Lorentz Breaking and $SU(2)_L \times U(1)_Y$ Gauge Invariance for Neutrino Decays

U. D. Jentschura, I. Nándori, and G. Somogyi

1Department of Physics, Missouri University of Science and Technology, Rolla, Missouri 65409, USA
2MTA–DE Particle Physics Research Group, P.O. Box 51, H–4001 Debrecen, Hungary
3University of Debrecen, P.O.Box 105, H–4010 Debrecen, Hungary
4MTA Atomki, P.O. Box 51, H–4001 Debrecen, Hungary

Conceivable Lorentz-violating effects in the neutrino sector remain a research area of great general interest, as they touch upon the very foundations on which the Standard Model and our general understanding of fundamental interactions is laid. Here, we investigate the relation of Lorentz violation in the neutrino sector in light of the fact that neutrinos and corresponding left-handed charged leptons form $SU(2)_L$ doublets under the electroweak gauge group. Lorentz-violating effects thus cannot be fully separated from questions related to gauge invariance. The model dependence of the effective interaction Lagrangians used in various recent investigations is investigated, with a special emphasis on neutrino splitting, otherwise known as neutrino-pair Cerenkov radiation, NPCR, and vacuum pair emission (electron-positron-pair Cerenkov radiation, LPCR). We investigate two scenarios in which Lorentz violating effects do not necessarily also break electroweak gauge invariance, the first, which involves a restricted set of gauge transformation, a subgroup of $SU(2)_L$, the second, where differential Lorentz violation is exclusively introduced by the mixing of the neutrino flavor and mass eigenstates.

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I. INTRODUCTION

Recently, tight bounds on Lorentz-violating parameters for neutrinos have been derived from astrophysical observation [1–3], based on on the notion that neutrino decay into electron-positron pairs (“lepton-pair Cerenkov radiation”) becomes kinematically allowed under a Lorentz noninvariance of the neutrino dispersion relation. One observes that a slight violation of Lorentz invariance at high energy would lead to a large virtuality of propagating neutrinos, exceeding the electron-positron pair production threshold.

Very recently, these calculations have been supplemented by an analysis of the neutrino splitting process [3] ($\nu \rightarrow \nu \bar{\nu} \nu$), which in contrast to charged-lepton-pair Cerenkov radiation [4, 5] ($\nu \rightarrow \nu e^+ e^-$) has negligible threshold and can serve to set even tighter bounds on the Lorentz-violating parameters.
On the theoretical side, corresponding calculations are mainly based on the notion that Lorentz noninvariance is restricted to the neutrino sector, while the Lorentz-violating parameter $\delta v = v^2 - 1$ is set equal to zero for electrons (and positrons). I.e., one assumes that the maximum attainable velocity for electrons is exactly equal to $v_e = c$, where $c$ is the speed of light [1, 2, 4–6].

One might argue, though, that electrons and neutrinos enter an $SU(2)_L$ doublet, so that in addition to Lorentz violation, also gauge symmetry is violated if one assumes a different propagation velocity for electrons (muons, tauons) and neutrinos in the high-energy limit.

Two models have thus been investigated by Bezrukov and Lee [5] in order to analyze the decay of superluminal neutrinos by electron-positron pair emission (“vacuum pair emission”), one (“model $\Gamma$”), in which the normal Lorentz metric enters the interaction Lagrangian, and another one (“model $\Pi$”), in which the same Lorentz-violating “metric”

$$\tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu} = \text{diag}(1, -v, -v, -v), \quad v > 1,$$

enters the dispersion relation of the decaying neutrino. (The “metric” is noncovariant, hence no distinction between upper and lower indices.) Potentially, this “metric” also enters the effective interaction Lagrangian describing the decay process, where $v \geq 1$ is a Lorentz-violating parameter, which can be different for the maximum velocities of the initial ($v = v_i$) and final ($v = v_f$) particles in the decay process. (In this paper, we have $\hbar = c = \epsilon_0 = 1$.)

In Ref. [3], an even more general approach is taken, and the metric entering the interaction Lagrangian is taken in the form

$$\tilde{g}^{\mu\nu}(v_{\text{int}}) = \text{diag}(1, -v_{\text{int}}, -v_{\text{int}}, -v_{\text{int}}). \quad (2)$$

where $v_{\text{int}}$ is not necessarily equal to $v_i$ or $v_f$. Here, we show in detail how the parameters of the models used by Cohen and Glashow [4], and Bezrukov and Lee [5], and by us in Ref. [3], are related to the gauge invariance under the $SU(2)_L$ group, and how the formulation of the gauge sector relates to the individual Lorentz-violating parameters of the neutrino flavor and mass eigenstates, and those of the charged leptons.

This is important for a general understanding of the relation of potential Lorentz symmetry breaking in the neutrino sector, to fundamental symmetries and to other conceivable non-standard interactions [7]. (Note that corresponding questions do not occur in Lorentz-symmetry conserving models [8].) The underlying question is the following: Is the existence of the LPCR process compatible with electroweak gauge invariance, or, do the tight bounds derived in [1–3] additionally depend on the possibly problematic assumption of a breaking of $SU(2)_L$ symmetry, in addition to Lorentz symmetry? These considerations are quite crucial for the clarification of the status of the derived astrophysical bounds on the Lorentz-violating parameters [1–3].

This paper is organized as follows. In Sec. II, we introduce a modified Dirac algebra, adapted to the description of Lorentz-violating neutrinos, to a generalized Dirac algebra. Our investigations continue in Sec. III with the discussion of a manifestly Lorentz- and gauge-symmetry breaking model for the interaction of (conceivable) superluminal neutrinos with electroweak gauge bosons; this model has recently been used in Refs. [4, 5]. In Sec. IV, we continue with the investigation of a model which breaks Lorentz symmetry and (partially) restores electroweak gauge symmetry, within a restricted electroweak symmetry group. We continue in Sec. V with the discussion of a Lorentz-breaking model which fully restores the electroweak symmetry group. Conclusions are reserved for Sec. VI, and some additional general remarks on the relation of (spontaneous) Lorentz-symmetry breaking and gauge invariance are relegated to Appendix A.

II. RELATION TO MODIFIED DIRAC ALGEBRA

Before we go in medias res, let us briefly discuss a connection of the common Lagrangian used in the description of superluminal, Lorentz-violating neutrinos, to a generalized Dirac algebra. According to Eq. (16) of Ref. [3], one may use, for a Lorentz-violating left-handed Dirac particle,

$$L = i\bar{\psi}\gamma^\mu \tilde{g}^{\mu\nu} \partial_\nu \psi, \quad (3)$$

where we assume that $\psi = \nu^{(m)}$ is a left-handed neutrino mass eigenstates, i.e., $[(1 - \gamma^5)/2] \psi = \psi$. (In the course of the current investigations, we attempt to keep the notation as concise as possible and avoid any superfluous superscripts, or subscripts.) We can write this as

$$L = \bar{\psi} \gamma^\mu \partial_\mu \psi, \quad (4)$$

where the $\tilde{\gamma}^\mu$ are given as

$$\tilde{\gamma}^0 = 1, \quad \tilde{\gamma}^i = v \gamma^i. \quad (5)$$
These fulfill the anti-commutator relation
\[ \{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \} = \tilde{g}^{\mu\nu} = \text{diag}(1, -v^2, -v^2, -v^2) , \] (6)
where we note the square of the velocity. One can in fact relate this formalism to so-called vierbein coefficients (cf. Refs. [9–14]). Namely, in a more general context, one can define the generalized Dirac matrices
\[ \tilde{\gamma}^\mu = e^\mu_A \gamma^A , \] (7)
where the Einstein summation convention is used, and \( \gamma^A \) with \( A = 0, 1, 2, 3 \) are the ordinary Dirac \( \gamma \) matrices, while the \( e^\mu_A \) take the role of the so-called “vierbein” in general relativity, with the property
\[ \tilde{g}^{\mu\nu} = e^A_\mu g_{AB} e^B_\nu = e^A_\mu e^A_\nu . \] (8)
This implies that the “vierbein” takes the role of the square root of the metric [10]. Capital Latin indices can be raised with the flat-space metric \( g^{AB} \). One can then easily show that
\[ \{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \} = e^\mu_A e^\nu_B \{ \gamma^A, \gamma^B \} = e^\mu_A e^\nu_B (2g^{AB}) = 2 \tilde{g}^{\mu\nu} . \] (9)

The analogy to the formalism of general relativity implies that \( \tilde{g}^{\mu\nu} \) takes the role of a modified Lorentz “metric”, but without curvature (because we assume that the coefficients are constant). The word “metric” should be understood with a grain of salt (hence the apostrophes), because it does not constitute a space-time metric in the sense of general relativity, that is used to measure space-time intervals, but rather, a mathematical object used to parameterize the dispersion relation of a Lorentz-violating particle. Because of the lack of curvature, the “metric” \( \tilde{g}_{\mu\nu} \) is still characterizing a flat “space-time”. For a truly curved space, the notation \( g_{\mu\nu} \) has been proposed in Refs. [9–14] in order to distinguish the curved-space quantities from the flat-space ones. (As a remark, we here note that the superluminal, Lorentz-violating neutrino model is different from Lorentz-conserving, \( \gamma^5 \) Hermitian models discussed in the literature [15].)

For a modified “metric” of the form (6), one can choose the vierbein coefficients as
\[ e^0_0 = 1 , \quad e^0_i = e^i_0 = 0 , \quad e^i_j = v \delta^i_j , \quad i, j = 1, 2, 3 . \] (10)

The modified Dirac equation describing the Lorentz violation can then be written as
\[ (i \tilde{\gamma}^\mu \partial_\mu - m) \psi = 0 . \] (11)
We here suppress the chirality projectors and assume that \( \psi \) is a (conceivably left-handed) fermion field. One can multiply from the left by the operator \( (i \tilde{\gamma}^\nu \partial_\nu + m) \), and use the operator identity
\[ (i \tilde{\gamma}^\nu \partial_\nu + m) (i \tilde{\gamma}^\mu \partial_\mu - m) = -\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu - m^2 . \] (12)
For the metric (6), one can use the identity
\[ -\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu - m^2 = E^2 - v^2 \vec{p}^2 - m^2 , \] (13)
where \( E \) is the energy and \( \vec{p} \) is the momentum operator. This leads to the dispersion relation,
\[ E = \pm \sqrt{\vec{p}^2 v^2 + m^2} . \] (14)
It then becomes straightforward to incorporate mass terms into the theory, in the form of a Lagrangian
\[ \mathcal{L} = \bar{\psi} (i\tilde{\gamma}^\mu \partial_\mu - m) \psi , \] (15)
which describes a Lorentz-violating particle with the dispersion relation (14).

### III. Lorentz Violation and Gauge (Non-)Invariance

In this section, we discuss how the coupling to the electroweak gauge sector has to be modified in order to obtain the effective interaction Lagrangian used by Cohen and Glashow [4], which is equivalent to “model I” used by Bezrukov and Lee [5].
Let us keep the notation as simple as possible, and start from the standard generalized Dirac Lagrangian (15), which we recall for convenience,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

and assume that $\psi$ stands for a (Majorana) neutrino field. (Questions related to the $SU(2)_L$ doublet will be answered below.) We can write this Lagrangian as

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu \partial_\mu - m + i(\tilde{\gamma}^\mu - \gamma^\mu)\partial_\mu]\psi,$$

where $Q$ is the Lorentz-violating perturbation and constitutes a special case of Eqs. (3) and (8) of Ref. [16].

In Refs. [17, 18], the $Q$ term is advocated to be the sub-Planck limit of a nonlocal theory with spontaneous Lorentz and CPT violation [17], or in more general term, as the low-energy limit of new physics originating from the Planck scale. Indeed, as we investigate possible violations of Lorentz invariance, we explore the limits of validity of our current understanding of fundamental quantum field theory. E.g., it is well known that Lorentz invariance is one of the assumptions underlying the proof of the CPT theorem [19]. A violation of Lorentz invariance therefore allows for violations of CPT, and indeed, some of the operators in the full ansatz for $Q$, as discussed in Ref. [16], are CPT odd. For a long time, one has held the belief that CPT violation automatically implies a violation of Lorentz invariance [20], while conversely, broken Lorentz invariance allows for, but does not require, broken CPT invariance [20]. Recently [21, 22], invoking additional concepts like space-time noncommutativity, the conclusions of Ref. [20] have been questioned, and it has been claimed that scenarios exist where CPT invariance is broken, but Lorentz invariance still holds. In general, the questions regarding the ultimate limits of the validity of our current understanding of fundamental physical laws must include bounds on Lorentz-violating terms, and terms that allow for other broken fundamental symmetries, like CPT.

Furthermore, the Lorentz-violating operators are assumed to be the sub-Planck limit of new physics originating at the Planck scale, where the fundamental interactions will be completely different from “low-energy” physics [where “low-energy” could even extend to the PeV scale, which is still three orders of magnitude below the (reduced) Planck scale of $\sqrt{1/(8\pi G)} = 2.4 \times 10^{18}$ GeV]. At “low” energy, the coupling to the electroweak proceeds by the substitution

$$\partial_\mu \rightarrow D_\mu,$$

where the operator $D_\mu$ constitutes the $SU(2)_L$ covariant derivative, applied to an $SU(2)_L$ doublet, as discussed below. It is therefore permissible, or, suggested, to experiment with the idea, that the substitution (18) applies only to the unperturbed Lagrangian in Eq. (17), but leaves the perturbative $Q$ term unchanged. In this case, the perturbative term does not participate in the electroweak interaction [$SU(2)_L$ doublet], while modifying the free propagation of the neutrino [once the $Q$ operator is written so that it applies only to the upper component of the $SU(2)_L$ doublet, i.e., only to the neutrino].

To be specific, let us start from the doublet

$$L_e = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix},$$

[see Eq. (12.227) of Ref. [23]], where $\nu_e$ is the electron neutrino field, and $e_L$ is the left-handed electron-positron field, and consider the coupling to the electroweak sector, as in Eq. (12.232) of [23], concentrating on the terms that couple to the electroweak gauge fields [in the Lorentz-covariant theory]

$$\mathcal{L}_G = \bar{L}_e(i\gamma^\mu D_\mu)L_e = \bar{L}_e \left[ i\gamma^\mu \left( \partial_\mu - g'/2 B_\mu - g \frac{\tau^2}{2} A_\mu \right) \right] L_e,$$

where the $B$ and the $A_\mu^i$ ($i = 1, 2, 3$) fields transform into the photon, the $Z_0$ and the $W^\pm$ gauge bosons under electroweak unification (for details, see the discussion below). The Pauli matrices are the $\tau_i$, and they act within the $SU(2)_L$ doublet. The charge $e$, and the electroweak couplings including the Weinberg angle are related to $g$ and $g'$ [see Eq. (32) below]. If we add to $\mathcal{L}_G$ the mass term

$$\mathcal{L}_M = -\bar{L}_e \cdot M \cdot L_e = \left( \begin{pmatrix} \bar{\nu}_e \\ \bar{\psi}_e \end{pmatrix} \begin{pmatrix} -m_\nu + Q & 0 \\ 0 & -m_e \end{pmatrix} \begin{pmatrix} \nu_e \\ \psi_e \end{pmatrix} \right),$$

(21)
then the metric to be used for the effective interaction (Fermi interaction) at the electroweak vertex remains the unperturbed Lorentz metric $g_{\mu\nu}$, while the propagation of free neutrinos acquires a Lorentz-breaking term $Q$, as specified in Eq. (17). In writing Eq. (21), we use an oscillation-free neutrino model, and assume, furthermore, that the neutrino mass term is of the Majorana type, i.e., $\nu_\tau = \nu_\tau^C$ where $C$ denote the charge conjugate. We also supplement the right-handed component of the electron field, $\bar{\psi}_e = e_L + e_R$, for the Dirac mass term of the electron. An inspection shows that the Lagrangian

$$L = L_G + L_M$$

(22)
directly leads to the interaction Lagrangian used by Cohen and Glashow [4] and in “model I” of Bezrukov and Lee [5]. Strictly speaking, the Lagrangian $L_G + L_M$ breaks electroweak gauge invariance due to the presence of partial (not covariant) derivative operators in $Q$, but the gauge and Lorentz-breaking terms enter at the same perturbative level, namely, at first order in $Q$ (see also Appendix A).

IV. LORENTZ–VIOLATION AND GAUGE COUPLING: ONE–FLAVOR MODEL

In this section, we investigate which Lagrangian should be used in the calculation of vacuum pair emission and neutrino splitting if we intend to preserve electroweak gauge invariance to the extent possible. We intend to show that it is possible to preserve $SU(2)_L$ gauge invariance under a restricted set of gauge transformations in the electroweak sector, specifically, the sector related to the $Z_0$ exchange, and still break Lorentz invariance differentially, i.e., with different values for the Lorentz-breaking parameters, for neutrinos compared to charged fermions. We first calculate this in an “oscillation-free” environment (using only one particle generation), where we first neglect the mixing of neutrino mass eigenstates, and weak interaction eigenstates, due to the off-diagonal entries of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix.

We again emphasize that $\tau_i$ matrices in Eq. (20) act in the $SU(2)_L$ doublet, while the $\tilde{\gamma}^\mu$ matrices act on the electrons and neutrinos separately. The first observation is that one can choose the free Lagrangian as follows (we ignore the mass terms which are irrelevant for the considerations that follow),

$$L_F \sim \begin{pmatrix} \bar{\nu}_e \\ e_L \end{pmatrix} \begin{pmatrix} i\tilde{\gamma}_e^\mu \partial_\mu & 0 \\ 0 & i\tilde{\gamma}_e^\mu \partial_\mu \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$$

(23)

for the $SU(2)_L$ doublet. In this case, it is immediately clear that free neutrinos and free electrons obtain different maximal velocities, according to the anti-commutation relations

$$\{\tilde{\gamma}_e^\mu, \tilde{\gamma}_e^\nu\} = 2 \text{diag}(1, -v_{\nu_e}^2, -v_{\nu_e}^2, -v_{\nu_e}^2),$$

(24a)

$$\{\tilde{\gamma}_e^\mu, \tilde{\gamma}_e^\nu\} = 2 \text{diag}(1, -v_e^2, -v_e^2, -v_e^2).$$

(24b)

As discussed in Sec. II, these lead to dispersion relations $E_e = \sqrt{p_e^2 + m_e^2}$, $E_{\nu_e} = \sqrt{p_{\nu_e}^2 + m_{\nu_e}^2}$ for the electron and electron neutrino, respectively.

The second observation is that one can replace the partial derivatives in Eq. (23) by covariant derivatives, according to Eq. (20). The covariant derivation, under the $SU(2)_L$ gauge group, is matrix-valued and the substitution $\partial_\mu \rightarrow D_\mu$ will lead to off-diagonal entries in Eq. (23), coupling the electron to the neutrino by what is later identified as the $W$ boson. Furthermore, the diagonal matrix (diagonal with regard to the $SU(2)_L$ doublet) with entries

$$\begin{pmatrix} i\tilde{\gamma}_e^\mu & 0 \\ 0 & \tilde{\gamma}_e^\mu \end{pmatrix}$$

(25)
does not necessarily commute with the $W$ interaction Lagrangian, which is proportional to the terms involving the $\tau_1$ and $\tau_2$ matrices in Eq. (20). However, one can formulate a restricted set of gauge transformations, which pertain only to the

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(26)

matrix in Eq. (20), and restrict the covariant derivative to

$$i\partial_\mu \rightarrow iD_\mu = i\partial_\mu - \frac{g'}{2} B_\mu + \frac{g}{2} \tau_3 A_{3,\mu},$$

(27)
The gauge coupling Lagrangian $\mathcal{L}_{Z,A}$ which is to be added to $\mathcal{L}_F$ under the restricted set of gauge transformations, reads as follows,

\[
\mathcal{L}_{Z,A} = \mathcal{L}_e \cdot G \cdot \mathcal{L}_e ,
\]

\[
G = \left( \begin{array}{cc} \frac{1}{2} \sigma^\mu \left( gA_{3,\mu} - g'B_\mu \right) & 0 \\ \frac{1}{2} \gamma_\nu \left( gA_{3,\mu} + g'B_\mu \right) & 0 \end{array} \right).
\]

(28a)

(28b)

Defining, as in Eq. (12.238) of Ref. [23], the $Z$ and $A_\mu$ fields as

\[
Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( -gA_{3,\mu} + g'B_\mu \right),
\]

(29)

\[
A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( gB_\mu + g'A_{3,\mu} \right),
\]

(30)

one obtains the following couplings,

\[
\mathcal{L}_{Z,A} = -\frac{e}{2} \left[ \tan \theta_W \left( \bar{\nu}_e \bar{\gamma}_\nu \nu_e + \bar{e}_L \gamma_\nu \nu_e \right) - \cot \theta_W \left( \bar{e}_L \gamma_\nu \nu_e - \bar{\nu}_e \gamma_\nu \nu_e \right) \right] Z_\mu
\]

\[
- e \bar{z}_e \gamma^\mu_L A_\mu \nu_e,
\]

(31)

where

\[
\frac{e}{\sqrt{g^2 + g'^2}}, \quad \tan \theta_W = \frac{g'}{g},
\]

\[
e = g' \cos \theta_W = g \sin \theta_W,
\]

(32)

and $\theta_W$ is the Weinberg angle, and $e$ is the electron charge. (Adding the right-handed component of the charged fermion field restores the full QED Lagrangian for the coupling of the electron-positron field.) The result (31) is exactly equivalent to the corresponding terms in Eq. (12.240) of Ref. [23], with the replacement $\gamma^\mu \rightarrow \tilde{\gamma}_\nu$ for the neutrino couplings to the $Z_0$ boson, and $\gamma^\mu \rightarrow \tilde{\gamma}_\nu$ for the electron couplings to the $Z_0$ boson. The resulting modified effective Fermi Lagrangian describing the coupling of electrons and neutrino is thus exactly the one of “model II” used by Bezrukov and Lee, and by us in Ref. [3]. (We recall that both neutrino as well as lepton-pair Cerenkov radiation proceed by $Z_0$ exchange.)

For the calculation of neutrino splitting [3], it means that, e.g., if the $\tilde{\gamma}_\nu$ for muon neutrinos are different from those of electrons neutrinos, $\tilde{\gamma}_e$, because of a different maximum velocity for the two species, then the neutrino splitting process becomes kinematically allowed (for $v_{\nu_\mu} = v_f = v_{\nu_e} = v_i$). Furthermore, the effective interaction Lagrangian describing the four-fermion vertex receives a correction from the two $Z_0$ vertices, leading to the appropriate replacement

\[
\psi_{\text{int}} = v_i v_f
\]

(33)

for the “metric” to be used in the effective Lagrangian in Eq. (18) of Ref. [3], within a gauge-invariant formulation. The same is done in “model II” of Ref. [5]. Here, $v_i$ is the Lorentz-violating velocity parameter for the initial (oncoming) particle, while $v_f$ is that of the emitted (final) particle.

In the context of Lorentz-breaking, one often finds that symmetry groups are broken down to smaller subgroups (see also the discussion in Appendix A). Here, we observe that the Lorentz-breaking terms change the gauge group from $SU(2)_L \times U(1)$ to $U(1)_L \times U(1)$.

V. LORENTZ–VIOLATION AND GAUGE COUPLING: THREE–FLAVOR MODEL

In the above considerations, we have shown that it is possible to formulate differential Lorentz violation in the same $SU(2)_L$ doublet, to obtain different Lorentz-breaking parameters for electron neutrinos as compared to (left-handed) electrons, without breaking $SU(2)_L$ gauge invariance, within a restricted set of gauge transformation. This consideration required the use of different $\tilde{\gamma}^\mu$ matrices for the upper and lower components of the same $SU(2)_L$ doublet. One might ask the question if different Lorentz-breaking parameters could be obtained for different neutrino species, as compared to electrons, and among the neutrino mass eigenstates, even if one uses the same $\tilde{\gamma}^\mu$ matrices for
the upper and lower components of the same \(SU(2)_L\) doublet, and only assumes a dependence of the \(\tilde{\gamma}^\mu\) matrices on the fermion generations. In contrast to the model discussed in Sec. II, we here preserve full \(SU(2)_L\) gauge invariance.

We thus start from the Lagrangian (ignoring the free mass terms)

\[
\mathcal{L}_{3G} = \left( \bar{\nu}_e \right) \begin{pmatrix} i \tilde{\gamma}_\nu^{\mu} D_\mu & 0 \\ 0 & i \tilde{\gamma}_\nu^{\mu} D_\mu \end{pmatrix} \left( \nu_e \right) + (e \leftrightarrow \mu) + (e \leftrightarrow \tau),
\]

(34)

where “3G” refers to the three generations, and we assume a uniform Lorentz violation within the first generation, and uniform within the second generation, but with different overall parameters,

\[
\tilde{\gamma}^\rho_{\nu_e} = \tilde{\gamma}^\rho_{\nu_\mu} = \tilde{\gamma}^\rho_{\nu_\tau} = \tilde{\gamma}^\rho_{\nu_\tau}.
\]

(35)

As the matrix

\[
\begin{pmatrix} i \tilde{\gamma}_\nu^{\mu} & 0 \\ 0 & \tilde{\gamma}_\nu^{\mu} \end{pmatrix}
\]

(36)

is proportional to the unit matrix [from within the \(SU(2)_L\) doublet], full gauge invariance is preserved.

Invoking neutrino oscillations, we can write the mass term as

\[
\mathbf{M} = \bar{\nu}_k^{(m)} m_k \nu_k^{(m)},
\]

(37)

where the \(\nu_k^{(m)}\) are the neutrino mass eigenstates \((k = 1, 2, 3)\) is summed over). In the free theory, we end up with a Lagrangian

\[
\mathcal{L}_F = i \bar{\nu}_k^{(m)} \gamma^{(m),\mu}_{kj} \partial_\mu \nu_k^{(m)} - \bar{\nu}_k^{(m)} m_k \nu_k^{(m)},
\]

(38)

where the mass eigenstates \(\nu_k^{(m)}\) and the flavor eigenstates \(\nu_j^{(f)}\) are related by

\[
\nu_k^{(m)} = U_{kj} \nu_j^{(f)},
\]

(39)

where \(U_{kj}\) is the PMNS matrix. The emergence of the PMNS matrix for both Dirac as well as Majorana neutrinos is discussed in detail in Ref. [24]. Again, neutrino mass \((m)\) and flavor \((f)\) eigenstates are distinguished based on their superscript. Of course, the mass-basis matrices

\[
\tilde{\gamma}^{(m),\mu}_{kj} = U_{kl} \gamma^{(f),\mu}_{lj} U_{lj}^+
\]

(40)

are effective, Lorentz-violating, modified Dirac matrices describing the (possibly off-diagonal, \(k \neq j\)) Lorentz violation in the neutrino mass eigenstate basis.

Two limiting cases are of interest: In the high-energy limit, one can neglect the mass term in Eq. (38), and observes that in this limit, the flavor eigenstates become equal to the mass eigenstates. Then,

\[
\mathcal{L}_F \approx i \bar{\nu}_k^{(f)} \gamma^{(f),\mu}_{kj} \partial_\mu \nu_k^{(f)},
\]

(41)

where the \(\gamma^{(f),\mu}_{kj}\) are diagonal in the flavor basis [see Eq. (34), with \(f = e, \mu, \tau\)]. Under this assumption, vacuum pair emission (Refs. [4, 5]) is kinematically forbidden in the high-energy region, because the charged fermions offer the same Lorentz-violating parameters as the corresponding neutrino flavors, but neutrino splitting (Ref. [3]) is kinematically allowed, because of the differences among the Lorentz-violating parameters for the different neutrino flavors, which happen to approximate the mass (energy) eigenstates under the given assumptions. Consequences of the latter assumptions for a Lorentz-violating, but entirely gauge-invariant theory, will be studied in Ref. [25].

By contrast, in the low-energy limit, the Lorentz-violating parameters play a subordinate role as compared to the mass terms, and the energy splitting for equal momenta, among the neutrinos, is given in the mass eigenstate basis. In that limit, an inspection shows that the dominant terms in the free Lagrangian (38) are just the diagonal ones in the mass basis,

\[
\mathcal{L}_F \approx i \bar{\nu}_k^{(m)} \gamma^{(m),\mu}_{kj} \partial_\mu \nu_k^{(m)} - \bar{\nu}_k^{(m)} m_k \nu_k^{(m)},
\]

(42)
where \( \hat{\gamma}_k^{(m),\mu} = \hat{\gamma}_k^{(m),\mu} \) (no summation over \( k \)). Under these assumptions, The maximal attainable velocities \( v_k^{(m)} \) of the mass eigenstates are thus related of the flavor eigenstates, by the relation

\[
\sum_{\ell=1}^{3} U_{k\ell} v_{\ell}^{(f)} U_{kk}^{-1} = v_k^{(m)} \quad \text{(no summation over } k). \tag{43}
\]

The transition among the two regimes characterized by the Lagrangians (41) and (42) occurs at a momentum scale of the order of

\[
|\vec{p}| = \sqrt{\delta m^2 / \delta_{f_1 f_2}}, \tag{44}
\]

where \( \delta m^2 \) is a typical neutrino mass square difference, and of course, \( \delta_{f_1 f_2} \) is a typical delta-parameter difference among the Lorentz-violating parameters for the different neutrino flavors (We set \( v^2 = 1 + \delta \), in accordance with Refs. [3–5]). For a parameter estimates of two different neutrino flavors of \( \delta_{f_1 f_2} \sim 10^{-20} \) and \( \delta m^2 \sim 10^{-3} \text{eV}^2 \), the transition should occur at momenta on the order of \( 10^8 \ldots 10^9 \text{eV} \). (We here refer to bounds on Lorentz-violating parameters from laboratory-based experiments [26, 27], which are less strict than those derived from astrophysical observations [28, 29]; the latter, though, are under less stringent external control.)

The conclusion is that, for the Lorentz-breaking but gauge-invariant formulation of the neutrino splitting process, the appropriate choice [for the low-energy region, as measured by Eq. (44)] for the Lorentz-violating parameter in the effective interaction Lagrangian is [see Eq. (18) of Ref. [3]]

\[
v_{\text{int}} = v_i^{(m)} v_f^{(m)} \tag{45}
\]

where the velocities of the initial and final mass eigenstates are denoted as \( v_i^{(m)} \) and \( v_f^{(m)} \), respectively. Furthermore, it is clear that the effective velocities \( v_k^{(m)} \) for the neutrino mass eigenstates, under the given assumptions, will be different from those of the electrons, which are given (due to the absence of mass mixing among the charged leptons) by \( v_i^{(f)} \), thus kinematically the vacuum-pair emission process [again, for the low-energy region, as measured by Eq. (44)].

### VI. NEUTRINO SPLITTING

In order to fix ideas, we recall here, from Ref. [3], the result for the decay rate \( \Gamma \) and the energy loss rate \( dE/dx \), for the neutrino-splitting process, setting \( v_i^2 = 1 + \delta_i \) and \( v_f^2 = 1 + \delta_f \), within the “gauge-invariant” model discussed here in Sec. IV. The results read as

\[
\Gamma_{\nu_i \rightarrow \nu_f, \nu_i \bar{\nu}_f} = b \frac{G_F^2}{192\pi} k_1^5, \tag{46a}
\]

\[
\frac{dE_{\nu_i \rightarrow \nu_f, \nu_i \bar{\nu}_f}}{dx} = -b' \frac{G_F^2}{192\pi} k_1^6, \tag{46b}
\]

where

\[
b = \frac{17}{210} (\delta_i - \delta_f) \left[ (\delta_i - \delta_f)^2 + \frac{7}{11} \delta_{\text{int}}^2 \right], \tag{47a}
\]

\[
b' = \frac{11}{168} (\delta_i - \delta_f) \left[ (\delta_i - \delta_f)^2 + \frac{4}{11} \delta_{\text{int}}^2 \right], \tag{47b}
\]

and \( \delta_{\text{int}} \) given by Eq. (33) for the model discussed here in Sec. IV. For the three-flavor model discussed in Sec. V, in the low-energy region [measured in terms of Eq. (44)], the appropriate choice is [see Eq. (45)]

\[
\delta_{\text{int}} = \delta_i^{(m)} + \delta_f^{(f)}, \tag{48}
\]

where \([v_i^{(m)}]^2 = 1 + \delta_i^{(m)} \) and \([v_f^{(m)}]^2 = 1 + \delta_f^{(m)} \). Consequences for Lorentz-violating parameters, in view of the absence of an appreciable energy threshold for the neutrino splitting process, are discussed in Ref. [3].
VII. CONCLUSIONS

In this paper, we have investigated the assumptions underlying the model dependent interaction Lagrangians used in [3–5] for the formulation of the lepton-pair (LPCR) and neutrino-pair Cerenkov radiation (NPCR, see Sec. VI) processes, which have led to very tight bounds on the Lorentz-violating parameters in the neutrino sector [1–3]. The main results can be summarized as follows.

Conclusion (i). The model used by Cohen and Glashow [4], and “model I” of Bezrukov and Lee [5], can be traced to an interaction Lagrangian which breaks electroweak gauge invariance, in addition to Lorentz invariance (see Sec. III). However, this breaking proceeds on the same perturbative level on which the Lorentz-breaking terms themselves are formulated [see Eq. (21)]. A discussion on the implications with respect to fundamental symmetries is given in Sec. III of this article.

Conclusion (ii). “Model II” of Bezrukov and Lee, used in the formulation of the LPCR process in Ref. [5], and also used by us in Ref. [3], is gauge invariant under a restricted set of gauge transformations, within the $SU(2)_L$ gauge group. The use of non-uniform modified Dirac matrices, within the same $SU(2)_L$ doublet, is crucial to this observation [see Eq. (23)]. The derivation goes through even in an “oscillation-free” environment where one neglects the off-diagonal entries of the PMNS matrix, in the neutrino sector. The result given in Eq. (33) clarifies the “gauge-invariant” Lagrangian used in “model II” [see Eq. (4) of Ref. [5]].

Conclusion (iii). If one invokes neutrino oscillations, then the situation is even more favorable for the gauge-invariant models (see Sec. V). One can use uniform modified Dirac matrices within the same $SU(2)_L$ doublet, but assumes different Lorentz-violating parameters between generations [see Eq. (34)]. By assuming only a generation dependence, one obtains differential Lorentz violation among the neutrino mass eigenstates, and between neutrinos and charged leptons, without breaking $SU(2)_L$ gauge invariance. Under these assumptions [see Eq. (45)], it is useful to keep the Lorentz-violating parameters $\gamma_{\mu}$ that enters the interaction Lagrangian, separate from the ones of the initial and final states, as is done in Ref. [3].

We have thus clarified the cryptic remark of the “gauge invariance” of “model II” of Bezrukov and Lee [see Eq. (4) of Ref. [5]], and provided additional motivation for the functional form of the various model-dependent interaction Lagrangians used in Refs. [3–5].

The very stringent bounds on the Lorentz-violating parameters in the neutrino sector, based on astrophysical observations [1–3], thus do not require models in which electroweak gauge invariance is broken. This observation is quite crucial because it implies that one cannot “argue away” the tight bounds derived in Refs. [1–3] for the Lorentz-breaking parameters in the neutrino sector, based on the notion that the preservation of electroweak gauge invariance would otherwise preclude the existence of the decay processes on which the bounds are based.

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Appendix A: Spontaneous Lorentz–Symmetry Breaking: Models and Implications

Although the ansatz of the current paper is completely phenomenological, and we do not discuss the possible mechanism behind Lorentz violation in any greater detail, it is still instructive to mention a specific model of spontaneous Lorentz invariance violation, which has been discussed in rather great detail in the literature.

Namely, according to Refs. [30–32], the photon could be formulated as the Nambu-Goldstone boson linked to spontaneous Lorentz invariance violation. (This ansatz was originally formulated before electroweak unification.) Interest in this approach has recently been revived, and the theory has been worked out in greater detail [30–41]. Both Abelian as well as a non-Abelian gauge theories have been discussed [34]. In the case of an Abelian gauge theory, one assumes that the gauge field $A_\mu$ obtains a non-vanishing vacuum expectation value according to [see text after Eq. (1) of Ref. [40]],

$$\langle A_\mu \rangle = n_\mu \, M \, ,$$

(A1)

where $M$ is a (possibly large) energy scale at which the breaking of Lorentz symmetry occurs. The Lorentz group restricts itself to $SO(1, 2)$ if $n_\mu$ is space-like ($n_\mu n^\mu = -1$), and into $SO(3)$ if $n_\mu$ is timelike ($n_\mu n^\mu = 1$).
The dynamical constraint [see Eq. (1) of Ref. [40]]
\[ A_\mu A^\mu = n^2 M^2, \]  
(A2)
is imposed on the \( A_\mu \) field. One then parameterizes the \( A_\mu \) field as [see Eq. (3) of Ref. [40]]
\[ A_\mu = a_\mu + \frac{n_\mu}{n^2} (n \cdot A), \]  
(A3)
where the \( a_\mu \) takes the role of the photon field. The following Lagrangian is eventually obtained after an expansion in leading order in \( 1/M \) [see Eq. (3) of Ref. [40]],
\[ L(a, \psi) = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} \delta(n \cdot a)^2 \\
+ \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e a_\mu \bar{\psi} \gamma^\mu \psi \\
- \frac{1}{4} f_{\mu\nu} h^{\mu\nu} \frac{n^2 a_\rho a^\rho}{M} + \frac{e n^2 a_\rho a^\rho}{2M} \bar{\psi} (\gamma \cdot n) \psi. \]  
(A4)
Here, \( a_\mu \) takes the role of the (quantized) electromagnetic field, while \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \) is the field strength tensor. Also, \( h^{\mu\nu} = \eta^{\mu\rho} \partial_\rho - \eta^{\nu\rho} \partial_\rho \) is an oriented Lorentz-violating tensor. The orthogonality condition \( n \cdot a = 0 \) is explicitly introduced in the Lagrangian through a gauge-fixing term with parameter \( \delta \). Note that the Lagrangian (A4) is obtained after a suitable redefinition of the fermion field, as given explicitly in Eq. (6) of Ref. [40].

We note that the sum of the terms
\[ \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e a_\mu \bar{\psi} \gamma^\mu \psi \]  
(A5)
in Eq. (A4) add up to the gauge-invariant quantum electrodynamic interaction (\( e \) is the electron charge).

In the limit \( M \to \infty \), the entire Lagrangian (A4) approximates the ordinary QED Lagrangian. However, for finite \( M \), the fifth and the sixth term on the right-hand side of Eq. (A4), which are initially generated by spontaneous Lorentz breaking in the electromagnetic sector, explicitly break electromagnetic gauge invariance, in addition to breaking Lorentz invariance. The fifth term generates a three-photon vertex, while the sixth term generates a two-fermion, two-photon interaction (see, e.g., Ref. [37]).

In Eq. (22) of Ref. [37], it is shown that the contribution of both of the Lorentz-breaking terms to the electron-photon scattering amplitude vanishes (due to mutual cancelations) if we take the matrix element between on-shell spinors. Around Eq. (32) of Ref. [37], it is argued that the same cancelation occurs for the one-loop amplitude, if the specific photon propagator integral given in Eq. (32) of Ref. [37] is evaluated in dimensional regularization. These considerations show that the Lorentz-violating terms in Eq. (A4) do not necessarily lead to observable effects at low-energy.

The generalization to spontaneous Lorentz-symmetry breaking in non-Abelian gauge fields involves the assumption [see Eq. (9) of Ref. [34]]
\[ \langle A_\mu^i \rangle = n_\mu^i M \]  
(A6)
where the upper index \( i \) describes the component within the non-Abelian gauge group, e.g., \( SU(N) \), in which case \( i = 1, \ldots, N \). For the Lorentz-breaking terms to vanish in the low-energy limit, one then has to make additional assumptions regarding the masses of the particles in a given \( SU(N) \) multiplet; e.g., according to Eq. (19) of Ref. [34], one needs to assume these masses to be equal.

For the context of the current paper, two observations are relevant:

1. The approach taken in Refs. [34–41] starts from a spontaneous Lorentz symmetry breaking at some high-energy scale \( M \), involving a gauge boson field. This assumption is quite natural, because symmetry breaking for a vector field automatically singles out a specific direction in space-time (it would not necessarily do so for a spinor). However, as a comparison to Eq. (A4) shows, for the case of spontaneous symmetry breaking in the gauge boson sector, the fermion sector is largely unaffected by the Lorentz-symmetry breaking, which initially occurs only in the gauge boson sector. [We observe that the third and fourth term in Eq. (A4) add up to the fully Lorentz-covariant Lagrangian for the electromagnetically coupled electron.] Hence, the ansatz discussed in Refs. [30–41] is not directly applicable to the models constrained by our calculations, which pertain to Lorentz violation in the fermion (neutrino) sector.

2. A very important observation can be made. Namely, Lorentz violation and gauge invariance violation are intimately intertwined. The term
\[ \frac{en^2 a_\rho a^\rho}{2M} \bar{\psi} (\gamma \cdot n) \psi \]  
(A7)
in Eq. (A4) is manifestly non-gauge invariant. We remember that a gauge transformation in quantum electrodynamics works as \( a_\mu \rightarrow a_\mu - \partial_\mu A \) and \( \psi \rightarrow e^{i\lambda} \psi \), where \( A = A(x) \) is the gauge function, and \( e \) is the electron charge. Under this gauge transformation, the term \( (A7) \) is manifestly non-invariant. Lorentz violation has thus created a term that violates gauge invariance, on the perturbative level (in first order in the 1/\( M \) expansion). Analogously, the model used by Cohen and Glashow in Ref. [4] assumes a breaking of gauge invariance on the perturbative level, in the latter case, of the electroweak gauge symmetry. Based on our comparison with the approach taken in Refs. [30–41], this is a perfectly permissible assumption.
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