SEMISTABLE HIGGS BUNDLES AND REPRESENTATIONS OF ALGEBRAIC FUNDAMENTAL GROUPS: POSITIVE CHARACTERISTIC CASE

GUITANG LAN, MAO SHENG, AND KANG ZUO

Abstract. Let $k$ be an algebraic closure of finite fields with odd characteristic $p$ and a smooth projective scheme $X/W(k)$. Let $X^0$ be its generic fiber and $X$ the closed fiber. For $X^0$ a curve Faltings conjectured that semistable Higgs bundles of slope zero over $X^0$ correspond to genuine representations of the algebraic fundamental group of $X^0_{\mathbb{C}_p}$ in his $p$-adic Simpson correspondence [3]. This paper intends to study the conjecture in the characteristic $p$ setting. Among other results, we show that isomorphism classes of rank two semistable Higgs bundles with trivial chern classes over $X^0$ are associated to isomorphism classes of two dimensional genuine representations of $\pi_1(X^0)$ and the image of the association contains all irreducible crystalline representations. We introduce intermediate notions strongly semistable Higgs bundles and quasi-periodic Higgs bundles between semistable Higgs bundles and representations of algebraic fundamental groups. We show that quasi-periodic Higgs bundles give rise to genuine representations and strongly Higgs semistable are equivalent to quasi-periodic. We conjecture that a Higgs semistable bundle is indeed strongly Higgs semistable.

1. Introduction

N. Hitchin [4] introduced rank two stable Higgs bundles over a compact Riemann surface $X$ and showed that they correspond naturally to irreducible representations of the fundamental group $\pi_1(X)$ by solving a Yang-Mills equation, which generalizes the earlier works by Donaldson, Uhlenbeck-Yau for polystable vector bundles. Later C. Simpson obtained the full correspondence for any polystable Higgs bundles over arbitrary dimensional complex projective manifolds. In [3] G. Faltings established the correspondence between Higgs bundles and generalized representations of $\pi_1(X)$ over $p$-adic fields. He conjectured that semistable Higgs bundles under his functor shall correspond to usual $p$-adic representations of $\pi_1(X)$. In this paper we intend to study Faltings’s conjecture in the characteristic $p$ setting.

Let $k$ be the algebraic closure of finite fields of odd characteristic $p$. Let $X/W(k)$ be a smooth projective $W := W(k)$-scheme and $X/k$ its closed fiber. In this paper, if not specified, a Higgs bundle over $X$ means a system of Hodge bundles $(E = \oplus_{i+j=n}E^{i,j}, \theta = \oplus_{i+j=n}\theta^{i,j})$, where $E$ is a vector bundle over $X$, $\theta$ is a morphism of $O_X$-modules satisfying

$$\theta^{i,j} : E^{i,j} \to E^{i-1,j+1} \otimes \Omega_X,$$

$$\theta \wedge \theta = 0.$$

For simplicity, we assume throughout that $n \leq p-2$. Fix an ample divisor $H \subset X$ over $W$. The Higgs semistability of $(E, \theta)$ is referred to the $\mu$-semistability with respect to $H \subset X$, the reduction of $H$.

This work is supported by the SFB/TR 45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG, and partially supported by the University of Science and Technology of China.
Theorem 1.1 (Corollary 3.9 and Corollary 4.2). There is a functor from the category of quasi-periodic Higgs-de Rham sequences of type \((e, f)\) to the category of crystalline representations of \(\pi_1(X'_0)\) into \(\text{GL}(F_{pf})\), where \(X' := X \times_W \mathcal{O}_K\) for a totally ramified extension \(\text{Frac}(W) \subset K\) with ramification index \(e\). There is also a functor in the opposite direction. These two functors are equivalence of categories in the case \(e = 0\) and quasi-inverse to each other.

Consequently, we obtain the following

Corollary 1.2 (Corollary 5.2). Under the above functors, there is one to one correspondence between the isomorphism classes of irreducible crystalline \(F_{pf}\)-representations of \(\pi_1(X^0)\) and the isomorphism classes of periodic Higgs stable bundles of period \(f\).

The leading term of a quasi-periodic Higgs-de Rham sequence is a quasi-periodic Higgs bundle. We show that

Theorem 1.3 (Theorem 2.5). A quasi-periodic Higgs bundle is strongly Higgs semistable with trivial chern classes. Conversely, A strongly Higgs semistable bundle with trivial chern classes is quasi-periodic.

Strongly semistable vector bundles are strongly semistable Higgs bundles with trivial Higgs fields. As a semistable bundle need not be strongly semistable, the notion of strongly semistability should be replaced by the strongly Higgs semistability. The next result supports our viewpoint.

Theorem 1.4 (Theorem 2.6). A rank two semistable Higgs bundle is strongly Higgs semistable.

We would like to make the following

Conjecture 1.5. A semistable Higgs bundle is strongly Higgs semistable.

As an application of the above results, we obtain the following

Corollary 1.6 (Theorem 5.6). Any isomorphism class of rank two semistable Higgs bundles with trivial chern classes over \(X\) is associated to an isomorphism class of crystalline representations of \(\pi_1(X^0)\) into \(\text{GL}_2(k)\). The image of the association contains all irreducible crystalline representations of \(\pi_1(X^0)\) into \(\text{GL}_2(k)\).

The plan of our paper is arranged as follows: in Section 2 we introduce the notions strongly Higgs semistable bundles which generalizes the notion of strongly semistable vector bundles in the paper [7] of Lange-Stuhler and quasi-periodic Higgs bundles which generalizes the notion of periodic Higgs subbundles introduced in [11]. We show that a strongly Higgs semistable with trivial chern classes is equivalent to a quasi-periodic Higgs bundle, and a rank two semistable Higgs bundle is strongly Higgs semistable. We conjecture that semistable Higgs bundles of arbitrary rank are strongly Higgs semistable. In Section 3 we show in Theorem 3.1 that there is a one to one correspondence between the strict \(p\)-torsion category \(\mathcal{M}_F^{\nabla}_{[0,n],f}(X/W)\) of Faltings with endomorphism \(F_{pf}\) and the category of periodic Higgs-de Rham sequences of type \((0, f)\).

In Section 4, we extend the construction for periodic Higgs bundles to quasi-periodic Higgs bundles. In Section 5, we give some complements and applications of the above theory.

Acknowledgements: Arthur Ogus has recently pointed to us that the inverse Cartier transform in the paper [13] for the nilpotent Higgs bundles coincides with the construction in [9]. Christopher Deninger has drawn our attention to the work [6], and Adrian Langer has helped us understanding [6]. We thank them heartily.
2. Strongly semistable Higgs bundles

In this paper, a vector bundle over \( X \) means a torsion free coherent sheaf of \( O_X \)-module. A Higgs-de Rham sequence over \( X \) is a sequence of form

\[
\begin{align*}
(E_0, \theta_0) & \xrightarrow{C_0^{-1}} (H_0, \nabla_0) \xrightarrow{\text{Gr}_{Fil_0}} (E_1, \theta_1) \xrightarrow{C_0^{-1}} (H_1, \nabla_1) \xrightarrow{\text{Gr}_{Fil_1}} \cdots
\end{align*}
\]

In the sequence, \( C_0^{-1} \) is the inverse Cartier transform constructed in [13] (see also [9]). A. Ogus remarked that the exponential twisting of [9] is equivalent to the more general construction in [13] and the equivalence is implicitly implied by Remark 2.10 loc. cit. \( Fil_i \) is a decreasing filtration on \( H_i \) with the property \( Fil_i^0 = H_i \) and \( Fil_i^{n+1} = 0 \) and such that \( \nabla_i \) obeys the Griffiths transversality with respect to it.

**Definition 2.1.** A Higgs bundle \((E, \theta)\) is called strongly Higgs semistable if it appears in the leading term of a Higgs-de Rham sequence whose Higgs terms \((E_i, \theta_i)\)s are all Higgs semistable.

Recall that [7] a vector bundle \( E \) is said to be strongly semistable if \( F_X^n E \) is semistable for all \( n \in \mathbb{N} \). Clearly, a strongly semistable vector bundle \( E \) is strongly Higgs semistable: one takes simply the Higgs-de Rham sequence as

\[
\begin{align*}
(E_0, 0) & \xrightarrow{C_0^{-1}} (F_X^1 E, \nabla_{can}) \xrightarrow{\text{Gr}_{Fil_{tr}}} (E_1, 0) \xrightarrow{C_0^{-1}} (F_X^2 E, \nabla_{can}) \xrightarrow{\text{Gr}_{Fil_{tr}}} \cdots
\end{align*}
\]

where \( \nabla_{can} \) is the canonical connection in the theorem of Cartier descent and \( Fil_{tr} \) is the trivial filtration.

**Definition 2.2.** A Higgs bundle \((E, \theta)\) is called periodic if it appears in the leading term of a periodic Higgs-de Rham sequence, that is, there exists a natural number \( f \) such that there is an isomorphism of Higgs bundles

\[
(E_f, \theta_f) \cong (E_0, \theta_0),
\]

which via \( C_0^{-1} \) induces inductively a filtered isomorphism of de Rham bundles

\[
(H_{f+i}, \nabla_{f+i}, Fil_{f+i}) \cong (H_i, \nabla_i, Fil_i),
\]

and hence also an isomorphism of Higgs bundles for all \( i \in \mathbb{N} \),

\[
(E_{f+i}, \theta_{f+i}) \cong (E_i, \theta_i).
\]

The minimal number \( f \geq 1 \) is called the period of the sequence. One understands a periodic Higgs-de Rham sequence of period \( f \) through the following diagram:

\[
\begin{align*}
(E_0, \theta_0) \xrightarrow{C_0^{-1}} (H_0, \nabla_0) \xrightarrow{\text{Gr}_{Fil_0}} (H_{f-1}, \nabla_{f-1}) \xrightarrow{C_0^{-1}} (E_f, \theta_f) \xrightarrow{\text{Gr}_{Fil_{f-1}}} \cdots
\end{align*}
\]

In general, we make the following
A Higgs bundle $(E, \theta)$ is called quasi-periodic if it appears in the leading term of a quasi-periodic Higgs-de Rham sequence, i.e., it becomes periodic after a nonnegative integer $e \geq 0$.

We add a simple lemma which follows directly from the construction of $C_0^{-1}$ via the exponential function [9].

**Lemma 2.4.** Let $(E, \theta)$ be a nilpotent Higgs bundle (not necessarily a system of Hodge bundles) with exponent $\leq p - 1$. It holds that $\det C_0^{-1}(E, \theta) = F_X^e \det E$. Consequently,

$$\deg C_0^{-1}(E, \theta) = p \deg E.$$

**Proof.** It follows from the fact that in the determinant, the exponential twisting appeared in the construction of $C_0^{-1}(E, \theta)$ is simply the identity. 

**Theorem 2.5.** A quasi-periodic Higgs bundle is strongly Higgs semistable with trivial Chern classes. Conversely, a strongly Higgs semistable bundle with trivial Chern classes is quasi-periodic.

**Proof.** One observes that, in a Higgs-de Rham sequence, $c_l(E_{i+1}) = p^l c_l(E_i), i \geq 0$. This forces the Chern classes of a quasi-periodic Higgs bundle to be trivial. By Lemma 2.4, a degree $\lambda$ Higgs subbundle (not necessarily subsystem of Hodge bundles) in $(E_i, \theta_i)$ gives rise to a degree $p\lambda$ Higgs subbundle in $(E_{i+1}, \theta_{i+1})$. This implies that, in a Higgs-de Rham sequence of a quasi-periodic Higgs bundle, each Higgs term $(E_i, \theta_i)$ contains no Higgs subbundle of positive degree. So $(E_i, \theta_i)$ is Higgs semistable. Thus we have shown the first statement.

Assume $X$ has a model over a finite field $k' \subset k$. Let $M_{r,ss}(X)$ be the moduli space of $S$-equivalence classes of rank $r$ semistable Higgs bundles with trivial Chern classes over $X$. After A. Langer [6] and C. Simpson [8], it is a projective variety over $k'$. For a strongly Higgs semistable bundle $(E, \theta)$ over $X$ with trivial Chern classes, we consider the set of $S$-isomorphism classes $\{(E_i, \theta_i), i \in \mathbb{N}_0\}$, where $(E_i, \theta_i)s$ are all Higgs terms in a Higgs-de Rham sequence for $(E, \theta)$. Note that the operators $C_0^{-1}$ and $Gr_{Fil_k}$ do not change the definition field of objects. Thus, if the leading term $(E_0, \theta_0) = (E, \theta)$ is defined over a finite field $k'' \supset k'$, all terms in a Higgs-de Rham sequence are defined over $k''$. This implies that the above sequence is a sequence of $k''$-rational points in $M_{r,ss}(X)$ and hence finite. So we find two integers $e$ and $f$ such that $[(E_e, \theta_e)] = [(E_{e+f}, \theta_{e+f})]$. If $(E_e, \theta_e)$ is Higgs stable, then there is a $k''$-isomorphism of Higgs bundles $(E_e, \theta_e) \cong (E_{e+f}, \theta_{e+f})$. If it is only Higgs semistable, we obtain only a $k''$-isomorphism between their gradings. But we do find a $k''$-isomorphism of Higgs bundles after a certain finite field extension $k'' \subset k'''$; there exists a finite field extension $k'''$ of $k''$ such that $(E_e, \theta_e)$ admits a Jordan-Hölder (abbreviated as JH) filtration defined over $k'''$. The operator $Gr_{Fil_k} \circ C_0^{-1}$ transports this JH filtration into a JH filtration on $(E_{e+1}, \theta_{e+1})$ defined over the same field $k'''$. Then this holds for any Higgs term $(E_i, \theta_i), i \geq e$. Without loss of generality, we assume that there are only two stable components in the gradings. Then the isomorphism classes of extensions over two stable Higgs bundles are described by a projective space over a finite field. Since there are finitely many $S$-equivalence classes in $\{(E_i, \theta_i), i \geq e\}$ and over each $S$-equivalence class there are only finitely many $k'''$-isomorphism classes, there exists a $k'''$-isomorphism $(E_e, \theta_e) \cong (E_{e+f}, \theta_{e+f})$ after possibly choosing another $e, f$. It determines via $C_0^{-1}$ an isomorphism of flat bundles between $(H_e, \nabla_e)$ and $(H_{e+f}, \nabla_{e+f})$. This isomorphism defines a filtration $Fil'_e$ on $H_{e+f}$ from the filtration $Fil_l$ on $H_e$, which may differs.
from the original one. Put
\[(E_{e+f+1}', \theta_{e+f+1}') = Gr_{Fil_{e+f}}(H_{e+f}, \nabla_{e+f}).\]

One has then a tautological isomorphism between \((E_{e+1}, \theta_{e+1})\) and \((E_{e+f+1}', \theta_{e+f+1}').\)
Continuing the construction, we show that a strongly semistable Higgs bundle with trivial Chern classes can be putted into the leading term of a quasi-periodic Higgs-de Rham sequence, hence quasi-periodic. This shows the converse statement. □

**Theorem 2.6.** A rank two semistable Higgs bundle is strongly Higgs semistable.

**Proof.** Let \((E, \theta)\) be a rank two semistable Higgs bundle over \(X/k\). Note first that, for the reason of rank, \(\theta^2 = 0\). Hence the operator \(C_{0}^{-1}\) applies. Denote \((H, \nabla)\) for \(C_{0}^{-1}(E, \theta)\), and \(HN\) the Harder-Narasimhan filtration on \(H\). We need to show that the graded Higgs bundle \(Gr_{HN}(H, \nabla)\) is semistable. If \(H\) is semistable, there is nothing to prove: in this case, the \(HN\) is trivial and hence the induced Higgs field is zero, and \(Gr_{HN}(H, \nabla) = (H, 0)\) is Higgs semistable. Otherwise, the \(HN\) filtration is of form \(0 \to L_1 \to H \to L_2 \to 0\).

**Claim 2.7.** \(L_1 \subset H\) is not \(\nabla\)-invariant.

**Proof.** We can assume that \(\theta \neq 0\). Otherwise, by the Cartier descent, it follows that \(L_1 \cong F_1^* G_1\) for a rank one sheaf \(G_1 \subset E\) whose degree is positive, which contradicts with the semistability of \(E\). Write \(E = E^{1,0} \oplus E^{0,1}\) and \(\theta : E^{1,0} \to E^{0,1} \otimes \Omega_X\) is nonzero. By the local construction of \(C_0^{-1}\), the \(p\)-curvature of \(\nabla\) is nilpotent and nonzero. As \(L_1\) is of rank one, it follows that the \(p\)-curvature of \(\nabla|_{L_1}\) is zero. Again by the construction of \(C_0^{-1}\), \(\nabla\) preserves the rank one subsheaf \(L'_1 := C_0^{-1}(E^{0,1}, 0)\) and the restriction \(\nabla|_{L'_1}\) has also the \(p\)-curvature zero property. Let \(C \subset X\) be a generic curve. Then the nonzeroness of \(\theta\) implies that \(E^{0,1}|_C\) has negative degree. So is \(L'_1|_C\). As \(L_1\) has positive degree, they are not the same rank one subsheaf of \(H\). Therefore, over a nonempty open subset \(U \subset C\), one has \(H = L_1 \oplus L'_1\). It contradicts the nonzeroness of the \(p\)-curvature of \(\nabla\). □

Then it follows that
\[\theta' = Gr_{HN} \nabla : L_1 \to L_2 \otimes \Omega_X\]
is nonzero. Let \(L \subset Gr_{HN}H = L_1 \oplus L_2\) be a Higgs sub line bundle. As \(\theta'|_L = 0\), the composite
\[L \hookrightarrow L_1 \oplus L_2 \to L_1\]
is zero. Hence the natural map \(L \to L_2\) is nonzero and it follows that
\[\deg L \leq \deg L_2 < 0.\]
In this case, \(Gr_{HN}(H, \nabla)\) is Higgs stable. □

We would like to make the following

**Conjecture 2.8.** A semistable Higgs bundle is strongly Higgs semistable.
3. A Higgs Correspondence

In this section we aim to establish a Higgs correspondence between the category of Higgs-de Rham sequences of periodic Higgs bundles over \( X/k \) and the (modified) strict \( p \)-torsion category \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \), \( n \leq p-2 \) (abbreviated as \( \mathcal{M}_p \)) introduced by Faltings [1]. Here strict means that each object in the category is annihilated by \( p \).

We introduce first the category \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \), a modification of the Faltings category \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \). For each \( f \in \mathbb{N} \), let \( \mathbb{F}_p \) be the unique extension of \( \mathbb{F}_p \) in \( k \) of degree \( f \). An object in \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \) (abbreviated as \( \mathcal{M}_f \)) is a five tuple \((H, \nabla, \text{Fil}, \Phi, \iota)\), where \((H, \nabla, \text{Fil}, \Phi)\) is object in \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \) and

\[
t : \mathbb{F}_p \ni \text{End}_{\mathcal{M}_p}(H, \nabla, \text{Fil}, \Phi)
\]

is an embedding of \( \mathbb{F}_p \)-algebras. A morphism is a morphism in \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \) respecting the endomorphism structure. Clearly, the category \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \) for \( f = 1 \) is just the original \( \mathcal{M}_{[0,n]}^\mathcal{F}(X/W) \).

On the Higgs side, we define the category \( \mathcal{HB}_{n,(0,f)}(X/k) \) (abbreviated as \( \mathcal{HB}_{(0,f)} \)) of the periodic Higgs-de Rham sequences of type \((0,f)\) as follows: an object is a tuple \((E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{f-1}, \phi)\) where \((E, \theta)\) is a Higgs bundle on \( X/k \), \( \text{Fil}_i, 0 \leq i \leq f-1 \) is a decreasing filtration on \( C_0^{-1}(E_i, \theta_i) \) satisfying \( \text{Fil}_i = C_0^{-1}(E_i, \theta_i), \text{Fil}_{i+1}^n = 0 \) and the Griffiths transversality such that \( \text{Gr}_{\text{Fil}_i}(H_i, \nabla_i) \) is torsion free with \((E_0, \theta_0) = (E, \theta)\) and \((E_i, \theta_i) := \text{Gr}_{\text{Fil}_{i-1}}(H_{i-1}, \nabla_{i-1})\) inductively defined, and \( \phi \) is an isomorphism of Higgs bundles

\[
\text{Gr}_{\text{Fil}_{f-1}} \circ C_0^{-1}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).
\]

The information of such a tuple is encoded in the following diagram:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {\((E_0, \theta_0)\)};
\node (B) at (1.5,0) {\((E_f, \theta_f)\)};
\node (C) at (0,-1) {\((H_0, \nabla_0)\)};
\node (D) at (1.5,-1) {\((H_{f-1}, \nabla_{f-1})\)};
\node (E) at (1.5,-2) {\((E_{f-1}, \theta_{f-1})\)};
\node (F) at (0,-2) {\((E_0, \theta_0)\)};
\node (G) at (1.5,-2) {\((E_f, \theta_f)\)};
\draw[->] (A) to node [midway, above] {$\phi$} (G);
\draw[->] (A) to node [midway, below] {$\phi_i$} (F);
\draw[->] (B) to node [midway, above] {$\phi_{f-1}$} (G);
\draw[->] (B) to node [midway, below] {$\phi_{f-1}$} (F);
\draw[->] (A) to node [midway, right] {$\text{Gr}_{\text{Fil}_0}$} (C);
\draw[->] (B) to node [midway, right] {$\text{Gr}_{\text{Fil}_{f-1}}$} (D);
\draw[->] (C) to node [midway, left] {$C_0^{-1}$} (A);
\draw[->] (D) to node [midway, left] {$C_0^{-1}$} (B);
\end{tikzpicture}
\end{array}
\]

Note that \((E, \theta)\) of a tuple in the category is indeed periodic. A morphism between two objects is a morphism of Higgs bundles respecting the additional structures. As an illustration, we explain a morphism in the category \( \mathcal{HB}_{(0,1)} \) in detail: let \((E_i, \theta_i, \text{Fil}_i, \phi_i), i = 1, 2\) be two objects and

\[
f : (E_1, \theta_1, \text{Fil}_1, \phi_1) \rightarrow (E_2, \theta_2, \text{Fil}_2, \phi_2)
\]

a morphism. By the functoriality of \( C_0^{-1} \), the morphism \( f \) of Higgs bundles induces a morphism of flat bundles:

\[
C_0^{-1}(f) : C_0^{-1}(E_1, \theta_1) \rightarrow C_0^{-1}(E_2, \theta_2).
\]

It is required to be compatible with the filtrations, and the induced morphism of Higgs bundles is required to be compatible with \( \phi \), that is, there is a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_{\text{Fil}_1} C_0^{-1}(E_1, \theta_1) & \xrightarrow{\phi_1} & (E_1, \theta_1) \\
\downarrow \text{Gr}_{C_0^{-1}(f)} & & \downarrow f \\
\text{Gr}_{\text{Fil}_2} C_0^{-1}(E_2, \theta_2) & \xrightarrow{\phi_2} & (E_2, \theta_2).
\end{array}
\]
**Theorem 3.1.** There is a one to one correspondence between the category $\mathcal{MF}_{[0,n],f}(X/W)$ and the category $\mathcal{HB}_{n,(0,f)}(X/k)$.

To show the theorem, we choose and fix a small affine covering $\{U_i\}$ of $X$, together with an absolute Frobenius lifting $F_{U_i}$ on each $U_i$. By modulo $p$, the covering induces an affine covering $\{U_i\}$ for $X$. We show first a special case of the theorem.

**Proposition 3.2.** There is a one to one correspondence between the Faltings category $\mathcal{MF}_{[0,n]}(X/W)$ and the category $\mathcal{HB}_{n,(0,1)}(X/k)$.

Let $(H, \nabla, \Fil, \Phi)$ be an object in $\mathcal{MF}$. Put $(E, \theta) := Gr_{Fil}(H, \nabla)$. The following lemma gives a functor $\mathcal{GR}$ from the category $\mathcal{MF}$ to the category $\mathcal{HB}_{(0,1)}$.

**Lemma 3.3.** There is a filtration $Fil_{\exp}$ on $C_0^{-1}(E, \theta)$ together with an isomorphism of Higgs bundles

$$\phi_{\exp} : Gr_{Fil_{\exp}}(C_0^{-1}(E, \theta)) \cong (E, \theta),$$

which is induced by the Hodge filtration $Fil$ and the relative Frobenius $\Phi$.

**Proof.** By Proposition 5.1, we showed that the relative Frobenius induces a global isomorphism of flat bundles

$$\tilde{\Phi} : C_0^{-1}(E, \theta) \cong (H, \nabla).$$

So we define $Fil_{\exp}$ on $C_0^{-1}(E, \theta)$ to be the inverse image of $Fil$ on $H$ by $\tilde{\Phi}$. It induces tautologically an isomorphism of Higgs bundles

$$\phi_{\exp} = Gr(\tilde{\Phi}) : Gr_{Fil_{\exp}}(C_0^{-1}(E, \theta)) \cong (E, \theta).$$

Next, we show that the functor $C_0^{-1}$ induces a functor in the opposite direction. Given an object $(E, \theta, \Fil, \phi) \in \mathcal{HB}_{(0,1)}$, it is clear to define the triple

$$(H, \nabla, \Fil) = (C_0^{-1}(E, \theta), \Fil).$$

What remains is to produce a relative Frobenius $\Phi$ from the $\phi$. Following Faltings [1] Ch. II. d), it suffices to give for each pair $(U_i, F_{U_i})$ an $O_{U_i}$-morphism

$$\Phi(U_i, F_{U_i}) : F_{U_i}^* Gr_{Fil} H|_{U_i} \to H|_{U_i}$$

satisfying

1. strong $p$-divisibility, that is, $\Phi(U_i, F_{U_i})$ is an isomorphism,
2. horizontal property,
3. over each $U_i \cap U_j$, $\Phi(U_i, F_{U_i})$ and $\Phi(U_j, F_{U_j})$ are related via the Taylor formula.

Recall [9] that over each $U_i$ we have the identification (chart)

$$\alpha_i := \alpha(U_i, F_{U_i}) : (F_{U_i}^* E|_{U_i}, d + \frac{dF_{U_i}}{p} F_{U_i}^* \theta|_{U_i}) \cong C_0^{-1}(E, \theta)|_{U_i}.$$

We define $\Phi(U_i, F_{U_i})$ to be the composite

$$F_{U_i}^* Gr_{Fil} H|_{U_i} \xrightarrow{F_{U_i}^* \phi} F_{U_i}^* E|_{U_i} \xrightarrow{\alpha_i} C_0^{-1}(E, \theta)|_{U_i} = H|_{U_i}.$$

By construction, $\Phi(U_i, F_{U_i})$ is strongly $p$-divisible. By Proposition 5 loc. cit., the transition function between $\alpha_i$ and $\alpha_j$ is given by the Taylor formula. It follows that $\Phi(U_i, F_{U_i})$ and $\Phi(U_j, F_{U_j})$ are interrelated by the Taylor formula.

**Lemma 3.4.** Each $\Phi(U_i, F_{U_i})$ is horizontal with respect to $\nabla$. 

We define first a natural isomorphism $A$ over $U$. The commutativity of the second diagram follows now from that of the last diagram.

The above lemma provides us with the functor $\text{C}_0^{-1}$ in the opposite direction. Now we can prove Proposition 3.2.

Proof. The equivalence of categories follows by providing natural isomorphisms of functors:

$$\mathcal{Gr} \circ \text{C}_0^{-1} \cong \text{Id}, \quad \text{C}_0^{-1} \circ \mathcal{Gr} \cong \text{Id}.$$
We start with an object \((E, \theta, \mathcal{F}i\ell, \Phi)\) in the category \(\mathcal{M}F\). We call it \(\mathcal{A}(H, \nabla, \mathcal{F}i\ell, \Phi)\). It is straightforward to verify that \(\mathcal{A}\) is indeed a transformation. Conversely, a natural isomorphism \(\mathcal{B}\) from \(\mathcal{G}\mathcal{R} \circ C_0^{-1}\) to \(\text{Id}\) is given as follows: for \((E, \theta, \mathcal{F}i\ell, \phi)\), put

\[
(H, \nabla, \mathcal{F}i\ell, \Phi) = C_0^{-1}(E, \theta, \mathcal{F}i\ell, \phi) \quad (E', \theta', \mathcal{F}i\ell', \phi') = \mathcal{G}\mathcal{R}(H, \nabla, \mathcal{F}i\ell, \Phi).
\]

Then \(\phi : Gr_{\mathcal{F}i\ell} \circ C_0^{-1}(E, \theta) \cong (E, \theta)\) induces an isomorphism from \((E', \theta', \mathcal{F}i\ell', \phi')\) to \((E, \theta, \mathcal{F}i\ell, \phi)\) in \(\mathcal{H}\mathcal{B}_{(0,1)}\), which we define to be \(\mathcal{B}(E, \theta, \mathcal{F}i\ell, \phi)\). It is direct to check that \(\mathcal{B}\) is a natural isomorphism.

Before moving to the proof of Theorem 3.1 in general, we shall introduce an intermediate category, the category of periodic Higgs-de Rham sequences of type \((0, 1)\) with endomorphism structure \(\mathbb{F}_p\): an object is a five tuple \((E, \theta, \mathcal{F}i\ell, \phi, \iota)\), where \((E, \theta, \mathcal{F}i\ell, \phi)\) is object in \(\mathcal{H}\mathcal{B}_{(0,1)}\) and \(\iota : \mathbb{F}_p \to \text{End}_{\mathcal{H}\mathcal{B}_{(0,1)}}(E, \theta, \mathcal{F}i\ell, \phi)\) is an embedding of \(\mathbb{F}_p\)-algebras. We denote this category by \(\mathcal{H}\mathcal{B}_f\). A direct consequence of Proposition 3.2 is the following

**Corollary 3.5.** The category \(\mathcal{M}F_{\nabla, f}^{[0,n]}(\mathbf{X}/W)\) is equivalent to the category \(\mathcal{H}\mathcal{B}_f\) of Higgs-de Rham sequences of type \((0, 1)\) with endomorphism structure \(\mathbb{F}_p\).

Corollary 3.5 and the following proposition finish the proof of Theorem 3.1

**Proposition 3.6.** There is a one to one correspondence between the category \(\mathcal{H}\mathcal{B}_{(0,f)}\) of periodic Higgs-de Rham sequences of type \((0, f)\) and the category \(\mathcal{H}\mathcal{B}_f\) of periodic Higgs-de Rham sequences of type \((0, 1)\) with endomorphism structure \(\mathbb{F}_p\).

We start with an object \((E, \theta, \mathcal{F}i_0, \ldots, \mathcal{F}i_{f-1}, \phi)\) in \(\mathcal{H}\mathcal{B}_{(0,f)}\). Put

\[
(G, \eta) := \bigoplus_{i=0}^{f-1}(E_i, \theta_i)
\]

with \((E_0, \theta_0) = (E, \theta)\). As the functor \(C_0^{-1}\) is compatible with direct sum, one has the identification

\[
C_0^{-1}(G, \eta) = \bigoplus_{i=0}^{f-1} C_0^{-1}(E_i, \theta_i).
\]

We equip the filtration \(\mathcal{F}i\ell\) on \(C_0^{-1}(G, \eta)\) by \(\bigoplus_{i=0}^{f-1} \mathcal{F}i\ell_i\) via the above identification. Also \(\phi\) induces a natural isomorphism of Higgs bundles \(\tilde{\phi} : Gr_{\mathcal{F}i\ell} C_0^{-1}(G, \eta) \cong (G, \eta)\) as follows: as

\[
Gr_{\mathcal{F}i\ell} C_0^{-1}(G, \eta) = \bigoplus_{i=0}^{r-1} Gr_{\mathcal{F}i\ell_i} C_0^{-1}(E_i, \theta_i),
\]

we require that \(\tilde{\phi}\) maps the factor \(Gr_{\mathcal{F}i\ell_i}(E_i, \theta_i)\) identically to the factor \((E_{i+1}, \theta_{i+1})\) for \(0 \leq i \leq f-2\) (assume \(f \geq 2\) to avoid the trivial case) and the last factor \(Gr_{\mathcal{F}i\ell_{f-1}}(E_{f-1}, \theta_{f-1})\) isomorphically to \((E_0, \theta_0)\) via \(\phi\). Thus the so constructed four tuple \((G, \eta, \mathcal{F}i\ell, \tilde{\phi})\) is an object in \(\mathcal{H}\mathcal{B}_{(0,1)}\).

**Lemma 3.7.** For an object \((E, \theta, \mathcal{F}i_0, \ldots, \mathcal{F}i_{f-1}, \phi)\) in \(\mathcal{H}\mathcal{B}_{(0,f)}\), there is a natural embedding of \(\mathbb{F}_p\)-algebras

\[
\iota : \mathbb{F}_p \to \text{End}_{\mathcal{H}\mathcal{B}_{(0,1)}}(G, \eta, \mathcal{F}i\ell, \tilde{\phi}).
\]

Thus the extended tuple \((G, \eta, \mathcal{F}i\ell, \tilde{\phi}, \iota)\) is an object in \(\mathcal{H}\mathcal{B}_f\).
Proof. Without loss of generality, we assume \( f = 2 \). Choose a primitive element \( \xi \) in \( \mathbb{F}_p^* / \mathbb{F}_p \) once and for all. To define the embedding \( \iota \), it suffices to specify the image \( s := \iota(\xi) \), which is defined as follows: write \( (G, \eta) = (E_0, \theta_0) \oplus (E_1, \theta_1) \). Then \( s = m_\xi \oplus m_{\xi^p} \), where \( m_{\xi^i}, i = 0, 1 \) is the multiplication map by \( \xi^i \). It defines an endomorphism of \((G, \eta)\) and preserves \( Fil \) on \( C_0^{-1}(G, \eta) \). Write \((Gr_{Fil} \circ C_0^{-1})(s)\) to be the induced endomorphism of \( Gr_{Fil} C_0^{-1}(G, \eta) \). It remains to verify the commutativity

\[
\tilde{\phi} \circ s = (Gr_{Fil} \circ C_0^{-1})(s) \circ \tilde{\phi}.
\]

In terms of a local basis, it boils down to the equation

\[
\begin{pmatrix}
0 & 1 \\
\phi & 0
\end{pmatrix}
\begin{pmatrix}
\xi & 0 \\
0 & \xi^p
\end{pmatrix}
= 
\begin{pmatrix}
\xi^p & 0 \\
0 & \xi
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
\phi & 0
\end{pmatrix},
\]

which is clear. \( \square \)

Conversely, given an object \((G, \eta, Fil, \phi, \iota)\) in the category \( \mathcal{HB}_f \), we can associate it an object in \( \mathcal{HB}_{(0, f)} \) as follows: the endomorphism \( \iota(\xi) \) decomposes \((G, \eta)\) into eigenspaces:

\[
(G, \eta) = \bigoplus_{i=0}^{f-1} (G_i, \eta_i),
\]

where \( (G_i, \eta_i) \) is the eigenspace to the eigenvalue \( \xi^i \). The isomorphism \( C_0^{-1}(\iota(\xi)) \) induces the eigen-decomposition of the de Rham bundle as well:

\[
(C_0^{-1}(G, \eta), Fil) = \bigoplus_{i=0}^{f-1} (C_0^{-1}(G_i, \eta_i), Fil_i).
\]

Under the decomposition, the isomorphism \( \phi : Gr_{Fil} C_0^{-1}(G, \eta) \cong (G, \eta) \) decomposes into \( \bigoplus_{i=0}^{f-1} \phi_i \) such that

\[
\phi_i : Gr_{Fil_i} C_0^{-1}(G_i, \eta_i) \cong (G_{i+1 \mod f}, \theta_{i+1 \mod f}).
\]

Put \((E, \theta) = (G_0, \theta_0)\).

Lemma 3.8. The filtrations \( \{Fil_i\} \)s and isomorphisms of Higgs bundles \( \{\phi_i\} \)s induce inductively the filtration \( \widetilde{Fil}_i \) on \( C_0^{-1}(E_i, \theta_i), i = 0, \ldots, f - 1 \) and the isomorphism of Higgs bundles

\[
\tilde{\phi} : Gr_{\widetilde{Fil}_f^{-1}} (E_{f-1}, \theta_{f-1}) \cong (E, \theta).
\]

Thus the extended tuple \((E, \theta, \widetilde{Fil}_0, \ldots, \widetilde{Fil}_f, \tilde{\phi})\) is an object in \( \mathcal{HB}_{(0, f)} \).

Proof. Again we shall assume \( f = 2 \). The filtration \( \widetilde{Fil}_0 \) on \( C_0^{-1}(E_0, \theta_0) \) is just \( Fil_0 \). Via the isomorphism

\[
C_0^{-1}(\phi_0) : C_0^{-1} Gr_{Fil_0} C_0^{-1}(G_0, \eta_0) \cong C_0^{-1}(G_1, \eta_1),
\]

we obtain the filtration \( \widetilde{Fil}_1 \) on \( C_0^{-1}(E_1, \theta_1) \) from the \( Fil_1 \). Finally we define \( \tilde{\phi} \) to be the composite:

\[
Gr_{Fil_1} (E_1, \theta_1) = Gr_{Fil_1} C_0^{-1} Gr_{Fil_0} C_0^{-1}(E, \theta) \xrightarrow{Gr_{Fil_1}, C_0^{-1}(\phi_0)} Gr_{Fil_1} C_0^{-1}(G_1, \eta_1) \xrightarrow{\phi_1} (E, \theta).
\]

We come to the proof of Proposition 3.6.
Proof. Note first that Lemma 3.7 gives us a functor $E$ from $\mathcal{HB}_{(0,f)}$ to $\mathcal{HB}_f$, while Lemma 3.8 gives a functor $F$ in the opposite direction. We show that they give an equivalence of categories. It is direct to see that

$$ F \circ E = Id. $$

So it remains to give a natural isomorphism $\tau$ between $E \circ F$ and $Id$. Again we assume that $f = 2$ in the following argument. For $(E, \theta, Fil, \phi, \iota)$, put

$$ F\{(E, \theta, Fil, \phi, \iota)\} = (G, \eta, Fil_0, Fil_1, \tilde{\phi}), \quad E(G, \eta, Fil_0, Fil_1, \tilde{\phi}) = (E', \theta', Fil', \phi', \iota'). $$

Notice that $(E', \theta') = (G, \eta) \oplus Gr_{Fil_0}C_0^{-1}(G, \eta)$, we define an isomorphism of Higgs bundles by

$$ Id \oplus \phi_0 : (E', \theta') = (G, \eta) \oplus Gr_{Fil_0}C_0^{-1}(G, \eta) \cong (E_0, \theta_0) \oplus (E_1, \theta_1) = (E, \theta). $$

It is easy to check that the above isomorphism gives an isomorphism $\tau(E, \theta, Fil, \phi, \iota)$ in the category $\mathcal{HB}_f$. The functorial property of $\tau$ is easily verified. \hfill $\Box$

Faltings showed that the (contravariant) functor $D$ $[1]$ from $\mathcal{MF}_{[0,n]}(X/W)$ to the category of continuous $F_p$-representations of $\pi_1(X^0)$ is fully faithful. The image is closed under subobject and quotient, and its object is called dual crystalline sheaf. In our paper we take the dual of $D$ (cf. page 43 loc. cit.) without changing the notation. A crystalline $F_p$-representation is a crystalline $F_p$-representation $\mathbb{V}$ with an embedding of $F_p$-algebras $F_p \hookrightarrow \text{End}_{\pi_1(X^0)}(\mathbb{V})$.

**Corollary 3.9.** There is an equivalence of categories between the category of crystalline $F_p$-representations of $\pi_1(X^0)$ and the category of periodic Higgs-de Rham sequences of type $(0, f)$.

**Proof.** Under the functor $D$, an $F_p$-endomorphism structure on an object of $\mathcal{MF}$ is mapped to an $F_p$-endomorphism structure on the corresponding $F_p$-representation, and vice versa. The result is then a direct consequence of Theorem 3.1. \hfill $\Box$

Let $\rho$ be a crystalline $F_p$-representation of $\pi_1(X^0)$, and $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$ the corresponding periodic Higgs-de Rham sequence of type $(0, f)$. For

$$ (E_f, \theta_f) = Gr_{Fil_{f-1}}(H_{f-1}, \nabla_{f-1}), $$

$C_0^{-1}(\phi)$ induces the pull-back filtration $C_0^{-1}(\phi)^*Fil_0$ on $C_0^{-1}(E_f, \theta_f)$ and an isomorphism of Higgs bundles $GrC_0^{-1}(\phi)$ on the gradings. It is easy to check that

$$ (E_1, \theta_1, Fil_1, \cdots, Fil_{f-1}, C_0^{-1}(\phi)^*Fil_0, GrC_0^{-1}(\phi)) $$

is an object in $\mathcal{HB}_{(0,f)}$, which is called the shift of $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$. For any multiple $lf, l \geq 1$, we can lengthen $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$ to an object of $\mathcal{HB}_{(0,lf)}$: as above, we can inductively define the induced filtration on $(H_j, \nabla_j), f \leq j \leq lf - 1$ from $Fil_{0S}$ via $\phi$. One has the induced isomorphism of Higgs bundles $GrC_0^{-1}(\phi)^{(lf)} : (E_{(l'f+1)f}, \theta_{(l'f+1)f}) \cong (E_{lf}, \theta_{lf}), 0 \leq l' \leq l - 1$. The isomorphism $\phi_l : (E_{lf}, \theta_{lf}) \cong (E_0, \theta_0)$ is defined to be the composite of them. The obtained object $(E, \theta, Fil_0, \cdots, Fil_{lf-1}, \phi_l)$ is called the $l$-th lengthening of $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$. The following result is obvious from the construction of the above correspondence.

**Proposition 3.10.** Let $\rho$ and $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$ be as above. Then the followings are true:

(i) The shift of $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$ corresponds to $\rho^0 = \rho \otimes_{F_p, \sigma} F_p^{l'}$, the $\sigma$-conjugation of $\rho$. Here $\sigma \in \text{Gal}(F_p^l/F_p)$ is the Frobenius element.
For \( l \in \mathbb{N} \), the \( l \)-th lengthening of \((E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)\) corresponds to the base extension \( \rho \otimes_{F_{pf}} F_{p^l} \).

We remind also the reader of the following result.

**Corollary 3.11.** Periodic Higgs bundles are locally free.

**Proof.** Let \((E, \theta)\) be a periodic Higgs bundle. Then a Higgs-de Rham sequence for it gives an object in the category \( \mathcal{HB}_{(0,f)} \) for a certain \( f \). Let \((H, \nabla, Fil, \Phi, \iota)\) be the corresponding object in \( \mathcal{MF}_f \). The proof of Theorem 2.1 [1] (cf. page 32 loc. cit.) says that \( Fil \) is a filtration of locally free subsheaves of \( H \) and the grading \( Gr_{Fil} H \) is also locally free. It follows immediately that \((E, \theta)\) is locally free. \( \square \)

4. **Quasi-periodic Higgs bundles**

A quasi-periodic Higgs-de Rham sequence of of type \((e, f)\) is a tuple

\[(E, \theta, Fil_0, \cdots, Fil_{e+f-1}, \phi),\]

where \( \phi \) is an isomorphism of Higgs bundles

\[\phi : Gr_{Fil_{e+f-1}}(H_{e+f-1}, \nabla_{e+f-1}) \cong (E_e, \theta_e).\]

It follows from Corollary 3.11 that the Higgs bundles \((E_i, \theta_i), e \leq i \leq e + f - 1\) are locally free. They form the category \( \mathcal{HB}_{n,(e,f)}(X/k) \).

We are going to associate a quasi-periodic Higgs-de Rham sequence of type \((e, f)\) with an object in a Faltings category. We recall first the strict \( p \)-torsion category \( \mathcal{MF}_{[0,n]}^\nabla(X_V/R_V) \), which is based on the category introduced by Faltings in §3-§4 [2]. For \( V \) a totally ramified extension of \( W(k) \), Faltings §2 [2] introduced the base ring \( R_V \) as follows: a uniformizer \( \pi \) of \( V \) has the minimal polynomial

\[f(T) = T^e + \sum_{0 \leq i \leq e} a_i T^i \in W[T].\]

It defines the \( W \)-algebra morphism \( W[[T]] \to V, T \mapsto \pi \) and \( R_V \) is defined to be the PD-hull of \( V \). One has an excellent lifting \( X/k \) over \( R_V \), that is, one takes \( X \times W R_V \), the base change of \( X/W \) to \( R_V \). Put \( \mathcal{X} = X \times_W R_V/p = X \times_k R_V/p \). It depends only on the ramification index \( e \) of \( V \), not on \( V \) itself. The sheaf of \( k \)-algebras \( \mathcal{O}_X \) admits a natural filtration \( Fil_{\mathcal{O}_X} \). The composite of the natural maps

\[k = W/p \to R_V/p \xrightarrow{T \mapsto 0} k\]

is the identity. It induces the commutative diagram of \( k \)-schemes

\[\begin{array}{ccc}
X & \xrightarrow{\mu} & \mathcal{X} \\
\downarrow{id} & & \downarrow{\lambda} \\
X & & \\
\end{array}\]

An object of the category \( \mathcal{MF}_{[0,n]}^\nabla(X_V/R_V) \) is a four tuple \((H, \nabla, Fil, \Phi)\), where \((H, Fil)\) is a locally filtered-free \( \mathcal{O}_X \)-module of finite rank, with a local basis consisting of homogenous elements of degrees between 0 and \( n \), \( \nabla : H \to H \otimes \Omega_X/k \) an integrable connection satisfying the Griffiths transversality, the relative Frobenius \( \Phi \) is strongly \( p \)-divisible (i.e. \( \Phi \) locally over \( \mathcal{U}_i \subset \mathcal{X} \) induces an isomorphism \( F_{\mathcal{U}_i} Gr^n_{Fil} H \cong H|_{\mathcal{U}_i} \)) and horizontal with respect to \( \nabla \).
Lemma 5.1. The morphism $\lambda$ induces a functor $\lambda^*$ from $\mathcal{HB}_{(e,f)}$ to $\mathcal{MF}_{[0,n],f}(X_V/R_V)$ and the morphism $\mu$ a functor $\mu^*$ from $\mathcal{MF}_{[0,n],f}(X_V/R_V)$ to the category $\mathcal{HB}_{(0,f)}$. 

Proof. For $(E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{e+f-1}, \phi)$, we take $(E', \theta') = \bigoplus_{i=0}^{e+f-1} (E_i, \theta_i)$. Then $\text{Fil}_i s$ and $\phi$ induces naturally an object $(E', \theta', \text{Fil}_0', \cdots, \text{Fil}_{e'}', \phi')$ in $\mathcal{HB}_{(e,1)}$. Thus it suffices to show the above statement for $f = 1$.

Put $H = \lambda^*H_0$, $\nabla = \lambda^*\nabla_0$ and $\text{Fil} = \text{Fil}_{O_K} \otimes \lambda^*\text{Fil}_0$. Note that one has a natural isomorphism of $O_X$-modules $F_{U_i}^* \text{Gr}^{H_0}_{\text{Fil}_i} H \cong \lambda^*F_{U_i}^* \text{Gr}^{H_0}_{\text{Fil}_i} H$. We define the relative Frobenius $\Phi$ on $H$ via the above isomorphism composed with $\lambda^*\Phi_{(U_i,F_{U_i})}$, where $\Phi_{(U_i,F_{U_i})} : F_{U_i}^* \text{Gr}^{H_0}_{\text{Fil}_i} H \rightarrow H_{e|U_i}$ appeared in the paragraph before Lemma 3.3. This gives us the functor $\lambda^*$ from $\mathcal{HB}_{(e,1)}$ to $\mathcal{MF}_{[0,n]}(X_V/R_V)$. Conversely, given an object $(H, \nabla, \text{Fil}, \Phi) \in \mathcal{MF}_{[0,n]}(X_V/R_V)$, the tuple $(\mu^*H, \mu^*\nabla, \mu^*\text{Fil}, \mu^*\Phi)$ is naturally an object in $\mathcal{MF}_{[0,n]}(X/W)$: over $U_i$, $\Phi$ gives an isomorphism $F_{U_i}^* \text{Gr}^{H_0}_{\text{Fil}_i} H \cong H_{U_i}$. Pulling back the isomorphism via $\mu$, we get $F_{U_i}^* \mu^* \text{Gr}^{H_0}_{\text{Fil}_i} H \cong \mu^* H_{U_i}$. As there is a natural $O_X$-modules isomorphism $\text{Gr}^{H_0}_{\text{Fil}_i} \mu^* H \cong \mu^* \text{Gr}^{H_0}_{\text{Fil}_i} H$, we have an isomorphism $F_{U_i}^* \mu^* \text{Fil}_{U_i} \mu^* H_{U_i} \cong \mu^* H_{U_i}$, which shows that $\mu^* \Phi$ is indeed a relative Frobenius. We define $\mu^*(H, \nabla, \text{Fil}, \Phi) \in \mathcal{HB}_{(0,1)}$ to be the object associated to $(\mu^*H, \mu^*\nabla, \mu^*\text{Fil}, \mu^*\Phi)$.

\[ \square \]

Corollary 4.2. There is a functor from the category of quasi-periodic Higgs-de Rham sequences of type $(e, f)$ to the category of crystalline representations of $\pi_1(X^0)$ into $\text{GL}(\mathbb{F}_{p^f})$, where $X^0$ is the generic fiber of $X := X \times_W O_K$ for a totally ramified extension $\text{Frac}(W) \subset K$ with ramification index $e$. There is also a functor in the converse direction.

Proof. The first part follows from the above functor $\lambda^*$ and the proof of Theorem 5. i) [2]. To provide a functor in the opposite direction, we use the functor $\mu^*$ together with choosing an additional embedding of the category $\mathcal{HB}_{(0,f)}$ into $\mathcal{HB}_{(e,f)}$. This can be done as follows: for an object $(E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{f-1}, \phi) \in \mathcal{HB}_{(0,f)}$, let $l \in \mathbb{N}$ be the minimal number with $e \leq lf$. Then there is a unique object $(E', \theta', \text{Fil}_0', \cdots, \text{Fil}_{e+f-1}', \phi')$ in $\mathcal{HB}_{(e,f)}$ obtained from its $l + 1$-th lengthening which satisfies the equality

\[ (E_i', \theta_i') = (E_{lf-e+i}, \theta_{lf-e+i}), \quad 0 \leq i \leq e + f. \]

\[ \square \]

5. Applications

Given a periodic Higgs-de Rham sequence

\[ (H_0, \nabla_0) \xrightarrow{C_0^{-1}} \text{Gr}^{H_0}_{\text{Fil}_0} \xrightarrow{C_0^{-1}} (H_1, \nabla_1) \xrightarrow{\cdots} \]

we make the following observation:

Lemma 5.1. If $(E, \theta) = (E_0, \theta_0)$ is Higgs stable, then there is a unique periodic Higgs-de Rham sequence for $(E, \theta)$ up to isomorphism.

Proof. Let $f \in \mathbb{N}$ be the period of the sequence. Thus there is an isomorphism $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ such that the tuple $(E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{f-1}, \phi)$ makes an object in
We show that the datum $Fil_i$, $0 \leq i \leq f - 1$ and $\phi$ are uniquely determined up to isomorphism. By Theorem 3.1, there is a corresponding object
\[(H, Fil, \nabla, \Phi, i) \in MF_f\]
satisfying $Gr_{Fil}(H, \nabla) = \bigoplus_{i=1}^f (E_i, \theta_i)$. Because it holds that
\[(Gr_{Fil} \circ C_0^{-1})^j(E_i, \theta_f) = (E_i, \theta_i), 1 \leq i \leq f - 1,\]
each $(E_i, \theta_i)$ is also Higgs stable by Corollary 4.4 \[\square\]. Now we show inductively that $Fil_i$ is unique. This is because of the fact that there is a unique filtration on a flat bundle which satisfies the Griffiths transversality and its grading is Higgs stable. Now we consider $\phi$. For another choice $\varphi$, one notes that $\varphi \circ \phi^{-1}$ is an automorphism of $(E, \theta)$. As it is stable, one must have $\varphi = \lambda \phi$ for a nonzero $\lambda$ in $k$. It is easy to see there is an isomorphism in $\mathcal{H}(0, f)$:
\[(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi) \cong (E, \theta, Fil_0, \cdots, Fil_{f-1}, \lambda \phi).\]
\[\square\]

Because of the above lemma, the period of a periodic Higgs stable bundle is well defined. We make then the following statement.

**Corollary 5.2.** Under the equivalence of categories in Corollary 3.3, there is one to one correspondence between the isomorphism classes of irreducible crystalline $\mathbb{F}_{p^f}$-representations of $\pi_1(X^0)$ and the isomorphism classes of periodic Higgs stable bundles of period $f$.

The first examples of periodic Higgs stable bundles are the rank two Higgs subbundles of uniformizing type arising from the study of the Higgs bundle of a universal family of abelian varieties over the good reduction of a Shimura curve of PEL type (see \[\square\]). In that case, one 'sees' the corresponding representations because of the existence of extra endomorphisms in the universal family. The above result gives a vast generalization of this primitive example.

When a periodic Higgs bundle $(E, \theta)$ is only Higgs semistable, the above uniqueness statement is no longer true. We shall make the following

**Assumption 5.3.** For each $0 \leq i \leq f - 1$, the filtration $Fil_i$ on $H_i$ is preserved by any automorphism of $(H_i, \nabla_i)$.

An isomorphism $\varphi : (E_f, \theta_f) \cong (E_0, \theta_0)$ induces
\[(Gr_{C_0^{-1}})^nf(\varphi) : (E_{(n+1)f}, \theta_{(n+1)f}) \cong (E_nf, \theta_nf).\]

For $-1 \leq i < j$, we define
\[\varphi_{i,j} = (Gr_{C_0^{-1}})^{(i+1)f}(\varphi) \circ \cdots \circ (Gr_{C_0^{-1}})^{jf}(\varphi) : (E_{(j+1)f}, \theta_{(j+1)f}) \cong (E_{(i+1)f}, \theta_{(i+1)f}).\]

For $i = -1$ put $\varphi_{j} = \varphi_{j,-1}$.

**Lemma 5.4.** For any two isomorphisms $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$, there exists a pair $(i, j)$ with $0 \leq i < j$ such that $\phi_{j,i} \circ \varphi_{j,i}^{-1} = id$.

**Proof.** If we denote $\tau_s = \phi_s \circ \varphi_s^{-1}$, then $\tau_s$ is an automorphism of $(E_0, \theta_0)$. Moreover, each element in the set $\{\tau_s\}_{s \in \mathbb{N}}$ is defined over the same finite field in $k$. As this is a finite set, there are $j > i \geq 0$ such that $\tau_j = \tau_i$. So the lemma follows. \[\square\]
Proposition 5.5. Assume \( 5.3 \). Let \((i, j)\) be a pair given by Lemma \( 5.4 \) for two given isomorphisms \( \varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0) \). Then there is an isomorphism in \( \mathcal{B}(0, (j-i)_f) \):

\[
(E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{j-i-1}, \varphi_{j-i-1}) \cong (E, \theta, \text{Fil}_0, \cdots, \text{Fil}_{j-i-1}, \phi_{j-i-1}).
\]

Proof. Put \( \beta = \phi_i \circ \varphi_i^{-1} : (E_0, \theta_0) \cong (E_0, \theta_0) \). We shall check that it induces an isomorphism in \( \mathcal{B}(0, (j-i)_f) \). By Assumption \( 5.3 \), \( C^{-1}_0 (GrC^{-1}_0)^m(\beta) \) for \( m \geq 0 \) always respects the filtrations. We need only to check that \( \beta \) is compatible with \( \phi_{j-i-1} \) as well as \( \phi_{j-i-1} \). So it suffices to show that the following diagram is commutative:

\[
\begin{array}{ccc}
E_{(j-i)_f} & \xrightarrow{\varphi_{j-i-1}} & E_0 \\
\downarrow \varphi_{j,i-1} & & \downarrow \varphi_i \\
E_{(j+1)_f} & & E_{(i+1)_f} \\
\downarrow \phi_{j,i-1} & & \downarrow \phi_i \\
E_{(j-i)_f} & \xrightarrow{\phi_{j,i-1}} & E_0
\end{array}
\]

And it suffices to show that the following diagram is commutative:

\[
\begin{array}{ccc}
E_{(j-i)_f} & \xrightarrow{\varphi_{j,i-1}} & E_0 \\
\downarrow \varphi_{j,i-1} & & \downarrow \varphi_i \\
E_{(j+1)_f} & & E_{(i+1)_f} \\
\downarrow \phi_{j,i-1} & & \downarrow \phi_i \\
E_{(j-i)_f} & \xrightarrow{\phi_{j,i-1}} & E_0
\end{array}
\]

In the above diagram, the anti-clockwise direction is

\[
\phi_{j-i-1} \circ \phi_{j-j-i-1} \circ \varphi_{j-j-i-1}^{-1} \circ \varphi_{j-j-i-1}^{-1} = \phi_j \circ \varphi_j^{-1} = \phi_i \circ (\phi_{j,i} \circ \varphi_{j,i}^{-1}) \circ \varphi_i.
\]

By the requirement for \((i, j)\), we have \( \phi_{j,i} \circ \varphi_{j,i}^{-1} = id \), so the anti-clockwise direction is \( \phi_i \circ \varphi_i \), which is exactly the clockwise direction. So \( \beta \) is shown to be compatible with \( \phi_{j-i-1} \) and \( \varphi_{j-i-1} \). \( \square \)

We deduce some consequences from the above result.

Theorem 5.6. Any isomorphism class of rank two semistable Higgs bundles with trivial chern classes over \( X \) is associated to an isomorphism class of crystalline representations of \( \pi_1(X^0) \) into \( \text{GL}_2(k) \). The image of the association contains all irreducible crystalline representations of \( \pi_1(X^0) \) into \( \text{GL}_2(k) \).

Proof. The second statement follows from Theorem \( 5.2 \). Let \((E, \theta)\) be a rank two semistable Higgs bundle with trivial \( c_1 \) and \( c_2 \) over \( X \). By Theorems \( 2.6 \) and \( 2.5 \) it is a quasi-periodic Higgs bundle. Recall that we use the HN-filtration in the proof. Hence
we obtain the quasi-periodic Higgs-de Rham sequence for \((E, \theta)\). Let \(e \in \mathbb{N}_0\) be the minimal number such that \((Gr_{HN} \circ C_0^{-1})^e(E, \theta)\) is periodic and say its period is \(f \in \mathbb{N}\). Thus from \((E, \theta)\) we obtain in the above way an object
\[
((Gr_{HN} \circ C_0^{-1})^e(E, \theta), Fil_0 = HN, \ldots, Fil_{f-1} = HN, \phi)
\]
in \(\mathcal{HB}_{(0, f)}\), which is unique up to the choice of \(\phi\). Let \(\rho\) be the corresponding representation by Theorem 8.9. As \(HN\)'s clearly satisfy the Assumption 5.3, it follows from Proposition 5.5 that the isomorphism class of \(\rho \otimes k\) is independent of the choice of \(\phi\). It is clear that an isomorphic Higgs bundle to \((E, \theta)\) is associated to the same isomorphism class of crystalline representations. This shows the first statement.

Next, we want to compare the classical construction of Katz and Lange-Stuhler (see §4 [5] and §1 [7]) using an Artin-Schreier cover with the one in the current paper. Namely, we consider the isomorphism classes of vector bundles \(E\) over \(X\) satisfying \(F^{\geq f}_E \cong E\) for an exponent \(f \in \mathbb{N}\). By Proposition 1.2 and Satz 1.4 in [7] (see also §4.1 [5]), they are in bijection with the isomorphism classes of representations \(\pi_1(X) \rightarrow GL(k)\). Let \([\rho_{KLS}]\) be the isomorphism class of representations \(\pi_1(X) \rightarrow GL(k)\) corresponding to the isomorphism class of \(E\). Let \(E\) be such a bundle over \(X\) with an isomorphism \(\phi : F^{\geq f}_E \cong E\). It gives rise to a tuple \((E, 0, Fil_{tr}, \cdots, Fil_{tr}, \phi)\), an object in \(\mathcal{HB}_{(0, f)}\). Then by Theorem 8.9, there is a corresponding crystalline representation \(\rho : \pi_1(X^0) \rightarrow GL(\mathbb{F}_p)\). After Proposition 5.5, the isomorphism class of \(\rho \otimes k\), \(k\) is independent of the choice of \(\phi\). The following result follows directly from the construction of the representation due to Faltings [1].

**Lemma 5.7.** Let \(\tau\) be a crystalline representation of \(\pi_1(X^0)\) into \(GL(\mathbb{F}_p)\) and \((H, \nabla, Fil, \Phi)\) the corresponding object in \(\mathcal{MF}^{\nabla}_{[0, n]}(X/W)\). If the filtration \(Fil\) is trivial, namely, \(Fil^0H = H, Fil^1H = 0\), then \(\tau\) factors through the specialization map \(sp : \pi_1(X^0) \rightarrow \pi_1(X)\).

**Proof.** Let \(U_i = SpecR\) be a small affine subset of \(X\), and \(\Gamma = \text{Gal}(\bar{R}/R)\) the Galois group of maximal extension of \(R\) étale in characteristic zero (cf. Ch. II. b) [11]). Let \(R^{ur} \subset \bar{R}\) be the maximal subextension which is étale over \(R\) and \(\Gamma^{ur} = \text{Gal}(R^{ur}/R)\). By the local nature of the functor \(D\) (cf. Theorem 2.6 loc. cit.), it is to show that the representation \(D(H_i)\) of \(\Gamma\), constructed from the restriction \(H_i := (H, \nabla, Fil, \Phi)|_{U_i} \in \mathcal{MF}^{\nabla}_{[0, n]}(R)\), factors through the natural quotient \(\Gamma \rightarrow \Gamma^{ur}\). To that we have to examine the construction of \(D(H_i)\) carried in pages 36-39 loc. cit. (see also pages 40-41 for the dual object). First of all, we can choose a basis \(f\) of \(H_i\) which is \(\nabla\)-flat. Because \(Fil\) is trivial, \(\Phi\) is a local isomorphism. So for any basis \(e\) of \(H_i\), \(f = \Phi(e \otimes 1)\) is then a flat basis of \(H_i\). The construction of module \(D(H_i) \subset H_i \otimes \bar{R}/p\) does not use the connection, but the definition of \(\Gamma\)-action does (see page 37 loc. cit.). A basis of \(D(H_i)\) is of form \(f \otimes x\), where \(x\) is a set of tuples in \(\bar{R}/p\) satisfies the equation \(x^p = Ax\), where \(A\) is the matrix of \(\Phi\) under the basis \(f\) (i.e. \(\Phi(f \otimes 1) = Af\)). Now that \(A\) is invertible, the entries of \(x\) lie actually in \(R^{ur}/p\). Since \(f\) is a flat basis, the action of \(\Gamma\) on \(f \otimes x\) coincides the natural action of \(\Gamma\) on the second factor. Thus it factors through the quotient \(\Gamma \rightarrow \Gamma^{ur}\).

By the above lemma, \(\rho\) factors as
\[
\pi_1(X^0) \xrightarrow{sp} \pi_1(X) \rightarrow GL(\mathbb{F}_p).
\]

**Theorem 5.8.** Let \(\rho_F : \pi_1(X) \rightarrow GL(\mathbb{F}_p)\) be the induced representation from \(\rho\). Then \(\rho_F \otimes k\) is in the isomorphism class \([\rho_{KLS}]\).
Proof. We can assume that $E$ as well as $\phi$ are defined over $X|k'$ for a finite field $k'$. Then we obtain from Proposition 4.1.1 [5] or Satz 1.4 [7] a representation $\rho_{KLS} : \pi_1(X) \to \text{GL}(\mathbb{F}_p')$. We are going to show that $\rho_F$ and $\rho_{KLS}$ are isomorphic $\mathbb{F}_p'$-representations. For $f = 1$, this follows directly from their constructions: Katz and Lange-Stuhler construct the representation by solving $\phi$-invariant sections through the equation $x^p = Ax$, which it is exactly what Faltings does in the case of trivial filtration by the above description of his construction. For a general $f$, Katz and Lange-Stuhler solve locally the equation $x^{p^f} = Ax$, which is equivalent to a system of equations of form

$$x_0^p = x_1, \ldots, x_{f-2}^p = x_{f-1}, x_{f-1}^p = Ax_0.$$  

To examine our construction, we take a local basis $e_0 = e$ of $E_0 = E$ and put $e_i = F_X^i e$, a local basis of $E_i$ for $0 \leq i \leq f - 1$. Write $\phi(e_{f-1}) = Ae_0$. Put $\tilde{e} = (e_0, \ldots, e_{f-1})$, and $\tilde{x} = (x_1, \ldots, x_{f-1})$. Then the $\tilde{\phi}$ in Lemma 3.8 has the expression $\tilde{\phi}(\tilde{e}) = \tilde{A}\tilde{e}$ with

$$\tilde{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ \phi & 0 & \cdots & 0 \end{pmatrix}.$$  

One notices that the equation $\tilde{x}^p = \tilde{A}\tilde{x}$ written into components is exactly the above system of equations. Thus one sees that the $\mathbb{F}_p'$-representation $\rho_F$ corresponding to $(E, 0, \text{Fil}_{tr}, \cdots, \text{Fil}_{tr}, \phi)$ by Corollary 3.9 is isomorphic to $\rho_{KLS}$ as $\mathbb{F}_p'$-representations.

It may be noteworthy to deduce the following

Corollary 5.9. Let $\tau$ be a crystalline representation of $\pi_1(X^0)$ with the corresponding object $(H, \nabla, \text{Fil}, \Phi) \in \mathcal{MF}_{(0,n)}(X/W)$. Then $\tau$ factors through the specialization map iff the filtration Fil is trivial.

Proof. One direction is Lemma 5.7. It remains to show the converse direction. Let $\tau_0$ be the induced representation of $\pi_1(X)$ from $\tau$. As it is of finite image, one constructs directly from $\tau_0$ a vector bundle $E$ over $X$ such that $F_X^* E \cong E$. Choosing such an isomorphism, we obtain a representation of $\pi_1(X)$ and then a representation $\tau'$ of $\pi_1(X^0)$ by composing with the specialization map. By Theorem 5.8, $\tau' \otimes \mathbb{F}_p'$ is isomorphic to $\tau \otimes \mathbb{F}_p'$ for a certain $f \in \mathbb{N}$. It follows from Proposition 3.10 (ii) that the filtration Fil is trivial.

We conclude the paper by providing many more examples beyond the rank two semistable Higgs bundles and strongly semistable vector bundles.

Proposition 5.10. Let $(H, \nabla, \text{Fil}, \Phi) \in \mathcal{MF}_{[0,n]}(X/W)$. Then any Higgs subbundle $(G, \theta) \subset \text{Gr}_{Fil}(H, \nabla)$ of degree zero is strongly Higgs semistable with trivial chern classes.

Proof. Put $(E, \theta) = \text{Gr}_{Fil}(H, \nabla)$. Proposition 0.2 [10] says that $(E, \theta)$ is a semistable Higgs bundle of degree zero. Note that the operator $\text{Gr}_{Fil} \circ C_0^{-1}$ does not change the degree, rank and definition field of $(G, \theta)$, and as there are only finitely many Higgs subbundles of $(E, \theta)_0$ with the same degree, rank and definition field as $(G, \theta)$, there exists a pair $(e, f)$ of nonnegative integers with $s > r$ such that

$$(\text{Gr}_{Fil} \circ C_0^{-1})^s(G, \theta) = (\text{Gr}_{Fil} \circ C_0^{-1})^r(G, \theta).$$
holds. Thus \((G, \theta)\) is quasi-periodic and strongly Higgs semistable with trivial Chern classes by Theorem 2.5. □

References

[1] G. Faltings, Crystalline cohomology and \(p\)-adic Galois-representations, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25-80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[2] G. Faltings, Integral crystalline cohomology over very ramified valuation rings, Journal of the AMS, Vol. 12, no. 1, 117-144, 1999.

[3] G. Faltings, A \(p\)-adic Simpson correspondence, Advances in Mathematics 198 (2005), 847-862.

[4] N. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3), 55, 1987, 59-126.

[5] N. Katz, \(p\)-adic properties of modular schemes and modular forms, Lecture Notes in Mathematics, Vol. 350, Springer, Berlin, 1973.

[6] A. Langer, Moduli spaces of sheaves in mixed characteristic, Duke Math. J. 124 (2004), no. 3, 571-586.

[7] H. Lange and U. Stuhler, Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, Math. Z. 156 (1977), 73-83.

[8] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, II. Inst. Hautes Études Sci. Publ. Math. No. 79, 47-129 (1994), No. 80, 5-79 (1995).

[9] G.-T. Lan, M. Sheng, K. Zuo, An inverse Cartier transform via exponential in positive characteristic, 2012, Preprint.

[10] M. Sheng, H. Xin, K. Zuo, A note on the characteristic \(p\) nonabelian Hodge theory in the geometric case, arXiv: 1202.3942, 2012.

[11] M. Sheng, K. Zuo, Periodic Higgs subbundles in mixed characteristic, Preprint, 2012.

[12] M. Sheng, J.-J. Zhang, K. Zuo, Higgs bundles over the good reduction of a quaternionic Shimura curve, J. reine angew. Math., DOI 10.1515, 2011.

[13] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic \(p\), Publ. Math. Inst. Hautes études Sci. 106 (2007), 1-138.

E-mail address: lan@uni-mainz.de

Institut für Mathematik, Universität Mainz, Mainz, 55099, Germany

E-mail address: msheng@ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China

E-mail address: zuok@uni-mainz.de

Institut für Mathematik, Universität Mainz, Mainz, 55099, Germany