Spectral Turán-Type Problems on Cancellative Hypergraphs

Zhenyu Ni\textsuperscript{a} Lele Liu\textsuperscript{b} Liying Kang\textsuperscript{c}

Submitted: Aug 29, 2023; Accepted: May 1, 2024; Published: May 17, 2024
© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

Let $G$ be a cancellative 3-uniform hypergraph in which the symmetric difference of any two edges is not contained in a third one. Equivalently, a 3-uniform hypergraph $G$ is cancellative if and only if $G$ is $\{F_4, F_5\}$-free, where $F_4 = \{abc, abd, bcd\}$ and $F_5 = \{abc, abd, cde\}$. A classical result in extremal combinatorics states that the maximum size of a cancellative hypergraph is achieved by the balanced complete tripartite 3-uniform hypergraph, which was firstly proved by Bollobás and later by Keevash and Mubayi. In this paper, we consider spectral extremal problems for cancellative hypergraphs. More precisely, we determine the maximum $p$-spectral radius of cancellative 3-uniform hypergraphs, and characterize the extremal hypergraph. As a by-product, we give an alternative proof of Bollobás’ result from spectral viewpoint.

Mathematics Subject Classifications: 05C35, 05C50, 05C65

1 Introduction

Consider an $r$-uniform hypergraph (or $r$-graph for brevity) $G$ and a family of $r$-graphs $\mathcal{F}$. We say $G$ is $\mathcal{F}$-free if $G$ does not contain any member of $\mathcal{F}$ as a subhypergraph. The Turán number $ex(n, \mathcal{F})$ is the maximum number of edges of an $\mathcal{F}$-free hypergraph on $n$ vertices. Determining Turán numbers of graphs and hypergraphs is one of the central problems in extremal combinatorics. For graphs, the problem was asymptotically solved for all non-bipartite graphs by the celebrated Erdős-Stone-Simonovits Theorem [5, 6]. By contrast with the graph case, there is comparatively little understanding of the hypergraph Turán number. We refer the reader to the surveys [8, 11, 14].

\textsuperscript{a}School of Mathematics and Statistics, Hainan University, Haikou 570228, P.R. China (995264@hainanu.edu.cn).

\textsuperscript{b}School of Mathematical Sciences, Anhui University, Hefei 230601, P.R. China (liu@ahu.edu.cn).

Corresponding author.

\textsuperscript{c}Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China (lykang@shu.edu.cn).
In this paper we consider spectral analogues of Turán-type problems for $r$-graphs. For $r = 2$, the picture is relatively complete, due in large part to a longstanding project of Nikiforov, see e.g., [15] for details. However, for $r \geq 3$ there are very few known results. In [12], Keevash, Lenz and Mubayi determined the maximum $p$-spectral radius of any $3$-graph on $n$ vertices not containing the Fano plane when $n$ is sufficiently large. They also obtained a $p$-spectral version of the Erdős-Ko-Rado theorem on $t$-intersecting $r$-graphs. Recently, Ellingham, Lu and Wang [4] showed that the $n$-vertex outerplanar $3$-graph of maximum spectral radius is the unique $3$-graph whose shadow graph is the join of an isolated vertex and the path $P_{n-1}$. Gao, Chang and Hou [9] studied the spectral extremal problem for $K_{r+1}^+$-free $r$-graphs among linear hypergraphs, where $K_{r+1}^+$ is the $r$-expansion of the complete graph $K_{r+1}$, i.e., $K_{r+1}^+$ is obtained from $K_{r+1}$ by enlarging each edge of $K_{r+1}$ with $(r - 2)$ new vertices disjoint from $V(K_{r+1})$ such that distinct edges of $K_{r+1}$ are enlarged by distinct vertices. Generalizing Gao-Chang-Hou’s result, She, Fan, Kang and Hou [18] considered the linear spectral Turán type problems for the expansion of a color critical graph.

To state our results precisely, we need some basic definitions and notations. A $3$-graph is tripartite or 3-partite if it has a vertex partition into three parts such that every edge has exactly one vertex in each part. Let $T_3(n)$ be the complete 3-partite 3-graph on $n$ vertices with part sizes $\left\lfloor n/3 \right\rfloor, \left\lfloor (n+1)/3 \right\rfloor, \left\lfloor (n+2)/3 \right\rfloor$, and $t_3(n)$ be the number of edges of $T_3(n)$. That is,

$$t_3(n) = \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor.$$ 

We call an $r$-graph $G$ cancellative if $G$ has the property that for any edges $A$, $B$, $C$ whenever $A \cup B = A \cup C$, we have $B = C$. Equivalently, $G$ is cancellative if $G$ has no three distinct triples $A$, $B$, $C$ satisfying $B \Delta C \subseteq A$, where $\Delta$ is the symmetric difference. For graphs, the condition is equivalent to saying that $G$ is triangle-free. Moving on to 3-graphs, we observe that $B \Delta C \subseteq A$ can only occur when $|B \cap C| = 2$ for $B \neq C$. This leads us to identify the two non-isomorphic configurations that are forbidden in a cancellative 3-graph: $F_3 = \{abc, abd, bcd\}$ and $F_5 = \{abc, abd, cde\}$.

It is well-known that the study of Turán numbers dates back to Mantel’s theorem, which states that $ex(n, K_3) = \left\lfloor n^2/4 \right\rfloor$. As an extension of the problem to hypergraphs, Katona conjectured (c.f. [1]), and Bollobás proved the following result.

**Theorem 1** ([1]). A cancellative 3-graph on $n$ vertices has at most $t_3(n)$ edges, with equality only for $T_3(n)$.

In [10], Keevash and Mubayi presented a new proof of Bollobás’ result, and further proved a stability theorem for cancellative hypergraphs.

The aim of this paper is to establish a $p$-spectral analogue of Theorem 1. One motivation for studying this problem is that the spectral radius of an $r$-graph $G$ is an upper bound for the average degree of $G$, and hence any upper bound on the spectral radius also gives an upper bound on the size of $G$. It is important to note that we will not rely on the usage of Theorem 1 to accomplish the proof of our main result (Theorem 10), although one can simplify the proof with the help of Theorem 1. As a by-product, we give an
alternative proof of Bollobás’ result for the case $3 \mid n$ from spectral viewpoint. Our main results can be summarized as follows.

**Theorem 2.** Let $p \geq 1$ and $G$ be a cancellative 3-graph on $n$ vertices.

1. If $p \geq 3$, then $\lambda^{(p)}(G) \leq \lambda^{(p)}(T_3(n))$, with equality if and only if $G = T_3(n)$.

2. If $p = 1$ and $G$ has at least one edge, then $\lambda^{(1)}(G) = 2/9$.

## 2 Preliminaries

In this section we introduce definitions and notation that will be used throughout the paper, and give some preliminary lemmas.

Given an $r$-graph $G = (V(G), E(G))$ and a vertex $v$ of $G$. The **link** $L_G(v)$ is the $(r - 1)$-graph consisting of all $S \subseteq V(G)$ with $|S| = r - 1$ and $S \cup \{v\} \in E(G)$. The **degree** $d_G(v)$ of $v$ is the size of $L_G(v)$. As usual, we denote by $N_G(v)$ the neighborhood of a vertex $v$, i.e., the set formed by all the vertices which form an edge with $v$. In the above mentioned notation, we will skip the index $G$ whenever $G$ is understood from the context.

The **shadow graph** of $G$, denoted by $\partial(G)$, is the graph with $V(\partial(G)) = V(G)$ and $E(\partial(G))$ consisting of all pairs of vertices that belong to an edge of $G$, i.e., $E(\partial(G)) = \{e : |e| = 2, e \subseteq f \text{ for some } f \in E(G)\}$. For more definitions and notation from hypergraph theory, see e.g., [2].

For any real number $p \geq 1$, the $p$-spectral radius was introduced by Keevash, Lenz and Mubayi [12] and subsequently studied by Nikiforov [16, 17]. Let $G$ be an $r$-graph of order $n$, the polynomial form of $G$ is a multi-linear function $P_G(x) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ as

$$P_G(x) = r! \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$ 

The **p-spectral radius** of $G$ is defined as

$$\lambda^{(p)}(G) := \max_{\|x\|_p = 1} P_G(x),$$

where $\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$.

For any real number $p \geq 1$, we denote by $S_{p,+}^{n-1}$ the set of all nonnegative real vectors $x \in \mathbb{R}^n$ with $\|x\|_p = 1$. If $x \in \mathbb{R}^n$ is a vector with $\|x\|_p = 1$ such that $\lambda^{(p)}(G) = P_G(x)$, then $x$ is called an **eigenvector** corresponding to $\lambda^{(p)}(G)$. Note that $P_G(x)$ can always reach its maximum at some nonnegative vectors. By Lagrange’s method, we have the **eigen-equations** for $\lambda^{(p)}(G)$ and $x \in S_{p,+}^{n-1}$ as follows:

$$\lambda^{(p)}(G)x_i^{p-1} = (r - 1)! \sum_{\{i, i_2, \ldots, i_r\} \in E(G)} x_{i_2} \cdots x_{i_r} \text{ for } x_i > 0.$$
It is worth mentioning that the $p$-spectral radius $\lambda_p(G)$ shows remarkable connections with some hypergraph invariants. For instance, $\lambda_1(G)/r!$ is the Lagrangian of $G$, $\lambda_p(G)/(r-1)!$ is the usual spectral radius introduced by Cooper and Dutle [3], and $\lambda_\infty(G)/r!$ is the number of edges of $G$ (see [16, Proposition 2.10]).

Given two vertices $u$ and $v$, we say that $u$ and $v$ are equivalent in $G$, in writing $u \sim v$, if transposing $u$ and $v$ and leaving the remaining vertices intact, we get an automorphism of $G$.

**Lemma 3** ([16]). Let $G$ be a uniform hypergraph on $n$ vertices and $u \sim v$. If $p > 1$ and $x \in \mathbb{S}_{p-1}$ is an eigenvector to $\lambda_p(G)$, then $x_u = x_v$.

### 3 Cancellative hypergraphs of maximum $p$-spectral radius

The aim of this section is to give a proof of Theorem 2. We split it into Theorem 10 – Theorem 18, which deal with $p = 3$, $p > 3$ and $p = 1$, respectively.

#### 3.1 General properties on cancellative hypergraphs

We start this subsection with a basic fact.

**Lemma 4.** Let $G$ be a cancellative hypergraph, and $u, v$ be adjacent vertices. Then $L(u)$ and $L(v)$ are edge-disjoint graphs.

**Proof.** Assume by contradiction that $e \in E(L(u)) \cap E(L(v))$. Since $u$ and $v$ are adjacent in $G$, we have $\{u, v\} \subseteq e_1 \in E(G)$ for some edge $e_1$. Hence, $e_2 = e \cup \{u\}$, $e_3 = e \cup \{v\}$, and $e_1$ are three edges of $G$ such that $e_2 \Delta e_3 \subseteq e_1$, a contradiction. $\square$

Let $G$ be a 3-graph and $v \in V(G)$. We denote by $E_v(G)$ the collection of edges of $G$ containing $v$, i.e., $E_v(G) = \{e : v \in e \in E(G)\}$. For a pair of vertices $u$ and $v$ in $G$, we denote by $T_v^u(G)$ a new 3-graph with $V(T_v^u(G)) = V(G)$ and

$$E(T_v^u(G)) = (E(G) \setminus E_v(G)) \cup \{(e \setminus \{u\}) \cup \{v\} : e \in E_u(G) \setminus E_v(G)\}.$$  

**Lemma 5.** Let $G$ be a cancellative 3-graph. Then $T_v^u(G)$ is also cancellative for any $u, v \in V(G)$.

**Proof.** Suppose to the contrary that there exist three edges $e_1, e_2, e_3 \in T_v^u(G)$ such that $e_1 \Delta e_2 \subseteq e_3$. Recalling the definition of $T_v^u(G)$, we deduce that $u, v$ are non-adjacent in $T_v^u(G)$, and $(e \cup \{u\}) \setminus \{v\} \in E(G)$ for any $e \in E_v(T_v^u(G))$. On the other hand, since $G$ is cancellative, we have $v \in e_1 \cup e_2 \cup e_3$. Denote by $\alpha$ the number of edges $e_1, e_2, e_3$ containing $v$. It suffices to consider the following three cases.

**Case 1.** $\alpha = 3$. We have $v \in e_1 \cap e_2 \cap e_3$. Hence, $e_1' = (e_1 \cup \{u\}) \setminus \{v\}$, $e_2' = (e_2 \cup \{u\}) \setminus \{v\}$ and $e_3' = (e_3 \cup \{u\}) \setminus \{v\}$ are three edges in $G$ with $e_1' \Delta e_2' \subset e_3'$. This contradicts the fact that $G$ is cancellative.

**Case 2.** $\alpha = 2$. Without loss of generality, we assume $v \in (e_1 \cap e_2) \setminus e_3$ or $v \in (e_1 \cap e_3) \setminus e_2$. If $v \in (e_1 \cap e_2) \setminus e_3$, then $e_3 \in E(G)$. It follows that $e_1' = (e_1 \cup \{u\}) \setminus \{v\}$,
Let \( e_1, e_2, e_3 \) are three edges of \( G \) with \( e_1' \triangle e_2' \subset e_3 \), which is a contradiction. If \( v \in (e_1 \cap e_3) \setminus e_2 \), then \( e_2 \in E(G) \). It follows that \( e_1' = (e_1 \cup \{u\}) \setminus \{v\} \), \( e_2 \) and \( e_3' = (e_3 \cup \{u\}) \setminus \{v\} \) are three edges of \( G \) with \( e_1' \triangle e_2 \subset e_3' \), a contradiction.

**Case 3.** \( \alpha = 1 \). Without loss of generality, we assume \( v \in e_3 \setminus (e_1 \cup e_2) \). Then \( e_1 \in E(G) \) and \( e_2 \in E(G) \). We immediately obtain that \( e_1, e_2 \) and \( e_3' = (e_3 \cup \{u\}) \setminus \{v\} \) are three edges of \( G \) with \( e_1 \triangle e_2 \subset e_3' \). This is a contradiction and proves Lemma 5. \( \square \)

The following result can be found in [16, Proposition 7.2].

**Lemma 6 ([16]).** Let \( p > 1 \) and \( G \) be a complete 3-partite 3-graph. Then

\[
\lambda^{(p)}(G) = \frac{6}{\sqrt{27}} \cdot (|E(G)|)^{1-1/p}.
\]

**Proof.** Assume that \( V_1, V_2 \) and \( V_3 \) are the vertex classes of \( G \) with \( n_i := |V_i| \) and \( n_1 \geq n_2 \geq n_3 \). Let \( \mathbf{x} \in \mathbb{R}_{p+}^{n_i} \) be an eigenvector corresponding to \( \lambda^{(p)}(G) \). By Lemma 3, for \( i = 1, 2, 3 \) we denote \( a_i := x_v \) for \( v \in V_i \), and set \( \lambda := \lambda^{(p)}(G) \) for short. In light of eigenequation (2), we find that

\[
\begin{align*}
\lambda a_1^{p-1} &= 2n_2n_3a_2a_3, \\
\lambda a_2^{p-1} &= 2n_1n_3a_1a_3, \\
\lambda a_3^{p-1} &= 2n_1n_2a_1a_2,
\end{align*}
\]

from which we obtain that \( a_i = (3n_i)^{-1/p}, i = 1, 2, 3 \). Therefore,

\[
\lambda = 2 \cdot (27 \cdot n_1n_2n_3)^{1-1/p} = \frac{6}{\sqrt{27}} \cdot (|E(G)|)^{1-1/p}.
\]

This completes the proof of Lemma 6. \( \square \)

### 3.2 Extremal \( p \)-spectral radius of cancellative hypergraphs

Let \( \text{SPEX}_p(n, \{F_1, F_3\}) \) be the set of all 3-graphs attaining the maximum \( p \)-spectral radius among cancellative hypergraphs on \( n \) vertices. If \( p = 3 \), we will denote it by \( \text{SPEX}(n, \{F_1, F_3\}) \) for short. Given a vector \( \mathbf{x} \in \mathbb{R}^n \) and a set \( S \subset [n] := \{1, 2, \ldots, n\} \), we write \( \mathbf{x}(S) := \prod_{i \in S} x_i \) for short. The support set \( S \) of a vector \( \mathbf{x} \) is the set of the indices of non-zero elements in \( \mathbf{x} \), i.e., \( S = \{i \in [n] : x_i \neq 0\} \). Also, we denote by \( x_{\min} := \min\{|x_i| : i \in [n]\} \) and \( x_{\max} := \max\{|x_i| : i \in [n]\} \).

**Lemma 7.** Let \( p > 1 \), \( G \in \text{SPEX}_p(n, \{F_1, F_3\}) \), and \( \mathbf{x} \in \mathbb{R}_{p+}^{n_i} \) be an eigenvector corresponding to \( \lambda^{(p)}(G) \). If \( u, v \) are two non-adjacent vertices, then \( x_u = x_v \).

**Proof.** Assume \( u \) and \( v \) are two non-adjacent vertices in \( G \). Since \( G \) is a cancellative 3-graph, we have \( T^*_u(G) \) is also cancellative by Lemma 5. It follows from (1) and (2) that

\[
\begin{align*}
\lambda^{(p)}(T^*_u(G)) &\geq 6 \sum_{e \in E(G)} \mathbf{x}(e) - 6 \sum_{e \in E_u(G)} \mathbf{x}(e) + 6 \sum_{e \in E_v(G)} \mathbf{x}(e \setminus \{v\}) \cdot x_u \\
&= \lambda^{(p)}(G) - 3\lambda^{(p)}(G)x_u^p + 3\lambda^{(p)}(G)x_v^{p-1}x_u \\
&= \lambda^{(p)}(G) + 3\lambda^{(p)}(G)(x_v^{p-1} - x_u^{p-1}) \cdot x_u,
\end{align*}
\]

THE ELECTRONIC JOURNAL OF COMBINATORICS 31(2) (2024), #P2.32

5
which yields that \( x_u \geq x_v \). Likewise, we also have \( x_v \geq x_u \) by considering \( T_u^w(G) \). Hence, \( x_u = x_v \), completing the proof of Lemma 7. 

\[ \square \]

**Lemma 8.** Let \( p > 1 \), \( G \in \text{SPEX}_p(n, \{ F_4, F_5 \}) \), and \( u, v \) be two non-adjacent vertices. Then there exists a cancellative 3-graph \( H \) such that

\[
\lambda(H) = \lambda(G), \quad \text{and } d_H(w) \leq d_G(w), \quad w \in V(G).
\]

**Proof.** Assume that \( x \in \mathbb{S}^{n-1} \) is an eigenvector corresponding to \( \lambda(G) \). By Lemma 7, \( x_u = x_v \). Without loss of generality, we assume \( d_G(u) \geq d_G(v) \). In view of (1) and (2), we have

\[
\lambda(G)(T_u^w(G)) \geq 6 \sum_{e \in E(G)} x(e) - 6 \sum_{e \in E_u(G)} x(e) + 6 \sum_{e \in E_v(G)} x(e \setminus \{v\}) \cdot x_u
\]

\[ = \lambda(G) + 3\lambda(G)(x_v^{p-1} - x_u^{p-1}) \cdot x_u
\]

\[ = \lambda(G). \]

Observe that \( T_u^w(G) \) is a cancellative 3-graph and \( G \in \text{SPEX}_p(n, \{ F_4, F_5 \}) \). We immediately obtain that \( \lambda(G)(T_u^w(G)) = \lambda(G) \). It is straightforward to check that \( H := T_u^w(G) \) is a cancellative 3-graph satisfying (3), as desired. 

\[ \square \]

Next, we give an estimation on the entries of eigenvectors corresponding to \( \lambda(G) \).

**Lemma 9.** Let \( G \in \text{SPEX}_p(n, \{ F_4, F_5 \}) \) and \( x \in \mathbb{S}^{n-1}_p \) be an eigenvector corresponding to \( \lambda(G) \). If \( 1 < p \leq 3 \), then

\[
x_{\min} > \left( \frac{3}{4} \right)^{2(p-1)} \cdot x_{\max}.
\]

**Proof.** Suppose to the contrary that \( x_{\min} \leq \left( \frac{3}{4} \right)^{2(p-1)} \cdot x_{\max} \). Let \( u \) and \( v \) be two vertices such that \( x_u = x_{\min} \) and \( x_v = x_{\max} > 0 \). Then we have

\[
\left( 1 + \frac{x_u}{x_v} \right) \left( \frac{x_u}{x_v} \right)^{p-1} \leq \left( 1 + \left( \frac{3}{4} \right)^{2(p-1)} \right) \left( \frac{3}{4} \right)^2 \leq \frac{7}{4} \cdot \frac{9}{16} < 1,
\]

which implies that

\[
x_v^p - x_u^p > x_u^{p-1} x_v. \tag{4}
\]

On the other hand, by eigenequations we have

\[
2 \sum_{e \in E_u(G) \setminus E_v(G)} x(e) \geq \lambda(G)(x_v^p - x_u^p). \tag{5}
\]

Now, we consider the cancellative 3-graph \( T_u^w(G) \). In light of (1) and (5), we have

\[
\lambda(G)(T_u^w(G)) \geq 6 \sum_{e \in E(G)} x(e) - 6 \sum_{e \in E_u(G)} x(e) + 6 \sum_{e \in E_v(G) \setminus E_u(G)} x(e \setminus \{v\}) \cdot x_u
\]

\[
\geq \lambda(G) - 3\lambda(G)x_u^p + 3\lambda(G)(x_v^p - x_u^p) \cdot \frac{x_u}{x_v}
\]

\[ > \lambda(G) + 3\lambda(G) \left( -x_u^p + x_u^{p-1} x_v \cdot \frac{x_u}{x_v} \right)
\]

\[ = \lambda(G),
\]
where the third inequality is due to (4). This contradicts the fact that $G$ has maximum $p$-spectral radius over all cancellative hypergraphs. \hfill \square

Now, we are ready to give a proof of Theorem 2 for $p = 3$.

**Theorem 10.** Let $G$ be a cancellative 3-graph on $n$ vertices. Then $\lambda^{(3)}(G) \leq \lambda^{(3)}(T_3(n))$ with equality if and only if $G = T_3(n)$.

**Proof.** According to Lemma 8, we assume that $G^* \in \text{SPEX}(n, \{F_4, F_5\})$ is a 3-graph such that $L_{G^*}(u) = L_{G^*}(v)$ for any non-adjacent vertices $u$ and $v$.

Our first goal is to show $G^* = T_3(n)$ by Claim 11 – Claim 13. Assume that $x \in S^{n-1}_{3,+}$ is an eigenvector corresponding to $\lambda^{(3)}(G^*)$; $u_1$ is a vertex in $G^*$ such that $x_{u_1} = x_{\text{max}}$ and $u_2$ is a vertex with $x_{u_2} = \max\{x_v : v \in N_{G^*}(u_1)\}$. Let $U_1 := V(G^*) \setminus N_{G^*}(u_1)$ and $U_2 := V(G^*) \setminus N_{G^*}(u_2)$. Since $u_2 \in V(G^*) \setminus U_1$, there exists a vertex $u_3$ such that \{u_1, u_2, u_3\} $\in E(G^*)$. Let $U_3 = V(G^*) \setminus N_{G^*}(u_3)$. Recall that for any non-adjacent vertices $u$ and $v$ we have $L_{G^*}(u) = L_{G^*}(v)$. Hence, the sets $U_1$, $U_2$ and $U_3$ are well-defined.

**Claim 11.** The following statements hold:

1. $d_{G^*}(u_1) > n(n-1)/9$;
2. $d_{G^*}(u_2) > n(n-1)/12$;
3. $d_{G^*}(v) > n(n-1)/16$, $v \in V(G^*)$.

**Proof of Claim 11.** Since $T_3(n)$ is a cancellative 3-graph, it follows from Lemma 6 that

$$\lambda^{(3)}(G^*) \geq \lambda^{(3)}(T_3(n)) = 2 \cdot (t_3(n))^{2/3},$$

which is equivalent to

$$27 \cdot \left(\frac{\lambda^{(3)}(G^*)}{2}\right)^{3/2} \geq 27 \cdot t_3(n) = \begin{cases} n^3, & n \equiv 0 \pmod{3}, \\ (n-1)^2(n+2), & n \equiv 1 \pmod{3}, \\ (n-2)(n+1)^2, & n \equiv 2 \pmod{3}. \end{cases}$$

By simple algebra we see

$$\lambda^{(3)}(G^*) \geq \frac{2 \cdot ((n-2)(n+1)^2)^{2/3}}{9} > \frac{2n(n-1)}{9}. \quad (6)$$

(1). By eigenequation with respect to $u_1$, we have

$$\lambda^{(3)}(G^*)x_{u_1}^2 = 2 \sum_{\{u_1, i, j\} \in E(G^*)} x_i x_j \leq 2d_{G^*}(u_1)x_{u_1}^2.$$

Combining with (6), we get

$$d_{G^*}(u_1) \geq \frac{\lambda^{(3)}(G^*)}{2} > \frac{n(n-1)}{9}. \quad (7)$$

THE ELECTRONIC JOURNAL OF COMBINATORICS 31(2) (2024), #P2.32
(2). Since $L_{G^*}(u) = L_{G^*}(v)$ for any pair $u, v \in U_1$, we immediately obtain that $|(e \setminus \{u_2\}) \cap U_1| \leq 1$ for each $e \in E_{u_2}(G^*)$ by the definition of $U_1$. It follows from $x_{u_2} = \max\{x_v : v \in V(G^*) \setminus U_1\}$ that
\[
\lambda^{(3)}(G^*)x_{u_2}^2 = 2 \sum_{\{u_2, i, j\} \in E(G^*)} x_i x_j \leq 2d_{G^*}(u_2)x_{u_1}x_{u_2},
\]
which, together with Lemma 9 for $p = 3$, gives
\[
d_{G^*}(u_2) \geq \frac{x_{u_2}}{x_{u_1}} \cdot \frac{\lambda^{(3)}(G^*)}{2} \\
\geq \frac{3}{4} \cdot \frac{\lambda^{(3)}(G^*)}{2} \\
> \frac{1}{12} n(n - 1).
\]
The last inequality is due to (6).

(3). Let $v$ be an arbitrary vertex in $V(G^*)$. Then
\[
\lambda^{(3)}(G^*)x_{v}^2 = 2 \sum_{\{v, i, j\} \in E(G^*)} x_i x_j \leq 2d_{G^*}(v)x_{u_1}^2.
\]
Hence, by Lemma 9 and (6) we have
\[
d_{G^*}(v) \geq \left(\frac{x_v}{x_{u_1}}\right)^2 \cdot \frac{\lambda^{(3)}(G^*)}{2} > \frac{1}{16} n(n - 1),
\]
as desired. $\square$

Next, we consider the graph $H = L_{G^*}(u_1) \cup L_{G^*}(u_2) \cup L_{G^*}(u_3)$. Let $\phi : E(H) \to [3]$ be a mapping such that $\phi(f) = i$ if $f \in E_{G^*}(u_i)$, $i \in [3]$. By Lemma 4, $\phi$ is an edge coloring of $H$. For convenience, we denote $L := V(G^*) \setminus (U_1 \cup U_2 \cup U_3)$.

**Claim 12.** If $L \neq \emptyset$, then there is no rainbow star $K_{1,3}$ in the induced subgraph $H[L]$ with the coloring $\phi$.

**Proof of Claim 12.** Suppose to the contrary that there exist $v_1, v_2, v_3, v_4 \in L$ with $\phi(v_1v_2) = 1$, $\phi(v_1v_3) = 2$ and $\phi(v_1v_4) = 3$. We first show that $\{v_1, v_2, v_3, v_4\}$ induces a clique in $\partial(G^*)$ by contradiction. Without loss of generality, we assume $v_2v_3 \notin E(\partial(G^*))$. Then $L_{G^*}(v_2) = L_{G^*}(v_3)$. Since $\phi(v_1v_2) = 1$ and $\phi(v_1v_3) = 2$, we have $\{u_1, v_1, v_2\} \in E(G^*)$ and $\{u_2, v_1, v_3\} \in E(G^*)$. This implies that $e_1 = \{u_1, u_2, u_3\}$, $e_2 = \{u_1, v_1, v_2\}$ and $e_3 = \{u_2, v_1, v_3\}$ are three edges in $G^*$ with $e_2 \cap e_3 \subseteq e_1$, which is impossible.

On the other hand, since $L = V(G^*) \setminus (U_1 \cup U_2 \cup U_3)$, we have $v_iu_j \in E(\partial(G^*))$ for any $i \in [4]$, $j \in [3]$. Therefore, every pair of vertices in $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}$ is contained in an edge of $G^*$. Consider the graph
\[
H' := \left(\bigcup_{i=1}^{3} L_{G^*}(u_i)\right) \cup \left(\bigcup_{i=1}^{4} L_{G^*}(v_i)\right).
\]
By Claim 11, we have
\[
|E(H')| = \sum_{1 \leq i \leq 3} d_{G^*}(u_i) + \sum_{1 \leq j \leq 4} d_{G^*}(v_j) > \left( 1 + \frac{3}{4} + 5 \times \frac{9}{16} \right) \cdot \frac{1}{9}n(n-1) = \frac{73}{144}n(n-1) > \left( \frac{n}{2} \right).
\]
a contradiction completing the proof of Claim 12. \(\Box\)

**Claim 13.** \(L = \emptyset\).

**Proof of Claim 13.** Suppose to the contrary that \(L \neq \emptyset\). For \(i = 1, 2, 3\), let \(L_i\) be the set of vertices in \(L\) which is not contained in an edge with coloring \(i\). By Claim 12, we have \(L = L_1 \cup L_2 \cup L_3\). Without loss of generality, we assume \(L_1 \neq \emptyset\). Let \(w\) be a vertex in \(L_1\). Then there exists an edge \(f \in G^*\) such that \(f = \{u_1, w, w'\}\), where \(w' \in U_2 \cup U_3\). If \(w' \in U_2\), then \(f' = \{u_1, u_3, w'\} \in E(G^*)\). Since \(G^*\) is cancellative, \(w\) is not a neighbor of \(u_3\) in \(G^*\). This implies that \(w \in U_3\), a contradiction to \(w \in L\). Similarly, if \(w' \in U_3\), then \(w \in U_2\), which is also a contradiction. \(\Box\)

Now, we continue our proof. By Claim 13, we immediately obtain that \(G^*\) is a complete 3-partite 3-graph with vertex classes \(U_1, U_2\) and \(U_3\). Hence, \(G^* = T_3(n)\) by Lemma 6.

Finally, it is enough to show that \(G = T_3(n)\) for any \(G \in \text{SPEX}(n, \{F_4, F_5\})\). According to Lemma 8 and Claim 13, we can transfer \(G\) to the complete 3-partite 3-graph \(T_3(n)\) by a sequence of switchings \(T_u^i(\cdot)\) that keep the spectral radius unchanged. Let \(T_1, \ldots, T_s\) be such a sequence of switchings \(T_u^i(\cdot)\) which turn \(G\) into \(T_3(n)\). Consider the 3-graphs \(G = G_0, G_1, \ldots, G_s = T_3(n)\) in which \(G_i\) is obtained from \(G_{i-1}\) by applying \(T_i\). Let \(z \in S^{n-1}_{3,+}\) be an eigenvector corresponding to \(\lambda^{(3)}(G_{s-1})\) and \(T_u^i(G_{s-1}) = T_3(n)\), and denote
\[
A := V(G_{s-1}) \setminus \{N_{G_{s-1}}(v) \cup \{u\} \cup \{v\}\}.
\]
Hence, we have \(L_{G_{s-1}}(w) = L_{G_{s-1}}(v) = L_{T_3(n)}(v)\) for each \(w \in A\). In what follows, we shall prove \(L_{G_{s-1}}(u) = L_{G_{s-1}}(v)\), and therefore \(G_{s-1} = T_3(n)\). If \(L_{G_{s-1}}(u) \neq L_{G_{s-1}}(v)\), there exists an edge \(e = v_1v_2 \in L_{G_{s-1}}(u) \setminus L_{G_{s-1}}(v)\) since \(z_u = z_v\) by Lemma 7. Let \(M_1\) and \(M_2\) be two subsets of \(V(G_{s-1})\) such that \(M_1 \cup M_2 = N_{G_{s-1}}(v)\) and \(L_{G_{s-1}}(v) = K_{|M_1|,|M_2|}\). If \(\{v_1, v_2\} \subset N_{G_{s-1}}(v)\), then \(\{v_1, v_2\} \subset M_1\) or \(\{v_1, v_2\} \subset M_2\). It follows that there exists a vertex \(w \in N_{G_{s-1}}(v)\) such that \(f_1 := \{v, w, v_1\} \in E(G_{s-1})\) and \(f_2 := \{v, w, v_2\} \in E(G_{s-1})\).

However, \(f_1 \Delta f_2 \subset \{u, v_1, v_2\} \in E(G_{s-1})\), a contradiction. So we obtain \(\{v_1, v_2\} \cap A \neq \emptyset\). Without loss of generality, we assume \(v_1 \in A\). Then \(L_{G_{s-1}}(v_1) = L_{G_{s-1}}(v)\), i.e., \(uv_2 \in L_{G_{s-1}}(v)\). Thus, \(u \in N_{G_{s-1}}(v)\), a contradiction. This implies that \(G_{s-1} = T_3(n)\). Likewise, \(G_{i-1} = G_i\) for each \(i \in [s-1]\), and therefore \(G = T_3(n)\). This completes the proof of the theorem. \(\Box\)
According to Theorem 10, we can give an alternative proof of Bollobás’ result for

\[ n \equiv 0 \pmod{3}. \]

**Corollary 14.** Let \( G \) be a cancellative 3-graph on \( n \) vertices with \( n \equiv 0 \pmod{3} \). Then \( |E(G)| \leq t_3(n) \) with equality if and only if \( G = T_3(n) \).

**Proof.** Denote by \( z \) the all-ones vector of dimension \( n \). In view of (1), we deduce that

\[
\lambda^{(3)}(G) \geq \frac{P_G(z)}{\|z\|_3^3} = \frac{6|E(G)|}{n}.
\]

On the other hand, by Theorem 10 we have

\[
\lambda^{(3)}(G) \leq \lambda^{(3)}(T_3(n)) = 2 \cdot (t_3(n))^{2/3}.
\]

As a consequence,

\[
|E(G)| \leq \frac{n}{3} \cdot (t_3(n))^{2/3} = t_5(n).
\]

Equality may occur only if \( \lambda^{(3)}(G) = 2 \cdot (t_3(n))^{2/3} = \lambda^{(3)}(T_3(n)) \), and therefore \( G = T_3(n) \) by Theorem 10.

Next, we will prove Theorem 2 for the case \( p > 3 \) as stated in Theorem 16.

**Lemma 15 ([16]).** Let \( p \geq 1 \) and \( G \) be an \( r \)-graph with \( m \) edges. Then the function

\[
f_G(p) := \left( \frac{\lambda^{(p)}(G)}{r!m} \right)^p
\]

is non-increasing in \( p \).

**Theorem 16.** Let \( p > 3 \) and \( G \) be a cancellative 3-graph on \( n \) vertices. Then \( \lambda^{(p)}(G) \leq \lambda^{(p)}(T_3(n)) \) with equality if and only if \( G = T_3(n) \).

**Proof.** Assume that \( p > 3 \) and \( G \) is a 3-graph in \( \text{SPEX}_p(n, \{F_4, F_5\}) \) with \( m \) edges. It is enough to show that \( G = T_3(n) \). By Lemma 15, we have

\[
\left( \frac{\lambda^{(p)}(G)}{6m} \right)^p \leq \left( \frac{\lambda^{(3)}(G)}{6m} \right)^3,
\]

which, together with \( \lambda^{(3)}(G) \leq 2 \cdot (t_3(n))^{2/3} \) by Theorem 10 and Lemma 6, gives

\[
\lambda^{(p)}(G) \leq (6m)^{1-3/p} \cdot (\lambda^{(3)}(G))^{3/p} \leq 2^{3/p} \cdot (6m)^{1-3/p} \cdot (t_3(n))^{2/p}.
\]

On the other hand, we have

\[
\lambda^{(p)}(G) \geq \lambda^{(p)}(T_3(n)) = \frac{6}{\sqrt[3]{21}} \cdot (t_3(n))^{1-1/p}.
\]

We immediately obtain \( m \geq t_3(n) \). The result follows from Theorem 1. \( \square \)
Finally, we shall give a proof of Theorem 2 for the remaining case \( p = 1 \). In what follows, we always assume that \( x \in \mathbb{S}^{n-1}_+ \) is an eigenvector such that \( x \) has the minimum possible number of non-zero entries among all eigenvectors corresponding to \( \lambda^{(1)}(G) \). Before continuing, we need the following result.

**Lemma 17** ([7]). Let \( G \) be an \( r \)-graph and \( S \) be the support set of \( x \). Then for each pair vertices \( u \) and \( v \) in \( S \), there is an edge in \( G[S] \) containing both \( u \) and \( v \).

**Theorem 18.** Let \( G \) be a cancellative 3-graph. Then \( \lambda^{(1)}(G) = 2/9 \).

**Proof.** Assume that \( G \) is a cancellative 3-graph with support set \( S \). Let \( H := G[S] \). By Lemma 17, for any \( u, v \in S \) there is an edge in \( H \) containing both \( u \) and \( v \). Hence, each pair of edges of \( H \) has at most one common vertex by \( H \) being cancellative. So the shadow graph of \( H \) is the complete graph \( K_{|S|} \). Since \( H \) is cancellative, the link graphs \( L_H(u) \) and \( L_H(v) \) are edge-disjoint graphs for any distinct vertices \( u, v \in S \). It follows from (2) that

\[
|S| \cdot \lambda^{(1)}(G) = 2 \sum_{uv \in E(\partial(H))} x_u x_v \leq 1 - \frac{1}{|S|},
\]

where the last inequality follows from Motzkin–Straus Theorem [13]. On the other hand, set

\[
z_v = \begin{cases} 1/|S|, & v \in S, \\ 0, & \text{otherwise.} \end{cases}
\]

We immediately have

\[
\lambda^{(1)}(G) \geq 6 \sum_{e \in E(H)} z(e) = 2 \sum_{v \in V(H)} \left( z_v \cdot \sum_{f \in L_H(v)} z(f) \right) = \frac{|S| - 1}{|S|^2},
\]

where the last inequality follows from the fact that \( d_H(v) = (|S| - 1)/2 \) for \( v \in V(H) \). Combining with (8) we get

\[
\lambda^{(1)}(G) = \frac{|S| - 1}{|S|^2}.
\]

Clearly, \((|S| - 1)/|S|^2\) attains its maximum at \(|S| = 3\) when \(|S| \geq 3\). Hence, we see \( \lambda^{(1)}(G) \leq 2/9 \). Finally, noting that \( \lambda^{(1)}(G) \) is at least the Lagrangian of an edge \( K_3^{(3)} \), i.e.,

\[
\lambda^{(1)}(G) \geq \lambda^{(1)}(K_3^{(3)}) = \frac{2}{9},
\]

we obtain \( \lambda^{(1)}(G) = 2/9 \), as desired. \( \square \)

**Remark 19.** For an \( r \)-graph \( G \) on \( n \) vertices, it is well-known that \( \lambda^{(1)}(G)/r! \) is the Lagrangian of \( G \). In [19], Yan and Peng present a tight upper bound on \( \lambda^{(1)}(G) \) for \( F_5 \)-free 3-graphs, see [19] for details.
Acknowledgements

We are grateful to the referees for their careful readings and helpful comments that improved the presentation of this paper. The first author was partially supported by Hainan Provincial Natural Science Foundation of China (No. 122QN218) and the National Nature Science Foundation of China (No. 12201161). The third author was supported by the National Natural Science Foundation of China (No. 12331012).

References

[1] B. Bollobás. Three-graphs without two triples whose symmetric difference is contained in a third. *Discrete Math.*, 8: 21–24, 1974.
[2] A. Bretto, Hypergraph Theory: An Introduction, Springer, 2013.
[3] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436: 3268–3299, 2012.
[4] M.N. Ellingham, L. Lu and Z. Wang. Maximum spectral radius of outerplanar 3-uniform hypergraphs. *J. Graph Theory*, 100 (4): 671–685, 2022.
[5] P. Erdős and A.H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52: 1087–1091, 1946.
[6] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1: 51–57, 1966.
[7] P. Frankl and V. Rödl. Hypergraphs do not jump. *Combinatorica*, 4: 149–159, 1984.
[8] Z. Füredi, Turán type problems, in Surveys in Combinatorics, Cambridge University Press, Cambridge, pp. 253–300, 1991.
[9] G. Gao, A. Chang, and Y. Hou. Spectral radius on linear $r$-graphs without expanded $K_{r+1}$. *SIAM J. Discrete Math.*, 36 (2): 1000–1011, 2022.
[10] P. Keevash and D. Mubayi. Stability theorems for cancellative hypergraphs. *J. Combin. Theory Ser. B*, 92: 163–175, 2004.
[11] P. Keevash. Hypergraph Turán problems, in Surveys in Combinatorics, Cambridge University Press, Cambridge, pp. 83–139, 2011.
[12] P. Keevash, J. Lenz, and D. Mubayi. Spectral extremal problems for hypergraphs. *SIAM J. Discrete Math.*, 28 (4): 1838–1854, 2014.
[13] T. Motzkin and E. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canad. J. Math.*, 17: 533–540, 1965.
[14] D. Mubayi and J. Verstraëte. A survey of Turán problems for expansions, in Recent Trends in Combinatorics, IMA Vol. Math. Appl. 159, Springer, pp. 117–143, 2016.
[15] V. Nikiforov. Some new results in extremal graph theory, in Surveys in Combinatorics, Cambridge University Press, Cambridge, pp. 141–181, 2011.
[16] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457: 455–535, 2014.
[17] V. Nikiforov. Some extremal problems for hereditary properties of graphs. *Electron. J. Combin.*, 21, #P1.17, 2014.

[18] C. She, Y. Fan, L. Kang and Y. Hou. Linear spectral Turán problems for expansions of graphs with given chromatic number. Preprint available at arXiv:2211.13647, 2023.

[19] Z. Yan and Y. Peng. $\lambda$-perfect hypergraphs and Lagrangian densities of hypergraph cycles. *Discrete Math.*, 342: 2048–2059, 2019.