Quantum Effect and Curvature Strength of Naked Singularities

Umpei MIYAMOTO,1,∗) Hideki MAEDA2,**) and Tomohiro HARADA3,***)

1Department of Physics, Waseda University, Tokyo 169-8555, Japan
2Advanced Research Institute for Science and Engineering, Waseda University, Tokyo 169-8555, Japan
3Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK

(Received November 20, 2004)

There are many solutions to the Einstein field equations that demonstrate naked singularity (NS) formation after regular evolution. It is possible, however, that such a quantum effect as particle creation prevents NSs from forming. We investigate the relation between the curvature strength and the quantum effects of NSs in a very wide class of spherical dust collapse. Through a perturbative calculation, we find that if the NS is very strong, the quantum particle creation diverges as the Cauchy horizon is approached, while if the NS is very weak, the creation should be finite. In the context of cosmic censorship, strong NSs will be subjected to the backreaction of quantum effects and may disappear or be hidden behind horizons, while weak NSs will not.

§1. Introduction

The cosmic censorship hypothesis (CCH) presents one of the most important unsolved problems in general relativity.1) Its validity is often assumed in the analysis of physical phenomena in strong gravitational fields. There are two versions of this hypothesis. The weak hypothesis states that all singularities in gravitational collapse are hidden within black holes. This version implies the future predictability of the spacetime outside the event horizon. The strong hypothesis asserts that no singularities visible to any observer can exist. This version states that all physically reasonable spacetimes are globally hyperbolic. Despite several attempts, neither a general proof nor a precise mathematical formulation of the hypothesis has yet been obtained. By contrast, some solutions of the Einstein field equation with regular initial conditions evolving into spacetimes containing naked singularities (NSs) have been found.2,3)

For a naked-singular spacetime to provide a counterexample of the CCH, it is at least necessary that the Cauchy horizon (CH) be stable. Although the CCH was originally stated in the classical context, CHs may be unstable due to the backreaction of quantum effects, such as particle creation, i.e., particle creation may prevent NSs from forming. Study of such a possibility can be traced back to the pioneering works of Ford and Parker4) and Hiscock, Williams and Eardley.5) Ford and Parker considered the particle creation during the formation of a shell-crossing NS to obtain

∗) E-mail: umpei@gravity.phys.waseda.ac.jp
**) E-mail: hideki@gravity.phys.waseda.ac.jp
***) E-mail: T.Harada@qmul.ac.uk
a finite amount of flux.\textsuperscript{4)} Hiscock et al. considered the formation of a shell-focusing NS in the collapse of a null-dust fluid to obtain a diverging amount of flux.\textsuperscript{5)} Subsequently, such quantum phenomena have been studied in models of a self-similar dust,\textsuperscript{6)–8)} a self-similar null dust,\textsuperscript{8), 9)} and an analytic dust\textsuperscript{10)} models, for which the luminosities are found to diverge as negative powers of the time remaining to the CHs. The analytic model describes the spherical dust collapse with an analytic initial density profile with respect to locally Cartesian coordinates. The analyticity of the initial density profile and the self-similarity are incompatible in the spherically symmetric dust model. It is argued that the quantum radiation from a strong NS, such as a shell-focusing one, must diverge as the CH is approached,\textsuperscript{6), 9)} although this has not yet been proved. See Ref. 11) for a recent review of the quantum and classical radiation processes during NS formation.

Recently, two of the present authors computed the quantum radiation in spherically symmetric self-similar collapse without specifying the collapsing matter.\textsuperscript{8)} It was found that in the generic class of self-similar spacetimes resulting in a NS, the luminosity of particle creation diverges as the inverse square of the time remaining to the CH. Moreover, there was another interesting result, leading us to the present study. In the self-similar collapse of a massless scalar field, described by the Roberts solution,\textsuperscript{12)} the luminosity remains finite at the CH. This result can be interpreted in terms of the curvature strength of the NS along the CH. Although NSs forming in generic spherically symmetric self-similar spacetimes are known\textsuperscript{13)} to satisfy the strong curvature condition (SCC)\textsuperscript{14)} along the CH, the NS appearing in the Roberts solution does not satisfy even the limiting focusing condition (LFC),\textsuperscript{15)} which is weaker than the SCC.\textsuperscript{∗)} The relation between the curvature strength and quantum effect of NSs has already been suggested in Ref. 10). It demonstrates the weak divergence of the quantum radiation in the so-called analytic dust model, in which the forming NS is known to be weak.\textsuperscript{17), 18)} However, a comprehensive understanding of the relation between the curvature strength and the quantum effects of NSs has not yet been realized. The purpose here is to show how the amount of quantum radiation during the formation of NSs depends on such properties of singularities as the curvature strength. This analysis will help us obtain knowledge about the instability of CH, which should be predicted by a full semiclassical theory, taking into account the backreaction of quantum fields on gravity. In addition, it is shown here how the manner in which the quantized scalar fields are coupled to gravity changes the amount of quantum radiation. The dependence on the manner of coupling is important, because the CHs exhibit a semiclassical instability, caused by all fundamental quantum fields.

The organization of this paper is as follows. In §2.1, we introduce a class of the Lemaître-Tolman-Bondi (LTB) solutions\textsuperscript{19)} that result in the formation of a NS. Then, the class of LTB solutions is divided into three sub-classes, depending on

\textsuperscript{∗)} Following the work of Clarke and Królak,\textsuperscript{16)} consider a geodesic (N), affinely parameterized by κ, with tangent vector kμ, and terminating at or emanating from a singularity, where κ = 0. If

\[ \lim_{\kappa \to 0} \kappa^2 R_{\mu \nu} k^\mu k^\nu = 0 \]

and

\[ \lim_{\kappa \to 0} \kappa R_{\mu \nu} k^\mu k^\nu = 0 \]

where \( R_{\mu \nu} \) is the Ricci tensor, then the SCC and LFC are satisfied along N, respectively. Since the quantity \( R_{\mu \nu} k^\mu k^\nu \) for the Roberts solution indeed vanishes along the CH, the NS satisfies neither the SCC nor the LFC.
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the curvature strength of the NSs in §2.2. In §3, we calculate a map of null rays passing near the NS. This plays crucial roles in estimating quantum emission in the geometrical optics approximation. In §4, the luminosity and total energy of emitted particles are computed. Section 5 is devoted to discussion. Throughout this paper, the units for which \( c = G = \hbar = 1 \) and the signature \((-,+,+,+)^*\) for spacetime metrics are used.

§2. Naked singularity in spherically symmetric dust collapse

2.1. Lemaître-Tolman-Bondi solutions admitting a naked singularity

The LTB solution,\(^{19}\) which describes the collapse of a dust fluid, is written in a comoving coordinate system as

\[
\begin{align*}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{\dot{R}^2}{1 + f(r)} \mathrm{d}r^2 + R^2(t,r) \mathrm{d}\Omega^2, \\
\dot{R}^2 &= \frac{\mathcal{F}(r)}{\dot{R}} + f(r), \\
\rho &= \frac{\mathcal{F}'}{8\pi R^2 \dot{R}},
\end{align*}
\]

where \( \rho \) is the energy density, \( \mathrm{d}\Omega^2 \) is the line element of a unit two-dimensional sphere, and the prime and dot denote the partial derivatives with respect to \( r \) and \( t \), respectively. Since we are concerned with the collapse of a dust fluid, we require \( \dot{R} < 0 \). The arbitrary functions \( \mathcal{F}(r) \) and \( 1 + f(r) > 0 \) are twice the conserved Misner-Sharp mass and the specific energy, respectively. In this paper, we only consider the case \( f = 0 \), which is called ‘marginally bound collapse’. In this case, Eq. (2.1) is integrated to give

\[
R^3 = \frac{9}{4} \mathcal{F}(r) \left[ t - t_s(r) \right]^2,
\]

where \( t_s(r) \) is an arbitrary function of \( r \). The time \( t = t_s(r) \) corresponds to the moment when a dust shell at \( r \) meets the shell-focusing singularity, which is defined by \( R = 0 \). It is possible to choose \( t_s(r) = r \) by scaling \( r \). LTB solutions can describe the formation of a shell-focusing NS from regular spacetimes. It has been shown that the shell-focusing singularity existing at \( R = 0 \) with \( r > 0 \) is completely spacelike,\(^{20}\) and therefore our investigation will be confined to the singularity at \( (t,r) = (0,0) \).

Here, we introduce a class of marginally bound LTB solutions in which the leading-order term of the mass function \( \mathcal{F} \) near the regular center takes the form

\[
\mathcal{F}(r) = \frac{4\lambda^3}{9(\mu + 1)^3} r^{3\mu + 1} + o(r^{3\mu + 1}),
\]

where \( \mu \geq 0 \) and \( \lambda > 0 \) are constants.\(^*\) In Appendix A, it is shown that this class

\(^*\) This form of \( \mathcal{F} \) is chosen so that the initial density profile can be expanded in a power series in \( R \) [see Eq. (2.5)] and the singularity located at \( (t,r) = (0,0) \) is at least locally naked (see Appendix A). The nontrivial form of the factor and power in Eq. (2.3) are just for the simplicity of Eq. (2.4). Of course, other choices of \( \mathcal{F} \) can allow the singularity to be naked.
of LTB spacetimes results in the formation of a shell-focusing NS, which could be globally naked and therefore violate the weak version of CCH. Radial null geodesics are described near the center by

$$\frac{dt}{dr} = \pm R' \simeq \pm \lambda r^\mu \mathcal{F}(t/r), \quad (2.4)$$

where

$$\mathcal{F}(z) \equiv \left[ 1 - \frac{3\mu + 1}{3(\mu + 1)} z \right] (1 - z)^{-1/3}. \quad (2.4)$$

Here, the upper and lower signs correspond to outgoing and ingoing null geodesics, respectively.

We now consider the initial regular density profile near the regular center. The density profile at the initial time $t = t_{\text{in}} < 0$ is written,

$$\rho(t_{\text{in}}, r) = \frac{1}{6\pi t_{\text{in}}^2} \left[ 1 + 2 \left( 1 + \frac{F(r)}{r F'} \right) \frac{r}{t_{\text{in}}} + O \left( \left( r/t_{\text{in}} \right)^2 \right) \right].$$

Therefore, the initial density profile in terms of physical radius $R \propto r^{\mu + 1/3}$ takes the form

$$\rho_{\text{in}}(R) = \rho_0 + \rho_1 R^\gamma + \cdots, \quad (2.5)$$

where

$$\gamma \equiv \frac{3}{3\mu + 1}, \quad \rho_0 \equiv \frac{1}{6\pi t_{\text{in}}^2}, \quad \rho_1 \equiv -\frac{(3\mu + 2)(\mu + 1)^{3/3\mu + 1}}{3(3\mu + 1)\pi \lambda^{3/3\mu + 1}(t_{\text{in}})^{(9\mu + 5)/(3\mu + 1)}. \quad (2.6)}$$

The parameter $\gamma$ is in the region $0 < \gamma \leq 3$ for $\mu \geq 0$. In particular, the analytic and self-similar LTB models are the cases with $\gamma = 2$ ($\mu = 1/6$) and $\gamma = 3$ ($\mu = 0$), respectively.

2.2. Curvature strength of the naked singularities

The curvature strength of spacetime singularities is defined in the hope that weak convergence would imply the extendibility of the spacetime in a distributional sense. In this context, Tipler defined the strong curvature condition (SCC), while Królikak defined a weaker condition, which we call the limiting focusing condition (LFC). The sufficient and necessary conditions for the singularities in spherically symmetric spacetimes with vanishing radial pressure satisfying the LFC or SCC are given in simple forms. Suppose a singularity is naked, and the relation between the circumferential radius $R$ and the Misner-Sharp mass $m$ is given by

$$R \simeq 2y_0 m^\beta \quad (2.7)$$

near the singularity along the null geodesics terminating at or emanating from the NS. The constant $\beta$ is in the region

$$1/3 < \beta \leq 1. \quad (2.8)$$
Table I. The relations among the curvature strength of naked singularities, the luminosity, and the energy of scalar fields near the Cauchy horizons (a) for the non-conformally coupled scalar fields ($\xi \neq 1/6$) and (b) for the conformally coupled scalar field ($\xi = 1/6$). The constant $\gamma$ parameterizes the initial density profile of the dust fluid. SCC implies LFC.

(a)

| $\gamma$ | 0     | $3/4$ | 1     | $3/2$ | 3     |
|----------|-------|-------|-------|-------|-------|
| Strength | -     | Weak  | LFC   | SCC   |       |
| Luminosity | -     | Finite| Divergent | Finite| Divergent |
| Energy   | -     | Finite| Divergent |       |       |

(b)

| $\gamma$ | 0     | $3/4$ | $3/2$ | 2     | 3     |
|----------|-------|-------|-------|-------|-------|
| Strength | -     | Weak  | LFC   | SCC   |       |
| Luminosity | -     | Finite| Divergent |       |       |
| Energy   | -     | Finite| Divergent |       |       |

The constants $y_0$ and $\beta$ are determined so that there is a positive finite limit $y_0 \equiv \lim_{m \to 0} R/(2m^\beta)$ along those null geodesics. If the geodesic satisfies the “gravity-dominance condition”$^{18}$ the sufficient and necessary conditions are summarized as the following theorem.$^{18}$ For the radial null geodesic terminating from or emanating from the NS, if and only if $1/3 < \beta < 1/2$ is satisfied, neither the SCC nor the LFC holds, and if and only if $1/2 < \beta < 1$ is satisfied, the LFC holds but the SCC does not, and if and only if $\beta = 1$, both the SCC and the LFC hold.

For the LTB spacetimes we consider, from Eq. (A.2), we find that $R \propto r^{\mu+1}$ holds along the outgoing null geodesic emanating from the NS, i.e., along the CH.$^*$ Also, we find that $m = F(r)/2 \propto r^{3\mu+1}$ holds along the outgoing null geodesic near the singularity. Therefore, the constant $\beta$ is obtained as

$$\beta = \frac{\mu + 1}{3\mu + 1} = \frac{2\gamma + 3}{9}. \quad (2.9)$$

With the above theorem, proved by Harada et al., the NS for $0 < \gamma < 3/4$ satisfies neither the LFC nor the SCC along the outgoing null geodesic emanating from the NS. The NS for $3/4 \leq \gamma < 3$ does not satisfy the SCC, but it does satisfy the LFC. The NS for $\gamma = 3$ satisfies both the LFC and the SCC. (See also Table I for the relation between $\gamma$ and the curvature strength of NSs.)

§3. Map of null rays passing near the naked singularity

3.1. Local map

The global map $v = G(u)$ is defined as the relation between the moments when a null ray leaves $I^-$ and when it terminates at $I^+$ after passing through the regular

$^*$ The null ray given by Eq. (A-2) is an asymptotic solution of the null geodesic equation with the boundary condition $t(0) = 0$. In fact, Eq. (A-2) is the earliest null ray that emanates from the singularity. This was proved by Christodoulou,$^{20}$ and that proof can be easily generalized to other cases.
A null singularity occurs at the spacetime point \((u_0,v_0)\), and it is visible from \(I^+\), where \((u,v)\) is a suitable double null coordinate system. An outgoing null ray \(u = \text{constant}\) can be traced backward in time from \(I^+\) to \(I^-\). It turns out to be an ingoing null ray \(v = G(u)\) after passing the regular center, located at \(r = 0\), with \(t < 0\). The outgoing null ray \(u = u_0\) and the ingoing null ray \(v = v_0\) represent the CH and the null ray that terminates at the NS, respectively.

This global map plays crucial roles in estimating the quantum radiation with the geometrical optics approximation. The global map cannot be obtained without solving the null geodesic equation from \(I^-\) to \(I^+\) globally.

However, the main properties of the global map should be determined by the behavior of null rays near the singularities, since the particle creation should be caused by the curvature around singularities. From this point of view, Tanaka and Singh proposed an alternative map, which we call the local map. They considered an observer on a comoving shell who sends ingoing null rays. These null rays are reflected at the regular center, and then return to the same comoving observer. A radial null geodesic crosses a comoving shell located at a fixed comoving radius \(r\) before and after the reflection at the center. Thus these null rays define a map between the sending time and the receiving time measured by the proper time for the comoving observer. (See Fig. 2 for a schematic illustration of the local map.)

The local map is conjectured to have the same structure as the global map, because there are no singular features in the map between the proper time on a comoving shell at a finite distance and those measured by the null coordinates naturally defined at infinity. This conjecture has been confirmed for the self-similar dust model and the analytic dust model, which is a non-self-similar spacetime. Two of the present authors generalized the local map to general self-similar spacetimes, and the validity of such a local map was confirmed for the self-similar Vaidya model. Therefore, one can assume that the local map has the same structure as the global map in the LTB spacetimes.

In this section, we calculate the local map by solving the radial null geodesic
Fig. 2. A schematic spacetime diagram of the region filled with dust, with the illustration of the local map defined in §3. A pair consisting of an ingoing null ray \( t_-(r) \) and an outgoing null ray \( t_+(r) \) is depicted (solid curve). It passes near the NS, located at \( (t,r) = (0,0) \). Null rays terminating at and emanating from the NS, \( t_-^{\text{crit}}(r) \) and \( t_+^{\text{crit}}(r) \), are also depicted (dashed curves). The latter is the CH. A comoving observer is located at \( r = r_0 = \text{[constant]} \). The local map is defined as the relation between \( t_-(r_0) \) and \( t_+(r_0) \).

To make the notation simple, we replace the coordinates \( r \) and \( t \) with \( r' \) and \( t' \) given by

\[
\begin{align*}
  r' &= \lambda^{1/\mu}r, \\
  t' &= \lambda^{1/\mu}t,
\end{align*}
\]

and we write \( r' \) and \( t' \) as \( r \) and \( t \) hereafter in this section. Then, the null geodesic equation (2.4) becomes

\[
\frac{dt}{dr} = \pm r^\mu F(t/r). \tag{3.1}
\]

Before beginning the calculation, it is convenient to summarize the calculational scheme for obtaining the local map that we implement in the following subsections. To obtain the local map, we have to obtain solutions of Eq. (3.1) near \( r = 0 \), which correspond to radial null rays passing through the center at \( t = t(0) = -t_0 \) (\( t_0 > 0 \)), and then take the limit \( t_0 \to 0 \). We cannot expect, however, to have general exact solutions to this equation. Hence, here we adopt the following scheme to obtain the local map. First, we apply three different approximation regimes, A, B and C, and find three kinds of expressions, \( t = t^A(r) \), \( t = t^B(r) \) and \( t = t^C(r) \), respectively. Next, we show that these three regimes have an overlapping region, where all three approximations are valid and the obtained approximate solutions
Fig. 3. A schematic illustration of the regions in which the regimes A, B and C are valid. The regions corresponding to the regimes A, B and C are $0 < r < \eta A t_0^{1/(1+\mu)}$, $t_0/\eta B < r < \eta A t_0^{1/(1+\mu)}$ and $D/\eta C \sim t_0/\eta C \lesssim r \lesssim \eta C^{1/\mu}$, respectively, where $\eta X \ll 1$ ($X = A, B, C$) is a constant independent of $t_0$. The region for B is included in that for A. The regions for B and C exist if $t_0 < (\eta A \eta B)^{(\mu+1)/\mu}$ and $t_0 < (\eta C^{(\mu+1)/\mu}$ are satisfied, respectively. The regions for B and C overlap if $t_0 < (\eta A \eta C)^{(\mu+1)/\mu}$. All these conditions are satisfied for $t_0 < \min[(\eta A \eta B)^{(\mu+1)/\mu}, (\eta A \eta C)^{(\mu+1)/\mu}]$. In the limit $t_0 \to 0$, the regions for A and B shrink to zero, but the region for C, where the comoving observer should be located, remains finite.

can be matched with each other. Finally, we calculate the local map, that is, we calculate the relation between the sending time and the receiving time of the null ray at a comoving observer near the center.

In §3.3, we consider approximation regime A, in which $0 \leq r < \eta A t_0^{1/(1+\mu)}$ is satisfied. Here, $\eta A$ ($\ll 1$) is a positive constant, independent of $t_0$. In this regime, we can treat the center, and then we can relate the ingoing and outgoing null rays, which reach the center at the same time, $t = -t_0$. In §3.4, we consider approximation regime B, in which $t_0/\eta B < r < \eta A t_0^{1/(1+\mu)}$ is satisfied. Here, $\eta B$ ($\ll 1$) is a positive constant. This regime can exist only when $t_0 < (\eta A \eta B)^{(\mu+1)/\mu}$. Although regime B is completely included in regime A, regime B enables us to have an explicit expression for solutions, and it is therefore essential to obtain the local map. In §3.5, we consider approximation regime C, where $t/r \ll 1$ is assumed. When we choose $t/r$ to be $O(\eta C)$, where $\eta C$ ($\ll 1$) is a sufficiently small constant, it turns out that the approximation is valid for $t_0/\eta C \lesssim r \lesssim \eta C^{1/\mu}$. Regimes A and B obviously have an overlapping region. When we take the limit $t_0 \to 0$, regime B and regime C also have an overlapping region, and regime C is valid at finite radius. (See also Fig. 3 for the illustration of regions, where each regime is valid.) In §3.6, we implement the matching between the approximate solutions $t = t^B(r)$ and $t = t^C(r)$ in the overlapping region. In §3.7, the local map is finally obtained for a comoving observer at a finite radius in the region for regime C. This is the generalization of the result of Tanaka and Singh.21)
3.3. Regime A: $0 \leq r < \eta_A t_0^{1/(1+\mu)}$

To find a null geodesic for $0 \leq r < \eta_A t_0^{1/(1+\mu)}$, we set

$$\frac{r}{t_0^{1/(\mu+1)}} = \epsilon \zeta,$$

$$\frac{r}{t_0} = \frac{\zeta}{\delta},$$

where $\epsilon$ and $\delta$ are constants, and $\zeta$ is variable for $r$. Here, we assume that $\epsilon < \eta_A$ and $\zeta$ is $O(1)$. The quantity $\delta$ is introduced for later convenience and is not necessarily small in regime A. From Eqs. (3.2) and (3.3), the following relations hold:

$$t_0 = \epsilon^{(\mu+1)/\mu} \delta^{(\mu+1)/\mu},$$

$$r = \epsilon^{(\mu+1)/\mu} \delta^{1/\mu} \zeta.$$  

The null geodesic $t = t^A(r)$ can be expanded in $\epsilon$ as follows:

$$t^A(r) = -t_0 + \sum_{n=1}^{\infty} \epsilon^{(\mu+1)(n\mu+1)/\mu} t_n^A(\zeta),$$

where $t_n^A(\zeta)$ ($n = 1, 2, \cdots$) are functions of $\zeta$ of order unity. Substituting Eq. (3.6) into Eq. (3.1), we obtain the differential equations

$$\frac{dt_1^A}{d\zeta}(\zeta) = \pm \delta^{(\mu+1)/\mu} \zeta^\mu F(-\delta/\zeta),$$  

$$\frac{dt_2^A}{d\zeta}(\zeta) = \pm \delta^{\mu-1} F'(-\delta/\zeta) t_1^A(\zeta),$$  

$$\frac{dt_3^A}{d\zeta}(\zeta) = \pm \frac{1}{2} \delta^{(\mu-1)/\mu} \zeta^{-2} F''(-\delta/\zeta) (t_1^A)'(\zeta) \pm \delta^{\mu-1} F'(-\delta/\zeta) t_2^A(\zeta),$$

and so on, where $F'$ and $F''$ denote derivatives of $F$ with respect to its argument. Since we have $t^A(0) = -t_0$, $t_n^A(0) = 0$ must be satisfied for $n \geq 1$. Equation (3.7) can be integrated immediately to give

$$t_1^A(\zeta) = \pm \frac{1}{\mu+1} \delta^{(\mu+1)/\mu} \zeta^{\mu+1}(1 + \delta/\zeta)^{2/3}.$$

With $t_1^A(\zeta)$ obtained above, Eq. (3.8) is integrated to give

$$t_2^A(\zeta) = \frac{1}{\mu+1} \delta^{(2\mu+1)/\mu} \int_0^\zeta \hat{\zeta}^{2\mu} F'(-\delta/\hat{\zeta})(1 + \hat{\delta}/\hat{\zeta})^{2/3} d\hat{\zeta}$$

$$= \delta^{(\mu+1)(2\mu+1)/\mu} I_2(\zeta/\delta),$$

where

$$I_2(y) \equiv \frac{1}{\mu+1} \int_0^y x^{2\mu} F'(-1/x)(1 + 1/x)^{2/3} dx.$$
This integration cannot be expressed in terms of elementary functions. In a similar way, one can write $t_3^A(\zeta)$ in the integral form

$$t_3^A(\zeta) = \pm \int_0^\zeta \frac{1}{2(\mu + 1)^2} \delta(3\mu + 1)/\zeta^3 \frac{3\mu}{\zeta^3} (\hat{\zeta})(1 + \delta/\hat{\zeta})^4/3$$

$$+ \delta(2\mu^2 + 4\mu + 1)/\zeta^{\mu - 1} F'(-\delta/\hat{\zeta}) I_2(\hat{\zeta}/\delta) d\hat{\zeta},$$

where

$$I_3(y) \equiv \int_0^y \left[ \frac{1}{2(\mu + 1)^2} x^{3\mu} F''(-1/x)(1 + 1/x)^{4/3} + x^{\mu - 1} F'(-1/x) I_2(x) \right] dx. \quad (3.14)$$

It should be noted that we can safely take the limit $r \to 0$ in this regime, because we do not assume that $\delta$ is small.

3.4. Regime B: $t_0/\eta_B < r < \eta_B t_0^{1/(1+\mu)}$

For the approximation regime B, we additionally assume that $\delta < \eta_B$. This also requires that $t_0 \ll 1$, from Eq. (3.4). The approximate solution $t = t^B(r)$ can be obtained by making approximation $t = t^A(r)$ with $\delta \ll 1$. Hence, we define $t^B_n(\zeta)$ as the function which is obtained by approximating $t^A_n(\zeta)$ with $\delta \ll 1$. Thus, we have

$$t_1^B = \pm \frac{1}{\mu + 1} \delta(\mu + 1)/\zeta^{\mu + 1} \left[ 1 + \frac{2}{3} \zeta^{-1} \delta + O(\delta^2) \right]. \quad (3.15)$$

The approximate form of $t_2^A$ for $\delta \ll 1$ can be obtained by using the asymptotic form of $I_2(y)$ for large $y$ in Eq. (3.13) given by

$$I_2(y) = C_2 - \frac{2\mu}{3(\mu + 1)^2(2\mu + 1)} y^{2\mu + 1} [1 + O(1/y)],$$

where $C_2$ is a constant. Except for the term $C_2$, all terms are completely determined by integrating the expanded integrand. Therefore, $t_2^B(\zeta)$ is obtained as

$$t_2^B(\zeta) = C_2 \delta(\mu + 1)(2\mu + 1)/\mu - \frac{2\mu}{3(\mu + 1)^2(2\mu + 1)} \delta(2\mu + 1)/\zeta^{2\mu + 1} [1 + O(\delta)]. \quad (3.16)$$

Similarly, the asymptotic form of $I_3$ is given by

$$I_3(y) = C_3 + \left[ \frac{2\mu^2 + \mu + 1}{9(\mu + 1)^3(2\mu + 1)(3\mu + 1)} y^{3\mu + 1} - \frac{2C_2}{3(\mu + 1)} y^{\mu} \right] [1 + O(1/y)],$$

and hence $t_3^B(\zeta)$ is

$$t_3^B(\zeta) = \pm C_3 \delta(\mu + 1)(3\mu + 1)/\mu \pm \frac{2\mu^2 + \mu + 1}{9(\mu + 1)^3(2\mu + 1)(3\mu + 1)} \delta(3\mu + 1)/\zeta^{3\mu + 1} [1 + O(\delta)]$$

$$\pm \frac{2C_2}{3(\mu + 1)} \delta(2\mu^2 + 4\mu + 1)/\zeta^{\mu} [1 + O(\delta)]. \quad (3.17)$$
Then, an approximate solution is obtained from Eqs. (3.4)-(3.6) and (3.15)-(3.17) as

\[
t^B(r) = -t_0 + C_2 t_0^{2\mu+1} \pm C_3 t_0^{3\mu+1} + \left[ \pm \frac{1}{\mu+1} r^{\mu+1} - \frac{2\mu}{3(\mu+1)^2(2\mu+1)} r^{2\mu+1} \pm \frac{2\mu^2 + \mu + 1}{9(\mu+1)^3(2\mu+1)(3\mu+1)} r^{3\mu+1} \right] \times [1 + O(\delta)].
\] (3.18)

For later use, we have obtained explicit expression up to third order. It is straightforward to compute higher orders. It should be noted that we cannot take the limit \( r \to 0 \) in (3.18).

3.5. Regime C: \( t/r \ll 1 \)

Suppose \( t/r \ll 1 \). Here, we introduce the condition \( \eta_C \ll 1 \), which controls the order of \( t/r \); specifically \( t/r = O(\eta_C) \). In this approximation regime, we can expand \( f \) in \( t/r \) on the right-hand side of Eq. (3.1) and obtain the expanded form of solutions.

Let us consider the critical outgoing and ingoing null geodesics \( t = t_\pm^{\text{crit}}(r) \), which emanate from and terminate at the NS, i.e., \( t = r = 0 \). Then \( t = t_\pm^{\text{crit}}(r) \) gives the CH, by definition. If we assume that \( t/r \) is \( O(\eta_C) \), these geodesics are obtained by expanding the null geodesic equation (3.1) in power series in \( r \) with the boundary condition \( t(0) = 0 \):

\[
t_\pm^{\text{crit}}(r) = \pm \frac{1}{\mu+1} r^{\mu+1} - \frac{2\mu}{3(\mu+1)^2(2\mu+1)} r^{2\mu+1} \pm \frac{2\mu^2 + \mu + 1}{9(\mu+1)^3(2\mu+1)(3\mu+1)} r^{3\mu+1} + O(\eta_C^4). \] (3.19)

Here, the upper (lower) sign corresponds to an emanating (terminating) null ray. For \( 0 < r \lesssim \eta_C^{1/\mu} \), \( t/r \) is \( O(\eta_C) \) on the critical null rays, and this approximation is valid. That is, the approximation regime C is valid for \( 0 < r \lesssim \eta_C^{1/\mu} \) on the critical null rays.

However, we are interested in null rays that correspond to times slightly before the times of these critical null rays. We expand the solution as

\[
t = t^C(r) = \sum_{n=1}^{\infty} t^C_n(r), \] (3.20)

where we assume that \( t^C_n(r)/r \) is \( O(\eta_C^n) \). Substituting Eq. (3.20) into Eq. (3.1) and expanding the right-hand side, the following differential equations are obtained:

\[
\frac{dt^C}{dr} = \pm r^\mu, \] (3.21)

\[
\frac{dt^C_2}{dr} = \mp \frac{2\mu}{3(\mu+1)} r^{\mu-1} t^C_1(r), \] (3.22)

\[
\frac{dt^C_3}{dr} = \mp \frac{2\mu}{3(\mu+1)} r^{\mu-1} t^C_2(r) \mp \frac{\mu - 1}{9(\mu+1)} r^{\mu-2} (t^C_1)^2(r). \] (3.23)
These equations are integrated to yield

\[
\begin{align*}
    t^C_1(r) &= D_\pm \pm \frac{1}{\mu + 1} r^{\mu + 1}, \\
    t^C_2(r) &= \pm \frac{2}{3(\mu + 1)} D_\pm r^\mu - \frac{2\mu}{3(\mu + 1)^2(2\mu + 1)} r^{2\mu + 1}, \\
    t^C_3(r) &= \pm \frac{1}{9(\mu + 1)} D^2_\pm r^{\mu - 1} + \frac{1}{9\mu(\mu + 1)} D_\pm r^{2\mu} \\
    &\quad \pm \frac{2\mu^2 + \mu + 1}{9(\mu + 1)^3(2\mu + 1)(3\mu + 1)} r^{3\mu + 1},
\end{align*}
\]

(3.24) (3.25) (3.26)

where \( D_\pm \) is an integration constant that comes from the integration of Eq. (3.21). The integration constants for Eqs. (3.22) and (3.23) are set to zero. From Eqs. (3.20) and (3.24)–(3.26), the solution takes the following form:

\[
t^C(r) = D_\pm \\
    \pm \frac{1}{\mu + 1} r^{\mu + 1} \\
    \pm \frac{2}{3(\mu + 1)} D_\pm r^\mu - \frac{2\mu}{3(\mu + 1)^2(2\mu + 1)} r^{2\mu + 1} \\
    \pm \frac{1}{9(\mu + 1)} D^2_\pm r^{\mu - 1} + \frac{1}{9\mu(\mu + 1)} D_\pm r^{2\mu} \mp \frac{2\mu^2 + \mu + 1}{9(\mu + 1)^3(2\mu + 1)(3\mu + 1)} r^{3\mu + 1} \\
    \pm O(\eta^3_C).
\]

(3.27)

Now, we can examine in what region the solution (3.27) is valid. In order for the expansion (3.20) to be valid, \( t^C_n(r)/r \) must be \( O(\eta^n_C) \). This is the case if

\[
\max \left( r^\mu, \frac{D_\pm}{r} \right) = O(\eta_C).
\]

If \( D_\pm \) is sufficiently small, the above condition implies

\[
D_\pm/\eta_C \lesssim r \lesssim \eta_C^{1/\mu}.
\]

This is the region in which approximation regime C is valid for the null geodesic passing through the center at \( t(0) = -t_0 < 0 \). The relation between \( D_\pm \) and \( t_0 \) is elucidated in the following subsection.

3.6. Matching the approximation regimes

Since we are interested in the null rays that are close to the critical null geodesics, we take the limit \( t_0 \to 0 \). The region for approximation regime B is completely included in that for regime A, and the matching is trivially implemented. If we assume \( D_\pm = O(t_0) \), the region in which regime C is valid must have an overlap with the region in which regime B is valid if \( t_0 \lesssim (\eta_A \eta_C)^{(\mu + 1)/\mu} \). This means that we can relate the integration constants \( D_\pm \), which appear in the regime C solution \( t = t^C(r) \), to \( t_0 \), which appears in the regime B solution \( t = t^B(r) \), by matching these two solutions in the overlapping region.
The different expressions $t = t^B(r)$ and $t = t^C(r)$ for the null geodesic are obtained independently in §§3.4 and 3.5. We find that the solution $t = t^B(r)$ given by Eq. (3.18) coincides with the solution $t = t^C(r)$ given by Eq. (3.27) at several lowest orders if the constant terms satisfy the following relation:

$$D_\pm \simeq -t_0 + C_2r_0^{2\mu+1} \pm C_3r_0^{3\mu+1}. \quad (3.28)$$

This verifies the validity of the assumption that $D_\pm$ is sufficiently small and $D_\pm$ is $O(t_0)$ in the limit $t_0 \to 0$. This also implies that the approximation regime $C$ is valid for $t_0/\eta_C \lesssim r \lesssim \eta_C^{1/\mu}$. Here, let us see the condition on $t_0$ which must be satisfied for the matching. $t_0$ has to satisfy

$$t_0 < \min \left( (\eta_A\eta_B)^{(\mu+1)/\mu}, (\eta_A\eta_C)^{(\mu+1)/\mu} \right), \quad (3.29)$$

so that the region for the regime $B$ and the overlapping region for the regimes $B$ and $C$ can exist. It is possible to take such a small $t_0$, because we are interested in the limit $t_0 \to 0$. (See also Fig. 3.)

### 3.7. Obtaining the local map

To obtain the local map for this spacetime, we consider a comoving observer at $r = r_0$, where $r_0$ satisfies $r_0 \lesssim \eta_C^{1/\mu}$. As time proceeds, this observer approaches the ingoing critical null ray $t = t_{\text{crit}}^-(r)$ and therefore enters the region $t_0/\eta_C \lesssim r \lesssim \eta_C^{1/\mu}$, where approximation regime $C$ is valid. Then, $t_{\pm}(r_0) = t_{\pm}^C(r_0)$ are regarded as the sending time and receiving time of the null geodesic, where the signs $\pm$ are introduced to distinguish the outgoing and ingoing null rays. From Eq. (3.27), the constant $D_\pm$ is specified as

$$-D_\pm = t_{\pm}^\text{crit}(r_0) - t_{\pm}(r_0) + O(t_0\eta_C),$$

where $t_{\pm}^\text{crit}(r_0)$ is the moment when the observer crosses the null geodesic (3.19). Therefore, if we take $\eta_C$ to be sufficiently small, $D_-$ ($D_+$) can be interpreted as the time difference between the moments when the observer emits (receives) the null ray and crosses the null ray terminating at (emanating from) the NS. (See also Fig. 2.)

Since we consider a set consisting of an ingoing null ray which reaches the center at $t = -t_0$ and an outgoing null ray which can be regarded as a reflected ray of the former at $t = -t_0$, we pick up both the ingoing and outgoing null rays with the same value for $t_0$. When we eliminate $t_0$ from Eq. (3.28) for both signs, we have the relation between $D_+$ and $D_-$

$$D_- \simeq D_+ - 2C_3(-D_+)^{3\mu+1},$$

which can be rewritten in terms of the sending and receiving times $t_{\pm}(r_0)$ at $r = r_0$ for sufficiently small $r_0$ as

$$t_{\pm}^C(r_0) \simeq t_{\pm}^\text{crit}(r_0) - \left[ t_{\pm}^\text{crit}(r_0) - t_{\pm}^C(r_0) \right] - 2C_3 \left[ t_{\pm}^\text{crit}(r_0) - t_{\pm}^C(r_0) \right]^{3\mu+1}. \quad (3.30)$$

Equation (3.30) is the very local map, relating the moments when the comoving observer locating at $r = r_0$ sends the ingoing null ray and receives the reflected
outgoing null rays. If we revive $\lambda$, the final result becomes

$$t^C_{-}(r_0) \simeq t_{\text{crit}}^{-}(r_0) - \left[ t_{\text{crit}}^{+}(r_0) - t^C_{+}(r_0) \right] - 2C_3\lambda^3 \left[ t_{\text{crit}}^{+}(r_0) - t^C_{+}(r_0) \right]^{\mu+1}. \tag{3.31}$$

Now, note that the comoving observer must be in the region $t_0/\eta_C \lesssim r_0 \lesssim \eta_1^{1/\mu} \lambda^{-1/\mu}$. This implies that the asymptotic structure of the local map in the limit $t_0 \to 0$, therefore the main feature of the global map, is determined by only the behavior of the null rays in the small but finite region $0 < r \lesssim \eta_1^{1/\mu} \lambda^{-1/\mu}$.

§4. Luminosity and energy of particle creation

We consider the massless scalar fields coupled to the scalar curvature $R$ as

$$\left( \Box - \xi R \right) \phi = 0, \tag{4.1}$$

where $\xi$ is an arbitrary real constant. In particular, the scalar fields with $\xi = 0$ and $\xi = 1/6$ are minimally and conformally coupled ones, respectively. The stress-energy tensor of the scalar field in an asymptotically flat region is given by

$$T^{(\xi)}_{\mu\nu} \simeq \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} \eta_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi - \xi \nabla_{\mu} \nabla_{\nu} \phi^2 + \xi \eta_{\mu\nu} \Box \phi^2,$$

where $\eta_{\mu\nu}$ is a flat metric. To calculate the luminosity of emitted particles we need an estimate of the vacuum expectation value of the above stress-energy tensor. A suitable regularization is required in the calculation of the vacuum expectation value, because the stress-energy tensor is quadratic in fields at the same point. The regularization for minimally and conformally coupled scalar fields is obtained in Ref. 4) using the point-splitting regularization scheme. This can easily be generalized to the above more general scalar fields to give

$$L^{(\xi)}_{lm}(u) = \frac{1}{4\pi} \left[ \left( \frac{1}{4} - \xi \right) \left( \frac{G''}{G'} \right)^2 + \left( \xi - \frac{1}{6} \right) \frac{G'''}{G'} \right], \tag{4.2}$$

where the prime denotes differentiation with respect to $u$, and $l (= 0, 1, \cdots)$ and $m (= 0, \pm 1, \cdots, \pm l)$ are angular momentum and magnetic quantum numbers, respectively. The luminosity is the sum of all these modes, but it diverges, because the luminosity given in (4.2) is independent of $(l, m)$. Such a divergence is due to the fact that we ignored the back scattering by the potential barrier, which will reduce the emission for highly rotational modes. Hereafter, we omit the quantum numbers $(l, m)$, and it should simply be kept in mind that the above expression holds only for small $l$. The total energy of emitted particles is estimated by integrating the luminosity with respect to $u$:

$$E^{(\xi)}(u) \equiv \int_{-\infty}^{u} L^{(\xi)}(u') du'.$$
4.1. Quantum radiation in the non-self-similar LTB spacetimes: $0 < \gamma < 3$

As described in §3, we assume that the local map and the global map have the same structure. This means that from Eq. (3.31), the asymptotic form of the global map will take the form

$$G(u) \simeq v_0 - A(u_0 - u) - Ag\lambda^3(u_0 - u)^{3\mu + 1},$$

where $u = u_0$ and $v = v_0$ are the CH and the ingoing null ray that terminate at the NS, respectively, and $A$ and $g$ are constants. Keeping in mind that

$$\mu = \frac{3 - \gamma}{3 \gamma},$$

one can calculate the luminosity as

$$L(\xi) \simeq \left(\xi - \frac{1}{6}\right) \frac{3(3 - \gamma)(3 - 2\gamma)}{4\pi \gamma^3} g\omega_s^{(3-\gamma)/\gamma} (u_0 - u)^{-3+3/\gamma}$$

$$+ \left[\xi - \frac{7\gamma - 15}{36(\gamma - 2)}\right] \frac{27(\gamma - 2)(3 - \gamma)}{4\pi \gamma^4} g^2 \omega_s^{2(3-\gamma)/\gamma} (u_0 - u)^{-4+6/\gamma}$$

$$+ O\left((u_0 - u)^{-5+9/\gamma}\right),$$

where we have defined the “frequency” of the NSs as

$$\omega_s \equiv \lambda^{1/\mu} = \lambda^{3\gamma/(3-\gamma)}. \quad (4.4)$$

In Appendix B, we show that the frequency (4.4) is identical to that defined in Ref. 10). Depending on whether or not $\xi = 1/6$, the leading-order term of the luminosity changes. Let us examine the time dependence of the luminosity and the total energy of emitted particles in detail for the cases $\xi \neq 1/6$ and $\xi = 1/6$.

In the case $\xi \neq 1/6$, the first term in Eq. (4.3) dominates, except in the special case $\gamma = 3/2$. For $0 < \gamma \leq 1$, the leading term vanishes as $u \to u_0$. For $1 < \gamma < 3$ and $\gamma \neq 3/2$, the luminosity diverges as the CH is approached as a negative power of the time remaining to the CH. The special case $\gamma = 3/2$, in which the factor of the first term in Eq. (4.3) vanishes, is divided into two cases, depending on whether $\xi \neq 1/4$ or $\xi = 1/4$. If $\xi \neq 1/4$, the second term in Eq. (4.3) survives as a finite constant. If $\xi = 1/4$, the factor of the second term also vanishes. Therefore, higher-order terms contribute to the luminosity, which is finite for $\gamma = 3/2$. Therefore, in both cases that $\gamma = 3/2$, the luminosity remains finite at the CH. Now, let us examine the total energy of emitted particles. For $3/2 < \gamma < 3$, the leading term is

$$E(\xi) \simeq -\left(\xi - \frac{1}{6}\right) \frac{3(3 - \gamma)}{4\pi \gamma^2} g\omega_s^{(3-\gamma)/\gamma} (u_0 - u)^{-2+3/\gamma},$$

and therefore the total energy diverges as the CH is approached. In particular, in the case of an analytic LTB solution ($\gamma = 2$), the energy diverges as $(u_0 - u)^{-1/2}$, which coincides with the result in Ref. 10). For $0 < \gamma < 3/2$, the total energy remains finite as the CH is approached. In the special case $\gamma = 3/2$, the energy also remains finite. [See also Table I(a).]

In the case of the conformally coupled scalar field, which is defined by $\xi = 1/6$, the second term in Eq. (4.3) dominates. It is found that in the case $0 < \gamma \leq 3/2$, the
luminosity remains finite at most. While, for $3/2 < \gamma < 3$, the luminosity diverges, due to the negative power of the time remaining to the CH. Let us examine the total energy of the emitted particles. For $2 < \gamma < 3$, we have

$$E^{(1/6)} \simeq \frac{(3-\gamma)^2}{16\pi\gamma^3(\gamma-2)}g^2\omega_s^2(3-\gamma)/\gamma(u_0-u)^{-3+6/\gamma},$$

which diverges as the CH is approached. For $\gamma = 2$, we have

$$E^{(1/6)} \simeq \frac{3}{256\pi}g^2\omega_s^2\ln[\omega_s(u_0-u)]^{-1},$$

and therefore the energy diverges logarithmically. This too is identical to the result in Ref. 10). If $0 < \gamma < 2$, the energy remains finite, at most. [See also Table I(b).]

It follows that the radiation resulting from a conformally coupled scalar field is milder than that resulting from non-conformally coupled scalar fields for a given value of $\gamma$. Such a consequence appears to result from the fact that the coupling of the conformal scalar field to gravity is weaker than that of the other scalar fields. In this relation, it should be pointed out that a conformally coupled scalar field must have non-zero and finite mass to be created in the early universe, while non-conformally coupled scalar particles are created regardless of their mass.\textsuperscript{22)}

4.2. Quantum radiation in a self-similar LTB spacetime: $\gamma = 3$

The global map for self-similar LTB spacetimes ending in NS formation is calculated analytically in Ref. 6). Subsequently, its main property was re-produced with the local-map method.\textsuperscript{8), 21)} The global map for null rays passing near the CH is given by

$$G(u) \simeq v_0 - B(u_0-u)^\alpha \left[1 + q(u_0-u) + O((u_0-u)^2)\right],$$

where $\alpha$, $B$ and $q$ are constants. The terms in the square brackets in Eq. (4.8) together form an analytic function of $(u_0-u)$.\textsuperscript{*} The constant $\alpha$ depends only on the parameter $\lambda$ in Eq. (2.3), and it has been shown to be greater than unity for the region of $\lambda$ in which the singularity is naked.\textsuperscript{6)} Using the global map (4.8), we can compute the luminosity and energy of the particle creation as

$$L^{(\xi)} \simeq \frac{(\alpha-1)(\alpha+1-12\xi)}{48\pi}(u_0-u)^{-2} + \frac{\alpha^2-1}{24\pi\alpha}q(u_0-u)^{-1},$$

$$E^{(\xi)} \simeq \frac{(\alpha-1)(\alpha+1-12\xi)}{48\pi}(u_0-u)^{-1} + \frac{\alpha^2-1}{24\pi\alpha}q\ln q^{-1}(u_0-u)^{-1}.$$

The first term in both Eqs. (4.9) and (4.10) dominates, except in the special case that $\xi = (\alpha+1)/12$. Therefore, the luminosity and energy generically diverge as

\textsuperscript{*} In previous works on particle creation during NS formation in the self-similar LTB spacetime, only the constant term in the square brackets in Eq. (4.8) was considered.\textsuperscript{6), 8), 21)} This was sufficient to obtain results. It is easy to calculate the higher-order terms using the local-map method and to show that they constitute an analytic function near the CH. It is possible, however, that the emergence of the scale $q$ in Eq. (4.8) from such a scale invariant spacetime as self-similar LTB solution indicates the breakdown of the local-map method. It is not clear to what extent the local-map method is valid.
the inverse square and the inverse of the time remaining to the CH, respectively. However, in the special case $\xi = (\alpha + 1)/12$, each second term in Eqs. (4.9) and (4.10) dominates so that the power and energy diverge inversely and logarithmically, respectively. This case, however, should be regarded as an accidental cancellation in the sense that the coupling constant $\xi$ happens to be a special value $\alpha$, which is determined by the details of the collapse. [See also Tables I(a) and (b).]

§5. Discussion

In this paper we have considered particle creation during the formation of shell-focusing NSs in a wide class of spherical dust collapse, which is described by the marginally bound LTB solutions. Each solution has a different initial density profile, and the resulting NSs have a variety of curvature strengths along the CHs. The luminosity and energy of particle creation have been estimated for each LTB solution and each scalar field that couples to scalar curvature in the linear form. The results are summarized in Tables I(a) and (b).

We first mention the validity of the approximations which have been made in this article. Next, we discuss the relations between the quantum radiation and the curvature strength of the NSs and also the manner in which the scalar fields couple to gravity. Finally, we discuss the implications of the present result to the CCH.

The analysis has been based on three assumptions: the validity of the local-map method, the geometrical optics approximation, and quantum field theory in curved spacetime. The validity of each approximation seems to be open to debate. (See Ref. 10 for discussion of the geometric optics approximation and Refs. 28) and 8) for quantum field theory in curved spacetime.) Here, we focus on the local-map method, on which our entire analysis is based. The point is that the crucial factor of particle creation, the redshift of particles, must be determined by the geometry near the NS, while in Hawking radiation, the redshift is determined by the event horizon, which is a global object. It is unlikely that the global map has a structure that differs from that of the local map, since there is no singular feature in the map between the moments on the comoving observer at a finite distance and that of the null coordinates naturally defined at infinity. Indeed, in the models of the self-similar LTB, the analytic LTB and the self-similar Vaidya, the local-map method yields the correct results, which are obtained with the global map. For this reason, we have assumed the validity of the local-map method.

From the results, it is found that following statements hold for the generic naked-singular LTB spacetimes: the SCC along the CH is a sufficient condition for the luminosity and energy of the created scalar particles to diverge as the CH is approached; while, violation of the LFC is a sufficient condition for the luminosity and energy to be finite at the CH; if the NS does not satisfy the SCC but does satisfy the LFC, the luminosity and energy can be either divergent or finite. We only consider dust collapse for simplicity. However, the above statements regarding the curvature strength and the quantum radiation should be independent of the collapsing matter, because particle creation is a purely kinematic phenomenon and not directly related to the Einstein field equations. Therefore, we conjecture that the above statements hold.

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for spherically symmetric collapsing spacetimes with *any kind of collapsing matter*. Of course, its validity should be verified or examined with known solutions ending in NS formation. The self-similar models which have been examined to this time, the collapse of a null-dust fluid,\(^8\),\(^9\),\(^23\) a massless scalar field,\(^8\),\(^12\) and a perfect fluid,\(^8\),\(^24\) support this conjecture. There are many examples to be investigated: NS formation in counter-rotating particles,\(^18\),\(^25\) non-self-similar null dust,\(^23\) null strange quark matter,\(^26\) various matter fields in the critical collapse\(^27\) and so on. Here, we also mention the manner in which scalar fields couple to gravity. Although the quantum radiation due to the conformally coupled scalar field is less than that of other scalar fields, including the minimally coupled scalar field, the dependence of the amount of radiation on the manner of coupling is not so drastic as to modify the above statements.

Next, we consider the implications of our results for the CCH. The diverging radiation from strong NSs corresponds to an instability of the strong NS formation. The system will enter into a phase where the backreaction from the quantum field to spacetime plays an important role. While, the finite radiation from the weak NSs corresponds to a stability of the weak NS formation. This is striking, because the weak NSs seem to need another mechanism if they are to hide behind horizons. Of course, we cannot dismiss the possibility that the effect of backreaction suppresses the quantum radiation and that strong NSs appear, all things considered.

In the present analysis, we did not find a necessary and sufficient condition on the curvature strength of NSs for the quantum radiation to be divergent or finite. We believe that a new definition of the strength of (naked) singularities should be proposed from the viewpoint of the behavior of quantum fields on spacetimes rather than the viewpoint of the behavior of classical particles. Such a philosophy can also be seen in the wave-probe approach to NSs,\(^29\) which is based on the theory of dynamics in non-globally hyperbolic spacetimes developed first by Wald.\(^30\) With regard to this point, there is room for further investigation.

**Acknowledgements**

We are grateful to Kei-ichi Maeda for his continuous encouragement. U. M. thanks L. H. Ford for a helpful comment on the coupling of scalar fields and also thanks T. P. Singh and D. A. Konkowski for useful comments on the Cauchy horizon instability. Thanks are due to A. Ishibashi for thoughtful comments on the implications and applications of the results. U. M. was supported by Waseda University Grant for Special Research Projects. H. M. was supported by a Grant for The 21st Century COE Program (Holistic Research and Education Center for Physics Self-Organization Systems) at Waseda University. T. H. was supported by JSPS.

**Appendix A**

---

Nakedness of the Singularity---

In order to determine whether or not the singularity is naked, we investigate the future-directed outgoing null geodesics emanating from the singularity at \((t, r) =\)
(0, 0). We find the asymptotic solutions that obey a power law near the center\(^3\)) in the form

\[ t \simeq X_0 r^p, \quad (A.1) \]

where \(X_0 > 0\) and \(p \geq 1\) are constants. The latter condition is due to the fact that the orbit of the shell-focusing singularity is \(t = t_s(r) = r\). After some straightforward calculations, for \(\mu > 0\) we find the asymptotic solution

\[ t \simeq \frac{\lambda}{\mu + 1} r^{\mu + 1}. \quad (A.2) \]

With Eq. (A.2) and the fact that the apparent horizon, which is defined by \(F = R\), behaves as \(t = t_{ah}(r) = r - 2F(r)/3 \simeq r\) for \(\mu > 0\) near the center, the singularity is at least locally naked. In the case of the self-similar case (\(\mu = 0\)), a similar treatment is possible, and the singularity is known to be naked for small values of \(\lambda\).\(^3\)) We consider the situation in which the collapsing dust ball is attached to an outer vacuum region at a comoving radius \(\tilde{r} = \text{[constant]}\), within which the null ray (A.2) is outside the apparent horizon. Then, the singularity is globally naked, and the weak version of CCH is violated.

**Appendix B**

---

**Frequency of the Singularity**

---

It is helpful to compare the gauge used in this paper with one used in much of the literature. This comparison shows that \(\omega_s\) defined in §4 coincides with the characteristic frequency of the singularity introduced in Ref. 10), up to a numerical factor. Let us denote the comoving coordinates by \((\tilde{t}, \tilde{r})\), in which \(\tilde{r}\) is chosen to coincide with the physical radius \(R\) at the initial regular epoch \(\tilde{t} = 0\), i.e., we have \(R(0, \tilde{r}) = \tilde{r}\). We assume that the mass function \(F(\tilde{r})\) can be expanded near the regular center as

\[ F(\tilde{r}) = F_1 \tilde{r}^a + F_2 \tilde{r}^b + \cdots, \]

where \(a\) and \(b\) are constants satisfying \(a < b\). Then the initial density profile can be written

\[ \rho(0, \tilde{r}) = \frac{a F_1}{8\pi} \tilde{r}^{a-3} + \frac{b F_2}{8\pi} \tilde{r}^{b-3} + \cdots. \]

(B.1)

Comparing Eq. (B.1) with Eqs. (2.5) and (2.6), we obtain the powers and coefficients of \(F(\tilde{r})\) as

\[ a = 3, \quad b = \frac{3(3\mu + 2)}{3\mu + 1}, \]

\[ F_1 = \frac{4}{9t_{in}^2}, \quad F_2 = -\frac{8(\mu + 1)^{3/3\mu + 1}}{9\lambda^{3/(3\mu + 1)}(-t_{in})^{(9\mu + 5)/(3\mu + 1)}}. \]

It is found that the power \(b\) satisfies \(3 < b \leq 6\) for \(\mu \geq 0\). In Ref. 10), Harada et al. determined the characteristic frequency of the naked singularity in the analytic model.
\( \mu = 1/6 \) using physical consideration. It is easy to repeat their computations for general values of \( \mu > 0 \). One possible quantity, composed only of \( F_1 \) and \( F_2 \), which is independent of the choice of the initial time slice and has the dimension of frequency is

\[
F_1^{(9\mu+5)/(6\mu)} (-F_2)^{-(3\mu+1)/(3\mu)} = (\mu + 1)^{-1/\mu} \left( \frac{2}{3} \right)^{(9\mu+5)/(3\mu)} \left( \frac{9}{8} \right)^{(3\mu+1)/(3\mu)} \omega_s, \tag{B.2}
\]

where we have used Eq. (4.4). In terms of \( \gamma \), the quantity (B.2) can be written as follows:

\[
\Omega_\gamma (\lambda) \equiv F_1^{(2\gamma+9)/(2(3-\gamma))} (-F_2)^{-3/(3-\gamma)}.
\]

In the case of the analytic LTB model (\( \gamma = 2 \)), we have

\[
\Omega_2 (\lambda) = F_1^{13/2} (-F_2)^{-3},
\]

which coincides with the frequency defined in Ref. 10), up to a numerical factor. This shows that it is valid to define \( \omega_s \equiv \lambda^{1/\mu} \) as the frequency of a singularity. In the self-similar LTB solution (\( \mu = 0 \)), such a quantity does not exist, because of the scale-invariant nature of self-similar spacetimes.

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