Spherical tilings by congruent quadrangles over pseudo-double wheels (III) — the essential uniqueness in case of convex tiles

Yohji Akama\textsuperscript{a, }∗, Yudai Sakano\textsuperscript{a}

\textsuperscript{a}Mathematical Institute
Graduate School of Science, Tohoku University
Sendai, Miyagi 980-0845 JAPAN

Abstract

In [B. Grünbaum, G. C. Shephard, Spherical tilings with transitivity properties, in: The geometric vein, Springer, New York, 1981, pp. 65–98], they proved “for every spherical normal tiling by congruent tiles, if it is isohedral, then the graph is a Platonic solid, an Archimedean dual, an $n$-gonal bipyramid ($n \geq 3$), or an $n$-gonal trapezohedron (i.e., the pseudo-double wheel of $2n$ faces).” In the classification of spherical monohedral tilings, one naturally asks an “inverse problem” of their result: \textit{For a spherical monohedral tiling of the above mentioned topologies, when is the tiling isohedral?} We prove that for any spherical monohedral quadrangular tiling being topologically a trapezohedron, if the number of faces is 6, or 8, if the tile is a kite, a dart or a rhombi, or if the tile is convex, then the tiling is isohedral.

\textbf{Keywords:} spherical tiling, quadrangle, pseudo-double wheel, isohedral.

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1. Introduction

In [1], we proved that twelve copies of some spherical \textit{concave} quadrangle organize two spherical tilings such that the two tilings have the same plain graph (the \textit{pseudo-double wheel} of twelve faces [2]), but one spherical tiling (See the right bottom of Figure 1) is isohedral (i.e. the symmetry group acts transitively on the tiles) while the other not (See Figure 2). The latter spherical non-isohedral tiling by congruent \textit{concave} quadrangles over a pseudo-double wheel is a counterexample of the inverse of Grünbaum-Shephard’s result [3] “for every spherical \textit{normal} tiling by congruent tiles, if it is isohedral, then the graph is a

\textsuperscript{∗}Corresponding author

Email addresses: akama@math.tohoku.ac.jp (Yohji Akama), my.sailingday.0827spc@gmail.com (Yudai Sakano)

URL: http://www.math.tohoku.ac.jp/akama/stcq/ (Yohji Akama)
Platonic solid, an Archimedean dual \([4]\), an \(n\)-gonal bipyramid \((n \geq 3)\), or an \(n\)-gonal trapezohedron (i.e., the pseudo-double wheel of \(2n\) faces).” So we ask an “inverse problem” of their result:

**Question 1.** For a spherical monohedral tiling with the graph being a Platonic solid, an Archimedean dual, an \(n\)-gonal bipyramid or an \(n\)-gonal trapezohedron, when is it isohedral?

According to [5], when the tile is a kite, a dart, or a rhombi, every spherical tiling over a pseudo-double wheel is isohedral. Moreover, by checking the number of tiles and the vertex types in the complete table of spherical monohedral triangular tilings [6], we can prove that

**Theorem 1.** If a spherical tiling by congruent triangles is topologically a Platonic solid, an Archimedean dual, an \(n\)-gonal bipyramid, then the tiling is isohedral.

To state our partial answer of this inverse problem, recall that we cannot divide the tile of a spherical monohedral quadrangular tiling into two congruent triangles, if and only if the tile is either type 2 or type 4 of Figure 3 (Figure 3), since the tile of a spherical monohedral quadrangular tiling has necessarily equilateral adjacent edges [7, Proposition 1].

**Theorem 2.** Given a spherical tiling consisting \(T\) of \(F\) congruent quadrangles of type 2 or 4, such that

- \(F = 6\) or \(8\); or,
- the tile is convex and the graph of the tiling is pdw\(_F\) with \(F \geq 10\).

Then \(T\) has chart \(P\_F\) as in Figure 4 or the mirror, and \(T\) is isohedral.

In [8], we proved that such a tiling \(T\) of Theorem 2 indeed exists. Actually, we proved that for every spherical tilings by congruent possibly concave quadrangles over pseudo-double wheels, we explicitly represented every inner angles and every edge-length by the inner angle \(\gamma\) (and the inner angle \(\alpha\), resp.) of the tile.

Our result is a first step toward the classification of the spherical tilings by congruent quadrangles, because a quadrangulation of the sphere is mechanically obtained exactly from a pseudo-double wheel, by means of finite number of applications of two local expansions of a graph [2]. See Proposition 3.

This paper is organized as follows: In the next section, we present a setting to classify the spherical monohedral quadrangular tilings in terms of planar graphs, angle- and length-assignments to the graph, and matching theory. In Section 4, we derive Theorem 2 from Theorem 3. In Section 5 we prove Theorem 2. In the final section, we conjecture that Question 1 holds for convex tiles, based on Theorem 4. Then we propose to classify a reasonable class of spherical monohedral quadrangular tilings. Finally we show that the use of forbidden patterns is a natural approach to classify spherical tilings by congruent type-2 convex quadrangles, based on computer experiment [9].
Figure 1: The chart $\mathcal{P}_F$ is defined by the left figure, where the graph is a pseudo-double wheel of $F$ faces (Definition 2). The whirl at each vertex indicates the cyclic order for the edges incident to the vertex. The thick edges have length $b$, the dotted edges have length $c$, and the other edges have length $a$. When the tile is type 2, we have $a = c \neq b$; when the tile is type 4, the lengths $a, b, c$ are mutually distinct. The right column are examples of spherical tilings by 6, 8, 10, 12 congruent type-2 quadrangles, from top to bottom. They are isohedral, because of the axis through the pole and the center, and each axis through the midpoint of each non-meridian edge and the center. The twelve tiles of the last tiling organizes a non-isohedral tiling given in Figure 2.

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2. A setting to classify the spherical monohedral quadrangular tilings

We say a tiling by polygons is edge-to-edge, if the vertices and edges of tiles match. Unless otherwise stated, a tiling always implies an edge-to-edge spherical tiling of the sphere, and is identified modulo the special orthogonal group $SO(3)$. We say two tilings are mirror image to each other, if they are different but identified modulo the orthogonal group $O(3)$.

Definition 1. 1. A map is $M = (V, E, \{A_v\}_{v \in V})$ such that

- $G = (V, E)$ is a nonoriented graph, where $V$ is a finite set of vertices and $E$ is a finite set of edges. An edge is a nonoriented pair of distinct vertices.
- $A_v$ is the set of angles around $v$, that is, a set $\{(v_1, v, v_2), (v_2, v, v_3), \ldots, (v_n-1, v, v_n), (v_n, v, v_1)\}$ such that $v_1, \ldots, v_n$ is the list of vertices adjacent to $v$. We write an angle $(u, v, w)$ by $\angle uvw$. 

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Figure 2: The chart $\mathcal{A}$ of the spherical non-isohedral tiling by twelve congruent concave quadrangles. Thick (resp. thin) edges of the chart correspond to edges of length $b$ (resp. $a$) of the tiling (see Theorem 3). The left upper figure is the view from a 2-fold rotation axis through the midpoint between the vertices $v_0$ and $v_1$. The left middle figure is from a 2-fold rotation axis through the midpoint between the vertices $v_3$ and $v_4$. The left bottom is from the other 2-fold rotation axis through the poles. The twelve tiles of the tiling organizes an isohedral tiling give in Figure 1 (right bottom).

Definition 2 (Pseudo-double wheel \cite{2}). For an even number $F \geq 6$, a pseudo-double wheel pdw$_F$ with $F$ faces is a map such that

- the graph is obtained from a cycle $(v_0, v_1, v_2, \ldots, v_{F-1})$, by adjoining a new vertex $N$ to each $v_{2i}$ ($0 \leq i < F/2$) and then by adjoining a new vertex $S$ to each $v_{2i+1}$ ($0 \leq i < F/2$). We identify the suffix $i$ of the vertex $v_i$ modulo $F$.

- The inner angles at each vertex $v$ is defined naturally by the cyclic order at $v$. The cyclic order at the vertex $N$ is defined as follows: the edge $Nv_{2i+2}$ is next to the edge $Nv_{2i}$. The cyclic order at the vertex $v_{2i}$ ($0 \leq i < F/2$) is: the edge $v_{2i}N$ is next to the edge $v_{2i}v_{2i+1}$, which is next to the edge $v_{2i}v_{2i-1}$. The cyclic order at the vertex $S$ is: the edge $Sv_{2i+1}$ is next to the edge $Sv_{2i-1}$. The cyclic order at the vertex $v_{2i+1}$ ($0 \leq i < F/2$) is: the edge $v_{2i+1}S$ is next to the edge $v_{2i+1}v_{2i}$, which is next to the edge $v_{2i+1}v_{2i+2}$.

We call each edge $Nv_{2i}$ northern, each edge $Sv_{2i+1}$ southern, and the other edges non-meridian. The number of edges is $2F$.

The form of a pseudo-double wheel is a graph consisting of the vertices, edges and faces of Figure 1.

The graph of a spherical tiling by congruent quadrangles is a simple quadrangulation of the sphere such that the minimum degree three. Here a simple
Figure 3: Two quadrangles of type 2 and a quadrangle of type 4. The second quadrangle of type 2 is obtained from the first quadrangle of type 2 by swapping \((\alpha, \beta)\) and \((\delta, \gamma)\) with the length assignment unchanged. The chirality of the first quadrangle of type 2 is opposite to that of the second quadrangle of type 2. The variable \(a, b, c\) are for edge-lengths and they have different values. The variables \(\alpha, \beta, \gamma, \delta\) are for angles.

A quadrangulation of the sphere is a finite simple graph embedded on the sphere such that every face is bounded by a walk of four edges [2]. Let \(Q_2\) be the class of simple quadrangulations of the sphere such that the minimum degree three. Let \(Q_3\) be the class of 3-connected simple quadrangulations of the sphere. By the theorem of Steinitz, \(Q_3\) is the class of quadrangle-faced polytopal graphs. A polytopal graph is the graph of some polytope, where a polytope is, by definition, a bounded region obtained by a finite number of half spaces. According to Table 2 (simple quadrangulations with minimum degree three) and Table 3 (3-connected quadrangulations) of [2], there is a simple spherical quadrangulation \(G\) of twelve quadrangular faces with the minimum degree three but \(G\) is not polytopal. By an stcq graph, we mean a graph of some spherical tiling by congruent quadrangles. We do not know whether every polytopal graph is an stcq graph or not. Anyway, a quadrangulation of \(Q_2\) is a candidate of an stcq graph.

Proposition 3 ([2, Theorem 2, Theorem 3]). Let \(i\) be 2 or 3. For every simple quadrangulation \(G \in Q_i\), there is a sequence \(G_0, G_1, \ldots, G_k = G\) of quadrangulations in \(Q_i\) such that \(G_0\) is a pseudo-double wheel and, for each \(i, G_{i+1}\) can be obtained from \(G_i\) by applying two local expansions. Actually, all such are generated by a program \texttt{plantri} [10]. See Table 1 as for \(Q_2\). Specifically, the quadrangulations of \(Q_i\) with 6 or 8 faces are exactly \(\text{pdw}_6\) or \(\text{pdw}_8\).

Definition 4 (Chart). 1. A chart of a tiling consists of following data:
   - A map \(M = ((V, E), \{A_v\}_{v \in V})\).
   - A length-assignment \(L : E \to \mathbb{R}_{>0}\).
   - An angle-assignment \(K\), which is a function from \(\bigcup_{v \in V} A_v\) to the set of affine combination of the variables \(\alpha, \beta, \gamma, \delta\) over \(\mathbb{R}\) subject to
     \[
     \sum_{a \in A_v} K(a) = 2 [\pi\text{rad}] \text{ for each } v \in V. \quad (1)
     \]
2. We say a chart is of type \(t\), if each face is of type \(t\).
3. The mirror image of a chart \( \mathcal{M} = ((V, E), \{A_v\}_{v \in V}, L, K) \) is a chart \( \mathcal{M}^R = ((V, E), \{A^R_v\}_{v \in V}, L, K^R) \) such that \( A^R_v := \{ \angle uvw ; \angle wvu \in A_v \} \) and \( K^R(\angle uvw) = K(\angle wvu) \).

When we embed naturally the map \( \text{pdw}_F \) into the sphere, the angles at each vertex have positive values with respect to the axial vector from the center to the vertex.

**Definition 5 (Vertex types).** The types of angles are, by definition, variables \( \alpha, \beta, \gamma, \delta \). If the angles around a vertex \( v \) are exactly \( n_\alpha \) angles of type \( \alpha \), \( n_\beta \) angles of type \( \beta \), \( n_\gamma \) angles of type \( \gamma \) and \( n_\delta \) angles of \( \delta \), then we say the vertex type of \( v \) is \( n_\alpha \alpha + n_\beta \beta + n_\gamma \gamma + n_\delta \delta \).

**Lemma 6.** In a tiling by \( F \) congruent quadrangles, every tile has area \( 4\pi/F \), which is the total of the inner angles subtracted by \( 2\pi \). In other words

\[
\alpha + \beta + \gamma + \delta - 2 = \frac{4}{F}.
\]

Hence there is no vertex of type \( \alpha + \beta + \gamma + \delta \).

**Remark 1.** Classification of spherical tilings by \( F \geq 10 \) congruent quadrangles of type 2 or 4 is partially mechanizable. For each even number \( F \geq 10 \), generate all 2-connected simple quadrangulations \( G \) of the sphere such that the minimum degree is three and the number of faces is \( F \) (cf. Proposition 3). Each tile of a spherical monohedral quadrangular tiling of type 2 or type 4 matches an adjacent tile at the edge of length \( b \). All such tile-matchings are generated mechanically, as the perfect matchings \([11]\) of the dual graph of the tiling’s graph \( G \). If a graph with even number of faces are embeddable into an orientable, connected compact 2-fold, and all the faces are quadrangles, then the dual graph has a perfect matching \([12]\). The extreme points of the edge of length \( b \) have angle \( \alpha \) and \( \delta \). By respecting this constraint, we can mechanically generate all possible angle-assignments. Every angle-assignment generates a system of equations, that is, equation \([2]\) and equations \([1]\). If for every tile-matching, no such systems of equations has a positive solution \( \alpha, \beta, \gamma, \delta < 1 \), then the spherical quadrangulation is not realizable by a spherical tiling by \( F \) congruent convex quadrangles of type 2 or 4.

3. Two forbidden (type-2/type-4) length-assignments and proof of Theorem 2

Theorem 2 is derived from the following:

**Theorem 3.** Given a spherical tiling \( \mathcal{T} \) by \( F \) congruent quadrangles of type 2 or 4.

1. If the tile is convex or \( F = 6, 8 \), then the left pattern of Figure 4 and the mirror image are impossible for the length-assignment of the tiling \( \mathcal{T} \).
Figure 4: For each $t = 2, 4$, the two figures and their mirror images are impossible as the length-assignment of a spherical tiling by congruent quadrangle of type $t$, if the tiles are convex, or if the number of faces is 8. Here thick edges are of length $b$ while the others are of length $a$ or $c$. A triangle indicates that nonnegative number of edges may occur at this position in the cyclic order around the vertex (but they need not); When the number $F$ of tiles is 8, for both patterns, the pair of vertices designated by $i + 6, i + 7$ can be identical to the pair of vertices designated by $i - 2, i - 1$. But in this case, all triangle symbols are empty. See Theorem [3].

2. If the tile is convex or $F = 6, 8$, then the right pattern in Figure 4 and the mirror images are forbidden for the length-assignment of the tiling $T$.

Remark 2. By an automorphism of a map $M$, we mean any automorphism $h$ of the graph that preserves the cyclic orders of the vertices. Here we let $h$ send any angle $\angle uvw$ to $\angle h(u)h(v)h(w)$. For a chart $A = (M, L, K)$ and an automorphism $h$ of the map $M$, let $h(A)$ be a chart $(M, L \circ h^{-1}, K \circ h^{-1})$.

The correspondence of the vertices of Figure 4

$$v_{i+k} \mapsto v_{i+5-k} \quad (-2 \leq k \leq 7)$$

preserves the cyclic orders of edges at each vertex, if the designated triangles in the figure are regarded as edges. Moreover the correspondence preserves the length-assignment of Figure 4 (left).

As mentioned in the caption of Figure 4, the two patterns in Figure 4 generalize the two length-assignments (Figure 5) for pdw such that there is a meridian edge of length $b$ and every face has only one edge of length $b$. In the right figure, a meridian edge of length $b$ is followed by two “consecutive” non-meridian edges of length $b$. The left figure contains two “consecutive” meridian edges of length $b$. The two figures are not realizable by a spherical tiling by 8 congruent quadrangles of type 2 or 4, according to [13].
Figure 5: The length-assignments of pdw₆ such that there is a meridian edge of length b and every face has only one edge of length b. Here 8 and 9 are the north pole and the south pole.

By applying Theorem 3 repeatedly to a slightly rotated map around the axis through the north pole and the south pole, the only allowed length-assignment over pdw₆ satisfies the assumption of Theorem 4 ([1, Theorem 2]). That is, diagrammatically speaking with Figure 2 the length b edges appear “alternatingly” in the meridian edges and the non-meridian edges, in the northern hemisphere, and in the southern hemisphere. Here is an excerpt of [1, Theorem 2]:

Theorem 4. Assume a chart A satisfies the following assumptions:

(I) the map of the chart is pdw₆ for some F ≥ 10, and

(II) there is b > 0 such that all the edges Nv₆i, v₆i₊₁S and v₆i₊₃v₆i₊₄ have length b for each nonnegative integer i < F/6 while the other edges do lengths ≠ b.

Then there exists a spherical non-isohedral tiling T by congruent quadrangles, uniquely up to special orthogonal transformation. Moreover the tile is a concave quadrangle of type 2. The tiling T realizes A, where the length-assignment and the angle-assignment are Figure 2 (F = 12 in particular).

By Theorem 4 ([1, Theorem 2]), the only allowed chart in question is exactly that of a spherical tiling by F = 12 congruent convex quadrangles of type 2. However, the tile is convex, because of the assumption of Theorem 2. Hence every spherical tiling by F ≥ 8 congruent convex quadrangles of type 2 or 4 over pdw₆, every edge of length b is not meridian. We can easily show the same holds for F = 6, as in [13]. Because we can prove the following in the rest of this section, Theorem 2 follows from Theorem 3.

Theorem 5. Given a spherical tiling by F ≥ 6 congruent possibly concave quadrangles of type t (t ∈ {2, 4}) such that the map is pdw₆ and all the edges of length b are non-meridian. Then, the chart modulo mirror image is necessarily P₆.

Various spherical tilings by congruent concave quadrangles over pseudo-double wheels are depicted in [8].
From now on, the numbers designated at vertices in the figures and the symbols $v_i$’s have nothing to do with the vertices $v_i$’s introduced in the definition of pseudo-double wheels (Definition 2).

Convention 1. Each displayed vertex is distinct from the others; Edges that are completely drawn must occur in the cyclic order given in the picture; Half-edges indicate that an edge must occur at this position in the cyclic order around the vertex; A triangle indicates that one or more edges may occur at this position in the cyclic order around the vertex (but they need not); If neither a half-edge nor a triangle is present in the angle between two edges in the picture, then these two edges must follow each other directly in the cyclic order of edges around that vertex.

For any property $P(\alpha, \beta, \gamma, \delta)$ for the inner angles $\alpha, \beta, \gamma, \delta$ of the tile, the conjugate property $P^*(\alpha, \beta, \gamma, \delta)$ is, by definition, a property $P(\delta, \gamma, \beta, \alpha)$.

![Figure 6](image-url)

Figure 6: The left is the chart $Q_F$ of type 2. The chart $Q_F$ is obtained from the chart $P_F$ by swapping the inner angles $\alpha \leftrightarrow \delta, \beta \leftrightarrow \gamma$ in $F/2$ number of faces. The right is the chart of type 2 considered in Case 1 of the proof of Theorem 5.

The rest of this section is devoted to the proof of Theorem 5.

Any non-meridian edge of length $b$ is not adjacent to another non-meridian edge of length $b$, because for any $t = 2, 4$, no pair of edge of length $b$ are adjacent in a tile of type $t$.

If the tile is of type 4, then all the edges of length $c$ are non-meridian. Otherwise, a meridian edge of length $c$ is adjacent to another non-meridian edge of length $b$ or to a meridian edge of length $b$. In the former case, the two edges of length $b$ are adjacent, which is impossible. The latter case is impossible as we assumed all the edges of length $b$ are non-meridian. Thus we have $P_F$.

Hereafter we assume $t = 2$. We have only to prove the case the dotted edges are of length $a$ in Figure 1, because the other case is just a mirror image and is proved similarly.

Thus the type of the pole $N$ is $m\beta + n\gamma$ and the type of the other pole $S$ is $i\beta + j\gamma$ for some $m, n, i, j$. As the valences of the both poles are equal, we have $m + n = i + j = F/2$. Then we have $m - i = -(n - j)$. 
If the two vertex types are different, we have \(m - i \neq 0\) or \(n - j \neq 0\). Therefore \(m\beta + n\gamma = i\beta + j\gamma\) implies \(\beta = \gamma = 4/F\). Then the tile becomes a isosceles trapezoid, which implies \(\alpha = \delta\). Thus by swapping \((\alpha, \beta)\) and \((\delta, \gamma)\) of suitable tiles, the resulting chart is \(\mathcal{P}_F\), and is still equivalent to the original one.

We consider the other case where the two vertex types of the poles \(N\) and \(S\) are equal, i.e., \(m = i\) and \(n = j\).

If \(m = 0\), then we have (1). If \(n = 0\), then by swapping inner angles \((\alpha, \beta) \leftrightarrow (\delta, \gamma)\) of all the tiles of type 2, we have \(\mathcal{P}_F\).

Consider the case where there is a vertex of type \(\alpha + \beta + \delta\), or a vertex of type \(\alpha + \gamma + \delta\). Then Lemma 6 implies \(\beta = 4/F\) or \(\gamma = 4/F\). In the former case, we have \(m\beta + n\gamma = 2\). As \(m + n = F/2\), we have \(4n/F - n\gamma = 0\) and so \(\gamma = \beta = 4/F\). In the latter case \(\gamma = 4/F\), we have \(\beta = \gamma = 4/F\) similarly.

Hence the quadrangle of type 2 is an isosceles trapezoid, and thus \(\alpha = \delta\). So in all tiles contributing the angle \(\gamma\) to the vertex type of the pole \(N\) or \(S\), if we swap \((\alpha, \beta) \leftrightarrow (\delta, \beta)\) in a chart, then the resulting tiling is congruent to the original tiling. Therefore, the tiling has chart \(\mathcal{P}_F\) of type 2 or the mirror image.

Next, consider the other case, that is, the case where \(mn \neq 0\) and any vertex type is neither \(\alpha + \beta + \delta\) nor \(\alpha + \gamma + \delta\). We have three cases.

Case 1. Some angle \(\beta\) is adjacent to another angle \(\beta\) at the pole \(N\); Case 1′. Some angle \(\gamma\) is adjacent to another angle \(\gamma\) at the pole \(N\); or Case 2. Angles \(\beta\) and angles \(\gamma\) appear alternatingly around the pole \(N\).

In all cases, we fix the location of edges of length \(b\):

\[
\text{an edge } v_{2i}v_{2i+1} \text{ has length } b \quad (0 \leq i \leq (F - 2)/2).
\]  

Once the conclusion of Theorem 5 is established in Case 1, the same argument with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped establishes the conclusion also in Case 1′.

Without loss of generality, we can assume, as in Figure 6 (left), \(\angle v_2Nv_{2i+2}\) is \(\gamma\) \((i = 0); \beta \quad (i = 1)\), both in Case 1 and Case 2. By this and the property 3 with \(i = 0\), we have \(\angle Nv_2v_{2i+1} = \beta \quad (i = 0); \alpha \quad (i = 1)\). By 3,

\[
\angle v_{2j-1}v_{2j}v_{2j+1} = \alpha, \text{ or } \delta.
\]  

Because no vertex has type \(\alpha + \beta + \delta\), \(\angle v_3v_2v_1 = \alpha\). Hence

\[
\angle v_3Sv_1 = \gamma.
\]  

So, by \(\angle v_3v_2v_1 = \alpha\) and by \(\angle Nv_2v_{2i+1} = \beta \quad (i = 0); \alpha \quad (i = 1)\), we have

\[
2\alpha + \beta = 2.
\]  

On the other hand, by 3 with \(i = 1\), we have

\[
\angle Nv_4v_3 = \gamma.
\]  

Now consider Case 1 (See Figure 6 (right)). Without loss of generality, we can assume \(\angle v_4Nv_6 = \beta\). Then the property 3 with \(i = 2\) implies \(\angle Nv_4v_5 = \alpha\).
Because of (4) and the absence of the vertex type $\alpha+\gamma+\delta$, we have $\angle v_3v_4v_5 = \alpha$. By this, (7) and $\angle Nv_4v_5 = \alpha$, we have $2\alpha + \gamma = 2$. By this and (9), the angle $\beta$ is equal to $\gamma$. Thus, as above, the chart is $P_F$ or the mirror image.

In Case 2, the number $F$ of tiles is a multiple of four and is greater than or equal to 8. Moreover angle $\beta$ occurs in the type of vertex $N$ same time as angle $\gamma$. So, without loss of generality, we assume for each positive integer $k \leq F/4$, it holds that $\angle v_{4k+3}v_{4k+4}v_{4k+5} = \beta$ and

$$\angle v_{4k}Nv_{4k+2} = \gamma.$$ (8)

As with (5), we can derive (7) and $\angle v_{4k+1}Nv_{4k+3} = \gamma$. Since $\angle v_{4k+3}v_{4k+4}v_{4k+5} = \beta$ by (3) and (8), we have $\angle Nv_{4k}v_{4k+1} = \beta$. This chart is $Q_F$ of type 2, as in Figure 6 (left).

The chart $Q_F$ is not realizable by some spherical tiling by congruent concave quadrangles of type 2. Assume $Q_F$ is. Then the edge length $a$ is greater than $\pi/2$. Otherwise, the vertices $v_{2i}$’s are on the northern hemisphere and the vertices $v_{2i-1}$ on the southern, from which $\alpha, \delta < \pi$ follows. Hence, as $\beta, \gamma < \pi$ by $F \geq 8$, the tile is convex on the contrary to the premise. So $a > \pi/2$. Thus, the tile $Nv_2v_3v_4$ implies $\delta > 1$ but the tile $Nv_0v_1v_2$ does $\delta < 1$, which is a contradiction.

No spherical tiling by congruent convex quadrangles of type 2 realizes the chart $Q_F$ of Figure 6 (left). Assume otherwise. Apply the following lemma:

**Lemma 7 ([8, Lemma 2]).** If there is a quadrangle $ABCD$ of type 2 such that the vertex $B$ is located on a pole while the length between the opposite vertex $D$ and the other pole is equal to the lengths of edges $BC$ and $CD$, then $\angle BCD + \angle DBC$ is the angle of line. If the quadrangle $ABCD$ is convex, then $\angle DBA < \angle CBA$.

**Proof.** Consider the vertex $\overline{C}$ antipodal to $C$. Then, $\overline{CB\overline{C}D}$ is a lune, so $\angle BCD = \angle \overline{B\overline{C}D}$. Because $DB = B\overline{C} = D\overline{C}$, we have a regular triangle, so $\angle \overline{CDB} = \angle B\overline{C}D$. Hence $\angle BCD + \angle DBC = \angle \overline{CDB} + \angle DBC = \angle \overline{CDC}$ is the angle of line. \hfill $\square$

Then by Lemma 7, $\angle v_{F-1}Nv_0 = \pi - \gamma\pi$ because the tile is of type 2 while $\angle v_{F-1}Nv_0 < \beta\pi$ because the tile is convex. Therefore, $\beta\pi + \gamma\pi > \angle v_{F-1}Nv_0 + \gamma\pi = \pi$. The vertex $N$ has a vertex type $\beta+F/4(\beta+\gamma)$ greater than 2. This is a contradiction.

Hence, whether the tile is convex or concave, no spherical tiling by congruent quadrangles of type 2 realizes the chart $Q_F$. Thus the chart should be $P_F$. This completes the proof of Theorem 5.

Hence the derivation of Theorem 2 from Theorem 5 is complete.

4. Proof of Theorem 3

The following Subsection 4.1 and Subsection 4.2 respectively prove Theorem 3 (1) and (2). For each subsection, we will prove a lemma that generates
all candidates of angle-assignments on the length-assignment, and then will reject them.

We try to isolate the tile’s convexity assumption from the proof of Theorem 3 because by knowing how concave tile can escape from the two forbidden patterns, there might be a chance to find a new spherical tiling by concave quadrangles over a pseudo-double wheel.

We use the following lemmas. A quadrangle of type 2 or type 4 satisfies the following disjunctions on inner angle’s equality.

Lemma 8. 
1. In a quadrangle of type 2 or type 4, \( \alpha \neq \beta \) or \( \gamma \neq \delta \).

2. In a quadrangle of type 4, \( \alpha \neq \delta \) or \( \beta \neq \gamma \).

Proof. Assume otherwise. Consider a lune of which boundary contains the two edge \( \alpha \delta \) and \( \beta \gamma \) of the quadrangle. Then we have an isosceles triangle beside the edge \( \alpha \beta \) of the tile because \( \alpha = \beta \), and we have an isosceles triangle beside the edge \( \gamma \delta \) of the tile because \( \gamma = \delta \). Because the two sides of any lune has equal length, the length \( b \) of the edge \( \alpha \delta \) should be equal to the length \( a \) of the edge \( \beta \gamma \). \( \Box \)

Lemma 9 ([1, Lemma 2]). In a quadrangle of type 2, \( \beta \neq \delta \) and \( \alpha \neq \gamma \). In a quadrangle of type 4, \( \alpha \neq \gamma \).

Lemma 10 ([1, Lemma 5]). Suppose a spherical tiling by congruent quadrangles of type 2 has (i) a 3-valent vertex \( v \) incident to 3 edges of length \( a \), and (ii) a 3-valent vertex incident to an edge of length \( a \) and to an edge of length \( b \). Then \( \alpha \neq \delta \) and \( \beta \neq \gamma \).

4.1. The proof of Theorem 3

By determining the vertex type of the vertices \( F \) and \( F + 1 \) in the following Lemma 13, we generate ten cases by fact \( h \) given in Remark 2 is an automorphism of the map Figure 4 (left). Then we will reject all the cases in Lemma 14.

Definition 11. In Figure 4 (left), let \( A \) be the sum of the types of the four designated inner angles \( \angle v_{i-1}v_Fv_i, \angle v_{i+2}v_Fv_{i+2}, \angle v_{i+4}v_Fv_{i+4}, \) and \( \angle v_{i+6}v_Fv_{i+6} \) around the vertex \( F \), and let \( B \) be the sum of the types of the four designated inner angles \( \angle v_{i-1}v_Fv_{i+1}, \angle v_{i+3}v_Fv_{i+3}, \angle v_{i+5}v_Fv_{i+5}, \) and \( \angle v_{i+7}v_Fv_{i+7} \) around the vertex \( F + 1 \).

Lemma 12. If \( F = 8 \) or the tile is convex, then \( 2\alpha + 2\delta \notin \{A, B\} \).

Proof. Otherwise by considering the vertex types of the vertices \( F \) and \( F + 1 \), we have \( 2\alpha + 2\delta \leq 2 \) (the equality holds when the number of faces is 8), and thus \( 2\alpha \leq 1 \) or \( 2\delta \leq 1 \). Here the vertex types of \( i \) and \( i + 4 \) is \((\alpha + X + \delta, \alpha + X' + \delta)\), \((2\alpha + X, X' + 2\delta)\) or \((X + 2\delta, 2\alpha + X')\) for some \( X, X' \in \{\beta, \gamma\} \). Thus we have \( \beta \geq 1 \) or \( \gamma \geq 1 \). But the tiling in question is edge-to-edge when \( F = 8 \) or else against the convexity of the tile. \( \Box \)
Lemma 13. Suppose the chart is of the form Figure A (left). Then if $2\alpha + 2\delta \notin \{A, B\}$, then $A = B = \alpha + 3\delta$ or $3\alpha + \delta$ and the tile is a quadrangle of type 2. In this case, the list of such patterns modulo the automorphism of Remark 3 consists of ten patterns in Figure 7 and those with $\alpha$ and $\delta$ swapped.

Figure 7: The ten patterns in this figure and those with $\alpha$ and $\delta$ swapped forms the complete list of charts of the form Figure A (left) satisfying $F = 8$ or the convexity of the tile. The central point (infinite point) is $v_F(v_F+1)$ of Lemma 13. We will prove all of them are forbidden in Subsection 4.1. In the first four cases, the eight designated inner angles is unchanged by the automorphism given in Remark 2.

**Proof.** We claim

$$A \notin \{4\alpha, 4\delta\}. \quad (9)$$

Consider the case where the tile is a quadrangle of type 4. If $A = 4\alpha$, then the vertex $i$ is incident to two edges of length $c$, which is a contradiction. If $A = 4\delta$, then the edge $v_i + 2v_F$ has length $c$. Therefore the edge $v_{F+1}v_{i+3}$ does so and the angle $\angle v_{i+1}v_{i+2}v_{i+3}$ as well as the angle $\angle v_{i+3}v_{i+4}v_{i+5}$ has type $\beta$. As the edge $v_{i+5}v_{F+1}$ and the edge $v_{i+1}v_{F+1}$ have length $b$ and $A = 4\delta$, the angle $\angle v_Fv_{i+2}v_{i+1}$ and the angle $\angle v_{i+3}v_{i+2}v_F$ have type $\gamma$. By comparing the type $\beta + 2\gamma$ of vertices $i + 2$ and that $2\alpha + \beta$ of $i + 4$, we have $\alpha = \gamma$, which contradicts against Lemma 9.

Consider the case where the tile is a quadrangle of type 2. Assume $A = 4\alpha$. Then the types of the vertex $i + 2$ and the vertex $i + 3$ are both

$$2\beta + \gamma. \quad (10)$$


Otherwise the type of vertex \(i + 2\) is \(3\beta\) and thus that of vertex \(i + 3\) is \(2\gamma + Y\) for some \(Y \in \{\beta, \gamma\}\). Then \(\beta = \gamma = 2/3\). As the vertex \(i\) is 3-valent and incident to a length-\(a\) edge and a length-\(b\) edge, we have a contradiction against Lemma \([11]\).

Hence the vertex \(i + 4\) has type \(2\delta + \gamma\). By \([10]\), we have \(\beta = \delta\), which contradicts against Lemma \([9]\). We can similarly prove \(A \neq 4\delta\) for the tile being a quadrangle of type 2. This completes the proof of \([9]\).

We can prove similarly \(B \notin \{4\alpha, 4\delta\}\) by using the automorphism \(h\) given in Remark \([2]\). So, by Lemma \([12]\) and \([11]\),

\[
\{A, B\} \subseteq \{3\alpha + \delta, \alpha + 3\delta\}.
\]

We prove that the tile is not a quadrangle of type 4. Assume otherwise. Then none of \(A\) and \(B\) is \(3\alpha + \delta\), because two occurrences of inner angle \(\alpha\) necessarily share an edge of length \(b\), so the other end-point is incident to two edges of length \(c\), a contradiction. Moreover we have \(A \neq \alpha + 3\delta\). Assume otherwise. When \(\angle v_F v_{i+2} v_F \neq \delta\), the angle is \(\alpha\). By \(A = \alpha + 3\delta\), we have \(\angle v_{i+4} v_F v_{i+2} = \delta\) and \(v_{i+2} v_F = c\), which is a contradiction. Therefore \(\angle v_F v_{i+2} = \delta\). Thus \(v_F v_{i+2} = c = v_F + v_{i+3} = v_{i+1} v_{i+1}\). It follows that the vertex \(i + 2\) has type \(\beta + 2\gamma = 2\). On the other hand, the assumption \(A = \alpha + 3\delta\) implies \(\angle v_{i+4} v_F v_{i+2} = \delta\) or \(\angle v_{i+4} v_F v_{i+6} = \delta\), because the tile is of type 4. In the former case, the vertex \(i\) has type \(2\alpha + \beta\) while in the latter case, the vertex \(i + 4\) has type \(2\alpha + \beta\). By comparing it with the type of the vertex \(i + 2\), we have \(\alpha = \gamma\) in either case. This contradicts against Lemma \([9]\). Finally, we have \(B \neq \alpha + 3\delta\). Otherwise, by a similar argument, the length of edges \(v_{i+1} v_{i+2} v_F\) and \(v_{i+6} v_F\) are all \(c\), the type of the vertex \(i + 3\) is \(2\gamma + \beta\), the type of the angles \(\angle v_{i+1} v_{i+2} v_F\) and \(\angle v_{i+4} v_{i+5} v_{i+4}\) are both \(\alpha\), and the types of \(\angle v_{i+1} v_{i+2} v_F\) and \(v_{i+4} v_{i+5} v_{i+6}\) are both \(\beta\). Since we assumed \(B = \alpha + 3\delta\), the type of \(\angle v_{i+4} v_{i+1} v_{i+1}\) or the type of \(\angle v_{i+7} v_{i+1} v_{i+5}\) is \(\delta\), from which \(2\alpha + \beta\) follows. Therefore \(\alpha = \gamma\). This contradicts against Lemma \([9]\). Hence the tile is of type 2, and thus we can apply Lemma \([10]\). As the vertices \(vi\) and \(v_{i+2}\) satisfy the conditions \((ii)\) and \((i)\) of Lemma \([10]\), we have \(A \neq \delta\). Then \(A = B\), because otherwise \([11]\) implies \(\alpha = \delta\).

The four designated inner angles around the vertex \(F\) bijectively correspond to those around the vertex \(F + 1\) via the automorphism \(h\) given in Remark \([2]\). The distinguished type, say \(\delta\), is aligned to exactly one of the four designated inner angles around the vertex \(F\), and so is to exactly one of the four designated inner angles around the vertex \(F + 1\). Hence, there are \(\binom{4}{2} = 6\) pairs of non-\(h\)-equivalent inner angles and are 4 pairs of \(h\)-equivalent inner angles. This ends the proof of Lemma \([13]\).

The following lemma forbids the ten patterns of Figure \([7]\).

**Lemma 14.** A chart of a spherical tiling by congruent quadrangles of type 2 is none of the three patterns \((1), (2), (3)\) presented in Figure \([8]\) and is none of the three \((1), (2), (3)\) with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped in the same Figure.

Theorem \([3]\) \([11]\) is proved through rejecting all the ten charts in Figure \([7]\) as well as all the ten patterns with \(\alpha\) and \(\delta\) swapped, as follows: Case 9 in Figure \([7]\).
is impossible by the pattern (1) of Figure 8, Case 2 by the pattern (2), Case 4 by applying the same pattern (2) with \( v_F, v_{F+1} \) being (the infinite point, the central point), and the other cases by the pattern (3). All the ten patterns in Figure 7 with \( \alpha \) and \( \delta \) swapped are rejected similarly.

The rest of this subsection is devoted to the proof of Lemma 14. (1) We prove that the pattern (1) of Figure 8 is impossible for a type-2 chart, by establishing the following: there do not exist two type-2 tiles \( v_i v_{i+1} v_{F+1} v_{i-1} \) and \( v_F v_i v_{i-1} v_{i-2} \) such that they match at the length-b edge \( v_i v_{i-1} \), (i) the opposite edge \( v_{i+1} v_{F+1} \) to the matching edge \( v_i v_{i-1} \) in the first tile has length b, and an adjacent edge \( v_i v_F \) to the matching edge \( v_i v_{i-1} \) in the second tile does length b; (ii) the two inner angles at an extreme point \( v_F \) of the edge \( v_i v_{i+1} v_{F+1} v_{i-1} \) have type \( \alpha \); and (iii) \( \angle v_{i-2} v_{i-1} v_{F+1} = \beta \).

Assume some spherical tiling by congruent quadrangles of type 2 satisfies all the three conditions. By the condition (i) and the type of the first two angles of the condition (ii), the two inner angles at the other extreme point \( v_i \) of the edge \( v_i v_F \), namely, \( \angle v_F v_i v_{i+1} \) and \( \angle v_F v_i v_{i-1} \), have type \( \delta \). The two inner angles are adjacent to the inner angle \( \angle v_{i+1} v_i v_{i-1} \) of type \( \gamma \), because of the first tile \( v_i v_{i+1} v_{F+1} v_{i-1} \) and because the last angle \( \angle v_{i-1} v_{F+1} v_{i+1} \) of the condition (ii) has type \( \alpha \). Therefore \( \gamma + 2 \delta = 2 \). On the other hand, by \( v_{i+1} v_F + b \) and the condition (ii), we have \( \angle v_F v_{i+1} v_{i-1} v_i = \beta \). Because of the second tile \( v_F v_i v_{i-1} v_{i-2} \) and because \( \angle v_i v_F v_{i-2} = \alpha \) by the condition (ii), we have \( \angle v_i v_{i-1} v_{i-2} = \gamma \). By this and the condition (iii), we have \( \gamma + 2 \beta = 2 \). Therefore \( \beta = \delta \), which contradicts against Lemma 9. By the argument above with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped, we have \( \gamma = \alpha \), which also contradicts against Lemma 9.

Similarly, we can prove that the pattern (1) of Figure 8 with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped is impossible for a type-2 chart.

(2) We prove that the pattern (2) in Figure 8 is impossible for a type-2 chart. Assume otherwise. Then there exist four tiles \( v_F v_i v_{i+2} v_{i+1} v_i \), \( v_{i+2} v_{i+3} v_{F+1} v_{i+1} \), \( v_{i-1} v_i v_{i+1} v_{F+1} \) and \( v_i v_{i-1} v_{i-2} v_F \) such that

![Figure 8](image-url)
(i) The two vertices \( v_{i-1} \) and \( v_{i+2} \) are 3-valent.

(ii) The two edges \( v_i v_F \) and \( v_{i+1} v_{F+1} \) have length \( b \).

(iii) The angle \( \angle v_{i-2} v_F v_i \), as well as the two designated angles around the vertex \( v_{F+1}, \angle v_{i+1} v_{F+1} v_{i-1}, \) and \( \angle v_{i+3} v_{F+1} v_{i+1}, \) have types \( \alpha \).

(iv) \( \angle v_i v_F v_{i+2} \) has type \( \delta \).

By the conditions (iii) and (iv), the vertex \( i \) has type
\[
\alpha + \gamma + \delta = 2 \tag{12}
\]
while the vertex \( i + 1 \) has type
\[
\beta + 2\delta = 2 \tag{13}
\]
By the conditions (i) and (ii), the lengths of the two edges \( v_{i+2} v_{i+3} \) and \( v_i v_{F+1} \) are \( a \) and \( \angle v_{F+1} v_{i+2} v_{i+3} \) has type \( Y \in \{ \beta, \gamma \} \). So the vertex \( i + 2 \) has type
\[
Y + 2\gamma = 2, \quad (Y \in \{ \beta, \gamma \}) \tag{14}
\]
When \( Y = \beta \), (13) implies \( \gamma = \delta \), (12) becomes \( \alpha + 2\delta = 2 \), and (14) implies \( \alpha = \beta \). This contradicts against Lemma 8.

When \( Y = \gamma \), the equation (14) implies \( \gamma = 2/3 \). By condition (iii), the two edges \( v_{i-1} v_{i+1} \) and \( v_{i-1} v_{i-2} \) have length \( a \). Hence the vertex \( i - 1 \) is 3-valent by condition (ii) and has type \( 2\beta + \gamma \) or \( \beta + 2\gamma \). In either case, \( \beta = \gamma = 2/3 \). This contradicts against Lemma 10.

Similarly, we can prove that the pattern (2) with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped is impossible for a type-2 chart.

(3) We prove the chart (3) of Figure 8 is impossible for a type-2 chart. Assume otherwise. Because the types of \( \angle v_{i+1} v_{F+1} v_{i-1} \) and \( \angle v_{F+1} v_{i+1} v_{i-1} \) are equal and because the four edges \( v_i v_F, v_{i+1} v_{F+1}, v_{F+1} v_{i-1}, \) and \( v_{F+1} \) have length \( b \), the type of \( \angle v_{i-1} v_i v_{i+1} \) and that of \( \angle v_{i-1} v_i v_{i+1} \) are equal. Since the sum of the types of four angles around the vertex \( F \), \( \angle v_{i-1} v_F v_i \), \( \angle v_i v_F v_{i+1} \), \( \angle v_{i+1} v_F v_{i+2} \), \( \angle v_{i-1} v_F v_{i+1} \), \( \angle v_i v_F v_{i+1} \), \( \angle v_{i+1} v_F v_{i+2} \) is \( 3\alpha + \delta \) or \( \alpha + 3\delta \), we have \( \alpha = \delta \).

The vertex \( v_{i-1} \) is 3-valent and incident to three edges of length \( a \). There are two type-2 tiles \( v_{i-1} v_{i+1} v_{F+1} \) and \( v_i v_{i-1} v_{i-2} v_F \). Moreover, the vertex \( v_{i+1} \) is 3-valent and incident to a length-\( a \) edge and a length-\( b \) edge. By Lemma 10 \( \alpha = \delta \). We have a contradiction.

Similarly, we can prove that the pattern (3) with \((\alpha, \beta)\) and \((\delta, \gamma)\) swapped is impossible for a type-2 chart. This completes the proof of Lemma 14.

Hence, we have proved Theorem 3 (1). □
Figure 9: The ten patterns in this figure and those with the designated \( \beta \)'s and the designated \( \gamma \)'s swapped forms the complete list of charts of the form Figure 4 (right). See Lemma 15. All charts are subject to common naming convention of the vertices.

4.2. The proof of Theorem 3

**Lemma 15.** If a chart of a spherical tiling by congruent quadrangles is of the form Figure 4 (right), then the vertices \( i - 1 \) and \( i + 2 \) have the same type, and the tile is of type 2. In this case, the list of such charts constitutes ten patterns of Figure 9 and those with \( (\alpha, \beta) \) and \( (\delta, \gamma) \) swapped.

**Proof.** We first verify that the tile is a quadrangle of type 2. Assume otherwise. Then the tile is a quadrangle of type 4. If \( \angle v_i v_{i+2} \) of Figure 4 (right) has type \( \delta \), then the edge \( v_F v_{i+2} \) has length \( c \), but is not incident to an edge of length \( b \) in a quadrangle \( v_F v_i v_{i+4} v_{i+3} v_{i+2} \), which is a contradiction. Therefore \( \angle v_i v_{i+2} \) has type \( \alpha \), and thus \( \angle v_F v_{i+1} \) has type \( \delta \). Because two edges of length \( c \) are not adjacent, \( \angle v_F v_i v_{i+1} \) has type \( \delta \) and \( \angle v_{i-1} v_i v_{i+1} \) does \( \gamma \). On the other hand, the edge \( v_{i+2} v_{i+3} \) of the quadrangle \( v_i v_{i+1} v_{i+1} v_{i+3} v_{i+2} \) has length \( c \), since the edge is not adjacent to the \( b \)-length edge \( v_{i+1} v_{i+2} \) of the quadrangle. Therefore the edge \( v_{i+4} v_F \) has length \( c \) and \( \angle v_{i+2} v_{i+4} \) has type \( \gamma \). Hence the type of \( v_F \) subsumes the type \( \alpha + \gamma + \delta \) of \( v_i \) and thus is greater than 2, which is a contradiction. Hence the tile is of type 2.

The vertex \( v_{i+2} \) is 3-valent and is incident to three length-\( a \) edges. Furthermore, the vertex \( v_i \) is 3-valent and is incident to a length-\( a \) edge and to a length-\( b \) edge. Hence Lemma 10 implies \( \beta \neq \gamma \).
Lemma 10. Because Lemma 15 implies the tile is of type 2.

By enumerating all the possibilities of the types of \(v_{i-1}\) and \(v_{i+2}\), we have ten patterns of Figure 9 and the ten patterns with \(\beta\) and \(\gamma\) swapped. \(\square\)

To finish the proof of Theorem 8 we reject all ten cases in Figure 9. Case 0 is impossible because the vertex \(v_i\) has type \(\alpha + \gamma + \delta\) which is a subtype of the type of the vertex \(F\).

Lemma 16. If a spherical tiling by \(F\) congruent quadrangles satisfies \(F = 8\) or the convexity of the tile, then Case 3 of Figure 9 is impossible.

**Proof.** The angle \(\angle v_{i+2}Fv_{i+4}\) has type \(\beta\) because \(\angle v_{i+3}v_{i+2}v_F\) has type \(\gamma\) and the edge \(v_{i+2}v_F\) has length \(a\). Moreover the angle has the same type as the angle \(\angle v_{i+2}v_Fv_{i+4}\) because otherwise, we have \(\alpha + \beta + \gamma + \delta \leq 2\), which contradicts against Lemma 6. We have \(\alpha + 2\beta + \delta \leq 2\) (the equality holds when the number of faces is 8) and \(\alpha + \gamma + \delta = 2\) because the edge \(v_iF\) has length \(b\). Since \(2\beta + \gamma = 2\) from Figure 9 (Case 3), we have \(\beta \geq 1\) (the equality holds when the number of faces is 8), which contradicts against the convexity of the tile (against fact tiling is edge-to-edge, when the number of faces is 8). \(\square\)

The other nine cases are rejected without using the tile’s convexity assumption, as follows:

Each of Case 0, 2, 4 is impossible because the vertex \(v_i\) has type \(\alpha + \gamma + \delta\), \(\alpha + \beta + \gamma + \delta\), \(\alpha + \gamma + \delta\) which are subtypes of the type of the vertex \(F\), in respective case.

Case 1 and Case 7 both contradict against Lemma 14 (1), and Case 9 does against Lemma 14 (1) applied to \(v_{F+1}\).

In Case 6, the type of the vertex \(i + 1\) is \(\alpha + \beta + \delta\). By comparing the type \(2\alpha + \gamma\) of the vertex \(i\) and the type \(2\beta + \gamma\) of the vertex \(i - 1\), we have \(\alpha = \beta\). By this, the type \(2\beta + \gamma\) of the vertex \(i - 1\) is equal to \(\alpha + \beta + \gamma\). So by comparing it with the type \(\alpha + \beta + \gamma\) of the vertex \(i + 1\), we have \(\gamma = \delta\) as well as \(\alpha = \beta\). Since Lemma 15 implies the tile is of type 2, we have a contradiction against Lemma 8.

In Case 8, by comparing the type \(\alpha + \gamma + \delta\) of the vertex \(i + 1\) and the type \(\alpha + \beta + \delta\) of the vertex \(i\), we have \(\beta = \gamma\), which contradicts against Lemma 10 because Lemma 15 implies the tile is of type 2.

In Case 5, the type of vertex \(i\) is \(2\alpha + \beta = 2\). The type of \(\angle v_{i+4}v_Fv_{i+6}\) is \(\beta\) or \(\gamma\). First consider the case \(\angle v_{i+4}v_Fv_{i+6}\) is \(\beta\).

\(\angle v_{F+1}v_{i+3}v_{i+4}\) has type \(\delta\). Otherwise it has \(\alpha\). By comparing the types of the vertex \(i\) and the vertex \(i + 3\), we have \(\beta = \gamma\), which contradicts against Lemma 10 because Lemma 15 implies the tile is of type 2.

We see that \(\angle v_{F+1}v_{i+5}v_{i+4}\) has type \(\beta\) and \(\angle v_{i+6}v_{i+5}v_{i+4}\) does \(\delta\). So \(\angle v_{i+6}v_{i+5}v_{F+1}\) does \(\alpha\). Otherwise \(\gamma\) must be 0 because the type of vertex \(v_F\) is less than or equal to \(\gamma + \beta + 2\delta\).
Then $\beta = \gamma$ by comparing the types of vertices $i + 4$ and $i + 5$. Because the tile of type 2 by Lemma 15, we have a contradiction against Lemma 10. So $\angle v_{i+4}v_Fv_{i+6}$ does not have type $\beta$ in Case 5.

Next consider the case $\angle v_{i+4}v_Fv_{i+6}$ has type $\gamma$ in Case 5. Then $\angle v_{i+5}v_{i+4}v_F$ has type $\beta$ and $\angle v_Fv_{i+4}v_{i+3}$ does $\delta$, and $\angle v_{i+3}v_{i+4}v_{i+5}$ has type $\alpha$ or $\delta$. By comparing the types of $i$ and $i + 4$, we have $\alpha = \delta$, which contradicts against Lemma 10 because the tile is of type 2 by Lemma 15.

All the arguments are still valid even if we swap $(\alpha, \beta)$ and $(\delta, \gamma)$. So this completes the proof of Theorem 3.

5. Discussion

By mostly combinatorial argument, we have derived Theorem 2, a spherical geometric theorem.

Every spherical tiling by congruent convex quadrangles over a pseudo-double wheel is isohedral, by Theorem 2 for type-2 and type-4 tiles, by [5] for a tile being a kite or a rhombus. But some by congruent concave quadrangles over a pseudo-double wheel is not isohedral (Theorem 4). So we conjecture that the “inverse” of Grünbaum-Shephard’s result holds if the tile is convex. That is,

Conjecture 1. For every normal spherical monohedral tiling topologically a Platonic solid, an Archimedean dual, an $n$-gonal bipyramid, or an $n$-gonal trapezohedron ($n \geq 3$), if the tile is convex, then the tiling is isohedral.

Because of Theorem 1, the conjecture is completely settled down if we work it out for the non-triangle-faced Platonic solids and such Archimedean duals. The classification of spherical tiling by twelve congruent pentagons [14] seems useful for the solution. We hope the partial mechanization described in Remark 4 followed by trigonometric arguments is a feasible strategy.

5.1. Classification of spherical monohedral quadrangular tilings

The classification of spherical monohedral quadrangular tilings solves “spherical Hilbert’s eighteenth problem”: Enumerate spherical anisohedral triangles, anisohedral quadrangles and anisohedral pentagons. By an anisohedral tile, we mean a tile that admits a spherical monohedral quadrangular tiling but not a spherical isohedral tiling. There are two infinite series of spherical anisohedral triangles but none of the other triangles, kites, darts, rhombi are anisohedral [5].

The only spherical tilings by congruent quadrangles of type 2 or 4 over pseudo-double wheels are $\mathcal{P}_F$ or $\mathcal{A}$, if we can drop the tile’s convexity assumption from Theorem 3 specifically from Lemma 12 and Lemma 16. If there is, however, another spherical monohedral tiling by congruent concave quadrangles of type 2 or type 4 with the map being a pseudo-double wheel, then such a tiling necessarily has a meridian edge of length $b$ by Theorem 5.

Brinkmann’s group observed that the graphs of spherical monohedral quadrangular tilings can be enumerating efficiently if the number of distinct degrees
of vertex is known (See Table 1). Brinkmann conjectured that there are at most three distinct vertex types in a chart of any spherical monohedral quadrangular tiling, motivated by linear algebraic (or linear programming) argument that independent equations of $\alpha, \beta, \gamma, \delta$ arise from the distinct vertex types. Indeed, the conjecture holds for every spherical monohedral tiling such that the tile is a kite, a dart, or a rhombus, according to the classification [5].

Even if we can enumerate all graphs of spherical monohedral type-2/type-4 quadrangular tiling by computers, to show that spherical tilings by congruent concave quadrangles exist requires difficult trigonometric arguments because there is a spherical concave quadrangle $Q$ such that there are continuously many spherical non-congruent quadrangles $Q'$ with the same cyclic list of inner angles of $Q$, according to [8].

We propose to classify

1. the spherical tilings by congruent tilings such that the tile is possibly concave and the graphs are the spherical monohedral (kite/dart/rhombic)-faced tilings [5]; and

2. the spherical tilings by congruent convex quadrangles of type 2.

Computer experiments using [10, 2] show that for sufficiently high number of faces more than 40% of spherical quadrangulations can already be excluding as the map of a spherical tiling by congruent convex type-2 quadrangles by checking for some small forbidden substructures [9].

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