In the framework of zeta-function approach the Casimir energy for three simple model system: single delta potential, step function potential and three delta potentials is analyzed. It is shown that the energy contains contributions which are peculiar to the potentials. It is suggested to renormalize the energy using the condition that the energy of infinitely separated potentials is zero which corresponds to subtraction all terms of asymptotic expansion of zeta-function. The energy obtained in this way obeys all physically reasonable conditions. It is finite in the Dirichlet limit and it may be attractive or repulsive depending on the strength of potential. The effective action is calculated and it is shown that the surface contribution appears. The renormalization of the effective action is discussed.

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I. INTRODUCTION

Recently there was great interest to Casimir effect for semi-transparent boundaries \cite{4, 6, 13, 19, 20, 23}. It was suggested to consider the potential with zero support instead of the rigid Dirichlet condition. This potential models the semi-transparent boundary condition. The space is not divided into separate parts by Dirichlet boundaries and some modes of field may go through the boundary. Firstly, this kind of calculations were made in Refs. \cite{3, 17, 21}. It was claimed that in the limit of infinite strength of potential, which was called Dirichlet limit \cite{13}, the energy is divergent and it does not coincide with that obtained for Dirichlet boundary condition. The authors of \cite{13} noted that in framework of QED it is impossible to obtain finite result. This divergence appears in the energy only. The force, which is the derivative of energy with respect the position of plate, is finite \cite{20} in the Dirichlet limit.

The same problem was emphasized in Ref. \cite{6}. Singular potential brings additional surface contributions to heat kernels coefficients \cite{5} and to the effective action and there is no universal method to make unambiguous renormalization procedure and fix all parameters. Necessity to consideration of the surface contributions to energy in framework of bag model was noted in Ref. \cite{1, 18}. From the point of view of field theory the delta-like potential may be considered as brane theory and the parameter of potential strength is regarded as brane’s mass \cite{12}. In framework of quantum field theory this parameter has to be renormalized, too. Renormalization group equation for brane with co-dimension greater then unit was obtained in Ref. \cite{12}. It was noted that for renormalization we have to introduce into theory not merely brane mass but also brane tension \cite{6, 11} and others parameters \cite{12} the number of which depends on the bulk action. There is another analogy of system under consideration with field theory in curved space-time with singular scalar curvature. For example, in space-time of short-throat wormhole \cite{15} the space-time is everywhere flat except the throat where the scalar curvature is singular. The strength of the delta-potential corresponds to non-conformal coupling constant which has to be renormalized too \cite{22}.

In this paper we reexamine this problem in framework of zeta-regularization approach considering in details some simple model system in Sec. \ref{sec:models} namely, single delta potential, step function potential and three delta potentials. The main features of these models are summarized in Sec. \ref{sec:zeta}. It was noted that the singularities appeared in the limit of infinite strength are connected with brain (potential) itself. There we suggest a method to extract physically reasonable expression for energy which obey all conditions we need. It is shown that the same result may be obtained by Lukosz renormalization procedure. The Casimir energy may possesses the maximum and the Casimir force may be attractive as well as repulsive. Close analogy with ”surface energy” is noted. The effective action in framework of zeta-regularization for $\phi^4$ theory with delta potential is considered in Sec. \ref{sec:effective}. Recently similar approach was used by Toms in Ref. \cite{24} for free scalar field. The same ambiguous of renormalization procedure in surface term \cite{6} is recovered. We observe that this ambiguous is connected with some surface quantity peculiar to the potential (brane) itself. By using approach of Sec. \ref{sec:models} we obtain physically reasonable result which is coincides with that obtained in Sec. \ref{sec:zeta} We note that the Lukosz renormalization procedure takes off all singularities of effective action.

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II. THE ZETA-REGULARIZATION APPROACH

Let us consider the massive scalar field in \( N + 1 \) dimensional Minkowski space-time with potential \( V \) which depends on single coordinate \( x \), only. In the framework of zeta-regularization approach \([1, 2, 7]\) the regularized energy is defined by following expression

\[
E^{(N)}(s) = \frac{1}{2} \mu^{2s} \zeta(s - \frac{1}{2}, D_N)
\]

(1)

where

\[
\zeta(s, D_N) = \sum_{(n)} \omega_{(n)}^{-2s}
\]

is the zeta-function of operator

\[
D_N = -\triangle_N + m^2 + V
\]

with eigen-values \( \omega_{(n)}^2 \). Parameter \( \mu \) with dimension of mass has introduced to keep right dimension of the energy. In accordance with Ref. \([2]\) one defines the Casimir energy by relation

\[
E^c = \lim_{s \to 0} (E^{(N)}(s) - E^{\text{div}}_d(s)),
\]

(2)

where

\[
E^{\text{div}}_d(s) = \lim_{m \to \infty} E^{(N)}(s) = \left( \frac{\mu}{m} \right)^{2s} \frac{1}{2(4\pi)^{N/2}} \sum_{n=0}^{N+1} B_n^N m^{N+1-n} \frac{\Gamma(s + \frac{n-N-1}{2})}{\Gamma(s - \frac{1}{2})}
\]

is the divergent part of energy, where \( B_n^N \) are the heat kernel coefficients of operator \( D_N \). This expression obeys to the physical reasonable condition

\[
\lim_{m \to \infty} E^c = 0.
\]

(3)

The spectrum of this operator is numerated by one discrete number \( n \), and continuous numbers \( k_{N-1} \in (-\infty, +\infty) \)

\[
\omega^2 = k_n^2 + k_{N-1}^2 + m^2.
\]

By integrating the definition of zeta function over continuous variables \( k_{N-1} \) we obtain the relation

\[
\zeta(s - \frac{1}{2}, D_N) = \frac{1}{(4\pi)^{\frac{N-1}{2}}} \frac{\Gamma(s - \frac{N}{2})}{\Gamma(s - \frac{1}{2})} \zeta(s - \frac{N}{2}, D_1)
\]

(4)

by using which we have to solve now the one dimensional problem only. We note that the using above relation in Eq. (4) gives the energy per unit square of plate, that is surface density of energy. To calculate the zeta-function we will use approach \([2]\) in framework of which the zeta-function is represented in the form below

\[
\zeta(s, D_1) = \frac{\sin(\pi s)}{\pi} \int_{m}^{\infty} dk(k^2 - m^2)^{-s} \frac{\partial}{\partial k} \ln \Psi(ik, R, R')
\]

(5)

where the spectrum of energy \( k^2 = m^2 - \omega^2 \) is found from relation \( \Psi(k, R, R') = 0 \). Let us proceed to consideration of different forms of potential.

A. Single singular potential \( V = \lambda_0 \delta(x) \)

We consider a singular potential \( V = \lambda_0 \delta(x) \) and two \( N - 1 \) dimensional Dirichlet plates at points \( x = R \) and \( x = -R' \). Because of Eq. (4) we need to consider the imaginary energies only. Let us denote \( m^2 - \omega^2 = k^2 \) and consider the one dimensional problem in imaginary axis \( k \to ik \). In this case the eigen-problem equation reads

\[
\left[ -\frac{\partial^2}{\partial x^2} + k^2 + \lambda_0 \delta(x) \right] \phi = 0.
\]
In two domains $x \in (0, R)$ and $x \in (-R', 0)$ we obtain the general solution of this equation

$$
\phi^+ = C_1^+ e^{kx} + C_2^+ e^{-kx},
$$
$$
\phi^- = C_1^- e^{kx} + C_2^- e^{-kx}.
$$

Four constants $C_1^\pm$ and $C_2^\pm$ obey to four homogenous equations

$$
\phi^+(0) = \phi^-(0),
$$
$$
\phi^+(0) = \phi^-(0) + \lambda_s \phi^+(0),
$$
$$
\phi^+(R) = 0,
$$
$$
\phi^-(-R') = 0.
$$

The solution of this system exists if and only if the determinant of the system equals to zero. This determinant is the function $\Psi(ik, R, R')$ which is used in Eq. (5):

$$
\Psi(ik, R, R') = \frac{1}{k} \left[ e^{-kR'} A' + e^{kR} A \right],
$$

where

$$
A' = \frac{1}{k} \left[ \lambda_s e^{kR} + (2k - \lambda_s) e^{-kR} \right],
$$
$$
A = \frac{1}{k} \left[ \lambda_s e^{-kR} - (2k + \lambda_s) e^{kR} \right].
$$

It is easy to see from Eq. (6) by equating $\Psi(ik, R, R')$ to zero that the boundary states appear if

$$
\lambda_s < -\frac{R + R'}{RR'}.
$$

This boundary state is localized in the potential (brane) and in the case of whole space $R = R' \to \infty$ it has the form below [17]:

$$
\phi(x) = Ce^{-|\lambda_s x|}.
$$

Because the possible boundary states have already incorporated in Eq. (5) we assume arbitrary sign of $\lambda_s$. Nevertheless we suppose that the inequality

$$
\lambda_s > -2m
$$

is valid. In opposite case we can not use the theory in present form (see Ref. [17]).

To find the heat kernel coefficients which are defined as asymptotic expansion over mass,

$$
\zeta(s, D_1)^{as} = \frac{1}{\sqrt{4\pi}} \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} B_{2n} m^{-2s+1-n} \Gamma\left(s - \frac{1}{2} + \frac{n}{2}\right),
$$

we take the asymptotic expansion of zeta function [12] over $k \to \infty$. One has

$$
\frac{\partial}{\partial k} \ln \Psi(k, R, R')^{as} = R + R' - \frac{\lambda_s}{k(2k + \lambda_s)} = R + R' - \frac{1}{k} \sum_{l=1}^{\infty} \frac{(-1)^l \lambda_s^l}{2^l k^{l+1}}.
$$

By using this expression we obtain heat kernel coefficients

$$
B_0 = R + R',
$$
$$
B_n = \left( \frac{\lambda_s}{2} \right)^{2n-1} \left( \frac{\sqrt{\pi}}{\Gamma(n + \frac{1}{2})} \right),
$$
$$
B_{n+\frac{1}{2}} = \left( \frac{\lambda_s}{2} \right)^{2n} \left( \frac{\sqrt{\pi}}{n!} \right),
$$
$$
n = 1, 2, 3, \ldots.
Then, using Eqs. 4 and 5 we arrive at the following expression for Casimir energy

\[ E_c^{(N)}[\lambda_s, R, R'] = -\frac{1}{2(4\pi)^2} \Gamma\left(\frac{N}{2} + 1\right) \int_m^{\infty} dk (k^2 - m^2)^{N/2} Z_N[k, \lambda_s, R, R'], \tag{11} \]

where

\[ Z_N[k, \lambda_s, R, R'] = \frac{\partial}{\partial k} \ln \Psi(k, R, R') - (R + R') + \frac{1}{k} - \sum_{i=1}^{N} (-1)^i \frac{\lambda_i}{2k^{i+1}}. \tag{12} \]

Let us consider some limiting cases. In massless limit \( m \to 0 \) the energy is divergent

\[ E_c^{(N)}[\lambda_s, R, R'] \bigg|_{m=0} \approx \frac{(-1)^N}{(4\pi)^2} B_{\frac{N}{2}} \ln \frac{m}{\lambda_s} \tag{13} \]

in agreement with Ref. 4. The energy in whole space \((R = R' \to \infty)\),

\[ E_c^{(N)}[\lambda_s, R \to \infty, R' \to \infty] = \frac{1}{2(4\pi)^2} \Gamma\left(\frac{N}{2} + 1\right) \int_m^{\infty} dk (k^2 - m^2)^{N/2} \left[ \frac{\lambda_s}{k(2k + \lambda_s)} + \sum_{i=1}^{N} (-1)^i \frac{\lambda_i}{2k^{i+1}} \right] \tag{14} \]

is finite but it is ill defined in the Dirichlet limit, \( \lambda_s \to \infty \). Here the \( _2F_1 \) is the hypergeometric function. The leading divergent term in this limit coincides with that in Eq. 13. The energy is well defined in the limit \( \lambda_s \to 0 \). In this case

\[ Z_N[k, \lambda_s \to 0, R, R'] = \frac{2(R + R')}{e^{2k(R+R')} - 1} \]

which corresponds to the Casimir energy for two Dirichlet plates at points \(-R\) and \(R\).

Let us consider the energy for particular dimensions \( N = 1, 3 \) in manifest form. Then for \( N = 1 \) we obtain

\[ E_c^{(1)}[\lambda_s, R, R'] = -\frac{m}{2\pi} \int_1^{\infty} dx (x^2 - 1)^{1/2} Z_1[mx, \lambda_s, R, R'], \]

where

\[ Z_1[k, \lambda_s, R, R'] = \left\{ 8k^3(R + R') + 2k\lambda_s \left[ -2 + e^{2kR} + e^{2kR'} + 2kR(e^{2kR} - 1) + 2kR(e^{2kR'} - 1) \right] + (e^{2kR} - 1)(e^{2kR'} - 1)\lambda_s^2 \right\}^{-1}. \]

In the limit \( R = R' \to \infty \)

\[ E_c^{(1)}[\lambda_s, R \to \infty, R' \to \infty] = -\frac{m}{4\pi} \left( \pi - \frac{\lambda_s}{m} - \sqrt{4 - \frac{\lambda_s^2}{m^2}} \arccos \frac{\lambda_s}{2m} \right). \]

This expression coincides exactly with that obtained in Ref. 13. It is ill defined in the Dirichlet limit \( \lambda_s \to \infty \):

\[ E_c^{(1)}[\lambda_s \to \infty, R \to \infty, R' \to \infty] \approx \frac{\lambda_s}{4\pi} (1 - \ln \frac{\lambda_s}{2m} - \frac{m}{4}). \]

The force acting on the plate coincides with that obtained in Ref. 20 in massless case. We have to take limit \( R' \to \infty \) and then take the derivative with respect \( R \) with sign minus. After integrating by parts we obtain the force in the form below

\[ F[\lambda_s, R, R' \to \infty] = -\frac{\partial}{\partial R} E^{(1)} = -\frac{1}{4\pi R^2} \int_0^{\infty} \frac{ydy}{(\lambda_s^2 + 1)e^y - 1}. \tag{15} \]
To compare with Ref. \cite{20} we have to take limit $\lambda' \to \infty$ in Eq. (2.13) of this paper and change the notations of parameters $a \to R$ and $\lambda \to \lambda_s R$. As was noted in Ref. \cite{20} this expression is well defined in the Dirichlet limit $\lambda_s \to \infty$.

In three dimensional case, $N = 3$, one has

$$E^{(3)}_c[\lambda_s, R, R'] = -\frac{m^4}{12\pi^2} \int_1^\infty dx(x^2 - 1)^{3/2}Z_3[mx, \lambda_s, R, R'],$$

where

\[
Z_3[k, \lambda_s, R, R'] = \left\{32k^5(R + R') + 8k^3\lambda_s \left[-2 + e^{2kR} + e^{2kR'} + 2kR'(e^{2kR} - 1) + 2kR(e^{2kR'} - 1)\right] + 4k^2\lambda_s^2 \left[-2 + e^{2kR} + e^{2kR'}\right] + 2k\lambda_s^3 \left[-2 + e^{2kR} + e^{2kR'}\right] + (e^{2kR} - 1)(e^{2kR'} - 1)\lambda_s^4\right\}^{-1}. 
\]

In the limit $R = R' \to \infty$ we obtain the following expression:

$$E^{(3)}_c[\lambda_s \to \infty, R \to \infty, R' \to \infty] = \frac{m^3}{576\pi^2} \left[24\pi - 24\frac{\lambda_s}{m} + 9\frac{\lambda_s^2}{m^2} + 8\frac{\lambda_s^3}{m^3} - 6\left(\frac{\lambda_s^3}{m^3}\right)^{1/2} \arccos \frac{\lambda_s}{m}\right],$$

which is divergent in the Dirichlet limit $\lambda_s \to \infty$

$$E^{(3)}_c[\lambda_s \to \infty, R \to \infty, R' \to \infty] \approx \frac{\lambda_s^3}{72\pi^2} - \frac{m\lambda_s^2}{64\pi} - \frac{m^2\lambda_s}{32\pi^2} + \frac{m^3}{24\pi} + \frac{1}{16\pi^2} \left(-\frac{\lambda_s^3}{6} + m^2\lambda_s\right)\ln \frac{\lambda_s}{m}.$$  

\section{B. A single step function potential}

Let us regularize delta function by step function by relation

$$\delta(x) = \lim_{\epsilon \to 0^+} \left\{ \begin{array}{ll} 0, & |x| > \epsilon, \\ 1, & |x| < \epsilon. \end{array} \right.$$ 

and consider the energy for finite value of the regularization parameter $\epsilon$. In this case we get the following expression for function $\Psi$ for $R, R' > \epsilon$:

$$\Psi^{out}_{\epsilon}(ik, R, R') = \frac{1}{k} \left[e^{-kR'A'_\epsilon} + e^{kR'A_{\epsilon}}\right],$$

where

$$A'_\epsilon = e^{kR}\sinh(2\epsilon k\epsilon) - e^{-kR+2\epsilon} \left[\sinh(2\epsilon k\epsilon) - 2\cosh(2\epsilon k\epsilon)\right],$$

$$A_\epsilon = -e^{-kR-2\epsilon} \left[\sinh(2\epsilon k\epsilon) + 2\cosh(2\epsilon k\epsilon)\right] + e^{-kR}\sinh(2\epsilon k\epsilon).$$

and $k_{\epsilon}^2 = k^2 + \lambda_s/2\epsilon$. In the limit $\epsilon \to 0$ this function tends to that obtained in last section and given by Eq. (6):

$$\lim_{\epsilon \to 0^+} \Psi^{out}_{\epsilon}(ik, R, R') = \Psi(ik, R, R').$$  

(17)

The first three heat kernel coefficients are the same as for delta potential case, the rest coefficients are divergent in the limit $\epsilon \to 0$:

$$B^{out}_0 = R + R', \quad B^{out}_{\frac{n}{2}} = -\sqrt{\pi},$$

$$B^{out}_n = \frac{2(-1)^n}{n!} \left(\frac{\lambda_s}{2}\right)^n \epsilon^{-n+1},$$

$$B^{out}_{\frac{n+1}{2}} = (-1)^n \left[\frac{2\Gamma(n + \frac{1}{2})}{n!} - \sqrt{\pi}\right] \left(\frac{\lambda_s}{2}\right)^n \epsilon^{-n}, \quad n = 1, 2, 3, \ldots.$$  

(18)
To find the function $\Psi$ inside the potential we have to assume $R, R' < \epsilon$ and consider additional Dirichlet boundaries far from the potential at points $x = \pm H$. In the end we tend these boundaries to infinity: $H \gg R, R, \epsilon$. Because we have three domains confined by Dirichlet boundaries the function $\Psi$ is the product of three functions:

$$\Psi^\infty(ik, R, R') = \Psi^1_\epsilon \Psi^2_\epsilon \Psi^3_\epsilon,$$

where

$$\Psi^1_\epsilon = \frac{e^{-(\epsilon-R')k}}{4k \epsilon} \left\{ e^{k(H-\epsilon)} \left[ e^{-2(\epsilon-R')k_\epsilon (k_\epsilon - k)} + e^{-k(H-\epsilon)} \left[ e^{-2(\epsilon-R')k_\epsilon (k_\epsilon + k)} \right] \right] \right\},$$

$$\Psi^2_\epsilon = \sinh(k_\epsilon (R + R')) \frac{1}{k_\epsilon},$$

$$\Psi^3_\epsilon = \Psi^1_\epsilon(R' \Rightarrow R).$$

The heat kernel coefficients have the form below

$$B^{in}_0 = 2H, \quad B^{in}_1 = -3\sqrt{\pi},$$

$$B^{in}_n = \frac{2(-1)^n}{n!} \left( \frac{\lambda_s}{2} \right)^n \epsilon^{n+1},$$

$$B^{in}_{n+1} = (-1)^n \left[ \frac{2\Gamma(n + \frac{1}{2})}{n!} - 3\sqrt{\pi} \right] \left( \frac{\lambda_s}{2} \right)^n \epsilon^{-n}, \quad n = 1, 2, 3, \ldots.$$

The difference with above calculated heat kernel coefficients appears at terms with half-integer indices:

$$B^{in}_{n+\frac{1}{2}} - B^{out}_{n+\frac{1}{2}} = (-1)^n \frac{2\sqrt{\pi}}{n!} \left( \frac{\lambda_s}{2} \right)^n \epsilon^{-n}, \quad n = 0, 1, 2, \ldots.$$

In the framework of zeta-regularization approach the energy has the following form:

$$E^{(N)}_\epsilon[\lambda_s, R, R', \epsilon] = \frac{1}{2(4\pi)^{\frac{N}{2}}} \frac{1}{\Gamma(\frac{N}{2} + 1)} \int_m^{\infty} dk (k^2 - m^2)^{N/2} Z_N[k, \lambda_s, R, R', \epsilon],$$

where

$$Z^{out}_N[k, \lambda_s, R, R', \epsilon] = \frac{\partial}{\partial k} \ln \Psi^{out}_\epsilon(k, R, R') - (R + R') + \frac{1}{k}$$

$$- 2 \sum_{l=1}^{[\frac{N}{2}]} \frac{(-1)^l \Gamma(l + \frac{1}{2})}{l!} \frac{\lambda_s^l}{2^l k^{2l+1} \epsilon^{l-1}} + \sum_{l=2}^{[\frac{N}{2}]} \frac{(-1)^l \lambda_s^l}{2^l k^{2l+1} \epsilon^{l-1}} \left[ \frac{2}{\sqrt{\pi} l!} - 1 \right],$$

$$Z^{in}_N[k, \lambda_s, R, R', \epsilon] = \frac{\partial}{\partial k} \ln \Psi^{in}_\epsilon(k, R, R') - 2H + \frac{3}{k}$$

$$- 2 \sum_{l=1}^{[\frac{N}{2}]} \frac{(-1)^l \Gamma(l + \frac{1}{2})}{l!} \frac{\lambda_s^l}{2^l k^{2l+1} \epsilon^{l-1}} + \sum_{l=2}^{[\frac{N}{2}]} \frac{(-1)^l \lambda_s^l}{2^l k^{2l+1} \epsilon^{l-1}} \left[ \frac{2}{\sqrt{\pi} l!} - 3 \right].$$

In the last expression the limit $H \rightarrow \infty$ is assumed. Because of relation the energy is divergent in the limit $\epsilon \rightarrow 0$ starting with dimension $N = 3$. In the limit of whole space $R = R' = L/2 \rightarrow \infty$, the energy is finite. It is ill defined in the Dirichlet limit, $\lambda_s \rightarrow \infty$, or/and $m \rightarrow 0$:

$$E^{(N)}_\epsilon[\lambda_s, R, R', \epsilon]|_{m \rightarrow 0} \approx \frac{(-1)^N}{(4\pi)^{\frac{N}{2}}} B_{N+1} \frac{m}{\lambda_s}$$

where the heat kernel coefficient is given by Eqs. and .

The energy calculated for this model potential has additional peculiarity for $R, R' \approx \epsilon$. The energy is divergent in the limit $R \rightarrow \epsilon$ or $R' \rightarrow \epsilon$, but it is finite for $R = \epsilon$ or $R' = \epsilon$. Indeed, let us consider the asymptotic expansion of function

$$\partial_x \ln \Psi^{out}_\epsilon(ik, \epsilon, \epsilon).$$
The half-integer coefficients obtained by this expression are different from \[18\] starting from \(\frac{3}{2}\) coefficient:

\[
B_0 = 2\epsilon, \quad B_\frac{3}{2} = -\sqrt{\pi},
\]

\[
B_n = \frac{2(-1)^n}{n!} \left(\frac{\lambda_s}{2}\right)^n \epsilon^{-n+1},
\]

\[
B_{n+\frac{1}{2}} = (-1)^{n+1} \sqrt{\pi} \left(\frac{\lambda_s}{2}\right)^n \epsilon^{-n}, \quad n = 1, 2, 3, \ldots.
\]

It is easy understand the origin of this problem. To obtain the heat kernel coefficients \[18\] we assumed that \(R, R' > \epsilon\) and for this reason we turn down all terms as \(e^{-k(R-\epsilon)}\) and \(e^{-k(R'-\epsilon)}\) and after this we set \(R = R' = \epsilon\). We may turn down these kind of terms for arbitrary small, but non-zero, value of \(R - \epsilon\) and \(R' - \epsilon\). We obtained the following expression for asymptotic expansion

\[
\partial_x \ln \Psi_e^\text{out}(ik, R, R')^\text{as}_{R, R' = \epsilon} = -\frac{2}{k} + \frac{2k\epsilon}{k^2} + \frac{2k\epsilon}{k^3}.
\]

If we put \(R = R' = \epsilon\) at the beginning we obtain another expression for asymptotic expansion

\[
\partial_x \ln \Psi_e^\text{out}(ik, \epsilon, \epsilon)^\text{as} = -\frac{2}{k} + \frac{2k\epsilon}{k^2} - \frac{\lambda_s}{2ck^3} + \ldots.
\]

To reveal this divergence in manifest form let us use another form of asymptotic expansion. We keep all terms as \(e^{-k(R-\epsilon)}\) and \(e^{-k(R'-\epsilon)}\) and will consider asymptotic expansion with these terms as constants:

\[
\partial_x \ln \Psi_e^\text{out}(ik, R, R')^\text{as} = (R + R') - \frac{1}{k} - \frac{\lambda_s}{4\epsilon k^2} \left[2\epsilon - e^{-2k(R-\epsilon)}(R - \epsilon) - e^{-2k(R'-\epsilon)}(R' - \epsilon)\right]
\]

\[
+ \frac{\lambda_s}{4\epsilon k^2} \left[e^{-2k(R-\epsilon)} + e^{-2k(R'-\epsilon)}\right]
\]

\[
+ \frac{\lambda_s^2}{32\epsilon^2 k^4} \left[6\epsilon + (e^{-4k(R-\epsilon)} - 2e^{-2k(R-\epsilon)})(R - \epsilon) + (e^{-4k(R'-\epsilon)} - 2e^{-2k(R'-\epsilon)})(R' - \epsilon)\right] + \ldots
\]

From this expression we observe that additional terms appears. The contributions of them to even degree of \(k\) (heat kernel coefficients with integer indices) are insufficient. For \(R, R' \neq \epsilon\) they are exponentially small, but for \(R, R' = \epsilon\) they are zero. The contributions to odd degrees of \(k\) (heat kernel coefficients with half-integer indices) are important. They are exponentially small for \(R, R' \neq \epsilon\), but they are constant for \(R, R' = \epsilon\). The first non-trivial contribution starts from \(1/k^3\). Therefore this kind of expansion reproduces right the expansion for \(R, R' \neq \epsilon\) as well as for \(R, R' = \epsilon\).

Let us use this expansion to calculate the zeta-function and the energy. It allows us to find in manifest form the divergence as the boundary position \(R\) close to boundary of potential \(\epsilon\). In the 3-dimensional case the divergent contribution is due to the following term

\[
-\frac{\lambda_s}{12\pi^2 \epsilon} \int_1^\infty (k^2 - m^2)^{3/2} e^{-2k(R-\epsilon)} + e^{-2k(R'-\epsilon)} \frac{dk}{k^3}.
\]

For \(R - \epsilon \ll 1\) we obtain the following behavior of energy close to boundary of potential

\[
E_e^{(3)}[\lambda_s, R \approx \epsilon, R', \epsilon] \approx -\frac{\lambda_s}{12\pi^2 \epsilon |R - \epsilon|}.
\]

The application of this approach to internal function \[19\] gives the same result but with opposite sign.

It is obvious that this divergence connects with model under consideration. If we adopt another kind of regularization for delta-function as a consequence of analytic functions we obtain no divergence except for infinitely close position of plates each to other as should be the case for Casimir force. For position of Dirichlet plates exactly on the boundary we obtain the following finite expression for energy

\[
E_e^{(3)}[\lambda_s, R = \epsilon, R' = \epsilon, \epsilon] = -\frac{1}{12\pi^2} \int_m^\infty dk (k^2 - m^2)^{3/2} \left[-2\epsilon + \frac{1}{k} - \frac{k}{k^2} + \frac{2k\epsilon}{k^3} \coth(2\epsilon k_e) + \frac{\lambda_s}{2k^2} - \frac{\lambda_s}{2ck^3} - \frac{3\lambda_s^2}{16\epsilon k^4}\right].
\]
C. Three singular potentials \( V = \lambda_2 \delta(x + R') + \lambda_3 \delta(x) + \lambda_4 \delta(x - R) \)

Let us consider three delta-potentials

\[
V(x) = \lambda_2 \delta(x + R') + \lambda_3 \delta(x) + \lambda_4 \delta(x - R)
\]

and the field with Dirichlet boundary condition at surfaces \( x = -l, L \), where \( l, L > R', R \).

The \( \Psi \) function has the form below

\[
\Psi(k, R, R', L, l, \lambda_2, \lambda_1, \lambda_2) = \frac{e^{k(l+L)}}{k^4} \left\{ \begin{array}{c}
(2k - \lambda_1) (2k - \lambda_2) e^{-2k(l+L)} \\
(2k - \lambda_1) (2k - \lambda_2) e^{-2k(l+L)} - \lambda_2 (2k + \lambda_1) (2k - \lambda_2) e^{-2kl} \\
(2k - \lambda_1) (2k - \lambda_2) e^{-2k(l+L')} + \lambda_2 (2k - \lambda_1) \lambda_2 e^{-2k(l+L'-R')} + \lambda_2 \lambda_1 \lambda_2 e^{-2k(L+R'-R)} \\
\lambda_1 \lambda_2 e^{-2k(l-\lambda_1)} - \lambda_2 (2k - \lambda_1) (2k + \lambda_2) e^{-2kL} - \lambda_1 \lambda_2 e^{-2k(L'-R')} \\
(2k + \lambda_1) (2k + \lambda_2) e^{-2k(L'-R')} + (2k + \lambda_1) (2k + \lambda_2) \end{array} \right.
\]

For \( \lambda_1 = \lambda_2 = 0 \) this function coincides to that considered in the first section. The following relation is valid:

\[
\Psi(k, R, R', L, l, 2\lambda_2, \lambda_1, 0, \lambda_2 = 0) = \Psi(k, R = 0, R' = 0, L, l, \lambda = 0, \lambda, \lambda)
\]

as should be the case. But the energy does not obey this kind of relation. Indeed, let us consider the case of two potentials and put \( \lambda = 0 \). One has

\[
\Psi(k, R, R', L, l, \lambda = 0, \lambda_1, \lambda_2) = \frac{2e^{2kL}}{k^3} \left\{ \begin{array}{c}
(2k - \lambda_1) (2k - \lambda_2) e^{-4kL} - \lambda_1 (2k - \lambda_2) e^{-2k(L+R)} \\
(2k - \lambda_1) \lambda_2 e^{-2k(L+R')} - \lambda_1 \lambda_2 e^{-2k(L'-R')} + \lambda_1 \lambda_2 e^{-2k(2L'-R')} \\
(2k + \lambda_1) \lambda_2 e^{-2k(L'-R')} - \lambda_1 (2k + \lambda_2) e^{-2k(L-R)} + (2k + \lambda_1) (2k + \lambda_2) \end{array} \right.
\] (28)

In this case

\[
Z_N[k, 0, \lambda_1, \lambda_2, R, R'] = \frac{\partial}{\partial k} \ln \Psi(k, R, R', L, l, 0, \lambda_1, \lambda_2) - (L + l) + \frac{1}{k} - \sum_{l=1}^{N} (-1)^{l} \frac{\lambda_1^l + \lambda_2^l}{2^l k^{l+1}}.
\]

The difference has the following form:

\[
\triangle Z = - \sum_{l=2}^{N} (-1)^{l} \frac{2\lambda_1^l - (2\lambda_2)^l}{2^l k^{l+1}},
\]

and it is non-zero starting from dimension \( N = 3 \). This fact originates from renormalization procedure.

In the limit of whole space, \( L = l \to \infty \), we get from \( \frac{25}{25} \)

\[
\Psi(k, R, R', L, l, \lambda = 0, \lambda_1, \lambda_2) = \frac{2e^{2kL}}{k^3} \left\{ \begin{array}{c}
- \lambda_1 \lambda_2 e^{-2k(R'+R)} + (2k + \lambda_1) (2k + \lambda_2) \end{array} \right.
\] (29)

If we then put potentials to infinity \( R = R' \to \infty \) we obtain non-zero result

\[
Z_N[k, 0, \lambda_1, \lambda_2, R, R'] = - \frac{\lambda_1}{k(2k + \lambda_1)} - \frac{\lambda_2}{k(2k + \lambda_2)} - \sum_{l=1}^{N} (-1)^{l} \frac{\lambda_1^l + \lambda_2^l}{2^l k^{l+1}}
\]

\[
= - \sum_{l=N+1}^{\infty} (-1)^{l} \frac{\lambda_1^l + \lambda_2^l}{2^l k^{l+1}}.
\]
This is in contradiction with expected result that the energy has to be zero. We observe that in this limit the energy is the sum of energy of two single plates. Therefore we conclude that this energy is connected with potential itself and it has the additivity property provided the infinite distances between potentials (without interaction). For \( J \) plates in the limit of infinite distances we get

\[
Z_N = -\sum_{i=1}^{J} \frac{\lambda_{s_i}}{k(2k + \lambda_{s_i})} - \sum_{i=1}^{J} \sum_{l=1}^{N} (-1)^l \frac{\lambda_{s_l}}{2^l k^{l+1}}.
\]

Therefore, with each \( j \)-th singular potential \( \lambda_{s_j} \delta(x-R_j) \) we may connect the ”surface” energy by relation

\[
E_{s_j}^{(N)} = \frac{m^{N+1}}{2(4\pi)^2} \frac{1}{\Gamma(\frac{N}{2} + 1)} \int_{1}^{\infty} dx (x^2 - 1)^{N/2} \left[ \frac{\lambda_{s_j}}{m(x^{2} + \lambda_{s_j})} + \sum_{l=1}^{N} (-1)^l \frac{\lambda_{s_j}^l}{2^l m^{l+1} x^{l+1}} \right].
\]

This energy is ill defined in the limits \( m \to 0 \) or/and \( \lambda_{s_j} \to \infty \).

### III. Renormalization Procedure

Let us summarize useful information from above section. First of all, we should like to note that the divergence of above energies is connected with renormalization procedure only. The regularized expression for energy \( \mathcal{E}_s \) is well defined in the limit \( \lambda_s \to \infty \) or \( \epsilon \to 0 \). The renormalization procedure brings a problem. The point is that for renormalization we subtract finite number of terms (see \( \mathcal{E}_{\lambda} \) or \( \mathcal{E}_{\epsilon} \)) of series expansion over small value \( \lambda_s/k \ll 1 \) or \( \lambda_s/\epsilon k^2 \ll 1 \). After renormalization we use expressions obtained for \( \lambda_s \to \infty \) or \( \epsilon \to 0 \). Obviously that the main divergence originate from the last term of truncated series. Indeed, we must consider the asymptotic expansion over \( k \) the integrand

\[
\frac{\partial}{\partial k} \ln \Psi(k, R, R')^{as} = R + R' - \frac{1}{k} - \frac{\lambda_s}{k(2k + \lambda_s)} + O(e^{-kR}, e^{-kR'}). \tag{31}
\]

The last term is finite in the limit \( \lambda_s \to \infty \) and the difference

\[
\frac{\partial}{\partial k} \ln \Psi(k, R, R') - \frac{\partial}{\partial k} \ln \Psi(k, R, R')^{as} \tag{32}
\]

is finite in this limit, too. But if we use truncated series expansion

\[
\frac{\partial}{\partial k} \ln \Psi(k, R, R')^{as} = R + R' - \frac{1}{k} + \sum_{l=1}^{N} (-1)^l \frac{\lambda_{s_l}^l}{2^l k^{l+1}}
\]

the difference \( \mathcal{E}_{\lambda}^{as} \) is infinite in the Dirichlet or massless limit. The main divergence comes from the last term which gives the contribution to heat kernel coefficient \( B_{\mathcal{E}_{\lambda}} \) which in turn gives the conformal anomaly.

Second, with each potential connects the inner quantity given by Eq. \( \mathcal{E}_{\lambda} \) which we called ”surface” energy. It is defined as energy of single delta-potential provided the boundaries are in infinity. This quantity possesses an additive property: the ”surface” energy of set delta-potentials is the sum of ”surface” energies of each delta-potential provided the infinite distance between potentials. We note also that this quantity is connected with potential itself. In self-consistent theory which takes into account gravitational field of brane \( \mathcal{E}_{\mathcal{E}} \), this energy will give some contribution to gravitational field of brane.

For this reason let us consider another renormalization procedure. Instead of extracting terms which will survive in the limit \( m \to \infty \) we subtract all terms in asymptotic expansion of zeta function. It is easy to see that this procedure corresponds to subtracting ”surface” energy of delta potentials, which is defined for \( R, R' \to \infty \). It seems reasonable that the Casimir energy is zero if all potentials are in infinity. Indeed, all terms of asymptotic expansion in \( \mathcal{E}_{\lambda} \) may be obtained by taking the limits \( R, R' \to \infty \):

\[
\frac{\partial}{\partial k} \ln \Psi(k, R, R'), R, R' \to \infty = R + R' - \frac{1}{k} - \frac{\lambda_s}{k(2k + \lambda_s)} + O(e^{-kR}, e^{-kR'}). \tag{33}
\]

It is easy to see that the condition \( \mathcal{E}_{\lambda} \) is still valid for this kind of renormalization. In this case the energy for single delta-potential considered in Sec. \( \mathcal{E}_{\lambda} \) has the form \( \mathcal{E}_{\lambda} \):

\[
\bar{E}_{c}^{(N)}[\lambda_s, R, R'] = -\frac{1}{2(4\pi)^2} \frac{1}{\Gamma(\frac{N}{2} + 1)} \int_{m}^{\infty} dk (k^2 - m^2)^{N/2} Z_N[k, \lambda_s, R, R'], \tag{33}
\]
where function

\[ \tilde{Z}_N[k, \lambda_s, R, R'] = \frac{2 \left( 4k^2(R + R') + \lambda_s \left[ -2 + e^{2kr} (1 + 2kR') + e^{2kr'} (1 + 2kR) \right] + \lambda_s^2 \left[ (e^{2kr} - 1)R + (e^{2kr'} - 1)R' \right] \right)}{(2k + \lambda_s) \left[ 2(k^{2e^{2k(r+R')}} - 1) + (e^{2kr} - 1)(e^{2kr'} - 1)\lambda_s \right]} \]

does not depend on the dimension of space. It is easy to see that the difference of these energies is that divergent expression \[14\] which we called "surface" energy \[30\]:

\[ E_c^{(N)}[\lambda_s, R, R'] - \tilde{E}_c^{(N)}[\lambda_s, R, R'] = E_c^{(N)}[\lambda_s, R \to \infty, R' \to \infty] = E_s^{(N)}. \]

The renormalization procedure suggested by Lukosz \[16\] (see also \[8\]) gives the same result. Indeed, according with this approach we have to include our system to large box, for example with \[\lambda\] as should be the case. For all reasonable cases. In one-dimensional case \[M\] divided into three domains: \[M_1 : \{ x \in [-H, -R'] \}, M_2 : \{ x \in [-R', R] \} \text{ and } M_3 : \{ x \in [R, H] \}. \] The renormalized energy is given by equation

\[ \Delta E_c^{(N)} = E_c^{(N)}(M_1) + E_c^{(N)}(M_2) + E_c^{(N)}(M_3) - E_c^{(N)}(M), \]

where the last term is calculated for whole space but without internal boundaries. In the end we have to tend \(H \to \infty\). It is very easy to apply this formula in our case. Because in domains \(M_1\) and \(M_2\) there is no potential we may use our formulas with \(\lambda_s = 0\). Therefore we obtain

\[ \Delta E_c^{(N)} = E_c^{(N)}[\lambda_s = 0, -R', H] + E_c^{(N)}[\lambda_s, R, R'] + E_c^{(N)}[\lambda_s = 0, H, -R] - E_c^{(N)}[\lambda_s, H, H] \]

\[ = -\frac{1}{2(4\pi)^2 \Gamma(\frac{4}{2} + 1)} \int_m^\infty dx(k^2 - m^2)^{3/2} \frac{\partial}{\partial k} \ln \frac{k^2 \Psi(ik, -R', H)\lambda_s = 0 \Psi(ik, R, R')\Psi(ik, H, -R)}{\Psi(ik, H, H)}. \]

In the limit \(H \to \infty\) we obtain the expression for the energy which coincides exactly with that obtained above and given by Eq. \[43\].

Let us consider some special cases of energy obtained. In massless limit the energy is finite for any dimensions. In the Dirichlet limit \(\lambda_s \to \infty\),

\[ \tilde{Z}_N[k, \lambda_s \to \infty, R, R'] = \frac{2R}{e^{2kr} - 1} + \frac{2R'}{e^{2kr'} - 1}, \]

the energy, as expected, is finite and it is the sum of two Casimir energies for domains \((-R', 0)\) and \((0, R')\):

\[ \tilde{E}_c^{(N)}[\lambda_s \to \infty, R, R'] = \tilde{E}_c^{(N)}[\lambda_s, 0, R'] + \tilde{E}_c^{(N)}[\lambda_s, R, 0], \]

as should be the case. For \(\lambda_s = 0\)

\[ \tilde{Z}_N[k, \lambda_s = 0, R, R'] = \frac{2(R + R')}{e^{2kr(R' + R)} - 1}, \]

which gives the Casimir energy for two plates in points \(-R'\) and \(R\). Therefore, this energy has satisfactory behavior for all reasonable cases. In one-dimensional case \(N = 1\) the force acting for one plate \((R' \to \infty)\) due to singular potential coincides with that obtained by Milton in Ref. \[20\] in massless case and it is given by Eq. \[15\]. This fact is evident because the difference of two energies is constant which does not depend on the distance to boundary.

The dependence of the Casimir energy \[33\] for three dimensional case is plotted in Fig. \[11\] for \(R = R' = L/2\) as a function of \(Lm\) for different values of \(\lambda_s\) and the fixed mass \(m\). We observe that the energy is negative for \(\lambda_s > 0\). It is easy to see this statement from Eq. \[34\]: in this case all terms of \(Z\) are positive. The slope of curves is negative for all position of plates. It means that two plates are attracted as in usual Casimir case. For negative \(\lambda_s\) a maximum appears at position \(L_m\) and the energy in the maximum grows with increasing \(|\lambda_s|\). For distances smaller then \(L_m\) the plates are attracted but for \(L > L_m\) the force is repulsive. The repulsive character of Casimir force due to delta potential was observed in Ref. \[22\]. Because of the present theory is valid for \(\lambda_s / m > -2\) we obtain from Eq. \[17\] that for \(Lm > 2\) the boundary state appears. The energy of this state \(E = \sqrt{m^2 - k^2}\) is found from following equation

\[ \tanh \frac{kL}{2} = \frac{2k}{\lambda_s}. \]

It is easy to show from Eq. \[34\] that the energy is positive for large distance between plates \(mL \to \infty\) and for \(0 > \lambda_s / m > -2\). Therefore for large distances between plates the Casimir force is repulsive and boundary state appears localized at the position of potential. There is a close analogy of this point with widely discussed now
"surface states" \[14\]. It was shown the repulsive character of "surface states" which may be regarded as boundary states \[14\].

In the case of step function potential the Lukosz regularization or renormalization procedure suggested above give the following expression for energy \((R, R') > \epsilon)\)

\[
\tilde{E}^{(3)}(\lambda_s, R, R', \epsilon) = -\frac{1}{12\pi^2} \int_{m}^{\infty} dk (k^2 - m^2)^{3/2} \frac{\partial}{\partial k} \ln \frac{k^2 \Psi_{\text{out}}^r(ik, -R', H)_{\lambda_s=0} \Psi_{\text{out}}^r(ik, R, R') \Psi_{\text{out}}^r(ik, H, -R)_{\lambda_s=0}}{\Psi_{\text{out}}^r(ik, H, H)_{\epsilon \to \infty}}. 
\]

For the case of position of plates inside the potential, \(R, R' < \epsilon\) we put the system in large box with size \(2H\). We have three domains: \(x \in (-H, -R')\), \(x \in (-R', R)\), \(x \in (R, H)\). For renormalization we have to subtract the energy without internal boundaries at \(R\) and \(-R'\). This energy is defined by function \(\Psi_{\text{out}}^r\) because the boundaries at \(\pm H\) are outside the potential. Therefore one has:

\[
\tilde{E}^{(3)}(\lambda_s, R, R', \epsilon) = -\frac{1}{12\pi^2} \int_{m}^{\infty} dk (k^2 - m^2)^{3/2} \left[ \frac{\partial}{\partial k} \ln k^2 \Psi_{\text{out}}^r(ik, R, R') \right]_{\epsilon \to \infty} - \frac{\lambda_s}{\epsilon k^3}. 
\]

Due to relation \[17\] it is obvious that in the limit \(\epsilon \to 0\) the energy \[35\] is finite and it coincides with energy calculated above for \(\delta\) potential

\[
\tilde{E}^{(3)}(\lambda_s, R, R', \epsilon \to 0) = \tilde{E}^{(3)}(\lambda_s, R, R').
\]

In the limit \(\lambda_s \to \infty\) one has

\[
E^{(3)}(\lambda_s \to \infty, R, R', \epsilon) = -\frac{1}{12\pi^2} \int_{m}^{\infty} dk (k^2 - m^2)^{3/2} \left[ \frac{2(R - \epsilon)}{e^{2k(R - \epsilon)} - 1} + \frac{2(R' - \epsilon)}{e^{2k(R' - \epsilon)} - 1} \right] 
\]

which is exact the sum of two Casimir energies for domains \(x \in [-R', -\epsilon]\) and \(x \in [\epsilon, R]\) as should be the case. For zero potential \(\lambda_s = 0\) the expression \[36\] reproduce the Casimir energy for two boundaries at points \(x = -R', R\).

The numerical calculations of Casimir energy is reproduced in Fig. \[2\] for \(R = R' = L/2, \lambda_s = 1, \epsilon = 0.1\). We observe the divergence of energy at the boundary of potential \(L = 2\epsilon = 0.2\) which was noted in Sec. \[11\]. For all position of Dirichlet plates the Casimir force is attractive. In the right figure we show details of the energy close to this boundary. The smaller \(\epsilon\) the closer the position of this divergence to origin and the closer external part to the case calculated for delta-function case. For small \(\epsilon\) and negative \(\lambda_s\) the Fig. \[1\] is reproduced.

In the case of three delta functions in framework of this procedure we obtain the following expression for the function \(\tilde{Z}\):

\[
\tilde{Z}_N = \Psi(k, R, R', L, l, \lambda_s, \lambda_{s1}, \lambda_{s2}) \frac{e^{-k(l+L)}k^4}{(2k + \lambda_s)(2k + \lambda_{s1})(2k + \lambda_{s2})}. 
\]

The same expression may be obtained by Lukosz renormalization procedure. We have to use this approach for each solitary potential provided that others are in the infinity. This expression is well defined in all limits we need. For example, in Dirichlet limit \(\lambda_s, \lambda_{s1}, \lambda_{s2} \to \infty\) the energy is the sum of fore Casimir energies for plates in points \(x = -l, -R', 0, R, L\).

FIG. 1: The Casimir energy for two plates in positions \(x = \pm L\) with singular potential \(V = \lambda_s \delta(x)\) between them for \(\lambda_s = -1.9m\) (thick curve), \(\lambda_s = 0\) (middle curve), and \(\lambda_s = 4m\) (thin curve).
FIG. 2: The Casimir energy for 3-dimensional case and for $R = R' = L/2, \lambda_s = 1, \epsilon = 0.1$. The boundary of potential is at point $Lm = 0.2$. There is a divergence in this point which is plotted in detail in the right figure.

IV. EFFECTIVE ACTION POINT OF VIEW

Let’s consider the same problem from the point of view of effective action. One has the scalar field $\phi$ living in $(3 + 1)$ dimensional space-time with singular potential concentrated at the surface $\Sigma$. To make clear regularization procedure we consider a nonlinear theory with the following action

$$S = \frac{1}{2} \int_M \left( (\nabla \phi)^2 + m^2 \phi^2 + \frac{\lambda}{6} \phi^4 + \Lambda \right) d^4 x + \frac{1}{2} \int_\Sigma (\Lambda_s + \lambda_s \phi^2) d^3 x. \tag{38}$$

Here the $\Lambda$ is cosmological constant and $\Lambda_s$ is surface tension. As was noted in Refs. \[6, 11\] we have to include the surface terms at the beginning for correct renormalization procedure. In fact, the action (38) describes a brane with co-dimension one in four dimensional space-time with tension $\Lambda_s$ and mass of brane $\lambda_s$. The renormalization group of brane with co-dimension more then unit was considered in Ref. \[12\]. Similar problem for free scalar field without self-action was considered in Ref. \[24\].

To calculate the energy of quantum fluctuation we consider the field at finite temperature $T = 1/\beta$ and in the end of calculation we take the limit of zero temperature $\beta \to \infty$. For this reason we proceed to the Euclidean regime $t \to -i \tau$ with $\tau \in [0, \beta]$. Let us consider the Casimir problem – two Dirichlet plates at points $x = R$ and $x = -R'$. The singular potential is concentrated at point $x = 0$.

We divide the field on classical part and quantum fluctuations: $\phi = \phi_{cl} + \phi$. In one loop approximation and framework of zeta regularization approach we have the following expression for action

$$S = S_{cl} + S_q = \frac{1}{2} \int_M \left( (\nabla \phi_{cl})^2 + m^2 \phi_{cl}^2 + \frac{\lambda}{6} \phi_{cl}^4 + \Lambda \right) + \frac{1}{2} \int_\Sigma (\Lambda_s + \lambda_s \phi_{cl}^2) - \frac{1}{2} m^2 \Gamma(s) \zeta(s, D_4),$$

where

$$D_4 = -\triangle_4 + m^2 + \lambda \phi_{cl}^2 + \lambda_s \delta(x). \tag{39}$$
In the limit \( s \to 0 \)
\[
S_q = -\frac{1}{2} \left( \frac{1}{s} + \ln \mu^2 \right) \zeta(0, D) - \frac{1}{2} \zeta'(0, D),
\]
where \( \mu^2 = \bar{\mu}^2 e^{-\gamma} \).

Due to the relation
\[
\zeta(s, D_1) = \frac{1}{4\pi} \frac{\zeta(s - 1, D_2)}{s - 1}
\]
we consider two dimensional eigenvalue problem
\[
\left[ -\frac{\partial}{\partial x} - \frac{\partial}{\partial \tau} + M^2 + \lambda_s \delta(x) \right] \phi = \omega^2 \phi
\]
with \( M^2 = m^2 + \lambda_s \varphi^2_{cl} \). Taking into account the periodicity over \( \tau \) we obtain
\[
\left[ -\frac{\partial}{\partial x} + \left( \frac{2\pi n}{\beta} \right)^2 + M^2 - \omega^2 + \lambda_s \delta(x) \right] \phi = 0,
\]
where \( n = 0, \pm 1, \pm 2, \ldots \) Let us denote \( \left( \frac{2\pi n}{\beta} \right)^2 + M^2 - \omega^2 = p^2 \) and consider the solution of above equation in imaginary axis \( p = ik \). In this case the equation reads
\[
\left[ -\frac{\partial^2}{\partial x^2} + k^2 + \lambda_s \delta(x) \right] \phi = 0. \tag{41}
\]

This equation has already considered in last section. Using the results we arrive at the following expression for zeta function
\[
\zeta(s, D_2) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk (k^2 - M_n^2)^{-s} \frac{\partial}{\partial k} \ln \Psi(k, R, R'), \tag{42}
\]
where \( M_n^2 = M^2 + (2\pi n/\beta)^2 \).

Taking into account this expression we obtain heat kernel coefficients
\[
B_0 = \Omega, \quad B_\pm = -\sqrt{\pi} \Sigma,
\]
\[
B_n = - \left( \frac{\lambda_s}{2} \right)^{2n-1} \sqrt{\frac{\pi \Sigma}{\Gamma(n + \frac{1}{2})}},
\]
\[
B_{n+ \frac{1}{2}} = \left( \frac{\lambda_s}{2} \right)^{2n} \sqrt{\frac{\pi \Sigma}{n!}}, \quad n = 1, 2, 3, \ldots,
\]
where \( \Omega = (R + R') \beta \) is the volume and \( \Sigma = \beta \) is the square of single boundary surface. These heat kernel coefficients may be obtained by multiplication to \( \Sigma \) the coefficients obtained for delta potentials \([10]\). We observe that the zeroth coefficient is volume of the system and all others coefficients are proportional to square of boundary. To note this moment it is better to define the density \( b_n \) of heat kernel coefficients by relations \( B_0 = b_0 \Omega \) and \( B_n = b_n \Sigma, \ n > 0 \).

In calculations of heat coefficients the asymptotic relation \([11]\)
\[
\sum_{n=-\infty}^{+\infty} M_n^{-1-2s} |_{\beta \to \infty} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s + \frac{1}{2} - \frac{1}{2})}{\Gamma(s + \frac{1}{2})} m^{1-1-2s} \beta
\]
was used.

To obtain the effective action we have to calculate \( \zeta(-1, D_2) \) and \( \zeta'(-1, D_2) \). For this aim we will use the formula \([12]\). We subtract from and add to integrand the first five terms of asymptotic expansion of it and represent the zeta function in the form
\[
\zeta(s, D_2) = \zeta_a(s, D_2) + \zeta_b(s, D_2) \tag{43}
\]
\[
= \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk (k^2 - M_n^2)^{-s} Z[k, \lambda_s, R, R']
\]
\[
+ \frac{1}{4\pi} \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} B_\pm m^{-2s+n} \Gamma(s - 1 + \frac{n}{2}).
\]
Using this equation we calculate the zeta function at point $-1$:

$$
\zeta_a(-1, D_2) = 0,
\zeta_b(-1, D_2) = \frac{1}{4\pi} \left\{ -\frac{1}{2} m^4 \Omega b_0 + \Sigma [m^2 b_1 - b_2] \right\}
$$

and its derivative in this point

$$
\zeta'_a(-1, D_2) = -\sum_{n=-\infty}^{\infty} \int_{M_n}^{\infty} dk (k^2 - M_n^2) \mathcal{Z}_0[k, \lambda, R, R'],
\zeta'_b(-1, D_2) = \frac{\ln m^2}{2 \pi} \left\{ -\frac{1}{2} m^4 \Omega b_0 - \Sigma [m^2 b_1 - b_2] \right\} - \frac{1}{4\pi} \left\{ \frac{4\sqrt{2}}{3} m^3 b_1 - 2\sqrt{2} \pi m b_2 - b_2 \right\}.
$$

To calculate $\zeta'_a(-1, D_2)$ we use the Abel-Plana formula

$$
\sum_{n=1}^{\infty} f[n] = \int_0^\infty f[x]dx - \frac{1}{2} f[0] + i \int_0^\infty \frac{f[iy] - f[-iy + \epsilon]}{e^{2\pi y} - 1} dy.
$$

In the limit $\beta \to \infty$ we get

$$
\zeta'_a(-1, D_2) = -\frac{2m^4 \beta}{3\pi} \int_0^\infty dx (x^2 - 1)^{3/2} \mathcal{Z}_0[mx, \lambda, R, R'] + O(e^{-\beta}).
$$

Then, using Eq. (40) we express four dimensional zeta function in therms of two dimensional:

$$
\zeta(0, D_4) = -\frac{1}{16\pi^2} \zeta(-1, D_2),
\zeta'(0, D_4) = -\frac{1}{4\pi} \zeta'(-1, D_2) - \frac{1}{4\pi} \zeta(-1, D_2).
$$

Taking into account these equations and putting all terms together we obtain the following expression for effective action

$$
S = \Omega \mathcal{S}^{\text{eff}}_{\Omega} + \Sigma \mathcal{S}^{\text{eff}}_{\Sigma} + \beta \mathcal{E}^{(3)}_c + O(e^{-\beta}),
$$

where

$$
\mathcal{S}^{\text{eff}}_{\Omega} = b_0 \left\{ \frac{M^4}{64\pi^2 s} - \frac{3M^4}{128\pi^2} + \frac{M^4}{64\pi^2} \ln \frac{M^2}{\mu^2} \right\} + \frac{1}{2} m^2 \varphi_{cl}^2 + \frac{\lambda}{12} \varphi_{cl}^4 + \frac{1}{2} \Lambda,
\mathcal{S}^{\text{eff}}_{\Sigma} = \frac{M^2 b_1 - b_2}{32\pi^2 s} - \frac{M^2 b_1}{24\pi^3 s} + \frac{M^2 b_1}{32\pi^2} + \frac{M^2 b_1}{24\pi^3} + \frac{M^2 b_1 - b_2}{16\pi^3/2} - \frac{M^2 b_1 - b_2}{32\pi^2} \ln \frac{M^2}{\mu^2} + \frac{1}{2} \Lambda + \frac{1}{2} \Lambda_s + \frac{1}{2} \lambda_s \varphi_{cl}^2,
$$

and $\mathcal{E}^{(3)}_c$ is given by Eq. (19), and $M^2 = m^2 + \lambda \varphi_{cl}^2$. The last term in Eq. (44) can not be regarded as volume or surface contribution.

From above expression for effective action we observe that one has to renormalize not merely parameters of volume part of action but also parameters of surface contribution, too. To renormalize effective action we use minimal subtraction procedure and make the following shift of constants

$$
m^2 \to m^2 + \frac{\lambda m^2}{16\pi^2 s}, \lambda \to \lambda + \frac{3\lambda^2}{16\pi^2 s}, \Lambda \to \Lambda + \frac{m^4}{32\pi^2 s}, \lambda_s \to \lambda_s - \frac{\lambda b_1}{16\pi^2 s}, \Lambda_s \to \Lambda_s - \frac{m^2 b_1 - b_2}{16\pi^2 s}.
$$

The renormalization of the surface parameters depends on the details of potential (brane) which are encoded in heat kernel coefficients. The same renormalization condition was used in Ref. [4]. We may realize our model as a model of scalar field in curved space-time but with singular scalar curvature as, for example, in Ref. [15] for wormhole space-time. In this case the parameter $\Lambda_s$ is proportional to radius of wormhole’s throat $a$ and $\lambda_s$ plays the role of non-conformal coupling $\lambda_s \sim a \xi$ and it has to be renormalized, too, as was noted in Ref. [22]. Therefore the effective action takes the following form

$$
\mathcal{S}^{\text{eff}}_{\Omega} = -\frac{3M^4}{128\pi^2 s} + \frac{M^4}{64\pi^2} \ln \frac{M^2}{\mu^2} + \frac{1}{2} m^2 \varphi_{cl}^2 + \frac{\lambda}{12} \varphi_{cl}^4 + \frac{1}{2} \Lambda,
\mathcal{S}^{\text{eff}}_{\Sigma} = \frac{M^2 b_1}{24\pi^3 s} + \frac{M^2 b_1}{32\pi^2} + \frac{M^2 b_1}{16\pi^3} - \frac{M^2 b_1 - b_2}{32\pi^2} \ln \frac{M^2}{\mu^2} + \frac{1}{2} \Lambda + \frac{1}{2} \lambda_s \varphi_{cl}^2.
$$
To fix the values of parameters of model both volume and surface parts we have the only free parameter $\mu$. To remove the renormalization unambiguous in effective potential we may use the standard relation for effective potential
\[
S^{eff}_\Omega(0)'' = m^2, \tag{46}
\]
by using which we obtain that $\ln m^2/\mu^2 = 1$. This value of $\mu$ we exploit in surface part of effective action. Therefore there is no way to fix surface parameters of the theory and ambiguous in renormalization procedure in surface term appears which was noted in Ref. [6]. The similar problem appears in Ref. [10] where effective action was calculated in space-time of cosmic string. We arrive at the following expression for action for zero value of classical field
\[
\left. S^{eff}_\Omega(0) = -\frac{m^4}{128\pi^2} + \frac{1}{2}\Lambda, \right. \tag{47}
\]
where the last term is calculated for whole space but without internal boundaries. In the end we have to tend $H \to \infty$.

In this case additional contribution appears as
\[
\sum_{n=0}^{\infty} \int_{M_n}^{\infty} dk (k^2 - M_n^2)^{-s} \tilde{Z}[k, \lambda_s, R, R'] \to \infty.
\]

We may use another approach to calculate zeta function at point $s = -1$ and subtract from integrand all terms of asymptotic expansion
\[
\zeta(s, D_2) = \tilde{\zeta}_0(s, D_2) + \tilde{\zeta}_0(s, D_2)
= \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty} \int_{M_n}^{\infty} dk (k^2 - M_n^2)^{-s} \tilde{Z}[k, \lambda_s, R, R']
+ \frac{1}{4\pi} \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} B_{2s} m^{-2s+2} \Gamma(s - 1 + \frac{n}{2}). \tag{48}
\]

In this case additional contribution appears as surface term in action:
\[
S = \Omega S^{eff}_\Omega + \sum S^{eff}_\Sigma + \beta E^{(3)}_{c} + O(e^{-m\beta}), \tag{49}
\]
where
\[
\Delta S^{eff}_\Sigma(0) \approx \frac{m^3}{12\pi} - \frac{m^2 \lambda_s}{32\pi^2} + \frac{5\lambda_s^3}{576\pi^2} + \frac{1}{2}\Lambda_s + \frac{1}{16\pi^2} \left( -\frac{\lambda_s^3}{6} + \frac{m^2 \lambda_s}{6}\right) \ln \frac{\lambda_s}{m}.
\]

The Lukosz renormalization procedure considered in above sections gives the same finite result without a renormalization. We will apply this procedure to action before renormalization of constants. In accordance with this approach we have to include our system to large box, for example with $x = \pm H$, where $H > R, R'$. The space is divided into three parts: $M_1 : \{x \in (-H, -R')\}, M_2 : \{x \in (-R', R)\}$ and $M_3 : \{x \in (R, H)\}$. The contribution to energy from boundaries is given by equation
\[
\Delta S = S(M_1) + S(M_2) + S(M_3) - S(M),
\]
where the last term is calculated for whole space but without internal boundaries. In the end we have to tend $H \to \infty$. It is very easy to apply this formula in our case. One has
\[
\Delta S = 2\beta S^{eff}_\Sigma(\lambda_s = 0) - \frac{\beta}{12\pi^2} \int_{m}^{\infty} dx (k^2 - m^2)^{3/2} \ln \frac{k^2 \Psi(k, -R', H)|_{\lambda_s=0} \Psi(k, R, R') \Psi(k, H, -R)|_{\lambda_s=0}}{\Psi(k, H, H)}.
\]
where
\[ S_{eff}^{\Sigma}(\lambda_s = 0) = \frac{M^3 b_1/2}{24 \pi^{3/2}} + \frac{1}{2} \lambda_s + \frac{1}{2} \lambda_s \phi_{cl}^2. \]

In the limit \( H \to \infty \) we obtain the following expression
\[ \Delta S = 2 \beta S_{eff}^{\Sigma}(\lambda_s = 0) + \beta \tilde{E}_c^{(3)}[\lambda_s, R, R']. \]

and \( \tilde{E}_c^{(3)}[\lambda_s, R, R'] \) is given by Eq. (33). Therefore we observe that this procedure takes off a volume part, all pole terms and the renormalization parameter \( \mu \). Therefore, in framework of this renormalization the energy (with dimension surface energy density) has the following form
\[ E = \left. \frac{\partial \Delta S}{\partial \beta} \right|_{\beta \to \infty} = -\frac{M^3 b_1/2}{12 \pi^{3/2}} + \Lambda_s + \tilde{E}_c^{(3)}[\lambda_s, R, R']. \]

First two terms gives tension of brane with quantum corrections. The last term completely coincides with that obtained in Sec. III by subtracting all terms of asymptotic expansion and given by Eq. (34).

V. CONCLUSION

Let us summarize our results and observations. In framework of zeta-function approach we considered three model potentials namely, single singular potential, step function potential and three singular potentials. We note problems which were appeared in connection with these potentials: (i) the Dirichlet limit, \( \lambda_s \to \infty \), is ill defined and the energy is divergent in this limit, (ii) the energy calculated for step regularized delta-function is divergent in the sharp limit, \( \epsilon \to 0 \), when the step function tends to the delta function, (iii) in massless limit \( m \to 0 \) the energy is divergent, too. All of these divergencies has the same structure – the energy is logarithmical divergent with second heat kernel coefficient as a factor (for 3 + 1 dimensional case).

In the model of three delta potentials we observe that the energy of this system for infinite separation of potentials transforms to a sum of energies each of which is peculiar to single potential (brane) itself which we called "surface" energy. To obtain physically reasonable result we suggest to define the Casimir energy as energy without this "surface" contribution. In this case the Casimir energy defined in this way for infinitely separated potentials (empty space) is zero. The same result may be obtained by using Lukosz renormalization procedure. We showed also, that this kind of renormalization correspond to subtracting from zeta-function all terms of its asymptotic expansion. The problem appears if we truncate this series and subtract the finite terms of series. The Casimir energy calculated in this way is well defined for all physical situations and in the Dirichlet limit the delta-function transforms to Dirichlet boundary as should be the case.

Next, we considered in details the effective action for single delta potential in (3 + 1) dimensional case in framework of zeta-regularization approach. To consider renormalization we use the non-linear \( \phi^4 \) model with brane part containing the brane tension and brane’s mass. The effective action has surface contributions except the volume part (effective potential). The renormalization of the surface part is ambiguous. It is possible to subtract pole terms by renormalization of the brane parameters. But there is no universal way to fix these parameters. The application the Lukosz renormalization procedure takes off all singularities of the model and gives the Casimir energy which is coincide with that obtained by suggested procedure.

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