Breaking the cosmological background degeneracy by two-fluid perturbations in $f(R)$ gravity

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Abstract. One of the exact solutions of $f(R)$ theories of gravity in the presence of different forms of matter exactly mimics the $\Lambda$CDM solution of general relativity at the background level. In this work we study the evolution of scalar cosmological perturbations in the covariant and gauge-invariant formalism and show that although the background in such a model is indistinguishable from the standard $\Lambda$CDM cosmology, this degeneracy is broken at the level of first-order perturbations. This is done by predicting different rates of structure formation in $\Lambda$CDM and the $f(R)$ model both in the complete and quasi-static regimes.

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1. Introduction

The ΛCDM (or Concordance) Model of cosmology [1] is one of the greatest successes of General Relativity (GR). It reproduces beautifully all the main observational results [2] such as the dimming of type Ia Supernovae [3, 4, 5, 6], Cosmic Microwave Background (CMB) radiation anisotropies [7, 8], Large Scale Structure formation [9, 10, 11], baryon acoustic oscillations [12] and weak lensing [13]. But the ΛCDM model has many serious shortcomings as well, most notably the fine-tuning problem associated with the cosmological constant and the so-called coincidence problems [14].

Among the leading alternatives to the ΛCDM paradigm are \( f(R) \) models of gravity. These models are based on modifications of the standard Einstein-Hilbert action and naturally admit the currently assumed expansion history of the Universe such as the early inflationary epoch [15] and the late-time accelerated expansion. Several recent lines of research have therefore focused on the viability of these theories as alternatives to dark energy and their cosmological and astrophysical applications [16, 17, 18, 19, 20, 21, 22, 23] (see the recent reviews [24, 25, 26, 27] and references therein for more examples) but such studies come at the cost of high complexity of the physics involved.

One approach in this direction is the technique of cosmological reconstruction where it is assumed that the expansion history of the Universe is known exactly, and the field equations are inverted to deduce what class of theories will give rise to this particular cosmological evolution [2, 28, 29, 30, 31, 32, 33, 34, 35]. In [2] it was shown that the ΛCDM expansion history can be mimicked exactly by an \( f(R) \) model that describes a universe filled with a minimally-coupled and non-interacting, massless scalar field and dust-like matter. This means that if one has to discriminate between these two models, one has to go beyond the level of the Friedmann-Lemaître-Robertson-Walker (FLRW) background and study how the perturbations of matter in these models grow. If the predicted rates of structure formation appear to be different, then that is one way of breaking the background degeneracy that exists between ΛCDM and the mimicking \( f(R) \) model.

The present paper is organized as follows: in Sec.2 we give a summary of the background cosmological evolution as determined by the Concordance Model. We then give a reconstruction of the model of \( f(R) \) gravity that gives the exact background expansion history as the Concordance Model.

Sec.3 deals with the formulation of the covariant density, expansion and curvature perturbations and their corresponding evolution equations, and the analysis of the matter power spectrum produced, followed by a brief discussion of the quasi-static approximation in Sec. 4.

Finally in Sec.5 we discuss the results and give conclusions of the work.

Natural units (\( \hbar = c = k_B = 8\pi G = 1 \)) will be used throughout this paper, and Latin indices run from 0 to 3. The mathematical operators \( \nabla \) (or \( ; \)), \( \tilde{\nabla} \) and the comma derivative , represent the usual covariant derivative, the spatial covariant derivative, and
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partial differentiation, respectively. We use the $(-, +, +, +)$ signature and the Riemann tensor is defined by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd} \Gamma^a_{ce} - \Gamma^f_{bd} \Gamma^a_{cf},$$  \hspace{1cm} (1)$$

where the $\Gamma^a_{bd}$ are the Christoffel symbols (i.e., symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae} \left( g_{be,d} + g_{ed,b} - g_{bd,e} \right).$$ \hspace{1cm} (2)$$

The Ricci tensor is obtained by contracting the first and the third indices of the Riemann tensor:

$$R_{ab} = g^{cd} R_{cadb}.$$ \hspace{1cm} (3)$$

Unless otherwise stated, an overdot $\dot{}$ represents differentiation with respect to cosmic time $t$ whereas $f', f''$, etc., are shorthands for first, second, etc., derivatives of the function $f(R)$ with respect to the Ricci scalar

$$R = R^a_a, \hspace{1cm} (4)$$

and $f$ is used as a shorthand for $f(R)$.

2. The background spacetime

If we write the generalized Einstein-Hilbert action for $f(R)$-gravitational interactions

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ f(R) + 2L_m \right],$$ \hspace{1cm} (5)$$

and apply the variational principle of least action with respect to the metric $g_{ab}$, we obtain a generalization of the Einstein Field Equations (EFEs) given as

$$G_{ab} = \frac{T^m_{ab}}{f'} + \frac{1}{f'} \left[ \frac{1}{2} g_{ab} (f - R f') + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right] \equiv T_{ab},$$ \hspace{1cm} (6)$$

where $T^m_{ab}$ is the usual energy-momentum tensor (EMT) of standard matter given by

$$T^m_{ab} \equiv \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g_{ab}},$$ \hspace{1cm} (7)$$

and the remaining terms on the right-hand side of Eqn (6)

$$T^R_{ab} \equiv \frac{1}{f'} \left[ \frac{1}{2} g_{ab} (f - R f') + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right],$$ \hspace{1cm} (8)$$

can be interpreted as the effective EMT for the curvature fluid.
Because of the arbitrary function introduced in the Lagrangian of Eqn (5), there is more freedom to explain cosmic acceleration (inflation and late-time acceleration)\cite{36} and large-scale structure formation without the inclusion of exotic matter and energy (see \cite{15, 16, 27, 37, 38, 39, 40} and references therein).

However, not all functional forms of these models can be viable cosmological models: a wide range of them can be ruled out based on observations, cosmological and astrophysical, while others can be rejected because of theoretical pathologies.

The EMTs of standard matter and that of the total fluid are conserved, i.e.,
\[
T^{m:ab} = 0, \quad T^{b:ab} = 0 ,
\]
but the effective EMTs of matter \(\tilde{T}^{m:ab} \equiv \frac{T^{m:ab}}{f'}\) and curvature \(T^{R:ab}\) are not individually conserved \cite{41}:
\[
\tilde{T}^{m:ab} = \frac{T^{m:ab}}{f'} - \frac{f''}{f'^2} T^{m:ab} R^{ab}, \quad (10)
\]
\[
T^{R:ab} = \frac{f''}{f'^2} T^{m:ab} R^{ab}. \quad (11)
\]

For a spatially flat, homogeneous and isotropic (FLRW) background, the matter energy-momentum tensor is given by
\[
T^{ab} = \mu_m u^a u^b + p_m h^{ab} + q^m u^a u^b + q^m u^a + \pi^m_{ab} , \quad (12)
\]
where \(\mu_m, p_m, q^m, \pi^m_{ab}\) denote the standard matter energy density, pressure, heat flux and anisotropic pressure, respectively. Here \(u^a\) is the 4-velocity of fundamental observers:
\[
u^a = \frac{dx^a}{dt} , \quad (13)
\]
whereas
\[
h_{ab} = g_{ab} + u_a u_b \quad (14)
\]
is the projection tensor into the tangent 3-spaces orthogonal to \(u^a\). In the standard 1 + 3-covariant approach, the 4-velocity vector \(u^a\) is used to define the covariant time derivative for any tensor \(S^{a..b}_{c..d}\) along an observer’s worldlines:
\[
\nabla_{c..d} = u^e \nabla e S^{a..b}_{c..d} , \quad (15)
\]
and the tensor \(h_{ab}\) is used to define the fully orthogonally projected covariant derivative \(\nabla\) for any tensor \(S^{a..b}_{c..d}\):
\[
\nabla e S^{a..b}_{c..d} = h^a_{f..h} h^b_{g..r} h^q_{p..d} \nabla e S^{f..g}_{p..q} , \quad (16)
\]
with total projection on all the free indices. The covariant derivative of the timelike vector \(u^a\) is decomposed into its irreducible parts as
\[
\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c , \quad (17)
\]
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where $A_a = \dot{u}_a$ is the acceleration, $\Theta = \nabla_a u^a$ is the expansion, $\sigma_{ab} = \nabla_{(a} u_{b)}$ is the shear tensor and $\omega^a = \varepsilon^{abc}\nabla_b u_c$ is the vorticity vector.

The energy conservation equation for standard matter is given by

$$\dot{\mu}_m = -3H (\mu_m + p_m) ,$$

where $H = \frac{\dot{a}}{a} = \frac{1}{3} \Theta$ is the Hubble expansion parameter and $a = a(t)$ is the cosmological scale factor. For flat FLRW spacetimes, the background cosmological expansion history is described by the Raychaudhuri equation

$$\dot{H} + H^2 = -\frac{1}{6f'} \left[ \mu_m + 3p_m + f - f'R + 3Hf'' \dot{R} + 3f''' \ddot{R} + 3f''\dddot{R} \right] ,$$

the Friedmann equation

$$H^2 = \frac{1}{6f'} \left[ 2\mu_m + 2f' - f - 6Hf'' \dot{R} \right] ;$$

and the equation for the Ricci scalar,

$$R = 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = 6\dot{H} + 12H^2 .$$

In the $\Lambda$CDM paradigm, current observations seem to suggest that the Friedmann equation is very well constrained by the relation (20)

$$H^2 = \frac{1}{3} (\mu_d + \Lambda) ,$$

where $\mu_d \geq 0$ is the energy density of pressureless matter (dust) and $\Lambda$ is the cosmological constant. Planck’s most recent results [42] give a lower bound of the current value of the Hubble parameter at

$$H_0 = 67.3 \pm 1.2 \text{km s}^{-1} \text{Mpc}^{-1} ,$$

as well as the current fractional energy densities of dust (cold dark matter plus baryonic matter) and dark energy (assumed to be due to the cosmological constant ) at

$$\Omega_{0d} \equiv \frac{\mu_{0d}}{3H_0^2} = 0.317 , \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2} = 0.683 .$$

From Eqn (22) we see that the first and second time derivatives of the scale factor $a(t)$ can be given as [2]

$$\dot{a} = \sqrt{\frac{\mu_{0d}}{3a} + \frac{\Lambda}{3}} ; \quad \ddot{a} = \frac{1}{2} (\dot{a}^2)a = \frac{2\Lambda a^3 - \mu_{0d}}{6a^2} .$$

Using these relations in Eqn (21) one can show the following interrelation between the scale factor, the Ricci scalar and their derivatives:

$$R(a) = \frac{4\Lambda a^3 + \mu_0}{a^3} , \quad a(R) = \left( \frac{\mu_{0d}}{R - 4\Lambda} \right)^{1/3} ,$$

$$H(R) = \frac{1}{a(R)} \sqrt{\frac{\mu_0}{3a(R)} + \frac{\Lambda}{3}} , \quad \dot{R} = R_{,a}(a(R)) \sqrt{\frac{\mu_0}{3a(R)} + \frac{\Lambda}{3}} .$$
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If we substitute all the above background quantities (expressed in terms of the Ricci scalar) into the Friedmann equation (20), we obtain

\[ 6(R - 3\Lambda)(R - 4\Lambda)f'' - (R - 6\Lambda)f' - f + 2\mu_m = 0 , \tag{28} \]

the solution of which mimics the ΛCDM expansion history exactly.

Let us now consider matter comprising dust-like matter plus a non-interacting stiff fluid (or a massless scalar field), i.e.,

\[ \mu_m = \rho_d + \rho_s , \tag{29} \]

with their present energy densities \( \rho_{0d} \) and \( \rho_{0s} \) and barotropic equations of state given by \( w_d = 0 \) and \( w_s = 1 \), respectively.

The energy and momentum conservation equations for the two-fluid system in the energy frame of matter (where the four-velocity vector field is aligned with the four-velocity of the total matter, i.e., \( u^a = u_m^a \)) is given as

\[ \dot{\rho}_d + \rho_d \dot{\Theta} = 0 , \tag{30} \]
\[ \dot{\rho}_s + 2\dot{\Theta}\rho_s = 0 . \tag{31} \]

From the conservation equations, the total matter density is

\[ \rho_m(a) = \frac{\rho_{0d}}{a^3} + \frac{\rho_{0s}}{a^6} \Rightarrow \rho_m(R) = (R - 4\Lambda) + \frac{\rho_{0s}}{\rho_{0d}^2}(R - 4\Lambda)^2 . \tag{32} \]

Substituting this into the Friedmann equation (20), we get the following particular solution [2]:

\[ f(R) = R + \alpha R^2 - 2\Lambda , \tag{33} \]

where

\[ \alpha = -\frac{2}{9} \frac{\rho_{0s}}{\rho_{0d}^2} . \tag{34} \]

Thus the theory of gravity described by (33) has an exact ΛCDM solution in the background level for a non-interacting two-fluid system of dust and a stiff matter like a massless scalar field. However, it is interesting to note that \( \alpha < 0 \), if both the dust and scalar field densities are positive. Such theories have ghosts which exactly compensate for the massless scalar field and this is the reason that the solution does not depend on the scalar field dynamics. On the other hand if we consider massless ghost field with the dust then \( \alpha > 0 \) and the ghost field compensates for the extra degrees of freedom of the fourth-order gravity.

3. Dynamics of scalar perturbations of the two-fluid system

In what follows, we investigate the stability of the background described in the previous section with respect to generic linear inhomogeneous and anisotropic perturbations by
linearizing the most general propagation and constraint equations for this $f(R)$ theory around the $\Lambda CDM$ solution. This is done using the $1+3$ covariant approach to perturbations [43, 44, 45, 46, 41, 47, 48], where quantities that vanish in the background spacetime are considered to be first order and are automatically gauge-invariant locally by virtue of the Stewart-Walker lemma [49]. In order to write down the set of linear equations we first need to choose a physically motivated frame $u^a$. The natural choice is the one which is along the four-velocity of the total matter.

The usual covariant and gauge-invariant inhomogeneity variables of the total matter and expansion are given by [45, 43]

$$D^m_a = \frac{a \tilde{\nabla}_a \mu_m}{\mu_m}, \quad Z_a = a \tilde{\nabla}_a \Theta,$$  

where the information about our deviation from standard GR is carried by the following dimensionless gradient quantities [41, 50]:

$$R_a = a \tilde{\nabla}_a R, \quad \Re_a = a \tilde{\nabla}_a \dot{R}.$$  

For the dust and stiff-component fluids, we have

$$D^d_a = \frac{a \tilde{\nabla}_a \mu_d}{\mu_d}, \quad D^s_a = \frac{a \tilde{\nabla}_a \mu_s}{\mu_s}$$

where $\mu_m = \mu_d + \mu_s$ and $\mu_m D^m_a = \mu_d D^d_a + \mu_s D^s_a$. The set of linearized evolution equations describing the perturbations in the dust-stiff fluid mixture is given by the first-order coupled system of equations [51, 41]:

$$\dot{D}^d_a + Z_a = 0,$$  

$$\dot{D}^s_a - \Theta D^s_a + 2Z_a = 0,$$  

$$\dot{Z}_a - \left( \frac{\ddot{R}}{f} - \frac{2}{3} \Theta \right) Z_a + \frac{\mu_m}{f} D^m_a - \Theta \frac{f''}{f} \Re_a + \frac{f''}{f} \tilde{\nabla}^2 \Re_a - \left( \frac{1}{2} - \frac{1}{2} \frac{f f''}{f'^2} - \frac{f'' \mu_m}{f'^2} - \dot{R} \Theta \left( \frac{f''}{f} \right)^2 + \dot{R} \Theta \frac{f''}{f} \right) \Re_a = 0,$$  

$$\dot{\Re}_a - \Re_a = 0,$$  

$$\dot{\Re}_a + \left( \Theta + 2 \frac{\ddot{R}}{f''} \right) \Re_a + \dot{R} Z_a - \frac{\mu_m}{3 f''} D^m_a - \tilde{\nabla}^2 \Re_a + \left( \frac{\ddot{R}}{f''} + \frac{\dot{R}^2 f''}{f''} + \Theta \frac{f''}{f''} + \frac{f'}{3 f''} - \frac{R}{3} \right) \Re_a = 0.$$

This system of 5-coupled partial differential equations involving vectorial gradients is difficult to solve. However, using the technique of uniquely decomposing a vector into divergence-free (solenoidal) and curl-free (irrotational) parts and applying harmonic decomposition, the above system can be rendered solvable.
3.1. Evolution of the scalar perturbations

If we take the spherically symmetric (trace) of gradient quantities (vectors) defined in (35) and (36), we obtain variables that characterize the evolution of the spherically symmetric part of the gradients. Since matter on cosmological scales is generally thought to follow spherical clustering, these new variables, given below, are what we are interested in knowing the evolutions of:

\[
\Delta_m = a \widetilde{\nabla}^a D^m_a, \quad \Delta_d = a \widetilde{\nabla}^a D^d_a, \quad \Delta_s = a \widetilde{\nabla}^a D^s_a,
\]
\[
\mathcal{Z} = a \widetilde{\nabla}^a Z_a, \quad \mathcal{R} = a \widetilde{\nabla}^a R_a, \quad \mathcal{M} = a \widetilde{\nabla}^a M_a.
\] (43)

Given below are the evolution equations of the harmonically decomposed (see [44] for a detailed treatment of harmonics) gradient variables:

\[
\dot{\Delta}^k_m + \mathcal{Z}^k = 0, \quad \dot{\Delta}^k_s - \Theta \Delta^k_s + 2 \mathcal{Z}^k = 0,
\]
\[
\dot{\mathcal{Z}}^k - (\dot{R} f'' f' - \frac{2}{3} \Theta) \mathcal{Z}^k + \frac{\mu m}{f'} \Delta^k_m - \Theta \frac{f''}{f'} \mathcal{M}^k = 0,
\]
\[
\dot{\mathcal{R}}^k = -\mathcal{M}^k = 0, \quad \dot{\mathcal{M}}^k = \left( \Theta + 2 \dot{R} \frac{f''}{f''} - \frac{2}{3} \Theta \right) \mathcal{M}^k + \dot{R} \mathcal{Z}^k - \frac{\mu m}{3 f''} \Delta^k_m
\]
\[
\quad + \left[ \frac{k^2}{a^2} + \dot{R} \frac{f''}{f''} + \frac{\dot{R}^2}{3} \frac{f''}{f''} + \Theta \dot{R} \frac{f''}{f''} + \frac{f'}{3 f''} - \frac{R}{3} \right] \mathcal{R}^k = 0.
\] (46) (47) (48)

where any separable scalar gradient is defined in terms of its harmonic components by

\[
X = \sum_k X^k(t) Q_k(x)
\] (49)

and using the Laplace-Beltrami operator

\[
\widetilde{\nabla}^2 Q = -\frac{k^2}{a^2} Q
\] (50)

with the wave number \( k = \frac{2 \pi a}{\lambda} \) and \( \dot{Q}_x = 0 \).

In redshift space, the evolutions of the perturbations can be given by...
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\begin{align}
(1 + z)^2 \frac{d\Delta_k}{dz} &= \mathcal{Z}^k, \\
(1 + z)^2 \frac{d\Delta_k}{dz} &= -\Theta \Delta_s^k + 2\mathcal{Z}^k, \\
(1 + z)^2 \frac{d\mathcal{Z}^k}{dz} &= -(\dot{R} \frac{f''}{f'} - \frac{2}{3} \Theta) \mathcal{Z}^k + \frac{\mu_m f_m}{f'} \Delta_m^k - \Theta \frac{f''}{f'} \mathcal{R}^k \\
&\quad - \left[ \frac{1}{2} + \frac{f'' k^2}{f'} a^2 \right] \frac{1}{2} \frac{f f''}{f' a^2} + \frac{f'' \mu_m}{f' a^2} - \dot{R} \Theta \left( \frac{f''}{f'} \right)^2 - \hat{R} \Theta \frac{f''}{f'} \right] \mathcal{R}^k, \\
(1 + z)^2 \frac{d\mathcal{R}^k}{dz} &= -\mathcal{R}^k, \\
(1 + z)^2 \frac{d\mathcal{R}^k}{dz} &= \left( \Theta + 2 \hat{R} \frac{f''}{f'} \right) \mathcal{R}^k + \dot{R} \mathcal{Z}^k - \frac{\mu_m}{3f''} \Delta_m^k \\
&\quad + \left[ \frac{k^2}{a^2} + \hat{R} \frac{f''}{f'} + \hat{R}^2 \frac{(f''')}{f'} + \Theta \hat{R} \frac{f''}{f'} + \frac{f'}{3f''} \frac{R}{3} \right] \mathcal{R}^k,
\end{align}

where the usual definition of redshift $1 + z = \frac{a_0}{a}$ is used.$\S$

To solve the set of scalar perturbation equations numerically, we redefine the following normalized quantities:

\[
\mathcal{Z}(z) = H_0 Z_n(z), \quad \mathcal{R}(z) = H_0^2 \mathcal{R}_n(z), \quad \mathcal{R}(z) = H_0^3 \mathcal{R}_n(z)
\]

\[
H(z) = H_0 h(z), \quad k = H_0 \kappa.
\]

Using the above quantities we are able to rewrite Equations (51-55) as a system of five ODEs:

$\S$ For practical purposes, we have normalized $a_0 = 1$ today.
(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \frac{d\Delta_n^k}{dz}} = Z_n^k(z), \quad (57)

(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \frac{dZ_n^k}{dz}} = -3\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \Delta_n^k + 2Z_n^k(z)}, \quad (58)

(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \frac{dZ_n^k(z)}{dz}} = \left[2\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A} - \frac{36\Omega_{0d}\Omega_{0s}(1 + z)^3}{27\Omega_{0d}^2 - 4\Omega_{0s}}\right]Z_n^k(z)

+ \frac{3(1 + z)^3\Omega_{0d}\Delta_n^k + (1 + z)^3\Omega_{0d}\Delta_s^k}{\left[1 - \frac{4\Omega_{0d}}{9\Omega_{0d}^2} ((1 + z)^3\Omega_{0d} + 4\Omega_A)\right]} + \frac{4\Omega_{0s}}{9\Omega_{0d}^2 - 4\Omega_{0s}}((1 + z)^3\Omega_{0d} + 4\Omega_A)\mathcal{R}_n^k

- \left[\frac{1}{2} - \frac{4\Omega_{0s}(1 + z)^2\kappa^2}{2\Omega_s \left[2(1 + z)^3\Omega_{0d} + 6\Omega_A - \frac{2\Omega_s ((1 + z)^3\Omega_{0d} + 4\Omega_A)^2}{3\Omega_{0d}^2}\right]} - \frac{4\Omega_{0s} \left[3(1 + z)^3(\Omega_{0d} + (1 + z)^3\Omega_{0s})\right]}{27\Omega_{0d}^2 \left[1 - \frac{4\Omega_{0s}}{9\Omega_{0d}^2}((1 + z)^3\Omega_{0d} + 4\Omega_A)\right]^2} \mathcal{R}_n^k = 0, \quad (59)

(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \frac{dR_n^k}{dz}} = -\mathcal{R}_n^k, \quad (60)

(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \frac{d\Delta_n^k}{dz}} = 3\sqrt{(1 + z)^3\Omega_{0d} + \Omega_A \Delta_n^k} - 9\Omega_{0d}(1 + z)^3h(z)Z_n^k

+ \frac{81\Omega_{0d}^2(1 + z)^3[\Omega_{0d}\Delta_n^k + \Omega_{0s}(1 + z)^3\Delta_s^k]}{4\Omega_{0s}}

+ \left[(1 + z)^2\kappa^2 - \frac{27\Omega_{0d}^2 - 4\Omega_{0s} ((1 + z)^3\Omega_{0d} + 4\Omega_A)}{4\Omega_{0s}} - (1 + z)^3\Omega_{0d} - 4\Omega_A\right] \mathcal{R}_n^k. \quad (61)

The $k$-dependence of the amplitudes of the perturbations at a given redshift is depicted by the plots of the power spectrum defined as [52]:

$$\langle \Delta_d(k_1)\Delta_d(k_2) \rangle = P(k_1)\delta(k_1 + k_2), \quad (62)$$

where $k_1$ and $k_2$ are wavevectors of Fourier components of the solutions of the above system. Since isotropy of the perturbations is assumed, we simply write $P(k_1)$ instead of $P(k_1)$. The power spectrum of the fluctuations in GR (plus $\Lambda$CDM) is scale-invariant. Thus any deviation from such invariance in our $f(R)$ models calls for a closer analysis of structure formation scenarios in such models. It is also a powerful way of putting tight constraints on the viability of the $f(R)$ gravitational models in question.

The following plots (Figs. (1)-(7)) show how the growth of the dust perturbations depend on scale for different values of $\Omega_s$ when $\alpha < 0$. The vertical axis shows the normalized power spectrum $P(\kappa) = \left[\frac{\Delta_d(\kappa)}{\Delta_d(\kappa=0)}\right]^2$ today ($z = 0$). Note the general drop in
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power for the \( R + \alpha R^2 - 2\Lambda \) model as opposed to the scale indifference of the perturbations in GR.

**Figure 1.** Power spectrum for \( \Omega_s = 10^{-1} \).

**Figure 2.** Power spectrum for \( \Omega_s = 10^{-2} \).

**Figure 3.** Power spectrum for \( \Omega_s = 10^{-3} \).

**Figure 4.** Power spectrum for \( \Omega_s = 10^{-4} \).

**Figure 5.** Power spectrum for \( \Omega_s = 10^{-5} \).

**Figure 6.** Power spectrum for \( \Omega_s = 0 \). Note that \( \Omega_s = 0 \) corresponds to GR+ΛCDM; hence the scale-invariance.
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Figure 7. Power spectrum for different values of $\Omega_s = 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$.

In the case of $\alpha > 0$, we observe a flat spectrum (Figs. (8)-(13)) on the longer scale regimes and the spectra start to rise on smaller and smaller scales (larger and larger $\kappa$ values).

Figure 8. Power spectrum for $\Omega_s = -10^{-1}$.

Figure 9. Power spectrum for $\Omega_s = -10^{-2}$.

Figure 10. Power spectrum for $\Omega_s = -10^{-3}$.

4. The quasi-static approximation

It has been shown [51, 53, 54] that on small scales, the contributions from the temporal fluctuations of $\mathcal{R}^k$ can be neglected, i.e., $\dot{\mathcal{R}}^k \simeq 0 \Rightarrow \mathcal{R}^k \simeq 0$, because they are quickly damped away. If we apply this approximation to Eqns (57)-(61) the resulting sub-
horizon \((\kappa^2 \gg H^2)\) perturbation equations reduce to

\[
(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00}\frac{d\Delta^k_d}{dz}} = Z^k_n(z),
\]

\[
(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00}\frac{d\Delta^k_n}{dz}} = -3\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00}\Delta^k_s + 2z^k_n(z)},
\]

\[
(1 + z)\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00}\frac{dZ^k_n(z)}{dz}} = \left[2\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00} - \frac{36\Omega_{0d}\Omega_{0s}(1 + z)^3}{27\Omega_{0d}^2 - 4\Omega_{0s}}} \right] Z^k_n(z)
\]

\[
+ \frac{3(1 + z)^3}{1 - \frac{4\Omega_{00}}{9\Omega_{0d}}((1 + z)^3\Omega_{0d} + 4\Omega_{00})} \left[\frac{\Omega_{0d}\Delta^k_d + (1 + z)^3\Omega_{0s}\Delta^k_s}{\Omega_{0d} + 4\Omega_{00}}\right] - \frac{1}{2} - \frac{4\Omega_{0s}(1 + z)^2\kappa^2}{3}\left[9\Omega_{0d}^2 - 4\Omega_{0s}((1 + z)^3\Omega_{0d} + 4\Omega_{00})\right]
\]

\[
2\Omega_{0s}\left[3(1 + z)^3\Omega_{0d} + 6\Omega_{00} - \frac{2\Omega_{0s}(1 + z)^3\Omega_{0d} + 4\Omega_{00})}{3\Omega_{0d}^2}\right] - \frac{4\Omega_{0s}[3(1 + z)^3(\Omega_{0d} + (1 + z)^3\Omega_{0s})]}{27\Omega_{0d}^2}\left[1 - \frac{4\Omega_{0s}}{9\Omega_{0d}}((1 + z)^3\Omega_{0d} + 4\Omega_{00})\right]^2
\]

\[
+ \frac{16\Omega_{0s}^2((1 + z)^3(\Omega_{0d} + 4\Omega_{00}))}{27\Omega_{0d}^2(1 - \frac{4\Omega_{0s}}{9\Omega_{0d}}((1 + z)^3\Omega_{0d} + 4\Omega_{00})^2)}\times \frac{9\Omega_{0d}(1 + z)^3\sqrt{(1 + z)^3\Omega_{0d} + \Omega_{00}}}{4\Omega_{0s}(1 + z)^2\kappa^2 - 9\Omega_{0d}^2} Z^k_n(z)
\]

\[
- \frac{81\Omega_{0s}^2((1 + z)^3(\Omega_{0d}^2 + \Omega_{0s}^2 + (1 + z)^3\Delta^k_s))}{4\Omega_{0s}^2(1 + z)^2\kappa^2 - 9\Omega_{0d}^2}
\].

One can see from the following plots that the quasi-static solutions (blue dashed lines) of the dust perturbations are indeed a good approximation to the solution of the full equations (red solid lines) for the \(R + \alpha R^2 - 2\Lambda\) model. In these plots, the vertical axis \(\Delta_d(z) \equiv D\) corresponds to the amplitudes of the dust perturbations.

We observe from these plots (Figs.14)-(19)) that for large enough \(\kappa\) values (smaller scales), the quasi-static and full solutions are indistinguishable both for \(\alpha < 0\ (\Omega_{0s} > 0)\) and \(\alpha > 0\ (\Omega_{0s} < 0)\), thus confirming the validity of the quasi-static approximation for our present model.
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5. Discussion and Conclusion

Using the scheme of first order covariant perturbations, we have shown that it is possible to break the degeneracy that exists between ΛCDM and the reconstructed $f(R)$ models which, at the background level, describe exactly the same cosmic expansion history. Using different values for the characteristic parameter $\alpha$ (or correspondingly different $\Omega_{0s}$ by virtue of Eqn (34)), we have shown that first-order perturbations show that structures in these two (i.e., ΛCDM and $R + \alpha R^2 - 2\Lambda$) models evolve at different rates, the former independently of the wavenumber $\kappa$, the latter in accordance with the $\kappa$-dependence of Eqns (53) and (55). In the long wavelength regime, we observe flat power spectrum in both models. Moreover, the peculiar drop in power in the short-wavelength regime in some $f(R)$ theories [52, 54] is a feature we have observed in the physically
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interesting (α < 0, Ω_s > 0) models we considered. In the case of ghost solutions (α > 0, Ω_s < 0), while the background degeneracy still remains broken, the power spectra rise, rather than drop, with decreasing scale (increasing κ-value). We have made an application of the quasi-static technique of approximation for the solutions of the perturbation equations. The technique appears to be a very good approximation to the full short-wavelength solutions for the models we studied.

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