Classification of finite dimensional simple Lie algebras in prime characteristics

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Abstract. We give a comprehensive survey of the theory of finite dimensional Lie algebras over an algebraically closed field of prime characteristic and announce that the classification of all finite dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 3$ is now complete. Any such Lie algebra is up to isomorphism either classical or a filtered Lie algebra of Cartan type or a Melikian algebra of characteristic 5.

Unless otherwise specified, all Lie algebras in this survey are assumed to be finite dimensional. In the first two sections, we review some basics of modular Lie theory including absolute toral rank, generalized Winter exponentials, sandwich elements, and standard filtrations. In Section 3, we give a systematic description of all known simple Lie algebras of characteristic $p > 3$ with emphasis on graded and filtered Cartan type Lie algebras. We also discuss the Melikian algebras of characteristic 5 and their analogues in characteristics 3 and 2. Our main result (Theorem 7) is stated in Section 4 which also contains formulations of several important theorems frequently used in the course of classifying simple Lie algebras. The main principles of our proof of Theorem 7, with emphasis on the rank two case, are outlined in Section 5. As suggested by the referee, we mention in Section 6 some interesting open problems related to the subject.

We would like to thank the referee for careful reading and valuable comments.

1. The beginnings

The theory of Lie algebras over a field $F$ of characteristic $p > 0$ was initiated by Jacobson, Witt and Zassenhaus. In [37], Jacobson investigated purely inseparable field extensions $E/F$ of the form $E = F(c_1, \ldots, c_n)$ where $c_i^p \in F$ for all $i \leq n$. Although such field extensions do not possess nontrivial $F$-automorphisms, Jacobson developed for them a version of Galois theory. The role of Galois automorphisms in his theory was played by $F$-derivations.

The set $\text{Der}_F E$ of all $F$-derivations of $E$ carries the following three structures:

- a natural structure of a vector space over $E$,
- a natural $p$-structure given by the $p$th power map $D \mapsto D^p$,
• a Lie algebra structure given by the commutator product.

Let \( \mathfrak{F} \) denote the set of all subfields of \( E \) containing \( F \) and \( \mathfrak{L} \) the set of all \( E \)-subspaces of \( \text{Der}_F E \) stable under the \( p \)th power map and Lie bracket in \( \text{Der}_F E \). Both sets \( \mathfrak{F} \) and \( \mathfrak{L} \) are partially ordered by inclusion. Given a subset \( X \) in \( \text{Der}_F E \) we let \( E^X \) denote the subfield of \( E \) consisting of all \( \alpha \in E \) satisfying \( x(\alpha) = 0 \) for all \( x \in X \).

**Theorem 1 (J 37).** The map \( \mathfrak{L} \ni L \mapsto E^L \in \mathfrak{F} \) is an order-reversing bijection between \( \mathfrak{L} \) and \( \mathfrak{F} \).

Jacobson singled out the \( p \)-structure above as being of major importance for Lie theory.

**Definition 1 (J 37).** A Lie algebra \( L \) over \( F \) is called restrictable if for any \( x \in L \) the derivation \( (\text{ad} \, x)^p \) of \( L \) is inner.

Any restrictable Lie algebra \( L \) carries a \( p \)-mapping \( x \mapsto x^p \) which enjoys the three following properties:

1. \((\lambda x)^p = \lambda^p x^p,\)
2. \((\text{ad} \, x)^p = \text{ad} \, x^p,\)
3. \((x + y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y) t^{i-1},\) where \( s_i(x, y) \in L \) are such that
   \[
   \sum_{i=1}^{p-1} s_i(x, y) t^{i-1} = (\text{ad} \, (tx + y))^{p-1}(x)
   \]

(here \( x, y \in L, \lambda \in F, \) and \( t \) is a variable). Such a \( p \)-mapping is uniquely determined up to a \( p \)-linear map from \( L \) into its center \( z(L) \). It is therefore unique for any restrictable Lie algebra \( L \) with \( z(L) = \{0\} \). Once the mapping \([p]\) is fixed, the pair \((L, [p])\) is called a restricted Lie algebra. If \( I \) is a restricted ideal of \( L \), then the quotient Lie algebra \( L/I \) carries a natural \( p \)-mapping given by \((x + I)^{p} = x^{p} + I\) for all \( x \in L \). We mention for completeness that the Lie algebras of linear algebraic groups over \( F \) are all equipped with canonical \( p \)-mappings, hence carry canonical restricted Lie algebra structures.

From now on we assume that \( F \) is algebraically closed. Some time before 1939 Witt discovered (for any \( p > 3 \)) a \( p \)-dimensional simple Lie algebra with no finite dimensional analogues in characteristic 0. The Witt algebra \( W(1; 1) \) has basis \( \{e_{-1}, e_0, e_1, \ldots, e_{p-2}\} \) over \( F \) and the Lie product in \( W(1; 1) \) is given by

\[
[e_i, e_j] = \begin{cases} 
(j - i) e_{i+j} & \text{if } -1 \leq i + j \leq p - 2, \\
0 & \text{otherwise}.
\end{cases}
\]

As Witt himself never published his example, we have only indirect information about his discovery. Zassenhaus generalized Witt’s example by considering a subgroup \( G \) of order \( p^n \) in the additive group of \( F \) and by giving a \( p^n \)-dimensional vector space \( W_G := \bigoplus_{g \in G} F e_g \) a Lie algebra structure via \([e_g, e_h] := (g - h) e_{g+h}\) for all \( g, h \in G \). Such Lie algebras are often referred to as Zassenhaus algebras.

In [Z 39], Zassenhaus investigated irreducible representations of nilpotent Lie algebras over fields of prime characteristics. This paper is the starting point of the modular representation theory of Lie algebras.

In [Cha 41], Chang described all irreducible representations of the Witt algebra \( W(1; 1) \). According to [Cha 41], Witt used the following realization of the Lie algebra \( W(1; 1) \): Let \( O(1; 1) \) denote the truncated polynomial algebra \( F[X]/(X^p) \),
and let $x$ be the image of $X$ in $\mathcal{O}(1; 1)$. Give $\mathcal{O}(1; 1)$ an algebra structure by setting 
\[ \{f, g\} := f(dx/dx) - g(dx/dx) \] 
for all $f, g \in \mathcal{O}(1; 1)$. It is readily seen that the map 
\[ e_i \mapsto x_i^{+1} \] 
extends to an algebra isomorphism $W(1; 1) \cong \mathcal{O}(1; 1, \{\cdot, \cdot\})$. For 
$i \in \mathbb{F}_p$ set $u_i = (1 + x)^{i+1}$. Then \[ \{u_i, u_j\} = (j - i)u_{i+j} \] 
for all $i, j \in \mathbb{F}_p$. This shows that $W(1; 1)$ is isomorphic to the Zassenhaus algebra associated with the additive subgroup $\mathbb{F}_p \subset F$.

2. Some basics

This section is a short introduction into the general theory of modular Lie algebras with emphasis on results and techniques used in Classification Theory. Most of the results discussed here are valid for any prime $p$.

2.1. Maximal tori in restricted Lie algebras. Let $\mathfrak{g}$ be a restricted Lie algebra over $F$. An element $x \in \mathfrak{g}$ is called \emph{semisimple} (respectively, \emph{nilpotent}) if $x$ lies in the restricted subalgebra of $\mathfrak{g}$ generated by $x^p$ (respectively, if $x^p = 0$ for $e \geq 0$). For any $x \in \mathfrak{g}$ there exist unique commuting $x_s$ and $x_n$ in $\mathfrak{g}$ such that $x_s$ is semisimple, $x_n$ is nilpotent, and $x = x_s + x_n$. We denote by $\mathfrak{g}_s^0$ the set of all $y \in \mathfrak{g}$ such that $(\text{ad} x)^\dim \mathfrak{g}(y) = 0$, and define $\dim \mathfrak{g}(x) := \min \{\dim \mathfrak{g}_s^0 | x \in \mathfrak{g}\}$. If $\dim \mathfrak{g}_s^0 = \dim \mathfrak{g}(x)$ then $\mathfrak{g}_s^0$ is a Cartan subalgebra of $\mathfrak{g}$ (this is a standard fact of Lie theory).

An element $t \in \mathfrak{g}$ is called \emph{toral} if $t^p = t$. A restricted subalgebra $t$ of $\mathfrak{g}$ is called \emph{toral} (or a \emph{torus} of $\mathfrak{g}$) if the $p$-mapping is invertible on $t$. Any toral subalgebra of $\mathfrak{g}$ is abelian and admits a basis consisting of toral elements. Set 
\[ MT(\mathfrak{g}) := \max \{\dim t | t \text{ is a torus in } \mathfrak{g}\}. \]

A torus $t$ of $\mathfrak{g}$ is called \emph{maximal} if the inclusion $t \subset t'$ with $t'$ toral implies $t = t'$. The centralizer $\mathfrak{c}_\mathfrak{g}(t)$ of any maximal torus in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and, conversely, the semisimple elements of any Cartan subalgebra of $\mathfrak{g}$ lie in its center and form a maximal torus in $\mathfrak{g}$. The reader should be warned, however, that maximal tori (and their centralizers) in a restricted Lie algebra may have different dimensions (see [ST 77]). In other words, there may exist maximal tori in $\mathfrak{g}$ of dimension less that $MT(\mathfrak{g})$.

Let $t$ be a maximal torus of $\mathfrak{g}$, $\mathfrak{h} = \mathfrak{c}_\mathfrak{g}(t)$, and let $V$ be a finite dimensional restricted $\mathfrak{g}$-module (this means that $\rho_V(x^p) = \rho_V(x)^p$ for any $x \in \mathfrak{g}$ where $\rho_V$ denotes the corresponding representation). Since $\rho_V(t)$ is abelian and consists of semisimple elements, $V$ decomposes into weight spaces relative to $t$:
\[ V = \bigoplus_{\lambda \in t^*} V_\lambda, \qquad V_\lambda = \{v \in V | t \cdot v = \lambda(t)v \ \forall t \in t\}. \]

The set of $t$-weights $\{\lambda \in t^* | V_\lambda \neq 0\}$ of $V$ will be denoted by $\Gamma^w(V, t)$. It is worth mentioning that if $t$ is a toral element of $t$ then $\lambda(t) \in \mathbb{F}_p$ for any $\lambda \in \Gamma^w(V, t)$. Set $\Gamma(V, t) = \Gamma^w(V, t) \setminus \{0\}$. For $\mathbb{F}_p$-independent linear functions $\mu_1, \ldots, \mu_k \in \Gamma(V, t)$ define
\[ V(\mu_1, \ldots, \mu_k) := \bigoplus_{(i_1, \ldots, i_k) \in \mathbb{F}_p^k} V_{i_1 \mu_1 + \cdots + i_k \mu_k}. \]
The subspace $V(\mu_1, \ldots, \mu_k)$ is called a $k$-\emph{section} of $V$.

If $V$ is an algebra over $F$ (not necessarily associative or Lie) and $\mathfrak{g}$ acts on $V$ as derivations then $V(\mu_1, \ldots, \mu_k)$ is a subalgebra of $V$. If $V = \mathfrak{g}$, the adjoint
Let \( h = c_g(t) \) be a regular Cartan subalgebra of \( g \). In [Win 69], Winter proved that for any \( x \in g, \) satisfying \( x^{[p]} = 0 \) the exponential operator \( \exp x \in \text{GL}(g) \) maps the root space decomposition of \( g \) relative to \( h \) onto that of another regular Cartan subalgebra, denoted \( h_x \). To appreciate this result one should keep in mind that in characteristic \( p \) the condition \( x^{[p]} = 0 \) does not always guarantee that \( \exp x \) is an automorphism of \( g \) (for example, consider the case where \( g = W(1; 1) \) and \( x = e_{-1} \)).

In [Wil 83], Wilson assigned a generalized exponential operator to any root vector \( x \in g, \) such that \( x^{[p]} \in t \). Inspired by Wilson’s construction, the first author assigned generalized exponential operators to all root vectors in \( g, \) see [P 86b]. Generalized exponential operators and resulting switchings of regular Cartan subalgebras in \( g \) play an important rôle in Classification Theory.

Let \( \xi \in \text{Hom}_{F_\gamma}(F, F) \) be such that \( \xi^p - \xi = \text{Id}_F \). As \( F \) is algebraically closed, it is straightforward to see that \( \xi : F \rightarrow F \) exists and is uniquely determined up to a linear map from \( F \) to \( \mathbb{F}_p \). Given \( x \in g, \gamma \in \Gamma, \) we denote by \( m = m(x) \) the least positive integer \( k \) with \( x^{[p]^k} \in t \) (such an integer exists because \( t \) is a maximal torus in \( g \)). Set

\[
q(x) = \begin{cases} 
\sum_{i=1}^{m-1} x^{[p]^i} & \text{for } m > 1, \\
0 & \text{for } m = 1.
\end{cases}
\]

Note that \( q(x) \in h \). Define the \textit{generalized Winter exponential} \( E_{x,\xi} \in \text{GL}(g) \) by setting

\[
E_{x,\xi}(y) = -\sum_{i=0}^{p-1} \prod_{j=1}^{p-1} \left( (\xi(\alpha(x^{[p]^j})) + j) \text{Id}_g - \text{ad } q(x) \right) (\text{ad } x)^i(y)
\]

for all \( y \in g, \) where \( \alpha \in \Gamma \cup \{0\}, \) and extending to \( g \) by linearity (our convention here is that \( g_0 = h \)). Notice that if \( x^{[p]} = 0 \) then \( E_{x,\xi} = \exp x \). In general, \( E_{x,\xi} \) is a polynomial in \( \text{ad } x \); see [P 89].

According to [P 86b], \( h_x = E_{x,\xi}(h) \) is a regular Cartan subalgebra of \( g \) and

\[
g = h_x \oplus \sum_{\alpha \in \Gamma} E_{x,\xi}(g_{\alpha})
\]

is the root space decomposition of \( g \) relative to \( h_x \). For \( t \in t \) set

\[
t_x := t - \gamma(t)(x + q(x)).
\]

The subspace \( t_x = \{ t_x | t \in t \} \) coincides with the unique maximal torus in \( h_x \); see [P 86b]. The set of roots \( \Gamma(g, t_x) \) of \( g \) relative to \( t_x \) has the form \( \Gamma(g, t_x) = \{ \alpha_{x,\xi} | \alpha \in \Gamma \} \subset t_x^* \) where

\[
\alpha_{x,\xi}(t_x) = \alpha(t) - \xi(\alpha(x^{[p]^m})) \gamma(t) \quad (\forall t_x \in t_x).
\]

If \( h' = E_{y,\xi}(h) \) for some \( y \in \bigcup_{\xi \in \Gamma} g, \), we say that \( h' \) is obtained from \( h \) by an \textit{elementary switching}. By [P 89], any two regular Cartan subalgebras of \( g \) can be
obtained each from another by a finite sequence of elementary switchings. This result has the following important consequence:

**Proposition 2** (P-St 99). Let \( t_1 \) and \( t_2 \) be two tori of maximal dimension in \( g \), \( V \) a finite dimensional restricted \( g \)-module, \( \Delta_1 = \Gamma^w(V, t_i) \), and \( Q_i \) the \( \mathbb{F}_p \)-span of \( \Delta_1 \) in \( t_i \), where \( i = 1, 2 \). Then there exists a linear isomorphism of \( \mathbb{F}_p \)-spaces \( \psi : Q_1 \rightarrow Q_2 \) such that \( \psi(\Delta_1) = \Delta_2 \) and \( \dim V_\mu = \dim V_{\psi(\mu)} \) for all \( \mu \in \Delta_1 \).

As a consequence one obtains that \( \mathbb{F}_p \delta_1 \subset \Delta_1 \) for some \( \delta_1 \in t_1^* \) if and only if \( \mathbb{F}_p \delta_2 \subset \Delta_2 \) for some \( \delta_2 \in t_2^* \). Also, \( 0 \in \Delta_1 \) if and only if \( 0 \in \Delta_2 \).

### 2.2. Absolute toral rank

It is often useful to view a Lie algebra as a subalgebra of a restricted Lie algebra.

**Definition 2** (St-F 88). Let \( L \) be a Lie algebra. A triple \( (\mathcal{L}, [p], i) \) where \( \mathcal{L} \) is a restricted Lie algebra with p-mapping \([p] : \mathcal{L} \rightarrow \mathcal{L} \) and \( i : L \hookrightarrow \mathcal{L} \) is an injective Lie algebra homomorphism, is called a p-envelope of \( L \) if the restricted Lie subalgebra of \( \mathcal{L} \) generated by \( i(L) \) coincides with \( \mathcal{L} \).

The Lie algebra \( L \) is often identified with \( i(L) \subset \mathcal{L} \). We list below a few basic properties of p-envelopes. All proofs can be found in [St-F 88] [St 04].

**2.2.1.** Let \( (\mathcal{L}, [p], i) \) and \( (\mathcal{L}', [p]', i') \) be two p-envelopes of \( L \). Then there exists an isomorphism of restricted Lie algebras \( \psi : \mathcal{L}/\mathfrak{z}(\mathcal{L}) \rightarrow \mathcal{L}'/\mathfrak{z}(\mathcal{L}') \) such that \( \psi \circ \pi \circ i = \pi' \circ i' \) where \( \pi \) and \( \pi' \) denote the canonical homomorphisms of restricted Lie algebras \( \mathcal{L} \rightarrow \mathcal{L}/\mathfrak{z}(\mathcal{L}) \) and \( \mathcal{L}' \rightarrow \mathcal{L}'/\mathfrak{z}(\mathcal{L}') \).

**2.2.2.** A p-envelope \( (\mathcal{L}, [p], i) \) of \( L \) is called minimal if \( \mathfrak{z}(\mathcal{L}) \) is contained in \( \mathfrak{z}(i(L)) \). Any \( L \) admits a minimal p-envelope, and any two minimal p-envelopes of \( L \) are isomorphic as ordinary Lie algebras.

**2.2.3.** Suppose \( L \) is semisimple. Then \( L \) has one “obvious” minimal p-envelope, namely, the restricted Lie subalgebra of \( \text{Der} L \) generated by \( \text{ad} L \). This p-envelope is semisimple. Any two semisimple p-envelopes of \( L \) are isomorphic as restricted Lie algebras.

**Definition 3.** Let \( (\mathcal{L}, [p], i) \) be a p-envelope of \( L \). The absolute toral rank of \( L \), denoted \( TR(L) \), is the maximal dimension of tori in the restricted Lie algebra \( \mathcal{L}/\mathfrak{z}(\mathcal{L}) \). In other words,

\[
TR(L) := MT(\mathcal{L}/\mathfrak{z}(\mathcal{L})).
\]

In view of (2.2.1), this definition is independent of the choice of a p-envelope of \( L \). For \( L \) semisimple, \( TR(L) = MT(L_p) \) where \( L_p \) stands for the restricted Lie subalgebra of \( \text{Der} L \) generated by \( \text{ad} L \) (see (2.2.3)). We shall need a few basic properties of \( TR(L) \) all of which can be found in [St 04].

**2.2.4.** \( L \) is nilpotent if and only if \( TR(L) = 0 \).

**2.2.5.** If \( I \) is an ideal of \( L \) then \( TR(L/I) + TR(I) \leq TR(L) \).

**2.2.6.** Let \( T \) be a torus of maximal dimension in a finite dimensional p-envelope of \( L \) and let \( \gamma_1, \ldots, \gamma_k \) be \( \mathbb{F}_p \)-independent roots in \( \Gamma(L, T) \). Then

\[
TR(L(\gamma_1, \ldots, \gamma_k)) \leq k.
\]

In particular, \( TR(L(\alpha)) \leq 1 \) for any \( \alpha \in \Gamma(L, T) \) and \( TR(L(\alpha, \beta)) \leq 2 \) for any two \( \alpha, \beta \in \Gamma(L, T) \).
2.3. Sandwich elements. Given an arbitrary Lie algebra $L$ over a field we define $S(L) := \{ s \in L \mid (\text{ad} s)^2 = 0 \}$. The set $S(L)$ plays a crucial rôle in Kostrikin’s work on the restricted Burnside problem (see [Ko 90]). If $2L = L$ and $s \in S(L)$, then $(\text{ad} s)(\text{ad} x)(\text{ad} s) = 0$ for any $x \in L$. Because of this property the elements of $S(L)$ are often referred to as sandwich elements (the term is due to Kostrikin). As an example, $S(W(1;1)) = \bigoplus_{2i \neq p} F e_i$. In general, $S(L)$ is not closed under vector addition however. If $2L = L$, then $S(L)$ is closed under Lie multiplication (see [Ko 90] for more detail).

Assume until the end of this subsection that $\mathrm{char} F = p > 2$ and let $L$ be finite dimensional over $F$. Let $c \in S(L)$ and $x \in L$. Since $(\text{ad} c)(\text{ad} x)(\text{ad} c) = 0$ we have $((\text{ad} c)(\text{ad} x))^2 = 0$. This implies that $\text{tr} (\text{ad} c)(\text{ad} x) = 0$. As a consequence, $L$ is contained in the radical of the Killing form of $L$. The Lie algebras over $F$ containing nonzero sandwich elements are called strongly degenerate (the term is due to Kostrikin). It follows from the preceding remark that the Killing form of any strongly degenerate simple Lie algebra over $F$ is identically zero.

By the Engel–Jacobson theorem, the linear span $\langle S \rangle$ of $S$ is a nilpotent Lie subalgebra of $L$. So $S$ is invariant under all automorphisms of $L$ the same is true for the normalizer of $S$ in $L$. As a consequence, every strongly degenerate simple Lie algebra $L$ contains a proper nonzero subalgebra invariant under all automorphisms of $L$. (This remark also shows that in characteristic 0 the equality $S(L) = \{ 0 \}$ is equivalent to the semisimplicity of $L$.) For $p > 3$, the Lie algebras $L$ over $F$ with $S(L) = \{ 0 \}$ are closely related to the Lie algebras of semisimple algebraic groups over $F$; see the discussion in (3.1) for more detail.

In [Ko–S 66], Kostrikin and Shafarevich conjectured that for $p > 5$ the normaliser of $\langle S \rangle$ in any strongly degenerate simple Lie algebra $L$ is a maximal subalgebra of $L$. In his PhD thesis and a subsequent series of preprints, S.A. Kirillov verified this conjecture for all known finite dimensional simple Lie algebras of characteristic $p > 3$. Unfortunately, all attempts to find an a priori proof of the conjecture failed.

2.4. Standard filtrations. Let $L$ be a simple Lie algebra over $F$ and $L(0)$ a maximal subalgebra of $L$. Let $L(-1)$ be a subspace of $L$ such that $L(0) \subset L(-1)$ and $[L(0), L(-1)] \subset L(-1)$, and assume further that $L(-1)/L(0)$ is an irreducible $L(0)$-module. Following Weisfeiler [We 68], we define the standard filtration of $L$ associated with the pair $(L(0), L(-1))$ by setting

\begin{align*}
L_{(i+1)} &= \{ x \in L_{(i)} \mid [x, L_{(-1)}] \subset L(0) \}, & i \geq 0, \\
L_{(-i-1)} &= [L_{(-i)}, L_{(-1)}] + L_{(-1)}, & i > 0.
\end{align*}

Since $L(0)$ is a maximal subalgebra of $L$ this filtration is exhaustive. Since $L$ is simple, the filtration is separating. So there are $s_1 > 0$ and $s_2 \geq 0$ such that

$L = L_{(-s_1)} \supset \ldots \supset L_{(0)} \supset \ldots \supset L_{(s_2+1)} = \{ 0 \}.$

By construction, all subspaces $L_{(i)}$ of $L$ are invariant under the action of the restricted subalgebra of $\text{Der} L$ generated by $\text{ad} L(0)$. A standard filtration is called long if $L_{(1)} \neq \{ 0 \}$.

Now let $G = \bigoplus_{i \in \mathbb{Z}} G_i$ be a graded Lie algebra, that is $[G_i, G_j] \subset G_{i+j}$ for all $i, j \in \mathbb{Z}$. The following four conditions occur very frequently in Classification Theory:

1. $G_{-1}$ is an irreducible and faithful $G_0$-module;
2. $G_{-i} = [G_{-i+1}, G_{-1}]$ for all $i \geq 1$;
(g3) if $x \in G_i$, $i > 0$, and $[x, G_{-1}] = (0)$, then $x = 0$;
(g4) if $x \in G_{-1}$, $i > 0$, and $[x, G_k] = (0)$ for all $k > 0$, then $x = 0$.

The graded Lie algebra $gr L = \bigoplus_{i=-s_1}^{s_2} gr_i L$, where $gr_i L = L_i(L)/L_{(i+1)}$, corresponding to the standard filtration above satisfies the conditions (g1), (g2), (g3).

The quotient of $gr L$ by its largest ideal contained in $\sum_{i < -1} gr_i L$ satisfies all four conditions (g1) – (g4).

3. Classes of simple Lie algebras

The main conjecture on the structure of finite dimensional simple Lie algebras over algebraically closed fields of characteristic $p$ is known as the generalized Kostrikin–Shafarevich conjecture. It states the following:

For $p > 5$, any finite dimensional simple Lie algebra over $F$ is either classical or isomorphic to one of the filtered Lie algebras of Cartan type.

This conjecture is due to Kac Kac 71 Kac 74, who formulated it for $p > 3$ (see also Ko 71). Our next goal is to give a detailed description of the Lie algebras mentioned in the generalized Kostrikin–Shafarevich conjecture.

3.1. Classical Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ the corresponding root system, and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a basis of simple roots in $\Phi$. For $\alpha, \beta \in \Phi$ set $\langle \beta, \alpha^\vee \rangle = 2(\beta|\alpha)/(\alpha|\alpha)$, where, as usual, $(\cdot|\cdot)$ denotes a scalar product on the $\mathbb{R}$-span of $\Phi$ invariant under the Weyl group of $\Phi$.

**Theorem 3 (Che 56).** The Lie algebra $\mathfrak{g}$ has a basis

$$\mathcal{B} = \{e_\alpha | \alpha \in \Phi\} \cup \{h_i| 1 \leq i \leq l\}$$

such that the following conditions hold:

1. $[h_i, h_j] = 0$, $1 \leq i, j \leq l$.
2. $[h_i, e_\beta] = \langle \beta, \alpha_i^\vee \rangle e_\beta$, $1 \leq i \leq l$, $\beta \in \Phi$.
3. $[e_\alpha, e_{-\alpha}] = h_\alpha$ is a $\mathbb{Z}$-linear combination of $h_1, \ldots, h_l$.
4. Let $\alpha, \beta \in \Phi$, $\beta \neq \pm \alpha$, and let $\{\beta - qa, \ldots, \beta + ra\}$ be the $\alpha$-string through $\beta$. Then $[e_\alpha, e_\beta] = 0$ if $\alpha + \beta \notin \Phi$ and $[e_\alpha, e_\beta] = \pm(q+1)e_{\alpha + \beta}$ if $\alpha + \beta \in \Phi$.

Moreover, $q \in \{0, 1, 2\}$ if $\alpha + \beta \in \Phi$.

The $\mathbb{Z}$-span $\mathfrak{g}_\mathbb{Z}$ of $\mathcal{B}$ is a $\mathbb{Z}$-form in $\mathfrak{g}$ closed under taking Lie brackets. Therefore, $\mathfrak{g}_F := \mathfrak{g}_\mathbb{Z} \otimes_{\mathbb{Z}} F$ is a Lie algebra over $F$ with basis $\mathcal{B} \otimes 1$ and structure constants obtained from those for $\mathfrak{g}_\mathbb{Z}$ by reducing modulo $p$. For $p > 3$, the Lie algebra $\mathfrak{g}_F$ fails to be simple if and only if the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ has type $A_l$ where $l = mp - 1$ for some $m \in \mathbb{N}$. If $\Phi$ has type $A_{mp-1}$ then $\mathfrak{g}_F \cong \mathfrak{sl}(mp)$ has a one-dimensional center (consisting of scalar matrices) and the Lie algebra $\mathfrak{g}_F/\mathfrak{z}(\mathfrak{g}_F) \cong \mathfrak{psl}(mp)$ is simple. The simple Lie algebras over $F$ thus obtained are called classical.

All classical Lie algebras are restricted with $p$th power map given by $(e_\alpha \otimes 1)^{[p]} = 0$ and $(h_i \otimes 1)^{[p]} = h_i \otimes 1$ for all $\alpha \in \Phi$ and $1 \leq i \leq l$. As in characteristic $0$, they are parametrized by Dynkin diagrams of types $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. We stress that, by abuse of characteristic $0$ notation, the classical simple Lie algebras over $F$ include the Lie algebras of simple algebraic $F$-groups of exceptional types. All classical simple Lie algebras are closely related to simple algebraic groups over $F$. 


A Lie algebra $L$ of characteristic $p > 3$ is called \textit{almost classical} if
\[ \text{ad } g \subset L \subset \text{Der } g \]
where $g$ is a direct sum of classical simple Lie algebras. One of the examples of such algebras is the Lie algebra $\mathfrak{pgl}(n) := \mathfrak{gl}(n)/FI_n$. When $p$ does not divide $n$, we have that $\mathfrak{pgl}(n) \cong \mathfrak{s}(n)$ as Lie algebras. However, $\mathfrak{pgl}(mp) \not\cong \mathfrak{s}(mp)$, because for $p > 2$ the Lie algebra $\mathfrak{s}(mp)$ is perfect with a 1-dimensional center, while the Lie algebra $\mathfrak{pgl}(mp)$ is centerless and $[\mathfrak{pgl}(mp), \mathfrak{pgl}(mp)] = \mathfrak{psl}(mp)$ is an ideal of codimension 1 in $\mathfrak{pgl}(mp)$. It is easy to see that the Lie algebra $\mathfrak{pgl}(mp)$ is almost classical.

All almost classical Lie algebras are semisimple, but the case of $\mathfrak{pgl}(mp)$ shows that they are not always direct sums of classical simple Lie algebras. Kostrikin conjectured in [Ko 63, Ko 71] that for $p > 5$ a Lie algebra $L$ over $F$ is almost classical if and only if $[\mathfrak{sl}(L)] = \{0\}$ (a closely related conjecture can be found in the last section of [Ko-S 66]). Kostrikin’s conjecture was proved in [P 86a] for $p > 5$ and in [P 86c] for $p = 5$.

3.2. Graded Lie algebras of Cartan type. In [Ko-S 69], Kostrikin and Shafarevich gave a unified description of a large class of nonclassical simple Lie algebras over $F$. Their construction was motivated by classical work of E. Cartan [C 09] on infinite dimensional, simple transitive pseudogroups of transformations. To define finite dimensional modular analogues of complex Cartan type Lie algebras Kostrikin and Shafarevich replaced formal power series algebras over $\mathbb{C}$ by divided power algebras over $F$.

Let $\mathbb{N}_0^m$ denote the additive monoid of all $m$-tuples of nonnegative integers. For $\alpha, \beta \in \mathbb{N}_0^m$ define $(^\alpha_\beta) = (\begin{smallmatrix} \alpha(1) \\
\beta(1) & \cdots & \alpha(m) 
\end{smallmatrix})$ and $\alpha! = \prod_{i=1}^m \alpha(i)!$. For $1 \leq i \leq m$ set $
\epsilon_i = (\delta_{1i}, \ldots, \delta_{mi})$ and $\mathbf{1} = \epsilon_1 + \cdots + \epsilon_m$.

Give the polynomial algebra $F[X_1, \ldots, X_m]$ its standard coalgebra structure (with all $X_i$ being primitive) and denote by $\mathcal{O}(m)$ the graded dual of $F[X_1, \ldots, X_m]$, a commutative associative algebra over $F$. It is well-known (and easily seen) that $\mathcal{O}(m)$ has basis $\{x^\alpha \mid \alpha \in \mathbb{N}_0^m\}$ and the product in $\mathcal{O}(m)$ is given by
\[ x^\alpha x^\beta = (^{\alpha}_\beta) x^{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^m. \]

We write $x_i$ for $x^{\alpha_i} \in \mathcal{O}(m)$, $1 \leq i \leq m$. For each $m$-tuple $\underline{\alpha} \in \mathbb{N}^m$ we denote by $\mathcal{O}(m; \underline{\alpha})$ the $F$-span of all $x^\alpha$ with $0 \leq \alpha(i) < p^{\alpha_i}$ for $i \leq m$. This is a subalgebra of $\mathcal{O}(m)$ of dimension $p^{\underline{\alpha}}$ where $|\underline{\alpha}| = n_1 + \cdots + n_m$. Note that $\mathcal{O}(m; \underline{0})$ is isomorphic to the truncated polynomial algebra $F[X_1, \ldots, X_m]/(X_1^{p^{n_1}}, \ldots, X_m^{p^{n_m}})$.

Assigning degree $|\alpha| = \alpha(1) + \cdots + \alpha(m)$ to each $x^\alpha \in \mathcal{O}(m)$ gives rise to a grading of the algebra $\mathcal{O}(m)$, called \textit{standard}. Each $\mathcal{O}(m; \underline{\alpha})$ is a graded subalgebra of $\mathcal{O}(m)$. The $k$th graded component of $\mathcal{O}(m)$ is denoted by $\mathcal{O}(m)_k$. The subspaces $\mathcal{O}(m)(k) := \bigoplus_{\alpha \geq k} \mathcal{O}(m)$, form a decreasing filtration of $\mathcal{O}(m)$, called the \textit{standard filtration}. The completion of $\mathcal{O}(m)$ relative to its standard filtration is denoted by $\hat{\mathcal{O}}(m)$. The elements of $\hat{\mathcal{O}}(m)$ are the infinite formal sums of the form $\sum \lambda_\alpha x^\alpha$ with $\lambda_\alpha \in F$. The algebra $\mathcal{O}(m)$ is linearly compact and $\mathcal{O}(m)$ is canonically embedded into $\hat{\mathcal{O}}(m)$. The subspaces $\mathcal{O}(m)(k) := \{\sum_{\alpha \geq k} \lambda_\alpha x^\alpha \mid \lambda_\alpha \in F\}$ and $\mathcal{O}(m)_k := \mathcal{O}(m)_k$ induce a decreasing filtration and topological grading of $\mathcal{O}(m)$, respectively. These are, again, called \textit{standard}.\]
There is a family of continuous maps \( \{ y \mapsto y^{(s)} \mid s \in \mathbb{N}_0 \} \) from \( \mathcal{O}((m))_{(1)} \) into \( \mathcal{O}((m)) \), called divided power maps, such that

\[
(x^{(0)}) = 1 \quad \text{for all} \quad x \in \mathcal{O}((m))_{(1)};
\]

\[
(x^{(s)}) = \left( (s \alpha)!/(\alpha!)^s s! \right) x^{s \alpha} \quad \text{for all} \quad \alpha \neq (0, \ldots, 0);
\]

\[
(\lambda x)^{(s)} = \lambda^s x^{(s)} \quad \text{for all} \quad \lambda \in F, x \in \mathcal{O}((m))_{(1)};
\]

\[
(x + y)^{(s)} = \sum_{i=0}^{s} x^{(i)} y^{(s-i)} \quad \text{for all} \quad x, y \in \mathcal{O}((m))_{(1)}.
\]

A continuous automorphism \( \phi \) (respectively, derivation \( D \)) of the topological algebra \( \mathcal{O}((m)) \) is called admissible (respectively, special) if \( \phi(x^{(s)}) = (\phi(x))^{(s)} \) (respectively, \( D(x^{(s)}) = x^{(s)} D(x) \)) for all \( x \in \mathcal{O}((m))_{(1)} \) and all \( s \in \mathbb{N}_0 \). For \( 1 \leq i \leq m \), the \( i \)th partial derivative \( \partial_i \) of \( \mathcal{O}((m)) \) is defined as the special derivation of \( \mathcal{O}((m)) \) with the property that \( \partial_i(x^{(s)}) = x^{s-i} \partial_i \) if \( \alpha(i) > 0 \) and 0 otherwise. Each admissible automorphism of \( \mathcal{O}((m)) \) respects the standard filtration of \( \mathcal{O}((m)) \). Each finite dimensional subalgebra \( \mathcal{O}(m; \mathfrak{n}) \) is stable under the partial derivatives \( \partial_1, \ldots, \partial_m \).

The set \( W((m)) \) of all special derivations of \( \mathcal{O}((m)) \) is an infinite dimensional Lie subalgebra of \( \text{Der} \mathcal{O}((m)) \) and an \( \mathcal{O}((m)) \)-module, via \( fD(x) = fDx \) for all \( f \in \mathcal{O}((m)) \) and \( D \in W((m)) \). Since each \( D \in W((m)) \) is uniquely determined by its values \( D x_1, \ldots, D x_m \), the Lie algebra \( W((m)) \) is a free \( \mathcal{O}((m)) \)-module with basis \( \partial_1, \ldots, \partial_m \). The subspaces

\[
W((m))_k := \bigoplus_{i=1}^{m} \mathcal{O}((m))_{k+1} \partial_i \quad \text{and} \quad W((m))_{(k)} := \bigoplus_{i=1}^{m} \mathcal{O}((m))_{(k+1)} \partial_i
\]

for \( k \geq -1 \) form a topological grading and decreasing filtration of \( W((m)) \), respectively. Needless to say, both are called standard. Note that

\[
[W((m))_{(i)}, W((m))_{(j)}] \subset W((m))_{(i+j)} \quad \text{for all} \quad i \geq -1, j \geq 0.
\]

The group \( \text{Aut}_c \mathcal{O}((m)) \) of all admissible automorphisms acts on \( W((m)) \) by the rule \( D \mapsto D^\phi := \phi^{-1} D \phi \), where \( \phi \in \text{Aut}_c \mathcal{O}((m)) \) and \( D \in W((m)) \), and respects the standard filtration of \( W((m)) \).

The general Cartan type Lie algebra \( W(m; \mathfrak{n}) \) is the \( \mathcal{O}(m; \mathfrak{n}) \)-submodule of \( W((m)) \) generated by the partial derivatives \( \partial_1, \ldots, \partial_m \). The Lie algebra \( W(m; \mathfrak{n}) \) is a subalgebra \( \text{Der} \mathcal{O}(m; \mathfrak{n}) \). When \( \mathfrak{n}_r = 1 \), it is isomorphic to the full derivation algebra of \( F[X_1, \ldots, X_m]/(X_1^p, \ldots, X_m^p) \), a truncated polynomial ring in \( m \) variables. In the literature, \( W(m; \mathfrak{n}) \) is often referred to as a Lie algebra of Witt type. Since \( W(m; \mathfrak{n}) \) is obviously a free \( \mathcal{O}(m; \mathfrak{n}) \)-module of rank \( m \), we have that \( \dim W(m; \mathfrak{n}) = m p^r \mathfrak{n}_r \). The Lie algebra \( W(m; \mathfrak{n}) \) is simple unless \( (p, m) = (2, 1) \). If \( \mathfrak{n}_r \neq 1 \) and \( \mathfrak{n}_r \neq 1 \) then \( \partial_i^r \neq 0 \) on \( \mathcal{O}(m; \mathfrak{n}) \). Since \( \partial_i^r \) is not a special derivation of \( \mathcal{O}((m)) \) it follows that \( W(m; \mathfrak{n}) \) is restrictable if and only if \( \mathfrak{n}_r = 1 \).

Give the \( \mathcal{O}((m)) \)-module

\[
\Omega^1((m)) := \text{Hom}_{\mathcal{O}((m))} \left( W((m)), \mathcal{O}((m)) \right)
\]

a \( W((m)) \)-module structure by setting \( (D \alpha)(D') := D(\alpha(D')) - \alpha([D, D']) \) for all \( D, D' \in W((m)) \) and \( \alpha \in \Omega^1((m)) \), and define \( d : \mathcal{O}((m)) \rightarrow \Omega^1((m)) \) by the rule \( (df)(D) = Df \) for all \( D \in W((m)) \) and \( f \in \mathcal{O}((m)) \). Notice that \( d \) is
a homomorphism of $W((m))-\text{modules and } \Omega^1((m))$ is a free $\mathcal{O}((m))-\text{module with basis } dx_1, \ldots, dx_m$. Let

$$\Omega((m)) = \bigoplus_{0 \leq k \leq m} \Omega^k((m))$$

be the exterior algebra, over $\mathcal{O}((m))$, on $\Omega^1((m))$. Then $\Omega^0((m)) = \mathcal{O}((m))$ and each graded component $\Omega^k((m))$, $k \geq 1$, is a free $\mathcal{O}((m))-\text{module with basis } \{dx_1 \wedge \cdots \wedge dx_i | 1 \leq i_1 < \cdots < i_k \leq m\}$. The elements of $\Omega((m))$ are called \textit{differential forms} on $\mathcal{O}((m))$.

The map $d$ extends (uniquely) to a zero-square linear operator of degree 1 on $\Omega((m))$ such that

$$d(f \omega) = (df) \wedge \omega + f d(\omega), \quad d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge d(\omega_2)$$

for all $f \in \mathcal{O}((m))$ and all homogeneous $\omega, \omega_1, \omega_2 \in \Omega((m))$. For $D \in W((m))$, we have that $D(f \omega) = (Df)\omega + f D(\omega)$. It follows that each $D \in W((m))$ extends to a derivation of the $F$-algebra $\Omega((m))$. All such derivations commute with $d$. The group $\text{Aut}_c \mathcal{O}((m))$ acts on $\Omega^1((m))$ by the rule

$$(\phi \omega)(D) := \phi(\omega(D))$$

for all $\phi \in \text{Aut}_c \mathcal{O}((m)), \omega \in \Omega^1((m)), D \in W((m))$. Moreover,

$$\phi(f \omega) = \phi(f)\phi(\omega) \quad \text{and} \quad \phi \circ d = d \circ \phi$$

for all $\phi \in \text{Aut}_c \mathcal{O}((m)), \omega \in \Omega((m)), f \in \mathcal{O}((m))$. It follows that the action of $\text{Aut}_c \mathcal{O}((m))$ on $\Omega^1((m))$ extends to an embedding $\text{Aut}_c \mathcal{O}((m)) \hookrightarrow \text{Aut}_F \Omega((m))$. It can be shown that

$$D^\phi(\omega) = \phi^{-1}(D(\phi(\omega))$$

for all $D \in W((m)), \phi \in \text{Aut}_c \mathcal{O}((m)), \omega \in \Omega((m))$.

Each $m$-tuple $\alpha$ of nonnegative integers induces a grading of the algebra $\mathcal{O}(m)$ defined by assigning $\deg(x^\alpha) = r(1)\alpha(1) + \cdots + r(m)\alpha(m)$ to each monomial $x^\alpha \in \mathcal{O}(m)$. Such a grading, in turn, induces (topological) gradings and decreasing filtrations of the algebras $\mathcal{O}(m; \underline{n}), \mathcal{O}(m)), W(m; \underline{n}),$ and $W((m))$. It also induces a topological grading of the algebra $\mathcal{O}(m))$ which extends that of $\mathcal{O}(m)) = \Omega^0((m))$. The differential $d$ of $\mathcal{O}(m))$ preserves all components of this grading. The gradings and filtrations thus obtained are all said to be of \textit{type $\underline{r}$}. In this new terminology, the standard gradings and filtrations defined above are all of type $\underline{1}$.

As in the characteristic 0 case, the three differential forms below are of particular interest:

$$\omega_S := dx_1 \wedge \cdots \wedge dx_m, \quad m \geq 3,$$
$$\omega_H := \sum_{i=1}^r dx_i \wedge dx_{i+r}, \quad m = 2r \geq 2,$$
$$\omega_K := dx_{2r+1} + \sum_{i=1}^r (x_{i+r}dx_i - x_i dx_{i+r}), \quad m = 2r + 1 \geq 3.$$

These forms give rise to the following Lie algebras:

$$S((m)) := \{D \in W((m)) | D(\omega_S) = 0\},$$
\textit{special Lie algebra},

$$H((m)) := \{D \in W((m)) | D(\omega_H) = 0\},$$
\textit{Hamiltonian Lie algebra},

$$K((m)) := \{D \in W((m)) | D(\omega_K) \in \mathcal{O}((m))\omega_K\},$$
\textit{contact Lie algebra}.
Define Lie algebras $CS((m))$ and $CH((m))$ by setting

$$CS((m)) := \{ D \in W((m)) \mid D(\omega_S) \in F\omega_S \},$$

$$CH((m)) := \{ D \in W((m)) \mid D(\omega_H) \in F\omega_H \}. $$

Obviously, $CX((m))^{(1)} \subseteq X((m))$ for $X \in \{S, H\}$. For $X \in \{W, S, CS, H, CH\}$, set $\mathfrak{L}_X = \epsilon_1 + \cdots + \epsilon_m = 1$. For $X = K$, set $\mathfrak{L}_X = \epsilon_1 + \cdots + \epsilon_{m-1} + 2\epsilon_m = 1 + \epsilon_m$. For $X \in \{W, S, CS, H, CH, K\}$ and $\underline{n} \in \mathbb{N}^m$, define

$$X(m; \underline{n}) = X((m)) \cap W(m; \underline{n}).$$

Each $X(m; \underline{n})$ is a graded subalgebra of the Lie algebra $X((m))$ regarded with its grading of type $\mathfrak{L}_X$. The graded components of $X(m; \underline{n})$ are denoted by $X(m; \underline{n})_i$, $i \in \mathbb{Z}$. Note that $X(m; \underline{n})_i = (0)$ for $i \leq -2$ if $X \neq K$. Also, $\dim K(m; \underline{n})_{-2} = 1$ and $K(m; \underline{n})_i = (0)$ for $i \leq -3$.

Suppose $p > 3$. In [Ko-S 69], it was shown that the Lie algebras $S(m; \underline{n})^{(1)}$, $H(m; \underline{n})^{(1)}$ and $K(m; \underline{n})^{(1)}$ are simple for $m \geq 3$ and that so is $H(2; \underline{n})^{(2)}$. Moreover, $K(m; \underline{n}) = K(m; \underline{n})^{(1)}$ unless $p \mid (m + 3)$. For $X \in \{W, S, CS, H, CH, K\}$ any $\mathfrak{L}_X$-graded Lie subalgebra of $X(m; \underline{n})$ containing $X(m; \underline{n})^{(\infty)}$ is called a finite dimensional graded Lie algebra of Cartan type. According to [Ko-S 66] the Lie algebra $X(m; \underline{n})^{(\infty)}$ is restrictable if and only if $\underline{n} = 1$.

The original Kostrikin–Shafarevich conjecture [Ko-S 66] of 1966 states the following:

For $p > 5$, any finite dimensional restrictable simple Lie algebra over $F$ is either classical or isomorphic to one of the Lie algebras $W(m; \underline{1})$, $m \geq 1$, $S(m; \underline{1})^{(1)}$, $m \geq 3$, $H(m; \underline{1})^{(2)}$, $m \geq 2$, $K(m; \underline{1})^{(1)}$, $m \geq 3$.

### 3.3. Filtered Lie algebras of Cartan type

In order to give a unified description of all known finite dimensional simple Lie algebras of characteristic $p > 5$, Kac [Kac 74] and Wilson [Wil 69, Wil 76] introduced certain filtered deformations of finite dimensional graded Lie algebras of Cartan type. A streamlined treatment of these algebras is given [St 04].

We first outline Wilson’s original approach. Let $X \in \{W, S, H, K\}$, $\underline{n} \in \mathbb{N}^m$, and let $\Phi$ be an admissible automorphism of $O((m))$. For $X = K$ assume further that $\Phi$ respects the $\mathfrak{L}_X$-filtration of $O((m))$ (if $X \neq K$ this assumption is fulfilled automatically). Define

$$X(m; \underline{n}; \Phi) := (\Phi^{-1} \circ X((m)) \circ \Phi) \cap W(m; \underline{n}).$$

It is clear from the definition that $X(m; \underline{n}; \text{Id}) = X(m; \underline{n})$ and $W(m; \underline{n}; \Phi) = W(m; \underline{n})$.

**Definition 4 ([Wil 76]).** The Lie algebra $X(m; \underline{n}; \Phi)^{(\infty)}$ is called a filtered Lie algebra of Cartan type if $X(m; \underline{n}; \Phi)$ satisfies the following two conditions:

1. $X(m; \underline{n}; \Phi) \cap W(m; \underline{n})^{(2+\delta_{X,K}),X} \neq (0)$;
2. $X(m; \underline{n}; \Phi) + (\Phi \circ X((m)) \circ \Phi^{-1}) \cap W(m; \underline{n})^{(1+\delta_{X,K}),X} = \Phi \circ X((m)) \circ \Phi^{-1}$.

Here $W(m; \underline{n})^{(k),X}$ denotes the $k$th component of the $\mathfrak{L}_X$-filtration of $W(m; \underline{n})$.

The embedding of a filtered Cartan type Lie algebra $X(m; \underline{n}; \Phi)^{(\infty)}$ into the Lie algebra $W(m; \underline{n})$ regarded with its filtration of type $\mathfrak{L}_X$, induces a natural filtration of $X(m; \underline{n}; \Phi)^{(\infty)}$. The corresponding graded algebra is isomorphic to a graded Cartan type Lie algebra (possibly of type $CS$ or $CH$) containing $X(m; \underline{n})^{(\infty)}$ as a
minimal ideal. The subalgebra $L_{(0)} = X(m; \mathfrak{n}; \Phi)_{(\infty)} \cap W(m; \mathfrak{n})_{(0)}$ is called the standard maximal subalgebra of $L = X(m; \mathfrak{n}; \Phi)_{(\infty)}$. For $p > 3$, this subalgebra can be characterized as the unique proper subalgebra of maximal dimension in $L$; see [Kr 71, Sk 95, St 04]. As a consequence, $L_{(0)}$ is stable under all automorphisms of $L$. For $p > 3$, each Cartan type Lie algebra $L$ is simple ([Wil 76]).

The following important abstract characterization of filtered Cartan type Lie algebras is due to Wilson [Wil 76]. Let $\mathcal{L}$ be a Lie algebra over $F$ and let $\mathcal{L}_0$ be a subalgebra of $\mathcal{L}$. Then we have a natural representation $\rho : \mathcal{L}_0 \to \mathfrak{gl}(\mathcal{L}/\mathcal{L}_0)$ of the Lie algebra $\mathcal{L}_0$ given by

$$(\rho(x))(y + \mathcal{L}_0) = [x, y] + \mathcal{L}_0 \quad \text{for all } x \in \mathcal{L}_0, \ y \in \mathcal{L}.$$ 

**Theorem 4 (Wilson’s Theorem).** Let $\mathcal{L}$ be a simple Lie algebra over $F$ and suppose that $\text{char } F = p > 3$. Then $\mathcal{L}$ is isomorphic to a finite dimensional filtered Cartan type Lie algebra if and only if $\mathcal{L}$ is strongly degenerate and contains a maximal subalgebra $\mathcal{L}_0$ such that either $\mathcal{L}_0$ has codimension 1 in $\mathcal{L}$ or else $\rho(\mathcal{L}_0)$ contains a linear transformation $Y$ of rank 1 such that $[Y, [Y, \rho(\mathcal{L}_0)]] \neq (0)$.

Kac’s approach [Kac 74] to filtered Cartan type Lie algebras pushed further by Skryabin in [Sk 86, Sk 90, Sk 91, Sk 93, Sk 95] involved more general differential forms in $\Omega((m))$. Combined with Wilson’s theorem it eventually led to a complete classification of filtered Lie algebras of Cartan type.

Recall that the algebra $\Omega((m))$ is linearly compact. Given a unital associative subalgebra $B$ of $\Omega((m))$ we let $W(B)$ and $W(B)_{(0)}$ denote the normalizers of $B$ and $B \cap \Omega((m))_{(0)}$ in $W((m))$, respectively. We denote by $B^*$ the group of invertible elements of $B$. Following [Sk 91] we say that $B$ is an admissible subalgebra of $\Omega((m))$ if $B$ is closed in $\Omega((m))$ and $W(B)_{(0)}$ has codimension $m$ in $W((B))$. This definition is inspired by a crucial definition in [Kac 74]. Any finite dimensional subalgebra of $\Omega((m))$ of the form $\phi(\Omega((m); \mathfrak{n}))$ with $\phi \in \text{Aut}_F \Omega((m))$ and $\mathfrak{n} \in \mathbb{N}^m$ is admissible. Conversely, given a finite dimensional admissible subalgebra $B \subset \Omega((m))$ there are an automorphism $\phi \in \text{Aut}_F \Omega((m))$ and a tuple $\mathfrak{n} \in \mathbb{N}^m$ such that $B = \phi(\Omega((m); \mathfrak{n}))$; see [Sk 91].

To ease notation we set $\Omega = \Omega((m))$, $\Omega^k = \Omega^k((m))$ for $0 \leq k \leq m$, and put $\Omega_{\text{even}} := \bigoplus_{k \geq 0} \Omega^{2k}$. Observe that $\Omega_{\text{even}}$ is a commutative algebra over $F$ and $W((m))$ acts on $\Omega_{\text{even}}$ as derivations. The subspace $\Omega_{\text{even}}^+ := \Omega((m))_{(1)} \oplus \bigoplus_{k \geq 1} \Omega^{2k}$ is a maximal ideal of $\Omega_{\text{even}}$ which intersects trivially with $(\Omega_{\text{even}})^W((m)) = \mathcal{F}$, and is well-known that the first cohomology group $H^1(W((m)), \Omega_{\text{even}})$ vanishes; see [Sk 91, Theorem 7.5] for example. According to [Sk 91, Proposition 1.2] this implies that there exists a unique system of divided powers $\omega \mapsto \omega^{(s)}$, $s \geq 0$, on $\Omega^\text{even}$ with respect to which $W((m))$ acts on $\Omega_{\text{even}}^+$ as special derivations. It has the property that $\omega^{(s)} \in \Omega^{2js}$ whenever $\omega \in \Omega^{2i}$ and $s \geq 1$.

Recall that a differential form $\omega \in \Omega$ is called closed if $d\omega = 0$. Following [Kac 74] we say that $\omega \in \Omega^m$ is nondegenerate if $m \geq 2$ and $\omega = \varphi dx_1 \wedge \ldots \wedge dx_m$ for some $\varphi \in \Omega((m))^*$. We call $\omega \in \Omega^2$ nondegenerate if $m = 2r \geq 2$, $\omega$ is closed, and the form $\omega^{(r)} \in \Omega^m$ is nondegenerate (if $m = 2$ this is consistent with the previous definition). Finally, we say that $\omega \in \Omega^1$ is nondegenerate if $m = 2r + 1 \geq 3$ and $(d\omega)^{(r)} \wedge \omega \in \Omega^m$ is nondegenerate.

Given a finite dimensional admissible subalgebra $B$ of $\Omega((m))$ we let $\Omega(B) = \bigoplus_{k=0}^{m} \Omega^k(B)$ denote the $B$-subalgebra of $\Omega$ generated (over $B$) by $dB$. For $f \in \Omega((m))_{(1)}$ we set $\exp f := \sum_{i \geq 0} f^{(i)}$, an element in $\Omega((m))^*$. Let $s(B)$ (respectively,
\( \mathfrak{h}(B) \) denote the set of all nondegenerate forms \( \omega \in \Omega^m \) (respectively, \( \omega \in \Omega^2 \)) such that \( \omega = (\exp u) \omega' \) for some \( \omega' \in \Omega(B) \) and \( u \in \mathcal{O}((m)) \) satisfying \( du \in \Omega^1(B) \). Let \( \mathfrak{k}(B) \) denote the set of all nondegenerate forms in \( \Omega^1(B) \).

For \( \omega \in \mathfrak{s}(B) \) define the Lie algebras
\[
S(B, \omega) := \{ D \in W(B) \mid D\omega = 0 \},
\]
\[
CS(B; \omega) := \{ D \in W(B) \mid D\omega \in F\omega \}.
\]

For \( \omega \in \mathfrak{h}(B) \) define the Lie algebras
\[
H(B; \omega) := \{ D \in W(B) \mid D\omega = 0 \},
\]
\[
CH(B; \omega) := \{ D \in W(B) \mid D\omega \in F\omega \}.
\]

For \( \omega \in \mathfrak{k}(B) \) define the Lie algebra
\[
K(B; \omega) := \{ D \in W(B) \mid D\omega \in B\omega \}.
\]

It is proved in \cite{Kac74, Skryabin93, Skryabin95} that except for two cases in characteristic 2 the Lie algebras
\[
W(B), S(B; \omega)^{(1)}, H(B; \omega)^{(2)}, K(B; \omega)^{(1)}
\]
are simple. Dimensions and explicit bases of the Lie algebras \( X(B; \omega), X(B; \omega)^{(1)} \) and \( X(B; \omega)^{(2)} \) are found in \cite{Skryabin95, Kirillov89, Kirillov90} (see also \cite{B-K-K93}). The most accessible reference, by far, is \cite{Stud94} Sect. 6.

Any simple Lie algebra \( L = X(B; \omega)^{(\infty)} \) is naturally filtered and \( \text{gr} L \), the corresponding graded algebra, is isomorphic to a graded Lie algebra of Cartan type. In view of Wilson’s theorem this implies that for \( p > 3 \) each \( X(B; \omega)^{(\infty)} \) is isomorphic to a filtered Cartan type Lie algebra. The converse is also true: for \( p > 3 \) any filtered Cartan type Lie algebra \( X(m; \omega; \Phi)^{(\infty)} \) is isomorphic to one of \( X(B; \omega)^{(\infty)} \) where \( B = \phi(\mathcal{O}(m; \underline{n})) \) for some \( \phi \in \text{Aut}, \mathcal{O}(m) \) (see \cite{Kac74, Ku89, Skryabin95}).

The \( p \)-structure of filtered Cartan type Lie algebras is described by the following theorem.

**Theorem 5** \cite{Kac74, Skryabin95}. Let \( B = \phi(\mathcal{O}(m; \underline{n})) \) where \( \phi \in \text{Aut}, \mathcal{O}(m) \).

1. The Lie algebras \( W(B), CS(B; \omega), CH(B; \omega), K(B; \omega) \) are restrictable if and only if \( \underline{n} = 1 \).
2. The Lie algebras \( S(B; \omega) \) and \( H(B; \omega) \) are restrictable if and only if \( \underline{n} = 1 \) and \( \omega \in \Omega(B) \).
3. The Lie algebras \( S(B; \omega)^{(1)}, H(B; \omega)^{(1)} \) and \( H(B; \omega)^{(2)} \) are restrictable if and only if \( \underline{n} = 1 \) and \( \omega \in d\Omega(B) \).

It follows from Theorem 5 that for \( p > 3 \) the Lie algebra \( X(m; \underline{n}; \Phi)^{(\infty)} \) is restrictable if and only if it is isomorphic to one of \( W(m; 1), S(m; 1)^{(1)}, H(m; 1)^{(2)}, K(m; 1)^{(1)} \).

The realizations of filtered Cartan type Lie algebras just described are very useful in view of Kac’s Isomorphism Theorem which was later refined by Skryabin; see \cite{Kac74, Ku89, Skryabin95}. Let \( B \) (respectively, \( B' \)) be an admissible subalgebra of \( \mathcal{O}((m)) \) (respectively, \( \mathcal{O}((m')) \)) and \( X, X' \in \{ W, S, H, K \} \). Slightly abusing notation we set \( W(B; \omega) = W(B) \) and likewise for \( W(B') \). We call a linear map
\[
\sigma : X(B; \omega)^{(\infty)} \longrightarrow X'(B'; \omega')^{(\infty)}
\]
standard if $\sigma(D) = \psi \circ D \circ \psi^{-1}$ for all $D \in X(B; \omega)^{(\infty)}$, where $\psi: \mathcal{O}(m) \sim \mathcal{O}(m')$ is a continuous isomorphism of divided power algebras satisfying $\psi(B) = B'$ and $\psi(\omega) = C\omega'$ with $C \in F^*$ for $\omega \in \mathfrak{s}(B) \cup \mathfrak{h}(B)$ and $C \in B'^*$ for $\omega \in \mathfrak{k}(B)$.

Clearly, any standard map is a Lie algebra isomorphism. Also, if $\sigma: X(B; \omega)^{(\infty)} \rightarrow X'(B'; \omega')^{(\infty)}$ is a standard isomorphism then necessarily $m = m'$ and $X = X'$.

**Theorem 6 (Isomorphism Theorem).** Let $B, B'$ and $X, X'$ be as above. Then with eight exceptions in characteristic 2 and three exceptions in characteristic 3 any isomorphism between the Lie algebras $X(B; \omega)^{(\infty)}$ and $X'(B'; \omega')^{(\infty)}$ is standard.

In our further discussion of the Lie algebras $X(B; \omega)$ we shall assume (without loss of generality) that $B = \mathcal{O}(m; \underline{n})$. In this special case, $X(B; \omega)$ is denoted by $X(m; \underline{n}; \omega)$. The corresponding set $\mathfrak{x}(B)$ of nondegenerate forms will be denoted by $\mathfrak{x}(m; \underline{n})$. We shall also assume (as we may) that $\underline{n}$ is a partition of $|\underline{n}|$, that is $n_1 \geq \ldots \geq n_m$. Let $G(m; \underline{n})$ denote the set-wise stabilizer of $\mathcal{O}(m; \underline{n})$ in $\text{Aut}(\mathcal{O}(m))$. This is a connected algebraic group with a large unipotent radical; see [Wil 71] for more detail. It follows from Theorem 5 that with a few exceptions in characteristics 2 and 3 the Lie algebras $X(m; \underline{n}; \omega)^{(\infty)}$ and $X'(m'; \underline{n}'; \omega')^{(\infty)}$ are isomorphic if and only if $m = m'$, $\underline{n} = \underline{n}'$ and $g\omega = C\omega'$ for some $g \in G(m; \underline{n})$, where $C \in F^*$ for $X \in \{S, H\}$ and $C \in \mathcal{O}(m; \underline{n})^*$ for $X = K$. Let $\mathfrak{S}(\underline{n})$ denote the group of all permutations $\pi$ of $\{1, 2, \ldots, m\}$ such that $n_{\pi i} = n_i$ for all $i$.

The orbits of $G(m; \underline{n})$ on $\mathfrak{s}(m; \underline{n})$ are described in [T 78] and [Wil 80]. Let $I(\underline{n})$ denote the subset of $\{1, 2, \ldots, m\}$ consisting of 1 and all $k$ with $n_k < n_{k-1}$. Set $\delta(\underline{n}) = (p^{n_1} - 1, \ldots, p^{n_m} - 1)$. According to [T 78, Wil 80], each $\omega \in \mathfrak{s}(m; \underline{n})$ is conjugate under $G(m; \underline{n})$ to a nonzero scalar multiple of precisely one form in the set

$$\{(\exp x_1) \omega_S | i \in I(\underline{n})\} \cup \{\omega_S, (1 - x^{\delta(\underline{n})}) \omega_S\}.$$

As a consequence, for $p > 2$ and $\underline{n} \in \mathbb{N}^m$ fixed, there are only finitely many filtered Lie algebras of type $S(m; \underline{n}; \Phi)^{(\infty)}$ up to isomorphism.

The orbits of $G(m; \underline{n})$ on $\mathfrak{k}(m; \underline{n})$ are described in [K-K 86b] in the simplest case $\underline{n} = \underline{1}$ and in [Sk 86] for any $\underline{n}$. Let $\mathcal{D}_k$ denote the set of all decompositions of $\{1, 2, \ldots, m\}$ into a disjoint union of the form

$$I = \{i_0\} \sqcup \{i_1, i_1'\} \sqcup \ldots \sqcup \{i_r, i_r'\}, \quad i_k < i_k',
$$

(different orderings of the subsets within the union are not distinguished). The group $\mathfrak{S}(\underline{n})$ acts on the set $\mathcal{D}_k$. Given $I \in \mathcal{D}_k$ define

$$\omega_{K, I} := dx_{i_0} + \sum_{k=1}^r x_{i_k} dx_{i_k'},$$

an element in $\mathfrak{k}(m; \underline{n})$. It is proved in [Sk 86] that for $p > 2$ each $\omega \in \mathfrak{k}(m; \underline{n})$ is conjugate under $G(m; \underline{n})$ to $f\omega_{K, I}$ for some $f \in \mathcal{O}(m; \underline{n})^*$ and $I \in \mathcal{D}_k$. Moreover, the orbit of $f\omega_{K, I}$ under $G(m; \underline{n})$ intersects with $\mathcal{O}(m; \underline{n})^*\omega_{K, I'}$ for $I' \in \mathcal{D}_k$ if and only if there is a $\pi \in \mathfrak{S}(\underline{n})$ such that $\pi(I) = I'$.

Thus for $p > 2$ any filtered Cartan type Lie algebra $K(m; \underline{n}; \omega)^{(\infty)}$ is isomorphic to a graded Cartan type Lie algebra $K(m; \underline{n}'; \omega')^{(1)}$ (here $|\underline{n}'| = |\underline{n}|$ but in general $\underline{n}'$ need not be a partition of $|\underline{n}|$). It follows that for $p > 2$ and $\underline{n} \in \mathbb{N}^m$ fixed, there are only finitely many Cartan type Lie algebras $K(m; \underline{n}; \Phi)^{(\infty)}$ up to isomorphism.

The orbit set $\mathfrak{h}(m; \underline{n})/G(m; \underline{n})$ is studied in [Kac 74, K-K 86a, B-G-O-S-W, Sk 86, Sk 90]. In the simplest case $\underline{n} = \underline{1}$ it is described in [K-K 86a]. For an
arbitrary \( n \), a reasonably small set of representatives for each \( G(m; n) \)-orbit in \( h(m; n) \) is found in [B-G-O-S-W]. A complete description of \( h(m; n)/G(m; n) \) is given in [Sk 86, Sk 90].

Set \( h_1(m; n) := h(m; n) \cap \Omega(\mathcal{O}(m; n)) \) and \( h_2(m; n) := h(m; n) \setminus h_1(m; n) \). Both \( h_1(m; n) \) and \( h_2(m; n) \) are \( G(m; n) \)-stable. The orbit sets \( h_2(m; n)/G(m; n) \) and \( k(m + 1; n)/(G(m + 1; n)) \) are somewhat similar to each other. Let \( D_h \) denote the set of all decompositions

\[
I = \{i_1, i_1'\} \cup \ldots \cup \{i_r, i_r'\}, \quad i_k < i_k',
\]

of \( \{1, 2, \ldots, m\} \) into a disjoint union of pairs (different orderings of the pairs within the union are not distinguished). The group \( \mathcal{G}(n) \) acts on the set \( D_h \). Given \( i \in \{1, 2, \ldots, m\} \) and \( I \in D_h \) define

\[
\omega_{H,i,I} := d\left( \exp x_i \sum_{k=1}^r x_k \, dx_{i_k'} \right),
\]

an element in \( h_2(m; n) \). It is proved in [Sk 86, Sk 90] that for \( p > 2 \) each \( \omega \in h_2(m; n) \) is conjugate under \( G(m; n) \) to \( \omega_{H,i,I} \) for some \( I \in D_h \) and \( i \in \{1, 2, \ldots, m\} \). Moreover, the \( G(m; n) \)-orbit of \( \omega_{H,i,I} \) intersects with \( F^r \omega_{H,j,I'} \) for \( I' \in D_h \) and \( j \in \{1, 2, \ldots, m\} \) if and only if there is a \( \pi \in \mathcal{G}(n) \) such that \( \pi(I) = I' \) and \( \pi_i = j \). As a consequence, for \( p > 2 \) and \( n \in \mathbb{N}^m \) fixed, there are only finitely many isomorphism classes of Lie algebras of the form \( H(m; n; \omega) \) with \( \omega \in h_2(m; n) \).

The orbit set \( h_1(m; n)/G(m; n) \) is much more complicated. It is no longer discrete, for \( m \geq 4 \), and this allows one to exhibit multiparameter families of pairwise nonisomorphic simple Lie algebras of dimensions \( p^{|n|} - 2 \) and \( p^{|n|} - 1 \). This phenomenon was first discovered by Kac who disproved an earlier conjecture of Kostrikin stating that no such families could exist for \( p > 3 \) (see [Ko 71]).

Let \( J_l(\lambda) \) denote the Jordan block of order \( l \) with eigenvalue \( \lambda \in F \). Let \( O_l \) (respectively, \( E_l \)) denote the zero (respectively, identity) matrix of order \( l \). Let

\[
C_l = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0
\end{bmatrix},
\]

a monomial matrix of order \( l \). Given \( d, s \in \mathbb{N} \) and \( \lambda \in F \) define

\[
C_{d,s}(\lambda) = \begin{bmatrix}
O_s & E_s & \cdots & O_s \\
\vdots & \ddots & \ddots & \vdots \\
O_s & O_s & \cdots & E_s \\
J_{d}(\lambda) & O_s & \cdots & O_s
\end{bmatrix},
\]

a block-monomial matrix of order \( ds \). Let \( \mathcal{H}_m \) denote the set of all pairs of block-diagonal, skew-symmetric matrices

\[
(A, B) = \left( \text{diag}(A_1, \ldots, A_k), \text{diag}(B_1, \ldots, B_k) \right)
\]

of order \( m = 2r = 2r_1 + \ldots + 2r_k \geq 2 \) such that

\[
A_i = \begin{bmatrix}
O_{r_i} & E_{r_i} \\
-E_{r_i} & O_{r_i}
\end{bmatrix},
\]
and $B_i$ is one of

$$
\begin{bmatrix}
O_{r_i} & J_{r_i}(0) \\
-J_{r_i}(0) & O_{r_i}
\end{bmatrix},
\begin{bmatrix}
O_{r_i} & C_{d_i,s_i}(\lambda) \\
-C_{d_i,s_i}(\lambda) & O_{r_i}
\end{bmatrix},
\begin{bmatrix}
O_{r_i} & C_{r_i} \\
-C_{r_i} & O_{r_i}
\end{bmatrix},
$$

where $\lambda \neq 0$ and $r_i = d_is_i$. To each $(A, B) = ((a_{ij}), (b_{ij})) \in \mathcal{H}_m$ one associates a differential form $\omega_{A,B} \in \Omega(m)$ by setting

$$\omega_{A,B} = \sum_{i<j} \left( a_{ij} + b_{ij}x_i^{(p^n-1)}x_j^{(p^n-1)} \right) dx_i \wedge dx_j.$$  

It is straightforward to see that $\omega_{A,B} \in \mathfrak{h}_1(m; \mathbb{N})$. One of the main results in Sk 86 (see also Sk 90) says that any $\omega \in \mathfrak{h}_1(m; \mathbb{N})$ is conjugate under $G(m; \mathbb{N})$ to one of $\omega_{A,B}$ with $(A, B) \in \mathcal{H}_m$ (this holds in all prime characteristics). Skryabin also found a necessary and sufficient condition for two forms $\omega_{A,B}$ and $\omega_{A',B'}$ to be conjugate under $G(m; \mathbb{N})$. It involves an equivalence relation on the set of all pairs of sequences of natural numbers, finite of equal length or periodic; see Sk 90 for more detail.

### 3.4. Melikian algebras and their relatives

In this subsection we assume that $p \in \{2, 3, 5\}$. Around 1980, Melikian (a PhD student of Kostrikin at the time) discovered a new series of finite dimensional simple Lie algebras $M(m,n)$ of characteristic 5 depending on two parameters $m, n \in \mathbb{N}$.

Suppose char $F = 5$. In M 80, M 82, the algebra $M(m,n)$ is described as a graded Lie algebra $L = \bigoplus_{i=-2}^1 L_i$ of dimension $5^{m+n+1}$ whose graded subalgebra $L_{-2} \oplus L_{-1}$ is isomorphic to a five dimensional Heisenberg Lie algebra and $L_0^{(1)} \cong W(1; \mathbb{N})$ as Lie algebras. Moreover, $L_0 = L_0^{(1)} \oplus \mathfrak{g}(L_0)$, $\mathfrak{g}(L_0) = Fz$, $(\text{ad } z)|_{L_k} = k \cdot \text{Id}_{L_k}$ for all $k \in \mathbb{Z}$, and $L_{-1} \cong \mathfrak{h}(1; \mathbb{N})/F$ as $W(1; \mathbb{N})$-modules. It is shown in M 82 that each Melikian algebra is strongly degenerate and the only restrictable algebra in the family is $M(1,1)$ (see also Ku 90 and St 04 where all derivations of $M(m,n)$ are determined).

It is stated in M 80 that $M(1,1)$ is neither a classical Lie algebra nor a Lie algebra of Cartan type. In M 82, Melikian outlines a proof of this statement relying on properties of $\mathbb{Z}$-gradings in the contact Lie algebra $K(3; \mathbb{N})$. An alternative proof will be given below. Melikian’s work showed that the assumption that $p > 5$ in the generalized Kostrikin–Shafarevich conjecture could not be relaxed.

A few years later Ermolaev observed that $\mathfrak{g} = M(m,n)$ admits a more natural $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \geq -3} \mathfrak{g}(i)$ that satisfies the conditions (g1), (g2), (g3) of (2.4) and has the property that $\bigoplus_{i \leq 1} \mathfrak{g}(i)$, regarded as a local Lie algebra, is isomorphic to the local Lie algebra associated with a depth 3 grading of a Lie algebra $\mathcal{L}$ of type $G_2$. In particular, the nonpositive part $\bigoplus_{i \leq 0} \mathfrak{g}(i)$ of $\mathfrak{g}$ is isomorphic to a maximal parabolic subalgebra of $\mathcal{L}$. This observation enabled Kuznetsov to give in Ku 91 an explicit description of $M(m,n)$.

Set $n := (m,n)$ and define

$$\begin{align*}
G_0 &:= \bigoplus_{i \equiv 0 (\text{mod } 3)} \mathfrak{g}(i), \\
G_1 &:= \bigoplus_{i \equiv 1 (\text{mod } 3)} \mathfrak{g}(i), \\
G_2 &:= \bigoplus_{i \equiv 2 (\text{mod } 3)} \mathfrak{g}(i).
\end{align*}$$

Then $\mathfrak{g} = G_0 \oplus G_1 \oplus G_2$ is a $(\mathbb{Z}/3\mathbb{Z})$-grading of $\mathfrak{g}$. According to Ku 91,

$$G_0 \oplus G_1 \oplus G_2 = W(2; \mathbb{N}) \oplus \mathfrak{h}(2; \mathbb{N}) \oplus \mathfrak{W}(2; \mathbb{N}).$$
as vector spaces. Moreover, \(G_0\) is identified with \(W(2; \mathbb{N})\) as Lie algebras, \(G_1\) is identified with \(O(2; \mathbb{N})\) as vector spaces, and \(G_2\) is identified with \(\tilde{W}(2; \mathbb{N}) = \{D \mid D \in W(2; \mathbb{N})\}\), a vector space copy of \(W(2; \mathbb{N})\). The Lie product in \(g\) is given by

\[
\begin{align*}
[D, E] &= \widetilde{[D, E]} + 2\text{div}(D) \bar{E}, \\
[D, f] &= D(f) - 2\text{div}(D) f, \\
[f_1\bar{h}_1 + f_2\bar{h}_2, g_1\bar{h}_1 + g_2\bar{h}_2] &= f_1g_2 - f_2g_1, \\
[f, \bar{E}] &= f\bar{E}, \\
[f, g] &= 2(f\bar{D}_g - g\bar{D}_f), \quad D_h = \partial_1(h)\partial_2 - \partial_2(h)\partial_1,
\end{align*}
\]

for all \(D, E \in W(2; \mathbb{N}), f, g, h, f_1, g_1 \in O(2; \mathbb{N})\). Here \(\text{div} : W(2; \mathbb{N}) \to O(2; \mathbb{N})\) is the linear map taking \(f_1\partial_1 + f_2\partial_2\) to \(\partial_1(f_1) + \partial_2(f_2)\). It follows from the above formulæ that the Lie subalgebra of \(M(m, n)\) generated by the graded components \(\mathfrak{g}(\pm 1)\) is isomorphic to a classical Lie algebra of type \(G_2\).

Assume for a contradiction that \(M(1, 1)\) is either classical or of Cartan type. Since \(M(1, 1)\) is strongly degenerate, simple, and restrictable it must be isomorphic to one of \(W(m; \mathbb{L}), S(m; \mathbb{L})^{(1)}, H(m; \mathbb{L})^{(2)}, K(m; \mathbb{L})^{(1)}\). Since \(\dim M(1, 1) = 125\), there is only one option, namely, \(M(1, 1) \cong K(3; \mathbb{L})\). Using the above multiplication table one can observe that \(t_6 := F_1 \cdot x_1\partial_1 + F_1 \cdot x_2\partial_2\) is a torus in \(M(1, 1)\) whose centralizer \(h\) is a five dimensional Cartan subalgebra of \(M(1, 1)\) with the property that \([h, [h, h]] = t_6\) (see [P 94] for more detail). However, all Cartan subalgebras in \(K(3; \mathbb{L})\) are abelian, as can be deduced from [Dem 72] and [St 04] (7.5). Thus \(M(1, 1) \not\cong K(3; \mathbb{L})\), and so \(M(1, 1)\) is neither classical nor of Cartan type.

Although the Melikian algebras have sporadic nature and can survive as Lie algebras only at characteristic 5, they have some relatives in characteristics 3 and 2. This was discovered by Skryabin [Sk 92] and Brown [Br 95].

Suppose \(\text{char} F = 3\). Each Skryabin algebra \(g\) is equipped with a \(Z\)-grading \(g = \bigoplus_{i \geq -4} g_i\) satisfying the conditions (g1), (g2), (g3) of (2.4) and one of the three conditions below:

1) \(g_i = (0)\) for \(i \leq -3\) and \(g_0 = \mathfrak{g}(g_{-1})\),

2) \(g_i = (0)\) for \(i \leq -3\) and \(g_0 = s\mathfrak{g}(g_{-1})\),

3) \(g_i = (0)\) for \(i \leq -5\) and \(g_0 = \mathfrak{g}(g_{-1}), \dim g_{-4} = 3\).

Moreover, \(\dim g_{-1} = 3\) and \(g_{-2} \cong \wedge^2 g_{-1}\) in all cases, and \(g_{-3} \cong \wedge^3 g_{-1}\) in case 3. In cases 1) and 2), each Skryabin algebra admits a natural \((\mathbb{Z}/2\mathbb{Z})\)-grading \(g = G_0 \oplus G_1\) such that \(G_0\) is either \(W(3; \mathbb{N})\) or \(S(3; \mathbb{N})^{(1)}\) and \(G_1\) is a nice irreducible \(G_0\)-module. In case 3), each Skryabin algebra admits a natural \((\mathbb{Z}/4\mathbb{Z})\)-grading \(g = G_0 \oplus G_1 \oplus G_2 \oplus G_3\) such that \(G_0 = W(3; \mathbb{N})\) and each \(G_i\) with \(i \neq 0\) is a nice \(G_0\)-module. In all cases, the Lie bracket in \(g\) is given by explicit formulæ involving classical operations with differential forms (see [Sk 92] for more detail).

Now suppose \(\text{char} F = 2\). In [Br 95], Brown constructed three series of simple Lie algebras over \(F\) one of which relates closely with the Melikian series.

Following [Br 95] consider the \((\mathbb{Z}/3\mathbb{Z})\)-graded algebra \(L = L_0 \oplus L_1 \oplus L_2\) such that \(L_0 = W(2; \mathbb{N}), L_1 = O(2; \mathbb{N})\), and \(L_2 = \{fu \mid f \in O(2; \mathbb{N})\}\), a second vector space copy of \(O(2; \mathbb{N})\). The multiplication function \([\cdot, \cdot] : L \times L \to L\) satisfies the
identity $[x, x] = 0$, agrees with the Lie bracket of $W(2; n)$, and has the following properties:

$$
[D, fu] = \text{div}(fD)u, \quad [D, f] = D(f), \quad [fu, gu] = 0,
$$

$$
[fu, g] = fDg, \quad [f, g] = Dg(fu)
$$

(here $f, g \in \mathcal{O}(2; n)$, $D \in W(2; n)$, and $Dg$ has the same meaning as before). It is shown in [Br 95] that $L$ is a Lie algebra carrying a natural $\mathbb{Z}$-grading $L = \bigoplus_{i \geq -4} L_i$ such that $L_{-4} = z(L)$. The Lie algebra $g := (L/z(L))^{(1)}$ is denoted by $G_{2}(2; n)$. It is simple, has dimension $2|n| + 2$, and inherits from $L$ a natural $\mathbb{Z}$-grading $g = \bigoplus_{i \geq -3} g_i$ satisfying the conditions $(g1), (g2), (g3)$ of (2.4). Moreover, $g_0 \cong g([-1], 1)$, $\dim g_{-1} = \dim g_{-3} = 2$, and $g_{-2} \cong \wedge^2 g_{-1}$.

The 14-dimensional Lie algebra $G_{2}(2; 1)$ is not restrictable but can be obtained by reducing modulo 2 a nonstandard $\mathbb{Z}$-form of a complex Lie algebra of type $G_2$ (see [Br 95] for more detail).

4. Classification theorems

One of the main goals of this survey is to announce the following theorem which, in particular, confirms the original Kostrikin–Shafarevich conjecture in full generality; see [P-St 06].

**Theorem 7 (Classification Theorem).** Let $L$ be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$. Then $L$ is either a classical Lie algebra or a filtered Lie algebra of Cartan type or one of the Melikian algebras.

Our proof of Theorem 7 relies on several earlier classification results which we are going to formulate. From now on we assume that $\text{char } F = p > 3$.

The following useful characterization of classical Lie algebras is due to Seligman and Mills:

**Theorem 8 ([M-Se 57]).** A Lie algebra $L$ over $F$ is a direct sum of classical simple Lie algebras if and only if the following conditions hold:

1. $L$ is perfect and $z(L) = (0)$;

2. $L$ contains an abelian Cartan subalgebra $H$ such that
   
   a. $L = H \oplus \sum_{\alpha \neq 0} L_\alpha$ where $L_\alpha = \{ x \in L \mid [h, x] = \alpha(h)x \quad (\forall h \in H) \}$;
   
   b. if $L_\alpha \neq (0)$, then $\dim [L_\alpha, L_{-\alpha}] = 1$;
   
   c. if $L_\alpha \neq (0)$ and $L_\beta \neq (0)$, then $L_{\alpha + k\beta} = (0)$ for some $k \in \mathbb{F}_p$.

A short proof of the Seligman–Mills theorem based on the Kac–Moody theory can be found in [S 80].

The following important theorem allows one to recognize certain filtered simple Lie algebras:

**Theorem 9 (Recognition Theorem).** Let $L$ be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$. Let

$L = L_{(s')} \supset \ldots \supset L_{(0)} \supset \ldots \supset L_{(s)} \supset (0), \quad [L_{(i)}, L_{(j)}] \subseteq L_{(i+j)},$

be a filtration of $L$ satisfying the following conditions:

a. $s, s' \geq 1$ and $s' \leq s$;
(b) \(L(0)/L(1)\) is a direct sum of ideals each of which is either classical simple or \(\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{pgl}(n)\) with \(p/n\) or abelian;

(c) \(L(-1)/L(0)\) is an irreducible \(L(0)\)-module;

(d) for all \(j \leq 0\), if \(x \in L(j)\) and \([x, L(1)] \subseteq L(j+2)\), then \(x \in L(j+1)\);

(e) for all \(j \geq 0\), if \(x \in L(j)\) and \([x, L(-1)] \subseteq L(j)\), then \(x \in L(j+1)\).

Then \(L\) is either classical or is isomorphic as a filtered algebra to a Lie algebra of Cartan type or a Melikian algebra regarded with their natural filtrations.

The Recognition Theorem incorporates Wilson’s theorem [Wil 76] and earlier results of Kostrikin and Shafarevich [Ko-S 69]. Kac was the first to formulate a version of this theorem for graded Lie algebras, and he made in \([Ko-S 69]\) many deep and important observations towards its proof. One of Kac’s original assumptions on the pair \((L_{-1}, L(0))\) was relaxed by Benkart–Gregory in [B-G 89]. The first complete proof of the Recognition Theorem for graded Lie algebras was obtained only very recently by Benkart–Gregory–Premet; see [B-G-P]. Theorem 9 is a consequence of this result; see [St 04] Section 5 for more detail.

Theorems 8 and 9 are fundamental, and most of the classification proofs rely on them at some stage.

Given a nilpotent Lie subalgebra \(H\) of \(L\) we denote by \(H^\text{tor}_p\) the unique maximal torus in the \(p\)-envelope of \(H\) in \(\text{Der} L\). We say that \(H\) is \textit{triangular} if \(\text{ad} h\) is a nilpotent linear operator for any \(h \in H^{(1)}\) (this is the same as to say that \(\text{ad} H\) stabilizes a flag of subspaces in \(L\)).

We list below a few other classification results which are invoked frequently. All of them share the assumption that \(L\) is a finite dimensional simple Lie algebra over \(F\).

4.1. \textit{Kaplansky} [Kap 58]: If \(p > 3\) and \(L\) contains a one dimensional Cartan subalgebra \(Ft\) with \(\text{ad} t\) toral, then \(L\) is either \(\mathfrak{sl}(2)\) or \(W(1; 1)\).

4.2. \textit{Demushkin} [Dem 70, Dem 72], \textit{Strade} [St 04] (7.5): If \(L\) is a restricted Lie algebra of Cartan type, then all maximal tori of \(L\) have the same dimension and split into finitely many conjugacy classes under the action of \(\text{Aut} L\).

4.3. \textit{Kuznetsov} [Ku 76], \textit{Weisfeiler} [We 84], \textit{Skryabin} [Sk 97], \textit{Strade} [St 04]: If \(p > 3\) and \(L\) contains a solvable maximal subalgebra, then either \(L \cong \mathfrak{sl}(2)\) or \(L \cong W(1; n)\).

4.4. \textit{Wilson} [Wil 77], \textit{Premet} [P 94]: If \(H\) is a nontriangular Cartan subalgebra of \(L\), then \(p = 5\) and there exist \(F\)-independent \(\alpha, \beta \in \Gamma(L, H^\text{tor}_p)\) and an ideal \(R(\alpha, \beta)\) of the 2-section \(L(\alpha, \beta)\) such that

\[
L(\alpha, \beta)/R(\alpha, \beta) \cong M(1, 1).
\]

4.5. \textit{Wilson} [Wil 78], \textit{Premet} [P 94]: If \(p > 3\) and \(L\) contains a Cartan subalgebra \(H\) with \(\dim H^\text{tor}_p = 1\), then \(L\) is one of \(\mathfrak{sl}(2), W(1; 2), H(2; 2; \Phi)\).

4.6. \textit{Block–Wilson} [B-W 82], \textit{Wilson} [Wil 83]: Suppose \(L\) is restrictable and \(p > 7\). If \(L\) contains a toral Cartan subalgebra, then either \(L\) is classical or \(L \cong W(n; 1)\).

4.7. \textit{Benkart–Osborn} [B-O 84]: If \(L\) contains a one dimensional Cartan subalgebra and \(p > 7\), then \(L\) is either \(\mathfrak{sl}(2)\) or \(W(1; 2)\) or \(L \cong H(2; 2; \Phi)\) and \(\dim L = p|x|\).
The following result of Block–Wilson marked the first real breakthrough in solving the classification problem for $p > 7$.

**Theorem 10** ([B-W 88]). The original Kostrikin–Shafarevich conjecture is true for $p > 7$.

Relying heavily on an important intermediate result of [B-W 88] and the classification techniques of Block–Wilson the second author was able to generalize Theorem 10, with some support of R.L. Wilson (see [St 89b, St 91, St 92, St 93, B-O-St 94, St 94, St 98]).

**Theorem 11** (Strade 1998). The generalized Kostrikin–Shafarevich conjecture is true for $p > 7$.

Large parts of the proof of Theorem 11 go through for $p > 3$ and are incorporated into our proof of Theorem 7.

**5. Principles of the classification**

Let $L$ be a simple Lie algebra over $F$ (recall that char $F = p > 3$). As in the characteristic 0 case we hope to get more insight into the structure of $L$ by looking at the root space decomposition of $L$ relative to its Cartan subalgebra $h$. However, most of the classical results are no longer valid in our situation. For example, a $(2m + 1)$-dimensional Heisenberg Lie algebra over $F$ admits irreducible representations of dimension $p^m$. This implies that Lie’s theorem on solvable Lie algebras fails in characteristic $p$. The Killing form of any strongly degenerate simple Lie algebra over $F$ vanishes (see (2.3)). Since all finite dimensional Cartan type Lie algebras over $F$ are strongly degenerate, Cartan’s criterion is no longer valid in characteristic $p$ either. Cartan subalgebras of $L$ need not be conjugate under the automorphism group Aut $L$ and, in fact, may have different dimensions (see our discussion in (2.1)). In characteristic 5, one can even expect $L$ to possess nontriangulable Cartan subalgebras (see (4.4) and our discussion in (3.4)).

In general, a nonrestrictable Lie algebra does not possess a Jordan–Chevalley decomposition. To fix that we embed $L \cong \text{ad} L$ into its semisimple $p$-envelope $\mathcal{L}$ (see (2.2.3)). The Lie algebra $\mathcal{L} \subset \text{Der} L$ is restricted, hence admits a Jordan–Chevalley decomposition. By construction, $\mathcal{L}^{(1)} \subseteq L$ (and $\mathcal{L} = L$ if and only if $L$ is restrictable). We choose a torus $T$ of maximal dimension in $L$ and take a close look the root space decomposition

$$L = H \oplus \sum_{\alpha \in \mathcal{V}(L,T)} L_\alpha$$

of $L$ relative to $T$. Although the subalgebra $H = \{ x \in L \mid [t, x] = 0 \ \forall t \in T \}$ is nilpotent it is not always a Cartan subalgebra of $L$ (if $L$ is nonrestrictable, it may even happen that $H = (0)$). We wish to gather as much information as we can on the structure of 1- and 2-sections of $L$ relative to $T$. In characteristic 0, such information eventually allows one to determine the global structure of $L$.

In characteristic $p$, the local analysis is much more involved. There are a number of reasons for that. To mention just a few, the 1-sections of $L$ relative to $T$ are no longer “reductive” and their irreducible representations are hard to describe. Some tori of maximal dimension in $\mathcal{L}$ are unsuitable for our purposes, and a lot of effort is spent on optimizing a randomly chosen $T$ by using generalized Winter exponentials; see (2.1). In the course of the proof one has to make various
sophisticated choices of maximal subalgebras, carry out detailed computations in Lie algebras of small rank, and study central extensions of such algebras and their irreducible representations.

For any \( \alpha \in \Gamma(L, T) \) the semisimple quotient \( L[\alpha] \) of the 1-section \( L(\alpha) \) is either zero or \( \mathfrak{sl}(2) \) or \( W(1; 1) \) or the inclusion

\[
H(2; 1)^{(2)} \subset L[\alpha] \subset H(2; 1)
\]

holds (this follows from (4.5)). Accordingly we call \( \alpha \) solvable, classical, Witt or Hamiltonian. It is not difficult to show that the radical of \( L(\alpha) \) is \( T \)-stable. So \( T \) acts as derivations on \( L[\alpha] \) and \( L[\alpha]^{(2)} \). Following Block–Wilson we say that \( \alpha \) is a proper root if either \( L[\alpha] \in \{0, \mathfrak{sl}(2)\} \) or \( L[\alpha] \) is of Cartan type and the standard maximal subalgebra of \( L[\alpha]^{(2)} \) is \( T \)-invariant. If \( \alpha \) is not a proper root we say that \( \alpha \) is improper.

The main intermediate result of \([B-W 88]\) is a classification of all simple Lie algebras of absolute toral rank 2 for \( p > 7 \). Combining this classification with a version of (4.4) for \( p > 7 \), Block and Wilson succeeded to describe the semisimple quotients of all 2-sections in a restricted simple Lie algebra. Having achieved that they proceed as follows:

The description of the quotients \( L[\alpha] \) mentioned above implies that each 1-section \( L(\alpha) \) contains a unique subalgebra \( Q(\alpha) \) with \( H \subset Q(\alpha) \) and \( \dim Q(\alpha) = \dim L(\alpha) - e(\alpha) \), where

\[
e(\alpha) = \begin{cases} 
0 & \text{if } \alpha \text{ is solvable or classical}, \\
1 & \text{if } \alpha \text{ is Witt}, \\
2 & \text{if } \alpha \text{ is Hamiltonian}.
\end{cases}
\]

The subalgebra \( Q(\alpha) \) is solvable if \( \alpha \) is solvable or Witt, and \( Q(\alpha)/\operatorname{rad} Q(\alpha) \cong \mathfrak{sl}(2) \) if \( \alpha \) is classical or Hamiltonian. In all cases, \( Q(\alpha) \) is \( T \)-invariant if and only if \( \alpha \) is a proper root of \( L \). Generalized Winter exponentials are now used to “optimize” \( T \).

A torus \( T \subset \mathcal{L} \) is called optimal if \( \dim T = MT(\mathcal{L}) \) and the number of proper roots in \( \Gamma(L, T) \) is maximal possible. Using their description of the semisimple quotients \( L(\alpha, \beta)/\operatorname{rad} L(\alpha, \beta) \) Block and Wilson prove that in the restricted case all roots of \( L \) relative to an optimal torus \( T \subset \mathcal{L} \) are proper. They then look again at the 2-sections of \( L \) relative to \( T \) to prove that the \( T \)-invariant subspace

\[
Q = Q(L, T) := \sum_{\alpha \in \Gamma(L, T)} Q(\alpha)
\]

is a Lie subalgebra of \( L \). The rest of the proof is straightforward. If \( Q = L \), Block and Wilson show that the Seligman–Mills theorem applies to \( L \). So \( L \) is classical in this case. If \( Q \neq L \), they show that \( Q \) can be embedded into a maximal subalgebra satisfying the conditions of the Recognition Theorem.

For an arbitrary simple \( L \), the second author used the Block–Wilson classification of simple Lie algebras of rank 2 to obtain a list of all possible \( T \)-semisimple quotients of the 2-sections of \( L \) (this list is longer than in the restricted case). He then succeeded to optimize \( T \) in \( \mathcal{L} \) and in the joint work with Benkart and Osborn \([B-O-St 94]\) constructed a large Lie subalgebra \( Q = Q(L, T) \) of \( L \). However, the final parts of the proof in the general case are much more involved; see \([St 91, St 93, St 94, St 98]\). Essentially, this is due to the fact that \( \mathcal{L} \) is no longer simple. Since optimal tori may lie outside \( L \) some 3-sections have to be thoroughly investigated.
It turned out that if all regular Cartan subalgebras of $L$ are triangulable, then the final parts of the second author’s classification go through for $p > 3$ after a proper modification. This modification is carried out in [P-St 04, P-St 06], thus settling the remaining case $p = 7$ of the generalized Kostrikin-Shafarevich conjecture. If $L$ contains a nontriangulable regular Cartan subalgebra, then [P-St 04, Theorem A] and (4.4) imply that $p = 5$ and one of the semisimple quotients $L(\alpha, \beta)/\rad L(\alpha, \beta)$ is isomorphic to $M(1, 1)$. This situation is investigated in [P-St 07], the last paper of the series. The main result of [P-St 07] states that $L$ is then isomorphic to a Melikian algebra $M(m, n)$.

The hardest part of our proof of Theorem 7 is the classification of the simple Lie algebras of absolute toral rank 2 and the description of the 2-sections of $L$ relative to $T$. The former is obtained in [P-St 97, P-St 99, P-St 01] while the latter is carried out in [P-St 04]. Below we outline our arguments in the rank 2 case.

(A) From now on we assume that $L$ is a nonclassical simple Lie algebra of absolute toral rank 2 and $L$ is the semisimple $p$-envelope of $L$; see (2.2.3) and Definition 3 in (2.2). In view of (4.5) we may assume that for any maximal torus $T \subset L$ the centralizer $H = c_L(T)$ has the property that $\dim H^\tor_p = 2$ (in particular, it can be assumed that all maximal tori in $L$ are two dimensional). Finally, we may assume that all simple Lie algebras $g$ with $TR(g) = 2$ and $\dim g < \dim L$ are known.

Our ultimate goal is to prove that $L$ admits a filtration satisfying the conditions of the Recognition Theorem. However, at the beginning of the investigation any long filtration invariant under a two dimensional torus in $L$ would do. Thus we have to address the following

**Problem.** Find a long standard filtration in $L$ stable under the action of a maximal torus in $\mathcal{L}$.

This problem is solved in [P-St 97] by producing a root sandwich in $L$, that is a nonzero sandwich element $c \in L$ such that $[T, c] \subset Fc$ for some torus $T$ of maximal dimension in $\mathcal{L}$. The set of all such sandwiches is denoted by $S(L, T)$. Adopting the method used in [P 86a, P 86c] for proving Kostrikin’s conjecture we first show that under some mild assumptions on a 1-section of $L$ there exists a nonzero $x \in H \cup \bigcup_{\gamma \in \Gamma(L, T)} L_\gamma$

such that $(\ad x)^3 = 0$. Then we use some techniques from [B 77, Ko 67] and the theory of finite dimensional Jordan algebras to find a root sandwich $c$. More precisely, we prove

**Theorem 12 ([P-St 97]).** Let $g$ be a simple Lie algebra of absolute toral rank 2 over $F$. Then either $g$ is classical or $g \cong H(2; 1; \Phi)^{(2)}$ with $\dim g = p^2 - 1$ or there exists a two dimensional torus $t$ in the semisimple $p$-envelope of $g$ such that $S(g, t) \neq \emptyset$.

Having found a root sandwich $c \in L$ we now observe that any maximal subalgebra $L(0)$ of $L$ containing $H + c_L(c)$ gives rise to a long $T$-invariant filtration of $L$. Indeed, let $L(-1)$ be any $L(0)$-stable subspace of $L$ such that $L(-1) \supset L(0)$ and $L(-1)/L(0)$ is an irreducible $L(0)$-module. The $L(0)$-module $L(-1)$ is $H$-stable, hence $T$-stable (for $T = H^\tor_p$). Therefore, so are all components of the standard
filtration associated with the pair \((L_{-1}), L_{(0)}\): see (2.4) for more detail. Since \([L_{-1}, c] \subset cL_{(0)} \subset L_{(0)}\) we have that \(0 \neq c \in L_{(1)}\).

(B) Next we investigate the graded Lie algebra \(G := \text{gr} L\). Let \(M(G)\) denote the largest ideal of \(G\) contained in \(\sum_{i \leq -1} G_i\), and \(\bar{G} := G/M(G)\). By a theorem of Weisfeiler [WC 78], the Lie algebra \(G\) is semisimple and has a unique minimal ideal, denoted \(A(\bar{G})\). Furthermore, \(\bar{G}\) inherits a natural grading from \(G\) which satisfies the conditions (g1) - (g4). Note that finite dimensional semisimple Lie algebras over \(F\) need not be direct sums of simple ideals (in fact, simple ideals may not exist at all). The structure of semisimple modular Lie algebras was determined by Block in \([B 69]\). The following important theorem describes the structure of a semisimple Lie algebra with a unique minimal ideal:

**Theorem 13 (Block’s Theorem).** Let \(g\) be a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic \(p > 0\) and suppose that \(g\) contains a unique minimal ideal, I say. Then there exist an \(r \in \mathbb{N}_0\) and a simple Lie algebra \(s\) such that \(I \cong s \otimes \mathcal{O}(r; 1)\) as Lie algebras. Moreover, \(g \cong \text{Ad}_I g\) and \((\text{Ad} g) \otimes \mathcal{O}(r; 1) \subset \text{Ad}_I g \subset (\text{Der} g) \otimes \mathcal{O}(r; 1) \rtimes \text{Id}_s \otimes W(r; 1)\).

In a sense, the above-mentioned theorem of Weisfeiler can be regarded as a graded version of Block’s theorem; see \([ST 04]\) (3.5) for more detail. In \([P-St 99]\), we show that our maximal subalgebra \(L_{(0)}\) can be chosen such that

- either \(G_{2} \neq (0)\) or \([G_{-1}, G_{1}], G_{1} \neq (0)\).

In this case, Weisfeiler’s theorem says that \(A(\bar{G}) = \bigoplus_i A(\bar{G})_i\) where \(A(\bar{G})_i = A(\bar{G}) \cap \bar{G}_i\) and there exist a graded simple Lie algebra \(S = \bigoplus_i S_i\) and an integer \(m \geq 0\) such that

\[A(\bar{G}) \cong S \otimes \mathcal{O}(m; 1) = \bigoplus_{i \in \mathbb{Z}} (S_i \otimes \mathcal{O}(m; 1))\]

as graded Lie algebras. In \([P-St 99]\) we show that \(m \leq 1\). Moreover, we prove that if \(m = 1\), then the absolute toral rank of \(S\) drops. In view of (2.2.4) and (4.5) the equality \(m = 1\) implies that \(S\) is one of \(\mathfrak{sl}(2), W(1; 1), H(2; 1)\).

We first consider the case where \(m = 1\). By Block’s theorem, we then have an embedding

\[\bar{G} \hookrightarrow (\text{Der} S) \otimes \mathcal{O}(1; 1) \rtimes \text{Id}_S \otimes W(1; 1)\]

In view of a conjugacy theorem proved in \([P-St 99]\) along comes an induced embedding of tori

\[T \hookrightarrow T_0 \otimes 1 + \text{Id}_S \otimes Fz\partial, \quad z \in \{x, 1 + x\},\]

where \(T_0\) is a one dimensional torus in \(\text{Der} S\). We then show that \(T\) and \(L_{(0)}\) can be chosen such that \(S \cong H(2; 1)\) as graded Lie algebras, where \(H(2; 1)\) is regarded with its grading of type \(\frac{1}{2}\); see (3.2). In particular, \(S_0 \cong \mathfrak{sl}(2)\) and \(S_{-k} = (0)\) for \(k \geq 2\). We also show that \(M(G) = (0)\). This information enables us to conclude, eventually, that \(p = 5\) and \(L \cong M(1, 1)\).

(C) From now on we may assume that \(m = 0\). Using the inequality \(TR(G) \leq TR(L)\), proved in \([SK 98]\), we show that \(TR(S) = 2\). We now wonder whether \(S\) is listed in the Classification Theorem.

First we observe that the root sandwich \(c \in L_{(1)}\) gives rise to a nonzero sandwich element of \(G\) contained in the graded component \(G_i\) for some \(i \geq 1\). Since \(M(G) \subset \ldots\)
\( \bigoplus_{i<1} G_i \), it follows that the Lie algebra \( \bar{G} \) must be strongly degenerate. Since \( S \subset \bar{G} \subset \text{Der} \ S \), it follows that \( S \) is not a classical Lie algebra.

Next we observe that the quotient space \( \bar{M} := M(G)/M(G)^2 \) is a \( \bar{G} \)-module, hence an \( S \)-module. Let \( S_p \) denote the \( p \)-envelope of \( S \) in \( \text{Der} \ S \). We show in [P-St 01] that any composition factor \( V \) of the \( S \)-module \( \bar{M} \) can be viewed in a natural way as a restricted \( S_p \)-module and \( T \) can be identified with a two dimensional torus in \( S_p \). Since \( H \subset L_{(0)} \) it must be that

\[ 0 \not\in \Gamma^w(V,T). \]

On the other hand, we show in [P-St 01] that if \( S \) is isomorphic to one of \( S(3; \underline{1})^{(1)}, H(4; \underline{1})^{(2)}, K(3; \underline{1}), M(1,1), H(2;(2,1))^{(2)} \), then \( T \) has weight 0 on any finite dimensional restricted \( S_p \)-module. This implies that \( M(G) = (0) \) if \( S \) is one of these Lie algebras. A slight modification of the argument shows that \( M(G) = (0) \) if \( S \) is one of \( W(2; \underline{1}), W(1; 2), H(2; \underline{1}; \Phi)^{(2)} \). Since \( TR(S) = 2 \) we deduce the following:

if \( S \) is known, then \( M(G) = (0) \).

Suppose \( S \) is known. Then \( \bar{G} \cong G = \text{gr} \ L \), hence \( L \) is a filtered deformation of \( \bar{G} \subset \text{Der} \ S \). So there exists a Lie algebra \( \mathfrak{L} \) over the polynomial ring \( F[t] \) such that

\[ \mathfrak{L}/(t - \lambda)\mathfrak{L} \cong L \text{ if } \lambda \neq 0, \text{ and } \mathfrak{L}/t\mathfrak{L} \cong \bar{G} \supset S. \]

Suppose \( S \) is a Melikian algebra. Since \( TR(S) = 2 \) we then have \( S = M(1,1) \). By [Ku 90], all derivations of \( M(1,1) \) are inner (see also [St 04 (7.1)]). So it must be that \( G \cong S \). We already mentioned in (3.4) that \( M(1,1) \) contains a two dimensional torus \( t_0 \) whose centralizer \( \mathfrak{h} \) is a nontriangulable Cartan subalgebra of \( M(1,1) \). As \( TR(S) = 2 \), the Cartan subalgebra \( \mathfrak{h} \) is regular in \( S \). As all regular Cartan subalgebras of a finite dimensional restricted Lie algebra have the same dimension (see (2.1)) we can lift \( \mathfrak{h} \) to a nontriangulable Cartan subalgebra of minimal dimension in \( \mathfrak{L} \otimes_{F[t]} F(t) \). We then use a deformation argument to show that \( L \) contains a nontriangulable Cartan subalgebra as well. Using (4.4) we finally conclude that \( L \cong M(1,1) \).

Suppose \( S \cong X(m; \underline{n})^{(2)} \) where \( X \in \{ W, S, H, K \} \). Any grading of a Lie algebra \( \mathfrak{g} \) is induced by the action of a one dimensional torus of the algebraic group \( \text{Aut} \mathfrak{g} \). Each such torus is contained in a maximal torus of \( \text{Aut} \mathfrak{g} \). The conjugacy theorem for maximal tori of algebraic groups enables us to prove that any grading of \( S \) is obtained by assigning certain integral weights to the elements of a generating set of the divided power algebra \( \mathcal{O}(m; \underline{n}) \). This procedure also describes the gradings of \( \text{Der} \ S \) and provides valuable information on gradings of \( G \) (for \( G \) can be regarded as a graded subalgebra of \( \text{Der} \ S \)). It turns out that very few gradings of \( G \) can satisfy the conditions \( (g1), (g2), (g3) \). Taking graded Cartan type Lie algebras of rank 2 one at a time we show that our choice of \( L_{(0)} \) (and \( T \)) forces the grading of \( S \cong X(m; \underline{n})^{(2)} \) to be standard. At this point Wilson’s theorem enables us to conclude that \( L \) is a filtered Lie algebra of Cartan type.

If \( S \) is a filtered Cartan type Lie algebra not considered before, then that is one of type \( H(2; \underline{1}; \Phi)^{(2)} \), then \( S \) is nonrestrictable of dimension \( p^2 - 1 \) or \( p^2 \). In this case, \( S_p \) is known to possess a two dimensional toral Cartan subalgebra \( t \) with the property that \( \dim S_\gamma = 1 \) for all \( \gamma \in \Gamma(S, t) \). This information and an intermediate result of [B-W 82] (applicable for \( p > 3 \) in view of (4.5)) allow us to show that \( L \) too is of Cartan type.
(D) It remains to consider the case where $S \cong G = \text{gr} L$ is a minimal counterexample to our theorem. At this stage we may also assume that passing from $L$ to $G$ always produces unknown simple graded Lie algebras (subject to certain conditions on $T$ and $L(0)$). We use this as a technical tool for improving $L(0)$ and obtaining more information on the structure of $\Gamma \cap G(T)$. Given $K = \Gamma(G)$ we set $K_\alpha := \{ x \in G_\alpha \mid \alpha(x, G_{-\alpha}) = 0 \}$ and denote by $K'(G, T, \alpha)$ the Lie subalgebra of $G$ generated by all $K_\alpha$ with $\alpha \in \mathbb{F}_p^*$. It follows from the main result of 

\[ P-St 99 \]

that $K'(G, T, \alpha)$ is a triangulable subalgebra of $G$.

The most important task for us now is to determine the graded component $G_0$. From [Sk 97] we know that the radical of $G_0$ is abelian, while (4.3) entails that $G_0$ is nonsolvable. Thus if $\text{rad} G_0 \neq (0)$, then $\tilde{G}_0 := G_0/\text{rad} G_0$ has absolute toral rank 1. Moreover, it follows from (4.5) that $\tilde{G}_0$ is either $\text{sl}(2)$ or $W(1; 1)$ or the inclusion $H(2; 1)' \subset \tilde{G}_0 \subset H(2; 1)$ holds. Combining some representation theory with the fact that $K'(G, T, \alpha)$ is triangulable (see [P-St 99]), we show after a detailed analysis that either $G_0 \cong W(1; 1) \times \mathcal{O}(1; 1)$ (a natural semidirect product) or the radical of $G_0$ is one dimensional and central, and the extension

$$0 \to \text{rad} G_0 \to G_0 \to \tilde{G}_0 \to 0$$

splits. If $G_0$ is semisimple with a unique minimal ideal $I$, then Block’s theorem says that $I \cong \mathfrak{s} \otimes \mathcal{O}(r; 1)$ for some simple Lie algebra $\mathfrak{s}$. If $r > 0$ we prove that $\mathfrak{s}$ has absolute toral rank 1 and there are a vector space $V$ over $F$ and a linear isomorphism

$$G_0 \sim V \otimes \mathcal{O}(k; \underline{1})$$

such that $\mathcal{O}(k; \underline{1}) \cong \mathcal{O}(r; 1)$ as algebras and the action of $G_0$ on $G_{-1}$ is induced by a Lie algebra embedding

$$G_0 \hookrightarrow \mathfrak{gl}(V) \otimes \mathcal{O}(k; \underline{1}) \rtimes \text{Id}_V \otimes W(k; \underline{1}).$$

Moreover, $\pi(G_0)$, the image of $G_0$ under the canonical projection

$$\pi : \mathfrak{gl}(V) \otimes \mathcal{O}(k; \underline{1}) \rtimes \text{Id}_V \otimes W(k; \underline{1}) \to W(k; \underline{1}),$$

is transitive, that is has the property that $\pi(G_0) + W(k; \underline{1})(0) = W(k; \underline{1})$. Using the simplicity of $G$ and Cartan prolongation techniques inspired by earlier work of Kuznetsov (see e.g. Ku 76) we show that $\pi(G_0)$ is an $\mathcal{O}(k; \underline{1})$-submodule of $W(k; \underline{1})$. The transitivity of $\pi(G_0)$ now forces $\pi(G_0) = W(k; \underline{1})$, while toral rank considerations yield $k = 1$, $\underline{1} = \underline{1}$. This enables us to prove that

$$G_0 \cong \mathfrak{s} \otimes \mathcal{O}(1; 1) \rtimes \text{Id}_\mathfrak{s} \otimes W(1; 1),$$

where $\mathfrak{s}$ is either $\mathfrak{sl}(2)$ or $W(1; 1)$. As a consequence, we obtain that $G_0$ belongs to a short list of known linear Lie algebras.

Considering algebras from this list one at a time we show that $T$ can be chosen such that all roots in $\Gamma(G, T)$ are proper. This allows us to obtain much better estimates for $\dim G_i \gamma$ with $i < 0$ and $\gamma \in \Gamma(G, T)$. We use this new information to show that either $G_0$ is a classical Lie algebra of rank 2 or the $p$-envelope $\mathfrak{Z}_0$ of $G_0$ in $\text{Der} G$ is isomorphic to $\mathfrak{gl}(2)$ as restricted Lie algebras.

Let $G'$ denote the Lie subalgebra of $G$ generated by $G_{\pm 1}$ and $M(G')$ the maximal graded ideal of $G'$ contained in $\sum_{i<0} G_i$. If $M(G') \neq (0)$ we combine the Recognition Theorem with some representation theory of Cartan type Lie algebras to show that $\mathfrak{Z}_0 \cong \mathfrak{gl}(2)$ and $G'/M(G')$ is classical of type $A_2$, $C_2$ or $G_2$. We then use the representation theory of algebraic groups to show that this cannot happen.
As a result, \( M(G') = (0) \). Then the Recognition Theorem applies to \( G \) itself, showing that \( G \) is known. This contradiction proves the Classification Theorem in the rank 2 case.

6. Some open problems

The classification problem in characteristics 2 and 3 is wide open. Since our knowledge of finite dimensional simple Lie algebras over algebraically closed fields of characteristics 2 and 3 is very limited, it is not clear at present whether a complete classification of such algebras can ever be achieved. As indicated in our discussion at the end of (3.3) the classification of Hamiltonian forms in \( h_1(m, n) \) was reduced by Skryabin to a certain problem of linear algebra. Luckily, the problem turned out to be tame. But if it turned out to be wild, we would never have a complete classification in characteristic \( p > 3 \).

The first three items will address issues in characteristics 2 and 3.

**Conjecture 1.** The automorphism group of any finite dimensional simple Lie algebra over an algebraically closed field of characteristic \( p > 0 \) is infinite.

For \( p > 3 \), one can easily deduce Conjecture from the results in [P 86a, P 86c] or, alternatively, from Theorem 7. However, the conjecture remains wide open for \( p \in \{2, 3\} \).

In [Sk 98], Skryabin proved that any finite dimensional simple Lie algebra of absolute toral rank one over an algebraically closed field of characteristic 3 is isomorphic to either \( sl(2) \cong W(1; 1) \) or \( psl(3) \cong H(2; 1) \). He also proved in loc. cit. that no finite dimensional simple Lie algebras of absolute toral rank 1 exist in characteristic 2.

**Problem 1.** Classify all finite dimensional simple Lie algebras of absolute toral rank two over algebraically closed fields of characteristics 2 and 3.

In characteristic 2, strong results closely related to Problem 1 are obtained by A. Grishkov and the first author (work in progress). We are unaware of any ongoing work on the characteristic 3 case of Problem 1.

As mentioned in [B-G-P], it would be very useful to have a version of the Recognition Theorem for graded Lie algebras of characteristics 2 and 3.

**Problem 2.** Classify all finite dimensional graded Lie algebras \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) over algebraically closed fields of characteristics 2 and 3 that satisfy the conditions (g1) - (g4) of (2.4) and have the property that \( G_0 \) is isomorphic to the Lie algebra of a reductive group.

The last four items will deal, mainly, with the case where \( p > 3 \).

**Problem 3.** Determine the absolute toral rank of all finite dimensional simple Lie algebras over algebraically closed fields of characteristic \( p > 3 \).

One should stress here that the value of \( TR(L) \) is known for many simple Lie algebras \( L \). In particular, it is known for all restricted Lie algebras of Cartan type. Some results related to Problem 3 can be found in [B-K-K 95]. The most interesting open case of Problem 3 is the case where \( L = H(m; \omega; \omega_{A,B}) \) and \( \omega_{A,B} \in h_1(m; n) \) is such that \( \det B = 0 \).

**Problem 4.** Determine the automorphism groups of all finite dimensional simple Lie algebras over algebraically closed fields of characteristic \( p > 3 \). In particular, is it true that any finite dimensional simple Lie algebra \( L \) admits a \( \mathbb{Z} \)-grading \( L = \)}
$\oplus_{i \in \mathbb{Z}} L_i$ with $L_0 \neq L$? Equivalently, is it true that the connected component of the algebraic group $\text{Aut} L$ is not unipotent?

There are many examples of simple Lie algebras with solvable automorphism groups; in fact, such algebras occur in all four Cartan series. Probably, the most interesting open case of Problem 4 is the case where $L = H(m; n; \omega_{A,B})^{(2)}$ and $\omega_{A,B} \in \mathfrak{h}_1(m; n)$ is such that $\det B \neq 0$.

The next problem was suggested to the first author by R. Guralnick.

**Question 1** (cf. [G-K-P-S], Question 2.3). Is it true that any finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 0$ can be generated by two elements?

For $p \in \{2, 3\}$ Question 1 is out of reach at the moment. However, for $p > 3$, finite dimensional simple Lie algebras are likely to enjoy a much stronger property which is nowadays referred to as “one and a half generation”.

**Problem 5** (cf. [G-K-P-S], Question 2.4). Let $L$ be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$. Use Theorem 7 to prove that for any nonzero $x \in L$ there is $y \in L$ such that $L = \langle x, y \rangle$.

The simplicity assumption on $L$ in Problem 5 is crucial. Indeed, analyzing the semidirect products

$$\mathcal{L}(g, m) := (\text{Id}_g \otimes \mathcal{D}) \rtimes (g \otimes \mathcal{O}(m; 1)),$$

where $g$ is a finite dimensional simple Lie algebra over $F$ and $\mathcal{D}$ is the commutative subalgebra of $W(m; 1)$ spanned by $\partial_1, \ldots, \partial_m$, one can observe that for any natural number $n$ there exists a finite dimensional semisimple Lie algebra $L$ over $F$ such that the set

$$S_n(L) := \{x \in L \mid \langle x, y_1, \ldots, y_n \rangle \text{ is solvable for all } y_1, \ldots, y_n \in L\}$$

is nonzero. More precisely, it is not hard to see that for the semisimple Lie algebra $L = \mathcal{L}(g, n + 1)$ one has

$$g \otimes x_1^{p-1} \cdots x_n^{p-1} x_{n+1}^{p-1} \subseteq S_n(L).$$

This is in sharp contrast with the situation for finite groups; see [G-K-P-S] Theorem 1.1 for more detail.

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