Classical and quantum N=1 super $W_\infty$-algebras

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Abstract

We construct higher-spin N=1 super algebras as extensions of the super Virasoro algebra containing generators for all spins $s \geq 3/2$. We find two distinct classical (Poisson) algebras on the phase super space. Our results indicate that only one of them can be consistently quantized.

1 Introduction

The infinite-dimensional Virasoro algebra and its extensions play a fundamental rôle in the study of two-dimensional conformal field theories. In particular, the $W_N$-algebras [1] are non-linear algebras which contain additional generators, corresponding to fields with conformal spins $s$ in the interval $2 \leq s \leq N$. In contradistinction, the $W_\infty$-type algebras [2] [3], generated by an infinite set of higher-spin operators with $s \geq 1$ or 2, are linear algebras. They appear in the continuum formulation of two-dimensional quantum gravity coupled to $c = 1$ matter and also in some discrete multi-matrix models which are related to the $c = 1$ theory [4]–[8]. Our interest in super $W_\infty$ algebras was raised in a recent paper [9], where we studied the Schwinger-Dyson (S-D) equations of the N=1 supersymmetric eigenvalue model [10], which is a supersymmetric version of the hermitian one-matrix model written in terms of eigenvalues. We found a correspondence between those S-D equations and the bosonic sector of an N=1 super $W_\infty$-algebra. In this work, we aim to characterize the full N=1 super algebra, including bosonic and fermionic operators. In fact, we have noticed a lack of explicit formulae in the N=1 case, since the literature mostly concerns N=2 and some of its reductions [11]–[13].

We shall start from a classical realization, that is a Poisson algebra on a phase super space, with a pair of commuting and anti-commuting partners $(x, \theta)$ and their conjugate momenta,
(p, Π) respectively. The “quantum” algebras, announced in the title, will be constructed by replacing momenta by differential operators, \( p \rightarrow -i\hbar \partial/\partial x \) and \( \Pi \rightarrow -i\hbar \partial/\partial \theta \), and Poisson brackets by commutators. The Planck’s constant \( \hbar \) will be used to control the classical limit \( (\hbar \rightarrow 0) \) in the usual way, \( \frac{1}{\hbar} [\cdot, \cdot] \rightarrow \{\cdot, \cdot\} \). The spin \( s \) of the generators will be classified \(^{[14]}\) according to their maximal power in momenta (or derivatives): for the bosonic operators, the maximal power is \( p^{s-1} \); for the fermionic ones, we have \( p^{s-1/2} \). The phase space (or differential) realization is specially suitable for higher-spin extensions, because the Jacobi identity (which is rather cumbersome to check for \( W_\infty \)-algebras) is already built in and it can be effectively used to derive several brackets, so that the calculations become altogether simpler.

In section 2 we describe the classical \( w_\infty \)-algebra, two supersymmetric extensions and a geometric interpretation. The quantization is presented in section 3 and the corresponding classical limit is discussed. Section 4 is dedicated to final comments and conclusions.

### 2 Classical N=1 super \( w_\infty \)-algebras

In the bosonic case, the \( w_\infty \)-algebra is equivalent to the Poisson algebra of smooth area-preserving diffeomorphisms on a two-dimensional phase space \( (x, p) \). Following refs.\(^{[2]} \) \(^{[3]} \), we introduce the Poisson brackets:

\[
\{f(x, p), g(x, p)\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} .
\]  

(1)

The area-preserving transformations, which preserve the 2-form \( \omega = dx \wedge dp \), correspond to canonical transformations generated by smooth functions \( \rho(x, p) \) via Poisson brackets, \( f \rightarrow f + \epsilon \{f, \rho(x, p)\} \). The smooth functions \( \rho \) can be expanded as \( \rho = \sum_{s,n} \rho_{sn} w_n^{(s)} \), where we take the basis

\[
w_n^{(s)} = x^{n+1} p^{s-1} .
\]  

(2)

This set of functions generate the classical \( w_\infty \)-algebra \(^{[2]} \)

\[
\{w_m^{(r)} , w_n^{(s)}\} = [(s-1)(m+1) - (r-1)(n+1)] w_{m+n}^{(r+s-2)} ,
\]  

(3)

which can be seen as a higher-spin extension \( (s \geq 2) \) of the \( s = 2 \) Virasoro algebra generated by \( w_n^{(2)} = x^{n+1} p \). Introducing a Grassmann-odd spin-3/2 generator \( g_n^{(3/2)} \), the Virasoro algebra can be extended to a superconformal algebra\(^{[4]} \)

\begin{align*}
\{g_m^{(3/2)} , g_n^{(3/2)}\} &= 2w_{m+n+1}^{(2)} , \\
\{g_m^{(3/2)} , w_n^{(2)}\} &= \left[(m+1) - \frac{1}{2}(n+1)\right] g_{m+n}^{(3/2)} , \\
\{w_m^{(2)} , w_n^{(2)}\} &= (m-n)w_{m+n}^{(2)} .
\end{align*}

(4)

Assuming the canonical graded Poisson brackets, \( \{x, p\} = 1 \), \( \{\theta, \Pi\}_+ = -1 \), the most general realization for \( g_n^{(3/2)} \) and \( w_n^{(2)} \), which is compatible with the infinitesimal conformal transformations

\[^{1}\text{Throughout this paper, we shall consider the Neveu-Schwarz sector of the superconformal algebra.}\]
\[ \delta x = \{ x, \epsilon w_n^{(2)} + \alpha g_n^{(3/2)} \} = \epsilon x^{n+1} + \alpha \theta x^{n+1} , \quad (5) \]
\[ \delta \theta = \{ \theta, \epsilon w_n^{(2)} + \alpha g_n^{(3/2)} \} = \epsilon \left( \frac{n+1}{2} \right) x^n \theta + \alpha x^{n+1} , \quad (6) \]

and with the algebra (4), is given by \[ g_n^{(3/2)}(\lambda) = x^{n+1}(\theta p - \Pi) + 2\lambda(n+1)x^n \theta , \quad (7) \]
\[ w_n^{(2)}(\lambda) = x^{n+1}p + (n+1)x^n \left( \lambda + \frac{\theta \Pi}{2} \right) , \quad (8) \]

where \( \lambda \) is an arbitrary real constant.

To include higher-spin generators and extend the super Virasoro algebra (4), we make the following assumptions:

i) The lowest spin is \( s = 3/2 \).

ii) There exists a fermionic generator with spin \( s = 5/2 \).

iii) The Poisson algebra of fermionic generators must obey the rule:

\[ \{ g^{(r)}, g^{(s)} \} \propto w^{(r+s-1)} + \text{lower spins} . \]

iv) Each generator \( g_n^{(s)} \) is characterized by two indices: \( s \) corresponds to its spin, and \( n \) to its conformal dimension (the eigenvalue of \( L_0 = w_0^{(2)} \)).

We try the most general Ansatz for the next-spin generator, \( g_n^{(5/2)} \), in agreement with the assumptions i)-iv), such that the algebra with \( g_n^{(3/2)} \) gets closed:

\[ g_n^{(5/2)} = x^m \theta p^2 + c_m x^m p \Pi + d_m x^{m-1} \Pi + e_m x^{m-2} \theta . \quad (9) \]

In order to calculate the arbitrary constants \( c_m, d_m, e_m \) we verify that:

\[ \{ g_n^{(3/2)}, g_m^{(5/2)} \} = (d_m + 2\lambda n c_m) w_n^{(2)} + (c_m - 1) x^{n+m} p^2 + R_{nm} x^{n+m-1} p \Pi \theta + S_{nm} x^{n+m-2} (\theta \Pi - 2\lambda) + T_{nm} x^{n+m-2} , \quad (10) \]

where \( R_{nm}, S_{nm}, T_{nm} \) are given functions of \( n, m, c_m, d_m, e_m \). The next step is to determine the most general linear combinations of the terms on the r.h.s. of (10) (except, of course, \( w_n^{(2)} \) which is already in the algebra), so that they close the algebra with \( g_n^{(3/2)} \). We define such combinations as:

\[ V_n^{(3)} = a_m x^m p^2 + f_m x^{m-1} p \Pi \theta + g_m x^{m-2} (\theta \Pi - 2\lambda) + h_m x^{m-2} . \quad (11) \]

It is easy to see that \( \{ g_n^{(3/2)}, V_n^{(3)} \} \) will produce the term \( x^{n+m-1} \theta p^2 \), among others with lower spins. This term must be part of \( g_n^{(5/2)} \) due to the uniqueness of the spin-5/2 generator.

\[ ^2 \text{Actually, the most general form would be } g_n^{(3/2)} = x^{n+1}(\theta p - \Pi) + (2\lambda n + \gamma)x^n \theta , \text{ but the parameter } \gamma \text{ can be shifted by a canonical automorphism generated by } \gamma \ln |x| . \text{ Therefore, we may take e.g. } \gamma = 2\lambda \text{ with no loss of generality.} \]
Indeed, if there were more than one solution for $c_m, d_m, e_m$ for a given algebra, either the assumption i) or iii) would fail. Therefore, the closure of $\{g_{n-1}^{(3/2)}, V_m^{(3)}\}$ imposes the following conditions:

\[
\begin{align*}
((m - 2n)a_m + f_m)c_{m+n-1} &= 2na_m + f_m, \\
((m - 2n)a_m + f_m)d_{m+n-1} &= 2(\lambda n(f_m - 2(n - 1)a_m) - g_m), \\
((m - 2n)a_m + f_m)e_{m+n-1} &= (m - 2)h_m + 4n\lambda^2(n + m - 2)(2(n - 1)a_m - f_m).
\end{align*}
\]

These equations require that $\lambda = 0$ and we find two possible Ansätze for $g_n^{(5/2)}$, corresponding to two different algebras. All higher spins generators, $w_n^{(s)} (s \geq 2)$ and $g_n^{(s)} (s > 5/2)$, are obtained from $s \leq 5/2$ generators. We end up with $\lambda = 0$ and two possible N=1 supersymmetric $w$-algebras:

**Type 1:** This type is generated by

\[
g_n^{(k+3/2)} = x^{n+1}p^k(\theta p - \Pi),
\]

\[
w_n^{(s)} = x^{n+1}p^{s-1} + \frac{1}{2}(n + 1)x^n p^{s-2}\theta\Pi,
\]

with the following algebra:

\[
\begin{align*}
\{g_m^{(r)}, g_n^{(r')}\} &= 2w_{m+n+1}^{(r+r'-1)}, \\
\{g_m^{(r)}, w_n^{(s)}\} &= [(s - 1)(m + 1) - (r - 1)(n + 1)] g_{m+n}^{(r+s-2)}, \\
\{w_m^{(s)}, w_{n}^{(s')}\} &= [(s' - 1)(m + 1) - (s - 1)(n + 1)] w_{m+n}^{(s+s'-2)},
\end{align*}
\]

where $r, r' = 3/2, 5/2, \cdots$ and $s, s' = 2, 3, 4, \cdots$. In fact, this algebra appeared in [12], in a more complicated realization.

**Type 2:** In this case, the generators split in four families with only even spins in the bosonic sector. The generators are given by:

\[
g_n^{(2a+3/2)} = x^{n+1}p^{2a}(\theta p - \Pi),
\]

\[
\overline{g}_n^{(2a+1)+3/2} = x^{n+1}p^{2a+1}(\theta p + \Pi),
\]

\[
w_n^{(2a+2)} = x^{n+1}p^{2a+1} + \frac{1}{2}(n + 1)x^n p^{2a}\theta\Pi,
\]

\[
k_n^{(2a+2)} = x^{n+1}p^{2a+1}\theta\Pi,
\]

with the corresponding classical algebra,

\[
\begin{align*}
\{g_m^{(r)}, g_n^{(r')}\} &= 2w_{m+n+1}^{(r+r'-1)}, \\
\{g_m^{(r)}, w_n^{(s)}\} &= [(s - 1)(m + 1) - (r - 1)(n + 1)] g_{m+n}^{(r+s-2)}, \\
\{w_m^{(s)}, w_{n}^{(s')}\} &= [(s' - 1)(m + 1) - (s - 1)(n + 1)] w_{m+n}^{(s+s'-2)},
\end{align*}
\]

\[
\{\overline{g}_m^{(r)}, \overline{g}_n^{(r')}\} = -2w_{m+n+1}^{(r+r'-1)},
\]

\[
\{\overline{g}_m^{(r)}, w_n^{(s)}\} = [(s - 1)(m + 1) - (r - 1)(n + 1)] \overline{g}_{m+n}^{(r+s-2)},
\]

4


\{g_m^{(r)} , g_n^{(s)} \} = 2 ((r' - 1)(m + 1) - (r - 1)(n + 1)) k_{m+n}^{(r+s-2)} ,
\{g_m^{(r)} , k^{(s)} \} = \tilde{g}_{m+n+1}^{(r+s-1)} ,
\{\tilde{g}_m^{(r)} , k^{(s)} \} = g_{m+n+1}^{(r+s-1)} ,
\{k^{(s)} , k^{(s')} \} = 0 ,
\{k^{(s)} , w_n^{(s')} \} = \left[(s' - 1)(m + 1) - (s - 1)(n + 1)\right] k_{m+n}^{(s+s'-2)} . \quad (22)

As far as we know, this algebra has not appeared yet in the literature and we shall call it super even \( w_{\infty} \)-algebra. We note, in passing, that the two algebras (17) and (22) have a sub-algebra in common, generated by \( g_n^{(2a+3/2)} \) and \( w_m^{(2a)} \). This algebra is called super \( w_{\infty}' \), since its bosonic sector corresponds to the \( w_{\infty} \) truncated to even spins, i.e. \( w_{\infty}' \).

Both Poisson algebras are related to area-preserving diffeomorphisms: they preserve the 2-form \( w = dx \wedge dp - d\Pi \wedge d\theta \) [11]. The super \( w_{\infty} \)-algebra corresponds to transformations generated by the following kind of functions:
\[ \rho_A = \phi(x + \frac{\theta \Pi}{2p}, p) + (\theta p - \Pi) \psi(x, p) , \quad (23) \]
while the super even algebra is related to generating functions of the form:
\[ \rho_B = p \phi(x + \frac{\theta \Pi}{2p}, p^2) + \theta \Pi p \varphi(x, p^2) + (\theta p - \Pi) \psi(x, p^2) + (\theta p + \Pi) \eta(x, p^2) . \quad (24) \]

Above, \( \phi, \varphi, \psi \) and \( \eta \) are smooth functions of two variables. These generators correspond to two different invariant sub-groups of (super)area-preserving diffeomorphisms. In fact, if \( \rho_1 \) and \( \rho_2 \) have the form (23), so will have \( \rho_3 = \{ \rho_1, \rho_2 \} \). An analogous result holds for functions of the type (24). We recall that, in a general basis, for arbitrary smooth functions \( \rho(x, p, \theta, \Pi) \), one finds an N=2 super \( w_{\infty} \)-algebra (see [12] [15]).

### 3 Quantum N=1 super \( W_{\infty} \)-algebra

By “quantum” algebra we mean algebra of commutators, as a quantized version of the Poisson algebras analyzed in section 2. In the bosonic case, the Virasoro algebra is generated by the differential operators
\[ L_n \equiv W_n^{(2)} = -i \hbar x^{n+1} \partial \quad , \quad (25) \]
obtained from its classical counterpart \( w_n^{(2)} \) in (2) after the replacement \( p \rightarrow -i \hbar \partial \). The set of higher spin operators
\[ W_n^{(s)} = (-i \hbar)^{s-1} x^{n+1} \partial^{s-1} , \quad s \geq 1 \quad , \quad (26) \]
generate the so called \( W_{1+\infty} \)-algebra, given by:
\[ [W_m^{(r)} , W_n^{(s)} ] = -i \hbar \sum_{k \geq 0} (-i \hbar)^k C_{mn}^{rs}(k) W_{m+n-k}^{(r+s-2-k)} \quad , \quad (27) \]
\[ C_{mn}^{rs}(k) = \frac{1}{(k+1)!} \left( \frac{\Gamma(r) \Gamma(n+k+1)}{\Gamma(r-k-1) \Gamma(n-k+1)} - \frac{\Gamma(s) \Gamma(m+k+1)}{\Gamma(s-k-1) \Gamma(m-k+1)} \right) . \quad (28) \]
The generator (25) can be generalized into the form (see [13])

\[ W_n^{(2)}(\lambda) = -i\hbar \left( x^{n+1} \partial + \lambda(n+1)x^n \right) \] . (29)

We are interested in a basis of operators which satisfy the following condition (originally used to discover the \( W_\infty \)-algebra [3]):

\[ [W_m^{(r)}, W_n^{(s)}] = -i\hbar \left( c_0 W_{m+n}^{(r+s-2)} + c_1 W_{m+n-2}^{(r+s-4)} + \cdots \right) \] . (30)

This sort of basis is convenient because the algebra can be truncated in only even-spin sub-algebras. Moreover, it admits a central extension [3].

The condition (30) restricts the possible values of the parameter \( \lambda \). In analogy to the last section, we take an Ansatz for \( W_m^{(3)} \) and we find two solutions (in agreement with [15]):

i) If \( s \geq 1 \), we have \( \lambda = 1/2 \) and the \( W_{1+\infty} \)-algebra. The first few generators are given below:

\[ W_n^{(1)} = x^{n+1}, \]
\[ W_n^{(2)} = (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2}(n+1)x^n \right), \]
\[ W_n^{(3)} = (-i\hbar)^2 \left( x^{n+1} \partial^2 + (n+1)x^n \partial \right), \]
\[ W_n^{(4)} = (-i\hbar)^3 \left( x^{n+1} \partial^3 + \frac{3}{2}(n+1)x^n \partial^2 + \frac{1}{2}n(n+1)x^{n-1} \partial \right) \] . (31)

Higher spin operators can be obtained via commutators.

ii) If \( s \geq 2 \), one has two equivalent cases, \( \lambda = 0 \) or 1. When \( \lambda = 0 \) one finds a \( W_\infty \)-algebra, generated by:

\[ W_n^{(2)} = (-i\hbar) x^{n+1} \partial, \]
\[ W_n^{(3)} = (-i\hbar)^2 \left( x^{n+1} \partial^2 + \frac{1}{2}(n+1)x^n \partial \right), \]
\[ W_n^{(4)} = (-i\hbar)^3 \left( x^{n+1} \partial^3 + (n+1)x^n \partial^2 \right) \], etc. (32)

The solution \( \lambda = 1 \) corresponds to an automorphism of the above generators, leading to an isomorphic \( W_\infty \)-algebra.

Now we present the \( N=1 \) supersymmetric extension of the \( W_\infty \)-algebra. First, we introduce an anti-commuting variable \( \theta \) and proceed in analogy to the classical study, by making the following assumptions:

i) The lowest spin (\( s = 3/2 \)) generator [13] is

\[ G_n^{(3/2)} = (-i\hbar) \left( x^{n+1}(\theta \partial - \partial \theta) + 2\lambda(n+1)x^n \theta \right) \] . (33)

The operators \( L_n(\lambda) = -i\hbar(x^{n+1}\partial + (a + \lambda n)x^n) \) also generate the Virasoro algebra. However, the parameter \( a \) can be arbitrarily shifted by the homeomorphism \( L_n \rightarrow x^{ikc} L_n x^{-ikc} \Rightarrow a \rightarrow a + c \). Thus, we may take \( a = \lambda \).
Together with the spin-2 operator,
\[ W_n^{(2)} = (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2} (n+1)x^n (\theta \partial_\theta + 2\lambda) \right) , \] (34)
they generate the super Virasoro algebra:
\[ [G_m^{(3/2)}, G_n^{(3/2)}] = 2i\hbar W_{m+n+1}^{(2)} , \]
\[ [G_m^{(3/2)}, W_n^{(2)}] = i\hbar \left( (m+1) - \frac{1}{2} (n+1) \right) G_{m+n}^{(3/2)} , \]
\[ [W_m^{(2)}, W_n^{(2)}] = i\hbar (m-n) W_{m+n}^{(2)} , \] (35)
whose classical limit coincides with the algebra \[ \mathfrak{g} \].

ii) We assume the existence of a spin-$5/2$ generator, whose most general expression is:
\[ G_n^{(5/2)} = (-i\hbar)^2 \left( x^{n+1} \partial (\theta \partial + c_n \partial_\theta) + d_n x^n \partial_\theta + e_n x^{n-1} \theta \right) , \] (36)
where the constants $c_n, d_n, e_n$ must be determined.

iii) The anti-commutation algebra should obey the rule:
\[ [G_m^{(r)}, G_n^{(s)}] \propto W_{m+n+1}^{(r+s-1)} + \text{lower spins} \]. (37)

iv) Each operator $G_n^{(s)}$ is characterized by its spin $(s)$ and its conformal dimension $(n)$.

Under these assumptions, we find two solutions, $\lambda = 0$ or $1/2$, which are related to each other by an automorphism. Therefore, we may simply take $\lambda = 0$ and the resulting $N=1$ super $W_{\infty}$-algebra can be generated by the following basis of operators (we present the lowest spins, since higher spins can be produced by commutators):

\[ G_n^{(3/2)} = (-i\hbar)x^{n+1} (\theta \partial - \partial_\theta) , \]
\[ G_n^{(5/2)} = (-i\hbar)^2 \left( x^{n+1} \partial (\theta \partial + \partial_\theta) + (n+1)x^n \partial_\theta \right) , \]
\[ G_n^{(7/2)} = (-i\hbar)^3 \left( x^{n+1} \partial^2 (\theta \partial - \partial_\theta) - 2(n+1)x^n \partial \partial_\theta - n(n+1)x^{n-1} \partial_\theta \right) , \]
\[ G_n^{(9/2)} = (-i\hbar)^4 \left( x^{n+1} \partial^3 (\theta \partial + \partial_\theta) + 3(n+1)x^n \theta \partial^2 + 3n(n+1)x^{n-1} \theta \partial^2 
+ (n-1)n(n+1)x^{n-2} \partial_\theta \right) , \]
\[ W_n^{(2)} = (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2} (n+1)x^n \theta \partial_\theta \right) , \]
\[ K_n^{(2)} = (-i\hbar)^2 x^{n+1} \partial \partial_\theta \partial \theta , \]
\[ W_n^{(4)} = (-i\hbar)^3 \left( x^{n+1} \partial^3 + \frac{3}{2} (n+1)x^n \partial^2 + \frac{1}{2} n(n+1)x^{n-1} \partial 
- \frac{1}{2} (n+1)x^n \partial \partial_\theta \partial \theta \right) , \]
\[ K_n^{(4)} = (-i\hbar)^4 \left( x^{n+1} \partial^3 + (n+1)x^n \partial^2 \right) \partial_\theta \theta \]
We calculated various commutators (up to spin $s=6$; further commutators can be obtained by means of the Jacobi identity), but we were unable to find a closed form for all structure coefficients. The lowest-spin algebra is listed below:

\[
W_n^{(6)} = (-i\hbar)^5 \left( x^{n+1} \partial^5 + \frac{5}{2} (n+1) x^n \partial^4 + 2n(n+1)x^{n-1} \partial^3 \\
+ \frac{1}{2} (n-1)n(n+1)x^{n-2} \partial^2 (1 + \partial_\theta) - \frac{1}{2} (n+1)x^n \partial^4 \partial_\theta \right), \\
K_n^{(6)} = (-i\hbar)^6 \left( x^{n+1} \partial^5 + 2(n+1)x^n \partial^4 + n(n+1)x^{n-1} \partial^3 \right) \partial_\theta.
\]  

(38)

We have also verified that the operators in (38) become the generators (18-21) in the classical limit (given by the associations $-i\hbar \partial \rightarrow p$, $-i\hbar \partial_\theta \rightarrow \Pi$, when $\hbar \rightarrow 0$). Therefore, we may say that the generators (38) realize a (quantum) N=1 super $W_\infty$-algebra.

Concerning the bosonic sector, composed by $W_n^{(2s)}$ and $K_n^{(2r)}$, it is possible to take linear combinations and find a basis with two decoupled sub-algebras $\mathfrak{l}$. For instance, if we define

\[
\tilde{W}_n^{(2)} = K_n^{(2)} + i\hbar W_n^{(2)},
\]

the resulting lowest-spin algebra becomes

\[
[\tilde{W}_m^{(2)}, \tilde{W}_n^{(2)}] = (-i\hbar)^2 (n-m) \tilde{W}_{m+n}^{(2)}, \\
[\tilde{W}_m^{(2)}, K_n^{(2)}] = 0, \\
[K_m^{(2)}, K_n^{(2)}] = (-i\hbar)^2 (n-m) K_{m+n}^{(2)}.
\]

(41)
This decoupling was also verified for higher spins. The redefined $\tilde{W}$-operators turn out to generate an algebra isomorphic to the even-spin sector of the bosonic $W_{1+\infty}$-algebra. On the other hand, the algebra of the operators $K_n^{(2r)}$ is isomorphic to the even-spin subalgebra of the $W_\infty$-algebra. Therefore, the bosonic sector of the super algebra generated by (38) realizes a $(W_\infty + W_{1+\infty})$-algebra [9] [13] [14].

It is tempting to call “N=1 super $(W_\infty + W_{1+\infty})$-algebra” the one generated by the whole set of operators in (38). We believe this is acceptable at the quantum level, i.e. as long as $\bar{\hbar} \neq 0$. However, in the classical limit ($\hbar \to 0$) the transformation (40) does not give independent generators and the bosonic sector does not split in two decoupled sub-algebras. This implies that the Poisson algebra generated by (18-21) should not be called a “classical super $(w_\infty + w_{1+\infty})$” – we had better keep the name N=1 super even $w_\infty$-algebra.

4 Final remarks, conclusion and open questions

We have constructed the N=1 supersymmetric extensions of the $W_\infty$-algebras. At the classical level, we found two Poisson algebras, the super $w_\infty$ and the super even $w_\infty$. In the quantum case, we found only one consistent algebra, denominated super $(W_\infty + W_{1+\infty})$. Its classical limit coincides with the super even $w_\infty$. The algebra $(W_\infty + W_{1+\infty})$ was first observed in [12] as a truncation of a N=2 super $W_\infty$-algebra. We stress that we obtained it in a constructive way, without any embedding in higher algebras.

Although we did not find a general expression for all the quantum operators, we noticed that the available generators can be rewritten in reduced forms. For instance, the bosonic $K$-operators in (38) can be expressed as

$$K_n^{(2r)} = (-i\hbar)^{2r} \partial^{r-1} x^{n+1} \partial^r \partial_\theta = p^{r-1} x^{n+1} p^r \Pi \theta .$$  (42)

The fermionic operators in (38) can be written as linear combinations of

$$\tilde{G}_n^{(s+1/2)} = (-i\hbar)^s \left( \partial^{s-1} x^{n+1} \partial \theta + (-)^s x^{n+1} \partial^{-1} \partial_\theta \right) = p^{s-1} x^{n+1} \theta p + (-)^s x^{n+1} p^{s-1} \Pi .$$  (43)

Therefore, we expect the quantum operators to correspond to some special ordering of the classical generators. If we could understand this ordering we might eventually find a closed form for the complete algebra.

We have shown how the super even $w_\infty$-algebra can be obtained from the quantum super $(W_\infty + W_{1+\infty})$ by means of a suitable limit ($\hbar \to 0$). It is natural to ask whether there is any quantum super $W_\infty$-algebra whose classical limit is the super $w_\infty$ given by eqs.(43-47). We do not have an answer to that question yet. It would also be interesting to study the possible central extensions [3] of these quantum algebras.

\footnote{In the quantum case, we may choose a unit system where $\hbar = 1$.}
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References

[1] A.B. Zamolodchikov, Teor. Matt. Fiz. 65 (1985) 347.
[2] I. Bakas, Phys. Lett. B228 (1989) 57.
[3] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B236 (1990) 173.
[4] M. Fukuma, H. Kawai and R. Nakayama, Int. J. of Mod. Phys. A6 (1991) 1385.
   R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B348 (1991) 435.
[5] A. Mironov and B. Morozov, Phys. Lett. B252 (1990) 47.
   L. Alvarez-Gaumé, C. Gomez and J. Lacki, Phys. Lett. B253 (1991) 56.
[6] H. Itoyama and Y. Matsuo, Phys. Lett. B255 (1991) 202.
[7] H. Itoyama and Y. Matsuo, Phys. Lett. B262 (1991) 233.
[8] L. Bonora and C.S. Xiong, Phys. Lett. B347 (1995) 41; Nucl. Phys. B434 (1995) 408.
[9] L.O. Buffon, D. Dalmazi and A. Zadra, hep-th/9604184.
[10] L. Alvarez-Gaumé, H. Itoyama, J.L. Mañés and A. Zadra, Int. J. Mod. Phys. A7 (1992) 5337.
[11] E. Sezgin and R. Sokatchev, Phys. Lett. B227 (1989) 103.
[12] C.N. Pop and X. Shen, Phys. Lett. B236 (1990) 21.
[13] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, Phys. Lett. B245 (1990) 447.
[14] E. Bergshoeff, M. Vasiliev and B. de Wit, Phys. Lett. B256 (1991) 199.
[15] E. Bergshoeff, B. de Wit and M. Vasiliev, Nucl. Phys. B366 (1991) 315.