Proof of Riemann hypothesis

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To my wife Luisi

Abstract We prove the Riemann hypothesis

We use the following definition of the zeta function: $z = \sigma + iT$, $\sigma \in (0,1)$ (1):

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \frac{1}{\Gamma(z)} \int_{0}^{\infty} \frac{x^{z-1}}{e^{x} + 1} dx.$$  \hfill (1)

The Riemann hypothesis states that all nontrivial zeroes of $\zeta(z)$ are concentrated on the line $\sigma = 1/2$.

The Riemann hypothesis was a famous Hilbert problem (number eight of 23). It is also one of the seven Clay Millennium Prize Problems. It was formulated in Riemann's 1859 Manuscript [1]. Elegant, crisp, falsifiable, and far-reaching, this conjecture is the epitome of what a good conjecture should be [2].

It is the opinion of many mathematicians that the Riemann hypothesis is probably the most important open problem in pure mathematics [3].

In this paper, we prove this hypothesis. We highlight the main steps of the proof.

- We use the integral representation (1) of Riemann zeta function (RZ) in the critical strip.
- We consider square of modulus $K(\sigma, T)$ (2) of the integral in the definition (1) which has the same set of zeroes in the critical strip as (RZ).
- We make transformations of the formula for $K(\sigma, T)$.
- We consider only the case $T > 1.3 \times 10^{21}$, all other cases are known or can be reduced to this one.
- We know that if zero of RZ in the critical strip has coordinates $(\sigma_0, T)$, then point with coordinates $(1 - \sigma_0, T)$ is also zero of RZ.
- We need to prove that second derivative of $K(\sigma, T)$ over $\sigma$ is nonnegative at least for $T > 1.3 \times 10^{21}$.
- From previous item it follows that $K(\sigma, T)$ is U-convex over $\sigma$ and from item before previous it follows that $K(\sigma, T)$ has roots only on the line $\sigma = 1/2$.
- To prove that $K^{(2)}(\sigma, T)$ is positive we make further transformations of its formula and reduce the problem to the proof of correlation inequality (7).
- This correlation inequality is FKG inequality if we prove that function $f$ in (6) is monotone decreasing with $h$.
- The prove of last fact is the most technical. We consider this function as the sum of functions (see (7)) and approximate this sum by integral (16). We do this not in all range of $x$ but for sufficiently small $x < [T^{3/2}]/T$. For large $x$ we estimate the sum in another way (38).
We prove the convexity of $K(\sigma, T)$ for $\sigma \in (1/(60 \ln(T)), 1 - (60 \ln(T)))$. For other values of $\sigma$ RH follows from the result of Ford, which uses Vinogradov's theorem. To consider such restrictions for $\sigma$ in our considerations is necessary to make proper estimations in formulas (36), (38).

In the rest proof we use Lagrange interpolation formula to estimate the terms in formula (19). The main dominating (negative) term arise from the estimation (39), which shows that function $f(h, T)$ is decreasing in the interval $h \in (0, 1)$.

In Appendix we derive different relations using software Mathematica.

Remarks 1. Note that our proof is vanish for Hurwitz zeta function (HZ). One of the causes is that there is no mirror symmetry between the roots of HZ to the line $\sigma = 1/2$ (see corresponding item above). Symmetry is important in our proof because otherwise from convexity of $K(\sigma, t)$ does not follows that zeroes can belong only the middle line.

2. Note that we use software Mathematica to produce cumbersome formulas in Appendix and don't encourage the reader to do this by hand.

3. Riemann hypothesis is consequence of Generalized Riemann hypothesis, but we consider them apart introducing full prove of Riemann hypothesis.

Proof

We assume that $T > 1.3 \times 10^{21}$. For smaller (positive) values of $T$, zeroes of the zeta function lie on the line $\sigma = 1/2$. This is shown in (6). The square of the modulus of the integral in the definition of the zeta function is as follows:

$$K(\sigma, T) = \frac{1}{2} \int_0^\infty \int_0^\infty \frac{x^{\sigma-1}y^{\sigma-1} + x^{\sigma-1}y^{\sigma-1}}{(e^x+1)(e^y+1)} \, dx \, dy.$$  \hspace{1cm} (2)

Changing variable $y \to c^2/x$, we have

$$K(\sigma, T) = 2 \int_0^\infty \int_0^\infty \frac{c^{2\sigma-1}\cos(2T \ln \frac{x}{c})}{x(e^{c^2/x}+1)(e^x+1)} \, dx \, dc.$$  

Changing variable $2T \ln \frac{x}{c} \to u$, we have

$$K(\sigma, T) = \frac{1}{T} \int_0^\infty \int_0^\infty \frac{c^{2\sigma-1}\cos(u)\, du \, dc}{(e^{ce^{-u/(2T)}}+1)(e^{ce^{u/(2T)}}+1)}$$

$$= \frac{1}{T} \left(2 \int_0^\infty y^{2\sigma-1} \int_0^\infty \frac{\cos^2(x/2)\, dx \, dy}{(e^{ye^{x/(2T)}}+1)(e^{ye^{-x/(2T)}}+1)} \right.$$  

$$\left. - \int_0^\infty y^{2\sigma-1} \int_0^\infty \frac{dx \, dy}{(e^{ye^{x/(2T)}}+1)(e^{ye^{-x/(2T)}}+1)} \right).$$

Then

$$\left( K(\sigma, T) \right)^{(2)}_{\sigma \sigma} = \frac{1}{T} \left(2 \int_0^\infty y^{2\sigma-1}((\ln(y))^2 \int_0^\infty \frac{\cos^2(x/2)\, dx \, dy}{(e^{ye^{x/(2T)}}+1)(e^{ye^{-x/(2T)}}+1)} \right.$$  

$$\left. - \int_0^\infty y^{2\sigma-1}((\ln(y))^2 \int_0^\infty \frac{dx \, dy}{(e^{ye^{x/(2T)}}+1)(e^{ye^{-x/(2T)}}+1)}) \right).$$
We are going to prove the inequality

\[(K(\sigma, T))_{\sigma, T}^{(2)} \geq 0\]  \hspace{1cm} (3)

for \(\sigma > \frac{1}{60\ln(T)}\). (For \(\sigma \leq \frac{1}{60\ln(T)}\), there are no zeroes of the seta function on a critical strip. See below for an explanation). From inequality \(\text{(3)}\), the proof of Riemann hypothesis follows. With the root \(\sigma_0 \in (0, 1)\), function \(K(\sigma, T)\) has the root \(1 - \sigma_0\), and we have the fact that the convex function could not have root away from \(\sigma_0 = 1/2\). To prove the inequality \(\text{(3)}\), it is sufficient to prove the inequality

\[
\int_0^\infty y^{2\sigma-1}(\ln(y))^2 \int_0^\infty \frac{\cos^2(\pi x/2) dx dy}{(e^{y^\pi/(2T)} + 1)(e^{y^{\pi/(2T)} - \pi x/(2T)} + 1)} \geq \frac{1}{2} \int_0^1 \int_0^\infty \frac{\cos^2(\pi x/2) dx dy}{(e^{y^\pi/(2T)} + 1)(e^{y^{\pi/(2T)} - \pi x/(2T)} + 1)}.
\]  \hspace{1cm} (4)

The last inequality is equivalent to the inequality

\[
\int_0^1 \bar{f}(h, T) \cos^2(\pi h/2) dh \geq \frac{1}{2} \int_0^1 \bar{f}(h, T) dh,
\]  \hspace{1cm} (5)

where

\[
\bar{f}(h, T) = \sum_{i=0}^{\infty} \int_0^\infty y^{2\sigma-1}(\ln(y))^2 \bar{\lambda}(i, y, h, T) dy,
\]  \hspace{1cm} (6)

and

\[
\bar{\lambda}(x, y, h, T) = \frac{1}{(e^{y^\pi/(2T)} + 1)(e^{y^{\pi/(2T)} - \pi x/(2T)} + 1)} + \frac{1}{(e^{y^{\pi/(2T)} + \pi x/(2T)} - \pi x/(2T)} + 1\). (8)
\]

We divide the interval \([0, \infty)\) into the parts \(a_i = [\pi i, \pi(i + 1)], and i = 0, 1, \ldots\) and write the inequality \(\text{(7)}\) as follows

\[
\int_0^\infty \sum_{i=0}^{\infty} y^{2\sigma-1}(\ln(y))^2 \int_0^{i+1} \frac{\cos^2(\pi x/2) dx dy}{(e^{y^\pi/(2T)} + 1)(e^{y^{\pi/(2T)} - \pi x/(2T)} + 1)} \geq \frac{1}{2} \sum_{i=0}^{\infty} y^{2\sigma-1}(\ln(y))^2 \int_0^{i+1} \frac{dy}{(e^{y^\pi/(2T)} + 1)(e^{y^{\pi/(2T)} - \pi x/(2T)} + 1)}.
\]  \hspace{1cm} (7)

The last inequality is equivalent to the inequality \(\text{(5)}\). (To show this, subdivide each interval \([\pi i, \pi(i + 1)]\) into subintervals \([\pi i, \pi i + \pi/2], [\pi i + \pi/2, \pi i + \pi]\), and consider the integration (changing variables \(x to h\) from \(\pi i \rightarrow \pi i + \pi/2\) on the first subinterval, and \(\pi(i + 1) \rightarrow \pi(i + \pi/2)\) on each second subinterval.)

For inequality \(\text{(5)}\) to be valid it is sufficient to prove two inequalities

\[
\int_0^1 \int_0^\infty y^{2\sigma-1}(\ln(y))^2 \bar{\lambda}(0, y, h, T) \cos^2(\pi h/2) dh dy \geq \frac{1}{2} \int_0^1 \int_0^\infty y^{2\sigma-1}(\ln(y))^2 \bar{\lambda}(0, y, h, T) dh dy
\]  \hspace{1cm} (8)

and

\[
\int_0^1 f(h, T) \cos^2(\pi h/2) dh dy \geq \frac{1}{2} \int_0^1 f(h, T) dh,
\]  \hspace{1cm} (9)

and

\[
\int_0^1 f(h, T) \cos^2(\pi h/2) dh dy \geq \frac{1}{2} \int_0^1 f(h, T) dh,
\]  \hspace{1cm} (9)
where

\[ f(h, T) = \sum_{i=1}^{\infty} \int_0^\infty y^{2\sigma-1}(\ln(y))^2 \bar{\lambda}(i, y, h, T) dy. \] (10)

At first we prove inequality (8). We need to prove that

\[ \int_0^\infty \int_0^1 y^{2\sigma-1}(\ln(y))^2 \bar{\lambda}(0, y, h, T) \cos(\pi h) dh dy \geq 0. \]

Integrating in parts over \( h \) in the lhs of the last inequality we obtain equivalent inequality.

\[ \int_0^\infty \int_0^1 y^{2\sigma-1}(\ln(y))^2 (\bar{\lambda}(0, y, h, T)) (1) h \sin(\pi h) dh dy \leq 0. \] (11)

Define

\[ \rho(y, h, T) = \frac{1}{(e^{ye^h/\pi} + 1)(e^{y+e^{-h}/\pi} + 1)}. \]

Then

\[ \bar{\lambda}(0, y, h, T) = \rho(y, h, T) + \rho(y, 2 - h, T) \]

and inequality (11) is equivalent to the inequality

\[ \int_0^\infty \int_0^1 y^{2\sigma-1}(\ln(y))^2 (\rho(y, h, T)) (1) h \sin(\pi h) dh dy \leq 0. \]

We will use Lagrange interpolation

\[ (\rho(y, h, T))_h(1) = (\rho(y, 0, T))_h(1) + (\rho(y, 0, T))_h(2)h + \frac{1}{2}(\rho(y, \theta, T))_h(3)h^2. \]

Because

\[ (\rho(y, 0, T))_h(1) = 0, (\rho(y, 0, T))_h(2) = -\left(e^{y} + y + 1\right) \frac{2e^{y} y}{(e^{y} + 1)^3}, \]

we have

\[ (\rho(y, h, T))_h(1) = -\left(e^{y} + y + 1\right) \frac{2e^{y} y}{(e^{y} + 1)^3} \frac{\pi^2}{(2T)^2} h + \frac{1}{2}(\rho(y, \theta, T))_h(3)h^2. \] (13)

Last formula shows that to prove inequality (11) it is sufficient to prove the inequality

\[ -\frac{\pi^2}{(2T)^2} \int_0^\infty y^{2\sigma-1}(\ln(y))^2 (e^{y} + y + 1) \frac{2e^{y} y}{(e^{y} + 1)^3} \left(\int_1^2 h \sin(\pi h) dh \right) dh \]

\[ - \int_0^1 h \sin(\pi h) dh \right) dy - \int_0^\infty y^{2\sigma-1}(\ln(y))^2 (\rho(y, \theta, T))_h(3) \int_0^2 h^2 dh dy \geq 0. \] (14)

Using inequality (50) from Appendix and integrating over \( h \) we reduce the proof of inequality (14) to the proof of the inequality

\[ \frac{5}{\pi} \frac{\pi^2}{(2T)^2} \int_0^\infty y^{2\sigma-1}(\ln(y))^2 (e^{y} + y + 1) \frac{2e^{y} y}{(e^{y} + 1)^3} dy \geq \frac{10^4}{T^3} \] (15)
Because
\[
\int_0^\infty y^{2\sigma}(\ln(y))^2(e^y + 1)\,dy > \frac{2e^y}{(e^y + 1)^3} \int_1^\infty y^{2\sigma+2}(3e^y)\,dy
\]
\[
> \frac{1}{4} \int_1^\infty e^{-y}\,dy > \frac{1}{4e},
\]
the inequality (15) to be valid it is sufficient that the following inequality is true
\[
\frac{5}{\pi} \frac{\pi^2}{(2T)^2} \frac{1}{4e} > \frac{10^4}{T^3}.
\]
This inequality true for \( T > 1.3 \times 10^{21} \). Hence inequality (8) is true.

Now we prove the inequality (5). Inequality (5) is an FKG inequality (11) if we can show that
\[
\text{function } f(h, T) \text{ is monotone decreasing with } h \in [0, 1].
\]
Function \( \cos^2 \) is monotone decreasing with \( h \in (0, 1] \) and
\[
\int_0^1 \cos^2(\pi h/2)\,dh = \frac{1}{2}.
\]
Define
\[
\gamma(x, y, h, T) = \frac{1}{(e^{ye^{\pi/(2T)}+\pi x/hT/(2T)} + 1)(e^{ye^{-\pi/(2T)}-\pi x/hT/(2T)} + 1)}.
\]
and
\[
\lambda(x, y, h, T) = \gamma(x, y, h, T) + \gamma(x + 1/T, y, -h, T).
\]
We already noted that to prove the inequality (5), it is sufficient to prove that function \( (f(h, T))_h \) is negative. We will prove this for \( T > 1.3 \times 10^{21} \).

To prove that \( (f(h, T))_h \) is negative, we use the estimation of the sum by an integral (8):
\[
\left| \int_a^b \varphi(x)\,dx - \frac{b - a}{n} \sum_{i=1}^n \varphi \left( a + \frac{2i - 1}{2n} (b - a) \right) \right| \leq \frac{(b - a)^3}{24n^2} \max_{x \in [a, b]} |\varphi''(x)|. \tag{16}
\]
We assume \( a = 0, b = \frac{n}{T} \), \( \varphi(x) = (\lambda(x, y, h, T))_h \), \( n = [T^{3/2}] \), and have the estimation:
\[
\left| \int_0^{[T^{3/2}]/T} ((\lambda(x, y, h, T))_h \,dx - \frac{1}{T} \sum_{i=1}^{[T^{3/2}]/T} \lambda \left( \frac{(2i - 1)}{2T}, y, h, T \right) \right|_h \right| \leq \frac{[T^{3/2}]}{24[T^{3/2}]^2} \max_{x \in [0, \sqrt{T} + 1]} \left| (\lambda(x, y, h, T))_h \right|_h. \tag{17}
\]
Because
\[
\int_0^{[T^{3/2}]/T} \varphi(x, y, h, T)\,dx = \frac{1}{2T} \left( \gamma([T^{3/2}]/T, y, h, T) - \gamma([T^{3/2}]/T + 1/T, y, -h, T) - (\gamma(0, y, h, T) - \gamma(1/T, y, -h, T)) \right), \tag{18}
\]
we have
\[
\sum_{i=1}^{T^{3/2}} \left( \lambda \left( \frac{2i-1}{2T}, y, h, T \right) \right)_{h}^{(1)}
\]
\[
- \frac{1}{2T} \left( \gamma ([T^{3/2}]/T, y, h, T) - \gamma ([T^{3/2}]/T + 1/T, y, -h, T) - (\gamma (0, y, h, T) - \gamma (1/T, y, -h, T)) \right)
\]
\[
\leq \frac{2}{24T^{3/2}} \max_{x \in [0, \sqrt{T} + 1]} |((\lambda (x, y, h, T))_{h}^{(1)})_{xx}^{(2)}|.
\]

Next, we need the (upper) bound for
\[
\max_{x \in [0, \sqrt{T} + 1]} |((\lambda (x, y, h, T))_{h}^{(1)})_{xx}^{(2)}|.
\]

Because
\[
|((\lambda (x, y, h, T))_{h=1}^{(1)})_{xx}^{(2)}| = 0,
\]
we have
\[
((\lambda (x, y, h, T))_{h}^{(1)})_{xx}^{(2)} = ((\lambda (x, y, \theta, T))_{h=1}^{(2)})_{xx}^{(2)}(h - 1),
\]
for some \( \theta \in [0, 1] \). Thus, we should find the upper bound uniform over a choice of the variables for the value
\[
|((\lambda (x, y, h, T))_{h=1}^{(2)})_{xx}^{(2)}|.
\]

The following result is due to Ford (2002) \cite{10}.

**Theorem 1.** If \( \sigma + iT \) is a zero of the Riemann zeta function, then
\[
\sigma \leq 1 - \frac{1}{57.54 (\ln T)^{2/3}(\ln \ln T)^{1/3}}.
\]

In the Appendix, we prove the inequality
\[
\int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} |((\lambda (x, y, h, T))_{h}^{(1)})_{xx}^{(2)}| dy < \frac{10^{8} (\ln T)^{4/3}(\ln \ln T)^{2/3}}{T^{2}}.
\]

Hence
\[
\int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} |((\lambda (x, y, h, T))_{h}^{(1)})_{xx}^{(2)}| dy < \frac{10^{8} (\ln T)^{4/3}(\ln \ln T)^{2/3}(1 - h)}{T^{2}}.
\]

Proof of this theorem uses Vinogradovs mean-value theorem \cite{12}.

We rough condition (20) as follows:
\[
\sigma < 1 - \frac{1}{60 \ln(T)}.
\]

Hence, we can assume that \( \sigma > 1/(60 \ln(T)) \).
Inequality (19) now can be written as follows:

\[
\frac{1}{T} \sum_{i=1}^{[T^{3/2}]} \left( \lambda \left( \frac{(2i-1)}{2T}, y, h, T \right) \right)_{y}^{(1)} \leq \frac{1}{2T} \left( \gamma \left( \frac{[T^{3/2}]}{T}, y, T, -h, T \right) - \gamma \left( \frac{[T^{3/2}]}{T} + 1/T, y, -h, T \right) - (\gamma(0, y, h, T) - \gamma(1/T, y, -h, T)) \right) \\
\leq (1-h) \frac{4 \times 10^3 \times (\ln T)^{4/3} (\ln \ln T)^{2/3}}{T^{7/2}} e^{-y e^{\pi x} + y^4 e^{4\pi x}}
\]

It is left to find the proper bound for

\[
\gamma \left( \frac{[T^{3/2}]}{T}, y, h, T \right) - \gamma \left( \frac{[T^{3/2}]}{T} + 1/T, y, -h, T \right) - (\gamma(0, y, h, T) - \gamma(1/T, y, -h, T)).
\]

We will use the following interpolation formula

\[
\gamma \left( \frac{[T^{3/2}]}{T}, y, h, T \right) - \gamma \left( \frac{[T^{3/2}]}{T} + 1/T, y, -h, T \right) = (\gamma(0, y, h, T) - \gamma(1/T, y, -1, T))_{h}^{(1)} (h-1)
\]

\[
\gamma(0, y, h, T) - \gamma(1/T, y, -h, T) = (\gamma(0, y, 1, T) - \gamma(1/T, y, -1, T))_{h}^{(1)} (h-1)
\]

\[
\frac{1}{6}(\gamma(0, y, \theta, T) - \gamma(1/T, y, -\theta, T))_{h}^{(3)} (h-1)^3
\]

Define \( R = 1/T \). We have

\[
(\gamma(0, y, h, T) - \gamma(1/T, y, -h, T))_{h=1}^{(1)} = ((\gamma(0, y, h, T) - \gamma(1/T, y, -h, T))_{h=1}^{(1)}) R = \theta_1, \frac{1}{T},
\]

\(\theta_1 \in [0, 1]\).

Now we find the upper bound for the value

\[
\int_{0}^{\infty} y^{2\sigma-1} (\ln(y))^2 (\gamma(0, y, h, T) - \gamma(1/T, y, -h, T)) dy.
\]

We have

\[
\gamma(0, y, h, T) = \frac{1}{(e^{y e^{\pi T/2 T} + h e^{\pi T/2 T}} + 1)(e^{y e^{\pi T/2 T} - h e^{\pi T/2 T}} + 1)}.
\]

\[
\gamma(1/T, y, -h, T) = \frac{1}{(e^{y e^{3\pi T/2 T} - h e^{3\pi T/2 T}} + 1)(e^{y e^{3\pi T/2 T} + h e^{3\pi T/2 T}} + 1)}.
\]

Next we have

\[
((\gamma(0, y, 1, T = 1/R) - \gamma(1/T, y, -1, T = 1/R))_{h=1}^{(1)}) R
\]

\[
= \pi R \left( \frac{2\pi y^2 e^{-\pi R y} - 2\pi R}{(e^{-\pi R y} + 1)^3 (e^{\pi R y} + 1)} - \frac{2\pi y^2 e^{-\pi R y} + e^{\pi R y}}{(e^{-\pi R y} + 1)^2 (e^{\pi R y} + 1)^2} + \frac{2\pi y^2 e^{2\pi R y} + 2\pi R}{(e^{-\pi R y} + 1)^3 (e^{\pi R y} + 1)^3} \right)
\]

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Thus we have

\[
\frac{y e^{e^{-Ry-\pi R}}} \left( e^{e^{-Ry+\pi R}} \right)^2 \left( e^{e^{Ry}} + 1 \right) - \frac{y e^{e^{Ry-\pi R}}} \left( e^{e^{-Ry-\pi R}} \right)^2 \left( e^{e^{Ry}} + 1 \right)
\]

First we need to show the validness of the following inequality

\[
\frac{e^{ye^{-\pi T}/T} e^{-2\pi/T}} {e^{ye^{-\pi T}/T} + 1} + \frac{e^{ye^{\pi/T} e^{2\pi/T}}}{e^{ye^{\pi/T} + 1} + 1} + \frac{2e^{y(e^{-\pi/T} + e^{\pi/T})}}{(e^{ye^{-\pi T}/T} + 1)(e^{ye^{\pi/T} + 1})} = 2\eta(0, \eta, -1, T) > 100.
\]

To prove it we define \( a = \frac{e^{ye^{-\pi T}/T} e^{-2\pi/T}} {e^{ye^{-\pi T}/T} + 1}, b = \frac{e^{ye^{\pi/T} e^{2\pi/T}}}{e^{ye^{\pi/T} + 1} + 1} \). Then we can rewrite last inequality as

\[
b e^{-2\pi/T} + a e^{2\pi/T} + 2ab - 2b^2 e^{-2\pi/T} - 2a^2 e^{2\pi/T} > -\frac{100}{T}.
\]

We need to show that

\[
a + b + 2ab - 2a^2 - 2b^2 \geq 0, \quad a, b \in [0.5, 1]
\]

This is true because \( \text{lhs} \) is \( \cap \)-convex over \( a \) and \( b \) and \( \min \) achieved at the end point(s). Rest term in the middle expression from (30) exceeds \(-\frac{100}{T}\). From this inequality easily follows inequality (28).

From the other hand we can write new inequality

\[
\frac{y e^{e^{-Ry+\pi R}}}{(e^{e^{Ry}} + 1)^2} = \frac{y e^{Ry+\pi R}}{(e^{e^{Ry}} + 1)^2} < 0
\]

Thus using this inequality and inequality (28) we obtain

\[
((\gamma(0, y, 1, T) - \gamma(1, y, -1, T))_{h}^{(1)})_{R}^{(1)} < -\frac{\pi^2}{2T} ye^{-2.1y}
\]

Thus we have

\[
\int_{0}^{\infty} y^{2\sigma-1} (\ln y)^2 (\gamma(0, y, 1, T) - \gamma(1/T, y, -1, T))_{h}^{(1)}_{R} dy < -\frac{\pi^2}{2T} \int_{1}^{e} \left( \frac{y - 1}{e - 1} \right)^2 e^{-2.1y} dy < -\frac{0.198}{T}
\]
Second, we find the bound for the value (last line in (25)):

\[
\frac{1}{6} \int_0^\infty y^{2\sigma-1} (\ln(y))^2 \left( \gamma(0, y, \theta, T) - \gamma(1/T, y, -\theta, T) \right)_{hh}^3 dy.
\] (34)

In Appendix we prove inequality (32):

\[
\frac{1}{6} \int_0^\infty y^{2\sigma-1} (\ln(y))^2 |\gamma(0, y, \theta, T) - \gamma(1/T, y, -\theta, T)|_{hh}^3 dy < \frac{2000}{T^3} \left( \int_0^1 (\ln(y))^2 dy + \int_1^\infty y^6 e^{-y} dy \right) < \frac{2000}{T^3} (2 + 6!) < \frac{1.5 \cdot 10^7}{T^3}
\] (35)

When \( x = [T^{3/2}]/T, [T^{3/2}]/T + 1/T \), we have

\[
\left( \gamma(x, y, h, T) \right)_{hh}^{(1)} = \frac{\pi}{2T} \left( -\frac{e^{ye^{\pi/(2T) + \pi x + h\pi/(2T)}} y e^{\pi/(2T) + \pi x + h\pi/(2T)}}{(e^{ye^{\pi/(2T) + \pi x + h\pi/(2T)}})^2} + \frac{e^{-\pi/(2T) - \pi x - h\pi/(2T)}}{(e^{ye^{-\pi/(2T) - \pi x - h\pi/(2T)}} + 1)^2} \right) < e^{-ye^{\sqrt{T}} y e^{\sqrt{T}}}
\]

Hence

\[
|\gamma([T^{3/2}]/T, y, h, T) - \gamma([T^{3/2}]/T + 1/T, y, -h, T)|_{hh}^{(1)} < 20 ye^{\sqrt{T}} e^{-ye^{\sqrt{T}}}
\]

Multiple 20 here is redundant.

Integrating we have at last

\[
\int_0^\infty y^{2\sigma-1} (\ln(y))^2 \left( \gamma([T^{3/2}]/T, y, h, T) - \gamma([T^{3/2}]/T + 1/T, y, -h, T) \right)_{hh}^{(1)} dy
\] (36)

< \frac{20}{6} \int_0^\infty y^{2\sigma-1} (\ln(y))^2 ye^{\sqrt{T}} e^{-ye^{\sqrt{T}}} dy

< 20 e^{-2\sqrt{T}} \int_0^\infty z^{2\sigma} e^{-z (\ln(z) - \sqrt{T})^2} dz

< 20 e^{-\sqrt{T}/(30 \ln(T))} \left( \int_0^1 ((\ln z)^2 - 2 \sqrt{T} \ln(z) + T) dz + \int_1^\infty z^4 e^{-z} + T e^{-z} dz \right)

< 20 e^{-\sqrt{T}/(30 \ln(T))} \left( 2 + 2 \sqrt{T} + T + 5! + T \right) < e^{-\sqrt{T}/(30 \ln(T))} T^2.

Here in third line we make substitution \( ye^{\sqrt{T}} = z \) and in the forth line set \( \sigma = 1/(60 \ln T) \). Last we estimate the value of the sum of \( \lambda \), which is not included in the sum from (19). Because \( \left( \lambda \left( \frac{2i-1}{2T}, y, 1, T \right) \right)_{hh}^{(1)} = 0 \), we have the expansion

\[
\left( \lambda \left( \frac{2i-1}{2T}, y, h, T \right) \right)_{hh}^{(1)} = \left( \lambda \left( \frac{2i-1}{2T}, y, \theta, T \right) \right)_{hh}^{(2)} (h - 1)
\]
and hence
\[ \left| \left( \lambda \left( \frac{2i - 1}{2T}, y, h, T \right) \right) \right|_h^{(1)} < \left| \left( \lambda \left( \frac{2i - 1}{2T}, y, \theta, T \right) \right) \right|_{hh}^{(2)} (1 - h). \] (37)

In Appendix we show that
\[ \left| (\lambda(x, y, h, T))_{hh}^{(2)} \right| < 20e^{-ye^{x}}(ye^{x} + y^{2}e^{2x}). \]

Using previous inequality we obtain a chain of inequalities:
\[
\begin{align*}
\frac{1}{T} \sum_{i=\lceil T^{3/2} \rceil + 1}^{\infty} \int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} \left| \left( \lambda \left( \frac{2i - 1}{2T}, y, h, T \right) \right) \right|_{h}^{(1)} dy &< 20 \sum_{i=\lceil T^{3/2} \rceil + 1}^{\infty} \int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} \left( e^{-ye^{i/T}}(ye^{i/T} + y^{2}e^{2i/T}) \right) dy (1 - h) \\
&< 30 \sum_{i=\lceil T^{3/2} \rceil + 1}^{\infty} \int_{0}^{\infty} (z^{2\sigma} + z^{2\sigma + 1})e^{-2\pi i ((\ln(z) - i/T)^{2}e^{z} - z)} dz (1 - h) \\
&< 10^{4}T^{3} \ln(T)e^{-\sqrt{T}/30 \ln(T)} (1 - h).
\end{align*}
\] (38)

Here in the third line we make substitution $ye^{i/T} \rightarrow z$ and set $\sigma = 1/(60 \ln T)$.

Combining the inequalities (24), (33), (35), and (38), we obtain the bound
\[
\begin{align*}
\int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} \left( \frac{1}{T} \sum_{i=1}^{\infty} \left( \lambda \left( \frac{2i - 1}{2T}, y, h, T \right) \right) \right)_{h}^{(1)} dy &< \\
\frac{1}{2T} \int_{0}^{\infty} y^{2\sigma - 1} (\ln(y))^{2} \left( \gamma([T^{3/2}]/T, y, h, T) - \gamma([T^{3/2}]/T + 1/T, y, -h, T) \right) \\
- (\gamma(0, y, h, T) - \gamma(1/T, y, -h, T)) \right) dy &< \\
+10^{8} \frac{10^{8}(\ln T)^{4/3}(\ln \ln T)^{2/3}(1 - h)}{T^{7/2}} \\
&< \frac{1}{T} \left( \frac{10^{8} * (\ln T)^{4/3}(\ln \ln T)^{2/3}}{T^{5/2}} - \frac{0.198}{T^{2}} + \frac{1.5 \cdot 10^{7}}{T^{3}} + T^{5} e^{-\sqrt{T}/(30 \ln(T))} + 10^{4}T^{5} e^{-\sqrt{T}/(30 \ln(T))} \right) (1 - h)
\end{align*}
\] (39)

From this bound, it easily follows that (at least) for $T > 1.3 \cdot 10^{21}$ function $(f(h, T))_{h}^{(1)}$ is negative. This completes the proof.

**Remark**

Next we introduce the Ramanujans $\tau$- Dirichlet Series Conjecture. Proof is similar (and even simpler) that the proof of Riemann hypothesis and we will not show all details of calculus.

The Ramanujan $\tau$- function is defined in terms of its generating function
\[ g(z) = z \prod_{i=1}^{\infty} (1 - z^{i})^{24} = \sum_{i=1}^{\infty} \tau_{n} z^{n}. \] (41)
We consider the associated Dirichlet series
\[ f(s) = \sum_{n=1}^{\infty} \tau_n n^{-s}. \quad (42) \]

We have
\[ f(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} g(e^{-x}) \, dx. \quad (43) \]

We have functional equations
\[ z^6 g(e^{-2\pi z}) = \left(\frac{1}{z}\right)^6 g(e^{-2\pi/z}) \quad (44) \]
and
\[ (2\pi)^s \Gamma(6-s) f(6-s) = (2\pi)^{-s} \Gamma(6+s) f(6+s). \quad (45) \]

Ramanujan \( \tau \)-Dirichlet Series Conjecture [14] says that all nontrivial zeroes of \( f(s), s = \sigma + iT \) in the critical strip \( \sigma \in [5.5, 6.5] \) lie on the line \( \sigma = 6 \).

From equation (45) it follows that if \( s = \sigma + iT \) is zero of \( f \) then \( s = \sigma - iT \) and \( s = 12 - \sigma \pm iT \) are also zeroes of \( f \).

Thus to compute the Conjecture it is sufficient to prove \( |\Gamma(s) f(s)|^2 \) is \( U\)-convex over \( \sigma \in [5.5, 6.5] \) when \( T > 0 \) is fixed. We will explicitly prove this fact for \( T > 10^5 \), for smaller \( T \geq 0 \) we can check this fact using software Mathematika and modern computer.

Using standard calculus we obtain the formula
\[ K(\sigma, T) = |\Gamma(s) f(s)|^2 \]
\[ = 2 \int_{0}^{\infty} \int_{0}^{\infty} y^{2\sigma-1} \cos(Tx) g(e^{-ye^{x/2}}) g(e^{-ye^{x/2}}) \, dxdy. \quad (46) \]

We are going to prove that \( (K(\sigma, T))^{(2)}_{\sigma} \geq 0 \) when \( T > 10^5 \), \( \sigma \in [5.5, 6.5] \).

Hence we need to prove that
\[ (K(\sigma, T))^{(2)}_{\sigma} \]
\[ = 16 \left( \int_{0}^{\infty} \int_{0}^{\infty} y^{2\sigma-1} (\ln(y))^2 \cos^2(x/2) g(e^{-ye^{x/(2T)}}) g(e^{-ye^{-x/(2T)}}) \, dxdy \right. \\
\[ - \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} y^{2\sigma-1} g(e^{-ye^{x/(2T)}}) g(e^{-ye^{-x/(2T)}}) \, dxd \left. \right) \geq 0. \quad (47) \]

or
\[ \int_{0}^{1} \rho(h, T) \cos^2(\pi h/2) \, dh \geq \frac{1}{2} \int_{0}^{\infty} \rho(h, T) \, dh, \quad (48) \]
where
\[ \rho(h, T) = \sum_{i=0}^{\infty} \int_{0}^{\infty} y^{2\sigma-1} (\ln(y))^2 \tilde{\beta}(i, y, h, T) \, dy, \quad (49) \]
and
\[ \tilde{\beta}(x, y, h, T) = \gamma(x, y, h, T) + \gamma(x + 1, y, -h, T), \]
\[ \gamma(x, y, h, T) = g(e^{-ye^{\pi x/T + h\pi/(2T)}}) g(ye^{-\pi x/T - h\pi/(2T)}). \]
To prove Generalized Riemann hypothesis consider Dirichlet $L$- function

$$
\xi(s = \sigma + iT) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{i=1}^{\infty} \chi(i) e^{-ix} x^{s-1} dx
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{p=0}^{\infty} e^{-Kpx} \sum_{i=1}^{K} \chi(i) e^{-ix} x^{s-1} dx
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{1 - e^{-Kx}} \sum_{i \leq K: \gcd(K,i) = 1} \chi(i) e^{-ix} x^{s-1} dx
$$

where $\chi$ is Dirichlet character, $K$ is its period. It is known that

$$
\sum_{i=1}^{K} \chi(i) = 0.
$$

Zeroes of $\xi$ in the critical strip coincide with zeroes of

$$
\tilde{K}(\sigma, T) = \left| \int_0^\infty \frac{1}{1 - e^{-Kx}} \sum_{i \leq K: \gcd(K,i) = 1} \chi(i) e^{-ix} x^{s-1} dx \right|^2.
$$

In the same way as in the case of RH we consider second derivative

$$
(\tilde{K}(\sigma, T))^{(2)} = \int_0^\infty y^{2\sigma-1} (\ln(y))^2 \int_0^\infty \sum_{i=1}^{K} e^{-ye^{2/(2T)}(\xi(i)/T+(1-\sigma)(\xi(i)/T)}
$$

$$
\sum_{i=1}^{K} e^{-ye^{-x/(2T)}(\xi(i)/T)} \cos(x) dx dy.
$$

Here $\xi(j) = \ln(\chi(j))/i, i = \sqrt{-1}$ if $\chi(j) \neq 0$ and $\xi(j) = \infty$ otherwise. Analysis of this expression is (almost) the same as in the case of $K(\sigma, T)$, we skip the detailed proof at this stage.
Appendix

\[
(\rho(y, h, T))_{hhh}^{(2)} = \\
\frac{3e^{3e^{-\frac{h}{2T}y}}y^{-\frac{h}{2T}\pi y^3}}{4 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^4 \left(1 + e^{e\frac{h}{2T}y}\right) T^3} + \frac{3e^{\frac{h}{2T} + e^{-\frac{h}{2T}y} + \frac{h}{2T} y^3 y^3}}{4 \left(1 + e^{-\frac{h}{2T}y}\right)^4 \left(1 + e^{e\frac{h}{2T}y}\right) T^3}
\]

\[
\frac{e^{2e^{-\frac{h}{2T}y} - \frac{h}{2T} y^2} \left(-e^{\frac{h}{2T} \pi y} - \frac{\pi}{2T}\right) y^2}{2 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^3 \left(1 + e^{e\frac{h}{2T}y}\right) T^2} + \frac{e^{e^{-\frac{h}{2T}y} + \frac{h}{2T} y^2} \left(-e^{\frac{h}{2T} \pi y} - \frac{\pi}{2T}\right) y^2}{2 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^3 \left(1 + e^{e\frac{h}{2T}y}\right) T^2}
\]

\[
\frac{e^{-\frac{h}{2T} y + e^{-\frac{h}{2T}y} + \frac{h}{2T} y^2} \left(-e^{\frac{h}{2T} \pi y} + \frac{\pi}{2T}\right) y^2}{2 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^3 \left(1 + e^{e\frac{h}{2T}y}\right) T^2} + \frac{e^{\frac{h}{2T} + 2e^{-\frac{h}{2T}y} + \frac{h}{2T} y^2} \left(-e^{\frac{h}{2T} \pi y} + \frac{\pi}{2T}\right) y^2}{2 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^3 \left(1 + e^{e\frac{h}{2T}y}\right) T^2}
\]

\[
8 \left(1 + e^{e^{-\frac{h}{2T}y}}\right)^2 \left(1 + e^{e\frac{h}{2T}y}\right) T^3
\]

Hence

\[
|\rho(y, h, T)_{hhh}| < 100e^{-y}(y + y^3) \frac{1}{T^3}
\]

and

\[
\int_0^\infty y^{2\sigma - 1} (\ln(y))^2 |\rho(y, h, T)_{hhh}| dy < 100e^{-y} y^{2\sigma} (\ln(y))^2 (1 + y^2) dy < \frac{10^4}{T^3}.
\]

(50)
Hence when \( x = i > T^{3/2} - 1 \) we have

\[
|\langle \lambda(x, y, h, T) \rangle^{(2)}_{hh} | < 100e^{-y^2} (ye^i + y^2e^{2i}).
\] (51)

Next we have

\[
\langle \lambda(x, y, h, T) \rangle^{(2)}^{(2)}_{xx} = \]

\[
\left( e^{-\frac{2\pi h}{T}} - 2\pi x + e^{-\frac{2\pi h}{T}} \pi x - \frac{3\pi}{2} \right) y^4 + \left( 1 + e^{\frac{2\pi h}{T}} - \frac{3\pi}{2} \right) y^2 T^2
\]
\[
6e^{-\frac{2\pi h}{\gamma} + 4\pi x + 4e^{\frac{2\pi h}{\gamma}} + \pi x - \frac{2\pi y}{\gamma}} \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right)^5 - 2T^2
\]

\[
3e^{-\frac{2\pi h}{\gamma} - 3\pi x + 3e^{-\frac{2\pi h}{\gamma} - \frac{2\pi y}{\gamma}} - \frac{3\pi}{\gamma} \pi^3 \left(3 - e^{-\frac{2\pi h}{\gamma} - \frac{2\pi y}{\gamma}}\right) y^3}
\]

\[
2 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x - \frac{2\pi y}{\gamma}}\right)^4 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right) T^2
\]

\[
e^{-\frac{2\pi h}{\gamma} - \pi x + 2e^{-\frac{2\pi h}{\gamma} - \pi x - \frac{3\pi}{\gamma} \pi} y + e^{-\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - \frac{3\pi}{\gamma} \pi^3 \left(-2e^{-\frac{2\pi h}{\gamma} - \frac{2\pi y}{\gamma}} + 2\pi\right) y^3}
\]

\[
2 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x - \frac{2\pi y}{\gamma}}\right)^3 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right)^2 T^2
\]

\[
e^{\frac{2\pi h}{\gamma} - \pi x + 2e^{\frac{2\pi h}{\gamma} - \pi x - \frac{3\pi}{\gamma} \pi} y + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - \frac{3\pi}{\gamma} \pi^3 \left(2e^{\frac{2\pi h}{\gamma} + \pi x - \frac{3\pi}{\gamma} \pi} y - 2\pi\right) y^3}
\]

\[
2 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x - \frac{2\pi y}{\gamma}}\right)^3 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right)^2 T^2
\]

\[
e^{-\frac{2\pi h}{\gamma} - \pi x + 2e^{\frac{2\pi h}{\gamma} - \pi x - \frac{3\pi}{\gamma} \pi} y + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - \frac{3\pi}{\gamma} \pi^3 \left(e^{\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - e^{\frac{2\pi h}{\gamma} - \frac{3\pi}{\gamma} \pi} y\right) y^3}
\]

\[
2 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x - \frac{2\pi y}{\gamma}}\right)^3 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right)^2 T^2
\]

\[
e^{-\frac{2\pi h}{\gamma} - \pi x + 2e^{\frac{2\pi h}{\gamma} - \pi x - \frac{3\pi}{\gamma} \pi} y + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - \frac{3\pi}{\gamma} \pi^3 \left(-e^{\frac{2\pi h}{\gamma} + \pi x + \frac{3\pi}{\gamma} \pi} y - e^{\frac{2\pi h}{\gamma} - \pi x - \frac{3\pi}{\gamma} \pi} y\right) y^3}
\]

\[
2 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x - \frac{2\pi y}{\gamma}}\right)^3 \left(1 + e^{\frac{2\pi h}{\gamma} + \pi x + \frac{2\pi y}{\gamma}}\right)^2 T^2
\]
\[
3e^{-\frac{3h}{T}+3\pi x+3e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+\frac{2T}{\pi}y}^3 (2e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+2\pi y)^3 \\
+ 2 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 T^2 \\
3e^{-\frac{3h}{T}+3\pi x+3e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+\frac{2T}{\pi}y}^3 \left(2e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+2\pi y+\pi\right)^3 \\
+ 2 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^3 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^3 T^2 \\
3e^{-\frac{3h}{T}+3\pi x+3e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+\frac{2T}{\pi}y}^3 \left(3e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+3\pi\right)^3 \\
+ 2 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 T^2 \\
3e^{-\frac{3h}{T}}-3\pi x+3e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}-\frac{3T}{2T}y+\frac{2T}{\pi}y}^3 \left(-e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}+\frac{3T}{2T}y+\pi\right)^3 \\
+ \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 \left(1+e^{-\frac{3h}{T}+\pi x+\frac{2T}{\pi}y}\right)^4 T
\]
\[
2e^{-\frac{\pi h}{27}} - \pi x + 2e^{\frac{\pi h}{27}} - \pi x - \frac{\pi x}{27} y + e^{\frac{\pi h}{27}} + \pi x + \frac{\pi h}{27} y - \frac{\pi}{27} \pi^3 \left( -e^{\frac{\pi h}{27}} + \pi x + \frac{\pi h}{27} y - \frac{\pi}{27} \right) y^3 
\]
\[
\begin{align*}
&\frac{e^{-\frac{\pi h}{4T}} - 2\pi x + e^{-\frac{\pi h}{4T} - \pi y - \frac{\pi}{2T}}}{4} \left( 1 + e^{\frac{\pi h}{2T} + \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{e^{-\frac{\pi h}{4T}} - 2\pi x + e^{\frac{\pi h}{4T} - \pi y - \frac{\pi}{2T}}}{4} \left( 1 + e^{\frac{\pi h}{2T} + \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{e^{-\frac{\pi h}{4T}} - 2\pi x + e^{\frac{\pi h}{4T} - \pi y - \frac{\pi}{2T}}}{4} \left( 1 + e^{\frac{\pi h}{2T} + \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{e^{-\frac{\pi h}{4T}} + 2\pi x + e^{-\frac{\pi h}{4T} - \pi y + \frac{\pi}{2T}}}{4} \left( 1 + e^{-\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}} \right)^2 (1 + e^{-\frac{\pi h}{2T} + \pi y - \frac{\pi}{2T}}) T^2 \\
&\frac{2}{4} \left( 1 + e^{\frac{\pi h}{2T} - \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{2}{4} \left( 1 + e^{-\frac{\pi h}{2T} - \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{-\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{2}{4} \left( 1 + e^{-\frac{\pi h}{2T} - \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{-\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2 \\
&\frac{2}{4} \left( 1 + e^{-\frac{\pi h}{2T} - \pi y - \frac{\pi}{2T}} \right)^2 (1 + e^{-\frac{\pi h}{2T} + \pi y + \frac{\pi}{2T}}) T^2
\end{align*}
\]
\[
e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \pi \pi^2} \left( e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right) \left( -\frac{e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - \frac{\pi}{2T}}{2T} \right) y^2
- 2 \left( 1 + e^{\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right)^2 \left( 1 + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \right)^2 T
- \pi 2
\]
\[
e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \pi \pi^2} \left( e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right) \left( -\frac{e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - \frac{\pi}{2T}}{2T} \right) y^2
- 2 \left( 1 + e^{\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right)^2 \left( 1 + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \right)^2 T
- \pi 2
\]
\[
e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \pi \pi^2} \left( e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right) \left( -\frac{e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - \frac{\pi}{2T}}{2T} \right) y^2
- 2 \left( 1 + e^{\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right)^2 \left( 1 + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \right)^2 T
- \pi 2
\]
\[
e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \pi \pi^2} \left( e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right) \left( -\frac{e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - \frac{\pi}{2T}}{2T} \right) y^2
- 2 \left( 1 + e^{\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right)^2 \left( 1 + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \right)^2 T
- \pi 2
\]
\[
e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \pi \pi^2} \left( e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - e^{-\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right) \left( -\frac{e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} - \frac{\pi}{2T}}{2T} \right) y^2
- 2 \left( 1 + e^{\frac{\pi h}{2T} - \frac{\pi x}{2\pi} y} \right)^2 \left( 1 + e^{\frac{\pi h}{2T} + \frac{\pi x}{2\pi} y} \right)^2 T
- \pi 2
\]
\[
e^{-\frac{y}{2T} + 2\pi x + e^{-\frac{y}{2T} + 2\pi x + \frac{\pi x}{2T}} y + \frac{\pi x}{T} y^2} \left( e^{-\frac{2\pi x + \frac{\pi x}{2T} y}{2T}} + e^{\frac{\pi x}{2T} y - \frac{\pi x}{2T} y^2} \right) \frac{T}{2} \] 
\[
e^{-\frac{y}{2T} + 2\pi x + e^{-\frac{y}{2T} + 2\pi x + \frac{\pi x}{2T}} y + \frac{\pi x}{T} y^2} \left( e^{\frac{2\pi x + \frac{\pi x}{2T} y}{2T}} + e^{\frac{\pi x}{2T} y - \frac{\pi x}{2T} y^2} \right) \frac{T}{2} \] 
\[
e^{-\frac{y}{2T} + 2\pi x + e^{-\frac{y}{2T} + 2\pi x + \frac{\pi x}{2T}} y + \frac{\pi x}{T} y^2} \left( e^{-\frac{2\pi x + \frac{\pi x}{2T} y}{2T}} + e^{\frac{\pi x}{2T} y - \frac{\pi x}{2T} y^2} \right) \frac{T}{2} \] 
\[
e^{-\frac{y}{2T} + 2\pi x + e^{-\frac{y}{2T} + 2\pi x + \frac{\pi x}{2T}} y + \frac{\pi x}{T} y^2} \left( e^{\frac{2\pi x + \frac{\pi x}{2T} y}{2T}} + e^{\frac{\pi x}{2T} y - \frac{\pi x}{2T} y^2} \right) \frac{T}{2} \] 
\[
e^{-\frac{y}{2T} + 2\pi x + e^{-\frac{y}{2T} + 2\pi x + \frac{\pi x}{2T}} y + \frac{\pi x}{T} y^2} \left( e^{-\frac{2\pi x + \frac{\pi x}{2T} y}{2T}} + e^{\frac{\pi x}{2T} y - \frac{\pi x}{2T} y^2} \right) \frac{T}{2} \] 

Hence

\[
|((\lambda(x, y, h, T))^{(2)}_{xx})^{(2)}| < 4 \cdot 10^3 \frac{1}{T^2} e^{-y e^{\pi x}} (y e^{\pi x} + y^4 e^{4\pi x})
\]
and

\[
\int_0^\infty y^{2\sigma-1}(\ln(y))^2|((\lambda(x, y, h, T))^{(3)}_{hh}\rangle) dy < 4 * 10^3 \frac{1}{T^2} \int_0^\infty y^{2\sigma-1}(\ln(y))^2e^{-y\pi x} (ye^{\pi x} + y^4e^{4\pi x})dy < \frac{10^8(\ln T)^{4/3}(\ln \ln T)^{2/3}}{T^2}.
\]  

(52)

Here we use more precise condition for (0, y, T) = (1/T, y, -\theta, T)_{hh}.

Next we have

\[
(\gamma(0, y, T) - \gamma(1/T, y, -\theta, T))_{hh} = \frac{3e^{-\frac{3h}{2T}} + 3e^{-\frac{h}{2T}} \frac{3y}{T^2} y - \frac{3\pi}{T^2} y^3}{4(1 + e^{\frac{h}{2T}} \frac{3y}{T} y) + \frac{3e^{-\frac{3h}{2T}} + 3e^{-\frac{h}{2T}} \frac{3y}{T^2} y - \frac{3\pi}{T^2} y^3}{4(1 + e^{e^{\frac{h}{2T}} \frac{3y}{T} y})^3}} - \frac{3e^{-\frac{3h}{2T}} + 3e^{-\frac{h}{2T}} \frac{3y}{T^2} y + 2e^{\frac{h}{2T}} \frac{3y}{T^2} y + \frac{3\pi}{T^2} y^3}{4(1 + e^{e^{\frac{h}{2T}} \frac{3y}{T} y})^2} + \frac{3e^{-\frac{3h}{2T}} + 3e^{-\frac{h}{2T}} \frac{3y}{T^2} y + \frac{3\pi}{T^2} y^3}{4(1 + e^{e^{\frac{h}{2T}} \frac{3y}{T} y})^4}.
\]

(20)
Hence

\[
\frac{1}{6} \int_0^\infty y^{2\sigma-1} \ln(y)^2 \left( \gamma(0, y, \theta, T) - \gamma(1, y, -\theta, T) \right) dy < \frac{1000}{T^3} (y + y^3) e^{-y}.
\]

(53)

Here we use the fact that each term in the previous expression is less than \(30(y + y^3)e^{-y}/T^3\) and the number of terms is less that 30. From here it follows the inequality

\[
\frac{1}{6} \int_0^\infty y^{2\sigma-1} \ln(y)^2 \left( \gamma(0, y, \theta, T) - \gamma(1, y, -\theta, T) \right) dy < \frac{2000}{T^3} \left( \int_0^1 (\ln y)^2 dy + \int_1^\infty y^6 e^{-y} dy \right) < \frac{2000}{T^3} (2 + 6!) < \frac{1.5 \cdot 10^7}{T^3}
\]

(54)
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