Perturbed Self-Similar Massless Scalar Field in Spherically Symmetric Spaceimes

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Abstract

In this paper, we investigate the linear perturbations of the spherically symmetric spacetimes with kinematic self-similarity of the second kind. The massless scalar field equations are solved which yield the background and an exact solutions for the perturbed equations. We discuss the boundary conditions of the resulting perturbed solutions. The possible perturbation modes turn out to be stable as well as unstable. The analysis leads to the conclusion that there does not exist any critical solution.

Keyword: Linear Perturbations, Self-Similar Solutions

1 Introduction

In General Relativity, gravitational collapse of a realistic body is one of the important problems. At the threshold of black hole formation, the matter is just about to form a black hole and an infinitesimally small perturbation can either cause the matter to disperse to infinity or to form a black hole. The dynamics close to the threshold exhibits interesting behavior such as power law scaling of length scales, self-similarity of the solutions and universality.

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typical setup is to take an initial matter distribution parameterized by a single parameter. The critical solution is self-similar, i.e., it repeats itself on ever decreasing length scales. The fluid density and pressure increase by many order of magnitudes and in some cases, the fluid velocity becomes extremely relativistic. Another way of investigating the critical collapse is to use the fact that the critical solution is continuously self-similar. Self-similarity leads to the set of partial differential equations into a set of ordinary differential equations that can then be solved quite easily numerically to a very high precision. One then performs a perturbation analysis around the critical solution to determine the scaling exponent.

Critical phenomena in gravitational collapse were discovered by Choptuik [1,2] in the spherically symmetric collapse of massless scalar field. Since then the phenomenon has been observed in a variety of matter sources. There are two steps to find critical solutions: Firstly, find a generic family (or families) of solutions, defined by a parameter, say $p$ such that when $p > p^*$ the collapse forms black holes and when $p < p^*$, it does not. Secondly, the perturbations of the solution $p = p^*$ are performed to investigate the spectrum of their modes. If the solution has only one unstable mode, then this solution is a critical solution and the exponent $\gamma$ is given by

$$\gamma = \frac{1}{|\sigma|}, \quad (1)$$

where $\sigma$ is the unstable mode [3].

Garfinkle [4] found a class, say, $S[n]$, of exact solutions to the Einstein massless scalar field equations in (2 + 1)-dimensions. He showed that in the strong field regime the $n = 4$ solution fits very well with the numerical critical solution found by Pretorious and Choptuik [5]. Later, Garfinkle and Gundlach [6] studied their linear perturbations and found that only the solution with $n = 2$ has one unstable mode, while the one with $n = 4$ has three. They further required that no matter field should come out of the already formed black holes. This additional condition seems physically quite reasonable and has been widely used in the investigation of black hole perturbations [7]. Hirschmann et al. [8] systematically studied the critical gravitational collapse of a scalar field. They surveyed all the analytic, continuously self-similar solutions and also examined their perturbations considering their global structure. Clement and Fabbri [9] investigated analytical treatment of critical collapse in 2 + 1-dimensional AdS spacetime. Cavaglia et.al. [10] analysed approximately self-similar critical collapse in 2 + 1-dimensions.
Miguelote et al. [11] studied the gravitational collapse of self-similar perfect fluid in 2 + 1-dimensional spacetimes with circular symmetry. They also studied the linear perturbations of homothetic self-similar stiff fluid solutions. Frolov [12] studied the perturbations of the continuously self-similar critical solution of the gravitational collapse of a massless scalar field. Brandt et al. [13] studied the gravitational collapse of spherically symmetric perfect fluid with kinematic self-similarity. They also studied the solutions of the Einstein field equations found by Benoit and Coley [14] and concluded that some of the solutions represent gravitational collapse. Wang [15] studied the critical collapse of a cylindrical symmetric scalar field. He introduced the notion of homothetic self-similarity to four-dimensional spacetimes and then presented a class of exact solutions to the Einstein massless scalar field equations. Wang et al. [16] studied plane symmetric self-similar solutions to Einstein’s four-dimensional theory of gravity and explored the local and gravitational conditions.

Recently, Chan et al. [17] investigated the solution of the Einstein massless scalar field equations with kinematic self-similarity of the second kind in the (2+1)-dimensional spacetimes with circular symmetry. They discussed their local and global properties and also found that some of these solutions represent gravitational collapse of the scalar field. The same authors [18] studied the linear perturbations of the (2+1)-dimensional circularly symmetric solution with kinematic self-similarity of the second kind. They obtained an exact solution for the perturbation equations and the possible perturbation modes and showed that the background solution is a stable.

In this paper, we investigate the linear perturbations of the spherically symmetric spacetimes with kinematic self-similarity of the second kind. We analyze these solutions to see whether the solution is stable or not. The possible perturbation modes are also discussed. The paper has been organized as follows. In the next section, we shall consider the general spherically symmetric spacetimes and calculate self-similar variable of the second kind. Section 3 is devoted to the linear perturbation of the field equations. In sections 4 and 5, we find the possible solutions of the linear perturbation equations and check the boundary conditions for the perturbed solutions respectively. Section 6 is focused for discussion of the results.
2 Self-Similarity of the Second Kind

In this section, we calculate the self-similar variable of the second kind for spherically symmetric spacetimes. Further, we write down the non-zero components of the Ricci tensor in terms of the self-similar variable. The most general form of the spherically symmetric metric is given by [19]

$$ds^2 = e^{2\Phi(t,r)}dt^2 - e^{2\Psi(t,r)}dr^2 - r^2 S^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (2)

where $\Phi(t,r)$, $\Psi(t,r)$, $S(t,r)$ are arbitrary functions of $t, r$. The non-zero components of the Ricci tensor are

$$R_{00} = e^{2(\Phi-\Psi)}\left[\Phi_r(\Phi_r - \Psi_r + \frac{2S_r}{S} + \frac{2}{r}) + \Phi_{r r}\right] - \frac{2S_{tt}}{S} + \frac{2\Phi_t S_t}{S} + \Phi_t \Psi_t - \Psi_t^2 - \Psi_{tt},$$  \hspace{1cm} (3)

$$R_{01} = \frac{2\Psi_t}{r} + \frac{2\Phi_t S_t}{S} + \frac{2\Phi_r S_r}{S} - \frac{2S_t}{r S} - \frac{2S_{rt}}{S},$$  \hspace{1cm} (4)

$$R_{11} = e^{2(\Psi-\Phi)}\left[\Psi_{tt} + \Psi_t(\Psi_t + \frac{2S_t}{S} - \Phi_t)\right] - \Phi_{rr} + \Phi_r \Psi_r - \Phi_r^2 - \frac{2S_{rr}}{S} - \frac{4S_r}{r S} + \frac{2\Psi_r S_r}{S} + \frac{2\Psi_r}{r},$$  \hspace{1cm} (5)

$$R_{22} = e^{-2\Phi}(r^2 S_t^2 + r^2 SS_{tt} - r^2 SS_t \Phi_t + r^2 SS_t \Psi_t) - 4r SS_r - r^2 S_r^2 \Phi_r - r^2 S_r^2 \Psi_r + 1,$$  \hspace{1cm} (6)

$$R_{33} = R_{22} \sin^2 \theta,$$  \hspace{1cm} (7)

where the subscripts $t$ and $r$ mean differentiation w.r.t $t$ and $r$ respectively.

We consider here solutions with kinematic self-similarity of the second kind for which the self-similar variable $x$ turns out to be

$$x = \ln\left(\frac{r}{(-t)^{\frac{1}{\alpha}}}\right), \quad \tau = -\ln(-t),$$  \hspace{1cm} (8)

or inversely

$$r = e^{\frac{\alpha x - \tau}{\alpha}}, \quad t = -e^{-\tau},$$  \hspace{1cm} (9)

where $\alpha$ is a dimensionless constant. The components of the Ricci tensor in terms of the self-similar variable can be written as follows

$$R_{00} = \frac{e^{2(\Phi-\Psi)}}{r^2}\left\{\Phi_x(\Phi_x - \Psi_x + \frac{2S_x}{S} + 1) + \Phi_{xx}\right\} - \frac{1}{\alpha^2 t^2}\left\{\alpha^2[\Psi_{\tau \tau}\right\}$$
\[ R_{01} = -\frac{2}{\alpha r} \{ \alpha [\Psi (1 + S_x) + S_x (\Phi - 1) - \frac{S_{xx}}{S}] \\
+ \Psi (1 + \frac{S_x}{S}) + \frac{S_x}{S} (\Phi - 1) - \frac{S_{xx}}{S} \}, \tag{10} \]
\[ R_{11} = \frac{e^{2(\psi - \Phi)}}{\alpha^2 r^2} \{ \alpha^2 [\Psi + \Psi - \Phi + 2 \frac{S_x}{S}] \\
+ \alpha [2 \Psi + \Psi (1 + 2 \Psi - \Phi + 2 \frac{S_x}{S}) + \Psi (\Phi + 2 \frac{S_x}{S})] \\
+ \Psi + \Psi (\Psi - \Phi + 2 \frac{S_x}{S}) \} + 1 \frac{1}{r^2} \{ \Psi (\Psi - \Phi + 1) \\
- \Phi + 2 \Psi (1 + \frac{S_x}{S}) - 2 \frac{S}{S} (S_{xx} + S_x) \}, \tag{11} \]
\[ R_{22} = r^2 S^2 \left( \frac{e^{-2\Phi}}{\alpha^2 r^2 S} [\alpha^2 (S_x (1 + \Psi - \Phi + \frac{S_x}{S}) + S_{xx})] \\
+ \alpha (S_x (\Psi - \Phi + 2 \frac{S_x}{S}) + S_x (1 + \Psi - \Phi) + 2 S_{xx}) \\
+ S_x (\Psi - \Phi + \frac{S_x}{S}) + S_{xx} \right] - \frac{e^{-2\Psi}}{r^2} \frac{1}{S} (S_x (3 + \Phi_x \\
- \Psi + \frac{S_x}{S}) + S_{xx} + \Phi_x - \Psi + 1) \} + 1, \tag{13} \]
\[ R_{33} = R_{22} \sin^2 \theta. \tag{14} \]

## 3 Linear Perturbation of the Field Equations

This section is devoted to set up the Einstein field equations by using linear perturbation. For this purpose, we take

\[
\begin{align*}
\Phi(\tau, x) &= \Phi_0(x) + \epsilon \Phi_1(x) e^{k\tau}, \\
\Psi(\tau, x) &= \Psi_0(x) + \epsilon \Psi_1(x) e^{k\tau}, \\
S(\tau, x) &= S_0(x) + \epsilon S_1(x) e^{k\tau}, \\
\phi(\tau, x) &= \phi_0(x) + \epsilon \phi_1(x) e^{k\tau},
\end{align*}
\tag{15}
\]
where \( \epsilon \) is a very small real constant and \( k \) is an arbitrary constant. The quantities with subscripts 0 and 1 denote background self-similar solutions and perturbations respectively. It is understood that there may be many perturbation modes for different values (possibly complex) of the constant \( k \). The general perturbation will be the sum of these individual modes. The modes with \( \text{Re}(k) > 0 \) grow as \( \tau \to \infty \) and are referred to as unstable modes while the ones with \( \text{Re}(k) < 0 \) decay and are referred to as stable modes. By definition, critical solutions will have one and only one unstable mode.

We take the following background solution \cite{18}

\[
\begin{align*}
\Phi_0(x) &= 0, \quad \Psi_0(x) = -\frac{1}{2} \alpha x, \\
S_0(x) &= \frac{2}{2 - \alpha} e^{-\frac{4}{\alpha} x}, \quad \phi_0(x) = 2q \ln(-t) \tag{16}
\end{align*}
\]

and the apparent horizon is given by

\[
\begin{align*}
r_{AH}(t) &= [(2 - \alpha)\sqrt{-t}]^{\frac{2}{z-\alpha}}, \quad \alpha < 2, \tag{17}
\end{align*}
\]

where \( q = \pm \frac{1}{\sqrt{8}} \). For a massless scalar field \( \phi \), the Einstein field equations are

\[
\begin{align*}
R_{ab} &= \kappa \phi_a \phi_b, \quad a, b = 0, 1, 2. \tag{18}
\end{align*}
\]

Here we choose units such that \( \kappa = 1 \) for the sake of simplicity. Using Eq.(15) in Eqs.(10)-(14), it follows that

\[
R_{ab} = R_{ab}(\tau, x, \epsilon) \tag{19}
\]

If we take \( R_{ab} \) as a function of \( \epsilon \) only and expand it in terms of \( \epsilon \), it follows that

\[
R_{ab}(\tau, x, \epsilon) = \frac{1}{(-t)^2} \{ R_{ab}(0)(x) + \epsilon R_{ab}^{(1)}(x) e^{k\tau} + O(\epsilon^2) \}. \tag{20}
\]

The non-vanishing components of the Ricci tensor up to first order in \( \epsilon \) are

\[
\begin{align*}
R_{00}^{(1)}(x) &= e^{2(\Phi_0 - \Psi_0 - x + \frac{5}{2})} \{ \Phi'_0(2\Phi'_1 - \Psi'_1) - \Phi'_1 \Psi'_0 + \Phi''_0 \\
&+ \Phi'_1 + 2(\Phi_1 - \Psi_1)[\Phi'_0(\Phi'_0 - \Psi'_0) + \Phi'_0 + \Phi'_0] + \frac{1}{S_0}[\Phi'_0 S'_0 S_1 + \Phi'_1 S'_0 S_1] \\
&+ 4(\Phi_1 - \Psi_1) \Phi'_0 S'_0 + 2\Phi'_0 S'_1 + 2\Phi'_1 S'_0 \} + \frac{e^{2r}}{\alpha^2} \{ \Phi'_0(\alpha k \Psi_1 \tag{21}
\end{align*}
\]

\[
\begin{align*}
&+ \Psi'_1 + \Psi'_0(\alpha k \Phi_1 + \Phi'_1) - 2\Psi'_0(\alpha k \Psi_1 + \Psi'_1) - \alpha^2 k^2 \Psi_1
\end{align*}
\]
\[-2\alpha k \Psi' - \Psi'' - \alpha^2 k^2 \Psi_1 - \alpha \Psi' + \frac{1}{S_0} [2 \Phi'_0 (\alpha k S_1 + S'_1) + 2 S'_0 (\alpha k \Phi_1 + \Phi'_1) - 2 \alpha^2 k^2 S_1 - 2 \alpha^2 k S_1 - 4 \alpha k S'_1 - 2 S''_1] - 2 \alpha^2 k S_1 - 2 \alpha S'_1 - \frac{S_1}{S_0} (2 S'_0 \Phi'_0 - 2 S''_0 - 2 \alpha S'_0) \}, \tag{21}

\begin{align*}
R_{01}^{(1)}(x) &= -2 e^{\frac{\alpha k \tau - x}{\alpha S_0}} \left[ - \frac{S_1}{S_0} (S'_0 - \Phi'_0 S'_0 - \Psi'_0 S'_0 + S''_0) \\
&- \Phi'_0 (\alpha k S_1 + S'_1) - \Psi'_0 S'_1 - S'_0 (\alpha k \Psi_1 + \Psi'_1) + \Phi_k S'_1 + S''_1 - S_0 (\alpha k \Psi_1 + \Phi'_1) + \alpha k S_1 + S'_1 \right], \tag{22}

R_{11}^{(1)}(x) &= \frac{e^{2(\Psi_0 - \Phi_0 + \tau)}}{\alpha^2} \left[ 2 (\Psi_1 - \Phi_1) (\Psi'_0 + \Psi'' + \alpha \Psi'_0) \\
&- \Phi'_0 \Psi'_0 + 2 \frac{\Psi'_0 S'_0}{S_0} + 2 \alpha k \Psi'_0 \Psi_1 + 2 \Psi'_0 \Psi'_1 + \alpha^2 k^2 \Psi_1 \\
&+ 2 \alpha k \Psi'_1 + \Psi'' + \alpha^2 k \Psi_1 + \alpha \Psi'_1 - \alpha k \Phi'_0 \Psi_1 - \Phi'_0 \Psi'_1 \\
&- \alpha k \Phi_1 \Psi'_0 - \Phi'_1 \Psi'_0 + S'_0 (2 \alpha k S_1 \Psi'_0 + S_0 S'_1 \Psi'_0) \\
&+ \alpha k S_0 S'_0 (\Psi_1 + S_0 \Psi'_0 - S' S_0 \Psi'_0)] - e^{2(\Psi - x)} [-\Phi'_0 \Psi'_0 \\
&+ \Phi'_1 - 2 \Phi'_0 \Phi'_1 - \Phi'_0 \Psi'_1 - \frac{2}{S_0} (S'_1 \Psi'_0 + S_0 \Psi'_1) \\
&+ S_0 \Psi'_1 - S''_1 - S'_1 + 2 \frac{S_1}{S_0} (S'_0 \Psi'_0 - S''_0 - S'_0) \right], \tag{23}

R_{22}^{(1)}(x) &= -e^{-2 \Psi_0} \left[ 2 (S_1 - S_0 \Psi_1) (\Phi'_0 S'_0 + \Phi'_0 S_0 - \Psi'_0 S'_0 - \Psi'_0 S_0) \\
&+ S''_0 + 3 S'_0 + S_0 + \frac{S_3^2}{S_0} + \Phi'_0 S_0 S'_1 - \Psi'_0 S_0 S'_1 + 2 S'_0 S'_1 \\
&+ 3 S_0 S'_1 - \Phi'_0 S'_0 S_1 - \Psi'_0 S_0 S_1 - 3 S_0 S_1 + \Phi'_1 S_0 S_0' \\
&- \Psi'_0 S_0 S'_0 + S_0 S'_1 - S''_0 S_1 + \Phi'_1 S'_0 - \Psi'_0 S'_0 - \frac{S_3^2 S'_1}{S_0} \right] + e^{2(\frac{\alpha k \tau + x}{\alpha} - \Phi_0)} \left[ \alpha k (\alpha S_0 S_1 + \alpha k S_0 S_1 + 2 S'_0 S_1 - \Phi'_0 S_0 S_1 \\
&+ S_0 (\Psi'_0 S_1 - \Phi'_1 S'_0 + \Psi'_1 S'_0 + 2 S'_1)) + \alpha S_0 S'_1 + 2 S'_0 S'_1 \\
&+ S_0 S'_0 \Phi'_0 + S_0 S'_0 \Psi'_0 - S_0 S'_0 \Phi'_1 + S_0 S'_0 - \frac{S_3^2 S'_1}{S_0} \right] + (S_1 - 2 \Phi_1 S_0) (S'_0 + S'_0 \Phi'_0 + S''_0 + \frac{S_3^2 S'_0}{S_0}) \}, \tag{24}

R_{33}^{(1)}(x) &= R_{22}^{(1)}(x) \sin^2 \theta. \tag{25}
Now we can calculate the quantities $A_{ab} \equiv \phi_a \phi_b$ using Eq.(15) in Eqs.(10)-(14). Thus

$$A_{ab}(\tau, x, \epsilon) = \frac{1}{(-t)^2} \{ A_{ab}^{(0)}(x) + \epsilon A_{ab}^{(1)}(x)e^{k\tau} + O(\epsilon^2) \}. \quad (26)$$

The perturbed part is given by

$$A_{00}^{(1)}(x) = -\frac{e^{2\tau}}{\alpha} [4q(\alpha k \phi_1 + \phi'_1)],$$
$$A_{01}^{(1)}(x) = -e^{\frac{\alpha + 1}{\alpha} \tau-x} (2q \phi'_1),$$
$$A_{11}^{(1)}(x) = 0,$$
$$A_{22}^{(1)}(x) = 0,$$
$$A_{33}^{(1)}(x) = 0. \quad (27)$$

The linear perturbation equations can be written as

$$R_{ab}^{(1)}(x) = A_{ab}^{(1)}(x). \quad (28)$$

Using Eqs.(27) and (28), it turns out that

$$4\alpha q(\alpha k \phi_1 + \phi'_1) = \frac{3}{2} \alpha^2 k \Phi_1 + \frac{3}{2} \alpha \Phi'_1 + \alpha^2 k^2 \Psi_1 + 2\alpha k \Psi'_1 + \Psi''_1 + \frac{2}{4} e^{\frac{1}{2} \alpha x} [(2k^2 + 2k + \frac{1}{2}) \alpha^2 S_1 + 2\alpha(2k + 1) S'_1 + 2S''_1], \quad (29)$$
$$0 = \Phi'_1, \quad (30)$$
$$2\alpha q \phi'_1 = (\alpha - 2)(\alpha k \Psi_1 + \Psi'_1) + \alpha \Phi'_1 + \frac{2}{2} e^{\frac{1}{2} \alpha x} [\alpha(2k + 1) S_1 + (2\alpha k + \alpha + 2) S'_1 + 2S''_1], \quad (31)$$
$$0 = 2\alpha^2 k(k - 1) \Psi_1 + 2\alpha(2k - 1) \Psi'_1 + 2\Psi''_1 + \alpha(\alpha k \Phi_1 + \Phi'_1) + \alpha^2 (\Psi_1 - \Phi_1) - \frac{\alpha(2 - \alpha)}{2} e^{\frac{1}{2} \alpha x} [\alpha(2k + 1) S_1 + 2S'_1], \quad (32)$$
$$0 = 2(2 - \alpha) \Psi'_1 + (\alpha - 2) \Phi'_1 + \Phi''_1 - (2 - \alpha) e^{\frac{1}{2} \alpha x} [\alpha S_1 + (\alpha + 2) S'_1 + 2S''_1], \quad (33)$$
\[
0 = \alpha (\alpha k \Phi_1 + \Phi_1') - \alpha (\alpha k \Psi_1 + \Psi_1') \\
- \alpha^2 \Phi_1 - \frac{2 - \alpha}{2} e^{\frac{1}{2} \alpha x} [\alpha^2 k (2k - 1) S_1] \\
+ \alpha (4k - 1) S_1' + S_1''], \quad (34)
\]
\[
0 = (\alpha - 2)(\Psi_1' - \Phi_1') - \Psi_1 (\alpha^2 - 4 \alpha) \\
+ 4) - \frac{2 - \alpha}{2} e^{\frac{1}{2} \alpha x} [(4 - \alpha) S_1] \\
+ (6 - \alpha) S_1' + 2S_1'']. \quad (35)
\]

4 Solutions of the Linear Perturbation Equations

Now we solve system of the perturbed Eqs.(29)-(35). From Eq.(30), we have
\[
\Phi_1 = b, \quad (36)
\]
where \(b\) is an integration constant. Multiplying Eq.(35) by 2, adding in Eq.(33) and using Eq.(30), we obtain
\[
(\alpha - 2) \Psi_1 + 2e^{\frac{1}{2} \alpha x} [S_1 + 2S_1' + S_1''] = 0. \quad (37)
\]
Using Eq.(36) in Eq.(34), it becomes
\[
0 = \alpha^2 (k - 1) b - \alpha \Psi_1' - \alpha^2 k \Psi_1 \\
+ \frac{2 - \alpha}{2} e^{\frac{1}{2} \alpha x} [\alpha^2 k (2k - 1) S_1 + \alpha (4k - 1) S_1' + S_1'']. \quad (38)
\]
From Eq.(37), we have
\[
\Psi_1 = \frac{2}{2 - \alpha} e^{\frac{1}{2} \alpha x} [S_1 + 2S_1' + S_1'']. \quad (39)
\]
Eliminating \(\Psi_1\) and \(\Psi_1'\) from Eqs.(38) and (39), it turns out that
\[
AS_1 + BS_1' + CS_1'' + DS_1''' + E e^{-\frac{1}{2} \alpha x} = 0, \quad (40)
\]
where
\[
A = 2\alpha^2 (2k + 1) - \alpha^2 k (2 - \alpha)^3 (2k - 1), \quad (41)
\]
\[
B = 2\alpha (4\alpha k + 2\alpha + 2) - \alpha (4k - 1)(2 - \alpha)^3, \quad (42)
\]
\[
C = 2\alpha (2\alpha k + \alpha + 4) - (2 - \alpha)^3, \quad (43)
\]
\[
D = 4\alpha, \quad (44)
\]
\[
E = -2b\alpha^2 (k - 1)(2 - \alpha). \quad (45)
\]
It can be found that the solution with self-similarity of the second kind is identical to the solution with first kind for the same type of fluid. This means that the spacetime can have two different kinds of self-similarities, i.e., there exist two vector fields $\xi^\mu_1$, $\xi^\mu_2$, where $\xi^\mu_1$ describes self-similarity of the first kind and $\xi^\mu_2$ of the second kind. This happens when the spacetimes has high symmetry. As the background solution with self-similarity of the second kind becomes identical to the solution with self-similarity of the first kind, we take $\alpha = 1$. Thus Eqs.(41)-(45) become

$$
A = -2k^2 + 5k + 2, \quad (46)
$$
$$
B = 4k + 9, \quad (47)
$$
$$
C = 4k + 9, \quad (48)
$$
$$
D = 4, \quad (49)
$$
$$
E = -2bk + 2b \quad (50)
$$

and Eq.(40) takes the form

$$
(-2k^2 + 5k + 2)S_1 + (4k + 9)S_1' + (4k + 9)S_1'' + 4S_1'' + (-2bk + 2b)e^{-\frac{1}{2}x} = 0. \quad (51)
$$

This has the following solution

$$
S_1(x) = c_1 e^{Ux} - \frac{8b(k - 1)}{8k^2 - 16k + 3} e^{-\frac{1}{2}x}, \quad (52)
$$

where $c_1$ is an integration constant and

$$
U = \frac{1}{12} P - \frac{B}{12A} + \frac{1}{12} A, \quad (53)
$$
$$
P = -9 - 4k, \quad (54)
$$
$$
B = 27 - 24k - 16k^2, \quad (55)
$$
$$
A = \sqrt{[297 - 756k + 288k^2 - 64k^3 + 6\sqrt{3}\sqrt{\Delta}]^\frac{1}{2}}, \quad (56)
$$
$$
\Delta = 999 - 4644k + 6984k^2 - 3936k^3 + 1600k^4 - 512k^5. \quad (57)
$$

5 Boundary Conditions for the Perturbed Solutions

In this section, we shall discuss some geometrical and physical conditions [20] needed to be imposed for the spherically symmetry. For gravitational collapse, we impose the following conditions:
There must exist a symmetry axis which can be expressed by

\[ X \equiv \sqrt{|\xi^a_{(\theta)} \xi^b_{(\theta)} g_{ab}|} \rightarrow 0 \]  \hspace{1cm} (58)

as \( r \rightarrow 0 \), where we have chosen the radial coordinate such that the axis is located at \( r = 0 \) and \( \xi^a_{(\theta)} \) is a Killing vector with a close orbit which is given by \( \xi^a_{(\theta)} \partial_a = \partial_\theta \).

The spacetime near the symmetry axis is locally flat which can be written by [19]

\[ X_a X_b g^{ab} \rightarrow -1 \]  \hspace{1cm} (59)

as \( r \rightarrow 0 \). It is mentioned here that solutions failing to satisfy this condition may also be acceptable. Since we are mainly interested in gravitational collapse, we assume that this condition strictly holds at the beginning of the collapse so that we can be sure that the singularity to be founded later on the axis is due to the collapse.

Closed timelike curves can be easily introduced in spacetimes with spherical symmetry. For their absence the condition,

\[ \xi^a_{(\theta)} \xi^b_{(\theta)} g_{ab} < 0, \]  \hspace{1cm} (60)

must hold in the whole spacetime. In addition to these conditions, we also require that the spacetime is asymptotically flat in the radial direction. For self-similar solutions, this condition cannot be satisfied unless we restrict their validity only up to a maximal radius say, \( r = r_0(t) \), and join them with others in the region \( r > r_0(t) \) which are asymptotically flat as \( r \rightarrow \infty \). Here we simply assume that the self-similar solutions are valid in the whole spacetime.

The boundary conditions at the event horizon \( r_{AH} \), given by Eq.(17), take the following form

\[ r_{AH} = -t \]  \hspace{1cm} (61)

and the corresponding metric becomes

\[ ds^2_{AH} = -4(-t)^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  \hspace{1cm} (62)

This shows that the apparent horizon is singular only at \( t = 0 \) and the final state of the collapse is marginally naked singularity. Now we discuss the following three cases:

(\text{i}) \hspace{0.5cm} \Delta > 0, \hspace{1cm} (\text{ii}) \hspace{0.5cm} \Delta = 0, \hspace{1cm} (\text{iii}) \hspace{0.5cm} \Delta < 0.
Case (i): In this case, using Eq.(52) in (39), we have
\[ \Psi_1 = -\frac{4b(k - 1)}{8k^2 - 16k + 3} + 2(1 + 2U + U^2)C_1 e^{(U + \frac{1}{2})x}. \] (63)

For the first boundary condition, given by Eq.(58), we calculate the quantity \( \sqrt{X} = rS_1 \) by using Eq.(8) in (52) so that
\[ S_1 = c_1 \left( \frac{r}{-t} \right)^U - \frac{8b(k - 1)}{8k^2 - 16k + 3} \left( \frac{r}{-t} \right)^{-\frac{1}{2}}, \] (64)
\[ rS_1 = c_1 \frac{r^{U+1}}{(-t)^U} - \frac{8b(k - 1)}{8k^2 - 16k + 3} (-rt)^{-\frac{1}{2}}. \] (65)

For \( rS_1 \to 0 \) as \( r \to 0 \), all the exponents of \( r \) must be greater than zero. It can be checked from Eqs.(53)-(56) that \( U + 1 \) is positive for \(-\infty < k < 1\) but turns out to be complex otherwise. Thus the first condition is satisfied for \(-\infty < k < 1\) but is not fulfilled otherwise. Consequently, these perturbations are limited by the boundary conditions for \( 1 < k < \infty \).

Case (ii): For \( \Delta = 0 \), \( k \to 1.70872 \) along with Eqs.(52) and (39), we have
\[ \Psi_1 = 2.8876b + 0.2043e^{-0.8196x}. \] (66)

The boundary conditions, given by Eqs.(58)-(60), can be applied by making use of Eq.(8) in Eq.(52)
\[ S_1 = \left( \frac{r}{-t} \right)^{-1.3196} + 5.7752b \left( \frac{r}{-t} \right)^{-\frac{1}{2}}, \] (67)
\[ rS_1 = r^{-0.3196}(-t)^{1.3196} + 5.7752br^{\frac{1}{2}}(-t)^{\frac{1}{2}}. \] (68)

Since all the exponents of \( r \) are not positive, the condition given by Eq.(58) is not satisfied. Again perturbations are limited by the boundary conditions.

Case (iii): Here we use Eq.(52) in Eq.(39) so that
\[ \Psi_1 = -\frac{4b(k - 1)}{8k^2 - 16k + 3} + (1 + U_1)c_1 (\cos U_2 + i \sin U_2) e^{U_3x}, \] (69)

where
\[ U_1 = \frac{1}{12}(-9 - 4k) - \frac{27 - 24k - 16k^2}{12(297 - 756k + 288k^2 - 64k^3 - 108\Delta)^{\frac{1}{2}}}. \]
\[ U_2 = \{ \frac{27 - 24k - 16k^2}{12(297 - 756k + 288k^2 - 64k^3 - 108\Delta)^{\frac{1}{3}}} \} x, \] (70)

\[ U_3 = \left\{ \frac{1}{12} (t - 4k) - \frac{27 - 24k - 16k^2}{12(297 - 756k + 288k^2 - 64k^3 - 108\Delta)^{\frac{1}{3}}} \right\} x. \] (71)

For the boundary condition, given by Eq.(58), we require that \( \sqrt{X} = rS_1 \) and \( S_1 \) is real only if \( c_1 = 0 \). Thus we obtain

\[ S_1 = -\frac{8b(k-1)}{8k^2-16k+3} \left( \frac{r}{-t} \right)^{\frac{1}{2}}, \] (73)

\[ rS_1 = -\frac{8b(k-1)}{8k^2-16k+3} (-rt)^{\frac{1}{2}}. \] (74)

Clearly, \( rS_1 \to 0 \) as \( r \to 0 \), the condition, given by Eq.(58), is satisfied only if \( k \neq 1 \pm \sqrt{\frac{3}{2}} \). For the second boundary condition, given by Eq.(59), we have

\[ X_aX_bg^{ab} = -\frac{8192b^4(k-1)^4r}{(8k^2-16k+3)^4} \left( br + \frac{\Psi_1}{t} \right). \] (75)

Thus \( X_aX_bg^{ab} \to 0 \) as \( r \to 0 \).
6 Concluding Remarks

We have investigated the linear perturbations of the spherically symmetric spacetimes with kinematic self-similarity of the second kind. The self-similar variable and the background solution found for the spherically symmetric spacetimes become identical to that of the circularly symmetric metric [18]. However, the linearly perturbed solution obtained for the spherically symmetric metric is different from the exact solution for the circularly symmetric metric. The boundary conditions for all possible values of $\Delta$ are discussed for this linearly perturbed solution.

For $\Delta > 0$ and $\Delta = 0$, the first boundary condition, given by Eq.(58), is not satisfied. This shows that the perturbations are limited by the boundary conditions in both these cases. For $\Delta < 0$, the boundary conditions are satisfied only for $k \neq 1 \pm \sqrt{5/8}$ which admits both stable and unstable modes for the perturbation. The stable modes are for $k < 0$ and the unstable modes are for $k > 0$ but $k \neq 1 \pm \sqrt{5/8}$. The unstable modes for the perturbation imply that it is not a critical solution.

Acknowledgment

The author would like to thank referee for the constructive comments.

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