Dual approaches for defects condensation

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\textbf{A B S T R A C T}

We review two methods used to approach the condensation of defects phenomenon. Analyzing in details their structure, we show that in the limit where the defects proliferate until occupy the whole space these two methods are dual equivalent prescriptions to obtain an effective theory for the phase where the defects (like monopoles or vortices) are completely condensed, starting from the fundamental theory defined in the normal phase where the defects are diluted.

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1. Introduction

The quantum field theory description of a physical system relies on a proper identification of its degrees of freedom which are then interpreted as excited states of the fields defining the theory. However it is sometimes the case that the theory may contain important structures which are not described in this way and cannot be expressed in a simple manner in terms of the fields appearing in the Lagrangian, having a non-local expression in terms of them. These structures appear under certain conditions as defects; prescribed singularities of the fields defining the theory. A general conjecture \cite{2} claims that defects are described by a dual formulation in which they appear as excitations of the dual field, but this can be proved only in some particular instances. Nevertheless, much can be gained just with the information that these structures appear as singularities of the fundamental fields even without knowing their precise dynamics. A pressing question is if it is possible to address, with this limited information, the situation in which the collective behavior of defects becomes the dominant feature of a theory. It is one of the purposes of this work to discuss an extreme case of sorts. We want to present a general proposal of how to describe a situation in which the singularities of the fields proliferate defining a new vacuum for the system. In this picture the new degrees of freedom are recognized as excitations of the established condensate of defects.

This view is supported by the fact that if we are interested only in the low lying excitations it is perfectly reasonable to take the condensate as given, not worrying how it was set on, and construct an effective field theory describing the excitations. It is well known for instance that the pions, which can be recognized as excitations of the chiral symmetry breaking condensate composed of quark–antiquark pairs, can be described by an effective field theory without knowing about QCD. Even though we need not know the details of how the condensate is formed it is important to stress that the condensate defines the vacuum and carries vital information about the symmetry content used in the construction of the effective theory. It is in this way also bound to have an effect in all the other fields comprising the system. The example of a superconducting medium also comes to mind, where the condensate vacuum endows the electromagnetic excitations with a mass. This same idea is employed on the electroweak theory where a condensate is the only consistent way to give mass to the force carriers, the W and Z, and in fact to account for all the masses of the standard model. This is an example where the properties of the condensate itself are not completely established and still a matter of debate. The currently accepted view is that its low lying excitations are the Higgs particles, still to be detected, described by a scalar field.

More akin to our take on the condensate concept, as a collective behavior of defects, is the dual superconductor model of confine-
ment which is based on the superconductor phenomenology [1].

It is expected that the QCD vacuum at low energies is a chro-

magnetic condensate leading to the confinement of color charges

immersed in this medium. In dual superconductor models of color

confinement, magnetic monopoles appear as topological defects in

points of the space where the abelian projection becomes sin-

gular [9]. There are in fact many other examples in which the

condensation of defects is responsible for drastic changes in the

system by defining the new vacuum of the theory. We may men-

tion vortices in superfluids and line-like defects in solids which are

responsible for a great variety of phase transitions [6]. All these in-

stances point to the importance of getting a better understanding

of the condensation phenomenon.

In all these examples there are some general features of the con-
condensates which can tell a lot about what to expect of the sys-
tem when condensation sets in without the precise knowledge of

how this happened. These general features are what we intend to

explore in this Letter. The main inspiration for this work comes

from the study of two particular approaches to this problem: one

is the Abelian Lattice Based Approach (ALBA) discussed by Banks,

Myerson and Kogut in [3] within the context of relativistic lattice

field theories and latter also by Kleinert in [5] in the condensed

matter context. The other one is the Julia–Toulouse Approach (JTA)

introduced by Julia and Toulouse in [4] within the context of or-

dered solid-state media and later reformulated by Quevedo and

Trugenberger in the relativistic field theory context [7].

The ALBA was used, for example, by Banks, Myerson and Kogut

to study phase transitions in abelian lattice gauge theories [3].

A few years latter Kleinert obtained a disorder field theory for

the superconductor from which he established the existence of a

tricritical point separating the first-order from the second-order

superconducting phase transitions [5]. In this Letter we shall be

using the notations in the recent book by Kleinert [6].

Developing in the work of Julia and Toulouse, Quevedo and

Trugenberger studied the different phases of field theories of com-

 pact antisymmetric tensors of rank $h-1$ in arbitrary space–time
dimensions $D=d+1$. Starting in a coulombic phase, topological
defects of dimension $d-h-1$ ($d-h-1$)-branes may condense
leading to a confining phase. In that work one of the applications
of the JTA was the explanation of the axion mass. It was known
that the QCD instantons generate a potential which gives mass to
the axion. However, the origin of this mass in a dual description
were a puzzle. When the JTA is applied it is clear that the conden-
sation of instantons is responsible for the axion mass.

Recently, some of us and collaborators have made a proposal that
the JTA would be able to explain the dual phenomenon to radiative corrections [10] and used this idea to compute the fermionic determinant in the QED$_3$ case. This result was immedi-
ately extended to consider the use of the JTA to study QED$_3$ with
magnetic-like defects. By a careful treatment of the symmetries of the system we suggested a geometrical interpretation for some debatable issues in the Maxwell–Chern–Simons-monopole system,
such as the induction of the non-conserved electric current to-
gether with the Chern–Simons term, the deconfinement transition
and the computation of the fermionic determinant in the presence
of Dirac string singularities [11]. It is important to point that the
main signature of the JTA is the rank-jumping of the field ten-
sor due to the defects condensation. However, this discontinuous
change of the theory still puzzles a few. It is another goal of this
investigation to shed some light in this matter.

In the present work we hope to help clarify the above men-
tioned issues focusing in the analysis of the structure of these two
methods, i.e., JTA and ALBA, by working out an explicit ex-
ample. Introducing a new Generalized Poisson’s Identity (GPI) for
$p$-branes in arbitrary space–time dimensions and the novel con-
cept of Poisson-dual branes we show that in the specific limit where
the defects proliferate until they occupy the whole space these two
approaches are dual equivalent prescriptions to obtain an effective
theory for the phase where the defects are completely
condensed, starting from the fundamental theory defined in the
normal phase where the defects are diluted.

2. Setting the problem

The example we will work here is the Maxwell theory in the
Presence of monopoles that eventually condense, which serves as
an abelian toy model that simulates quark confinement.

The Maxwell field $A_{\mu}$ minimally coupled to electric charges $e$
and non-minimally coupled to magnetic monopoles $g$ is described
by the following action:

$$S = S_{\text{M}}^M + S_{\text{int}} = -\int d^4x \frac{1}{4} (F_{\mu\nu} - F_{\mu\nu}^M)^2 - \int d^4x j_{\mu} A^\mu, \tag{1}$$

where $F^\mu_{\nu} = \delta^{\mu}(x; L')$ ( $\delta^{\mu}(x; L)$) a $\delta$-distribution that localizes
the world line $L'$ ($L$) of the electric (magnetic) charge $e$ ($g$).

$F_{\mu\nu}^M = \delta^{\mu}(x; S) := \frac{e}{2} \delta^{\mu\nu\alpha\beta} \delta_{\alpha\beta}(x; S)$ is the magnetic Dirac brane,
with $\delta_{\mu\nu}(x; S)$ a $\delta$-distribution that localizes the world surface $S$ of
the Dirac string coupled to the monopole [8] and has the current
$J_{\mu}$ in its border. The field $A_{\mu}$ experiences a jump of discontinu-
ity as it crosses $S$, hence $F_{\mu\nu}$ has a $\delta$-singularity over $S$ [13] that
exactly cancels the one in $F_{\mu\nu}^M$ such that $F_{\mu\nu} - F_{\mu\nu}^M = \delta^{\mu\nu}$ is the
regular combination which expresses the observable fields $E$
and $B$. As we shall see, the quantum field theory associated to this
action has two different kinds of local symmetries: the first one is
the usual electromagnetic gauge symmetry, $A_{\mu}(x) \rightarrow A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu} A(x)$, with integrable $A$, i.e., $[\partial_{\mu}, \partial_{\nu}] A = 0$. The second one
represents the freedom of moving the unphysical surface $S$
over the space:

$$F_{\mu\nu}^M \rightarrow F'_{\mu\nu} = F_{\mu\nu}^M + \partial_{\mu} A^{\mu}_{\text{M}} - \partial_{\nu} A^{\mu}_{\text{M}},$$

where $A^{\mu}_{\text{M}} = \delta^{\mu}(x; V)$ is the magnetic Dirac brane, $A_{\mu}(x) \rightarrow A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu} A(x)$, with integrable $A$, i.e., $[\partial_{\mu}, \partial_{\nu}] A = 0$. The second
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where $A^{\mu}_{\text{M}} = \delta^{\mu}(x; V)$ is the magnetic Dirac brane,
Extremizing $\tilde{S}$ with respect to $f_{\mu\nu}$ we get $f_{\mu\nu} = \epsilon_{\mu\nu\sigma\rho} f^{\sigma\rho}_{\mu\nu}$ and substituting that in (3) we reobtain the original action (1) while extremizing $\tilde{S}$ with respect to $A_\mu$, we get the condition $\epsilon_{\mu\nu\sigma\rho} f^{\sigma\rho}_{\mu\nu} = \eta^{\mu}_{\nu} j^\mu$, which can be solved by $f_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} f^{\sigma\rho}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} (F_{\rho} - F_{\sigma}^E)$. We introduced the dual vector potential $A_\mu$ in $\tilde{F}_{\mu\nu} := \eta^{\mu}_{\nu} A_\nu - \eta^{\nu}_{\mu} A_\mu$ and the electric Dirac brane $F^E_{\mu\nu}$ that localizes the world surface of the electric Dirac string coupled to the electric charge. Substituting this result in (3) and discarding an electric brane–magnetic brane contact term that does not contribute to the partition function due to the Dirac quantization condition, we obtain the dual action:

$$\tilde{S} = \tilde{S}_E^F + \tilde{S}_{int} = \int d^4x \left[ -\frac{1}{4} \epsilon_{\mu\nu\sigma\rho} \tilde{F}^{\sigma\rho}_{\mu\nu} - \tilde{A}^\mu j_\mu \right],$$

(4)

where the couplings are inverted relatively to the ones in the original action (1); here the dual vector potential $\tilde{A}_\mu$ couples minimally with the monopole and non-minimally with the electric charge.

### 3. Abelian lattice based approach

We are now in position to consider monopole condensation by applying the ALBA to the dual Maxwell action (4). The main goal of this approach is to obtain an effective action for the condensed phase in the dual picture. The ALBA is based on the observation that upon condensation, the magnetic defects initially described by $\delta$-distributions are elevated to the field category describing the long-wavelength fluctuations of the magnetic condensate. The condition triggering the complete condensation of the defects is given by the disappearance of the Poisson-dual brane (defined below) coming from a Generalized Poisson’s Identity (see the discussion in Appendix A). We shall say that the branes $L$ and $V$ (or the associated currents $\delta_\mu(x; L)$ or $\delta_\mu(x; V)$) are Poisson-dual to each other. Using (7) we can rewrite (6) as:

$$Z^C[\tilde{A}_\mu] = \int \mathcal{D}\eta_\mu \sum_{\{V\}} \delta \left[ g \left( \frac{\eta_\mu}{g} - \delta_\mu(x; L) \right) \right] \times \exp \left\{ i \int d^4x \left[ -\frac{e}{2} \eta^{\mu}_2 + \eta_\mu \tilde{A}^\mu \right] \right\} \times e^{i \int d^4\tilde{x} \delta_\mu(x; V) \frac{g}{\pi} \int D\tilde{\mu}} \times \exp \left\{ i \int d^4x \left[ -\frac{e}{2} \eta^{\mu}_2 + \eta_\mu \tilde{A}^\mu \right] \right\} = \sum_{\{V\}} \int D\tilde{\mu} \int \mathcal{D}\eta_\mu \exp \left\{ i \int d^4x \left[ -\frac{e}{2} \eta^{\mu}_2 + \eta_\mu \tilde{A}^\mu \right] \right\}.$$

(8)

In the first line we introduced the auxiliary field $\eta_\mu$, which will replace the $\delta$-distribution current in the condensed phase as discussed above. In the second line we exponentiated the current conservation condition through use of the $\tilde{\mu}$ field and also made use of the GPI to bring into the game the Poisson-dual current $\tilde{\mu}^{\nu} = 2\pi \delta_\mu(x; V)$. We also made an integration by parts and discarded a constant multiplicative factor since it drops out in the calculation of correlation functions.

Integrating the auxiliary field $\eta_\mu$ in the partial partition function (8) and substituting the result back in the complete partition function (5) we obtain, as the effective total action for the condensed phase in the dual picture, the London limit of the Dual Abelian Higgs Model (DAHM):

$$\tilde{S}_{DAHM}^E = \int d^4x \left[ -\frac{1}{4} \epsilon_{\mu\nu\sigma\rho} \tilde{F}^{\sigma\rho}_{\mu\nu} + \frac{m_A^2}{2} \left( \partial^2_{\mu} \tilde{\theta} - \tilde{\theta}^{\nu}_{\mu} - g \tilde{A}_\mu \right)^2 \right],$$

(9)

where we defined $m_A^2 := \frac{1}{e^2}$. This effective action is the main result of this approach. In the next section we shall dualize this result and one could be concerned with the fact that (9) constitutes a nonrenormalizable theory, thus requiring a cutoff in order to be well defined as an effective quantum theory. However, one can always think of its UV completion, in this case the complete DAHM, which is renormalizable, and then take its dual, taking the London limit afterwards [9]. At least in the case considered here, the result is exactly the same one obtains by directly dualizing the London limit of the DAHM, thus justifying the procedure we shall adopt in the next section.

Considering now that a complete condensation of monopoles takes place we let their worldlines $L$ proliferate and occupy the whole space, implying that $\tilde{\theta}^\mu_{\nu} \to 0$ as seen from (7) and the discussion afterwards (notice that $\tilde{\theta}^\mu_{\nu}$ appears as a vortex-like defect for the scalar field $\tilde{\theta}$ describing the magnetic condensate, being a parameter that controls the monopole condensation). Integrating the Higgs field $\tilde{\theta}$ we get a transverse mass term for $\tilde{A}_\mu$ (Higgs mechanism) such that the electric field has a finite penetration depth $\lambda = \frac{1}{m_A} = \sqrt{\frac{e}{G}}$ in the DSC: this is the dual Meissner effect. Integrating now the field $\tilde{A}_\mu$ we obtain after some algebra the effective action:

$$\tilde{S}_L^E := \int d^4x \left[ -\frac{1}{2} \epsilon_{\mu\nu\sigma\rho} F^{\sigma\rho}_{\mu\nu} + \frac{1}{4} f^{\mu\nu}_{\delta\mu\nu} - \tilde{A}^\mu A_\mu \right].$$

(3)
\[
\tilde{S}_{\text{eff}} = \int d^4x \left[ -\frac{m^2}{4} F_{\mu \nu}^2 \tilde{F}_{\mu \nu} - \frac{1}{2} \tilde{F}_{\mu \nu}^2 - \frac{1}{2} \tilde{F}_{\mu \nu} \right]. \tag{10}
\]

The first term in (10) is responsible for the charge confinement: it spontaneously breaks the electric brane symmetry such that the electric Dirac string \( F_{\mu \nu} \) acquires energy becoming physical and constitutes now the electric flux tube connecting two charges of opposite sign immersed in the DSC. The flux tube has a thickness equal to the penetration depth of the electric field in the DSC: \( \lambda = \frac{1}{m} = \sqrt{e_c} \). The shape of the Dirac string is no longer irrelevant: the stable configuration that minimizes the energy is that of a straight tube (minimal space). Substituting in the first term of (10) such a solution for the string term, \( \tilde{F}_{\mu \nu} = 2 \epsilon_{\mu \nu \alpha \beta} \frac{1}{\lambda} (\partial^\alpha j^\beta - \partial^\beta j^\alpha) \), where \( j^\mu := (0, \vec{R} := \vec{R}_1 - \vec{R}_2) \) is a straight line connecting \( e_1 \) in \( \vec{R}_1 \) and \(-e_2 \) in \( \vec{R}_2 \), and taking the static limit we obtain a linear confining potential between the electric charges [9]. We also note that eliminating the magnetic condensate (i.e., taking the limit \( m = 0 \)) we recover the diluted phase with no confinement: the interaction between the electric currents in (10) becomes of the long-range (Coulomb) type and the confining term goes to zero (in terms of the flux tube we see that it acquires an infinite thickness such that the electric field is no longer confined and occupies the whole space).

In summary, the supplementing of the dual action with a kinetic term for the magnetic Dirac branes which respects the local symmetries of the system, the subsequent use of the GN (A6) and the consideration of the limit where the Poisson-dual current \( \partial^\alpha \) goes to zero gives us a proper condition for the complete condensation of monopoles, leading to confinement, as viewed from the dual picture.

4. Julia–Toulouse approach

Now we want to analyze the monopole condensation within the direct picture, where the defects couple non-minimally with the gauge field \( A_\mu \).

Using the Dirac quantization condition we can rewrite (1) as:

\[
S = \int d^4x \left[ -\frac{1}{4} F_{\mu \nu}^2 - \frac{1}{4} F_{\mu \nu} \epsilon_{\mu \nu \alpha \beta} \tilde{F}_{\alpha \beta} \right]. \tag{11}
\]

Julia and Toulouse made the crucial observation that if the monopoles completely condense we have a complete proliferation of the magnetic strings associated to them, hence the field \( A_\mu \) cannot be defined anywhere in the space. This implies that \( F_{\mu \nu} \) can no longer be written in terms of \( A_\mu \). The JTA consists in the rank-jump ansatz of taking the object \( F_{\mu \nu} \) as being the fundamental field describing the condensed phase. Hence \( F_{\mu \nu} \) acquires a new meaning and becomes the field describing the magnetic condensate. Defining \( \Lambda_{\mu \nu} := m \Delta_{\mu \nu} \) and supplementing (11) with a kinetic term of the form \( \frac{1}{4} \epsilon_{\mu \nu \alpha \beta} (\partial_\mu \Lambda_{\alpha \beta} + \partial_\nu \Lambda_{\alpha \beta} + \partial_\alpha \Lambda_{\beta \mu} + \partial_\beta \Lambda_{\mu \nu})^2 \) for the new field \( \Lambda_{\mu \nu} \), we obtain as the effective action for the condensed phase, in the direct picture, the massive Kalb–Ramond action:

\[
S_{\text{K–R}} = \int d^4x \left[ -\frac{1}{2} (\partial_\mu \Lambda_{\mu \nu})^2 + \frac{m^2}{2} \tilde{\Lambda}_{\mu \nu}^2 + \frac{m^2}{2} \tilde{F}_{\mu \nu} \right], \tag{12}
\]

where \( \tilde{\Lambda}_{\mu \nu} = \frac{1}{\epsilon} \epsilon_{\mu \nu \alpha \beta} \Lambda_{\alpha \beta} \).

Notice that in implementing the JTA the fundamental field of the theory experiences a rank-jump through the phase transition: we started with a 1-form in the normal phase and finished with a 2-form in the completely condensed phase. The rank-jump is a general feature of the JTA since in implementing this prescription we always use the ansatz of reinterpreting the kinetic term with non-minimal coupling for the field describing the dilute phase as being a mass term for the new field describing the condensate formed in the phase where the defects proliferate until occupy the whole space.

Let us now apply the duality transformation in (9). For this we introduce an auxiliary field \( f_{\mu \nu} \) such that the master action reads:

\[
S_{\text{Master}} := \int d^4x \left[ -\frac{1}{2} f_{\mu \nu} (\tilde{F}_{\mu \nu} - \tilde{F}_{\mu \nu}) + \frac{1}{4} F_{\mu \nu}^2 + \frac{m^2}{2} (\partial_\mu \theta - \partial_\mu g - g \partial_\mu \Lambda)^2 \right]. \tag{13}
\]

Extremizing (13) with respect to \( f_{\mu \nu} \) we get \( f_{\mu \nu} = \tilde{F}_{\mu \nu} \) and substituting this result back in the master action we recover (9). On the other hand, extremizing (13) with respect to \( \Lambda_{\mu \nu} \) we obtain:

\[
\Lambda_{\mu \nu} = -\frac{1}{m^2} (\partial_\mu f_{\mu \nu} + \frac{1}{g} (g^\nu \partial_\mu \tilde{F}_{\mu \nu})).
\]

Substituting (14) in (13), it follows that:

\[
S_{\text{Master}} = \int d^4x \left[ -\frac{1}{2m^2} (\partial_\mu F_{\mu \nu})^2 + \frac{1}{4} F_{\mu \nu}^2 + \frac{m^2}{2} \tilde{A}_{\mu \nu} \tilde{F}_{\mu \nu} + \frac{1}{g} (\partial_\mu \tilde{A}_{\mu \nu} - \partial_\mu \tilde{F}_{\mu \nu}) \right].
\]

where we integrated by parts and considered the antisymmetry of \( F_{\mu \nu} \) in order to use \( \partial_\mu \partial_\nu f_{\mu \nu} = 0 \).

Defining now \( f_{\mu \nu} := m \Lambda_{\mu \nu} \) and making the identification \( m_{\lambda} \equiv m \Lambda \), we get as the dual action to (9) the massive Kalb–Ramond action in the presence of vortices, a generalization of the result obtained by Quevedo and Trugenberger in [7]:

\[
S_{\text{VR}} = \int d^4x \left[ -\frac{1}{2} (\partial_\mu \tilde{A}_{\mu \nu})^2 + \frac{m^2}{4} \tilde{A}_{\mu \nu}^2 + \frac{m^2}{2} \tilde{A}_{\mu \nu} \tilde{F}_{\mu \nu}
+ \frac{m^2}{2g} (\partial_\mu \tilde{A}_{\mu \nu} - \partial_\mu \tilde{F}_{\mu \nu}) \right].
\]

More precisely, this extension consists in the construction of an action for the case with an incomplete condensate that is however already described by a rank-jumped tensor. If we now take the limit \( \tilde{F}_{\mu \nu} \rightarrow 0 \) in (16) we recover exactly the massive Kalb–Ramond action (12) obtained in [7] through the application of the JTA to (1). That establishes the duality between the JTA and the ALBA in the limit where the Poisson-dual current goes to zero, which physically corresponds to the limit of complete condensation of the defects. However, (16) with \( \tilde{F}_{\mu \nu} \neq 0 \) displays a new and important result, which is a consequence of this formalism, showing that the rank-jump which is the signature of the JTA also occurs in the partial condensation process with the presence of vortex-like defects.

5. Conclusion

We established the equivalence through duality of two different approaches developed to handle defects, represented by magnetic monopoles in the example worked here, in the physically interesting context where the defects dominate the dynamics of the system. It was clearly shown that the two approaches are complementary, being different descriptions of the same phenomenon in the limit where the Poisson-dual current vanishes which characterizes the complete condensation of the defects. Indeed, within the formalism here called as ALBA the transition becomes smoother since the Poisson-dual current \( \tilde{F}_{\mu \nu} \) appears as a parameter that controls the proliferation of the magnetic defects. On the other hand,
within the formalism referred to as JTA the phase transition is
signalized by a rank-jump of the tensor field and seems to be
a discontinuous phenomenon. However, the duality JTA–ALBA brings
a new possibility.

It is important to say that this dual equivalence was possible
due to a suitable interpretation of the generalization of the Pois-
son identity developed here. We clearly showed that this identity
is an essential tool to use in the context of defects condensation:
the proliferation of the branes in one of the sides of the identity
is accompanied by the dilution of the branes of complementary di-
ension in the other side of the identity. Due to this observation
we were able to identify the signature of the complete condensa-
tion of defects in the dual picture (ALBA) with the vanishing of
the Poisson-dual current. As the main result, we showed that in
this specific limit, when the Poisson-dual current is set to zero,
the ALBA and the JTA are two dual equivalent prescriptions for
describing condensation of defects.

As the final remark we point out the fact that when we con-
consider nonzero configurations of the Poisson-dual current \( \tilde{\theta}_\mu^V \)
we allow the description of an intermediary region interpo-
lating between the diluted and the completely condensed phases.
As discussed, this corresponds to the presence of vortex-like defects in
the condensate. It is possible to see that this new phase with
the presence of vortices (\( \tilde{\theta}_\mu^V \neq 0 \)), just like in the extreme case
where the complete monopole condensation sets in, is also
described within the direct picture by a rank-jumped action. The
JTA as originally described by Quevedo and Trugenberger, there-
fore, will describe the physically interesting extreme case where
all defects are condensed.

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Appendix A. Generalized Poisson’s Identity (GPI)

In this appendix we generalize the reasoning used in [6] in or-
der to account for an ensemble of p-branes in arbitrary space–time
dimensions.

Let us consider a d-dimensional hypercubical lattice with
spacings \( a \). Attribute to each site \( x = (x_1, x_2, \ldots, x_d) \), \( x_1, x_2, \ldots, x_d \in a\mathbb{Z} \)
of the lattice a configuration
\[
\theta_j^V(x) := 2\pi \frac{n_i(x)}{a^p} , \quad (A.1)
\]
where \( p \leq d, p,d \in \mathbb{N} \) and \( i \) is a set of \( k \leq d, k \in \mathbb{N} \)
indices each one of them running from 1 to \( d \) and \( n_i(x) \in \mathbb{Z} \).

The Poisson’s summation formula is given by
\[
\sum_{n \in \mathbb{Z}} e^{2\pi \text{i} nf} = \sum_{m \in \mathbb{Z}} \delta(f - m), \quad (A.2)
\]
where \( f \) is a integrable function.

Using (A.2) for each pair \( (x,i) \) it follows that
\[
\sum_{n_i(x) \in \mathbb{Z}} \exp \left[ 2\pi \text{i} \sum_x a^d \frac{n_i(x)}{a^p} f_i(x) \right] = \sum_{|n_i(x)| \in \mathbb{Z}} \prod_{(x,i)} \delta \left( f_i(x) - \frac{n_i(x)}{a^{d-p}} \right) , \quad (A.3)
\]
where we have used the fact that the exponential argument must
be nondimensional, hence \( a^{d-p+1/2} = a^0 = 1 \Rightarrow \{ f \} = a^{p-d} \).

The continuum limit corresponds to make the number \( N \) of lat-
tice sites go to infinity while keeping the lattice hypervolume \( V_d \)
fixed which gives the condition \( a \to 0 \). In this limit we formally
define the Poisson-dual current by
\[
\theta_j^V(x; \xi^p) := \lim_{a \to 0} \lim_{N \to \infty} \frac{\theta_j^V(x)}{a^p} = 2\pi \frac{n_i(x)}{V_d} . \quad (A.4)
\]

The object \( \theta_j^V(x; \xi^p) \) has dimension \( a^{-p} \) and is singular over a
region \( \xi^p \) of dimension \( p \) on the lattice where \( n_i(x \in \xi^p) :\neq 0 \).
In the rest of the lattice, where \( n_i(x \notin \xi^p) := 0 \), we have from (A.1)
that \( \theta_j^V(x) = 0 \) such that \( \theta_j^V(x; \xi^p) \) vanishes outside the region \( \xi^p \).
Thus we identify the object \( \theta_j^V(x; \xi^p) \) with a delta configuration that
localizes the \( p \)-brane \( \xi^p \):
\[
\theta_j^V(x; \xi^p) = 2\pi \delta_i(x; \xi^p) . \quad (A.5)
\]

Hence in the continuum limit the identity (A.3) is given by
\[
\sum_{\{x^d-p\}} e^{2\pi \text{i} \int d^k \delta_i(x; \xi^p) f_i(x)} = \sum_{\{x^d-p\}} \delta[f_i(x) - \delta_i(x; \xi^d)] \int D\tau_i e^{i \int \delta x \cdot \tau_i} , \quad (A.6)
\]
which is the GPI.

The brane proliferation–dilution interpretation of the GPI (A.6)
follows from the following reasoning: if \( \{x^d-p\} \to \emptyset \) then \( \delta_i(x; \xi^d) \to 0 \) (there are no \( \{x^d-p\} \) branes in the space to be local-
ized) and
\[
\sum_{\{x^d-p\}} \delta[f_i(x) - \delta_i(x; \xi^d)] \to \delta[f_i] \equiv \int D\tau_i e^{i \int \delta x \cdot \tau_i} , \quad (A.7)
\]

Comparing (A.6) and (A.7) we see that in the limit of dilution of
the \( \{x^d-p\} \) branes we have \( \theta_j^V(x; \xi^p) = 2\pi \delta_i(x; \xi^p) \to \tau_i \) and
\[
\sum_{\{x^d-p\}} \int D\tau \to \int D\gamma_i , \quad \text{which corresponds to the proliferation of the } \{\xi^p\} \text{ branes.}
\]

Conversely, in the limit of proliferation of the \( \{x^d-p\} \) branes,
\[
\theta_j^V(x; \xi^d) = 2\pi \delta_i(x; \xi^d) \to \gamma_i \text{ and } \sum_{\{x^d-p\}} \int D\gamma_i \text{ we have}
\]
\[
\sum_{\{x^d-p\}} \delta[f_i(x) - \delta_i(x; \xi^d)] \to \int D\gamma_i \delta[f_i - \gamma_i] = \emptyset . \quad (A.8)
\]

Comparing (A.6) and (A.8) we see that in the limit of proliferation
of the \( \{x^d-p\} \) branes we have \( \theta_j^V(x; \xi^p) = 2\pi \delta_i(x; \xi^p) \to \emptyset \)
which corresponds to the dilution of the \( \{\xi^p\} \) branes.

It is important to notice that the information about which brane
configurations are accessible by the system in the brane sums in
the GPI (A.6) is not present in this formulation, being an external
input controlled by hands as when we considered previously, for
example, the extreme cases of prolific or diluted accessible brane
configurations.

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