move’ the Hankel transform from the Schwartz space on which it acts to a different space using a Fourier transform which we think of as an ‘intertwining operator’. The main part of the proof is the computation of the effect of this ‘intertwining operator’ on the Hankel transform. The operator is built in such a way that the Hankel transform is replaced with an operator which sends a function \( \phi \) on the real line to the function \( |x|^{-\nu-1/2} e^{\text{sgn}(x)\pi i/2}(1/\nu - 1) \). It is easy to see that this operator is self-reciprocal and from this follows the same result for the Hankel transform.

In the case where \( \nu \) is a positive integer, the above operator gives the action of the Weyl element of \( \text{SL}(2, \mathbb{R}) \) or \( \text{GL}(2, \mathbb{R}) \) on the discrete series representations. The operator which moves us from the above action to the Hankel transform is an honest intertwining operator for a discrete series representation which moves us from a model for the induced space to the Kirillov model. Using the above results, we can completely determine the ‘smooth’ space of this Kirillov model and give an explicit action in this model.

Our paper is divided as follows. In Section 2, we describe the Schwartz space for the general Hankel transform of order \( \nu \) with \( \Re(\nu) > -1 \) and prove that the Hankel transform preserves this space. In Section 3, we define our ‘intertwining operator’ and compute its composition with the Hankel transform. Using this, we prove the inversion formula for the Hankel transform. In Sections 4–7, we turn our attention to integer order and to the discrete series representations. We compute the full image of the induced space into the ‘Kirillov space’ and find two invariant subspaces. We use the Fréchet topology to show that the invariant subspaces that we find are precisely the smooth spaces of the various discrete series representations of \( \text{SL}(2, \mathbb{R}) \). In Section 8, we describe the Kirillov representation of \( \text{GL}(2, \mathbb{R}) \).

### 2. A Schwartz space for the classical Hankel transform

In this section, we describe the Schwartz space [1,5,22] for the Hankel transform and show that this space is invariant under the Hankel transform.

Assume that \( \Re(\nu) > -1 \) and \( x > 0 \), and let \( J_\nu(x) \) be the classical J-Bessel function defined by

\[
J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}.
\]

Let \( f \) be a complex-valued function on \([0, \infty)\). Define the Hankel transform of order \( \nu \) by

\[
\mathcal{H}_\nu(f)(y) = \int_0^\infty f(x) \sqrt{xy} J_\nu(2\sqrt{xy}) \frac{dx}{x}.
\]

It is possible by a simple change of variable to go from this Hankel transform to the ‘classical’ Hankel transform used in [20]. Let \( S([0, \infty)) \) be the Schwartz space of functions on \([0, \infty)\). That
is, $f : [0, \infty) \to \mathbb{C}$ is in $S([0, \infty))$ if $f$ is smooth on $[0, \infty)$ and $f$ and all its derivatives are rapidly decreasing at $\infty$. Let

$$S_\nu([0, \infty)) = \{ f : [0, \infty) \to \mathbb{C} | f(x) = x^{1/2+\nu/2}f_1(x) \text{ and } f_1 \in S([0, \infty)) \}.$$  

Since $J_\nu(z)$ is of order $z^\nu$ when $z$ is near zero, it follows that the integral defining $\mathcal{H}_\nu(f)$ above converges absolutely when $f \in S_\nu([0, \infty))$ and $\text{Re}(\nu) > -1$. It follows from [5] that when $\nu$ is real, $\nu > -1$, $H_\nu$ is a linear isomorphism of $S_\nu([0, \infty))$ satisfying $H_\nu^2 = \text{Id}$. Moreover, $H_\nu$ is an $L_2([0, \infty), dx/x)$ isometry. We will give a simple proof that $H_\nu$ is a linear isomorphism of $S_\nu([0, \infty))$ satisfying $H_\nu^2 = \text{Id}$ when $\text{Re}(\nu) > -1$. We start by showing that $H_\nu$ preserves $S_\nu([0, \infty))$.

Let $D$ be the differential operator

$$D = x \frac{d}{dx}.$$  

**Proposition 2.1** Assume that $\text{Re}(\nu) > -1$. Then,

(a) $D$ maps $S_\nu([0, \infty))$ into $S_\nu([0, \infty))$,  
(b) $\mathcal{H}_\nu(f)(y)$ is rapidly decreasing in $y$,  
(c) $\mathcal{H}_\nu(Df)(y) = -D(\mathcal{H}_\nu(f))(y)$ and  
(d) $y^{-1/2-\nu/2} \mathcal{H}_\nu(f)(y)$ is smooth on $[0, \infty)$.

**Proof** (a) is immediate. (b)–(d) are standard using differentiation under the integral, integration by parts and the formula: $(d/dx)(y^{-1/2}x^{(\nu+1)/2}J_{\nu+1}(2\sqrt{xy})) = x^{\nu/2}J_{\nu}(2\sqrt{xy})$, which follows from [13, (5.3.5)].  

**Corollary 2.2** Assume that $\text{Re}(\nu) > -1$. Then, $\mathcal{H}_\nu$ maps $S_\nu([0, \infty))$ into itself.

**Proof** Let $f \in S_\nu([0, \infty))$. It follows from (d) that $\mathcal{H}_\nu(f)(y)$ is smooth on $(0, \infty)$ and from (a), (b) and (c) that $D^n(\mathcal{H}_\nu(f))(y)$ is rapidly decreasing for $n = 0, 1, 2, \ldots$. It follows immediately that $\mathcal{H}_\nu(f)(y)$ and all its derivatives are rapidly decreasing. By (d), $g(y) = y^{-1/2-\nu/2} \mathcal{H}_\nu(f)(y)$ is smooth on $[0, \infty)$; hence, we can write $\mathcal{H}_\nu(f)(y) = y^{1/2+\nu/2}g(y)$ for $g \in S([0, \infty))$.  

3. The inversion formula for the Hankel transform

We now turn to prove the self-reciprocity of the Hankel transform. The crucial idea is to define an operator (‘intertwining operator’) $T_\nu$ from the Schwartz space above to another space of smooth functions (but not Schwartz in the usual sense), which we will call $I_\nu$, and to compute the composition of the Hankel transform with this operator. We will find an operator $\mathcal{W}_\nu$ on $I_\nu$ so that the following diagram commutes:

$$\begin{array}{ccc}
S_\nu & \xrightarrow{\mathcal{H}_\nu} & S_\nu \\
T_\nu \downarrow & & \downarrow T_\nu \\
I_\nu & \xrightarrow{\mathcal{W}_\nu} & I_\nu
\end{array}$$

We now define the above operators. Let $f \in S_\nu([0, \infty))$. Extend $f$ to $(-\infty, \infty)$ by setting it to be zero on the negative reals. We denote this extension again by $f$. Assume $\text{Re}(\nu) > -1$. Let
Let $f \in S_v([0, \infty))$, then $T_v \circ \mathcal{H}_v(f) = \mathcal{W}_v \circ T_v(f)$.

Note that this theorem is precisely the statement that the above diagram is commutative. We regard it as our main theorem since it allows us to move from the complicated Hankel transform $\mathcal{H}_v$ to the simple operator $\mathcal{W}_v$.

**Proof** The proof is based on the Weber integral [13, p. 132]:

$$
\int_0^\infty u^{\nu+1} e^{-au^2} J_\nu(\beta u) \, du = \frac{\beta^\mu}{(2\alpha)^{\mu+1}} \frac{e^{-\beta^2/4\alpha}}{\sqrt{\pi} \Gamma(\mu+1)},
$$

where $\text{Re}(\mu) > -1$, $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$. Assume that $\text{Re}(\nu) > -1$. It follows from the dominated convergence theorem that if $f \in S_v([0, \infty))$, then

$$
T_v(f)(z) = \lim_{\epsilon \to 0^+} (2\pi)^{-1/2} \int_0^\infty y^{-1/2+\nu/2} f(y) e^{iyz} e^{-\epsilon y} \, dy.
$$

Hence,

$$
T_v \circ \mathcal{H}_v(f)(z) = \lim_{\epsilon \to 0^+} (2\pi)^{-1/2} \int_0^\infty y^{-1/2+\nu/2} e^{iyz} e^{-\epsilon y} \int_0^\infty f(x) \sqrt{xy} J_\nu(2\sqrt{xy}) \frac{dx}{x} \, dy
$$

$$
= \lim_{\epsilon \to 0^+} (2\pi)^{-1/2} \int_0^\infty f(x) x^{-1/2} \int_0^\infty y^{\nu/2} e^{-(iz+\epsilon)y} J_\nu(2\sqrt{xy}) \, dy \, dx
$$

$$
= \lim_{\epsilon \to 0^+} 2(2\pi)^{-1/2} \int_0^\infty f(x) x^{-1/2} \left( \int_0^\infty u^{\nu+1} e^{-(iz+\epsilon)u^2} J_\nu(2\sqrt{ux}) \, du \right) \, dx
$$

$$
= (3.2) \lim_{\epsilon \to 0^+} (2\pi)^{-1/2} \int_0^\infty f(x) x^{-1/2+\nu/2} (-iz+\epsilon)^{-\nu-1} e^{-x/(\epsilon-iz)} \, dx
$$

$$
= (2\pi)^{-1/2} |z|^{-\nu-1} e^{\text{sgn}(z)\pi i(\nu+1)/2} \int_0^\infty f(x) x^{-1/2+\nu/2} e^{ix/-z} \, dx
$$

$$
= |z|^{-\nu-1} e^{\text{sgn}(z)\pi i(\nu+1)/2} T_v(f) \left( -\frac{1}{z} \right).
$$

As a corollary, we obtain the inversion formula of the Hankel transform on the Schwartz space. For the inversion formula on a bigger space, see [20].
**Theorem 3.2**  Assume $\operatorname{Re}(\nu) > -1$ and $f \in S_\nu([0, \infty))$. Then,
\[ \mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = f. \]

**Proof**  This follows immediately from Theorem 3.1 and the self-reciprocity of $\mathcal{W}_\nu$,
\[ \mathcal{W}_\nu \circ \mathcal{W}_\nu = \text{Id}, \]
which is easy to check. The argument is as follows. Let $f \in S_\nu([0, \infty))$. Then,
\[ T_\nu \circ \mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ T_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ \mathcal{W}_\nu \circ T_\nu(f) = T_\nu(f). \]
Since $T_\nu$ is one to one, it follows that $\mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = f$. □

We can also define the space $I_\nu$ (which we think of as an induced representation space).
\[ I_\nu = \{ \phi : \mathbb{R} \to \mathbb{C} \mid \phi \text{ is smooth on } \mathbb{R} \text{ and } \mathcal{W}_\nu(\phi) \text{ is smooth on } \mathbb{R} \}. \tag{3.4} \]

**Proposition 3.3**  $T_\nu$ maps $S_\nu([0, \infty))$ into $I_\nu$.

**Proof**  Let $f \in S_\nu([0, \infty))$. We will show that $T_\nu(f) = \phi \in I_\nu$. Let $g(x) = x^{-1/2} + x^{\nu/2}f(x)$. Then, $T_\nu(f) = \tilde{g}$. Since $x^ng(x)$ is absolutely integrable for every positive integer $n$, it follows from standard Fourier analysis that $\phi = \tilde{g}$ is smooth. Now apply $T_\nu$ to $\mathcal{H}_\nu(f)$. Since $\mathcal{H}_\nu(f) \in S_\nu([0, \infty))$, it follows from the same argument that $T_\nu(\mathcal{H}_\nu(f))$ is smooth. But from Theorem 3.1, this is precisely $\mathcal{W}_\nu(T_\nu(f)) = \mathcal{W}_\nu(\phi)$. □

### 3.1. $L^2$ isometry of the real Hankel transform

The $L^2$ isometry of the Hankel transform is well known and follows from the Plancherel formula for the Hankel transform [1].

Let $\nu$ be real and $\nu > -1$. For $f_1, f_2 \in S_\nu([0, \infty))$, we let
\[ \langle f_1, f_2 \rangle = \int_0^\infty f_1(x)\overline{f_2(x)} \frac{dx}{x}. \]

**Proposition 3.4**  Assume $\nu$ is real and $\nu > -1$. Let $f_1, f_2 \in S_\nu([0, \infty))$. Then,
\[ \langle \mathcal{H}_\nu(f_1), \mathcal{H}_\nu(f_2) \rangle = \langle f_1, f_2 \rangle. \]

The proof is immediate using Fubini’s theorem and the fact that $J_\nu(x)$ takes real values when $\nu$ and $x$ are real.

### 4. The induced space for the discrete series

In this section, we define an action of the group $G = \text{SL}(2, \mathbb{R})$ on the space $I_d$, where $d$ is a positive integer. This is the smooth part of the full induced space of the discrete series representation. Using the results described in Section 1, we will compute the image of this space under the inverse of the intertwining map $T_d$ defined in (3.1). The image gives us a ‘Kirillov space’ for the full induced space. It is well known that the full induced space is reducible and has two closed invariant irreducible subspaces. We will later show that the image of one of them is the space $S_d([0, \infty))$. In this section, we show that the image of $S_d([0, \infty))$ under the operator $T_d$ is invariant.
Let $G = \text{SL}(2, \mathbb{R})$. Let $N$ and $A$ be the subgroups of $G$ defined by
\[
N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \quad A = \left\{ s(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{R}^* \right\}.
\]

Let
\[
w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad r(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.
\]

4.1. **The asymptotic description of the space $I_d$**

Let $d$ be a positive integer. The space $I_d$ is the space of smooth functions $\phi : \mathbb{R} \to \mathbb{C}$ such that
\[
W_d(\phi)(x) = (i)^{d+1} x^{-d-1} \phi\left(\frac{1}{x}\right)
\]
is smooth. This is also the space of smooth functions $\phi$ such that $x^{-d-1} \phi(-1/x)$ is smooth. The following property of this space is immediate.

**Lemma 4.1** The differential operators $d/dx$ and $D = x(d/dx)$ map $I_d$ into itself.

We shall now give another description of $I_d$.

4.1.1. **Asymptotic expansions**

We recall some results from the theory of asymptotic expansions. For the proofs, see [4,6,16]. Let $\phi : \mathbb{R} \to \mathbb{C}$. Let $a_0, a_1, \ldots$ be complex numbers. We say that $\phi$ has the asymptotic expansion
\[
\phi(x) \approx \sum_{m=0}^{\infty} a_m x^{-m}
\]
at infinity if $\phi(x) - \sum_{m=0}^{N} a_m x^{-m} = O(|x|^{-N-1})$ when $|x| \to \infty$ for all non-negative integers $N$ where the implied constant is dependent on $N$. (Note that we have grouped together $\infty$ and $-\infty$, which is not needed in a more general definition.) It is easy to see that the constants $a_m$ are determined uniquely by $\phi$ although $\phi$ is not determined uniquely by $\{a_m\}$. (For example, $\phi(x) + e^{-|x|}$ will have the same asymptotic expansion as $\phi$.) The following results are well known [4].

**Lemma 4.2** Let $N$ be a non-negative integer and $c$ be a real constant. Then, $\phi(x) = (x + c)^{-N}$ has an asymptotic expansion of the form
\[
(x + c)^{-N} \approx x^{-N} - Ncx^{-N-1} + \ldots
\]

**Corollary 4.3** If $\phi(x)$ has an asymptotic expansion, then $\phi(x + c)$ has an asymptotic expansion for every real constant $c$.

**Proposition 4.4** Assume that $\phi$ is differentiable and both $\phi$ and $\phi'$ have asymptotic expansions. Then, the asymptotic expansion of $\phi'$ is the derivative term by term of the asymptotic expansion of $\phi$. That is,
\[
\phi'(x) \approx \sum_{m=0}^{\infty} -ma_m x^{-m-1}.
\]
The following theorem follows from [4, Theorem 3 and (14)].

**Theorem 4.5** Assume that $\phi$ is smooth on $\mathbb{R}$, and let $\gamma(x) = \phi(1/x)$. Then, $\phi$ and all its derivatives have asymptotic expansions on $\mathbb{R}$ if and only if $\gamma$ is smooth at $x = 0$ (hence if and only if $\gamma$ is smooth on $\mathbb{R}$).

**Proposition 4.6** The space $I_d$ is the space of smooth functions $\phi$ such that $\phi$ and all its derivatives have asymptotic expansions and such that the asymptotic expansion for $\phi$ is of the form

$$\phi(x) \approx a_{d+1} x^{-d-1} + \cdots.$$  \hfill (4.1)

**Proof** Let $\phi \in I_d$, and let $\gamma(x) = x^{-d-1} \phi(-1/x)$. Since $\gamma$ is smooth and since $\phi(x) = x^{-d-1}\gamma(-1/x)$ is smooth, it follows from Theorem 4.5 that $\phi$ has the required asymptotic expansion. Now assume that $\phi$ (and its derivatives) has the asymptotic expansion (4.1). Then, $\alpha(x) = x^{d+1} \phi(x)$ is smooth and has an asymptotic expansion. It follows from Theorem 4.5 that $\gamma(x) = \alpha(-1/x) = x^{-d-1} \phi(-1/x)$ is smooth and hence $\phi \in I_d$.

We now define a representation $\pi_d$ of $G = SL(2, \mathbb{R})$ on $I_d$ in the following way: let $\phi \in I_d$.

$$(w\phi)(x) = (i)^{d+1} W_d(x) \phi(x) = x^{-d-1} \phi \left( \frac{1}{x} \right), \quad (4.2)$$

$$(n(y)\phi)(x) = \phi(x + y), \quad (4.3)$$

$$(s(z)\phi)(x) = z^{-d-1} \phi(z^{-2} x). \quad (4.4)$$

The fact that $\phi(x+y) \in I_d$ follows from Corollary 4.3. It is easy to see that these maps give isomorphisms of $I_d$. More generally, we have

$$\left( \pi_d \begin{pmatrix} p & q \\ r & s \end{pmatrix} \phi \right)(x) = (rx + p)^{-d-1} \phi \left( \frac{sx + q}{rx + p} \right), \quad (4.5)$$

where the above matrix is in $G$. It is easy to check that this gives a representation of $G$, that is, $\pi_d(g_1 g_2) = \pi_d(g_1) \pi_d(g_2)$ for all $g_1, g_2 \in G$.

We shall now define an ‘intertwining operator’ $M_d$ on $I_d$ and compute its image. This operator is the inverse of the operator $T_d$ and will take us back to the ‘Kirillov model’. For $\phi \in I_d$, we define

$$M_d(\phi)(y) = |y|^{-(d+1)/2} \hat{\phi}(y) = |y|^{-(d+1)/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x) e^{-i y x} \, dx.$$  

We view $M_d(\phi)(y)$ as a function on $\mathbb{R} - \{0\}$. Its behaviour at zero is central to the description of this mapping. Let $\tilde{S}_d = M_d(I_d)$.

**Lemma 4.7** Let $f \in \tilde{S}_d$. Then, $f$ is differentiable for every $y \neq 0$. The operator $D = y(d/dy)$ maps $\tilde{S}_d$ into itself.

**Proof** We first assume that $d > 1$. The case $d = 1$ will be proved later. Let $\phi \in I_d$. Then, $x \phi(x) \in L^1(\mathbb{R})$ and hence $\hat{\phi}$ is differentiable. We have $(D\phi)^\vee(y) = -\hat{\phi}(y) - D\hat{\phi}(y)$ and

$$M_d(D(\phi))(y) = |y|^{-(d+1)/2} D\hat{\phi}(y) = -|y|^{-(d+1)/2} \hat{\phi}(y) - |y|^{-(d+1)/2} D\hat{\phi}(y).$$

Since $D\phi \in I_d$, it follows that $|y|^{-(d+1)/2} D\hat{\phi}(y) \in \tilde{S}_d$. Now $D(|y|^{-(d+1)/2} \hat{\phi}(y)) = ((-d + 1)/2) |y|^{-(d+1)/2} \hat{\phi}(y) + |y|^{-(d+1)/2} D\hat{\phi}(y)$. Since both summands belong to $\tilde{S}_d$, we get our conclusion. \hfill $\blacksquare$
Corollary 4.8  If \( f \in \tilde{S}_d \), then \( f \) is smooth at every \( y \neq 0 \).

Lemma 4.9  Let \( f \in \tilde{S}_d \). Then, \( f \) is a Schwartz function, that is, \( f \) and all its derivatives are rapidly decreasing at \( \infty \) and \( -\infty \).

Proof  Let \( \phi \in I_d \) be such that \( f = M_d(\phi) \). Since \( \phi \) and all its derivatives are in \( L^1(\mathbb{R}) \), it follows that \( \tilde{\phi} \) is rapidly decreasing; hence, \( f(y) = |y|^{-(d+1)/2} \tilde{\phi}(y) \) is rapidly decreasing. Since \( D(f) \in \tilde{S}_d \), it follows that \( D(f) \) is rapidly decreasing; hence, \( f' \) is rapidly decreasing. Using induction and the fact that \( D^m(f) \) is rapidly decreasing, we get that \( f^{(m)} \) is rapidly decreasing.

We shall now describe the behaviour at zero of the functions in our space \( \tilde{S}_d \). We will also complete the case \( d = 1 \) that was left out in the proof of Lemma 4.7. We let \( \phi_0(x) = (1 + x^2)^{-(d-1)/2} \) and \( \phi_j(x) = \phi_0^{(j)} \), the \( j \)th derivative of \( \phi_0 \). It is easy to check that \( \phi_0 \in I_d \); hence by Lemma 4.1, \( \phi_j \in I_d \) for every positive integer \( i \). We shall compute the functions \( f_j = (\tilde{\phi}_j) \). From [9, 3.771 (2)], it follows that

\[
\begin{align*}
  f_0(y) &= \frac{2^{(d+1)/2}}{\Gamma((d-1)/2)} |y|^{d/2} K_{d/2}(|y|), \\
  f_j(y) &= (-i)^j y^j f_0(y) = \frac{2^{(d+1)/2}}{\Gamma((d-1)/2)} (-i)^j y^j |y|^{1/2} K_{d/2}(|y|).
\end{align*}
\]

Lemma 4.10  \( f_j(y) \) are smooth on \( y \neq 0 \). They are smooth on the right and on the left at \( y = 0 \). \( f_0(y) \) is continuous at \( y = 0 \). \( f_j(y) \) is \( j \) times differentiable at \( x = 0 \) and satisfies \( f_j^{(r)}(0) = 0 \) for \( r = 0, 1, \ldots, j - 1 \) and \( f_j^{(j)}(0) \neq 0 \).

Proof  \( f_0(0) \neq 0 \) because it is an integral of a positive function. It follows that \( f_j^{(j)}(0) \neq 0 \).

Proof of Lemma 4.7 for \( d = 1 \)  We need to prove that \( M_1(\phi)(y) \) is differentiable at \( y \neq 0 \). The rest of the proof will follow the same lines as in the proof of Lemma 4.7. Let \( \phi \in I_1 \). Then, \( \phi \) has an asymptotic expansion of the form \( \phi(x) \approx \alpha_2 x^{-2} + \cdots \). Let \( \alpha(x) = \phi(x) - \alpha_2 x_0(x) = \phi(x) - \alpha_2 (1 + x^2)^{-1} \). Then, \( \alpha(x) \) and \( \alpha(x) \) are in \( L^1 \); hence, \( \alpha \) is differentiable. Now \( \phi(y) = \alpha(y) + a_2 \phi_0(y) = \hat{\alpha}(y) + 2a_2 e^{-|y|} \) is differentiable. The same argument will be used in the proof of the following corollary.

Corollary 4.11  Let \( f \in \tilde{S}_d \). Then, \( |y|^{-(d+1)/2} f(y) \) is smooth on the right and on the left at \( y = 0 \).

Proof  There exist \( \phi \in I_d \) such that \( |y|^{(d-1)/2} \phi(y) \) is smooth on the left and on the right at \( y = 0 \). For each integer \( m > 0 \), we can write \( \phi = \alpha_m + \sum_{j=0}^{m} a_{j+d-1} \phi_j \) so that \( \alpha_m \) satisfies that \( x^d \alpha_m(x) \in L^1(\mathbb{R}) \) for \( r = 0, 1, \ldots, m + d - 1 \). It follows that \( \alpha_m \) is \( m + d - 1 \) times differentiable on \( \mathbb{R} \). But \( \phi = \hat{\alpha}_m + \sum_{j=0}^{m} a_{j+d-1} \phi_j \) and since \( f_j \) is smooth on the right and on the left at \( y = 0 \), it follows that \( \phi \) is \( m + d - 1 \) smooth on the right and on the left at \( y = 0 \). Since this is true for all \( m \), we get the conclusion.

Lemma 4.12  Let \( f \in \tilde{S}_d \). Then, \( |y|^{(d-1)/2} f(y) \) has continuous \( d - 1 \) derivatives at \( y = 0 \).

Proof  There exist \( \phi \in I_d \) such that \( |y|^{(d-1)/2} \phi(y) \) is smooth. From the asymptotics of \( \phi \), it follows that \( \phi(x), x\phi(x), \ldots, x^{d-1} \phi(x) \) are all in \( L^1(\mathbb{R}) \). The conclusion follows.
We will show that Lemma 4.9, Corollary 4.11 and Lemma 4.12 give a complete description of \( S_d \). To do that, we define the inverse map \( T_d \) by \( T_d(f) = (|y|^{-1/2} f)^+ \). That is,

\[
T_d(f)(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |y|^{(d-1)/2} f(y) e^{iyz} dy.
\]

Note that if \( f \) vanishes on \( (-\infty, 0) \), then this is the same as \( T_d \) that was defined in (3.1).

**Proposition 4.13** Let \( f \) be such that \(|y|^{-(d-1)/2} f(y)\) is smooth on \([0, \infty)\) and on \((-\infty, 0)\) (but not necessarily smooth at \( y = 0 \)) and such that \( f \) and all its derivatives are rapidly decreasing. Then, \( f \in S_d \).

**Proof** It is sufficient to show that \( T_d(f) \in I_d \) since \( f = M_d(T_d(f)) \) and hence is in \( S_d \). To do that, we write \( f = f_+ + f_- \), where \( f_+ \) vanishes on \((0, -\infty)\) and is smooth on \([0, \infty)\) and \( f_- \) vanishes on \((0, \infty)\) and is smooth on \((-\infty, 0)\). (Note that \( f(0) = 0 \).) Now \( f_+ \in S_d([0, \infty)) \) and, by Proposition 3.3, \( T_d(f_+ ) \in I_d \). Arguments similar to those in the proof of Proposition 3.3 will show that \( T_d(f_-) \in I_d \) and hence \( T_d(f) \in I_d \).

**Theorem 4.14** \( \tilde{S}_d \) is the set of smooth functions \( f \) on \( \mathbb{R} - \{0\} \) such that \( f \) and all its derivatives are rapidly decreasing at \( \pm \infty \) and such that \(|y|^{(d-1)/2} f(y)\) is smooth on the right and on the left at \( y = 0 \) and has \( d - 1 \) continuous derivatives at \( y = 0 \). (When \( d = 1 \), it means that \(|y|^{d-1} f(y) = f(y)\) is continuous.)

**Proof** By Lemma 4.9, Corollary 4.11 and Lemma 4.12, \( \tilde{S}_d \) is contained in the space of functions satisfying the above conditions. Now assume that \( f \) is as described above and we will show that it is in \( \tilde{S}_d \), that is, it is the image under \( M_d \) of a function in \( I_d \). Let \( \tilde{f} = |y|^{(d-1)/2} f(y) \). By the properties of \( \{f_j\} \) in Lemma 4.10, we can find it using the triangulation argument constants \( c_0, \ldots, c_{d-1} \) so that

\[
\tilde{h} = \tilde{f} - \sum_{j=0}^{d-1} c_j \tilde{f}_j
\]

vanishes at zero and all its first \( d - 1 \) derivatives vanish at zero. It follows that \( h(y) = |y|^{-(d+1)/2} \tilde{h} \) satisfies the conditions of Proposition 4.13 and hence is in \( \tilde{S}_d \). Hence, \( f = h + \sum_{j=0}^{d-1} c_j f_j \) is in \( \tilde{S}_d \).

Since \( I_d \) is isomorphic to \( \tilde{S}_d \) under the isomorphism \( M_d \), it follows that the action of \( G \) on \( I_d \) induces an action of \( G \) on \( \tilde{S}_d \). Let \( I_d^+ \) be the subspace of \( I_d \), which is given by

\[
I_d^+ = T_d(S_d([0, \infty))). \tag{4.8}
\]

By Corollary 2.2, \( \mathcal{H}_d \) preserves the space \( S_d([0, \infty)) \). Hence, by Theorem 4.14, \( w \) preserves the space \( I_d^+ \). It is easy to see using (4.3) and (4.4) that the induced action of \( n(y) \) and \( s(z) \) on \( \tilde{S}_d \) stabilizes the space \( S_d([0, \infty)) \). Hence, it follows that \( n(y) \) and \( s(z) \) stabilize the space \( I_d^+ \). Since \( w, n(y) \) and \( s(z) \) generate \( G \) (when \( y \) and \( z \) vary), we get the following corollary.

**Corollary 4.15** \( I_d^+ \) is a \( G \) invariant subspace of \( I_d \).

Let

\[
S_d((-\infty, 0)) = \{ f : (-\infty, 0], \rightarrow \mathbb{C}, f = x^{(d+1)/2} f_j(x) \text{ and } f_j \in S((-\infty, 0]) \}. \tag{4.9}
\]
For \( y \in (-\infty, 0] \) and \( f \in S_d(( -\infty, 0]) \), we define

\[
\mathcal{H}_d(f)(y) = \int_{-\infty}^0 f(x) \sqrt{\frac{J_d(2 \sqrt{|xy|})}{|x|}} \, dx.
\]

It is clear that \( S_d(( -\infty, 0]) \) is a subspace of \( \tilde{S}_d \). It follows from Corollary 2.2 that \( \mathcal{H}_d \) stabilizes \( S_d(( -\infty, 0]) \) and that the induced action of \( n(y) \) and \( s(z) \) given by

\[
(n(y)f)(x) = e^{iyx}f(x), \quad (s(z)f)(x) = f(z^2x)
\]

also stabilizes \( S_d(( -\infty, 0]) \). Define

\[
I^+_d = T_d(S_d(( -\infty, 0])).
\]

Then, \( I^-_d \) is a \( G \) invariant subspace of \( I_d \). We show in Section 6 that \( I^+_d \) is a closed subspace under the Fréchet topology on \( I_d \) and hence (from the theory of \( (g, K) \) modules) \( I^+_d \oplus I^-_d \) is the unique maximal closed invariant subspace of \( I_d \).

5. The Kirillov model of the discrete series

The Kirillov model is a particular realization of certain representations of \( \text{SL}(2, \mathbb{R}) \) or \( \text{GL}(2, \mathbb{R}) \) (or, more generally, representations of \( \text{GL}(n, F) \), where \( F \) is a local field) with a prescribed action of the Borel subgroup. In this section, we describe the Kirillov model of the discrete series representations of \( G = \text{SL}(2, \mathbb{R}) \). Our main theorem of this section and this paper is a description of the smooth space of the Kirillov model. This theorem will be stated in this section and proved in the following two sections.

5.1. The action of the group in the Kirillov model

Let \( d \) be a positive integer. Using the map \( T_d \) and its inverse \( M_d \), we can move the action of \( G \) from the space \( I^+_d \) to the space \( S_d \). We shall denote this action by \( R^+_d \). That is, if \( g \in G \), then the action of \( g \) on \( f \in S_d \) is given by

\[
R^+_d(g)(f) = M_d(gT_d(f)).
\]

Thus, we obtain the following formulas. For each \( f \in S_d([0, \infty)) \), we have

\[
\begin{align*}
(R^+_d(n(y))f)(x) &= e^{iyx}f(x), \\
(R^+_d(s(z))f)(x) &= f(z^2x), \\
(R^+_d(w)f)(x) &= i^{-(d+1)}\mathcal{H}_d(f)(x).
\end{align*}
\]

Let

\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

be a basis for the Lie algebra. The action of the Lie algebra on the above Kirillov model is given by

\[
(R^+_d(X)f)(x) = ixf(x)(R^+_d(H)f)(x) = 2xf'(x)(R^+_d(Y)f)(x) = -i\frac{d^2 - 1}{4x}f(x) + ixf''(x).
\]
Another (non-isomorphic) action can be obtained by considering the action of $G$ on $S_d((−∞, 0])$ (see (4.9) and the remark below it) and moving this action to $S_d([0, ∞)).$ The action is given by

$$\left( R_d^−(n(y))f \right)(x) = e^{-ixy}(R_d^−(s(z))f)(x) = f(x^2)(R_d^−(w)f)(x) = id + H_d(f)(x).$$ (5.5)

It is easy to see that the action of $n(y)$ and $s(z)$ extends to the space $L^2((0, ∞), dx/x).$ It follows from Corollary 3.3 that the action of $w$ also extends to $L^2((0, ∞), dx/x).$ We denote again by $R_d^±$ the representation on the space $H_d = L^2((0, ∞), dx/x)$ obtained in such a way. This is called the Kirillov Hilbert representation of $G.$

**Proposition 5.1** $R_d^±$ is a strongly continuous unitary representation on $L^2((0, ∞), dx/x).$

*Proof* It is sufficient to show that the map $g \rightarrow R_d^±(g)f$ is continuous at $g = w$ for every $f \in L^2((0, ∞), dx/x)$ (see [12, p. 11]). This is easy to show for a characteristic function of an interval using the formulas in (5.1) and by approximation for a general function. ■

**Proposition 5.2** $R_d^±$ is an irreducible representation.

*Proof* This representation is already irreducible when restricted to the Borel subgroup (see [12, Proof of Proposition 2.6]). ■

Our main theorem of this section is as follows.

**Theorem 5.3** The space of smooth vectors of $H_d = L^2((0, ∞), dx/x)$ under the action of $R_d^±$ is $S_d([0, ∞)).$ That is, $\mathcal{H}_\infty^d = S_d([0, ∞)).$

To prove Theorem 5.3, we will need to compare the Fréchet topologies on the induced space and on the Kirillov space.

6. The Fréchet topology

In this section, we describe the Fréchet topology on the different smooth spaces that we are considering. This topology will play a role in determining the smooth space of the Kirillov model.

Let $G = \text{SL}(2, \mathbb{R}).$ Let $d$ be a positive integer and let

$$\text{Ind}_d = \{ F : G \rightarrow \mathbb{C} | F \text{ is smooth and } F(n(x)s(a)g) = a^{d+1}F(g), \text{ for all } x \in \mathbb{R}, a \in \mathbb{R}^* \}. \quad (6.1)$$

$G$ acts on this space by right translations and $\mathfrak{g} = \text{Lie}(G)$ acts by left invariant differential operators and induces an action of the enveloping algebra $U(\mathfrak{g})$ on $\text{Ind}_d.$ There is an $L^2$ norm defined on this space by

$$||F||^2 = \frac{1}{2\pi} \int_0^{2\pi} |F(r(\theta))|^2 d\theta. \quad (6.2)$$

The space $\text{Ind}_d$ is given by the Fréchet topology defined by the seminorms

$$||F||_D = ||D(F)||, \quad D \in U(\mathfrak{g}).$$
We will now show that the space $\text{Ind}_d$ is isomorphic as a vector space to the space $I_d$ defined in (3.4). For each $F \in \text{Ind}_d$, define

$$\phi_F(x) = F(wn(x)). \quad (6.3)$$

We have

$$wn(x) = s((1 + x^2)^{-1/2})n(-x)r(\theta_x), \quad (6.4)$$

where $\theta_x \in (0, \pi)$ is determined by the equalities $\sin(\theta_x) = (1 + x^2)^{-1/2}$ and $\cos(\theta_x) = -x(1 + x^2)^{-1/2}$. From this, we get the following well-known proposition.

**Proposition 6.1** The mapping $F \to \phi_F$ is an isomorphism of vector spaces.

**Proof** Given $\phi \in I_d$, we define

$$F_{\phi}(r(\theta)) = (1 + x^2)^{(d+1)/2} \phi(x) \quad (6.5)$$

for $\theta \in (0, \pi)$ and define

$$F_{\phi}(r(\pi)) = F_{\phi}(-I) = \lim_{x \to \infty} (1 + x^2)^{(d+1)/2} \phi(x) = (w\phi)(0).$$

$F_{\phi}$ is defined on all of $G$ via the equivariance property

$$F_{\phi}(n(y)s(a)r(\theta)) = \text{sgn}(a)(d+1)F_{\phi}(r(\theta)) \quad (6.6)$$

and the fact that each $g \in G$ can be written uniquely in the form $g = n(y)s(a)r(\theta)$ for $\theta \in (0, \pi]$.

It is easy to check that $\phi \to F_{\phi}$ is the inverse map to $F \to \phi_F$. ■

The isomorphism $F \to \phi_F$ defined above induces a Fréchet topology on the space $I_d$. To give this topology explicitly, we need the following.

**Proposition 6.2** Let $F \in \text{Ind}_d$ and $\phi_F : \mathbb{R} \to \mathbb{C}$ as defined above. Then,

$$||F||^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 + x^2)^d|\phi_F(x)|^2 dx.$$

**Proof** This is immediate using (6.3) and (6.4). ■

Hence, we define a Fréchet topology on $I_d$ using the seminorms

$$||\phi||_{d,D} = \int_{-\infty}^{\infty} (1 + x^2)^d|\phi(x)|^2 dx, \quad D \in U(g).$$

It is useful to replace the seminorms given in the Fréchet topology of $\text{Ind}_d$ with an equivalent set of seminorms which gives the topology of uniform convergence of functions and derivatives. That is, for $F \in \text{Ind}_d$, define $||F||_{\infty} = \max_{\theta} |F(\theta)|$ to be the $L$-infinity norm on $\text{Ind}_d$. It is well known [18, Theorem 1.8] that the set of seminorms

$$||F||_{\infty,D} = ||D(F)||_{\infty}, \quad D \in U(g),$$

is an equivalent set of seminorms to the above set and gives the same Fréchet topology. Moreover, let

$$\partial = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \partial(F)(r(\theta)) = \frac{\partial}{\partial \theta} F(r(\theta)).$$

Then, it is sufficient to have the seminorms

$$||F||_{\infty,n} = ||\partial^n(F)||_{\infty}.$$

Hence, we get the following corollary.
Corollary 6.3 The Fréchet topology on $I_d$ is given by the set of seminorms
\[ ||\phi||_{\infty,d,D} = |(1 + x^2)^{(d+1)/2}(D\phi)(x)|_\infty, \quad D \in U(g), \]
or by the subset of seminorms
\[ ||\phi||_{\infty,d,n} = |(1 + x^2)^{(d+1)/2}(\partial^n\phi)(x)|_\infty, \quad n = 0, 1, \ldots. \]

Proposition 6.4 The functional $l_{\lambda,n}$ on $I_d$ defined by
\[ l_{\lambda,n}(\phi) = \int_{-\infty}^{\infty} t^n \phi(t) e^{-i\lambda t} \, dt \]
is continuous for $n = 0, \ldots, d - 1$ and $\lambda \in \mathbb{R}$.

Proof Assume that $\phi_m \in I_d$ satisfy $\lim_{m \to \infty} \phi_m = 0$ in the Fréchet topology. Then, $||(1 + t^2)^{(d+1)/2}\phi_m)||_\infty \to 0$ and hence there exist a sequence of positive constants $c_m$ such that
\[ |\phi_m(t)| \leq c_m (1 + t^2)^{-(d+1)/2}, \quad \text{for all } t \in \mathbb{R}, \]
and $\lim_{m \to \infty} c_m = 0$. It follows that
\[ |l_{\lambda,n}(\phi_m)| \leq c_m \int_{-\infty}^{\infty} |t|^n (1 + t^2)^{-(d+1)/2} \, dt. \]
The above integral converges by the assumption on $n$ and hence we have that $\lim_{m \to \infty} l_{\lambda,n}(\phi_m) = 0$. 

Let $I^+_d$ and $I^-_d$ be the subspaces of $I_d$ defined in (4.8) and (4.11). Then, it is easy to see that
\[ I^+_d = \{ \phi \in I_d | l_{\lambda,n}(\phi) = 0, \quad \text{for all } 0 \leq n \leq d - 1 \text{ and } \lambda \leq 0 \} \]
and
\[ I^-_d = \{ \phi \in I_d | l_{\lambda,n}(\phi) = 0, \quad \text{for all } 0 \leq n \leq d - 1 \text{ and } \lambda \geq 0 \}. \]

Corollary 6.5 $I^+_d$ and $I^-_d$ are the closed invariant subspaces of $I_d$. They are irreducible as Fréchet representations.

Proof By Proposition 6.4, they are closed as the intersection of closed subspaces. By Corollary 4.15 and the discussion given below it, they are $G$ invariant subspaces. By the theory of $(g, K)$ modules [8, p. 2.8, Theorem 2], $I_d$ has exactly three non-trivial closed invariant subspaces: two irreducible (lowest weight and highest weight) subspaces and their sum. It follows that $I^+_d$ and $I^-_d$ are irreducible.

Our next goal is to show that $S_d([0, \infty))$ is a space of smooth vectors in $H_d = L^2([0, \infty), \, dx/\lambda)$. That is, our first step in showing that $H^\infty_d = S_d([0, \infty))$ is to show that $S_d([0, \infty)) \subseteq H^\infty_d$. To do that, we will need to compare the $L^2$ inner product on $S_d([0, \infty))$ with the standard (compact) inner product on $\text{Ind}_d$. Combining the isomorphism $T_d$ from $S_d$ to $I_d$ and the isomorphism $\phi \to F_\phi$ defined in (6.5) and (6.6) from $I_d$ to $\text{Ind}_d$, we get a one-to-one mapping from $S_d([0, \infty))$ to $\text{Ind}_d$ and a subspace of $\text{Ind}_d$. This infinite-dimensional subspace has two different inner products. The standard inner product on the induced space (6.2) and the inner product induced from the $L^2$
inner product on $S_d$. Following [8, (288)], we will give an explicit formula for this induced inner product and compare the two.

We shall define an invariant ‘norm’ on $\text{Ind}_d$ (it is in fact a norm on the invariant subspaces):

$$||F||^2_I = \frac{1}{2\pi} \int_0^{2\pi} A(F)(r(\theta))\overline{F}(r(\theta)) \, d\theta,$$

where $A(F)$ is the intertwining operator defined by

$$A(F)(g) = \int_{-\infty}^{\infty} f(w^{-1}n(x)g) \, dx, \quad g \in G, \ f \in \text{Ind}_d.$$

**Remark 6.6** Since $F(r(\theta - \pi)) = (-1)^d F(r(\theta))$ for every $\theta$ and since the same holds for $A(F)$, we have that

$$||F||_I = \frac{1}{\pi} \int_0^{\pi} A(F)(r(\theta))\overline{F}(r(\theta)) \, d\theta.$$

For $x \in \mathbb{R}$, we define $\theta_x \in (0, \pi)$ by $\theta_x = \cot^{-1}(-x)$ and we have $w_n(x) = s((1 + x^2)^{-1/2})n(-x)r(\theta_x)$ and $w^{-1}n(x) = s(1 + x^2)^{-1/2})n(-x)r(\theta_x)$. Hence, using the change of variable $\theta = \cot^{-1}(x)$, we have

$$A(F)(g) = \int_0^\pi F(r(\theta)g)(\sin(\theta))^{d-1} \, d\theta$$

and in particular $|A(F)(r(\theta))| \leq \pi ||F||_\infty$. Hence, we proved the following

**Lemma 6.7**

$$||F||_I \leq \pi ||F||_\infty.$$

**Proposition 6.8** Let $\phi \in \text{Ind}_d$. Then,

$$\frac{(-2\pi i)^d}{2((d - 1)!) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn}(y)y^{d-1} \phi(x - y)\overline{\phi(x)} \, dy \, dx = ||M(\phi)||^2_2,} \tag{6.7}$$

where $||M(\phi)||_2$ is the $L^2$ norm of $M(\phi)$ in the space $L^2((0, \infty), dx/x)$.

**Proof** The proof is the same as that in [8, pp. 1.62–1.64] using the Tate identity:

$$\int_{-\infty}^{\infty} \phi(y)\text{sgn}(y)y^{d-1} \, dy = \frac{2((d - 1)!)}{(-2\pi i)^d} \int_{-\infty}^{\infty} \phi(x)x^{-d} \, dx,$$

which is valid for every $\phi \in \text{Ind}_d$. 

For $f \in S_d$, we attach $F \in \text{Ind}_d$ by $F = F_{\text{Ind}_d}(f)$, where $F_\phi$ is defined in (6.5) and (6.6).

**Proposition 6.9** Let $f \in S_d$. Then,

$$||f||_2 = \frac{2((d - 1)!)}{(-2\pi i)^d} ||f||_I.$$
Proof

\[ ||F||_2^2 = \frac{1}{\pi} \int_0^{\pi} A(F)(r(\theta)) \overline{F(r(\theta))} \, d\theta \]

\[ = \frac{1}{\pi} \int_0^{\pi} A(F)(s(\sin(\theta))n(\cot(\theta))r(\theta)) \overline{F(s(\sin(\theta))n(\cot(\theta))r(\theta))} \, d\theta \]

\[ = \int_{-\infty}^{\infty} A(F)(\overline{wn(x)}) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(w^{-1}n(y)wn(x)) \overline{F(wn(x))} \, dy \, dx \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn}(y)y^{d-1} \phi_F(x-y)\overline{\phi_F(x)} \, dy \, dx = ||f||_2^2. \]

\[ \square \]

Corollary 6.10 Let \( f \) and \( F \) be as described above, then there exists a constant \( c \) (depending on \( d \) but not on \( f \)) such that \( ||f||_2 \leq c||F||_\infty \).

Corollary 6.11 \( S_d \) is a space of smooth functions in the Hilbert representation space \( \mathcal{H}_d \) of \( R_d \).

Proof By Theorem 1.8 given in [18], the smooth vectors in the Hilbert representation associated with \( \text{Ind}_d \) are the smooth functions on \( G \) satisfying (6.1) (i.e. the space \( \text{Ind}_d \) is the space of smooth vectors in the appropriate \( L^2 \) space). Hence, the map \( f \) to \( F \) given above is a \( G \) invariant map sending the space \( S_d([0, \infty)) \) to a space of smooth vectors in \( \text{Ind}_d \). Using Corollary 6.10 and the definition of smooth vectors, we get our result.

\[ \square \]

7. \((g, K)\) modules and Fréchet space isomorphism

In this section, we show that \( S_d([0, \infty)) \) is the smooth space of the representation \( R_d^\pm \) on the space \( \mathcal{H}_d = L^2((0, \infty), dx/x) \). To do that, we will show that the operator \( M_d \) from \( L^2_d \) to \( S_d([0, \infty)) \) is an isomorphism of the \((g, K)\) modules of \( K \)-finite vectors and also an isomorphism between the spaces of smooth vectors.

7.1. \( K \)-finite vectors

Using the map \( M_d \), we can find \( K \)-finite vectors in \( \mathcal{H}_d = L^2((0, \infty), dy/y) \).

Lemma 7.1 Every function of the form \( y^{(d+1)/2}p(y)e^{-y} \), where \( p(y) \) is a polynomial, is a \( K \)-finite vector in \( \mathcal{H}_d \).

Proof We let \( F_0 \in \text{Ind}_d \) be defined by \( F_0(r(\theta)) = e^{-((d+1)\theta/2)} \) and extended to \( G \) as in (6.6). Then, \( \phi(x) = \phi F_0(x) = (1 + x^2)^{-(d+1)/2}e^{(d+1)\tan^{-1}(x)} \), and by [9, 3.944 (5),6)], we have that \( M_d(\phi) = (\sqrt{2\pi/d!})y^{(d+1)/2}e^{-y} \). Now the \( n \)th derivative \( \phi(x) \) is also a \( K \)-finite vector since it is the application \( n \) times of the differential operator \( X \in g \). Since \( M_d(\phi^{(n)}) = \lambda y^n y^{(d+1)/2}e^{-y} \) for a non-zero constant \( \lambda \), it follows that \( p(y)y^{(d+1)/2}e^{-y} \) is a \( K \)-finite vector in \( \mathcal{H}_d \) for every polynomial \( p(y) \) and we prove the lemma.

\[ \square \]

The Laguerre orthogonal polynomials \( L_n^d(x) \) are defined by the formula

\[ L_n^d(x) = e^{x^{-d}} \frac{d^n}{dx^n} (e^{-x}x^{n+d}). \]
It is well known (see [13, (4.21.1) and 4.23, Theorem 3]) that for a fixed integer \( d \geq 0 \), the set of functions \( \phi_n(x) = (n!/(n + d)!)^{1/2} x^{d/2} L_d^n(x) \), \( n = 0, 1, \ldots \), is a complete orthonormal system for \( L^2((0, \infty), dx) \). Hence, we get the following.

**Proposition 7.2.** Let \( d \) be a positive integer. The set of functions \( e_n(x) = (n!/(n + d)!)^{1/2} x^{(d+1)/2} e^{-x} \), \( n = 0, 1, \ldots \), is a complete orthonormal system of \( K \)-finite vectors (in fact, \( K \) eigenfunctions) for \( \mathcal{H}_d = L^2((0, \infty), dx/x) \).

**Proof.** By Lemma 7.1, the functions \( e_n(x) \) are all \( K \)-finite. By the remark on the Laguerre polynomials, it follows that the set of functions \( x^{-1/2} e_n(x) \), \( n = 0, 1, \ldots \), is a complete orthonormal set for \( L^2((0, \infty), dx) \). If \( f(x) \in L^2((0, \infty), dx/x) \) is orthogonal to all functions \( e_n(x), n = 0, 1, \ldots \), then \( x^{-1/2} f(x) \in L^2((0, \infty), dx) \) is orthogonal to \( x^{-1/2} e_n(x), n = 0, 1, \ldots \), and hence is the zero function.

**Corollary 7.3.** The set of all \( K \)-finite vectors in \( \mathcal{H}_d \) is the set of functions \( x^{(d+1)/2} e^{-x} p(x) \), where \( p(x) \) is a polynomial.

**Proof.** Otherwise, we would be able to find a \( K \)-finite eigenfunction which is orthogonal to all the \( e_n \), which is a contradiction.

**Corollary 7.4.** The \((g, K)\) module of \( K \)-finite vectors in \( L^+_d \) is isomorphic to the \((g, K)\) module of \( K \)-finite vectors in \( \mathcal{H}_d \) and both are irreducible.

**Proof.** The operator \( T_d \) is a one-to-one intertwining operator between the \((g, K)\) module of the \( K \)-finite vectors in \( L^+_d \) and the \((g, K)\) module of the \( K \)-finite vectors in \( \mathcal{H}_d \). By the above Corollary, it is onto it.

Our main result of this paper is as follows.

**Theorem 7.5.** The space \( S_d((0, \infty)) \) is the space of smooth vectors in \( \mathcal{H}_d \), that is, \( S_d((0, \infty)) = \mathcal{H}_d^\infty \).

**Proof.** By Corollary 6.10, the operator \( M_d \) is a smooth intertwining operator between \( L^+_d \) and \( \mathcal{H}_d^\infty \) whose image is \( S_d((0, \infty)) \). By Corollary 7.4, \( M_d \) restricts to a \((g, K)\) isomorphism of the spaces of \( K \)-finite vectors. By a theorem of Casselman and Wallach [19, Theorem 11.6.7], there is a unique continuous extension of such an isomorphism to an isomorphism of the smooth spaces of each representation. It follows that \( M_d \) is that extension and that \( M_d \) is onto \( \mathcal{H}_d^\infty \) and hence \( S_d((0, \infty)) = \mathcal{H}_d^\infty \).

8. The Kirillov model for the discrete series representations of \( GL(2, \mathbb{R}) \)

In this section, we use our previous results to describe the Kirillov model and, in particular, the smooth space of the Kirillov model of the discrete series representations of \( GL(2, \mathbb{R}) \).

The discrete series representations of \( GL(2, \mathbb{R}) \) are parametrized by two real characters \( \chi_1(t) = |t|^{s_1} \text{sgn}(t)^{m_1} \) and \( \chi_2(t) = |t|^{s_2} \text{sgn}(t)^{m_2} \), where \( s_1, s_2 \in \mathbb{C} \) and \( m_1, m_2 \in \{0, 1\} \) are such that

\[
|t|^{s_1 - s_2} \text{sgn}(t)^{m_1 - m_2} = t^d \text{sgn}(t)
\]  

for some positive integer \( d \). The smooth space of the discrete series is a subspace of the induced representation \( \text{Ind}(\chi_1, \chi_2) \), which is given by the space of smooth functions
To obtain the smooth space of the Kirillov model, we define a mapping \( M \) the central character of this representation by \( \omega \). It is easy to see that the mapping \( F \to \phi_F \) in (6.3) gives an isomorphism between \( \text{Ind}(\chi_1, \chi_2) \) and \( I_d \), where \( d \) is the positive integer given by (8.1). Moreover, the induced action of the subgroup \( \text{SL}(2, \mathbb{R}) \) on \( I_d \) is given by (4.5). It follows from Corollary 6.5 and the theory of \((g, K)\) modules [8, p. 2.7, 2.8] that the space \( I_d^+ \oplus I_d^- \) is an irreducible closed subspace of \( I_d \) under the action of \( \text{GL}(2, \mathbb{R}) \).

To obtain the smooth space of the Kirillov model, we define a mapping \( M_{\chi_1, \chi_2} \) from \( I_d \) by

\[
M_{\chi_1, \chi_2}(\phi)(y) = |y|^{(d+1)/2} \text{sgn}(y)^{m_2} \hat{\phi}(y).
\]

**Remark 8.1** The mapping \( F \to M_{\chi_1, \chi_2}(\phi_F)(y) \) from the subspace of the induced representation \( \text{Ind}(\chi_1, \chi_2) \) is identical to the map

\[
F \to W_F \left( \begin{array}{cc} |y|^{1/2} \text{sgn}(y) & 0 \\ 0 & |y|^{-1/2} \end{array} \right)
\]

defined in [8, (75)], where \( W_F \) is the Whittaker function associated with \( F \). This space of functions can be called a ‘normalized’ Kirillov model. It is related to the ‘standard’ Kirillov model, \( F \to W_F \left( \begin{array}{c} y \\ 0 \\ 1 \end{array} \right) \), via the relation in [8, (75)], which is

\[
W_F \left( \begin{array}{cc} |y|^{1/2} \text{sgn}(y) & 0 \\ 0 & |y|^{-1/2} \end{array} \right) = \omega(|y|^{1/2}) W_F \left( \begin{array}{c} y \\ 0 \\ 1 \end{array} \right).
\]

It is easy to see that \( M_{\chi_1, \chi_2}(\phi)(y) = \text{sgn}(y)^{m_2} M_d(\phi)(y) \). Since \( M_d \) maps the space \( I_d^+ \) onto \( S_d((0, \infty)) \) and the space \( I_d^- \) onto \( S_d((\infty, 0]) \), it follows that \( M_{\chi_1, \chi_2} \) does the same. Hence, \( M_{\chi_1, \chi_2} \) sends the space \( I_d^+ \oplus I_d^- \) onto the space \( K_d \), which can be described as follows: \( K_d \) is the space of smooth functions \( f : (\mathbb{R} - \{0\}) \to \mathbb{C} \) such that \( f \) and all its derivatives are rapidly decreasing at \( \pm \infty \) and such that the function \( g(x) = |x|^{(d+1)/2} f(x) \) is smooth on the right and on the left at \( x = 0 \). An example for such a function is the function \( |x|^{(d+1)/2} e^{-|x|} \).

The proof for these assertions is the same as the proof of Theorem 3.1. We define an irreducible representation of \( \text{GL}(2, \mathbb{R}) \) on an \( L^2 \) space and using the various Fréchet topologies show that the smooth space of this representation is \( K_d \). We now describe the Hilbert space and the action of \( \text{GL}(2, \mathbb{R}) \) on this space. The details of the proofs are left to the reader.

Let \( \mathcal{V}_d = L^2(\mathbb{R}, dx/|x|) \). We define a representation \( R_{\chi_1, \chi_2} \) on \( \mathcal{V}_d \) by

\[
R_{\chi_1, \chi_2} \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right) f \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) = e^{iyx} f(x),
\]

\[
R_{\chi_1, \chi_2} \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) f \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) = |a|^{(s_1 + s_2 + 1)/2} f(ax) = \omega(|a|^{1/2}) f(ax),
\]

\[
R_{\chi_1, \chi_2} \left( \begin{array}{cc} b & 0 \\ 0 & b \end{array} \right) f \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) = |b|^{s_1 + s_2 + 1} \text{sgn}(b)^{m_1 + m_2} f(x) = \omega(b) f(x),
\]

\[
R_{\chi_1, \chi_2} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) f \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) = \text{sgn}(y)^{m_2 + d + 1} \int_{-\infty}^{\infty} f(x)|y| \, dx/|x|,
\]
where
\[
\tilde{j}_d(x) = \begin{cases} 
(i)^{-(d+1)}\sqrt{x}J_d(2\sqrt{x}) & \text{if } x > 0, \\
0 & \text{if } x < 0,
\end{cases}
\] (8.2)
and the integral is defined as an $L^2$ extension of a unitary operator on the space $K_d$.

**Theorem 8.2** The representation $R_{\chi_1, \chi_2}$ on the space $L^2(\mathbb{R}, dx/|x|)$ is strongly continuous and irreducible. It is unitary if the central character $\omega = \chi_1\chi_2$ is unitary.

**Theorem 8.3** The smooth space of the representation $R_{\chi_1, \chi_2}$ on $L^2(\mathbb{R}, dx/|x|)$ is the space $K_d$.

**Acknowledgements**

This research was partially supported by the Center of Excellence of the Israel Science Foundation (grant no. 1691/10).

**References**

[1] N.B. Andersen, *Real Paley–Wiener theorems for the Hankel transform*, J. Fourier Anal. Appl. 12(1) (2006), pp. 17–25.
[2] E.M. Baruch and Z. Mao, *Bessel identities in the Waldspurger correspondence over the real numbers*, Israel J. Math. 145 (2005), pp. 1–81.
[3] J.W. Cogdell and I. Piatetski-Shapiro, *The Arithmetic and Spectral Analysis of Poincaré Series*, Perspectives in Mathematics Vol. 13, Academic Press, Boston, MA, 1990.
[4] J.G. van der Corput, *Asymptotic developments III. Again on the fundamental theorem of asymptotic series*, J. Anal. Math. 9 (1961/1962), pp. 195–204.
[5] A.J. Duran, *On Hankel transform*, Proc. Amer. Math. Soc. 110(2) (1990), pp. 417–424.
[6] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.
[7] L.M. Gel’fand, M.I. Graev, and I.I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, Generalized Functions Vol. 6, Academic Press, Boston, MA, 1990, translated from the Russian by K.A. Hirsch, Reprint of the 1969 edition.
[8] R. Godement, *Notes on Jacquet–Langlands Theory*, The Institute for Advanced Study, Princeton, NJ, 1970.
[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier/Academic Press, Amsterdam, 2007.
[10] H. Hankel, *Die Cylinderfunctionen erster und zweiter Art*, Math. Ann. 1(3) (1869), pp. 467–501.
[11] A.A. Kirillov, *Infinite-dimensional representations of the complete matrix group*, Dokl. Akad. Nauk SSSR 144 (1962), pp. 37–39.
[12] A.W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Mathematical Series Vol. 36, Princeton University Press, Princeton, NJ, 1986.
[13] N.N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972, Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.
[14] T.M. MacRobert, *Fourier integrals*, Proc. Roy. Soc. Edinburgh 51 (1931), pp. 116–126.
[15] Y. Motohashi, *A note on the mean value of the zeta and L-functions. XII*, Proc. Japan Acad. Ser. A Math. Sci. 78(3) (2002), pp. 36–41.
[16] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], Computer Science and Applied Mathematics, New York, London, 1974.
[17] N.Ja. Vilenkin and A.U. Klimyk, *Representation of Lie Groups and Special Functions. Vol. I*, Mathematics and Its Applications (Soviet Series) Vol. 72, Kluwer Academic, Dordrecht, 1991.
[18] R. Wallach, *Real Reductive Groups. I*, Academic Press, Boston, MA, 1988.
[19] R. Wallach, *Real Reductive Groups. II*, Pure and Applied Mathematics Vol. 132, Academic Press, Boston, MA, 1992.
[20] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.
[21] H. Weyl, *Singular integralgleichungen*, Math. Ann. 66 (1908), pp. 273–324.
[22] A.H. Zemanian, *A distributional Hankel transformation*, SIAM J. Appl. Math. 14 (1966), pp. 561–576.