ON $\alpha$-EMBEDDED SETS AND EXTENSION OF MAPPINGS

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Abstract. We introduce and study $\alpha$-embedded sets and apply them to generalize the Kuratowski Extension Theorem.

1. Introduction

A subset $A$ of a topological space $X$ is called functionally open (functionally closed) if there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}((0,1])$ ($A = f^{-1}(0)$).

Let $G_0^\alpha(X)$ and $F_0^\alpha(X)$ be the collections of all functionally open and functionally closed subsets of a topological space $X$, respectively. Assume that the classes $G_\xi^\alpha(X)$ and $F_\xi^\alpha(X)$ are defined for all $\xi < \alpha$, where $0 < \alpha < \omega_1$. Then, if $\alpha$ is odd, the class $G_\alpha^\alpha(X)$ ($F_\alpha^\alpha(X)$) consists of all countable intersections (unions) of sets of lower classes, and, if $\alpha$ is even, the class $G_\alpha^\alpha(X)$ ($F_\alpha^\alpha(X)$) consists of all countable unions (intersections) of sets of lower classes. The classes $F_\alpha^\alpha(X)$ for odd $\alpha$ and $G_\alpha^\alpha(X)$ for even $\alpha$ are said to be functionally additive, and the classes $F_\alpha^\alpha(X)$ for even $\alpha$ and $G_\alpha^\alpha(X)$ for odd $\alpha$ are called functionally multiplicative. If a set belongs to the $\alpha$'th functionally additive and to the $\alpha$'th functionally multiplicative class simultaneously, then it is called functionally ambiguous of the $\alpha$'th class.

For every $0 \leq \alpha < \omega_1$ let

$$B_\alpha^\alpha(X) = F_\alpha^\alpha(X) \cup G_\alpha^\alpha(X)$$

and let

$$B^\alpha(X) = \bigcup_{0 \leq \alpha < \omega_1} B_\alpha^\alpha(X).$$

If $A \in B^\alpha(X)$, then $A$ is said to be a functionally measurable set.

If $P$ is a property of mappings, then by $P(X,Y)$ we denote the collection of all mappings $f : X \to Y$ with the property $P$. Let $P(X) (P^*(X))$ be the collection of all real-valued (bounded) mappings on $X$ with a property $P$.

By the symbol $C$ we denote, as usually, the property of continuity.

Let $K_0(X,Y) = C(X,Y)$. For an ordinal $0 < \alpha < \omega_1$ we say that a mapping $f : X \to Y$ belongs to the $\alpha$'th functional Lebesgue class, $f \in K_\alpha(X,Y)$, if the preimage $f^{-1}(V)$ of an arbitrary open set $V \subseteq Y$ is of the $\alpha$'th functionally additive class in $X$.

A subspace $E$ of $X$ is $P$-embedded ($P^*$-embedded) in $X$ if every (bounded) function $f \in P(E)$ can be extended to a (bounded) function $g \in P(X)$. 

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A subset $E$ of $X$ is said to be $z$-embedded in $X$ if every functionally closed set in $E$ is the restriction of a functionally closed set in $X$ to $E$. It is well-known that

$$E - C\text{-embedded} \Rightarrow E - C^*\text{-embedded} \Rightarrow E - z\text{-embedded}.$$  

Recall that sets $A$ and $B$ are completely separated in $X$ if there exists a continuous function $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

The following theorem was proved in [2, Corollary 3.6].

**Theorem 1.1** (Blair-Hager). *A subset $E$ of a topological space $X$ is $C$-embedded in $X$ if and only if $E$ is $z$-embedded in $X$ and $E$ is completely separated from every functionally closed set in $X$ disjoint from $E$.***

It is natural to consider $P$- and $P^*$-embedded sets if $P = K_\alpha$ for $\alpha > 0$. In connection with this we introduce and study a class of $\alpha$-embedded sets which coincides with the class of $z$-embedded sets when $\alpha = 0$. In Section 3 we generalize the notion of completely separated sets to $\alpha$-separated sets. Section 4 deals with ambiguously $\alpha$-embedded sets which play the important role in the extension of bounded $K_\alpha$-functions. In the fifth section we prove an analog of the Tietze-Uryshon Extension Theorem for $K_\alpha$-functions. Section 6 concerns the question when $K_1$-embedded sets coincide with $K_1^*$-embedded sets. The seventh section presents a generalization of the Kuratowski Theorem [11, p. 445] on extension of $K_\alpha$-mappings with values in Polish spaces.

### 2. $\alpha$-EMBEDDED SETS

Let $0 \leq \alpha < \omega_1$. A subset $E$ of a topological space $X$ is $\alpha$-embedded in $X$ if for any set $A$ of the $\alpha$’th functionally additive (multiplicative) class in $E$ there is a set $B$ of the $\alpha$’th functionally additive (multiplicative) class in $X$ such that $A = B \cap E$.

**Proposition 2.1.** *Let $X$ be a topological space, $0 \leq \alpha < \omega_1$ and let $E \subseteq X$ be an $\alpha$-embedded set of the $\alpha$’th functionally additive (multiplicative) class in $X$. Then every set of the $\alpha$’th functionally additive (multiplicative) class in $E$ belongs to the $\alpha$’th functionally additive (multiplicative) class in $X$.***

**Proof.** For a set $C$ of the $\alpha$’th functionally additive (multiplicative) class in $E$ we choose a set $B$ of the $\alpha$’th functionally additive (multiplicative) class in $X$ such that $C = B \cap E$. Then $C$ belongs to the $\alpha$’th functionally additive (multiplicative) class in $X$ as the intersection of two sets of the same class. \hfill $\square$

**Proposition 2.2.** *Let $X$ be a topological space, $E \subseteq X$ and

(i) $X$ is perfectly normal, or
(ii) $X$ is completely regular and $E$ is its Lindelöf subset, or
(iii) $E$ is a functionally open subset of $X$, or
(iv) $X$ is a normal space and $E$ is its $F_\alpha$-subset,

then $E$ is $0$-embedded in $X$.***
Proof. Let $G$ be a functionally open set in $E$.

(i). Choose an open set $U$ in $X$ such that $G = E \cap U$. Then $U$ is functionally open in $X$ by Vedenissoff’s theorem \[31\] p. 45.

(ii). Let $U$ be an open set in $X$ such that $G = E \cap U$. Since $X$ is completely regular, $U = \bigcup_{s \in S} U_s$, where $U_s$ is a functionally open set in $X$ for each $s \in S$.

Notice that $G$ is Lindelöf, provided $G$ is $F_\alpha$ in the Lindelöf space $E$ \[31\] p. 192. Then there exists a countable set $S_0 \subseteq S$ such that $G \subseteq \bigcup_{s \in S_0} U_s$. Let $V = \bigcup_{s \in S_0} U_s$.

Then $V$ is functionally open in $X$ and $V \cap E = G$.

(iii). Consider continuous functions $\varphi : E \rightarrow [0, 1]$ and $\psi : X \rightarrow [0, 1]$ such that $G = \varphi^{-1}((0, 1])$ and $E = \psi^{-1}((0, 1])$. For each $x \in X$ we set

$$f(x) = \begin{cases} \varphi(x) \cdot \psi(x), & x \in E, \\ 0, & x \in X \setminus E. \end{cases}$$

Since $\varphi(x) \cdot \psi(x) = 0$ on $\overline{E} \setminus E$, $f : X \rightarrow [0, 1]$ is continuous. Moreover, $G = f^{-1}((0, 1])$. Hence, the set $G$ is functionally open in $X$.

(iv). Let $\tilde{G}$ be an open set in $X$ such that $G = \tilde{G} \cap E$. Since $G$ is functionally open in $E$, $G$ is $F_\alpha$ in $E$. Consequently, $G$ is $F_\alpha$ in $X$, provided $E$ is $F_\alpha$ in $X$. Since $X$ is normal, for every $n \in \mathbb{N}$ there exists a continuous function $f_n : X \rightarrow [0, 1]$ such that $f_n(x) = 1$ if $x \in F_\alpha$ and $f_n(x) = 0$ if $x \in X \setminus \tilde{G}$. Then the set $V = \bigcup_{n=1}^{\infty} f_n^{-1}((0, 1])$ is functionally open in $X$ and $V \cap E = G$. \hfill \Box

Examples [2, 3] and [2, 4] show that none of the conditions (i)-(iv) on $X$ and $E$ in Proposition 2.2 can be weaken.

Recall that a topological space $X$ is said to be perfect if every its closed subset is $G_\delta$ in $X$.

Example 2.3. There exist a perfect completely regular space $X$ and its functionally closed subspace $E$ which is not $\alpha$-embedded in $X$ for every $0 \leq \alpha < \omega_1$.

Consequently, there is a bounded continuous function on $E$ which cannot be extended to a $\mathcal{K}_\alpha$-function for every $\alpha$.

Proof. Let $X$ be the Niemyski plane \[31\] p. 22], i.e. $X = \mathbb{R} \times [0, +\infty)$, where a base of neighborhoods of $(x, y) \in X$ with $y > 0$ form open balls with the center in $(x, y)$, and a base of neighborhoods of $(x, 0)$ form the sets $U \cup \{(x, 0)\}$, where $U$ is an open ball which tangent to $\mathbb{R} \times \{0\}$ in the point $(x, 0)$. It is well-known that the space $X$ is perfect and completely regular, but is not normal.

Denote $E = \mathbb{R} \times \{0\}$. Since the function $f : X \rightarrow \mathbb{R}$, $f(x, y) = y$, is continuous and $E = f^{-1}(0)$, the set $E$ is functionally closed in $X$.

Notice that every function $f : E \rightarrow \mathbb{R}$ is continuous. Therefore, $|\mathcal{B}_\alpha^*(E)| = 2^{2\aleph_0}$ for every $0 \leq \alpha < \omega_1$. On the other hand, $|\mathcal{B}_\alpha^*(X)| = 2^{\aleph_0}$ for every $0 \leq \alpha < \omega_1$, provided the space $X$ is separable. Hence, for every $0 \leq \alpha < \omega_1$ there exists a set $A \in \mathcal{B}_\alpha^*(E)$ which can not be extend to a set $B \in \mathcal{B}_\alpha^*(X)$.
Observe that a function \( f : E \to [0,1] \), such that \( f = 1 \) on \( A \) and \( f = 0 \) on \( E \setminus A \), is continuous on \( E \). But there is no \( K_\alpha \)-function \( f : X \to [0,1] \) such that \( g|_E = f \), since otherwise the set \( B = g^{-1}(1) \) would be an extension of \( A \). \( \square \)

**Example 2.4.** There exist a compact Hausdorff space \( X \) and its open subspace \( E \) which is not \( \alpha \)-embedded in \( X \) for every \( 0 \leq \alpha < \omega_1 \).

*Proof.* Let \( X = D \cup \{ \infty \} \) be the Alexandroff compactification of an uncountable discrete space \( D \) \([5, p. 169]\) and \( E = D \). Fix \( 0 \leq \alpha < \omega_1 \) and choose an arbitrary uncountable set \( A \subseteq E \) with uncountable complement \( X \setminus A \). Evidently, \( A \) is functionally closed in \( E \). Assume that there is a set \( B \) of the \( \alpha \)'th functionally multiplicative class in \( X \) such that \( A = B \cap E \). Clearly, \( B = A \cup \{ \infty \} \). Moreover, there exists a function \( f : X \to \mathbb{R} \) of the \( \alpha \)'th Baire class such that \( B = f^{-1}(0) \) \([9, Lemma 2.1]\). But every continuous function on \( X \), and consequently every Baire function of the class \( \alpha \) on \( X \) satisfies the equality \( f(x) = f(\infty) \) for all but countably many points \( x \in X \), which implies a contradiction. \( \square \)

**Proposition 2.5.** Let \( 0 \leq \alpha \leq \beta < \omega_1 \) and let \( X \) be a topological space. Then every \( \alpha \)-embedded subset of \( X \) is \( \beta \)-embedded.

*Proof.* Let \( E \) be an \( \alpha \)-embedded subset of \( X \). If \( \beta = \alpha \), the assertion of the proposition is obvious. Suppose the assertion is true for all \( \alpha \leq \beta < \xi \) and let \( A \) be a set of the \( \xi \)'th functionally additive class in \( E \). Then there exists a sequence of sets \( A_n \) of functionally multiplicative classes \( < \xi \) in \( E \) such that \( A = \bigcup_{n=1}^{\infty} A_n \).

According to the assumption, for every \( n \in \mathbb{N} \) there is a set \( B_n \) of a functionally multiplicative class \( < \xi \) in \( X \) such that \( A_n = B_n \cap E \). Then the set \( B = \bigcup_{n=1}^{\infty} B_n \) belongs to the \( \xi \)'th functionally additive class in \( X \) and \( A = B \cap E \). \( \square \)

The inverse proposition is not true, as the following result shows.

**Theorem 2.6.** There exist a completely regular space \( X \) and its 1-embedded subspace \( E \subseteq X \) which is not 0-embedded in \( X \).

*Proof.* Let \( X_0 = [0,1] \), \( X_s = \mathbb{N} \) for every \( s \in (0,1] \), \( Y = \prod_{s \in (0,1]} X_s \) and

\[
X = [0,1] \times Y = \prod_{s \in [0,1]} X_s.
\]

Then \( X \) is completely regular as a product of completely regular spaces \( X_s \). Let \( A_1 = (0,1] \) and \( A_2 = \{0\} \).

For \( i = 1,2 \) we consider the set

\[
F_i = \bigcap_{n \neq i} \{ y = (y_s)_{s \in [0,1]} \in Y : |\{s \in (0,1) : y_s = n\}| \leq 1 \}.
\]

Obviously, \( F_1 \cap F_2 = \emptyset \) and the sets \( F_1 \) and \( F_2 \) are closed in \( Y \).
Let

\[ B_1 = A_1 \times F_1, \ \ \ B_2 = A_2 \times F_2 \text{ and } E = B_1 \cup B_2. \]

It is easy to see that the sets \( B_1 \) and \( B_2 \) are closed in \( E \), and consequently they are functionally clopen in \( E \).

**Claim 1.** The set \( B_i \) is 0-embedded in \( X \) for every \( i = 1, 2 \).

**Proof.** Let \( C \) be a functionally open set in \( B_i \).

Let us consider the set

\[ H = \{ x = (x_s)_{s \in [0,1]} \in X : |\{ s \in [0,1] : x_s \neq 1 \}| \leq \aleph_0 \}. \]

Then the set \([0,1] \times F_i \) is closed in \( H \) for every \( i = 1, 2 \). Since \( H \) is the \( \Sigma \)-product of the family \( (X_s)_{s \in [0,1]} \) (see [5] p. 118), according to [10] the space \( H \) is normal. Consequently, \([0,1] \times F_i \) is normal as closed subspace of normal space for every \( i = 1, 2 \). Clearly, \( B_1 \) is functionally open in \([0,1] \times F_1 \). Hence, \( B_1 \) is 0-embedded in \([0,1] \times F_1 \) according to Proposition 2.2(iii). Then \( C \) is functionally open in \([0,1] \times F_1 \) by Proposition 2.2. Notice that the set \([0,1] \times F_1 \) is 0-embedded in \( H \) by Propositions 2.2(iv). Hence, there exists a functionally open set \( C' \) in \( H \) such that \( C' \cap ([0,1] \times F_1) = C \). It follows from [3] that \( H \) is 0-embedded in \( X \). Then there exists a functionally open set \( C'' \) in \( X \) such that \( C'' \cap H = C'' \). Evidently, \( C'' \cap B_1 = C \). Therefore, the set \( B_1 \) in 0-embedded in \( X \).

Analogously, it can be shown that the set \( B_2 \) is 0-embedded in \( X \), using the fact that \( B_2 \) is 0-embedded in \([0,1] \times F_2 \) according to Proposition 2.2(iv).

**Claim 2.** The set \( E \) is not 0-embedded in \( X \).

**Proof.** Assuming the contrary, we choose a functionally closed set \( D \) in \( X \) such that \( D \cap E = B_1 \). Then \( D = f^{-1}(0) \) for some continuous function \( f : X \to [0,1] \).

It follows from [5] p. 117 that there exists a countable set \( S = \{0\} \cup T \), where \( T \subseteq (0,1] \), such that for any \( x = (x_s)_{s \in [0,1]} \) and \( y = (y_s)_{s \in [0,1]} \) of \( X \) the equality \( x|_S = y|_S \) implies \( f(x) = f(y) \). Let \( y_0 \in Y \) be such that \( y_0|_T \) is a sequence of different natural numbers which are not equal to 1 or 2. We choose \( y_1 \in F_1 \) and \( y_2 \in F_2 \) such that \( y_0|_T = y_1|_T = y_2|_T \). Then

\[ f(a, y_0) = f(a, y_1) = f(a, y_2) \]

for all \( a \in [0,1] \). We notice that \( f(0, y_1) = 0 \). Therefore, \( f(0, y_0) = 0 \). But \( f(a, y_2) > 0 \) for all \( a \in A_2 \). Then \( f(a, y_0) > 0 \) for all \( a \in A_2 \). Hence, \( A_1 = (f^{y_0})^{-1}(0) \), where \( f^{y_0}(a) = f(a, y_0) \) for all \( a \in [0,1] \), and \( f^{y_0} \) is continuous. Thus, the set \( A_1 = (0,1] \) is closed in \([0,1] \), which implies a contradiction.

**Claim 3.** The set \( E \) is 1-embedded in \( X \).

**Proof.** Let \( C \) be a functionally \( G_\delta \)-set in \( E \). We put

\[ E_1 = A_1 \times Y, \ \ E_2 = A_2 \times Y. \]

Then the set \( E_1 \) is functionally open in \( X \) and the set \( E_2 \) is functionally closed in \( X \). For \( i = 1, 2 \) let \( C_i = C \cap B_i \). Since for every \( i = 1, 2 \) the set \( C_i \) is functionally
Let $B_i$ be a subset of $X$. By Proposition 2.5, there exists a functionally $G_\delta$-set $\tilde{C}_i$ in $X$ such that $\tilde{C}_i \cap B_i = C_i$. Let

$$\tilde{C} = (\tilde{C}_1 \cap E_1) \cup (\tilde{C}_2 \cap E_2).$$

Then $\tilde{C}$ is functionally $G_\delta$ in $X$ and $\tilde{C} \cap E = C$. \hfill $\square$

3. $\alpha$-separated sets and $\alpha$-separated spaces

Let $0 \leq \alpha < \omega_1$. Subsets $A$ and $B$ of a topological space $X$ are said to be $\alpha$-separated if there exists a function $f \in K_\alpha(X)$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

Remark that $0$-separated sets are also called completely separated [5, p. 42].

Lemma 3.1 (Lemma 2.1 [8]). Let $X$ be a topological space, $\alpha > 0$ and let $A_\alpha \subseteq X$ be a subset of the $\alpha$'th functionally additive class. Then there exists a sequence $(A_n)_{n=1}^\infty$ such that each $A_n$ is functionally ambiguous of the class $\alpha$ in $X$, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $A = \bigcup_{n=1}^\infty A_n$.

Proof. Since $A$ belongs to the $\alpha$'th functionally additive class, $A = \bigcup_{n=1}^\infty B_n$, where each $B_n$ belongs to the functionally multiplicative class $< \alpha$ in $X$. Therefore, each $B_n$ is functionally ambiguous of the class $\alpha$. Let $A_1 = B_1$ and $A_n = B_n \setminus \bigcup_{k<n} B_k$ for $n > 1$. Then $(A_n)_{n=1}^\infty$ is the required sequence. \hfill $\square$

Lemma 3.2 (Lemma 2.2 [8]). Let $X$ be a topological space, $\alpha \geq 0$ and let $A_n$ belongs to the $\alpha$'th functionally additive class in $X$ for every $n \in \mathbb{N}$ with $X = \bigcup_{n=1}^\infty A_n$.

Then there exists a sequence $(B_n)_{n=1}^\infty$ of mutually disjoint functionally ambiguous sets of the class $\alpha$ in $X$ such that $B_n \subseteq A_n$ and $X = \bigcup_{n=1}^\infty B_n$.

Proof. If follows from Lemma 3.1 that for every $n \in \mathbb{N}$ there exists a sequence $(F_{n,m})_{m=1}^\infty$ such that each $F_{n,m}$ is functionally ambiguous of the class $\alpha$ in $X$, $F_{n,m} \cap F_{n,k} = \emptyset$ for $m \neq k$ and $A_n = \bigcup_{m=1}^\infty F_{n,m}$. Let $k : \mathbb{N}^2 \to \mathbb{N}$ be a bijection.

Set

$$C_{n,m} = F_{n,m} \setminus \bigcup_{k(p,s) \lt k(n,m)} F_{p,s}.$$ 

Evidently, $\bigcup_{n,m=1}^\infty C_{n,m} = X$. Let $B_n = \bigcup_{m=1}^\infty C_{n,m}$. Then $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n = X$ and $B_n \subseteq \bigcup_{n=1}^\infty F_{n,m} = A_n$. Notice that each $C_{n,m}$ is functionally ambiguous of the class $\alpha$. Therefore, $B_n$ belongs to the functionally additive class $\alpha$ for every $n$. Moreover, $B_n \cap B_m = \emptyset$ for $n \neq m$. Since $X \setminus B_n = \bigcup_{k \neq n} B_k$, $B_n$ is functionally ambiguous of the class $\alpha$. \hfill $\square$
Lemma 3.3. Let $0 \leq \alpha < \omega_1$ and let $A$ be a subset of the $\alpha$’th functionally multiplicative class of a topological space $X$. Then there exists a function $f \in K_\alpha^*(X)$ such that $A = f^{-1}(0)$.

Proof. For $\alpha = 0$ the lemma implies from the definition of a functionally closed set.

Let $\alpha > 0$. Since the set $B = X \setminus A$ is of the $\alpha$’th functionally additive class, there exists a sequence of functionally ambiguous sets $B_n$ of the $\alpha$’th class in $X$ such that $B = \bigcup_{n=1}^{\infty} B_n$ and $B_n \cap B_m = \emptyset$ for all $n \neq m$ by Lemma 3.1. Define a function $f : X \to [0,1]$, 

$$f(x) = \begin{cases} 0, & \text{if } x \in A, \\ \frac{1}{n}, & \text{if } x \in B_n. \end{cases}$$

Take an arbitrary open set $V \subseteq [0,1]$. If $0 \not\in V$ then $f^{-1}(V)$ is of the $\alpha$’th functionally additive class as a union of at most countably many sets $B_n$. If $0 \in V$ then there exists such a number $N$ that $\frac{1}{n} \in V$ for all $n > N$. Then the set $X \setminus f^{-1}(V) = \bigcup_{n=1}^{N} B_n$ belongs to the $\alpha$’th functionally multiplicative class. Hence, $f^{-1}(V)$ is of the $\alpha$’th functionally additive class in $X$. Therefore, $f \in K_\alpha^*(X)$.

Proposition 3.4. Let $0 \leq \alpha < \omega_1$ and let $X$ be a topological space. Then any two disjoint sets $A$ and $B$ of the $\alpha$’th functionally multiplicative class in $X$ are $\alpha$-separated.

Proof. By Lemma 3.3 we choose functions $f_1, f_2 \in K_\alpha(X)$ such that $A = f_1^{-1}(0)$ and $B = f_2^{-1}(0)$. For all $x \in X$ let 

$$f(x) = \frac{f_1(x)}{f_1(x) + f_2(x)}.$$ 

It is easy to see that $f \in K_\alpha(X)$, $f(x) = 0$ on $A$ and $f(x) = 1$ on $B$.

Let $0 \leq \alpha < \omega_1$. A topological space $X$ is $\alpha$-separated if any two disjoint sets $A, B \subseteq X$ of the $\alpha$’th multiplicative class in $X$ are $\alpha$-separated. It follows from Urysohn’s Lemma [5] p. 41 that a topological space is 0-separated if and only if it is normal. Proposition 3.4 implies that every perfectly normal space is $\alpha$-separated for each $\alpha \geq 0$. It is naturally to ask whether there is an $\alpha$-separated space for $\alpha \geq 1$ which is not perfectly normal.

Example 3.5. There exists a completely regular 1-separated space which is not perfectly normal.

Proof. Let $D = D(m)$ be a discrete space of the cardinality $m$, where $m$ is a measurable cardinal number [6] 12.1. According to [6] 12.2, $D$ is not a realcompact space. Let $X = vD$ be a Hewitt realcompactification of $D$ [5] p. 218. Then $X$ is an extremally disconnected $P$-space, which is not discrete [6] 12H. Thus, there exists a point $x \in X$ such that the set $\{x\}$ is not open. Then $\{x\}$, being a closed
set, is not a $G_δ$-set, since $X$ is a $P$-space (i.e. a space in which every $G_δ$-subset is open). Therefore, the space $X$ is not perfect.

If $A$ and $B$ are disjoint $G_δ$-subsets of $X$, then $A$ and $B$ are open in $X$. Notice that in an extremally disconnected space any two disjoint open sets are completely separated \[1, 1H\]. Consequently, $A$ and $B$ are 1-separated, since every continuous function belongs to the first Lebesgue class.

Clearly, every ambiguous set $A$ of the class 0 in a topological space (i.e., every clopen set) is a functionally ambiguous set of the class 0. If $A$ is an ambiguous set of the first class, i.e. $A$ is an $F_{α}$- and a $G_δ$-set, then $A$ need not be a functionally $F_{α}$- or a functionally $G_δ$-set. Indeed, let $X$ be the Niemytski plane, $E$ be a set which is not of the $G_δα$-type in $\mathbb{R}$ and let $A = E \times \{0\}$ be a subspace of $X$. Then $A$ is closed and consequently $G_δ$-subset of $X$, since the Niemytski plane is a perfect space. Assume that $A$ is a functionally $F_{α}$-set in $X$. Then $A = \bigcup_{n=1}^{∞} A_n$, where $A_n$ is a functionally closed subset of $X$ for every $n \in \mathbb{N}$. According to [13, Theorem 5.1], a closed subset $F$ of $X$ is a functionally closed set in $X$ if and only if the set $\{x \in \mathbb{R} : (x, 0) \in F\}$ is a $G_δ$-set in $\mathbb{R}$. It follows that for every $n \in \mathbb{N}$ the set $A_n$ is a $G_δ$-subset of $\mathbb{R}$, which implies a contradiction.

**Theorem 3.6.** Let $0 ≤ α < ω_1$ and let $X$ be an $α$-separated space.

1. Every ambiguous set $A \subseteq X$ of the class $α$ is functionally ambiguous of the class $α$.

2. For any disjoint sets $A$ and $B$ of the $(α + 1)$'th additive class in $X$ there exists a set $C$ of the $(α + 1)$'th functionally multiplicative class such that $A \subseteq C \subseteq X \setminus B$.

3. Every ambiguous set $A$ of the $(α + 1)$'th class in $X$ is a functionally ambiguous set of the $(α + 1)$'th class.

4. Any set of the $α$'th multiplicative class in $X$ is $α$-embedded.

**Proof.** (1) Since the set $B = X \setminus A$ belongs to the $α$'th multiplicative class in $X$, there exists a function $f \in K_α(X)$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. Then $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Hence, the sets $A$ and $B$ are of the $α$'th functionally multiplicative class. Consequently, $A$ is a functionally ambiguous set of the class $α$.

(2) Choose two sequences $(A_n)_{n=1}^{∞}$ and $(B_n)_{n=1}^{∞}$, where $A_n$ and $B_n$ belong to the $α$'th multiplicative class in $X$ for every $n \in \mathbb{N}$, such that $A = \bigcup_{n=1}^{∞} A_n$ and $B = \bigcup_{n=1}^{∞} B_n$. Since $X$ is $α$-separated, for every $n, m \in \mathbb{N}$ there exists a function $f_{n,m} \in K_α(X)$ such that $A_n \subseteq f_{n,m}^{-1}(1)$ and $B_m \subseteq f_{n,m}^{-1}(0)$. Set $C = \bigcap_{n=1}^{∞} \bigcup_{m=1}^{∞} f_{n,m}^{-1}((0, 1])$. 


Then the set \( C \) is of the \((\alpha + 1)\)th functionally multiplicative class in \( X \) and \( A \subseteq C \subseteq X \setminus B \).

(3) Let \( A \subseteq X \) be an ambiguous set of the \((\alpha + 1)\)th class. Denote \( B = X \setminus A \). Since \( A \) and \( B \) are disjoint sets of the \((\alpha + 1)\)th additive class in \( X \), according to (2) there exists a set \( C \subseteq X \) of the \((\alpha + 1)\)th functionally multiplicative class such that \( A \subseteq C \subseteq X \setminus B \). It follows that \( A = C \), consequently \( A \) is of the \((\alpha + 1)\)th functionally multiplicative class. Analogously, it can be shown that \( B \) is also of the \((\alpha + 1)\)th functionally multiplicative class. Therefore, \( A \) is a functionally ambiguous set of the \((\alpha + 1)\)th class.

(4) If \( \alpha = 0 \) then \( X \) is a normal space. Therefore, any closed set \( F \) in \( X \) is 0-embedded by Proposition 2.2.

Let \( \alpha > 0 \) and let \( E \subseteq X \) be a set of the \(\alpha\)th multiplicative class in \( X \). Choose any set \( A \) of the \(\alpha\)th functionally multiplicative class in \( E \). Since the set \( E \setminus A \) belongs to the \(\alpha\)th functionally additive class in \( E \), there exists a sequence of sets \( B_n \) of the \(\alpha\)th functionally multiplicative class in \( E \) such that \( E \setminus A = \bigcup_{n=1}^{\infty} B_n \).

Then for every \( n \in \mathbb{N} \) the sets \( A \) and \( B_n \) are disjoint and belong to the \(\alpha\)th multiplicative class in \( X \). Since \( X \) is \(\alpha\)-separated, we can choose a function \( f_n \in K_\alpha(X) \) such that \( A \subseteq f_n^{-1}(0) \) and \( B_n \subseteq f_n^{-1}(1) \). Let \( \hat{A} = \bigcap_{n=1}^{\infty} f_n^{-1}(0) \).

Then the set \( \hat{A} \) belongs to the \(\alpha\)th functionally multiplicative class in \( X \) and \( \hat{A} \cap E = A \).\hfill \( \Box \)

**Proposition 3.7.** A topological space \( X \) is normal if and only if every its closed subset is 0-embedded.

**Proof.** We only need to prove the sufficiency. Let \( A \) and \( B \) be disjoint closed subsets of \( X \). Then \( A \) is a functionally closed subset of \( E = A \cup B \). Since \( E \) is closed in \( X \), \( E \) is a 0-embedded set. Therefore, there is a functionally closed set \( \hat{A} \) in \( X \) such that \( A = E \cap \hat{A} \). Then \( B \) is a functionally closed subset of the closed set \( D = \hat{A} \cup B \). Since \( D \) is 0-embedded in \( X \), there exists a functionally closed set \( \hat{B} \) in \( X \) such that \( B = D \cap \hat{B} \). It is easy to check that \( \hat{A} \cap \hat{B} = \emptyset \). If \( f : X \to [0,1] \) be a continuous function such that \( \hat{A} = f^{-1}(0) \) and \( \hat{B} = f^{-1}(1) \), then the sets \( U = f^{-1}([0,1/2]) \) and \( V = f^{-1}((1/2,1]) \) are disjoint and open in \( X \), \( A \subseteq U \) and \( B \subseteq V \). Hence, \( X \) is a normal space.\hfill \( \Box \)

An analog of the previous proposition takes place for hereditarily \(\alpha\)-separated spaces. We say that a topological space \( X \) is **hereditarily \(\alpha\)-separated** if every its subspace is \(\alpha\)-separated.

**Proposition 3.8.** Let \( 0 \leq \alpha < \omega_1 \) and let \( X \) be a a hereditarily \(\alpha\)-separated space. If every subset of the \((\alpha + 1)\)th multiplicative class in \( X \) is \((\alpha + 1)\)-embedded, then \( X \) is \((\alpha + 1)\)-separated.

**Proof.** Let \( A, B \subseteq X \) be disjoint sets of the \((\alpha + 1)\)th multiplicative class. Then \( A \) is ambiguous of the class \((\alpha + 1)\) in \( E = A \cup B \). Since \( E \) belongs to the \((\alpha + 1)\)th multiplicative class in \( X \), \( E \) is \((\alpha + 1)\)-embedded. Moreover, \( E \) is \(\alpha\)-separated as
a subspace of the hereditarily \( \alpha \)-separated space \( X \). According to Theorem 3.6(3) \( A \) is functionally ambiguous of the \((\alpha+1)\)th class in \( E \). Therefore, there is a set \( \hat{A} \) of the \((\alpha+1)\)th functionally multiplicative class in \( X \) such that \( A = E \cap \hat{A} \).
Then \( B \) is a functionally ambiguous subset of the class \((\alpha+1)\) in \( D = \hat{A} \cup B \).
Since \( D \) belongs to the \((\alpha+1)\)th multiplicative class in \( X \), \( D \) is \((\alpha+1)\)-embedded. Therefore, there exists a set \( \hat{B} \) of the \((\alpha+1)\)th functionally multiplicative class in \( X \) such that \( B = D \cap \hat{B} \). It is easy to check that \( \hat{A} \cap \hat{B} = \emptyset \).
Hence, the sets \( \hat{A} \) and \( \hat{B} \) are \((\alpha+1)\)-separated by Proposition 3.4. Then \( A \) and \( B \) are \((\alpha+1)\)-separated too. \( \square \)

Remark that the Alexandroff compactification of the real line \( \mathbb{R} \) endowed with the discrete topology is a hereditarily normal space which is not 1-separated.

We give some examples below of \( \alpha \)-separated subsets of a completely regular space.

**Proposition 3.9.** Let \( X \) be a completely regular space and \( A, B \subseteq X \) are disjoint sets. Then

(a) if \( A \) and \( B \) are Lindelöf \( G_\delta \)-sets, then they are 1-separated;

(b) if \( A \) is a Lindelöf hereditarily Baire space and \( B \) is a functionally \( G_\delta \)-set, then \( A \) and \( B \) are 1-separated;

(c) if \( A \) is Lindelöf and \( B \) is an \( F_\sigma \)-set, then \( A \) and \( B \) are 2-separated.

**Proof.** (a). Let \( A = \bigcap_{n=1}^{\infty} U_n \), where \( U_n \) is an open set in \( X \) for every \( n \in \mathbb{N} \).
Since \( X \) is completely regular, \( U_n = \bigcup_{s \in S_n} U_{s,n} \) for every \( n \in \mathbb{N} \), where all the sets \( U_{s,n} \) are functionally open in \( X \). Then for every \( n \in \mathbb{N} \) there is a countable set \( S_{n,0} \subseteq S_n \) such that \( A \subseteq \bigcup_{s \in S_{n,0}} U_{s,n} \), since \( A \) is Lindelöf. Let \( V_n = \bigcup_{s \in S_{n,0}} U_{s,n} \), \( n \in \mathbb{N} \). Obviously, every \( V_n \) is a functionally open set and \( A = \bigcap_{n=1}^{\infty} V_n \). Hence, \( A \) is a functionally \( G_\delta \)-subset of \( X \). Analogously, \( B \) is also a functionally \( G_\delta \)-set. Therefore, the sets \( A \) and \( B \) are 1-separated by Proposition 3.4.

(b). According to \[7, Proposition 12\] there is a functionally \( G_\delta \)-set \( C \) in \( X \) such that \( A \subseteq C \subseteq X \setminus B \). Taking a function \( f \in K_1(X) \) such that \( C = f^{-1}(0) \) and \( B = f^{-1}(1) \), we obtain that \( A \) and \( B \) are 1-separated.

(c). Let \( X \setminus B = \bigcap_{n=1}^{\infty} U_n \), where \((U_n)_{n=1}^{\infty}\) is a sequence of open subsets of \( X \).
Then \( U_n = \bigcup_{s \in S_n} U_{s,n} \) for every \( n \in \mathbb{N} \), where all the sets \( U_{s,n} \) are functionally open in \( X \). Since \( A \) is Lindelöf, \( A \subseteq V_n = \bigcup_{s \in S_{n,0}} U_{s,n} \), where the set \( S_{n,0} \) is countable for every \( n \in \mathbb{N} \). Denote \( C = \bigcap_{n=1}^{\infty} V_n \). Then \( C \) is a functionally \( G_\delta \)-set in \( X \) and \( A \subseteq C \subseteq X \setminus B \). Since \( C \) is a functionally ambiguous set of the second class, \( A \) and \( B \) are 2-separated. \( \square \)
The following example shows that the class of separation of sets $A$ and $B$ in Proposition 3.9(c) can not be made lower.

**Example 3.10.** There exist a metrizable space $X$ and its disjoint Lindelöf $F_\sigma$-subsets $A$ and $B$, which are not 1-separated.

**Proof.** Let $X = \mathbb{R}$, $A = \mathbb{Q}$ and $B$ is a countable dense subsets of irrational numbers. Assume that $A$ and $B$ are 1-separated, i.e. there exist disjoint $G_\delta$-sets $C$ and $D$ in $\mathbb{R}$ such that $A \subseteq C$ and $B \subseteq D$. Then $\overline{C} = \overline{D} = \mathbb{R}$, which implies a contradictions, since $X$ is a Baire space. □

4. **Ambiguously $\alpha$-embedded sets**

Let $0 < \alpha < \omega_1$. A subset $E$ of a topological space $X$ is ambiguously $\alpha$-embedded in $X$ if for any functionally ambiguous set $A$ of the class $\alpha$ in $E$ there exists a functionally ambiguous set $B$ of the class $\alpha$ in $X$ such that $A = B \cap E$.

**Proposition 4.1.** Let $0 < \alpha < \omega_1$ and let $X$ be a topological space. Then every ambiguously $\alpha$-embedded set $E$ in $X$ is $\alpha$-embedded in $X$.

**Proof.** Take a set $A \subseteq E$ of the $\alpha$'th functionally additive class in $E$. Then $A$ can be written as $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n$ is a functionally ambiguous set of the class $\alpha$ in $E$ for every $n \in \mathbb{N}$ by Lemma 3.1. Then there exists a sequence of functionally ambiguous sets $B_n$ of the class $\alpha$ in $X$ such that $A_n = B_n \cap E$ for every $n \in \mathbb{N}$. Let $B = \bigcup_{n=1}^{\infty} B_n$. Then the set $B$ belongs to the $\alpha$'th functionally additive class in $X$ and $B \cap E = A$. □

We will need the following auxiliary fact.

**Lemma 4.2** (Lemma 2.3 [8]). Let $0 < \alpha < \omega_1$ and let $X$ be a topological space. Then for any disjoint sets $A, B \subseteq X$ of the $\alpha$'th functionally multiplicative class in $X$ there exists a functionally ambiguous set $C$ of the class $\alpha$ in $X$ such that $A \subseteq C \subseteq X \setminus B$.

**Proof.** Lemma 3.2 implies that there are disjoint functionally ambiguous sets $E_1$ and $E_2$ of the class $\alpha$ such that $E_1 \subseteq X \setminus A$, $E_2 \subseteq X \setminus B$ and $X = E_1 \cup E_2$. It remains to put $C = E_2$. □

**Proposition 4.3.** Let $0 < \alpha < \omega_1$ and let $X$ be a topological space. Then every $\alpha$-embedded set $E$ of the $\alpha$'th functionally multiplicative class in $X$ is ambiguously $\alpha$-embedded in $X$.

**Proof.** Consider a functionally ambiguous set $A$ of the class $\alpha$ in $E$. Then there exists a set $B$ of the $\alpha$'th functionally multiplicative class in $X$ such that $A = B \cap E$. Since $E$ is of the $\alpha$'th functionally multiplicative class in $X$, the set $A$ is also of the same class in $X$. Analogously, the set $E \setminus A$ belongs to the $\alpha$'th functionally multiplicative class in $X$. It follows from Lemma 4.2 that there exists a functionally ambiguous set $C$ of the class $\alpha$ in $X$ such that $A \subseteq C$ and
Hence, \( C \cap (E \setminus A) = \emptyset \). Clearly, \( C \cap E = A \). Hence, the set \( E \) is ambiguously \( \alpha \)-embedded in \( X \).

**Example 4.4.** There exists a 0-embedded \( F_\sigma \)-set \( E \subseteq \mathbb{R} \) which is not ambiguously 1-embedded.

**Proof.** Let \( E = \mathbb{Q} \). Obviously, \( E \) is a 0-embedded set. Consider any two disjoint \( A \) and \( B \) which are dense in \( E \). Then \( A \) and \( B \) are simultaneously \( F_\sigma \)- and \( G_\delta \)-sets in \( E \). Assume that there exists an \( F_\sigma \)- and \( G_\delta \)-set \( C \) in \( \mathbb{R} \) such that \( A = E \cap C \). Since \( A \subseteq C \) and \( B \subseteq \mathbb{R} \setminus C \), the sets \( C \) and \( \mathbb{R} \setminus C \) are dense in \( \mathbb{R} \). Moreover, the sets \( C \) and \( \mathbb{R} \setminus C \) are \( G_\delta \) in \( \mathbb{R} \). It implies a contradiction, since \( \mathbb{R} \) is a Baire space.

**Example 4.5.** There exists a Borel non-measurable ambiguously 1-embedded subset of a perfectly normal compact space.

**Proof.** Let \( X \) be the "two arrows" space (see [5, p. 212]), i.e. \( X = X_0 \cup X_1 \), where \( X_0 = \{(x,0) : x \in (0,1)\} \) and \( X_1 = \{(x,1) : x \in [0,1)\} \). The topology base on \( X \) is generated by the sets

\[
((x - \frac{1}{n}, x) \times \{0\}) \cup ((x - \frac{1}{n}, x) \times \{1\}) \text{ if } x \in (0,1] \text{ and } n \in \mathbb{N}
\]

and

\[
((x, x + \frac{1}{n}) \times \{0\}) \cup ((x, x + \frac{1}{n}) \times \{1\}) \text{ if } x \in [0,1) \text{ and } n \in \mathbb{N}.
\]

For a set \( A \subseteq X \) we denote

\[A^+ = \{x \in [0,1] : (x,1) \in A\} \text{ and } A^- = \{x \in [0,1] : (x,0) \in A\}.
\]

It is not hard to verify that for every open or closed set \( A \subseteq X \) we have \( |A^+ \Delta A^-| \leq \aleph_0 \). It follows that \( |B^+ \Delta B^-| \leq \aleph_0 \) for any Borel measurable set \( B \subseteq X \).

Let \( E = X_0 \). Since \( E^+ = \emptyset \) and \( E^- = (0,1) \), the set \( E \) is non-measurable. We show that \( E \) is an ambiguously 1-embedded set. Indeed, let \( A \subseteq E \) be an \( F_\sigma \)- and \( G_\delta \)-subset of \( E \). Then \( B = E \setminus A \) is also an \( F_\sigma \)- and \( G_\delta \)-subset of \( E \). Let \( \tilde{A} \) and \( \tilde{B} \) be \( G_\delta \)-sets in \( X \) such that \( \tilde{A} = \tilde{A} \cap E \) and \( \tilde{B} = \tilde{B} \cap E \). The inequalities \( |\tilde{A}^+ \Delta \tilde{A}^-| \leq \aleph_0 \) and \( |\tilde{B}^+ \Delta \tilde{B}^-| \leq \aleph_0 \) imply that \( |C| \leq \aleph_0 \), where \( C = \tilde{A} \cap \tilde{B} \). Hence, \( C \) is an \( F_\sigma \)-set in \( X \). Moreover, \( C \) is a \( G_\delta \)-set in \( X \). Therefore, \( \tilde{A} \setminus C \) and \( \tilde{B} \setminus \tilde{C} \) are \( G_\delta \)-sets in \( X \). According to Lemma 4.2 there is an \( F_\sigma \)- and \( G_\delta \)-set \( D \) in \( X \) such that \( \tilde{A} \setminus C \subseteq D \) and \( D \cap (\tilde{B} \setminus \tilde{C}) = \emptyset \). Then \( D \cap E = A \).

5. **Extension of real-valued \( K_\alpha \)-functions**

Analogs of Proposition 5.1 and Theorem 5.3 for \( \alpha = 1 \) were proved in [7].

**Proposition 5.1.** Let \( X \) be a topological space, \( E \subseteq X \) and \( 0 < \alpha < \omega_1 \). Then the following conditions are equivalent:

(i) \( E \) is \( K_\alpha \)-embedded in \( X \);

(ii) \( E \) is ambiguously \( \alpha \)-embedded in \( X \);
(iii) \((X, E, [c, d])\) has the \(K_\alpha\)-extension property for any segment \([c, d] \subseteq \mathbb{R}\).

**Proof.** (i) \(\implies\) (ii). Take an arbitrary functionally ambiguous set \(A\) of the class \(\alpha\) in \(E\) and consider its characteristic function \(\chi_A\). Then \(\chi_A \in K_\alpha(E)\), as is easy to check. Let \(f \in K_\alpha(X)\) be an extension of \(\chi_A\). Then the sets \(f^{-1}(1)\) and \(f^{-1}(0)\) are disjoint and belong to the \(\alpha\)'th functionally multiplicative class in \(X\).

According to Lemma \ref{lem:extension} there exists a functionally ambiguous set \(B\) of the class \(\alpha\) in \(X\) such that \(f^{-1}(1) \subseteq B\) and \(B \cap f^{-1}(0) = \emptyset\). It remains to notice that \(B \cap E = f^{-1}(1) \cap E = \chi_A^{-1}(1) = A\). Hence, \(E\) is an ambiguously \(\alpha\)-embedded set in \(X\).

(ii) \(\implies\) (iii). Let \(f \in K_\alpha(E, [c, d])\). Define

\[
\begin{align*}
    h_1(x) &= \begin{cases} 
    f(x), & \text{if } x \in E, \\
    \inf f(E), & \text{if } x \in X \setminus E,
    \end{cases} \\
    h_2(x) &= \begin{cases} 
    f(x), & \text{if } x \in E, \\
    \sup f(E), & \text{if } x \in X \setminus E,
    \end{cases}
\end{align*}
\]

Then \(c \leq h_1(x) \leq h_2(x) \leq d\) for all \(x \in X\).

We prove that for any reals \(a < b\) there exists a function \(h \in K_\alpha(X)\) such that

\[
h_2^{-1}([c, a]) \subseteq h^{-1}(0) \quad \text{and} \quad h_1^{-1}([b, d]) \subseteq h^{-1}(1).
\]

Fix \(a < b\). Without loss of generality we may assume that

\[
\inf f(E) \leq a < b \leq \sup f(E).
\]

Denote

\[
A_1 = f^{-1}([c, a]), \quad A_2 = f^{-1}([b, d]).
\]

Then \(A_1\) and \(A_2\) are disjoint sets of the \(\alpha\)'th functionally multiplicative class in \(E\).

Using Lemma \ref{lem:extension} we choose a functionally ambiguous set \(C\) of the class \(\alpha\) in \(E\) such that \(A_1 \subseteq C\) and \(C \cap A_2 = \emptyset\). Since \(E\) is an ambiguously \(\alpha\)-embedded set in \(X\), there exists such a functionally ambiguous set \(D\) of the class \(\alpha\) in \(X\) that \(D \cap E = C\). Moreover, by Proposition \ref{prop:extension} there exist sets \(B_1\) and \(B_2\) of the \(\alpha\)'th functionally multiplicative class in \(X\) such that \(A_i = E \cap B_i\) when \(i = 1, 2\). Let

\[
\hat{A}_1 = D \cap B_1, \quad \hat{A}_2 = (X \setminus D) \cap B_2.
\]

Then the sets \(\hat{A}_1\) and \(\hat{A}_2\) are disjoint and belong to the \(\alpha\)'th functionally multiplicative class in \(X\). Moreover, \(A_1 = E \cap \hat{A}_1\) and \(A_2 = E \cap \hat{A}_2\). According to Proposition \ref{prop:extension} there is a function \(h \in K_\alpha^*(X)\) such that

\[
h^{-1}(0) = \hat{A}_1 \quad \text{and} \quad h^{-1}(1) = \hat{A}_2.
\]

According to \ref{thm:extension} Theorem 3.2 there exists a function \(g \in K_\alpha(X)\) such that

\[
h_1(x) \leq g(x) \leq h_2(x)
\]

for all \(x \in X\). Clearly, \(g\) is an extension of \(f\) and \(g \in K_\alpha(X, [c, d])\).

(iii) \(\implies\) (i). Let \(f \in K_\alpha^*(E)\) and let \(|f(x)| \leq C\) for all \(x \in E\). Consider a function \(g \in K_\alpha(X)\) which is an extension of \(f\). Define a function \(r: \mathbb{R} \to [-C, C]\),

\[
r(x) = \min\{C, \max\{x, -C\}\}.
\]

Obviously, \(r\) is continuous. Let \(h = r \circ g\). Then \(h \in K_\alpha^*(X)\) and \(h|_E = f\). Hence, \(E\) is \(K_\alpha^\ast\)-embedded in \(X\). \(\qed\)
Lemma 5.2. Let $0 < \alpha < \omega_1$, $X$ be a topological space and let $E \subseteq X$ be such an $\alpha$-embedded set in $X$ that for any set $A$ of the $\alpha$'th functionally multiplicative class in $X$ such that $E \cap A = \emptyset$ the sets $E$ and $A$ are $\alpha$-separated. Then $E$ is an ambiguously $\alpha$-embedded set.

Proof. Consider a functionally ambiguous set $C$ of the class $\alpha$ in $E$ and denote $C_1 = C$, $C_2 = E \setminus C$. Then there exist sets $\hat{C}_1$ and $\hat{C}_2$ of the $\alpha$'th functionally multiplicative class in $X$ such that $\hat{C}_i \cap E = C_i$ when $i = 1, 2$. Then the set $A = \hat{C}_1 \cap \hat{C}_2$ is of the $\alpha$'th functionally multiplicative class in $X$ and $A \cap E = \emptyset$. Let $h \in K_\alpha(X)$ be a function such that $E \subseteq h^{-1}(0)$ and $A \subseteq h^{-1}(1)$. Denote $H = h^{-1}(0)$ and $H_i = H \cap \hat{C}_i$ when $i = 1, 2$. Since $H_1$ and $H_2$ are disjoint sets of the $\alpha$'th functionally multiplicative class in $X$, by Lemma 4.2 there is a functionally ambiguous set $D$ of the class $\alpha$ in $X$ such that $H_1 \subseteq D \subseteq X \setminus H_2$. Obviously, $D \cap E = C$. \qed

Theorem 5.3. Let $0 < \alpha < \omega_1$ and let $E$ be a subset of a topological space $X$. Then the following conditions are equivalent:

(i) $E$ is $K_\alpha$-embedded in $X$; (ii) $E$ is $\alpha$-embedded in $X$ and for any set $A$ of the $\alpha$'th functionally multiplicative class in $X$ such that $E \cap A = \emptyset$ the sets $E$ and $A$ are $\alpha$-separated.

Proof. $(i) \implies (ii)$. Let $C \subseteq E$ be a set of the $\alpha$'th functionally multiplicative class in $E$. Then by Lemma 3.3 we choose a function $f \in K_\alpha(E)$ such that $C = f^{-1}(0)$. If $g \in K_\alpha(X)$ is an extension of $f$, then the set $B = g^{-1}(0)$ belongs to the $\alpha$'th functionally multiplicative class in $X$ and $B \cap E = C$. Hence, $E$ is an $\alpha$-embedded set in $X$.

Now consider a set $A$ of the $\alpha$'th functionally multiplicative class in $X$ such that $E \cap A = \emptyset$. According to Lemma 3.3 there is a function $h \in K_\alpha(X)$ such that $A = h^{-1}(0)$. For all $x \in E$ let $f(x) = \frac{1}{h(x)}$. Then $f \in K_\alpha(E)$. Let $g \in K_\alpha(X)$ be an extension of $f$. For all $x \in X$ let $\varphi(x) = g(x) \cdot h(x)$. Clearly, $\varphi \in K_\alpha(X)$. It is not hard to verify that $E \subseteq \varphi^{-1}(1)$ and $A \subseteq \varphi^{-1}(0)$.

$(ii) \implies (i)$. Remark that according to Lemma 5.2 the set $E$ is ambiguously $\alpha$-embedded in $X$.

Let $f \in K_\alpha(E)$ and let $\varphi : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Using Proposition 5.1 to the function $\varphi \circ f : E \to [-1, 1]$ we have that there exists a function $h \in K_\alpha(X, [-1, 1])$ such that $h|_E = \varphi \circ f$. Let

$$A = h^{-1}(-1) \cup h^{-1}(1).$$

Then $A$ belongs to the $\alpha$'th functionally multiplicative class in $X$ and $A \cap E = \emptyset$. Therefore, there exists a function $\psi \in K_\alpha(X)$ such that $A \subseteq \psi^{-1}(0)$ and $E \subseteq \psi^{-1}(1)$. For all $x \in X$ define

$$g(x) = \varphi^{-1}(h(x) \cdot \psi(x)).$$

Remark that $g \in K_\alpha(X)$ and $g|_E = f$. \qed
Corollary 5.4. Let $0 < \alpha < \omega_1$ and let $E$ be a subset of the $\alpha$'th functionally multiplicative class of a topological space $X$. Then the following conditions are equivalent:

(i) $E$ is $K_\alpha$-embedded in $X$;
(ii) $E$ is $\alpha$-embedded in $X$.

6. $K_1^*$-embedding versus $K_1$-embedding

A family $\mathcal{U}$ of non-empty open sets of a space $X$ is called a $\pi$-base \footnote{This footnote is not visible in the original text.} if for any non-empty open set $V$ of $X$ there is $U \in \mathcal{U}$ with $V \subseteq U$.

Proposition 6.1. Let $X$ be a perfect space of the first category with a countable $\pi$-base. Then there exist disjoint $F_\sigma$- and $G_\delta$-subsets $A$ and $B$ of $X$ which are dense in $X$ and $X = A \cup B$.

Proof. Let $(V_n : n \in \mathbb{N})$ be a $\pi$-base in $X$ and $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n$ is a closed nowhere dense subset of $X$ for every $n \geq 1$. Let $E_1 = X_1$ and $E_n = X_n \setminus \bigcup_{k<n} X_k$ for $n \geq 2$. Then $E_n$ is a nowhere dense $F_\sigma$- and $G_\delta$-subset of $X$ for every $n \geq 1$, $E_n \cap E_m = \emptyset$ if $n \neq m$, and $X = \bigcup_{n=1}^{\infty} E_n$.

Let $m_0 = 0$. We choose a number $n_1 \geq 1$ such that $(\bigcup_{n=1}^{n_1} E_n) \cap V_1 \neq \emptyset$ and let $A_1 = \bigcup_{n=1}^{n_1} E_n$. Since $\overline{X \setminus A_1} = X$, there exists a number $m_1 > n_1$ such that $(\bigcup_{n=n_1+1}^{m_1} E_n) \cap V_1 \neq \emptyset$. Set $B_1 = \bigcup_{n=n_1+1}^{m_1} E_n$. It follows from the equality $\overline{X \setminus (A_1 \cup B_1)} = X$ that there exists $n_2 > m_1$ such that $(\bigcup_{n=m_1+1}^{n_2} E_n) \cap V_2 \neq \emptyset$.

Further, there is such $m_2 > n_2$ that $(\bigcup_{n=n_2+1}^{m_2} E_n) \cap V_2 \neq \emptyset$. Let $A_2 = \bigcup_{n=m_1+1}^{n_2} E_n$ and $B_2 = \bigcup_{n=n_2+1}^{m_2} E_n$. Repeating this process, we obtain the sequence of numbers

$m_0 < n_1 < m_1 < \cdots < n_k < m_k < n_{k+1} < \ldots$

and the sequence of sets

$A_k = \bigcup_{n=m_k+1}^{n_k} E_n, \quad B_k = \bigcup_{n=n_k+1}^{m_k} E_n, \quad k \geq 1,$

such that $A_k \cap V_k \neq \emptyset$ and $B_k \cap V_k \neq \emptyset$ for every $k \geq 1$.

Let $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. Clearly, $X = A \cup B$, $A \cap B = \emptyset$ and $\overline{A} = \overline{\overline{B}} = X$. Moreover, $A$ and $B$ are $F_\sigma$-sets in $X$. Therefore, $A$ and $B$ are $F_\sigma$- and $G_\delta$-subsets of $X$. \qed

We say that a topological space $X$ hereditarily has a countable $\pi$-base if every its closed subspace has a countable $\pi$-base.

**Proposition 6.2.** Let $X$ be a hereditarily Baire space, $E$ be a perfectly normal ambiguously $1$-embedded subspace of $X$ which hereditarily has a countable $\pi$-base. Then $E$ is a hereditarily Baire space.

**Proof.** Assume that $E$ is not a hereditarily Baire space. Then there exists a nonempty closed set $C \subseteq X$ of the first category. Notice that $C$ is a perfectly normal space with a countable $\pi$-base. According to Proposition 6.1 there exist disjoint dense $F_\sigma$- and $G_\delta$-subsets $A$ and $B$ of $C$ such that $C = A \cup B$. Since $C$ is $F_\sigma$- and $G_\delta$-set in $E$, the sets $A$ and $B$ are also $F_\sigma$ and $G_\delta$ in $E$. Therefore there exist disjoint functionally $F_\sigma$- and $G_\delta$-subsets $\tilde{A}$ and $\tilde{B}$ of $X$ such that $A = \tilde{A} \cap E$ and $B = \tilde{B} \cap E$. Notice that the sets $\tilde{A}$ and $\tilde{B}$ are dense in $C$. Taking into account that $X$ is hereditarily Baire, we have that $C$ is a Baire space. It follows a contradiction, since $\tilde{A}$ and $\tilde{B}$ are disjoint dense $G_\delta$-subsets of $C$. \qed

Remark that there exist a metrizable separable Baire space $X$ and its ambiguously $1$-embedded subspace $E$ which is not a Baire space. Indeed, let $X = (\mathbb{Q} \times \{0\}) \cup (\mathbb{R} \times (0, 1])$ and $E = \mathbb{Q} \times \{0\}$. Then $E$ is closed in $X$. Therefore, any $F_\sigma$- and $G_\delta$-subset $C$ of $E$ is also $F_\sigma$ and $G_\delta$ in $X$. Hence, $E$ is an ambiguously $1$-embedded set in $X$.

**Theorem 6.3.** Let $X$ be a hereditarily Baire space and let $E \subseteq X$ be its perfect Lindelöf subspace which hereditarily has a countable $\pi$-base. Then $E$ is $K_1^*$-embedded in $X$ if and only if $E$ is $K_1$-embedded in $X$.

**Proof.** Since the sufficiency is obvious, we only need to prove the necessity.

According to Proposition 6.1 the set $E$ is ambiguously $1$-embedded in $X$. Using Proposition 6.2 we have $E$ is a hereditarily Baire space. Since $E$ is Lindelöf, Proposition 5.9 (b) implies that $E$ is $1$-separated from any functionally $G_\delta$-set $A$ of $X$ such that $A \cap E = \emptyset$. Therefore, by Theorem 5.3 the set $E$ is $K_1$-embedded in $X$. \qed

7. A GENERALIZATION OF THE KURATOWSKI THEOREM

K. Kuratowski [11, p. 445] proved that every mapping $f \in K_\alpha(E,Y)$ has an extension $g \in K_\alpha(X,Y)$ if the case $X$ is a metric space, $Y$ is a Polish space and $E \subseteq X$ is a set of the multiplicative class $\alpha > 0$.

In this section we will prove that the Kuratowski Extension Theorem is still valid if $X$ is a topological space and $E$ is a $K_\alpha$-embedded subset of $X$.

We say that a subset $A$ of a space $X$ is discrete if any point $a \in A$ has a neighborhood $U \subseteq X$ such that $U \cap A = \{a\}$.

**Theorem 7.1** (Theorem 2.11 [8]). Let $X$ be a topological space, $Y$ be a metrizable separable space, $0 \leq \alpha < \omega_1$ and $f \in K_\alpha(X,Y)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ such that

(i) $f_n \in K_\alpha(X,Y)$ for every $n$;
Proposition 7.2. Let \( A \) be a partition of \( X \) and let \( \alpha \) be an \( \alpha \)-separated from any disjoint with it set of the \( \alpha \)'th functionally multiplicative class in \( X \) and let \( (A_n : n \in \mathbb{N}) \) be a partition of \( E \) by functionally ambiguous sets of the class \( \alpha \) in \( E \). Then there is a partition \( (B_n : n \in \mathbb{N}) \) of \( X \) by functionally ambiguous sets of the class \( \alpha \) in \( X \) such that \( A_n = E \cap B_n \) for every \( n \in \mathbb{N} \).

Proof. According to Proposition 5.2 for every \( n \in \mathbb{N} \) there exists a functionally ambiguous set \( D_n \) of the class \( \alpha \) in \( X \) such that \( A_n = D_n \cap E \). By the assumption there exists a function \( f \in K_\alpha(X) \) such that \( E \subseteq f^{-1}(0) \) and \( X \setminus \bigcup_{n=1}^{\infty} D_n \subseteq f^{-1}(1) \).

Let \( D = f^{-1}(0) \). Then the set \( X \setminus D \) is of the \( \alpha \)'th functionally additive class in \( X \). Then there exists a sequence \( (E_n)_{n=1}^{\infty} \) of functionally ambiguous set of the class \( \alpha \) in \( X \) such that \( X \setminus D = \bigcup_{n=1}^{\infty} E_n \). For every \( n \in \mathbb{N} \) denote \( C_n = E_n \cup D_n \). Then all the sets \( C_n \) are functionally ambiguous of the class \( \alpha \) in \( X \) and \( \bigcup_{n=1}^{\infty} C_n = X \).

Let \( B_1 = C_1 \) and \( B_n = C_n \setminus \left( \bigcup_{k<n} C_k \right) \) for \( n \geq 2 \). Clearly, every \( B_n \) is a functionally ambiguous set of the class \( \alpha \) in \( X \), \( B_n \cap B_m = \emptyset \) if \( n \neq m \) and \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n = X \). Moreover, \( B_n \cap E = A_n \) for every \( n \in \mathbb{N} \). \( \square \)
Let $0 \leq \alpha < \omega_1$, $X$ and $Y$ be topological spaces and $E \subseteq X$. We say that a collection $(X, E, Y)$ has the $K_{\alpha}$-extension property if every mapping $f \in K_{\alpha}(E, Y)$ can be extended to a mapping $g \in K_{\alpha}(X, Y)$.

**Theorem 7.3.** Let $0 < \alpha < \omega_1$ and let $E$ be a subset of a topological space $X$. Then the following conditions are equivalent:

(i) $E$ is $K_{\alpha}$-embedded in $X$;

(ii) $(X, E, Y)$ has the $K_{\alpha}$-extension property for any Polish space $Y$.

**Proof.** Since the implication (ii) $\Rightarrow$ (i) is obvious, we only need to prove the implication (i) $\Rightarrow$ (ii). Let $Y$ be a Polish space with a metric $d$ which generates its topological structure such that $(Y, d)$ is complete and let $f \in K_{\alpha}(E, Y)$.

It follows from Theorem 7.1 that there exists a sequence of mappings $f_n \in K_{\alpha}(E, Y)$ which is uniformly convergent to $f$ on $E$. Moreover, for every $n \in \mathbb{N}$ the set $f_n(E) = \{y_{i_n, n} : i_n \in I_n\}$ is at most countable and discrete. We may assume that each $f_n(E)$ consists of distinct points.

For every $n \in \mathbb{N}$ and for each $(i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n$ let

$$B_{i_1, \ldots, i_n} = f_1^{-1}(y_{i_1, 1}) \cap \cdots \cap f_n^{-1}(y_{i_n, n}).$$

Then for each $i_1 \in I_1, \ldots, i_n \in I_n$ the set $B_{i_1, \ldots, i_n}$ is functionally ambiguous of the class $\alpha$ in $E$ and the family $(B_{i_1, \ldots, i_n} : i_1 \in I_1, \ldots, i_n \in I_n)$ is a partition of $E$ for every $n \in \mathbb{N}$. By Proposition 7.2 we choose a sequence of systems of functionally ambiguous sets $D_{i_1, \ldots, i_n}$ of the class $\alpha$ in $X$ such that

1. $D_{i_1, \ldots, i_n} \cap E = B_{i_1, \ldots, i_n}$ for every $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n$;
2. $(D_{i_1, \ldots, i_n} : i_1 \in I_1, \ldots, i_n \in I_n)$ is a partition of $X$ for every $n \in \mathbb{N}$.

For all $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n$ let

3. $D_{i_1, \ldots, i_n} = \emptyset$, if $B_{i_1, \ldots, i_n} = \emptyset$.

Notice that the system $(B_{i_1, \ldots, i_n, i_{n+1}} : i_{n+1} \in I_{n+1})$ forms a partition of the set $B_{i_1, \ldots, i_n}$ for every $n \in \mathbb{N}$.

For all $i_1 \in I_1$ let

$$C_{i_1} = D_{i_1}.$$

Assume that for some $n \geq 1$ the system $(C_{i_1, \ldots, i_n} : i_1 \in I_1, \ldots, i_n \in I_n)$ of functionally ambiguous sets of the class $\alpha$ in $X$ is already defined and

(A) $B_{i_1, \ldots, i_n} = E \cap C_{i_1, \ldots, i_n};$
(B) $(C_{i_1, \ldots, i_n} : i_1 \in I_1, \ldots, i_n \in I_n)$ is a partition of $X$;
(C) $C_{i_1, \ldots, i_n} = \emptyset$ if $B_{i_1, \ldots, i_n} = \emptyset;$
(D) $(C_{i_1, \ldots, i_{n-1}, i_n} : i_n \in I_n)$ is a partition of the set $C_{i_1, \ldots, i_{n-1}}$.

Fix $i_1, \ldots, i_n$. Since the set $K = C_{i_1, \ldots, i_n} \setminus \bigcup_{k \in I_{n+1}} D_{i_1, \ldots, i_n, k}$ is of the $\alpha$’th functionally multiplicative class in $X$ and $K \cap E = \emptyset$, there exists a set $H$ of the $\alpha$’th functionally multiplicative class in $X$ such that $E \subseteq H \subseteq X \setminus K$. Using [8 Lemma 2.1] we obtain that there exists a sequence $(A_k)_{k=1}^\infty$ of disjoint functionally
ambiguous sets of the class $\alpha$ in $X$ such that
\[ C_{i_1,\ldots,i_n} \setminus H = \bigcup_{k=1}^{\infty} A_k. \]

Let
\[ M_{i_1,\ldots,i_n,i_{n+1}} = \emptyset, \quad \text{if} \quad D_{i_1,\ldots,i_n,i_{n+1}} = \emptyset, \]
and
\[ M_{i_1,\ldots,i_n,i_{n+1}} = (A_{i_{n+1}} \cup D_{i_1,\ldots,i_n,i_{n+1}}) \cap C_{i_1,\ldots,i_n}, \quad \text{if} \quad D_{i_1,\ldots,i_n,i_{n+1}} \neq \emptyset. \]

Now let
\[ C_{i_1,\ldots,i_n,1} = M_{i_1,\ldots,i_n,1}, \]
and
\[ C_{i_1,\ldots,i_n,i_{n+1}} = M_{i_1,\ldots,i_n,i_{n+1}} \setminus \bigcup_{k<i_{n+1}} M_{i_1,\ldots,i_n,k} \quad \text{if} \quad i_{n+1} > 1. \]

Then for every $n \in \mathbb{N}$ the system $(C_{i_1,\ldots,i_n} : i_1 \in I_1, \ldots, i_n \in I_n)$ of functionally ambiguous sets of the class $\alpha$ in $X$ has the properties (A)–(D).

For each $n \in \mathbb{N}$ and $x \in X$ let
\[ g_n(x) = y_{i_n,n}, \]
if $x \in C_{i_1,\ldots,i_n}$. It is not hard to prove that $g_n \in K_\alpha(X,Y)$.

We show that the sequence $(g_n)_{n=1}^{\infty}$ is uniformly convergent on $X$. Indeed, let $x_0 \in X$ and $n, m \in \mathbb{N}$. Without loss of generality, we may assume that $n \geq m$. By the property (B), $x_0 \in C_{i_1,\ldots,i_n} \cap C_{j_1,\ldots,j_m}$. It follows from (B) and (D) that $i_1 = j_1, \ldots, i_m = j_m$. Take an arbitrary point $x$ from the set $B_{i_1,\ldots,i_n}$, the existence of which is guaranteed by the property (C). Then $f_m(x) = y_{i_m,m} = g_m(x_0)$ and $f_n(x) = y_{i_n,n} = g_n(x_0)$. Since the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent on $E$,\[ \lim_{n,m \to \infty} d(y_{i_m,m}, y_{i_n,n}) = 0. \]
Hence, the sequence $(g_n)_{n=1}^{\infty}$ is uniformly convergent on $X$.

Since $Y$ is a complete space, for all $x \in X$ define $g(x) = \lim_{n \to \infty} g_n(x)$. According to the property (A), $g(x) = f(x)$ for all $x \in E$. Moreover, $g \in K_\alpha(X,Y)$ as a uniform limit of functions from the class $K_\alpha$.

8. Open problems

**Question 8.1.** Does there exist a completely regular not perfectly normal space in which any functionally $G_\delta$-set is $1$-embedded?

**Question 8.2.** Does there exist a completely regular not perfectly normal space in which any set is $1$-embedded?

**Question 8.3.** Do there exist a normal space and its functionally $G_\delta$-subset which is not $1$-embedded?

**Question 8.4.** Do there exist a topological space $X$ and its subspace $E$ such that $E$ is $K_1^\ast$-embedded and is not $K_1$-embedded in $X$?
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References

[1] Blair R. Filter characterization of $z$, $C^*$, and $C$-embeddings, Fund. Math. 90 (1976), 285–300.
[2] Blair R., Hager A. Extensions of zero-sets and of real-valued functions, Math. Zeit. 136 (1974), 41–52.
[3] Corson H. Normality in subsets of product spaces, Amer. J. Math. 81 (1959), 785–796.
[4] Encyclopedia of General Topology. Edited by Klaas Pieter Hart, Jun-iti Nagata and Jerry E. Vaughan, Elsevier (2004).
[5] Engelking R. General Topology. Revised and completed edition. Heldermann Verlag, Berlin (1989).
[6] Gillman L., Jerison M. Rings of continuous functions, Van Nostrand, Princeton (1960).
[7] Kalenda, O., Spurný, J. Extending Baire-one functions on topological spaces, Topol. Appl. 149 (2005), 195–216.
[8] Karlova O. Baire classification of mappings which are continuous with respect to the first variable and of the $\alpha$'th functionally class with respect to the second variable, Mathematical Bulletin NTSH, 2 (2005), 98–114 (in Ukrainian).
[9] Karlova O. Classification of separately continuous functions with values in $\sigma$-metrizable spaces, Applied General Topology 13 (2) (2012), 167–178.
[10] Kombarov A., Malykhin V. On $\Sigma$-products, DAN SSSR, 213 (1973), 774–776 (in Russian).
[11] Kuratowski K. Topology, V.1, Moscow: Mir (1966) (in Russian).
[12] Lukeš J., Malý J., Zajíček L. Fine Topology Methods in Real Analysis and Potential Theory, Springer-Verlag, (1986).
[13] Ohta H. Extension properties and the Niemytski plane, Applied General Topology, 1 (1) (2000), 45–60.