Discernibility of Topological Variations for Networked LTI Systems

Yuqing Hao, Qingyun Wang, Zhisheng Duan, Senior Member, IEEE, and Guanrong Chen, Fellow, IEEE

Abstract—In this article, the discernibility of topological variations for networked linear time-invariant (LTI) systems is investigated, where the network topology is general, and the nodes have identical higher dimensional dynamics. A necessary and sufficient condition on the discernibility is derived, revealing how the topological variations, node-system dynamics, and inner interactions altogether affect the discernibility of the network. Compared with the existing conditions in (Patil et al., 2019) and (Roy et al., 2020), which require the network topology to be undirected, this condition is more general. Furthermore, the discernibility of topological variations for multiagent systems is revised. A new necessary and sufficient condition is established, and the indiscernible space is completely characterized. Differing from the condition provided in (Roy et al., 2020), this condition removes the requirements on the multiagent system and has broader applicability. The effectiveness of the results is demonstrated by several examples.

Index Terms—Discernibility, indiscernible state, networked systems, topological variation.

I. INTRODUCTION

In the last two decades, the study of complex networks has received extensive attention, particularly from the communities in physics, sociology, mathematics, biology, information technology, and engineering. It has flourishing advances in science and broad applications in such areas as smart grids [1], global transportation systems [2], networked robotics [3], and social networks [4], [5].

In a networked system, topological structure plays a fundamental role on the dynamics of the network. In this respect, it is already known that topological variations can have a major impact on the network stability and performance [6]–[8]. For example, in multiagent systems, link failures may affect the convergence rate of consensus or even destroy agreement when the network connectivity is lost [9], [10]. More importantly, topological variations may affect the network secure and reliable operation [11]. Therefore, detecting topological variations in networked systems is crucial in the management and recovery operations.

In recent years, this subject, as well as the more general problem of network identification, have received compelling attention. Some efficient methods have been proposed in the literature, e.g., [12]–[23]. The authors of [12] utilized Wiener filters to reconstruct the topology of a dynamical network, where the nodes represent scalar signals, and the edges correspond to transfer functions connecting different node signals. Then, the identification of a single module in such a network was investigated in [13] using prediction error methods. In [14], the authors reconstructed gene networks by Bayesian network methodologies. The authors of [15] studied topology identification using power spectral analysis. Apart from network identification, several techniques have been proposed for the detection of topological variations. In [17], the authors developed maximum a posteriori detection for link failures in network synchronization processes. The authors in [16] proposed a method to detect and isolate link failures based on observed jumps in the derivatives of the output responses of some nodes in the network. In [18], the authors generalized the hidden Markov model to model-changing networks, and provided an algorithm for detecting topological changes in dynamical networks. Note that, most, if not all, results on the detection of topological variations focus on developing detection algorithms. However, whether topological variations can actually be detectable is even more fundamental [28]. The theoretical barriers to the detection problem, irrespective of any specific detection algorithm, are still not well understood. Recently, the notion of discernibility has become a focal topic for investigation.

The authors of [24] investigated the detectability of single link failures in a multiagent network under the agreement protocol, and then the results were generalized to the multilink-failure case in [25]. In [26], the authors established the conditions under which it is possible to detect node or link disconnections for integrator networks. The authors in [27] studied the discernibility of topological changes in networks of linear dynamical systems. The results were derived under the assumption that the network topological structures are undirected graphs. Very recently, in [28], the authors established a necessary and sufficient condition on the discernibility of topological changes in a network of differential-algebraic equations (DAEs), with the indiscernible initial states characterized.

In this article, the discernibility of topological variations for networked LTI systems is considered. The contribution of this article is threefold. First, the network topology is general, directed, and weighted. The node systems have identical higher dimensional linear dynamics. Differing from [27], [28], and [31], this article allows directed network topologies. Second, a necessary and sufficient condition on the discernibility of topological variations is derived in terms of generalized eigenvectors of a few matrices with lower dimensions. Compared with the results in [28], this lower-dimensional criterion is easier to verify, which explicitly illustrates how the topological variations, node-system dynamics, and inner interactions altogether affect the discernibility.

Manuscript received 8 March 2021; accepted 18 December 2021. Date of publication 23 December 2021; date of current version 28 December 2022. This work was supported in part by the National Natural Science Foundation of China under Grant 12172020, Grant 11802006, Grant 11932003, and Grant 21212002, in part by the Beijing Natural Science Foundation under Grant 2122010, and in part by the Hong Kong Research Grants Council under the GRF Grant CityU 11206320. Recommended by Associate Editor M. Kanat Camlibel. (Corresponding author: Yuqing Hao.)

Yuqing Hao and Qingyun Wang are with the Department of Dynamics and Control, Beihang University, Beijing 100191, China, and also with the Beijing Advanced Discipline for Unmanned Aircraft System, Beihang University, Beijing 100191, China (e-mail: haoyq@buaa.edu.cn; nmqingyun@163.com).

Zhisheng Duan is with the State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Engineering Science, College of Engineering, Peking University, Beijing 100871, China (e-mail: duanzs@pku.edu.cn);

Guanrong Chen is with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong (e-mail: gchen@ee.cityu.edu.hk).

Digital Object Identifier 10.1109/TAC.2021.3137791

0018-9286 © 2021 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See https://www.ieee.org/publications/rights/index.html for more information.
of the network. Third, the discernibility of topological variations for multiagent systems is revisited. A new necessary and sufficient condition is established with all the indiscernible states fully characterized. Compared with the condition provided in [31], this condition removes the requirements on the multiagent system, thus is more general.

The remainder of this article is organized as follows: Some preliminaries and the model description are given in Section II. A necessary and sufficient condition on the discernibility of topological variations is developed in Section III. The discernibility of topological changes for multiagent systems is revisited in Section IV, with a necessary and sufficient condition derived. Finally, Section V concludes this article.

II. PRELIMINARIES AND MODEL DESCRIPTION

Some preliminaries and the model description are introduced in this section.

A. Notation and Lemmas

Let \( \mathbb{N} \) be the field of integers, \( \mathbb{R} \) and \( \mathbb{C} \) be the fields of real and complex numbers, respectively. Let \( I_n \) be the identity matrix of size \( n \times n \), \( e_i \) be the vector with all zero entries except for \( e_{ii} = 1 \), \( 0_n \) be the \( n \times 1 \) vector with all zero entries, \( 1_n \) be the \( n \times 1 \) vector containing all ones, \( 0_{n \times m} \) be the \( n \times m \) matrix with all zero entries, and \( \text{diag}(a_1, a_2, \ldots, a_n) \) be a diagonal matrix with diagonal entries \( a_1, a_2, \ldots, a_n \). Moreover, let \( \sigma(A) \) denote the set of all eigenvalues of matrix \( A \), and \( A \otimes B \) be the Kronecker product of matrices \( A \) and \( B \).

A directed and weighted graph \( G = (V, E, W) \) consists of a node set \( V = \{1, \ldots, n\} \), an edge set \( E \subseteq V \times V \), and a weight matrix \( W = [w_{ij}] \in \mathbb{R}^{n \times n} \). Note that, \( (j, i) \in E \) if and only if \( w_{ij} \neq 0 \). For node \( i \in V \), the neighbor set is \( N(i) = \{ j \in V \mid (i, j) \in E \} \). Similarly, the in-set of node \( i \in V \) is \( N_i(i) = \{ j \in V \mid (j, i) \in E \} \).

The adjacency matrix of graph \( G \) is denoted by \( A(G) = [a_{ij}] \in \mathbb{R}^{n \times n} \), where \( a_{ij} = w_{ij} \) if \( (j, i) \in E \) and \( a_{ij} = 0 \) otherwise.

**Lemma 1 [29]:** An LTI dynamical system \( \dot{x} = Ax \), denoted by a pair of matrices \((A, C)\), is unobservable if and only if there exists an eigenvalue-eigenvector pair \((\lambda, v)\) of \( A \), such that \( C v = 0 \).

**Definition 1 [30]:** A vector \( x_m \) is called an \( m \)-th order generalized eigenvector of matrix \( A \) corresponding to the eigenvalue \( \lambda \) if \((A - I)^m x_m = 0\) and \((A - \lambda I)^{m-1} x_m \neq 0\). Also, \( x_1, x_2, \ldots, x_g \) form a Jordan chain of \( A \) with top vector \( x_1 \), where the maximum value for \( g \) is called the length of this Jordan chain.

B. Model Description

Consider a network consisting of \( N \) identical LTI node systems, with a general directed and weighted topology \( G = (V, E, W) \), in the following form:

\[
\dot{x}_i = A x_i + \sum_{j=1}^{N} w_{ij} H x_j, \quad i = 1, 2, \ldots, N
\]

(1)

where \( x_i \in \mathbb{R}^n \) is the state vector, \( A \in \mathbb{R}^{n \times n} \) is the state matrix describing the dynamics of the node systems; \( w_{ij} \in \mathbb{R} \) represents the coupling strength between nodes \( i \) and \( j \); \( H \in \mathbb{R}^{n \times n} \) denotes the inner coupling matrix describing the interactions among components of \( x_j \). Assume that, \( w_{ii} = 0 \), and \( w_{ij} \neq 0 \) if there is an edge from node \( j \) to node \( i \), otherwise \( w_{ij} = 0 \), for all \( i, j = 1, 2, \ldots, N \). To avoid trivial situations, always assume \( N \geq 2 \) in this article.

Denote

\[
L = [w_{ij}] \in \mathbb{R}^{N \times N}
\]

(2)

which represents the network topology of the networked systems (1). Let \( X = [x_1^T, x_2^T, \ldots, x_N^T]^T \) be the whole state of the networked systems. Then, the networked systems (1) can be rewritten in a compact form as

\[
\dot{X} = \Phi X
\]

(3)

where

\[
\Phi = I_N \otimes A + L \otimes H.
\]

(4)

III. MAIN RESULTS

The effect of topological changes on the system dynamics is investigated. In particular, the interest is in characterizing the topological change that does not change the network dynamics (for certain initial states). A topological change caused by a removal/addition of an edge, or a change in the edge weight, results in a new system

\[
\dot{\tilde{X}} = \Phi \tilde{X}
\]

(5)

with

\[
\Phi = I_N \otimes A + L \otimes H.
\]

(6)

Now, the concept of indiscernible initial state is introduced.

**Definition 2 [28]:** Consider the networked system (3)–(4). An initial state \( X_0 \in \mathbb{R}^{Nn} \) is called indiscernible with respect to the topological change \( L \rightarrow \tilde{L} \) if and only if for all solutions \( X \) of (3) and all solutions \( \tilde{X} \) of (5), the following implication holds:

\[
X(0) = X_0 \Rightarrow \dot{X}(t) = \dot{\tilde{X}}(t) \quad \forall t \in \mathbb{R}.
\]

Note that, \( X_0 = 0_{Nn} \) is always an indiscernible initial state (independent of the specific topological variation), which is called the trivial indiscernible initial state. According to whether a nontrivial indiscernible initial state exists, the topological changes can be classified into two groups as follows.

**Definition 3 [28]:** For the networked system (3)–(4), a topological change \( L \rightarrow \tilde{L} \) is called always discernible if there is no (nontrivial) indiscernible initial state. But, if there exists a nontrivial indiscernible initial state, the topological change is called possibly indiscernible.

**Lemma 2:** Consider the networked system (3)–(4). A topological change \( L \rightarrow \tilde{L} \) is always discernible if and only if system

\[
\dot{X} = \Phi X, \quad Y = [(L - \tilde{L}) \otimes H] X
\]

is observable.

**Proof:** Necessity It was shown in [27], that discernibility is equivalent to the observability of the pair \((\Delta, \Gamma)\), where \( \Delta = \text{diag}(\Phi, \Phi) \) and \( \Gamma = [I - I] \). Note that, \((\Delta, \Gamma)\) is observable if and only if there exists no eigenvalue-eigenvector pair of \( \Delta \), denoted as \((\mu, \eta)\), such that \( \Gamma \eta = 0 \). Let \( \eta_1 = [\eta_1^T, \eta_2^T]^T \), where \( \eta_1, \eta_2 \in \mathbb{C}^{Nn} \). Then, there exists no eigenpair \((\mu, [\eta_1^T, \eta_2^T]^T)\) satisfying

\[
\begin{bmatrix}
\Phi \\
\Phi
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} = \mu
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
\]

and \([I - I]
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} = 0\). It follows that \( \Phi \) and \( \Phi \) have no common eigenpair, i.e., there exists no eigenpair \((\mu, \eta_1)\) satisfying \( \Phi \eta_1 = \mu \eta_1 \) and \( \Phi \eta_1 = \mu \eta_1 \). From the above equations, one concludes that there exists no eigenvalue-eigenvector pair of \( \Phi \), denoted as \((\mu, \eta_1)\), such that \( (\Phi - \Phi) \eta_1 = 0 \). Thus, the pair \((\Phi, \Phi - \Phi)\) is observable. Since \( \Phi - \Phi = (L - \tilde{L}) \otimes H \), it follows that system \( \dot{X} = \Phi X, \quad Y = [(L - \tilde{L}) \otimes H] X \) is observable.
Sufficiency: The sufficiency can be proved similarly, thus is omitted.

In what follows, a necessary and sufficient condition on the discernibility of topological variations is derived.

Let $T \in \mathbb{C}^{N \times N}$ be a nonsingular matrix such that $T^{-1}LT = J = \text{blockdiag}\{J_1, J_2, \ldots, J_s\}$, where $J$ is the Jordan form of $L$, and the $i$th Jordan block $J_i = \lambda_i I_\alpha + N_{\alpha_i} \in \mathbb{C}^{\alpha \times \alpha}$ with $N_{\alpha_i}$ being a nilpotent matrix, for $i = 1, \ldots, s$. The Jordan chain of $L$ associated with the eigenvalue $\lambda_i$ is denoted as $t_i^1, t_i^2, \ldots, t_i^\alpha$, where $t_i^1$ is the top vector, and $\alpha$ is the length of the Jordan chain.

**Definition 4:** Let $A \in \mathbb{C}^{N \times N}$, $H \in \mathbb{C}^{N \times N}$, and $\lambda$ be an eigenvalue of $A$. If vectors $X_1, X_2, \ldots, X_q$ satisfy that $(\lambda - A)X_i = 0$ and $(\lambda I - A)X_{i+1} = HX_i$, for $i = 1, \ldots, q - 1$, then $\lambda$ is the eigenvalue of $A$ corresponding to the eigenvalue $\lambda$, where $X_1$ is the top vector, and the maximum value for $\lambda$ is the length of this generalized Jordan chain.

Denote the eigenvalues of $A + \lambda_i H$ as $\mu_i$, with the corresponding generalized Jordan chain about $H$ denoted as $\xi_i^1, \xi_i^2, \ldots, \xi_i^\alpha_j$, where $\xi_i^j$ is the top vector, and $\gamma_i$ is the length, for $j = 1, \ldots, p_i$, $i = 1, \ldots, s$.

In what follows, the eigenvectors of $\Phi$ are expressed through the generalized eigenspaces of matrices with lower dimensions.

**Theorem 1:** Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ be the set of the eigenvalues of $L$, and $M_1 = \{\mu_1^1, \ldots, \mu_1^{p_1}\}$ be the set of the eigenvalues of $A + \lambda_1 H$. Then, $\sigma(\Phi) = \{\mu_1^1, \ldots, \mu_1^{p_1}, \mu_2^1, \ldots, \mu_2^{p_2}, \ldots, \mu_s^1, \ldots, \mu_s^{p_s}\}$. Moreover, the eigenvectors of $\Phi$ corresponding to $\mu_i^j$ are $\eta_i^j = t_i^\gamma_i \otimes \xi_i^\gamma_i, \eta_i^j = t_i^\gamma_i \otimes \xi_i^\gamma_i, \ldots, \eta_i^j = t_i^\gamma_i \otimes \xi_i^\gamma_i$.

The proof follows from Theorem 1, Lemmas 1 and 2 easily, thus is omitted.

**Remark 1:** Theorem 2 provides a precise and efficient criterion to determine the discernibility of topological changes for large-scale networked systems, using the generalized eigenvectors of a few matrices of low dimensions. Differing from the conditions given in [27] and [28], which require the network topology to be undirected, the new condition here removes the requirement, thus is more general.

**Remark 2:** The notion of discernibility can be generalized to the case, where topological variations are detected by observing the output trajectories. If only $y_i = Cx_i$ can be observed for each node, then the output of the networked system (3)-(4) is $Y = (I_N \otimes C)X = \Psi X$. The problem of verifying the discernibility of topology variations can be generalized to the problem of determining under what conditions $\Phi$ and $\Phi$ cannot give rise to exactly the same output dynamics, which is called $\Psi$-discernibility. With the help of the generalized inverse matrix of the observability matrix, a necessary and sufficient condition on the $\Psi$-discernibility of topological variations can be established with a similar method. From the condition, it can be concluded that for a topological change, the discernibility is necessary for the $\Psi$-discernibility. Thus, the discernibility of topological variations is the fundamental limitation to the problem of detecting topological changes from measurements, which holds even in the most favorable situation where the whole network state is available for measurement.

The following example demonstrates the effectiveness of the above results.

**Example 1:** Consider a simple network of three connected identical nodes, shown in (a) of Fig. 1, with $w_{21} = w_{32} = w_{31} = 1$, and $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

It can be easily verified that $L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The eigenvalues of $L$ are $\lambda_1 = 0, \lambda_2 = 1$, and $\lambda_3 = -1$, with the corresponding eigenvectors $t_1 = e_1 - e_3, t_2 = e_2 + e_3$ and $t_3 = e_2 - e_3$, respectively. Then, the eigenvalue of $A + \lambda_1 H$ is $A + \mu_1^1 = 1$, with the corresponding eigenvector $\xi_1^1 = e_2$, and the eigenvalue of $A + \lambda_2 H = A + H$ is $\mu_2^1 = 2$, with the corresponding eigenvector $\xi_1^2 = e_2$. The eigenvalue of $A + \lambda_3 H = A - H$ is $\mu_3^1 = 0$, with the corresponding eigenvector $\xi_1^3 = e_2$.

Assume that the edge from node 3 to node 2 is removed. Then, the new topology matrix is $\tilde{L} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is easy to verify that $[(L - \tilde{L}) \otimes H]n_{11}^1 \neq 0$, $[(L - \tilde{L}) \otimes H]n_{21}^1 \neq 0$, and $[(L - \tilde{L}) \otimes H]n_{31}^1 \neq 0$. 

![Fig. 1. Network topologies. (a) $G$. (b) $\tilde{G}$.](image-url)
Therefore, the topological change \( L \rightarrow \tilde{L} \) for this networked system is always discernible.

In the following, some more intuitive and easily verified conditions on the discernibility of topological variations are presented, which reveal how the topological change (described by the matrix \( L - \tilde{L} \)), the node-system dynamics \( A \), and the inner interactions (presented by the matrix \( H \)) altogether affect the discernibility of the network.

**Corollary 1:** If the topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4) is always discernible, then the topology matrices \( L \) and \( \tilde{L} \) have no common eigenvalue–eigenvector pair.

**Proof:** It follows from Theorem 2 that if the topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4) is always discernible, then \((L, L - \tilde{L})\) is observable. If \((L, L - \tilde{L})\) is observable, then the two matrices \( L \) and \( \tilde{L} \) have no common eigenvalue–eigenvector pair.

The effectiveness of this corollary is demonstrated by the following example.

**Example 2:** Consider a simple network of three connected identical nodes, shown in (a) of Fig. 2, with \( w_{21} = w_{32} = w_{23} = 1 \), and \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

It can be easily verified that \( L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Assume that the edge from node 2 to node 3 is removed. Then, the new topology matrix is \( \tilde{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

Since \( L \) and \( \tilde{L} \) have a common eigenvalue–eigenvector pair \((0, e_1 - e_3)\), one can easily deduce that the topological change \( L \rightarrow \tilde{L} \) for this networked system is possibly indiscernible.

Actually, one cannot detect the topological change when there exists some initial state \( x_0 \) satisfying \( e^{\lambda t} x_0 = e^{\tilde{\lambda} t} x_0 \). An instance of this situation is \( x_0 = [0, 1, 0, 0, 0, -1]^T \), with which the two networked systems will generate exactly the same trajectory.

It is revealed by Corollary 1 that the topological variation plays a crucial role for discernibility. In what follows, link and node disconnections are considered, respectively, with some easily verified discernibility conditions derived.

**Corollary 2:** (Link disconnection) Consider a topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4), which results from the disconnection of edge \((i, j)\) \( \in E \). If there exists an eigenvector \( v \) of \( L \) satisfying \( v_i = 0 \), then the topological change is possibly indiscernible.

**Proof:** Consider the eigenvalue–eigenvector pair \((\lambda, v)\) of \( L \), and recall that \( \tilde{L} = L - \sum_{j \in N(i)} w_{ji} e_j e_i^T \). Then, \( \tilde{L} v = L v - \sum_{j \in N(i)} w_{ji} e_j e_i^T v \). Thus, \( L \) and \( \tilde{L} \) have a common eigenvalue–eigenvector pair. According to Corollary 1, it is easy to show that this topological change is possibly indiscernible.

**Corollary 3:** (Node disconnection) Consider a topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4), which results from the disconnection of node \( i \) \( \in V \). If there exists an eigenvector \( v \) of \( L \) satisfying \( v_i = 0 \) and \( \sum_{k \in N(i)} w_{ik} v_k = 0 \), then this topological change is possibly indiscernible.

**Proof:** Consider the eigenvalue–eigenvector pair \((\lambda, v)\) of \( L \), and recall that \( \tilde{L} = L - \sum_{j \in N(i)} w_{ji} e_j e_i^T - \sum_{k \in N(i)} w_{ik} e_i e_k^T \). Then, \( \tilde{L} v = L v - \sum_{j \in N(i)} w_{ji} e_j e_i^T v - \sum_{k \in N(i)} w_{ik} e_i e_k^T v = \).

Since \( v_i = 0 \) and \( \sum_{k \in N(i)} w_{ik} v_k = 0 \), one has \( \tilde{L} v = L v = \lambda v \). Thus, \( L \) and \( \tilde{L} \) have a common eigenvalue–eigenvector pair. According to Corollary 1, it is easy to show that this topological change is possibly indiscernible.

The above corollaries show how the topological variations affect the discernibility. In what follows, the effect of node-system dynamics and inner interactions on the discernibility of topological change is revealed.

**Corollary 4:** If the topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4) is always discernible, then \((A, H)\) is observable.

**Proof:** Let \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_s\} \) be the set of the eigenvalues of \( L \). It follows from Theorem 2 that if the topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4) is always discernible, then \((A + \lambda_i \Lambda, H)\) is observable, for \( i = 1, \ldots, s \). It is easy to show that if \((A + \lambda_i \Lambda, H)\) is observable, then \((A, H)\) is observable.

The effectiveness of Corollary 4 is illustrated by the following example.

**Example 3:** Consider a simple network of three connected identical nodes, shown in (a) of Fig. 3, with \( w_{21} = w_{32} = w_{13} = 1 \), and \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

It can be easily verified that \( L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Assume that the edge from node 3 to node 1 is removed. Then, the new topology matrix is \( \tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

Since \((A, H)\) is not observable, one can easily deduce that the topological change for this networked system is possibly indiscernible.

Indeed, one can find an indiscernible initial state \( x_0 = [0, 1, 0, 1, 0, 1]^T \), with which the two networked systems will generate exactly the same trajectory.

For the case that the topology matrix \( L \) is diagonalizable, the discernibility condition can be recovered from Theorem 2 as a special case.

**Corollary 5:** Let \( L \) be diagonalizable with the eigenvalue set \( \Lambda = \{\lambda_1, \ldots, \lambda_N\} \). Then, the topological change \( L \rightarrow \tilde{L} \) for the networked system (3)-(4) is always discernible if and only if the following two conditions hold simultaneously.

1. \( L \) and \( \tilde{L} \) have no common eigenvalue–eigenvector pair.
2. If matrices \( A + \lambda_i H, A + \lambda_k H, \ldots, A + \lambda_p H \) (\( \lambda_i \in \Lambda \), for \( k = 1, \ldots, p, p > 1 \)) have a common eigenvalue \( \sigma \), then \( [(L - \tilde{L}) t_{1p}] \otimes (H \xi_{1p}^{\gamma_{1p}}), \ldots, [(L - \tilde{L}) t_{1p}] \otimes (H \xi_{1p}^{\gamma_{1p}}) \) are linearly independent, where \( t_{ik} \) is the eigenvector of \( L \) corresponding to the eigenvalue \( \lambda_i \); \( \gamma_{ik} \geq 1 \) is the geometric multiplicity of \( \sigma \) for \( A + \lambda_i H; \xi_{ip}^{\gamma_{ip}} (i = 1, \ldots, \gamma_{ip}) \) are the eigenvectors of \( A + \lambda_i H \) corresponding to \( \sigma, k = 1, 2, \ldots, p \).
For the case that the topology matrix $L$ is diagonalizable, the topological change $L \to \tilde{L}$ is always discernible if and only if the pair $(\Phi, (L - \tilde{L}) \otimes H)$ is observable. Thus, the simplified discernibility condition can be established similarly.

**Remark 3:** A necessary and sufficient condition on the discernibility of topological variations in heterogeneous DAE networks was established in [28], which is in terms of the common eigenspaces of the networks. Compared with this condition, the lower dimensional criteria established in [28], which is in terms of the common eigenspaces of the Laplacian matrices of connected networks, will not be affected by the coupling, and hence any topological variation for the multiagent system (7)–(8) is possibly indiscernible. This indiscernible initial state corresponds to the situation, where all nodes have the same initial condition. If the topology matrices $L$ and $\tilde{L}$ corresponding to the $i$th agent is, then there exist nontrivial indiscernible states if and only if the Laplacian matrices $L$ and $\tilde{L}$ have a common eigenvalue–eigenvector pair. Accordingly, any $\lambda \in \sigma(L) \cap \sigma(\tilde{L})$, let $\Xi(\lambda)$ be a matrix whose columns form a basis of the span of eigenvectors corresponding to $\lambda$, which is common to $L$ and $\tilde{L}$. Then, the set of all the real indiscernible states is given by

$$T = \{0\} \bigoplus_{\lambda \in \sigma(L) \cap \sigma(\tilde{L})} \text{span}\{\Xi(\lambda) \otimes \Im\}.$$  

The above condition imposes additional requirements on the multiagent system, thus limits its applicability. It is worth noting that the conditions proposed in Section III can also be used to deal with the discernibility of topological variations for multiagent systems. Based on the results in Section III, a new discernibility condition for the multiagent system (7)–(8) is established as follows. Compared with the condition provided in [31], the new condition here removes the requirements, thus is more general.

**Corollary 6:** Let $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ be the set of the eigenvalues of $L$. Then, there exists a nontrivial indiscernible initial state with respect to the topological change $L \to \tilde{L}$ for the multiagent system (7)–(8) if and only if one of the following conditions holds.

1) $L$ and $\tilde{L}$ have a common eigenvalue-eigenvector pair.

2) There exists a common eigenvalue $\sigma$ of matrices $A - \lambda \_1B, A - \lambda \_2B, \ldots, A - \lambda \_pB (\lambda \_i \in \Lambda,$ for $k = 1, \ldots, p, p > 1$), such that $[(L - \tilde{L})t_{11}] \otimes (B_{\xi_1}e_{11}), \ldots, [(L - \tilde{L})t_{1p}] \otimes (B_{\xi_1}e_{1p}), \ldots$ are linearly independent, where $t_{1k}$ is the eigenvector of $L$ corresponding to the eigenvalue $\lambda \_k, \xi_1 = 1$ is the geometric multiplicity of $\sigma$ for $A - \lambda \_1B; \xi_1 = 1, \ldots, p$, are the eigenvectors of $A - \lambda \_kB$ corresponding to $\sigma, k = 1, 2, \ldots, p$.

**Remark 4:** Note that, Laplacian matrices of connected networks always have the same eigenvalue–eigenvector pair $(0, 1 \_N\_N)$. Any topological variation for the multiagent system (7)–(8) is possibly indiscernible. This indiscernible initial state corresponds to the situation, where all the agents start with the same initial state. As a consequence, the diffusive coupling is zero, and the topological variations will not affect the dynamics of the whole network.

**Remark 5:** It follows from Corollary 6 that if $(A, B)$ is unobservable, then any topological change $L \to \tilde{L}$ for the multiagent system (7)–(8) is possibly indiscernible. Recall that the controllability of $(A, B)$ is required in [31] for the discernibility of topological variations. The condition here clarifies that the observability of $(A, B)$, instead of the controllability, is necessary for the discernibility. If $(A, B)$ is not observable, then there exists an eigenvector $\xi$ of $A$, such that $B\xi = 0$. If one sets the initial states of the agents to $\alpha \xi$, where $\alpha$ is taken as different scalars, the diffusive coupling between any two agents will be annihilated by $\alpha B$ in $\tilde{L}$. In this situation, the dynamics of the network will not be affected by the coupling, and hence any topological variation is not discernible.
In the following, the indiscernible space will be characterized explicitly and completely. Before moving on, a useful lemma is introduced.

**Lemma 3 [28]:** Consider a multiagent system with dynamics given by (7)–(8) and a topological change $L \to \tilde{L}$. Let

$$\Theta = \left\{ v \in \mathbb{C}^{Nn} \mid \exists (v_1, v_2, \ldots, v_k) \text{ common Jordan chain of } F \text{ and } \tilde{F} \text{ for the same eigenvalue } \lambda \in \mathbb{C}, \text{ and } v = v_i \text{ for some } i \in \{1, 2, \ldots, k\} \right\}$$

be the set of the vectors that appear in a common Jordan chain of $F$ and $\tilde{F}$. Then, $X_0 \subseteq \mathbb{R}^{Nn}$ is an indiscernible initial state for the topological change $L \to \tilde{L}$ if and only if it is in the span of all common Jordan chains of $F$ and $\tilde{F}$, i.e.,

$$X_0 \in \text{span } \Theta \cap \mathbb{R}^{Nn}.$$  

**Theorem 4:** Consider a multiagent system with dynamics described by (7)–(8) and a topological change $L \to \tilde{L}$. Let

$$\Gamma = \left\{ v \in \mathbb{C}^{Nn} \mid v_1, v_2, \ldots, v_m \text{ denotes the Jordan chain of } F \text{. There exists } 1 \leq r \leq m \text{ such that } \left[(L - \tilde{L}) \otimes B\right] v_{r+1} \neq 0 \text{, so that } v = v_i \text{ for } i \in \{1, \ldots, r\} \right\}$$

be the set of the unobservable states of $(F, (L - \tilde{L}) \otimes B)$. Then, $X_0 \subseteq \mathbb{R}^{Nn}$ is an indiscernible initial state for the topological change $L \to \tilde{L}$ if and only if it is in the span of all the unobservable states of $(F, (L - \tilde{L}) \otimes B)$, i.e.,

$$X_0 \in \text{span } \Gamma \cap \mathbb{R}^{Nn}.$$  

**Proof:** It is easy to verify that $\text{span } \Gamma = \text{span } \Theta$. Therefore, the conclusion follows from Lemma 3.

**Remark 6:** The unobservable states of $(F, (L - \tilde{L}) \otimes B)$ are not limited to the eigenvectors of $F$, which are orthogonal to $(L - \tilde{L}) \otimes B$. They are the vectors in the Jordan chains of $F$, which are orthogonal to $(L - \tilde{L}) \otimes B$. Only when all the unobservable eigenvalues are semisimple, the span of the unobservable eigenvectors is the whole unobservable space.

In what follows, the Jordan chains of $F$ are expressed through the generalized eigenspaces of some matrices with lower dimensions.

**Theorem 5:** Let $A = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be the set of the eigenvalues of $L$, and $M = \{\mu_1, \mu_2, \ldots, \mu_N\}$ be the set of the eigenvalues of $A - \lambda_i B$. Then, $\sigma(F) = \{\mu_1, \mu_2, \ldots, \mu_N\}$.

Moreover, the Jordan chain of $F$ corresponding to the eigenvalue $\mu_i$ is $\psi_{ij} = t_i \otimes \xi_{ij}$, where $t_i$ is the eigenvector of $L$ corresponding to the eigenvalue $\lambda_i; \xi_{ij}, \ldots, \xi_{ij}$ is the Jordan chain of $A - \lambda_i B$ associated with $\mu_i$, $j = 1, \ldots, q_i$, $i = 1, \ldots, N$.

Based on a method similar to Theorem 1, one can prove this theorem easily.

**Remark 7:** If $(A, B)$ is not observable, denote the set of the unobservable eigenvalues as $\Delta$. For any $\beta \in \Delta$, let $\Xi(\beta)$ be a matrix whose columns form a basis of the span of the corresponding unobservable states. Then

$$\bigoplus_{\beta \in \Delta} \text{span}\{I_N \otimes \Xi(\beta)\} \subset T_F.$$  

**Remark 8:** In this article, some lower dimensional discernibility conditions are established by analyzing the forms of the eigenvectors of the network model’s state matrix. A similar eigenvector analysis for networked systems was utilized to check the network controllability in [7], [8], [32]–[34]. Differing from [7], [8], [32]–[34], the Jordan chains of the network model’s state matrix are also expressed here to characterize the indiscernible space. Note that, the indiscernible space was described in [28] by using the vectors in the common Jordan chains of $F$ and $\tilde{F}$. Here, all the indiscernible states are characterized by (generalized) eigenvectors of a few matrices with lower dimensions. This lower dimensional characterization is easier to use, which would be more convenient for related theoretical analysis and practical applications.

Next, the effectiveness of the above results is illustrated by analyzing the counterexample in [31].

Note that, the only common eigenvalue–eigenvector pair of $L$ and $\tilde{L}$ is $(0, I)$, and $(L - 1) I \otimes I_4$. Hence, one has $T_F = \text{span}\{I_4 \otimes I_4, I_4 \otimes I_3\}$. It is clear that $A - 1B, A - 2B, A - 3B, A - 4B$ have the only common eigenvalue $1$. By simple calculation, it is easy to obtain that

$$T_F = \text{span}\{I_4 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\}.$$  

Therefore, the set of all the nontrivial indiscernible states is given by

$$T_{\text{ind}} = \text{span}\{I_4 \otimes I_3\} \oplus \text{span}\left\{I_4 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}.$$
Consider a multiagent system consisting of $N$ connected agents is considered. It is revealed by this example that the results of this article are effective and easier to use, when applied to large-scaled networks.

**Example 4:** Consider a multiagent system consisting of $N$ identical agents connected by a path graph, shown in (a) of Fig. 4, with $w_{12} = w_{23} = \cdots = w_{N-1}N = w_{N(N-1)} = 1$, and $A = \begin{bmatrix} 1 & 2 & 0 \ 0 & 6 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \ 0 & 1 \end{bmatrix}$.

It can be easily verified that

$$
\mathcal{L} = \begin{bmatrix}
-1 & 1 & -1 & 0 & \cdots & -1 \\
1 & -1 & 1 & -1 & \cdots & 0 \\
-1 & 1 & -1 & 1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
$$

The eigenvalues of $\mathcal{L}$ are $\lambda_k = 2 - 2\cos(k\pi/N)$, with the corresponding eigenvectors $e_k = \{\cos \frac{2\pi}{N} \cos \frac{4\pi}{N} \cdots \cos \frac{2(k-1)\pi}{N}\}$, where $k = 0, 1, \cdots, N-1$. Assume that the edge connecting nodes 1 and 3 is added. Then, the new network is shown in (b) of Fig. 4, with the Laplacian matrix $\mathcal{L}_1 = \mathcal{L} - \begin{bmatrix} 0 & 1 & 0 \ 1 & 0 & 1 \ 0 & 1 & 0 \end{bmatrix}$.

It is easy to verify that an eigenpair of $\mathcal{L}_1$, denoted as $(\lambda_k, e_k)$, is also the eigenpair of $\mathcal{L}$ if and only if $[e_k]_1 = [e_k]_2 = [e_k]_3$, where $[e_k]_i$ denotes the $i$th component of the $k$th eigenvector, $i = 1, 3$. From the forms of the eigenvectors of $\mathcal{L}$, it follows that $\mathcal{L}$ and $\mathcal{L}_1$ have a common eigenpair if and only if $\cos \frac{k\pi}{N} = \cos \frac{3k\pi}{N}$, for some $k \in \{0, 1, \cdots, N-1\}$. It implies that $k\pi/3 = (3k\pi)/N + 2m\pi$, where $m \in \mathbb{N}$. Then, one has $k = -mN$ or $k = (2/3)mN$.

It is obvious that any two matrices of $A - \lambda_0 B$, $A - \lambda_1 B$, $\cdots$, $A - \lambda_{N-1} B$ have no common eigenvalue. Thus, $T_p = \emptyset$. Therefore, for the case where $N$ is an integral multiple of 3, the set of all the nontrivial indiscernible states is given by $T_{ind} = \text{span}\{1_N \otimes I_2, \cos \frac{\pi}{3} \cos \frac{3\pi}{3} \cos \frac{\pi}{3} \cdots \cos \frac{3(N-1)\pi}{3} \otimes I_2\}$, while for the case where $N$ is not an integral multiple of 3, the set of all the nontrivial indiscernible states is given by $T_{ind} = \text{span}\{1_N \otimes I_2\}$.

**Remark 9:** The existence of indiscernible initial states for homogeneous DAE networks was investigated in [28], with a sufficient condition established. However, only the forms of some indiscernible states were formulated therein. The results in this article fully describe the indiscernible space, with a necessary and sufficient discernibility condition derived. Clearly, the analysis here is more comprehensive.

**V. CONCLUSION**

This article has investigated the conditions under which a topological variation in networked LTI systems can be detected. The considered network topology is general, directed, and weighted. A condition on the discernibility of topological variations has been established, which is necessary and sufficient. Compared with the conditions given in [28] and [31], which require the network topology to be undirected, this condition is applicable also to directed topologies. Moreover, the discernibility of topological variations for multiagent systems has been revisited. A new necessary and sufficient condition has been derived, and all the indiscernible states have been completely characterized. Differing from the condition established in [31], this condition removes the requirements on the multiagent system, thus is more general.

In future studies, the discernibility of topological variations for networked heterogeneous systems will be investigated.

**REFERENCES**

[1] G. H. Wen, X. H. Yu, Z. W. Liu, and W. W. Yu, “Adaptive consensus-based robust strategy for economic dispatch of smart grids subject to communication uncertainties,” IEEE Trans. Ind. Informat., vol. 14, no. 6, pp. 2484–2496, Jun. 2018.

[2] J. Banavar, F. Colaiori, A. Flammini, A. Maritan, and A. Rinaldo, “Topology of the fittest transportation network,” Phys. Rev. Lett., vol. 84, no. 20, pp. 4745–4748, 2000.

[3] H. B. Du, G. H. Wen, Y. Y. Cheng, Y. G. He, and R. T. Jia, “Distributed finite-time cooperative control of multiple high-order nonholonomic mobile robots,” IEEE Trans. Neural Netw. Learn. Syst., vol. 28, no. 12, pp. 2998–3006, Dec. 2017.

[4] N. E. Friedkin and E. C. Johnsen, “Influence networks and opinion change,” Adv. Group Processes, vol. 16, no. 1, pp. 1–29, 1999.

[5] R. Hegselmann and U. Krause, “Opinion dynamics and bounded confidence: Models, analysis and simulation,” Simulation, vol. 5, no. 3, pp. 1–24, 2002.

[6] G. H. Wen, W. W. Yu, G. Q. Hu, J. D. Cao, and X. H. Yu, “Pinning synchronization of directed networks with switching topologies: A multiple lyapunov functions approach,” IEEE Trans. Neural Netw. Learn. Syst., vol. 26, no. 12, pp. 3239–3250, Dec. 2015.

[7] Y. Q. Hao, Z. S. Duan, and G. R. Chen, “Further on the controllability of networked MIMO LTI systems,” Int. J. Robust Nonlinear Control, vol. 28, no. 5, pp. 1778–1788, 2018.

[8] O. Ofati-Saber and R. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Trans. Autom. Control, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[9] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Trans. Autom. Control, vol. 50, no. 9, pp. 949–970, Sep. 2005.

[10] Y. Q. Hao, Q. Y. Wang, Z. S. Duan, and G. R. Chen, “The role of reverse edges on consensus performance of chain networks,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 51, no. 3, pp. 1757–1765, Mar. 2021.

[11] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, “Catastrophic cascade of failures in interdependent networks,” Nature, vol. 464, no. 7291, pp. 1025–1028, 2010.
[12] D. Materassi and M. V. Salapaka, “On the problem of reconstructing an unknown topology via locality properties of the Wiener filter,” *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1765–1777, Jul. 2012.

[13] P. M. J. Van den Hof, A. Dankers, P. S. C. Heuberger, and X. Bombois, “Identification of dynamic models in complex networks with prediction error methods–basic methods for consistent module estimates,” *Automatica*, vol. 49, pp. 2994–3006, 2013.

[14] A. Mittal, A. Kassim, and T. Tan, *Bayesian Network Technologies: Applications and Graphical Models*. Hershey, PA, USA: IGI Publishing, 2007.

[15] S. Shahrampour and V. M. Preciado, “Topology identification of directed dynamical networks via power spectral analysis,” *IEEE Trans. Autom. Control*, vol. 60, no. 8, pp. 2260–2265, Aug. 2015.

[16] M. A. Rahimian and V. M. Preciado, “Detection and isolation of failures in directed networks of LTI systems,” *IEEE Trans. Control Netw. Syst.*, vol. 2, no. 2, pp. 183–192, Jun. 2015.

[17] R. Dhal, J. A. Torres, and S. Roy, “Detecting link failures in complex network processes using remote monitoring,” *Physica A*, vol. 437, pp. 36–54, 2015.

[18] J. Costanzo, D. Materassi, and B. Sinopoli, “Using Viterbi and Kalman to detect topological changes in dynamic networks,” in *Proc. Amer. Control Conf.*, 2017, pp. 5410–5415.

[19] G. Cavraro, R. Arghandeh, G. Barchi, and A. von Meier, “Distribution network topology detection with time-series measurements,” in *Proc. IEEE PES Innov. Smart Grid Technol. Conf.*, 2015, pp. 1–5.

[20] P. Barooah, “Distributed cut detection in sensor networks,” in *Proc. 47th IEEE Conf. Decis. Control*, 2008, pp. 1097–1102.

[21] H. H. M. Weerts, J. Linder, M. Enqvist, and P. M. J. Van den Hof, “Abstractions of linear dynamic networks for input selection in local module identification,” *Automatica*, vol. 117, 2020, Art. no. 108975.

[22] M. Dimovska and D. Materassi, “A control theoretic look at Granger causality: Extending topology reconstruction to networks with direct feedthroughs,” *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 699–713, Feb. 2021.

[23] M. T. Schaub, S. Segarra, and J. N. Tsitsiklis, “Blind identification of stochastic block models from dynamical observations,” *SIAM J. Math. Data Sci.*, vol. 2, no. 2, pp. 335–367, 2020.

[24] M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, “Characterization of link failures in multi-agent systems under the agreement protocol,” in *Proc. Amer. Control Conf.*, 2012, pp. 5258–5263.

[25] M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, “Detectability of multiple link failures in multi-agent systems under the agreement protocol,” in *Proc. IEEE Conf. Decis. Control*, 2012, pp. 118–123.

[26] G. Battistelli and P. Tesi, “Detecting topology variations in dynamical networks,” in *Proc. IEEE Conf. Decis. Control*, 2015, pp. 3349–3354.

[27] G. Battistelli and P. Tesi, “Detecting topology variations in networks of linear dynamical systems,” *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 1287–1299, Sep. 2018.

[28] D. Patil, P. Tesi, and S. Trem, “Indiscernible topological variations in DAE networks,” *Automatica*, vol. 101, pp. 280–289, 2019.

[29] K. M. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ, USA: Prentice Hall, 1996.

[30] M. Roman, *Advanced Linear Algebra*. New York, NY, USA: Springer, 2005.

[31] S. Roy, M. Xue, G. Battistelli, and P. Tesi, “Comment on ‘detecting topology variations in networks of linear dynamical systems’,” *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 1, pp. 187–188, Mar. 2020.

[32] M. Xue and S. Roy, “Comments on ‘upper and lower bounds for controllable subspaces of networks of diffusively coupled agents’,” *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2306–2306, Jul. 2018.

[33] M. Xue and S. Roy, “Input-output properties of linearly-coupled dynamical systems: Interplay between local dynamics and network interactions,” in *Proc. IEEE Conf. Decis. Control*, 2017, pp. 487–492.

[34] M. Xue and S. Roy, “Modal barriers to controllability in networks with linearly coupled homogeneous subsystems,” *IEEE Trans. Autom. Control*, vol. 66, no. 12, pp. 6187–6193, Dec. 2021.