Lepton Flavour Violating Leptonic/Semileptonic Decays of Charged Leptons in the Minimal Supersymmetric Standard Model

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ABSTRACT: We consider the leptonic and semileptonic (SL) lepton flavour violating (LFV) decays of the charged leptons in the minimal supersymmetric standard model (MSSM). The formalism for evaluation of branching fractions for the SL LFV charged-lepton decays with one or two pseudoscalar mesons, or one vector meson in the final state, is given. Previous amplitudes for the SL LFV charged-lepton decays in MSSM are improved, for instance the $\gamma$-penguin amplitude is corrected to assure the gauge invariance. The decays are studied not only in the model-independent formulation of the theory in the frame of MSSM, but also within the frame of the minimal supersymmetric $SO(10)$ model within which the parameters of the MSSM are determined. The latter model gives predictions for the neutrino-Dirac Yukawa coupling matrix, once free parameters in the model are appropriately fixed to accommodate the recent neutrino oscillation data. Using this un-ambiguous neutrino-Dirac Yukawa couplings, we calculate the LFV leptonic and SL decay processes assuming the minimal supergravity scenario. A very detailed numerical analysis is done to constrain the MSSM parameters. Numerical results for SL LFV processes are given, for instance for $\tau^- \to e^- (\mu^-) \pi^0$, $\tau^- \to e^- (\mu^-) \eta$, $\tau^- \to e^- (\mu^-) \eta'$, $\tau^- \to e^- (\mu^-) \rho^0$, $\tau^- \to e^- (\mu^-) \phi$, $\tau^- \to e^- (\mu^-) \omega$, etc.

KEYWORDS: Supersymmetry Phenomenology, Beyond Standard Model, GUT
1. Introduction

The neutrino oscillation experiments gave a first experimental evidence beyond the standard model (SM) of the electroweak interactions. In the SM, neutrinos are massless, purely left-handed particles, so there is no leptonic analogy of the Cabibbo-Kobayashi-Maskawa (CKM) matrix in the SM. The neutrino oscillation experiments proved that the neutrinos do mix and that they do have mass. The mixing matrix in the lepton sector, the Maki-Nakagawa-Sakata (MNS) matrix has bi-large mixing structure, indicating that the source of the lepton-flavour mixing is different from the corresponding mixing in the quark sector. The lepton-flavour mixing observed in neutrino oscillation is the first confirmation that
the lepton flavour is not conserved quantity. Therefore, experimental observation of the other lepton-flavour violating (LFV) processes is naturally expected. Theoretical study of such processes has a long history before the observation of neutrino oscillations. The model independent study of the operators using the SM fields [1, 2, 3] shows that there are no LFV operators of dimension less or equal to four. There is one dimension five LFV operator that induces neutrino oscillations. The LFV decays can be induced only with the operators of dimension six or more. As the new physics is expected to appear at the scale much larger than the electroweak scale \( \sim 246 \text{ GeV} \), the LFV decay effects are expected to be much more suppressed than the neutrino oscillation effects. Model independent study of the LFV processes gives the limits on LFV which every model has to satisfy. A model dependent analysis depends on the structure of the model, but are much more predictive than the corresponding model independent analysis. Therefore, the both approaches are indispensable for a theoretical study of LFV. Although the leptonic LFV processes have been studied extensively both in model independent way and using various models [4, 5, 6], the semileptonic (SL) LFV processes have been studied only in few models [7, 8, 9, 10, 11, 12, 13].

The supersymmetric (SUSY) models have much nicer theoretical properties than their non-SUSY counterparts. For example, quadratically divergent contributions to the Higgs boson mass from heavy (e.g. GUT scale) particles cancel with its SUSY partners and as a result the gauge-hierarchy problem is much better resolved. The supersymmetrization of the SM cannot be done without additional assumptions. For instance, in the supersymmetric version of the SM there are dimension four operators violating both the lepton number (\( \mathcal{L} \)) and the baryon number (\( \mathcal{B} \)), leading to very fast proton decay. That led us to the introduction of a discrete \( \mathbb{Z}_2 \) symmetry, so called, \( R \)-parity to forbid such undesirable terms. The SUSY breaking also has to be done in such a way not to induce too large flavour violation effects. There are few successful SUSY breaking mediation mechanisms, such as gravity mediation [14], gauge mediation [15], anomaly mediation [16], gaugino mediation [17], radion mediation [18], etc. The best established among them is the minimal supergravity model (mSUGRA) [14] which assumes that the SUSY breaking occurs in the hidden sector at very high scale, which communicates with the visible sector (containing SM) with flavour-blind gravitational interactions. The induced soft SUSY breaking mass terms are requested to be universal at the SUSY breaking mediation scale (say, the (reduced) Planck scale), and are therefore flavour-diagonal. The magnitude of the soft SUSY breaking mass terms obtained is in the range to induce potentially observable consequences in the visible sector. The renormalization group (RG) flow from the (reduced) Planck scale to the mass scale of the right-handed neutrinos, induces the flavour non-diagonal terms in the SUSY soft breaking terms for the sleptons, through the flavor-non diagonal Dirac-neutrino Yukawa matrices they contain [19]. They can lead to considerable LFV effects, which, depending on the model parameters can be in the range of the forthcoming LFV experiments [20].

In this paper, we assume the MSSM with three right-handed neutrinos as the low-energy effective theory below the GUT scale. In such a framework, neutrino oscillation data suggest the existence of very massive right-handed neutrinos which give rise to the
small left-handed neutrino masses through the see-saw mechanism [21]. In SO(10) models, the required right-handed neutrinos can be naturally embedded into the common multiplet together with the SM particles for each generation. In this paper, the minimal renormalizable SUSY SO(10) model [22, 23, 24, 25, 26, 27] will be taken as a starting theoretical frame. One of the advantageous points of this model is the automatically conserved $R$-parity defined as $R = (-1)^{3(B-L)+2S}$, where $S$ represents the spin of a field. Namely, the $SO(10)$ model discussed here spontaneously breaks the gauged $B - L$ symmetry by two units, leading to an automatic $R$-parity conservation. The breaking of $SO(10)$ group to the SM gauge group, $SU(3)_C \times SU(2)_L \times U(1)_Y$ [28, 29] and its phenomenological consequences [6, 30, 31] has already been discussed in our previous publications.

The main goal of this paper is an analysis of neutrinoless SL LFV decays of charged leptons within the MSSM model where parameters are obtained from the underlying $SO(10)$ model. At the same time we intend to see how the previous phenomenological analyses constrain the LFV parameters. The paper consists basically of three parts which are given in three sections. In section two, we give the MSSM form factors comprised in the LFV amplitudes at the quark-lepton level. We rederive these form factors, because some of them were not derived completely in the previous literature. In section three, the charged-lepton SL LFV amplitudes at lepton-meson level are derived using a simple hadronization procedure for the quark currents. The branching ratios corresponding to these amplitudes are also given. In section four, the minimal renormalizable SUSY $SO(10)$ model is described, the parameters of the MSSM model are derived from the $SO(10)$ model, too. Using the predicted Yukawa couplings in the minimal $SO(10)$ model, we perform a numerical estimation for the SL LFV processes. The last section is devoted to a summary. In Appendix A, we give our notation for the mass matrices of the neutralino, chargino and sfermions. The MSSM Lagrangian for fermion-sfermion-(gaugino, Higgsino) interaction and the trilinear interactions with $Z$ boson are given in Appendix B and C, respectively. In appendix D we present the loop functions needed to evaluate the SL LFV processes. The quark content of meson states, essential for the hadronization of quark currents, is listed in Appendix E, together with constants that define the hadronized quark current in $\gamma$-penguin and $Z$-boson-penguin amplitude.

2. Effective Lagrangian for LFV Interactions

2.1 Sources for LFV Interactions

Even though the soft SUSY breaking parameters are flavour blind at the scale of the SUSY breaking mediation, the LFV interactions in the model can induce the LFV sources at low-energy through the renormalization effects. In the following analyses we assume the mSUGRA scenario [14] as the SUSY breaking mediation mechanism. At the original scale of the SUSY breaking mediation we impose the boundary conditions on the soft SUSY breaking parameters, which are characterized by only five parameters: $m_0$, $M_{1/2}$, $A_0$, $B$ and $\mu$. Here, $m_0$ is the universal scalar mass, $M_{1/2}$ is the universal gaugino mass, and $A_0$ is the universal coefficient of the trilinear couplings. The parameters in the Higgs potential, $B$ and $\mu$, are determined at the electroweak scale so that the Higgs doublets obtain the correct
electroweak symmetry breaking VEV’s through the radiative breaking scenario \[32\]. The soft SUSY breaking parameters at low energies are obtained through the RGE evolutions with the boundary conditions at the GUT scale.

Although the SUSY breaking mediation scale is normally taken to be the (reduced) Planck scale or the string scale (\(\sim 10^{18} \text{ GeV}\)), in the following calculations we impose the boundary conditions at the GUT scale (\(\sim 10^{16} \text{ GeV}\)), and analyze the RGE evolutions from the GUT scale to the electroweak scale. This ansatz is the same as the one in the so-called constrained MSSM (CMSSM).

The effective theory which we analyze below the GUT scale is the MSSM with the right-handed neutrinos. The superpotential in the leptonic sector is given by

\[
W_Y = Y_u^{ij}(u_R^c)_{ij}q_Hu + Y_d^{ij}(d_R^c)_{ij}q_Hd \\
+ Y_\nu^{ij}(\nu_R^c)_{ij}e_Hu + Y_e^{ij}(e_R^c)_{ij}e_Hd + \frac{1}{2}M_{Rij}(\nu_R^c)_{ij}(\nu_R^c)_{ij} + \mu H_dH_u ,
\]

(2.1)

where the indices \(i, j\) run over three generations, \(H_u\) and \(H_d\) denote the up-type and down-type MSSM Higgs doublets, respectively, and \(M_{Rij}\) is the heavy right-handed Majorana neutrino mass matrix. We work in the basis where the charged-lepton Yukawa matrix \(Y_e\) and the mass matrix \(M_{Rij}\) are real-positive and diagonal matrices: \(Y_e^{ij} = Y_e \delta_{ij}\) and \(M_{Rij} = \text{diag}(M_{R1}, M_{R2}, M_{R3})\). Thus, the LFV is originated from the off-diagonal components of the neutrino-Dirac-Yukawa coupling matrix \(Y_\nu\). The soft SUSY breaking terms in the leptonic sector is described as

\[
-L_{\text{soft}} = \bar{\nu}_i^c (m_{\nu}^2)_{ij} \nu_j + \bar{u}^c_{Ri} (m_u^2)_{ij} u_R + \bar{d}^c_{Ri} (m_d^2)_{ij} d_R \\
+ \bar{\ell}_i (m_e^2)_{ij} \ell_j + \bar{\nu}_{Ri} (m_\nu^2)_{ij} \nu_R + \bar{e}^c_{Ri} (m_e^2)_{ij} e_R \\
+ m_{H_u}^2 H_u^\dagger H_u + m_{H_d}^2 H_d^\dagger H_d + \left( B_{\mu} H_d H_u + \frac{1}{2} B_\nu M_{Rij} \nu_R^{i\dagger} \nu_R^j + h.c. \right) \\
+ \left( A_{\nu}^{ij} \bar{\nu}_{Ri} \nu_j H_u + A_{\nu}^{ij} \bar{\nu}_{Ri} \nu_j H_d + h.c. \right) \\
+ \left( A_{\nu}^{ij} \bar{\nu}^{i\dagger}_{Ri} \nu_j H_u + A_{\nu}^{ij} \bar{\nu}^{i\dagger}_{Ri} \nu_j H_d + h.c. \right) \\
+ \left( \frac{1}{2} M_1 \tilde{B} \tilde{B} + \frac{1}{2} M_2 \tilde{W}^a \tilde{W}^a + \frac{1}{2} M_3 \tilde{G}^a \tilde{G}^a + h.c. \right) .
\]

(2.2)

As discussed above, we impose the universal boundary conditions at the GUT scale such that

\[
(m_{\nu}^2)_{ij} = (m_u^2)_{ij} = (m_d^2)_{ij} = m_0^2 \delta_{ij} ,
\]

\[
(m_e^2)_{ij} = (m_\nu^2)_{ij} = (m_e^2)_{ij} = m_0^2 \delta_{ij} ,
\]

\[
m_{H_u}^2 = m_{H_d}^2 = m_0^2 ,
\]

\[
A_u^{ij} = A_0 Y_u^{ij} , \quad A_d^{ij} = A_0 Y_d^{ij} ,
\]

\[
A_\nu^{ij} = A_0 Y_\nu^{ij} , \quad A_\nu^{ij} = A_0 Y_\nu^{ij} ,
\]

\[
M_1 = M_2 = M_3 = M_{1/2} ,
\]

(2.3)
and evolve the soft SUSY breaking parameters to the electroweak scale according to their RGE’s. The $\mu$ parameter and the $B$ parameter are determined at the electroweak scale so as to minimize the Higgs potential,

$$|\mu|^2 = \frac{m_{H_d}^2 - m_{H_u}^2 \tan^2 \beta}{\tan^2 \beta - 1} - \frac{1}{2} M_Z^2,$$

$$B\mu = -\frac{1}{2} (m_{H_d}^2 + m_{H_u}^2 + 2|\mu|^2) \sin 2\beta \left( = \frac{1}{2} m_A^2 \sin 2\beta \right). \quad (2.4)$$

The LFV sources in the soft SUSY breaking parameters such as the off-diagonal components of $(m_{\tilde{\ell}}^2)_{ij}$ and $A_{\ell i}^0$ are induced by the LFV interactions through the neutrino Dirac Yukawa couplings. For example, the LFV effect most directly emerges in the left-handed slepton mass matrix through the RGE’s such as

$$\mu \frac{d}{d\mu} (m_{\tilde{\ell}}^2)_{ij} = \mu \frac{d}{d\mu} (m_{\tilde{\ell}}^2)_{ij} \big|_{\text{MSSM}} + \frac{1}{16\pi^2} \left( m_{\tilde{\ell}}^2 Y^\dagger Y + Y^\dagger Y m_{\tilde{\ell}}^2 + 2Y^\dagger m_{\tilde{\ell}}^2 Y^\dagger + 2m_{\tilde{\ell}}^2 Y^\dagger + 2A_{\ell i}^0 A_{\ell j}^0 \right)_{ij} \quad (2.5)$$

where the first term on the right-hand side denotes the MSSM term with no LFV. In the leading-logarithmic approximation, the off-diagonal components ($i \neq j$) of the left-handed slepton mass matrix are estimated as

$$(\Delta m_{\tilde{\ell}}^2)_{ij} \sim -\frac{3m_0^2 + A^2_0}{8\pi^2} \left( Y^0 L Y \right)_{ij}, \quad (2.6)$$

where the distinct thresholds for the right-handed Majorana neutrinos are taken into account by the matrix $L_{ij} = \log \left( \frac{M_{\nu_R}}{M_{\tilde{\ell}_i}} \right) \delta_{ij}$. We can see that the neutrino-Dirac-Yukawa coupling matrix plays the crucial role in calculations of the LFV processes.

### 2.2 Effective Lagrangian in terms of quark fields and LFV form factors

In any model containing standard model as the low-energy effective theory, an effective Lagrangian for the SL LFV decays of a lepton contains only three terms: the photon-penguin, the $Z$-boson-penguin and the box term,

$$i\mathcal{L}_{\text{eff}} (\ell_i \rightarrow \ell_j + \bar{q} + q') = i\mathcal{L}^\gamma_{\text{eff}} + i\mathcal{L}^Z_{\text{eff}} + i\mathcal{L}^{\text{box}}_{\text{eff}}. \quad (2.7)$$

These terms have the following generic structure,

$$i\mathcal{L}^\gamma_{\text{eff}} (x) = -ie^2 \int d^4 y \bar{\ell}_j(x) \left[ (-\partial^2 \gamma_{\mu} + \phi_x \partial_x \phi_{x\mu}) D(x-y)(\mathcal{P}^L_{1\gamma} \mathcal{P}_L + \mathcal{P}^R_{1\gamma} \mathcal{P}_R) \right. \left. + \sigma_{\mu \nu} D(x-y)(\mathcal{P}^{L\nu}_{2\gamma} \mathcal{P}_L + \mathcal{P}^{R\nu}_{2\gamma} \mathcal{P}_R) \right] \ell_i(x) \times \sum_{q=u,d,s} Q_q \bar{q}(y) \gamma^\mu q(y), \quad (2.8)$$

$$i\mathcal{L}^Z_{\text{eff}} (x) = i\frac{g^2}{m_Z^2} \bar{\ell}_j(x) \gamma_{\mu} (\mathcal{P}^L_Z \mathcal{P}_L + \mathcal{P}^R_Z \mathcal{P}_R) \ell_i(x)$$
\[ i \mathcal{L}_{\text{eff}}^{\text{box}} = i \sum_{q_u, q_d, s} \bar{q}(i \sigma_{\mu\nu} P_L \ell_i)(\bar{q}_a \gamma_\mu P_L q_b) + \bar{q}(i \sigma_{\mu\nu} P_R \ell_i)(\bar{q}_a P_R q_b) \]

\[ \times \left[ B_{\ell_i q_u q_b}^{L}(\bar{q}_a \gamma_\mu P_L \ell_i)(\bar{q}_a \gamma_\mu P_L q_b) + B_{\ell_i q_u q_b}^{R}(\bar{q}_a \gamma_\mu P_R \ell_i)(\bar{q}_a \gamma_\mu P_R q_b) \right], \]

where \( s_W = \sin \theta_W \) and \( c_W = \cos \theta_W \), \( Q_q \) is the quark charge in units of \( e \), and \( I_{3q} \) is weak quark isospin. \( D(x - y) \) is the Green function for the massless scalar particle, contained in the photon propagator. The structure of the photon-penguin term in the effective Lagrangian is a consequence of the gauge invariance. Especially, the first term contained in the photon propagator. The structure of the photon-penguin term in the form factors is contained in the form factors.

2.2.1 The photon-penguin form factors

The amplitude for \( \ell_i \rightarrow \ell_j \gamma^* \) for an off-mass-shell photon process is obtained from the corresponding part of the effective Lagrangian neglecting the quark current and the photon propagator,

\[ \mathcal{M}_{\mu} = i T_{\mu} = -e \bar{\ell}_j \left[ (q^2 \gamma_\mu - q_u \gamma_\mu \gamma_5)(P_{1\gamma} + P_{1\gamma}^R) + i \sigma_{\mu\nu} q^\nu (P_{2\gamma}^L + P_{2\gamma}^R) \right] u_\ell_i. \]

The amplitude is written without photon polarization vector.

In the MSSM the photon-penguin amplitude has two contributions, a chargino and a neutralino contribution. That reflects in the structure of the form factors,

\[ P_{a\gamma}^{L,R} = P_{a\gamma}^{(C)L,R} + P_{a\gamma}^{(N)L,R}, \quad a = 1, 2 \]

with \( C \) and \( N \) subscript denoting the chargino and neutralino part of a form factor.

Because of the gauge invariance, the zeroth-order and the first-order term in Taylor expansion in momenta and masses of incoming and outgoing particles are equal zero. Here, the second order term in Taylor expansion is presented, and higher order terms are neglected.

The neutralino contributions are

\[ P_{1\gamma}^{(N)L} = \frac{i}{576 \pi^2} N_{AX}^R N_{iAX}^R \left( \frac{1}{m_{\tilde{e}_X}^2} \right) \times \frac{11(x_{AX}^0)^3 - 18(x_{AX}^0)^2 + 9(x_{AX}^0) - 2 - 6(x_{AX}^0)^3 \ln(x_{AX}^0)}{(1 - x_{AX}^0)^4}, \]

\[ P_{1\gamma}^{(N)R} = P_{1\gamma}^{(N)R} |_{L \leftrightarrow R}, \]
\[ \mathcal{P}_{2\gamma}^{(N)L} = \frac{i}{32\pi^2} \left[ N_{jAX}^{R(e)} N_{\chi_i AX}^{R(e)*} (-1)m_j \frac{1}{m_{\tilde{x}_X}^2} \right. \\
\times 2(x_{AX}^0)^3 + 3(x_{AX}^0)^2 - 6(x_{AX}^0) + 1 - 6(x_{AX}^0)^2 \ln(x_{AX}^0) \\
\left. \quad \times 6(1 - x_{AX}^0)^4 \right] \\
+ N_{jAX}^{L(e)} N_{\chi_i AX}^{L(e)*} (-1)m_i \frac{1}{m_{\tilde{x}_X}^2} \right. \\
\times 2(x_{AX}^0)^3 + 3(x_{AX}^0)^2 - 6(x_{AX}^0) + 1 - 6(x_{AX}^0)^2 \ln(x_{AX}^0) \\
\left. \quad \times 6(1 - x_{AX}^0)^4 \right], \quad (2.15) \\
\mathcal{P}_{2\gamma}^{(N)R} = \mathcal{P}_{2\gamma}^{(N)L} |_{L \rightarrow R}, \quad (2.16) \\
\]

where \( x_{AX}^0 = M_{\chi_A}^2 / m_{\tilde{x}_X}^2 \). The chargino contributions are given by

\[ \mathcal{P}_{1\gamma}^{(C)L} = \frac{i}{576\pi^2} C_{jAX}^{R(e)} C_{\chi_i AX}^{R(e)*} \frac{1}{m_{\tilde{\nu}_X}^2} \]
\[ \times \left[ 16 - 45(x_{AX}^-) + 36(x_{AX}^-)^2 - 7(x_{AX}^-)^3 + 6(2 - 3(x_{AX}^-)) \ln(x_{AX}^-) \right], \]
\[ \mathcal{P}_{1\gamma}^{(C)R} = \mathcal{P}_{1\gamma}^{(C)L} |_{L \rightarrow R}, \quad (2.17) \]

\[ \mathcal{P}_{2\gamma}^{(C)L} = \frac{i}{32\pi^2} \left[ C_{jAX}^{R(e)} C_{\chi_i AX}^{R(e)*} \frac{1}{m_{\tilde{\nu}_X}^2} \right. \\
\times \left[ 2 + 3(x_{AX}^-) - 6(x_{AX}^-)^2 + (x_{AX}^-)^3 + 6(x_{AX}^-) \ln(x_{AX}^-) \right] \\
\left. \quad \times 6(1 - x_{AX}^-)^4 \right] \\
+ C_{jAX}^{L(e)} C_{\chi_i AX}^{L(e)*} \frac{1}{m_{\tilde{\nu}_X}^2} \right. \\
\times \left[ 2 + 3(x_{AX}^-) - 6(x_{AX}^-)^2 + (x_{AX}^-)^3 + 6(x_{AX}^-) \ln(x_{AX}^-) \right] \\
\left. \quad \times 6(1 - x_{AX}^-)^4 \right], \quad (2.18) \\
\mathcal{P}_{2\gamma}^{(C)R} = \mathcal{P}_{2\gamma}^{(C)L} |_{L \rightarrow R}, \quad (2.19) \\
\]

where \( x_{AX}^- = M_{\chi_A}^2 / m_{\tilde{\nu}_X}^2 \). The form factor contributions are written in the same way as in Ref. [20], including the loop functions into the expressions, to make the comparison with the results of Ref. [20] easy. Both chargino and neutralino parts of the form factors agree with the corresponding form factors in the reference [20] if the terms proportional to the mass of the lighter mesons \( m_j \) are neglected. Nevertheless, these terms cannot be neglected, because the constants \( N_{jAX}^{L,R(e)} \) and \( C_{jAX}^{L,R(e)} \) (see Appendix B) also depend on the lepton masses is such a way that in some cases the term proportional to \( m_j \) is larger than the term proportional to the mass of decaying lepton \( m_i \).
2.2.2 The Z-penguin form factors

The amplitude for $\ell_i \rightarrow \ell_j Z^*$, for the off-mass-shell Z-boson process, (obtained analogously as the photon-penguin amplitude) is

$$\mathcal{M}_\mu^Z = \frac{i}{(4\pi)^2} g_{\mu\nu} \left[ \gamma_\mu P_L(\mathcal{P}_Z^{(C)L} + \mathcal{P}_Z^{(N)L}) + \gamma_\mu P_R(\mathcal{P}_Z^{(C)R} + \mathcal{P}_Z^{(N)R}) \right] u_i, \quad (2.21)$$

where $\mathcal{P}_Z^{(C)L,R}$ and $\mathcal{P}_Z^{(N)L,R}$ are the chargino and neutralino part of the total form factors, $\mathcal{P}_Z^{L,R}$. Here are the expressions for these form factors:

$$\mathcal{P}_Z^{(C)L} = C_{jBX}^{R(e)} C_{iAX}^{R(e)*} \left[ E_{BA}^{L(\tilde{\chi}^-)}(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) \right]$$

$$- 2E_{BA}^{R(\tilde{\chi}^-)} F_2(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) + \delta ABC Z_{e_j} f_1(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) \right]$$

$$+ \left\{ C_{jBX}^{R(e)} E_{BA}^{L(\tilde{\chi}^-)}(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) \right\}, \quad (2.22)$$

$$\mathcal{P}_Z^{(N)L} = N_{jBY} N_{iAX}^{R(e)*} \left[ - 2D_{YX}^{(e)} F_2(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) + \delta ABC Z_{e_j} f_1(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) \right]$$

$$+ \left\{ N_{jBX}^{L(\tilde{\chi}^-)} E_{BA}^{R(e)}(m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_A}^2, m_{\tilde{\chi}_B}^2) \right\}, \quad (2.23)$$

$$\mathcal{P}_Z^{(C)R} = \mathcal{P}_Z^{(C)L}(L \leftrightarrow R), \quad (2.24)$$

$$\mathcal{P}_Z^{(N)R} = \mathcal{P}_Z^{(N)L}(L \leftrightarrow R). \quad E_{BA}^{R(\tilde{\chi}^0)} = - E_{BA}^{L(\tilde{\chi}^0)}. \quad (2.25)$$

$G_{Z\tilde{e}}$ and $G_{Z\tilde{e}}^R$ are constants appearing in the SM $Ze_i e_j$ vertices,

$$L_{Ze_i e_j} = - g\gamma_\mu \delta_{ij} \left[ G_{Z\tilde{e}}^L P_L + G_{Z\tilde{e}}^R P_R \right]$$

$$- g\gamma_\mu \delta_{ij} \left\{ - \frac{1}{2} \frac{2}{c_W} s_W^2 \right\} P_L + \left[ \frac{s_W^2}{c_W} \right] P_R \right\}, \quad (2.26)$$

while $E_{BA}^{R(\tilde{\chi}^-)}$ and $E_{BA}^{L(\tilde{\chi}^0)}$ are constants in the Z-boson—chargino and Z-boson—neutralino vertices, and $D_{YX}^{(e)}$ is a constant in the Z-boson—selectron vertex. These constants are defined in Appendix C. $F_1(a, b, c)$ and $F_2(a, b, c)$ are loop functions contained in the triangle-diagram part of the amplitude, and $f_1$ and $f_2$ are the loop functions coming from the self-energy part of the amplitude. They are given in Appendix D.

The terms in Eqs. (2.22) and (2.23) which have a corresponding contribution in the photon amplitude (the leading order photon-penguin amplitude comes from six Feynman diagrams, while the Z-boson-penguin amplitude has eight Feynman-diagram contributions) have been compared by replacing Z-boson vertices with corresponding photon vertices, and an agreement was found. The remaining two Feynman diagram contributions which are embraced by curly brackets in Eqs. (2.22) and (2.23) have been checked carefully. The new terms in $\mathcal{P}_Z^{(C)\ell}$ in comparison with Ref. [20] are third (self-energy—type term) and fourth term. Further, neither of our terms in $\mathcal{P}_Z^{(C)R}$ does agree with amplitude in Ref. [20], although the expression in the curly brackets is almost equal to it.
2.2.3 The box form factors

The box contribution to the SL LFV $\ell \to \ell', q_bq_b$ amplitude comes from two box-diagrams in the leading order of perturbation theory. The box-amplitude reads

$$\mathcal{M}_{box} = \frac{i}{(4\pi^2)} \sum_{q_aq_b=\bar{u}d, dd, s\bar{s}, d\bar{s}} \left[ B_{\bar{q}_aq_b}^{L}(\bar{u}_{\ell}\gamma^\mu PLu_{\ell})(\bar{u}_{q_a}\gamma_\mu PLv_{q_b}) + B_{\bar{q}_aq_b}^{R}(\bar{u}_{\ell}\gamma^\mu PRu_{\ell})(\bar{u}_{q_a}\gamma_\mu PRv_{q_b}) ight. \\
+ B_{3\bar{q}_aq_b}^{L}(\bar{u}_{\ell}\gamma^\mu PLu_{\ell})(\bar{u}_{q_a}\gamma_\mu PRv_{q_b}) + B_{3\bar{q}_aq_b}^{R}(\bar{u}_{\ell}\gamma^\mu PRu_{\ell})(\bar{u}_{q_a}\gamma_\mu PLv_{q_b}) \\
+ B_{4\bar{q}_aq_b}^{L}(\bar{u}_{\ell}\gamma^\mu PLu_{\ell})(\bar{u}_{q_a}\gamma_\mu PLv_{q_b}) + B_{4\bar{q}_aq_b}^{R}(\bar{u}_{\ell}\gamma^\mu PRu_{\ell})(\bar{u}_{q_a}\gamma_\mu PRv_{q_b}) \\
+ B_{5\bar{q}_aq_b}^{L}(\bar{u}_{\ell}\gamma^\mu PLu_{\ell})(\bar{u}_{q_a}\gamma_\mu PLv_{q_b}) + B_{5\bar{q}_aq_b}^{R}(\bar{u}_{\ell}\gamma^\mu PRu_{\ell})(\bar{u}_{q_a}\gamma_\mu PRv_{q_b}) \\
\left. + B_{6\bar{q}_aq_b}^{L}(\bar{u}_{\ell}\gamma^\mu PLu_{\ell})(\bar{u}_{q_a}\gamma_\mu PLv_{q_b}) + B_{6\bar{q}_aq_b}^{R}(\bar{u}_{\ell}\gamma^\mu PRu_{\ell})(\bar{u}_{q_a}\gamma_\mu PRv_{q_b}) \right].$$

(2.27)

The very rich structure of the box-diagram amplitude is a consequence of the Fierz transformation of the terms with product of lepton-quark and quark-lepton vector and axial-vector currents. All currents permitted by the Dirac algebra do appear. Each box-amplitude form factor has a chargino (C) and a neutralino (N) contribution.

$$B_{\bar{q}_aq_b}^{L,R} = B_{\bar{q}_aq_b}^{(N)L,R} + B_{\bar{q}_aq_b}^{(C)L,R},$$

(2.28)

$$B_{3\bar{q}_aq_b}^{L,R} = B_{3\bar{q}_aq_b}^{(N)L,R} + B_{3\bar{q}_aq_b}^{(C)L,R}.$$  

(2.29)

Here and in the following equations $\bar{q}_aq_b$ assume the values appearing in the sum in Eq. (2.27). Neutralino contributions read

$$B_{1\bar{q}_aq_b}^{(N)L} = \frac{1}{4} d_2(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \\
+ \frac{1}{2} d_0(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) M_{\chi^0_A}^0 M_{\chi^0_B}^0 N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)},$$

(2.30)

$$B_{2\bar{q}_aq_b}^{(N)L} = -\frac{1}{4} d_2(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \\
- \frac{1}{2} d_0(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) M_{\chi^0_A}^0 M_{\chi^0_B}^0 N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)},$$

(2.31)

$$B_{3\bar{q}_aq_b}^{(N)L} = d_0(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) M_{\chi^0_A}^0 M_{\chi^0_B}^0 \left\{ - \frac{1}{2} N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \\
- \frac{1}{2} N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \right\},$$

(2.32)

$$B_{4\bar{q}_aq_b}^{(N)L} = \frac{1}{2} d_2(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) \left\{ - \frac{1}{2} N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \\
- \frac{1}{2} N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \right\},$$

(2.33)

$$B_{5\bar{q}_aq_b}^{(N)L} = \frac{1}{8} d_0(M^2_{\chi^0_A}, M^2_{\chi^0_B}, m_{\tilde{e}_X}^2, m_{\tilde{\chi}_0^0}^2) M_{\chi^0_A}^0 M_{\chi^0_B}^0 \left\{ N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \\
- N_{iAX}^{R(e)} N_{jBX}^{R(e)} N_{bBY}^{R(e)} N_{aAY}^{R(e)} \right\},$$

(2.34)

$$B_{1\bar{q}_aq_b}^{(N)R} = B_{1\bar{q}_aq_b}^{(N)L} \bigg|_{L \to R} \quad (i = 1, \ldots, 4)$$

(2.35)

$$B_{3\bar{q}_aq_b}^{(N)R} = B_{3\bar{q}_aq_b}^{(N)L} \bigg|_{L \to R}. \quad (2.36)$$
The chargino contributions are

\[
\mathcal{B}^{(C)L}_{1q_{a}q_{b}} = \frac{1}{4} d_2 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) C_{iAX}^R C_{jBX}^R C_{aBY}^{R(q)_*} C_{bAY}^{R(q)} \delta_{qd} \\
+ \frac{1}{2} d_0 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) M_{\chi_A} M_{\chi_B} C_{iAX}^R C_{jBX}^R C_{aBY}^{R(q)_*} C_{bAY}^{R(q)} \delta_{q'u},
\]

\[
\mathcal{B}^{(C)L}_{2q_{a}q_{b}} = -\frac{1}{4} d_2 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{qd} \\
- \frac{1}{2} d_0 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) M_{\chi_A} M_{\chi_B} C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{q'u},
\]

\[
\mathcal{B}^{(C)L}_{3q_{a}q_{b}} = d_0 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) M_{\chi_A} M_{\chi_B} \left\{ -\frac{1}{2} C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{qd} \\
- \frac{1}{2} C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{q'u} \right\},
\]

\[
\mathcal{B}^{(C)L}_{3q_{a}q_{b}} = d_2 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) \left\{ -\frac{1}{2} C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{qd} \\
- \frac{1}{2} C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{q'u} \right\},
\]

\[
\mathcal{B}^{(C)L}_{4q_{a}q_{b}} = \frac{1}{8} d_0 (M_{\chi_A}^2, M_{\chi_B}^2, m_{\bar{F}_X}^2, m_{\bar{q}'_Y}^2) M_{\chi_A} M_{\chi_B} \left\{ C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{qd} \\
- C_{iAX}^R C_{jBX}^R C_{aBY}^{L(q)_*} C_{bAY}^{L(q)} \delta_{q'u} \right\},
\]

\[
\mathcal{B}^{(C)L}_{4q_{a}q_{b}} = \mathcal{B}^{(C)L}_{1q_{a}q_{b}} |_{L \to R} \quad (i = 1, \ldots, 4)
\]

where for \( q = u(d) \), \( q' = d(u) \). The sum over paired indices is assumed. The Kronecker function \( \delta_{q'u} \) denotes that \( q \) quark is one of up (\( u \), \( c \) or \( t \)) quarks [one of the down quarks]. The loop functions \( d_0 \) and \( d_2 \) are evaluated by neglecting the momenta of incoming and outgoing particles. They are listed in Appendix D.

### 3. Amplitudes and branching ratios

#### 3.1 Hadronization of currents

The effective Lagrangians (the matrix elements) for photon and Z-boson part of the amplitudes for SL LFV lepton decays comprise vector and axial-vector currents, but box amplitude contains all possible quark currents permitted by Dirac algebra, that is scalar-, pseudoscalar-, vector-, axial-vector- and tensor-quark currents. To perform calculation of charged-lepton SL LFV decay rates these currents have been converted into meson currents comprising mesons that appear in the possible final products of the charged-lepton SL LFV decays we study here. The hadronization procedure we use here is not exact in the sense that we do not include the sea-quark and gluon content of the meson fields, but it is precise enough to give much better than order of magnitude decay rates of the processes considered here. The quark content of the meson states is given in Appendix E. The hadronization of axial-vector current is achieved through PCAC (see e.g. (34, 35, 36)); for normalization of pseudoscalar coupling constants used here and for further details see (37)). The hadronization of the vector current is achieved using vector-meson-dominance assumption (see (35, 36)); for normalization of the vector meson-decay constant and details
see [37, 38]). Hadronization of scalar currents is achieved by comparing the quark sector of the SM Lagrangian and the corresponding effective meson Lagrangian [38] (for applications in content of LFV and details see [8]). Hadronization of the pseudoscalar current is obtained by the same procedure as for the scalar current. The results obtained by using this procedure is equal to the result obtained by using equation of motion for current-quark masses (see e.g. [39]) and results for hadronization of axial-vector current up to the difference of up and down quark masses or up to the difference of pseudoscalar decay constants. The hadronization of the tensor-quark currents is obtained by comparing the derivative of tensor-quark current with vector-quark current and using the equations of motion for current quark masses. The difference between terms, one containing the derivative of the incoming quark field and the other containing derivative of the outgoing quark field, have been neglected. The error expected from this approximation is proportional to the amount of breaking of $SU(3)_{\text{flavour}}$ symmetry. The tensor currents are proportional to the current quark masses, and therefore give smaller contribution than the other quark currents. Therefore, the error introduced by this approximation in the total SL LFV amplitude is small.

Here we summarize the basic quantities needed to describe the hadronization of quark currents.

1. The pseudoscalar meson decay constants [43] ($f_P$, $P = \pi^0, \eta, \eta', K^0, \bar{K}^0$);
2. The constants $\gamma_V$ [37] ($V = \rho^0, \phi, \omega, K^*, \bar{K}^*$) defining the vector meson decay constants ($f_V \sim m_V^2/\gamma_V$);
3. The mixing angles $\theta_P$ and $\theta_V$ [43] defining the physical meson-nonet states in terms of the unphysical singlet and octet meson states;
4. The parameter $r$ [38] ($m_u, m_d$ and $m_s$ are current quark masses),

$$r = \frac{2m_{\pi^+}^2}{m_u + m_d} = \frac{2m_{K^0}^2}{m_d + m_s} = \frac{2m_{K^+}^2}{m_u + m_s},$$

that appears in the hadronization procedure for scalar and pseudoscalar currents.

Having the identification of the quark currents with the corresponding meson currents, achieved by the above hadronization procedure, one can write down the effective Lagrangian as a sum of terms with an incoming lepton field $\ell_i$ and outgoing lepton field $\ell_j$ and pseudoscalar meson (P) or vector meson (V) field(s). This Lagrangian directly gives the amplitudes for the $\ell_i \rightarrow \ell_j P(V)$ processes. Amplitudes with a pseudoscalar meson in the final state contributions come from the pseudoscalar and axial-vector coupling part of the effective Lagrangian, while the amplitudes with vector mesons have vector and tensor coupling contributions. Only the scalar coupling gives no contribution to the one-meson processes in the final state, $\ell_i \rightarrow \ell_j P(V)$. They contribute only to the processes with two pseudoscalar mesons in the final state, $\ell_i \rightarrow \ell_j P_1 P_2$.

### 3.2 Vector-meson–pseudoscalar-meson interactions

The processes with two pseudoscalar mesons in the final state are generated by the scalar-quark-current part of the effective Lagrangian, and vector-quark- and tensor-quark-current of the effective Lagrangian. The scalar-quark-current part of the effective Lagrangian
produces two pseudoscalar fields directly. The vector-quark- and tensor-quark-current parts produce a resonant vector meson state \((V)\), which decays into two pseudoscalar mesons \((P)\). The \(VPP\) interactions necessary for description of the \(VPP\) interactions appearing in the charged-lepton SL LFV decays are described by the part of the meson Lagrangian containing these \(VPP\) vertices \cite{8}

\[
\mathcal{L}_{VPP} = -\frac{ig_{\rho\pi}}{2} \left\{ \rho^{0,\mu} \left( 2\pi^{+\mu} \partial^{\mu} \pi^{-} + K^{+\mu} \partial^{\mu} K^{-} - K^{0\mu} \partial^{\mu} K^{0} \right) \\
+ \sqrt{3} s_v \omega^{\mu} (K^{+\mu} K^{-} + K^{0\mu} K^{0}) + \sqrt{3} c_v \phi^{\mu} (K^{+\mu} K^{-} + K^{0\mu} K^{0}) \\
+ K^{0,\mu} \left( -\sqrt{2} \pi^{+\mu} \partial^{\mu} K^{-} + \pi^{0\mu} \partial^{\mu} K^{0} \right) + \sqrt{3} c_p K^{0\mu} \partial^{\mu} \eta + \sqrt{3} s_p K^{0\mu} \partial^{\mu} \eta' \right\} + \ldots \tag{3.2}
\]

This Lagrangian is a part of the nonlinear \((U(3)_{L} \times U(3)_{R})/U(3)_{V}\) symmetric sigma-model Lagrangian. \(U(3)_{V}\) symmetry corresponds to the vector mesons in the linear realization of the gauge equivalent \((U(3)_{L} \times U(3)_{R})_{\text{global}} \times U(3)_{V}\) linear sigma-model \cite{41,40}. One can include the \((U(3)_{L} \times U(3)_{R})/U(3)_{V}\) breaking terms, too \cite{41}. That was applied to SL LFV tau-lepton decays in Ref. \cite{2}, but for the estimates of the charged-lepton SL LFV decays it is unnecessary complication, and we will not consider it here.

From Eq. (3.2) one can read the \(c_{VP_{1}P_{2}}\) couplings in terms of \(g_{\rho\pi}\) coupling. For instance, \(c_{\rho K^{+}K^{-}} = \frac{1}{2} g_{\rho\pi}\).

When the amplitudes with vector-meson resonance(s) are formed, the square of the vector meson mass in the \(m_{V}^{2}/\gamma_{V}\), appearing in every vector meson decay constant, has to be replaced with \((m_{V}^{2} - im_{V}^{\gamma V})/\gamma_{V}\), where \(\gamma_{V}\) is the decay width of the vector meson \cite{2}.

### 3.3 Charged-lepton SL LFV with one meson in the final state

Now we can write down all amplitudes we are interested in. (Notice that the axial-vector mesons \((A)\) are not included. They decay into three pseudoscalar mesons, and therefore the amplitudes are much more complicated. New vertices with \(A - V - P\) couplings should be included, kinematics is much more involved \(etc\). The scalar mesons are also not included. They also lead to complications.). They are given as follows.

\[
i\mathcal{M}_{\ell_{i} \rightarrow \ell_{j} V} = i \bar{u}_{\ell_{j}} \left[ \left( \gamma_{\mu} - \frac{g_{\mu}\gamma \cdot q}{q^{2}} \right) P_{L} \mathcal{P}_{j}^{ij V} + \left( \gamma_{\mu} - \frac{g_{\mu}\gamma \cdot q}{q^{2}} \right) P_{R} \mathcal{P}_{j}^{ij V} \right] P_{V} \mathcal{P}_{\gamma_{V}}^{ij V} \\
+ \frac{i \sigma_{\mu\nu} P_{L} q^{\nu}}{q^{2}} P_{2\gamma_{V}}^{ij V} + \frac{i \sigma_{\mu\nu} P_{R} q^{\nu}}{q^{2}} P_{2\gamma_{V}}^{ij V} \\
+ \gamma_{\mu} P_{L} \left( \mathcal{B}_{Z_{V}}^{ij V} + \mathcal{B}_{L_{1}}^{ij V} \right) + \gamma_{\mu} P_{R} \left( \mathcal{B}_{Z_{V}}^{ij V} + \mathcal{B}_{1R}^{ij V} \right) \\
+ i \sigma_{\mu\nu} P_{L} q^{\nu} \mathcal{B}_{2L}^{ij V} + i \sigma_{\mu\nu} P_{R} q^{\nu} \mathcal{B}_{2R}^{ij V} \right] u_{\ell_{i}} \gamma_{V}^{\mu}. \tag{3.3}\]

\[
i\mathcal{M}_{\ell_{i} \rightarrow \ell_{j} P} = i \bar{u}_{\ell_{j}} \left[ \left[ \gamma_{\mu} P_{L} \left( \mathcal{P}_{j}^{ij P} + \mathcal{B}_{L_{1}}^{ij P} \right) + \gamma_{\mu} P_{R} \left( \mathcal{P}_{j}^{ij P} + \mathcal{B}_{1R}^{ij P} \right) \right] q^{\mu} \\
+ \left[ P_{L} \mathcal{B}_{2L}^{ij P} + P_{R} \mathcal{B}_{2R}^{ij P} \right] \right] u_{\ell_{i}}. \tag{3.4}\]
The form factors in Eqs. (3.3) and (3.4) are defined as follows:

\[ P_{a\gamma L,R}^{ijV} = -e^2 P_{a\gamma}^{L,R} \frac{m_{V}}{\sqrt{2} \gamma_{V}} k_{V}^{ij}, \quad (a = 1, 2), \quad (V = \rho^{0}, \phi, \omega) \]  

(3.5)

\[ P_{ZL,R}^{ijV} = \frac{g_{Z}^{2}}{m_{Z}^{2} \gamma_{V}} P_{Z}^{L,R} \left( \frac{m_{V}^{2}}{\sqrt{2} \gamma_{V}} k_{V}^{ij}, \quad (V = \rho^{0}, \phi, \omega) \right) \]  

(3.6)

\[ B_{1L,R}^{ijV} = \frac{m_{V}^{2}}{\sqrt{2} \gamma_{V}} \left[ k_{V}^{ij} \left( B_{1\bar{u}u}^{L,R} + B_{1\bar{u}u}^{L,R} \right) + k_{V}^{ij} \left( B_{1\bar{d}d}^{L,R} + B_{1\bar{d}d}^{L,R} \right) \right] + k_{V}^{ij} \left( B_{1\bar{d}d}^{L,R} + B_{1\bar{d}d}^{L,R} \right) \]  

(3.7)

\[ B_{2L,R}^{ijV} = \frac{2 \gamma_{V}}{\sqrt{2}} \left[ k_{V}^{ij} \left( m_{s} - m_{d} \right) \right] B_{4sd}^{L,R} \]  

(3.8)

\[ P_{ZL,R}^{ijL} = \frac{g_{Z}^{2}}{m_{Z}^{2} \gamma_{V}} P_{Z}^{L,R} \left( \sqrt{2} f_{P} \right) k_{P}^{ij}, \quad (P = \pi^{0}, \eta, \eta') \]  

(3.9)

\[ B_{1L,R}^{ijP} = s_{L,R} \left( \sqrt{2} f_{P} \right) \frac{1}{2} \left[ k_{P}^{ij} \left( B_{1\bar{u}u}^{L,R} + B_{1\bar{u}u}^{L,R} \right) + k_{P}^{ij} \left( B_{1\bar{d}d}^{L,R} + B_{1\bar{d}d}^{L,R} \right) \right] + k_{P}^{ij} \left( B_{1\bar{d}d}^{L,R} + B_{1\bar{d}d}^{L,R} \right) \]  

(3.10)

\[ B_{2L,R}^{ijP} = s_{L,R} \left( -\frac{1}{2} \right) \left( \sqrt{2} f_{P} \right) \frac{1}{2} \left[ k_{P}^{ij} \left( B_{3\bar{u}u}^{L,R} + B_{3\bar{u}u}^{L,R} \right) + k_{P}^{ij} \left( B_{3\bar{d}d}^{L,R} + B_{3\bar{d}d}^{L,R} \right) \right] + k_{P}^{ij} \left( B_{3\bar{d}d}^{L,R} + B_{3\bar{d}d}^{L,R} \right) \]  

(3.11)

In Eqs. (3.10) and (3.11), \( s_{L} = 1 \) and \( s_{R} = -1 \). The constants \( k_{V}^{ij}, k_{V}^{ij}, k_{P}^{ij}, k_{P}^{ij} \) are defined in Appendix E.

A branching ratio for the processes \( \ell_{i} \rightarrow \ell_{j} P \) with unpolarized initial and final particles reads

\[ B(\ell_{i} \rightarrow \ell_{j} P) = \frac{1}{8 \pi} \frac{1}{m_{i}^{2}} \frac{1}{\Gamma_{i}} \frac{1}{2} \lambda^{\frac{1}{2}}(m_{i}^{2}, m_{j}^{2}, m_{P}^{2}) \]  

\[ \times \left[ \left| P_{ZL}^{ijP} + B_{1L}^{ijP} \right|^{2} + \left| P_{ZR}^{ijP} + B_{1R}^{ijP} \right|^{2} \right] i_{P1} \]  

\[ + \left| B_{2L}^{ijP} \right|^{2} + \left| B_{2R}^{ijP} \right|^{2} \]  

(3.12)
where $\Gamma_{l_i}$ is a total decay rate of the lepton $l_i$ and

$$i_{P1} = \frac{1}{2} \left( (m_i^2 - m_j^2)^2 - (m_i^2 + m_j^2)m_P^2 \right),$$

$$i_{P2} = \frac{1}{2}(m_i^2 + m_j^2 - m_P^2),$$

$$i_{P3} = m_P^2,$$

$$i_{P4} = \frac{1}{2}(m_i^2 + m_P^2 - m_j^2),$$

$$i_{P5} = \frac{1}{2}(m_i^2 - m_P^2 - m_j^2),$$  \hspace{1cm} (3.13)

and

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$  \hspace{1cm} (3.14)

A branching ratio for the processes $\ell_i \to \ell_jV$ with unpolarized initial and final particles reads.

$$B(\ell_i \to \ell_jV) = \frac{1}{8\pi} \frac{1}{\Gamma_{l_i}} \frac{1}{m_i^2} \lambda_1^2(m_i^2, m_j^2, m_V^2)$$

$$\times \left[ \left( |\mathcal{P}_{1\gamma L}^{ijV} + \mathcal{P}_{1\gamma L}^{ijV} + \mathcal{B}_{ijV}^{ijV}|^2 + |\mathcal{P}_{1\gamma R}^{ijV} + \mathcal{P}_{Z R}^{ijV} + \mathcal{B}_{1R}^{ijV}|^2 \right) i_{V1} ight.$$  

$$+ \left( \frac{\mathcal{P}_{2\gamma L}^{ijV}}{m_V^2} + \mathcal{B}_{2L}^{ijV} \right) \left( \frac{\mathcal{P}_{2\gamma R}^{ijV}}{m_V^2} + \mathcal{B}_{2R}^{ijV} \right)^2 \right) i_{V2}$$

$$+ \left( \frac{\mathcal{P}_{1\gamma L}^{ijV} + \mathcal{P}_{1\gamma L}^{ijV} + \mathcal{B}_{1L}^{ijV}}{m_V^2} \right) \left( \frac{\mathcal{P}_{1\gamma R}^{ijV} + \mathcal{P}_{Z R}^{ijV} + \mathcal{B}_{1R}^{ijV}}{m_V^2} \right)^* + c.c. \right) (-m_im_j)$$

$$+ \left( \frac{\mathcal{P}_{2\gamma L}^{ijV}}{m_V^2} + \mathcal{B}_{2L}^{ijV} \right) \left( \frac{\mathcal{P}_{2\gamma R}^{ijV}}{m_V^2} + \mathcal{B}_{2R}^{ijV} \right)^* + c.c. \right) (-m_i^2m_j)$$

$$+ \left( \frac{\mathcal{P}_{1\gamma L}^{ijV} + \mathcal{P}_{1\gamma L}^{ijV} + \mathcal{B}_{1L}^{ijV}}{m_V^2} \right) \left( \frac{\mathcal{P}_{2\gamma L}^{ijV}}{m_V^2} + \mathcal{B}_{2L}^{ijV} \right)^*$$

$$+ \left( \frac{\mathcal{P}_{1\gamma R}^{ijV} + \mathcal{P}_{Z R}^{ijV} + \mathcal{B}_{1R}^{ijV}}{m_V^2} \right) \left( \frac{\mathcal{P}_{2\gamma R}^{ijV}}{m_V^2} + \mathcal{B}_{2R}^{ijV} \right)^* + c.c. \right) (m_ji_{V3})$$

$$+ \left( \frac{\mathcal{P}_{1\gamma L}^{ijV} + \mathcal{P}_{1\gamma L}^{ijV} + \mathcal{B}_{1L}^{ijV}}{m_V^2} \right) \left( \frac{\mathcal{P}_{2\gamma R}^{ijV}}{m_V^2} + \mathcal{B}_{2R}^{ijV} \right)^*$$

$$+ \left( \frac{\mathcal{P}_{2\gamma L}^{ijV}}{m_V^2} + \mathcal{B}_{2L}^{ijV} \right) \left( \frac{\mathcal{P}_{2\gamma R}^{ijV}}{m_V^2} + \mathcal{B}_{2R}^{ijV} \right)^* + c.c. \right) (m_i^2i_{V4}) \right], \hspace{1cm} (3.15)$$

where

$$i_{V1} = \frac{1}{2m_V^2} \left[ m_V^2(m_i^2 + m_j^2) + (m_i^2 - m_j^2)^2 - 2m_i^4 \right].$$
$$iV_2 = \frac{1}{2} \left( m_i^2 - m_j^2 \right)^2 - \frac{1}{2} m_V^2 \left( m_i^2 + m_j^2 \right) - \frac{1}{2} m_V^4,$$
$$iV_3 = \frac{1}{2} \left( m_i^2 - m_j^2 + m_V^2 \right),$$
$$iV_4 = \frac{1}{2} \left( m_i^2 - m_j^2 - m_V^2 \right). \quad (3.16)$$

3.4 SL LFV decays of a lepton with two pseudoscalar mesons in the final state

Amplitude for a general $\ell_i \to \ell_j P_1 P_2$ decay rate is a sum of scalar-coupling contribution and resonance contributions (coming from vector- and tensor-coupling contributions)

$$i\mathcal{M}_{\ell_i \to \ell_j P_1 P_2} = i\mathcal{M}_{\ell_i \to \ell_j P_1 P_2}^{\text{res}} + i\mathcal{M}_{\ell_i \to \ell_j P_1 P_2}^{\ell_i \to \ell_j P_1 P_2}, \quad (3.17)$$

where

$$\mathcal{M}_{\ell_i \to \ell_j P_1 P_2}^{\ell_i \to \ell_j P_1 P_2} = i\bar{u}_{\ell_j} \left[ D_{i1L}^{P_1 P_2} \left( \phi_2 - \phi_1 \right) - \frac{m_2^2 - m_1^2}{q^2} \delta \right] P_L$$
$$+ D_{i1R}^{P_1 P_2} \left( \phi_2 - \phi_1 \right) - \frac{m_2^2 - m_1^2}{q^2} \delta P_R$$
$$+ E_{i1L}^{P_1 P_2} i\sigma_{\mu\nu} P_L (p_1 - p_2)^\mu q^\nu + E_{i1R}^{P_1 P_2} i\sigma_{\mu\nu} P_R (p_1 - p_2)^\mu q^\nu \right] u_{\ell_i}. \quad (3.18)$$

$$i\mathcal{M}_{\ell_i \to \ell_j P_1 P_2}^{\text{res}} = i\bar{u}_{\ell_j} \left[ P_L A_{i1L}^{\ell_i \to \ell_j P_1 P_2} + P_R A_{i1R}^{\ell_i \to \ell_j P_1 P_2} \right] u_{\ell_i}. \quad (3.19)$$

In Eq. (3.18) $D_{i1L,R}^{P_1 P_2}$ and $E_{i1L,R}^{P_1 P_2}$ are form factors built from the trilinear $c V P_1 P_2$ couplings (defined by the Lagrangian (3.2)), normalized vector-meson propagators,

$$\frac{m_V^2 - i m_V \Gamma_V}{q^2 - m_V^2 + i m_V \Gamma_V} \quad (3.20)$$

and form factors for $\ell_i \to \ell_j V$ processes divided by the the mass of the resonant vector meson, e.g.

$$\tilde{P}^{ijV}_{1\gamma L,R} = \frac{P^{ijV}_{1\gamma L,R}}{m_V^2}. \quad (3.21)$$

The expressions for the $D_{i1L,R}^{P_1 P_2}$ and $E_{i1L,R}^{P_1 P_2}$ form factors are

$$D_{i1L,R}^{P_1 P_2} = \sum_V \left( \tilde{P}^{ijV}_{1\gamma L,R} + \tilde{P}^{ijV}_{Z L,R} + \tilde{B}^{ijV}_{1\gamma L,R} \right) \frac{m_V^2 - i m_V \Gamma_V}{q^2 - m_V^2 + i m_V \Gamma_V} c_{CP} P_1 P_2, \quad (3.22)$$

$$E_{i1L,R}^{P_1 P_2} = \sum_V \left( \frac{\tilde{P}^{ijV}_{2L,R}}{q^2} + \tilde{B}^{ijV}_{2L,R} \right) \frac{m_V^2 - i m_V \Gamma_V}{q^2 - m_V^2 + i m_V \Gamma_V} c_{CP} P_1 P_2. \quad (3.23)$$

The sum goes over neutral vector mesons only ($V = \rho^0, \phi, \omega, K^{*0}$, $\bar{K}^{*0})$. The coefficients of the non-resonant part of the amplitude, the constants $A_{i1P_1 P_2}$, are defined as the coefficients of the $P_1 P_2$ product of fields contained in the matrix-valued operator

$$\frac{r}{4} \sum_{q_a, q_b = u, d, s} \left( \Pi^2 \right)_{q_b q_a} \left( B_{3q_a q_b}^{L,R} + \tilde{B}_{3q_a q_b}^{L,R} \right) \quad (3.24)$$
where \( \Pi \) is a matrix of pseudoscalar fields

\[
\Pi = \begin{pmatrix}
\pi^0 + \frac{1}{\sqrt{3}} \eta_8 + \frac{\sqrt{2} \pi^0}{\sqrt{3}} \\
\sqrt{2} \pi^- \\
\sqrt{2} K^-
\end{pmatrix}
\begin{pmatrix}
\sqrt{2} \pi^+ \\
-\pi^0 + \frac{1}{\sqrt{3}} \eta_8 + \frac{\sqrt{2} \pi^0}{\sqrt{3}} \\
\sqrt{2} K^0
\end{pmatrix}
\begin{pmatrix}
\sqrt{2} K^+ \\
\frac{2}{\sqrt{3}} \eta_8 + \frac{\sqrt{2} \pi^0}{\sqrt{3}}
\end{pmatrix}.
\]

(3.25)

For example,

\[
A_{1\pi^0}^{L,R} = \frac{r}{4} \left[ \frac{1}{2} \left( \mathcal{B}_{3\mu}^{L,R} + \mathcal{B}_{3\delta}^{L,R} + \mathcal{B}_{3d}^{L,R} + \mathcal{B}_{3d}^{R} \right) \right].
\]

(3.26)

Having the amplitudes one can easily evaluate the branching fractions. We assume that incoming and outgoing particles are not polarised.

\[
B(\ell_i \to \ell_j P_1 P_2) = \frac{1}{(2\pi)^2} \frac{1}{\Gamma_{\ell_i}} \frac{1}{32m_i^4} \int_{(m_1-m_2)^2}^{(m_1+m_2)^2} ds_{12} \int_{s_{j2}^{min}}^{s_{j2}^{max}} ds_{j2} |\mathcal{M}_{\ell_i \to \ell_j P_1 P_2}|^2
\]

\[
= \frac{1}{(2\pi)^2} \frac{1}{32m_i^4} \int_{(m_1-m_2)^2}^{(m_1+m_2)^2} ds_{12} \times \left[ \left( |D_{1L}^{P_1 P_2}|^2 + |D_{1R}^{P_1 P_2}|^2 \right) \bar{I}_1 \\
+ \left( |E_{1L}^{P_1 P_2}|^2 + |E_{1R}^{P_1 P_2}|^2 \right) \bar{I}_2 \\
+ \left( |A_{1L}^{P_1 P_2}|^2 + |A_{1R}^{P_1 P_2}|^2 \right) \bar{I}_3 \\
+ \left( \left( D_{1L}^{P_1 P_2} \right)^* (D_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_4 \\
+ \left( \left( E_{1L}^{P_1 P_2} \right)^* (E_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_5 \\
+ \left( \left( A_{1L}^{P_1 P_2} \right)^* (A_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_6 \\
+ \left( \left( D_{1L}^{P_1 P_2} \right)^* (E_{1L}^{P_1 P_2})^* + \left( D_{1R}^{P_1 P_2} \right)^* (E_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_7 \\
+ \left( \left( D_{1L}^{P_1 P_2} \right)^* (E_{1R}^{P_1 P_2})^* + \left( D_{1R}^{P_1 P_2} \right)^* (E_{1L}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_8 \\
+ \left( \left( D_{1L}^{P_1 P_2} \right)^* (A_{1L}^{P_1 P_2})^* + \left( D_{1R}^{P_1 P_2} \right)^* (A_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_9 \\
+ \left( \left( D_{1L}^{P_1 P_2} \right)^* (A_{1R}^{P_1 P_2})^* + \left( D_{1R}^{P_1 P_2} \right)^* (A_{1L}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_{10} \\
+ \left( \left( E_{1L}^{P_1 P_2} \right)^* (A_{1L}^{P_1 P_2})^* + \left( E_{1R}^{P_1 P_2} \right)^* (A_{1R}^{P_1 P_2})^* + \text{c.c.} \right) \bar{I}_{11} \right].
\]

(3.27)

The Mandelstam variables are defined as \( s_{ab} = (p_a - p_b)^2 \), e.g. \( s_{12} = (p_1 - p_2)^2 \). The kinematical bounds on the Mandelstam variables \( s_{j2}^{min} \) and \( s_{j2}^{max} \) are well known [43]. The \( \bar{I} \) integrals read,

\[
\bar{I}_1 = 2s_{j2}^{max} + 2s_{j2}^{min}(e_1 + e_2) + (2e_1e_2 - e_3e_4) \Gamma,
\]

\[
\bar{I}_2 = -2s_{j2}^{max}(e_{10}) + 2s_{j2}^{max}(e_5e_{10} + e_6e_{10} - e_6e_7 - e_8e_7) \\
+ 2((e_5e_6e_7 + e_8e_9e_7 - e_5e_9e_{10} - e_8e_6e_{11}) + e_3(e_{11}e_{10} - e_7^2)) \Gamma,
\]

\[
\bar{I}_3 = (e_5e_7)(e_5e_7 + e_8e_7 - e_5e_9e_{10} - e_8e_6e_{11} - e_3(e_{11}e_{10} - e_7^2)) \Gamma.
\]
\[\bar{I}_3 = e_3 \Gamma,\]
\[\bar{I}_4 = m_i m_j e_4 \Gamma,\]
\[\bar{I}_5 = m_i m_j (e_{10} e_{11} - e_7^2) \Gamma,\]
\[\bar{I}_6 = m_i m_j \Gamma,\]
\[\bar{I}_7 = m_j (-e_1 e_6) \Gamma,\]
\[\bar{I}_8 = m_i (e_1 e_8) \Gamma,\]
\[\bar{I}_9 = m_j (\overline{s_{j2}} + (e_2) \Gamma),\]
\[\bar{I}_{10} = m_i (\overline{s_{j2}} + (e_1) \Gamma),\]
\[\bar{I}_{11} = -s_{j2} (e_8 - e_6) + (e_8 e_9 - e_5 e_6) \Gamma,\]

where

\[s_{j2} = \int_{s_{j2}^{\min}}^{s_{j2}^{\max}} ds_{j2} s_{j2}.\]  

(3.28)

(3.29)

The quantities \(e_i\) read

\[e_1 = -e_5 - \frac{m_2 - m_1}{s_{12}} e_8,\]
\[e_2 = -e_9 - \frac{m_2 - m_1}{s_{12}} e_6,\]
\[e_3 = \frac{1}{2} (m_i^2 + m_j^2 - s_{12}),\]
\[e_4 = e_{11} - \frac{(m_2^2 - m_1^2)^2}{s_{12}},\]
\[e_5 = m_2^2 + \frac{1}{2} (m_i^2 + m_j^2 - s_{12}),\]
\[e_6 = \frac{1}{2} (m_i^2 - m_j^2 + s_{12}),\]
\[e_7 = m_1^2 - m_2^2,\]
\[e_8 = \frac{1}{2} (m_i^2 - m_j^2 - s_{12}),\]
\[e_9 = m_1^2 + \frac{1}{2} (m_i^2 + m_j^2 - s_{12}),\]
\[e_{10} = s_{12},\]
\[e_{11} = 2m_1^2 + 2m_2^2 - s_{12},\]
\[e_{12} = -e_4.\]  

(3.30)

4. Minimal SO(10) model and its predictions

Now we are ready to estimate the SL LFV processes. In order to perform a concrete evaluation for the SL LFV processes, we need an information on the Yukawa couplings. In this paper, we make use of the minimal \(SO(10)\) model, as an example which gives a precise information for the neutrino-Dirac-Yukawa couplings. We begin with an overview of the minimal SUSY \(SO(10)\) model proposed in [22] and recently analysed in detail in Ref. [23, 24, 25, 26, 27]. Even when we concentrate our discussion on the issue how to reproduce
the realistic fermion mass matrices in the \(SO(10)\) model, there are lots of possibilities for introduction of Higgs multiplets. The minimal supersymmetric \(SO(10)\) model is the one where only one \(10\) and one \(126\) Higgs multiplet have Yukawa couplings (superpotential) with \(16\) matter multiplets. Therefore, the quark and lepton mass matrices can be described as

\[
M_u = c_{10} M_{10} + c_{126} M_{126}, \\
M_d = M_{10} + M_{126}, \\
M_D = c_{10} M_{10} - 3 c_{126} M_{126}, \\
M_e = M_{10} - 3 M_{126}, \\
M_R = c_R M_{126},
\]

(4.1)

where \(M_u, M_d, M_D, M_e\) and \(M_R\) denote up-type quark, down-type quark, neutrino Dirac, charged-lepton and right-handed neutrino Majorana mass matrices, respectively. Note that all the quark and lepton mass matrices are characterized by only two basic mass matrices, \(M_{10}\) and \(M_{126}\), and three complex coefficients \(c_{10}, c_{126}\) and \(c_R\).

The mass matrix formulas in Eq. (4.1) lead to the GUT relation among the quark and lepton mass matrices,

\[
M_e = c_d (M_d + \kappa M_u),
\]

(4.2)

where

\[
c_d = -\frac{3 c_{10} + c_{126}}{c_{10} - c_{126}},
\]

(4.3)

\[
\kappa = -\frac{4}{3 c_{10} + c_{126}}.
\]

(4.4)

Without loss of generality, we can start with the basis where \(M_u\) is real and diagonal, \(M_u = D_u\). Since \(M_d\) is the symmetric matrix, it can be described as \(M_d = V_{\text{CKM}}^* D_d V_{\text{CKM}}\) by using the CKM matrix \(V_{\text{CKM}}\) and the real diagonal mass matrix \(D_d\). \(^1\) Considering the basis-independent quantities, \(\text{tr}[\tilde{M}_e^\dagger M_e]\), \(\text{tr}[(\tilde{M}_e^\dagger M_e)^2]\) and \(\text{det}[\tilde{M}_e^\dagger M_e]\), and eliminating \(|c_d|\), we obtain two independent equations,

\[
\left(\frac{\text{tr}[\tilde{M}_e^\dagger M_e]}{m_e^2 + m_\mu^2 + m_\tau^2}\right)^2 = \frac{\text{tr}[(\tilde{M}_e^\dagger M_e)^2]}{m_e^4 + m_\mu^4 + m_\tau^4},
\]

(4.5)

\[
\left(\frac{\text{tr}[\tilde{M}_e^\dagger M_e]}{m_e^2 + m_\mu^2 + m_\tau^2}\right)^3 = \frac{\text{det}[\tilde{M}_e^\dagger M_e]}{m_e^2 m_\mu^2 m_\tau^2},
\]

(4.6)

where \(\tilde{M}_e \equiv V_{\text{CKM}}^* D_d V_{\text{CKM}}^\dagger + \kappa D_u\). With input data of six quark masses, three angles and one CP-phase in the CKM matrix and three charged-lepton masses, we can solve the above

\(^1\)In general, \(M_d = U^* D_d U^\dagger\) by using a general unitary matrix \(U = e^{i\alpha} e^{i\beta\text{T}_3} e^{i\gamma\text{T}_3} V_{\text{CKM}} e^{i\beta\text{T}_3} e^{i\gamma\text{T}_3}\). We omit the diagonal phases to keep the number of the free parameters in the model as small as possible.
equations and determine $\kappa$ and $|c_d|$, but one parameter, the phase of $c_d$, is left undetermined \[23\]. The original basic mass matrices, $M_{10}$ and $M_{126}$, are described by

\begin{align*}
M_{10} &= \frac{3 + |c_d|e^{i\sigma}}{4}V_{\text{CKM}}^* D_d V_{\text{CKM}}^† + \frac{|c_d|e^{i\sigma}\kappa}{4}D_u, \\
M_{126} &= \frac{1 - |c_d|e^{i\sigma}}{4}V_{\text{CKM}}^* D_d V_{\text{CKM}}^† - \frac{|c_d|e^{i\sigma}\kappa}{4}D_u,
\end{align*}

as the functions of $\sigma$, the phase of $c_d$, with the solutions, $|c_d|$ and $\kappa$, determined by the GUT relation.

Now let us solve the GUT relation and determine $|c_d|$ and $\kappa$. Since the GUT relation of Eq. (4.2) is valid only at the GUT scale, we first evolve by renormalization group equations (RGE's) the data from the weak scale to the corresponding quantities at the GUT scale for a given $\tan \beta$. Then we and use them as input data at the GUT scale. Note that it is non-trivial to find a solution of the GUT relation, since the number of free parameters (fourteen) is almost the same as the number of inputs (thirteen). The solution of the GUT relation exists only if we take appropriate input parameters. By using the experimental data at the $M_Z$ scale [42], we get the following absolute values for charged fermion masses (in units of GeV) and the CKM matrix at the GUT scale, $M_G$, with $\tan \beta = 45$, $m_s = 0.072$ and $\delta = 1.518$:

- $m_u = 0.00103$, $m_c = 0.299$, $m_t = 133$
- $m_d = 0.00170$, $m_s = 0.0263$, $m_b = 1.55$
- $m_e = 0.000411$, $m_\mu = 0.0868$, $m_\tau = 1.69$

and

\[
V_{\text{CKM}} = \begin{pmatrix}
0.975 & 0.222 & 0.000146 - 0.00279i \\
-0.222 - 0.000121i & 0.974 + 0.000129i & 0.0320 \\
0.00697 - 0.00272i & -0.0312 - 0.000626i & 0.999
\end{pmatrix},
\]

in the standard parameterization. The phases of the fermion masses are not determined by the diagonalization procedure. Here the masses are chosen be real. The signs of the fermion masses have been chosen to be equal, $-$ for $m_u$, $m_c$, $m_d$ and $m_s$, and $+$ for $m_t$ and $m_b$. So determined masses at the GUT scale are used as input parameters in order to solve Eqs. (4.5) and (4.6). As an example, here we show one of two solutions,

\[
\kappa = 0.0134 - 0.000791i, \quad |c_d| = 6.39. \tag{4.9}
\]

Once the parameters, $|c_d|$ and $\kappa$, are determined, we can describe all the fermion mass matrices as a functions of $\sigma$ from the mass matrix formulas of Eqs. (4.1), (4.7) and (4.8). Interestingly, in the minimal $SO(10)$ model even light Majorana neutrino mass matrix, $M_\nu$, can be determined as a function of the phase $\sigma$ and $c_R$ through the seesaw mechanism $M_\nu = -M^T_D M^{-1}_R M_D$. 

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Now we give an example of the neutrino-Dirac-Yukawa coupling matrix which can fit the GUT relation. In the basis where both the charged-lepton and right-handed Majorana neutrino mass matrix are diagonal with real and positive eigenvalues, the neutrino Dirac Yukawa coupling matrix at the GUT scale for fixed $\sigma = 3.223$ [rad] is found to be

$$Y_{\nu} = \begin{pmatrix}
0.000310 + 0.00348i & -0.000894 - 0.000249i & 0.0447 + 0.0531i \\
0.00590 - 0.0103i & -0.0164 - 0.0427i & 0.308 + 0.116i \\
0.00215 + 0.00126i & 0.0558 - 0.0559i & -0.381 + 0.604i
\end{pmatrix}. \quad (4.10)$$

Using these Yukawa coupling constant matrices, we proceed with numerical calculations. In evaluating the LFV branching ratios, we first solve the RGE’s for the soft SUSY breaking parameters and the Yukawa couplings in the MSSM to determine the masses and mixings for the SUSY particles. Then we input these data into the formulae presented in the previous sections.

In the following, we list the input parameters what we used for the hadronization processes. For the pseudoscalar meson decay constants, we take as an input the following values \[ f_{\pi^0} = 0.119\text{[GeV]} \], \[ f_{\eta} = 0.131\text{[GeV]} \], \[ f_{\eta'} = 0.118\text{[GeV]} \], \quad (4.11) \]
and for the vector-meson decay constants we use the values, extracted from the vector meson decay, \[ V \rightarrow e^+e^- \],

$$\gamma_{\rho^0} = 2.518, \quad \gamma_\phi = 2.993, \quad \gamma_\omega = 3.116. \quad (4.12)$$

The mixing angles between singlet and octet states for the vector mesons and for the pseudoscalar mesons that we use are \[ \theta_V = 35^\circ, \quad \theta_P = -17.3^\circ. \quad (4.13) \]

In order to investigate the dependence of SL LFV branching ratios on model parameters we plot their dependence on \( m_{\tilde{\tau}} \); \( \tau \rightarrow e\pi^0 \), \( \tau \rightarrow e\eta \), \( \tau \rightarrow e\eta' \), \( \tau \rightarrow \mu\pi^0 \), \( \tau \rightarrow \mu\eta \), and \( \tau \rightarrow \mu\eta' \) graphs in Fig. 1, \( \tau \rightarrow e\rho^0 \), \( \tau \rightarrow \mu\rho^0 \) \( \tau \rightarrow e\phi \), \( \tau \rightarrow \mu\phi \) \( \tau \rightarrow e\omega \), and \( \tau \rightarrow \mu\omega \) graphs in Fig 2, \( \tau \rightarrow e\pi^+\pi^- \), \( \tau \rightarrow \mu\pi^-\pi^+ \), \( \tau \rightarrow eK^0\bar{K}^0 \), and \( \tau \rightarrow \mu K^0\bar{K}^0 \), \( \tau \rightarrow eK^+K^- \), and \( \tau \rightarrow \mu K^+K^- \) graphs in Fig 3, and \( \tau \rightarrow e\gamma \) and \( \tau \rightarrow \mu\gamma \) graphs in Fig 4. In Fig 5. we plot the \( m_{\tilde{\tau}} \) dependence of the quantities used to determine the allowed region of \( m_{\tilde{\tau}} \) mass, the MSSM contribution to the anomalous magnetic moment $\Delta a_\mu$ and $\mu \rightarrow e\gamma$ branching ratio. The parameters are chosen so as to satisfy the WMAP constraint on the cold dark matter (CDM) relic density \[ \Omega_{\text{CDM}}h^2 = 0.1126. \quad (4.14) \]

We can transmute this value into the approximate relation between \( m_0 \) and \( M_{1/2} \), such as

$$m_0 \text{[GeV]} = \frac{9}{28} M_{1/2} \text{[GeV]} + 150 \text{[GeV]}, \quad (4.15)$$
Figure 1: The branching ratios for SL LFV decays $\tau \to e\pi^0/e\eta/e\eta'/\mu\pi^0/\mu\eta/\mu\eta'$ as a function of mass of lightest charged sfermion $m_{\tilde{\tau}_R}$. As an input we have taken $\tan \beta = 45$, $\mu < 0$, and $A_0 = 0$. The maximal values for the branching ratios of the $\tau \to e/\mu \pi^0$
Figure 2: The branching ratios for SL LFV decays $\tau \to e\rho^0/e\phi/e\omega/\mu\rho^0/\mu\phi/\mu\omega$ as a function of mass of lightest charged sfermion $m_{\tilde{\tau}_R}$. The input parameters are as in Fig. 1.

processes are found to be

$$\text{BR}(\tau \to e\pi^0) \simeq 1.7 \times 10^{-14},$$

(4.16)
Figure 3: The branching ratios for SL LFV decays $\tau \rightarrow e\pi^+\pi^-/eK^0\bar{K}^0/eK^+K^-/\mu\pi^+\pi^-/\mu K^+K^-/\mu K^0\bar{K}^0$ as a function of mass of lightest charged sfermion $m_{\tilde{\tau}^R}$. The input parameters are as in Fig. 1.

$$BR(\tau \rightarrow \mu\pi^0) \simeq 2.4 \times 10^{-12},$$  \hspace{1cm} (4.17)
and for the $\tau \rightarrow e/\mu \eta$ processes $\text{BR}(\tau \rightarrow e/\mu \eta) \simeq 0.15 \times \text{BR}(\tau \rightarrow e/\mu \pi^0)$, with suitably chosen CMSSM parameters which can realize the neutralino dark matter scenario by the WMAP data. It can be realized with the following set of parameters: $\tan \beta = 45$, $\mu > 0$ and $A_0 = 0$, with $M_{1/2} = 600$ [GeV], $m_0 = 343$ [GeV]. This parameter set can also predict the muon $g - 2$ within the range of the recent result of Brookhaven E821 experiment and also provides the $\tau \rightarrow \mu \gamma$ and $\mu \rightarrow e \gamma$ branching ratios close to the current experimental bound [4]. The ratio between two processes $\tau \rightarrow e/\mu \pi^0$ and $\tau \rightarrow e/\mu \eta$ is a result of the dominance of the $Z$-boson-penguin amplitude in these processes, and reflects the difference in the form factors and the mixings between the singlet state and the octet state of the $\eta$. 

**Figure 4:** The branching ratios for decays $\tau \rightarrow e\gamma/\mu\gamma$ as a function of mass of lightest charged sfermion $m_{\tilde{f}_R}$. The input parameters are as in Fig. 1.

**Figure 5:** The anomalous magnetic moment and branching ratios for $\mu \rightarrow e\gamma$ as a function of mass of lightest charged sfermion $m_{\tilde{f}_R}$. These quantities restrict the region of $m_{\tilde{f}_R}$ values. The present experimental upper limits are represented by horizontal red lines. The input parameters are as in Fig. 1.
mesons,

\[
\frac{\text{BR}(\tau \to e/\mu \eta)}{\text{BR}(\tau \to e/\mu \pi)} \sim \left( \frac{f_\eta}{f_\pi} \right)^2 \times \left( \frac{c_P}{\sqrt{3}} + \frac{s_P}{\sqrt{6}} \right)^2 \sim 0.15. \tag{4.18}
\]

We can also see the correlation between the branching ratios for the processes \( \tau \to e/\mu \rho^0 \) and \( \tau \to e/\mu \gamma \) as \( \text{BR}(\tau \to e/\mu \rho^0) \simeq 5.5 \times 10^{-3} \times \text{BR}(\tau \to e/\mu \gamma) \). It can be estimated on the basis of the photon-penguin-amplitude dominance in these amplitudes, giving

\[
\frac{\text{BR}(\tau \to e/\mu \rho^0)}{\text{BR}(\tau \to e/\mu \gamma)} \sim \frac{1}{2} \left( \frac{e_{\gamma\rho}}{\gamma_{\rho\rho}} \right)^2 \sim 7 \times 10^{-3}. \tag{4.19}
\]

When we impose the constraints from discrepancy of \( \tau \) and \( e^+e^- \) data in the muon \( g-2 \) measurements \cite{10} and from upper limits on the \( \mu \to e\gamma \) branching ratio, we obtain that the model permits the of \( m_{\tilde{\tau}_R} \) values satisfying,

\[
m_{\tilde{\tau}_R} > 204 \text{ [GeV]}. \tag{4.20}
\]

The lower bound comes from the muon \( g-2 \) constraint. The \( \mu \to e\gamma \) gives also the lower bound but it is below the lower bound from the muon \( g-2 \) constraint. The \( g-2 \) curve has uprising behaviour above \( m_{\tilde{\tau}_R} = 340 \) [GeV], but at \( m_{\tilde{\tau}_R} = 1000 \) [GeV] it is almost independent on \( m_{\tilde{\tau}_R} \) and has a value \( 5.4 \times 10^{-10} \), slightly below the present experimental \( g-2 \) uncertainty. Therefore, one can expect that the improvement of the \( g-2 \) measurements will give the upper limit on \( m_{\tilde{\tau}_R} \), too. Using the lower bound on \( m_{\tilde{\tau}_R} \) values one can find the theoretical upper bounds for all leptonic and SL LFV branching ratios. Leptonic and dominant SL LFV decays are given in Table I.

5. Summary

The evidence for the neutrino masses and flavour mixings implies the non-conservation of the lepton-flavour symmetry. Thus, the LFV processes in the charged-lepton sector are expected. In supersymmetric model based on the minimal \( SO(10) \) model, the values for the rates of the LFV processes are generally still several orders of magnitudes below the accessible current experimental bounds. In this paper, we have presented the detailed theoretical description for the SL LFV decays of the charged leptons with one or two pseudoscalar mesons or one vector meson in the final state. Also, some previous formulae have been corrected. The \( \gamma \)-penguin amplitude is corrected to assure the gauge invariance, the \( Z \)-penguin amplitude is corrected, new box contributions to the box amplitude has been found and previously neglected terms are given.

To evaluate the decay rates of the LFV processes within the MSSM, the parameters and the LFV interactions of the MSSM have to be specified. It has been shown \cite{23} that the minimal SUSY \( SO(10) \) model can simultaneously accommodate all the observed quark-lepton mass matrix data involving the neutrino oscillation data with appropriately fixed free parameters. In this model, the neutrino-Dirac-Yukawa coupling matrix are completely determined, and its off-diagonal components are the primary source of the lepton-flavour violation in the basis where the charged-lepton and the right-handed neutrino mass matrices
are real and diagonal. Using this Yukawa coupling matrix, we have calculated the rate of the LFV processes assuming the mSUGRA scenario. The analytical formulae of various SL LFV processes, $\ell_i \rightarrow \ell'_j P$, $\ell_i \rightarrow \ell_j V$, $\ell_i \rightarrow \ell_j PP$ are given. Using these formulae, we have numerically evaluated $\ell_i \rightarrow \ell'_j P$, $\ell_i \rightarrow \ell_j V$ and $\ell_i \rightarrow \ell_j P_1 P_2$ branching ratios. Among these, the branching ratios of $\tau \rightarrow \mu \gamma$ and $\mu \rightarrow e \gamma$ may be interesting for the near future experiments. The typical CMSSM parameters used in calculations can realize the neutralino dark matter scenario by the WMAP data.

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| Process         | Theor. upper bound | Exp. upper bound |
|-----------------|--------------------|------------------|
| $\mu \rightarrow e\gamma$ | $6.9 \times 10^{-13}$ | $1.2 \times 10^{-11}$ |
| $\tau \rightarrow e\gamma$ | $3.1 \times 10^{-12}$ | $1.0 \times 10^{-9}$ |
| $\tau \rightarrow \mu\gamma$ | $4.1 \times 10^{-10}$ | $4.5 \times 10^{-9}$ |
| $\tau \rightarrow e\pi^0$ | $1.7 \times 10^{-14}$ | $1.9 \times 10^{-7}$ |
| $\tau \rightarrow \mu\pi^0$ | $2.4 \times 10^{-12}$ | $4.3 \times 10^{-7}$ |
| $\tau \rightarrow e\eta$ | $2.9 \times 10^{-15}$ | $2.3 \times 10^{-7}$ |
| $\tau \rightarrow \mu\eta$ | $4.2 \times 10^{-13}$ | $2.3 \times 10^{-7}$ |
| $\tau \rightarrow e\eta'$ | $2.8 \times 10^{-15}$ | $10 \times 10^{-7}$ |
| $\tau \rightarrow \mu\eta'$ | $3.7 \times 10^{-13}$ | $4.1 \times 10^{-7}$ |
| $\tau \rightarrow e\rho^0$ | $2.3 \times 10^{-14}$ | $2.0 \times 10^{-6}$ |
| $\tau \rightarrow \mu\rho^0$ | $2.8 \times 10^{-12}$ | $6.3 \times 10^{-6}$ |
| $\tau \rightarrow e\phi$ | $1.4 \times 10^{-14}$ | $6.9 \times 10^{-6}$ |
| $\tau \rightarrow \mu\phi$ | $1.7 \times 10^{-12}$ | $7.0 \times 10^{-6}$ |
| $\tau \rightarrow e\omega$ | $1.6 \times 10^{-15}$ | $-$ |
| $\tau \rightarrow \mu\omega$ | $1.5 \times 10^{-13}$ | $-$ |
| $\tau \rightarrow e\pi^+\pi^-$ | $4.2 \times 10^{-14}$ | $8.7 \times 10^{-7}$ |
| $\tau \rightarrow \mu\pi^+\pi^-$ | $5.8 \times 10^{-12}$ | $2.8 \times 10^{-7}$ |
| $\tau \rightarrow eK^0\bar{K}^0$ | $4.3 \times 10^{-15}$ | $2.2 \times 10^{-6}$ |
| $\tau \rightarrow \muK^0\bar{K}^0$ | $5.3 \times 10^{-13}$ | $3.4 \times 10^{-6}$ |
| $\tau \rightarrow eK^+K^-$ | $6.7 \times 10^{-15}$ | $3.0 \times 10^{-7}$ |
| $\tau \rightarrow \muK^+K^-$ | $5.1 \times 10^{-13}$ | $11.7 \times 10^{-7}$ |

Table 1: Theoretical upper bounds $\ell \rightarrow \ell'\gamma$ processes and dominant SL LFV processes. The upper bound is obtained from the muon $g - 2$ constraint. We referred the experimental data mainly from [44] and partly from [45].
A. Notation for the MSSM Lagrangian

Here we summarize our notation necessary for defining the masses of the sparticles in the MSSM Lagrangian.

\[ v \equiv \sqrt{\langle H_0^0 \rangle^2 + \langle H_d^0 \rangle^2} = 174.1 \text{ [GeV]}, \]  
(A.1)  

and

\[ \tan \beta \equiv \frac{\langle H_0^0 \rangle_u}{\langle H_0^0 \rangle_d}. \]  
(A.2)  

Then the charged fermion mass matrices are given by

\[ M_{ij}^u = -Y_{ij}^u v \sin \beta, \]  
(A.3)  

\[ M_{ij}^d = Y_{ij}^d v \cos \beta, \]  
(A.4)  

\[ M_{ij}^e = Y_{ij}^e v \cos \beta. \]  
(A.5)  

The mass matrix of the charginos is written as:

\[ \mathcal{L} = -\left( \begin{array}{cc} \overline{W}^-_R & \overline{H}^-_R \\ H^-_R & -W^-_R \end{array} \right) M_{\tilde{\chi}^\pm} \left( \begin{array}{c} \overline{W}^-_L \\ H^-_d \end{array} \right) + h.c., \]  

\[ M_{\tilde{\chi}^\pm} = \left( \begin{array}{c} M_2 \\ \sqrt{2}M_W \sin \beta \emptyset \mu \end{array} \right). \]  
(A.6)  

The mass matrix of the neutralinos is written as:

\[ \mathcal{L} = -\frac{1}{2} \left( \begin{array}{ccc} \overline{B}_L & \overline{W}_L^3 & \overline{H}^0_{dL} \\ \overline{W}_L^3 & H^0_{dL} & \overline{H}^0_{dL} \end{array} \right) M_{\tilde{\chi}^0} \left( \begin{array}{c} \overline{B}_L \\ \overline{W}_L^3 \\ \overline{H}^0_{uL} \end{array} \right) + h.c., \]  

\[ M_{\tilde{\chi}^0} = \left( \begin{array}{ccc} M_1 & 0 & -M_Z \sin \theta_W \cos \beta \\ 0 & M_2 & M_Z \cos \theta_W \cos \beta \\ -M_Z \sin \theta_W \sin \beta & M_Z \cos \theta_W \sin \beta & -\mu \end{array} \right). \]  
(A.7)  

The mass matrices of the squarks are written as follows:

\[ \mathcal{L} = -(m^2_{\tilde{u}})^{ij} \tilde{u}_i \tilde{u}_j - (m^2_{\tilde{d}})^{ij} \tilde{d}_i \tilde{d}_j, \]
\[
\begin{align*}
m^2_{\tilde{d}} &= \begin{pmatrix}
m^2_{\tilde{\nu}} + M^d_{\nu} & -A^d_{\nu} v \sin \beta - M^d_{\nu} \mu^* \cot \beta \\
-A^d_{\nu} v \sin \beta - M^d_{\nu} \mu \cot \beta & m^2_{\nu} + M^d_{\nu} - M^d_W \cos 2\beta \sin^2 \theta_W
\end{pmatrix} \\
&+ \begin{pmatrix}
M^2_Z \cos 2\beta (\frac{1}{2} - \frac{2}{3} \sin^2 \theta_W) & 0 \\
0 & \frac{2}{3} M^2_Z \cos 2\beta \sin^2 \theta_W
\end{pmatrix} \mathbf{1}_{3\times3},
\end{align*}
\]

\[
\begin{align*}
m^2_{\tilde{u}} &= \begin{pmatrix}
m^2_{\tilde{\nu}} + M^u_{\nu} & -A^u_{\nu} v \sin \beta - M^u_{\nu} \mu^* \cot \beta \\
-A^u_{\nu} v \sin \beta - M^u_{\nu} \mu \cot \beta & m^2_{\nu} + M^u_{\nu} - M^u_W \cos 2\beta \sin^2 \theta_W
\end{pmatrix} \\
&+ \begin{pmatrix}
M^2_Z \cos 2\beta (\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W) & 0 \\
0 & -\frac{1}{3} M^2_Z \cos 2\beta \sin^2 \theta_W \mathbf{1}_{3\times3}
\end{pmatrix},
\end{align*}
\]

(A.8)

The mass matrices of the sleptons are written as follows:

\[
\begin{align*}
\mathcal{L} &= -(m^2_{\tilde{\nu}})^{ij} \tilde{\nu}_i \tilde{\nu}_j - (m^2_{\tilde{\nu}})^{ij} e_i^\dagger e_j, \\
m^2_{\tilde{\nu}} &= m^2_{\tilde{\nu}} + \frac{1}{2} M^2_Z \cos 2\beta \mathbf{1}_{3\times3}, \\
m^2_{\tilde{e}} &= \begin{pmatrix}
m^2_{\tilde{\nu}} + M^e_{\nu} & A^e_{\nu} v \cos \beta - M^e_{\nu} \mu^* \tan \beta \\
A^e_{\nu} v \sin \beta - M^e_{\nu} \mu \tan \beta & m^2_{\nu} + M^e_{\nu} - M^e_W \cos 2\beta \sin^2 \theta_W
\end{pmatrix} \\
&+ \begin{pmatrix}
M^2_Z \cos 2\beta (\frac{1}{2} + \sin^2 \theta_W) & 0 \\
0 & -\frac{2}{3} M^2_Z \cos 2\beta \sin^2 \theta_W \mathbf{1}_{3\times3}
\end{pmatrix}.
\end{align*}
\]

(A.9)

They are diagonalized with unitary matrices as follows:

\[
\begin{align*}
O_R M^\pm_R O^\dagger_L &= \text{diag}(M_{\tilde{\chi}^+}, M_{\tilde{\chi}^-}), \\
O^*_N M^0_N O^\dagger_N &= \text{diag}(M_{\tilde{\chi}^0}, M_{\tilde{\chi}_1^0}, M_{\tilde{\chi}_2^0}, M_{\tilde{\chi}_3^0}, M_{\tilde{\chi}_4^0}), \\
U_{\tilde{f}} m^2_{\tilde{\nu}} U_{\tilde{f}}^\dagger &= \text{diag}(m^2_{\tilde{\nu}_1}, \ldots, m^2_{\tilde{\nu}_6}), \quad (f = u, d, e), \\
U_{\tilde{\nu}} m^2_{\tilde{\nu}} U_{\tilde{\nu}}^\dagger &= \text{diag}(m^2_{\tilde{\nu}_1}, m^2_{\tilde{\nu}_2}, m^2_{\tilde{\nu}_3}).
\end{align*}
\]

(A.10)

B. Lagrangian for fermion-sfermion-gaugino/Higgsino interactions in MSSM

The LFV interactions in the MSSM include the fermion-sfermion-gaugino/Higgsino vertices. These vertices, and corresponding coupling constants \( (C_{iAX}^{L,R}(f), N_{iAX}(f), f = \nu, e, u, d) \) are defined by the following Lagrangian

\[
\mathcal{L} = \overline{u_i} \left[ C_{iAX}^{L(u)} P_L + C_{iAX}^{R(u)} P_R \right] \tilde{\chi}_A^+ d_X + \overline{d_i} \left[ C_{iAX}^{L(d)} P_L + C_{iAX}^{R(d)} P_R \right] \tilde{\chi}_A^0 u_X \\
+ \overline{\nu_i} C_{iAX}^{R(\nu)} P_R \tilde{\chi}_A^0 e_X + \overline{e_i} \left[ C_{iAX}^{L(e)} P_L + C_{iAX}^{R(e)} P_R \right] \tilde{\chi}_A^0 \tilde{\nu}_X \\
+ \overline{u_i} \left[ N_{iAX}^{L(u)} P_L + N_{iAX}^{R(u)} P_R \right] \tilde{\chi}_A^0 \tilde{u}_X + \overline{d_i} \left[ N_{iAX}^{L(d)} P_L + N_{iAX}^{R(d)} P_R \right] \tilde{\chi}_A^0 \tilde{d}_X \\
+ \overline{\nu_i} N_{iAX}^{R(\nu)} P_R \tilde{\chi}_A^0 \tilde{\nu}_X + \overline{e_i} \left[ N_{iAX}^{L(e)} P_L + N_{iAX}^{R(e)} P_R \right] \tilde{\chi}_A^0 \tilde{e}_X \\
+ \text{h.c.}
\]

\[
\equiv \overline{u_i} P_L \tilde{\chi}_A^0 d_X \left[ g \left\{ -\frac{m_{\tilde{u}_i}}{\sqrt{2} M_W \sin \beta} (O_R)_{22} (U^*_d)_{X_i} \right\} \right]
\]
+ \mathcal{L}_f - g \sum_f \mathcal{T} \left[ \mathcal{L}_3^f \left( \frac{1}{c_W} - Q_f \frac{g_2}{c_W} \right) + \mathcal{A} \left( s_W Q_f \right) \right] f, \quad \mathcal{L}_b = -g \sum_b \left( \bar{b}^\dagger i \gamma^\mu \gamma_5 b \right) \left[ Z^\mu \left( \frac{1}{c_W} - Q_f \frac{g_2}{c_W} \right) + A^\mu \left( s_W Q_f \right) \right].
(Q_A) and third component of the weak isospin (I_{3A}) are not in general flavour-diagonal in
the mass basis, because the mixed states may have different I_{3A} values.

For LFV processes the interaction of photon and Z-boson with charginos, neutralinos
and sfermion fields is needed. The Lagrangians for the corresponding weak-basis fields is

easily written knowing the charges and I's and sfermion fields. The Lagrangians for the corres-
ponding weak-basis fields is

of the sfermion field in the mass basis, because the mixed states may have different
components of the weak-basis fields for charginos and neutralinos

In this Appendix the loop functions appearing in the Z-boson amplitude and the box
amplitudes are listed.

D. Loop functions

In this Appendix the loop functions appearing in the Z-boson amplitude and the box
amplitudes are listed.

Z-boson loop functions

The Z-boson amplitude comprises two-loop functions from the triangle-diagram part of
the amplitude, F_1(a,b,c) and F_2(a,b,c), and two-loop functions from the self-energy part of
the amplitude, f_1(a,b) and f_2(a,b) are

\begin{align}
F_1(a,b,c) &= -\frac{1}{b-c} \left[ \frac{a \ln a - b \ln b}{a-b} - \frac{a \ln a - c \ln c}{a-c} \right], \\
F_2(a,b,c) &= 3 - \frac{1}{b-c} \left[ \frac{a^2 \ln a - b^2 \ln b}{a-b} - \frac{a^2 \ln a - c^2 \ln c}{a-c} \right],
\end{align}

Here, in definition of D_{XY} index i is summed over generation indices 1, 2, 3. Q^f is a charge
of the sfermion f, and I_{3f} is its third component of isospin in the weak basis. The charges
and third components of the isospin of the weak-basis fields for charginos and neutralinos
are explicitly written in the definitions of constants E^{L,R}_{AB}(\chi^\pm) and E^{L,R}_{AB}(\chi^0).
\[ f_1(a, b) = \frac{1}{2} \left( \frac{\ln a}{2} - \frac{a^2 + b^2 + 2b^2 \ln a - \ln b}{4(a - b)^2} \right), \quad (D.3) \]

\[ f_2(a, b) = \frac{1}{2} \left( \frac{\ln b}{2} + \frac{a^2 - b^2 + 2a^2 \ln b - \ln a}{4(a - b)^2} \right). \quad (D.4) \]

These loop functions are evaluated by neglecting the momenta of incoming and outgoing particles. The functions \( F_1 \) and \( F_2 \) are symmetric regarding replacement of their arguments \((a, b, c)\). In the limit of two equal argument the functions \( F_1 \) and \( F_2 \) have the following form,

\[ F_1(a, b, b) = \frac{a - b - a \ln a + a \ln b}{(a - b)^2}, \quad (D.5) \]

\[ F_2(a, b, b) = \frac{1}{4} - \frac{\ln b}{4} + \frac{a^2 - b^2 + 2a^2 \ln b - \ln a}{8(a - b)^2}. \quad (D.6) \]

The arguments of logarithms appearing in the \( F_1 \) and \( F_2 \) can be divided by any constant, what can be used to redefine these functions as functions of two variables, for instance \( b/a \) and \( c/a \). The unpleasant \( \ln b \) term in \( f_2 \) function can be replaced with \( \ln(b/a) \) because of unitarity cancellations in the sum \( N_{jBY}^{R(e)} N_{iAX}^{R(e^*)} \), and therefore \( f_1 \) and \( f_2 \) can be expressed in terms of one variable \((b/a)\) only.

**Box-loop functions**

Box amplitude contains two-loop functions, \( d_0 \) and \( d_2 \).

\[ d_0(x, y, z, w) = \frac{x \ln x}{(y - x)(z - x)(w - x)} + \frac{y \ln y}{(x - y)(z - y)(w - y)} \]

\[ + \frac{z \ln z}{(x - z)(y - z)(w - z)} + \frac{w \ln w}{(x - w)(y - w)(z - w)}, \quad (D.7) \]

\[ d_2(x, y, z, w) = \frac{1}{4} \left\{ \frac{x^2 \ln x}{(y - x)(z - x)(w - x)} + \frac{y^2 \ln y}{(x - y)(z - y)(w - y)} \right. \]

\[ + \frac{z^2 \ln z}{(x - z)(y - z)(w - z)} + \left. \frac{w^2 \ln w}{(x - w)(y - w)(z - w)} \right\}. \quad (D.8) \]

As mentioned before, they are also by evaluated neglecting the momenta of incoming and outgoing particles.

**E. Meson states and quark currents**

Meson states are assumed to contain valence quarks only. The quark-antiquark \((q_a q_b^\ast)\) content of the pseudoscalar-meson states is given in the table below. The quark-antiquark content of the vector-meson states is obtained replacing the fields \( K^+, K^0, \pi^+, \pi^0, \pi^- \), \( \bar{K}^0 K^-, \eta_8, \eta_1, \eta \) and \( \eta' \) by fields \( K^{*+}, K^{*0}, \rho^+, \rho^0, \rho^- \), \( K^{*0} \bar{K}^{*-} \), \( \phi_8, \phi_1, \phi \) and \( \omega \), and the angle \( \theta_P \) by the angle \( \theta_V \). From the quark content of the meson fields one can
find the meson content of e.g. axial vector (A) and vector (V) quark currents, (factor of proportionality, Lorentz indices and spinor structures are neglected),

\[
\begin{align*}
\langle \bar{u} u \rangle_A &\sim \left(\frac{c_P}{\sqrt{6}} - \frac{s_P}{\sqrt{3}}\right) \eta^\dagger + \left(\frac{s_P}{\sqrt{6}} + \frac{c_P}{\sqrt{3}}\right) \eta'\dagger + \frac{1}{\sqrt{2}} \pi^0\dagger \equiv k^{0}_{uu} \eta^\dagger + k^{0}_{uu} \eta'\dagger + k^{0}_{uu} \pi^0\dagger \\
\langle \bar{d} d \rangle_A &\sim \left(\frac{c_P}{\sqrt{6}} - \frac{s_P}{\sqrt{3}}\right) \eta^\dagger + \left(\frac{s_P}{\sqrt{6}} + \frac{c_P}{\sqrt{3}}\right) \eta'\dagger - \frac{1}{\sqrt{2}} \rho^0\dagger \equiv k^{0}_{dd} \eta^\dagger + k^{0}_{dd} \eta'\dagger + k^{0}_{dd} \pi^0\dagger \\
\langle \bar{s} s \rangle_A &\sim \left(-\frac{2c_P}{\sqrt{6}} - \frac{s_P}{\sqrt{3}}\right) \eta^\dagger + \left(-\frac{2s_P}{\sqrt{6}} + \frac{c_P}{\sqrt{3}}\right) \eta'\dagger \equiv k^{0}_{ss} \eta^\dagger + k^{0}_{ss} \eta'\dagger \quad (E.1) \\
\langle \bar{u} u \rangle_V &\sim \left(\frac{c_V}{\sqrt{6}} - \frac{s_V}{\sqrt{3}}\right) \phi^\dagger + \left(\frac{s_V}{\sqrt{6}} + \frac{c_V}{\sqrt{3}}\right) \omega^\dagger + \frac{1}{\sqrt{2}} \rho^0\dagger \equiv k^{0}_{uu} \phi^\dagger + k^{0}_{uu} \omega^\dagger + k^{0}_{uu} \rho^0\dagger \\
\langle \bar{d} d \rangle_V &\sim \left(\frac{c_V}{\sqrt{6}} - \frac{s_V}{\sqrt{3}}\right) \phi^\dagger + \left(\frac{s_V}{\sqrt{6}} + \frac{c_V}{\sqrt{3}}\right) \omega^\dagger - \frac{1}{\sqrt{2}} \rho^0\dagger \equiv k^{0}_{dd} \phi^\dagger + k^{0}_{dd} \omega^\dagger + k^{0}_{dd} \rho^0\dagger \\
\langle \bar{s} s \rangle_V &\sim \left(-\frac{2c_V}{\sqrt{6}} - \frac{s_V}{\sqrt{3}}\right) \phi^\dagger + \left(-\frac{2s_V}{\sqrt{6}} + \frac{c_V}{\sqrt{3}}\right) \omega^\dagger \equiv k^{0}_{ss} \phi^\dagger + k^{0}_{ss} \omega^\dagger \quad (E.2)
\end{align*}
\]

Here \( s_V = \sin \theta_V \), and \( c_V = \cos \theta_V \), and the numerical factors are normalizations of mesons expressed in terms quark fields. The combinations of constants \( k^{P,V}_{q_i q_b} \) are contained in expressions for all quark currents (from the scalar-quark current to the tensor-quark current) and we introduce them to abbreviate the expressions for box form factors.

The photon-penguin and the Z-boson–penguin amplitudes contain combinations of constants \( k^{P,V}_{q_i q_b} \) given in Table 3. For instance, \( k^{0}_{\gamma} = Q_u k^{0}_{uu} + Q_d k^{0}_{dd} + Q_s k^{0}_{ss} \).
| k | $\rho^0$ | $\phi$ | $\omega$ |
|---|---|---|---|
| $k_\gamma^V$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}c_V$ | $\frac{1}{\sqrt{2}}s_V$ |
| $k_Z^V$ | $\frac{1}{\sqrt{2}}c_2W$ | $c_Vc_2W + \frac{s_V}{2\sqrt{3}}$ | $\frac{s_V}{\sqrt{6}}c_2W - \frac{c_V}{2\sqrt{3}}$ |

Table 3. Combinations of $k_{\bar{q}_aq_b}^{P,V}$ constants appearing in photon-penguin and the Z-boson–penguin $\ell \to \ell' P(V)$ amplitudes.
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