Breathing skyrmions in chiral antiferromagnets

S. Komineas\(^1,2\) and P. E. Roy\(^3\)

\(^1\)Department of Mathematics and Applied Mathematics, University of Crete, 70013 Heraklion, Crete, Greece
\(^2\)Institute of Applied and Computational Mathematics, FORTH, Heraklion, Crete, Greece
\(^3\)Hitachi Cambridge Laboratory, Hitachi Europe Limited, Cambridge CB3 0HE, United Kingdom

(Dated: March 9, 2022)

Breathing oscillations of skyrmions in chiral antiferromagnets can be excited by a temporal modification of the Dzyaloshinskii-Moriya (DM) interaction \[^1,2\]. They are commonly observed in chiral ferromagnetic films, typically extending a few or tens of nanometers laterally, and their nontrivial topology makes them robust against perturbations. Skyrmions exhibit particle-like dynamics \[^3\] which, together with their small size, lead to properties such as a low driving threshold current \[^4\]. These qualities have given rise to a range of proposed applications such as their use as the constituent information carriers in racetrack memory/logic devices \[^5-9\], memristor elements in artificial synapses for neuromorphic computing architectures \[^10\], in spintronics-based transistor device concepts \[^11\], and as spin-wave scatterers in magnonic computing and logic devices \[^12\].

Most of the existing work is focused on ferromagnetic skyrmions. On the other hand, skyrmions are expected to exist also in antiferromagnets (AFM) for essentially the same reasons as in ferromagnets. Antiferromagnetic materials have advantages such as high operational frequencies in the THz range and robustness against external magnetic field perturbations. Furthermore, unlike in ferromagnets and despite their topological nature, skyrmions in AFMs do not present a Hall angle in their dynamical behavior. These properties have prompted proposals to replace ferromagnetic skyrmions by their antiferromagnetic counterparts \[^11,13-18\]. Recent experimental observations of stable skyrmions in AFMs \[^19,20\] give further motivation for the study of AFM skyrmions for future device implementations.

The presence of the chiral DM interaction is crucial for the dynamics of solitons, in addition to its role in their stabilization. A skyrmion breathing mode arises in chiral ferromagnets \[^21\] due to the breaking of the conservation of the total magnetization perpendicular to the film. A similar mode exists also for antiferromagnetic skyrmions. In the latter case, independent oscillations of the skyrmion radius and of its chirality are possible \[^22\].

We are motivated by the emerging kinetic energy in the AFM continuum \[^23,24\] and we propose an effective potential for the skyrmion oscillation dynamics. This leads to a systematic approach in order to understand details of these dynamical modes, such as the frequency of oscillations, by a combination of analytical and numerical methods. The form of the effective potential leads to the observation that an expansion of the skyrmion can eventually lead to its annihilation by a subsequent collapse due to internal breathing dynamics. This counter-intuitive method shows that a large external force is not necessary in order to annihilate the topological texture; instead skyrmion annihilation is obtained by small controlled changes of the DM or anisotropy parameters that can be induced, e.g., by a voltage pulse \[^25\], in combination with internal dynamics.

It would be an ideal situation to have methods for skyrmion creation and annihilation by controlled small perturbations while at the same time the skyrmion remains robust to all usual perturbations. If this is achieved the potential for skyrmions as functional objects would be significantly enhanced. In this context, our method shows the way for controlled skyrmion annihilation by small perturbations. In addition, the successful analytical arguments presented may be useful in studies for a method towards controlled individual skyrmion generation using only mild forces.

Sec. \[^11\] discusses the energy in an antiferromagnet in the discrete and in the continuum model. Sec. \[^11\] shows how skyrmion annihilation via breathing dynamics can be achieved. In Sec. \[^15\] we derive the effective potential for breathing oscillations and give a theoretical description of the annihilation dynamics. In Sec. \[^16\] we derive the frequency of small amplitude breathing oscillations in the case of skyrmions with a large and a small radius.
Sec. VI discusses helicity oscillations for the skyrmion. Sec. VII contains our concluding remarks.

II. SKYRMION ENERGY

We consider a square lattice of spins in a material with the usual exchange, perpendicular anisotropy and Dzyaloshinskii-Moriya interaction. The magnetic energy of the lattice is

\[ E^d = \sum_{i,j} J_n \mathbf{S}_{i,j} \cdot (\mathbf{S}_{i+1,j} + \mathbf{S}_{i,j+1}) + \frac{k}{2} [1 - (\mathbf{S}_{i,j})_z^2] \]

\[ + D [\hat{\mathbf{e}}_2 \cdot (\mathbf{S}_{i,j} \times \mathbf{S}_{i+1,j}) - \hat{\mathbf{e}}_1 \cdot (\mathbf{S}_{i,j} \times \mathbf{S}_{i,j+1})] \]

where the spin variables are assumed normalized \(|\mathbf{S}_{i,j}| = 1\). We will typically use, in numerical simulations, the parameter values

\[ J_n = 2 \times 10^{-21} \text{ Joule}, \quad D = 0.047J_n, \quad k = 0.01J_n. \]  

The equation of motion for the spins is

\[ \frac{\partial \mathbf{S}_{i,j}}{\partial t} = -\gamma \mathbf{S}_{i,j} \times \mathbf{F}_{i,j} + \alpha \mathbf{S}_{i,j} \times \mathbf{S}_{i,j} \times \frac{\partial \mathbf{S}_{i,j}}{\partial t}, \]

\[ \mathbf{F}_{i,j} = -\frac{1}{\mu_0 \mu_s} \frac{\partial E^d}{\partial \mathbf{S}_{i,j}} \]

where \( \mathbf{F} \) is the effective field, \( \gamma = g_e \mu_B \mu_0 / h = 2.211 \times 10^6 \text{ m A}^{-1}\text{s}^{-1} \) is the gyromagnetic ratio, and \( \alpha \) is the damping parameter. We choose the saturation magnetization \( \mu_s = 4\mu_B \), where \( \mu_B \) is the Bohr magneton. The material parameters have been chosen to resemble what is expected for a range of antiferromagnetic oxides [26, 27].

A continuum model is obtained if we consider a lattice of spin tetramers where the normalized Néel vector \( \mathbf{n}_{\alpha,\beta} \) is defined at tetramer sites \((\alpha, \beta)\) [24]. The distance between tetramer sites is defined to be \( 2\epsilon \) where \( \epsilon \equiv \sqrt{k/J} \) is a small parameter. In the limit \( \epsilon \rightarrow 0 \), a continuous Néel vector field \( \mathbf{n} = \mathbf{n}(x, y, \tau) \) is obtained, with \( \mathbf{n}^2 = 1 \), where \( x, y \) and \( \tau \) are scaled space and time variables [28, 29, 30, 31]. The energy in the continuum is (see also Refs. [30, 31])

\[ E = T + V_\lambda \]  

where the kinetic energy is

\[ T = \frac{1}{2} \int \mathbf{n}^2 \, d^2x \]  

and the dot denotes differentiation with respect to the scaled time \( \tau \), and the potential energy

\[ V_\lambda = E_{\text{ex}} + E_{\text{DM}} + E_{\text{an}} \]  

which includes exchange, DM, and anisotropy contributions,

\[ E_{\text{ex}} = \frac{1}{2} \int (\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n}) \, d^2x \]

\[ E_{\text{DM}} = \lambda \int \epsilon_{\mu\nu} \hat{\mathbf{e}}_\mu \cdot (\partial_\nu \mathbf{n} \times \mathbf{n}) \, d^2x \]  

\[ E_{\text{an}} = \frac{1}{2} \int (1 - n_\alpha^2) \, d^2x \]

with \( \lambda = D/\sqrt{k/J_n} \) being a scaled DM parameter, and all energy components being in units of \( J_n \). Symbols \( \partial_\mu, \partial_\nu \), with \( \mu, \nu = 1, 2 \), denote differentiation with respect to \((x, y)\), respectively, \( \epsilon_{\mu\nu} \) is the antisymmetric tensor, and the summation convention for repeated indices is adopted. Note that \( x \) is a scaled coordinate and actual distances (in physical units) are given by

\[ ax/\epsilon, \quad ay/\epsilon \]  

where \( a \) is the distance between neighboring spins. The unit of time is

\[ \tau_0 = \frac{\mu_s}{g_e \mu_B 2\sqrt{2k/J_n}} = 0.373 \text{ ps} \]

where the numerical value corresponds to the parameter values in Eq. (2) and \( \mu_s = 4\mu_B \).

The potential energy \( V_\lambda \) is identical in form to the energy for a ferromagnet with corresponding interactions. It is thus known that the ground state is uniform (Néel) for \( \lambda < 2/\pi \) while it is the spiral for \( \lambda > 2/\pi \) [2]. Isolated skyrmions are localised excitations on a uniform background. On the other hand, the presence of the kinetic term in Eq. (4) opens possibilities that are not there in ferromagnets. We will study oscillations of the skyrmion.

We consider an axially-symmetric skyrmion configuration for the Néel vector, written conveniently in terms of the spherical variables

\[ \Theta = \Theta(r, \tau), \quad \Phi = \phi + \chi(t) \]

where \((r, \phi)\) are polar coordinates and \( \chi \) is an angle that may be called the helicity. We will assume that \( \chi \) may depend on time only or it is a constant. The kinetic energy of Eq. (5) takes the form

\[ T = \frac{1}{2} \int \left( \dot{\Theta}^2 + \sin^2 \Theta \dot{\chi}^2 \right) 2\pi r \, dr. \]

The exchange and anisotropy energies depend on \( \Theta \) only,

\[ E_{\text{ex}} = \frac{1}{2} \int \left[ (\Theta')^2 + \frac{\sin^2 \Theta}{r^2} \right] 2\pi r \, dr, \]  

\[ E_{\text{an}} = \frac{1}{2} \int \sin^2 \Theta \, 2\pi r \, dr \]

where the prime denotes differentiation with respect to the space variable \( r \). The DM term is written as

\[ E_{\text{DM}} = \lambda \cos \chi \, \tilde{E}_{\text{DM}}, \]

\[ \tilde{E}_{\text{DM}} = \int \left( \Theta' + \frac{\cos \Theta \sin \Theta}{r} \right) 2\pi r \, dr. \]
In the continuum theory, given by

\[ \Theta = \frac{\pi}{R} \]

\[ \Theta = \frac{\pi}{R} \]

It is an auxiliary field netization vector \( \pi / \)
\[ \Theta = \frac{\pi}{R} \]

be at the point where the magnetization points in-plane,
\[ R \]

static skyrmion as a function of its radius
\[ R \]

shows the numerically calculated energy from Eq. (6) of
\[ \lambda \]

meter
\[ \lambda \]

for
\[ \chi \]

V
\[ V \]

parameter values in Eq. (2)).
\[ V \]

1
\[ V \]

47
\[ V \]

57 and corresponding equilibrium radii
\[ V \]

14
\[ V \]

71
\[ V \]

39. The radius
\[ R \]

of the magnetization of Eq. (14) as
\[ A \]

\[ m = \frac{\epsilon}{2\sqrt{2}} n \times \dot{n} \]

1
\[ m \]

n
\[ m \]

component) during the simulation are shown
\[ m \]

3
\[ m \]

2
\[ m \]

\[ T = \frac{4}{\epsilon^2} \int m^2 \, d^2 x \] (15)

In the following, we will make use of its discretized form
\[ T = 16 \sum_{\alpha,\beta} m^2_{\alpha,\beta} \] (16)

that gives the kinetic energy as a sum over the lattice of tetramers.

### III. SKYRMION ANNIHILATION VIA BREATHING

The fact that the energy of an infinitesimally small skyrmion is finite, \( E = 4\pi \), as shown in Fig. 1 suggests the possibility to annihilate the skyrmion by a finite force. This idea will be combined with the dynamics of the breathing mode that is known to exist in chiral magnets [21], as will be explained in the following.

We assume a material with parameter value \( \lambda_0 \) that gives a static skyrmion with radius \( R_0 \) and energy \( V_0 \). We further assume a method to expand this skyrmion and produce one with a larger radius \( R > R_0 \) and, naturally, a larger energy \( V > V_0 \). We expect that the skyrmion energy, when this is out of equilibrium, can be chosen \( V > 4\pi \) if \( R \) is large enough. Using such a large radius skyrmion as an initial state, we anticipate that the breathing dynamics can reduce the skyrmion radius down to \( R \to 0 \) since the energy of the oscillator is larger than the potential energy of a \( \rightarrow 0 \) skyrmion.

We implement the above ideas in a numerical simulation of a spin lattice with 500 \( \times \) 500 sites. We consider a static skyrmion for the parameter values in Eq. (2). We propagate in time the dynamical equations (3) including damping with \( \alpha = 0.0025 \). The system is subjected to a voltage pulse that initially modifies the DM parameter to the value \( D = 0.057 J_n (\lambda = 0.57) \) as shown in Fig. 2. The evolution of the skyrmion radius and discrete energy (1) are shown in Fig. 2b. Four snapshot of the skyrmion profile (\( n_3 \) component) during the simulation are shown in Fig. 2c. At time \( t_0 \), the initial skyrmion is shown. The skyrmion expands during the voltage pulse, and it is shown at the end of the pulse at time \( t_1 \). Subsequently, breathing dynamics induces shrinking of the skyrmion, as shown at time \( t_2 \). The skyrmion is eventually annihilated at time approximately 12 ps. After annihilation, low amplitude waves in the form of radiation are found to spread radially, away from the original skyrmion center, as shown at time \( t_3 \). In Fig. 2d, we see that the skyrmion radius is increasing during the voltage pulse. The pulse is switched off at the time that the skyrmion has maximum radius. Then, the radius is decreasing due to breathing dynamics until the skyrmion shrinks to a point and is annihilated. The energy is decreasing due to damping. There is a step-like increase of the energy when the voltage pulse is switched off, due to the drop of the DM parameter. This brings the energy to a value greater than \( 4\pi \). When annihilation of the skyrmion happens, the discrete energy is somewhat lower than \( 4\pi \) (which is the prediction of the continuum theory for minimum skyrmion energy). Certainly, the prediction based on the continuum model is not expected to be quantitatively correct when the skyrmion is concentrated in a few lattice sites, just before annihilation. The energy dissipation conti-
FIG. 2. Skyrmion annihilation via breathing. (a) The parameter $D$ is shown vs time. A square pulse of 7 ps is modulating the DM parameter to the value $D = 0.057 J_n (\lambda = 0.57)$. After the pulse is switched off, it is $D = 0.047 J_n (\lambda = 0.47)$. (b) The total energy from Eq. (1) vs time is shown by a blue solid line. The skyrmion radius is shown by a red dashed line. It is given in units of lattice spacing $a$. (c) Spatial distribution of $n_3$ at time points $t_i$ (indicated on the energy graph). Snapshot $t_0$ shows the initial skyrmion (static profile for parameter values of Eq. (2)), snapshot $t_1$ shows the skyrmion when the pulse is switched off, snapshot $t_2$ shows the skyrmion before annihilation, snapshot $t_3$ shows the generated waves after the skyrmion has been annihilated. Only part of the simulation space is shown.

ues faster after the skyrmion annihilation. We stop the simulation when radiation reaches the boundaries of the numerical mesh.

We proceed to a detailed study of the skyrmion annihilation dynamics. At the time that the skyrmion is concentrated in a point, the kinetic energy should be equal to $T = V - 4\pi > 0$. This seems incompatible with the intuitive expectation that $\dot{R} \to 0$ at $R = 0$. The apparent contradiction can be resolved if we estimate the kinetic energy for a skyrmion of small radius where the profile is approximated by a Belavin-Polyakov (BP) configuration [32]. A BP skyrmion with a time dependent radius is given by

$$\tan\left(\frac{\Theta}{2}\right) = \frac{R(t)}{r}.$$  

(17)

This is valid from $r = 0$ up to a distance $r \sim O(1/\ln R)$ when $R \ll 1$ (this is based on Eq. (26) of Ref. [30]). The kinetic energy is

$$T = \frac{1}{2} \int \dot{\Theta}^2 2\pi r \, dr \approx -4\pi \ln R \bar{R}^2$$  

(18)

where we have taken the limits in the integral (18) from $r = 0$ to $r \sim 1/\ln R$ and we have only kept the dominant term for $\bar{R} \to 0$. The skyrmion profile decays exponentially for larger distances $r$ and we thus neglect this contribution to $T$. The quantity $m = -8\pi \ln \bar{R}$ can be considered as the mass of the breathing skyrmion and it is diverging for $\bar{R} \to 0$. This behavior is connected with the well-known divergence of the integrated magnetization for the BP skyrmion. A nonzero kinetic energy implies

$$\dot{\bar{R}} \sim \frac{1}{\sqrt{-\ln \bar{R}}}, \quad \bar{R} \to 0.$$  

(19)

Based on Eq. (19), we anticipate that $\dot{\bar{R}} \to 0$ as $\bar{R} \to 0$ while at the same time the kinetic energy remains strictly positive due to the mass term. We finally mention that the time it takes to achieve the annihilation via breathing is finite as can be found by integrating in time $\dot{\bar{R}}$ in Eq. (19).

Fig. 3 shows the kinetic energy (16) for the numerical simulation in Fig. 2 until the skyrmion is annihilated. The skyrmion radius is shown in the same figure for comparison. The kinetic energy starts from zero, reaches a maximum and returns to a very small value as the skyrmion radius approaches a maximum. At that time, the pulse is switch off and the skyrmion radius starts decreasing rapidly until the skyrmion annihilates. The kinetic energy has a nonzero value at the time of annihilation. The numerical results confirm the predictions of
the previous paragraphs that are based on the continuum model.

Methods usually employed for the creation or annihilation of complex topological textures involve driving them at material boundaries or forcing them to shrink down to the atomic size due to large fields. By contrast, the annihilation of the skyrmion described in this section is obtained due to a mild perturbation. Shrinking follows by the natural breathing dynamics and a singularity formation happens in a finite time interval. It should also be noted that the nontrivial topology of the skyrmion is not an obstacle in the singularity formation and eventually in the annihilation process.

The proposed dynamics can be induced in various ways. (i) One could temporarily increase the scaled parameter $\lambda$ (by modifying the DM or anisotropy parameters), as presented in Fig. 2. (ii) One might also decrease the parameter, thus initiating breathing, but, in this case, a sufficiently low value should be maintained until the skyrmion annihilates. (iii) Alternatively, an external magnetic field would act as easy-plane anisotropy in an antiferromagnet effectively reducing the easy-axis anisotropy parameter and thus increasing the dimensionless parameter $\lambda$. This case would require a separate study due to the more involved dynamics introduced by the external field $\left[23, 24\right]$. (iv) Modification of the kinetic energy of the skyrmion could also give further alternatives in order to initiate annihilation dynamics. This could be achieved by inducing magnetization in the antiferromagnetic lattice.

IV. NONLINEAR BREATHING MODE

We proceed to a systematic study of the breathing dynamics in the linear and the nonlinear regime that will lead to quantitative predictions for the breathing and for the annihilation dynamics. We consider, in this section, modes with helicity $\chi = 0$ and a dynamic skyrmion profile

$$\Theta = \Theta(r, t), \quad \Phi = \phi. \quad (20)$$

The kinetic energy in Eq. (11) reduces to

$$T = \frac{1}{2} \int \dot{\Theta}^2 2\pi r \, dr \quad (21)$$

and the potential energy is

$$V_{\lambda}(\Theta) = E_{\text{ex}} + \lambda \tilde{E}_{\text{DM}} + E_{\text{an}} \quad (22)$$

with $E_{\text{ex}}, \tilde{E}_{\text{DM}}, E_{\text{an}}$ defined in Eqs. (12), (13).

We aim to study breathing oscillations, that is, a periodic change of the skyrmion profile $\Theta(r, t)$. We assume a material with parameter $\lambda_0$. In order to invoke breathing dynamics, we change the parameter to a value $\lambda \neq \lambda_0$ (for example, by applying a voltage) and assume that the skyrmion profile eventually relaxes to $\Theta_{\lambda}$. When the parameter is restored to the value $\lambda_0$, breathing dynamics is initiated.

In order to make progress analytically, we make a simplifying assumption that is supported by numerical simulations. As the skyrmion radius changes (oscillates) during breathing, we assume that the profile adiabatically adjusts to the static skyrmion profile $\Theta_{\lambda}$ for the corresponding radius $R = R(\lambda)$. Under this assumption, the potential energy that the skyrmion experiences during the breathing motion is given by (22) applied for $\lambda = \lambda_0$,

$$V_{\lambda_0}(\Theta_{\lambda}) = E_{\text{ex}} + \lambda_0 \tilde{E}_{\text{DM}} + E_{\text{an}}$$

$$= V_{\lambda} + (\lambda_0 - \lambda)\tilde{E}_{\text{DM}} \quad (23)$$

where all terms are evaluated for $\Theta = \Theta_{\lambda}$. The energy components for each value of the parameter $\lambda$ and the corresponding value of radius $R$ can be calculated numerically.

In Fig. 1 we plot (by solid lines) the potential energy (23) for the equilibrium profiles $\Theta_{\lambda}$ as a function of their radius $R$. Three lines are plotted for the cases $\lambda_0 = 0.4, 0.47, 0.57$, which correspond to static skyrmion solutions with radius $R_0 = 0.71, 1.14, 2.39$ respectively. As expected, every solid line has a minimum at the corresponding value of the radius.

The main features of the potential wells in Fig. 1 can be anticipated. We first consider the case of small radius, that is, $R \to 0$, or, equivalently, $\lambda \to 0$ [32, 33]. We have $E_{\text{ex}} \to 4\pi$, $\tilde{E}_{\text{DM}}, E_{\text{an}} \to 0$ and this gives $V_{\lambda} \to 4\pi$. We also have $\tilde{E}_{\text{DM}} \sim -8\pi R$, and thus Eq. (23) gives

$$V_{\lambda_0} \to 4\pi, \quad \text{as} \quad R \to 0. \quad (24)$$

In the case of large radius, we have $V_{\lambda} \sim O(R^{-1})$ and $\tilde{E}_{\text{DM}} = -4\pi R$ as $R \to \infty$, or, equivalently, $\lambda \to 2/\pi$ [36]. Thus, Eq. (23) gives

$$V_{\lambda_0} \sim 4\pi \left(1 - \frac{\pi\lambda_0}{2}\right) R, \quad \text{as} \quad R \to \infty \quad (25)$$

where we have used $\lambda = 2/\pi + O(R^{-2})$. Eqs. (24), (25) confirm the features of the potentials shown in Fig. 1 that are crucial for describing breathing dynamics.

The potential wells shown in Fig. 1 give rise to oscillating motion around the minimum $V_0 = V_{\lambda_0}(\Theta_{\lambda_0})$. We demonstrate this by performing a simulation similar to that in Sec. [11] but we now retain the voltage pulse for a longer time, 13 ps. We use a smaller damping parameter $\alpha = 0.001$ in order to demonstrate clearer the oscillating motion. Fig. 1 shows the skyrmion radius and the kinetic energy during the motion. An oscillating motion starts in the potential well $V_{\lambda_0=0.57}$. The skyrmion radius has passed the maximum when the pulse is switched off. Then, a new oscillating motion sets in in the potential well $V_{\lambda_0=0.47}$. The oscillations eventually die out due to damping. The kinetic energy $T$ is close to zero at both turning points of the oscillation, i.e., at the minimum and maximum radii.
the skyrmion radius corresponding to the static profile and they represent the one extremum of the oscillating radius of the initial profiles are plotted by blue circles for which the potential takes the value \( V = 0 \) in Eq. (3), for a range of initial profiles \( \Theta_{\lambda} \). We proceed to systematic simulations of large amplitude oscillations. We start the simulation with \( \lambda = 0 \) in Eq. (9) that corresponds to an angular frequency \( \omega_0 = 0.3 \). This is in agreement with the analytical result (35) given in the next section. The frequency is decreasing for large amplitude oscillations. This is anticipated since the potential in Fig. 1 is slower than parabolic.

We return to the issue of skyrmion annihilation and we can now expand upon the results of Sec. III. Based on the potential wells shown in Fig. 1, one can start breathing dynamics either pushing towards a radius smaller than that at the potential minimum or to a larger radius (as we have shown in Fig. 2). Furthermore, one can imagine a combination of the two possibilities. That is, one may modulate \( \lambda \) periodically around the value \( \lambda_0 \), thus pushing the skyrmion radius to values smaller and larger than \( R_0 \) periodically. Using the appropriate frequency for the \( \lambda \) modulation, this is expected to lead to resonance, i.e., large amplitude oscillations for the skyrmion radius, and to its eventual annihilation. A small amplitude modulation of \( \lambda \) will be sufficient for the resonance phenomenon.

V. SMALL BREATHING OSCILLATIONS

The breathing motion with a small amplitude will be studied analytically, based on the potential (23) plotted in Fig. 1. We will study separately the case of a skyrmion of large radius and a skyrmion of small radius.

A. Skyrmions of large radius

Let us consider a large value of \( \lambda_0 \) so that the corresponding static skyrmion has a large radius. The skyrmion radius \( R(t) \) will vary with time during the breathing motion. The skyrmion profile is approximated as a domain wall centered at the position of the radius,

\[
\Theta = 2 \arctan \left( e^{-z} \right), \quad z \equiv r - R(t)
\]

and we assume that \( \chi = 0 \) during the motion. The kinetic energy (11) gives

\[
T = \frac{\dot{R}^2}{2} \int_0^\infty \text{sech}^2(r - R) \, (2\pi r \, dr).
\]

For \( R \gg 1 \), we may extend the lower limit to \( -\infty \) (with an exponentially small error) and obtain

\[
T \approx \frac{\dot{R}^2}{2} \int_{-\infty}^{\infty} \text{sech}^2 z \, (2\pi R \, dz) = 2\pi R \dot{R}^2.
\]
FIG. 5. The Néel field $\mathbf{n}$ and the magnetization field $\mathbf{m}$ of the skyrmion during breathing. (a) The Néel field for the initial skyrmion. (b) The magnetization field during skyrmion expansion, at time $t_1$ indicated in Fig. 4. (c) The magnetization field during skyrmion shrinking, at time $t_2$. The magnetization points azimuthally in both cases as anticipated by Eq. (26).

FIG. 6. For $\lambda_0 = 0.47$ ($D = 0.047$), we show the maximum and minimum radii of skyrmion breathing oscillations, when the initial skyrmion profile is $\Theta_\lambda$. Lines show theoretical results based on Eq. (22) and Fig. 1. Circles show results of numerical simulations. The simulations are initiated with skyrmion profiles at the blue circles (they give the one extremum of the oscillation). The other extremum of the oscillation is shown by red circles. (The radius $R$ is in scaled units and the actual length is given, as in Eq. (8), by $aR/\epsilon$, with $\epsilon = 0.1$ for parameter values [2].)

For calculating the potential energy [23], we will use the results [36]

\[
V_\lambda = \frac{4\pi^2\lambda_2}{R} + O\left(R^{-3}\right), \quad \lambda_2 = 0.3057 \\
E_{DM} = -4\pi R + O\left(R^{-1}\right)
\]  

where $R$ is the radius of the static skyrmion for DM parameter $\lambda$, approximated by [36]

\[
\lambda = \frac{2}{\pi} - \frac{\lambda_2}{R^2} + O\left(R^{-4}\right).
\]  

Our objective is to evaluate the potential [23] around the radius $R_0$ that corresponds to the parameter $\lambda_0$. We set

\[
R = R_0(1 + \delta)
\]

and we use Eq. (30) to obtain

\[
\frac{\lambda_0}{\lambda} - 1 \approx -\frac{\pi\lambda_2}{R_0^2} \left(1 - \frac{3}{2}\delta\right)\delta, \quad \delta \ll 1.
\]

Inserting Eq. (32) and (31) in Eq. (29), we have

\[
V_\lambda \approx \frac{4\pi^2\lambda_2}{R_0} (1 - \delta + \delta^2)
\]

\[
\frac{\lambda_0 - \lambda}{\lambda} E_{DM} \approx \frac{4\pi^2\lambda_2}{R_0} \delta - \frac{1}{2}\delta^2.
\]

Substituting the two last equations in the potential energy [23], we obtain the parabolic form

\[
V_{\lambda_0}(\Theta_\lambda) = \frac{4\pi^2\lambda_2}{R_0} \left(1 + \frac{1}{2}\delta^2\right).
\]  

Eqs. (28), (33) give the Lagrangian

\[
L = 2\pi R_0^3\delta^2 - \frac{2\pi^2\lambda_2}{R_0} \delta^2
\]

which implies harmonic oscillations with angular frequency

\[
\omega_b = \sqrt{\frac{\pi\lambda_2}{R_0^2}} \approx \sqrt{\frac{\pi}{\lambda_2}} \left(\frac{2}{\pi} - \lambda_0\right).
\]

Fig. 7 shows the results of numerical simulations for breathing oscillations of small amplitude. For $\lambda_0$ close to the value $2/\pi$, the numerical results confirm Eq. (35). The dependence of the breathing frequency on $R_0$ has been obtained in Ref. [22], but the numerical factor in Eq. (35) introduces a correction to that result. This modification originates in the corrected dependence of the radius on the DM parameter given in Eq. (30) compared to an earlier result obtained in Ref. [35].
Using the latter, the potential \( A_5 \) is written as

\[
\text{of the asymptotic results of Ref. [32]. The parameter } \lambda (A_5) \text{ derived in Appendix A. That is an improvement by [32, 33] and the skyrmion radius } R \text{ approximation formulas). Inserting (31) in (38), we find the quadratic been possible to obtained by using previously available is obtained thanks to Eq. (A5) while it would not have }

\[
\text{that is valid for small } R \text{ and } R_0. \text{ Eq. (38) has a minimum at } R = R_0 \text{ as desired (note that the minimum is obtained thanks to Eq. (A5) while it would not have been possible to obtained by using previously available formulas). Inserting (31) in (38), we find the quadratic approximation}
\]

\[
V_{\lambda_0}(R) = 4\pi \left[ 1 - R(R \ln R - 2R \ln R_0) + \frac{1}{2} R^2 \right]
\]

\[
\text{that is valid for small } R \text{ and } R_0. \text{ Eq. (38) has a minimum at } R = R_0 \text{ as desired (note that the minimum is obtained thanks to Eq. (A5) while it would not have been possible to obtained by using previously available formulas). Inserting (31) in (38), we find the quadratic approximation}
\]

\[
V_{\lambda_0} \approx 4\pi \left( 1 + R_0^2 \ln R_0 - R_0^2 \ln R_0 \delta^2 \right).
\]

Eqs. (39) and (39) give the Lagrangian

\[
L = -4\pi R_0^2 \ln R_0 \left( \delta^2 - \delta \right).
\]

We thus find that small breathing oscillations for skyrmions of small radius have an angular frequency

\[
\omega_b = 1.
\] (41)

Note that this is equal to the frequency \[48\] for helicity oscillations discussed in the next Section.

The numerical results shown in Fig. 7 indicate that \( \omega_b \) takes a value of \( O(1) \) for skyrmions of small radius and it is thus consistent with the result in Eq. (41). We cannot simulate the dynamics of skyrmions for very small \( \lambda \) due to their very small size and we thus do not present a precise numerical result for \( \omega_b \) as \( \lambda \to 0 \).

\section{VI. Helicity Oscillations}

For completeness, we consider an oscillation mode where the radial skyrmion profile remains unchanged while the helicity depends on time

\[
\Theta = \Theta(r), \quad \Phi = \phi + \chi(t).
\] (42)

However, this assumption is not consistent with the equations of motion derived from the energy functional. Indeed, simulations show that Eq. (42) does not give a good approximation for the dynamical profile. When we start from a static skyrmion profile \( \Theta_\lambda \) and choose uniform helicity \( \chi \neq 0 \), we obtain a complicated motion that seems to combine breathing and helicity oscillations. Nevertheless, Eq. (42) will prove its usefulness as it will lead to an approximation for the equation of motion for \( \chi \) and to the frequency of the observed oscillations. We therefore argue that this assumption is useful and we proceed to use it in the following calculations.

The kinetic energy (41) reduces to the expression

\[
T = \frac{1}{2} \chi^2 \int \sin^2 \Theta (2\pi r dr) = \chi^2 E_{an}
\] (43)

and the potential energy is

\[
V_\lambda = E_{ex} + \lambda \cos \chi \tilde{E}_{DM} + E_{an}
\] (44)

where \( E_{ex}, \tilde{E}_{DM}, E_{an} \) depend on \( \Theta \) but not on \( \chi \), as seen in Eqs. (12), (13). Omitting terms independent of \( \chi \), the Lagrangian is

\[
L = T - V = \chi^2 E_{an} - \lambda \cos \chi \tilde{E}_{DM}.
\] (45)

By a standard scaling argument for the minimizer \( \Theta_\lambda \), a virial relation is obtained [38].

\[
2E_{an} + E_{DM} = 0 \Rightarrow \lambda \tilde{E}_{DM} = -2E_{an}.
\] (46)

The latter is used in Eq. (45) to give

\[
L = 2E_{an} \left( \frac{1}{2} \chi^2 + \cos \chi \right)
\] (47)
Lagrangian \( [47] \) describes a pendulum. For small \( \chi \ll 1 \), it gives harmonic oscillations with angular frequency

\[
\omega_h = 1 \quad (48)
\]

that is a period \( T = 2\pi \) (or \( T = 2.3 \) ps when we use the time unit in Eq. (9)). Large amplitude oscillations will have a smaller frequency as in the case of a pendulum. The value in Eq. (48) agrees with the results of numerical simulations.

Note that oscillations of helicity, for a small amplitude, are found to have the same frequency \( [48] \) as small breathing oscillations for skyrmions of small radius \( [41] \).

**VII. CONCLUDING REMARKS**

We have studied breathing oscillations of skyrmions in chiral antiferromagnets using analytical arguments and calculations, within a continuum model, that are valid in the nonlinear regime. The predictions are confirmed and the results are extended by systematic numerical simulations within the original discrete spin model. A significant result is the prediction, confirmed by simulations, that the forced expansion of the skyrmion radius invokes breathing oscillations that can lead to the skyrmion annihilation. This counter-intuitive process offers an alternative to the typically employed methods of forced skyrmion suppression. It can prove to be a more convenient method as it only requires mild forcing. This is sufficient because the annihilation is actually brought about, not by the forcing itself, but by the invoked internal dynamics. Furthermore, the phenomenon is interesting also from a theoretical perspective because it involves the creation of a singularity in finite time. Finally, we give an analytical calculation based on an energetic method for the frequency of small amplitude oscillations.

In the development of the theoretical arguments, we have given details about the kinetic energy of the continuum model for an antiferromagnet. This is actually an emergent kinetic energy that originates in the exchange energy of the discrete model. We derive the surprising result that the kinetic energy can be tuned to a non-zero value at the point of the singularity formation and skyrmion annihilation.

Given the counter-intuitive dynamical behaviour of the breathing skyrmion, it will be interesting to consider further dynamical phenomena of this system. For example, it is interesting to investigate the skyrmion domain wall velocity during breathing. If this could reach the maximum velocity allowed for a traveling wall, that is \( \dot{v}_{\text{max}} = \sqrt{1 - (\pi \lambda/2)^2} \) \([39]\), then an instability of the Néel state will occur that may lead to the spontaneous formation of a spiral.

**ACKNOWLEDGEMENTS**

This work was supported by the project “ThunderSKY” funded by the Hellenic Foundation for Research and Innovation and the General Secretariat for Research and Innovation, under Grant No. 871.

**Appendix A: Approximation of the potential for small radius**

The available asymptotic formulae for the energy at small \( \lambda \) are \([32]\)

\[
V_\lambda(\Theta_\lambda) = 4\pi \left( 1 + \frac{\lambda^2}{\ln \lambda} \right) + o \left( \frac{\lambda^2}{\ln \lambda} \right),
E_{\text{DM}} = 8\pi \frac{\lambda^2}{\ln \lambda} + O \left( \frac{\lambda^2}{\ln \lambda^2} \right).
\]

Substituting in the potential energy \([23]\), we obtain

\[
V_{\lambda_0} = 4\pi \left[ 1 + (2\lambda_0 - \lambda) \frac{\lambda}{\ln \lambda} \right].
\]

The derivative of this potential is

\[
V'_{\lambda_0} = 4\pi \left[ 2 \frac{\lambda_0 - \lambda}{\ln \lambda} - 2 \frac{\lambda_0 - \lambda}{(\ln \lambda)^2} \right].
\]

We have \( V'_{\lambda_0}(\lambda_0) \neq 0 \), due to the term \( 1/(\ln \lambda)^2 \), and thus no minimum is implied at \( \lambda = \lambda_0 \). The only way to obtain a formula that gives \( V'_{\lambda_0}(\lambda_0) = 0 \) up to terms \( 1/(\ln \lambda)^2 \) is to write

\[
E_{\text{ex}} = 4\pi \left[ 1 + \frac{1}{2} \frac{\lambda^2}{(\ln \lambda)^2} \right], \quad E_{\text{DM}} = 8\pi \frac{\lambda^2}{\ln \lambda},
\]

that gives

\[
V_{\lambda_0}(\lambda) = 4\pi \left( 1 + 2\lambda_0 - \lambda \right) \frac{\lambda}{\ln \lambda} + \frac{1}{2} \frac{\lambda^2}{(\ln \lambda)^2}.
\]

We have

\[
V'_{\lambda_0} = 4\pi \left[ 2 \frac{\lambda_0 - \lambda}{\ln \lambda} + 2 \frac{\lambda_0 - \lambda}{(\ln \lambda)^2} \right] + O \left( \frac{\lambda}{(\ln \lambda)^3} \right).
\]

and thus \( V_{\lambda_0} \) has the desired behaviour at \( \lambda = \lambda_0 \).

Formula \( [A4] \) for \( E_{\text{ex}} \) is an improvement over previous results \([36]\) and it is necessary in order to find the oscillation frequency for breathing oscillations in Sec. \( \nabla B \).

---

\[1\] A. Bogdanov and D. Yablonskii. Thermodynamically stable “vortices” in magnetically ordered crystals. the mixed state of magnets. Zh. Eksp. Teor. Fiz., 95:178–182, 1989.
[2] A. N. Bogdanov and A. Hubert. Thermodynamically stable magnetic vortex states in magnetic crystals. JMMM, 138:255–269, 1994.
[3] Stavros Komineas and Nikos Papanicolaou. Skyrmion dynamics in chiral ferromagnets. Phys. Rev. B, 92:064412, Aug 2015.
[4] Wanjun Jiang, Gong Chen, Kai Liu, Jiadong Zang, Suzanne G.E. te Velthuis, and Axel Hoffmann. Skyrmions in magnetic multilayers. Physics Reports, 704:1–49, 2017. Skyrmions in Magnetic Multilayers.
[5] A. Fert, V. Cros, and J. Sampaio. Skyrmions on the track. Nature Nanotechnol., 8:152–156, 2013.
[6] R. Tomasello, E. Martinez, R. Zivieri, L. Torres, M. Carpentieri, and G. Finocchio. A strategy for the design of skyrmion racetrack memories. Sci. Rep, 4:6784, 2014.
[7] Wataru Koshiba, Yoshih Kaneko, Junichi Iwasaki, Masashi Kawasaki, Yoshinori Tokura, and Naoto Nagao. Memory functions of magnetic skyrmions. Japanese Journal of Applied Physics, 54(5):053001, apr 2015.
[8] X. Zhang, M. Ezawa, and Y. Zhou. Magnetic skyrmion logic gates: conversion, duplication and merging of skyrmions. Sci. Rep, 5:9400, 2015.
[9] Shijiang Luo, Min Song, Xin Li, Yue Zhang, Jeongmin Hong, Xiaofei Yang, Xuecheng Zou, Nuo Xu, and Long You. Reconfigurable skyrmion logic gates. Nano Lett., 18:1180—1184, 2018.
[10] Kyung Mee Song, Jae-Seung Jeong, Biao Pan, Xichao Zhang, and Yan Zhou. Dynamics of an antiferromagnetic skyrmion in a racetrack with a defect. Phys. Rev. B, 100:144439, Oct 2019.
[11] Z. Jin, T. T. Liu, W. H. Li, X. M. Zhang, Z. P. Hou, D. Y. Chen, Z. Fan, M. Zeng, X. B. Lu, X. S. Gao, M. H. Qin, and J.-M. Liu. Dynamics of antiferromagnetic skyrmions in the absence or presence of pinning defects. Phys. Rev. B, 102:054419, Aug 2020.
[12] H. Diego Gao, Shangand Rosales, Flavia A. Gómez Albarracín, Vladimir Tsurkan, Gurutinder Kaur, Tom Fennell, Paul Steffens, Martin Boehm, Petr Cermák, Astrid Schneidewind, Eric Ressouche, Daniel C. Cabra, Christian Rüegg, and Oksana Zaharko. Fractional antiferromagnetic skyrmion lattice induced by anisotropic couplings. Nature, 586:37–41, 2020.
[13] Harion Jani, Jheng-Cyuan Lin, Jiachao Chen, Jack Harrison, Francesco Maccherozzi, Jonathon Schad, Saurav Prakash, Chang-Beom Eom, A. Ariando, T. Venkatesan, and Paolo G. Radaelli. Antiferromagnetic half-skyrmions and bimerons at room temperature. Nature, 590:74–79, 2021.
[14] Christoph Schütte and Markus Garst. Magnon-skyrmion scattering in chiral magnets. Phys. Rev. B, 90:094423, Sep 2014.
[15] T. Archer, C. D. Pemmaraju, S. Sanvito, C. Franchini, T. Schneidewind, Eric Ressouche, Daniel C. Cabra, Chris, Simon. Unraveling the role of dipolar versus exchange interactions in transition metal oxides. Journal of Physics D: Applied Physics, 54(5):053001, apr 2021.
[16] Priya Gopal, Riccardo De Gennaro, Marta Silva dos Santos Gusamo, Rabih Al Rahal Orabi, Haihang Wang, Stefano Curtarolo, Marco Fornari, and Marco Buongiorno Nardelli. Improved electronics structure and magnetic properties of MnFe2O4 by the spin hall effect. Phys. Rev. B, 90:094423, Sep 2014.
[17] Helen V. Gomonay and Vadim M. Loktev. Spin transfer torque in high-spin density ferromagnets. J. Magn. Magn. Mater., 116:147203, Apr 2016.
[18] I. V. Bar'yakhtar and B. A. Ivanov, and N Papanicolaou. Vortex dynamics in chiral ferromagnets. Phys. Rev. B, 99:184429, May 2019.
[19] T. Archer, C. D. Pemmaraju, S. Sanvito, C. Franchini, T. Schneidewind, Eric Ressouche, Daniel C. Cabra, Chris, Simon. Unraveling the role of dipolar versus exchange interactions in transition metal oxides. Journal of Physics D: Applied Physics, 54(5):053001, apr 2021.
[20] A. N. Bogdanov and D. A. Yablonskii. Contribution to the theory of inhomogeneous states of magnets in the region of magnetic-field-induced phase transitions. mixed state of antiferromagnets. Sov. Phys. JETP, 60:142,1989.
[21] A. N. Bogdanov and A. Shestakov. Vortex states in antiferromagnetic crystals. Phys. Solid State, 40:1350, 1998.
[22] Stavros Komineas, Christof Melcher, and Stephanos Venakides. The profile of chiral skyrmions of small radius. Nonlinearity, 33:2295–3408, May 2020.
[23] Anne Bernard-Mantel, Cyrill B. Muratov, and Thilo M. Simon. Unraveling the role of dipolar versus...
Dzyaloshinskii-Moriya interactions in stabilizing compact magnetic skyrmions. *Phys. Rev. B*, 101:045416, Jan 2020.

[34] Anne Bernand-Mantel, Cyrill B. Muratov, and Theresa M. Simon. A quantitative description of skyrmions in ultrathin ferromagnetic films and rigidity of degree ±1 harmonic maps from $\mathbb{R}^2 \to S^2$. *Archive for Rational Mechanics and Analysis*, 239(1):219–299, 2021.

[35] S. Rohart and A. Thiaville. Skyrmion confinement in ultrathin film nanostructures in the presence of Dzyaloshinskii-Moriya interaction. *Phys. Rev. B*, 88:184422, Nov 2013.

[36] Stavros Komineas, Christof Melcher, and Stephanos Venakides. Chiral skyrmions of large radius. *Physica D: Nonlinear Phenomena*, 418:132842, 2021.

[37] Stavros Komineas and Nikos Papanicolaou. Traveling skyrmion in chiral antiferromagnets. *SciPost Phys.*, 8:086, 2020.

[38] A. Bogdanov and A. Hubert. The properties of isolated magnetic vortices. *Phys. Stat. Sol. B*, 186:527, 1994.

[39] Riccardo Tomasello and Stavros Komineas. Vortex propagation and phase transitions in a chiral antiferromagnetic nanostripe. *Phys. Rev. B*, 104:064438, Aug 2021.