Motion of Spinning and Spinning Deviation in Riemannian Geometry

Magd E. Kahil¹,²*, Samah A. Ammar²**, and Shymaa A. Refaey²***

¹Faculty of Engineering, Modern Sciences and Arts University, Giza, Egypt
²Women’s College for Arts, Science and Education, Ain Shams University, Cairo, Egypt

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Abstract—Equations of motion of spinning density for extended objects and the corresponding deviation equations are derived. The problem of motion for a variable mass of a spinning extended object is obtained. Spinning fluids may be considered as a special case to express the motion of spinning density for extended objects. Meanwhile, the spinning density tensor can be expressed in terms of the tetrad formalism of general relativity to be regarded as a gauge theory of gravity. The equations of spinning and spinning deviation density tensors have been derived using a specific type of Bazanski Lagrangian.

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1. INTRODUCTION

Spinning motion is regarded as one of the actual features of the characteristic behavior for objects in nature, which has led many authors to focus on the cause of the spinning process. Would it be eligible to include its internal properties or discard them as a step of simplification? From this perspective, it is vital to begin with the Mathisson–Papapetrou equations of spinning motion [1]

\[ DP^\alpha \frac{Ds}{Ds} = \frac{1}{2} R^\alpha_{\beta\gamma\delta} S^{\gamma\delta} U^\beta, \tag{1} \]

where \( P^\alpha \) is the momentum of a particle,

\[ P^\alpha = \left( mU^\alpha + U_\mu \frac{DS^{\alpha\mu}}{Ds} \right), \]

\( R^\alpha_{\beta\gamma\delta} \) is the Riemannian curvature, \( U^\alpha = \frac{dx^\alpha}{ds} \) is the unit tangent vector, \( s \) is a parameter varying along the curve, and \( S^{\gamma\delta} \) is the spin tensor. A spinning object with precession (Gyroscopic motion) can be described using the following equation:

\[ DS^{\mu\nu} \frac{Ds}{Ds} = P^\mu U^\nu - P^\nu U^\mu, \tag{2} \]

If \( P^\alpha = mU^\alpha \), then Eqs. (1) and (2) become the Papapetrou equations (for short)

\[ DU^\alpha \frac{Ds}{Ds} = \frac{1}{2m} R^\alpha_{\beta\gamma\delta} S^{\gamma\delta} U^\beta. \tag{3} \]

And

\[ DS^{\mu\nu} \frac{Ds}{Ds} = 0, \tag{4} \]

provided that [2]

\[ S^{\mu\nu} = \bar{s}(U^\mu \Psi^\nu - U^\nu \Psi^\mu), \tag{5} \]

where \( \bar{s} \) is the spin magnitude, \( U^\alpha \) is the 4-vector velocity, and \( \Psi^\alpha \) is the geodesic deviation vector. Equation (3) can be obtained from the geodesic equations [3]

\[ DU^\alpha \frac{Ds}{Ds} = 0 \tag{6} \]

and the geodesic deviation equations

\[ D^2 \Psi^\alpha \frac{Ds}{Ds^2} = R^\alpha_{\mu\nu\rho} U^\mu U^\nu \Psi^\rho. \tag{7} \]

If we apply the following transformation of paths for different parameters [2],

\[ \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{d\tau} + \beta \frac{D\Psi^\alpha}{D\tau}, \tag{8} \]

where \( \tau \) is a parameter describing another trajectory, and \( \beta \) is an arbitrary parameter.

Thus, by using a covariant derivative with respect to the parameter \( s \) on both sides and taking into consideration that \( D\tau/Ds = 1 \), we get

\[ DU^\alpha \frac{Ds}{Ds} = DU^\alpha \frac{D\tau}{D\tau} + \beta \frac{D^2 \Psi^\alpha}{D\tau^2}. \tag{9} \]

Using the geodesic equations (6) and the geodesic deviation equations (7) as well as Eq. (3), one can
obtain Eq. (1), while if one considers the Frenkel condition
\[ S_{\mu\nu} U^\nu = 0, \]
covariantly differentiated on both sides, after some manipulations one can get Eq. (4).

Thus, due to the extension of the spin tensor from pole-dipole moments to multipole moments for extended objects, this may lead to examining its corresponding propagation equation [4]. Such an equation may be obtained by means of introducing the spin tensor density \( S^{\alpha\beta\gamma} \) as a third-order skew-symmetric tensor viable to describe extended objects. These equations play a vital role in astrophysics and early cosmology, becoming a good candidate for describing a spinning fluid and also for describing the status of an accretion disc orbiting a compact gravitational field as in AGN [5]. Also, it contributes to understanding the problem of motion of quark–gluon heavy ion collisions in the early Universe [6].

In our present work, we are going to derive the equations of spinning density tensor and spin deviation density tensor in different cases as described in Riemannian geometry in GR. Accordingly, the spin density tensor is related to the thermodynamic variables [7],
\[ T d\dot{s} = dE + \rho d\left(\frac{1}{\rho}\right) - \frac{1}{2} \omega_{\mu\nu} ds^{\mu\nu}, \]
where \( T \) is the temperature, \( \dot{s} \) is the entropy, \( E \) is the energy density, \( \omega_{\mu\nu} \) is the spin angular velocity, and \( s^{\mu\nu} \) the spin density. Thus, in self-consistent theories described, the entropy become conserved, such that
\[ \frac{d\dot{s}}{ds} \approx 1. \] (11)
The first law of thermodynamics becomes
\[ \frac{dE}{ds} + \frac{d\rho}{ds} - \frac{1}{2} \omega_{\mu\nu} \frac{ds^{\mu\nu}}{ds} = 1. \] (12)
Meanwhile, it is worth mentioning that in the presence of strong spinning, a viscosity should be included to define the spin density tensor [8]. It is well known that spinning fluids are dominating properties of nature, which may be found to describe the problem of motion of particles in an accretion disc as a gyrodynamics fluid. This is a counterpart of the Papapetrou equation [9]. Owing to the spin density tensor, an interaction between a spinning motion and thermodynamic variables may be found in Eq. (12) [10].

Thus, it is well known that \( S^{\alpha\beta\gamma} \) is a third-order tensor, viable to define extended objects. Nevertheless, from a progenitor case, the spin density tensor is bounded to be skew-symmetric in the last two indices, it comes to arise for expressing a spinning fluid element as a confined case of an extended object. Yet, one may find out that the Weyssenhoff tensor [6] is the most eligible candidate for expressing a spin fluid element, i.e.,
\[ S^{\rho\mu\nu} = S^{\mu\nu} U^\rho. \] (13)
Differentiating both sides of (13) with a covariant derivative, we get
\[ \frac{DS^{\rho\mu\nu}}{Ds} = \frac{DS^{\mu\nu}}{Ds} U^\rho + \frac{DU^\rho}{Ds} S^{\mu\nu}. \] (14)
Now, one uses the following Lagrangian [11]:
\[ L = g_{\mu\nu} U^\mu \frac{D\Psi^\nu}{Ds} + S_{\mu\nu} \frac{D\Psi^{\mu\nu}}{Ds}. \] (15)
By taking variation with respect to the vector \( \Psi^\rho \), one gets (6),
\[ \frac{DU^\rho}{Ds} = 0. \]
Also, taking variation with respect to the spinning deviation tensor \( \Psi^{\rho\delta} \), we get (4),
\[ \frac{DS^{\rho\delta}}{Ds} = 0. \] (16)
Consequently, substituting from (6) and (16) into (14), one obtain
\[ \frac{DS^{\rho\delta\lambda}}{Ds} = 0, \] (17)
which is an equation for the spin density tensor. Accordingly, we may obtain equations analogously by means of its corresponding Bazanski Lagrangian stemmed from its original formalism [3] and its modification in GR [12], to become
\[ L = S_{\alpha\mu\nu} \frac{D\Psi^{\alpha\mu\nu}}{Ds}, \] (18)
such that by taking variation with respect to \( \Psi^{\rho\delta\lambda} \),
\[ \frac{DS^{\rho\delta\lambda}}{Ds} = 0. \] (19)
If we apply the commutation relation such that
\[ (S^{\rho\mu\nu}_{\beta\alpha} - S^{\rho\mu\nu}_{\beta\alpha}) U^\alpha \Psi^\beta = S^{\sigma\mu\nu} R^{\beta}_{\sigma\alpha\gamma} U^\alpha \Psi^\beta \] (20)
and
\[ S^{\rho\mu\nu}_{\beta\delta} \Psi^\delta = \Psi^{\rho\mu\nu}_{\beta\delta} U^\delta, \] (21)
we obtain its corresponding spin density deviation tensor equation
\[ \frac{D^2\Psi^{\rho\delta\lambda}}{Ds^2} = S^{\sigma\delta\lambda} R^{\rho}_{\sigma\alpha\beta} U^\alpha \Psi^\beta. \] (22)
The importance of the spin density deviation tensor is to examine the stability conditions of an accretion...
Thus, by operating the Euler-Lagrange equations, we suggest the equivalent Bazanski Lagrangian to be

\[ D^2 \hat{\Psi}^{\rho \mu \nu} = \hat{S}^{\delta [\mu \nu} R^{\rho \delta}_{\alpha \beta} U^\alpha \Psi^\beta + f^{\rho \mu \nu \delta} \hat{\Psi}^\delta. \]

Consequently, its corresponding spin density deviation tensor equation can be obtained by applying in a similar way the commutation relations as given in (21) and (22) as follows:

\[ D^2 \hat{S}^{\rho \mu \nu} = \hat{S}^{\delta [\mu \nu} R^{\rho \delta}_{\alpha \beta} U^\alpha \Psi^\beta + f^{\rho \mu \nu \delta} \hat{\Psi}^\delta. \]

2. SPINNING DENSITY TENSOR AND SPINNING DENSITY DEVIATION TENSOR EQUATIONS: PAPAPETROU-LIKE EQUATIONS

In this section, we are going to examine a massive density spin tensor able to describe an orbiting extended object for a compact object. Accordingly, the Weyssenhoff spin vector may be amended to be expressed as follows:

\[ \hat{S}^{\rho \mu \nu} = S^{\mu \nu} P^\rho, \] (23)

where \( P^\rho \) is the momentum in which it is relating to \( \hat{S}^{\rho \mu \nu} \) in the following sense:

\[ \hat{S}^{\rho \mu \nu} = S^{\mu \nu} (m U^\rho + U_\delta D S^{\rho \delta}_{\bar{S}}). \] (24)

Covariantly differentiating both sides for (23), we get

\[ \hat{D} \hat{S}^{\rho \mu \nu} = \hat{D} S^{\mu \nu} P^\rho + \hat{D} P^\rho S^{\mu \nu}. \] (26)

We suggest the equivalent Bazanski Lagrangian to be

\[ L = \hat{S}^{\rho \mu \nu} \frac{D \hat{\Psi}^{\rho \mu \nu}}{D s} + f^{\rho \mu \nu} \hat{\Psi}^{\rho \mu \nu}. \]

Thus, by operating the Euler-Lagrange equations with respect to the deviation tensor \( \hat{\Psi}^{\rho \mu \nu} \), which have the form

\[ \frac{d}{d s} \frac{\partial L}{\partial \hat{\Psi}^{\rho \mu \nu}} - \frac{\partial L}{\partial \hat{\Psi}^{\rho \mu \nu}} = 0, \] (28)

we get

\[ \frac{D \hat{S}^{\rho \mu \nu}}{D s} = f^{\rho \mu \nu}, \] (29)

i.e.,

\[ f^{\rho \mu \nu} = \frac{D S^{\mu \nu}}{D s} P^\rho + \frac{D P^\rho}{D s} S^{\mu \nu}. \]

3. SPINNING DENSITY TENSOR AND SPINNING DENSITY DEVIATION TENSOR EQUATIONS: A VARIABLE MASS

Consider a massive spin density tensor whose mass is not constant but is a function of the parameter \( s \), so that its corresponding Weyssenhoff tensor becomes

\[ \hat{S}^{\rho \mu \nu} = m(s) U^\rho S^{\mu \nu}. \] (31)

Differentiating both sides, we obtain:

\[ \frac{D \hat{S}^{\rho \mu \nu}}{D s} = \frac{D (m(s) U^\rho)}{D s} S^{\mu \nu} + m(s) U^\rho \frac{D S^{\mu \nu}}{D s}. \]

Thus we can suggest a Lagrangian obtained for a spinning variable mass object:

\[ L = m(s) g_{\mu \nu} U^\rho \frac{D \hat{\Psi}^{\rho \mu \nu}}{D s} + (m(s) U^\rho \frac{D P^\rho}{D s} S^{\mu \nu} + \frac{1}{2} R^{\rho \alpha \beta \gamma} S^{\alpha \beta \gamma} \hat{\Psi}^{\rho \mu \nu} + \frac{1}{2} R^{\rho \alpha \beta} S^{\alpha \beta} \hat{\Psi}^{\rho \mu \nu}). \] (33)

where \( m(s) \) is the mass function [14]. Applying variation,

\[ \frac{d}{d s} \frac{\partial L}{\partial \hat{\Psi}^{\rho \mu \nu}} - \frac{\partial L}{\partial \hat{\Psi}^{\rho \mu \nu}} = 0, \] (34)

to the above Lagrangian, we get

\[ \frac{D U^\delta}{D s} = m(s) g_{\mu \nu} (g^{\delta \rho} - U^\delta U^\rho) \]

\[ + \frac{1}{2 m(s)} R^{\delta \rho \alpha \beta} S^{\rho \alpha \beta}. \] (35)

Also, in a similar way, variation with respect to \( \hat{\Psi}^{\delta \sigma} \) leads to

\[ \frac{D \hat{S}^{\delta \sigma}}{D s} = 0. \] (36)
Thus the spinning density tensor with a variable mass obeys the equation
\[
\frac{D\hat{S}^{\rho\mu}}{Ds} = \left(\frac{m(s)}{m(s)}(g^{\rho\mu} - U^\sigma U^\rho) \right) + \frac{1}{2m(s)}R^\rho_{\lambda\alpha\beta}S^{\lambda\alpha\beta}S^{\mu\nu}. \tag{37}
\]
Accordingly, its corresponding Bazanski Lagrangian may be expressed as
\[
L = \hat{S}_{\rho\mu\nu} \frac{D\hat{\Psi}^{\rho\mu\nu}}{Ds} + \hat{f}_{\rho\mu\nu}\hat{\Psi}^{\rho\mu\nu}, \tag{38}
\]
where
\[
\hat{f}_{\rho\mu\nu} = \left(\frac{m(s)}{m(s)}(g^{\rho\mu} - U^\sigma U^\rho) \right) + \frac{1}{2m(s)}R^\rho_{\lambda\alpha\beta}S^{\lambda\alpha\beta}S^{\mu\nu}. \tag{39}
\]
Varying with respect to \(\hat{\Psi}^{\rho\mu\nu}\), we obtain
\[
\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \hat{f}_{\rho\mu\nu}. \tag{40}
\]
Following the same procedure as given in Eq. (21), we obtain the deviation spinning density equation that becomes
\[
\frac{D^2\tilde{\Psi}^{\rho\mu\nu}}{Ds^2} = (\hat{f}_{\rho\mu\nu})_{,\delta}\tilde{\Psi}^{\delta} + S^{\xi\mu\nu}R^\rho_{\xi\alpha\beta}U^\alpha\tilde{U}^{\beta}. \tag{41}
\]
This equation may be applied in studying the stability of a variable spinning disk which may work to explain the effect of mass excess in a region orbiting a compact object. Such an illustration may give rise to examining the effect of dark matter in the accretion disk of a compact object.

4. SPINNING DENSITY TENSOR AND SPINNING DENSITY DEVIATION TENSOR EQUATIONS: A SPINNING FLUID

In this section we are going to suggest that the relation between a variable mass and a spinning fluid with
\[
\tilde{S}^{\rho\mu\nu} = (p + \rho)(s)U^\rho S^{\mu\nu}, \tag{42}
\]
where \(m(s) = (p + \rho)\). Accordingly, the pressure is turning to be the only parameter of \(s\), while the density is becoming constant. Owing to this suggestion, the spin fluid behaves like a spinning variable mass, such that
\[
\frac{dm(s)}{dx^\rho} = \frac{dp(s)}{dx^\rho}. \tag{43}
\]
Such an equivalence may give rise to suggesting the following Lagrangian:
\[
L = (p + \rho)(s)g_{\mu\nu}U^\mu \frac{D\tilde{\Psi}^{\nu}}{Ds} D(S^{\rho\mu\nu})_{,\delta} \tilde{\Psi}^{\delta} + S^{\xi\mu\nu}R^\rho_{\xi\alpha\beta}U^\alpha\tilde{U}^{\beta}. \tag{44}
\]
This equation may be written as
\[
\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \hat{f}_{\rho\mu\nu}. \tag{45}
\]
Accordingly, its corresponding Bazanski Lagrangian may be written as
\[
L = \hat{S}_{\rho\mu\nu} \frac{D\hat{\Psi}^{\rho\mu\nu}}{Ds} + \hat{f}_{\rho\mu\nu}\hat{\Psi}^{\rho\mu\nu}, \tag{46}
\]
where \(\hat{\Psi}^{\rho\mu\nu}\) is the spin deviation tensor of \(\hat{S}^{\rho\mu\nu}\), and
\[
\hat{f}_{\rho\mu\nu} = \left(\frac{p,\sigma}{p + \rho}g^{\rho\mu} - U^\rho U^\mu \right) + \frac{1}{2(p + \rho)}R^\rho_{\lambda\alpha\beta}S^{\lambda\alpha\beta}S^{\mu\nu}. \tag{47}
\]
Varying it with respect to \(\hat{\Psi}^{\rho\mu\nu}\), we obtain
\[
\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \hat{f}_{\rho\mu\nu}. \tag{48}
\]
Following the same procedure given by (21) and (22), we get
\[
\frac{D^2\tilde{\Psi}^{\rho\mu\nu}}{Ds^2} = (\hat{f}_{\rho\mu\nu})_{,\delta}\tilde{\Psi}^{\delta} + S^{\xi\mu\nu}R^\rho_{\xi\alpha\beta}U^\alpha\tilde{U}^{\beta}. \tag{49}
\]
4.1. Modified Forms of Spin Density

In this part, we suggest a modified form of the spin density tensor in Riemannian geometry to be:
(a) \(P^\alpha = mU^\alpha\),
\[
S^{\alpha\beta\gamma} = \frac{1}{3!}(S^{\beta\gamma}U^\alpha + S^{\gamma\alpha}U^\beta + S^{\alpha\beta}U^\gamma). \tag{50}
\]
Differentiating covariantly both sides, we get
\[
\frac{DS^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!} \left( \frac{DS^{\beta\gamma}}{Ds}U^\alpha + S^{\beta\gamma}\frac{DU^\alpha}{Ds} + \frac{S^{\gamma\alpha}}{Ds}U^\beta \right). \tag{51}
\]
\( S^{\gamma \alpha} \frac{DU^\beta}{D_s} + \frac{DS^{\alpha \beta}}{D_s} U^\gamma + S^{\alpha \beta} \frac{DU^\gamma}{D_s} \). \quad (51)

(i) For \( \frac{DU^\alpha}{D_s} = 0 \) and \( \frac{DS^{\alpha \beta}}{D_s} = 0 \), substituting in (50), we obtain

\[
\frac{DS^{\alpha \beta \gamma}}{D_s} = 0. \quad (52)
\]

(ii) For \( \frac{DU^\alpha}{D_s} = \frac{1}{2m} R^{\alpha}_{\beta \gamma \delta} S_{\beta \gamma} U^\beta \) and \( \frac{DS^{\alpha \beta}}{D_s} = 0 \), Eq. (50) can be rewritten as

\[
\frac{DS^{\alpha \beta \gamma}}{D_s} = \frac{1}{3!} \left( \frac{1}{2m} R^{\alpha}_{\mu \nu \rho} S^{\mu \nu \rho} U^\mu S^{\beta \gamma} + \frac{1}{2m} R^{\beta}_{\mu \nu \rho} S^{\mu \nu \rho} U^\mu S^{\gamma \alpha} + \frac{1}{2m} R^{\gamma}_{\mu \nu \rho} S^{\mu \nu \rho} U^\mu S^{\alpha \beta} \right), \quad (53)
\]

i.e.,

\[
\frac{DS^{\alpha \beta \gamma}}{D_s} = \frac{1}{3!} \left( \frac{1}{2} R^{\alpha}_{\mu \nu \rho} S^{\beta \gamma} + R^{\beta}_{\mu \nu \rho} S^{\gamma \alpha} + R^{\gamma}_{\mu \nu \rho} S^{\alpha \beta} \right), \quad (54)
\]

which becomes

\[
\frac{DS^{\alpha \beta \gamma}}{D_s} = \frac{1}{3!} \left( R^{\alpha}_{\mu \nu \rho} S^{\beta \gamma} \right) S^{\mu \nu \rho}. \quad (55)
\]

(b) \( P^\alpha \neq m U^\alpha \),

\[
\tilde{S}^{\alpha \beta \gamma} = \frac{1}{3!} \left( S^{\beta \gamma} P^\alpha + S^{\gamma \alpha} P^\beta + S^{\alpha \beta} P^\gamma \right). \quad (56)
\]

Differentiating covariantly both sides, we get

\[
\frac{D \tilde{S}^{\alpha \beta \gamma}}{D_s} = \frac{1}{3!} \left( \frac{DS^{\beta \gamma}}{D_s} P^\alpha + S^{\beta \gamma} \frac{DP^\alpha}{D_s} + \frac{DS^{\alpha \beta}}{D_s} P^\gamma + \frac{DS^{\alpha \gamma}}{D_s} P^\alpha - P^\gamma \right).
\]

However, if \( \frac{DP^\alpha}{D_s} = \frac{1}{2} R^{\alpha}_{\beta \gamma \delta} S^\beta \gamma U^\beta \) and

\[
\frac{DS^{\alpha \beta}}{D_s} = P^\alpha U^\beta - P^\beta U^\alpha, \text{ then we obtain}
\]

\[
\frac{D \tilde{S}^{\alpha \beta \gamma}}{D_s} = \frac{1}{3!} \left( \left( P^\beta U^\gamma - P^\gamma U^\beta \right) P^\alpha + \frac{1}{2} R^{\alpha}_{\mu \nu \rho} S^{\mu \nu \rho} S^{\beta \gamma} \right) + \left( P^\gamma U^\alpha - P^\alpha U^\gamma \right) P^\beta + \frac{1}{2} R^{\beta}_{\mu \nu \rho} S^{\mu \nu \rho} S^{\gamma \alpha} + \left( P^\alpha U^\beta - P^\beta U^\alpha \right) P^\gamma + \frac{1}{2} R^{\gamma}_{\mu \nu \rho} S^{\mu \nu \rho} S^{\alpha \beta}, \quad (57)
\]

in which it can be

\[
\frac{D \tilde{S}^{\alpha \beta \gamma}}{D_s} = \frac{1}{12} S^{\mu \nu \sigma} (S^{\alpha \beta} R^{\gamma}_{\mu \nu \sigma} + S^{\beta \gamma} R^{\alpha}_{\mu \nu \sigma}). \quad (59)
\]

Their corresponding deviation equation is obtained similarly to Eq. (48).

5. SPINNING AND SPINNING DENSITY

DEVIATION EQUATIONS FOR A GAUGE THEORY OF GRAVITY: TETRAD FORMALISM

5.1. A Class of Gauge Theories of Gravity in Riemannian Geometry

The problem of studying the microscopic structure of particles led many relativists to get a paradigm shift towards gauge theories of gravity. From this perspective, the building blocks of this type of theories stands for an analogy between the quantities used in current gauge theories and space-time. It is of interest to revisit primarily GR to be descriptive by gauge theory and their counterpart in non-Riemannian geometry [15, 16]. Yet, Collins et al. (1989) described GR as a gauge theory of gravity subject to the gauge potential vectors \( e^\mu_a \) and \( \omega^{ij}_m \) to represent translational and rotational gauge potentials, respectively. From this perspective, the problem of invariance of any quantities must be a covariant derivative invariant under general coordinate transformations (GCT) and local Lorentz transformation (LLT) that are expressible in terms of a gauge potential of translation and rotation in the following way [17]. The equations of physics will contain derivatives of tensor fields, and it is, therefore, necessary to define the covariant derivatives of tensor fields under the transformations GCT and LLT, one must define two types of connection fields to be associated with each of them. Accordingly, the Christoffel symbol \( \{ \alpha \}_{\mu \nu} \) is referred to GCT while the spin connection \( \omega_{ab} \) is related to LLT.

Accordingly,

\[
D_{\mu} e^m_{\nu} \overset{\text{def}}{=} e^m_{\nu,\mu} - \{ \alpha \}_{\mu \nu} e^m + \omega^{m}_{\mu \nu} e^n \quad (60)
\]

and

\[
D_{\rho} D_{\mu} e^m_{\nu} = D_{\mu} D_{\rho} e^m_{\nu}. \quad (61)
\]

Using this concept, it turns out that GR may be expressed in terms of connecting \( e^\mu_a \), \( \omega^{ij}_m \), and \( \{ \alpha \}_{\mu \nu} \) together:

\[
g_{\mu \nu} \overset{\text{def}}{=} e^a_{\mu \nu} b \eta_{ab}, \quad (62)
\]

\[
\{ \alpha \}_{\mu \nu} \overset{\text{def}}{=} \frac{1}{2} g^{\alpha \sigma}(g_{\nu \sigma, \alpha} + g_{\alpha \nu, \sigma} - g_{\alpha \nu, \sigma}). \quad (63)
\]

And for an arbitrary vector \( A_{\mu} \)

\[
A_{\mu ; cd} - A_{\mu ; dc} = \tilde{R}^{\alpha}_{\mu \rho c} A_{\alpha}. \quad (64)
\]
where $\tilde{R}^a_{\mu\nu\rho\sigma}$, the curvature tensor, may be defined due
to the gauge approach, in terms of the spin
connection $\omega_{\mu}^{\alpha}$:

$$\tilde{R}^c_{\mu\nu} = \omega^c_{\mu\nu} - \omega^c_{\mu\rho} + \omega^c_{\nu\rho} - \omega^c_{\rho\sigma} - \omega^c_{\mu\sigma}.$$

From this perspective, one can figure out the effect of both mesh indices and world indices on describing the curvature tensor.

5.2. Spinning and Spinning Deviation Equation of an Object in a Class of Gauge Theory in Riemannian Geometry

In a similar way to [17], one can find out the equations of motion for a spinning object with precession in a class of gauge theories by taking variation with respect to $\Psi^\mu$ and $\Psi^{\mu\nu}$ simultaneously in the following Lagrangian:

$$L = g_{\mu\nu} P^\mu \frac{D\Psi^\nu}{Ds} + \frac{1}{2} \tilde{R}^a_{\nu\rho\sigma} S^{ab} U^\nu \Psi^\mu$$

$$+ e^{\alpha}_{c} e^{b}_{\beta} S_{ab} \frac{D\Psi^{\mu\nu}}{Ds} + (P_{\mu} U_{\nu} - P_{\nu} U_{\mu}) \Psi^{\mu\nu},$$

to obtain

$$\frac{DP^\alpha}{Ds} = \frac{1}{2} \tilde{R}^a_{\nu\rho\sigma} S^{ab} U^\nu.$$

(66)

Multiply both sides by $e^c_{\alpha}$, we get

$$\frac{DP^c}{Ds} = \frac{1}{2} \tilde{R}^a_{\nu\rho\sigma} S^{ab} U^\nu,$$

(67)

$$\frac{DS^{\alpha\beta}}{Ds} = (P^\alpha U^\beta - P^\beta U^\alpha),$$

(68)

i.e.,

$$\frac{D e^{\alpha}_{c} e^{b}_{\beta} S^{ab}}{Ds} = (P^\alpha U^\beta - P^\beta U^\alpha),$$

(69)

$$e^{\alpha}_{c} e^{b}_{\beta} S^{ab} + S^{ab} \frac{D(e^{\alpha}_{c} e^{b}_{\beta})}{Ds}$$

$$= (P^\alpha U^\beta - P^\beta U^\alpha).$$

(70)

Multiplying both sides by $e^d_{\alpha} e^e_{\beta}$, we find

$$\frac{D e^{\alpha}_{c} e^{d}_{\beta} e^{e}_{\gamma} S^{ab}}{Ds} + e^{d}_{\alpha} e^{e}_{\gamma} S^{ab} \frac{D(e^{\alpha}_{c} e^{b}_{\beta})}{Ds}$$

$$+ e^{e}_{\gamma} (P^\alpha U^\beta - P^\beta U^\alpha),$$

(71)

provided that

$$\frac{D(e^{\alpha}_{c} e^{b}_{\beta})}{Ds} = 0,$$

(72)

and consequently,

$$\frac{DS^{cd}}{Ds} = (P^c U^d - P^d U^c).$$

(73)

Using the commutation relations as mentioned above, we obtain their corresponding deviation equations

$$\frac{D^2 \Psi^{\mu\nu}}{Ds^2} = S^{\rho\nu}[\tilde{R}^a_{\rho\beta} U^\beta \Psi^\delta + (P^\nu U^d - P^d U^\nu),$$

(74)

5.3. Spinning Density and Spinning Density Deviation Equations for a Class of Gauge Theories

Let us define the spin tensor

$$\vec{S}^{\alpha\beta\gamma} = S^{\alpha\beta\gamma} P^a.$$

(75)

Differentiating both sides covariantly, we get

$$\frac{DS^{\alpha\beta}}{Ds} = \frac{DS^{\alpha\beta}}{Ds} P^a + \frac{DP^a}{Ds} S^{\alpha\beta}.$$

(76)

Substituting (66) and (72) in (75), we obtain

$$\frac{DS^{\alpha\beta}}{Ds} = f^{\alpha\beta},$$

(77)

where

$$f^{\alpha\beta} = p^a (p^b U^c - p^c U^b) + \frac{1}{2} \tilde{R}^{\rho\sigma}_{\mu\nu} S^{\mu\nu j} S^{\sigma\rho j}.$$

We suggest the equivalent Bazanski Lagrangian

$$L = \vec{S}_{\alpha\beta\gamma} \frac{D\vec{\Psi}^{\alpha\beta\gamma}}{Ds} + f^{\alpha\beta} \vec{\Psi}^{\alpha\beta\gamma},$$

(78)

whence after taking variation with respect to its corresponding deviation tensor

$$\frac{D\vec{S}_{\alpha\beta\gamma}}{Ds} = f^{\alpha\beta\gamma},$$

(79)

Similarly, their corresponding geodesic deviation equation may be found as follows:

$$\frac{D^2 \vec{\Psi}^{\alpha\beta\gamma}}{Ds^2} = S^{\rho\nu}[\tilde{R}^a_{\rho\beta} U^\beta \vec{\Psi}^\delta + (P^\nu U^d - P^d U^\nu),$$

(80)

6. DISCUSSION AND CONCLUDING REMARKS

The spinning density tensor may be used to express a spinning fluid. From this perspective, the 3rd rank skew-symmetric tensor turns to be specified to become $S^{\alpha\beta\gamma}$ skew-symmetric in the last two indices in an approach relating the spin density tensor with the Weyssenhoff tensor which is describing a spinning fluid. The problem of spinning density is vital to describe an extended object which can be expressible to a plasma fluid and heavy ion collisions [18].

In the present work, we have focused on the spinning fluid tensor and its deviation as a special case of applying a spinning density tensor. In principle, a spinning fluid tensor is a special case of spin density
as it is skew-symmetric in the last two indices, as expressed by the Weyssenhoff tensor. This description led many authors to geometrizing it by means of using the Riemann-Cartan geometry instead of the Riemannian geometry, and this led to a transition from GR to the Einstein-Cartan theory of gravity [10, 19]. This type of geometry has employed a new role of orthonormal frames used to describe its internal composition. This gives rise to an insightful vision to examine the internal structure of a fluid element using a rotational potential vector \( \Gamma^{i}_{j\mu} \) called the spin connection, able to describe space-time torsion besides the translational potential vector \( e^{a}_{\alpha} \). It is worth mentioning that Hehl et al. were able to include the spin tensor not only in the energy-momentum tensor \( T_{\mu\nu} \) but also in the angular momentum equations \( \Omega^{\mu}_{\nu\rho} \) in terms of space-time torsion, which is present in Poincare gauge theory (PGT) [16, 20].

Moreover, the tetrad formalism uses orthonormal frames in order to express the spin density tensor as shown in Eq. (74), provided that the curvature tensor is defined in terms of the spin connection [19]. Using this type of amended curvature with symmetric affine connection, in the torsionless case we have obtained its corresponding spin density equation (76) and the spin deviation density tensor equation (79) as expressed in a mixed way with anholonomic indices besides holonomic ones.

Nevertheless, an extended work to examine the problem of motion in Clifford space [21], a spinning density tensor may be described in Clifford spaces [22] in the following way, such that

\[
\hat{U}^{\mu} = \frac{\partial s}{\partial x^{\mu}}, \quad \hat{S}^{\mu\nu} = \frac{\partial s}{\partial x^{\alpha\beta}},
\]

and

\[
\hat{S}^{\mu\nu\rho} = \frac{\partial s}{\partial x^{\alpha\beta\gamma}}
\]

in terms of holographic coordinates such that \( s = s(x^{\mu}, x^{\mu\nu}, x^{\mu\nu\rho}, \ldots) \). However, from this perspective, replacing vectors with polyvectors for expressing the torsion and curvature tensors is viable to describe the internal structure of spinning density tensors in Riemannian geometry. This issue will be studied in detail in our future work as well.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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