BICOVARIANT CALCULUS ON TWISTED ISO($N$), QUANTUM POINCARE GROUP AND QUANTUM MINKOWSKI SPACE

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Abstract

A bicovariant calculus on the twisted inhomogeneous multiparametric $q$-groups of the $B_n, C_n, D_n$ type, and on the corresponding quantum planes, is found by means of a projection from the bicovariant calculus on $B_{n+1}, C_{n+1}, D_{n+1}$. In particular we obtain the bicovariant calculus on a dilatation-free $q$-Poincaré group $ISO_q(3,1)$, and on the corresponding quantum Minkowski space.

The classical limit of the $B_n, C_n, D_n$ bicovariant calculus is discussed in detail.
1 Introduction

We present a bicovariant differential calculus on the inhomogeneous multiparametric quantum groups of the $B_n, C_n, D_n$ type, and on the corresponding quantum planes. Our main motivation being an exhaustive study of the differential calculus on the quantum Poincaré group found in refs. [1], we mainly focus our discussion on the orthogonal inhomogeneous $q$-groups $ISO_q(N)$. All the quantities relevant to their differential calculus are explicitly constructed. The results are then directly applied to the $q$-Poincaré group $ISO_q(3,1)$, and to the quantum Minkowski space that emerges as the quantum coset $Fun_q(ISO(3,1)/SO(3,1))$.

The technique used in deriving the differential calculus on $ISO_q(N)$ or $ISp_q(N)$ is based on a projection from the bicovariant calculus on $ISO_q(N+2)$ or $ISp_q(N+2)$. This technique was first proposed in [2] and applied to find the quantum inhomogeneous groups $IGL_q(N)$ and the corresponding bicovariant calculus. Their multiparametric extensions were treated in [3]. Other references on inhomogeneous $q$-groups can be found in [4].

The projection method was then applied to the multiparametric $SO_{q,r=1}(N+2)$ to obtain the bicovariant calculus on $ISO_{q,r=1}(N)$, where $r = 1$ corresponds to the “minimal deformations”, or twistings, with diagonal $R$-matrix [5]. The gauging of the resulting deformed $q$-Poincaré “Lie algebra” leads to the $q$-gravity theory discussed in the same references. The structure of the multiparametric inhomogeneous $q$-groups $ISO_{q,r}(N)$ obtained via the projection technique was studied in detail in ref. [1], where a dilatation-free $q$-Poincaré group depending on one real deformation parameter was found. Absence of dilatations requires $r = 1$.

In the present paper the bicovariant calculus on $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ with $r = 1$ is obtained after a detailed study of the homogeneous orthogonal and symplectic $q$-groups in the $r \to 1$ limit. The necessity of taking $r = 1$ is discussed. The functionals of the universal enveloping algebra $U(SO_{q,r=1}(N+2))[U(Sp_{q,r=1}(N+2))]$ relevant for the construction of a bicovariant calculus are analyzed and “projected” to well defined functionals on $ISO_{q,r=1}(N)[ISp_{q,r=1}(N)]$. The differential calculus is found and explicitly formulated in terms of these “projected” functionals, that in the commutative limit become the tangent vectors to the inhomogeneous orthogonal [symplectic] groups. In this general setting we are able to retrieve all the results of [6] (where these functionals were only given in terms of their action on the adjoint $q$-group elements) in a direct way, and to clarify important points. For example we will easily see how the typical “abundance” of left-invariant one-forms can be lifted in the $r = 1$ case.

In our framework the bicovariant calculus on the orthogonal multiparametric quantum plane follows almost as a corollary.

In Section 2 we briefly review the basics of $B_n, C_n, D_n$ multiparametric quantum
groups, mainly to establish notations. In Section 3 we recall the $R$-matrix formulation of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ of ref. [4], and discuss the real forms yielding $ISO_{q,r}(n, n; \mathbb{R})$, $ISO_{q,r}(n, n + 1; \mathbb{R})$, $ISp_{q,r}(n, \mathbb{R})$ and $ISO_{q,r}(n + 1, n - 1; \mathbb{R})$, this last being the one used in [3, 4] to obtain the quantum Poincaré group $ISO_{q,r}(3, 1)$. The universal enveloping algebra and the bicovariant calculus on multiparametric $B_n, C_n, D_n$ $q$-groups (and their real forms) are given in Section 4 and 5 respectively. In Section 6 we examine the case $r = 1$. We clarify some issues related to the classical limit and see how in this limit some tangent vectors become linearly independent, thus providing the correct classical dimension of the tangent space. A similar mechanism occurs for the left-invariant one-forms. In Section 7 the bicovariant calculus on $ISO_{q,r=1}(N)$ and its real forms is constructed. Finally Section 8 deals with the differential calculus on the orthogonal quantum plane $\text{Fun}_{q,r=1}(ISO(N)/SO(N))$.

2 $B_n, C_n, D_n$ multiparametric quantum groups

The $B_n, C_n, D_n$ multiparametric quantum groups are freely generated by the noncommuting matrix elements $T^a_{\ b}$ (fundamental representation) and the identity $I$. The noncommutativity is controlled by the $R$ matrix:

$$R^{ab}_{\ ef} T^e_{\ d} T^f_{\ c} = T^b_{\ d} T^a_{\ c} R^{ef}_{\ cd}$$

which satisfies the quantum Yang-Baxter equation

$$R^{a_1 b_1}_{\ a_2 b_2} R^{a_2 c_1}_{\ a_3 c_2} R^{b_2 c_2}_{\ b_3 c_3} = R^{b_1 c_1}_{\ b_2 c_2} R^{a_1 c_2}_{\ a_2 c_3} R^{a_2 b_2}_{\ a_3 b_3},$$

a sufficient condition for the consistency of the “RTT” relations (2.1). The $R$-matrix components $R^{ab}_{\ cd}$ depend continuously on a (in general complex) set of parameters $q_{ab}, r$. For $q_{ab} = r$ we recover the uniparametric $q$-groups of ref. [3]. Then $q_{ab} \to 1, r \to 1$ is the classical limit for which $R^{ab}_{\ cd} \to \delta^a_c \delta^b_d$; the matrix entries $T^a_{\ b}$ commute and become the usual entries of the fundamental representation. The multiparametric $R$ matrices for the $A, B, C, D$ series can be found in [7] (other ref.s on multiparametric $q$-groups are given in [3, 4]). For the $B, C, D$ case they read:

$$R^{ab}_{\ cd} = \delta^a_c \delta^b_d [\frac{1}{q_{ab}} + (r - 1) \delta^{ab} + (r^{-1} - 1) \delta^{ab}] (1 - \delta^{an_2}) + \delta^a_n \delta^n_{n_2} \delta^2_{c} \delta^2_{d} + (r - r^{-1})[\theta^{ab}_{c} \delta^a_d \delta^b_c - \epsilon_a \epsilon_r \rho_a - \rho_c \delta^a \delta^b \delta_c \delta_d]$$

where $\theta^{ab} = 1$ for $a > b$ and $\theta^{ab} = 0$ for $a \leq b$; we define $n_2 \equiv \frac{N + 1}{2}$ and primed indices as $a' \equiv N + 1 - a$. The indices run on $N$ values ($N=$dimension of the fundamental representation $T^a_{\ b}$), with $N = 2n + 1$ for $B_n[SO(2n + 1)], N = 2n$ for $C_n[Sp(2n)], D_n[SO(2n)]$. The terms with the index $n_2$ are present only for the $B_n$ series. The $\epsilon$ and $\rho$ vectors are given by:

$$\epsilon_a = \begin{cases} +1 & \text{for } B_n, D_n, \\ +1 & \text{for } C_n \text{ and } a \leq n, \\ -1 & \text{for } C_n \text{ and } a > n. \end{cases}$$
Moreover the following relations reduce the number of independent $q_{ab}$ parameters [7]:

\[ q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}; \quad q_{ab} = \frac{r^2}{q_{ab}} = q_{a'b'} \]

(2.6) where (2.7) also implies $q_{aa'} = r$. Therefore the $q_{ab}$ with $a < b \leq \frac{N}{2}$ give all the $q$'s.

It is useful to list the nonzero complex components of the $R$ matrix (no sum on repeated indices):

\[
R^{ab}_{aa} = r, \quad a \neq n_2 \\
R^{ab}_{aa'} = r^{-1}, \quad a \neq n_2 \\
R^{ab}_{n_2 n_2} = 1 \\
R^{ab}_{ab} = \frac{r}{q_{ab}}, \quad a \neq b, a' \neq b \\
R^{ab}_{ba} = r - r^{-1}, \quad a > b, a' \neq b \\
R^{ab}_{a'a'} = (r - r^{-1})(1 - \epsilon a^{b a - p_{a'}}), \quad a > a' \\
R^{ab}_{b'b'} = -(r - r^{-1})\epsilon a^{b b - p_{b'}}, \quad a > b, a' \neq b
\]

(2.8)

where $\epsilon = \epsilon_a \epsilon_{a'}$, i.e. $\epsilon = 1$ for $B_n, D_n$ and $\epsilon = -1$ for $C_n$.

**Remark 2.1**: The matrix $R$ is upper triangular, that is $R^{ab}_{cd} = 0$ if $[a = c$ and $b < d]$ or $a < c$, and has the following properties:

\[ R_{q,r}^{-1} = R_{q^{-1},r^{-1}}; \quad (R_{q,r})^{ab}_{cd} = (R_{q,r})^{ca'd'a'c'b'}; \quad (R_{q,r})^{ab}_{cd} = (R_{p,r})^{dc}_{pa} \]

(2.9)

where $q, r$ denote the set of parameters $q_{ab}, r$, and $p_{ab} \equiv q_{ba}$. The inverse $R^{-1}$ is defined by $(R^{-1})^{ab}_{cd}R^{cd}_{ef} = \delta^a_e \delta^b_f = R^{ab}_{cd}(R^{-1})^{cd}_{ef}$. Eq. (2.9) implies that for $|q| = |r| = 1$, $R = R^{-1}$.

**Remark 2.2**: Let $R_r$ be the uniparametric $R$ matrix for the $B, C, D$ q-groups. The multiparametric $R_{q,r}$ matrix is obtained from $R_r$ via the transformation [8, 7] 

\[ R_{q,r} = F^{-1}R_r F^{-1} \]

(2.10)

where $(F^{-1})^{ab}_{cd}$ is a diagonal matrix in the index couples $ab, cd$:

\[ F^{-1} \equiv diag(\sqrt{\frac{r}{q_{11}}}, \sqrt{\frac{r}{q_{12}}}, ... \sqrt{\frac{r}{q_{NN}}}) \]

(2.11)
and \(ab, cd\) are ordered as in the \(R\) matrix. Since \(\sqrt{q_{ab}} = (\sqrt{q_{ba}})^{-1}\) and \(q_{aa'} = q_{bb'}\), the non diagonal elements of \(R_{q,r}\) coincide with those of \(R_r\). The matrix \(F\) satisfies \(F_{12}F_{21} = 1\) i.e. \(F^{ab}_{ef}F^{fe}_{de} = \delta^{a}_{c}\delta^{b}_{d}\), the quantum Yang-Baxter equation \(F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}\) and the relations \((R_r)_{12}F_{13}F_{23} = F_{23}F_{13}(R_r)_{12}\). Note that for \(r = 1\) the multiparametric \(R\) matrix reduces to \(R = F^{-2}\).

**Remark 2.3:** Let \(\hat{R}\) the matrix defined by \(\hat{R}_{ab}^{cd} \equiv R_{ba}^{cd}\). Then the multiparametric \(\hat{R}_{q,r}\) is obtained from \(\hat{R}_r\) via the similarity transformation

\[
\hat{R}_{q,r} = F\hat{R}_r F^{-1} \tag{2.12}
\]

The characteristic equation and the projector decomposition of \(\hat{R}_{q,r}\) are therefore the same as in the uniparametric case:

\[
(\hat{R} - rI)(\hat{R} + r^{-1}I)(\hat{R} - \epsilon r^{\epsilon N}I) = 0 \tag{2.13}
\]

\[
\hat{R} - \hat{R}^{-1} = (r - r^{-1})(I - K) \tag{2.14}
\]

\[
\hat{R} = rP_S - r^{-1}P_A + \epsilon r^{\epsilon N}P_0 \tag{2.15}
\]

with

\[
P_S = \frac{1}{r + r^{-1}}[\hat{R} + r^{-1}I - (r^{-1} + \epsilon r^{\epsilon N})P_0]
\]

\[
P_A = \frac{1}{r + r^{-1}}[-\hat{R} + rI - (r - \epsilon r^{\epsilon N})P_0]
\]

\[
P_0 = Q_N(r)K
\]

\[
Q_N(r) \equiv (C_{ab}C^{ab})^{-1} = \frac{1 - r^2}{(1 - r^{N+1})(1 - r^{N+1} + 1 + \epsilon)} , \quad K^{ab}_{cd} = C^{ab}C_{cd}
\]

\[
I = P_S + P_A + P_0 \tag{2.16}
\]

To prove (2.14) in the multiparametric case note that \(F_{12}K_{12}F_{12}^{-1} = K_{12}\). Orthogonality (and symplecticity) conditions can be imposed on the elements \(T^a_{\ b}\), consistently with the \(RTT\) relations (2.7):

\[
C^{bc}T^a_{\ b}T^d_{\ c} = C^{ad}, \quad C_{ac}T^a_{\ b}T^c_{\ d} = C_{bd}I \tag{2.17}
\]

where the (antidiagonal) metric is :

\[
C_{ab} = \epsilon_a r^{-\rho_a} \delta_{ab'} \tag{2.18}
\]

and its inverse \(C^{ab}\) satisfies \(C^{ab}C_{bc} = \delta^a_c = C_{ab}C^{ba}\). We see that for the orthogonal series, the matrix elements of the metric and the inverse metric coincide, while for the symplectic series there is a change of sign: \(C^{ab} = \epsilon C_{ab}\). Notice also the symmetry \(C_{ab} = C_{ba'}\).

The consistency of (2.17) with the \(RTT\) relations is due to the identities:

\[
C_{ab}\hat{R}^{bc}_{\ de} = (\hat{R}^{-1})^{cf}_{\ ad}C_{fe}, \quad \hat{R}^{bc}_{\ de}C^{ea}_{\ f} = C^{bf}(\hat{R}^{-1})^{ca}_{\ f} \tag{2.19}
\]
These identities hold also for $\hat{R} \rightarrow \hat{R}^{-1}$ and can be proved using the explicit expression (2.8) of $R$.

We note the useful relations, easily deduced from (2.15):

$$C_{ab}\hat{R}_{cd}^{ab} = \epsilon r^{-N} C_{cd}^{ab}, \quad C_{ab}\hat{R}_{cd}^{ab} = \epsilon r^{-N} C_{cd}^{ab} \quad (2.20)$$

The co-structures of the $B_n, C_n, D_n$ multiparametric quantum groups have the same form as in the uniparametric case: the coproduct $\Delta$, the counit $\varepsilon$ and the coinverse $\kappa$ are given by

$$\Delta(T^a_b) = T^a_b \otimes T^b_c \quad (2.21)$$
$$\varepsilon(T^a_b) = \delta^a_b \quad (2.22)$$
$$\kappa(T^a_b) = C^{ac} T^c_d C_{db} \quad (2.23)$$

Four conjugations (i.e. algebra antihomomorphism, coalgebra homomorphism and involution, satisfying $\kappa(\kappa(T^a_b)^*) = T$) can be defined, but only two of these can be extended to the corresponding inhomogeneous groups. These two are defined as follows:

- trivially as $T^a_b = T^a_b$. Compatibility with the $RTT$ relations (2.1) requires $\bar{R}_{q,r} = R_{q,r}^{-1} = R_{q-1,r-1}$, i.e. $|q| = |r| = 1$. Then the $CTT$ relations are invariant under $*$-conjugation. The corresponding real forms are $\text{SO}_{q,r}(n, n; \mathbf{R})$, $\text{SO}_{q,r}(n, n+1; \mathbf{R})$ (for $N$ even and odd respectively) and $\text{Sp}_{q,r}(n; \mathbf{R})$.

- on the orthogonal quantum groups $\text{SO}_{q,r}(2n, \mathbb{C})$, extending to the multiparametric case the one proposed by the authors of ref. [10] for $\text{SO}_q(2n, \mathbb{C})$. The conjugation is defined by:

$$\big(\bar{T}^a_b\big)^* = \mathcal{D}^a_c T^c_d \mathcal{D}^d_b \quad (2.24)$$

$\mathcal{D}$ being the matrix that exchanges the index $n$ with the index $n + 1$. This conjugation is compatible with the coproduct: $\Delta\big((T^a_b)^*\big) = (\Delta T^a_b)^*$; for $|r| = 1$ it is also compatible with the orthogonality relations (2.17) (due to $\bar{C} = C^T$ and also $\mathcal{D} C \mathcal{D}^T = C$) and with the antipode: $\kappa(\kappa(T^a_b)^*) = T$. Compatibility with the $RTT$ relations is easily seen to require

$$(\bar{R})_{n+n+1} = R^{-1}, \quad \text{i.e.} \quad \mathcal{D}_1 \mathcal{D}_2 R_{12} \mathcal{D}_1 \mathcal{D}_2 = \bar{R}_{12}^{-1} \quad (2.25)$$

which implies

i) $|q_{ab}| = |r| = 1$ for $a$ and $b$ both different from $n$ or $n + 1$;

ii) $q_{ab}/r \in \mathbb{R}$ when at least one of the indices $a, b$ is equal to $n$ or $n + 1$.

This conjugation leads to the real form $\text{SO}_{q,r}(n+1, n-1; \mathbf{R})$, and is in fact the one needed to obtain $\text{ISO}_{q,r}(3, 1; \mathbf{R})$, as discussed in refs. [3, 4] and later in this paper.

Remark 2.4: Using formula (2.3) or (2.8), we find that the $R^{AB}_{CD}$ matrix for the $\text{SO}_{q,r}(N+2)$ and $\text{Sp}_{q,r}(N+2)$ quantum groups can be decomposed in terms of
$SO_{q,r}(N)$ and $Sp_{q,r}(N)$ quantities as follows (splitting the index A as A=($\circ$, $a$, ●), with $a = 1, \ldots, N$):

$$R_{CD}^{AB} = \begin{pmatrix}
\circ \circ & \circ \bullet & \bullet \circ & \bullet \bullet & \circ d & \bullet d & c\circ & c\bullet & c d \\
\circ r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\circ 0 & r^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet 0 & f(r) & r^{-1} & 0 & 0 & 0 & 0 & 0 & -\epsilon C_{cd}\lambda r^{-\rho} \\
\bullet 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet b & 0 & 0 & 0 & 0 & \frac{r}{q_{ob}}\delta_{d}^{b} & 0 & 0 & 0 & 0 \\
a\circ & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{ao}}\delta_{c}^{a} & 0 & 0 \\
a\bullet & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{ae}}\delta_{e}^{a} & 0 \\
ab & 0 & -C_{ba}\lambda r^{-\rho} & 0 & 0 & 0 & 0 & 0 & R_{cd}^{ab} & 0
\end{pmatrix}$$

(2.26)

where $R_{cd}^{ab}$ is the $R$ matrix for $SO_{q,r}(N)$ or $Sp_{q,r}(N)$, $C_{ab}$ is the corresponding metric, $\lambda \equiv r - r^{-1}$, $\rho = \frac{N+1-\epsilon}{2}$ ($r^{\rho} = C_{\bullet \circ}$) and $f(r) \equiv \lambda(1 - \epsilon r^{-2\rho})$. The sign $\epsilon$ has been defined after eq. s (2.8).

### 3 The quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$

An $R$-matrix formulation for the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ was obtained in ref. [1], in terms of the $R_{CD}^{AB}$ matrix for the $SO_{q,r}(N+2)$ and $Sp_{q,r}(N+2)$ quantum groups. It was found that the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are freely generated by the non-commuting matrix elements $T_{AB}^{A} [A=($$\circ$, $a$, ●), with $a = 1, \ldots, N$)] and the identity $I$, modulo the relations:

$$T_{\circ}^{a} = T_{\bullet}^{b} = T^{\bullet \circ} = 0, \quad (3.1)$$

the RTT relations

$$R_{EF}^{AB} T_{C}^{E} T_{D}^{F} = T_{D}^{B} T_{E}^{A} R_{CD}^{EF}, \quad (3.2)$$

and the orthogonality (symplecticity) relations

$$C^{BC} T_{B}^{A} T_{C}^{D} = C^{AD}, \quad C^{AC} T_{B}^{A} T_{D}^{C} = C_{BD} \quad (3.3)$$

The co-structures of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are simply given by:

$$\Delta(T_{B}^{A}) = T_{C}^{A} \otimes T_{B}^{C}, \quad \kappa(T_{B}^{A}) = C^{AC} T_{C}^{D} C_{DB}, \quad \epsilon(T_{B}^{A}) = \delta_{B}^{A}. \quad (3.4)$$

After decomposing the indices A=($\circ$, $a$, ●), and defining:

$$u \equiv T_{\circ}^{\circ}, \quad v \equiv T_{\bullet}^{\bullet}, \quad z \equiv T_{\bullet}^{\circ}, \quad x^{a} \equiv T_{\bullet}^{a}, \quad y_{a} \equiv T_{a}^{\circ} \quad (3.5)$$
the relations (3.2) and (3.3) become

\[
R^{ab}_{\phantom{ab}ef} T^e_c T^f_d = T^b_f T^a_e R^{ef}_{\phantom{ef}cd} \tag{3.6}
\]

\[
T^a_b C^{bc} T^d_c = C^{ad} I \tag{3.7}
\]

\[
T^a_b C_{ac} T^c_d = C_{bd} I \tag{3.8}
\]

\[
T^b_d x^a = \frac{r}{q_{d*}} R^{ab}_{\phantom{ab}ef} x^e T^f_d \tag{3.9}
\]

\[
P^a_{\phantom{a}cd} x^c_d = 0 \tag{3.10}
\]

\[
T^b_d v = \frac{q_{b*}}{q_{d*}} v T^b_d \tag{3.11}
\]

\[
x^b v = q_{b*} v x^b \tag{3.12}
\]

\[
u u = u v = I \tag{3.13}
\]

\[
u x^b = q_{b*} x^b u \tag{3.14}
\]

\[
u T^b_d = \frac{q_{b*}}{q_{d*}} T^b_d u \tag{3.15}
\]

\[
y_b = -r^a T^a_b C_{ac} x^c u \tag{3.16}
\]

\[
z = -\frac{1}{(r-\rho + \epsilon r^{a'-2})} x^b C_{ba} x^a u \tag{3.17}
\]

where \( q_{a*} \) are \( N \) complex parameters related by \( q_{a*} = r^2 / q_{d*} \), with \( a' = N + 1 - a \).

Note that in the symplectic case, \( x^b C_{ba} x^a = 0 \) so that the constraint (3.17) reads \( z = 0 \). The matrix \( P_A \) in eq. (3.10) is the \( q \)-antisymmetrizer for the \( B, C, D \) \( q \)-groups given by (cf. (2.16)):}

\[
P^a_{\phantom{a}cd} = -\frac{1}{r + r^{-1}} (\hat{R}^{ab}_{\phantom{ab}cd} - r \delta^a_c \delta^b_d + \frac{r - r^{-1}}{\epsilon r^{N-1-\epsilon}} C^{ab} C_{cd}). \tag{3.18}
\]

The last two relations (3.16) - (3.17) are constraints, showing that the \( T^A_B \) matrix elements in eq. (3.2) are really a redundant set. This redundancy is necessary if we want to express the \( q \)-commutations of the \( \text{ISO}_{q,r}(N) \) and \( \text{ISp}_{q,r}(N) \) basic group elements as \( RTT = TTR \) (i.e. if we want an \( R \)-matrix formulation). Remark that, in the \( R \)-matrix formulation for \( IGL_{q,r}(N) \), all the \( T^A_B \) are independent [2, 3]. Here we can take as independent generators the elements

\[
T^a_b, x^a, v, u \equiv v^{-1} \text{ and the identity } I \quad (a = 1, \ldots N) \tag{3.19}
\]

The co-structures on the \( \text{ISO}_{q,r}(N) \) generators can be read from (3.4) after decomposing the indices \( A = o, a, \bullet \):

\[
\Delta(T^a_b) = T^a_c \otimes T^c_b, \quad \Delta(x^a) = T^a_c \otimes x^c + x^a \otimes v, \tag{3.20}
\]

\[
\Delta(v) = v \otimes v, \quad \Delta(u) = u \otimes u, \tag{3.21}
\]
It is a Hopf algebra epimorphism because \( H = \text{Ker}(P) \) is a Hopf ideal. Then any element of \( S_{q,r}(N + 2)/H \) is of the form \( P(a) \) and

\[
P(a) + P(b) \equiv P(a + b) ; \quad P(a)P(b) \equiv P(ab) ; \quad \mu P(a) \equiv P(\mu a) , \quad \mu \in \mathbb{C}
\]

\[
\Delta(P(a)) \equiv (P \otimes P)\Delta_{N+2}(a) ; \quad \varepsilon(P(a)) \equiv \varepsilon_{N+2}(a) ; \quad \kappa(P(a)) \equiv P(\kappa_{N+2}(a))
\]

Eq.s (3.26) - (3.17) have been obtained in [4] by taking the \( P \) projection of the \( RTT \) and \( CTT \) relations of \( S_{q,r}(N + 2) \), with the notation \( u \equiv P(T^a_o), \quad v \equiv P(T^*_{\bullet b}), \quad z \equiv P(T^\circ_{\bullet}), \quad x^a \equiv P(T^a_{\bullet}), \quad y_a \equiv P(T^a_o), \quad T^a_b \equiv P(T^a_b) ; \quad I \equiv P(I) ; \quad 0 \equiv P(0) \), cf. (3.3).

Note 3.2: From the commutations (3.14) - (3.15) we see that one can set \( u = I \) only when \( q_{a\bullet} = 1 \) for all \( a \). From \( q_{a\bullet} = r^2/q_{\bullet \bullet} \), cf. eq. (2.7), this implies also \( r = 1 \).

Note 3.3: eq.s (3.10) are the multiparametric (orthogonal or symplectic) quantum plane commutations. They follow from the \((a_{\bullet}, b_{\bullet})\) \( RTT \) components and (3.17).

Finally, the two real forms of \( S_{q,r}(N + 2) \) mentioned in the previous section are inherited by \( IS_{q,r}(N) \), with the following conditions on the parameters:

- \( |q_{ab}| = |q_{a\bullet}| = |r| = 1 \) for \( ISO_{q,r}(n, n; \mathbb{R}) \), \( ISO_{q,r}(n, n+1; \mathbb{R}) \) and \( ISp_{q,r}(n; \mathbb{R}) \).
• For $ISO_{q,r}(n + 1, n - 1; R)$: $|r| = 1$; $|q_{a\bullet}| = 1$ for $a \neq n, n + 1$; $|q_{ab}| = 1$ for $a$ and $b$ both different from $n$ or $n + 1$; $q_{ab}/r \in R$ when at least one of the indices $a, b$ is equal to $n$ or $n + 1$; $q_{a\bullet}/r \in R$ for $a = n$ or $a = n + 1$.

In particular, the quantum Poincaré group $ISO_{q,r}(3, 1; R)$ is obtained by setting $|q_{1\bullet}| = |r| = 1$, $q_{2\bullet}/r \in R$, $q_{12}/r \in R$.

According to Note 3.1, a dilatation-free $q$-Poincaré group is found after the further restrictions $q_{\bullet \bullet} = q_{2\bullet} = r = 1$. The only free parameter remaining is then $q_{12} \in R$.

4 Universal enveloping algebra $U(S_{q,r}(N + 2))$

The universal enveloping algebra of $S_{q,r}(N+2)$, i.e. the algebra of regular functionals $E$ on $S_{q,r}(N + 2)$, is generated by the functionals $L^\pm$, and the counit $\varepsilon$.

The $L^\pm$ linear functionals on $S_{q,r}(N+2)$ are defined by their value on the matrix elements $T^A_B$:

$$L^\pm_B(T^C_D) = (R^\pm)^{AC}_{BD}, \quad L^\pm_B(I) = \delta^A_B \quad (4.1)$$

with

$$(R^+)^{AC}_{BD} \equiv R^{CA}_{DB}, \quad (R^-)^{AC}_{BD} \equiv (R^{-1})^{AC}_{BD}. \quad (4.2)$$

To extend the definition (4.1) to the whole algebra $S_{q,r}(N + 2)$ we set

$$L^\pm_B(ab) = L^\pm_C(a)L^\pm_B(b) \quad \forall a, b \in S_{q,r}(N + 2). \quad (4.3)$$

The commutations between $L^\pm_B$ and $L^\pm_C$ are given by:

$$R_{12}L^\pm_2L^\pm_1 = L^\pm_1L^\pm_2R_{12}, \quad R_{12}L^\pm_2L^\pm_1 = L^\pm_1L^\pm_2R_{12}, \quad (4.4)$$

where as usual the product $L^\pm_2 L^\pm_1$ is the convolution product $L^\pm_2 L^\pm_1 \equiv (L^\pm_2 \otimes L^\pm_1)\Delta$.

Note 4.1: $L^+$ is upper triangular and $L^-$ is lower triangular. Proof: apply $L^+$ and $L^-$ to the $T$ elements and use the upper and lower triangularity of $R^+$ and $R^-$, respectively.

The $L^\pm_B$ elements satisfy orthogonality conditions analogous to (3.3):

$$C^{AB}L^{\pm C}_B L^{\pm D}_A = C^{DC}\varepsilon, \quad C^{AB}L^{\pm B}_C L^{\pm A}_D = C^{DC}\varepsilon \quad (4.5)$$

as can be verified by applying them to the $q$-group generators, and using (2.13). They provide a quantum inverse for $L^\pm_B$:

$$(L^\pm_B)^{-1} = C^{DA}L^{\pm C}_D C_{BC} \quad (4.6)$$

The co-structures of the algebra generated by the functionals $L^\pm$ and $\varepsilon$ are defined by:

$$\Delta'(L^\pm_B)(a \otimes b) \equiv L^\pm_B(ab) = L^\pm_A(a)L^\pm_G_B(b), \quad (4.7)$$
\[ \varepsilon'(L^\pm_A B) \equiv L^\pm_A B(I) ; \quad \kappa'(L^\pm_A B)(a) \equiv L^\pm_A B(\kappa(a)) = (L^\pm_A B)^{-1}(a), \quad (4.8) \]

so that

\[ \Delta'(L^\pm_A B) = L^\pm_A G \otimes L^\pm_C B, \quad (4.9) \]
\[ \varepsilon'(L^\pm_A B) = \delta^A_B ; \quad \kappa'(L^\pm_A B) = (L^\pm_A B)^{-1} = C^{DA} L^\pm_C D C_{BC}. \quad (4.10) \]

The *-conjugation on \( S_{q,r}(N + 2) \) induces a *-conjugation on \( U(S_{q,r}(N + 2)) \) in two possible ways (we denote them as * and \( \sharp \)-conjugations):

\[ \phi^*(a) \equiv \overline{\phi(\kappa(a))} ; \quad \phi^\sharp(a) \equiv \overline{\phi(\kappa^{-1}(a^*))} \quad (4.11) \]

where \( \phi \in U(S_{q,r}(N + 2)) \), \( a \in S_{q,r}(N + 2) \), and the overline denotes the usual complex conjugation. Both * and \( \sharp \) can be shown to satisfy all the properties of Hopf algebra involutions. It is not difficult to determine their action on the basis elements \( L^\pm_A B \). The two \( S_{q,r}(N + 2) \) *-conjugations of the previous section induce respectively the following conjugations on the \( L^\pm_A B \):

\[ (L^\pm_A B)^* = L^\pm_A B \quad (4.12) \]
\[ (L^\pm_A B)^\sharp = \mathcal{D}^D_C L^\pm_C D \mathcal{D}^D_B \quad (4.13) \]

To find \( (L^\pm_A B)^\sharp \) one uses the general formula \( (\phi)^\sharp = \kappa'^2 \left[ (\phi)^* \right] \), deducible from the compatibility of both conjugations with the antipode: \( \kappa'^{-1}(\phi^*) = [\kappa'(\phi)]^* \), \( \kappa'^{-1}(\phi^\sharp) = [\kappa'(\phi)]^\sharp \).

## 5 Bicovariant calculus on \( S_{q,r}(N + 2) \)

The bicovariant differential calculus on the uniparametric \( q \)-groups of the \( A, B, C, D \) series can be formulated in terms of the corresponding \( R \)-matrix, or equivalently in terms of the \( L^\pm \) functionals. This holds also for the multiparametric case. In fact all formulas are the same, modulo substituting the \( q \) parameter with \( r \) when it appears explicitly (typically as \( \frac{1}{q-q^{-1}} \)).

We briefly recall how to construct a bicovariant calculus. The general procedure can be found in ref. [12, 13], or, in the notations we adopt here, in ref. [14]. It realizes the axiomatic construction of ref. [15].

As in the uniparametric case [12], the functionals

\[ f_{A_1 B_1}^{A_2 B_2} = \kappa'(L^{+B_1}_{-A_1}) L^{-A_2}_{B_2}. \quad (5.1) \]

and the elements of \( A = S_{q,r}(N + 2) \):

\[ M_{B_2 A_1}^{B_1 A_2} = T_{B_1 A_1}^{B_2} \kappa(T^{A_2}_{B_2}). \quad (5.2) \]
satisfy the following relations, called bicovariant bimodule conditions, where for simplicity we use the adjoint indices \( i, j, k, \ldots \) with \( i = _B^A, \; i = _A^B \):

\[
\Delta'(f^i_j) = f^i_k \otimes f^k_j \; ; \; \; \varepsilon'(f^i_j) = \delta^i_j, \tag{5.3}
\]

\[
\Delta(M_j^i) = M_l^i \otimes M_l^i \; ; \; \; \varepsilon(M_j^i) = \delta^i_j, \tag{5.4}
\]

\[
M_j^i(a \ast f^i_k) = (f_j^i \ast a) M_i^k \tag{5.5}
\]

The star product between a functional on \( A \) and an element of \( A \) is defined as:

\[
\chi \ast a \equiv (id \otimes \chi)\Delta(a), \; \; a \ast \chi \equiv (\chi \otimes id)\Delta(a), \; \; a \in A, \; \chi \in A' \tag{5.6}
\]

Relation (5.5) is easily checked for \( a = T^A_B \) since in this case it is implied by the RTT relations; it holds for a generic \( a \) because of property (5.3).

The space of quantum one-forms is defined as a right \( A \)-module \( \Gamma \) freely generated by the symbols \( \omega^A_1^2 \):

\[
\Gamma \equiv \{a^A_1 \omega^A_2^1\} \; , \; \; a^A_1 \in A \tag{5.7}
\]

**Theorem 5.1** (due to Woronowicz: see Theorem 2.5 in the last ref. of [15], p. 143): because of the properties (5.3), \( \Gamma \) becomes a bimodule over \( A \) with the following right multiplication:

\[
\omega^A_1^2 a = (f^A_1 B^B_2 \ast a) \omega^B_2 \tag{5.8}
\]

in particular:

\[
\omega^A_1^2 T^R_S = (R^{-1})^T B_1 C^A_1 (R^{-1})^T B_2 S T^R T^R T^B_1 \omega^B_2 \tag{5.9}
\]

Moreover, because of properties (5.4), we can define a left and a right action of \( A \) on \( \Gamma \):

\[
\Delta_L : \Gamma \rightarrow A \otimes \Gamma \; ; \; \; \Delta_L(a \omega^A_1^2 b) \equiv \Delta(a) (I \otimes \omega^A_1^2) \Delta(b), \tag{5.10}
\]

\[
\Delta_R : \Gamma \rightarrow \Gamma \otimes A \; ; \; \; \Delta_R(\omega^A_1^2 b) \equiv \Delta(A) (\omega^B_2 \otimes M^B_2 A^A_1) \Delta(b). \tag{5.11}
\]

These actions commute, i.e. \( (id \otimes \Delta_R)\Delta_L = (\Delta_L \otimes id)\Delta_R \) because of (5.3), and give a bicovariant bimodule structure to \( \Gamma \).

The **exterior derivative** \( d : A \rightarrow \Gamma \) can be defined via the element \( \tau \equiv \sum_A \omega^A_A \in \Gamma \). This element is easily shown to be left and right-invariant:

\[
\Delta_L(\tau) = I \otimes \tau \; ; \; \; \Delta_R(\tau) = \tau \otimes I \tag{5.12}
\]

and defines the derivative \( d \) by

\[
da = \frac{1}{r - r^{-1}}[\tau a - a \tau]. \tag{5.13}
\]

The factor \( \frac{1}{r - r^{-1}} \) is necessary for a correct classical limit \( r \rightarrow 1 \). It is immediate to prove the Leibniz rule

\[
d(ab) = (da)b + a(db), \; \forall a, b \in A. \tag{5.14}
\]
Another expression for the derivative is given by:

\[ da = (\chi_{A_1}^{A_2} \times a) \omega_{A_2} \]  

(5.15)

where the linearly independent elements

\[ \chi_B^A = \frac{1}{r - r^{-1}} [f^A_C B - \delta^A_B \varepsilon] \]  

(5.16)

are the tangent vectors such that the left-invariant vector fields \( \chi^A_B \) are dual to the left-invariant one-forms \( \omega_{A_2}^{A_1} \). The equivalence of (5.13) and (5.15) can be shown by using the rule (5.8) for \( \tau a \) in the right-hand side of (5.13).

Using (5.15) we compute the exterior derivative on the basis elements of \( S_{q,r}(N + 2) \):

\[ d T^A_B = \frac{1}{r - r^{-1}} [(R^{-1})^{CR}_{ET} (R^{-1})^{TE}_{SB} T^A_C - \delta^A_B \omega^S_R \equiv T^A_C X^{CB}_R \omega^S_R \]  

(5.17)

where we have

\[ X^{A_1 B_1}_{A_2 B_2} \equiv \frac{1}{r - r^{-1}} [(R^{-1})^{A_1 B_1}_{ET} (R^{-1})^{TE}_{B_2 A_2} - \delta^B_{B_2} \delta^A_{A_1}] = z K^{A_1 B_1}_{A_2 B_2} - (\hat{R}^{-1})^{A_1 B_1}_{A_2 B_2} \]

(5.18)

with \( z \equiv e^r N^{-1} \), \( K^{A_1 B_1}_{A_2 B_2} = C^{A_1 B_1}_{A_2 B_2} \). [From (2.14), the second equality in (5.18) is easily proven.] Every element \( \rho \) of \( \Gamma \), which by definition is written in a unique way as \( \rho = A_1^{A_2} A_2 \omega^{A_2} \), can also be written as

\[ \rho = \sum_k a_k d b_k \]  

(5.19)

for some \( a_k, b_k \) belonging to \( A \). This can be proven directly by inverting the relation (5.17). The result is an expression of the \( \omega \) in terms of a linear combination of \( \kappa(T) d T \), as in the classical case:

\[ \omega_{A_1}^{A_2} = Y_{A_1 B_1}^{A_2 B_2} \kappa(T^{B_1}_C) d T^{C}_{B_2} \]  

(5.20)

where \( Y \) satisfies \( X^{A_1 B_1}_{A_2 B_2} Y_{B_1 C_1}^{B_2 C_2} = \delta^A_{C_1} \delta^B_{C_2} \), \( Y_{A_1 B_1}^{A_2 B_2} X_{B_1 C_1}^{B_2 C_2} = \delta^A_{C_1} \delta^A_{C_2} \) and is given explicitly by

\[ Y_{A_1 B_1}^{A_2 B_2} = \alpha [(z - \lambda) C_{A_1 B_1} C_{A_2 B_2} + C_{A_1 D} R^{D A_2}_{C B_1} C^{C B_2} - \frac{\lambda}{z(z - 1)} D^A_{A_1} (D^{-1})^{B_2}_{B_1}] \]

(5.21)

with \( \alpha = \frac{1}{z(z - 1)} \) and \( D^E_C \equiv C^{E F} C_{C F} \). The \( r = 1 \) limit of (5.17) is discussed in the next section.

Due to the bi-invariance of \( \tau \) the derivative operator \( d \) is compatible with the actions \( \Delta_L \) and \( \Delta_R \):

\[ \Delta_L (adb) = \Delta(a)(id \otimes d) \Delta(b) \quad \Delta_R (adb) = \Delta(a)(d \otimes id) \Delta(b) \]  

(5.22)
these two properties express the fact that \( d \) commutes with the left and right action of the quantum group, as in the classical case.

**Remark 5.1:** The properties (5.14), (5.19) and (5.22) of the exterior derivative (5.15) realize the axioms of a first-order bicovariant differential calculus [15].

The tensor product between elements \( \rho, \rho' \in \Gamma \) is defined to have the properties \( \rho a \otimes \rho' = \rho \otimes a \rho' \), \( a(\rho \otimes \rho') = (a\rho) \otimes \rho' \) and \( (\rho \otimes \rho')a = \rho \otimes (\rho'a) \). Left and right actions on \( \Gamma \otimes \Gamma \) are defined by:

\[
\Delta_L(\rho \otimes \rho') \equiv \rho_1 \rho' \otimes \rho_2 \otimes \rho'_2, \quad \Delta_L : \Gamma \otimes \Gamma \rightarrow A \otimes \Gamma \otimes \Gamma \tag{5.23}
\]

\[
\Delta_R(\rho \otimes \rho') \equiv \rho_1 \otimes \rho'_1 \otimes \rho_2 \rho'_2, \quad \Delta_R : \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma \otimes A \tag{5.24}
\]

where \( \rho_1, \rho_2, \) etc., are defined by:

\[
\Delta_L(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in A, \rho_2 \in \Gamma; \quad \Delta_R(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in \Gamma, \rho_2 \in A.
\]

The extension to \( \Gamma^{\otimes n} \) is straightforward.

The exterior product of one-forms is consistently defined as:

\[
\omega_{A_1}^{A_2} \wedge \omega_{D_1}^{D_2} \equiv \omega_{A_1}^{A_2} \otimes \omega_{D_1}^{D_2} - \Lambda_{A_1 A_2}^{A_2 D_2} |_{C_1 D_1} C_1 B_1 \omega_{C_1}^{C_2} \otimes \omega_{B_1}^{B_2} \tag{5.25}
\]

where the \( \Lambda \) tensor is given by:

\[
\Lambda_{A_1 A_2}^{A_2 D_2} |_{C_1 D_1} C_1 B_1 = \int_{A_1 A_2 B_1} M_{C_1 D_1}^{C_2 D_2} = \\
= d^{f_2} c^{f_2} (R^{f_2} B_1 G_1 | R^{-1} C_1 G_1 E_1 A_1 | R^{-1}) A_2 E_1 G_2 D_1 R^{G_2 D_2} B_2 F_2 \tag{5.26}
\]

This matrix satisfies the characteristic equation:

\[
(\Lambda + r^2 I) (\Lambda + r^{-2} I) (\Lambda + e r^{1-N} I) (\Lambda + e^{-r^{-1}+1} N I) \times \\
(\Lambda - e^{-r^{-1}+1} N I) (\Lambda - e^{r^{-1}+1} N I) (\Lambda - I) = 0 \tag{5.27}
\]

due to the characteristic equation (2.13). For simplicity we will at times use the adjoint indices \( i, j, k, \ldots \) with \( i = A_B, \ j = A_B \). Define

\[
(P_I, P_J)^{a_2 d_2}_{a_1 d_1} b_1 \equiv d^{f_2} c^{f_2} (\hat{R}^{b_2 f_2} c_{g_1}^{g_1} (P_I)^{c_{g_1}^{g_1} a_2 e_1}_{a_1 e_1} (\hat{R}^{-1}) a_2 e_1 d_1 g_2 (P_J)^{d_2 g_2}_{b_2 f_2} \tag{5.28}
\]

where \( P_I = P_S, P_A, P_0 \) are given in (2.16). The \( (P_I, P_J) \) are themselves projectors, i.e.:

\[
(P_I, P_J) (P_K, P_L) = \delta_{IK} \delta_{JL} (P_I, P_J) \tag{5.29}
\]

Moreover

\[
(I, I) = I \tag{5.30}
\]

From (5.25) we find

\[
\omega^i \wedge \omega^j = -Z^{ij}_{k\ell} \omega^k \wedge \omega^\ell \tag{5.31}
\]
with
\[ Z = (P_S, P_S) + (P_A, P_A) + (P_0, P_0) - I \]  
see ref. [16]. The inverse of Λ always exists, and is given by

\[ (\Lambda^{-1})_{A_1 A_2}^{D_1 D_2} |_{B_1 B_2} \mathcal{C}_1 \mathcal{C}_2 = f_{D_1 D_2}^{B_1 B_2} (T^{A_2 \mathcal{C}_2} \mathcal{C}_1) = R_i^{B_1 B_2 C_1} (R_i^{A_2 \mathcal{C}_2}) \mathcal{C}_1 \mathcal{C}_2 (d^{-1})_i \mathcal{C}_1 \mathcal{C}_2 (d_1) \]  
(5.33)

Note that for \( r = 1 \), \( \Lambda^2 = I \) and \( (\Lambda + I)(\Lambda - I) = 0 \) replaces the seventh-order spectral equation (5.20). In this special case, one finds the simple formula:

\[ \omega^i \wedge \omega^j = -\Lambda^{i j} \wedge \omega^j \]  
i.e. \( Z = \Lambda \).  
(5.34)

Using the exterior product we can define the **exterior differential** on \( \Gamma \):

\[ d : \Gamma \to \Gamma \wedge \Gamma ; \quad d(a_k d b_k) = d a_k \wedge d b_k \]  
(5.35)

which can easily be extended to \( \Gamma^{\wedge n} \) \((d : \Gamma^{\wedge n} \to \Gamma^{\wedge(n+1)} \) \( \Gamma^{\wedge n} \) being defined as in the classical case but with the quantum permutation (braid) operator \( \Lambda \) [15]). The definition (5.35) is equivalent to the following:

\[ d\hat{\theta} = \frac{1}{r - r^{-1}} [\tau \wedge \theta - (-1)^k \theta \wedge \tau], \]  
(5.36)

where \( \theta \in \Gamma^{\wedge k} \). The exterior differential has the following properties:

\[ d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta' \quad ; \quad d(d\theta) = 0 , \]  
(5.37)

\[ \Delta_L (\theta d\theta') = \Delta_L (\theta) (d \otimes d) \Delta_L (\theta') \quad ; \quad \Delta_R (\theta d\theta') = \Delta_R (\theta) (d \otimes id) \Delta_R (\theta') , \]  
(5.38)

where \( \theta \in \Gamma^{\wedge k} \), \( \theta' \in \Gamma^{\wedge n} \).

The **q-Cartan-Maurer equations** are found by using (5.36) in computing \( d \omega^{C_2}_{C_1} \):

\[ d\omega^{C_2}_{C_1} = \frac{1}{r - r^{-1}} (\omega^B_B \wedge \omega^{C_2}_{C_1} + \omega^{C_1}_{C_1} \wedge \omega^B_B ) \equiv -\frac{1}{2} \mathcal{C}_1^{A_1 A_2 B_2} | C_1 \mathcal{C}_2 \mathcal{C}_2 \wedge \omega^B_B \]  
(5.39)

with:

\[ \mathcal{C}_1^{A_1 A_2 B_2} | C_1 \mathcal{C}_2 = -\frac{2}{(r - r^{-1})} \{ Z^{C_2}_{B_2} C_1 | A_1 A_2 B_1 + \delta^{A_1}_{C_1} \delta^{C_2}_{A_2} \delta^{B_1}_{B_2} \} \]  
(5.40)

To derive this formula we have used the flip operator \( Z \) on \( \omega^B_B \wedge \omega^{C_2}_{C_1} \).

Finally, we recall that the \( \chi \) operators close on the **q-Lie algebra**:

\[ \chi_i \chi_j - \Lambda^{kl}_{ij} \chi_k \chi_l = \mathcal{C}^{k}_{ij} \chi_k \]
(5.41)
where the $q$-structure constants are given by

$$C_{jk}^i = \chi_k(M_j^i)$$ explicitly: \[ C_{A1}^{A2} \frac{B_1}{B_2} | C_2 = \frac{1}{r - r^{-1}} [-\delta_{B2}^{B1} \delta_{A1}^{A2} + \Lambda \frac{B}{B} \frac{C_1}{C_2} | A_1 A_2 B_1 B_2]. \] (5.42)

The $C$ structure constants appearing in the Cartan-Maurer equations are in general related to the $C$ constants of the $q$-Lie algebra [13]:

$$C_{jk}^i = \frac{1}{2} [C_{jk}^i - \Lambda_{rs}^j C_{rs}^i].$$ (5.43)

In the particular case $\Lambda^2 = I$ (i.e. for $r = 1$) it is not difficult to see that in fact $C = C$; and that the $q$-structure constants are $\Lambda$-antisymmetric:

$$C_{jk}^i = -\Lambda_{rs}^j C_{rs}^i.$$ (5.44)

The $\chi$ and $f$ operators close on the algebra (5.41) and

$$\Lambda_{ij} f^i_p f^j_q = f^m_p f^m_q \Lambda_{ij}^{pq},$$ (5.45)

$$C_{mn} f^m_j f^n_k + f^i_j \chi_k = \Lambda_{pq}^{jk} f^i_q + C_{jk}^i f^i_l,$$ (5.46)

$$\chi_k f^n_l = \Lambda^{ij}_{kl} f^n_i \chi_j,$$ (5.47)

This algebra is sufficient to define a bicovariant differential calculus on $A$ (see e.g. [17]), and will be called “bicovariant algebra” in the sequel. By applying the relations defining the bicovariant algebra to the element $M_r^s$ we can express them in the adjoint representation:

$$C_{ri}^n C_{nj}^s = \Lambda^{kl}_{ij} C_{rk}^n C_{nl}^s = C_{ij}^k C_{rk}^s \quad (q\text{-Jacobi identities})$$ (5.48)

$$\Lambda_{ij}^{nk} \Lambda_{rp}^{lk} \Lambda_{is}^{pq} = \Lambda_{ij}^{nk} \Lambda_{rl}^{ms} \Lambda_{kj}^{ij} \Lambda_{pq}^{xl} \quad (Yang-Baxter)$$ (5.49)

$$C_{mn} f^m_j f^n_k + \Lambda^{il}_{ij} C_{lk}^s = \Lambda_{pq}^{jk} \Lambda^{is}_{il} l_q C_{rp}^l + C_{jk}^m \Lambda^{is}_{rl} C_{ni}^m$$ (5.50)

Using the definitions (5.16) and (5.1) it is not difficult to find the co-structures on the functionals $\chi$ and $f$:

$$\Delta'(\chi_i) = \chi_j \otimes f^j_i + \varepsilon \otimes \chi_i; \quad \Delta'(f^i_j) = f^i_k \otimes f^k_j,$$ (5.52)

$$\varepsilon'(\chi_i) = 0; \quad \varepsilon'(f^i_j) = \delta^i_j,$$

$$\kappa'(\chi_i) = -\chi_j \kappa'(f^j_i); \quad \kappa'(f^k_j) f^j_i = \delta^k_j \varepsilon = f^k_j \kappa'(f^j_i).$$

Note that in the $r, q \to 1$ limit $f^i_j \rightarrow \delta^i_j \varepsilon$, i.e. $f^i_j$ becomes proportional to the identity functional and formula (5.8), becomes trivial, e.g. $\omega^1 a = a \omega^1$ [use $\varepsilon* a = a$].

**Note 5.1:** The formulae characterizing the bicovariant calculus have been written in the basis $\{\chi_{AB}^A\}, \{\omega_C^D\}$ because of the particularly simple expression of the $f_{A}^{BC} D$
and $\chi^A_B$ functionals in terms of $L^{\pm A}_B$, see (5.1) and (5.10). Obviously the calculus is independent from the basis chosen. If we consider the linear transformation

$$\omega^i \rightarrow \omega'^i = X^i_{\ j} \omega^j$$

(where we use adjoint indices $i = A_i^2$, $j = B_1^1$), from the exterior differential

$$da = (\chi_i \ast a) \omega^i = (\chi'_i \ast a) \omega'^i$$

we find

$$\chi_i \rightarrow \chi'_i = \chi_j(X^{-1})^j_i,$$

and from the coproduct rule (5.52) of the $\chi_i$ we find $f^i_{\ j} \rightarrow f'^i_{\ j} = X^i_{\ l} f^l_{\ m} (X^{-1})^m_{\ j}$; while from (5.11) we have $M^i_{\ j} \rightarrow M'^i_{\ j} = (X^{-1})^i_{\ l} M^l_{\ m} X^m_{\ j}$.

A useful change of basis is obtained via the following transformation:

$$\omega^{A_1}_{A_2} \rightarrow \psi^{A_1}_{A_2} = X^{A_1 B_1}_{A_2 B_2} \omega^{B_1}_{B_2}$$

$$\chi^{A_1}_{A_2} \rightarrow \psi^{A_1}_{A_2} = \chi^{B_1}_{B_2} \chi^{A_2}_{A_1}$$

where $X$ and its (second) inverse $Y$ are defined in (5.18) and (5.21). Using (5.20) it is immediate to see that

$$\psi^{A_1}_{A_2} = \kappa(T^{A_1}_{C}) dT^{C}_{A_2} .$$

We also have:

$$\psi^{A_1}_{A_2} (T^{B_1}_{B_2}) = \psi^{A_1}_{A_2} (\tilde{T}^{B_1}_{B_2}) = \delta^{B_1}_{A_1} \delta^{A_2}_{B_2} \quad \text{where} \quad \tilde{T}^{B_1}_{B_2} \equiv T^{B_1}_{B_2} - \delta^{B_1}_{B_2} I .$$

Formula (5.56) follows from $\psi^{A_2}_{A_1} (I) = 0$ and:

$$\psi^{A_1}_{A_2} = \kappa(T^{A_1}_{C}) dT^{C}_{A_2} = \kappa(T^{A_1}_{C}) (\psi^{B_2}_{A_1} * T^{C}_{A_2}) \psi^{B_1}_{B_2}$$

$$\psi^{A_1}_{A_2} = \kappa(T^{A_1}_{C}) T^{C}_{D} \psi^{B_2}_{B_1} (T^{D}_{A_2}) \psi^{B_1}_{B_2} = \psi^{B_2}_{B_1} (T^{A_1}_{A_2}) \psi^{B_1}_{B_2} .$$

The analogue of the coordinates $\tilde{T}^{B_1}_{B_2}$ in the old basis is given by

$$x^{B_2}_{B_1} \equiv Y^{C_1}_{B_1 C_2} T^{C_1}_{C_2}, \quad \chi^{A_1}_{A_2} (x^{B_2}_{B_1}) = \delta^{A_1}_{B_1} \delta^{B_2}_{A_2} .$$

Compatibility of the conjugation defined in (4.11) with the differential calculus requires $(\chi_i)^* \ast \omega^i$ to be a linear combination of $(\kappa^*)^{-2}(\chi_i)$, or $(\chi_i)^* \ast \omega^i$ to be a linear combination of the $\chi_i$. This follows from Theorem 1.10 (Woronowicz) of last ref. in [15], and from equations (4.11) with $\phi = \chi$.

---

1We recall from [15] (13) that the quantum group elements (coordinates) $x^i$ such that

$$x^i \in \text{Ker} \varepsilon \quad \text{and} \quad \chi_i(x^j) = \delta^1_i$$

are uniquely defined by these two conditions. Notice, by the way, that $f^i_{\ j}(a) = \chi_j(x^i a)$.
From the definitions (5.1), (5.16), it is straightforward to find how the ∗ and ♦ conjugations act on the tangent vectors χ. Both the conjugations (4.12) and (4.13) satisfy the criteria given above for its compatibility with the differential calculus. Indeed the conjugation (4.13) yields [use (5.16), (5.1), (4.10), (4.4 )]:

\[
(\chi^A_B)^* = -\epsilon r^{-1-N} \chi^C_D D^F_B D^A_G R^{DG}_{FE} D^E_C
\]

(5.59)

with \( D^E_C \equiv C^E F C^C F \). To find \( \chi^* \) corresponding to (4.12) just take \( D^A_B = \delta^A_B \) in (5.59). The criterion given above for the compatibility with the differential calculus is fulfilled since \((\kappa')^{-2}(\chi_i)\) is a linear combination of \( \chi_i \):

\[
\kappa'^2(\chi^A_B) = (D^{-1})^A_C \chi^C_D D^B_B
\]

(5.60)

as can be seen from (5.16) and \( \kappa'^2(L^\pm A_B) = (D^{-1})^A_C L^\pm C_D D^B_B \), cf. (4.10).

Using the inversion formulae (5.20) one finds the induced conjugation on the left-invariant one-forms:

\[
(\omega^A_{B_2})^* = -\mathcal{D}^F_{B_2} \mathcal{D}^{B_2}_G C^{MG}_{AC} (Y_{A_2 B_1}^{A_2 B_2}) X^{AC_1}_{MC_2} \omega^C_{C_2}.
\]

(5.61)

6 Differential calculus on \( S_{q,r=1}(N+2) \)

As discussed in Section 3, we have obtained the quantum inhomogeneous groups \( IS_{q,r}(N) \) via the projection

\[
P : \quad S_{q,r}(N+2) \longrightarrow \frac{S_{q,r}(N+2)}{H} = IS_{q,r}(N)
\]

(6.1)

with \( H=\text{Hopf ideal in } S_{q,r}(N+2) \) defined in Section 3. As a consequence, the universal enveloping algebra \( U(IS_{q,r}(N)) \) is a Hopf subalgebra of \( U(S_{q,r}(N+2)) \), and contains all the functionals that annihilate \( H = \text{Ker}(P) \).

Let us consider now the \( \chi \) functionals in the differential calculus on \( S_{q,r}(N+2) \). Decomposing the indices we find:

\[
\chi^a_b = \frac{1}{r - r^{-1}} [f^{\alpha}_{\alpha} b - \delta^a_b \epsilon] + \frac{1}{r - r^{-1}} f^\bullet\bullet_a_b
\]

(6.2)

\[
\chi^a_\circ = \frac{1}{r - r^{-1}} f^{\alpha}_{\alpha} \circ
\]

(6.3)

\[
\chi^\circ_b = + \frac{1}{r - r^{-1}} [f^{\circ}_{\circ} b + f^\bullet\circ_b]
\]

(6.4)

\[
\chi^\circ = + \frac{1}{r - r^{-1}} f^\bullet\circ
\]

(6.5)

\[
\chi^\bullet_b = \frac{1}{r - r^{-1}} f^\bullet\bullet b
\]

(6.6)

\[
\chi^\circ_\circ = \frac{1}{r - r^{-1}} [f^{\circ}_{\circ} \circ - \epsilon] + \frac{1}{r - r^{-1}} [f^{\circ}_{\circ} + f^\bullet\circ_\circ]
\]

(6.7)
where we have indicated the terms that do and do not annihilate the Hopf ideal \( H \). We see that only the functionals \( \chi_b^\bullet \), \( \chi_\circ^\bullet \) and \( \chi^\bullet \) do annihilate \( H \), and therefore belong to \( U(IS_{q,r}(N)) \). The resulting bicovariant differential calculus [18] contains dilatations and translations, but does not contain the tangent vectors of \( S_{q,r}(N) \), i.e. the functionals \( \chi_a^\bullet \). Indeed these contain \( f_{\bullet}^\bullet a \), in general not vanishing on \( H \). We can, however, try to find restrictions on the parameters \( q, r \) such that \( f_{\bullet}^\bullet a(H) = 0 \).

As we will see, this happens for \( r = 1 \). For this reason we consider in the following the particular multiparametric deformations called “minimal deformations” or twistings, corresponding to \( r = 1 \).

We first examine what happens to the bicovariant calculus on \( S_{q,r}(N + 2) \) in the \( r = 1 \) limit. The \( R \) matrix is given by, cf. (2.8):

\[
\begin{align*}
R_{AB}^{AB} &= q_{AB}^{-1} + O(\lambda) \\
R_{BA}^{AB} &= \lambda \quad A > B, A' \neq B \\
R_{AA'}^{AA'} &= \lambda (1 - \epsilon \rho_A - \rho_{A'}) \quad A > A' \\
R_{BB'}^{AA'} &= -\lambda \epsilon \rho_B + O(\lambda^2) \quad A > B, A' \neq B
\end{align*}
\]

where \( O(\lambda^n) \) indicates an infinitesimal of order \( \geq \lambda^n \); the \( q_{AB} \) parameters satisfy:

\[
q_{AB} = q_{AB}^{-1} = q_{A'B}^{-1} = q_{BA}^{-1} \quad q_{AA} = q_{AA'} = 1
\]

up to order \( O(\lambda) \). Note that the components \( R_{AA'}^{AA'} \) are of order \( O(\lambda^2) \) for the orthogonal case \( (\epsilon = 1) \) and of order \( O(\lambda) \) for the symplectic case \( (\epsilon = -1) \). The \( RTT \) relations simply become:

\[
T_{A_2}^{B_1} T_{A_3}^{B_2} = \frac{q_{B_1 B_2} T_{A_3}^{B_1}}{q_{A_2 A_3}} T_{A_1}^{B_2}. \quad (6.16)
\]

For \( r = 1 \) the metric is \( C_{AB} = \epsilon_A \delta_{AB'} \) and therefore we have \( C_{AB} = \epsilon C_{BA} \). Using the definition (1.1), it is easy to see that

\[
L_{-A}^A (T_{-D}^C) = \delta_{D}^C q_{AC} + O(\lambda) \quad (6.17)
\]

\[2\] By \( \lim_{r \to 1} a \), where the generic element \( a \in S_{q,r}(N + 2) \) is a polynomial in the matrix elements \( T_{A}^{B} \) with complex coefficients \( f(r) \) depending on \( r \), we understand the element of \( S_{q,r=1}(N + 2) \) with coefficients given by \( \lim_{r \to 1} f(r) \). The expression \( \lim_{r \to 1} \phi = \varphi \), where \( \phi \in U(S_{q,r}(N + 2)) \) and \( \varphi \in U(S_{q,r=1}(N + 2)) \) means that \( \lim_{r \to 1} \phi(a) = \varphi(\lim_{r \to 1} a) \) for any \( a \in S_{q,r}(N + 2) \) such that \( \lim_{r \to 1} a \) exists. Finally, the left invariant one-forms \( \omega^i \) are symbols, and therefore \( \lim_{r \to 1} a_i \omega^i \equiv (\lim_{r \to 1} a_i) \omega^i \) [see (5.4)].
Moreover, the following relations hold:

\[ \Delta(\pm L_A \otimes T_B) = \pm \lambda \quad A \neq B, A' \neq B; A < B \text{ for } L^+, A > B \text{ for } L^- \]

\[ L^\pm_{A'}(T_A') = \pm \lambda [1 - e^{\pm (\rho_A - \rho_{A'})}] \quad A < A' \text{ for } L^+, A > A' \text{ for } L^- \]

\[ L^\pm_{AB}(T^{A'}_{B'}) = \mp \lambda \epsilon_{ABC} + O(\lambda^2) \quad A \neq B, A' \neq B; A < B \text{ for } L^+, A > B \text{ for } L^- \]

all other \(L^\pm(T)\) vanishing. Relations (6.18) and (6.20) imply that for any generator \(T^D_C\) we have \(L^\pm_{AB}(T^D_C) = -\epsilon_{ABC}L^\pm_{A'}(T^D_{C'}) + O(\lambda^2)\) with \(A \neq B, A \neq B'\).

In general, since

\[ \Delta(\pm A) = \Delta(\pm A) \otimes \Delta(\pm A) \]

\[ \Delta(\pm B) = \Delta(\pm B) + \Delta(\pm A) \otimes L^\pm_{AB} + O(\lambda^2), \quad A \neq B \]

we find that

\[ L^\pm_{A} = O(1) \]

\[ L^\pm_{A} = O(\lambda), \quad A \neq B, A \neq B' \]

\[ L^\pm_{A'} = O(\lambda^2) \text{ for } SO_q, \quad O(\lambda) \text{ for } Sp_q \]

where, by definition, \(\phi = O(\lambda^n)\) (\(\phi\) being a functional) means that for any element \(a \in S_q, r(N + 2)\) with well-defined classical limit, we have \(\phi(a) = O(\lambda^n)\).

Moreover, the following relations hold:

\[ L^\pm_{A} = L^\pm_{A} + O(\lambda), \quad \kappa(\pm B) = \epsilon_{ABC}L^\pm_{A'} + O(\lambda) \quad \text{and therefore}, \quad \kappa^2 = id + O(\lambda) \]

Similarly, one can prove the relations involving the \(f\) functionals (no sum on repeated indices):

\[ f_{AA}^{ABA} = \epsilon + O(\lambda) \]

\[ f_{AB}^{ABA} = O(1) \quad \text{and} \quad f_{A'B'}^{ABA} = f_{B'B}^{ABA'} + O(\lambda) \]

\[ f_{CA}^{A} = O(\lambda^2), \quad C \neq A \]

\[ f_{C}^{BA} = O(\lambda^2) \quad |A < B, C \neq B| \text{ or } |A > B, C \neq A| \]

[hint: check (6.24)-(6.29) first on the generators, then use the coproduct in (5.52)].

From the last relation we deduce

\[ \chi_{BA}^{A} = \frac{1}{\lambda} f_{A'B'}^{ABA} + O(\lambda), \quad A < B \]

\[ \chi_{BA}^{A} = \frac{1}{\lambda} f_{A'}^{ABA} + O(\lambda), \quad A > B \]

and from (5.20) and (5.28) one has

\[ \chi_{A}^{A} = \frac{1}{\lambda} [f_{A'A'}^{ABA} - \epsilon] \]

Next one can verify that

\[ \chi_{A}^{BA}(T_{BA}) = -q_{BA} + O(\lambda) \]

\[ \chi_{A}^{BA}(T_{BA'}) = \epsilon_{A'B'} + O(\lambda) \quad \left\{ A \neq B, A \neq B' \right\} \]

\[ \chi_{A}^{BA}(T_{CD}) = 0 \quad \text{otherwise} \]

\[ \left\{ A \neq B, A \neq B' \right\} \]
∀ \ T^C_D, \ \chi^A_A(T^C_D) = -\chi^{A'}_A(T^C_D) + O(\lambda). \hspace{1cm} (6.34)

Eq.s (6.33) yield the relation between \( \chi \) functionals:

∀ \ T^C_D, \ \chi^{B'}_{A'}(T^C_D) = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A_B(T^C_D) + O(\lambda), \ A \neq B, A \neq B'. \hspace{1cm} (6.35)

It is not difficult to prove that the coproduct rule in (5.52) is compatible with (6.35) and (6.34) making them valid on arbitrary polynomials in the \( T^{AB} \) elements:

\chi^{B'}_{A'} = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A_B + O(\lambda), \ A \neq B, A \neq B'; \hspace{1cm} \chi^A_A = -\chi^{A'}_A + O(\lambda). \hspace{1cm} (6.36)

Finally:

\chi^A_A = O(\lambda) \text{ for } SO_q, \ O(1) \text{ for } Sp_q, \ A \neq A'. \hspace{1cm} (6.37)

Summarizing, in the \( r \rightarrow 1 \) limit, only the following \( \chi \) functionals survive:

\chi^A_A \equiv \lim_{r \rightarrow 1} \frac{1}{\lambda} [f^A_A - \epsilon] \hspace{1cm} (6.38)

\chi^A_B \equiv \lim_{r \rightarrow 1} \frac{1}{\lambda} f^A_A, \ A > B, A \neq B' \hspace{1cm} (6.39)

\chi^A_B \equiv \lim_{r \rightarrow 1} \frac{1}{\lambda} f^B_A, \ A < B, A \neq B' \hspace{1cm} (6.40)

\chi^{A'}_{A'} \equiv \lim_{r \rightarrow 1} \frac{1}{\lambda} \sum_C f^C_A = 0 \text{ for } SO_q, \neq 0 \text{ for } Sp_q \hspace{1cm} (6.41)

Notice that (6.36) and (6.37) are all contained in the formula:

\chi^{B'}_{A'} = -\frac{\epsilon_A \epsilon_B}{q_{BA}} \chi^A_B + O(\lambda) \hspace{1cm} (6.42)

thus in the \( r \rightarrow 1 \) limit there are \( (N + 2)(N + 1)/2 \) tangent vectors for \( SO_q(N + 2) \) and \( (N + 2)(N + 3)/2 \) tangent vectors for \( Sp_q(N + 2) \), exactly as in the classical case.

The \( r = 1 \) limit of (5.17) reads:

\[ dT^A_B = -\sum_C T^A C q_{CB} (\omega^C_B - \epsilon_B \epsilon_C q_{BC} \omega^C_{B'}) , \hspace{1cm} (6.43) \]

and therefore, for \( r = 1 \), \( \omega \) appears only in the combination

\[ \Omega^B_A \equiv \omega^B_A - \epsilon_A \epsilon_B q_{AB} \omega^{A'}_{B'} , \hspace{1cm} (6.44) \]

Only \( (N + 2)(N + 1)/2 \) \( [(N + 2)(N + 3)/2 \) for \( Sp_q(N + 2) \)] of the \( (N + 2)^2 \) one forms \( \Omega^B_A \) are linearly independent because [compare with (6.42)]:

\[ \Omega^{A'}_{B'} = -\frac{\epsilon_A \epsilon_B}{q_{AB}} \Omega^B_A . \hspace{1cm} (6.45) \]
In the sequel, instead of considering the left module of one-forms freely generated by $\omega_A^B$, we consider the submodule $\Gamma$ freely generated by $\Omega_A^B$ with $A < B$ for $SO_q$ and $A' \leq B$ for $Sp_q$. In fact only this submodule will be relevant for the $r = 1$ differential calculus. As in the classical case $\mathbb{R}$, in order to simplify notations in sums we often use $\chi_{AB}$ and $\Omega_A^B$ without the restriction $A' \leq B$ see for ex. (6.50) below. The bimodule structure on $\Gamma$, see Theorem 5.1, is given by the $r \to 1$ limit of the $f^i_j$ functionals. These are diagonal in the $i,j$ indices [i.e. they vanish for $i \neq j$, see (6.26)-(6.29)] and still satisfy the property (5.3). We have:

$$
\Omega_A^B a = (\omega_A^B - \epsilon_A^B q_{AB} \omega_B^{A'})a = (f_A^{BCD} * a) \omega_C^D - \epsilon_A^B q_{AB} (f_B^{A'D'}_{C'} * a) \omega_{D'}^{C'}
$$

(6.46)

where in the last equality we have used (6.27) and no sum is understood. We see that the bimodule structure is very simple since it does not mix different $\Omega$'s. Moreover, relation (6.43) is invertible and yields:

$$
\Omega_A^B = -q_{AB} \kappa (T_B^C) dT_A^C ;
$$

(6.47)

in the limit $q_{AB} = 1$, the $\Omega_A^B$ are to be identified with the classical one-forms, and indeed for $q_{AB} = 1$ eq. (6.47) reproduces the correct classical limit $\Omega = -g^{-1} dg$ for the left-invariant one-forms on the group manifold.

The bimodule commutation rule (6.47) yields a formula similar to (5.9). Replacing the values of the $R$ matrix for $r = 1$ we find the commutations:

$$
\Omega_{A_2}^B T^R_{S} = \frac{q_{A_2}^S T^R_{A_1}}{q_{A_1}^S} \Omega_{A_1}^B
$$

(6.48)

For $r = 1$ the coproduct on the $\chi$ functionals reads

$$
\Delta'(\chi_A^B) = \chi_A^B \otimes f_A^{BA} + \epsilon \otimes \chi_A^B \quad \text{no sum on repeated indices.}
$$

(6.49)

cf. (5.52). We then consider the $r = 1$ limit of (5.15) and therefore define the exterior differential by:

$$
da \equiv \frac{1}{2} (\chi_A^B * a) \Omega_A^B = \sum_{A' \leq B} (\chi_{A'}^B * a) \Omega_{A'}^B, \quad \forall a \in A ,
$$

(6.50)

\footnote{To make closer contact with the classical case one may define:

$$
\Omega^{AB} \equiv \Omega_C^B C^{CA} = \epsilon_A^C \Omega_A^B ; \quad \chi_{AB} \equiv C_{AC} \chi^C_B = \epsilon_A^C \chi_{A'B} ,
$$

and retrieve the more familiar $q$-antisymmetry:

$$
\Omega^{AB} = -\epsilon q_{AB} \Omega^{BA} ; \quad \chi_{AB} = -\epsilon q_{AB} \chi_{BA} .
$$}
where in the second expression we have used the basis of linear independent tangent vectors \( \{ \chi^A_B \}_{A' \leq B} \) and dual one-forms \( \{ \Omega_B^A \}_{A' \leq B} \) (notice that in the \( SO_q \) case we have \( A' < B \) because \( \chi^A_A = \Omega_A^A = 0 \)). The Leibniz rule is satisfied for \( d \) defined in (6.50) because of (6.49) and (6.46). Moreover any \( \rho = a^A_B \Omega_B^A \in \Gamma \) can be written as \( \rho = \sum_k a_k db_k \) [use (6.47)].

We now introduce a left and a right action on the bimodule \( \Gamma \) of one-forms:

\[
\Delta_L(a\Omega_B^A) \equiv \Delta(a)I \otimes \Omega_B^A , \quad (6.51)
\]
\[
\Delta_R(a\Omega_B^A) \equiv \Delta(a)(\Omega_C^D \otimes M^C_{DA}B) . \quad (6.52)
\]

where \( M^C_{DA}B = T^C_A \kappa(T^B_D) \). [Using (6.44) one can check that this is the \( \rho = 1 \) limit of (5.10) and (5.11).] Relation (6.52) is well defined i.e. \( \Delta_R(\Omega_B^A) = \Delta_R(-q_{AB}^C \Omega^A_C) \) because \( \epsilon_F \epsilon_E \epsilon_{F'F} \epsilon_{E'E} M^{F'E'}_{B'B} = \epsilon_{A'B}^F q_{AB} \epsilon_{B'B}^{F'} \). Since in the \( \rho = 1 \) case the bicovariant bimodule conditions (5.3), (5.4) and (5.5) are still satisfied, it is easy to deduce that \( \Delta_L \) and \( \Delta_R \) give a bicovariant bimodule structure to \( \Gamma \).

The differential (6.50) gives a bicovariant differential calculus if it is compatible with \( \Delta_L \) and \( \Delta_R \), i.e. if:

\[
\Delta_L(ab \omega) = \Delta(a)(\omega \otimes d)\Delta(b) , \quad (6.53)
\]
\[
\Delta_R(ab \omega) = \Delta(a)(d \otimes \omega)\Delta(b) . \quad (6.54)
\]

The proof of the compatibility of \( d \) with \( \Delta_L \) is straightforward, just use (6.50) and the coassociativity of the coproduct \( \Delta \). In order to prove (6.54) it is sufficient to prove the following

**Theorem 6.1:**

\[
\Delta_R(db) = (d \otimes id)\Delta(b) \quad \forall \ b . \quad (6.55)
\]

**Proof:** We first review how the theorem is proved in the \( \rho \neq 1 \) case. On the left hand side we have

\[
\Delta_R(db) = \Delta_R[(\chi_i \ast b)\omega^j] = \Delta[b_1 \chi_i(b_2)]\omega^j \otimes M^i_j = b_1 \omega^j \otimes b_2 \chi_i(b_3)M^i_j \quad (6.56)
\]

with \( (\Delta \otimes id)\Delta(b) = (id \otimes \Delta)\Delta(b) \equiv b_1 \otimes b_2 \otimes b_3 \) [cf. (5.2)], while for the right hand side

\[
(d \otimes id)\Delta(b) = db_1 \otimes b_2 = b_1 \chi_j(b_2)\omega^j \otimes b_3 = b_1 \omega^j \otimes \chi_j(b_2)b_3 . \quad (6.57)
\]

Therefore (6.53) holds if and only if

\[
b_1 \omega^j \otimes b_2 \chi_i(b_3)M^i_j = b_1 \omega^j \otimes \chi_j(b_2)b_3 \quad (6.58)
\]

and this last relation is equivalent to

\[
b_1 \chi_i(b_2)M^i_j = \chi_j(b_1)b_2 , \quad \text{i.e.} \quad \chi_i \ast b = (b \ast \chi_j)\kappa(M^i_j) \quad (6.59)
\]

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as one can verify by applying $m(\kappa \otimes id)\Delta_L \otimes id$ ($m$ denotes multiplication) to (6.58), and using the linear independence of the $\omega^i$. Now formula (6.59) holds also in the limit $r = 1$. Indeed if we consider $b$ to be a polynomial in the $T^A_B$ with well behaved coefficients in the $r \to 1$ limit, then $\lim_{r \to 1}[b_1 \chi_i(b_2)]^j = \lim_{r \to 1} [\chi_j(b_1)]b_2$ i.e. $b_1[\lim_{r \to 1} \chi_i(b_2)]^j = [\lim_{r \to 1} \chi_j(b_1)]b_2$ so that relation (6.59) remains valid for $r = 1$, cf. (6.38)-(6.41). At this point one can prove Theorem 6.1 in the $r = 1$ case simply by substituting $\Omega$ to $\omega$ in (6.56), (6.57) and (6.58). Since (6.59) holds for $r = 1$, then also (6.58) holds in this limit and the theorem is proved. □

Relation (6.59) has an important geometrical interpretation: the left invariant vector field $\chi_i^*$ (associated with the tangent vector $\chi_i$) can be expressed in terms of the right invariant vector fields $*\chi_i$ via the “deformed functions on the group” $\kappa(M_j^i)$ [19].

In virtue of Remark 5.1 we conclude that (6.50) defines a bicovariant differential calculus on $S_q(N + 2)$.

Note 6.1: in the right-hand side of (6.59) the sum on the indices $j$ is restricted to $C' \leq D$, thus using the basis $\{\chi^C_{\alpha} \}_{C' \leq D}$, provided one replaces $M$ by

$$
\begin{align*}
M^C_{DA}B & \equiv M^C_{DA}B - \epsilon_{CDE}q_{DCM^{D'}}_{C'A}B \quad \text{for} \quad C' \neq D, \quad A' \neq B \\
M^C_{\alpha A'B} & \equiv 0 \quad \text{for} \quad SO_q \\
M^C_{\alpha A'B} & \equiv 0, \quad M^C_{D'A'} & \equiv 0 \quad \text{for} \quad Sp_q
\end{align*}
$$

(6.60)

This is easily seen from (6.42). We can also write $\Delta_R(a\Omega_A^B) = \sum_{C' \leq D} \Delta(a)(\Omega_C^D \otimes M^C_{DA}B)$, cf. (6.32), thus using the basis $\{\Omega_C^D\}_{C' \leq D}$. According to the general theory [15], the elements $M^C_{DA}B$ with $C' \leq D$, $A' \leq B$ are by definition the adjoint elements for the differential calculus on $S_{q,r=1}(N + 2)$. Since the calculus is bicovariant [cf. (5.53), (6.54)] we know a priori that the $M^C_{DA}B$ with $C' \leq D$, $A' \leq B$ satisfy the properties (6.4) and (6.5). □

It is useful to express the bicovariant algebra (5.13), (5.14)-(5.17) in the $r \to 1$ limit. A direct proof in the $SO_q$ case is also instructive. We call $P_\gamma$ the “$q$-antisymmetric” projector defined by:

$$
P_{A\beta } = \frac{1}{2}(\delta_{A\beta C} - q_{ABC}\delta_{B\gamma}^{\gamma})^{D}_{\gamma} e_{D\alpha}^{\alpha}.
$$

Then one easily shows that $P_{A\beta C} = -q_{ABC}p_{A\beta C} = -q_{ABC}p_{A\beta C} = -q_{ABC}p_{A\beta C}$ and

$$
\begin{align*}
\Omega^\gamma P_{-j}^i & = \Omega^\gamma, \quad P_{-j}^i \chi_j = \chi_i, \quad P_{-j}^i f_{k}^j = f_{k}^j P_{-j}^i = P_{-j}^i f_{k}^j P_{-j}^i, \\
M_{-j}^i \equiv 2P_{-i}^j M_{i}^j = 2M_{-i}^j P_{-j}^i, \quad M_{-j}^i P_{-i}^j = M_{-i}^j P_{-j}^i = 2M_{-i}^j M_{-j}^i = 2M_{-i}^j M_{-j}^i, \quad P_{-j}^i = 2P_{-i}^j M_{-j}^i = 2M_{-i}^j P_{-j}^i, \quad P_{-j}^i = 2P_{-i}^j M_{-j}^i = 2M_{-i}^j P_{-j}^i,
\end{align*}
$$

where greek letters $\alpha, \beta$ represent adjoint indices $(A_1, A_2)$, $(B_1, B_2)$ with the restriction $A_1 < A_2$, $B_1 < B_2$. It is then straightforward to show that $\Delta(M_{-i}^j) = M_{-i}^\alpha \otimes M_{-\beta}^j$ and $\varepsilon(M_{-i}^\beta) = \delta_{\gamma}^\beta$. Applying $P_\gamma$ to (6.43) and using $f_i^j = 0$ unless $i = j$ cf. (6.26)-(6.29) one also proves $M_{-i}^\alpha (a * f_{i}^\alpha) = (f_{i}^\beta * a)M_{-i}^\beta$. These formulae hold in particular if all indices are greek, thus proving (6.4) and (6.5) for $SO_{q,r=1}(N + 2)$. 23
Therefore (5.45)-(5.47) read (no sum on repeated indices):

$$f^i_{\ j} f^j_{\ i} = f^i_{\ j} f^j_{\ i}$$

$$C_{jk}^i f^j_{\ j} f^k_{\ i} + f^i_{\ j} \chi_k = \Lambda^{k|j} f^i_{\ j} + C_{jk}^i f^i_{\ j}$$

$$\chi_k f^i_{\ j} = \Lambda^{k|j} f^i_{\ j} \chi_k$$

Explicitly the q-Lie algebra (5.41) reads:

$$X^{B_1}_{C_1} X^{B_2}_{C_2} = q^{B_1} q^{B_2} q^{C_1} q^{C_2} X^{B_1}_{C_1} X^{B_2}_{C_2} X^{B_1}_{C_1} X^{B_2}_{C_2}$$

The Cartan-Maurer equations are obtained by differentiating (6.47):

$$d\Omega^B_A = q_{AB} q_{BC} q_{CA} C_{CD} \Omega^B_C \wedge \Omega^D_A$$

The commutations between C's are easy to find using (6.34):

$$\Omega^{A_2}_{A_1} \wedge \Omega^{D_2}_{D_1} = \Omega^{A_1}_{A_1} \wedge \Omega^{D_1}_{D_1}$$

Finally, we turn to the *-conjugations given by equations (5.59) and (5.61). Their \( r \to 1 \) limit yields

$$(\Omega_A^B)^* = -q_{BA} D^C_A \Omega^D_B D^B_C; \quad (\chi_A^B)^* = -q_{CD} D^A_C \chi^B_D D^D_B,$$

and shows that we have a bicovariant *-differential calculus.

## 7 Differential calculus on \( ISO_{q,r=1}(N) \)

We reconsider now, in the \( r \to 1 \) limit, the functionals given in eqs (6.2)-(6.11). We list below the functionals among these that annihilate the Hopf ideal:

$$\chi^a_b = \frac{1}{r - r^{-1}} [f^c_{\ b} \delta^a_c]$$

$$\chi^a_\circ = \frac{1}{r - r^{-1}} f^c_{\ a}$$

$$\chi^a_\bullet = \frac{1}{r - r^{-1}} f^c_{\ \bullet}$$

$$\chi^\circ_\circ = \frac{1}{r - r^{-1}} [f^\circ_{\ \circ} - \varepsilon]$$

$$\chi^\bullet_\bullet = \frac{1}{r - r^{-1}} [f^\bullet_{\ \bullet} - \varepsilon]$$

(7.1)
Note that in the \( r \to 1 \) limit \( \chi_{o}^{\bullet} \) vanishes for \( SO_{q,r=1}(N+2) \), and does not vanish in the case \( Sp_{q,r=1}(N+2) \). We treat here the orthogonal case, both for simplicity and because we are more interested for physical reasons to orthogonal (rather than symplectic) inhomogeneous q-groups. The reader can easily extend our discussion to the symplectic case, and include the \( \chi_{o}^{\bullet} \) tangent vector (besides the diagonal \( \chi_{a}^{a'} \)).

Taking all the \( \chi' \)'s given in (7.1) one obtains a differential calculus containing dilatations (because of the presence of \( \chi_{o}^{\circ} \) and \( \chi_{o}^{\bullet} \)). It is however possible to exclude the generators \( \chi_{o}^{\circ} \) and \( \chi_{o}^{\bullet} \) from the list, and obtain a dilatation-free bicovariant differential calculus. This we will discuss in the rest of this section, while, in a more general setting, the case with dilatations is discussed in ref. [13].

For \( r = 1 \) the \( \chi' \)'s in (7.1) are not independent, cf. relation (6.42) of previous section, and we have:

\[
\chi_{a'}^{b} = -q_{ab} \chi_{b}^{a}, \quad \chi_{o}^{b} = -\frac{1}{q_{b}} \chi_{b}^{\circ}, \quad \chi_{o}^{\circ} = -\chi_{o}^{\bullet}
\]  

(7.2)

Therefore we consider \( \chi_{a}^{b} \) as a candidate basis for the tangent vectors on \( ISO_{q,r=1}(N) \). We will show that these \( \chi \) functionals indeed define a bicovariant differential calculus on \( ISO_{q,r=1}(N) \).

Theorem 7.1: the functionals \( f_{i}^{j} \), obtained from those of \( SO_{q,r=1}(N+2) \) by restricting the indices to \( i = ab, \bullet b \), annihilate the Hopf ideal \( H \).

Proof: According to the results of the previous section, the only non-vanishing functionals with indices \( i = ab, \bullet b \) are

\[
\begin{align*}
\tilde{f}_{a_{1}b_{1}}^{a_{2}b_{2}} &= \kappa(L^{+b_{1}}_{a_{1}})L^{-a_{2}}_{a_{1}} \times L^{-a_{2}}_{b_{2}} \times \tilde{f}_{a_{1}b_{1}}^{a_{2}b_{2}} \\
\tilde{f}_{a_{1}b_{1}}^{a_{2}b_{2}} &= \kappa(L^{+b_{1}}_{a_{1}})L^{-a_{2}}_{a_{1}} \times L^{-a_{2}}_{b_{2}} \times \tilde{f}_{a_{1}b_{1}}^{a_{2}b_{2}}
\end{align*}
\]  

(7.3)

To prove the theorem, first one checks directly that the functionals (7.3) vanish on the generators \( T \) of the ideal \( H \), i.e. on \( T = T_{o}^{a}, T_{b}^{\bullet}, T_{o}^{\bullet} \). This extends to any element of the form \( aTb \ (a, b \in SO_{q,r=1}(N+2)) \), i.e. to any element of \( H = Ker(P) \), because of the property (5.3) which in the \( SO_{q,r=1}(N+2) \) reads \( \Delta_{N+2}^{a}(f_{i}^{j}) = f_{i}^{j} \otimes f_{i}^{j} \) since the functionals \( f_{i}^{j} \) vanish when \( i \neq j \).

Thus the functionals \( \chi_{i} \) and \( f_{i}^{j} \) with \( i = ab, \bullet b \), which we denote collectively by the symbol \( f \), all vanish on \( H \). Then these functionals are well defined on the quotient \( ISO_{q,r=1}(N) = SO_{q,r=1}(N+2)/Ker(H) \), in the sense that the “projected” functionals

\[
\tilde{f} : ISO_{q,r=1}(N) \rightarrow C, \quad \tilde{f}(P(a)) = f(a), \quad \forall a \in SO_{q,r=1}(N+2)
\]  

(7.4)

are well defined. Indeed if \( P(a) = P(b) \), then \( f(a) = f(b) \) because \( f(Ker(P)) = 0 \). This holds for any functional \( f \) vanishing on \( Ker(P) \).
The product $fg$ of two generic functionals vanishing on $Ker P$ also vanishes on $Ker P$, because $Ker P$ is a co-ideal (see ref. [1]): $fg(Ker P) = (f \otimes g) \Delta_{N+2}(Ker P) = 0$. Therefore $\bar{fg}$ is well defined on $ISO_{q,r=1}(N)$; moreover, [use (3.28)] $\bar{fg}[P(a)] \equiv fg(a) = (\bar{f} \otimes \bar{g})\Delta(P(a)) \equiv \bar{f}\bar{g}[P(a)]$, and the product of $\bar{f}$ and $\bar{g}$ involves the coproduct $\Delta$ of $ISO_{q,r=1}(N)$.

There is a natural way to introduce a coproduct on the $\bar{f}$'s:

$$\Delta' \bar{f} = \bar{f} i \otimes \bar{f} i \equiv \bar{f} i [P(a)P(b)] = \bar{f} [P(ab)] = f(ab) = \Delta_N^2 (f) [a \otimes b] \ . \quad (7.5)$$

It is then straightforward to show, from the relations (5.52) for $SO_{q,r=1}(N + 2)$ i.e. $\Delta_{N+2} i = x_i \otimes f^i + \varepsilon \otimes x_i$, $\Delta'_{N+2} i = f^i \otimes f^i$, that

$$\Delta' \bar{f} i = \bar{f} i \otimes \bar{f} i \quad \text{i.e.} \quad \bar{f} i [P(a)P(b)] = \bar{f} i [P(a)\bar{f} i [P(b)] (7.6)$$

$$\Delta' x_i = \bar{x} i \otimes \bar{x} i + \varepsilon \otimes \bar{x} i \quad (7.7)$$

with $i$ adjoint index running over the set of indices $ab, \bullet b$. With abuse of notations we will simply write $f$ instead of $\bar{f}$, and the $f$ in (7.3) will be seen as functionals on $ISO_{q,r=1}(N)$.

Consider now the elements $M_{-}^{i,j} \in ISO_{q,r=1}(N)$ obtained by projecting with $P$ those of $SO_{q,r=1}(N+2)$ and with the restriction $i = ab, \bullet b$ on the adjoint indices. The effect of the projection is to replace the coinverse in $SO_{q,r=1}(N+2)$, i.e. $\kappa_{N+2}$, with the coinverse $\kappa$ of $IS_{q,r=1}(N)$ (see the last of (3.28)). The nonvanishing elements are:

$$M_{-}^{b_1 b_2 a_1 a_2} = T_{a_1 b_1}^{b_2} \kappa(T^{a_2 b_2}) - q_{b_2 b_1} T_{a_1}^{b_1} \kappa(T^{a_2 b_2})$$

$$M_{-}^{\bullet b_1 a_2} = x^{b_2} \kappa(T^{a_2 b_2}) - q_{b_2 b_1} x^{b_2} \kappa(T^{a_2 b_2})$$

$$M_{\bullet}^{\bullet b_2 a_2} = \nu \kappa(T^{a_2 b_2}) \quad (7.8)$$

In the sequel greek letters will denote adjoint indices $\alpha = (a_1, a_2)$ with $a_1' < a_2$, and $\alpha = (\bullet, a_2)$.

**Theorem 7.2**: the left $A$-module $[A = ISO_{q,r=1}(N)] \Gamma$ freely generated by the symbols $\Omega^\alpha$ is a bicovariant bimodule over $ISO_{q,r=1}(N)$ with the right multiplication (no sum on repeated indices):

$$\Omega^\alpha a = (f^\alpha a) \Omega^\alpha, \quad a \in ISO_{q,r=1}(N) \quad (7.9)$$

[where the $f^\alpha_\beta$ are found in (7.3) and the $*$-product is computed with the co-product $\Delta$ of $ISO_{q,r=1}(N)$] and with the left and right actions of $ISO_{q,r=1}(N)$ on $\Gamma$ given by:

$$\Delta_L(a_\alpha \Omega^\alpha) \equiv \Delta(a_\alpha) I \otimes \Omega^\alpha \quad (7.10)$$

$$\Delta_R(a_\alpha \Omega^\alpha) \equiv \Delta(a_\alpha) \Omega^\beta \otimes M_{-}^{\alpha \beta} \quad (7.11)$$
the $M_{\beta}^\alpha$ being given in (7.8).

**Proof:** we prove the theorem by showing that the functionals $f$ and the elements $M_{\beta}^\alpha$ listed in (7.3) and (7.8) satisfy the properties (5.3)-(5.5) (cf. the theorem by Woronowicz discussed in the Section 4). Applying the projection $P$ to the $SO_{q,r}=1(N+2)$ relation $\Delta_{N+2}(M_{\beta}^\alpha) = M_{\gamma}^\alpha \otimes M_{\gamma}^\beta$, one verifies directly that the elements $M_{\beta}^\alpha$ in (7.8) do satisfy the properties (5.4). We have already shown that the functionals $f$ in (7.3) satisfy (5.3).

Consider now the last property (5.5). For $SO_{q,r}=1(N+2)$ it explicitly reads (cf. Note 6.1):

$$M_{A_1^1 A_2^1 B_1^2 B_2^1}^A(a \ast f_{A_1^1 A_2^1} C_{C_1^2}^C) = (f_{B_1^2 B_2^1 A_2^1} \ast a) M_{C_1^2 A_2^1}^C A_1^1 \quad \text{with } A_1^1 < A_2^1, B_1^2 < B_2^1, C_1^2 < C_2.$$  

(7.12)

Restrict the free indices to greek indices, and apply the projection $P$ on both members of the equation. It is then immediate to see that only the $f$’s in (7.3) and the $M$’s in (7.8) enter in (7.12).

We still have to prove that the $\ast$ product in (7.12) can be computed via the coproduct $\Delta$ in $ISO_{q,r}=1(N)$. Consider the projection of property (7.12), written symbolically as:

$$P[M_{\Delta}(f \otimes id)\Delta_{N+2}(a)] = P[(id \otimes f)\Delta_{N+2}(a)M] \quad \text{.}$$  

(7.13)

Now apply the definition (7.4) and the first of (3.28) to rewrite (7.13) as

$$P(M_{\Delta})(f \otimes id)\Delta(P(a)) = (id \otimes \bar{f})\Delta(P(a))P(M_{\Delta}) \quad \text{.}$$  

(7.14)

This projected equation then becomes property (5.5) for the $ISO_{q,r}=1(N)$ functionals $f$ and adjoint elements $M_\ast$, with the correct coproduct $\Delta$ of $ISO_{q,r}=1(N)$.

\[\Box\]

Using the general formula (7.9) we can deduce the $\Omega, T$ commutations for $ISO_{q,r}=1(N)$:

$$\Omega_{a_1^{a_2} T_{r_2}}^{a_2 T_{r_1}} s = \frac{q_{a_2 a_1}}{q_{a_1 s}} \Omega_{a_1}^{a_2}$$  

(7.15)

$$\Omega_{a_1^{a_2} x_{r_2}}^{a_2 x_{r_1}} s = \frac{q_{a_2 \cdot x_{a_1}}}{q_{a_1 \cdot x}} \Omega_{a_1}^{a_2}$$  

(7.16)

$$\Omega_{a_1^{a_2} u}^{a_2 u} s = \frac{q_{a_2 \bullet u}}{q_{a_2 \cdot u}} \Omega_{a_1}^{a_2}$$  

(7.17)

$$\Omega_{a_1^{a_2} T_{r_2}}^{a_2 T_{r_1}} s = q_{a_2 a_1} q_{a_2 a_1} \Omega_{a_1}^{a_2}$$  

(7.18)

$$\Omega_{a_1^{a_2} x_{r_2}}^{a_2 x_{r_1}} s = q_{a_2 \cdot x_{a_1}} \Omega_{a_1}^{a_2}$$  

(7.19)

$$\Omega_{a_1^{a_2} u}^{a_2 u} s = \frac{1}{q_{a_2 \cdot u}} \Omega_{a_1}^{a_2}$$  

(7.20)

**Note 7.1:** $u$ commutes with all $\Omega$'s only if $q_{a_2 \bullet u} = 1$ (cf. Note 3.2). This means that $u = I$ is consistent with the differential calculus on $ISO_{q,a,b,r=1,q_{a,b}=1}(N)$.  

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An exterior derivative on $ISO_{q,r=1}(N)$ can be defined as

$$da = (\chi_\alpha * a)\Omega^\alpha$$  \hfill (7.21)

where the $\chi_\alpha = \chi^a(\alpha < b), \chi_b^\alpha$ are given in (7.1). Due to the coproduct (7.7) and the commutations (7.9) this derivative satisfies the Leibniz rule. It is also compatible with the left and right action of $ISO_{q,r=1}(N)$ since (6.54) holds. This can be seen by noting that the key equation (6.59), which in the $SO_{q,r=1}(N+2)$ case reads [see (6.60), (6.42)]:

$$\chi^{A_2}_A * b = (b * \chi^{B_1}_B)\kappa(M_{\Lambda A_1^B B_1^A}) \quad \text{with} \quad A_2' < A_2, B_2' < B_2$$  \hfill (7.22)

becomes property (6.59) for the $ISO_{q,r=1}(N)$ functionals $\chi$ and adjoint elements $M_\cdot$, with the correct coproduct $\Delta$ of $ISO_{q,r=1}(N)$, once we restrict the free indices to greek indices and apply the projection $P$.

The exterior derivative on the generators $T^A_B$ is given by:

$$dT^a_b = -\sum_c T^a_c q_{cb} \Omega^c$$  \hfill (7.23)

$$dx^a = -\sum_c T^a_c q_{\cdot c} V^c$$  \hfill (7.24)

$$du = dv = 0$$  \hfill (7.25)

where we have defined $V^a \equiv \Omega^a_\cdot$. Again, for $q_{\cdot \cdot} = 1, u = v = I$ is a consistent choice.

Every element $\rho$ of $\Gamma$ can be written as $\rho = \sum_k a_k db_k$ for some $a_k, b_k$ belonging to $ISO_{q,r=1}(N)$. Indeed inverting (7.25) yields:

$$\Omega^c_a = -q_{ac} \kappa(T^b_c) dT^c_a$$  \hfill (7.26)

$$V^b = -\frac{1}{q_{\cdot \cdot}} \kappa(T^b_c) dx^c$$  \hfill (7.27)

Thus all the axioms for a bicovariant first order differential calculus on $ISO_{q,r=1}(N)$ are satisfied.

The exterior product of the left-invariant one-forms is defined as

$$\Omega^\alpha \wedge \Omega^\beta \equiv \Omega^\alpha \otimes \Omega^\beta - \Lambda^{\alpha\beta}_{\gamma\delta} \Omega^\gamma \otimes \Omega^\delta$$  \hfill (7.28)

where

$$\Lambda^{\alpha\beta}_{\gamma\delta} = f^{\alpha}_{\delta} (M_{\gamma}^{\beta})$$  \hfill (7.29)

This $\Lambda$ tensor can in fact be obtained from the one of $SO_{q,r=1}(N+2)$ by restricting its indices to the subset $ab, \cdot b$. This is true because when $i, l = ab, \cdot b$ we have $f^i i (KerP) = 0$ so that $f^i i$ is well defined on $ISO_{q,r=1}(N)$, and we can write
\[ f^i_j(M_{-k}^j) = \tilde{f}^i_j[P(M_{-k}^j)] \] (see discussion after Theorem 7.1). Then we can just specialize indices in equation (6.67) and deduce the \(q\)-commutations for the one-forms \(\Omega\) and \(V\):

\[
\Omega_{a_1}^{a_2} \wedge \Omega_{d_1}^{d_2} = -q_{a_1 d_2} q_{d_1 a_1} q_{a_2 d_2} \Omega_{d_1}^{a_2} \wedge \Omega_{a_1}^{a_2}
\] (7.30)

\[
\Omega_{a_1}^{a_2} \wedge V^{d_2} = -\frac{q_{a_2 \bullet}}{q_{a_1 \bullet}} q_{a_1 d_2} q_{d_2 a_2} V^{d_2} \wedge \Omega_{a_1}^{a_2}
\] (7.31)

\[
V^{a_2} \wedge V^{d_2} = -\frac{q_{a_2 \bullet}}{q_{d_2 \bullet}} q_{d_2 a_2} V^{d_2} \wedge V^{a_2}
\] (7.32)

The exterior differential on \(\Gamma^{\wedge n}\) can be defined as in Section 5 (eq. (5.35)), and satisfies all the properties (5.37)-(5.38).

The Cartan-Maurer equations

\[
d\Omega^a = -\frac{1}{2} C_{\beta \gamma}^{\alpha} \Omega^\beta \wedge \Omega^\gamma
\] (7.33)

can be explicitly written for the \(\Omega\) and \(V\) by differentiating eq.s (7.26) and (7.27):

\[
d\Omega^b_a = q_{a b} q_{c a} \Omega^b_c \wedge \Omega^a
\] (7.34)

\[
dV^b = q_{a \bullet} q_{b a} \Omega^b \wedge V^a
\] (7.35)

where the one-forms \(\Omega^a_b\) with \(a' > b\) are given by \(\Omega^b_a = -q_{ab} \Omega^a_{b'}\); i.e. we consider (as it is usually done in the classical limit), the one-forms \(\Omega^a_b\) to be "\(q\)-antisymmetric" \(\Omega^a_b = -q_{ab} \Omega^a_{b'}\), cf. eq. (6.45).

Using the values of \(\Lambda^{\alpha \beta \gamma} = f^\alpha_\beta (M_{-\gamma})\) and of the structure constants \(C_{\alpha \beta}^{\gamma} = \chi_k(M_{-\beta})\) we can explicitly write the "\(q\)-Lie algebra" of \(ISO_{q,r=1}(N)\). The \(\chi_{c_1} \chi_{b_2} \chi_{c_2} \) \(q\)-commutations read as in eq. (5.65) with lower case indices, and give the \(SO_{q,r=1}(N)\) \(q\)-Lie algebra; the remaining commutations are

\[
\chi^{c_1}_{c_2} \chi_{b_2} - \frac{q_{c_1 \bullet}}{q_{c_2 \bullet}} q_{b_2 c_1} q_{c_2 b_2} \chi_{b_2} \chi^{c_1}_{c_2} = \frac{q_{c_1 \bullet}}{q_{c_2 \bullet}} [C_{b_2 c_2} \chi^{c_1}_{c_2} - \delta^{c_1}_{b_2} q_{c_2 c_1} \chi_{c_2}]
\]

(7.36)

with the definition

\[
\chi_a \equiv \chi_a^\bullet
\] (7.37)

It is not difficult to verify that the \(C\) constants do coincide with the \(C\) constants appearing in the Cartan-Maurer equations (7.33)-(7.35).

The \(\ast\)-conjugation on the \(\chi\) functionals and on the one-forms \(\Omega\) can be deduced from (5.68):

\[
(\chi^a_b)^\ast = -q_{c d} \mathcal{D}^a_c \chi^c_d \mathcal{D}^d_b, \quad (\chi_b)^\ast = -(q_{d \bullet})^{-1} \chi_d \mathcal{D}^d_b
\] (7.38)
\[(\Omega^a_b)^* = -q_{ab}^c D^c_a \Omega^d_c \mathcal{D}_d^b, \quad (V^b)^* = -q_{\bullet \bullet} V^d \mathcal{D}_d^b \quad (7.39)\]

Remark 7.1: as discussed in [1] and at end of Section 3, a \(q\)-Poincaré group without dilatations (i.e. \(u = I\)) has only one free real parameter \(q_{12}\), which is the real parameter related to the \(q\)-Lorentz subalgebra. Then the formulas of this section can be specialized to describe a bicovariant calculus on the dilatation-free \(ISO_{q,r=1}(3,1)\) provided \(q_{\bullet \bullet} = 1\) and \(q_{12} \in \mathbb{R}\). It is however possible to have a bicovariant calculus without the dilatation generator \(\chi_{\bullet \bullet}\) on \(ISO_{q,r=1}(3,1)\) with \(u \neq I\). The possibility of having a dilatation-free \(q\)-Lie algebra describing a bicovariant calculus on a \(q\)-group containing dilatations \(u\) was already observed in the case of IGL \(q\)-groups, see ref. [3]. The \(q\)-Poincaré algebra presented in [4] corresponds to the case \(q \equiv q_{\bullet \bullet}, \quad q_{12} = 1\), for which the Lorentz subalgebra is undeformed and the \(q\)-Poincaré group contains \(u \neq I\). Finally, the bicovariant calculus that includes the dilatation generator \(\chi_{\bullet \bullet}\) is discussed in ref. [18].

8 The multiparametric orthogonal quantum plane as a \(q\)-coset space

In this section we derive the differential calculus on the orthogonal quantum plane

\[
Fun_{q,r=1}\left(\frac{ISO(N)}{SO(N)}\right), \quad (8.1)
\]

i.e. the \(ISO_{q,r=1}(N)\) subalgebra generated by the coordinates \(x^a\).

The coordinates \(x^a\) satisfy the commutations (3.10):

\[
x^a x^b = q_{ab} x^b x^a \quad (8.2)
\]

Note that the coordinates \(x^a\) do not trivially commute with the \(SO_{q,r=1}(N)\) \(q\)-group elements, but \(q\)-commute according to relations (3.9):

\[
T^b_d t^a = \frac{q_{ba}}{q_{\bullet \bullet}} x^a t^b + \varepsilon \otimes \chi^a_{\bullet \bullet} \quad (8.3)
\]

Lemma: \(\chi^b_c(a) = 0\) when \(a\) is a polynomial in \(x^a\) and \(v\), with all monomials containing at least one \(x^a\). This is easily proved by observing that no tensor exists with the correct index structure. In fact we can extend this lemma even to \(v \cdots v\), due to

\[
\chi^b_c(v) = 0 \quad (8.4)
\]

and the coproduct rule (7.7), reading explicitly (no sum on repeated indices):

\[
\Delta'(\chi^a_b) = \chi^a_b \otimes f^a_{ba} b + \varepsilon \otimes \chi^a_b, \quad \Delta'(\chi^a_{\bullet \bullet}) = \chi^a_{\bullet \bullet} \otimes f^a_{\bullet \bullet} b + \varepsilon \otimes \chi^a_{\bullet \bullet} \quad (8.5)
\]

Theorem 8.1: \(\chi^b_c * a = 0\) when \(a\) is a polynomial in \(x^a\).
Proof: we have \( \chi^b_c \ast a = (id \otimes \chi^b_c)(a_1 \otimes a_2) = a_1 \chi^b_c(a_2) \). We use here the standard notation \( \Delta(a) \equiv a_1 \otimes a_2 \). Since \( a_2 \) is a polynomial in \( x^a \) and \( v \) (use the coproduct rule (3.20)), and \( \chi^b_c \) vanishes on such a polynomial (previous Lemma), the theorem is proved. \( \square \)

Because of this theorem we can write the exterior derivative of an element of the quantum plane as

\[
d a = (\chi_c \ast a) V^c
\]  
(8.6)

Thus \( d a \) is expressed in terms of the “q-vielbein” \( V^c \).

The value and action of \( \chi_s \) on the coordinates is easily computed, cf. the definition in (7.1):

\[
\chi_c(x^a) = -q_c \delta^a_c, \quad \chi_c \ast x^a = -q_c T^a_c ;
\]

(8.7)

using (8.5) we find the deformed Leibniz rule

\[
\chi_c \ast (ab) = (\chi_c \ast a) f^c_{ab} + a \chi_c \ast b .
\]

(8.8)

From (8.7) the exterior derivative of \( x^a \) is:

\[
d x^a = -q_c T^a_c V^c = - V^c T^a_c
\]

(8.9)

[use (7.18)] and gives the relation between the \( q \)-vielbein \( V^c \) and the differentials \( dx^a \).

The \( x^a \) and \( V^b \) \( q \)-commute as (cf. (7.19)):

\[
V^a x^b = q_a \delta^a_b \ x^b V^a
\]

(8.10)

and via eq. (8.9) and (3.9) we find the \( dx^a, x^b \) commutations:

\[
dx^a x^b = q_{ab} x^b dx^a
\]

(8.11)

After acting on this equation with \( d \) we obtain the commutations between the differentials:

\[
dx^a \wedge dx^b = -q_{ab} dx^b \wedge dx^a .
\]

(8.12)

The commutations between the partial derivatives are given in eq.(7.36).

All the relations of this section are covariant under the \( SO_{q,r=1}(N) \) action:

\[
x^a \rightarrow T^a_{\ b} \otimes x^b .
\]

(8.13)

Notice that the partial derivatives \( \chi_c \), and in general all the tangent vectors \( \chi \) and vector fields \( \chi\ast \) of this paper have “flat” indices. To compare \( \chi_c\ast \) with partial derivative operators with “curved” indices, we need to define the operators \( \partial_s \):

\[
\partial_s (a) \equiv - \frac{1}{q_b} (\chi_b \ast a) \kappa(T^b_s) ,
\]

(8.14)
\[ da = \partial_s(a) \, dx^s \]  

which is equation (8.6) in “curved” indices. The action of \( \partial_s \) on the coordinates is

\[ \partial_s(x^a) = \delta^a_s I \quad \partial_s(bx^a) = b \partial_s(x^a) + q_{sa} \partial_s(b) \, x^a \]  

The tangent vector fields \( \chi_c \) of this paper and the partial derivatives \( \partial_s \) are derivative operators that act “from the right to the left” as it is seen from their deformed Leibniz rule (8.8), (8.16). This explains the inverted arrow on \( \partial_s \). We can also define derivative operators acting from the left to the right, as in refs [20], using the antipode \( \kappa \) which is antimultiplicative. For a generic quantum group the vectors \( -\kappa'^{-1}(\chi_i) \equiv -\chi_i \circ \kappa^{-1} \) act from the left and we also have

\[ da = (\chi_i * a)\omega^i = \omega^i (-\kappa'^{-1}(\chi_i) * a) \]  

as is seen from \( \kappa'(\chi_i) = -\chi_j \kappa'(f^j_i) \) and \( \kappa'^{-1}(f^k_j)f^i_k = \delta^i_j \) [third line of (5.52)].

We then define the partial derivatives

\[ \partial_s(a) \equiv \kappa^{-1}(T^i_s) \kappa'^{-1}(\chi_i) * a , \]  

so that

\[ da = dx^s \, \partial_s(a) . \]  

The action of \( \partial_s \) on the coordinates is

\[ \partial_s(x^a) = \delta^a_s I \quad \partial_s(x^a b) = \partial_s(x^a) b + q_{as} x^a \partial_s(b) \]  

From eqs. (8.14), (8.18) and (7.36), or directly from \( d^2 = 0 \) and \( dx^a \wedge dx^b = dx^a \otimes dx^b - q_{ab} dx^b \otimes dx^a \) [a consequence of (8.12)], one finds the following commutations between the “curved” partial derivatives:

\[ \partial_r \partial_s = q_{sr} \partial_s \partial_r , \quad \partial_r \partial_s = q_{rs} \partial_r \partial_s . \]  

Finally, we note that the transformation

\[ \xi^a = \frac{1}{\sqrt{2}}(x^a + x'^a) , \quad a \leq n \]  

\[ \xi^{n+1} = \frac{i}{\sqrt{2}}(x^n - x^{n+1}) \]  

\[ \xi^a = \frac{1}{\sqrt{2}}(-x^a + x'^a) , \quad a > n + 1 \]  

defines real coordinates \( \xi^a \) for the even dimensional orthogonal quantum plane \( Fun_{q,r=1}(ISO(n+1, n-1)/SO(n+1, n-1)) \) endowed with the conjugation ii)
discussed in Section 2. Moreover on this basis the metric becomes diagonal. Likewise it is possible to define antihermitian $\chi$ and real $\Omega, V$.

For $n = 2$ the results of this section immediately yield the bicovariant calculus on the $q$-Minkowski space, i.e. the multiparametric orthogonal quantum plane $\text{Fun}_{q,r=1}(\text{ISO}(3,1)/\text{SO}(3,1))$.

## A The Hopf algebra axioms

A Hopf algebra over the field $K$ is a unital algebra over $K$ endowed with the linear maps:

$$\Delta : A \to A \otimes A, \quad \varepsilon : A \to K, \quad \kappa : A \to A$$

(A.1)

satisfying the following properties $\forall a, b \in A$:

$$\begin{align*}
(\Delta \otimes \text{id})\Delta(a) &= (\text{id} \otimes \Delta)\Delta(a) \\
(\varepsilon \otimes \text{id})\Delta(a) &= (\text{id} \otimes \varepsilon)\Delta(a) = a \\
m(\kappa \otimes \text{id})\Delta(a) &= m(\text{id} \otimes \kappa)\Delta(a) = \varepsilon(a)I \\
\Delta(ab) &= \Delta(a)\Delta(b) ; \quad \Delta(I) = I \otimes I \\
\varepsilon(ab) &= \varepsilon(a)\varepsilon(b) ; \quad \varepsilon(I) = 1
\end{align*}$$

(A.2) \hspace{1cm} (A.3) \hspace{1cm} (A.4) \hspace{1cm} (A.5) \hspace{1cm} (A.6)

where $m$ is the multiplication map $m(a \otimes b) = ab$. From these axioms we deduce:

$$\kappa(ab) = \kappa(b)\kappa(a) ; \quad \Delta[\kappa(a)] = \tau(\kappa \otimes \kappa)\Delta(a) ; \quad \varepsilon[\kappa(a)] = \varepsilon(a) ; \quad \kappa(I) = I$$

(A.7)

where $\tau(a \otimes b) = b \otimes a$ is the twist map.

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