Embeddings of non-simply-connected 4-manifolds in 7-space. II. On the smooth classification

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(2020 Mathematics subject classification: 57R52; 57R67; 55R15)

1. Overview and main results

We consider smooth manifolds, embeddings and isotopies. For an n-manifold $P$ denote by $E^m(P)$ the set of isotopy classes of embeddings $P \to \mathbb{R}^m$. The group $E^m(S^n)$ acts on $E^m(P)$ by embedded connected sum $\#$ and its quotient by $E^m(P)$. Denote this action by # and its quotient by $E^m_{\#}(P)$.

Remark 1.1 (The action of knots in general). If the quotient $E^m_{\#}(P)$ is known for a closed n-manifold $P$, the description of $E^m(P)$ is reduced to the determination of knots $S^1 \to \mathbb{R}^7$ acting on $E^m(P)$.

Keywords: 4-manifolds; embeddings; higher-dimensional knots; embedded connected sum; isotopy
of the orbits of the embedded connected sum action of $E^m(S^n)$ on $E^m(P)$. For a general closed $n$-manifold $P$ describing the action by a non-zero group of knots $E^m(S^n)$ on $E^m(P)$ is a non-trivial task. For the cases when the quotient $E^m_{\#}(N)$ coincides with the set of PL embeddings up to PL isotopy, the quotient has been known since 1960s [18–20]. However, until recently no description of the action (or, equivalently, no classification of $E^m(P)$) was known for $E^m(S^n) \neq 0$ and $P$ not a disjoint union of homology spheres. For recent results see [3, 13, 14]. On the other hand, the description of the action in [2, 15, 16] is not hard, the hard part of the cases considered there is rather the description of the quotient $E^m_{\#}(P)$.

There are non-isotopic embeddings $g_1, g_2 : S^2 \to S^4$ and an embedding $f : \mathbb{R}P^2 \to S^4$ such that $f \# g_1$ is isotopic to $f \# g_2$ [21]; i.e. the action of the monoid $E^4(S^2)$ on $E^4(\mathbb{R}P^2)$ is not free.

Various authors have studied the analogous connected sum action of the group of homotopy $n$-spheres on the set of smooth $n$-manifolds homeomorphic to given manifold; see for example [10, 11, 23] and references there.

More motivation and background for this paper may be found in [12, 17, 19] and Part I [4, § 1]. In this paper $N$ is a closed connected oriented 4-manifold and $H_q := H_q(N; \mathbb{Z})$. We present a readily calculable classification (in the sense of [17, Remark 1.2]) of $E^7(N)$ when $H_1$ is torsion free (up to an indeterminacy in certain cases). See theorems 1.2, 1.6 and corollaries 1.5, 1.8 below. Our classification is complete when $H_2 = 0$ (see theorem 1.6 and corollary 1.8.b) or when the signature of $N$ is divisible neither by 64 nor by 9 (see theorem 1.6 and corollary 1.5). The classification requires finding a complete set of invariants and constructing embeddings realizing particular values of these invariants. The invariants we use are described in [4, lemma 1.3, § 2.2, § 2.3] and § 2. An overview of the proof of their completeness is given in [4, § 1.4] and in [5, remark 7.4] (using definitions recalled at the beginning of § 2).

The action of $E^7(S^4) \cong \mathbb{Z}_{12}$ on $E^7(N)$ was investigated in [14] and determined when $H_1 = 0$ in [3], which also classified $E^7(N)$ in this case. In [4] we described the quotient $E^7_{\#}(N)$ when $H_1 = 0$. Thus the main novelty of this paper is the description of this action for $H_1 \neq 0$.

Denote by $q_{\#} : E^7(N) \to E^7_{\#}(N)$ the quotient map.

Let us state our main result for $N = S^1 \times S^3$. For this identify $E^7(S^4)$ and $\mathbb{Z}_{12}$ by the isomorphism $\eta$ of [3] (recalled in a more general situation in § 2) and consider the following diagram (where the left triangle is not commutative):

$$\begin{array}{ccc}
\mathbb{Z}_{12} \times \mathbb{Z}^2 & \xymatrix{\ar[r]^\text{pr}_2 & \mathbb{Z}^2} \\
\# \times \tau \ar[d] \ar[u]^\# \times \tau & \tau \ar[d]_{\#} \ar@{=>}[ur]^{\tau_{\#} := q_{\#} \tau} \\
E^7(S^1 \times S^3) & E^7_{\#}(S^1 \times S^3).
\end{array}$$

The map $\tau$ is defined in [4, § 1.2]. We define the map $\# \times \tau$ by $(\# \times \tau)(a, l, b) := a \# \tau(l, b)$. 
Theorem 1.2. The map $\# \times \tau : \mathbb{Z}_{12} \times \mathbb{Z}^2 \to E^7(S^4 \times S^3)$ is a surjection such that

(a) for different pairs $l, b$ the sets $P_{l,b} := (\# \times \tau)(\mathbb{Z}_{12} \times (l, b))$ either are disjoint or coincide;

(b) $P_{l,b} = P_{l',b'}$ if $(l = l' \land b \equiv b' \mod 2l)$;

(c) $|P_{l,b}| = \begin{cases} 12 & l \neq 0 \\ 2 \gcd(b, 6) & l = 0. \end{cases}$

In theorem 1.2 the surjectivity of $\tau$ and parts (a) and (b) follow from [4, theorem 1.1]. The new part of theorem 1.2 is (c); this part follows from corollary 1.8.b below (because for $l \neq 0$ the group $\text{coker} \ l$ is finite, so $\text{div} \ b = 0$). Cf. [5, addendum 7.3].

Example 1.3. There is an embedding $f : S^1 \times S^3 \to S^7$ with $f(N) \subset S^6$ and a pair of non-isotopic embeddings $g_1, g_2 : S^4 \to S^7$ such that $f \# g_1$ and $f \# g_2$ are isotopic.

This example follows because $|P_{0,1}| = 2$ by theorem 1.2 and there is a representative of $\tau(0,1)$ whose image is in $S^6 \subset S^7$ [4, lemma 2.18]. Example 1.3 shows the necessity of the simple-connectivity assumption in the following result (which is [14, the effectiveness theorem 1.2]):

If $f : N \to S^7$ is an embedding of a spin simply-connected closed 4-manifold $N$, $f(N) \subset S^6$ and embeddings $g_1, g_2 : S^4 \to S^7$ are not isotopic, then $f \# g_1$ and $f \# g_2$ are not isotopic.

Before stating our main result for the general case in theorem 1.6 below we state the following corollaries of it.

Corollary 1.4. (of theorem 1.6.c; proved in § 3) Let $N$ be a closed connected orientable 4-manifold with torsion free $H_1$. Then the following statements are equivalent:

(i) for every embedding $f : N \to S^7$ and non-isotopic embeddings $g_1, g_2 : S^4 \to S^7$ the embeddings $f \# g_1$ and $f \# g_2$ are not isotopic (i.e. the action $\#$ of $E^7(S^4)$ on $E^7(N)$ is free);

(ii) $N$ is an integral homology 4-sphere.

The definition of the Boéchat-Haefliger invariant $\varkappa : E^7(N) \to H_2$ is recalled in § 2.

Corollary 1.5. (of theorem 1.6.c) Let $N$ be a closed connected orientable 4-manifold with torsion free $H_1$ and $f : N \to S^7$ an embedding.

(a) If $\varkappa(f)$ is neither divisible by 4 nor by 3, then for every embedding $g : S^4 \to S^7$ the embeddings $f \# g$ and $f$ are isotopic.
If \( \varkappa(f) \) is divisible by 4 but neither by 8 nor by 3, then there is a non-trivial embedding \( g_1 : S^4 \to S^7 \) such that for every embedding \( g : S^4 \to S^7 \) the embedding \( f \# g \) is isotopic to either \( f \) or \( f \# g_1 \).

Corollary 1.5 follows from corollary 1.8.bc (or from theorem 1.6.c and addendum 1.7 because \( 4 \mathbb{Z}_{\gcd(\varkappa(f), 24)} = 0 \) under the assumptions of corollary 1.5). The assumption of corollary 1.5.a is automatically satisfied when the signature of \( N \) is divisible neither by 16 nor by 9.

Before stating our main results for the general case, we establish some conventions, notation and definitions.

**Coefficients and intersections in manifolds.** Unless otherwise stated, we omit \( \mathbb{Z} \)-coefficients from the notation of (co)homology groups. We identify the coefficient group \( \mathbb{Z} \) as the \( \mathbb{Z} \)-group of a free abelian group denote by \( \text{div} \) the divisibility of \( \mathbb{Z} \). Let \( \rho_n \) be reduction modulo \( n \). The intersection \( x \cap y \) of a \( \mathbb{Z} \)-homology class \( x \) and a \( \mathbb{Z}_n \)-homology class \( y \) is defined as the \( \mathbb{Z}_n \)-homology class \( \rho_n x \cap y \).

Let \( \hat{u} := \text{gcd}(u, 24) \) for an element \( u \) of a free abelian group.

Let

\[
H^\text{DIFF}_2 := \{ u \in H_2 \mid \rho_2 u = \text{PD} w_2(N), \ u \cap_N u = \sigma(N) \} \subset H_2.
\]

The map \( \varkappa_\# : E^7_\#(N) \to H_2 \) is well-defined by the formula \( \varkappa = \varkappa_\# q_\# \) because of the additivity of \( \varkappa \) [4, lemma 2.3].

If \( H_1 = 0 \), then \( \varkappa_\# \) is an injection whose image is \( H^\text{DIFF}_2 \).

The image of \( \varkappa = \varkappa_\# q_\# \) is \( H^\text{DIFF}_2 \) and \( |\varkappa^{-1}(u)| = \hat{u} / \text{gcd}(u, 2) \) for each \( u \in H^\text{DIFF}_2 \).

Here the first sentence is easily deduced from [1], see [4, remark 2.20.e]. The second sentence is proved in [3]. Because of the second sentence we regard the Boéchat-Haefliger invariant as a map \( \varkappa : E^7(N) \to H^\text{DIFF}_2 \).

Our second main result is a generalization of the above statement to non-simply-connected 4-manifolds.

**Definition of div, \( B(H_3) \), \( \overline{l} \), a symmetric pair, \( K_{u,l} \), \( C_{u,l} \) and \( \cap_4 \).** For an element \( u \) of a free abelian group denote by \( \text{div} u \) the divisibility of \( u \), i.e. \( \text{div} 0 = 0 \) and \( \text{div} u \) is the largest integer which divides \( u \) for \( u \neq 0 \). For an element \( u \) of an abelian group \( G \) denote by \( \text{div} u \) the divisibility of \( [u] \in G/\text{Tors} G \).

Denote by \( B(H_3) \) the space of bilinear forms \( H_3 \times H_3 \to \mathbb{Z} \). For \( l \in B(H_3) \) denote by \( \overline{l} : H_3 \to H_1 \) the adjoint homomorphism uniquely defined by the property
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\[ l(x, y) = x \cap_N l'y. \]  

A pair \((u, l) \in H_2 \times B(H_3)\) is called symmetric if

\[ l(y, x) = l(x, y) + u \cap_N x \cap_N y \quad \text{for all} \quad x, y \in H_3. \]

For \(u \in H_2\), \(l \in B(H_3)\) and \(d := \text{div} u \in \mathbb{Z}\) denote

\[ K_{u,l} := \ker(2\rho_d l) \subset H_3 \quad \text{and} \quad C_{u,l} := \text{coker}(2\rho_d l). \]

If the pair \((u, l)\) is symmetric, then a bilinear map

\[ \cap_d: C_{u,l} \times K_{u,l} \rightarrow \mathbb{Z}_d \]

is well-defined by \([x] \cap_d y := x \cap_N y.\]

The maps \(\kappa, \lambda, \beta_{u,l}, \text{and} \ \eta_{u,l,b}, \theta_{u,l,b}\) of theorem 1.6 below are defined in \(\S\ 2.\) The definitions of \(\kappa, \lambda, \beta_{u,l}\) are recalled from [4, \S\ 2.2, 2.3] and the definitions of \(\eta_{u,l,b}\) and \(\theta_{u,l,b}\) are new.

**Theorem 1.6.** Let \(N\) be a closed connected orientable 4-manifold with torsion free \(H_1.\)

(a) The product

\[ \kappa \times \lambda: E^7(N) \rightarrow H_2^{DIFF} \times B(H_3) \]

has non-empty image consisting of symmetric pairs.

(b) For every \((u, l) \in \text{im}(\kappa \times \lambda)\) denote \(d := \text{div} u.\) Every map

\[ \beta_{u,l}: (\kappa \times \lambda)^{-1}(u, l) \rightarrow C_{u,l} \]

is surjective.

(c) For every \(b \in C_{u,l}\) every map

\[ \eta_{u,l,b}: \beta_{u,l}^{-1}(b) \rightarrow \frac{\mathbb{Z}_d}{\text{im} \theta_{u,l,b}} \]

is an injection whose image consists of all even elements. Moreover, the map

\[ \theta_{u,l,b}: K_{u,l} \rightarrow 4\mathbb{Z}_d \]

is a homomorphism and

\[ \theta_{u,l,b}(y) - \theta_{u,l,b'}(y) = 4\rho_d(b - b') \cap_d y \quad \text{for every} \quad b, b' \in C_{u,l} \quad \text{and} \quad y \in K_{u,l}. \]

(d) \(|\beta_{u,l}^{-1}(b)| = \frac{\hat{u}}{\gcd(u, 2)} \cdot |\text{im} \theta_{u,l,b}|.\)

\[ \text{1 Indeed, for each} \ x \in H_3 \text{and} \ y \in K_{u,l} \text{we have} \ 2\hat{\ell}x \cap_N y = 2l(x, y) \equiv 2l(y, x) = 2\hat{\ell}y \cap_N x \equiv 0. \]

Hence \(\text{im}(2\rho_d l) \cap_N K_{u,l} = \{0\} \subset \mathbb{Z}_d.\)
We call geometrically defined maps invariants. In particular, the maps \( \lambda \) and \( \varpi \) are invariants.

The maps \( \beta_{u,l} \) and \( \eta_{u,l,b} \) are relative invariants. See [5, the remark after theorem 1.6].

Parts (a) and (b) of theorem 1.6 follow from [4, theorem 1.3]. The new part of theorem 1.6 is (c), which is proven in § 2. Part (d) follows because by (c) \( \text{im} \eta_{u,l,b} = 2\hat{\mathbb{Z}}_d/\text{im} \theta_{u,l,b} \).

We remark that theorem 1.2 is not an immediate corollary of theorem 1.6, cf. [4, remarks 2.20.a and 2.24].

Addendum 1.7. In the notation of theorem 1.6, for each \( a \in \mathbb{Z}_{12} \), \( (u,l) \in \text{im}(\varpi \times \lambda)^{-1}(u,l) \) and \( b \in C_{u,l} \) and \( f \in \beta_{u,l}^{-1}(b) \)

\[
\eta_{u,l,b}(f \# a) = \eta_{u,l,b}(f) + [2a] \in \frac{\mathbb{Z}_d}{\text{im} \theta_{u,l,b}}.
\]

This follows from the definition of \( \eta_{u,l,b} \) (§ 2) and [4, lemma 4.3.b].

Corollary 1.8. For each \( (u,l) \in \text{im}(\varpi \times \lambda) \) let \( d := \text{div} u \). There is \( f_{u,l} \in (\varpi \times \lambda)^{-1}(u,l) \) such that for each \( f \in (\varpi \times \lambda)^{-1}(u,l) \) and \( a, a' \in \mathbb{Z}_{12} \), denoting \( b := \beta(f, f_{u,l}) \in C_{u,l} \) we have

(a) \( f \# a = f \# a' \Leftrightarrow a = a' \), provided either
\begin{itemize}
  \item \( u = 0 \) and \( \text{div} b \) is divisible by 6, or
  \item \( u \neq 0 \), \( 2\rho_d \bar{l} = 0 \) and \( u \) is divisible by 24 \( \text{ord}(4b) \);
\end{itemize}

(b) \( f \# a = f \# a' \Leftrightarrow a \equiv a' \mod 2 \text{gcd}(\text{div} b, 6) \), provided \( u = 0 \);

(c) \( f \# a = f \# a' \Leftrightarrow a \equiv a' \mod \frac{\text{ord}(4b)}{\text{gcd}(u, 2)^2} \), provided \( u \neq 0 \) and \( 2\rho_d \bar{l} = 0 \).

Part (a) follows from parts (b,c). Parts (b,c) are proven in § 3. Cf. remark 3.1 and [5, § 7 and corollary 1.9].

2. Definition of the invariants

In this paper we use conventions, notation and the following definitions of [4, § §2.1, 4.1].

- \( N \) is a closed connected oriented 4-manifold with torsion free \( H_1 \);
- \( f, f_0, f_1 : N \to S^7 \) are embeddings;
- \( C = C_f \) is the closure of the complement in \( S^7 \) to a sufficiently small tubular neighbourhood of \( f(N) \); the orientation on \( C \) is inherited from the orientation of \( S^7 \);

\[ \text{The class } u \text{ is divisible by } d \text{ and hence by the order } \text{ord}(4b) \text{ of } d \text{ in the } d\text{-group } C_{u,l}. \]
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• \( \partial \nu = \partial \nu_f : \partial C \to N \) is the sphere subbundle of the normal vector bundle of \( f \): the total space of \( \partial \nu \) is identified with \( \partial C \).

Consider the following diagram:

\[
\begin{array}{ccc}
H_{q-2}(N) & \xrightarrow{PD} & H^{6-q}(N) \\
\downarrow \partial \nu' & A & \downarrow AD \\
H_{q+1}(C, \partial) & \xrightarrow{\partial C} & H_q(\partial C) & \xrightarrow{i_C} & H_q(C) \\
\downarrow PD & A & \downarrow AD \\
H^{6-q}(C) & \xleftarrow{AD} & H_q(N)
\end{array}
\]

Here \( AD \) is Alexander duality and \( A = A_f, \hat{A} = \hat{A}_f \) are homology Alexander duality isomorphisms.

**Definition of \( \kappa[f] \) and \( \lambda[f] \).** Define

\[
\kappa[f] := A_f^{-1}(A_f[N] \cap_{C_f} A_f[N]) \in H_2.
\]

This is well-defined because under an isotopy of \( f \) the above class is changed continuously (or, alternatively, because by [4, lemma 3.2.\( \kappa' \),e] this definition is equivalent to the earlier definitions \([1, 3, 4, 14]\)).

Define

\[
\lambda[f](x, y) := x \cap_N \hat{A}_f^{-1}(A_f[N] \cap_{C_f} \hat{A}_f y)
\]

for each \( x, y \in H_3 \). This is well-defined because under an isotopy of \( f \) the right-hand factor in the above formula is changed continuously (or, alternatively, because by [4, lemma 3.2.\( \lambda' \)] this definition is equivalent to the earlier definition [4]).

We abbreviate the subscript \( f_k \) to just \( k \). For a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) define a closed oriented 7-manifold \( M = M_\varphi := C_0 \cup_\varphi (-C_1) \). We call a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) a **\( \pi \)-isomorphism** if \( M_\varphi \) is parallelizable. We shall omit the phrase ‘for a bundle isomorphism \( \varphi \)’ if its choice is clear from the context.

If \( P \) is a (compact oriented) codimension \( c \) submanifold of a manifold \( Q \) and either \( y \in H_k(Q) \) or \( y \in H_k(Q, \partial) \), denote

\[
r_{P, Q}(y) = r_P(y) = y \cap P := PD((PDy)|_P) \in H_{k-c}(P, \partial).
\]

A class \( Y \in H_5(M_\varphi) \) is a **joint Seifert class** if \( Y \cap C_k = A_k[N] \) for each \( k = 0, 1 \) [4, lemma 3.13.a]. A joint Seifert class \( Y \in H_5(M_\varphi) \) is called a **d-class** for an integer \( d \) if \( \rho_d Y^2 = 0 \) (or, equivalently, \( Y^2 \in dH_3(M_\varphi) \)).

In this section we assume that \( \kappa(f_0) = \kappa(f_1) \) and \( \lambda(f_0) = \lambda(f_1) \). Denote \( d := \text{div} \kappa(f_0) \).
**Definition of** $\beta(f_0, f_1)$ and $\beta_{u,l}$. By [4, lemmas 2.4 and 2.5] there is a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$ and a joint Seifert class $Y \in H_5(M_\varphi)$. Define

$$\beta(f_0, f_1) := [(i(\partial C_0, M_\varphi, \partial \nu_0)^{-1})^{-1} \rho_d Y^2] \in C_{\varphi(f_0), \lambda(f_0)}$$

using the composition $H_1(N; \mathbb{Z}_d) \xrightarrow{\partial \nu_0} H_3(\partial C_0; \mathbb{Z}_d) \xrightarrow{i(\partial C_0, M_\varphi)} H_3(M_\varphi; \mathbb{Z}_d)$.

The map $\beta : ([\varphi \times \lambda])^{-1}(u, l) \to C_{u,l}$ is well-defined by $\beta([f], [g]) := \beta(f, g)$, see below.

Take $[f'] \in (\varphi \times \lambda)^{-1}(u, l)$ and define

$$\beta_{u,l} : (\varphi \times \lambda)^{-1}(u, l) \to C_{u,l} \text{ by } \beta_{u,l}[f] := \beta([f], [f'])$$

The map $\beta_{u,l}$ depends on $[f']$ but we do not indicate this in the notation.

**Proof that the map $\beta$ is well-defined.** This holds by the transitivity of $\beta$ [4, lemma 2.10] and because $\beta(f_0, f_1) = 0$ for isotopic embeddings $f_0, f_1$. The latter is proved below analogously to the proof of $\beta(f, f) = 0$ in [4, § 3.7, proof of the additivity of $\beta$ (lemma 2.9)].

Namely, take an isotopy $F : S^7 \times I \to S^7 \times I$ between $F_0 = \text{id} S^7$ and a diffeomorphism $F_1 : S^7 \to S^7$ such that $F_1 \circ f_0 = f_1$. Take tubular neighbourhoods so that $F_1 \partial C_0 = \partial C_1$. Take $\varphi = F_1|\partial C_0$. Define $C_F$ and the class $z := A_F[N \times I] \in H_6(C_F, \partial)$ analogously to $C_I$ and $A_f[N]$. Then $M_\varphi = \partial C_F$ is parallelizable. Take $Y = \partial z \in H_5(\partial C_F)$.

Take any $k = 0, 1$ and an integer $s > 0$. The map $r_{N \times k} \circ \partial : H_{s+1}(N \times I) \to H_s(N)$ is an isomorphism. Hence by Alexander duality the map $r_{C_k} \circ \partial : H_{s+1}(C_k, \partial) \to H_s(C_k, \partial)$ is an isomorphism. We have $r_{C_k} \partial z^2 = (A_k[N])^2$. Since $(A_k[N])^2$ is divisible by $d$, the class $z^2$ is divisible by $d$, so $Y^2$ is divisible by $d$. Thus $\beta(f_0, f_1) = 0$.

**Definition of** $\eta(\varphi, Y)$ for a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$ and a $d$-class $Y \in H_5(M_\varphi)$, of $Y_{f,y}$ and $\theta(f, y)$. Since $\varphi : \partial C_0 \to \partial C_1$ is a $\pi$-isomorphism, $M_\varphi$ is spin. Take any normal spin structure on $M$ given by [4, lemma 4.2]. Since $M_\varphi$ is simply-connected, a normal spin structure on $M_\varphi$ is unique. Since $\Omega^{spin}_7(CP^\infty) = 0$ [9, lemma 6.1] there is a 8-manifold $W$ with a normal spin structure and $z \in H_6(W, \partial)$ such that $\partial W = M_\varphi$ and $\partial z = Y$. Consider the following fragment of the exact sequence of the pair $(W, \partial W)$:

$$H_4(\partial W; \mathbb{Z}_d) \xrightarrow{i_\nu} H_4(W; \mathbb{Z}_d) \xrightarrow{j_\nu} H_4(W, \partial; \mathbb{Z}_d) \xrightarrow{\partial} H_3(\partial W; \mathbb{Z}_d)$$

Since $\partial W \rho_d z^2 = \rho_d Y^2 = 0$, there is a class $\overline{z^2} \in H_4(W; \mathbb{Z}_d)$ such that $j_\nu \overline{z^2} = \rho_d z^2$. Denote by $p^*_W \in H_4(W, \partial)$ the spin characteristic class [4, § 3.1]. Define

$$\eta(\varphi, Y) = \eta(f_0, f_1, d, \varphi, Y) := \rho_d(\overline{z^2} \cap_W (z^2 - p^*_W)) \in \mathbb{Z}_d.$$ 

For $y \in H_3$ denote

$$Y_{f,y} := \partial(A_f[N] \times I) + i\hat{A}_{f,y} \in H_5(C_I \times I) \text{ and } \theta(f, y) := \eta(id \partial C_f, Y_{f,y}) \in \mathbb{Z}_d.$$
LEMMA 2.1 (proved in §§ 4, 5). (a) For every $f$ and $y$, $\theta(f, y)$ is divisible by $4$.
(b) The map $\theta(f, \cdot): K_{\infty(f), \lambda(f)} \lambda(f) \to \mathbb{Z}_d$ is a homomorphism, where $d := \text{div } \infty(f)$.
(c) For every $[f_0], [f_1] \in (\infty \times \lambda)^{-1}(u, l)$ and $y \in K_{u, l}$, we have for $d := \text{div } u$ that $\theta(f, y) = \theta(f_1, y) = 4\rho_\delta(\beta_0(y) \cap N y)$.

Definition of $\theta_{u, l, b}$. Take any $(u, l) \in \text{im}(\infty \times \lambda)$ and $b \in \text{C}_{u, l}$. Let $d := \text{div } u$. Define
$$\theta_{u, l, b}: K_{u, l} \to 4\mathbb{Z}_d$$
by $\theta_{u, l, b}(y) := \theta(f, y)$, where $[f] \in \beta_{u, l}^{-1}(b)$.

The map $\theta_{u, l, b}$ is well-defined (i.e. is independent of the choice of $f$) and is a homomorphism by lemma 2.1.ab and the transitivity of $\beta$ [4, lemma 2.10].

Definition of $\eta(f_0, f_1)$. Take representatives $f_0, f_1$ of two isotopy classes in $(\infty \times \lambda)^{-1}(u, l)$ such that $\beta(f_0, f_1) = 0$. By [4, lemma 2.5] there is a $\pi$-isomorphism $\varphi: \partial C_0 \to \partial C_1$. By [4, lemma 4.1] there is a $d$-class $Y \in H_5(M_\varphi)$ for $d := \text{div } \infty(f_0)$. Define
$$\eta(f_0, f_1) := [\eta(\varphi, Y)] \in \frac{\mathbb{Z}_d}{\text{im } \theta_{u, l, b}}.$$ This is well-defined by [4, lemma 4.3.c] and lemma 2.3.a below, and is even by [4, lemma 4.3.a].

LEMMA 2.2. Let $f_0, f_1, f_2: N \to S^7$ be embeddings and $\varphi_01: \partial C_0 \to \partial C_1$, $\varphi_{12}: \partial C_1 \to \partial C_2 \pi$-isomorphisms and $Y_01 \in H_5(M_{\varphi_01})$, $Y_{12} \in H_5(M_{\varphi_{12}})$ $d$-classes. Then $\varphi_02 = \varphi_{12} \varphi_01$ is a $\pi$-isomorphism and there is a $d$-class $Y_02 \in H_5(M_{\varphi_02})$ such that $\eta(\varphi_02, Y_02) = \eta(\varphi_01, Y_01) + \eta(\varphi_{02}, Y_{12})$.

This is proved analogously to [3, lemma 2.10], cf. [13, § 2, additivity property] (the property that $Y_02$ is a $d$-class is achieved analogously to [4, § 4.3, proof of lemma 4.6]).

LEMMA 2.3. Let $[f_0], [f_1] \in (\infty \times \lambda)^{-1}(u, l)$ be such that $\beta(f_0, f_1) = 0$. Denote $d := \text{div } u$. Take any $\pi$-isomorphism $\varphi: \partial C_0 \to \partial C_1$.

(a) The residue $\eta(f_0, f_1)$ is independent of the choice of a $d$-class $Y \in H_5(M_{\varphi})$.
(b) If $\eta(f_0, f_1) = 0$, then there is a $d$-class $Y \in H_5(M_{\varphi})$ such that $\eta(\varphi, Y) = 0 \in \mathbb{Z}_d$.

Proof of (a). Take any pair of $d$-classes $Y', Y'' \in H_5(M_{\varphi})$. Part (a) follows because
$$\eta(\varphi, Y') - \eta(\varphi, Y'') \overset{(1)}{=} \eta(\text{id } \partial C_0, Y) \overset{(2)}{=} \theta(f_0, y) = \theta(f_0, \beta_{(f_0, f')}(y)) \in \mathbb{Z}_d,$$
where
- equality (1) holds for some $d$-class $Y \in H_5(M_{f_0})$ by lemma 2.2;
- equality (2) holds for some $y \in K_{u, l}$ by the description of $d$-classes [4, lemma 4.7].
Proof of (b). Part (b) follows because

\[ 0 \overset{(1)}{=} \eta(\varphi, Y') - \theta_{u,l,\beta_{u,l}(f_0)}(y) = \eta(\varphi, Y') - \theta(f_0, y) \overset{(3)}{=} \eta(\varphi, Y) \in \mathbb{Z}_{\hat{d}}, \]

where

- equality (1) holds for some \( d \)-class \( Y' \in H_5(M_{\varphi}) \) and \( y \in K_{u,l} \) because \( \eta(f_0, f_1) = 0; \)
- equality (3) holds for some \( d \)-class \( Y \in H_5(M_{\varphi}) \) by lemma 2.2.

\[ \square \]

**Lemma 2.4 (Transitivity of \( \eta \)).** For any triple of embeddings \( f_0, f_1, f_2 : N \to S^7 \) having the same values of \( \varphi \)- and \( \lambda \)-invariants and the property that \( \beta(f_0, f_1) = \beta(f_1, f_2) = 0 \), we have \( \eta(f_2, f_0) = \eta(f_2, f_1) + \eta(f_1, f_0) \).

This follows by lemma 2.2.

**Theorem 2.5 (Isotopy classification).** If \( \lambda(f_0) = \lambda(f_1) \), \( \varphi(f_0) = \varphi(f_1) \), \( \beta(f_0, f_1) = 0 \) and \( \eta(f_0, f_1) = 0 \), then \( f_0 \) is isotopic to \( f_1 \).

**Proof.** The proof is analogous to the proof of [4, Isotopy Classification Modulo Knots theorem 2.8]. We only need to replace the second paragraph of that proof by the following sentence: ‘Since \( \eta(f_0, f_1) = 0 \), by lemma 2.3.b we can change \( Y \) and assume additionally that \( \eta(\varphi, Y) = 0 \).’

**Definition of \( \eta_{u,l,b} \).** Take any \( [f_0] \in \beta_{u,l}^{-1}(b) \). Define the map

\[ \eta_{u,l,b} : \beta_{u,l}^{-1}(b) \to \frac{\mathbb{Z}_{\hat{d}}}{\text{im } \theta_{u,l,b}} \quad \text{by} \quad \eta_{u,l,b}[f] := \eta(f, f_0). \]

The map \( \eta_{u,l,b} \) depends on \( f_0 \) but we do not indicate this in the notation.

**Proof that the map \( \eta_{u,l,b} \) is well-defined.** This holds by the transitivity of \( \eta \) (lemma 2.4) and because \( \eta(f_0, f_1) = 0 \) for isotopic embeddings \( f_0, f_1 \).

Let us prove the latter. Repeat the second paragraph from the above proof that the map \( \beta \) is well-defined. Take \( W = C_P \). Then \( W \) is parallelizable, so \( p_W^* = 0 \). In the proof that the map \( \beta \) is well-defined we have shown that \( z^2 \) is divisible by \( d \). Hence \( \eta(f_0, f_1) = p^*_{\lambda}(z^2 \cap_W z^2) = 0 \).

**Proof of theorem 1.6.c.** The property on \( \theta_{u,l,b} - \theta_{u,l,b'} \) holds by lemma 2.1.c. The map \( \eta_{u,l,b} \) is injective by the isotopy classification theorem 2.5. The image of this map consists of all even elements by [4, lemma 4.3.a] and addendum 1.7.

\[ \square \]

3. Proof of corollaries 1.4 and 1.8.bc

**Proof of corollary 1.4.** By theorem 1.6.c and addendum 1.7 (ii) \( \Rightarrow \) (i).

The other direction is implied by the following assertions.

(*) If the action of knots is free and \( H_1 \) is torsion free, then \( H_1 = 0 \).

(**) If the action of knots is free and \( H_1 = 0 \), then \( H_2 = 0 \).

**Proof of (**).** By theorem 1.6.a there is an embedding \( f_1 : N \to S^7 \) and \( (u, l_1) := (\varphi \times \lambda)(f_1) \) is a symmetric pair. If \( H_1 \neq 0 \), then there is a basis \( \{y_1, \ldots, y_n\} \) for
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$H_3$ with $n > 0$. Express $l_1 = l_1^{ij}$ as a matrix with respect to this basis. For any symmetric matrix $a^{ij}$ the pair $(u, l_1 - a)$ is again a symmetric pair. Take $a^{ij}$ to be the symmetric matrix with $a^{ij} = l_1^{ij}$ for $i \leq j$. Then $l := l_1 - a$ is strictly upper-triangular with respect to the chosen basis; i.e. $l^{ij} = 0$ for $i > j$. The pair $(u, l)$ is symmetric and $l(y_1) = 0$. By theorem 1.6.a there is an embedding $f$ with $(x \times \lambda)(f) = (u, l)$.

Since $y_1 \in K_{u,l}$ is primitive, by Poincaré duality there is an element $z' \in H_1$. Since the action of knots is free, by theorem 1.6.c and addendum 1.7 $u$ is divisible by $24$ and $\theta_{u,l,b} = 0$ for every $b \in C_{u,l}$. Then $d := \text{div } u$ is divisible by $24$, and so is $\tilde{d}$. Hence by theorem 1.6.c

$$0 - 0 = \theta_{u,l,b}(y_1) - \theta_{u,l,b}(y_1) = 4\rho_3'(x \cap_N y_1) = 4 \not\in \mathbb{Z}_{\tilde{d}}.$$  

This contradiction shows that $H_1 = 0$.

Proof of (**). Since the action of knots is free, by theorem 1.6.c and addendum 1.7 every element $u \in H^2_{DIFF}$ is divisible by $24$. Since $\rho_2 u = \text{PD} w_2(N)$, we obtain $w_2(N) = 0$, so the intersection form $\cap_N$ of $N$ is even. If $H_2 \neq 0$, then the intersection form of $N$ is indefinite [6, theorem 1]. Hence by [7, theorem 1.2.21] this form is isomorphic to $mH_+ \oplus nE_8$, where $H_+$ is the standard hyperbolic form with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the form $E_8$ is positive definite, so $m > 0$. By [7, lemma 1.2.20] $\sigma(N)$ is divisible by $4$. Then $u := (2, \sigma(N)/2) \oplus (m - 1)0 \oplus n0 \in H^2_{DIFF}$ and $u$ is not divisible by $24$. This contradiction shows that $H_2 = 0$. □

Remark 3.1. In corollary 1.8.bc each of the assumptions ‘$u = 0$’ and ‘$u \neq 0$ and $2\rho_3\tilde{d} = 0$’ can be replaced by each of the following successively weaker assumptions:

(1) $\rho_3'K_{u,l} \subset \rho_3'H_3$ is a direct summand, or

(2) every homomorphism $\rho_3'K_{u,l} \rightarrow 4\mathbb{Z}_{\tilde{d}}$ extends to $\rho_3'H_3$, or

(3) there is an element $\tilde{b} \in C_{u,l}$ such that $\theta_{u,l,\tilde{b}} = 0$.

Clearly, ‘either $u = 0$ or $2\rho_3\tilde{d} = 0$’ $\Rightarrow$ (1) $\Rightarrow$ (2).

Proof that (2) $\Rightarrow$ (3). Take any $b' \in C_{u,l}$. We have $\theta_{u,l,b'} = \theta_{u,l,b'}^+ : \rho_3'H_3 \rightarrow 4\mathbb{Z}_{\tilde{d}}$. Extend $\theta_{u,l,b'}^+$ to a homomorphism $\rho_3'H_3 \rightarrow 4\mathbb{Z}_{\tilde{d}}$. Since $H_3$ is free, $\rho_3'H_3$ is a free $\mathbb{Z}_{\tilde{d}}$-module. Hence the latter homomorphism is divisible by $4$. Then by Poincaré duality there is a class $x \in \rho_3'H_1$ such that $\theta_{u,l,b'}^+(z) = 4x \cap_N z$ for every $z \in \rho_3'K_{u,l}$. Let $\tilde{b} := \tilde{b'} + [\tilde{x}]$, where $\tilde{x} \in \rho_3'H_1$ is a lifting of $x$. Then by theorem 1.6

$$\theta_{u,l,\tilde{b}}(y) = \theta_{u,l,b'}(y) - 4\rho_3'([\tilde{x}] \cap_N y) = \theta_{u,l,b'}^+(\rho_3'y) - 4x \cap_N \rho_3'y = 0$$

for every $y \in K_{u,l}$. □
Proof of corollary 1.8.bc under the assumption (3) of remark 3.1. Define the element \( \beta_{u,l}(f) := \tilde{d} - \beta_{u,l}(f) \in C_{u,l} \). Then \( \theta'_{u,l,b} = \theta_{u,l,b} - b \) for every \( b \in C_{u,l} \), hence \( \theta'_{u,l,0} = 0 \). Therefore we may assume that \( \beta_{u,l} \) is chosen so that \( \theta_{u,l,0} = 0 \).

Take any \( b \in C_{u,l} \) and denote \( K := 4b \cap_{d} K_{u,l} \subset \mathbb{Z}_d \). So

\[
gcd(d, 2) \cdot |\beta_{u,l}^{-1}(b)| = \begin{cases} \frac{\hat{d}}{\im \theta_{u,l,b}} & (1) \\ \frac{\hat{d}}{\rho_d K} & (2) \end{cases} = [Z_d : \rho_d K] = \gcd(\hat{d}, [Z_d : K]),
\]

where equalities (1) and (2) hold by theorem 1.6.d. Now corollary 1.8.bc is implied by addendum 1.7 and the following lemma 3.2. \( \square \)

**Lemma 3.2.** Let \( V \) be a free \( \mathbb{Z} \)-module, \( d \) an integer, \( \rho_d : V \to V/dV \) the reduction mod \( d \) and \( m : V \to V^* \) a homomorphism whose polarization \( V \times V \to \mathbb{Z} \) has a symmetric mod \( d \) reduction. Then

(a) a bilinear map \( \cap_d : \ker(\rho_d m) \times \ker(\rho_d m) \to \mathbb{Z}_d \) is well-defined by \([x] \cap_d y := \rho_d x(y)\) for \( x \in V^* \).

(b) for every \( c \in \ker(\rho_d m) \)

\[ [Z_d : c \cap_d \ker(\rho_d m)] = \begin{cases} \div c & d = 0, \\ \frac{d}{\ord c} & d \neq 0. \end{cases} \]

This lemma is elementary and so possibly known. Part (a) is simple and is essentially proved in footnote 1. Part (b) is proved in [5, §3].

4. Proof of Lemma 2.1.b

Before reading the proof of lemma 2.1 we recommend reading the idea of the proof in [5, §6].

In this and the following section \( l = \lambda(f) = \lambda(f_0) = \lambda(f_1), u = \kappa(f) = \kappa(f_0) = \kappa(f_1) \) and \( d = \div u \). Denote \( 1_m := (1, 0, \ldots, 0) \in S^m, \Delta := 1_2 \times D^4 \times 1_1 \) and \( t := S^2 \times 0 \times S^2 \). For every \( y \in H_3 \) take the following objects constructed in [4, proof of lemma 4.8]: 6-manifolds \( V \subset C_f \) and \( \tilde{V} := V \cup S^2 \times S^3 \) \( (S^2 \times D^4 \times 1) \), an embedding \( \iota_2 : S^2 \times S^3 \times D^2 \to \Int C_f = \Int C_f \times \frac{1}{2}, 8 \)-manifolds \( W_\pm \subset C_f \times I \) and \( W := W_- \cup S^2 \times S^3 \times S^2 \) \( (S^2 \times D^4 \times S^2) \), classes \( Z \in H_6(W, \partial) \) and \( z := Z + [\tilde{V}] \in H_6(W, \partial) \). The objects are not uniquely constructed from \( y \), and we allow arbitrary choices in that construction.

**Definition of \( W', W'_- \text{ and } \iota' : W' \to W.** Let

\[ W'_- := C_f - \int_{\iota_2} \text{ and } W' := W'_- \cup_{\gamma} S^2 \times D^4 \times S^1 \]

(the manifold \( W' \) may be called the result of an \( S^1 \)-parametric surgery along \( \iota_2 \).)

Define an embedding \( W'_- \to W_\pm \) by \( x \mapsto x \times 1/2 \). We assume that this embedding
and the standard embedding $S^2 \times D^4 \times S^1 \rightarrow S^2 \times D^4 \times S^2$ (that is the product of the identity and the equatorial inclusion $S^1 \rightarrow S^2$) fit together to give an embedding $i': W' \rightarrow W$.

Observe that $\Delta, \hat{V} \subset W'$.

**Lemma 4.1.** For every $y \in H_3$ we have

$$z^2 \cap_{W} W_- \equiv 2i_{V, \partial}(Z \cap V) \in H_4(W_-, \partial)$$

(since $\partial V \subset \partial W_-$, the inclusion induces a map $i_{V, \partial}: H_4(V, \partial) \rightarrow H_4(W_-, \partial)$).

**Proof.** Since $\hat{V} \subset W'$, we have $[\hat{V}]^2 = 0 \in H_4(W, \partial)$. Also

$$Z^2 \cap W_- = (A_f[N] \times I)^2 \cap W_- = A_f \chi(f) \times I \cap W_- \equiv 0 \in H_4(W_-, \partial).$$

Hence

$$z^2 \cap W_- = (Z + [\hat{V}])^2 \cap W_- \equiv 2(Z \cap W_- [\hat{V}]) \cap W_- = 2i_{V, \partial}(Z \cap \hat{V} \cap W_-)$$

$$= 2i_{V, \partial}(Z \cap V).$$

□

*Proof of lemma 2.1.b.* In this proof a statement or a construction involving $k$ holds or is made for $k = 0, 1$. Given $y_k \in K_{u, J}$ construct the manifold $W_k$ as $W$ of [4, proof of lemma 4.8] by parametric surgery in $C_f \times [k - 1, k]$. We add the subscript $k$ to $W_-, W', t, \Delta, Z, \hat{V}, z$ constructed in [4, proof of lemma 4.8]. (So unlike in other parts of this paper, a subscript 0 for a manifold does not mean deletion of a codimension 0 ball from the manifold.) Define

$$W := W_0 \cup_{C_f \times 0} W_1$$

and

$$W_- := C_f \times [-1, 1] \setminus \text{Int im}(v_{3,0} \sqcup v_{3,1})$$

$$= W_0 \cup_{C_f \times 0} W_1.\text{ }.$$\text{ }$

The manifold $W$ just defined should not be confused with the manifolds which were previously denoted $W$ but are now denoted $W_0$ and $W_1$. The same remark holds for $W_-$ and for $Z, V, \hat{V}, z$ constructed below.

The spin structure on $W_-$ coming from $S^7 \times [-1, 1]$ extends to $W$. Clearly, we have $\partial W = \partial(C_f \times [-1, 1]) \cong M_f$ (for the ‘boundary’ spin structures on $\partial(C_f \times [-1, 1])$ and on $M_f$).

Since $H_5(t_k \times \Delta_k) = 0$, by the Poincaré dual to cohomological exact sequence of the pair $(W, W_-)$ (cf. diagram (*) in [4, proof of lemma 4.8]), $r_{W_-}: H_6(W, \partial) \rightarrow$
\( H_6(W_-, \partial) \) is an epimorphism. Take any 
\[
Z \in r_{W_-}^{-1}(A_f[N] \times [-1, 1] \cap W_-) \subset H_6(W, \partial).
\]

Denote
\[
V := V_0 \cup V_1, \quad \hat{V} := \hat{V}_0 \cup \hat{V}_1 \quad \text{and} \quad \hat{z} := Z + [\hat{V}] \in H_6(W, \partial).
\]

Since \( \partial_W Z = Y_{f,0} \) and \( \partial_W \hat{V}_k = i_0 \hat{W} \hat{A}_f y_k \), we have \( \partial z = Y_{f, y_0 + y_1} \). Thus the pair \((W, z)\) is a spin null-bordism of \((M_f, Y_{f, y_0 + y_1})\).

Since \( y_k \in K_{u,t} \), we have \( \partial z_k \equiv 0 \). Take any \( \frac{\hat{z}_k}{d} \in j_W^{-1} \rho_d z_k^2 \). Let
\[
\frac{\hat{z}_k}{d} := i_{W_0, W} \frac{\hat{z}_k}{d} + i_{W_1, W} \frac{\hat{z}_k}{d}.
\]

Then \( \frac{\hat{z}_k}{d} \cap W_k = \frac{\hat{z}_k}{d} \). Also
\[
j_W \frac{\hat{z}_k}{d} \cap W_- = \sum_{k=0}^1 j_W \frac{\hat{z}_k}{d} \cap W_- = \rho_d \sum_{k=0}^1 z_k^2 \cap W_k- \quad \text{and}
\]
\[
\sum_{k=0}^1 z_k^2 \cap W_k- (1) = 2 \sum_{k=0}^1 i_{V_0, W_k} (Z_k \cap V_k) = 2 i_{V, W_-} (Z \cap V) (3) = \frac{z^2}{d} \cap W_-.
\]

Here the congruences (1) and (3) modulo \( d \) hold by lemma 4.1 and analogously to lemma 4.1, respectively.

Hence by the Poincaré dual to cohomological exact sequence of the pair \((W, W_-)\) with coefficients \( \mathbb{Z}_d \) (cf. diagram (*) in [4, proof of lemma 4.8]) \( j_W \frac{\hat{z}_k}{d} - \rho_d z^2 = n_0 \rho_0[t_0] + n_1 \rho_1[t_1] \) for some \( n_0, n_1 \in \mathbb{Z}_d \). We have
\[
n_k[t_k] = (j_W \frac{\hat{z}_k}{d} - \rho_d z^2) \cap W_k = j_W \frac{\hat{z}_k}{d} - \rho_d z^2_k = 0 \in H_4(W_k, \partial; \mathbb{Z}_d).
\]

Therefore \( n_0 = n_1 = 0 \). So \( j_W \frac{\hat{z}_k}{d} = \rho_d z^2 \).

Since \( W_k := W_k - C_f \times [0, (2k - 1)/3) \) is a deformation retract of \( W_k \), the inclusion \( \hat{W}_k \to W_k \) induces an isomorphism on \( H_4 \). Clearly, \( z \cap W_k = z_k \), so \( z^2 \cap W_k = z_k^2 \). Hence
\[
\frac{\hat{z}^2}{d} \cap W (z^2 - p_W^*) = \sum_{k=0}^1 (\frac{\hat{z}^2}{d} \cap W_k) \cap W_k ((z^2 - p_W^*) \cap W_k) = \sum_{k=0}^1 z_k^2 \cap W_k (z_k^2 - p_W^*).
\]

So \( \eta(f, \cdot) \) is a homomorphism. \( \square \)

5. Proof of lemma 2.1.ac

Lemma 5.1. For every \( y \in H_3 \) we have:

(a) \( \partial(Z \cap V) = [\partial \Delta] - i_{\partial C_f \cap \partial V} \xi y \in H_3(\partial V) \), where \( \xi : N_0 \to \partial C_f \) is a weakly unlinked section for \( f \) (see definition in [4, § 2.2]);

(b) \( p_W^* = 2m[t] \in H_4(W, \partial) \) for some \( m \in \mathbb{Z} \).
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Lemma 5.1.b is essentially proved in the proof of [4, lemma 4.8].

Proof of (a). The equality follows because

\[ Z \cap V = (Z \cap W_-) \cap V = (A_f[N] \times I) \cap V = A_f[N] \cap V \in H_4(V, \partial) \quad \text{and} \quad \partial(A_f[N] \cap V) = A_f[N] \cap \partial V = i_{\partial V}(A_f[N] \cap v) \]

\[ - i_{\partial V}(A_f[N] \cap \nu_f^{-1}P) \overset{(3)}{=} [\partial \Delta] - [\xi P]. \]

Here \( P \) and \( v \) are defined in [4, proof of lemma 4.8]. Equality (3) follows because

- \( A_f[N] \cap \nu_f^{-1}P = [\xi P] \) by [4, lemma 3.2.a].
- \[ A_f[N] \cap \text{im} \, v = [v(1_2 \times S^3)] = [\partial \Delta] \] since \( (A_f[N] \cap \text{im} \, v) \cap \text{im} \, v [v(S^2 \times 1_3)] = A_f[N] \cap C_f S_f = 1. \)

Here equality (2) holds because \( v(S^2 \times 1_3) \) is homologous to \( S_f \) in \( C_f \).

\[ \square \]

Lemma 5.2. For every \( y \in K_u,l \) there is a class \( \hat{z}^2 \in H_4(W; \mathbb{Z}_d) \) such that

(a) \( \overline{z^2} := \hat{z}^2 + n[t] \in j_w^{-1} \rho_d \hat{z}^2 \subset H_4(W; \partial; \mathbb{Z}_d) \) for some \( n \in \mathbb{Z}_d; \)

(b) \( [t]^2 = (\hat{z^2})^2 = 0 \in \mathbb{Z}_d \) and \( [t] \cap \overline{z^2} = 2 \in \mathbb{Z}_d. \)

The proof is given later in this section.

Proof of lemma 2.1.a. The lemma follows by [4, lemma 4.8] and lemmas 5.1.b, 5.2. Indeed,

\[ \overline{z^2} \cap \overline{z^2} = \overline{z^2} \cap \overline{z^2} - \overline{z^2} \cap \overline{z^2}, \quad (\hat{z^2} + n[t]) \cap \overline{z^2} - \overline{z^2} \cap \overline{z^2} - \overline{z^2} \cap \overline{z^2}, \quad [t]^2 = (\hat{z^2})^2 = 0 \in \mathbb{Z}_d. \]

Here

- equality (2) holds by lemma 5.1.b and property (a) of lemma 5.2,
- equality (3) holds by property (b) of lemma 5.2.

\[ \square \]

In the proof of lemma 2.1.c we will use not only the statement of lemma 5.2 but also the following definition, which is also used in the proof of lemma 5.2.

Definition of \( a, s, \hat{z^2} \) for \( y \in K_u,l \). By lemma 5.1.a there is a representative

\[ a \in C_4(V) \quad \text{of} \quad Z \cap V \in H_4(V, \partial) \quad \text{such that} \quad \partial a = \partial \Delta - \xi P. \]

(Such a representative is obtained from a representative \( a' \in C_4(V) \) of \( Z \cap V \in H_4(V, \partial) \) such that \( \partial a' = \partial \Delta - \xi P + \partial a'' \) for some \( a'' \in C_4(\partial V) \) by the formula \( a := a' - a''. \))
Since \( y \in K_{u,l} \), by [4, lemma 3.2, \( \lambda \)] there is a chain
\[
s \in C_4(C_f \times 0; \mathbb{Z}_d)
\]
such that \( \partial s = 2\xi P \times 0 \).

Define
\[
\hat{z}^2 := \left[ 2a - 2\Delta - 2\xi P \times \left[ 0, \frac{1}{2} \right] + s \right] \in H_4(W; \mathbb{Z}_d).
\]

**Proof of lemma 5.2.** We have
\[
\rho_d z^2 \cap W_{-} \overset{(1)}{=} 2\rho_d iv_{V,W_{-}} (Z \cap V) = [2a]_{W_{-}, 0} = [2a - 2\xi P \times [0, \frac{1}{2}] + s]_{W_{-}, 0}
\]
\[
= \hat{z}^2 \cap W_{-} = j_w z^2 \cap W_{-},
\]

where equality (1) follows by lemma 4.1. Hence by the Poincaré dual to cohomological exact sequence of the pair \((W, W_{-})\) (cf. diagram (*) in [4, proof of lemma 4.8]) \( \rho_d z^2 = j_w (z^2 + n[t]) \) for some \( n \in \mathbb{Z}_d \). Thus property (a) holds.

Let us prove property (b). We have \( [t] \cap_{\hat{z}} z^2 = [t] \cap_{W_{-}} (z^2 \cap W_{-}) = [t] \cap_{W_{-}} [2a]_{W_{-}, 0} = 2[t] \cap_{\partial W_{-}} [\partial a]
\]
\[
= 2[t] \cap_{t \times \partial \Delta} [\partial \Delta] = 2.
\]

Here the homology classes are taken in the space indicated under ‘\( \cap \)’ (so \([t]\) has different meaning in different parts of the formula), and \( z^2 \cap W_{-} = [2a]_{W_{-}, 0} \) is proved in the proof of (a). \( \square \)

**Proof of lemma 2.1.c.** Take any bundle isomorphism \( \varphi: \partial C_0 \to \partial C_1 \) given by [4, lemma 2.5]. Take a closed oriented 3-submanifold \( P \subset N \) realizing \( y \in H_{f_0} = H_{f_1} \).

For \( k = 0, 1 \) construct the maps \( v_j, j = 0, 1, 2, 3, \) manifolds \( V_k \subset C_k, \hat{V}_k, W'_k \) and \( W_k \), chains \( a_k, s_k \) and classes \( Z_k, z_k, z^2_k \) as in [4, proof of lemma 4.8] and above. (So unlike in other parts of this paper, subscript 0 of a manifold does not mean deletion of a codimension 0 ball from the manifold.) Define
\[
W := W_0 \cup_{\varphi \times \text{id} I: \partial C_0 \times I \to \partial C_1 \times I} W_1.
\]

The manifold \( W \) just defined should not be confused with the manifolds which were previously denoted \( W \) but are now denoted \( W_0 \) and \( W_1 \). The same remark holds for \( z, Z, \hat{V} \) constructed below.
Consider the following segment of the Poincaré dual to cohomological Mayer-Vietoris sequence:

\[ H_6(W, \partial) \xrightarrow{r_{W_0} \oplus r_{W_1}} H_6(W_0, \partial) \oplus H_6(W_1, \partial) \xrightarrow{r_0 \oplus (-r_1)} H_4(\partial C_0). \]

Here \( r_k \) is the composition \( H_6(W_k, \partial) \xrightarrow{\partial} H_5(\partial W_k) \xrightarrow{r_{W_0}} H_4(\partial C_0) \). We have

\[ r_k Z_k = (\partial Z_k) \cap \partial C_0 = Y_{f_k} \cap \partial C_0 \]

\[ = \partial(Y_{f_k} \cap C_k) \stackrel{(4)}{=} \partial A_k[N] \stackrel{(5)}{=} \partial A_{1-k}[N] \stackrel{(6)}{=} r_{1-k} Z_{1-k} \in H_4(\partial C_0). \]

Here

- equality (4) holds by descriptions of joint Seifert classes [4, lemma 3.13.a];
- equality (5) holds by agreement of Seifert classes [4, lemma 3.5.a];
- equality (6) holds analogously to the previous set of equalities.

Hence there exists \( Z \in H_6(W, \partial) \) such that \( Z \cap W_k = Z_k \). Denote

\[ \hat{V} := \hat{V}_0 \bigcup_{\varphi: \nu_0^{-1} P \to \nu_1^{-1} P} \hat{V}_1 \subset W' \quad \text{and} \quad z := [\hat{V}] \in H_6(W, \partial). \]

Clearly, \( z \cap W_k = z_k \).

Take \( z_k^2 \in H_4(W; \mathbb{Z}_d) \) given by lemma 5.2. Then by lemmas 5.1.b and 5.2

\[ \overline{z_k^2} \cap W \overset{p_W^*}{\longrightarrow} 4m_k = \overline{z_k^2} \cap W \overset{p_W^*}{\longrightarrow} 4n_k = 2\overline{z_k^2} \cap W \overline{z_k^2} = 2\overline{z_k^2} \cap W \overline{z_k^2}. \]

Hence

\[ \eta(f_k, y) = \rho_{\partial}^*(\overline{z_k^2} \cap W_k (2\overline{z_k^2} - p_{W_k}^*)) = \rho_{\partial}^*(\overline{z_k^2} \cap W (2\overline{z_k^2} - p_{W_k}^*)). \]

Take a weakly unlinked section \( \xi_0: N_0 \to \partial C_0 \) of \( f_0 \). By [4, lemma 3.4] \( \xi_1 := \varphi \xi_0 \) is an unlinked section of \( f_1 \). Hence

\[ \partial a_1 - \partial \Delta_1 = -\xi_0 P = -\xi_0 P = -\partial a_0 - \partial \Delta_0 \quad \text{and} \quad \partial s_1 = 2\xi_1 P = 2\xi_0 P = \partial s_0. \]

Identify \( M_\varphi \) with \( M_\varphi \times 0 \subset \partial W \) and subsets of \( M_\varphi \) with the corresponding subsets of \( W \). Denote

\[ \hat{a} := [\Delta_0 - a_0 + a_1 - \Delta_1] \in H_4(\hat{V}; \mathbb{Z}_d) \quad \text{and} \quad s := [s_0 - s_1] \in H_4(M_\varphi; \mathbb{Z}_d). \]

Then by the definition of \( \overline{z_k^2} \)

\[ \overline{z_0^2} - \overline{z_1^2} = i_\varphi s - 2i\hat{a}, \quad \text{where} \quad i_\varphi := i_{M_\varphi, W} \quad \text{and} \quad i := i_{\hat{V}, W}. \]

We have \( i_\varphi s \cap W p_W^* = s \cap_{M_\varphi} p_{M_\varphi}^* = 0. \)
Since 
\[(z \cap M_\varphi) \cap M_\varphi S^2_{f_0} = (\partial z \cap C_0) \cap C_0 S^2_{f_0} = Y_{f_0} \cap C_0 S^2_{f_0} = 1,\]
z \cap M_\varphi is a joint Seifert class for \(\varphi\). Then 
\[i_\varphi s \cap_W z^2 = (s \cap \partial C_0) \cap \partial C_0 (z^2 \cap \partial C_0) = (2\xi_0 y \cap \partial C_0 v_0')\beta = 2\beta \cap_N y,
\]
where
- \(\beta \in H_1(N; \mathbb{Z}_d)\) is a lifting of \(\beta(f_0, f_1)\);
- equality (2) follows because we have \(s \cap \partial C_0 = 2[\xi_0 P] = 2\xi_0 y\) and because we have the identity \(z^2 \cap \partial C_0 = (z \cap M_\varphi)^2 \cap \partial C_0 = \nu_0' \beta\) by the definition of \(\beta(f_0, f_1)\).

We have 
\[z^2 \cap_W i_\hat{a} = (Z + [\hat{V}])^2 \cap_W i_\hat{a} = Z^2 \cap_W i_\hat{a} + 2Z \cap_W [\hat{V}] \cap_W i_\hat{a} = (Z \cap \hat{V})^2 \cap \hat{V} \hat{a} + 2i(Z \cap \hat{V}) \cap_W i_\hat{a} = ((\hat{a})_{V}^3 + 2(i\hat{a})^2 = (\hat{a})_{V}^3,\]
where
- equality (1) follows by the definition of \(z\);
- equalities (2) and (5) follow because \(\hat{V} \subset W',\) so \([\hat{V}] = 0\) and \((i\hat{a})^2 = 0;\)
- equality (3) is obvious;
- equality (4) follows because \(Z \cap \hat{V} = \hat{a}\) by the definition of \(a_0, a_1, \hat{a}\).

Therefore \(i_\hat{a} \cap_W (2z^2 - p_W^*) = 2((\hat{a})_{V}^3 - \hat{a} \cap \hat{V} p_V^*) \equiv 0\) by [22, theorem 5].

Now the lemma follows because 
\[(\hat{z}^2_0 - \hat{z}^2_1) \cap_W (2z^2 - p_W^*) = 2i_\varphi s \cap_W z^2 \
- i_\varphi s \cap_W p_W^* - 2i\hat{a} \cap_W (2z^2 - p_W^*) \equiv 4\beta \cap_N y.\]

\[\square\]

Acknowledgements
We would like to thank B. Owens for assistance with the literature on 4-manifolds and the anonymous referee for helpful remarks. We would like to thank the Hausdorff Institute for Mathematics and the University of Bonn for their hospitality and support during the early stages of this project. Supported in part by the Russian Foundation for Basic Research Grants No. 15-01-06302 and 19-01-00169, by Simons-IUM Fellowship and by the D. Zimin Dynasty Foundation.

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