Effective resistances of two-dimensional resistor networks

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Abstract
We investigate the behavior of two-dimensional resistor networks, with finite sizes and different kinds (rectangular, hexagonal, and triangular) of lattice geometry. We construct a network by having a network unit repeat itself \(L_x\) times in the \(x\)-direction and \(L_y\) times in the \(y\)-direction. We study the relationship between the effective resistance (\(R_{\text{eff}}\)) of the network on dimensions \(L_x\) and \(L_y\). The behavior is simple and intuitive for a network with rectangular geometry; however, it becomes non-trivial for other geometries which are solved numerically. We find that \(R_{\text{eff}}\) depends on the ratio \(L_x/L_y\) in all three studied networks. We also check the consistency of our numerical results experimentally for small network sizes.

Keywords: circuit analysis, resistor network, electrical experiment, Kirchhoff’s laws

(Some figures may appear in colour only in the online journal)

1. Introduction

Resistor network problems have been widely studied in various contexts, starting from textbook physics and competitive tests [1] to electrical engineering [2], condensed matter physics [3], and statistical physics [4]. Since many regular electrical networks take the shape of meshes, similar to the lattices in solid state crystals (not to be confused with two-port lattice filters), it is intriguing to find out the equivalent or effective resistance (\(R_{\text{eff}}\)) of such networks. A lattice is a set of a periodic arrangement of atoms or molecules (called the basis) in a crystal [5]. The positions of those bases or building blocks are termed sites [5]. This concept is extended to the case where we have a periodic arrangement of electrical elements (e.g. a single or cluster of resistors) on an electrical network. We call the network a lattice and the junctions on the net-

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work are treated as sites of the lattice. There have been extensive studies on two-dimensional lattices in order to investigate percolation-based conductivity [6] in such systems, and methods like effective medium theory [6–9] and Green’s function method [6, 10–13] have been formulated to solve the problems. However, most of these studies focus on infinite systems with stochastic resistance distribution (random resistor network). Recently, many studies have been conducted for finite size networks, particularly methods based on the Laplacian matrix formalism [12, 14, 15], nodal potential difference equation-based approach [16], and a more sophisticated recursion transform or RT approach formulated by Tan and other colleagues [17–25]. However, a systematic study of the $R_{\text{eff}}$ dependence on dimensional parameters is found in the literature. Moreover, our paper does not compete with the methodologies mentioned above, rather it mostly desires to bring attention toward $R_{\text{eff}}$ characteristics for various lattice geometries, evaluated straightforwardly by solving a set of electrical equations. Numerics has been discussed in great detail, looking at its pedagogical and educational appeal toward students with a background of the basic circuit theory (primarily Kirchhoff’s laws, typically offered in high schools and undergraduate colleges). Many of the finite size network studies focus on ‘two-point geometry’ (electric bias connects to two nodes only), whereas we use the ‘bus-bar’ geometry (electric bias connects to two edges of the lattice keeping an open boundary condition for the lattice of finite size) [26]. Such a network geometry is routinely considered in random resistor networks to study percolation-based conduction on a sheet or to model a metal-to-insulator transition on a bulk inhomogeneous sample connected between two electrical leads in experiments [27, 28]. Network geometry dependence of $R_{\text{eff}}$ brings the connection to the graph theory [29, 30] and generalization of $Y$–$\Delta$ or star–polygon transformation can open doors of future research [31]. The network geometries studied in our papers are easy to solve numerically once the correct equations are formulated. Given resources, the network models can be constructed by students easily and by knowing the dependence of $R_{\text{eff}}$ on the geometry, a device can be designed whose resistance can be controlled by tuning its dimensions.

Our paper is organized in the following way. We first explain the generic resistor network setup and then discuss the analytical solution for the rectangular geometry. Then we discuss the numerical formulation and the $R_{\text{eff}}$ dependence on the dimensions, obtained from our numerical results for rectangular, hexagonal, and triangular resistor networks. As a summary, we compare these results for these three different geometries, and finally we describe a small experiment to test our theoretical findings.

2. Generic resistor network configuration

We describe below our generic setup for various lattice geometries.

(a) Define a two-dimensional (2D) lattice network. Though the geometry varies from lattice to lattice, we define the size of each lattice by two Cartesian lengths $L_x$ and $L_y$. For a rectangular lattice, $L_xL_y$ becomes the number of the lattice sites as well.

(b) Each site attaches to a resistance of value $R$ spreading along the direction of its neighborhood sites.

(c) Apply a bias $V$ at one edge of the lattice (say in the direction of the length $L_x$) and ground the other edge (hence the lattice acts like an active medium attached to a battery or applied voltage). Only one site from each unit cell (a minimal region that completes the lattice when repeated across the lattice area) of the lattice is attached to the bias or grounding and such sites must be equivalent for all unit cells of the lattice.
Typically, electrical networks with resistors and biases are solved by using one of Kirchhoff’s circuit laws (originally announced by Gustav Kirchhoff in 1845) [32, 33], which is often termed mesh-current or nodal analysis by electrical engineers [2]. Of Kirchhoff’s voltage and current laws, it is more convenient to use the current law that states that the total current at a circuit junction (lattice site in our case) must be zero. Hence the key equation for the Kirchhoff’s current law (KCL) at a site or node \((i, j)\) in 2D Cartesian coordinate is as follows:

\[
\sum_k I_{ij}(k) = 0 \Rightarrow \sum_k \left[ \frac{(V_{ik} - V_{ij})}{R_{ik}} + \frac{(V_{kj} - V_{ij})}{R_{jk}} \right] = 0, \tag{1}
\]

where \(k\) denotes all nearest-neighbor nodes (lattice sites, voltage or grounding connection) to site \((i, j)\). We discuss the implementation of this in the following sections, where we formulate them in the form of a matrix equation for various network geometries.

### 3. Rectangular resistor network

As the most common 2D geometry, we begin with a finite size rectangular lattice defined by lengths \(L_x\) and \(L_y\). Following the setup defined in the previous section, a voltage \(V\) is applied at one end of the lattice, in the direction of the length \(L_x\) (see figure 1), while the other end is grounded. Resistances, each with value \(R\), are connected to each site in all four directions. Our objective is to find out the effective resistance \((R_{\text{eff}})\) for the geometry and how \(R_{\text{eff}}\) depends on the dimensions \(L_x\) and \(L_y\).

#### 3.1. Analytical solution

To find out the effective resistance in the lattice, for a moment, we assume there are no resistive connections in the \(y\) (vertical) direction. Thus for a lattice of size \(L_x \times L_y\), sites are connected
only in the \(x\)-direction (see figure 2). There are total \(L_y\) branches of parallel resistances, with each branch consisting of a set of resistances in series. Now in each set of resistances in series, we note that the equivalent resistance between two adjacent sites is \(R + R = 2R\) (resistance \(R\) on the left of one site and on the right of the other site). Thus we find \(L_x - 1\) number of resistances of value \(2R\) between the first and last lattice sites and two resistances of value \(R\) on the left and right ends of the lattice. Thus in total, we have \(L_x\) number of resistances of value \(2R\) in series on each branch. Thus the equivalent resistance of each branch is \(2RL_x\). Since there exist \(L_y\) such branches in parallel, the overall effective resistance of this simplified circuit is as follows:

\[
R_{\text{eff}}^{\text{simp}} = \left[ \frac{1}{2RL_x} + \ldots \text{(\(L_y\) times)} \right]^{-1} = \frac{2RL_x}{L_y}.
\]  

(2)

We know that when two resistances \((R_1\) and \(R_2\)) are connected in series, as shown in figure 3(a), the potential drop after the first resistance \(R_1\) (i.e. in the middle of \(R_1\) and \(R_2\)) will be given by

\[
V - V_1 = \frac{V R_1}{R_1 + R_2},
\]

which gives the potential at the middle of \(R_1\) and \(R_2\):

\[
V_1 = V \left( 1 - \frac{R_1}{R_1 + R_2} \right).
\]

(4)

Extending this argument to our case, the potential at a junction \((i, j)\) of the circuit shown in figure 2 will be

\[
V_{i,j} = V \left( 1 - \frac{(2i - 1)R}{2RL_x} \right) = V \left( 1 - \frac{2i - 1}{2L_x} \right).
\]

(5)

This is because there are equivalent resistances of value \((i - 1) \times 2R + R = (2i - 1)R\) \(((i - 1)\) resistances with value \(2R\) plus a single resistance with value \(R\)) to the left of site \((i, j)\). Equation (5) shows that the potential at any branch is independent of the \(y\)-coordinate.
Thus, even if we were to connect the sites with resistances in the $y$-direction (which was our original network to begin with), no current would flow in the $y$-direction for the same $x$-coordinate. This means that the original network, with all the lattice sites joined, is equivalent to the network with sites joined only in the $x$-direction. Since the two networks are equivalent, the effective resistances of the original rectangular network will be the same as the one in equation (6):

$$R_{\text{rect}}^{\text{eff}} = 2R \frac{L_x}{L_y} = R \frac{z L_x}{2 L_y},$$

where we attempt to write the formula in a more generic form by looking at the coordination number $z$ (number of nearest-neighbor sites, $z = 4$ for a rectangular lattice). We can easily note that a balanced Wheatstone bridge [2] with resistance $R$ on each of its branches is the $L_x = 1$ and $L_y = 2$ case of the rectangular network (see figure 3(b)). There, by applying equation (6), we get $R_{\text{eff}} = 2R \cdot 1/2 = R$, which is supposed to be the desired result for the bridge network.

### 3.2. Numerical formulation

In our rectangular lattice of size $L_x \times L_y$, we can mark out nine distinct kinds of lattice sites.

- Left bottom corner site ($i = 1, j = 1$)
- Left top corner site ($i = 1, j = L_y$)
- Right bottom corner site ($i = L_x, j = 1$)
- Right top corner site ($i = L_x, j = L_y$)
- Left non-corner edge sites ($i = 1, j \in [2, L_y - 1]$)
- Bottom non-corner edge sites ($i \in [2, L_x - 1], j = 1$)
- Right non-corner edge sites ($i = L_x, j \in [2, L_y - 1]$)
- Top non-corner edge sites ($j = L_y, i \in [2, L_x - 1]$)
- Non-border inner sites ($i \in [2, L_x - 1], j \in [2, L_y - 1]$)

The KCLs for the above nine kinds of sites follow.
(a) **Left bottom corner site** \(i = 1, j = 1:\)

\[
\frac{V - V_{i1}}{R} + \frac{V_{21} - V_{i1}}{2R} + \frac{V_{12} - V_{i1}}{2R} = 0.
\]

\(7\)

(b) **Left top corner site** \(i = 1, j = L_y:\)

\[
\frac{V - V_{i1L_y}}{R} + \frac{V_{2L_y} - V_{i1L_y}}{2R} + \frac{V_{1L_y-1} - V_{i1L_y}}{2R} = 0.
\]

\(8\)

(c) **Right bottom corner site** \(i = L_x, j = 1:\)

\[
\frac{V_{L_x-1,1} - V_{L_x1}}{2R} + \frac{V_{L_x1} - V_{L_x1}}{R} + \frac{V_{L_x2} - V_{L_x1}}{2R} = 0.
\]

\(9\)

(d) **Right top corner site** \(i = L_x, j = L_y:\)

\[
\frac{V_{L_x-1,L_y} - V_{L_xL_y}}{2R} + \frac{V_{L_xL_y} - V_{L_xL_y}}{R} + \frac{V_{L_xL_y-1} - V_{L_xL_y}}{2R} = 0.
\]

\(10\)

(e) **Left non-corner edge site** \(i = 1, j = 2 \text{ to } L_y - 1:\)

\[
\frac{V - V_{i1}}{R} + \frac{V_{2j} - V_{i1}}{2R} + \frac{V_{i1,j-1} - V_{i1,j}}{2R} + \frac{V_{i1,j+1} - V_{i1,j}}{2R} = 0.
\]

\(11\)

(f) **Right non-corner edge site** \(i = L_x, j = 2 \text{ to } L_y - 1:\)

\[
\frac{V_{L_x-1,j} - V_{L_x,j}}{2R} + \frac{V_{L_x,j} - V_{L_x,j}}{R} + \frac{V_{L_x,j-1} - V_{L_x,j}}{2R} + \frac{V_{L_x,j+1} - V_{L_x,j}}{2R} = 0.
\]

\(12\)

(g) **Bottom non-corner edge site** \(i = 2 \text{ to } L_x, j = 1:\)

\[
\frac{V_{i-1,1} - V_{i1}}{2R} + \frac{V_{i+1,1} - V_{i1}}{2R} + \frac{V_{i2} - V_{i1}}{2R} = 0.
\]

\(13\)

(h) **Top non-corner edge site** \(i = 2 \text{ to } L_x, j = L_y:\)

\[
\frac{V_{i-1,L_y} - V_{iL_y}}{2R} + \frac{V_{i+1,L_y} - V_{iL_y}}{2R} + \frac{V_{iL_y-1} - V_{iL_y}}{2R} = 0.
\]

\(14\)

(i) **Non-border inner site** \(i = 2 \text{ to } L_x - 1, j = 2 \text{ to } L_y - 1:\)

\[
\frac{V_{i-1,j} - V_{i,j}}{2R} + \frac{V_{i+1,j} - V_{i,j}}{2R} + \frac{V_{i,j-1} - V_{i,j}}{2R} + \frac{V_{i,j+1} - V_{i,j}}{2R} = 0.
\]

\(15\)

We can rearrange the above equations by collecting the coefficients of \(V_{ij}.\)

(a) **Left bottom corner site** \(i = 1, j = 1:\)

\[
\left[\frac{1}{R} + \frac{1}{2R} + \frac{1}{2R}\right] V_{i1} - \frac{1}{2R} V_{21} - \frac{1}{2R} V_{12} = \frac{V}{R}.
\]

\(16\)

(b) **Left top corner site** \(i = 1, j = L_y:\)

\[
\left[\frac{1}{R} + \frac{1}{2R} + \frac{1}{2R}\right] V_{i1L_y} - \frac{1}{2R} V_{2L_y} - \frac{1}{2R} V_{1L_y-1} = \frac{V}{R}.
\]

\(17\)

(c) **Right bottom corner site** \(i = L_x, j = 1:\)

\[
\left[\frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R}\right] V_{L_x1} - \frac{1}{2R} V_{L_x-1,1} - \frac{1}{2R} V_{L_x2} = 0.
\]

\(18\)
(d) **Right top corner site** → \( i = L_x, j = L_y \):

\[
\left[ \frac{1}{2R} + \frac{1}{R} + \frac{1}{2R} \right] V_{L_x,L_y} - \frac{1}{2R} V_{L_x-1,L_y} - \frac{1}{2R} V_{L_x,L_y-1} = 0. \tag{19}
\]

(e) **Left non-corner edge site** → \( i = 1, j = 2 \) to \( L_y - 1 \):

\[
\left[ \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} \right] V_{1,j} - \frac{1}{2R} V_{2,j} - \frac{1}{2R} V_{1,j-1} - \frac{1}{2R} V_{1,j+1} = \frac{V}{R}. \tag{20}
\]

(f) **Right non-corner edge site** → \( i = L_x, j = 2 \) to \( L_y - 1 \):

\[
\left[ \frac{1}{2R} + \frac{1}{R} + \frac{1}{2R} \right] V_{L_x,j} - \frac{1}{2R} V_{L_x-1,j} - \frac{1}{2R} V_{L_x,j-1} - \frac{1}{2R} V_{L_x,j+1} = 0. \tag{21}
\]

(g) **Bottom non-corner edge site** → \( i = 2 \) to \( L_x, j = 1 \):

\[
\left[ \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} \right] V_{i,1} - \frac{1}{2R} V_{i-1,1} - \frac{1}{2R} V_{i+1,1} - \frac{1}{2R} V_{i,2} = 0. \tag{22}
\]

(h) **Top non-corner edge site** → \( i = 2 \) to \( L_x, j = L_y \):

\[
\left[ \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} \right] V_{i,L_y} - \frac{1}{2R} V_{i-1,L_y} - \frac{1}{2R} V_{i+1,L_y} - \frac{1}{2R} V_{i,L_y-1} = 0. \tag{23}
\]

(i) **Non-border inner site** → \( i = 2 \) to \( L_x - 1, j = 2 \) to \( L_y - 1 \):

\[
\left[ \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} \right] V_{i,j} - \frac{1}{2R} V_{i-1,j} - \frac{1}{2R} V_{i+1,j} - \frac{1}{2R} V_{i,j-1} - \frac{1}{2R} V_{i,j+1} = 0. \tag{24}
\]

Now, the \( V_{ij} \) values constitute a \( L_x \times L_y \) matrix, but if we linearize (see appendix A for details) it as a column vector \( \mathbf{V} \) of length \( N_L \equiv L_x L_y \), the above equations can be represented in matrix notation as

\[
\mathbf{G} \mathbf{V} = \mathbf{I}, \tag{25}
\]

where \( \mathbf{I} \) is a column vector of size \( N_L \), whose values are given by the right-hand side of equation (16) to equation (24) and \( \mathbf{G} \) is a \( N_L \times N_L \) square matrix consisting of the coefficients of the variables in the equations. Since \( \mathbf{G} \) has units \( 1/R \), we call it the conductance matrix. Since the potential at any lattice site \((i, j)\) depends only on its neighboring sites, the matrix \( \mathbf{G} \) is generally sparse, and can be solved using a sparse matrix solver numerically. The method is very similar to the typical transfer matrix method used in circuit analysis [34] and also similar to the method used in the context of a disordered resistor network [35].

Once the above matrix system is solved and we know the potential at all lattice sites, the effective resistance can be determined by dividing the total applied voltage by the net current flowing through the lattice in the direction of the applied voltage. It can be observed that the net current can be determined using the potential of the end sites of the lattice. The net current, in this case, would be given by

\[
I_{\text{net}} = \frac{\sum_{j=1}^{L_y} V_{L_x,j}}{R}. \tag{26}
\]
The effective resistance can then be determined as

\[ R_{\text{eff}} = \frac{V}{I_{\text{net}}} \]  

(27)

In practice, we choose \( R = 1 \) and \( V = 1 \).

3.3. Results

We first plot \( R_{\text{eff}} \) against \( L_x \) for several fixed values of \( L_y \). As expected from the analytical solution expressed in equation (6), \( R_{\text{eff}} \) grows linearly as \( L_x \) increases, and the slope of the linear curve drops at a larger value of \( L_y \) (see figure 4(a)).

When \( L_x \) is kept constant, \( R_{\text{eff}} \) decreases as \( L_y \) increases and \( R_{\text{eff}} \) vs \( 1/L_y \) plots show linear relationships, establishing that \( R_{\text{eff}} \propto L_x/L_y \). Now, to find the proportionality constant, we define

\[ r \equiv \frac{R_{\text{eff}}L_y}{RL_x} \]  

(28)

which according to equation (6) should be equal to \( z/2 = 2 \). Both figures 5(a) and (b) show that \( r \) is a constant when \( L_x \) and \( L_y \) are varied respectively, keeping the other dimension as a fixed parameter. The value of the constant is 2, and hence we see that the numerical results agree very well with our analytical formula.

4. Hexagonal network model

Now we consider the hexagonal- or graphene-type [36] honeycomb lattice network. Out of two possible orientations, we select a hexagonal lattice which has armchair edges (shapes like \( \ldots \backslash \wedge \wedge / \ldots \)) in the \( x \)-direction and zigzag edges (shapes like \( \ldots /\vee/\vee \ldots \)) in the \( y \)-direction (see figure 6) and dub this the armchair hexagonal lattice, borrowing nomenclature from the graphene nanoribbon literature [37].

4.1. Numerical formulation

Here for our convenience, we break the sites into two categories—(i) \( M \)-type sites, sitting at the middle corners of a hexagon, which connect to the bias and grounding, and (ii) \( S \)-type
Figure 5. Plot of $r = R_{R0}/(RL_x)$ as (a) $L_x$ and (b) $L_y$ is varied while other parameters are kept fixed. Both show $R_{R0} = 2$, which it is independent of dimensions $L_x$ and $L_y$.

Figure 6. A resistor network on a hexagonal lattice geometry.

sites, sitting on the top or bottom sides of a hexagon. We add extra indices 0 and 1 to specify $M$ and $S$ sites respectively. Now we can see there must always be equal and even numbers of $M$ and $S$ sites in the $x$-direction in a lattice with complete hexagons. The number of $M$ and $S$ sites ($L_M$ or $L_S$) sets the measurement of the length $L_x$: $L_x = L_M^x = L_S^x$. On the other hand, the number of voltage connections determines the length $L_y$: $L_y = L_M^y, L_y = L_S^y + 1 = L_y + 1$. The total number of sites can be determined as $N_{site} = L_M^x L_M^y + L_S^x L_S^y = L_x L_y + L_x(L_y + 1) = L_x(2L_y + 1)$. We can distinguish six kinds of sites in this system.

- Left border sites ($i = 1, j = 1$ to $L_x$, $k = 0$)
- Right border sites ($i = L_x, j = 1$ to $L_y$, $k = 0$)
- Top border sites ($i = 1$ to $L_x$, $j = L_y + 1$, $k = 1$)
- Bottom border sites ($i = 1$ to $L_x$, $j = 1$, $k = 1$)
- $M$-type inner sites ($i = 2$ to $L_x - 1, j = 1$ to $L_y$, $k = 0$)
- S-type inner sites \((i = 1 \text{ to } L_x, j = 2 \text{ to } L_y, k = 1)\)

The KCL for the above six kinds of sites would be as follows.

(a) **Left border sites** \(i = 1, j = 1 \text{ to } L_y, k = 0:\)

\[
\frac{V - V_{i,j,0}}{R} + \frac{V_{i+1,j,1} - V_{i,j,0}}{2R} + \frac{V_{i,j,1} - V_{i,j,0}}{2R} = 0. \tag{29}
\]

(b) **Right border sites** \(i = L_x, j = 1 \text{ to } L_y, k = 0:\)

\[
\frac{-V_{2L_x,j,0}}{R} + \frac{V_{2L_x,j+1,l} - V_{2L_x,j,0}}{2R} + \frac{V_{2L_x,j+1,l} - V_{2L_x,j,0}}{2R} = 0. \tag{30}
\]

(c) **Top border sites** \(i = 1 \text{ to } L_x, j = L_y + 1, k = 1:\)

if \(i = \text{odd},\)

\[
\frac{V_{i,j,0} - V_{i,j+1,1}}{2R} + \frac{V_{i+1,j+1,1} - V_{i,j+1,1}}{2R} = 0. \tag{31}
\]

if \(i = \text{even},\)

\[
\frac{V_{i,j,0} - V_{i,j+1,1}}{2R} + \frac{V_{i-1,j+1,1} - V_{i,j+1,1}}{2R} = 0. \tag{32}
\]

(d) **Bottom border sites** \(i = 1 \text{ to } L_x, j = 1, k = 1:\)

if \(i = \text{odd},\)

\[
\frac{V_{i,j,0} - V_{i,j+1,1}}{2R} + \frac{V_{i+1,j+1,1} - V_{i,j+1,1}}{2R} = 0. \tag{33}
\]

if \(i = \text{even},\)

\[
\frac{V_{i,j,0} - V_{i,j+1,1}}{2R} + \frac{V_{i-1,j+1,1} - V_{i,j+1,1}}{2R} = 0. \tag{34}
\]

(e) **M-type inner sites** \(i = 2 \text{ to } L_x - 1, j = 1 \text{ to } L_y, k = 0:\)

if \(i = \text{odd},\)

\[
\frac{V_{i-1,j,0} - V_{i,j,0}}{2R} + \frac{V_{i,j+1,0} - V_{i,j,0}}{2R} + \frac{V_{i,j+1,1} - V_{i,j,0}}{2R} = 0. \tag{35}
\]

if \(i = \text{even},\)

\[
\frac{V_{i+1,j,0} - V_{i,j,0}}{2R} + \frac{V_{i,j+1,0} - V_{i,j,0}}{2R} + \frac{V_{i,j+1,1} - V_{i,j,0}}{2R} = 0. \tag{36}
\]

(f) **S-type inner sites** \(i = 1 \text{ to } L_x, j = 2 \text{ to } L_y, k = 1:\)

if \(i = \text{odd},\)

\[
\frac{V_{i+1,j,1} - V_{i,j,1}}{2R} + \frac{V_{i,j+1,1} - V_{i,j,1}}{2R} + \frac{V_{i,j+1,0} - V_{i,j,1}}{2R} = 0. \tag{37}
\]

if \(i = \text{even},\)

\[
\frac{V_{i-1,j,1} - V_{i,j,1}}{2R} + \frac{V_{i,j+1,1} - V_{i,j,1}}{2R} + \frac{V_{i,j+1,0} - V_{i,j,1}}{2R} = 0. \tag{38}
\]

Note that here we have two types of sites, namely \(M\) and \(S\), and that though we use \((i, j)\) to denote the two-dimensional location of the different types of sites, a particular kind of site belongs to a particular type. Hence one type’s \(i\) or \(j\) should not coincide with another type’s \(i\) or \(j\), and this distinction is managed by the index \(k\). As before, the above equations can be represented in matrix notation and are solved using a sparse matrix solver.
4.2. Results

Like in the previous case, we first plot $R_{\text{eff}}$ as $L_x$ is varied, keeping $L_y$ fixed at different values. Even for a hexagonal lattice, $R_{\text{eff}}$ seems to increase linearly with $L_x$, as seen in figure 7(a). We then plot $R_{\text{eff}}$ as $L_y$ is varied, keeping $L_x$ fixed at different values. The result is shown in figure 7(b). Though $R_{\text{eff}}$ decreases with increasing $L_y$, like in the rectangular lattice case, $R_{\text{eff}}$ vs $1/L_y$ plots are not exactly linear.

To see the actual dependence, we again plot the ratio $r = R_{\text{eff}} L_y / (R L_x)$ against $L_x$ and $L_y$ while keeping other parameters fixed. We found a few interesting observations: (i) when $L_y$ is fixed, the ratio $r$ becomes independent of $L_x$; (ii) when $L_x$ is fixed, $r$ becomes universal and it approaches a constant in the thermodynamic limit\(^3\) ($L_y \to \infty$). These two observations, as shown in figure 7, let us arrive at the conclusion that the ratio is a sole function of $L_y$:

$$r = \alpha(L_y).$$

(39)

This leads to an empirical formula for the effective resistance:

$$R_{\text{eff}}^{\text{hex}} = \alpha(L_y) \frac{L_x}{L_y}.$$  

(40)

Now we further note that $\alpha(z, L_y)$ approaches $z$ as $L_y \to \infty$, where $z = 3$ is the lattice coordination number for a hexagonal lattice. Since $\alpha(z, L_y)$ has to be dimensionless to keep equation (39) physically consistent, a convenient guess could be

$$\alpha(L_y) = z e^{-c/L_y},$$

(41)

which implies

$$\ln \alpha(L_y) = \ln z - c L_y^{-1},$$

(42)

\(^3\)The thermodynamic limit is attributed to a very large or infinite size system where the average thermodynamic or equilibrium properties dominate over the individual elements of the system. Conventional equilibrium-based thermodynamics cannot be applied to a small finite size system and hence the limit is called thermodynamic [38]. The limit has significance to the study of percolation-based conductivity through a resistor network [39].
Figure 8. Plot of $r = R_{\text{eff}} L_y/(RL_x)$ as (a) $L_y$ is varied for different values of $L_x$ and (b) as $L_x$ is varied for different values of $L_y$ for an armchair hexagonal lattice. (c) $\ln r$ plotted against $1/L_y$ to verify the formula given in equation (41) for $\alpha$ where $r = \alpha(z, L_y)$. 
where $c$ is a constant. Now figure 8(c) plots $\ln \alpha$ against $L_y^{-1}$ for various $L_x$ and we can see when $L_y^{-1}$ approaches zero (thermodynamic limit), $\ln \alpha$ approaches $\ln z = \ln 3 \simeq 1.1$ vindicating our guessed formula for $\alpha$ in equation (41).

5. Triangular network model

We now move to the case where the lattice is triangular. The network considered is shown in figure 9. Clearly this lattice is similar to a rectangular lattice, with the exception that diagonal sites in one particular direction are also connected via an equivalent resistance $2R$.

5.1. Numerical formulation

For a triangular lattice, we have nine kinds of lattice sites.

- Left bottom corner site ($i = 1, j = 1$)
- Left top corner site ($i = 1, j = L_y$)
- Right bottom corner site ($j = 1, i = L_x$)
- Right top corner site ($i = L_x, j = L_y$)
- Left non-corner edge sites ($i = 1, j \in [2, L_y - 1]$)
- Right non-corner edge sites ($i = L_x, j \in [2, L_y - 1]$)
- Bottom non-corner edge sites ($j = 1, i \in [2, L_x - 1]$)
- Top non-corner edge sites ($j = L_y, i \in [2, L_x - 1]$)
- Non-border inner sites ($i \in [2, L_x - 1], j \in [2, L_y - 1]$)

The KCL, which relates the potential at any site $(i, j)$ with its neighboring site, for the above nine kinds of sites would be as follows.

(a) **Left bottom corner site** $\rightarrow i = 1, j = 1$:

$$\frac{V - V_{1,1}}{R} + \frac{V_{2,1} - V_{1,1}}{2R} + \frac{V_{1,2} - V_{1,1}}{2R} = 0. \quad (43)$$
(b) Left top corner site $\rightarrow i = 1, j = L_x$:
\[
\frac{V - V_{1,i}}{R} + \frac{V_{2,i} - V_{1,i}}{2R} + \frac{V_{L_x} - V_{1,i}}{2R} + \frac{V_{L_x,1} - V_{1,i}}{2R} = 0. \tag{44}
\]

(c) Right bottom corner site $\rightarrow i = L_x, j = 1$:
\[
\frac{V_{L_x,1} - V_{L_x,i}}{2R} + \frac{V_{L_x,1} - V_{L_x,i}}{2R} + \frac{V_{L_x - 1,1} - V_{L_x,i}}{2R} = 0. \tag{45}
\]

(d) Right top corner site $\rightarrow i = L_x, j = L_y$:
\[
\frac{V_{L_x,1} - V_{L_x,i}}{2R} - \frac{V_{L_x,1} - V_{L_x,i}}{2R} + \frac{V_{L_x,1} - V_{L_x,i}}{2R} = 0. \tag{46}
\]

(e) Left non-corner edge site $\rightarrow i = 1, j = 2$ to $L_y - 1$:
\[
\frac{V - V_{1,i}}{R} + \frac{V_{2,i} - V_{1,i}}{2R} + \frac{V_{1,i-1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} + \frac{V_{2,i-1} - V_{1,i}}{2R} = 0. \tag{47}
\]

(f) Right non-corner edge site $\rightarrow i = L_x, j = 2$ to $L_y - 1$:
\[
\frac{V_{L_x,1} - V_{L_x,i}}{2R} + \frac{V_{L_x,1} - V_{L_x,i}}{2R} + \frac{V_{L_x,i+1} - V_{L_x,i}}{2R} + \frac{V_{L_x,1} - V_{L_x,i}}{2R} = 0. \tag{48}
\]

(g) Bottom non-corner edge site $\rightarrow i = 2$ to $L_x$, $j = 1$:
\[
\frac{V_{1,1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} + \frac{V_{2,i} - V_{1,i}}{2R} + \frac{V_{1,i-1} - V_{1,i}}{2R} = 0. \tag{49}
\]

(h) Top non-corner edge site $\rightarrow i = 2$ to $L_x$, $j = L_y$:
\[
\frac{V_{1,1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} + \frac{V_{i,i-1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} = 0. \tag{50}
\]

(i) Non-border inner site $\rightarrow i = 2$ to $L_x - 1$, $j = 2$ to $L_y - 1$:
\[
\frac{V_{1,1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} + \frac{V_{i,i-1} - V_{1,i}}{2R} + \frac{V_{1,i+1} - V_{1,i}}{2R} + \frac{V_{i,i-1} - V_{1,i}}{2R} = 0. \tag{51}
\]

5.2. Results

After solving the above equations in the matrix form, we plot $R_{eff}$ against $L_x$, keeping $L_y$ fixed at various values. We again note that $R_{eff}$ increases with $L_x$ and $1/L_y$ (see figure 10(a)). However, unlike the rectangular or hexagonal lattice network, the dependence of $R_{eff}$ is not strictly linear, rather it only becomes linear in $L_x$ with a large value of $L_x$ or $L_y$ (see figure 10(c)).

Now we look at the ratio $r = R_{eff}(L_y)/(R_{eff})$ and find that it depends non-trivially on both $L_x$ and $L_y$. As can be noted from figures 11(a) and (b), $r$ approaches a constant value only when
Figure 10. Plot of $R_{\text{eff}}$ against (a) $L_x$ for different values of $L_x$ and (b) $1/L_y$ (inset: against $L_y$) for different values of $L_x$ for a triangular lattice. (c) $R_{\text{eff}}/L_x$ vs $L_x$ plots show that $R_{\text{eff}}$ depends linearly on $L_x$ only at large $L_x$ or $L_y$. (d) $R_{\text{eff}}/L_y$ vs $L_y$ plots show that $R_{\text{eff}}$ does not strictly depend linearly on $1/L_y$.

$L_x/L_y$ (when $L_x$ varied, $L_y$ fixed) or $L_y/L_x$ (when $L_y$ varied, $L_x$ fixed) is significantly large (i.e. $L_x$ or $L_y$ approaches the thermodynamic limit compared to the other dimension). However, unlike the earlier two lattice cases, we could not trivially solve any empirical function or formula for the $R_{\text{eff}}$ dependence on $L_x$ and $L_y$ for the triangular lattice network. We presume that this non-triviality arises because of the diagonal resistance dependence of the circuit current which is absent in hexagonal and rectangular lattice networks. Also, one should note that the hexagonal lattice is a brick-wall lattice [11], which is a shifted version of a rectangular lattice and hence both lattices bear similarity in the current flow distribution, and in that sense, the triangular lattice network is entirely unique. Only the natural expectation that the effective resistance will be independent of dimension at a large value of $L_x$ and $L_y$ has been reflected in the three different lattice networks discussed in our paper.

6. Summary

Now in table 1, we briefly summarize the $R_{\text{eff}}$ dependence on the dimensions for various network geometries discussed in previous sections.

Our numerical codes (in Python) are freely available to the public on the GitHub repository: https://github.com/hbaromega/2D-Resistor-Network.
Figure 11. Plot of $R_{\text{eff}}$ as (a) $L_x$ is varied for different values of $L_y$ and (b) $L_y$ is varied for different values of $L_x$ for a triangular lattice-based network.

Table 1. Table for dependence of $R_{\text{eff}}$ on various 2D lattice geometries.

| Network geometry | $L_x$ ($L_z$ Fixed) | $L_y$ ($L_z$ Fixed) | Formula |
|------------------|----------------------|---------------------|---------|
| Case A: Rectangular | $\propto L_x$ | $\propto 1/L_y$ | $R_{\text{eff}}L_x/L_y$ |
| Case B: Hexagonal (armchair) | $\propto L_x$ | $\propto 1/L_y$ at $L_x \ll L_y$ | $R_{\text{eff}}L_x/L_y$ |
| Triangular | $\propto L_x$ at $L_x \ll L_y$ | Not strictly $\propto 1/L_y$ | $R_{\text{eff}}L_x/L_y$ |

7. Experiment: determining the effective resistance of a resistor network

Our theoretical findings can be easily verified by setting up simple circuits made up of resistors of equal magnitudes. We first constructed a $2 \times 2$ rectangular or square network (case A) and a $2 \times 2$ armchair hexagonal network (case B) on a breadboard using equal resistors of resistance 100 $\Omega$. We measured the effective resistances of the networks using a MECO 603 digital multimeter and compared it with our theoretical results. In cases A and B, $R_{\text{eff}}$ should be $2R = 200 \Omega$ and $2.7143R = 271.43 \Omega$ respectively (see sections 3 and 4). Our multimeter readings show 200 $\Omega$ for case A and 271 $\Omega$ for case B respectively, showing consistent agreement with our theoretical predictions (figures 12(c) and (d)).

We then connected the networks to a DC power supply (Keltronix Series 8000, India) and determined the effective resistance by measuring the voltage and current across the circuit (figure 13(a)). The power supply has $\pm 0.01\%$ line and load regulations over a constant voltage setup (maintained in our experiment) and a maximum 1 mV (rms) AC ripple, as mentioned by the manufacturer. Tables 2 and 3 show the readings obtained for the $2 \times 2$ square and hexagonal networks.

From tables 2 and 3, we plot the $I$–$V$ (current vs voltage) curves and fit each of them with linear regression lines using the least square method [40]. The slopes of the regression lines estimate the values of conductance, $G_{\text{eff}} = 1/R_{\text{eff}}$. We find $G_{\text{eff}} = 0.0051 \Omega^{-1}$ and $G_{\text{eff}} = 0.0037 \Omega^{-1}$, implying $R_{\text{eff}} = 196.08 \Omega$ and $R_{\text{eff}} = 270.27 \Omega$ for case A (square) and B (hexagonal) respectively. The values are slightly off from the theoretical values: 1.96% below for case A and 0.43% below for case B. In case A, the root mean square error and coefficient of determination ($R^2$ score) [40] of the regression line are $5.76 \times 10^{-4}$ and 0.999831 respectively. The same for case B are $1.6 \times 10^{-7}$ and 0.999139 respectively.
Figure 12. (a) The setup of a $2 \times 2$ square network and (b) $2 \times 2$ hexagonal network on a breadboard. The multimeter’s reading of $R_{\text{eff}}$ for (c) the square and (d) hexagonal networks. Each resistor in the networks has resistance $R = 100 \ \Omega$.

The values reflect that the regression lines have reasonably high accuracy. The lines, however, show very small finite intercepts of values 0.00158 and 0.00226 A. These offset values possibly originate from the resistances in the circuit connection (breadboard and wire connections to the power supply) since we have already checked that the networks accurately produce the theoretical result when measured separately with a multimeter. The least counts of the voltage and current of our instrument were 0.01 V ($\Delta V$) and 0.001 A ($\Delta I$) respectively. From them, we estimated the maximum permissible error in resistance ($\Delta R$) relative to the determined resistance from Ohm’s law ($R = V/I$): $\Delta R/R = \pm |\Delta V/V + \Delta I/I|$ and found it to range from $\pm 1.34\%$ to $\pm 6.21\%$ in case A and from $\pm 1.76\%$ to $\pm 8.08\%$ in case B.

Thus as inference, we can say that our experiment validates the theory within very low error bars. The Python codes of our experimental plots and regression analysis can be found at https://github.com/hbaromega/2D-Resistor-Network/tree/master/EXPT.

8. Outlook

The detailed but simple derivations of finite size lattice networks of three distinct geometries and the discussed simple experiment on a breadboard setup offer a very easy and effective way to teach network analysis to students or even adults since the background requirement is minimal (only Kirchhoff’s laws and a programming language). Therefore, this can be added to one of the earlier proposed curricula [41] in this regard. Later, such studies can be
Figure 13. (a) Setup for determining the effective resistance using a DC voltage regulator. (b) Current vs voltage plots for square and hexagonal lattices. The slopes $s$ of the curves estimate the values of $R_{\text{eff}}$ ($R_{\text{eff}} = 1/G_{\text{eff}}$). Here we find $R_{\text{eff}} = 196.08$ $\Omega$ for the square network and $R_{\text{eff}} = 270.27$ $\Omega$ for the hexagonal network.
## Table 2. Case A: Rectangular/square case.

| $V$ (V) | $I$ (A) |
|---------|---------|
| 3.01    | 0.017   |
| 3.48    | 0.019   |
| 4.01    | 0.022   |
| 4.49    | 0.024   |
| 5.01    | 0.027   |
| 5.63    | 0.03    |
| 6.01    | 0.032   |
| 6.51    | 0.035   |
| 7.06    | 0.037   |
| 7.47    | 0.04    |
| 7.95    | 0.042   |
| 8.47    | 0.045   |
| 9.16    | 0.048   |
| 9.49    | 0.05    |
| 10.13   | 0.053   |
| 10.53   | 0.055   |
| 11.06   | 0.058   |
| 11.49   | 0.06    |
| 12.3    | 0.064   |
| 13.02   | 0.068   |
| 13.5    | 0.07    |
| 14.06   | 0.073   |
| 14.71   | 0.076   |
| 15.1    | 0.078   |
Table 3. Case B: Hexagonal (armchair) case.

| V (V) | I (A) |
|-------|-------|
| 2.54  | 0.013 |
| 3.52  | 0.015 |
| 4     | 0.017 |
| 4.55  | 0.019 |
| 5     | 0.021 |
| 5.52  | 0.023 |
| 6.15  | 0.025 |
| 6.52  | 0.026 |
| 7.08  | 0.029 |
| 7.5   | 0.03  |
| 8.06  | 0.032 |
| 8.48  | 0.034 |
| 9.02  | 0.036 |
| 9.5   | 0.038 |
| 10.06 | 0.04  |
| 10.45 | 0.041 |
| 11.05 | 0.043 |
| 11.59 | 0.045 |
| 12.09 | 0.047 |
| 12.46 | 0.049 |
| 13.02 | 0.051 |
| 13.49 | 0.053 |
| 14.03 | 0.055 |
| 14.74 | 0.057 |
| 15.1  | 0.059 |

connected to graph theory since graphs offer visual appeal for one’s learning and conceptual comprehensibility.

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Appendix A. Linearization of V matrix

The KCL equations contain two-dimensional $V_{ij}$ elements. For the rectangular or triangular lattice, when we linearize it to a one-dimensional vector or column matrix, we take one of the following two mappings.

Mapping 1:

$$V_{1,1}, \ldots, V_{L_x,1} \rightarrow V_{1}, \ldots, V_{L_x}.$$  

$$V_{1,2}, \ldots, V_{L_x,2} \rightarrow V_{L_x+1}, \ldots, V_{2L_x}.$$
\[ V_{1,y}, \ldots, V_{L_x,y} \rightarrow V_{(L_y-1)L_x+1, \ldots, V_{L_x L_y}}. \]

Generically, \((i, j) \rightarrow (j - 1)L_x + i.\) \hspace{1cm} (A.1)

Mapping 2:

\[ V_{1,y}, \ldots, V_{1,y} \rightarrow V_{1, \ldots, V_{y}}. \]
\[ V_{2,y}, \ldots, V_{2,y} \rightarrow V_{L_y+1, \ldots, V_{2L_y}}. \]
\[ \vdots \]
\[ V_{L_x,y}, \ldots, V_{L_x,y} \rightarrow V_{(L_y-1)L_x+1, \ldots, V_{L_x L_y}}. \]

Generically, \((i, j) \rightarrow (i - 1)L_y + j.\) \hspace{1cm} (A.2)

Now a typical equation such as equation (16) looks like

\[ \alpha V_1 + \beta V_2 + \ldots + \gamma V_{L_x} = I_1, \]

which can be recast as

\[ G_{11} V_1 + G_{12} V_2 + \ldots + G_{1L_x} V_{L_x} = I_1. \]

Generically this can be written as

\[ G_{ii} V_1 + G_{i2} V_2 + \ldots + G_{iL_x} V_{L_x} = I_i, \]

which builds the matrix form:

\[
\begin{bmatrix}
G_{11} & G_{12} & \ldots & G_{1N_L} \\
G_{21} & G_{22} & \ldots & G_{2N_L} \\
\vdots & \vdots & \ddots & \vdots \\
G_{N_L,1} & G_{N_L,2} & \ldots & G_{N_L,N_L}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_{N_L}
\end{bmatrix}
= 
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_{N_L}
\end{bmatrix},
\]

where \(N_L \equiv L_x L_y.\)

A.1. Finding the corresponding row and column of \(G,\) given the row of \(I\)

Since each lattice site follows a particular KCL depending on its neighborhood and voltage connection, the rank of that lattice (reflected by the row or index \(i\) of current vector \(I\) in equation (A.5)) in the mapped 1D array will denote the row of \(G\) and the index of \(V\) (which is a vector or column matrix) will yield the column of \(G.\)

A.2. Mapping in the hexagonal lattice case

Since we introduce another index \(k\) in the armchair hexagonal lattice, we extend the linear mapping as

\[(i, j, k) \rightarrow i + (j - 1)L_x + kL_x L_y.\]

One can check that the mapping conserves the total number of sites \(N_{\text{site}} = L_x (2L_y + 1).\)
\( k = 0 \) case:

\[
\begin{align*}
V_{1,0} & \rightarrow V_1, V_{Lx,0}, \\
V_{1,2,0} & \rightarrow V_{Lx+1}, V_{2Lx,0}, \\
& \vdots \\
V_{1,Lx,0} & \rightarrow V_{(Lx-1)Lx+1}, V_{LxLx}.
\end{align*}
\]  

(A.8)

\( k = 1 \) case:

\[
\begin{align*}
V_{1,1} & \rightarrow V_{LxLx}, V_{1,Lx+1,1}, \\
V_{1,2,1} & \rightarrow V_{(Lx+1)Lx+1}, V_{2LxLx}, \\
& \vdots \\
V_{1,Lx,1} & \rightarrow V_{(2Lx-1)Lx+1}, V_{2LxLx}, \\
V_{1,Lx+1,1} & \rightarrow V_{2LxLx+1}, V_{(2Lx+1)Lx}.
\end{align*}
\]  

(A.9)

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