Universal Approximation Theorems

Anastasis Kratsios *

October 9, 2019

Abstract

The universal approximation theorem established the density of specific families of neural networks in the space of continuous functions and in certain Bochner spaces, defined between any two Euclidean spaces. We extend and refine this result by proving that there exist dense neural network architectures on a larger class of function spaces and that these architectures may be written down using only a small number of functions. We prove that upon appropriately randomly selecting the neural networks architecture’s activation function we may still obtain a dense set of neural networks, with positive probability. This result is used to overcome the difficulty of appropriately selecting an activation function in more exotic architectures.

Conversely, we show that given any neural network architecture on a set of continuous functions between two T0 topological spaces, there exists a unique finest topology on that set of functions which makes the neural network architecture into a universal approximator. Several examples are considered throughout the paper.

Keywords: Universal Approximation Theorems, Artificial Neural Networks, Hypercyclic Operators, Deep Learning, Randomization, Brownian-Volterra Networks, Dynamical Systems.

Mathematics Subject Classification (2010): 68T01, 68T05, 92B20, 47A16, 30L05.

1 Introduction

Neural networks have their origins in [55] and in [65], wherein the author pioneered a method for emulating the behaviour of the human brain using machines. Arguably the second major wave in the theory of neural networks happened following the universal approximation theorems of [39], [17], and of [38], which demonstrated that certain neural network architectures are capable of approximating any continuous function between any two Euclidean spaces, uniformly on compacts. This series of papers initiated the theoretical justification of the empirically observed performance of neural networks, which up until that point was argued only by analogy with the Komologorov-Arnold Representation Theorem of [47].

*Department of Mathematics, ETH Zürich, HG G 32.3, Rämistrasse 101, 8092 Zürich. email: anastasis.kratsios@math.ethz.ch

This research was supported by the ETH Zürich Foundation.
Since then, neural networks have found ubiquitous use in a number of areas, ranging from machine learning and computer vision to mathematical finance and engineering. However, the applicability of neural network methods to the problem arising in those areas can be unclear at times, for two reasons. The first theoretical difficulty is that the functions which are being approximated by neural networks may not lie within the scope of the currently available universal approximation theorems. For example, the space of Borel probability measures on $[-M, M]^d$ for some $M > 0$ and some positive integer $d$, a space which arises in applied probability theory and falls outside the scope of the contemporary universal approximation theorems. The second difficulty is that the approximation of a continuous function by neural networks may need to be successfully achieved in a topology which is strictly finer than the topology of uniform convergence on compacts.

In this work, we establish the existence of universally approximating architectures that address both these concerns. In fact, we first establish the existence of universally approximating architectures, for any separable function space which is homeomorphic to a suitable convex body in an infinite-dimensional Fréchet space, to a finite-dimensional compact Riemannian manifold or, under certain additional conditions, to a retract of a separable Banach space. Moreover, we show that in the first two cases these architectures can be described entirely by exactly two functions and, in the later case, the architecture may be entirely described by exactly three functions.

As a partial converse to the aforementioned problems, we show that for any neural network architecture between two topological spaces $X$ and $Y$, there exists a unique finest topology on any non-empty subset of $C(X; Y)$, for which that architecture is a universal approximator.

Concrete applications of our results include a number of classical spaces such as the Gel’fand-Pettis spaces $P^1_\mu(B(X); Y)$, the Bochner spaces $L^1_\mu(B(X); Y)$, the spaces of entire functions between two Fréchet spaces $H_{bc}(X; Y)$, and the Fréchet space of infinitely differentiable functions $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$, and various others. The requirements on the locally-convex spaces $X, Y$ and the Borel measure $\mu$ are described within the main body of this paper.

Next, we review the classical definition of a neural network and discuss a necessary generalization which will be required in order to define neural networks in non-linear spaces.

### 1.1 Feed-Forward Neural Networks

Let us begin by recalling the classical definition of a feed-forward neural network between Euclidean spaces, as concisely formulated in [29].

**Definition 1.1 (Feed-Forward Neural).** Let $d$ and $D$ be positive integers. A feed-forward neural network, is a continuous function $F : \mathbb{R}^d \rightarrow \mathbb{R}^D$ admitting the following representation

$$F(x) = W_{n+1} \circ \sigma \circ W_n \cdots \circ \sigma \circ W_1(x),$$  

(1.1)

where for $i = 1, \ldots, H + 1$, $W_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^{d_{i+1}}$, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous bounded function applied component-wise, and $d = d_1, \ldots, d_{n+1} = D \in \mathbb{N}$. The map $\sigma \circ W_i$ is called a Layer.

Since we are concerned with developing a universal approximation theory for neural networks on general function spaces, we must first rigorously define what we mean by a "function space."
Definition 1.2 (Function Space). Let $X, Y$ be non-empty sets. A function space is a pair $(\mathcal{X}, 0)$ of a non-empty $T_0$ topological space $\mathcal{X}$, whose elements are equivalence classes of functions from $X$ to $Y$ and a distinguished point $0 \in \mathcal{X}$. We call 0 the zero-function.

Returning to Definition 1.1, remark that it was unnecessary for each of the layers to take values in a different space since each $\mathbb{R}^d_i \subseteq \ell^2$. Therefore, a multi-layer feed-forward neural network (also called a deep feed-forward neural network) may be built by repeatedly composing maps of the form

$$\iota_d \circ (\sigma \circ W) \circ \pi_d;$$

where $\sigma$ and $W : \mathbb{R}^d \to \mathbb{R}^\tilde{d}$ are as in Definition 1.1, $\pi_d$ is the projection of $\ell^2$ onto $\mathbb{R}^d$, $\iota$ is the inclusion of $\mathbb{R}^\tilde{d}$ into $\ell^2$, and $d, \tilde{d}$ are non-negative integers.

Suppose now that we would like to define a deep feed-forward neural network on an arbitrary function space $\mathcal{X}$. In this case, Definition 1.1 does not apply, since affine maps cannot be defined if either $X$ or $Y$ fails to be a linear space. This difficulty is overcome by allowing neural network layers to be taken from any suitable set of functions in $\mathcal{X}$. Moreover, in general a deep neural network cannot be built up from several layers by composition if $X \neq Y$; because in this case function composition is ill-defined. Instead, any associative map $\circ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ may be used to combine layers and form deep neural networks in $\mathcal{X}$. For example, if $Y$ has a well-defined additive structure then $\circ$ may be taken to be point-wise addition in $Y$. By incorporating these alterations into Definition 1.1, we may extend the definition of a deep feed-forward neural network to general function spaces.

In order to concisely formulate our definitions, we will introduce some notation. Let $f, g \in \mathcal{X}$, $A$ be a non-empty subset of $\mathcal{X}$, define the function $f \circ g \in \mathcal{X}$ and the subsets $A^\circ, A^+$ by:

$$f \circ g \triangleq \circ (f, g)$$

$$A^\circ \triangleq \{ f \in \mathcal{X} : (\exists f_1, \ldots, f_n \in A) f = f_n \circ \cdots \circ f_1 \},$$

$$A^+ \triangleq A \cup \{ f \in \mathcal{X} : f \text{ is constant} \}.$$

If there are no constant functions in $\mathcal{X}$ then note that $A^+ = A$.

Definition 1.3 ((Feed-Forward) Neural Networks). Let $\mathcal{X}$ be a function space, $\circ$ be an associative map from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}$, and $\mathcal{F} \subseteq \mathcal{X}$ be a non-empty subset. Then:

(i) A shallow (feed-forward) neural network is an element of $\mathcal{F}$,

(ii) A deep (feed-forward) neural network is an element of $\mathcal{NN}^\circ \triangleq (\mathcal{F}^+)^\circ$

(iii) The pair $(\mathcal{F}, \circ)$ is referred to as the architecture of $\mathcal{NN}^\circ$,

(iv) The map $\circ$ is called a networking function.

For every $f \in \mathcal{NN}^\circ$ the functions $f_1, \ldots, f_n \in \mathcal{F}^+$ appearing in the representation

$$f = f_n \circ \cdots \circ f_1$$

will be called layers of $f$. 3
Remark 1.4 (Notation). For simplicity, we will make the following notational conventions. We will write \( \mathcal{F} \) instead of \((\mathcal{F}, \ominus)\) and we will write \( \mathcal{X} \) in place of \((\mathcal{X}, 0)\). For the remainder of this paper, \( X \) and \( Y \) will be used to denote non-empty sets underlying the function space \( \mathcal{X} \).

Remark 1.5 (Conventions). In the case where composition between elements of \( \mathcal{X} \) is well-defined, we adopt the convention that \( \ominus \triangleq \circ \) (unless specified stated otherwise).

If \( X \) and \( Y \) are topological spaces, then until otherwise specified, the set \( C(X; Y) \) will always be equipped with the compact-open topology. When \( \mathcal{X} \) is a topological vector space, then its zero-function will always be taken to be its zero-vector.

The central object of study in this paper are architectures which satisfy the following extension of the universal approximation theorem.

Definition 1.6 (Universal Approximator). An architecture \( \mathcal{F} \) is called a universal approximator in \( \mathcal{X} \) if \( \mathcal{N} \mathcal{F} \) is dense in \( \mathcal{X} \).

This paper focuses on characterizing, representing, and establishing the existence of universal approximators on most separable function spaces. Our paper is organized as follows.

1.2 Organization of Paper

Section 2 begins by showing that if \( \mathcal{X} \) is homeomorphic to a separable Banach space, then every universal approximator \( \mathcal{F} \) on \( \mathcal{X} \) must admit the following representation

\[
\mathcal{F} = \{ \phi_i^n(\sigma_i) \}_{i \in I},
\]

for some family of topologically transitive maps \( \{ \phi_i \}_{i \in I} \subseteq C(\mathcal{X}; \mathcal{X}) \) and some family of functions \( \{ \sigma_i \}_{i \in I} \subseteq \mathcal{X} \), such that each \( \sigma_i \) has a dense orbit under \( \phi_i \). We then use this to show that every universal approximator on a separable function space which is homeomorphic to an infinite-dimensional Banach space contains a smaller universal approximator which is entirely described by a dynamical system. We call these smaller universal approximators microgenerated architectures and these will be used to infer the universal approximation property of the larger architecture.

We show that a wide range of function spaces, including all infinite-dimensional separable Fréchet spaces, admit a universal approximator which is also microgenerated. We show that for the function spaces studied in [38], the functions \( \sigma_i \) are precisely the activation functions that correspond to universal approximators. Some of these results rely on the Arens-Eells space of [3], which we overview within Appendix B.

In Section 3, we investigate the converse question and show that any architecture on a non-empty subset \( \mathcal{X} \subseteq C(X; Y) \) is a universal approximator when \( \mathcal{X} \) is endowed with a correct topology. Moreover, there is a unique finest topology on \( \mathcal{X} \) for which the architecture is a universal approximator.

In general, it may be difficult to identify a ”good activation function”. In Section 4, we show that an appropriate randomization of the activation function will yield a universal approximator, with positive probability. These results are used both to construct universal approximators on several spaces not covered by the classical theory and to construct universal approximators with other new properties.
2 Universal Approximation Theorems via Dynamical Systems

In this section, we exploit the connection between dynamical systems and neural networks to obtain a general universal approximation theory. The connection between neural networks and discrete dynamical systems on $\mathbb{R}^d$ first appeared with the introduction of echo-state networks in [40], with their universal approximation capabilities recently being established in [30].

Analogously, the connection between neural networks with a continuum of layers, ordinary differential equations, and continuous dynamical systems on $\mathbb{R}^d$ was first explored in [11] and since then in [12], amongst others. The universal approximation capabilities of such "infinitely deep" architectures, on specific spaces of continuous functions, is also well-known and dates back to the PhD thesis of [76] and the results of [49].

Our first result characterizes universal approximators on any separable function spaces $\mathcal{X}$, which is homeomorphic to an infinite-dimensional Banach space. Our characterization theorem shows any universal approximator on $\mathcal{X}$ is equivalent to a family of discrete dynamical systems on $\mathcal{X}$, each possessing a dense orbit in $\mathcal{X}$. A direct consequence of this result is that any universal approximator on an infinite-dimensional Banach space is always well-described by some dynamical system. This consequence reduces the proof of universal approximation of an architecture to the construction of a topologically transitive dynamical system generating part of the architecture.

Our second result establishes the existence of a universal approximator on any separable function space $\mathcal{X}$ which is homeomorphic to a convex body in some infinite-dimensional Fréchet space. The existence of the universal approximator is obtained by showing that any convex body on an infinite-dimensional Fréchet space admits a dynamical system which possesses a dense orbit. By a convex body in a Fréchet space, we mean a convex subset having a non-empty interior.

Both these results rely on dynamical systems which are determined by hypercyclic operators. A hypercyclic operator on a separable topological vector space $E$, as defined in [33, Definition 2.15], is a continuous linear operator $T : E \to E$ for which there exists some $x \in E$ such that the set $\{T^n(x)\}_{n \in \mathbb{N}}$ is dense in $E$. The set $\{T^n(x)\}_{n \in \mathbb{N}}$ is called the orbit of $x$ under $T$. For more details on hypercyclic operators, the authors recommend [33, Chapter 2] for a comprehensive treatment of the subject.

Remark 2.1. We adopted the convention that $T^0 = 1_E$, where $1_E$ is the identity map on $E$.

2.1 Microgenerated Architectures

Theorem 2.2. Let $\mathcal{X}$ be a separable function space and let $\mathcal{F}$ be an architecture on $\mathcal{X}$. If $\mathcal{X}$ is homeomorphic to an infinite-dimensional Banach space, then the following are equivalent

(i) $\mathcal{F}$ is a universal approximator in $\mathcal{X}$,

(ii) For any $\sigma \in \mathcal{N}^{\mathcal{F}}$ there exists a map $\phi \in C(\mathcal{X}; \mathcal{X})$ such that

$$\mathcal{F}(\sigma, \phi) \triangleq \{\phi^n(\sigma)\}_{n \in \mathbb{N}} \subseteq \mathcal{N}^{\mathcal{F}},$$

and $\mathcal{F}(\sigma, \phi)$ is dense in $\mathcal{X}$. 

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Moreover, if $\mathcal{X}$ is an infinite-dimensional Banach space then the map $\phi$ can be taken to be a hypercyclic (bounded linear) operator from $\mathcal{X}$ to itself.

**Proof.** Let us first fix some notation. We denote the homeomorphism $\Phi : \mathcal{X} \to E$ onto a Banach space $E$. Note that since $\mathcal{X}$ is separable, then $E$ is also separable.

First, suppose that (ii) holds and fix any $\sigma \in \mathcal{N}\mathcal{N}^{\phi}$. Since $\mathcal{F}(\sigma, \phi) \subseteq \mathcal{N}\mathcal{N}^{\phi}$ and $\mathcal{F}(\sigma, \phi)$ is dense in $\mathcal{X}$ then

$$\mathcal{X} = \overline{\mathcal{F}(\sigma, \phi)} \subseteq \overline{\mathcal{N}\mathcal{N}^{\phi}} \subseteq \mathcal{X},$$

where $\overline{X}$ denotes the closure of a subset $X \subseteq \mathcal{X}$ in $\mathcal{X}$. Therefore, $\mathcal{F}$ is a universal approximator.

Suppose now that (i) holds. Let us begin by fixing $\sigma \in \mathcal{X}$ and setting $x \triangleq \Phi(\sigma)$. We will need a dense linearly independent subset of $E$, which we now construct. Note that since $E$ is a separable Banach space, then it is second-countable; that is there exists a countable base $\{U_n\}_{n \in \mathbb{N}}$ of $E$. Since $\{U_n\}_{n \in \mathbb{N}}$ is a base for $E$, it must cover $E$. Therefore there is some $U_N \in \{U_n\}_{n \in \mathbb{N}}$ containing $x$. Without loss of generality, let us reorder $\{U_n\}_{n \in \mathbb{N}}$ such that $x \in U_0$. Furthermore, with no loss of generality, we assume that for each for each $n \in \mathbb{N}$, the open set $U_n$ is non-empty.

By induction, we will build a countable linearly-independent subset $\{x_n\}_{n \in \mathbb{N}}$ of $E$ which satisfies

$$x_n \in U_n \cap \Phi \left( \mathcal{N}\mathcal{N}^{\phi} \right) \quad \text{and} \quad x_0 = x; \quad (\forall n \in \mathbb{N}). \quad (2.1)$$

The base case holds by taking $x_0 = x$. For the inductive step, suppose that $\{x_n\}_{n=0}^i$ satisfies (2.1) for $0, \ldots, i \in \mathbb{N}$. Since $\text{span} \left( \{x_n\}_{n=0}^i \right)$ is finite-dimensional and $E$ is infinite-dimensional then $\text{span} \left( \{x_n\}_{n=0}^i \right)$ is a closed subset of $E$ with empty interior. Therefore $U_n$ cannot be contained in $\text{span} \left( \{x_n\}_{n=0}^i \right)$; hence $U_n - \text{span} \left( \{x_n\}_{n=0}^i \right)$ is a non-empty open subset of $E$.

Since $\mathcal{N}\mathcal{N}^{\phi}$ is dense in $\mathcal{X}$ and since $\Phi$ is a homeomorphism, then $\Phi (\mathcal{N}\mathcal{N}^{\phi})$ is dense in $\mathcal{X}$. Therefore, every non-empty open subset of $\mathcal{X}$ contains an element of $\Phi (\mathcal{N}\mathcal{N}^{\phi})$. Whence, there exists an

$$x_{n+1} \in U_n - \text{span} \left( \{x_n\}_{n=0}^i \right). \quad (2.2)$$

This completes the inductive step.

Since, $U_n$ is a base for the topology on $E$, then (2.1) implies that $\{x_n\}_{n \in \mathbb{N}}$ is linearly-independent and dense in $E$. Therefore, [33, Theorem 8.24] implies that there exists a hypercyclic operator $T$ on $E$ such that

$$T^n(x_0) = x_n. \quad (2.3)$$

Since $\Phi$ is a homeomorphism and since $x = \Phi(\sigma)$, then setting $\phi \triangleq \Phi \circ T \circ \Phi^{-1}$ and noting that

$$\mathcal{X} = \Phi^{-1} \left( \{x_n\} \right) = \Phi^{-1} \left( \{T^n(x_1)\} \right) = \Phi^{-1} \circ T^n \circ \Phi(\sigma) \quad (\forall n \in \mathbb{N}),$$

shows that $\mathcal{F}(\sigma, \phi)$ is dense in $\mathcal{X}$.

It now only remains to show that $\mathcal{F}(\sigma, \phi) \subseteq \mathcal{F}$. Indeed, for every $n \in \mathbb{N}$, $x_n \in \Phi (\mathcal{N}\mathcal{N}^{\phi})$. Moreover, since $\Phi$ is invertible, then (2.3) implies that for every $n \in \mathbb{N}$, we have that

$$\Phi^{-1} \circ T^n \circ \Phi(\sigma) = \Phi^{-1} \circ T(x_0) = \Phi^{-1}(x_n) \in \mathcal{N}\mathcal{N}^{\phi}.$$

This completes the proof. \qed
Next, we show that neural networks architectures tend to be over-parameterized since they must contain a universal approximator which depends only on two functions.

**Corollary 2.3.** Suppose that $\mathcal{X}$ is separable and homeomorphic to an infinite-dimensional Banach space. If $\mathcal{F}$ is a universal approximator on $\mathcal{X}$, then for every $\sigma \in \mathcal{N}^{\mathcal{F}}$, there a continuous map $\phi \in C(\mathcal{X}; \mathcal{X})$ satisfying:

(i) The architecture $\mathcal{F}(\sigma, \phi) \triangleq \{\phi^n(\sigma)\}_{n \in \mathbb{N}}$ is a universal approximator on $\mathcal{X}$,

(ii) $\{\phi^n(\sigma)\}_{n \in \mathbb{N}} \subseteq \mathcal{N}^{\mathcal{F}}$.

**Proof.** By Theorem 2.2 (iv), if $\mathcal{F}$ is a universal approximator then for any $\sigma \in \mathcal{N}^{\mathcal{F}}$, the architecture $\mathcal{F}(\sigma, \phi)$ is itself dense in $\mathcal{X}$ and $\mathcal{F}(\sigma, \phi) \subseteq \mathcal{N}^{\mathcal{F}}$. In particular, $\mathcal{N}^{\mathcal{F}(\sigma, \phi)}$ must be dense in $\mathcal{X}$ since $\mathcal{F}(\sigma, \phi) \subseteq \mathcal{N}^{\mathcal{F}(\sigma, \phi)}$ and since $\mathcal{F}(\sigma, \phi)$ is dense in $\mathcal{X}$. Therefore, $\mathcal{F}(\sigma, \phi)$ is a universal approximator on $\mathcal{X}$. \qed

Since the study of universal approximators can be reduced to the study of architectures of the form $\{\phi^n(\sigma)\}_{n \in \mathbb{N}}$, we introduce the following terminology.

**Definition 2.4 (Microgenerated Architecture).** An architecture $\mathcal{F}$ on $\mathcal{X}$ is said to be generated (resp. microgenerated) if there exists an indexing set $I$ (resp. an indexing set of cardinality 1), $\{\phi_i\}_{i \in I} \subseteq C(\mathcal{X}; \mathcal{X})$, and $\{\sigma_i\}_{i \in I} \subseteq \mathcal{X}$ such that

(i) $\mathcal{N}^{\mathcal{F}} = \bigcup_{i \in I} \{\phi_i^n(\sigma_i)\}$,

(ii) There exists some $i^* \in I$, such that set $\{\phi_i^{n*}(\sigma_{i^*})\}_{n \in \mathbb{N}}$ is dense in $\mathcal{X}$.

The functions $\{\phi_i\}_{i \in I}$ are called generators of $\mathcal{F}$.

**Remark 2.5.** Microgenerated networks are shown to play an analogous role to a linearly independent set in linear algebra, a sub-base in topology, or a monotone class in measure theory, but within the scope of neural network theory.

Next, we show how to apply microgenerated architectures in order to deduce that an architecture on a function space is a universal approximator. The key is that microgenerated architectures are typically much more analytically tractable since we may use all the tools from the theory of hypercyclic operators; whereas the same is not true for general architectures.

### 2.1.1 Examples on High-Dimensional Linear Spaces

Our first batch of examples focuses on extending classical neural network architectures, akin to Definition 1.1, to larger classes of function spaces from those treated in [38]. We will make extensive use of tensor products between locally convex spaces (LCSs).

Therefore, before continuing our analysis we briefly overview some key facts about topological tensor products on LCSs; beginning with the algebraic tensor product. The algebraic tensor
product of two linear spaces $X$ and $Y$, denoted by $X \otimes Y$, is represented as the solution to the universal problem illustrated by Figure 1, where $Z$ is any other linear space.

When $X, Y$ and $Z$ are LCSs then $f, g$ is required to be continuous; in which case there are multiple solutions to the universal problem illustrated by Figure 1. Each such solution is found by first choosing a suitable topology on $X \otimes Y$ which makes the map $\otimes : X \times Y \to X \otimes Y$ continuous, then showing that a family of semi-norms can represent this topology, and then completing $X \otimes Y$ with respect to that family of semi-norms.

The LCS obtained by completing the finest topology on $X \otimes Y$ which makes $\otimes$ continuous is called the *projective tensor product* and is denoted by $X \otimes_\pi Y$. Likewise, the LCS obtained by completing the weakest topology on $X \otimes Y$ which makes $\otimes$ continuous is called the *injective tensor product*, and is denoted by $X \otimes_\iota Y$. Throughout this section, we routinely consider the completion of $X \otimes Y$ with respect to a *tensor norm topology*, for some tensor norm $a$, we denote this completed LCS by $X \otimes_a Y$. The details of tensor norms on LCS are summarized in [20, Section 35]; but all we need to know here are the following three facts. First, the tensor norm topology on $X \otimes Y$ is at-least as fine as the topology of the injective tensor product on $X \otimes Y$ but no finer than the topology of the projective tensor product on $X \otimes Y$. Second, if $X$ and $Y$ are separable then so is $X \otimes_a Y$. Third, if $f : X \to Z_1$ and $g : Y \to Z$ are continuous linear maps then $f \otimes g$ extends to a continuous linear map from $X \otimes_a Y$ to $Z$. The authors also recommend the lecture notes of [48] on the subject and the book [67, Chapter 6] for the details in the special case where $X, Y, Z$ are Banach Spaces.

Derived from the results of [38], it is known that the feed-forward neural networks of Definition 1.1 can be used to learn any function between Euclidean space, whose components lie in the Bochner space $L^1_\mu(B(\mathbb{R}^d); \mathbb{R})$ for some finite Borel measure $\mu$ on $\mathbb{R}^d$. In infinite-dimensions, the component-wise approach to integration is consistent with the Gel’fand-Pettis integral (also called the weak integral) introduced in [26]. For any separable LCS space $X$ and any finite Borel measure $\mu$ on $X$, we will denote the space of Gel’fand-Pettis integrable functions from $X$ to $Y$ by $P^1_\mu(B(X); Y)$. The next result generalizes the universal approximation theorem of [38] to the infinite-dimensional setting.

**Proposition 2.6.** Let $X$ be a separable Banach space, $Y$ be a separable LCS space with $\dim(Y) \geq 2$, and $\mu$ be an inner-regular finite Borel measure on $X$. Then there exists a dense subset $D \subseteq L^1_\mu(B(X); \mathbb{R}) \otimes Y$ such that for every $\sum_{i=1}^n \sigma_i \otimes y_i \in D$, the architecture

$$\mathcal{F} \triangleq \left\{ \sum_{i=1}^n \sigma_i(Ax + b)y_i : (A \in B(X; X))(b \in X) \right\},$$

is both dense in $P^1_\mu(B(X); Y)$ and is a universal approximator on $P^1_\mu(B(X); Y)$; where $B(X; X)$ is the set of bounded linear operators on $X$. 

\[ \begin{align*}
X \times Y & \xrightarrow{\otimes} X \otimes Y \\
\forall f \times g \text{bilinear} & \Rightarrow \exists f \otimes g \text{linear}
\end{align*} \]

Figure 1: Universal Property of Tensor Products.
Proof. Denote by $\| \cdot \|_X$ the norm on $X$, fix $b \in X - \{0\}$, and define the affine map

$$W : X \to X \quad \quad \quad x \mapsto x + b.$$ 

Since $W$ is affine with affine inverse, then $W$ is injective and bi-measurable. For every $x \in X$, we have that

$$\|W^n(x)\|_X \geq n\|b\|_X - \|x\|_X \xrightarrow{n \to \infty} \infty,$$

whence $W$ is unbounded on $X$. Since any non-empty compact subset $K \subset X$ is bounded in $X$ then there must be some $N_K \in \mathbb{N}$ such that

$$W^n(K) \cap K = \emptyset, \quad (\forall n \geq N_K). \quad (2.5)$$

Since $X$ is separable and a Banach space then it is Polish. Since $\mu$ is an inner-regular Borel measure on a Polish space then, for every $\epsilon > 0$ there exists some non-empty compact subset $K_{\epsilon} \subset X$ such that

$$\mu(X - K_{\epsilon}) < \epsilon. \quad (2.6)$$

Combining (2.5) and (2.6) it follows that

$$\mu(X - K_{\epsilon}) < \epsilon, \quad W^n(K_{\epsilon}) \cap K_{\epsilon} = \emptyset \quad (\forall n \geq N_{K_{\epsilon}}). \quad (2.7)$$

Since $L^1_{\mu}(\mathcal{B}(X); \mathbb{R})$ is an ”admissible Banach space” (in the terminology of [6, Page 2]), $(X, \mathcal{B}(X), \mu)$ is a finite measure space, and $T_W(f) \triangleq f \circ W$ is a bounded linear operator on $L^1_{\mu}(\mathcal{B}(X); \mathbb{R})$ then (2.7) implies that $T_W$ satisfies [6, Corollary 2.2 (D2)]; whence $T_W$ is topologically mixing.

By [33, Proposition 2.40] since $T_W$ is topologically mixing then $T_W \oplus T_W$ is hypercyclic. By [54, Proposition 1.14 (2) and (3)], for any tensor norm which is finer that the projective tensor norm, $T_W \otimes 1_F$ is hypercyclic. Hence, there exists a dense subset $D \subset L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y$ such that for every $\sum_{i=1}^{n} \sigma_i(x) y_i \in D$

$$\left\{ T_W \otimes_{\epsilon} 1_Y \left( \sum_{i=1}^{n} \sigma_i(x) \otimes y_i \right) \right\} = \left\{ \sum_{i=1}^{n} \sigma_i(x + nb) \otimes y_i \right\}. \quad (2.8)$$

is dense in $L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y$. In particular, this holds for $L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y$, since $L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y$ is coarser that $L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y$ on the subset $L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes F$.  

In the first line of the proof of [67, Proposition 3.13], we find that the extension of the canonical map:

$$L^1_{\mu}(\mathcal{B}(X); \mathbb{R}) \otimes_{\epsilon} Y \to P^1_{\mu}(\mathcal{B}(X); Y)$$

$$\sum_{i=1}^{n} \sigma_i(x) \otimes y_i \mapsto \sum_{i=1}^{n} \sigma_i(x)y_i,$$

is an isometric isomorphism. Whence (2.8) implies that

$$\left\{ (T_W \otimes_{\epsilon} 1_Y)^k \left( \sum_{i=1}^{n} \sigma_i(x) \otimes y_i \right) \right\}_{k \in \mathbb{N}} = \left\{ T_W^k \otimes_{\epsilon} 1_Y \left( \sum_{i=1}^{n} \sigma_i(x) \otimes y_i \right) \right\}_{k \in \mathbb{N}}$$

$$= \left\{ \sum_{i=1}^{n} \sigma_i(x + kb) y_i \right\}_{k \in \mathbb{N}}. \quad (2.9)$$
is dense in $P^1_\mu(B(X); Y)$. In particular, (2.4) must be dense in $P^1_\mu(B(X); Y)$; whence $F$ is a universal approximator on the Gel’fand-Pettis space $P^1_\mu(B(X); Y)$.

The most common alternative to the Gel’fand-Pettis integral is the Bochner integral, also called the strong integral. We may obtain the following alternative infinite-dimensional extension of the universal approximation theorem of [39], formulated for the Bochner integral. However, we must first impose an additional constraint on $Y$, namely that it is a nuclear space. Introduced in the PhD thesis of [34], nuclear spaces can be characterized as LCSs which behave exactly like finite-dimensional Banach spaces with respect to topological tensor products; that is given a LCS $X$ and a nuclear space $Y$, the projective and injective tensor products $X \otimes_\pi Y$ and $X \otimes_\epsilon Y$ are isometrically isomorphic.

**Corollary 2.7.** Let $X$ be a separable Banach space, $Y$ be a nuclear space of dimension $\text{dim}(Y) \geq 2$, and $\mu$ be an inner-regular finite Borel measure on $X$. Then there exists a dense subset $D \subseteq L^1_\mu(B(X); \mathbb{R}) \otimes Y$ such that for every $\sum_{i=1}^n \sigma_i \otimes y_i \in D$, the architecture

$$F \triangleq \left\{ \sum_{i=1}^n \sigma_i (Ax + b)y_i : (A \in B(X; X))(b \in X) \right\},$$

(2.10)

is dense in $L^1_\mu(B(X); Y)$ and is a universal approximator on $L^1_\mu(B(X); Y)$; where $B(X; X)$ is the set of bounded linear operators on $X$.

**Proof.** Since $Y$ is nuclear, then $L^1_\mu(B(X); \mathbb{R}) \otimes_\pi Y$ and $L^1_\mu(B(X); \mathbb{R}) \otimes_\epsilon Y$ are isometrically isomorphic. Since any isometric isomorphism is a homeomorphism and since the image of a dense set under a homeomorphism is itself dense, then the image of (2.10) is dense in $L^1_\mu(B(X); \mathbb{R}) \otimes_\pi Y$. In [67, Chapter 2], it is shown that $L^1_\mu(B(X); \mathbb{R}) \otimes_\pi Y$ is isometrically isomorphic to the Bochner space $L^1_\mu(B(X); Y)$ under the map extending the canonical map $\sum_{i=1}^n \sigma_i \otimes y_i \mapsto \sum_{i=1}^n \sigma_i y_i$. The result follows upon applying Proposition 2.6.

Immediately from Corollary 2.7 we obtain a number of examples.

**Example 2.8.** Let $d \geq 2$ and let $\mu$ be an inner-regular Borel probability measure on $\mathbb{R}^d$. The Schwartz space $S(\mathbb{R}^n)$, also called the space of rapidly decaying functions, defined by

$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \| f \|_{\alpha, \beta} < \infty \ \forall \alpha, \beta \in \mathbb{N}^n \},$$

with topology generated by the semi-norms (multi-indexed by $\alpha$ and $\beta$)

$$\| f \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|,$$

is a nuclear space; see [34]. Therefore, Corollary 2.7 implies that there exists a dense subset $D \subseteq L^1_\mu(B(\mathbb{R}^d); \mathbb{R}) \otimes S(\mathbb{R}^n)$ such that for every $\sum_{i=1}^n \sigma_i \otimes f_i \in D$ the architecture

$$\left\{ \sum_{i=1}^n \sigma_i (Ax + b)f_i : (A \in \text{Mat}_{d \times d}(\mathbb{R}))(b \in \mathbb{R}^d) \right\},$$

is a universal approximator on the Bochner space $L^1_\mu(B(\mathbb{R}^d); S(\mathbb{R}^n))$; where $\text{Mat}_{d \times d}(\mathbb{R})$ is the set of $d \times d$ matrices with entries in $\mathbb{R}$.
Example 2.9. Let $d$ be a positive integer. The signature method of [53] is concerned with learning functions defined on $\mathbb{R}$ and taking values in the space $T(\mathbb{R}^d)$ uniformly on compacts, where $T(\mathbb{R}^d)$ is defined by

$$T(\mathbb{R}^d) \triangleq \bigoplus_{n \in \mathbb{N}} (\mathbb{R}^d)^{\otimes n};$$

and where $(\mathbb{R}^d)^{\otimes 0} \triangleq \mathbb{R}$ and the $\oplus$ denotes the direct sum of locally convex spaces. Since each $(\mathbb{R}^d)^{\otimes n}$ is a finite-dimensional Banach space and all finite-dimensional Banach spaces are isometrically isomorphic, then it can be shown that $(\mathbb{R}^d)^{\otimes n}$ is isometrically isomorphic to $\mathbb{R}^{nd}$; whence (2.11) is isometrically isomorphic to

$$\bigoplus_{n \in \mathbb{N}} \mathbb{R}^{nd}. \quad (2.12)$$

Since every finite-dimensional Banach space is nuclear and by [34, Theorem 3.3] the countable direct sum of nuclear spaces is again nuclear; whence (2.12) implies that $T(\mathbb{R}^d)$ is itself a nuclear space and can be identified with (2.12).

Therefore, by Corollary 2.7, there exists a dense subset $D \subseteq L_\mu(\mathbb{R}; \mathbb{R}) \otimes T(\mathbb{R}^d)$ such that for every $\sum_{i=1}^{n} \sigma_i \otimes y_i \in D$, the architecture

$$\left\{ \sum_{i=1}^{n} \sigma_i(ax + b)y_i : a, b \in \mathbb{R} \right\}$$

is a universal approximator on $L_\mu(\mathbb{R}; T(\mathbb{R}^d))$, for every finite Borel measure $\mu$ on $\mathbb{R}$.

The space of holomorphic functions on $\mathbb{C}$ is a classical object, with many applications in probability theory ranging from characteristic functions of random-variables, to the Lévy-Khintchine representation theorem in the study of Lévy processes, and to the affine transform formula of [23, 45] in the study of affine processes.

Spaces of holomorphic functions are not limited to one or finitely many dimensions, as in [21] holomorphic functions between separable LCSs were introduced. Appealing to [58, Theorem 7], a continuous function $\sigma : X \rightarrow Y$ is holomorphic if and only if for every $\psi \in Y'$ the function

$$\mathbb{C} \rightarrow \mathbb{C}$$

$$z \mapsto \psi \circ \sigma(x + zb),$$

is holomorphic, in the univariate sense, on the set $\{z \in \mathbb{C} : x + zb \in X\}$ for every $x, b \in X$. The Fréchet space of all holomorphic functions from $X$ to $Y$, with the topology of uniform convergence on compacts is denoted by $H_{bc}(X; Y)$.

In [5, Theorem 5.3], it is proven that $H_{bc}(X; Y)$ is isometrically isomorphic to $H_{bc}(X, \mathbb{C}) \otimes Y$, granted that $X, X', Y$ all have Schauder bassises. From hereon out, we will identify the spaces $H_{bc}(X, \mathbb{C}) \otimes Y$ and $H_{bc}(X, Y)$ and under this identification, every element $\sigma \in H_{bc}(X, Y)$ can be represented as

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x) \otimes y_i,$$

where $\sigma_1, \ldots, \sigma_n \in H_{bc}(X, \mathbb{C})$ and $y_1, \ldots, y_n \in Y$.  

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Proposition 2.10 (Space of Holomorphic Functions). Let $X$ and $Y$ be separable complex LCSs and suppose that $X, X', Y$ have Schauder bassises. Then, for any networking function $\odot$ on $H_{bc}(X; Y)$, there is a dense subset $D \subseteq H_{bc}(X; Y)$ such that for every $\sigma \in D$ the architecture:

$$\mathcal{F} \triangleq \{\sigma(Ax + b) : A \in B(X, X) \text{ and } b \in X\},$$

is a universal approximator on $H_{bc}(X; Y)$; where $B(X, X)$ is the set of bounded linear operators from $X$ to itself.

Proof. Since $X'$ is separable, [4, Corollary 2.2] for any $b \in X - \{0\}$ the translation operator $T_b$ defined by

$$T_b : H_{bc}(X; \mathbb{C}) \rightarrow H_{bc}(X; \mathbb{C})$$

$$\sigma(x) \mapsto \sigma(x + b),$$

is hypercyclic on $H_{bc}(X; \mathbb{C})$. By [54, Proposition 1.14 (i-ii)] $T_b \otimes 1_Y$ is hypercyclic on $H_{bc}(X; \mathbb{C})$.

On [69, page 115], it is shown that since $X, X', Y$ have Schauder bassises, then each one possesses the approximation property. Therefore, by [5, Theorem 5.3], there is a linear isometric isomorphism

$$\Phi_1 : H_{bc}(X; \mathbb{C}) \otimes \epsilon Y \rightarrow H_{bc}(X; \mathbb{C} \otimes \epsilon Y).$$

(2.14)

Since $\mathbb{C} \otimes \epsilon X$ is the completion of $\mathbb{C} \otimes Y$ with respect to the topology defining the injective tensor product on the algebraic tensor product $\mathbb{C} \otimes Y$ and since $\mathbb{C} \cong Y$, then (2.14) implies that $H_{bc}(X; Y)$ and $H_{bc}(X; \mathbb{C}) \otimes \epsilon Y$ are isometrically isomorphic. Denote the isometric isomorphism from $H_{bc}(X; \mathbb{C}) \otimes \epsilon Y$ to $H_{bc}(X; Y)$ by $\Phi$.

By [33, Proposition 1.40], since $H_{bc}(X; Y)$ and $H_{bc}(X; \mathbb{C}) \otimes \epsilon Y$ are quasi-conjugate since they are isometrically isomorphic; hence $\Phi \circ (T_b \otimes 1_Y)$ is hypercyclic on $H_{bc}(X; Y)$. By the Birkhoff Transitivity Theorem ([33, Theorem 2.19]), there exists a dense subset $D \subseteq H_{bc}(X; Y)$ such that for every $\sigma \in D$ the set

$$\{\Phi((T_b \otimes \epsilon 1_Y)^n(\sigma))\}_{n \in \mathbb{N}} = \left\{\Phi \left(\sum_{i=1}^{n} \sigma_i(x + (n \cdot b)) \otimes y_i\right)\right\}_{n \in \mathbb{N}}$$

(2.15)

is dense in $H_{bc}(X; Y)$; where $\Phi^{-1}(\sigma) = \sum_{i=1}^{n} \sigma_i(x + (n \cdot b)) \otimes y_i$. In particular, (2.15) implies that (2.13) is a universal approximator in $H_{bc}(X; Y)$ (for any networking function). \square

2.1.2 Examples on Finite-Dimensional Linear Spaces

Neural networks are commonly used to learn smooth functions, uniformly on compacts. For example, in recent times, several authors such as [41, 31], have effectively used artificial neural networks to solve high-dimensional PDE problems numerically. Their motivation stems from the fact that neural network architectures are typically better suited to high-dimensional problems than most currently available numerical methods for PDEs. The application of neural networks to PDEs is either justified by the results of [38] or instead through [56] (if one desires convergence with a Sobolev topology).
A natural question which arises is: if we do not view the set of smooth functions from \( \mathbb{R}^d \) to itself, as a subspace of \( C(\mathbb{R}^d; \mathbb{R}^d) \) equipped with the subspace topology, but instead we consider it with the "usual" finer topology induced by the family of semi-norms
\[
\{ p_n(\sigma) \triangleq \max_{0 \leq k \leq n} \sup_{\|x\|_d \leq n} \|\sigma^{(k)}(x)\| \mathbb{R}^d \}_{n \in \mathbb{N}},
\]
does one still achieve universal approximation? The next result provides an affirmative answer to this question.

**Proposition 2.11** (Space of Smooth Functions). Let \( d \) be a positive integer and fix \( w \in C^\infty(\mathbb{R}^d; \mathbb{R}) \) satisfying
\[
w(x) \neq 0, \quad (\forall x \in \mathbb{R}). \tag{2.16}
\]
Then there exists a dense subset \( D \subseteq C^\infty(\mathbb{R}^d; \mathbb{R}^d) \) such that for every \( \sigma \in D \), the architecture
\[
\left\{ \prod_{i=0}^n w^i(A_i x) \sigma(W_i x) : n \in \mathbb{N}, \text{ and } A_1, W_1, \ldots, A_n, W_n \in \text{Aff}(\mathbb{R}^d; \mathbb{R}^d) \right\}, \tag{2.17}
\]
is a universal approximator on \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \); where \( \text{Aff}(\mathbb{R}^d; \mathbb{R}^d) \) is the set of affine maps from \( \mathbb{R}^d \) to itself. In particular, the result holds for \( w(x) = 1 \).

**Remark 2.12.** If one does not take \( w = 1 \) in Proposition 2.11, then Proposition 2.11 provides an example of a universal approximator which lightly deviates from the traditional mold suggested by Definition 1.1.

**Proof.** For every \( b \in \mathbb{R}^d - \{0\} \) the affine map \( W \), from \( \mathbb{R}^d \) to itself, defined by
\[
W : \mathbb{R}^d \rightarrow \mathbb{R}^d
x \mapsto x + b,
\]
is smooth, injective, satisfies \( det(J[W]) = det(I) \neq 0 \); where \( J[W] \) is the Jacobian matrix of \( W \), and for every compact subset \( K \subseteq \mathbb{R}^d \) there must exist some \( N \in \mathbb{N} \) satisfying
\[
W^n(K) \cap K = \emptyset, \quad (\forall n \geq N). \tag{2.18}
\]
Moreover, since \( w \) satisfies (2.16), then Theorem [64, Theorem 3.2.9] implies that the operator \( C_{w,W} \) on \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \) defined by:
\[
C_{w,W} : C^\infty(\mathbb{R}^d; \mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^d)
\sigma(x) \mapsto w(x)(\sigma \circ W(x)),
\]
is topologically mixing; whence \( C_{w,W} \) is hypercyclic. By the Birkhoff Transitivity Theorem ([33, Theorem 2.19]). There exists a dense subset \( D \subseteq C^\infty(\mathbb{R}^d; \mathbb{R}^d) \) such that for every \( \sigma \in D \), the orbit
\[
\left\{ C_{w,W}^n(\sigma) \right\}_{n \in \mathbb{N}} = \left\{ \prod_{i=0}^n w^i(W^{i-1}x) \sigma(W^i x) \right\}, \tag{2.19}
\]
is dense in \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \) and where \( W^{-1} \triangleq 0 \). In particular, (2.19) implies that for every \( \sigma \in D \) the architecture of (2.17) is dense in \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \).
Next, we illustrate how our theory may be used to obtain new types of universal approximation results in the classical setting of deep-feed forward neural networks, as formulated in Definition (1.1). Specifically, using Corollary 2.7, it is shown that the collection of classical deep feed-forward neural networks having bounded height but arbitrary depth are universal approximators in the Bochner space $L_1^1(B(R^d); R^d)$.

In fact, this result still holds true if additional constraints are imposed on the affine functions making up each layer of the classical deep feed-forward neural networks of Definition (1.1). The statement of the next result relies on the definition of a translation function $t_b$, where $b \in R^d$ is defined by

$$t_b : R^d \to R^d$$

$$x \mapsto x + b.$$

**Corollary 2.13** (Deep Neural-Networks of Bounded Height with Layer Constraints). Fix a positive integer $d$, a finite Borel measure $\mu$ on $R^d$, let $\circ$ be the composition map $\circ$, and $M$ be such that there exists some $b \in R^d - \{0\}$ such that

$$\{t_{nb}\}_{n \in N} \subseteq M \subseteq \text{Aff}(R^d; R^d).$$

Then there exists a dense and connected $G_\delta$-subset $D$ of the Bochner space $L_1^1(B(R^d); R^d)$, such that for every $\sigma \in D$, the following set of deep feed-forward neural networks is dense in $L_1^1(B(R^d); R^d)$

$$\{ f \in L_1^1(\mathcal{B}(R^d); R^d) : (\exists k \in N) (\exists W_1, \ldots, W_{k+1} \in M) f = W_{k+1} \circ \sigma \circ W_k \circ \cdots \circ \sigma \circ W_1 \}.$$  \hspace{1cm} (2.20)

In particular, $M$ may be taken to be all of $\text{Aff}(R^d; R^d)$.

The proof of Corollary 2.13 relies on a technical appended Lemma.

**Proof.** For every $W \in M$ define the map $\phi_W$ by

$$\phi_W : L_1^1(\mathcal{B}(R^d); R^d) \to L_1^1(\mathcal{B}(R^d); R^d)$$

$$f \mapsto f \circ W,$$  \hspace{1cm} (2.21)

and recall that, as shown in the proof of Proposition 2.6, for any $b \in R^d - \{0\}$ each of the maps $\phi_b$ are hypercyclic on $L_1^1(\mathcal{B}(R^d); R^d)$. By the Birkhoff Transitivity Theorem ([33, Theorem 2.19]), it follows that each $\phi_b$ is topologically transitive on $L_1^1(\mathcal{B}(R^d); R^d)$. From Theorem 2.2, we know that there exists a dense subset $D \subseteq L_1^1(\mathcal{B}(R^d); R^d)$ such that for every $\sigma \in D$, the set

$$\mathcal{F} \triangleq \{ \phi_b^n(\sigma) : n \in N, b \in R^d - \{0\} \},$$

is dense in $L_1^1(\mathcal{B}(R^d); R^d)$ where, as before, $\phi_b \triangleq \phi_{tb}$. Whence $\mathcal{N}\mathcal{F}$ is dense in $L_1^1(\mathcal{B}(R^d); R^d)$.

Furthermore, by definition $f \in \mathcal{N}\mathcal{F}$ if and only if $f$ is of the form

$$f = \phi_b^n(\sigma),$$

for some $n \in N$. 
Moreover, by applying Lemma A.1, we find that for any \( \sigma \in D \), we have that
\[
\{ \phi^n_b(\sigma) \}_{n \in \mathbb{N}} = \{ \phi_{nb}(\sigma) \}_{n \in \mathbb{N}}.
\]
Upon applying the assumption that \( \{ t_{nb} \}_{n \in \mathbb{N}} \subseteq \mathcal{M} \) and the fact that \( 1_{\mathbb{R}^d} = \phi^0 = \phi_{0b} \in \mathcal{M} \) implies that
\[
\{ \phi^n_b(\sigma) \}_{n \in \mathbb{N}} \subset \{ \phi_W(\sigma) : W \in \mathcal{M} \}
\]
\[
\subseteq \bigcup_{k \in \mathbb{N}; k > 0} \{ \phi_{W_k}(\sigma) \circ \cdots \circ \phi_{W_1}(\sigma) : W_1, \ldots, W_k \in \mathcal{M} \}
\]
\[
= \bigcup_{k \in \mathbb{N}; k > 0} \{ 1_{\mathbb{R}^d} \circ \phi_{W_k}(\sigma) \circ \cdots \circ \phi_{W_1}(\sigma) : W_1, \ldots, W_k \in \mathcal{M} \}
\]
\[
\subseteq \bigcup_{k \in \mathbb{N}; k > 0} \{ W_{k+1} \circ \phi_{W_k}(\sigma) \circ \cdots \circ \phi_{W_1}(\sigma) : W_1, \ldots, W_{k+1} \in \mathcal{M} \}
\]
\[
\subseteq L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d).
\]
Since the set (2.22) is dense in \( L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) \), then (2.23) implies that
\[
L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) = \overline{\{ \phi^n_b(\sigma) \}_{n \in \mathbb{N}}}.
\]
\[
\subseteq \bigcup_{k \in \mathbb{N}; k > 0} \{ W_{k+1} \circ \phi_{W_k}(\sigma) \circ \cdots \circ \phi_{W_1}(\sigma) : W_1, \ldots, W_{k+1} \in \mathcal{M} \}
\]
\[
\subseteq L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d).
\]

Example 2.14. Assume the setting of Corollary 2.13. The following set of deep feed-forward neural networks is dense in \( L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) \)
\[
\left\{ f \in L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) : \left( \exists W_1, \ldots, W_{n+1} \in \text{Aff}^+\left(\mathbb{R}^d; \mathbb{R}^d\right) \right) f = W_{n+1} \circ \sigma \circ W_n \circ \cdots \circ \sigma \circ W_1 \right\};
\]
where \( \text{Aff}^+\left(\mathbb{R}^d; \mathbb{R}^d\right) \subseteq \text{Aff}\left(\mathbb{R}^d; \mathbb{R}^d\right) \) is defined by
\[
\text{Aff}^+\left(\mathbb{R}^d; \mathbb{R}^d\right) \triangleq \left\{ f \in \text{Aff}\left(\mathbb{R}^d; \mathbb{R}^d\right) : f(x) = Ax + b; \det(A) \neq 0 \text{ and } b \in \mathbb{R}^d \right\}.
\]

2.1.3 Examples on Non-Linear Spaces
Probability measures are foundational to all parts of statistical inference, probability theory, and many parts of machine learning. The following result describes the construction of a universal approximator on the space of Borel probability measures \( \mathcal{P}(X) \) on a non-empty separable and compact metric space \( X \), equipped with the topology of weak convergence in measure. Indeed, by the Riesz-Markov-Kakutani representation theorem we may view \( \mathcal{P}(X) \) as a function space by identifying it with the weak*-closed convex subset \( \hat{\mathcal{P}}(X) \) of \( C(X; \mathbb{R})' \), defined by:
\[
\hat{\mathcal{P}}(X) \triangleq \{ f \in C(X; \mathbb{R}), \ f(X) = 1 \}.
\]
Therefore, $\mathcal{P}(X)$ is well-within the scope of our theory.

In order to introduce our universal approximator on $\mathcal{P}(X)$ we must first define the Ruelle-Fröbenius-Perron self-maps on $\mathcal{P}(X)$, introduced by [66]. For each $f \in C(X;X)$, the Ruelle-Fröbenius-Perron self-map $f_{FP}$ induced by $f$, is defined as:

$$f_{FP}(\mu)(A) \triangleq \mu(f^{-1}[A]), \quad (\forall A \in \mathcal{B}(X)). \tag{2.25}$$

**Proposition 2.15.** Let $\phi \in C(X,X)$ and $F \subseteq C(X,X)$ be such that:

(i) For every pair of non-empty open subsets $U,V \subseteq X$ there exists $N \in \mathbb{N}$ satisfying $\phi^n(U) \cap V \neq \emptyset$, for all $n \geq N$,

(ii) For every $\{x_i\}_{i=1}^k, \{y_i\}_{i=1}^k \subseteq X$ there exists some $f \in F$ satisfying $f(x_i) = y_i$ for each $i \in \{1, \ldots, k\}$.

Then for every $\{x_i\}_{i \in \mathbb{N}} \subseteq X$ and every networking function $\circ$ on $\mathcal{P}(X)$, the architecture

$$\left\{ (\phi^n \circ f)_{FP} \left( \sum_{i=1}^k q_i \delta_{x_i} \right) : f \in F, q_i \in \mathbb{Q} \cap [0,1], \sum_{i=1}^k q_i = 1, k \in \mathbb{N}^+, n \in \mathbb{N} \right\}, \tag{2.26}$$

is dense in $\mathcal{P}(X)$ and consequently is a universal approximator on $\mathcal{P}(X)$; where $\mathbb{N}^+$ denotes the set of positive integers.

**Proof.** Set $n = 0$ and let $k \in \mathbb{N}^+$. For any $q_1, \ldots, q_k \in [0,1] \cap \mathbb{Q}$ satisfying $\sum_{i=1}^k q_i = 1$, any $f \in F$, and any distinct $x_1, \ldots, x_k \in X$, by using (2.25) we compute:

$$(\phi^n \circ f)_{FP} \left( \sum_{i=1}^k q_i \delta_{x_i} \right) = (\phi^0 \circ f)_{FP} \left( \sum_{i=1}^k q_i \delta_{x_i} \right) = f_{FP} \left( \sum_{i=1}^k q_i \delta_{x_i} \right) = \sum_{i=1}^k q_i \delta_{f(x_i)}. \tag{2.27}$$

Since $X$ is separable, let $\tilde{X}$ be a countable dense subset of $X$. By assumption (ii), for every distinct $y_1, \ldots, y_k \in \tilde{X}$ there exists some $f \in F$ satisfying

$$f(x_i) = y_i. \tag{2.28}$$

Combining (2.27) and (2.28) we find that:

$$\left\{ \sum_{i=1}^k q_i \delta_{y_i} : q_i \in \mathbb{Q} \cap [0,1], \sum_{i=1}^k q_i = 1, \{y_i\}_{i=1}^k \subseteq \tilde{X}, k \in \mathbb{N}^+, n \in \mathbb{N} \right\} \subseteq \mathcal{F}. \tag{2.29}$$

In the proof of [63, Theorem 1.1] it is shown that if $X$ is separable and complete then the right-hand side of (2.29) is dense in $(\mathcal{P}(X),d_{LP})$ where $d_{LP}$ is the Lévy-Prokhorov metric on $\mathcal{P}(X)$ (viewed now as a set), defined by

$$d_{LP}(\mu,\nu) := \inf \left\{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^c) + \varepsilon \text{ and } \nu(A) \leq \mu(A^c) + \varepsilon \text{ for all } A \in \mathcal{B}(X) \right\}.$$ 

By the results of [63], the metric $d_{LP}$ on $\mathcal{P}(X)$ induces a topology equivalent to the topology of weak convergence of measures. Therefore, $\mathcal{F}$ is dense in $\mathcal{P}(X)$. 

\[ \square \]
Remark 2.16. Though the function $\phi$ did not enter into the proof of Proposition 2.15, it will ensure that this architecture is compatible with later results in this paper.

Let us give a concrete representation of the universal approximators of Proposition 2.15 in the case where $X$ is the $d$-dimensional hyper-cube $[-M,M]^d$. We metrize $X$ by restricting the Euclidean metric on $\mathbb{R}^d$. Our representation makes use of the metric projection $P_{[-M,M]^d}$ defined by:

$$P_{[-M,M]^d} : \mathbb{R}^d \rightarrow [-M,M]^d$$

$$x \mapsto \text{argmin}_{y \in [-M,M]^d} \|y - x\|^2 = \left(\min \{M, \max\{-M,x_i\}\}\right)_{i=1}^d,$$

where the right-hand side of (2.30) is computed in [7, Lemma 6.26 (ii)].

Corollary 2.17. Fix $M > 0$, a positive integer $d$, and fix a $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, non-constant, and continuous function. Set:

(i) $\phi((x_i)_{i=1}^d) \equiv \left(2M \left(1 - \frac{1}{M^i}\right) - \frac{1}{2}\right)_{i=1}^d$,

(ii) $F$ be the collection of functions $f$ which admit the representation

$$f = P_{[-M,M]^d} \circ W_2 \circ \sigma \circ W_1,$$

where $W_2, W_1$ are composable affine functions, $\sigma$ is applied component-wise, and $P_{[-M,M]^d}$ is the metric projection of $\mathbb{R}^d$ onto $[-M,M]^d$.

Then the architecture (2.26) is a universal approximator on $\mathcal{P}([-M,M]^d)$ and the shallow networks are dense in $\mathcal{P}([-M,M]^d)$.

Proof. By [39, Theorem 2.5], we find that the set of functions

$$\{W_2 \circ \sigma \circ W_1 : W_2, W_1 \text{ are composable and affine}\},$$

satisfy Proposition 2.15 (ii). Since $P_{[-M,M]^d}$ is the identity on $[-M,M]^d$ and $P_{[-M,M]^d}$ restricts the co-domain of the maps of (2.31) from $\mathbb{R}^d$ to $[-M,M]^d$, then $F$ satisfies Proposition 2.15 (ii) and each $f \in F$ maps $[-M,M]^d$ into itself.

Next we show that $\phi$ satisfies (i) of Proposition 2.15; following [33], we call this property topologically mixing. By [33, Example 1.39], the tent map

$$T : [0,1] \rightarrow [0,1]$$

$$x \mapsto 1 - 2 \left|x - \frac{1}{2}\right|,$$

is topologically mixing on $[0,1]$. By induction, we now show that $T^d$ is topologically mixing on $[0,1]^d$ for every positive integer $d$. We have just seen the base case for $d = 1$. Next for the inductive step, suppose that the claim is true for some positive integer $d \geq 1$. By [33, Proposition 1.42 (v)], $T^{d+1} = T \times T^d$ is topologically mixing since both $T$ and $T^d$ are; thus the induction holds and $T^d$ is topologically mixing for every positive integer $d$. 

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Next, define the map $\Phi : [-M, M]^d \to [0, 1]^d$ by
\[
\Phi^{-1}(x)_i = 2M(x_i - \frac{1}{2}); \quad (\forall i \in \{1, \ldots, d\}).
\]
Then $\Phi$ is a homeomorphism and in we compute that
\[
\phi = \Phi^{-1} \circ T^d \circ \Phi.
\]
Where-from, $\phi$ and $T$ satisfy the (quasi-conjugacy relation):
\[
T^d \circ \Phi = \Phi \circ \phi. \tag{2.32}
\]
Since $T^d$ is topologically mixing and since (2.32) holds then by [33, Proposition 1.40] it follows that $\phi$ is topologically mixing; ie: Proposition 2.15 (i) holds. Thus, the conclusion is obtained by applying Proposition 2.15. \qed

The space $\mathcal{P}(X)$ provides an example of a familiar space in which the usual concatenation of activation functions and affine functions is ill-defined. Nevertheless, Definition (1.3) allows us to obtain a universal approximator on $\mathcal{P}(X)$ by only slightly deviating from the mold provided by Definition 1.1. In this way, Proposition 2.15 and Corollary 2.17 demonstrate the necessity of turning to non-classical neural networks architectures in order to successfully solve problems set in various rather conventional function spaces. Next, we prove that a wide range of function spaces admit microgenerated universal approximators.

2.2 Existence

The existence of universal approximators is established by the following two theorems. The first covers most conceivable function spaces, and the second covers those function spaces which may not be covered by the former, granted that its topology is induced by a metric.

Theorem 2.18 (Existence I). Let $\mathcal{X}$ be a separable function space and let $\circ$ be an networking function on $\mathcal{X}$. If $\mathcal{X}$ is homeomorphic to either

(a) A closed (or an open) convex body in an infinite-dimensional Fréchet space,
(b) A $d$-dimensional connected and compact Riemannian manifold, with $d \geq 2$

then it follows that

(i) There exists a dense and connected $G_\delta$-subset $D \subseteq \mathcal{X}$ and $\phi \in C(\mathcal{X}; \mathcal{X})$ such that, for each $\sigma \in D$ the architecture $\mathcal{F} \triangleq \{\phi^n(\sigma)\}_{n \in \mathbb{N}}$ is a universal approximator. Moreover, the shallow networks are dense in $\mathcal{X}$.

(ii) For every pair of non-empty open subsets $V, U \subseteq \mathcal{X}$, there exists some $N \in \mathbb{N}$ for which
\[
\phi^n(U) \cap V \neq \emptyset; \quad (\forall n \geq N). \tag{2.33}
\]
If $\mathcal{X}$ is not separable then there does not exist an architecture on $\mathcal{X}$ satisfying both (i) and (ii).
The next existence theorem covers the case where the function space is metric and that its topology is similar to a linear space, in the following senses.

**Definition 2.19** (Metric Function Space). A function space $\mathcal{X}$ is said to be a metric function space, if the topology of $\mathcal{X}$ is induced by a (possibly incomplete) metric $d_{\mathcal{X}}$.

**Assumption 2.20.** There exists an injective Lipschitz map $\iota: \mathcal{X} \rightarrow E$ into some separable Banach space $E$ which admits a continuous retract, i.e.: there exists some continuous function $j: E \rightarrow \mathcal{X}$ satisfying

$$j \circ \iota = 1_{\mathcal{X}}. \quad (2.34)$$

**Theorem 2.21** (Existence II). Let $\mathcal{X}$ be a separable metric function space satisfying Assumption 2.20 and let $\ominus$ be a networking function on $\mathcal{X}$. Then

(i) $\delta$ admits a continuous left-inverse, which we denote by $\beta$,

(ii) There is a dense subset $D \subseteq \mathcal{A}(\mathcal{X})$ such that for every $F \in D$ the architecture

$$\mathcal{F} \triangleq \{\beta \circ \phi^n(F)\}_{n \in \mathbb{N}},$$

is a universal approximator on $\mathcal{X}$; where $\phi \in C(\mathcal{A}(\mathcal{X}), \mathcal{A}(\mathcal{X}))$ is a topologically transitive map.

**Remark 2.22.** Theorem 2.21 is a partial extension of Theorem 2.2 (a) since the inclusion map of any closed convex body in a Banach space, equipped with the induced metric, admits a continuous left-inverse. However, in this case the representation of the universal approximator becomes more complex than the one obtained in Theorem 2.2. Hence, the purpose of Theorem 2.21 is to address metric function spaces not covered by Theorem 2.2.

**Proof of 2.18.** Let us first consider the case where $\mathcal{X}$ is homeomorphic to a closed (or an open) convex body in a separable infinite-dimensional Fréchet space. Let $\Phi_1: \mathcal{X} \rightarrow C$ be a homeomorphism onto a closed (or open) convex body $C$ in a separable infinite-dimensional Fréchet space $E$. Since $C$ is a closed (or open) convex body and $E$ is a separable infinite-dimensional Fréchet space, then [8, Theorem 1] implies that there exists a homeomorphism $\Phi_2: C \rightarrow E$. Therefore the map $\Phi \triangleq \Phi_2 \circ \Phi_1$ is a homeomorphism from $\mathcal{X}$ onto $E$.

Since $E$ is a separable infinite-dimensional Fréchet space, then the Ansari-Bernal Theorem applies, [33, Theorem 8.9]; whence there exists a hypercyclic operator $T$ on $E$, such that for every pair of non-empty open subsets $\tilde{U}, \tilde{V} \subseteq E$, there is some $N \in \mathbb{N}$ satisfying

$$T^n \left(\tilde{U}\right) \cap \tilde{V} \neq \emptyset; \quad (\forall n \geq N). \quad (2.35)$$

By Lemma A.2, setting $\phi \triangleq \Phi^{-1} \circ T \circ \Phi$ yields (i).

For (ii), first observe that since $\Phi$ is a homeomorphism, then it maps open sets in $E$ to open sets in $\mathcal{X}$ and visa-versa. Therefore, there exist unique open sets $U, V \subseteq \mathcal{X}$ such that $\Phi(U) = \tilde{U}$ and $\Phi(V) = \tilde{V}$. Hence, for every $n \geq N$ it follows that

$$\phi^n(U) \cap V = \Phi^{-1} \circ T^n \circ \Phi(U) \cap V = \Phi^{-1} \circ T^n(\tilde{U}) \cap V. \quad (2.36)$$
Since $\Phi$ and $\Phi^{-1}$ are bijections then they map non-empty sets to non-empty sets; whence by (2.35) and (2.36) we have that
\[
\Phi(\phi^n(U) \cap V) = \Phi \left( \Phi^{-1} \circ T^n(\tilde{U}) \cap V \right) = T^n(\tilde{U}) \cap \Phi(V) = T^n(\tilde{U}) \cap \tilde{V} \neq \emptyset,
\]
for every $n \geq N$. This establishes (ii).

Now, consider the case where $\mathcal{X}$ is homeomorphic to a connected, compact $d$-dimensional Riemannian manifold $(M,g)$ with $d \geq 2$. Let $\Phi : \mathcal{X} \to (M,g)$ be such a homeomorphism. Since $(M,g)$ is a Riemannian manifold, it is a class $C^1$-manifold. Since $(M,g)$ is a compact $C^1$-manifold of dimension at-least 2 then [2, Theorem 2], implies that the set of topologically mixing maps are dense in the space $Diff^1(M,g)$, where $Diff^1(M,g)$ denotes the class of $C^1$-diffeomorphisms of $(M,g)$ onto itself equipped with the $C^1$-topology (see [9] for details on this topology). Most notably for our purposes, since the identity map is an element of $Diff^1(M,g)$, then $Diff^1(M,g)$ is non-empty, hence there must exist a topologically transitive map $T : (M,g) \to (M,g)$. Therefore (ii) holds for $(M,g)$.

Since $(M,g)$ is compact, then it must be geodesically-complete (see [43, Section 1.5] for this result as well as a discussion on geodesic-completeness). Since $(M,g)$ is connected and geodesically-complete, then the Hopf-Rinow Theorem (of [37]) implies that $(M,g)$ can be equipped with a metric $d_g$ which makes $(M,d_g)$ into a complete metric space. Moreover, since $(M,g)$ is of dimension at-least 2 and it is connected, then it cannot contain any isolated points. Therefore, the general version of the [33, Birkhoff Transitivity Theorem; Theorem 1.16] implies that (i) must hold for $T$ on $(M,g)$. By setting $\phi \triangleq \Phi^{-1} \circ T \circ \Phi$ and proceeding analogously to case (a), we find that (i) and (ii) must hold for $\mathcal{X}$.

Lastly, consider the case where $\mathcal{X}$ is not separable. Since $\mathcal{F}$ is countable it cannot be dense in $\mathcal{X}$. Therefore, there does not exist an architecture satisfying (i). \hfill \Box

**Proof of 2.21.** We first establish the existence of a continuous retract $\beta$ of $\delta$. Since $\iota$ is Lipschitz, then by Compilation B.3 (iv), there must exist some continuous linear map $I : \mathcal{E}(\mathcal{X}) \to E$ satisfying
\[
I \circ \delta = \iota. \tag{2.37}
\]
Combining (2.37) and (2.34), we find that $\beta \triangleq (j \circ I)$ satisfies
\[
\beta \circ \delta = j \circ (I \circ \delta) = j \circ \iota = 1_\mathcal{X}. \tag{2.38}
\]
Both $j$ and $I$ are continuous, therefore $\beta$ is a retract of $\delta$. Since $\delta$ is the right-inverse of $\beta$, then $\beta$ is also a surjection onto $\mathcal{X}$. Since $\beta$ is a continuous surjection then $\beta$ maps any dense subset of $\mathcal{E}(\mathcal{X})$ to a dense subset of $\mathcal{X}$.

By Compilation B.3 (i), $\mathcal{E}(\mathcal{X})$ is a separable Banach space. Therefore, by Theorem 2.18, there exists a topologically mixing map $\phi \in C(\mathcal{X}; \mathcal{X})$ and a dense subset $D \subseteq \mathcal{E}(\mathcal{X})$, such that for every $F \in D$, $\mathcal{F} \triangleq \{\phi^n(F)\}_{n \in \mathbb{N}}$ is a universal approximator on $\mathcal{E}(\mathcal{X})$; that is $\mathcal{N}\mathcal{N}^\mathcal{F}$ is dense in $\mathcal{E}(\mathcal{X})$. Whence,
\[
\mathcal{F} \triangleq \beta \left( \{\phi^n(F)\}_{n \in \mathbb{N}} \right) = \{\beta \circ \phi^n(F)\}_{n \in \mathbb{N}},
\]
is dense in $\mathcal{X}$. Therefore, for any networking map $\circ$, $\mathcal{N}\mathcal{N}^\mathcal{F}$ is a universal approximator on $\mathcal{X}$. \hfill \Box

Next, we present an abstract applications of these existence Theorems.
2.2.1 Two Examples

Let us consider a class of function spaces covered by Theorem 2.21 but not by Theorem 2.2. Hadamard manifolds form a particularly well-behaved class of Riemannian manifolds, which include both the Euclidean spaces and the hyperbolic spaces. Since their inception, Hadamard spaces have found a number of exciting applications outside of differential geometry; some of these applications include, the study of morphing in computer vision as studied in [59] and the description of the geometry associated to any 1-factor stochastic volatility model in mathematical finance, as described in [35].

Briefly, a Hadamard manifold is a geodesically complete and simply-connected $d$-dimensional Riemannian manifold of non-positive sectional curvature; that is a Riemannian with no holes in which any two points must be connected by a minimal length path, and this minimal length path must be indefinitely extendable along the manifold. The statement about the curvature can essentially be interpreted as formalizing the requirement that all triangles on a Hadamard manifold have interior angles which must sum up to at-most $180^\circ$.

Using Theorem 2.21, we obtain a surprisingly simple universal approximation theorem for function spaces that possess the structure of a Hadamard manifold.

**Corollary 2.23.** Fix a positive integer $d$, $m$ be the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$, let $X$ be a function space which is also a $d$-dimensional Hadamard manifold, and let $\mu$ be the Lebesgue measure on $\mathbb{R}^d$. Then there exists a surjective Lipschitz map $\beta : L_1^1(\mathbb{R}^d; \mathbb{R}^d) \to X$ such that for every universal approximator $F$ on $L_1^1(\mathbb{R}^d; \mathbb{R}^d)$,

$$\left\{ f \in X : \exists F \in \mathbb{N}^n F \right\},$$

is a universal approximator on $X$.

**Proof.** Since $X$ is Hadamard, then it is simply connected and geodesically complete. Therefore by Hopf-Rinow Theorem (of [37]) it is complete as a metric space. Since any simply connected topological space is connected, then $X$ is connected. Since $X$ is connected and complete as a metric space, then the Cartan-Hadamard Theorem (see [43, Corollary 6.9.1] for a modern formulation) implies that $X$ is diffeomorphic to some Euclidean space $\mathbb{R}^d$; denote that diffeomorphism by $\Phi_1 : X \to \mathbb{R}^d$; moreover, $\Phi_1$ is 1-Lipschitz.

On [28, Page 4] it is shown that there exists a bounded linear surjection $\beta^\mathbb{R}^d : \mathbb{A}(\mathbb{R}^d) \to \mathbb{R}^d$; by [28, Lemma 2.4] we know that $\beta^\mathbb{R}^d$ is a continuous left-inverse of $\delta^\mathbb{R}^d$.

By [16], there exists a linear isometric isomorphism $\Phi^2 : \mathbb{A}(\mathbb{R}^d) \to L_1^1(\mathbb{R}^d; \mathbb{R}^d)/N$ where $N$ is a closed linear (strict) subspace of $L_1^1(\mathbb{R}^d; \mathbb{R}^d)$. Furthermore, the quotient map $q : L_1^1(\mathbb{R}^d; \mathbb{R}^d)/N \to L_1^1(\mathbb{R}^d; \mathbb{R}^d)$ is a continuous linear map; whence $q$ is a bounded linear map.

Since all bounded linear maps are Lipschitz then we may conclude that

$$\beta \triangleq \Phi^1 \circ \beta^\mathbb{R}^d \circ \Phi^2 \circ q : L_1^1(\mathbb{R}^d; \mathbb{R}^d) \to X,$$

is a surjective Lipschitz map, since it is the composition of Lipschitz surjections. Since every Lipschitz map is continuous and the image of a dense subset under a continuous surjection is itself dense then

$$\left\{ \beta \circ \delta^n(\tilde{F}) \right\}_{n \in \mathbb{N}},$$

(2.39)
is dense in \(X\), since \(NN^F\) is dense in \(L^1_0(\mathbb{R}^d;\mathbb{R}^d)\), because \(F\) is a universal approximator.

Lastly, let us consider an application of our results to a family of spaces arising in harmonic analysis. Let \(m\) denote the Lebesgue measure on \(\mathbb{R}\). The function space \(L_m^\infty(\mathcal{B}(\mathbb{R});\mathbb{R})\), of essentially bounded functions, plays a central role in many areas of mathematics and statistics. The assumption of essential boundedness is often too restrictive for most applications, moreover the non-separability of \(L_m^\infty(\mathcal{B}(\mathbb{R});\mathbb{R})\) makes it analytically intractable. For instance, the classical universal approximation theorems of [38] fail for \(L_m^\infty(\mathcal{B}(\mathbb{R});\mathbb{R})\). Introduced in [42], the space of functions of \textit{bounded mean oscillation} (BMO) serves as the standard replacement for \(L_m^\infty(\mathcal{B}(\mathbb{R});\mathbb{R})\) when the assumption of essential boundedness is too stringent.

The functions making up BMO are the collections \(m\)-locally-integrable functions on \(\mathbb{R}\) which satisfy

\[
\sup_{x \in \mathbb{R}, r > 0} \int_{y \in \text{Ball}(x;r)} f(y) \, dy \triangleq \sup_{x \in \mathbb{R}, r > 0} \frac{1}{\mu(\text{Ball}(x;r))} \int_{y \in \text{Ball}(x;r)} \left| f(y) - \bar{f}_{\text{Ball}(x;r)}(y) \right| \, dy < \infty,
\]

where \(\bar{f}_{\text{Ball}(x;r)}\) denotes the mean of \(f\) on \(\text{Ball}(x;r)\) and \(\mu\) denotes the Lebesgue measure on \(\mathbb{R}\). Similarly to \(L_m^\infty(\mathbb{R};\mathbb{R})\), BMO can be endowed with the structure of a Banach space; however BMO also fails to be separable. See [18] for more details on BMO.

The non-separability, as well as a number of other technical shortcomings of BMO, motivated [68] to instead consider its subspace of functions of \textit{vanishing mean oscillation} on \(\mathbb{R}\), denoted \(VMO(\mathbb{R};\mathbb{R})\). The space \(VMO(\mathbb{R};\mathbb{R})\) is characterized by the additional requirement that a BMO function \(f\) satisfy

\[
\lim_{r \downarrow 0} \int_{y \in \text{Ball}(x;r)} f(y) \, dy \to 0 \quad (\forall x \in \mathbb{R}).
\]

Hence, the next result is an abstract analogue of a universal approximation theorem on \(L_m^\infty(\mathbb{R};\mathbb{R})\). The interested reader is referred to [68] for more details on \(VMO(\mathbb{R};\mathbb{R})\) and on \(BMO(\mathbb{R};\mathbb{R})\).

**Corollary 2.24.** There exists a microgenerated architecture in \(VMO(\mathbb{R};\mathbb{R})\) which satisfies Theorem 2.2 (i)-(iv), where the networking function \(\cup\) is taken to be point-wise addition in \(\mathbb{R}\).

**Proof.** In [68] it can be seen that \(VMO(\mathbb{R};\mathbb{R})\) defines a separable Banach space. The result follows by applying Theorem 2.2.

In the next section, we provide a partial converse to the previous results by showing that for any suitable topological space \(X\) any architecture \(\emptyset \neq \mathcal{F} \subseteq C(X;X)\), there exists a finest non-trivial topology on any non-empty subset of \(C(X;Y)\) for which \(\mathcal{F}\) is a universal approximator.

### 3 A Converse to the Universal Approximation Theorem

The classical universal approximation property of neural networks, as developed in [17, 39], states that certain neural network architectures are capable of approximating any function in \(C(\mathbb{R}^d;\mathbb{R}^D)\) \textit{uniformly on compacts}. Observe that this result does not rule out the possibility that these neural networks are capable of approximating any function in \(C(\mathbb{R}^d;\mathbb{R}^D)\) with respect to
stronger topologies. We want to examine the maximum level of precision which any specific architecture can guarantee when approximating continuous functions between two spaces. We can obtain a concrete lower-estimate of this topology by turning to the following modification of the compact-open topology on $C(X; Y)$; details on the compact-open topology are found in [46].

Definition 3.1 (The $\tau_{\mathcal{F}} \cdot \star$ Topology). Let $X, Y$ be non-empty topological spaces, $\emptyset \subset X \subseteq C(X; Y)$ and $\mathcal{F}$ be an architecture on $\mathcal{X}$. Define the topology $\tau_{\mathcal{F}} \cdot \star$ as being generated by the following sub-base

$$B_{\mathcal{F}} \triangleq \left\{ V_{\mathcal{F}}(K, U) : K \text{ compact in } X, U \text{ open in } Y \right\}$$

$$V_{\mathcal{F}}(K, U) \triangleq \left\{ f \in \mathcal{N}_{\mathcal{F}} : f(K) \subseteq U \right\}.$$ (3.1)

The following result interprets the maximal level of precision of the architecture $\mathcal{F}$ as the finest topology on $X$, for which $\mathcal{N}_{\mathcal{F}}$ is dense in $X$. To better understand why this is an appropriate way to measure an architecture’s approximation capabilities consider two extreme examples. On the one hand, if $\mathcal{N}_{\mathcal{F}}$ is dense with respect to the discrete topology on $X$ then $\mathcal{N}_{\mathcal{F}}$ can exactly reproduce any function with perfect precision. On the other end of the spectrum, if $\mathcal{N}_{\mathcal{F}}$ is only dense with respect to the trivial topology on $X$ then it can only approximate any function if all functions are deemed indistinguishable; in this case $\mathcal{N}_{\mathcal{F}}$ cannot accurately approximate any function.

Both of these extremes are atypical since one can verify that the first case can only happen if $X = \mathcal{N}_{\mathcal{F}}$ and the second case only occurs if $\mathcal{F}$ contains only one function which is fixed under the networking function $\circ$ and there no constant functions in $\mathcal{X}$. Luckily, both these situations are extreme; thus most architectures lie in between these two cases.

Theorem 3.2 (Universal Approximation Capabilities of Neural Network). Let $\mathcal{X}$ be a non-empty set of equivalence classes of functions between two non-empty sets $X$ and $Y$ and $\mathcal{F}$ be an architecture on $\mathcal{X}$. Then there exists a topology $\tau_{\mathcal{F}}$ on $\mathcal{X}$, such that:

(i) **Universal Approximation**: The closure of $\mathcal{N}_{\mathcal{F}}$ in $\mathcal{X}$ with respect to $\tau_{\mathcal{F}}$ is $\mathcal{X}$,

(ii) **Maximality**: If $\tau$ is another topology on $\mathcal{X}$ for which the $\tau$-closure of $\mathcal{N}_{\mathcal{F}}$ is $\mathcal{X}$, then $\tau \subseteq \tau_{\mathcal{F}}$,

(iii) **Inherited Universality**: If $\tau$ is coarser than $\tau_{\mathcal{F}}$, then the closure of $\mathcal{N}_{\mathcal{F}}$ with respect to $\tau$ is $\mathcal{X}$,

Suppose moreover, that $X$ and $Y$ are topological spaces, $Y$ is a $T_0$ space containing at-least 2 points, and that $\mathcal{X} \subseteq C(X; Y)$. Then:

(v) **Concrete Lower-Bound**: $\tau_{\mathcal{F}} \cdot \star$ is at-least as course as $\tau_{\mathcal{F}}$. In particular, $\mathcal{N}_{\mathcal{F}}$ is dense in $\mathcal{X}$, with respect to $\tau_{\mathcal{F}} \cdot \star$.  

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(vi) **Non-Triviality**: $\tau^{F,*}$ is not the trivial topology. Furthermore, if $\mathcal{X} = C(X;Y)$ then $\tau^{F,*}$ is coarser than the restriction of the compact-open topology on $C(X;Y)$ to $\mathcal{X}$. Moreover, in the extreme case where

$$\mathcal{NN}^{F} = \mathcal{X} = C(X;Y),$$

then $\tau^{F,*}$ agrees with the compact-open topology on $C(X;Y)$. In particular, in this case $\tau^{F,*}$ is not discrete.

**Proof.** Let $\{\tau_{i}\}_{i \in I}$ be the set of topologies on $\mathcal{X}$ under which $\mathcal{NN}^{F}$ is dense in $\mathcal{X}$. The set $\{\tau_{i}\}_{i \in I}$ is non-empty since it must contain the trivial topology on $\mathcal{X}$. For every $i \in I$, let

$$f_{i} : \mathcal{X} \to (\mathcal{X}, \tau_{i})$$

$$f_{i}(x) \mapsto x,$$

be the (set-theoretic) identity map. By [10, Book 1, Proposition 2.3.4], the initial (or weak) topology with respect to $\{f_{i}\}_{i \in I}$, exists and is characterized as the finest topology on $\mathcal{X}$ making each $f_{i}$ continuous. Let $\tau^{F}$ be this topology.

Applying [10, Theorem 2.1.1], we observe that the inverse image of a dense set under a continuous map is itself dense. Since each $f_{i}$ is continuous from $(\mathcal{X}, \tau)$ to $(\mathcal{X}, \tau_{i})$, then $\mathcal{NN}^{F}$ is dense on $\mathcal{X}$ with respect to $\tau^{F}$. This gives (i).

By definition of $\tau^{F}$, the maps $f_{i}$ are continuous. Since each $f_{i}$ is the identity map, [10, Book 1, Proposition 2.2.3] implies that $\tau_{i}$ is coarser than $\tau^{F}$. Statement (ii) then follows by definition of the indexing set $I$.

For (iii), note that if $\tau$ is coarser than $\tau^{F}$, then every closed set in $\tau$ is also closed in $\tau^{F}$. Therefore, by definition of the closure of $\mathcal{NN}^{F}$ with respect to $\tau$ and by (i), we have that

$$\mathcal{X} = \bigcap_{\mathcal{NN}^{F} \subseteq K, K \in \tau^{F}} K \subseteq \bigcap_{\mathcal{NN}^{F} \subseteq K, K \in \tau^{F}} K = \overline{\mathcal{NN}^{F}}^{\tau^{F}} \subseteq \mathcal{X};$$

where $\overline{\mathcal{NN}^{F}}^{\tau^{F}}$ denotes the $\tau$-closure of $\mathcal{NN}^{F}$. Hence, $\mathcal{NN}^{F}$ is dense in $\mathcal{X}$ with respect to $\tau$.

For (iv), since $B^{F}$ is a sub-base for the topology $\tau^{F,*}$, every open set in $\tau^{F,*}$ is an arbitrary union and finite intersection of the sets $\{V^{F}(K,U) : K \text{ compact in } X, U \text{ open in } Y\}$ as well as with $\mathcal{X}$, $\emptyset$. By definition, the elements of $V^{F}(K,U)$ are all in $\mathcal{NN}^{F}$. Note also that $\mathcal{NN}^{F} \cap \mathcal{X} = \mathcal{NN}^{F}$. Therefore, every non-empty open set in $\tau^{F}$ contains an element of $\mathcal{NN}^{F}$. Hence, $\mathcal{NN}^{F}$ is dense in $\mathcal{X}$ if the latter is equipped with $\tau^{F,*}$. Applying (ii), we obtain the conclusion to (iv).

For (v), let $c_{1}, c_{2}$ be distinct points in $Y$. Since $Y$ is a $T_{0}$ space, then there exist a non-empty open subset $U \subseteq Y$ containing $c_{1}$ but not containing $c_{2}$. By definition, $\mathcal{NN}^{F}$ contains the constant functions. Hence, for any compact subset $K$ of $X$, the maps

$$f_{c_{1}}(x) \overset{\Delta}{=} c_{1}$$
$$f_{c_{2}}(x) \overset{\Delta}{=} c_{2},$$
satisfy \( f_{c_1}(K) \subseteq U \) and \( f_{c_2}(K) \nsubseteq U \). Therefore, the sub-base element \( V^\mathcal{F}(K; U) \) is neither equal to \( \emptyset \) nor equal to \( \mathcal{X} \). Hence, the topology \( \tau^{\mathcal{F}} : \tau^* \) is non-trivial since it contains open sets other than \( \{ \emptyset, \mathcal{X} \} \). Moreover, if \( \mathcal{N}^{\mathcal{F}} \neq \mathcal{X} \), then this topology is not Hausdorff; whence it is not non-discrete.

In the extreme case where \( \mathcal{N}^{\mathcal{F}} = \mathcal{X} = C(X; Y) \). Comparing the standard sub-base for the compact-open topology on \( C(X; Y) = \mathcal{X} \) with the sub-base for \( \tau^{\mathcal{F}} : \tau^* \) given by (3.1), implies that that both sub-bases are one and the same. Since any two topologies coincide if their sub-base elements coincide, we have that, in this case, \( \tau^{\mathcal{F}} : \tau^* \) is precisely the compact-open topology on \( \mathcal{X} \).

Theorem 3.2 provides a theoretical guarantee that any architecture on a non-empty subset of continuous functions between reasonable topological spaces must be a universal with respect to some non-trivial topology, as quantified by \( \tau^{\mathcal{F}} : \tau^* \). Furthermore, Theorem 3.2 reinterprets Theorem 2.18 and the classical universal approximation theorems of [17, 39, 38, 62] as describing lower-estimates on \( \tau^{\mathcal{F}} : \tau^* \). This is because these universal approximation theorems guarantee that certain architectures are universal approximators with respect to some specific topology on a set of (equivalence classes) of functions, but these results do not guarantee that there is no finer topology on that set of (equivalence classes) of functions for which it is still a universal approximator.

4 Universal Approximation Capabilities of Randomly Activated Architectures

Consider the function space \( \mathcal{X} = P^1_\mu(B(X); Y) \), where \( X, Y \) are separable Banach spaces and \( \mu \) is a finite Borel measure on \( X \). Proposition 2.6 showed that the architecture

\[
\left\{ \sum_{i=1}^{n} \sigma_i(Ax + b)y_i : (A \in B(X; X))(b \in X) \right\} = \{ \phi_{A,b}(\sigma) \}_{(A,b) \in I},
\]

is a universal approximator for any \( \sigma \in D \), where \( \phi_{A,b} \) are the composition operators defined by

\[
\phi_{A,b}: P^1_\mu(B(X); Y) \to P^1_\mu(B(X); Y)
\]

\[
\sigma \mapsto \sigma \circ (Ax + b),
\]

where \( A \in B(X, X), b \in X, \) and \( I = B(X, X) \times X \). Thus, the right-hand side of (4.1) implies that \( \sigma \) is a "good activation function", if and only if, the orbit of \( \sigma \) under some \( \phi_{A,b} \) is dense in \( P^1_\mu(B(X); Y) \); with the analogous conclusion being true for the architectures studied by Propositions 2.7, 2.10, and 2.11, and consequentially in [39]. Therefore, the right-hand side of (4.1) will be used a definition of an activation function for neural network architectures which do not fit into the classical mold of (1.1).

Definition 4.1 (Activation Function). Let \( \mathcal{F} = \{ \phi^n_i(\sigma_i) \}_{i \in I, n \in \mathbb{N}} \) be a generated architecture on \( \mathcal{X} \). The functions \( \{ \sigma_i \}_{i \in I} \) are called activation functions if \( \mathcal{F} \) is a universal approximator on \( \mathcal{X} \).

Remark 4.2. If \( \mathcal{F} \) a microgenerated architecture on \( \mathcal{X} \) (ie: \( \mathcal{F} = \{ \phi^n(\sigma) \}_{n \in \mathbb{N}} \)) and \( \mathcal{X} \) is a separable LCSs then \( \sigma \) is an activation function if and only if \( \sigma \) is a hypercyclic vector for \( \phi \).
Though Theorems 2.18 and 2.2 proved that activation functions are dense in \( X \), they did not explicitly describe a method for selecting one. In the next section we will exploit the hypercyclicity property of \( \phi \) in order to randomly generate a suitable activation functions with positive probability.

The next section considers the problem of appropriately choosing an activation function.

### 4.1 Universal Approximation by Randomization

A popular technique from computer science, which allows one to probabilistically overcome the difficulty present in specific tasks is randomization. In [25], it was shown that randomization could also be used to overcome the computational burden of training very large networks. This method has since then, overcome the computational shortcomings of its time and has found many applications ranging from image processing tasks in [75], to general classification tasks in [1], and even to cloud computing tasks in [72]. The theoretical aspects of random neural network theory have also found notable examination in recent contributions by [61, 52], amongst many others.

Each of these methods focused on using randomization to overcome computational difficulties. Instead, we will use randomization to choose a activation function \( \sigma \in X \).

In order to do this, we will appeal to topologically transitive maps on a function space \( X \). A topologically transitive (resp. topologically mixing) map on \( X \) is any continuous \( \phi \in C(X; X) \) satisfying: for any non-empty open subsets \( U, V \subseteq X \), there exists some \( N \in \mathbb{N} \) for which
\[
\phi^N(U) \cap V \neq \emptyset
\]
(resp. satisfying (2.33)).

**Definition 4.3** (Randomly Activated Architecture). Fix a probability space \((\Omega, \Sigma, \mathbb{P})\), a function space \( X \), and a non-empty indexing set \( I \). Let \( \varphi \triangleq \{ \phi_i \}_{i \in I} \) be a family of maps in \( C(X; X) \) one of which is topologically transitive, \( \cup \) be a networking function, and let \( \sigma : (\Omega, \Sigma) \to (X, B(X)) \) be a random element satisfying the non-degeneracy condition
\[
\mathbb{P}(\sigma \in V) > 0; \quad (\exists U \subseteq X \text{ open})(\forall \emptyset \neq V \subseteq U \text{ open}). \tag{4.2}
\]

The randomly activated architecture induced by \( \varphi \) and \( \sigma \) is defined to be the following collection of \( X \)-valued random elements:
\[
\mathcal{F}^{\varphi, \sigma} \triangleq \{ \phi^n_i(\sigma) \}_{n,i} \in \mathbb{N} \times I. \tag{4.3}
\]

We call \( \sigma \) a random activation function and say that \( \mathcal{F}^{\varphi, \sigma} \) is microgenerated if \( I \) is a singleton.

Randomly activated architectures are universal approximators in the following sense.

**Theorem 4.4.** In the notation of Definition 4.3, let \( X \) be a separable function space and let \( \mathcal{F}^{\varphi, \sigma} \) be a randomly activated architecture on \( X \). Then for every non-empty open subset \( V \subseteq X \),
\[
0 < \sup_{i \in I} \mathbb{P} \left( \bigcup_{n \in \mathbb{N}} \{ \omega \in \Omega : \phi^n_i(\sigma(\omega)) \in V \} \right); \quad (\forall \emptyset \neq V \subseteq X \text{ open}). \tag{4.4}
\]
Moreover, if $\mathcal{X}$ is homeomorphic to a separable infinite-dimensional Banach space, $\varphi$, $\sigma$, and $\mathcal{N}^{\mathbb{R}^{\times}}$ exist and can be assumed to be microgenerated.

**Proof.** Suppose that $\mathcal{X}$ is an arbitrary separable function space. Let $I$ be a non-empty indexing set and $\varphi \triangleq \{\phi_i\}_{i \in I}$ be a set of topologically transitive maps on $\mathcal{X}$ and $\sigma$ be an $\mathcal{X}$-valued random element defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ which satisfies the non-degeneracy condition (4.2).

Since there is a topologically transitive map $\phi \in \{\phi_i\}_{i \in I}$, then for every non-empty open subset $V$ of $\mathcal{X}$ and every non-empty open neighborhood $U$ of 0 there exists $N \in \mathbb{N}$ satisfying (4.7). Moreover, we compute that

$$
(\phi^N)^{-1} [\phi^N(U) \cap V] = (\phi^N)^{-1} [\phi^N(U)] \cap (\phi^N)^{-1} [V] = U \cap (\phi^N)^{-1} [V].
$$

(4.5)

Since $\phi$ is continuous, then so is $\phi^N$; hence $(\phi^N)^{-1} [V]$ is an open subset of $\mathcal{X}$. Since, $U$ and $V$ are open subsets in $\mathcal{X}$ then the right-hand side of (4.5) is an open subset of $\mathcal{X}$. Since $\phi$ is topologically transitive, then (2.33) and our choice of $N \in \mathbb{N}$ implies that the right-hand side of (4.5) is a non-empty subset of $U$. Hence (4.8) and (4.5) imply that

$$
0 < \mathbb{P}(\sigma \in U \cap (\phi^N)^{-1} [V]) = \mathbb{P}(\phi^N(\sigma) \in \phi^N(U) \cap V) \leq \mathbb{P}(\phi^N(\sigma) \in V) \leq \mathbb{P}\left( \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega : \phi^n_i(\sigma) \in V\} \right) \leq \sup_{i \in I} \mathbb{P}\left( \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega : \phi^n_i(\sigma) \in V\} \right).
$$

(4.6)

This establishes (4.4).

Now, we show existence in the case were $\mathcal{X}$ is homeomorphic to a separable infinite-dimensional Banach space. Let $C_0([0, 1]; \mathbb{R})$ denote the classical Wiener space comprised only of the functions in $C([0, 1]; \mathbb{R})$ which take the value zero at the infimum over $[0, 1]$. Let $\mathbb{P}$ denote the Wiener measure on $(C_0([0, 1]; \mathbb{R}), \mathcal{B}(C_0([0, 1]; \mathbb{R})))$. Since $\mathbb{P}$ is a strictly positive and locally finite probability measure, then for every non-empty open subset $V \subseteq C_0([0, 1]; \mathbb{R})$ we have that

$$
0 < \mathbb{P}(V) \leq 1.
$$

Therefore, $\mathbb{P}$ satisfies the non-degeneracy condition (4.2); with $U \triangleq C_0([0, 1]; \mathbb{R})$.

Let $\Phi_1 : \mathcal{X} \to E$ be a homeomorphism onto some infinite-dimensional Banach space $E$. Since $\mathcal{X}$ is separable and $\Phi_1$ is a homeomorphism, then $E$ is separable. In [32], it is shown that the classical Wiener space $C_0([0, 1]; \mathbb{R})$ is also an infinite-dimensional separable Banach space. Therefore, by [44] there must be a homeomorphism $\Phi_2 : E \to C_0([0, 1]; \mathbb{R})$. Hence $\Phi \triangleq \Phi_2 \circ \Phi_1 : \mathcal{X} \to C_0([0, 1]; \mathbb{R})$ is a homeomorphism. By the [33, Ansari-Bernal Theorem; Theorem 8.9] there exists a topologically mixing continuous linear map $T$ from $C_0([0, 1] : \mathbb{R})$.
to itself, since $C_0([0,1];\mathbb{R})$ is an infinite-dimensional separable Banach space. By the Birkhoff Transitivity Theorem, see [33, Theorem 2.19], $T$ is topologically transitive; that is, for every pair of non-empty open subsets $U, V \subseteq C_0([0,1];\mathbb{R})$ there exists some $N \in \mathbb{N}$ for which

$$T^N(U) \cap V \neq \emptyset. \quad (4.7)$$

Since $\Phi$ is a homeomorphism, then by [33, Proposition 1.40] we deduce that $\varphi$, defined by

$$\varphi \triangleq \Phi^{-1} \circ T \circ \Phi,$$

is a topologically transitive map from $X$ to itself. Set $\varphi \triangleq \{ \phi \}$ and $\sigma = 1_E$.

The Wiener measure $\mathbb{P}$ is non-atomic and of full support on $C_0([0,1];\mathbb{R})$, see [60, Chapter 3] for example. Since $\Phi$ is a homeomorphism, then so is $\Phi^{-1}$; whence the push-forward measure

$$\tilde{\mathbb{P}} \triangleq (\Phi^{-1})_* (\mathbb{P})$$

must be strictly positive and of full-support on $X$, since $\mathbb{P}$ is and since $\Phi$ is a homeomorphism. Therefore, the law of $\sigma$ is strictly positive and of full-support on $X$. Since $\tilde{\mathbb{P}}$ is also Borel then for every non-empty open subset $\tilde{U}$ of $C_0([0,1];\mathbb{R})$ we have that

$$0 < \tilde{\mathbb{P}} \left( \sigma \in \tilde{U} \right). \quad (4.8)$$

This establishes the three existence claims, in the case where $X$ is homeomorphic to a separable infinite-dimensional Banach space.

Remark 4.5. For any randomly activated network $\mathcal{F}^{\varphi,\sigma}$ and every $i \in I$ the generator $\phi_i$ is continuous; whence the set $\bigcup_{n \in \mathbb{N}} \{ \phi_i^n(\sigma) \}$ is $\Sigma$-measurable.

However, if $\mathcal{F}^{\varphi,\sigma}$ is not microgenerated or if the networking function is not measurable then there is no guarantee that the sets $\bigcup_{n \in \mathbb{N}, i \in I} \{ \phi_i^n(\sigma) \}$ or $\{ \{ \phi_i^n(\sigma) \}^+ \}$ are $\Sigma$-measurable. Hence, (4.4) avoids issues of measurability of these larger sets by exploiting the generators of $\mathcal{F}^{\varphi,\sigma}$ to give meaning to the statement”.

4.1.1 An Application: Learning an $\mathbb{R}$-Valued Path

Next, we consider an application of Theorem 4.4. We focus on the classical Wiener space $C_0([0,1];\mathbb{R})$, which is the subspace of $C([0,1];\mathbb{R})$ comprised only of the function which take the value zero at the infimum over $[0,1]$; it is equip $C_0([0,1];\mathbb{R})$ with the subset topology. Let $\mathbb{P}$ denote the Wiener measure on $C_0([0,1];\mathbb{R})$ and let $B_t$ denote the canonical coordinate process

$$B_t(w) = w_t;$$

that is, $B_t$ is a Brownian motion on $((C_0([0,1];\mathbb{R});\mathcal{B}((C_0([0,1];\mathbb{R})),\mathbb{P})$ and $\mathbb{P}$ satisfies the non-degeneracy condition (4.2). See the corresponding chapter in [14] for more details on Brownian motion.
Example 4.6 (Brownian-Volterra Networks). Let \( g : [0, 1] \to [0, 1] \) be an increasing differentiable function with integrable derivative, which strictly dominates the identity map, ie:

\[
g(x) > x; \quad (\forall x \in (0, 1)),
\]

and let \( \odot : C_0([0, 1]; \mathbb{R}) \times C_0([0, 1]; \mathbb{R}) \to C_0([0, 1]; \mathbb{R}) \) be point-wise addition. On [15, page 44] it is shown that the Volterra composition operator, defined by

\[
V_g : C_0([0, 1]; \mathbb{R}) \to C_0([0, 1]; \mathbb{R})
\]

\[
f \mapsto \int_0^g(t) f(x) \, dx,
\]

is a linear operator and hence \( V_g \) must be a bounded linear operator. In [57] it was shown that the operator \( V_g \) is hypercyclic on \( C_0([0, 1]; \mathbb{R}) \). By the Birkhoff Transitivity Theorem, see [33, Theorem 2.19], every hypercyclic operator on a separable Fréchet space is topologically transitive; whence \( V_g \) is topologically transitive. Therefore,

\[
\mathcal{G}^{BV\text{Net}:g} \triangleq \left\{ k \int_0^{g(t)} \int_0^{g(x_1)} \cdots \int_0^{g(x_{n-1})} 1_{C_0([0,1];\mathbb{R})}(\omega) x_n \, dx_n \cdots dx_1 \right\}_{n \in \mathbb{N}, k \in \mathbb{R}},
\]

\[
= \left\{ k \int_0^{g(t)} \int_0^{g(x_1)} \cdots \int_0^{g(x_{n-1})} B_{x_n}(\omega) \, dx_n \cdots dx_1 \right\}_{n \in \mathbb{N}, k \in \mathbb{R}}
\]

is a randomly activated architecture on \( C_0([0, 1]; \mathbb{R}) \), for any networking function \( \odot \) on \( C_0([0, 1]; \mathbb{R}) \); where the iterated integrals are computed \( \omega \)-wise; where the random activation function \( \sigma \) is then to be the identity \( 1_{C_0([0,1];\mathbb{R})} \) on \( C_0([0, 1]; \mathbb{R}) \).

In particular, \( \mathcal{G}^{BV\text{Net}:g} \) must satisfy the conclusion of Theorem 4.4. That is, for every \( f \in C_0([0, 1]; \mathbb{R}) \), and every \( \epsilon > 0 \) it follows that:

\[
0 < \sup_{k \in \mathbb{R}} \mathbb{P} \left\{ \omega \in C_0([0, 1]; \mathbb{R}) : \bigcup_{n \in \mathbb{N}} \left\{ d_{\infty} \left( f, k \int_0^{g(t)} \int_0^{g(x_1)} \cdots \int_0^{g(x_{n-1})} B_{x_n}(\omega) \, dx_n \cdots dx_1 \right) < \epsilon \right\} \right\},
\]

where \( B \) is a Brownian motion defined on the classical Wiener space.

Next, we will numerically illustrate the effectiveness of randomly activated networks by implementing and comparing the Brownian-Volterra network against a number of leading alternatives.

We generate time-series data from an OU process depending on a fractional stochastic volatility process with Hurst parameter \( H = 0.1 \), this will play the role of our response variable; denote this time-series by \( Y \triangleq (y_t)_{t=0}^N \). Our choice of Hurst parameter is taken from the seminal work of [24]. Our regressor is a time-series of a discretized fractional Brownian motion path with a different Hurst parameter of \( H = 0.3 \); this choice mimics the uncertainty present in choosing a model in practice; denote this time-series by \( X \triangleq (x_t)_{t=0}^N \). We take \( N = 1000 \) and use the last 500 data points as our test set, with the remainder being used for the training and validation sets.

We will regress \( X \) onto \( Y \) using the following regression methods:

(i) Ordinary least squares (OLS),

(ii) Least absolute deviations (LAD),

(iii) Least squares with robust standard errors (LSCV),

(iv) Generalized least squares (GLS),

(v) Stochastic gradient descent (SGD),

(vi) Support vector regression (SVR),

(vii) Random forest (RF),

(viii) Neural network (NN),

(ix) Bayesian neural network (BNN).
(ii) Local regression (loc.Reg),

(iii) P-Splines (pSpln),

(iv) Feed-forward Neural Network (FF-ANN),

(v) Elastic-Net regularization of [77] applied to a large collection of randomly generated fractional Brownian motion paths (including $X$) with various Hurst parameters (Rres-Net),

(vi) A large deep network with layers populated by centered Gaussian random variables (Rand-Net),

(vii) The Brownian-Volterra network.

In this way, we may compare our method against standard linear and non-linear regression methods (OLS, loc.Reg, sSpln, FF-ANN) as well as other natural candidates for a random architecture generation (Rres-Net, Rand-Net). The test set performance of all the methods and each method’s computation time is summarized in the Table 1. The plots of the entire predicted time-series are illustrated in Figure 2. Each of the regression methods rapidly loses reliability on the test set, however, the Brownian-Volterra network most closely tracks the time-series and it best reflects its shape on the test set on the first 100 data points of the test set. Each of the methods then breakdown beyond that point, as expected, due to the drastic change in the shape and trajectory of the OU process’ path.

![Comparison Of Regression Methods](image)

**Figure 2**: Comparison between Various Regression Methods for Predicting Time-series data.
Table 1: Error Statistics: Simulated Data - OU Process with Rough Stochastic Volatility.

| Method      | Mean Res. | 95 L | 95 U | Rel. MSE | Run-Time (Sec.) |
|-------------|-----------|------|------|----------|-----------------|
| OLS         | 1.443e-01 | 1.866e-01 | 2.182e-01 | 5.106e-02 | 9.737e-04 |
| Loc.Reg     | 1.326e-01 | 1.579e-01 | 1.821e-01 | 3.291e-02 | 1.707e-03 |
| pSpln       | 1.354e-01 | 1.606e-01 | 1.822e-01 | 3.343e-02 | 5.342e-02 |
| FF-ANN      | 1.402e-01 | 1.660e-01 | 1.878e-01 | 3.488e-02 | 1.293e+00 |
| Rres-Net    | 1.926e-01 | 2.291e-01 | 2.675e-01 | 6.975e-02 | 1.924e+00 |
| Rand-Net    | 1.599e-01 | 1.990e-01 | 2.253e-01 | 5.215e-02 | 7.384e-01 |
| BV-Net      | -2.541e-02 | 7.508e-03 | 3.480e-02 | 1.264e-02 | 1.171e+01 |

The "Mean Res." column in Table 1 represents the mean residual, the "95 L" and "95 U" columns are record the 95% bias-corrected bootstrapped confidence intervals (see [19] for details), the "Rel. MSE" column records the estimated relative mean-squared error, and the "Run-Time (Sec.)" column record the time it took to run a given method on our machine.

Table 1 underlines the fact that the performance of a randomly activated network is surprisingly satisfactory even when compared to the leading non-parametric methods.

Remark 4.7. For this result to be part of our framework, it is crucial that \( g(t) > t \) on \((0, 1)\). If this is not the case, then the Volterra-composition operator fails to by hypercyclic, as discussed in [36]. Thus if one prefers to use the map \( g(t) = t \) then other techniques outside the scope of this paper must be employed.

The results form this section equipped us with many tools for explicitly building and for proving the existence of universal approximators on a large number of spaces. Nevertheless, certain spaces are outside the scope of Theorem 2.2. Those spaces include certain non-linear spaces, such as the ones studied in [50, 51], as well as non-separable function spaces such as \( L_{loc}^1 \).

We now summarize the contributions made in this article.

5 Conclusion

We have extended the definition of neural network architectures to arbitrary function spaces and gained a new perspective on universal approximation through Theorem 2.2. This result characterized every universal approximator on any function space, which is homeomorphic to a separable Banach space, in terms of a family of hypercyclic operators on that function space. Corollary 2.3 then showed that the study of universal approximator on such a function space could be successfully undertaken by only focusing on microgenerated architectures. Following this perspective, universal approximation theorems were obtained for various fundamental function spaces, including \( P_\mu^1(B(X); Y) \), \( L_\mu^1(B(X); Y) \), \( H_{bc}(X; Y) \), \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \), and various other well-studied function spaces.

As a further consequence, a refinement of the results of [39] was also obtained in Corollary 2.13 which stated that, for a generic family of activation functions, classical neural networks of bounded height but of arbitrary depth are universal approximators on \( \mathbb{R}^d \). Moreover, this
held true even when restrictions were placed on the weight matrices forming each layer of the neural network.

Together, Theorem 3.2 and Corollary 2.23 established the existence of microgenerated universal approximators on any function spaces which is either homeomorphic to a closed convex body Fréchet spaces, a compact and connected Riemannian manifold of dimension $d \geq 2$, or a simply connected Hadamard manifold. Likewise, 2.21 established the existence of universal approximators, entirely described by only three functions, on any separable metric function space which is a retract of a Banach space.

Theorem 3.2 investigated the converse question. We proved that for any two non-empty $T_0$ topological spaces $X$, $Y$ with at-least two points, and any architecture $\mathcal{F}$ on a non-empty subset of $C(X;Y)$, there existed a unique finest topology making $\mathcal{F}$ a universal approximator on $X$.

Using generated networks, we gave meaning to activation functions for our general neural networks. Theorem 2.18 showed that in general activation functions are dense in many function spaces, and in Theorem 4.4 it was shown that upon randomizing the activation function, an architecture is still a universal approximator with positive probability. This technique was used to surmount the difficulty of choosing an activation function for more exotic function spaces. We introduced the Volterra networks on $C_0([0,1];\mathbb{R})$ as an example of randomly activated networks which were seen to be universal approximators with positive probability.

The author would like to thank the ETH Zürich foundation for its support.

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A Technical Lemma(s)

**Lemma A.1.** Let \( b \in \mathbb{R}^d - \{0\} \), \( n \in \mathbb{N} \), and let \( \phi_b \triangleq \phi_{tb} \). Then for every \( \sigma \in L^1_\mu(B(\mathbb{R}^d); \mathbb{R}) \)
\[
\phi^n_b(\sigma) = \phi_{nb}(\sigma).
\]

**Proof.** We proceed by induction. For the case where \( n = 0 \), note that \( \phi^0_b \) was defined to be the identity \( 1_{L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d)} \) on \( L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) \); therefore
\[
\phi^0_{0b}(\sigma)(x) = \sigma(0b + x) = \sigma(x) = 1_{L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d)}(\sigma)(x) = \phi^0(\sigma)(x).
\]
Whence, \( \phi^0_b = \phi_{0b} \). For the case where \( n > 0 \), then by the induction hypothesis, for every \( \sigma \in L^1_\mu(B(\mathbb{R}^d); \mathbb{R}^d) \) and every \( b \in \mathbb{R}^d - \{0\} \) we find that
\[
\phi^n_b(\sigma)(x) = \phi_b \circ \phi^{n-1}_b(\sigma)(x) = \phi_b \circ \phi_{(n-1)b}(\sigma)(x) = \phi_b(\sigma(x + (n-1)b)) = \sigma(x + (n-1)b + b) = \sigma(x + nb) = \phi_{nb}(\sigma)(x).
\]
This concludes the induction hypothesis.

**Lemma A.2.** Let $\mathcal{X}$ be a function space, $E$ be a Fréchet space, $\Phi : \mathcal{X} \to E$ be a homeomorphism, and $T$ be a hypercyclic operator on $E$. For every $n \in \mathbb{N}$ define the map $\phi^n \triangleq \Phi^{-1} \circ T^n \circ \Phi$. Then the there is a dense $G_\delta$-subset $D \subseteq \mathcal{X}$ such that for every $\sigma \in D$ the architecture

\[ \{ g \in \mathcal{X} : \{ \phi^n(\sigma) \}_{n \in \mathbb{N}} \}, \]

is dense in $\mathcal{X}$. Moreover, for every $n, m \in \mathbb{N}$,

\[ \phi^n \circ \phi^m = \phi^{n+m}. \]

**Proof.** By the Birkhoff Transitivity Theorem, see [33, Theorem 2.19], since $T$ is hypercyclic then the set of vectors $x \in E$ such that $\{ T^n(x) \}_{n \in \mathbb{N}}$ is dense in $E$, is itself a dense $G_\delta$-subset of $E$; let us denote this set by $HC(T)$. Moreover, by [33, Corollary 2.56] $HC(T)$ is connected in $E$. Since $\Phi$ is a homeomorphism then $D \triangleq \Phi(HC(T))$ is a dense and connected $G_\delta$-subset of $\mathcal{X}$ and for each $\sigma \in D$

\[ \mathcal{X} = \Phi^{-1} \left( \{ \Phi^n(\sigma) \}_{n \in \mathbb{N}} \right) \triangleq \{ \Phi^{-1} \circ T^n \circ \Phi(\sigma) \}_{n \in \mathbb{N}}, \]

where $\bar{X}$ denotes the closure of a subset $X$ of $\mathcal{X}$ in $\mathcal{X}$. This establishes the first result. For the claim about the composition of $\phi^n$ with $\phi^m$, note that:

\[ \phi^n \circ \phi^m = (\Phi^{-1} \circ T^n \circ \Phi) \circ (\Phi^{-1} \circ T^m \circ \Phi) = \Phi^{-1} \circ (T^n \circ T^m) \circ \Phi = \Phi^{-1} \circ T^{n+m} \circ \Phi = \phi^{n+m}. \]

\[ \square \]

## B A Brief Overview of the Arens-Eells Space

In Corollary 2.17, we had identified the set of finite signed Borel measures on $[-M, M]^d$ with the continuous dual space of $C([-M, M]^d; \mathbb{R})$ endowed with the weak*-topology. Similarly, if $\mathcal{X}$ is any locally compact, then the space $\mathcal{M}(\mathcal{X})$ of finite signed Borel measures on $\mathcal{X}$ with finite total variation, metrized by the total-variation metric, can be identified with the continuous dual space of $C_0(\mathcal{X}; \mathbb{R})$ endowed with the weak*-topology. This identification is provided by the Riesz-Markov-Kakutani Representation Theorem which guarantees that every continuous linear functional $F \in C_0(\mathcal{X}; \mathbb{R})'$ can be identified with a measure $\mu \in \mathcal{M}(\mathcal{X})$ by integration

\[ F[g] = \int_{f \in \mathcal{X}} g(f) d\mu_F(f) \quad (\forall g \in C(\mathcal{X}; \mathbb{R})), \quad (B.1) \]

and visa-versa. In [71] and in [70], the authors interpreted elements of $C_0(\mathcal{X}; \mathbb{R})'$ as *generalized functions* (and later distributions) since these functionals can encode properties which cannot be captured by regular functions, such as positive mass at a point, as pioneered in [22].

This construction does not apply to arbitrary metric function spaces $\mathcal{X}$, since in general $\mathcal{X}$ need not be locally compact. Therefore, if $\mathcal{X}$ is not locally compact then the Riesz-Markov-Kakutani Representation Theorem does not apply, whence the spaces $C_0(\mathcal{X}; \mathbb{R})'$ and $\mathcal{M}(\mathcal{X})$ can no longer be identified. Even when these two spaces do coincide, $C_0(\mathcal{X}; \mathbb{R})'$ is inconvenient to
work with since it fails to properly encode many of the metric properties of \( \mathcal{X} \). Intuitively, this is because non-Lipschitz continuous functions disregard the metric geometry of \( \mathcal{X} \). For instance, it is not clear if \( C_0(\mathcal{X}; \mathbb{R})' \) contains an isometric copy of \( \mathcal{X} \). Instead, we will consider a sort of metric-theoretic completion of \( \mathcal{M}(\mathcal{X}) \) discovered in [3].

The Arens-Eells space over \( \mathcal{X} \) is a universal Banach space which offers a solution to these two obstacles; it is defined as follows. First, replace \( C_0(\mathcal{X}; \mathbb{R}) \) by the Banach space \( \text{Lip}_0(\mathcal{X}; \mathbb{R}) \) whose elements are Lipschitz functions from \( \mathcal{X} \) to \( \mathbb{R} \) which send \( 0_\mathcal{X} \in \mathcal{X} \) to \( 0 \in \mathbb{R} \) and whose norm assigns to any \( g \in \text{Lip}_0(\mathcal{X}; \mathbb{R}) \) to its minimal Lipschitz constant \( \text{Lip}(g) \), defined by:

\[
\text{Lip}(g) \triangleq \sup_{f, h \in \mathcal{X}: f \neq h} \frac{|g(f) - g(h)|}{d_\mathcal{X}(f, h)}.
\]

As shown in [27], an isometric embedding of \( \mathcal{X} \) into \( \text{Lip}_0(\mathcal{X}; \mathbb{R})' \) is obtained by mapping every function \( f \in \mathcal{X} \) to the Dirac delta "function" \( \delta_f \), which acts on elements in \( \text{Lip}_0(\mathcal{X}; \mathbb{R}) \) via the point-evaluation map

\[
g \mapsto \int_{h \in \mathcal{X}} g(h) d\delta_f(h) = g(f).
\]

The map \( f \mapsto \delta_f \) will be denoted by \( \delta \) (or by \( \delta^\mathcal{X} \) when necessary). Note, \( \delta \) sends \( 0_\mathcal{X} \) to the zero vector in \( \text{Lip}(\mathcal{X}; \mathbb{R})' \). The Arens-Eells space is the Banach space which is defined as the closure of the span \( \{ \delta_f : f \in \mathcal{X} \} \) in \( \text{Lip}_0(\mathcal{X}; \mathbb{R})' \). We will denote its norm by \( \| \cdot \|_{KR} \) for reasons which will be made clear shortly.

**Definition B.1** (Arens-Eells Space). Let \( \mathcal{X} \) be a metric function space. The Banach subspace span \( \{ \delta_f : f \in \mathcal{X} \} \) of \( \text{Lip}_0(\mathcal{X}; \mathbb{R})' \) is called the Arens-Eells space over \( \mathcal{X} \) and is denoted by \( \mathcal{AE}(\mathcal{X}) \).

**Remark B.2.** In non-linear Banach space theory, Arens-Eells spaces are called Lipschitz-Free spaces. This is because, in the Banach case, any Arens-Eells space arises from the application of the left-adjoint of the forgetful functor from \( \text{Ban} \) to \( \text{Met}^* \), to a pointed metric space in \( \text{Met}^* \).

We summarize some key properties of the Arens-Eells space in the following compilation of relevant results from the theories of non-linear Banach spaces and metric spaces. The results are adapted to the setting of metric function spaces.

**Compilation B.3.** Let \( \mathcal{X}, \mathcal{Y} \) be a separable metric function spaces, and let \( E \) be a separable Banach space. Then \( \mathcal{AE}(\mathcal{X}) \) satisfies the following properties:

(i) [73, 74, Theorem 3.3; Theorem 3.2]: \( \mathcal{AE}(\mathcal{X}) \) is a separable Banach space and is isometrically isomorphic to a pre-dual of \( \text{Lip}_0(\mathcal{X}; \mathbb{R}) \). Moreover, if \( \mathcal{X} \) has finite diameter then this pre-dual is unique up to linear isometric isomorphism,

(ii) [27, Page 2]: The norm on \( \mathcal{AE}(\mathcal{X}) \) is equivalent to the following variant of the Kantorovich-Rubinstein (a.k.a. Wasserstein) metric, defined on any \( F \in \mathcal{AE}(\mathcal{X}) \) by

\[
\| F \|_{KR} \triangleq \inf \left\{ \sum_{i=1}^{k} |a_i| d_\mathcal{X}(y_i, z_i) : F = \sum_{i=1}^{k} |a_i| (\delta_{y_i} - \delta_{z_i}) \right\},
\]
(iii) [27, Page 91]: For any Lipschitz map \( f: \mathcal{X} \to \mathcal{Y} \) taking \( 0_{\mathcal{X}} \) to \( 0_{\mathcal{Y}} \), there exists a continuous linear map \( F: \mathcal{A}(\mathcal{X}) \to \mathcal{A}(\mathcal{Y}) \) satisfying
\[
F \circ \delta^\mathcal{X} = \delta^\mathcal{Y} \circ f,
\]

In particular, if \( f \) is a surjective isometry, then \( F \) is a linear isometric isomorphism.

(iv) [73, Theorem 3.6]: For any Lipschitz map \( f: \mathcal{X} \to E \) there exists a unique bounded linear operator \( F: \mathcal{A}(\mathcal{X}) \to E \) which lifts \( f \), that is
\[
f = F \circ \delta.
\]

Moreover, \( \text{Lip}(f) = \|F\|_{\text{op}} \) where \( \| \cdot \|_{\text{op}} \) is the operator norm on the Banach space of bounded linear operators between \( \mathcal{A}(\mathcal{X}) \) and \( E \). Furthermore, if \( f \) is an isometric embedding then \( F \) is a linear isometry.

The next example, provides a concrete interpretation of the Arens-Eells space of a non-empty compact subset of separable metric function space.

**Example B.4.** Let \( K \) be a compact subset of a separable metric function space \( \mathcal{X} \), such that \( 0^{\mathcal{X}} \in K \). [73, Theorem 3.19] shows that the map
\[
\mathcal{M}^0(K) \to \mathcal{A}(K)
\]
\[
\nu \mapsto EV_\nu(g) \triangleq \int_{f \in \mathcal{X}} g(f) d\mu, \tag{B.3}
\]
is a bijection; where \( g \in \text{Lip}_0(\mathcal{X}; \mathbb{R}) \) and where \( \mathcal{M}^0(K) \) is the set of finite signed Borel measures \( \mu \) on \( K \) satisfying \( \mu(K) = 0 \). Therefore, the Arens-Eells space \( \mathcal{A}(K) \) can be identified with the set of signed Borel measures in \( \mathcal{M}^0(K) \) with the metric topology induced by
\[
d_{KR}(\mu, \nu) \triangleq \| EV_\mu - EV_\nu \|_{KR}.
\]

For more details on the Arens-Eells space, we refer the author to [73, 13].