Brane-like singularities with no brane

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We use a method of linearization to study the emergence of the future cosmological singularity characterized by finite value of the cosmological radius. We uncover such singularities that keep Hubble parameter finite while making all higher derivatives of the scale factor (starting out from the $\ddot{a}$) diverge as the cosmological singularity is approached. Since such singularities has been obtained before in the brane world model we name them the "brane-like" singularities. These singularities can occur during the expanding phase in usual Friedmann universe filled with both a self-acting, minimally coupled scalar field and a homogeneous tachyon field. We discover a new type of finite-time, future singularity which is different from type I-IV cosmological singularities in that it has the scale factor, pressure and density finite and nonzero. The generalization of $w$-singularity is obtained as well.

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I. INTRODUCTION

Starting out from the discovery of the cosmic acceleration \cite{1} there have been constructed many models of the dark energy, including the very unusual ones: the phantom energy, the tachyon cosmologies, the brane worlds etc. Consideration of these models results in some unexpected conclusions about possibility of new cosmological doomsday scenarios: the Big Rip singularity (BRS) \cite{2}, the Big Freeze singularity (BFS) \cite{3, 4}, the Sudden Future singularity (SFS) \cite{5}, the Big Boost singularity (BBtS) \cite{6}, and the Big Break singularity (BBS) \cite{7, 8}.

In all these models the evolution ends with the curvature singularity, $|\ddot{a}(t)| \to \infty$, reachable in a finite proper time, say as $t \rightarrow t_s$. BRS and BFS both take place in the phantom models but with the different equations of state. In particular, BRS takes place if $w = p/\rho = \text{const} < -1$ whereas BFS occurs for the dark energy in the form of a phantom generalized Chaplygin gas. Models with the SFS, BFS, BBtS and BBS singularities are characterized by a finite value of the cosmological radius but different values of Hubble expansion parameter $H_s = H(t_s)$ and different signs of (divergent) expression $\ddot{a}/a$ (cf. also \cite{6}):

\begin{align*}
    a_s < \infty, & \quad H_s = +\infty, \quad \ddot{a}_s/a_s = +\infty, \quad \text{(BRS)} \\
    a_s < \infty, & \quad H_s = +\infty, \quad \ddot{a}_s/a_s = +\infty, \quad \text{(BFS)} \\
    a_s < \infty, & \quad 0 < H_s < \infty, \quad \ddot{a}_s/a_s = -\infty, \quad \text{(SFS)}
\end{align*}

whereas BFS occurs for the dark energy in the form of a phantom generalized Chaplygin gas. Models with the SFS, BFS, BBtS and BBS singularities are characterized by a finite value of the cosmological radius but different values of Hubble expansion parameter $H_s = H(t_s)$ and different signs of (divergent) expression $\ddot{a}/a$ (cf. also \cite{6}):

\begin{align*}
    a_s < \infty, & \quad 0 < H_s < \infty, \quad \dot{a}_s/a_s = +\infty, \quad \text{(BBtS)} \\
    a_s < \infty, & \quad H_s = 0, \quad \dot{a}_s/a_s = -\infty. \quad \text{(BBS)}
\end{align*}

Remark 1. One of classifications of singularities for the modified gravity was given in \cite{11}; for the classification and discussion concerned with avoiding the singularities in the alternative gravity dark energy models cf. \cite{11}. Another classification of finite-time future singularities (Type I-IV singularities) is presented in \cite{12}. According to this classification, the BRS is a singularity of type I, BFS is of type III, SFS and BBtS are type II and BBS - type II with $\rho_s = \rho(t_s) = 0$ (although this is a quite non-trivial special case of a type II singularities). Our classification doesn’t contain singularities of IV type (for $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow 0, |p| \rightarrow 0$ and higher derivatives of $H$ diverge) but as we shall see in Sec. VI, the classification of Ref. \cite{12} is not exactly complete too: the type IV is the special case of a more general type of singularities.

Remark 2. Another type of "singularity" - so called $w$-singularity was obtained in \cite{13}. This "singularity" has a finite scale factor, vanishing energy density and pressure, and the singular behavior manifesting itself only in a time-dependent barotropic index $w(t)$. The $w$-singularities seem to be most similar to the type IV but are different nonetheless since they do not lead to any divergence of higher order derivatives of $H$ \cite{13}.

One surmises that $w$-singularity is not a correct physical singularity since all the physical values (i.e. density, pressure and higher derivatives of the scale factor or Hubble roots) are finite. Moreover, the definition of $w$-singularity from the \cite{13} is an incomplete one. To show this let us consider the following form of the scale factor

\begin{equation}
    a(t) = a_s - A(t_s - t)^m. \quad (1)
\end{equation}

(1) is the special case of the general form of the scale factor from the \cite{13} (with $B = 0, A = a_s, C/t_s^m = -A$, \cite{12}).
$D = 1, \, n = m$). One can show that for $t \to t_s$:

(a) type III singularity if $0 < m < 1$;

(b) type II singularity if $1 < m < 2$;

(c) $w$-singularity if $m > 2$.

The case $m = 1$ results in a model with the constant barotropic index $w = -1/3$. The case $m = 2$ is the most interesting one because

$$\rho \to 0, \quad p \to \frac{4A}{3a_s} \neq 0, \quad |w| \to \infty,$$

and

$$\frac{d^{2n} H}{dt^{2n}} = 0, \quad \frac{d^{2n+1} H}{dt^{2n+1}} \sim \frac{A^{n+1}}{a_s^{n+1}} < \infty,$$

at $t = t_s$. Thus we have some generalization of $w$-singularity such that the pressure is non-vanishing and finite at $t = t_s$.

The BRS and BFS have been obtained in the phantom cosmologies (BRS for the phantom perfect fluid with equation of state $p/\rho = w = \text{const} < -1$ and BFS for the phantom Chaplygin models. Throughout the paper we’ll stick to the metric units with $8\pi G/3 = c = 1$). The BBtS is connected to the effect of the conformal anomaly that drives the expansion of the Universe to a maximal value of the Hubble constant, after which the solution becomes complex. The BBS takes place in tachyon models.

Unlike BRS, BFS and BBtS altogether, the BBS and SFS are violating just the dominant energy condition ($\rho \geq 0, \, -p \leq \rho \leq \rho$). It is also possible to obtain some generalization of these singularities. In particular, generalization of the Sudden Future singularities (the so called Generalized Sudden Future singularities or GSFS) are singularities such that one has the derivative of pressure $p^{(m-2)}$ singularity which accompanies the blow-up of the $m$-th derivative of the scale factor $a^{(m)}$. These singularities are possible in theories with higher-order curvature quantum corrections and corresponds to classification in this paper.

Despite the fact that there has recently been a great inflow of articles, elaborating on the aforementioned singularities, an absolute majority of them has been of a mathematical nature, while the physical reasons for arousal of such singularities still remain less than clear. A remarkable exception is the article, which has introduced for the first time a new type of cosmological singularities located on the brane (for discussion about the soft singularities on brane with the quantum corrections cf. 15). These singularities are characterized by the fact that while the Hubble parameter and scale factor remain finite, all higher derivatives of the scale factor ($\ddot{a}$ etc.) diverge as the cosmological singularity is approached. These singularities may be obtained as the result of embedding of $(3+1)$-dimensional brane in the bulk and this is why these singularities will be henceforth referred to as the "brane-like" singularities. We’ll define the "brane-like" singularities in a following fashion: we’ll say that singularity is of a "brane-like" type if at the instance of its occurrence both scale factor and density remain finite and nonzero, while all the higher order derivatives of scale factor (starting with the second order) become altogether singular, i.e. $a \to a_s, \, \rho \to \rho_s, \, 0 < a_s \ll \infty, \, 0 < \rho_s < \infty, \, d^n a_s/dt^n = \infty$ for $n > 1$.

Evidently, the class of "brane-like" singularities includes the singularities of Type II (with $\rho_s \neq 0$) or SFS and BBtS. Moreover, BBS will also be of this type whenever we are talking about the models with the constant positive curvature, since at the singularity point $\rho_s = 1/a_s^2$.

The physical nature of "brane-like" singularities emergence is quite clear: in the simple case with $Z_2$ reflection symmetry and the identical cosmological constants on the two sides of the brane, the dynamical equation contains few additional terms. One of them is the square root of the sum of contributions of density (on the brane), tension, cosmological constant and the "dark radiation" (the last one arises due to the projection of the bulk gravitational degrees of freedom onto the brane). This sum is not positively defined and might become negative during the cosmological evolution. Thus, the solution of the cosmological equations can’t be continued beyond the point where this sum turns to zero and what we end up with at this point is nothing but a "brane-like" singularity. Since the existence of such singularities is natural in the brane physics, it won’t be against the logic to assume that the appearance of "brane-like" singularities in usual Friedmann cosmology (SFS or BBtS) might be an evidence of validity of the brane hypothesis. Therefore it is interesting to consider "brane-like" singularities without a brane (i.e. in FLRW cosmology) to establish the particular form of potential and the equation of state that will result in such singularities during cosmological dynamics. Such potential and the equation of state may altogether be useful for answering the big cosmological question: Don’t we really live on the brane?

Furthermore, such singularities may actually result in very unusual models. In fact, let’s consider the universe which contains a "brane-like" singularity. If the universe is filled with a scalar field while the Hubble parameter $H(t_s) = H_s$ and the scale factor $a(t_s) = a_s$ are finite at the singular point $(H_s < \infty, \, a_s < \infty)$ then the value of the scalar field $\phi(t_s) = \phi_s$ might be finite as well. On the other hand, quantum corrections of higher order (in N-loops approximation) depend on the higher derivatives. If higher derivatives of scale factor diverge then this will also be the case for the scalar field. So one can expect that since all higher derivatives of scale factor and field alike diverge as the cosmological singularity is approached, then the quantum effects will be dominating for $t \to t_s$. This will be the case in spite of the fact that both density $\rho_s$ and scale factor $a_s$ will be finite and that $\rho_s$ might be small and $a_s$ – very large.

It may seem that quantum corrections will be dominating because the pressure $|p| \to \infty$ as the cosmological singularity is approached. This is not the case for the singularities of the IV type being "brane-like" by definition. Moreover, in Sec. VI we’ll construct the singularities of
even more general type that will violate the classifications of [12] since \( \rho_s \) and \( p_s \) will be finite and nonzero and all higher derivatives will diverge.

In this paper one constructs "brane-like" singularities in Friedmann-Lemaitre-Robertson-Walker universe filled with the usual self-acting, minimally coupled scalar field or homogeneous tachyon field. In this cosmology we'll also construct the singularities (with finite scale factor) where that Hubble variable vanishes and all higher derivatives of the scale factor diverge as the cosmological singularity is approached. That type of singularities is the generalization of the Big Break singularities and there will also be those of the "brane-like" type for the case of a constant positive curvature. We will calculate both self-acting potential \( V(\phi) \) and tachyon potential \( V(T) \) that result in appearance of such singularities. Additionally, we'll present the equation of state for such models. Moreover, the classification of singularities from the Ref. [12] will be complemented.

This paper contains a discussion of a simple but useful method which allows one to construct exact solutions of the cosmological Friedmann equations filled with both self-acting, minimally coupled scalar field and a homogeneous tachyon field. The method itself will be denoted as the method of linearization and it has been previously suggested in [16] (see also [17], [18], [19]). We'll give a brief discussion of this method in the next Section. In Sec. III we construct exact solutions with "brane-like" singularities. Then we consider field models (both tachyon and the usual minimally coupled scalar field) near the singularity (Sec. IV). Finally, in Sec. V we show that method of linearization allows one to obtain exact forms of potentials in tachyon models. In particular, we'll show that tachyon model that has been discussed in detail in [17] is one of the simplest models in framework of the method of linearization. In Sec. VI we construct the new type of singularities which are some kind of generalization of the type IV singularities. The discussion is concluded in Sec. VII.

II. THE METHOD OF LINEARIZATION

Let us write the Friedmann equations as

\[
\left( \frac{\dot{a}}{a} \right)^2 = \rho - \frac{k}{a^2}, \quad 2\frac{\ddot{a}}{a} = -\left( \rho + 3p \right).
\]

The crucial point of this paper is the fact that a Friedmann equations admits a linearizing substitution and can therefore be studied via the different powerful mathematical methods which were specifically developed for the linear differential equations. This is the reason we call our approach the method of linearization [16]:

**Proposition.** Let \( a = a(t) \) (with \( p = p(t), \rho = \rho(t) \)) be a solution of (2). Then for the case \( k = 0 \) the function \( \psi_n = a^n \) is the solution of the Schrödinger equation

\[
\frac{\psi_n}{\psi_n} = U_n,
\]

with potential

\[
U_n = n^2 \rho - \frac{3n}{2} (\rho + p).
\]

For example:

\[
U_1 = -\frac{\rho + 3p}{2}, \quad U_2 = \rho - 3p, \quad U_3 = \frac{9}{2} (\rho - p),
\]

or

\[
U_{1/2} = -\frac{1}{2} \left( \rho + \frac{3p}{2} \right), \quad U_{-1} = \frac{5\rho + 3p}{2}
\]

and so on.

**Remark 3.** If the universe is filled with scalar field \( \phi \) whose Lagrangian is

\[
L = \frac{\phi^2}{2} - V(\phi) = K - V,
\]

then the expression (4) will be

\[
U_n = n(n - 3)K + n^2V.
\]

In particular case \( n = 3 \) \( U_3 = 9V(\phi) \). This particular case has been extensively studied in [17], [18].

**Remark 4.** For small values of \( n \ll 1 \) one gets \( U_n \sim -3n(\rho + p)/2 \); for example, if \( n = 0.01 \) then

\[
U_n \sim -(0.0149\rho + 0.015p) \sim -0.015(\rho + p).
\]

Therefore one can use \( U_n < 0 \) to check whether the weak energy condition is violated [21]. If, on the contrary, \( n \gg 1 \) then \( U_n \sim n^2\rho \).

**Remark 5.** If \( k = \pm 1 \) then the Proposition is valid only for the case \( n = 0, 1 \).

As we have shown in [19], the representation of the Einstein-Friedmann equations as a second-order linear differential equation [2] allows for a usage of an arbitrary (known) solution for construction of another, more general solution parameterized by a set of 3N constants, where \( N \) is an arbitrary natural number. The large number of free parameters should prove itself useful for constructing a theoretical model that agrees satisfactorily with the results of astronomical observations. In particular, \( N = 3 \) solutions in the general case already exhibit inflationary regimes [10]. Unlike the previously studied two-parameter solutions (see [17], [18]), these three-parameter solutions might describe an exit from inflation without any fine tuning of parameters as well as the several consecutive inflationary regimes.

In the next Section we will show that the method of linearization is indeed an effective one for construction of a "brane-like" singularity.
III. "BRANE-LIKE" SINGULARITY

Assume
\[ U(t) = \frac{\kappa u_s^2}{(t - t_s)^\alpha}, \tag{6} \]
with \( u_s^2 = \text{const} > 0, \kappa = \pm 1, \alpha > 0. \) For simplicity one has omitted the index \( n \): \( U_n \to U(t), \psi_n \to \psi. \) For \(|t - t_s| \gg 1\) the potential \( U(t) \to 0 \) so \( \psi(t) \sim t \) and \( a(t) \sim t^{1/n} \). If \( n = 2 \) we have a universe filled with radiation (\( u = 1/3 \)), and for \( n = 3/2 \) we have a dust universe with \( u = 0 \).

Now let us consider the solution of the \( U \) at \( t \to t_s \):
\[ \psi(t) = \psi_s + \sum_{k=1}^{\infty} c_k (t - t_s)^{(2-\alpha)} \tag{7} \]
where
\[ \psi_s = a_s^{1/n} = \frac{(2 - \alpha)(\alpha - 1)}{\kappa u_s^2}, \tag{8} \]
and
\[ c_k = \frac{\kappa^{k-1} u_s^{2(k-1)}}{(2 - \alpha)(2 - \alpha - 1)^k k! \prod_{m=0}^{k-2} ((k - m)(2 - \alpha) - 1)}, \tag{9} \]
with \( c_1 = 1. \) For \( \alpha < 2 \) the series \( \psi \) is convergent for any \( t \) (including \( t = t_s \)) since its radius is
\[ R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right| = +\infty. \]
If \( \alpha > 2 \) then the first two terms of \( \psi \) are
\[ \psi = \psi_s + \frac{1}{(t - t_s)^{\alpha - 2}}, \]
so the function \( \psi(t) \) will be singular at \( t \to t_s \) and we have either Big Rip (\( n > 0 \)) or Big Crunch (\( n < 0 \)) at \( t = t_s. \) If \( \alpha = 2 \) the general solution of the \( \psi \) will be of the form
\[ \psi = \psi_s + \frac{1}{(t - t_s)^{\alpha - 2}}, \tag{10} \]
with the arbitrary constants \( C_{1,2} \) and \( L = \sqrt{1 + 4\kappa u_s^2}/2. \) For any \( C_1 \) and \( C_2 \) the solution \( \psi \) results in Big Rip or Big Crunch singularity as well. Finally, if \( \alpha = 1 \) then the series \( \psi \) will be
\[ \psi = (t_s - t)^2 \left[ 1 + \frac{\kappa u_s^2}{2} (t_s - t) + \frac{\alpha^4}{12} (t_s - t)^2 + \frac{\kappa u_s^6}{144} (t_s - t)^3 + \frac{\alpha^8}{2880} (t_s - t)^4 + \ldots \right], \]
so there is no cosmological singularity and \( a \) is finite.

Thus, in framework of our investigation one must consider only \( 0 < \alpha < 1 \) and \( 1 < \alpha < 2. \) Keeping only the two first terms in \( \psi \) and using
\[ H = \frac{\dot{\psi}}{n\psi}, \quad \frac{\ddot{a}}{a} = \frac{\ddot{\psi}}{n\psi} - (n - 1)H^2, \tag{11} \]
one gets
\[ \psi \sim \frac{(2 - \alpha)(\alpha - 1)}{\kappa u_s^2} - (t_s - t)^{2-\alpha}, \]
\[ H = \frac{\kappa u_s^2}{n(\alpha - 1)} (t_s - t)^{1-\alpha}, \tag{12} \]
\[ \frac{\ddot{a}}{a} = \frac{u_s^2}{n(t_s - t)^\alpha} \left[ \kappa - u_s^2(n - 1)(n(\alpha - 1)^2 (t_s - t)^{2-\alpha} \right]. \]
Using \( \psi \) one can show that the conditions \( \psi_s > 0 \) and \( H > 0 \) will hold if \( n > 0. \) So one has two cases:
(i) \( \alpha = -1 \) then \( 0 < \alpha < 1 \) and one gets BBS;
(ii) \( \alpha = +1 \) then \( 1 < \alpha < 2 \) and one gets BFS.
To obtain SFS one should use another solution of \( \psi \):
\[ \psi = \psi_s - n\psi_s H_s(t_s - t) + \frac{\kappa u_s^2\psi_s}{(2 - \alpha)(1 - \alpha)} (t_s - t)^{2-\alpha} + \sum_{k=1}^{\infty} c_{2k} (t_s - t)^{(2-\alpha)(k+1)} + \sum_{k=0}^{\infty} c_{2k+1} (t_s - t)^{(2-\alpha)(k+1)+1}, \tag{13} \]
where \( H_s = \text{const} > 0, \psi_s = a_s^{1/n} = \text{const} > 0 \) and
\[ c_{2k} = \frac{(\kappa u_s^2)^{k+1} \psi_s}{(2 - \alpha)(k+1)! \prod_{m=0}^{k+1} [m(2 - \alpha) - 1]}, \]
\[ c_{2k+1} = -\frac{(\kappa u_s^2)^{k+1} nH_s \psi_s}{(2 - \alpha)(k+1)! \prod_{m=1}^{k+1} [m(2 - \alpha) + 1]}. \]
Using \( \psi \) we get
\[ H(t_s) = H_s, \quad \left( \frac{\ddot{a}}{a} \right)_{t \to t_s} = \frac{\kappa u_s^2}{n(t_s - t)^\alpha} - (n - 1)H_s^2, \]
so the solution \( \psi \) contains the SFS at \( t \to t_s \) for \( \alpha < 1. \)
At last, let’s consider the \( m \). After differentiation one gets
\[ \frac{d^m\psi}{dt^m} = (-1)^m \sum_{k=1}^{\infty} c_k \prod_{l=0}^{m-1} [k(2 - \alpha) - l] (t_s - t)^{k(2-\alpha)-m}, \tag{14} \]
so we have a singularity in \( \psi \) for
\[ k < \frac{m}{2 - \alpha}. \tag{15} \]
Since \( 0 < \alpha < 1 \) then
\[ \frac{1}{2} < \frac{1}{2 - \alpha} < 1. \]

Therefore for \( m > 1 \) the expression \( \psi \) diverge as the cosmological singularity is approached. Thus we have obtained the solution which contains a kind of generalization of the Big Break singularity (the scale factor
remains finite and the Hubble parameter vanishes as singularity is approached). In the case of positive curvature one has to choose \( n = 1 \) (cf. Remark 5) and we have a "brane-like" singularity (the density is finite and positive whereas all higher derivatives of the scale factor, starting out from the second one, diverge as the cosmological singularity is approached). In the next Section we'll present a couple of models with the self-acting and minimally coupled scalar fields or with the homogeneous tachyon fields \( T = T(t) \) described by Sens or Born-Infeld type Lagrangians which result in such behavior.

The similar investigation can be done for the (13). This singularity is characterized by the fact that Hubble parameter remains finite instead of vanishing as the cosmological singularity is approached. This solution describes the appearance of a "brane-like" singularity in the flat universe.

### IV. FIELD MODELS

If the universe is filled with a self-acting and minimally coupled scalar field with Lagrangian [5] then the energy density and pressure are

\[
\rho = K + V, \quad p = K - V,
\]

therefore

\[
V = \frac{1}{2}(\rho - p), \quad K = \frac{1}{2}(\rho + p). \tag{16}
\]

Using (3) one can write

\[
K = \frac{\psi^2 - \psi^2}{3n\psi^2} = \frac{(w + 1)\psi^2}{2n^2\psi^2}, \tag{17}
\]

\[
V = \frac{n\psi^2 + (3 - n)\psi^2}{3n^2\psi^2} = \frac{(1 - w)\psi^2}{2n^2\psi^2}, \tag{18}
\]

where

\[
w = \frac{p}{\rho} = -1 + \frac{2n}{3} \left(1 - \frac{\psi^2}{\psi^2}\right). \tag{19}
\]

The second model is the universe filled with a homogeneous tachyon field \( T = T(t) \) described by the Sens Lagrangian density [20]:

\[
L = -V(T)\sqrt{1 - g_{00}T^2}, \tag{20}
\]

and in a flat Friedmann universe with metric \( ds^2 = dt^2 - a^2(t)dr^2 \) we have density and pressure

\[
\rho = \frac{V(T)}{\sqrt{1 - T^2}}, \quad p = L, \tag{21}
\]

so

\[
\dot{T}^2 = 1 + w, \quad V = \frac{\psi^2}{n^2\psi^2}\sqrt{-w}, \tag{22}
\]

where \( w = p/\rho \) is defined by (19).

The expression for \( V \) (22) holds iff \( w < 0 \). If \( w > 0 \) one should introduce a new field theory based on a Born-Infeld type action with Lagrangian [7]

\[
L = W(T)\sqrt{T^2 - 1}, \tag{23}
\]

so

\[
\rho = \frac{W(T)}{\sqrt{T^2 - 1}}, \quad p = L,
\]

and

\[
\dot{T}^2 = 1 + w, \quad W = \frac{\psi^2}{n^2\psi^2}\sqrt{-w}. \tag{24}
\]

It is interesting that models (20) and (23) can be connected via the so-called "transgression of the boundaries" [8].

Now, using the potential (20) and the solution (7) we get

\[
w = -\frac{2n(\alpha - 1)^2}{3\kappa u_s^2(t_s - t)^{2 - \alpha}}, \tag{25}
\]

Therefore, for the case \( \kappa = +1 \) we have \( w < 0 \) (BFS) and \( w \to -\infty \) as the cosmological singularity is approached. Using (17) and (22) one concludes that this will be the case for the phantom (both usual \( \phi \) and tachyon \( T \) fields only - the case which we leave out of consideration in this paper.

For the case \( \kappa = -1 \) and \( 0 < \alpha < 1 \) one has a Big Break singularity with \( w \to +\infty \). For the model (5) we get

\[
\phi^2 = \frac{2u_s^2}{3n(t_s - t)^\alpha} > 0,
\]

and

\[
p \to \frac{2u_s^2}{3n(t_s - t)^\alpha} \to +\infty, \quad H \to 0,
\]

as the cosmological singularity is approached. So all energy conditions are satisfied and \( H_s = 0 \) as it should be for a BBS. If \( k = +1 \) we have a "brane-like" singularity with

\[
\rho_s = \frac{u_s^4}{(2 - \alpha)^2(1 - \alpha)^2}. \tag{26}
\]

At \( t \to t_s \) the potential \( V = V(\phi) \) and field \( \phi = \phi(t) \) are given by expressions

\[
V(\phi) = -\frac{u_s^2}{3n} \left(\frac{3n(2 - \alpha)}{8u_s}\right)^{-\alpha/(2 - \alpha)} (\phi - \phi_s)^{-2\alpha/(2 - \alpha)}, \tag{27}
\]

\[
\phi = \phi_s + \frac{2u_s}{2 - \alpha} \sqrt{\frac{2}{3n}} (t_s - t)^{1-\alpha/2},
\]

and
and \( V(\phi) \to -\infty, \phi \to \phi_* \) at \( t \to t_* \).

For the solution (13) we have the same potential (27) and

\[
\phi = \frac{2\kappa u_s^2}{3nH_s^2(t_s - t)}
\]

instead of (25).

In the case of tachyon cosmology one should use the model (23). It easy to see that \( T^2 > 0 \) and

\[
T(t) = T_s + 2\sqrt{\frac{2n_1 - \alpha}{3\alpha u_s}}(t_s - t)^{n/2} \to T_s,
\]

\[
W(\Phi) = \sqrt{\frac{2}{3n}} \Phi^{2/\alpha - 3} \to 0,
\]

with

\[
\Phi = \frac{1}{2} \sqrt{\frac{3}{2n_1 - \alpha}} (T - T_s).
\]

Since \( 0 < \alpha < 1 \) then \( 2/\alpha - 3 > -1 \). For example, if \( \alpha = 2/5 \) then

\[
W(T) = \frac{5\sqrt{6}u_s^2}{12500n^{5/2}} (T - T_s)^2,
\]

and for the \( \alpha = 2/7 \)

\[
W(T) = \frac{21\sqrt{6}u_s^2}{12500n^{7/2}} (T - T_s)^4.
\]

Finally, let us consider the equation of state which results in the potential (27). It was shown in [7] that the equation of state

\[
\rho + p = \gamma \rho^\lambda,
\]

results in dynamics which might be described by the self-acting potential is the form

\[
V(\phi) = Q^{-2/(\lambda - 1)} - \frac{\gamma}{2} Q^{-2\lambda/(\lambda - 1)},
\]

with

\[
Q = \frac{3\sqrt{7}(\lambda - 1)(\phi - \phi_*)}{2},
\]

where \( \gamma > 0, \lambda > 1 \). When \( \phi \to \phi_* \) we have

\[
V(\phi) \to -\frac{\gamma}{2} Q^{-2\lambda/(\lambda - 1)} \to -\infty.
\]

Near the singularity, the correct equation of state for the solution (7) in case of a positive curvature has the form which looks similar to (29):

\[
\rho + 3p = \gamma (\rho - \rho)^{-|\lambda|},
\]

where

\[
|\lambda| = \frac{\alpha}{2(1 - \alpha)},
\]

and \( 0 < \alpha < 1 \).

V. TACHYON POTENTIALS

The method of linearization proves to be extremely useful in finding the potentials of the exact solvable tachyon models. In particular, as we shall see, the tachyon model which was discussed in detail in [7] is one of the simplest models in framework of the method of linearization.

Let’s consider Eq. (3) with potential \( U = 0 \). The solution of this equation \( \psi = Ct + C' \to Ct \) by the translation \( t \to t' = Ct/C \). In this case we get

\[
w = \frac{p}{\rho} = -1 + \frac{2n}{3}, \quad p = \frac{2n - 3}{3n^2t^2}, \quad T^2 = w + 1,
\]

so

\[
T = \pm \sqrt{\frac{2n}{3} (t - t_s) + T_s},
\]

and

\[
V(T) = \frac{2\sqrt{9 - 6n}}{n [2(T - T_s) \pm \sqrt{6nt_s}]}.
\]

If \( n = 3(1 + k)/2 \) with \(-1 < k < +1 \) (so \( 0 < n < 3 \)) and \( t_s = 0 \) then (32) has exactly the form of one of the potentials from the first paper [7] (with \( T_0 \to T_s \)).

The case with \( U = \mu^2 = \text{const} > 0 \) is a more interesting example. The solution of the (3) with Big Bang singularity at \( t = 0 \) is \( \psi = C \sinh(\mu t) \). So

\[
w = -1 + \frac{2n}{3 \cosh^2 \mu t}, \quad p = \frac{\mu^2(2n - 3 \cosh^2 \mu t)}{2n^2 \sinh^2 \mu t}.
\]

Using (23) one gets

\[
T(t) = \pm \sqrt{\frac{2n}{3\mu^2} \arctan(\sinh \mu t) + T_0},
\]

and

\[
V(T) = \frac{\mu^2 \cosh \mu t}{3n^2 \sinh^2 \mu t} \sqrt{9 \cosh^2 \mu t - 6n}.
\]

Introducing \( \Lambda = \mu^2/n^2, k = 2n/3 - 1 \) we get

\[
V(T) = \frac{\Lambda}{\sin^2 \frac{\xi}{\Lambda}} \sqrt{1 - (1 + k) \cos^2 \frac{\xi}{\Lambda}},
\]
where

\[ \xi = \frac{3}{2} \sqrt{A(1 + k)T}. \]

(33) is the basic tachyon model of the paper [7].

It is not difficult to construct many other integrable tachyon models using the simple, solvable potentials of (33). Another fruitful way of doing it lies in a use of the Darboux transformation to those initial potentials \( U = 0, U = \mu^2 \). This, however, is out of scope of the paper.

VI. GENERALIZED TYPE IV SINGULARITIES

The singularities of type IV (according to the classification of Ref. [12]) have the following behavior: for \( t \to t_s, a \to a_s, \rho \to \rho_s, \rho \to p_s \) and the higher derivatives of \( H \) diverge \((0 < a_s < \infty)\). In this section we present a new type of singularities:

**Generalized type IV:** For \( t \to t_s, a \to a_s, \rho \to \rho_s, \rho \to p_s \) and higher derivatives (starting out from the third one) of \( H \) diverge and \( 0 < a_s < \infty, 0 < \rho_s < \infty, 0 < |p_s| < \infty \).

Let’s put \( n = 3, \psi = a^3 \) and (in parametric form)

\[ \psi = A + \frac{\kappa B}{4} (4 \cos \eta - 2 \cos^2 \eta - \cos^4 \eta - 4 \log(1 + \cos \eta)), \]

\[ t = t_s + \frac{1}{\kappa} \left( \log \left| \tan \frac{\eta}{2} \right| + \cos \eta \right), \tag{34} \]

where \( A, B, \kappa \) are constants, \( 0 \leq \eta \leq \pi; \eta = 0 \) corresponds to \( t = -\infty, \eta = \pi/2 \) to \( t = t_s \) (singularity) and \( \eta = \pi \) to \( t = +\infty \). After the differentiation we get (a dot denotes the derivative with respect to cosmic time \( t \) rather than to parameter \( \eta \))

\[ \dot{\psi} = \kappa^2 B \left( 1 - \cos^3 \eta \right), \]

\[ \ddot{\psi} = 3\kappa^3 B \sin^2 \eta, \]

\[ \cdots \quad \psi = 6B\kappa^4 \sin^2 \eta, \]

\[ \cdots \quad \psi = \frac{6\kappa^5 B \sin^2 \eta(\cos^2 \eta + 1)}{\cos \eta}, \]

and so on.

At \( t = t_s \)

\[ \psi = A, \quad \dot{\psi} = \kappa^2 B, \quad \ddot{\psi} = 3\kappa^3 B \]

and, starting out from the third one, all higher derivatives diverge. At \( t = +\infty \)

\[ \psi = \text{sign}(\kappa B) \times \infty, \quad \dot{\psi} = 2\kappa^2 B, \quad \ddot{\psi} = 0. \]

Therefore at \( t = t_s \)

\[ \rho_s = \frac{\kappa^4 B^2}{9A^2}, \quad p_s = \frac{\kappa^4 (B - 2A)}{9A^2}, \quad w_s = 1 - \frac{6A}{\kappa B}. \]

Thus we have a generalization of a type IV singularity at \( t_s \), where the density and pressure are finite and nonzero whereas all higher derivatives of \( H \) diverge.

It is convenient to introduce a new parameter \( s \):

\[ s = \frac{6A}{\kappa B}; \]

so

\[ \rho_s = \frac{4\kappa^2}{s^2}; \quad p_s = \frac{4\kappa^2 (1 - s)}{s^2}; \quad w_s = 1 - s. \]

If \( 4/3 < s < 2 \) then \(-1 < w_s < -1/3\); if \( s = 2 \) then \( w_s = -1\); if \( s > 2 \) then \( w_s < -1 \). To obtain the initial Big Bang singularity at \( t = t_i \), \( -\infty < t_i < t_s \) one should put

\[ s < s_i = 6 \log 2 - \frac{3}{2} \sim 2.659, \]

or

\[ w_s > w_i = -1.659. \]

In the initial Big Bang singularity, the barotropic index \( w = +\infty \), on the other hand, \( w(\eta) \) is the monotonously decreasing function. These properties result in a following conclusion: if

\[ \frac{4}{3} < s < s_i, \]

then after Big Bang the model (34) will go through the usual expansion with damping, but starting out from some moment it will experience an accelerated expansion up to a future generalized type IV singularity.

VII. CONCLUSION

In this paper we have discussed a simple method of construction of exact solutions of the Friedmann equations with finite scale factor singularities. Despite simplicity, the method allows for acquirement of solutions characterized by the extremely interesting properties.

The main results of this work are:

(i) we have obtained a new type of finite-time, future singularities which seem to be most similar to the type IV
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[1] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [arXiv:astro-ph/9805201]; S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [arXiv:astro-ph/9812133].

[2] A.A. Starobinsky, Grav. Cosmol. 6, 157 (2000) [arXiv:astro-ph/9912054]; R.R. Caldwell, Phys. Lett. B545 (2002) 23 [arXiv:astro-ph/9908168]; M. Kamionkowski and N.N. Weinberg, Phys. Rev. Lett. 91 (2003) 071301 [arXiv:astro-ph/0302506]; P.F. Gonzalez-Diaz, Phys. Lett. B586 (2004) 1 [arXiv:astro-ph/0312579]; P.F. Gonzalez-Diaz, Phys.Rev.Lett. 93 (2004) 071301 [arXiv:astro-ph/0404045]; P.F. Gonzalez-Diaz, Phys.Rev. D69 (2004) 103512 [arXiv:astro-ph/0311244]; S.M. Carroll, M. Hoffman and M. Trodden, Phys. Rev. D68 (2003) 023500 [arXiv:astro-ph/0301273]; S. Nojiri and S.D. Odintsov, Phys. Rev. D70 (2004) 103522 [arXiv:hep-th/0408170]; A.V. Yurov, P.M. Moruno, P.F. Gonzalez-Diaz, Nucl.Phys. B759 (2006) 320 [arXiv:astro-ph/0606529].

[3] M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, arXiv:gr-qc/0612135; M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, arXiv:0707.2390 [gr-qc]; A. V. Yurov, A. V. Astashenok and P.F. Gonzalez-Diaz, Grav. Cosmol. 14 No.3, 205 (2008), arXiv:0705.4108 [astro-ph]; F. Cannata, A. Yu. Kamenshchik and D. Regoli, Phys. Lett. B670 (2009) 241, arXiv:0801.2348 [gr-qc].

[4] Y. Shtanov and V. Sahni, Class.Quant.Grav. 19 (2002) L101 [arXiv:gr-qc/0204040].

[5] J.D. Barrow, Class. Quant. Grav. 21 (2004) L79 [arXiv:gr-qc/0403084]; J.D. Barrow, Class. Quant. Grav. 21 (2004) 5619 [arXiv:gr-qc/0409062]; J.D. Barrow, Class. Quant. Grav. 22 (2005) 1563 [arXiv:gr-qc/0411045]; M.P. Dabrowski, Phys.Rev. D71 (2005) 103505 [arXiv:gr-qc/0410033]; M.P. Dabrowski, Phys.Lett. B625 (2005) 184 [arXiv:gr-qc/0505069]; J.D. Barrow and S.Z. Lip, arXiv:0901.1026 [gr-qc].

[6] A.O. Barvinsky, C. Deffayet and A. Y. Kamenshchik, JCAP 0805 (2008) 020, arXiv:0801.2063 [hep-th].

[7] V. Corini, A. Yu. Kamenshchik, U. Moschella and V. Pasquier, Phys. Rev. D69 (2004) 123512 [arXiv:hep-th/0311111]; Z. Keresztes, L. A. Gergely, V. Gorini, U. Moschella, A. Yu. Kamenshchik, Phys. Rev. D70 (2009) 083504, arXiv:0901.2292 [gr-qc].

[8] A. Kamenshchik, C. Kiefer and B. Sandhofer, Phys.Rev. D76 (2007) 064032, arXiv:0705.1688 [gr-qc].

[9] S. Nojiri and S.D. Odintsov, arXiv:0903.5231 [hep-th]; S. Nojiri and S.D. Odintsov, Phys.Rev. D72 (2005) 023003 [hep-th/0505215].

[10] S. Nojiri, S. D. Odintsov Phys.Rev.D78 (2008) 046006, arXiv:0804.3519 [hep-th]; K. Bamba, S. Nojiri and S D Odintsov, JCAP0810:045 (2008), arXiv:0807.2575 [hep-th].

[11] S. Capozziello, M. De Laurentis, S. Nojiri, S.D. Odintsov, arXiv:0903.2753 [hep-th].

[12] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D71 (2005) 063004 [arXiv:hep-th/0501025].

[13] M.P. Dabrowski and T. Denkiewicz, Phys. Rev. D76 (2002) 063521, arXiv:0705.1688 [gr-qc].

[14] M.P. Dabrowski, T. Denkiewicz and M.A. Hendry, Phys.Rev. D75 (2007) 123524, arXiv:0704.1383 [astro-ph].

[15] S. Nojiri and S D Odintsov, Phys.Lett. B595 (2005) 1 [arXiv:hep-th/0505215].

[16] S. Nojiri and S D Odintsov, Phys.Lett. B595 (2005) 1 [arXiv:hep-th/0505215].
[18] A.V. Yurov, [astro-ph/0305019]; A.V. Yurov and S.D. Vereshchagin, Theor. Math. Phys. 139 (2004) 787 [arXiv:hep-th/0502099].
[19] A. V. Yurov and A. V. Astashenok, Theor. Math. Phys. 158 (2009) 261, [arXiv:0902.1979] [astro-ph.CO].
[20] A. Sen, JHEP 0204 (2002) 048 [arXiv:hep-th/0203211]; A. Sen, JHEP 0207 (2002) 065 [arXiv:hep-th/0203265].

[21] At the same time, one shall keep in mind that this equation is nothing but approximate. To ensure (with the help of $\psi_n$) that the weak energy condition will indeed be violated, one can use exact equation $\sigma_n = -3n(\rho + p)/2$, where $\sigma_n = \psi_n/\psi_n$. 