Reduced Order Modeling by Modal Identification Method and POD-Galerkin approach of the heated circular cylinder wake in mixed convection

Laurent Cordier\textsuperscript{1,3}, Manuel Girault\textsuperscript{2}, Daniel Petit\textsuperscript{2}

\textsuperscript{1} Pprime Institute, CNRS - Université de Poitiers - ENSMA, Département Fluides, Thermique, Combustion, CEAT, Université de Poitiers, 43, route de l’aérodrome, F86036 Poitiers Cedex, France
\textsuperscript{2} Pprime Institute, CNRS - ENSMA - Université de Poitiers, Département Fluides, Thermique, Combustion, ENSMA, Téléport 2, 1, avenue Clément Ader, BP 40109, F86961 Futuroscope Chasseneuil Cedex, France

E-mail: laurent.cordier@univ-poitiers.fr, manuel.girault@ensma.fr, daniel.petit@ensma.fr

Abstract. When dealing with control of thermal systems, the first step in the design of a model-based controller is the development of a model which accurately captures heat transfer dynamics. Classical high-fidelity models directly derived from the application of conservation laws on the discretized space domain, lead to sets of Ordinary Differential Equations in time whose size is usually too huge for practical implementation of real-time control. A variety of techniques have been developed for building low order models, involving a small number of degrees of freedom compared to high-fidelity models. Here, we focus on two approaches: Modal Identification Method, mainly used in heat transfer, and a POD-Galerkin method, based on Proper Orthogonal Decomposition and traditionally employed in fluid mechanics. The objective of this work is to compare on a simple 2D mixed convection problem, the accuracy of the reduced order models derived by both methods for describing the flow dynamics. Results are presented for the heated circular cylinder wake at a Reynolds number equal to 200 and a Richardson number equal to 2. Velocity and temperature fields at different time instants, computed with a reference Finite Element model, are used as data for both approaches.

1. Introduction

Optimization and control of thermal systems nowadays have a major place in the field of energy management and in many industrial processes. Indeed, thermal control is not only crucial for the industrial competition but it may impact severely our environment. In the design of a model-based controller, the first step is the development of a model which accurately captures the physical behaviour of the thermal system. This high-fidelity model, derived directly from the application of the conservation laws to the system, is then used after discretization in space as a dynamical model for solving the control problem. However, the size of the discretized set of Ordinary Differential Equations is in general so huge that it is nearly impossible to deal with real situations where the phenomena are complex. This situation is even worse when we are interested in real-time thermal control.
control applications. Not surprisingly, a lot of attention has been paid to reducing the costs of the nonlinear state equations by using reduced order models (i.e. models involving a low number of degrees of freedom) for describing the flow dynamics. A variety of techniques are available for model order reduction. Here, we are focusing on two approaches: Modal Identification Method (MIM), mainly used in thermal systems [1,2], and a POD-Galerkin method (POD-G), based on Proper Orthogonal Decomposition [3] and traditionally employed in fluid mechanics [4]. Briefly, the MIM consists in firstly defining the structure of the reduced order model, secondly generating some input-output data characteristic of the system dynamics, thirdly adjusting the model parameters using optimization techniques so that the model outputs fit the output data [1]. The POD-G approach is also a three steps procedure. First, a matrix of snapshots of the state vector (fields of variables) is created. Second, the POD basis is determined with a singular value decomposition of this matrix. It can be shown at this level that the POD decomposition is optimal in an energetic sense for representing the data contained in the original database. Then, a Galerkin projection of the high-fidelity model onto the subspace spanned by the POD modes is done, leading to the reduced order model [3]. POD-G reduced order models were used for solving optimization problems for natural convection flows. Let us cite for instance the estimation of the strength of a heat source [5], the control of a thermally driven flow of an electrically conducting fluid in a 2D cavity using a magnetic field as actuator [6], and the suppression of Rayleigh-Bénard convection in a 2D cavity by adapting the heat flux profile at the bottom wall [7]. As a preliminary step of deriving a model-based controller for mixed convection flows, this work focuses on the derivation of low order models. The objective is hence to compare on a simple 2D configuration of mixed convection described in section 2, the accuracy of the reduced order models derived by both POD-G (section 4) and MIM (section 5). Results are presented in section 6 for the heated circular cylinder wake at a Reynolds number equal to 200 and a Richardson number equal to 2. For this configuration of mixed convection described in section 2, the accuracy of the reduced order models is crucial and special care must be taken, as presented in section 3.

2. The studied problem: 2D mixed convection past a heated cylinder

We consider an incompressible unsteady mixed convection flow of a Newtonian fluid in an open domain \( \Omega \), written with the Boussinesq approximation. Such problem is governed by the conservation equations for mass, momentum and energy:

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla (\mathbf{v} \otimes \mathbf{v}) = -\nabla p_m + \nabla (\nabla \cdot \mathbf{v}) - \beta (T - T_\infty(t)) \mathbf{g}
\]

(1)

\[
\frac{\partial T}{\partial t} + \nabla (T \mathbf{v}) = \frac{k_{th}}{\rho_c C_p} \nabla \cdot \nabla T
\]

(2)

\[
\forall x \in \Omega, \forall t \in [0; t_f] : \mathbf{v} = \mathbf{v}(x, t), \quad p_m = p_m(x, t) \quad \text{and} \quad T = T(x, t)
\]

are respectively the velocity vector, driving pressure and temperature at position \( x \) and time \( t \). The temperature \( T_\infty \) of the fluid entering the domain is supposed to be function of time. Initially, the whole domain is assumed to be at \( T_{x,0} = T_\infty(t = 0) \). \( \nu \) (m\(^2\).s\(^{-1}\)) is the kinematic viscosity of the fluid, \( C_p \) (J.kg\(^{-1}\).K\(^{-1}\)) its specific heat and \( k_{th} \) (W.m\(^{-1}\).K\(^{-1}\)) its thermal conductivity. These properties are assumed to be uniform and constant throughout the domain, and evaluated at \( T_{x,0} \). Density \( \rho \) (kg.m\(^{-3}\)) is the only property supposed to vary with temperature. In the Boussinesq approximation, density is considered constant (equal to \( \rho_\infty = \rho(T_{x,0}) \)) in all the terms but the buoyancy term, for which density variation depends linearly on temperature variation, thus leading to the \( -\beta (T - T_\infty) \mathbf{g} \) term in (2). \( \mathbf{g} \) (m.s\(^{-2}\)) is the gravity vector and \( \beta \) (K\(^{-1}\)) the thermal expansion coefficient.

Although (1), (2) and (3) are written for the general 3D case, a 2D mixed convection flow past a heated circular cylinder of diameter \( D \) in an open domain \( \Omega \) is here considered, as shown in Figure 1.
Taking the cylinder centre as the origin of coordinates, the spatial domain is \( \Omega = [-5D; 20D] \times [-10D; 10D] \). The boundary \( \Gamma \) of \( \Omega \) is partitioned in the following sub-boundaries: \( \Gamma_{\text{in}} \) is the inlet boundary where the fluid at temperature \( T_\infty \) enters the domain with a velocity \( v_\infty \), \( \Gamma_{\text{out}} \) is the union of all boundaries where the fluid exits the domain, \( \Gamma_{\text{cd}} \) and \( \Gamma_{\text{cn}} \) are the parts of the cylinder boundary where a prescribed temperature \( T_p \) and heat flux density \( \varphi \) are respectively considered. In Figure 1, \( \mathbf{n} \) is defined as the outward unit vector normal to each considered sub-boundary.

![Figure 1. General schematic configuration.](image)

### 3. Low order model formulation

#### 3.1. General approximation of velocity and temperature fields

It is assumed that the temperature field may be written as a sum of functions, each one being the product of a function of space by a function of time:

\[
v(x, t) \approx \sum_{i=1}^{N_v} \phi_i(x) a_i(t)
\]

where the spatial functions \( \phi_i \) are a truncation of a basis of the Hilbert space \( L_2(\Omega) \) with \( d \) the dimension of the problem. Similarly, we suppose that the temperature field may be written as:

\[
T(x, t) \approx \sum_{i=1}^{N_T} \psi_i(x) b_i(t)
\]

where the spatial functions \( \psi_i \) are a truncation of a basis of the Hilbert space \( L_2(\Omega) \). Of course, our goal will be to find a low order model, hence corresponding to small values of \( N_v \) and \( N_T \) in the expansions (4) and (5).

#### 3.2. Galerkin projection for the momentum equation

Let us now define \( \mathbf{R}_v(x, t) \) as the residual of the momentum equation (2):

\[
\mathbf{R}_v(x, t) = \frac{\partial \mathbf{u}}{\partial t} + \nabla (\mathbf{u} \otimes \mathbf{u}) + \frac{\nabla p_m}{\rho_v} - \mathbf{v} \nabla (\mathbf{v} \cdot \mathbf{n}) + \beta (T - T_\infty) \mathbf{g}
\]

After inserting decompositions (4) and (5) into (6), the residual \( \mathbf{R}_v(x, t) \) is forced to be orthogonal to the subspace of \( [L_2(\Omega)]^d \) spanned by the spatial functions \( \phi_i \). The Galerkin projection of (2) onto \( \phi_i \) hence writes:

\[
(\mathbf{R}_v(x, t), \phi_k(x))_\Omega = \int_\Omega \mathbf{R}_v(x, t). \phi_k(x) d\Omega = 0 \quad \forall k \in [1; N_v]
\]

where the inner product \((.,.)_\Omega\) is defined as \((u, v)_\Omega = \int_\Omega u v d\Omega = \int_\Omega (\sum_{i=1}^{d} u_i v_i) d\Omega.\)
For introducing explicitly the boundary conditions in the model, the diffusion, convection and pressure terms are integrated by parts, using Green formulas. Equation (7) may be finally written as:

\[
\sum_{i=1}^{N_{\text{p}}} (I_{\text{p}})_{ki} \frac{da_i(t)}{dt} - \sum_{i=1}^{N_{\text{p}}} (L_{\text{p}})_{ki} a_i(t) - \sum_{i=1}^{N_{\text{p}}} \sum_{j=1}^{N_{\text{p}}} (Q_{\text{p}})_{kij} a_i(t) a_j(t) - \sum_{\ell=1}^{N_{\text{r}}} (E_{\text{p}})_{k\ell} b_\ell(t) + \mathcal{P}_k - (V_{\text{p}})_{k} v_x(t) - (W_{\text{p}})_{k} v_y(t) - (K_{\text{p}})_{k} \mathcal{T}_x(t) = 0 \quad \forall k \in [1; N_{\text{p}}]
\]

The expression of the coefficients \((I_{\text{p}})_{ki}, (L_{\text{p}})_{ki}, (Q_{\text{p}})_{kij}, etc.)\) is given in Appendix A.

3.3. Galerkin projection for the energy equation

Similarly, the Galerkin projection associated with the residual \(\mathcal{R}_T(x, t)\) of the energy equation (3) writes:

\[
\int_\Omega \mathcal{R}_T(x, t) \psi_k(x) \, d\Omega = 0 \quad \forall k \in [1; N_T]
\]

After integration by parts of the diffusion and convection terms in (9), we finally get:

\[
\sum_{i=1}^{N_{\text{T}}} (I_{\text{T}})_{ki} \frac{db_i(t)}{dt} - \sum_{i=1}^{N_{\text{T}}} (L_{\text{T}})_{ki} b_i(t) - \sum_{i=1}^{N_{\text{T}}} \sum_{j=1}^{N_{\text{T}}} (Q_{\text{T}})_{kij} b_i(t) a_j(t) - (P_{\text{T}})_{k} \psi(t) - (K_{\text{T}})_{k} \mathcal{T}_x(t) = 0 \quad \forall k \in [1; N_{\text{T}}]
\]

All the coefficients \((I_{\text{T}})_{ki}, (L_{\text{T}})_{ki}, (Q_{\text{T}})_{kij}, etc.)\) are given in Appendix B.

Note: In the case of several independent boundary conditions of same type but with different time-varying values, similar terms are introduced, each one associated to a specific boundary condition.

4. Proper Orthogonal Decomposition

In the following, POD is presented from a discrete viewpoint (used in practice). Let us consider a data set \(\{u^m(x_i) = u(x_i, t_m), \quad i = 1, ..., N_x, \quad m = 1, ..., N_t\}\) composed of \(N_t\) snapshots taken over a time interval, each snapshot being a flow realization, i.e. a field defined on a spatial mesh with \(N_x\) nodes. \(u(x_i, t_m)\) may thus be a vector field such as velocity \(v(x_i, t_m)\) or a scalar field such as temperature \(T(x_i, t_m)\). Let us call \(P_S\) the orthogonal projection onto a subspace \(S\) of reduced dimension \(N_{\text{POD}} \ll \min(N_x, N_t)\). We introduce an average operator \(\mathbb{E}(\cdot)\) over the \(N_t\) snapshots, for instance an ensemble average \(\mathbb{E}(u) = \frac{1}{N_t} \sum_{m=1}^{N_t} u^m\). The aim of POD is to find a basis \(\Phi_k(x_i)\) \((i = 1, ..., N_x; \quad k = 1, ..., N_{\text{POD}})\) of the subspace \(S\) such as minimizing the following error:

\[
E(\|u - P_S u\|^2_\Omega) = \frac{1}{N_t} \sum_{m=1}^{N_t} \|u^m - P_S u^m\|^2_\Omega = \frac{1}{N_t} \sum_{m=1}^{N_t} \left\| u^m(x) - \sum_{k=1}^{N_{\text{POD}}} \Phi_k(x)a_k^{SP}(t_m) \right\|^2_\Omega \quad (11)
\]

where \(a_k^{SP}(t_m)\) are the temporal coefficients of the POD expansion (the superscript \(SP\) standing for Snapshot-POD). Those \(a_k^{SP}(t_m)\) are specific to the data set and, in particular, are hence different of the \(a_k(t)\) in (8) or the \(b_k(t)\) in (10). It can be shown ([3] for instance) that the minimization problem (11) leads to an eigenvalue problem whose solutions are the \(\Phi_k(x)\). This historical approach is called direct method. However, when the input data come from numerical simulations, it is much more efficient to use an alternate way of computing the POD eigenfunctions. This method, known as method of snapshots [3], consists of writing the POD modes as linear combinations of the snapshots:

\[
\Phi_k(x_i) = \frac{1}{N_t \times \lambda_i} \sum_{m=1}^{N_t} a_k^{SP}(t_m) u^m(x_i) \quad k = 1, ..., N_{\text{POD}}
\]

It can be shown ([3] for instance) that the vectors \(a_k^{SP} = (a_k^{SP}(t_1) \cdots a_k^{SP}(t_{N_t}))^T\) are determined as the solutions of a new eigenvalue problem given by:
\[ C a_{SP}^k = \lambda_k a_{SP}^k \quad k = 1, \ldots, N_{POD} \]

where \( C \in \mathbb{R}^{N_t \times N_t} \) is a temporal correlation matrix such that:

\[ C_{ij} = \frac{1}{N_t} \left( \mathbf{u}^i(\mathbf{x}), \mathbf{u}^j(\mathbf{x}) \right)_\Omega = \frac{1}{N_t} \int_{\Omega} \mathbf{u}(\mathbf{x}, t_i). \mathbf{u}(\mathbf{x}, t_j) d\Omega \]

Spectral theory guarantees that the eigenvalues \( \lambda_k \) of \( C \) are positive real numbers. By convention, these eigenvalues are reordered such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_{POD}} \geq 0 \). Moreover, we can choose

1. the spatial eigenfunctions \( \Phi_k \) to be orthonormal with respect to the spatial inner product leading to \( a_{SP}^k(t_m) = \left( \mathbf{u}^m(\mathbf{x}), \Phi_k(\mathbf{x}) \right)_\Omega \), and
2. the temporal eigenfunctions \( a_{SP}^k(t) \) to be orthogonal.

Once the \( a_{SP}^k \) \( k = 1, \ldots, N_{POD} \) are computed by solving (13), the spatial eigenfunctions \( \Phi_k \) can be obtained through (12) and then used to compute, through discrete integrations, the coefficients of the matrices and vectors arising from the Galerkin projections described in sections 3.2 and 3.3, thus defining entirely the POD-G reduced-order models given by (8) and (10).

5. Modal Identification Method

MIM consists in building a model similar in form to (8) and (10), but without prior computation of functions \( \phi_i \) and \( \psi_i \). Here, the low order model (LOM) is identified through an optimization procedure [1]. Thanks to a Galerkin projection of the Poisson equation for pressure, the pressure term \( \mathcal{P}_k \) in (8) may be written as additional contributions of quadratic terms in \( a_i a_j \) and linear terms in \( b_i \). Moreover, matrices \( I_v \) and \( I_r \) are assumed to be invertible. In addition, (8) is rewritten so that contributions \( a_i a_j \), \( i = 1, \ldots, N_v; j = 1, \ldots, N_p \) are counted without repetition. In the case of steady boundary conditions, one therefore gets from (8) and (10) the final form of the dynamical LOM to be identified through MIM:

\[ \dot{\mathbf{a}}(t) = L_a \mathbf{a}(t) + Q_a \Psi(\mathbf{a}(t)) + R \mathbf{b}(t) + \mathbf{v}_a \]

and

\[ \dot{\mathbf{b}}(t) = L_b \mathbf{b}(t) + Q_b \Pi(\mathbf{a}(t), \mathbf{b}(t)) + \mathbf{v}_b \]

where nonlinear vectors \( \Psi(\mathbf{a}) \) of size \( N_v \) and \( \Pi(\mathbf{a}, \mathbf{b}) \) of size \( N_R \) are expressed as:

\[ \Psi(\mathbf{a}) = \begin{pmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_{N_v} & a_2^2 & a_2 a_3 & \cdots & a_2 a_{N_v} & \cdots & a_{N_v} a_1 & a_{N_v} a_2 & \cdots & a_{N_v} a_{N_v} \end{pmatrix}^T \]

\[ \Pi(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_{N_T} & a_2 b_1 & \cdots & a_2 b_{N_T} & \cdots & a_{N_v} b_1 & \cdots & a_{N_v} b_{N_T} \end{pmatrix}^T \]

with \( L_a \in \mathbb{R}^{N_R \times N_v}, Q_a \in \mathbb{R}^{N_R \times N_v}, R \in \mathbb{R}^{N_R \times N_T}, V_a \in \mathbb{R}^{N_v}, L_b \in \mathbb{R}^{N_R \times N_T}, Q_b \in \mathbb{R}^{N_R \times N_R}, V_b \in \mathbb{R}^{N_R} \).

According to (4), velocity components at original mesh nodes \( x_j \), \( j = 1, \ldots, N_x \) are computed as:

\[ v_k(x_j, t) = \sum_{i=1}^{N_v} (\phi_i(x_j) a_i(t)) = \sum_{i=1}^{N_v} \left[ H_p^{(k)} \right]_{ji} a_i(t), \quad k = 1, \ldots, d \]

and according to (5), temperature at original mesh nodes \( x_j \), \( j = 1, \ldots, N_x \) is obtained from:

\[ T(x_j, t) = \sum_{i=1}^{N_T} \psi_i(x_j) b_i(t) = \sum_{i=1}^{N_T} [H_T]_{ji} b_i(t) \]

In MIM, the determination of all the components of \( L_a, Q_a, R, V_a, L_b, Q_b, V_b, H_T \) and \( H_p^{(k)}, k = 1, \ldots, d \), gathered in vector \( \theta \), is recast in a parameter estimation problem. We chose \( N_v = N_T = N_M \) in the present work. Unknown parameter vector \( \theta \) is hence estimated through the minimization of a quadratic functional \( J_{id(NM)}^{(NM)}(\theta) \) built on an output error, defined on \( N_t \) time instants:

\[ J_{id(NM)}^{(NM)}(\theta) = \sum_{i=1}^{N_t} \sum_{j=1}^{N_x} \sum_{k=1}^{d} \left( v_k(x_j, t_i; \theta) - v^{FE}_k(x_j, t_i) \right)^2 + \left( T(x_j, t_i; \theta) - T^{FE}(x_j, t_i) \right)^2 \]

where:
\( v_k(x_j, t_i; \theta) \) and \( T(x_j, t_i; \theta) \) are respectively the \( k \)th velocity component and the temperature at position \( x_j \) and time \( t_i \), computed by the LOM and which depend on \( \theta \).

\( v_k^{FE}(x_j, t_i) \) and \( T^{FE}(x_j, t_i) \) are the corresponding data fields computed with the reference FE model.

The minimization of \( \mathcal{J}_{id}^{(NM)}(\theta) \) defined by (18) is repeated for successive values of \( N_{Nt} \) until no improvement in the data fitting process can be obtained or when the wished accuracy has been reached. The optimization algorithms used in the identification procedure are the Particle Swarm Optimization method and the Ordinary Least Squares method.

6. Results and discussion

In sections 2 and 3, the heated cylinder wake flow was introduced in the general framework of boundary conditions functions of time. Hereafter, the numerical results are presented in the simplified case of steady boundary conditions with a Dirichlet boundary condition all around the cylinder. This configuration is then fully characterized by three dimensionless quantities, usually the Reynolds number \( Re = v_c D/\nu \), the Prandtl number \( Pr = \nu/\alpha \) with \( \alpha = k_{th}/\rho_\infty \) the fluid thermal diffusivity, and the Richardson number \( Ri = g \beta D (T_p - T_\infty)/\nu^2 \). Two cases were considered: an isothermal flow at \( Re = 100 \) and a mixed convection flow at \( Re = 200 \), \( Ri = 2 \) and \( Pr = 0.708 \). Equations (1), (2), (3) along with associated boundary conditions have been solved in dimensionless form using the FE solver FlexPDE\textsuperscript{©}, which constitutes our reference model. For the isothermal case, the number of nodes \( N_x \) in the mesh is 3058, whereas 6880 have been employed for the mixed convection case.

For the isothermal case, more than 5 hours have been required on a personal computer for the computation of \( N_t=1600 \) time steps with the reference FE model. The last 800 snapshots have been used as data for building LOMs through MIM. In Figure 2 and Figure 3 are displayed the velocity norms computed by the reference FE model and an order 7 LOM respectively, for a particular instant. The model involving only 7 degrees of freedom has thus allowed us to reproduce the velocity field on the original mesh. On the same computer, only 12 minutes have been needed to compute 1600 time steps using this LOM. It should be underlined that the main part of computing time has been required to perform the projection (16) of state vector \( a(t) \in \mathbb{R}^2 \) onto the original mesh. In fact, only 0.125 s have been used to compute \( a(t) \) at all 1600 instants by solving (14). Such a LOM hence appears of great interest for real-time applications. Moreover, the LOM allows extending the calculation and remains stable.

For the mixed convection case, about 8.5 hours have been needed to compute 1600 time steps with the reference FE model. Again, the last 800 snapshots (velocity and temperature fields) have been used to identify the LOMs by MIM. The norm of velocity fields computed with the FE model and an order 7 LOM are shown respectively in Figure 4 and Figure 5 for a specific instant, whereas corresponding temperature fields are displayed in Figure 6 and Figure 7. Only 23 minutes have been required to compute velocity and temperature fields on the original mesh at 1600 time steps. Again, the LOM allows extending the computation to a larger time horizon and remains stable.

In order to assess the quality of the LOM, we have computed the mean quadratic relative error between fields obtained with the reference FE model and the LOM. For the \( v_1 \) component of velocity, this error is defined as:

\[
E_{v_1} = \sqrt{\frac{\sum_{i=1}^{N_t} \sum_{j=1}^{N_x} (v_1^{LOM}(x_j, t_i) - v_1^{FE}(x_j, t_i))^2}{\sum_{i=1}^{N_t} \sum_{j=1}^{N_x} (v_1^{FE}(x_j, t_i))^2}}
\]

Similar quantities are defined for \( v_2 \) and the temperature \( T \). In the above mentioned mixed convection case, the values are \( E_{v_1} = 1.42 \times 10^{-2} \), \( E_{v_2} = 8.93 \times 10^{-2} \) and \( E_T = 7.78 \times 10^{-2} \).
Simulation results obtained with the POD-G low order model will be presented during the conference.

**7. Conclusions and prospects**

Two model reduction approaches have been applied on a 2D mixed convection flow past a heated circular cylinder: the Modal Identification Method, and the POD-Galerkin method, based on Proper Orthogonal Decomposition. The presented case is a flow with steady boundary conditions corresponding to a Reynolds number $Re = 200$ and a Richardson number $Ri = 2$. Simulations made
with a reference Finite Element model have been used as data for both approaches. The obtained low order models have proved to be able to reproduce the flow behaviour. Although the presented results correspond to a unique set of steady boundary conditions, the low order model formulation has been developed for time-varying boundary conditions, opening the way to future works: first we will consider building low order models able to work in a range of steady boundary conditions (leading to a range of dimensionless parameters $Re$ and $Ri$), before trying to handle the case of time-varying boundary conditions in a long-term control-oriented framework.

8. References

[1] Girault M, Petit D and Videcoq E 2011 Identification of low-order models and their use for solving inverse boundary problems Thermal Measurements and Inverse Techniques ed H R B Orlande et al (Boca Raton: CRC Press-Taylor and Francis Group) chapter 13 pp 457–506.

[2] Rouizi Y, Girault M, Favennec Y and Petit D 2010 Model reduction by the Modal Identification Method in forced convection: application to a heated flow over a backward-facing step Int. J. Therm. Sci. 49 1354-68.

[3] Cordier L and Bergmann M 2008 Proper Orthogonal Decomposition: an overview Lecture series 2002-04, 2003-03 and 2008-01 on post-processing of experimental and numerical data Von Karman Institute for Fluid Dynamics.

[4] Bergmann M and Cordier L 2008 Optimal control of the cylinder wake in the laminar regime by trust-region methods and POD reduced-order models J. Comput. Phys. 227 7813-40.

[5] Park H M and Jung W S 2001 The Karhunen-Loève Galerkin method for the inverse natural convection problems Int. J. Heat Mass Transfer 44 155-67.

[6] Park H M and Lee W J Feedback control of natural convection 2001 Comput. Meth. Appl. Mech. Eng. 191 1013-28.

[7] Park H M and Sung M C 2003 Stabilization of two-dimensional Rayleigh–Bénard convection by means of optimal feedback control Physica D 186 185-204.

Appendix A. Coefficients of the Galerkin projection for the momentum equation (8)

$\forall k \in [1; N_p], \forall i \in [1; N_p], \forall j \in [1; N_p], \forall \ell \in [1; N_p]: \quad (l_p)_{ki} = \int_{\Omega} \phi_i(x) \cdot \phi_j(x) \, d\Omega$

$(L_p)_{ki} = \nu \int_{\Omega} \nabla \phi_i(x) \cdot \nabla \phi_j(x) \, d\Omega + \nu \int_{\Gamma_{\text{out}}} \left( \nabla \phi_i(x) \cdot n \right) \cdot \phi_j(x) \, d\Gamma - \nu \int_{\Gamma_{\text{out}}} \left( \nabla \phi_j(x) \cdot n \right) \cdot \phi_i(x) \, d\Gamma$

$(Q_p)_{kij} = \int_{\Omega} \left( \phi_i(x) \Phi_j(x) \right) \nabla \phi_k(x) \, d\Omega - \int_{\Gamma_{\text{out}}} \left( \phi_j(x) \cdot n \right) \left( \phi_i(x) \cdot \phi_k(x) \right) \, d\Gamma$

$(E_p)_{\ell} = -\beta \int_{\Omega} \psi_{\ell}(x) \cdot g \cdot \phi_k(x) \, d\Omega \quad (V_p)_{k} = \nu \int_{\Gamma_{\text{in}}} \left( \nabla \phi_k(x) \cdot n \right) \cdot n \, d\Gamma$

$(W_p)_{k} = -\int_{\Gamma_{\text{in}}} \phi_k(x) \cdot n \, d\Gamma \quad (K_{g})_{k} = \beta \int_{\Omega} g \cdot \phi_k(x) \, d\Omega \quad \text{and} \quad P_{k} = \frac{1}{\rho_\infty} \int_{\Omega} \nabla p_m \cdot \phi_k(x) \, d\Omega$

Appendix B. Coefficients of the Galerkin projection for the energy equation (10)

$\forall k \in [1; N_T], \forall i \in [1; N_T], \forall j \in [1; N_T]: \quad (l_T)_{ki} = \int_{\Omega} \psi_i(x) \cdot \psi_j(x) \, d\Omega$

$(L_T)_{ki} = \frac{k_{th}}{\rho_\infty c_p} \int_{\Omega} \psi_i(x) \cdot \nabla \psi_j(x) \, d\Omega + \int_{\Gamma_{\text{in}}} \psi_i(x) \cdot \nabla \psi_j(x) \, d\Gamma - \int_{\Gamma_{\text{out}}} \psi_i(x) \cdot \nabla \psi_j(x) \, d\Gamma$

$(Q_T)_{kij} = \int_{\Omega} \psi_i(x) \cdot \psi_j(x) \cdot \Phi_k(x) \, d\Omega - \int_{\Gamma_{\text{in}}} \psi_j(x) \cdot \Phi_k(x) \cdot \nabla \psi_i(x) \, d\Gamma$

$(P_T)_{k} = \frac{1}{\rho_\infty c_p} \int_{\Gamma_{\text{in}}} \psi_k(x) \, d\Gamma \quad (K_{T_p})_{k} = \frac{k_{th}}{\rho_\infty c_p} \int_{\Gamma_{\text{cd}}} \nabla \psi_k(x) \cdot n \, d\Gamma$

$(K_{T_m})_{k} = -\frac{k_{th}}{\rho_\infty c_p} \int_{\Gamma_{\text{in}}} \nabla \psi_k(x) \cdot n \, d\Gamma$