Ensemble Steering, Weak Self-Duality, and the Structure of Probabilistic Theories

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Abstract

In any probabilistic theory, we may say a bipartite state $\omega$ on a composite system $AB$ steers its marginal state $\omega^B$ if, for any decomposition of $\omega^B$ as a mixture $\omega^B = \sum p_i \beta_i$ of states $\beta_i$ on $B$, there exists an observable $\{a_i\}$ on $A$ such that the conditional states $\omega_B|a_i$ are exactly the states $\beta_i$. This is always so for pure bipartite states in quantum mechanics, a fact first observed by Schrödinger in 1935. Here, we show that, for weakly self-dual state spaces (those isomorphic, but perhaps not canonically isomorphic, to their dual spaces), the assumption that every state of a system is steered by some bipartite state on two copies of that system, of a composite amounts to the homogeneity of the cone of unnormalized states. If the state space is actually self-dual, and not just weakly so, this implies (via the Koecher-Vinberg Theorem) that it is the self-adjoint part of a formally real Jordan algebra, and hence, quite close to being quantum mechanical.

1 Introduction

The founders of quantum mechanics were already well aware that some of its most non-classical (and seemingly paradoxical) aspects are naturally understood in terms of information. The Bohrian notion of complementarity, for example, can be understood in terms of one type of knowledge about a system precluding another. The fact

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that measurement must disturb the state of a system, also stressed by Bohr, again
gives fundamental status to a notion closely connected to information, the notion of
measurement, whose essence is the acquisition of some sort of information about a
system. Schrödinger was particularly inclined to take this point of view, as evidenced
by his description of entanglement:

The best possible knowledge of a total system does not necessarily include
total knowledge of all its parts, not even when these are fully separated
from each other and at the moment are not influencing each other at all.

[28]

The development of quantum information theory has rekindled interest in the possibility of characterizing quantum theory in operational or information-theoretic terms. Characteristically, this newer work has focused on finite-dimensional systems, and has emphasized considerations involving composite systems. It has become clear that many properties of quantum systems, e.g., the existence and basic properties of entangled states, are much better understood as generically non-classical, rather than specifically quantum, phenomena, in the sense that they arise in arbitrary non-classical probabilistic theories [12, 4, 18, 23, 24, 31]. There is therefore a premium on identifying operationally meaningful properties of bipartite quantum states that are, so to say, parochial—that is, properties that are not generic in this way. An example is the principle of information causality, recently introduced in [27]. Ideally, one would like to find a small set of such principles that pick out quantum mechanics on the nose, but failing this, it is still of interest to identify principles that define a small neighborhood of theories close to quantum mechanics.

A property of entangled quantum states that struck Schrödinger as especially odd is the fact that an observer controlling one component of such a state can steer the other system into any statistical ensemble for its (necessarily, mixed) marginal state, simply by choosing to measure a suitable observable [28, 21].

It is rather discomforting that the [quantum] theory should allow a system to be steered or piloted into one or the other type of state at the experimenter’s mercy in spite of his having no access to it. [28]

What Schrödinger found discomforting is now understood to be an important information theoretic feature of quantum mechanics. This became clear when Bennett and Brassard [13], in the same paper that introduced quantum key distribution, considered a natural quantum scheme for another important cryptographic primitive, bit commitment, and showed that ensemble steering can be used to break it. [1]

[1] In the scheme, the two possible values Alice can commit to are represented by two distinct ensembles for the same density matrix; she is to send samples from the ensemble to Bob in order
In this paper, we connect the possibility of ensemble steering with two very special geometric properties shared by finite-dimensional classical and quantum state spaces. First, such state spaces are self-dual: their cones of (un-normalized) effects are canonically isomorphic to their dual cones of (un-normalized) effects, meaning that the isomorphism defines an inner product. Secondly, they are homogeneous: their groups of order-isomorphisms act transitively on the interiors of their positive cones. These two properties come close to characterizing finite-dimensional quantum and classical state spaces: according to a celebrated theorem, due to Koecher [25] and Vinberg [30], finite-dimensional homogeneous, self-dual cones are precisely the cones of positive elements of formally real Jordan algebras. Once one has gone this far, two further axioms (local tomography, and the existence of qubits) suffice to recover QM uniquely. Here, we establish that in any probabilistic theory in which universal self-steering is possible (meaning that every state is the marginal of a bipartite state steering for that marginal) state spaces must be homogeneous and weakly self-dual, meaning that the cones of un-normalized states and un-normalized effects must be isomorphic, but perhaps not canonically so. This reduces the gap between the generic “self-steering” theory and quantum mechanics, largely to that between weak and strong self-duality.

A brief outline of the rest of this paper is as follows. In Section 2, we review, for the reader’s convenience, the mathematical framework in which we work, a variant of the generalized probability theory proposed by Mackey [26] over 50 years ago, and now quite standard in foundational work in quantum theory. In particular, we discuss what we mean by a composite system, and by a probabilistic “theory”. (A more detailed treatment of some of this material, can be found, e.g., in [4].) In Section 3, we introduce weakly self-dual state spaces, and establish some of their properties. In particular, we show (as a case of a result that we establish for general state spaces) that, for an irreducible state space \(A\), any bipartite state on \(A \otimes A\) corresponding to an order-isomorphism between \(A\) and its dual is pure. In Section 4, we connect this to the possibility of purifying a state, showing that weakly self-dual homogeneous state spaces are precisely those in which every interior state (i.e., state not on the boundary of the state space) arises as the marginal of such an isomorphism state. Section 5 discusses steering per se, illustrating the idea with several examples, and establishing an order-theoretic necessary and sufficient condition for the steering of one marginal, suitable quotient of the other. Section 6 summarizes our main results and suggests a number of questions for further work.

to commit, and later reveal which states she drew so that Bob can check that she used the claimed ensemble. However, by sending, not a draw from the ensemble but one system of a pure bipartite entangled state with the specified density matrix, and keeping the other system, she can realize either ensemble after she’s already sent the systems to Bob by making measurements on her entangled system, enabling her to perfectly mimic commitment to either bit.
2 The ordered linear spaces formalism

This section provides a quick summary of the formalism of abstract state spaces used in [4, 5, 6, 10, 11]. In the interest of brevity, we omit detailed motivation for the definitions below, referring the interested reader to [4]. Suffice it to say here that any model of a probabilistic system characterized by states and observables in the usual way [16, 17], fits naturally into this very general framework. Although much of what we do below can be extended to a more general context, we restrict ourselves here to finite-dimensional systems. Thus, we assume—generally, without further comment—that all vector spaces in what follows are finite dimensional.

An ordered linear space (OLS) is a real vector space $V$ equipped with a partial ordering compatible with the linear structure in the sense that it satisfies $x \leq y \Rightarrow x + z \leq y + z$ and $x \leq y \Rightarrow tx \leq ty$ for all $x, y, z \in V$ and all non-negative scalars $t$. Any such ordering is determined by the pointed convex cone $V_+$, called the positive cone, of vectors $x$ with $0 \leq x$, since $x \leq y$ iff $y - x \in V_+$; conversely, any pointed convex cone induces an ordering on $V$ in this way. If $a$ and $b$ are elements of an ordered linear space $V$ with $a \leq b$, we write $[a, b]$ for the set of vectors $x \in V$ with $a \leq x \leq b$. It is easy to see that this set is convex.

If a pointed convex cone is also generating (i.e., spans $V$, so that $V = V_+ - V_+$) and closed, it is called regular. Henceforth, we mean by “ordered linear space” one whose positive cone is regular. Examples include the space $\mathbb{R}^X$ of real-valued functions on a set $X$, ordered pointwise on $X$, and the space $\mathcal{L}(\mathbb{H})$ of Hermitian operators on a (finite-dimensional) Hilbert space $\mathbb{H}$, with the usual operator-theoretic order, i.e., $a \geq 0$ iff $a = b^*b$ for some $b \in \mathcal{L}(\mathbb{H})$.

We say that a linear map $\varphi : V \to W$ between ordered linear spaces $V$ and $W$ is positive iff it is order-preserving, or equivalently iff it takes the positive cone of $V$ into that of $W$, i.e., $\varphi(x) \in B_+$ for all $x \in V_+$. In particular, a linear functional $f \in V^*$ is positive iff $f(x) \geq 0$ for all $x \in V_+$. An order isomorphism between ordered linear spaces $V$ and $W$ is a positive linear isomorphism $\varphi : V \to W$ with positive inverse—that is, $\varphi$ is an order isomorphism if it is bijective and satisfies $\varphi(x) \geq 0$ in $W$ iff $x \geq 0$ in $V$. An order-isomorphism from an OLS to itself is an order-automorphism (or just automorphism). The set $\mathcal{L}_+(V, W)$ of positive linear mappings from an OLS $V$ to an OLS $W$ is a pointed, closed convex cone in the space $\mathcal{L}(V, W)$ of all linear maps from $V$ to $W$; where $V$ and $W$ are finite-dimensional, this cone is also generating. In the special case where $W = \mathbb{R}$, we write $\mathcal{L}_+(V, \mathbb{R})$ as $V^*_+$, referring to this as the dual cone to $V$.

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2A convex cone in a real vector space $V$ is a convex set $K \subseteq V$ closed under multiplication by non-negative scalars. If $K \cap -K = \{0\}$, the cone is said to be pointed.
An order unit in an ordered linear space $V$ is a vector $u \in V_+$ such that, for every $x \in V_+$, there is some positive scalar $t$ with $x \leq tu$. An order unit on $V$ is an order unit in $V^*$—equivalently (in finite dimensions), a strictly positive functional $u \in A^*$, that is, one with $u(\alpha) > 0$ for $\alpha > 0$. (In other words, $u$ is in the interior of $A^*_+$.)

For example, if $A = \mathbb{R}^X$, the functional $u(f) = \sum_{x \in X} f(x)$ is an order unit. For $A = \mathcal{L}(\mathbb{H})$, the trace is an order unit.

A face of a cone $V_+$ is a sub-cone $F_+$ with the property that if $x, y \in V_+$ with $x + y \in F_+$, then $x, y \in F_+$.

In particular, if $F_+$ is a face and $0 \leq x \leq y \in F_+$, then $x \in F_+$ as well. The smallest face containing a given element $y \in V_+$ is denoted $\text{Face}(y)$. When this coincides with the ray generated by $y$, we say that $y$ is extremal in $V_+$ (note this is not the same thing as saying $y$ is an extreme point of $V_+$—only $0$ is that). A proper face, i.e., one not the whole cone, is contained in the topological boundary of the cone. An easy exercise shows that $y$ is an order-unit in the OLS $F := F_+ - F_+$ spanned by $F_+$, iff $F_+ = \text{Face}(y)$. Note also that the intersection of faces is a face, and that a face of a face of $V_+$ is a face of $V_+$.

If $A$ and $B$ are ordered linear spaces, there is a natural ordering on their direct sum, namely, $(A \oplus B)_+ = \{ x + y | x \in A_+, y \in B_+ \}$. We refer to $A \oplus B$, with this ordering, as the ordered direct sum of $A$ and $B$. An ordered linear space $V$ is irreducible iff there exists no non-trivial decomposition $V$ of $V$ as an ordered direct sum. Every OLS in finite dimension is a direct sum of irreducible ones. An OLS is simplicial iff it can be represented as an ordered direct sum of one-dimensional subspaces.

**Definition 2.1.** By an abstract state space, we mean a pair $(A, u_A)$ where $A$ is an ordered linear space and $u_A$ is a distinguished order-unit on $A$. We refer to a positive element of $A$ with $u_A(\alpha) = 1$ as a normalized state. The set of all normalized states is a compact convex subset of $A_+$, which we denote by $\Omega_A$.

In quantum mechanics, the relevant example is $A = \mathcal{L}(\mathbb{H})$, as described above, with $u_A(a) = \text{Tr}(a)$; thus, $\Omega_A$ is the set of density matrices of $\mathbb{H}$. However, any finite-dimensional compact convex set can be represented, in an essentially canonical way, as $\Omega_A$ for a suitable abstract state space $(A, u_A)$, so the definition allows for arbitrarily general models.

Of course, the picture thus far is incomplete: we also need some way to describe the results of measurements performed on a system. To this end, note that if $a$ is an outcome of some measurement, then, for any state $\alpha \in \Omega_A$, there should be a

3 A subcone of $V_+$ is, of course, a subset of $V_+$ that is itself a cone.

4 The set $\Omega_A$ is a base for the positive cone $A_+$: a convex set $S$ such that every non-zero $\alpha \in A_+$ is a positive scalar multiple of a unique vector in $S$. In the case $S = \Omega_A$, the vector is $\alpha/u_A(\alpha)$. Indeed, what we are calling an abstract state space is essentially the same thing as a ordered vector space with a distinguished cone-base, i.e., what we might call a (finite-dimensional) cone-base space.
well-defined probability $\alpha(a)$ to obtain $a$ as the result of measuring the system when
the latter is in state $\alpha$. In order to maintain consistency with our intuitive notion
that convex combinations of states reflect randomized preparations, we must require
that $\alpha \mapsto \alpha(a)$ be affine, i.e., that it preserve convex combinations. It can be shown
that any affine functional on $\Omega_A$ extends uniquely to a positive linear functional on
$A$; thus, we are led to the following

**Definition 2.2.** An effect on an abstract state space $(A, u_A)$ is a positive functional
$a \in A^*$ such that $a \leq u_A$—equivalently, $a \in A^*$ is an effect iff $0 \leq a(\alpha) \leq 1$ for every
normalized state $\alpha \in \Omega_A$.

Note that $0$ and $u_A$ are, respectively, the smallest and largest effects on $A$, and the
set of all effects on $A$ is precisely the interval $[0, u_A]$. From the discussion above,
we see that every measurement outcome will correspond to (or define) an effect on
$A$. We make the further assumption here that the converse holds, i.e., that every
effect represents a measurement outcome. Accordingly, a discrete observable on
$A$ is a family $\{a_x\}_{x \in X}$ of effects, indexed by a finite set $X$ (a “value space”), with
$\sum_{x \in X} a_x(\alpha) = 1$ for all $\alpha \in \Omega$, i.e., with $\sum_x a_x = u_A$. In the classical case where
$A = \mathbb{R}^S$ for a finite set $S$, an observable in this sense corresponds to a “fuzzy” random
variable, while in the quantum case, with $A = \mathcal{L}(\mathbb{H})$, an effect is a positive operator
between $0$ and $1$, and an observable is a discrete POVM (positive operator-valued
measure). A common type of observable has $X = \{1, 2, ..., n\}$; such an observable
amounts to a set $a_1, ..., a_n$ of effects with $\sum_i a_i = u_A$.

The formalism sketched above accommodates composite systems. Let $A$ and $B$ be
abstract state spaces, with order-units $u_A \in A^*$, $u_B \in B^*$ and normalized state spaces
$\Omega_A$ and $\Omega_B$. We write $A \otimes_{\text{max}} B$ for the space of bilinear forms on $A^* \times B^*$, ordered
by the cone of forms nonnegative on products $a \otimes b$ of positive elements (i.e. $a \in A_+^*$,
$b \in B_+^*$), with order unit $u_A \otimes u_B$. We write $A \otimes_{\text{min}} B$ for the same space, ordered
by the (generally, much smaller) cone generated by the product states $\alpha \otimes \beta$, where
$\alpha \in A_+$ and $\beta \in B_+$.

States in $A \otimes_{\text{max}} B$ satisfy a natural no-signaling condition, namely, that the marginal
states of $A$ and $B$ are well-defined, not depending on which observable may be mea-
ured on the other wing. Conversely, it can be shown [24, 31] that a joint probability
assignment to measurement outcomes associated with $A$ and $B$ that satisfies this
non-signaling requirement, necessarily extends to a positive bilinear form on $A^* \times B^*$,
hence, to an element of $A \otimes_{\text{max}} B$. Thus, the maximal tensor product captures all
non-signaling states—at least, insofar as we regard bipartite states as determined by
joint probability assignments to pairs of local measurement outcomes. This last as-
sumption, sometimes called local tomography or local observability, is well-known to be
violated by real quantum mechanics in which the composite of two systems described
by $n$-dimensional real Hilbert spaces is taken to be the state space of the tensor
product of the two Hilbert spaces. Attempts to similarly describe the composite of two quaternionic quantum systems are even more problematic: the state space over a tensor product of quaternionic “Hilbert spaces” is too small to accommodate even the product states. Local tomography is therefore often suggested [3, 20, 12, 4, 15, 14] as a possible axiom for quantum theory. We shall adopt it here as a working assumption.

More generally, we can consider the space \((A^* \otimes B^*)^*\) of bilinear forms on \(A^* \times B^*\), equipped with any cone lying between the minimal and maximal ones, as “a tensor product” of \(A\) and \(B\). (Note that as we are in finite dimensions, \((A^* \otimes B^*)^*\) and \(A \otimes B\) are isomorphic as vector spaces. In what follows, we shall write \(AB\), generically, for such a composite.\(^5\)

If \(F_+\) is a face of \(A_+\), then any choice of composite \(AB\) induces a canonical choice for a composite of \(F\) and \(B\), which we denote by \(FB\). It consists of those states whose \(A\)-marginals lie in \(F\), and is easily seen to be a face of \(AB\). For \(G\) a face of \(B\), \(AG\) is defined similarly. We also write \(FG\), for the face \(FB \cap AG\).

For the purposes of this paper, we may understand by the phrase physical theory, a class of abstract state spaces, closed under the formation of such a product, so as to allow the representation of composite systems. In a more complete treatment of this idea, one may take a theory to be a category of abstract state spaces, with morphisms corresponding to the processes allowed by the theory. For some further development of this idea, see [10, 11]. In accordance with our standing assumption, we here consider only finite-dimensional theories, i.e., those consisting of finite-dimensional state spaces.

3 Weak self-duality

A bipartite state on a composite system \(AB\), represented by a positive bilinear form \(\omega : A^* \times B^* \to \mathbb{R}\), can also be represented by a positive map \(\hat{\omega} : A^* \to B = B^{**}\) defined by \(\hat{\omega}(a)(b) = \omega(a, b)\). Note that we then have \(\hat{\omega}(u_A) = \omega^B\), the \(B\) marginal of \(\omega\). Notice also that the adjoint map \(\hat{\omega}^* : B^* \to A^{**} = A\) represents the same state, evaluated in the opposite order, i.e, \(\hat{\omega}^*(b)(a) = \hat{\omega}(a)(b) = \omega(a, b)\). Hence, \(\hat{\omega}^*(u_B) = \omega^A\). (Conversely, any positive linear map taking \(u_A\) to a normalized state of \(B\) defines a normalized bipartite state in \(A \otimes_{\text{max}} B\).)

Definition 3.1. An abstract state space \(A\) is weakly self-dual iff there exists an order-isomorphism \(\eta : A^* \simeq A\).

\(^5\)We might, abandoning local tomography, also consider still larger spaces, of which \(A \otimes B\) is only a quotient. Indeed, this is necessary to accommodate the states on the usual tensor product of real Hilbert spaces, as a composite state space.
Multiplying by a sufficiently large or small positive scalar if necessary, one can assume in the above definition that \( \eta(u_A) =: \alpha_o \in \Omega_A \), i.e., \( \eta \) defines a bipartite state. It follows that \( \eta^{-1}(\alpha_o) = u_A \), so \( \eta^{-1} \) is a bipartite effect. [HB: Not clear to me this follows. More argument needed...perhaps define isomorphism effect earlier and use its properties?]

**Definition 3.2.** A bipartite state \( \omega \) in \( A \otimes_{\text{max}} B \) is an *isomorphism state* iff \( \hat{\omega} : A^* \to B \) is an order isomorphism.

The existence of a composite containing isomorphism states is far from guaranteeing the weak self-duality of \( A \) or \( B \); indeed, for *any* state space \( B \), there is a state space \( A \) whose positive cone is isomorphic to the dual of \( B \)’s, hence for which \( A \otimes_{\text{max}} B \) contains isomorphism states. But the existence of an isomorphism state in \( A \otimes A \) obviously *does* imply that \( A \) is weakly self-dual. We call such a state an *automorphism state*.

If \( \omega \) is a state on \( AB \), and \( \tau : A \to A \) and \( \eta : B \to B \) are automorphisms of \( A \) and \( B \), respectively, then \( \eta \circ \hat{\omega} \circ \tau^* \) defines another isomorphism state, with

\[
(\eta \circ \hat{\omega} \circ \tau^*)(a)[b] = \omega(\tau a, \eta^* b).
\]

**Theorem 3.3.** Let \( A \) be an irreducible ordered linear space. Then automorphisms of \( A \) lie on extremal rays of the cone \( L_+(A, A) \) of positive maps from \( A \) to \( A \).

**Proof of Theorem:** Let \( \chi \) be an automorphism on \( A_+ \). Suppose \( \chi = \psi + \mu \), where \( \psi, \mu : A \to A \) are positive maps. Let \( x \neq 0 \) be extremal in \( A_+ \); then \( \chi(x), \psi(x), \mu(x) \) are also extremal, whence, as \( \chi(x) = \psi(x) + \mu(x) \), there are constants \( c_x, d_x \geq 0 \) with

\[
\psi(x) = c_x \chi(x) \tag{1}
\]
\[
\mu(x) = d_x \chi(x) \tag{2}
\]
\[
c_x + d_x = 1 \tag{3}
\]

We will show that \( c_x \) and \( d_x \) are independent of \( x \), so that \( \psi = c \chi \) and \( \mu = d \chi \).

It will be sufficient to prove this for \( x \) ranging over the elements of a basis \( E = \{ x_i \} \) for \( A \) consisting of extremal elements of \( A_+ \). Let \( S \subseteq E \) be maximal with respect to the property that

\[
c_x = c'_x \text{ and } d_x = d'_x
\]

for all \( x, x' \in S \). We claim that \( S = E \). To see this, suppose \( y \) is any extremal element of \( A_+ \), and let \( S_y \) be the support of \( y \) in \( E \). Expanding \( y \) in the basis \( E \), we have \( y = \sum_i t_i \chi(x_i) \). From [1] and [2], we have \( \psi(y) = c_y \chi(y) = c_y \sum_i t_i \chi(x_i) \) and \( \mu(y) = d_y \chi(y) = d_y \sum_i t_i \chi(x_i) \), with \( c_y + d_y = 1 \) from [3]. Alternatively, expanding \( y \)
before applying $\psi$ and $\mu$ gives
$$\psi(y) = \sum_i t_i \psi(x_i) = \sum_i t_i c_x \chi(x_i), \quad \mu(y) = \sum_i t_i \mu(x_i) = \sum_i \alpha i d_x \chi(x_i).$$
Since $\{x_i\}$ is a basis and $\chi$ an automorphism, $\{\chi(x_i)\}$ is a basis as well. But the expansion of an element in a basis is unique. So $c \to \mu$:
$$A \ni \sum_i t_i \chi(x_i) = \sum_i t_i c_x \chi(x_i),$$
$$d \ni \sum_i t_i \chi(x_i) = \sum_i t_i d_x \chi(x_i),$$
and thus for all $i$, either $t_i = 0$ or $c_i = c_x$ and $d_i = d_x$. Thus, if $S_y \cap S \neq \emptyset$, the set $S_y \cup S$ again enjoys the property that the coefficients $c_i$ and $d_i$ are constant; since $S$ is maximal with respect to this property, $S_y \subseteq S$.

Letting $K(S)$ denote the cone $A_+ \cap \text{span}(S)$, we now see that every extremal point of $A_+$ lies either in $K(S)$ or in $K(E \setminus S)$. That is, $A_+$ is the direct convex sum of $K(S)$ and $K(E \setminus S)$. As $A_+$ is irreducible, we must have $K(E \setminus S) = \{0\}$, i.e., $S = E$. This completes the proof.

Example 3.4. Automorphisms need not be extremal in reducible cones.

Consider the cone in two dimensions with extreme rays along the positive $x$ and $y$ axes. Consider the convex base with extreme points $(0,1)$ and $(1,0)$, and let $\chi$ be the automorphism such that $\chi(x,y) = (2x,y)$. Let $\varphi, \mu$ be automorphisms such that $\varphi(x,y) = (x/2,y/2)$, $\mu(x,y) = (3x/2,y/2)$. Then for all $(x,y)$, $\chi(x,y) = \varphi(x,y) + \mu(x,y)$. But $\varphi$ and $\mu$ are not multiples of $\chi$. Thus the automorphism $\chi$ is not extremal.

Note that if $A \simeq B$, say by an isomorphism $\eta : B \simeq A$, then there is an order-isomorphism $\mathcal{L}(A,B) \simeq \mathcal{L}(A,A)$ given by $\varphi \mapsto \eta \circ \varphi$, where $\varphi : A \to B$. Thus, Theorem 3.3 tells us that order-isomorphisms (if any exist) are extremal in the cone of positive linear maps between any two finite-dimensional ordered linear spaces. In particular, if $A$ and $B$ are abstract state spaces, then if we interpret positive linear maps $A^* \to B$ as bipartite states between $A$ and $B$, we have

**Corollary 3.5.** If $A$ is an irreducible abstract state space, then isomorphism states (if any exist) are pure in $A \otimes_{\text{max}} B$.

We will say a positive map $\varphi : A \to B$ between ordered linear spaces $A$ and $B$ factors isomorphically through a face $F_+$ of $A_+$ if there exists a positive, idempotent linear map $p : A \to F$ such that $p(A_+) = F_+$, and $\varphi = \varphi' \circ p$, where $\varphi' : F_+ \to Y$ is an order-isomorphism from $F$ onto the span of a face of $Y$. We have the following extension of Theorem 3.3

**Corollary 3.6.** Let $\omega$ be a state in $A \otimes_{\text{max}} B$. If $\hat{\omega} : A^* \to B$ factors isomorphically through an irreducible face of $A^*$ then it lies on an extremal ray in the cone of positive maps from $A^*$ to $B$.

**Proof:** Let $A$ and $B$ be finite-dimensional ordered linear spaces with regular cones $A_+$ and $B_+$. Suppose a positive surjection $\varphi : A \to B$ factors as $\varphi = \varphi' \circ p$ where $p : A \to F$ is a positive idempotent projecting $A$ onto the span of a face $F_+$ of $A_+$. 


Assume $\varphi'$ is an order-isomorphism, hence, extremal in $L_+(F, B)$. We’d like to show that $\varphi$ is extremal in $L_+(A, B)$.

To this end, let $\varphi = \alpha + \beta$ where $\alpha, \beta \in L_+(A, B)$. It will be enough to show $\alpha$ and $\beta$ are multiples of one another. Let $x \in A$ be extremal. We can decompose $x$ as $x = x_0 + x_1$ where $x_1 \in \text{im}(P)$ and $x_0 \in \ker(P) = \ker(\varphi)$. Now

$$\alpha(x_0) + \beta(x_0) = \varphi(x_0) = 0.$$ 

Hence, $\alpha = -\beta$ on $\ker(P)$. Also,

$$\alpha(x_1) + \beta(x_1) = \varphi'(x_1)$$

for all $x_1 \in \text{im}_+(p)$, so, by the extremality of $\varphi'$ and $x_1$, we have

$$\alpha(x_1) = c\varphi'(x_1) \text{ and } \beta(x_1) = d\varphi'(x_1)$$

for all $x_1 \in \text{im}(P)$, with $c, d \geq 0$ and $c + d = 1$. We wish to show, then, that $\alpha(x_0) = \beta(x_0) = 0$ for all $x \in A$—equivalently, for all $x_0 \in \ker(\varphi) = \ker(p)$. To see this, let $y \in F_+$ be extremal. Since $p$ takes $A_+$ onto $F_+$, we can find some $x_1 \in A_+$ with $p(x_1) = y$. Since $\varphi' : F_+ \cong B$ is an order-isomorphism, we have $\varphi(x) = \varphi'(p(x_1))$ extremal in $B_+$ for any $x = x_1 + x_0$ with $x_0$ any element of $\ker p$. From the discussion above, we have

$$\alpha(x_1) = \varphi(x) - \alpha(x_0) = \varphi(x) + \beta(x_0), \text{ and also } \beta(x_1) = \varphi(x) - \beta(x_0).$$

As $\alpha$ and $\beta$ are positive maps, $\alpha(x_1)$ and $\beta(x_1)$ lie in $B_+$. Since $\varphi(x)$ is extremal and $\varphi(x) \pm \beta(x_0) \geq 0$, it must be that $\beta(x_0)$ is a non-negative multiple of $\varphi(x)$. A similar argument shows that $\alpha(x_0)$ is a non-negative multiple of $\varphi(x)$. But then, as $\alpha(x_0) = -\beta(x_0)$, we must have $\alpha(x_0) = \beta(x_0) = 0$.

\section{Purification}

An important fact about quantum states is that they can be purified: any state is the marginal of a pure bipartite state. One would like to know to what extent this is true more generally. Within the general framework developed above, we can already obtain, fairly easily, some remarkably strong results in this direction.

Given an abstract state space $(A, u)$, we can turn $A^*$ into an abstract state space by using any interior state $\alpha_o \in A_+$ as the order unit. We shall write $A^\bullet$, generically, for such a state space $(A^*, \alpha_o)$, leaving the choice of $\alpha_o$ tacit. Since the latter choice is, in general, not at all canonical, there are generally many non-isomorphic state spaces
related to $A$ in this way, none of which has any special status as “the dual” of $A$. Nevertheless, these spaces are useful. For one thing, the identity map $A \to A$ can be interpreted as an isomorphism (hence, by Theorem 3.3, pure if $A$ is irreducible) state in $A^\dagger \otimes_{\max} A$ having $\alpha_o$ as its $A$-marginal. In this sense, every state interior to $A$ has a purification. In general, however, the “ancilla” $A^\dagger$ in terms of which $\alpha_o \in A$ is purified, depends on $\alpha_o$.

**Theorem 4.1.** The following are equivalent:

(a) $A$ is homogeneous;

(b) Every normalized state in the interior of $A_+$ is the $A$-marginal of an isomorphism state in $B \otimes_{\max} A$, where $B$ is any (fixed) state space order-isomorphic to $A^\ast$.

This gives us a physical interpretation of homogeneity: first, that the various “dual” abstract state spaces $A^\dagger = (A^\ast, \alpha_o)$ are all isomorphic, not only as ordered linear spaces but as abstract state spaces (although not necessarily in a canonical way), and second, as telling us that in the irreducible case, all interior states of $A$ can be purified to isomorphism states using a fixed ancilla, namely, any choice of $A^\dagger$. (Note, too, that because the dual of a homogeneous cone is homogeneous, condition (b) is also equivalent to the homogeneity of $A^\ast$.)

**Proof:** (a) $\Rightarrow$ (b) Consider an order-isomorphism $\eta : B^\ast \to A$. Define $\eta(u_B) =: \alpha_o$, then since $u_B$ is in the interior of $B_+^\ast$, $\alpha_o$ belongs to the interior of $A_+$. Since the latter is homogeneous, for any normalized state $\alpha$ we can find some order-isomorphism $\tau : A \simeq A$ with $\alpha = \tau(\alpha_o)$; thus, $\alpha = (\tau \circ \eta)(u_B)$. Note that as $\alpha \in \Omega_A$, it follows that $\tau \circ \eta$ defines a normalized bipartite state, with marginal $\alpha$.

(b) $\Rightarrow$ (a): Let $\alpha, \beta$ be any two elements of $\text{int} \ A_+$. Let $t\alpha, s\beta$ be the normalized versions of $\alpha$ and $\beta$, with $t, s > 0$. Let $\omega_\alpha, \omega_\beta$ be the isomorphism states on $B \otimes_{\max} A$ with $A$-marginals $t\alpha, s\beta$ respectively, whose existence is guaranteed by (b). That is, $\hat{\omega}_\alpha(u_B) = t\alpha$, $\hat{\omega}_\beta(u_B) = s\beta$. The automorphism $(s/t)\hat{\omega}_\beta \circ \hat{\omega}_\alpha^{-1}$ takes $\alpha$ to $\beta$, so $A$ is homogeneous.

In the case that $A$ is weakly self-dual, we can use an order-isomorphism $\eta : A^\ast \simeq A$ to identify $A$ with $A^\dagger$, using $\alpha_o = \eta(u_A)$. Applying the preceding Theorem, we have

**Corollary 4.2.** For any irreducible state space $A$, the following are equivalent:

(a) $A$ is weakly self-dual and homogeneous;

(b) Every normalized state in the interior of $A_+$ is the marginal of an isomorphism state in $A \otimes_{\max} A$. 
Remark: Recently, Chiribella, D’Ariano and Perinotti [15] have examined the consequences of assuming, as an axiom, that all states dilate to a pure state that is unique up to a reversible transformation on one marginal. We note that if \( \alpha \in \Omega_A \) can be achieved as the marginal of two isomorphism states \( \omega \) and \( \mu \) in the same composite \( BA \) (so \( B^* \cong A \), say via an isomorphism \( \sigma \)), so that \( \alpha = \hat{\omega}(u_B) = \hat{\mu}(u_B) \), then \( \tau := \hat{\omega} \circ \hat{\mu}^{-1} \) is a unit-preserving order-automorphism of \( B^* \), and \( \omega(a,b) = \mu(\tau(a),b) \).

We are using the convention \( \hat{\omega} : B^* \to A \); \( \tau \) is a reversible map acting on \( B^* \) (note that unit preservation is the condition dual to base preservation). So for the set of isomorphism states having \( \alpha \) as \( A \)-marginal, the dilation condition is met; in constructing a composite, uniqueness can be ensured by including no other pure states with \( \alpha \) as marginal, in the extreme generators of the composite cone.

5 Steering

By an ensemble for a state \( \beta \in B \), we mean a finite set of \( \beta_i \in B_+ \) such that \( \sum_i \beta_i = \beta \). Note that we defined ensembles not as lists of probabilities and associated normalized states, but as lists of unnormalized states; the two definitions are equivalent, as the norms \( u_B(\beta_i) \) of the \( \beta_i \) encode the probabilities. Indeed, from \( \sum_i \beta_i = \omega_B \) and the positivity of the \( \beta_i \), it follows that \( u_B(\beta_i) \) must be probability weights, \( 0 \leq u_B(\beta_i) \), \( \sum_i u_B(\beta_i) = 1 \). If instead \( \sum_i \beta_i \leq \omega_B \) is required, we have a subensemble for \( \omega_B \).

As discussed in Section 1, pure quantum-mechanical states have the interesting property that any ensemble for either marginal state can be realized as the conditional states arising from a suitable choice of observable on the other wing of the system. Generalizing, we are led to the following

**Definition 5.1.** A bipartite state \( \omega \in A \otimes_{\text{max}} B \) is steering for its \( B \) marginal iff, for every ensemble (convex decomposition) \( \omega^B = \sum_i \beta_i \), where \( \beta_i \) are un-normalized states of \( B \), there exists an observable \( E = \{x_i\} \) on \( A \) with \( \beta_i = \hat{\omega}(x_i) \). We say that \( \omega \) is bistereering iff it’s steering for both marginals.

If \( \alpha \) is any state on \( A \) and \( \beta \) is a pure state on \( B \), then \( \omega = \alpha \otimes \beta \) is trivially steering for \( \omega_B = \beta \) since the latter has no non-trivial ensembles. If \( \alpha \) is mixed, then \( \omega \) will not be steering for \( \omega_A \). On the other hand, any pure product state is steering, as is any isomorphism state. In light of Theorem 4.2, this might suggest that steering states are always pure. But that is not correct. Indeed, any classical bipartite state exhibiting perfect correlation between its marginals is steering for both its marginals. (This is closely related to Example 2.4 above.)

\(^6\) Although we will not need it, there is a unique norm, called the base norm, that agrees with the linear functional \( u_B \) on \( B_+ \), so the terminology is justified.
It follows almost immediately from the definition, that if \( \omega \) is steering for its \( B \)-marginal, \( \hat{\omega}(A_+) \) is a face of \( B_+ \). Indeed, we have

**Lemma 5.2.** If \( \omega \) is steering, then \( \hat{\omega}(A_+) = \text{Face}(\omega^B) \).

**Proof:** Suppose that \( \beta_1, \beta_2 \in B_+ \) with \( \beta_1 + \beta_2 \in \hat{\omega}(A_+) \)—say, \( \beta_i = \hat{\omega}(a_i) \) where \( a_i \in A^*_+ \). Since \( u_A \) is an order unit for \( A^*_+ \), we can find a scalar \( t > 0 \) such that

\[
t(a_1 + a_2) \leq u_A,
\]

whence,

\[
t\beta_1 + t\beta_2 = \hat{\omega}(t(a_1 + a_2)) \leq \hat{\omega}(u_A) = \omega^B.
\]

Thus, \( \{t\beta_1, t\beta_2\} \) is a sub-ensemble for \( \omega^B \). Since \( \omega \) steers \( \omega^B \), \( t\beta_1 \) and \( t\beta_2 \)—and hence, also \( \beta_1 \) and \( \beta_2 \)—lie in \( \hat{\omega}(A_+) \). This shows that \( \hat{\omega}(A_+) \) is a face of \( B_+ \). Notice that the argument also shows that \( \omega^B \) is an order unit for \( \hat{\omega}(A_+) \); hence, the latter is exactly \( \text{Face}(\omega^B) \). \( \square \)

The converse, however, is not true: a state may satisfy \( \hat{\omega}(A_+) = \text{Face}(\omega^B) \) but not be steering, as in the following example.

**Example 5.3.** \( A_+ \) is the simplicial cone \( \mathbb{R}^3_+ \), \( B_+ \) is \( \mathbb{R}^2_+ \), with the usual order units \((1,1,1)\) and \((1,1)\) respectively, and \( \omega(a,b) \) is determined by the following table of probabilities:

\[
\begin{array}{ccc}
x & y & z \\
1/4 & 0 & 1/4 \\
1/4 & 1/4 & 1/4
\end{array}
\]

The row index ranges over the values \( x := (1,0) \), \( y := (0,1) \) for \( a \), the column index \( b \) ranges over the values \( x := (1,0,0) \), etc..

We see that \( \hat{\omega}(x) = (1/4,0) \), \( \hat{\omega}(y) = (0,1/4) \), and \( \hat{\omega}(z) = (1/4,1/4) \), while \( \hat{\omega}(u) = \hat{\omega}(x + y + z) = (1/2,1/2) \). The image of the order interval \([(0,0,0),(1,1,1)]\) of \( \mathbb{R}^3_+ \) under \( \hat{\omega} \) is the hexagon with vertices \((0,0),(1/4,0),(0,1/4),(1/2,1/4),(1/4,1/2),(1/2,1/2)\). This is a proper subset of the interval \([0,\pi((1,1,1))]\), which is the square with vertices \((0,0),(1/2,0),(0,1/2),(1/2,1/2)\). So as claimed, even though \( \pi(u) \) is the desired order unit in \( \mathbb{R}^2_+ \), the image \( \pi([0,u]) \) of the unit interval in \( \mathbb{R}^3 \), while it generates \( \mathbb{R}^2_+ \), is not a unit interval (for any ordering of \( \mathbb{R}^2 \)).

In particular, the ensemble \( (0,1/2), (1/2,0) \) for \( \omega^B = (1/2,1/2) \) cannot be represented as \( \hat{\omega}(a_1), \hat{\omega}(a_2) \) for any elements \( a_1, a_2 \in [0,u_A] \)—much less with \( a_1, a_2 \) summing to \( u_A \). Accordingly, \( \omega \) is not steering. \( \triangle \)

In fact, the condition that \( \omega \) be steering for its \( B \)-marginal places a very strong and subtle constraint on \( \hat{\omega} \). If \( X \) and \( Y \) are partially ordered sets, an order-preserving
surjection \( p : X \to Y \) is a quotient map iff, for all \( y_1, y_2 \in X \), \( y_1 \leq y_2 \) iff \( y_i = p(x_i) \) for some \( x_1 \leq x_2 \) in \( X \). We shall say that \( p \) is a strong quotient map iff it has the property that every chain \( y_1 \leq y_2 \leq \cdots \leq y_n \) in \( Y \) is the image of some chain \( x_1 \leq x_2 \leq \cdots \leq x_n \) in \( X \), i.e., \( y_1 = p(x_1), y_2 = p(x_2), \ldots, y_n = p(x_n) \). Evidently, a strong quotient map is a quotient map (just apply the definition to chains of length 2), but the converse is, in general, false.

**Theorem 5.4.** Let \( \omega \) be a bipartite state in \( AB \). Then \( \omega \) is steering for its \( B \) marginal iff \( \hat{\omega} : [0, u_A] \to [0, \omega^B] \) is a strong quotient map of ordered sets.

**Proof:** First, suppose \( \omega \) is steering, and let \( 0 \leq \beta_1 \leq \beta_2 \leq \cdots \beta_{k-1} \omega^B \). Then \( \{ \beta_1, \beta_2 − \beta_1, \ldots, \beta_{k-1} − \beta_{k-2}, \omega^B − \beta_{k-1} \} \) is an ensemble for \( \omega^B \); as \( \omega \) is steering, there is an observable \( \{ a_1, a_2, a_3, \ldots, a_k \} \) on \( A \) with \( \hat{\omega}(a_1) = \beta_1, \hat{\omega}(a_2) = \beta_2 − \beta_1, \ldots, \hat{\omega}(a_k) = \omega^B − \beta_{k−1} \). In particular, defining \( b_j = \sum_{i<j} a_j \), we have \( \hat{\omega}(b_j) = \beta_j \); since \( b_1 \leq b_2 \leq \cdots \beta_k \), \( \hat{\omega} \) induces a quotient map of ordered sets \( [0, u_A] \to [0, \omega^B] \). For the converse, suppose \( \hat{\omega} \) induces such a quotient map. For any ensemble \( \{ \eta_1, \ldots, \eta_k \} \) for \( \omega^B \), define \( \beta_1 = \eta_1, \beta_2 = \eta_2 + \eta_1, \ldots, \beta_j = \sum_{i<j} \beta_i \) for \( j \leq k−1 \). By definition of a strong quotient map, there are then elements \( b_1 < b_2 < \cdots < b_{k−1} \) in \( [0, u_A] \) with \( \hat{\omega}(b_j) = \beta_j \). Setting \( a_j = b_j − (\sum_{i<j} b_i) \), we have \( \hat{\omega}(a_j) = \eta_j \) and \( \sum_{j<k−1} a_j = b_k \). Thus, \( \{ a_1, \ldots, a_{k−1}, a_k := u_A − b_k \} \) is the desired observable steering to \( \{ \eta_j \} \).

**Remarks:** (i) We suspect, but so far have been unable to prove, that a quotient map of order-intervals \( [0, u] \to [0, v] \) is necessarily a strong quotient.

(ii) An obvious sufficient condition for \( \hat{\omega} : [0, u_A] \to [0, \omega^B] \) to be a quotient map of ordered sets is for there to exist an affine section \( \sigma : [0, \omega^B] \to [0, u_A] \). However, as Example A.3 in Appendix A shows, this is not necessary for steering.

It follows from Theorem 5.4 that the ordering of \( \text{Face}(\omega^B) = \hat{\omega}(A_+) \) is exactly the quotient linear ordering induced by the linear surjection \( \hat{\omega} \), i.e., \( \beta_1 \leq \beta_2 \) in \( \text{Face}(\omega^B) \) iff \( \beta_i = \hat{\omega}(a_i) \) for some \( a_1, a_2 \in A \) with \( a_1 \leq a_2 \). It also follows that if \( \hat{\omega} \) is injective, it is an order isomorphism. This last point is important enough to record as

**Corollary 5.5.** Let \( \omega \) be steering for \( \omega^B \), where \( \omega^B \) is interior to \( B_+ \), so that \( \text{Face}(\omega^B) = B_+ \). If \( \hat{\omega} \) is injective (non-singular), then \( \hat{\omega} \) is an order isomorphism. If \( B_+ \) (and therefore \( A_+ \)) is irreducible, therefore, by Theorem 5.3, it is pure in \( A \otimes_{\max} B \).

In other words, if \( A \) and \( B \) have the same dimension, then the states that are steering for an interior marginal are precisely the isomorphism states (and hence steering for both marginals).

We are now in a position to make good on the claim made in the introduction.
Definition 5.6. A probabilistic theory supports universal steering if, for every system $B$ in the theory and every state $\beta \in B$, there exists a system $A_\beta$ and a bipartite state $\omega$ in $A_\beta \otimes B$ that steers its $B$-marginal $\omega^B = \beta$. A theory supports uniform universal steering if, for every system $B$ in the theory, there exists a system $A_B$ such that for every state $\beta \in A$, there exists a state $\omega$ in $A_B \otimes B$ that steers its $B$-marginal $\beta$. A probabilistic theory supports universal self-steering if, for every system $A$ in the theory, every state $\alpha \in A$ can be represented as a marginal of some bipartite state on two copies of $A$—that is, some state $\omega \in AA$—steering for that marginal. (That is, it supports uniform universal steering with $A_B \simeq A$.)

Corollary 5.5 combined with Theorem 4.1 establish

Proposition 5.7. In any theory that supports universal uniform steering, every irreducible, finite-dimensional state space in the theory is homogeneous.

In light of Corollary 4.2 we also have

Proposition 5.8. In any theory that supports universal self-steering, every irreducible, finite-dimensional state space in the theory is homogeneous and weakly self-dual.

If a theory supports universal self-steering, and also has the property that every direct summand of a state space is again a state space belonging to the theory (a reasonable requirement, at least in finite dimensional settings), then every finite-dimensional state space in the theory is a direct sum of homogeneous, weakly self-dual factors, hence, homogeneous and weakly self-dual.

An interesting question is to what extent the gap between universal steering and uniform universal steering is a genuine one. One might investigate this question by looking for examples of state spaces for which each state can be steered, but that are not homogeneous.

Remark: A particularly strong “steering” axiom would require that, for every state $\alpha$ of every system $A$ in the theory, there exist a steering state $\omega$ on a composite $AA$ of two copies of $A$, having both marginals equal to $\alpha$. Such a theory must be “mono-entropic” in the sense that measurement and mixing entropies of states coincide, as

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7It is easily seen that direct sums of homogeneous or weakly self-dual cones are, respectively, homogeneous or weakly self-dual. For weak self-duality, one just proves that any sum of isomorphisms $\alpha_i : A_i \to B_i$ of direct summands, is an isomorphism of the direct sums $\oplus_i A_i$ and $\oplus_i B_i$. For homogeneity, one uses the fact that the interior of the positive cone of a direct sum consists precisely of sums of interior points of the positive cones of all summands. Then to get from any such interior point $x = \sum_i x_i$ to any other $y = \sum_i y_i$, one uses a sum of automorphisms $\alpha_i$ of the summands $A_i$, chosen (as homogeneity of each $A_i$ ensures is possible) such that $\alpha_i(x_i) = y_i$. 

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discussed in [7]; states in such a theory must also be spectral, in the sense of [32]. Further elaboration of these points can be found in Appendix B of [7].

6 Conclusion and discussion

We have shown that the state spaces of any probabilistic theory that allows for uniform universal ensemble steering, in the sense that for every system $A$ in the theory, there’s another system $B$ such that every state on system $A$ can be steered by some state in the composite $BA$, are homogeneous. We say that system $B$ steers system $A$, in this case. If we require systems to be self-steering (i.e., that each system $A$ steer itself), they must be homogeneous and weakly self-dual. If one could motivate the stronger assumption that these state spaces are strongly self-dual, then the Koecher–Vinberg Theorem [25, 30], together with the Jordan–von Neumann–Wigner classification theorem [22], would imply that all state spaces are those of formally real Jordan algebras. In our finite dimensional setting, this means that their normalized state spaces are convex direct sums of sets affinely isomorphic to the unit-trace elements in the cones of positive semidefinite matrices in a real, complex, or quaternionic matrix algebra, or to Euclidean balls, or to the unit-trace $3 \times 3$ positive semidefinite matrices over the octonions.

From here, our standing assumption of local tomography (that bipartite states are determined by the probabilities they assign to product effects) restricts the possibilities further. A theorem of Hanche-Olsen [19] asserts that any JB-algebra $A$ (which includes all formally real Jordan algebras, at least in finite dimension) whose vector-space tensor product with the self-adjoint part of $M_2(\mathbb{C})$—that is, with a qubit—can be made into a JB tensor product, is isomorphic to the self-adjoint part of a (complex) $\mathbb{C}^*$-algebra. In other words, it is essentially quantum-mechanical. As we will establish elsewhere, Hanche-Olsen’s requirements for a JB tensor product impose on the cones associated with the three JB-algebras in question, exactly the operational requirements we’ve imposed on a composite of state spaces. Thus, Hanche-Olsen’s result implies that if a homogeneous, self-dual state space has a locally tomographic homogeneous, self-dual composite with a qubit, then it is the state space of a $C^*$-algebra — so, a direct sum of the state spaces of standard quantum theory.

Therefore, if a self-dual physical theory makes room for qubits, and permits universal self-steering, its state spaces are essentially quantum-mechanical, in the fairly standard sense that uses the complex field, but extended to allow for superselection sectors (direct summands).

These considerations trace a route, within a broad landscape of locally tomographic
non-signaling theories, from the single information-theoretic feature of quantum states that most puzzled Schrödinger—the possibility of steering—to the full mathematical apparatus of (finite-dimensional) $C^*$-algebraic quantum mechanics. This route is interrupted by a gap: that between weak and strong self-duality. There may be ways to bridge this gap. One strategy for doing so can be found in [32].

On the other hand, it would also be interesting to see whether a complete and self-consistent theory can be constructed using only weakly self-dual state spaces, that still allows for universal steering. An essential step towards constructing such a theory would be to find a class of weakly, but not strongly, self-dual, homogeneous state spaces that is closed under some reasonable non-signaling tensor product. A plausible idea is to include all steering states, but as discussed in Appendix B, it does not work. We are investigating other possibilities based on keeping a rich supply of steering states. Of course, should it prove that any category of homogeneous, weakly-self-dual state spaces that admits a reasonable tensor product must be strongly self-dual then the gap mentioned above will turn out to have been only an illusion, and quantum theory, in the $C^*$-algebraic sense, will be naturally characterized, at least in finite dimension, in terms of steering and the existence of a locally tomographic nonsignaling tensor product.

Finally, we mention that additional information-processing properties provide some motivation for weak self-duality, both via steering, and more directly. For example, in [8] it was shown that an exponentially secure bit commitment protocol, based on the nonuniqueness of convex decomposition in nonclassical state spaces, exists in any theory which has at least some nonclassical state spaces, coupled only by the minimal tensor product (so that there is no entanglement). But in a nonclassical theory in which all states can be steered, this type of bit commitment protocol cannot exist. This does not provide a tight connection between steering and no-bit-commitment; other types of bit commitment protocols might be able to coexist with steering. But it is suggestive. Another connection is between weak self-duality and teleportation: if it is possible, to conclusively teleport a system through a copy of itself with nonzero success probability, then it must be weakly self-dual [6].

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8. We note that our notion of tensor product is less innocuous than might be evident, as it requires the positivity of bipartite states on all product effects, which, for example, need not be required in a theory in which only a subcone of $A_+^*$ is considered to represent operationally relevant effects on which positivity is to be required.

9. This also requires that all composites $AB, A'B'$ of isomorphic systems $A \simeq A', B \simeq B'$, contained in the theory, be isomorphic.
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A Examples for Section 5

Example A.1. An example where there is a section that enables steering. Consider the following state in the maximal tensor product of two state spaces with square base. We’ll view this as the state space of two two-outcome tests \(\{a, a'\}\) and \(\{b, b'\}\), and also identify \(a, a'\), \(b, b'\) with the atomic effects in the dual cone. We’ll label each of the four vertices of the normalized state space by the two atomic effects it makes certain: \(ab, ab', a'b, a'b'\). Writing, e.g. \(ab \otimes a'b'\) for a bipartite product state, the state:

\[
\omega = \frac{1}{2}(ab \otimes ab + a'b \otimes a'b)
\]

has

\[
\hat{\omega}(a) = \frac{1}{2}ab, \quad \hat{\omega}(a') = \frac{1}{2}a'b,
\]

so measuring \(\{a, a'\}\) on the first system gives the ensemble \(\{\frac{1}{2}ab, \frac{1}{2}a'b\}\) for the second-system marginal \(\omega^B = \frac{1}{2}(ab + a'b)\). Face(\(\omega^B\)) is generated as nonnegative linear combinations of \(ab\) and \(a'b\), and is thus a two-dimensional ordered subspace of the three-dimensional state space of the second system. Its unit interval is the square that is the convex hull of \((0, 0), ab/2, a'b/2, (ab + a'b)/2\). The quotient of the first system space, \(A\), by the kernel of the linear map \(\hat{\omega}\), can be represented by setting up the state space to have 90° opening angle between opposite extremal rays, and taking the orthogonal (90°) projection onto the plane normal to the ray \(b'\), i.e. the projection along the ray generated by \(b'\). This indeed gives a two-dimensional classical state space, i.e. one isomorphic to Face(\(\omega^B\)). Moreover, there is an affine section \(\sigma\) of this quotient map into \(A^*_+\), given by inverting the relations (6), thus allowing us to map the unit interval \([0, \omega^B]\) in Face(\(\omega^B\)) to the diagonal cross section, in the \(a, a'\) plane, of the unit interval of \(A^*_+\). \(\sigma \circ \pi\) is indeed a positive projection on \(A^*_+\), namely the orthogonal projection onto this plane.

For the state (5), the mentioned section is the only affine section over Face(\(\omega^B\)). A slight modification, however, gives an example in which many affine sections exist.

Example A.2. An example in which there are many affine sections over the image of the order unit. Let

\[
\hat{\omega} = \frac{1}{2}(x \otimes ab + y \otimes a'b)
\]

with \(x\) any state in the line segment \([ab, ab']\) and \(y\) any state in the segment \([a'b, a'b']\) respectively. This still gives a state steering for the same marginal \(\omega^B = \frac{1}{2}(ab + a'b)\) as in the preceding example. If we choose \(x = ab, y = a'b'\), that is,

\[
\hat{\omega} = \frac{1}{2}(ab \otimes ab + a'b' \otimes a'b),
\]

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we can steer the marginal using either of the two observables \(\{a, a'\}, \{b, b'\}\). In this case, there are two distinct sections over \(\text{Face}(\omega^B)\) that enable steering. △

In the example of Eq. (8), the kernel of \(\hat{\omega}\) is generated, not by \(b'\) as before, but by \(a'b - ab'\), and the natural way of attempting to represent the quotient map, namely by projection onto a subspace complementary to the kernel in the original space, if we project orthogonally in the natural geometry, projects onto the subspace spanned by \(ab, a'b\), and the order unit, giving a slice of the order interval that includes the diagonal of the square.

Although the kernel of this map is of course one-dimensional, its positive kernel, which we’ve claimed must be an exposed face of the state cone, is the trivial such exposed face: the zero subspace. This example shows that quotients whose positive kernel is trivial are not necessarily uninteresting or ill-behaved.

**Example A.3. An affine section of the order-quotient map is not necessary for steering.** Let \(A \simeq \mathbb{R}^4\) be an abstract state space whose normalized states form a cube; its dual cone is a regular polyhedral cone in \(\mathbb{R}^4\) with octahedral base. The atomic effects (largest effects on extremal rays of the effect cone) are the vertices of such an octahedral base. The example has \(B \simeq \mathbb{R}^3\) ordered by a cone with regular hexagonal base; the center of the hexagon is a state having three distinct two-state extremal ensembles, each composed of a pair of opposite vertices of the hexagon (with weights 1/2 on each one). \(\hat{\omega}^{AB}\) is chosen to take the six vertices of the octahedron of atomic effects in \(A^*\) to the six vertices of the hexagon of states normalized to 1/2, in such a way that opposite vertex-pairs are mapped to opposite vertex-pairs. This is clearly positive and linear, and maps the order-unit (twice the center of the octahedron of atomic effects) to the center of the hexagon of normalized states (which is twice the center of the hexagon of states normalized to 1/2). \(\text{Face}(\hat{\omega}^B)\) is the entire hexagonal cone in \(\mathbb{R}^3\), so \(\hat{\omega}\) is itself (technically, is a representative of) the quotient \(\pi\). All three ensembles for \(\omega^B\) consist of extremal states, and because they have unique \(\omega\)-preimages, a section of \(\hat{\omega}\), we know that a section of \(\pi\) must take these, the vertices of a hexagon, to the vertices of the octahedron, matching opposites to opposites. But no affine map can do this. The affine span of the hexagonal points has affine dimension 2, while that of the octagonal ones has affine dimension 3. △

### B The Steering Product

A physical *theory*, as distinct from a model of a single physical system, should allow some device whereby several such models can be combined to yield a model of a composite system. It is natural to suppose that such a theory constitutes a category
equipped with a well-behaved tensor product; that is, a symmetric monoidal category
as described in e.g. [1]. This suggests the following problem: given two weakly self-
dual state spaces $A$ and $B$, does there exist a reasonable model for a composite state
space $AB$ that is again weakly self-dual? In light of the connection between weak self-
duality and steering, one might consider building a weakly self-dual tensor product
by including all steering states, but as we’ll see, this does not work, even where the
factors are quantum-mechanical.

Note that for non-simplicial weakly self-dual factors, neither the maximal nor minimal
tensor product will be weakly-self-dual, since $(A(Ω) ⊗_{max} A(Ω'))^* ≇ A(Ω)^* ⊗_{min}
A(Ω')^* ≇ A(Ω) ⊗_{min} A(Ω') ≇ A(Ω) ⊗_{max} A(Ω')$.

If $A$ and $A'$ are order-isomorphic (alternatively, and notationally easier, $A = A'$; hereafter this will be assumed), one candidate for the weakly self-dual tensor prod-
tuct, namely, the convex hull of the pure tensor states and the isomorphism states.
However, this is unsatisfactory in various ways, not least that it degenerates to the
minimal tensor product if $A$ and $B$ are not isomorphic; also, using this observation,
it’s easy to see that this tensor product is not associative.

A better candidate is the steering product.

**Definition B.1.** Let $A ⊗_{str} B$, called the “steering product” of $A$ and $B$, be $A ⊗ B$,
ordered by the cone generated by all steering states in $A ⊗_{max} B$.

As remarked above, pure product states are steering; hence, this is a valid tensor
product in our sense.

**Conjecture B.2.** $⊗_{str}$ is associative.

Suppose $η_A : A^* → A$ and $η_B : B^* → B$ are order-isomorphisms implementing the
weak self-duality of $A$ and $B$. We can use these to convert a state $ω : A^* → B$ to an
effect

$$η_{AB}(ω) := η_B^{-1}ωη_A : A → B^*$$

This gives us a linear map $η_{AB} : A ⊗_{max} B → A^* ⊗_{max} B^* = (A ⊗_{min} B)^*$. Evidently,
this is positive. It is also easy to check that $η_{AB}$ takes product states to product effects,
and isomorphism states to isomorphism effects.

**Question 1.** If $ω$ is steering, is $η_{AB}(ω)$ also steering (in some suitable dual sense)?

**Fact B.3.** If $A = B = L_h(\mathbb{C}^2)$—that is, if $A$ and $B$ are two qubits, then $A ⊗_{str} B = A ⊗_{max} B$ (which is not weakly self-dual).

**Proof:** To see this one uses the fact that states in the maximal tensor product all
correspond to positive maps that are decomposable, i.e., sums of completely positive
and co-completely positive maps. The extremal ones are all either product states, or isomorphism states, since the automorphism group of a qubit—indeed, of any quantum system—is generated by the maps $X \mapsto AXA^\dagger$ for nonsingular $A$, and any transpose map $X \mapsto X^t$. ■

One might ask whether if $A$ and $B$ are weakly self-dual, $A \otimes_{\text{str}} B$ is too. This is not so, as the above example of two qubits, combined with observation that the maximal tensor product of non-simplicial weakly self-dual cones is not weakly self-dual, show.

Same question if $A$ and $B$ are homogeneous.

**Question 2.** Is the steering-for-both-marginals product equal to the steering-for-one-marginal product? This would follow, for example, from the proposition that every steering-for-one-marginal state is a convex combination of steering-for-both-marginals states.

**Question 3.** Is the steering-for-both-marginals product, perhaps under conditions like homogeneity of both state spaces, equal to the topological closure of the cone generated by automorphisms?

We don’t know the answer for sure even in the quantum case, because this raises the question whether states corresponding to nondecomposable maps can be steering. We conjecture that they cannot, and that therefore the answer is “yes” in the quantum case.