Cross-Overs of Bramson’s Shift at the Transition Between Pulled and Pushed Fronts

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Abstract

The Bramson logarithmic shift of the position of pulled fronts is a universal feature common to a large class of monostable traveling wave equations. As one varies the non-linearities it so happens that one can observe, at some critical non-linearity, a transition from pulled fronts to pushed fronts. At this transition the Bramson shift is modified. In the limit where time goes to infinity and the non-linearity becomes critical, the position of the front exhibits a cross-over. The goal of the present note is to give the expression of this cross-over function, for a particular model which is exactly soluble, with the hope that this expression would remain valid for more general traveling wave equations at the transition between pulled and pushed fronts. Other cross-over functions are also obtained, for this particular model, to describe the dependence on initial conditions or the effect of a cut-off.

Keywords Fisher–KPP equation · Travelling waves · Bramson shift · Cut-offs

1 Introduction

Since its introduction by Fisher [25], and Kolmogorov et al. [33] the F-KPP equation

\[
\frac{du}{dt} = \frac{d^2u}{dx^2} + u(1-u)
\]

(1)

has played a central role in the theory of partial differential equations [3, 28] and of the Branching Brownian motion [35] (see also [14, 18, 20, 21]). When the initial condition \(u(x, 0) \equiv u_0(x)\) is a step function

\[
u_0(x) = \begin{cases} 
1 & \text{for } x \leq 0 \\
0 & \text{for } x > 0
\end{cases}
\]

(2)

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or when it decays fast enough as for example

\[
u_0(x) = \begin{cases} 
1 & \text{for } x \leq 0 \\
\exp(-sx) & \text{for } x > 0 \quad \text{with } s > 1
\end{cases}
\]  

(3)

it is known since the work of Bramson [4, 11, 12, 29] that in the long time limit the solution becomes a front

\[u(x, t) \simeq W(x - \mu_t)\]  

(4)

located at position

\[\mu_t \simeq 2t - \frac{3}{2} \ln t + A(\{u_0(x)\}) + o(1).\]  

(5)

The traveling wave shape \(W\) in (4) is a solution of an ordinary non-linear differential equation

\[2W'(z) + W''(z) + W(z)(1 - W(z)) = 0\]  

(6)

which satisfies \(W(-\infty) = 1\) \(W(\infty) = 0\) and the shift \(A(\{u_0(x)\})\) depends on the initial condition \(u(x, 0)\) (and also on the choice made to fix the solution of (6), for example \(W(0) = \frac{1}{2}\)). What is remarkable is that the log \(t\) term in (5) does not depend on the initial condition [11, 12] (as long as it decays fast enough). In fact this Bramson shift \(-\frac{3}{2}\ln t\) remains the same for all monostable traveling wave equations of the form

\[\frac{du}{dt} = \frac{d^2u}{dx^2} + g(u)\]  

(7)

when the non-linearity \(g(u)\) satisfies

\[g(0) = 0; \quad g'(0) = 1; \quad g(1) = 0;\]  

(8)

and

\[0 < g(u) < u \quad \text{for } 0 < u < 1\]  

(9)

When the position of the traveling wave is given by (5) one says that the front is pulled [41]. Thus when the non-linearity satisfies condition (9), one is in the case of "pulled fronts". When (9) is not satisfied, keeping \(g(u) > 0\) for \(u\) in \((0, 1)\), but with some values of \(u\) where \(g(u) > u\), the long time asymptotics may still given by (4, 5) for steep enough initial conditions (2, 3) (in which case the front is still pulled) or may lead to a different long time asymptotics (called "pushed fronts") with the position \(\mu_t\) of the front becoming

\[\mu_t = vt + C(\{u_0(x)\}) + o(1) \quad \text{with } v > 2\]  

(10)

(for a general non-linearity \(g(u)\) as in (7) the shape \(W\) of the wave front is solution of \(vW' + W'' + g(W) = 0\) with \(v = 2\) in the pulled case and some \(v > 2\) in the pushed case).

For example it is well known [7, 26, 28] that for

\[g(u) = u(1 - u)(1 + 2au)\]  

(11)

the non linearity satisfies the condition (9) for \(a \leq \frac{1}{2}\) but the front remains pulled for \(a < 1\). For \(a > 1\) the front becomes pushed and, for steep enough initial conditions, the asymptotic velocity \(v = \sqrt{a} + \sqrt{\frac{1}{a}}\) with, for \(a > 1\), a front shape given by \(W(x) = \left(1 + \exp[\sqrt{a}x]\right)^{-1}\). (Other examples can be found in [2]).
At the transition between the pulled case and the pushed case \([1, 2, 5]\) (i.e. for \(a = 1\) for the example \((11)\)), the long time asymptotics of \(\mu_t\) is modified to become

\[
\mu_t \simeq 2t - \frac{1}{2} \ln t + B(\{u_0(x)\}) + o(1)
\]

(12)

where, the logarithmic term is now \(-\frac{1}{2} \log t\) instead of \(-\frac{3}{2} \log t\) in \((5)\).

One goal of the present work is to try to understand the cross-over between the three different asymptotics \((5, 10, 12)\). Let us imagine that the non-linearity \(g_a(u)\) in \((7)\) depends on a parameter \(a\) and that there is a critical value \(a_c\) of this parameter which separates the pulled case for \(a < a_c\) from the pushed case for \(a > a_c\). (This critical value \(a_c = 1\) for the example \((11)\)). Already one could guess that, in \((5)\), the constants \(A(\{u_0(x)\}) \rightarrow \infty\) as \(a \rightarrow a^-\) and \(C(\{u_0(x)\}) \rightarrow -\infty\) as \(a \rightarrow a^+\). But what to expect for \(\mu_t\) in the regime where \(|a - a_c|\) is very small and \(t\) is very large?

We are going to obtain the asymptotics of \(\mu_t\) in this cross-over regime for a system \([9]\) which can be viewed as the hydrodynamic limit of the \(N\)-BBM or of the \(L\)-BBM \([13, 16, 17, 19, 23, 34, 40]\). In this simple model \([9]\) the evolution of \(u\) is given by

\[
\frac{du}{dt} = \frac{d^2 u}{dx^2} + u \quad \text{for} \quad x > \mu_t
\]

\[
\mu_t = 1
\]

\[
\partial_x u(\mu_t, t) = -a \quad \text{with} \quad a > 0
\]

(13)

This is a free boundary problem \([17, 31]\) and, given the initial condition \(u_0(x)\), the evolution \((13)\) determines both \(u(x, t)\) and \(\mu_t\). (To avoid complicated discussions, we consider here only decreasing initial conditions such that \(\mu_0 = 0\), \(u_0(0) = 1\) and \(u(\infty) = 0\)). This problem was studied in \([9]\), the parameters \(\alpha\) and \(\beta\) in \([9]\) taking here the values \(\alpha = 1\) and \(\beta = -a\).

In this simple model, as in its earlier lattice version \([15]\), the non-linearity comes only from the boundary condition at \(x = \mu_t\) (see \((13)\) which can be interpreted as \(\frac{du}{dt} = 0\) for \(u \geq 1\)). The reason why explicit calculations are possible for this example is that there exists an exact relation established in \([9]\) (see Equation (6) of \([9]\)) between the position \(\mu_t\) and the initial condition \(u_0(x)\) which can be written here as

\[
1 + r \int_0^\infty dz \, u_0(z) \, e^{rz} = (1 - ar) \int_0^\infty dt \, e^{\mu_t - (1+r^2)t}
\]

(14)

(This relation remains valid when \(r\) varies as long as the integrals on both sides of \((14)\) converge.)

Based on this equation it has been shown (see Equations (36,39,40) of \([9]\)) that for a step initial condition \((2)\) (as well as for steep enough initial conditions as in \((3)\))

For \(a < 1\)

\[
\mu_t = 2t - \frac{3}{2} \log t + A(\{u_0(x)\}) - \frac{3 \sqrt{\pi}}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)
\]

\[
a = 1 \quad \mu_t = 2t - \frac{1}{2} \log t + B(\{u_0(x)\}) - \frac{\sqrt{\pi}}{2 \sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)
\]

\[
a > 1 \quad \mu_t = \left(\frac{2 + (a - 1)^2}{a}\right) t + C \left(\{u_0(x)\}\right) + o(1)
\]

(15)

where \(A, B, C\) are time-independent constants whose expressions are not known and depend on the initial condition \(u_0\).
What we are going to show in Sect. 2 is that, in the scaling regime where
\[ a - 1 = \epsilon \ll 1 \quad \text{and} \quad t = O(\epsilon^{-2}), \]  
\[ \mu_t = 2t - \frac{1}{2} \log t + \Psi_1(\sqrt{t}) + B(u_0(x)) + o(1) \]  
where
\[ \Psi_1(z) = \log \left[ 1 + 2z e^{z^2} \int_\sqrt{\epsilon}^\infty dv e^{-v^2} \right] \]  
and
\[ B(u_0(x)) = \log \left[ 1 + \int_0^\infty dz e^z u_0(z) \right] - \frac{1}{2} \log \pi \]

We will see (using (A.8)) that the limits \( \epsilon \sqrt{t} \to \pm \infty \) are consistent with (15).

In Sect. 3 we will study the cross-over due to initial conditions, i.e. when the decay rate \( s \to 1 \) in (3). We will discuss separately the pulled case (i.e. when \( a < 1 \) in (13)) and the transition point \( a = 1 \) between the pushed and the pulled cases.

Lastly, in Sect. 4, we will discuss the effect of a cut-off on the velocity of the traveling wave (where one imposes the additional boundary condition that \( u(\mu_t + L) = 0 \)). In this case too, there a crossover near the transition between the pulled and the pushed cases when the cut-off parameter \( L \to \infty \) at the same time as the distance to the transition point \( \epsilon = a - 1 \to 0 \), keeping the product \( L \epsilon = O(1) \).

2 Cross-Over of the Bramson Shift at the Transition Between Pushed and Pulled Fronts

All our analysis is based on the exact relation (14) (which was established in [9])) between the initial condition \( u_0(x) \) and the position of the front \( \mu_t \). Assuming that \( a = 1 + \epsilon \) as in (13) and (16) one can rewrite (14) as
\[ \left[ 1 + r \int_0^\infty dz u_0(z) e^{rz} \right] \frac{1}{1 - r - \epsilon r} = \int_0^\infty dt e^{\mu_t - (1+\epsilon r^2)t} \]  
(20)

As in [9] the main idea is to equate the singular parts of the two sides of (20) when \( r \to 1 \). If one expands the left hand side in powers of \( \epsilon \), one gets, by keeping at each order in \( \epsilon \) the most singular term at \( r = 1 \),
\[ \left[ 1 + r \int_0^\infty dz u_0(z) e^{rz} \right] \frac{1}{1 - r - \epsilon r} \overset{(\text{sing})}{=} \left[ 1 + \int_0^\infty dz u_0(z) e^z \right] \sum_{n=0}^{\infty} \frac{\epsilon^n}{(1-r)^{n+1}} \]  
(21)

(\( \overset{(\text{sing})}{=} \) means that, at each order in \( \epsilon \), the most singular terms in the limit \( r \to 1 \) are the same on both sides of \( \overset{(\text{sing})}{=} \)). Then using the fact that
\[ \frac{1}{(1-r)^{n+1}} = \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^\infty dt t^{-\frac{n+1}{2}} e^{-(1-r^2)t} \]  
(22)
one gets for the left-hand side of (21)

\[
\left[1 + r \int_0^\infty dz \ u_0(z) \ e^{rz} \right] \frac{1}{1 - r - \epsilon r} \overset{\text{(sing)}}{=} \left[1 + \int_0^\infty dz \ u_0(z) \ e^{r^2} \right] \int_0^\infty dt \ \frac{G_1(\epsilon \sqrt{t})}{\sqrt{t}} \ e^{-(1-r)^2t}
\]

where \( G_1(z) \) is defined by

\[
G_1(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma\left(\frac{n+1}{2}\right)}
\]

On the other hand, as far as the singular part at \( r = 1 \) is concerned, one could replace the lower bound in the integral on the right hand side of (20) by any time \( t_0 \)

\[
\int_0^{\infty} dt \ e^{r(\mu_t - (1+r^2)t)} \overset{\text{(sing)}}{=} \int_{t_0}^{\infty} dt \ e^{r(\mu_t - (1+r^2)t)}
\]

(since the integral from 0 to \( t_0 \) is an analytic function of \( r \)). Therefore, as \( t_0 \) can be arbitrary large, it is clear that the singular part at \( r = 1 \) of the right hand side of (20) is fully determined by the long time asymptotics of \( \mu_t \). One has for the right hand side of (20)

\[
\int_0^{\infty} dt \ e^{r(\mu_t - 2t) - (1-r)^2t} \overset{\text{(sing)}}{=} \int_0^{\infty} dt \ e^{r(\mu_t - 2t) - (1-r)^2t} \left[1 + (r - 1)(\mu_t - 2t) + \cdots\right]
\]

(26)

where, again, in the last line, we keep only the first term because we look for the most singular term in \( r \) at each order in powers of \( \epsilon \).

Equating the left hand side (23) and the right hand side of (26) of (20) for a whole neighborhood of \( r = 1 \) leads to the following large \( t \) asymptotics for \( \mu_t \)

\[
e^{\mu_t - 2t} \simeq \left[1 + \int_0^\infty dz \ u_0(z) \ e^{rz} \right] \frac{G_1(\epsilon \sqrt{t})}{\sqrt{t}}
\] (27)

which can be rewritten as in (17, 18, 19) using the expression (A.7) of \( G_1(z) \) established in the Appendix A. One can also note that the \( z \to \pm \infty \) asymptotics (A.8) of the function \( G_1 \) allow to recover both the pulled and the pushed regimes (15).

As explained in Appendix B, by analyzing the next singular terms at \( r = 1 \) in the equation (20), one can get cross-over functions of the next terms in the asymptotics of \( \mu_t \), i.e. vanishing corrections to (17) in the long time limit [1, 2, 9, 10, 24, 27, 30, 36, 39].

### 3 Cross-Over Due to Initial Conditions

In this section we analyze the cross-over due to the initial condition both in the pulled case and at the boundary between the pulled and the pushed case. As we will see the analysis is very similar to what was done in Sect. 2.

1. **The pulled case (\( a < 1 \) and \( s \to 1 \)):
In the pulled case, (i.e. for $a < 1$ in the problem (13)), we want to consider the cross-over regime where $s \to 1$ in the initial condition (3) and $t \to \infty$. If we write

$$s = 1 + \varphi \quad \text{with} \quad \varphi \ll 1$$

(28)

the relation (14) between the initial condition (3) and the position of the front $\mu_t$ becomes

$$\frac{1 + \varphi}{(1 - ar)(1 + \varphi - r)} = \int_0^\infty dt \, e^{r\mu_t - (1+r^2)t}$$

(29)

Now if, at each order in powers of $\varphi$, we look for the most singular term at $r = 1$ on the left hand side of (29) one gets

$$\frac{1 + \varphi}{(1 - ar)(1 + \varphi - r)} (\text{sing}) = \frac{1}{(1 - a)(1 + \varphi - r)}$$

(30)

Copying what was done in (21, 22, 23) one gets for the left hand side of (29)

$$\frac{1 + \varphi}{(1 - ar)(1 + \varphi - r)} (\text{sing}) = \frac{1}{(1 - a)(1 + \varphi - r)}$$

(31)

The analysis of the right hand side of (29) is exactly the same (see (26)) as in Sect. 2. Therefore one gets for the pulled case

$$e^{\mu_t - 2t} \simeq \left[ \frac{1}{1 - a} \right] \frac{G_1(-\varphi \sqrt{t})}{\sqrt{t}}$$

(32)

so that, using the expression (A.7) one obtains

$$\mu_t = 2t - \frac{1}{2} \log t - \log(1 - a) + \Psi_1(-\varphi \sqrt{t}) + o(1)$$

(33)

where the function $\Psi_1$ is given in (18).

2. The boundary between the pulled and the pushed cases ($a = 1$ and $s \to 1$):

Let us now look at the case $a = 1$ in (13), i.e. at the transition between the pushed and the pulled case. The left hand side of (20) becomes

$$\frac{1 + \varphi}{(1 - r)(1 + \varphi - r)} (\text{sing}) = \sum_{n \geq 0} (-1)^n \frac{\varphi^n}{(1 - r)^{n+2}} = \int_0^\infty dt G_2(-\varphi \sqrt{t}) \, e^{-(1-r^2)t}$$

(34)

where we have used (23) and the function $G_2$ is defined by

$$G_2(z) = \sum_{n \geq 0} \frac{z^n}{r^{n+1}}$$

(35)

The expression of the most singular terms in the right hand side of (20) remains given by (26), so that, equating the most singular parts (34) and (26) of the two sides of (20) one gets

$$\mu_t = 2t + \Psi_2(-\sqrt{\varphi t})$$

(36)
where the function $\Psi_2$ is given by (see (A.9))

$$\Psi_2(z) = \log \left[ \frac{2}{\sqrt{\pi}} e^{\frac{z^2}{2}} \int_{-\infty}^{z} e^{-t^2} \, dt \right]$$  \hspace{1cm} (37)

### 4 The Effect of a Cut-Off on the Velocity of the Traveling Wave

For the F-KPP equation it is well established that the first correction to the velocity of the traveling wave due to a weak white noise [8, 37, 38] is the same as the correction due to a cut-off [6, 13, 22]. In this section we calculate this correction due to a cut-off for the model (13). We will analyze successively the pulled case, the pushed case and the cross-over near the transition between pulled and pushed fronts.

A traveling wave in the problem (13) moving at velocity $v$ is a solution $u$ of the form

$$u(x, t) = W(x - vt) = \begin{cases} A \exp \left[ -\gamma (x - vt) \right] + B \exp \left[ -\gamma^{-1} (x - vt) \right] & \text{for } x \geq vt \\ 1 & \text{for } x \leq vt \end{cases}$$  \hspace{1cm} (38)

where the amplitudes $A$ and $B$ satisfy

$$W(0) = A + B = 1; \quad W'(0) = -A \gamma - B \gamma = -a.$$  \hspace{1cm} (39)

These equations (39) determine $A$ and $B$ in terms of $\gamma$ with a velocity $v$ and a front position $\mu_t$ given by

$$v = \gamma + \frac{1}{\gamma}; \quad \mu_t = vt$$  \hspace{1cm} (40)

If one introduces a cut-off by requiring that the front vanishes at position $x = vt + L$, one gets an additional equation which fixes $\gamma$ and therefore the velocity $v$

$$W(L) = A \exp[-\gamma L] + B \exp \left[ -\gamma^{-1} L \right] = 0.$$  \hspace{1cm} (41)

For

$$a = 1 + \epsilon$$  \hspace{1cm} (42)

$\gamma$ should therefore satisfy

$$\exp \left( L \frac{1 - \gamma^2}{\gamma} \right) = -\gamma \frac{1 + \epsilon - \gamma}{1 - \gamma - \epsilon \gamma}.$$  \hspace{1cm} (43)

To determine the large $L$ correction to the velocity $v$ one simply has to calculate the large $L$ dependence of the solution $\gamma$ of (43). One can distinguish the following three situations:

1. **The pulled case ($\epsilon < 0$):**
   In absence of a cut-off, i.e. in the limit $L \to \infty$, the velocity $v \to 2$ and $\gamma \to 1$. For large $L$, the solution of (43) is

   $$\gamma = 1 \pm i \frac{\pi}{L} + \cdots$$  \hspace{1cm} (44)

   and one recovers the expected correction to the velocity for pulled fronts [6, 13, 22]

   $$v = 2 - \frac{\pi^2}{L^2} + \cdots.$$  \hspace{1cm} (45)
2. The pushed case ($\epsilon > 0$):

In the pushed case, $A$ in (38) vanishes when there is no cut-off, i.e. for $L \to \infty$, so that $\gamma = \frac{1}{a} = \frac{1}{1+\epsilon}$. For large $L$, the solution of (43) is

$$\gamma = \frac{1}{1+\epsilon} + \frac{2\epsilon + \epsilon^2}{(1+\epsilon)^3} \exp\left[-\frac{2\epsilon + \epsilon^2}{1+\epsilon}L\right] + \cdots$$

(46)

and the correction to the velocity is exponentially small in $L$

$$v \simeq 2 + \frac{\epsilon^2}{1+\epsilon} - \frac{(2\epsilon + \epsilon^2)^2}{(1+\epsilon)^3} \exp\left[-\frac{2\epsilon + \epsilon^2}{1+\epsilon}L\right] + \cdots$$

(47)

as expected in the pushed case [32].

3. The cross-over between the pulled and the pushed case:

For $\epsilon$ small and large $L$, keeping the product $L\epsilon$ of order 1, if one writes

$$\gamma \simeq 1 + i \frac{\chi}{L}$$

(48)

and solves (43) in the large $L$ limit, $\chi$ is solution of

$$\chi = (L\epsilon) \tan(\chi)$$

(49)

(one should choose the solution such that $\chi \to \frac{\pi}{2}$ as $L\epsilon \to 0$: for other choices like $(\frac{(2n+1)\pi}{2}$ with $n = 1, 2, 3 \cdots$ the traveling wave $W(x)$ would not remain positive in the interval $(0, L)$). Therefore

$$v = 2 - \frac{\chi^2}{L^2} + \cdots$$

(50)

where $\chi$ is the function of $L\epsilon$, solution of (49). For example for $L\epsilon$ small one gets

$$\chi = \frac{\pi}{2} - \frac{2}{\pi} L\epsilon - \frac{8}{\pi^3} (L\epsilon)^2 + \left(\frac{8}{3\pi^3} - \frac{64}{\pi^3}\right) (L\epsilon)^3 + O((L\epsilon)^4)$$

(51)

giving

$$v = 2 + \frac{1}{L^2} \left[-\frac{\pi^2}{4} + 2(L\epsilon) + \frac{4}{\pi^2} (L\epsilon)^2 + \left(\frac{8}{3\pi^2} - \frac{32}{\pi^2}\right) (L\epsilon)^3 + O((L\epsilon)^4)\right].$$

(52)

In the particular case where $\epsilon = 0$, i.e. right at the transition between the pulled and the pushed case, $\chi = \frac{\pi}{2}$ and the correction to the velocity becomes

$$v \simeq 2 - \frac{\pi^2}{4L^2}$$

(53)

in contrast to the expression (45) for the pulled case.

For $L\epsilon \to -\infty$, one also gets

$$v = 2 - \frac{\pi^2}{L^2} \left(1 + \frac{2}{L\epsilon} + \frac{3}{L^2\epsilon^2} + \cdots\right)$$

(54)

which matches with (45).
As the product $L\epsilon$ varies from $-\infty$ to 1 the solution $\chi$ of (49) decreases from $\pi$ to $0$ (taking the value $\frac{\pi}{2}$ when $L\epsilon = 0$). For $L\epsilon > 1$, $\chi$ solution of (49) becomes imaginary: $\chi = i\chi'$ and the velocity becomes

$$v = 2 + \frac{\chi'^2}{L^2} + \cdots$$

(55)

where $\chi'$ is solution of

$$\chi' = L\epsilon \tanh \chi' \quad \text{with} \quad 0 < \chi'$$

(56)

(There is no singularity at $L\epsilon = 1$. The solutions (52) and (55) for $L\epsilon < 1$ and $L\epsilon > 1$ are analytic continuations of each other.)

For $L\epsilon \to +\infty$, one has

$$\chi' = L\epsilon (1 - 2e^{-L\epsilon} + \cdots)$$

(57)

$$v = 2 + \epsilon^2 - 4\epsilon^2 e^{-2L\epsilon} + \cdots$$

(58)

which also matches with (49).

## 5 Conclusion

The goal of the present work was to derive explicit expressions (17, 18, 19) of the cross-over functions describing the position of the front near the transition between pulled and pushed fronts for the model (13). For the same model, other cross-over functions have been obtained for the dependences on initial conditions (33, 36, 37) or for the shift of velocity (49, 50, 53, 55, 56) due to a cut-off.

All these cross-over functions were obtained for the very specific model (13). Of course it would be interesting to know their degree of universality, in particular if they could also describe the same regimes for other non-linear traveling wave equations, such as those discussed in [1, 2].

Another interesting question would be to understand the effect of a weak noise on the velocity of traveling waves. For pulled fronts it is well established [8, 38] that the leading correction (50) to the velocity due to a weak noise can be predicted by a cut-off argument [13]. Does the same cut-off argument give the correct correction (53) due a weak noise at the transition between pulled and pushed front and even in the whole cross-over regime?

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### Appendix A

In this appendix we obtain explicit expressions of the sums of the series which appear in (22, 32, 35). We are going to show that the function $G_\alpha(z)$ defined by

$$G_\alpha(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma \left( \frac{n+\alpha}{2} \right)}$$

(A.1)
can be written, for $\alpha > 2$, in terms of an incomplete Gamma function as

$$G_\alpha(z) = e^{z^2}z^{2-\alpha} \int_0^z e^{-y} dy \left[ \frac{y^{\frac{\alpha-4}{2}}}{\Gamma(\frac{\alpha-2}{2})} + \frac{y^{\frac{\alpha-3}{2}}}{\Gamma(\frac{\alpha-1}{2})} \right]$$  \hspace{1cm} (A.2)

To do so, one can check that the series in (A.1) satisfies

$$\frac{dG_\alpha}{dz} = 2zG_\alpha - \frac{\alpha - 2}{z}G_\alpha + \frac{2}{z \Gamma(\frac{\alpha-2}{2})} + \frac{2}{\Gamma(\frac{\alpha-1}{2})}$$  \hspace{1cm} (A.3)

and one gets for $\alpha > 2$

$$G_\alpha(z) = 2e^{z^2}z^{2-\alpha} \int_0^z e^{-u^2} du \left[ \frac{u^{\alpha-3}}{\Gamma(\frac{\alpha-2}{2})} + \frac{u^{\alpha-2}}{\Gamma(\frac{\alpha-1}{2})} \right]$$  \hspace{1cm} (A.4)

which, is equivalent to (A.2).

Moreover it is also easy to check that for, any $\alpha$, the sum in (A.1) satisfies

$$G_\alpha(z) = \frac{1}{\Gamma(\frac{\alpha}{2})} + zG_{\alpha+1}(z)$$  \hspace{1cm} (A.5)

Therefore (A.2) together with (A.5) gives a closed expression of $G_\alpha$ for any $\alpha$.

Using the explicit expression (A.2) or the differential equation (A.3) it is easy to extract the large $z$ asymptotic series of $G_\alpha$:

$$G_\alpha(z) \simeq 2z^{2-\alpha} e^{z^2} - \sum_{n \geq 1} \frac{1}{z^n \Gamma(\frac{\alpha-n}{2})} \hspace{1cm} \text{as } z \to +\infty$$  \hspace{1cm} (A.6)

$$G_\alpha(z) \simeq -\sum_{n \geq 1} \frac{1}{z^n \Gamma(\frac{\alpha-n}{2})} \hspace{1cm} \text{as } z \to -\infty$$

In the particular cases $\alpha = 1$ and $\alpha = 2$, one then gets from (A.2, A.6)

$$G_1(z) = \frac{1}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} e^{z^2} \int_{-\infty}^z e^{-t^2} dt$$  \hspace{1cm} (A.7)

with the following asymptotics

For $z \to -\infty$ \hspace{1cm} $G_1(z) \simeq \frac{1}{\sqrt{\pi}} z^2 + O\left(\frac{1}{z^4}\right)$

For $z \to 0$ \hspace{1cm} $G_1(z) = \frac{1}{\sqrt{\pi}} + z + O(z^2)$

For $z \to +\infty$ \hspace{1cm} $G_1(z) \simeq 2ze^{z^2} + \frac{1}{\sqrt{\pi}} z^2 + O\left(\frac{1}{z^4}\right)$  \hspace{1cm} (A.8)

and

$$G_2(z) = \frac{2}{\sqrt{\pi}} e^{z^2} \int_{-\infty}^z e^{-t^2} dt$$  \hspace{1cm} (A.9)
with the following asymptotics

For \( z \to -\infty \)
\[
G_2(z) \simeq -\frac{1}{\sqrt{\pi} z} + O\left(\frac{1}{z^3}\right)
\]

For \( z \to 0 \)
\[
G_2(z) = 1 + \frac{2}{\sqrt{\pi} z} + O(z^2)
\]

For \( z \to +\infty \)
\[
G_2(z) \simeq 2e^{z^2} - \frac{1}{\sqrt{\pi} z} + O\left(\frac{1}{z^3}\right)
\]

(A.10)

**Appendix B**

The goal of this appendix is to indicate how to push further the analysis of the most singular terms at \( r = 1 \) of the equality (20), by looking now at the two most singular terms in order to study vanishing corrections to the front position [1, 2, 9, 10, 24, 30, 36, 39]. To simplify the discussion we will only consider the case of a step initial condition, i.e. to the case where \( u_0(z) = 0 \) for \( z > 0 \). Looking, at each order in \( \epsilon \), for the two most singular terms in the left hand side of (20) one gets

\[
\left[ 1 + r \int_0^{\infty} dz \ u_0(z) \ e^{rz} \right] \frac{1}{1 - r - \epsilon r} \overset{(\text{sing})}{=} \sum_{n \geq 0} \epsilon^n \left[ \frac{1}{(1-r)^{n+1}} - \frac{n}{(1-r)^n} \right]
\]

(B.1)

Let us now assume, for large \( t \) and \( \epsilon \sqrt{t} = O(1) \), the following generalization of (27)

\[
e^{\mu t - 2t} = \frac{1}{\sqrt{t}} \sum_{n \geq 0} b_n(\epsilon \sqrt{t})^n + \frac{\log t}{t} \sum_{n \geq 1} c_n(\epsilon \sqrt{t})^n + \frac{1}{t} \sum_{n \geq 0} d_n(\epsilon \sqrt{t})^n + o\left(\frac{1}{t}\right)
\]

(B.2)

Using the identity (22)

\[
\int_{t_0}^{\infty} dt \ t^{\frac{n-1}{2}} e^{-(1-r)^2 t} \overset{(\text{sing})}{=} \frac{\Gamma\left(\frac{n+1}{2}\right)}{(1-r)^{n+1}}
\]

(B.3)

as well as

\[
\int_{t_0}^{\infty} dt \ t^{\frac{n-1}{2}} \log t \ e^{-(1-r)^2 t} \overset{(\text{sing})}{=} -2 \frac{\Gamma\left(\frac{n+1}{2}\right) \log(1-r)}{(1-r)^{n+1}} + \frac{\Gamma'\left(\frac{n+1}{2}\right)}{2(1-r)^{n+1}}
\]

(B.4)

and

\[
\int_{t_0}^{\infty} dt \ t^{-1} e^{-(1-r)^2 t} \overset{(\text{sing})}{=} -2 \log(1-r)
\]

(B.5)
one gets for the two most singular terms of the right hand side of (20)

\[ \int_0^\infty dt \, e^{\nu t - (1-r)^2 t} \text{(sing)} = \sum_{n \geq 0} b_n e^n \Gamma \left( \frac{n+1}{2} \right) \left( \frac{n+1}{2} \right) + \sum_{n \geq 1} c_n e^n \left( -2 \log(1-r) \Gamma \left( \frac{n}{2} \right) + \Gamma' \left( \frac{n}{2} \right) \right) \]

\[ \left[ -2d_0 \log(1-r) + \sum_{n \geq 1} d_n e^n \Gamma \left( \frac{n}{2} \right) \right] - (1-r) \left[ \frac{1}{2} \sum_{n \geq 0} b_n e^n \left( -2 \log(1-r) \Gamma \left( \frac{n+1}{2} \right) + \Gamma' \left( \frac{n+1}{2} \right) \right) \right] \]

\[ + \sum_{n \geq 0} f_n e^n \Gamma \left( \frac{n+1}{2} \right) \]

where the \( f_n \)'s are the coefficients of the series

\[ \sum_{n \geq 0} f_n \z^n = \left( \sum_{n \geq 0} b_n \z^n \right) \log \left( \sum_{n \geq 0} b_n \z^n \right) \]

Then equating the two most singular parts of the two sides of (20) one gets

\[ b_n = \frac{1}{\Gamma \left( \frac{n+1}{2} \right)} \text{ for } n \geq 0 \]

\[ c_n = -\frac{1}{2\Gamma \left( \frac{n}{2} \right)} \text{ for } n \geq 1 \]

\[ d_0 = -\frac{1}{2} \]

\[ d_n = \frac{1}{2} \Gamma' \left( \frac{n}{2} \right) - \frac{1}{2} \frac{\Gamma' \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} - \frac{1}{2} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma' \left( \frac{n+1}{2} \right)} - \frac{n}{\Gamma \left( \frac{n}{2} \right)} \text{ for } n \geq 1 \]

Except for the \( d_n \)'s one can get closed expressions of the series (see Appendix A)

\[ \sum_{n \geq 0} b_n \z^n = G_1(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma \left( \frac{n+1}{2} \right)} \]

\[ \sum_{n \geq 1} c_n \z^n = G_0(z) = -\frac{z}{2} G_1(z) \]

This gives the dominant vanishing correction to (17) for the step initial condition (2)

\[ \mu_t = 2t - \frac{1}{2} \log t + \Psi_1(\epsilon \sqrt{t}) \]

\[ -\frac{1}{2} \log \pi - \frac{\epsilon \log t}{2} + \sum_{n \geq 0} d_n(\epsilon \sqrt{t})^n \frac{1}{\sqrt{t} \, G_1(\epsilon \sqrt{t})} + o \left( \frac{1}{\sqrt{t}} \right) \] (B.6)
(Note that the term $\epsilon \log t$ is in fact of order $\log t / \sqrt{t}$ since $\epsilon \sqrt{t} = O(1)$ in the large $t$ limit.) Given the values of $b_0$ and of $d_0$ it is easy to see that this is consistent with (15) when $\epsilon = 0$ i.e. when $a = 1$. On the other hand, I was not able to estimate the series $d_n$ for $z \to -\infty$ and so could not check if (B.6) is consistent with (15) in the limit $\epsilon \to -\infty$.

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