SOLVABILITY IN ABSTRACT EVOLUTION EQUATIONS WITH COUNTABLE TIME DELAYS IN BANACH SPACES: GLOBAL LIPSCHITZ PERTURBATION

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(Communicated by Cristina Pignotti)

Abstract. This paper deals with the solvability in the semilinear abstract evolution equation with countable time delays,

\[
\begin{aligned}
\frac{du}{dt}(t) + Au(t) &= F(u(t), (u(t - \tau_n))_{n \in \mathbb{N}}), \quad t > 0, \\
u(t) &= u_0(t), \quad t \in \bigcup_{n \in \mathbb{N}} [-\tau_n, 0],
\end{aligned}
\]

in a Banach space \(X\), where \(-A\) is a generator of a \(C_0\)-semigroup with exponential decay and \(F : X \times X^N \to X\) is Lipschitz continuous. Nicaise and Pignotti (J. Evol. Equ.; 2018;18;947–971) established global existence and exponential decay in time for solutions of the above equation with finite time delays in Hilbert spaces under global or local Lipschitz conditions. The purpose of the present paper is to generalize the result to the case of countable time delays in Banach spaces under a global Lipschitz condition.

1. Introduction. There are many ordinary and partial differential equations with time delays modeling various phenomena (see e.g., Inoue, Miyakawa and Yoshida [11], Lightbourne and Rankin [15], Pao [25, 26], Ruan and Zhao [30] for diffusion models; Li and Guo [13, 14] for chemotaxis models). From a mathematical point of view it is important to give stability results for such equations (see e.g., Alabau-Boussouira, Nicaise and Pignotti [1], Ammari, Nicaise and Pignotti [2], Freedman and Zhao [8], Friesecke [9], Guesmia [10], Liu [16], Nicaise and Pignotti [18, 19, 20, 21, 22, 23], Oliva [24], Pignotti [28], Said-Houari and Soufyane [31]), whereas sometimes results induce instability (see Bátkai and Piazzera [4], Datko [6], Datko, Lagnese and Polis [7], Nicaise and Pignotti [17], Xu, Yung and Li [32]). In 2015, Nicaise and Pignotti [21] studied exponential stability for the abstract evolution equation with linear dependency on the delayed state,

\[
\begin{aligned}
\frac{dU}{dt}(t) &= AU(t) + G(U(t)) + kBU(t - \tau), \quad t > 0, \\
U(0) &= U_0, \quad BU(t - \tau) = g(t), \quad t \in (0, \tau),
\end{aligned}
\]

2020 Mathematics Subject Classification. Primary: 34K30; Secondary: 34K20.
Key words and phrases. Abstract evolution equations, time delays, global existence, exponential decay.

The first author is supported by Grant-in-Aid for Scientific Research (C), No. 16K05182.
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in a Hilbert space $\mathcal{H}$, where $A$ is a generator of a $C_0$-semigroup with exponential decay, $B$ is a bounded linear operator from $\mathcal{H}$ into itself, $G$ is a Lipschitz continuous operator on $\mathcal{H}$, $\tau > 0$, $U^0 \in \mathcal{H}$ and $g \in C([0, \tau]; \mathcal{H})$. The main goal in [21] was to establish a general result for stability by employing an iterative argument regarding the delay term as a known datum on the intervals $(0, \tau), (\tau, 2\tau), \ldots$ After that, Nicaise and Pignotti [23] succeeded in extending the result in [21] by analyzing the following more general model with nonlinear dependency not only on the state but also on delayed states,

\[
\begin{cases}
\frac{dU}{dt}(t) = AU(t) + \sum_{n=1}^{N} F_n(U(t), U(t - \tau_n)), & t > 0, \\
U(t - \tau) = U_0(t), & t \in (0, \tau],
\end{cases}
\]

where $\mathcal{H}$ and $A$ are the same as above, $N$ is a positive finite natural number, $\tau_n$ are time delays such that $0 < \tau_n < \tau = \tau_1$, $n = 2, \ldots N$ and the nonlinear operators $F_n : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfy some Lipschitz conditions and moreover $U_0 \in C([0, \tau]; \mathcal{H})$.

The purpose of this paper is to extend the above results to a further general model with countable time delays, i.e.,

\[
\begin{cases}
\frac{du}{dt}(t) + Au(t) = F(u(t), (u(t - \tau_n))_{n \in \mathbb{N}}), & t > 0, \\
u(t) = u_0(t), & t \in I := \bigcup_{n \in \mathbb{N}} [-\tau_n, 0],
\end{cases}
\]  

(1.1)

in a Banach space $X$ with norm $\| \cdot \|$, where $-A$ is a generator of a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$, $\tau_n > 0$, $n \in \mathbb{N}$ and $F : X \times X^\mathbb{N} \to X$ satisfies a Lipschitz condition explained later. Here we emphasize that the problem (1.1) allows both finite and infinite delayed states. Indeed, setting $\tau_{\max} := \max\{\tau_n | n \in \mathbb{N}\}$ if the maximum exists and $\tau_{\sup} := \sup\{\tau_n | n \in \mathbb{N}\} \leq \infty$, we see that

\[I = \bigcup_{n \in \mathbb{N}} [-\tau_n, 0] = \begin{cases} 
[-\tau_{\max}, 0] & \text{if the maximum exists,} \\
(-\tau_{\sup}, 0) & \text{otherwise.}
\end{cases}\]

Throughout this paper, we assume that there exist a constant $\gamma_0 \geq 0$ and a non-negative $\ell^1$-sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that

\[\|F(u, (v_n)_{n \in \mathbb{N}}) - F(u^*, (v^*_n)_{n \in \mathbb{N}})\| \leq \gamma_0 \|u - u^*\| + \sum_{n \in \mathbb{N}} \gamma_n \|v_n - v^*_n\| \]  

(1.2)

for all $(u, (v_n)_{n \in \mathbb{N}}), (u^*, (v^*_n)_{n \in \mathbb{N}}) \in X \times \ell^1(\mathbb{N}; X)$, where

\[\ell^1(\mathbb{N}; X) := \left\{(v_n)_{n \in \mathbb{N}} \in X^\mathbb{N} \mid \sum_{n \in \mathbb{N}} \gamma_n \|v_n\| < \infty\right\} .\]

Before stating the main results, we define mild solutions of the problem (1.1).

**Definition 1.1.** Given $u_0 \in C(I; X) \cap L^\infty(I; X)$, a function $u \in C([0, \infty); X)$ is called a *mild solution* of the problem (1.1) if $u$ satisfies the integral equation

\[u(t) = e^{-tA}u_0(0) + \int_0^t e^{-(t-s)A} F(u(s), (\pi(s - \tau_n))_{n \in \mathbb{N}}) \, ds, \quad t \geq 0, \]  

(1.3)

where

\[\pi(t) := \begin{cases} 
\pi(t) & \text{for } t > 0, \\
u_0(t) & \text{for } t \in I.
\end{cases}\]
Now the main results read as follows. The first one asserts well-posedness of the problem (1.1) under mild conditions.

**Theorem 1.1.** Assume that \(-A\) generates a \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) with (possibly exponentially growing) estimate
\[
\|e^{-tA}\|_{L(X)} \leq Me^{\omega t} \quad (t \geq 0) \quad \text{with some } \omega > 0, \ M > 0 \quad (1.4)
\]
and \(F\) satisfies the condition (1.2). Then for all \(u_0 \in C(I;X) \cap L^\infty(I;X)\) there exists a unique mild solution \(u \in C([0,\infty);X)\) of the problem (1.1).

The second is concerned with exponential stability in the problem (1.1) under strict conditions.

**Theorem 1.2.** Suppose that \(-A\) generates a \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) having the (exponentially decaying) estimate
\[
\|e^{-tA}\|_{L(X)} \leq Me^{-\omega t} \quad (t \geq 0) \quad \text{with some } \omega > 0, \ M > 0 \quad (1.5)
\]
and \(F\) satisfies \(F(0,(0)_{n \in \mathbb{N}}) = 0\) and the condition (1.2). Assume further that
\[
\gamma_0 + \sum_{n \in \mathbb{N}} \gamma_n e^{\omega \tau_n} < \frac{\omega}{M} \quad (1.6)
\]
and there exists \(p \in [1,\infty]\) such that
\[
\sum_{n \in \mathbb{N}} \gamma_n^{1-1/p} e^{\omega \tau_n} < \infty. \quad (1.7)
\]
Then for all \(u_0 \in C(I;X)\) fulfilling that the function \(t \to e^{\omega t}u_0(t)\) belongs to \(L^p(I;X) \cap L^\infty(I;X)\), there exists a unique mild solution \(u \in C([0,\infty);X)\) of the problem (1.1). Moreover, the solution \(u\) has the decay estimate
\[
\|u(t)\| \leq M \left[\|u_0(0)\| + \left(\sum_{n \in \mathbb{N}} \gamma_n^{1-1/p} e^{\omega \tau_n} \right) \left(\int_I \|e^{\omega s}u_0(s)\|_p^p \, ds \right)^{1/p} \right] e^{-mt} \quad (1.8)
\]
for all \(t \geq 0\), where \(m > 0\) is a constant given by
\[
m := M \left[\frac{\omega}{M} - \left(\gamma_0 + \sum_{n \in \mathbb{N}} \gamma_n e^{\omega \tau_n} \right) \right]. \quad (1.9)
\]

**Remark 1.1.** The conditions (1.6) and (1.7) extend the corresponding previous conditions assumed in [23, (2.6) and (2.7)]. Indeed, if \(p = \infty\) and \(\tau_{\sup} < \infty\), then the conditions (1.6) and (1.7) are reduced to
\[
\gamma_0 + \left(\sum_{n \in \mathbb{N}} \gamma_n \right) e^{\omega \tau_{\sup}} < \frac{\omega}{M},
\]
and
\[
\left(\sum_{n \in \mathbb{N}} \gamma_n \right) \tau_{\sup} e^{\omega \tau_{\sup}} < \infty,
\]
which are derived from the smallness condition for \(\tau_{\sup}^\prime\):
\[
\tau_{\sup} < \frac{1}{\omega} \log \left(\frac{\omega}{M\gamma} - \frac{\gamma_0}{\gamma} \right) \quad \text{with } \gamma := \sum_{n \in \mathbb{N}} \gamma_n.
\]

**Remark 1.2.** As studied in [23], we need a delicate argument in the case that \(F\) is locally Lipschitz continuous. The detail will be discussed by the second author [33].
The strategy of the proof of the main results is to apply the semigroup theory as in Pazy [27] to the corresponding integral equation written as the so-called Duhamel’s formula or variation of parameters formula. One of the differences and difficulties in comparison with the iterative argument employed by Nicaise and Pignotti [21, 23] is whether or not one constructs the intervals $(0, \tau_{\text{min}})$, $(\tau_{\text{min}}, 2\tau_{\text{min}})$, \ldots, where \(\tau_{\text{min}} := \min\{\tau_1, \tau_2, \ldots, \tau_N\}\). The advantage of the iterative argument is that one can regard the delay term as a known datum on the intervals shorter than the delay times. However, in our case having countable delay times, there is a possibility that \(\tau_{\text{inf}} := \inf\{\tau_1, \tau_2, \ldots\} = 0\), which means that one cannot construct even the first interval \((0, \tau_{\text{inf}})\). To overcome the difficulty, we go back to original arguments in abstract semilinear evolutions equations in the semigroup theory found in [27]. In particular, we choose the suitable function space such that one can prove existence, uniqueness and exponential decay at the same time. Moreover, such arguments do not require the inner product in a Hilbert space. As a consequence, the results and proofs of the present paper act on an arbitrary Banach space.

This paper is organized as follows. In Section 2 we will prove Theorem 1.1 by using the contraction mapping principle in a suitable Banach space. Section 3 will be devoted to the proof of Theorem 1.2. In Section 4 we will deal with some concrete examples of the abstract evolution equations.

2. Proof of Theorem 1.1. Assume that \(-A\) generates a \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) with estimate (1.4) and \(F\) satisfies the condition (1.2). Let \(\omega' > \omega\) be a constant which will be fixed later. Then we define a Banach space \(\mathcal{Y}\) as follows:

\[
\mathcal{Y} := \left\{ u \in C([0, \infty); X) \mid \sup_{t \in [0, \infty)} e^{-\omega' t}\|u(t)\| < \infty \right\},
\]

\[
\|u\|_{\mathcal{Y}} := \sup_{t \in [0, \infty)} e^{-\omega' t}\|u(t)\| \quad \text{for} \ u \in \mathcal{Y}.
\]

Given \(u_0 \in C(I; X) \cap L^\infty(I; X)\), in view of the integral equation (1.3), it suffices to find a fixed point of the mapping \(\Phi : \mathcal{Y} \to \mathcal{Y}\) defined as

\[
(\Phi u)(t) := e^{-tA}u_0(0) + \int_0^t e^{-(t-s)A}F(u(s), (\bar{\pi}(s - \tau_n))_{n \in \mathbb{N}}) \, ds \quad (t \geq 0) \quad \text{for} \ u \in \mathcal{Y},
\]

where we can write down \(\bar{\pi}(s - \tau_n)\) as follows:

\[
\bar{\pi}(s - \tau_n) = \begin{cases} 
  u(s - \tau_n) & \text{if} \ s > \tau_n, \\
  u_0(s - \tau_n) & \text{if} \ s \leq \tau_n.
\end{cases}
\]

We first let \(u \in \mathcal{Y}\) and then we verify that \(\Phi u \in \mathcal{Y}\). From the definition of \(\Phi\) we can invoke (1.4) and (1.2) with \(u^* = 0\) and \(u^*_n = 0 \ (\forall \ n \in \mathbb{N})\) to obtain

\[
\|(\Phi u)(t)\| \leq Me^{\omega t}\|u_0(0)\| + M \int_0^t e^{\omega(t-s)}\|F(u(s), (\bar{\pi}(s - \tau_n))_{n \in \mathbb{N}})\| \, ds
\]

\[
\leq Me^{\omega t}\|u_0(0)\|
\]

\[
+ M \int_0^t e^{\omega(t-s)} \left(\|F(0, (0)_{n \in \mathbb{N}})\| + \gamma_0\|u(s)\| + \sum_{n \in \mathbb{N}} \gamma_n\|\bar{\pi}(s - \tau_n)\|\right) \, ds
\]

(2.1)
for all \( t \geq 0 \). Hence we have

\[
\| (\Phi u)(t) \| \leq M e^{\omega t} \| u_0(0) \| + M \int_0^t e^{\omega(t-s)} \left( \| F(0, (0)_{n \in \mathbb{N}}) \| + \gamma_0 \| u(s) \| \right) \, ds \\
+ M \sum_{n \in \mathbb{N}} \gamma_n \int_0^{\min(\tau_n, t)} e^{\omega(t-s)} \| u_0(s - \tau_n) \| \, ds \\
+ M \sum_{n \in \mathbb{N}} \gamma_n \int_{\min\{\tau_n, t\}}^t e^{\omega(t-s)} \| u(s - \tau_n) \| \, ds \\
\leq M e^{\omega t} \| u_0(0) \| + M \int_0^t e^{\omega(t-s)} \left( \| F(0, (0)_{n \in \mathbb{N}}) \| + \gamma_0 \| u(s) \| \right) \, ds \\
+ M \sum_{n \in \mathbb{N}} \gamma_n \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \int_0^{\min\{\tau_n, t\}} e^{\omega(t-s)} \, ds \\
+ M \sum_{n \in \mathbb{N}} \gamma_n \int_{\min\{0, t-\tau_n\}}^{t-\tau_n} e^{\omega(t-s-\tau_n)} \| u(s) \| \, ds \\
\leq M e^{\omega t} \| u_0(0) \| + M \| F(0, (0)_{n \in \mathbb{N}}) \| \int_0^t e^{\omega(t-s)} \, ds \\
+ M \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \sum_{n \in \mathbb{N}} \gamma_n \int_0^{\tau_n} e^{\omega(t-s)} \, ds + \tilde{M} \int_0^t e^{\omega(t-s)} \| u(s) \| \, ds
\]

for all \( t \geq 0 \), where

\[
\tilde{M} := M \left( \gamma_0 + \sum_{n \in \mathbb{N}} \gamma_n e^{-\omega \tau_n} \right) . \tag{2.2}
\]

Recalling that \((\gamma_n)_{n \in \mathbb{N}}\) is an \( \ell^1 \)-sequence in (1.2), we have \( \tilde{M} < \infty \). Hence we infer

\[
\| (\Phi u)(t) \| \leq M e^{\omega t} \| u_0(0) \| + M e^{\omega t} \| F(0, (0)_{n \in \mathbb{N}}) \| \int_0^t e^{-\omega s} \, ds \\
+ M e^{\omega t} \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \sum_{n \in \mathbb{N}} \gamma_n \int_0^{\tau_n} e^{-\omega s} \, ds + \tilde{M} e^{\omega t} \| u \|_{\gamma} \int_0^t e^{(\omega' - \omega)s} \, ds \\
\leq M e^{\omega t} \| u_0(0) \| + M e^{\omega t} \| F(0, (0)_{n \in \mathbb{N}}) \| \frac{1 - e^{-\omega t}}{\omega} \\
+ M e^{\omega t} \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \sum_{n \in \mathbb{N}} \gamma_n \frac{(1 - e^{-\omega \tau_n})}{\omega} + \tilde{M} e^{\omega t} \| u \|_{\gamma} e^{(\omega' - \omega)t} \frac{1}{\omega' - \omega},
\]

which leads to

\[
e^{-\omega' t} \| (\Phi u)(t) \| \leq M e^{-(\omega' - \omega)t} \| u_0(0) \| + \frac{M}{\omega} e^{-(\omega' - \omega)t} \| F(0, (0)_{n \in \mathbb{N}}) \| \\
+ \frac{M}{\omega} e^{-(\omega' - \omega)t} \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \sum_{n \in \mathbb{N}} \gamma_n (1 - e^{-\omega \tau_n}) + \frac{\tilde{M}}{\omega' - \omega} \| u \|_{\gamma} \\
\leq M \| u_0(0) \| + \frac{M}{\omega} \| F(0, (0)_{n \in \mathbb{N}}) \| \\
+ \frac{M}{\omega} \| u_0 \|_{L^\infty(\tau; \mathcal{X})} \sum_{n \in \mathbb{N}} \gamma_n (1 - e^{-\omega \tau_n}) + \frac{\tilde{M}}{\omega' - \omega} \| u \|_{\gamma} \tag{2.3}
\]
Then by the definition of $\Phi$ and (1.4) we deduce from (1.2) that

$$
\Phi u \in \mathcal{Y} \text{ for all } u \in \mathcal{Y}.
$$

We next show that $\Phi : \mathcal{Y} \to \mathcal{Y}$ is a contraction mapping. To see this we let $u, v \in \mathcal{Y}$. Then by the definition of $\Phi$ and (1.4) we deduce from (1.2) that

$$
\begin{align*}
\| (\Phi u)(t) - (\Phi v)(t) \| & \leq M \int_0^t e^{\omega(t-s)} \| F(u(s), (\pi(s - \tau_n))_{n \in \mathbb{N}}) - F(v(s), (\pi(s - \tau_n))_{n \in \mathbb{N}}) \| \, ds \\
& \leq M \int_0^t e^{\omega(t-s)} \left( \gamma_0 \| u(s) - v(s) \| + \sum_{n \in \mathbb{N}} \gamma_n \| \pi(s - \tau_n) - \pi(s - \tau_n) \| \right) \, ds
\end{align*}
$$

for all $t \geq 0$. Comparing this with (2.1), we can repeat a similar calculation in the derivation of (2.3) and arrive at

$$
e^{-\omega t'} \| (\Phi u)(t) - (\Phi v)(t) \| \leq \frac{M}{\omega' - \omega} \| u - v \|_{\mathcal{Y}}$$

for all $t \geq 0$. This means that if we choose $\omega'$ as

$$\omega' > \omega + \frac{M}{\omega' - \omega},$$

then $\Phi : \mathcal{Y} \to \mathcal{Y}$ is a contraction mapping such that

$$
\| \Phi u - \Phi v \|_{\mathcal{Y}} \leq \delta \| u - v \|_{\mathcal{Y}} \text{ with } \delta := \frac{M}{\omega' - \omega} < 1.
$$

Therefore, applying the contraction mapping principle to $\Phi$, we see that there exists a fixed point of $\Phi$, which is a solution of the integral equation (1.3), that is, the mild solution of the problem (1.1). We finally show uniqueness of mild solutions to the problem (1.1). To this end, given $u_0 \in C(I; X) \cap L^\infty(I; X)$, we let $u, v$ be two mild solutions to (1.1), which solve the integral equation (1.3), i.e., $u = \Phi u, v = \Phi v$. Then a similar calculation as above gives

$$
\begin{align*}
\| u(t) - v(t) \| & \leq M \int_0^t e^{\omega(t-s)} \| F(u(s), (\pi(s - \tau_n))_{n \in \mathbb{N}}) - F(v(s), (\pi(s - \tau_n))_{n \in \mathbb{N}}) \| \, ds \\
& \leq M \gamma_0 \int_0^t e^{\omega(t-s)} \| u(s) - v(s) \| \, ds \\
& \quad + M \sum_{n \in \mathbb{N}} \gamma_n \int_{\min\{0, t - \tau_n\}}^{t-\tau_n} e^{\omega(t-s-\tau_n)} \| u(s) - v(s) \| \, ds \\
& \leq \frac{M}{\omega'} \int_0^t e^{\omega(t-s)} \| u(s) - v(s) \| \, ds,
\end{align*}
$$

which can be rewritten as

$$
e^{-\omega t} \| u(t) - v(t) \| \leq \frac{M}{\omega'} \int_0^t e^{-\omega s} \| u(s) - v(s) \| \, ds,$$

for all $t \geq 0$. Thus we conclude from Gronwall’s lemma that $e^{-\omega t} \| u(t) - v(t) \| \equiv 0$, which implies that $u(t) \equiv v(t)$. This proves uniqueness of mild solutions to the problem (1.1). \qed
3. **Proof of Theorem 1.2.** In this section we assume that $-A$ generates a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ with the decay estimate (1.5) and $F$ satisfies $F(0,(0)_{n \in \mathbb{N}}) = 0$ and (1.2). Moreover, we suppose that (1.6) and (1.7) for some $p \in [1, \infty]$. We pick $u_0 \in C(I;X)$ such that $t \to e^{\omega t}u_0(t)$ belongs to $L^p(I;X) \cap L^\infty(I;X)$. Let $\omega_*$ be a constant such that $0 < \omega_* < \omega$, which will be fixed later. Then, changing the sign of the exponential function in the proof of Section 2, we define a similar Banach space denoting by the same symbol $\mathcal{Y}$ as follows:

$$\mathcal{Y} := \left\{ u \in C([0,\infty); X) \mid \sup_{t \in [0,\infty)} e^{\omega_* t} \| u(t) \| < \infty \right\},$$

$$\| u \|_{\mathcal{Y}} := \sup_{t \in [0,\infty)} e^{\omega_* t} \| u(t) \| \text{ for } u \in \mathcal{Y}.$$

Then we shall again find a fixed point of the mapping $\Phi : \mathcal{Y} \to \mathcal{Y}$ defined in Section 2. We first let $u \in \mathcal{Y}$ and we show that $\Phi u \in \mathcal{Y}$. Invoking the conditions (1.5) and (1.2) and noting that $F(0,(0)_{n \in \mathbb{N}}) = 0$, we have

$$\| (\Phi u)(t) \|$$

$$\leq Me^{-\omega t} \| u_0(0) \| + M \int_0^t e^{-\omega(t-s)} \left( \gamma_0 \| u(s) \| + \sum_{n \in \mathbb{N}} \gamma_n \| \pi(s-\tau_n) \| \right) ds$$

$$\leq Me^{-\omega t} \| u_0(0) \| + M \sum_{n \in \mathbb{N}} \gamma_n \int_0^{\min\{\tau_n,t\}} e^{-\omega(t-s)} \| u_0(s-\tau_n) \| ds$$

$$+ M \gamma_0 \int_0^t e^{-\omega(t-s)} \| u(s) \| ds + M \sum_{n \in \mathbb{N}} \gamma_n \int_0^{\min\{\tau_n,t\}} e^{-\omega(t-s)} \| u(s-\tau_n) \| ds,$$

and hence,

$$\| (\Phi u)(t) \|$$

$$\leq Me^{-\omega t} \| u_0(0) \| + M \sum_{n \in \mathbb{N}} \gamma_n \int_{-\tau_n}^{\min\{0,t-\tau_n\}} e^{-\omega(t-\tau_n)} \cdot e^{\omega s} \| u_0(s) \| ds$$

$$+ Me^{-\omega \gamma_0 \int_0^t e^{\omega s} \| u(s) \| ds + M \sum_{n \in \mathbb{N}} \gamma_n \int_{\min\{0,t-\tau_n\}}^{t-\tau_n} e^{-\omega(t-s-\tau_n)} \| u(s) \| ds$$

$$\leq Me^{-\omega t} \| u_0(0) \| + Me^{-\omega t} \| e^{\omega t}u_0(t) \|_{L^p(I;X)} \sum_{n \in \mathbb{N}} \gamma_n e^{\omega \tau_n} \left( \int_{-\tau_n}^0 ds \right)^{1-1/p}$$

$$+ \tilde{M} e^{-\omega t} \int_0^t e^{\omega s} \| u(s) \| ds$$

(3.1)

for all $t \geq 0$, where $\tilde{M} > 0$ is a constant defined by (2.2). Therefore it follows that

$$\| (\Phi u)(t) \| \leq Me^{-\omega t} \| u_0(0) \| + Me^{-\omega t} \| e^{\omega t}u_0(t) \|_{L^p(I;X)} \sum_{n \in \mathbb{N}} \gamma_n e^{\omega \tau_n}$$

$$+ \tilde{M} e^{-\omega t} \| u \|_{\mathcal{Y}} \frac{e^{(\omega - \omega_*)t} - 1}{\omega - \omega_*},$$
which together with \( \omega_* < \omega \) leads to
\[
e^{\omega t} \| (\Phi u)(t) \| \leq M \| u_0(0) \| + M \| e^{\omega t} u_0(t) \|_{L^p(I;X)} \sum_{n \in \mathbb{N}} \gamma_n r_n^{1-1/p} e^{\omega \tau_n} + \frac{\tilde{M}}{\omega - \omega_*} \| u \|_Y
\]
for all \( t \geq 0 \). Since the right-hand side is finite by virtue of (1.7), this implies that
\[
\Phi u \in \mathcal{Y} \quad \text{for all } u \in \mathcal{Y}.
\]
In order to show that \( \Phi : \mathcal{Y} \to \mathcal{Y} \) is a contraction mapping, we let \( u, v \in \mathcal{Y} \). Then almost the same calculation as above derives
\[
e^{\omega t} \| (\Phi u)(t) - (\Phi v)(t) \| \leq \frac{\tilde{M}}{\omega - \omega_*} \| u - v \|_Y
\]
for all \( t \geq 0 \). This says that if we choose \( \omega_* \) as
\[
0 < \omega_* < \omega - \tilde{M},
\]
which is possible by (1.6), then \( \Phi : \mathcal{Y} \to \mathcal{Y} \) is a contraction mapping such that
\[
\| \Phi u - \Phi v \|_Y \leq \delta \| u - v \|_Y \quad \text{with } \delta := \frac{\tilde{M}}{\omega - \omega_*} < 1.
\]
Thus the contraction mapping principle provides a fixed point of \( \Phi \), which is a solution of the integral equation (1.3), that is, the mild solution of the problem (1.1). In order to show uniqueness of mild solutions to the problem (1.1), given \( u_0 \in C(I;X) \cap L^\infty(I;X) \), we let \( u, v \) be two mild solutions to (1.1), that is, \( u(t) = (\Phi u)(t) \) and \( v(t) = (\Phi v)(t) \). Then a similar calculation as above yields
\[
e^{\omega t} \| u(t) - v(t) \| \leq \tilde{M} \int_0^t e^{\omega s} \| u(s) - v(s) \| \, ds
\]
for all \( t \geq 0 \). Therefore the Gronwall lemma says that \( e^{-\omega t} \| u(t) - v(t) \| \equiv 0 \), which implies \( u(t) \equiv v(t) \). This proves uniqueness of mild solutions to the problem (1.1). We finally prove the decay estimate (1.8). Since the mild solution \( u \) satisfies \( u(t) = (\Phi u)(t) \), by a calculation similar to (3.1), we have
\[
\| u(t) \| \leq M e^{-\omega t} \| u_0(0) \| + M e^{-\omega t} \| e^{\omega t} u_0(t) \|_{L^p(I;X)} \sum_{n \in \mathbb{N}} \gamma_n e^{\omega \tau_n} \left( \int_{-\tau_n}^0 ds \right)^{1-1/p} + \tilde{M} e^{-\omega t} \int_0^t e^{\omega s} \| u(s) \| \, ds
\]
for all \( t \geq 0 \). Hence it follows that
\[
e^{\omega t} \| u(t) \| \leq MK + \tilde{M} \int_0^t e^{\omega s} \| u(s) \| \, ds,
\]
where
\[
K := \| u_0(0) \| + \left( \sum_{n \in \mathbb{N}} \gamma_n r_n^{1-1/p} e^{\omega \tau_n} \right) \| e^{\omega t} u_0(t) \|_{L^p(I;X)}.
\]
Since \( m = \omega - \tilde{M} \) in view of (1.9) and (2.2), the above inequality is rewritten as
\[
e^{\omega t} \| u(t) \| \leq MK + (\omega - m) \int_0^t e^{\omega s} \| u(s) \| \, ds
\]
for all \( t \geq 0 \). Therefore the Gronwall lemma leads to \( \| u(t) \| \leq MKe^{-mt} \) for all \( t \geq 0 \), which warrants the desired decay estimate (1.8). \( \square \)
4. **Examples.** In this section we give two concrete examples of abstract evolution equations with countable time delays. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be globally Lipschitz continuous functions satisfying $f_n(0) = 0$.

4.1. **The semilinear heat equation with countable time delays.** We consider the following problem:

$$
\begin{cases}
    u_t(x, t) - \Delta u(x, t) = \sum_{n=1}^{\infty} f_n(u(x, t - \tau_n)), & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\
    u(x, t) = u_0(x, t), & x \in \Omega, \ t \in I := \bigcup_{n \in \mathbb{N}} [-\tau_n, 0],
\end{cases}
$$

where $\tau_n > 0$ is the time delay and $u_0 \in C(I; X)$ is the initial data, where $X := L^q(\Omega)$, $1 < q < \infty$. The Dirichlet Laplacian $\Delta$ in $X$ generates a $C_0$-semigroup with decaying estimate (1.5) (see e.g., [29, Proposition 48.5]). Therefore we can apply Theorem 1.2 to guarantee well-posedness and exponential decay in the Banach space $X = L^q(\Omega)$ under some conditions for $\tau_n$.

4.2. **The damped wave equation with countable time delays.** Let us consider the following problem:

$$
\begin{cases}
    u_{tt}(x, t) - \Delta u(x, t) + a(x)u_t(x, t) = \sum_{n=1}^{\infty} f_n(u(x, t - \tau_n)), & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\
    u(x, t) = u_0(x, t), \ u_t(x, t) = u_1(x, t), & x \in \Omega, \ t \in I := \bigcup_{n \in \mathbb{N}} [-\tau_n, 0],
\end{cases}
$$

where $\tau_n > 0$ is the time delay and the damping coefficient $a \in L^\infty(\Omega)$ fulfills that $a(x) \geq a_0 > 0$ a.e. on some open subset of $\Omega$ satisfying some control geometric properties (see e.g., [3]). The initial datum $(u_0, u_1)$ is taken in $C(I; H^1_0(\Omega) \times L^2(\Omega))$. Setting

$$
X := H^1_0(\Omega) \times L^2(\Omega),
$$

$$
U := \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad A := \begin{pmatrix} 0 & -1 \\ -\Delta & a \end{pmatrix},
$$

$$
F(U(t), (U(t - \tau_n))_{n \in \mathbb{N}}) := \begin{pmatrix} \sum_{n=1}^{\infty} f_n(u(x, t - \tau_n)) \\ 0 \end{pmatrix},
$$

we can rewrite the above problem as (1.1). It is well-known that $-A$ generates a $C_0$-semigroup with decaying estimate (1.5) (see e.g., [12, 34]). Thus we can obtain existence, uniqueness and exponential decay under some conditions for $\tau_n$ as in Theorem 1.2.

**Acknowledgments.** The authors would like to thank the referees for their fruitful comments and helpful suggestions on improving this paper.
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Received February 2020; revised June 2020.

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