STABILITY OF AXIALLY-SYMMETRIC SOLUTIONS TO INCOMPRESSIBLE MAGNETOHYDRODYNAMICS WITH NO AZIMUTHAL VELOCITY AND WITH ONLY AZIMUTHAL MAGNETIC FIELD

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Abstract. Incompressible viscous axially-symmetric magnetohydrodynamics is considered in a bounded axially-symmetric cylinder. Vanishing of the normal components, azimuthal components and also azimuthal components of rotation of the velocity and the magnetic field is assumed on the boundary. First, global existence of regular special solutions, such that the velocity is without the swirl but the magnetic field has only the swirl component, is proved. Next, the existence of global regular axially-symmetric solutions which are initially close to the special solutions and remain close to them for all time is proved. The result is shown under sufficiently small differences of the external forces. All considerations are performed step by step in time and are made by the energy method. In view of complicated calculations estimates are only derived so existence should follow from the Faedo-Galerkin method.

1. Introduction. In this paper we consider axially-symmetric solutions to the magnetohydrodynamics (mhd) equations

\[ \begin{align*}
  v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= -\nabla \frac{H^2}{2} + H \cdot \nabla H + f, \\
  \text{div } v &= 0, \\
  H_t + v \cdot \nabla H - H \cdot \nabla v - \mu \Delta H &= 0, \\
  \text{div } H &= 0,
\end{align*} \]

where \( v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3 \) is the velocity of the fluid, \( p = p(x,t) \in \mathbb{R} \) is the pressure, \( H = (H_1(x,t), H_2(x,t), H_3(x,t)) \in \mathbb{R}^3 \) is the magnetic field, \( f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3 \) is the external force field, \( \nu > 0 \) is the constant viscosity coefficient, \( \mu > 0 \) is the resistivity, \( x = (x_1, x_2, x_3) \) are the Cartesian coordinates.

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Equations (1.1)–(1.4) are considered in a cylindrical domain \( \Omega \subset \mathbb{R}^3 \) with the axis of symmetry equal to the \( x_3 \)-axis. Let \( S \) be the boundary of \( \Omega \).

The boundary \( S \) is split into two parts, \( S = S_1 \cup S_2 \), where \( S_1 \) is parallel to the \( x_3 \)-axis and \( S_2 \) is perpendicular to it. We have that \( S_2 = S_2(-a) \cup S_2(a) \), where \( a > 0 \) is given and \( S_2(b) \) meets the \( x_3 \)-axis at \( x_3 = b \), \( b \in \{-a, a\} \).

Since problem (1.1)–(1.4) is considered in a bounded domain \( \Omega \) we assume the following boundary conditions

\[
v \cdot \vec{n} = 0 \quad \text{on} \quad S,
\]

\[
\text{azimuthal components of} \ v \ \text{and} \ \text{rot} \ v \ \text{vanish on} \ S,
\]

\[
H \cdot \vec{n} = 0 \quad \text{on} \quad S,
\]

\[
\text{azimuthal components of} \ H \ \text{and} \ \text{rot} \ H \ \text{vanish on} \ S,
\]

where \( \vec{n} \) is the unit outward vector normal to \( S \). The boundary conditions (1.5), (1.6) and (1.7), (1.8) are the Navier boundary conditions for velocity and the magnetic field, respectively. Finally, we add the initial conditions

\[
v\big|_{t=0} = v(0), \quad H\big|_{t=0} = H(0).
\]

There is a huge literature concerning the problem of global existence of regular solutions to incompressible viscous mhd. The first result on local existence, uniqueness and global existence for small data was established by Duvaut and Lions [4]. Moreover, we recall the paper of Sermange and Temam [12], where global existence of regular two-dimensional solutions is proved.

Finally, we recall results on regularity of solutions to incompressible mhd equations. The two-dimensional case is treated in [2, 7, 19] and the three-dimensional case in [3, 5, 6, 10].

Recently there appear papers concerning non-resistivity incompressible mhd. In [13] global small solutions to three-dimensional mhd system is proved. In [11] global small solutions of 2-D incompressible mhd is shown. Moreover, Z. Lei in [9] proved existence of long time solutions to incompressible axially-symmetric mhd system with zero diffusivity in the equation for the magnetic field. The received result is weak because the estimate increases very strongly to infinity as time passes to infinity. Probably, this feature is relevant to the Cauchy problem only.

In this paper we are interested to prove existence of incompressible axially-symmetric mhd with the following properties:

1. We are going to show existence of global regular solutions which are estimated by a bound independent of time. This gives a possibility of showing stability, periodicity and finally existence of stationary solutions.
2. We want to show existence of global non-small solutions. This can be realized by showing existence of global regular large special solutions and subsequently proving existence of solutions which remain close to these special solutions for all time.
3. Finally, we want to consider mhd system with an external force that is non-vanishing and nondecreasing in time.

Looking at the results of Lei [9] we see that to obtain the above aims we have to consider viscous and resistivity mhd. Moreover, we think that the mhd motions should be considered in a bounded axially-symmetric domain with appropriate boundary conditions. We need such boundary conditions so that the interior motion is separated from any exterior influence. The proposed boundary conditions are a little more restricted than the slip boundary conditions (see [18]). Hence much
stronger restrictions are imposed on the angular component of velocity because the slip boundary conditions imply only that

\[ v_{\varphi,r} \left( S_1 \right) = \frac{1}{R} v_\varphi, \quad v_{\varphi,z} \left( S_2 \right) = 0 \]

(see [18, Ch. 4, Sect. 2, Lemma 2.1]) but we assume that \( v_\varphi \mid_S = 0 \).

For the non-swirl Navier-Stokes system \((v_\varphi = 0)\) the considered boundary conditions are the same as in celebrated paper of O.A. Ladyzhenskaya [8]. The imposed boundary conditions for the magnetic field have a deep physical meaning. Vanishing of the normal component of the magnetic field \(H\) on the boundary and assumption \(B = \mu H\) means that the flux of the magnetic induction \(B\) through the boundary equals zero. Hence the lines of the magnetic induction are parallel to the boundary so the considered motion is separated from the exterior region.

In view of \((1.13)_2\), vanishing of the angular component of the magnetic field \(H\) implies that current \(J\) is parallel to \(S\). Hence there is no flux of charges through the boundary. Finally, we assume vanishing angular component of curl of \(H\) so that the angular component of current vanishes on \(S\) (see \((1.13)_2\)).

Since the Navier-Stokes and magnetohydrodynamics equations have special regular global solutions it is natural to examine their stability. Thanks to this we are also able to prove existence of global regular solutions which are close in some spaces to the special solutions for all time. To prove stability we transform the considered equations to linear equations for perturbations of considered quantities with regular coefficients depending on the special solutions. Then it is possible to derive a differential inequality guaranteeing non-increasing in time of sufficiently small initial perturbations (see \((4.2)\)). A vast literature concerning stability of special solutions to the Navier-Stokes equations is presented in [14, 17]. We have to emphasize that derivation of the differential inequality \((4.2)\) for magnetohydrodynamics equations is far from trivial. The technique presented in this paper was already developed in [16, 17, 14, 15].

This paper is organized in the following way. In Introduction axially-symmetric magnetohydrodynamics equations are formulated (see \((1.26)-(1.32)\)). The question of existence of solutions is presented in Remark 4. Separately equations for special solutions such that velocity is without swirl and magnetic field has only swirl component are presented (see \((1.34)-(1.38)\)).

Moreover, the equations for perturbations are described by \((1.39)-(1.48)\). Finally, the main results are presented at the end of Introduction. In Section 3 we prove existence of global regular special solutions (see Lemmas 3.1–3.10) and Theorem 3.11. The differential inequality necessary for showing stability is derived in Proposition 1. Then, Proposition 2 proves stability.

The aim of this paper is to prove stability of axially-symmetric solutions with non-swirl velocity and swirl magnetic field (we denote them as special solutions) in a set of general axially-symmetric solutions. Moreover, we have to prove global existence of regular non-swirl velocity and swirl magnetic field of axially-symmetric solutions bounded by constants independent of time. To examine axially-symmetric solutions we introduce the cylindrical coordinates \(r, \varphi, z\) by the relations

\[ x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z. \]  

(1.10)

Next, we use the ortho-normal basis

\[ \bar{e}_r = (\cos \varphi, \sin \varphi, 0), \quad \bar{e}_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad \bar{e}_z = (0, 0, 1). \]  

(1.11)
Then the cylindrical coordinates of $v$, $\omega = \text{rot} \, v$, $H$, $j = \text{rot} \, H$, $f$ are defined by
\[
v(x, t) = v_r(r, z, t)e_r + v_\varphi(r, z, t)e_\varphi + v_z(r, z, t)e_z, \\
H(x, t) = H_r(r, z, t)e_r + H_\varphi(r, z, t)e_\varphi + H_z(r, z, t)e_z, \\
p(x, t) = p(r, z, t), \\
f(x, t) = f_r(r, z, t)e_r + f_\varphi(r, z, t)e_\varphi + f_z(r, z, t)e_z
\]
and curl and current density have the forms
\[
\omega(x, t) = \text{rot} \, v(x, t) = \omega_r(r, z, t)e_r + \omega_\varphi(r, z, t)e_\varphi + \omega_z(r, z, t)e_z \\
= -v_{r, z}e_r + (v_{r, r} - v_{z, r})e_\varphi + \frac{1}{r}(rv_\varphi)_{, r} e_z, \\
j(x, t) = \text{rot} \, H = j_r(r, z, t)e_r + j_\varphi(r, z, t)e_\varphi + j_z(r, z, t)e_z \\
= -H_{r, z}e_r + (H_{r, r} - H_{z, r})e_\varphi + \frac{1}{r}(rH_\varphi)_{, r} e_z.
\]
Let us recall that the swirls of velocity and the magnetic field are denoted by
\[
u_0 = rv_\varphi, \quad H_0 = rH_\varphi.
\]
Finally, using the cylindrical coordinates, we describe domain $\Omega$. Let $R > 0$ and $a > 0$ be given. Then
\[
\Omega = \{ x \in \mathbb{R}^3 : r < R, |z| < a, \varphi \in [0, 2\pi] \}, \\
S_1 = \{ x \in \mathbb{R}^3 : r = R, |z| < a, \varphi \in [0, 2\pi] \}, \\
S_2 = \{ x \in \mathbb{R}^3 : r < R, z \in (-a, a), \varphi \in [0, 2\pi] \}.
\]
Then the equations describing axially-symmetric incompressible viscous magneto-hydrodynamics have the form
\[
v_r, t + v^' \cdot \nabla' v_r - \nu \left( \nabla^2 - \frac{1}{r^2} \right) v_r - \frac{1}{r} v_{r}^2 + p_r, \\
= H' \cdot \nabla' H_r - \frac{1}{r} H_{\varphi}^2 + f_r, \\
v_\varphi, t + v^' \cdot \nabla' v_\varphi - \nu \left( \nabla^2 - \frac{1}{r^2} \right) v_\varphi + \frac{1}{r} v_\varphi v_r, \\
= H' \cdot \nabla' H_\varphi + \frac{1}{r} H_\varphi H_r + f_\varphi, \\
v_z, t + v^' \cdot \nabla' v_z - \nabla^2 v_z + p_z = H' \cdot \nabla' H_z + f_z,
\]
where $v^' = (v_r, v_\varphi)$, $H' = (H_r, H_\varphi)$, $v^' \cdot \nabla' = v_r \partial_r + v_\varphi \partial_\varphi$, $H' \cdot \nabla' = H_r \partial_r + H_\varphi \partial_\varphi$, $\nabla^2 = \partial_r^2 + \partial_\varphi^2 + \frac{1}{r} \partial_r$, and
\[
H_{r, t} + v^' \cdot \nabla' H_r - \mu \left( \nabla^2 - \frac{1}{r^2} \right) H_r = H' \cdot \nabla' v_r, \\
H_{\varphi, t} + v^' \cdot \nabla' H_\varphi - \mu \left( \nabla^2 - \frac{1}{r^2} \right) H_\varphi + \frac{1}{r} v_\varphi H_r, \\
= H' \cdot \nabla' v_\varphi + \frac{1}{r} v_r H_\varphi, \\
H_{z, t} + v^' \cdot \nabla' H_z - \mu \nabla^2 H_z = H' \cdot \nabla' v_z, \\
(rv_r)_{, r} + (rv_z)_{, z} = 0, \\
(rH_r)_{, r} + (rH_z)_{, z} = 0.
\]
Equations (1.17) imply existence of the stream function \( \psi \) and the magnetic stream function \( \phi \). Then it follows

\[
v_r = -\psi_{,z}, \quad v_z = \frac{1}{r}(r\psi)_{,r}, \quad H_r = -\phi_{,z}, \quad H_z = \frac{1}{r}(r\phi)_{,r}.
\]

(1.18)

Therefore, the axially-symmetric magnetohydrodynamics motions are described by the following system of equations

\[
v_{\varphi,t} + v' \cdot \nabla' v_\varphi - \nu \left( \nabla^2 - \frac{1}{r^2} \right) v_\varphi + \frac{1}{r} v_\varphi v_r
= H' \cdot \nabla' H_\varphi + \frac{1}{r} H_\varphi H_r + f_\varphi, \quad v_\varphi|_S = 0,
\]

(1.19)

\[
\omega_{\varphi,t} + v' \cdot \nabla' \omega_\varphi - \nu \left( \nabla^2 - \frac{1}{r^2} \right) \omega_\varphi - \frac{1}{r} v_r \omega_\varphi - \frac{1}{r} (v_\varphi^2)_{,z}
= H' \cdot \nabla' j_\varphi - \frac{1}{r} H_r j_\varphi - \frac{1}{r} (H_r^2)_{,z} + F_\varphi, \quad \omega_\varphi|_S = 0,
\]

(1.20)

\[
\nabla^2 \psi = \omega_\varphi, \quad \psi|_S = 0,
\]

(1.21)

\[
H_\varphi|_S = 0,
\]

\[
j_{\varphi,t} + v' \cdot \nabla' j_\varphi - \mu \left( \nabla^2 - \frac{1}{r^2} \right) j_\varphi - H' \cdot \nabla' \omega_\varphi
= (v_{r,r} - v_{z,z})(H_{r,z} + H_{z,r}) + (H_{z,z} - H_{r,r})(v_{r,z} + v_{z,r}), \quad j_\varphi|_S = 0,
\]

(1.23)

\[
\nabla^2 \phi = j_\varphi, \quad \phi|_S = 0,
\]

(1.24)

where \( F_\varphi = f_{r,z} - f_{z,r} \).

To prove stability of the special solutions we have to formulate equations for general and for special axially-symmetric solutions, separately. The first will be denoted by the upper index 1 and the second by 2. Since we are going to work in a class of smooth solutions we recall the following compatibility conditions at \( r = 0 \) for axially-symmetric mhd equations

\[
v_\varphi(0, z, t) = \omega_\varphi(0, z, t) = \psi(0, z, t) = 0,
\]

(1.25)

\[
H_\varphi(0, z, t) = j_\varphi(0, z, t) = \phi(0, z, t) = 0.
\]

It is natural for axially-symmetric motions to replace equations (1.19)–(1.24) for \( v_\varphi, H_\varphi, \omega_\varphi, j_\varphi, \psi, \phi \) by equations for \( v_\varphi/r, H_\varphi/r, \omega_\varphi/r, j_\varphi/r, \psi/r, \phi/r \).

Therefore we introduce the quantities with the lower index 1 as follows

\[
v_\varphi(r, z, t) = rv_1(r, z, t), \quad \omega_\varphi(r, z, t) = r\omega_1(r, z, t),
\]

\[
\psi(r, z, t) = r\psi_1(r, z, t), \quad H_\varphi(r, z, t) = rH_1(r, z, t),
\]

(1.26)

\[
j_\varphi(r, z, t) = rj_1(r, z, t), \quad \phi(r, z, t) = r\phi_1(r, z, t).
\]

Then the equations for general axially-symmetric mhd equations have the form

\[
\frac{1}{v_1}_{,t} + \frac{v'}{v_1} \cdot \nabla' v_1 - \frac{1}{r} \psi_{1,z} v_1 - \nu \left( \frac{1}{v_1}_{,rr} + \frac{1}{v_1}_{,zz} + \frac{3}{r} \frac{1}{v_1}_{,r} \right)
= H' \cdot \nabla' H_1 - \frac{1}{r} \phi_{1,z} H_1 + f_1,
\]

(1.27)
\[ \frac{1}{\omega_{1,t}} + \frac{1}{v'} \cdot \nabla \omega_1 - \nu \left( \frac{1}{\omega_{1,rr}} + \frac{3}{r^2} \omega_{1,r} \right) - \frac{1}{(v^2_1)_z} = \frac{1}{H'} \cdot \nabla j_1 - \frac{1}{(H^2_1)_z} + F_1, \]  
\[ \frac{1}{H_{1,t}} + \frac{1}{v'} \cdot \nabla H_1 - \mu \left( \frac{1}{H_{1,rr}} + \frac{3}{r^2} H_{1,r} \right) = \frac{1}{H'} \cdot \nabla \psi_1, \]  
\[ \frac{1}{j_{1,t}} + \frac{1}{v'} \cdot \nabla j_1 - \psi_{1,z} j_1 - \mu \left( \frac{1}{j_{1,rr}} + \frac{3}{r^2} j_{1,r} \right) = \frac{1}{H'} \cdot \nabla \omega_1, \]  
\[ \frac{1}{\phi_{1,t}} + \frac{1}{v'} \cdot \nabla \phi_1 - \psi_{1,z} \phi_1 - \mu \left( \frac{1}{\phi_{1,rr}} + \frac{3}{r^2} \phi_{1,r} \right) = \frac{1}{H'} \cdot \nabla \psi_1, \]  
\[ \frac{1}{\omega_{1,t}} + \frac{1}{v'} \cdot \nabla \omega_1 - \nu \left( \frac{1}{\omega_{1,rr}} + \frac{3}{r^2} \omega_{1,r} \right) - \frac{1}{(v^2_1)_z} = \frac{1}{H'} \cdot \nabla j_1 - \frac{1}{(H^2_1)_z} + F_1, \]  
\[ \frac{1}{H_{1,t}} + \frac{1}{v'} \cdot \nabla H_1 - \mu \left( \frac{1}{H_{1,rr}} + \frac{3}{r^2} H_{1,r} \right) = \frac{1}{H'} \cdot \nabla \psi_1, \]  
\[ \frac{1}{j_{1,t}} + \frac{1}{v'} \cdot \nabla j_1 - \psi_{1,z} j_1 - \mu \left( \frac{1}{j_{1,rr}} + \frac{3}{r^2} j_{1,r} \right) = \frac{1}{H'} \cdot \nabla \omega_1, \]  
\[ \frac{1}{\phi_{1,t}} + \frac{1}{v'} \cdot \nabla \phi_1 - \psi_{1,z} \phi_1 - \mu \left( \frac{1}{\phi_{1,rr}} + \frac{3}{r^2} \phi_{1,r} \right) = \frac{1}{H'} \cdot \nabla \psi_1, \]  
where \( \frac{1}{v'} \) and \( \frac{1}{H'} \) are reexpressed as

\[ \frac{1}{v'} = -\frac{1}{r^2 \psi_1}, \quad \frac{1}{H'} = -\frac{1}{r^2 \phi_1}. \]

We assume the following special solutions to axially-symmetric mhd equations

\[ \frac{2}{v^2} = 0, \quad \frac{2}{H^2} = 0, \quad \frac{2}{\phi} = 0, \quad \frac{2}{v_1} = 0, \]  
\[ \frac{2}{\omega_1} = r \omega_1, \quad \frac{2}{H_1} = r H_1, \quad j_1 = 0. \]

Then the special solutions satisfy the system of equations

\[ \frac{2}{\omega_{1,t}} + \frac{2}{v'} \cdot \nabla \omega_1 - \nu \left( \Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) = -2 \frac{2}{H_{1,t} z} + \frac{2}{F_1}, \quad \frac{2}{\omega_1} = 0, \]  
\[ \frac{2}{H_{1,t}} + \frac{2}{v'} \cdot \nabla H_1 - \mu \left( \Delta H_1 + \frac{2}{r} H_{1,r} \right) = 0, \quad \frac{2}{H_1} = 0, \]  
\[ \frac{2}{\psi_{1,t}} + \frac{2}{v'} \cdot \nabla \psi_1 - \frac{2}{\psi_{1,r}} \psi_{1,rr} + \frac{3}{r} \psi_{1,r} = \frac{2}{\omega_1}, \quad \frac{2}{\psi_1} = 0, \]  
\[ \frac{2}{\omega_{1,t}} + \frac{2}{v'} \cdot \nabla \omega_1 - \nu \left( \Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) = -2 \frac{2}{H_{1,t} z} + \frac{2}{F_1}, \quad \frac{2}{\omega_1} = 0, \]  
\[ \frac{2}{H_{1,t}} + \frac{2}{v'} \cdot \nabla H_1 - \mu \left( \Delta H_1 + \frac{2}{r} H_{1,r} \right) = 0, \quad \frac{2}{H_1} = 0, \]  
\[ \frac{2}{\psi_{1,t}} + \frac{2}{v'} \cdot \nabla \psi_1 - \frac{2}{\psi_{1,r}} \psi_{1,rr} + \frac{3}{r} \psi_{1,r} = \frac{2}{\omega_1}, \quad \frac{2}{\psi_1} = 0, \]

To prove stability of the special solutions we introduce the notation

\[ v_1 = v_1 = v_1 = v_1, \quad \omega_1 = \omega_1 = \omega_1, \quad H_1 = H_1 = H_1, \]  
\[ v' = v' - v', \quad j_1 = j_1 - j_1 = j_1, \quad H_1 = H_1 - H_1, \]  
\[ \psi_1 = \psi_1 - \psi_1, \quad \phi_1 = \phi_1 - \phi_1 = \phi_1, \quad F_1 = F_1 - F_1, \]  
\[ f_1 = f_1 - f_1 = f_1. \]
The above quantities describe a distance between general and special axially-symmetric solutions. Finally, we formulate the equations for the differences

\[ \nu_{1,t} + \nu' \cdot \nabla' v_1 + \frac{2}{r} \nabla' v_1 - 2(\psi_{1,z} v_1 + \psi_{1,z} v_1) - \nu \left( \Delta v_1 + \frac{2}{r} v_{1,r} \right) \]  

(1.40)

\[ = H' \cdot \nabla' H_1 + H' \cdot \nabla' H_1 - 2 \phi_{1,z} H_1 - 2 \phi_{1,z} H_1 + f_1, \]

\[ \omega_{1,t} + \nu' \cdot \nabla \omega_1 + \nu' \cdot \nabla \omega_1 + \frac{2}{r} \nabla \omega_1 - \nu \left( \Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) \]  

(1.41)

\[ = (v_2^2)_z + H' \cdot \nabla j_1 - 2 H_1 H_{1,z} - 2 H_1 H_{1,z} - 2 H_1 H_{1,z} + F_1, \]

\[ H_{1,t} + v' \cdot \nabla' H_1 + v' \cdot \nabla' H_1 + \mu (\Delta H_1 + \frac{2}{r} H_{1,r}) = H' \cdot \nabla' v_1, \]  

(1.42)

\[ j_{1,t} + v' \cdot \nabla' j_1 + v' \cdot \nabla' j_1 - (\psi_{1,z} + \psi_{1,z}) j_1 - \mu (\Delta j_1 + \frac{2}{r} j_{1,r}) \]

\[ = H' \cdot \nabla \omega_1 + H' \cdot \nabla \omega_1 - \phi_{1,z} j_1 \]

(1.43)

\[ + \frac{1}{r} (v_{r,r} - v_{z,r} + \frac{2}{v_{r,r}} - v_{z,r})(H_{r,r} + H_{z,r}) \]

\[ + \frac{1}{r} (H_{r,r} - H_{r,r})(v_{r,r} + v_{z,r} + \frac{2}{v_{r,r}} + v_{z,r}). \]

Moreover, we assume the following boundary conditions for solutions to (1.40)–(1.43),

\[ v_1|_S = 0, \quad H_1|_S = 0, \quad \omega_1|_S = 0, \quad j_1|_S = 0 \]  

(1.44)

and also the initial conditions

\[ v_1|_{t=0} = v_1(0), \quad H_1|_{t=0} = H_1(0), \quad \omega_1|_{t=0} = \omega_1(0), \quad j_1|_{t=0} = j_1(0). \]  

(1.45)

Finally, coordinates of \( v' \) and \( H' \) are calculated from the relations

\[ v_r = -(r \psi_1)_z, \quad v_z = \frac{1}{r} (r^2 \psi_1)_r, \quad H_r = -(r \phi_1)_z, \quad H_z = \frac{1}{r} (r^2 \phi_1)_r, \]

(1.46)

and \( \psi_1, \phi_1 \) are solutions to the elliptic problems

\[ - \left( \Delta \psi_1 + \frac{2}{r} \psi_{1,r} \right) = \omega_1, \quad \psi_1|_S = 0 \]  

(1.47)

and

\[ - \left( \Delta \phi_1 + \frac{2}{r} \phi_{1,r} \right) = j_1, \quad \phi_1|_S = 0. \]  

(1.48)

Now, we formulate the main results of this paper. From Section 3 we have

**Theorem 1.1.** Let \( T > 0 \) be given. Suppose that \( \tilde{\omega}_1(0) \in L_2(\Omega), \tilde{H}_1(0) \in H^4(\Omega), \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \| \tilde{F}(t) \|_{L_2(\Omega)}^2 dt < \infty. \) Then the following estimates for solutions to problem (1.34)–(1.38) hold:

\[ \| \tilde{H}_1(t) \|_{L_2(\Omega)}^2 + \mu \int_{kT}^{t} \| \tilde{H}_1(t') \|_{H^1(\Omega)}^2 dt' \leq c \| \tilde{H}_1(0) \|_{L_2(\Omega)}^2 \equiv c \bar{A}_1^2, \]  

(1.49)

\[ \| \tilde{\omega}_1(t) \|_{L_2(\Omega)}^2 + \nu \int_{kT}^{t} \| \tilde{\omega}_1(t') \|_{H^1(\Omega)}^2 dt' \]

\[ \leq c \left( \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \| \tilde{F}(t) \|_{L_2(\Omega)}^2 dt + \| \tilde{\omega}_1(0) \|_{L_2(\Omega)}^2 \right) \equiv c \bar{A}_2^2. \]  

(1.50)
\[
\frac{2}{t} \left\| H_1(t) \right\|_{H^1(\Omega)}^2 + \mu \int_{kT}^{t} \frac{2}{t} \left\| H_1(t') \right\|_{H^2(\Omega)}^2 dt' \leq c(A_2^2 + \tilde{A}_2^2)A_1^2 + c\left\| H_1(0) \right\|_{H^1(\Omega)}^2, \quad (1.51)
\]
for \( t \in [kT, (k + 1)T] \) and any \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

**Remark 1.** In this paper estimates (1.49)–(1.50) and also existence of solutions to problem (1.34)–(1.38) are proved (see Theorem 3.11). The existence of solutions to problem (1.34)–(1.38), in the spaces determined by the l.h.s. norms of (1.49)–(1.50), is proved step by step in time by some combination of the Faedo-Galerkin and successive approximations methods (see Theorem 3.11). Some comments on existence of solutions to problem (1.27)–(1.32) can be found in Remark 4.

The aim of this paper is to prove stability of special solutions described by problem (1.34)–(1.38) in a set of axially-symmetric solutions. Therefore, to prove stability we are looking for solutions to problem (1.39)–(1.48) which describe disturbances to the special solutions. The stability means existence of small solutions to problem (1.39)–(1.48) for all time with data sufficiently small. We proved stability in Section 4, where appropriate estimates are derived. Hence, Propositions 1 and 2 imply

**Theorem 1.2.** Let \((v, H)\) be a solution to (1.39)–(1.48). Let \( k \in \mathbb{N}_0 \). Let \( T > 0 \) be given. Let

\[
X^2(t) = \left| v_1(t) \right|_{L^4(\Omega)}^4 + \left\| H_1(t) \right\|_{L^4(\Omega)}^4 + \left\| \omega_1(t) \right\|_{L^4(\Omega)}^4 + \left\| j_1(t) \right\|_{L^4(\Omega)}^4
\]
\[
+ \left\| H_1(t) \right\|_{L^2(\Omega)}^2,
\]
\[
A^2(t) = \frac{2}{t} \left\| \omega_1(t) \right\|_{H^1(\Omega)}^2 + \frac{2}{t} \left\| \omega_1(t) \right\|_{L^2(\Omega)}^2 + \frac{2}{t} \left\| H_1(t) \right\|_{H^2(\Omega)}^2,
\]
\[
G^2(t) = \left\| f_1(t) \right\|_{L^{4/3}(\Omega)}^4 + \left\| F_1(t) \right\|_{L^{4/3}(\Omega)}^4.
\]
Let \( A_0^2 \) be a finite number such that

\[
\sup_{k \in \mathbb{N}_0} \int_{kT}^{(k + 1)T} A^2(t) dt \leq A_0^2.
\]

Let \( \gamma_s > 0 \) be so small that

\[
\nu_s - c_0(\gamma_s^2 + \nu_s^4 + \gamma_s^8) \geq c_s,
\]
where \( c_s \in (0, \nu_s), \nu_s = \min\{\nu, \mu\}, \) and \( c_0 \) is some constant. Let \( \gamma \in (0, \gamma_s] \), let \( T \) be so large that

\[
\frac{c_s}{2} T \geq A_0^2.
\]

Let \( \alpha \) be so small and \( T \) so large that

\[
\alpha \exp(A_0^2) + \exp \left( - \frac{c_s}{2} T \right) \leq 1.
\]

Assume that

\[
c_0 \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k + 1)T} G^2(t) dt \leq \alpha \gamma,
\]
\[
X^2(0) \leq \gamma, \quad c_0 G^2(t) \leq \frac{c_s}{2} \gamma, \quad t \in \mathbb{R}_+.
\]

Then

\[
X^2(t) \leq \gamma \quad \text{for all} \quad t \in \mathbb{R}_+.
\]

From Theorems 1.1 and 1.2 we have
Theorem 1.3. Let the assumptions of Theorems 1.1 and 1.2 hold. Then there exists a solutions to problem (1.18)–(1.24) such that
\[ v = \ddot{v} + \dot{v}, \quad H = \dot{H} + \ddot{H}, \]
where \((\ddot{v}, \dddot{H})\) is described by Theorem 1.1 and \((\ddot{v}, \dot{H})\) is a solution to problem (1.39)–(1.48) from Theorem 1.2.

Remark 2. Since the calculations in this paper are very complicated we restrict our considerations to derive estimates only. All estimates are proved by the energy method. This means that existence should follow from the Faedo-Galerkin method.

2. Notation and auxiliary results. Let \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). By \(L_p(\Omega), p \in [1, \infty], \Omega \subset \mathbb{R}^n\) we denote the Lebesgue space of integrable functions and by \(H^s(\Omega), s \in \mathbb{N}_0, \Omega \subset \mathbb{R}^n\), the Sobolev space of functions with the finite norm
\[ ||u||_{H^s(\Omega)} = \left( \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha_x u(x)|^2 dx \right)^{1/2}, \]
where \(D^\alpha_x = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_i \in \mathbb{N}_0, i = 1, \ldots, n\). Let \(u = (u_1, \ldots, u_n)\) be a vector. Then \(|u| = \sqrt{|u_1|^2 + \cdots + |u_n|^2}\).

By \(L_p^2(\Omega \times (kT, (k+1)T)), k, l \in \mathbb{N}_0\), we denote a space of functions with the following finite norm
\[ ||u||_{L_p^2(\Omega \times (kT, (k+1)T))} = ||u||_{L_\infty(kT, (k+1)T; H^1(\Omega))} + ||u||_{L_2(kT, (k+1)T; H^{l+1}(\Omega))}. \]

We use the interpolations
\[ ||u||_{L_p(\Omega)} \leq ||u||_{L_\infty(\Omega)}^{\theta} ||u||_{L_2(\Omega)}^{1-\theta}, \quad \theta = 3 \left( \frac{1}{2} - \frac{1}{p} \right), \quad 2 \leq p \leq 6 \tag{2.1} \]
and (see [1, Ch. 3, Sect. 15])
\[ ||u||_{L_p(\Omega)} \leq c ||u||_{H^k(\Omega)} ||u||_{L_2(\Omega)}^{1-\theta}, \quad \theta = \frac{3}{k} \left( \frac{1}{2} - \frac{1}{p} \right), \tag{2.2} \]

Lemma 2.1 (the Poincaré inequality). Let \(u \in H^1(\Omega), |u|_S = 0\). Then there exists a constant \(c_p\) such that
\[ c_p ||u||_{L_2(\Omega)} \leq ||\nabla u||_{L_2(\Omega)}^2. \tag{2.3} \]

Under similar proof to the proof of Lemma 3.3 we have

Lemma 2.2. Assume that \(j_1 \in L_2(\Omega)\). Then there exists a solution to problem (1.48) such that \(\phi_1 \in H^2(\Omega)\) and
\[ ||\phi_1||_{H^2(\Omega)} \leq c ||j_1||_{L_2(\Omega)}. \tag{2.4} \]

Let us consider the elliptic problem (1.24). Assume that \(j_\varphi \in H^3(\Omega)\). Then there exists a solution to problem (1.24) such that \(\phi \in H^3(\Omega)\) and
\[ ||\phi||_{H^3(\Omega)} \leq c ||j_\varphi||_{H^1(\Omega)}. \tag{2.5} \]

Since \(\Omega\) is bounded, we have

Remark 3. Let \(H' = (H_r, H_z)\) be defined by (1.46)_{3,4}. Then (2.4) yields
\[ ||H'||_{H^1(\Omega)} \leq c ||j_1||_{L_2(\Omega)}. \tag{2.6} \]

Remark 4. Using the methods presented in the proof of Theorem 3.11 we can also prove existence of solutions to problem (1.26)–(1.32).
3. Special solutions. In this Section we derive estimates for special solutions and prove global in time existence of regular special solutions. For this purpose we consider system (1.35)–(1.38) with appropriate boundary and initial conditions. To simplify presentation we skip the upper index 2. Therefore, the special solutions satisfy

\[
\begin{align*}
\omega_{1,t} + v' \cdot \nabla' \omega_1 - \nu \left( \Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) &= -2H_1 H_{1,z} + F_1, \\
H_{1,t} + v' \cdot \nabla' H_1 - \mu \left( \Delta H_1 + \frac{2}{r} H_{1,r} \right) &= 0, \\
- \left( \psi_{1,rr} + \psi_{1,zz} + \frac{3}{r} \psi_{1,r} \right) &= \omega_1, \\
v_r &= -(r \psi_1)_z, \quad v_z = \frac{1}{r} (r^2 \psi_1)_r,
\end{align*}
\]

with the boundary conditions

\[
\omega_1|_{S} = 0, \quad H_1|_{S} = 0, \quad \psi_1|_{S} = 0, \quad v \cdot \vec{n}|_{S} = 0
\]

and the initial conditions

\[
v|_{t=0} = v(0), \quad \omega_1|_{t=0} = \omega_1(0), \quad H_1|_{t=0} = H_1(0), \quad \psi_1|_{t=0} = \psi_1(0).
\]

**Lemma 3.1.** Let \( T > 0 \) be given. Assume that

\[
H_1(0) \in L_{\sigma}(\Omega), \quad \sigma \in [2, \infty).
\]

Then solutions to (3.2), (3.5.2.4) satisfy

\[
\begin{align*}
\|H_1(t)\|_{L_{\sigma}(\Omega)}^2 &+ 2\mu \int_{kT}^{T} dt' \int_{\Omega} |\nabla H_1^\sigma|^2 dt' dx \\
&\leq \exp(-2\mu c_p kT) \|H_1(0)\|_{L_{\sigma}(\Omega)}^2, \quad t \in [kT, (k+1)T],
\end{align*}
\]

where \( k \in \mathbb{N}_0, c_p \) is the following constant from the Poincaré inequality (2.3), \( 2\mu \leq \frac{2\mu(c_p-1)}{\sigma} \) for \( \sigma \in [2, \infty) \).

**Proof.** Multiplying (3.2) by \( H_1|H_1|^{\sigma-2} \), integrating over \( \Omega \), using the boundary conditions (3.5.2.4), we obtain

\[
\frac{1}{\sigma} \frac{d}{dt} \|H_1\|_{L_{\sigma}(\Omega)}^\sigma + \frac{4\mu(\sigma-1)}{\sigma^2} \int_{\Omega} |\nabla H_1^\sigma|^2 dx = 0.
\]

Employing that \( 2\mu \leq \frac{4\mu(c_p-1)}{\sigma} \leq 4\mu \) for \( \sigma \in [2, \infty] \), we derive from (3.9) the inequality

\[
\frac{d}{dt} \|H_1\|_{L_{\sigma}(\Omega)}^\sigma + 2\mu \int_{\Omega} |\nabla H_1^\sigma|^2 dx \leq 0.
\]

The Poincaré inequality yields

\[
\frac{d}{dt} \|H_1\|_{L_{\sigma}(\Omega)}^\sigma + 2\mu c_p \|H_1\|_{L_{\sigma}(\Omega)}^\sigma \leq 0.
\]

Hence, inequality (3.11) implies

\[
\frac{d}{dt} \|H_1(t)\|_{L_{\sigma}(\Omega)}^\sigma \exp(2\mu c_p t) \leq 0.
\]

Integrating (3.12) with respect to time from \( kT \) to \( t \in (kT, (k+1)T) \), gives

\[
\|H_1(t)\|_{L_{\sigma}(\Omega)}^\sigma \leq \exp(-2\mu c_p (t-kT)) \|H_1(kT)\|_{L_{\sigma}(\Omega)}^\sigma.
\]
Setting $t = (k + 1)T$ yields
\[
\|H_1((k + 1)T)\|_{L_\sigma(\Omega)}^2 \leq \exp(-2\mu c_p T)\|H_1(kT)\|_{L_\sigma(\Omega)}^2. \tag{3.14}
\]
From iteration we have
\[
\|H_1(kT)\|_{L_\sigma(\Omega)}^2 \leq \exp(-2\mu c_p kT)\|H_1(0)\|_{L_\sigma(\Omega)}^2. \tag{3.15}
\]
Integrating (3.10) with respect to time from $kT$ to $t \in (kT, (k + 1)T)$ and using (3.15) imply (3.8). This concludes the proof. \[\square\]

**Lemma 3.2.** Let $T > 0$ be given. Assume that $A_3^2 = \frac{A_3^2}{1-\exp(-\frac{4c_p^* T}{T})} + \|\omega_1(0)\|_{L_2(\Omega)}^2$, where
\[
A_1^2 = \sup_k \frac{4c_0}{c_p^*} \int_{kT}^{(k+1)T} \|F_1(t)\|_{L_{4/5}(\Omega)}^2 dt, \quad A_2^2 = T\|H_1(0)\|_{L_4(\Omega)}^4 + A_1^2 \tag{3.16}
\]
and $A_i$, $i = 1, 2, 3$, are bounded. Then
\[
\|\omega(t)\|_{L_2(\Omega)}^2 \leq \frac{\nu c^*_p}{4} \int_{kT}^{t} \|\omega(t')\|_{H_1(\Omega)}^2 dt' \leq A_2^2 + A_3^2 \equiv A_4^2, \tag{3.17}
\]
where $t \in (kT, (k + 1)T], k \in \mathbb{N}_0$, $c^*_p = \min\{1, c_p\}$ and $c_0$ appears below (3.21).

**Proof.** Multiplying (3.1) by $\omega_1$, integrating over $\Omega$, integrating by parts and using the boundary conditions (3.5) yield
\[
\frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \nu \|\nabla \omega_1\|_{L_2(\Omega)}^2 = \int_{\Omega} H_1^2 \omega_1 dx + \int_{\Omega} F_1 \omega_1 dx. \tag{3.18}
\]
Applying the Hölder and the Young inequalities to the first term on the r.h.s. of (3.18) implies
\[
\frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \frac{\nu}{2} \|\nabla \omega_1\|_{L_2(\Omega)}^2 \leq \frac{1}{2\nu} \int_{\Omega} H_1^2 dx + \int_{\Omega} F_1 \omega_1 dx. \tag{3.19}
\]
We apply the Poincaré inequality to the second term on the l.h.s. and set $c^*_p = \frac{2}{c_p + 1}$. Then (3.19) takes the form
\[
\frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \frac{\nu}{4} c^*_p \|\omega_1\|_{H_1(\Omega)}^2 \leq \frac{1}{2\nu} \|H_1\|_{L_4(\Omega)}^4 + \int_{\Omega} F_1 \omega_1 dx. \tag{3.20}
\]
Estimating the last term on the r.h.s. gives
\[
\frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \frac{\nu}{8} c^*_p \|\omega_1\|_{H_1(\Omega)}^2 \leq \frac{1}{2\nu} \|H_1\|_{L_4(\Omega)}^4 + \frac{2c_0}{\nu c^*_p} \|F_1\|_{L_{4/5}(\Omega)}^2. \tag{3.21}
\]
where $c_0$ is the constant from imbedding $\|u\|_{L_4(\Omega)}^2 \leq c_0 \|u\|_{H_1(\Omega)}^2$. From (3.21) we have
\[
\frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \frac{\nu}{4} c^*_p \|\omega_1\|_{L_2(\Omega)}^2 \leq \frac{1}{\nu} \|H_1\|_{L_4(\Omega)}^4 + \frac{4c_0}{\nu c^*_p} \|F_1\|_{L_{4/5}(\Omega)}^2. \tag{3.22}
\]
Next (3.22) implies
\[
\frac{d}{dt} \left(\|\omega_1\|_{L_2(\Omega)}^2 \exp \left(\frac{\nu c^*_p t}{4}\right)\right) \leq \frac{1}{\nu} \left(\|H_1\|_{L_4(\Omega)}^4 + \frac{4c_0}{c^*_p} \|F_1\|_{L_{4/5}(\Omega)}^2\right) \exp \left(\frac{\nu c^*_p t}{4}\right). \tag{3.23}
\]
Integrating (3.23) with respect to time from \( t = kT \) to \( t \in (kT, (k+1)T) \) yields

\[
\|\omega(t)\|_{L^2_2(\Omega)}^2 \leq \exp \left( -\frac{\nu}{4c_p^*}t \right) \mathcal{K}_4(t) \|\omega(0)\|_{L^2_2(\Omega)}^2 \frac{1}{\nu} \|H_1(0)\|_{L^4(\Omega)}^4 + \frac{4c_0}{\nu c_p^*} \|F_1(t)\|_{L_{6/5}(\Omega)}^2 dt' + \exp \left( -\frac{\nu}{4c_p^*}T \right) \|\omega(1)\|_{L^2_2(\Omega)}^2.
\]

Simplifying we get

\[
\|\omega(t)\|_{L^2_2(\Omega)}^2 \leq \int_{kT}^{(k+1)T} \mathcal{K}_4(t) \|\omega(0)\|_{L^2_2(\Omega)}^2 \frac{1}{\nu} \|H_1(0)\|_{L^4(\Omega)}^4 + \frac{4c_0}{\nu c_p^*} \|F_1(t)\|_{L_{6/5}(\Omega)}^2 dt + \exp \left( -\frac{\nu}{4c_p^*}T \right) \|\omega(1)\|_{L^2_2(\Omega)}^2.
\]

Using (3.15) yields

\[
\|\omega(1)\|_{L^2_2(\Omega)}^2 \leq \int_{kT}^{(k+1)T} \mathcal{K}_4(t) \|\omega(0)\|_{L^2_2(\Omega)}^2 \frac{1}{\nu} \|H_1(0)\|_{L^4(\Omega)}^4 + \frac{4c_0}{\nu c_p^*} \|F_1(t)\|_{L_{6/5}(\Omega)}^2 dt + \exp \left( -\frac{\nu}{4c_p^*}T \right) \|\omega(1)\|_{L^2_2(\Omega)}^2.
\]

Hence, iteration implies

\[
\|\omega(1)\|_{L^2_2(\Omega)}^2 \leq \frac{A_2^2}{1 - \exp \left( -\frac{\nu}{4c_p^*}T \right)} + \|\omega(1)\|_{L^2_2(\Omega)}^2 \equiv A_3^2.
\]

Let us express (3.21) in the form

\[
\frac{d}{dt} \|\omega\|_{L^2_2(\Omega)}^2 + \nu c_p^* \|\omega\|_{H^1(\Omega)}^2 \leq \frac{1}{\nu} \|H_1\|_{L^4(\Omega)}^4 + \frac{4c_0}{\nu c_p^*} \|F_1\|_{L_{6/5}(\Omega)}^2.
\]

Integrating (3.29) with respect to time from \( t = kT \) to \( t \in (kT, (k+1)T) \), \( k \in \mathbb{N}_0 \), and using (3.8) we obtain (3.17). This concludes the proof. \( \square \)

From [16] we have

**Lemma 3.3.** Assume that \( \omega_1 \in L_\infty(\mathbb{R}_+; L^2_2(\Omega)) \). Then there exists a solution to problem (3.3), (3.5) such that \( \psi_1 \in H^3(\Omega) \) and

\[
\|\psi_1\|_{L_\infty(\mathbb{R}_+; H^2(\Omega))} \leq c \|\omega_1\|_{L_\infty(\mathbb{R}_+; L^2_2(\Omega))}.
\]

Let us consider the elliptic problem (1.21). Let \( k \in \mathbb{N}_0 \). Assume that \( \omega_\infty \in L^2_2(\mathbb{R}_+; L^2_2(\Omega)) \). Then there exists a solution to problem (1.21) such that \( \psi \in L^2_2(\mathbb{R}_+; L^2_2(\Omega); H^3(\Omega)) \) and

\[
\|\psi\|_{L^2_2(\mathbb{R}_+; L^2_2(\Omega); H^3(\Omega))} \leq c \|\omega_\infty\|_{L^2_2(\mathbb{R}_+; L^2_2(\Omega); H^3(\Omega))}.
\]
From Lemmas 3.1–3.3 we have

**Lemma 3.4.** Assume that $H_1(0) \in L_4(\Omega)$, $F_1 \in L_2(kT, (k + 1)T; L_{6/5}(\Omega))$, $k \in \mathbb{N}_0$, $\omega_1(0) \in L_2(\Omega)$. Then there exists a solution to problem (3.1)–(3.6) such that $H_1 \in V_2^0(\Omega \times (kT, (k + 1)T))$, $\nu' = (v_r, v_z)$ and

$$
\|H_1^2\|^2_{V_2^0(\Omega \times (kT, (k + 1)T))} \leq c\|H_1(0)\|_{L_4(\Omega)}^4, \tag{3.32}
$$

$$
\|\nu'\|_{V_2^0(\Omega \times (kT, (k + 1)T))} \leq c\left(\sup_k \left\|F_1 \right\|_{L_2(kT, (k + 1)T; L_{6/5}(\Omega))} + \|\omega_1(0)\|_{L_2(\Omega)}\right) \leq A_4^2, \quad k \in \mathbb{N}_0.
$$

To increase regularity of $H_1$ described by (3.32) of Lemma 3.4 we need the following result

**Lemma 3.5.** Assume that for all $k \in \mathbb{N}_0$, $\omega_1 \in L_\infty(kT, (k + 1)T; L_2(\Omega))$, $H_1 \in L_2(kT, (k + 1)T; L_2(\Omega))$ and $H_1(kT) \in H^1(\Omega)$. Then solutions to problem (3.2), (3.5) satisfy

$$
\|H_1(t)\|^2_{H^1(\Omega)} + \mu \int_{kT}^t \|H_1(t')\|^2_{H^2(\Omega)} dt' \leq c\left(\sup_{t \in [kT, (k + 1)T]} \left\|\omega_1(t)\right\|_{L_4(\Omega)}^2 + \|H_1(kT)\|^2_{H^1(\Omega)}\right) \tag{3.33}
$$

for $t \in [kT, (k + 1)T]$, $k \in \mathbb{N}_0$.

**Proof.** Multiplying (3.2) by $H_1$, integrating over $\Omega$ and using boundary conditions (3.5) yields

$$
\frac{d}{dt}\|H_1\|^2_{L_2(\Omega)} + \mu\|H_1\|^2_{H^1(\Omega)} \leq 0.
$$

To obtain an estimate for derivatives of $H_1$ we introduce the partition of unity

$$
\varphi^{(1)}(r) = 1 \quad \text{for} \quad r \leq r_0, \quad \varphi^{(1)}(r) = 0 \quad \text{for} \quad r \geq 2r_0,
$$

$$
\varphi^{(2)}(r) = 1 \quad \text{for} \quad r \geq 2r_0, \quad \varphi^{(2)}(r) = 0 \quad \text{for} \quad r \leq r_0.
$$

Let $H_1^{(i)} = H_1 \varphi^{(i)}$. Multiplying (3.2) by $\varphi^{(i)}$ implies

$$
H_1^{(i)} + v' \cdot \nabla H_1^{(i)} - v_r H_1^{(i)} - \mu \Delta H_1^{(i)} - \frac{2\mu}{r} H_1^{(i)} + 2\mu H_1 r \varphi^{(i)} + \frac{2\mu}{r} H_1 \varphi^{(i)} = 0,
$$

where the dot denotes derivative with respect to $r$. Differentiating (3.35) with respect to $r$ gives

$$
H_1^{(i)} + v' \cdot \nabla H_1^{(i)} + v' \cdot \nabla H_1^{(i)} - (v_r H_1^{(i)})_r
$$

$$
- \mu \left(\Delta H_1^{(i)} + \frac{2}{r} H_1^{(i)} r - \frac{3}{r^2} H_1^{(i)}_r\right)
$$

$$
= -\mu \left(2H_1 r \varphi^{(i)} + H_1^{(i)} + \frac{2}{r} H_1 \varphi^{(i)}\right)_r.
$$

(3.36)
Multiply (3.36) by $H_{1,r}^{(1)}$, integrate over $\Omega$ and use that $H_{1,r}^{(1)}|_{\partial(\Omega \cap \text{supp } \varphi^{(1)})} = 0$. Then we get

$$
\frac{1}{2} \frac{d}{dt} \|H_{1,r}^{(1)}\|_{L^2(\Omega)}^2 + \mu \|\nabla H_{1,r}^{(1)}\|_{L^2(\Omega)}^2 + \int_{\Omega} v_{r,\varphi} \cdot \nabla H_{1,r}^{(1)} H_{1,r}^{(1)} \, dx \\
- \int_{\Omega} (v_r H_{1,r}^{(1)}) H_{1,r}^{(1)} \, dx - 2\mu \int_{\Omega} \frac{1}{r} H_{1,r}^{(1)} H_{1,r}^{(1)} \, dx \\
+ 3\mu \int_{\Omega} \frac{1}{r^2} \|H_{1,r}^{(1)}\|^2 \, dx = -\mu \int_{\Omega} \left( 2H_{1,r} \varphi^{(1)} + H_{1,r}^{(1)} H_{1,r}^{(1)} \right) H_{1,r}^{(1)} \, dx.
\tag{3.37}
$$

Now, we examine the particular terms in (3.37).

The third term on the l.h.s. of (3.37) can be expressed in the form

$$
\int_{\Omega} v_{r,r} |H_{1,r}^{(1)}|^2 \, dx + \int_{\Omega} v_{z,r} H_{1,r}^{(1)} H_{1,r}^{(1)} \, dx \equiv I_1^1 + I_1^2,
$$

where $v_r = -(\psi_1)_z$, $v_z = \psi_1 + 2\psi_1$, so $v_{r,r} = -r\psi_1_{r,z} - \psi_1_z$, $v_{z,r} = r\psi_1_{r,r} + 3\psi_1_r$.

Applying the Hölder inequality and interpolation yields

$$
|I_1^1| \leq \|v_{r,r}\|_{L^2(\Omega)} \|H_{1,r}^{(1)}\|_{L^2(\Omega)} \leq \epsilon \|\psi_1\|_{H^2(\Omega)} \|\nabla^2 H_{1,r}^{(1)}\|_{L^2(\Omega)} \|H_{1,r}^{(1)}\|_{L^2(\Omega)}^{1/4} \\
\leq \epsilon \|\nabla^2 H_{1,r}^{(1)}\|_{L^2(\Omega)} + c/\epsilon \|\psi_1\|_{H^2(\Omega)} \|H_{1,r}^{(1)}\|_{L^2(\Omega)}.
$$

Similarly, we have

$$
|I_1^2| \leq \|v_{z,r}\|_{L^2(\Omega)} \|\nabla H_{1,r}^{(1)}\|_{L^2(\Omega)} \\
\leq \epsilon \|\nabla^2 H_{1,r}^{(1)}\|_{L^2(\Omega)} + c/\epsilon \|\psi_1\|_{H^2(\Omega)} \|H_{1,r}^{(1)}\|_{L^2(\Omega)}.
$$

Integrating by parts in the fourth term on the l.h.s. of (3.37) gives the expression

$$
\int_{\Omega} v_r H_{1,r} \varphi^{(1)} (H_{1,r}^{(1)} r, r) \, drdz
$$

which is estimated by

$$
\epsilon \left( \int_{\Omega} |H_{1,r}^{(1)}|^2 \, dx + \int_{\Omega} |H_{1,r}^{(1)}| \, dx \right) + c/\epsilon \int_{\Omega} v_r^2 H_{1,r}^2 |\varphi^{(1)}|^2 \, dx \equiv I_2,
$$

where it is used that $\text{supp } \varphi^{(1)}$ is located in a positive distance from the axis of symmetry. The last term in $I_2$ is bounded by

$$
eq c \|\psi_r\|_{L^2(\Omega)}^2 \|H_{1,r}^2\|_{L^2(\Omega)} \leq c \|\psi_r\|_{L^2(\Omega)} \|H_{1,r}^2\|_{L^2(\Omega)} \|H_{1,r}^2\|_{L^2(\Omega)} \|H_{1,r}^2\|_{L^2(\Omega)}.
$$

The fifth term on the l.h.s. of (3.37) equals

$$
-\mu \int_{\Omega} (|H_{1,r}^{(1)}|^2)_{r=0} \, drdz = \mu \int_{-a}^a H_{1,r}^{(1)} \, drdz.
$$

Finally, integrating by parts in the term on the r.h.s. of (3.37) yields

$$
\int_{\Omega} \left( 2H_{1,r} \varphi^{(1)} + H_{1,r}^{(1)} H_{1,r}^{(1)} + \frac{2}{r} H_{1,r} \varphi^{(1)} \right) (H_{1,r}^{(1)} r, r) \, drdz
$$

which we estimate by

$$
\epsilon (\|H_{1,r}^{(1)}\|_{L^2(\Omega)}^2 + \|H_{1,r}^{(1)}\|_{L^2(\Omega)}^2) + c/\epsilon \|H_{1,r}^2\|_{L^2(\Omega)}^2.
$$
Employing the above estimates in (3.37), using the Poincaré inequality in the second term on the l.h.s. of (3.37) we obtain for sufficiently small $\varepsilon$ the inequality

$$
\frac{d}{dt}\left\| H^{(1)}_{1,r} \right\|^2_{L^2(\Omega)} + \left\| H^{(1)}_{1,r} \right\|^2_{H^1(\Omega)} \leq c\left\| \omega \right\|^2_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)} + \varepsilon \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)} \left\| \omega \right\|^2_{L^2(\Omega)} + c\left\| H^{(1)}_{1} \right\|^2_{H^1(\Omega)}.
$$

(3.38)

Next, we shall obtain an estimate for derivatives with respect to $z$. Multiplying (3.35) by $-H^{(1)}_{1,z}$ and integrating the result over $\Omega$ yields

$$
\frac{1}{2}\frac{d}{dt}\left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + \mu \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + \mu \int_{\Omega} H^{(1)}_{1,r} H^{(1)}_{1,z} \, dx

+ \mu \int_{\Omega} \int_{r}^{1} H^{(1)}_{1,r} H^{(1)}_{1,z} \, dx - \int_{\Omega} \nabla H^{(1)}_{1,z} \cdot \nabla H^{(1)}_{1,r} \, dx + \int_{\Omega} v_{r} H^{(1)}_{1,z} \, dx

+ 2\mu \int_{\Omega} \int_{r}^{1} H^{(1)}_{1,z} \, dz = \mu \int_{\Omega} \left( 2H^{(1)}_{1,r} + H^{(1)}_{1,z} \right) H^{(1)}_{1,z} \, dx.
$$

(3.39)

Now we examine the particular terms in (3.39). The third term on the l.h.s. is bounded by

$$
\varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| H^{(1)}_{1,r} \right\|^2_{L^2(\Omega)}.
$$

Using that $H^{(1)}_{1,r}|_{S_2} = 0$ we can integrate by parts in the fourth and in the last terms to get

$$
-3\mu \int_{\Omega} H^{(1)}_{1,z} L \, dz = -3\mu \int_{\Omega} \left( \langle H^{(1)}_{1,z} \rangle \right)_r \, dz = \frac{3\mu}{2} \int_{-a}^{a} \left( H^{(1)}_{1,z} \right)^2 \, dz.
$$

We can integrate by parts with respect to $z$ in the fifth term on the l.h.s. of (3.39) using that $v_{r} \cdot \nabla H^{(1)}_{1}|_{S_2} = 0$ which holds in view of $\partial_z H^{(1)}_{1}|_{S_2} = 0$ and $v_{z}|_{S_2} = 0$. Therefore, the term takes the form

$$
\int_{\Omega} v_{r} \cdot \nabla H^{(1)}_{1,z} \, dx + \int_{\Omega} v_{z} \cdot \nabla H^{(1)}_{1,z} \, dx,
$$

where the first integral vanishes and the second is bounded by

$$
\varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| v_{r} \right\|^2_{L^2(\Omega)} \left\| \nabla H^{(1)}_{1} \right\|^2_{L^2(\Omega)}

\leq \varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} \left\| \nabla H^{(1)}_{1} \right\|^2_{L^2(\Omega)}

\leq \varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{H^2(\Omega)} + c/\varepsilon \left\| \omega \right\|^2_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{H^1(\Omega)}.
$$

The sixth term on the l.h.s. of (3.39) is bounded by

$$
\varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| v_{r} \right\|^2_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)}

\leq \varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| \omega \right\|^2_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)}

\leq \varepsilon \left( \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + \left\| H^{(1)} \right\|^2_{H^1(\Omega)} \right) + c/\varepsilon \left\| \omega \right\|^2_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)}.
$$

Finally the term on the r.h.s. of (3.39) is bounded by

$$
\varepsilon \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + c/\varepsilon \left\| H^{(1)}_{1} \right\|^2_{H^1(\Omega)}.
$$

Employing the above estimates in (3.39) and using that $\varepsilon$ is sufficiently small we get

$$
\frac{d}{dt}\left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} + \mu \left\| H^{(1)}_{1,z} \right\|^2_{L^2(\Omega)} \leq c\left\| H^{(1)}_{1} \right\|^2_{L^2(\Omega)}

+ \varepsilon \left\| H^{(1)}_{1} \right\|^2_{H^2(\Omega)} + c/\varepsilon \left\| \omega \right\|^4_{L^2(\Omega)} \left\| H^{(1)}_{1} \right\|^2_{H^1(\Omega)} + c\left\| H^{(1)}_{1} \right\|^2_{H^1(\Omega)}.
$$

(3.40)
Adding (3.34), (3.38) and (3.40) appropriately, we derive
\[
\frac{d}{dt} \| H_1^{(1)} \|^2_{H^1(\Omega)} + \mu \| H_1^{(1)} \|^2_{H^2(\Omega)} \leq c \| \omega \|^8_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)} + c \| \omega \|^4_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)}. \tag{3.41}
\]

To examine (3.2) in a neighborhood located in a positive distance from the axis of symmetry we multiply (3.2) by \( \varphi^{(2)} \) and obtain the equation
\[
H_1^{(2)} + v' \cdot \nabla H_1^{(2)} - v_r H_1 \varphi^{(2)} - \mu \Delta H_1^{(2)} - \frac{2\mu}{r} H_1^{(2)}
= -\mu \left( 2H_1, \varphi^{(2)} + H_1 \varphi^{(2)} + \frac{2}{r} H_1 \varphi^{(2)} \right). \tag{3.42}
\]

Multiplying (3.42) by \( -\Delta H_1^{(2)} \), integrating over \( \Omega \) and using the boundary condition \( H_1^{(2)}|_{\partial(\Omega \cap \text{supp} \varphi^{(2)})} = 0 \) we have
\[
\frac{d}{dt} \| \nabla H_1^{(2)} \|^2_{L^2(\Omega)} + \mu \| \Delta H_1^{(2)} \|^2_{L^2(\Omega)} \leq c \| v' \cdot \nabla H_1^{(2)} \|^2_{L^2(\Omega)} + c \| v_r H_1 \|^2_{L^2(\Omega)} + c \| H_1 \|^2_{H^1(\Omega)}. \tag{3.43}
\]

The first term on the r.h.s. is bounded by
\[
\| v' \|^2_{L^6(\Omega)} \| \nabla H_1^{(2)} \|^2_{L^3(\Omega)} \leq c \| \omega \|^2_{L^3(\Omega)} \| \nabla H_1^{(2)} \|^2_{L^2(\Omega)} \| \nabla H_1^{(2)} \|^2_{L^2(\Omega)} \leq \varepsilon \| \nabla H_1^{(2)} \|^2_{L^2(\Omega)} + c/\varepsilon \| \omega \|^2_{L^2(\Omega)} \| H_1^{(2)} \|^2_{H^1(\Omega)},
\]
and the second by
\[
\| v_r \|^2_{L^6(\Omega)} \| H_1 \|^2_{L^3(\Omega)} \leq c \| \omega \|^2_{L^3(\Omega)} \| H_1 \|^2_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)} \leq \varepsilon \| H_1 \|^2_{L^3(\Omega)} + c/\varepsilon \| \omega \|^2_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)}.
\]

Using that the problem
\[
\Delta H_1^{(2)} = \Delta H_1^{(2)}, \quad H_1^{(2)}|_{\partial (\Omega \cap \text{supp} \varphi^{(2)})} = 0
\]
implies the estimate
\[
\| H_1^{(2)} \|^2_{H^2(\Omega)} \leq c \| \Delta H_1^{(2)} \|^2_{L^2(\Omega)}
\]
we obtain from (3.43) for sufficiently small \( \varepsilon \) the inequality
\[
\frac{d}{dt} \| \nabla H_1^{(2)} \|^2_{L^2(\Omega)} + \mu \| H_1^{(2)} \|^2_{H^1(\Omega)} \leq c \| \omega \|^2_{L^2(\Omega)} \| H_1^{(2)} \|^2_{H^1(\Omega)} + c \| H_1 \|^2_{H^1(\Omega)}. \tag{3.44}
\]

Now, from (3.34), (3.41) and (3.44) we derive the inequality
\[
\frac{d}{dt} \| H_1 \|^2_{H^1(\Omega)} + \mu \| H_1 \|^2_{H^2(\Omega)} \leq c \| \omega \|^2_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)} + c \| \omega \|^2_{L^2(\Omega)} \| H_1 \|^2_{L^2(\Omega)}. \tag{3.45}
\]

Integrating (3.45) with respect to time from \( t = kT \) to \( t \in (kT, (k + 1)T] \) we derive (3.33). This concludes the proof. □

In Lemma 3.5 quantity \( \| H_1(kT) \|^2_{H^1(\Omega)} \) is not estimated in terms of data. To make it we need.
Lemma 3.6. Let $k \in \mathbb{N}_0$. Assume that
\[
B_1^2 = \sup_{k \in \mathbb{N}_0} \left[ \sup_{t \in [kT,(k+1)T]} \left( \| \omega_1(t) \|^8_{L^2(\Omega)} + \| \omega_1(t) \|^4_{L^2(\Omega)} \right) \right],
\]
\[
\cdot \int_{kT}^{(k+1)T} \| H_1(t) \|^2_{L^2(\Omega)} dt < \infty,
\]
where $H_1$ is estimated in (3.8) and $\omega_1$ in (3.17). Assume that $H_1(0) \in H^1(\Omega)$. Then the following estimate holds
\[
\| H_1(kT) \|^2_{H^1(\Omega)} \leq \frac{cB_1^2}{1 - e^{-\mu T}} + \exp(-\mu kT)\| H_1(0) \|^2_{H^1(\Omega)}. \tag{3.46}
\]

Proof. Expressing (3.45) in the form
\[
\frac{d}{dt} \| H_1 \|^2_{H^1(\Omega)} + \mu \| H_1 \|^2_{H^1(\Omega)} \leq c(\| \omega_1 \|^8_{L^2(\Omega)} + \| \omega_1 \|^4_{L^2(\Omega)}) \| H_1 \|^2_{L^2(\Omega)}
\]
we have
\[
\frac{d}{dt} (\| H_1 \|^2_{H^1(\Omega)} \exp(\mu t)) \leq c(\| \omega_1 \|^8_{L^2(\Omega)} + \| \omega_1 \|^4_{L^2(\Omega)}) \| H_1 \|^2_{L^2(\Omega)} \exp(\mu t). \tag{3.47}
\]

Integrating (3.47) with respect to time from $t = kT$ to $t = (k+1)T$ and using definition of $B_1$ we get
\[
\| H_1((k+1)T) \|^2_{H^1(\Omega)} \leq cB_1 + \exp(-\mu T)\| H_1(kT) \|^2_{H^1(\Omega)}. \tag{3.48}
\]
Iterating (3.48) yields (3.46). This concludes the proof.

Remark 5. To prove existence of solutions to problem (3.1)–(3.6) we express (3.1), (3.2) in the form
\[
\omega_{1,t} - \nu \left( \Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) = -v' \cdot \nabla \omega_1 - 2H_1 H_{1,z} + F_1 = F_2,
\]
\[
\omega_1|_{t=0} = \omega_1(0), \quad \omega_1|_{S} = 0 \tag{3.49}
\]
and
\[
H_{1,t} - \mu \left( \Delta H_1 + \frac{2}{r} H_{1,r} \right) = -v' \cdot \nabla' H_1 \equiv F_3,
\]
\[
H_1|_{t=0} = H_1(0), \quad H_1|_{S} = 0 \tag{3.50}
\]

For given $F_2$ and $F_3$ we prove existence of solutions to problems (3.49) and (3.50) in the norms of $\omega_1$ and $H_1$ introduced in Lemmas 3.2 and 3.5 by the Faedo-Galerkin method. Then by the method of successive approximations and [16] we can prove existence of solutions to problem (3.1)–(3.6) in the normed appeared in Lemmas 3.1–3.6.

Now we describe the statement more precisely.

To prove existence of solutions to problem (3.1)–(3.6) we use the following method of successive approximations. Let $\psi_{1n}$ be given. Then $v_n = v_{zn} e_r + v_{zn} e_z = -(r \psi_{1n}) e_r + \frac{1}{r} (r^2 \psi_{1n}) e_z$. From this construction we have that $\text{div } v_n = 0$. Then we calculate
\[
H_{1n+1,t} + v_n \cdot \nabla H_{1n+1} - \mu \left( \Delta H_{1n+1} + \frac{2}{r} H_{1n+1,r} \right) = 0 \tag{3.51}
\]
\[
H_{1n+1}|_{t=0} = H_1(0), \quad H_{1n+1}|_{S} = 0,
\]
Lemma 3.7. Assume that \( H_1(0) \in H^1(\Omega) \cap L_\sigma(\Omega), \ v'_n \in C(\mathbb{R}_+; H^1(\Omega)), \ v'_n = (v_r, v_z). \) Then there exists a solution to problem (3.51) such that \( H_{n+1} \in L_\infty(\mathbb{R}_+; H^1(\Omega)) \cap L_2(kT, (k+1)T; H^2(\Omega)) \cap L_\sigma(\mathbb{R}_+, L_\sigma(\Omega)), k \in \mathbb{N}_0 \) and the estimate holds

\[
\|H_{n+1}\|_{L_\infty(\mathbb{R}_+; H^1(\Omega))} + \|H_{n+1}\|_{L_2(kT, (k+1)T; H^2(\Omega))} \leq C(\|v'_n\|_{L_2(\Omega)}^4 + 1) \exp(-c_0kT)\|H(0)\|_{L_2(\Omega)}, \|H_{n+1}(t)\|_\sigma \leq c_1(0)\|H(0)\|_\sigma.
\]

Proof. We skip indices 1, n, n + 1. Take the fundamental basis \( \{\varphi_k\}_{k=1}^{\infty} \) in \( H^1(\Omega) = \{u \in H^1(\Omega) : u|_S = 0\} \) which is orthonormal in \( L_2(\Omega). \)

Then we are looking for approximate solutions

\[
H_n(x, t) = \sum_{k=1}^{n} c_{kn}(t)\varphi_k(x),
\]

where \( c_{kn}(t) \in C^1 \) and \( \varphi_k(x) \) are smooth.

Moreover, we assume that \( H_n \) is a solution to the integral identity

\[
(H_{n,t}, \eta) + (v \cdot \nabla H_n, \eta) + \mu(\nabla H_n, \nabla \eta) - 2\mu \left( \frac{1}{r} H_{n,r}, \eta \right) = 0,
\]

where

\[
(u, v) = \int_{\Omega} u(x)v(x)dx
\]

is the scalar product in \( L_2(\Omega). \) We assume that (3.56) holds for any smooth function from \( H^1(\Omega). \) To show existence of the approximate solutions (3.55) we set \( \eta = \varphi_i, \) so we obtain the following system of ordinary differential equations

\[
(H_{n,t}, \varphi_i) + (v \cdot \nabla H_n, \varphi_i) + \mu(\nabla H_n, \nabla \varphi_i) - 2\mu \left( \frac{1}{r} H_{n,r}, \varphi_i \right) = 0,
\]

where \( i = 1, \ldots, k. \) Hence for \( v \) continuous with respect to time we have existence \( c_{kn}, k = 1, \ldots, n. \) To obtain an estimate we multiply (3.58) by \( c_{in}, \) summ over \( i \) from 1 to \( n. \) Then we get

\[
\frac{d}{dt}|H_n|^2 + \mu|\nabla H_n|^2 = 0.
\]

For \( \eta = H_n|_{H_n}|^{\sigma-2} \) we obtain from (3.56) the equality

\[
\frac{1}{\sigma} \frac{d}{dt}|H_n|^\sigma + \frac{4(\sigma-1)}{\sigma^2}|\nabla H_n|^{\sigma/2} = 0,
\]

where we used the simplified notation \( |H|_\sigma = \|H\|_{L_\sigma(\Omega)}. \)
Applying the Poincaré inequality we have the estimate
\[
|H_n(t)|_\sigma^2 + \int_0^t |H_n|_\sigma^2 dt' + \int_0^t |\nabla|H_n|^2|_2^2 dt' \leq c|H_n(0)|_\sigma^2.
\] (3.61)
Moreover, (3.60) implies the inequality
\[
\frac{d}{dt}|H_n|_\sigma^2 + c_0|H_n|_\sigma^2 \leq 0
\] (3.62)
so
\[
|H_n(t)|_\sigma \leq \exp(-c_0t)|H_n(0)|_\sigma.
\] (3.63)
Applying the considerations leading to (3.13) and (3.15) we obtain
\[
|H_n(kT)|_\sigma^2 \leq \exp(-c_0kT)|H_1(0)|_\sigma^2
\] (3.64)
and
\[
|H_n(t)|_\sigma^2 \leq \|H_1(0)\|_{L_\sigma(\Omega)}^2, \quad t \in (kT, (k+1)T], \quad k \in \mathbb{N}_0.
\] (3.65)
Consider (3.56). Use the partition of unity introduced between (3.34) and (3.35) and the notation from (3.35).

Setting \( \eta = \varphi^{(1)} \partial_r H_n^{(1)} \), we obtain (3.37), where \( H_1^{(1)} \) is replaced by \( H_n^{(1)} \).

Hence performing the considerations leading to (3.38), where \( v \) is treated as given, we obtain in the place of (3.38) the inequality
\[
\frac{d}{dt}\|H_n\|_{L_2(\Omega)}^2 + \mu\|H_n\|_{H^1(\Omega)}^2 \leq c\|v_{r,r}\|_{L_2(\Omega)}^8\|H_n\|_{L_2(\Omega)}^2
\]
\[
+ \varepsilon\|H_n\|_{L_2(\Omega)}^2 + c\|v_r\|_{L_2(\Omega)}^4\|H_n\|_{L_2(\Omega)}^2 + c\|H_n\|_{H^1(\Omega)}^2.
\] (3.66)
Similarly, instead of (3.40) we obtain the inequality
\[
\frac{d}{dt}\|H_n\|_{L_2(\Omega)}^2 + \mu\|H_n\|_{L_2(\Omega)}^2 \leq c\|H_n\|_{L_2(\Omega)}^2
\]
\[
+ \varepsilon\|H_n\|_{L_2(\Omega)}^2 + c\|v_r\|_{L_2(\Omega)}^4\|H_n\|_{L_2(\Omega)}^2 + c\|H_n\|_{H^1(\Omega)}^2.
\] (3.67)
From (3.60) for \( \sigma = 2 \) and the Poincaré inequality we get
\[
\frac{d}{dt}\|H_n\|_{L_2(\Omega)}^2 + c_0\|H_n\|_{H^1(\Omega)}^2 \leq 0.
\] (3.68)
Adding appropriately (3.66), (3.67) and (3.68) we obtain the inequality
\[
\frac{d}{dt}\|H_n\|_{H^1(\Omega)}^2 + \mu\|H_n\|_{H^2(\Omega)}^2 \leq c\|v_{r,r}\|_{L_2(\Omega)}^8\|H_n\|_{L_2(\Omega)}^2
\]
\[
+ c\|v_r\|_{L_2(\Omega)}^4\|H_n\|_{L_2(\Omega)}^2 + c\|H_n\|_{H^1(\Omega)}^2.
\] (3.69)
Let \( \eta \) in (3.56) be equal to \( \eta = -\varphi^{(2)} \Delta H_n^{(2)} \). Then instead of (3.44) we have
\[
\frac{d}{dt}\|H_n\|_{H^1(\Omega)}^2 + \mu\|H_n\|_{H^2(\Omega)}^2 \leq c\|v'\|_{L_2(\Omega)}^8\|H_n\|_{L_2(\Omega)}^2
\]
\[
+ c\|H_n\|_{H^1(\Omega)}^2
\] (3.70)
where \( v' = (v_r, v_z) \). From (3.68), (3.69) and (3.70) we have
\[
\frac{d}{dt}\|H_n\|_{H^1(\Omega)}^2 + \mu\|H_n\|_{H^2(\Omega)}^2 \leq c\|v_{r,r}\|_{L_2(\Omega)}^8\|H_n\|_{L_2(\Omega)}^2
\]
\[
+ c\|v'\|_{L_2(\Omega)}^4\|H_n\|_{L_2(\Omega)}^2 + c\|H_n\|_{L_2(\Omega)}^2.
\] (3.71)
Integrating (3.68), (3.71) with respect to time from $kT$ to $t \in [kT, (k+1)T]$, $k \in \mathbb{N}_0$, and using estimate (3.65) we have
\[
\|H_n(t)\|_{H^1(\Omega)} + \mu \int_{kT}^{t} \|H_n'(t')\|_{H^2(\Omega)} dt' + \|H_n(t)\|_{L^2(\Omega)} \leq \varphi(\|v\|_{L^\infty(\mathbb{R}^+; H^1(\Omega))}, \|H(0)\|_{H^1(\Omega)}), \quad t \in [kT, (k+1)T],
\]
(3.72)
where $\varphi$ is an increasing positive function.

Consider estimates (3.60), (3.64), (3.68) and (3.72). Passing with $n \to \infty$ we get existence of solutions to problem (3.51) such that $H_{1n} \in L^\infty(\mathbb{R}^+; H^1(\Omega)) \cap L^2(kT, (k+1)T; H^2(\Omega)) \cap L^\infty(\mathbb{R}^+; L^\sigma(\Omega))$, $k \in \mathbb{N}_0$ and estimate (3.54) holds. This concludes the proof.

Lemma 3.8. Consider problem (3.52). Let the assumptions of Lemma 3.2 hold. Then there exists a solution to problem (3.52) such that $\omega_{1n+1} \in V_0^2(\Omega \times (kT, (k+1)T))$, $k \in \mathbb{N}_0$, and the estimate holds
\[
\|\omega_{1n+1}\|_{V_0^2(\Omega \times (kT, (k+1)T))} \leq cA_2^4,
\]
(3.73)
where $A_2^4$ is introduced in (3.17).

Proof. We use the idea of the approximate solutions from the proof of Lemma 3.7.

Lemma 3.9. Consider problem (3.53). Assume that $\omega_{1n+1}$ satisfies the same assumptions as $\omega_1$ in Lemma 3.3. Then there exists $\psi_{1n+1}$ and satisfies (3.30) and (3.31).

Proof. Consider problem (3.53). Drop indices $1, n+1$. Then we are looking for the approximate solutions to (3.53) in the form
\[
\psi_n = \sum_{i=1}^{n} c_{in}(t) \varphi_i(x)
\]
which are solutions to the identity
\[
- (\Delta \psi_n, \eta) = (\omega, \eta)
\]
(3.74)
which holds for any smooth function $\eta \in H^2(\Omega)$.

Setting $\eta = \Delta \varphi_i$, multiplying by $c_{in}$ and adding we get the estimate
\[
\|\psi_n\|_{H^2(\Omega)} \leq c\|\omega\|_{L^2(\Omega)}.
\]
(3.75)
So for $\omega = 0$ we get that $\psi_n = 0$ so by the Fredholm theorem we have existence of approximate solution.

Next by (3.75) we have weak convergence $\psi_n \rightharpoonup \psi$ in $H^2(\Omega)$ and strong in $H^1(\Omega)$. Therefore, the lemma is proved.

Next we show convergence. For this we introduce the differences
\[
\bar{H}_{1n} = H_{1n} - H_{1n-1}, \quad \bar{\omega}_{1n} = \omega_{1n} - \omega_{1n-1}, \quad \bar{\psi}_{1n} = \psi_{1n} - \psi_{1n-1},
\]
(3.76)
which are solutions to the problems
\[
\bar{H}_{1n,t} + \bar{v}_{n-1} \cdot \nabla \bar{H}_{1n} + \bar{v}_{n-1} \cdot \nabla H_{1n-1} - \mu \left( \Delta \bar{H}_{1n} + \frac{2}{r} \bar{H}_{1n,r} \right) = 0
\]
\[
\bar{H}_{1n,t}|_{t=0} = 0, \quad \bar{H}_{1n}|_{S} = 0,
\]
(3.77)
boundary conditions we have

\[ \omega_{1n,t} + v_{n-1} \cdot \nabla \omega_{1n} + \bar{v}_{n-1} \cdot \nabla \omega_{1n-1} - \nu \left( \Delta \omega_{1n} + \frac{2}{r} \omega_{1n,r} \right) \]

(3.78)

\[ = -2(H_{1n}\bar{H}_{1n,r} + \bar{H}_{1n}H_{1n-1,r}), \]

\[ \omega_{1n}|_{t=0} = 0, \quad \bar{\omega}_{1n}|_{t=0} = 0, \]

(3.79)

Lemma 3.10. Consider problem (3.77). Assuming that \( H_1(0) \in L_4(\Omega) \), \( \omega_1(0) \in L_2(\Omega) \). Consider any interval \([kT, (k+1)T]\), \( k \in \mathbb{N}_0 \). Then for any sufficiently small interval \((t, t+h) \subset [kT, (k+1)T]\) there exists a constant \( \gamma < 1 \) such that

\[ \sup_t \| \bar{v}_{n-1} \|^2_{H^1(\Omega)} \leq \gamma \sup_t \| \bar{v}_{n-1} \|^2_{H^1(\Omega)}. \]

(3.80)

Proof. Multiplying (3.77) by \( H_{1n} \) and integrate the result over \( \Omega \) and use the boundary conditions we have

\[ \frac{1}{2} \frac{d}{dt} \| H_{1n} \|^2_{L^2(\Omega)} + \mu \| \nabla H_{1n} \|^2_{L^2(\Omega)} + \mu \int_{-a}^a |H_{1n}|_{r=0}^2 \, dz \]

\[ = -\int_{\Omega} \bar{v}_{n-1} \cdot \nabla H_{1n-1} \, dx = \int_{\Omega} \bar{v}_n \cdot \nabla \bar{H}_{1n-1} \, dx. \]

Applying the Hölder and Young inequalities to the term on the r.h.s. yields

\[ \frac{d}{dt} \| H_{1n} \|^2_{L^2(\Omega)} + \mu \| H_{1n} \|^2_{H^1(\Omega)} \leq c \| \bar{v}_{n-1} \|^2_{L^4(\Omega)} \| H_{1n} \|^2_{L^4(\Omega)} \]

\[ \leq c \| \bar{v}_{n-1} \|^2_{L^4(\Omega)} \exp(-c_0 t) \| H_{1}(0) \|^2_{L^4(\Omega)}. \]

(3.81)

Integrating (3.81) with respect to time from \( t \) to \( t+h \) we get

\[ \| \bar{H}_{1n}(t) \|^2_{L^2(\Omega)} + \mu \int_t^{t+h} \| \bar{H}_{1n}(t') \|^2_{H^1(\Omega)} \, dt' \]

\[ \leq c \sup_t \| \bar{v}_{n-1} \|^2_{L^4(\Omega)} \| H_{1}(0) \|^2_{L^4(\Omega)} \int_t^{t+h} \exp(-c_0 t') \, dt'. \]

(3.82)

Multiplying (3.78) by \( \bar{\omega}_{1n} \), integrating over \( \Omega \) and using the boundary condition yields

\[ \frac{d}{dt} \| \bar{\omega}_{1n} \|^2_{L^2(\Omega)} + \nu \| \nabla \bar{\omega}_{1n} \|^2_{L^2(\Omega)} \leq \int_{\Omega} \bar{v}_{n-1} \cdot \nabla \bar{\omega}_{n-1} \bar{\omega}_{1n} \, dx \]

\[ + \int_{\Omega} (H_{1n}\bar{H}_{1n,r} + \bar{H}_{1n}H_{1n-1,r}) \bar{\omega}_{1n} \, dx. \]

(3.83)

The first term on the r.h.s. of (3.83) is treated in the following way

\[ \left| \int_{\Omega} \bar{v}_{n-1} \cdot \nabla \bar{\omega}_{n-1} \bar{\omega}_{1n} \, dx \right| = \left| \int_{\Omega} \bar{v}_{n-1} \cdot \nabla \bar{\omega}_{n-1} \omega_{1n} \, dx \right| \leq \varepsilon \| \nabla \bar{\omega}_{1n} \|^2_{L^2(\Omega)} \]

\[ + c/\varepsilon \sup_t \| \bar{v}_{n-1} \|^2_{L^4(\Omega)} \| \omega_{n-1} \|^2_{L^4(\Omega)} \]

and the second term on the r.h.s. of (3.83) is estimated by

\[ \varepsilon \| \bar{\omega}_{1n} \|^2_{L^2(\Omega)} + c/\varepsilon (\| H_{1n} \|^2_{L^4(\Omega)} \| \bar{H}_{1n,r} \|^2_{L^2(\Omega)} + \| \bar{H}_{1n} \|^2_{L^4(\Omega)} \| H_{1n-1,r} \|^2_{L^4(\Omega)}). \]

Employing the estimates in (3.83) and using (3.73) for \( n-1 \) and (3.54) yields

\[ \frac{d}{dt} \| \bar{\omega}_{1n} \|^2_{L^2(\Omega)} + \nu \| \bar{\omega}_{1n} \|^2_{H^1(\Omega)} \leq c \sup_t \| \bar{v}_{n-1} \|^2_{L^4(\Omega)} + c \| \bar{H}_{1n} \|^2_{H^1(\Omega)}. \]

(3.84)
Integrating the inequality with respect to time implies
\[
\|\bar{\omega}_1\|_{L^2(\Omega)}^2 + \nu \int_t^{t+h} \|\bar{\omega}_1(n(t'))\|_{H^1(\Omega)}^2 \leq ch \sup_t \|\bar{v}_{n-1}\|_{L^4(\Omega)}^2
\]
\[+ c \int_t^{t+h} \|\bar{H}_1(n(t'))\|_{L^4(\Omega)}^2 \, dt' \tag{3.85}\]
\[
\leq ch \sup_t \|\bar{v}_{n-1}\|_{L^4(\Omega)}^2 + c(\exp(-c_0t) - \exp(-c_0(t + h))) \sup_t \|\bar{v}_{n-1}\|_{L^4(\Omega)}^2,
\]
where (3.82) was used. Consider the problem
\[- \Delta \bar{\psi}_{1n} = \bar{\omega}_1n, \quad \bar{\psi}_{1n}|_S = 0. \tag{3.86}\]
Then (3.86) and (3.85) imply
\[
\sup_t \|\bar{v}_n\|_{H^1(\Omega)}^2 \leq c[h + \exp(-c_0t) - \exp(-c_0(t + h))] \sup \|\bar{v}_{n-1}\|_{L^4(\Omega)}^2.
\tag{3.87}\]
From (3.87) we derive
\[
\sup_t \|\bar{v}_n\|_{H^1(\Omega)}^2 \leq \gamma \sup_t \|\bar{v}_{n-1}\|_{H^1(\Omega)}^2,
\tag{3.88}\]
where \(\gamma < 1\) for \(h\) sufficiently small. Inequality (3.88) implies convergence. This concludes the proof. \(\square\)

Summarizing the above considerations we get

**Theorem 3.11.** Assume that \(H_1(0) \in L_\sigma(\Omega), \sigma \geq 4, \omega_1(0) \in L_1(\Omega), \nu \in L_2(kT,(k + 1)T; L_{6/5}(\Omega)), k \in \mathbb{N}_0\). Then there exists a solution to problem (3.1)–(3.6) such that
\[
H_1 \in L_\infty(\mathbb{R}^+; L_\sigma(\Omega)), \quad \omega_1 \in V_2^0(\Omega \times (kT, (k + 1)T)),
\nu' \in L_\infty(\mathbb{R}^+; H^1(\Omega)).
\]
Assuming additionally that \(H_1(0) \in H^1(\Omega)\) we obtain that \(H_1 \in V_2^1(\Omega \times (kT, (k + 1)T)).\)

4. Stability of the special solutions. To prove stability of the special solutions (3.1)–(3.6) we first derive a differential inequality for solutions to problem (1.40)–(1.48). Hence we introduce the quantities
\[
X^2(t) = \|v_1(t)\|_{L^4(\Omega)}^4 + \|H_1(t)\|_{L^4(\Omega)}^4 + \|\omega_1(t)\|_{L^4(\Omega)}^2 + \|j_1(t)\|_{L^2(\Omega)}^2
\]
\[+ \|v_1(t)\|_{L^1(\Omega)}^2 + \|H_1(t)\|_{L^1(\Omega)}^2,
\]
\[
Y^2(t) = \|\nabla v_1^2(t)\|_{L^2(\Omega)}^2 + \|\nabla H_1^2(t)\|_{L^2(\Omega)}^2 + \|\omega_1(t)\|_{H^1(\Omega)}^2
\]
\[+ \|j_1(t)\|_{H^1(\Omega)}^2 + \|v_1(t)\|_{H^2(\Omega)}^2 + \|H_1(t)\|_{H^2(\Omega)}^2,
\tag{4.1}\]
\[
A^2(t) = \|\bar{H}_1\|_{H^2(\Omega)}^2 + \|\bar{\omega}_1\|_{H^1(\Omega)}^2,
\]
\[
G^2(t) = \|f_1(t)\|_{L^{4/3}(\Omega)}^4 + \|f_1(t)\|_{L^{6/5}(\Omega)}^2 + \|F_1(t)\|_{L^{6/5}(\Omega)}^2.
\]

**Proposition 1.** Assume that all quantities in (4.1) are finite. Then there exists a constant \(c_0\) such that
\[
\frac{d}{dt} X^2 + \nu_* Y^2 \leq c_0(X + X^2 + X^4 + X^8 + Y^2)X^2 + c_0 A^2(X^2 + X^4) + c_0 G^2. \tag{4.2}\]
where \(\nu_* = \min\{\nu, \mu\}.\)
Lemma 4.1. Assume that the following quantities are finite

\[ X_i^1(t) = \|v_1(t)\|_{L_4(\Omega)} + \|H_1(t)\|_{L_4(\Omega)}^4, \]

\[ Y_i^1(t) = \|\nabla v_1^2(t)\|_{L_2(\Omega)}^2 + \|\nabla H_1^2(t)\|_{L_2(\Omega)}^2, \]

\[ X_i^2(t) = \|\omega(t)\|_{L_2(\Omega)}^2 + \|J_1(t)\|_{L_2(\Omega)}^2, \]

\[ A_i^1(t) = \|\phi_1(t)\|_{L_2(\Omega)}^2 + \|\bar{H}_1(t)\|_{H_2(\Omega)}^2, \]

\[ G_i^2(t) = \|f_1\|_{L_{4/3}(\Omega)}. \]

Then the following inequality holds

\[ \frac{d}{dt} X_i^2 + \nu_s Y_i^2 \leq cX_i^2 X_i^2(1 + X_i^2) + cA_i^2(X_i^2 + X_i^2) + cG_i^2. \]

Proof. Multiplying (4.3) by \( v_1 |v_1|^2 \), integrating the result over \( \Omega \) and using the boundary conditions (1.44)1, we obtain

\[ \frac{1}{4} \frac{d}{dt} \|v_1\|_{L_4(\Omega)}^4 + 3\nu \int \Omega |\nabla v_1^4|^2 dx + \frac{\nu}{2} \int \Omega |v_1|^4 \partial_z |v_1|^4 dx = 2 \int \Omega (\psi_1 z v_1^4 + \bar{\psi}_1 z v_1^4) dx \]

\[ + \int \Omega (H' \cdot \nabla H_1 + H' \cdot \nabla \bar{H}_1) v_1 |v_1|^2 dx - 2 \int \Omega (\phi_1 z H_1 + \phi_1 z \bar{H}_1) v_1 |v_1|^2 dx \quad (4.3) \]

Integrating by parts and using the boundary conditions (1.44)1 the first term on the r.h.s. of (4.3) equals

\[ -2 \int \Omega (\psi_1 + \bar{\psi}_1) v_1^4 z dx = -4 \int \Omega (\psi_1 + \bar{\psi}_1) v_1^2 \partial_z v_1^2 dx \equiv I_1. \]

Hence,

\[ |I_1| \leq \frac{\epsilon}{2} \|\partial_z v_1^2\|_{L_4(\Omega)}^2 + \frac{8}{\epsilon} \int \Omega (\psi_1^2 + \bar{\psi}_1^2) v_1^2 dx. \]

In view of boundary conditions (1.44)1 and integration by parts the second term on the r.h.s. of (4.3) takes the form

\[ -\frac{3}{2} \int \Omega (H_1 + \bar{H}_1) H' \cdot \nabla v_1^2 v_1 dx \equiv I_2. \]

Hence

\[ |I_2| \leq \frac{\epsilon}{2} \int \Omega |\nabla |v_1|^2|^2 dx + \frac{9}{8\epsilon} \int \Omega (H_1^2 + \bar{H}_1^2) H' |v_1|^2 dx, \]

where the second integral is bounded by

\[ \frac{9}{8\epsilon} (\|H_1\|_{L_4(\Omega)}^2 + \|\bar{H}_1\|_{L_4(\Omega)}^2) \|H'\|_{L_2(\Omega)}^2 \|v_1\|_{L_2(\Omega)}^2. \]

The third term on the r.h.s. of (4.3) is bounded by

\[ 2 \int \Omega |\phi_1 z| (|H_1| + |\bar{H}_1|) |v_1|^3 dx \]

\[ \leq 2 \|v_1\|_{L_2(\Omega)}^3 \|\phi_1 z\|_{L_2(\Omega)} (\|H_1\|_{L_4(\Omega)} + \|\bar{H}_1\|_{L_4(\Omega)}). \]

Applying the Hölder inequality the last term on the r.h.s. of (4.3) is bounded by

\[ \|f_1\|_{L_{4/3}(\Omega)} \|v_1\|_{L_2(\Omega)}^3. \]
Employing the above estimates in (4.3) and using that $\varepsilon = \nu$ we get

\[
\frac{1}{4} \frac{d}{dt} \|v_1\|_{L^4(\Omega)}^4 + 2\nu \|
abla v_1^2\|_{L^2(\Omega)}^2 \leq \frac{8}{\nu} (\|\psi_1\|_{L_\infty(\Omega)}^2 + \|\psi_1\|_{L_\infty(\Omega)}^2) \|v_1\|_{L^4(\Omega)}^4
\]

\[
+ \frac{9}{8\nu} \|H_1\|_{L^4(\Omega)}^2 + \|\tilde{H}_1\|_{L^4(\Omega)}^2 \|H_1\|_{L^4(\Omega)}^2 \|v_1\|_{L^4(\Omega)}^4
\]

\[
+ \|\phi_{1,1}\|_{L^2(\Omega)} \|H_1\|_{L^4(\Omega)} + \|\tilde{H}_1\|_{L^4(\Omega)}
\]

\[
+ \|f_1\|_{L^4(\Omega)} \|v_1\|_{L^4(\Omega)}^3
\]

(4.4)

For solutions to problems (1.47), (1.48) and (3.3) we have the estimates (see Lemma 3.3)

\[
\|\psi_1\|_{L_\infty(\Omega)} \leq c \|\psi_1\|_{H^2(\Omega)} \leq c \|\omega_1\|_{L^2(\Omega)},
\]

\[
\|\tilde{\psi}_1\|_{L_\infty(\Omega)} \leq c \|\tilde{\psi}_1\|_{H^2(\Omega)} \leq c \|\tilde{\omega}_1\|_{L^2(\Omega)},
\]

\[
\|\phi_{1,1}\|_{L^2(\Omega)} \leq c \|j_1\|_{L^2(\Omega)}.
\]

The Poincaré inequality yields

\[
\|v_1\|_{L^4(\Omega)}^4 \leq c \|
abla v_1^2\|_{L^2(\Omega)}^2
\]

(4.6)

so the Sobolev imbedding implies

\[
\|v_1\|_{L^4(\Omega)} \leq c \|v_1\|_{H^1(\Omega)} \leq c \|
abla v_1^2\|_{L^2(\Omega)}.
\]

(4.7)

Finally, (1.46)–(1.48) imply

\[
\|H_1\|_{L^6(\Omega)} \leq c \|H_1\|_{H^1(\Omega)} \leq c \|\phi_{1,1}\|_{H^2(\Omega)} \leq c \|j_1\|_{L^2(\Omega)},
\]

\[
\|v'_1\|_{L^6(\Omega)} \leq c \|v'_1\|_{H^1(\Omega)} \leq c \|\psi_1\|_{H^2(\Omega)} \leq c \|\omega_1\|_{L^2(\Omega)}.
\]

(4.8)

In view of (4.5)–(4.8) we derive from (4.4) the inequality

\[
\frac{d}{dt} \|v_1\|_{L^4(\Omega)}^4 + \nu \|
abla v_1^2\|_{L^2(\Omega)}^2 \leq c(\|\omega_1\|_{L^2(\Omega)}^2 + \|\tilde{\omega}_1\|_{L^2(\Omega)}^2) \|v_1\|_{L^4(\Omega)}^4
\]

\[
+ c \|H_1\|_{L^4(\Omega)} \|j_1\|_{L^2(\Omega)} + \|\tilde{H}_1\|_{L^4(\Omega)} \|j_1\|_{L^2(\Omega)} + c \|f_1\|_{L^4(\Omega)}.
\]

(4.9)

Now we obtain an analogous estimate for $H_1$. Multiplying (1.42) by $H_1 |H_1|^2$, integrating over $\Omega$ and by parts applying the boundary conditions (4.4)2, we derive

\[
\frac{d}{dt} \|H_1\|_{L^4(\Omega)}^4 + \mu \|
abla H_1^2\|_{L^2(\Omega)}^2 \leq \int_\Omega v_1' \cdot \nabla H_1^2 H_1^2 dx + \int_\Omega H_1 \cdot \nabla v_1 H_1^3 dx.
\]

(4.10)

The first term on the r.h.s. of (4.10) is bounded by

\[
\|v'_1\|_{L^4(\Omega)} \|
abla H_1^2\|_{L^2(\Omega)} \|H_1\|_{L^4(\Omega)}^2 \leq \varepsilon \|H_1\|_{L^4(\Omega)}^2 + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^4 \|
abla H_1\|_{L^2(\Omega)}^2,
\]

where (4.8) for $v'$ is used. Integrating by parts in the second term on the r.h.s. of (4.10) yields

\[
\int_\Omega H_1 \cdot \nabla v_1 H_1^3 dx = \frac{3}{2} \int_\Omega H_1 \cdot \nabla H_1^2 H_1 v_1 dx \leq \varepsilon \int_\Omega \|
abla H_1^2\|^2 dx
\]

\[
+ c/\varepsilon \int_\Omega H_1^2 v_1^2 dx,
\]

where the second integral is bounded by

\[
\frac{c}{\varepsilon} \|v_1\|_{L^4(\Omega)} \|H_1\|_{L^4(\Omega)}^2 \|H_1^2\|_{L^4(\Omega)} \leq \varepsilon \|v_1\|_{L^4(\Omega)}^2 + c/\varepsilon \|H_1\|_{L^4(\Omega)}^4 \|j_1\|_{L^2(\Omega)}^4,
\]
where (4.8) was used. Using the above estimates in (4.10) and assuming that \( \varepsilon \) is sufficiently small we get

\[
\frac{d}{dt} \|H_1\|_{L_4(\Omega)}^4 + \mu \|\nabla H_1^2\|_{L_2(\Omega)}^2 \leq c \|\nabla H_1\|_{L_2(\Omega)}^4 \|\omega_1\|_{L_2(\Omega)}^4
\]

+ \varepsilon \|v_1\|_{L_2(\Omega)}^4 + c/\varepsilon \|H_1\|_{L_4(\Omega)}^4 \|j_1\|_{L_2(\Omega)}^4. \tag{4.11}
\]

Adding (4.9) and (4.11), using that \( \varepsilon \) is sufficiently small and employing notation from the assumptions of the lemma we have

\[
\frac{d}{dt} X_1^2 + \nu S_2 \leq c X_1^2 X_2^2 (1 + X_2^2) + c A_1^2 (X_1^2 + X_2^2) + c G_1^2. \tag{4.12}
\]

This proves Lemma 4.1. \( \square \)

Next, we shall obtain an inequality with \( X_2^3 \) under the time derivative.

**Lemma 4.2.** Assume that the following quantities are finite

\[
X_3^2 = \|\omega_1\|_{L_4(\Omega)}^2 + \|v_1\|_{L_2(\Omega)}^2 + \|H_1\|_{L_2(\Omega)}^2,
\]

\[
Y_2^2 = \|\omega_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2,
\]

\[
A_2^2 = \|\omega_1\|_{L_4(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2,
\]

\[
G_2^2 = \|f_1\|_{L_6(\Omega)}^2 + \|F_1\|_{L_{6/5}(\Omega)}^2.
\]

Then the following differential inequality holds

\[
\frac{d}{dt} X_3^2 + \nu \frac{d}{dt} Y_2^2 \leq \varepsilon \|j_1\|_{H^1(\Omega)}^2 + c / \varepsilon \|j_1\|_{L_2(\Omega)}^6 + c (A_2^2 + A_2^2) X_3^2 + c (X_3^2 + (Y_2^2 + A_2^2) \|j_1\|_{L_2(\Omega)} + G_2^2). \tag{4.13}
\]

**Proof.** Multiplying (1.41) by \( \omega_1 \), integrating the result over \( \Omega \) and using the boundary conditions (1.44), we have

\[
\frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L_2(\Omega)}^2 + \nu \|\nabla \omega_1\|_{L_2(\Omega)}^2 + \nu \int_{-a}^a \omega_1^2 |z = 0| dz
\]

\[
= - \int_{\Omega} \nu' \cdot \nabla \omega_1^2 \omega_1 dx + \int_{\Omega} (v_1^2)_z \omega_1 dx + \int_{\Omega} H' \cdot \nabla j_1 \omega_1 dx \tag{14.4}
\]

\[
- 2 \int_{\Omega} (H_1 H_1, z + H_1 H_1, z + H_1 H_1, z) \omega_1 dx + \int_{\Omega} F_1 \omega_1 dx.
\]

Integrating by parts and using the boundary conditions (1.44) in the first term on the r.h.s. of (14.4), it equals

\[
\int_{\Omega} \nu' \cdot \nabla \omega_1^2 \omega_1 dx \equiv I_3,
\]

so

\[
|I_3| \leq \varepsilon \int_{\Omega} |\nu' \omega_1^2| dx + c / \varepsilon \int_{\Omega} \nu^2 \omega_1^2 dx.
\]

Integrating by parts in the second term on the r.h.s. of (14.4) we have

\[
- \int_{\Omega} \nu^2 \omega_1^2 dx
\]

which is bounded by

\[
\varepsilon \|\omega_1\|_{L_2(\Omega)}^2 + c / \varepsilon \|v_1\|_{L_4(\Omega)}^4.
\]
Similarly, integrating by parts in the third term on the r.h.s. of (4.14) yields

$$- \int_{\Omega} H' \cdot \nabla \omega_1 j_1 dx$$

which is bounded by

$$\varepsilon \| \nabla \omega_1 \|_{L^2(\Omega)}^2 + c/\varepsilon \int_{\Omega} H'^2 j_1^2 dx.$$

The last but one term on the r.h.s. of (4.14) equals

$$- \int_{\Omega} (H_1^2)_{,x} \omega_1 dx - 2 \int_{\Omega} (H_1 H_1')_{,x} \omega_1 dx = \int_{\Omega} H_1^2 \omega_1 dx + 2 \int_{\Omega} H_1 H_1' \omega_1 dx$$

which is estimated by

$$\varepsilon \| \omega_1, x \|_{L^2(\Omega)}^2 + c/\varepsilon \left( \| H_1 \|_{L^4(\Omega)}^4 + \int_{\Omega} H_1^2 \tilde{H}_1^2 dx \right).$$

Finally, the last term on the r.h.s. of (4.14) is estimated by

$$\varepsilon \| \omega_1 \|_{L^2(\Omega)}^2 + c/\varepsilon \| F_1 \|_{L^2_{w, s}(\Omega)}^2.$$

Employing the above estimates in (4.14) and using that $\varepsilon$ is sufficiently small yields

$$\frac{d}{dt} \| \omega_1 \|_{L^2(\Omega)}^2 + \nu \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq c \int_{\Omega} v'^2 w_1^2 dx + c \| \omega_1 \|_{L^4(\Omega)}^4 + c \int_{\Omega} H_1^2 \tilde{H}_1^2 dx + c \| F_1 \|_{L^2_{w, s}(\Omega)}^2 \tag{4.15}$$

Now we estimate the terms from the r.h.s. of (4.15). The first term on the r.h.s. of (4.15) is bounded by

$$\| v' \|_{L^2(\Omega)}^2 \omega_1 \|_{L^2(\Omega)}^2 \leq c \| \omega_1 \|_{L^2(\Omega)}^2 \| \omega_1 \|_{L^2(\Omega)}^2,$$

where (1.46) and estimate (4.8)$_2$ for solutions to (1.47) are used. The last but one term on the r.h.s. of (4.15) is bounded by

$$\left| \int_{\Omega} H_1 H_1' \tilde{H}_1' dx \right| \leq \| H_1 \|_{L^4(\Omega)} \| H_1 \|_{L^2(\Omega)} \| \tilde{H}_1 \|_{L^2(\Omega)}^2 \leq c \| H_1 \|_{H^1(\Omega)}^2 + c/\varepsilon \| H_1 \|_{H^1(\Omega)}^2 \| H_1 \|_{L^2(\Omega)}^2$$

and the third term on the r.h.s. of (4.15) by

$$\| H' \|_{L^2(\Omega)}^2 \| j_1 \|_{L^2(\Omega)} \leq c \| j_1 \|_{L^4(\Omega)} \| j_1 \|_{L^4(\Omega)} \| j_1 \|_{L^2(\Omega)} \leq c \| j_1 \|_{L^2(\Omega)}^2 + c/\varepsilon \| j_1 \|_{L^2(\Omega)}^2.$$

Employing the above estimates in (4.15) we derive the inequality

$$\frac{d}{dt} \| \omega_1 \|_{L^2(\Omega)}^2 + \nu \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq \varepsilon \| j_1 \|_{L^2(\Omega)}^2 + c/\varepsilon \| j_1 \|_{L^2(\Omega)}^2 \leq c \| \omega_1 \|_{L^2(\Omega)}^2 \| \omega_1 \|_{L^2(\Omega)}^2$$

which appears we need additional differential inequality. For this purpose we multiply (1.42) by $H_1$ and integrate the
result over $\Omega$. Then we get
\[
\frac{1}{2} \frac{d}{dt} \|H_1\|_{L^2(\Omega)}^2 + \mu \|H_1\|_{H^1(\Omega)}^2 \leq - \int_{\Omega} v' \cdot \nabla H_1 H_1 dx + \int_{\Omega} H' \cdot \nabla v_1 H_1 dx
\]
\[
= \int_{\Omega} v' \cdot \nabla H_1 H_1^2 dx - \int_{\Omega} H' \cdot \nabla H_1 v_1 dx.
\]
Applying the Hölder and Young inequalities to the r.h.s. of the above inequality gives
\[
\frac{d}{dt} \|H_1\|_{L^2(\Omega)}^2 + \mu \|H_1\|_{H^1(\Omega)}^2 \leq c \|H_1\|_{H^1(\Omega)}^2 \|\omega_1\|_{L^2(\Omega)}^2 + c \|j\|_{L^2(\Omega)}^2 \|v_1\|_{H^1(\Omega)}^2. \quad (4.17)
\]
Multiplying (1.40) by $v_1$ and integrating over $\Omega$ yields
\[
\frac{1}{2} \frac{d}{dt} \|v_1\|_{L^2(\Omega)}^2 + \nu \|\nabla v_1\|_{L^2(\Omega)}^2 = 2 \int_{\Omega} \psi_{1,z} v_1^2 dx + 2 \int_{\Omega} \psi_{1,z} \phi_1^2 dx
\]
\[
+ \int_{\Omega} H' \cdot \nabla H_1 v_1 dx + \int_{\Omega} H' \cdot \nabla \phi_1^2 H_1 v_1 dx - 2 \int_{\Omega} \phi_{1,z} H_1 v_1 dx \quad (4.18)
\]
\[-2 \int_{\Omega} \phi_{1,z} H_1 v_1 dx + \int_{\Omega} f_1 v_1 dx.
\]
Now, we estimate the particular terms from the r.h.s. of (4.18). The first two terms are bounded by
\[
2 \|v_1\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} (\|\psi_{1,z}\|_{L^3(\Omega)} + \|\phi_{1,z}\|_{L^3(\Omega)})
\]
\[
\leq \varepsilon \|v_1\|_{H^1(\Omega)} + c/\varepsilon (\|\omega_1\|_{L^2(\Omega)}^2 + \|\phi_1\|_{L^2(\Omega)}^2) \|v_1\|_{L^2(\Omega)}^2.
\]
The third and fourth terms are bounded by
\[
\|v_1\|_{L^2(\Omega)} \|H'\|_{L^3(\Omega)} (\|\nabla H_1\|_{L^2(\Omega)} + \|\nabla \phi_1\|_{L^2(\Omega)})
\]
\[
\leq \varepsilon \|v_1\|_{H^1(\Omega)}^2 + c/\varepsilon \|j_2\|_{L^2(\Omega)}^2 (\|H_1\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2).
\]
The fifth and sixth terms are estimated by
\[
\|v_1\|_{L^2(\Omega)} \|\phi_{1,z}\|_{L^3(\Omega)} (\|H_1\|_{L^2(\Omega)} + \|\phi_1\|_{L^2(\Omega)})
\]
\[
\leq \varepsilon \|v_1\|_{H^1(\Omega)}^2 + c/\varepsilon \|j_1\|_{L^2(\Omega)}^2 (\|H_1\|_{L^2(\Omega)}^2 + \|\phi_1\|_{L^2(\Omega)}).
\]
Finally, the last term is bounded by
\[
\varepsilon \|v_1\|_{L^2(\Omega)}^2 + c/\varepsilon \|f_1\|_{L^2(\Omega)}^2.
\]
Employing the above estimates in (4.18) and using that $\varepsilon$ is sufficiently small we get
\[
\frac{d}{dt} \|v_1\|_{L^2(\Omega)}^2 + \nu \|v_1\|_{H^1(\Omega)}^2 \leq c (\|\omega_1\|_{L^2(\Omega)}^2 + \|\phi_1\|_{L^2(\Omega)}^2) \|v_1\|_{L^2(\Omega)}^2
\]
\[
+ c (\|H_1\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2) \|j\|_{L^2(\Omega)}^2 + c \|f_1\|_{L^2(\Omega)}^2. \quad (4.19)
\]
From (4.16), (4.17) and (4.19) we have
\[
\frac{d}{dt}(\|\omega_1\|_{L^2(\Omega)}^2 + \|v_1\|_{L^2(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2) + \nu_\epsilon(\|\omega_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^1(\Omega)}^2)
+ \|H_1\|_{H^1(\Omega)}^2 \leq \epsilon \|j_1\|_{L^2(\Omega)}^2 + c/\epsilon \|j_1\|_{L^2(\Omega)}^6 + c\|\omega_1\|_{L^2(\Omega)}^2 \|\omega_1\|_{L^2(\Omega)}^2
+ \epsilon \|\omega_1\|_{L^2(\Omega)}^2 \|\omega_1\|_{L^2(\Omega)}^2 \|\omega_1\|_{L^2(\Omega)}^2 + c\|j_1\|_{L^2(\Omega)}^2 \|v_1\|_{H^1(\Omega)}^2) \tag{4.20}
\]
\[
+ c(\|\omega_1\|_{L^2(\Omega)}^2 + \|\omega_1\|_{L^2(\Omega)}^2)\|v_1\|_{L^2(\Omega)}^2 + c(\|H_1\|_{H^1(\Omega)}^2)
+ \|H_1\|_{H^1(\Omega)}^2 \|j_1\|_{L^2(\Omega)}^2 + c(\|F_1\|_{L^2_{0,0}(\Omega)}^2 + \|j_1\|_{L^2(\Omega)}^2)\).
\]
In view of assumptions of this lemma inequality (4.20) takes the form of (4.13). This concludes the proof.

From (4.16) and the above inequality we obtain (4.13). This concludes the proof of Lemma 4.2.

**Lemma 4.3.** Assume that the following quantities are finite
\[
X_j^2 = \|\omega_1\|_{L^2(\Omega)}^2 + \|j_1\|_{L^2(\Omega)}^2,
Y_j^2(t) = \|\omega_1(t)\|_{H^1(\Omega)}^2 + \|j_1(t)\|_{H^1(\Omega)}^2,
A_j^2(t) = \|\omega_1(0)\|_{H^1(\Omega)}^2.
\]
Assume that the quantities introduced in Lemmas 4.1 and 4.2 are also finite.

Then
\[
\frac{d}{dt}\|j_1\|_{L^2(\Omega)}^2 + \mu\|j_1\|_{H^1(\Omega)}^2 \leq \epsilon \|\omega_1\|_{H^1(\Omega)}^2 + c/\epsilon(\|X_j^2 + X_j^4 + X_j^8\|X_j^2) + c(A_j^2 + A_j^4)X_j^2. \tag{4.21}
\]

**Proof.** Multiplying (1.43) by \(j_1\), integrating the result over \(\Omega\) and by parts, using the boundary conditions (1.44), yield
\[
\frac{d}{dt}\|j_1\|_{L^2(\Omega)}^2 + \mu\|\nabla j_1\|_{L^2(\Omega)}^2 + \mu \int_{-a}^{a} j_1^2_{|r=0}dz = \int_{\Omega} (\psi_{1,z} + \psi_{1,z})J_1^2dx
+ \int_{\Omega} (H' \cdot \nabla \omega_1 + H' \cdot \nabla \omega_1)j_1 dx - \int_{\Omega} \phi z J_1^2dx
+ \int_{\Omega} \frac{1}{r}(v_{r,r} - v_{z,z} + v_{r,z} - \overline{v}_{r,z})(H_{r,r} + H_{z,r})j_1 dx
+ \int_{\Omega} \frac{1}{r}(H_{z,z} - H_{r,r})(v_{r,z} + v_{z,r} + \overline{v}_{r,z} + \overline{v}_{z,r})j_1 dx. \tag{4.22}
\]
Now, we estimate the particular terms from the r.h.s. of (4.22). The first term on the r.h.s. of (4.22) can be expressed in the form
\[
-2\int_{\Omega} (\psi_1 + \psi_1)j_1 j_{1,z}dx
\]
which is bounded by
\[
\epsilon \|j_1\|_{L^2(\Omega)}^2 + c/\epsilon \int_{\Omega} (\psi_1^2 + \psi_1^2)j_1^2dx.
\]
The Hölder inequality and Lemma 3.3 applied to the second term yield the bound
\[
(\|\psi_1\|_{L^\infty(\Omega)}^2 + \|\psi_1\|_{L^2(\Omega)}^2)\|j_1\|_{L^2(\Omega)}^2 \leq c(\|\omega_1\|_{L^2(\Omega)}^2 + \|\omega_1\|_{L^2(\Omega)}^2)\|j_1\|_{L^2(\Omega)}^2.
\]
Similarly, the third term on the r.h.s. of (4.22) can be estimated by
\[ \varepsilon \| j_{1,z} \|^2_{L^2(\Omega)} + c/\varepsilon \| j_1 \|^4_{L^2(\Omega)}. \]
Integrating by parts the second term on the r.h.s. of (4.22) equals
\[ - \int_\Omega (H' \cdot \nabla j_1 \omega_1 + H' \cdot \nabla j_1 \omega_1^2) dx \]
which is estimated by
\[ \varepsilon \| \nabla' j_1 \|^2_{L^2(\Omega)} + c/\varepsilon \int_\Omega (H'^2 \omega_1^4 + H'^2 \omega_1^2) dx. \]
Applying the Hölder inequality the second integral is bounded by
\[ c (\| H' \|^2_{L^2(\Omega)} \| \omega_1 \|^2_{L^4(\Omega)} + \| H' \|^2_{L^2(\Omega)} \| \omega_1 \|^2_{L^4(\Omega)}) \equiv I_1. \]
In view of (1.46), (1.48) and Remark 3 we have
\[ I_1 \leq c \| j_1 \|^2_{L^2(\Omega)} (\| \omega_1 \|^2_{L^4(\Omega)} + \| \omega_1 \|^2_{L^4(\Omega)}) \equiv I_2. \]
Employing the interpolation
\[ \| u \|_{L^2(\Omega)} \leq c \| u \|^{1/2}_{L^6(\Omega)} \| u \|^{1/2}_{L^4(\Omega)} \]
we have
\[ I_2 \leq \varepsilon \| \omega_1 \|^2_{L^6(\Omega)} + c/\varepsilon \| j_1 \|^4_{L^6(\Omega)} \| \omega_1 \|^2_{L^4(\Omega)} + \| \omega_1 \|^2_{L^4(\Omega)} \| j_1 \|^2_{L^6(\Omega)}. \]
Next we estimate the fourth term on the r.h.s. of (4.22). For simplicity, we first consider the following part of this term
\[ J_1 = \int_\Omega \frac{1}{r} (v_{r,r} - v_{z,z}) (H_{r,z} + H_{z,r}) j_1 dx. \]
To examine \( J_1 \) we use the potential \( \psi_1 \) and \( \phi_1 \) introduced in (1.46). Then we have
\[ J_1 = \int_\Omega \frac{1}{r} \left( - (r \psi_1)_{,rz} - \frac{1}{r} (r^2 \psi_1),r \right) \left( - (r \phi_1)_{,zz} + \left( \frac{1}{r} (r^2 \phi_1),r \right) \right) j_1 dx \]
\[ = \int_\Omega \frac{1}{r} (-3 \psi_1, z - 2 \phi_1, r z) (-r \phi_1, z z + r \phi_1, r r + 3 \phi_1, r) j_1 dx \]
\[ = - \int_\Omega \frac{9}{r} \psi_1, z \phi_1, r + 3 \psi_1, z (- \phi_1, z z + \phi_1, r r) + 6 \phi_1, r \psi_1, r z \]
\[ + 2 r \psi_1, r z (- \phi_1, z z + \phi_1, r r) \]
Now, we examine the particular terms in \( J_1 \). The first term is bounded by
\[ \left| \int_\Omega \frac{1}{r} \frac{1}{r^{1-\alpha}} \psi_1, z \phi_1, r j_1 dx \right| \leq \left( \int_\Omega \int_0^R \frac{1}{r^{12\alpha}} r dr dz \right)^{1/12} \left| \frac{1}{r^{1-\alpha}} \psi_1, z \right|_{L^2(\Omega)} \cdot \| \phi_1, r \|_{L^2(\Omega)} \| j_1 \|_{L^4(\Omega)} = J_2. \]
To have the first factor bounded we need that \( \alpha < \frac{1}{6} \). Applying the Hardy inequality to the second factor and also Lemmas 2.2, 3.3 and Remark 3 yield
\[ J_2 \leq c \| \phi_1, r \|_{L^2(\Omega)} \| \phi_1, r \|_{L^6(\Omega)} \| j_1 \|_{L^4(\Omega)} \leq c \| \omega_1 \|_{L^4(\Omega)} \| j_1 \|_{L^4(\Omega)} \cdot \| j_1 \|^{3/4}_{L^6(\Omega)} \| j_1 \|^{1/4}_{L^2(\Omega)} \leq \varepsilon \| j_1 \|^2_{L^2(\Omega)} + c/\varepsilon \| \omega_1 \|^3_{L^2(\Omega)} \| j_1 \|^2_{L^2(\Omega)}. \]
The second term in $J_1$ is bounded by

$$3\psi_{1,\tau}L_2(\Omega)(\phi_{1,zz}\|L_2(\Omega) + \phi_{1,rr}\|L_2(\Omega))\|j_1\|L_2(\Omega)$$

$$\leq c\|\omega_1\|L_2(\Omega)\|j_1\|L_2(\Omega)\|j_1\|L_2(\Omega)/\|j_1\|L_2(\Omega)$$

$$\leq c\|\phi_{1}\|L_2(\Omega)\|j_1\|L_2(\Omega)\|j_1\|L_2(\Omega) + c/\|\omega_1\|L_2(\Omega)^{1/3}\|j_1\|L_2(\Omega)^2,$$

where Lemmas 2.2, 3.3 and Remark 3 were again used. Similarly, the third term in $J_1$ is estimated by

$$6\|\phi_{1,rr}\|L_2(\Omega)\|\psi_{1,rr}\|L_2(\Omega)\|j_1\|L_2(\Omega) \leq c/\|\omega_1\|L_2(\Omega)^{1/3}\|j_1\|L_2(\Omega)^2.$$

Finally the last term in $J_1$ is expressed in the form

$$2\int_\Omega \psi_{1,\tau}((-\phi_{,zzz} + \phi_{,rrr} - 2\phi_{1,rr})j_1 + (-\phi_{,zz} + \phi_{,rr} - 2\phi_{1,r})j_1) dx \equiv J_3.$$

Continuing, we have

$$|J_3| \leq \|\psi_{1,\tau}\|L_2(\Omega)(\|\phi\|H^2(\Omega) + \|\phi_1\|H^2(\Omega))\|j_1\|L_2(\Omega)$$

$$+ \|\psi_{1,\tau}\|L_2(\Omega)(\|\phi_{1,zz}\|L_2(\Omega) + \|\phi_{1,rr}\|L_2(\Omega) + \|\phi_{1,\tau}\|L_2(\Omega))\|j_1\|L_2(\Omega)$$

$$\leq \varepsilon_1\|\phi\|H^3(\Omega) + \|\phi_1\|H^3(\Omega) + c/\varepsilon_1\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega) + \varepsilon_2\|j_1\|L_2(\Omega)$$

$$+ c/\varepsilon_1\|\omega_1\|L_2(\Omega)^2\|\phi_{1,zz}\|L_2(\Omega) + \|\phi_{1,rr}\|L_2(\Omega) + \|\phi_{1,\tau}\|L_2(\Omega))$$

$$\equiv J_4,$$

where Lemma 3.3 is used. Applying interpolation (2.1) the second term in $J_4$ is bounded by

$$c/\varepsilon_1\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega) \leq \varepsilon_3\|\phi_{1,\tau}\|L_2(\Omega)^2 + c/\varepsilon_1\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega)^2.$$

In view of (2.2) we estimate the fourth term in $J_4$ by

$$c/\varepsilon_2\|\omega_1\|L_2(\Omega)^2\|\phi\|H^3(\Omega) + \|\phi_1\|H^3(\Omega)$$

$$\leq \varepsilon_4\|\phi\|H^4(\Omega) + c/\varepsilon_2\|\omega_1\|L_2(\Omega)^2\|\phi\|H^3(\Omega) + c/\varepsilon_2\|\omega_1\|L_2(\Omega)^2\|\phi_1\|H^3(\Omega)$$

$$\leq \varepsilon_4\|\phi\|H^4(\Omega) + c/\varepsilon_2\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega)^2 + c/\varepsilon_2\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega)^2$$

where Lemma 3.3 is used. In view of the above estimates we derive

$$J_4 \leq c/\|\phi\|H^4(\Omega) + c/\|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega)^2 = J_5.$$

Summarizing, we get

$$|J_1| \leq \varepsilon\|\phi\|H^4(\Omega) + c/\varepsilon(\|\omega_1\|L_2(\Omega)^{8/5} + \|\omega_1\|L_2(\Omega)^{4/3} + \|\omega_1\|L_2(\Omega)^2\|j_1\|L_2(\Omega)^2).$$

(4.23)

Next, we consider the following part of the fourth term from the r.h.s. of (4.22)

$$J_6 = \int_\Omega \frac{1}{r}(\nabla_{r,\tau} - \nabla_{z,\tau})(H_{r,zz} + H_{z,rr})j_1 dx.$$

Using the potential $\psi_1$ and $\phi_1$ the integral $J_6$ takes the form

$$J_6 = -\int_\Omega \frac{9}{r^2}\psi_{1,zz}\phi_{1,r} + \frac{2}{r}\psi_{1,zz}\phi_{1,rr} + 6\phi_{1,rr}\phi_{1,r}$$

$$+ 2r\psi_{1,zz}\phi_{1,rr} - \phi_{1,zz}\phi_{1,rr} + \phi_{1,rr}r dx.$$
Comparing $J_6$ with $J_7$ we see that $\psi_1$ is replaced by $\psi_2$. Therefore, instead of (4.23) we derive the estimate

$$|J_6| \leq \varepsilon \|j_1\|_{H^4(\Omega)}^2 + c/\varepsilon(\|\omega_1\|_{L^2(\Omega)}^{8/5} + \|\omega_1\|_{L^2(\Omega)}^{4/3} + \|\omega_1\|_{L^2(\Omega)}^{1/3})^2 1_{H^1(\Omega)}.$$

Finally, we examine the last term on the r.h.s. of (4.22). Employing the potential $\psi_1$ and $\phi_1$ introduced in (1.46) it takes the form

$$K = \int_{\Omega} \left( \frac{1}{r} (r^2 \phi_1)_r + (r \phi_1)_r \right) \{ - (r \psi_1)_z + \left( \frac{1}{r^2} (r^2 \psi_1)_r \right)_r \} dx \equiv K_1 + K_2,$$

where

$$K_1 = \int_{\Omega} \left[ \frac{9}{r} \phi_1 z \psi_1, r + 6 \phi_1, rz \psi_1, r + 3 \phi_1, z (-\psi_1, zz + \psi_1, rz) + 2r \phi_1, rz (-\psi_1, zz + \psi_1, rz) \right] dx.$$

We estimate the first term in $K_1$ in a similar way as the first term in $J_1$ was estimated. Denoting it by $K_3$, we have

$$|K_3| \leq c \|\psi_1, rr\|_{L^2(\Omega)} \|\phi_1, z\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)} 1_{H^1(\Omega)} \leq \varepsilon \|j_1\|_{L^2(\Omega)}^2 + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{8/5} \|j_1\|_{L^2(\Omega)}^2.$$  

The second term in $K_1$ is bounded by

$$6 \|\phi_1, rz\|_{L^2(\Omega)} \|\psi_1, r\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)} \leq \varepsilon \|j_1\|_{L^2(\Omega)}^2 + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{4/3} \|j_1\|_{L^2(\Omega)}^2.$$  

and the third term in $K_1$ by

$$3 \|\phi_1, z\|_{L^2(\Omega)} (\|\psi_1, zz\|_{L^2(\Omega)} + \|\psi_1, rr\|_{L^2(\Omega)}) \|j_1\|_{L^2(\Omega)} \leq \varepsilon \|j_1\|_{L^2(\Omega)}^2 + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{4/3} \|j_1\|_{L^2(\Omega)}^2.$$  

Finally, the last term in $K_1$ can be expressed in the form

$$2 \int_{\Omega} r \phi_1, r (j_1, z) (\psi_1, zz - \psi_1, r r) j_1, z \| dx \equiv K_4.$$

Since in view of Lemma 3.3 the following estimate

$$\|\psi_1\|_{H^3(\Omega)} \leq c\|\omega_1\|_{H^1(\Omega)}$$

does not hold, we express $K_4$ in the form

$$K_4 = 2 \int_{\Omega} \phi_1, r (\psi_1, zz - \psi_1, rr) j_1 dx + 4 \int_{\Omega} \phi_1, r \psi_1, rz j_1 dx + 2 \int_{\Omega} r \phi_1, r (\psi_1, zz - \psi_1, rr) j_1, z dx \equiv K_4^1 + K_4^2 + K_4^3.$$  

Now, we estimate the particular integrals in $K_4$. First

$$|K_4^1| \leq c \|\phi_1, r\|_{L^2(\Omega)} \|\psi_1, zz\|_{L^2(\Omega)} + \|\psi_1, rr\|_{L^2(\Omega)}) \|j_1\|_{L^2(\Omega)} \leq \varepsilon \|j_1\|_{H^3(\Omega)}^2 + c/\varepsilon \|\phi_1, r, rz\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)}^2 = K_4^4.$$  

Using that $\Omega$ is bounded the first term in $K_4^3$ is estimated by

$$c\|\omega_1\|_{H^1(\Omega)}^2 \leq c\|\omega_1\|_{H^1(\Omega)}^2.$$  

(4.25)
The second term in $K^4$, in view of (2.1), the Sobolev imbedding and Lemma 3.3, is bounded by
\[ c\|j_1\|_{L^2(\Omega)}\|j_1\|_{L^6(\Omega)}\|j_1\|_{L^2(\Omega)} \leq \varepsilon\|\psi_1\|_{H^1(\Omega)} + c/\varepsilon\|j_1\|_{L^2(\Omega)}. \]
Summarizing, we have
\[ |K^4_1| \leq \varepsilon(\|\omega_1\|_{H^1(\Omega)} + \|j_1\|_{H^1(\Omega)}) + c/\varepsilon\|j_1\|_{L^2(\Omega)}. \]
Similarly, we derive
\[ |K^4_2| \leq c\|\phi_{1,r}\|_{L^6(\Omega)}\|\psi_1,zz - \psi_1,rr\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)} \leq \varepsilon\|j_1\|_{L^2(\Omega)}\|\omega_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)} \leq \varepsilon\|j_1\|_{H^1(\Omega)} + c/\varepsilon\|\omega_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)}. \]
Finally,
\[ |K^4_3| \leq 2 \int_\Omega |\phi_{1,r} - \phi_{1}| |\psi_1,zz - \psi_1,rr| |j_1| dx \leq \varepsilon\|j_1\|_{L^2(\Omega)} + c/\varepsilon\|\psi_1\|_{H^2(\Omega)}(\|\phi_{1,r}\|_{L^2(\Omega)} + \|\phi_1\|_{L^\infty(\Omega)}) \equiv K^5_4. \]
Applying (2.2) and Lemma 3.3 the second term in $K^4$ is bounded by
\[ c\|\omega_1\|_{L^2(\Omega)}\|\phi_{1,zz}\|_{H^1(\Omega)} + c\|\omega_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)} \leq \varepsilon\|j_1\|_{H^1(\Omega)} + c/\varepsilon\|\omega_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)} \leq \varepsilon\|j_1\|_{H^1(\Omega)} + c/\varepsilon\|\omega_1\|_{L^2(\Omega)}\|j_1\|_{L^2(\Omega)}. \]
where (4.25) for $j$ is used. Summarizing the above estimates we arrive to the following bound
\[ |K_4| \leq \varepsilon(\|\omega_1\|_{H^1(\Omega)} + \|j_1\|_{H^1(\Omega)}) + c/\varepsilon(\|\omega_1\|_{L^2(\Omega)} + \|j_1\|_{L^2(\Omega)}) \leq \varepsilon\|\omega_1\|_{L^2(\Omega)} + \|j_1\|_{L^2(\Omega)} + \|\omega_1\|_{L^2(\Omega)}^4 + \|j_1\|_{L^2(\Omega)}^4. \]
Collecting all estimates for $K_1$ we get
\[ |K_1| \leq \varepsilon\|\omega_1\|_{H^1(\Omega)} + \|j_1\|_{H^1(\Omega)}^2 + c/\varepsilon(\|\omega_1\|_{L^2(\Omega)} + \|j_1\|_{L^2(\Omega)}^4). \]
To obtain an estimate for $K_2$ we recall that $\psi_1$ in $K_1$ is replaced by $\psi_1$. Then we express $K_2$ in the form
\[ K_2 = \int_\Omega \left[ \frac{9}{4} \phi_{1,zz}^2 + 6 \phi_{1,rr} \psi_1,rr + 3 \phi_{1,zz} (\psi_1,zz + \psi_1,rr) \right] j_1 dx. \]
Hence, the first term in $K_2$ is bounded by
\[ \varepsilon\|j_1\|_{L^4(\Omega)} + c/\varepsilon\|\omega_1\|_{L^2(\Omega)}^2 \|j_1\|_{L^2(\Omega)}^2, \]
the second and the third by
\[ \varepsilon\|j_1\|_{L^4(\Omega)} + c/\varepsilon\|\omega_1\|_{L^2(\Omega)}^2 \|j_1\|_{L^2(\Omega)}^2. \]
The last term in $K_2$, denoted by $K_5$, has the following similar to $K_4$ form,

$$K_5 = 2 \int_\Omega \phi_{1r} \frac{2}{\phi_{1r}} (\psi_{zzz} - \psi_{rrz}) j_1 dx + 4 \int_\Omega \phi_{1r} \psi_{rzz} j_1 dx + 2 \int_\Omega \phi_{1r} (\psi_{zzz} - \psi_{rr}) j_1, dx \equiv K_5^1 + K_5^2 + K_5^3,$$

where

$$|K_5^1| \leq c \|\phi_{1r}\|_{L^2(\Omega)} \left( \frac{2}{\phi_{1r}} \right) \|\psi\|_{H^1(\Omega)} \|j_1\|_{L^2(\Omega)} \frac{1}{\|j_1\|_{L^2(\Omega)}},$$

$$|K_5^2| \leq c \|\phi_{1r}\|_{L^2(\Omega)} \|\psi_{rzz}\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)},$$

$$|K_5^3| \leq c \|\phi_{1r}\|_{L^2(\Omega)} \|\psi_{zzz}\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)}.$$

Summarizing, we have

$$|K_2| \leq \varepsilon \|j_1\|_{H^1(\Omega)} + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{8/5} + \|\omega_1\|_{H^1(\Omega)}^{4/3} + \|\omega_1\|_{L^2(\Omega)}^{2} \|j_1\|_{L^2(\Omega)}.$$

Finally, we use all the above estimates in the r.h.s. of (4.22). Assuming that $\varepsilon$ is sufficiently small we get

$$\frac{d}{dt} \|j_1\|_{L^2(\Omega)} + \mu \|j_1\|_{H^1(\Omega)} \leq \varepsilon \|\omega_1\|_{H^1(\Omega)} + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{8/5} + \|\omega_1\|_{H^1(\Omega)}^{4/3} + \|\omega_1\|_{L^2(\Omega)} \|j_1\|_{L^2(\Omega)}$$

(4.28)

where we used the estimates

$$\|\omega_1\|_{L^2(\Omega)}^{8/5} \leq \varepsilon + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{4/3}, \quad \|\omega_1\|_{L^2(\Omega)}^{4/3} \leq \varepsilon + c/\varepsilon \|\omega_1\|_{L^2(\Omega)}^{2}.$$

This concludes the proof of Lemma 4.3. \qed

**Proof of Proposition 1.** Let us introduce the quantities

$$\hat{X}_1^2 = \|v_1\|_{L^2(\Omega)}^2 + \|H_1\|_{L^2(\Omega)}^2 + \|\omega_1\|_{L^2(\Omega)}^2 + \|j_1\|_{L^2(\Omega)}^2,$$

$$\hat{Y}_1^2 = \|v_1\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2 + \|\omega_1\|_{H^1(\Omega)}^2 + \|j_1\|_{H^1(\Omega)}^2,$$

$$\hat{A}_1^2 = \|H_1\|_{H^1(\Omega)}^2 + \|\omega_1\|_{H^1(\Omega)}^2,$$

$$\hat{G}_1^2 = \|f_1\|_{W_0^1(\Omega)}^2 + \|F_1\|_{L^2(\Omega)}^2.$$

From (4.13) and (4.21) it follows

$$\frac{d}{dt} \hat{X}_1^2 + \nu \hat{Y}_1^2 \leq c(\hat{X}_1^2 + \hat{X}_1^4 + \hat{X}_1^8 + \hat{Y}_1^2 + \hat{A}_1^2) \hat{X}_1^2 + c\hat{G}_1^2.$$

(4.30)

In view of notation (4.1) we obtain from (4.30) and Lemma 4.1 inequality (4.2). This concludes the proof of Proposition 1. \qed

Finally, we prove
Proposition 2. Let $k \in \mathbb{N}_0$. Let $X, A, G$ be defined in (4.1). Let $A_0$ be a finite constant. Assume that $\sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} c_0 A^2(t) dt \leq A_0^2 < \infty$

$$X^2(0) \leq \gamma, \quad c_0 G^2(t) \leq \frac{c_*}{2} \gamma, \quad t \in \mathbb{R}_+,$$

where $\gamma \in (0, \gamma_*]$, where $\gamma_*$ is so small that

$$\nu_* / 2 - c_0 (\gamma_* + \gamma_*^2 + \gamma_*^4 + \gamma_*^8) \geq c_*, \quad c_* \in (0, \nu_*)/2), \quad c_0 \gamma_* \leq \nu_*/2,$$

and

$$\frac{c_*}{2} T \geq A_0^2, \quad c_0 \int_{kT}^{(k+1)T} G^2(t) dt \leq \alpha \gamma, \quad \alpha \exp(A_0^2) + \exp \left(-\frac{c_*}{2} T \right) \leq 1.$$

Then

$$X^2(t) \leq \gamma \quad \text{for any} \quad t \in \mathbb{R}_+.$$

Proof. We express (4.2) in the form

$$\frac{d}{dt} X^2 \leq -X^2 (\nu_* / 2 - c_0 (\gamma_* + \gamma_*^2 + \gamma_*^4 + \gamma_*^8)) + c_0 A^2 X^2 + c_0 G^2. \quad (4.31)$$

Since the coefficients of (4.29) depend on the special solution determined step by step in time, we consider (4.29) in the time interval $[kT, (k+1)T], \, k \in \mathbb{N}_0$. Assuming that

$$X^2(kT) \leq \gamma, \quad c_0 G^2(t) \leq \frac{c_*}{2} \gamma \quad \text{for} \quad t \in [kT, (k+1)T] \quad (4.32)$$

and introducing the quantity

$$Z^2(t) = \exp \left(-\int_{kT}^{t} c_0 A^2(t') dt' \right) X^2(t), \quad t \in [kT, (k+1)T]$$

we obtain from (4.31) the inequality

$$\frac{d}{dt} Z^2 \leq -c_* Z^2 + c_0 G^2, \quad (4.33)$$

where $\bar{G}^2(t) = G^2(t) \exp \left(-\int_{kT}^{t} c_0 A^2(t') dt' \right)$. Suppose that

$$t_* = \inf \left\{ t \in (kT, (k+1)T] : X^2(t) > \gamma \right\}$$

$$= \inf \left\{ t \in (kT, (k+1)T] : Z^2(t) > \gamma \exp \left(-\int_{kT}^{t} c_0 A^2(t') dt' \right) \right\} \geq kT.$$

Hence for $t \in (0, t_* - 1)$ inequality (4.33) holds. Clearly, we have

$$Z^2(t_*) = \gamma \exp \left(-\int_{kT}^{t_*} c_0 A^2(t) dt \right) \quad \text{and} \quad Z^2(t) > \gamma \exp \left(-\int_{kT}^{t} c_0 A^2(t) dt \right) \quad (4.34)$$

for $t > t_*$. Then (4.33) yields

$$\frac{d}{dt} Z^2|_{t=t_*} \leq c_* \left(-\gamma + \frac{\gamma}{2} \right) \exp \left(-\int_{kT}^{t_*} c_0 A^2(t) dt \right) < 0, \quad (4.35)$$

contrary with (4.34). Hence

$$Z^2(t) \leq \gamma \exp \left(-\int_{kT}^{t} c_0 A^2(t') dt' \right) \quad \text{for} \quad t > t_*.$$

(4.36)
From the definition of $Z^2(t)$ we have

$$X^2(t) \leq \gamma \exp \left( \int_{t_*}^t c_0 A^2(t') dt' \right) \text{ for } t > t_*.$$

For $\gamma \leq \gamma_*$ inequality (4.31) takes the form

$$\frac{d}{dt} X^2 + c_* X^2 \leq c_0 A^2 X^2 + c_0 G^2. \quad (4.37)$$

Integrating (4.37) with respect to time from $t = kT$ to $t = (k+1)T$ gives

$$X^2((k+1)T) \leq \exp \left( \int_{kT}^{(k+1)T} c_0 A^2(t) dt \right) \int_{kT}^{(k+1)T} c_0 G^2 dt$$

$$+ \exp \left( -c_* T + \int_{kT}^{(k+1)T} c_0 A^2(t) dt \right) X^2(kT) \quad (4.38)$$

$$\leq \exp(A_0^2) \int_{kT}^{(k+1)T} c_0 G^2(t) dt + \exp(-c_* T + A_0^2) X^2(kT),$$

where the second inequality follows from the assumptions of the lemma.

Hence $X^2((k+1)T) \leq \gamma$. Then induction proves the lemma. \hfill \Box

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