We show that for neural network functions that have width less or equal to the input dimension all connected components of decision regions are unbounded. The result holds for continuous and strictly monotonic activation functions as well as for ReLU activation. This complements recent results on approximation capabilities of Hanin and Sellke (2017) and connectivity of decision regions of Nguyen et al. (2018) for such narrow neural networks. Further, we give an example that negatively answers the question posed in Nguyen et al. (2018) whether one of their main results still holds for ReLU activation. Our results are illustrated by means of numerical experiments.

Keywords: Expressive Power, Approximation by Network Functions, Neural Networks, Decision Regions, Width of Neural Networks

1. Introduction

In recent years machine learning experienced a remarkable evolution mainly due to the progress achieved with deep neural networks, c.f. Krizhevsky et al. (2012), Hinton et al. (2012), Nguyen et al. (2018), He et al. (2016) and Schmidhuber (2015), Goodfellow et al. (2016) for an overview and theoretical background. The need for theoretical foundations accompanying the practical success has been recognised a while ago. As a part of this, the approximation properties, or expressiveness, of neural network functions have attracted intense interest in recent research. The central result in this field is the classic universal approximation theorem, which states that any reasonable (continuous or Borel measurable) function can be approximated with arbitrary accuracy (in terms of uniform approximation or $L_p$ norms) by neural network functions that have only one hidden layer for nearly every activation function c.f. Cybenko (1989), Hornik (1991). The limitation of practical relevance of the latter result is that it does not give bounds for the needed width of the network. On the other hand, from empirical observations it turned out that depth has a significant impact on the performance of neural networks which is why a lot of research has been dedicated to the effect of depth on the expressive power of neural networks, c.f. Telgarsky
(2016), Telgarsky (2015), Mhaskar and Poggio (2016), Montufar et al. (2014), Raghu et al. (2016), Rolnick and Tegmark (2017), Lin et al. (2017), Cohen et al. (2016).

It is however clear that besides depth a neural network needs a certain width in order to be a universal approximator, c.f. Remark 9. The importance of width is for instance pointed out in Nguyen and Hein (2017). Recently, the needed minimum width to guarantee certain approximation properties has been investigated in Lu et al. (2017), Hanin and Sellke (2017) and Hanin (2017) for network functions with ReLU activation. For instance, in Hanin and Sellke (2017) it is shown that the required width for a universal approximator (with respect to continuous functions) is equal to input dimension plus one. In the latter works the authors also give estimates on the depth of networks with low width to approximate certain classes of continuous functions. Giving insights in a related direction, it has recently been shown in Nguyen et al. (2018) that for a certain class of activation functions a width larger than the input dimension is needed to learn disconnected decision regions.

In this paper, we follow the work of Hanin and Sellke (2017) and Nguyen et al. (2018). Our results complement what is found in the latter works.

Let us introduce the following notation. For a set $D \subset \mathbb{R}^d$ we denote by $D^0$ the set of interior points, by $\partial D = D \setminus D^0$ the boundary of $D$ and for $f : \mathbb{R}^d \to \mathbb{R}^m$ we set $\|f\|_D := \sup\{\|f(x)\|_2 : x \in D\}$ (the norm $\|\cdot\|_2$ being the Euclidian norm in the image space). We consider neural network functions $F : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}}$ where $d_{in}$ is called the input dimension and $d_{out}$ the output dimension. Our network functions have the following form $F := W_L \circ A_{L-1} \circ \ldots \circ A_1$ where $A_j(x) = \sigma(W_j x + b_j)$ with $W_j \in \mathbb{R}^{d_j \times d_{j-1}}$ (weights), $b_j \in \mathbb{R}^{d_j}$ (bias) and $\sigma : \mathbb{R} \to \mathbb{R}$ the activation function. We emphasize that $\sigma$ and the preimage $\sigma^{-1}$ are understood to be applied elementwise when applied to vectors or subsets of $\mathbb{R}^d$. The widely applied activation, called rectified linear unit shortly ReLU is defined by $t \mapsto \max\{t, 0\}$. We set $d_0 := d_{in}$ and $d_L = d_{out}$ and call $d_j$ the width of layer $j = 1, \ldots, L$. The width of the network is defined as $\omega := \omega_F = \max\{d_j : j = 1, \ldots, L\}$ and $L$ is called the depth of the network. Adapting the notation from Hanin and Sellke (2017) to our needs, we define $\omega_{\min}(\sigma, d_{in}, d_{out})$ to be the minimum width such that for every continuous $f : [a, b]^{d_{in}} \to \mathbb{R}^{d_{out}}$ and $\varepsilon > 0$ there exists a network function $F : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}}$ with activation function $\sigma$ and $\omega_F \leq \omega_{\min}(\sigma, d_{in}, d_{out})$ such that $\|f - F\|_{[a, b]^{d_{in}}} < \varepsilon$, where $a < b$ are some real numbers.

2. Related work

It is a fundamental observation that the expressiveness of a classifier function is closely related to the notion of decision regions.

Definition 1 For a network function $F = (F_1, \ldots, F_{d_{out}}) : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}}$ and $j \in \{1, \ldots, d_{out}\}$, the set $C_j := \{x \in \mathbb{R}^{d_{in}} : F_j(x) > F_k(x), \text{ for all } k \neq j\}$ is called a decision region (for class $j$). If $K \subset \mathbb{R}^{d_{in}}$, then $C_j \cap K$ is called the decision region (of class $j$) in $K$.

Definition 2 A set $C \subset \mathbb{R}^d$ is said to be connected if there exist no disjoint open sets $U, V \subset \mathbb{R}^d$ such that $C \subset U \cup V$ and $C \cap U$ and $C \cap V$ are non-empty. Let $K \subset \mathbb{R}^d$ be some compact set, then $C$ is said to be connected in $K$, if such sets $U, V$ do not exist for $C \cap K$. 

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It is known that open sets can be decomposed into a disjoint family of connected open sets. In what follows, an element of this family is called connected component. These components are maximal in the sense that they are no proper subset of another connected proper subset of the original set.

It should be noted that the notion of connectivity is not equivalent to the more intuitive but more restrictive term of path-connectivity (c.f. Definition 3). In practical settings it is however reasonable to assume that each connected component is also path-connected.

**Definition 3** A set $C \subset \mathbb{R}^d$ is said to be path-connected if for all $x_1, x_2 \in C$ there exists a continuous path $\gamma : [0, 1] \to C$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Let $K \subset \mathbb{R}^d$ be some compact set, then $C$ is said to be path-connected in $K$, if for all $x_1, x_2 \in C \cap K$ there exists a continuous path $\gamma : [0, 1] \to C \cap K$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

In Fawzi et al. (2017) the decision regions of deep neural networks are investigated empirically. By experiments with ImageNet networks, the authors observe that two samples that are predicted to belong to the same class can be connected by a continuous path, where the path is found by a dedicated algorithm provided in this work. The latter article further gives some interesting insight regarding the local curvature of decision boundaries. In contrast to that experimentally driven work, we focus on the theoretical aspects. In particular, our work is more related to Nguyen et al. (2018) and Hanin and Sellke (2017) which also start from theoretical considerations. The connectivity of corresponding decision regions is investigated in Nguyen et al. (2018) from theoretical perspective. A central result therein is the following.

**Theorem 4 (Theorem 3.10, Nguyen et al. (2018))** Let $F$ be a neural network function such that $d_{in} = d_1 \geq d_2 \geq \ldots d_L = d_{out}$ and each weight matrix has full rank. If the activation function $\sigma$ is continuous, strictly monotonically increasing and satisfies $\sigma(\mathbb{R}) = \mathbb{R}$, then every decision region is connected.

In Hanin and Sellke (2017) the explicit approximation of ReLU network functions is investigated. They prove estimates on minimum width needed to guarantee the universal approximation property. Precisely, Theorem 1 in Hanin and Sellke (2017) states that

$$d_{in} + 1 \leq \omega_{min}(\text{ReLU}, d_{in}, d_{out}) \leq d_{in} + d_{out}. \quad (1)$$

Interestingly, the latter results restrict the expressive power of neural network functions when $\omega_F \leq d_{in}$ holds. However, the former one assumes surjective activation functions, whereas the second result is stated for ReLU. In the following section we take up the above results and partially close this gap.

Very recently, we got aware that our investigations are very much related to parts of the interesting work of Lin and Jegelka (2018), which mainly focuses on expressiveness of ResNet networks.

### 3. Results

In our main result we show that in case of $\omega_F \leq d_{in}$ and continuous and strictly monotonic activation functions or ReLU activation the components of the decision regions are unbounded.
This implies that they intersect the boundary of the natural bounding box of input data. This result is closely related to Theorem 4 as it limits the possible topology of decision regions in terms of connectivity. One the other hand, this result implies in a straightforward way that the lower estimate in (1) from Hanin and Sellke (2017) holds for a wider class of activation functions. In that sense our result complements these results. We further give explicit examples that show that Theorem 4 from Nguyen et al. (2018) does not hold for the stronger connectivity in the sense of connectivity with respect to a compact input domain, c.f. Definition 2. Based on this example, we finally construct a case that shows that Theorem 4 does not hold for ReLU activation.

We exploit the basic observation of the following lemma and show that for certain narrow neural networks this can be continued to the input domain. The content of the lemma is well-known. We give a short proof for interested readers.

**Lemma 5** Let \( A \in \mathbb{R}^{n \times d} \) with \( n \leq d \), \( b \in \mathbb{R}^n \) and \( x \in \mathbb{R}^d \) with \( Ax < b \). Then there exists a non-zero \( v \in \mathbb{R}^d \) such that \( A(x + \lambda v) < b \) for all \( \lambda \geq 0 \)

The geometrical interpretation of the previous lemma is that a convex set in \( \mathbb{R}^d \), that is described by less than \( d + 1 \) hypersurfaces cannot enclose a point. It directly follows that for every compact set \( K \) such that if \( C = \{ x \in K : Ax < b \} \neq \emptyset \) then \( C \cap \partial K \neq \emptyset \).

**Proof** In case where \( A \) is regular let \( w \) be the unique solution of \( Aw = b \). One directly verifies that \( v = x - w \) has the desired property. Otherwise, we set \( v \) equal to one of the (non-zero) vectors that are orthogonal to rows of \( A \).

In the what follows, it will be convenient to argue with non-singular weight matrices. The following remark clarifies that this is justified in our setting.

**Remark 6** Consider a neural network function \( F : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}} \) with \( \omega_F \leq d_{in} \) and \( L \) layers. Then, possibly by padding with zeros rows and zero components, we can consider \( x \mapsto W_j x + b_j \), \( j = 1, ..., L - 1 \) as mapping from \( \mathbb{R}^{d_{in}} \) to \( \mathbb{R}^{d_{in}} \) so that \( W_j \in \mathbb{R}^{d_{in} \times d_{in}} \). It is further clear that on a compact set \( K \subset \mathbb{R}^d \), \( x \mapsto W_j x + b_j \) can be approximated with arbitrary accuracy by a mapping \( x \mapsto \tilde{W}_j x + \tilde{b}_j \) with non-singular \( \tilde{W}_j \in \mathbb{R}^{d_{in} \times d_{in}} \) with respect to the norm \( \| \cdot \|_K \). Together with the continuity of \( \sigma \) we can thus approximate \( F \) by some network function \( \tilde{F} : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}} \) with \( L \) layers, the same \( \sigma \) as activation function and where each weight matrix \( \tilde{W}_j \in \mathbb{R}^{d_{in} \times d_{in}} \) is non-singular for \( j = 1, ..., L - 1 \). Further, for a fixed decision region of \( F \), say \( C_1 \), one verifies that one can adjust the parameters \( \tilde{W}_L, \tilde{b}_L \) in the final layer in a way that the corresponding decision region \( \tilde{C}_1 \) of \( \tilde{F} \) satisfies \( \tilde{C}_1 \cap K \subset C_1 \cap K \).

Our main result states as follows.

**Theorem 7** Let \( F : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}} \) be a neural network function with continuous and strictly monotonic activation function \( \sigma \) or \( \sigma \) = ReLU and \( \omega_F \leq d_{in} \). Then for every decision region \( C_j, j = 1, ..., d_{out} \), the non-empty connected components of \( C_j \) are unbounded.

**Proof** For a general \( d_{in} \)-dimensional box \( K = [a, b]^{d_{in}} \), \( a < b \), we show that each connected component of the decision regions intersect \( \partial K \). We consider \( C_1 \), the decision region for the first class (c.f. Definition 1), for which we fix a connected component that intersects
We denote the intersection of this component with K by C. That is, C is non-empty and open in K and \( C \cap K^c \neq \emptyset \). We assume that each weight matrix \( W_j, j = 1, ..., L - 1 \) is in \( \mathbb{R}^{d_{in} \times d_{in}} \) and that is non-singular. By Remark 6, this covers the remaining cases. By the definition of \( C_1 \) and by continuity we have that \( \tilde{C} := A_{L-1} \circ ... \circ A_1(C) \) is a connected subset of \( \Omega = \{ y \in \mathbb{R}^{d_{in}} : b_{1,L} - b_{k,L} > (w_{k,L} - w_{1,L})y, \ k = 2, ..., d_{out} \} \) where \( w_{j,L} \) denotes the \( j \)-th row of \( W_L \) and \( b_{j,L} \) is the \( j \)-th component of \( b_L \).

First, consider the case when \( \sigma \) is continuous and strictly monotonic. In this case, our assumptions give that \( A_{L-1} \circ ... \circ A_1 \) is injective and hence the Invariance Domain Theorem gives that \( \tilde{C} \) has interior points, namely the images of the interior points from C under that mapping. By Lemma 5, there exists an element \( y_0 \in \partial \tilde{C} \cap \Omega \). Otherwise the hypersurfaces defined by \( b_{1,L} - b_{k,L} = (w_{k,L} - w_{1,L})y \) for \( k = 2, ..., d_{out} \) would enclose the interior points of \( \tilde{C} \). Again by the Invariance Domain Theorem we have that \( A_{L-1} \circ ... \circ A_1 \) maps interior points to interior points and this allows us to deduce the existence of an \( x_0 \in \partial C \) such that \( y_0 = A_{L-1} \circ ... \circ A_1(x_0) \). If \( x_0 \in \partial K \) the proof for this case is finished. Otherwise \( x_0 \) is an interior point in \( K \). Since \( y_0 \) is an interior point of \( \Omega \), the continuity of \( A_{L-1} \circ ... \circ A_1 \) implies the existence of a small \( \varepsilon > 0 \) such that \( B = \{ x : \| x - x_0 \|_2 < \varepsilon \} \) is a subset of \( K \) and such that \( A_{L-1} \circ ... \circ A_1(B) \subset \Omega \). Hence, \( C \cap B \subset C_1 \), but \( B \) is not a subset of \( C \) which contradicts the fact that \( C \) is a connected component of \( C_1 \).

Now the case \( \sigma = \text{ReLU} \) is considered. If for all \( x \in C \), we have \( W_jA_{j-1} \circ ... \circ A_1(x) + b_j > 0 \) for \( j = 2, ..., L - 1 \), then \( A_{L-1} \circ ... \circ A_1 \) constitutes a linear affine and invertible map from \( C \) to \( \tilde{C} \). Thus, following the same line of arguments as in the previous case, we obtain that \( K \cap C \neq \emptyset \). Otherwise, there exist an \( x_0 \in C \) and a smallest \( l \in \{1, ..., L - 1\} \) with corresponding \( k \in \{1, ..., d_{in}\} \) such that \( w_{k,l}y_0 + b_{k,l} \leq 0 \) where \( y_0 := A_{l-1} \circ ... \circ A_1(x_0) \) if \( l > 1 \) and \( y_0 = x_0 \) otherwise, and where \( w_{k,l} \) denotes the \( l \)-th row of \( W_l \) and \( b_{k,l} \) is the \( l \)-th component of \( b_l \). Without loss of generality say \( k = 1 \). Then by the definition of ReLU, the classification does not change on \( y_l = y_0 - t e_1, t > 0 \) where \( e_1 = (1, 0, ..., 0)^T \). More precisely, \( A_{l-1} \circ ... \circ A_1(\sigma(y_0)) = A_{l-1} \circ ... \circ A_1(\sigma(y_l)) \in \Omega \) for all \( t > 0 \). Since we have chosen \( l \) to be minimal, the mapping \( A_{l-1} \circ ... \circ A_1 \), for \( l > 1 \) or identity otherwise, is linear affine and invertible on \( C \). Its preimage of the half line \( \{ y_0 - t e_1 : t \geq 0 \} \) thus intersects \( \partial C \) in some point \( w \). Since by the preceding consideration \( A_{L-1} \circ ... \circ A_1(w) \in \Omega \), we can conclude that \( w \in \partial K \) by the same arguments as in the cases above.

With Theorem 7 we can now extend the lower bound of (1) from Hanin and Sellke (2017) to a wider class of activation functions.

**Corollary 8** For network functions with continuous and strictly monotonic activation function \( \sigma \) or \( \sigma = \text{ReLU} \) the lower estimate \( d_{in} \prec \omega_{\min}(\sigma, d_{in}, d_{out}) \) holds.

**Proof** For the purpose to show the result by contradiction, let \( f : K := [0, 1]^{d_{in}} \rightarrow \mathbb{R} \) be continuous with \( f(x_h) = -1 \) where \( x_h = (1/2, ..., 1/2)^T \) and \( f = 1 \) on the boundary of \( K \) and assume that
\[
\| f - F \|_K < 1/2. \tag{2}
\]
Then necessarily $F(x_h) < 0$ and $F(x_b) > 0$ for all $x_b$ in $\partial K$. By Theorem 7 the preimage of $(-\infty, 0)$ under $F$ (the decision region $F < 0$) is either empty or intersects the boundary of $K$ and hence gives a contradiction, since $F > 1/2$ holds on $\partial K$. □

**Remark 9** It is easily seen that in general $\omega_{\min}(\sigma, d_{in}, d_{out}) < d_{in}$ is impossible for all $\sigma$. Indeed, in this case $W_1$ would have non trivial kernel and therefore every function that is non-constant on all subspaces of $\mathbb{R}^{d_{in}}$, such as $x \mapsto \prod_{j=1}^{d_{in}} x_j^2$, cannot be approximated with arbitrary accuracy.

We now formulate an example that shows that, despite the restrictions given by Theorem 7 and Theorem 4, decision regions can be disconnected as subset of a compact input domain (c.f. Definition 2).

**Example 1** Let $K = [-1,1] \times [-1,1]$ and $\sigma$ be the ReLU activation function. The weights and bias in the first layer are set as follows: $W_1$ is the rotation matrix with angle $\alpha = -\pi/4$, i.e.:

$$W_1 = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix},$$ (3)

and $b_1 = \sqrt{2}(1,-1/2)^T$. The parameters for the (output) second layer are as follows: $W_2 = 1/\sqrt{2} (1,-4)^T$, $b_2 = -1/4$. The complete model now is written as

$$F : K \to \mathbb{R}, \ x \mapsto W_2 \sigma(W_1 x + b_1) + b_2.$$ (4)

We further set

$$A_1 : K \to \mathbb{R}^2, \ x \mapsto \sigma(W_1 x + b_1).$$ (5)

One easily verifies that $A_1(K)$ and the decision hypersurface defined by $W_2 x + b_2 = 0$ are as depicted in Figure 1. Then the region in $K$ that is mapped to $(-\infty, 0)$ is not connected in $K$ as depicted in Figure 2.

It is straightforward to adapt the above example to the leaky ReLU activation $\sigma_{\beta}(t) = \max\{t, \beta t\}, \ 0 < \beta < 1$. The important point is that a convex excerpt of the activation function can be used to create an internal corner in the image of $K$ under the mapping that corresponds to the first layer, c.f. Figure 1 on the right.

We further extend Example 1 to the case that the mapping starts at $\mathbb{R}^2$ rather than $K$. The purpose is to clarify a question that has been left as an open problem in Nguyen et al. (2018). Precisely, they ask whether their condition that $\sigma$ maps $\mathbb{R}$ surjectively to $\mathbb{R}$ in Theorem 4 can be dropped and still holds for mappings that map to a proper subset of $\mathbb{R}$ such as ReLU. The following shows that this is not the case in general.

**Example 2** Let $\sigma$ be the ReLU activation function and $W_1 = I_2$ the identity in $\mathbb{R}^2$ and $b_1 = (0,0)^T$. Then $A_1 : \mathbb{R}^2 \to \mathbb{R}^2, \ x \mapsto \sigma(W_1 x + b_1)$ maps $\mathbb{R}^2$ to $Q_1 := \{(x_1,x_2)^T \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Let $W_2$ be the rotation matrix for angle $\alpha = -3/4\pi$ (c.f. (3)), and $b_2 = \sqrt{2}(1,1/2)^T$. The resulting image of $\mathbb{R}^2$ under $G_2 : \mathbb{R}^2 \to \mathbb{R}^2, \ x \mapsto \sigma(W_2 A_1(x) + b_2)$
Figure 1: Example 1: drawing of $W_1K + b_1$ (left) and $\sigma(W_1K + b_1)$ right ($x_1$ horizontal, $x_2$ vertical). The red line depicts the decision hypersurface defined by $W_2x + b_2 = 0$

Figure 2: Drawing of the set $K$ in Example 1 with crosshatched blue area for the preimage of $(-\infty, 0)$ under $F$ ($x_1$ horizontal, $x_2$ vertical)
Figure 3: Example 2 ($x_1$ horizontal, $x_2$ vertical): Left, crosshatched blue area depicts the preimage of $[0, 1/\sqrt{2}] \times \{0\} \cup [3/\sqrt{2}, \infty) \times \{0\}$ under $G_2$. Right: crosshatched blue area depicts the preimage of $(-\infty, 0)$ under $F$

is similar to Figure 1 right, except that the right bar is continued to $+\infty$. One verifies that the preimage of these bars is as depicted in Figure 3 left. Following Example 1, we set $W_3 = 1/\sqrt{2} (1, -4)^T$, $b_3 = 1/4$ and define $F : \mathbb{R}^2 \to \mathbb{R}$ as $x \mapsto W_3 G_2(x) + b_3$. Now the preimage of $(-\infty, 0)$ under $F$ is not connected in $\mathbb{R}^2$ as sketched in Figure 3 right.

The conclusion from the previous examples can be summarized as follows.

**Corollary 10** From Examples 1 and 2 it immediately follows that in general

1. for a network function with ReLU or leaky ReLU activation function and width not exceeding the input dimension, the decision areas are not necessarily connected with respect to a bounded input domain (c.f. Definition 2).

2. In Theorem 4 the condition that the activation function maps $\mathbb{R}$ surjectively to $\mathbb{R}$ can in general not be dropped. In particular, Theorem 4 doesn’t hold for ReLU activation.

4. Experiments

We tested the above results by means of some numerical experiments.

We applied a fully connected neural networks with ReLU activation and two hidden layers of varying width on the MNIST data. The softmax function has been applied after the linear output layer and the cross-entropy was taken as objective function. The models have been trained with the Adam optimizer and a batch-size of 100. The results on the achieved test accuracy is given in Figure 4.

We also performed a parameter study to check the theoretical bound of our theorem by means of a toy example. We defined a dataset consisting of two n-spheres both being
centered at the origin. The inner n-sphere (first class) has a radius of \( n^{\frac{1}{2}} \) and the outer n-sphere (second class) a radius of \( n \). We increase the radius of the n-sphere with the dimension in order to avoid numerical problems when we go to higher dimensions. The two and three dimensional dataset is shown in Fig. 5. The neural network we wanted to train should separate the inner sphere from the outer sphere and we considered a trained network to be successful only when we reached a test accuracy of 100%. According to Theorem 7, this should only be possible if \( \omega_F > d_{in} \). If \( \omega_F \leq d_{in} \), we should always get an unbounded decision region and consequently not obtain a test accuracy of 100%. We generated \( 10^7 \) uniformly distributed points on each sphere for the training data and \( 25 \cdot 10^5 \) uniformly distributed points on each sphere for the test data. Each network was trained for 10 epochs using a batch size of \( 10^4 \) and the Adam optimizer with a learning rate of 0.001. For the cost function we used the cross entropy. We trained neural networks of different sizes with fully connected layers and ReLU activation functions only. We combined the following different possible settings:

1. Input dimension: 2, 3, 8 and 20,

2. Number of fully connected layers: 1, 2, 10,

3. All layers of width \( d_{in} \) and layers of width \( d_{in} + 1 \).

Each one of the 24 possible combinations of settings was repeated for 100 trainings of 10 epochs in order to have a meaningful representation of the setting. For each training, we stored the maximal obtained accuracy on the test dataset as the performance of the training. For input data of dimension two and three, Fig. 6, and Fig. 7, respectively, illustrate the successful training of a network with \( \omega_F > d_{in} \) and an example of an unsuccessful training of a network with \( \omega_F \leq d_{in} \). As expected, the networks with \( \omega_F \leq d_{in} \) learn an unbounded
Figure 5: Input datasets in the two and three dimensional case. The dataset consists of two classes: the inner sphere (orange) labeled as 0 and the outer sphere (blue) labeled as 1. Left: 1-sphere with radius 0.5 for the inner sphere and radius 1.0 for the outer sphere. Right: 2-sphere with radius 1.0 for the inner sphere and radius 2.0 for the outer sphere.

decision region. The summary of the complete parameter study is illustrated in Fig. 8. Successful training was only possible when $\omega_F > d_{in}$, although some of the settings which could have been successful did not achieve 100% test accuracy.

5. Conclusion

For a wide class of activation functions, we have shown that for neural network functions that have a maximum width less or equal than the input dimension the connected components of the decision regions are unbounded. Hence, for such networks the decision regions intersect the boundary of a natural input domain. This links some recent results from Hanin and Sellke (2017) and Nguyen et al. (2018), where for such narrow neural networks limitations regarding their expressive power for the case of ReLU activation are achieved in the first work, and the connectivity of decision regions is restricted for the case of continuous, monotonically increasing and surjective $\sigma : \mathbb{R} \to \mathbb{R}$ in the second work.

We illustrated our findings by numerical experiments with spherical data where it was observed that the input dimension is the critical threshold for network width in order to achieve 100% accuracy. However, in experiments on MNIST data we could not detect a limitation on the performance for such narrow networks. This raises the question to what extent limitations in terms of connectivity imply a crucial restriction in practical applications. From theoretical perspective, it would be interesting to know if Theorem 7 still holds for other types of activation function, like non-continuous and oscillating activations.
Figure 6: Trained networks for input dimension two. Left: Decision region for a network with one hidden layer of size $d_{in} + 1$ obtaining 100% test accuracy. Right: Decision region for a network with one hidden layer of size $d_{in}$. We see that the decision regions needs to be unbounded.

Figure 7: Trained networks for input dimension three. Left: Decision region for a network with one hidden layer of size $d_{in} + 1$ obtaining 100% test accuracy. Right: Decision region for a network with one hidden layer of size $d_{in}$. We see that the decision regions needs to be unbounded.
Figure 8: Results of the complete test run (24 combination of settings trained 100 times for 10 epochs). Left: Input data dimension $d_{in}$ plotted against maximal network width $\omega_F$. Green points are instances for which the network obtained at least once 100% accuracy on the test set. Blue points are instances for which the network never obtained 100% test accuracy. We clearly see the trend that the network is only capable of learning correct decision regions when $\omega_F > d_{in}$. Right: For each input data dimension (2, 3, 8 and 20), we tested three different network sizes (colored in the same color from left to right: 1, 2 and 10). The height of the bar plot represents the percentage of networks which obtained 100% accuracy on the test set out of 100 repeated trainings. For $d_{in} = 8$ and $d_{in} = 20$ some of the network settings did never achieve 100% although success should be possible. That is why some bars vanish completely. We only plot networks for which $\omega_F > d_{in}$, since otherwise the percentage is zero.
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