NOTE ON DEDEKIND TYPE DC SUMS

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Abstract. In this paper we study the Euler polynomials and functions and derive some interesting formulae related to the Euler polynomials and functions. From those formulae we consider Dedekind type DC(Dahee-Changhee)sums and prove reciprocity laws related to DC sums.

1. Introduction/Preliminaries

The Euler numbers are defined as

\[ \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi, \quad (\text{see [1-31]}), \]

and the Euler polynomials are also defined as

\[ \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi, \quad (\text{see [4, 5, 6]}). \]

The first few of Euler numbers are 1, \(-\frac{1}{2}, 0, \frac{1}{4}, \) and \(E_{2k} = 0\) for \(k = 1, 2, 3, \ldots\). From (1) and (2), we can easily derive the following.

\[ E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l}, \quad \text{where} \quad \binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!}, \quad (\text{see [4, 5, 6]}). \]
In (1), it is easy to see that $E_0 = 1$, $E_n(1) + E_n = 2\delta_{0,n}$, where $\delta_{0,n}$ is the Kronecker symbol. That is, $E_n(1) = -E_n$ for $n = 1, 2, 3, \ldots$, (see [4, 5]). We denote $\bar{E}_n(x)$ the $n$-th Euler function given by the Fourier expansion.

$$\bar{E}_n(x) = m!2 \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi ix}}{((2n+1)\pi i)^{m+1}},$$  

which, for $0 \leq x < 1$, reduces the $n$-th Euler polynomials.

By (3), we easily see that

$$\frac{dE_n(x)}{dx} = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = nE_{n-1}(x).$$

From (4), we note that

$$\int_0^x E_n(t) \, dt = \frac{1}{n+1}E_{n+1}(x), \text{ (see [4]).}$$

By the definitions of the Euler numbers and the Euler polynomials, we easily see that

$$2 \sum_{k=0}^{n-1} (-1)^k e^{kt} = \sum_{l=0}^{\infty} ((-1)^n E_l(n) + E_l) \frac{t^l}{l!}.$$ 

Thus, we have

$$2 \sum_{k=0}^{n-1} (-1)^k k^l = (-1)^n E_l(n) + E_l.$$ 

It is well known that the classical Dedekind sums $S(h, k)$ first arose in the transformation formula of the logarithm of Dedekind eta-function (see [17, 25, 26, 28]). If $h$ and $k$ are relative prime integers with $k > 0$, then Dedekind sum is defined as

$$S(h, k) = \sum_{u=1}^{k-1} (((\frac{u}{k}))(\frac{hu}{k})), \text{ (see [17, 25, 26, 28])}$$

where $((x))$ is defined as

$$((x)) = x - [x] - \frac{1}{2}, \text{ if } x \text{ is not an integer,}$$

$$= 0, \text{ otherwise,}$$
where \([x]\) is the largest integer \(\leq x\), (cf. [17]).

Generalized Dedekind sums \(S_p(h, k)\) are defined as

\[
S_p(h, k) = \sum_{a=1}^{k-1} \frac{a}{k} \bar{B}_p\left(\frac{ah}{k}\right), \tag{9}
\]

where \(h\) and \(k\) are relative prime positive integers and \(\bar{B}_p(x)\) are the \(p\)-th Bernoulli functions, which are defined as

\[
\bar{B}_p(x) = B_p(x - [x]) = -p!(2\pi i)^{-p} \sum_{m=-\infty, m\neq 0}^{\infty} m^{-p} e^{2\pi imx}, \tag{see [17, 25, 26]}
\]

where \(B_p(x)\) are the \(p\)-th ordinary Bernoulli polynomials.

Recently Y. Simsek have studied \(q\)-Dedekind type sums related to \(q\)-zeta function and basic \(L\)-series (see [31, 18]). He also studies \(q\)-Hardy-Berndt type sums associated with \(q\)-Genocchi type zeta and \(q\)-\(l\)-functions related to previous author’s paper (see [18, 31]). In this paper we consider Dedekind type DC(Daehee-Changhee) sums as follows.

\[
T_p(h, k) = 2 \sum_{u=1}^{k-1} (-1)^{u-1} \frac{u}{k} \bar{E}_p\left(\frac{hu}{k}\right), \quad (h \in \mathbb{Z}_+),
\]

where \(\bar{E}_p(x)\) are the \(p\)-the Euler functions. Note that \(T_p(h, k)\) is the similar form of generalized Dedekind type sums. Finally, we prove the following reciprocity law for an odd \(p\):

\[
k^pT_p(h, k) + h^pT_p(k, h)
\]

\[
= 2 \sum_{u=0}^{k-1} \left( kh(E + \frac{u}{k}) + k(E + h - [\frac{hu}{k}]) \right)^p + (hE + kE)^p + (p + 2)E_p,
\]

where \(h, k\) are relative prime positive integers and

\[
(Eh + Ek)^{n+1} = \sum_{l=0}^{n+1} \binom{n+1}{l} E_l h^l E_{n+1-l} k^{n+1-l}.
\]
2. ON THE RECIPROcity LAW FOR DEDEKIND TYPE DC SUMS

In this section, we assume \( p \in \mathbb{N} \) with \( p \equiv 1 \) mod 2. By the definition of the Euler polynomials, we see that

\[
E_p(x + y) = \sum_{s=0}^{p} \binom{p}{s}(x + y)^{p-s}E_s = \sum_{s=0}^{p} \binom{p}{s}E_s \sum_{k=0}^{p-s} \binom{p-s}{k}x^ky^{p-s-k}
\]

\[
= \sum_{s=0}^{p} \binom{p}{s} \sum_{j=0}^{s} \binom{s}{j}E_jx^{s-j}y^{p-s} = \sum_{s=0}^{p} \binom{p}{s}E_s(x)y^{p-s}.
\]

From (2), we can also derive

\[
E_p(mx) = m^p \sum_{s=0}^{m-1} E_p(x + \frac{s}{m})(-1)^s.
\]

By (5), we easily see that

\[
\int_0^1 xE_p(x)dx = \frac{E_{p+1}(1)}{p+1} - \frac{E_{p+1}(1)}{p+1} + \frac{E_{p+1}}{p+1} = 0,
\]

and

\[
\int_0^1 xE_p(x)dx = \sum_{s=0}^{p} \binom{p}{s}E_s \int_0^1 x^{p-s+1}dx = \sum_{s=0}^{p} \binom{p}{s} \frac{E_s}{p-s+2}.
\]

By (12) and (13), we obtain the following lemma.

**Lemma 1.** For \( p \in \mathbb{N} \) with \( p \equiv 1 \) mod 2, we have

\[
\sum_{s=0}^{p} \binom{p}{s}E_s \frac{1}{p-s+2} = \frac{E_{p+1}}{p+1} = 0.
\]

For \( s \in \mathbb{N} \) with \( s \equiv 0 \) mod 2 and \( s < p \), we have

\[
\frac{d^s(xE_p(x))}{(dx)^s} \bigg|_{x=1} = s! \binom{p}{s}E_{p-s}(1) = -s! \binom{p}{s}E_{p-s},
\]

and, from (3), we note that

\[
\frac{d^s(xE_p(x))}{(dx)^s} \bigg|_{x=1} = s! \sum_{v=0}^{p-s} \binom{p-v+1}{s} \binom{p}{v}E_v
\]

\[
= s! \sum_{v=0}^{p} \binom{p}{v} \binom{p-v+1}{s} E_v.
\]

By (14) and (15), we obtain the following theorem.
Theorem 2. For $s \in \mathbb{N}$ with $s \equiv 0 \mod{2}$ and $s > p$, we have

\begin{equation}
\sum_{v=0}^{p} \binom{p}{v} \binom{p-v+1}{s} E_v = -\binom{p}{s} E_{p-s} = \binom{p}{s} E_{p-s}(1).
\end{equation}

Let us define Dedekind type DC sums as follows.

\begin{equation}
T_p(h, k) = 2 \sum_{u=1}^{k-1} (-1)^{u-1} \frac{u}{k} \overline{E}_p \left( \frac{hu}{k} \right), \quad (h \in \mathbb{Z}^+),
\end{equation}

where $\overline{E}_p(x)$ is the $p$-th Euler function.

For $m \equiv 1 \mod{2}$, we have

\begin{equation}
T_p(1, m) = 2 \sum_{u=1}^{m-1} (-1)^{u-1} \frac{u}{m} \sum_{v=0}^{p} \binom{p}{v} E_v \left( \frac{u}{m} \right)^{p-v}
\end{equation}

By (7) and (18), we obtain the following theorem.

Theorem 3. For $m \equiv 1 \mod{2}$, we have

\begin{equation}
T_p(1, m) = \sum_{v=0}^{p} \binom{p}{v} E_v m^{-(p+1-v)} \left( E_{p-v+1}(m) - E_{p-v+1} \right).
\end{equation}

From (3) we can also derive

\begin{equation}
E_{p-v+1}(m) - E_{p-v+1} = \sum_{i=0}^{p-v} \binom{p-v+1}{i} m^{p+1-v-i} E_i
\end{equation}

so that we find

\begin{equation}
T_p(1, m) = \sum_{v=0}^{p} \binom{p}{v} m^{-(p+1-v)} E_v \sum_{i=0}^{p-v} \binom{p-v+1}{i} m^{p+1-v-i} E_i
\end{equation}

\begin{equation}
= \frac{1}{m^p} \sum_{v=0}^{p} \binom{p}{v} E_v \sum_{i=0}^{p-v} \binom{p-v+1}{i} E_i m^{p-i}.
\end{equation}

Therefore, we obtain the following corollary.
Corollary 4. For \( m \equiv 1 \pmod{2} \), we have

\[
\begin{align*}
\sum_{v=0}^{p} \binom{p}{v} E_v \sum_{i=0}^{p-v} \binom{p-v+1}{i} E_i m^{p-i}.
\end{align*}
\]

Interchanging the order of summation in (22), we obtain

\[
\begin{align*}
m^p T_p(1, m) &= \sum_{i=0}^{p-2} \binom{p}{v} E_v \sum_{i=0}^{p-v} \binom{p-v+1}{i} E_i m^{p-i} \\
&+ \binom{p+1}{p} E_p + \sum_{v=0}^{p} \binom{p}{v} E_v m^p + \sum_{v=0}^{1} \binom{p}{v} E_v \left( \binom{p-v+1}{p-1} E_{p-1} m \right) \\
&= \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} E_v \left( \binom{p-v+1}{i} E_i m^{p-i} \right) + (p+1) E_p + \sum_{v=0}^{p} \binom{p}{v} E_v m^p.
\end{align*}
\]

Therefore, we obtain the following proposition.

**Proposition 5.** For \( m \in \mathbb{N} \) with \( m \equiv 1 \pmod{2} \), we have

\[
\begin{align*}
m^p T_p(1, m) &= \sum_{v=0}^{p} \binom{p}{v} E_v m^p + \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} E_v \left( \binom{p-v+1}{i} E_i m^{p-i} \right) + (p+1) E_p.
\end{align*}
\]

In the sum over \( i \), the only non-vanishing terms are those for which the index \( i \) is odd. Hence, since \( i < p \) in this sum we may use (3) and Theorem 2 to obtain

\[
\begin{align*}
&= \sum_{i=0}^{p} \binom{p}{i} E_{p-i} (1) E_i m^{p-i} + p E_p.
\end{align*}
\]

Therefore, we obtain the following theorem.

**Theorem 6.** For odd \( p \) with \( p > 1 \), \( m \in \mathbb{Z}_+ \) with \( m \equiv 1 \pmod{2} \), we have

\[
\begin{align*}
m^p T_p(1, m) &= \sum_{i=0}^{p} \binom{p}{i} E_{p-i} (1) E_i m^{p-i} + p E_p.
\end{align*}
\]
Now we employ the symbolic notation as $E_n(x) = (E + x)^n$. It is easy to show that
\begin{align*}
\sum_{u=0}^{k-1} (-1)^u \sum_{s=0}^{p} \binom{p}{s} h^s E_s \left( \frac{u}{k} \right) E_{p-s} \left( h - \frac{hu}{k} \right) &= k^p \sum_{u=0}^{k-1} (-1)^u \left( (E + \frac{u}{k}) + (E + h - \frac{hu}{k}) \right)^p \\
&= k^p \sum_{u=0}^{k-1} (-1)^u \left( Eh + E + h + \frac{1}{2} - \frac{1}{2} + huk^{-1} - \frac{hu}{k} \right)^p \\
&= k^p \sum_{u=0}^{k-1} (-1)^u \left( Eh + E + h + \frac{1}{2} + \bar{E}_1 \left( \frac{hu}{k} \right) \right)^p.
\end{align*}

Now as the index $u$ range through the values $u = 0, 1, 2, \cdots, k - 1$, the product $hu$ range through a complete residue system modulo $k$ since $(h, k) = 1$ and due to the periodicity of $\bar{E}_1(x)$, the term $\bar{E}_1 \left( \frac{hu}{k} \right)$ may be replaced $\bar{E}_1 \left( \frac{u}{k} \right)$ without alternating the sum over $u$. For $k \in \mathbb{Z}_+$ with $k \equiv 1 \mod 2$, we have
\begin{align*}
(25) &= k^p \sum_{u=0}^{k-1} (-1)^u \left( E + Eh + h + \frac{1}{2} + \bar{E}_1 \left( \frac{u}{k} \right) \right)^p = k^p \sum_{u=0}^{k-1} (-1)^u \left( (E + \frac{u}{k}) + h(E + 1) \right)^p \\
&= k^p \sum_{u=0}^{k-1} (-1)^u \sum_{s=0}^{p} \binom{p}{s} E_s \left( \frac{u}{k} \right) h^{s-p} E_{p-s} \left( 1 \right) \\
&= \sum_{s=0}^{p} \binom{p}{s} k^{p-s} \left( k^s \sum_{u=0}^{k-1} (-1)^u E_s \left( \frac{u}{k} \right) \right) h^{s-p} E_{p-s} \left( 1 \right) = \sum_{s=0}^{p} \binom{p}{s} k^{p-s} E_s h^{s-p} E_{p-s} \left( 1 \right).
\end{align*}

Therefore, we obtain the following theorem.

**Theorem 7.** Let $h, k$ be natural numbers with $(h, k) = 1$. For odd $p$ with $p > 1$, and $k \equiv 1 \mod 2$, we have
\begin{align*}
\sum_{s=0}^{p} \binom{p}{s} h^{s-p} E_s h^{s-p} E_{p-s} \left( 1 \right) = k^p \sum_{u=0}^{k-1} (-1)^u \sum_{s=0}^{p} \binom{p}{s} h^s E_s \left( \frac{u}{k} \right) E_{p-s} \left( h - \frac{hu}{k} \right).
\end{align*}

Let $T$ be the sum of
\begin{align*}
T &= k^p T_p(h, k) + h^p T(k, h) \\
&= 2k^p \sum_{u=1}^{k-1} (-1)^{u-1} \frac{u}{k} \bar{E}_p \left( \frac{hu}{k} \right) + 2h^p \sum_{v=0}^{h-1} (-1)^{v-1} \frac{v}{h} \bar{E}_p \left( \frac{kv}{h} \right).
\end{align*}
We assume first that $p > 1$ and $h, k \in \mathbb{N}$ with $h \equiv 1 \mod 2$, and $k \equiv 1 \mod 2$.

\begin{equation}
\bar{E}_p\left(\frac{h}{k} u\right) = h^p \sum_{v=0}^{h-1} (-1)^v \bar{E}_p\left(\frac{u+v}{k}\right),
\end{equation}

and

\begin{equation}
\bar{E}_p\left(\frac{k}{h} v\right) = k^p \sum_{u=0}^{k-1} (-1)^u \bar{E}_p\left(\frac{v+u}{h}\right).
\end{equation}

From (26) and (27), we can easily derive the following (28).

\begin{equation}
T = (hk)^p \frac{2}{k} \sum_{u=1}^{k-1} (-1)^{u-1} \frac{u}{h} \sum_{v=0}^{h-1} (-1)^v \bar{E}_p\left(\frac{u+v}{k}\right)
+ (hk)^p \frac{2}{h} \sum_{v=1}^{h-1} (-1)^{v-1} \frac{v}{h} \sum_{u=0}^{k-1} (-1)^u \bar{E}_p\left(\frac{v+u}{h}\right)
= (hk)^p \frac{2}{hk} \sum_{u=0}^{k-1} \sum_{v=0}^{h-1} (-1)^{u+v-1} \left(\frac{uh+vk}{hk}\right) E_p\left(\frac{u}{k} + \frac{v}{h}\right).
\end{equation}

Therefore, we obtain the following theorem.

**Theorem 8.** Let $h, k \in \mathbb{N}$ with $h \equiv 1 \mod 2$ and $k \equiv 1 \mod 2$. For $p > 1$, we have

\[ k^p T_p(h, k) + h^p T(k, h) = 2(hk)^p \sum_{u=0}^{k-1} \sum_{v=0}^{h-1} (-1)^{u+v-1}(uh+vk)(hk)^{-1} E_p\left(\frac{u}{k} + \frac{v}{h}\right). \]

Now as the indices $u$ and $v$ run through the range $u = 0, 1, 2, \cdots, k-1$, $v = 0, 1, 2, \cdots, h-1$, respectively, the linear combination $uh+vk$ ranges through a complete residue system modulo $hk$, and each term $uh+vk$ satisfies the inequalities $0 \leq uh+vk < 2hk$. If we define the sets

\[ A = \{uh+vk|0 \leq uh+vk < hk\}, B = \{uh+vk|hk+1 \leq uh+vk < 2hk\}, \]

\[ C = \{\lambda|0 \leq \lambda \leq hk-1\}. \]

Let $h, k \in \mathbb{N}$ with $h \equiv 1 \mod 2$ and $k \equiv 1 \mod 2$. From (28), we note that
(29) \[ T = (hk)^p \left( 2 \sum_{\lambda \in A} \frac{\lambda}{hk} (-1)^{\lambda-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) + 2 \sum_{\lambda \in B} \frac{\lambda}{hk} (-1)^{\lambda-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) \right). \]

Now if \( y \in B \), then \( y = hk + \lambda \), where \( \lambda \in \mathbb{C} \), but \( \lambda \notin A \) (for if \( \lambda \in A \) then we have \( \lambda \equiv y \mod hk \)), but \( A \cup B \) forms a complete residue system modulo \( hk \). Hence, we have

(30) \[ 2 \sum_{y \in B} \frac{y}{hk} (-1)^{y-1} E_p \left( \frac{y}{hk} \right) = 2 \sum_{\lambda \in C \setminus A} (-1)^{\lambda-1} E_p \left( \frac{\lambda}{hk} \right) + 2 \sum_{\lambda \in C \setminus A} \frac{\lambda}{hk} (-1)^{\lambda-1} E_p \left( \frac{\lambda}{hk} \right). \]

By (29) and (30), we see that

\[
T = (hk)^p \left\{ 2 \sum_{\lambda \in A} \frac{\lambda}{hk} (-1)^{\lambda-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) + 2 \sum_{\lambda \in C \setminus A} (-1)^{\lambda-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) \right. \\
\left. + 2 \sum_{\lambda \in C \setminus A} \frac{\lambda}{hk} (-1)^{\lambda-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) \right\} \\
= (hk)^p \left\{ 2 \sum_{\lambda=0}^{hk-1} (-1)^{\lambda} \frac{\lambda}{hk} \tilde{E}_p \left( \frac{\lambda}{hk} \right) + 2 \sum_{\lambda=0}^{hk-1} (-1)^{\lambda} \tilde{E}_p \left( \frac{\lambda}{hk} \right) \right. \\
\left. - 2 \sum_{u=0}^{h-1} \sum_{0 \leq uh + vk < hk} (-1)^{u+v-1} \tilde{E}_p \left( \frac{uh + vk}{hk} \right) \right\}. \\
\]

It is easy to see that

\[
2 \sum_{\lambda=0}^{hk-1} \tilde{E}_p \left( \frac{\lambda}{hk} \right) (-1)^{\lambda-1} = 2(hk)^{-p} \tilde{E}_p (0) = 2(hk)^{-p} E_p. \\
\]

Hence, we have

(31) \[ T = (hk)^p \left( T_p(1, kh) + 2(hk)^{-p} E_p - S \right), \]
where

\[
S = 2 \sum_{0 \leq u \leq k-1} \sum_{0 \leq v \leq h-1} (-1)^{u+v-1} E_p\left(\frac{uh + vk}{hk}\right)
\]

\[
= 2 \sum_{0 \leq u \leq k-1} \sum_{0 \leq v \leq h-1} (-1)^{u+v-1} E_p\left(\frac{u}{k} + \frac{v}{h}\right).
\]

From the definition of \( S \), we note that

\[
S = 2 \sum_{u=0}^{k-1} \sum_{v=0}^{[h - \frac{hu}{k}]} (-1)^{u+v-1} E_p(\frac{u}{k} + \frac{v}{h}) = 2 \sum_{u=0}^{k-1} \sum_{v=0}^{[h - \frac{hu}{k}]} (-1)^{u+v-1} \left( E + \frac{u}{k} + \frac{v}{h} \right)^p
\]

\[
= 2 \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} E_s(\frac{u}{k}) \sum_{v=0}^{[h - \frac{hu}{k}]} (-1)^{v} v^{p-s}
\]

\[
= \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} E_s(\frac{u}{k}) \left( 2 \sum_{v=0}^{[h - \frac{hu}{k}]} (-1)^{v} v^{p-s} \right)
\]

\[
= \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} E_s(\frac{u}{k}) \left( (1) h - \left[ \frac{hu}{k} \right] \right) + \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} E_s(\frac{u}{k}) E_{p-s}
\]

\[
= \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} E_s(\frac{u}{k}) E_{p-s}(h - \left[ \frac{hu}{k} \right]) - h^{-p} \sum_{s=0}^{p} \binom{p}{s} h^{s-k-s} E_{p-s} E_s.
\]

Returning to (31), we have

\[
T = (hk)^p \{ T_p(1, kh) + 2(kh)^{-p} E_p - \sum_{s=0}^{p} \binom{p}{s} h^{s-p} \sum_{u=0}^{k-1} (-1)^{u-1} \left[ \frac{hu}{k} \right] E_{p-s} \left( \frac{u}{k} \right) \}
\]

\[
+ h^{-p} \sum_{s=0}^{p} \binom{p}{s} h^{s-k-s} E_{p-s} E_s \}.
\]
By Theorem 6 we see that

\[ T = \sum_{s=0}^{p} \binom{p}{s} E_s E_{p-s} (1)(hk)^{p-s} - \sum_{s=0}^{p} \binom{p}{s} h^s k^p \sum_{u=0}^{k-1} (-1)^{u-\left[\frac{hu}{k}\right]} E_s \left(\frac{u}{k}\right) E_{p-s} \left(h - \left[\frac{hu}{k}\right]\right) \]

\[ + (p + 2) E_p + \sum_{s=0}^{p} \binom{p}{s} h^s k^{p-s} E_s E_{p-s}. \]

From Theorem 7, we can also derive the following equation (32).

\[ T = \sum_{s=0}^{p} \binom{p}{s} k^p h^s \sum_{u=0}^{k-1} E_s \left(\frac{u}{k}\right) E_{p-s} \left(h - \left[\frac{hu}{k}\right]\right) \left(1 - (-1)^{u-\left[\frac{hu}{k}\right]}\right) \]

\[ + \sum_{s=0}^{p} \binom{p}{s} h^s k^{p-s} E_s E_{p-s} + (p + 2) E_p \]

(32)

Therefore, we obtain the following theorem.

**Theorem 9.** Let \( h, k \in \mathbb{N} \) with \( h \equiv 1 \) mod 2 and \( k \equiv 1 \) mod 2 and let \((h, k) = 1\). For \( p > 1 \), we have

\[ k^p T_p(h, k) + h^p T_p(k, h) \]

\[ = 2 \sum_{u=0}^{k-1} \left( kh(E + \frac{u}{k}) + k(E + h - \frac{hu}{k}) \right)^p + (hE + kE)^p + (p + 2) E_p, \]

where

\[ (hE + kE)^p = \sum_{s=0}^{p} \binom{p}{s} h^s E_s E_{p-s}. \]

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