ON THE UNIFORM SQUEEZING PROPERTY AND THE SQUEEZING FUNCTION

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1. Introduction

In [7, 8] and [11], the concept called holomorphic-homogeneous-regular and equivalently the uniformly-squeezing, respectively, for complex manifolds has been introduced. This concept was essential for estimation of several invariant metrics. See the above cited papers for details.

Let $\Omega$ be a complex manifold of dimension $n$. The squeezing function $\sigma_{\Omega} : \Omega \to \mathbb{R}$ of $\Omega$ is defined as follows: for each $p \in \Omega$ let

$$F(p, \Omega) := \{ f : \Omega \to \mathbb{B}^n, \ 1\text{-}1 \text{ holomorphic, } f(p) = 0 \},$$

where:

- $\mathbb{B}^n(p; r) = \{ z \in \mathbb{C}^n : \|z - p\| < r \}$, and
- $\mathbb{B}^n = \mathbb{B}^n(0; 1) = \mathbb{B}^n((0, \ldots, 0); 1)$.

Then

$$\sigma_{\Omega}(p) = \sup \{ r : \mathbb{B}^n(0, r) \subset f(\Omega), \text{ for some } f \in F(p, \Omega) \}.$$ 

Furthermore, the squeezing constant $\hat{\sigma}_{\Omega}$ for $\Omega$ is defined by

$$\hat{\sigma}_{\Omega} := \inf_{p \in \Omega} \sigma_{\Omega}(p).$$

Definition 1.1 (Liu-Sun-Yau [7, 8]; Yeung [11]). A complex manifold $\Omega$ is called holomorphic homogeneous regular (HHR), or equivalently uniformly squeezing (USq), if $\hat{\sigma}_{\Omega} > 0$.

Notice that the property HHR (i.e., USq) is preserved by biholomorphisms. The squeezing function and squeezing constants are also biholomorphic invariants.

These concepts have been developed in order for the study of completeness and other geometric properties such as the metric equivalence of the invariant metrics including Carathéodry, Kobayashi-Royden, Teichmüller, Bergman, and Kaehler-Einstein metrics. It is obvious that the examples of HHR/USq manifolds include bounded homogeneous domains. In case the manifold is biholomorphic to a bounded domain and the holomorphic automorphism orbits accumulate at every boundary point, such as in the case of the Bers embedding of the Teichmüller space, again USq/HHR property holds. A bit less obvious example may be the bounded strongly convex...
domains (as the majority of them do not possess any holomorphic automorphisms except the identity map), proved by S.-K. Yeung [11]. But there, one of the most standard examples, such as the bounded convex domains and the bounded strongly pseudoconvex domains were left untouched.

Indeed the starting point of this article is to show

**Theorem 1.1.** All bounded convex domains in $\mathbb{C}^n$ ($n \geq 1$) are HHR (i.e., USq).

The concept of squeezing function $\sigma_\Omega$ defined above plays an important role, and moreover it appeals to us that the further investigations on this function should be worthwhile. One immediate observation is that if, $\sigma_\Omega(p) = 1$ for some $p \in \Omega$, then $\Omega$ is biholomorphic to the unit open ball (I). In the light of studies on the asymptotic behavior of several invariant metrics of the strongly pseudoconvex domains, perhaps the following question is natural to pose:

**Question 1.1.** If $\Omega$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, would $\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1$ hold for every boundary point $p \in \partial \Omega$?

While we do not know the solution at the time of this writing, fortunately, we are able to present the following result.

**Theorem 1.2.** If $\Omega$ is a bounded domain in $\mathbb{C}^n$ with a $C^2$ strongly convex boundary, then $\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1$ for every $p \in \partial \Omega$.

The proof-arguments also clarify and simplify some previously-known theorems; those shall be mentioned in the final section as remarks.

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2. BOUNDED CONVEX DOMAINS ARE HHR/USQ MANIFOLDS

The aim of this section is to establish Theorem 2.1 stated below. Not only does this theorem cover the case left untreated in [11], but our method is different. (See also [11] on this matter). Our method uses a version of the “scaling method in several complex variables” initiated by S. Pinchuk [9]. In fact, we use the version presented in [4], modified for the purpose of studying the asymptotic boundary behavior of holomorphic invariants.

**Theorem 2.1.** Every convex Kobayashi hyperbolic domain in $\mathbb{C}^n$ is HHR/USq.

Note that all bounded domains are Kobayashi hyperbolic, and every convex Kobayashi hyperbolic domain is biholomorphic to a bounded domain.
But the bounded realization may not in general be convex. In that sense this theorem is more general than Theorem 1.1.

**Proof.** We proceed in 5 steps.

**Step 1. Set-up.** Let $\Omega$ be a convex hyperbolic domain in $\mathbb{C}^n$. Suppose that $\Omega$ is not HHR/USq. Then there exists a sequence $\{q_j\}$ in $\Omega$ converging to a boundary point, say $q \in \partial \Omega$ such that

$$\lim_{j \to \infty} S_\Omega(q_j) = 0.$$ 

Needless to say, it suffices to show that such a sequence cannot exist.

**Step 2. The $j$-th orthonormal frame.** Let $\langle \cdot, \cdot \rangle$ represent the standard Hermitian inner product of $\mathbb{C}^n$, and let $\|v\| = \sqrt{\langle v, v \rangle}$. For every $q \in \mathbb{C}^n$ and a complex linear subspace $V$ of $\mathbb{C}^n$, denote by

$$B^V(q, r) = \{p \in \mathbb{C}^n : p - q \in V \text{ and } \|p - q\| < r\}.$$ 

Now let $q \in \Omega$ and define the positive number $\lambda(q, V)$ by

$$\lambda(q, V) = \max\{r > 0 : B^V(q, r) \subset \Omega\}.$$ 

This number is finite for each $(q, V)$, whenever $\dim V > 0$, since $\Omega$ is Kobayashi hyperbolic.

Fix the index $j$ momentarily. Then we choose an orthonormal basis for $\mathbb{C}^n$, with respect to the standard Hermitian inner product $\langle \cdot, \cdot \rangle$. First consider

$$\lambda_1^j := \lambda(q_j, \mathbb{C}^n).$$

Then there exists $q^{1*}_j \in \partial \Omega$ such that $\|q^{1*}_j - q_j\| = \lambda_1^j$. Let

$$e_1^j = \frac{q^{1*}_j - q_j}{\|q^{1*}_j - q_j\|}.$$ 

Then consider the complex span $\text{Span}_\mathbb{C}\{e_1^j\}$, and let $V^1$ be its orthogonal complement in $\mathbb{C}^n$. Then take

$$\lambda_2^j := \lambda(q_j, V^1)$$

and $q^{2*}_j \in \partial \Omega$ such that $q^{2*}_j - q_j \in V^1$ and $\|q^{2*}_j - q_j\| = \lambda_2^j$. Then let

$$e_2^j := \frac{q^{2*}_j - q_j}{\|q^{2*}_j - q_j\|}.$$ 

With $e_1^j, e_2^j, \ldots, e_\ell^j$ and $\lambda_1^j, \lambda_2^j, \ldots, \lambda_\ell^j$ chosen, the next element $e_{\ell+1}^j$ is selected as follows. Denote by $V^\ell$ the complex orthogonal complement of $\text{Span}_\mathbb{C}\{e_1^j, e_2^j, \ldots, e_\ell^j\}$. Then

$$\lambda_{\ell+1}^j := \lambda(q_j, V^\ell)$$
and \(q^*_{j+1} \in \partial \Omega\) such that \(q^*_{j+1} - q_j \in V^\ell\) and \(\|q^*_{j+1} - q_j\| = \lambda^\ell_{j+1}\). Let
\[
e_j^{\ell+1} := \frac{q^*_{j+1} - q_j}{\|q^*_{j+1} - q_j\|}.
\]
By induction, this process yields an orthonormal set \(e^1_j, \ldots, e^n_j\) for \(\mathbb{C}^n\) and the positive numbers \(\lambda^1_j, \ldots, \lambda^n_j\).

**Step 3. Stretching complex linear maps.** Let \(\hat{e}^1_j, \ldots, \hat{e}^n_j\) denote the standard orthonormal basis for \(\mathbb{C}^n\), i.e.,
\[
\hat{e}^1 = (1, 0, \ldots, 0), \hat{e}^2 = (0, 1, 0, \ldots, 0), \ldots, \hat{e}^n = (0, \ldots, 0, 1).
\]
Define the stretching linear map \(L_j : \mathbb{C}^n \to \mathbb{C}^n\) by
\[
L_j(z) = \sum_{k=1}^n \frac{\langle z - q_j, e^k_j \rangle}{\lambda^k_j} \hat{e}^k
\]
for every \(z \in \mathbb{C}^n\). Note that, for each \(j\), \(L_j\) maps \(\Omega\) biholomorphically onto its image.

**Step 4. Supporting hyperplanes.** Notice that
\[
L_j(q^*_1) = 0 = (0, \ldots, 0), L_j(q^*_1) = \hat{e}^1, \ldots, L_j(q^*_n) = \hat{e}^n.
\]
We shall consider the supporting hyperplanes, say \(\Pi^k_j (k = 1, \ldots, n)\), of \(L_j(\Omega)\) at points \(L_j(q^*_k)\), \(k = 1, \ldots, n\), respectively.

**Substep 4.1. The supporting hyperplane \(\Pi^1_j\):** Recall that \(L_j(q^*_1) = \hat{e}^1 = (1, 0, \ldots, 0)\). Due to the choice of \(q^*_1\) the supporting hyperplane of \(\Omega\) at \(q^*_1\) must also support the sphere tangent to the boundary \(\partial \Omega\). Consequently the supporting hyperplane \(\Pi^1_j\) of \(L_j(\Omega)\) must support a smooth surface (an ellipsoid) tangent to \(L_j(\partial \Omega)\) at \(\hat{e}^1\). Thus the equation for this hyperplane \(\Pi^1_j\) is
\[
\text{Re} (z_1 - 1) = 0
\]
(independently of \(j\), being perpendicular to \(\hat{e}^1\) consequently). We also note that
\[
L_j(\Omega) \subset \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \text{Re} z_1 < 1\}.
\]

**Substep 4.2. The rest of supporting hyperplanes \(\Pi^k_j\), for \(k \geq 2\):** First consider the case \(k = 2\). Then the supporting hyperplane \(\Pi^2_j\) passes through \(L_j(q^*_2) = \hat{e}^2 = (0, 1, \ldots, 0)\). Since the restriction of \(\Omega\) to \(V^1\) contains the sphere in \(V^1\) tangent to the restriction of \(\partial \Omega\) at the point \(\hat{e}^2\), the supporting hyperplane \(\Pi^2_j\) restricted to \(L_j(V^1)\) takes the equation \(\{(z_2, \ldots, z_n) \in \mathbb{C}^{n-1} : \text{Re} (z_2 - 1) = 0\}\). Hence
\[
\Pi^2_j = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \text{Re} (a^2_1 z_1 + a^2_2 (z_2 - 1)) = 0\}
\]
for some \((a_{j}^{2,1}, a_{j}^{2,1}) \in \mathbb{C}^2\) with \(\left|a_{j}^{2,1}\right|^2 + \left|a_{j}^{2,2}\right|^2 = 1\) and \(a_{j}^{2,2} > 0\). We also have that

\[
L_{j}(\Omega) \subset \{(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}: \Re (a_{j}^{1}z_{1} + a_{j}^{2}(z_{2} - 1)) < 0\}.
\]

For \(k \in \{3, \ldots, n\}\), one deduces inductively that the supporting hyperplane \(\Pi_{j}^{k}\) passes through the point \(\hat{e}^{k}\), and that

\[
\Pi_{j}^{k} = \{(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}:
\Re (a_{j}^{1}z_{1} + \cdots + a_{j}^{k,k-1}z_{k-1} + a_{j}^{k,k}(z_{k} - 1)) = 0,
\]

with \(a_{j}^{k,k} > 0\) and \(\sum_{\ell=1}^{k} |a_{j}^{k,\ell}|^2 = 1\). Also,

\[
L_{j}(\Omega) \subset \{(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}:
\Re (a_{j}^{1}z_{1} + \cdots + a_{j}^{k,k-1}z_{k-1} + a_{j}^{k,k}(z_{k} - 1)) < 0\}.
\]

Substep 4.3. Polygonal envelopes: We add this small substep for convenience. From the discussion by far in this Step, we have the \(j\)-th polygonal envelope (of \(L_{j}(\Omega)\))

\[
\Sigma_{j} := \{(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}:
\Re z_{1} < 1
\Re (a_{j}^{1,1}z_{1} + a_{j}^{2,2}(z_{2} - 1)) < 0
\vdots
\Re (a_{j}^{n,1}z_{1} + \cdots + a_{j}^{n,n-1}z_{n-1} + a_{j}^{n,n}(z_{n} - 1)) < 0\}
\]

Step 5. Bounded realization. Notice that, for every \(k \in \{1, \ldots, n\}\), the disc

\[
D_{j}^{k} := \{z = (z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}: \langle z - q_{j}, e_{j}^{\ell} \rangle = 0, \forall \ell \neq k; \|z - q_{j}\| < \lambda_{j}^{k}\}
\]

is contained in \(\Omega\). Hence, every \(L_{j}(\Omega)\) contains the discs \(D_{j}^{k} : \zeta \in \mathbb{C}, |\zeta| < 1\) for every \(k = 1, \ldots, n\). Since \(\Omega\) is convex and since \(L_{j}\) is linear, \(L_{j}(\Omega)\) is also convex. Therefore, the “unit acorn”

\[
A := \{(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}: |z_{1}| + \cdots + |z_{n}| < 1\}
\]

is contained in \(L_{j}(\Omega)\). This restricts the unit normal vectors \(n_{j}^{k} := (a_{j}^{k,1}, \ldots, a_{j}^{k,k}, 0, \ldots, 0) \in \mathbb{C}^{n}\) for every \(k = 2, \ldots, n\). Namely, there is a positive constant \(\delta > 0\) independent of \(j\) and \(k\) such that \(a_{j}^{k,k} \geq \delta\) for every \(j, k\).
Now taking a subsequence (of $q_j$), we may assume that the sequence of unit vectors \( \{n_j^k\}_{j=1}^{\infty} \) converges for every $k \in \{2, \ldots, n\}$. Let us write
\[
\lim_{j \to \infty} n_j^k = n^k = (a_{k,1}, \ldots, a_{k,k}, 0, \ldots, 0)
\]
for each $k = 1, 2, \ldots, n$.

Consider the maps
\[
B_j(z_1, \ldots, z_n) = (\zeta_1, \ldots, \zeta_n)
\]
defined by
\[
\zeta_1 = z_1, \\
\zeta_2 = a_{j,1}^1 z_1 + a_{j,2}^2 z_2, \\
\vdots \\
\zeta_n = a_{j,1}^n z_1 + \ldots + a_{j,n}^n z_n.
\]

Then it follows that
\[
B_j \circ L_j(\Omega) \subset B_j(\Sigma_j) = \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n: \Re \zeta_1 < 1, \Re \zeta_2 < a_{j,2}^2, \ldots, \Re \zeta_n < a_{j,n}^n\}
\]

Now we consider the Cayley transformation, for each $j$,
\[
\Phi_j(z_1, \ldots, z_n) = \left( \frac{z_1}{2 - z_1}, \frac{z_2}{2a_{j,2}^2 - z_2}, \ldots, \frac{z_n}{2a_{j,n}^n - z_n} \right).
\]

Then $\Phi_j \circ B_j(\Sigma_j) \subset D^n$, where $D^n$ denote the unit polydisc in $\mathbb{C}^n$ centered at the origin. Also, there exists a positive constant $\delta' \in (0, \delta)$ such that $\Phi_j \circ B_j(\Sigma_j) \subset D^n$ contains the ball of radius $\delta'$ centered at the origin 0.

Since $\Phi_j \circ B_j \circ L_j(q_j) = (0, \ldots, 0)$ for every $j$, we now conclude that the squeezing function satisfies
\[
\sigma_{\Omega}(q_j) \geq \frac{\delta'}{\sqrt{n}}
\]
This estimate, which holds for every sequence $q_j$ approaching the boundary, yields the desired contradiction at last. Thus the proof is complete. \(\square\)

3. **Boundary behavior of squeezing function on strongly convex domains**

Consider first the following

**Definition 3.1.** Let $\Omega$ be a domain in $\mathbb{C}^n$. A boundary point $p \in \partial \Omega$ is said to be *spherically-extreme* if

1. the boundary $\partial \Omega$ is $C^2$ smooth in an open neighborhood of $p$, and
2. there exists a ball $B^n(c(p); R)$ in $\mathbb{C}^n$ of some radius $R$, say, centered at some point $c(p)$ such that $\Omega \subset B^n(c(p); R)$ and $p \in \partial \Omega \cap \partial B^n(c(p); R)$.

The main goal of this section is to establish
Theorem 3.1. If a domain $\Omega$ in $\mathbb{C}^n$ admits a spherically-extreme boundary point $p$, say, in a neighborhood of which the boundary $\partial \Omega$ is $C^2$ smooth, then
$$\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1.$$ 

Proof. Since every boundary point of a $C^2$ strongly convex bounded domain is spherically-extreme, this theorem implies Theorem 1.2. The rest of this section is devoted to the proof of Theorem 3.1, which we shall proceed in seven steps.

Step 1: Sphere Envelopes. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with a boundary point $p \in \partial \Omega$ such that
(i) $\partial \Omega \cap B^n(p; r_0)$ is $C^2$-smooth for some $r_0 > 0$, and
(ii) $p$ is a spherically-extreme boundary point of $\Omega$.

Then there exist positive constants $r_1, r_2$ and $R$ with $r_0 > r_1 > r_2$ such that every $q \in \Omega \cap B^n(p; r_2)$ admits points $b(q) \in \partial \Omega \cap B^n(p; r_1)$ and $c(q) \in \mathbb{C}^n$ satisfying the conditions
(iii) $\|q - b(q)\| < \|q - z\|$ for any $z \in \partial \Omega - \{b(q)\}$, and
(iv) $\|c(q) - b(q)\| = R$ and $\Omega \subset B^n(c(q); R)$.

Figure 1. Sphere envelopes

Notice that (iii) says that $b(q)$ is the unique boundary point that is the closest to $q$, and that the constant $R$ in (iv) is independent of the choice of $q \in B^n(p; r_2)$.

Step 2: Centering. From this stage we shall exploit the familiar notation
$$z = (z_1, \ldots, z_n),$$
$$z' = (z_2, \ldots, z_n),$$
$$u = \Re z_1,$n
$$v = \Im z_1.$$ 

For each $q \in \Omega \cap B^n(p, r_2)$, choose a unitary transform $U_q$ of $\mathbb{C}^n$ such that
the map \( A_q(z) := U_q(z - b(q)) \) satisfies the following conditions:

\[
A_q(q) = (\lambda_q, 0, \ldots, 0)
\]

for some \( \lambda_q > 0 \), and

\[
A_q(\Omega) \subset B_n((R, 0, 0, \ldots, 0); R) = \{ z \in C^n : |z_1 - R| + \|z'\|^2 < R^2 \}.
\]

Then there exists a positive constant \( r_3 < r_2 \) such that

\[
z \in A_q(\Omega) \cap B^n(0, r_3) \quad \iff \|z\| < r_3 \quad \text{and} \quad 2u > H_{b(q)}(z') + K_{b(q)}(v, z') + R_{b(q)}(v, z')
\]

where:

- \( H_{b(q)} \) is a quadratic positive-definite Hermitian form such that there exists a constant \( c_0 > 0 \), independent of \( q \), satisfying

\[
H_{b(q)}(z') \geq c_0 \|z'\|^2
\]

and

- there exists a constant \( C > 0 \), independent of \( q \in B^n(p; r_3) \cap \Omega \), such that

\[
|K_{b(q)}(v, z')| \leq C(|v|^2 + |v||z'|),
\]

whenever \( z \in B^n(0, r_3) \). Furthermore, we have

\[
|R_{b(q)}(v, z')| = o(|v|^2 + \|z'\|^2).
\]

In particular, the choice of \( r_3 \) can allow us the estimate

\[
|R_{b(q)}(v, z')| \leq \frac{c_0}{2}(|v|^2 + \|z'\|^2).
\]

Notice that

\[
\lim_{\Omega \ni q \to p} b(q) = p, \quad \lim_{\Omega \ni q \to p} H_{b(q)}(z') = H_p(z'),
\]

and

\[
\lim_{\Omega \ni q \to p} A_q = I \quad \text{(the identity map)}.
\]
This last and an inductive construction yield that for each integer $m > 2$
there exists a strictly-increasing integer-valued function $k(m)$ such that
\begin{align}
\mathbb{B}^n(0; r_3/(2k(m))) &\subset A_q(\mathbb{B}^n(p; r_3/k(m))) \subset \mathbb{B}^n(0; r_3/m),
\end{align}
whenever $q \in \mathbb{B}^n(p, \frac{r_3}{2k(m)})$.

**Step 3: The Cayley transform.** The Cayley transform considered here is the map
\begin{align}
\kappa(z) := \left(\frac{1 - z_1}{1 + z_1}, \frac{\sqrt{2}z_2}{1 + z_1}, \ldots, \frac{\sqrt{2}z_n}{1 + z_1}\right),
\end{align}
well-defined except at points of $Z = \{z \in \mathbb{C}^n : z_1 = -1\}$. Notice that this transform maps the open unit ball $\mathbb{B}^n(0; 1)$ biholomorphically onto the Siegel half space
\begin{align}
S_0 := \{z \in \mathbb{C}^n : 2\Re z_1 > \|z\|^2\}.
\end{align}
Moreover, $\kappa \circ \kappa = 1$ and consequently, $\kappa(S_0) = B^n(0, 1)$. Notice also that, if we denote by $1 = (1, 0, \ldots)$ and $-1 = (-1, 0, \ldots)$, then we have $\kappa(1) = (0, \ldots, 0)$, $\kappa((0, \ldots, 0)) = 1$, $\kappa(-1) = \infty$ and $\kappa(\infty) = -1$.

**Step 4: Stretching.** Let $q \in \Omega \cap \mathbb{B}^n(p; \frac{r_3}{2k(m)})$. If we let $m$ tend to infinity. Then of course $A_q(q) = (\lambda_q, 0, \ldots, 0)$ approaches $A_q(b(q)) = (0, \ldots, 0)$ and so $\lambda_q$ approaches zero. For simplicity, denote by $\lambda = \lambda_q$, suppressing the notation $q$. But $\lambda$ is still dependent upon $q$. Note that
\begin{align}
A_q(\mathbb{B}^n(c(q); R)) = \{z \in \mathbb{C}^n : 2R \Re z_1 > \|z\|^2\}.
\end{align}
Define the map $\Lambda_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ by
\begin{align}
\Lambda_\lambda(z) := \left(\frac{z_1}{\lambda}, \frac{z_2}{\sqrt{\lambda}}, \ldots, \frac{z_n}{\sqrt{\lambda}}\right),
\end{align}
the stretching map, introduced originally by Pinchuk (cf. [9]).
Recall (3.13). This stretching map transforms $A_q(\Omega) \cap \mathbb{B}^n(0; \frac{r_3}{2k(3)})$ to the domain $\Lambda_\lambda(A_q(\Omega) \cap \mathbb{B}^n(0; \frac{r_3}{2k(3)}))$ so that
\begin{align}
z \in A_\lambda \circ A_q(\Omega) \cap \mathbb{B}^n(0; \frac{r_3}{\sqrt{\lambda k(3)}})
\iff \|z\| < \frac{r_3}{\sqrt{\lambda k(3)}} \quad \text{and} \quad 2u > H_{b(q)}(z') + \frac{1}{\lambda} K_{b(q)}(\lambda v, \sqrt{\lambda} z') + \frac{1}{\lambda} R_{b(q)}(\lambda v, \sqrt{\lambda} z').
\end{align}
On the other hand, notice that
\begin{align}
\|\frac{1}{\lambda} K_{b(q)}(\lambda v, \sqrt{\lambda} z')\| &\leq C\sqrt{\lambda}(\|v\|^2 + \|v\|\|z'\|)
\end{align}
and that
\begin{align}
\|\frac{1}{\lambda} R_{b(q)}(\lambda v, \sqrt{\lambda} z')\| &\leq \frac{1}{\lambda} o((\|\lambda v\|^2 + \|\sqrt{\lambda} z'\|^2)) = \frac{1}{\lambda} o(\lambda)
\end{align}
on $\mathbb{B}^n(0; \rho)$ for any fixed constant $\rho > 0$. Notice that both terms approach zero as $\lambda$ tends to zero. Thus, these terms can become sufficiently small if we limit $q$ to be contained in $\mathbb{B}^n(p; \frac{r_2}{2k(m)})$ for some sufficiently large $m$.

**Step 5: Set-convergence.** This step is in part heuristic; and the heuristics appearing, especially which concern set-convergences, in this step are not used in the proof, strictly speaking. We include this step because they seem to help us to grasp the logical structure of the proof. On the other hand, the constructions in (3.13)–(3.15) shall be used in the proof-arguments, especially in Step 7.

The main role of the stretching map $\Lambda_{\lambda}$, as $\lambda \searrow 0$ is to rescale the domains successively, letting them to converge to the set-limits. For instance if one considers

$$\Lambda_{\lambda}(A_{q}(\Omega) \cap B^n(0, r_3))$$

then, one can see that $\Lambda_{\lambda}(B^n(0, r_3))$ contains $B^n(0, r_2/\sqrt{\lambda})$, a very large ball, which exhausts $\mathbb{C}^n$ successively as $\lambda$ approaches zero. In the meantime within that large ball, $\Lambda_{\lambda}(A_{q}(\Omega))$ is restricted only by the inequality

$$2u > H_{\mu}(z') + \bar{K}_{\lambda}(v, z')$$

where $\bar{K}_{\lambda} = o(\lambda)$ is small enough to be negligible. One can imagine that indeed the “limit domain” of this procedure should be

(3.13) $\hat{\Omega} := \{z \in \mathbb{C}^n : 2u > H_{\mu}(z')\}$.

Here, of course, $H_{\mu}(z')$ is the quadratic positive-definite Hermitian form which appears in the defining inequality of $\Omega$ about the boundary point $p$ (understood as the origin):

$$2\text{Re } z_1 > H_{\mu}(z') + o(|\text{Im } z_1| + \|z'\|^2).$$

Notice that

$$\kappa(\hat{\Omega}) = \{z \in \mathbb{C}^n : |z_1|^2 + H_{\mu}(z') < 1\},$$

and hence there is a $\mathbb{C}$-linear isomorphism

(3.14) $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$

that maps $\kappa(\hat{\Omega})$ biholomorphically onto the unit ball $\mathbb{B}^n(0; 1)$ with $L(1) = 1$.

Before leaving this step we remark that, since $\Omega \subset \mathbb{B}^n(c(q); R)$ whenever $q \in \mathbb{B}^n(p; r_2)$, $A_q(\Omega) \subset A_q(\mathbb{B}^n(c(q); R)) = \mathbb{B}^n((R, 0, \ldots, 0); R)$. This in turn implies that

(3.15) $\Lambda_{\lambda} \circ A_q(\Omega) \subset \Lambda_{\lambda}(\mathbb{B}^n((R, 0, \ldots, 0); R)) \subset \mathcal{E} := \{z \in \mathbb{C}^n : 2R \text{Re } z_1 > \|z'\|^2\}$.

The last inclusion follows by (3.10).

**Step 6: Auxiliary domains.** Let $\delta > 0$ be given. Consider the domains

(3.16) $G_{\delta} := \{z \in \mathbb{C}^n : 2u > -\delta|v| + (1 - \delta)H_{\mu}(z')\}$,
\begin{align}
F_\delta &:= \{ z \in \mathbb{C}^n : 2u > \delta |v| + (1 + \delta)H_{b(q)}(z') \} \\
\mathcal{H}_q &:= \{ z \in \mathbb{C}^n : 2u > H_{b(q)}(z') \},
\end{align}

in addition to \( \hat{\Omega} \) and \( \mathcal{E} \) introduced in (3.13) and (3.15).

A straightforward computation checks that the image \( \kappa(G_\delta) \) of \( G_\delta \) via the Cayley transform \( \kappa \) introduced earlier is
\begin{equation}
\kappa(G_\delta) = \{ z \in \mathbb{C}^n : |z_1|^2 - \delta^2 \frac{|z_1 - \bar{z}_1|}{2} + (1 - \delta)H_{b(q)}(z') < 1 \}.
\end{equation}

Hence, there exists \( \delta_0 > 0 \) that, for every \( \delta \) with \( 0 < \delta < \delta_0 \), \( \kappa(G_\delta) \) is a bounded domain. Notice also that this domain is arbitrarily close to the domain \( \kappa(H_{b(q)}) \) as \( \delta_0 \) becomes arbitrarily small. It follows therefore that, for every \( \epsilon > 0 \), there exists \( \delta_0 > 0 \) such that
\begin{equation}
L \circ \kappa(G_\delta) \subset \mathbb{B}^n(0; 1 + \epsilon)
\end{equation}
whenever \( 0 < \delta < \delta_0 \). Moreover, observe that the stretching map \( \Lambda_\lambda \)

preserves all such domains as
\[ F_\delta, G_\delta, \hat{\Omega}, \mathcal{E} \text{ and } \mathcal{H}_q. \]
Let us now define the expression

\[ G(z) := L \circ \kappa \circ \Lambda \circ A_q(z) \]

for \( z \in \mathbb{C}^n - (\Lambda \circ A_q)^{-1}(Z) \). [The set \( Z \) has been defined in (3.8). Notice that this expression \( G \) depends upon \( q \in \mathbb{B}^n(0; r_2) \), for instance; see Figure 3 in Step 4 for an illustration.] In particular, this \( G \) maps \( \Omega \) onto its image \( G(\Omega) \) biholomorphically.

**Step 7: Proof of Theorem 3.1** Our present goal is to show the following

**Claim.** For any \( \epsilon \) with \( 0 < \epsilon < 1/2 \), there exists an integer \( m > 0 \) such that

\[ \mathbb{B}^n(0; 1 - \epsilon) \subset G(\Omega) \subset \mathbb{B}^n(0; 1 + \epsilon) \]

whenever \( q \in \Omega \cap B^n(p, \frac{r_3}{2km}) \).

Since \( G(q) = 0 \), this implies that the squeezing function \( \sigma_{\Omega} \) satisfies

\[ \sigma_{\Omega}(q) \geq \frac{1 - \epsilon}{1 + \epsilon}. \]

Notice that this completes the proof of Theorem 3.1.

Therefore we are only to establish this claim. Start with \( \mathbb{B}^n(0; 1 - \epsilon) \).

Notice first, by the definition of \( F_{\delta} \), that for every \( \delta > 0 \) there exists \( m_1 > 0 \) such that

\[ F_{\delta} \cap \mathbb{B}^n(0; r_2/m) \subset A_q(\Omega) \cap \mathbb{B}^n(0; r_2/m), \]

for any \( m > m_1 \).

Also,

\[ \kappa^{-1} \circ L^{-1}(\mathbb{B}^n(0; 1 - \epsilon)) \subset \subset \kappa^{-1} \circ L^{-1}(\mathbb{B}^n(0; 1)) = \hat{\Omega}. \]

As discussed in (T4)–(3.7), \( L \circ \kappa(H_q) \) is sufficiently close to \( L \circ \kappa(\hat{\Omega}) \) which is the unit ball, whenever \( q \in \mathbb{B}^n(p, \frac{r_3}{2km}) \) and \( m \) is sufficiently large. Therefore there exist an integer \( m_2 > m_1 \) such that \( (L \circ \kappa)^{-1}(\mathbb{B}^n(0; 1 - \epsilon)) \subset \subset H_q \) whenever \( q \in \mathbb{B}^n(p; r_3/m_2) \).

As in (3.19), a direct computation yields

\[ \kappa(F_{\delta}) = \{ z \in \mathbb{C}^n : |z_1|^2 + \frac{\delta}{2}|z_1 - \bar{z}_1| + (1 + \delta)H_{b(q)}(z') < 1 \}. \]

Now, consider the set \( L \circ \kappa \circ \Lambda(\mathcal{F}_\delta) \) for each \( \delta > 0 \). (Recall that \( \Lambda(\mathcal{F}_\delta) = \mathcal{F}_\delta \) as remarked in the line below (3.20).) These domains increase monotonically as \( \delta \searrow 0 \) (since \( \mathcal{F}_\delta \)'s do) in such a way that the union \( \bigcup_{0 < \delta < \delta_0} L \circ \kappa \circ (\mathcal{F}_\delta) \) becomes arbitrarily close to \( \mathbb{B}^n(0; 1) \) as \( m \) is sufficiently large. Consequently there exists a constant \( \delta > 0 \) such that \( \mathbb{B}^n(0; 1 - \epsilon) \subset \subset L \circ \kappa \circ (\mathcal{F}_\delta) \). Moreover there is an integer \( m_3 > m_2 \) such that

\[ \Lambda^{-1}_\lambda(\kappa^{-1} \circ L^{-1}(\mathbb{B}^n(0; 1 - \epsilon))) \subset \mathbb{B}^n(0; r_3/k(m_1)), \]
as $\Lambda^{-1}$ scales down the compact subsets (since $\lambda < r_3/m_2$, sufficiently small) to a small set near the origin. Hence, we have

$$\Lambda^{-1}(\kappa^{-1} \circ L^{-1}(B^n(0; 1 - \epsilon)) \subset F_\delta \cap B^n(0; r_3/k(m_1)) \subset \Omega.$$ 

Consequently,

$$\begin{align*}
B^n(0; 1 - \epsilon) &\subset L \circ \kappa \circ \Lambda^{-1}(F_\delta \cap B^n(0; r_3/k(m_1))) \\
&= L \circ \kappa \circ \Lambda^{-1}(A_q(\Omega)) \\
&= G(\Omega),
\end{align*}$$

as long as $q \in B^n(p; r_3/2k(m_2))$.

Now we show that $G(\Omega) \subset B^n(0; 1 + \epsilon)$. Consider

$$\Omega' := \Omega - B^n(p, r_2).$$

Notice that there exists an integer $\ell >> 1$ such that

$$A_q(\Omega') \subset A_q(\Omega) - B^n(0; r_2/\ell) \subset E - B^n(0; r_2/\ell).$$

Now, there exists an integer $m_4 > 3$ such that, if $m > m_4$ and $q \in B^n(p, r_3/m)$, then

$$\Lambda(\mathcal{E} - B^n(0; r_2/k)) \subset \{ z \in \mathcal{E} : \text{Re} z_1 > \frac{r_2}{r_3} \cdot \frac{m_4}{\ell} \}.$$

This implies that there exists $m_4$ such that

$$G(\Omega') \subset L \circ \kappa(\{ z \in \mathcal{E} : \text{Re} z_1 > \frac{r_2}{r_3} \cdot \frac{m_4}{\ell} \}) \subset (B^n(-1; \rho(m_4)))$$

for some $\rho(m)$ which approaches zero as $m$ tends to infinity; a direct computation with the Cayley transform and the choice of $L$ (cf. (3.14)) verify this immediately. Therefore, choosing $m_4$ sufficiently large, we arrive at

$$G(\Omega') \subset B^n(-1; \epsilon).$$

For the $\epsilon$ given above, there exists $\delta$ such that

$$L \circ \kappa(G_\delta) \subset B^n(0; 1 + \epsilon).$$
Fix this $\delta$. Then, recall how the auxiliary domain $G_\delta$ was defined in (3.16).

Given any $\delta > 0$, according to (3.4)–(3.6), there exists $\rho > 0$ such that

$$A_q(\Omega) \cap B^n(0; \rho) \subset G_\delta.$$

On the other hand, we can go back to (3.26) and require that $r_2/\ell < \rho/2$.

Then we have

(3.29) $$A_q(\Omega) \cap B^n(0; 2r_2/\ell) \subset G_\delta.$$ 

Since there exists an integer $m_5 > 0$ such that $A_q(B^n(p; r_2/\ell) \subset B^n(0; 2r_2/\ell)$, we have that

$$G(\Omega - \Omega') \subset L \circ \kappa \circ \Lambda_\lambda (A_q(\Omega) \cap B^n(0; 2r_2/\ell)).$$
This implies

\[ G(\Omega - \Omega') \subset L \circ \kappa \circ \Lambda_{1}(A_{q}(\Omega) \cap \mathbb{B}^{n}(0; 2r_{2}/\ell)) \]

by (3.29)

\[ G(\Omega - \Omega') \subset L \circ \kappa \circ \Lambda_{1}(G_{d}) \]

by the sentence following (3.20)

\[ \subset \mathbb{B}^{n}(0; 1 + \epsilon). \]

By (3.27) and (3.30) we have that

\[ G(\Omega) \subset \mathbb{B}^{n}(0; 1 + \epsilon). \]

This completes the proofs of Claim and Theorem 3.1.

\[ \square \]

4. Remarks

In this final section we present several remarks.

4.1. On the spherically-extreme points. Pertaining to Question 1.1, one of the naturally rising question would be whether one may re-embed (the closure of) the bounded strongly pseudoconvex domain so that the pre-selected boundary point becomes spherically extreme. Recent paper by Diederich-Fornaess-Wold [2] says that the answer to this question is affirmative. Owing to this new result, Theorem 3.1 now implies the following

**Theorem 4.1.** If \( \Omega \) is a bounded domain in \( \mathbb{C}^{n} \) with a \( C^{2} \)-smooth strongly pseudoconvex boundary, then

\[ \lim_{\Omega \ni z \to \partial \Omega} \sigma_{\Omega}(z) = 1. \]

On the other hand, a more ambitious try may be that one would like to re-embed the domain using the automorphisms of \( \mathbb{C}^{n} \) to achieve the same goal. But this cannot work. Here is a counterexample to such a try:

**Example 4.1.** Consider the domain \( U \) which is the open 1/10- tubular neighborhood of the circle \( S := \{(e^{it}, 0) \in \mathbb{C}^{2} : t \in \mathbb{R}\} \). This domain is strongly pseudoconvex. Let \( p = (9/10, 0) \). Clearly \( p \in \partial U \). If there were \( \psi \in \text{Aut}(\mathbb{C}^{2}) \) that makes \( \psi(p) \) spherically-extreme for \( \psi(U) \), then consider the analytic disc \( \Sigma := \psi(\Delta) \) where \( \Delta := \{(z, 0) : |z| \leq 1\} \). Since \( \Delta \) crosses \( \partial U \) transversally at \( \psi(p) \), \( \Sigma \) crosses the sphere envelope at \( \psi(p) \) and extends to the exterior of the sphere. On the other hand the boundary of \( \Sigma \) remains inside \( \psi(U) \) and hence inside the sphere. Now let the sphere expand radially from its center, and let it stop at the radius beyond which cannot have intersection with the holomorphic disc \( \Sigma \). Then the sphere is tangent to a point to \( \Sigma \) at an interior point keeping the whole disc inside the sphere. The maximum principle now implies that \( \Sigma \) should be entirely on the sphere. But the boundary of \( \Sigma \) is strictly inside the sphere, which is a contradiction. This implies that \( p \) cannot be made spherically-extreme via any re-embedding by an automorphism of \( \mathbb{C}^{n} \).
Acknowledgement: This example was obtained after a valuable discussion between the first named author and Josip Globevnik. The first named author would like to express his thanks to Josip Globevnik for pointing out such possibility.

4.2. On the exhaustion theorem by Fridman-Ma. The main theorem by Buma Fridman and Daowei Ma in [3] had obtained the conclusion of Theorem 3.1 in the special case \( \Omega \ni q \rightarrow p \) transversely to the boundary \( \partial \Omega \). However, that is not sufficient to prove Theorem 3.1; it is indeed necessary to consider all possible sequences approaching the boundary. In [3] they need not consider the point sequences approaching the boundary tangentially, as their interest was only on the holomorphic exhaustion of the ball by the biholomorphic images of a bounded strongly pseudoconvex domain. On the other hand, our proof of Theorem 3.1 gives a proof to their theorem as well; one only need to use \((1 + \epsilon)^{-1}G(z)\) instead of \(G\). [Recall that \(G\) depends upon \(q\). Letting \(q\) converge to \(p\) and \(\epsilon\) tend to zero, one gets a sequence of maps that exhausts the unit ball holomorphically.]

4.3. Plane domain cases. For domains in \(\mathbb{C}\), several theorems have been obtained by F. Deng, Q. Guan and L. Zhang in [1]. Theorem 3.1 obviously includes many of those results, as every boundary point of a plain domain with \(C^2\) smooth boundary is spherically-extreme.

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