ZALCMAN CONJECTURE FOR CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT. Let $A$ denote the class of analytic functions in the unit disk $D$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $S$ denote the class of functions $f \in A$ which are univalent (i.e., one-to-one). In 1960s, L. Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for $n \geq 2$, which implies the famous Bieberbach conjecture $|a_n| \leq n$ for $n \geq 2$. For $f \in S$, Ma [21] proposed a generalized Zalcman conjecture $|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1)$ for $n \geq 2, m \geq 2$. Let $U$ be the class of functions $f \in A$ satisfying $\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| < 1$ for $z \in D$. and $F$ denote the class of functions $f \in A$ satisfying $\text{Re} \left( (1-z)^2 f'(z) \right) > 0$ in $D$. In the present paper, we prove the Zalcman conjecture and generalized Zalcman conjecture for the class $U$ using extreme point theory. We also prove the Zalcman conjecture and generalized Zalcman conjecture for the class $F$ for the initial coefficients.

1. Introduction and Preliminaries

Let $H$ denote the class of analytic functions in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$. Here $H$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $D$. Let $A$ denote the class of functions $f \in H$ such that $f(0) = 0$ and $f'(0) = 1$. Let $S$ denote the class of functions $f \in A$ which are univalent (i.e., one-to-one) in $D$. If $f \in S$ then $f(z)$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  

A function $f \in A$ is called starlike (convex respectively) if $f(D)$ is starlike with respect to the origin (convex respectively). Let $S^*$ and $C$ denote the class of starlike and convex functions in $S$ respectively. It is well-known that a function $f \in A$ is in $S^*$ if, and only if, $\text{Re} \left( z f'(z)/f(z) \right) > 0$ for $z \in D$. Similarly, a function $f \in A$ is in $C$ if, and only if, $\text{Re} \left( (1 + z f''(z)/f'(z)) > 0 \right)$ for $z \in D$. From the above it is easy to see that $f \in C$ if, and only if, $z f' \in S^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in S^*$, a function $f \in A$ is said to be close-to-convex with argument $\alpha$ and with respect to $g$.

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if
\[ \text{Re} \left( e^{i\alpha} \frac{z f'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D}. \]

Let \( \mathcal{K}_\alpha(g) \) denote the class of all such functions, and
\[
\mathcal{K}(g) := \bigcup_{\alpha \in (-\pi/2, \pi/2)} \mathcal{K}_\alpha(g) \quad \text{and} \quad \mathcal{K}_\alpha := \bigcup_{g \in S^*} \mathcal{K}_\alpha(g)
\]
be the classes of close-to-convex functions with respect to \( g \), and close-to-convex function with argument \( \alpha \), respectively. The class
\[
\mathcal{K} := \bigcup_{\alpha \in (-\pi/2, \pi/2)} \mathcal{K}_\alpha = \bigcup_{g \in S^*} \mathcal{K}(g)
\]
is the class of all close-to-convex functions. It is well-known that every close-to-convex function is univalent in \( \mathbb{D} \). Geometrically, \( f \in \mathcal{K} \) means that the complement of the image domain \( f(\mathbb{D}) \) is the union of non-intersecting half-lines. These standard classes are related by the proper inclusions
\[ C \subset S^* \subset K \subset S. \]

For \( 0 < \lambda \leq 1 \), let \( \mathcal{U}(\lambda) \) be the class of functions \( f \in A \) satisfying
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| < \lambda \quad \text{for} \quad z \in \mathbb{D}.
\]
Since \( f'(z)(z/f(z))^2 \neq 0 \) in \( \mathbb{D} \), it follows that every function in the class \( \mathcal{U}(\lambda) \) is non-vanishing in \( \mathbb{D} \setminus \{0\} \). We set \( \mathcal{U} := \mathcal{U}(1) \). It is known that functions in \( \mathcal{U}(\lambda) \) are locally univalent and functions in the class \( \mathcal{U} \) are univalent (see [3]). Furthermore, Aksentev [1] and Ozaki and Nunokawa [24] have shown that functions in \( \mathcal{U}(\lambda) \) are univalent, i.e., \( \mathcal{U}(\lambda) \subseteq S \) for \( 0 < \lambda \leq 1 \).

It is worth to mention here some important definitions and results related to the subsets of \( \mathcal{H} \).

**Definition 1.1.** A set \( \mathcal{F} \subseteq \mathcal{H} \) is normal if each sequence \( \{f_n\} \) in \( \mathcal{F} \) has a subsequence \( \{f_{n_k}\} \) which converges to a function \( f \in \mathcal{H} \) uniformly on every compact subset of \( \mathbb{D} \).

**Definition 1.2.** A set \( \mathcal{F} \subseteq \mathcal{H} \) is locally bounded if for each point \( a \in \mathbb{D} \) there are constants \( M \) and \( r > 0 \) such that for all \( f \in \mathcal{F} \),
\[
|f(z)| \leq M \quad \text{for} \quad |z - a| < r.
\]
That is, \( \mathcal{F} \) is locally bounded if, about each point \( a \in \mathbb{D} \) there is a disk on which \( \mathcal{F} \) is uniformly bounded.

**Lemma 1.2.** [8, Lemma 2.8, p. 153] A set \( \mathcal{F} \subseteq \mathcal{H} \) is locally bounded if, and only if, for each compact set \( K \subset \mathbb{D} \) there is a constant \( M \) such that
\[
|f(z)| \leq M
\]
for all \( f \in \mathcal{F} \) and \( z \in K \).

**Theorem 1.3.** [8, Montel’s Theorem 2.9, p. 153] A family \( \mathcal{F} \subseteq \mathcal{H} \) is normal if, and only if, is locally bounded.

**Corollary 1.4.** [8, Corollary 2.10, p. 154] A set \( \mathcal{F} \subseteq \mathcal{H} \) is compact if, and only if, it is closed and locally bounded.
Theorem 1.5. \[\text{[10, Theorem 2.6 (Growth Theorem)]}\] For each \( f \in \mathcal{S} \),
\[
\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.
\]
For each \( z \in \mathbb{D}, \ z \neq 0 \), equality occurs if, and only if, \( f \) is a suitable rotation of Koebe function.

For \( f, g \in \mathcal{H} \), we say that \( f \) is subordinate to \( g \), written as \( f \prec g \) or \( f(z) \prec g(z) \), if there exists an analytic function \( \omega : \mathbb{D} \to \mathbb{D} \) with \( \omega(0) = 0 \) such that \( f(z) = g(\omega(z)) \) for \( z \in \mathbb{D} \). Furthermore, if \( g \) is univalent in \( \mathbb{D} \) then \( f \prec g \) if, and only if, \( f(0) = g(0) \) and \( f(\mathbb{D}) \subseteq g(\mathbb{D}) \). If \( \mathcal{G} \subseteq \mathcal{H} \), we use the notation \( s(\mathcal{G}) = \{ f : f \prec g \text{ for some } g \in \mathcal{G} \} \). If \( \mathcal{G} \) is a compact subset of \( \mathcal{H} \) then it is not difficult to show that \( s(\mathcal{G}) \) is compact subset of \( \mathcal{H} \) (for instance, see \[\text{[11, Lemma 5.19]}\]).

Suppose \( X \) is a linear topological vector space and \( V \subseteq X \). A point \( x \in V \) is called an extreme point of \( V \) if it has no representation of the form \( x = ty + (1-t)z \), \( 0 < t < 1 \) as a proper convex combination of two distinct points \( y, z \in V \). We denote \( EV \) the set of extreme points of \( V \). The convex hull of a set \( V \subseteq X \) is the smallest convex set containing \( V \). The closed convex hull denoted by \( \overline{co}V \) is defined as the intersection of all closed convex sets containing \( V \). That is, the closed convex hull of \( V \) is the smallest closed convex set containing \( V \), which is the closure of the convex hull of \( V \). The Krein-Milman Theorem asserts that every compact subset of a locally convex topological space is contained in the closed convex hull of its extreme points (see, for instance, \[\text{[9]}\]). For a general reference and for many important results on this topic, we refer to \[\text{[14]}\].

As a first step for application of the knowledge of extreme point of these classes Brickman et al. \[\text{[6]}\] pointed out the following general results.

Theorem A. Let \( \mathcal{G} \) be a compact subset of \( \mathcal{H} \) and \( J \) be a complex-valued continuous linear functional on \( \mathcal{H} \). Then \( \max\{\text{Re} J(f) : f \in \overline{co} \mathcal{G}\} = \max\{\text{Re} J(f) : f \in \mathcal{G}\} = \max\{\text{Re} J(f) : f \in E \overline{co} \mathcal{G}\} \).

Definition 1.3. If \( F \) is a convex subset of \( \mathcal{H} \) and \( J : \mathcal{H} \to \mathbb{R} \) then \( J \) is called convex on \( F \) provided that \( J(tf + (1-t)g) \leq tJ(f) + (1-t)J(g) \) whenever \( f, g \in F \) and \( 0 \leq t \leq 1 \).

Theorem B. Let \( \mathcal{G} \) be a compact subset of \( \mathcal{H} \) and \( J \) be a real-valued, continuous and convex functional on \( \overline{co} \mathcal{G} \). Then \( \max\{J(f) : f \in \overline{co} \mathcal{G}\} = \max\{J(f) : f \in \mathcal{G}\} = \max\{J(f) : f \in E \overline{co} \mathcal{G}\} \).

The proof of these two results can be found in \[\text{[14, Theorem 4.5, Theorem 4.6]}\]. In order to solve such linear extremal problems over \( \mathcal{G} \), it suffices to solve them over the smaller class \( E \overline{co} \mathcal{G} \). This reduction thereby becomes an effective technique for solving various linear extremal problems. Using this technique we solve the Zalcman conjecture for the class \( \mathcal{U} \).

In 1960s, L. Zalcman posed a conjecture that if a function \( f \in \mathcal{S} \) is given by \[\text{(1.1)}\]
\[
|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad \text{for } n \geq 2,
\]
the equality holds only for the Koebe function $k(z) = z/(1 - z)^2$ or its rotation. It is important to note that the remarkable Zalcman conjecture implies the celebrated Bieberbach conjecture $|a_n| \leq n$ for $f \in S$ (see [7]). A well-known consequence of the area theorem shows that (1.6) holds good for $n = 2$ (see [10]). For $f \in S$, Kruskhal has proved the Zalcman conjecture for $n = 3$ (see [17]) and recently for $n = 4, 5, 6$ (see [18]). For a simple and elegant proof of Zalcman conjecture for the case $n = 3$, we refer to [18]. The Zalcman conjecture for functions in the class $S$ is still open for $n > 6$. However, using complex geometry and universal Teichmüller spaces Kruskhal has proved it for all $n \geq 2$ in his unpublished work [19].

The Zalcman conjecture has been proved affirmatively for certain special subclasses of $S$, such as starlike functions, typically real functions, close-to-convex functions [7, 20] and an observation also demonstrates that the Zalcman conjecture is asymptotically true (see [11]). Recently, Abu Muhana et al. [2] solved Zalcman conjecture for the class $F$ consists of the family of functions $f \in A$ satisfying the condition $\Re (1 + z f''(z)/f'(z)) > -1/2$ for $z \in \mathbb{D}$. Functions in the class $F$ are known to be convex in some direction (and hence close-to-convex and univalent) in $\mathbb{D}$. In 1986, Brown and Tsaо [7] proved the Zalcman conjecture for the starlike functions and typically real functions. In 1988, Ma [20] proved that the Zalcman conjecture for close-to-convex functions. For basic properties of starlike functions, typically real functions and close-to-convex functions we refer to [10, 30].

In 1999, Ma [21] proposed a generalized Zalcman conjecture for $f \in S$ that for $n \geq 2, m \geq 2$,

$$|a_n a_m - a_{n+m-1}| \leq (n - 1)(m - 1),$$

which is still an open problem. Ma [21] has proved this generalized Zalcman conjecture for classes $S^*$ and $S_R$. Here $S_R$ denote the class of all functions in $S$ with real coefficients. In 2017, Ravichandran and Verma [27] proved it for the classes of starlike and convex functions of given order and for the class of functions with bounded turning.

In the present paper, we prove the Zalcman conjecture and generalized Zalcman conjecture for the class $U$ using extreme point theory. We also prove the Zalcman conjecture and generalized Zalcman conjecture for the class $F$ for the initial coefficients. The organization of the paper is as follows. In Section 2 we prove that the class $U(\lambda)$ for $0 < \lambda \leq 1$ is compact. In particular the class $U$ is compact. In Section 3 we will characterize the closed convex hull of the class $U$ and its extreme points. Then by using extreme point theory, we prove the Zalcman conjecture in Section 3 and generalized Zalcman conjecture for the class $U$ in Section 4. We prove the Zalcman conjecture and generalized Zalcman conjecture for the class $F$ for the initial coefficients in Section 5.

Before we prove our main results we recall some important results which will play vital role in our proofs. In 2016, Obradović et al. [23] prove the following interested result.
Proposition 1.7. \[23\] If \( f \in \mathcal{U}(\lambda) \) for \( 0 < \lambda \leq 1 \), then for \( z \in \mathbb{D} \),

\[
\frac{f(z)}{z} < \frac{1}{(1-z)(1-\lambda z)}.
\]

Let

\[
\mathcal{R} := \left\{ F \in \mathcal{H} : \cos(F) = \int_{|x|=1} F(xz) \, d\mu(x) : \mu \in \mathcal{A} \right\},
\]

where \( \mathcal{A} \) denote the set of probability measure on \( \partial \mathbb{D} \). We recall the following well-known result of Hallenbeck et al. [15].

Lemma 1.8. [15] \( \frac{1}{(1-z)^{\alpha+i\beta}} \in \mathcal{R} \) if, and only if, \( \alpha \geq 1 \) and \( \beta = 0 \).

Let \( \mathcal{P} \) denote the class of all analytic functions \( p \) in \( \mathbb{D} \) with \( p(0) = 1 \) satisfying \( \text{Re} \, p(z) > 0 \) in \( \mathbb{D} \). Functions in the class \( \mathcal{P} \) are called the Carathéodory functions and can be expressed as

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.
\]

Lemma 1.10. [14] \( p \in \mathcal{P} \) if, and only if, there is a probability measure \( \mu \) on \( \partial \mathbb{D} \) such that

\[
p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} \, d\mu(x).
\]

Equivalently, in view of the Lemma 1.10 for \( p \in \mathcal{P} \) given by (1.9) can be written as

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n = \int_{0}^{2\pi} \frac{1+e^{it}z}{1-e^{it}z} \, d\nu(t).
\]

On comparing both the sides of (1.11) we obtain

\[
c_n = 2 \int_{0}^{2\pi} e^{int} \, d\nu(t).
\]

Lemma 1.13. \[26\] Lemma 2.3, p. 507] If \( p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P} \), then for all \( n, m \in \mathbb{N} \),

\[
|\lambda c_n c_m - c_{n+m}| \leq \begin{cases} 
2, & 0 \leq \lambda \leq 1 \\
2|2\lambda - 1|, & \text{elsewhere}.
\end{cases}
\]

If \( 0 < \lambda < 1 \), the inequality is sharp for the function \( p(z) = (1+z^{n+m})/(1-z^{n+m}) \).

In other cases, the inequality is sharp for the function \( p(z) = (1+z)/(1-z) \).

2. Compactness of the set \( \mathcal{U}(\lambda) \)

Theorem 2.1. For \( 0 < \lambda \leq 1 \), the class \( \mathcal{U}(\lambda) \) is compact.
We aim to show that \( g_n \to g \) uniformly on every compact subset of \( \mathbb{D} \). Let
\[
\frac{z}{f_n(z)} f'_n(z) - 1 \quad \text{and} \quad \frac{z}{f(z)} f'(z) - 1.
\]
We aim to show that \( h_n \to h \) uniformly on every compact subset of \( \mathbb{D} \). Let
\[
h_n(z) = \frac{f_n(z)}{z} \quad \text{and} \quad h(z) = \frac{f(z)}{z}.
\]
Then \( h_n(z) \neq 0 \) and \( h(z) \neq 0 \), for \( z \in \mathbb{D} \) and \( n \in \mathbb{N} \). Now we prove that \( h_n \to h \) uniformly on \( \overline{D} = \{ z \in \mathbb{C} : |z| \leq r \} \), \( 0 < r < 1 \), where \( D = \{ z : |z| < r \} \). Let
\[
M_n = \sup_{z \in \overline{D}} |h_n(z) - h(z)| = \sup_{z \in \overline{D}} \left| \frac{f_n(z) - f(z)}{z} \right|.
\]
Since \( h_n(z) - h(z) \) is analytic function in \( D \) and continuous on \( \overline{D} \), so by maximum modulus theorem
\[
\max\{|h_n(z) - h(z)| : z \in \overline{D}\} = \max\{|h_n(z) - h(z)| : z \in \partial D\}.
\]
Since \( f_n \) converges to \( f \) uniformly on \( \overline{D} \) therefore, \( M_n \) converges to 0 as \( n \) tends to \( \infty \). Hence \( h_n \) converges to \( h \) uniformly on \( \overline{D} \). Therefore, \( h_n \) converges to \( h \) uniformly on every compact subset of \( \mathbb{D} \). Since \( h_n \to h \) uniformly on every compact subset of \( \mathbb{D} \) and \( h_n(z) \neq 0 \), \( h(z) \neq 0 \), for \( z \in \mathbb{D} \) and \( n \in \mathbb{N} \), it is not difficult to show that \( 1/h_n \) converges to \( 1/h \) uniformly on every compact subset of \( \mathbb{D} \).

If a sequence of continuous functions \( f_n \) converge uniformly to a continuous function \( f \) on some compact set, then \( f_n \) is uniformly bounded on that compact set. In view of the above discussion, it is easy to see that \( (1/h_n)^2 \) converges to \( (1/h)^2 \) uniformly on every compact subset of \( \mathbb{D} \). Also, \( f'_n \) converges to \( f' \) uniformly on every compact subset of \( \mathbb{D} \). Hence \( (1/h_n)^2 f'_n \) converges to \( (1/h)^2 f' \) uniformly on every compact subset of \( \mathbb{D} \). Therefore \( g_n \) converges to \( g \) uniformly on every compact subset of \( \mathbb{D} \).

Since \( |g_n(z)| < \lambda \) for each \( n \in \mathbb{N} \), we prove that \( |g(z)| \leq \lambda \). Suppose not, then there exists \( z_0 \in \mathbb{D} \) such that \( |g(z_0)| > \lambda \). Let \( \epsilon = |g(z_0)| - \lambda \). Then there exists \( N \in \mathbb{N} \) such that
\[
|g_n(z) - g(z)| < \epsilon = |g(z_0)| - \lambda
\]
for \( n \geq N \) and \( z \in \mathbb{D} \). In particular,
\[
|g_N(z_0) - g(z_0)| < |g(z_0)| - \lambda.
\]
Therefore,
\[
|g(z_0)| - |g_N(z_0)| \leq |g_N(z_0) - g(z_0)| < |g(z_0)| - \lambda,
\]
shows that \( |g_n(z_0)| > \lambda \) which is a contradiction and hence \( |g(z)| \leq \lambda \) in \( \mathbb{D} \). If there exists some point \( z_0 \in \mathbb{D} \) such that \( |g(z_0)| = \lambda \) then by Maximum Modulus theorem, \( g \) must be a constant function, which is a contradiction. Therefore, \( |g(z)| < \lambda \) for \( z \in \mathbb{D} \) and hence \( f \in \mathcal{U}(\lambda) \). This shows that \( \mathcal{U}(\lambda) \) is closed. In view of Theorem 1.3.4...
it is easy to observe that the class $S$ is locally bounded. Since the class $U(\lambda) \subseteq S$ for $0 < \lambda \leq 1$, it follows that the class $U(\lambda)$ is also locally bounded. Thus, $U(\lambda)$ is compact. □

In particular, for $\lambda = 1$, the class $U$ is compact.

### 3. Zalcman Conjecture for the class $U$

**Theorem 3.1.** $\overline{\mathcal{U}}$ consists of all functions represented by

$$f(z) = \int_{|x|=1} \frac{z}{(1 - xz)^2} \, d\mu(x),$$

where $\mu \in \Lambda$. Here $\Lambda$ denotes the set of probability measure on $\partial \mathbb{D}$. Further, $\overline{\mathcal{U}}$ consists functions of the form

$$f(z) = \frac{z}{(1 - z)^2}, \quad |x| = 1.$$

**Proof.** Let $f \in U$ then in view of Proposition 1.7 we have

$$\frac{f(z)}{z} < \frac{1}{(1 - z)^2}.$$

Let $F(z) = 1/(1 - z)^2$ then from Lemma 1.8 we obtain

$$\overline{\cos}(F) = \left\{ \int_{|x|=1} F(xz) \, d\mu(x) : \mu \in \Lambda \right\} \quad \text{and} \quad E\overline{\cos}(F) = \{ F(xz) : |x| = 1 \}.$$

Let

$$\mathcal{G} := \left\{ \int_{|x|=1} \frac{z}{(1 - xz)^2} \, d\mu(x) : \mu \in \Lambda \right\}$$

then our aim is to prove that $\overline{\mathcal{U}} = \mathcal{G}$. To prove this, we first prove that $\mathcal{G}$ is convex and compact. The fact that $\mathcal{G}$ is convex follows from the convexity of the set of probability measure on $\partial \mathbb{D}$ i.e., convexity of $\Lambda$. To prove that $\mathcal{G}$ is compact, in the view of Corrolary 1.4, we will show that $\mathcal{G}$ is closed and locally bounded. The fact that $\mathcal{G}$ is closed follows from the weak-star compactness of the set of probability measure on $\partial \mathbb{D}$. To see that $\mathcal{G}$ is locally bounded, let $|z| = r < 1$ and $f \in \mathcal{G}$, then

$$f(z) = \int_{|x|=1} \frac{z}{(1 - xz)^2} \, d\mu(x).$$

Since $1 - |xz| \leq |1 - xz|$, we have $(1 - r)^2 \leq |1 - xz|^2$ for $|x| = 1$ and hence

(3.2)$$\frac{|z|}{|1 - xz|^2} \leq \frac{r}{(1 - r)^2}.$$

In view of (5.2), we obtain

$$|f(z)| = \left| \int_{|x|=1} \frac{z}{(1 - xz)^2} \, d\mu(x) \right| \leq \frac{r}{(1 - r)^2}.$$

Therefore $\mathcal{G}$ is locally bounded.
If \( f \in \mathcal{U} \) then \( f(z)/z \in s(F) \) which implies that \( f(z)/z \in \overline{\mathcal{C}_s}(F) \). That is,
\[
\frac{f(z)}{z} = \int_{|x|=1} \frac{1}{(1-xz)^2} \, d\mu(x)
\]
for some \( \mu \in \wedge \) and hence,
\[
f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2} \, d\mu(x)
\]
for some \( \mu \in \wedge \). Therefore \( f \in \mathcal{G} \) and hence \( \mathcal{U} \subseteq \mathcal{G} \). Since \( \mathcal{G} \) is closed and convex and \( \overline{\mathcal{G}} \) is the smallest closed convex set containing \( \mathcal{U} \), it follows that \( \overline{\mathcal{G}} \subseteq \mathcal{G} \). Since
\[
E\mathcal{G} = \left\{ \frac{z}{(1-xz)^2} : |x| = 1 \right\}
\]
and for each \( x \) such that \( |x| = 1 \), functions of the form \( z/(1-xz)^2 \) belong to \( \mathcal{U} \), it follows that \( E\mathcal{G} \subseteq \mathcal{U} \). Since \( \mathcal{G} \) is compact and convex, we conclude that \( \overline{\mathcal{G}} = \mathcal{G} \) and hence by Krein-Milman Theorem, it follows that
\[
\mathcal{G} \subseteq \overline{\mathcal{E}\mathcal{G}} \subseteq \overline{\mathcal{G}} \mathcal{U}.
\]
Therefore, \( \mathcal{G} \subseteq \overline{\mathcal{G}} \mathcal{U} \) and hence \( \overline{\mathcal{G}} \mathcal{U} = \mathcal{G} \) and
\[
E\overline{\mathcal{G}} \mathcal{U} = \left\{ \frac{z}{(1-xz)^2} : |x| = 1 \right\}.
\]
This completes the proof. \( \square \)

Let \( f \in \mathcal{S} \) be given by (1.1). Then for fixed \( n \in \mathbb{N} \), define the functional \( \phi : \mathcal{S} \to \mathbb{C} \) defined by, \( \phi(f) = a_n^2 - a_{2n-1} \). The rotations of \( f \in \mathcal{S} \) be given by
\[
g(z) = e^{-i\theta} f(e^{i\theta} z) = z + \sum_{n=2}^{\infty} A_n z^n,
\]
where \( A_n = a_n e^{i(n-1)\theta} \). Since \( \mathcal{S} \) is rotationally invariant, \( g \in \mathcal{S} \). A simple computation shows that
\[
\phi(g) = A_n^2 - A_{2n-1} = a_n^2 e^{2i(n-1)\theta} - a_{2n-1} e^{2i(n-1)\theta} = e^{2i(n-1)\theta} \phi(f).
\]
This shows that \(|\phi(g)| = |\phi(f)|\). That is, \(|a_n^2 - a_{2n-1}| \) is invariant under rotations. Since \( \mathcal{U} \) is rotationally invariant, maximizing \(|a_n^2 - a_{2n-1}| \) over \( \mathcal{U} \) is equivalent to maximizing \( \text{Re} (a_n^2 - a_{2n-1}) \) over \( \mathcal{U} \). It is easy to show that
\[
\text{Re} (a_n^2 - a_{2n-1}) = \text{Re} (a_n^2) - \text{Re} (a_{2n-1}) = (\text{Re} (a_n))^2 - (\text{Im} (a_n))^2 - \text{Re} (a_{2n-1}) \leq (\text{Re} (a_n))^2 - \text{Re} (a_{2n-1}).
\]
In view of (3.3), we maximize \( (\text{Re} (a_n))^2 - \text{Re} (a_{2n-1}) \) over \( \mathcal{U} \) to prove the Zalcman conjecture for the class \( \mathcal{U} \).

**Theorem 3.4.** Let \( f \in \mathcal{U} \) be given by (1.1). Then \(|a_n^2 - a_{2n-1}| \leq (n-1)^2 \) for \( n \geq 2 \). This inequality is sharp with equality for the Koebe function and its rotations i.e., functions of the form \( f(z) = z/(1-xz)^2 \) where \( |x| = 1 \).
Proof. Since $U \subseteq \mathcal{S}$, for the case $n = 2$, the proof of Zalcman Conjecture holds good (see [10]) as consequence of the area theorem. For the case $n = 3$, the Zalcman Conjecture has been proved by Krushkal [18] for the class $S$. Therefore, it suffices to prove the Zalcman Conjecture for $n \geq 4$ for the class $U$. For this, for fixed $n \in \mathbb{N}$ we define the functional $J : \mathcal{S} \rightarrow \mathbb{R}$ by

$$J(f) = (\text{Re}(a_n))^2 - \text{Re}(a_{2n-1}).$$

We first prove that $J$ is convex on $\overline{\mathcal{U}}$. Let $f, g \in \overline{\mathcal{U}}$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. In view of Definition [1,3] we show that $J(tf + (1-t)g) \leq tJ(f) + (1-t)J(g).$ Let $tf(z) + (1-t)g(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where $A_n = ta_n + (1-t)b_n$. A computation shows that

$$J(tf + (1-t)g) = (\text{Re}(A_n))^2 - \text{Re}(A_{2n-1})$$

$$= (\text{Re}(ta_n + (1-t)b_n))^2 - \text{Re}(ta_{2n-1} + (1-t)b_{2n-1})$$

$$= (\text{Re}(ta_n + (1-t)b_n))^2 - t\text{Re}(a_{2n-1}) - (1-t)\text{Re}(b_{2n-1})$$

$$= t^2(\text{Re}(a_n))^2 + (1-t)^2(\text{Re}(b_n))^2 + 2t(1-t)\text{Re}(a_n)\text{Re}(b_n)$$

$$- t(\text{Re}(a_n))^2 - (1-t)(\text{Re}(b_n))^2 + t(\text{Re}(a_n))^2$$

$$+ (1-t)(\text{Re}(b_n))^2 - t\text{Re}(a_{2n-1}) - (1-t)\text{Re}(b_{2n-1})$$

$$= t^2(\text{Re}(a_n))^2 - t(\text{Re}(a_n))^2(1-t)^2(\text{Re}(b_n))^2 - (1-t)(\text{Re}(b_n))^2$$

$$+ 2t(1-t)\text{Re}(a_n)\text{Re}(b_n)t ((\text{Re}(a_n))^2 - \text{Re}(a_{2n-1}))$$

$$+ (1-t)((\text{Re}(b_n))^2 - \text{Re}(b_{2n-1}))$$

$$= t(t-1)(\text{Re}(a_n))^2 + t(t-1)(\text{Re}(b_n))^2 - 2t(t-1)\text{Re}(a_n)\text{Re}(b_n)$$

$$+ tJ(f) + (1-t)J(g)$$

$$= t(t-1)(\text{Re}(a_n) - \text{Re}(b_n))^2 + tJ(f) + (1-t)J(g)$$

$$= tJ(f) + (1-t)J(g) - t(1-t)(\text{Re}(a_n) - \text{Re}(b_n))^2$$

$$\leq tJ(f) + (1-t)J(g).$$

and hence $J$ is a convex functional. In view of Theorem [13] Theorem [2,1] and Theorem [3,1] we consider the function $f_0$ of the form

$$f_0(z) = \frac{z}{(1-xz)^2} = z + \sum_{n=2}^{\infty} A_n z^n,$$

where $|x| = 1$ and $A_n = nx^{n-1}$. Therefore, $\phi(f_0) = A_n^2 - A_{2n-1} = (n^2 - 2n + 1)x^{2n-2}$ and hence $|\phi(f_0)| = (n-1)^2$. For $x = e^{i\theta}$, a simple computation shows that

$$J(f_0) = (\text{Re}(a_n))^2 - \text{Re}(a_{2n-1})$$

$$= n^2(\text{Re}(x^{n-1}))^2 - (2n-1)\text{Re}(x^{2n-2})$$

$$= n^2 \cos^2(n-1)\theta - (2n-1)\cos(2n-1)\theta$$

$$= n^2 \cos^2(n-1) - (2n-1)(2\cos^2(n-1)\theta - 1)$$

$$= \cos^2(n-1)\theta(n^2 - 4n + 2) + 2n - 1$$

$$\leq n^2 - 4n + 2 + 2n - 1 \quad \text{(since } n^2 - 4n + 2 > 0 \text{ for } n \geq 4.)$$

$$= n^2 - 2n + 1 = (n-1)^2.$$
Therefore, \((\text{Re}(a_n))^2 - \text{Re}(a_{2n-1}) \leq (n-1)^2\) and hence \(\text{Re}(a_n^2 - a_{2n-1}) \leq (n-1)^2\). This implies that \(|a_n^2 - a_{2n-1}| \leq (n-1)^2\) and the equality holds for the function of the form \(f_0(z) = z/(1-xz)^2\), where \(|x| = 1\). This completes the proof. \(\square\)

4. Generalized Zalcman Conjecture for the class \(\mathcal{U}\)

Let \(p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}\) then from (1.12), it is easy to see that for all \(n, m \in \mathbb{N}\),

\[
\lambda c_{n-1}c_{m-1} - c_{n+m-1} = 2 \left( 2\lambda \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\nu(t) - \int_0^{2\pi} e^{i(n+m-2)t} \, d\nu(t) \right).
\]

By using Lemma 1.13, we obtain

\[
\begin{align*}
(4.1) & \quad \left| 2\lambda \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\nu(t) - \int_0^{2\pi} e^{i(n+m-2)t} \, d\nu(t) \right| \\
& \leq \begin{cases} 
1, & 0 \leq \lambda \leq 1 \\
|2\lambda - 1|, & \text{elsewhere}.
\end{cases}
\end{align*}
\]

We now prove the generalized Zalcman conjecture for the class \(\mathcal{U}\).

**Theorem 4.2.** Let \(f \in \overline{\mathcal{U}}\) be given by (1.1). Then for \(n, m \geq 2\)

\[
|a_n a_m - a_{n+m-1}| \leq \begin{cases} 
n + m - 1, & \text{if } (n, m) \text{ is } (2, n), (m, 2), (3, 3), (3, 4), (4, 3) \\
(n-1)(m-1), & \text{otherwise.}
\end{cases}
\]

The second inequality is sharp and the equality holds for the Koebe function and its rotations.

**Proof.** Let \(f \in \overline{\mathcal{U}}\) be given by \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\). Then from Theorem 3.1 there exists a probability measure \(\mu\) on \(\partial \mathbb{D}\) such that

\[
f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2} \, d\mu(x).
\]

Equivalently, there exists a probability measure \(\nu\) on \([0, 2\pi]\) such that

\[
f(z) = \int_0^{2\pi} \frac{z}{(1-e^{it}z)^2} \, d\nu(t)
\]

which can be written as,

\[
(4.3) \quad z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} n \left( \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \right) z^n.
\]
By comparing both the sides of (4.3), we obtain

\[(4.4)\]
\[a_n = n \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t).\]

Using (4.4), we obtain

\[a_n a_m - a_{n+m-1} = nm \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\nu(t) - (n+m-1) \int_0^{2\pi} e^{i(n+m-2)t} \, d\nu(t)\]

which can be written as

\[(4.5)\]
\[a_n a_m - a_{n+m-1} = n + m - 1 \left[ \frac{nm}{n + m - 1} \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\nu(t) \right.\]
\[\left. - \int_0^{2\pi} e^{i(n+m-2)t} \, d\nu(t) \right].\]

Comparing (4.4) and (4.5), we obtain

\[(4.6)\]
\[\lambda = \frac{nm}{2n + 2m - 2}.\]

For \(\lambda \leq 1\), we see that (4.6) implies \(nm - 2n - 2m + 2 \leq 0\) which is equivalent to \((n - 2)(m - 2) \leq 2\). If any of \(n, m\) is 2 and for the pairs \((n, m) = (3, 3), (3, 4), (4, 3)\) the inequality \((n - 2)(m - 2) \leq 2\) holds. In all other choices of \(n, m\) we get \(\lambda > 1\).

Therefore, we have

\[(4.7)\]
\[\left| \frac{nm}{n + m - 1} \int_0^{2\pi} e^{i(n-1)t} \, d\nu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\nu(t) - \int_0^{2\pi} e^{i(n+m-2)t} \, d\nu(t) \right|\]

\[\leq \begin{cases} 1, & \text{if } (n, m) \text{ is } (2, n), (n, 2), (3, 3), (3, 4), (4, 3) \\ \frac{(n-1)(m-1)}{n+m-1}, & \text{other pairs of } (n, m). \end{cases}\]

Therefore from (4.5) and (4.7), we obtain

\[|a_n a_m - a_{n+m-1}| \leq \begin{cases} n + m - 1, & \text{if } (n, m) \text{ is } (2, n), (m, 2), (3, 3), (3, 4), (4, 3) \\ (n - 1)(m - 1), & \text{otherwise}. \end{cases}\]

The second inequality is sharp and equality holds for the Koebe function and its rotations. \(\square\)
5. ZALCMAN CONJECTURE FOR CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

In this section, we consider the class of close-to-convex functions with argument 0 and with respect to Koebe function \( k(z) = z/(1 - z)^2 \). More precisely, let

\[
F = \{ f \in \mathcal{A} : \Re (1 - z)^2 f'(z) > 0, \quad z \in \mathbb{D} \}.
\]

Clearly the functions in \( F \) are convex in the positive direction of the real axis. The region of variability of the class \( F \) has been studied by Ponnusamy et al in [25]. In 2017, Ali and Vasudevarao [4] obtained the sharp logarithmic coefficient for functions in the class \( F \).

Our aim is to solve the Zalcman conjecture for \( n = 2 \) and generalized Zalcman Conjecture for \( n = 2, m = 3 \).

**Theorem 5.1.** Let \( f \in F \) given by (1.1). Then

(i) \(|a_2^2 - a_3| \leq 1\)

(ii) \(|a_2 a_3 - a_4| \leq 2\).

These inequalities are sharp with equality for the Koebe function \( k(z) = z/(1 - z)^2 \) and its rotations.

**Proof.** For \( f \in F \), let \( g(z) = (1 - z)^2 f'(z) \). Clearly \( \Re g(z) > 0 \). Since \( \Re g(z) > 0 \) in \( \mathbb{D} \), there exists an analytic function \( \phi : \mathbb{D} \to \mathbb{D} \) such that

\[
\phi(z) = \frac{g(z) - 1}{g(z) + 1}.
\]

Clearly, \( \phi(0) = 0 \). Let

\[
\phi(z) = \sum_{n=1}^{\infty} c_n z^n = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots.
\]

In view of the Schwarz Lemma, we get \( |c_1| \leq 1 \). From (1.1) we have

\[
g(z) = (1 - z)^2 f'(z) = (1 + z^2 - 2z)(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}).
\]

From (5.2), we have

\[
\phi(z)(g(z) + 1) = g(z) - 1.
\]

A simple computation shows that

\[
(5.3) \qquad a_2 = 1 + c_1
\]

\[
a_3 = \frac{1}{3} \left( 2c_1^2 + 4c_1 + 2c_2 + 3 \right)
\]

\[
a_4 = \frac{1}{4} \left( 2c_1^3 + 4c_1^2 + 4c_1 c_2 + 6c_1 + 4c_2 + 2c_3 + 4 \right).
\]
Therefore using (5.3) we have
\[ a_2^2 - a_3 = (1 + c_1)^2 - \frac{1}{3}(2c_1^2 + 4c_1 + 2c_2 + 3) \]
\[ = \frac{1}{3}(3(1 + 2c_1 + c_1^2) - 2c_1^2 - 4c_1 - 2c_2 - 3) \]
\[ = \frac{1}{3}(3 + 6c_1 + 3c_1^2 - 2c_1^2 - 4c_1 - 2c_2 - 3) \]
\[ = \frac{1}{3}(c_1^2 + 2c_1 - 2c_2) \]
which yields
\[ |a_2^2 - a_3| = \frac{1}{3}(|c_1^2 + 2c_1 - 2c_2|) \leq \frac{1}{3}(|c_1|^2 + 2|c_1| + 2|c_2|) . \]

It is well-known that $|c_n| \leq 1 - |c_1|^2$ for $n \geq 2$. Therefore,
\[ (5.4) \quad |a_2^2 - a_3| \leq \frac{|c_1|^2 + 2|c_1| + 2(1 - |c_1|^2)}{3} = \frac{2|c_1| + 2 - |c_1|^2}{3} . \]

Let $x = |c_1|$ and $|c_1| \leq 1$. Let $h(x) = 2x + 2 - x^2$, where $0 \leq x \leq 1$. Clearly, $h$ is non-negative in $[0, 1]$ and $h'(x) = 2 - 2x$ is also non-negative in $[0, 1]$. Therefore, $h$ is an increasing function and hence $h(1) = 3$ is the maximum value. Hence $2|c_1| + 2 - |c_1|^2 \leq 3$. In view of this and (5.3), we obtain
\[ |a_2^2 - a_3| \leq 1 . \]

A simple computation using (5.3) gives
\[ a_2a_3 - a_4 = \frac{1}{3}((1 + c_1)(2c_1^2 + 4c_1 + 2c_2 + 3) - \frac{1}{4}(2c_1^3 + 4c_1^2 + 4c_1c_2 + 6c_1 + 4c_2 + 2c_3 + 4) \]
\[ = \frac{1}{3}(2c_1^3 + 6c_1^2 + 2c_1c_2 + 7c_1 + 2c_2 + 3) - \frac{1}{4}(2c_1^3 + 4c_1^2 + 4c_1c_2 + 6c_1 + 4c_2 + 2c_3 + 4) \]
\[ = \frac{1}{12}(2c_1^3 + 12c_1^2 - 4c_1c_2 + 10c_1 - 4c_2 - 6c_3) \]
\[ = \frac{1}{6}(c_1^3 + 6c_1^2 - 2c_1c_2 + 5c_1 - 2c_2 - 3c_3) . \]

Therefore,
\[ |a_2a_3 - a_4| \leq \frac{1}{6}(|c_1^3 + 6|c_1|^2 + 2|c_1||c_2| + 5|c_1| + 2|c_2| + 3|c_3|) \]
\[ \leq \frac{1}{6}(|c_1|^3 + 6|c_1|^2 + 2|c_1|(1 - |c_1|^2) + 5|c_1| + 2(1 - |c_1|^2) + 3(1 - |c_1|^2)) \]
\[ = \frac{1}{6}(-|c_1|^3 + |c_1|^2 + 7|c_1| + 5) . \]

Thus we obtain
\[ (5.5) \quad |a_2a_3 - a_4| \leq \frac{1}{6}(-|c_1|^3 + |c_1|^2 + 7|c_1| + 5) . \]
Let $x = |c_1|, |c_1| ≤ 1$ and $L(x) = -x^3 + x^2 + 7x + 5$, where $0 ≤ x ≤ 1$. Then $L'(x) = -3x^2 + 2x + 7$. It is easy to see that both the functions $L$ and $L'$ are non-negative in $[0, 1]$. Therefore, $L$ is increasing in $[0, 1]$ and $L(1) = 12$ is the maximum value. Hence $-|c_1|^3 + |c_1|^2 + 7|c_1| + 5 ≤ 12$. Using (1.3), we obtain $|a_2a_3 - a_4| ≤ 2$.

This completes the proof. □

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