Quantum de Moivre–Laplace theorem for noninteracting indistinguishable particles in random networks

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Received 13 March 2017, revised 30 August 2017
Accepted for publication 22 September 2017
Published 17 November 2017

Abstract
The asymptotic form of the average probability to count \( N \) indistinguishable identical particles in a small number \( r \ll N \) of binned-together output ports of a \( M \)-port Haar-random unitary network, proposed recently in Shchesnovich (2017 Sci. Rep. 7 31) in a heuristic manner with some numerical confirmation, is presented with mathematical rigour and generalised to an arbitrary (mixed) input state of \( N \) indistinguishable particles. It is shown that, both in the classical (distinguishable particles) and quantum (indistinguishable particles) cases, the average counting probability into \( r \) output bins factorises into a product of \( r - 1 \) counting probabilities into two bins. This fact relates the asymptotic Gaussian law to the de Moivre–Laplace theorem in the classical case and similarly in the quantum case where an analogous theorem can be stated. The results have applications to the setups where randomness plays a key role, such as the multiphoton propagation in disordered media and the scattershot Boson sampling.

Keywords: identical particles, random networks, quantum statistics

1. Introduction

Recently it was argued [1] that the probability to count \( N \) indistinguishable particles, bosons or fermions, impinging at a unitary \( M \)-port, in binned-together output ports, if averaged over the Haar-random unitary matrix representing the multiport or, for a fixed multiport, over the input configurations of the particles, takes the asymptotic Gaussian form as \( N \to \infty \) with the particle density \( N/M \) being constant. The quantum statistics of bosons or fermions shows up in the Gaussian law precisely through the particle density.
The purpose of the present work is two-fold. First, a mathematically rigorous formulation and proof of the asymptotic Gaussian law is given, substituting the heuristic derivation of [1]. To this end, a new technical tool, a factorisation of the average $r$-bin counting probability into a product of probabilities for some binary cases (two bins), is employed. The factorisation reduces the classical asymptotic Gaussian law for the $r$-bin partition to the de Moivre–Laplace theorem [2]. It is shown that a similar reduction exists in the quantum case as well, suggesting an interpretation of the respective asymptotic Gaussian law as a quantum version of the de Moivre–Laplace theorem with the events (particle counts) correlated due to indistinguishability of identical particles. Second, the asymptotic results are shown to hold for arbitrary (mixed) input states of $N$ identical particles impinging on a random unitary multiport, thereby significantly extending the scope of applicability of the asymptotic Gaussian law.

Mathematically, the results below can be thought of as a generalisation of the well-known asymptotic result for the multinomial distribution, originally due to works of de Moivre, Lagrange and Laplace (for a historical review, see [2]). The multinomial distribution applies when the identical particles are sent one at a time through the multiport. The asymptotic Gaussian law exposes the effect of the quantum statistics of identical particles on their behaviour [3–6], as opposed to the interference effects [7, 8], and is applicable to the setups where randomness plays a key role, such as multiphoton propagation through disordered media [9–11]. It should be stressed that the complexity of behaviour of bosonic particles in linear unitary networks asymptotically challenges the digital computers, which is the essence of the Boson sampling idea [12, 13] with the proof-of-principle experiments [3, 14–20]. For instance, the asymptotic Gaussian law is applicable to the scattershot Boson sampling [13, 19], where randomness in the setup is due to the heralded photon generation in random input ports.

The rest of the text is organised as follows. Section 2 contains a brief statement of the problem and a rigorous formulation of the main results in the form of two theorems. The theorems are proven in section 3, where the binary classical case is briefly reminded in section 3.1 and the $r$-bin case is considered in section 3.2. The $r$-bin quantum case is analysed in section 3.3, where similarities of the classical and quantum cases are highlighted. Appendices A–C contain mathematical details of the proof. In section 4 the results in theorems 1 and 2 are generalised to the arbitrary (mixed) input state of indistinguishable particles. Finally, in section 5 a brief summary of the results is given.

2. The counting probability of identical particles in a random multiport with binned-together output ports

Consider a unitary quantum $M$-port network (i.e. where there are $M$ input and output ports) described by the unitary matrix $U$ connecting the input $|k,\text{in}\rangle$ and output $|l,\text{out}\rangle$ states as follows $|k,\text{in}\rangle = \sum_{l=1}^{M} U_{kl} |l,\text{out}\rangle$. Let us partition the output ports into $r$ bins having $K \equiv (K_1, \ldots, K_r)$ ports. We are interested in the average probability of counting $N$ noninteracting identical particles, impinging at the network input, in the binned-together output ports (here and below the term ‘average’ means the average over the Haar-random unitary matrix $U$; where necessary we will use the notation $\langle \ldots \rangle$). We consider binned-together output ports, since in the quantum case the average probability of an output configuration of indistinguishable bosons is exponentially small and, therefore, hard to estimate experimentally (see below).

Let us differentiate by $\sigma$ the three cases of identical particles, where bosons correspond to $\sigma = +$, fermions to $\sigma = -$, and distinguishable particles to $\sigma = 0$. The term ‘distinguishable particles’ applies here to quantum particles in orthogonal states with respect to the degrees of freedom not affected by a multiport [21–24], such as the arrival time in the case of photons.
(however, the separation into the internal and operating modes depends on the detection scheme, e.g. the time-resolving detection scheme of [25, 26] acts as an unitary transform on the internal degrees of freedom, turning them into the operating modes). A working model of distinguishable particles is the case of identical particles sent through network one at a time. The average probability for a single particle from input port \( k \) to land into bin \( i \) reads [1] \( p_i = \sum_{l \in K_i} |\langle U_{k,l} \rangle|^2 = q_i \equiv K_i / M \), since \( |\langle U_{k,j} \rangle|^2 = 1 / M \) where \( |\langle U_{k,j} \rangle|^2 \) is the probability of the transition \( k \rightarrow l \). For identical particles sent one at a time through a random multiport, the average probability to count \( n \equiv (n_1, \ldots, n_r) \) particles in the output bins becomes a multinomial distribution:

\[
P^{(0)}(\mathbf{n}|\mathbf{K}) = \frac{N!}{\prod_{i=1}^r n_i!} \prod_{i=1}^r q_i^{n_i}. \tag{1}
\]

Equation (1) is also the exact average probability in a fixed unitary multiport and uniform averaging over all possible input configurations of distinguishable particles, but is only an approximation to the exact average probability in the case of a fixed input of \( N \) distinguishable particles in a Haar-random multiport. The exact average probability in the latter case has an extra factor due to the correlations between the matrix elements \( |\langle U_{k,l} \rangle|^2 \) (see, for more details, [1]).

Now let us now consider indistinguishable particles (assuming in the case of fermions up to one particle per network port). For the input \( \mathbf{k} = (k_1, \ldots, k_N) \) and output \( \mathbf{l} = (l_1, \ldots, l_r) \) configurations the average probability of the transition \( \mathbf{k} \rightarrow \mathbf{l} \) is just the inverse of the number of Fock states of \( N \) bosons (fermions) in \( M \) ports: \( p^{(+)}(\mathbf{l}|\mathbf{k}) = \frac{\gamma M!}{M+N-1} \cdots M \left( p^{(-)}(\mathbf{l}|\mathbf{k}) = \frac{\gamma M!}{M+(N+1)} \right) \). This observation leads to the following quantum equivalent of equation (1) (with the upper signs for bosons and the lower ones for fermions) [1]:

\[
P^{(\pm)}(\mathbf{n}|\mathbf{K}) = \frac{N!}{(M+N+1) \cdots M} \prod_{i=1}^r \frac{(K_i \pm n_i + 1) \cdots K_i}{n_i!} \\
= P^{(0)}(\mathbf{n}|\mathbf{K}) \prod_{i=1}^r \left[ \prod_{l=0}^{n_i-1} [1 \pm l/K_i] \right] \prod_{l=0}^{N-1} [1 \pm l/M] \equiv P^{(0)}(\mathbf{n}|\mathbf{K}) Q^{(\pm)}(\mathbf{n}|\mathbf{K}). \tag{2}
\]

As in the classical case, the probability formula (2) applies also to a fixed unitary multiport with the averaging performed uniformly over the input configurations \( \mathbf{k} \) allowed by the quantum statistics [1]. Moreover, as shown in section 4 below, the average probability in equation (2) actually applies for an arbitrary mixed input state of \( N \) indistinguishable particles.

Our focus on the binned-together output ports for a large multiport and large number of particles has to be explained. Assuming that only polynomial in \( N \) number of experimental runs is accessible (due to the decoherence or as in verification protocols for the Boson sampling [3, 5, 12, 19, 27]), an exponentially small in \( N \) probability cannot be estimated. For a fixed particle density \( \alpha = N/M \), the probability of a particular configuration of indistinguishable bosons at the output of a random \( M \)-port is asymptotically exponentially small in \( N \), on average (see also [27, 28]):

\[
p^{(+)}(\mathbf{l}|\mathbf{k}) = \sqrt{2 \pi N (1 + \alpha)} e^{-\gamma N} \left[ 1 + O \left( \frac{1}{N} \right) \right], \quad \gamma = \ln \left( \frac{1}{\alpha} \right) + \left( 1 \pm \frac{1}{\alpha} \right) \ln (1 \pm \alpha). \tag{3}
\]

Though the density \( \alpha \) is not fixed in the statement of theorem 1 below, it is natural to consider the asymptotic limit at a fixed density, i.e. when both the number of particles and the number
of ports tend to infinity (for bosons, there is also the high-density case $M = O(1)$ as $N \to \infty$, see corollary 1 below).

The key point used to prove the asymptotic Gaussian law, discovered in [1], is that the $r$-bin case can be factorised into a set of $r - 1$ binary ($r = 2$) cases. Indeed, both in the classical and quantum cases, there is a factorisation of the average counting probability into binned-together output ports (see appendix A):

$$P^{(\sigma)}(n|K) = P^{(\sigma)}(n_1, N - n_1|K_1, M_1 - K_1)P^{(\sigma)}(n_2, N - n_2|K_2, M_2 - K_2) \ldots P^{(\sigma)}(n_r, N - n_r|K_r, M_r - K_r),$$

(4)

where $N_1 = N, M_1 = M$ and for $s = 2, \ldots, r - 1$

$$N_s = N - \sum_{i=1}^{s-1} n_i, \quad M_s = M - \sum_{i=1}^{s-1} K_i.$$  

(5)

Equation (4) shows the key role played by the binary case, for which we will use also another notation $P^{(\sigma)}_{N,M}(n|K) = P^{(\sigma)}(n, N - n|K, M - K)$, with the independent variables as the arguments. Below we will need the following definitions (for $s \geq 2$):

$$\tilde{q}_s = \frac{K_s}{M_s} = \frac{q_s}{1 - \sum_{i=1}^{s-1} q_i}, \quad \tilde{x}_s = \frac{n_s}{N_s} = \frac{x_s}{1 - \sum_{i=1}^{s-1} x_i}, \quad x_i = \frac{n_i}{N}.$$  

(6)

where $0 \leq \tilde{q}_s \leq 1$ takes the place of $q_s$ in the $s$th factor of the binary factorisation of the average classical probability for the $r$-bin case in equation (4), i.e.

$$P^{(0)}_{N,M}(n|K) = \frac{N!}{n! (N - n)!} q_1^{n_1} (1 - q_1)^{N - n_1}.$$  

When there is such $C > 0$ (independent of $X$) that $Y \leq CX$, we will use the notation $Y = O(X)$. The following two theorems state the main results.

**Theorem 1.** Consider the Haar-random unitary $M$-port with the output ports binned together into $r$ of subsets consisting of $K_1, \ldots, K_r$ ports. Then, as $N, M \to \infty$ for constant $r$ and $q_i = K_i/M > 0, i = 1, \ldots, r$, the average probability to count $n = (n_1, \ldots, n_r)$ identical particles into the output bins, such that:

$$|n_i - N\tilde{q}_i| \leq AN^{\frac{1}{2} - \epsilon}, \quad A > 0, \quad 0 < \epsilon < \frac{1}{6},$$

(7)

for some constant $A$, has the following asymptotic form (here $\delta_{\sigma,\pm} = 1$ when $\sigma = \pm$ and zero otherwise)

$$P^{(\sigma)}(n|K) = \exp \left\{ -N \sum_{i=1}^{r} \frac{(x_i - \tilde{q}_i)^2}{2(1 + \sigma \alpha) N^2} \prod_{i=1}^{r} \sqrt{q_i} \right\} \left\{ 1 + O \left( \frac{1 - \alpha \delta_{\sigma,-}}{N^{3\epsilon}} + \frac{\alpha \delta_{\sigma,+}}{N^2} \right) \right\}.$$  

(8)

An important note is in order. In the course of the proof (see section 3) it is also established that the $r$-bin asymptotic Gaussian on the right hand side of equation (8) satisfies the same factorisation as the average counting probability, equation (4), to the error of the asymptotic approximation in equation (8).

For a finite density $\alpha$, in the quantum case the error in equation (8) scales as $O(N^{-3\epsilon})$. In this case we get $x_i = q_i$ in the limit $N \to \infty$ for bosons, fermions, and distinguishable particles. In the usual presentation of the classical result $\epsilon = 1/6$ [29], with this choice the error in
equation (8) scales as $O(N^{-\frac{1}{2}})$ (this choice, however, invalidates the error estimate in theorem 2, equation (9) below, thus it is not allowed).

Since not all particle counts are covered by equation (7), theorem 1 does not guarantee the asymptotic Gaussian to be an uniform approximation for all $n$. However, if equation (7) is violated, then the respective particle counts occur with an exponentially small probability (asymptotically undetectable in an experiment with only a polynomial in $N$ number of runs). This is stated in the following theorem.

**Theorem 2.** The average probability of the particle counts $n = (n_1, \ldots, n_r)$ violating equation (7), as $N, M \to \infty$ for a constant $\alpha = N/M$ satisfies the following upper bound:

$$P_{\sigma(n|K)} = O\left(N^{\frac{\sigma^2}{2(1+\sigma\alpha)}}N^{\frac{1}{2}\epsilon}\right),$$

where $A$ is from equation (7) and $s(0) = 1/2$ (distinguishable particles), $s(+) = 1/2$ (bosons) and $s(−) = 5/2$ (fermions).

In [1] the high-density limit for bosons was mentioned, realised for $N \to \infty$ and $M = O(1)$. This case is a corollary to theorem 1.

**Corollary 1.** As $N \to \infty$ and a fixed $M \gg 1$ the average probability to count $n = (n_1, \ldots, n_r)$ identical particles in $r$ bins with $K_1, \ldots, K_r$ output ports of a Haar-random unitary $M$-port, such that equation (7) is satisfied, has the following approximate asymptotic form:

$$P_{\sigma}(+)\left(n|K\right) = \frac{M^{\frac{-1}{2}}}{(2\pi)^{\frac{r-1}{2}}} \prod_{i=1}^{r} \sqrt{q_i} \exp \left\{-\frac{M \sum_{i=1}^{r} (x_i - q_i)^2}{2q_i} \right\} \left\{1 + O\left(\frac{1}{M} + \frac{1}{N^{3\epsilon}}\right)\right\}. \tag{10}$$

Corollary 1 tells us that in the limit $N \to \infty$ the relative particle counting variables $x_1, \ldots, x_r$ are approximated by the continuous Gaussian random variables:

$$x_i = q_i + \frac{\xi_i}{\sqrt{M}},$$

where $\xi_1, \ldots, \xi_r$ are random variables satisfying the constraint $\sum_{i=1}^{r} \xi_i = 0$ with a Gaussian joint probability density (similar to the classical case, see [29]):

$$\rho = \frac{\exp\left\{-\sum_{i=1}^{r} \frac{\xi_i^2}{2q_i}\right\}}{(2\pi)^{\frac{r-1}{2}}} \prod_{i=1}^{r} \sqrt{q_i}. \tag{12}$$

(The factor $\left(\frac{M}{N}\right)^{\frac{1}{2}}$ in equation (10) allows to approximate the sum $\sum_{n} P_{\sigma}(+)\left(n\right) = 1$ by the integral ($I_M$) of $\rho$ (12) over $\xi_1, \ldots, \xi_r$ with $0 \leq \xi_i \leq \sqrt{M}$. The latter is exponentially close to $1$ for $M \gg 1$, we have $I_M = 1 - e^{-O(M)}$.)

3. Proof of the theorems

3.1. The binary classical case

Let us first consider the classical binary case $r = 2$. Denote by $K_2$ the Kullback–Leibler divergence:
Using Stirling’s formula $n! = \sqrt{2\pi(n + \theta_n)(n/e)^n}$, where $\frac{1}{6} < \theta_n < 1.77$ for $n \geq 1$ and $\theta_0 = \frac{1}{\pi}$ [30], we have:

$$P_{N,M}(n|K) \leq \frac{\exp\left\{-NK_2(x|q)\right\}}{\sqrt{2\pi Nq(1-q)}} \left[1 + \mathcal{O}\left(\frac{1}{N^{\frac{1}{2}}+\epsilon}\right)\right].$$

(14)

Expanding the Kullback–Leibler divergence (13) by using equation (7):

$$K_2(x|q) = \frac{(x-q)^2}{2q(1-q)} + \mathcal{O}\left(\frac{1}{N^{\epsilon}}\right),$$

(15)

and substituting the result in equations (14) we get (8) for the binary classical case.

To show equation (9) consider the first line of equation (14), valid for all $0 \leq x \leq 1$, and observe that $(x + \theta_n/N)(1 - x + \theta_{N-n}/N) \geq (2\pi N)^{-2}$. We obtain:

$$P_{N,M}(n|K) \leq 2\pi \sqrt{N} \exp\left\{-NK_2(x|q)\right\} \left[1 + \mathcal{O}\left(\frac{1}{N}\right)\right].$$

Using Pinsker’s inequality [31]:

$$K_2(x|q) \geq (x-q)^2$$

(17)

and that by equation (7) $|x - q| > AN^{-\frac{1}{2}-\epsilon}$ for $\epsilon < 1/6$, we obtain the required scaling of equations (9) from (16):

$$P_{N,M}(n|K) \leq 2\pi \sqrt{N} \exp\left\{-A^2(1-q)^{\frac{1}{2}}\right\} \left[1 + \mathcal{O}\left(\frac{1}{N}\right)\right]$$

$$= \mathcal{O}\left(\sqrt{N} \exp\left\{-A^2N^{\frac{1}{2}-\epsilon}\right\}\right).$$

(18)

### 3.2. The r-bin classical case

Let us consider the average probability for the general $r$-bin classical case. We can employ the factorisation into $r-1$ binary probabilities given by equations (4) and (5). First of all, let us show the equivalence of equation (7) to the following set of conditions for some constant $A$ (see equation (6)):

$$|n_l - N\tilde{q}_l| \leq \tilde{A}N^{\frac{1}{2}-\epsilon}, \quad \tilde{A} > 0, \quad l = 1, \ldots, r - 1.$$  

(19)
To this end it is enough to observe that:

\[
n_l - N\bar{q}_l = n_l - Nq_l + \sum_{i=1}^{l-1} (n_i - Nq_i), \quad l = 1, \ldots, r - 1.
\]  

(20)

Indeed, the relations in equation (20) are invertible, whereas \( n_r - Nq_r = -\sum_{i=1}^{r-1} (n_i - Nq_i) \).

To prove the classical \( r \)-bin case in theorems 1 and 2 one can proceed as follows. If equation (7) is satisfied for all \( i \), then so is equation (19). Using equation (14) for the binary case into equation (4) we have:

\[
P^{(0)}(n|K) = \prod_{l=1}^{r-1} P^{(0)}_{N|M_l}(n_l|K_l) = \prod_{l=1}^{r-1} \exp \left\{ -N|K|_2(\bar{x}_l|\bar{q}_l) \right\} \sqrt{2\pi N\bar{q}_l(1-\bar{q}_l)} \left[ 1 + O \left( \frac{1}{N^{3+\epsilon}} \right) \right].
\]

(21)

From equation (6) we get:

\[
N\bar{q}_l(1 - \bar{q}_l) = Nq_l \frac{1 - \sum_{i=1}^{l-1} q_i}{1 - \sum_{i=1}^{l-1} q_i} \left( 1 - \frac{\sum_{i=1}^{l-1} n_i / N - q_i}{1 - \sum_{i=1}^{l-1} q_i} \right),
\]

(22)

therefore the denominator in equation (21) becomes equal to that in equation (8) to the necessary error, i.e.

\[
\prod_{l=1}^{r-1} N\bar{q}_l(1 - \bar{q}_l) = N^{r-1} \prod_{l=1}^{r} q_l \left[ 1 + O \left( \frac{1}{N^{3+\epsilon}} \right) \right],
\]

(23)

(since \( 1/3 + \epsilon > 3\epsilon \) for \( \epsilon < 1/6 \), the error conforms with that in equation (8)). In its turn, the term in the exponent in equation (21) can be reshaped using the following identity for the Kullback–Leibler divergence (see appendix B):

\[
\sum_{l=1}^{r} N|K|_2(\bar{x}_l|\bar{q}_l) = N \sum_{l=1}^{r} x_l \ln \left( \frac{x_l}{q_l} \right) \equiv NK_r(x|q).
\]

(24)

By expanding both sides of equation (24) into powers of \( x_l - q_l \) and comparing the terms to the leading order in \( N \) we obtain the following asymptotic identity:

\[
\sum_{l=1}^{r-1} N|K|_2(\bar{x}_l - \bar{q}_l)^2 = N \sum_{l=1}^{r-1} \frac{(x_l - q_l)^2}{2q_l(1-\bar{q}_l)} + O \left( \frac{1}{N^{3\epsilon}} \right).
\]

(25)

Substituting equations (23) and (25) into equations (21) we obtain (8) for the \( r \)-bin classical case.

To show (9) for the \( r \)-bin classical case, let us select the factorisation (4) such that \( i = 1 \) is the first violation of equation (7). Then the probability \( P^{(0)}_{N|M}(n_1|K_1) \) appearing in equation (4) satisfies equation (9), as proven above in the binary case. This observation results in equation (9) for the \( r \)-bin classical case and concludes the proof of the theorems in the classical case.

One important note. From the proof of the theorems for the \( r \)-bin case it is clear that the asymptotic Gaussian for the \( r \)-bin case is a product of the asymptotic Gaussians for the \( r - 1 \) binary cases to the same accuracy as in equation (8), i.e. due to the equivalence of equations (7) and (19) the general case follows from the binary case.
3.3. The quantum case

Let us consider the quantum factor $Q(\pm)(n|K)$ (recall that $+$ is for bosons and $-$ is for fermions) introduced in the second line in equation (2), which accounts for the correlations between the indistinguishable particles due to their quantum statistics. By the following asymptotic identity [1] (see also appendix C):

$$\prod_{l=0}^{n} \left[ 1 \pm \frac{l}{m} \right] = \left( 1 \pm \frac{n}{m} \right)^{n \pm m + 1/2} e^{-n} \left[ 1 + \mathcal{O} \left( \frac{n}{m(m \pm n)} \right) \right]$$

(26)

when $N, M \to \infty$ we get for $n_i$ satisfying equation (7) (i.e. $n_i = N q_i + \mathcal{O}(N^{-3/4}) \to \infty$)

$$Q(\pm)(n|K) \equiv \frac{\prod_{i=1}^{r} \left( \prod_{n=1}^{N-1} \left[ 1 \pm l/K_i \right] \right)}{\prod_{n=1}^{N} \left[ 1 \pm l/M \right]} = \frac{\prod_{i=1}^{r} \left( \prod_{n=0}^{N-1} \left[ 1 \pm l/K_i \right] \right)}{\prod_{n=1}^{N} \left[ 1 \pm l/M \right]} \frac{1 \pm \frac{l}{M}}{1 \pm \frac{l}{N} \left( 1 \pm \frac{1}{M} \right)^{1/2}} \left[ 1 + \mathcal{O} \left( \sum_{i=1}^{r} \frac{n_i}{K_i (K_i - n_i)} + \frac{N}{M (M \pm N)} \right) \right].$$

(27)

Note that for fermions $n_i \leq K_i$ (for $n_i > K$ quantum factor is equal to zero). Now, let us clarify the order of the error in equation (27). From equation (7) we get:

$$K_i \pm n_i = (M \pm N) q_i \pm [N q_i - n_i] \geq \left| \frac{1}{\alpha} - q_i N - \alpha N^{2/3} \epsilon \right|.$$

Thus we can estimate:

$$\frac{n_i}{K_i (K_i \pm n_i)} = \mathcal{O} \left( \frac{\alpha^2}{(1 \pm \alpha) N} \right).$$

Taking this into account, let us rewrite equation (27) as follows:

$$Q(\pm)(n|K) = \frac{\exp \left[ (N \pm M) K_i (X_i^{(\pm)} | q_i) \right]}{(1 \pm \epsilon)^{r-1} \prod_{t=1}^{r} \sqrt{\frac{X_i^{(\pm)}}{q_i}}} \left[ 1 + \mathcal{O} \left( \frac{\alpha^2}{(1 \pm \alpha) N} \right) \right].$$

(28)

where $K_i$ is defined in equation (24) and we have introduced new variables $X_i^{(\pm)}$

$$X_i^{(\pm)} \equiv \frac{K_i \pm n_i}{M \pm N} = \frac{q_i \pm \alpha x_i}{1 \pm \alpha}, \quad 0 \leq X_i^{(\pm)} \leq 1, \quad X_i^{(\pm)} - q_i = \frac{\pm \alpha}{1 \pm \alpha} (x_i - q_i),$$

(29)

(analogs of $x_i$ (6) in the quantum case). Now, if equation (7) is satisfied, we can separate the leading order in the quantum factor by expanding the Kullback–Leibler divergence $K_i (X_i^{(\pm)} | q_i)$ (similar as in equation (25)), whereas in the denominator in equation (28) we have $X_i^{(\pm)} / q_i = 1 + \mathcal{O} \left( \frac{\alpha^{N^{1/3} \epsilon}}{1 + \alpha} \right).$ By selecting the leading order error for $0 < \epsilon < 1/6$ we get (recall that $\sigma = +$ for bosons and $\sigma = -$ for fermions):

$$Q(\sigma)(n|K) = \frac{\exp \left( N \frac{\sigma - \sigma_+}{1 + \sigma_+} \sum_{t=1}^{r} \frac{q_t (q_t - q_i)^2}{2q_t} \right)}{(1 + \sigma_+)^{r-1} \prod_{t=1}^{r} \frac{X_i^{(\pm)}}{q_t}} \left[ 1 + \mathcal{O} \left( \frac{\alpha \delta_{\sigma} \alpha^{N^{2/3} \epsilon}}{N} + \frac{\alpha^3}{(1 + \sigma_\alpha)^{3 N^{2/3} \epsilon}} \right) \right].$$

(30)

For bosons the first error term on the right hand side of equation (30) can dominate the second in the high-density case $\alpha \to \infty$, thus we have to keep it. To obtain equation (8) in the
quantum case one can just multiply the result of equation (8) for the classical probability, proven in section 3.2, by the quantum factor in equation (30) and select the leading order error terms (observing the possibility that \( \alpha \to \infty \) for bosons and \( \alpha \to 1 \) for fermions). This proves theorem 1 in the quantum case.

One important observation is in order. The \( r \)-bin quantum factor \( \mathcal{Q}^{(r)}(\vec{F}) \) of equation (27) is simply a product of the \( r-1 \) binary quantum factors \( \mathcal{Q}_{N,M}^{(r)} \) defined similar as in equation (27), but with \( M_i \) and \( N_i \) as in the factorisation formula (4) and \( \tilde{X}_{i}^{(\pm)} \) defined as in equation (6). This fact simply follows from the factorisation formula (4) valid in the classical and quantum cases. The same factorisation is valid also for the leading order of the respective quantum factors, up to the error term in equation (30). In fact, one could prove equation (8) in the general \( r \)-bin case using the binary case, similar as it was done in section 3.2. Indeed, there exists the following identity for the Kullback–Leibler divergence (an analog of equation (24)):

\[
\sum_{i=1}^{r-1} (N_i \pm M_i) \mathcal{K}_2(\tilde{X}_{i}^{(\pm)}|\tilde{q}_i) = (N \pm M) \sum_{i=1}^{r} X_{i}^{(\pm)} \ln \left( \frac{X_{i}^{(\pm)}}{q_i} \right) = (N \pm M) \mathcal{K}_r(X^{(\pm)}|\vec{q}),
\]

which is proved via the same steps as the respective identity (24) in the classical case appendix B.

Let us now prove theorem 2 in the quantum case. The quantum result in equation (9) can be shown by reduction to the binary case, similar as in section 3.2, where \( i = 1 \) is the index of the first violation of equation (7). Consider the respective quantum probability \( P_{N,M}^{(+)}(n_1|K_1) = P_{N,M}^{(0)}(n_1|K_1) \mathcal{Q}_{N,M}^{(r)}(n_1|K_1) \) which enters the factorisation (4) in the quantum case (below we drop the subscript 1 for simplicity).

Let us first focus on the case of bosons. From equation (C.10) of appendix C for \( M \gg 1 \) we get (see equations (13) and (29)):

\[
\mathcal{Q}_{N,M}^{(+)}(n|K) < (1 + \alpha) \exp \left\{ N \left( 1 + \frac{1}{\alpha} \right) \mathcal{K}_2(X^{(+)}|q) \right\} \left[ 1 + O \left( \frac{1}{M} \right) \right].
\]

The quantum probability \( P_{N,M}^{(+)}(n|K) = P_{N,M}^{(0)}(N|K) \mathcal{Q}_{N,M}^{(+)}(n|K) \) involves a combination of two Kullback–Leibler divergencies, see equations (16) and (32):

\[
NK_2(x|q) - N \left( 1 + \frac{1}{\alpha} \right) \mathcal{K}_2(X^{(+)}|q) = N \left[ \mathcal{K}_2(x|X^{(+)} + \frac{1}{\alpha} \mathcal{K}_2(q|X^{(+)}) \right].
\]

By using Pinsker’s inequality (17) and equation (29) we obtain:

\[
\mathcal{K}_2(x|X^{(+)} + \frac{1}{\alpha} \mathcal{K}_2(q|X^{(+)})) > \left( X^{(+)} - x \right)^2 + \frac{1}{\alpha} \left( X^{(+)} - q \right)^2 = \frac{(x - q)^2}{1 + \alpha}.
\]

Finally, taking into account that by our assumption \(|x - q| > AN^{-1/3-\epsilon}\) and that \( \alpha \) is kept constant in theorem 2, from equations (16), (32)–(35) we get the required estimate:

\[
P_{N,M}^{(+)}(n|K) < 2\pi \sqrt{N}(1 + \alpha) \exp \left\{ - \frac{A^2}{1 + \alpha} N^{1/2 - 2\epsilon} \right\} \left[ 1 + O \left( \frac{1}{N} \right) \right].
\]
\[ p^{(-)}(n|K) = \frac{N!}{(M - N + 1) \ldots M} \frac{(M - K - [N - K] + 1) \ldots (M - K)}{(N - K)!} \]

\[ = \left( \frac{N}{M} \right)^{K-1} \prod_{l=1}^{K-1} \frac{1 - l/K}{1 - l/M} < \left( \frac{N}{M} \right)^{K} = \exp \left\{ -N \frac{\ln(1/\alpha)}{q} \right\}. \quad (36) \]

i.e. it falls faster with \( N \) than the estimate in equation (9). Consider now the opposite case \( n \leq K - 1 \) and \( N - N \leq M - K - 1 \). We have in this case \( \chi^{(-)}, 1 - \chi^{(-)} \geq 1/(M - N) = N \frac{\alpha}{(1 - \alpha)N} \). Therefore, from equation (C.10) of appendix C we get:

\[ Q_{N,M}^{(-)}(n|K) < \frac{q(1 - q)}{(1 - \alpha)^2N^2} \exp\left\{ -N \frac{\ln(1/\alpha)}{q} \right\} \left[ 1 + \mathcal{O}\left( \frac{1}{M} \right) \right] \]

\[ = \mathcal{O}\left( N^2 \exp \left\{ -A^2 \frac{1}{1 - \alpha} N^{1 - 2\epsilon} \right\} \right). \quad (37) \]

where we have expanded the Kullback–Leibler divergence as in equations (15), used (29) and Pinsker’s inequality (17) together with the assumption \( |x - q| > AN^{-1/3-\epsilon} \) for \( \epsilon < 1/6 \). Recalling the respective classical bound (18) we get equation (9) for fermions. This concludes the proof of theorem 2 in the quantum case.

Finally, as in the classical case, in the quantum case the asymptotic Gaussian for the \( r \)-bin partition in equation (8) is a product of the asymptotic Gaussians for the binary partitions, which appear in equation (4), to the accuracy of the approximation in equation (8), due to the analogous identity, equation (31). This fact relates the statements of theorems 1 and 2 to those of the binary case via the equivalence of equations (7) and (19).

4. Arbitrary (mixed) input state

In section 2 in the formulation of theorems 1 and 2 we have assumed a Fock input state \( |n, m\rangle = |n_1, \ldots, n_M; m\rangle \) of \( N \) indistinguishable identical particles. However, it is easy to see that the theorems generalise to an arbitrary input state:

\[ \rho = \sum_{n,m} \rho_{n,m} |n, m\rangle \langle m, n|, \]

where the summation is over \( |n| = |m| = N \) \(|n| \equiv n_1 \ldots n_M\). Let us consider bosons first. Using the expansion of the input Fock state \( |n, m\rangle \) over the output \( |s, out\rangle \) [12]:

\[ |n, m\rangle = \sum_s \frac{1}{\sqrt{n!s!}} \text{per}(U|n|s))|s, out\rangle. \quad (39) \]

where the summation is over all \( |s| = N \), \( n! \equiv n_1 \ldots n_M! \), and \( \text{per}(\ldots) \) denotes the matrix permanent [32], in our case of the \( N \)-dimensional submatrix of the \( M \)-port matrix \( U \) built on the rows and columns corresponding to the occupations \( n \) and \( s \), respectively. Given the input state in equation (38), the average probability to detect an output configuration \( |l\rangle \), corresponding to occupations \( s \), reads:

\[ p^{(+)}(l|\rho) = \frac{1}{s!} \sum_{n,m} \frac{\rho_{n,m}}{\sqrt{n!m!}} \left( \text{per}(U|n|s)) \left( \text{per}(U|m|s)) \right)^* \right). \quad (40) \]
Let us evaluate the average by expanding the matrix permanents:

\[
\langle \text{per}(U[n]|s) \rangle \langle \text{per}(U[m]|s) \rangle^* = \left( \sum_{\mu_1,\ldots,\mu_N} \prod_{i=1}^N U_{\mu_i(1),i}^* U_{\mu_i(2),i}^* \right)
\]

\[
= \sum_{\mu_1,\ldots,\mu_N} \sum_{\nu_1,\ldots,\nu_N} \mathcal{W}(\tau_{\nu}) \prod_{i=1}^N \delta_{\nu_1(i),\mu_1(i)} \delta_{\nu_2(i),\mu_2(i)}
\]

\[
= \delta_{n,m} \sum_{\mu_1,\ldots,\mu_N} \sum_{\nu_1,\ldots,\nu_N} \sum_{\chi,\mu,\nu} \mathcal{W}(\tau_{\nu}) \delta_{\mu_1,\mu_1^{-1}} \chi \delta_{\tau,\mu}
\]

\[
= \frac{\delta_{n,m} \mathcal{W}[n]|n|!}{(M+N-1)!} \ldots(M!}
\]

(41)

where \( k = (k_1,\ldots,k_N) \) and \( k' = (k'_1,\ldots,k'_N) \) are the input ports corresponding to the occupations \( n \) and \( m \), respectively, \( l = (l_1,\ldots,l_N) \) are the output ports corresponding to the occupations \( s \), \( S_N \) is the group of permutations of \( N \) elements (the symmetric group), whereas \( S_N \equiv S_{n_1} \otimes \ldots \otimes S_{n_N} \), \( \mathcal{W} \) is the Weingarten function of the unitary group [33, 34], and \( \delta_{n,m} \equiv \prod_{i=1}^N \delta_{n_i,m_i} \). We have used the known expression for the last sum on the right hand side of equation (41) (derived in the supplemental material to [5]).

Equation (41) tells us that the non-diagonal elements of the mixed state in equation (38) do not contribute to the average probability, if the averaging is performed over the Haar-random unitary matrix \( U \) (the ratio on the right hand side of equation (41) is the average probability \( p^{(+)}(l|k) \), see section 2). Since theorems 1 and 2 hold for any Fock input state \( |n, in \rangle \), we conclude that they hold for the general input of equation (38).

For fermions, the analogs of equations (40) and (41) (in this case \( m_i, n_i, s_i \leq 1 \)) are obtained by replacing the permanent by the determinant, which results in the appearance of the sign functions \( \text{sgn}(\mu_{1,2}) \) and \( \text{sgn}(\mu) \) (\( \mu = \mu_{1,2} \)) in equation (41) where there are \( \mu_{1,2} \) and \( \mu \). In this case the last summation in equation (41) reads \( N! \sum_{\mu} \text{sgn}(\mu) \mathcal{W}(\mu) = \frac{N!}{M-1} \). (see the supplemental material to [5]) i.e. we get the average probability \( p^{(+)}(l|k) \). The same conclusion holds.

5. Conclusion

We have given a rigorous formulation of the results on the asymptotic form of the average counting probability of identical particles in the binned-together output ports of a Haar-random multiport, presented recently in [1] with only a heuristic derivation and some numerical evidence. The key observation was that, both in the classical and quantum cases, there is a convenient factorisation of the average probability for the \( r \)-bin case into \( r-1 \) average counting probabilities for the two-bin case. Moreover, the results of [1] were extended to an arbitrary mixed input state of \( N \) indistinguishable particles.

In the classical case, we have shown that the de Moivre–Laplace theorem, which provides an asymptotic form of the binary average counting probability, actually applies also to the \( r \)-bin case via the above factorisation. The asymptotic Gaussian form also satisfies the mentioned factorisation to an error of the same order as in the Moivre–Laplace theorem. Finally, though we have considered a physical model involving a random unitary multiport, where the probabilities of \( r \) events are rationals (each probability equal to a fraction of the respective
number of ports), the results apply for a general multinomial distribution with arbitrary such probabilities (since the factorisation is derived for the general probabilities).

Our primary interest, however, was the quantum case, when there are correlations between the identical particles due to their quantum statistics. We have formulated and proven a quantum analog of the de Moivre–Laplace theorem for the indistinguishable identical bosons and fermions (and generalised it to the $r$-bin case), where again the binary case applies to the $r$-bin case by the above mentioned factorisation (and, similarly to the classical case, the asymptotic Gaussian also satisfies the same factorisation to the order of the approximation error). Therefore, besides giving a rigorous formulation of the recently discovered quantum asymptotic Gaussian law and extending them to the arbitrary input state, we have also provided an illuminating insight on how the general $r$-bin case reduces to the binary case.

Our results have immediate applications for the counting probability (in the binned-together output ports) of identical particles propagating in disordered media and chaotic cavities, and also for the scattershot version of the Boson sampling.

Acknowledgments

The research was supported by the National Council for Scientific and Technological Development (CNPq) of Brazil, grant: 304129/2015-1, and by the S\'ao Paulo Research Foundation (FAPESP), grant: 2015/23296-8.

Appendix A. The factorisation of the counting probability

Consider first the classical case. We have (for general $q_1, \ldots, q_r$):

$$P(n_1, \ldots, n_r | q_1, \ldots, q_r) = \frac{N!}{\prod_{i=1}^{r} n_i!} \prod_{i=1}^{r} q_i^{n_i} = \frac{N!}{n_1!(N-n_1)!} q_1^{n_1} (1-q_1)^{N-n_1}$$

$$\times \left(\frac{N-n_1}{n_2}\right) \prod_{i=2}^{r} \left( \frac{q_i}{1-q_i} \right)^{n_i} = \ldots$$

$$= P(n_1, N-n_1 | q_1, 1-q_1) \ldots P(n_{r-1}, N-n_{r-1} | \tilde{q}_{r-1}, 1-\tilde{q}_{r-1}),$$

(A.1)

where the dots denote the sequential factorisation (similar to that in the first line), where we have taken into account the definitions in equations (5) and (6) and that:

$$\frac{q_2}{1-q_1} = \tilde{q}_2, \quad \frac{q_3}{(1-q_1)(1-q_2)} = \frac{q_3}{1-q_1-q_2} = \tilde{q}_3, \ldots$$

Equation (A.1) implies the stated factorisation for the classical probability $P^{(0)}(n|K)$.

Now let us consider the quantum case. We have:

$$p^{(B,F)}(n_1, \ldots, n_r | K_1, \ldots, K_r) = \frac{N!}{\prod_{i=1}^{r} n_i!} \left( \frac{(M-1)!}{(M\pm N\mp 1)!} \right) \prod_{i=1}^{r} \frac{(K_i \pm n_i \mp 1)!}{(K_i-1)!}$$

$$= \frac{N!}{n_1!(N-n_1)!} \left( \frac{(M-1)!}{(M\pm N\mp 1)!} \right) \frac{(K_1 \pm n_1 \mp 1)!}{(K_1-1)!} \ldots$$

$$\times p^{(B,F)}(n_2, \ldots, n_r | K_2, \ldots, K_r) = \ldots$$

$$= p^{(B,F)}(n_1, N-n_1 | K_1, M-K_1) \ldots p^{(B,F)}(n_{r-1}, N_{r-1} - n_{r-1} | K_{r-1}, M_{r-1} - K_{r-1}),$$

(A.2)
where again the sequential factorisation was employed and equation (5) (for instance, in the second factorisation we have \( \sum_{i=2}^{r} K_i = M - K_1 = M_2 \) and \( \sum_{i=2}^{r} n_i = N - n_1 = N_2 \)).

**Appendix B. An identity for the Kullback–Leibler divergence**

Let us rewrite the Kullback–Leibler divergence in equation (21) (using also equation (6)) as follows:

\[
N \left\{ \sum_{i=1}^{r} x_i \ln \left( \frac{x_i}{q_i} \right) + Z_i \ln \left( \frac{Z_i}{Q_i} \right) - Z_{i-1} \ln \left( \frac{Z_{i-1}}{Q_{i-1}} \right) \right\}, \tag{B.1}
\]

where we have noted

\[
Z_l \equiv 1 - \sum_{i=1}^{l} x_i, \quad Q_l \equiv 1 - \sum_{i=1}^{l} q_i. \tag{B.2}
\]

Now it is easy to see that due to the form of the last two terms in equation (B.1) the sum of the Kullback–Leibler divergencies as in equation (B.1) with \( l = 1, \ldots, r-1 \) which appear in equation (21) give:

\[
\sum_{l=1}^{r-1} N_l K_2(x_l | q_l) = N \left\{ \sum_{l=1}^{r-1} x_l \ln \left( \frac{x_l}{q_l} \right) + Z_{r-1} \ln \left( \frac{Z_{r-1}}{Q_{r-1}} \right) \right\} = N \sum_{l=1}^{r} x_l \ln \left( \frac{x_l}{q_l} \right), \tag{B.3}
\]

since \( Z_{r-1} = x_r \) and \( Q_{r-1} = q_r \).

**Appendix C. Asymptotic form of \( \prod_{l=1}^{n} \left[ 1 \pm \frac{l}{m} \right] \)**

We will use the second-order Euler’s summation formula [35]:

\[
\sum_{l=1}^{n} f(l) = \int_{1}^{n} dx f(x) + \frac{f(n) + f(1)}{2} + \frac{f^{(1)}(n) - f^{(1)}(1)}{12} - \frac{1}{2} \int_{1}^{n} dx P_2(x) f^{(2)}(x), \tag{C.1}
\]

where \(|P_2(x)| \leq 1/6\). Setting \( f(x) = \ln(1 \pm x/m) \) (in our case \( 1 \leq x \leq n \)), we observe that

\[
f^{(1)}(n) - f^{(1)}(1) = \frac{1}{n \pm m} - \frac{1}{1 \pm m} = O \left( \frac{n}{m(m \pm n)} \right) \tag{C.2}
\]

and

\[
\left| \int_{1}^{n} dx P_2(x) f^{(2)}(x) \right| \leq \frac{|f^{(1)}(n) - f^{(1)}(1)|}{6}. \tag{C.3}
\]

Then by integrating:

\[
\int_{1}^{n} dx \ln \left( 1 \pm \frac{l}{m} \right) = (n \pm m) \ln \left( 1 \pm \frac{n}{m} \right) - (1 \pm m) \ln \left( 1 \pm \frac{1}{m} \right) - n + 1, \tag{C.4}
\]

and using equations (C.1)–(C.3) we obtain equation (26).

The above can be used to find the upper and lower bounds using that the two involved functions \( f^{(\pm)} = \ln (1 \pm x/m) \) are strictly monotonous. Let us first consider \( f^{(+)}(x) \). By the geometric consideration similar to that of [35] one can easily establish that:
\[
\int_1^n \, dx \, f'^+(x) < \sum_{l=1}^n f'(l) < \int_1^n \, dx \, f'^+(n). \tag{C.5}
\]

Taking \( \hat{f}'(x) \equiv -f'(-x) \) and using equation (C.2) we get:
\[
\int_1^n \, dx \, f'^-(-x) + f'^-(n) < \sum_{l=1}^n f'^-(l) < \int_1^n \, dx \, f'^-(-x). \tag{C.6}
\]

Equations (C.4)–(C.6) allow to get the announced bounds. First of all, for \( m \gg 1 \) (in our case \( K_i \) or \( M \) take place of \( m \)) we can approximate:
\[
\left( 1 \pm \frac{1}{m} \right)^{1 \pm m} = \exp \left\{ \left( 1 \pm m \right) \sum_{p=1}^{\infty} \frac{(-1)^{p-1} (\pm 1)^p}{p^m} \right\} = \exp \left\{ 1 + O \left( \frac{1}{m} \right) \right\}. \tag{C.7}
\]

Therefore, from equations (C.4)–(C.7) we obtain (the upper digit in the parenthesis is for the plus sign, while the lower choice is for the minus sign):
\[
\prod_{l=1}^n \left( 1 \pm \frac{1}{m} \right) < \left( 1 \pm \frac{n}{m} \right)^{1 \pm m} e^{-n} \left[ 1 + O \left( \frac{1}{m} \right) \right], \tag{C.8}
\]
\[
\prod_{l=1}^n \left( 1 \pm \frac{1}{m} \right) \geq \left( 1 \pm \frac{n}{m} \right)^{1 \pm m} e^{-n} \left[ 1 + O \left( \frac{1}{m} \right) \right]. \tag{C.9}
\]

Let us now find the bounds on the quantum factor in equation (27). Using the definition of the Kullback–Leibler divergence (24) and the quantities \( X_{\pm i} \) defined in equation (29) for \( M \gg 1 \) we obtain from equations (C.8) and (C.9) (with \( \sigma = + \) for bosons, the upper line in the parenthesis, and \( \sigma = - \) for fermions, the lower line in the parenthesis):
\[
Q^{(\sigma)} < Q'^{(\sigma)} \left\{ \frac{1 + \alpha}{(1 - \alpha)^{-\tau} \prod_{l=1}^\infty \left( \frac{X'^{-}(l)}{\psi} \right)^{-1}} \right\} \left[ 1 + O \left( \frac{1}{M} \right) \right], \tag{C.10}
\]
\[
Q^{(\sigma)} > Q'^{(\sigma)} \left\{ \frac{(1 + \alpha)^{-\tau} \prod_{l=1}^\infty \left( \frac{X'^{+}(l)}{\psi} \right)^{-1}}{1 - \alpha} \right\} \left[ 1 + O \left( \frac{1}{M} \right) \right], \tag{C.11}
\]

with
\[
Q'^{(\pm)} \equiv \exp \left\{ (N \pm M)K_r \langle X^{(\pm)}(q) \rangle \right\} \tag{C.12}
\]
and fixed \( q_i = K_i/M \) as \( M \to \infty \). Equation (C.10) will be of use in the proof of equation (9) in theorem 2.

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