Symmetry protection of measurement-based quantum computation in ground states

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Abstract. The two-dimensional cluster state, a universal resource for measurement-based quantum computation, is also the gapped ground state of a short-ranged Hamiltonian. Here, we examine the effect of perturbations to this Hamiltonian. We prove that, provided the perturbation is sufficiently small and respects a certain symmetry, the perturbed ground state remains a universal resource. We do this by characterizing the operation of an adaptive measurement protocol throughout a suitable symmetry-protected quantum phase, relying on generic properties of the phase rather than any analytic control over the ground state.

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1. Introduction

A quantum computer relies on quantum entanglement to achieve computational speedups. In the traditional, circuit-based model for quantum computation, the required entanglement is built up throughout the course of the computation through application of entangling gates coupling two or more qubits at a time. Alternatively, in the model of measurement-based quantum computation (MBQC) [1, 2], universal quantum computation is achieved solely through single-particle operations (specifically, single-particle measurements) on a fixed entangled resource state, independent of the quantum algorithm being performed.

Since the initial discovery that the two-dimensional (2D) cluster state is a universal resource for MBQC [1], much effort has been devoted to characterizing other universal resource
states. Many of the universal resource states so far identified [1, 3–8] have been projected entangled pair states (PEPS) [9] of small bond dimension. The tensor network structure of these states facilitates the analysis of measurements, which might otherwise be an intractable problem. Another advantage of such states is that under appropriate conditions [10], they are unique (possibly gapped) ground states of local frustration-free Hamiltonians on spin lattices. This suggests a method of constructing the resource state by cooling an appropriate interacting spin system [11, 12].

However, if we wish to adopt this viewpoint of the resource state for MBQC as the ground state of a quantum spin system, it would be too restrictive to confine ourselves to states in which the effect of measurements can be determined analytically from the tensor-network structure. A generic local Hamiltonian, or even an arbitrarily small generic local perturbation to a PEPS parent Hamiltonian, will not have such a property. Therefore, it is desirable to develop an understanding of MBQC in ground states of spin systems that does not rely on analytic control of the ground state. For this reason, there has been interest in relating MBQC to forms of quantum order which, as parameters of the Hamiltonian are varied, can disappear only at a quantum phase transition [13–15].

In this paper, we will use such a connection between MBQC and quantum order to give a precise characterization of the operation of MBQC in the ground states of a large class of perturbations to the 2D cluster model. This will allow us to give a rigorous proof that such perturbed ground states remain universal resources for MBQC provided that the perturbation is sufficiently small. Our proof relies in part on an extension of the relationship introduced in [15] between MBQC and symmetry-protected topological (SPT) order [16–18], a form of quantum order characterizing quantum systems that cannot be smoothly deformed into a product state while a certain symmetry is enforced. If the perturbation to the 2D cluster model respects an appropriate symmetry, then the perturbed ground state will still possess non-trivial SPT order, and we will show that this gives us sufficient information about the ground state to characterize the implications of the perturbation for MBQC. Our result therefore holds independently of any analytic solution for the perturbed ground state.

Our proof of universality is in the same spirit as [19]. There, it was shown that, whereas measurements on the cluster state simulate quantum circuits, measurements on a noisy cluster state simulate the same circuits, but with added noise. Here, our task is complicated by the highly correlated nature of the ‘errors’ in the resource state that result from a change in the Hamiltonian. Nevertheless, we will show how to exploit the additional structure resulting from SPT order to establish an effective noise model for ground states of appropriate perturbed cluster models. Therefore, universal quantum computation can be achieved (for sufficiently small perturbations, corresponding to sufficiently weak noise in the effective circuit model) by choosing a measurement protocol which simulates a fault-tolerant quantum circuit. The universality is then a consequence of the threshold theorem [20] for fault-tolerant quantum computation with noisy quantum circuits.

1.1. Summary of results

Our ultimate goal in this paper is to prove the universality for a MBQC of a class of perturbations of the 2D cluster state. However, in order to reach this goal, most of this paper will be devoted to a further elucidation of the relationship between SPT order and MBQC. For simplicity of presentation, we will first explore this relationship in one-dimensional (1D) systems. It has
already been shown that in a class of quantum phases characterized by SPT order, the structure implied by SPT order leads to the perfect operation of the identity gate in MBQC [15]. Here, we consider the 1D cluster model, which lies in the simplest of the SPT phases considered in [15], and characterize the operation of non-trivial (i.e. not the identity) gates in the presence of a perturbation which respects the symmetry protecting this SPT phase. We obtain the following.

**Theorem 1 (Effective noise model in one dimension).** Consider a measurement protocol which in the exact 1D cluster model would simulate a sequence of gates. In the perturbed resource state, the same measurement protocol simulates the same gate sequence, but with additional noise associated with each non-trivial gate. So long as the non-trivial gates are sufficiently separated from each other by identity gates, this effective noise has no correlations between different time steps, i.e. it is Markovian.

The proof of theorem 1 will be divided into two stages. First, in section 2 we will establish theorem 1 for ground states that are pure finitely correlated states (pFCS), a special case of matrix-product states (MPS). For such states, both the manifestations of SPT order [17, 18], and the effect of measurements [3, 4] can be understood straightforwardly in terms of the tensor-network structure. The ideas leading to theorem 1 can thus be understood most directly in this context. Second, in section 3 we will prove theorem 1 for arbitrary ground states within the SPT phase.

The extension of these ideas to the 2D cluster model will be considered in section 4. We will construct an appropriate symmetry group, such that the following result is satisfied for symmetry-respecting perturbations.

**Theorem 2 (Effective noise model in two dimensions).** Consider a measurement protocol that in the exact 2D cluster model would simulate a sequence of gates. In the perturbed resource state, the same measurement protocol simulates the same gate sequence, but with additional noise associated with each gate. So long as the non-trivial gates are sufficiently separated from each other by identity gates, this effective noise has no correlations between different time steps, or between different gates taking place at the same time step, i.e. it is local and Markovian.

Combined with the existing results on fault tolerance in the circuit model [20], theorem 2 will imply the main result of this paper:

**Theorem 3.** For sufficiently small symmetry-respecting perturbations, the perturbed ground state remains a universal resource for MBQC.

### 2. The effective noise model construction: finitely correlated states

In this section, we will prove our effective noise model result, theorem 1, for a restricted class of ground states of infinite 1D chains. Specifically, we consider pFCS [21, 22]. A pFCS can be considered as the thermodynamic limit of the translationally invariant MPS $|\Psi_N\rangle$ generated by the MPS tensor $A$, on finite chains of $N$ sites with periodic boundary conditions, e.g.

$$|\Psi_5\rangle = \begin{array}{cccccc} A & A & A & A & A \end{array}$$

(1)
(here and throughout this paper, we use a graphical notation to represent the contraction of tensors, e.g. see [23, 24]). The MPS tensor $A$ must satisfy an additional condition known as injectivity, which is related to the exponential decay of the correlation functions; each of the finite-chain states $|\Psi_N\rangle$ (for sufficiently large $N$) is then the unique gapped ground state of a local frustration-free Hamiltonian [25].

We have several motivations for considering this class of ground states. Firstly, it is widely believed that pFCS capture the essential physics of gapped ground states of infinite 1D translationally invariant spin chains in general. (Note that, although the theorem regarding the efficient approximation of ground states of finite spin chains by MPS [26, 27] could be regarded as supporting this belief, we cannot use this theorem to draw any rigorous conclusions for our purposes here, since it does not hold that the MPS tensor $A$ can be kept fixed independently of the system size for a constant accuracy.) Secondly, the ideas leading to our effective noise model result find their simplest and most physically meaningful expression in this context. Finally, the proof presented here will play a dual role in our paper, as it can also be applied to arbitrary quantum states, provided that they satisfy a few extra criteria in common with pFCS. Thus, in order to establish the effective noise model result for general ground states, which we do in section 3, it will suffice to provide a separate proof of these criteria.

The outline of this section is as follows. We begin in sections 2.1 and 2.2 by reviewing the properties of the 1D cluster model and the nature of the SPT phase in which it is contained. In section 2.3, we review the results of [15] regarding the structure shared by pFCS ground states throughout the whole SPT phase. In section 2.4, we prove a key result: the standard adaptive measurement protocol acting on a ground state in the phase is equivalent to a non-adaptive dual process acting on a ‘topologically disentangled’ version of the ground state, which we refer to as the dual state. In section 2.5, we give a characterization of the dual process in the case that the original resource state is the exact cluster state. Finally, in section 2.6 we exploit the short range of the correlations in pFCS to construct the effective noise model for any pFCS ground states within the SPT phase, establishing theorem 1 for the case of pFCS ground states.

2.1. The one-dimensional (1D) cluster model in the absence of perturbations

Here we recall the properties of the 1D cluster model in the absence of perturbations. The Hamiltonian is

$$H = - \sum_i Z_{i-1} X_i Z_{i+1},$$

(2)

where $X_i$ denotes the Pauli $X$ operator acting on the $i$th site, and similarly for $Z_i$. With appropriate boundary conditions, the system has a unique ground state (the cluster state), and an energy gap of 2, independent of the system size.

This model has a global $Z_2 \times Z_2$ symmetry generated by the symmetry operations $\prod_{i \text{ even}} X_i$ and $\prod_{i \text{ odd}} X_i$. We consider this symmetry to be on-site, which is to say it acts on states as a unitary representation $U(g)$ of the symmetry group $G = Z_2 \times Z_2 = \{1, x, y, z\}$ (with $y = xz$), such that $U(g)$ acts as $U(g) = [u(g)]^{\otimes N}$, where $N$ is the number of sites (we group qubits into two-qubit sites in order to ensure this condition is satisfied; see figure 1). As we will see in section 2.2, the cluster model lies in a non-trivial SPT phase with respect to this symmetry, so that the cluster state cannot be smoothly deformed into a product state without breaking the symmetry [28].
The generators of the on-site $Z_2 \times Z_2$ symmetry of the 1D cluster model. Each dot is a qubit, and the shaded areas constitute two-qubit sites.

The 1D cluster state can be represented as a pFCS [4]. For our purposes we will take the MPS tensor $A_c$ to have the form

\[
A_c[++] = \mathbb{I}, \quad A_c[+-] = X, \quad A_c[-+] = Z, \quad A_c[--] = XZ = -iY.
\]

This is expressed with respect to a particular basis for a two-qubit site, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Here, and throughout this paper, we use the notation $A[\psi]$ to refer to the linear operator obtained from the MPS tensor $A$ by interpreting $|\psi\rangle$ as a linear operator (acting on states from the right), where $|\psi\rangle^*$ denotes the tensor obtained by complex conjugation from the rank-1 tensor corresponding to the state $|\psi\rangle$.

The MPS representation for the cluster state plays a crucial role in the correlation space picture [3, 4] for the operation of the cluster state as a quantum computational wire [29]. When a projective measurement is performed on a site, giving the outcome $|\psi\rangle$, this is interpreted as inducing an evolution $A[\psi]$ on a ‘correlation system’. In the case of the cluster state, for any qubit rotation $U$ about the $x$- or $z$-axis, one can find a product basis $\{|\alpha\rangle\}$ for a two-qubit site, such that

\[
A_c[\alpha] = B_\alpha U,
\]

where $B_\alpha$ is an outcome-dependent unitary byproduct operator. This byproduct can be accounted for by adjusting the basis for future measurements depending on the outcome of the current one.

2.2. Symmetry-protected topological order in finitely correlated states

Here, we will review the results of [17, 18] on the manifestation of SPT order in pFCS, and demonstrate that the 1D cluster model indeed lies in a non-trivial SPT phase with respect to the $Z_2 \times Z_2$ symmetry.

Consider some ground state which is invariant under the on-site representation $U(g) = [u(g)]^\otimes N$ of some symmetry group $G$, and which can be represented as a pFCS, as in equation (1). The tensor $A$ can be taken to obey a symmetry condition [18, 30]

\[
A[u(g)^\dagger |\psi\rangle] = \beta(g) W(g)^\dagger A[\psi] W(g),
\]

where $\beta(g)$ is a 1D linear representation of the symmetry group $G$, and $W(g)$ is a projective unitary representation of the symmetry group $G$. This means that

\[
W(g_1) W(g_2) = \omega(g_1, g_2) W(g_1 g_2)
\]

(6)
for some function $\omega$, called the factor system of the projective representation, which maps pairs of group elements to phase factors. By blocking sites, we can ensure that $\beta(g) = 1$ (however, for simplicity we will assume that $\beta(g) = 1$ without blocking). Equation (5) can then be represented graphically as

$$
\begin{array}{c}
| A \rangle \\
\mu(g)
\end{array}
= [W(g) | A | W(g)^\dagger].
$$

Observe that $W(g)$ can be multiplied by a $g$-dependent phase factor without affecting equation (7); a set of factor systems related by such a transformation is referred to as a cohomology class. The arguments of [17, 18] show that two such pFCS ground states correspond to the same cohomology class if and only if they are in the same symmetry-protected phase. Non-trivial cohomology classes (those not containing the trivial factor system $\omega(g_1, g_2) = 1$) correspond to phases with non-trivial SPT order.

As an example, consider the cluster model, and its $Z_2 \times Z_2$ symmetry. The on-site representation $u(g)$ of the symmetry is generated by

$$
\begin{align*}
&u(x) = X \otimes \mathbb{I}, \\
&u(z) = \mathbb{I} \otimes X,
\end{align*}
$$

and the MPS tensor is given by equation (3). It can be shown that the symmetry condition equation (7) is satisfied with the projective representation $W = V_P$, where $V_P$ is the Pauli representation

$$
V_P(1) = \mathbb{I}, \quad V_P(x) = X, \quad V_P(z) = Z, \quad V_P(y) = Y.
$$

This projective representation has non-trivial cohomology class, so that the cluster model lies in a non-trivial symmetry-protected phase.

2.3. Symmetry-respecting perturbations to the cluster state

Suppose we now consider a perturbation to the cluster Hamiltonian equation (2), such that the perturbed model still respects the $Z_2 \times Z_2$ symmetry and admits a pFCS ground state. Unless the perturbation is large enough to induce a phase transition, the MPS tensor $A$ corresponding to the perturbed pFCS ground state should still satisfy the symmetry condition equation (7), for some projective representation $W(g)$ with the same factor system as the Pauli projective representation $V$ (equation (10)).

The general form of the MPS satisfying these symmetry conditions was established in [15]. Here we will briefly review the relevant results from [15]. We observe that the Pauli representation satisfies a property which we refer to as maximal non-commutativity.

**Definition 1.** A projective representation $W(g)$ of an Abelian group $G$ is called maximally non-commutative if the subgroup $Z_W(G) \equiv \{g \in G : W(g) \text{ commutes with } W(h) \text{ for all } h \in G\}$ (which we can think of as the ‘projective centre’ of $G$) is the trivial subgroup.

Note that the subgroup $Z_W(G)$ is actually determined by the factor system $\omega$, since $W(g)W(h) = \omega(g, h)W(gh) = \omega(g, h)\omega(h, g)^{-1}W(h)W(g)$. Furthermore, it is the same for all factor systems within a given cohomology class. Much of the discussion in this paper can be

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applied to any SPT phase characterized by a finite Abelian symmetry group and a maximally non-commutative cohomology class.

An important consequence of maximal non-commutativity of a factor system is [15].

**Lemma 1.** For each maximally non-commutative factor system \( \omega \) of a finite Abelian group \( G \), there exists a unique (up to unitary equivalence) irreducible projective representation with factor system \( \omega \). The dimension of this irreducible representation is \( \sqrt{|G|} \).

Specifically, the Pauli representation \( V_P \) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) is the unique irreducible projective representation corresponding to its factor system. In general, throughout this paper, we will use \( V(g) \) to denote the unique irreducible projective representation for the current factor system. A consequence of lemma 1 is that, for a tensor satisfying the symmetry condition equation (7), the bond space decomposes as a tensor product of a \( \sqrt{|G|} \)-dimensional protected subsystem in which \( W(g) \) acts irreducibly as \( V(g) \) and a junk subsystem in which \( W(g) \) acts trivially, i.e.

\[
W(g) = V(g) \otimes I.
\]  

Thus the tensor \( A \) appearing the MPS representation of ground states in the symmetry-protected phase satisfies the symmetry condition

\[
\begin{array}{ccc}
A & V(g) & V(g)^\dagger \\
\hline
\hline
\hline
\end{array}
\]

Here we use a thick line (____) to represent the protected subsystem, and a dashed line (-----) to represent the junk subsystem. The protected subsystem enjoys several nice properties for storing and manipulating logical information in a quantum computation, as we now show.

Suppose we perform a projective measurement on one site in a simultaneous eigenbasis \( \{ |i \rangle \} \) (which is \( |++\rangle, |+-\rangle, |--\rangle, |--\rangle \) for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) cluster state symmetry), and obtain the outcome \( |i \rangle \). Then the resulting state on the remaining sites is found by replacing the original MPS tensor at the measured site by

\[
\begin{array}{ccc}
A & V(g) & V(g)^\dagger \\
\hline
\hline
\hline
\end{array}
\]

Now we make use of another consequence of maximal non-commutativity [15].

**Lemma 2.** Let \( u(g) \) be a linear on-site representation of a finite Abelian symmetry group \( G \), and let \( \omega \) be a maximally non-commutative factor system of \( G \). Then for each basis element \( |i \rangle \) in a simultaneous eigenbasis \( \{ |i \rangle \} \) of \( u(g) \), there exists a group element \( g_i \) such that

\[
\chi_i(g)V(g) = V(g_i)V(g)V(g_i)^\dagger, \quad \forall g \in G
\]

for any projective representation \( V(g) \) with factor system \( \omega \), where \( \chi_i(g) \) is the scalar representation of \( G \) such that \( u(g)|i \rangle = \chi_i(g)|i \rangle \).

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For the example of the cluster state symmetry, we have $g_{++} = 1$, $g_{+-} = x$, $g_{-+} = z$, $g_{--} = y$, as can readily be verified directly.

As was shown in [15], lemma 2 in conjunction with the symmetry condition equation (12) implies the decomposition $A[i] = V(g_i) \otimes \tilde{A}[i]$, represented graphically as

\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overline{V}(g_i)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\tilde{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i^*
\end{array}
\end{array}
\end{array}
\] (15)

for some tensor $\tilde{A}$. Another way of writing this result is that

\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overline{V}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\tilde{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i^*
\end{array}
\end{array}
\end{array}
\] (16)

where we have defined the tensor

\[
\begin{array}{c}
\begin{array}{c}
\overline{V}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\sum_i \left( \begin{array}{c}
\overline{V}(g_i)
\end{array} \right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i^*
\end{array}
\end{array}
\] (17)

Note that, from a quantum circuit perspective, this tensor can also be interpreted as a unitary controlled operation $\sum_i |i\rangle\langle i| \otimes V(g_i)$ coupling a site to an ancilla particle; hence the choice of notation. Conversely, any MPS tensor of the form equation (16) for some tensor $\tilde{A}$ will satisfy the symmetry condition equation (12). Following [31], we refer to the tensor $\tilde{A}$ as the degeneracy tensor; and to the tensor of equation (17), which is determined entirely by the symmetry, as the structural tensor.

From equation (15), we see that, in the correlation space picture, measuring in a simultaneous eigenbasis $\{|i\rangle\}$ leads to an evolution on the protected subsystem of correlation space given by an outcome-dependent unitary $V(g_i)$; this evolution is determined by the symmetry (hence the same throughout the SPT phase), and decoupled from the junk subsystem. Viewing the unitaries $V(g_i)$ as outcome-dependent byproducts of the measurements, which can straightforwardly be accommodated in a deterministic evolution using the standard techniques of MBQC, we say that the identity gate operates perfectly throughout the SPT phase. However, the result of measurement in any other basis is not fixed by the symmetry, and in general leads to the protected subsystem being coupled to the junk subsystem, so that the operation of other measurement-based gates will not be a robust property of the symmetry-protected phase.

2.4. The dual picture for measurement-based quantum computation (MBQC) on a 1D resource state

In order to deal with the randomness of measurement outcomes, the measurement protocol for MBQC with the cluster state needs to be adaptive: the outcome of the measurement on
one site will affect the measurement basis on other sites arbitrarily far away. In analysing the effect of this protocol when acting on a perturbed resource state, we would like to make an argument based on the locality of the perturbed Hamiltonian, but the non-local adaptivity of the measurement protocol poses a difficulty. Therefore, in this section, we develop an alternative characterization of the effect of the cluster state adaptive measurement protocol acting on a ground state in the symmetry-protected phase. We will show that this protocol is equivalent to a dual process acting on a related state, which we call the dual state. We will show that this dual process simply consists of a sequence of unitary interactions between selected sites (those corresponding to the locations of non-trivial gates) and an ancilla particle, with no adaptivity.

In our discussion of the dual process, we will represent a pFCS ground state on an infinite chain as a formal tensor network

\[ |\Psi\rangle = \cdots A A A A A \cdots. \tag{18} \]

This is not, of course, the mathematically rigorous way to describe pFCS, but we find it useful for facilitating understanding. In appendix A, we will discuss how to formulate similar arguments in the rigorous pFCS framework. Later on (in section 3), we will also be interested in finite chains; in that case, the arguments of this subsection can be applied more directly, given appropriate boundary conditions (specifically, the boundary conditions at the right edge should be as depicted in equation (50)).

The usefulness of the exact 1D cluster state (with MPS tensor \( A_c \)) as a quantum computational wire results from the fact that, for each gate \( U \) in a certain set, there exists a basis \( \{ |\alpha\rangle \} \) such that

\[ \frac{A_c}{|\alpha\rangle} = \frac{U B_\alpha}{|\alpha\rangle}, \tag{19} \]

where \( B_\alpha \) is the outcome-dependent unitary byproduct operator. When we measure one site projectively and obtain the outcome \( |\alpha\rangle \), the original MPS tensor \( A \) is replaced at the measured site by equation (19) in the tensor-network description of the resultant state.

In the case of the exact cluster state, the effect of the byproduct operator can be accounted for by adjusting the measurement basis for future measurements. This fact turns out to be closely related to the non-trivial SPT order, as we now demonstrate. Our discussion relies on the observation that, in the cluster state, the byproduct operators are Pauli operators. That is to say, it is always the case that \( B_\alpha \) is a scalar multiple of \( V(g_\alpha) \) for some \( g_\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_2 \). Hence, we can make use of the symmetry condition (which can be derived from equation (12))

\[ \frac{B_\alpha}{A_c} = \frac{A_c}{B_\alpha}, \tag{20} \]

where \( b_\alpha = u(g_\alpha) \). Applying this condition repeatedly shows that the byproduct operator can be displaced arbitrarily far to the right. In our formal tensor-network picture for an infinite chain, we consider that this process is continued indefinitely, so that the byproduct operator ‘disappears
out to infinity’, and is replaced with $b_\alpha$ acting on all sites to the right of the one on which the measurement took place, i.e.

$$\begin{align*}
  &B_\alpha A_c A_c A_c A_c \cdots = A_c A_c A_c A_c \cdots

to infinity’, and is replaced with $b_\alpha$ acting on all sites to the right of the one on which the measurement took place, i.e.

$$\begin{align*}
  &B_\alpha A_c A_c A_c A_c \cdots = A_c A_c A_c A_c \cdots

to infinity’, and is replaced with $b_\alpha$ acting on all sites to the right of the one on which the measurement took place, i.e.

$$\begin{align*}
  &B_\alpha A_c A_c A_c A_c \cdots = A_c A_c A_c A_c \cdots
\end{align*}$$

Hence, whenever we obtain the ‘wrong’ outcome for a measurement (i.e. the corresponding byproduct operator $B_\alpha$ is not the identity), we can recover the ‘correct’ resultant state by applying the correction $b_\alpha^\dagger$ to all the remaining sites on the right (equivalently, we can simply adjust the measurement basis for measurements on those sites).

Let us now examine what happens when we perform the same adaptive measurement protocol on a resource state that is not the exact cluster state. Consider a pFCS ground state contained with the same SPT phase as the cluster state, characterized by the Pauli representation of the group $Z_2 \times Z_2$. We will keep using the same measurement protocol as for the exact cluster state. (Our argument could be generalized to any pFCS ground state contained within any SPT phase characterized by a finite Abelian symmetry group $G$ and a maximally non-commutative cohomology class, so long as the adaptive correction appearing in the measurement protocol takes the same form as for the cluster state, i.e. application of $u(g_\alpha)$ to the sites on the right for some group elements $g_\alpha \in G$.) The resource state is then of the form

$$\begin{align*}
  |\Psi\rangle = \cdots A A A A A \cdots
\end{align*}$$

with the MPS tensor $A$ of the form equation (16). We now repeat the above argument, in reverse. We make use of the symmetry condition equation (12) in the form

$$\begin{align*}
  &A = B_\alpha^\dagger A B_\alpha
\end{align*}$$

from which we obtain

$$\begin{align*}
  \cdots A A A A A \cdots = \cdots A B_\alpha^\dagger A A A \cdots
\end{align*}$$
Therefore, we have shown that the process we actually perform, i.e. applying the measurement-dependent correction to the sites on the right of the one measured, is equivalent to a different process, in which the measurement-dependent correction is applied in the internal bond space of the MPS, as depicted in the right-hand side of equation (24). In a physical system, of course, we do not have direct access to the internal bonds of a tensor network state, so we could never perform the latter process directly; nevertheless, the two are equivalent.

Following the measurement and the adaptive correction, which we think of as being performed internally, as in the right-hand side of equation (24), the outcome of the measurement can be ‘forgotten’, i.e. we describe the resultant state of the system as the mixture of the right-hand side of equation (24) for all possible measurement outcomes. Without affecting the reduced state on the remaining unmeasured sites, for notational convenience we replace this mixture with a coherent superposition, i.e.

\[
\text{(25)}
\]

where we have defined the tensor

\[
\text{(26)}
\]

which we can also interpret as a unitary coupling $\sum_\alpha |\alpha\rangle \langle \alpha| \otimes B^\dagger_\alpha = \sum_\alpha |\alpha\rangle \langle \alpha| \otimes V(g_\alpha)^\dagger$ coupling a site to an ancilla particle. We now use the $k$ index throughout the paper to distinguish the ‘$G$’s resulting from different measurement operations. (The label $k$ refers to the site at which the measurement is being performed; we include this label to reflect the dependence on the measurement basis $\{|\alpha\rangle\}$ and byproduct operators $B_\alpha$, which will in general be different for each site at which a measurement is performed.)

Now, using the expression equation (16) for the MPS tensor $A$, we can write

\[
\text{(27)}
\]

where we have defined the tensor $G_k$ (which can also be interpreted as a unitary coupling between a site and the ancilla particle) according to

\[
\text{(28)}
\]
Figure 2. (a) After the adaptive measurement sequence, we can treat the resultant state as having the form shown. This is equivalent (b) to building $|\Psi\rangle$ from the dual state $|\tilde{\Psi}\rangle$ (the shaded box; see equation (29)) by unitary couplings to an ancilla particle.

We are now in a position to define our dual process. Suppose we perform a sequence of such adaptive measurements at successive sites, which at each site is described by the insertion of the tensor equation (26), as in equation (25). There will be a different coupling $G_k$ associated with each site $k$. As shown in figure 2, we find that the original adaptive measurement process, applied to the resource state $|\Psi\rangle$, is equivalent to a dual process applied to the dual state $|\tilde{\Psi}\rangle$. The dual state $|\tilde{\Psi}\rangle$ is defined to be the state built from the degeneracy tensor $\tilde{A}$, with the structural tensor discarded:

$$|\tilde{\Psi}\rangle = \cdots - A - A - A - A - \cdots .$$  \hspace{1cm} (29)

The dual process comprises a series of consecutive unitary interactions $G_k$ between individual sites $k$ and an ancilla particle.

There are several reasons why this ‘dual picture’ is a useful way to understand the operation of MBQC in 1D ground states. Firstly, the dual process lacks the long-range measurement adaptivity that is a characteristic of the original adaptive measurement protocol. Secondly, the perfect operation of the identity gate is automatically built in, because, for sites $k$ at which the adaptive measurement process at the given site is the one that corresponds in the exact cluster state to the identity gate (i.e. the measurement basis is the simultaneous eigenbasis $\{|i\rangle\}$ of the
symmetry, and the byproduct operators are $B_i = V(g_i)$, where the $g_i$ are the group elements appearing in equation (15), the corresponding coupling is trivial, $G_k = \mathbb{I}$.

The final motivation for the dual picture is that the dual state on which it is based has some physical significance in its own right, and retains some key properties of the original resource state. For example, if the original MPS tensor $A$ generates a pFCS, then so does $\tilde{A}$, and the respective correlation lengths obey the inequality $\tilde{\xi} \leq \xi$ (see appendix A). Additionally, in appendix B we will show how our dual state can be obtained from the original ground state through a generalization of the unitary that was introduced by Kennedy and Tasaki [32, 33] to transform the SPT Haldane phase [34–36] into a local symmetry-breaking phase; this unitary has recently been described as a ‘topological disentangler’ [37], and in some sense we can think of the dual state as being a topologically disentangled version of the original resource state.

2.4.1. The dual process for initialization and readout in the 1D cluster state. Above we only discussed measurement sequences corresponding to unitary gates in correlation space. A complete scheme for using a 1D resource state as a quantum computational wire also includes measurement sequences corresponding to initialization (i.e. discarding the current state in correlation space and replacing it with a fixed state), and readout (i.e. making the state in correlation space available as the physical state of one qubit). We now describe briefly how the measurement protocols used on the 1D cluster state for these purposes can be accommodated in our framework.

Initialization. The initialization procedure involves measuring a site in the computational basis $\{\ket{00}, \ket{01}, \ket{10}, \ket{11}\}$. In this basis, the MPS tensor $A_c$ for the exact cluster state takes the form

$$A_c[00] = \ket{0}\bra{0}, \quad A_c[01] = \ket{1}\bra{0}, \quad A_c[10] = \ket{0}\bra{1}, \quad A_c[11] = -\ket{1}\bra{1}. \quad (30, 31, 32, 33)$$

The randomness of measurement outcomes can therefore be accounted for by applying the appropriate outcome-dependent correction operator in correlation space following the measurement: $B_{00}^\dagger = B_{10}^\dagger = \mathbb{I}$, $B_{01}^\dagger = B_{11}^\dagger = X$. Since the correction operators are Pauli operators, the above discussion applies without change.

Readout. The standard readout procedure for the cluster state involves measuring the second qubit of a two-qubit site in the computational basis, then applying an outcome-dependent correction operator to the first qubit, which acts as the output. Provided that we are only interested in the final state of the output qubit, this procedure is equivalent to a coherent correction operator coupling the two qubits in the site (specifically, it is a controlled-Z gate $C_Z = \ket{0}\bra{0} \otimes \mathbb{I} + \ket{1}\bra{1} \otimes Z$). Carrying through a similar argument to that given above for unitary gates, section 2.4, we obtain the same result, but with the interaction $G_k$ in the dual process between the site $k$ in question and the ancilla particle given by

$$G_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{array}{c} \Downarrow \\ \mathcal{Y} \\ \bullet \end{array} \quad (34)$$
2.5. Matrix-product states (MPS) of minimal bond dimension and the dual picture

As an example of the general formalism introduced in section 2.4, here we will examine the form of the couplings \( G_k \) appearing in the dual process (equation (28)), in the particular case that the resource state is an MPS with bond dimension \( D = \sqrt{|G|} \), where \( G \) is the symmetry group characterizing the symmetry-protected phase. Given that the dimension of the protected subsystem is \( \sqrt{|G|} \) (by lemma 1), this is the smallest possible value of \( D \), and corresponds to the absence of a junk subsystem (or, more precisely, a junk subsystem of dimension 1).

In particular, the 1D cluster state is of this type. In general, the MPS tensor \( A \) for such an MPS must be of the form
\[
A[i] = \tilde{A}[i] V(g_i),
\]
where the \( \tilde{A}[i] \) here are scalars. It follows that the dual of such a state is a product state, with each site in the state \( |\phi\rangle = \sum_i \tilde{A}[i]|i\rangle \). (We choose the normalization for the MPS tensor \( A \) so that \( \langle \phi|\phi\rangle = 1 \).) Therefore, the effect of the dual process acting on the dual state results from a series of independent interactions of the form
\[
\begin{array}{c}
G_k \\
|\phi\rangle
\end{array}
\]
We recall that, in the correlation space picture of quantum computational wires, a quantum state can serve as a resource for executing a unitary gate \( U \) if there exists some basis \( \{|\alpha\rangle\} \) such that
\[
A[\alpha] = \beta_\alpha B_\alpha U,
\]
for some set of unitary byproduct operators \( B_\alpha \) and scalars \( \beta_\alpha \). We will now show how this property manifests itself in the dual picture, for the class of states considered here. We make use of the representation for the MPS tensor \( A \) as
\[
\begin{array}{c}
A \\
\phi
\end{array}
\]
It follows that, at a site \( k \) measured in the basis \( \{|\alpha\rangle\} \), with the byproduct operators \( B_\alpha \), we have
\[
\begin{array}{c}
G_k \\
|\phi\rangle
\end{array} = \begin{array}{c}
A \\
\phi
\end{array}
\]
\[
\begin{array}{c}
A \\
\phi
\end{array} = \begin{array}{c}
U \\
\phi
\end{array}
\]
where \( |\varphi\rangle = \sum_\alpha \beta_\alpha |\alpha\rangle \). (It can be shown that our choice of normalization ensures that \( \langle \varphi|\varphi\rangle = 1 \).) Here the first equality follows from equation (38) and the definition of \( G \); and the
second inequality follows by equation (37). Thus, we have shown that in the dual picture the gate $U$ simply acts on the ancilla particle.

Next we will do a similar analysis for the initialization and readout procedures specific to the 1D cluster state.

**Initialization.** Recall the discussion of initialization in section 2.4. We make use of the form of the MPS tensor $A_C$ in the computational basis, equations (30)–(33), multiplied by the appropriate normalization factor as discussed above. Thus, for the site $k$ at which initialization takes place, we find that

$$\Gamma_1 = \frac{1}{\sqrt{2}} (|00\rangle \langle 0| + |01\rangle \langle 0| + |10\rangle \langle 1| - |11\rangle \langle 1|)$$

(thanks to our choice of normalization, we find that $\Gamma$ is an isometry, i.e. $\Gamma^\dagger \Gamma = I$). Therefore, applying the measurement sequence for initialization leads to the ancilla system getting initialized in the state $|0\rangle$, as we would expect.

**Readout.** From the definition of the operator $G_k$ in the case of sites $k$ at which readout takes place (equation (34)), we find that (here we separate a site into its two constituent qubits, each denoted by a thick line)

$$|I\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

is the canonical maximally-entangled state. Thus the state of the ancilla qubit indeed gets transferred onto the output qubit.

### 2.6. MBQC on a perturbed resource state simulates a noisy quantum circuit

In the previous subsection, we saw how measurements on an MPS of minimal bond dimension correspond to quantum gates. Now we will consider what happens when we perform the same measurement sequences on a perturbed resource state, assuming that the perturbed state remains within the same SPT phase. We will find that measurements on such a perturbed cluster state simulate the same quantum circuit, but with noisy gates. The noise is described by application of a completely positive, trace preserving (CPTP) noise superoperator following each gate.

In section 2.5, we were able to treat each gate independently in the case of the unperturbed cluster state because the dual state was a product state, $|\tilde{\Psi}\rangle = |\phi\rangle \otimes \tilde{N}$. This will no longer be true once we introduce perturbations, but we still want to treat gates independently. Towards this
Figure 3. (a) As long as the locations of non-trivial gates (sites \( k \) and \( l \) in this diagram) are separated by a distance much greater than the correlation length \( \tilde{\xi} \), the reduced state on those locations will be approximately a product state, and the dual process then reduces (b) to a sequence of independent interactions.

end, we recall that for a site for which the corresponding sequence is that for the identity gate, the associated coupling \( G_k \) in the dual picture between that site and the ancilla particle is trivial. Therefore, such a site can be traced out from the beginning without affecting the final state of the output qubit. That is to say, we only need to consider the reduced state \( \tilde{\rho} = \text{Tr}_{\text{trivial sites}} |\tilde{\Psi}\rangle \langle \tilde{\Psi}| \) on the remaining sites, which are those corresponding to non-identity gates (we refer to these as the non-trivial sites). We are free to choose our measurement protocol to ensure that the distance between any two non-trivial sites is much greater than the correlation length \( \tilde{\xi} \). For pFCS, it is then straightforward to show that \( \tilde{\rho} \) is approximately a product state \( \tilde{\rho}_{\text{prod}} = \otimes_k \tilde{\rho}_k \) over the non-trivial sites, or more precisely

\[
\|\tilde{\rho} - \tilde{\rho}_{\text{prod}}\|_1 \leq mf(R),
\]

where \( \| \cdot \|_1 \) denotes the trace norm, \( m \) is the number of non-trivial sites, \( R \) is the minimum distance between any two non-trivial sites, and \( f(R) \) is a function related to the transfer channel of the pFCS, behaving asymptotically as \( f(R) = \mathcal{O}[\exp(-R/\tilde{\xi})] \) for large \( R \), where \( \tilde{\xi} \) is the correlation length associated with the pFCS.

We first consider the case where \( \tilde{\rho} = \tilde{\rho}_{\text{prod}} \) exactly. Because \( \tilde{\rho} \) is then a product state, we find, as in the previous subsection, that the dual process acting on the dual state is again effectively a sequence of independent interactions, this time of the form

\[
(46)
\]

(see figure 3). Let us suppose that \( G_k \) results from the measurement sequence corresponding to a unitary gate \( U_k \). Then, after tracing out the physical site, equation (48) corresponds to an
evolution on the ancilla qubit described by the CPTP map

\[ A_k(\sigma) = \text{Tr}_{\text{physical site}} \left( G_k(\sigma \otimes \tilde{\rho}_k) G_k^\dagger \right). \]  

(47)

As we saw in section 2.5, in the absence of perturbations to the cluster state, \( \tilde{\rho}_k = \ket{\phi}\bra{\phi} \) and \( A_k = U_k \), where \( U_k(\sigma) = U_k \sigma U_k^\dagger \). In general we can write \( A_k = \mathcal{E}_k \circ U_k \), where \( \mathcal{E}_k \) is a noise superoperator for which it is straightforward to show that

\[ \| \mathcal{E}_k - I \|_\diamond \leq \| \tilde{\rho}_k - \ket{\phi}\bra{\phi} \|_1, \]  

(48)

where \( \| \cdot \|_\diamond \) is the diamond norm on superoperators [38]. The cases where \( G_k \) corresponds to initialization or readout are analogous. Therefore we have shown (in the case \( \tilde{\rho} = \tilde{\rho}_{\text{prod}} \)) that the measurement protocol on the perturbed cluster state reproduces the desired quantum circuit, except that each gate (as well as the initialization and readout steps) is accompanied by some associated noise. Furthermore, if the perturbation is sufficiently small, then the reduced states \( \tilde{\rho}_k \) will be close to \( \ket{\phi}\bra{\phi} \) (see appendix D for the proof), so that the noise will be weak, in the sense that \( \mathcal{E}_k \) is close to the identity superoperator in the diamond norm.

In the general case, in which \( \tilde{\rho} \) and \( \tilde{\rho}_{\text{prod}} \) are not equal, but are \( \epsilon \)-close in the trace distance, we just need to observe that the reduced state of the output qubit following the dual process can be obtained from \( \tilde{\rho} \) by application of some CPTP superoperator, which we call \( B \). From the contractivity property of the trace distance, it follows that \( \| B(\tilde{\rho}) - B(\tilde{\rho}_{\text{prod}}) \|_1 \leq \| \tilde{\rho} - \tilde{\rho}_{\text{prod}} \|_1 \leq \epsilon \). Therefore, the effective noisy quantum circuit description correctly describes the final state of the output qubit up to an accuracy \( \epsilon \). Note that, because the bound equation (45) depends on the number of non-trivial gates \( m \), it will be necessary to have the separation \( R \) scale with \( m \) in order to obtain a fixed accuracy \( \epsilon \), but only logarithmically; indeed, the minimum separation required to achieve an accuracy \( \epsilon \) scales like \( R_{\min} / \tilde{\xi} = O[\log(m/\epsilon)] \).

2.7. Summary of section 2

In section 2, we have presented, within the context of pFCSs, the main ideas leading to our effective noise model construction. Our discussion has hinged around the ‘dual state’ which we associated with each ground state carrying the appropriate SPT order. Loosely speaking, we can think of the entanglement in SPT-ordered ground states as comprising ‘topological’ and ‘non-topological’ components intertwined. The topological component is fixed throughout the phase and is responsible for the distinctive characteristics of the SPT phase, such as the degeneracy in the entanglement spectrum [39], the diverging localizable entanglement length [40, 41] and the perfect operation of the identity gate. One can think of the dual state as being obtained from the original ground state by a topological disentangler, ‘separating out’ the topological component of the entanglement and leaving only the non-topological component [37].

In this paper, the importance of the dual state is due to the following fact, which we established in section 2.4: the cluster state adaptive measurement protocol, when applied to an SPT-ordered ground state, couples in a natural way to the topological component of the entanglement, and the effect is thus equivalent to a ‘dual process’ (with a simpler structure) acting on the dual state. This result gives rise to an effective quantum circuit description describing the outcome of the measurement protocol applied to any SPT-ordered ground state (e.g. see figure 3(a)). The action of non-trivial gates is determined by an interaction with a single site in the dual state, and perturbations to the dual state give rise to noisy gates. As long as the locations of non-trivial gates are sufficiently separated, the reduced state on the sites relevant
for the gate operation will be a product state, and this corresponds to independent noise acting on each gate in the effective circuit description.

3. The effective noise model construction: general ground states

In this section, we will extend our characterization of the effective noise model to any ground state within the symmetry-protected phase, without reference to FCSs. Instead of starting from scratch, we will build on the results of section 2, as follows. We formulate a condition that we believe (on physical grounds) to be satisfied for any system within the symmetry-protected phase. We will show that this condition leads to a construction for the dual state of any ground state in the SPT phase, independently of the pFCS formalism. Furthermore, given an exact MPS representation for the dual state (which always exists, albeit possibly with a bond dimension exponentially large in the system size), we show that one can construct a corresponding MPS representation for the original ground state, such that the arguments of section 2 can be applied without significant change. In order to establish the approximate factorization condition equation (45) in the case of general ground states, we will show that the dual state is (like the original resource state) the gapped ground state of a local Hamiltonian, which can be constructed in a straightforward way from the original Hamiltonian. This will allow us to establish the approximate factorization condition equation (45) without assuming that the dual state has a pFCS structure.

3.1. Symmetry-protected topological order and boundary conditions

Because we are considering general ground states, we can no longer make direct use of the characterization of SPT order in FCSs of [17, 18]. Instead, we adopt the perspective in which SPT order is related to the fractionalized edge modes associated with open boundary conditions [36]. Our discussion will, out of necessity, be physically motivated rather than mathematically rigorous, but will suggest the formulation of the precise assumptions under which the rigorous results of this paper can be proven.

Consider a 1D chain with open boundary conditions, with symmetry-respecting interactions such that, in the bulk, there is no symmetry-breaking and a finite energy gap for excitations. It is still possible that the energy gap for edge excitations may be much smaller than the bulk gap (or even zero). We denote by $\mathcal{P}$ the subspace comprising the low-lying edge states. If the chain is sufficiently long, we expect that the gap in the bulk should ensure that $\mathcal{P}$ decomposes a tensor product of degrees of freedom associated with the left and right edges respectively, $\mathcal{P} = \mathcal{P}_l \otimes \mathcal{P}_r$, and the restriction $H_\mathcal{P}$ of the Hamiltonian $H$ onto this subspace is a sum of terms acting on each edge: $H_\mathcal{P} = h_1 \otimes I + I \otimes h_1$.

For our purposes, we are interested in how the edge states transform under the symmetry. Because $\mathcal{P}$ is spanned by a set of energy eigenspaces, and the Hamiltonian commutes with the representation of the symmetry $U(g)$, it follows that $\mathcal{P}$ must be an invariant subspace for $U(g)$. We write the operation of $U(g)$ restricted to $\mathcal{P}$ as $U_\mathcal{P}(g)$. We expect that, for sufficiently long chains, the symmetry should act independently on the respective edge states, i.e.

$$U_\mathcal{P}(g) = V_l(g) \otimes V_r(g).$$

By assumption, $U(g)$, and hence $U_\mathcal{P}(g)$, is a linear representation of the symmetry group $G$. It follows that $V_l(g)$ and $V_r(g)$ are in general projective representations of $G$, and if $V_l(g)$ has factor
system $\omega$, then $V_i(g)$ must have factor system $\omega^{-1}$. We are free to transform $V_i(g) \rightarrow \beta(g)V_i(g)$, $V_i(g) \rightarrow \beta^{-1}(g)V_i(g)$ for any $g$-dependent phase factors $\beta(g)$ without affecting equation (49), but the cohomology class $[\omega]$ is uniquely determined. Furthermore, it is intuitively clear that any continuous symmetry-respecting variation in the Hamiltonian cannot change the cohomology class $[\omega]$, except at a phase transition (where the gap closes in the bulk, and the left- and right- edge modes need no longer be non-interacting). Therefore, we have an alternative characterization of the SPT phase corresponding to a cohomology class $[\omega]$: it comprises the systems where a left edge is associated with emergent edge states transforming projectively under the symmetry with cohomology class $[\omega]$. This is a generalization of the well-known observation that systems in the $SO(3)$-invariant Haldane phase have emergent spin-1/2 degrees of freedom at the edges [42, 43].

In non-trivial SPT phases, the edge interactions $h_l$ and $h_r$ (and therefore the overall Hamiltonian $H$) will always have degenerate ground states, due to the fact that non-trivial projective representations cannot be 1D. On the other hand, we expect, at least in the case that the symmetry group $G$ is Abelian, that a non-degenerate ground state can be recovered by introducing terminating particles at the left and right edges, transforming projectively under the symmetry with factor systems $\omega^{-1}$ and $\omega$ respectively (see figure 4). This is because, loosely speaking, these terminating particles can couple to the edge modes, with the composite system at each edge transforming under a linear representation (and therefore, in the case of an Abelian symmetry group, generically having a non-degenerate ground state). For example, the ground state of a spin chain in the Haldane phase can be made non-degenerate through coupling to spin-1/2 particles at the edges. Conversely, if the terminating particles do not transform with the cohomology classes $[\omega^{-1}]$ and $[\omega]$ respectively, then the degeneracy cannot be removed completely because there is still a non-trivial projective symmetry transformation at each edge.

**Figure 4.** (a) The low-lying energy subspace $\mathcal{P}$ of a 1D chain with open boundary conditions, in an SPT phase characterized by the cohomology class $[\omega]$, decomposes as a tensor product of ‘emergent edge modes’ (red) associated with each end, transforming projectively under the symmetry. (b) The degeneracy can be removed by coupling terminating particles (green) at each end, leading to an effective coupling to the edge modes.
Thus, we have arrived at yet another characterization of SPT order, which we state as a conjecture in the absence of a rigorous proof.

**Conjecture 1.** A 1D chain respecting an on-site representation of an Abelian symmetry group $G$ is in the SPT phase characterized by cohomology class $[\omega]$ if and only if the following condition is satisfied.

**Condition 1.** The finite-chain ground state can be made non-degenerate and gapped by the inclusion of symmetry-respecting interactions coupling the left and right edges of the chain to terminating particles transforming projectively under the symmetry, with factor systems $\omega$ and $\omega^{-1}$, respectively.

In any case, in the remainder of this section, we will consider systems satisfying condition 1. Specifically, all the results will apply to finite chains with the appropriate edge couplings imposed to ensure a non-degenerate gapped ground state. This will prove convenient for our analysis, but the properties of the system in the bulk should not, of course, depend on the boundary conditions.

Note also that, in the case of a system with the interactions governed by the parent Hamiltonian of a pFCS (generated by an MPS tensor satisfying the symmetry condition equation (7) corresponding to the symmetry-protected phase), condition 1 can easily be established directly. Furthermore, the stability theorem of [44] ensures that condition 1 remains true for sufficiently small symmetry-respecting perturbations of such models, regardless of the validity of conjecture 1.

### 3.2. The general construction for the dual state; exact MPS representation of symmetry-protected topological-ordered ground states

Recall that in section 2.4, we defined the dual state in the context of pFCS. Here, we will give an analogous construction for the dual state corresponding to a general ground state within a symmetry-protected phase, provided that the phase is characterized by a finite Abelian group $G$ and a maximally non-commutative cohomology class $[\omega]$. The construction applies to a finite chain, with the appropriate boundary conditions as discussed in section 3.1. This construction will then allow us to express the original ground state as an MPS, with the MPS tensors satisfying an appropriate symmetry condition.

We consider a finite chain coupled to terminating particles, such that the overall system is invariant under the symmetry $U(g) = V^*(g) \otimes [\mu(g)]^\otimes N \otimes V(g)$. Here we have taken the right terminating particle to transform under $V(g)$, the unique irreducible projective representation with factor system $\omega$, and the left terminating particle under $V^*(g)$ ($V^*(g)$ is the operator obtained from $V(g)$ by complex conjugation of the matrix elements in some basis; observe that $V^*(g)$ is a projective representation of $G$ with factor system $\omega^{-1}$).

The natural analogues in the current setting (finite chains, with the specific choice of boundary conditions) of the pFCS ground states which we considered in section 2 are states of the form

$$|\Psi\rangle = \begin{array}{cccccc}
L & A & A & A & A & R
\end{array}$$

(50)
for some end vectors \( |L \rangle \) and \( |R \rangle \), and where the MPS tensor \( A \) satisfies the symmetry condition equation (12) (which can be shown to ensure the invariance of the state under \( U(g) \)). Given the decomposition equation (16) for the MPS tensor \( A \), it follows that the dual state can be obtained from the original ground state by a sequence of unitary interactions between individual sites and the terminating particle on the right (see figure 5); we can think of the overall unitary transformation \( \mathcal{D} \) as a ‘topological disentangler’. Specifically, we have \( \mathcal{D} |\Psi \rangle = |\tilde{\Psi} \rangle \otimes |I \rangle \), where \( |I \rangle = \sum_{k=1}^{D} |k \rangle |k \rangle \) is the canonical maximally entangled state between the terminating particles.

We will now show that, for a general gapped symmetry-respecting ground state \( |\Psi \rangle \) (not necessarily in the MPS form equation (50)), it remains the case that \( \mathcal{D} |\Psi \rangle = |\tilde{\Psi} \rangle \otimes |I \rangle \) for some state \( |\tilde{\Psi} \rangle \) on the non-terminating sites; this will serve as the definition of the dual state \( |\tilde{\Psi} \rangle \) for general ground states.

We observe that the original ground state \( |\Psi \rangle \) must be invariant under the global symmetry operation \( U(g) \), i.e.

\[
U(g) |\Psi \rangle = \alpha(g) |\Psi \rangle.
\]  

(Without loss of generality, we can set \( \alpha(g) = 1 \) by absorbing it into the definition of the symmetry\(^2\).) This implies that the state \( \mathcal{D} |\Psi \rangle \) is invariant under \( \mathcal{D} U(g) \mathcal{D}^\dagger \). Let us examine what

\(^2\) Specifically, we replace the action of the symmetry \( V(g) \) on the right terminating particle with its equivalent under a unitary transformation (by lemma 1 of [15]), \( \alpha^{-1}(g) V(g) \).
this ‘dual’ symmetry looks like. We observe that

\[
D_1[u(g) \otimes V(g)]D_1^\dagger = \sum_i \chi_i(g)|i\rangle \otimes V(g)\rangle V(g)\langle i\langle V(g) \otimes V(g) \langle i\langle \rangle (52)
\]

\[
= \sum_i |i\rangle \langle i\langle V(g) \otimes V(g) \langle i\langle \rangle (53)
\]

\[
= I \otimes V(g),
\]

where

\[
D_1 = \sum_i |i\rangle \langle i\langle V(g) \otimes \langle i\langle \rangle (55)
\]

is the interaction from which \( D \) is built; to get to equation (53), we made use of equation (14). From this, one can show that

\[
DU(g)D^\dagger = V^*(g) \otimes I_{\otimes N} \otimes V(g).
\]

(56)

It it straightforward to show (using the irreducibility of \( V \)) that invariance of a state under the right-hand side of equation (56) implies that it must be of the form \( D|\Psi\rangle = |\tilde{\Psi}\rangle \otimes |I\rangle \) for some state \( |\tilde{\Psi}\rangle \), as required.

It is now straightforward to construct an appropriate MPS representation for a general ground state. Indeed, let us consider an MPS representation for the dual state \( |\tilde{\Psi}\rangle \) of the form

\[
|\tilde{\Psi}\rangle = \tilde{A}_1 \cdots \tilde{A}_2 \cdots \tilde{A}_3 \cdots \tilde{A}_4 \cdots \tilde{A}_5
\]

(57)

We choose this representation to be exact; this may require the bond dimension to be very large (scaling exponentially in the system size), but that is of no importance to us. Then we have

\[
D|\Psi\rangle = |I\rangle \otimes |\tilde{\Psi}\rangle
\]

(58)

Now we can apply the inverse transformation \( D^\dagger \) to obtain

\[
|\Psi\rangle = \tilde{\tilde{A}}_1 \cdots \tilde{\tilde{A}}_2 \cdots \tilde{\tilde{A}}_3 \cdots \tilde{\tilde{A}}_4 \cdots \tilde{\tilde{A}}_5
\]

(60)

This is a representation of \(|\Psi\rangle\) as an MPS, with each of the shaded regions corresponding to an MPS tensor \( A \) of the form equation (16), and hence satisfying the symmetry condition corresponding to the symmetry-protected phase. In addition, we should take note of the boundary conditions at the right edge. These boundary conditions ensure that the arguments of section 2.4 apply without any need to invoke an infinite-system limit.
3.3. The dual state as the ground state of a local Hamiltonian

In the previous subsection, we constructed the dual state for any ground state in the symmetry-protected phase. The original ground state is, by assumption, the gapped ground state of a local Hamiltonian. In this subsection we will show that this is also true of the dual state. That is, starting from the original Hamiltonian $H$, we construct another local Hamiltonian $\tilde{H}$ for which the dual state is the gapped ground state.

We start by proving a useful fact about the unitary transformation $D$ introduced in the previous section: although it is in general non-local, it maps symmetry-respecting local observables (i.e. those supported on a small set of sites of finite size) to local observables. Indeed, let us consider some local observable $h$; we will show that $DhD^\dagger$ is also local. For concreteness, we suppose that $h$ acts on two adjacent sites somewhere in the bulk. Now, observe that

$$h' = V(h)V'$$

where

$$\tilde{h}' = \tilde{V}(h)\tilde{V}'$$

By means of equations (62) and (54), it can be verified that if $h$ commutes with the symmetry, i.e.

$$V(g)hV(g) = hV(g)V(g)$$

then

$$\tilde{V}(g)\tilde{h}\tilde{V}(g) = \tilde{h}\tilde{V}(g)\tilde{V}(g)$$

Since $V(g)$ is an irreducible projective representation, equation (64) implies (by Schur’s lemma) that $\tilde{h}$ acts trivially on the terminating particle, i.e.

$$\tilde{h} = \tilde{h}$$
From equations (61) and (65), we get the equality pictured. This shows that \( Dh = \tilde{h} D \), or equivalently \( DhD^\dagger = \tilde{h} \).

Now, using equations (60) and (65), we find that \( DhD^\dagger = \tilde{h} \), where \( \tilde{h} \) acts on the same two sites as \( h \) (see figure 6). Thus, although the duality transformation \( D \) is non-local, we have shown that it maps local symmetry-respecting operators to local operators, as promised. The exception is operators \( h \) at the left edge, which act non-trivially on the left terminating particle; in that case, the above argument breaks down, but we can observe directly from the structure of \( D \) that \( DhD^\dagger \) is supported on the union of the support of \( h \) and the right terminating particle. For operators \( h \) acting non-trivially on the right terminating particle, the argument must be adjusted, but the conclusion that \( DhD^\dagger \) is supported on the support of \( h \) still holds.

We are now in a position to construct the Hamiltonian for which the dual state \( |\tilde{\Psi}\rangle \) is the gapped ground state. We observe that \( DhD^\dagger \) has \( |I\rangle \otimes |\tilde{\Psi}\rangle \) as its gapped ground state; however, it includes terms acting non-trivially on the terminating particles. We define a Hamiltonian acting only on the intermediate sites according to \( \tilde{H} = \langle I|DhD^\dagger|I\rangle \equiv \mathcal{F}(H) \); by the locality result proven above, each local interaction term in \( H \) corresponds to a local term in \( \tilde{H} \) supported on the same set of sites. It can be shown that \( |\tilde{\Psi}\rangle \) is the unique ground state of \( \tilde{H} \), and that the gap is at least as large as that of \( DhD^\dagger \), or equivalently \( H \).

### 3.4. The factorization condition for general ground states

Recall that the other condition that needed to be satisfied in order to apply the arguments of section 2 for general ground states was that the factorization condition for the reduced density...
operator $\hat{\rho}$ on the non-trivial sites in the dual state

$$\hat{\rho} \approx \bigotimes_k \hat{\rho}_k \equiv \hat{\rho}_{\text{prod}} \quad (66)$$

should be satisfied when the non-trivial sites are sufficiently separated from each other. Recall that, for the case of pFCS, one can prove the bound

$$\|\hat{\rho} - \hat{\rho}_{\text{prod}}\|_1 \leq m f(R), \quad (67)$$

with $m$ the number of non-trivial sites, and $f(R)$ a function scaling asymptotically as $f(R) = \mathcal{O}[\exp(-R/\tilde{\xi})]$, where $\tilde{\xi}$ is the correlation length in the dual state. We conjecture that equation (67) should be a general property of all gapped ground states of local Hamiltonians. However, we have only been able to rigorously prove the weaker bound

$$\|\hat{\rho} - \hat{\rho}_{\text{prod}}\|_1 \leq m d^2 f(R), \quad (68)$$

where $f(R)$ is as before, and $d$ is the dimension of the Hilbert space at each site; see appendix C for the proof.

Note that if we assume only the weaker bound equation (68), then the separation between non-trivial sites will need to scale more rapidly with the number of gates $m$: we find that the minimum separation $R_{\text{min}}$ required for an accuracy $\epsilon$ scales like

$$R_{\text{min}}/\tilde{\xi} = \mathcal{O}(m) + \mathcal{O}([\log(1/\epsilon)]). \quad (69)$$

This still implies that the number of measurements need scale only as a polynomial in the number of non-trivial gates.

3.5. Non-zero temperature

The formulation of the dual state as the ground state of a dual Hamiltonian extends naturally to non-zero temperature: under the topological disentangler $D$, the thermal state of the original Hamiltonian $H$ maps to the thermal state of a dual Hamiltonian $\tilde{H}$. Furthermore, it can be shown that an appropriate adaptive measurement protocol acting on the thermal state of the original Hamiltonian is equivalent to a non-adaptive dual process (of the same form as in the zero-temperature case), acting on the thermal state of the dual Hamiltonian.

However, it does not appear possible to construct a Markovian effective noise model for non-zero temperature using the same techniques as for zero temperature. The reason is that our arguments were based on the assumption that the reduced state $\hat{\rho}_k$ on each of the non-trivial sites in the dual state does not differ greatly from its value in the dual of the unperturbed resource state. This is indeed the case for small local perturbations to the Hamiltonian (as we prove in appendix D), but it need not be true for non-zero temperature. For example, consider the 1D Ising model, with Hamiltonian

$$- \sum_i Z_i Z_{i+1} + Z_1 \quad (70)$$

(we include the $Z_1$ term to select out a unique ground state). In this model, it can be shown (e.g. using the transfer matrix method) that the reduced state on a single spin changes discontinuously as soon as the temperature is switched on (this is closely related to the disappearance of the magnetic order in the 1D Ising model at non-zero temperature). Given the structure of the dual Hamiltonian as discussed in appendix B, there is good reason to believe that it will exhibit a similar phenomenon.
The difficulty of treating thermal states in our framework should not be surprising, as the dual process has the perfect operation of the identity gate built in, whereas the cluster model is not expected to have a long-range identity gate at non-zero temperature. On the other hand, there exists a measurement protocol for a three-dimensional cluster model that retains the perfect operation of the identity gate at sufficiently small non-zero temperatures [45]. Therefore, if the dual process description could be extended to measurement protocols such as this one, then it might be expected that the dual Hamiltonian would possess an ordered phase that persists at non-zero temperature, such that the local reduced state varies continuously with temperature up to the phase transition.

4. Two-dimensional systems and fault tolerance

The equivalence we demonstrated in sections 2 and 3, between MBQC on perturbed resource states and noisy quantum circuits, opens up the possibility of exploiting the results in the literature on fault-tolerant quantum computation with noisy quantum circuits. Here, we will extend the results of the previous sections to the 2D cluster model, which, unlike the 1D models considered previously, is a universal resource for quantum computation. We will again find that, provided the perturbation to the Hamiltonian respects a certain symmetry, MBQC using the perturbed ground state as a resource is equivalent to a noisy quantum circuit. We will show that the noise in this effective circuit description has no correlations in time (as in the previous section), nor any correlations in space. This reduction to local, Markovian noise will allow us to invoke the threshold theorem to deduce that, provided the perturbation respects the symmetry and is sufficiently small, the perturbed ground state remains a universal resource for MBQC.

It should be emphasized that, although we make use of the theory of fault-tolerant quantum computation, our final result cannot be described as a fault tolerance result for MBQC, since it applies only to symmetry-respecting perturbations, and we assume noiseless operation of the measurement protocol.

4.1. The ‘quasi-1D’ model

Here, we make a first attempt at generalizing the 1D results to a 2D model that is universal for quantum computation. The ground state of the model we introduce here is not strictly a universal resource for MBQC unless we allow non-single-qubit measurements; however the discussion here will serve as a stepping stone for consideration of the 2D cluster model in section 4.2.

In the absence of perturbations, the 2D model we consider involves $N$ uncoupled 1D cluster states arranged in the second dimension, as shown in figure 7. The Hamiltonian acting on each chain is simply the 1D cluster Hamiltonian. For generality we also assume the existence of some uncoupled qubits, each with an associated term $-X$ in the Hamiltonian (i.e. the ground state is $|+\rangle$). In order to treat this 2D model within the same framework that we have developed for 1D systems, we will consider an entire $N_v \times 2$ block (where $N_v$ is the extent in the vertical direction) to be a single ‘site’, as shown in figure 7(a); hence we can consider the lattice to comprise a 1D chain of such ‘sites’. The unperturbed ground state, which we denote $|\Psi_Q\rangle$, then has an MPS representation that is essentially a tensor product of several copies of the 1D cluster state MPS representation, with a correlation system comprising $N$ qubits. Each chain contributes a separate $Z_2 \times Z_2$ symmetry, so that the model is invariant under a symmetry group $G = (Z_2 \times Z_2)^N = \{(g_1, \ldots, g_N) | g_1, \ldots, g_N \in Z_2 \times Z_2\}$. The projective representation
The first step in generalizing the 1D results to 2D models involves consideration of a ‘quasi-1D’ model, which consists of a 1D cluster Hamiltonian acting on each of $N$ qubit chains arranged in the vertical dimension, as well as a term favouring the $|+\rangle$ states on the uncoupled qubits. The model has a $(Z_2 \times Z_2)^N$ symmetry, arising from the $Z_2 \times Z_2$ symmetry associated with each of the $N$ chains. We can treat this model as ‘quasi-1D’ by defining our sites (shown as green shaded areas) so that they span the vertical dimension.

Figure 7. The first step in generalizing the 1D results to 2D models involves consideration of a ‘quasi-1D’ model, which consists of a 1D cluster Hamiltonian acting on each of $N$ qubit chains arranged in the vertical dimension, as well as a term favouring the $|+\rangle$ states on the uncoupled qubits. The model has a $(Z_2 \times Z_2)^N$ symmetry, arising from the $Z_2 \times Z_2$ symmetry associated with each of the $N$ chains. We can treat this model as ‘quasi-1D’ by defining our sites (shown as green shaded areas) so that they span the vertical dimension.

Of this symmetry in correlation space is the $N$-qubit generalization of the Pauli representation, namely

$$V((g_1, \ldots, g_N)) = V_P(g_1) \otimes \cdots \otimes V_P(g_N),$$

(71)

where $V_P$ is the single-qubit Pauli representation of $Z_2 \times Z_2$, given by equation (10). It can be checked that this projective representation is maximally non-commutative, and because it has dimension $2^N = \sqrt{|G|}$, it must be the unique irreducible projective representation corresponding to its factor system (by lemma 1 from section 2.3).

Now, it is easy to see that, where $S$ is the set of gates which can be executed in correlation space by a single-site measurement (up to Pauli byproducts) in the 1D cluster state, we can execute any tensor product

$$s_1 \otimes s_2 \otimes \cdots \otimes s_N, \quad s_1, \ldots, s_N \in S$$

(72)

(up to Pauli byproducts) in correlation space by a single-site measurement in our 2D model; we just do the corresponding measurements on each chain separately. We can also find a measurement basis for a columnar site which induces entangling gates between two qubits in correlation space; however, this measurement basis clearly cannot correspond to single-qubit measurements, since the two chains would then remain uncoupled. For reasons that will become clear when we consider the relation of the present model to the 2D cluster state in section 4.2, we will only consider entangling gates between nearest-neighbour qubits in correlation space, for which we construct the measurement basis in a particular way, as follows.

We define the on-site unitary $u$, which involves applying controlled-Z gates between neighbouring qubits to turn our original resource state $|\Psi_Q\rangle$ into another graph state $|\Psi'_Q\rangle$ in which the two chains of interest are coupled, as shown in figure 8. If $A[\cdot]$ is the MPS tensor for $|\Psi_Q\rangle$ at the given site, then $A'[\cdot] = A[u^\dagger(\cdot)]$ is the MPS tensor for $|\Psi'_Q\rangle$. Using the measurement sequences described in [46], it can be shown that there exists a measurement basis $\{|a\rangle\}$ for a columnar site, corresponding to single-qubit measurements, such that $A'|a\rangle = B_a U$, where $U$ is an entangling two-qubit gate, and the $B_a$ are outcome-dependent Pauli byproducts. It follows
Figure 8. In the ‘quasi-1D’ model, two-qubit gates in correlation space cannot be done with single-qubit measurements. However, after applying controlled-$Z$ gates between neighbouring qubits in order to couple two chains, an entangling gate can be performed in correlation space by means of single-qubit measurements.

that this two-qubit gate can be performed in correlation space (up to the same Pauli byproducts) by measuring in the basis \( \{ |\alpha\rangle \} \).

From the above considerations, we see that the model we are discussing can be considered as a generalization of the 1D cluster state in which \( N \) qubits can be propagated in correlation space, acted on by entangling gates between nearest neighbour qubits as well as single-qubit gates. In the presence of symmetry-respecting perturbations to the Hamiltonian, the arguments of sections 2 and 3 can still be applied for any finite \( N \). However, if we want to exploit the locality of the perturbation in the vertical direction as well as the horizontal, we need to make some additional arguments. First, we observe that (by lemma 1) the protected subsystem of correlation space (which corresponds to the ancilla system appearing in the dual picture of MBQC) will have dimension \( 2^N \), and by identifying the action of the symmetry within the protected subsystem with equation (71), we can decompose the protected subsystem into \( N \) qubits, one associated with each chain.

Our argument now hinges on two observations. Firstly, the dual Hamiltonian of which the dual state is the gapped ground state, as constructed in section 3.3, is in fact a sum of interactions that are local on the original 2D lattice. Secondly, the unitary couplings \( G_k \) appearing in the dual process, which a priori could couple an entire columnar site to the entire \( N \)-qubit ancilla system, in fact acts trivially outside an appropriately localized area (see figure 10). These observations both follow from the form of the interaction

\[
D_1 = \sum_i |i\rangle \otimes V(g_i)
\]

between a columnar site and the ancilla system. (Recall that \( D_1 \) and its inverse appeared in the development of the dual picture in section 2.4, as well as in the construction of the duality transformation \( \mathcal{D} \) from which the dual Hamiltonian \( \mathcal{H} \) was obtained in section 3.3.) It is easily seen that in the present quasi-1D setup, \( D_1 \) simply comprises a number of applications of the corresponding operator \( D_1^{(1)} \) for the 1D cluster chain (see figure 9).

Now, consider a quantum circuit comprising a sequence of gates, and let \( Q_k \) be the sets of physical (not ancilla) qubits acted on by the corresponding couplings \( G_k \) in the dual process. Just as in the 1D case, we expect that if \( R \equiv \min_{k_1, k_2} \text{dist}(Q_{k_1}, Q_{k_2}) \) is much larger than the correlation length \( \tilde{\xi} \) for the dual state, then the reduced state \( \text{Tr}_{\bigcup Q_k} |\tilde{\Psi}\rangle \langle \tilde{\Psi}| \) on \( \bigcup Q_k \) should be approximately a product state over the \( Q_k \)s. Thus, arguing as in the 1D case (see section 2.6), we find that performing the measurement sequence on a perturbed resource state corresponds

\[
\text{New Journal of Physics 14} (2012) 113016 (http://www.njp.org/)
\]
to a noisy quantum circuit, with the noise described by a noise superoperator $\mathcal{E}_k$ following each gate. Furthermore, $\mathcal{E}_k$ acts non-trivially only on the same qubits that were acted on by the corresponding gate in the original noiseless quantum circuit. The strength of the noise, as given by $\|\mathcal{E}_k - I\|_\diamond$, is determined by the deviation (in the trace norm) of the reduced density operator on $Q_k$ from its unperturbed value (see equation (48) in section 2.6), which should be small for small perturbations.

Let us now estimate the required scale-up in the size of the resource state. We only consider in detail the case of local quantum circuits (i.e. containing only gates acting between nearest-neighbour qubits). As in the 1D case (section 3.4), according to the rigorous factorization result proved in appendix C, the minimum separation $R_{\text{min}}$ between any of the $Q_k$s required for an accuracy $\epsilon$ scales like

$$R_{\text{min}}/\tilde{\xi} = \mathcal{O}(m) + \mathcal{O}[(1/\epsilon)].$$

(74)

The required scale-up can be expressed in terms of $R_{\text{min}}$, as follows. First, we must ensure that, at each time step, all non-trivial gates are separated by a distance of at least $R_{\text{min}}$. This leads to a scale-up by a factor of $\sim R_{\text{min}}^2$ in the number of time steps. Then, the buffering between horizontal locations at which nontrivial gates take place implies another factor of $R_{\text{min}}$ scale-up. Hence, the total scale-up factor is $s \sim R_{\text{min}}^2$. On the other hand, if the quantum circuit that we want to simulate is not already local, then translating it into a local circuit will introduce
additional overhead (still scaling at worst polynomially in the number of qubits in the quantum circuit).

4.2. The two-dimensional cluster model

Now we will return to the model we are actually interested in: the 2D cluster model on a square lattice. Investigations of the effect on this model of perturbations [13, 47–49] have demonstrated a variety of results depending on the perturbation. Here, we will focus on perturbations respecting an appropriate symmetry. When this symmetry is enforced, the cluster model lies in a robust SPT phase, within which the identity gate is protected and the effective noise model construction of this paper can be applied.

In order to achieve our goal, we will establish an equivalence between the 2D cluster model and a ‘quasi-1D’ model of the type considered in the previous section. The basic idea is to define a (local) duality transformation $U$ (not the same as the duality transformation $D$ we have considered previously) which relates the two models. Specifically, we define

$$U = \prod_{(i,j) \in L} (CZ)_{ij}, \quad (75)$$

where $(CZ)_{ij}$ is the controlled-Z gate acting on qubits $i$ and $j$, and the product is over an appropriate set $L$ comprising nearest-neighbour pairs of qubits. By an appropriate choice of $L$, we can ensure that applying $U$ to the 2D cluster Hamiltonian turns it into a model of the type we considered in the previous section.

Now, let $H$ be a perturbation to the 2D cluster Hamiltonian. Then $UHU^\dagger$ is a perturbation to the quasi-1D model, and the arguments of the preceding section can be applied provided that the perturbation respects the appropriate symmetry. Furthermore, the result (in terms of statistics of measurement outcomes) of performing the adaptive measurement protocol described in section 4.1 on the ground state of $UHU^\dagger$, involving measuring the observables $\hat{o}$, must be the same as the effect of performing the same protocol on the ground state of $H$, but measuring the observables $U^*\hat{o}U$. We will now examine in detail this corresponding measurement protocol for perturbations of the 2D cluster model.

In the quasi-1D resource state, there is a set of ‘redundant’ qubits which never need to be measured. It turns out to be convenient to assume, however, that we do measure those qubits, in the $z$ basis, and that we do this before any other measurements. We observe that all the measurements that are performed on the quasi-1D state (after applying controlled-Z gates to couple chains where we want to perform a two-qubit gate) are all single-qubit, and are either in the $z$ basis, i.e. measuring $Z$, or in the $x$-$y$ plane, i.e. measuring $\sigma_\theta = (\cos \theta)X + (\sin \theta)Y$ for some angle $\theta$. Hence, the corresponding observables to measure in the 2D cluster state are either of the form $Z_j$ (for some qubit $j$), or

$$\sigma^{(j)}_\theta \prod_{k \in N_j} Z_k \quad (76)$$

for some qubit $j$ and angle $\theta_j$, and where $N_j$ is some set of neighbouring redundant qubits. But, since we measured the redundant qubits first, they are all now in eigenstates of $Z$. Therefore, labelling the measured values of $Z$ on the redundant qubits by $\{z_j\}$, we see that measuring $U\hat{o}_jU^\dagger$ is equivalent to measuring $\sigma^{(j)}_\theta(\prod_{k \in N_j} z_k)$, which in turn is equivalent to measuring $\sigma^{(j)}_\theta$ and reinterpreting the measurement outcomes based on the value of $\prod_{k \in N_j} z_k$. Therefore, we
Figure 11. Two possible layouts for the 1D chains (red lines) on a 2D lattice. Each layout is associated with a measurement protocol for MBQC on the 2D cluster state, and with a symmetry group (a representative generator of which is shown in each case). For any layout, we can construct an effective noise model corresponding to performing the associated measurement protocol on a perturbed cluster state, provided that the perturbation respects the associated symmetry. (a) Horizontal and (b) diagonal.

have shown that the measurement protocol on the 2D cluster state can be implemented using only single-qubit measurements and adaptivity. It can be checked that the measurement protocol so constructed is essentially the same as the usual one for the 2D cluster state on a square lattice, which is described, e.g. in [1, 46].

Finally, let us discuss the required symmetry. The duality transformation $U$ can be used to relate the $(\mathbb{Z}_2 \times \mathbb{Z}_2)^\times$ symmetry which protects the quasi-1D model to a corresponding one in the 2D cluster model. The form of the generators of the latter symmetry is shown in figure 11(a). Let us remark that we can also make similar arguments in the case that the 1D chains are arranged on the 2D square lattice in an unconventional way, for example diagonally [13, 50] as shown in figure 11(b). The advantage of the diagonal layout is that the symmetry (one of the generators of which is depicted in figure 11(b)) takes a particularly simple form, due to the fact that every non-chain qubit neighbours an even number of chain qubits, and so the $Z$'s that would normally appear on non-chain qubits all cancel. In particular, this symmetry commutes with an especially simple and physically meaningful perturbation, namely a uniform magnetic field in the $x$ direction, i.e. $V = B \sum_i X_i$. (The effect of such a perturbation has been studied numerically in [49, 51]; the SPT cluster phase persists up to a first-order phase transition at $|B| = 1$.)
4.3. Perturbed ground states are universal resources

Let us summarize the conclusions which we obtain from the considerations in sections 4.1 and 4.2 by stating them as a theorem. In combination with the threshold theorem of fault-tolerant quantum computation, this theorem will allow us to deduce that sufficiently small symmetry-respecting perturbations to the 2D cluster Hamiltonian retain ground states which can serve as universal resources for MBQC.

We consider perturbations to the exact cluster Hamiltonian $H_C$ on the 2D square lattice, which we can take to be a sum of local commuting terms, with an energy gap to the first excited state of 2. Suppose now we consider a perturbed Hamiltonian $H = H_C + V$, and $V$ is a perturbation of the form

$$V = \sum_{u \in \Lambda} V_u,$$

where $\Lambda$ is the set of all lattice sites, and each $V_u$ is an interaction term supported on the set $B(u, r)$ of sites within some fixed distance $r$ (more generally, interactions decaying exponentially with distance would not present an obstacle to our arguments). We define the local strength of the perturbation by

$$J \equiv \max_u \| V_u \|.$$  

(78)

The cluster Hamiltonian $H_C$ belongs to a class of Hamiltonians for which it has been shown [52, 53] that the gap is stable to local perturbations, i.e. there exists a threshold $\eta > 0$ (depending only on $r$), such that the gap of the perturbed Hamiltonian is at least 1, provided that $J \leq \eta$.

Let us assume that the perturbation $V$ respects an appropriate symmetry group, constructed according to the procedure described in section 4.2 (such as the one of the symmetry groups depicted in figure 11). Suppose then we want to use the perturbed ground state $|\Psi_1\rangle$ to simulate a local quantum circuit containing $N$ qubits, $T$ time steps, and $m$ gates, with the gates drawn from the gate set $S$ comprising single-qubit rotations, a two-qubit entangling gate (as constructed in section 4.1), and the non-unitary RESTART gate (which corresponds to the reinitialization of a qubit). We obtain the following result.

**Theorem 4.** Provided $J \leq \eta$, we can find an appropriate measurement protocol on the ground state $|\Psi\rangle$ such that the final reduced state on the output qubits is $\epsilon$-close in the trace norm to the outcome of the quantum circuit, with added noise. In each time step $t$ of the equivalent circuit process, the appropriate gates are applied, followed by a noise process described by a superoperator $\mathcal{E}_t$. This superoperator can be written as a tensor product $\mathcal{E}_t = \bigotimes_{A} \mathcal{E}_{t,A}$, where the product is over ‘locations’, i.e. sets of qubits coupled by a gate in the time step $t$ (each qubit not coupled by a gate in the time step $t$ also counts as a location, but $\mathcal{E}_{t,A} = I$ in that case). Thus, the noise has no correlations in space (other than those due to gates acting between qubits) or time. Furthermore, the noise operator $\mathcal{E}_{t,A}$ at each location and time is close to the identity superoperator in the diamond norm:

$$\| \mathcal{E}_{t,A} - I \|_\Diamond \leq cJ,$$

(79)

for some constant $c$ (dependent only on $r$). The number of qubits measured $n$ satisfies

$$n = NT \{O(m) + O[\log(1/\epsilon)]\}.$$ 

(80)
Proof. The only aspect that we have not previously discussed is the bound equation (79). Following the same argument as in the 1D case (section 2.6), we find (using the analogue of equation (48)) that the deviation $\|E_{t,A} - I\|_\diamond$ is bounded above by $\Delta_X \equiv \|\tilde{\rho}_X - \tilde{\rho}_{X,0}\|_1$, where $X$ is the set of qubits in the 2D lattice that affect the operation of the gate in question, and $\tilde{\rho}_X$ and $\tilde{\rho}_{X,0}$ are the reduced states on $X$ of the perturbed dual state $|\tilde{\Psi}\rangle$ and the unperturbed dual state $|\Psi_0\rangle$ respectively. Physically, it should be clear that $\Delta_X$ will be small for small perturbations; in appendix D we demonstrate that, so long as $J \leq \eta$, the inequality $\Delta_X \leq cJ$ holds for some constant $c$ depending only on $r$. $\square$

Now that we have shown that perturbations in the Hamiltonian correspond to noisy quantum circuits, we can invoke the threshold theorem of fault-tolerant quantum computation [20, 54, 55]. For our purposes, the most suitable version is theorem 13 of [20], which we can state as follows.

**Theorem 5.** Let us assume a noise model as described in theorem 4. Then there exists a threshold $\eta' > 0$ and a constant $\alpha$ such that, so long as $\|E_{t,A} - I\|_\diamond \leq \eta'$ for all $A,t$, then the following properties hold. For any $\epsilon > 0$, and any local quantum circuit $C$ made from gates drawn from $S$ (with $N$ qubits, $T$ time steps, and $m$ gates), there exists another local circuit $C'$ with gates drawn from $S$, such that $C'$ with noise produces the same result (in terms of the probability distribution for the final readout, and up to an error $\epsilon$) as $C$ without noise. The scale-up factors for the number of qubits, the number of time steps, and the number of gates are all bounded by $(\text{const.}) \times \log^a (m/\epsilon)$.

Combining theorems 4 and 5, we obtain the following.

**Theorem 6.** Consider the perturbed model $H = H_C + V$ as described above. Then there exists some threshold $\eta'' = \min\{\eta, \eta'/c\} > 0$ (depending only on $r$) with the following property. Provided that $J < \eta''$, then for any local quantum circuit $C$ (with $N$ qubits, $T$ time steps, and $m$ gates), with gates drawn from $S$, we can find an appropriate measurement protocol on the perturbed ground state $|\tilde{\Psi}\rangle$ such that the result is equivalent (in terms of the probability distribution for the final readout, and up to an error $\epsilon$ which can be made arbitrarily small) to the outcome of the original quantum circuit. As $m \to \infty$ with $\epsilon$ held fixed, the number of measured qubits $n$ satisfies

$$n \leq NT \times O\left(m \log^3 \frac{m}{\epsilon}\right).$$

This is sufficient to show that the perturbed ground states remain universal resources, which is theorem 3 as stated in section 1.1; it is the main result of this paper.

5. Conclusion

In this paper, we have developed a framework to characterize the effectiveness of measurement protocols for MBQC with SPT-ordered ground states of quantum spin systems. This has allowed us to prove the universality for MBQC of the ground states of perturbed versions of the 2D cluster Hamiltonian, provided that the perturbation is sufficiently small and respects an appropriate symmetry.

The type of SPT order that we have presented here is that present in 1D systems, which is related to a non-trivial factor system (also known as a two-cocycle). It is for this reason that,
in order to establish universality in 2D systems, we had to treat them as ‘quasi-1D’ and assume an extensive symmetry group \((Z_2 \times Z_2)^{\times N}\), which grows with the vertical extent of the system. For standard, non-extensive symmetries in two dimensions, SPT orders can be related to three-cocycles \([56, 57]\), but it remains to be seen whether similar connections can be drawn between such 2D SPT order and MBQC.

Finally, we note that if MBQC in ground states of quantum spin systems is to be a robust form of quantum computation, then it must be possible in the presence of arbitrary (not necessarily symmetry-respecting) local perturbations to the Hamiltonian, as well as at non-zero temperature. Non-symmetry-respecting perturbations break the symmetry that is essential to our argument; the difficulty of extending our treatment to non-zero temperature was discussed in section 3.5. Nor have we considered the effect of non-ideal measurements, or of decoherence of the resource state taking place during the course of the measurement protocol. Therefore, it remains an open question whether fault-tolerant MBQC is possible with such imperfections.

Acknowledgments

We thank I Schwarz for bringing to our attention the connection between SPT order and the nature of the adaptive measurement protocol to correct for ‘wrong’ outcomes (see section 2.4). We acknowledge support from the ARC via the Centre of Excellence in Engineered Quantum Systems (EQuS), project number CE110001013.

Appendix A. The dual finitely correlated state

A pFCS \([21, 22]\) is the thermodynamic limit of the translationally invariant MPS generated by a fixed MPS tensor \(A\). The nature of the correlations can be expressed through the transfer channel

\[
\mathcal{A}(\sigma) = \sum_k A[k]\sigma A[k]^\dagger
\]  

(A.1)

(here the sum is over some basis \(|k\rangle\) for the site Hilbert space; it can be shown that this definition of \(\mathcal{A}\) is independent of the choice of basis). In its canonical form, a pure FCS is further characterized by the following properties.

(a) \(\mathcal{A}\) is unital, i.e. \(\mathcal{A}(\mathbb{I}) = \mathbb{I}\).

(b) There exists a density operator \(\Lambda\) such that \(\mathcal{A}^\dagger(\Lambda) = \Lambda\).

(c) Defining \(a\) to be the largest magnitude eigenvalue of \(\mathcal{A}\) other than the one corresponding to the eigenvector \(\Lambda\), we have that \(|a|\) is strictly less than 1.

The correlation length is then defined by \(\xi \equiv -1/\log|a|\), and the eigenvalues of \(\Lambda\) correspond to the entanglement spectrum obtained from a cut in an infinite chain.

Now we restrict ourselves to pFCS generated by tensors \(A\) satisfying the decomposition equation (16). We define the CPTP superoperator \(\mathcal{V}_g\) according to \(\mathcal{V}_g(\sigma) = [V(g) \otimes \mathbb{I}]\sigma [V(g) \otimes \mathbb{I}]^\dagger\); it can be checked that \(\mathcal{V}_g\) is a linear representation (in the space of superoperators) of the symmetry group \(G\), and that it commutes with \(A^\dagger\) for all \(g \in G\). Therefore, since \(\Lambda\) is the unique eigenvector of \(A^\dagger\) with eigenvalue 1, it must satisfy \(\mathcal{V}_g(\Lambda) = \chi(g)\Lambda\) for some scalars \(\chi(g)\). The fact that \(\mathcal{V}_g\) is trace preserving ensures that \(\chi(g) = 1\). Hence we find that \(\Lambda\) commutes
with $V(g) \otimes \mathbb{1}$; it follows by Schur’s Lemma that $\Lambda$ factorizes as $\Lambda = \Omega \otimes \tilde{\Lambda}$ for some density operator $\tilde{\Lambda}$, and where $\Omega = \mathbb{1}/\sqrt{|G|}$ is the maximally mixed state on the protected subsystem (recall that the $\sqrt{|G|}$ is the dimension of the protected subsystem). It follows that there is a $\sqrt{|G|}$-fold degeneracy in the entanglement spectrum throughout the SPT phase, generalizing the two-fold degeneracy in the $(Z_2 \times Z_2)$-protected Haldane phase [39].

We can define the transfer channel corresponding to the dual FCS (generated by $\tilde{A}$) according to

$$\tilde{A}(\rho) = \sum_k \tilde{A}[k] \rho \tilde{A}[k]^\dagger.$$  \hspace{1cm} (A.2)

Observe that $\mathcal{A}$ unital implies that $\tilde{A}$ is also unital, and that

$$\mathcal{A}^\dagger (\Omega \otimes \sigma) = \Omega \otimes \tilde{A}^\dagger (\sigma)$$  \hspace{1cm} (A.3)

for any operator $\sigma$ acting on the junk subsystem. Hence, any eigenvalue of $\tilde{A}$ must also be an eigenvalue of $\mathcal{A}$. From this we can see that the dual FCS is also a pure FCS, and $\tilde{\Lambda}$ is the unique fixed point of $\tilde{A}^\dagger$; thus, the entanglement spectrum of the dual state is the same as that of the original state, but with the $\sqrt{|G|}$-fold degeneracy removed\(^3\). The respective correlation lengths obey the inequality $\bar{\xi} \leq \xi$.

We now outline how the arguments of section 2.4 can be put on a rigorous footing within the pFCS formalism. Given everything that we have established so far, it can be shown that the reduced state $\rho_n$ of the original FCS on a block of $n$ adjacent sites can be obtained from the corresponding reduced state $\tilde{\rho}_n$ of the dual FCS according to the quantum circuit shown in figure A.1(a). Assuming that we choose $n$ large enough that all measurements take place within this block of $n$ sites, we can then make arguments similar to those of section 2.4, and we find that the result of the adaptive measurement protocol is equivalent to a sequence of interactions between the dual state and an ancilla particle as shown in figure A.1(b), with the unitary interactions $G_k$ defined as they were previously.

Appendix B. Connection with the Kennedy–Tasaki transformation

The Kennedy–Tasaki (KT) transformation [32, 33] is a non-local unitary transformation which transforms a spin-1 chain in the $Z_2 \times Z_2$ symmetry-protected Haldane phase into a system where the symmetry is spontaneously broken in the bulk. In this section, adapting [37], we will define a generalized version of the KT transformation, which can be applied to any system in the non-trivial SPT phase with respect to an on-site representation $U(g) = [u(g)]^\otimes N$ of the symmetry group $G = Z_2 \times Z_2$ (like the original KT transformation, our generalization is defined for finite chains with open boundary conditions and no terminating particles). We will show that, when the ground state of the original system can be expressed as a pFCS, the ground state of the KT-transformed system is essentially the same as the state which we have referred to throughout this paper as the ‘dual state’. We expect that for general ground states the situation should be qualitatively similar.

Observe that the symmetry group is generated by two commuting elements $x$ and $z$; hence for any $g \in G$, we can write $g = x^{m(g)} z^{n(g)}$ for some $m(g)$ and $n(g)$ taking values of 0 or 1. We will choose to write the unique non-trivial irreducible projective representation as

\(^3\) A similar property was found numerically for the KT transformation in [37]. We will discuss the connection between that transformation and our ‘dual state’ in appendix B.

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Figure A.1. (a) The quantum circuit generating the reduced state on \( n \) sites for the original FCS. The ancilla particle is initially in the maximally mixed state \( \Omega \), and the symbol ■ indicates that it should be traced out at the end. (b) The adaptive measurement protocol acting on the original FCS is equivalent to a dual process acting on the dual state, comprising a sequence of unitary interactions \( G_k \) coupling the dual state to an ancilla particle.

\[ V(g) = X^{m(g)} Z^{n(g)} \], where \( X \) and \( Z \) are the appropriate Pauli operators. This is a rephasing of the Pauli representation \( V_P \) defined in equation (10); thus the factor system is different to, but in the same cohomology class as, that of \( V_P \). (The construction of the dual state does depend on the specific choice of representative factor system for a cohomology class, although in a fairly trivial way; the present choice is the one that will ensure that the KT transformation reproduces the dual state exactly.)
The generalized KT transformation $\mathcal{D}_{\text{KT}}$ is then defined as follows:

$$\mathcal{D}_{\text{KT}} = \prod_{k<l} D_{kl}, \quad \text{(B.1)}$$

where $D_{kl}$ is a unitary coupling the two sites $k$ and $l$ according to

$$D_{kl} \equiv \sum_i |i\rangle\langle i| \otimes u(x^{m(g)}) . \quad \text{(B.2)}$$

Note that all the operators $D_{kl}$ appearing in the product equation (B.1) commute. In the case that the particles are spin-1s, with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ rotation symmetry consisting of $\pi$ rotations about the $x$, $y$, and $z$ particles (the simultaneous eigenbasis of the symmetry for a single site is then $\{|x\rangle, |y\rangle, |z\rangle\}$, where $|\alpha\rangle, \alpha = x, y, z$ is the 0 eigenstate of the spin component operator $S_\alpha$), the transformation $\mathcal{D}_{\text{KT}}$ reduces to the standard KT transformation. Specifically, equations (B.1) and (B.2) correspond to equations (4) and (5) in [37].

For open boundary conditions, we expect there to be an approximate (becoming exact in the limit as the length of the chain goes to infinity) four-fold degeneracy, due to the two-fold degeneracy associated with each edge. An appropriate analogue of the SPT pFCS for this choice of boundary conditions is as follows: the low-energy subspace is spanned by states of the form

$$|\Psi(L, R)\rangle = \begin{array}{c}
L^L
\end{array} A A A A \begin{array}{c}
R^R
\end{array}, \quad \text{(B.3)}$$

where the MPS tensor $A$ obeys the usual symmetry condition, where $|L^L, L^R\rangle$ are fixed end vectors, while $|L\rangle$ and $|R\rangle$ are allowed to vary (within the 2D irrep space) in order to generate the four-dimensional low-energy subspace.

Now we want to examine what happens to a state of the form equation (B.3) under $\mathcal{D}_{\text{KT}}$. Suppose we set $|R\rangle = |+\rangle$, $|L\rangle = |0\rangle$. It is then straightforward to show [using the symmetry condition equation (12) and the decomposition equation (15)] that applying all the pairwise interactions $D_{1l}$ involving the first site gives the result

$$\left(\prod_{l>1} D_{1l}\right) |\Psi(0, +)\rangle = \begin{array}{c}
0
\end{array} A A A A + \begin{array}{c}
R^R
\end{array}. \quad \text{(B.4)}$$

Continuing this process, we find that

$$\mathcal{D}_{\text{KT}} |\Psi(0, +)\rangle = \begin{array}{c}
L^L
\end{array} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \begin{array}{c}
R^R
\end{array}. \quad \text{(B.5)}$$

which is the dual state. To obtain the other states within the low-energy subspace, it is sufficient to observe that, using the symmetry condition on the tensor $A$, we have

$$|\Psi(1, +)\rangle = [u(x)]^{\otimes N} |\Psi(0, +)\rangle, \quad \text{(B.6)}$$

$$|\Psi(0, -)\rangle = [u(z)]^{\otimes N} |\Psi(0, +)\rangle, \quad \text{(B.7)}$$
and that $D_{\text{KT}}$ commutes with $u(g)$ for all $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, the other basis states for the KT-transformed system can be obtained from the dual state by application of a symmetry operation $[u(g)]^\otimes N$ for some $g \in G$. The dual state thus represents one of the four distinct symmetry-breaking states within the low-energy subspace of the transformed system.

Let us also note that, although they relate to different boundary conditions, the unitary transformation $D$ that we introduced in section 3.2 is equivalent to the KT transformation $D_{\text{KT}}$, in the following sense. Indeed, an important property of $D_{\text{KT}}$ is that for any local symmetry-respecting observable $\hat{\sigma}$, $D_{\text{KT}} \hat{\sigma} D_{\text{KT}}^\dagger$ remains local and symmetry-respecting. It turns out that this also holds for $D$, in the case of observables $\hat{\sigma}$ acting in the bulk (the fact that $D \hat{\sigma} D^\dagger$ is local was established in section 3.3; it can be shown that $D \hat{\sigma} D^\dagger$ still respects the on-site symmetry as well). Thus, like $D_{\text{KT}}$, the transformation $D$ can be applied to yield a local, symmetry-respecting Hamiltonian $\tilde{H}$ in the bulk. It can be shown that $\tilde{H}$ is precisely the KT-transformed Hamiltonian [58]. Thus, in line with the results for pFCS described above, we expect $\tilde{H}$ to have four degenerate, locally distinguishable symmetry-breaking ground states in the bulk. On the other hand, when the transformation $D$ is applied to the edge interactions (those which couple the ends of the chain to the terminating particles), the result need not respect the symmetry. Therefore, including the edge interactions favours one of the symmetry-breaking ground states over the others and leads to the non-degeneracy of the dual state $|\tilde{\Psi}\rangle$.

### Appendix C. The factorization condition for ground states of a local Hamiltonian

In this appendix, we will show how to derive the approximate factorization result equation (67) for a non-degenerate gapped ground state $|\Psi\rangle$ of a local Hamiltonian. Our main tool is the existing theorem on the exponential decay of correlation functions for such a ground state [59–61]. This theorem states that there exists a correlation length $\xi$ and a function $f(x)$, with $f(x) = O[\exp(-x/\xi)]$ as $x \to \infty$, such that for any sets of lattice sites $X$ and $Y$, and observables $A_X$ and $B_Y$ supported on $X$ and $Y$ respectively, we have

$$\frac{|\langle A_X B_Y \rangle_\Psi - \langle A_X \rangle_\Psi \langle B_Y \rangle_\Psi|}{\|A_X\| \|B_Y\|} \leq f(\text{dist}(X, Y)) \min\{|X|, |Y|\},$$

(C.1)

where $\langle \cdot \rangle_\Psi$ denotes the expectation value of an observable with respect to $|\Psi\rangle$, and $|X|$ denotes the number of sites contained in the set $X$.

Suppose that instead of two sets of lattice sites, we have $m$ disjoint sets $X_1, \ldots, X_m$. Let $N = \sum_{k=1}^m |X_k|$ be the total number of lattice sites contained within all of the $X_k$s. We can obtain the following corollary.

**Lemma C.1.** For any observables $A_{X_1}, \ldots, A_{X_m}$ supported on the respective sets

$$\frac{|\langle A_{X_1} \cdots A_{X_m} \rangle_\Psi - \langle A_{X_1} \rangle_\Psi \cdots \langle A_{X_m} \rangle_\Psi|}{\|A_{X_1}\| \cdots \|A_{X_m}\|} \leq f(R)N,$$

(C.2)

where $R$ is the smallest distance between any two of the $X_k$’s, i.e. $R = \min_{k \neq l} \text{dist}(X_k, X_l)$. 

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Proof. Without loss of generality, we can assume that $\| A_{X_k} \| = 1$ for all $k$. Observing that $| \langle A_{X_k} \rangle | \leq \| A_{X_k} \| = 1$, we have that

$$| \langle A_{X_1} \cdots A_{X_m} \rangle \psi - \langle A_{X_1} \rangle \psi \cdots \langle A_{X_m} \rangle \psi | \leq | \langle A_{X_1} \cdots A_{X_m} \rangle \psi - \langle A_{X_1} \rangle \psi \langle A_{X_2} \cdots A_{X_m} \rangle \psi |$$

$$+ | \langle A_{X_2} \cdots A_{X_m} \rangle \psi - \langle A_{X_2} \rangle \psi \cdots \langle A_{X_m} \rangle \psi |.$$  \hspace{1cm} (C.3)

The two-body result implies that the first term in the right-hand side of equation (C.3) is bounded by $|X_1| f(R)$. Continuing in this way, the lemma follows by induction. \hspace{1cm} □

Now we want to show that the reduced state on the union of the $X_k$s is close to a product state. To do this, we make use of the following lemma. We remind the reader that here we make use of both the trace norm $\| A \|_1 = \text{Tr} \sqrt{A^* A}$ and the spectral norm $\| A \| = \max_{\langle \psi | \psi \rangle = 1} \| A | \psi \rangle \|$. They are both distinct from the norm induced by the Hilbert–Schmidt inner product.

Lemma C.2. Consider a vector space of dimension $D$. Then there exists a basis $\{ E_i | i = 1, \ldots, D^2 \}$ for the space of linear operators supported on the site, orthonormal with respect to the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{Tr} (A^* B)$, and comprising Hermitian operators $E_i$ such that such that $\| E_i \|_1 \| E_i \| = 1$.

Proof. Given a basis $\{ | m \rangle \}, m = 1, \ldots, d$, such a set can be constructed comprising the operators $| m \rangle \langle m |$, and $(1/\sqrt{2}) (| m \rangle \langle n | + | n \rangle \langle m |)$ and $(i/\sqrt{2}) (| m \rangle \langle n | - | n \rangle \langle m |)$ for $m \neq n$. \hspace{1cm} □

Now we are ready to prove the main result.

Theorem C.1. Let $| \Psi \rangle$ be the non-degenerate gapped ground state of a local Hamiltonian. Let $\rho$ be the reduced state of $| \Psi \rangle$ on $\bigcup_{k=1}^m X_k$, and let $\rho_k$ be the reduced state on $X_k$. Then

$$\| \rho - \rho_{\text{prod}} \|_1 \leq N f(R) d^{2N},$$

where $\rho_{\text{prod}} = \bigotimes_k \rho_k$. \hspace{1cm} (C.4)

Proof. Recall that for a linear operator $P$, $\| P \|_1 = \max_{\| A \| = 1} | \text{Tr} (AP) |$, where the maximization is over all linear operators $A$ with unit spectral norm. Now, we can expand

$$A = \sum_{i_1, \ldots, i_m} \alpha_{i_1, \ldots, i_m} E_{i_1} \otimes \cdots \otimes E_{i_m},$$

where the tensor product is over the sets $X_1, \ldots, X_m$; the $E_i$ are as constructed in lemma C.2; and the scalars $\alpha_{i_1, \ldots, i_m}$ are determined by $\alpha_{i_1, \ldots, i_m} = \text{Tr} [ A^*(E_{i_1} \otimes \cdots \otimes E_{i_m}) ]$, which implies (since $\| A \| = 1$) that

$$| \alpha_{i_1, \ldots, i_m} | \leq \| E_{i_1} \|_1 \cdots \| E_{i_m} \|_1.$$  \hspace{1cm} (C.6)

Now, note that

$$\text{Tr} (A \rho) = \sum_{i_1, \ldots, i_m} \alpha_{i_1, \ldots, i_m} \langle E_{i_1} \cdots E_{i_m} \rangle \rho,$$

$$\text{Tr} (A \rho_{\text{prod}}) = \sum_{i_1, \ldots, i_m} \alpha_{i_1, \ldots, i_m} \langle E_{i_1} \rangle \rho_1 \cdots \langle E_{i_m} \rangle \rho_m.$$  \hspace{1cm} (C.7)

$$\text{Tr} (A \rho_{\text{prod}}) = \sum_{i_1, \ldots, i_m} \alpha_{i_1, \ldots, i_m} \langle E_{i_1} \rangle \rho_1 \cdots \langle E_{i_m} \rangle \rho_m.$$  \hspace{1cm} (C.8)
Hence, by lemma C.1, we find that

\[ |\text{Tr}[A(\rho - \rho_{\text{prod}})]| \tag{C.9} \]

\[ \leq N f(R) \sum_{i_1, \ldots, i_m} |\alpha_{i_1, \ldots, i_m}| \| E_{i_1} \| \cdots \| E_{i_m} \| \tag{C.10} \]

\[ \leq N f(R) \sum_{i_1, \ldots, i_m} (\| E_{i_1} \|_1 \cdots \| E_{i_m} \|_1)(\| E_{i_1} \| \cdots \| E_{i_m} \|) \tag{C.11} \]

\[ = N f(R)d^{2N}, \tag{C.12} \]

since \( \| E_k \|_1 \| E_k \| = 1 \) for all \( k \).

\[ \square \]

**Appendix D. Local perturbations perturb continuously**

Physically, it should be clear that small perturbations in a gapped local Hamiltonian lead to small variations in the reduced state obtained from the ground state on a finite region of the lattice. Here we will give a rigorous proof of this fact, as follows.

**Theorem D.1.** Let \( H(s) \) be a differentiable path of Hamiltonians of the form

\[ H(s) = \sum_{u \in \Lambda} H_u(s), \tag{D.1} \]

where the sum is over all the lattice sites \( u \) in a finite-dimensional lattice \( \Lambda \), and \( H_u(s) \) is supported on the set \( B(u, r) \) of sites within some fixed distance \( r \) of \( u \). Suppose that for \( 0 \leq s \leq S \), the Hamiltonian \( H(s) \) has a unique ground state \( |\Psi(s)\rangle \), and there is a uniform lower bound \( \gamma > 0 \) on the gap. Then there exists a constant \( c \) (dependent only on the lattice geometry, on \( r \), and on \( \gamma \)) such that for any set \( X \) of lattice sites, we have

\[ \| \rho_X(S) - \rho_X(0) \|_1 \leq c|X|J' S, \tag{D.2} \]

where \( \rho_X(s) \) is the reduced state on \( X \), i.e. \( \rho_X(s) = \text{Tr}_X |\Psi(s)\rangle \langle \Psi(s)| \), and \( J' \equiv \max_{u \in \Lambda, s \in [0, S]} \| \partial_s H_u(s) \| \).

**Proof.** The proof relies on the following consequence of the theory of quasiadiabatic continuation \([62, 63]\): under the given assumptions, there exists a family of Hamiltonians \( \mathcal{H}(s) \) such that

\[ \frac{d}{ds}|\Psi(s)\rangle = \mathcal{H}(s)|\Psi(s)\rangle, \tag{D.3} \]

where \( \mathcal{H}(s) \) can be written as

\[ \mathcal{H}(s) = \sum_{u \in \Lambda} \mathcal{H}_u(s), \tag{D.4} \]
such that for any site \( u \), \( \mathcal{H}_u(s) \) can be approximated by an observable \( \overline{\mathcal{H}}_u(s) \) supported on \( X^c \) (the complement of \( X \)), with error
\[
\| \overline{\mathcal{H}}_u(s) - \mathcal{H}_u(s) \| \leq J' f \left( \frac{\text{dist}(u, X)}{\gamma} \right),
\]  
where \( f \) is a rapidly decaying function (dependent only on \( r \)).

Now, for \( s \in [0, S] \), we have (where \( \rho(s) \equiv |\Psi(s)\rangle \langle \Psi(s)| \))
\[
\partial_s \rho_X(s) = i \text{Tr}_{X^c} [\mathcal{H}(s), \rho(s)].
\]

Hence
\[
\| \partial_s \rho_X(s) \|_1 = \max_{\|A_X\|=1} |\text{Tr}(A_X \text{Tr}_{X^c} [\mathcal{H}(s), \rho(s)])| \quad \text{(D.7)}
\]
\[
= \max_{\|A_X\|=1} |\text{Tr}(A_X [\mathcal{H}(s), \rho(s)])|
\]
\[
= \max_{\|A_X\|=1} |\text{Tr}([A_X, \mathcal{H}(s)]\rho(s))|
\]
\[
\leq \max_{\|A_X\|=1} \|[A_X, \mathcal{H}(s)]\|
\]
\[
\leq \max_{\|A_X\|=1} \sum_{u \in \Lambda} \|[A_X, \mathcal{H}_u(s)]\|
\]

(D.11)

Here the maximization is over all linear operators \( A_X \) supported on \( X \) with unit spectral norm. We have made use of the fact that for any linear operator \( P \), \( \|P\|_1 = \max_{\|A\|=1} |\text{Tr}(AP)| \), with \( \|A\| \) the spectral norm.

Now, for any operator \( A_X \) supported on \( X \), we have that \([A_X, \overline{\mathcal{H}}_u(s)] = 0\) since \( A_X \) and \( \overline{\mathcal{H}}_u(s) \) are supported on disjoint subsets. Hence, using equation (D.5) (and the fact that \( \|A_X\| = 1 \)), we find that
\[
\|[A_X, \mathcal{H}_u(s)]\| \leq 2\|A_X\|\|\overline{\mathcal{H}}_u(s) - \mathcal{H}_u(s)\|
\]
\[
\leq 2J' f \left( \frac{\text{dist}(u, X)}{\gamma} \right),
\]
so that
\[
\| \partial_s \rho_X(s) \|_1 \leq 2J' \sum_{u \in \Lambda} f \left( \frac{\text{min}_{x \in X} \text{dist}(u, x)}{\gamma} \right)
\]

(D.14)

We can bound the sum according to
\[
\sum_{u \in \Lambda} f \left( \frac{\text{dist}(u, X)}{\gamma} \right)
\]
\[
= \sum_{u \in \Lambda} f \left( \frac{\text{min}_{x \in X} \text{dist}(u, x)}{\gamma} \right)
\]
\[
\leq \sum_{u \in \Lambda} \sum_{x \in X} f \left( \frac{\text{dist}(u, x)}{\gamma} \right)
\]

(D.17)
\begin{equation}
= \sum_{x \in X} \sum_{u \in \Lambda} f \left( \frac{\text{dist}(u, x)}{\gamma} \right)
\end{equation}
(D.18)

\begin{equation}
\leq |X| \max_{x \in X} \sum_{u \in \Lambda} f \left( \frac{\text{dist}(u, x)}{\gamma} \right)
\end{equation}
(D.19)

\begin{equation}
\leq |X| c/2,
\end{equation}
(D.20)

where in the last step the rapid decay of \( f \) ensures that the sum is bounded by a constant \( c/2 \) dependent on the lattice geometry, \( \gamma \), and \( r \).

Hence, by the triangle inequality for the trace norm, we have

\begin{equation}
\| \rho_X(S) - \rho_X(0) \|_1 \leq \int_0^S \| \partial_s \rho_X(s) \|_1 \, ds \leq c |X| J' S,
\end{equation}
(D.21)

as required. \( \square \)

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