Distributed Bi-level Energy Allocation Mechanism with Grid Constraints and Hidden User Information

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Abstract— A novel distributed energy allocation mechanism for Distribution System Operator (DSO) market through a bi-level iterative auction is proposed. With the locational marginal price at the substation node known, the DSO runs an upper level auction with aggregators as intermediate agents competing for energy. This DSO level auction takes into account physical grid constraints such as line flows, transformer capacities and node voltage limits. This auction mechanism is a straightforward implementation of projected gradient descent on the social welfare (SW) of all home level agents. Aggregators, which serve home level agents - both buyers and sellers, implement lower level auctions in parallel, through proportional allocation and without asking for utility functions and generation capacities that are considered private information. The overall bi-level auction is shown to be efficient and weakly budget balanced.

Index Terms—microgrid; agents; trading; auction; bid; social welfare, fairness

NOMENCLATURE

Some of the symbols used are listed here for convenience.

| Symbol | Description |
|--------|-------------|
| $\mathcal{N}$ | Set of nodes, excluding root |
| $\mathcal{A}$ | Set of nodes with aggregators |
| $\delta$ | Maximum allowable per unit (pu) voltage deviation |
| $P_{0}$, $Q_{0}$ | Active, Reactive power from root node in pu |
| $V_{0}$ | Voltage at root in pu |
| $k, l$ | The $k^{th}$, $l^{th}$ nodes, $k, l \in \mathcal{N}$ |
| $\mathcal{D}(k)$ | Set of downstream nodes of $k$ (immediate & separated) |
| $u(k)$ | Index of immediate upstream node of node $k$ |
| $\mathcal{U}(k)$ | Index of all upstream nodes of $k$ to root, $k \in \mathcal{U}(k)$ |
| $r_{k}, x_{k}$ | Resistance and reactance of line $(u(k), k)$ in pu |
| $p_{k}$ | Real power injected into $k$ in pu, $k \notin \mathcal{A}$ $\Rightarrow p_{k} = 0$ |
| $q_{k}$ | Reactive power injected into $k$ in pu, $k \notin \mathcal{A}$ $\Rightarrow q_{k} = 0$ |
| $P_{k}$ | Active power flowing through line $(u(k), k)$ in pu |
| $Q_{k}$ | Reactive power flowing through line $(u(k), k)$ in pu |
| $\Delta V_{k}$ | Voltage drop through line $(u(k), k)$ in pu |
| $V_{k}$ | Voltage at node $k$ in pu |
| $S_{k}$ | MVA limit of line going into node $k$ (line $k$) in pu |
| $\theta_{k}$ | Fraction of $p_{k}$ as $q_{k}$ at node $k$ when $k \in \mathcal{A}$ |
| $c_{k}$ | Per unit cost of energy of node $k$ in (\$/pu), $k \in \mathcal{A}$ |
| $\mathcal{N}_{B}^{k}$ | Set of buyers at node $k$, $k \in \mathcal{A}$ |
| $\mathcal{N}_{S}^{k}$ | Set of sellers at node $k$, $k \in \mathcal{A}$ |
| $i, j$ | The $i^{th}$ buyer and $j^{th}$ seller. |
| $g_{ij}^{k}$ | Max generation of $j^{th}$ seller in pu at node $k$ |
| $d_{ij}^{k}$ | Demand in pu delivered to the $i^{th}$ buyer at node $k$ |
| $s_{ij}^{k}$ | Supply in pu by to the $j^{th}$ seller at node $k$ |
| $b_{ij}^{k}$ | Total bid money for demand $d_{ij}^{k}$ by the $i^{th}$ buyer |
| $u_{ij}^{k}, v_{ij}^{k}$ | Utility functions of buyer $i$ and seller $j$ at node $k$ |
| $c_{0}$ | Price function at root (substation) node in \$/pu |

I. INTRODUCTION

The proliferation of Renewable Energy Resources (RES) at the distribution level is reshaping the market structure of the Distribution System Operators (DSO). The electricity sector has evolved from a highly regulated system operated by vertically integrated utilities to a relatively decentralized system based more fully on market valuation and allocation mechanisms [1]. A RES owner with a surplus of energy (prosumer) is anticipated to participate in such mechanisms more strategically while seeking profit [2], [3]. Specifically, it sells energy with the objective of maximizing its payoff, i.e. the sum of the monetary gain from supplying an amount of energy and utility it gains from retaining any surplus energy that is not traded. In a similar manner, the payoff of an energy consumer in the double auction, i.e. a buyer, is the difference between its utility gained from consuming energy and the cost of procuring that energy.

DSOs on the other hand, are expected to leverage the available local resources in order to capture additional value by optimizing the system for least cost operation while maintaining the physical system operation constraints [4]. One of the key challenges for efficient energy distribution mechanisms is its design in such a way that can motivate active participation of customers [5]. Without active participation of customers in such energy distribution mechanisms, the benefits of smart grid is not fully realized [6]. Therefore, efficient mechanisms that ensure the optimal operation within distribution system constraints while maintaining incentives for customers to participate are needed.

The existing energy grid literature mostly focuses on DLMP based pricing that is typically determined in a centralized fashion by the Lagrange multiplier of the distribution bus energy balance constraint [7], [8], [9], [10], [11], [12]. It was shown by [13] that the introduction of price responsive customers causes distribution line congestion issues. DLMP based pricing is used in [9] as means of dynamic pricing tariffs to alleviate distribution system congestion. Similar to the work of authors in [8] and [9] the recent work in [10] proposes two benchmark pricing methodologies, namely DLMP and iDLMP, for a congestion free energy management by buildings.
providing flexible demand. Aggregators in this model have contracts with flexible buildings to decide their reserve and energy schedule by interacting with the DSO in a cost optimal manner in order to avoid any congestion in the distribution grid. The iterative DLMP, i.e. iDLMP, is determined based on standard dual decomposition methods [14] on the DSO level to alleviate the need for data transfer among the DSO and aggregators.

In contrast to the research discussed above, we focus on an efficient, i.e. SW maximizing, energy distribution mechanism that is completely congruent with the physical topology of the radial distribution grid. In order to run such an efficient energy market in a distributed manner, the DSO maintains an aggregator agent at each node. The aggregators implement auctions with their own sets of consumers and prosumers, in parallel.

A novel bi-level distribution auction mechanism is proposed that converges to a socially optimal solution, while maintaining physical grid constraints. The lower level auction, referred to as the Aggregator Level Auction (ALA), is conducted by the local aggregators assigned to each distribution node among downstream consumers and prosumers. The ALA reflects what happens in practice in a real grid as it automatically establishes equilibrium conditions among these downstream agents, without any need for their private information, i.e. their utility functions and generation capacities. Efficiency is attained at the aggregator level indirectly through the ALA, which we show to be SW maximizing, obviating the need for any physically unrealizable explicit optimization. At equilibrium, an aggregator can act as a net seller or buyer depending on the sum of the demands and supply availabilities of its own consumers and prosumers.

The upper level auction, referred to as the DSO Level Auction (DLA), is implemented iteratively by the DSO among aggregators competing for the share of energy that the DSO receives from the wholesale market. The goal of the DSO’s auction is to determine the optimal amount to draw from the wholesale market and allocate it among competing aggregators in an efficient manner while maximizing the global SW and maintaining physical grid constraints such as voltage, line, and transformer limits.

II. FRAMEWORK

Consider a radial distribution network with \( N \) nodes excluding root as shown in Fig.1. The proposed bi-level auction mechanism is implemented in two levels among aggregators assigned to primary distribution nodes by the DSO, i.e. DLA, and among strategic consumers and prosumers, i.e. agents, residing on a lateral feeder by the aggregators, i.e. ALA. In the ALA, each agent’s objective is to maximize its own profit by participation in its aggregator’s auction. Each aggregator’s objective is to maximize its agents’ SW without access to their private utility functions and generation capacities as shown in Fig.2 by hidden lines. The aggregators make this possible by using double-sided proportional allocation [3] and participation in the DLA by competing with other aggregators in order to get their optimal demand/supply share of real power \( p_k \). The DSO’s objective is to implement the DLA iteratively until equilibrium is established and the maximum global SW is attained.

As shown in Fig.2, at each iteration of the DLA, aggregator \( k \) receives real power \( p_k \), implements its ALA and submits its’ per unit price \( c_k \) for \( p_k \). The price \( c_k \) is obtained as the market equilibrium price of the ALA. Upon return of the new set of prices \( c_k \) by each aggregator \( k \in A \), the DSO reruns the DLA to find the new supply \( p_k \) while maintaining physical grid constraints and remaining budget balanced, i.e. do not lose any money. This procedure continues until convergence is achieved.

A. Distribution System Constraints

The DSO needs to make sure that grid constraints are not violated. It is assumed in this research that distribution system is balanced and all quantities can be represented per phase at each consumer or prosumer node. The simplified version of DistFlow equations [15], [16] that has been extensively used in literature [17], [18], [19] is used to set up the grid constraints during each iteration of the DLA. Simplified DistFlow is given by the following set of equations,

\[
P_{k+1} = P_k - p_k, \quad (1)
\]

\[
Q_{k+1} = Q_k - q_k, \quad (2)
\]

\[
V_{k+1} = V_k - \frac{r_k p_{k+1} + x_k q_{k+1} V_k}{V_0}, \quad (3)
\]

where \( V_0 \) is the root node pu voltage. Since a typical distribution system is tree structured (see Fig.3), with \( \Delta V_k = \frac{r_k p_k + x_k q_k}{V_0} \) as
the voltage drop at line segment entering node \( k \), the simplified DistFlow equations in Eqns. (1) – (3) can be rewritten for \( k \) as,

\[
P_k = p_k + \sum_{l \in D(k)} p_l, \tag{4}
\]

\[
Q_k = q_k + \sum_{l \in D(k)} q_l, \tag{5}
\]

\[
V_k = V_0 - \sum_{l \in U(k)} \Delta V_l. \tag{6}
\]

Here, \( D(k) \) is the set of immediate and separated downstream and \( U(k) \) is the set of upstream (including \( k \)) nodes of \( k \) to root node. For instance in Fig.3, \( D(17) = \{18,19,20,21\} \), \( U(27) = \{1,2,27\} \). Furthermore, the following system architecture matrices are defined,

\[
[A]_{kl} = \begin{cases} 
1 & k = \text{aggregator } l \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
[D]_{kl} = \begin{cases} 
1 & l \in D(k) \text{ or } k = l \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
[U]_{kl} = \begin{cases} 
1 & l \in U(k) \\
0 & \text{otherwise}. 
\end{cases}
\]

In the equations above, \( A \) is an \( N \times A \) matrix and \( D, U \) are \( N \times N \) matrices associated with the spatial topology of the radial distribution network. The matrix \( A \) is the node-aggregator matrix that has an entry of unity (‘1’) at every column on the row, i.e. node, with which it is associated. The matrices \( D \) and \( U \) corresponds to the descendant and ancestor nodes. Every row (node) of \( D \) and \( U \) has a unit entry where the corresponding column (node) is its descendant or ancestor, and a zero elsewhere. The voltage drop in line \( k \) is \( \Delta V_k = r_k P_k + x_k Q_k \), so that,

\[
\Delta V = V_0^{-1} (r \odot P + x \odot Q) \tag{7}
\]

where \( \odot \) is the elementwise (Hadamard) product. Hence, using matrices defined above, Eqns. (4) – (6) can be written as follow,

\[
P = DA\mathbf{p}, \tag{8}
\]

\[
Q = DA\mathbf{q}, \tag{9}
\]

\[
V = V_0 \mathbf{1}_N - U \Delta V. \tag{10}
\]

Note that it is assumed that homes are furnished with smart meters and inverters [20], [21] in case of sellers, that are capable of communicating their reactive power supply/demand to/from the aggregators at the end of each ALA. Every home may have a different power factor, but revealed to the aggregator. Hence, each aggregator can form and communicate their reactive power need as aggregate fraction of their real power demand/supply during each iteration of the DLA, the fraction of \( p_k \) that upon multiplication gives their reactive power supply/demand \( q_k \), i.e. \( q_k = \theta_k p_k \). In addition to \( c_k \), this fraction denoted as \( \theta_k \) is communicated to the DSO and is used by the DLA in order to account for grid constraints.

With \( \theta = \{\theta_k\}_{k \in A} \) and \( \mathbf{q} = \theta \odot \mathbf{p} \) as elementwise product of \( \theta \) and \( \mathbf{p} \), using Eqn. (7), Eqns. (8) – (10) can be rewritten as,

\[
P = DA\mathbf{p}, \tag{11}
\]

\[
Q = DA(\theta \odot \mathbf{p}). \tag{12}
\]

\[
V = V_0 \mathbf{1}_N - \frac{1}{V_0} \mathbf{U}(r \odot DA\mathbf{p} + x \odot DA(\theta \odot \mathbf{p})). \tag{13}
\]

In the above version of simplified DistFlow equations, the real/reactive branch flows and node voltages are entirely given as a function of nodes real power injection \( \mathbf{p} \), substation per unit voltage \( V_0 \) and distribution grid topology. With the latter two known to the DSO, it can implement the DLA to determine \( \mathbf{p} \) by using Eqns. (11) – (13) to set up grid constraints. The physical grid constraints are explained below.

A.1 Transformer Capacity: Since the distribution transformer(s) at the substation node has limited capacity, the total amount of apparent power that the DSO can draw from the wholesale market is bounded and can be expressed as,

\[
p^T \mathbf{J}^1 A \mathbf{p} + p^T \mathbf{J}^0 B \mathbf{p} \leq S_k^2. \tag{14}
\]

Here, \( J_A^1 = J_A^1 \mathbf{1}^T \) and \( J_B^0 = \theta \mathbf{1}^T \).

A.2 Line Limits: The total apparent power at line \( k \) cannot exceed its MVA limit \( S_k \), so that \( P_k^2 + Q_k^2 \leq S_k^2 \). Defining the matrix \( E_k \) as a square matrix consisting of all but one zero entries with the only non-zero entry of unity appearing at the \( k \)th row and the \( k \)th column, it is seen that,

\[
p^T E_k^2 P + Q^T E_k Q \leq S_k^2.
\]

Whence, using Eqns. (11) and (12), the following line limit constraint is obtained,

\[
p^T Z_k \mathbf{p} \leq S_k^2, \quad \forall k \in N. \tag{15}
\]

Each matrix \( Z_k \) above is given by,

\[
Z_k = A^T E_k^2 D_k A + \text{diag} \theta \mathbf{1}^T D_k E_k D_k \text{diag} \theta. \tag{16}
\]

A.3 Node Voltage Limits: The voltage at node \( k \) is given by Eqn. (13), which is entirely written in terms of \( \mathbf{p} \) and other grid parameters that is known to the DSO. The total voltage deviation at node \( k \) must not exceed a numerical value of \( \delta \) (typically 0.05), yielding the following constraint,

\[
1_N - \delta \leq V \leq 1_N + \delta. \tag{17}
\]

Whence using Eqn. (13) in (17), and upon further simplification the following bounds on the power vector \( \mathbf{p} \) are obtained,

\[
\mathbf{1} \leq M \mathbf{p} \leq \mathbf{1}. \tag{18}
\]

In the above expression, the lower and upper bounds are,

\[
\mathbf{1} = (V_0 - 1) \mathbf{1}_N - \delta, \quad \mathbf{1} = (V_0 - 1) \mathbf{1}_N + \delta.
\]

The matrix \( \mathbf{M} \) that shows the sensitivity of voltage deviation to power injection \( \mathbf{p} \) is equal to,

\[
\mathbf{M} = \mathbf{M}_p + \mathbf{M}_q \text{diag} \theta. \tag{19}
\]

Where, \( \mathbf{M}_p = V_0^{-1} \mathbf{U} \odot \mathbf{DA}, \quad \mathbf{M}_q = V_0^{-1} \mathbf{U} \odot \mathbf{DA}. \)

A.4 Budget Balance: Similar to [22], [13], [10], the market price at the substation node is considered coupled with the demand drawn by the grid from the wholesale market and is modeled as,

\[
c_0 = c_0^b + \beta_0 \sum_k p_k. \tag{20}
\]

Here, \( c_0^b \) is the base demand price and \( \beta_0 \) is the sensitivity coefficient that can be obtained using statistics of historical data of locational marginal price as explained in [22]. Note that during modeling \( c_0 \), the DSO can add a reasonable fixed
amount to \(c^0\) to account for any grid usage tariff. Despite that, the goal is that the DSO shall not lose any money at the end of the mechanism. That is, the total reimbursement to the wholesale market and aggregators that supply energy shall not exceed that paid by the aggregators that receive energy. In other words, \(c^0 \mathbf{1}_b^T \mathbf{p} + c_0 \mathbf{1}_p^T \mathbf{p} \geq 0\). Replacing \(c_0\) from Eqn. (20) yields the following DSO budget balance constraint,

\[
c^0 \mathbf{1}_b^T \mathbf{p} - c^p \mathbf{p} + \beta_0 p^T \mathbf{j}_d^T \mathbf{p} \leq 0. \tag{21}
\]

### B. DSO Level Auction

#### B.1 Social Welfare Problem

The feasible set \(\mathcal{F}\) consists of all power vectors \(\mathbf{p}\) that meet constraints in Eqns. (14), (15), (18) and (21) pertaining to substation transformers MVA limit, lines MVA limit, nodes voltage limits, and DSO budget balance, i.e.

\[
\mathcal{F} = \left\{ \mathbf{p} \mid \begin{array}{l}
\mathbf{p}^T \mathbf{Z}_k^p \mathbf{p} \leq S^2_k, \\
\forall k \in \mathcal{N}: \mathbf{p}^T \mathbf{Z}_k^p \mathbf{p} \leq S^2_k, \\
\forall k \in \mathcal{N}: \mathbf{b}^T \mathbf{p} \leq \mathbf{b}_k^T, \\
c^0 \mathbf{1}_b^T \mathbf{p} - c^p \mathbf{p} + \beta_0 p^T \mathbf{j}_d^T \mathbf{p} \leq 0
\end{array} \right\}. \tag{22}
\]

The goal of DSO is to solve the SW problem as stated below. Maximize w.r.t. \(\mathbf{d}^k\) and \(\mathbf{s}^k\): 

\[
\Omega(\mathbf{d}^k, \mathbf{s}^k, \mathbf{p}) = \sum_{k \in \mathcal{A}} \theta_k (\mathbf{d}^k, \mathbf{s}^k, \mathbf{p}). \tag{23}
\]

Subject to:

\[
\begin{array}{l}
\mathbf{p} \in \mathcal{F}, \\
\mathbf{p} = \left[ \mathbf{1}^T_{N_B} \mathbf{d}^k - \mathbf{1}^T_{N_S} \mathbf{s}^k \right]_{k \in \mathcal{A}}, \\
\mathbf{s}^k \leq \mathbf{g}^k.
\end{array} \tag{24}
\]

Each term \(\theta_k\) in (23) is given by,

\[
\theta_k (\mathbf{d}^k, \mathbf{s}^k | \mathbf{p}_k) = \mathbf{1}^T_{N_B} \mathbf{u}^k + \mathbf{1}^T_{N_S} \mathbf{v}^k, \tag{26}
\]

where \(\mathbf{u}^k\) and \(\mathbf{v}^k\) are the hidden utility functions of buyers and sellers in aggregator \(k\). Under the assumption that the utility functions \(\mathbf{u}^k\) and \(\mathbf{v}^k\) are concave, the DSO SW optimization problem is a convex optimization problem. Note that in the DSO SW problem, \(\mathbf{d}^k\) and \(\mathbf{s}^k\) are local variable vectors pertaining to aggregator \(k\)’s buyers and sellers demand and supply whereas \(\mathbf{p}\) is the global variable vector of injections to all aggregators. Due to lack of access to the utility functions \(\mathbf{u}^k\), \(\mathbf{v}^k\), and for computational efficiency [23], [24], the DSO implements a distributed algorithm by decomposing its original problem into a master and sub-problems and solves it in a distributive fashion. The master problem is solved by the DLA and the sub-problems are solved by the ALAs locally.

#### B.2 Distributed Implementation

At the lower level, each aggregator \(k\), implements the following sub-problem in parallel by means of the ALA:

Maximize w.r.t. \(\mathbf{d}^k, \mathbf{s}^k: \ \theta_k (\mathbf{d}^k, \mathbf{s}^k, \mathbf{p})
\]

Subject to Eqns. (24) and (25).

At the upper level, each iteration of the DLA realizes a projected gradient descent step of the decomposed SW problem with \(\mathbf{p}\) as the global variable. The DSO sends to each aggregator \(k\) the power \(p_k\) and receives the gradient direction \(\lambda = \nabla_{\mathbf{p}} \Omega\) from it. It will be shown in Proposition 7 that \(\lambda = c\) the vector of costs towards which the ALA converges. Next, the vector \(\mathbf{p}\) is incrementated by an amount \(\varepsilon \lambda\) to \(\mathbf{p}'\), where \(\varepsilon\) is the gradient step size. The vector \(\mathbf{p}'\) is then projected onto the feasible region \(\mathcal{F}\), after the constraint parameters are updated according to Eqns. (16) and (19). Parameter updates are required because the reactive power changes according to the numerical value of the projection \(\mathbf{p}'\), which in turn causes the fraction \(\theta\) to change.

**Algorithm DLA:**

\[
\mathbf{p} \leftarrow \text{initial} \\
\text{Repeat:} \\
\quad \text{Send to aggregators } k \in \mathcal{A}: p_k \\
\quad \text{Receive from aggregators } k \in \mathcal{A}: \lambda_k = c_k a_k \\
\quad \text{If } \lambda_k a_k \neq T \text{ then exit loop} \\
\quad \mathbf{p}' = \mathbf{p} + \varepsilon \lambda \\
\quad \text{Compute constraint parameters: } \theta, \mathbf{M}, \mathbf{Z}_k \\
\quad \mathbf{p} = \text{argmin}_{\mathbf{p}^\prime} \| \mathbf{p} - \mathbf{p}' \| \\
\text{Until convergence} \\
\text{Output: } \mathbf{p}, \mathbf{c}, \mathbf{[d^k]}_{k \in \mathcal{A}}, \mathbf{[s^k]}_{k \in \mathcal{A}}
\]

**Proposition 1:** The DLA is an efficient mechanism.

**Proof:** The DLA is a straightforward implementation of projected gradient descent of the SW optimization problem, whose convergence has been well studied [25], [26], [24].

The DLA algorithm terminates when any aggregator \(k\) returns a flag \(a_k = F\) defined in the following, indicating that the constraints in Eqns. (24) and (25) were not satisfied,

\[
a_k = \begin{cases} \top & \text{if } p_k = \mathbf{1}^T_{N_B} \mathbf{d}^k - \mathbf{1}^T_{N_S} \mathbf{s}^k, \\
F & \text{if } p_k \neq \mathbf{1}^T_{N_B} \mathbf{d}^k - \mathbf{1}^T_{N_S} \mathbf{s}^k. \end{cases} \tag{27}
\]

### C. Aggregator Level Auction

#### C.1 Virtual Bidding

Price anticipation is an undesirable effect that occurs in proportional auctions with relatively few bidders [3], [27]. In such cases, the bidders are aware of the sensitivity of the equilibrium cost, i.e. that \(\frac{\partial c_k}{\partial \delta^p} \neq 0\) for a buyer, and \(\frac{\partial c_k}{\partial \delta^T} \neq 0\) for a seller. As the bidders place bids to maximize their individual payoffs, price anticipation leads to inefficiency. In [3], virtual bidding was shown to approach price-taking conditions for isolated microgrids by lowering the market powers of the bidders. Virtual bidding involves the presence of a virtual agent, which acts as both a seller and a buyer. Unlike other agents, the virtual bidder is merely an algorithmic entity that is incorporated within the aggregator, ergo has access to \(\mathbf{b}^k\) and \(\mathbf{s}^k\). Before addressing the implementation of the ALA, the following propositions are established.

**Proposition 2:** Due to virtual bidding, the ALA can be arbitrarily close to price taking mechanism. In other words, the following expressions hold,

\[
c_k = \frac{\mathbf{1}^T_{N_B} \mathbf{b}^k}{p_k + \mathbf{1}^T_{N_S} \mathbf{s}^k}. \tag{28}
\]
\[
\frac{\partial c_k}{\partial b^k_i} = 0, \quad \frac{\partial c_k}{\partial s^k_j} = 0. \tag{29}
\]

**Proof:** The virtual bidder bids a large amount of energy \( s_0 \), which it buys for a total amount \( c_0 s_0 \). Here \( c_0 = \frac{1}{p_k + s_0 + 1} \) is the desired cost under price taking. The actual price, \( c_k = \frac{c_0 s_0 + T_k b^k_k}{p_k + s_0 + 1} \), is computed by the aggregator. It can be readily established that \( \lim_{s_0 \to 0} c_k \lim_{s_0 \to 0} s_0 + 1 = 0 \), justifying Eqn. (28). Likewise \( \lim_{s_0 \to 0} \frac{\partial c_k}{\partial s_0} = 0 \) and \( \lim_{s_0 \to 0} \frac{\partial c_k}{\partial s_0} = 0 \), so that Eqn. (29) is valid in the limiting case.

**Proposition-3:** Due to virtual bidding, the ALA is strongly budget and energy balanced.

**Proof:** It follows from Eqn. (28) that \( c_k p_k + c_k s^k T_B S^k = \frac{1}{p_k + s_0 + 1} b^k \). The RHS is the total monetary amount that the aggregator receives from the buyers. The LHS is the sum of the payment that the aggregator makes to the DSO and the reimbursement amount given to the sellers. This establishes strong budget balance. Energy balance is established under the proportional allocation auction paradigm [28], where the energy to each buyer to be proportional to its bid, i.e. \( d^k = \frac{1}{c_k} b^k \). Energy balance immediately follows from Eqn. (28). Note that this satisfies the constraint in in Eqn. (24).

### C.2 Distributed Implementation

The ALA receives \( p_k \) from the DSO, and initializes the cost \( c_k \) (see Fig. 2). In each step, it sends \( c_k \) to the sellers and receives \( s^k \). Using proportional allocation, it determines \( d^k \) which is communicated to the buyers. The buyers return their bids \( b^k \). The cost \( c_k \) is determined as a two-step procedure using virtual bidding.

#### Algorithm ALA(k):

Receive from DSO: \( p_k \)
Initialize: \( c_k \)
Repeat:
- Send to sellers: \( c_k \)
  - Receive from sellers: \( s^k \)
  - \( d^k = \frac{1}{c_k} b^k \)
- Send to buyers: \( d^k \)
  - Receive from buyers: \( b^k \)
  - \( c^k = \frac{1}{p_k + s_0 + 1} b^k \)
- Send to DSO: \( c_k \)

#### D. Equilibrium Analysis

**D.1 Bidding Strategies:** Buyers and sellers’ bidding strategies are established by means of the following propositions.

**Proposition-4:** Under virtual bidding, the bid \( b^k_i \) placed by each buyer \( i \) is such that its marginal utility equals its per unit cost, \( \frac{\partial u^k_i}{\partial a^k_i} = c_k \).

**Proof:** The buyer’s payoff \( u^k_i \) is maximized when its derivative is zero, i.e. \( \frac{\partial u^k_i}{\partial a^k_i} = 0 \). Under virtual bidding, Eqn. (29) holds so that Eqn. (30) is satisfied.

**Proposition-5:** Under virtual bidding, the bid \( s^k_j \) placed by each seller \( j \) is such that if the seller does not bid its entire generation \( s^k_j < g^j \), its marginal utility, with \( y_j \geq 0 \) being a positive scalar quantity, equals its per unit cost, \( \frac{\partial u^j_k}{\partial s^j_k} = c_k \).

**Proof:** In fact, \( y^j_k \) is a dual variable as shall be seen here. The seller’s payoff \( v^j_k \) is maximized under the constraint \( s^k_j \leq g^j \). Lagrangian for this problem is \( L_j(s^k_j) = v^j_k(s^k_j - g^j) + c_k s^k_j \). At stationarity, \( \frac{\partial L_j}{\partial s^k_j} = 0 \). Under virtual bidding, Eqn. (29) holds so that Eqn. (31) is satisfied.

**D.2 Aggregator Equilibrium:** During each iteration of DLA, an aggregator establishes equilibrium conditions to return cost \( c_k \).

**Proposition-6:** The equilibrium of the aggregator \( k \)’s auction maximizes the social welfare \( \Theta_k(d^k, s^k, p) \) as defined in Eqn. (15) with respect to \( d^k, s^k \) under the energy balance constraint, \( p_k = \frac{1}{N^B} d^k - \frac{1}{N^S} s^k \) and with no seller selling more energy than its generated capacity, \( s^k \leq g^k \).

**Proof:** The statement above defines a constrained optimization problem with the following Lagrangian \( L_k(d^k, s^k) \).

\[
L_k(d^k, s^k) = \Theta_k(d^k, s^k) - \lambda_k(s^k - g^k) - \lambda_k(1 - T_B d^k - 1 - T_S s^k - p_k). \tag{32}
\]

The constrained optimum satisfies the ALA’s energy balance condition. Stationarity at the optimum implies that \( V_{d^k} \lambda_k = 0 \) and \( V_{s^k} \lambda_k = 0 \). From Eqn. (26), it is that \( V_{d^k} \lambda_k = \lambda_k 1_{N^B} \) and \( V_{s^k} \lambda_k = -\lambda_k 1_{N^S} \), so that the optimum of \( \Theta_k \) coincides with the auction equilibrium when \( \lambda_k = c_k \) in Eqns. (30) and (31) and with \( \lambda_k \) being the vector of entries \( \gamma^j_k \) in Eqn. (31).

**Proposition-7:** The cost vector \( c \) returned by ALA is also the gradient of the overall SW function given by Eqn. (23), i.e.
\[ V_p \Omega([d^k]_{k=0}^{d^k} [s^k]_{k=0}^{s^k}, p) = c. \] (33)

**Proof:** From (30) and (31), it is seen that \( \lambda_k = c_k \). It follows that \( \frac{\partial}{\partial p_k} \Theta_k(d^k, s^k) = c_k \). The expression in Eqn. (33) directly follows from Eqn. (23).

### III. SIMULATION RESULTS

Simulation results corroborates the theory presented. A modified IEEE 37 node system (Fig.3) with a base value of 100kVA is used to simulate the bi-level mechanism in Matlab. Seventeen aggregators with different number of buyer and seller agent, shown in Table I, were generated and assigned to the nodes with load (nodes circled in Fig.3). The total number of agents considered was 483 out of which 303 were buyers and 180 were sellers. The utility functions of the agents were assumed to follow concave logarithmic curves according to the equations \( u_i(d_i) = x_i \log(y_i d_i + 1) \) and \( v_j(g_j - s_j) = x_j \log(y_j (g_j - s_j) + 1) \). The quantities \( x_i, y_i, x_j \) and \( y_j \) in these equations were different for each agent, and were generated randomly and adjusted so that agents’ marginal utilities are scaled to reasonable per unit prices. The generation \( g_j \) for sellers were also drawn at random, uniformly in the interval [0.1, 0.5] pu.

**TABLE 1: Aggregator assignment and number of buyers/sellers.**

| \( A_{agg} \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Node | 1 | 8 | 12 | 13 | 17 | 18 | 22 | 23 | 25 | 26 | 27 | 29 | 30 | 31 | 33 | 35 | 36 |
| \( N_b \) | 14 | 11 | 25 | 27 | 20 | 8 | 21 | 27 | 22 | 21 | 5 | 13 | 7 | 19 | 10 | 26 | 27 |
| \( N_s \) | 17 | 3 | 5 | 21 | 22 | 10 | 15 | 6 | 4 | 22 | 3 | 9 | 4 | 19 | 5 | 6 | 9 |

Four scenarios were simulated to investigate the effect of price and capacity at the substation node on the distribution of energy and its associated cost to each aggregator, grid constraints, customer’s action and overall SW. These scenarios (I, II, III, and IV) were generated based on substation node price and capacity parameters in Eqn.(20) and (14) respectively as follows.

I) \( c_b^0 = 80 \) pu (high), \( \beta_0 = 40 \) pu^2 (elastic), \( S_0 = 25 \) pu,

II) \( c_b^0 = 20 \) pu (low), \( \beta_0 = 30 \) pu^2 (elastic), \( S_0 = 25 \) pu,

III) \( c_b^0 = 20 \) pu (low), \( \beta_0 = 10 \) pu^2 (elastic), \( S_0 = 25 \) pu,

IV) \( c_b^0 = 20 \) pu (low), \( \beta_0 = 0 \) pu^2 (inelastic), \( S_0 = 40 \) pu.

In case I, price was intentionally increased dramatically to draw minimum energy from the substation bus and observe energy trade among aggregators. In IV, price was made inelastic to demand in the distribution grid and the transformer capacity was increased to 40 pu in order draw high amount of energy from the substation node and observe activation of the physical grid constraints during DLA.

Matlab’s fmincon function was used to solve constraint minimization problem at each iteration of the DLA. Under each scenario, Fig.4 shows the energy injection \( p_k \) to each aggregator \( k \) and cost \( c_k \) that they pay. The energy \( p_k \) and its’ per unit cost \( c_k \) refers to the efficient solution of the DLA and ALA under equilibrium. In other words, the amounts \( p_k \) depicted for each scenario are the solutions at equilibrium when the global SW given by Eqn. (23) has stabilized at its constraint maximum. Furthermore, the costs \( c_k \) are the stable market-clearing price of the ALA at equilibrium and is different for each aggregator, making the DLA price heterogeneous.

In scenario I, due to high base price \( c_b^0 \) and high sensitivity \( \beta_0 \), the aggregators are willing to trade among them at different prices ranging from 469 pu/ to 592 pu. In this case, many aggregators, especially those that have higher number of sellers, decide to sell energy in the DLA as supply is scarce and they are able to sell at high cost \( c_k \). The DSO in this scenario draws only 1.72 pu at a \( c_b \) of 869 pu/ from the wholesale market at which it is strongly budget balanced, i.e. constraint in Eqn.(21) is active and DSO makes zero profit. In scenario II, due to lower

![Fig. 3. Modified IEEE 37 node system with aggregator numbering.](image)

**TABLE 3.** DLA outcome for aggregators’ share of energy (in pu) and its associated costs (in pu).

| \( k \) | \( p_k \) | \( c_k \) | \( \% \) |
|---|---|---|---|
| 1 | 419 | 525 | 87.63 pu |
| 2 | 21.87 pu | 30.14 pu | 87.63 pu |
| 3 | 10.83 pu | 20 | 1.72 pu |

![Fig.4. DLA outcome for aggregators’ share of energy (in pu) and its associated costs (in pu).](image)

Each scenario, Fig.4 shows the energy injection \( p_k \) to each aggregator \( k \) and cost \( c_k \) that they pay. The energy \( p_k \) and its per unit cost \( c_k \) refers to the efficient solution of the DLA and ALA under equilibrium. In other words, the amounts \( p_k \) depicted for each scenario are the solutions at equilibrium when the global SW given by Eqn. (23) has stabilized at its constraint maximum. Furthermore, the costs \( c_k \) are the stable market-clearing price of the ALA at equilibrium and is different for each aggregator, making the DLA price heterogeneous.
respectively. Due to low $c_0$ in both scenarios, all aggregators trade (import) at lower costs ranging from 330¢/pu to 457¢/pu in scenario III and 233¢/pu to 448¢/pu in scenario IV. In scenarios III, the DSO still makes no money. However, in scenario IV it makes 4992 cents acting as an arbitrager. We emphasize that the proposed mechanism is an efficient bi-level auction in which the goal is to maximize the overall SW. The DSO’s budget is merely set as a constraint. As pointed out earlier, DSO can accommodate its profit by means of a constant tariff in the pricing model in Eqn. (20).

Note that the amount of extraction/injection from/to each aggregator node depends on the substation node price $c_0$, number of buyer and seller agents, sellers’ generation capacity $g_j^k$, and marginal utilities $u_i^k$ and $v_j^k$ of each aggregator $k$. In general, aggregators with available surplus energy and lower equilibrium price $c_k$ supply more $p_k$ to the rest of the network while those that have higher $c_k$ supply less. Similarly, aggregators with deficit energy are assigned more $p_k$ if their equilibrium price $c_k$ are higher.

Convergence to global optimum SW (sum of utilities of 483 agents) under each case is shown in Fig.5. More injection from the wholesale market, i.e. higher $\sum_p$, results to higher SW. This is because the sellers utility functions argument in Eqn. (26) is given by $g_j - s_j$ and more supply from the wholesale market means less supply from the sellers. This results to higher sellers’ utility and higher SW.

![Fig.5. Optimum SW under each scenario. In scenarios other than III, some aggregator $k$ returns $a_k = F$ and the DLA terminates.](image)

Node pu voltages at the end of DLA are shown in Fig.6 and each line segments’ real ($P_k$), reactive ($Q_k$), and apparent ($S_k$) power flows, within line MVA limits ($S_k$) are shown in Fig.7. Note that as we move from scenario I to IV, node voltages decrease and line flows increase as more energy is supplied through the substation node. Negative line flows vanish and more positive line flows appear. For example, the flows in line segments going into nodes 13 and 17 are negative in scenario I and II and positive in III, and IV. In scenario I, since the price $c_0$ is high and as a result supply is scarce, with high number of sellers aggregators 4 and 5 on nodes 13 and 17 (A4 and A5 in Fig.3) supply to A3 on node 12 that has 25 buyers and only 5 sellers as well as some other upstream aggregators. In scenario II however, A4 on node 13 starts to import (see Fig. 4) and A5 still exports feeding both A4 and A3. This makes flow in line going to node 17 and 13 negative. Notice that A3 imports from both A5 and other upstream aggregators in this case.

In scenarios III and IV, all flows are highly positive causing higher voltage drops. Note that for scenario IV, we intentionally increased the substation transformer capacity from nominal 25 to 40 pu to allow higher line flows as the substation transformer constraint hits the limit first before any line does. The DLA proceeds and maximizes the overall SW until it hits the line limit constraints given by Eqn. (15). As seen, line segments 2, 8, and 27 hit the limit and no further trading is allowed.

![Fig.6. Node pu voltages within given bounds.](image)

In order to illustrate the result of ALAs, auction outcome for A6 on node 18 under scenarios I and IV are tabulated in Table 2. In scenario I, A6 exports 0.834 pu to the grid that is equal to the sum of supplies ($\Sigma s_i^p$) by its sellers minus sum of demands of its buyers ($\Sigma d_i^p$). The equilibrium price $c_6$ is 533¢/pu at which the buyers and sellers’ marginal utilities $u_i^6$ and $v_j^6$ stabilize. In this case, because of high prices, all sellers are willing to trade and declare nonzero $s_i^6$. In scenario IV, A6 imports 0.958 pu at a lower equilibrium price of 273¢/pu. Buyers demand increase and for sellers only those with marginal utilities $v_j^6$ equal to $c_6$ get to supply a nonzero $s_i^6$. Notice that seller number 3, 6, 7, and 8 with $v_j^6 > c_6$ have been allocated zero $s_i^6$ by A6 allowing these sellers to consume all their generation $g_j^6$ themselves and result to higher overall SW.
TABLE 2: ALA outcome for A6 at node 18 under scenarios I and IV.

| Buyer Agents |  
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $p_i$ (pu)   | $c_i$ (pu)  | $V_i$ (pu)  |  
| I           | IV          | I           | IV          | I           | IV          | I           | IV          | I           | IV          | I           | IV          |  
| -0.834      | 0.958       | 553         | 273         | 1.032       | 0.972       |  

| Seller Agents |  
|---------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|  
| $d_i$ (pu)    | $c_i$ (pu)  | $u_i$ (pu)  |  
| I            | IV          | I            | IV          | I            | IV          | I            | IV          | I            | IV          | I            | IV          |  
| 0.115         | 0.234       | 553          | 273         | 553          | 273         |  
| 0.105         | 0.214       | 553          | 273         | 553          | 273         |  
| 0.118         | 0.240       | 553          | 273         | 553          | 273         |  
| 0.079         | 0.160       | 553          | 273         | 553          | 273         |  
| 0.131         | 0.267       | 553          | 273         | 553          | 273         |  
| 0.126         | 0.257       | 553          | 273         | 553          | 273         |  
| 0.081         | 0.165       | 553          | 273         | 553          | 273         |  
| 0.123         | 0.251       | 553          | 273         | 553          | 273         |  

IV. CONCLUSION

We present a globally efficient bi-level energy allocation mechanism that is implemented by DSO at the upper and aggregators at the lower level to maximize SW while not violating any grid constraints. The upper level auction is price heterogeneous among aggregators while the lower level auction is price uniform among home agents. Through distributive lower level auction with proportional allocation, the DSO is shown to achieve global efficiency while not asking for any private information of the home agents as well as mitigating the effect of price anticipation. Future research can be conducted on extending the current model to multiple time slots. Aggregators that do not include any energy storage or conventional generators can implement an auction during each timeslot independently of the others. For aggregators that do so, temporal constraints, such as battery state of charge update would need to be taken into account.

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