On the error term in the prime geodesic theorem for SL4

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Abstract. In this paper we pay our particular attention to the error term in the prime geodesic theorem for compact symmetric spaces represented as double coset spaces of the special linear group of order four over real numbers. It is known that in the case of compact locally symmetric Riemannian manifolds of strictly negative sectional curvature, the corresponding error term depends on classification of Riemannian symmetric spaces of real rank one. In particular, the error term is a function depending on the dimension of the underlying locally symmetric space. In this research we prove that the error term in the case at hand is a function depending on the degree of the polynomial that appears in the functional equation of the corresponding Selberg zeta function.

1. Introduction
In [2] and [12], the authors derived two main results: a length spectrum for compact symmetric spaces represented as quotients of the Lie group $SL_4(\mathbb{R})$, and its application in totally quartic fields with no real quadratic subfield.

Length spectrum (prime geodesic theorem) is given by

$$
\pi(x) = 2 \text{li}(x) + O \left( x^{\frac{3}{4}} (\log x)^{-1} \right)
$$

as $x \to +\infty$, where $\text{li}(x) = \int_2^x \frac{dt}{\log t}$, and $\pi(x)$ is the counting function described in detail below.

The Selberg zeta function (the Ruelle zeta function) is usually applied in the proof of the prime geodesic theorem (see, e.g., [15], [7], [8], [13], [11], [3]-[5], etc.)

In fact, such functions are applied in a way analogous to the way the Riemann zeta function is applied in the proof of the prime number theorem (see, e.g., [1], [10], [17], etc.)

The prime geodesic theorem stated above is then applied in order to prove an asymptotic formula for class numbers of orders in totally complex quartic fields with no real quadratic subfield. More precisely, it is proved that

$$
\pi_S(x) \sim \frac{e^{4x}}{8x}
$$

as $x \to +\infty$, where

$$
\pi_S(x) = \sum_{\mathcal{O} \in \mathcal{O}^r(S) \atop R(\mathcal{O}) \leq x} \lambda_S(\mathcal{O}) h(\mathcal{O}).
$$
Here, $S$ is a finite, non-empty set of prime numbers containing an even number of elements, $O^c(S)$ is the subset of isomorphy classes of orders in fields in $C^c(S)$, where $C^c(S) \subset C(S)$ is the subset of fields with no real quadratic subfield. Furthermore, $R(O)$ resp. $h(O)$ denote the regulator resp. the class number of the order $O$.

For a field $F \in C(S)$ and an order $O \in O_F(S)$, the constant $\lambda_S(O) (= \lambda_S(F))$ is defined by

$$\lambda_S(O) = \prod_{p \in S} f_p(F),$$

where $f_p(F)$ is the inertia degree of $p$ in $F$.

$C(S)$ is the set of all totally complex quartic fields $F$ such that all primes $p \in S$ are non-decomposed in $F$.

Finally, $O(S)$ is the union of all $O_F(S)$, where $F$ ranges over $C(S)$, and $O_F(S)$ is the set of all isomorphism classes of orders in $F$ which are maximal at all $p \in S$.

Note that for long time it was not possible to separate the class number and the regulator in the summation (see, e.g., [6], [16]). However, in [14], the author proved that such a separation is actually possible.

In this paper we pay attention to the error term $O \left( x^{\frac{3}{4}} \left( \log x \right)^{-1} \right)$ in (1). We prove that this error term should actually be replaced by the error term $O \left( x^{\frac{1}{2}} \frac{D}{\pi} \left( \log x \right)^{-1} \right)$, where $D$ is the degree of the polynomial that appears in the functional equation of the Selberg zeta function in the case at hand.

2. Preliminaries

As it is usual, we define the counting functions

$$\psi_j(x) = \int_0^x \psi_{j-1}(t) \, dt,$$

$$j \in \mathbb{N},$$

where $\mathcal{E}_P(\Gamma)$ is the set of all conjugacy classes $[\gamma]$ in $\Gamma$, and $\chi_1(\gamma)$ is the first higher Euler characteristics of the symmetric space $X^\gamma = \Gamma \setminus G, K$. 

Namely, our object of research is the symmetric space $X = \Gamma \setminus G, K$, where $G = SL_4(\mathbb{R}), K = SO(4)$, and $\Gamma$ is a discrete and co-compact subgroup of $G$.

More precisely, it is initially required $K$ to be the maximal compact subgroup of $G$. Therefore, $K = SO(4)$.

In particular, $G, \Gamma$ and $\Gamma$ are the centralizers of $\gamma$ in $G$ and $\Gamma$, respectively, and $K = K \cap G$. 

$P$ is a parabolic with Langlands decomposition $P = MAN$, where

$$M = S \begin{pmatrix} SL_2^+(\mathbb{R}) & SL_2^+(\mathbb{R}) \\ \end{pmatrix},$$

$$A = \left\{ \begin{pmatrix} a & \alpha \\ \alpha \alpha^{-1} & \alpha^{-1} \\ \end{pmatrix} : \alpha > 0 \right\},$$

$$\psi_0(x) = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \chi_1(\Gamma \gamma) \mathcal{L}_\gamma,$$
\[ N = \begin{pmatrix}
I_2 & \text{Mat}_2(\mathbb{R}) \\
0 & I_2
\end{pmatrix}. \]

We use \( \gamma_0 \) to denote primitive elements.

If it happens that \( \gamma \) and \( \gamma_0 \) appear together in the same formula, we shall mean that \( \gamma_0 \) is the primitive element corresponding to \( \gamma \).

It is assumed that for \( [\gamma] \in \mathcal{E}_P(\Gamma) \), \( \gamma \) is conjugate in \( G \) to an element \( a_\gamma b_\gamma \in A^- B \), where

\[ A^- = \left\{ \begin{pmatrix}
a & a \\
a^{-1} & a^{-1}
\end{pmatrix} : 0 < a < 1 \right\} , \]

\[ B = \begin{pmatrix}
SO(2) \\
SO(2)
\end{pmatrix}. \]

Thus, \( a_\gamma \) is a matrix in \( A^- \).

Besides this notation, we write \( a_\gamma \) also for the top left entry in the matrix \( a_\gamma \) itself.

Consequently, we define the length \( l_\gamma \) of \( \gamma \) to be \( 8 \log a_\gamma \).

Finally, we define the counting function

\[ \pi(x) = \sum_{[\gamma] \in \mathcal{E}_P^p(\Gamma), e^{l_\gamma} \leq x} \chi_1(\Gamma_\gamma), \]

where \( \mathcal{E}_P^p(\Gamma) \) is the set of primitive classes in \( \mathcal{E}_P(\Gamma) \).

The Ruelle zeta function attached to \( X_\Gamma \) will be denoted by \( R_{\Gamma, 1}(s) \), and the corresponding Selberg zeta function will be denoted by \( Z_{P, \wedge^q \mathfrak{n}}(s), q \in \{0, 1, ..., 4\} \), where \( \wedge^q \) denotes the exterior product, and \( \mathfrak{n} \) is the complexified Lie algebra of \( \mathcal{N} \),

\[ \bar{\mathcal{N}} = \begin{pmatrix}
I_2 & 0 \\
\text{Mat}_2(\mathbb{R}) & I_2
\end{pmatrix}. \]

As it is usual for this kind of research, we apply the higher order differential operator (and its properties)

\[ \Delta_k^+ f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x + (k - i) h), \]

for some \( k \in \mathbb{Z} \), and \( h \) an arbitrary constant.

For \( t > 0 \), let \( N(t) \) denote the number of poles and zeros of \( Z_{P, \wedge^q \mathfrak{n}}(s), q \in \{0, 1, ..., 4\} \) at points \( \frac{1}{2} + ix \), where \( 0 < x < t \).

By Lemma 3.1.2 in [12], \( N(t) = O(t^D) \).
3. Main result

The following theorem represents the main result of our research.

Theorem 1. Let $X_{\Gamma}$ be as above. Then,

$$\pi (x) = 2 \text{li} (x) + \mathcal{O} \left( x^{1-\frac{1}{\pi}} \left( \log x \right)^{-1} \right)$$

as $x \to +\infty$, where $\text{li} (x) = \int_{2}^{x} \frac{dt}{\log t}$.

Proof. By [12, (12)],

$$\psi_k (x) = \sum_{\alpha \in S_k} c_k (\alpha) x^{\alpha+k},$$

where $S_k$ is the set of poles of

$$\frac{R_{\Gamma,1}^{\prime} (s)}{R_{\Gamma,1} (s)} s^{-1} (s+1)^{-1} \ldots (s+k)^{-1},$$

and $c_k (\alpha)$ is the residue at $\alpha$.

We may write

$$\psi_k (x) = \sum_{\alpha \in S_k} \text{Res}_{s=\alpha} \left( \frac{R_{\Gamma,1}^{\prime} (s)}{R_{\Gamma,1} (s)} s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right).$$

Since $\frac{R_{\Gamma,1}^{\prime} (s)}{R_{\Gamma,1} (s)} = \sum_{q=0}^{4} (-1)^q \frac{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})}{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})}$, it follows that

$$\psi_k (x) = \sum_{q=0}^{4} (-1)^q \sum_{\alpha \in S_k,q} \text{Res}_{s=\alpha} \left( \frac{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})}{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})} s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right),$$

where $S_{k,q}$ denotes the set of poles of

$$\frac{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})}{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})} s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k}.$$

We may write

$$\psi_k (x) = \sum_{q=0}^{4} (-1)^q \sum_{\alpha \in S_{k,q}} c_\alpha (q,k),$$

where

$$c_\alpha (q,k) = \text{Res}_{s=\alpha} \left( \frac{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})}{Z_{P,\Lambda}\bar{n} (s+\frac{q}{4})} s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right).$$
As it is known, the Selberg zeta function \( Z_{P,A^n}(s + \frac{q}{4}) \) has a double zero at \( 1 - \frac{q}{4} \), while the remaining poles and zeros of \( Z_{P,A^n}(s + \frac{q}{4}) \) lie in \( [-\frac{q}{4}, \frac{3}{4} - \frac{q}{4}] \cup \{ \frac{1}{4} - \frac{q}{4} + i \mathbb{R} \} \).

Note that the values \( 0, -1, \ldots, -k \) are single poles of

\[
s^{-1} (s + 1)^{-1} \ldots (s + k)^{-1} x^{s+k}.
\]

Also note that \( 0 \) may appear as a simple pole of \( \frac{Z'_{P,A^n}(s + \frac{q}{4})}{Z_{P,A^n}(s + \frac{q}{4})} \), \( q \in \{0, 1, \ldots, 3\} \), i.e., as a singularity of \( Z_{P,A^n}(s + \frac{q}{4}) \), \( q \in \{0, 1, \ldots, 3\} \). Moreover, \( 0 \) is a simple pole of \( \frac{Z'_{P,A^n}(s+1)}{Z_{P,A^n}(s+1)} \) (since \( 0 \) is double zero of \( Z_{P,A^n}(s + 1) \)). Finally, \(-1\) may appear as a simple pole \( \frac{Z'_{P,A^n}(s+1)}{Z_{P,A^n}(s+1)} \), i.e., as a singularity of \( Z_{P,A^n}(s + 1) \).

Denote by \( I_q \) the set of values \( j \in \{0, -1, \ldots, -k\} \) such that \( j \) is a singularity of \( Z_{P,A^n}(s + \frac{q}{4}) \).

Put \( I'_q = I_k \setminus I_q \), where \( I_k = \{0, -1, \ldots, -k\} \).

Obviously, \( 0 \in I_4 \), while it can appear as an element of \( I_q \), \( q \in \{0, 1, \ldots, 3\} \). Moreover, \(-1\) can appear as an element \( I_4 \). Note that \( \{-2, -3, \ldots, -k\} \subseteq I'_q \) for \( q \in \{0, 1, \ldots, 4\} \).

Now, \( I_k = I_q \cup I'_q \).

If \( j \in I_q \), then \( j \) is a pole of order two of

\[
\frac{Z'_{P,A^n}(s + \frac{q}{4})}{Z_{P,A^n}(s + \frac{q}{4})} s^{-1} (s + 1)^{-1} \ldots (s + k)^{-1} x^{s+k}.
\]

Otherwise, if \( j \in I'_q \), then \( j \) is a simple pole.

Besides the set \( I_q \) of singularities of \( Z_{P,A^n}(s + \frac{q}{4}) \), the set of the remaining singularities \( s^q \) of \( Z_{P,A^n}(s + \frac{q}{4}) \) will be denoted by \( S^q \).

Hence, the elements of \( S^q \) are also simple poles of

\[
\frac{Z'_{P,A^n}(s + \frac{q}{4})}{Z_{P,A^n}(s + \frac{q}{4})} s^{-1} (s + 1)^{-1} \ldots (s + k)^{-1} x^{s+k}.
\]

Now, we calculate the residues given above.

We write

\[
\frac{Z'_{P,A^n}(s + \frac{q}{4})}{Z_{P,A^n}(s + \frac{q}{4})} = \frac{o_q^z}{s-z} \left( 1 + \sum_{i=1}^{+\infty} a_{i,z}^q (s-z)^i \right),
\]

where \( z \) is a singularity of \( Z_{P,A^n}(s + \frac{q}{4}) \), \( o_q^z \) is the order of \( z \), and \( a_{i,z}^q \)'s are the corresponding coefficients.

Let \( s^q \in S^q \). We derive,
\[ c_{q}^{l} (q, k) = \lim_{s \to q} (s - s^{q}) \frac{Z'_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})}{Z_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})} s^{-1} (s + 1)^{-1} ... (s + k)^{-1} x^{s+k} \]

\[ = \lim_{s \to q} (s - s^{q}) \frac{a_{q}^{q}}{s - s^{q}} \left( 1 + \sum_{i=1}^{+\infty} a_{i, s^{q}}^{q} (s - s^{q})^{i} \right) s^{-1} (s + 1)^{-1} ... (s + k)^{-1} x^{s+k} \]

\[ = a_{q}^{q} (s^{q})^{-1} (s^{q} + 1)^{-1} ... (s^{q} + k)^{-1} x^{s+k}. \]

Suppose that \(-j \in I_{q}\). Now,

\[ c_{-j} (q, k) = \lim_{s \to -j} \frac{d}{ds} \left( (s + j)^{2} \frac{Z'_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})}{Z_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})} s^{-1} (s + 1)^{-1} ... (s + k)^{-1} x^{s+k} \right). \]

Note that

\[ (s + j)^{2} \frac{Z'_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})}{Z_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})} s^{-1} (s + 1)^{-1} ... (s + k)^{-1} x^{s+k} \]

\[ = a_{q}^{q} \left( 1 + \sum_{i=1}^{+\infty} a_{i,-j}^{q} (s + j)^{i} \right) s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} x^{s+k} \]

\[ = a_{q}^{q} s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} x^{s+k} \]

Hence,

\[ \frac{d}{ds} \left( (s + j)^{2} \frac{Z'_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})}{Z_{P, \Lambda^{q} \tilde{A}} (s + \frac{q}{4})} s^{-1} (s + 1)^{-1} ... (s + k)^{-1} x^{s+k} \right) \]

\[ = a_{q}^{q} s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} x^{s+k} \log x - \]

\[ a_{q}^{q} s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} \times \]

\[ \left( s^{-1} + (s + 1)^{-1} + ... + (s + j - 1)^{-1} + (s + j + 1)^{-1} + ... + (s + k)^{-1} \right) x^{s+k} + \]

\[ a_{q}^{q} a_{1,-j}^{q} s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} x^{s+k} + \]

\[ a_{q}^{q} a_{1,-j}^{q} (s + j) \frac{d}{ds} \left( s^{-1} (s + 1)^{-1} ... (s + j - 1)^{-1} (s + j + 1)^{-1} ... (s + k)^{-1} x^{s+k} \right) + ... \]

We conclude,

\[ c_{-j} (q, k) = a_{q}^{q} \prod_{l=0}^{k} (-j + l)^{-1} x^{-j+k} \log x - \]

\[ a_{q}^{q} \prod_{l=0}^{k} (-j + l)^{-1} \left( - \sum_{l=0}^{k} (-j + l)^{-1} + a_{1,-j}^{q} \right) x^{-j+k}. \]
Finally, let $-j \in I_q^\prime$. Then,

$$c_{-j} (q, k) = \lim_{s \to -j} \left( s + j \right) \frac{Z_{P, A_n}^\prime (s + \frac{3}{2})}{Z_{P, A_n}^\prime (s + \frac{3}{2})} s^{-1} (s + 1)^{-1} \ldots (s + k)^{-1} x^{s+k}$$

$$= \frac{Z_{P, A_n}^\prime (-j + \frac{3}{2})}{Z_{P, A_n}^\prime (-j + \frac{3}{2})} \prod_{l=0}^{k} (-j + l)^{-1} x^{-j+k}.$$ 

Put $S^q_q = S^q \cap \mathbb{R}$, and $S^q_{\frac{1}{2} - \frac{q}{2}} = S^q \setminus S^q_{\mathbb{R}}$.

Let $z \in S^q_{\frac{1}{2} - \frac{q}{2}}$.

Since $S^q_{\frac{1}{2} - \frac{q}{2}} \subset S^q$, it follows that

$$h^{-k} \Delta^+_k c_z (q, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} o^q_z z^{-1} (z + 1)^{-1} \ldots (z + k)^{-1} (x + (k - i) h)^{z+k}$$

$$= h^{-k} o^q_z z^{-1} (z + 1)^{-1} \ldots (z + k)^{-1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x + (k - i) h)^{z+k}$$

$$= O \left( h^{-k} |z|^{-k-1} x^{\frac{1}{2} - \frac{q}{2} + k} \right) = O \left( h^{-k} |z|^{-k-1} x^{\frac{1}{2} + k} \right) \tag{2}$$

Moreover,

$$h^{-k} \Delta^+_k c_z (q, k) = h^{-k} \int_x^{x+h} \int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} \int_{t_3}^{t_3+h} \int_{t_4}^{t_4+h} \left( o^q_z z^{-1} (z + 1)^{-1} \ldots (z + k)^{-1} t_1^{z+k} \right)^{(k)} dt_1 \ldots dt_4$$

$$= h^{-k} o^q_z z^{-1} \int_x^{x+h} \int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} \int_{t_3}^{t_3+h} \int_{t_4}^{t_4+h} t_1^{\frac{1}{2} - \frac{q}{2}} dt_1 \ldots dt_4.$$ 

Here,

$$\left| h^{-k} \Delta^+_k c_z (q, k) \right| \leq h^{-k} |o^q_z| |z|^{-1} \int_x^{x+h} \int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} \int_{t_3}^{t_3+h} \int_{t_4}^{t_4+h} h (t_2 + h)^{\frac{1}{2} - \frac{q}{2}} dt_2 \ldots dt_4$$

$$= h^{-k} |o^q_z| |z|^{-1} h \int_x^{x+h} \int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} \int_{t_3}^{t_3+h} \int_{t_4}^{t_4+h} h (t_3 + h_2)^{\frac{1}{2} - \frac{q}{2}} dt_3 \ldots dt_4$$

$$= \ldots$$

$$= |o^q_z| |z|^{-1} (x + h_{k+1} + h_k + \ldots + h_2)^{\frac{1}{2} - \frac{q}{2}}$$
Since \( h_i \in [0, h] \) for \( i \in \{2, 3, \ldots, k + 1\} \), and \( h \leq \frac{\varepsilon}{2} \), we obtain that

\[
h^{-k} \Delta^+_k c_z(q, k) = O \left( |z|^{-1} x^{\frac{1}{2} - \frac{q}{4}} \right) = O \left( |z|^{-1} x^{\frac{1}{2}} \right) \quad (3)
\]

Now, we estimate the sum

\[
\sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} h^{-k} \Delta^+_k c_z(q, k) .
\]

We may write

\[
\sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} h^{-k} \Delta^+_k c_z(q, k) = \sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} h^{-k} \Delta^+_k c_z(q, k) + \sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} h^{-k} \Delta^+_k c_z(q, k) .
\]

Here, we apply (2) resp. (3) if \(|z| > M\) resp. \(|z| \leq M\).

Hence,

\[
\sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} h^{-k} \Delta^+_k c_z(q, k) = O \left( x^{\frac{1}{2}} \sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} |z|^{-1} \right) + O \left( h^{-k} x^{\frac{1}{2} + k} \sum_{z \in S^{\frac{q}{2} - \frac{q}{4}}} |z|^{-k-1} \right) = O \left( x^{\frac{1}{2}} \int_{|z| > M} t^{-1} dN(t) \right) + O \left( h^{-k} x^{\frac{1}{2} + k} \int_{M}^{+\infty} t^{-k-1} dN(t) \right) = O \left( x^{\frac{1}{2}} M^{D-1} \right) + O \left( h^{-k} x^{\frac{1}{2} + k} M^{D-k-1} \right) .
\]

Now, we estimate \( h^{-k} \Delta^+_k c_1(0, k) \).

By previous calculations, we know that

\[
c_1(0, k) = 2 \left( (k + 1)! \right)^{-1} x^{1+k} .
\]

For any \( k \) times differentiable function \( f \), we have

\[
h^{-k} \Delta^+_k f(x) = h^{-k} \int_{x}^{x+h} \int_{t_k}^{t_k+h} \ldots \int_{t_2}^{t_2+h} f^{(k)}(t_1) dt_1 \ldots dt_k \]

\[
= h^{-k} \int_{x}^{x+h} \int_{t_k}^{t_k+h} \ldots \int_{t_3}^{t_3+h} h f^{(k)}(t_2 + h) dt_2 \ldots dt_k
\]
\[
\begin{align*}
\frac{x+h}{x} & = h^{-k} h \int \int \cdots \int f^{(k)} (t_3 + h_3 + h_2) \, dt_3 \cdots dt_k \\
& = \cdots \\
& = f^{(k)} (x + h_{k+1} + h_k + \cdots + h_2).
\end{align*}
\]

Since \( h_i \in [0, h] \) for \( i \in \{2, 3, \ldots, k+1\} \), we may write

\[
h^{-k} \Delta_k^+ f (x) = f^{(k)} (\tilde{x}),
\]

where \( \tilde{x} \in [x, x + kh] \).

Thus, \( h^{-k} \Delta_k^+ c_1 (0, k) = h^{-k} \Delta_k^+ 2 ((k + 1)!)^{-1} x^{1+k} = 2 ((k + 1)!)^{-1} \left( \tilde{x}^{1+k} \right)^{(k)} = 2 \tilde{x} \)

for some \( \tilde{x} \in [x, x + kh] \).

Since \( h \leq \frac{x}{2} \), it follows that

\[
h^{-k} \Delta_k^+ c_1 (0, k) = O (x).
\]

In other words,

\[
h^{-k} \Delta_k^+ c_1 (0, k) = P x + Q
\]

for some \( P \) and \( Q \).

It is not so hard to determine \( P \) and \( Q \) explicitly.

We have,

\[
\Delta_k^+ c_1 (0, k) = \Delta_k^+ 2 ((k + 1)!)^{-1} x^{1+k}
\]

\[
= \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2 ((k + 1)!)^{-1} \left( x + (k - i) \right)^{1+k}
\]

\[
= \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2 ((k + 1)!)^{-1} \sum_{j=0}^{1+k} \binom{1+k}{j} x^{1+k-j} ((k - i) \, h)^j.
\]

We fix some \( k \).

Let \( k = mD \) for some even \( m \).

We obtain,

\[
\Delta_{mD}^+ c_1 (0, mD) = \sum_{i=0}^{mD} (-1)^i \binom{mD}{i} 2 ((mD + 1)!)^{-1} \sum_{j=0}^{1+mD} \binom{1+mD}{j} x^{1+mD-j} ((mD - i) \, h)^j.
\]

Hence,

\[
\begin{align*}
P & = h^{-mD} \sum_{i=0}^{mD} (-1)^i \binom{mD}{i} 2 ((mD + 1)!)^{-1} \binom{1+mD}{mD} ((mD - i) \, h)^{mD} \\
& = 2 ((mD + 1)!)^{-1} \binom{1+mD}{mD} \sum_{i=0}^{mD} (-1)^i \binom{mD}{i} (mD - i)^{mD}
\end{align*}
\]
\[= 2 \left( (mD + 1)! \right)^{-1} \frac{(1 + mD)!}{1!(mD)!} \sum_{i=0}^{mD} (-1)^i \frac{(mD)!}{(mD - i)!} (mD - i)^{mD} \]

\[= 2 \sum_{i=0}^{mD} (-1)^i \frac{1}{(mD - i)!} (mD - i)^{mD} = 2. \]

Furthermore,

\[Q = h^{-mD} \sum_{i=0}^{mD} (-1)^i \binom{mD}{i} 2 \left( (mD + 1)! \right)^{-1} \frac{(1 + mD)}{1 + mD} \left( (mD - i) h \right)^{1+mD} \]

\[= 2 h \left( (mD + 1)! \right)^{-1} \sum_{i=0}^{mD} (-1)^i \binom{mD}{i} \left( (mD - i) \right)^{1+mD} \]

\[= 2h \left( (mD + 1)! \right)^{-1} \sum_{i=0}^{mD} (-1)^i \frac{(mD)!}{(mD - i)!} \left( (mD - i) \right)^{1+mD} \]

\[= 2h (mD + 1)^{-1} \sum_{i=0}^{mD} (-1)^i \frac{1}{(mD - i)!} (mD - i)^{1+mD} = hmD. \]

Hence,

\[h^{-mD} \Delta^{+}_{mD} c_1 (0, mD) = 2x + mDh. \]

Consequently,

\[h^{-mD} \Delta^{+}_{mD} c_1 (0, mD) = 2x + O(h). \] (5)

By (4),

\[\sum_{q=0}^{4} (-1)^q \sum_{z \in S_{\frac{1}{2}+\frac{4}{M}}} h^{-mD} \Delta^{+}_{mD} c_z (q, mD) = O \left( x^{\frac{1}{2}M^{D-1}} \right) + O \left( h^{-mD} x^{\frac{1}{2}+mD} M^{D-mD-1} \right). \] (6)

Since the left sides in (5) and (6) are obviously summands of \(h^{-mD} \Delta^{+}_{mD} \psi_{mD} (x)\), and \(\psi_0 (x) \leq h^{-mD} \Delta^{+}_{mD} \psi_{mD} (x)\), it is clear that the error terms on the right hand sides of (5) and (6) play important role in determining the error term in \(\psi_0 (x)\), and hence in determining the error term in the prime geodesic theorem in the case at hand.

We want to determine \(h\) and \(M\) such that

\[h = x^{\frac{1}{2}M^{D-1}} = h^{-mD} x^{\frac{1}{2}+mD} M^{D-mD-1}. \]

Put \(h = x^\alpha, M = x^\beta\).

Hence,

\[h = x^\alpha, \]

\[x^{\frac{1}{2}M^{D-1}} = x^{\frac{1}{2}+\beta D-\beta}, \]

\[h^{-mD} x^{\frac{1}{2}+mD} M^{D-mD-1} = x^{-amD+\frac{1}{2}+mD+\beta D-\beta mD-\beta}. \]
We require that
\[ \alpha = \frac{1}{2} + \beta D - \beta = -\alpha m D + \frac{1}{2} + m D + \beta D - \beta m D - \beta. \]

We obtain, \( \beta = \frac{1}{4} \). Then, \( \alpha = \frac{1}{2} + \beta D - \beta = 1 - \frac{1}{4} \).

Thus, \( h = x^{1 - \frac{1}{4}} \), \( M = x^{\frac{1}{4}} \).

In this scenario, the error term in (5) and (6) is determined uniquely, i.e., it is given by \( O \left( x^{1 - \frac{1}{4}} \right) \).

Now, we estimate
\[ \sum_{q=0}^{4} (-1)^q \sum_{j=-2}^{-k} h^{-k} \Delta_k^+ c_{-j} (q, k). \]

Since \( h^{-k} \Delta_k^+ c_{-j} (q, k) = 0 \) for \( -j \in \{-2, -3, \ldots, -k\} \), it follows that
\[ \sum_{q=0}^{4} (-1)^q \sum_{j=-2}^{-k} h^{-k} \Delta_k^+ c_{-j} (q, k) = 0. \] (7)

Now, we estimate
\[ \sum_{q=0}^{4} (-1)^q h^{-k} \Delta_k^+ c_{-1} (q, k) = \sum_{q=0}^{3} (-1)^q h^{-k} \Delta_k^+ c_{-1} (q, k) + h^{-k} \Delta_k^+ c_{-1} (4, k). \]

Since \( h^{-k} \Delta_k^+ c_{-1} (q, k) = 0 \) for \( q \in \{0, 1, \ldots, 3\} \), it follows that
\[ \sum_{q=0}^{4} (-1)^q h^{-k} \Delta_k^+ c_{-1} (q, k) = h^{-k} \Delta_k^+ c_{-1} (4, k). \]

If \(-1 \notin I_4\), then \( h^{-k} \Delta_k^+ c_{-1} (4, k) = 0. \)

Suppose that \(-1 \in I_4.\)

Now,
\[ c_{-1} (4, k) = o_1 \prod_{l=0}^{k} (-1 + l)^{-1} x^{-k-1} \log x \]
\[ o_1 \prod_{l=0}^{k} (-1 + l)^{-1} \left( -\sum_{l=1}^{k} (-1 + l)^{-1} + o_1 \Delta_{-1,1} \right) x^{-k-1}. \]

Since \( (x^{-k-1} \log x)^{(k)} = \frac{(k-1)!}{x} \), it follows that

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\[ h^{-k} \Delta_k^{+} c_{-1}(4, k) = o^{q}_{4}(-1)^{-1} \frac{1}{\tilde{x}_{-1,4,k}} \]

for some \( \tilde{x}_{-1,4,k} \in [x, x + kh] \).

We conclude,

\[ \sum_{q=0}^{4} (-1)^q h^{-k} \Delta_k^{+} c_{-1}(q,k) = O(x^{-1}). \tag{8} \]

Finally, we estimate

\[ \sum_{q=0}^{4} (-1)^q h^{-k} \Delta_k^{+} c_0(q,k). \]

If \( 0 \in I_q \), then

\[ h^{-k} \Delta_k^{+} c_0(q,k) = \frac{Z'_{P,A^q}}{Z_{P,A^q}} \left( \frac{\bar{q}}{4} \right). \]

Suppose that \( 0 \in I_q \). Now,

\[ c_0(q,k) = o^q_{0} \prod_{l=0}^{k} (l)^{-1} x^k \log x - o^q_{0} \prod_{l=0}^{k} (l)^{-1} \left( -\sum_{l=0}^{k} (l)^{-1} + a^{q}_{1,0} \right) x^k. \]

Note that \( (x^k \log x)^{(k)} = k! \log x + k! \sum_{l=1}^{k} \frac{1}{l}, (x^k)^{(k)} = k! \).

Hence,

\[ h^{-k} \Delta_k^{+} c_0(q,k) = o^q_{0} (k!)^{-1} \left( k! \log \tilde{x}_{0,q,k} + k! \sum_{l=1}^{k} \frac{1}{l} \right) + o^q_{0} (k!)^{-1} \left( -\sum_{l=1}^{k} \frac{1}{l} + a^{q}_{1,0} \right) k! \]

\[ = o^q_{0} \log \tilde{x}_{0,q,k} + o^q_{0} \sum_{l=1}^{k} \frac{1}{l} - o^q_{0} \sum_{l=1}^{k} \frac{1}{l} + o^q_{0} a^{q}_{1,0} \]

\[ = o^q_{0} \log \tilde{x}_{0,q,k} + o^q_{0} a^{q}_{1,0}. \]

for some \( \tilde{x}_{0,q,k} \in [x, x + kh] \).

It immediately follows that

\[ \sum_{q=0}^{4} (-1)^q h^{-k} \Delta_k^{+} c_0(q,k) = O(\log x). \tag{9} \]

It remains to estimate

\[ \sum_{q=0}^{4} (-1)^q \sum_{s^q \in S^{q}_{R}} h^{-k} \Delta_k^{+} c_{s^q}(q,k). \]
We have,

\[ h^{-k} \Delta^+ c_{sv} (q,k) = o_{sv} (s^q)^{-1} \tilde{x}_{sv,q,k} \]

for some \( \tilde{x}_{sv,q,k} \in [x, x + kh] \).

Since \( s^4 \leq -\frac{1}{4} \) for \( s^4 \in S^4_R \), \( s^3 \leq \frac{1}{4} \) for \( s^3 \in S^3_R \), \( s^2 \leq \frac{1}{2} \) for \( s^2 \in S^2_R \), \( s^1 \leq \frac{3}{4} \) for \( s^1 \in S^1_R \), and \( s^0 \leq \frac{3}{4} \) for \( s^0 \in S^0_R \), it follows that

\[
\sum_{q=0}^{4} (-1)^q \sum_{s^q \in S^q_R} h^{-k} \Delta^+ c_{sv} (q,k) = O \left( x^{\frac{3}{4}} \right). \tag{10}
\]

Now, taking \( k = mD \), \( m \) even, \( h = x^{1-\frac{1}{2m}} \), \( M = x^{\frac{1}{2m}} \), and combining (5)-(10) with \( \psi_0 (x) \leq h^{-mD} \Delta^+ mD \psi_m (x) \), we obtain that

\[ \psi_0 (x) \leq 2x + O \left( x^{1-\frac{1}{2m}} \right). \]

Similarly,

\[ \psi_0 (x) \geq 2x + O \left( x^{1-\frac{1}{2m}} \right). \]

Hence,

\[ \psi_0 (x) = 2x + O \left( x^{1-\frac{1}{2m}} \right). \]

As it is known, this equality yields that

\[ \pi (x) = 2 \text{li} (x) + O \left( x^{1-\frac{1}{2m}} (\log x)^{-1} \right) \]

as \( x \to +\infty \).

This completes the proof. \( \square \)

4. Remarks

The author in \[12, p. 64\], derived that

\[ \Delta \left( \frac{2}{(2D+1)!} x^{2D+1} \right) = ax + b \]

for some \( a, b \in \mathbb{R} \).

Then, it was not so hard to calculate \( a \) and \( b \) explicitly.

While it was done for the \( a \), the \( b \) was considered as a constant, and hence as a non-important term in further calculations. This approach led to the conclusion that the error term \( O \left( x^{\frac{3}{4}} \right) \) could be achieved via remaining two error terms \( O \left( K^{D-1} x^{\frac{1}{2}} \right) \) and \( O \left( K^{-D-1} x^{2D+\frac{1}{2}} d^{-2D} \right) \).

Recently \[9\], we have shown that this is really possible. Actually, we have deduced that \( O \left( x^{\frac{3}{4}} \right) \) can be achieved if we take \( K = x^{\frac{1}{4D-1}} \) and \( d = x^{\frac{10D-5}{10D-4}} \).
However, as it can be seen from the proof of our main result in this paper, the \( b \) (\( Q \) in our case) must be taken into account in calculations since it does not represent an arbitrary constant. More precisely, it represents the error term \( O(h) \).

Thus, the error terms \( O(h), O\left(x^{\frac{1}{2}}M^{D-1}\right) \) and \( O\left(h^{-mD}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \) are responsible for achieving our \( O\left(x^{1-\frac{3}{2}D}\right) \).

Regarding the corresponding results in [12], [2] and [9], it is enough to replace \( \frac{3}{4} \) by \( 1 - \frac{1}{2D} \) in the final form of the prime geodesic theorem.

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