Bilinear Approach to $N = 2$ Supersymmetric KdV equations

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Abstract

The $N = 2$ supersymmetric KdV equations are studied within the framework of Hirota’s bilinear method. For two such equations, namely $N = 2, a = 4$ and $N = 2, a = 1$ supersymmetric KdV equations, we obtain the corresponding bilinear formulations. Using them, we construct particular solutions for both cases. In particular, a bilinear Bäcklund transformation is given for the $N = 2, a = 1$ supersymmetric KdV equation.
1 Introduction

The theory of supersymmetric integrable systems has been an extensive research field for more than twenty years. As a consequence, many supersymmetric integrable equations have been studied and a number of interesting properties has been established. Among them, the most celebrated supersymmetric system is the supersymmetric Korteweg-de Vries (KdV) equation [1][2]. It has been shown that, as its bosonic analogue, the supersymmetric KdV (SKdV) equation is a bi-Hamiltonian system [3], has Darboux and Bäcklund transformations [4][5], can be casted into bilinear form [7][8][9], etc.

In the literature, there exist more than one supersymmetric extensions for the KdV equation. The most interesting ones are the N=2 supersymmetric KdV equations. The system was originally introduced by Laberge and Mathieu [10][11]. It reads as

\[ \phi_t = -\phi_{xxx} + 3(\phi D_1 D_2 \phi)_x + \frac{1}{2}(a - 1)(D_1 D_2 \phi^2)_x + 3a \phi^2 \phi_x \]

(1)

where \( \phi = \phi(x,t,\theta_1,\theta_2) \) is a superboson function depending on temporal variable \( t \), spatial variable \( x \) and its fermionic counterparts \( \theta_i (i = 1, 2) \). \( D_1 \) and \( D_2 \) are the super derivatives defined by \( D_1 = \partial_{\theta_1} + \theta_1 \partial_x \), \( D_2 = \partial_{\theta_2} + \theta_2 \partial_x \) and \( a \) is a parameter. In the sequel, we will refer to (1) as the SKdV\(_a\) equation. This one-parameter family of equations (1) is integrable only for certain values of the parameter \( a \). Indeed, Laberge and Mathieu in [10] have shown that for both \( a = -2 \) and \( a = 4 \), there exist Lax operators and Hamiltonian structures and infinite conservation laws. Then they [11] introduced \( S\!KdV_1 \) equation as a Hamiltonian equation with the \( N = 2 \) superconformal algebra as a second Hamiltonian structure. The Lax representations are given for \( S\!KdV_{-2} \) and \( S\!KdV_1 \) equations in [12]. Kupershmidt (see ref. [11]) observed that \( S\!KdV_4 \) equation is actually a bi-Hamiltonian system while Oevel and Popowicz [3] constructed the bi-Hamiltonian structures for both \( S\!KdV_{-2} \) and \( S\!KdV_4 \) equations based on \( r \)-matrix theory. A Painlevé analysis is performed for the \( S\!KdV_a \) equations in [13]. We also remark that \( N = 2 \) SKdV systems can be represented in terms of \( N = 1 \) Lax operators [14][15].

The purpose of this paper is to study the \( S\!KdV_a \) equation from the viewpoint of Hirota’s method. Recently, Hirota’s method has been applied to the supersymmetric integrable systems and the equations considered includes \( S\!KdV \) equation [9][8], supersymmetric MKdV equation [16], supersymmetric classical Boussinesq equation or supersymmetric two-boson equation [17]. As in the classical case, Hirota’s method can be adopted not only for constructing solutions, but also can be used for derivations of other properties. Therefore, this approach is very effective in the study of supersymmetric systems.

The paper is organized as follows. In section 2, we will transform the \( S\!KdV_4 \) equation into bilinear form and construct its solitons. In section 3, we first convert the \( S\!KdV_1 \) equation into bilinear form, then making use of this bilinear form a Bäcklund transformation and soliton solutions are constructed for this system. And a Lax representation can be worked out for the \( S\!KdV_1 \) equation. Final section contains a brief discussion.
2 SKdV$_4$ equation

In this section, we will consider the SKdV$_4$ and show that it can be converted into a bilinear form. Our strategy to do so is first to rewrite this equation in terms of N=1 formalism, then we embed it into the hierarchy of the supersymmetric two-boson system. Based on this connection, we provide a proper bilinear form for the SKdV$_4$. We also construct soliton solutions for this system.

2.1 Bilinear form

From (1), our SKdV$_4$ equation reads as

$$\phi_t = \left[-\phi_{xx} + 3\phi D_1 D_2 + \frac{3}{2} D_1 D_2 \phi^2 + 4\phi^3\right]_x,$$  

(2)

let

$$\phi = v + \theta_2 \beta$$

where $v = v(t, x, \theta_1)$ is a bosonic (even) function while $\beta = \beta(t, x, \theta_1)$ is a fermionic (odd) one. Then the SKdV$_4$ equation (2) in components takes the following form

$$v_t = \left[-v_{xx} + 6v D \beta + 3(D v) \beta + 4v^3\right]_x,$$  

(3a)

$$\beta_t = \left[-\beta_{xx} - 6v D v_x - 3v_x D v + 3\beta D \beta + 12v^2 \beta\right]_x,$$  

(3b)

where and hereafter we use $D = D_1$ for simplicity. Suppose that

$$v = \frac{1}{2} i u, \quad \beta = -\alpha + \frac{1}{2} D u$$

where $i = \sqrt{-1}$, then the system (3a-3b) is transformed into

$$u_t = \left[-u_{xx} + 3\alpha D u - 6u D \alpha - u^3 + 3uu_x\right]_x,$$  

(4a)

$$\alpha_t = \left[-\alpha_{xx} - 3\alpha D \alpha - 3u^2 \alpha - 3u\alpha_x\right]_x.$$  

(4b)

It has been shown that above system and the supersymmetric two-boson (sTB) equation share the same hierarchy [15]. The system (4a-4b) in fact is the third flow while the sTB is the second one. So we recall the sTB equation [18]

$$u_{t_2} = (-u_x + u^2 + 2D \alpha)_x,$$  

(5a)

$$\alpha_{t_2} = (\alpha_x + 2u \alpha)_x.$$  

(5b)

As shown in [17], through the following dependent variables transformations

$$u = -\left(\ln \frac{f}{g}\right)_x = -\varphi_x, \quad \alpha = (D \ln g)_x = D \rho_x,$$  

(6)
the sTB equation is brought into the bilinear form

\[ (D_t + D_x^2)f \cdot g = 0, \quad (7a) \]
\[ S(D_t + D_x^2)f \cdot g = 0, \quad (7b) \]

where the super Hirota derivative is defined as:

\[ SD_m^mD^n_x f \cdot g = (D_{\theta_1} - D_{\theta_2}) \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right)^m \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n f(x_1, t_1, \theta_1)g(x_2, t_2, \theta_2) \bigg|_{x_1=x_2=x, \theta_1=\theta_2=\theta}. \]

Now substituting the transformations (6) into the eqs. (5a) and (5b), we obtain

\[ \varphi_{t_2} = -\varphi_{xx} - \varphi_x^2 - 2\rho_{xx}, \quad (8a) \]
\[ \rho_{t_2} = \rho_{xx} - 2D^{-1}(\varphi_x D \rho_x). \quad (8b) \]

Observing that the eqs. (5a,5b) or (8a,8b) and the eqs. (4a,4b) are the second and third flows respectively, we now convert the latter into bilinear form.

With the help of the transformations (6), the system (4a)-(4b) is rewritten as

\[ \varphi_t + \varphi_{xxx} + 3(D\varphi_x)(D\rho_x) + 6\varphi_x\rho_{xx} + \varphi_x^3 + 3\varphi_x\varphi_{xx} = 0, \]
\[ (D\rho_t + D\rho_{xxx} + 3\varphi^2_x D\rho_x - 3\varphi_x D\rho_{xx} + 3\rho_{xx} D\rho_x)_{xx} = 0, \]

integrating the above equations once and taking zero as the integration constants, we obtain

\[ \varphi_t + \varphi_{xxx} + 3(D\varphi_x)(D\rho_x) + 6\varphi_x\rho_{xx} + \varphi_x^3 + 3\varphi_x\varphi_{xx} = 0, \quad (9a) \]
\[ D\rho_t + D\rho_{xxx} + 3\varphi^2_x D\rho_x - 3\varphi_x D\rho_{xx} + 3\rho_{xx} D\rho_x = 0. \quad (9b) \]

For (9a), we have

\[
0 = \varphi_t + \frac{1}{4}(\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx}) \\
+ \frac{3}{4}[\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx} + 4(D\varphi_x)(D\rho_x)] \\
= \varphi_t + \frac{1}{4}(\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx}) \\
- \frac{3}{4}\{\varphi_x(-\varphi_{xx} - \varphi_x^2 - 2\rho_{xx}) + (-\varphi_{xx} - \varphi_x^2 - 2\rho_{xx})x + 2\rho_{xx} - 2D^{-1}(\varphi_x D \rho_x)_x\} \\
\overset{8a,8b}{=} \varphi_t + \frac{1}{4}(\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx}) - \frac{3}{4}(\varphi_x\varphi_t + \varphi_x t + 2\rho_{xt}) \\
= \frac{1}{fg}(D_t - \frac{3}{4}D_x D_t + \frac{1}{4}D_x^3)f \cdot g
\]

(10)
and for (9b), we have

\[
0 = \frac{1}{2} \left\{-[\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx} + 3(D\varphi_x)(D\rho_x)]D\varphi \\
-3\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx} + 3(D\varphi_x)(D\rho_x)] + 2D\rho_t \\
+\frac{3}{4}[\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx} + 4(D\varphi_x)(D\rho_x)]D\varphi \\
+\frac{3}{4}([\varphi_{xx} + \varphi_x^2 + 2\rho_{xx}]D\varphi_x + \frac{3}{4}\varphi_xD(\varphi_{xx} + \varphi_x^2 + 2\rho_{xx})) \\
-\frac{3}{2}\varphi_xD[\rho_{xx} - 2D^{-1}(\varphi_xD\rho_x)] + \frac{3}{2}(\varphi_{xx} + \varphi_x^2 + 2\rho_{xx})D\rho_x \\
+\frac{3}{4}D([\varphi_{xx} + \varphi_x^2 + 2\rho_{xx}] + \frac{1}{4}((\varphi_x^3 + 3\varphi_x\varphi_{xx} + 6\varphi_x\rho_{xx} + \varphi_{xxx}))D\varphi \\
+3\varphi_xD\varphi_{xx} + D\varphi_{xxx} + 2D\rho_{xxx} + 3(\varphi_{xx} + \varphi_x^2 + 2\rho_{xx})D\varphi_x \\
+6(\varphi_{xx} + \varphi_x^2 + 2\rho_{xx})D\rho_x] \right\}
\]

\[
= \frac{1}{2fg} \left\{ f \cdot g - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3 \right\}
\]

therefore, our $SKdV_4$ equation assumes the following bilinear form

\[
(D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)f \cdot g = 0,
\]

\[
S(D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)f \cdot g = 0.
\]

### 2.2 Solutions

For a given system, Hirota’s bilinear form is ideal for constructing particular solutions. Next we shall show that a class of solutions can be calculated for the $SKdV_4$ equation. We take

\[
f = \varepsilon f_1, \quad g = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots,
\]

substituting the above expressions into the eqs. (12a) and (12b) and collecting the alike power terms, we have

\[
\varepsilon^1 : \quad (D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)(f_1 \cdot 1) = 0,
\]

\[
S(D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)(f_1 \cdot 1) = 0,
\]

and for $i \geq 1$,

\[
\varepsilon^{i+1} : \quad (D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)(f_1 \cdot g_i) = 0,
\]

\[
S(D_t - \frac{3}{4}D_xD_t + \frac{1}{4}D_x^3)(f_1 \cdot g_i) = 0.
\]
From (13), we get
\[ f_{1t} - \frac{3}{4} f_{1xx} + \frac{1}{4} f_{1xxx} = 0, \quad D(f_{1t} - \frac{3}{4} f_{1xx} + \frac{1}{4} f_{1xxx}) = 0. \tag{15} \]
From eqs. (7a) and (7b), we may take [17],
\[ f_{1t} = -f_{1xx}, \quad g_{i,t_2} = g_{i,xx} - 2kg_{i,x}. \tag{16} \]
substituting the above expression into (15), we get \( f_{1t} = -f_{1xxx} \). Therefore, \( f_1 \) assumes the following form
\[ f_1 = e^{kx-k^2t_2-k^3t+\theta\xi} \tag{17} \]
where \( k \) is an usual constant and \( \xi \) is a Grassmann odd constant. Then, substituting eqs. (16) and (17) into (14), we obtain
\[ g_{i,t} + 3k^2 g_{i,x} - 3kg_{i,xx} + g_{i,xxx} = 0, \]
\[ (D - \xi - \theta k)(g_{i,t} + 3k^2 g_{i,x} - 3kg_{i,xx} + g_{i,xxx}) = 0. \]
Accordingly we can choose
\[ g_i = e^{k_i x+k_i(k_i-2k)t_2+k_i(3k_k-k_i^2-3k^2)t+\theta\xi_i} \]
so, we have
\[ f = e^{kx-k^2t_2-k^3t+\theta\xi}, \quad g = 1 + \sum_{i=1}^{N} e^{k_i x+k_i(k_i-2k)t_2+k_i(3k_k-k_i^2-3k^2)t+\theta\xi_i}. \tag{18} \]
These solutions have the remarkable property that they allow fusion and fission to take place [7, 8].

3 SKdV_{1} equation

In above section, we succeeded to construct the bilinear form and a class of solutions for the SKdV_{4} equation. Now we turn to the SKdV_{1} equation and study it from the viewpoint of Hirota’s bilinear method. We will show that this system enjoys a simple bilinear form and a remarkable Bäcklund transformation.

3.1 Bilinear form

As in the case of the SKdV_{4} equation, we will work in the context of \( N = 1 \) formalism. Therefore, let
\[ \phi = v + \theta_2 \beta \]
then from the system (1), our SKdV\textsubscript{1} equation in component reads as

\begin{align}
v_t &= \left[-v_{xx} + 3vD\beta + v^3\right]_x, \quad (19a) \\
\beta_t &= \left[-\beta_{xx} - 3vDv_x + 3\beta D\beta + 3v^2\beta\right]_x. \quad (19b)
\end{align}

Now we introduce the following dependent variable transformations

\begin{align}
v &= i \left(\ln \frac{f}{g}\right)_x = i\varphi_x, \quad \beta = -(D \ln f g)_x = -D\rho_x, \quad (20)
\end{align}

substituting the above expressions into the eqs. (19a-19b), we obtain

\begin{align}
\varphi_t + \varphi_{xxx} + 3\varphi_x\rho_{xx} + \varphi^3_x &= 0, \quad (21a) \\
D\rho_t + D\rho_{xxx} + 3\varphi_xD\varphi_{xx} + 3\rho_{xx}D\rho_x + 3\varphi^2_xD\rho_x &= 0, \quad (21b)
\end{align}

for (21a), we have

\begin{align}
\varphi_t + \varphi^3_x + \varphi_{xxx} + 3\varphi_x\rho_{xx} = \frac{1}{fg} (D_t + D^3_x) f \cdot g = 0, \quad (22)
\end{align}

and for (21b), we have

\begin{align}
0 &= D\rho_t + D\rho_{xxx} + 3\varphi_xD\varphi_{xx} + 3\rho_{xx}D\rho_x + 3\varphi^2_xD\rho_x \\
&= D\rho_t + D\rho_{xxx} + 3\varphi_xD\varphi_{xx} + 3\rho_{xx}D\rho_x + 3\varphi^2_xD\rho_x + (\varphi_t + \varphi_{xxx} + 3\varphi_x\rho_{xx} + \varphi^3_x)D\varphi \\
&= \frac{1}{fg} S(D_t + D^3_x) f \cdot g. \quad (23)
\end{align}

From the above eqs. (22/23), we obtain the bilinear form for the SKdV\textsubscript{1} equation

\begin{align}
(D_t + D^3_x) f \cdot g &= 0, \quad (24a) \\
S(D_t + D^3_x) f \cdot g &= 0. \quad (24b)
\end{align}

**Remark:** This bilinerization is particularly simple and can be considered as a direct generalization of the bilinear form of the supersymmetric two-boson system (7a/7b). However, it is interesting that these two systems do not belong to the same integrable hierarchy.

### 3.2 Bäcklund transformation

Integrable systems often possess Bäcklund transformations, which may be used to construct solutions. Also, Bäcklund transformation is considered as a characteristic of integrability for a given system. In this section, we will derive a bilinear BT for the SKdV\textsubscript{1} system. We follow the paper [16] and our results are summarized in the following
Proposition 1 Suppose that \((f, g)\) is a solution of eqs. (24a) and (24b), then \((f', g')\) satisfying the following relations,

\[
\begin{align*}
D_x g \cdot f' - D_x f \cdot g' &= \mu g f' - \mu f g', \\
SD_x g \cdot f' + SD_x f \cdot g' &= \mu S g \cdot f' + \mu S f \cdot g', \\
(D_t + D_x^2 - 3\mu D_x^2 + 3\mu^2 D_x) g \cdot g' &= 0, \\
(D_t + D_x^2 - 3\mu D_x^2 + 3\mu^2 D_x) f \cdot f' &= 0,
\end{align*}
\]

is another solution of (24a) and (24b), where \(\mu\) is an ordinary constant.

Proof. We consider the following

\[
\begin{align*}
\mathbb{P}_1 &= [(D_t + D_x^2) f \cdot g'] g' - f g[(D_t + D_x^2) f \cdot g'], \\
\mathbb{P}_2 &= [S(D_t + D_x^2) f \cdot g'] f' - f g[S(D_t + D_x^2) f \cdot g'].
\end{align*}
\]

We will show that above eqs. (25a)-(25d) imply \(\mathbb{P}_1 = 0\) and \(\mathbb{P}_2 = 0\). The case of \(\mathbb{P}_1\) can be verified as in [10], so we will concentrate on \(\mathbb{P}_2\) next. We will use various bilinear identities
which are presented in Appendix A.

\[ P_2 \begin{align*}
A.1 & \equiv S[(D_t g \cdot g') \cdot f' - g g' \cdot (D_t f \cdot f')] + (S f \cdot g)(D_t f' \cdot g') - (D_t f \cdot g)(S f' \cdot g') \\
-3D_x[(S D_x f \cdot g') \cdot (D_x g \cdot f') + (S D_x g \cdot f') \cdot (D_x f \cdot g')] + (S f \cdot g)(D_x^2 f' \cdot g') \\
- (D_x^2 f \cdot g)(S f' \cdot g') + S[(D_x^2 f \cdot f') \cdot g g' - f f' \cdot (D_x^2 g \cdot g')] \\
= & -3D_x[(S D_x f \cdot g') \cdot (D_x g \cdot f') + (S D_x g \cdot f') \cdot (D_x f \cdot g')] \\
+ S[(D_t + D_x^3) g \cdot g' \cdot f f' + [(D_t + D_x^3) f \cdot f'] \cdot g g'] \\
+ (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g'[(D_t + D_x^3) f \cdot g]
\end{align*}
\]

\[ A.3 \equiv -3D_x[(S D_x f \cdot g') \cdot (D_x g \cdot f') + (S D_x g \cdot f') \cdot (D_x f \cdot g')] \\
+ 3\mu D_x[(S D_x g \cdot f') \cdot g f' + (D_x g \cdot f') \cdot (S g' \cdot f) - (S g \cdot f') \cdot (D_x g' \cdot f)] \\
- g f' \cdot (S D_x g' \cdot f)] - 3\mu^2 D_x[(S g \cdot f') \cdot g f' + g f' \cdot (S g' \cdot f)] \\
+ (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g')[D_t + D_x^3) f \cdot g]
\]

\[ A.4 \equiv -3D_x[(S D_x f \cdot g') \cdot (D_x f \cdot g' + \mu g f' - \mu f g')] + (S D_x g \cdot f')(D_x g \cdot f' - \mu f g' + \mu f g')] \\
+ 3\mu D_x[(S D_x g \cdot f') \cdot g f' + (D_x f \cdot g' + \mu f g') \cdot (S g' \cdot f) \\
+ (S g \cdot f') \cdot (D_x g \cdot f' - \mu f g' + \mu f g') - g f' \cdot (S D_x g' \cdot f)] \\
- 3\mu^2 D_x[(S g \cdot f') \cdot g g' + g f' \cdot (S g' \cdot f)] \\
+ (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g')[D_t + D_x^3) f \cdot g]
\]

\[ = 3D_x[(D_x f \cdot g' - \mu f g') \cdot (S D_x - \mu S) f \cdot g'] \\
+ 3D_x[(D_x g \cdot f' - \mu g f') \cdot (S D_x - \mu S) g \cdot f'] \\
+ (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g')[D_t + D_x^3) f \cdot g]
\]

\[ 25a \equiv 3D_x[(D_x g \cdot f' - \mu g f') \cdot (S D_x f \cdot g' - \mu S f \cdot g' + SD_x g \cdot f' - \mu S g \cdot f')] \\
+ (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g')[D_t + D_x^3) f \cdot g]
\]

\[ 25b \equiv (S f \cdot g)[(D_t + D_x^3) f' \cdot g'] - (S f' \cdot g')[D_t + D_x^3) f \cdot g].
\]

Since \((f, g)\) is a solution, it satisfies \((D_t + D_x^3) f \cdot g = 0\). Also, taking account of \(P_1 = 0\), it
yields that \((D_t + D_x^3) f' \cdot g' = 0\). Therefore, we finally have \(P_2 = 0\) and the proof is completed.

Next we will demonstrate that a spectral problem can be derived from the above BT.

For this purpose, we assume

\[ m = f'/f, \quad n = g'/g, \]
then by simple manipulation, from eqs. (25a-25d) we have

\[ m_x + m\varphi_x + n\varphi_x - n_x + \mu m - \mu n = 0, \]  
\[ mD\varphi_x + 2\varphi_x Dn + nD\varphi_x - 2Dn_x - 2\mu Dn - mD\rho_x - nD\rho_x = 0, \]  
\[ n_t - 3n_x\varphi_{xx} + 3n_x\rho_{xx} + n_{xxx} - 3\mu n\varphi_{xx} + 3\mu n\rho_{xx} + 3\mu n_{xx} + 3\mu^2 n_x = 0, \]  
\[ m_t + 3m_x\varphi_{xx} + 3m_x\rho_{xx} + m_{xxx} + 3\mu m\varphi_{xx} + 3\mu m\rho_{xx} + 3\mu m_{xx} + 3\mu^2 m_x = 0. \]  

\[ D(m+n)x + \mu D(m+n) + (m+n)D\rho_x + \varphi_x D(m-n) + \mu(m-n)D\varphi + (m-n)\varphi_x D\varphi = 0. \]  

(27)

To obtain a more compact form, we introduce

\[ U = m - n, \quad V = m + n, \]

in these variables, and the eqs. (26a-26d) can be rewritten simply as

\[ U_x + \varphi_x V + \mu U = 0, \]  
\[ \varphi_x DU + DV_x + \mu DV + VD\rho_x = 0, \]  
\[ V_t + 3\mu\rho_{xx}V + 3\mu\varphi_{xx}U + V_{xxx} + 3\varphi_{xx}U_x + 3\rho_{xx}V_x + 3\mu V_x + 3\mu^2 V_x = 0, \]  
\[ U_t + 3\mu\rho_{xx}U + 3\mu\varphi_{xx}V + U_{xxx} + 3\varphi_{xx}V_x + 3\rho_{xx}U_x + 3\mu U_x + 3\mu^2 U_x = 0. \]

(28a-28d)

Therefore we have the following

**Proposition 2** The compatibility condition of (28a)-(28d) are the SKdV1 equations (19a) and (19b).

**Proof:** Direct calculations.

### 3.3 Solutions

Since our SKdV1 system (19a,19b) has (24a,24b) as its Hirota’s bilinear form, we may adopt the standard perturbation method to find its possible soliton-like solutions. By tedious but straightforward calculation, we find one-soliton, two-soliton and three-soliton solutions, which are listed in the following

**One-soliton:**

\[ f = 1 + e^{\eta + \theta \xi}, \quad g = 1 - e^{\eta + \theta \xi}. \]

where \( \eta = kx - k^3 t + c_0. \)

**Two-soliton:**

\[ f = 1 + e^{\eta_1 + \xi_1} + e^{\eta_2 + \xi_2} + A_{12} e^{\eta_1 + \xi_1 + \xi_2}, \quad g = 1 - e^{\eta_1 + \xi_1} - e^{\eta_2 + \xi_2} + A_{12} e^{\eta_1 + \eta_2 + \xi_1 + \xi_2}. \]
Three-soliton:

\[
\begin{align*}
f &= 1 + e^{\eta_1 + \theta_1} + e^{\eta_2 + \theta_2} + e^{\eta_3 + \theta_3} \\
&\quad + A_{12} e^{\eta_1 + \eta_2 + \theta_1 + \xi_1 + \xi_2} + A_{13} e^{\eta_1 + \eta_3 + \theta_1 + \xi_1 + \xi_3} + A_{23} e^{\eta_2 + \eta_3 + \theta_2 + \xi_2 + \xi_3} \\
&\quad + (m_{13} m_{23} A_{12} + m_{12} m_{32} A_{13} + m_{12} m_{13} A_{23}) e^{\eta_1 + \eta_2 + \eta_3 + \theta_1 + \xi_1 + \xi_2 + \xi_3}, \\
g &= 1 - e^{\eta_1 + \theta_1} - e^{\eta_2 + \theta_2} - e^{\eta_3 + \theta_3} \\
&\quad + A_{12} e^{\eta_1 + \eta_2 + \theta_1 + \xi_1 + \xi_2} + A_{13} e^{\eta_1 + \eta_3 + \theta_1 + \xi_1 + \xi_3} + A_{23} e^{\eta_2 + \eta_3 + \theta_2 + \xi_2 + \xi_3} \\
&\quad - (m_{13} m_{23} A_{12} + m_{12} m_{32} A_{13} + m_{12} m_{13} A_{23}) e^{\eta_1 + \eta_2 + \eta_3 + \theta_1 + \xi_1 + \xi_2 + \xi_3},
\end{align*}
\]

where \(\eta_i = k_i x - k_i^3 t + c_i\), \(m_{ij} = \frac{k_i - k_j}{k_i + k_j}\) and

\[
A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right) \left(\frac{k_j - k_i + 2\xi_i \xi_j}{k_i + k_j} + 2\theta \frac{k_i \xi_j - k_j \xi_i}{k_i + k_j}\right).
\]

4 Discussions

In this paper, we study the N=2 SKdV equations within the framework of the Hirota bilinear method. For two of the three integrable cases, namely SKdV\(_4\) and SKdV\(_1\) equations, we succeed in obtaining their bilinear forms. We also construct the solutions for both equations and find a simple Bäcklund transformation for the SKdV\(_1\) equation.

Our results are presented in the N=1 form, so it is interesting to find if it is possible to study these N=2 equations within the N=2 form. Also, there is another N=2 SKdV equation–SKdV\(_{-2}\) and working out its bilinear form is an open problem.

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Appendix: some bilinear identities

In this appendix, some relevant bilinear identities are listed. Proofs of these identities are straightforward so are omitted. Here a, b, c and d are arbitrary even functions of the independent variables x, t, and \(\theta\).
\[(SD_xa \cdot b)cd - ab(SD_xc \cdot d) = S[(D_xb \cdot d) \cdot ac - bd \cdot (D_xa \cdot c)]\]
\[+ (Sa \cdot b)(D_xc \cdot d) - (D_xa \cdot b)(Sc \cdot d), \quad (A.1)\]
\[(SD_x^3a \cdot b)cd - ab(SD_x^3c \cdot d) = -3D_x[(SD_xa \cdot d) \cdot (D_xb \cdot c) + (SD_xb \cdot c) \cdot (D_xa \cdot d)]\]
\[+ S[(D_x^3a \cdot c) \cdot bd - ac \cdot (D_x^3b \cdot d)]\]
\[+ (Sa \cdot b)(D_x^3c \cdot d) - (D_x^3a \cdot b)(Sc \cdot d), \quad (A.2)\]
\[S[(D_x^2a \cdot b) \cdot cd - ab \cdot (D_x^2c \cdot d)] = D_x[(SD_xa \cdot c) \cdot bd + (D_xa \cdot c) \cdot (Sb \cdot d)]\]
\[+ D_x[(-Sa \cdot c) \cdot (D_xb \cdot d) - ac \cdot (SD_xb \cdot d)], \quad (A.3)\]
\[S[(D_xa \cdot b) \cdot cd - ab \cdot (D_xc \cdot d)] = D_x[(Sa \cdot d) \cdot bc + ad \cdot (Sb \cdot c)]. \quad (A.4)\]

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