Towards a Background Independent Quantum Gravity in Eight Dimensions

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Abstract

We start a program of background independent quantum gravity in eight dimensions. We first consider canonical gravity a la "Kaluza-Klein" in $D = d + 1$ dimensions. We show that our canonical gravity approach can be applied to the case of self-dual gravity in four dimensions. Further, by using our previously proposed classical action of Ashtekar self-dual gravity formalism in eight dimensions, we proceed to develop the canonical approach in eight dimensions. Our construction considers different $SO(8)$ symmetry breakings. In particular, the breaking $SO(8) = S^7_R \times S^7_L \times G_2$ plays an important role in our discussion.

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1. Introduction

Considering the motivation for background independent quantum gravity [1] one finds that most of the arguments can be applied not only to four dimensions but to any higher dimensional gravitational theory based in Einstein-Hilbert action. For instance, the statement that ”gravity is geometry and therefore there should no be background metric” is also true in a higher dimensional gravitational theory based in Einstein-Hilbert action. Similar conclusion can be obtained thinking in a non-perturbative context. So, why to rely only in four dimensions when one considers background independent quantum gravity? Experimental evidence of general relativity in four dimensions is established only at the classical, but not at the quantum level. Thus at present, in the lack of experimental evidence of quantum gravity any argument concerning the dimensionality of the spacetime should be theoretical.

A possibility for setting four dimensions comes from the proposal of self-dual gravity [2]-[3]. One starts with the observation that the potential (playing by the three dimensional scalar curvature) in the Hamiltonian constraint is difficult to quantize. In the case of four dimensions it is shown that such a potential can be avoided by introducing new canonical variables [4] which eventually are obtained via self-dual gravity [2]-[3]. In turn, self-dual gravity seems to make sense only in four dimensions since in this case the dual of a two form (the curvature) is again a two form. This argument is based on the definition of the duality concept in terms of the completely antisymmetric density $\epsilon_{A_0 A_1 A_2 A_3}$ which takes values in the set \{-1, 0, 1\}. The Riemann curvature $R^{AB}$ is a two form. Thus the dual $^{*}R_{A_0 A_1 A_2 A_3} = \frac{1}{2} \epsilon_{A_0 A_1 A_2 A_3 A_4 A_5 A_6 A_7} R^{A_4 A_5 A_6 A_7}$ is a two form only for $D = 4$. Hence, in trying to define the self-dual object $^{+}R^{AB}$ one discovers that only in four dimensions one can establish the combination $^{+}R^{AB} = \frac{1}{2} (R^{AB} - i^{*}R^{AB})$.

The definition of duality in terms of the $\epsilon$-symbol is not, however, the only possibility. A number of authors [5]-[8] have shown that duality also makes sense through a definition in terms of the $\eta$-symbol. In fact, the $\eta$-symbol is very similar to the $\epsilon$-symbol in four dimensions; is a four index completely antisymmetric object and take values also in the set \{-1, 0, 1\}. However, the $\eta$-symbol lives in eight dimensions rather than in four. Moreover, while the $\epsilon$-symbol in four dimensions can be connected with quaternions, the $\eta$-symbol is related to the structure constants of octonions (see [9]-[10] and Refs. therein). Thus, in eight dimensions we can also introduce the dual $^{*}R_{A_0 A_4 A_5} = \frac{1}{2} \eta_{A_0 A_4 A_5 A_6 A_7 A_8} R^{A_6 A_7 A_8}$ and consequently the self-dual object $^{+}R^{AB} = \frac{1}{2} (R^{AB} - i^{*}R^{AB})$. 
\begin{align}
\frac{1}{4}(R^{AB} + & \star R^{AB}) \text{ (see section 6 for details). It remains to prove whether by}
\text{using this new kind of duality we can also avoid the potential in terms of the}
\text{scalar Riemann curvature in the Hamiltonian constraint which is inherent to}
\text{any higher dimensional theory as we shall see in section 2. In this work we}
\text{show that in fact duality in terms of the } \eta \text{-symbol avoids also such a potential.}
\text{Our strategy is first to develop canonical gravity } a \, l a \, "Kaluza-Klein" \text{ and}
\text{then to discuss self-dual gravity in four dimensions. This allows us to follow a}
\text{parallel program in eight dimensions and in this way to determine the canonical}
\text{constraints of self-duality gravity in eight dimensions.}
\end{align}

The above comments can be clarified further with the help of group theory. We recall that in four dimensions the algebra \( \text{so}(1,3) \) can be written as \( \text{so}(1,3) = su(2) \times su(2) \). So, the curvature \( R^{AB} \) can be decomposed additively \cite{2}:
\[ R^{AB}(\omega) = +R^{AB}(+\omega) + -R^{AB}(-\omega) \]
where \( +\omega \) and \( -\omega \) are the self-dual and anti-self-dual parts of the spin connection \( \omega \). In an Euclidean context this is equivalent to write the norm group of quaternions \( O(4) \) as \( O(4) = S^3 \times S^3 \), where \( S^3 \) denotes the three sphere. The situation in eight dimensions is very similar since \( O(8) = S^7 \times S^7 \times G_2 \), with \( S^7 \) denoting the seven sphere, suggesting that one can also define duality in eight dimensions, but modulo the exceptional group \( G_2 \) \cite{11}-\cite{12}.

In turn, these results in the context of group theory are connected with the famous Hurwitz theorem which establishes that any normed algebra is isomorphic to the following: real, complex, quaternion and octonion algebra (see \cite{10} and Refs. therein). Considering duality, one learns that it is reasonable to define it for quaternions and octonions via the generalized vector product \cite{11}. In this sense, the classical approach of Ashtekar formalism in eight dimensions proposed in Refs. \cite{13}-\cite{15} has some kind of uniqueness. In this work we give some steps forward on the program of developing quantum gravity in eight dimensions. Specifically, in sections 6, by using self-dual gravity defined in terms of the \( \eta \)-symbol we develop a canonical gravity in eight dimensions. We find the eight dimensional canonical Diffeomorphism and Hamiltonian constraints and we outline, in the final section, a possible physical quantum states associated with such constraints.

2. Canonical gravity \textit{a la } "Kaluza-Klein"

Let us start with a brief review of canonical gravity. We shall use some kind of "Kaluza-Klein" mechanism for our review. One of the advantage of this method is that one avoids the use of a time-like vector field. This allows us to describe, in straightforward way, canonical self-dual gravity at the level of the
action for both four and eight dimensions. Although our canonical method resembles the one used in Ref. [16] our approach contains complementary descriptions and computations.

We shall assume that the vielbein field \( e^{(A)}_\mu = e^{(A)}_\mu(t,x) \), on a \( D = d + 1 \)-manifold \( M^D \), can be written in the form

\[
e^{(A)}_\mu = \begin{pmatrix} E_0^{(0)}(t,x) & E_0^{(a)}(t,x) \\ 0 & E_i^{(a)}(t,x) \end{pmatrix}.
\] (1)

Although in writing (1) we do not consider any kind of dimensional reduction or compactification, this form of \( e^{(A)}_\mu \) is in a sense inspired by the Kaluza-Klein mechanism. The inverse \( e_\nu^{(A)} e^{(A)}_\mu = \delta_\nu^\mu \), with \( \delta_\mu^\nu \) denoting the Kronecker delta. We find

\[
e^{(A)}_\mu = \begin{pmatrix} E_0^{(0)}(t,x) & E_0^{(a)}(t,x) \\ 0 & E_i^{(a)}(t,x) \end{pmatrix},
\] (2)

with \( E_0^{(0)} = 1/E_0^{(0)}, E_0^{(a)} = -E_0^{(a)}E_i^{(a)}/E_0^{(0)} \) and \( E_i^{(a)}E_i^{(a)} = \delta_i^i \). In the above the indices \( (A) \) and \( \mu \) of \( e^{(A)}_\mu \) denote frame and target spacetime indices respectively.

In general, the metric \( \gamma_{\mu\nu} \) is defined in terms of \( e^{(A)}_\mu \) in the usual form

\[
\gamma_{\mu\nu} = e^{(A)}_\mu e^{(B)}_\nu \eta_{(AB)}. \tag{3}
\]

Here, \( \eta_{(AB)} \) is a flat \((d+1)\)-metric. We shall write \( e^{(A)}_\mu = e^{(B)}_\mu \eta_{(AB)}, e^{(A)}_\mu = e^{(B)}_\mu \eta^{(AB)} \) and also \( e^{(A)}_\mu = \gamma_{\mu\nu} e^{(A)}_\nu \) and \( e^{(A)}\mu = \gamma^{\mu\nu} e^{(A)}_\nu \), where \( \eta^{(AB)} \) is the inverse of \( \eta_{(AB)} \).

In the particular case in which \( e^{(A)}_\mu \) is written as (1) \( \gamma_{\mu\nu} \) becomes

\[
\gamma_{\mu\nu} = \begin{pmatrix} -N^2 + g_{ij}N^iN^j & N_i \\ N_j & g_{ij} \end{pmatrix},
\] (4)

where \( N = E_0^{(0)}, N_i = E_0^{(a)}E_i^{(b)}\delta_{(ab)}, g_{ij} = E_i^{(a)}E_j^{(b)}\delta_{(ab)} \) and \( N^i = g^{ij}N_j \), with \( g^{ik}g_{kj} = \delta_i^j \). Here the symbol \( \delta_{(ab)} \) also denotes a Kronecker delta.

We also find that

\[
\gamma^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2}N^i \\ N^{-2}N_j & g^{ij} - N^{-2}N^iN_j \end{pmatrix}.
\] (5)

We observe that (4) and (5) provide the traditional ansatz for canonical gravity. So, \( N \) and \( N_i \) admit the interpretation of lapse function and shift vector, respectively. Thus, in terms of \( N \) and \( N_i \), (1) and (2) become
\[ e_\mu^{(A)} = \begin{pmatrix} N & E_i^{(a)} N^i \\ 0 & E_i^{(a)} \end{pmatrix} \] (6)

and
\[ e^{(A)}_\mu = \begin{pmatrix} N^{-1} & -N^{-1}N^i \\ 0 & E_i^{(a)} \end{pmatrix}. \] (7)

For later calculations it is convenient to write
\[ E_i^{(a)} = E_i^{(a)} \eta_{ab}, \ E_i^{(a)} = E_i^{(a)} \eta_{ab} \] and also\[ E_i^{(a)} = g_{ij} E_j^{(a)}; \ E_i^{(a)} = g_{ij} E_j^{(a)}. \] Observe that although \( e_i^{(a)} \neq E_i^{(a)} \). This is because when we consider the \( e \) notation we raise and lower indices with the metric \( \gamma \), while in the case of the \( E \) notation we raise and lower indices with the metric \( g \). In fact, this is one of the reasons for distinguishing \( e \) and \( E \) in the ansatz (1) and (2).

We shall assume that \( e_\mu^{(A)} \) satisfies the condition
\[ \partial_\mu e^{(A)}_\nu - \Gamma^\alpha_{\mu\nu} e^{(A)}_\alpha + \omega^{(AB)}_\mu e_{\nu(B)} = 0. \] (8)

Here, \( \Gamma^\alpha_{\mu\nu}(\gamma) = \Gamma^\alpha_{\nu\mu}(\gamma) \) and \( \omega^{(AB)}_\nu = -\omega^{(BA)}_\nu \) denote the Christoffel symbols and the spin connection respectively. The expression (8) determines, of course, a manifold with a vanishing torsion. Using (8), it is not difficult to see that \( \omega_{(ABC)} = e_\mu(\alpha) \omega_{\mu(BC)} = -\omega_{(ACB)} \) can be written in terms of
\[ F^{(A)}_{\mu\nu} = \partial_\mu e_\nu^{(A)} - \partial_\nu e_\mu^{(A)} \] (9)
in the following form
\[ \omega_{(ABC)} = \frac{1}{2} \left[ F_{(ABC)} + F_{(CAB)} + F_{(CBA)} \right], \] (10)

where
\[ F_{(ABC)} = e_\mu^{(A)} e_\nu^{(B)} F_{\mu\nu(\alpha)} = -F_{(BAC)}. \] (11)

Considering (6), (7) and (9) we find
\[ F_{0i(0)} = \partial_i N, \] (12)
\[ F_{ij(0)} = 0, \] (13)
\[ F_{0i(a)} = \partial_0 E_i(a) - \partial_i E_j(a) N^j - E_j(a) \partial_i N^j \] (14)
and
\[ F_{ij(a)} = \partial_i E_{j(a)} - \partial_j E_{i(a)}. \]  

(15)

Our aim is to obtain the different components of \( \omega_{\mu(BC)} \) knowing the expressions (12)-(15). For this purpose we first observe that (13) implies

\[ F_{(ab0)} = 0. \]  

(16)

Thus, (10) leads to the following splitting

\[ \omega_{(00a)} = F_{(a00)}, \]  

(17)

\[ \omega_{(0ab)} = \frac{1}{2} \left[ F_{(0ab)} - F_{(0ba)} \right], \]  

(18)

\[ \omega_{(a0b)} = \frac{1}{2} \left[ F_{(a0b)} + F_{(b0a)} \right], \]  

(19)

and

\[ \omega_{(abc)} = \frac{1}{2} \left[ F_{(abc)} + F_{(cab)} + F_{(cba)} \right]. \]  

(20)

Since

\[ \omega_{i(0a)} = E_i^{\ (b)} \omega_{(b0a)}, \]  

(21)

\[ \omega_{0(bc)} = N \omega_{(0bc)} + E_i^{\ (a)} N^i \omega_{(abc)}, \]  

(22)

\[ \omega_{0(0b)} = N \omega_{(00b)} + E_i^{\ (a)} N^i \omega_{(a0b)}, \]  

(23)

and

\[ \omega_{i(abc)} = E_i^{\ (a)} \omega_{(abc)}, \]  

(24)

by means of (6)-(7) we get

\[ \omega_{i(0a)} = \frac{N^{-1}}{2} E_i^{\ (b)} \left[ E_i^{\ (b)} j F_{j0(a)} - E_i^{\ (b)} j N^k F_{jk(a)} \right. \]  

\[ + E_i^{\ (a)} j F_{j0(b)} - E_i^{\ (a)} j N^k F_{jk(b)} \left], \right. \]  

(25)

\[ \omega_{0(bc)} = \frac{N^{-1}}{2} \left[ E_i^{\ (b)} i F_{0i(c)} - N^i E_i^{\ (b)} j F_{ij(c)} - E_i^{\ (b)} i F_{0i(b)} + N^i E_i^{\ (b)} j F_{ij(b)} \right. \]  

\[ + E_i^{\ (a)} N^i \omega_{(abc)}, \right. \]  

(26)
and
\[
\omega_{0(0b)} = NF_{(00)} + \frac{N^{-1}}{2} E_k^{(a)} N^k [E^{(a)}_i F_{0(b)} - E^{(a)}_i N^j F_{ij(b)} \\
+ E^{(b)}_i F_{0(a)} - E^{(b)}_i N^j F_{ij(a)}].
\] (27)

Consequently, using (12)-(15) it is not difficult to obtain the results
\[
\omega_{i(0a)} = \frac{N^{-1}}{2} E_j^{(a)} \left[ -\partial_0 g_{ij} + D_i N_j + D_j N_i \right],
\] (28)
\[
\omega_{0(bc)} = \frac{N^{-1}}{2} \left[ E_i^{(b)} \partial_0 E_i^{(c)} - E_i^{(c)} \partial_0 E_i^{(b)} - (E_i^{(b)} E_j^{(c)} - E_i^{(c)} E_j^{(b)}) D_i N_j \right],
\] (29)
and
\[
\omega_{0(ab)} = -E_i^{(b)} \partial_i N + \frac{N^{-1}}{2} N^i E_j^{(b)} \left[ -\partial_0 g_{ij} + D_i N_j + D_j N_i \right],
\] (30)
where \(D_i\) denotes covariant derivative in terms of the Christoffel symbols \(\Gamma^i_{jk} = \Gamma^i_{jk}(g)\).

With the help of (28), (29) and (30), we are now ready to compute the Riemann tensor
\[
R_{\mu\nu(AB)} = \partial_\mu \omega_{\nu(AB)} - \partial_\nu \omega_{\mu(AB)} + \omega_{\mu(AC)} \omega_{\nu(B)}^{(C)} - \omega_{\nu(AC)} \omega_{\mu(B)}^{(C)},
\] (31)

But before we do that let us first observe that
\[
R_{ij(0a)} = D_i \omega_{j(0a)} - D_j \omega_{i(0a)},
\] (32)
where
\[
D_i \omega_{j(0a)} = \partial_i \omega_{j(0a)} - \Gamma^k_{ij}(g) \omega_{k(0a)} - \omega_j^{(c)} \omega_i^{(a)}.
\] (33)

We also obtain
\[
R_{ij(ab)} = \tilde{R}_{ij(ab)} + \omega_{i(0a)} \omega_{j(0b)} - \omega_{j(0a)} \omega_{i(0b)},
\] (34)
\[
R_{0i(0a)} = \partial_0 \omega_{i(0a)} - \partial_i \omega_{0(0a)} + \omega_{0(0c)} \omega_i^{(c)} - \omega_i^{(0c)} \omega_0^{(c)}.
\] (35)

and
\[
R_{0i(ab)} = \partial_0 \omega_{i(ab)} - \partial_i \omega_{0(ab)} + \omega_{0(ac)} \omega_i^{(c)} - \omega_i^{(ac)} \omega_0^{(c)} + \omega_{0(0a)} \omega_i^{(0b)} - \omega_i^{(0a)} \omega_{0(b)}.
\] (36)
Here,
\[ \tilde{R}_{ij(ab)} = \partial_i \omega_j(ab) - \partial_j \omega_i(ab) + \omega_i(ac) \omega_j^{(c)}(b) - \omega_j(ac) \omega_i^{(c)}(b). \]  
(37)

It becomes convenient to write
\[ K_{ij} = \frac{N^{-1}}{2} (-\partial_0 g_{ij} + D_i N_j + D_j N_i). \]  
(38)

So, by using (28)-(30) we get
\[ R_{ij(ab)} = \tilde{R}_{ij(ab)} + \left[ E_{(a)}^k E_{(b)}^l K_{ik} K_{jl} - E_{(a)}^k E_{(b)}^l K_{ij} K_{kl} \right], \]  
(39)

\[ R_{0i(0a)} = \partial_0 (E_{(a)}^k) K_{ik} + E_{(a)}^k \partial_0 K_{ik} - \frac{1}{2} E^{(c)k} K_{ik} [E_{(c)}^l \partial_0 E_{(l(a)} - E_{(c)}^l \partial_0 E_{(l(a)}], \]  
(40)

and
\[ R_{0i(ab)} = \partial_0 \omega_{i(ab)} + \left( -E_{(a)}^j \partial_j N + N_j E_{(a)}^k K_{jk} \right) \left( E_{(b)}^l K_{il} \right) \]  
\[ - \left( E_{(a)}^l K_{il} \right) \left( -E_{(b)}^j \partial_j N + N_j E_{(b)}^k K_{jk} \right) - D_i \partial_0 \omega_{0(ab)}. \]  
(41)

Let us now consider the scalar curvature tensor
\[ R = e_{(A)}^\mu e_{(B)}^\nu R_{\mu\nu}^{(AB)}. \]  
(42)

By virtue of (7) we have
\[ R = 2N^{-1} E_{(a)}^i R_{0i(0a)} - 2N^{-1} N^i E_{(a)}^j R_{ij(0a)} + E_{(a)}^i E_{(b)}^j R_{ij(ab)} \]  
(43)
or
\[ R = -2N^{-1} E^{(a)i} R_{0i(0a)} + 2N^{-1} N^i E^{(a)j} R_{ij(0a)} + E_{(a)}^i E_{(b)}^j R_{ij(ab)}. \]  
(44)

Therefore, substituting (32), (37), (39) and (40) into (44), we find
\[ R = -N^{-1} \partial_0 (g_{ij}) K^{ij} - 2N^{-1} \partial_0 (g_{ij} K^{ij}) + 2N^{-1} E^{(a)i} D_i \omega_{0(0a)} \]  
\[ + 2N^{-1} N^i E^{(a)j} \left( D_i (E^{(a)k} K_{jk}) - D_j (E^{(a)k} K_{ik}) \right) \]  
\[ + E_{(a)}^i E_{(b)}^j \tilde{R}_{ij}^{(ab)} + E_{(a)}^i E_{(b)}^j \left[ E^{(a)k} E^{(b)l} K_{ik} K_{jl} - E^{(a)k} E^{(b)l} K_{jk} K_{il} \right], \]  
(45)
where we considered the expression \( g^{ij} = E^{(a)}_i E^{(a)}_j \) and the property \( K_{ij} = K_{ji} \). By using the fact that
\[
D_i E_j^{(a)} = \partial_i E_j^{(a)} - \Gamma^k_{ij}(g) E_k^{(a)} + \omega^{(a)}_i E_j^{(a)} = 0,
\]

we find that (45) is reduced to
\[
R = N^{-1} \{-\partial_0(g_{ij}) K^{ij} - 2\partial_0(g_{ij} K^{ij}) + 2D_i(E^{(a)} \omega_{0(a)}) \\
+ 2N^i D_j[i(g^{kl} K_{kl}) - g^{ij} K_{ik}] + \tilde R + g^{ij} K_{ij} g^{kl} K_{kl} - K_{ij} K^{ij}.
\]

In this way we see that the action
\[
S_D = \int_{M^D} D\sqrt{-\tilde{\gamma}} R = \int_{M^D} \sqrt{g} NR = \int_{M^D} \tilde{E} NR
\]
becomes
\[
S_D = \int_{M^D} \tilde{E} \{-\partial_0(g_{ij}) K^{ij} - 2\partial_0(g_{ij} K^{ij}) \\
-(D_j N_i + D_i N_j)[g^{ij}(g^{kl} K_{kl}) - K^{ij}] + N(\tilde R + g^{ij} K_{ij} g^{kl} K_{kl} - K_{ij} K^{ij})
\]
\[
+ D_j \{+2\tilde{E}((E^{(a)} \omega_{00(a)}) - N_i[g^{ij}(g^{kl} K_{kl}) - K^{ij}]\}
\]

where \( \tilde{E} \) is the determinant of \( E_i^{(a)} \). But according to (38) we have
\[
D_j N_i + D_i N_j = D_j N_i + D_i N_j = 2NK_{ij} + \partial_0(g_{ij}).
\]

Thus, up to a surface term (48) yields
\[
S_D = \int_{M^D} \tilde{E} \{-\partial_0(g_{ij}) K^{ij} - 2\partial_0(g_{ij} K^{ij}) \\
-(2NK_{ij} + \partial_0 g)[g^{ij}(g^{kl} K_{kl}) - K^{ij}] + N(\tilde R + g^{ij} K_{ij} g^{kl} K_{kl} - K_{ij} K^{ij})\}.
\]

Simplifying this expression we get
\[
S_D = \int_{M^D} \tilde{E} \{-\partial_0(g_{ij} K^{ij}) - \partial_0(g_{ij}) g^{ij}(g^{kl} K_{kl}) \\
+ N(\tilde R + K_{ij} K^{ij} - g^{ij} K_{ij} g^{kl} K_{kl})\}.
\]

Since \( \partial_0 \tilde{E} = \frac{1}{2} \tilde{E} \partial_0(g_{ij}) g^{ij} \) we can further simplify (51) in the form
\[ S_D = \int_{M^D} \{-2\partial_0(\tilde{E}g_{ij}K^{ij}) + \tilde{E}\{N(\tilde{R} + K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl})\}\}. \quad (52) \]

So up to a total time derivative we end up with

\[ S_D = \int_{M^D} L = \int_{M^D} \tilde{E}N(\tilde{R} + K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl}) \]
\[ = \int_{M^D} \sqrt{g}N(\tilde{R} + K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl}). \quad (53) \]

This is of course the typical form of the action in canonical gravity (see Refs. in [17] and references therein).

Let us now introduce the canonical momentum conjugate to \( g_{ij} \),

\[ \pi^{ij} = \frac{\partial L}{\partial \partial_0 g_{ij}}. \quad (54) \]

Using (38) and (53) we obtain

\[ \pi^{ij} = -\tilde{E}(K^{ij} - g^{ij}g^{kl}K_{kl}). \quad (55) \]

Thus, by writing (53) in the form

\[ S_D = \int_{M^D} \{2\tilde{E}N(K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl}) \]
\[ + \tilde{E}N\{\tilde{R} - (K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl})\}\}. \quad (56) \]

we see that, in virtue of (55), the first term in (56) can be written as

\[ 2\tilde{E}N(K_{ij}K^{ij} - g^{ij}K_{ij}g^{kl}K_{kl}) = -2N K_{ij} \pi^{ij} \]
\[ = -(-\partial_0 g_{ij} + D_i N_j + D_j N_i) \pi^{ij}, \quad (57) \]

where once again we used (38). Thus, by considering (55) and (57) we find that up to surface term \( S_D \) becomes

\[ S_D = \int_{M^D} \{\partial_0 g_{ij} \pi^{ij} + 2N_i D_j \pi^{ij} \]
\[ + \tilde{E}N\{\tilde{R} - \frac{1}{E^2}(\pi^{ij} \pi^{ij} - \frac{1}{D-2}g^{ij} \pi^{ij}g^{kl} \pi_{kl})\}\}. \quad (58) \]

We see that \( N \) and \( N^i \) play the role of Lagrange multiplier and therefore from (58) it follows that the Diffeomorphism and Hamiltonian constraints are

\[ H^i \equiv 2D_j \pi^{ij} \quad (59) \]
and
\[ H \equiv \tilde{E}\{ \tilde{R} - \frac{1}{E^2}(\pi_{ij}\pi^{ij} - \frac{1}{D-2}g^{ij}\pi_{ij}g^{kl}\pi_{kl}) \}, \tag{60} \]
respectively. The expression (60) can also be written as
\[ H = \sqrt{g}\tilde{R} - \frac{1}{\sqrt{g}}(\pi_{ij}\pi^{ij} - \frac{1}{D-2}g^{ij}\pi_{ij}g^{kl}\pi_{kl}). \tag{61} \]
Even with a rough inspection of the constraint (61) one can expect that ”the potential term” \( \tilde{R} \) presents serious difficulties when we make the transition to the quantum scenario;
\[ \hat{H}^i \mid \psi > = 0 \tag{62} \]
and
\[ \hat{H} \mid \psi > = 0. \tag{63} \]
We would like to remark that according to our development this is true not just in four dimensions but in an arbitrary dimension \( D \).

3.- Palatini formalism

Similar conclusion, in relation to the quantization of ”the potential term” \( \tilde{R} \), can be obtained if we use the so called Palatini formalism. In this case the variables \( E_{\mu}^{(A)} \) and \( \omega_{\nu}^{(AB)} \) are considered as independent variables. We start again with the action (47), namely \( S_D = \int_{M_D} \tilde{E}NR \), with \( R \) given by (44). Substituting (32), (34) and (35) into (47) we find
\[ S_D = \int_{M_D} \tilde{E}\{-2E^{(a)i}[\partial_0\omega_{i(0a)} - \partial_i\omega_{0(0a)} + \omega_{0(0c)}\omega_i^{(c)}(a) - \omega_i(0c)\omega_0^{(c)}(a)] \]
\[ + 2N^iE^{(a)j}[D_i\omega_{j(0a)} - D_j\omega_{i(0a)}] \]
\[ + NE^{(a)i}E^{(b)j}[\tilde{R}_{ij(ab)} + \omega_{i(0a)}\omega_{j(0b)} - \omega_{j(0a)}\omega_{i(0b)}], \tag{64} \]
which can also be written as
\[ S_D = \int_{M_D}\{-2\tilde{E}E^{(a)i}\partial_0\omega_{i(0a)} + NE^{(a)i}E^{(b)j}\tilde{R}_{ij(ab)} + \omega_{i(0a)}\omega_{j(0b)} \]
\[ - \omega_{j(0a)}\omega_{i(0b)}] - 2\tilde{E}E^{(a)i}D_i\omega_{0(0a)} + 2N^iE^{(a)j}[D_i\omega_{j(0a)} - D_j\omega_{i(0a)}]. \tag{65} \]
The last two terms in (65) can be used for obtaining the formula $D_i E_j^{(a)} = 0$ as a field equation. So if we focus in the first two terms in (65) we see that the quantities $\tilde{E} E^{(a)}_i$ and $\omega_i^{(0a)}$ can be considered as conjugate canonical variables, with $\tilde{E} E^{(a)}_i$ playing the role of a conjugate momentum to $\omega_i^{(0a)}$, while the expression

$$H = E^{(a)i} E^{(a)j} \left[ \tilde{R}_{ij}^{(ab)} + \omega_i^{(0a)} \omega_j^{(0b)} - \omega_j^{(0a)} \omega_i^{(0b)} \right]$$

(66)

plays the role of a Hamiltonian constraint. So when we proceed to quantize the system we again expect to find some difficulties because of the term $\tilde{R} = E^{(a)}_i E^{(a)}_j \tilde{R}_{ij}^{(ab)}$. Once again, this is true in any dimension $D$.

4.- Self-dual formalism in four dimensions

In four dimensions something interesting happens if instead of (47) one considers the alternative action \[2\]-\[3\]

$$+ S_4 = \frac{1}{2} \int_{M^4} e e^{(A)}_\mu e^{(B)}_\nu + R^{(AB)}_{\mu\nu}. \tag{67}$$

Here,

$$\pm R^{(AB)}_{\mu\nu} = \frac{1}{2} \pm M^{(AB)}_{(CD)} R^{(CD)}_{\mu\nu}, \tag{68}$$

with

$$\pm M^{(AB)}_{(CD)} = \frac{1}{2} \left( \delta^{(AB)}_{(CD)} \mp i \epsilon^{(AB)}_{(CD)} \right)$$

(69)

is the self(anti-self)-dual sector of $R^{(AB)}_{\mu\nu}$. The symbol $\delta^{(AB)}_{(CD)} = \delta^{(A)}_{(C)} \delta^{(B)}_{(D)} - \delta^{(B)}_{(C)} \delta^{(A)}_{(D)}$ denotes a generalized delta. (Observe that the presence of the completely antisymmetric symbol $\epsilon^{(AB)}_{(CD)}$ in (60) is an indication that the spacetime dimension is equal to four.) Since $+ R^{(AB)}_{\mu\nu}$ is self-dual, that is

$$\frac{1}{2} \epsilon^{(AB)}_{(CD)} + R^{(CD)}_{\mu\nu} = i + R^{(AB)}_{\mu\nu}, \tag{70}$$

we find that $+ S$ can be written as

$$+ S_4 = \frac{1}{2} \int_{M^4} E \left\{ 2 E^{(0)}_j E^{(0)}_i + R^{(0a)}_0 + 2 E^{(0)}_i E^{(a)}_j + R^{(ab)}_{ij} \right\}$$

$$- i \frac{1}{2} E^{(a)}_i E^{(b)}_j \epsilon^{abc} + R_{ij}^{(0c)} \right\}$$

(71)
showing that only \( R_{\mu\nu}^{(0a)} \) is needed. Here we used the definition \( \varepsilon^{abc} \equiv \varepsilon^{0abc} \).

A fine point is that up to the Bianchi identities for \( R_{\mu\nu}^{(AB)} \), \( +S_4 \) is equivalent to \( S_4 \). If we use the \( 3+1 \) decomposition (6) and (7) we find that (71) becomes

\[
+ S_4 = -\int_{M^4} \tilde{E} \{ 2E_{(a)}^i + R_{0i}^{(0a)} - 2N E_{(a)}^j + R_{ij}^{(0a)} - \frac{i}{2} NE_{(a)}^i E_{(b)}^j \varepsilon_{bc}^a + R_{ij}^{(0c)} \}.
\]  

According to (35), we discover that the first term in (72) establishes that \( \tilde{E} E_{(a)}^i \) can be understood as the canonical momentum conjugate to \( +\omega_i^{(0a)} \). Thus one can interpret the second and the third terms in (64) as the canonical constraints,

\[
+H^i = -2\tilde{E} E_{(a)}^j + R_{ij}^{(0a)} = 0
\]  

and

\[
+H = -\frac{1}{2} \tilde{E} E_{(a)}^i E_{(b)}^j \varepsilon_{abc} + R_{ij(0c)} = 0,
\]  

(see Ref. [42]). Comparing (66) and (74) one sees that the term \( \tilde{R} = E^{(a)i} E^{(b)j} \tilde{R}_{ij(ab)} \) is not manifest in (74). At first sight one may expect that this reduced result of the Diffeomorphism and Hamiltonian constraints may induce a simplification at the quantum level. However, it is known that there are serious difficulties for finding the suitable representation for the corresponding associated states with (73) and (74). This is true, for instance, when one tries to find suitable representation of the reality condition associated with the connection.

One of the key ingredients to achieve the simpler constraint (74) is, of course, the self-duality of \( +R_{\mu\nu}^{(AB)} \). This mechanism works in four dimensions because of the lemma; the dual of a two form is another two form. This is, of course, true because we are using the \( \varepsilon \)-symbol to define duality. Thus, in higher dimensions this lemma is no longer true. However, in eight dimensions there exist another possibility to define duality as we shall see in section 6.

5.- Generalization of self-dual formalism in four dimensions

In this section we shall apply the canonical formalism to the action [18]-[19]

\[
S_4 = -\frac{1}{16} \int_{M^4} \varepsilon^{\mu\nu\alpha\beta} + \mathcal{R}_{\mu\nu}^{(AB)} + \mathcal{R}_{\alpha\beta}^{(CD)} \varepsilon^{(ABCD)},
\]  

(75)
which is a generalization of (67). Here,

\[ R_{\mu\nu}^{(AB)} = R_{\mu\nu}^{(AB)} + \Sigma_{\mu\nu}^{(AB)} \quad (76) \]

with \( R_{\mu\nu}^{(AB)} \) defined in (31) and

\[ \Sigma_{\mu\nu}^{(AB)} = e_{\mu}^{(A)} e_{\nu}^{(B)} - e_{\mu}^{(B)} e_{\nu}^{(A)}. \quad (77) \]

In fact, by substituting (76) and (77) into (75) one can show that the action (75) is reduced to three terms: topological invariant term, cosmological constant term and the action (67).

By using (70) it is not difficult to see that (75) can be decomposed as

\[ S_4 = -\frac{i}{2} \int_{M^4} \varepsilon^{\mu\nu\alpha\beta} + R_{\mu\nu}^{(0a)} + R_{\alpha\beta}^{(0a)}. \quad (78) \]

Further decomposition gives

\[ S_4 = -i \int_{M^4} \varepsilon^{ijk} + R_{0i}^{(0a)} + R_{jk}^{(0a)}. \quad (79) \]

Considering (76) we obtain

\[ S_4 = -i \int_{M^4} \{ \varepsilon^{ijk} + \Sigma_{0i}^{(0a)} + R_{jk}^{(0a)} + \Sigma_{ij}^{(0a)} \}. \quad (80) \]

Using (32) and (35) one sees that the first term is a surface term as expected, while the last term is a cosmological constant term. Thus, by focusing only in the second and third terms we get

\[ + S_4 = -i \int_{M^4} \{ \varepsilon^{ijk} + \Sigma_{0i}^{(0a)} + R_{jk}^{(0a)} + \Sigma_{ij}^{(0a)} \}. \quad (81) \]

which can be reduced to

\[ + S_4 = -i \int_{M^4} \{ \frac{1}{2} N \varepsilon^{ijk} E_i^{(a)} + R_{jk}^{(0a)} + \frac{i}{2} N^l \varepsilon^{ijk} \varepsilon_{(bc)}^{(a)} E_i^{(b)} E_k^{(c)} + R_{jk}^{(0a)} \}

\[ - \frac{i}{2} \varepsilon^{ijk} \varepsilon_{(bc)}^{(a)} E_j^{(b)} E_k^{(c)} + R_{0i}^{(0a)} \}. \quad (82) \]

In turn, it is straightforward to prove that this action reduces to the action (72). So, the constraints (73) and (74) can also be written as
\[ H = -\frac{i}{2} \varepsilon^{ijk} E_{(a)} + R_{0ia} = 0 \]  

(83)

and

\[ H_l = \frac{1}{2} \varepsilon^{ijk} \varepsilon^{(a)}_{(bc)} E_{(b)} E_{(c)} + R_{0ia} = 0. \]  

(84)

It is interesting to observe the simplicity of the present construction in contrast to the development of sections 3 and 4.

6. Self-dual formalism in eight dimensions

One of the key ingredients for achieving the simpler route in the derivation of the constraints (83) and (84) is, of course, the self-duality of \( + R^{(AB)}_{\mu\nu} \). This works in four dimensions because the dual of a two form is another two form. However, in higher dimensions this line of thought is difficult to sustain except in eight dimensions. In fact, one can attempt to generalize the formalism of section 4 to higher dimensions using BF technics [22] but the self-dual property is lost as it was described in section 4. On the other hand in eight dimensions one may take recourse of the octonionic structure constants and define a self-dual four form \( \eta^{\mu\nu\alpha\beta} \) which can be used to construct similar approach to the one presented in section 4 as it was proved in Refs. [13] and [14]. The aim of this section is to pursuing this idea by exploring the possibility of bringing the formalism to the quantum scenario.

Our starting point is the action [13]

\[ S_8 = \frac{1}{192} \int_{M^8} e \eta^{\mu\nu\alpha\beta} + \mathcal{R}^{(AB)}_{\mu\nu} + \mathcal{R}^{(CD)}_{\alpha\beta} \eta^{(ABCD)}. \]  

(85)

Here, the indices \( \mu, \nu, ..etc \) are ”spacetime” indices, running from 0 to 7, while the indices \( A, B, ..etc \) are frame indices running also from 0 to 7. (Just by convenience in what follows, we shall assume an Euclidean signature.) The quantity \( e \) is the determinant of the eight dimensional matrix \( e^{(A)}_{\mu} \).

In addition, we have the following definition:

\[ \mathcal{R}^{(AB)}_{\mu\nu} = R^{(AB)}_{\mu\nu} + \Sigma^{(AB)}_{\mu\nu}, \]  

(86)

with

\[ R_{\mu\nu(AB)} = \partial_{\mu} \omega_{\nu(AB)} - \partial_{\nu} \omega_{\mu(AB)} + \omega_{\mu(AC)} \omega^{(C)}_{\nu (B)} - \omega_{\mu(BC)} \omega^{(C)}_{\nu (A)} \]  

(87)
and

\[ \Sigma_{\mu\nu}^{(AB)} = e_\mu^{(A)} e_\nu^{(B)} - e_\mu^{(B)} e_\nu^{(A)}. \quad (88) \]

The \( \eta \)-symbol \( \eta^{(ABCD)} \) is a completely antisymmetric object, which is related with the octonion structure constants \( \eta_{(abc0)} = \psi_{abc} \) and its dual \( \eta_{(abcd)} = \varphi_{(abcd)} \), satisfying the self-dual (anti-self-dual) formula

\[ \eta^{(ABCD)} = \frac{\varsigma}{4!} \varepsilon^{(ABCD)EFGH} \eta^{(EFGH)}. \quad (89) \]

For \( \varsigma = 1 \), \( \eta^{(ABCD)} \) is self-dual (and for \( \varsigma = -1 \) is anti-self-dual). Moreover, \( \eta \)-symbol satisfies the relations [20]-[21] (see also Refs. [5] and [6]),

\[ \eta^{(ABCD)} \eta^{(EFCD)} = 6 \delta^{(EF)}_{(AB)} + 4 \eta^{(EF)}_{(AB)}, \quad (90) \]

\[ \eta^{(ABCD)} \eta^{(EBCD)} = 42 \delta^E_A, \quad (91) \]

and

\[ \eta^{(ABCD)} \eta^{(ABCD)} = 336. \quad (92) \]

Finally, by introducing the dual of \( R_{\mu\nu}^{(AB)} \) in the form

\[ \star R_{\mu\nu}^{(AB)} = \frac{1}{2} \eta^{(AB)}_{(CD)} R_{\mu\nu}^{(CD)}, \quad (93) \]

we define the self-dual \( + R_{\mu\nu}^{(AB)} \) and anti-self-dual \( - R_{\mu\nu}^{(AB)} \) parts of \( R_{\mu\nu}^{(AB)} \) in the form

\[ + R_{\mu\nu}^{(AB)} = \frac{1}{4} (R_{\mu\nu}^{(AB)} + \star R_{\mu\nu}^{(AB)}) \quad (94) \]

and

\[ - R_{\mu\nu}^{(AB)} = \frac{1}{4} (3 R_{\mu\nu}^{(AB)} - \star R_{\mu\nu}^{(AB)}), \quad (95) \]

respectively. Since

\[ ** R_{\mu\nu}^{(AB)} = 3 R_{\mu\nu}^{(AB)} + 2 \star R_{\mu\nu}^{(AB)}, \quad (96) \]

we see that

\[ ** R_{\mu\nu}^{(AB)} = 3^+ R_{\mu\nu}^{(AB)} \quad (97) \]

and

\[ ** R_{\mu\nu}^{(AB)} = - R_{\mu\nu}^{(AB)}. \quad (98) \]
Thus, up to a numerical factor we see that $+\mathcal{R}_{\mu
u}^{(AB)}$ and $-\mathcal{R}_{\mu
u}^{(AB)}$ play, in fact, the role of the self-dual and anti-self-dual parts, respectively of $\mathcal{R}_{\mu
u}^{(AB)}$. It turns out to be convenient to write (94) as [12]

$$+\mathcal{R}_{\mu\nu}^{(AB)} = \frac{1}{2} + \Lambda^{(AB)}_{(CD)} \mathcal{R}_{\mu\nu}^{(CD)},$$

where

$$+\Lambda^{(AB)}_{(CD)} = \frac{1}{4} (\delta^{(AB)}_{(CD)} + \eta^{(AB)}_{(CD)}).$$

While, (95) can be written in the form

$$-\mathcal{R}_{\mu\nu}^{(AB)} = \frac{1}{2} - \Lambda^{(AB)}_{(CD)} \mathcal{R}_{\mu\nu}^{(CD)},$$

with

$$-\Lambda^{(AB)}_{(CD)} = \frac{1}{4} (3\delta^{(AB)}_{(CD)} - \eta^{(AB)}_{(CD)}).$$

The objects $\pm \Lambda$ admit an interpretation of projection operators. In fact, one can prove that the objects $+\Lambda$ and $-\Lambda$, given in (100) and (102) respectively, satisfy [12]

$$+\Lambda + -\Lambda = 1,$$

$$+\Lambda - -\Lambda = -\Lambda + -\Lambda = 0,$$

$$+\Lambda^2 = +\Lambda,$$

and

$$-\Lambda^2 = -\Lambda.$$  

Here, $\pm \Lambda^2$ means $\frac{1}{4} \pm \Lambda^{(AB)}_{(CD)} \Lambda^{(EF)}_{(GH)} \delta^{(ABEF)}$.

Finally, the object $\eta^{\mu\nu\alpha\beta}$ is a completely antisymmetric tensor determined by the relation

$$\eta_{\mu\nu\alpha\beta} \equiv e_\mu^{(A)} e_\nu^{(B)} e_\alpha^{(C)} e_\beta^{(D)} \eta_{(ABCD)}.$$  

Before we explore the consequences of (85) let us try to understand the volume element structure in (85) from alternative analysis. For this purpose it turns out convenient to define the quantity
\[ \hat{e} \equiv \frac{1}{4!} \hat{\eta}^{\mu\nu\alpha\beta} e^{(A)}_\mu e^{(B)}_\nu e^{(C)}_\alpha e^{(D)}_\beta \eta_{(ABCD)}, \]  

(108)

where, \( \hat{\eta}^{\mu\nu\alpha\beta} \) takes values in the set \([-1, 0, 1]\) and has exactly the same octonionic properties as \( \eta_{(ABCD)} \) (specified in (89)-(92)). The formula (108) can be understood as the analogue of the determinant for \( e^{(A)}_\mu \) in four dimensions. Thus, by using the octonionic properties (89)-(92) for \( \eta_{(ABCD)} \), such as the self-duality relation

\[ \eta^{(ABCD)} = \frac{1}{4!} \epsilon^{(ABCDEFGH)\eta_{(FGH)}}, \]  

(109)

from (107) one can prove that up to numerical constants \( a = \frac{1}{5} \) and \( b = \frac{1}{3} \) one obtains

\[ \hat{e} \hat{\eta}^{\mu\nu\alpha\beta} = a \hat{\eta}^{\mu\nu\alpha\beta} + b \hat{\eta}^{\mu\nu\tau\lambda} \eta^{\alpha\beta}_{\tau\lambda}, \]  

(110)

which proves that at least \( \hat{\eta}^{\mu\nu\alpha\beta} \sim \hat{e} \eta^{\mu\nu\alpha\beta} \). The expression (110) means that there are two terms in (85), one which can be written as

\[ S_8 \sim \frac{1}{192} \int_{M^8} \epsilon^{\mu\nu\alpha\beta\lambda\rho\sigma\tau} \eta^{\lambda\rho\sigma\tau} + R^{(AB)}_{\mu\nu} + R^{(CD)}_{\alpha\beta} \eta_{(ABCD)}, \]  

(111)

In four dimensions the corresponding ratio \( \frac{\hat{e}}{e} \) gives \( \frac{\hat{e}}{e} = 1 \). However, the situation is more subtle in eight dimensions because we can not set \( \frac{\hat{e}}{e} = 1 \) and this suggests an exotic volume element mediated in part by the exceptional group \( G_2 \). This is suggested in part because the quantities \( \hat{\eta}^{\mu\nu\alpha\beta} \) and \( \eta_{(ABCD)} \) are only \( G_2 \)-invariant rather than \( SO(8) \)-invariant.

Now considering (107) and (109) one observes that \( \eta^{\mu\nu\alpha\beta} \) is also self-dual in eight dimensions, that is

\[ \eta^{\mu\nu\alpha\beta} = \frac{1}{4!} \epsilon^{\mu\nu\alpha\beta\lambda\rho\sigma\tau} \eta^{\lambda\rho\sigma\tau}, \]  

(112)

which implies that the action (85) can also be written as

\[ S_8 = \frac{1}{(192)4!} \int d^8x \ e^{\lambda\rho\sigma\tau\mu\nu\alpha\beta} \eta^{\lambda\rho\sigma\tau} + R^{(AB)}_{\mu\nu} + R^{(CD)}_{\alpha\beta} \eta_{(ABCD)} \]  

(113)

or

\[ S_8 = \frac{1}{(192)4!} \int d^8x \ \epsilon^{\mu\nu\alpha\beta\lambda\rho\sigma\tau} \eta^{\lambda\rho\sigma\tau} + R^{(AB)}_{\mu\nu} + R^{(CD)}_{\alpha\beta} \eta_{(ABCD)}, \]  

(114)
since
\[ \varepsilon^{\mu\alpha\beta\lambda\rho\sigma} = \frac{1}{e} \varepsilon^{\mu\nu\alpha\beta\lambda\rho\sigma\tau}. \]  
(115)

Here, we recall that the quantity \( e \) denotes the usual determinant of \( e^A_\mu \) in eight dimensions. The expression (114) allows us to write (85) in the alternative form
\[ S_8 = \frac{1}{(192)4!} \int_{M^8} \eta \wedge +\mathcal{R}^{(AB)} \wedge +\mathcal{R}^{(CD)} \eta(ABCD). \]  
(116)

Now, since
\[ +\mathcal{R}_{\mu\nu}^{(AB)} = + R_{\mu\nu}^{(AB)} + +\Sigma_{\mu\nu}^{(AB)}, \]  
(117)
on one finds that the action (85) becomes
\[ S_8 = \frac{1}{192} \int_{M^8} e(T + K + C), \]  
(118)

with
\[ T = \eta^{\mu\nu\alpha\beta} + R_{\mu\nu}^{(AB)} + R_{\alpha\beta}^{(CD)} \eta(ABCD); \]  
(119)
\[ K = 2\eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(AB)} + R_{\alpha\beta}^{(CD)} \eta(ABCD); \]  
(120)

and
\[ C = \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(AB)} + \Sigma_{\alpha\beta}^{(CD)} \eta(ABCD). \]  
(121)

It turns out that the \( T \) term can be identified with a topological invariant in eight dimensions. In fact, it can be considered as the "gravitational" analogue of the topological term of \( G_2 \)-invariant super Yang-Mills theory [23];
\[ S_{YM} = \int_{M^8} \eta^{\mu\nu\alpha\beta} F_{\mu\nu}^a F_{\alpha\beta}^b g_{ab}, \]  
(122)

where \( F_{\mu\nu}^a \) is the Yang-Mills field strength and \( g_{ab} \) is the group invariant metric. Similarly, \( K \) should lead to a kind of gravity in eight dimensions. Finally, \( C \) may be identified with the analogue of a cosmological constant term. It is worth mentioning that, in general, the \( \varepsilon \)-symbol is Lorentz invariant in any dimension, but in contrast the \( \eta \)-symbol is only \( SO(7) \)-invariant and therefore one must have that the action (85) is only \( SO(7) \)-invariant.

For our purpose we shall focus in the \( K \)-sector of (118), namely
\[ + S_8 = \frac{1}{96} \int_{M^8} \epsilon \rho^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(AB)} + R_{\alpha\beta}^{(CD)} \eta_{(ABCD)}, \quad (123) \]

which in virtue of (97) can also be written as

\[ + S_8 = \frac{1}{16} \int_{M^8} \epsilon \rho^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(AB)} + R_{\alpha\beta(AB)}. \quad (124) \]

We are ready to develop a canonical decomposition of (124). We get

\[ + S_8 = \frac{1}{16} \int_{M^8} \epsilon \{2\eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(0a)} + R_{\alpha\beta (0a)} + \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(ab)} R_{\alpha\beta (ab)} \}, \quad (125) \]

which can be written as

\[ + S_8 = \frac{1}{2} \int_{M^8} \epsilon \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{(0a)} + R_{\alpha\beta (0a)}. \quad (126) \]

Here we used the property \( \eta^{(0a)cd} \eta^{(0bcd)} = \eta^{(ac)bd} \eta^{(bcd)} = \psi^{abcd} = 6 \delta^b_a \), which can be derived from (80), and we considered the fact that \( + R_{\alpha\beta (bc)} = \eta^{(a)}_{(bc)} + R_{\alpha (bc)} \). A further decomposition of (126) gives

\[ + S_8 = \int_{M^8} \tilde{E} \{ \eta^{ijk} + \Sigma_{0i}^{(0a)} + R_{jk(0a)} + \eta^{ijk} + \Sigma_{ij}^{(0a)} + R_{0k(0a)} \}, \quad (127) \]

which can be reduce to

\[ + S_8 = \int_{M^8} \tilde{E} \left\{ \frac{1}{4} N \eta^{ijk} E_i^{(a)} + R_{jk(0a)} + \frac{1}{4} N l \eta^{ijk} \eta_{(bc)}^{(a)} E_i^{(b)} E_l^{(c)} + R_{jk(0a)} \right. \\
+ \left. \frac{1}{4} \eta^{ijk} \eta_{(0a)} E_j^{(b)} E_k^{(c)} + R_{0i}^{(0a)} \right\}. \quad (128) \]

So, the constraints derived from the action (128) are

\[ \mathcal{H} = \frac{1}{4} \tilde{E} \eta^{ijk} E_i^{(a)} + R_{jk(0a)} = 0 \quad (129) \]

and

\[ \mathcal{H}_l = \frac{1}{4} \tilde{E} \eta^{ijk} \eta_{(bc)}^{(a)} E_i^{(b)} E_l^{(c)} + R_{jk(0a)} = 0. \quad (130) \]

Observe that the term \( \tilde{R} = E^{(a)} E^{(b)} \tilde{R}_{ij(ab)} \) is not manifest in (129) and therefore, once again, one may expect some simplification at the quantum level.
Therefore, this shows that the introduction of the self-dual curvature tensor 
\[ +R_{\mu\nu}^{(AB)} \] using the \( \eta \)-symbol makes sense in eight dimensions. However, once again, this possible quantum simplification is an illusion because the need of the reality condition for the connection may lead to some difficulties for finding suitable representation which implements such a reality condition.

One may wonder whether the same construction may be achieved by considering the anti-self-dual sector via the anti-self-dual curvature tensor 
\[ -R_{\mu\nu}^{(AB)} \]. In order to give a possible answer to this question one requires to analyze the formalism from the perspective of octonionic representations of the group \( SO(8) \). Let us first recall the case of four dimensions in connection with the norm group of the quaternions, namely \( SO(4) \). In this case one has the decomposition
\[ SO(4) = S^3 \times S^3, \] (131)  
which, in turn, allows the result
\[ [^+ J_{(AB)}, ^- J_{(AB)}] = 0, \] (132)  
where \( \pm J_{(AB)} \) are the self-dual and anti-self-dual components of the generator \( J_{(AB)} \) of \( SO(4) \). As a consequence of this one has the splitting
\[ R_{\mu\nu}^{(AB)} = ^+ R_{\mu\nu}^{(AB)}(\pm \omega) - ^- R_{\mu\nu}^{(AB)}(-\omega). \] (133)  
This means that there is not mixture between the self-dual and anti-self-dual components of \( R_{\mu\nu}^{(AB)} \) and consequently one may choose to work either with the self-dual sector or anti-self-dual sector of \( R_{\mu\nu}^{(AB)} \).

The case of eight dimensions is more subtle because the decomposition \( \pm R_{\mu\nu}^{(AB)} \) of \( R_{\mu\nu}^{(AB)} \), according to the expressions (94) and (95), is connected to the splitting of the 28 independents generators \( J_{(AB)} \) of \( SO(8) \) in 7 generators \( ^+ R J_{(AB)} \equiv (\Lambda J)_{(AB)} \) and 21 generators \( ^- R J_{(AB)} \equiv (-\Lambda J)_{(AB)} \) which do not commute, that is, the generators \( ^+ R J_{(AB)} \) and \( ^- R J_{(AB)} \), corresponding to \( S^7_R \equiv SO(8)/SO(7)_R \) and \( SO(7)_R \) respectively do not satisfy the expression (132). In turn, this means that we can not write \( R_{\mu\nu}^{(AB)} \) as in (133). The situation can be saved by considering beside the right sector, \( S^7_R \) and \( SO(7)_R \), corresponding to the value \( \varsigma = 1 \) in the expression (89), the left sector \( S^7_L \equiv SO(8)/SO(7)_L \) and \( SO(7)_L \) corresponding to the value \( \varsigma = -1 \) in (89). In fact, with this tools at hand one finds the possibility to combine the generators \( ^+ R J_{(AB)} \) and \( ^+ L J_{(AB)} \) of \( S^7_R \) and \( S^7_L \) respectively, rather than \( ^+ R J_{(AB)} \) and \( ^- R J_{(AB)} \) or \( ^+ L J_{(AB)} \) and \( ^- L J_{(AB)} \), according to the \( SO(8) \)-decomposition
\[ SO(8) = S^7_R \times S^7_L \times G_2, \] (134)
which is a closer decomposition to (131) (see [12] for details). In this case the analogue of (133) will be
\[
R_{\mu\nu}^{(AB)} = R_{\mu\nu}^{(R)}(L_\omega) + R_{\mu\nu}^{(L)}(R_\omega),
\]
modulo the exceptional group \(G_2\). We should mention that just by convenience in our formalism above we wrote \(R_{\mu\nu}^{(R)}\) as \(R_{\mu\nu}^{(L)}\), but in general it is necessary to keep in mind the distinction between \(R_{\mu\nu}^{(R)}(R_\omega)\) and \(R_{\mu\nu}^{(L)}(L_\omega)\). What it is important is that one may choose to work either with the \(R_{\mu\nu}^{(R)}(R_\omega)\) sector or \(R_{\mu\nu}^{(L)}(L_\omega)\) sector of \(R_{\mu\nu}^{(AB)}\) in the group manifold \(SO(8)/G_2\).

7. Toward a background independent quantum gravity in eight dimensions and final comments

Having the canonical constraints (129) and (130) we become closer to our final goal of developing quantum gravity in eight dimensions. In fact in this section we shall outline possible quantum physical states \(|\Psi\rangle\) associated with the corresponding Hamiltonian operators \(\mathcal{H}'\) and \(\mathcal{H}'_l\) (associated with (129) and (130) respectively) via the expressions
\[
\mathcal{H}' |\Psi\rangle = 0 \quad \text{(136)}
\]
and
\[
\mathcal{H}'_l |\Psi\rangle = 0. \quad \text{(137)}
\]
Of course, even from the beginning one may have the feeling that the physical solutions of (136) and (137) will be more subtle than in the case of four dimensions. This is in part due to the fact that the topology in eight dimensions is less understood than in three or four dimensions. Nevertheless some progress in this direction has been achieved [24].

In order to describe the physical states, which solves (136) and (137), one may first write the canonical commutations relations:
\[
[\hat{A}^{(a)}_i(x), \hat{A}^{(b)}_j(y)] = 0,
\]
\[
[\hat{E}^{(a)}_i(x), \hat{E}^{(b)}_j(y)] = 0,
\]
\[
[\hat{E}^{(a)}_i(x), \hat{A}^{(b)}_j(y)] = \delta^i_j \delta^b_a \delta^7(x, y).
\]

22
Here, we have made the symbolic transition $+\omega_i^{(0a)} \rightarrow A_i^{(a)}$ and consider $A_i^{(a)}$ as a \textit{spin}(7) gauge field. We choose units such that $\hbar = 1$. It is worth mentioning that by introducing the analogue generalized determinant (107) for $E_i^{(a)}$ one may write the conjugate momentum $\hat{E}_i^{(a)}(x)$ explicitly in terms of $\hat{E}_i^{(a)}$. The next step is to choose a representation for the operators $\hat{A}_i^{(a)}$ and $\hat{E}_i^{(a)}$ of the form

\begin{equation}
\hat{A}_i^{(a)}\Psi(A) = A_i^{(a)}\Psi(A),
\end{equation}

\begin{equation}
\hat{E}_i^{(a)}\Psi(A) = \frac{\delta\Psi(A)}{\delta A_i^{(a)}}.
\end{equation}

Using these relations one discovers that the quantum constraints can be solved by Wilson loops wave functions

\begin{equation}
\Psi_\gamma(A) = tr P \exp \int_\gamma A
\end{equation}

labelled by the loops $\gamma$.

Of course these quantum steps are completely analogous to the case of four dimensions [25]-[27]. However they are necessary if one wants to go forward in our quantum program. We believe that interesting aspects in this process can arise if one looks for a physical states in terms of the analogue of the Chern-Simons states in four dimensions. The reason is because Chern-Simons theory is linked to instantons in four dimensions via the topological term $\int_M tr c^{\mu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$, while in eight dimensions the topological term should be of the form $\int_M tr \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}$. Surprisingly this kind of topological terms have already been considered in the literature in connection with $G_2$-instantons (see [23] and references therein).

The present work just describes the first steps towards the construction of background independent quantum gravity in eight dimensions. We certainly may have in the route many of the problems of the traditional Ashtekar formalism in four dimensions such as the issue of time. However one of the advantage that may emerge from the present formalism is the possibility to bring many new ideas from twelve dimensions via the transition $10 + 2 \rightarrow (3 + 1) + (7 + 1)$ [28]. In fact twelve dimensions is one of the most interesting proposals for building $M$-theory [29]. An example of this, Smolin [30]-[31] (see also Refs [32] and [33]) has described the possibility to construct background independent quantum gravity in the context of topological $M$-theory by obtaining Hitchin’s 7 seven dimensional theory, which in principle seems to admit background independent formulation, from the classical limit of $M$-theory, namely eleven
dimensional supergravity. The idea is focused on an attempt of reducing the eleven dimensional manifold $M^{1+10}$ in the form

$$M^{1+10} \rightarrow R \times \Sigma \times S^1 \times R^3.$$  

(141)

Here, $\Sigma$ is a complex six-dimensional manifold. Considering that the only degree of freedom is the gauge field three form $A$ which is pure gauge $A = d\beta$ and therefore locally trivial $dA = 0$, the Smolin’s conjecture is that the Hitchin’s action can be derived from the lowest dimensional term that can be made from $d\beta$ on $R \times \Sigma$ of the corresponding effective action (see Ref. [30] for details).

Observing that $\Sigma \times S^1$ is a seven dimensional manifold and since, via the octonion structure, the solution $0 + 8$ is related to the seven sphere solution of eleven dimensional supergravity one is motivated to conjecture that there must be a connection between our approach of incorporating Ashtekar formalism in the context of $M$-theory and the Smolin’s program. In turn, $M$-theory has motivated the study of many mathematical structures such as oriented matroid theory [34] (see Refs [35]-[39]). Thus we see as interesting physical possibility a connection between matroid theory and Ashtekar formalism. The reason for this is that symbols $\varepsilon^{\mu\nu\alpha\beta}$ and $\eta^{\mu\nu\alpha\beta}$ may be identified with two examples of four rank chirotopes [40] and therefore it is necessary to find a criterion for the uniqueness of these symbols from these perspective [41].

Finally, so far in this article we have focused on the Euclidean case via the possible representations for $SO(8)$. For further research it may be interesting to investigate the Lorentzian case associated with the group $SO(1,7)$. Since $SO(7)$ is a subgroup of $SO(1,7)$ one finds that (up to some modified numerical factors) most of the algebraic relations for octonions given in (89)-(92) are similar. For instance, the self-duality relation (89) should be modified with $\varsigma = \pm i$ instead of $\varsigma = \pm 1$. Thus, the discussion at the end of section 6 should be slightly modified. However, the transition from Euclidean to Lorentzian signature at the level of the action (85), and its corresponding quantum theory, may be more complicated. In this case the usual Wick rotation may be not enough procedure as in canonical gravity in four dimensions [43] and therefore it may be necessary to consider a modified action with free parameters controlling the signature of the spacetime.

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