Interaction and decay of Kelvin waves in the Gross-Pitaevskii model

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Abstract

By solving numerically the governing Gross-Pitaevskii equation, we study the dynamics of Kelvin waves on a superfluid vortex. After determining the dispersion relation, we monitor the turbulent decay of Kelvin waves with energy initially concentrated at large length scales. At intermediate length scales, we find that the decay is consistent with scaling predictions of theoretical models. Finally we report the unexpected presence of large-length scale phonons in the system.
Introduction. Hydrodynamic turbulence may occur in any fluid, and is characterised by energy transfer throughout the length scales. From a dynamical point of view, it is an out-of-equilibrium state of a large number of degrees of freedom; its complete description is possible only using a statistical approach [1, 2]. The main properties of turbulence are understood in terms of Richardson’s energy cascade. Energy, initially injected or stocked at large length scale $D$, undergoes a cascade process, and it is transferred locally through the scales in the so-called transparency window. The cascade, driven by the nonlinear term of the governing Navier-Stokes equation, halts at a characteristic small length scale $\eta$, called the Kolmogorov scale. At this small scale the diffusive term of the equation becomes more important than the nonlinear term, and energy is dissipated by viscous effects.

If a constant flux of energy is injected into the system and homogeneous isotropic turbulence develops, the statistical steady-state which is reached is characterized by the famous Kolmogorov energy spectrum $E(k) \sim k^{-5/3}$ (where $k$ is the wavenumber modulus) in the transparency window $1/D \ll k \ll 1/\eta$, as confirmed experimentally and numerically in the past few decades [1].

In this respect, pure superfluid turbulence, also called quantum turbulence, is less understood. The superfluid is different from an ordinary (classical) fluid in two respects: the viscosity is zero and vorticity is concentrated in discrete filaments, each carrying one quantum of circulation $\kappa$ [3]. Quantum turbulence is an apparently disordered tangle of such filaments. The superfluid behaves similarly to a classical fluid, but only at length scales larger than the mean inter-vortex distance $\ell$. At such large scales, metastable bundles of quantum vortices [4, 5] appear spontaneously in the vortex tangle and evolve like classical eddies [6]. This qualitative picture has been confirmed by the observation of the Kolmogorov spectrum in laboratory experiments [7–9] and in numerical simulations using different theoretical models [9–12]. However, how energy is transferred to smaller scales and finally dissipated by phonon emission [13] is still a matter of debate.

It is widely agreed that the major contribute to this energy transfer to smaller scales arises from the interaction between helical vortex oscillation modes, called Kelvin waves [3, 14, 15]. It is thought that a Kelvin wave cascade process, similar to the classical Richardson cascade, takes place, in which the final energy sink is acoustic rather than viscous. In the last decade competing theories have been proposed to explain how Kelvin waves interact, and to account for the Kelvin waves cascade different exponents have been predicted for the Kelvin wave-
action spectrum \( n(k) \sim k^{-\alpha} \). By using a scale-to-scale balance argument, Vinen et al. [16] obtained \( \alpha = 3 \). In the framework of wave turbulence theory [2, 17], Kozik and Svistunov [18] considered a 6-wave interaction process leading to \( \alpha = 17/5 = 3.4 \), while L’vov and Nazarenko [19] considered a 4-wave process interacting with a large amplitude Kelvin wave field giving \( \alpha = 11/3 = 3.6 \). Recently, using a tilting symmetry argument, Sonin proposed a variable exponent \( 3 \leq \alpha \leq 17/5 \) that depends on the number of interacting Kelvin waves considered [20]. Although the numerical values of the predicted exponent are relatively close to each other (hence difficult to distinguish numerically [21]), the prefactors predicted by these theories differ by orders of magnitude. Clearly, better understanding of Kelvin waves dynamics is crucial to explain the large-wavenumber regime of the energy cascade in a pure superfluid at low temperatures.

The aim of this report is to present direct observation of the interaction of Kelvin waves using the Gross-Pitaevskii equation (GPE). The GPE describes quantitatively the dynamics of a Bose-Einstein condensate, and, qualitatively, models a pure superfluid. Previous results on this problem have been obtained using the vortex filament model based on the Biot-Savart law [21, 22]. When using the GPE, quantities indirectly related to Kelvin waves have been monitored, such as the incompressible kinetic energy spectrum [23–26]. The GPE has three advantages on the vortex filament model. Firstly, it resolves scales that are comparable to the vortex core; secondly, it includes the generation of sound waves (phonons); finally, it includes vortex reconnection (hence there is no ambiguity when vortex strands become close to each other). The disadvantage of the GPE is the limited number of length scales which are available to the numerical solution in three dimensions. Our work on Kelvin waves is complementary to Krstulovic’s one [27], who reported evidence in favour of the L’vov-Nazarenko spectrum of the Kelvin waves cascade.

The model and the numerical integration. The GPE describes a weakly interacting Bose-Einstein condensate. It is also used to model a generic superfluid, and, using the Madelung transformation, can be turned into an equation for an inviscid barotropic compressible fluid [11]. Here we consider the dimensionless GPE

\[
    i\partial_t \psi + \frac{1}{2} \nabla^2 \psi - \frac{1}{2} |\psi|^2 \psi = 0,
\]

where the mean density is \( \langle \rho \rangle = |\psi|^2 = 1 \). In these units, the healing length (the characteristic length scale) is \( \xi = 1 \), the sound speed is \( c = 1 \), and the quantum of circulation is
\[
\kappa = 2\pi.
\]

To study isolated Kelvin wave dynamics on a single vortex line aligned along the x-axis, we consider a computational box with periodic boundary conditions in x and anti-periodic (reflective) in y and z. As we are interested in small amplitude Kelvin waves, our discretization is \(\Delta x = \xi, \Delta y = \Delta z = \xi/4\); our typical grid has resolution \(256 \times 129 \times 129\) points. This choice is a compromise between finite size effects (due to the reflective boundaries), the smallest scale to be resolved, and the time of the simulations. The GPE is integrated in time using a standard split-step method: the linear terms are evolved exactly in Fourier space, and the nonlinear terms are computed exactly in physical space. The numerical error arises from the splitting technique \([26, 28]\). The discrete Fourier trasform (DFT) algorithm is applied in the x-direction, whereas in the y- and z-directions we use the cosine Fourier transform (CFT), also known as real odd discrete Fourier transform \([29]\). The chosen time step \(\Delta t = 5 \times 10^{-3}\) is much smaller than the fastest linear time period, which is \(T_{k_{\text{max}}} \simeq 8 \times 10^{-2}\).

The technique to prepare the initial condition is the following. The initial wave-function is defined as

\[
\psi(x, y, z, t_0) = \Psi_{2D}[y - \bar{y}(x, t_0), z - \bar{z}(x, t_0)],
\]

where \(\Psi_{2D}(Y, Z) = \sqrt{\rho(R)} e^{i\theta(Y, Z)}\) is the two-dimensional wave-function with a quantum vortex positioned in the origin of the YZ-plane (perpendicular to the x-axis), with \(R = \sqrt{Y^2 + Z^2}; \rho(R) = R^2 (a_1 + a_2 R^2) / (1 + b_1 R^2 + b_2 R^4)\), and \(\theta(Y, Z) = \text{atan2}(Z, Y)\). The function \(\text{atan2}(...)\) is the extension of the arctangent function whose principal value is in the range \((-\pi; \pi]\); the coefficients \(a_1 = 11/32, a_2 = 11/384, b_1 = 1/3,\) and \(b_2 = 11/384\) arise from a second order Padé approximation \([30]\). Thus, on each slice \(x = \text{const}\), a vortex is created with centre at \([\bar{y}(x, t_0), \bar{z}(x, t_0)]\). This idea of building three-dimensional vortex lines starting from a two-dimensional wave-function was previously used to create distorted vortex rings \([31]\) and vortex knots \([32, 33]\).

The Kelvin wave state of the vortex line is defined as

\[
w(x, t_0) = \bar{y}(x, t_0) + i \bar{z}(x, t_0) = \int A(k, t_0) e^{ikx} dk,
\]

where \(w(0, t_0) = w(L_x, t_0)\) to satisfy the periodic boundary condition in \(x\). The complex coefficient \(A(k, t)\) is the Fourier transform in \(x\) of \(w(x, t)\) and the Kelvin wave-action spectrum is simply \(n(k, t) = |A(k, t)|^2\).
FIG. 1. (Colors online) Iso-surface of the density field at the threshold level $\rho_{th} = 0.2$ showing the initial condition of a Kelvin wave state characterized by a Gaussian spectrum peaked at large scales with random phases (corresponding initial spectrum shown in Fig. [3]).

All our initial conditions are defined from a Kelvin wave spectrum, setting $A(k, t_0) = \sqrt{n(k, t_0)} \exp[i\varphi(k)]$ with $\varphi(k)$ randomly distributed in $[0, 2\pi)$, and then going back to physical space to obtain $w(x, t_0)$. An example of a Kelvin wave state characterized by a Gaussian spectrum peaked at large scales with random phases is shown in Fig. [1].

How to identify the Kelvin wave state during the time evolution? We compute the vortex centre position on each slice $x = \text{constant}$ using a simple method of weighted centre of mass below a chosen density threshold $\rho = \rho_{th}$. The resulting coordinates of the vortex centre are

$$
\bar{y} = \frac{\int y \tilde{\rho} H(\tilde{\rho}) \, dS}{\int \tilde{\rho} H(\tilde{\rho}) \, dS}, \quad \bar{z} = \frac{\int z \tilde{\rho} H(\tilde{\rho}) \, dS}{\int \tilde{\rho} H(\tilde{\rho}) \, dS},
$$

(4)

with $\tilde{\rho} = \rho_{th} - \rho$, $H(\ldots)$ is Heaviside’s step function and $dS = dx dy$. By setting $\rho_{th} = 0.3$ and assuming axisymmetric density around the vortex core, we estimate the vortex centre with accuracy an order of magnitude bigger than the size of a discretization grid cell, $\Delta y \Delta z$. For completeness, we mention that a more sophisticated but numerically expensive technique is described in [27].

The dispersion relation. The dispersion relation is the most important quantity for Kelvin waves dynamics. The relation describes how each Kelvin wave component evolves indepen-
dently of the other. The dispersion relation obtained on the thin-filament approximation is

\[ \omega(k) = \frac{\kappa}{2\pi a_0^2} \left[ 1 - \sqrt{1 + \frac{|k|a_0 K_0(|k|a_0)}{K_1(|k|a_0)}} \right], \quad (5) \]

where \( a_0 \) is the vortex core cut-off parameter and \( K_n(...) \) is a modified Bessel function of order \( n \). In the limit of long Kelvin wavelength \( |k|a_0 \ll 1 \) it is well approximated by

\[ \omega(k) \simeq -\frac{\kappa k^2}{4\pi} \log \left( \frac{1}{|k|a_0} \right) - c, \quad \text{with} \quad c = 0.116... \]

at a first order correction. A similar relation was derived starting from the GPE rather than the Biot-Savart law in the same long-wavelength limit setting \( a_0 = \xi \); the most recent estimate of the correction in this case is given by \( c = 0.003187 \). [35]

The dispersion relation is determined in the following way. We consider an initial condition in which many Kelvin wave modes are excited, and compute the evolution to determine one period; in this way we determine the relation between frequency and wavenumber \( k \).

We consider an initial energy equipartitioned spectrum \( n(k,t_0) = T/\omega(k) \), where \( T \) represents the “temperature” of the Kelvin waves, and, for simplicity, the dispersion \( \omega(k) = \kappa k^2/4\pi \) does not include either the logarithmic or the constant correction. We are free to choose \( T \) in order to initially satisfy the quasi-small steepness condition \( |k|A(k,t_0) \leq 1 \) and to make sure that the excited vortex line is far enough from the reflective boundaries of the computational domain. We thus set \( T = 1 \) and phases randomly distributed in \([0, 2\pi)\) and the resulting initial condition has averaged steepness \(|k|A(k,t_0) \simeq 0.4\).

The dispersion relation which we obtain evolving the governing GPE model for sufficiently long turnover times is shown in Fig. 2. Equation (5), widely used in the literature, fits well the computed dispersion only for small wavenumbers, while for \(|k|\xi \geq 1\) the simpler expression \( \omega_{est}(k) = \kappa k^2/4\pi = k^2/2 \) showing free-particle behaviour better fits the data. The failure of dispersion (5) is expected at the scale \( k = 1/a_0 \) as Biot-Savart assumes that the vortex core size \( a_0 \) is infinitesimal compare to the Kelvin wave amplitudes and Fig. 2 shows exactly the crossover between the two regimes. Our result is consistent with the numerical results of Roberts [35], and does not exhibit the negative frequency shift \( \omega_0 \) observed in [36, 37]. It is important to notice that the energy equipartitioned initial spectrum is not the statistical steady state of the system as during the evolution we observed changes in its shape.

The Kelvin wave decay turbulence. In order to understand the interaction of Kelvin waves
FIG. 2. (Color online) Kelvin wave dispersion relation $\omega$ vs $k$ evaluated for an initial condition characterized by the Kelvin wave spectrum $n(k,t_0) = T/\omega(k)$ with $T = 1$, $\omega(k) = \kappa k^2/4\pi$, and phases randomly distributed in $[0,2\pi)$. The blue dashed line corresponds to the theoretical prediction coming from the Biot-Savart filament model \(^{[5]}\) with $a_0 = \xi$. The red line is the estimated free-particle dispersion $\omega_{\text{est}}(k) = \kappa k^2/4\pi$. Inset: a zoom at the low Kelvin wavenumbers.

which eventually leads to a turbulent energy cascade, we simulate the free decay and monitor the expected transfer of energy from large to small scales. We set up an initial Gaussian-shaped spectrum $n(k,t_0) = A/\left(\sigma\sqrt{2\pi}\right) \exp\left[-(|k| - \mu)^2/(2\sigma^2)\right]$, with $A = 2$, $\mu = 10 \Delta k_x$, and $\sigma = 3 \Delta k_x$, where $\Delta k_x = 2\pi/256 \Delta x$ being the smallest Kelvin wave mode. With this choice, almost all the initial energy is contained in the largest length scales. We perform five numerical simulations with different initial random phases; the resulting directional- and ensemble-averaged spectral evolution $\langle n(k,t) \rangle$ is shown in Fig. 3. The value of time cited in the figure refers not to the dimensionless units used to solve the GPE, but rather to Kelvin wave turnover period $T_p = 2\pi/\omega_{\text{est}}(k_p)$, where $k_p = \mu$ is the wavenumber at which the initial large-scale spectrum peaks. In these units, the nonlinear evolution is rapid: starting from the initial concentration at the mesocales, energy is visibly shifted to the largest wavenumber.
FIG. 3. (Color online) Averaged Kelvin wave spectrum \( \langle n(k,t) \rangle \) plotted vs wavenumber \( k \) on a log-log scale at different times \( t \) during the evolution. The spectrum evolves from the initial Gaussian shape and approximately acquires power-law dependence at intermediate wavenumbers \( 0.5 \leq |k| \leq 2 \). Inset: the final spectrum \( \langle n(k,5T_p) \rangle \) vs wavenumber, compensated by Vinen’s \( (k^{-3}) \), Kozik and Svistunov’s \( (k^{-17/5}, \text{KS}) \), and L’vov and Nazarenko’s \( (k^{-11/3}, \text{LN}) \) scalings respectively.

already at \( t \approx 2.5T_p \). During the successive stage of the evolution, the power-law scaling appears approximately in the wavenumber range interval \( 0.5 \leq |k| \leq 2 \). The inset shows the final averaged spectrum \( \langle n(k,5T_p) \rangle \) compensated by the scalings predicted by Vinen et. al, Kozik and Svistunov, and L’vov and Nazarenko. It is apparent that, as found using the vortex filament model [21], all the predicted scalings are close to the numerical results.

To gain physical insight into the Kelvin waves interaction, we compute the spectrum of the incompressible kinetic energy, which is only a fraction of the total (conserved) energy of the system. Following [11], we split the superfluid’s kinetic energy into compressible and incompressible parts, associated to the presence phonons and quantum vortices respectively. Fig.s 4 and 5 display the (directional- and ensemble-averaged) one-dimensional kinetic energy
FIG. 4. (Color online) Averaged parallel compressible and incompressible kinetic energy spectra $\langle E^c(k_{\parallel}, t) \rangle$ and $\langle E^i(k_{\parallel}, t) \rangle$ vs wavenumber $k$, plotted on log-log scale at different times.

spectra, where we distinguish between the directions parallel ($k_{\parallel}$) and perpendicular ($k_{\perp}$) to the unperturbed vortex line (in practice, $k_{\parallel} = k_x$, whereas $k_{\perp}$ is the Fourier transform of the radius coordinate in the $YZ$ plane, averaged over the angular coordinate).

The numerical results suggest the following three conclusions. Firstly, it is clear in Fig. 4 that the parallel compressible energy component (initially negligible at all length scales compared to the incompressible component), at the final time $t = 5T_p$ becomes dominant in the range $|k|\xi \geq 0.7$. We conclude that energy is transferred from Kelvin waves to phonons mostly at these smaller scales.

Secondly, the perpendicular incompressible energy component, shown in Fig. 5, does not evolve in time in a significant way. Likely, this quantity is completely dominated by the vortex core and not by the different Kelvin wave modes which are excited long the vortex line.

Thirdly, both parallel and perpendicular compressible energy components seem to equilibrate to a power-law behavior at scales $|k|\xi \leq 1$, but are strongly reduced at smaller scales.
This result is somehow unexpected, because the scenario which is described in the literature seems to imply that phonons should be mainly produced at length scales which are much smaller than the length scales which initially contained the energy. We do not know if the mechanism which generates large scale phonons is due to the presence of the oscillating vortex or to phonon interaction [38]. We have verified that the effect is present also when we start from a different initial condition (smaller amplitude and less compressible energy).

Conclusions. We have performed numerical simulations of interacting Kelvin waves using the GPE and an idealized vortex configuration which consists of only one perturbed vortex line. The method which we have developed to numerically produce the desired Kelvin wave spectrum on the vortex line and track the vortex core position is general and can be used for other studies.

Our first step was to determine the dispersion relation of Kelvin waves, starting from an initial condition where all Kelvin modes are excited and (in the first approximation) energy is equipartitioned. As expected, we have found that the dispersion relation often used in the
literature (Eq. 5) is valid only for long waves ($|k|a_0 < 0.2$, where we set $a_0 = \xi$).

The second step was to determine the turbulent decay of an initial spectrum with most of the energy stored at large scales. By running different simulations with different initial random phases and ensemble-averaging, we have obtained a power-law spectrum which is consistent with the predictions of the competing theories for the Kelvin wave turbulence. Unfortunately, as the inertial range is too narrow, we are unable to draw any strong conclusions about which theory is more consistent with our data (unlike what done in [27]).

The third step was to extract the parallel and perpendicular compressible and incompressible contributions to the kinetic energy. We found that, at small length scales, the compressible parallel component becomes stronger than the incompressible component, confirming that phonons play a crucial role in the decay. Unexpectedly, we observe the formation of large scale phonons in both energy components.

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