Penrose limit of a non–supersymmetric RG fixed point

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Abstract

We extend the BMN duality between IIB superstring theory on a pp–wave background and a sector of $\mathcal{N} = 4$ super Yang-Mills theory to the non–supersymmetric and unstable background built by Romans as a compactification on a $U(1)$ bundle over $\mathbb{CP}^2$ with 3–form and 5–form field strength fluxes. We obtain a stable theory with the fewest number of supercharges (e.g. 16) allowed by this kind of solutions and make conjectures on the dual gauge theory.
1 Introduction

A duality between type IIB strings on the maximally supersymmetric pp–wave \cite{1} and a sector of $\mathcal{N} = 4$ super Yang-Mills theory has been recently proposed. In \cite{3} Berenstein, Maldacena, and Nastase show how string theory on the maximally supersymmetric pp–wave can be obtained from $AdS_5 \times S^5$ as the Penrose limit \cite{4} along a null geodesic.

This background is particularly interesting because the superstring theory can be solved on it using the GS superstring formalism and light–cone gauge \cite{5}, \cite{6}. All the string states are massive

$$H = -p_+ = \sum_{n=-\infty}^{+\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}$$

(1.1)

where $N_n$ is the occupation number of the $n$th normal mode of the bosonic and fermionic fields. Choosing a null geodesic breaks $SU(4)_R$ to the $U(1)_R$ of rotations along the geodesic. BMN argued that string states on the pp–wave limit of $AdS_5 \times S^5$ are dual to a sector of $\mathcal{N} = 4$ SYM which is composed of both chiral and non–chiral operators with large dimension $\Delta$, large $U(1)_R$ charge $J$ and fixed $\Delta - J$

$$\Delta - J = -\frac{p_+}{\mu} = \sum_{n=-\infty}^{+\infty} N_n \sqrt{1 + \frac{4\pi g_s N n^2}{J^2}}$$

(1.2)

$$\Delta + J = -\mu p_-$$

where $R^4 \equiv 4\pi \alpha' g_s N$ is the common radius of $AdS_5$ and $S^5$ and the expressions are valid for $\frac{\Delta-J}{J} \ll 1$.

BMN explicitly identified the non–chiral operators corresponding to the string spectrum (1.1). Expression (1.2) can be expanded as a perturbation series in the constant $g_s N = \frac{\lambda}{J^2}$, which can be interpreted as an effective coupling constant. In this way, perturbative calculations may be performed in the non–perturbative regime $\lambda \gg 1$, and comparison with the superstring results can be carried out.

After \cite{3} the Penrose limit has been applied to other, less symmetric, models, and some results were obtained on the non–perturbative behaviour of a large variety of gauge theories. In this paper we go in this same direction and consider the large $\Delta$ and $J$ (along with large $N$ and $\lambda$) limit of a non–supersymmetric unstable gauge theory which is obtained as the IR fixed point of the renormalization group flow from $\mathcal{N} = 4$ SYM deformed through a mass term for one of the fermions in the adjoint of the gauge group \cite{7} \cite{8} \cite{9}. The dual to this theory is the compactification built in \cite{10} as a $U(1)$ fibration over $\mathbb{C}P^2$. The background has three– and five–form field strengths turned on, and is unstable.
After taking the Penrose limit, we find a pp–wave background with constant NS–NS and R–R field strengths. pp–wave backgrounds with 2–form fields have been considered in [11], [12], and [13] for a different and supersymmetric fixed–point. Despite the original instability, our solution is stable and has the minimal number of supersymmetries allowed for IIB pp–waves (e.g. 16). This has some interesting consequences. First of all, the bosons and fermions have different masses, which gives the theory a non–vanishing, finite and positive zero–point energy. Moreover, in the perturbative expansion of the anomalous dimension of the dual operators, we find that some scalars have as the first order correction a term proportional to

\[(\Delta - J)_n = \sqrt{2} \left( 1 \pm n \sqrt{\frac{\pi}{8}} \sqrt{\lambda_{eff}} + \frac{\pi}{16} n^2 \lambda_{eff} + O(\lambda_{eff}^{3/2}) \right)\]  

(1.3)

where \(\lambda_{eff} = \frac{g_s^2 M^N}{\mathcal{F}}\).

We also find that the perturbation series for some fermionic states starts from the second order

\[(\Delta - J)_n = \frac{1}{\sqrt{2}} \left( 1 + \frac{\pi^2}{16} n^4 \lambda_{eff}^2 + O(\lambda_{eff}^3) \right)\]  

(1.4)

as was also found for a different model in [13].

This paper is organized as follows. In section 2 we review the construction of the supergravity solution of [10], first following the original paper, and then in a different way that makes the symmetries more evident. In section 3 we take the Penrose limit of this solution. In section 4 we quantize the string theory on the constant NS–NS and R–R fields pp–wave, and give an approximation to the zero–point energy. In the last section we discuss the field theory dual to the pp–wave string theory.

## 2 The SU(3) × U(1) supergravity solution

The 2–dimensional complex projective space \(\mathbb{CP}^2\) is defined as the subset of \(\mathbb{C}^3 - \{0\}\) with the identification \((z_1, z_2, z_3) \simeq (\lambda z_1, \lambda z_2, \lambda z_3)\) with \(\lambda\) any complex number different from zero. The metric of this space is given by

\[ds^2 = \frac{dr^2 + r^2 \sigma_3^2}{(1 + \Lambda r^2)^2} + \frac{r^2 (\sigma_1^2 + \sigma_2^2)}{1 + \Lambda r^2}\]

where \(\Lambda\) is the cosmological constant and \(\sigma_i, i = 1, 2, 3\) are the SU(2)–invariant forms satisfying \(d\sigma_1 = 2\sigma_2 \wedge \sigma_3\) and permutations. It is possible to define a connection \(A\) that satisfies

\[d\eta + \left(\frac{1}{4} \omega_{ab} \Gamma^{ab} - i e A\right) \eta = 0\]  

(2.1)
where $e$ is the charge of the spinor $\eta$ (we use the same charge normalization as \[10\]):

$$A = A_\mu dx^\mu = 3 \frac{C_{\mu\nu} dx^\mu}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $z_1 = x_1 + i x_2$, $z_2 = x_3 + i x_4$ and $z_1^2 + z_2^2 + z_3^2 = \frac{6}{\Lambda}$. The solution to equation (2.1) is given by

\[
\begin{cases}
  e = \frac{1}{2} \\
  (\Gamma_{12} - \Gamma_{34})\eta = 0 \\
  (i + \Gamma_{12})\eta = 0 \\
  \partial_i \eta = 0 \quad i = 1, 2, 3, 4
\end{cases}
\]  

(2.2)

and imposing $\eta^\dagger \eta = 1$, $\eta$ can be determined up to a phase $\alpha$. If we define the charge conjugate spinor $\chi \equiv C\eta^*$, we can build a complex $(2,0)$–form $K \equiv \chi^\dagger \Gamma_{ij} \eta e^i \wedge e^j$ ($\{e^i\}$ ($i = 1, \ldots, 4$) are the vielbeins of $\mathbb{C}P^2$) which will be useful later to define the complex three–form on the 10–dimensional background. Given (2.2) we find that

\[
\begin{cases}
  K_{12} = K_{34} = 0 \\
  K_{13} = -K_{24} = iK_{23} = iK_{14} = e^{i\alpha}
\end{cases}
\]  

(2.3)

where $\alpha$ is an arbitrary real constant.

The connection $A$ can be used to build a $U(1)$ bundle over $\mathbb{C}P^2$. We refer to this space as $M_5$. Its metric is given by

$$ds_{M_5}^2 = ds_{\mathbb{C}P^2}^2 + c^2(d\tau - A)^2$$  

(2.4)

for a constant $c$ to be determined later.

Given $M_5$ we can build a compactification of IIB supergravity of the form $AdS_5 \times M_5$ \[10\]

$$ds^2 = ds_{AdS_5}^2 + ds_{M_5}^2$$

$$G_3 = \frac{2}{R} e^{-i\tau} K_{ij} \eta^* e^i \wedge e^j \wedge e^5$$

$$F_5 = -\frac{1}{\sqrt{2R}} (e^\mu \wedge e^\nu \wedge e^\sigma \wedge e^\tau + e^m \wedge e^n \wedge e^p \wedge e^r)$$  

(2.5)

where $\{e^\mu\}$ and $\{e^m\}$ are the vielbeins of $AdS_5$ and $M_5$ respectively, and $\Lambda_{\mathbb{C}P^2} = \frac{1}{e^2} = \frac{8}{R^2}$.

We choose a coordinate system in which the $AdS$ metric is given by

$$ds_{AdS}^2 = R^2(-\cosh^2 \rho \ dt^2 + d\rho^2 + \sinh^2 \rho \ d\Omega_3)$$

4
and the complex coordinates of $\mathbb{CP}^2$ are

\[
\begin{align*}
  z_1 &= \cot \omega \cos \frac{\theta + \phi}{2} e^{i \frac{\theta - \phi}{2}} \\
  z_2 &= \cot \omega \sin \frac{\theta - \phi}{2}
\end{align*}
\]  

(2.6)

The ten–dimensional metric takes the form

\[
\begin{align*}
  ds_{10}^2 &= R^2 (-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3) + \frac{R^2}{8} (d\tau + \frac{3}{2} \cos^2 \omega (d\psi + \cos \theta \, d\phi))^2 + \\
  &\quad + \frac{3}{4} R^2 (d\omega^2 + \sin^2 \omega \cos^2 \omega \sigma_1^2 + \cos^2 \omega (\sigma_1^2 + \sigma_2^2))
\end{align*}
\]  

(2.7)

where

\[
\begin{align*}
  \sigma_1 + i\sigma_2 &= -\frac{i}{2} e^{i\psi} (d\theta - i \sin \theta \, d\phi) \\
  \sigma_3 &= \frac{1}{2} (d\psi + \cos \theta \, d\phi)
\end{align*}
\]

while for the complex three–form we obtain

\[
\begin{align*}
  G_3 &= \frac{3}{8\sqrt{2}} e^{i\alpha} e^{-i(\tau + \psi)} R^2 \cos \omega (d\theta + i \sin \theta \, d\phi) \wedge (d\omega + \frac{i}{2} \sin \omega \cos \omega (d\psi + \cos \theta \, d\phi)) \wedge \\
  &\quad \wedge (d\tau + \frac{3}{2} \cos^2 \omega (d\psi + \cos \theta \, d\phi))
\end{align*}
\]  

(2.8)

The symmetry of this solution is $SU(3) \times U(1)^1$. The $SU(3)$ is a spatial symmetry, it comes from the symmetry of the compact space $M_5$, while the $U(1)$ is a subgroup of the $SU(1,1) \subset SL(2, \mathbb{R})$ symmetry of chiral supergravity [14] and can be associated with translations of the $\alpha$ phase of the complex 3–form.

This solution has no supersymmetries, and is unstable. In fact there is a scalar mode in the 6 of $SU(3)$ with $m^2 = -\frac{40}{9}$ (in units of the $AdS$ radius) which is below the Breitenlohner–Freedman bound [15].

There is another way to build the solution $AdS_5 \times M^5$. This background is the ten–dimensional lifting of the solution corresponding to one of the critical points of the scalar potential of five–dimensional gauged supergravity [16]. Each of these critical points is characterized by the expectation value of a set of scalar fields. After making the proper changes, we can use the same formulas as [17] for the embedding of the five–dimensional solution into chiral ten–dimensional supergravity.

The general formula is

\[
\begin{align*}
  ds_{10}^2 &= \Omega^2 ds_{AdS}^2 + ds_5^2
\end{align*}
\]  

(2.9)

\footnote{The $U(1)$ factor doesn’t appear in the original work of Romans [10]. It was first argued to belong to the symmetry group of this solution in [14].}
where
\[ \Omega^2 = \xi \cosh \chi \]
\[ ds_5^2(\alpha, \chi) = \frac{a^2 \text{sech} \chi}{\xi} \left( dx^I Q_{IJ}^{-1} dx^J \right) + \frac{a^2 \sinh \chi \tanh \chi}{\xi^3} \left( x^I J_{IJ} dx^J \right)^2 \]
\[ Q_{IJ} = \text{diag} \left( e^{-2\nu}, e^{-2\nu}, e^{-2\nu}, e^{4\nu}, e^{4\nu} \right) \]
\[ J_{IJ} = -J_{JI} \quad \text{and} \quad J_{14} = J_{23} = J_{65} = 1 \]
\[ \xi^2 = x^I Q_{IJ} x^J \]
and \( R_0 \) is the radius of the round \( S^5 \) compactification, while \( \chi \) and \( \nu \) are the scalar fields which determine the five–dimensional solution. In our case they are
\[ \nu = 0 \quad \text{and} \quad \chi = \frac{1}{2} \log \left( 2 - \sqrt{3} \right) \] (2.10)

New complex coordinates are defined based on the structure of \( J \)
\[ u^1 = x^1 + ix^4, \quad u^2 = x^2 + ix^3, \quad u^3 = x^5 - ix^6 \] (2.12)
which transform in the 3 of \( SU(3) \). We parametrize them as
\[ u^1 = \cos \omega e^{i(P + \frac{\psi + \phi}{2})} \cos \frac{\theta}{2} \]
\[ u^2 = \cos \omega e^{i(P + \frac{\psi - \phi}{2})} \sin \frac{\theta}{2} \]
\[ u^3 = \sin \omega e^{iP} \] (2.13)

so that if we define \( z_1 = \frac{u^1}{u^3} \) and \( z_2 = \frac{u^2}{u^3} \), these coordinates give the right parametrization of \( \mathbb{CP}^2 \). Now we only need to make two remarks in order to obtain our solution. First of all the radius \( R_0 \) of \( AdS_5 \) in the round \( S^5 \) compactification is different from our radius \( R \): (2.9), (2.10) and (2.11) show that
\[ R_0 \text{ sech} \chi = \frac{2}{3} R^2 \] (2.14)
Also the coordinate \( P \) has to be rescaled in order to obtain the metric of the fibration over \( \mathbb{CP}^2 \) as we wrote it above. The right substitution is \( P = \frac{\tau}{3} \).

This way of obtaining the solution is very useful when considering the symmetries of the theory. If we write the metric (2.10) and complex three–form in terms of the \( u \) coordinates
\[ ds_5^2 = \frac{2}{3} R^2 \sum_{k=1}^{3} \left( du^k d\bar{u}^k + \frac{1}{8} (u^k d\bar{u}^k - \bar{u}^k du^k)^2 \right) \] (2.15)
\[ G_3 = i \frac{9}{4\sqrt{2}} e^{i\alpha} d\bar{u}^1 \wedge d\bar{u}^2 \wedge d\bar{u}^3 \] (2.16)
the solution is manifestly \( SU(3) \times U(1) \) symmetric.
3 The Penrose limit

3.1 Taking the limit

We now consider the Penrose limit of (2.7). We consider the null geodesic $\rho = \omega = \theta = 0$ and scale our coordinates as

$$
\rho = \frac{r}{R}, \quad \omega = \frac{2}{\sqrt{3} R} \eta, \quad \theta = \frac{4}{\sqrt{3} R} \chi
$$

(3.1)

We define

$$
\beta = \frac{\psi + \phi}{2}, \\
x^+ = \frac{1}{2} \left( t + \frac{1}{2 \sqrt{2}} (\tau + 3 \beta) \right), \\
x^- = \frac{R^2}{2} \left( t - \frac{1}{2 \sqrt{2}} (\tau + 3 \beta) \right)
$$

(3.2)

and expand the metric keeping only $O(1)$ terms

$$
ds^2 = -4 dx^+ dx^- - r^2 (dx^+)^2 + dr^2 + r^2 d\Omega_3 + d\eta^2 + \eta^2 d\beta^2 + d\chi^2 + \chi^2 d\phi^2 + 2 \sqrt{2} dx^+ (\eta^2 d\beta + \chi^2 d\phi^2)
$$

(3.3)

There is a way to write this metric which is far more intuitive in view of the quantization of the string theory on this background. If we change coordinates

$$
\varphi_1 = \phi - \sqrt{2} \left( x^+ - \frac{x^-}{R^2} \right), \\
\varphi_2 = \beta - \sqrt{2} \left( x^+ - \frac{x^-}{R^2} \right)
$$

(3.4)

the metric reads

$$
ds^2 = -4 dx^+ dx^- + \sum_{i=1}^8 (dx^i)^2 - \left( \sum_{i=1}^4 (x^i)^2 + 2 \sum_{i=5}^8 (x^i)^2 \right) (dx^+)^2
$$

(3.5)

where $x_1, \ldots, x_4$ are along the spatial coordinates of the AdS part of the pp-wave and we defined $\chi e^{i \varphi_1} = z_1 = x_5 + i x_6$ and $\eta e^{i \varphi_2} = z_2 = x_7 + i x_8$.

We take the same limit on the 3– and 5–forms, and obtain

$$
G_3 = 2 e^{i \alpha} dx^+ \wedge dz_1 \wedge dz_2, \\
F_5 = -\frac{1}{\sqrt{2}} dx^+ \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8)
$$

(3.6)
The metric and the forms we have obtained from the Penrose limit of the $SU(3) \times U(1)$ solution are, as one would expect, a solution of the equations of motion of supergravity. They indeed satisfy the relation [13]
\[ \text{tr}A = -8f^2 - 2|b|^2 \]
where $A_{ij}$ are the masses of the bosonic zero modes that can be read off the coefficients of the $(dx^+)^2$ terms in the metric, e.g. $A_{ij} = \text{diag}(1, 1, 1, 2, 2, 2, 2)$, and $b$ and $f$ are the coefficients of $G_3$ and $F_5$ respectively.

As usual, the solution preserves the 16 supercharges $\Gamma^+ \epsilon = 0$ [2]. In Appendix B we check that there are no other supersymmetries.

4 String spectrum

4.1 Bosonic sector

The string theory on a pp–wave background is exactly solvable even when there are nontrivial NS–NS and R–R fields. The bosonic spectrum doesn’t get contributions from the R–R fields [3], but feels only the graviton and the NS–NS 3–form field strength. The model we consider and its quantization are similar to [11], [13] and [18].

We introduce a mass parameter $m$ by scaling $x^+$ and $x^-$ as
\[ x^+ \rightarrow mx^+, \quad x^- \rightarrow \frac{x^-}{m} \quad (4.1) \]
Both $x^+$ and $x^-$ have now the dimension of a length. We also decompose the complex 3–form $G_3$ as
\[ G_3 = H_3^{NS} + iF_3^{RR} \]
The background is then given by
\[ ds^2 = -4dx^+dx^- + \sum_{i=1}^{8} (dx^i)^2 - m^2 \left( \sum_{i=1}^{4} (x^i)^2 + 2 \sum_{i=5}^{8} (dx^i)^2 \right) (dx^+)^2 \quad (4.2) \]
\[ H_3^{NS} = 2m dx^+ \wedge (\cos \alpha (dx^5 \wedge dx^7 - dx^6 \wedge dx^8) - \sin \alpha (dx^6 \wedge dx^7 + dx^5 \wedge dx^8)) \]
\[ F_3^{RR} = 2m dx^+ \wedge (\cos \alpha (dx^6 \wedge dx^7 + dx^5 \wedge dx^8) + \sin \alpha (dx^5 \wedge dx^7 - dx^6 \wedge dx^8)) \]
\[ F_5 = -\frac{m}{\sqrt{2}} dx^+ \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \]
and the action for the bosonic sector is [18]
\[ S_B = \frac{1}{2\pi \alpha'} \int d\tilde{\tau} \int_0^{2\pi} d\sigma \tilde{L}_B \quad (4.3) \]
where, after fixing the gauge \( x^+ = 2\alpha'p^+\tilde{\tau} \), the “lagrangian” is given by

\[
\tilde{\mathcal{L}}_B = \frac{1}{4} \left( \tilde{\partial}_0 X^i \right)^2 - m^2 (\alpha'p^+)^2 \sum_i c_i X^i - \frac{1}{4} \left( X^{i'} - 2\alpha'p^+ H_{i+j} X^j \right) X^{j'} - 2\alpha'p^+ \left( \tilde{\partial}_0 X^- - X^{-i} \right)
\]

and \( c_i = (1,1,1,2,2,2) \). \( \tilde{\mathcal{L}}_B \) doesn’t have the dimensions of an energy. This is because \( \tilde{\tau} \) is not a time–sized variable. A natural choice for defining a new time coordinate is suggested by the gauge–fixing condition

\[
\tau = 2\alpha'p^+\tilde{\tau} = x^+ \Rightarrow \tilde{\partial}_0 = \frac{\partial}{\partial \tilde{\tau}} = 2\alpha'p^+ \frac{\partial}{\partial \tau} = 2\alpha'p^+ \partial_0
\]

We also require that the lagrangian and the gauge–fixing condition are consistent, thus we rescale \( \tilde{\mathcal{L}}_B \) in such a way that its functional derivative with respect to \( \dot{X}^- \) gives \( p_- = -2p^+ \). We thus get for the lagrangian and action, respectively

\[
\mathcal{L}_B = \frac{\tilde{\mathcal{L}}_B}{2\alpha'^2 p^+} = \frac{p^+}{2} \sum_{i=1}^{8} \dot{X}^i X^i - \frac{m^2 p^+}{2} \sum_{i=1}^{8} c_i X^i - \frac{1}{8(\alpha'^2 p^+)} \sum_{i,j=1}^{8} \left( \delta_{ij} X^{i'} - 2\alpha'p^+ H_{i+j} X^j \right) X^{j'} +
\]

\[
-2p^+ \dot{X}^- + \frac{1}{\alpha'} X^{-i}
\]

and

\[
S_B = \frac{1}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \mathcal{L}_B
\]

We may now proceed in the evaluation of the bosonic string spectrum: if we explicitly substitute \( H_{i+j} \) into (4.6) then

\[
\mathcal{L}_B = \frac{p^+}{2} \sum_{i=1}^{8} \dot{X}^i X^i - \frac{m^2 p^+}{2} \sum_{i=1}^{8} c_i X^i - \frac{1}{8(\alpha'^2 p^+)} \sum_{i,j=1}^{8} X^{i'2} + \frac{m}{2\alpha'} \left( \cos \alpha \left( X^5 X^{7'} + X^7 X^5' \right) - X^7 X^5' - X^6 X^8' + X^8 X^6' \right) \sin \alpha \left( X^6 X^{7'} - X^7 X^6' + X^5 X^{8'} - X^8 X^5' \right) \right) +
\]

\[
-2p^+ \dot{X}^- + \frac{1}{\alpha'} X^{-i}
\]

Four bosonic modes are independent harmonic oscillators, while the other four are magnetically coupled through the \( H_3 \) field. The term in the \( X^- \) field only gives a constraint on the physical states, since it is linear in the field. We introduce three parameters

\[
k \equiv \frac{p^+}{2} \quad a \equiv \frac{1}{8(\alpha'^2 p^+)} \quad g \equiv \frac{m}{2\alpha'}
\]
to avoid as much confusion as possible in the following expressions. We study the equations of motion of the fields \((X^5, X^6, X^7, X^8)\); the ones for the four independent fields \((X^1, X^2, X^3, X^4)\) are obtained by substituting \(2m^2 \to m^2\) and \(g \to 0\) in the expressions for the interacting degrees of freedom.

The Euler–Lagrange equations read

\[
\begin{align*}
-\ddot{X}^5 &= 2m^2 X^5 - \frac{a}{k} X^{5''} - \frac{g}{k} \left( \cos \alpha X^7' - \sin \alpha X^8' \right) \\
-\ddot{X}^6 &= 2m^2 X^6 - \frac{a}{k} X^{6''} + \frac{g}{k} \left( \cos \alpha X^8' + \sin \alpha X^7' \right) \\
-\ddot{X}^7 &= 2m^2 X^7 - \frac{a}{k} X^{7''} + \frac{g}{k} \left( \cos \alpha X^5' - \sin \alpha X^6' \right) \\
-\ddot{X}^8 &= 2m^2 X^8 - \frac{a}{k} X^{8''} - \frac{g}{k} \left( \cos \alpha X^6' + \sin \alpha X^5' \right)
\end{align*}
\] (4.10)

We build a \textquotedblleft vector\textquotedblright \(X(\sigma, \tau) \equiv \{X^i(\sigma, \tau)\}\) with \(i = 5, 6, 7, 8\) and expand it in Fourier modes

\[
X(\sigma + 2\pi, \tau) = X(\sigma, \tau) \quad X(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} c_n(\tau) e^{in\sigma}
\] (4.11)

From the reality of \(X\) it follows that \(\overline{c_{-n}}(\tau) = c_n(\tau)\). The system of equations of motion becomes

\[
-\ddot{c}_n(\tau) = T_n c_n(\tau)
\] (4.12)

where

\[
T_n = \begin{pmatrix}
2m^2 + \frac{an^2}{k} & 0 & -\frac{ig}{k} \cos \alpha & \frac{ig}{k} \sin \alpha \\
0 & 2m^2 + \frac{an^2}{k} & \frac{ig}{k} \sin \alpha & \frac{ig}{k} \cos \alpha \\
\frac{ig}{k} \cos \alpha & -\frac{ig}{k} \sin \alpha & 2m^2 + \frac{an^2}{k} & 0 \\
-\frac{ig}{k} \sin \alpha & -\frac{ig}{k} \cos \alpha & 0 & 2m^2 + \frac{an^2}{k}
\end{pmatrix}
\] (4.13)

The eigenvalues of this matrix are

\[
k^+_n = 2m^2 + \frac{an^2}{k} + \frac{g}{k} n
\] (4.14)

\[
k^-_n = 2m^2 + \frac{an^2}{k} - \frac{g}{k} n
\] (4.15)

each with multiplicity 2. Since \(T_n\) is a self–adjoint operator, \(T_n^\dagger = T_n\), we can choose a set of orthonormal eigenvectors of \(T_n\) as a basis for the four–dimensional space of the
Quantization of the bosonic fields is then achieved as usual by promoting \( \sigma, \tau \), we define the momentum vector \( \Pi(\sigma, \tau) \) to operators (the Fourier coefficients are also promoted to operators \( A_n^{\pm(i)} \) and \( A_n^{\pm(i)} \)) and imposing canonical commutation relations on them

\[
[X^i(\sigma, \tau), \Pi_j(\sigma', \tau)] = i \delta^i_j \delta(\sigma - \sigma')
\]
By using (4.23) and (4.24) we can calculate the commutation relations for the \( A_n^{\pm(i)} \) and \( A_n^{\pm(i)} \) operators. Since we will interpret them as creation and annihilation operators, we normalize their commutators to one. We thus define (\( \omega_n^{\pm} \) is always strictly positive)

\[
a_n^{\pm(i)} \equiv \sqrt{\frac{8\pi k \omega_n^{\pm}}{\omega_n^{\pm}}} A_n^{\pm(i)}
\]

so that

\[
\begin{align*}
[a_n^{+}, a_n^{+}] & = \delta_{mn} \delta^{ij} \\
[a_n^{-}, a_n^{-}] & = \delta_{mn} \delta^{ij}
\end{align*}
\]

with all other commutators equal to zero.

Substituting (4.22) and \( \Pi (\sigma, \tau) \) into the hamiltonian (\( L_{B}^{5,6,7,8} \) stands for the part of the lagrangian involving only the fields \( X^5, \cdots, X^8 \))

\[
H_{B}^{5,6,7,8} = \int_{0}^{2\pi} d\sigma \ (\Pi \cdot X - L_{B}^{5,6,7,8})
\]

we find that

\[
H_{B}^{5,6,7,8} = \sum_{n=-\infty}^{+\infty} \left( \omega_n^{+} \left( a_n^{(1)} a_n^{(1)} + a_n^{(2)} a_n^{(2)} \right) + \omega_n^{-} \left( a_n^{-} a_n^{-} + a_n^{-} a_n^{-} \right) \right)
\]

We introduce the number operators

\[
\begin{align*}
N_n^0 & \equiv a_n^{0(1)} a_n^{0(1)} + a_n^{0(2)} a_n^{0(2)} + a_n^{0(3)} a_n^{0(3)} + a_n^{0(4)} a_n^{0(4)} \\
N_n^+ & \equiv a_n^{+(1)} a_n^{+} + a_n^{+} a_n^{+} \\
N_n^- & \equiv a_n^{-} a_n^{-} + a_n^{-} a_n^{-}
\end{align*}
\]

where the “0” quantities can be easily obtained from the ± ones via the substitution mentioned above. The complete bosonic hamiltonian is given by

\[
H_{B} = \sum_{n=-\infty}^{+\infty} \left( \omega_n^{0} N_n^0 + \omega_n^{+} N_n^+ + \omega_n^{-} N_n^- \right)
\]

### 4.2 Fermionic sector

It comes out that in light–cone gauge, using the GS formalism, it is possible to quantize the fermionic sector of superstring theory on a pp–wave with non–zero 3–forms and 5–forms [6], [18]. The only contribution to the fermionic part of the lagrangian is given by the supercovariant kinetic term for the two GS spinors

\[
\tilde{\mathcal{L}}_F = i \left( \eta^{ab} \delta_{1J} - \epsilon^{ab} \rho_{31J} \right) \bar{\theta}_a \sigma^m \tilde{\sigma}^I \Gamma_m (\tilde{D}_b)^{JK} \theta^K
\]
where

\[
\hat{D}_a = \partial_a + \frac{1}{4} \tilde{\partial}_a x^k \left[ \left( \omega_{mnk} - \frac{1}{2} H_{mnk} \rho_3 \right) \Gamma_{mn} - \left( \frac{1}{12} F_{mn} \Gamma_{mn} \rho_1 + \frac{1}{120} F_{mn} \Gamma_{mn} \rho_0 \right) \Gamma_k \right] \quad (4.32)
\]

\[
\rho_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.33)
\]

and \(\theta^1\) and \(\theta^2\) are 32-component space–time Majorana spinors; since we are considering IIB superstrings they have the same chirality. We choose \(\Gamma_{11} \theta^I = \theta^I\) to be consistent with the choices made for the background gravitino and dilatino in Appendix B.

Fixing the light–cone gauge

\[
x^+ = 2 \alpha' p^+ \tilde{\tau} = \tau \quad \text{and} \quad \Gamma^+ \theta^I = 0 \quad (4.34)
\]

the supercovariant derivative can be simplified to

\[
\hat{D}_{b}^{JM} \theta^M = \partial_b \theta^I - m \alpha' p^+ b_0 \left[ (\cos \alpha (\Gamma^{57} - \Gamma^{68}) - \sin \alpha (\Gamma^{67} + \Gamma^{58})) \rho_3^{JM} \theta^M + 
+ (\cos \alpha (\Gamma^{67} + \Gamma^{58}) + \sin \alpha (\Gamma^{57} - \Gamma^{68})) \rho_1^{JM} \theta^M + 
- \frac{1}{\sqrt{2}} (\Gamma^{1234} + \Gamma^{5678}) \rho_0^{JM} \theta^M \right] \quad (4.35)
\]

Using some gamma–matrices algebra and the chirality of the \(\theta\) spinors it comes out that the only non–zero contribution to the sum over \(m\) in (4.31) comes from the + term. As explained in Appendix A we use the light–cone gauge and the chirality of \(\theta^I\) to reduce the degrees of freedom of the spinors, and write the lagrangian using 8–component spinors \(S^I\) in place of the 32–component ones \(\theta^I\). Taking the same normalization as for the bosonic lagrangian, we eventually get

\[
\mathcal{L}_F = \frac{2i}{\alpha'} \left\{ -2 \alpha' p^+ (S^1 \partial_0 S^1 + S^2 \partial_0 S^2) - S^1 \partial_1 S^1 + S^2 \partial_1 S^2 + m \alpha' p^+ \left[ S^1 (\cos \alpha (\gamma^{57} - 
- \gamma^{68}) - \sin \alpha (\gamma^{67} + \gamma^{58})) S^1 - S^2 (\cos \alpha (\gamma^{57} - \gamma^{68}) - \sin \alpha (\gamma^{67} + \gamma^{58})) S^2 + 
+ 2 S^1 (\cos \alpha (\gamma^{67} + \gamma^{58}) + \sin \alpha (\gamma^{57} - \gamma^{68})) S^2 + 2 \sqrt{2} S^2 \gamma^{5678} S^1 \right] \right\} \quad (4.36)
\]

We can now quantize the fermionic sector. First of all we write the equations of

\footnote{We explain our conventions on gamma–matrices and spinors in Appendix A}
motion

\[ -i \dot{S}^1 = \frac{i}{2\alpha' p^+} \partial_1 S^1 - \frac{im}{2} \left( [\cos \alpha (\gamma^{57} - \gamma^{68}) - \sin \alpha (\gamma^{67} + \gamma^{58})] S^1 + [\cos \alpha (\gamma^{67} + \gamma^{58}) - \sin \alpha (\gamma^{57} - \gamma^{68})] S^2 - \sqrt{2} \gamma^{5678} S^1 \right) \]

\[ -i \dot{S}^2 = -\frac{i}{2\alpha' p^+} \partial_1 S^2 - \frac{im}{2} \left( [\cos \alpha (\gamma^{57} - \gamma^{68}) - \sin \alpha (\gamma^{67} + \gamma^{58})] S^2 + [\cos \alpha (\gamma^{67} + \gamma^{58}) + \sin \alpha (\gamma^{57} - \gamma^{68})] S^1 + \sqrt{2} \gamma^{5678} S^1 \right) \]  

(4.37)

and decompose the spinorial fields in Fourier modes

\[ S^I(\sigma + 2\pi, \tau) = S^I(\sigma, \tau) \]  

(4.38)

\[ S^I(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} S^I_n(\tau) e^{in\sigma} \]  

(4.39)

Writing the equations of motion for the normal modes we notice that the $F_5$ interaction couples every component of one spinor to the same component of the other spinor, without mixing different components, while the 3–form interactions leave four modes unaffected while coupling magnetically all the other components: $H_3$ acts within the same spinor, while $F_3$ mixes components of the first spinor with components of the second one, and viceversa. We then have four components of each spinor ($S^1_{n(3)}$, $S^1_{n(4)}$, $S^1_{n(5)}$, $S^1_{n(6)}$ and $S^2_{n(3)}$, $S^2_{n(4)}$, $S^2_{n(5)}$, $S^2_{n(6)}$) which are coupled only through the 5–form as

\[ -i \dot{S}^{1(k)} = -\frac{n}{2\alpha' p^+} S^1_{n(k)} + \frac{im}{\sqrt{2}} S^2_{n(k)} \]

\[ -i \dot{S}^{2(k)} = \frac{n}{2\alpha' p^+} S^2_{n(k)} - \frac{im}{\sqrt{2}} S^1_{n(k)} \]  

(4.40)

Their frequencies can be found by diagonalizing the matrix

\[ \left( \begin{array}{cc} -\frac{n}{2\alpha' p^+} & \frac{im}{\sqrt{2}} \\ -\frac{im}{\sqrt{2}} & \frac{n}{2\alpha' p^+} \end{array} \right) \]  

(4.41)

which gives

\[ \omega_n^{(0)} = \pm \sqrt{\frac{m^2}{2} + \frac{n^2}{(2\alpha' p^+)^2}} \]  

(4.42)

The other eight components feel both the complex 3–form and the 5–form

\[ -i \dot{\bar{S}}_n = T_n \bar{S}_n \]  

(4.43)
where \( \tilde{S}_n \equiv \left( S_n^{1(1)}, S_n^{1(2)}, S_n^{1(7)}, S_n^{1(8)}, S_n^{2(1)}, S_n^{2(2)}, S_n^{2(7)}, S_n^{2(8)} \right) \) and

\[
T_n = \begin{pmatrix}
-\frac{n}{2\alpha' p^+} & 0 & -im\beta(\alpha) & 0 & -im\beta(\alpha) & 0 \\
0 & -\frac{n}{2\alpha' p^+} & 0 & -im\beta(\alpha) & 0 & -im\beta(\alpha) \\
im\beta(\alpha) & 0 & -\frac{n}{2\alpha' p^+} & 0 & -i\beta(\alpha) & 0 \\
0 & im\beta(\alpha) & 0 & -\frac{n}{2\alpha' p^+} & 0 & -i\beta(\alpha) \\
-\frac{im}{\sqrt{2}} & 0 & m\beta(\alpha) & 0 & im\beta(\alpha) & 0 \\
m\beta(\alpha) & -\frac{im}{\sqrt{2}} & 0 & -im\beta(\alpha) & 0 & im\beta(\alpha) \\
0 & -m\beta(\alpha) & 0 & im\beta(\alpha) & 0 & -\frac{2\alpha' p^+}{2} \\
0 & 0 & -m\beta(\alpha) & 0 & -\frac{2\alpha' p^+}{2} & 0
\end{pmatrix}
\] (4.44)

and we have defined \( \beta(\alpha) \equiv \cos \alpha + i \sin \alpha \). The frequencies of the normal modes can be obtained, again, by diagonalizing this matrix. The result is

\[
\omega_n^- = \pm \sqrt{\left( \frac{n^2}{(2\alpha' p^+)^2} + \frac{5}{2} m^2 - 2m \sqrt{\frac{n^2}{(2\alpha' p^+)^2} + m^2} \right) (4.45)}
\]

\[
\omega_n^+ = \pm \sqrt{\left( \frac{n^2}{(2\alpha' p^+)^2} + \frac{5}{2} m^2 + 2m \sqrt{\frac{n^2}{(2\alpha' p^+)^2} + m^2} \right) (4.46)}
\]

To get the hamiltonian we organize the 16 eigenvalues in the following way

\[
\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_0 \\
\omega_5 = \omega_6 = \omega^-_n \\
\omega_7 = \omega_8 = \omega^+_n
\] (4.47)

and the other eight ones are defined in the same way but with the minus sign. The Majorana condition on the spinors \( S^1 \) and \( S^2 \) gives a relation between the positive– and negative–frequency components of the Fourier expansion \( \Theta_n^1 \). We obtain then

\[
S(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} \sum_{j=1}^{8} \left( \Theta_n^j w_n^j e^{i\omega_n^j \tau} + \overline{\Theta}_n^j \overline{w}_n^j e^{-i\omega_n^j \tau} \right) e^{i n \sigma}
\] (4.48)

where \( w_n^j \) are the eigenvalues of the kinetic matrix built from the blocks (4.41) and (4.44). We promote the \( \Theta_n^j \) and \( \overline{\Theta}_n^j \) to operators \( \Theta_n^j \) and \( \Theta_n^j \) and define the new operators \( b_n^j \) and \( b_n^j \) by requiring that imposing the canonical anti-commutation relations for the spinor fields and momentums

\[
\left\{ S^{I,j}(\sigma, \tau), \Pi_{j'}^{I'}(\sigma', \tau) \right\} = i\delta_{ij} \delta(\sigma - \sigma') \delta^{IJ} \quad \text{for} \quad i, j = 1, \cdots, 8
\] (4.49)
is equivalent to realizing a Clifford algebra with $b_n^j$ and $b_n^j\dagger$

$$\left\{b_m^i, b_n^j\right\} = \delta_{mn}\delta^{ij} \quad (4.50)$$

The hamiltonian is given by

$$H_F = \sum_{n=-\infty}^{+\infty} \sum_{i=1}^{8} \omega_i^n S_n^i S_n^i \quad (4.51)$$

**4.3 Spectrum**

We summarize here the results we obtained in the two preceding subsections on the spectrum of superstring theory on this minimally supersymmetric pp-wave. The total hamiltonian is given by

$$H = H_B + H_F = \sum_{n=-\infty}^{+\infty} \left(\omega_n^{B,0} N_n^{B,0} + \omega_n^{B,+} N_n^{B,+} + \omega_n^{B,-} N_n^{B,-} + \omega_n^{F,0} N_n^{F,0} + \omega_n^{F,+} N_n^{F,+} + \omega_n^{F,-} N_n^{F,-}\right) \quad (4.52)$$

and

$$\omega_n^{B,0} = \sqrt{m^2 + \frac{n^2}{(2\alpha'p^+)^2}}$$

$$\omega_n^{B,+} = \sqrt{2m^2 + \frac{n^2}{(2\alpha'p^+)^2} + \frac{mn}{\alpha'p^+}}$$

$$\omega_n^{B,-} = \sqrt{2m^2 + \frac{n^2}{(2\alpha'p^+)^2} - \frac{mn}{\alpha'p^+}}$$

$$\omega_n^{F,0} = \sqrt{\frac{m^2}{2} + \frac{n^2}{(2\alpha'p^+)^2}}$$

$$\omega_n^{F,-} = \sqrt{\frac{n^2}{(2\alpha'p^+)^2} + \frac{5}{2}m^2 - 2m\sqrt{\frac{n^2}{(2\alpha'p^+)^2} + m^2}}$$

$$\omega_n^{F,+} = \sqrt{\frac{n^2}{(2\alpha'p^+)^2} + \frac{5}{2}m^2 + 2m\sqrt{\frac{n^2}{(2\alpha'p^+)^2} + m^2}}$$

The physical states will also have to satisfy the constraint

$$P = \sum_{n=-\infty}^{+\infty} nN_n = 0 \quad (4.54)$$
which follows from the equations of motion of the coordinates. We define a vacuum \( |0\rangle \) which is annihilated by all the bosonic and fermionic destruction operators, and build string states by applying the creation operators on it, taking care of satisfying the constraint \( (4.54) \).

As can be noted from the masses \( (4.53) \) of the bosonic modes, the tachyon that made the ten–dimensional background unstable is not present in the string spectrum after the Penrose limit. This could have been guessed even before calculating the frequencies of the bosonic modes, and is due to the fact that when we put the superstring theory on the pp–wave limit of a supergravity compactification, we are keeping only those states in string theory which have very large momentum along the geodesic we take the Penrose limit on, i.e. we are considering high–order Kaluza–Klein states. We argue that the behaviour of the scalars of our theory is not much different (at least qualitatively) from the \( \text{AdS}_5 \times S^5 \) solution, where the KK–angular momentum contribution to the mass of the scalars in any representation of \( SU(4) \) is eventually dominant over the other contributions \( [19] \).

Thus, in the limit of large \( J \), the KK contribution to the energy of the states will drive the mass of the tachyon to a positive value. This is analogous to what happens in the Penrose limit of type IIB string theory on \( \text{AdS}_5 \times T^{p,q} \) \( [20] \) and of type 0 string theory on \( \text{AdS}_5 \times S^5 \) \( [21] \).

4.4 Zero–point energy

Despite the solution has 16 supersymmetries, the masses of the bosons and fermions are not equal, and the string theory will have a non–vanishing zero–point energy given by

\[
E_0 = \sum_{n=-\infty}^{+\infty} E_{0,n} = \sum_{n=-\infty}^{+\infty} (2\omega_n^{B,0} + \omega_n^{B,-} + \omega_n^{B,+} - 2\omega_n^{F,0} - \omega_n^{F,-} - \omega_n^{F,+})
\]

(4.55)

Other examples of the same phenomenon include \( [20], [21] \) and \( [22] \). The series \( (4.55) \) is convergent and we can approximate it with the integral

\[
\int_{-\infty}^{+\infty} dx \left( 2 \sqrt{m^2 + \frac{x^2}{(2\alpha' p^+)^2}} - 2 \sqrt{\frac{m^2}{2} + \frac{x^2}{(2\alpha' p^+)^2}} + \sqrt{2m^2 + \frac{x^2}{(2\alpha' p^+)^2} + \frac{mx}{\alpha' p^+}} + \right.
\]

\[
\left. + \sqrt{2m^2 + \frac{x^2}{(2\alpha' p^+)^2} - \frac{mx}{\alpha' p^+}} - \frac{x^2}{(2\alpha' p^+)^2} + \frac{5}{2} m^2 - 2m \sqrt{\frac{x^2}{(2\alpha' p^+)^2} + m^2} \right)
\]

(4.56)
where we have substituted the discrete variable $n$ with a continuous one $x$. This integral can be evaluated numerically, and gives a positive result

$$E_0 \sim 2m^2 \alpha' p^+$$

(4.57)

It is interesting to compare this with another non-supersymmetric example [21]. There the zero-point energy $E_0/m$ vanishes for $m\alpha' p^+ \to +\infty$ (perturbative limit in the dual gauge theory), while is unbounded from below in the limit $m\alpha' p^+ \to 0$ (supergravity limit). In this case we have the opposite behaviour, $E_0/m$ goes to 0 in the supergravity limit and diverges in the perturbative limit.

5 Dual gauge theory

The gauge theory dual to the compactification of [10] is $\mathcal{N} = 4$ $SU(4)_R$ super Yang-Mills deformed by a mass term for one of the four fermions in the adjoint of the gauge group $SU(N)$ [7] [8]. The compactification we are considering is indeed obtained from $AdS_5 \times S^5$ by turning on a complex 3-form. Complex 3-forms are in the $10$ of $SU(4)$ and couple to boundary operators which are bilinears in the four fermions that belong to the spectrum of $\mathcal{N} = 4$ SYM. In particular the 3-form $\omega_{3}$ we turned on is in the singlet representation of $SU(3) \subset SU(4)$ and thus can only couple to a mass term for one of the fermions. We choose $\lambda_4$ as the fermion that gets a mass. Thus the gauge theory lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + m \text{Tr} (\lambda_4 \lambda_4 + \bar{\lambda}_4 \bar{\lambda}_4)$$

(5.1)

The fermion mass term is a relevant operator which drives a RG flow from the $\mathcal{N} = 4$ UV fixed point to an IR fixed point where all supersymmetries are broken. Nothing will prevent scalar fields to gain mass through radiative corrections and the effective infrared theory will be made up only of three massless fermions and $SU(N)$ gauge fields, all in the adjoint of the gauge group. The authors of [9] argued that, as a consequence of the instability of this fixed point, the chiral symmetry $SU(3)$ is dynamically broken down to $SO(3)$. Since our model is stable after the Penrose limit, we believe that the subset of operators we are considering and their symmetries are well described by the pp-wave background.

Let us consider the symmetries preserved by the limit. In section 3.1 we redefined
the angle coordinates of the original space

\[ t = mx^+ + \frac{x^-}{mR^2} \]

\[ \phi = \sqrt{2} \left( \frac{mx^+ - x^-}{mR^2} \right) + \varphi_1 \]

\[ \psi = \sqrt{2} \left( \frac{mx^+ - x^-}{mR^2} \right) + 2\varphi_2 - \varphi_1 \]

\[ \tau = -\sqrt{2} \left( \frac{mx^+ - x^-}{mR^2} \right) - 3\varphi_2 \]

thus the Hamiltonian and the light–cone momentum are given by

\[ H = -\frac{p^+}{m} = i \frac{\partial}{\partial x^+} = \frac{i}{m} \frac{\partial}{\partial t} - i\sqrt{2} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \phi} \right) = \Delta - \sqrt{2} (J_{\tau-\psi} - J_{\phi}) \]

\[ 2p^- = -mp_- = im \frac{\partial}{\partial x^-} = \frac{1}{R^2} \left( \frac{i}{\partial t} + i\sqrt{2} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \phi} \right) \right) = \frac{\Delta + \sqrt{2} (J_{\tau-\psi} - J_{\phi})}{R^2} \]

where \( J_{\tau-\psi} \equiv i \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \psi} \right) \) and \( J_{\phi} \equiv i \frac{\partial}{\partial \phi} \). As manifest from (2.13) \( J_{\tau-\psi} - J_{\phi} \) is a \( U(1) \) generator of \( SU(3) \).

The boundary operator \( \text{Tr} \lambda_\phi \lambda_\psi + \text{h.c.} \) under \( SU(4) \rightarrow SU(3) \) decomposes as

\[ 10 \rightarrow 1 + 3 + 6 \]  \hspace{1cm} (5.4)

As we have already said, the singlet operator \( \text{Tr} \lambda_4 \lambda_4 \) couples to\(^3 d\tilde{u}_1 \wedge d\tilde{u}_2 \wedge d\tilde{u}_3 \), while \( \text{Tr} \lambda_i \lambda_j \) to \( d\bar{u}_i \wedge d\bar{u}_j \wedge d\bar{u}_k \). From this we obtain the charges of Table 1 where \( J \equiv J_{\tau-\psi} - J_{\phi} \). By looking at the charges it is evident that at the \( n = 0 \) level the symmetry conserved by the Penrose limit is \( SU(2) \times U(1) \subset SU(3) \), where \( SU(2) \) rotates the fermions \( (\lambda_2, \lambda_3) \).

Since there are no supersymmetries, and the supergravity approximation gives results only on some low–lying states, we can only make conjectures on the spectrum of the fields, which we leave for future work to verify. The \( SU(3) \) symmetry of the original theory ensures that the fields \( \lambda_1, \lambda_2, \lambda_3 \) and, separately, \( \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \) will have the same dimension. Thus looking at Table 1 it seems a good guess to choose as the building block of the vacuum the field \( \bar{\lambda}_1 \). Its charge is the largest among all the fields, which means that if we take as a first approximation to the energy of the operator the sum of its \( \Delta \) and \( J \), \( \text{Tr}\bar{\lambda}_1^{2J} \) will be the one with the smallest energy. Moreover \( \bar{\lambda}_1 \) is a singlet of the \( SU(2) \) symmetry group of the free theory, feature we would expect from the vacuum. Thus we take

\[ |0\rangle = \text{Tr} \left( \bar{\lambda}_1^{2J} \right) \]  \hspace{1cm} (5.5)

\(^3\)The coefficients in front of the 3–forms are not relevant in the calculation of the charges of the fermions, and we will ignore them. The couplings will be valid up to a constant coefficient.
Table 1: The charges of the fermionic massless fields which make up the spectrum of the effective theory at the IR fixed point.

| $J_{\tau-\psi}$ | $J_\phi$ | $J$ |
|-----------------|----------|-----|
| $\lambda_1$    | $\frac{1}{3}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ |
| $\lambda_1$    | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{2}{3}$ |
| $\lambda_2$    | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{2}{3}$ |
| $\lambda_2$    | $\frac{1}{3}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ |
| $\lambda_3$    | $\frac{1}{3}$ | $0$ | $-\frac{1}{3}$ |
| $\lambda_3$    | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ |

The four bosonic states with $H = 1$ are obtained, as usual in this kind of theories, by applying the gauge–covariant derivative along one of the four space–time directions to a scalar pair $(\bar{\lambda}_1 \lambda_1)$. The covariant derivative adds a unit to the dimension, while leaving the charge unchanged.

$$H = 1 \quad \rightarrow \quad \text{Tr} \left( \bar{\lambda}_1 2J^{-2} D_i (\bar{\lambda}_1 \lambda_1) \right)$$ (5.6)

We argue the four bosonic states with $H = \sqrt{2}$ are obtained by substituting one of the scalars $(\bar{\lambda}_1 \lambda_1)$ with one of the Goldstone bosons of the symmetries that were broken in $SU(3) \rightarrow SU(2) \times U(1)$. The four generators of these broken symmetries give rise to the operators

$$\text{Tr} \left( \bar{\lambda}_1 2J^{-2} (\bar{\lambda}_1 \lambda_2 + \bar{\lambda}_2 \lambda_1) \right) \quad \text{Tr} \left( \bar{\lambda}_1 2J^{-2} (\bar{\lambda}_1 \lambda_3 + \bar{\lambda}_3 \lambda_1) \right)$$

$$\text{Tr} \left( i\bar{\lambda}_1 2J^{-2} (\bar{\lambda}_1 \lambda_2 - \bar{\lambda}_2 \lambda_1) \right) \quad \text{Tr} \left( i\bar{\lambda}_1 2J^{-2} (\bar{\lambda}_1 \lambda_3 - \bar{\lambda}_3 \lambda_1) \right)$$ (5.7)

This identification is strengthened by the following consideration: if we substitute a $\bar{\lambda}_1$ field with a $\bar{\lambda}_2$ or $\bar{\lambda}_3$ fermion, we are subtracting a $2/3$ charge and adding a $-1/3$ one. Since these three fermions have the same $\Delta$, if we naively add up the charges and dimension of the new state we find that, because of (5.3), $H = \sqrt{2}$.

The fermionic states should be built by adding a fermion to one of the states. The six states with $H = 1/\sqrt{2}$ could be realized by adding to (5.5) one of the fermions $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$. Each of them has two degrees of freedom, since they are Weyl fermions, giving a total of six states. If this guess will prove to be right, the other two states with $H = 3/\sqrt{2}$ are to be expected to come from the insertion of a $\bar{\lambda}_4$ (or the corresponding combination of fields in the effective theory). When we integrate it out in the IR fixed point, this field is presumably substituted by a trilinear in the other three fermions (it is the easiest way to build a state with $J = 0$), giving an idea of why the energy of the two states $\text{Tr} \left( \bar{\lambda}_1 2J \bar{\lambda}_4 \right)$ should be three times the energy of the states $\text{Tr} \left( \bar{\lambda}_1 2J \bar{\lambda}_k \right)$. 

20
We define $\lambda_{\text{eff}} = \frac{\lambda}{J^2}$. The stringy operators are probably obtained from the $n = 0$ ones by adding phases as in [3]. The dimensions have a common zero–point contribution

$$(\Delta - J)_0 = \frac{E_0}{m} \sim \frac{1}{\sqrt{\lambda_{\text{eff}}}} \quad (5.8)$$

From (4.53), (5.3) and $R^4 = 4\pi gN\alpha'^2$, the perturbative expansion for the dimension of the single impurity operators reads

$$(\Delta - J)^{B,0}_n = \sqrt{1 + \frac{\pi n^2 \lambda}{J^2}} = 1 + \frac{\pi}{4} n^2 \lambda_{\text{eff}} + O(\lambda_{\text{eff}}^2)$$

$$(\Delta - J)^{B,-}_n = \sqrt{2} \sqrt{1 - \sqrt{\frac{\pi n \sqrt{\lambda}}{2J} + \frac{\pi n^2 \lambda}{4J^2}}} = \sqrt{2} \left(1 - n \sqrt{\frac{\pi}{8} \sqrt{\lambda_{\text{eff}}} + \frac{\pi}{16} n^2 \lambda_{\text{eff}} + O(\lambda_{\text{eff}}^{3/2})}\right)$$

$$(\Delta - J)^{B,+}_n = \sqrt{2} \sqrt{1 + \sqrt{\frac{\pi n \sqrt{\lambda}}{2J} + \frac{\pi n^2 \lambda}{4J^2}}} = \sqrt{2} \left(1 + n \sqrt{\frac{\pi}{8} \sqrt{\lambda_{\text{eff}}} + \frac{\pi}{16} n^2 \lambda_{\text{eff}} + O(\lambda_{\text{eff}}^{3/2})}\right)$$

$$(\Delta - J)^{F,0}_n = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\pi n^2 \lambda}{J^2}} = \frac{1}{\sqrt{2}} \left(1 + \frac{\pi}{2} n^2 \lambda_{\text{eff}} + O(\lambda_{\text{eff}}^2)\right)$$

$$(\Delta - J)^{F,-}_n = \sqrt{\frac{5}{2} - 2} \sqrt{1 + \frac{\pi n^2 \lambda}{2J^2} + \frac{\pi n^2 \lambda}{2J^2}} = \frac{1}{\sqrt{2}} \left(1 + \frac{\pi^2}{16} n^4 \lambda_{\text{eff}}^2 + O(\lambda_{\text{eff}}^3)\right)$$

$$(\Delta - J)^{F,+}_n = \sqrt{\frac{5}{2} + 2} \sqrt{1 + \frac{\pi n^2 \lambda}{2J^2} + \frac{\pi n^2 \lambda}{2J^2}} = \frac{3}{\sqrt{2}} \left(1 + \frac{\pi}{9} n^2 \lambda_{\text{eff}} + O(\lambda_{\text{eff}}^2)\right) \quad (5.9)$$

Some comments are in order. First of all we notice that the zero–point contribution to $\Delta - J$ is divergent in the perturbative limit. $(\Delta - J)_0$ is a constant common to all operators, and perturbation theory should still allow us to calculate the difference between the dimension of an operator and that of the vacuum. We also find that in the expansions for the $H = \sqrt{2}$ scalars the first contribution is of order $\sqrt{\lambda_{\text{eff}}}$. This seems to be an original feature of our model, and from [23] we believe it suggests that the quantity $e^{i\frac{2\pi}{J}} - e^{-i\frac{2\pi}{J}}$ should appear in the leading coefficient of the perturbation expansion, indicating that moving an impurity in one direction or the other, in Feynman graphs, should give different contributions. It must also be noticed that the dimension of two fermionic modes doesn’t get corrections from one–loop graphs, and its expansion starts at the second order. This same feature was found in [13].

We leave for future work a firmer analysis of the gauge theory.

**Acknowledgements**

We would like to thank Luciano Girardello and Alberto Zaffaroni for suggesting the problem under study and for useful discussions. This work was partially supported by
Appendix: $\Gamma$ matrices and conventions

The metric signature is $\eta^{MN} = (-, +, +, \ldots, +)$. When we write antisymmetric forms in component notation we use the following normalization

$$\omega_p = \frac{1}{p!} \omega_1 \ldots \omega_p \, dx^1 \wedge \ldots \wedge dx^p$$

We adopt the same conventions as [5] for $\Gamma$–matrices and indicate with $\Gamma^M$ the $32 \times 32$–component gamma–matrices, and with $\tilde{\gamma}^M$ the $16 \times 16$–component ones.

$$\Gamma^M = \begin{pmatrix} 0 & \tilde{\gamma}^M \\ \tilde{\gamma}^M & 0 \end{pmatrix}$$

(A.1)

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} \tilde{\gamma}^M \tilde{\gamma}^N + \tilde{\gamma}^N \tilde{\gamma}^M = 2\eta^{MN}$$

(A.2)

$$\tilde{\gamma}^M = (1, \tilde{\gamma}^i, \tilde{\gamma}^9) \quad \tilde{\gamma}^M = (-1, \tilde{\gamma}^i, \tilde{\gamma}^9)$$

(A.3)

where $M, N = 0, 1, 2, \ldots, 9$ and $i, j = 1, 2, 3, \ldots, 8$.

We adopt the Majorana representation, $C = \Gamma^0$, so we can choose all $\tilde{\gamma}^M$ to be real and symmetric, and assume the normalization

$$\Gamma_{11} \equiv \Gamma^0 \Gamma^1 \ldots \Gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(A.4)

We define $\Gamma^{M_1 \ldots M_p}$ as the antisymmetrized product of $\Gamma$ matrices with the same normalization as forms

$$\Gamma^{M_1 \ldots M_p} = \frac{1}{p!} \varepsilon_{M_1 M_2 \ldots M_p} \Gamma^{M_1} \Gamma^{M_2} \ldots \Gamma^{M_p} = \Gamma^{M_1} \Gamma^{M_2} \ldots \Gamma^{M_p}$$

where the last equation is valid if and only if all the indices are different (otherwise the matrix equals zero).

32–→ 8–component spinor decomposition

Because of the normalization we chose for $\Gamma_{11}$ and the condition on the space–time GS spinors: $\Gamma_{11} \theta^I = \theta^I$, the 32–component spinor has only 16 non–zero components

$$\theta^I = \begin{pmatrix} \tilde{\theta}^I \\ 0 \end{pmatrix}$$

(A.5)
where $\tilde{\theta}^I$ is a 16–component Majorana spinor. Moreover, not all of the components of $\tilde{\theta}^I$ are physical degrees of freedom, since we still have to take into account the light–cone gauge. We define

$$\Gamma^+ \equiv \frac{\Gamma^0 + \Gamma^9}{2} = \begin{pmatrix} 0 & \tilde{\gamma}^+ \\ \tilde{\gamma}^- & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^- \equiv \frac{\Gamma^0 - \Gamma^9}{2} = \begin{pmatrix} 0 & \tilde{\gamma}^- \\ \tilde{\gamma}^+ & 0 \end{pmatrix} \quad (A.6)$$

but now

$$\Gamma^+ \theta = \theta \quad \Rightarrow \quad \Gamma^0 \theta = \begin{pmatrix} 0 \\ -\theta \end{pmatrix} \quad \text{and} \quad \Gamma^9 \theta = \begin{pmatrix} 0 \\ \tilde{\gamma}^9 \tilde{\theta} \end{pmatrix} \quad (A.7)$$

thus

$$\Gamma^+ \theta = 0 \quad \Leftrightarrow \quad \tilde{\gamma}^+ \tilde{\theta} = 0 \quad \Leftrightarrow \quad \gamma^9 \tilde{\theta} = \tilde{\theta}$$

$$\Gamma^- \theta = 0 \quad \Leftrightarrow \quad \tilde{\gamma}^- \tilde{\theta} = 0 \quad \Leftrightarrow \quad \tilde{\gamma}^9 \tilde{\theta} = -\tilde{\theta} \quad (A.8)$$

If we decompose the 16–component spinors as $\tilde{\theta} = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$ then

$$\chi^+ = 0 \quad \Leftrightarrow \quad \tilde{\gamma}^+ \tilde{\theta} = 0 \quad \text{and} \quad \chi^- = 0 \quad \Leftrightarrow \quad \tilde{\gamma}^+ \tilde{\theta} = 0 \quad (A.9)$$

and we can represent $\tilde{\gamma}^9$ as $\begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}$.

The light–cone and chirality conditions are thus equivalent in our representation to imposing that only the first 8 components (which constitute a spinor $S$ on their own) of a 32–component spinor are non–zero. A representation of the $SO(8)$ algebra of gamma–matrices $\gamma^i$ can be built on these spinors (we will do this in the next subsection), and in particular it is found that for $i_1, \ldots, i_p = 1, \ldots, 8$

$$\Gamma^{i_1 \cdots i_p} \theta \rightarrow \gamma^{i_1 \cdots i_p} S \quad (A.10)$$

$$\Gamma^0 \theta \rightarrow -S \quad \Gamma^9 \theta \rightarrow S \quad (A.11)$$

In particular

$$\gamma^{12 \cdots 8} S = S \quad (A.12)$$

but since $(\gamma^{1234})^2 = 1$ we have

$$\gamma^{1234} S = \gamma^{5678} S \quad (A.13)$$

**$SO(8)$ gamma–matrices algebra representation on 8–component spinors**

To build a realization of the $SO(8)$ gamma–matrices algebra on the 8–component spinors we define the combinations $(j = 1, 2, 3, 4)$

$$a_j = \frac{1}{2} (\gamma^{2j-1} - i \gamma^{2j})$$

$$a^+_j = \frac{1}{2} (\gamma^{2j-1} + i \gamma^{2j}) \quad (A.14)$$
Because of the algebra \( \{ \gamma^i, \gamma^j \} = 2 \delta^{ij} \) the operators which we just defined realize a Clifford algebra

\[
\{ a_j^\dagger, a_k \} = \delta_{jk} \quad \{ a_j^\dagger, a_k^\dagger \} = 0 \quad \{ a_j, a_k \} = 0
\]

and in particular \( a_j^\dagger a_j^\dagger = a_j a_j = 0 \). We can interpret the \( a_j \) and \( a_j^\dagger \) as creation and annihilation operators on an 8–dimensional vector space. We define the vacuum \( |0\rangle \) as the state which is annihilated by all the \( a_j \) operators and build vectors by applying the \( a_j^\dagger \) operators on it and represent spinors as

\[
\begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6 \\
S_7 \\
S_8
\end{pmatrix} =
\begin{pmatrix}
|0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle \\
\begin{pmatrix} a_j^\dagger a_j^\dagger \end{pmatrix} |0\rangle
\end{pmatrix}
\]

(B.16)

\[ds^2 = -4 e^+ e^- + \sum_{i=1}^8 (e^i)^2 \]

We find that the only non–zero components of the spin connection \( \omega_M^{NP} \) are

\[\omega^{-i} = -\omega^{i-} = \frac{1}{2} A_{ii} x^i \, dx^+\]
From (3.6)

\[ \mathcal{G} = 2e^{i\alpha} (\Gamma^5 + i\Gamma^6) (\Gamma^7 + i\Gamma^8) \Gamma^+ \]

\[ \mathcal{F} = -\frac{1}{\sqrt{2}} (\Gamma^{1234} + \Gamma^{5678}) \Gamma^+ \]

(B.5)

The equation involving the variation of the dilatino is easy to solve and gives

\[ (1 + i\Gamma^{56}) (1 + i\Gamma^{78}) (1 - \Gamma^{09}) \epsilon = 0 \]

(B.6)

If we represent the spinors in a basis of eigenvectors of the Lorentz operators \{\Gamma^{09}, i\Gamma^{12}, i\Gamma^{34}, i\Gamma^{56}, i\Gamma^{78}\}, then a generic spinor can be written as \((\pm, \pm, \pm, \pm, \pm)\). The solutions of (B.6) are

1. \((+, \pm, \pm, \pm, \pm)\)
2. \((-+, \pm, \pm, \pm, \pm)\)
3. \((-+, \pm, \pm, +, -)\)
4. \((-+, \pm, \pm, -, +)\)

(B.7)

The condition \(\Gamma_{11} \epsilon = \epsilon\) imposes, moreover, that there be an even number of "\(-\)" eigenvalues. Not all of these solutions are supersymmetries of the background: the more involved gravitino equation must also be satisfied. Because of (B.1), (B.4) and \((\Gamma^+)^2 = 0\) it is found that

\[ \partial_- \epsilon = 0 \]

\[ \partial_i \epsilon = -i\Omega_i \epsilon - i\Lambda_i \epsilon \]

(B.8)

where

\[ \Omega_i \equiv -\frac{1}{4\sqrt{2}} \Gamma^+ (\Gamma^{1234} + \Gamma^{5678}) \Gamma_i \]

\[ \Lambda_i \equiv \frac{i}{8} e^{i\alpha} (2 (\Gamma^5 + i\Gamma^6) (\Gamma^7 + i\Gamma^8) \Gamma^+ \Gamma_i + \Gamma^i (\Gamma^5 + i\Gamma^6) (\Gamma^7 + i\Gamma^8) \Gamma^+) \]

Again because of \((\Gamma^+)^2 = 0\) we have that \(\Omega_i \Omega_j = \Omega_i \Lambda_j = \Lambda_i \Lambda_j = 0\) for any \(i, j = 1, \ldots, 8\) and then \(\partial_i \partial_j \epsilon = 0\): \(\epsilon\) can only depend linearly on the \(x^i\)’s

\[ \epsilon = \chi - i \sum_{j=1}^{8} x^j (\Omega_j \chi + \Lambda_j \chi^*) = \]

\[ = \left( 1 - i \sum_{j=1}^{8} x^j \Omega_j \right) \chi - i \sum_{j=1}^{8} x^j \Lambda_j^* \]

(B.9)

where \(\chi = \chi(x^+)^\alpha\) is a positive chirality spinor to be determined via the \(\delta \psi_+ = 0\) equation and (B.6). Substitution of (B.9) into (B.6) gives

\[ (1 + i\Gamma^{56}) (1 + i\Gamma^{78}) (1 - \Gamma^{09}) \chi = 0 \]

(B.10)
thus \( \chi \) will be one of (B.7). Substituting (B.9) into \( \delta \psi_+ = 0 \) and setting the constant term and the 8 terms linear in \( x^i \) separately equal to zero, we find that the following equations must be solved

\[
a) \quad \partial_+ \chi = -\frac{i}{2\sqrt{2}} \left( \Gamma^{1234} + 5^{5678} \right) \Gamma^+ \Gamma^- \chi + \frac{1}{4} e^{i\alpha} \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \left( 1 - \Gamma^+ \Gamma^- \right) \chi^* \\
b) \quad \Omega_j (\partial_+ \chi) + \Lambda_j (\partial_+ \chi^*) + \frac{i}{2} A_{jj} \Gamma_j \Gamma^+ \chi - \frac{i}{2\sqrt{2}} \left( \Gamma^{1234} + 5^{5678} \right) \left( \Omega_j \chi + \Lambda_j \chi^* \right) + \\
+ \frac{1}{2} e^{i\alpha} \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \left( \Omega_j \chi^* + \Lambda^*_j \chi \right) = 0
\]

(B.11)

We then substitute \( a \) into \( b \), and find that \( \chi \) must satisfy

\[
\frac{1}{2} e^{i\alpha} \Omega_j \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \chi^* + \frac{1}{2} e^{-i\alpha} \Lambda_j \left( \Gamma^5 - i\Gamma^6 \right) \left( \Gamma^7 - i\Gamma^8 \right) \chi + i A_{jj} \Gamma_j \Gamma^+ \chi + \\
- \frac{i}{\sqrt{2}} \left( \Gamma^{1234} + 5^{5678} \right) \left( \Omega_j \chi + \Lambda_j \chi^* \right) + e^{i\alpha} \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \left( \Omega_j \chi^* + \Lambda^*_j \chi \right) = 0
\]

(B.12)

We notice that if \( \Gamma^+ \chi = 0 \), which corresponds to the first of (B.7), then equation (B.12) is satisfied: our solution has at least 16 supersymmetries. Let us now consider cases (2)-(4): after some gamma–matrices algebra, we rewrite (B.12) for \( j = 1, 2, 3, 4 \) as

\[
i \left( A_{jj} + \frac{1}{2} \right) \Gamma^+ \Gamma^j \chi = -\frac{i}{16} \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \Gamma^+ \Gamma^j \left( \Gamma^5 - i\Gamma^6 \right) \left( \Gamma^7 - i\Gamma^8 \right) \chi + \\
+ \frac{e^{i\alpha}}{4\sqrt{2}} \left( \Gamma^{1234} + 5^{5678} \right) \Gamma^+ \Gamma^j \left( \Gamma^5 + i\Gamma^6 \right) \left( \Gamma^7 + i\Gamma^8 \right) \chi^* \quad \text{ (B.13)}
\]

The left–hand side can never be zero, because we are considering the case in which \( \Gamma^+ \chi \neq 0 \), thus there can be a solution to this equation only if the right–hand side of the equation doesn’t vanish. Let us consider the three cases (2)-(4) of (B.7)

\[
(2) \rightarrow \begin{cases} \Gamma^5 - i\Gamma^6 \neq 0 \\ \Gamma^7 - i\Gamma^8 \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} (\Gamma^5 + i\Gamma^6) \chi^* \neq 0 \\ (\Gamma^7 + i\Gamma^8) \chi^* \neq 0 \end{cases}
\]

(3) \rightarrow \begin{cases} (\Gamma^5 - i\Gamma^6) \chi = 0 \\ (\Gamma^7 - i\Gamma^8) \chi \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} (\Gamma^5 + i\Gamma^6) \chi^* = 0 \\ (\Gamma^7 + i\Gamma^8) \chi^* \neq 0 \end{cases}

(4) \rightarrow \begin{cases} (\Gamma^5 - i\Gamma^6) \chi \neq 0 \\ (\Gamma^7 - i\Gamma^8) \chi = 0 \end{cases} \quad \text{and} \quad \begin{cases} (\Gamma^5 + i\Gamma^6) \chi^* \neq 0 \\ (\Gamma^7 + i\Gamma^8) \chi^* = 0 \end{cases}

\footnote{This is a general result for type IIB superstrings on a pp–wave [2].}
Thus we see that (3) and (4) cannot be solutions of (B.13). The only possible solution is (2). The first term on the right-hand side of (B.13) evaluated on $\chi = (-, \pm, \pm, -, -)$ gives $i \Gamma^+ \Gamma^j \chi$. As we already mentioned the condition $\Gamma_{11} \chi = \chi$ implies that either one of the two $\pm$'s must be a $-$ and the other a $+$. It follows then that $\Gamma^{1234} \chi = \chi$, while $\Gamma^{5678} \chi = -\chi$ and $\left( \Gamma^{1234} + \Gamma^{5678} \right) \chi = 0$. The second term on the right-hand side of (B.13) becomes $2\sqrt{2}e^{i\alpha} \Gamma^+ \Gamma^j \Gamma^{57} \chi\ast$, and equation (B.13) reads ($A_{jj} = 1$ for $i, j = 1, 2, 3, 4$)

$$\chi = -2\sqrt{2}i e^{i\alpha} \Gamma^{57} \chi\ast$$

which has no solutions.

References

[1] R. Gueven, Plane Wave Limits and T-Duality, Phys. Lett. B482 (2000) 255; hep-th/0005061.

[2] M. Blau, J. Figueroa-O’Farrill, C. Hull, G. Papadopoulos, A new maximally supersymmetric background of IIB superstring theory, JHEP 0201 (2002) 047; hep-th/0110242. Penrose limits and maximal supersymmetry, hep-th/0201081. M. Blau, J. Figueroa-O’Farrill, G. Papadopoulos, Penrose limits, supergravity and brane dynamics; hep-th/0202111.

[3] D. Berenstein, J. M. Maldacena and H. Nastase, Strings in flat space and pp waves from $N = 4$ super Yang Mills, JHEP 0204, 013 (2002); hep-th/0202021.

[4] R. Penrose, Any spacetime has a plane wave as a limit, Differential geometry and relativity, Reidel, Dordrecht, 1976, pp.271-275.

[5] R. R. Metsaev, Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background, Nucl. Phys. B 625, 70 (2002); hep-th/0112044.

[6] R. R. Metsaev and A. A. Tseytlin, Exactly solvable model of superstring in plane wave Ramond-Ramond background, Phys. Rev. D 65, 126004 (2002); hep-th/0202109.

[7] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics, JHEP 9812, 022 (1998); hep-th/9810126.

[8] J. Distler and F. Zamora, Non-supersymmetric conformal field theories from stable anti-de Sitter spaces, Adv. Theor. Math. Phys. 2, 1405 (1999); hep-th/9810206.
[9] J. Distler and F. Zamora, *Chiral symmetry breaking in the AdS/CFT correspondence*, JHEP **0005**, 005 (2000); [hep-th/9911040](https://arxiv.org/abs/hep-th/9911040).

[10] L. J. Romans, *New Compactifications Of Chiral N=2 D = 10 Supergravity*, Phys. Lett. B **153**, 392 (1985).

[11] R. Corrado, N. Halmagyi, K. D. Kennaway and N. P. Warner, *Penrose limits of RG fixed points and pp-waves with background fluxes*, [hep-th/0205314](https://arxiv.org/abs/hep-th/0205314).

[12] E. G. Gimon, L. A. Pando Zayas and J. Sonnenschein, *Penrose limits and RG flows*, [hep-th/0206033](https://arxiv.org/abs/hep-th/0206033).

[13] D. Brecher, C. V. Johnson, K. J. Lovis and R. C. Myers, *Penrose limits, deformed pp-waves and the string duals of N = 1 large N gauge theory*, [hep-th/0206045](https://arxiv.org/abs/hep-th/0206045).

[14] M. Gunaydin, L. J. Romans and N. P. Warner, *Compact And Noncompact Gauged Supergravity Theories In Five-Dimensions*, Nucl. Phys. B **272**, 598 (1986).

[15] P. Breitenlohner and D. Z. Freedman, *Positive Energy In Anti-De Sitter Backgrounds And Gauged Extended Supergravity*, Phys. Lett. B **115**, 197 (1982).

[16] A. Khavaev, K. Pilch and N. P. Warner, *New vacua of gauged N = 8 supergravity in five dimensions*, Phys. Lett. B **487**, 14 (2000); [hep-th/9812035](https://arxiv.org/abs/hep-th/9812035).

[17] K. Pilch and N. P. Warner, *A new supersymmetric compactification of chiral IIB supergravity*, Phys. Lett. B **487**, 22 (2000); [hep-th/0002192](https://arxiv.org/abs/hep-th/0002192).

[18] J. G. Russo and A. A. Tseytlin, *On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds*, JHEP **0204**, 021 (2002); [hep-th/0202179](https://arxiv.org/abs/hep-th/0202179).

[19] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, *The Mass Spectrum Of Chiral N=2 D = 10 Supergravity On S**5**, Phys. Rev. D **32**, 389 (1985).

[20] N. Itzhaki, I. R. Klebanov and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, JHEP **0203**, 048 (2002); [hep-th/0202153](https://arxiv.org/abs/hep-th/0202153).

[21] F. Bigazzi, A. L. Cotrone, L. Girardello and A. Zaffaroni, *pp-wave and nonsupersymmetric gauge theory*, [hep-th/0205296](https://arxiv.org/abs/hep-th/0205296).

[22] S. Frolov and A. A. Tseytlin, *Semiclassical quantization of rotating superstring in AdS(5) x S**5*, JHEP **0206**, 007 (2002); [hep-th/0204226](https://arxiv.org/abs/hep-th/0204226).

[23] D. J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in N = 4 Yang-Mills theory*, [hep-th/0205066](https://arxiv.org/abs/hep-th/0205066).
[24] S. S. Gubser, *Supersymmetry and F-theory realization of the deformed conifold with three-form flux*, hep-th/0010010.