THE SCHUR FUNCTOR OF UNIVERSAL ENVELOPING PRE-LIE ALGEBRA

VLADIMIR DOTSENKO AND OISIN FLYNN-CONNOLLY

ABSTRACT. Recent work of the first author and Tamaroff implies that the embedding of the Lie operad into the operad of pre-Lie algebras has the PBW property, and consequently, a functorial PBW type theorem holds for universal enveloping pre-Lie algebras of Lie algebras, generalising Segal’s PBW theorem. We compute the corresponding Schur functor in terms of combinatorics of rooted trees.

1. INTRODUCTION

The algebraic structure of a pre-Lie algebra, or a right-symmetric algebra, is formally defined as a vector space $V$ equipped with a binary operation $\triangleleft$ satisfying the identity

$$(a_1 \triangleleft a_2) \triangleleft a_3 - a_1 \triangleleft (a_2 \triangleleft a_3) = (a_1 \triangleleft a_3) \triangleleft a_2 - a_1 \triangleleft (a_3 \triangleleft a_2).$$

Any pre-Lie algebra is a Lie algebra with respect to the commutator $[a_1, a_2] := a_1 \triangleleft a_2 - a_2 \triangleleft a_1$, so one may consider universal enveloping pre-Lie algebras of Lie algebras. They have been first studied by Segal in [Seg94] who was motivated by geometric questions such as the classification problem of left invariant flat affine structures on Lie groups; the relevance of pre-Lie algebras for such purposes was discovered by Vinberg [Vin63]. Segal found a certain basis of nonassociative words for the universal enveloping algebra; his answer does not depend on the Lie algebra structure on $g$, and therefore one can say that an analogue of the classical Poincaré–Birkhoff–Witt theorem holds, along the lines of the general definition of [MS14], or the category theory upgrade of that definition proposed by the first author and Tamaroff in [DT18]. As one of the applications of the theory developed in [DT18], an earlier result [Dot19] of the first author was used to establish that a functorial Poincaré–Birkhoff–Witt theorem holds for the universal enveloping pre-Lie algebras of Lie algebras. This means the following strengthening of Segal’s result: the underlying vector space of the universal enveloping algebra is obtained from the underlying vector space of the original Lie algebra by applying a certain endofunctor (in fact, a Schur functor). However, that result relies on computational methods for shuffle operads and, consequently, does not give an explicit formula for the endofunctor. The classical Poincaré–Birkhoff–Witt isomorphism $U(g) \cong S(g)$ deals with quotients of free associative algebras, and therefore considers algebras spanned by associative words in generators. It is well known that pre-Lie algebras are spanned by trees whose vertices are labelled by generators: implicit in the works of Cayley [Cay57], it has been proved rigorously in [CL01, DL02]. The main result of this paper describes the Schur functor of the universal enveloping pre-Lie algebra of a Lie algebra in a way that is reminiscent of the classical Poincaré–Birkhoff–Witt isomorphism and at the same time highlights the appealing combinatorics of rooted trees. Namely, we prove the following theorem:
Theorem (Th. 1). Let \( g \) be a Lie algebra, and \( U_{\text{PreLie}}(g) \) its universal enveloping pre-Lie algebra. There is a vector space isomorphism

\[
U_{\text{PreLie}}(g) \cong RT_{\neq 1}(S(g)),
\]

where \( RT_{\neq 1} \) is the species of rooted trees for which no vertex has exactly one child; moreover, these isomorphisms can be chosen in a way that is natural with respect to Lie algebra morphisms.

Our proof completely bypasses the very technical argument of [Dot19]; instead, we use a modification of the homological criterion of freeness [DT18, Prop. 4.1] that allows us to utilise the underlying rooted tree structure, bringing in standard techniques for working with graph complexes [Kon19].

We also prove, by a similar method, a simpler result concerning associative universal enveloping algebras of pre-Lie algebras. For an operad \( \mathcal{P} \) and a \( \mathcal{P} \)-algebra \( V \), the associative universal enveloping algebra, also known as universal multiplicative enveloping algebra, is the associative algebra \( UA(V) \) whose category of left modules is equivalent to the category of operadic \( V \)-modules. For a pre-Lie algebra \( V \), this object was studied in [KU04, Th. 1] by means of noncommutative Gröbner bases, and its monomial basis was constructed. In this paper, we use the approach to associative universal enveloping algebras of [Kho18] to prove the following functorial version of the description of associative universal enveloping algebras of pre-Lie algebras:

Theorem (Th. 2). Let \( V \) be a pre-Lie algebra, and \( UA(V) \) its associative universal enveloping algebra. There is a vector space isomorphism

\[
UA(V) \cong T(V) \otimes S(V).
\]

Moreover, these isomorphisms can be chosen in a way that is natural with respect to pre-Lie algebra morphisms.

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2. Functorial PBW theorems

All operads in this paper are defined over a field \( k \) of characteristic zero, and are assumed weight graded and connected. When writing down elements of operads, we use Latin letters as placeholders; any nontrivial signs (in case one wishes to, say, work with homologically graded algebras) would only arise from applying operations to arguments via the usual Koszul sign rule. This is a short note, and we do not intend to overload it with excessive recollections. We refer the reader to [LV12] for relevant information on symmetric operads, Koszul duality, and operadic twisting cochains, and to [BLL98] for information on combinatorics of species; in this section we only offer a brief recollection of the recent results on functorial Poincaré–Birkhoff–Witt theorems. In fact, there exist two kinds of universal enveloping algebras, and two kinds of functorial Poincaré–Birkhoff–Witt theorems.

2.1. Universal enveloping \( Q \)-algebras of \( \mathcal{P} \)-algebras. The universal enveloping algebras of the first kind are defined whenever one is given a morphism of operads

\[
\phi: \mathcal{P} \to Q.
\]
Such a morphism leads to a natural functor $\phi^*$ from the category of $\mathcal{Q}$-algebras to the category of $\mathcal{P}$-algebras (pullback of the structure). This functor admits a left adjoint $\phi_!$ computed via the relative composite product formula $[\text{KM}01, \text{Rez}96]$

$$\phi_!(V) = \mathcal{Q} \circ_{\mathcal{P}} V,$$

where $V$ in the latter formula is regarded as a “constant analytic endofunctor” (a symmetric sequence supported at arity zero); this left adjoint is called the universal enveloping $\mathcal{Q}$-algebra of the $\mathcal{P}$-algebra $V$.

In joint work with Tamaroff $[\text{DT}18]$, the first author gave a categorical definition of what it means for the datum $(\mathcal{P}, \mathcal{Q}, \phi)$ to have the PBW property: by definition, one requires that there exists an endofunctor $\mathcal{X}$ such that the underlying object of the universal enveloping $\mathcal{Q}$-algebra of any $\mathcal{P}$-algebra $V$ is isomorphic to $\mathcal{X}(V)$ naturally with respect to $\mathcal{P}$-algebra morphisms. According to $[\text{DT}18, \text{Th. 3.1}]$, the datum $(\mathcal{P}, \mathcal{Q}, \phi)$ has the PBW property if and only if the right $\mathcal{P}$-module action on $\mathcal{Q}$ via $\phi$ is free; in this case, the endofunctor $\mathcal{X}$ that generates that right module satisfies the above condition:

$$\phi_!(V) \equiv \mathcal{X}(V)$$

naturally with respect to $\mathcal{P}$-algebra morphisms.

2.2. Associative universal enveloping algebras of $\mathcal{P}$-algebras. The universal enveloping algebra of the second kind is defined for any algebra $V$ over an operad $\mathcal{P}$: it is an associative algebra $\mathcal{U}A(V)$ whose category of left modules is equivalent to the category of $V$-modules defined by means of operad theory; in the “pre-operad” literature, this object is often referred to as the universal multiplicative enveloping algebra. Let us briefly summarise the relevant background information here, following $[\text{Fre}09, \text{Kho}18]$. First, one considers a particular type of $\{1, 2\}$-coloured operads, namely those whose structure operations can either have all inputs and the output of colour 1 or all inputs but one of colour 1 and the remaining input as well as the output of colour 2. Such an operad is a pair $(\mathcal{Q}, \mathcal{R})$, where $\mathcal{Q}$ is a usual operad, and $\mathcal{R}$ is a right $\mathcal{Q}$-module in the category of twisted associative algebras, or in other words, a $\text{Ass} \cdot \mathcal{Q}$-bimodule.

For a usual operad $\mathcal{P}$, one can consider the derivative $\partial(\mathcal{P})$ defined by

$$\partial(\mathcal{P})(I) := \mathcal{P}(I \sqcup \{\star\}),$$

and define a $\{1, 2\}$-coloured analytic endofunctor $(\mathcal{P}, \partial(\mathcal{P}))$, where by definition the input $\star$ and the output of $\partial(\mathcal{P})$ are of colour 2. This endofunctor has a $\{1, 2\}$-coloured operad structure arising from the operad structure on $\mathcal{P}$, and algebras $(V, M)$ over this coloured operad are precisely a $\mathcal{P}$-algebra $V$ and a $V$-module $M$. As we assume all operads connected, the augmentation $\mathcal{P} \rightarrow I$ of the operad $\mathcal{P}$ may be used to make the pair $(\mathcal{P}, \partial(I))$ a $\{1, 2\}$-coloured operad. The unit $\eta: I \rightarrow \mathcal{P}$ of the operad $\mathcal{P}$ gives rise to a morphism of two-coloured operads

$$\psi: (\mathcal{P}, \partial(I)) \rightarrow (\mathcal{P}, \partial(\mathcal{P})),$$

and if one denotes by $\mathcal{K}$ the trivial $V$-module (that is, the module on which all the operations of the augmentation ideal of $\mathcal{P}$ vanish), we have

$$(V, \mathcal{U}A(V)) \equiv (\mathcal{P}, \partial(\mathcal{P}))(\mathcal{P}, \partial(I))(V, \mathcal{K}).$$

From this observation and $[\text{DT}18, \text{Th. 3.1}]$, it immediately follows that a functorial PBW type theorem for associative universal enveloping algebras holds if
and only if the endofunctor \( \partial(\mathcal{P}) \) is a free right \( \mathcal{P} \)-module; in this case, the endofunctor \( \mathcal{Y} \) that generates that right module satisfies \( \mathcal{Y}(A) \cong UA(A) \) naturally with respect to \( \mathcal{P} \)-algebra morphisms.

3. The universal enveloping pre-Lie algebra of a Lie algebra

In this section, we prove the main result of this paper, the functorial version of the Poincaré–Birkhoff–Witt theorem for pre-Lie algebras [Seg94, Th. 2] which gives a precise description of the underlying vector space of the universal enveloping algebra via a combinatorially defined analytic endofunctor.

**Theorem 1.** Let \( g \) be a Lie algebra, and \( U_{\text{PreLie}}(g) \) its universal enveloping pre-Lie algebra. There is a vector space isomorphism

\[ U_{\text{PreLie}}(g) \cong RT_{\neq 1}(S(g)), \]

where \( RT_{\neq 1} \) is the species of rooted trees for which no vertex has exactly one child; moreover, these isomorphisms can be chosen in a way that is natural with respect to Lie algebra morphisms.

**Proof.** One of the tools for proving freeness of modules over weight graded operads is the homological criterion of freeness [DT18, Prop. 4.1] that states that a weight graded right \( \mathcal{P} \)-module \( M \) over a connected weight graded operad \( \mathcal{P} \) is free if and only if the homology of the complex

\[ (\mathcal{M} \circ \kappa \mathcal{B}(\mathcal{P}), d) \]

is concentrated in degree zero. Here the differential of the complex combines the differential of the bar complex \( \mathcal{B}(\mathcal{P}) \) of the augmentation ideal of \( \mathcal{P} \) with the twisting cochain \( \kappa : \mathcal{B}(\mathcal{P}) \rightarrow \mathcal{P} \) arising from the identity map on \( \mathcal{P} \). If the operad \( \mathcal{P} \) is Koszul, one may replace the bar complex by its homology given by the Koszul dual cooperad; this results in a much smaller complex

\[ K_{\ast}(\mathcal{M}, \mathcal{P}) := (\mathcal{M} \circ \tau \mathcal{P}, d), \]

where the whole differential comes from the twisting cochain \( \tau : \mathcal{P} \rightarrow \mathcal{P} \) arising from the identity map on the generators. As in [DT18, Prop. 4.1], if the homology of this complex is concentrated in degree zero, it represents the Schur functor that freely generates \( \mathcal{M} \) as a right \( \mathcal{P} \)-module.

According to [CL01, DL02], the underlying endofunctor of the operad \( \text{PreLie} \) is the linearisation of the species of rooted trees \( RT \). Since the operad \( \text{Lie} \) is known to be Koszul, as we just explained, to establish that this operad is free as a right \( \text{Lie} \)-module, one has to consider the chain complex

\[ (1) \quad K_{\ast}(\text{PreLie}, \text{Lie}) := (\text{PreLie} \circ \tau \text{Lie}, d). \]

Since the Koszul dual cooperad \( \text{Lie}^e \) is one-dimensional in each arity, the \( n \)-th component \( K_{\ast}(\text{PreLie}, \text{Lie})(n) \) of this chain complex has a basis of rooted trees whose vertices are decorated by disjoint subsets of \( \{1, \ldots, n\} \). The differential is reminiscent of the usual graph complex differential [Kon19]: it is equal to the sum of all possible ways to split a vertex of a tree into two vertices connected with an edge and to distribute the subset labelling that vertex between the two new vertices; such an element appears with the sign arising from the decomposition maps in the cooperad \( \text{Lie}^e \).
To compute the homology of this complex, we shall utilise a fairly standard filtration argument. Let us define the frame of a rooted tree as the longest path starting from the root and consisting of vertices that have exactly one child (the last point of the frame is the first vertex with at least two children or a leaf). Clearly, the differential either increases the cardinality of the frame by one, or leaves it unchanged. Let us define for each $d$ the space $F^d K_\ast(\text{PreLie}, \text{Lie})(n)$ as the span of all decorated trees for which the sum of the size of the frame and the homological degree does not exceed $d$. These spaces form an increasing filtration of the complex $K_\ast(\text{PreLie}, \text{Lie})(n)$. This complex is finite-dimensional, therefore we may use the spectral sequence associated to the filtration. In the associated graded complex, the differential must increase the size of the frame at each step. The frame of a rooted tree can be identified with an element of $\text{Ass}^\circ \text{Lie}^\circ (J)$, where $J$ is the set of vertices of the frame, therefore, the frame of a decorated rooted tree from $K_\ast(\text{PreLie}, \text{Lie})(n)$ can be identified with an element of $\text{Ass}^\circ \text{Lie}^\circ (J)$, since the associated graded differential only modifies the frame, the associated graded chain complex is isomorphic to the direct sum of complexes with the fixed set $J$ decorating the frame. For each such complex, the homology of the $\text{Ass}^\circ \text{Lie}^\circ (J)$ is simply $\text{Com}(J)$ concentrated in degree zero, since the operad $\text{Ass}$ is free as a right $\text{Lie}$-module, and the generators may be identified with the underlying endofunctor of $\text{Com}$. The remaining part of the differential deals separately with the trees grafted at the last vertex of the frame, and an inductive argument applies, showing that the homology is concentrated in degree zero and is isomorphic to the endofunctor $\text{RT}_{\neq 1} \circ \text{Com}$. As a consequence, the underlying object of $\text{U}_{\text{PreLie}}(g)$ is naturally isomorphic to

$$\text{PreLie}^\circ \text{Lie}(g) \cong (\text{RT}_{\neq 1} \circ \text{Com} \circ \text{Lie})^\circ (g) \cong \text{RT}_{\neq 1} \circ \text{Com}(g) \cong \text{RT}_{\neq 1}(S(g)),$$

as required. \hfill \Box

4. The associative universal enveloping algebra of a pre-Lie algebra

In this section, we prove a functorial version of the result of [KU04, Th. 1]. We remark that this result is mentioned in [Kho18, Th. 5.4]; however, the proof of that paper utilises shuffle operads, and therefore no filtrations of algebras can be obtained from filtrations of operads, and no direct conclusion about Schur functors can be made.

**Theorem 2.** Let $V$ be a pre-Lie algebra, and $\text{UA}(V)$ its associative universal enveloping algebra. There is a vector space isomorphism

$$\text{UA}(V) \cong T(V) \otimes S(V).$$

Moreover, these isomorphisms can be chosen in a way that is natural with respect to pre-Lie algebra morphisms.

**Proof.** Let us first outline the combinatorial intuition behind the answer. Once again, we remark that the underlying endofunctor of the operad $\text{PreLie}$ is the linearisation of the species of rooted trees $\text{RT}$. Therefore, instead of the endofunctor $\delta(\text{PreLie})$ we may consider the species $\delta(\text{RT})$ of rooted trees where all of the vertices but one are labelled. Such a tree has a spine which consists of the path from the root to the unlabelled vertex and all the children of the unlabelled vertex. The vertices on the path from the root to the unlabelled vertex
are ordered, and the children of the unlabelled vertex are unordered, so spines are described by the species product $\text{Ord} \times \text{Set}$, where $\text{Ord}$ is the species of linear orders (whose linearisation is the operad $\text{Ass}_+$ of unital associative algebras, and $\text{Set}$ is the species of sets (whose linearisation is the operad $\text{Com}_+$ of unital commutative algebras). Clearly, any rooted tree from $\partial(RT)$ can be viewed as the result of insertion of labelled rooted trees into the labelled vertices of the spine, so we have an isomorphism of endofunctors

$$\partial(RT) \cong (\text{Ord} \times \text{Set}) \circ RT,$$

which after linearisation gives us

$$\partial(\text{PreLie}) \cong (\text{Ass}_+ \otimes \text{Com}_+) \circ \text{PreLie},$$

so there exists some right module isomorphism $\partial(\text{PreLie}) \cong \mathcal{Y} \circ \text{PreLie}$ for a certain Schur functor $\mathcal{Y}$, we recover $\mathcal{Y} \cong \text{Ass}_+ \otimes \text{Com}_+$. From that, one deduces that

$$\text{UA}(V) \cong \partial(\text{PreLie}) \circ_{\text{PreLie}} (V) \cong ((\text{Ass}_+ \otimes \text{Com}_+) \circ \text{PreLie}) \circ_{\text{PreLie}} (V) \cong (\text{Ass}_+ \otimes \text{Com}_+)((V) \cong T(V) \otimes S(V),$$

as required. Existence of the Schur functor $\mathcal{Y}$ is established in [Kho18, Th. 5.4] using the shuffle criterion of freeness [Dot13, Th. 4], so in principle this completes the proof; however, we would also like to give a simple self-contained argument that establishes the module freeness from scratch and is somewhat reminiscent of our proof of the main result of the paper.

According to [CL01], the rooted trees interpretation of the operad $\text{PreLie}$ corresponds to the operad structure on the linearisation of the species of rooted trees defined in a simple combinatorial way: if $T_1 \in RT(n)$, and $T_2 \in RT(m)$, then for $1 \leq i \leq n$ the tree $T_1 \circ_i T_2$ is given by

$$T_1 \circ_i T_2 = \sum_{f: \text{in}(T_1, i) \to \text{vert}(T_2)} T_1 \circ_f T_2.$$ 

Here $\text{in}(T_1, i)$ are the incoming edges of the vertex $i$ in $T_1$ and $\text{vert}(T_2)$ is the set of all vertices of $T_2$; the tree $T_1 \circ_f T_2$ is obtained by replacing the vertex $i$ of the tree $T_1$ by the tree $T_2$, and grafting the subtree corresponding to the input $v$ of $i$ at the vertex $f(v)$ of $T_2$. Let us consider the filtration $F^k \partial(\text{PreLie})$ by the length of the spine, so that $F^k \partial(\text{PreLie})$ is the span of all trees with the spine of length at least $k$. The combinatorial operad structure above shows that this filtration respects the right pre-Lie module structure: insertion can either keep the spine length the same or increase it. The associated graded module $\text{gr}_F \partial(\text{PreLie})$ has a simpler structure, for which insertion at each vertex $i$ along the spine is a simple combinatorial insertion (this may be compared with the nonassociative permutative operad structure [Liv06]). Thus, each tree $T$ from $\text{gr}_F \partial(\text{PreLie})$ is obtained by combinatorial insertions of appropriate trees into vertices of the spine, and so we have the right module isomorphism

$$\text{gr}_F \partial(\text{PreLie}) \cong (\text{Ass}_+ \otimes \text{Com}_+) \circ \text{PreLie},$$

which by a standard spectral sequence argument leads to an isomorphism

$$\partial(\text{PreLie}) \cong (\text{Ass}_+ \otimes \text{Com}_+) \circ \text{PreLie}.$$ 

From this, we proceed as above to show that $\text{UA}(V) \cong T(V) \otimes S(V)$ naturally in $V$. \qed
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