Origin of the $U(1)$ field mass in superconductors

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Abstract. Recently, a new theory for superconductivity has been put forward, in which the persistent current generation is attributed to the emergent singularities of the electronic wave function that are created by the spin-twisting itinerant circular motion of electrons. The persistent current generated by this mechanism behaves in every respect like supercurrent in superconductors, yielding the flux quantum $\hbar/e$ and the Josephson frequency $2eV/h$, where $h$ is Planck’s constant, $-e$ is the electron charge, and $V$ is the voltage across the Josephson junction. The mass generation of the $U(1)$ gauge field (or the Meissner effect) in the new theory is due to the emergence of topological objects, ‘instantons’ generated by the single-valued requirement of the wave function in the presence of the emergent singularities.

The current standard theory of superconductivity is based on the BCS theory, and explains the emergence of superconductivity as due to the global $U(1)$ gauge symmetry breaking realized by the Cooper pair formation. The $U(1)$ field mass generation is believed to be due to this global $U(1)$ gauge symmetry breaking. However, the feasibility of this mechanism has been questioned since no known interaction can prepare the global $U(1)$ symmetry broken state from the normal state. We argue here that the $U(1)$ mass generation in the BCS superconductor can be attributed to the one by the instanton mentioned above if the Rashba spin-orbit interaction is added. Then, the occurrence of persistent current generation becomes due to the instanton formation, and the role of the Cooper pair formation is to stabilize the instanton by providing an energy gap for perturbative excitations. Upon forming the Cooper pair, the instanton is stabilized and persistent current generation becomes possible. Thus, the superconducting transition temperature coincides with the Cooper pair formation temperature.

1. Introduction

The superconducting state is currently regarded as the global $U(1)$ gauge symmetry broken one brought about by the Cooper pair formation [1]. This point is most clearly seen in the Ginzburg-Landau theory [2], where the superconducting order parameter is given by

$$\langle \psi_\uparrow \psi_\downarrow \rangle$$

where $\psi_\sigma$ is the field operator for electrons with spin $\sigma$, and $\langle \hat{O} \rangle$ means the expectation value of the operator $\hat{O}$ in the superconducting state.

The non-zeroeness of the above order parameter violates the global $U(1)$ gauge invariance since it is not invariant with respect to the global $U(1)$ gauge change

$$\psi_\sigma \rightarrow e^{ic} \psi_\sigma,$$  \hspace{1cm} (2)

where $c$ is a constant.
In elementary particle theory, a relativistic extension of the Ginzburg-Landau model, the abelian Higgs model, is often used to explain the mass generation in the gauge field [3]. The appearance of the mass here corresponds to the occurrence of the Meissner effect in superconductors.

The current understanding of superconductivity is based on the Cooper pair formation in the BCS theory [4], and the $U(1)$ field mass generation (thus, the occurrence of the Meissner effect) is attributed to the global $U(1)$ gauge symmetry breaking [1]. In this understanding, the Cooper pair formation is the main cause of the occurrence of superconductivity. However, the experimental facts for the cuprate superconductivity indicate that the Cooper pair formation may not be the main cause of the occurrence of superconductivity [5, 6, 7, 8, 9]. Also the validity of the assumed global $U(1)$ gauge symmetry breaking in the current standard theory has been questioned; we list three reasons against the global $U(1)$ gauge symmetry breaking mechanism below.

- The symmetry given in Eq. (2) is actually a redundancy in the quantum theory. Namely, $\psi_\alpha$ and $e^{ie\psi_\alpha}$ describes the same physics, and the phase change is not a symmetry transformation [10].
- No known interaction can mix states with different charged states, thus, it is impossible to prepare a state in which the phase of the operator in Eq. (1) is physically meaningful [11, 12, 13]. In other words, the assumed superconducting state cannot be reached from a high temperature non-superconducting state by time propagation of the quantum state (which contains the superconductor and its environment) by the Hamiltonian with known interactions.
- If a superconducting state is reachable from a fixed electron number normal state by reducing the temperature through the emission of radiation without contact with anything, the reached superconducting state has the fixed number of electrons, thus, Eq. (1) is zero.

If we insist that the theory of superconductivity must preserve the global $U(1)$ invariance, we need to consider a different $U(1)$ field mass generation mechanism for superconductivity. One candidate is the one put forward by ‘t Hooft for QCD (quantum chromodynamics) that utilizes the topological objects, ‘instantons’ [14]. Actually, similar topological objects have been already considered in the $E \otimes e$ dynamical Jahn-Teller problem [15] (we will call them also ‘instantons’), and they are known to generate persistent motion even in the ground state. Thus, the $U(1)$ mass generation in superconductivity may be actually due to the appearance of ‘instanton’, and the persistent current generation is also explained by it.

Let us give a short description for the persistent current generation in the dynamical Jahn-Teller problem. For the $E \otimes e$ Jahn-Teller system, the exact solution for the whole system is obtained; low energy states correspond to those with a half-integral pseudorotational angular momentum. The same problem can be solved using the Born-Oppenheimer approximation. In this approximation, the high-energy electronic problem is solved first and the effective Hamiltonian for the low energy vibrational motion is obtained. The potential energy surface for the vibrational motion has a degenerate point and the real-valued electronic wave functions show sign-change when they are transported around the degenerate point. Since we know that the total wave function is a single-valued function of the coordinates, the vibrational wave function has to have the compensating sign-change. This problem can be handled by introducing a gauge potential, a pure gauge potential with a singularity at the degenerate point [16]. By calculating the energy levels with including the gauge potential in the low energy effective Hamiltonian, the low energy spectrum that agrees with the exact one is obtained. This example indicates that there arise situations where a gauge potential (or a topological term) must be added to the low energy effective Hamiltonian. We call this pure gauge with singularities ‘instanton’ in the following.
The above problem has received a renewed attention when the Berry phase was formulated by Berry [17]. The Berry connection for the $E \otimes e$ is actually the ‘instanton’ mentioned above. It is noteworthy that an analogous effect has been investigated in the context of topological insulators and persistent current generation in them is one of the hottest topics of materials science [18].

Actually, a new theory for superconductivity that uses this ‘instanton’ has been already put forward. In this theory, the doubly degenerate $E$ electronic state of the Jahn-Teller problem is replaced by the electron spin degree of freedom. The sign change of wave functions found in the real electronic functions in $E \otimes e$ problem corresponds to the sign change of wave functions for electrons moving circularly with twisting their spin directions. The center of the circular motion is a singularity of the wave function, and the single-valued requirement of the wave function around it gives rise to a pure gauge potential with singularities. The persistent current generated by this gauge potential behaves in every respect like supercurrent in superconductors, yielding the flux quantum $h/2e$ and the Josephson frequency $2eV/h$, where $h$ is Planck’s constant, $-e$ is the electron charge, and $V$ is the voltage across the Josephson junction [19, 8].

A notable point about the new superconductivity theory is that it is in accordance with the experimental fact that to observe the ac Josephson effect a dc current feeding is necessary [5, 20] (see Fig. 1). Josephson’s prediction is the appearance of ac current by applying a dc voltage across the superconductor-insulator-superconductor (or Josephson) junction [21]; however, a simple application of a dc voltage is impossible since the voltage across the junction becomes zero due to the dc Josephson effect. A clear-cut observation of the ac Josephson effect in reality is the observation of plateaus in the $I$-$V$ plot, where a dc current is fed and a microwave radiation is applied (see, https://www.youtube.com/watch?v=Z9vBPQrG0U4&nohtml5=False, for such an experiment). In some experiments, radiation that contained a frequency component of the Josephson frequency was detected under the condition that a dc current was fed [22, 23, 24]. In any case, the dc current feeding to the junction is an essential ingredient for the observation of the effect (Fig. 1 (b)); however, this dc current feeding is lacking in the situation considered in the Josephson’s derivation and textbook derivations (Fig. 1 (a)).

Recent re-derivations of the ac Josephson effect under the real experimental situation by maintaining the gauge invariance indicate there are two physically different contributions to the Josephson frequency (see Eq. (46) in [5] and Eq. (26) in [20]): one is from the chemical potential difference between the two leads connected to the junction which arises due to the fact that the electric current enters the junction from one end and exists to the other end, and these two ends are contacted with the leads that have different chemical potentials; the other contribution comes from the electric field existing in the region between two superconductors of the junction. These two contributions are the same in the magnitude since the voltage across the junction caused by the electric field balances the chemical potential difference of the leads in the stationary condition. It is important to note that these two contributions are physically different since the chemical potential arises from the many body effect with including the Pauli exclusion principle, while the contribution from the electric field arises from the electromagnetic interaction between the electron and the electric field. Actually, what is included in the Josephson’s prediction is the contribution from the electric field, only. He expressed it with the chemical potential difference; here, the fact is implicitly used that the magnitude of the the contribution from the chemical potential difference and that from the electric field are the same. If the dc current feeding boundary condition is adopted for his $2e$ carrier tunneling formalism, the Josephson frequency becomes $4eV/h$, which disagrees with the experiment. This indicates that the actual "ac Josephson effect" is not really due to the Cooper pair tunneling. On the other hand, a satisfactory explanation is given by the new theory [5, 20].

In this paper, we take up the $U(1)$ field mass generation problem in superconductors. We argue that it is attributed to the emergence of ‘instanton’, which we denote as $\mathbf{A}^{\text{loc}} = \frac{e}{20} \nabla \chi$
Figure 1. Schematic set-up for the Josephson effect experiment. (a) Situation considered by Josephson and found in textbooks [25, 26]. Only current flow between the two superconductors $S_L$ and $S_R$ is taken into account. (b) Actual experimental situation. A dc current flow through the leads connected to the junction exists. Experiments indicate the dc current is an essential ingredient for the observation of the Josephson effects. If the dc current exists the contribution described by the dotted arrows arise in addition to the contribution given by the solid arrow.

with $q = -e$. Note that the present paper is not the report of new results; most of the results presented here are already published. Our intention is to provide a seed for further discussion in fundamental aspects of superconductivity, and also to notify the importance of the results and ideas originally come from the Jahn-Teller problem are relevant to the understanding on the phenomenon of superconductivity.

2. The appearance of effective gauge potential: the abelian Higgs model and a model with an $\mathcal{E} \otimes e$ Jahn-Teller like coupling from spin-vortex formation

First, we summarize the well-known Higgs mechanism for the $U(1)$ (or abelian) gauge field model [3]. The Lagrangian for this model is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left[ (\partial_\mu - iq A_\mu) \phi \right] \left[ (\partial_\mu - iq A_\mu) \phi \right] + \mu^2 \phi^4 - \lambda (\phi^* \phi)^2, \quad (3)$$

where $q$ is the charge on the particle for $\phi$.

Using the polar coordinates $\phi = r e^{i \theta}$, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + r^2 (\partial_\mu \theta - q A_\mu)^2 + (\partial \rho)^2 + \mu^2 \rho^2 - \lambda \rho^4 \quad (4)$$

The effective vector potential $B_\mu$ is given by

$$B_\mu = A_\mu - q^{-1} \partial_\mu \theta \quad (5)$$

This effective potential is gauge invariant under the gauge transformation

$$\phi \rightarrow e^{i \alpha} \phi; \quad q A_\mu \rightarrow q A_\mu + \partial_\mu \alpha \quad (6)$$

If we write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \approx \partial_\mu B_\nu - \partial_\nu B_\mu \quad (7)$$

by neglecting the contribution

$$q^{-1} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \theta, \quad (8)$$
we obtain the following Lagrangian for the $U(1)$ gauge field,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + q^2 \rho^2 B^2_{\mu}. \quad (9)$$

The neglect of the term in Eq. (8) means that the range of the variation of $\theta$ is limited so that it can be considered as a single-valued function. Actually, $\theta$ is an angular variable with period $2\pi$; thus, the term in Eq. (8) can give rise to a delta function type contribution. However, we neglect it in the following.

The potential energy term in the Lagrangian is given by

$$\mathcal{E}_{\text{pot}} = -\mu^2 \rho^2 + \lambda \rho^4. \quad (10)$$

This gives rise to a Mexican hat like shape for the potential energy function with a minimum energy trough at

$$\rho_{\text{min}} = \sqrt{\frac{\mu^2}{2\lambda}}. \quad (11)$$

A symmetry breaking is realized by the non-zero expectation value for $\varphi$

$$|\langle \varphi \rangle| = \rho_{\text{min}} \quad (12)$$

Then, the gauge field Lagrangian is approximated as

$$\mathcal{L}_{\text{gauge}} \approx -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 B^2_{\mu}, \quad M^2 = 2 q^2 \rho^2_{\text{min}}, \quad (13)$$

where $M$ is considered as the mass of the gauge field.

From the above Lagrangian, the “Maxwell equations”

$$\partial_{\mu} F^{\mu\nu} = -j^\nu \quad (14)$$

are obtained, where the electric current $j^\nu$ is given by

$$j^\nu = -M^2 B_\nu = -M^2 (A_\nu - q^{-1} \partial_\nu \theta) \quad (15)$$

This can be identified as the London equation, leading to the Meissner effect. The above electric current is gauge invariant because $B_\nu$ is gauge invariant thanks to the angular variable $\theta$.

In superconductivity $\varphi$ is taken as

$$\varphi = \psi^+ \psi \quad (16)$$

and $q$ is set to $q = -2e$, where $-e$ is the electron charge. The non-zero expectation value for $\varphi$ corresponds to Eq. (1).

The above mentioned Higgs mechanism for the $U(1)$ field mass generation in superconductors is one of the essential ingredients of the current standard theory of superconductivity. However, as is explained in Introduction, there are reasons to seek for a different mechanism. In the following, we consider the mass generation by introducing an $E \otimes e$ like coupling term arising from spin vortex formation [8, 19].
First, we identify $\varphi$ to a two-component electron field given by

$$
\varphi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
$$

(17)

Next add the following $E \otimes e$ like coupling term between the spin vortex $(S_x, S_y, S_z)$ and the electron spin $\frac{1}{2} \varphi^\dagger \sigma \varphi$,

$$
JS_x \varphi^\dagger \sigma_x \varphi + JS_y \varphi^\dagger \sigma_y \varphi = (\psi_+^\dagger, \psi_-^\dagger) \begin{pmatrix} 0 & ke^{-i\xi} \\ ke^{i\xi} & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
$$

(18)

where $JS_x = k \cos \xi$ and $JS_y = k \sin \xi$. Here, we consider the case where the spin-vortex is lying in the $xy$ plane, $(S_x, S_y, 0)$, and the system is uniform in the $z$ direction. $k$ and $\xi$ are functions of $x$ and $y$, where $\xi$ describes the rotation angle of the spin in the $xy$ plane. Actually, $(S_x, S_y, 0)$ also arises from the spin of electrons, thus, it is obtained by a self-consistent calculation.

The spin-vortex is characterized by the winding number for $\xi$. Let us define the winding number for $\varphi$ along loop $C$,

$$
w_C[\xi] = \frac{1}{2\pi} \oint_C \nabla \xi \cdot d\mathbf{r}.
$$

(19)

When it is not zero, spin-vortices exist. Centers of spin-vortices are singularities where $\xi$ is not defined. Actually, those points are zeros of the field operator. In order to obtain a new field operator that has zeros at the centers of spin-vortices, we perform the following unitary transformation on $\varphi$

$$
U \varphi = U \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}; \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\xi}{2}} & e^{-i\frac{\xi}{2}} \\ e^{-i\frac{\xi}{2}} & -e^{i\frac{\xi}{2}} \end{pmatrix},
$$

(20)

where $U$ diagonalizes the matrix in Eq. (18) as

$$
U \begin{pmatrix} 0 & ke^{-i\xi} \\ ke^{i\xi} & 0 \end{pmatrix} U^{-1} = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}
$$

(21)

The corresponding unitary transformation on the kinetic energy gives rise to a centrifugal term, which makes the centers of spin-vortices zeros of the new field operator $U \varphi$.

The matrix $U$ is multi-valued if the winding number $w_C[\xi]$ is odd for some loop $C$. If we insist that the new field operator must be single-valued, this $U$ is not allowed. To make the new field operator single-valued, we add a phase factor $e^{i\frac{\xi}{2}}$ on $U$, and modify the unitary matrix as

$$
U' = e^{i\frac{\xi}{2}} U
$$

(22)

Thus, the resulting new field operator $\varphi'$ is given by

$$
\varphi' = U' \varphi
$$

(23)

To make the transformation matrix $U'$ single-valued, $\chi$ must satisfy the constraint

$$
w_C[\xi] + w_C[\chi] = \text{even number for any } C
$$

(24)

The added phase in Eq. (22) modifies the gauge potential in the following manner

$$
A'^\mu_{\nu} = A_{\mu} + \frac{1}{2q} \partial_{\mu} \chi
$$

(25)
since a part of the unitary transformation involves the following transformation
\[ e^{i\frac{2}{\hbar} (\partial_\mu - iqA_\mu)}e^{-i\frac{2}{\hbar} (\partial_\mu - iqA_\mu)} = \partial_\mu - iq(A_\mu + \frac{1}{2q}\partial_\mu \chi). \] (26)

Note that \( q = -e \), the electron charge; thus, the phase \( \frac{1}{2}\chi \) (period of \( \chi \) is \( 2\pi \)) is associated for each electron. In contrast to this, the phase \( \theta \) (period of \( \theta \) is \( 2\pi \)) is associated for Cooper pair in \( B_\mu \) in Eq. (5).

The multi-valuedness \( \chi \) plays an essential role for the single-valuedness of \( U' \). It needs to satisfy the constrain in Eq. (24). Such \( \chi \) is given in the following form
\[ \chi = 2\pi \sum_{a=1}^{N_{\text{center}}} w_a \tan^{-1} \frac{y - y_a}{x - x_a} + f(x, y) \] (27)
where \( N_{\text{center}} \) is the number of the centers of the spin-twisting, \((x_a, y_a)\) are zeros of the field operator, \( w_a \) are integers that satisfy the constraint in Eq. (24), and \( f \) is single-valued function.

The gradient of function \( f \), \( \nabla f \), is determined from the minimal energy condition under the constraint in Eq. (24). Let us consider the chase where \( \chi \) is independent of time, and the electromagnetic gauge potential is given by \( A_\mu = (0, A) \). The energy minimization under the constraint is achieved from the stationary condition for the following functional
\[ F[A + \frac{1}{2q} \nabla \chi] = E[A + \frac{1}{2q} \nabla \chi] + \sum_{\ell} \lambda_\ell \left( \oint_{C_\ell} \nabla \chi \cdot dr - 2\pi w_{C_\ell} \right) \] (28)
where \( \lambda_\ell \) are Lagrange multipliers; \( N_{\text{loop}} \) is the number of independent loops [8, 19].

The stationary condition
\[ \frac{\delta F}{\delta \nabla \chi} = 0 \] (29)
yields
\[ j = -\frac{\delta E}{\delta A} = -2q \delta E \delta \nabla \chi = 2q \sum_{\ell} \lambda_\ell \delta \nabla \chi = 2q \sum_{\ell} \lambda_\ell \delta(x; C_\ell), \] (30)
where \( \delta(x; C_\ell) \) is one of the delta functions introduced by Kleinert [27] given by
\[ \delta(x; C_\ell) = \oint_{C_\ell} \delta^3(x - y) dy \] (31)

Since \( \nabla \chi \) is obtained by optimizing the combination \( A + \frac{1}{2q} \nabla \chi \), the change of the gauge in \( A \) is absorbed by the change in \( \nabla \chi \) (actually, the change in \( \nabla f \)). Therefore, \( A_{\text{eff}} \) is gauge invariant like \( B_\mu \).

Let us calculate the Meissner current in the wave function formalism since it is more transparent. In the wave function formalism [19, 8], the wave function for the ground state is given in the following form,
\[ \Psi(r^{(1)}, \cdots, r^{(N)}) = \Psi_0(r^{(1)}, \cdots, r^{(N)})e^{-i \sum_{\alpha=1}^{N} \chi(r^{(\alpha)})} \] (32)
where \( r^{(j)} \) is the coordinate of the \( j \)th electron and \( N \) is the number of electrons. \( \Psi_0 \) is a currentless multi-valued wave function, where the multi-valuedness arises from the spin-twisting
of the itinerant electrons. The phase factor $e^{-\frac{i}{2} \sum_{\alpha=1}^{N} \chi^{(\alpha)}}$ arises to make the wave function single-valued by compensating the sign change brought about by the spin-twisting.

Using the above wave function, the current density is obtained as

$$j(x) = -\frac{q^2 \rho(x)}{m} \left( A + \frac{1}{2q} \nabla \chi \right) = -\frac{q^2 \rho}{m} A^{\text{eff}}$$  \hspace{1cm} (33)$$

where

$$\rho(x) = \int d^3r_1 \cdots d^3r_N \left| \Psi_0(x, r_1, \ldots, r_N) \right|^2$$  \hspace{1cm} (34)$$

The equation (33) should be compared with Eq. (15); $A_{\mu}^{\text{eff}} = (0, A^{\text{eff}})$ replaces the gauge invariant potential $B_{\mu}$ in Eq. (5). It is shown that the electromagnetic field acquired a mass due to the appearance of $\nabla \chi$ with fixing the number of electrons.

3. Instanton and $U(1)$ mass generation

There are several different definitions for instanton [28], thus, first, we clarify what we mean by instanton.

Our instanton is

(i) It is a gauge potential given as a pure gauge $iq^{-1}U \partial_\mu U^{-1}$ in the asymptotic region, where $U$ is a unitary matrix.

(ii) It is characterized by a topological integer.

(iii) It is a classical solution to equations derived from the stationary condition for some action.

From the $\chi$ given in Eq. (27), we can construct an instanton or a collection of instantons,

$$A^{\text{fic}}_{\mu} = \frac{1}{2q} \partial_\mu \chi = iq^{-1}e^{i\frac{q}{2}\chi} \partial_\mu e^{-i\frac{q}{2}\chi},$$  \hspace{1cm} (35)$$

where $U = e^{i\frac{q}{2}\chi}$, and it is a classical solution to equations derived from the stationary condition for Eq. (28). The equation (33) indicates that this instanton produces a mass for the electromagnetic field.

This instanton gives rise to a collection of flux tubes in the $z$-direction

$$F^{\text{fic}}_{xy} = \partial_x A^{\text{fic}}_y - \partial_y A^{\text{fic}}_x = \frac{\pi}{q} \sum_{a=1}^{N_{\text{inst}}} w_a \delta(x - x_a) \delta(y - y_a)$$  \hspace{1cm} (36)$$

We attribute an instanton for each flux tube; thus, the above collection of flux tubes is attributed to arise from a collection of the same number ($N_{\text{inst}} = N_{\text{center}}$) of instantons.

This field strength can be generalized using the formula by Nambu developed for the dual string [29],

$$F^{\text{fic}}_{\mu\nu} = -\sum_a \frac{\pi w_a}{2q} \epsilon^{\mu\nu\lambda\rho} \int d\tau^a d\sigma^a \frac{\partial (y_{\mu}^a, y_{\nu}^a)}{\partial (\tau^a, \sigma^a)} \delta^4(x - y^a)$$  \hspace{1cm} (37)$$

where $\epsilon^{\mu\nu\lambda\rho}$ is the Levi-Civita antisymmetric symbol, $y_{\mu}^a(\tau^a, \sigma^a)$ represents the position of a point on the world sheet swept out by the $a$th flux tube (or string).

In a more general form, Eq. (33) is given by

$$j_\nu = -m_{\nu}^2 A^{\text{eff}}_{\nu}$$  \hspace{1cm} (38)$$
It is interesting to note that Nambu derived the solution for \( F_{\mu\nu}^{\text{eff}} = \partial_{\mu} A_{\nu}^{\text{eff}} - \partial_{\nu} A_{\mu}^{\text{eff}} \) by assuming the string field \( F_{\mu\nu}^{\text{nc}} \). The equation for \( A_{\nu}^{\text{eff}} \) is obtained from the conservation of electric charge

\[
\partial_{\nu} j^{\nu} = -m_0^2 \partial_{\nu} A_{\nu}^{\text{eff}} = 0.
\]

In the present case, \( A_{\nu}^{\text{bc}} \), thus also \( A_{\nu}^{\text{eff}} \) for \( A_{\nu} \) with an arbitrary gauge, is specified from the condition in Eq. (29); further, the conservation of electric charge is automatically satisfied.

4. An \( E \otimes e \) like coupling arising from the Rashba spin-orbit coupling

A connection between the new theory based on the instanton and the standard theory will be made if the Cooper pair formation stabilizes the formation of the instanton. If this is achieved, the global \( U(1) \) gauge symmetry breaking can be removed from the superconductivity theory.

In order to realize the instanton formation in the BCS superconductor, actually, an additional interaction is needed. We found the Rashba spin-orbit interaction [30] can produce the required instanton [31].

Taking the direction of the internal electric field in the \( z \) direction, the Rashba interaction is expressed as

\[
\begin{pmatrix}
0 & k(\partial_x - i \partial_y) \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi^\dagger_+ \\
\psi^\dagger_-
\end{pmatrix} = k(\partial_x - i \partial_y)
\begin{pmatrix}
\psi^\dagger_+ \\
\psi^\dagger_-
\end{pmatrix}
\]

(40)

In a magnetic field, electrons perform cyclotron motion. Taking the center of the cyclotron motion as the origin, we introduce the polar coordinates \( \rho, \phi \) through \( x = \rho \cos \phi, y = \rho \sin \phi \). Then, the above term becomes

\[
\begin{pmatrix}
0 & ke^{-i\phi} (\partial_\rho + \frac{i}{\rho} \partial_\phi) \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi^\dagger_+ \\
\psi^\dagger_-
\end{pmatrix}
\]

(41)

This term plays the role of Eq. (18).

By setting \( \xi = \phi \) in Eq. (20), the above term is transformed as

\[
U \begin{pmatrix}
0 & ke^{-i\phi} (\partial_\rho + \frac{i}{\rho} \partial_\phi) \\
0 & 0
\end{pmatrix} U^{-1} = k \begin{pmatrix}
\partial_\rho + \frac{1}{2\rho} & \frac{i}{\rho} \partial_\phi \\
-\frac{i}{\rho} \partial_\phi & -\partial_\rho - \frac{1}{2\rho}
\end{pmatrix}
\]

(42)

This indicates that \( k(\partial_\rho + \frac{1}{2\rho}) \) in the above plays the role of \( k \) in Eq. (2). Since the winding number of \( \xi \) along this cyclotron orbit is odd, the phase \( e^{i\frac{\xi}{2}} \) must be added to \( U \). Thus, the transformation matrix becomes \( U' \) in Eq. (22).

A model calculation for a simple cylindrical-symmetry system was performed, previously [31]. It indicates that if the applied magnetic field is sufficiently weak compared with the Rashba interaction, spin-twisting itinerant motion is stabilized; thus, instantons appear.

5. Concluding remarks

In the abelian Higgs model, it is assumed that \( \langle \varphi \rangle \neq 0 \) is realized in the ground state. However, if we apply it to superconductivity by taking \( \varphi = \psi^\dagger_+ \psi^\dagger_- \), the assumption will not be fulfilled since such a ground state cannot be reached from the high temperature normal state by known interactions. Using the state with \( \langle \varphi \rangle \neq 0 \) is a convenient tool to incorporate electron pairing correlation [32]. However, using the phase of \( \langle \varphi \rangle \) as a physically meaningful variable is another assumption that violates the global \( U(1) \) gauge invariance. If the global \( U(1) \) gauge invariance is not violated in reality, it will yield artifacts. Actually, it leads to a wrong value for the ac
Josephson frequency. This point was not apparent since the correct Josephson frequency formula was obtained by Josephson by assuming the boundary condition that is different from the real experimental one. The re-derivations using the real experimental boundary condition indicate that there is an additional contribution which is lacking in the Josephson’s derivation. If this contribution is included, the Josephson frequency with $q = 2e$ yields $4eV/h$; this indicates that the observed ac Josephson effect cannot be attributed to the Cooper pair tunneling [5, 20]. We have argued that the origin of the ac Josephson effect is the appearance of $\chi$ in Eq. (32) which gives rise to the instanton $A_{\text{fic}}$.

It is now a common knowledge that the effective low energy Hamiltonian may contain a topological term (or instanton), and we have argued that the superconductivity is a phenomenon of the emergence of instanton. In this theory, the global $U(1)$ gauge symmetry breaking is removed from the theory of superconductivity. It is noteworthy that the new theory can be related to the Ginzburg-Landau theory; the low energy physics described by the phase $\chi$ and its conjugate variable, superfluid density $\rho$, yields the equation for the ‘wave function’ $\Psi = \rho^{1/2}e^{-i\chi}$, which can be identified as the Ginzburg-Landau equation [8].

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