Research article

Existence of solutions for $q$-fractional differential equations with nonlocal Erdélyi-Kober $q$-fractional integral condition

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Abstract: In this paper, we obtain sufficient conditions for the existence, uniqueness of solutions for a fractional $q$-difference equation with nonlocal Erdélyi-Kober $q$-fractional integral condition. Our approach is based on some classical fixed point techniques, as Banach contraction principle and Schauder’s fixed point theorem. Examples illustrating the obtained results are also presented.

Keywords: $q$-fractional differential equations; Erdélyi-Kober $q$-fractional integral; fixed point theorems; Riemann-Liouville fractional derivatives

Mathematics Subject Classification: 39A13, 34B18, 34A08

1. Introduction

The aim of this paper is to establish the existence and uniqueness of solutions for the following nonlinear Riemann-Liouville $q$-fractional differential equation subject to nonlocal Erdélyi-Kober $q$-fractional integral conditions

$$
\begin{align*}
D_q^\alpha x(t) + f(t, x(t), D_q^\delta x(t)) &= 0, \quad t \in (0, T), \\
x(0) &= 0, \quad a x(T) = \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} x(\xi_i),
\end{align*}
$$

where $D_q^\alpha$ and $D_q^\delta$ are the fractional $q$-derivative of Riemann-Liouville type of order $\alpha$ and $\delta$ on $(0, T)$ respectively, $1 < \alpha < 2$, $0 < \delta < 1$, $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_q^{\eta_i, \mu_i, \beta_i}$ denotes the Erdélyi-Kober fractional $q$-integral of order $\mu_i$ on $(0, T)$, $\mu_i > 0$, $\beta_i > 0$, $\eta_i \in \mathbb{R}$ and $\xi_i \in (0, T)$, $a$, $\lambda_i$ ($i = 1, 2, \ldots, n$) are some given constants.

The $q$-calculus or quantum calculus is an old subject that was initially developed by Jackson [1], basic definitions and properties of $q$-calculus can be found in [2]. The fractional $q$-calculus had its
Here we recall some definitions and fundamental results on fractional $q$-integral and fractional $q$-derivative. See the references [4–7] for complete theory.
For \( q \in (0, 1) \), define \([a]_q = \frac{1-q^a}{1-q} \), \( a \in \mathbb{R} \). The \( q \)-factorial function is defined as \((a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \ a, b \in \mathbb{R} \). If \( n \) is a positive integer. If \( v \) is not a positive integer, then \((a-b)^{(v)} = a^v \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{n+v}} \).

If \( b = 0 \), then \( a^{(v)} = a^v \). The \( q \)-gamma function is defined by \( \Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{(\alpha+1)}}, \ \alpha > 0 \), and satisfies \( \Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha) \).

The \( q \)-derivative of a function \( f \) is defined by \( (D_q f)(t) = \frac{(t-f(q^t))}{(1-q)^{(\alpha-1)}}, \ (D_q f)(0) = \lim_{t \to 0}(D_q f)(t) \). The \( q \)-integral of a function \( f \) defined on the interval \([0, b]\) is given by \( (I_q f)(t) = \int_0^t f(s)ds_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(q^i t), \ t \in [0, b] \).

Some results about operator \( D_q \) and \( I_q \) can be found in references [4]. Let us define fractional \( q \)-derivative and \( q \)-integral and outline some of their properties [4, 6, 8].

**Definition 1** (\([4]\)) Let \( \alpha \geq 0 \) and \( f \) be a function. The fractional \( q \)-integral of Riemann-Liouville type is given by \((I_q^0 f)(t) = f(t)\) and
\[
(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)ds_q s, \ \alpha > 0, \ t \in [0, b].
\]

**Definition 2** (\([6]\)) The fractional \( q \)-derivative fractional of Riemann-Liouville type of order \( \nu \geq 0 \) is defined by \( D_q^\nu f(t) = f(t) \) and
\[
D_q^\nu f(t) = D_q^l I_q^{l-\nu} f(t), \ \nu > 0,
\]
where \( l \) is the smallest integer greater than or equal to \( \nu \).

**Definition 3** (\([24]\)) For \( 0 < q < 1 \), the Erdélyi-Kober fractional \( q \)-integral of order \( \mu \geq 0 \) with \( \beta > 0 \) and \( \eta \in \mathbb{R} \) of a continuous function \( f : (0, \infty) \to \mathbb{R} \) is defined by
\[
P_q^{\mu, \beta, \eta} f(t) = \frac{\beta \tau^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^t (\tau^\beta - s^\beta q^{(\mu-1)})^\beta f(s)ds_q s.
\]
provided the right side is pointwise defined on \( \mathbb{R}^+ \).

**Remark 1** For \( \beta = 1 \) the above operator is reduced to the \( q \)-analogue Kober operator
\[
P_q^{\mu, \eta} f(t) = \tau^{-\eta(\mu+1)} \int_0^t (\tau - s)^{(\mu-1)} f(s)ds_q s
\]
that is given in [4]. For \( \eta = 0 \) the \( q \)-analogue Kober operator is reduced to the Riemann-Liouville fractional \( q \)-integral with a power weight:
\[
P_q^\mu f(t) = \tau^{-\mu} \int_0^t (\tau - s)^{(\mu-1)} f(s)ds_q s, \ \mu > 0.
\]

**Lemma 1** (\([4]\)) Let \( \alpha, \beta \in \mathbb{R}^+ \) and \( f \) be a continuous function on \([0, b] \). The Riemann-Liouville fractional \( q \)-integral has the following semi-group property
\[
P_q^\beta f(t) = P_q^\alpha f(t) = P_q^{\alpha+\beta} f(t).
\]
Lemma 2 ([8]) Let \( f \) be a \( q \)-integrable function on \([0, b]\). Then the following equality holds

\[
D_q^\alpha I_q^p f(t) = f(t), \quad \text{for } \alpha > 0, \ t \in [0, b].
\]

Lemma 3 ([4]) Let \( \alpha > 0 \) and \( p \) be a positive integer. Then for \( t \in [0, b] \) the following equality holds

\[
I_q^p D_q^\alpha f(t) = D_q^\alpha I_q^p f(t) - \sum_{k=0}^{p-1} \frac{p^{a-p+k}}{\Gamma_q(a+k)} D_q^k f(0).
\]

Lemma 4 ([24]) For \( f(t) = t^k \) and \( \beta > 0, \ \mu, \eta, \lambda \in \mathbb{R}, \ 0 < q < 1, \) then

\[
I_q^{\eta; \mu, \beta} t^k = \beta \left[ \frac{1}{\Gamma_q} \right] \frac{\Gamma_q(\eta + 1 + \frac{1}{\beta})}{\Gamma_q(\mu + \eta + 1 + \frac{1}{\beta})} t^k.
\]

3. Main results

In this section, we will give the main results of this paper. Let the space \( E = \{ x \in C([0, T], \mathbb{R}), D_q^\beta x \in C([0, T], \mathbb{R}) \} \) be endowed with the norm \( \| x \| = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |D_q^\beta x(t)| \).

It is known that the space \( E \) is a Banach space. To obtain our main results, we need the following lemma.

Lemma 5 Let \( h(t) \in C([0, T], \mathbb{R}) \). Then for any \( t \in [0, T] \), the solution of the following problem

\[
\begin{align*}
D_q^\alpha x(t) + h(t) &= 0, \quad t \in (0, T), \\
x(0) &= 0, \quad ax(T) = \sum_{i=1}^{n} \lambda_i I_q^{\eta_i; \mu_i, \beta_i} x(\xi_i).
\end{align*}
\]

is given by

\[
x(t) = -I_q^\alpha h(t) + \frac{t^{\alpha-1}}{M} \left( aI_q^\alpha h(T) - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i; \mu_i, \beta_i} h(\xi_i) \right),
\]

where \( M = aT^{\alpha-1} - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i; \mu_i, \beta_i} \varepsilon_i^{\alpha-1} = aT^{\alpha-1} - \sum_{i=1}^{n} \lambda_i \beta_i \left[ \frac{1}{\Gamma_q} \right] \frac{\Gamma_q(\eta_i + 1 + \frac{1}{\beta_i})}{\Gamma_q(\mu_i + \eta_i + 1 + \frac{1}{\beta_i})} \varepsilon_i^{\alpha-1} \neq 0. \)

Proof. Applying the operator \( I_q^\alpha \) on both sides of the first equation of (3.1) for \( t \in (0, T) \) and using Lemma 1 and Lemma 2, we have

\[
x(t) = -I_q^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.
\]

Applying the initial value condition \( x(0) = 0 \), we get \( c_2 = 0 \). By the boundary value condition, we have

\[
-\sum_{i=1}^{n} \lambda_i I_q^{\eta_i; \mu_i, \beta_i} h(\xi_i) + c_1 \sum_{i=1}^{n} \lambda_i I_q^{\eta_i; \mu_i, \beta_i} \varepsilon_i^{\alpha-1} = -aI_q^\alpha h(T) + c_1 aT^{\alpha-1}.
\]
that is
\[
c_1 = \frac{aI_q^\alpha h(T) - \sum_{i=1}^{n} \lambda_i p_{i,\mu,\beta} I_q^{\alpha} h(\xi_i)}{aT^{\alpha-1} - \sum_{i=1}^{n} \lambda_i p_{i,\mu,\beta} I_q^{\alpha} \xi_i^{-1}} = \frac{aI_q^\alpha h(T) - \sum_{i=1}^{n} \lambda_i p_{i,\mu,\beta} I_q^{\alpha} h(\xi_i)}{M}.
\]

Substituting \(c_1, c_2\) to (3.3), we obtain the solution (3.2). This completes the proof.

Using the Lemma 5, we can define an operator \(Q : E \to E\) as follows:
\[
Qx(t) = -I_q^\alpha f(s, x(s), D_q^\beta x(s))(t) + \frac{p^{\alpha-1}}{M}(aI_q^{\alpha-1} f(s, x(s), D_q^\beta x(s))(T) - \sum_{i=1}^{n} \lambda_i p_{i,\mu,\beta} I_q^{\alpha} f(s, x(s), D_q^\beta x(s))(\xi_i)),
\]
where
\[
I_q^\alpha f(s, x(s), D_q^\beta x(s))(\tau) = \frac{1}{\Gamma_q(\alpha)} \int_{0}^{\tau} (\tau - s)^{\alpha-1} f(s, x(s), D_q^\beta x(s))d_q s
\]
and \(\tau \in \{t, T, \xi_1, \xi_2, \ldots, \xi_n\}\). Then, the existence of solutions of system (1.1) is equivalent to the problem of fixed point of operator \(Q\) in (3.4).

In the following, we will use some classical fixed point techniques to give our main results.

**Theorem 1** Suppose that there exists a function \(L(t) : [0, T] \to \mathbb{R}^+\) \(-q\)-integrable such that
\[
|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L(t)(|x - \bar{x}| + |y - \bar{y}|),
\]
for each \(x, \bar{x}, y, \bar{y} \in \mathbb{R}\). Then problem (1.1) has an unique solution on \([0, T]\) if
\[
L_1 + L_2 + \frac{T^{\alpha-1}}{|M|} \left(1 + \frac{T^{-\delta} \Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)}\right) (|a|L_2 + \sum_{i=0}^{n} |\lambda_i| |\beta_i| \frac{1}{\beta_i} \frac{L_i \Gamma_q(\eta_i + 1)}{\Gamma_q(\mu_i + \eta_i + 1)}) < 1, \tag{3.5}
\]
where \(L_1 = \sup_{t \in [0, T]} I_q^\alpha L(t), L_2 = \sup_{t \in [0, T]} I_q^{\alpha-1} L(t), L_3 = \sup_{t \in [0, T]} I_q^{-\delta} L(t)\).

**Proof.** The conclusion will follow once we have shown that the operator \(Q\) defined (3.4) is contractively with respect to a suitable norm on \(E\).

For any functions \(x, y \in E\), we have
\[
\left\| (Qx)(t) - (Qy)(t) \right\| \leq \left| \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)}(f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s)))ds \right|
+ \frac{\varepsilon_1^\alpha}{M} \left| \frac{a}{\Gamma_q(\alpha - 1)} \int_0^T (T - qs)^{(\alpha - 2)}(f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s)))ds \right|
+ \sum_{i=0}^n |\lambda_i| \left| \left( \frac{\beta \xi_i}{\Gamma_q(\mu_i)} \right) \int_0^\xi (\xi - s^\beta q)^{\mu_i - 1} \right|
\frac{1}{\Pi_q(\alpha)} \int_0^\xi (s - q\tau)^{(\alpha - 1)}(f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s)))d_\tau ds \right|
\leq \frac{\| x - y \|}{\Gamma_q(\alpha)} \int_0^\xi (t - qs)^{(\alpha - 1)}L(s)ds + \frac{\varepsilon_1^\alpha}{M} \left| \frac{a}{\Gamma_q(\alpha - 1)} \int_0^T (T - qs)^{(\alpha - 2)}L(s)ds \right|
+ \sum_{i=0}^n |\lambda_i| \left| \left( \frac{\beta \xi_i}{\Gamma_q(\mu_i)} \right) \int_0^\xi (\xi - s^\beta q)^{\mu_i - 1} \right|
\frac{1}{\Pi_q(\alpha)} \int_0^\xi (s - q\tau)^{(\alpha - 1)}L(s)\tau d_\tau ds \right| \| x - y \|
\leq \frac{\| x - y \|}{\Pi_q(\alpha)} \left( L_1 + \frac{\varepsilon_1^\alpha}{|\Pi|} (|a|L_2 + L_1 \sum_{i=0}^n |\lambda_i| \left| \left( \frac{\beta \xi_i}{\Gamma_q(\mu_i)} \right) \int_0^\xi (\xi - s^\beta q)^{\mu_i - 1} \right| \right)
\leq \left( L_1 + \frac{\varepsilon_1^\alpha}{|\Pi|} (|a|L_2 + L_1 \sum_{i=0}^n |\lambda_i| \left| \left( \frac{\beta \xi_i}{\Gamma_q(\mu_i)} \right) \int_0^\xi (\xi - s^\beta q)^{\mu_i - 1} \right| \right) \| x - y \|.
\]

on the other hand

\[
D_q^\delta(Qx)(t) = -D_q^\delta I_q^\alpha f(s, x(s), D_q^\delta x(s))(t) + \frac{1}{M} \left( aI_q^\alpha f(s, x(s), D_q^\delta x(s))(T) \right.
- \sum_{i=1}^n \lambda_i I_q^{\alpha-\mu_i+\beta_i}I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i) \left. \right) \right)
\]

\[
= -D_q^\delta I_q^\alpha f(s, x(s), D_q^\delta x(s))(t) + \frac{1}{M} \left( aI_q^\alpha f(s, x(s), D_q^\delta x(s))(T) \right.
- \sum_{i=1}^n \lambda_i I_q^{\alpha-\mu_i+\beta_i}I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i) \left. \right) \Gamma_q(\alpha) \Gamma_q(\alpha - \delta) \mu_i^{-\delta} \]

Thus

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Volume 5, Issue 6, 6537–6551.
Corollary mapping principle, the problem (1.1) has an unique solution on $[0, T]$. The existence results for nontrivial solution for problem (1.1) are presented. For convenience, we denote

\[ |D_q^\beta(Qx(t)) - D_q^\beta(Qy(t))| \leq |f_q^{a-\delta}(f(s, x(s), D_q^\beta x(s))(t) - f(s, y(s), D_q^\beta y(s))))(t)\]

\[ + \frac{\Gamma_q(\alpha)T^{a-\delta}}{|M|\Gamma_q(\alpha - \delta)}(a|f_q^{a-1}(f(s, x(s), D_q^\beta x(s))(T) - f(s, y(s), D_q^\beta y(s))))(T)| + \sum_{i=1}^n |\lambda_i|l_i^{q_{\mu_i, \xi_i}} L_q(\xi_i))\]

\[ \leq \frac{T^{a-\delta}L(s)(t)}{|M|\Gamma_q(\alpha - \delta)}(|a|L_2 + \sum_{i=1}^n |\lambda_i|\beta_i^1 \max |L_q(\eta_i + 1)| x - y |,\]

which implies that

\[ ||Qx - Qy|| \leq \left\{ L_1 + L_3 + \frac{T^{a-\delta}}{|M|}(1 + \frac{T^{2a-2}}{|M|\Gamma_q(\alpha - \delta)} + \sum_{i=1}^n |\lambda_i|\beta_i^1 \max |L_q(\eta_i + 1)| x - y \right\} .\]

Thus the operator $Q$ is a contraction in view of the condition (3.5). By Banach’s contraction mapping principle, the problem (1.1) has a unique solution on $[0, T]$. This completes the proof.

Corollary 1 Assume that there exists $L_0 > 0$ such that

\[ |f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L_0(|x - \bar{x}| + |y - \bar{y}|),\]

for each $t \in [0, T]$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}$. Then the problem (1.1) has a unique solution whenever

\[ \left( \frac{T^a}{\Gamma_q(\alpha + 1)} + \frac{T^{a-\delta+1}}{\Gamma_q(\alpha - \delta + 1)} + \sum_{i=1}^n |\lambda_i|l_i^{q_{\mu_i, \xi_i}} s^\alpha(\xi_i) \right) L_0 < 1,\]

where $l_i^{q_{\mu_i, \xi_i}} s^\alpha(\xi_i) = \beta_i^1 \frac{\Gamma_q(n+1+\frac{\alpha-1}{\delta})}{\Gamma_q(\mu_i + \eta_i + 1+\frac{\alpha-1}{\delta})} \xi_i ^\alpha$.

In the following Theorem, the existence results for nontrivial solution for problem (1.1) are presented. For convenience, we denote

\[ \zeta = \sum_{i=1}^n |\lambda_i|l_i^{q_{\mu_i, \xi_i}} s^\alpha(\xi_i) = \sum_{i=1}^n |\lambda_i|\beta_i^1 \max |L_q(\eta_i + 1)| \frac{\Gamma_q(\eta_i + 1)}{\Gamma_q(\mu_i + \eta_i + 1)|,\]

\[ \varphi_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha)} \int_0^t (t -qs)^{\alpha-1} l_i(s)ds, i = 1, 2, 3,\]

\[ \sigma_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha - 1)} \int_0^t (t -qs)^{\alpha-2} l_i(s)ds, i = 1, 2, 3,\]

\[ \varsigma_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha - \delta)} \int_0^t (t -qs)^{\alpha-\delta-1} l_i(s)ds, i = 1, 2, 3,\]

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where \( l_i(t), i = 1, 2, 3 \) are defined in Theorem 2.

**Theorem 2** Let \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume there exist three nonnegative continuous functions \( l_i(t), i = 1, 2, 3 \) such that

\[
|f(t, x(t), D_q^r x(t))| \leq l_1(t) + l_2(t)|x(t)|^\alpha + l_3(t)|D_q^r x(t)|^\beta, \quad 0 < r, \quad i = 1, 2,
\]

and \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \). Then the problem (1.1) has a nontrivial solution.

**Proof.** We shall use Schauder’s fixed point theorem to prove our theorem. Define \( E_d = \{ x : x \in E, ||x|| \leq d \} \), where \( d \geq \max \left\{ 3\zeta_1 + |a|\sigma_1 \rho + a_1 + \rho \zeta_2, 3\zeta_2 + |a|\sigma_2 \rho + a_2 + \rho \zeta_2, 3\zeta_3 + |a|\sigma_3 \rho + a_3 + \rho \zeta_3 \right\} \) \( \rho = \frac{T^{-1}}{|M|} + \frac{T^{-1}}{|M|} \frac{\Gamma_\alpha(\alpha - 1)}{\Gamma_\alpha(\alpha - 1)} \). Note that \( E_d \) is a closed, bounded and convex subset of the Banach space \( E \). We now show that \( Q : E_d \to E_d \). In fact, for \( x \in E_d \) we have that

\[
|f(t, x(t), D_q^r x(t))| \leq l_1(t) + l_2(t)|x(t)|^\alpha + l_3(t)|D_q^r x(t)|^\beta.
\]

Thus

\[
|Q x(t)| \leq \frac{1}{\Gamma_\alpha(\alpha)} \left\{ |s|^{\alpha-1} \int_0^t (t - s)^{(\alpha-1)} l_1(s) + l_2(s)|x(t)|^\alpha + l_3(s)|D_q^r x(t)|^\beta ds + M \int_0^t (t - s)^{(\alpha-1)} l_1(s) + l_2(s)|x(t)|^\alpha + l_3(s)|D_q^r x(t)|^\beta ds \right. \]

\[
+ \sum_{i=1}^n |\lambda_i| \beta \epsilon^{\beta_i(\mu_i - 1)} \int_0^s (s^{\beta_i} - s^{\beta_i})(s^{\beta_i})^{(\mu_i - 1) - 1} \left. \int_0^{s} (s^{\beta_i} - s^{\beta_i})^{(\mu_i - 1) - 1} \int_0^{s} (s^{\beta_i} - s^{\beta_i})^{(\mu_i - 1) - 1} \right\}
\]

\[
\leq (\zeta_1 + \zeta_2 + \zeta_3)(1 + \zeta_2 + \zeta_3)T^{-1} l_1(t) + T^{-1} l_2(t)|x(t)|^\alpha + l_3(t)|D_q^r x(t)|^\beta.
\]

\[
|D_q^r Q x(t)| \leq (\zeta_1 + \zeta_2 + \zeta_3)(1 + \zeta_2 + \zeta_3)T^{-1} l_1(t) + T^{-1} l_2(t)|x(t)|^\alpha + l_3(t)|D_q^r x(t)|^\beta.
\]

From the two inequalities above, we get

\[
\|Q x \| = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |D_q^r x(t)|
\]

\[
\leq (\zeta_1 + |a|\sigma_1 \rho + a_1 + \rho \zeta_2) + (\zeta_2 + |a|\sigma_2 \rho + a_2 + \rho \zeta_2) + (\zeta_3 + |a|\sigma_3 \rho + a_3 + \rho \zeta_3)\frac{d}{3} + \frac{d}{3} + \frac{d}{3} = d.
\]
Hence, $Q$ maps $E_d$ into $E_d$. Also, it is easy to check that $Q$ is continuous, since $f$ is continuous. For each $x \in E_d$ and each $0 < t_1 < t_2 < T$, we have

$$
|Q(t_2) - Q(t_1)| \leq \frac{1}{\Gamma_q(\alpha)} \int_0^{t_2} \left| (t_2 - t_1)^{(\alpha-1)} - (t_1 - t_2)^{(\alpha-1)} \right| f(s, x(s), D_q^\alpha x(s)) d_q s \\
+ \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - q s)^{(\alpha-1)} f(s, x(s), D_q^\alpha x(s)) d_q s \\
+ \frac{t_2^\alpha - t_1^\alpha - \frac{1}{2}}{M} \left( \| a |I_q^\alpha f(s, x(s), D_q^\alpha x(s)) \| \right)
+ \sum_{i=1}^n |\lambda_i I_{q_i}^{\mu_i, \beta_i} f(s, x(s), D_q^\alpha x(s)) (\xi_i) |.
$$

Let $t_2 \rightarrow t_1$, we get $\| Q(t_2) - Q(t_1) \| \rightarrow 0$. Thus, $Q$ is uniformly bounded and equicontinuous. The theorem of Arzelà-Ascoli implies that $Q$ is completely continuous. By Schauder’s fixed point theorem, $Q$ has a fixed point in $E_d$. Clearly $x = 0$ is not a fixed point because $f(t, 0, 0) \neq 0$ for $t \in [0, T]$. Hence, the problem (1.1) has at least one nontrivial solution. This proves the theorem.

**Remark 2** In the Theorem 2, if $r_i > 1, i = 1, 2$, we may choose $l_2(t)$, $l_3(t)$ and $d$ such that

$$
|f(t, x(t), D_q^\delta x(t))| \leq l_2(t) |x(t)|^{r_1} + l_3(t) |D_q^\delta x(t)|^{r_2},
$$

and

$$
0 < d \leq \min \left\{ \frac{1}{2(\sigma_2^2 + |a|\sigma_2^\rho + q_2(1 + \rho)\xi)}, \frac{1}{2(\sigma_3^2 + |a|\sigma_3^\rho + q_3(1 + \rho)\xi)} \right\}.
$$

For $r_i = 1, i = 1, 2$, we have the following theorem.
**Theorem 3** Let \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function. Assume there exist three nonnegative continuous functions \( l_i(t), i = 1, 2, 3 \) such that

\[
|f(t, x(t), D_q^\delta x(t))| \leq l_1(t) + l_2(t)|x(t)| + l_3(t)|D_q^\delta x(t)|,
\]

and \( \max\left\{ s_2 + |a| \sigma_2 \rho + g_2(1 + \rho) \zeta, (s_3 + |a| \sigma_3 \rho + g_3(1 + \rho) \zeta) \right\} < \frac{1}{\lambda} \). Further, assume that \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \). Then the problem (1.1) has at least one nontrivial solution.

**Proof.** Let \( d > 3(s_1 + |a| \sigma_1 \rho + g_1(1 + \rho) \zeta) \). The proof is similar to Theorem 2, so it is omitted. This completes the proof.

Although Theorems 2 and Theorem 3 provide some simple conditions on the existence of solution of problem (1.1) and Theorem 1 provides a condition on the existence and uniqueness on the solution of problem (1.1), the following theorem provides an easily verifiable condition for the existence of a nontrivial solution for the problems (1.1).

**Theorem 4** Assume that \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) with \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \) and \( x \in E \). Suppose that

\[
\lim_{\|x\| \rightarrow 0} \frac{\max_{\|x\| \rightarrow 0} |f(t, x(t), D_q^\delta x(t))|}{\|x\|} = 0 \tag{3.6}
\]

holds. Then problem (1.1) has at least one nontrivial solution.

**Proof.** Choose a constant \( A \) such that

\[
A \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{|a| T^{2\alpha-2}}{|M| \Gamma_q(\alpha)} + \frac{T^{a-\delta}}{|M| \Gamma_q(\alpha - \delta)} + \frac{T^{a-1-\delta}}{|M| \Gamma_q(\alpha - \delta)} \right) < 1,
\]

where \( \kappa = \sum_{i=1}^n |\lambda_i| |\bar{\eta}_i| \frac{1}{\Gamma_q(\beta_i + \frac{1}{\beta_i})} \). By the condition (3.6), there exists a constant \( c_1 \) such that

\[
|f(t, x(t), D_q^\delta x(t))| < A \|x\| \text{ for any } t \in [0, T] \text{ and } \|x\| \geq c_1.
\]

Since \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), we can find another constant \( A_1 > 0 \) such that \( |f(t, x(t), D_q^\delta x(t))| < A_1 \), for \( t \in [0, T] \) and \( \|x\| \leq c_1 \). Let \( c = \max \{ \frac{1}{\lambda_1}, c_1 \} \), then for any \( \|x\| \leq c \), we have \( |f(t, x(t), D_q^\delta x(t))| < Ac \). Set

\[
E_c = \{ x \in E : \|x\| \leq c \}.
\]
Then for any \( x \in E_c \), we have

\[
|Qx(t)| \leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)} Acd_q s + \frac{\tau^{\alpha-1}}{M} \left( \frac{|a|}{\Gamma_q(\alpha - 1)} \int_0^t (T - qs)^{(\alpha - 2)} Acd_q s \right.
\]

\[
+ \sum_{i=1}^n |\lambda_i| \frac{\beta_i e^{-\beta_i (\eta_i + \mu_i)}}{\Gamma_q(\mu_i)} \int_0^t \left( e^{\beta_i (s - q)^{(\mu_i - 1)}} - s^{\beta_i} q^{(\mu_i - 1)} s^{\beta(\eta_i + 1) - 1} \right)
\]

\[
\frac{1}{\Gamma_q(\alpha - 1)} \int_0^t (s - q\tau)^{(\alpha - 1)} Acd_q \tau d_q s \leq \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha-1}|a|}{|M| \Gamma_q(\alpha + \delta)} + \frac{\kappa}{\Gamma_q(\alpha + 1)} \right) A \cdot c.
\]  

(3.7)

\[
|D_q^\rho(Qx)(t)| \leq \frac{1}{\Gamma_q(\alpha - \delta)} \int_0^t (t - qs)^{(\alpha - \delta - 1)} Acd_q s + \frac{\Gamma_q(\alpha) T^{\alpha-\delta}}{|M| \Gamma_q(\alpha - \delta)} \left( \frac{|a|}{\Gamma_q(\alpha - 1)} \int_0^t (T - qs)^{(\alpha - 2)} Acd_q s \right.
\]

\[
+ \sum_{i=1}^n |\lambda_i| \frac{\beta_i e^{-\beta_i (\eta_i + \mu_i)}}{\Gamma_q(\mu_i)} \int_0^t \left( e^{\beta_i (s - q)^{(\mu_i - 1)}} - s^{\beta_i} q^{(\mu_i - 1)} s^{\beta(\eta_i + 1) - 1} \right)
\]

\[
\frac{1}{\Gamma_q(\alpha - \delta + 1)} \int_0^t (s - q\tau)^{(\alpha - 1)} Acd_q \tau d_q s \leq \left( \frac{T^{\alpha-\delta}}{\Gamma_q(\alpha - \delta + 1)} + \frac{|a| T^{\alpha-\delta}}{|M| \Gamma_q(\alpha - \delta)} + \frac{T^{\alpha-1-\delta} \kappa}{|M| \Gamma_q(\alpha - \delta)} \right) A c.
\]  

(3.8)

Thus

\[
||Qx|| = \max_{t \in [T]} |Qx(t)| + \max_{t \in [T]} |D_q^\rho Qx(t)|
\]

\[
\leq \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha-1}|a|}{|M| \Gamma_q(\alpha + \delta)} + \frac{\kappa}{\Gamma_q(\alpha + 1)} \right) A c + \frac{T^{\alpha-\delta}}{\Gamma_q(\alpha - \delta + 1)} + \frac{|a| T^{\alpha-\delta}}{|M| \Gamma_q(\alpha - \delta)} + \frac{T^{\alpha-1-\delta} \kappa}{|M| \Gamma_q(\alpha - \delta)} A c
\]

\[
= A \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{|a| T^{\alpha-\delta}}{|M| \Gamma_q(\alpha + \delta)} + \frac{T^{\alpha-1-\delta} \kappa}{|M| \Gamma_q(\alpha - \delta)} c < c. \right)
\]

(3.9)

From (3.9), we obtain \( Q(E_c) \subset E_c \). By Schauder fixed point theorem, \( Q \) has at least one fixed point in \( E_c \). Clearly, \( x = 0 \) is not a fixed point because \( f(t, 0, 0) \neq 0 \). Therefore, problem (1.1) has at least one nontrivial solution, which completes the proof.

4. Example

In this section, we illustrate the results obtained in the last section.
Example 1 Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional $q$-integral conditions:

$$
\begin{align*}
\begin{cases}
    D^\frac{1}{2} \frac{1}{7} x(t) + \frac{t^2}{100} \left( \frac{|x(t) + D^\frac{1}{2} \frac{1}{7} x(t)|}{1 + 2|x(t) + D^\frac{1}{2} \frac{1}{7} x(t)|} + \cos t + 2 \right) + 1 = 0, \\
x(0) = 0, \\
    5x(1) = \frac{1}{100} \int_{\frac{1}{2}}^{\frac{1}{2}} x(t) \, dt + \frac{3}{100} \int_{\frac{1}{2}}^{\frac{1}{2}} x(t) \, dt,
\end{cases}
\end{align*}
$$

where $q = \frac{1}{2}, \delta = \frac{1}{4}, T = 1, \alpha = \frac{3}{2}, a = 3, \lambda_1 = \frac{1}{100}, \lambda_2 = \frac{3}{100}, \beta_1 = \frac{1}{10}, \beta_2 = \frac{1}{5}, \eta_1 = \frac{3}{8}, \eta_2 = \frac{1}{8}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \mu_1 = \frac{5}{4}, \mu_1 = \frac{7}{4}$, and $f(t, x(t), D_{q}^\gamma x(t)) = \frac{t^2}{100} \left( \frac{|x(t) + D_{q}^\gamma x(t)|}{1 + 2|x(t) + D_{q}^\gamma x(t)|} + \cos t + 2 \right) + 1$.

By computation, we deduce that

$$
|f(t, x(t), D_{q}^\gamma x(t)) - f(t, y(t), D_{q}^\gamma y(t))| \leq \frac{t^2}{2} \left( |x(t) - y(t)| + |D_{q}^\gamma x(t) - D_{q}^\gamma y(t)| \right),
$$

then, the first condition is satisfied with $L(t) = \frac{t^2}{2}$.

$$
M = 5 - \frac{1}{100} \times 10 \left[ 10 + \frac{1}{2} \Gamma_\frac{1}{2} \left( \frac{1}{2} + 1 + \frac{30}{2} \right) \right] - \frac{3}{100} \times 10 \left[ 5 \Gamma_\frac{1}{2} \left( \frac{1}{2} + 1 + 1 + \frac{15}{2} \right) \right]
$$

$> 5 - \frac{1}{500} \times \left[ 5 \times \left( 1 - \left( \frac{1}{2} \right)^{10} \right) \left( \frac{1}{4} \right)^{\frac{1}{2}} \right] - \frac{3}{50} \times 5 \left( 1 - \left( \frac{1}{2} \right)^{3} \right) \left( \frac{1}{2} \right)^{\frac{1}{2}}$

$> 5 - \frac{6}{500} \approx 4.986 \neq 0$.

$$
L_1 = \sup_{r \in [0, 1]} \frac{1}{\Gamma_q \left( \frac{3}{2} \right)} \int_{0}^{r} (t - q s) \left( \frac{1}{2} - 1 \right) s^2 \frac{d q s}{2} = \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right),
$$

$$
L_2 = \sup_{r \in [0, 1]} \frac{1}{\Gamma_q \left( \frac{3}{2} - 1 \right)} \int_{0}^{r} (t - q s) \left( \frac{1}{2} - 2 \right) s^2 \frac{d q s}{2} = \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right),
$$

$$
L_3 = \sup_{r \in [0, 1]} \frac{1}{\Gamma_q \left( \frac{3}{2} - \frac{1}{4} \right)} \int_{0}^{r} (t - q s) \left( \frac{1}{2} - 1 \right) s^2 \frac{d q s}{2} = \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right),
$$

$$
L_1 + L_3 + \frac{T^{\alpha - 1}}{|M|} \left( 1 + \frac{T^{-p} \Gamma_q (\alpha - p)}{\Gamma_q (\alpha - p)} \right) \left( |a| L_2 + \sum_{i=0}^{n} |a_i| \beta_i \left( \frac{1}{\beta_i} L_3 \Gamma_q (\eta_i + 1) \right) \right)
$$

$$
\leq \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right) + \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right) + \frac{1}{4} \frac{1}{100} \times 5 \times \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right) + \frac{1}{100} \times 10 \times \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right) + \frac{1}{2} \Gamma_q \left( \frac{3}{2} \right) \Gamma_q (\frac{1}{2})
$$

$$
\approx 0.894875 < 1.
$$
Hence, by Theorem 1, the boundary value problem (1.1) has an unique solution on \([0, 1]\).

**Example 2** Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional \(q\)-integral conditions:

\[
\begin{align*}
D_q^\frac{3}{2}x(t) + \frac{1}{2\pi} \sin(\pi|x|)^\left(\frac{|x|+|D_q^\delta x|}{|x|+|D_q^\delta x|+1}\right) + 1 &= 0, \quad t \in (0, 1) \\
x(0) &= 0, \\
5x(1) &= \frac{1}{10} I_q^{\frac{3}{4} + \frac{1}{2}} x\left(\frac{1}{2}\right) + \frac{3}{10} I_q^{\frac{3}{4} + \frac{1}{2}} x\left(\frac{1}{2}\right).
\end{align*}
\]

Here, \(q = \frac{1}{2}, \delta = \frac{1}{4}\). \(f(t, x(t), D_q^\delta x(t)) = \frac{1}{2\pi} \sin(\pi|x|)^\left(\frac{|x|+|D_q^\delta x|}{|x|+|D_q^\delta x|+1}\right) + 1,\)

\[
\frac{|f(t, x(t), D_q^\delta x(t))|}{\|x\|} = \frac{\frac{1}{2\pi} \sin(\pi|x|)^\left(\frac{|x|+|D_q^\delta x|}{|x|+|D_q^\delta x|+1}\right) + 1}{\|x\|} \leq \frac{\frac{1}{2}(|x| + |D_q^\delta x|)^{\frac{1}{2}} + 1}{\|x\|} \rightarrow 0, \text{ as } \|x\| \rightarrow \infty.
\]

Therefore, the conclusion of Theorem 4 implies that problem (1.1) has at least one solution on \([0, 1]\).

### 5. Conclusions

In this work, we utilize Banach contraction principle and Schauder’s fixed point theorem to research the existence, uniqueness of solutions for a \(q\)-fractional differential equation with nonlocal Erdélyi-Kober \(q\)-fractional integral condition and in which the nonlinear term contains a fractional \(q\)-derivative of Riemann-Liouville type. Some existence and uniqueness results of solutions are obtained, we also provide an easily verifiable condition for the existence of nontrivial solution for the problem (1.1).

### Acknowledgments

This research was Supported by Science and Technology Foundation of Guizhou Province (Grant No. [2016]7075, [2019]1162), by the Project for Young Talents Growth of Guizhou Provincial Department of Education under(Grant No.Ky[2017]133), and by the project of Guizhou Minzu University under (Grant No.16yjrcxm002).

### Conflict of Interest

The authors declare no conflict of interest.

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