Angular Momentum Mixing in a Non-spherical Color Superconductor

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We study the angular momentum mixing effects in the color superconductor with non-spherical pairing. We first clarify the concept of the angular momentum mixing with a toy model for non-relativistic and spinless fermions. Then we derive the gap equation for the polar phase of dense QCD by minimizing the CJT free energy. The solution of the gap equation consists of all angular momentum partial waves of odd parity. The corresponding free energy is found to be lower than that reported in the literature with \( p \)-wave only.

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I. INTRODUCTION

The properties of quark matter at extreme conditions have been an active research area both theoretically and experimentally. At high temperature, the quark-gluon plasma (QGP) has long been searched by colliding two nuclei at sufficiently high energy. On the other side, we expect that quark matter becomes color superconducting through a phase transition at high baryon density but low temperature\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\), which is the typical condition inside compact stars.

In a typical metallic superconductor, the electrons pair with equal chemical potential near the Fermi surface. The situation with a quark matter, however, is much more complicated. While the quark-quark interaction favors pairing between quarks of different flavors, the mass difference among \( u \), \( d \) and \( s \) together with the charge neutrality requirement induces a substantial mismatch among their Fermi momenta at the baryon density inside a compact star and thereby reduces the available phase space for Cooper pairing. A number of exotic color superconductivity phases in the presence of mismatch have been proposed in the literature\(^7\)\(^8\)\(^9\)\(^10\)\(^11\), but a consensus point of view of the true ground state has not been reached. The single flavor pairing\(^12\)\(^13\)\(^14\)\(^15\)\(^16\), which is free from the Fermi momentum mismatch, is an interesting alternative in this circumstance and will be considered here. Since the quark-quark interaction is attractive in the color anti-triplet channel, the color wave function of the pair is anti-symmetric. For the equal helicity pairing to be considered in this article, the parity of the orbital wave function has to be odd as required by the Pauli principle. Except for the color-spin-lock phase examined in \(^12\), the energy gap will not be spherical. The odd parity prevents the diquark wave function from realizing the full pairing potential. The energy scale of the color superconductivity is therefore reduced.

At ultra-high baryon densities, asymptotic freedom of QCD ensures the validity of the weak coupling expansion, which has been carried out for CSC by a number of authors\(^6\)\(^17\)\(^18\)\(^19\)\(^20\)\(^21\)\(^22\)\(^23\)\(^24\)\(^25\)\(^26\)\(^27\). The dominant pairing interaction is mediated by one-gluon exchange and can be decomposed into partial waves, as is shown in Eq.\(^{[29]}\) below. Quantitative results of the transition temperature in the equal helicity channel with an arbitrary angular momentum has been obtained from the first principle\(^21\)\(^22\), and read

\[
T_c^{(J)} = 512 \pi^3 \left( \frac{2}{N_f} \right)^{5/2} \frac{\mu}{g^5} \exp \left[ - \frac{3\pi^2}{\sqrt{2}g} + \gamma - \frac{1}{8}(\pi^2 + 4) + 3c_J \right]
\]

where \( N_f \) is the number of flavor, \( N \) is the number of colors, \( \mu \) is the chemical potential, \( g \) is the running coupling constant of QCD and \( \gamma (= 0.5772\ldots) \) is the Euler constant. The \( J \)-dependent constant

\[
c_J = \begin{cases} 
0, & \text{for } J = 0, \\
-2 \sum_{n=1}^{J} \frac{1}{n} & \text{for } J > 0.
\end{cases}
\]
The equal helicity pairing of odd parity picks up $T_c^{(1)}$ as the transition temperature. We have $T_c^{(1)} = e^{-6}T_c^{(0)} \approx 2.479 \times 10^{-3}T_c^{(0)}$. Natural analogy is drawn with the superfluidity of He$^3$. But important difference between the pairing potential in quark matter and that in He$^3$ has to be considered before ascertaining the angular dependence of the energy gap. The forward singularity of the one-gluon exchange renders the pairing strength equal for all partial waves (the same leading order term inside the bracket of (1)). The paring potential in He$^3$, however, is entirely in the channel of $J = 1$.

The transition temperature was determined from the pairing instability of the diquark scattering amplitude in the normal phase. In a perturbative treatment, the Dyson-Schwinger equation for the scattering amplitude is linear and the partial wave decomposition in Eq.(1) is legitimate. This is not the case with the gap equation below as the transition temperature. We have

$$\phi_M = \phi_{2SC} f_M(\hat{p})$$

where $\phi_{2SC}$ is the gap function of 2SC in the absence of the mismatch, $\hat{p}$ the direction of the relative momentum of the two quarks in a Cooper pair and the angular dependent factor

$$f_M(\hat{p}) = \sum_{J=1,3,5,\ldots} b_J Y_{JM}(\hat{p})$$

with $J$ the total angular momentum of the Cooper pair. Carrying the formulation of He$^3$ over to QCD amounts to drop all higher multipoles except that of $J = 1$, which will not satisfy the gap equation of QCD. It was argued in the literature that $b_1 = O(1)$ but $b_{J} = O(\mu)$ for $J \neq 1$. This, as will be shown below, is not the case. Instead, we find that the function $f_M(\hat{p})$ satisfies a nontrivial integral equation and thus $b_J = O(1)$ for all odd $J$’s. Therefore the angular momentum mixing does occurs in the subleading order of the gap function. The angular momentum mixing will modify all non-spherical "spin-1" CSC examined in the literature, we shall focus our attention in this paper to the equal helicity pairing with zero azimuthal quantum number, i.e. the analog of the polar phase of He$^3$. The subscript $M$ of $\phi_M$ and $f_M$ will be suppressed below. Even though this phase is unstable, it is the simplest one to illustrate the mixing mechanism.

The current work is organized as follows. In the next section, we shall clarify the concept of the angular momentum mixing with a toy model of non-relativistic and spinless fermions. In the Sect. III the gap equation for the single flavor CSC will be derived by minimizing the CJT free energy of QCD. This gap equation will be reduced to an nonlinear integral equation for the function $f(\hat{p})$ in the subsequent section and the numerical solution will be presented in the Sect. V. We conclude the paper in the Sect. VI. Some technical details are deferred to the Appendices. Our units are $\hbar = c = k_B = 1$ and 4-vectors are denoted by capital letters, $K \equiv K' = (k_0, \hat{k})$ with $k_0$ the Matsubara energy, which becomes continuous at $T_c = 0$. Throughout the article, we shall follow the definition of the leading order and the subleading order in $\hat{p}$. Upon taking the logarithm of the transition temperature or the magnitude of the gap function, the order $O(\frac{1}{\hat{p}})$ will be referred to as the leading one and the $O(1)$ term to the subleading one.

II. A TOY MODEL WITH ANGULAR MOMENTUM MIXING

To clarify the concept of the angular momentum mixing, we consider a toy model of nonrelativistic and spinless fermions. In terms of the creation and annihilation operators, the model Hamiltonian reads

$$H = \sum_{\hat{p}} \epsilon_{\hat{p}} a_{\hat{p}}^\dagger a_{\hat{p}} - \frac{\lambda}{4\Omega} \sum_{\hat{p},\hat{p}'} V(\hat{p} \cdot \hat{p}') a_{\hat{p}}^\dagger a_{-\hat{p}'} a_{\hat{p}-\hat{p}'}$$

where $\epsilon_{\hat{p}} = \frac{\hbar^2}{2m} - \mu$ with $m$ the mass and $\mu$ the chemical potential, $\lambda > 0$ is a coupling constant, $\Omega$ is the normalization volume and the summation $\sum_{\hat{p},\hat{p}'}$ extends to states with $|\epsilon_{\hat{p}}| < \omega_D$ and $|\epsilon_{\hat{p}'}| < \omega_D$ with $\omega_D$ a UV cutoff (Debye frequency for electronic superconductors). The angular dependent form factor $V(\hat{p} \cdot \hat{p}')$ can be expanded in series of the Legendre polynomials,

$$V(\hat{p} \cdot \hat{p}') = \sum_{J=0}^{\infty} (2J + 1) V_J P_J(\hat{p} \cdot \hat{p}')$$

where $P_J$ is the Legendre polynomial of degree $J$, and $V_J$ is the coupling constant for the interaction of $J$-th partial wave.}


Introducing the order parameter
\[ \chi(p) = < |a_p a_p^\dagger | > \] (7)
with \( | > \) the ground state and expanding the interaction term of (13) to the linear order of the fluctuation \( a_p a_p^\dagger - < |a_p a_p^\dagger | > \), we end up with the mean-field Hamiltonian
\[ H_{MF} = \frac{1}{2} \sum_{\vec{p}, |\epsilon_p| < \omega_D} \chi^*(\vec{p}) \phi(\vec{p}) + \sum_{\vec{p}} \epsilon_p a_p^\dagger a_p - \frac{1}{2} \sum_{\vec{p}, |\epsilon_p| < \omega_D} \left[ \phi^*(\vec{p}) a_p a_p^\dagger + \phi(\vec{p}) a_p^\dagger a_p^\dagger \right], \] (8)
where we have introduced the gap function via
\[ \phi(\vec{p}) = \frac{\lambda}{2\Omega} \sum_{\vec{p}, |\epsilon_p| < \omega_D} V(\vec{p} \cdot \vec{p}') \chi(p'). \] (9)
We have \( \chi(-\vec{p}) = -\chi(\vec{p}) \) and \( \phi(-\vec{p}) = -\phi(\vec{p}) \) following from their definitions. Upon a Bogoliubov transformation, we find that
\[ \chi(p) = \frac{\phi(p)}{2\epsilon_p} \] (10)
with \( \epsilon_p = \sqrt{\epsilon_p^2 + |\phi(p)|^2} \) and the ground state energy
\[ E_0 = \sum_{\vec{p}, \epsilon_p < 0} \epsilon_p + \Omega F \] (11)
with \( F \) the condensation energy density given by
\[ F = \frac{1}{2} \sum_{\vec{p}, |\epsilon_p| < \omega_D} \left[ \frac{\phi^*(\vec{p}) \phi(\vec{p})}{2\epsilon_p} + |\epsilon_p| - \epsilon_p \right]. \] (12)
Substituting (11) into (9), we obtain the gap equation
\[ \phi(\vec{p}) = \frac{\lambda}{4\Omega} \sum_{\vec{p}, |\epsilon_p| < \omega_D} V(\vec{p} \cdot \vec{p}') \frac{\phi(p')}{\epsilon_p}. \] (13)
In the weak coupling limit, \( \omega_D << \mu \) and \( \lambda D_F << 1 \) with \( D_F = \frac{\pi^2}{\sqrt{2\pi^2}} \) the density of states on the Fermi surface, but the magnitude of \( V(\vec{p} \cdot \vec{p}') \) remains of order one. We have
\[ \sum_{\vec{p}, |\epsilon_p| < \omega_D} = \Omega \int_{|\epsilon_p| < \omega_D} \frac{d^3\vec{p}}{(2\pi)^3} \simeq \frac{\Omega D_F}{4\pi} \int d^2\vec{p} \int_{-\omega_D}^{\omega_D} d\epsilon. \] (14)
Also, the support of the gap function extends only to a narrow band around the Fermi surface. We may ignore the dependence of \( \phi \) on the magnitude \( p = |\vec{p}| \) and switch the argument of \( \phi \) from \( \vec{p} \) to \( \vec{p}' \). Following (14), the integrations over \( p \) in (12) and (13) can be carried out readily and we end up with
\[ F = -\frac{D_F}{16\pi} \int d^2\vec{p} |\phi(\vec{p})|^2. \] (15)
and
\[ \phi(\vec{p}) = \frac{\lambda D_F}{8\pi} \int d^2\vec{p} V(\vec{p} \cdot \vec{p}') \phi(\vec{p}') \ln \frac{2\omega_D}{|\phi(\vec{p}')|}. \] (16)
The gap equation (16) is nonlinear because of the logarithm.
In what follows, we consider two extremes of \( V(\vec{p} \cdot \vec{p}') \), each of which gives rise to an exact solution to the gap equation (16). We present only the solution that is invariant under time reversal, i.e. the one with zero azimuthal quantum number.
**Case I:**

\[ V(\hat{p} \cdot \hat{p}^\prime) = 3P_1(\hat{p} \cdot \hat{p}^\prime) = 3\hat{p} \cdot \hat{p}^\prime. \]  

It corresponds to the partial wave expansion (6) with \( v_J = 1 \) and \( v_J = 0 \) for \( J \neq 1 \). The angular dependence of the pairing force in \( \text{He}^3 \) is of this type. The gap equation reads

\[ \phi(\hat{p}) = \frac{3\lambda D_F}{8\pi} \int d^2 \hat{p}^\prime \hat{p} \cdot \hat{p}^\prime \phi(\hat{p}^\prime) \ln \frac{2\omega_D}{|\phi(\hat{p}^\prime)|} \]  

and its solution of zero azimuthal quantum number is given by

\[ \phi(\hat{p}) = \phi_0 \cos \theta = \phi_0 P_1(\cos \theta) \]

with \( \theta \) the angle with respect to a prefixed direction in space and

\[ \phi_0 = 2\omega_D e^{-\frac{D_F}{\pi} \frac{1}{2}}. \]

The condensation energy density

\[ F = -\omega_D^2 D_F e^{-\frac{D_F}{\pi} \frac{1}{2}} \simeq -0.6492\omega_D^2 D_F e^{-\frac{D_F}{\pi} \frac{1}{2}}. \]

This solution corresponds to the polar phase of \( \text{He}^3 \) [23]. Since the gap function contains only the partial wave of \( J = 1 \), there is no angular momentum mixing. The additional term in the exponent of (20), \( \frac{1}{3} \), comes from the logarithm of (18).

**Case II:**

\[ V(\hat{p} \cdot \hat{p}^\prime) = 4\pi \delta^2(\hat{p} - \hat{p}^\prime) = \sum_J (2J + 1)P_J(\hat{p} \cdot \hat{p}^\prime). \]

This corresponds to a singularity of the two body scattering amplitudes in the forward direction. We have \( v_J = 1 \) for all \( J \) in (9). The last step of (22) follows from the addition theorem and the completeness of the spherical harmonics. The gap equation (16) becomes

\[ 1 = \frac{\lambda D_F}{2} \ln \frac{2\omega_D}{|\phi(\hat{p})|} \]  

which implies a constant \( |\phi(\hat{p})| \) and yields a solution of odd parity and zero azimuthal quantum number.

\[ \phi(\hat{p}) = 2\omega_D e^{-\frac{D_F}{\pi} \frac{1}{2}} \text{sign}(\cos \theta) = 2\omega_D e^{-\frac{D_F}{\pi} \frac{1}{2}} \sum_{n=0}^\infty (-1)^n(4n + 3) \frac{(2n - 1)!!}{2^{n+1}(n + 1)!} P_{2n+1}(\cos \theta). \]

The condensation energy density in this case reads

\[ F = -\omega_D^2 D_F e^{-\frac{D_F}{\pi} \frac{1}{2}}. \]

We refer to this case as the case with the angular momentum mixing because the gap function (24) contains all partial waves. Carrying the solution of the case I to the case II amounts to drop all partial waves other than that of \( J = 1 \) and would lead to a lower magnitude of the condensation energy (21).

The case with QCD is similar to the case II above since the forward singularity of the diquark scattering renders the pairing strength of all partial waves equal to the leading order. The running coupling constant \( g \) of QCD corresponds to \( \lambda \) here and the angular momentum mixing shows up in the \( O(1) \) term of \( \ln |\phi| \). Therefore we expect angular momentum mixing to the subleading order of the angular dependence of the gap function. Besides being an ultra relativistic system, the CSC of QCD differs from the toy model considered above in two aspects. The forward singularity of QCD also brings about the energy dependence of the gap, so the gap equation (13) will be replaced by the Eliashberg equation derived by minimizing the CJT effective action of QCD. Secondly, the pairing strength of each partial wave does fall off with an increasing \( J \) in the sub-leading order of the pairing potential. It is this falling off that makes the amount of the angular momentum mixing numerically small for the solution considered in this article.
III. DERIVATION OF THE GAP EQUATION FROM THE CJT FREE ENERGY

The QCD Lagrangian for one flavor of massless quark is given by

\[ \mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu + \mu \gamma_0) \psi - \frac{1}{4} G^{\alpha\nu}_a G^a_{\mu\nu} + \text{renormalization counterterms} \] (26)

where, \( \psi \) is the quark spinor in Dirac and color space and \( \bar{\psi} = \psi^\dagger \gamma_0 \). The covariant derivative acting on the fermion field is \( D_\mu = \partial_\mu + igT^a A^a_\mu \), where \( g \) is the running coupling constant, \( A^a_\mu \) is the gauge potential, \( T^a = \frac{1}{2} \lambda^a \) \((a = 1, \ldots, 8)\) is the \( a \)-th \( SU(3)_c \) generator with \( \lambda^a \) the \( a \)-th Gell-Mann matrix. \( G^{a\mu\nu}_a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \) is the field strength tensor. Introducing the Nambu-Gorkov spinor \( \Psi = (\psi, \psi_C) \), \( \bar{\Psi} = (\bar{\psi}, \bar{\psi}_C) \) (27)

where \( \psi_C = C \bar{\psi}^T \) is the charge-conjugate spinor and \( C \equiv i\gamma^2\gamma^0 \), the CJT effective action reads\[6, 30\]

\[ \Gamma[D, S] = -\frac{1}{2} \left\{ \text{Tr} \ln D^{-1} + \text{Tr} (D_0^{-1} D - 1) - \text{Tr} S^{-1} - \text{Tr} (S_0^{-1} S - 1) - 2\Gamma_2[D, S] \right\} \] (28)

where \( D \) and \( S \) are the full gluon and quark propagators, \( D_0^{-1} \) and \( S_0^{-1} \) are the inverse tree-level propagators for gluons and quarks, respectively. \( \Gamma_2 \) is the sum of all two-particle irreducible(2PI)vacuum diagrams built with \( D, S \) and the tree-level quark-gluon vertex \( \Gamma \). We have

\[ \Gamma_2 = -\frac{1}{4} \text{Tr} (D \Gamma S \bar{\Gamma} S) + ..., \] (29)

where the first term corresponds to the sunset diagram of Fig.1 and the contribution from \( ... \) is beyond the subleading order of the gap function\[6\].

The stationary points of the CJT effective action are determined by

\[ \frac{\delta\Gamma}{\delta D} = 0, \quad \frac{\delta\Gamma}{\delta S} = 0 \] (30)

which gives rise to the Dyson-Schwinger equation for gluons and quarks,

\[ \Pi = -2 \frac{\delta\Gamma_2}{\delta D}, \quad \Sigma = 2 \frac{\delta\Gamma_2}{\delta S} \] (31)

where \( \Pi \) and \( \Sigma \) are the gluon and quark self-energy defined via \( D^{-1} = D_0^{-1} + \Pi \) and \( S^{-1} = S_0^{-1} + \Sigma \). Instead of solving the two equations of (31) simultaneously, we shall reduce the CJT effective free energy with the aid of the first equation, leaving the gap function arbitrary. The gap equation (which is the Nambu-Gorkov off diagonal part of the second equation of (31)) will be derived after the CJT free energy is fully simplified under the weak coupling approximation.

Substituting the first equation of (31) into (28), the second term in Eq. (28) cancels the last term. We have

\[ \Gamma[D, S] = -\frac{1}{2} [\text{Tr} \ln D^{-1} - \text{Tr} \ln S^{-1} - \text{Tr} (S_0^{-1} S - 1)] \] (32)
In Nambu-Gorkov space, the inverse free quark propagator is
\[ S_0^{-1} \equiv \begin{pmatrix} [G^+_{0}]^{-1} & 0 \\ 0 & [G^-_{0}]^{-1} \end{pmatrix} \] (33)

where
\[ [G^\pm_{0}]^{-1} = (p_0 \pm \mu)\gamma_0 - \vec{\gamma} \cdot \vec{p}. \] (34)

On writing the quark self-energy
\[ \Sigma \equiv \begin{pmatrix} \Sigma^+ & \Phi^- \\ \Phi^+ & \Sigma^- \end{pmatrix} \] (35)

the full quark propagator,
\[ S = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix} \] (36)

can be obtained explicitly by inverting the matrix \( S_0^{-1} + \Sigma \).

For the single flavor pairing, the simplest choice of the off-diagonal block of Eq. (35) reads
\[ \Phi^+(P) = i\phi_\gamma \lambda_2 \] (37)

and \( \Phi^+ = \Phi^- \) (see Theorem 2 in [26]), where \( \lambda_2 \) is the 2nd Gell-Mann matrix and \( \phi \) is a function of the energy and the momentum, i.e. \( \phi = \phi(p_0, \vec{p}) \). \( \phi \) is even in \( p_0 \) and odd in \( \vec{p} \). By using the energy projectors of massless fermions \( \Lambda_{\pm}^5 = (1 \pm \gamma_0 \vec{\gamma} \cdot \vec{p})/2 \) and ignoring the contribution from the wave-function renormalization, the NG blocks of the propagator (36) take the form
\[ G^\pm = \frac{p_0 + (p \mp \mu)}{p_0^2 - (p \pm \mu)^2 - \phi^2 \lambda_2^2} \Lambda_p \gamma_0 \pm \frac{p_0 (p \pm \mu)}{p_0^2 - (p \pm \mu)^2 - \phi^2 \lambda_2^2} \Lambda_p \gamma_0 \] (38)

\[ \Xi^\pm = \frac{i\phi \lambda_2 \gamma_5}{p_0^2 - (p \pm \mu)^2 - \phi^2 \lambda_2^2} \Lambda_p + \frac{i\phi \lambda_2 \gamma_5}{p_0^2 - (p \pm \mu)^2 - \phi^2 \lambda_2^2} \Lambda_p. \] (39)

Because of the \( \lambda_2 \) of (37), the excitation in the third color direction is ungapped.

Now, we proceed to simplify the CJT free energy under the weak coupling approximation. Denote by \( \Gamma_n \) the free energy density of the normal phase, we have
\[ \Gamma = \Gamma_n + \Omega F \] (40)

where the condensate energy density
\[ F = -\frac{1}{2\Omega} \left[ \text{Tr} \ln D^{-1} - \text{Tr} \ln D_n^{-1} - \text{Tr} \ln S^{-1} + \text{Tr} \ln S_0^{-1} - \text{Tr} (S_0^{-1} S - 1) \right] \] (41)

is the part of \( \Gamma \) responsible to the gap equation. Following the procedure of [31, 32], we approximate
\[ \text{Tr} \ln D^{-1} - \text{Tr} \ln D_n^{-1} \simeq \text{Tr} \{ D_n \delta \Pi \} \] (42)

where
\[ \delta \Pi = \Pi - \Pi_n, \] (43)

with \( \Pi_n \) the hard-dense-loop (HDL) resummed gluon self-energy in normal phase and \( D_n \) the corresponding HDL gluon propagator. In the Coulomb gauge, the HDL gluon propagator is
\[ D_{n,00}(K) = D_t(K), \quad D_{n,0i}(K) = D_{n,00}(K) = 0, \quad D_{n,ij} = (\delta_{ij} - \hat{k}_i \hat{k}_j) D_t(K) \] (44)

where \( D_t(K) \) are the longitudinal and transverse propagators respectively and are diagonal in adjoint color space, i.e. \( D_t^{ab}_{\lambda \lambda} = \delta^{ab} D_{t,\lambda \lambda} \). Consequently, we only need the 00-component, \( \Pi^{00}(K) \), and the transverse projection of the ij-components,
\[ (\delta_{ij} - \hat{k}_i \hat{k}_j) \Pi^{ij}(K) = \Pi^{ii}(K) - \hat{k}_i \hat{k}_j \Pi^{ij}(K) \] (45)
The gluon self-energy in super phase reads

\[
\Pi_{\mu\nu}^{ab}(K) = \frac{1}{2} T \sum_{P,P'} \text{Tr} [\hat{\Gamma}_a S(P) \hat{\Gamma}_b S(P')] \]

(46)

where \( K = P - P' \) and

\[
\hat{\Gamma}_a = \begin{pmatrix}
\Gamma_a & 0 \\
0 & \hat{\Gamma}_a
\end{pmatrix}
\]

(47)

with \( \Gamma_a^\mu = \gamma^\mu T_a \) and \( \hat{\Gamma}_a^\mu = -\gamma^\mu T_a^T \). Substituting Eq. (36) into Eq.(46), we find that Nambu-Gorkov space,

\[
\Pi_{\mu\nu}^{ab}(K) = \frac{1}{2} T \sum_{P,P'} \{ \text{Tr}[\Gamma_a^\mu G^+(P)\Gamma_b^\mu G^+(P')] + \text{Tr}[\Gamma_a^\mu G^-(P)\Gamma_b^\mu G^-(P')] \\
+ \text{Tr}[\Gamma_a^\mu \Xi^-(P)\Gamma_b^\mu \Xi^+(P')] \}
\]

(48)

Since the HDL gluon propagators are diagonal in color space, we only need the diagonal terms of Eq.(48) to deal with Eq.(12). The explicit form of each diagonal term of (48) reads

\[
\text{Tr}[\Gamma_a^\mu G^+(P)\Gamma_b^\mu G^+(P')] = \frac{g^2}{2} T \sum_{P,P'} \text{Tr}[\gamma^\mu \Lambda_p^+ \gamma_0^\mu \Lambda_p^+ \gamma_0] w_a^+(P, P'),
\]

(49a)

\[
\text{Tr}[\Gamma_a^\mu G^-(P)\Gamma_b^\mu G^-(P')] = \frac{g^2}{2} T \sum_{P,P'} \text{Tr}[\gamma^\mu \Lambda_p^- \gamma_0^\mu \Lambda_p^- \gamma_0] w_a^-(P, P'),
\]

(49b)

\[
\text{Tr}[\Gamma_a^\mu \Xi^-(P)\Gamma_b^\mu \Xi^+(P')] = -\frac{g^2}{2} T \sum_{P,P'} \text{Tr}[\gamma^\mu \gamma_5 \Lambda_p^+ \gamma_5 \Lambda_p^- \gamma_0] w_a(P, P'),
\]

(49c)

\[
\text{Tr}[\Gamma_a^\mu \Xi^+(P)\Gamma_b^\mu \Xi^-(P')] = -\frac{g^2}{2} T \sum_{P,P'} \text{Tr}[\gamma^\mu \gamma_5 \Lambda_p^- \gamma_5 \Lambda_p^+ \gamma_0] w_a(P, P').
\]

(49d)

where the repeated color indexes on LHS are not to be summed. The quantities \( w^\pm \) and \( w \) on RHS of Eqs. [49a–49d] are given by

\[
w_a^\pm = \begin{cases}
1 \frac{p_0 \pm \epsilon_p g_0 \pm \epsilon_{p'}}{2 p_0 - \epsilon_p^2 p_0^2 - \epsilon_{p'}^2}, & a = 1, 2, 3 \\
\frac{1}{4} \left( \frac{p_0 \pm \epsilon_p g_0 \pm \epsilon_{p'}}{p_0^2 - \epsilon_p^2 p_0^2 - \epsilon_{p'}^2} + \frac{p_0 \pm \epsilon_p g_0 \pm \epsilon_{p'}}{p_0^2 - \epsilon_p^2 p_0^2 - \epsilon_{p'}^2} \right), & a = 4, \ldots, 7 \\
\frac{1}{6} \frac{p_0 \pm \epsilon_p g_0 \pm \epsilon_{p'}}{p_0^2 - \epsilon_p^2 p_0^2 - \epsilon_{p'}^2} + \frac{1}{3} \frac{p_0 \pm \epsilon_p g_0 \pm \epsilon_{p'}}{p_0^2 - \epsilon_p^2 p_0^2 - \epsilon_{p'}^2}, & a = 8
\end{cases}
\]

(50a)

and

\[
w_a = \begin{cases}
\frac{1}{2} \frac{\phi(P) \phi(P')}{(p_0^2 - \epsilon_p^2)(p_0^2 - \epsilon_{p'}^2)}, & a = 1, 2, 3 \\
0 & a = 4, \ldots, 7 \\
\frac{1}{6} \frac{\phi(P) \phi(P')}{(p_0^2 - \epsilon_p^2)(p_0^2 - \epsilon_{p'}^2)}, & a = 8
\end{cases}
\]

(50b)

where \( \epsilon_p = p - \mu \) and \( \epsilon_p = \sqrt{(p - \mu)^2 + \phi^2(P)} \). Since the dominant contributions in the weak coupling arise from the quasiparticles, we have ignored the contributions from the quasi-antiparticles in the calculations above. The trace
over Dirac space is straightforward

\[
\text{Tr} \left[ \gamma^0 A^\pm_\nu \gamma^0 A^\pm_\nu \gamma_0 \right] = -\text{Tr} \left[ \gamma^0 A^\pm_\nu \gamma^0 \gamma_5 A^\mp_\nu \gamma_0 \right] = 1 + \hat{p} \cdot \hat{p'}, \tag{51a}
\]

\[
\sum_i \text{Tr} \left[ \gamma^i A^\pm_\nu \gamma_0 \gamma^i A^\pm_\nu \gamma_0 \right] = \sum_i \text{Tr} \left[ \gamma^i \gamma_5 A^\pm_\nu \gamma^i \gamma_5 A^\mp_\nu \right] = 3 - \hat{p} \cdot \hat{p'}, \tag{51b}
\]

\[
\text{Tr} \left[ \gamma \cdot \vec{k} A^\pm_\nu \gamma_0 \gamma \cdot \vec{k} A^\pm_\nu \gamma_0 \right] = \text{Tr} \left[ \gamma \cdot \vec{k} \gamma_5 A^\pm_\nu \gamma \cdot \vec{k} \gamma_5 A^\mp_\nu \right] = \left( 1 + \hat{p} \cdot \hat{p'} \right) \frac{(p - p')^2}{k^2}. \tag{51c}
\]

It can be shown that the contribution from Eq. (50a) to \( F \) is suppressed by an order \( g \) relative to that from Eq. (50b) and will be ignored here. We neglect also the dependence of the gap function on the magnitude of the momentum, but keep the dependence on the energy and the momentum orientation. Then the integrals over \( p \) and \( p' \) can be carried out easily. Since we are only interested in the zero temperature, the Matsubara sum becomes an integral over the Euclidean energy. We find

\[
\text{Tr}[D_\nu \delta \Pi] = -\frac{6g^2 \mu^4}{32\pi^4} \int d\nu \int d\nu' \int d^2 \hat{p} \int d^2 \hat{p'} \frac{\phi(\nu, \hat{p}) \phi(\nu', \hat{p'})}{\sqrt{(\nu^2 + \phi^2(\nu, \hat{p}))((\nu')^2 + \phi^2(\nu', \hat{p'}))}} \times \left[ D_\nu(\nu - \nu', \theta) + D_\nu(\nu - \nu', \theta) \right], \tag{52}
\]

where \( \cos \theta = \hat{p} \cdot \hat{p'} \). Making use of the Nambu-Gorkov formalism in Eq. (33 - 39), the rest terms of the condensate energy density Eq. (52) can be evaluated readily.

\[
\frac{1}{\Omega} \text{Tr} \ln S^{-1} - \text{Tr} \ln S_0^{-1} = -\frac{4\mu^2}{(2\pi)^3} \int d\nu \int d^2 \hat{p} \left[ |\nu| - \sqrt{\nu^2 + \phi^2(\nu, \hat{p})} \right] \tag{53}
\]

\[
\frac{1}{\Omega} \text{Tr}(S_0^{-1} S - 1) = -\frac{4\mu^2}{(2\pi)^3} \int d\nu \int d^2 \hat{p} \frac{\phi^2(\nu, \hat{p})}{\sqrt{\nu^2 + \phi^2(\nu, \hat{p})}} \tag{54}
\]

The final expression of the condensation energy density reads

\[
F = -\frac{3g^2 \mu^4}{32\pi^4} \int d\nu \int d\nu' \int d^2 \hat{p} \int d^2 \hat{p'} V(\nu - \nu', \theta) \frac{\phi(\nu, \hat{p}) \phi(\nu', \hat{p'})}{\sqrt{(\nu^2 + \phi^2(\nu, \hat{p}))((\nu')^2 + \phi^2(\nu', \hat{p'}))}} + \frac{2\mu^2}{(2\pi)^3} \int d\nu \int d^2 \hat{p} \frac{\nu^2}{\sqrt{\nu^2 + \phi^2(\nu, \hat{p})}} \tag{55}
\]

where \( V \) contains the contribution from both magnetic and electric gluons, i.e.

\[
V = D_\nu(\nu - \nu', \theta) + D_\nu(\nu - \nu', \theta) \tag{56}
\]

The gap equation can be derived by minimizing \( F \) with respect to the gap function \( \phi(\nu, \hat{p}) \),

\[
\frac{\delta F}{\delta \phi} = 0 \tag{57}
\]

and we end up with

\[
\phi(\nu, \hat{p}) = \frac{g^2 \mu^2}{24\pi^3} \int d\nu' \int d^2 \hat{p'} V(\nu - \nu', \theta) \frac{\phi(\nu', \hat{p'})}{\sqrt{(\nu')^2 + \phi^2(\nu', \hat{p'})}}. \tag{58}
\]

A consistent derivation of the gap equation up to the subleading order requires both the contribution from \( w_n^\pm \) and that from the diagonal block of \( \left[ A^\pm_\nu \right] \) to be kept. The net result is to replace the first term inside the square root on RHS of (58) by \( \nu^2 / Z^2(\nu') \) with \( Z(\nu) \) the wave function renormalization of the normal phase. But it will not interfere with the angular dependence of the gap function to the subleading order as will be shown in the next section.
IV. THE INTEGRAL EQUATION FOR THE ANGULAR DEPENDENCE OF THE GAP

Although the pairing strength are equal to the leading order of the QCD running coupling constant, similar to the case II of the toy model, the subleading terms fall off with an increasing $J$. This makes the solution to the gap equation (58) highly nontrivial. In what follows, we shall isolate the energy dependence and the angle dependence of the pairing potential $V$. A differential equation with respect to the Matsubara energy will be derived from (58) that fixes the gap function up to an arbitrary function of the angle. This function will be determined then by (58) with $\phi$ a known function of the Matsubara energy.

Proceeding with the partial wave analysis, we expand $V(\nu - \nu', \theta)$ in series of Legendre polynomials [22]:

$$V(\nu - \nu', \theta) = \frac{1}{6\mu^2} \ln \frac{\omega_c}{|\nu - \nu'|} \sum_j (2J + 1) P_J(\cos \theta) + \frac{1}{2\mu^2} \sum_{J>0} (2J + 1) c_J P_J(\cos \theta)$$

(59)

where $\omega_c = \frac{1024\sqrt{\pi}^4\mu}{N_f^2 g^5}$ and $c_J$ is given by Eq. (2). Using the completeness relation

$$\sum_j (2J + 1) P_J(\cos \theta) = 4\pi \delta^2(\hat{p} - \hat{p}')$$

(60)

and the identity (proved in the Appendix A)

$$\int d^2 \hat{p}' \sum_{j=1}^{\infty} (2J + 1) c_J P_J(\hat{p} \cdot \hat{p}') f(\hat{p}') = 2 \int d^2 \hat{p}' \frac{f(\hat{p}') - f(\hat{p})}{|1 - \hat{p} \cdot \hat{p}'|}$$

(61)

with $f(\hat{p})$ an arbitrary function of $\hat{p}$, the gap equation (58) becomes

$$\phi(\nu, x) = \hat{g}^2 \int_0^{\omega_0} d\nu' \left\{ \frac{1}{2} \left( \ln \frac{\omega_c}{\nu - \nu'} + \ln \frac{\omega_c}{|\nu - \nu'|} \right) \frac{\phi(\nu', x)}{\sqrt{\nu'^2 + \phi^2(\nu', x)}} 
+ 3 \int_{-1}^1 dx' \frac{1}{|x - x'|} \left[ \frac{\phi(\nu', x')}{\sqrt{\nu'^2 + \phi^2(\nu', x')}} - \frac{\phi(\nu', x)}{\sqrt{\nu'^2 + \phi^2(\nu', x')}} \right] \right\}$$

(62)

where $\hat{g}^2 = g^2/(18\pi^2)$, $x = \hat{p} \cdot \hat{z}$ with $\hat{z}$ a fixed spatial direction and a UV cutoff, $\omega_0 \sim q\mu$ is introduced. In deriving (62), we have assumed that the gap depends on $x$ only, so the integration over the azimuthal angle of $\hat{p}'$ can be carried out explicitly. The gap equation (62) can be further simplified by using the approximation of Son [17]

$$\ln \frac{\omega_c}{|\nu - \nu'|} \simeq \ln \frac{\omega_c}{|\nu_> \nu|}$$

(63)

with $\nu_> = \max(\nu, \nu')$. It is convenient to introduce

$$\xi = \ln \frac{\omega_c}{\nu}, \quad a = \ln \frac{\omega_c}{\omega_0}$$

(64)

On writing $\phi = \phi(\xi, x)$ and

$$\Phi(\xi, x) = \hat{g}^2 \int_\xi^{\omega_c} \frac{d\xi'}{\sqrt{1 + \frac{\phi^2(\xi', x)}{\omega^2}}} e^{2\xi'}$$

(65)

the gap equation (62) becomes

$$\phi(\xi, x) = \xi \Phi(\xi, x) - \int_\xi a \frac{d\Phi}{d\xi} + 3 \int_{-1}^1 dx' \frac{\Phi(a, x') - \Phi(a, x)}{|x - x'|}$$

(66)

Taking the derivative of both sides with respect to $\xi$, we find

$$\frac{d\phi}{d\xi} = \Phi(\xi, x)$$

(67)
which implies the boundary condition

$$\frac{d\phi}{d\xi} \to 0$$  \quad (68)$$
as $\xi \to \infty$ for all $x$. Another derivative of (67) yields the ordinary differential equation

$$\frac{d^2 \phi}{d\xi^2} + \frac{g^2 \phi}{\sqrt{1 + \frac{\phi^2}{\omega^2} e^{2\xi}}} = 0$$  \quad (69)$$
which is universal for all $x$. It follows from Eq.(66) that the gap equation is equivalent to

$$a\Phi(a, x) - \phi(a, x) + 3 \int_{-1}^{1} \frac{\Phi(a, x') - \Phi(a, x)}{|x - x'|} = 0$$  \quad (70)$$
The solution to (69) subject to the condition (68) contains an arbitrary function of $x$ to be determined by (70). No further approximation has been made up to now.

The solution to the differential equation (69) proceeds in the same way as that for a spherical gap. To the leading order, the equation can be approximated by a linear one,

$$\frac{d^2 \phi^{(0)}}{d\xi^2} + \bar{g}^2 \theta(b - \xi)\phi^{(0)} = 0$$  \quad (71)$$
where $b(x)$ is to be determined by the condition $\frac{\phi^{(b, x)}}{\omega_c} e^b = 1$. Its solution that satisfies the boundary condition (68) and the continuity up to the first order derivative reads

$$\phi^{(0)}(\xi, x) = \begin{cases} 
\phi_0(x) \cos \bar{g}[b(x) - \xi], & \text{for } \xi < b(x), \\
\phi_0(x), & \text{for } \xi \geq b(x). 
\end{cases}$$  \quad (72)$$
where

$$b(x) = \ln \frac{\omega_c}{|\phi_0(x)|}.$$  \quad (73)$$
It follows from Eq.(67) then that

$$\Phi(\xi, x) = \begin{cases} 
\bar{g} \phi_0(x) \sin \bar{g}[b(x) - \xi], & \text{for } \xi < b(x), \\
0, & \text{for } \xi \geq b(x). 
\end{cases}$$  \quad (74)$$
The angle dependent factor $f(\bar{p})$ introduced in Eq.(3) is defined by

$$f(x) \equiv \frac{\phi_0(x)}{\Delta_0} = O(1)$$  \quad (75)$$
where $\Delta_0$ is the s-wave gap given by

$$\frac{\pi}{2} - \bar{g} \ln \frac{2\omega_c}{\Delta_0} = 0.$$  \quad (76)$$
where the contribution from the wave-function renormalization is ignored. Up to the subleading order, the differential equation (69) reads

$$\frac{d^2 \phi^{(1)}}{d\xi^2} + \bar{g}^2 \theta(b - \xi)\phi^{(1)} = \bar{g}^2 \left[\theta(b - \xi) - \frac{1}{\sqrt{1 + \frac{\phi^2}{\omega^2} e^{2\xi}}}\right]\phi^{(0)}.$$  \quad (77)$$
We find that

$$\phi^{(1)}(\xi, x) = \phi^{(1)}(\xi, x) + A(\xi, x)u(\xi, x) - B(\xi, x)v(\xi, x),$$  \quad (78)$$
where \( u(\xi, x) \) and \( v(\xi, x) \) are the two linearly independent solutions to the Eq.\( (71) \),

\[
\begin{align*}
    u(\xi, x) &= \begin{cases} 
        \cos \tilde{g}(b(x) - \xi), & \text{for } \xi < b(x), \\
        1, & \text{for } \xi \geq b(x).
    \end{cases} \\
    v(\xi, x) &= \begin{cases} 
        -\sin \tilde{g}(b(x) - \xi), & \text{for } \xi < b(x), \\
        \tilde{g}, & \text{for } \xi \geq b(x).
    \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
    A(\xi, x) &= \tilde{g} \int_{\xi}^{\infty} \frac{d\xi'}{1 + \frac{\omega_c^2(\xi', x)}{\omega_c^2} e^{2\xi'}} v(\xi', x) \phi^{(0)}(\xi', x), \\
    B(\xi, x) &= -\tilde{g} \int_{\xi}^{\infty} \frac{d\xi'}{1 + \frac{\omega_c^2(\xi', x)}{\omega_c^2} e^{2\xi'}} u(\xi', x) \phi^{(0)}(\xi', x).
\end{align*}
\]

At the point \( \xi = a \), we have

\[
A(a, x) = 1 + O(\tilde{g}), \quad B(a, x) \simeq \tilde{g} \ln 2
\]

Therefore

\[
\phi^{(1)}(a, x) \simeq \phi_0(x) \left[ \cos \tilde{g}(b - a) - \tilde{g} \ln 2 \sin \tilde{g}(b - a) \right]
\]

to the subleading order. Since \( \tilde{g}(b - a) = \pi/2 + O(g) \) according to Eq.\( (76) \), we have

\[
\Phi(a, x) = \tilde{g} \phi_0(x) + O(g),
\]

and

\[
\phi^{(1)}(a, x) \simeq \phi_0(x) \left[ \frac{\pi}{2} - \tilde{g}(b - a) - \tilde{g} \ln 2 \right] + O(g)
\]

Substituting Eqs.\( (84) \) and \( (85) \) into Eq.\( (70) \), we obtain the gap equation to the subleading order

\[
- \left[ \frac{\pi}{2} - \tilde{g} \ln \frac{\omega_c}{|\phi_0(x)|} - \tilde{g} \ln 2 \right] \phi_0(x) + \int_{-1}^{1} dx' \frac{\phi_0(x') - \phi_0(x)}{|x - x'|} = 0
\]

Then the integral equation for \( f(x) \),

\[
f(x) \ln |f(x)| - 3 \int_{-1}^{1} dx' \frac{f(x') - f(x)}{|x - x'|} = 0
\]

follows from \( (76) \).

Few comments are in order: 1) The spherical gap, \( f(x) = 1 \) is a trivial solution to Eq.\( (87) \) and there is no angular momentum mixing. 2) The ”spin-1” gap, carried over from the polar phase of \( \text{He}^3 \), \( f(x) \propto x \), fails to satisfy this equation. 3) Eq.\( (87) \) conserves the parity. In another word, its solution can be either an even or an odd function of \( x \). 4) If the wavefunction renormalization is restored, there will be an additional subleading term on RHS of \( (77) \) and an additional subleading term on RHS of \( B \) of eq.\( (82) \). This term, when substitute into Eq.\( (70) \), will cancel the corresponding contribution to \( \Delta_0 \)

\[
\Delta_0 = \pi e^{-\gamma T_c^{(0)}} = \frac{2048\pi^4 \mu}{N_f^2 g^5} e^{-\frac{\pi^2}{2\sqrt{2}g}} e^{\frac{2^7 + 1}{5}}.
\]

leaving the integral equation \( (87) \) intact.
V. THE NUMERICAL RESULTS OF THE ANGULAR DEPENDENCE

The solution to the integral equation Eq. (87) can be obtained from a variational principle. Upon substitution of Eq. (72) with \( \phi_0(x) = \Delta_0 f(x) \) into Eq. (55), the condensate energy density becomes a functional of \( f \) (details in Appendix B), i.e.

\[
F = \frac{\mu^2 \Delta_0^2}{2\pi^2} F[f],
\]

where

\[
F[f(x)] = \int_{-1}^{1} dx f^2(x) \left[ \ln|f(x)| - \frac{1}{2} \right] + \frac{3}{2} \int_{-1}^{1} dx \int_{-1}^{1} dx' \frac{(f(x) - f(x'))^2}{|x - x'|}
\]

\[
= 2 \int_{0}^{1} dx f^2(x) \left[ \ln|f(x)| - \frac{1}{2} \right] + 3 \int_{0}^{1} dx \int_{0}^{1} dx' \left\{ \frac{(f(x) - f(x'))^2}{|x - x'|} + \frac{(f(x) + f(x'))^2}{x + x'} \right\}
\]

with the last equality following from the odd parity of \( f(x) \), i.e. \( f(-x) = -f(x) \). Readers may easily verify that the variational minimum of Eq. (90) does solve Eq. (87).

Before the numerical solution, we consider a trial function

\[
f(x) = cx
\]

and substitute it into the target functional (90). The minimization yields

\[
c = e^{-\frac{\Delta_0^2}{8}} = e^{-6.4} \approx 3.459 \times 10^{-3},
\]

at which

\[
F[f] \approx -3.989 \times 10^{-6}.
\]

The trial function (91) is what people carried over from the polar phase of He\(^3\). The "-6" of the exponent of (92) comes from the pairing strength of the \( p \)-wave and the "\( \frac{1}{8} \)" stems from the logarithm of (77). The latter contribution was reported in [13]. The trial function (91) with \( p \)-wave alone is not optimal. The free energy will be lowered further by including higher partial waves of odd \( J \) as we shall see.

To find the variational minimum, we discretize the integral of Eq. (90) by dividing the domain \( x \in (0,1) \) into \( N(>>1) \) intervals with

\[
x_j = (j + \frac{1}{2}) \Delta x, \quad j = 0, 1, 2, \ldots, N - 1
\]

where \( \Delta x = 1/N \). We have then \( \mathcal{F} = \lim_{N \to \infty} \mathcal{F}_N \) with

\[
\mathcal{F}_N = 2\Delta x \sum_j f_j^2 \left( \ln f_j - \frac{1}{2} \right) + 6\Delta x^2 \sum_j f_j^2 \frac{x_j^2}{x_j} + 3\Delta x^2 \sum_{i,j,i \neq j} \left\{ \frac{(f_i - f_j)^2}{|x_i - x_j|} + \frac{(f_i + f_j)^2}{x_i + x_j} \right\}
\]

where we have dropped the limit \( x' \to x \) of the first term inside the curly bracket of Eq. (90). \( \mathcal{F}_N \) is a function of \( N \) variables. The stationary condition

\[
\frac{\partial \mathcal{F}}{\partial f_j} = 0
\]

yields

\[
f_j \ln f_j + 3\Delta x \left[ \frac{2}{x_j} + \sum_{i \neq j} \left( \frac{1}{|x_i - x_j|} + \frac{1}{x_i + x_j} \right) \right] f_j - 3\Delta x \sum_{i \neq j} \left( \frac{1}{|x_i - x_j|} + \frac{1}{x_i + x_j} \right) f_i = 0
\]

which is a discrete version of Eq. (87). Regarding \( f_i \)'s as given, the equation for \( f_j \) is of the form

\[
(\ln f_j + a) f_j - b = 0
\]
\[ J = 1 \quad J = 3 \quad J = 5 \]

\[
\begin{array}{ccc}
 b_J & 3.413 \times 10^{-3} & -2.328 \times 10^{-4} & 7.409 \times 10^{-5} \\
\end{array}
\]

TABLE I: The first three expansion coefficients of the gap function according to Legendre polynomials.

FIG. 2: The angular dependence of the gap function with angular momentum mixing. The dashed line and the solid line are the initial configuration and the final numerical results respectively.

with \( a \) and \( b \) positive. It has one and only one solution for \( f_j > 0 \).

We start with the trial function (91), \( f_j = e^{-17/3}x \) as an initial configuration and update each \( f_j \) by solving Eq. (98).

This way we lower the value of the target functional \( F \) in each step and approach the solution to (87) eventually. The process converges rapidly and our numerical solution to (87) is shown as solid line in Fig. 2, which depart from the trial function (dashed line) slightly. We find the minimum value of the target functional

\[
F[f] \simeq -4.130 \times 10^{-6},
\]

which drops from (93) by 3.5 percent.

It is instructive to examine the angular momentum contents of our solution in the partial wave expansion

\[
f(x) = \sum_{J=\text{odd}} b_J P_J(x).
\]

The coefficients of the first three partial waves, \( J = 1, 3, 5 \), calculated by substituting the numerical solution into the formula

\[
b_J = \frac{2J+1}{2} \int_{-1}^{1} dx f(x) P_J(x)
\]

are displayed in Table I. While the gap function contains all partial waves of odd \( J \), the component of \( J = 1 \) is the biggest. This is anticipated because the pairing strength of the all partial waves are equal in leading order but fall off with an increasing \( J \) in the subleading order as is shown in the partial wave expansion (59).
VI. CONCLUDING REMARKS

In summary we have explored the angular dependence of the gap function for a non-spherical pairing of CSC. Because of the equal strength of the pairing potential mediated by one-gluon exchange for all partial waves to the leading order of QCD running coupling constant and the nonlinearity of the gap equation, a non-spherical gap function cannot be restricted to one angular momentum channel only. Other multipoles are bound to show up, which renders the angular dependence of the gap nontrivial. On the other hand, the pairing strength to the subleading order decreases with increasing angular momentum $J$. The mixing effect will not be as big as that in the soluble toy model we introduced for the purpose of clarification.

For the single flavor CSC, we worked out the angular momentum mixing effect explicitly for the gap function with zero azimuthal quantum number at zero temperature. An nonlinear integral equation for the nontrivial angular dependence was derived and its solution was obtained numerically. The gap function in this case reads

$$
\phi = \begin{cases} 
\Delta_0 f(\hat{p} \cdot \hat{z}) \cos \bar{g} \left( \ln \frac{\nu}{\Delta_0|f(\hat{p} \cdot \hat{z})|} \right), & \text{for } \nu > \Delta_0|f(\hat{p} \cdot \hat{z})|, \\
\Delta_0 f(\hat{p} \cdot \hat{z}), & \text{for } \nu \leq \Delta_0|f(\hat{p} \cdot \hat{z})|. 
\end{cases}
$$

(102)

where $\Delta_0$ is given by Eq.(88) and $f(\hat{p} \cdot \hat{z})$ is plotted in Fig.2.

The drop of the free energy of the modified polar phase by the mixing, however, is numerically small. The magnitude of its condensation energy is smaller than that of the CSL phase by a factor of 1.48 instead of the factor 1.54 reported in [13]. The CSL phase remains stable. In this sense our results at the moment is of theoretical values only. There are many other candidate pairing states between quarks of the same flavor [12, 13]. Among them are the states with a nonzero azimuthal quantum number and the pairing between quarks of opposite helicities. The former is analogous to the $A$ phase of $\text{He}_3$ and may be present in a compact star with a strong magnetic field. The pairing force in the unequal-helicity channel is stronger [12, 13, 22]. The angular momentum mixing effect is generic in all nonspherical pairing states and the integral equation (87) can be readily generalized to these cases. There may be phenomenological implications of the angular momentum mixing. A systematic survey of the angular momentum mixing effect in all ”spin-1” CSC states covered in [13] will be reported in another paper.

Another place where the angular momentum mixing shows up is the CSC-LOFF state in the presence of Fermi momentum mismatch. It has been speculated [28] that the forward singularity will increase the upper limit of the mismatch value that supports a LOFF pairing. The new threshold was found in [29], motivated by the nearly equal pairing strength of all partial wave channels. The same mechanism works for the gap equation of LOFF pairing. Its free energy will be lowered by the angular momentum mixing and the lower edge of the LOFF window is expected to be shifted to a lower value of the mismatch parameter.

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APPENDIX A: THE DERIVATION OF EQUATION (61)

The integral formula of $c_J$ is [22]

$$
c_J = \int_{-1}^{1} dx \frac{P_J(x) - 1}{1 - x} \quad (A1)
$$

It is convenient to introduce

$$
c'_J = \int_{-1}^{1} dx \frac{P_J(x) - 1}{1 - x + \epsilon} = \int_{-1}^{1} dx \frac{P_J(x)}{1 - x + \epsilon} - \ln \frac{2 + \epsilon}{\epsilon} \quad (A2)
$$
where $\epsilon(>0)$ is an infinitesimal quantity. We have $\lim_{\epsilon \to 0^+} c'_j = c_J$. For the first term on RHS, we expand

$$
\frac{1}{1 - x + \epsilon} = \sum_J a_J P_J(x)
$$

(A3)

according to

$$
a_J = \frac{2J + 1}{2} \int_{-1}^{1} \frac{dx}{1 - x + \epsilon} P_J(x)
$$

(A4)

Therefore Eq. (A2) reads

$$
c'_J = \frac{2}{2J + 1} a_J - \frac{\ln 2 + \epsilon}{2}
$$

(A5)

Evaluating the summation in Eq. (61) is straightforward now

$$
\sum_{J=1}^{\infty} (2J + 1)c'_J P_J(\hat{p} \cdot \hat{p}') \to \sum_{J=1}^{\infty} 2a_J P_J(\hat{p} \cdot \hat{p}') - \sum_{J=1}^{\infty} (2J + 1)P_J(\hat{p} \cdot \hat{p}')\ln \frac{2 + \epsilon}{\epsilon}
$$

$$
= \frac{2}{1 - \hat{p} \cdot \hat{p}' + \epsilon} - 4\pi \delta^2(\hat{p} - \hat{p}')\ln \frac{2 + \epsilon}{2}
$$

(A6)

Then for an arbitrary function $f(\hat{p})$,

$$
\int d^2\hat{p}' \sum_{J=1}^{\infty} (2J + 1)c'_J P_J(\hat{p} \cdot \hat{p}') f(\hat{p}') = 2 \int d^2\hat{p}' \frac{f(\hat{p}')}{1 - \hat{p} \cdot \hat{p}' + \epsilon} - 4\pi f(\hat{p})\ln \frac{2 + \epsilon}{\epsilon}
$$

$$
= 2 \int d^2\hat{p}' \frac{f(\hat{p}') - f(\hat{p})}{1 - \hat{p} \cdot \hat{p}' + \epsilon} + 2 \int d^2\hat{p}' \frac{f(\hat{p})}{1 - \hat{p} \cdot \hat{p}' + \epsilon} - 4\pi f(\hat{p})\ln \frac{2 + \epsilon}{\epsilon}
$$

$$
= 2 \int d^2\hat{p}' \frac{f(\hat{p}') - f(\hat{p})}{1 - \hat{p} \cdot \hat{p}' + \epsilon}
$$

(A7)

The Eq. (61) is obtained by taking the limit $\epsilon \to 0^+$.

**APPENDIX B: THE CONDENSATION ENERGY DENSITY WITH THE ANGULAR MOMENTUM MIXING**

In this appendix, we shall derive the expression Eq. (55) of the condensation energy density with the angular momentum mixing. Substituting Eq. (59) into the first term of Eq. (55), we find

$$
F_1 = -\frac{3g^2 \mu^2}{32\pi^4} \int d\nu' \int d\nu \int d^2\hat{p}' \int d^2\hat{p} \left[ \frac{1}{16\pi^2} \ln \frac{\omega_c}{|\nu - \nu'|} \sum_{l=0}^{\infty} (2l + 1)P_l(\hat{p} \cdot \hat{p}') \phi(\nu, \hat{p}) \phi(\nu', \hat{p}') \right]
$$

$$
= -\frac{g^2 \mu^2}{16\pi^2} \left\{ \int d\nu' \int d\nu \int d^2\hat{p} \ln \frac{\omega_c}{|\nu - \nu'|} \frac{\phi(\nu, \hat{p})\phi(\nu', \hat{p})}{\sqrt{(\nu^2 + \phi^2(\nu, \hat{p}))((\nu')^2 + \phi^2(\nu', \hat{p}))}} \right.\left. -\frac{3}{4\pi} \int d\nu' \int d\nu \int d^2\hat{p}' \int d^2\hat{p} \frac{1}{1 - \hat{p} \cdot \hat{p}'} \left[ \frac{\phi(\nu, \hat{p})}{\sqrt{\nu^2 + \phi^2(\nu, \hat{p})}} - \frac{\phi(\nu', \hat{p})}{\sqrt{(\nu')^2 + \phi^2(\nu', \hat{p})}} \right] \right\}
$$

(B1)

Because of the evenness of $\phi(\nu, \hat{p})$ in $\nu$, we have

$$
F_0 = -\frac{g^2 \mu^2}{4\pi^3} \left\{ \int d^2\hat{p} \int d^2\hat{p}' \frac{\omega_0}{\sqrt{\nu^2 + \phi^2(\nu, \hat{p})}} \frac{\phi(\nu, \hat{p})\phi(\nu', \hat{p})}{\sqrt{(\nu')^2 + \phi^2(\nu', \hat{p})}} \right.\left. -\frac{3}{4\pi} \int d^2\hat{p}' \int d^2\hat{p} \frac{1}{1 - \hat{p} \cdot \hat{p}'} \left[ \frac{\phi(\nu, \hat{p})}{\sqrt{\nu^2 + \phi^2(\nu, \hat{p})}} - \frac{\phi(\nu', \hat{p})}{\sqrt{(\nu')^2 + \phi^2(\nu', \hat{p})}} \right] \right\}
$$

(B2)
where the approximation (B3) has been applied to the forward logarithm. For the gap function of zero azimuthal quantum number, $\phi(\nu, \bar{p})$ depends only on $x \equiv \bar{p} \cdot \hat{x}$. We find that

$$F_1 = -\frac{\mu^2}{2\pi^2 g^2} \left\{ \frac{1}{a} \int \frac{dx}{\xi} \int \frac{d\xi'}{\xi} \frac{d^2\Phi(\xi, x)}{d\xi} \frac{d^2\Phi(\xi', x)}{d\xi} + \frac{3}{2} \int \frac{dx}{|x - x'|} \int \frac{1}{|x - x'|} \left[ \Phi(a, x') - \Phi(a, x) \right]^2 \right\}$$  \hspace{1cm} (B3)

where, $\Phi(\xi, x)$ has been defined in Eq. (63) and $\xi$ and $a$ have been defined in (64). The integral over $\xi'$ followed by the integral by part over $\xi$ leads to

$$F_1 = -\frac{\mu^2}{2\pi^2 g^2} \left\{ -a \int \frac{dx}{\xi} \frac{d^2\Phi(\xi, x)}{d\xi} + \frac{3}{2} \int \frac{dx}{|x - x'|} \int \frac{1}{|x - x'|} \left[ \Phi(a, x') - \Phi(a, x) \right]^2 \right\}$$  \hspace{1cm} (B4)

Making use of Eq. (65) and (85), we have

$$\int_{\alpha}^{\infty} d\xi (\xi, x) = -\frac{\pi}{2} \phi^2(\xi) - g\phi^2(\xi) - g\ln2 \int_{\alpha}^{\infty} d\xi \left[ \phi^2(\xi, x) \right] d\xi$$ \hspace{1cm} (B5)

and thus

$$-a\Phi^2(a, x) - \int_{\alpha}^{\infty} d\xi \phi^2(\xi, x) = \frac{\pi}{2} \phi^2(a, x) - g\phi^2(\xi) - g\ln2 - \frac{\omega_0}{\nu^2 + \phi^2(\nu, x)}$$ \hspace{1cm} (B6)

Substituting $\phi_0(x) = \Delta_0 f(x)$ into (B4), we obtain that

$$F_1 = \frac{\mu^2 \Delta_0^2}{2\pi^2} \left\{ \int_{-1}^{1} dx f^2(x) \ln|f(x)| + \frac{3}{2} \int_{-1}^{1} dx \frac{1}{|x - x'|} \left[ f(x) - f(x') \right] \right\} - \frac{\mu^2}{2\pi^2} \int_{0}^{\omega_0} d\nu \sqrt{\nu^2 + \phi^2(\nu, x)}$$ \hspace{1cm} (B7)

Then the condensate energy density with the angular momentum mixing reads

$$F = F_1 + \frac{\mu^2}{4\pi} \int d\nu \int d^2\bar{p} \left[ \frac{\phi^2(\nu, \bar{p})}{\sqrt{\nu^2 + \phi^2(\nu, \bar{p})}} \right]$$

$$= \frac{\mu^2 \Delta_0^2}{2\pi^2} \left\{ \int_{-1}^{1} dx f^2(x) \left[ \ln|f(x)| - \frac{1}{2} \right] \right\} + \frac{3}{2} \int_{-1}^{1} dx \frac{1}{|x - x'|} \left[ f(x) - f(x') \right]$$ \hspace{1cm} (B8)

The minimization of this free energy give rise to Eq. (57) of the text.

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