Abstract

The Levi-Civita connection and geodesic equations for a stationary spacetime are studied in depth. General formulae which generalize those for warped products are obtained. These results are applied to some regions of Kerr spacetime previously studied by using variational methods. We show that they are neither space-convex nor geodesically connected. Moreover, the whole stationary part of Kerr spacetime is not geodesically connected, except when the angular momentum is equal to zero (Schwarzschild spacetime).
1 Introduction

The existence of a Killing vector field $K$ on a spacetime $(M, \langle \cdot, \cdot \rangle)$ is specially useful to study its geometry. It is well–known that, around each point $p$ such that $K_p \neq 0$, coordinates $(t, x^1, \ldots, x^n)$ can be chosen such that $K = \partial_t$ and all the components $g_{ij}$ of the metric are independent of $t$; this justifies the name “stationary” for the spacetime when $K$ is timelike. The stationary observers along $K$ not only see a non-changing metric but also find a constant $E = \langle \partial_t, \gamma' \rangle$ for any geodesic $\gamma$; thus, photons and freely falling particles has constant energy $E$ for these observers. When a non-vanishing Killing vector field $K$ is irrotational, i.e. the orthogonal distribution $K^\perp$ is involutive, then a local warped product structure appears ($g_{ti} = 0$, for $i = 1, \ldots, n$); if, additionally, $K$ is timelike (static case), the observers along $K$ measure a metric with no cross terms between space and time.

Since the introduction by Bishop and O’Neill of Riemannian warped products [BO], warped structures has been widely studied. This includes the Lorentzian case, where the contributions by O’Neill [O-83] and Beem, Ehrlich and Powell [BEP] (see also [BEC]) have been especially relevant. Recall that there are many examples of warped products among classical relativistic spacetimes. Nevertheless, the non-warped stationary case becomes difficult (we refer to [Sa-97] for a summary of mathematical properties in this general case).

Among the more interesting problems in stationary spacetimes appear those related to geodesics, as geodesic completeness or geodesic connectedness. The former has been widely studied by Romero and one of the authors [RS-94], [RS-95]. The latter was studied from a variational viewpoint first by Benci, Fortunato and Giannoni [BF], [BFG-90], and then by several authors (see the book [Ma-94]). The more classical stationary spacetime is Kerr spacetime which, essentially, represents the stationary gravitational field outside a rotating star. This spacetime is characterized by the mass $m > 0$ and angular momentum $ma$ of the star; in the limit case $a = 0$ it becomes Schwarzschild (outer) spacetime, which is static. The more recent book by O’Neill [O-95] study specifically Kerr spacetime (including the extending —non-stationary— regions).

The aim of this article is to study geodesics in a stationary standard spacetime $M = \mathbb{R} \times M_0$, giving some applications to Kerr spacetime. It is organised as follows.

In Section 2 we give general formulas for the Levi-Civita connection, which are an extension of those for warped products. In fact, they remain valid if $K = \partial_t$ is not timelike, and they can be generalized to more general warped products with crossed terms (Theorem 1, Remark 1). Then, we study geodesic equations, generalizing those in [Sa-99] (Theorem 2, Remark 3). Finally, we give general formulas for the Hessian of a function $\phi$ independent of $t$, Hess$\phi$ (Theorem 3). This Hessian is directly involved in the problem of geodesic connectedness because of the following notion
of convexity. Let \( c \) be a regular value of \( \phi \) and \( D = \phi^{-1}((c, \infty)) \). The boundary \( \partial D = \phi^{-1}(c) \) is (time, light, space-like) convex if \( \text{Hess} \phi(\tilde{v}, \tilde{v}) \leq 0 \) for all (time, light, space-like) vectors \( \tilde{v} \) tangent to \( \partial D \). Under certain natural assumptions, the convexity of the boundary imply geodesic connectedness (see [Ma-94] or [Sa-01]).

In Section 3 we study some general properties of (the stationary part of) Kerr spacetime \( M^a \). The slow rotating \((a^2 < m^2)\) or limit \((a^2 = m^2)\) cases are specially interesting because of the properties of the boundary of the stationary part; moreover, Schwarzschild spacetime is also included as the limiting static case \( a = 0 \). Section 3 is centered on the cases \( a^2 \leq m^2 \); however, our study will also cover the fast rotating case \( a^2 > m^2 \) (Theorem 5). In particular, we study regions \( M^a_\epsilon \) introduced by Giannoni and Masiello [GM], [Ma-94]. Recall that, from a variational viewpoint, the boundary of \( M^a \) is singular and difficult to study, even in the static case [BFG-92]. Thus, regions \( M^a_\epsilon \), with smooth boundary but arbitrarily close to the boundary of \( M^a \), were introduced by these authors. For \( a \) and \( \epsilon \) small enough, they showed that the boundary of \( M^a_\epsilon \) is time and light convex, which allowed them to prove some results on the existence of causal geodesics. In Section 3 general formulas to study convexity are provided, and \( M^a_\epsilon \) is shown to be non-space convex, Corollary 2.

In Section 4 we use geodesic equations to prove directly that neither Kerr spacetime \( M^a \) for any \( a \neq 0 \) nor any region \( M^a_\epsilon \), \( \epsilon > 0 \) (including the case \( a = 0 \)) are geodesically connected.

Finally, in Section 5 we prove that the excluded case \((a = 0 = \epsilon, \text{Schwarzschild spacetime})\) is geodesically connected. Even though the geodesic connectedness of this spacetime has already been proven by using variational methods in [BFG-92], we include this proof because of several reasons: (1) it is completely different, based on topological arguments introduced by the authors in [FS-00], and fully adapted to this case, (2) it is easily translatable to Schwarzschild black hole (Remark 5), where variational methods seems to fail, and (3) it is a simple case of the more involved proof of the geodesic connectedness of outer Kerr spacetimes (where the causal character of \( \partial_t \) changes, if \( a \neq 0 \)), which will be the subject of a next article [FS-01].

2 Levi-Civita connection and geodesics

Let \((M_0, < \cdot, \cdot>_R)\) be a Riemannian manifold, \((\mathbb{R}, -dt^2)\) the set of the real numbers with its usual metric reversed, and \( \delta \) and \( \beta \) a vector field and a positive smooth function on \( M_0 \), respectively. A (standard) stationary spacetime is the product manifold \( M = \mathbb{R} \times M_0 \) endowed with the Lorentz metric

\[
< \cdot, \cdot >= -\beta(x) dt^2 + < \cdot, \cdot>_R + 2 < \delta, \cdot>_R dt.
\]
Choosing an orthonormal basis $B_0 = (e_1, \ldots, e_n)$ at some $T_xM_0$ the matrix of $<.,.>$ in $B = (\partial_t, e_1, \ldots, e_n)$ is
\[
\begin{pmatrix}
-\beta & \delta^1 & \cdots & \delta^n \\
\delta^1 & 1 & 0 \\
\vdots & \ddots & \ddots \\
\delta^n & 0 & 1 \\
\end{pmatrix}
\equiv
\begin{pmatrix}
-\beta & \delta^t \\
\delta & I_{dn} \\
\end{pmatrix}
\tag{2}
\]
where $\delta^t = (\delta^1, \ldots, \delta^n)$ are the components of $\delta$ in $B_0$, and $I_{dn}$ is the identity matrix $n \times n$. Putting $\Lambda = -1/(\beta + \|\delta\|^2_R)$, the inverse of (2) is
\[
\begin{pmatrix}
\Lambda & -\Lambda\delta^t \\
-\Lambda\delta & I_{dn} + \Lambda\delta \otimes \delta^t \\
\end{pmatrix}
\tag{3}
\]
where $\|\cdot\|_R$ denotes the $<.,.>_R$-norm.

From now on, $\nabla$ and $\nabla^R$ will denote the Levi-Civita connection of $<.,.>$ and $<.,.>_R$, respectively. For each vector field $V$ on $M_0$, $V \in \mathfrak{X}(M_0)$, its lifting to $M$ will be denoted $\bar{V}$: that is, $\bar{V}(t,x) = V_x$, $\forall(t,x) \in M$ (analogously, if necessary, for a vector field on $\mathbb{R}$).

**Theorem 1** Let $(\mathbb{R} \times M_0, <.,.>)$ be a stationary spacetime and let $V, W \in \mathfrak{X}(M_0)$. Then:

(i) $\nabla_{\partial_t}\partial_t = -\frac{1}{2}\Lambda <\delta, \nabla^R\beta>_R \partial_t + \frac{1}{2}\Lambda <\delta, \nabla^R\beta>_R \delta + \frac{1}{2}\nabla^R\beta$. \hspace{1cm} \tag{4}

(ii) $\nabla_{\bar{V}}\bar{W} = \frac{1}{2}\Lambda (<W, \nabla^R_W\delta>_R + <V, \nabla^R_W\delta>_R)\partial_t$
\[-\frac{1}{2}\Lambda (<W, \nabla^R_W\delta>_R + <V, \nabla^R_W\delta>_R)\delta + \nabla^R_W\bar{W}. \hspace{1cm} \tag{5}\]

(iii) $2\nabla_{\bar{V}}\partial_t = 2\nabla_{\partial_t}\bar{V} = -\Lambda(V(\beta) + <\delta, \nabla^R_V\delta>_R - <\nabla^R_\delta\delta, V>_R)\partial_t$
\[+\Lambda(V(\beta) + <\delta, \nabla^R_V\delta>_R - <\nabla^R_\delta\delta, V>_R)\delta + \nabla^R_\delta_<\delta, V>_R. \hspace{1cm} \tag{6}\]

where $^\sharp$ denotes the vector field on $M_0$ metrically associated to the corresponding 1-form (that is, $<Y, <\nabla^R_\delta\delta, V>_R^\sharp>_R = <\nabla^R_\delta\delta, V>_R$ for any $Y \in \mathfrak{X}(M_0)$).
Proof. First, recall Koszul’s formula

\[ 2 < \nabla_Y Z, X > = Y < Z, X > + Z < X, Y > - X < Y, Z > - < Y, [Z, X] > + < Z, [X, Y] > + < X, [Y, Z] > \]  \tag{7} 

for any vector fields \( Y, Z, X \in \mathfrak{X}(M) \).

(i) Clearly, \( 2 < \partial_t, \nabla_{\partial_t} \partial_t > = \partial_t < \partial_t, \partial_t > = 0 \). For any \( X \in \mathfrak{X}(M_0) \), as \( \partial_t < \partial_t, X > = \partial_t < \delta, X > = 0 \), thus:

\[ 2 < \nabla_{\partial_t} \partial_t, X > = 2 \partial_t < \partial_t, X > - X < \partial_t, \partial_t > = X(\beta) = < \nabla^R \beta, X >_R. \]

Therefore, the result follows multiplying the inverse matrix \((\mathfrak{I})\) by the components \((0, \frac{1}{2} \nabla^R \beta)\) of the 1-form associated to \( \nabla_{\partial_t} \partial_t \).

(ii) Fixed \( x_0 \in M \) there is no loss of generality if we assume \([V, W]_{x_0} = 0\). Let \( X \in \mathfrak{X}(M_0) \) satisfying \([X, W] = [X, V] = 0\) at \( x_0 \). So, we have at this point by using \((\mathfrak{I})\):

\[ 2 < \nabla_V W, X > = V < W, X >_R + W < X, V >_R - X < V, W >_R = 2 < \nabla^R_V W, X >_R \]

Moreover, using \( \nabla^R_W V + \nabla^R_V W = 2 \nabla^R_V W \) at \( x_0 \) and again by \((\mathfrak{I})\),

\[ 2 < \nabla_V W, \partial_t > = V < \partial_t, W >_R + W < \partial_t, V >_R = < W, \nabla^R_V \partial_t >_R + < V, \nabla^R_W \partial_t >_R + 2 < \partial_t, \nabla^R_V W >_R. \]

Finally, the components of \( \nabla_V W \) are obtained again by using \((\mathfrak{I})\).

(iii) Clearly

\[ 2 < \nabla_V \partial_t, \partial_t > = - V(\beta) = - < \nabla^R \beta, V >_R. \]

Taking \( X \in \mathfrak{X}(M_0) \) commuting with \( V \) at \( x_0 \), we have from \((\mathfrak{I})\)

\[ 2 < \nabla_V \partial_t, X > = \nabla < \partial_t, X > + \partial_t < \nabla_V X > - X < \nabla_V \partial_t > = V < \delta, X >_R - X < V, \delta >_R = < \nabla^R_V \delta, X >_R \]

and the result follows by using \((\mathfrak{I})\) again. \( \square \)

Remark 1 Let \( \nabla^R \delta(V, W) = < \nabla^R_V \delta, W >_R \), and consider its symmetric and skew-symmetric parts: \( \text{Sym}\nabla^R \delta(V, W) = (\nabla^R \delta(V, W) + \nabla^R \delta(W, V))/2 \); \( \text{Sk}\nabla^R \delta(V, W) = (\nabla^R \delta(V, W) - \nabla^R \delta(W, V))/2 \). In what follows, \( \text{rot}\delta = 2\text{Sk}\nabla^R \delta \). In Theorem \( \mathfrak{I} \) the
differences between the general stationary case and the static (warped) case are the following:

(i) In (4), the term:
\[
-\frac{1}{2} \Lambda < \delta, \nabla R \beta > R (\partial_t - \delta).
\]
Note that this term vanishes if \(\beta\) is constant. This happens for the conformal metric \(< \cdot, \cdot > / \beta\), which can be used to study null geodesics (recall that null geodesics are conformal-invariant, up to reparametrizations).

(ii) In (5), the term:
\[
\Lambda \text{Sym} \nabla R \delta(V, W)(\partial_t - \delta).
\]
This term vanishes if \(\nabla R \delta\) is skew symmetric, that is, if \(\delta\) is a Killing vector field.

(iii) In (6), the terms:
\[
\Lambda V(\beta) \delta - \Lambda \text{rot}(V, \delta)(\partial_t - \delta) + \text{rot}(V, \cdot)^\natural.
\]
These terms reduces to \(\Lambda V(\beta) \delta\) if \(\delta\) is irrotational, that is, locally a gradient vector field.

For geodesic equations, let \(\gamma(s) = (t(s), x(s))\) be a geodesic on \(M\) with initial condition \(\gamma'(0) = (t'(0), x'(0))\) and \(t'(0) \neq 0 = x'(0)\) (the modification otherwise would be straightforward). We can choose a vector field \(X\) (resp. \(T\)) on \(M_0\) (resp. \(\mathbb{R}\)) extending \(x'(s)\) (resp. \(t'(s)\)). Then, the vector field \(Z = T + X\) satisfies on \(\gamma\):
\[
0 = \nabla Z Z = \nabla_T T + \nabla_T X + \nabla_X T + \nabla_X X
\]
and we can use the equalities (4), (5) and (6) to rewrite this relation.

Note that on the geodesic \(\gamma\)
\[
\begin{align*}
X(x(s)) &= x'(s) \\
T(t(s)) &= t'(s) \partial_t
\end{align*}
\]
holds, and so
\[
\begin{align*}
\nabla_T T &= t' \partial_t(t') \partial_t + t'^2 \nabla_{\partial_t} \partial_t \\
\nabla_T X &= \nabla_X T = t' \nabla_{\partial_t} X.
\end{align*}
\]

In order to obtain an equation for \(x(s)\) we will use that, in the last member of (8), the sum of the components on \(M_0\) of the four vector fields must be 0. These components can be obtained from Theorem 4. Using also (10) and writing \(W^{M_0} = \) the projection of the vector field \(W\) on \(M_0\),
\[
\begin{align*}
(\nabla_T T)^{M_0} &= t'^2 (\nabla_{\partial_t} \partial_t)^{M_0} = t'^2 (\Lambda < \delta, \nabla R \beta > R \delta + \nabla R \beta) \\
(\nabla_T X + \nabla_X T)^{M_0} &= 2 t' (\nabla_X \partial_t)^{M_0} = \Lambda t' (X(\beta) + \text{rot}(X, \delta)) \delta + t' \text{rot}(X, \cdot)^\natural \\
(\nabla_X X)^{M_0} &= \nabla_R X - \Lambda \nabla R \delta(X, X) \delta.
\end{align*}
\]
Adding these three relations and composing with \( \gamma \), we have

\[
\nabla^R_{x'}x' = \Lambda \nabla^R \delta(x', x') \delta
- \Lambda t' (\langle \nabla^R \beta, x' \rangle_R + \text{rot}(x', \delta)) \delta
- t' \text{rot}\delta(x', \cdot)^2
- \frac{1}{2} t'^2 (\Lambda < \delta, \nabla^R \beta >_R \delta + \nabla^R \beta)
\] (11)

On the other hand, \( q = < \gamma', \gamma' > \) is constant for any geodesic \( \gamma \) and, as \( \partial_t \) is a Killing vector field, \( E = < \gamma', \partial_t > \) is constant too. That is, we also have the relations:

\[
\begin{align*}
\langle (t', x'), (t', x') \rangle &= -\beta t'^2 + 2 < \delta, x' >_R t' + < x', x' >_R = q \\
\langle \partial_t, (t', x') \rangle &= -\beta t' + < \delta, x' >_R = E.
\end{align*}
\] (12)

Let \( \mathcal{X}_{r,s}(M_0) \) be the space of \( r-covariant, s-contravariant \) tensor fields on \( M_0 \) \((\mathcal{X}(M_0) \equiv \mathcal{X}_{1,0}(M_0); \mathcal{X}_{1,1}(M_0) \) is identifiable to the space of endomorphism fields). Equation (11) can be written as

\[
\nabla^R_{x'}x' = t'^2 R_0(x) + t' R_1(x, x') + R_2(x, (x', x'))
\] (13)

with \( R_0 \in \mathcal{X}(M_0) \), \( R_1 \in \mathcal{X}_{1,1}(M_0) \), \( R_2 \in \mathcal{X}_{1,2}(M_0) \) putting:

\[
\begin{align*}
R_0(x) &= -\frac{1}{2} (\Lambda < \delta, \nabla^R \beta >_R \delta + \nabla^R \beta)(x) \\
R_1(x, x') &= -\Lambda (\langle \nabla^R \beta, x' \rangle_R + \text{rot}(x', \delta)) \delta(x) + \text{rot}(\cdot, x')^2 \\
R_2(x, (x', x')) &= \Lambda \text{Sym}\nabla^R \delta(x', x') \delta(x).
\end{align*}
\] (14)

**Remark 2** \( R_0 \) vanishes if \( \beta \) is constant, \( R_1 \) reduces to \( -\Lambda < \nabla^R \beta, x' >_R \delta(x) \) if \( \delta \) is irrotational, and \( R_2 \) vanishes if \( \delta \) is Killing.

Substituting in (13) the value of \( t' \) from the second equation in (12) a second order equation for the spacelike component \( x(s) \) is obtained. Then, the first relation (12) can be regarded as a first integral of this equation. Moreover, a geodesic can be reconstructed for any solution of this differential equation. Summing up:

**Theorem 2** Consider a curve \( \gamma(s) = (t(s), x(s)) \) in a stationary spacetime \((\mathbb{R} \times M_0, < ., >)\). The curve \( \gamma \) is a geodesic if and only if \( E = < \gamma', \partial_t > \) is a constant and, then, \( x(s) \) satisfy

\[
\nabla^R_{x'}x' = \overline{R}_0(x) + \overline{R}_1(x, x') + \overline{R}_2(x, x' \otimes x')
\] (15)

where:

\[
\begin{align*}
\overline{R}_0(x) &= \frac{E^2}{\beta^2} R_0(x) \\
\overline{R}_1(x, x') &= -\frac{E}{\beta} \left( 2 \frac{< \delta, x' >_R}{\beta} R_0(x) + R_1(x, x') \right) \\
\overline{R}_2(x, x' \otimes x') &= \frac{( < \delta, x' >_R )^2}{\beta^4} R_0(x) + \frac{< \delta, x' >_R}{\beta^2} R_1(x, x') + R_2(x, (x', x'))
\end{align*}
\]

and \( R_0, R_1, R_2 \) are as in (14).
Remark 3 This result extends the construction in [Sa-99] for the static case. Indeed, for any solution \(x(s)\) of (15), the second equation (12) yields the value of \(t(s)\). Moreover, all the geodesics can be reparametrized in such a way that \(E = 0, 1\). Thus, we have the following two cases:

(a) Case \(E = 0\). The geodesic \(\gamma\) is always spacelike and orthogonal to \(\partial_t\), and we can also assume that the value of \(q\) is fixed equal to 1. Recall that in this case \(R_0 = R_1 = 0\). In the static case, equation (15) is just the equation of the geodesics of \((M_0, \langle \cdot, \cdot \rangle)\); thus, these geodesics can be regarded as trivial. Nevertheless, when \(\delta\) is not null the differential equation becomes:

\[
\nabla^R_{x'} x' = \mathcal{L}_2(x, x' \otimes x'),
\]

(compare, for example, with [Ma-93]).

(b) Case \(E = 1\). Equation (15) becomes rather complicated in general. In the static case, \(x(s)\) satisfies the equation of a classical Riemannian particle under the potential \(V = -1/2\beta\), as studied in [Sa-99], and the constant \(q/2\) is the classical energy (kinetic plus potential) of this particle. In the general case this interpretation does not hold\(^1\), even though \(q\) is a constant of the motion.

Now, let us study the expression of the Hessian for the stationary spacetime. Let \(\phi\) be a smooth function defined on a stationary spacetime \(M = \mathbb{R} \times M_0\) and let \(\tilde{V}\) be a vector field on \(M\). At any point \(p = (t, x) \in M\)

\[
\text{Hess}_\phi(p)[\tilde{V}, \tilde{V}] = \tilde{V} \tilde{V} \phi(p) - \nabla_{\tilde{V}} \tilde{V}(\phi)(p).
\]

We can assume \(\tilde{V} = \tilde{T} + \tilde{V}\) being \(T\) and \(V\) vector fields on \(\mathbb{R}\) and \(M_0\), respectively, and suppose \(T_p = t'_0 \partial_t\). Next, if \(\phi\) is independent of \(t\); we obtain at \(p\)

\[
\text{Hess}_\phi(p)[\tilde{V}, \tilde{V}] = \tilde{V} \tilde{V} \phi - \nabla_{\tilde{V}} \tilde{V}(\phi) - \nabla_{\tilde{T}} \tilde{T}(\phi) - \nabla_{\tilde{T}} \tilde{T}(\phi) =
\]

\[
= \tilde{V} \tilde{V} \phi - \nabla_{\tilde{V}} \tilde{V}(\phi) - t'_0 \nabla_{\partial_t} \tilde{V}(\phi) - t'_0 \nabla_{\partial_t} \partial_t(\phi) - t'_0^2 \nabla_{\partial_t} \partial_t(\phi).\tag{16}
\]

Taking into account (4), (5) and (6) and using, again, that \(\phi\) is independent of \(t\) we have

\[
\begin{align*}
\nabla_{\tilde{T}} \tilde{V}(\phi) &= -\Lambda \text{Sym} \nabla^R \delta(W, W)^{(0)}(\phi) + \nabla^R \tilde{V}(\phi) \\
\nabla_{\partial_t} \partial_t(\phi) &= \frac{1}{2} \Lambda \nabla_{\tilde{T}} \beta > R \delta(\phi) + \frac{1}{2} \nabla^R \beta(\phi) + 2 \nabla_{\partial_t} \partial_t(\phi).\tag{17}
\end{align*}
\]

Finally, if we substitute (17) in (16) we obtain:

\(^1\)This is a reason for the different notation of the constants with respect to [Sa-99]: \((\lambda, \epsilon)\) in this reference becomes \((E, q)\) now.
Theorem 3 Let $\phi : \mathbb{R} \times M_0 \to \mathbb{R}$ be a function independent of $t$, and $\tilde{v} = t'_0 \partial_t + v$ a vector on $T_{(t,x)} M = T_t \mathbb{R} \times T_x M_0$. Then

$$
\text{Hess}_\phi(p)[\tilde{v}, \tilde{v}] = \text{Hess}_\phi^R(x)[v, v] - \Lambda \left( t'_0 v(\beta) + \frac{1}{2} t'_0 \delta(\beta) - \text{Sym} \nabla^R \delta(v, v) + t'_0 \text{rot}(v, \delta) \right) \delta(\phi) + t'_0 \text{rot}(\nabla^R \phi, v) - \frac{1}{2} t'_0 \nabla^R \beta(\phi).
$$

(18)

3 Kerr spacetime

Kerr spacetime represents the stationary axis-symmetric asymptotically flat gravitational field outside a rotating massive object. Let $m > 0$ and $a$ be two constants, such that $m$ represents the mass of the object and $ma$ the angular momentum as measured from infinity. In the Boyer-Lindquist coordinates $(t, r, \theta, \varphi)$, Kerr metric takes the form

$$
ds^2 = g_{4,4} dt^2 + g_{1,1} dr^2 + g_{2,2} d\theta^2 + g_{3,3} d\varphi^2 + 2g_{3,4} dtd\varphi
$$

(19)

with

$$
g_{1,1} = \frac{\lambda(r, \theta)}{\Delta(r)} \quad g_{3,3} = \left[ r^2 + a^2 + \frac{2ma \sin^2 \theta}{\lambda(r, \theta)} \sin^2 \theta \right] \quad g_{4,4} = -1 + \frac{2mr}{\lambda(r, \theta)}
$$

(20)

and being

$$
\lambda(r, \theta) = r^2 + a^2 \cos^2 \theta \quad \text{and} \quad \Delta(r) = r^2 - 2mr + a^2.
$$

So, (19) can be written as in (11) taking

$$
<.., >^R = g_{1,1} dr^2 + g_{2,2} d\theta^2 + g_{3,3} d\varphi^2
$$

$$
\delta = \frac{g_{3,4}}{g_{3,3}} \partial_{\varphi}
$$

$$
\beta(r, \theta) = -g_{4,4}
$$

(21)

The structure of Kerr spacetime depends on the physical constants of the object $m$ and $a$. In what follows, we will consider the case $a^2 \leq m^2$ (the case $a^2 > m^2$ is simpler and the conclusions for this case are summarized at the end of Section 4). The function $\Delta(r)$ has the zeroes $r_+ = m + \sqrt{m^2 - a^2}$ and $r_- = m - \sqrt{m^2 - a^2}$. The hypersurfaces $r = r_+, r = r_-$ are singular for (14); they are the event horizons. Outside the first one (outer Kerr spacetime, $r > r_+$) the metric is not stationary if $a^2 > 0$, because the sign of the coefficient of $-dt^2$

$$
\beta(r, \theta) = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}
$$
changes. Function $\beta(r, \theta)$ is null on the hypersurface

$$r = m + \sqrt{m^2 - a^2 \cos^2 \theta},$$

and positive in the region $M^a$ outside this limit,

$$M^a = \mathbb{R} \times \{ x \in \mathbb{R}^3 : r > m + \sqrt{m^2 - a^2 \cos^2 \theta} \}.$$  \(22\)

So, this region endowed with the metric \(13\) is stationary and is called stationary Kerr spacetime. Recall that if the rotating body covers the stationary limit hypersurface, then the gravitational field generated by the body is stationary (out of the body).

Next, we obtain an expression for the Hessian of a function as in Theorem 3, applicable to study the convexity of stationary regions type

$$M^a_\epsilon = \mathbb{R} \times \{ x \in \mathbb{R}^3 : r > m + \sqrt{m^2 + \epsilon - a^2 \cos^2 \theta} \} \quad \epsilon > 0,$$

as in [GM], [Ma-94]. Consider the function

$$\phi_a(r, \theta) = \frac{1}{2} (r^2 - 2mr + a^2 \cos^2 \theta).$$

Clearly

$$\partial M^a_\epsilon = \mathbb{R} \times \{ (r, \theta, \varphi) : \phi_a(r, \theta) = \frac{1}{2} \epsilon \}.$$

Since the radial component of the gradient of $\phi_a$ with respect to the Euclidean metric in $\mathbb{R}^3$ is equal to $r - m > 0$, we have that $\partial M^a_\epsilon$ is smooth.

We have not only that $\phi_a$ is independent of $t$ but also that is independent of $\varphi$ and, thus, $\delta(\phi_a) = 0$. So, as a consequence of Theorem 3, we obtain

**Corollary 1** For any vector $\tilde{v} = t'_0 \partial_t + v$ tangent to $M^a$,

$$\text{Hess}_{\phi_a}(p)[\tilde{v}, \tilde{v}] = \text{Hess}^{R}_{\phi_a}(x)[v, v] + t'_0 \text{rot} \delta(\nabla \phi_a, v) - \frac{1}{2} t'^2 \nabla^R \beta(\phi_a). \quad 24$$

Recall that $\partial M^a_\epsilon$ is (time, space or light) convex if $\text{Hess}_{\phi_a}(p)[\tilde{v}, \tilde{v}] \leq 0$ for any (time, space or light) vector $\tilde{v}$ tangent to $\partial M^a_\epsilon$. This can be checked from \(24\) because the three terms in the right-hand side are directly computable. Indeed, let $\gamma(s) = (r(s), \theta(s), \varphi(s))$ be a geodesic in $M_0$ such that $\gamma(0) = x$ and $\gamma'(0) = v(\equiv (r'_0, \theta'_0, \varphi'_0))$. Putting $h(s) = \phi_a(\gamma(s))$ we obtain $\text{Hess}^{R}_{\phi_a}(x)[v, v] = h''(0)$. On the other hand, since $\gamma(s)$ is a geodesic,

$$r'' = -\Gamma^1_{1,1} r'^2 - 2\Gamma^1_{1,2} r' \theta' - \Gamma^1_{2,2} \theta'^2 - \Gamma^1_{3,3} \varphi'^2,$$

$$\theta'' = -\Gamma^2_{1,1} r'^2 - 2\Gamma^2_{1,2} r' \theta' - \Gamma^2_{2,2} \theta'^2 - \Gamma^2_{3,3} \varphi'^2,$$

$$\varphi'' = -\Gamma^3_{1,1} r'^2 - 2\Gamma^3_{1,2} r' \theta' - \Gamma^3_{2,2} \theta'^2 - \Gamma^3_{3,3} \varphi'^2.$$  \(25\)
where $\Gamma^k_{i,j}$ are the Christoffel symbols for $<\ldots,>$. Thus, replacing (25) in $h''(0)$ and taking into account that $r'_0 = \frac{a^2 \sin 2\theta}{2(r-m)}$ (i.e., $\tilde{v}$ is tangent to $\partial M^\alpha$):

$$
\text{Hess}^R_{\phi_a}(x)[v,v] = \left(\frac{a^4 \sin^2 2\theta}{8(r-m)^2} - \Gamma^1_{1,1} \frac{a^4 \sin^2 2\theta}{4(r-m)} - \Gamma^1_{1,2} a^2 \sin 2\theta - \Gamma^1_{2,2} (r-m) - a^2 \cos 2\theta + \Gamma^2_{1,1} \frac{a^4 \sin^2 2\theta}{8(r-m)^2} + \Gamma^2_{1,2} a^2 \sin 2\theta + \Gamma^2_{2,2} \frac{a^2 \sin 2\theta}{2} \right)
$$

+ $\phi_0^2 \left(-\Gamma^1_{3,3} (r-m) + \Gamma^2_{3,3} \frac{a^2 \sin 2\theta}{2} \right).$  

A straightforward computation shows:

$$
\frac{1}{2} \nabla^R \beta(\phi_a) = \Gamma^1_{4,4} (r-m) - \Gamma^2_{4,4} \frac{a^2 \sin 2\theta}{2},
$$

$$
\text{rot} \delta(\nabla \phi_a, v) = \left(\Gamma^2_{3,4} a^2 \sin 2\theta - \Gamma^1_{3,4} 2(r-m)\right) \phi_0',
$$

where $\Gamma^k_{i,j}$ are the Christoffel symbols for $<\ldots,>$. Summing up, substituting (26) and (27) in (24), a general expression for the Hessian of a vector $\tilde{v} \equiv (t'_0, r'_0, \theta'_0, \phi'_0)$ tangent to $\partial M^\alpha$ is obtained in terms of $t'_0, \theta'_0, \phi'_0$. From this expression, one can study when the region $M^\alpha$ is (time, light or space) convex directly (compare with [Ma-94, Ch. 7]).

**Corollary 2** $M^\alpha$ is not space convex for any $a^2 \leq m^2, \epsilon > 0$.

**Proof.** From (24), if $\tilde{v} \in T_p M^\alpha$ is tangent to $M_0$ then $\text{Hess}_{\phi_a}(p)[\tilde{v},\tilde{v}]$ is the right-hand side of (24). So, if $p = (t = 0, r = m + \sqrt{m^2 + \epsilon}, \theta = \frac{\pi}{2}, \phi = 0) \in \partial M^\alpha$ and $\tilde{v} = \partial_\theta \in T_p \partial M^\alpha$:

$$
\text{Hess}_{\phi_a}(p)[\tilde{v},\tilde{v}] = -(r-m) \Gamma^1_{2,2} + a^2 = \frac{r(r-m) \Delta(r)}{\lambda(r,\theta)} + a^2 > 0. \quad \square
$$

4. **Non geodesic connectedness**

In this section we study the non geodesic connectedness of some regions of the slow ($a^2 < m^2$), extreme ($a^2 = m^2$) and fast ($a^2 > m^2$) Kerr spacetime.

**Theorem 4** Stationary Kerr spacetime $M^\alpha$ with $0 < a^2 \leq m^2$ is not geodesically connected.

**Proof.** The first integrals of the geodesic equations of Kerr spacetime are

$$
\lambda(r,\theta) \phi' = \frac{D(t)}{\sin^2 \theta} + a \frac{p(r)}{\Delta(r)}
$$

$$
\lambda(r,\theta) t' = a \frac{D(\theta)}{\sin^2 \theta} + (r^2 + a^2) \frac{p(r)}{\Delta(r)}
$$

$$
\lambda(r,\theta) r' = \Delta(r) (q r^2 - K) + \frac{p^2}{\sin^2 \theta}
$$

$$
\lambda(r,\theta) \theta' = K + q a^2 \cos^2 \theta - \frac{D(\phi)}{\sin^2 \theta}
$$

(28)
Thus choose $\theta$ for $\lambda \equiv \delta$ otherwise, either any zero of $r$, or not) regions of outer Kerr spacetime with $r > r_+ > 0$ and being $q$ (normalization of the geodesic; rest mass), $K$ (Carter constant), $L$ (angular momentum) and $E$ (energy measured by observers in $\partial_t$) constants (we follow the notation in \cite{O-95, Chapter 4}). If $\gamma(s)$ is a geodesic joining the points in the $z$–axis $p_0 \equiv (t_0 = 0, r_0, \theta_0 = 0)$ and $p_1 \equiv (t_1 = 0, r_1, \theta_1 = \pi)$, $(r_+ <) r_0 < r_1$, in particular, $\gamma$ reaches the $z$–axis and, from the last equation in (28), $L = 0$ (otherwise $\lambda(r, \theta)^2 \theta^2$ would be negative near the $z$–axis). Then, the equation for $t$ reduces to

$$\lambda(r, \theta)t' = E \left( \frac{(r^2 + a^2)^2}{(r - r_-)(r - r_+)} - a^2 \sin^2 \theta \right).$$

But

$$\frac{(r^2 + a^2)^2}{(r - r_-)(r - r_+)} - a^2 \sin^2 \theta$$

is positive in $r \in (r_+, +\infty)$ so if $\gamma$ satisfies $\Delta t = t_1 - t_0 = 0$, necessarily $E = 0$ and thus

$$\lambda(r, \theta)^2 r^2 = \Delta(r)(q r^2 - K),$$

$$\lambda(r, \theta)^2 \theta^2 = K + q a^2 \cos^2 \theta.$$  \hfill (29)

Even more, from (23) we can ensure that $q > 0$ and $0 \leq \frac{K}{q} \leq r_0^2$ because, otherwise, either $\lambda(r, \theta)^2 r^2$ or $\lambda(r, \theta)^2 \theta^2$ would be negative at some point of $\gamma$. So, any zero $r^*$ of $\frac{\lambda(r, \theta)^2 r^2}{q} = (r - r_-)(r - r_+)(r^2 - \frac{K}{q})$ cannot be greater than $r_0$. Now choose $r_1 < 2m$. Then if, say $\gamma(0) = r_0$, $\gamma(1) = r_1$, at the point $s_0 \in (0, 1)$ such that $\theta(s_0) = \frac{\pi}{2}$ we have $r(s_0) > 2m$ (see (22)) and $r'(s_0) \neq 0$. So, if $r'(s_0) > 0$ (resp. $< 0$) then $r(s_0) < r(1)$ (resp. $r(0) > r(s_0)$), in contradiction with $r_0, r_1 < 2m$. \hfill \Box

Note that the previous proof can be extended in order to prove that (stationary or not) regions $R$ of outer Kerr spacetime with $a^2 \leq m^2$ satisfying $r > r_+ + \nu$ ($\nu > 0$) are not geodesically connected\footnote{Recall that this proof cannot be extended to the case $\nu = 0$, which is geodesically connected \cite{FS-01}.}. In fact, as $q > 0$, $0 \leq \frac{K}{q} \leq r_0^2$ and, so, any zero of $\frac{\lambda(r, \theta)^2 r^2}{q}$ is not greater than $r_0$, if we reparametrize $\gamma$ by $r$ then either

$$\Delta \theta = \int_{r_0}^{r_1} \frac{\sqrt{\frac{K}{q} + a^2 \cos^2 \theta}}{\sqrt{(r - r_-)(r - r_+)(r^2 - \frac{K}{q})}} dr.$$
or, if \( r' \) vanishes at a point \( r^* ( < r_0) \), perhaps:

\[
\Delta \theta = \int_{r^*}^{r_0} \frac{\sqrt{K/q + a^2 \cos^2 \theta}}{(r - r_-)(r - r_+)(r^2 - K/q)} dr + \int_{r^*}^{r_1} \frac{\sqrt{K/q + a^2 \cos^2 \theta}}{(r - r_-)(r - r_+)(r^2 - K/q)} dr.
\]

As \( \gamma \) must lie in \( R \) then \( r_0, r_1, r^* > r_+ + \nu \); so, taking \( r_0, r_1 \) close enough to \( r_+ + \nu \) we obtain necessarily \( \Delta \theta \) small, which contradicts that \( \theta_1 - \theta_0 = \pi \).

Moreover, no region \( M_\epsilon^a \) (\( a^2 \leq m^2 \)), \( \epsilon > 0 \) is geodesically connected because of the following: (i) it lies in the region \( r > m + \sqrt{m^2 - a^2 + \epsilon} \), and (ii) the two points of this region non-connectable by geodesics found above, lie in \( M_\epsilon^a \). Summing up:

**Corollary 3** Regions (stationary or not) of outer Kerr spacetime with \( 0 \leq a^2 \leq m^2 \) determined by \( r > r_+ + \nu \) for some \( \nu > 0 \) are not geodesically connected.

Regions \( M_\epsilon^a \) (\( 0 \leq a^2 \leq m^2 \)) are not geodesically connected for any \( \epsilon > 0 \).

**Remark 4** In the fast Kerr spacetime regions \( M^a \) are again those with \( \beta > 0 \) (compare with (22)) and regions \( M_\epsilon^a \) do not have a natural sense. However, the same arguments work because \( \Delta(r) = r^2 - 2mr + a^2 \) admits a positive lower bound when \( r > 0 \) which only depends on \( a \). On the other hand, recall that in fast Kerr spacetime, one can consider \( r \in \mathbb{R} \) and, thus to check the non-geodesic connectedness of \( p_0 = (t_0, r_0 < 0, \theta_0 = 0) \) with \( p_1 = (t_1, r_1 > 2m, \theta_1 = \pi/2, \varphi_1) \), which lie in the stationary part. Summing up, we obtain

**Theorem 5** (i) Stationary fast Kerr spacetime is not geodesically connected (if we assume \( r > 0 \) as well as if \( r \in \mathbb{R} \)).

(ii) Regions (stationary or not) of fast Kerr spacetime determined by \( r > \nu \) for some \( \nu > 0 \) are not geodesically connected.

(iii) The whole fast Kerr spacetime (including non-stationary regions and \( r \in \mathbb{R} \)) is not geodesically connected.

### 5 Geodesic connectedness of Schwarzschild spacetime

In this section we prove that given two points in \( M^{a=0} \) there exist a geodesic joining them. Previously, we need the following technical result:
Lemma 1 Let \( \{f_n(x)\}_n \) be a sequence of continuous functions on \([a_n, b] \subseteq \mathbb{R}, a_n \to a < b\) satisfying \(0 < c \leq f_n(x) \leq C\) for all \(n\), and let \( \{p_n(x)\}_n \) be a sequence of polynomials with degree bounded in \(n\) satisfying for all \(n\): \(p_n(a_n) = 0\), \(p_n'(a_n) = S_n > 0\) and \(p_n^k(a_n) \geq 0\) for \(k \geq 2\).

(i) If \(\{S_n\}_n \to \infty\), then

\[
\int_{a_n}^{b} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \to 0.
\]

(ii) If \(\{S_n\}_n \to 0\) and \(p_n^k(a_n)\) admits an upper bound for \(k \geq 2\) and all \(n\), then

\[
\int_{a_n}^{b} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \to \infty.
\]

Proof. (i) Consider the sequence of polynomials \(\{q_n(x)\}_n\), \(q_n(x) = S_n(x - a_n) \leq p_n(x)\) defined on \([a_n, b]\). As \(\int_{a}^{b} \frac{C}{\sqrt{x - a}} \, dx < \infty\) we have

\[
\int_{a_n}^{b} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \leq \int_{a_n}^{b} \frac{C}{\sqrt{q_n(x)}} \, dx = \frac{1}{\sqrt{S_n}} \int_{a_n}^{b} \frac{C}{\sqrt{x - a_n}} \, dx \to 0.
\]

(ii) Because of the boundedness of \(p_n^k(a_n)\), there exists \(M > 0\) such that \(q_n(x) = S_n(x - a_n) + M(x - a_n)^2 \geq p_n(x)\) on \([a_n, b]\) for all \(n\). Then

\[
\int_{a_n}^{b} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \geq \int_{a_n}^{b} \frac{c}{\sqrt{q_n(x)}} \, dx = \int_{a_n}^{b} \frac{c}{\sqrt{S_n(x - a_n) + M(x - a_n)^2}} \, dx.
\]

But the sequence of last integrands converges uniformly on compact subsets of \((a, b)\) to the function \(\frac{c}{\sqrt{M(x-a)^2}}\). Therefore, as \(\int_{a}^{b} \frac{c}{\sqrt{M(x-a)^2}} = \infty\) we obtain that the limit in (30) is \(\infty\). \(\square\)

Theorem 6 Schwarzchild spacetime \(M^{a=0}\) is geodesically connected.

Proof. Given two arbitrary points \(p_0\) and \(p_1\), the spherical symmetry of \(M^{a=0}\) allows us to assume \(p_0 = (t_0, r_0, \varphi_0, \theta_0 = \frac{\pi}{2})\), \(p_1 = (t_1, r_1, \varphi_1, \theta_1 = \frac{\pi}{2})\), \(t_0 \leq t_1\); we can also assume \(r_0 \leq r_1\) (the modifications if \(r_0 > r_1\) are obvious). If we consider only geodesics \(\gamma(s)\) on the equatorial plane \(\theta \equiv \frac{\pi}{2}\), then its first integrals are obtained taking \(a = 0\), \(K \equiv L^2\) and \(\theta \equiv \frac{\pi}{2}\) in (28), that is:

\[
\begin{align*}
  r^2 \varphi' &= L \\
r^2 t' &= E - \frac{r^3}{r - 2m} \\
r^4 r' &= r(r - 2m)(qr^2 - L^2) + r^4 E^2.
\end{align*}
\]
Notice also that if \( t_0 = t_1 \) we can consider only geodesics with \( E = 0 \); otherwise we can normalize \( E = 1 \). Let \( s(r) \) be the inverse function (where it exists) of \( r(s) \) given by \((31)\); using \( r \) as parameter in the other two equations \((31)\):

\[
\frac{d\phi}{dr} = \frac{L}{\sqrt{r(r-2m)(qr^2-L^2) + r^4E^2}}, \\
\frac{dr}{dr} = \frac{E r^3}{(r-2m)\sqrt{r(r-2m)(qr^2-L^2) + r^4E^2}}
\]  

(32)

on a certain domain, being \( \epsilon \in \{\pm 1\} \). If one consider geodesics with \( r' \neq 0 \) at any point, then the geodesic can be reparametrized by \( r \) (recall \( |s(r_1) - s(r_0)| < \infty \)) and the increments \( \Delta t, \Delta \phi \) can be calculated integrating directly in \((32)\). Nevertheless, we are going to see that \( r'(s) \) vanishes exactly at one point \( s^* \), and \( r^* = r(s^*) \) satisfies \( 2m < r^* < r_0 \). Recall that the denominator in \((32)\)

\[
h(r) = r(r-2m)(qr^2-L^2) + r^4E^2
\]

(33)

will vanish at \( r^* \). As \( r(s) \) will go from \( r_0 \) to \( r^* \) then necessarily \( h'(r^*) > 0 \) (notice that this implies \( |s(r^*) - s(r_0)| < \infty \)). As later on \( r(s) \) will go from \( r^* \) to \( r_1 \), then \( h(r) > 0 \) if \( r^* < r < r_1 \); we will consider geodesics with \( h(r_1) > 0 \) too. Summing up, it is sufficient to find constants \( E, q, L^2 \) as well as \( r^* \in (2m, r_0) \) such that the following relations \((34)\) and \((36)\) hold:

\[
h(r^*) = 0, \quad h'(r^*) > 0, \quad h(r) > 0 \quad \text{on} \quad (r^*, r_1];
\]

(34)

putting

\[
\Delta t = \int_{r_0}^{r_1} -\frac{E r^3}{(r-2m)\sqrt{h(r)}} dr + \int_{r_0}^{r_1} -\frac{E r^3}{(r-2m)\sqrt{h(r)}} dr
\]

\[
\Delta \phi = \int_{r_0}^{r_1} \frac{L}{\sqrt{h(r)}} dr + \int_{r_0}^{r_1} \frac{L}{\sqrt{h(r)}} dr,
\]

(35)

then

\[
\Delta t = t_1 - t_0, \\
\Delta \phi = \varphi_1 - \varphi_0 + 2k\pi
\]

(36)

for some integer \( k \). Moreover, if \( t_0 = t_1 \) we can fix \( E = 0 \), if \( t_0 < t_1 \) we fix \( E = 1 \).

We will consider first the case \( t_0 < t_1 \) and, thus,

\[
h(r) = r(r-2m)(qr^2-L^2) + r^4.
\]

(37)

If we look for \( r^* \) such that \( h(r^*) = 0 \) and \( h'(r^*) = S > 0 \) then the following two relations for the constants \( q \) and \( L^2 \) are obtained:

\[
qr^*2 - L^2 = -\frac{r^3}{r-2m}, \\
q = \frac{r^*2}{2(r^*2-2m)^2} = \frac{r^*2}{2(r^*2-2m)} + \frac{s}{2(r^*2-2m)}.
\]

(38)
Taking into account the dependences of \( q \) on \((r^* - 2m)\) in (38), there exist \( r_k^* \in (2m, r_0) \) near enough to \( 2m \) such that if \( r^* \in (2m, r_k^*] \) then

\[
q > 0 \quad \forall S > 0
\]  

(note also that \( L^2 > 0 \) if \( q > 0 \)). Moreover, in order to apply Lemma [1],

\[
\begin{align*}
  h^2(r^*) &= 6q^* (r^* - 2m) + 2(qr^{*2} - L^2) + (4q + 12) r^{*2} \\
  h^3(r^*) &= 6q(r^* - 2m) + (18q + 24)r^* \\
  h^4(r^*) &= 24q + 24.
\end{align*}
\]

Clearly \( h^3(r^*), h^4(r^*) > 0 \) and, taking into account (38) again, \( h^2(r^*) > 0 \), so \( h(r) > 0 \) if \( r > r^* \). Summing up, it is sufficient to find an element of \( A \equiv \{(r^*, S) : r^* \in (2m, r_k^*], S \in (0, \infty)\} \) such that the corresponding \((q, L^2)\) given from (38) and the function \( h(r) \equiv h(r, q, L^2) \) in (37) satisfy (36) with \( \Delta t, \Delta \varphi \) as in (34) and \( E = 1 \).

Fix \( r^* \in (2m, r_k^*] \) and consider \( \{(r^*, S_n)\}_n, \{S_n\}_n \to \infty \), then taking

\[
  f_n(r) \equiv \frac{r^3}{r^2 - 2m} \quad \text{and} \quad p_n(r) \equiv h_n(r)
\]

with \( h_n \equiv h(r, q(r^*, S_n), L^2(r^*, S_n)) \), hypotheses of Lemma [1] (i) clearly hold on the interval \([a, b] = [r^*, r_1]\). Therefore,

\[
(\Delta t)_n = \int_{r^*}^{r_1} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr + \int_{r^*}^{r_1} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr \to 0.
\]

This also holds if we take a sequence \( \{r^*_n\} \to r^* \) and compute \( (\Delta t)_n \) for \( (r^*_n, S_n) \). Analogously, if we consider \( \{S_n\}_n \to 0 \) then, from (38) and (40), \( h^k(r^*) \) admits an upper bound for \( k \geq 2 \) and all \( n \) thus, from Lemma [1] (ii), \( (\Delta t)_n \to 0 \). In conclusion, given \( \{\epsilon_n\}_n, \epsilon_n > 0, \epsilon_n \searrow 0 \) there exists \( \{\delta_n\}_n, \delta_n > 0, \delta_n \searrow 0 \), such that

\[
\begin{align*}
  \Delta t(r^*, S) < t_1 - t_0 & \quad \text{when} \quad (r^*, S) \in [2m + \epsilon_n, r_k^*] \times [\frac{1}{\delta_n}, \infty) \\
  \Delta t(r^*, S) > t_1 - t_0 & \quad \text{when} \quad (r^*, S) \in [2m + \epsilon_n, r_k^*] \times (0, \delta_n]
\end{align*}
\]

Next, we use topological arguments based on Brouwer’s degree \( \text{deg} \) (see the general viewpoint in [16]). Essentially, we will prove that among the zeroes of \( \Delta t - t_1 + t_0 \) given by (41), there is a \( (r^*, S) \in A \) such that \( \Delta \varphi \) satisfies (36). First, we prove

**Lemma 2** There exists a connected subset \( C_n \) of zeroes of \( \Delta t - t_1 + t_0 \) such that

\[
C_n \cap ([2m + \epsilon_n] \times (0, \delta_n)) \neq \emptyset \quad \text{and} \quad C_n \cap ([m + \epsilon_n, r_k^*] \times (0, \delta_n)) \neq \emptyset
\]

for every \( n \in \mathbb{N} \).
Proof of Lemma 3. Applying [Nu, Lemma 3.4] to the function

\[ F_n : [2m + \epsilon_n, r_0^*] \times (\delta_n, \frac{1}{\delta_n}) \rightarrow X \equiv \mathbb{R} \]

\[ (r^*, S) \rightarrow \Delta t(r^*, S) - t_1 + t_0 + S \]

and using (41), it is sufficient to prove (in the notation of [Nu]):

\[ i_X(F_n, r_0^*, G) \equiv \deg(\Id - F_n, r_0^*, G, 0) \neq 0 \]

where \( F_n : \mathbb{R} \rightarrow \mathbb{R} \) is an affine map

\[ \hat{F}_n : (\delta_n, \frac{1}{\delta_n}) \rightarrow \mathbb{R} \]

\[ S \rightarrow 2\frac{1-\delta_n S}{1-\delta_n} - 1 + S \]

has obviously \( \deg(\Id - \hat{F}_n, G, 0) = 1 \), and

\[ \deg(\Id - F_n, r_0^*, G, 0) = \deg(\Id - \hat{F}_n, G, 0) \]

(with the map \( \lambda \mapsto \Id - F_n + \lambda(F_n, r_0^* - \hat{F}_n) \), \( \lambda \in [0,1] \) is a homotopy from \( \Id - F_n \) to \( \Id - \hat{F}_n \) without zeroes on the boundary from (41)) which concludes the proof.  

Therefore, we obtain some \((r^*, S_n) \in C_n \) with \( r^*_n = 2m + \epsilon_n \). Taking now in Lemma 1 (ii)

\[ f_n(r) \equiv 1 \quad \text{and} \quad p_n(r) \equiv \frac{h_n(r)}{L_n^2}, \]

one checks from (38) and (40) that its hypotheses hold on the intervals \([a_n, b] = [r^*_n, r_1]\), obtaining

\[ (\Delta \varphi)_n = \int_{r_n^*}^{r_0} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr + \int_{r_n^*}^{r_1} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr \rightarrow \infty. \]  \hspace{1cm} (42)

On the other hand, from (41), the points in \( C_n \) with \( r^* = r_L^* \) have \( S \in (\delta_1, \frac{1}{\delta_1}) \), thus \( L^2 \) is upper bounded for these points and all \( n \) (see (38)) and so is \( \Delta \varphi \). This fact, (42) and the connectedness of \( C_n \) imply the existence of \((r^*, S) \in A \) such that (36) holds, as required.

Finally, consider the case \( t_0 = t_1 \) and thus, put \( E = 0 \). Now (33) becomes

\[ h(r) \equiv r(r - 2m)(qr^2 - L^2). \]

By imposing \( h(r^*) = 0 \) and \( h'(r^*) = 1 \) we obtain the following values for the constants \( q \) and \( L^2 \),

\[ qr^* - L^2 = 0 \]

\[ q = \frac{1}{2r^*(r^* - 2m)}. \]  \hspace{1cm} (43)
If we take $\{r_n^*\}_n \to 2m$, (43) and formulas analogous to (40) imply that Lemma 1 (ii) can be applied to the functions

$$f_n(r) \equiv 1 \quad \text{and} \quad p_n(r) \equiv \frac{h_n(r)}{L_n^2},$$

on the intervals $[a_n, b] = [r_n^*, r_1]$. Thus, we obtain (42) and, so, the existence of $r^* \in (2m, r_0)$ such that (36) holds. □

**Remark 5** The technique previously used in the proof of the geodesic connectedness of (outer) Schwarzschild spacetime is translatable to Schwarzschild black hole. In fact, now, we would use geodesics such that $r'(s)$ vanishes at $s^*$, with $r^* = r(s^*)$ satisfying ($r_0 \leq r_1 < r^* < 2m$).

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