On the density matrix for the kink ground state of higher spin XXZ chain

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The exact computation of the correlation functions of 1D quantum integrable models has been one of the challenging problems. For the spin-1/2 XXZ chain, the correlation functions in the antiferromagnetic regime were found to be expressed in the multiple integral form. However, the exact evaluation of them is still a hard work, and has been only successful for some special anisotropy parameters.

In the ferromagnetic regime, there is a class of non-translationally invariant ground state which should be called as kink ground state. We have studied the correlation function of the kink ground state, and exactly calculated the density matrix (see also ref 7).

The kink ground state also exists for arbitrary spin and dimension. In this paper, we extend the analysis developed in our previous paper to higher spin 1D XXZ chain, and calculate the density matrix (note that the model we consider in this paper is not the integrable higher spin XXZ chain).

The Hamiltonian of the spin S 1D infinite XXZ chain is

\[ H = -\sum_{m \in \mathbb{Z}} \left( S^x_m S^x_{m+1} + S^y_m S^y_{m+1} + \Delta (S^z_m S^z_{m+1} - S^2) \right). \]

We shall consider the ferromagnetic regime \( \Delta > 1 \). For later convenience, we parametrize \( \Delta \) as \( \Delta = (q^{1/2} + q^{-1/2})/2 \). Then \( \Delta > 1 \) corresponds to \( 0 < q < 1 \). A kink ground state is the superposition of kinks which have the same center. For a (normalized) kink state \( \otimes_{x \in \mathbb{Z}} |m_x\rangle \) (\( m_x \in \{-S, \cdots, S\} \)), the center \( j - 1/2 \) (\( j \in \mathbb{Z} \)) is the position where

\[ \sum_{x < j} (S - m_x) = \sum_{x > j} (m_x + S), \]

holds. Let us denote the kink ground state whose center is at \( j - 1/2 \) by \( |\Psi_j\rangle \), and introduce the generating function of \( |\Psi_j\rangle \)

\[ |\Psi(z)\rangle = \bigotimes_{x \in \mathbb{Z}_{<0}} (\sum_{m_x = -S}^S (zq^{(1/2)x})^{m_x+S} c(m_x)^{1/2} |m_x\rangle) \]

where \( c(m_x) = (2S)!/(S - m_x)!/|S + m_x|! \). |\Psi_j\rangle is the coefficient of \( z^j \) of the expansion of \( |\Psi(z)\rangle \), i.e,

\[ |\Psi(z)\rangle = \sum_{j \in \mathbb{Z}} z^j |\Psi_j\rangle. \]

Let us focus on one of the kink ground states \(|\Psi_0\rangle\), and calculate the density matrix

\[ \langle \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi(z) \rangle := \frac{\langle \Psi_0 | \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}, \]

where \( E^{r_j s_j}_{x_j} | m_x \rangle = \delta_{m_x,x_j} | r_j \rangle \). \( x_j \) is the position of the site where the operator \( E^{r_j s_j}_{x_j} \) acts on, and is assumed to be \( x_j \neq x_k \) for \( j \neq k \). We only consider the case when \( \sum_{j=1}^n (\epsilon_j - \epsilon'_j) = 0 \) is satisfied: otherwise, the density matrix is zero. We calculate \( \langle \Psi(z) | \Psi(z) \rangle \) and \( \langle \Psi(z) \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi(z) \rangle \) to obtain the exact expression of the density matrix since

\[ \langle \Psi(z) | \Psi(z) \rangle = \sum_{j = -\infty}^{\infty} z^{2j} \langle \Psi_j | \Psi_j \rangle, \]

\[ \langle \Psi(z) \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi(z) \rangle = \sum_{j = -\infty}^{\infty} z^{2j} \langle \Psi_j | \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi_j \rangle, \]

holds, and \( \langle \Psi_0 | \Psi_0 \rangle, \langle \Psi_0 | \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi_0 \rangle \) can be extracted from \( \langle \Psi(z) | \Psi(z) \rangle \) and \( \langle \Psi(z) \prod_{j=1}^n E^{r_j s_j}_{x_j} | \Psi(z) \rangle \), which are easier to calculate. Let us first calculate \( \langle \Psi(z) | \Psi(z) \rangle \). Using the Jacobi triplet product identity

\[ (q;q)_\infty (-x q^{1/2}; q)_\infty (-x^{-1} q^{1/2}; q)_\infty = \sum_{j = -\infty}^{\infty} x^j q^{j^2}, \]

we have

\[ \langle \Psi(z) | \Psi(z) \rangle = (w q^{1/2} q^{1/2})^{2S} (w^{-1} q^{1/2} q^{1/2})^{2S} = \frac{1}{(q; q)_\infty^{2S}} \left( \sum_{j = -\infty}^{\infty} w^j q^{j^2} \right)^{2S} = \frac{1}{(q; q)_\infty^{2S}} \sum_{j = -\infty}^{\infty} A_j w^j, \]

where \( w = z^2 \), \( (a; q)_\infty := \prod_{j=0}^{\infty} (1 - a q^j) \) and

\[ A_j := \sum_{j_k = j, j_k \in \mathbb{Z}} \prod_{k=1}^{2S} \frac{q^{j_k^2}}{q^{j_k^2}}, \]
Using eq. (7) and the following identity\(^6\)
\[
\prod_{j=1}^{n} \frac{1}{1+ x w_{ij}} = \sum_{j=0}^{\infty} (-x)^j \prod_{i=1}^{n} \frac{u_{i}^{j+n-1}}{u_{i} - u_{i}^{j}},
\]
(9)
\[
\langle \Psi(z) | \prod_{j=1}^{n} E_{x}^{j} \rangle = \sum_{j=0}^{\infty} (-x)^j \prod_{i=1}^{n} \frac{u_{i}^{j+n-1}}{u_{i} - u_{i}^{j}},
\]
\[
\langle \Psi(z) | \prod_{j=1}^{n} E_{x}^{j} \rangle = \prod_{j=1}^{n} \langle c(\epsilon_j) c(\epsilon_j') \rangle \prod_{j=1}^{n} \frac{1}{(1 + w\zeta_j)^{2S}}
\]
\[
\times (-wq^{\frac{1}{2}} q^{2S} (-w^{-1} q^{\frac{1}{2}} q^{2S})
\]
\[
= \prod_{j=1}^{\infty} \prod_{j=1}^{\infty} \langle c(\epsilon_j) c(\epsilon_j') \rangle \prod_{j=1}^{n} \frac{1}{(q; q)^{2S}}
\]
\[
\times \sum_{j=-\infty}^{\infty} \left\{ \sum_{j_1 + j_2 = j} \prod_{k=1}^{2S} \left( (-1)^{j_1} \prod_{l=1}^{n} \frac{\zeta_{j_k + n} - \zeta_{j_l}}{\zeta_{j_k} - \zeta_{j_l}} \right) \right\}.
\]
(10)
where \( \zeta_j = q^{\frac{1}{2} + x_j} \), \( \epsilon_j' = \epsilon_j + q + 2S \) and
\[
B_j = \sum_{j_1 + j_2 = j} \prod_{k=1}^{2S} \left( \frac{\zeta_{j_k + n} - \zeta_{j_l}}{\zeta_{j_k} - \zeta_{j_l}} \right).
\]
From eqs. (8) and (10), one has
\[
\langle \Psi_0 | \Psi \rangle = \frac{A_0}{(q; q)^{2S}},
\]
(11)
\[
\langle \Psi_0 | \prod_{j=1}^{n} E_{x}^{j} \rangle = \prod_{j=1}^{n} \langle c(\epsilon_j) c(\epsilon_j') \rangle \prod_{j=1}^{n} \frac{1}{(q; q)^{2S}}
\]
\[
\times \sum_{j_1 + j_2 = j} \prod_{k=1}^{2S} \left( (-1)^{j_1} \prod_{l=1}^{n} \frac{\zeta_{j_k + n} - \zeta_{j_l}}{\zeta_{j_k} - \zeta_{j_l}} \right),
\]
(12)
leading to the exact expression of the density matrix
\[
\prod_{j=1}^{n} \langle E_{x}^{j} \rangle = \prod_{j=1}^{n} \langle c(\epsilon_j) c(\epsilon_j') \rangle \prod_{j=1}^{n} \frac{1}{(q; q)^{2S}}
\]
\[
\times \sum_{j_1 + j_2 = j} \prod_{k=1}^{2S} \left( (-1)^{j_1} \prod_{l=1}^{n} \frac{\zeta_{j_k + n} - \zeta_{j_l}}{\zeta_{j_k} - \zeta_{j_l}} \right).
\]
(13)
One can check that for \( S = 1/2 \), eq. (13) recovers the result in ref. 6. However, the expression of eq. (13) gets more complicated as the spin becomes higher.

Concentrating on the \( S = 1 \) case, we can derive some slightly easier expression by using
\[
\prod_{j=1}^{n} \frac{1}{(1 + x w_j)^2} = \sum_{j=0}^{\infty} (-x)^j X_{n,j},
\]
\[
X_{n,j} = \lim_{u_{i+n} - u_{i+1} \to 1} \sum_{i=1}^{2n} \prod_{j=1}^{n} \frac{u_{i}^{j+n-1}}{u_{i} - u_{i}^{j}},
\]
which is a special case of eq. (9). The result is
\[
\prod_{j=1}^{n} E_{x}^{j} = \prod_{j=1}^{n} \langle c(\epsilon_j) c(\epsilon_j') \rangle \prod_{j=1}^{n} \frac{1}{(q; q)^{2S}} \times \sum_{j=0}^{\infty} (-1)^j X_{n,j} q^{(j + \sum_{k=1}^{n} \epsilon_k)^2}
\]
\[
\times \delta_{\text{even}}^j \left( \sum_{k=1}^{n} \sum_{\epsilon_k = 1}^{|j|} C \delta_{\text{odd}}^j \right),
\]
(14)
where \( C = 2 \sum_{k=1}^{n} q^{(k-1/2)^2} / (1 + 2 \sum_{k=1}^{n} q^{2k}) \). From eq. (14), the magnetization and the spin-spin correlation functions can be easily calculated.
\[
\langle S^z_x \rangle = \sum_{j=0}^{\infty} (-1)^j (j + 1) q^{2j} \langle \zeta_j^2 \rangle
\]
\[
\times \delta_{\text{even}}^j \left( \sum_{k=1}^{n} \sum_{\epsilon_k = 1}^{|j|} C \delta_{\text{odd}}^j \right),
\]
(15)
\[
\langle S^z_x, S^z_x \rangle = \sum_{j=0}^{\infty} (-1)^j X_{2,j} \delta_{\text{even}}^j + C \delta_{\text{odd}}^j
\]
\[
\times \delta_{\text{even}}^j \left( \sum_{k=1}^{n} \sum_{\epsilon_k = 1}^{|j|} C \delta_{\text{odd}}^j \right),
\]
(16)
\[
\langle S^x_x, S^x_x \rangle = 4 \zeta_1 \zeta_2 \sum_{j=0}^{\infty} (-1)^j X_{2,j}
\]
\[
\times \delta_{\text{even}}^j \left( \sum_{k=1}^{n} \sum_{\epsilon_k = 1}^{|j|} C \delta_{\text{odd}}^j \right),
\]
(17)
where
\[
X_{2,j} = \frac{(j + 1)(\zeta_1^3 - \zeta_2^3) - (j + 3) \zeta_1 \zeta_2 (\zeta_1^3 - \zeta_2^3)}{(\zeta_1 - \zeta_2)^3}.
\]
From these expressions, one can easily show that the spin-spin correlation functions decay exponentially for large distances.
\[
\langle S^z_x, S^z_x \rangle - \langle S^z_x \rangle \langle S^z_x \rangle \sim A^{zz}(x_1) q^{x_2 + \frac{1}{2}} \text{ for } x_2 \gg 1,
\]
(18)
\[
A^{zz}(x_1) = 2 \sum_{j=0}^{\infty} (-1)^j q^{2j} \left( 1 - \zeta_1^j q^{j+1} \right)
\]
\[
\times (j \zeta_2^{j+1} + (j + 1) q^j \delta_{\text{even}}^j + C \delta_{\text{odd}}^j),
\]
(19)
\[
\langle S^z_x, S^z_x \rangle - A^{+-}(x_1) q^{x_2 + \frac{1}{2}} \text{ for } x_2 \gg 1,
\]
\[
A^{+-}(x_1) = 4 \zeta_1^j \sum_{j=0}^{\infty} (-\zeta_1^j)(j + 1)
\]
\[ \times \{ (\delta_j^{\text{even}} + C_\delta^{\text{odd}}) \zeta_1 q^{\frac{(j+2)^2}{4}} + (\delta_j^{\text{odd}} + C_\delta^{\text{even}}) q^{\frac{(j+1)^2}{4}} \} . \]

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