Central Limit Theorem for Multi-Point Functions of the 2D Discrete Gaussian Model at high temperature

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Abstract

We study microscopic observables of the Discrete Gaussian model (i.e., the Gaussian free field restricted to take integer values) at high temperature using the renormalisation group method. In particular, we show the central limit theorem for the two-point function of the Discrete Gaussian model by computing the asymptotic of the moment generating function \( \langle e^{z(\sigma(0)-\sigma(y))^{DG}} \rangle_{\beta,\mathbb{Z}^2} \) for \( z \in \mathbb{C} \) sufficiently small. The method we use has direct connection with the multi-scale polymer expansion used in [4, 5], where it was used to study the scaling limit of the Discrete Gaussian model. The method also applies to multi-point functions and lattice models of sine-Gordon type studied in [22].

1 Introduction

The Discrete Gaussian (DG) model is an effective interface model whose height variables are restrained to take integer values and the thermal fluctuation is controlled by the Dirichlet energy. Specifically, let \( \Lambda \subset \mathbb{Z}^d \) be a square domain such that \( 0 \in \Lambda \). The DG measure \( \mathbb{P}^{DG}_{\beta,\Lambda} \) on \( \Omega^{DG}_{\Lambda} = \{ \sigma \in \mathbb{Z}^{2\pi \Lambda} : \sigma(0) = 0 \} \) with periodic boundary condition is defined by

\[
P^{DG}_{\beta,\Lambda}(\sigma) = \frac{1}{Z^{DG}_{\beta,\Lambda}} \exp \left( -\frac{1}{4\beta} \sum_{x \sim y \in \Lambda} (\sigma_x - \sigma_y)^2 \right)
\]

where \( \beta > 0 \) is the temperature parameter and \( Z^{DG}_{\beta,\Lambda} \) is a normalisation constant. Also, \( x \sim y \) indicates they are adjacent to each other when the domain \( \Lambda \) is equipped with periodic structure and \( \sum_{x \sim y} \) counts every edge of \( \Lambda \) twice. The expectation in the measure is denoted either \( \mathbb{E}^{DG}_{\beta,\Lambda}(\cdot) \) or \( \langle \cdot \rangle^{DG}_{\beta,\Lambda} \).

The case \( d = 2 \) has been drawing special attention from the probability and mathematical physics communities due to the presence of the localisation-delocalisation phase transition and its relation to other models such as the lattice Coulomb gas model and the Villain XY model. For some introductory aspects of this phase transition and duality, one may consult [2, 31]. When \( \beta \ll 1 \), Peierls' argument shows that \( \sup_{x \in \Lambda} \text{Var}(\sigma_x) \) is bounded uniformly as \( |\Lambda| \to \infty \), see [6]. Whereas for the case \( \beta \gg 1 \), Fröhlich and Spencer were the first to use the spin-wave method in [22] to prove delocalisation, namely \( \text{Var}(\sigma_x - \sigma_y) \geq c(\beta) \log ||x - y||_2 \) for some \( c(\beta) > 0 \) and uniformly over \( |\Lambda| \). Since then, the delocalisation was proved on more general graphs [1, 32] using new strategies, and a number of works [25, 26, 33, 38, 41] developed different views on how to understand the delocalised phase. Also, it was proved recently in [4, 5] that the scaling limit of the DG model with \( \beta \gg 1 \) is in fact a usual continuum Gaussian Free Field with renormalised temperature using the renormalisation group argument.

Delocalisation transitions are widely observed phenomenon in 2D random surface models, as seen in (non-exhaustive lists) [7, 10, 30, 36, 43] for continuous-valued models and [9, 16, 17] for discrete-valued models.

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27,28,42 for discrete-valued models. Delocalisation in 2D is closely related to the absence of symmetry breaking in 2D systems with continuous symmetry, the Mermin-Wagner theorem [34,35]. Such type of results can also be found in [15,21,37]. One may also consult [24,40] for different probabilistic models, approaches and diverse lists of references on this phenomenon.

In these notes, we aim to further develop the method of [4,5] to understand the multi-point functions of the DG model in the delocalised phase, and thereby establish scaled Gaussianity of microscopic observables.

1.1. Infinite volume functions. Our main theorem about the two-point function of the DG model is stated in terms of the infinite volume DG measure. More precisely, in the high-temperature phase, a translation invariant infinite volume Gibbs measure with variance delocalisation does not exist, cf. [33]. To resolve this, one usually considers the gradient Gibbs measure obtained in this way is the unique (translation-)ergodic gradient Gibbs measure with tilt 0 in the high-temperature phase. Thus we see that studying the gradient of the DG model is a correct approach does not exist, cf. [33]. To resolve this, one usually considers the gradient Gibbs measure

\[ \langle F(\sigma) \rangle_{\beta, Z^2}^{\text{DG}} = \lim_{n \to \infty} \langle F(\sigma) \rangle_{\beta, \Lambda(n)}^{\text{DG}} \]

for some sequence of tori increasing to \( Z^2 \) as subsets of \( Z^2 \), and \( F \) is a bounded measurable function that only depends on finite number of points in \( Z^2 \). It is shown in [33] that the measure obtained in this way is the unique (translation-)ergodic gradient Gibbs measure with tilt 0 in the high-temperature phase. Thus we see that studying the gradient of the DG model is a correct way to understand the model in the delocalised phase.

Our first main theorem is about the two-point function in this infinite volume gradient Gibbs measure. We use the notations \( \mathbb{D}_r = \{ z \in \mathbb{C} : |z| < r \} \) and \( S_{a,b} = \{ x+y : x \in [-a,a], y \in \mathbb{D}_b \} \subset \mathbb{C} \).

**Theorem 1.1.** Let \( \sigma \sim P_{\beta, Z^2}^{\text{DG}} \) and \( R > 0 \). Then there exist \( \beta_0(R), \epsilon_f, \alpha > 0 \), \( s_\beta(\epsilon) = O(e^{-\epsilon R^2}) \), \( h_t > 0 \) and \( c(\beta) \in \mathbb{R} \) such that, whenever \( \beta \geq \beta_0 \) and \( z \in S_{R,h_t} \),

\[ \log \langle e^{-r(\sigma(0) - \sigma(y))}_{\beta, Z^2} \rangle^{\text{DG}} = \frac{\log \|y\|_2}{2\pi(1 + s_\beta^2(\beta))} (1 + r_{\beta}(z,y))^2 + f_\beta(z) \]  

(1.2)

for some complex analytic functions \( r_{\beta}(\cdot, y), f_\beta : S_{R,h_t} \to \mathbb{C} \) with \( r_{\beta}(z,y) = O_\beta(\|y\|_2^{-\alpha}) \) as \( \|y\|_2 \to \infty, f_\beta \) independent of \( y \) and \( f_\beta(0) = f'_\beta(0) = 0 \).

This theorem can be generalised to the sine-Gordon type models of [22] (see Section 1.3.2 and Remark 2.2) and multi-point functions (see Section 1.3.3). But we confine most of our discussion to the case of the Discrete Gaussian model and the two-point function for the sake of clarity. Comparing (1.2) with [25] Corollary 6.2, Corollary 2.2, \( s_\beta(\epsilon) \) also satisfies \( s_\beta(\epsilon) \geq \frac{\beta}{\pi^2} e^{-\epsilon R^2} \) for \( \epsilon = 4\pi^2/3 \). Theorem 1.1 can be viewed as a complement, in a special case, of the Fröhlich-Spencer-Park bounds ([20,22], see also [31]) saying that, for each discrete square torus \( \Lambda \),

\[ \langle e^{\beta(f,\phi)} \rangle_{\Lambda}^{\text{GFF}} \leq \langle e^{\beta^{-1/2}(f,\sigma)} \rangle_{\beta, Z^2}^{\text{DG}} \leq \langle e^{(f,\phi)} \rangle_{\Lambda}^{\text{GFF}} \]  

(1.3)

for sufficiently large \( \beta \), some \( r(\beta) \in (0,1) \) such that \( r(\beta) \to 1 \) as \( \beta \to \infty \) and \( f \in \mathbb{R}^\Lambda \) such that \( \sum_{x \in \Lambda} f(x) = 0 \), where \( \phi \sim P_{\Lambda}^{\text{GFF}} \), meaning that \( \phi \) has the distribution of the Gaussian free field on \( \Lambda \).

Our main theorem can also be compared with our earlier result about the scaling limits in [5], where the observable is 'mesoscopic' in the sense that it is given as a sequence of functions \( (f_n)_n \subset \mathbb{R}^\Lambda \) where \( f_n(x) \) approximate \( n^{-1} f(n^{-1} x) \) where \( f \in C_0(\mathbb{R}) \) with \( \int_{\mathbb{R}^2} f = 0 \). Then \( \langle e^{(f_n,\sigma)} \rangle_{\beta, Z^2}^{\text{DG}} \) is shown to converge to the moment generating function of the Gaussian free field. To obtain the Theorem 1.1, there are two major improvements to be made. The first is to refine the analysis so that the 'microscopic' observables are within the scope of the renormalisation group method and the second is to extend the domain of the observables in the complex direction.

An immediate consequence of Theorem 1.1 is obtained by expanding moment generating function in a Taylor series as a function of \( \beta \), so

\[ \text{Var}^{\text{DG}}_{\beta, Z^2} [\sigma(0) - \sigma(y)] = \frac{\beta}{2\pi(1 + s_\beta^2(\beta))} \log \|y\|_2 + c_\beta + O_\beta(\|y\|_2^{-\alpha}) \]  

(1.4)
This shows the unboundedness of the variance directly. Also, we can rescale the right-hand side of (1.2) to obtain the central limit for the rescaled variables, so let \( t_\beta(y) = (\beta \log \|y\|_2)^{-1/2} \).

**Corollary 1.2** (Central limit theorem). Under the assumptions of Theorem 1.1, for \( s \in \mathbb{C} \),

\[
\log \left( e^{\epsilon(s)y})_{\beta,\mathbb{Z}^2} \right) = \frac{1}{2\pi(1 + s_0^2(\beta))} \left( 1 + \frac{c(\beta)}{\log \|y\|_2} + O_\beta \left( \frac{\|y\|_2}{\log \|y\|_2} \right) \right) \left( \frac{s^2}{\log \|y\|_2} \right)^2 \tag{1.5}
\]

as \( \|y\|_2 \to \infty \) and \( s^2/\log \|y\|_2 \to 0 \).

When the field is allowed to take continuous values and the interaction potential satisfies uniform convexity condition, the same observable as in Corollary 1.2 was shown to satisfy the central limit theorem in [10], and recently, the local central limit theorem was also proved in [38]. These results are obtained by applying stochastic homogenisation on the Helffer-Sjöstrand representation of the field, but this technique cannot be applied to integer valued systems. Also, a central limit theorem was obtained for the square ice model in [42] using the Russo-Seymour-Welsh estimate on the level lines.

Another interesting observable is the fractional charge correlation (or the cosine correlation).

**Corollary 1.3** (Fractional charge correlation). Under the assumptions of Theorem 1.1 and if \( \eta \in (-h_1, h_1) \setminus \{0\} \), then

\[
\langle \cos (\beta^{-1/2} \eta(\sigma(0) - \sigma(y))) \rangle_{\beta,\mathbb{Z}^2}^{DG} = C(\beta, \eta)\|y\|_2^{-2\pi \beta \eta^2 / \log \|y\|_2} \left( 1 + O_\beta \left( \frac{\|y\|_2}{\log \|y\|_2} \right) \right) \tag{1.6}
\]

as \( \|y\|_2 \to \infty \).

The polynomial decay of the fractional charge correlation also characterises the delocalisation, and the polynomial lower and upper bounds were established in [22][23]. In the localised phase, the Debye screening (cf. [14] for the lattice sine-Gordon model and [6] for the Discrete Gaussian model) induces exponential decay of the truncated charge correlation. The same type of result was obtained for the dimer model in [16] (where this observable is called the electric correlator). A similar result was also obtained for the interacting dimer model in [27] but only after smoothing the point observables in an appropriate way. For the lattice sine-Gordon model at the critical point, the fractional charge correlation was computed in [19]. This will be discussed again in Section 1.3.2.

### 1.2. Finite volume functions.

Theorem 1.1 is proved by first observing what happens in finite volumes. In what follows, \( \Lambda_N \) is \( L^N \times L^N \) two-dimensional discrete torus with distinguished point 0, where \( N \in \mathbb{Z}_{>0} \) and \( L \in \mathbb{Z}_{\geq 2} \). We consider \( (\Lambda_N)_{N>0} \) as nested subsets of \( \mathbb{Z}^2 \) with 0 \( \in \Lambda_N \) and \( \Lambda_N \) is represented by a corresponding subset of \( \mathbb{Z}^2 \). However, the metric \( \text{dist}_p(x, y) \) \( (p \in [1, \infty]) \) is that of the discrete torus, and also often is denoted \( \|x - y\|_p \) (even though it is not a norm).

The correlation function in the infinite volume can be reduced to a result on the finite discrete torus, \( \Lambda_N \). The observable can also be generalised. We will always be talking about the two-point observable \( f \) that satisfies the following.

\( (A_1) \) Suppose \( f_1, f_2 \in \mathbb{R}^{\mathbb{Z}^2} \) are two functions with compact supports such that 0 \( \in \text{supp}(f_1) \cap \text{supp}(f_2) \) and \( \sum_{x \in \mathbb{Z}^2} (f_1(x) + f_2(x)) = 0 \). Set \( f = f_1 + T_y f_2 \) where \( T_y \) is the translation by \( y \in \mathbb{Z}^2 \), defined by \( T_y f_2(x) = f_2(x - y) \). Then set

\[
M = \max\{\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty}\}, \quad \rho = \max\{\text{diam}(\text{supp}(f_1)), \text{diam}(\text{supp}(f_2))\}. \tag{1.7}
\]

For later use, we record an alternative requirement on \( f \).

\( (A'_1) \) \( f \) satisfies \( (A_1) \) but is allowed to have \( \sum_x f(x) \neq 0 \).
Although \( f \) is a function defined on \( \mathbb{Z}^2 \), for \( L^N \) sufficiently large compared to \( \|y\|_\infty + \rho \), \( f \) can be always be considered as a function on \( \Lambda_N \). To be more formal, we will let \( \Lambda'_N = \left( -\frac{L^N-1}{2}, \frac{L^N-1}{2} \right] \cap \mathbb{Z}^2 \subset \mathbb{Z}^2 \) if \( L \) is odd and \( \Lambda'_N = \left( -\frac{(L^N-1)}{2(l-1)}, \frac{(L^N-1)}{2(l-1)} \right] \cap \mathbb{Z}^2 \) if \( L \) is even. Also, let \( \iota_N : \Lambda_N \to \mathbb{Z}_2 \) be an isometric embedding such that \( \iota_N|_{\Lambda'_N} : \Lambda'_N \to \Lambda'_N \) is a bijection and \( \iota_N(0) = 0 \). Whenever we refer to \( f \) on \( \Lambda_N \) below, it will always mean \( f \circ \iota_N \) in real.

Then the main theorem can be reduced to the following, in which translation invariant covariance matrix means \( C \) : \( \Lambda \times \Lambda \to \mathbb{R} \) such that \( C(x_1, x_2) = C(x_1 + z, x_2 + z) \) for any \( z \in \Lambda \) and \( (f, C f) \geq 0 \) for any \( f \in \mathbb{R}^\Lambda \) (when \( \Lambda \) is either \( \Lambda_N \), a two-dimensional torus, or \( \mathbb{Z}^2 \)).

**Theorem 1.4.** There exists a translation invariant covariance matrix \( C_{\beta, \Lambda_N} \) and \( \beta_0(R), L_0(R) > 0 \) for each \( R \geq 0 \) such that the following holds. Let \( \xi \in S_{R, h_1} \) with \( h_1 \equiv h_1(M, \rho, L) > 0, L \geq L_0 \beta \geq \beta_0 \) and \( \sigma \sim \mathbb{P}^{DG}_{\beta, \Lambda_N} \). Then for \( f = f_1 + T_y f_2 \) satisfying [\( \mathbb{A}_i \)]

\[
\log(\exp^{-1/2}(t, \sigma))_{\beta, \Lambda_N} = \frac{1}{2} \sum_{x \in \Lambda_N} (\sigma_x - \sigma_y)^2 \]  

where \( h^a_\beta \ (a \in \{1, 2\}) \) and \( \psi_{\beta, \Lambda_N} \) are analytic functions in \( S_{R, h_1} \) satisfying the following:

- for \( f_2 \equiv 0 \), we have \( h_\beta^{(1)}[0] = h_\beta^{(2)}[1, 0] = 0; \)
- \( |h_\beta^{(2)}(\xi, y)| = O_{\beta, R}(||y||_\infty^{-\alpha}) \) uniformly in \( \xi \in S_{R, h_1} \) for some \( \alpha > 0; \)
- \( \psi_{\beta, \Lambda_N}(\xi, y) = O_{\beta, R}(||\Lambda_N||^{-\alpha}) \) uniformly in \( \xi \in S_{R, h_1} \) and \( y; \)
- there exists a translation invariant covariance matrix \( C_{\beta, \mathbb{Z}^2} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R} \) such that \( (f, C_{\beta, \mathbb{Z}^2} f) = \lim_{N \to \infty} (f, C_{\beta, \Lambda_N} f) \) (independent of choice of \( L \)).

To prove Theorem 1.1 and Theorem 1.4 we rewrite the moment generating function in terms of a continuous-valued statistical physics model defined in Section 2. In Section 3, we prove the main theorems assuming an estimate on the modified model. This estimate is proved in Section 5, using the renormalisation group method. Construction of the renormalisation flow and its properties appear in Section 4. A detailed overview on the technical aspects of the proof will also appear in Section 6.4.

### 1.3. Remarks and conjectures.

**1.3.1. Finite range Discrete Gaussian model.** As observed in [4], these results can also be extended to any finite ranged Discrete Gaussian models defined by the measure

\[
\mathbb{P}^{J-DG}_{\beta, \Lambda_N} = \frac{1}{Z_{J-DG}^{\beta, \Lambda_N}} \exp \left( -\frac{1}{4f|f_\beta} \sum_{x, y \in J} (\sigma_x - \sigma_y)^2 \right), \quad \sigma \in \Omega_{\Lambda_N}^{DG} \]  

for any \( J \subset \mathbb{Z}^2 \) with compact support that is invariant under lattice rotations and reflections and includes the nearest-neighbour sites of the origin. However, the uniqueness of infinite volume measure \( (\cdot)_{J-DG}^{\beta, \mathbb{Z}^2} \) is not known, so only a specific choice of infinite volume limit is used in [5].

**1.3.2. Relation to the lattice sine-Gordon model.** By the observation made in [4] (see also [14,18]), the same results hold for the lattice sine-Gordon model defined by the measure

\[
\mathbb{P}^{SG}_{\beta, \Lambda_N} = \frac{1}{Z_{SG}^{\beta, \Lambda_N}} \exp \left( -\frac{1}{2f} (\sigma, -\Delta \sigma) + \frac{z}{2} \sum_{x \in \Lambda_N} \cos(\sigma(x)) \right) \]  

on \( \Omega_{\Lambda_N}^{SG} = \{ \sigma \in \mathbb{R}^{\Lambda_N} : \sigma(0) = 0 \} \) when \( \beta > 8\pi \) and activity \( z \) is sufficiently small. Also by Remark 2.2, the same holds for a much wider class of models of sine-Gordon type (that includes the sine-Gordon model with any activity \( z \in \mathbb{R} \) and \( \beta \) chosen sufficiently large depending on \( z \)).

For the lattice sine-Gordon model, near the critical point, Theorem 1.4 complements the method of Falco [19], where the same quantity is computed for \( f = \delta_0 - \delta_2 \) and \( \beta \in (0, 2\pi \beta_{1/2}^2) \). For the case when \( J \) has sufficiently long range for finite range models in Section 1.3.1 Falco’s...
method of using the observable fields also extends the range of \( \beta \) to \((0, 2\pi \beta^{1/2})\) when \( \beta \) is near the critical value, i.e., Corollary 13 holds for \( \eta \in (0, 2\pi \beta^{1/2}) \). (It was seen in [25] that the near-critical values of these models are within the scope.) However, it does not give the required bounds for the usual Discrete Gaussian model where we can implement the renormalisation group method only for large values of \( \beta \) (well above the critical point). Using the observable fields is actually more common in this context, for example, as in [3, 29, 39].

1.3.3. Multi-point functions. The proof of Theorem 1.4 also implies analogous results for the multi-point functions. Let \( \bar{y} = (y_i)_{1 \leq i < l} \subset \mathbb{Z}^2 \), \( d_{\bar{y}} = \min \{ ||y_i - y_i||^2 : i \neq i_2 \} \) and \((f_i)_{i=1}^l \subset \mathbb{R}^{2^2}\) be such that \( \sum_i \sum_x h_i(x) = 0 \). Then there is \( \beta_0(R) \) such that whenever \( \beta \geq \beta_0(R) \) and \( \bar{z} \in S_{R, h_i} \), we have

\[
\log \left( \exp \left( \beta^{-1/2} \sum_{i=1}^l T_{y_i, f_i, \sigma_i} \right) \right)_{\beta, \mathcal{Z}} = \frac{1}{2} \beta^2 \left( \sum_{i=1}^l T_{y_i, f_i, \sigma_i} \right)_{\beta, \mathcal{Z}} + \sum_{i=1}^l h^{(1)}_{\beta}(\bar{f}_i)(\bar{z}) + h^{(2)}_{\beta}(\{f_i\}_{i=1}^l)(\bar{z}, \bar{y})
\]

where \( h^{(2)}(\bar{z}, \bar{y}) = O_\beta(d_{\bar{y}}^{-\alpha}) \) is a function analytic on a bounded domain containing 0 and \( h^{(1)}_{\beta} \) is the function from Theorem 4.4.

The analogue of Theorem 1.4 also holds true for any multi-point function, i.e., for any \((y_i)_{i=1}^l \subset \mathbb{Z}^2\) and \((c_i)_{i=1}^l\) such that \( \sum_i c_i = 0 \), there exists \( \beta_0(R) \) such that whenever \( \beta \geq \beta_0(R) \) and \( \bar{z} \in S_{R, h_i} \),

\[
\log \left( \exp \left( \beta^{-1/2} \sum_{i=1}^l c_i \sigma(y_i) \right) \right)_{\beta, \mathcal{Z}} = \frac{\beta^2}{2\pi(1 + s_0(\beta))} \mathcal{L}(\bar{c}, \bar{y})(1 + r_\beta(\bar{z}, \bar{y})) + f_\beta(\bar{z})
\]

where

\[
\mathcal{L}(\bar{c}, \bar{y}) := - \sum_{i<j} c_i c_j \log ||y_i - y_j||^2
\]

and \( r_\beta(\bar{z}, \bar{y}) \) and \( f_\beta(\bar{z}) \) are some complex analytic functions such that \( r_\beta(\bar{z}, \bar{y}) = O_\beta(d_{\bar{y}}^{-\alpha}) \) and \( f_\beta(0) = f'_\beta(0) = 0 \). Results of form Corollary 12 also follow.

1.3.4. Gradient correlation. With better choice of \( h^{(1)}_{\beta} \), it is believable that the decay of \( h^{(2)}_{\beta}(f_1, f_2, \beta, \mathcal{Z}) \) can be made at any polynomial rate, i.e.,

\[
\text{for each } k > 0, \quad h^{(2)}_{\beta}(\bar{z}, y) = O_k(||y||^{-2k}) \quad \text{as } ||y|| \to \infty,
\]

although we can only show this for some \( k = \alpha \in (0, 1) \) with the current technology. We expect that a refinement of the renormalisation group method we use would yield such a bound. If we had \( h^{(2)}_{\beta}(\bar{z}, y) = O_k(||y||^{-2(1+\varepsilon)}) \) for some \( \varepsilon > 0 \), and \( f_1 = \delta_{\mu} - \delta_{0}, f_2 = (\delta_{\nu} - \delta_{0}) \) for some \( \mu, \nu \in \{ \pm e_1, \pm e_2 \} \), then (by the proof of the theorem)

\[
\log \left( \exp \left( \beta^{-1/2} \sum_{\theta \in T_{\mathcal{Z}}} \right) \right)_{\beta, \mathcal{Z}} = \log \left( \exp \left( \beta^{-1/2} \sum_{f_1} \right) \right)_{\beta, \mathcal{Z}} + \log \left( \exp \left( \beta^{-1/2} \sum_{f_2} \right) \right)_{\beta, \mathcal{Z}} + O(||y||^{-2(1+\varepsilon)})
\]

and in the limit \( ||y|| \to \infty \), one would have \((f_1, \mathcal{C}_{\beta, \mathcal{Z}}^2 T_{y, f_2}) \sim C(\beta) \nabla^\mu \nabla^{-\nu}(-\Delta)^{-1}(y, 0) + O(||y||^{-d-\varepsilon}) \) for some \( \varepsilon > 0 \). Therefore we have

\[
\text{Cov}(\nabla^\mu \sigma(0), \nabla^\nu \sigma(y)) = \frac{\beta}{2} C(\beta) \nabla^\mu \nabla^{-\nu}(-\Delta)^{-1}(y, 0) + O(||y||^{-d-\varepsilon}).
\]

Moreover, due to the duality relation between the Villain XY model and the DG model, this would imply the energy correlation of the Villain XY model.

For continuous-valued models, estimates of this type can be found in [12] when the potential is uniformly convex and in [11, 29] when the potential is a small perturbation of a uniformly convex function.
1.4. Notations.

1.4.1. Derivatives. There are three different derivatives taking places here. If \( D \subset \mathbb{C} \) is some domain and we have \( f(\varphi, \tau) \), a smooth function \( \mathbb{R}^{\Lambda_N} \times D \to \mathbb{C} \), we will use \( Df \) for the derivative in the argument \( \varphi \) and \( \partial_\tau f \) for the derivative in \( \tau \). Given \( \varphi \in \mathbb{R}^{\Lambda_N}, \nabla^\mu \varphi(x) = \varphi(x + \mu) - \varphi(x) \) will denote the discrete derivative in direction \( \mu \in \hat{e} := \{ \pm e_1, \pm e_2 \} \) \((e_1, e_2)\) is the stander basis of \( \mathbb{Z}^2 \). Any \( \vec{\mu} = (\mu_1, \cdots, \mu_n) \in \hat{e}^n \) is called a multi-index with \(|\vec{\mu}| = n\), and defines

\[
\nabla^\vec{\mu} \varphi = \nabla^{\mu_1} \cdots \nabla^{\mu_1} \varphi.
\]

For \( n \geq 0 \), \( \nabla^n \varphi \) will denote the collection \( (\nabla^\vec{\mu} \varphi)_{|\vec{\mu}|=n} \) and

\[
|\nabla \varphi(x)|^2 = \sum_{\mu \in \hat{e}} |\nabla^\mu \varphi(x)|^2.
\]

Then for \( j \geq 0 \) and \( X \subset \Lambda_N \), we define \( \|\nabla^n \varphi\|_{L_1^j(X)}, \|\nabla^n \varphi\|_{L_2^j(\partial X)}, \|\nabla^n \varphi\|_{L_\infty(X)} \) and \( \|\varphi\|_{C_j} \) as in [3] Definition 5.2.

1.4.2. Polymers. The notions of \( j \)-scale blocks \((B_j)\) and polymers \((P_j)\) are used. Let

\[
B_0^j = [-\ell_j(L), L^j - \ell_j(L) - 1]^2 \cap \mathbb{Z}^2
\]

where \( \ell_j(L) = \frac{L^{j-1}}{2} \) if \( L \) is odd and \( \ell_j(L) = \frac{L(L^{j-1})}{2(L^{j-1})} \) if \( L \) is even. Also in \( \Lambda_N \), let \( B_0^j \) denote an analogous object, whenever a distinguished point \( 0 \in \Lambda_N \) is specified. In particular, they satisfy \( \text{dist}_\infty(0, (B_0^j)^c) \geq \frac{L_j}{3} \). Either on \( \mathbb{Z}^2 \) or \( \Lambda_N \), \( B_j \) is the set of \( L^j \mathbb{Z}^2 \)-translations of \( B_0^j \) and \( P_j \) is the set of any union of blocks in \( B_j \). By the choice of \( B_0^j \), any \( j+1 \)-polymer is also a \( j \)-polymer for any \( j \geq 0 \). The following is a list of relevant definitions.

- For \( X \in P_j \), denote \( B_j(X) \) for the collection of \( B \in B_j \) such that \( B \subset X \). Also denote \( |X|_j = |B_j(X)| \).
- \( X, Y \in P_j \) are connected (or \( X \sim Y \)) if \( \text{dist}_\infty(X, Y) \leq 1 \). The set of connected polymers is denoted \( P_j^c \) and the connected components are denoted \( \text{Comp}_j(X) \).
- The set of small polymers, \( S_j \), are the collection of \( X \in P_j \setminus \{\emptyset\} \) such that \( |X|_j \leq 4 \) and \( X \in P_j^c \). For \( X \in P_j \), denote \( X^* \) (small set neighbourhood of \( X \)) for the union of small polymers intersecting \( X \).
- For \( X \in P_j \), denote \( \overline{X} \) for the smallest \( j \) and \( 1 \)-scale polymer containing \( X \).

Although \( f_1 \) and \( f_2 \) are supported on \( B_0^j \) by \([A_j]\) it is not necessarily true that \( T_{f_2} \) is supported on a single scale- \( j \) block. New notations are required to denote these blocks: for \( y = (y_1, y_2) \in \mathbb{Z}^2 \), let

\[
B^j_1 = \text{unique block in } B_j \text{ that contains } (y_1 - \ell_j(L), y_2 - \ell_j(L)) \quad (1.20)
\]

\[
B^j_2 = B^j_1 + Lj e_1, \quad B^j_3 = B^j_2 + L(1)e_2, \quad B^j_4 = B^j_1 + L^j (e_1 + e_2) \quad (1.21)
\]

so \( \text{supp}(T_{y} u_{j,2}) \subset \cup_{k=1}^{A} \cup_{k=1}^{A} \cup_{k=1}^{A} B^j_k \). For future use, we also define \( j \)-polymers

\[
P^j_y := \cup_{k=0}^{A} B^j_k, \quad Q^j_y := \cup_{k=1}^{A} B^j_k \quad (1.22)
\]

Smooth functions on polymers are called polymer activities, defined as the following.

**Definition 1.5.** Fix \( \Lambda_N, \beta > 0 \) and \( h_1 > 0 \).

- Given \( X \in P_j \), define \( \mathcal{N}_j(X) \) (polymer activity on \( X \)) and \( \mathcal{N}_j \) (polymer activity) at scale \( j \) according to [3] Definition 5.1. \( F \in \mathcal{N}_j \) is said to obey lattice symmetries if \( F(X, \varphi) = F(SX, \varphi \circ S) \) for \( S \) either a translation or rotation or reflection (either in \( \Lambda_N \) or \( \mathbb{Z}^2 \)). \( F \) is said to be periodic if for any \( n \in 2\pi \beta^{-1/2} \mathbb{Z} \), \( F \) satisfies \( F(X, \varphi + n \mathbf{1}) = F(X, \varphi) \).
• $\mathcal{N}_{j,t}(X)$ is the subset of $C^\infty(\mathbb{R}^\Lambda \times \mathbb{D}_{h_1})$ such that $F(\varphi; \tau)$ only depends on $(\varphi|_{X^*}, \tau)$ and $D^n F(\varphi; \tau)$ analytic in $\tau \in \mathbb{D}_{h_1}$ for each $n \geq 0$. Also, let $\mathcal{N}_{j,t}$, the $t$-polymer activities, be the set of collections $F = (F(X, \cdot, \cdot))_{X \in \mathcal{P}_j}$ such that $F(X, \cdot, \cdot) \in \mathcal{N}_{j,h_1}(X)$ for each $X \in \mathcal{P}_j$.

The analyticity of $D^n F(\cdot; \tau)$ is used only when $n = 0$ for $F \in \mathcal{N}_{j,t}$, but appending the case of general $n$ does not cause any extra difficulty. If $H$ is some function of $\mathcal{P}_j$, then we also use the notions of polymer powers,

$$H^X := \prod_{B \in \mathcal{B}_j(X)} H(B), \quad H^{[X]} := \prod_{Y \in \text{Comp}_j(X)} H(Y).$$

(1.23)

For $F$ in either $\mathcal{N}_j$ or $\mathcal{N}_{j,t}$, we can define a natural extension to any polymers $X \in \mathcal{P}_j$ by

$$F(X, \varphi) := F^{[X]}(\varphi) = \prod_{Z \in \text{Comp}_j(X)} F(Z, \varphi).$$

(1.24)

1.4.3. Gaussian integrals. We introduce special notations for Gaussian integrals: for a finite set $V$, a Gaussian random variable $X$ on $\mathbb{R}^V$ with mean $x \in \mathbb{R}^V$ and covariance $C \in \mathbb{R}^{V \times V}$ is denoted $X \sim N(x, C)$. If $F : \mathbb{R}^V \to \mathbb{R}$ is a measurable function, then denote $\mathbb{E}^{X}_N[F(X)]$ for the Gaussian expectation with $X \sim N(0, C)$. The covariance $C$ and random variable $X$ are omitted if they are clear from the context, e.g., by just denoting $\mathbb{E}[F(X)]$. Gaussian integrals are also called the fluctuation integrals.

2 Two-point function as a tilted expectation

This section shows how to reduce the two-point function in the DG model to a partition function of a modified statistical physics model with an external field and a shifted environment, which will be written in terms of an expectation with exponential tilting. Renormalisation group method will be used to control this tilted expectation, so we also introduce basic elements that are required to construct the renormalisation group.

2.1. Regularisation of the model. $\langle e^{\beta^{-1/2} f(M, \sigma)} \rangle_{\beta, m^2, N}^{DG}$ is computed by first approximating the model with a massive model (with also the state space rescaled) defined by

$$\langle F(\sigma) \rangle_{\beta, m^2, N} = \frac{\sum_{\sigma \in \mathcal{Z}_N} e^{-\frac{1}{2}(\sigma, (\tau_{\Lambda} + m^2) \sigma)} F(\sigma)}{\sum_{\sigma \in \mathcal{Z}_N} e^{-\frac{1}{2}(\sigma, (\tau_{\Lambda} + m^2) \sigma)}}$$

(2.1)
where $Z_{\beta}^{\Lambda_N} = (2\pi \beta^{-1/2})^{\Lambda_N}$. Now $\sigma$ need not be fixed to be $\sigma(0) = 0$ and the model can be considered as a discrete model with covariance $(-\Delta + m^2)^{-1}$ acting on the configuration space $Z_{\beta}^{\Lambda_N}$. Then the limit $m^2 \downarrow 0$ is taken, whose validity is already justified in \cite[Lemma 2.1]{4},

$$
(F(\beta^{-1/2} \sigma))^{DG}_{\beta, \Lambda_N} = \lim_{m^2 \downarrow 0} \langle F(\sigma) \rangle_{\beta, m^2, \Lambda_N}
$$

whenever $F: \mathbb{R}^{\Lambda_N} \rightarrow \mathbb{C}$ is such that $F(\psi) = F(\psi + n1)$ for any $n \in 2\pi \beta^{-1/2}\mathbb{Z}$, the constant field $1(x) \equiv 1$ and $F$ is uniformly integrable with respect to the measures appearing in the equation.

Secondly, the underlying discreteness of the model is smoothened by integrating the ‘ultralocal’ part of the covariance and the stiffness renormalisation is performed by extracting $\frac{1}{2} s(\varphi, -\Delta \varphi)$ from the covariance. These will leave us with a new covariances

$$
C(m^2) = (-\Delta + m^2)^{-1} - \gamma 1d, \quad C(s, m^2) = (C(m^2)^{-1} - s \Delta)^{-1}
$$

($\gamma$ and $s$ are taken sufficiently small so that $C(s, m^2)$ is still positive definite—see \cite[Section 3]{4} for specific conditions) and new statistical physics model on configuration space $\mathbb{R}^{\Lambda_N}$ via \cite[(2.30)–(2.31)]{4},

$$
\sum_{\sigma \in \mathbb{Z}^\Lambda_N} e^{-\frac{1}{2}(\sigma, (-\Delta + m^2)\sigma)} e^{i\hat{\mathcal{S}}(\hat{\mathcal{I}}, \sigma)} \propto \mathcal{E}_{C(s, m^2)} e^{i\hat{\mathcal{S}}(\varphi, -\Delta \varphi)} \sum_{x \in \Lambda} \hat{U}(\varphi(x) + \gamma \hat{\mathcal{I}}(x))
$$

where $\hat{U}(\theta) = \sum_{q=1}^{\infty} \hat{z}(q)(\theta) \cos(q\beta^{1/2} \theta)$ for any $\theta \in \mathbb{C}$ and some $\hat{z}(q) \in \mathbb{R}$ with

$$
|\hat{z}(q)(\theta)| = O(e^{-\frac{1}{2} \gamma \beta(1+q)}).
$$

Finally, after adding and subtracting $\frac{1}{2} s(\varphi + \gamma \hat{\mathcal{S}}, (-\Delta)(\varphi + \gamma \hat{\mathcal{S}})) - \frac{1}{2} s(\varphi, -\Delta \varphi)$ in the exponent of the right-hand side of \[(2.3),\] we obtain the following.

**Lemma 2.1.** For all $\beta > 0$, $\gamma \in (0, 1/3)$, $m^2 \in (0, 1)$, $s$ small, $\hat{f} \in \mathbb{C}$,

$$
\left\langle e^{i\hat{\mathcal{S}}(\hat{\mathcal{I}}, \sigma)} \right\rangle_{\beta, m^2} = e^{\frac{1}{2} s}\mathcal{E}_{C(s, m^2)} \left[ e^{i\hat{\mathcal{S}}(\varphi)} Z_0(\varphi + \gamma \hat{\mathcal{S}}) \right].
$$

where $\hat{f} = (1 + s\gamma \Delta)\hat{f}$ and

$$
Z_0(\varphi) = \exp \left( \frac{1}{2} s_0 |\nabla \varphi|^2 + \sum_{x \in \Lambda} \sum_{q \geq 1} \hat{z}(q) \cos(q\beta^{1/2} \varphi(x)) \right)
$$

with $s_0 = s$ and $\hat{z}(q) = \hat{z}(q)$.

**Remark 2.2** (Sine-Gordon model). The procedure that shows \[(2.4)\] can also be applied to show a similar statement on a generalised class of models of sine-Gordon type defined by

$$
\mathcal{E}_{G-SG}^{\beta, m^2}(d\sigma) = \frac{1}{Z_{G-SG}^{\beta, m^2}} e^{-H_{G-SG}^{\beta, m^2}(\sigma)} d\sigma,
$$

$$
H_{m^2}^{G-SG}(\sigma) = \frac{1}{2\beta}(\sigma, (-\Delta + m^2)\sigma) + \log \prod_{x \in \Lambda_N} \sum_{q \geq 0} \lambda_q \cos(q\sigma(x))
$$

where $\lambda_q \in \mathbb{R}$ satisfies growth condition $|\lambda_q| \leq C e^{(\gamma+\beta^2)q^2} \lambda_0$ for $\theta < \frac{1}{3} \gamma$ and when the sum $\sum_{q \geq 0} \left(\cdots\right)$ makes sense as a distribution. A similar growth condition appeared in \cite{22}, and the DG model can be considered to be the case $\lambda_0 = 1$, and $\lambda_q = 2$ for $q \neq 0$.

As a special case, one could think of the usual lattice sine-Gordon with any activity $z \in \mathbb{R}$, i.e., with Hamiltonian $\frac{1}{2\beta}(\sigma, (-\Delta + m^2)\sigma) + \sum_{x \in \Lambda_N} z \cos(\sigma(x))$. Indeed,

$$
e^{z \cos(x)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(z/2)^n}{k!(n-k)!} |\mathcal{R}[e^{i(2k-n)x}]| = \sum_{k=0}^{\infty} \sum_{q=-k}^{k} \frac{(z/2)^{2k+q}}{k!(k+q)!} |\mathcal{R}[e^{iqx}]|
$$

$$
= \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} I(q, k) \frac{(z/2)^{2k+q}}{k!(k+q)!} \cos(qx)
$$

(2.10)
where reparametrisation \( n - 2k = q \) was made in the second equality and \( I(q, k) = 2 \) if \( 1 \leq q \leq k \) and \( I(q, k) = 1 \) otherwise. By making a trivial bound \( \sum_{k < q} I(q, k) \frac{q!}{k!(k+q)!}(z/2)^{2k+q} \leq 2(z/2)^q e^{z^2/4} \), we see that the lattice sine-Gordon model with any activity \( z \in \mathbb{R} \) satisfies the condition above.

Now we explain why the growth condition \( |\lambda_q| \leq C e^{(\eta + \beta \theta)q^2} \lambda_0 \) with \( \theta < \frac{\gamma}{2} \) is sufficient. Indeed,
\[
\frac{1}{\sqrt{2\pi}\gamma} \int_{\mathbb{R}} \lambda_q \cos(qx) e^{-\frac{1}{2\pi}(y-x)^2} dx = \lambda_q e^{-\frac{1}{2}\beta\gamma q^2} \cos(qy) = O \left( e^{-\left(\frac{1}{2}\beta\gamma - \beta\theta - \eta\right)q^2} \right) \lambda_0 \cos(qy).
\]
(2.11)

If we consider the unital Banach algebra of periodic functions
\[
\hat{\ell}^!(c) = \{ f \in L^\infty(\mathbb{R}) : f(x + 2\pi) = f(x), \| f \|_{\hat{\ell}^!(c)} < \infty \},
\]
(2.12)

\[
\| f \|_{\hat{\ell}^!(c)} = \sum_{q \in \mathbb{Z}} e^{c|q|} |\hat{f}(q)|, \quad \hat{f}(q) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-iqx} dx,
\]
(2.13)

then \( f \to \log(1 + f) \) is a well-defined continuously differentiable function \( \{ \| f \|_{\hat{\ell}^!(c)} < 1 \} \to \hat{\ell}^!(c) \). Thus with sufficiently large \( \beta \) and the choice \( c = \beta(\gamma/2 - \theta) \), we see that
\[
\log \left( \lambda_0 + \frac{1}{\sqrt{2\pi}\gamma} \int_{\mathbb{R}} \sum_{q \neq 0} \lambda_q \cos(\beta^{1/2} qx) e^{-\frac{1}{2}(y-x)^2} dx \right) = \log \lambda_0 + \hat{U}^{G-SG}(y)
\]
(2.14)

where
\[
\hat{U}^{G-SG}(y) = \sum_{q=0}^{\infty} \tilde{z}^{(q)}_{G-SG} \cos(\beta^{1/2} qy), \quad |\tilde{z}^{(q)}_{G-SG}(\beta)| \leq c_1 e^{-c_2(1+q)\beta}
\]
(2.15)

for some \( c_1, c_2 > 0 \) that depend on \( \eta, \theta, \gamma \). (See [4, Lemma 2.2] for a similar argument.) Along with the covariance decomposition \( (-\Delta + m^2)^{-1} = C(m^2) + \gamma \text{id} \), this computation shows that
\[
Z_{\beta, m^2}^{G-SG} \propto \int_{(\mathbb{R}^N)^2} e^{-\frac{1}{2}(\varphi, C(m^2)^{-1}\varphi) - \frac{1}{2}|c|^2} \prod_{x \in \Lambda_N} \sum_{q \neq 0} \lambda_q \cos(\beta^{1/2} q(\varphi(x) + \zeta(x))) d\zeta d\varphi
\]
\[
= \int_{(\mathbb{R}^N)^2} e^{-\frac{1}{2}(\varphi, C(m^2)^{-1}\varphi) - \frac{1}{2}|c|^2} \sum_{x \in \Lambda_N} \hat{U}^{G-SG}(\varphi(x)) d\varphi
\]
\[
= \int_{(\mathbb{R}^N)^2} e^{-\frac{1}{2}(\varphi, C(s, m^2)^{-1}\varphi) + s\varphi(\varphi, \varphi - \Delta) + \varphi - \Delta) + \varphi + \sum_{x \in \Lambda_N} \hat{U}^{G-SG}(\varphi(x)) d\varphi.
\]
(2.16)

This can be used to prove a version of Lemma 2.1 with \( \tilde{z}(q) \) replaced by \( \tilde{z}^{(q)}_{G-SG} \).

### 2.2. Finite range decomposition.
We study the expectations in Lemma 2.1 using a multiscale analysis. Gaussian field \( \varphi \sim \mathcal{N}(0, C(s, m^2)) \) is decomposed into independent Gaussian random variables residing in different scales, and this is equivalent to the decomposition of the covariance \( C(s, m^2) \). [4, Section 3] constructs a constant \( \varepsilon > 0 \) and a collection of (rotation, reflection and translation invariant) covariance matrices \( \Gamma_{j}(m^2) \) for \( j \geq 0 \) and \( \Gamma_{N}^{\infty}(m^2) \), and \( \tau_{N}(m^2) = \mathbb{R}_{>0} \) such that, on \( \Lambda_N \),
\[
C(s, m^2) = \sum_{j=1}^{N-1} \Gamma_{j}(m^2) + \Gamma_{N}^{\infty}(m^2) + \tau_{N}(m^2) Q_N
\]
(2.17)

whenever \( s \in [-\varepsilon, \varepsilon] \) and \( m^2 \in (0, 1] \) such that satisfy the following.

\((\Gamma)\) \( \Gamma_{j}(x, y) = 0 \) whenever \( \text{dist}_{\infty}(x, y) \geq \frac{1}{4}L^j \) (giving the name finite range decomposition) and
\[
\sup_{x, y \in \Lambda_N} |\nabla^{\tilde{\mu}} \Gamma_{j}(m^2)(x, y)| \leq \begin{cases} C_{\mu} L^{-n(j-1)} & \text{if } |\tilde{\mu}| = n > 0 \\
C_{0} \log L & \text{if } |\tilde{\mu}| = 0
\end{cases}
\]
(2.18)

for any multi-index \( \tilde{\mu} = (\mu_1, \cdots, \mu_n) \in \mathbb{E}^n \). The same bound holds for \( \Gamma_{N}^{\infty} \) and the bounds are uniform in \( m^2 \).
(Γ2) \( \Gamma_j(0,x)'s \) are independent of \( \Lambda_N \) if \( j \leq N - 1 \). Note that this makes sense only because of (Γ1).

(Γ3) \( \Gamma_j(m^2) \) (respectively \( \Gamma^{A_N}_j(m^2) \)) is continuous in \( m^2 \in (0,1] \) and attains limit \( \Gamma_j(0) \equiv \lim_{m^2 \to 0} \Gamma_j(m^2) \) (respectively \( \Gamma^{A_N}_j(0) \equiv \lim_{m^2 \to 0} \Gamma^{A_N}_j(m^2) \)) that satisfies

\[
\Gamma_j(m^2) = 0 \quad (0,0) = \frac{1}{2\pi(1+s)}(\log L + O(1)L^{-j-1}) \quad j \leq N - 1
\]  

(2.19)

with \( O(1) \) independent of \( j, L \) (but this is not necessarily true for \( \Gamma^{A_N}_j(0) \)).

(Γ4) \( t_N(s,m^2) \in (\max\{m^2 - CL2^N, 0\}, m^2) \) for some \( C > 0 \) and \( Q_N \) is the matrix with all entries equal to \( L^{-2N} \).

One immediate consequence of (Γ3) is that the limit \( m^2 \downarrow 0 \) only needs to be taken carefully when integrating against the covariance \( t_N(m^2)Q_N \). Indeed, by this decomposition, the expectation \( \mathbb{E}C(s,m^2) \) can be decomposed as

\[
\mathbb{E}C(s,m^2)[F(\varphi)] = \mathbb{E}C(s,m^2) = \mathbb{E}C(s,m^2) + \mathbb{E}C(s,m^2) \sum_{j=1}^{N-1} \Gamma_j + \tilde{\Gamma}_N \]  

(2.20)

where \( \zeta, \varphi' \) are independent Gaussian random variables with \( \varphi' \sim \mathcal{N}(0,t_NQ_N), \zeta \sim \mathcal{N}(0, \sum_{j=1}^{N-1} \Gamma_j + \tilde{\Gamma}_N(0)) \) so that \( \zeta + \varphi' \sim \mathcal{N}(0,t_N(C(s,m^2))) \). But since \( \Gamma_j(m^2)'s \) and \( \Gamma^{A_N}_j(m^2) \) converge as \( m^2 \downarrow 0 \), we can also take the same limit for \( \zeta(m^2) \) in (2.20). (Certain integrability condition has to be verified to prove this limit, and including \( s \) in the definition of \( Z_0 \) sufficiently small will do this job.) Hence the moment generating function will have its final alternative expression, whereafter we always take \( m^2 = 0 \) in \( \Gamma_j \) and \( \Gamma^{A_N}_j \).

In the following, we denote the tilted measure by

\[
\mathbb{E}(\tau)[F(\varphi)] = \frac{\mathbb{E}[e^{\tau(\varphi,\tilde{\varphi})}F(\varphi)]}{\mathbb{E}[e^{\tau(\varphi,\tilde{\varphi})}]} = \mathbb{E}(\tau)[F(\varphi)] \quad \tau \in \mathbb{C}
\]  

(2.21)

for a random variable \( \varphi \) on \( \mathbb{R}^{A_N} \) (or \( \mathbb{R}^{Z^2} \)) whenever the integrands are integrable.

**Proposition 2.3.** For \( |s| \) sufficiently small, \( f \) as in (A1), \( \tilde{f} = (1 + s\gamma\Delta)f, \beta > 0, \tau \in \mathbb{R}, \tau \in \mathbb{C} \), \( \beta = \tau + \tau \), and \( \mathbb{E}(\tau)[F(\varphi)] \) defined by (2.21),

\[
ed^{3-1/2}(\tilde{f},\sigma)_{\beta,A_N} = e^{3z^2(\tilde{f},\tilde{C}(s))} \lim_{m^2 \downarrow 0} F_{N,m^2}[f](\zeta,\tau)
\]  

(2.22)

where

\[
F_{N,m^2}[f]|(\zeta,\tau) = \frac{\mathbb{E} \mathbb{E}[Z_0(\varphi' + \tilde{\zeta} + 3\gamma f + \tau \tilde{C}(s)^{\tilde{f}})]}{\mathbb{E} \mathbb{E}[Z_0(\varphi' + \tilde{\zeta})]} \hspace{1cm} \]  

(2.23)

with \( \tilde{C}(s) = \sum_{j=1}^{N-1} \Gamma_j(m^2 = 0) + \Gamma^{A_N}_j(m^2 = 0) \), \( \varphi' \sim \mathcal{N}(0, t_N(m^2)Q_N) \) and \( \zeta \sim \mathcal{N}(0, \tilde{C}(s)) \).

**Proof.** We start from (2.22) and (2.6), saying

\[
ed^{3-1/2}(\tilde{f},\sigma)_{\beta,A_N} = \lim_{m^2 \downarrow 0} e^{3z^2(\tilde{f},\tilde{C}(s))} \mathbb{E} \mathbb{E}[Z_0(\varphi' + \tilde{\zeta})]
\]  

(2.24)

Since we have decomposition \( \varphi = \varphi' + \zeta(m^2) \) for some independent Gaussian random variables \( \varphi' \sim \mathcal{N}(0,t_NQ_N), \zeta(m^2) \sim \mathcal{N}(0, \sum_{j=1}^{N-1} \Gamma_j(m^2) + \Gamma^{A_N}_j(m^2) \), and since \( t_NQ_Nf = 0 \), if we can justify

\[
\lim_{m^2 \downarrow 0} \mathbb{E} \mathbb{E}[Z_0(\varphi' + \tilde{\zeta})] = \lim_{m^2 \downarrow 0} \mathbb{E} \mathbb{E}[Z_0(\varphi' + \tilde{\zeta})] \hspace{1cm} \]  

(2.25)

(for \( \zeta \sim \mathcal{N}(0,\tilde{C}(s)) \)), then we are done by a change of variables \( \zeta \to \zeta + \tau \tilde{C}(s)^{\tilde{f}} \). But (2.25) is implied by (Γ3).
In the statement, the coalescence scale $Q$ (recall holds, the size of $\tilde{\eta}$ of the ratio of renormalisation group argument is used to prove the proposition. Assuming Proposition 3.1. In the second half of this section, we give a brief overview of how the coalescence scale is formally just $\log L$.

Remark 2.4. Note that $E_\tau$ is not a probability measure for general $\tau \in C$. Even if we have assumed that $F$ is an analytic function with nice decay properties so that

$$E_{(\tau, C(\gamma))} [F(\bar{\eta})] = E_{C(\gamma)} [F(\bar{\eta} + \tau C(\gamma))]$$

holds, the size of $\tilde{C}(\gamma)$ diverges logarithmically in $|A_N|$. We will see in later sections that such a divergence of the complex shift is not preferable in the technical point of view. Thus we use an alternative approach in the following sections, where we also make multi-scale decomposition of the complex shifts.

2.3. Choice of parameters. The RG map constructed in $[1]$ is sensitive to the choice of parameters that will appear in the following sections. Since the current work also depends on this RG map, the choice of these parameters will also play non-negligible role. But this will not be at the centre of our discussion, so we try to simplify our account on this problem in later chapters once we define these parameters in this section.

- $\gamma > 0$ is a constant related to the discrete Laplacian and $c_f = \frac{1}{2} \gamma$ is related to the DG model. They are introduced in $[1]$ Section 3.7.
- $L$ is a parameter defining the renormalisation group map. It is chosen $L \geq L_0(M, \rho, \tau)$ where $L$ satisfies the assumptions $[1]$ Theorem 7.7, Lemma 8.3, Theorem 9.1 and Proposition 5.2. Also it is restricted to take value of the form $L = \ell N'$ for some $\ell$ and $N'$ both sufficiently large as in Lemma A.2.
- $A, c_2, c_4, c_w, c_k, c_h, c_0, \gamma, h, h_1 > 0$ appear in the definition of norms introduced in Section $[1]$ and related definitions in Appendix $A$. We require $A \geq A_0(L)$ to satisfy the assumptions of $[1]$ Theorem 7.7 and Proposition 5.4. $c_2, c_4, c_w, c_k, c_h$ are chosen according to $[1]$ Section 5-7, $[2]$ Section 3] and Lemma B.3. $\kappa = \sqrt{c_k / \log L}$ and $\tilde{h} = \max \{1, c_h \sqrt{\beta} \}$. $c_1$ and $h_1$ are fixed so that $h_1 = c_1 \log L^{-3/2}$ and satisfy Lemma 4.8, Lemma 4.10 and Lemma 4.12.
- $\beta \geq \beta_0$ is chosen to satisfy the assumptions of $[1]$ Theorem 7.7] and Proposition 5.2.
- There are constants that might share the same symbol but differ from line to line—e.g. $C, C_1, C_2, C', C''$ etc.

Some conditions are omitted in the list of references above. This is either because the condition is abundant or it is only a minor modification to the condition introduced in $[1]$. The main difference in the choice of parameters in this work with that of $[1]$ will be due to the introduction of new parameters $M, \rho, R$. However, this dependence might not always be apparent in the writing.

3 Overview of the proof

In the first half of this section, we prove the main theorems Theorem 1.1 and Theorem 1.4 assuming Proposition 3.1. In the second half of this section, we give a brief overview of how the renormalisation group argument is used to prove the proposition.

In view of Proposition 2.3 control of the moment generating function is just due to the control of the ratio $F_{N,m^2}$. Thus it is the objective of Proposition 3.1 to show how $F_{N,m^2}$ is controlled. In the statement, the *coalescence scale* $j_{by}$ is used:

$$j_{by} = \min \{ j \geq 0 : (B^j_{t})^{***} \cap Q^j_{y} \neq \emptyset \}.$$  (3.1)

(Recall $Q^j_{y} = \cup_{i=1} A_i B^j_{t}$, and $(B^j_{t})^{***}$ is taking the small set neighbourhood three times.) The coalescence scale is formally just $\log_L \text{dist}_{\infty}(\text{supp}(u_{j,1}), y + \text{supp}(u_{j,2}))$, but stated in the language
of blocks. If either \( u_{j,1} \equiv 0 \) or \( u_{j,2} \equiv 0 \), we use convention \( j_0 y = \infty \). Since \( B_0^j \supset \text{supp}(u_{j,1}) \) and \( Q_0^y \supset \text{supp}(T_y u_{j,2}) \), this definition implies \( \text{supp}(u_{j,1}) \cap \text{supp}(T_y u_{j,2}) = \emptyset \) for \( j < j_0 y \) and there exists \( C > 0 \) such that

\[
C^{-1} \| y \|_2 \leq L^{j_0 y} \leq C \| y \|_2.
\]

**Proposition 3.1.** Let \( f \) be as in (A.3) Then there exists \( s_0(\beta) = O(e^{-c \beta}) \) such that the following holds when \( Z_0 \) is defined by (2.7) with \( s = s_0(\beta) \): under the assumptions of Theorem 1.4 if \( \beta = \tau + \tau \) for some \( \tau \in \mathbb{R} \) and \( \tau \in \mathbb{D}_{h_1} \), then

\[
\lim_{m \to 10} F_{N,m} f(\tau, \tau) = e^{g_{\infty}(1) [f_1(\tau) + g_{\infty}(f_2(\tau) + \sum_{j > j_0 y} \tilde{g}_j(2, t_1, T_y f_2(\tau)) (1 + \tilde{c}_N(\tau, y)))}
\]

where \( \tilde{g}_\infty[f_1(\tau), \tilde{g}_j(2, t_1, T_y f_2(\tau)) and \tilde{c}_N(\tau, y) \) are analytic functions in \( \tau \in \mathbb{D}_{h_1} \) and satisfy

\[
\| \tilde{g}_j(2, t_1, T_y f_2(\tau)) \|_{L^\infty(D_{h_1})} \leq O(L^{-\alpha j})
\]

\[
\| \tilde{c}_N(\tau, y) \|_{L^\infty(D_{h_1})} \leq O(L^{-\alpha N})
\]

for some \( \alpha > 0 \).

The proof will only appear in Section 8. One point to emphasize is that each one-point free energy \( \tilde{g}_\infty[f_1(i = 1, 2) \) is independent of choice of \( y \). This is not an obvious consequence of the renormalisation group method, as it has strong dependence on the choice of the gridding, i.e., the choice of configurations of \( B_j \).

### 3.1. Proof of the main theorems.

The main theorems are direct consequences of Proposition 3.1.

**Proof of Theorem 1.4** Let \( Z_0 \) be defined by (2.7) and \( s = s_0(\beta) \) as in Proposition 3.1 Then by Proposition 2.8 and Proposition 3.1

\[
\log(e^{\beta^{-1/2}(\sigma, f)} DG) = \frac{1}{2} \lambda^2 (f, \hat{C}(s_0(\beta))) \int + \frac{1}{2} \gamma^2 (f, \hat{f}) \]

\[
+ \tilde{g}_\infty(\tau) [f_1] + \tilde{g}_\infty(\tau) [f_2] + \sum_{j > j_0 y} \tilde{g}_j(2, t_1, T_y f_2(\tau)) + \log (1 + \tilde{c}_N(\tau, y))
\]

(3.6)

By (3.1), (3.5) and (3.2), they satisfy

\[
\sum_{j > j_0 y} \tilde{g}_j(2, t_1, T_y f_2(\tau)) \leq O(L^{-\alpha j_0 y}) \leq O(\| y \|_2^{-\alpha}), \quad \| \tilde{c}_N(\tau, y) \| \leq O(L^{-\alpha N}) \leq O(|\Lambda N|^{-\alpha}).
\]

(3.7)

If we let

\[
H_N(\beta, y) = \log(e^{\beta^{-1/2}(\sigma, f)} DG) = \frac{1}{2} \lambda^2 (f, \hat{C}(s_0(\beta))) \int + \frac{1}{2} \gamma^2 (f, \hat{f}) \]

\[
+ \tilde{g}_\infty(\tau) [f_1] + \tilde{g}_\infty(\tau) [f_2] + \sum_{j > j_0 y} \tilde{g}_j(2, t_1, T_y f_2(\tau)) + \log (1 + \tilde{c}_N(\tau, y)),
\]

then it is an analytic function of \( \beta \in S_{R, h_1} \) (well-defined, because it is independent of the decomposition \( \tau + \tau \)). It also follows from taking limits \( N, \| y \|_2 \rightarrow \infty \) that

\[
h^{(1)}_\beta(1)[f_1(\beta) + f^{(1)}(f_2(\beta)) = \frac{1}{2} \lambda^2 \sum_{i=1,2} (f_i, \hat{f}_i) + \lim_{N \rightarrow \infty} H_N(\beta, y)
\]

(3.9)

\[
h^{(2)}_\beta(1)[f_1(\beta) + f^{(1)}(f_2(\beta)) = \frac{1}{2} \lambda^2 (f_1 + T_y f_2, \hat{f}_1 + T_y f_2) - h^{(1)}_\beta(1)[f_1(\beta) - h^{(1)}_\beta(1)[f_2(\beta)] + \lim_{N \rightarrow \infty} H_N(\beta, y)
\]

(3.10)

\[
\psi_\beta(\beta, y) = H_N(\beta, y) - \lim_{N \rightarrow \infty} H_N(\beta, y)
\]

(3.11)
are also analytic functions of $z$, independent of the decomposition $z = \tau + \tau'$. (3.9) leaves infinitely many choices of each $h^{(1)}_3[f_j(z)] (i = 1, 2)$, but we choose the branch where $h^{(1)}_3[f_j(z)] = g^\infty[f_j(z)] + \frac{2}{5} \tilde{\sigma}(f_j, \tilde{\nu})$ for $z \in \mathbb{D}_{h_1}$. Thus we obtain (1.8) with the above choices of $h^{(1)}_3$, $h^{(2)}_3$, $\psi_{\beta,N}$ and

$$C_{\beta,N} = (1 + s_0^\beta(S\Delta)\bar{\tilde{C}}(s_0^\beta))(1 + s_0^\beta(S\Delta)).$$  \hspace{0.5cm} (3.12)

The required estimates follow from (3.7). Also, for any $g = g_1 + T g_2$ that satisfies the assumptions of (A) we have $|\langle g, \Gamma \rangle| = O(L^{-2}|y|)$ and $|\langle g, \hat{\Gamma}^\Lambda_N \rangle| = O(L^{-2}|y|)$ by (1.1) so $\lim_{N \to \infty}(g, C_{\beta,N} g)$ absolutely converges. Hence, if we let

$$C_{\beta,\mathbb{Z}^2} = \sum_{j=1}^{\infty} (1 + s_0^\beta(S\Delta)\bar{\tilde{C}}(s_0^\beta))(1 + s_0^\beta(S\Delta)).$$  \hspace{0.5cm} (3.13)

then $(f, C_{\beta,\mathbb{Z}^2} f)$ is well-defined and equals $\lim_{N \to \infty}(f, C_{\beta,N} f)$.

**Proof of Theorem 1.2** For fixed $y \in \mathbb{Z}^2$, let $f_y(x) = \delta_0(x) - \delta_y(x)$. By Theorem 1.4, we just need to compute the asymptotic of $(f_y, C_{\Lambda_N,\beta} f_y)$ as $\|y\|_2 \to \infty$. Since $\bar{\tilde{C}}(s_0^\beta) = \lim_{m \geq 0} C(s_0^\beta, m^2 - t_N(m^2)Q_N$, we have in the Fourier space

$$(f_y, C_{\mathbb{Z}^2,\beta} f_y) = \frac{1}{|\Lambda_N|} \sum_{p \in \Lambda_N^*} |1 - e^{-i p \cdot y}|^2 |1 - s_0^\beta(S\Delta)|^2 \lim_{m \to 0} \frac{(\lambda(p) + m^2)^{-1} - \gamma}{1 + s_0^\beta(S\Delta)(\lambda(p) + m^2)^{-1} - \gamma} - t_N \delta_0(p))$$  \hspace{0.5cm} (3.14)

where $\lambda(p) = 2 - \cos(p_1) - \cos(p_2)$ is the Fourier transformation of $-\Delta$ and $\Lambda_N^*$ is the Fourier dual lattice of $\Lambda_N$ and $s = s_0^\beta$. Since $1 - e^{-i p \cdot y} = 0$ when $p = 0$, we may ignore $t_N \delta_0(p)$ term. In the limit $N \to \infty$, this discrete sum converges to the integral

$$\int_{[-\pi, \pi]^2} dp |1 - e^{-i p \cdot y}|^2 |1 - s_0^\beta(S\Delta)|^2 \lim_{m \to 0} \frac{(\lambda(p) + m^2)^{-1} - \gamma}{1 + s_0^\beta(S\Delta)(\lambda(p) + m^2)^{-1} - \gamma}.$$  \hspace{0.5cm} (3.15)

From this representation, as $\|y\|_2 \to \infty$,

$$(f_y, C_{\mathbb{Z}^2,\beta} f_y) = C_1(\beta) + \frac{1}{1 + s_0^\beta(S\Delta)}(f_y, (-\Delta)^{-1} f_y) + O(\|y\|_2^{-2})$$

$$= C_2(\beta) + \frac{1}{\pi(1 + s_0^\beta(S\Delta))} \log \|y\|_2 + O(\|y\|_2^{-2})$$  \hspace{0.5cm} (3.16)

for some $C_1(\beta), C_2(\beta) > 0$. Also since $h^{(\alpha)}_3$ are analytic $h^{(2)}_3[\delta_0 - \delta_0(\cdot, y)$ is an even function, we can extract out the values at $\tilde{\tau} = 0$ to obtain

$$\log(e^{\beta^{-1/2}(\sigma(0) - \sigma(y)))_{\mathbb{Z}^2} = C_3(\beta, y) + \frac{1}{2 \pi(1 + s_0^\beta(S\Delta))} \tilde{\sigma}(\log \|y\|_2 + C_4(\beta) + \tilde{\sigma}(\cdot, y))$$

$$+ (h^{(1)}_3[\delta_0(\cdot, \tilde{\tau}) + h^{(1)}_3[-\delta_0(\cdot, \tilde{\tau}) - h^{(1)}_3[\delta_0(\cdot, y) - h^{(1)}_3[-\delta_0(\cdot, y)]$$

where

$$r_\beta(\cdot, y) = \frac{h^{(2)}_3[\delta_0 - \delta_0(\cdot, y) - h^{(2)}_3[\delta_0 - \delta_0(\cdot, y) = O_\beta(\|y\|_2^{-2}) \text{ as } \|y\|_2 \to \infty$$  \hspace{0.5cm} (3.17)

But since the left-hand side of (3.17) vanishes when $\tilde{\tau} = 0$, we should have $C_3(\beta, y) = 0$. 

\[\square\]
3.2. Tilted expectation as shifted environment. In the rest of the section, we give a brief overview of how Proposition 3.1 is proved.

As is explained in Remark 2.1, the complex shift will have to be dealt by making a multi-scale decomposition. Namely, we consider

\[ \tilde{\mathbb{E}}_{(\tau), \tilde{\mathcal{C}}(\delta)}[\mathcal{C}^\tau] = \mathbb{E}_{(\tau), \mathcal{F}_{N-1}}^{\mathcal{N}_N} \mathbb{E}_{(\tau), \mathcal{F}_{N-2}}^{\mathcal{N}_{N-1}} \cdots \mathbb{E}_{(\tau), \mathcal{F}_1}^{\mathcal{N}_1} [\mathbb{C}(\zeta_1 + \cdots + \zeta_N)]. \]  

(3.19)

It will turn out in Lemma 4.10 that, under mild conditions on some function \( F_j \) analytic on a neighbourhood of the real space, \( \mathbb{E}_{(\tau), \mathcal{F}_{j+1}}^{\mathcal{N}_j} [F_j(\varphi' + \zeta_j)] = \mathbb{E}_{(\tau), \mathcal{F}_{j+1}}^{\mathcal{N}_j} [F_j(\varphi' + \zeta_j + \tau \Gamma_{j+1} \tilde{f})] \), so the tilted expectation can be written as an ordinary expectation with shifted environment interchangeably. Hence we prepare a lemma that controls the complex shifts seen in each renormalisation group step.

**Definition 3.2.** For \( f = f_1 + T_y f_2 \) as in \([A_j]\) and \( s \in \mathbb{R} \), let \((u_{j,\alpha})_{j \geq 1}\) be defined by

\[ u_{j,\alpha} = \begin{cases} \gamma f_\alpha & \text{if } j = 0 \\ \Gamma_j (1 + s\gamma \Delta) f_\alpha & \text{if } 1 \leq j < N \\ \Gamma_j^N (1 + s\gamma \Delta) f_\alpha & \text{if } j = N. \end{cases} \]  

(3.20)

and \( u_j = u_{j,1} + T_y u_{j,2} \).

Since \( \tilde{\mathcal{C}}(s) = \sum_{j=1}^{N-1} \Gamma_j + \Gamma_j^N \), if we let \( \tilde{f}_\alpha = (1 + s\gamma \Delta) f_\alpha \), these definitions say

\[ \sum_{j=1}^{N} u_{j,\alpha} = \tilde{\mathcal{C}}(s) \tilde{f}_\alpha, \quad \alpha \in \{1, 2\}. \]  

(3.21)

They also satisfy the following estimates.

**Lemma 3.3.** If \( L \geq 12(\rho + 2) \), then there exists \( C_n > 0 \) for each \( n \geq 0 \) such that, for each multi-index \( \bar{\mu} = (\mu_1, \ldots, \mu_n) \in \hat{\mathbb{C}}^n \)

\[ \| \nabla^{\bar{\mu}} u_{j,\alpha} \|_{L^\infty} \leq \begin{cases} C_0 M \rho^2 \log L & \text{if } n = 0, \|\bar{\mu}\| = 0 \\ C_n M \rho^2 L^{-n(0\vee(j-1))} & \text{if } n = \|\bar{\mu}\| > 1 \end{cases} \]  

(3.22)

and \( \text{supp}(u_{j,\alpha}) \subset B^j_0 \), where \( B^j_0 \) is the unique \( j \)-scale block that contains 0.

**Proof.** When \( j = 0 \), the bounds are direct from the definition. When \( j \in \{1, \ldots, N-1\} \), for each \( \alpha \in \{1, 2\}, \)

\[ \| \nabla^{\bar{\mu}} \Gamma_j f_\alpha \|_{L^\infty} = \sup_{x \in \Lambda_N} \| \sum_{y \in \Lambda_N} \nabla^{\bar{\mu}} \Gamma_j(x - y) f_\alpha(y) \| \leq M \rho^2 \| \nabla^{\bar{\mu}} \Gamma_j \|_{L^\infty}, \]  

(3.23)

while by (11) \( \| \nabla^{\bar{\mu}} \Gamma_j \|_{L^\infty} \leq CL^{-n(j-1)} \) for \( n \geq 1 \) and \( \| \Gamma_j \|_{L^\infty} \leq C \log L \) for \( n = 0 \). The same method also bounds \( \| \nabla^{\bar{\mu}} \Gamma_j \Delta f_\alpha \|_{L^\infty} \), and these are enough to bound \( \| \nabla^{\bar{\mu}} u_{j,\alpha} \|_{L^\infty} \). Finally, for \( j = N, \nabla^{\bar{\mu}} \Gamma_N \) satisfies bounds analogous to (2.18), so we have the desired bounds.

To see that \( \text{supp}(u_{j,\alpha}) \subset B^j_0 \), recall \( \Gamma_j(x, y) = 0 \) whenever \( \text{dist}_\infty(x, y) \geq \frac{1}{4} L^j \), so \( u_{j,\alpha}(x) = 0 \) whenever \( \|x\|_\infty \geq \rho + 2 + \frac{1}{4} L^j \). But by the definition of \( B^j_0 \), we have \( \text{dist}_\infty(0, (B^j_0)^c) \geq \frac{L^j}{3} \), so \( u_{j,\alpha}(x) = 0 \) for any \( x \notin B^j_0 \) and whenever \( L \geq 12(\rho + 2) \). \( \square \)

Although the lemma is proved for general \( |\bar{\mu}| = n \geq 0 \), we actually only need this result for \( n \leq 2 \). That is, we only use the fact that

\[ \| u_{j,\alpha} \|_{C^2_j} = \max_{n=0,1,2} \sup_{\bar{\mu} \in \hat{\mathbb{C}}^n} L^{n\bar{\mu}} \| \nabla^{\bar{\mu}} u_{j,\alpha} \|_{L^\infty} \]  

(3.24)

is bounded by a uniform constant. The following assumption on \((u_j)_{j \geq 0}\)'s can be used as well as \([A_j]\) when \( f_j \)'s are not mentioned directly.
Let $M, \rho > 0$. The sequence $(u_j)_{j \geq 1}$ has decomposition $u_j = u_{j,1} + Tu_{j,2}$ where $u_{j,1}$ and $u_{j,2}$ satisfy the following: for $\alpha \in \{1, 2\}$, there exists $C > 0$ such that $\|u_{j,\alpha}\|_{C^0_{\mathbb{A}^N(j-1)}} \leq CM\rho^2 \log L$ for each $j \leq N$, and $u_{j,\alpha}$ is supported on the unique $B_0^j \in B_j$ such that $0 \in B_0^j$.

Note that $\text{supp}(u_{j,2}) \subset B_0^j$ also implies

$$\text{supp}(Tu_{j,2}) \subset Q_{y_j}^j. \quad (3.25)$$

3.3. Effective potentials. An important implication of the covariance decomposition (2.17) is that it can be used to define the renormalisation group (RG) flow. Given $0^{th}$ scale partition function function $Z_0 : \mathbb{C}^{A_N} \rightarrow \mathbb{C}$, (2.7), the partition functions of scale $j$ are $Z_j : \mathbb{R}^{A_N} \times \mathbb{D}_h \rightarrow \mathbb{C}$ defined recursively by

$$Z_j^0(\varphi', \zeta; \tau) = Z_0(\varphi' + \zeta + (\tau)u_0) \quad (3.26)$$

and

$$Z_{j+1}^j(\varphi'; \tau) = \begin{cases} \mathbb{E}_{\tau+1}[Z_j^j(\varphi' + \zeta; \tau)] & \text{if } 0 \leq j \leq N - 2 \\ \mathbb{E}_{\tau+1}[Z_{j-1}^j(\varphi' + \zeta; \tau)] & \text{if } j = N - 1 \end{cases} \quad (3.27)$$

where the tilted expectations acts on $\zeta$. $\log Z_j^j$ is also called an effective potential, as it represents the renormalised theory at scale $j$. Note that

$$\mathbb{E}_{\tau+1}[Z_j^j(\varphi' + \zeta; \tau)] = \mathbb{E}_{\tau+1}[Z_j^j(\varphi' + \zeta; \tau)], \quad (3.28)$$

and similar holds for $\mathbb{E}_{\tau+1}[\cdots]$, so by (3.21),

$$\mathbb{E}_{\tau+1}[Z_0(\varphi' + \zeta + 5\gamma f + \tilde{C}(s)\tilde{f})] = Z_N(\varphi'; \tau) \quad (3.29)$$

$$\mathbb{E}_{\tau+1}[Z_0(\varphi' + \zeta)] = Z_N^0(\varphi'; 0). \quad (3.30)$$

Thus $F_{N,m^2}$ and $Z_N^j$ are related by

$$F_{N,m^2} = \mathbb{E}_{\tau+1}[Z_N^j(\varphi'; \tau)/\mathbb{E}_{\tau+1}[Z_0^j(\varphi'; 0)]. \quad (3.31)$$

3.4. Renormalisation group coordinates. We seek for a representation of $Z_j^j(\varphi; \tau)$ given by $j$-scale polymer expansion

$$Z_j^j(\varphi; \tau) = e^{-E_j(\Lambda_N|+g_j^j(\Lambda_N; \tau)} \sum_{X \in \theta_j} e^{U_j(\Lambda_N|, \varphi + (\tau)u_j)} K_j^j(X, \varphi; \tau), \quad j \geq 1 \quad (3.32)$$

for RG coordinates $(E_j, g_j^j, U_j, K_j^j)$, elements of normed spaces defined in Section 4. We give only a brief overview here: $E_j$ is a real number, $g_j^j(\Lambda_N; \tau)$ is an analytic function of $\tau \in \mathbb{D}_h$, and

$$U_j(X, \varphi) = \sum_{x \in X} \frac{1}{2} s_j \|\nabla \varphi(x)\|^2 + W_j(X, \varphi) \quad (3.33)$$

$$W_j(X, \varphi) = \sum_{q \geq 1} z_j^{(q)} \cos (q^{1/2} \beta \varphi(x)) \quad (3.34)$$

for any $\varphi \in \mathbb{C}^A$ and some $s_j \in \mathbb{R}$, $(z_j^{(q)})_{q \geq 1} \subset \mathbb{R}$. The polymer activity $K_j^j(X, \varphi; \tau)$ parameterises the deviation of $Z_j^j$ from $e^{U_j}$ (up to a constant multiple). The domain of $\varphi$-component of $K_j^j$ is initially $\mathbb{R}^A$, but it will be extended to some strip of $\mathbb{C}^{A_N}$ containing $\mathbb{R}^{A_N}$, whose specification becomes important when we discuss the analyticity of $K_j^j$.

Section 4 gives an explicit construction of the RG coordinates in scale $j + 1$, given the coordinates in scale $j$. This is called the RG map. Namely, if there exists a specific choice of
(E_j, g_j, U_j, K_j)_{j \leq J} satisfying \((3.27)\) and \((3.32)\), then \((E_{j+1}, g'_{j+1}, U_{j+1}, K_{j+1}')\) with certain contraction properties is constructed via the RG map

\[
\Phi_{j+1}^*: (U_j, K_j^0, K_j^0) \rightarrow (E_{j+1} - E_j, g'_{j+1} - g_j, U_{j+1}, K_{j+1}').
\]

(3.35)

The construction is based on operations defined in Section \[5\]. Key analytic properties of \(\Phi_{j+1}^*\) are also stated Section \[6\] but we defer the technical proof to Section \[7\]. A reader not familiar with the content of \[4\] may skip that section on the first read.

In general, the existence of the maps \(\Phi_{j+1}^*\) is not guaranteed for arbitrary large choice of \(N\) and \(j \leq N\). The content of the first part of Section \[8\] is to prove such an existential theorem. This theorem relies on the ‘stable manifold’ theorem of \[1\]—also see Section \[5.1\] where the existential theorem is obtained with \(\gamma = \tau = 0, g'_j \equiv 0, s_j, (z_j^{(a)})_{a \geq 0}, K_j^0 \rightarrow 0\) (in norms specified later) and a specific initial condition. Thus we only have to show the same convergence is also valid with the same initial condition and general \(\gamma \in \mathbb{R}, \tau \in \mathbb{D}_{h_1}\).

The second half of Section \[8\] concludes the proof of Proposition \[3.1\]. By \((3.29), (3.30)\) and \((3.32)\), the ratio of partition functions in \((2.22)\) will converge to \(\lim_{j \to \infty} \exp(g'_j)\) as \(N \to \infty\). Since the multi-scale polymer expansion \((3.32)\) has strong dependence on the structure of the grids, the conclusion will be subject to the fact that the limit \(\lim_{j \to \infty} g'_j\) is free of the grid bias. Indeed, this will be seen to be the case using a simple argument involving the translation invariance of the system.

## 4 Polymer activities and norms

As in \((3.32)\), the partition function at scale \(j\) is parametrised by coordinates \(E_j, g_j, U_j\) and \(K_j^0\). While the form of \(U_j\) is restricted by \((3.33)\), \(K_j^0\) can be a polymer activity of fairly high degree of freedom. In this section, we recall the norms defined on the polymer activities \(U_j\) and \(K_j^0(X, \cdot, \cdot)\) from \[5\] and generalise them so we can encode information about analyticity in the parameter \(\tau\).

### 4.1. Norms

We recall the norm on polymer activities from \[4\] using the regulator \(G_j\) in the following definition. A number of parameters, \(A, h, c_h, c_w, c_2 > 0\) and \(\kappa = c_w (\log L)^{-1}\) will be appearing in the definition of the norm. The choice of these parameters are summarised in Section \[2.3\].

**Definition 4.1.** For \(X \in \mathcal{P}_j\) and \(\varphi \in \mathbb{R}^{\mathcal{N}}\), define the regulator at scale \(j\) by

\[
G_j(X, \varphi) = \exp \left( \kappa (\|\nabla_j \varphi\|^2_{L_2^j(X)} + c_2 \|\nabla_j \varphi\|^2_{L_2^j(\partial X)} + W_j(X, \nabla_j^2 \varphi)^2) \right)
\]

(4.1)

where

\[
W_j(X, \nabla_j^2 \varphi)^2 = \sum_{B \in \mathcal{B}_j(X)} \|\nabla_j^2 \varphi\|^2_{L_2^\infty(B^c)}.
\]

(4.2)

Also define

\[
w_j(X, \varphi)^2 = \sum_{B \in \mathcal{B}_j(X)} \max_{n=1,2} \|\nabla_j^n \varphi\|^2_{L_2\infty}.
\]

(4.3)

The role of \(w_j\) is to make the regulators submultiplicative. While it is in general \(G_j(X, \varphi)G_j(Y, \varphi) \not\leq CG_j(X \cup Y, \varphi)\) for \(X \cap Y = \emptyset\), it is true, by \[4\] Lemma 5.8, for \(X, Y \in \mathcal{P}_j\) such that \(X \cap Y = \emptyset\),

\[
e^{c_w w_j(X, \varphi)^2} G_j(Y, \varphi) \leq G_j(X \cup Y, \varphi)
\]

(4.4)

whenever \(c_w\) is chosen sufficiently small. In particular, we have \(e^{c_w w_j(X, \varphi)^2} \leq G_j(X, \varphi)\), and for this reason, \(e^{c_w w_j(X, \varphi)^2}\) is also called the *strong regulator*.

Then we define (semi-)norms on polymer activities.

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Definition 4.2. For $F \in \mathcal{N}$, define
\[
\|D^n F(\varphi)\|_{n,T_j(X,\varphi)} = \sup\{|D^n F(\varphi)(f_1, \cdots, f_n)| : \|f_k\|_{C^2_j(X^*)} \leq 1 \text{ for each } k\}
\] (4.5)
\[
\|F(X,\varphi)\|_{h,T_j(X,\varphi)} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \|D^n F(X,\varphi)\|_{n,T_j(X,\varphi)}
\] (4.6)

We also use polymer activities that are analytic in parameter $\tau$, and the corresponding (semi-)norms.

Definition 4.3. For $c_t > 0$ a constant, let $h_t = c_t (\log L)^{-3/2}$. Define the positive Wiener space $W^+ (\mathbb{D}_{h_t})$ to be the space of holomorphic functions $f : \mathbb{D}_{h_t} \to \mathbb{C}$ such that the Wiener norm
\[
\|f\|_{h_t,W} = \sum_{n=0}^{\infty} \frac{h_t^n}{n!} |\partial^n f(\tau=0)|
\] (4.7)
is finite. For $A' > 0$, $F \in \mathcal{N}$, and $\tilde{h} = (h, h_t)$, define
\[
\|F(X,\varphi;\cdot)\|_{\tilde{h},T_j(X,\varphi)} = \sum_{n=0}^{\infty} \frac{h^n_t}{n!} |\partial^n F(X,\varphi;\tau=0)|_{h,T_j(X,\varphi)}
\] (4.8)
\[
\|F(X,\cdot;\cdot)\|_{\tilde{h},T_j(X,\varphi)} = \sup_{\varphi \in \mathcal{R}^N} G_j(X,\varphi)^{-1} \|F(X,\cdot;\cdot)\|_{\tilde{h},T_j(X,\varphi)}
\] (4.9)
\[
\|F\|_{h_t,T_j,X'} = \sup_{X \in \mathcal{Y}_f(\lambda_N)} (A')^{1_1} \|F(X,\cdot;\cdot)\|_{\tilde{h},T_j(X)}.
\] (4.10)

When $A' = A$, then the suffix $A'$ will often be omitted, i.e., $\|K_j(X;\cdot)\|_{\tilde{h},T_j} = \|K_j(X;\cdot;\cdot)\|_{\tilde{h},T_j,X'}$.

\[
\|F(X,\varphi;\tau)\|_{\tilde{h},T_j(X,\varphi)} = \|\mathcal{K}_j(X, (\varphi, \tau))\|_{h_t,T_j(X,\varphi)}
\] (4.11)

where $\mathcal{K}_j$ is a polymer activity on $\mathcal{P}_j \times (\mathcal{R}^N \times \mathcal{R}^N)$ defined by $\mathcal{K}_j(X, (\varphi, \tau)) = K_j(X, \varphi; \tau)$. Using this observation, we can re-derive most of the results on $\|\cdot\|_{h_t,T_j(X,\varphi)}$, obtained in [4], for $\|\cdot\|_{\tilde{h},T_j(X,\varphi)}$. In this concern, we will mention that certain results of [4] can be translated into results here, often without justifications in full detail. Vice versa, most bounds proved for $\|\cdot\|_{\tilde{h},T_j(X,\varphi)}$ should also be true for $\|\cdot\|_{h_t,T_j(X,\varphi)}$ if there is no specific dependence on $\tau$.

For $U_j$ in a restricted form, we can define and use a stronger norm.

Definition 4.5. Let $c_f = \frac{1}{q}$. Given $(s_j, (z_j^{(q)})_{q \geq 1}) \in \mathbb{R} \times \mathbb{R}^N$ and $U_j$ given by (3.33)-(3.34), define
\[
\|U_j\|_{\Omega_j^q} = A \max \left\{|s_j|, \sup_{q \geq 1} e^{|z_j^{(q)}|} |z_j^{(q)}|\right\}.
\] (4.12)

It is known from [4] Lemma 7.4, Lemma 7.13 with the choice of $\beta$ and $c_h$ as in Section 2.3 that
\[
\left\|\frac{1}{2} s_j |\nabla \varphi|_{B,j}^2\right\|_{2h,T_j(B,\varphi)} \leq C |s_j| w_j(B, \varphi)^2, \quad \|W_j(B, \varphi)\|_{2h,T_j(B,\varphi)} \leq CA^{-1} \|W_j\|_{\Omega_j^q}.
\] (4.13)
($h$ is replaced by $2h$ here, which is okay if we choose $C$ sufficiently large and $\beta \geq 8c_f^{-1}$.)
4.2. Key inequalities. We highlight the most important properties of the norms defined in the previous section. Whenever we are talking about $j$-scale polymer activities, $\mathbb{E}$ means we are taking expectation over $\zeta \sim \mathcal{N}(0, \Gamma_{j+1})$ if $j+1 < N$ and $\zeta \sim \mathcal{N}(0, \Gamma_{j}^2)$ if $j+1 = N$. The non-random part of the field is often denoted $\varphi'$.

**Lemma 4.6** (Submultiplicativity of the seminorm). For $\varphi \in \mathbb{R}^{A_n}$ and $Y_1, Y_2, X \in \mathcal{P}_j$ such that $Y_1, Y_2 \subset X$, suppose $F_1 \in \mathcal{H}_j(Y_1)$, $F_2 \in \mathcal{H}_j(Y_2)$ with $\|F_\alpha(Y_\alpha, \varphi)\|_{\bar{h}_j(Y_\alpha, \varphi)} < \infty$, $\alpha \in \{1, 2\}$. Then

$$\|F_1(Y_1, \varphi ; \tau) F_2(Y_2, \varphi ; \tau)\|_{\bar{h}_j(Y, \varphi)} \leq \|F_1(Y_1, \varphi ; \tau)\|_{\bar{h}_j(Y_1, \varphi)} \|F_2(Y_2, \varphi ; \tau)\|_{\bar{h}_j(Y_2, \varphi)}. \quad (4.14)$$

Also for each $k \geq -1$,

$$\|e^{F_1(X, \varphi ; \tau)} - \sum_{m=0}^{k} \frac{1}{m!} (F_1(X, \varphi ; \tau))^m\|_{\bar{h}_j(X, \varphi)} \leq \sum_{m=k+1}^{\infty} \frac{1}{m!} \|F_1(X, \varphi ; \tau)^m\|_{\bar{h}_j(X, \varphi)} \quad (4.15)$$

with convention $\sum_{m=0}^{-1} (\cdots) \equiv 0$.

**Proof.** The first inequality is a result of the submultiplicativity of the $\|\|_{\bar{h}_j(X, \varphi)}$-seminorm, \cite[(5.22)]{4} and Remark 4.4. To see the second, consider the sequence of polymer activities

$$H_k(X, \varphi ; \tau) = \sum_{m=0}^{k} \frac{1}{m!} (F_1(X, \varphi))^m. \quad (4.16)$$

Then by the submultiplicativity, \cite[(4.14)]{4}. we have

$$\|H_k(X, \varphi ; \tau)\|_{\bar{h}_j(X, \varphi)} \leq \exp (\|F_1(X, \varphi ; \tau)\|_{\bar{h}_j(X, \varphi)}) \quad (4.17)$$

Since $e^{F_1}$ is a pointwise limit of $H_k$ and $D^m \partial_{\tau}^n H_k(X, \varphi ; \tau) \rightarrow D^m \partial_{\tau}^n e^{F_1}(X, \varphi ; \tau)$ as $k \rightarrow \infty$ for each $n, m \geq 0$, we have

$$\sum_{n,m \geq 0} \frac{h^n \bar{h}^m}{m!} \|D^m \partial_{\tau}^n e^{F_1(X, \varphi ; \tau)}\|_{\bar{h}_j(X, \varphi)} \leq \limsup_{k \rightarrow \infty} \|H_k(X, \varphi ; \tau)\|_{\bar{h}_j(X, \varphi)} \quad (4.18)$$

for any $N' > 0$ so we see in fact

$$\|e^{F_1(X, \varphi ; \tau)}\|_{\bar{h}_j(X, \varphi)} \leq \exp (\|F_1(X, \varphi ; \tau)\|_{\bar{h}_j(X, \varphi)}) < \infty. \quad (4.19)$$

Then \cite[(4.15)]{4} is obtained from the Taylor’s theorem applied on $e^{F_1}$.

The following inequalities control the norms of the polymer activities under renormalisation group maps.

**Lemma 4.7.** For any $X \in \mathcal{P}_j$, $\varphi \in \mathbb{R}^{A_n}$ and $0 \leq j \leq N - 1$,

$$\mathbb{E}[G_j(X, \varphi' + \zeta)] \leq 2^{2|X|} G_{j+1}(X, \varphi'). \quad (4.20)$$

If $j$ is as in \cite[(A)]{3} and $u_j$ is defined by Definition 3.2 then for some $C \equiv C(M, \rho, \varsigma)$,

$$\mathbb{E}[G_j(X, \varphi' + \zeta + \tau u_{j+1})] \leq C 2^{2|X|} G_{j+1}(X, \varphi'). \quad (4.21)$$

**Lemma 4.8.** Let $F \in \mathcal{H}_1(X)$ be such that $\|F(X; \tau)\|_{\bar{h}_j(X)} < \infty$ and $h_t \leq (C_1 \log L)^{-3/2}$ for sufficiently large $C_1 = C_1(M, \rho)$. Then $D_{h_t} \tau \mapsto \mathbb{E} \tau D_{h_t}^n [D^m F(X, \varphi' + \zeta; \tau)]$ is analytic in $\tau \in D_{h_t}$ for each $n \geq 0$ and satisfies

$$\|\mathbb{E}_{(\tau \rightarrow \tau')} [F(X, \varphi' + \zeta; \tau)]\|_{\bar{h}_j(X, \varphi')} \leq C_2 2^{2|X|} \|F(X; \tau)\|_{\bar{h}_j(X)} G_{j+1}(X, \varphi'). \quad (4.22)$$

for some $C_2 \equiv C_2(M, \rho, \varsigma) > 0$.

Both lemmas are proved in Appendix A.
4.3. Neutralisation. Suppose \( F \) is a \((2\pi\beta^{-1/2})\)-periodic polymer activity, i.e., for any \( n \in 2\pi\beta^{-1/2}\mathbb{Z} \), \( F \) satisfies \( F(X, \varphi + n1) = F(X, \varphi) \). As the DG model only has a gradient Gibbs measure that is translation invariant in the infinite volume and in the delocalised phase, effective observables are only written in terms of \( \nabla \varphi \). Thus we use an operation that isolates the part of the polymer activity that only depends on \( \nabla \varphi \). This can be achieved by simply taking the 0th component of the Fourier decomposition:

\[
\hat{F}_0(X, \varphi) := \int_0^1 F(X, \varphi + 2\pi\beta^{-1/2}y1)dy,
\]

also called the neutral part or the charge 0 part of \( F \). It can be observed from the definition that

\[
\|\hat{F}_0(X, \varphi)\|_{h,T_j(X,\varphi)} \leq \|F(X)\|_{h,T_j(X)} G_j(X, \varphi).
\]

4.4. Analyticity of polymer activities. We start with an observation made in [13], where boundedness of a polymer activity implies the (complex-)analyticity of the polymer activity on a neighbourhood of the real space.

**Proposition 4.9.** Let \( h, h_1 > 0 \), \( X \in \mathcal{B}_j \) and \( \|F(X)\|_{\tilde{h},T_j(X)} < +\infty \). Then \( F(X, \cdot ; \tau) \) can be extended to the domain \( S_h(X) = \{ \varphi + i\psi \in \mathbb{C}^{\Lambda_N} : \varphi(x), \psi(x) \in \mathbb{R}, \|\psi\|_{C^2_j(X)} < h \} \) where each \( D^nF(X, \cdot ; \tau) \) is complex analytic and satisfies

\[
|F(X, \varphi + \phi; \tau)| \leq \|F(X, \varphi; \tau)\|_{\tilde{h},T_j(X,\varphi;\tau)}
\]

whenever \( \varphi \in \mathbb{R}^{\Lambda_N}, \phi \in \mathbb{C}^{\Lambda_N}, \|\phi\|_{C^2_j(X)} < h \).

**Proof.** The proof is the same as that of [4, Proposition 5.6], just replacing the norm \( \|\cdot\|_{h,T_j} \) by \( \|\cdot\|_{\tilde{h},T_j} \). The inequality (4.25) follows from expanding out \( F(X, \varphi + \cdot ; \tau) \) in Taylor series (see [4, (5.23)]).

By the proposition, we can make complex shift of variables in each Gaussian integrals using the Cauchy’s integral theorem as long as the shift is not too large. This result is summarised in the next lemma, whose proof is be presented in Appendix A.2. Again, when working with \( j \)-scale polymer activities, \( \mathbb{E} \) is an expectation over \( \zeta \sim \mathcal{H}(0, \Gamma_{j+1}) \) if \( j + 1 \neq N \) and \( \zeta \sim \mathcal{H}(0, \Gamma_{j'}^N) \) if \( j + 1 = N \).

**Lemma 4.10** (Gaussian complex shift of variable). Let \( f \) be as in \([A_j]\), \( u_{j+1} \) be as in Definition 3.2 and \( F \in \mathcal{H}_j(X) \) with \( \|F(X)\|_{\tilde{h},T_j(X)} < \infty \). Then for \( h_1 < (C \log L)^{-1}h \) with \( C > 0 \) sufficiently large and \( \tau \in \mathbb{D}_{h_1} \),

\[
\mathbb{E}_{(\tau)}[F(X, \varphi' + \zeta)] = \mathbb{E}[F(X, \varphi' + \zeta + \tau u_{j+1})].
\]

As the norm \( \|\cdot\|_{\tilde{h},T_j} \) exploits the analyticity of \( F \) even further, we would have to study this a bit further.

**Lemma 4.11.** Let \( \tilde{h} = (h, h_1) > 0 \) and \( F \in \mathcal{H}_j(X) \) be such that \( \|F(X; \tau)\|_{\tilde{h},T_j(X)} < +\infty \) where \( X \in \mathcal{B}_j^c \). Then \( \mathbb{D}_{h_1} \ni \tau \mapsto \mathbb{E}[D^nF(X, \varphi' + \zeta; \tau)] \) is a complex analytic function for each \( n \geq 0 \).

**Proof.** Let \( \tau \in \mathbb{D}_{(1-\delta)\hat{h}_1} \) for some \( \delta > 0 \) and \((\tau_m)_m\) be a sequence converging to \( \tau \) with \( |\tau_m| < (1-\delta)\hat{h}_1 \). Then by the Cauchy’s integral formula, for each \( n, m \geq 0 \) and \( \delta' \in (0, \delta),

\[
\|\partial_{\tau} D^n F(X, \varphi; \tau_m)\|_{n,T_j(X,\varphi)} \leq \frac{1}{2\pi i} \int_{|z|=(1-\delta')\hat{h}_1} \frac{D^n F(X, \varphi; z)}{(z-\tau_m)^2} dz \|_{n,T_j(X,\varphi)} \leq \frac{(1-\delta')m!}{(\delta-\delta')^2 h_1 h^n} \|F(X; \cdot)\|_{\tilde{h},T_j(X)} G_j(X, \varphi)
\]
where we have used, for \( z \in \mathbb{D}_{h_1} \),

\[
\|D^n F(X, \varphi; z)\|_{n,T_j(X, \varphi)} \leq \sum_{k=0}^{\infty} \frac{|k^n|}{k!} \|\partial^n D^n F(X, \varphi; \tau)\|_{\tau=0, n,T_j(X, \varphi)} \leq \frac{n!}{h^n} \|F(X, \varphi; \cdot)\|_{\tilde{h}, T_j(X, \varphi)}.
\]

(4.28)

By Lemma 4.7, \( \mathbb{E}[G_j(X, \varphi' + \zeta)] \leq 2^{\|X\|} G_{j+1}(\overline{X}, \varphi') \) for each \( \varphi' \in \mathbb{R}^A \), so we have

\[
\|ED^n[F(X, \varphi' + \zeta; \tau_m) - F(X, \varphi' + \zeta; \tau)]\|_{n,T_j(X, \varphi')} = \left\| \int_{\tau_m}^{\tau} \mathbb{E}[\partial_{\tau} D^n F(X, \varphi' + \zeta; \tau')] d\tau' \right\|_{n,T_j(X, \varphi')}
\]

\[\leq O(\|\tau_m - \tau\|) \frac{(1 - \delta')n!}{(\delta - \delta')^2 h^n} 2^{\|X\|} \|F(X; \cdot)\|_{\tilde{h}, T_j(X) G_{j+1}(\overline{X}, \varphi')},
\]

(4.29)

and in particular, \( ED^n F(X, \varphi' + \zeta; \tau) \) is continuous in \( \tau \).

Now let \( \tilde{\gamma} \) be any piecewise \( C^1 \) curve in \( \mathbb{D}_{h_1} \), and we consider

\[\int_{\tilde{\gamma}} \mathbb{E}[D^n F(X, \varphi' + \zeta; \tau)] d\tau.
\]

(4.30)

Again by (4.28) and Lemma 4.7, we have \( \mathbb{E}[D^n F(X, \varphi' + \zeta; \tau)] \leq \frac{n!}{h^n} \|F(X)\|_{\tilde{h}, T_j(X) G_{j+1}(\overline{X}, \varphi')} \)

so \( D^n F(X, \varphi' + \zeta; \tau) \) is integrable under the product measure \( \mathbb{P}(d\zeta) \otimes d\tau \) of (4.30), and by the Fubini’s theorem,

\[\int_{\tilde{\gamma}} \mathbb{E}[D^n F(X, \varphi' + \zeta; \tau)] d\tau = \mathbb{E} \left[ \int_{\tilde{\gamma}} D^n F(X, \varphi' + \zeta; \tau) d\tau \right],
\]

(4.31)

but by the Cauchy’s integral theorem, \( \int_{\tilde{\gamma}} D^n F(X, \varphi' + \zeta; \tau) d\tau = 0 \), making the whole integral vanish. Hence by the Morera’s theorem, \( \mathbb{E}[D^n F(X, \varphi' + \zeta; \tau)] \) is also complex analytic on \( \mathbb{D}_{h_1} \).

**Lemma 4.12.** Let \( h, h_t > 0 \) be such that \( h_t < (C \log L)^{-1} h \) for sufficiently large \( C \). Let \( F \in \mathcal{H}_{j+1}(X) \) be such that \( \|F(X; \tau)\|_{(h', h_t), T_j(X)} < +\infty \) where \( h'' = h + h_t \|u_{j+1}\|_{C^2_j} \) and \( X \in \mathcal{G}_j^\delta \). If we define

\[
F'(X, \varphi; \cdot) : \mathbb{D}_{h_1} \to \mathbb{C}, \quad \tau \mapsto F(X, \varphi + \tau u_{j+1}; \tau),
\]

(4.32)

then \( D^n F'(X, \varphi; \cdot) \) and \( \mathbb{E}[D^n F'(X, \varphi' + \zeta; \cdot)] \) are complex analytic functions of \( \tau \in \mathbb{D}_{h_1} \), for each \( n \geq 0 \), and satisfies

\[
\|F'(X, \varphi; \tau)\|_{\tilde{h}, T_j(X, \varphi')} \leq \|F(X, \varphi; \tau)\|_{(h', h_t), T_j(X, \varphi')}
\]

(4.33)

\[
\|\mathbb{E}(\tau)F(X, \varphi' + \zeta; \tau)\|_{\tilde{h}, T_j(X, \varphi')} \leq \|\mathbb{E}[F(X, \varphi' + \zeta; \tau)]\|_{(h', h_t), T_j(X, \varphi')}.
\]

(4.34)

**Proof.** By Proposition 4.9, \( z \mapsto D^n F(X, \varphi + zu_{j+1}; \tau) \) is analytic whenever \( \|zu_{j+1}\|_{C^2_j(X, \tau)} < h \).

However, since \( \|u_{j+1}\|_{C^2_j(X, \tau)} \leq 2C(M, \rho)\kappa^{-1} \), this condition is satisfied whenever \( |z| \leq h_t < (2C)^{-1}kh \), making \( F'(X, \varphi; \tau) \) analytic in \( \tau \in \mathbb{D}_{h_1} \). Now by the Chain rule,

\[
\frac{d^n}{d\tau^n} D^n F'(X, \varphi; \tau) = \sum_{m=0}^{\infty} \binom{m}{k} D^{n+k} \partial^{m-k}_\tau F(X, \varphi + \tau u_{j+1}; \tau)(u_{j+1}^k)
\]

(4.35)

\((D \text{ and } \partial_\tau \text{ are partial derivatives})\) so

\[
\|F'(X, \varphi; \tau)\|_{\tilde{h}, T_j(X, \varphi')} \leq \sum_{n,m=0}^{\infty} \sum_{k=0}^{\infty} \frac{h^n h^m}{n! m!} \binom{m}{k} \|D^{n+k} \partial^{m-k}_\tau F(X, \varphi; 0)\|_{n+k,T_j(X, \varphi')} \|u_{j+1}\|^k
\]

(4.36)
where the second line follows from change of variable $m' = m - k$ and $\|u_{j+1}\| = \|u_{j+1}\|_{L^2(X^*)}$.

This yields (4.33), hence by Lemma 4.11 we also have that $\mathbb{E}[D^nF'(X, \varphi' + \zeta; \tau)]$ complex analytic in $\tau \in \mathbb{D}_{h_1}$ for each $n \geq 0$. Now (4.34) follows from Lemma 4.11 saying that $\mathbb{E}(\tau)[F'(X, \varphi' + \zeta; \tau)] = \mathbb{E}[F'(X, \varphi' + \zeta; \tau)]$ and applying the same type of argument on $\mathbb{E}[F'(X, \varphi' + \zeta; \tau)]$.

This lemma is usually used after setting $C_1 > 0$ sufficiently large so that $h'' \leq 2h$. Then if we bound the right-hand side of (4.34) using the definition of $\|\cdot\|_{\hat{h}, T_j(X)}$-norm, we have

$$\|\mathbb{E}(\tau)[F'(X, \varphi' + \zeta; \tau)]\|_{\hat{h}, T_j(X)} \leq C2^{Xj}\|F(X, \cdot; \tau)\|_{(2h, h_1, T_j(X))}G_{j+1}(\mathcal{X}, \varphi'),$$

which is similar to the conclusion of Lemma 4.8 but weaker. However, (4.34) has use of its own as it makes is easy to import inequalities from [4].

5 Localisation and reblocking

In this section, we treat important operations used to define the renormalisation group map and state their key algebraic and analytic properties.

5.1. Localisation operator. Localising a polymer activity $F(X, \varphi)$ isolates the local relevant terms. In the case of our interest, these terms firstly take the neutral part of the polymer activity and secondly take the constant terms and the second degree terms when expanded out in $\nabla \varphi$. Hence we require the localisation operator to approximate the Taylor polynomial of order 2 (see Lemma 4.12), but it attains a stronger algebraic property (see (6.12)). Explicit definition of the localisation is skipped and the proofs of the analytic properties are deferred to the appendix, except for some key lemmas, because they are essentially repetitions of discussions from [4]. Explicit definition of the localisation is skipped and the proofs of the analytic properties are deferred to the appendix, except for some key lemmas, because they are essentially repetitions of discussions from [4].

**Definition 5.1.** Let $F_1 \in \mathbb{N}_j$ be periodic, and let $\hat{F}_0$ be its neutral part. For $X \in \mathbb{S}_j$, $B \in \mathcal{B}_j(X)$, define $\text{Loc}^{(2)}_{X, B} F_1(X)$ and $\text{Loc}^{(2)}_{X} F_1(X)$ according to [4] Definition 6.4. For periodic $F_2 \in \mathbb{N}_{j, t}$, $\hat{F}_{2, 0}$ its neutral part and $X \in \mathbb{S}_j$, define

$$\text{Loc}^{(0)}_{X} F_2(X) = \hat{F}_{2, 0}(X, 0).$$

The following shows that $\text{Loc}^{(2)}_{X} \mathbb{E}(\tau)$ and $\text{Loc}^{(0)}_{X} \mathbb{E}(\tau)$ are bounded operators, and proved in Section A.3.

**Lemma 5.2.** Let $F_1 \in \mathbb{N}_{j, t}$ be periodic and let $X \in \mathbb{S}_j$, $B \in \mathcal{B}_j(X)$. If $\|F_1(X)\|_{\hat{h}, T_j(X)} < \infty$, then $\text{Loc}^{(0)}_{X} \mathbb{E}(\tau) F_1(X, \varphi' + \zeta; \tau)$ is an analytic function of $\tau \in \mathbb{D}_{h_1}$ and there is $C \equiv C(M, \rho, \tau)$ such that

$$\|\text{Loc}^{(0)}_{X} \mathbb{E}(\tau) F_1(X, \varphi' + \zeta; \tau)\|_{h, T_j(X)} \leq C \|F_1(X; \tau)\|_{\hat{h}, T_j(X)}.$$

If $F_2 \in \mathbb{N}_j$ is periodic with $\|F_2\|_{h, T_j} < \infty$, then there is $C' \equiv C'(M, \rho, L, \tau)$ such that

$$\|\text{Loc}^{(2)}_{X, B} \mathbb{E}(\tau) F_2(X, \varphi' + \zeta)\|_{\hat{h}, T_j(B, \varphi')} \leq C' \|F_2(X)\|_{h, T_j(X)} e^{c_{\text{loc}}\omega_j(B, \varphi')}^2.$$

5.1.1. Irrelevance of non-local terms. The following proposition claims that the terms that are not included in $\text{Loc}^{(k)} \mathbb{E}(\tau)$ ($k \in \{0, 2\}$) contracts upon the fluctuation integral $\mathbb{E}(\tau)$ with factor

$$\alpha_{\text{Loc}}^{(k)} = C(L^{-2} \log L)^{(k+1)/2} + C \min \left\{ 1, \sum_{q \geq 1} e^{2q \omega h} e^{-(q-1/2)3\tau_j + 0} \right\},$$

and thus we see that the localisations serve as an even better approximations of the polymer activity upon fluctuation integral and change of scale.
Proposition 5.3. There exists a constant \( c_h > 0 \) such that the following holds for all periodic \( F \in \mathcal{H}_{j,1} \) and \( X \in \delta_j \). Assume \( h \geq \max \{c_h \sqrt{\beta}, 1 \} \), \( h_1 \leq (C \log L)^{-1} h \), \( L \geq L_0(R) \) for \( C \) and \( L_0(R) \) sufficiently large and \( \Lambda' \geq 1 \). If \( \|F\|_{h,T_j,A'} < \infty \), then for all \( \varphi' \in \mathbb{R}^{\Lambda_N} \),

\[
\|\text{Loc}^{(0)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta; \tau) - \mathbb{E}_{(\tau)} F(X, \varphi' + \zeta; \tau)\|_{h,T_{j+1}(\mathbb{X}, \varphi')}
\leq a^{(0)}_{ \text{Loc}(\Lambda')} \|\mathbb{E}\|_{h,T_j,A'} G_{j+1}(\mathbb{X}, \varphi').
\] (5.5)

If we assume instead \( \|F(X, \cdot; 0)\|_{h,T_j,A'} < \infty \) then

\[
\|\text{Loc}^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta; 0) - \mathbb{E}_{(\tau)} F(X, \varphi' + \zeta; 0)\|_{h,T_{j+1}(\mathbb{X}, \varphi')}
\leq a^{(2)}_{ \text{Loc}(\Lambda')} \|\mathbb{E}\|_{h,T_j,A'} G_{j+1}(\mathbb{X}, \varphi').
\] (5.6)

This proposition is proved in Section B.3.

5.2. Rescale-reblocking. When we are given with a polymer expansion at scale \( j \), the easiest way to rewrite the expansion in scale \( j + 1 \) is to introduce the rescale-reblocking operator

\[
\mathcal{S}F(X) = \sum_{Y \in \mathcal{P}_j^c} F(Y), \quad X \in \mathcal{P}_j^c.
\] (5.7)

For general \( X \in \mathcal{P}_j^c \), let \( \mathcal{S}F(X) = \prod_{Z \in \text{Comp}_j(X)} SF(Z) \). The following proposition claims that the polymer activities on small polymers have dominant contribution on \( SF \).

Proposition 5.4. There exists a constant \( \eta > 0 \) such that the following holds. Let \( L \geq 5 \), \( \Lambda' \geq \Lambda_0(L) \) where \( \Lambda_0(L) \) is some function only polynomially large in \( L \). If \( F \in \mathcal{H}_{j,1} \) is supported on large sets (i.e., non-small sets) and \( \|F\|_{h,T_j,A'} \leq \varepsilon_{h}(\Lambda') \), then for \( X \in \mathcal{P}_j^c \),

\[
\|\mathbb{S}[\mathbb{E}_{(t+\tau)} F(X, \cdot + \zeta)]\|_{h,T_{j+1}(X), A'} \leq (L^{-1}(\Lambda')^{-1} |X|)^{-1} \|F\|_{h,T_j}.
\] (5.8)

This proposition is proved in Section B.3.

5.3. Irrelevance of the perturbations. When the polymer expansion (3.32) is perturbed by a field, then we may use field-reblocking operator to rewrite them in terms of a polymer expansion free of the perturbation. This operator is defined as the following.

Definition 5.5. Given \( v_j \in \mathbb{C}^{\Lambda_N} \) and \( K_j \in \mathcal{H}_{j,1} \), \( U_j \in \mathcal{H}_j \), define for \( X \in \mathcal{P}_j^c \),

\[
\mathcal{R}_j[v_j, U_j, K_j](X, \varphi) = \sum_{Y \in \mathcal{P}_j(X)} \left( e^{U_j(X, \cdot + v_j)} - e^{U_j(X, \cdot)} \right)^{X \setminus Y} K_j(Y, \varphi)
\] (5.9)

where \((\cdot)^{X \setminus Y}\) is defined according to (1.23). This can be extended to general \( Z \in \mathcal{P}_j \) by \( \mathcal{R}_j[Z] = \mathcal{R}_j^{[2]}(\cdot) \) (recall (1.22)).

Given \( u_j \) as in (A.4) \( \tau \in \mathbb{R}, \tau \in \mathbb{C} \), define

\[
K_j^\dagger = \mathcal{R}_j[\tau + \tau u_j, U_j, K_j].
\] (5.10)

The rewriting of the polymer expansion is a purely algebraic result, as stated in the next proposition. Also, Lemma 5.7 indicates that field-reblocking can be done without much cost on the norm, as long as the size of the perturbed field is not too large.

Proposition 5.6. Let \( K_j \in \mathcal{H}_{j,1} \) and let \( U_j \in \mathcal{H}_j \) be additive over blocks, i.e., \( U_j(X \cup Y) = U_j(X) + U_j(Y) \) for all \( X \cap Y = \emptyset, X, Y \in \mathcal{P}_j \). Let \( Z_j \) be given by (3.32) and define \( K_j'^{\dagger} \) according to Definition 5.7 (with \( K_j \) in place of \( K_j'^{\dagger} \)). Then for \( c = -E_j|\Lambda_N| + g_j(\Lambda_N; \tau) \),

\[
Z_j'(\varphi) = e^c \sum_{X \in \mathcal{P}_j} e^{U_j(X, \cdot, \varphi)} K_j'^{\dagger}(X, \varphi).
\] (5.11)

If \( K_j', U_j \) are \((2\pi\beta^{-1/2})\)-periodic, then so is \( K_j'^{\dagger} \). If \( u_j \) as in (A.4) and \( P_j^\dagger \) is given by (1.22), then \( K_j'(X) = K_j'^{\dagger}(X) \) whenever \( X \cap P_j^\dagger = \emptyset \).
Proof. Using polymer expansion

\[ e^{U_j(A \setminus X, \varphi + \tau u_j)} = \sum_{Y \in \mathcal{P}_j(A \setminus X)} (e^{U_j(\cdot, \varphi + \tau u_j)} - e^{U_j(\cdot, \varphi)})^Y e^{U_j(A \setminus (X \cup Y), \varphi)}, \tag{5.12} \]

and denoting \( Z = X \cup Y \), we can rewrite (3.32) as

\[ Z'_j(\varphi) = e^r \sum_{Z \in \mathcal{P}_j(A \setminus \varphi)} e^{U_j(A \setminus Z, \varphi)} \sum_{Y \in \mathcal{P}_j(Z)} (e^{U_j(\cdot, \varphi + \tau u_j)} - e^{U_j(\cdot, \varphi)})^Y K_j(Z \setminus Y, \varphi). \tag{5.13} \]

But because of factorisation property of \( K_j \), (1.24), we actually have

\[ \sum_{Y \in \mathcal{P}_j(Z)} (e^{U_j(\cdot, \varphi + \tau u_j)} - e^{U_j(\cdot, \varphi)})^Y K_j(Z \setminus Y, \varphi) = \prod_{Z' \in \text{Comp}_j(Z)} \sum_{Y \in \mathcal{P}_j(Z')} (e^{U_j(\cdot, \varphi + \tau u_j)} - e^{U_j(\cdot, \varphi)})^Y K_j(Z' \setminus Y, \varphi), \tag{5.14} \]

giving (5.11). Final remarks are also clear from the definition of the operation \( \mathcal{R}_j \)'s. \( \square \)

**Lemma 5.7.** Suppose \( u_j \) satisfies (A.2), \( U_j \) given in form (3.33), \( K_j \in \mathcal{H}_{2A} \) and \( h_t < (C \log L)^{-1} h_t \) for \( C \) sufficiently large. Then there exists \( \varepsilon_r = \varepsilon_r(M, \rho, A, L, z) > 0 \) such that whenever \( \max\{\|U_j\|_{\Omega_j^0}, \|K_j\|_{\mathcal{H}_{2A}}\} \leq \varepsilon_r \) and \( \tau \in \mathbb{D}_{h_t} \),

\[ \|K_j^\dagger\|_{\mathcal{H}_{2A}, \frac{1}{2}A^2} \leq \max \left\{ \|U_j\|_{\Omega_j^0}, \|K_j\|_{\mathcal{H}_{2A}} \right\}. \tag{5.15} \]

**Proof.** By Lemma (4.12) and (4.13) with the assumption \( h + h_t < \kappa(2M)^{-1} h + 2h < 2h \), one may write

\[ \|U_j(B, \varphi + (\tau + \tau) u_j) - U_j(B, \varphi + \tau u_j)\|_{\mathcal{H}_{2A}(B, \varphi)} \leq \|U_j(B, \varphi + \tau u_j)\|_{h+t} \|u_j\|_{\Omega_j^0(B, \varphi)} \leq CA^{-1}\|U_j\|_{\Omega_j^0} u_j(B, \varphi + \tau u_j)^2. \tag{5.16} \]

Also by (4.13),

\[ \|U_j(B, \varphi + \tau u_j) - U_j(B, \varphi + u_j)\|_{\mathcal{H}_{2A}(B, \varphi)} \leq CA^{-1}\|U_j\|_{\Omega_j^0} (1 + C' \tau) \|u_j\|_{\Omega_j^0}^2. \tag{5.17} \]

Then Lemma (4.6) gives

\[ \|e^{U_j(B, \varphi + (\tau + \tau) u_j)} - e^{U_j(B, \varphi)}\|_{\mathcal{H}_{2A}(B, \varphi)} \leq \|e^{U_j(B, \varphi)}\|_{\mathcal{H}_{2A}(B, \varphi)} \|e^{U_j(B, \varphi + (\tau + \tau) u_j)} - U_j(B, \varphi)\|_{\mathcal{H}_{2A}(B, \varphi)} - 1 \|K_j(B, \varphi)\| \leq CA^{-1}\|U_j\|_{\Omega_j^0} e^{C_1\|w_j(B, \varphi)^2 + C_2(\tau) \|u_j\|_{\Omega_j^0}^2}. \tag{5.18} \]

Hence, if \( x := \max\{\|U_j\|_{\Omega_j^0}, \|K_j\|_{\mathcal{H}_{2A}, \frac{1}{2}A^2} \} \) is sufficiently smaller than \( \min\{A, c_\omega \} \) and \( (C' \tau) \|u_j\|_{\Omega_j^0}^2 \), then by (4.14) and (4.4),

\[ \left\| \left( e^{U_j(\cdot, \varphi + (\tau + \tau) u_j)} - e^{U_j(\cdot, \varphi)} \right)^{X \setminus Y} K_j(Y, \varphi) \right\|_{\mathcal{H}_{2A}(X, \varphi)} \leq A^{-|X_j|} (C^X)^{|X \setminus Y|} e^{c_\omega \|w_j(X, \varphi)^2 \} x | Comp_j(Y)| G_j(Y, \varphi) \leq A^{-|X_j|} (C^X)^{|X \setminus Y|} e^{c_\omega \|w_j(X, \varphi)^2 \} x | Comp_j(Y)| G_j(X, \varphi) \tag{5.19} \]

for \( X, Y \in \mathcal{P}_j, Y \subset X \). By the definition of \( K_j^\dagger \), now with \( x \) sufficiently smaller than \( A \),

\[ \|K_j^\dagger(X, \tau)\|_{\mathcal{H}_{2A}(X)} \leq A^{-|X_j|} \sum_{Y \in \mathcal{P}_j(X)} (C^X)^{|X \setminus Y|} \|w_j(X, \varphi)^2 \} x | Comp_j(Y)| \leq (A/2)^{-|X_j|}. \tag{5.20} \]

Together with (4.13), we obtain the bound on \( K_j^\dagger \). \( \square \)
6 Renormalisation group map

Having $Z^j_\Omega$ defined by \((3.27)\), suppose there exists \((E_j, g^j, U_j, K^j)\) that satisfies \((3.29)\). Then by Proposition 5.6

\[ Z^j_\Omega(\varphi; \tau) = e^{-E_j|\Lambda_N|+g^j(\Lambda_N; \tau)} \sum_{X \in \delta_j(\Lambda_N)} e^{U_j(\Lambda_N \setminus X; \varphi)}(K^j_j)(X, \varphi; \tau). \]  

(6.1)

The objective of this section is to define \((E_{j+1}, g^j_{j+1}, U_{j+1}, K^j_j)\) as a function of given \((E_j, g^j, U_j, K^j)\), which is also called a single RG step. Each coordinates is given as an element of the normed spaces as the following.

**Definition 6.1.**

- Denote $U_j$ and $W_j$ to be implicit functions of the coupling constants \((s_j, (z^j)^\eta_{\eta \geq 1} \subset \mathbb{R} \times \mathbb{R}^\eta\) defined by \((3.33)\) and \((3.34)\). The space of $U_j$ of such form with $||U_j||_{\Omega^j} < \infty$ (see Definition 4.9) is called $\Omega^j_U$.

- Let $\Omega^j_K$ be the space of $F_j \in \mathcal{N}_j$ that is invariant under lattice symmetries and \((2\pi \beta)^{-1/2}$-periodic, and let $||F_j||_{\Omega^j_K} := ||F_j||_{h, T_j} < \infty$.

- $\Omega^j_{K,j,t}$ is the space of periodic $F_j \in \mathcal{N}_j$ such that $||F_j||_{\Omega_{K,j,t}} < \infty$ and $\Omega^j_{K,j,t}$ is the space of periodic $F_j \in \mathcal{N}_j$ such that $||F_j||_{\Omega_{K,j,t}, A/2} < \infty$. Then $\Omega^j_{K,j,t} \subset \Omega^j_{K,j}$ (respectively $\Omega^j_{K,j,t} \subset \Omega^j_{K,j,t}$) is the space of pairs $(F_j^2(\cdot; 0), F_j^2)$ (respectively $(F_j^2(\cdot; 0), F_j^2)$) such that $F_j^2(X, \varphi; \tau) = F_j^2(X, \varphi; 0)$ (respectively $F_j^2(X, \varphi; \tau) = F_j^2(X, \varphi; 0)$) whenever $X \in \mathcal{D}_j$ has $X \cap (P_j)^\ast \neq \emptyset$. Then denote

\[
\begin{align*}
||F_j^2||_{\Omega_{K,j,t}} &= \max\{||F_j^2(\cdot; 0)||_{h, T_j}, ||F_j^2||_{\Omega_{K,j,t}}\} \\
||F_j^2||_{\Omega_{K,j,t}} &= \max\{||F_j^2(\cdot; 0)||_{h, T_j}, ||F_j^2||_{\Omega_{K,j,t}, A/2}\}. 
\end{align*}
\]

(6.2)

(6.3)

- Define $\Omega_j = \Omega^j_U \times \Omega^j_K$, $\Omega_{j,t} = \Omega^j_U \times \Omega^j_{K,j,t}$ and $\Omega_{j,t,t}$ = $\Omega^j_U \times \Omega^j_{K,j,t,t}$ endowed with norms

\[
\begin{align*}
||U_j, F_j||_{\Omega_j} &= \max\{||U_j||_{\Omega^j_U}, ||F_j||_{\Omega^j_K}\} \\
||U_j, F_j^2, F_j^2||_{\Omega_{j,t}} &= \max\{||U_j||_{\Omega^j_U}, ||F_j^2||_{\Omega^j_K}, ||F_j^2||_{\Omega^j_K}\} \\
||U_j, F_j^2, F_j^2||_{\Omega_{j,t,t}} &= \max\{||U_j||_{\Omega^j_U}, ||F_j^2||_{\Omega^j_K}, ||F_j^2||_{\Omega^j_K}\}. 
\end{align*}
\]

(6.4)

(6.5)

(6.6)

An element of the space $\Omega_j$ or $\Omega_{j,t}$ or $\Omega_{j,t,t}$ is typically denoted $\omega^0_j$ or $\omega^1_j$ or $\omega^{1,1}_j$ respectively. Given an element $\omega^0_j = (U_j, K^0_j, K^1_j) \in \Omega_{j,t}$, we have $(K^1_j)$ defined by Definition 5.5. Then by Proposition 5.6 and Lemma 5.7 we see that $\omega^{1,1}_j := (U_j, K^0_j, (K^1_j)) \in \Omega_{j,t,t}$ and satisfy

\[
||\omega^{1,1}_j||_{\Omega_{j,t,t}} \leq C ||\omega^1_j||_{\Omega_{j,t}}. 
\]

(6.7)

The RG map will first be defined as a function $\omega^{1,1}_j \mapsto \omega^0_{j+1}$, and this map can also be considered as a map $\omega^1_j \mapsto \omega^0_{j+1}$.

The main theorem of this section is a combination of the existential statement of a single RG step with an analytic estimate.

**Theorem 6.2.** Let $L, A, \beta > 0$ be sufficiently large. Then there exists $\epsilon_{nl} \equiv \epsilon_{nl}(\beta, A, L) > 0$ (only polynomially small in $\beta$) such that a ‘renormalisation group map’ exists whenever $||\omega^{1,1}_j||_{\Omega_{j,t,t}} = ||(U_j, K^1_j)||_{\Omega_{j,t,t}} \leq \epsilon_{nl}$, i.e., there is

\[
\Phi^{1,1}_j = (E_{j+1}, g_{j+1}, U_{j+1}, K^1_{j+1}) : \{|\omega^{1,1}_j||_{\Omega_{j,t}} \leq \epsilon_{nl}\} \to \mathbb{R} \times W^+(D_{h_t}) \times \Omega_{j+1}^U \times \Omega_{j+1}^{K,j}. 
\]

(6.8)
such that, whenever

\[(E_{j+1}, g^j_{j+1}, U_{j+1}, \vec{K}^j_{j+1}) = (E_j + \xi_j, g^j_j + \xi_j, U_j, \vec{K}^j_j)(\omega_j)\]  \hspace{1cm} (6.9)

and \(Z_j, Z^2_{j+1}\) are defined by (3.32), then they satisfy (3.27).

Moreover, there is a decomposition \(L^j_{j+1} = L^j_{j+1} + M^{j+1}_{j+1}\) such that \(L^j_{j+1}\) is a linear function of \(\vec{K}^j_j\) and \(M^{j+1}_{j+1}\) is a differentiable function of \((U_j, \vec{K}^j_j)\) with

\[
\|L^j_{j+1}\|_{\Omega^j_{j+1}, x} \leq C_1(\|L^2\|_{\Omega^j_{j+1}, x} \|K^j_0(\cdot; 0)\|_{\Omega^j_{j+1}} + \alpha^{(0)}_j \|\vec{K}^j_j\|_{\Omega^j_{j+1}, x})
\]  \hspace{1cm} (6.10)

\[
\|M^{j+1}_{j+1}\|_{\Omega^{j+1}_{j+1}, x} \leq C_2(\|\omega_j\|_{\Omega^j_{j+1}, x})
\]  \hspace{1cm} (6.11)

for some \(C_1 \equiv C_1(\xi) > 0\) independent of \(\beta, A, L, C_2 \equiv C_2(\tau, \beta, A, L)\) and \((\alpha^{(2)}_j, \alpha^{(0)}_j)\) are as in (5.3).

We note that \(\Omega^j_{j+1}\) and \(\Omega^j_{j+1}\) are vector spaces of collections \((F(X, \cdot))_{X \in \theta_j}^j\), but not of \((F(X, \cdot))_{X \in \theta_j}\). To say that a function on \(\Omega^j_{j+1}\) or \(\Omega^j_{j+1}\) is linear means that it is linear in \((F(X, \cdot))_{X \in \theta_j}\).

As always, at scale \(j \in \{0, \cdots, N - 1\}, \mathbb{E}\) always mean the expectation taken over \(\zeta \sim \mathcal{N}(0, \Gamma_j^0)\) if \(j + 1 < N\) and \(\zeta \sim \mathcal{N}(0, \Gamma_N^0)\) if \(j + 1 = N\). Usually, the remaining non-random coordinate is denoted \(\varphi'\) and \(\varphi = \varphi' + \zeta\).

### 6.1. Choice of counterterm and vacuum energy.

When \((\tau, \tau) = (0, 0)\), \(Z^0_j(\varphi; 0)\) corresponds to the same object in [4], only yielding \((E_j, s_j, (z^{(q)}_j)_{q \geq 1})\) and \(K^0_j(\cdot; 0)\). In the RG language, the choice of \(\xi_{s_0} \varphi^2_{X, N} \) in (2.7) corresponds to the counterterm, and \(\lim_{j \to \infty} E_j\) is called the vacuum energy. For the RG analysis to work, counterterm is chosen so that \((s_j, (z^{(q)}_j)_{q \geq 1}, K^0_j(\cdot; 0)) \to 0\) in the norms of \(\Omega_j\). This is an important reference point for our analysis, as our analysis is a mere perturbation of the model with vanishing external field. Thus we recall the choice of the counterterm and the vacuum energy.

In the following, \((\xi^j_j, U^j_j) : \Omega_j \to \mathbb{R} \times \mathcal{O}^j_{j+1}, (U_j, K^0_j) \mapsto (E^j_j, U^j_j)\) are as in [4] Definition 7.8, so that they solve

\[
-\xi^j_j(U^j_j, K_j^0)|B| + U^j_j(U_j, K_j, B, \varphi') = \mathbb{E}U^j_j(B, \varphi' + \zeta) + \sum_{X \in \theta_j : X \supset B} \mathcal{O}^{(2)}_{X, B} \mathbb{E}K^0_j(X, \varphi' + \zeta).
\]  \hspace{1cm} (6.12)

The following theorem makes a specific choice of the counterterm by putting together [4] Theorem 7.5, Corollary 8.7, and also guarantees the existence of the RG flow of any length.

**Theorem 6.3.** Let \(\tau = \tau = 0\). Also, let \(\beta \geq \beta_0, L \geq L_0\) and \(A \geq A_0(L)\) be sufficiently large. Then there exists \(s^0_0(\beta) = \mathcal{O}(e^{-c_1 \beta})\) such that, for any \(N \geq 1\), when \(s = s^0_0(\beta)\) (for \(s\) in (2.3) and \(Z_0\) has initial conditions \(s_0 = s^0_0(\beta)\), \(z_0 = 0 \) (for \(s_0, 0\) in Lemma 2.4) and \(K^0_0(X) = 1_{X = 0}\), then there exists \((K^0_j)_{j \in \{1, \cdots, N\}} \in \mathcal{O}^N_{\theta_j}\) such that (3.27) and (3.32) hold with \(\tau = \tau = 0\) and \(Z^0_{j+1}(\cdot; 0)\) defined by \(E_{j+1} = E_j + \xi_{j+1}, g^j_{j+1} = 0, \) and \(U_{j+1} = U^j_j(U^j_j, K^0_j(\cdot; 0))\). Moreover, the renormalisation group coordinates satisfy

\[
\|U_j\|_{\Omega^j_j} \leq \mathcal{O}(e^{-c_1 \beta} L^{-\alpha j}), \quad \|K^0_j\|_{\Omega^j_j} \leq \mathcal{O}(e^{-c_1 \beta} L^{-\alpha j})
\]  \hspace{1cm} (6.13)

for \(\alpha > 0\) such that \(L^\alpha = \mathcal{O}(L^{-2(\alpha^{(2)}_j)} - 1)\), and the bounds are uniform in \(\Lambda_N\).

The initial condition appearing in this theorem will occur again multiple number of times, and \(s^0_0(\beta)\) is a fundamental quantity related to the DG model in the delocalisation phase, so we emphasize this again.

**\(\Phi_{1/c}\)** Given \(\beta \geq \beta_0\), the parameter \(s\) is set to be \(s^0_0(\beta)\), the initial coupling constants \(U_0(X, \varphi) = \frac{1}{2}s_0 \varphi^2_X + \sum_{x \in X} \sum_{q \geq 1} z^q_0 \cos(q^l/2 \varphi(x))\) are given by \(s_0 = s^0_0(\beta), z^q_0 = \tilde{z}^q_0(\beta)\), and the initial perturbation coordinate is \(K^0_0(X) = 1_{X = 0}\).
6.2. RG map. Given coordinates $U_j$ and $\tilde{K}_j^{z_+1} = (K_j^0, (K_j^1)^\dagger) \in \Omega_{j,t+\dagger}^K$, we use the map $(U_j, K_j^0(\cdot;0)) \mapsto (\xi_{j+1}, U_{j+1}, K_j^0(\cdot;0))$ defined according to (6.12) and Theorem 6.3. In the following, we recall $P_y^j = \cup_{k=0}^j B_y^k$ (see (17.22)).

Definition 6.4. For $0 \leq j < N$ and given $\tilde{K}_j^{z_+1} \in \Omega_{j,t+\dagger}^N$, define

$$g_{j+1}^{z_+1}(B, \tilde{K}_j^1; \tau) = 1_{B \subseteq (P_y^j)^*} \sum_{Y \subseteq B} \frac{1}{|Y \cap (P_y^j)^*|} \text{Loc}^{(0)}_Y \mathbb{E}_{(P_y^j)^*} [(K_j^1)^\dagger(Y, \xi; \tau) - K_j^0(Y, \xi; 0)]$$

for $B \in B_j$, and for $X \in \mathcal{P}_{j+1}$, let

$$g_{j+1}^{z_+1}(X, \tilde{K}_j^{z_+1}; \tau) = \sum_{B \in \mathcal{B}(X)} g_{j+1}^{z_+1}(B, \tilde{K}_j^{z_+1}; \tau).$$

It is an immediate consequence of Lemma 6.2 that, if $\|\tilde{K}_j^{z_+1}\|_{\Omega_{j,t+\dagger}^N} < \infty$, then we have that $g_{j+1}^{z_+1}(B, \tilde{K}_j^1; \tau)$ is an analytic function of $\tau \in \mathbb{D}_{h_{\cdot}}$ with bound

$$\|g_{j+1}^{z_+1}(B, \tilde{K}_j^{z_+1}; \tau)\|_{h_{\cdot}, W} \leq C(M, \rho, \tau) A^{-1}\|\tilde{K}_j^{z_+1}\|_{\Omega_{j,t+\dagger}^N}$$

for each $B \in B_j$. The renormalisation group map for $K$-coordinate is defined as follows.

Definition 6.5. Under the same assumptions, the map $\omega_j^{z_+1} = (U_j, \tilde{K}_j^{z_+1}) \mapsto \tilde{K}_j^{z_+1}$ is defined by

$$\mathcal{K}_j^{z_+1}(U_j, \tilde{K}_j^{z_+1}, X, \cdot; \tau) = \sum_{X_0, X_1, Z(B_{2^{n+1}})} e_{\delta_{j+1}[T]-\delta_{j+1}(T; \tau)} e_{\delta_{j+1}(X \setminus (T \cdot + (t + \tau) u_{j+1})}

\mathbb{E}_{(P_y^j)^*} [(K_j^1)^\dagger(X, \xi; \tau) - K_j^0(X, \xi; 0)]

\prod_{Z'' \in \text{Comp}_{j+1}(Z)} J_j^1(B_{2^{n+1}}, Z'', \cdot; \tau),$$

whenever the integral $\mathbb{E}_{(P_y^j)^*}$ converges, where $\sum^*$ is running over disjoint $(j + 1)$-polymers $X_0, X_1, Z$ such that $X \not\subset Z, B_{2^n} \in \mathcal{B}_{j+1}(Z'')$ for each $Z'' \in \text{Comp}_{j+1}(Z), T = X_0 \cup X_1 \cup Z$ and $X = \cup Z \cup B_{2^n} \cup X_0 \cup X_1$, and

$$Q_j^1(D, Y, \varphi'; \tau) = 1_{Y \subseteq \delta_j} \left( \text{Loc}^{(2)}_{Y,D} \mathbb{E}_{(P_y^j)^*} [(K_j^1(0, Y, \varphi' + \xi; 0)] + \frac{1}{|Y \cap (P_y^j)^*|} \text{Loc}_Y^{(0)} \mathbb{E}_{(P_y^j)^*} [(K_j^1)^\dagger(Y, \varphi' + \xi; \tau) - K_j^0(Y, \varphi' + \xi; 0)] \right)$$

$$J_j^1(B, X, \varphi'; \tau) = 1_{B \subseteq \mathcal{B}_j(X)} \sum_{D \in \mathcal{B}_j(Y)} \sum_{Y \subseteq \delta_j} Q_j^1(D, Y, \varphi'; \tau) (1_{Y \not= X} - 1_{B = X}).$$

$$\delta K_j^1(X, \varphi'; \tau) = \sum_{B \in \mathcal{B}_j(X)} J_j^1(B, X, \varphi'; \tau)$$

$$\mathcal{K}_j^1(X, \varphi' + \xi; \tau) = \sum_{Y \subseteq \delta_j} e_{U_j(X \setminus Y, \varphi' + \xi; \tau)} (K_j^1)^\dagger(Y, \varphi' + \xi; \tau)$$

for $D \in \mathcal{B}_j, B \in \mathcal{B}_{j+1}, Y \in \mathcal{P}_j$ and $X \in \mathcal{P}_{j+1}$.

Remark 6.6. As is apparent in the notation, if we set $\varepsilon = \tau = 0$ and $K_{j+1}^0 = \mathcal{K}_{j+1}^0(U_j, \tilde{K}_j^{0,1})$, then this exactly corresponds to the choice of $K_{j+1}^0$ in Theorem 6.3. See [1] Definition 7.9.

Also, by comparing the expressions for $\mathcal{K}_{j+1}^0$ and $\mathcal{K}_{j+1}^1$ and the proof of [1] Theorem 7.5, (6.27) follows with the RG coordinates at scale $j + 1$ given by (6.8), only with minor adaptations.
7 Proof of Theorem 6.2

As was mentioned in Remark 6.6, the algebraic part of Theorem 6.2 will not be discussed in detail. We focus on the proof of the bounds (6.10) and (6.11). The next theorem is essentially just a restatement of the estimates.

Theorem 7.1 (Estimate for remainder coordinate). Under the setting of Theorem 6.2 and on each \( X \in D_{j+1}^c \), the map \( K_{j+1}^r(\omega;\cdot,\tau) \) admits a decomposition

\[
K_{j+1}^r(\omega_j^{i \dagger}, X; \tau) = L_{j+1}^r(\omega_j^{i \dagger}, X; \tau) + M_{j+1}^r(\omega_j^{i \dagger}, X; \tau)
\]  

(7.1)

into polymer activities at scale \( j+1 \) such that the following holds provided \( L \geq L_0(\varepsilon), A \geq A_0(L) \):

(i) When \( X \in D_j^c \), the map \( \omega_j^{i \dagger} \mapsto L_{j+1}^r(\omega_j^{i \dagger}, X) \) is independent of \( U_j \), linear in \( \vec{K}_j^{r,\dagger} \) such that

\[
\|L_{j+1}^r(\omega_j^{i \dagger})\|_K^r \leq C_1 \left( L^2 \alpha_0^2 \|K_j^0(\cdot; 0)\|_{\Omega_j^r} + \alpha_0^0 \|\vec{K}_j^{r,\dagger}\|_{\Omega_j^r} \right)
\]  

(7.2)

for some constant \( C_1 \equiv C_1(\varepsilon) > 0 \) independent of \( \beta, A, L \), and \( (\alpha_0^2, \alpha_0^0) \) are as in (5.4).

(ii) The remainder map \( M_{j+1} \) satisfies \( M_{j+1} = O(\|\omega_j^{i \dagger}\|_K^r) \) in the sense that there exists \( C_2 = C_2(\varepsilon, \beta, A, L) > 0 \) such that \( M_{j+1}(\omega_j^{i \dagger}) \) is continuously Fréchet-differentiable and

\[
\|DM_{j+1}(\omega_j^{i \dagger})\|_K^r \leq C_2 \|\omega_j^{i \dagger}\|_K^r
\]  

(7.3)

and \( M_{j+1}(0, 0) = 0 \).

Proof of Theorem 6.2 Let \( (\xi_{j+1}, U_{j+1}) \) be given by (6.12). Given \( K_j^{r,\dagger} \) be given by (6.10), \( g_j^{r,\dagger} \equiv g_j^{r,\dagger}(\vec{K}_j^{r,\dagger}) \) be given by Definition 6.4 and \( \vec{K}_j^{r,\dagger} = K_j^{r,\dagger}(U_j, \vec{K}_j^{r,\dagger}) \) be given by Definition 6.5. Then \( \vec{g}_j^{r,\dagger} \in W^+(D_{h_j}) \) by the remark after Definition 6.3 and \( \|K_j^{r,\dagger}\|_{K_j^{r,\dagger}} < \infty \) by Theorem 7.1 as \( \|\omega_j^{i \dagger}\|_{K_j^{r,\dagger}} \leq \varepsilon_{n_l} \) would imply \( \|\omega_j^{i \dagger}\|_K^r \leq C\varepsilon_{n_l} \) for some \( C > 0 \). Also, since any \( X \in D_{j+1}^c \) such that \( X \cap (P_{j+1}^y)^* = \emptyset \) also satisfies \( X \cap (P_{j+1}^y)^* = \emptyset \), we see that each polymers appearing in (6.17) is disjoint from \( (\text{supp}(u_{j+1}P_{j+1}^y)^* \) (is taken in scale \( j \)) for such \( X \), which means that \( K_{j+1}^r(X; \tau) = K_{j+1}^0(X; 0) \) for any \( \tau \). Therefore we have \( \vec{K}_j^{r,\dagger} \in K_j^{r,\dagger} \).

Finally, (3.27) follows from the proof of [4] Theorem 7.5, only with minor adaptations to the current setting. \( \square \)

7.1 Proof of Theorem 7.1 (i). Given \( K_j^{r,\dagger}(\omega_j^{i \dagger}) \) defined according to (6.17), consider its formal expansion in linear order of \( (U_j, K_j^0(\cdot; 0), K_j^{r,\dagger}) \) by (1) replacing \( e^{e_j+1}u_{j+1}^T - \vec{g}_j^{r,\dagger}(T) + u_{j+1}(X \setminus T) + (\varepsilon + \tau) u_{j+1} \) by \( U_j-e_{j+1}|B|+\vec{g}_j^{r,\dagger}(B)+u_{j+1}(B)+(\varepsilon+\tau) u_{j+1} \) by \( U_j-e_{j+1}|B|+\vec{g}_j^{r,\dagger}(B)+u_{j+1}(B)+(\varepsilon+\tau) u_{j+1} \), (2) extracting out terms with

\[
\#(X_0, X_1, Z) = |X_0|_{j+1} + \left| \text{Comp}_{j+1}(X_1) \right| + \left| \text{Comp}_{j+1}(Z) \right| \leq 1,
\]  

(7.4)

and (3) approximating \( \vec{K}_j(X) \) by \( \mathbb{S}(K_j^{r,\dagger})(X) \). These give us the expression

\[
L_{j+1}^r(\omega_j^{i \dagger})(X, \varphi') := \sum_{Y, \varphi \in X} \left( 1_{Y \in D_{h_j}} \mathbb{E}_{(\varepsilon + \tau)}(K_j^{r,\dagger})(Y, \varphi' + \zeta) - 1_{Y \in D_{h_j}} \sum_{D \in D(Y)} Q_j(D, Y, \varphi') \right)
\]  

\[
+ \sum_{D \in D_{h_j}} \left( \mathbb{E}_{(\varepsilon + \tau)}[U_j(D, \varphi' + \zeta)] + \mathbb{E}_{j+1}[D] - \vec{g}_j^{r,\dagger}(D) \right.
\]

\[
- U_{j+1}(D, \varphi' + (\varepsilon + \tau) u_{j+1}) - \sum_{Y \in D_{h_j}} Q_j(D, Y, \varphi').
\]  

(7.5)
(See [4] (7.41)] for a detailed treatment for a similar computation.) Also by the definition of \( Q_j \) and change of variable (justified by Lemma 4.10),

\[
\begin{align*}
Y \supset D & \quad \sum_{Y \in \mathcal{S}_j} Q_j^i(D, Y, \varphi') \\
& = \sum_{Y \in \mathcal{S}_j} \text{Loc}_Y^{(2)} \mathbb{E}_{(t+\tau)}[K_j^0(Y, \varphi' + \zeta; 0)] + \sum_{Y \in \mathcal{S}_j} \frac{1_{D \in (P_j^q)_*}}{|Y \cap (P_j^q)_*|} \text{Loc}_Y^{(0)} \mathbb{E}_{(t+\tau)} D_j^i(Y, \varphi' + \zeta; \tau) \\
& = \sum_{Y \in \mathcal{S}_j} \text{Loc}_Y^{(2)} \mathbb{E}[K_j^0(Y, \varphi' + \zeta + (t + \tau)u_{j+1}; 0)] + \sum_{Y \in \mathcal{S}_j} \frac{1_{D \in (P_j^q)_*}}{|Y \cap (P_j^q)_*|} \text{Loc}_Y^{(0)} \mathbb{E}_{(t+\tau)} D_j^i(Y, \varphi' + \zeta; \tau)
\end{align*}
\]

(7.6)

where

\[
D_j^i(Y, \varphi' + \zeta; \tau) = (K_j^i)^\dagger(Y, \varphi; \tau) - K_j^0(Y, \varphi; 0).
\]

(7.7)

But by (6.12) and Definition [6.3], we see that the second line of (7.5) vanishes. Hence

\[
\mathcal{L}_{j+1}^i(\omega_{j}^{(1)}) = \sum_{b=1,2,3} \mathcal{L}_{j+1}^{i,(b)}(\omega_{j}^{(1)})
\]

(7.8)

where

\[
\begin{align*}
\mathcal{L}_{j+1}^{i,(1)} & = \sum_{Y \in \mathcal{S}_j} 1_{Y \in S_j} (1 - \text{Loc}_Y^{(2)}) \mathbb{E}_{(t+\tau)}[K_j^0(Y, \varphi' + \zeta; 0)] \\
\mathcal{L}_{j+1}^{i,(2)} & = \sum_{Y \in \mathcal{S}_j} 1_{Y \in S_j} (1 - \text{Loc}_Y^{(0)}) \mathbb{E}_{(t+\tau)}[D_j^i(Y, \varphi' + \zeta; \tau)] \\
\mathcal{L}_{j+1}^{i,(3)} & = \sum_{Y \in \mathcal{S}_j} 1_{Y \in S_j} \mathbb{E}_{(t+\tau)}[(K_j^i)^\dagger(Y, \varphi' + \zeta; \tau)] = \mathbb{E}[1_{Y \in S_j} \mathbb{E}_{(t+\tau)}[(K_j^i)^\dagger(Y, \varphi' + \zeta; \tau)]](X)
\end{align*}
\]

(7.9) - (7.11)

For \( b = 1 \), by Proposition 6.3, we have

\[
\|\mathcal{L}_{j+1}^{i,(1)}(X)\|_{\mathcal{H}_{\text{lin},j+1}} \leq CL^2 \alpha_{\text{Loc}}^{(2)} A^{-|X_j|} \|K_j^0(\cdot; 0)\|_{\mathcal{H}_j},
\]

(7.12)

where \( L^2 \) factor originates from the fact that there are at most \( O(L^2) \) small polymers \( Y \) such that \( Y = X \). For \( b = 2 \), we see from Proposition 6.3 that, \( (K_j^i)^\dagger(Y) \) \( = K_j^0(Y; 0) \), equivalently \( D^i(Y) = 0 \), unless \( Y \cap (P_j^q)_* \neq \emptyset \). So

\[
\mathcal{L}_{j+1}^{i,(2)} = \sum_{Y \in \mathcal{S}_j} 1_{Y \in S_j} (1 - \text{Loc}_Y^{(0)}) \mathbb{E}_{(t+\tau)}[D_j^i(Y, \varphi' + \zeta; \tau)]
\]

(7.13)

but then by Proposition 6.3, we have

\[
\|(1 - \text{Loc}_Y^{(0)}) \mathbb{E}_{(t+\tau)}[D_j^i(Y, \varphi' + \zeta; \tau)]\|_{\mathcal{H}_{\text{lin},j}} \leq C \alpha_{\text{Loc}}^{(0)} (A/2)^{-|X_j|} \|\omega_{j}^{(1)}\|_{\mathcal{H}_{j+1}}.
\]

(7.14)

Finally, for \( b = 3 \), Proposition 6.4 gives

\[
\|\mathcal{L}_{j+1}^{i,(3)}\|_{\mathcal{H}_{\text{lin},j+1}} \leq (2L^{-1} A^{-1})^{|X_j|+1} \|(K_j^i)^\dagger\|_{\mathcal{H}_{j}}.
\]

(7.15)

7.2. Bound on the non-linear part. The terms with order \( \geq 2 \) can be identified by following the linearisation process of \( \mathcal{R}_{j+1}^i \) backwards. If we denote

\[
U_{j+1}(X, \varphi') = -\delta_{j+1}|X| + \mathcal{g}_{j+1}(X) + U_{j+1}(X, \varphi' + (t + \tau)u_{j+1})
\]

(7.16)
and
\[
\mathcal{R}_j(U_j, K_j^{(1)}) = \mathcal{R}_j(\omega_j^{(1)}) = (\xi_{j+1}|X| - g_{j+1}^*(X), U_j, \overline{K}_j^{(1)}, K_j^\dagger, \xi K_j^\dagger, J_j^\dagger)(\omega_j^{(1)})
\] (7.17)
then \(M_{j+1} := \mathcal{R}_j^{(1)}(\omega_j^{(1)}) - L_j^{(1)}(\omega_j^{(1)})\) can be considered as a function
\[
M_{j+1}(\omega_j^{(1)}) = M_{j+1}(\mathcal{R}_j(\omega_j^{(1)}), X, \varphi')
\] (7.18)
This can be compared to [11 (7.52)-(7.55)]. The differentiability of \(M_{j+1}\) can be checked from regularity estimates of their components, as summarised following.

(R1) For \(B \in B_{j+1}, Z \in \mathcal{D}_{j+1}, \varphi \in \mathbb{R}^{\Lambda N}\) and \(k \in \{0, 1, 2\},\)
\[
\|U(B, \varphi; \tau)\|_{\mathcal{R}_j(B, \varphi)} \leq C(\delta, L)(1 + \delta c_w \kappa w_j(B, \varphi)^2)|x|\] (7.19)
\[
\|\mu(B, \varphi; \tau) - \frac{1}{m!}(\mu(B, \varphi; \tau))^m\|_{\mathcal{R}_j(B, \varphi)} \leq C(\delta, L)|e^{\delta c_w \kappa w_j(B, \varphi)^2}|x|^{k+1},
\] (7.20)
for \(U \in \{U_j, U_j^{(1)}\}\) and some \(C(L)\), and the same inequalities hold with \(U(B)\) and \(C(\delta, L)\) replaced by \(\xi_{j+1}|B|\) or \(-g_{j+1}^*(B)\) and \(C(L)\), respectively, but \(\delta\) set to 0.

(R2) With \(D\) the Fréchet derivative in \(x,\)
\[
\|D\mu(B, \varphi; \tau)\|_{\mathcal{R}_j(B, \varphi)} \leq C(L)e^{c_w \kappa w_j(B, \varphi)^2},
\] (7.21)
\[
\|D^2 \mu(B, \varphi; \tau)\|_{\mathcal{R}_j(B, \varphi)} \leq C(L)e^{c_w \kappa w_j(B, \varphi)^2},
\] (7.22)
\[
\|D\xi K_j(Z, \varphi; \tau)\|_{\mathcal{R}_j(Z, \varphi)} \leq C(A, L)A^{-(1+\eta)}|Z|^{j+1}e^{c_w \kappa w_j(Z, \varphi)^2},
\] (7.23)
\[
\|D\xi K_j(Z, \varphi; \tau)\|_{\mathcal{R}_j(Z, \varphi)} \leq C(A, L)A^{-(1+\eta)}|Z|^{j+1}G_j(Z, \varphi),
\] (7.24)
\[
U \in \{U_j, U_j^{(1)}, \xi_{j+1}|B| - g_{j+1}^*(B)\}, and in the case of \(U = \xi_{j+1}|B| - g_{j+1}^*(B)\), the factor \(e^{c_w \kappa w_j(B, \varphi)^2}\) can be omitted. The derivatives exist in the space of polymer activities with finite \(\|\cdot\|_{\mathcal{R}_j(B)}\)-norm for \(e^U, D\mu, J_j\), finite \(\|\cdot\|_{\mathcal{R}_j(Z)}\)-norm for \(\xi K_j\) and finite \(\|\cdot\|_{\mathcal{R}_j(Z)}\)-norm for \(K_j\).

**Lemma 7.2.** Under the assumptions of Theorem [7.1] for any \(\delta > 0\) and \(\beta\) sufficiently large, \(\varepsilon(\delta, L) = \varepsilon(\delta, \beta, L)\) (only polynomially small in \(L\) and \(\beta\) and \(\eta > 0\) such that \([7.13]-[7.25]\) (with constants possibly depending on \(\beta\)) hold for \(\mathcal{R}_j\) defined by \([7.14]\) with \(\|\omega_j^{(1)}\|_{\mathcal{R}_{j+1,1}} \leq \varepsilon(\delta, \beta, L)\)

The proof of this lemma will be given shortly. But before proving it, we emphasize that this lemma is enough to obtain the differentiability of \(M_{j+1}\), due to the analysis of [1].

**Proof of Theorem 7.1 (ii).** We apply [11 Lemma 7.12] with its assumptions verified by Lemma 7.2—the details of the definitions of the normed spaces differ, but they do not play any crucial role. Thus we obtain
\[
\|D\mathcal{M}_{j+1}(\mathcal{R}_j(\omega_j^{(1)}))\|_{\mathcal{R}_j(Z^{(1)})} \leq C_2(\tau, \beta, A, L)\|\omega_j^{(1)}\|_{\mathcal{R}_{j+1,1}},
\] (7.26)
\[
\square
\]

7.3. **Proof of Lemma 7.2.** We are now only left to prove Lemma 7.2. The proof of this lemma is divided into two parts, the first part discussing [R1] and the second part discussing [R2].

**Lemma 7.3.** Suppose we are in the setting of Lemma 7.2, so \(\|\omega_j^{(1)}\|_{\mathcal{R}_{j+1,1}} \leq \varepsilon(\delta, \beta, L)\) with \(\varepsilon(\delta, \beta, L) > 0\) sufficiently small. Then [R1] holds.

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Proof. For the case \( \Omega = U_j \) and \( \varepsilon_{j+1}|B| \), since \( U_j \) does not have any dependence on \( \tau \), the norm \( ||h, T_j(B, \psi)||_h \) can actually be replaced by \( ||h, T_j(B, \psi)|| \). However, after the replacement, these bounds are simply implications of [4, Lemma 7.11]. For the case \( \Omega = U^{j+1} \) and \(-g^{j+1}_j(B)\), again by [4, Lemma 7.11], for any \( \delta > 0 \), there exists \( C(\delta, \beta, L) > 0 \) such that

\[
||-\varepsilon_{j+1}|B| + U^{j+1}(B, \varphi)||_h \leq C(\delta, \beta, L) \left( 1 + \frac{1}{3}\delta c_w\kappa(L)w_j(B, \varphi)^2 \right)||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}},
\]

(7.27)

for \( B \in B_{j+1} \), and by (6.16),

\[
||g^{j+1}_j(B, K_j; \tau)||_h \leq CA^{-1}||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}}.
\]

(7.28)

Letting \( U^{j+1}_j(B) = U^{j+1}(B) - \varepsilon_{j+1}|B| + g^{j+1}_j(B) \), we have

\[
||U^{j+1}_j(B, \varphi)||_h \leq C'(\delta, \beta, L) \left( 1 + \frac{1}{3}\delta c_w\kappa(L)w_j(B, \varphi)^2 \right)||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}},
\]

(7.29)

and by Lemma 4.12

\[
||U^{j+1}_j(B, \varphi + \tau u^{j+1}_j)\|_h \leq ||U^{j+1}_j(B, \varphi)||_h \leq ||U^{j+1}_j(B, \varphi)||_h \leq ||U^{j+1}_j(B, \varphi)||_h \leq ||U^{j+1}_j(B, \varphi)||_h,
\]

(7.30)

where \( h'' = h + h_1 \). Again, by (4.15),

\[
\sum^{k-1}_{m=0} \frac{1}{m!} (\Omega(B, \varphi)^m)_h \leq 1 \|\Omega(B, \varphi)||_h \|e^{\|\Omega(B, \varphi)||_h T_j(B, \varphi)} \leq \frac{1}{C(\delta, \beta, L)},
\]

(7.33)

for \( k \in \{0, 1, 2\} \). Also again by (4.15) and (7.19), for \( ||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}} \leq \frac{1}{C(\delta, \beta, L)} \),

\[
e^{\|\Omega(B, \varphi)||_h T_j(B, \varphi)} \leq e^{1 + \frac{1}{3}\delta c_w\kappa(L)w_j(B, \varphi)^2} ||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}}.
\]

(7.34)

\[
\|\Omega(B, \varphi)||_h T_j(B, \varphi) \leq C''\|\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}} e^{1 + \frac{1}{3}\delta c_w\kappa(L)w_j(B, \varphi)^2} ||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}}.
\]

(7.35)

which concludes the proof after having inserted into (7.33).

\[ \Box \]

Lemma 7.4. Under the setting of Lemma 7.2, [R2] holds.

Proof. (7.21) and (7.22) for the cases \( \Omega = U_j \) or \( \varepsilon_{j+1}|B| \) are already proved by [4, Lemma 7.15]. Also for the case \( \Omega = U^{j+1} \), the proof of [4, Lemma 7.15] reveals that these bounds are derived purely from (7.19) and (7.20). Their differentiability are also consequences of the differentiability argument of the same reference.

To derive (7.23) and (7.24), we first look for a bound on \( Q^j_l \). But Lemma 5.2 bounds both terms of \( Q^j_l \) by

\[
||Q^j_l(B, Z, \varphi; \tau)||_h \leq C(L) ||\omega_j^{\frac{1}{1}}||_{\Omega_{j+1}} e^{C_w\kappa w_j(B, \varphi)^2}.
\]

(7.36)
Proposition 8.1. When integrating with respect to $E(\varphi, \tau)$, taking it guaranteed the existence of the renormalisation group flow for any $\varepsilon > 0$. The infinite volume limit.

Finally, to see the bound on $K_j$, define the function $\mathcal{F}$ on polymer activities by

$$\mathcal{F}(U_j, K)(X, \varphi; \tau) = \sum_{Y \in \mathcal{V}_j} e^{U_j(\Omega \setminus Y, \varphi') + \zeta} K(Y, \varphi; \tau)$$

(7.37)

so that $\mathcal{F}(U_j, K_j^\varepsilon) = K_j$. We already have a good control of $\mathcal{F}$, since [4, Lemma 7.16] gives that

$$\|D\mathcal{F}(U_j, K^0_j)(Z, \varphi)\|_{h_r(Z, \varphi)} \leq C(A, L)A^{-(1+\eta)}|Z|^{1+\eta}G_j(Z, \varphi)$$

(7.38)

(7.39)

(where $D$ is a derivative in $(U_j, K^0_j)$) for some $\eta > 0$, $C(A, L) > 0$ and sufficiently small $\|U_j, K^0_j\|_{\Omega_j}$. By Remark 4.4, we can in fact translate this into a result written in terms of $\|\cdot\|_{h_r(Z, \varphi)}$, giving

$$\|D\mathcal{F}(U_j, K_j^\varepsilon)(Z, \varphi; \tau)\|_{h_r(Z, \varphi)} \leq C(A, L)(A/2)^{-(1+\eta)}|Z|^{1+\eta}G_j(Z, \varphi)$$

This is exactly (7.29). The differentiability of $K_j$ also follows from the argument of [4, Lemma 7.16].

8 Renormalisation group computation of the two-point energy

In this section, we always assume that $f_1$ and $f_2$ satisfy [A_j] and $u_1$ and $u_2$ are defined according to (3.20) using $f_1$ and $f_2$, respectively. However, $f$ will be either $f_1 + T_y f_2$ or $f_1$ or $f_2$ or $T_y f_2$ depending on the situation. Hence for $g_j^r$ defined according to Definition 6.4, the dependence on $f_1, f_2$ will be denoted explicitly by putting $f$-arguments in square brackets whenever necessary, e.g., $g_j^r \equiv g_j^r[f_1 + T_y f_2]$.

If $j$ is sufficiently small compared to the separation between $u_{j,1}$ and $T_y u_{j,2}$, then one may claim that $u_{j,1}$ and $T_y u_{j,2}$ do not feel each other, since $\varphi|_{\text{supp}(u_{j,1})}$ and $\varphi|_{\text{supp}(u_{j,2})}$ are independent when integrating with respect to $E_{T_j}$. On the other hand, if $j$ becomes sufficiently large so that the range of $T_j$ is large compared to the separation, then the fluctuation integrals pick up extra terms due to this lack of independence. This can also be implemented in the RG computation by treating the case $j < j_{0y}$ and the case $j \geq j_{0y}$ differently for $g_{j+1}^r$, recalling that $j_{0y}$ is the coalescence scale

$$j_{0y} = \min \left\{ j > 0 : (B_j^r)_{3,3} \cap Q_j^r \neq \emptyset \right\}$$

(8.1)

8.1 The infinite volume limit. To understand the properties of the DG measure on $\mathbb{Z}^d$, it is necessary to understand the RG flow under taking limit $N \to \infty$. Since the infinite volume limit was constructed for the case $(\tau, \varepsilon) = (0, 0)$ by [4 Proposition 8.3], this result can be extended to $(\tau, \varepsilon) \in \mathcal{D}_h \times \mathbb{R}$ taking the $(\tau, \varepsilon) = (0, 0)$ as the reference point. However, this extension is not needed in full generality. Rather, we only need the information that the coupling constants and the free energy, $(U_j^{\Lambda_N}, e_j^{\Lambda_N}, g_j^{\Lambda_N})$, are independent of $\Lambda_N$. This can be seen by a short induction argument, taking it guaranteed the existence of the renormalisation group flow for any $j < N$.

Proposition 8.1. Let $L$ be sufficiently large. For each $N \geq j_{0y} + 2$, suppose $(e_j^{\Lambda_N}, g_j^{\Lambda_N}, U_j^{\Lambda_N})_{j \leq N}$ can be constructed on $\Lambda_N$ via the maps $(\Phi_j^{\Lambda_N})_{0 \leq j \leq N - 1}$ of (6.6) and initial condition $[\Phi_0]$. Then $(e_j^{\Lambda_N}, g_j^{\Lambda_N}, U_j^{\Lambda_N})$ are independent of $\Lambda_N$ when $j < N$. 

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Theorem 6.2. For this, assume as an induction hypothesis, that by Theorem 6.3, existence of the flow (\(\tau_{N,N'}(0) = 0\)) for \(j \leq N - 1\) using Definition 6.2 on \(\Lambda_N\), and also construct the corresponding objects on \(\Lambda_{N'}\). Then assume, as an induction hypothesis, that whenever \(j < N - 1\),

\[
\bar{K}_{j+1}^{\Lambda_N}(Y, \varphi \circ \tau_{N,N'}) = \bar{K}_{j+1}^{\Lambda_N'}(\tau_{N,N'}(Y), \varphi), \quad \varphi \in S_h(\Lambda_{N'})
\]

(8.2)

for any \(Y \in \mathcal{B}(\Lambda_N)\) such that \(Y \subset B_0^{N-1}\). This holds for \(j = 0\) by the assumptions on the initial condition. As an immediate consequence of (8.2), we have the same for \((K_{j+1}^{\Lambda_N})^1\) and \((K_{j+1}^{\Lambda_N'})^1\).

But since \(g_{j+1}^{'}, \epsilon_{j+1}^{'}, \text{ and } U_{j+1}\) only depend on \((K_{j+1}^{\Lambda_N})^1(Y)\) for \(Y \in \mathcal{B}(\Lambda_N)\) such that \(Y \cap (P_y^j)^* \neq \emptyset\) (to be precise, \(\epsilon_{j+1}^{'}, \text{ and } U_{j+1}\) are functions of \(K_{j+1}^{\Lambda_N}(X; 0)\) for \(X \in \mathcal{B}_j\), but since \(K_{j+1}^{\Lambda_N}(X; 0)\) is translation invariant, we only need information of \(K_0^{\Lambda_N}(Y; 0)\) for \(Y \in \mathcal{B}_j\) and \(Y \cap (P_y^j)^* \neq \emptyset\), and any such \(Y\) is contained in \([-R, R]^2\) where

\[
R = 5L^2 + L^{\text{loc}} \leq 6L^N - 2 \leq \frac{(L - 2)L^{N-2}}{2},
\]

we also have

\[
U_{j+1}^{\Lambda_N} = U_{j+1}^{\Lambda_N'}, \quad \epsilon_{j+1}^{'}, \quad \epsilon_{j+1}^{'}, \quad g_{j+1}^{'}, \quad g_{j+1}^{'},
\]

(8.4)

Also for \(X \in \mathcal{B}_{j+1}(\Lambda_N)\) such that \(X \subset B_0^{N-1}\), (6.17) and (8.4) indicate that \(\bar{K}_{j+1}^{\Lambda_N}(X)\) is independent of \(N\), completing the induction. The conclusion follows from (8.3).

By the proposition, (8.2) and (8.4), we can almost pretend as if the dependence of \(\Lambda_N\) is not present in the RG flow \((\bar{U}_{j+1}^{\Lambda_N}, \bar{g}_{j+1}^{'}, \bar{U}_{j}^{\Lambda_N}, \bar{K}_{j}^{\Lambda_N})))\). Hence we may drop the superscript \(\Lambda_N\) when \(j \leq N - 1\) (but keep it for \(j = N\), e.g. use \(\bar{g}_{j+1}^{'})\). Even more, we can now talk about the infinite sequence \((\bar{g}_{j}^{'})_{j \in \mathbb{Z}}\) without making reference to \(\Lambda_N\) and we may also state the decay of \(\bar{g}_{j}^{'}\) as \(j \rightarrow \infty\). We start this by justifying the existence of the RG flow of infinite length for sufficiently large \(\beta > 0\).

**Proposition 8.2.** Let \(\tau \in \mathbb{D}_{h_i}\). There exists \(\beta_0(\tau), L_0(\tau) > 0\) such that the renormalisation group flow, (6.8), is defined for any \(N > 0\) and all \(j \in \{0, 1, \cdots, N - 1\}\) at \(\beta \geq \beta_0(\tau)\) and \(L \geq L_0(\tau)\)

when the initial condition is given by (\(\Phi_\text{loc}\)).

**Proof.** By Theorem 6.3, existence of the flow \((\bar{U}_{j}^{\Lambda_N}, \bar{g}_{j}^{'}, \bar{K}_{j}^{\Lambda_N}(0; 0))\) for \(j \leq N - 1\) is guaranteed when \(\beta \geq \beta_0(\tau)\) is taken sufficiently large and the initial condition is given by (\(\Phi_\text{loc}\)). They also satisfy

\[
\|U_j\|_{\Omega_{N}}, \|K_j(0; 0)\|_{\Omega_N} \leq C e^{-c_j \beta} L^{-\alpha_j},
\]

(8.5)

for some \(C > 0\). So we only have to prove bounds on \(K_j^*(\tau)\) and use them as an input to Theorem 6.2. For this, assume as an induction hypothesis, that

\[
\|K_j\|_{\bar{h}, T_j} \leq C' e^{-c_j \beta} L^{-\alpha_j}, \quad j < N - 1
\]

(8.6)

for some \(C' \equiv C'(\tau) > 0\). When \(\beta \geq \beta_0(\tau)\) is taken sufficiently large, then \(K_j^*\) sits in the domain where Theorem 7.3 holds. Then by (6.10) and (6.11),

\[
\|K_{j+1}^*(\cdot; \cdot) - K_{j+1}^*(\cdot; 0)\|_{\bar{h}, T_j+1} = \|K_{j+1}^*(\cdot; \cdot) - K_{j+1}^*(\cdot; 0)\|_{\bar{h}, T_j+1} \leq C_1 \alpha_{\text{loc}}(0) \|K_j^*(\cdot; \cdot) - (K_j^*)^*(\cdot; 0)\|_{\bar{h}, T_j, A/2}
\]

(8.7)

and

\[
\|K_{j+1}^*(\cdot; 0) - K_{j+1}^*(\cdot; 0)\|_{\bar{h}, T_j+1} \leq C_1 \alpha_{\text{loc}}^0(0) \|K_j^*(\cdot; 0) - (K_j^*)^*(\cdot; 0)\|_{h, T_j, A/2}
\]

(8.8)
whenever \( \omega_j \uparrow \| \mathbf{P}_{j_{i+k}} \| \mathbf{P}_{j_{i+k}} \) is sufficiently small compared to \( \alpha^{(0)}_{\text{Loc}} \) this can be achieved by taking \( \beta \) sufficiently large in (8.3), and (8.4) and using Lemma 5.7. Also by Lemma 5.7,

\[
\|(K_j^i)^{i}(\cdot ; \cdot)\|_{h, T_j, A/2} \leq \|\omega_j^i \|_{\| \mathbf{P}_{j_{i+k}} \|}.
\]  

(8.9)

Hence combining these bounds with (8.5),

\[
\|K_j^{i+1}(\omega_j)(\cdot ; \cdot)\|_{h, T_j, A/2} \leq \left( C + 2C_1(C + C')L^\alpha \alpha^{(0)}_{\text{Loc}} \right) e^{-c_j \beta} L^{-\alpha(j+1)}.
\]  

(8.10)

But if we use the fact that \( L^\alpha \) is chosen in such a way that \( L^2 \alpha^{(2)}_{\text{Loc}} \leq O(L^{-\alpha}) \) (see Theorem 6.3) and \( \alpha^{(0)}_{\text{Loc}} \leq L^2(\log L)^{-1} \alpha^{(2)}_{\text{Loc}} \), we may take \( L \) and \( C'/C \) sufficiently large to obtain

\[
C + 2C_1(C + C')L^\alpha \alpha^{(0)}_{\text{Loc}} \leq C',
\]  

(8.11)

completing the induction.

\[\square\]

In fact, the computations seen in the proof also gives bound on the RG coordinates.

**Corollary 8.3.** Under the assumptions of Proposition 8.2 for each \( B \in B_j \) and some \( C > 0 \),

\[
\|K_j\|_{h, T_j}, \|g_j^i(B; \tau)\|_{h, W} \leq Ce^{-c_j \beta} L^{-\alpha(j)}.
\]  

(8.12)

**Proof.** The bound on \( K_j \) is from 6.6. Then the bound on \( g_j^i \) also follows from 6.16. \[\square\]

**8.2. Computation of the free energy.** Based on the RG analysis, we can now show that

\[
F_{N, m^2}[f](t, \tau) = \frac{\mathbb{E}(\tau)[Z_0(\phi(m^2) + \gamma \tau f + \tau \sum_{j=0}^N u_j)]}{\mathbb{E}[Z_0(\phi(m^2))]}.
\]  

(8.13)

exhibits a well-defined limit as \( m^2 \downarrow 0 \) and \( N \to \infty \) described in terms of the renormalisation group coordinates, where \( \phi(m^2) \sim \mathbb{N}(0, \tilde{C}(s_0^0(\beta)) + L_N(m^2)Q_N) \) (see Proposition 2.3).

**Proposition 8.4.** Let \( f = f_1 + T_2f_2 \) be as in (8.4). Under the assumptions of Proposition 8.2 if \( \hat{\beta} \in \mathbb{C}, \tau \in \mathbb{D}_{h_1} \) and \( \tau \in \mathbb{R} \) are such that \( \hat{\beta} = \tau + \tau \), then

\[
F_{N, m^2}[f] \xrightarrow{N \to \infty} \exp \left( \sum_{j=1}^\infty \sum_{B \in B_j((P_j)^*)} g_j^i(B; \tau)[f] \right) \text{ uniformly in } m^2 \in (0, 1]
\]  

(8.14)

where \( \hat{f} = f + s_0^0(\beta)\gamma \Delta f \).

**Proof.** Now let us prove 8.14. Recall \( \tilde{C}(s_0^0) = \sum_{j=1}^{N-1} \Gamma_j + \Gamma^\Lambda_N \) and

\[
Z_N^\Lambda(\phi'; \tau) = \mathbb{E}(\tau)[Z_0(\phi' + \gamma \tau f + \tau \sum_{j=0}^N u_j)] \quad \zeta \sim \mathbb{N}(0, \tilde{C}(s_0^0)).
\]  

(8.15)

When the initial condition of the iteration \( (\Phi_j^i)_{j \leq N} \) is given by \( [\Phi_{1C}] \) then Proposition 8.2 guarantees that we may apply this iteration any number of times, and hence successive application of Theorem 6.2 gives

\[
Z_N^\Lambda(\phi'; \tau) = e^{-E_N^\Lambda (|\Lambda_N|)} \exp \left( \sum_{j=1}^N \sum_{B \in B_j((P_j)^*)} g_j^\Lambda_N[f(B; \tau)] \right)
\]  

\[
\times \left( e^{E_N^\Lambda (\Lambda_N, \phi' + (\tau + \tau)u_N)} + K_{j_0}^\Lambda_N(\Lambda_N, \phi'; \tau) \right)
\]  

(8.16)

with \( u_N = \Gamma^\Lambda_N f \), and satisfy estimates

\[
\|g_j^\Lambda_N\|_{h, W}, \|U_{N}^{\Lambda N}\|_{\Omega_N}, \|K_{j_0}^{\Lambda N}\|_{h, T_j} \leq Ce^{-c_j \beta} L^{-\alpha N}.
\]  

(8.17)
But also by Proposition 5.6 and Lemma 5.7,
\[
e^{U_N^{\Lambda_N}(\Lambda_N, \varphi') + (\tau + u_N)} + K_N^{\Lambda_N}(\Lambda_N, \varphi'; \tau) = e^{U_N^{\Lambda_N}(\Lambda_N, \varphi')} + (K_N^{\Lambda_N})^t(\Lambda_N, \varphi'; \tau)
\]  
with
\[
\left\|(K_N^{\Lambda_N})^t(\Lambda_N, \varphi'; \tau)\right\|_{\mathbb{E}, \mathbb{E}^U(\Lambda_N, \varphi')} \leq C' e^{-\frac{\beta}{2} N^{1+\gamma} \Lambda_N} G_N(\Lambda_N, \varphi').
\]  
(8.19)

Finally, we commence the integral
\[
\mathbb{E}[Z_0(\varphi(m^2) + \varphi u_N)] = \mathbb{E}[\varphi' Z_N(\varphi' + \varphi u_N; \tau)], \quad \varphi' \sim \mathcal{N}(0, \tau N).\]  
(8.20)

Observe that, if we take \( Y \sim \mathcal{N}(0, \tau N L^{-N}) \), then \( Y \mathbf{1} \) has the same distribution as \( \varphi' \) (where \( 1 \) is the constant field taking value \( 1 \)), also see (8.14)). So \( G_N(\Lambda_N, \varphi') \equiv \) (constant) almost surely. Also \(|\nabla \varphi'|^2 = 0\) almost surely, so \( |U_N^{\Lambda_N}(\Lambda_N, \varphi')| \leq \|U_N^{\Lambda_N}\|_{\Omega_N^1} \). Therefore
\[
\mathbb{E}[\varphi' Z_N(\varphi'; \tau)] = e^{-\mathbb{E}[\Lambda_N]} \exp \left( \sum_{j=1}^{N} \sum_{B \in \mathcal{B}_j((P_j')^*)} \mathbf{g}_j^{\Lambda_N}[f](B; \tau) \right) (1 + O(L^{-\alpha N})).
\]  
(8.21)

Since \( \mathbf{g}_j^{\Lambda_N} \) is independent of \( \Lambda_N \) for \( N \geq \max\{j, 2j_0 + 2\} \), the convergence (8.14) holds by (8.17) and Corollary 8.3. The convergence is uniform in \( m^2 \) because the convergence rate only depends on \( O(L^{-\alpha N}) \) and bounds on \( \mathbf{g}_j^{\Lambda_N}[f](B; \tau) \)'s, which are independent of \( m^2 > 0 \).

In this proposition, the finite volume function was computed only to justify that the infinite volume limit exists. However, finite volume case is actually the case of our interest, so we record one-by-product in the next corollary. In what follows, we use the notation
\[
\mathbf{g}_j^{\Lambda_N}[f](\mathbb{Z}^2; \tau) := \sum_{B \in \mathcal{B}_j((P_j')^*)} \mathbf{g}_j^{\Lambda_N}[f](B; \tau)
\]  
(8.22)

although \( \mathbf{g}_j^{\Lambda_N}(B; \tau)[f] \) is actually defined on \( \Lambda_N \).

**Corollary 8.5.** Fix \( R > 0 \) and let \( \tau = \tau + \tau + \tau \) with \( \tau \in \mathbb{D}_{h_1} \) and \( \tau \in [-R, R] \). Then for sufficiently large \( L \) (depending on \( R \) and \( N \)) and under the same assumptions as in Proposition 8.4 but \( f \) satisfying (A.\( \mathbf{j} \))
\[
\lim_{m^2 \downarrow 0} F_{N, m^2}[f] = e^{\sum_{j=1}^{\infty} \mathbf{g}_j^{\Lambda_N}[f](\mathbb{Z}^2; \tau) (1 + \tilde{\psi}_N^{\tau}(\tau))}
\]  
(8.23)

where \( \tilde{\psi}_N^{\tau}(\tau) \) is an analytic function of \( \tau \) in \( \mathbb{D}_{h_1} \) and \( \|\tilde{\psi}_N^{\tau}(\tau)\|_{h_1, W} = O(L^{-\alpha N}) \) with the rate uniform on \( \tau \in [-R, R] \).

**Proof.** By Proposition 2.3, \( F_{N, m^2} \) admits a limit as \( m^2 \downarrow 0 \) when \( f \) satisfies (A.\( \mathbf{j} \)) and the limit is an analytic function. It follows from (8.15), uniformity of (8.21) in \( m^2 \) and definition of \( F_{N, m^2} \) that
\[
1 + \tilde{\psi}_N^{\tau}(\tau) := e^{-\sum_{j=1}^{\infty} \mathbf{g}_j^{\Lambda_N}[f](\mathbb{Z}^2; \tau)} \lim_{m^2 \downarrow 0} F_{N, m^2}[f]
\]  
(8.24)

satisfies
\[
1 + \tilde{\psi}_N^{\tau}(\tau) = e^{\sum_{j=\infty}^{\infty} \mathbf{g}_j^{\Lambda_N}[f](\mathbb{Z}^2; \tau)} \lim_{m^2 \downarrow 0} \mathbb{E}[U_N^{\Lambda_N}(\Lambda_N, Y \mathbf{1}) + (K_N^{\Lambda_N})^t(\Lambda_N, Y \mathbf{1}; \tau)] = 1 + O(L^{-\alpha N})
\]  
(8.25)

with \( Y \sim \mathcal{N}(0, \tau N (m^2) L^{-N}) \). It also satisfies \( \|\tilde{\psi}_N^{\tau}(\tau)\|_{h_1, W} = O(L^{-\alpha N}) \) as
\[
\left\|\mathbb{E}[\partial_{\tau}^n (K_N^{\Lambda_N})^t(\Lambda_N, Y \mathbf{1}; \tau)]\right\|_{h, T_j(\Lambda_N, 0)} \leq C\|\partial_{\tau}^n (K_N^{\Lambda_N})^t(\Lambda_N)\|_{h, T_j(\Lambda_N)}
\]  
(8.26)

for any \( n \geq 0 \). Thus we have (8.23).

The final remark follows from the observation that the estimates of the previous sections would still stay uniformly true even if we were looking at the regime \( |\tau| \leq R \).
8.3. One and two-point energies. Due to Corollary 8.3, the proof of Proposition 3.1 is complete once we show that the sum \( \sum_j g_j^f(Z^2; \tau) \) is not subject to a bias due to the multi-scale grid structure. In this section, we will see that this is the case, using translation invariance of \( F_{N,m^2} \).

8.3.1. One-point energy. For any \( j \geq 1 \), we have

\[
\sum_{B \in \mathcal{B}_j((B_0^2)^*)} g_j^f(B; \tau)[T_{y/2}] = \sum_{B \in \mathcal{B}_j((Q_0^2)^*)} g_j^f(B; \tau)[T_{y/2}] \quad (8.27)
\]

Hence, by (8.14), this implies

\[
\exp \left( \sum_{j=1}^{\infty} \sum_{B \in \mathcal{B}_j((Q_0^2)^*)} g_j^f(B; \tau)[T_{y/2}] \right) = \lim_{N \to \infty} F_{N,m^2}[T_{y/2}] \quad \text{uniformly in } m^2 > 0. \quad (8.28)
\]

Also if \( f = f_2 \), the same principles give

\[
\exp \left( \sum_{j \neq 1} \sum_{B \in \mathcal{B}_j((B_0^2)^*)} g_j^f(B; \tau)[f_2] \right) = \lim_{N \to \infty} F_{N,m^2}[f_2] \quad \text{uniformly in } m^2 > 0. \quad (8.29)
\]

But since \( F_{N,m^2}[T_{y/2}] \) is independent of the choice of \( y \), we also see that the expression on the left side of (8.28),

\[
\exp \left( \sum_{j=1}^{\infty} \sum_{B \in \mathcal{B}_j((Q_0^2)^*)} g_j^f(Z^2; \tau)[T_{y/2}] \right) = \exp \left( \sum_{j \geq 1} g_j^f(Z^2; \tau)[T_{y/2}] \right) \quad (8.30)
\]

should also be independent of \( y \), i.e.,

\[
\sum_{j \geq 1} g_j^f(Z^2; \tau)[T_{y/2}] = \sum_{j \geq 1} \sum_{B \in \mathcal{B}_j((B_0^2)^*)} g_j^f(B; \tau)[f_2]. \quad (8.31)
\]

In summary, we have the following.

**Proposition 8.6.** Let \( \tau \in \mathbb{D}_{h_1} \) and \( \beta > 0 \) be sufficiently large and \( f_1 \) be as in \( (A_f^2) \). Define the infinite volume one-point energy of \( f_1 \) by

\[
g_\infty^f[f_1](\tau) = \sum_{j \geq 1} g_j^f[f_1](Z^2; \tau). \quad (8.32)
\]

Then \( g_\infty^f[f_1](\tau) = \sum_{j \geq 1} g_j^f[T_{y/2}](Z^2; \tau) \) for any \( y \). This series converges absolutely with rate

\[
\left\| g_\infty^f[f_1](\tau) - \sum_{j=1}^{m} g_j^f[T_{y/2}](Z^2; \tau) \right\|_{h_1,W} \leq C(M, \rho, \tau)L^{-\alpha m} \quad (8.33)
\]

with rate uniform on \( \tau \in [-R, R] \) and \( y \in Z^2 \), and \( g_\infty^f[f_1](\tau) \in W^+(\mathbb{D}_{h_1}) \).

**Proof.** The first part follows from the discussion above. Also the second part is a consequence of the estimate of Corollary 8.3 and recalling that any uniform limit of analytic functions is also an analytic function.

8.3.2. Two-point energy. The two-point energy is defined to be the free energy in scales after \( j_0y \). But for doing so, we will have to make sure that the free energy before the scale \( j_0y \) can be expressed as sum of the one-point energies.

**Lemma 8.7.** Let \( j \leq j_0y - 1 \) and \( f = f_1 + T_{y/2} \) as in \( (A_f^2) \). Then \( g_j^f[f_1 + T_{y/2}](B) = g_j^f[f_1](B) \) for \( B \in \mathcal{B}_{j-1}((B_0^2)^{-1})^* \) and \( g_j^f[f_1 + T_{y/2}](B) = g_j^f[T_{y/2}](B) \) for \( B \in \mathcal{B}_{j-1}((Q_0^2)^{-1})^* \).
Proof. These follow from the ‘local dependence’ of the RG flow as in the proof of Proposition 8.1. For the first statement, we assume, as an induction hypothesis, that
\[ K_j^+(X)[f_1 + T_y f_2] = K_j^+(X)[f_1] \quad \text{for any } X \in \mathcal{P}_j, X \cap (Q_{j_0}^{j_0-1})^* = \emptyset, j' < j_{0y} - 1. \] (8.34)
We have \( X^* \cap \text{supp}(T_y u_{j+2}) = \emptyset \) for any such \( X \), so this would imply \( K_j^+(X)[f_1 + T_y f_2] = K_j^+(X)[f_1] \). Also, by definition of \( j_{0y} \), saying \( j' < j_{0y} - 1 \) would mean \( (B_{j_0}^{j_0-1})^* \cap (Q_{j_0}^{j_0-1})^* = \emptyset \), so \( g_{j+1}^*[f_1 + T_y f_2](B) = g_{j+1}[f_1](B) \) for any \( B \in \mathcal{B}_j((B_j^*)^*) \) by Definition 8.1. If we also had \( j' < j_{0y} - 1 \), \( Y \in \mathcal{P}_{j'+1} \), and \( Y \cap (Q_{j_0}^{j_0-1})^* = \emptyset \), since \( K_{j'+1}^+(Y) \) only depends on \( f_1 + T_y f_2 \) via \( u_{j+1}(B) \), \( (g_{j'+1}^*[X^*] : X^* \in \mathcal{P}_{j'}(X^*), X \in \mathcal{P}_j(X)) \), the induction proceeds. The conclusion was also obtained during the induction process.

The second statement follows from an identical argument.

As we have claimed, this lemma implies that the free energy before scale \( j_{0y} \) is just the sum of two one-point energies.

Corollary 8.8. If \( j < j_{0y} \), then
\[ g_j^*[f_1 + T_y f_2](\mathbb{Z}^2; \tau) = g_j^*[f_1](\mathbb{Z}^2; \tau) + g_j^*[f_2](\mathbb{Z}^2; \tau) \quad \text{(8.35)} \]

Thus we can see that the two-point energy, defined in the following lemma, has diminishing contribution in the limit \( \|y\|_2 \to \infty \).

Lemma 8.9. Let \( \tau \in \mathbb{D}_{h_1} \) and \( \beta > 0 \) be sufficiently large. Define
\[ \tilde{g}_x^2[f_1, T_y f_2](\tau) = \sum_{j=1}^{\infty} g_j^*[f_1 + T_y f_2](\mathbb{Z}^2; \tau) - g_j^*[f_1](\mathbb{Z}^2; \tau) - g_j^*[f_2](\mathbb{Z}^2; \tau). \] (8.36)

Then \( \tilde{g}_x^2[f_1, T_y f_2] \in W^+(\mathbb{D}_{h_1}) \) and satisfies the bound
\[ \|\tilde{g}_x^2[f_1, T_y f_2]\|_{h_1, W} = O(\|y\|_2^{-\alpha}). \] (8.37)

Proof. By Corollary 8.8 and Proposition 8.9 we have
\[ \tilde{g}_x^2[f_1, T_y f_2](\tau) = \sum_{j=j_{0y}}^{\infty} \left( g_j^*[f_1 + T_y f_2](\mathbb{Z}^2; \tau) - g_j^*[f_1](\mathbb{Z}^2; \tau) - g_j^*[f_2](\mathbb{Z}^2; \tau) \right). \] (8.38)

Hence by Corollary 8.3, the norm on \( \tilde{g}_x^2(\tau)[f_1, T_y f_2] \) is bounded by \( O(L^{-\beta j_{0y}}) \). But since \( L^{-j_{0y}} = O(\|y\|_2^{-1}) \), we have the desired conclusion.

Thus we are equipped with all components required to prove Proposition 8.1.

Proof of Proposition 8.1. Our aim is to rewrite the limit \( \lim_{m^2 \downarrow 0} F_{N,m^2}[^f] \). By Corollary 8.5, Proposition 8.6, and Lemma 8.9
\[ \lim_{m^2 \downarrow 0} F_{N,m^2}[^f] = e^{\tilde{g}_x^2[f_1] + \tilde{g}_x^2[f_2] + \sum_{j > j_{0y}} \tilde{g}_j^2[f_1, T_y f_2]} (1 + \tilde{g}_N) \] (8.39)

where
\[ \tilde{g}_j^2[f_1, T_y f_2] = g_j^*[f_1 + T_y f_2](\mathbb{Z}^2) - g_j^*[f_1](\mathbb{Z}^2) - g_j^*[f_2](\mathbb{Z}^2) \] (8.40)

and the required properties also follow from the same references.
A  Action of $\mathcal{E}(\tau)$ on polymer activities

A.1. Subdecomposition of the the field. We import some lemmas from [4] and [5], that use the idea of subdecomposing the field and the regulator. When $\zeta \sim N(0, \Gamma_{j+1})$, it is observed in [4] Section 4.3 that, for $N' \in \mathbb{N}$ and $I_{N'} = \{s = 0, \frac{1}{N'}, \ldots, 1 - \frac{1}{N'}\}$ such that $\ell := L^{1/N'} \in \mathbb{N}$, there exist covariances $(\Gamma_{j+s,j+s+(N'-1)})_{s \in I_{N'}}$ such that

$$\zeta = \sum_{s \in I_{N'}} \xi_{j+s,j+s'}, \quad \Gamma_{j+1} = \sum_{s \in I_{N'}} \Gamma_{j+s,j+s'} \quad (A.1)$$

(where $s' = s+(N'-1)$) with each $\xi_{j+s,j+s'} \sim N(0, \Gamma_{j+s,j+s'})$ independent and $(\Gamma_{j+s,j+s+(N'-1)})_{s \in I_{N'}}$'s satisfy properties similar to [11] [14]. All subsequent properties we need are summarized in the following lemmas, where we use

$$g_{j+s}(X, \varphi) = \exp \left( \kappa c_4 \sum_{a=0}^{2} W_{j+s}(X, \nabla^a \varphi) \right), \quad X \in \mathcal{B}_{j+s} \quad (A.2)$$

where $W_{j+s}$ (resp. $\mathcal{B}_{j+s}$) is a subscale analogue of [12] (resp. $\mathcal{B}_j$) and $c_4 > 0$ is chosen in the next lemma. Given $Y \in \mathcal{B}_j$, denote $Y_s$ the smallest polymer in $\mathcal{B}_{j+s}$ that contains $Y$, and $\|c_{j+s}(X)\|_L^2, \|\nabla \|_{L^2}^2(X)$ and $\|\nabla \|_{L^2}^2(\partial X)$ are as in [4] Section 5.3.

Lemma A.1. For $\alpha = 1, 2$, let $f_\alpha$ be as in (A.1) and let

$$u_{j+s,j+s',\alpha} = \Gamma_{j+s,j+s'} f_\alpha, \quad \mathfrak{f} > s, \ s \in I_{N'}, \ \mathfrak{f} \in (N')^{-1} + I_{N'} \quad (A.3)$$

Then $u_{j+s,j+s',\alpha}$ is supported on $B^0_{j+s', \mathfrak{f}}$, the unique $j + \mathfrak{f}$-block containing 0 and for each $n \geq 0$ and

$$\|\nabla^m u_{j+s,j+s',\alpha}\|_{L} \leq \begin{cases} C_0 M^2 \log L & \text{if } n = 0 \\ C_3 M^2 L^{-n(j+s)} & \text{if } n \geq 1 \end{cases} \quad (A.4)$$

Also, $u_{j+1,\alpha}$ defined by Definition [3.3] admits decomposition

$$u_{j+1,\alpha} = \sum_{s \in I_{N'}} u_{j+s,j+s+(N')^{-1},\alpha} \quad (A.5)$$

Proof. The proof is identical to Lemma [3.3] \square

Lemma A.2 ([5] Lemma 3.1]). Assume $0 \leq j < N$, $L = \ell^{N'}$. For $X \in \mathcal{B}_{j+s}$ and $\varphi, \xi_0, \xi_B \in \mathbb{C}^{\Lambda N}$ for each $B \in \mathcal{B}_{j+s}(X)$, define

$$\log G_{j+s}(X, \varphi, \xi_0, (\xi_B)_{B \in \mathcal{B}_{j+s}(X)}) \quad (A.6)$$

$$= \kappa \|\nabla (\varphi + \xi_0)\|_{L^2}^2 \left( + \kappa c_2 \|\nabla (\varphi + \xi_0)\|_{L^2}^2 \right) + \sum_{B \in \mathcal{B}_{j+s}(X)} \|\nabla^2 (\varphi + \xi_B)\|_{L^2}^2 \quad (A.6)$$

For any choice of $c_2$ small enough compared to 1, there exist $c_4 \equiv c_4(c_2)$ and an integer $\ell_0 = \ell_0(c_1, c_2)$, such that for all $\ell \geq \ell_0$, $N' \geq 1$, $s \in (0, \frac{1}{N'}, \ldots, 1 - \frac{1}{N'})$, $s' = s + (N')^{-1}$ and $\kappa > 0$, for $X \in \mathcal{B}_{j+s}$,

$$G_{j+s}(X, \varphi, \xi_0, (\xi_B)_{B \in \mathcal{B}_{j+s}(X)}) \leq \max_{a \in \{s \}} g_{j+s}(X_{s'}, \xi_0) G_{j+s}(X_{s'}, \varphi) \quad (A.7)$$

where $X_{s'}$ is the smallest $j + s'$-polymer containing $X$.

By the lemmas, one may decompose

$$\frac{G_{j}(X, \varphi')}{G_{j+1}(X, \varphi')} \leq \prod_{s \in I_{N'}} g_{j+s}(X_{s'}, \xi_{j+s,j+s'} + \tau u_{j+s,j+s'})$$

$$\leq e^{C(M_{\rho,1}(\ell)N' \log L) \prod_{s \in I_{N'}} g_{j+s}(X_{s'}, \xi_{j+s,j+s'})^2} \quad (A.8)$$
where \( u_{j+s,j'+s'} = u_{j+s,j'+s',1} + T_y u_{j+s,j'+s',2} \) and the second inequality follows from Lemma \( \text{A.4} \).

But since \( N' = \log L / \log \ell \),

\[
\frac{G_j(X, \varphi' + \zeta + i u_{j+1})}{G_{j+1}(X, \varphi')} \leq C(M, \rho, \tau, \ell) \prod_{s \in I_{N'}} g_{j+s}(X_{s'}, \xi_{j+s,j+s'})^2. \tag{A.9}
\]

The expectation of each \( (g_{j+s})^2 \) is controlled by the next lemma.

**Lemma A.3** ([1] Lemma 5.12].) The following hold for \( g_{j+s} \).

- There exists \( C > 0 \) such that for any \( X \in \mathcal{P}_{j+s} \) and \( \zeta \in \mathbb{R}^N \),

\[
g_{j+s}(X, \zeta) \leq \exp \left( \frac{1}{2} Q_{j+s}(X, \zeta) \right) := \exp \left( C_c \kappa \frac{4}{\ell} \sum_{a=0}^{\ell} \sum_{i} \| \nabla \phi_{j+s}^a \|_{L_2^2(X^*)}^2 \right). \tag{A.10}
\]

- For any \( c_\kappa > 0 \), any integer \( \ell \), there is \( c_\kappa = c_\kappa(c_\kappa, \ell) > 0 \) such that if \( \kappa(L) = c_\kappa(\log L)^{-1} \) then

\[
\mathbb{E}[e^{2Q_{j+s}(X, \zeta)}] \leq 2^{\mathcal{N}(\zeta)+1} |X_{j+s}|, \quad \zeta \sim \mathcal{N}(0, \Gamma_{j+s,j+s+N}) \quad \text{for } m^2 \in [0, \delta] \text{ and sufficiently small } \delta > 0.
\]

The statement of this lemma is actually slightly stronger than its original version, but the proof is exactly the same. There is an immediate corollary of these computations.

**Proof of Lemma A.3**. The first inequality is [1] Proposition 5.9. The second inequality follows from combining (A.10) with Lemma A.3. \( \square \)

### A.2. Complex-valued shift of variables

We prove Lemma [1.10] here.

**Lemma A.4.** Let \( X \in \mathcal{P}_j \), let \( F \) be a function such that \( \|F\|_{h, T_j(X)} < \infty \). Let \( v_2 \in \mathbb{R}^{N_N} \) be such that \( \|\Gamma_{j+s,j+s+N_2}v_2\|_{\mathcal{C}_j^2} < \sqrt{2} \) for each \( s, \mathcal{P} \in \{0, \ldots, 1 - (N')^{-1}\} \) and \( s < \mathcal{P} \). If \( \varphi' \in \mathbb{R}^{N_N} \), \( v = v_1 + iv_2 \) for some \( v_1 \in \mathbb{R}^{N_N} \), and \( \zeta \sim \mathcal{N}(0, \Gamma_{j+1}) \) (or \( \mathcal{N}(0, \Gamma_{N}^N) \) when \( j+1 = N \)) then

\[
\mathbb{E}[F(X, \varphi' + \zeta + \Gamma_{j+1}v)] = e^{-\frac{i}{2} \langle v, \Gamma_{j+1}v \rangle} \mathbb{E}[e^{i \langle \zeta, v \rangle} F(X, \varphi' + \zeta)]. \tag{A.12}
\]

**Proof.** Since \( X \) does not play any role, we will drop it at most places. By change of variable \( \zeta \to \zeta - \Gamma_{j+1}v_1 \), we have

\[
\mathbb{E}[F(\varphi + \zeta + \Gamma_{j+1}v_1 + i \Gamma_{j+1}v_2)] = e^{-\frac{i}{2} \langle v_1, \Gamma_{j+1}v_1 \rangle} \mathbb{E}[e^{i \langle \zeta, v_1 \rangle} F(\varphi + \zeta + i \Gamma_{j+1}v_2)]. \tag{A.13}
\]

The complex part \( iv_2 \) can be treated similarly by using the Cauchy’s theorem, so we aim to prove, for any \( m^2 > 0 \) sufficiently small,

\[
\mathbb{E}[e^{i \langle \zeta, v_1 \rangle} F(\varphi + \zeta + i C v_2)] = e^{-i \langle v_1, C v_2 \rangle + \frac{i}{2} \langle v_2, C v_2 \rangle} \mathbb{E}[e^{i \langle \zeta, v_1 \rangle} F(\varphi + \zeta)] \tag{A.14}
\]

where now \( C = \Gamma_{j+1} + N' m^2 \) and \( \zeta(m^2) \sim \mathcal{N}(0, C|X^*|) \). Once this is obtained, we can take the limit \( m^2 \to 0 \) to conclude. But we have to be more careful than the real-valued shift of variables, as we only have restricted amount of analyticity for \( F \).

Also having done the subscale decomposition \( \zeta(m^2) = \sum_{s=0}^{N'} \xi_s \) with \( \xi_s \sim \mathcal{N}(0, \Gamma_{j+s,j+s'+s} m^2) \) (where \( s' = s + (N')^{-1} \)), the proof of (A.14), up to an induction, reduces to proving

\[
\mathbb{E}[e^{i \langle \zeta, v_1 \rangle} F(\varphi' + \xi_0 + i (C_s + C_{s'}) v_2)] = e^{-i \langle v_1, C v_2 \rangle + \frac{i}{2} \langle v_2, C v_2 \rangle} \mathbb{E}[e^{i \langle \zeta, v_1 \rangle} F(\varphi' + \xi_0 + i C_{s'} v_2)] \tag{A.15}
\]

[1] Lemma 5.12
where $C_s = \Gamma_{j+\sigma} + m^2, C_{>s} = \sum_{\tau > s} C_{\tau}, \xi_s \sim \mathcal{N}(0, C_s)$ and $\varphi' = \varphi + \sum_{\tau \neq s} \xi_{\tau}$. After writing the expectation in integral form, this is equivalent to

$$\int_{\mathbb{R}^{X^*}} d\xi_s e^{-\frac{1}{2}(\xi_s, C_s^{-1}\xi_s)} e^{(\xi_s, v + iv_2)} F(\varphi' + \xi_s + i(C_s + C_{>s}) v_2)$$

$$= \int_{\mathbb{R}^{X^*} - iC_{v_2}} d\xi_s e^{-\frac{1}{2}(\xi_s, C_s^{-1}\xi_s)} e^{(\xi_s, v + iv_2)} F(\varphi' + \xi_s + i(C_s + C_{>s}) v_2) \quad (A.16)$$

After having done orthonormal change of basis of $(\delta_x : x \in X^*)$ to $(e_y : y \in A)$ (with $|A| = |X^*|$) such that $C_s v_2 = a e_{y_0}$ for some $a \in \mathbb{R}$ and $y_0 \in A$, and writing $(\xi_{s,y})_{y \in A}$ for the coordinates for $\xi_s$ in this basis (so $\xi_s = \sum_{y \in A} \xi_{s,y} e_y$), this is implied by

$$\int_{\mathbb{R}} d\xi_{s,y_0} e^{-\frac{1}{2}(\xi_{s}, C_s^{-1}\xi_{s})} e^{(\xi_{s}, v + iv_2)} F(\varphi' + \xi_s + i(C_s + C_{>s}) v_2)$$

$$= \int_{\mathbb{R} - i\alpha} d\xi_{s,y_0} e^{-\frac{1}{2}(\xi_{s}, C_s^{-1}\xi_{s})} e^{(\xi_{s}, v + iv_2)} F(\varphi' + \xi_s + i(C_s + C_{>s}) v_2). \quad (A.17)$$

By Proposition 4.10 and our assumptions on the size of norms of $C_s v_2$ and $C_{>s} v_2$, the function $\xi_{s,y_0} \rightarrow F(\sum_{y \in A} \xi_{s,y} e_y + \varphi' + i(C_s + C_{>s}) v_2)$ is analytic in an open neighbourhood of $\{\xi_{s,y_0} = a + ib : a \in \mathbb{R}, b \in [-\alpha, 0]\}$ when $\xi_{s,y} \in \mathbb{R}$ for each $y \neq y_0$, so the equality is again implied by Cauchy’s theorem on complex integrals if we could prove

$$\left| e^{-\frac{1}{2}(\xi_{s}, C_s^{-1}\xi_{s})} e^{(\xi_{s}, v + iv_2)} F(\varphi' + \xi_s + i(C_s + C_{>s}) v_2) \right| = o(1) \quad (A.18)$$

as $\|\xi_s + ite_{y_0}\|_{L^2(X^*)} \rightarrow \infty$, when $\xi_s \in \mathbb{R}^{X^*}$ and $t \in [-\alpha, 0]$. To bound this, we use 4.25 and Lemma A.2 obtaining

$$|F(\varphi' + \xi_s + C_s v_1 + ite_{y_0} + i(C_s + C_{>s}) v_2)| \leq \|F\|_{h,T_j(X)} G_j(X, \varphi + \xi_s + C_s v_1)$$

$$\leq C \|F\|_{h,T_j(X)} G_{j,s'}(X, \varphi) g_{j,s'}(X_{j,s'}, \xi_s)^2, \quad (A.19)$$

where $s' = s + (N')^{-1}$. While by Lemma A.3 $g_{j,s'}(X_{j,s'}, \xi_s)^2 \leq e^{\xi_{Q_{j,s'}}}$ for some quadratic form $Q_{j,s'}$ with $E[e^{2\xi_{Q_{j,s'}}}] < \infty$. This implies

$$e^{-\frac{1}{2}(\xi_{s}, C_s^{-1}\xi_{s})} |F(\varphi' + \xi_s + C_s v_1 + ite_{y_0} + i(C_s + C_{>s}) v_2)| = o(1) \quad \text{as} \quad \|\xi_s\|_{L^2(X^*)} \rightarrow \infty \quad (A.20)$$

and proves (A.18).

\textbf{Proof of Lemma 4.10} Recall, $u_{j+1} = \Gamma_{j+\tilde{f}}$ if $j \geq 0$. By definition,

$$\mathbb{E}_{(\tau)} [F(X, \varphi' + \zeta)] = e^{-\frac{1}{2}r^2(\tau_{j+1})} \mathbb{E}_{(\xi, \tau)} [e^{\xi \tau_{j+1}} F(X, \varphi' + \zeta)] \quad (A.21)$$

Also, Lemma A.10 implies

$$\|\tau \Gamma_{j+s, j+\tilde{f}} \|_{C_{\tilde{f}}} < h/2, \quad \tilde{f} > s, s \in I_{N'}, \tilde{f} \in (N')^{-1} + I_{N'}, \quad (A.22)$$

whenever $\tau < (2M \rho^2 \log L)^{-1} h/2$. This verifies the assumptions of Lemma A.10 completing the proof.

\textbf{A.3. Proof of Lemma 4.8}

\textbf{Lemma A.5} (Gaussian integration by parts). Given $j$, let $u_{j+1} = \Gamma_{j+\tilde{f}}$. Let $F$ be a polymer activity such that $\|F\|_{h,T_j(X)} < \infty$. Then

$$\mathbb{E} \left[ D^k F(X, \varphi' + \zeta)(u_{j+1}^{\tilde{f}}) \right] = \sum_{l=0}^{k} \binom{k}{l} (u_{j+1}, \tilde{f})^{l/2} \mathbb{E}_{(\xi, \tau)} [(\zeta_{j+s, \tilde{f}})^{k-l} F(X, \varphi' + \zeta)] \quad (A.23)$$

when $\text{He}_{2p+1}(0) = 0$ and $\text{He}_{2p}(0) = (-1)^p \frac{(2p)!}{2^{2p}}$ for $p \in \mathbb{Z}_{>0}$.  

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Proof. We drop $X$ so that $F(X, \varphi)$ is written as $F(\varphi)$. It is sufficient to prove this for $\zeta \sim \mathcal{N}(0, C|X|)$ and $u_{j+1} = C\tilde{f}$, where $C = \Gamma_{j+1} + m^2$ and $m^2 > 0$. Also, by integration by parts and up to a normalisation factor, $\mathbb{E}_C[D^k F(X, \varphi' + \zeta)(u^\otimes_{j+1})]$ can be written in terms of the Lebesgue integral

$$
\int_{\mathbb{R}^n} d\zeta \ e^{-\frac{1}{2}(\zeta, C^{-1}\zeta)} D^k F(\varphi' + \zeta)(u^\otimes_{j+1}) = \int_{\mathbb{R}^n} d\zeta (-D)^k e^{-\frac{1}{2}(\zeta, C^{-1}\zeta)} (u^\otimes_{j+1}) F(\varphi' + \zeta). \tag{A.24}
$$

The derivative on the right-hand side can be written as

$$
(-1)^k \frac{d^k}{ds^k} \bigg|_{s=0} e^{-\frac{1}{2}(\zeta + su_{j+1}, C^{-1}(\zeta + su_{j+1}))} = (-1)^k \frac{d^k}{ds^k} \bigg|_{s=0} e^{-\frac{1}{2}As^2 - Bs - E} \tag{A.25}
$$

where $A = (u_{j+1}, C^{-1}u_{j+1}) = (u_{j+1}, \tilde{f})$, $B = (\zeta, C^{-1}u_{j+1}) = (\zeta, \tilde{f})$ and $E = \frac{1}{2}(\zeta, C^{-1}\zeta)$. By the Leibniz formula,

$$
(-1)^k \frac{d^k}{ds^k} e^{-\frac{1}{2}As^2 - Bs - E} = \sum_{l=0}^k \binom{k}{l} \frac{d^l}{ds^l} (\tilde{f}) (As)^{k-l} e^{-\frac{1}{2}As^2 - Bs - E}, \tag{A.26}
$$

where $\tilde{f}(x)$ is the Hermite polynomial of degree $l$. Therefore we obtain

$$
\mathbb{E}_C \left[ D^k F(X, \varphi' + \zeta)(u^\otimes_{j+1}) \right]
\propto \sum_{l=0}^k \binom{k}{l} \langle u_{j+1}, \tilde{f} \rangle^{l/2} \tilde{H}_l(0) \int_{\mathbb{R}^n} d\zeta e^{-\frac{1}{2}(\zeta, C^{-1}\zeta)} (\zeta, \tilde{f})^{k-l} F(X, \varphi' + \zeta). \tag{A.27}
$$

To conclude, take limit $m^2 \rightarrow 0$. \hfill \Box

Lemma A.6. Let $\tilde{f}$ be as in \textbf{[A.1]} $\tilde{f} = (1 + s\gamma \Delta)f$ and $u_{j+1}$ be defined by Definition \textbf{[A.2]} If $\|D^n F(X)\|_{n,T_j(X)} < \infty$, there are some constants $C_1(M, \rho), C_2(M, \rho, \tau) > 0$ such that

$$
\left\| \mathbb{E}_{\tilde{f}} \left[ \left( \zeta | X, \tilde{f} \right)^k D^n F(X, \varphi' + \zeta + ru_{j+1}) \right] \right\|_{n,T_j(X, \varphi)}
\leq C_2 \frac{|X|}{(C_1 \log L) \bar{\mathcal{E}}^k \left( \left( \frac{k}{2} + 1 \right) \right) G_{j+1}(X, \varphi')} \|D^n F(X)\|_{n,T_j(X)}. \tag{A.28}
$$

Proof. Again, we drop $X$. Since $|D^n F(\varphi' + \zeta)| \leq \|D^n F\|_{n,T_j(X)} \mathcal{G}_j(X, \varphi' + \zeta)$ we just need to bound $\mathbb{E}[(\zeta | X, \tilde{f})^k \mathcal{G}_j(X, \varphi' + \zeta)]$. Firstly, after the subdecomposition $\zeta | X = \sum_{s=0}^{N' - 1} \xi_s | X$ (as in \textbf{[A.1]}) and using the Jensen’s inequality, we have

$$
| (\zeta | X, \tilde{f})^k | = \left| \sum_{s=0}^{N' - 1} (\xi_s | X, \tilde{f}) \right|^k \leq (N')^{k-1} \sum_{s=0}^{N' - 1} |(\xi_s | X, \tilde{f})|^k
\leq (N')^{k-1} \sum_{s=0}^{N' - 1} \|\tilde{f}\|_1^k \|\xi_s\|_{L^\infty(X)}^k \tag{A.29}
$$

But since

$$
\|\xi_s\|_{L^\infty(X)}^k \leq \left( c_4 \mathcal{K} \right)^{-\frac{k}{2}} \Gamma \left( \frac{k + 2}{2} \right) \exp \left( c_4 \mathcal{K} \|\xi_s\|_{L^\infty(X)}^2 \right) \tag{A.30}
$$

($\Gamma$ is the gamma function) and $\|\tilde{f}\|_1 \leq M_{\rho^2}$, we see for some $C \equiv C(M, \rho)$ that

$$
|(\zeta | X, \tilde{f})^k| \leq (N')^{k-1} \sum_{s=0}^{N' - 1} C^{k-1} \mathcal{K}^{-\frac{k}{2}} \left( \left( \frac{k}{2} + 1 \right) ! \right) e^{c_4 \mathcal{K} \|\xi_s\|_{L^\infty(X)}^2}. \tag{A.31}
$$

Secondly, \textbf{[A.1]} gives decomposition

$$
\mathcal{G}_j(X, \varphi' + \zeta + ru_{j+1}) \leq C(M, \rho, \tau, \ell) \prod_{s \in \mathcal{I}_{N'}^{(N')-1}} g_{j+(N')-1,s}(X_{j+(N')-1,s}, \xi_s)^2 \mathcal{G}_{j+1}(X, \varphi'). \tag{A.32}
$$
But since $\|c^{\epsilon}\|_{L^\infty(X)} \leq g_{j+(N'-1)s}(X, \xi)$, combining this with \ref{A.31}, we have
\[
\left|\left|\zeta\right|_{X, \hat{f}}\right|^k G_j(X, \varphi' + \zeta) \leq (N')^k C^{\epsilon \kappa - \frac{k}{2}} \left|\left| G_{j+1}(X, \varphi') \right|\left| \prod_{s=0}^{N'-1} g_{j+(N'-1)s}(X_{j+(N'-1)s}, \xi)^3. \right|
\]  
(A.33)

Also Lemma \ref{A.3} says
\[
E\left[ \prod_{s=0}^{N'-1} g_{j+(N'-1)s}(X_{j+(N'-1)s}, \xi)^3 \right] \leq 2^{\|X\|}, \tag{A.34}
\]
so the conclusion follows from recalling that $N' = \frac{\log L}{\log \tau}$ and $\kappa = c_\kappa(\log L)^{-1}$.

\begin{lemma}
\label{A.7}
Let $f$, $\hat{f}$ and $u_{j+1}$ be as in the previous Lemma. Let $h_1 \leq \frac{(C \log L)^{-3/2}}{2}$ for sufficiently large $C \equiv C(M, \rho)$ and $\|F(X)\|_{\tilde{n}, T_j(X)} < \infty$. Then
\[
\|E[F(X, \varphi' + \zeta + (\tau + \xi) u_{j+1}; \tau)]\|_{\tilde{n}, T_{j}(X, \varphi')} \leq C_2 2^{\|X\|} \|F(X)\|_{\tilde{n}, T_{j}(X)} G_{j+1}(X, \varphi') \tag{A.35}
\]
for some $C_2 \equiv C_2(M, \rho, \tau)$, when $\zeta \sim N(0, \Gamma_{j+1})$.
\end{lemma}

\begin{proof}
Again, we drop $X$. We prove this by making the bound
\[
\sum_{l=0}^{\infty} \frac{h_1^l}{l!} \left| \left| \frac{d^l}{dt^l} \right|_{\tau=0} D^{n}E[F(\varphi' + \zeta + (\tau + \xi) u_{j+1}; \tau)] \right|_{n, T_{j}(X, \varphi)} \leq C_2 2^{\|X\|} \sum_{l=0}^{\infty} \frac{h_1^l}{l!} \|\partial_t^l D^{n} F(\tau)\|_{n, T_{j}(X)} G_{j+1}(X, \varphi') \tag{A.36}
\]
for each $n \geq 0$ and $C$ independent of $n$. Also, since $n$ does not play any role in the proof, we will just prove this for the case $n = 0$. To show this, first make expansion
\[
\left| \left| \frac{d^l}{dt^l} \right|_{\tau=0} F(\varphi' + \zeta + (\tau + \xi) u_{j+1}; \tau) \right| = \sum_{k=0}^{l} \binom{l}{k} D^{l-k} \partial_t^k F(\varphi' + \zeta + \tau u_{j+1}; 0)(u_{j+1}^{l-k}) \tag{A.37}
\]
and by Lemma \ref{A.5}
\[
E \left[ D^{l-k} \partial_t^k F(\varphi' + \zeta + \tau u_{j+1}; 0)(u_{j+1}^{l-k}) \right] \tag{A.38}
\]
\[
= \sum_{m=0}^{l-k} \binom{l-k}{m} (u_{j+1}, \hat{f})^m \text{He}_m(0) E [\zeta | X', \hat{f}]^{l-k-m} \partial_t^k F(\varphi' + \zeta + \tau u_{j+1}; 0),
\]
while by Lemma \ref{A.6}
\[
|E [\zeta | X', \hat{f}]^{l-k-m} \partial_t^k F(\varphi' + \zeta + \tau u_{j+1}; 0)| \leq C_2 2^{\|X\|} (C_1 \log L)^{2(l-k-m)} \left( \left| \frac{l-k-m}{2} \right| ! \right) G_{j+1}(X, \varphi') \|\partial_t^k F\|_{0, T_{j}(X)} \tag{A.39}
\]
Combining these bounds, using $(u_{j+1}, \hat{f}) \leq C(M, \rho) \log L$ and $\text{He}_{2p-1}(0) = 0$, $\text{He}_{2p}(0) = (-1)^p \frac{(2p)!}{2^p p!}$ for $p \in \mathbb{Z}_{\geq 0}$, we have
\[
\sum_{l=0}^{\infty} \frac{h_1^l}{l!} \left| \left| \frac{d^l}{dt^l} \right|_{\tau=0} E[F(\varphi' + \zeta + (\tau + \xi) u_{j+1}; \tau)] \right|_{0, T_{j}(X, \varphi)} \leq C_2 2^{\|X\|} G_{j+1}(X, \varphi') \sum_{k=0}^{\infty} \frac{h_1^l + k}{k!} \sum_{p=0}^{\infty} \frac{2^{2p} (C_1 \log L)^{2l-2p} (2p)!}{2^p p! (l'-2p)!} \left( \left| \frac{l'-2p}{2} \right| ! \right) \|\partial_t^k F\|_{0, T_{j}(X)} \tag{A.40}
\]
\end{proof}
after reparametrising $l' = l - k$, $m = 2p$ and $C_3 = (C_1/C)^{1/2}$. This is bounded by
\[ C_2 2^{[X]} \sum_{k=0}^{\infty} \frac{h_t^k}{k!} \| \partial_{k}^{e} F(\cdot; \tau) \|_{0,T_j(X)G_{j+1}(X, \varphi')} \times \sum_{l' = 0}^{\infty} \left( \cdots \right) \] (A.41)
where
\[ \sum_{l' = 0}^{\infty} \left( \cdots \right) = \sum_{l' = 0}^{\infty} \left( (C_1 \log L)^{\frac{3}{2}} h_t \right)^{l'} \sum_{p=0}^{\left\lfloor \frac{l'}{2p} \right\rfloor} C_3^p \frac{(C_1 \log L)^{-2p} l'!}{(l' - 2p)! \left\lfloor \frac{l'}{2} \right\rfloor !}. \] (A.42)

But after using the trivial bound $\sum_{p=0}^{\left\lfloor \frac{l'}{2p} \right\rfloor} \frac{1}{p! (l' - 2p)!} \left\lfloor \frac{l'}{2} \right\rfloor ! \leq e$ and setting $h_t$ and $L$ so that $\frac{1}{2} C_3 (C_1 \log L)^{-2} \leq 1$ and $h_t \leq \frac{1}{2} (C_1 \log L)^{-3/2}$, we see that (A.41) is bounded by a constant that is independent of $L$.

**Proof of Lemma 4.8** By Lemma 4.10,
\[ \mathbb{E}_{(\tau)}[F(X, \varphi'; \tau)] = \mathbb{E}[u(\cdot, j) F(X, \varphi' + \zeta + \tau u_j + 1; \tau)] \exp \left( \frac{1}{2} \tau^2 (j, \Gamma_j + 1) \right) \]
\[ = \mathbb{E}[F(X, \varphi' + \zeta + (\tau + \tau) u_j + 1; \tau)]. \] (A.43)

Hence in fact
\[ \| \mathbb{E}_{(\tau)}[F(X, \varphi'; \tau)] \|_{h_t, T_j(X, \varphi')} = \| \mathbb{E}[F(X, \varphi' + \zeta + (\tau + \tau) u_j + 1; \tau)] \|_{h_t, T_j(X, \varphi')} \] (A.44)
and we obtain the bound (1.22) by applying Lemma [A.7]. Also the analyticity of $\tau \mapsto \mathbb{E}_{(\tau)}[D^n F(X, \varphi' + \zeta; \tau)]$ is a consequence of Lemma [4.12].

**A.4. Proof of Lemma 5.2** For the first inequality, since Loc$_X^{(0)}$ $\mathbb{E}_{(\tau)} F_1(X, \varphi' + \zeta) = \mathbb{E}_{(\tau)} \hat{F}_1(X, \zeta)$ and by (4.24) and Lemma [4.8],
\[ \| \mathbb{E}_{(\tau)} \hat{F}_1(X, \zeta; \tau) \|_{h_t, w} \leq \| \mathbb{E}_{(\tau)} F_1(X, \zeta; \tau) \|_{h_t, T_j(X)} \leq C' 2^{[X]} \| F_1(X; \tau) \|_{h_t, T_j(X)}. \] (A.45)

The analyticity follows from the analyticity statement in Lemma [4.8].

For the second, since Loc$_Z^{(2)}$ gives a polynomial of degree 2, we have
\[ \| \text{Loc}^{(2)}_{Z,B} \mathbb{E}_{(\tau)}[F_2(Z, \varphi' + \zeta)] \|_{h_t, T_j(B, \varphi')} \leq 4 \| \text{Loc}^{(2)}_{Z,B} \mathbb{E}_{(\tau)}[F_2(Z, \varphi' + \zeta)] \|_{(h/2, h_1), T_j(B, \varphi')} \] (A.46)
while by (4.34),
\[ \| \text{Loc}^{(2)}_{Z,B} \mathbb{E}_{(\tau)}[F_2(Z, \varphi' + \zeta)] \|_{(h/2, h_1), T_j(B, \varphi')} \leq \| \text{Loc}^{(2)}_{Z,B} \mathbb{E}[F_2(Z, \varphi' + \zeta + u_j + 1)] \|_{h_t, T_j(B, \varphi')} \] (A.47)
whenever $h_1 < C (\log L)^{-1} h$. But [4] (6.16) says
\[ \| \text{Loc}^{(2)}_{Z,B} \mathbb{E}[F_2(Z, \varphi' + \zeta + u_j + 1)] \|_{h_t, T_j(B, \varphi')} \leq C \log L \| F_2(Z) \|_{h_t, T_j(Z)} e^{C (\log L) w_j(B, \varphi' + u_j + 1)^2} \]
\[ \leq C' (L) \| F_2(Z) \|_{h_t, T_j(Z)} e^{C (\log L) w_j(B, \varphi')^2} \] (A.48)
so we obtain [5.3].
B Contraction estimates

B.0.1. Taylor expansion. Given a polymer activity $F(X) \in \mathcal{H}_j(X)$ on $X \in \mathcal{P}_j$, we define the Taylor polymer of degree $n$ by

$$\text{Tay}^n F(X, \varphi + \psi) = \frac{1}{n!} \sum_{x_1, \ldots, x_n \in X} \frac{\partial^n F(X, \psi)}{\partial \psi(x_1) \cdots \partial \psi(x_n)} \varphi(x_1) \cdots \varphi(x_n)$$

and $\text{Rem}_n F(X, \varphi + \psi) = F(X, \varphi + \psi) - \text{Tay}^n F(X, \varphi + \psi)$. (B.1)

Also for $F(X) \in \mathcal{H}_j(X)$ neutral, define

$$\text{Tay}_2^\delta F(X, \varphi + \psi) = \frac{1}{|X|} \sum_{x_0 \in X} \text{Tay}_2^\delta F(X, \delta \varphi + \psi)$$

and

$$\text{Rem}_2^\delta F(X, \varphi + \psi) = F(X, \varphi + \psi) - \text{Tay}_2^\delta F(X, \varphi + \psi)$$

where $\delta \varphi(x) := \varphi(x) - \varphi(x_0)$ is dependent of the choice of $x_0 \in X$.

B.0.2. Fourier decomposition. If $F : \mathcal{P}_j \times \mathbb{R}^\Lambda \to \mathbb{C}$ is a $(2\pi \beta^{-1/2})$-periodic function, i.e., $F(X, \varphi + y1) = F(X, \varphi)$ for any $X \in \mathcal{P}_j$, $y \in 2\pi \beta^{-1/2} \mathbb{Z}$ and $1$ is the constant field, we have defined the neutral part of $F$ in Section 4.3. In fact, this definition can be extended to Fourier components of any period,

$$\hat{F}_q(X, \varphi) = \frac{\sqrt{\beta}}{2\pi} \int_0^{2\pi} ds e^{-isq} F(X, \varphi + s), \quad q \in \mathbb{Z}.$$ (B.5)

This is also called the charge-$q$ component of $F$, and satisfy

$$F(X, \varphi + t) = \sum_{q \in \mathbb{Z}} e^{i\sqrt{\beta}qt} \hat{F}_q(X, \varphi), \quad t \in \mathbb{R}.$$ (B.6)

If $F = \hat{F}_q$ for some $q \in \mathbb{Z}$, then we call $F$ is a polymer activity of charge $q$.

As explained in Section 5.1, $\text{Loc}^{(k)} F$ is intended to be the approximation of $\text{Tay}_k F$ with better algebraic properties, so we can bound $(1 - \text{Loc}^{(k)} \mathcal{E}) F(X)$ by bounding each term of the following:

$$(1 - \text{Loc}^{(k)}_X \mathcal{E}) F(X, \varphi' + \zeta) = \text{Rem}_k F_0(X, \varphi' + \zeta) + \sum_{q \in \mathbb{Z}} \text{Loc}^{(k)}_X \mathcal{E} \hat{F}_q(X, \varphi' + \zeta).$$ (B.7)

Hence the proof of Proposition 5.3 can now be reduced to the following statements.

**Proposition B.1.** Let $X \in S_j$, $\bar{h} = (h, h_4)$ and let $F$ be a $(2\pi / \sqrt{\beta})$-periodic polymer activity such that $\|F\|_{\bar{h}, \mathcal{T}_j(X)} < \infty$. Let $L \geq L_0$, $h \geq c_h \sqrt{\beta}$ and $h_4 < (C_1 \log L)^{-1} h$ for $L_0, c_h$ and $C_1$ sufficiently large. Then the following hold for some $C = C(M, \rho, \tau) > 0$.

1. If $F$ has charge $q$ with $|q| \geq 1$, then for all $\varphi' \in \mathbb{R}^\Lambda_N$,

$$\left\| \text{Loc}^{(k)} \mathcal{E} F(X, \varphi' + \zeta) \right\|_{\bar{h}, \mathcal{T}_{j+1}(X, \varphi')} \leq C e^{2\sqrt{\pi q h} e^{-(!q!^{-1/2}) \mathcal{T}_{j+1}(0)}} \|F(X)\|_{\bar{h}, \mathcal{T}_j(X)} G_{j+1}(X, \varphi').$$ (B.8)

2. If $F$ is neutral, then for all $\varphi' \in \mathbb{R}^\Lambda_N$ and $m \in \{0, 2\}$,

$$\left\| \text{Rem}_m F(X, \varphi' + \zeta) \right\|_{\bar{h}, \mathcal{T}_{j+1}(X, \varphi')} \leq C L^{-m} (\log L)^{m+1} \|F(X)\|_{\bar{h}, \mathcal{T}_j(X)} G_{j+1}(X, \varphi').$$ (B.9)
Lemma B.2. Let $F$ be a neutral polymer activity such that $F(X, \varphi) = F(X, -\varphi)$ and $\|F(\cdot; \tau)\|_{h,T_j} < \infty$. Then for $X \in \mathcal{S}_j$,

$$\| \frac{\text{Loc}_X^{(2)}}{E_{(i+\tau)}(X, \varphi + \zeta; \tau)} - \text{Ray}_2 \mathbb{E}_{(i+\tau)}(X, \varphi + \zeta; \tau) \|_{h,T_j+1(X, \varphi')} \leq C L^{3} (\log L)(A')^{-1} \|F\|_{h,T_j+1} G_{j+1}(X, \varphi').$$  \hspace{1cm} (B.10)

for any $A' \geq 1$.

B.1. Proof of Proposition [B.1] (1).

Corollary B.3. Let $q \in \mathbb{Z} \setminus \{0\}$, $X \in \mathcal{S}_j$, $\xi(x) = \sqrt{\beta}(\Gamma_{j+1}(x - x_0) - \Gamma_{j+1}(0))$ and $h \geq c_h \sqrt{\beta}$ for $c_h$ sufficiently large. Let $j$ satisfy $\left|\mathcal{A}_j\right|$, $u_{j+1}$ defined by Definition B.2 and $|\tau| < h_t \leq (C \log L)^{-1} h$ for sufficiently large $C$. If $F$ is a $q$-charge polymer activity on $X$ with $\|F(X)\|_{h,T_j(X)} < \infty$, then

$$\mathbb{E}[F(X, \varphi + \zeta + \tau u_{j+1})] = e^{\frac{1}{2} \beta T_{j+1}(0)(2|q|-1)} \mathbb{E}[e^{-i\sqrt{\beta} \sigma_0} F(X, \varphi + \zeta + \tau u_{j+1} + i\sigma_0 \xi)]$$ \hspace{1cm} (B.11)

where $\sigma_0 = \text{sign}(q)$ and any $\varphi \in \mathbb{R}^{\mathcal{A}_N}$.

Proof. Again, omit $X$. Without loss of generality, we take $q > 0$. We apply Lemma A.4 with $v = i\sqrt{\beta} \sigma_0$, (and the assumptions are verified by [4] Lemma 6.11), to obtain

$$\mathbb{E}[e^{-i\sqrt{\beta} \sigma_0} F(X, \varphi + \zeta + \tau u_{j+1} + i\sqrt{\beta} \Gamma_{j+1}(x - x_0))] = e^{\frac{1}{2} \beta T_{j+1}(0) q} F(X, \varphi + \zeta + \tau u_{j+1} + i\xi).$$ \hspace{1cm} (B.12)

But since $F$ has charge $q$,

$$F((X, \varphi + \zeta + \tau u_{j+1} + i\sqrt{\beta} \Gamma_{j+1}(x - x_0))) = e^{\beta T_{j+1}(0)|q|} F(X, \varphi + \zeta + \tau u_{j+1} + i\xi).$$ \hspace{1cm} (B.13)

Lemma B.4. Let $h \geq c_h \sqrt{\beta}$ and $L \geq L_0$ for $L_0$ and $c_h$ sufficiently large. There exists $C \equiv C(M, \rho, \tau) > 0$ such that for $X \in \mathcal{S}_j$, and any charge-$q$ polymer activity $F$ with $|q| \geq 1$ and $\|F(X)\|_{h,T_j(X)} < \infty$, and all $\varphi' \in \mathbb{R}^{\mathcal{A}_N}$,

$$\|\mathbb{E}_{(i)}[F(X, \varphi' + \zeta)]\|_{2h,T_j+1(X, \varphi')} \leq C e^{2\sqrt{\tau} \|\xi\| C_1 \sqrt{L}} \|F(X)\|_{h,T_j(X)} G_{j+1}(X, \varphi').$$ \hspace{1cm} (B.14)

Proof. This almost follows from [4] Lemma 6.13. To be specific, by [4] (6.59) (applied with $r = 1$ there), for any $h' > 0$,

$$\|\mathbb{E}_{(i)}[F(X, \varphi' + \zeta)]\|_{h',T_j(X, \varphi')} \leq e^{-\frac{1}{2}|q|-1} \beta T_{j+1}(0) \mathbb{E}_{(i)}[\|F(X, \varphi' + \zeta)\|_{h'+\|\xi\|,T_j(X, \varphi')}].$$ \hspace{1cm} (B.15)

where $\xi(x) = \sqrt{\beta}(\Gamma_{j+1}(x - x_0) - \Gamma_{j+1}(0))$, $x \in X$ and $\|\xi\| \equiv \|\xi\|_{C_1} \leq C_1 \sqrt{\beta}$ for some $C_1 > 0$ (by [4] Lemma 6.11). Also by [4] (6.60),

$$\|\mathbb{E}_{(i)}[F(X, \varphi' + \zeta)]\|_{2h,T_j+1(X, \varphi')} \leq e^{2\sqrt{\tau} \|\xi\|} \|\mathbb{E}_{(i)}[F(X, \varphi' + \zeta)]\|_{2C_2 L^{-\frac{1}{2}} T_j(X, \varphi')}.$$ \hspace{1cm} (B.16)

for some $C_2 > 0$. Hence setting $h' = 2C_2 L^{-\frac{1}{2}} h$ in (B.15) and combining with (B.16) gives

$$\|\mathbb{E}_{(i)}[F(X, \varphi' + \zeta)]\|_{2h,T_j+1(X, \varphi')} \leq e^{-\frac{1}{2}|q|-1} \beta T_{j+1}(0) e^{2\sqrt{\tau} \|\xi\|} \|G_{j+1}(X, \varphi' + \zeta + \tau u_{j+1})\| F(X) \|2C_2 L^{-\frac{1}{2}} h + \|\xi\|,T_j(X).$$ \hspace{1cm} (B.17)

By Lemma [4.14] $\mathbb{E}[G_{j+1}(X, \varphi' + \zeta + \tau u_{j+1})] \leq C 2^{\|X\|} G_{j+1}(X, \varphi')$, and for $L$ and $c_h$ sufficiently large, $2CL^{-\frac{1}{2}} h + \|\xi\| \leq h$, so we have the desired bound.

Proof of Proposition [B.7] (1). The conclusion follows from combining (B.14) with (B.14).
B.2. Proof of Proposition B.1 (2).

Lemma B.5. For \( X \in \mathcal{D}_j, \varphi \in \mathbb{R}^{\Lambda N} \) and neutral \( F(X) \) such that \( \| F \|_{h,T_j} < \infty \) and \( m \in \{0, 2\} \), choose some \( x_0 \in X \) and let \( \delta \varphi'(x) = \varphi'(x) - \varphi'(x_0) \). Then

\[
\| \text{Rem}_m^{\delta \varphi'} \mathbb{E}_{(t)} F(X, \zeta + \delta \varphi') \|_{2h,T_j+1(X, \varphi')} \leq C L^{-(m+1)(\log L)^{m+1}/2} (A')^{-|X|/2} \| F \|_{h,T_j, A'} G_{j+1}(X, \varphi') \tag{B.18}
\]

for some \( C \equiv C(M, \rho, i) \).

Proof. The proof is very similar to \[4\] Lemma 6.16, just observing that the norm on the left-hand side can admit \( 2h \) instead of \( h \), and using Lemma B.7.

Proof of Proposition B.2 (2). Since \( \text{Rem}_m^{\varphi'} = \frac{1}{X} \sum_{x_0 \in X} \text{Rem}_m^{\delta \varphi'} \) where \( \delta \varphi'(x) = \varphi'(x) - \varphi'(x_0) \), it suffices to prove the same bound for \( \text{Rem}_m^{\varphi'} \mathbb{E}_{(t+\tau)} F(X, \zeta + \varphi'; \tau) \). Also by (B.13),

\[
\| \text{Rem}_m^{\varphi'} \mathbb{E}_{(t+\tau)} F(X, \zeta + \varphi'; \tau) \|_{\tilde{h}',T_j+1(X, \varphi')} \leq \| \text{Rem}_m^{\varphi'} \mathbb{E}_{(t)} F(X, \zeta + \varphi'; \tau) \|_{\tilde{h}',T_j+1(X, \varphi')} \tag{B.19}
\]

where \( \tilde{h}' = (2h, h_1) \), and by definition of \( \| \cdot \|_{\tilde{h}',T_j+1(X, \varphi')} \)

\[
\| \text{Rem}_m^{\varphi'} \mathbb{E}_{(t)} F(X, \zeta + \varphi'; \tau) \|_{\tilde{h}',T_j+1(X, \varphi')} \leq \sum_{k=0}^{\infty} \frac{\| \text{Rem}_m^{\varphi'} \mathbb{E}_{(t)} \partial^k_{\tau} F(X, \zeta + \varphi'; \tau) \|_{2h,T_j+1(X, \varphi')}}{k!} \tag{B.20}
\]

But Lemma B.5 bounds the right-hand side, giving the desired conclusion.

B.3. Proof of Proposition 5.3.

Proof of Lemma B.2. Since \( \text{Loc}_X^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) \) and \( \text{Tay}_2^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) \) are polynomials in \( \varphi' \) of degree 2, we have

\[
\| \text{Loc}_X^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) - \text{Tay}_2^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) \|_{h,T_j+1(X, \varphi')} \leq 4 \| \text{Loc}_X^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) - \text{Tay}_2^{(2)} \mathbb{E}_{(t+\tau)} F(X, \varphi' + \zeta) \|_{2h,T_j+1(X, \varphi')} \tag{B.21}
\]

Also, by (B.13), it is enough to prove

\[
\| \text{Loc}_X^{(2)} \mathbb{E}_{(t)} F(X, \varphi' + \zeta) - \text{Tay}_2^{(2)} \mathbb{E}_{(t)} F(X, \varphi' + \zeta) \|_{2h,T_j+1(X, \varphi')} \leq C L^{-(3\log L)(A')^{-|X|/2}} \| F \|_{h,T_j, A'} G_{j+1}(X, \varphi') \tag{B.22}
\]

for any neutral polymer activity \( F \). And this follows from (the proof of) \[4\] Lemma 6.16, which says

\[
\| \text{Loc}_X^{(2)} \mathbb{E}_{(t)} F(X, \varphi' + \zeta) - \text{Tay}_2^{(2)} \mathbb{E}_{(t)} F(X, \varphi' + \zeta) \|_{h,T_j+1(X, \varphi')} \leq C L^{-(3\log L)(A')^{-|X|/2}} \| F \|_{h,T_j, A'} G_{j+1}(X, \varphi') \leq C L^{-(3\log L)(A'/2)^{-|X|/2}} \| F \|_{h,T_j, A'} G_{j+1}(X, \varphi'). \tag{B.23}
\]

where the final inequality is due to Lemma B.7.

\[\square\]
Proof of Proposition 5.3. Take \( k = 0 \) or \( 2 \). After expanding in the Fourier series \( F = \sum_{q \in \mathbb{Z}} \hat{F}_q \), we have

\[
\| \text{Loc}_X^{(k)} E_{(t+r)} F(X, \varphi' + \zeta; \tau) - E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}
\leq \| \text{Loc}_X^{(k)} E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) - E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}
+ \sum_{q \in \mathbb{Z} \setminus \{0\}} \| E_{(t+r)} \hat{F}_q(X, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}.
\] (B.24)

First by Proposition B.1 (2) and Lemma B.2

\[
\| \text{Loc}_X^{(k)} E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) - E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}
\leq \| \text{Loc}_X^{(k)} E_{(t+r)} \hat{F}_0(X, \varphi' + \zeta; \tau) - \text{Rem}_k \|_{h, T_{j+1}(X, \varphi')}
+ \| \text{Rem}_k \|_{h, T_{j+1}(X, \varphi')}< CL^{-(k+1)}(\log L)^{\frac{1}{2}+}(A')^{-1}\|X\| \| F \|_{h, T_{j+1}(X, \varphi')}.
\] (B.25)

Secondly by Proposition B.1 (1), applied on the terms with \( \hat{F}_q \) \( q \neq 0 \), we have

\[
\| E_{(t+r)} \hat{F}_q(X, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}
\leq C e^{2\sqrt{3}h} e^{-(q-1/2)\beta T_{j+1}(0)} (A')^{-1}\|X\| \| F \|_{h, T_{j+1}(X, \varphi')}.
\] (B.26)

\[\Box\]

B.4. Proof of Proposition 5.4.

Lemma B.6. Let \( X \in \mathcal{B}_j \) and \( A' \geq A_0(L) \) where \( A_0(L) \) is some function only polynomially large in \( L \). Then

\[
\sum_{|Y| = X \setminus \delta_j} (A')^{-|Y|} \leq (CL^2(A')^{-1/2})^{\|X\|+2}
\] (B.27)

for some \( C > 0 \) and \( \eta > 0 \).

Proof. This is a bound already proved in the lines between \( [4] (6.115)--(6.118)) \).

\[\Box\]

Proof of Proposition 5.4. By definition of \( S \)-operator,

\[
\| S \{ E_{(t,r)} [F Y \notin \delta_j] \} (X, \varphi') \|_{h, T_{j+1}(X, \varphi')}
\leq \sum_{|Y| = X \setminus \delta_j} \| E_{(t,r)} F(Y, \varphi' + \zeta; \tau) \|_{h, T_{j+1}(X, \varphi')}\] (B.28)

but by Lemma B.8 this is bounded by some multiple of

\[
G_{j+1}(X, \varphi') \sum_{|Y| = X \setminus \delta_j} 2^{|Y|} \| F(Y) \|_{h, T_{j+1}(Y)} \leq G_{j+1}(X, \varphi') \| F \|_{h, T_{j+1}(X, \varphi')} \sum_{|Y| = X \setminus \delta_j} (A'/2)^{|Y|}. \] (B.29)

We then just need to apply Lemma B.6 and take \( A' \) sufficiently large (depending on \( L \)) to conclude.
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