An Efficient Algorithm for Optimal Solving a Shortest Vector Problem in Compute-and-Forward Design

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Abstract—We consider the problem of finding the optimal coefficient vector that maximizes the computation rate at a relay in the compute-and-forward scheme. Based on the idea of sphere decoding, we propose a highly efficient algorithm that finds the optimal coefficient vector. First, we derive a novel algorithm to transform the original quadratic form optimization problem into a shortest vector problem (SVP) using the Cholesky factorization. Instead of computing the Cholesky factor explicitly, the proposed transform the original quadratic form optimization problem into the compute-and-forward scheme. Based on the idea of sphere decoding, we propose a highly efficient algorithm that finds the optimal coefficient vector. Finally, by taking into account some useful properties of the optimal coefficient vector, we modify the Schnorr-Euchner search algorithm to solve the SVP. We show that the estimated average complexity of our new algorithm is \(O(n^{1.5} P^{0.5})\) flops for i.i.d. Gaussian channel entries with SNR \(P\) based on the Gaussian heuristic. Simulations show that our algorithm is not only much more efficient than the existing ones that give the optimal solution, but also faster than some best known suboptimal methods. Besides, we show that our algorithm can be readily adapted to output a list of \(L\) best candidate vectors for use in the compute-and-forward design. The estimated average complexity of the resultant list-output algorithm is \(O(n^{1.5} P^{0.5} \log L + nL)\) flops for i.i.d. Gaussian channel entries.

Index Terms—Wireless relay network, slow-fading, compute-and-forward, computation rate, Cholesky factorization, shortest vector problem, sphere decoding.

I. INTRODUCTION

In relay networks, compute-and-forward (CF) is a promising relaying strategy that can offer higher rates than traditional ones (e.g., amplify-and-forward, decode-and-forward), especially in the moderate SNR regime. The crucial idea of CF is the application of linear/lattice codes and physical layer network coding (PLNC). The received signal at a relay is the linear combination of a set of transmitted signals, where the linear combination coefficients form the channel vector from the involved sources to that relay. Through multiplying the channel vector by an amplifying factor, the obtained new channel vector can be close to a coefficient vector with all integer-valued entries. This means that after applying an appropriate amplifying factor to the received signal at a relay, it will be approximately an integer linear combination of the transmitted signals. Since the same linear code is used at the sources, an integer linear combination of valid codewords is still a valid codeword, which means the aforementioned integer linear combination of the transmitted signals is possible to be successfully decoded as a linear combination of the messages corresponding to the transmitted signals. Under certain conditions, with a sufficient number of such decoded linear combinations, the transmitted messages can be recovered.

Obviously, the amplifying factors and the integer-valued coefficient vectors need to be carefully designed. When Nazer and Gastpar proposed the CF scheme in [1], they defined the computation rate, which refers to the maximum transmission rate at the involved sources of a relay such that the combined signals at the relay can be reliably decoded. Transmission rate, which is the minimum computation rate over all relays, determines the system performance. The transmission rate becomes 0 if the coefficient matrix formed with rows being the coefficient vectors at the relays is not of full rank [1]. It has been pointed out that setting the amplifying factor at a relay as the minimum-mean-square-error (MMSE) coefficient can maximize the computation rate at that relay. The difficulty lies in the design of the coefficient vectors. To optimize the system performance, the coefficient vectors have to be designed jointly. However, this requires each relay (or the destination instead) to know the channel state information (CSI) at the other relays, which could incur too much communication overhead in practice for large networks. Also, the joint optimization problem could be far too complex to solve. One alternative is to firstly develop a search algorithm to find good coefficient vectors at each relay with the criterion being maximizing the computation rate at that relay, and then apply a certain strategy to coordinate relays in selection of the coefficient vectors. This is reasonable when only the local CSI is available at each relay, or when the network is large. Unfortunately, the problem is difficult even for finding the coefficient vector that maximizes the computation rate at one relay, as it turns out to be a shortest vector problem (SVP) in
a lattice. In this paper, we shall first focus on developing the search algorithm for finding the optimal coefficient vector at a relay (defined as the one that maximizes the computation rate at that relay). After that, we will show how to adapt our algorithm such that it can be used for solving the CF design problem.

The SVP of finding the optimal coefficient vector at a relay has attracted a lot of research interests, and various methods have been proposed to solve the problem. The Fincke-Pohst method [4] was modified in [5] to solve a different but related problem, leading to the optimal coefficient vector and some other suboptimal vectors. A branch-and-bound algorithm, which uses some properties of the optimal vector, was proposed in [6]. But it appears that this algorithm is not very efficient in this application. There are some more efficient methods that give suboptimal solutions. Three suboptimal methods were proposed in [7]: a method based on the complex LLL [8], a simple quantized search method, which has been improved in [9], and an iterative MMSE-based quantization method. Although the average complexity of the LLL algorithm [10] is polynomial if the entries of the basis vectors independently follow the normal distribution $\mathcal{N}(0,1)$ (see, e.g., [11, 12]), the complexity of the first method could be too high since it has been proved in [11] that in the MIMO context, the worst-case complexity of the LLL algorithm is not even finite. The last two methods are of lower complexity, but they may not offer the desirable performance-complexity tradeoff, especially when the dimension is large. Besides these, the suboptimal quadratic programming relaxation method in [13] and its improvement in [14], are of relatively low complexity. Although their performance in terms of the computation rate are better than that of the last two methods proposed in [7], the difference between their performance and that of the optimal methods becomes obvious as the dimension and/or the SNR get large.

In this paper (an earlier version of this paper has been posted on arXiv.org), we propose an efficient algorithm for finding the optimal coefficient vector that maximizes the computation rate at a relay. First, we will derive an efficient algorithm with only $O(n)$ flops to transform the problem to a SVP by fully exploiting the structure of the Gram matrix to perform its Cholesky factorization (we do not form the whole Cholesky factor $R$ explicitly). Note that the complexity of the regular algorithm for Cholesky factorization is $O(n^3)$. We will also propose some conditions that can be checked with $O(n)$ flops, under which $e_1$ (the first column of the $n \times n$ identity matrix) is the optimal coefficient vector. Then, we will propose a modified Schnorr-Euchner search algorithm to solve the SVP by taking advantage of the properties of the optimal solution. Based on the Gaussian heuristic, we show that the average complexity of our new algorithm is around $O(n^{1.5}p^{0.5})$ flops for i.i.d. Gaussian channel entries with SNR $P$. Numerical results will be given to show the effectiveness and efficiency of our algorithm. Besides, we will show how to modify the proposed algorithm such that it can output a list of good coefficient vectors for use in the CF design.

Preliminary results of this work have been partly presented in a conference paper [15]. Compared with [15], this work contains the following new contributions:

- We use a new method to perform the Cholesky factorization to transform the optimization problem into a SVP which reduces the complexity from $O(n^3)$ to $O(n)$.
- Some properties of the Cholesky factor $R$ are characterized.
- We provide some conditions which guarantee that $e_1$ is an optimal coefficient vector, and these conditions can be checked with $O(n)$ flops.
- Some new improvements on the modified Schnorr-Euchner search algorithm [15] are made which further accelerates the algorithm.
- In addition to providing more simulation results to demonstrate the effectiveness and efficiency of our algorithm, we show that the estimated average complexity of our new algorithm is $O(n^{1.5}p^{0.5})$ flops for i.i.d. Gaussian channel entries based on the Gaussian heuristic.
- We show how to adapt the proposed algorithm so that it can be applied in CF design.

An algorithm with the average complexity of $O(n^{2.5}p^{0.5})$ flops for i.i.d. Gaussian channel entries was proposed in [16]. This algorithm finds the optimal solution by solving an optimization problem with one variable over a bounded region, which is totally different from our proposed algorithm. Simulations in Section VI also indicate that our algorithm is much more efficient.

The rest of the paper is organized as follows. In Section II, we start with introducing the coefficient vector design problem in CF. Then, in Section III we develop a new algorithm to solve the problem. We analyze the complexity of our proposed method in Section IV and present some numerical results in Section V In Section VI we show how to modify our algorithm for CF design. Finally, conclusions are given in Section VII.

**Notation.** Let $\mathbb{R}^n$ and $\mathbb{Z}^n$ be the spaces of the $n$-dimensional column real vectors and integer vectors, respectively. Let $\mathbb{R}^{m \times n}$ and $\mathbb{Z}^{m \times n}$ be the spaces of the $m \times n$ real matrices and integer matrices, respectively. Boldface lowercase letters denote column vectors and boldface uppercase letters denote matrices, e.g., $t \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. For a vector $t$, $\|t\|_2$ denotes the $\ell^2$-norm of $t$ and $t^T$ denotes the transpose of $t$. For $t \in \mathbb{R}^n$, we use $\lfloor t \rfloor$ to denote its nearest integer vector, i.e., each entry of $t$ is rounded to its nearest integer (if there is a tie, the one with smaller magnitude is chosen). Let $t_i$ be the element with index $i$ and $t_{i:j}$ be the vector composed of elements with indices from $i$ to $j$. $\lfloor t_i \rfloor$ denotes the smallest integer larger than or equal to $t_i$. For a matrix $A$, let $a_{ij}$ be the element at row $i$ and column $j$, $A_{i:j,k}$ be the submatrix containing elements with row indices from $i$ to $j$ and column indices from $k$ to $l$, and $A_{i:j,k:l}$ be the vector containing elements with row indices from $i$ to $j$ and column index $k$. Let $0^n$ and $0^{m \times n}$ denote the $n$-dimensional zero column vector and $m \times n$ zero matrix, respectively. Let $e_k^n$ and $I^n$ denote the $k$-th column of an $n \times n$ identity matrix $I$ and $n$-dimensional vector with all of its entries being 1, respectively. Sometimes the superscripts are omitted if the dimensions are obvious.
II. Problem Statement

We consider the problem of finding the optimal coefficient vector that maximizes the computation rate (defined in (1)) at a relay in the CF scheme. The application scenario we focus on is the wireless relay network with slow-fading channels and additive white Gaussian noise (AWGN). Sources, relays, and destinations are linked with slow-fading channels, and AWGN exists at each receiver. For the ease of explanation, we will focus on the real-valued channel model in the sequel.

Definition 1: (Channel Model) As shown in Figure 1, each relay (indexed by \( i = 1, 2, \ldots, n \)) observes a noisy linear combination of the transmitted signals through the channel

\[
y_i = \sum_{j=1}^{n} h_{i}(j) x_j + z_i,
\]

where \( x_j \in \mathbb{R}^N \) with the power constraint \( \frac{1}{N} \| x_j \|_2^2 \leq P \) is the transmitted codeword from source \( j (j = 1, 2, \ldots, n) \), \( h_{i} \in \mathbb{R}^N \) is the channel vector to relay \( i \) (here \( h_{i}(j) \) denotes the \( j \)-th entry of \( h_{i} \)), \( z_i \in \mathbb{R}^N \) is the noise vector with entries being i.i.d. Gaussian, i.e., \( z_i \sim \mathcal{N}(0, I) \), and \( y_i \) is the signal received at relay \( i \).

![Fig. 1. Channel Model](image)

For relay \( i \) with the channel vector \( h_i \), let \( a_i \) be the chosen coefficient vector, the computation rate is calculated according to the following theorem (1).

Theorem 1: The computation rate at relay \( i \) is uniquely maximized by choosing the amplifying factor as the MMSE coefficient, which results in a computation rate

\[
R(h_i, a_i) = \frac{1}{2} \log^+ \left( \frac{1}{\|a_i\|^2} - \frac{P(h_i^T a_i)^2}{1 + P \|h_i\|_2^2} \right), \tag{1}
\]

where \( \log^+ (x) \) is the upper incomplete logarithm function with respect to base 2 and \( \log^+(x) \triangleq \max(\log(x), 0) \).

Also, we define the optimal coefficient vector for a relay as below.

Definition 2: (The Optimal Coefficient Vector) The optimal coefficient vector \( a_i^* \) for a relay with channel vector \( h_i \) is the one that maximizes the computation rate,

\[
a_i^* = \arg \max_{a_i \in \mathbb{Z}^n \setminus \{0\}} R(h_i, a_i). \tag{2}
\]

The optimization problem (2) can be further formulated as the following problem (3):

\[
a_i^* = \arg \min_{a_i \in \mathbb{Z}^n \setminus \{0\}} a_i^T G_i a_i, \tag{3a}
\]

\[
G_i = I - \frac{P}{1 + P \|h_i\|_2^2} h_i h_i^T. \tag{3b}
\]

Hereafter, we will focus on relay \( i \), and thus ignore the subscript “\( i \)”, e.g., \( h_i \) will be directly written as \( h \). In the next section, an efficient method to solve (3) based on sphere decoding will be provided.

III. Proposed Method

Define the scaled channel vector \( t \) as

\[
t = \sqrt{\frac{P}{1 + P \|h\|_2^2}} h. \tag{4}
\]

Then, (3) is equivalent to the following problem:

\[
a^* = \arg \min_{a \in \mathbb{Z}^n \setminus \{0\}} a^T G a, \tag{5a}
\]

\[
G \triangleq I - t t^T. \tag{5b}
\]

Obviously, \( ||t||_2 < 1 \) and \( G \) is symmetric positive definite. Throughout this paper, we assume \( h \neq 0 \); otherwise, it is trivial.

The problem in (5) can be solved via the following two steps:

- First, for a given \( t \), compute \( G \) and find its Cholesky factorization, i.e., find an upper triangular matrix \( R \) such that \( G = R^T R \). Then (5) is equivalent to the following shortest vector problem (SVP),

\[
a^* = \arg \min_{a \in \mathbb{Z}^n \setminus \{0\}} \| R a \|_2. \tag{6}
\]

- Second, use a search algorithm, such as the LLL-aided Schnorr-Euchner search strategy [17], to solve (6). We will explain the details later.

It is easy to see that for a given \( t \), computing \( G \) costs \( \mathcal{O}(n^2) \) flops. Besides, it is well-known that computing the Cholesky factorization of a general \( n \times n \) matrix \( R \) such that \( G = R^T R \) costs \( \mathcal{O}(n^3) \) flops. Moreover, the complexity of the LLL-aided Schnorr-Euchner search strategy [17] for solving (6) may be too high. Fortunately, we find out that it is possible to accelerate the aforementioned two steps as follows:

- First, take advantage of the special structure of \( G \) in (5b) to compute its Cholesky factorization and transform (5) to (6), but do not explicitly form \( G \), the whole \( R \) and the SVP.

- Second, investigate the properties of a solution \( a^* \) to (6) and take them into account to modify the Schnorr-Euchner search strategy [17] to find \( a^* \).

Obviously, if \( a^* \) is a solution of (6), so is \(-a^*\). For simplicity, we apply the following restriction.

Restriction 1: Throughout this paper, we restrict the solution \( a^* \) to (6) such that \( t^T a^* \geq 0 \).
A. Transformation of the problem

To transform (5) to the SVP, we need to find the Cholesky factor $R$ of $G$ in (5b). Besides the regular method, one can use the algorithm proposed in [18], which costs $2n^2 + O(n)$ flops, to get the Cholesky factor $R$. However, the complexity can be further reduced. Also, to analyze the complexity of our proposed search algorithm in Section IV, we need to know the diagonal entries of $R$. In this subsection, we will take into account the special structure of $G$ to achieve this goal with only $O(n)$ flops (we do not form the whole $R$ explicitly. If the whole $R$ is needed for other applications, it costs $n^2/2 + O(n)$ flops). Based on the diagonal entries of $R$ and by investigating their properties, we will also propose some conditions that can be checked with $O(n)$ flops, under which the optimal solution $a^*$ can be obtained immediately without using any search algorithm.

Our proposed algorithm to find the Cholesky factor $R$ of $G$ in (5b) is based on the following theorem:

**Theorem 2:** The Cholesky factor $R$ of $G$ in (5b) is given by:

$$
 r_{ij} = \begin{cases} 
 1 - \sum_{k=1}^{i} t_{ik}^2, & j = i \\
 \sqrt{1 - \sum_{k=1}^{i} t_{ik}^2} \sqrt{1 - \sum_{k=1}^{j} t_{kj}^2}, & i < j \leq n,
 \end{cases}
$$

where $1 \leq i \leq n$ and denote $\sum_{k=1}^{0} = 0$.

**Proof.** To prove the theorem, we show any element of $G$ is equal to the corresponding element of $R^T R$ in the same position, i.e., by (5b), we would like to show

$$
\sum_{i=1}^{k} r_{ik}^2 = 1 - t_{kk}^2, \quad 1 \leq k \leq n
$$

and

$$
\sum_{i=1}^{k} r_{ik} r_{ij} = -t_{kj}, \quad 1 \leq k < j \leq n.
$$

By (7), we have

$$
\sum_{i=1}^{k} r_{ik}^2 = r_{kk}^2 + \sum_{i=1}^{k-1} r_{ik}^2
$$

and

$$
\sum_{i=1}^{k} r_{ik} r_{ij} = -t_{kj}, \quad 1 \leq k < j \leq n.
$$

Thus, both (8) and (9) hold, completing the proof. □

We can use Theorem 2 to design an efficient algorithm to find $R$. To simplify notation, we introduce an $(n+1)$-dimensional vector variable $f$ Let

$$
f_0 = 1, \quad f_i = 1 - \sum_{l=1}^{i} t_{il}^2, \quad 1 \leq i \leq n.
$$

Then by (7), we have

$$
r_{ii} = \sqrt{f_i/f_{i-1}}, \quad 1 \leq i \leq n,
$$

and

$$
R_{i,i+1:n} = (-t_{i}/\sqrt{f_i f_{i-1}})t_{i+1:n}^T, \quad 1 \leq i \leq n.
$$

Note that $R_{i,i+1:n}$ is a scaled $t_{i+1:n}$.

After getting $R$, we will modify the Schnorr-Euchner search algorithm to solve (6). Later we will see that it is not necessary to form $R$ explicitly (we will give more details to explain this in Section IV-D, i.e., we do not need to compute the multiplication of $-t_{i}/\sqrt{f_{i} f_{i-1}}$ and $t_{i+1:n}$). Thus, the complexity of obtaining $R$ is $O(n)$ flops.

By (7), it is easy to see that $R$ has the following properties which are useful to analyze the complexity of our proposed search algorithm.

**Theorem 3:** For $1 \leq k \leq n$, the following inequalities hold:

$$
\sqrt{1 - \sum_{i=1}^{k} t_{ik}^2} \leq r_{kk} \leq \sqrt{1 - t_{kk}^2},
$$

and

$$
\prod_{i=k}^{n} r_{ii} = \frac{\sqrt{1 - \sum_{i=1}^{k} t_{ik}^2}}{\sqrt{1 - \sum_{i=1}^{k-1} t_{ik}^2}} \geq \sqrt{1 - \sum_{i=1}^{k} t_{ik}^2}.
$$

**Proof.** The first inequality in (11) follows direct from (7) and the basic fact that $1 - \sum_{i=1}^{k-1} t_{ik}^2 \leq 1$ for $1 \leq i \leq n$ (recall that we define $\sum_{i=1}^{0} = 0$).

The second inequality in (11) follows direct from (7) and some basic calculations.

The equality in (12) follows direct from (7), and the inequality in (12) follows from the basic fact that $1 - \sum_{i=1}^{k-1} t_{ik}^2 \leq 1$.
for $1 \leq i \leq n$. □

By Theorem 2, we have the following interesting result, which can be used to describe the geometry of the search space later.

Theorem 4: For $1 \leq i < j \leq n$, the eigenvalues of $R_{i:j,i:j}^T R_{i:j,i:j}$ are 1 with algebraic multiplicity $j - i$ and $f_j / f_{i-1}$, where $f$ is defined in (10).

Proof. We first prove

$$R_{i:n,i:n}^T R_{i:n,i:n} = I_{n-i+1} - \frac{t_{i:n}}{f_{i-1}} \frac{t_{i:n}^T}{f_{i-1}}.$$  \hspace{1cm} (13)

If $i = 1$, then by Theorem 2 and (10), (13) holds. So we only need to prove it holds for $i > 1$.

By Theorem 2 we have

$$[R_{i:n,i:n}^T R_{i:n,i:n}] = [I_{n-i+1} - \frac{t_{i:n}}{f_{i-1}} \frac{t_{i:n}^T}{f_{i-1}}].$$

The right bottom parts of both sides are the same, thus,

$$R_{i:n,i:n}^T R_{i:n,i:n} = I_{n-i+1} - t_{i:n}^T R_{i:n,i:n} - R_{i:n,i:n} R_{i:n,i:n}^T.$$

By Theorem 2 and (10), we have

$$R_{i:n,i:n} = (t_{i:n} \left[ \frac{t_1}{f_1 f_2} \frac{t_2}{f_2 f_3} \ldots \frac{t_{i-1}}{f_{i-2} f_{i-1}} \right]^T).$$

Thus, we obtain

$$R_{i:n,i:n}^T R_{i:n,i:n} = I_{n-i+1} - (1 + \sum_{k=1}^{i-1} \frac{t_k^2}{f_k f_{k-1}}) t_{i:n}^T t_{i:n}.$$  \hspace{1cm} (13)

By (10),

$$\frac{t_k^2}{f_k f_{k-1}} = \frac{1}{f_k} - \frac{1}{f_{k-1}}.$$

Therefore,

$$R_{i:n,i:n}^T R_{i:n,i:n} = I_{n-i+1} - \frac{t_{i:n}}{f_{i-1}} \frac{t_{i:n}^T}{f_{i-1}}.$$  \hspace{1cm} (13)

i.e., (13) holds.

From (13), we can immediately get

$$R_{i:j,i:j}^T R_{i:j,i:j} = I_{j-i+1} - \frac{t_{i:j}}{f_{j-1}} \frac{t_{i:j}^T}{f_{j-1}}.$$  \hspace{1cm} (13)

Thus, the eigenvalues of $R_{i:j,i:j}^T R_{i:j,i:j}$ are 1’s and

$$1 - \sum_{k=1}^{j-i} \frac{t_k^2}{f_k} = f_j / f_{i-1}.$$

Equivalently, we have

$$1 - \sum_{k=1}^{j-i} \frac{t_k^2}{f_k} = f_j / f_{i-1}.$$  \hspace{1cm} (13)

Thus, from (13) holds, we have

$$\min_{1 \leq i \leq n} \left( 1 - \sum_{j=1}^{i-1} \frac{t_j^2}{f_j} \right) = \sqrt{1 - t_1^2}.$$  \hspace{1cm} (13)

Therefore, the first inequality in (14) becomes an equality with $a^* = e_1$, so $e_1$ is a solution to (6).

It is easy to see that (13) can be checked by $O(n)$ flops.

Remark 1: From (59), we can see that

$$\min_{1 \leq i \leq n} \left( 1 - \sum_{j=1}^{i-1} \frac{t_j^2}{f_j} \right) = \min_{1 \leq i \leq n} \left( 1 - \sum_{j=1}^{i-1} \frac{t_j^2}{f_j} \right).$$

Thus, if $\max_{1 \leq i \leq n} |t_i| \neq |t_1|$, then $e_1$ cannot be an optimal solution. Thus, (15) does not hold. However, we can order the entries of $t$ such that after the ordering, $\max_{1 \leq i \leq n} |t_i| = |t_1|$. Clearly, doing this can increase the probability of (15) holds.

And $e_j$ with $j$ satisfying $\max_{1 \leq i \leq n} |t_i| = |t_j|$ (here $t$ is the vector before using the transformation) is an optimal solution.
Naturally, it is interesting to know how often does \((15)\) hold? We will do some simulations for this in the end of next subsection.

**B. Reordering the entries of \(t\)**

After getting \((6)\), a search algorithm, such as the Schnorr-Euchner search strategy \((17)\) can be used to solve it, i.e., finding the shortest nonzero vector of the lattice \(L(R)\), which is defined by

\[
L(R) = \{ Rz | z \in \mathbb{Z}^n \}.
\]

The columns of \(R\) form a basis of \(L(R)\) (note that the basis of a lattice is not necessary an upper triangular matrix, but it must be a full column rank matrix). For any \(n \geq 2\), \(L(R)\) has infinity many bases and any of two are connected by a unimodular matrix \(Z\), i.e., \(Z \in \mathbb{Z}^{n \times n}\) and \(\text{det}(Z) = \pm 1\). Specifically, for each given lattice basis matrix \(R \in \mathbb{R}^{n \times n}\), \(RZ\) is also a basis matrix of \(L(R)\) if and only if \(Z\) is unimodular, see, e.g., \((20)\). The process of selecting a good basis for a given lattice, given some criterion, is called lattice reduction. In many applications, it is advantageous if the basis vectors are short and close to be orthogonal \((20)\). For more than a century, lattice reduction have been investigated by many people and several types of reductions have been proposed, including the KZ reduction \((21)\) (an efficient KZ reduction algorithm can be found in \((22)\)), the Minkowski reduction \((23)\) (an efficient Minkowski reduction algorithm can be found in \((24)\)), the LLL reduction \((10)\) and Seysen’s reduction \((25)\) etc.

For efficiency, lattice reduction for \(R\) in \((6)\) is usually used to strive for

\[
r_{11} \leq r_{22} \leq \ldots \leq r_{nn}
\]

(\(17)\) to accelerate searching. Notice that \((17)\) may not be achievable. For more details on why \((17)\) is desirable for, readers are referred to, e.g., \((20)\) and \((26)\).

The LLL reduction \((10)\) is a commonly used reduction method to strive for \((17)\). However, for this application, it has two main drawbacks. First, its complexity is high. In fact, when it was shown in \((11)\) that in the MIMO context, the worst-case complexity is not even finite. For more details, see, e.g., \((10)\), \((27)\) and \((12)\). Also, from the simulation results in Section \(VI\) we will see that the complexity of the LLL reduction is even higher than that of our proposed algorithm. Second, it may destroy the structure of \(R\) and some properties of the optimal solution \(a^*\) to the reduced problem (we will explain this in the latter part of this subsection). In this subsection, we will propose a method to strive for \((17)\) without the above shortcomings.

From \((7)\), to strive for \((17)\), we permute the entries of \(t\). To make \(r_{11}\) as small as possible, we permute \(t\) such that \(|t_1|\) is the largest. Suppose that \(t_j, 1 \leq j \leq i\) have been fixed, then from \((7)\), \(r_{jj} 1 \leq j \leq i\) are fixed. To make \(r_{j+1,j+1}\) as small as possible, we permute the entries of \(t_j, i + 1 \leq j \leq n\) such that \(|t_{j+1}|\) is the largest. So after the permutations we have

\[
|t_1| \geq |t_2| \geq \ldots \geq |t_n|.
\]

(\(18)\)

Here we want to point out the above idea of reordering the entries of \(t\) is actually the same as that of SQRD \((28)\), a column reordering strategy for a general matrix in the box-constrained integer least squares (BILS) problem \((29), (30)\). It is interesting to note that if we use the idea of V-BLAST \((31)\), another column reordering strategy used in solving BILS problems \((32)\), we will get the same ordering of \(t\). In fact, by \((7)\),

\[
r_{nn}^2 = \frac{1 - |t|^2}{1 - |t|^2 + t_n^2}.
\]

Thus, to make \(r_{nn}\) as large as possible, we need to permute \(t\) such that \(|t_n|\) is the smallest. Suppose that \(t_j, i + 1 \leq j \leq n\) have been fixed, then from \((7)\), \(r_{jj}, i + 1 \leq j \leq n\) are fixed. By \((7)\),

\[
r_{ii}^2 = \frac{1 - |t|^2 + \sum_{j=i+1}^n t_j^2}{1 - |t|^2 + \sum_{j=i+1}^n t_j^2 + t_i^2}.
\]

Thus, to make \(r_{jj}\) as large as possible, we permute the entries of \(t_j, 1 \leq j \leq i\) such that \(|t_i|\) is the smallest. So after the permutations we also have \((18)\).

To make the search process faster, we also want to make \(t_i \geq 0\) for \(1 \leq i \leq n\). This can easily be done. In fact, when we determine the \(i\)-th entry of \(t\) in the permutation process, we can use a sign permutation matrix so that the new \(i\)-th entry is nonnegative. Thus, eventually we have

\[
t_1 \geq t_2 \geq \ldots \geq t_n \geq 0.
\]

(\(19)\)

The above process can be described mathematically as follows. For any given \(t\), it is easy to find a signed permutation matrix \(Z \in \mathbb{Z}^{n \times n}\) such that \(\bar{t} = Zt\) satisfying:

\[
\bar{t}_1 \geq \bar{t}_2 \geq \ldots \geq \bar{t}_n \geq 0.
\]

This transformation is a sorting process and the complexity is \(O(n \log(n))\), see \((13)\) for more details. Note that \(ZZ^T = I\).

Then, with \(\bar{a} = Za\), the optimization problem \((5)\) can be transformed to

\[
\bar{a}^* = \min_{\bar{a} \in \mathbb{Z}^n \setminus \{0\}} \bar{a}^T \bar{G} \bar{a},
\]

\[
\bar{G} = \bar{I} - \bar{u} \bar{u}^T.
\]

Obviously \(\bar{a}^* = Z^T \bar{a}^*\).

Therefore, for the sake of convenience, in our later analysis, we assume \(t\) satisfies \((19)\).

In addition to speeding up the search, ordering the entries of \(t\) like in \((19)\) has another important effect, i.e., by the results in \((6)\) and \((15)\), if \((19)\) holds, we can find a solution \(a^*\) to \((6)\) such that

\[
a_1^* \geq a_2^* \geq \ldots \geq a_n^* \geq 0.
\]

(\(20)\)

The order of the elements of the solution \(a^*\) in \((20)\) is a key property of the solution we restricted for \((6)\). It has been used in \((13)\) to find a suboptimal solution to \((4)\), but only the property that \(a_i \geq 0, 1 \leq i \leq n\) has been used in \((6)\) to solve \((6)\). In this paper, we will take full advantage of it in designing the search algorithm. Note that, if the LLL reduction is used for reducing \(R\) in \((6)\), then \((20)\) may not hold, which is the second drawback of using the LLL reduction in striving for
The motivation for reordering the entries of \( t \) in [6] and [13] is to obtain the property [20], which was (partially) used in their methods. Here we gave another motivation from the search point of view.

Under [19] and Theorem 2 we have the following interesting results:

**Theorem 6:** If [19] holds, then for \( 1 \leq i \leq n - 1 \)
\[
R_{i:i+1,i+1} \leq R_{i:i+2,i+2} \leq \ldots \leq R_{i:n,n}
\]
\[
\tag{21}
\]

**Proof.** By (7), for \( 1 \leq i < j \leq n \),
\[
\left\| R_{i:j} \right\|^2 = \sum_{k=1}^{j-1} r_{kk}^2 + r_{jj}^2
\]
\[
= \sum_{k=1}^{j-1} \left( \frac{t_{kk}^2}{(1 - \sum_{l=1}^{j-1} t_{ll}^2)(1 - \sum_{l=1}^{j-1} t_{ll}^2)} + \frac{1 - \sum_{l=1}^{j-1} t_{ll}^2}{(1 - \sum_{l=1}^{j-1} t_{ll}^2)} \right) + \frac{t_{jj}^2}{(1 - \sum_{l=1}^{j-1} t_{ll}^2)}
\]
\[
= 1 - \frac{t_{jj}^2}{(1 - \sum_{l=1}^{j-1} t_{ll}^2)}. 
\]

By the aforementioned equations, (7) and (19), it is easy to see that (21) holds. \( \square \)

From Theorem 5 we can see that if (15) holds, \( e_1 \) is a solution to (6). In the following, we use some simulations to show how often does (15) hold? For each \( n \) and \( P \), we randomly generate 10000 realizations of \( h \). Then, we compute \( t \) by (4) and transform it such that (19) holds. Tables I and II respectively show the total number of cases over 10000 realizations that (15) holds for \( n \) from 2 to 16 with step 2 and \( n \) from 100 to 800 with step 100. From Tables I and II we can see that, (15) holds with a high probability when both \( n \) and \( P \) are small; however, it holds with a very low probability when both \( n \) and \( P \) are very large.

| \( p \) | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|---|
| 0 dB | 3817 | 4048 | 4948 | 5255 | 5778 | 6461 | 3486 | 3486 |
| 10 dB | 6148 | 3767 | 1728 | 1222 | 1025 | 881 | 790 | 731 |
| 20 dB | 4903 | 944 | 413 | 223 | 146 | 101 | 90 | 59 |

**TABLE II**

Number of \( e_1 \) being the optimal solution over 10000 realizations of \( h \)

| \( p \) | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
|---|---|---|---|---|---|---|---|---|
| 0 dB | 4837 | 4838 | 4568 | 4065 | 3540 | 3324 | 2866 | 2589 |
| 10 dB | 196 | 64 | 35 | 15 | 9 | 5 | 2 | 0 |
| 20 dB | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**C. Schnorr-Euchner search algorithm**

We first introduce a depth-first tree search algorithm: the Schnorr-Euchner search algorithm [17], [20], a variation of the Fincke-Pohst search strategy [4], to solve a general SVP, which has the form of (6). Note that, the Schnorr-Euchner algorithm is generally more efficient than the Fincke-Pohst, for more details, see, e.g., [20]. Then we modify it by using the properties of \( R \) and the optimal solution \( \alpha^* \) to make the search process faster.

Let the optimal solution be within the following hyper-ellipsoid:
\[
\| Ra \|^2 < \beta^2, \tag{22}
\]
where \( \beta \) is a constant. Define
\[
d_n = 0, \quad d_k = -\frac{1}{r_{kk}} \sum_{j=k+1}^{n} r_{kj} a_j, \quad k = n - 1, \ldots, 1. \tag{23}
\]
Then (22) can be written as:
\[
\sum_{i=1}^{n} r_{ii}^2 (a_i - d_i)^2 < \beta^2
\]
which is equivalent to
\[
r_{kk}^2 (a_k - d_k)^2 < \beta^2 - \sum_{j=k+1}^{n} r_{jj}^2 (a_j - d_j)^2 \tag{24}
\]
Algorithm 1: Schnorr-Euchner search algorithm

**Input:** A nonsingular upper triangular matrix \( R \in \mathbb{R}^{n \times n} \)

**Output:** A solution \( \mathbf{a}^* \) to the SVP in (6)

1. (Initialization) Set \( k = n, \beta = +\infty \).
2. Compute \( d_k \) by using (23), set \( a_k = |d_k| \) and \( s_k = \text{sgn}(d_k - a_k) \) (see (23)).
3. (Main Step) If the inequality in (24) does not hold, then go to Step 4. Else if \( k > 1 \), set \( k = k - 1 \) and go to Step 2. Else \((k = 1)\), go to Step 5.
4. (Outside ellipsoid) If \( k = n \), terminate. Else, set \( k = k + 1 \) and go to Step 6.
5. (A valid point is found) If \( \mathbf{a} \) is a nonzero vector, then save \( \mathbf{a}^* = \mathbf{a} \), set \( \beta = \| Ra \|_2 \) and \( k = k + 1 \).
6. (Enumeration at level \( k \)) Set \( a_k = a_k + s_k \), \( s_k = -s_k - \text{sgn}(s_k) \) and go to Step 3.

the corresponding search radius is

\[
\beta = |r_{11}| = \sqrt{1 - t_1^2}, \quad (26)
\]

Note that reordering the entries of \( \mathbf{t} \) that makes (15) hold gives the smallest \( \beta \) among any other orderings. This shows one of the benefits of the reordering leading to (19). Also from (21), the reordering gives

\[
\beta = |r_{11}| = \min_{1 \leq i \leq n} \| R_{1:i,i} \|_2,
\]

which implies \( \beta \) is better than any other \( \beta_i \), for \( i = 2, \ldots, n \), as the former corresponds to the smallest residual. In the modified algorithm, we just start with \( \beta \) given by (26).

In Section III-A we mentioned that it is not necessary to form the entries of \( \mathbf{R} \) explicitly; in the following, we show how to compute \( r_{kk} \) and \( d_k \) for \( 1 \leq k \leq n \), which are needed in (24). By (7) and (10), we have

\[
r_{kk}^2 = f_k/f_{k-1}, \quad 1 \leq k \leq n. \quad (27)
\]

In the modified algorithm, we will use a \( n \)-dimensional vector \( \mathbf{q} \) to store \( r_{kk}^2 \), i.e., let \( q_k = r_{kk}^2 \).

By (7), (10) and (23),

\[
d_k = \frac{t_k}{f_k} \sum_{j=k+1}^{n} t_j a_j.
\]

Thus, for computational efficiency, we introduce an \((n + 1)\)-dimensional vector \( \mathbf{p} \) with \( p_{n+1} = 0 \) to store some computed quantities. Specifically, after \( a_k \), \( 1 \leq k \leq n \) is chosen in the search process, we assume

\[
p_k = p_{k+1} + t_k a_k, \quad 1 \leq k \leq n, \quad (28)
\]

which explains why \( p_{n+1} = 0 \). Therefore, we have

\[
d_k = \frac{t_k p_{k+1}}{f_k}, \quad 1 \leq k \leq n. \quad (29)
\]

Now we make the main modification to Algorithm 2 by using the property of \( \mathbf{a}^* \) in (20). Note that in the search process of finding an integer point \( \mathbf{a} \) in the hyper-ellipsoid, the entries of \( \mathbf{a} \) are determined in the following order: \( a_n, a_{n-1}, \ldots, a_1 \). When we enumerate candidates for \( a_n \) at level \( n \), we will only enumerate the non-negative integers. When we enumerate candidates for \( a_k \) at level \( k \) (note that at this point, \( a_n, a_{n-1}, \ldots, a_{k+1} \) have been chosen), we will only enumerate those greater than or equal to \( a_{k+1} \). By doing these we can prune a lot of nodes from the search tree to make the search process much faster.

For the users to implement the algorithm easily and for our later complexity analysis, we provide the pseudo code of the modified algorithm in Algorithm 2.

Here we make a few comments to Algorithm 2. To unify the enumeration strategies for level \( n \) and for any lower level, we set \( \mathbf{a} \) to be an \((n + 1)\)-dimensional vector with \( a_{n+1} = 0 \), so that \( a_k \geq a_{k+1} \) holds for \( k = n \). Since the optimal solution \( \mathbf{a}^* \) is \( n \)-dimensional, we save \( \mathbf{a}^* = a_{1:n} \) if \( a_{1:n} \neq 0 \) when a valid integer vector \( \mathbf{a} \) is found. To avoid enumerating any integer smaller than \( a_{k+1} \) at level \( k \), we introduced a flag variable “flag” in the algorithm to indicate whether the enumeration has reached the lower bound \( a_{k+1} \) for \( 1 \leq k \leq n \). In the algorithm \( s_k \) is the difference between the next integer candidate for \( a_k \) and the current value of \( a_k \) and it is used to get the next integer candidate for \( a_k \).

Remark 2: By Theorem 3 if (15) holds, then \( e_1 \) is the solution. Therefore, before using the Modified Schnorr-Euchner search algorithm to find the optimal solution, we can test whether (15) holds. If it holds, then return \( e_1 \), otherwise, we use the search algorithm to find the optimal solution. This can usually further improve the efficiency of the algorithm, especially when \( n \) or \( P \) is small. But from Table 1 we can see this case occurs in a very low probability when both \( n \) and \( P \) are very large. Thus, for simplicity, we do not incorporate it in Algorithm 2.

IV. COMPLEXITY ANALYSIS

In this section, we will analyze the complexity, in terms of flops, of the proposed method, and compare it with two optimal methods proposed in [6] and [16], and two suboptimal methods, which are the LLL reduction approach [7] and the quadratic programming relaxation (QPR) approach [13] and its improvement in [14]. In the following analysis, we focus on the case that the channel entries are i.i.d. Gaussian, i.e., \( \mathbf{h} \sim \mathcal{N} (\mathbf{0}, \mathbf{I}) \).

A. Complexity analysis for the modified Schnorr-Euchner search algorithm

In this subsection, we analyze the complexity of Algorithm 2. The approach is to first estimate the number of nodes visited in the search tree and then to count the number of arithmetic operations for each node.

Remark 3: It is difficult, if not impossible, to analyze the complexity of Algorithm 2 because the search radius \( \beta \) changes in the search process. Thus, we assume that the search radius \( \beta \) keeps unchanged in our following analysis to obtain an upper estimate of the complexity of Algorithm 2.

To illustrate our discussion, we display the search tree corresponding to Algorithm 2 with the assumption that \( \beta \) is a constant in Figure 2. Since there is not a true tree root, the
Algorithm 2. Finding the optimal coefficient vector based on sphere decoding

```
Input : A vector $t \in \mathbb{R}^n$ that satisfies $\|t\| < 1$ (see (4) and (19))
Output: A solution $\alpha^*$ to (5)
1 $f = 0^n, f_1 = 1 - t_1^2$ // see (10)
2 $q = 0^n, q_1 = f_1 / \|q_i = r_i^2 k_k \|$ see (27)
3 for $i = 2 : n$ do
4 $f_i = f_{i-1} - t_i^2 f_i$ // see (10)
5 $q_i = f_i / f_{i-1}$ // see (27)
6 $p = 0^{n+1}$ // see (21)
7 $d = 0^n$ // see (25)
8 $\sigma = 0^n$  // $\sigma_k \triangleq \sum_{i=k+1}^{n} r_i^2 (a_i - d_i)^2$ for $k < n$
9 $k = 1$
10 $a = e_{1,n}^{n+1}$ // intermediate solution
11 $a^* = e_{1,n}^{n+1}$
12 $\alpha^* = \frac{1}{\sigma}$ // see (10) and (24)
13 $\delta = q_1 / \|q_1\|$ // \(\delta \triangleq q_k (a_k - d_k)^2\)
14 $s = 1^n$ // flag variable
while true do
17 if $\alpha < \beta^2$ then
18 if $k > 1$ then
19 $p_k = p_{k+1} + t_k a_k$ // see (21)
20 $k = k - 1$
21 $\sigma_k = \alpha$
22 $d_k = t_k p_{k+1} / f_k$ // see (29)
23 $a_k = [d_k]$ // move flag
24 if $a_k < a_{k+1}$ then
25 $a_k = a_{k+1}$
26 flag$_k = 1$
27 $s_k = 1$
28 else
29 $s_k = \text{sgn}(d_k - a_k)$ // see (25)
30 $\delta = q_k (a_k - d_k)^2$
31 $\beta^2 = \alpha$
32 $a^* = a_{1:n}$
33 else
34 if $k < n$ then
35 $k = k + 1$
36 $a_k = a_k + s_k$
37 if $a_k = a_{k+1}$ then
38 flag$_k = 1$
39 $s_k = -s_k - \text{sgn}(s_k)$
40 else if flag$_k = 1$ then
41 $s_k = 1$
42 else
43 $s_k = -s_k - \text{sgn}(s_k)$
44 $\delta = q_k (a_k - d_k)^2$
45 return
```

To estimate the number of nodes at each level of the search tree, for $k = n, n - 1, \ldots, 1$, we define

$$E_k(\beta) = \{a_{k:n} \in \mathbb{Z}^{n-k+1} : a_k \geq a_{k+1} \geq \ldots \geq a_n \geq 0, \|R_{k:n,k:n} a_{k:n}\| < \beta\}. \quad (30)$$

Note that each non-leaf node at level $k$ in the search tree corresponds to an $a_{k:n} \in E_k(\beta)$, and each leaf node labeled by $x$ at level $k$ corresponds to the case that $a_{k:n} \notin E_k(\beta)$ with $a_{k+1:n} \in E_{k+1}(\beta) (k < n)$. Let $|E_k(\beta)|$ denote the number of elements belong to $E_k(\beta)$, then the number of non-leaf nodes at level $k$ in the search tree is $|E_k(\beta)|$. It is easy to argue that the number of leaf nodes at level $k$ in the search tree is exactly equal to $|E_{k+1}(\beta)|$. Thus the total number of nodes (including both the non-leaf and leaf nodes) at level $k$ is $|E_k(\beta)| + |E_{k+1}(\beta)|$.

From Algorithm 2 any node at level $k$ comes from two possibilities. One is that it is generated after its parent node at level $k + 1$ is generated. This process corresponds to lines 19-35 of Algorithm 2 and the cost is $O(1)$ flops. The number of such nodes is $|E_{k+1}(\beta)|$. The other is that it is generated after a leaf node at level $k - 1$ is generated. This process corresponds to lines 38-49 and the cost is also $O(1)$ flops. The number of such nodes is $|E_k(\beta)|$. Thus, the total cost for generating all nodes at level $k$ is

$$c_k = (|E_k(\beta)| + |E_{k+1}(\beta)|)O(1), \quad (31)$$

where we define $|E_{n+1}(\beta)| = 0$. Let $C(n)$ denote the total cost of the search tree, then, by (31), we obtain

$$C(n) = \sum_{k=1}^{n} c_k = O(1) \sum_{k=1}^{n} |E_k(\beta)|. \quad (32)$$

Obviously, $|E_n(\beta)| \leq \lfloor \beta / r_{nn} \rfloor$. However, it is hard to rigorously compute or estimate $|E_k(\beta)|$ for $k < n$ since the inequalities are involved in (30), so for $k = 1, 2, \ldots, n$, we define supersets:

$$F_k(\beta) = \{a_{k:n} \in \mathbb{Z}^{n-k+1} : \|R_{k:n,k:n} a_{k:n}\| < \beta\}, \quad (33)$$

![Fig. 2. An example search tree corresponding to Algorithm 2 with \(\beta\) being a constant and \(n = 4\)](image-url)
where $\beta$ is the initial search radius used in Algorithm 2 (see (26)).

Let $|F_k(\beta)|$ denote the number of elements belong to $F_k(\beta)$. Obviously, we have

$$|E_k(\beta)| \leq |F_k(\beta)|. \quad (34)$$

We apply the so-called Gaussian heuristic, which is widely used in the complexity analysis of sphere decoding methods (see, e.g., [33], [34], [20], [35]), to estimate $|F_k(\beta)|$. This method approximates $|F_k(\beta)|$ by the volume of the hyper-ellipsoid $\|R_{k,n,k,n}a_{k,n}\|_2 < \beta$, namely,

$$|F_k(\beta)| \approx \frac{\beta^{n-k+1}}{\Gamma((n-k+1)/2+1)} \prod_{i=k} V_{n-k+1}, \quad (35)$$

where $V_{n-k+1}$ denotes the volume of an $(n-k+1)$-dimensional unit Euclidean ball, i.e.,

$$V_{n-k+1} = \frac{\pi^{(n-k+1)/2}}{\Gamma((n-k+1)/2+1)} \quad (36)$$

with $\Gamma$ being the Gamma function.

By (12) and (26), we have

$$\frac{\beta^{n-k+1}}{\prod_{i=k} V_{n-k+1}} \leq \frac{1}{\sqrt{1-\|t\|_2^2}} \sqrt{1-\|t\|_2^2}$$

$$\leq \frac{1}{\sqrt{1-\|t\|_2^2}} \quad (37).$$

Since

$$\Gamma\left(\frac{n-k+1}{2} + 1\right) = \left\{ \begin{array}{ll} \frac{(n-k+1)!}{2(n-k+1)!} & \text{if } n-k \text{ is odd} \\ \frac{(n-k+1)!}{2(n-k+1)!} & \text{if } n-k \text{ is even} \end{array} \right.,$$

where

$$(n-k+1)! = 1 \times 3 \times 5 \times \cdots \times (n-k+1).$$

From (36), we have

$$V_{n-k+1} = \left\{ \begin{array}{ll} \frac{\pi^{(n-k+1)/2}}{\Gamma((n-k+1)/2+1)} & \text{if } n-k \text{ is odd} \\ \frac{\pi^{(n-k+1)/2}}{2(2\pi)^{(n-k+1)/2}} & \text{if } n-k \text{ is even} \end{array} \right.. \quad (38)$$

By (38), it is not hard to see that

$$\lim_{k \to \infty} V_k = 0.$$ 

Therefore, by (37), instead of using (35), we only need the following approximation, which is weaker than the Gaussian heuristic,

$$|F_k(\beta)| \approx \frac{1}{\sqrt{1-\|t\|_2^2}} \max\{V_{n-k+1}, 1\}. \quad (39)$$

By direct calculation, we have $V_{13} = 0.9106$, $V_{14} = 0.5993$. By (38), obviously, $V_i$ is decreasing with $i \geq 13$. Therefore, from the aforementioned equation, we obtain

$$V_i \leq 1, \ \forall \ i \geq 13.$$ 

By direct calculation, we have

$$\max_{1 \leq j \leq 12} V_j = V_5 = 5.2638.$$

Therefore, combining with (32), (34) and (39), we obtain (recall that $C(n)$ is the total cost of the search tree)

$$C(n) \approx \frac{O(n)}{\sqrt{1-\|h\|_2^2}} \quad (40).$$

By (4), we have

$$\frac{1}{\sqrt{1-\|h\|_2^2}} = \frac{1}{\sqrt{1-\frac{P\|h\|_2^2}{1+P\|h\|_2^2}}} = 1 + P\|h\|_2^2.$$ 

Thus, by (40), we obtain

$$C(n) \approx O(n) \sqrt{1 + P\|h\|_2^2}. \quad (41).$$

In the following, simulation results are provided to support that (41) holds for general $n$ and $h$ in (4). From (32), we only need to show $\sum_{k=1}^n |E_k(\beta)| \leq O(n) \sqrt{1 + P\|h\|_2^2}$. We consider the case that the channel vector $h \sim \mathcal{N}(0, I)$. For each $n$ and each $P$, we randomly generate 10000 realizations of $h$.

Table III displays the average and largest ratios of $\sum_{k=1}^n |E_k(\beta)|$ to $n\sqrt{1 + P\|h\|_2^2}$ over 10000 samples. “AR” and “LR” in Table III respectively denote average and largest ratios. From Table III we can see that $\sum_{k=1}^n |E_k(\beta)| < 2n\sqrt{1 + P\|h\|_2^2}$ in all the tests. Note that the number of nodes searched by Algorithm 2 cannot be larger than $\sum_{k=1}^n |E_k(\beta)|$ because the radius $\beta$ reduces whenever a valid integer vector is found in the search process.

In the following, we investigate the expected value of $C(n)$ when the entries of $h$ are independent and identically distributed following the normal distribution $\mathcal{N}(0, 1)$. It is easy to see that $\|h\|_2^2$ follows the Chi-squared distribution $\chi^2(n)$. Therefore, $E[\|h\|_2^2] = n$. Since $1 + P\|h\|_2^2$ is a concave function of $x$, by Jensen’s Inequality,

$$E[\sqrt{1 + P\|h\|_2^2}] \leq \sqrt{1 + P} E[\|h\|_2^2] = \sqrt{1 + nP}. \quad (42)$$

Therefore, by (41) and (42), it is easy to see that the complexity of Algorithm 2 is around $O(n^{1.5})$ flops.

B. Comparison of the complexity of the proposed method with other methods

It is easy to see that, for any given $h$, computing $t$ by (4) costs $O(n)$ flops. And for any fixed $t$, transform it such
that (12) holds costs $O(n \log(n))$ comparisons. Since the total complexity of Algorithm 2 is around $O(n^{1.5})$ flops, the total complexity of the whole method is $O(n^{1.5})$ flops for the test cases.

The complexity of the QPR in [13] and [14] is $O(n^3)$ and $O(n^{1.5})$ flops, respectively. The method based on LLL lattice reduction [7] uses the regular method, costing $O(n^3)$, to obtain the Cholesky factor $R$. The optimal method proposed in [4] needs to find the inverse of $n$ matrices and solving $n$ linear equations with the dimensions from 1 to $n$, so its complexity is not smaller than $O(n^3)$. The complexity of the optimal method proposed in [16] is $O(n^{2.5})$ flops. Therefore, it is expected that our optimal algorithm is faster than the LLL reduction based method, the QPR in [13] and the two optimal methods proposed in [7] and [16], and is faster than or has more or less the same speed as the QPR in [14].

V. NUMERICAL SIMULATIONS

In this section, we present the numerical results to illustrate the effectiveness and efficiency of our new method. We consider the case that the entries of the channel vector $h \in \mathbb{R}^n$ are i.i.d. Gaussian, i.e., $h \sim \mathcal{N}(0, I)$. The dimension $n$ of $h$ ranges from 2 to 16. For a given $n$, we randomly generate 10000 realizations of $h$ for each $P$ from 0 dB to 20 dB, and apply different methods to calculate the corresponding computation rates. To compare the effectiveness of different methods, we compute the average computation rates. To compare their efficiency, we record the running time.

The methods considered include our new method called the improved sphere decoding (ISD) method, the branch-and-bound (BnB) algorithm in [6], the optimal method proposed in [16] (abbreviated as SG named after the authors), the method based on LLL lattice reduction algorithm [7] (abbreviated as LLL), and the quadratic programming relaxation (QPR) approach [13]. The quality-complexity tradeoff factor $\delta$ in the LLL method is set as 0.75. A larger $\delta$ ($1/4 < \delta \leq 1$) can give a higher rate, but the running time will increase drastically as $\delta$ increases. The upper bound on the number of real-valued approximations, $K_u$, in the QPR method is set according to the criterion proposed in [14]. Exact values of $K_u$ used in the simulations are listed in Table IV.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $K_u$ | 2 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 4 |

Table IV: The upper bound on the number of real-valued approximations in QPR method

We first compare the average computation rates. Figure 3(a), 3(b), 3(c) show the average computation rates over 10000 samples with the dimension $n$ being 4, 8, and 16, respectively. The ISD method, the BnB method and the SG methods are optimal. As expected, numerical results show that they always provide the highest computation rate. The corresponding curves of these three methods in Figure 3 exactly overlap with each other. The QPR method and the LLL based method provide rates close to that of the optimal methods. However, as the dimension increases, their performance degrade.

Now we compare the running time. Figure 4(a), 4(b), 4(c) show the running time of simulating 10000 samples with $P$ being 0 dB, 10 dB, and 20 dB, respectively. For the optimal methods, it is obvious that our new ISD method is much more efficient than both the BnB method and SG method. It can also be observed that the ISD method is also faster than the LLL based method. Although the QPR method [14] is faster than our ISD method in Figure 4(c), it is a suboptimal method and
its performance degrades for high dimension (see Figure 3(c)).

VI. ADAPTATION FOR USE IN COMPUTE-AND-FORWARD DESIGN

In the previous sections, we mainly focused on the problem of finding the optimal coefficient vector for one relay, which is defined as the one that gives the highest computation rate at that relay. However, it is the overall transmission rate that determines the system performance, rather than the computation rate at a single relay. The coefficient vector that maximizes the computation rate at one relay may not be optimal for the whole system. Thus, in this section we show how to modify our algorithm so that it can be applied in the CF design. We will also analyze the complexity of the modified algorithm, and present numerical results of the running time to show the efficiency of the modified algorithm.

A. Adapting the proposed algorithm

In this subsection, we show how to adapt our algorithm for use in CF design.

The transmission rate is the minimum computation rate over all relays if the coefficient matrix, which is formed by the coefficient vectors at relays, is full column rank; and it is 0 if the coefficient matrix has rank deficiency [1]. One naive strategy is to let each relay choose the coefficient vector that maximizes the computation rate. However, it has been shown in [5] to be inherently suboptimal since there is a high probability that the formed coefficient matrix is not full rank, especially for low SNR. Instead, we can adopt the strategy (named “Wei-Chen” after the authors) proposed in [5]:

1) Each relay searches a list of candidate coefficient vectors, and then forwards the list to the destination.
2) The destination performs a search based on the received lists, and finds a good set of coefficient vectors that can form a full rank coefficient matrix, and then sends to each relay the coefficient vector be used.
3) Each relay chooses the coefficient vector to be the one it receives.

The Wei-Chen strategy effectively resolves the rank deficiency issue, and achieves close-to-optimal transmission rate.

To apply our algorithm along with the Wei-Chen strategy, we need to modify our search algorithm such that it outputs a list of best coefficient vectors providing the best rates. In fact, several slight modifications suffice to serve the purpose:

1) Discard the constraint on the candidates in (20). This is to include suboptimal candidates that do not satisfy (20) but provide close-to-optimal rates.
2) Enumerate vectors at level-1 in the natural order by setting the initial value as $a_1 = \lceil d_1 \rceil$ in (23) and the step fixed as $s_1 = 1$. Since $a$ and $-a$ result in the same rate, only one of them needs to be enumerated. The above way of enumeration at level 1 ensures that only one of $a$ and $-a$ is enumerated.
3) Initialize the radius $\beta$ in (22) as 1 and shrink $\beta$ appropriately when a new candidate is enumerated: if the number of candidates in the output list is less than the desired number, put the new candidate in the list, and do not shrink the radius; otherwise, replace the candidate who has the largest value of $\|Ra\|$ in (22) (the most suboptimal candidate) in the output list with the new candidate, and shrink the radius as the corresponding radius of the most suboptimal candidate in the updated list. It is sufficient to set the initial radius as 1 rather than a larger value, because coefficient vectors providing positive rates must have radius smaller than 1.

The pseudo-code of the algorithm with the above modifications is provided in Algorithm 3.
Algorithm 3. Finding $L$ best coefficient vectors based on sphere decoding

**Input:** A vector $t \in \mathbb{R}^n$ that satisfies $\|t\| < 1$ (see (4) and (19)), and the desired number $L$ of candidates.

**Output:** A list $\Omega$ containing $L$ (or less, see Remark 4) best integer vectors to $\{\Omega\}$, and a list $\Gamma$ of the corresponding objective values $\|Ra\|^2$ for $a \in \Omega$.

1. $f = 0^n$, $f_1 = 1 - t_1^2$ // see (10)
2. $q = 0^n$, $q_1 = f_1$ // $q_k = r_k^2$, see (27)
3. for $i = 2 : n$ do
   4. $f_i = f_{i-1} - t_i^2$ // see (10)
   5. $q_i = f_i / f_{i-1}$ // (27)
   6. $p = 0^{n+1}$ // see (28)
   7. $d = 0^n$ // see (29)
   8. $\sigma = 0^n$ // $\sigma_k \triangleq \sum_{i=k+1}^{n} r_i^2 (a_i - d_i)^2$ for $k < n$
   9. $\Omega = \emptyset$
   10. $\Gamma = \emptyset$
   11. $k = 1$
   12. $a = e_1^n$ // intermediate solution
   13. $\beta^2 = 1$ // see (22)
   14. $\delta = q_1$ // $\delta \triangleq q_k (a_k - d_k)^2$
   15. $s = 1$
   16. while true do
      17. $\alpha = \sigma_k + \delta$
      18. if $\alpha < \beta^2$ then
         19. if $k > 1$ then
            20. $p_k = p_{k-1} + t_k a_k$ // see (28)
            21. $k = k - 1$
            22. $\sigma_k = \alpha$
            23. $d_k = t_k p_k + f_k$ // see (29)
         24. if $k > 1$ then
            25. $a_k = \lfloor d_k \rfloor$
            26. $s_k = \text{sgn}(d_k - a_k)$ // see (25)
         27. else
            28. $a_1 = \lfloor d_1 \rfloor$
            29. $s_1 = 1$
            30. $\delta = q_k (a_k - d_k)^2$
         31. else
            32. if $|\Omega| = L$ then
               33. $m = \text{arg max}_i \gamma_i$ // $\Gamma = \{\gamma_i\}$
               34. $\omega_m = a$ // $\Omega = \{\omega_i\}$
               35. $\gamma_m = \alpha$
            36. else
               37. $\Omega = \{\Omega, a\}$
               38. $\Gamma = \{\Gamma, \alpha\}$
            39. if $|\Omega| = L$ then
               40. $\beta^2 = \text{max}_i \gamma_i$
               41. $a_1 = q_1 + s_1$
               42. $\delta = q_1 (a_1 - d_1)^2$
            43. else
               44. if $k < n$ then
                  45. $k = k + 1$
                  46. $a_k = a_{k-1} + s_k$
                  47. $s_k = -s_k - \text{sgn}(s_k)$
                  48. $\delta = q_k (a_k - d_k)^2$
               49. else
                  50. return

**Remark 4:** Note that Algorithm 3 may output a list containing less candidates than desired. The reason is there are cases that the number of candidates that give positive rates is less than the desired number, and it is meaningless to contain vectors with 0 rate in the list.

**Remark 5:** The integer vectors in the output list of Algorithm 3 need to be transformed as described in Section III-B to serve as the CF coefficient vectors.

### B. Complexity analysis and numerical results

In this subsection, we will first analyze the complexity of our modified search algorithm, and then give numerical results of running time to show its efficiency.

The modifications that transform Algorithm 2 with a single output to Algorithm 3 with list output increase the running time for every instance. However, the upper bound $O(n \sqrt{1 + P \|h\|_2^2}$ on the estimated number of tree nodes searched by Algorithm 2 still applies to Algorithm 3 since in the complexity analysis presented in Section IV we have already assumed the radius $\beta$ does not shrink during the search (see Remark 5). In Algorithm 3, updating the length-$L$ output list after a new candidate with better objective is found takes $O(n + \log(L))$ flops: $O(\log(L))$ flops for locating the entry to be updated, and $O(n)$ flops for updating the located entry by replacing the entry with the new candidate. Outputting the length-$L$ list takes $O(nL)$ flops. Thus, the average complexity of Algorithm 3 is estimated to be $O(n + \log(L))$ flops. Thus, the average complexity of Algorithm 3 is $O(n^2 + n^1.5 \log(L) + nL)$.

Now we present numerical results of the running time to show the efficiency of our algorithm. We consider the wireless relay network with $n$ sources and $n$ relays. Slow fading channel with entries being i.i.d. standard Gaussian, the estimated average complexity of Algorithm 3 is $O(n^2.5 + n^1.5 \log(L) + nL)$.

Figure 5a shows the running time in seconds of the two considered algorithms with SNR $P = 10$ dB for different dimension $n$. It is clear that our algorithm is much more efficient than the other. The improvement in efficiency grows dramatically as the dimension $n$ increases: at $n = 2$ our algorithm is more than 2 times faster than the other algorithm, while at $n = 8$ our algorithm is more than 60 times faster, and the running time saving goes beyond 98%!

Figure 5b shows the running time in seconds of the two considered algorithms with dimension $n = 8$ for different
SNR $P$. As can be seen, our algorithm is consistently much more efficient than the other algorithm, and the improvement is universally significant for SNR from as low as 0dB to as high as 20dB.

VII. CONCLUSIONS

Based on the idea of sphere decoding, in this paper, a new low-complexity algorithm, which gives the optimal coefficient vector that maximizes the computation rate for a relay in the compute-and-forward scheme is proposed. We derived an efficient algorithm to compute the Cholesky factorization by using the special structure of the Gram matrix. It transformed the problem into a SVP in $O(n)$ flops without explicitly forming the whole Cholesky factor matrix. Some conditions, under which $e_1$ is an optimal coefficient vector, have also been given, and can be checked in $O(n)$ flops. We then modified the Schnorr-Euchner search algorithm to solve the SVP by taking advantage of the properties of the optimal coefficient vector. We showed that the expected complexity of our new method is $O(n^{1.5})$ for i.i.d. Gaussian channel entries based on the Gaussian heuristic. Simulations showed that our optimal method is not only much more efficient than the existing ones that give the optimal computation rate, but is also more efficient than some best previously known methods that give the close-to-optimal rate. In addition, we demonstrated how to adapt our algorithm so that it can be applied in compute-and-forward design.

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