NORM OF THE HAUSDORFF OPERATOR ON THE REAL HARDY SPACE $H^1(\mathbb{R})$

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Abstract. Let $\varphi$ be a nonnegative integrable function on $(0, \infty)$. It is well-known that the Hausdorff operator $H_\varphi$ generated by $\varphi$ is bounded on the real Hardy space $H^1(\mathbb{R})$. The aim of this paper is to give the exact norm of $H_\varphi$. More precisely, we prove that

$$\|H_\varphi\|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} = \int_0^\infty \varphi(t) dt.$$ 

1. Introduction and main result

Let $\varphi$ be a locally integrable function on $(0, \infty)$. The Hausdorff operator $H_\varphi$ is defined for suitable functions $f$ by

$$H_\varphi f(x) = \int_0^\infty \left( \frac{x}{t} \right) \frac{\varphi(t)}{t} dt.$$ 

The Hausdorff operator is an interesting operator in harmonic analysis. There are many classical operators in analysis which are special cases of the Hausdorff operator if one chooses suitable kernel functions $\varphi$, such as the classical Hardy operator, its adjoint operator, the Cesàro type operators, the Riemann-Liouville fractional integral operator,... See the survey article [7] and the references therein. In the recent years, there is an increasing interest on the study of boundedness of the Hausdorff operator on the real Hardy spaces, see for example [1, 2, 4, 7, 8, 9, 10, 11, 12].

Let $\Phi$ be a function in the Schwartz space $S(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \Phi(x) dx \neq 0$. Set $\Phi_t(x) := t^{-1} \Phi(x/t)$. Following Fefferman and Stein [5, 13], we define the real Hardy space $H^1(\mathbb{R})$ as the space of functions $f \in L^1(\mathbb{R})$ such that

$$\|f\|_{H^1(\mathbb{R})} := \|M_\Phi f\|_{L^1(\mathbb{R})} < \infty,$$

where $M_\Phi f$ is the smooth maximal function of $f$ defined by

$$M_\Phi f(x) = \sup_{t > 0} |f * \Phi_t(x)|, \quad x \in \mathbb{R}.$$
Remark that $\| \cdot \|_{H^1(\mathbb{R})}$ defines a norm on $H^1(\mathbb{R})$, whose size depends on the choice of $\Phi$, but the space $H^1(\mathbb{R})$ does not depend on this choice.

Let $\varphi$ be a nonnegative function in $L^1_{\text{loc}}(0, \infty)$. Although, it was shown in [4] that $\mathcal{H}_\varphi$ is bounded on $H^1(\mathbb{R})$ if and only if $\varphi \in L^1(0, \infty)$, the exact norm $\| \mathcal{H}_\varphi \|_{H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})}$ is still unknown.

Our main result is as follows.

**Theorem 1.1.** Let $\varphi$ be a nonnegative function in $L^1(0, \infty)$. Then

$$\| \mathcal{H}_\varphi \|_{H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})} = \int_0^\infty \varphi(t) dt.$$  

In Theorem 1.1, it should be pointed out that the norm of the Hausdorff operator $\mathcal{H}_\varphi \left( \int_0^\infty \varphi(t) dt \right)$ does not depend on the choice of the above function $\Phi$. Moreover, it still holds when the above norm $\| \cdot \|_{H^1(\mathbb{R})}$ is replaced by

$$\| f \|_{H^1(\mathbb{R})} := \| f \|_{L^1(\mathbb{R})} + \| H(f) \|_{L^1(\mathbb{R})},$$

where $H(f)$ is the Hilbert transform $H$ of $f \in L^1(\mathbb{R})$ defined by

$$H(f)(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy, \quad \text{a.e. } x \in \mathbb{R}.$$ 

See the last section for details.

**Corollary 1.1.** Let $\varphi \in L^1(0, \infty)$. Then $\mathcal{H}_\varphi$ is bounded on $H^1(\mathbb{R})$, moreover,

$$\left| \int_0^\infty \varphi(t) dt \right| \leq \| \mathcal{H}_\varphi \|_{H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})} \leq \int_0^\infty |\varphi(t)| dt.$$ 

Throughout the whole article, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any $E \subset \mathbb{R}$, we denote by $\chi_E$ its characteristic function.

2. **Proof of Theorem 1.1**

Let $P$ be the Poisson kernel on $\mathbb{R}$, that is, $P(x) = \frac{1}{e^{2\pi t}+1}$ for all $x \in \mathbb{R}$. For any $t > 0$, set $P_t(x) := \frac{1}{x^2+t}$. The Poisson maximal function $M_P f$ of a function $f \in L^1(\mathbb{R})$ is then defined by

$$M_P f(x) = \sup_{t>0} |P_t \ast f(x)|, \quad x \in \mathbb{R}.$$
Let $\mathbb{C}_+$ be the upper half-plane in the complex plane. The Hardy space $H^1_0(\mathbb{C}_+)$ is defined as the set of all holomorphic functions $F$ on $\mathbb{C}_+$ such that

$$\|F\|_{H^1_0(\mathbb{C}_+)} := \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|\,dx < \infty.$$ 

The following two lemmas are classical and can be found in [3, 6, 13].

**Lemma 2.1.** Let $f \in L^1(\mathbb{R})$. Then the following conditions are equivalent:

(i) $f \in H^1(\mathbb{R})$.

(ii) $H(f) \in L^1(\mathbb{R})$.

(iii) $M_P f \in L^1(\mathbb{R})$.

Moreover, in that case,

$$\|f\|_{H^1(\mathbb{R})} \sim \|f\|_{L^1(\mathbb{R})} + \|H(f)\|_{L^1(\mathbb{R})} \sim \|M_P f\|_{L^1(\mathbb{R})}.$$ 

**Lemma 2.2.** Let $F \in H^1_0(\mathbb{C}_+)$. Then the boundary value function $f$ of $F$, which is defined by

$$f(x) = \lim_{y \to 0} F(x+iy), \quad \text{a.e. } x \in \mathbb{R},$$

is in $H^1(\mathbb{R})$. Moreover,

$$\|f\|_{H^1(\mathbb{R})} \sim \|f\|_{L^1(\mathbb{R})} = \|F\|_{H^1_0(\mathbb{C}_+)}$$

and $F(x+iy) = P_y * f(x)$ for all $x+iy \in \mathbb{C}_+$.

In order to prove Theorem 1.1, we also need the following key lemma.

**Lemma 2.3.** Let $\varphi$ be a nonnegative function in $L^1(0, \infty)$. Then

(i) $\mathcal{H}_\varphi$ is bounded on $H^1(\mathbb{R})$, moreover,

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \leq \int_0^\infty \varphi(t)\,dt.$$ 

(ii) If $\text{supp } \varphi \subset [0, 1]$, then

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} = \int_0^1 \varphi(t)\,dt.$$ 

**Proof.** (i) For any $f \in H^1(\mathbb{R})$, by the Fubini theorem, we have

$$M_\Phi(\mathcal{H}_\varphi f)(x) = \sup_{r>0} \left| \int_{\mathbb{R}} dy \int_0^\infty \frac{1}{r} \Phi \left( \frac{x-y}{r} \right) f \left( \frac{y}{t} \right) \varphi(t) \frac{dt}{t} \right|$$

$$= \sup_{r>0} \left| \int_0^\infty \Phi_{r/t} * f \left( \frac{x}{t} \right) \varphi(t) \frac{dt}{t} \right|$$
\[ \leq \mathcal{H}_\phi(M_\phi f)(x) \]

for all \( x \in \mathbb{R} \). Hence,

\[
\| \mathcal{H}_\phi f \|_{H^1(\mathbb{R})} = \| M_\phi(\mathcal{H}_\phi f) \|_{L^1(\mathbb{R})} \\
\leq \int_{\mathbb{R}} dx \int_{0}^{\infty} M_\phi f \left( \frac{x}{t} \right) \frac{\varphi(t)}{t} dt \\
= \int_{0}^{\infty} \varphi(t) dt \| M_\phi(f) \|_{L^1(\mathbb{R})} = \int_{0}^{\infty} \varphi(t) dt \| f \|_{H^1(\mathbb{R})}.
\]

This proves that \( \mathcal{H}_\phi \) is bounded on \( H^1(\mathbb{R}) \), moreover,

\[ (2.1) \quad \| \mathcal{H}_\phi \|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \leq \int_{0}^{\infty} \varphi(t) dt. \]

(ii) Let \( \delta \in (0, 1) \) be arbitrary. By (2.1), we see that

\[
\| \mathcal{H}_{\phi_\delta} \|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \leq \int_{0}^{\infty} \varphi_\delta(t) dt = \int_{\delta}^{1} \varphi(t) dt < \infty
\]

and

\[ (2.2) \quad \| \mathcal{H}_\phi - \mathcal{H}_{\phi_\delta} \|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \leq \int_{0}^{\infty} [\varphi(t) - \varphi_\delta(t)] dt = \int_{0}^{\delta} \varphi(t) dt < \infty, \]

where \( \varphi_\delta(t) := \varphi(t) \chi_{[\delta,1]}(t) \) for all \( t \in (0, \infty) \).

For any \( \varepsilon > 0 \), we define the function \( F_\varepsilon : \mathbb{C} \to \mathbb{C} \) by

\[
F_\varepsilon(z) = \frac{1}{(z + i)^{1+\varepsilon}}
\]

where \( \zeta^{1+\varepsilon} = |\zeta|^{1+\varepsilon} e^{(1+\varepsilon) \arg \zeta} \) for all \( \zeta \in \mathbb{C} \). Then, by Lemma 2.2,

\[ (2.3) \quad \| f_\varepsilon \|_{H^1(\mathbb{R})} \sim \| F_\varepsilon \|_{H^1_\phi(\mathbb{C}^+)} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + 1}} dx < \infty, \]

where \( f_\varepsilon \) is the boundary value function of \( F_\varepsilon \).

For all \( z = x + iy \in \mathbb{C}^+ \), by the Fubini theorem and Lemma 2.2, we get

\[
P_y \ast \left( \mathcal{H}_{\phi_\delta}(f_\varepsilon) - f_\varepsilon \int_{0}^{\infty} \varphi_\delta(t) dt \right)(x)
\]
\[
\begin{align*}
&= \int_0^\infty \frac{1}{(\xi + i)^{1+\varepsilon}} \frac{\varphi(t)}{t} dt - \frac{1}{(\xi + i)^{1+\varepsilon}} \int_0^\infty \varphi(t) dt \\
&= \int_\delta^1 (\phi_{\varepsilon,z}(t) - \phi_{\varepsilon,z}(1)) \varphi(t) dt,
\end{align*}
\]
where \(\phi_{\varepsilon,z}(t) := \frac{e^{\varepsilon\sqrt{x^2 + 1}}}{(\varepsilon + i t)^{1+\varepsilon}}\). For any \(t \in [\delta, 1]\), a simple calculus gives
\[
|\phi_{\varepsilon,z}(t) - \phi_{\varepsilon,z}(1)| \leq |t - 1| \sup_{s \in [\delta, 1]} |\phi_{\varepsilon,z}'(s)|
\]
\[
\leq \frac{\varepsilon \delta^{-2}}{\sqrt{x^2 + 1}} + \frac{(1 + \varepsilon) \delta^{-2}}{\sqrt{x^2 + 1}}.
\]
Therefore, by Lemma 2.1,
\[
\left\| \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty \varphi_\delta(t) dt \right\|_{H^1(\mathbb{R})} \lesssim \left\| M_P \left( \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty \varphi_\delta(t) dt \right) \right\|_{L^1(\mathbb{R})}
\]
\[
\leq \int_\delta^1 \varphi(t) dt \int_{-\infty}^\infty \left[ \frac{\varepsilon \delta^{-2}}{\sqrt{x^2 + 1}} + \frac{(1 + \varepsilon) \delta^{-2}}{\sqrt{x^2 + 1}} \right] dx.
\]
This, together with (2.3), yields
\[
\left\| \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty \varphi_\delta(t) dt \right\|_{H^1(\mathbb{R})} \lesssim \frac{1}{\| f_\varepsilon \|_{H^1(\mathbb{R})}} \int_\delta^1 \varphi(t) dt \left[ \varepsilon \delta^{-2} + \frac{(1 + \varepsilon) \delta^{-2}}{\int_{-\infty}^\infty \frac{1}{\sqrt{x^2 + 1}} dx} \right] \to 0
\]
as \(\varepsilon \to 0\). As a consequence,
\[
\int_\delta^1 \varphi(t) dt = \int_0^\infty \varphi_\delta(t) dt \leq \left\| \mathcal{H}_{\varphi_\delta} \right\|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})}.
\]
This, combined with (2.2), allows us to conclude that
\[
\left\| \mathcal{H}_{\varphi} \right\|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \geq \int_0^1 \varphi(t) dt - 2 \int_0^\delta \varphi(t) dt \to \int_0^1 \varphi(t) dt.
\]
as $\delta \to 0$ since $\int_0^1 \varphi(t) dt < \infty$. Hence, by (2.1),
\[ \|H\varphi\|_{H^1(\mathbb{R})} = \int_0^1 \varphi(t) dt. \]

Now we are ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.3
\[ (2.4) \quad \|H\varphi\|_{H^1(\mathbb{R})} \leq \int_0^\infty \varphi(t) dt. \]

For any $m > 0$, we define $\varphi_m(t) = \varphi(mt)\chi_{(0,1)}(t)$ for all $t \in (0, \infty)$. Then, by Lemma 2.3, we see that
\[ (2.5) \quad \|H\varphi_m - H\widehat{\varphi_m} \|_{H^1(\mathbb{R})} \leq \int_0^m \left[ \varphi(t) - \varphi_m \left( \frac{t}{m} \right) \right] dt = \int_m^\infty \varphi(t) dt < \infty. \]

Noting that
\[ \|f \left( \frac{\cdot}{m} \right) \|_{H^1(\mathbb{R})} = m \|f(\cdot)\|_{H^1(\mathbb{R})} \] and \[ H\varphi_m \left( \frac{\cdot}{m} \right) f = H\varphi_m f \left( \frac{\cdot}{m} \right) \]
for all $f \in H^1(\mathbb{R})$, Lemma 2.3 yields
\[ \|H\varphi_m \left( \frac{\cdot}{m} \right) \|_{H^1(\mathbb{R})} = m \|H\varphi_m \|_{H^1(\mathbb{R})} = m \int_0^1 \varphi(mt) dt = m \int_0^1 \varphi(t) dt. \]

Combining this with (2.5) allows us to conclude that
\[ \|H\varphi\|_{H^1(\mathbb{R})} \geq \int_0^\infty \varphi(t) dt - 2 \int_m^\infty \varphi(t) dt \to \int_0^\infty \varphi(t) dt \]
as $m \to \infty$ since $\int_0^\infty \varphi(t) dt < \infty$. Hence, by (2.4),
\[ \|H\varphi\|_{H^1(\mathbb{R})} = \int_0^\infty \varphi(t) dt, \]
which ends the proof of Theorem 1.1.
3. Appendix

The main purpose of this section is to show that the norm of the Hausdorff operator \( \mathcal{H}_\varphi \) in Theorem 1.1 \( (\int_0^\infty \varphi(t)dt) \) still holds even when one replaces \( \|f\|_{H^1(\mathbb{R})} = \|M_\varphi f\|_{L^1(\mathbb{R})} \) by some other equivalent norms on \( H^1(\mathbb{R}) \). Such norms can be defined via the nontangential maximal functions, atoms, the Hilbert transform, ... See Stein’s book \[13\].

Let \( \psi \) be a function in the Schwartz space \( S(\mathbb{R}) \) satisfying \( \int_\mathbb{R} \psi(x)dx \neq 0 \); or be the Poisson kernel \( P \) on \( \mathbb{R} \). Then, for \( f \in L^1(\mathbb{R}) \), we define the nontangential maximal function \( \mathcal{M}_\psi f \) of \( f \) by
\[
\mathcal{M}_\psi f(x) = \sup_{|x-y|<t} |\psi_t * f(y)|, \quad x \in \mathbb{R}.
\]

A function \( a \) is called an \( H^1 \)-atom related to the interval \( B \) if
- \( \text{supp } a \subset B \);
- \( \|a\|_{L^\infty(\mathbb{R})} \leq |B|^{-1} \);
- \( \int_\mathbb{R} a(x)dx = 0 \).

We define the Hardy space \( H^1_{at}(\mathbb{R}) \) as the space of functions \( f \in L^1(\mathbb{R}) \) which can be written as \( f = \sum_{j=1}^\infty \lambda_j a_j \) with \( a_j \)'s are \( H^1 \)-atoms and \( \lambda_j \)'s are complex numbers satisfying \( \sum_{j=1}^\infty |\lambda_j| < \infty \). The norm on \( H^1_{at}(\mathbb{R}) \) is then defined by
\[
\|f\|_{H^1_{at}(\mathbb{R})} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j \right\}.
\]

The following is classical and can be found in Stein’s book \[13\].

**Theorem 3.1.** Let \( f \in L^1(\mathbb{R}) \). Then the following conditions are equivalent:
(i) \( f \in H^1(\mathbb{R}) \).
(ii) \( \mathcal{M}_\psi f \in L^1(\mathbb{R}) \).
(iii) \( f \in H^1_{at}(\mathbb{R}) \).
(iv) \( H(f) \in L^1(\mathbb{R}) \).

Moreover, in that case,
\[
\|f\|_{H^1(\mathbb{R})} \sim \|\mathcal{M}_\psi f\|_{L^1(\mathbb{R})} \sim \|f\|_{H^1_{at}(\mathbb{R})} \sim \|f\|_{L^1(\mathbb{R})} + \|H(f)\|_{L^1(\mathbb{R})}.
\]

The main aim of this section is to establish the following.

**Theorem 3.2.** Let \( \varphi \) be a nonnegative function in \( L^1(0, \infty) \). Then
\[
\|\mathcal{H}_\varphi\|_{(H^1(\mathbb{R}), \|\cdot\|) \to (H^1(\mathbb{R}), \|\cdot\|)} = \int_0^\infty \varphi(t)dt,
\]
where \( \|\cdot\| \) is one of the four norms in Theorem 3.1.
Proof. By the proofs of Lemma 2.3 and Theorem 1.1, we see that
\[ \| \mathcal{H}_\varphi \|_{(H^1(\mathbb{R}), \| \cdot \|_*) \to (H^1(\mathbb{R}), \| \cdot \|_*)} \geq \int_0^\infty \varphi(t) dt \]
for any norm of the four norms in Theorem 3.1. So, it suffices to show (3.1)
\[ \| \mathcal{H}_\varphi \|_{(H^1(\mathbb{R}), \| \cdot \|_*) \to (H^1(\mathbb{R}), \| \cdot \|_*)} \leq \int_0^\infty \varphi(t) dt. \]

**Case 1:** \( \| f \|_* = \| f \|_{L^1(\mathbb{R})} + \| H(f) \|_{L^1(\mathbb{R})} \). For any \( f \in H^1(\mathbb{R}) \), we have
\[ \| \mathcal{H}_\varphi f \|_{L^1(\mathbb{R})} \leq \int_0^\infty \varphi(t) dt \| f \|_{L^1(\mathbb{R})} \]
and
\[ \| H(\mathcal{H}_\varphi f) \|_{L^1(\mathbb{R})} = \| \mathcal{H}_\varphi (H(f)) \|_{L^1(\mathbb{R})} \leq \int_0^\infty \varphi(t) dt \| H(f) \|_{L^1(\mathbb{R})} \]
by [8, Theorems 1 and 3]. This implies that (3.1) holds.

**Case 2:** \( \| f \|_* = \| f \|_{H^1_0(\mathbb{R})} \). Denote by \( BMO(\mathbb{R}) \) the John-Nirenberg space (see [13]) with the norm
\[ \| g \|_{BMO} := \sup_{B} \frac{1}{|B|} \int_B \left| g(x) - \frac{1}{|B|} \int_B g(y) dy \right| < \infty, \]
where the supremum is taken over all intervals \( B \subset \mathbb{R} \). It is well-known that \( BMO(\mathbb{R}) \) is the dual space of \( H^1(\mathbb{R}) \), moreover,
\[ \| g \|_{BMO} = \sup_{\| f \|_* \leq 1} \left| \int_{\mathbb{R}} f(x) g(x) dx \right|, \]
where the supremum is taken over all functions \( f \in H^1(\mathbb{R}) \) with \( \| f \|_* \leq 1 \). Therefore, by [1, Theorem 3] and a standard functional analysis argument,
\[ \| \mathcal{H}_\varphi \|_{(H^1(\mathbb{R}), \| \cdot \|_*) \to (H^1(\mathbb{R}), \| \cdot \|_*)} = \| \mathcal{H}_\varphi^* \|_{BMO \to BMO} \leq \int_0^\infty \varphi(t) dt, \]
where \( \mathcal{H}_\varphi^* \) is the conjugated operator of \( \mathcal{H}_\varphi \) defined on \( BMO(\mathbb{R}) \) by
\[ \mathcal{H}_\varphi^* g(x) := \int_0^\infty g(tx) \varphi(t) dt, \quad x \in \mathbb{R}. \]
Case 3: \( \|f\|_* = \|M_\psi f\|_{L^1(\mathbb{R})} \). For any \( f \in H^1(\mathbb{R}) \), the Fubini theorem gives
\[
M_\psi(H_\varphi f)(x) = \sup_{|y-x|<r} \left| \int_{\mathbb{R}} \frac{1}{r} \psi \left( \frac{y-z}{r} \right) f \left( \frac{z}{t} \right) \frac{\varphi(t)}{t} dt \right|
\]
\[
= \sup_{|y-x|<r} \left| \int_{0}^{\infty} \psi_{r/t} * f \left( \frac{y}{t} \right) \frac{\varphi(t)}{t} dt \right|
\]
\[
\leq H_\varphi(M_\psi f)(x)
\]
for all \( x \in \mathbb{R} \). Hence,
\[
\|H_\varphi f\|_* = \|M_\psi(H_\varphi f)\|_{L^1(\mathbb{R})}
\]
\[
\leq \|H_\varphi\|_{L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})} \|M_\psi f\|_{L^1(\mathbb{R})} = \int_{0}^{\infty} \varphi(t) dt \|f\|_*,
\]
which implies that (3.1) holds, and thus ends the proof of Theorem 3.2.

Finally, we give a new proof for a known result (see [4, Theorem 1.2]).

**Theorem 3.3.** Let \( \varphi \) be a nonnegative function in \( L^1_{\text{loc}}(0, \infty) \) satisfying that \( H_\varphi \) is bounded on \( H^1(\mathbb{R}) \). Then \( \varphi \in L^1(0, \infty) \).

**Proof.** By Lemma 2.1, the following function
\[
f(x) = \frac{x}{(x^2 + 1)^2}, \quad x \in \mathbb{R},
\]
is in \( H^1(\mathbb{R}) \) since \( f(x) \in L^1(\mathbb{R}) \) and \( H(f)(x) = \frac{x^2 - 1}{2(x^2 + 1)} \in L^1(\mathbb{R}) \). Hence,
\[
H_\varphi f(x) = \int_{0}^{\infty} \frac{\frac{z}{t}}{[(\frac{z}{t})^2 + 1]^2} \frac{\varphi(t)}{t} dt
\]
is in \( H^1(\mathbb{R}) \) since \( H_\varphi \) is bounded on \( H^1(\mathbb{R}) \). As a consequence,
\[
\int_{0}^{\infty} \frac{y}{(y^2 + 1)^2} dy \int_{0}^{\infty} \varphi(t) dt = \int_{0}^{\infty} dx \int_{0}^{\infty} \frac{\frac{x}{t}}{[(\frac{x}{t})^2 + 1]^2} \frac{\varphi(t)}{t} dt
\]
\[
\leq \|H_\varphi f\|_{L^1(\mathbb{R})} \lesssim \|H_\varphi f\|_{H^1(\mathbb{R})} < \infty,
\]
this implies that \( \varphi \in L^1(0, \infty) \).

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