ON THE SPECTRUM OF THE STOKES OPERATOR

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ABSTRACT. We prove Li–Yau-type lower bounds for the eigenvalues of the Stokes operator and give applications to the attractors of the Navier–Stokes equations.

1. Introduction

The monotonically ordered eigenvalues \( \{\mu_k\}_{k=1}^{\infty} \) of the scalar Dirichlet problem for the Laplacian in a bounded domain \( \Omega \subset \mathbb{R}^n \)

\[-\Delta \varphi_k = \mu_k \varphi_k, \quad \varphi_k|_{\partial \Omega} = 0 \]
satisfy the classical H.Weyl asymptotic formula

\[ \mu_k \sim \left( \frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \to \infty, \]

where \( |\Omega| \) is the \( n \)-dimensional Lebesgue measure of \( \Omega \) and \( \omega_n = \pi^{n/2}/\Gamma(1 + n/2) \) is the volume of the unit ball in \( \mathbb{R}^n \). This implies that

\[ \sum_{k=1}^{m} \mu_k \sim \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} m^{1+2/n} \quad \text{as } m \to \infty. \]

In fact,

\[ \sum_{k=1}^{m} \mu_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} m^{1+2/n}. \quad (1.1) \]

This remarkable sharp lower bound was proved in [14] and holds for all \( m = 1, 2, \ldots \) and for any domain with \( |\Omega| < \infty \).

In this paper we prove Li–Yau-type lower bounds for the spectrum \( \{\lambda_k\}_{k=1}^{\infty} \) of the Stokes operator:

\[-\Delta v_k + \nabla p_k = \lambda_k v_k, \quad \text{div } v_k = 0, \quad v_k|_{\partial \Omega} = 0, \quad (1.2)\]

where \( \Omega \subset \mathbb{R}^n, |\Omega| < \infty, n \geq 2 \). The asymptotic behavior of the eigenvalues is known [1] (\( n = 3 \), [17] (\( n \geq 2 \)):

\[ \lambda_k \sim \left( \frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \to \infty. \quad (1.3) \]

The main result of this paper proved in Section [2] is the following sharp lower bound for the spectrum of the Stokes operator:

\[ \sum_{k=1}^{m} \lambda_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} m^{1+2/n}. \]

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In addition, $\lambda_1 > \mu_1$. Then in Section 3 we apply this bound with $n = 2$ and the Lieb-Thirring inequality with improved constant to the estimates of the dimension of the attractors of the Navier-Stokes system with Dirichlet boundary conditions.

2. Li-Yau bounds for the spectrum of the Stokes operator

Throughout $\Omega$ is an open subset of $\mathbb{R}^n$ with finite $n$-dimensional Lebesgue measure $|\Omega|$: $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $|\Omega| < \infty$.

We recall the basic facts in the theory of the Navier-Stokes equations [5, 13, 19, 21]. We denote by $\mathcal{V}$ the set of smooth divergence-free vector functions with compact supports $\mathcal{V} = \{u : \Omega \rightarrow \mathbb{R}^n, u \in C_0^\infty(\Omega), \text{div } u = 0\}$ and denote by $H$ and $V$ the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $H^1(\Omega)$, respectively. The Helmholtz-Leray orthogonal projection $P$ maps $L^2(\Omega)$ onto $H$, $P : L^2(\Omega) \rightarrow H$. We have (see [19])

$L^2(\Omega) = H \oplus H^\perp$, $H^\perp = \{u \in L^2(\Omega), u = \nabla p, p \in L^2_{\text{loc}}(\Omega)\}, \quad V \subseteq \{u \in H^1_0(\Omega), \text{div } u = 0\},$

where the last inclusion becomes equality for a bounded $\Omega$ with Lipschitz boundary.

The Stokes operator $A$ is defined by the relation

$$(Au, v) = (\nabla u, \nabla v) \quad \text{for all } u, v \in V \quad (2.1)$$

and is an isomorphism between $V$ and $V'$. For a sufficiently smooth $u$

$$Au = -P\Delta u.$$ 

The Stokes operator $A$ is an unbounded self-adjoint positive operator in $H$ with compact inverse. It has a complete in $H$ and $V$ system of orthonormal eigenfunctions $\{v_k\}_{k=1}^\infty \in V$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^\infty$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$:

$$Av_k = \lambda_k v_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots. \quad (2.2)$$

Taking the scalar product with $v_k$ we have by orthonormality and (2.1) that

$$\lambda_k = \|\nabla v_k\|^2. \quad (2.3)$$

In case when $\Omega$ is a bounded domain with smooth boundary the eigenvalue problem (2.2) goes over to (1.2).

Our main goal is to prove uniform estimates for the Fourier transforms of orthonormal families of divergence-free vector functions (see Lemma 2.4).

Given a function $\varphi \in L^2(\Omega)$ we denote by $\hat{\varphi}(\xi)$ the Fourier transform of its extension by zero outside $\Omega$:

$$(\mathcal{F}\varphi)(\xi) = \hat{\varphi}(\xi) = \int e^{-ix\xi} \varphi(x) \, dx.$$ 

\textbf{Lemma 2.1.} Let the family $\{\varphi_k\}_{k=1}^m$ be orthonormal in $L^2$: $(\varphi_k, \varphi_l) = \delta_{kl}$. Then

$$\sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 \leq |\Omega|. \quad (2.4)$$

\textbf{Proof.} Denoting by $*$ the complex conjugate we have by orthonormality

$$0 \leq \int \left( e^{-ix\xi} - \sum_{k=1}^m \hat{\varphi}_k(\xi) \varphi_k(x) \right) \left( e^{-ix\xi} - \sum_{l=1}^m \hat{\varphi}_l(\xi) \varphi_l(x) \right)^* \, dx = |\Omega| - \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2. \quad \Box$$
Remark 2.1. Inequality (2.4) is nothing other than Bessel’s inequality applied to the function \( h(x) = e^{-i\xi x} \) with \( \|h\|_{L^2}^2 = |\Omega| \) and the orthonormal family \( \{\varphi_j(x)\}_{j=1}^m \). \( [14] \)

Next we observe that Lemma 2.1 still holds if we replace the orthonormality condition by suborthonormality.

**Definition 2.1.** A family \( \{\varphi_i\}_{i=1}^m \) is called suborthonormal if for any \( \zeta \in \mathbb{C}^m \)

\[
\sum_{i,j=1}^m \zeta_i \zeta^*_j (\varphi_i, \varphi_j) \leq \sum_{j=1}^m |\zeta_j|^2. \tag{2.5}
\]

**Remark 2.2.** This convenient and flexible notion of suborthonormality was introduced in \( [9] \) with real \( \zeta \in \mathbb{R}^m \) and is equivalent to the formally more general Definition 2.1.

**Lemma 2.2.** Let the family \( \{\varphi_k\}_{k=1}^m \) be suborthonormal. Then

\[
\sum_{k=1}^m |\varphi_k(\xi)|^2 \leq |\Omega|. \tag{2.6}
\]

**Proof.** As in Lemma 2.1 with (2.5) instead of orthonormality we have

\[
0 \leq \int \left( e^{-i\xi x} - \sum_{k=1}^m \hat{\varphi}_k(\xi) \varphi_k(x) \right) \left( e^{-i\xi x} - \sum_{l=1}^m \hat{\varphi}_l(\xi) \varphi_l(x) \right)^* \, dx = \]

\[
= |\Omega| - 2 \sum_{k=1}^m |\hat{\varphi}_k(\xi)|^2 + \sum_{k,l=1}^m \hat{\varphi}_k(\xi) \hat{\varphi}_l(\xi)^*(\varphi_k, \varphi_l) \leq |\Omega| - \sum_{k=1}^m |\varphi_k(\xi)|^2. \]

We now turn to orthonormal families of vector functions \( \{u_k\}_{k=1}^m, u_k = (u_k^1, \ldots, u_k^n) \).

**Lemma 2.3.** Let the family of vector functions \( \{u_k\}_{k=1}^m \) be orthonormal in \( L^2(\Omega) \) and let \( Q \) be an arbitrary orthogonal projection. Then the family \( \{Qu_k\}_{k=1}^m \) is suborthonormal.

**Proof.** We set \( u_k = v_k + w_k, v_k = Qu_k \) and \( w_k = (I - Q)u_k \). Then \( (v_k, v_l) = 0 \) for all \( k,l = 1, \ldots, n \) and \( (u_k, u_l) = (v_k, v_l) + (w_k, w_l) \). Therefore

\[
\sum_{k,l=1}^m \zeta_k \zeta^*_l (v_k, v_l) = \sum_{k,l=1}^m \zeta_k \zeta^*_l (u_k, u_l) - \sum_{k,l=1}^m \zeta_k \zeta^*_l (w_k, w_l) = \]

\[
= \sum_{k=1}^m |\zeta_k|^2 - \left\| \sum_{k=1}^m \zeta_k w_k \right\|^2 \leq \sum_{k=1}^m |\zeta_k|^2. \]

**Corollary 2.1.** If the family of vector functions \( \{u_k\}_{k=1}^m \) is orthonormal in \( L^2 \), then

\[
\sum_{k=1}^m |\bar{u}_k(\xi)|^2 \leq n|\Omega|. \tag{2.7}
\]

**Proof.** By Lemma 2.3 each family \( \{u'_k\}_{k=1}^m \) is suborthonormal \( j = 1, \ldots, n \), and (2.7) follows from Lemma 2.2.

The next lemma is the central point in the proof of the lower bounds for the spectrum and says that under the divergence-free condition the estimate (2.7) goes over to (2.8).
Lemma 2.4. If the family of vector functions \( \{u_k\}_{k=1}^m \) is orthonormal and \( u_k \in H^1_0(\Omega) \), \( \text{div} u_k = 0, \ k = 1, \ldots, m \), then
\[
\sum_{k=1}^m |\hat{u}_k(\xi)|^2 \leq (n - 1)|\Omega|.
\] (2.8)

Proof. We first observe that for all \( \xi \in \mathbb{R}^n_\rho \)
\[
\xi \cdot \hat{u}_k(\xi) = \xi \cdot \int e^{-i\xi x} u_k(x) \, dx = i \int u_k \cdot \nabla_x e^{-i\xi x} \, dx = -i \int e^{-i\xi x} \text{div} u_k \, dx = 0.
\]
Let \( \xi_0 \neq 0 \) be of the form:
\[
\xi_0 = (a, 0, \ldots, 0), \quad a \neq 0.
\] (2.9)
Since \( \xi_0 \cdot \hat{u}_k(\xi_0) = 0 \) for \( k = 1, \ldots, m \), which in view of Lemmas 2.3 and 2.2 proves the estimate (2.8) for \( \xi \) of the form (2.9):
\[
\sum_{k=1}^m |\hat{u}_k(\xi_0)|^2 = \sum_{j=2}^n \sum_{k=1}^m |\hat{u}_k(\xi_0)|^2 \leq (n - 1)|\Omega|.
\]

The general case reduces to the case (2.9) by the corresponding rotation. Let \( \rho \) be a rotation of \( \mathbb{R}^n \) about the origin represented by the orthogonal \((n \times n)-\)matrix \( \rho \) with entries \( \rho_{ij} \). Given a vector function \( u(x) = (u^1(x), \ldots, u^n(x)) \) we consider the vector function
\[
u(x) := \rho u(\rho^{-1}x), \quad x \in \rho \Omega.
\]
Let us calculate the divergence of \( u_{\rho}(x) \). Setting \( \rho^{-1}x = y, \ y_l = \sum_k (\rho^{-1})_{lk} x_k \) we have
\[
\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_l} \left( \sum_j \rho_{ij} u^j(y) \right) = \sum_j \rho_{ij} \sum_l \frac{\partial u^j(y)}{\partial y_l} \frac{\partial y_l}{\partial x_i} = \sum_j \rho_{ij} \sum_l \frac{\partial u^j(y)}{\partial y_l} (\rho^{-1})_{li}.
\]
Therefore
\[
\text{div} u_{\rho}(x) = \sum_{i,j,l} \rho_{ij} \frac{\partial u^j(y)}{\partial y_l} (\rho^{-1})_{li} = \sum_{i,j,l} \frac{\partial u^j(y)}{\partial y_l} (\rho^{-1})_{li} \rho_{lj} = \text{div} u(y).
\]
In addition,
\[
(u_{\rho}, v_{\rho}) = \int \rho u(\rho^{-1}x) \cdot \rho v(\rho^{-1}x) \, dx = \int u(\rho^{-1}x) \cdot v(\rho^{-1}x) \, dx = \int u(y) \cdot v(y) \, dy = (u, v).
\]
Combining this we obtain that the family \( \{u_k\}_{k=1}^m \) belongs to \( H^1_0(\rho \Omega) \), is orthonormal and \( \text{div}(u_k)_{\rho} = 0 \).

Next we calculate \( \hat{u}_{\rho} \) and show that
\[
\hat{u}_{\rho}(\xi) = \rho \hat{u}(\rho^{-1}\xi).
\] (2.10)
In fact,
\[
(Fu_{\rho})(\xi) = \hat{u}_{\rho}(\xi) = \int e^{i\xi x} u_{\rho}(x) \, dx = \rho \int e^{i\xi x} u(\rho^{-1}x) \, dx = \rho \int e^{i\rho^{-1}\xi y} u(y) \, dy = \rho \hat{u}(\rho^{-1}\xi).
\]

We now fix an arbitrary \( \xi \in \mathbb{R}^n, \ \xi \neq 0 \) and set \( \xi_0 = (|\xi|, 0, \ldots, 0) \). Let \( \rho \) be the rotation such that \( \xi = \rho^{-1}\xi_0 \). Then we have
\[
\sum_{k=1}^m |\hat{u}_k(\xi)|^2 = \sum_{k=1}^m |\hat{u}_k(\rho^{-1}\xi_0)|^2 = \sum_{k=1}^m |(\rho^{-1}u_k)_{\rho}(\xi_0)|^2 = \sum_{k=1}^m |(u_k)_{\rho}(\xi_0)|^2 \leq (n - 1)|\Omega|,
\]
where we have used (2.10) and the fact that inequality (2.8) has been proved for \( \xi \) of the form (2.9) for any orthonormal family of divergence-free vector functions. Finally, the estimate (2.8) is extended to \( \xi = 0 \) by continuity (observe that \( u_k \in L_1 \) since \( |\Omega| < \infty \) and hence the Fourier transforms \( \hat{u}_k \) are continuous.)

\[ \square \]

Remark 2.3. In fact, (2.8) holds under milder assumption that \( u_k \in H \), \( k = 1, \ldots, m \).

We need the following lemma from [14], whose proof we give for the sake of completeness.

Lemma 2.5. (See [14].) Let a function \( f(\xi), f : \mathbb{R}^n \to \mathbb{R} \) satisfy

\[ 0 \leq f(\xi) \leq M_1 \quad \text{and} \quad \int |\xi|^2 f(\xi) d\xi \leq M_2. \]

Then

\[ \int f(\xi) d\xi \leq (M_1 \omega_n)^{2/(2+n)} (M_2 (2 + n)/n)^{n/(2+n)}. \]  

(2.11)

**Proof.** We first observe that (2.11) turns into equality for a constant multiple of the characteristic function \( g(\xi) \) of any ball centered at the origin in \( \mathbb{R}^n \). We set

\[ g(\xi) = \begin{cases} M_1, & |\xi| \leq R; \\ 0, & |\xi| > R. \end{cases} \]

Then \( (|\xi|^2 - R^2)(f(\xi) - g(\xi)) \geq 0 \) so that

\[ R^2 \int (f(\xi) - g(\xi)) d\xi \leq \int |\xi|^2 (f(\xi) - g(\xi)) d\xi \leq 0, \]

where the second inequality holds provided that \( R \) is defined by the equality

\[ \int |\xi|^2 g(\xi) d\xi = M_2. \]

Hence

\[ \int f(\xi) d\xi \leq \int g(\xi) d\xi = (M_1 \omega_n)^{2/(2+n)} (M_2 (2 + n)/n)^{n/(2+n)}. \]

\[ \square \]

We can now formulate our main results.

**Theorem 2.1.** Suppose that the family of vector functions \( \{u_k\}_{k=1}^m \in H^1_0(\Omega) \) is orthonormal and, in addition, \( \text{div} \, u_k = 0 \), \( k = 1, \ldots, m \). Then

\[ \sum_{k=1}^m \|\nabla u_k\|^2 \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n}. \]  

(2.12)

**Proof.** We set

\[ f(\xi) = \sum_{k=1}^m |\hat{u}_k(\xi)|^2. \]

By Lemma 2.4 and the Plancherel theorem \( f \) satisfies

1. \( 0 \leq f(\xi) \leq (n-1)|\Omega|; \)
2. \( \int f(\xi) d\xi = (2\pi)^n m; \)
3. \( \int |\xi|^2 f(\xi) d\xi = (2\pi)^n \sum_{k=1}^m \|\nabla u_k\|^2. \)
Using Lemma 2.5 we find that
\[ (2\pi)^n m = \int f(\xi) \, d\xi \leq \left( (n-1)|\Omega| \omega_n \right)^{2/(2+n)} \left( (2\pi)^n \sum_{k=1}^m \|\nabla u_k\|^2 (2 + n)/n \right)^{n/(2+n)}, \]
which is (2.12).

\[ \square \]

**Theorem 2.2.** The eigenvalues \( \lambda_k \) of the Stokes operator satisfy the following lower bound:
\[ \sum_{k=1}^m \lambda_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n (n-1)|\Omega|} \right)^{2/n} m^{1+2/n}. \]  

**Proof.** Since \( V \subseteq \{ u \in H^1_0(\Omega), \, \text{div} \, u = 0 \} \) we can chose the first \( m \) eigenvectors for the \( u_k \)'s in (2.12) and taking into account (2.3) we obtain (2.13). \( \square \)

**Remark 2.4.** In view of the asymptotics (1.3) this lower bound is sharp in the sense that the inequality with the coefficient of \( m^{1+2/n} \) larger than in (2.13) cannot hold for a sufficiently large \( m \).

**Remark 2.5.** Weaker lower bounds based on the estimate (2.7)
\[ \sum_{k=1}^m \lambda_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n (n-1)|\Omega|} \right)^{2/n} m^{1+2/n} \]
have earlier been proved in [10] for \( n = 2, 3 \).

**Remark 2.6.** In fact, for any orthonormal family \( \{u_k\}_{k=1}^m \in V \) we have
\[ \sum_{k=1}^m \|\nabla u_k\|^2 \geq \sum_{k=1}^m \lambda_k. \]

**Corollary 2.2.** Each eigenvalue \( \lambda_k \) satisfies
\[ \lambda_k \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n (n-1)|\Omega|} \right)^{2/n} k^{2/n}, \]  
while \( \lambda_1 \) satisfies
\[ \lambda_1 > \mu_1 \geq \frac{n}{2 + n} \left( \frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n}. \]

**Proof.** The sequence \( \{\lambda_k\}_{k=1}^\infty \) is nondecreasing and (2.14) is obvious. Since \( V \subset H^1_0(\Omega) \),
\[ \mu_1 = \min_{u \in H^1_0(\Omega)} \frac{\|\nabla u\|^2}{\|u\|^2} \leq \min_{u \in V} \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_1 \]
and the second inequality in (2.15) is (1.1) with \( m = 1 \). Let us prove that \( \lambda_1 > \mu_1 \). Suppose that \( \mu_1 = \|\nabla u_0\|^2/\|u_0\|^2 \) for some \( u_0 \in H^1_0(\Omega) \). It is well known that \( \mu_1 \) is a simple eigenvalue with (up to a constant factor) eigenfunction \( \varphi_1 \). Therefore any such \( u_0 \) is of the form \( u_0(x) = (l_1 \varphi_1(x), l_2 \varphi_1(x), \ldots, l_n \varphi_1(x)) \) for some constants \( l_1, \ldots, l_n \), \(|l| > 0\). (Without loss of generality we can assume that \(|l| = 1\) ) Now \( \lambda_1 = \mu_1 \) if and only if \( u_0 \) so obtained satisfies, in addition, \( \text{div} \, u_0 = 0 \). Therefore \( \frac{\partial \varphi_1}{\partial t} = \text{div} \, u_0 = 0 \), and \( \varphi_1 \) is constant along the lines parallel to \( l \), which is impossible. \( \square \)
3. Applications to the Navier–Stokes system

We write the two-dimensional Navier–Stokes system as an evolution equation in $H$

$$\partial_t u + \nu A u + B(u, u) = f, \quad u(0) = u_0,$$

(3.1)

where $A = -P \Delta$ is the Stokes operator and $B(u, v) = P(\sum_{i=1}^{2} u^i \partial_i v)$. The equation (3.1) generates the semigroup $S_t : H \to H$, $S_t u_0 = u(t)$, which has a compact global attractor $A \in H$ (see, for instance, [2], [5], [7], [21] for the case of a domain with smooth boundary $\partial \Omega$, and [12], [18] for a nonsmooth domain). The attractor $A$ is the maximal strictly invariant compact set.

**Theorem 3.1.** The fractal dimension of $A$ satisfies the following estimate

$$\dim_F A \leq \frac{1}{(8\sqrt{3} \pi)^{1/2}} \left(\lambda_1 |\Omega|\right)^{1/2} \left\|\frac{f}{\lambda_1} \right\| < \frac{1}{4\pi^{3/4}} \left\|\frac{f}{\nu} \right\|.$$  

(3.2)

**Proof.** Since for the proof of (3.2) we need to use in [3] Theorem 4.1 the new improved constants in the Lieb–Thirring inequality (3.6) below and in the lower bound (2.12) for $n = 2$, the proof of the theorem will only be outlined. The solution semigroup $S_t$ is uniformly differentiable in $H$ with differential $L(t, u_0) : \xi \to U(t) \in H$, where $U(t)$ is the solution of the variational equation

$$\partial_t U = -\nu AU - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t, u_0)U, \quad U(0) = \xi.$$  

(3.3)

We estimate the numbers $q(m)$ (the sums of the first $m$ global Lyapunov exponents):

$$q(m) \leq \limsup_{t \to \infty} \sup_{u_0 \in A} \sup_{\{v_j\}_{j=1}^m \in V} \frac{1}{t} \int_0^t \sum_{j=1}^m (\mathcal{L}(\tau, u_0)v_j, v_j) d\tau,$$

(3.4)

where $\{v_j\}_{j=1}^m \in V$ is an arbitrary orthonormal system of dimension $m$ [2], [4], [5], [21].

$$\sum_{j=1}^m (\mathcal{L}(t, u_0)v_j, v_j) = \nu \sum_{j=1}^m \|\nabla v_j\|^2 - \int \sum_{j=1}^m \sum_{k,i=1}^2 v_j^k \partial_k u^i v_j^i dx \leq$$

$$-\nu \sum_{j=1}^m \|\nabla v_j\|^2 + 2^{-1/2} \int \rho(x) |\nabla u(t, x)| dx \leq$$

$$-\nu \sum_{j=1}^m \|\nabla v_j\|^2 + 2^{-1/2} \|\rho\| \|\nabla u\| \leq$$

$$-\nu \sum_{j=1}^m \|\nabla v_j\|^2 + 2^{-1/2} \left( c_{LT} \sum_{j=1}^m \|\nabla v_j\|^2 \right)^{1/2} \|\nabla u(t)\| \leq$$

$$-\nu \sum_{j=1}^m \|\nabla v_j\|^2 + \frac{c_{LT}}{4\nu} \|\nabla u(t)\|^2 \leq -\nu \frac{c_{sp} m^2}{2|\Omega|} + \frac{c_{LT}}{4\nu} \|\nabla u(t)\|^2,$$

Here we used the inequality $|\sum_{k=1}^2 v^k \partial_k u^i v^i| = |\nabla u v \cdot v| \leq 2^{-1/2} |\nabla u| v|^2$ [3] Lemma 4.1, and, finally, (2.12), written for $n = 2$ and the orthonormal family $\{v_j\}_{j=1}^m \in V$ as follows

$$\sum_{k=1}^m \|\nabla v_k\|^2 \geq \frac{c_{sp} m^2}{|\Omega|}, \quad c_{sp} = 2\pi.$$  

(3.5)
Using the well-known estimate
\[
\limsup_{t \to \infty} \sup_{u_0 \in A} \frac{1}{t} \int_0^t \| \nabla u(\tau) \|^2 d\tau \leq \frac{\| f \|^2}{\lambda_1 \nu^2} = \lambda_1 \nu^2 G^2, \quad G = \| f \| / \lambda_1 \nu^2
\]
for the solutions lying on the attractor we obtain for the numbers \( q(m) \):
\[
q(m) \leq -\frac{\nu csp m^2}{2|\Omega|} + \frac{\nu \lambda_1 c_{LT} G^2}{4}.
\]
It was shown in [4] (see also [5, 21]) and in [3], respectively, that both the Hausdorff and fractal dimensions of \( A \) are bounded by the number \( m_* \) for which \( q(m_*) = 0 \). This gives that
\[
\dim_F A \leq \left( \frac{c_{LT}}{2c_{sp}} \right)^{1/2} (\lambda_1 |\Omega|)^{1/2} G,
\]
which in view of (3.6) and (3.5) proves the first inequality in (3.2), while the second inequality follows from (2.15) with \( n = 2: \lambda_1 > 2\pi /|\Omega| \).

\[\square\]

Theorem 3.2. Let the family \( \{ v_j \}_{j=1}^m \in H^1_0(\Omega), \Omega \subseteq \mathbb{R}^2 \) be orthonormal and \( \text{div} v_j = 0, j = 1, \ldots, m \). Then the following inequality holds for \( \rho(x) = \sum_{k=1}^m |v_k(x)|^2 \):
\[
\| \rho \|^2 = \int \left( \sum_{j=1}^m |v_j(x)|^2 \right)^2 dx \leq c_{LT} \sum_{j=1}^m \| \nabla v_j \|^2, \quad c_{LT} \leq \frac{1}{2\sqrt{3}}.
\]

Proof. It was proved in [3, 11] that the best (by notational definition) constant \( c_{LT} \) in (3.6) satisfies
\[
c_{LT} \leq 4L_{1,2},
\]
where the constant \( L_{1,2} \) comes from the Lieb–Thirring spectral estimate [16]
\[
\sum_{\mu_j < 0} |\mu_j|^{7/2} \leq L_{\gamma,n} \int_{\mathbb{R}^n} f(x)^{\gamma + n/2} dx
\]
for the negative eigenvalues of the scalar Schrödinger operator \( -\Delta - f \) in \( \mathbb{R}^n, f \geq 0 \). For \( L_{\gamma,n} \) we always have
\[
L_{\gamma,n} \geq L_{\gamma,n}^{cl} := \frac{\Gamma(\gamma + 1)}{(4\pi)^{n/2} \Gamma(\gamma + n/2 + 1)}.
\]
It was recently shown in [6] that for \( n \geq 1 \)
\[
L_{\gamma,n} \leq R \cdot L_{\gamma,n}^{cl}, \quad R = \pi / \sqrt{3} = 1.8138 \ldots , \quad \gamma \geq 1,
\]
which improves the previous important estimate \( L_{\gamma,n} \leq 2L_{\gamma,n}^{cl} \) established in [8]. Hence \( c_{LT} \leq 4R L_{1,2}^{cl} = 1/(2\sqrt{3}) \). The proof is complete.

\[\square\]

Remark 3.1. The idea to use Lieb–Thirring inequalities in the context of the Navier–Stokes equations [15] has led to estimates of dimension that are linear with respect to the Grashof number \( G \) [20]. First explicit estimates for the dimension of the attractors were obtained in [10] and improved in [3]. The explicit constants in (3.2) are further improvements (by the factor \((2 \cdot (2/R))^{1/2} = 1.485 \ldots \)) of the corresponding constants in [3].

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