Empty axis-parallel boxes

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Abstract

We show that, for every set of \( n \) points in the \( d \)-dimensional unit cube, there is an empty axis-parallel box of volume at least \( \Omega(d/n) \) as \( n \to \infty \) and \( d \) is fixed. In the opposite direction, we give a construction without an empty axis-parallel box of volume \( O(d^2 \log d/n) \). These improve on the previous best bounds of \( \Omega(\log d/n) \) and \( O(2^{7d}/n) \) respectively.

1 Introduction

Dispersion. A box is a Cartesian product of open intervals. Given a set \( P \subset [0,1]^d \), we say that a box \( B = (a_1,b_1) \times \cdots \times (a_d,b_d) \) is empty if \( B \cap P = \emptyset \). Let \( m_d(n) \) be the largest number such that every \( n \)-point set \( P \subset [0,1]^d \) admits an empty box of volume at least \( m_d(n) \).

The quantity \( m(P) \) is called the dispersion of \( P \). The motivation for estimating \( m_d(n) \) came independently in several subjects. The earliest occurrence is probably in the work Rote and Tichy [9] who were motivated by the relations to \( \varepsilon \)-nets in discrete geometry on one hand, and with the relations to discrepancy theory on the other. The dispersion also arose in the problem of estimating rank-one tensors [2, 6, 8] and in Marcinkiewicz-type discretizations [11].

The obvious bound \( m_d(n) \geq 1/(n+1) \) was observed in several works, including [4, 2, 9]. The first non-trivial lower bound of \( m_d(n) \geq \frac{5}{4(n+5)} \) for \( d \geq 2 \) is due to Dumitrescu and Jiang [3]. In [4] they proved, for fixed \( b \) and \( d \), that \( (n+1)m_d(n) \geq (b+1)m_d(b) - o(1) \), which implies that the limit

\[
c_d \overset{\text{def}}{=} \lim_{n \to \infty} nm_d(n)
\]

exists. Indeed, for each \( b \), \( \liminf(n+1)m_d(n) \geq (b+1)m_d(b) \), and therefore \( \liminf(n+1)m_d(n) \geq \limsup(n+1)m_d(n) \).

The best lower bound on \( m_d(n) \) for fixed \( d \) is due to Aistleitner, Hinrichs and Rudolf [1], which is \( c_d \geq \frac{1}{d} \log_2 d \). In the same paper they present a proof, due to Larcher, that \( c_d \leq 2^{7d+1} \). In this note we show that the correct dependence of \( c_d \) on \( d \) is neither logarithmic nor exponential, but polynomial.

Theorem 1. The dispersion of \( n \)-point sets in \([0,1]^d\) satisfies

\[
m_d(n) \geq \frac{1}{n} \cdot \frac{2d}{e} \left( 1 - 4dn^{-1/d} \right) \quad \text{for all } d \text{ and all } n.
\]

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Theorem 2. For every $d \geq 3$ and every $n \geq 1$, there is a set of at most $n$ points in $[0,1]^d$ for which the largest empty box has volume at most $8000d^2 \log d/n$.

For very large $n$, we have a slightly better lower bound.

Theorem 3. Suppose numbers $0 < T < R_0$ satisfy $R_0 - T < \log R_0 / T$. Then

$$c_d \geq R_0 \exp\left(-\frac{1}{2d}(R_0 - T)\right).$$

In particular, $c_d \geq \frac{2d}{e}(1 + e^{-2d})$ for all $d$, and $c_2 \geq 1.50476$.

This improves on the aforementioned bound of $c_2 \geq \frac{5}{4}$ by Dumitrescu–Jiang. Very recently the upper bound of $c_2 \leq 1.8945$ was proved by Kritzinger and Wiart [7].

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2 Proofs of the lower bounds (Theorems 1 and 3)

Averaging argument. We first give a simple argument for Theorem 1. We will then show how to modify that argument to get Theorem 3. We start with the common part of the two arguments.

Let $R_0 > 0$ be a parameter to be chosen later, and set $\delta \overset{\text{def}}{=} \frac{1}{2}(R_0/n)^{1/d}$. Let $f : [0,R_0] \to \mathbb{R}_+$ be some weight function. We postpone the actual choice of $f$ until later. We adopt the convention that $f(R) = 0$ if $R \geq R_0$.

Let $B$ be the cube of volume $R_0/n$ centered at the origin, i.e., $B \overset{\text{def}}{=} [-\delta, \delta]^d$. Using $f$, we define a function on $\mathbb{R}^d$ by

$$F(x) \overset{\text{def}}{=} f(2^d r d n) \quad \text{for} \quad ||x||_\infty = r.$$ 

Because $f$ vanishes outside $[0,R_0]$, the function $F$ is supported on $B$. Put $M \overset{\text{def}}{=} n \int F$. Note that $M = 2^d n \int_{r=0}^\delta f(2^d r d n) dr = \int_{R_0}^{R_0} f(R) dR$. Because

$$\int_{t \in \mathbb{R}^d} \sum_{p \in P-t} F(p) = \sum_{p \in P} \int_{t \in \mathbb{R}^d} F(p-t) = \sum_{p \in P} \int_{x \in \mathbb{R}^d} F(x) = M,$$

it follows that there exists $t \in [\delta, 1-\delta]^d$ such that

$$\sum_{p \in P-t} F(p) \leq M/(1-2\delta)^d. \quad (2)$$

It suffices to find a large box inside $B$ that is empty with respect to the set $P' \overset{\text{def}}{=} (P-t) \cap B$, for then we may obtain an empty box of the same volume inside $[0,1]^d$ after translating by $t$.

To find the empty box, we trim the sides off $B$. Namely, for each point $p \in P'$ there is a coordinate of largest absolute value. If there is more than one such direction, break the tie arbitrarily. Call this coordinate dominant for $p$. For each $i \in [d]$, put

$$a_i \overset{\text{def}}{=} \min\{-p_i : i \text{ is dominant for } p \in P' \text{ and } p_i \leq 0\},$$

$$b_i \overset{\text{def}}{=} \min\{ p_i : i \text{ is dominant for } p \in P' \text{ and } p_i \geq 0\}.$$
Should the set in the definition of \( a_i \) be empty, we put \( a_i = \delta \). Similarly, should the set in the definition of \( b_i \) be empty, we put \( b_i = \delta \). The box

\[
B' \overset{\text{def}}{=} \prod_{i=1}^{d} (-a_i, b_i)
\]

is evidently disjoint from \( P - t \) and is contained in \( B \).

**Lemma 4.** The volume of \( B' \) is at least \( R_0 \frac{n}{n} \prod_{p \in P'} \sqrt[\|p\|_{\infty}]{\delta} \).

**Proof.** Fix any coordinate \( i \in [d] \).

Suppose first that the two sets in definitions of \( a_i \) and \( b_i \) are non-empty. Let \( p, q \in P - t \) be the points such that \( a_i = p_i \) and \( b_i = q_i \). By the AM–GM inequality

\[
\frac{a_i + b_i}{2\delta} \geq \sqrt{\frac{a_i b_i}{\delta^2}} = \sqrt{\|p\|_{\infty}} \cdot \sqrt{\|q\|_{\infty}}.
\]

(3)

Suppose next that only of the two sets in the definitions of \( a_i \) and \( b_i \) is non-empty. Say \( a_i = p_i \) for some \( p \in P - t \) and \( b_i = \delta \) (the other case being symmetric). Then by the similar application of the AM–GM inequality we obtain

\[
\frac{a_i + b_i}{2\delta} \geq \sqrt{\|q\|_{\infty}}.
\]

(4)

By taking the product of (3) and (4) as appropriate over all \( i \in [d] \), and noting that every point has only one dominant coordinate, we obtain

\[
\text{vol } B' = (2\delta)^d \cdot \prod_{i=1}^{d} \frac{a_i + b_i}{2\delta} \geq (2\delta)^d \prod_{p \in P'} \sqrt[\|p\|_{\infty}]{\delta}. \quad \square
\]

**Simple weight function (proof of Theorem 1).** The simplest choice of the constant \( R_0 \) and weight function \( f \) is

\[
R_0 \overset{\text{def}}{=} 2d,
\]

\[
f(R) \overset{\text{def}}{=} \log \frac{R_0}{R}.
\]

With this choice, \( M = \int_0^{R_0} f(R) \, dR = R_0 \) and \( F(x) = d \log \frac{\delta}{\|x\|_{\infty}} \) on \( B \). So, we may combine (2) with Lemma 4 to obtain

\[
\text{vol } B' \geq \frac{R_0}{n} \exp \left( -\frac{1}{2} \sum_{p \in P'} \log \frac{\delta}{\|p\|_{\infty}} \right) = \frac{R_0}{n} \exp \left( -\frac{1}{2d} \sum_{p \in P'} F(p) \right)
\]

\[
\geq \frac{R_0}{n} \exp \left( \frac{1}{2d} M(1 - 2\delta)^{-d} \right) = \frac{R_0}{n} \exp \left( - (1 - (R_0/n)^{1/d})^{-d} \right)
\]

\[
\geq \frac{R_0}{n} \exp \left( -(1 - 2n^{-1/d})^{-d} \right) \geq \frac{R_0}{n} \exp \left( - (1 - 2dn^{-1/d})^{-1} \right).
\]

Using \( \exp(-(1 - x)^{-1}) = e^{-1} \cdot \exp(-x - x^2 - \cdots) \geq e^{-1} \cdot (1 - x - x^2 - \cdots) \geq e^{-1}(1 - 2x) \) for \( x \in [0, 1/2] \), we may deduce that

\[
\text{vol } B' \geq \frac{1}{n} \cdot 2d \cdot e \left( 1 - 4dn^{-1/d} \right).
\]
Better weight function (proof of Theorem 3). Let $T$ and $R_0$ be as in the statement of Theorem 3. Since the aim is to prove a bound on $c_d$, we may assume that $n$ is sufficiently large. Define

$$f(R) \overset{\text{def}}{=} \begin{cases} \log \frac{R_0}{R} & \text{if } R \leq T, \\ \log \frac{R_0}{T} & \text{if } T < R \leq R_0. \end{cases} \quad (5)$$

It is readily computed that $M = R_0 - T$. Since $(1 - 2\delta)^d \to 1$, it follows that $M/(1 - 2\delta)^d \leq \log \frac{R_0}{T}$, for large enough $n$. Because of (2), this implies that for no point $x \in P'$ does it hold that $R \leq T$, where $R = 2^d n \|x\|_\infty$. So, $F(x) = d \log \frac{\delta}{\|x\|_\infty}$ for all $x \in P' \cap B$. So, we may proceed as before to obtain

$$\text{vol } B' \geq \frac{R_0}{n} \exp \left( -\frac{1}{2d} \sum_{p \in P'} F(p) \right) \geq \frac{R_0}{n} \exp \left( -\frac{1}{2d} M (1 - 2\delta)^{-d} \right).$$

Taking the limit $n \to \infty$, the bounds on $c_d$ follows.

The bound $c_d \geq 2^d (1 + e^{-2d})$ is obtained by choosing $R_0 = 2d$ and $T = R_0 \exp(-R_0)$. The bound $c_2 \geq 1.50476$ is obtained by choosing $R_0 = 3.69513$ and $T_0 = 0.101622$.

3 Proof of the upper bound (Theorem 2)

Construction outline. Our construction is a modification of the Hilton–Hammerseley construction. As in the Hilton–Hammerseley construction, we will select primes $p_1, \ldots, p_d$, each of which is associated to respective coordinate direction. As in the analysis of Hilton–Hammerseley construction, we will be interested in canonical boxes, which are the boxes$^1$ of the form

$$B = \prod_{i=1}^d \left[ a_i \left( \frac{a_i}{p_i^{k_i}}, \frac{a_i+1}{p_i^{k_i}} \right). \right.$$

for some integers $0 \leq a_i < p_i^{k_i}, i = 1, 2, \ldots, d$.

For a prime $p$ and a nonnegative integer $x$, consider the base-$p$ expansion of the number $x$, say $x = x_0 + x_1 p + \cdots + x_\ell p^\ell$. Put $r_p(x) \overset{\text{def}}{=} x_0 p^{\ell-1} + x_1 p^{\ell-2} + \cdots + x_\ell p^{-\ell-1}$; note that $r_p(x)$ is the number in $[0,1)$ obtained by reversing the base-$p$ digits of $x$. Define the function $r: \mathbb{Z}_{\geq 0} \to [0,1]^d$ by $r(x) \overset{\text{def}}{=} (r_p(x), \ldots, r_{pd}(x))$.

Our construction is broken into two stages. The set that we construct in the first stage is an $r$-image of a certain subset of $\mathbb{Z}_{\geq 0}$. (Note that the usual Hilton–Hammerseley construction is the $r$-image of the interval of length $n$.) This set has $O(nd \log d)$ elements and intersects almost all the boxes of volume about $1/n$. In the second stage of the construction, we show that $d + 1$ suitably chosen translates of the first set meet all the boxes of volume $1/n$.

$^1$Here and elsewhere in this section we work with half-open boxes. Since every half-open box contains an open box of the same volume, this does not impair the strength of our constructions, but doing so will be technically advantageous.
First stage. To simplify the proof, we will discretize the boxes we work with. We will do so by shrinking them slightly, so that $i$’th coordinates have terminating base-$p_i$ expansions.

With hindsight we choose $p_i$ to be the $(d + i)\text{th}$ smallest prime, for each $i = 1, 2, \ldots, d$. Put $\gamma \eqdef p_1p_2 \cdots p_d$, and let $n$ be an arbitrary integer divisible by $2\gamma^{11}$.

**Definition 5.** We say that a box $\beta$ is a good box if it is of the form

$$\beta = \prod_{i=1}^{d} \left[ \frac{a_i}{p_i^{k_i}} + \frac{b_i}{p_i^{k_i+3}}, \frac{a_i}{p_i^{k_i}} + \frac{c_i}{p_i^{k_i+3}} \right],$$

for some integers $0 \leq b_i < c_i \leq p_i^3$ and $k_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \ldots, d$, and whose volume is $1/4n \leq \text{vol}(\beta) \leq 1/n$. Let $B = \prod_i \left[ a_i/p_i^{k_i}, (a_i + 1)/p_i^{k_i} \right]$. We call $(B, \beta)$ a good pair.

Note that $\text{vol}(B) \leq \gamma^3/n$. In other words, a (discretized) box $\beta$ is good if it is contained in a canonical box $B$ that is not much larger than $\beta$. Note that the choice of $B$ in the definition of a good pair is, in general, not unique.

Suppose $B$ is a canonical box. Write it as $B = \prod_i \left[ a_i/p_i^{k_i}, (a_i + 1)/p_i^{k_i} \right]$, and consider $r^{-1}(B)$. The set $r^{-1}(B)$ consists of the solutions to the system

$$x \equiv a'_1 \pmod{p_1^{k_1}},$$

$$x \equiv a'_2 \pmod{p_2^{k_2}},$$

$$\vdots$$

$$x \equiv a'_d \pmod{p_d^{k_d}},$$

where $a'_i \eqdef r_{p_i}(a_i)p_i^{k_i}$, i.e., $a'_i$ is the integer obtained from $a_i$ by reversing its base-$p_i$ expansion.

By the Chinese Remainder theorem, the set $r^{-1}(B)$ is an infinite arithmetic progression with step $D(B) \eqdef p_1^{k_1}p_2^{k_2} \cdots p_d^{k_d} = 1/\text{vol}(B)$. Let $A(B)$ be the least element of $r^{-1}(B)$, so that

$$r^{-1}(B) = \{ A(B) + kD(B) : k \in \mathbb{Z}_{\geq 0} \}.$$ 

Given a good pair $(B, \beta)$, define

$$L_B(\beta) \eqdef \{ k \in \mathbb{Z}_+ : r\left( A(B) + kD(B) \right) \in \beta \}.$$ 

**Claim 1.** The set $\mathcal{L} \eqdef \{ L_B(\beta) : (B, \beta) \text{ is a good pair} \}$ is of size at most $\gamma^{12}$.

**Proof.** Let $(B, \beta)$ be a good pair. Write $B$ and $\beta$ in the form

$$B = \prod_{i=1}^{d} \left[ \frac{a_i}{p_i^{k_i}}, \frac{a_i + 1}{p_i^{k_i}} \right], \quad \beta = \prod_{i=1}^{d} \left[ \frac{a_i}{p_i^{k_i}} + \frac{b_i}{p_i^{k_i+3}}, \frac{a_i}{p_i^{k_i}} + \frac{c_i}{p_i^{k_i+3}} \right].$$

We know that $r\left( A(B) + kD(B) \right) \in \beta$ is equivalent to

$$A(B) + kD(B) \equiv a'_1 + p_1^{k_1}J_1 \pmod{p_1^{k_1+3}},$$

$$A(B) + kD(B) \equiv a'_2 + p_2^{k_2}J_2 \pmod{p_2^{k_2+3}},$$

$$\vdots$$

$$A(B) + kD(B) \equiv a'_d + p_d^{k_d}J_d \pmod{p_d^{k_d+3}},$$

where

$$J_i \eqdef \frac{a_i}{p_i^{k_i}} + \frac{b_i}{p_i^{k_i+3}} - \frac{a_i}{p_i^{k_i}} - \frac{c_i}{p_i^{k_i+3}}$$

for $i = 1, 2, \ldots, d$.
where the sets $J_i$ consist of base-$p_i$ reversals of the numbers in the interval $[b_i, c_i)$ (which are 3-digit long in base $p_i$).

On the other hand, we know that

$$A(B) + kD(B) \equiv a'_1 + (\alpha_1 + k\delta_1)p_{1}^{k_1} \pmod{p_{1}^{k_1+3}},$$

$$A(B) + kD(B) \equiv a'_2 + (\alpha_2 + k\delta_2)p_{2}^{k_2} \pmod{p_{2}^{k_2+3}},$$

$$\vdots$$

$$A(B) + kD(B) \equiv a'_d + (\alpha_d + k\delta_d)p_{d}^{k_d} \pmod{p_{d}^{k_d+3}}$$

for some $\alpha_i, \delta_i \in \mathbb{Z}/p_i^3\mathbb{Z}, i = 1, 2, \ldots, d$. There are at most $\gamma^6$ different choices for $(\alpha_i, \delta_i)_{i=1}^d$. Also, there are at most $\gamma^6$ different choices for $(b_i, c_i)_{i=1}^d$ satisfying $0 < b_i < c_i \leq p_i^3$. Since $L_B(\beta)$ is determined by $(\alpha_i, \delta_i, b_i, c_i)_{i=1}^d$, the claim is true.

To each canonical box $B$ of volume between $1/4n$ and $\gamma^3/n$ we assign a type, so that boxes of the same type behave similarly. Formally, let $A(B)$ be the unique multiple of $n/\gamma^4$ satisfying $0 \leq A(B) - A(B) < n/\gamma^4$. Similarly, let $D(B)$ be the unique multiple of $n/\gamma^{11}$ satisfying $0 \leq D(B) - D(B) < n/\gamma^{11}$. The type of $B$ is then the pair $T(B) \overset{\text{def}}{=} (A(B), D(B))$.

Note that, from $1/4n \leq \text{vol}(B) \leq \gamma^3/n$ and $D(B) = 1/\text{vol}(B)$ it follows that

$$\frac{n}{\gamma^3} - \frac{n}{\gamma^{11}} < D(B) \leq 4n. \quad (6)$$

**Claim 2.** The number of types is at most $\gamma^{16}$.

**Proof.** Since $A(B) < D(B) \leq 4n$, the number of types is at most $(\frac{4n}{n/\gamma^{11}})(\frac{4n}{n/\gamma^{11}}) = 16\gamma^{15} \leq \gamma^{16}$. \hfill \Box

For a type $T = (A, D)$, let $\mathcal{Y}(T) \overset{\text{def}}{=} \{A + kD : k \in \mathbb{Z}_{\geq 0}\}$ be the arithmetic progression generated by $A$ and $D$. Note that if $T = T(B)$, then $\mathcal{Y}(T)$ is an approximation to $r^{-1}(B)$. In particular, $\mathcal{Y}(T)$ and $r^{-1}(B)$ intersect any long interval that is not too far from the origin in approximately the same way.

For integers $a, b$, denote by $[a, b)$ the integer interval consisting of integers $x$ satisfying $a \leq x < b$. Our construction will be a union of intervals of length $n/\gamma^3$ whose left endpoints are in $[0, n\gamma^4)$.

We first estimate the difference between respective terms in $\mathcal{Y}(T)$ and $r^{-1}(B)$ inside $[0, n\gamma^4)$.

**Claim 3.** Suppose $T(B) = (A(B), D(B))$. Then for any integer $x \in [0, n\gamma^4)$ and any integer $k$, $A(B) + kD(B) \in [x, x + n/\gamma^3)$ implies $A(B) + kD(B) \in [x, x + n/\gamma^3)$.

**Proof.** For such $k$, since $A(B) + kD(B) < n\gamma^4 + n/\gamma^3$, from (6) we deduce that

$$k < \frac{n\gamma^4 + n/\gamma^3}{n/\gamma^3 - n/\gamma^{11}} \leq 2\gamma^7.$$

In view of $k \geq 0$, this implies that

$$0 \leq (A(B) + kD(B)) - (A(B) + kD(B)) \leq \frac{n}{\gamma^4} + 2\gamma^7 \cdot \frac{n}{\gamma^{11}} = \frac{3n}{\gamma^4},$$

and hence $A(B) + kD(B) \in [x, x + n/\gamma^3 + 3n/\gamma^4) \subseteq [x, x + n/\gamma^3]$. \hfill \Box
For a type \( \mathcal{T} \) and \( L \in \mathcal{L} \) that satisfy \( \mathcal{T} = \mathcal{T}(B) \) and \( L = L_B(\beta) \) for some good pair \((B, \beta)\), define

\[
\mathcal{Y}_T(L) \overset{\text{def}}{=} \{ A + kD : k \in L \}.
\]

With this definition, \( \mathcal{Y}_T(L) \) is the approximation to \( r^{-1}(\beta) \) induced by the approximation \( \mathcal{Y}(B) \) to \( r^{-1}(B) \).

**Claim 4.** The set \( \mathcal{Y}_T(L) = \mathcal{Y}_T(L) \cap [0, n\gamma^4] \) is of size at least \( \gamma^4/16 + 1 \).

**Proof.** Let \((B, \beta)\) be a good pair such that \( \mathcal{T} = \mathcal{T}(B) \) and \( L = L_B(\beta) \). The set \( L_B(\beta) \) is \( \gamma^3 \)-periodic, i.e., \( k \in L_B(\beta) \) implies \( k + \gamma^3 \in L_B(\beta) \). The intersection of any interval of length \( \gamma^3 \) with \( L_B(\beta) \) is of size exactly \( \gamma^3 \frac{\text{vol}(\beta)}{\text{vol}(B)} \). Since the preimage of \([0, n\gamma^4] \) under the map \( k \mapsto A + kD \) contains non-overlapping intervals of length \( \gamma^3 \), the size of \( \mathcal{Y}_T(L) \) is at least

\[
\frac{1}{2} n\gamma \text{vol}(B) \cdot \gamma^3 \frac{\text{vol}(\beta)}{\text{vol}(B)} = \frac{1}{2} n\gamma^4 \text{vol}(\beta) \geq \gamma^4/16 + 1. \tag*{\Box}
\]

**Claim 5.** Let \( x \) be chosen uniformly from \([0, n\gamma^4] \). Then \( \Pr[\mathcal{Y}_T(L) \cap [x, x + n/2\gamma^3] \neq \emptyset] \geq 1/32\gamma^3 \).

**Proof.** Let \( y \in \mathcal{Y}_T(L) \) be arbitrary. If \( y \notin [0, n/2\gamma^3] \), then \( \Pr[y \in [x, x + n/2\gamma^3]] = 1/2\gamma^3 \). Since \( D > n/\gamma^3 - n/\gamma^{11} \geq n/2\gamma^3 \), the set \( \mathcal{Y}_T(L) \) contains at most one element in the interval \([0, n/2\gamma^3] \). Hence

\[
\mathbb{E}[\mathcal{Y}_T(L) \cap [x, x + n/2\gamma^3]] \geq 1/32\gamma^3.
\]

Since elements of \( \mathcal{Y}_T(L) \) are at least \( D \) apart, \( \mathcal{Y}_T(L) \cap [x, x + n/2\gamma^3] \in \{0, 1\} \) for all \( x \). Therefore,

\[
\Pr[\mathcal{Y}_T(L) \cap [x, x + n/2\gamma^3] \neq \emptyset] = \mathbb{E}[\mathcal{Y}_T(L) \cap [x, x + n/2\gamma^3]] \geq 1/32\gamma^3. \tag*{\Box}
\]

Sample \( 900\gamma^3 \log \gamma \) elements uniformly at random from \([0, n\gamma^4] \), independently from one another. Let \( X \) be the resulting set. Then by the preceding claim

\[
\Pr[\mathcal{Y}_T(L) \cap (X + [0, n/2\gamma^3]) \neq \emptyset] \leq (1 - 1/32\gamma^3)^{900\gamma^3 \log \gamma} < \gamma^{-28}.
\]

From Claims 1 and 2 and the union bound it then follows that there exists a choice of \( X \) such that \( \mathcal{Y}_T(L) \cap (X + [0, n/2\gamma^3]) \) is non-empty whenever \( \mathcal{T} = \mathcal{T}(B) \), \( L = L_B(\beta) \) and \((B, \beta)\) is a good pair. In other words, for every \((B, \beta)\) there exist \( x \in X \) and an integer \( k \in L_B(\beta) \) such that \( A(B) + kD(B) \in [x, x + n/\gamma^3] \). By Claim 3 this implies that \( A(B) + kD(B) \in [x, x + n/\gamma^3] \) for the same \( x \) and \( k \), whereas the definition of \( L_B(\beta) \) implies that \( r(A(B) + kD(B)) \in B \). Because this holds for every good pair \((B, \beta)\), the set \( P \overset{\text{def}}{=} r(X + [0, n/\gamma^3]) \) meets every good box.

Note that \( |P| \leq |X| \cdot \frac{D}{\gamma} \leq 900 \log \gamma \cdot n \leq 3000d \log d \cdot n \) (since \( \log \gamma \leq d \log p_d \leq 3d \log d \)).
**Second stage.** So far we have worked with boxes whose coordinates are rational numbers with denominators of the form $p_i^{k_i}$. Given an arbitrary box, we shall shrink it down to a box of such form. We begin by describing this process.

A p-interval is an interval of the form $[a/p^k, b/p^k)$ for some integers $0 \leq a < b < p^k$. A canonical p-interval is an interval of the form $[a/p^k, (a + 1)/p^k)$ with $0 \leq a < p^k$. Note that canonical boxes are precisely the boxes that are Cartesian products of canonical intervals in appropriate bases. A p-interval $[a/p^k, b/p^k)$ is well-shrunk if $b - a < p^2$.

**Claim 6.** Every interval $[s, u]$ contains a well-shrunk p-interval of length at least $(1 - 2/p)\text{len}[s, u]$.

**Proof.** Let $k$ be the smallest integer satisfying $\text{len}[s, u] \geq p^{-k}$. Let $I$ be the largest interval of the form $I = [a/p^{k+1}, b/p^{k+1})$ contained in $[s, u]$. Then $\text{len}I \geq u - s - 2p^{-k-1} \geq (1 - 2/p)(u - s)$, and $b - a = p^{k+1}\text{len}I \leq p^{k+1}\text{len}[s, u] < p^2$. □

Call an interval $[s, u]$ p-bad if it contains a rational number with denominator $p^{k+1}$, where $\text{len}[s, u] < 2p^{-k-2}$ and $k \in \mathbb{Z}_{\geq 0}$.

**Claim 7.** A box $\alpha = \prod_i [s_i, u_i) \subset [0, 1]^d$ of volume $1/n$ fails to contain a good box only if, for some $i \in [d]$, the interval $[s_i, u_i)$ is $p_i$-bad.

**Proof.** For each $i \in [d]$, let $[s'_i, u'_i)$ be a well-shrunk $p_i$-interval contained in $[s_i, u_i)$ as above. Let $\beta \triangleq \prod_i [s'_i, u'_i)$. Note that $\text{vol}(\beta) \geq \text{vol}(\alpha)\prod_i (1 - 2/p_i) \geq 1/4n$.

Let $B = \prod_i [a_i/p_i^{k_i}, (a_i + 1)/p_i^{k_i})$ be the smallest canonical box containing $\beta$. Since the p-interval $[s'_i, u'_i)$ is contained in $[a_i/p_i^{k_i}, (a_i + 1)/p_i^{k_i})$, we may write it in the form

$$[s'_i, u'_i) = [a_i/p_i^{k_i} + b_i/p_i^{\ell_i}, a_i/p_i^{k_i} + c_i/p_i^{\ell_i}]$$

for some integers $0 \leq b_i < c_i < p_i^{k_i-\ell_i}$. Since $[s'_i, u'_i)$ is well-shrunk, $c_i - b_i < p_i^{\ell_i}$. If $(B, \beta)$ is not a good pair, there exists $i \in [d]$, such that $\ell_i \geq k_i + 4$. Fix such an $i$. By the minimality of $B$, the interval $[s'_i, u'_i)$ contains a rational number with denominator $p_i^{k_i+1}$. Since $[s_i, u_i)$ contains $[s'_i, u'_i)$, this rational number is also contained in $[s_i, u_i)$. As $\text{len}[s_i, u_i) \leq (c_i - b_i + 2)p^{-\ell_i} < (p^2 + 2)p^{-k_i-4} \leq 2p^{-k_i-2}$, the interval $[s_i, u_i)$ is $p_i$-bad. □

**Claim 8.** Let $\Delta = 1/p(p - 1)$. Suppose $[s, u) \subset [1/p, 1)$ is an arbitrary interval. Then at most one of its translates $[s, u), [s, u) - 2\Delta, \ldots, [s, u) - 2d\Delta$ is p-bad.

**Proof.** Suppose that, for some $r$, the interval $[s, u) - 2r\Delta$ contains rational number $a/p^{k+1}$ and is of length $\text{len}[s, u) < 2p^{-k-2}$. Then the interval $[s_r, u_r) \triangleq [s, u) - 2r\Delta - a/p^{k+1}$ contains 0 and is also of length len$I < 2p^{-k-2}$. Hence, $u_r < 2p^{-k-2}$, and so $(k + 2)'nd$ digit in the base-$p$ of $u_r$ is either 0 or 1. Note that it is the same as the $(k + 2)'nd$ digit of $u - 2r\Delta$.

Since the base-$p$ expansion of $\Delta$ is 0.011111 ... and $2d + 1 < p$, for at most one of the numbers $u, u - 2\Delta, \ldots, u - 2d\Delta$ is the $(k + 2)'nd$ digits equal to 0 or 1. Hence, at most one of the intervals $[s, u), [s, u) - 2\Delta, \ldots, [s, u) - 2d\Delta$ contains a rational number with denominator $p^{k+1}$. □

Let $P$ be the set constructed in the first stage. Let $v \in [0, 1]^d$ be the vector whose $i$’th coordinate is $v_i = 1/p_i(p_i - 1)$. Let $P' \triangleq \bigcup_{r=0}^{d}(P + 2rv)$. We claim that $P'$ meets every subbox of $\prod_i [1/p_i, 1]$ of volume $1/n$. 8
Indeed, suppose \( \alpha = \prod_i [s_i, u_i] \subset \prod_i [1/p_i, 1] \) is an arbitrary box of volume \( 1/n \). Then by the preceding claim, there exists \( r \in \{0, 1, \ldots, d\} \) such that for no \( i \in [d] \) is the interval \( [s_i, u_i] - 2r\Delta \) \( p \)-bad. Claim 7 tells us that the box \( \alpha - 2r\Delta \) contains a good box. Since the set \( P \) meets all good boxes, it follows that the \( P + 2r\Delta \) meets \( \alpha \). As \( P + 2r\Delta \subset P' \), the set \( P' \) indeed meets \( \alpha \).

Finally, we scale the box \( \prod_i [1/p_i, 1] \) onto \( [0, 1]^d \). This way, we turn the set \( P' \) into a set that meets every subbox of \([0, 1]^d\) of volume \( \frac{1}{d} \prod_d p_i/(p_i - 1) \leq 2/n \). This set has size \( |P'| \leq (d + 1) \cdot 3000d \log d \cdot n \).

This construction shows that \( m_d([3000d(d + 1) \log d \cdot n]) \leq 2/n \) for all \( n \) that are divisible by \( 2^7 \). Since the limit \( c_d = \lim_{n \to \infty} nm_d(n) \) exists, it then follows that \( c_d \leq 6000d(d + 1) \log d \), which, by the Dumintruc-Jiang inequality mentioned in the introduction, implies that \( m_d(b) \leq \frac{1}{b+1} \cdot 6000d(d + 1) \log d \) for all \( b \). Because \( 6000d(d + 1) \log d \leq 8000d^2 \log d \), the proof is complete.

4 Problems and remarks

- Because of the \( n^{-1/d} \) term, the bound in Theorem 1 is weak when the number of points \( n \) is small compared to the dimension \( d \). It is likely possible to replace the term \( n^{-1/d} \) with \( O_d(n^{-1}) \) by using a more sophisticated averaging argument. In our argument we considered an average of translates of a function supported on a fixed box \( B \). The error term \( n^{-1/d} \) is due to the points near the boundary of \([0, 1]^d \) receiving less weight than the rest. One can remedy this by using, in addition to the translates of \( B \), also elongated boxes of volume \( \text{vol}(B) \) to add weight in the regions near the boundary of \([0, 1]^d \). In this paper, we decided to sacrifice the slightly stronger bound for a simpler proof.

For constructions of low-dispersion sets that are good when \( n \) is small compared to \( d \), see [10, 13].

- The low-dispersion sets are used in [2],[6],[8, Theorem 11] to give algorithms to approximate certain one-dimensional tensors. Because of that, it would be useful to derandomize the construction in Theorem 2, as doing so would yield a deterministic algorithm for that problem.

Disperion has also been studied on the torus. In this variant of the problem, the boxes are products of toroidal intervals, which, in addition to the usual intervals \((a, b)\) for \( a < b \), include the sets of the form \((a, b) \equiv (a, 1] \cup [0, b)\) for \( a > b \). Denote the \( d \)-dimensional torus by \( \mathbb{T}^d \), and let \( m_d^T \) be the corresponding dispersion function, i.e., the largest number such that there is an empty box of volume \( m_d^T(n) \) among every \( n \)-point set on \( \mathbb{T}^d \). Ullrich [12] proved that \( m_d^T(n) \geq \min\{1, d/n\} \). The construction of Larcher, which was mentioned in the introduction, carries over verbatim to the torus, and so \( m_d(n) \leq 2^{7d+1}/n \). We can improve the base of exponent from \( 2^7 \) to \( e/2 \).

Proposition 6. The toroidal dispersion satisfies \( m_d^T(n) \leq 32000(e/2)^d d^3 \log d/n \), for all \( n \) divisible by \( d \).

Proof. Let \( P \) be the set obtained by invoking Theorem 2 with \( n/d \) in place of \( n \). Write \( P + u \) to denote the shift of \( P \) by vector \( u \), where the ‘shift’ is understood as a shift on \( \mathbb{T}^d \). Set \( v \equiv (1/d, 1/d, \ldots, 1/d) \in \mathbb{T}^d \), and consider the shifts \( P + rv \) for \( r \in \{0, 1, \ldots, d - 1\} \). We claim that the toroidal dispersion of \( \bigcup_{r=0}^{d-1}(P + rv) \) is at most \( 32000(e/2)^d d^3 \log d/n \). To prove this, it
suffices, for every toroidal box $B_0$ of volume $32000(e/2)^d d^3 \log d/n$, to find $r \in \{0, 1, \ldots, d - 1\}$ such that the toroidal box $B_r \overset{\text{def}}{=} B_0 - rv$ contain a usual box of volume $8000d^3 \log d/n$.

Write $\text{len}(a, b)$ for the length of a toroidal interval $(a, b)$. If $(a, b)$ is a toroidal interval, the largest usual interval contained in $(a, b) - x$ has length $\text{len}(a, b)$ if $x \notin (a, b)$ and $\max\{\text{len}(a, x), \text{len}(x, b)\}$ if $x \in (a, b)$. For a toroidal interval $(a, b)$, let $f_{(a,b)}$ be the function given by

$$f_{(a,b)}(x) \overset{\text{def}}{=} \begin{cases} 
\log \left( \max \{\frac{\text{len}(a, x)}{\text{len}(a, b)}, \frac{\text{len}(x, b)}{\text{len}(a, b)}\} \right) & \text{for } x \in (a, b), \\
0 & \text{for } x \notin (a, b). 
\end{cases}$$

If $B = \prod (a_i, b_i)$ is a toroidal box, the largest usual box contained in $B - (x_1, \ldots, x_d)$ has volume

$$\text{vol}(B) \exp \left( \sum f_{(a_i, b_i)}(x_i) \right).$$

We shall estimate $\frac{1}{d} \sum_{r=0}^{d-1} f_{(a_i, b_i)}(r/d)$ by comparing it to the respective integral: Since the function $f_{(a,b)}$ is unimodal with minimum at $x = (a + b)/2$, the total variation of $f_{(a,b)}$ is $f(a) + f(b) - 2f_{(a,b)}(\frac{a+b}{2}) = 2 \log 2$. Hence,

$$\frac{1}{d} \sum_{r=0}^{d-1} f_{(a,b)}(r/d) \geq \int_0^1 f_{(a,b)}(x) \, dx - 2 \log 2/d. \tag{7}$$

We can bound the integral in turn by $\int f_{(a,b)}(x) \, dx = (\log 2 - 1) \text{len}(a, b) \geq \log 2 - 1$. Summing (7) over each of the $d$ coordinate directions, we then obtain

$$\frac{1}{d} \sum_{r=0}^{d-1} \sum_{i=1}^{d} f_{(a_i, b_i)}(r/d) \geq d \log(2/e) - 2 \log 2.$$

Hence, given any toroidal box $B_0$, there exists $r \in \{0, 1, \ldots, d - 1\}$ such that the toroidal box $B_r = B_0 - rv$ contains a usual box of volume at least $\text{vol}(B_0) \cdot \frac{1}{d} (2/e)^d$. In particular, if $\text{vol}(B_0) \geq 32000(e/2)^d d^3 \log d/n$, then $B_r$ contains a usual box of volume $8000d^3 \log d/n$. \hfill \square

It might be that the toroidal dispersion is indeed larger than the usual dispersion. One evidence in that direction is that the VC dimension of boxes in the $[0, 1]^d$ is $2d$ whereas the VC dimension of toroidal boxes is asymptotic to $d \log_2 d$, as recently showed by Gillibert, Lachmann and Müllner [5].

- We suspect that the smallest dispersion of an $n$-point set is asymptotic to $\Theta(d \log d \cdot \frac{1}{n})$.

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