Kinematic model of the pursuit problem on a plane by the chase method

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Abstract. This article considers a kinematic, geometric model of the pursuit problem on a plane by the chase method, where the pursuer cannot instantly change the direction of movement, while moving at a constant modulo speed. The initial speed of the pursuer is not directed at the target when the pursuit begins. In order for the speed vector of the pursuer to be directed at the target after some time, we have developed a method that is based on following the trajectory that connects the pursuer and the target. This trajectory takes into account the inertia of the pursuer in the sense that the radius of curvature of the trajectory is not less than a certain threshold value. Based on the materials of this article, test programs were written and an animated image was made.

Keywords – target, pursuer, trajectory, approach, simulation.

1. Introduction

For the chase method in the problem of pursuit on a plane with a constant speed, it is characteristic that the speed vector of the pursuer exactly coincides with the direction to the target. In this formulation, the problem has both a continuous model to solve and a quasi-discrete one. In a continuous model, the solution is based on a numerical or analytical solution of a system of differential equations:

\[
\begin{cases}
(T_x(t) - P_x(t)) \cdot \frac{dP_x(t)}{dt} = (T_y(t) - P_y(t)) \cdot \frac{dP_y(t)}{dt} \\
\left(\frac{dP_x(t)}{dt}\right)^2 + \left(\frac{dP_y(t)}{dt}\right)^2 = V^2
\end{cases}
\]

where \(\begin{bmatrix} P_x \\ P_y \end{bmatrix}\) – pursuer's position

\(\begin{bmatrix} T_x \\ T_y \end{bmatrix}\) – target’s position

\(V\) – constant target speed value

Where the first equation of the system means that, the velocity vector of the pursuer is co-directed with the vector of the line connecting the pursuer and the target. The second equation of the system says that the velocity modulus is constant.

As a simple example of a quasi-discrete model, we can give one of the iterative schemes that calculates the coordinates of the pursuer's next position.

\[
P_{t+1} = P_t + V_p \cdot \Delta t \cdot (T_t - P_t),
\]

\(\forall t \in \text{pursuer's position}, T \in \text{target's position}, V_p \in \text{pursuer's position, \Delta t \in sampling period.}\)

As can be seen from the statement of the pursuit problem using the constant-speed chase method, the pursuer's velocity vector must always be directed at the target, even at the moment when the pursuit begins.
2. Statement of the problem

Let’s assume that at the initial moment of time the pursuer’s velocity \( V_p \) is not directed at the stationary target \( T \) (Figure 1). Due to the fact that our model has restrictions that the direction of the velocity vector cannot change instantly, and the radius of curvature of the trajectory cannot be less than a certain value, we have introduced a minimum radius of curvature of the trajectory \( r \).

![Figure 1. Simulation of the pursuer's trajectory](image)

From figure 1, we see that in order to reach the stationary target \( T \), the pursuer \( P \) must pass the arc \( (PP_t) \) and go to the straight segment \( [P_tT] \).

Our task is to develop a method for constructing the trajectory of the pursuer \( P \) when the target \( T \) is in motion.

3. Theory

Consider the iterative scheme \( P_{i+1} = P_i + V_p \cdot \Delta t \cdot (T_i - P_i) \), when the pursuer's velocity vector \( V_p \) is directed constantly at the target (Figure 2). But the point \( P_{i+1} \) will be considered as the point of intersection of a straight line \( (P_iT_i) \) with a circle of radius \( V_p \cdot \Delta t \), with the center at the point \( P_i \).

![Figure 2. The pursuer's speed vector is always directed at the target](image)

Figure 3 shows the result of modeling the pursuit problem by the chase method, according to the iterative scheme, with finding the intersection point of a circle \( (P_i, V_p \cdot \Delta t) \) and a straight line \( (P_iT_i) \), in the computer mathematics system.
This approach to trajectory design does not allow you to model when the speed of the pursuer at the time of the beginning of the pursuit is not directed at the target.

Consider the following iterative scheme. We assume that at time $t_i$, the target's position $T_i$, the pursuer's position $P_i$, and the vector equation $F_i(s)$ of the currently predicted pursuer's trajectory are known (Figure 4).

In this iterative process, our task is to calculate the coordinates $P_{i+1}$ of the pursuer's next step and perform affine transformations of the vector function $F_i(s)$ in order to find expressions for the function $F_{i+1}(s)$.

To find the coordinates of the point $P_{i+1}$, solve the equation $|F_i(s_{i+1}) - F_i(s_i)| = V_p \cdot \Delta t$ with respect to the parameter $s_{i+1}$.

When we developed the iterative process model, we set the initial coordinates $T$ and $P$, the initial velocity vector of the pursuer $V_p$. We also determined the vector function $F(s)$ at the time of the beginning of the pursuit.

Figure 3. Result of the pursuit simulation using the chase method

Figure 4. Simulating of pursuer's trajectory
pursuit. In figure 1, this is a composite curve from the arc \(PP_i\) and a straight segment \([P_iT]\), where the parameter is the arc length of this curve.

At the i-th iterative step, the following take place:

A circle is constructed with the center at the point \(P_i\) of radius \(V_p \cdot \Delta t\). The intersection point of this circle and the trajectory \(F_i(s)\) will be the point of the next step of the pursuer \(P_{i+1}\).

Then we find the intersection point \(P_{i+1}'\) a parametric vector function \(F_i(s)\) with a circle centered at \(T_i\) and radius \(|T_{i+1} - P_{i+1}|\).

Then, we form a local basis \((h_1', h_2')\) with the center of coordinates at the point \(P_{i+1}'\) and recalculate the function \(F_{i+1}(s)\). In the basis \((h_1', h_2')\), it will look like \(F_i'(s)\). The components of the basis \((h_1', h_2')\) are:

\[
\begin{align*}
h_1' &= \frac{T_i - P_{i+1}'}{|T_i - P_{i+1}'|} \\
h_2' &= \begin{bmatrix} h_{1'y} \\ h_{1'x} \end{bmatrix}
\end{align*}
\]

The function \(F_i'(s)\) will look like this:

\[
F_i'(s) = \begin{bmatrix} (F_i(s) - P_{i+1}') \cdot h_1' \\ (F_i(s) - P_{i+1}') \cdot h_2' \end{bmatrix}
\]

Form a basis \((h_1, h_2)\) with the center of coordinates at the point \(P_{i+1}\). The basis components \((h_1, h_2)\) will look like this:

\[
\begin{align*}
h_1 &= \frac{T_{i+1} - P_{i+1}}{|T_{i+1} - P_{i+1}|} \\
h_2 &= \begin{bmatrix} h_{1'y} \\ h_{1'x} \end{bmatrix}
\end{align*}
\]

Note that \(|T_{i+1} - P_{i+1}| = |T_i - P_{i+1}'|\). Hence, we can state that the local representation in the local basis \((h_1, h_2)\) centered at point \(P_{i+1}\) of the curve \(F_{i+1}(s)\) will coincide with the local representation of the basis \((h_1', h_2')F_{i+1}'(s) = F_i'(s)\).

The basis of the world coordinate system \((E_1, E_2)\) in the basis \((h_1, h_2)\) looks like this:
\begin{align*}
e_1 &= \begin{bmatrix} E_1 \cdot h_1 \\ E_1 \cdot h_2 \end{bmatrix} \\
e_2 &= \begin{bmatrix} E_2 \cdot h_1 \\ E_2 \cdot h_2 \end{bmatrix}
\end{align*}

Therefore, the equation of the line \( F_{i+1}(s) \) will be expressed as follows:

\[ F_{i+1}(s) = \frac{F'_{i+1}(s) \cdot e_1}{F'_{i+1}(s) \cdot e_2} + P_{i+1}. \]

Total for the i-th step of the iteration we have the following: step \( T_{i+1} \) is chosen by purpose step pursuer \( P_{i+1} \) is calculated as the intersection point of the circle \( (P_i, V \cdot \Delta t) \) and the previously calculated vector parametric line \( F_i(s) \), based on the current available data are used to generate new predicted trajectory of the pursuer \( F_i(s) \).

Based on the above material, a test program was written that shows a simple example of how to approach the chase method when the pursuer's speed vector is not directed at the target at the start of the pursuit.

The pursuer's trajectory at the same time has restrictions on the curvature. The radius of curvature of the trajectory cannot be less than a certain threshold value.

Figure 5 just shows the results of the program. Figure 5 is supplemented with a link to an animated image, where you can dynamically observe the process of chasing a target along a predefined trajectory.

4. The results of the experiments

In the test program written based on the materials of the article, we see that the trajectory of the pursuer is the envelope line of a certain one-parameter set of curves that are congruent with each other.

The position in space of each line is determined by the coordinates of the target and the coordinates of the pursuer. The target's coordinates for each line are fixed, that is, they coincide with one of the ends.

The coordinates of the pursuer move relative to the lines. This movement depends on the coordinates of the target. As a result, we get a one-parameter set of congruent lines that depend on the target position.

Modeling of each of the lines was performed with conjugating a circle of a given radius with a straight line leading to the target (Fig. 1). The radius of the circle is determined by the minimum radius of curvature of the pursuer's trajectory.

5. Discussion of results

It should be noted that you can use more than just the result of combining a circle and a straight line to model the predicted trajectories of the pursuer. To ensure that the target is reached at a given angle, it is possible to use two circles of the minimum radius of curvature of the trajectory, and mating lines between them.

A circle on the target's side would provide the desired angle to reach the target, and a circle on the pursuer's side would provide its initial direction.

Not only straight lines, but also cubic parabolas, cadioids, clotoids and other lines can be used as the interface line to ensure smoothness by the second derivative at the interface points.

6. Conclusions

In this article, we consider a kinematic model of the pursuit problem on a plane by the chase method, when the pursuer's speed is not directed at the target at the moment of the beginning of the pursuit.

This method can be used when developing a geometric model of group pursuit with simultaneous achievement of a goal or goals. It can also be used in the development of models, when the pursuer reaches the goal using the chase method at specified angles.
Based on the proposed models and algorithms, test programs for calculating trajectories are written in the MathCAD computer mathematics system. Program texts are available on the author's resource: http://dubanov.expomenta.ru. Links to animated images produced based on the results of the programs are available on the resource: https://www.youtube.com/watch?v=UQ5bVKjVqZ4.

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