Reflection positivity and quantum Griffiths' inequalities

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Abstract  
We propose a new method of constructing the quantum Griffiths inequality. From a viewpoint of operator inequalities, we first study the quantum rotor model. This viewpoint clarifies important connections between the reflection positivity and the Griffiths inequality. Next we apply our method to the spin-1/2 quantum Heisenberg model. Finally we point out a model-independent structure which governs the correlation inequalities.

1 Introduction  
The Ising model is the most fundamental model to illustrate the phenomena of phase transitions, since its formulation by Lenz \[17\]. Let \( \Lambda \) be a finite subset of \( \mathbb{Z}^d \). The system’s Hamiltonian is given by the function

\[
H_\Lambda(\sigma) = - \sum_{x,y \in \Lambda} J_{xy} \sigma_x \sigma_y
\]

for each \( \sigma = \{\sigma_x\}_{x \in \Lambda} \in \{-1,+1\}^\Lambda \). \( J_{xy} \) is a nonnegative coupling constant. The expected value of function \( f : \{-1,+1\}^\Lambda \to \mathbb{R} \) is

\[
\langle f(\sigma) \rangle_\beta = \frac{\sum_{\sigma \in \{-1,+1\}^\Lambda} f(\sigma) e^{-\beta H_\Lambda(\sigma)}}{Z_\beta},
\]

where \( Z_\beta \) is the normalization constant: \( Z_\beta = \sum_{\sigma \in \{-1,+1\}^\Lambda} e^{-\beta H_\Lambda(\sigma)} \). In \[13\], Griffiths discovered the following famous inequalities.

- First Griffiths inequality:

\[
\langle \sigma_A \rangle_\beta \geq 0
\]

for each \( A \subseteq \Lambda \), where \( \sigma_A = \prod_{x \in A} \sigma_x \).

- Second Griffiths inequality:

\[
\langle \sigma_A \sigma_B \rangle_\beta \geq \langle \sigma_A \rangle_\beta \langle \sigma_B \rangle_\beta
\]

for each \( A, B \subseteq \Lambda \).

\[1\]To be precise, this general formulation was established by Kelly and Sherman \[20\].
Since Griffiths’ discovery, a huge number of rigorous studies of the Ising ferromagnet has been successfully achieved by applying his inequalities. The fact that Griffiths’ inequalities are so useful indicates that the inequalities express the essence of correlations in the Ising system. Therefore it is natural to ask whether similar inequalities hold true for other models. To study this problem means trying to seek a model-independent or universal aspect of the notion of correlations. It is already known that Griffiths’ inequalities hold true for some classical models, for example, the plane rotor model. This suggests that our problem is certainly meaningful. Ginibre took the first important step in giving a general framework of Griffiths’ inequalities [11]. However we know only a few concrete examples of quantum (=noncommutative) models which satisfy Griffiths’ inequalities, see e.g., [5, 10, 15]. Our goal here is to present a general method of constructing the Griffiths inequality for both classical and quantum systems. To this end, we advance the technique of the operator inequalities associated with the self-dual cones.

We already know that the quantum Ising model and the quantum rotor model indeed satisfy Griffiths’ inequalities. Thus these two models can be regarded as role models for our purpose. A standard approach to prove the Griffiths inequality for these systems is to reduce the $d$-dimensional quantum systems to the corresponding $d+1$-dimensional classical systems, see e.g., [3, 21]. Since known proofs of the quantum Griffiths’ inequalities rely on results of the classical systems, it is difficult to extend these proofs to quantum models which can not be reduced to classical models. Considering this situation, we take the following steps:

(i) We prove the Griffiths inequality for the quantum Ising model and the quantum rotor model by a method of operator inequalities and grasp common mathematical structures underlying both models.

(ii) We seek similar structures in other models by our viewpoint of operator inequalities and construct the Griffiths inequality by analogy.

In [29], we carried out this program and constructed Griffiths’ inequalities for the Hubbard model and the bose Hubbard model by studying the quantum Ising model as a prototype. We actually confirmed the effectiveness of our method of operator inequalities. However our results of the Hubbrad model are restricted to the ground state expectation of observables or the zero temperature case. In this paper, we will explore the case of finite-temperature. Our technical idea is to combine the reflection positivity and the operator inequalities associated with the self-dual cones. As we pointed out in [24], the reflection positivity can be regarded as an operator inequality associated with a special self-dual cone. This viewpoint makes it possible to see a common mathematical structure among various quantum models. The reflection positivity originates from the axiomatic quantum field theory [30]. Glimm, Jaffe and Spencer first found an application of the reflection positivity to the rigorous study of the phase transition [12]. This idea was successfully developed by Dyson, Fröhlich, Israel, Lieb and Simon [4, 8, 9] and many others. Besides Lieb discovered a crucial application of the reflection positivity to many-electron systems, called the spin reflection positivity [22]. Recently Jaffe and Pedrocchi studied the topological order by the reflection positivity [18, 19].

In this paper, we achieved the following:
We propose an operator theoretical construction of the Griffiths inequality for the quantum rotor model. Our method clarifies a connection between the reflection positivity and the Griffiths inequality.

We construct quantum Griffiths’ inequalities for the spin-1/2 quantum Heisenberg model by the method established in (A).

Although we don’t give any concrete applications of our results in this paper, we expect that these inequalities would play important roles in the statistical physics as the original Griffiths’ inequalities did in the Ising system. Finally we emphasize the following. From a viewpoint of operator inequalities, we can find a common mathematical structure among the several models: the quantum Ising model, the quantum rotor model, the Hubbard model, the bose Hubbard model, the quantum spin-1/2 Heisenberg model, the Fröhlich polaron model, the Holstein-Hubbard model and so on [7, 14, 22, 27, 28]. This universal structure enables us to construct the Griffiths inequality in each model mentioned above. From this fact, we expect to understand a model-independent or general expression of the notion of the correlation, see Section 5.

This paper is organized as follows. In Section 2, we introduce several operator inequalities related with self-dual cones. In particular, we give a useful expression of the reflection positivity in terms of the operator inequality. This expression is essential for our purpose. In Section 3 we discuss the quantum rotor model. As mentioned above, the Griffiths’ inequality for this model is already known. Through analysis of this model, we show how operator theoretical approach works well. We also provide some nontrivial extensions of the inequalities for this model. In Section 4 we prove the Griffiths inequality for the spin-1/2 quantum Heisenberg model by applying operator inequalities. We will see that our idea is a natural generalization of the method established in Section 3. In Section 5 we give concluding remarks. In Section A we prove a useful lemma.

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2 Preliminaries

2.1 Positivity preserving operators

Let $(\mathfrak{H}, \langle \cdot | \cdot \rangle)$ be a complex Hilbert space and $\mathfrak{P}$ be a convex cone in $\mathfrak{H}$. We say that $\mathfrak{P}$ is self-dual if

$$\mathfrak{P} = \{ x \in \mathfrak{H} | \langle x | y \rangle \geq 0 \ \forall y \in \mathfrak{P} \}. \quad (2.5)$$

In this section, we always assume that $\mathfrak{P}$ is self-dual. Each element $x$ in $\mathfrak{P}$ is called positive w.r.t. $\mathfrak{P}$ and written as $x \geq 0$ w.r.t. $\mathfrak{P}$.

**Definition 2.1** We denote by $\mathcal{B}(\mathfrak{H})$ the set of all bounded linear operators on $\mathfrak{H}$. Let $A \in \mathcal{B}(\mathfrak{H})$. If $Ax \geq 0$ w.r.t. $\mathfrak{P}$ for all $x \in \mathfrak{P}$, then we say that $A$ preserves the positivity w.r.t. $\mathfrak{P}$ and write

$$A \geq 0 \ \text{w.r.t.} \ \mathfrak{P}. \quad (\text{This symbol was introduced by Miura} \ [21]. \ \text{See also} \ [1, 6, 16]$$
We note that
\[ A \succeq 0 \text{ w.r.t. } \mathfrak{P} \implies \langle x | Ay \rangle \geq 0 \quad \forall x, y \in \mathfrak{P}. \] \hfill (2.6)

**Remark 2.2** The following properties are useful.

(i) \( a \geq 0, A \succeq 0 \implies aA \succeq 0. \)

(ii) \( A \succeq 0, B \succeq 0 \implies A + B \succeq 0. \)

(iii) \( A \succeq 0, B \succeq 0 \implies AB \succeq 0. \) \hfill \(\diamondsuit\)

**Theorem 2.3** Let \( H \) be a self-adjoint operator bounded from below. Assume the following.

(i) There exists a complete orthonormal system \( \{x_n\}_{n \in \mathbb{N}} \) of \( \mathfrak{H} \) such that \( x_n \in \mathfrak{P} \) for all \( n \in \mathbb{N} \).

(ii) \( e^{-\beta H} \) is in trace class for all \( \beta > 0 \).

(iii) \( e^{-\beta H} \succeq 0 \text{ w.r.t. } \mathfrak{P} \) for all \( \beta > 0 \).

Let \( \langle \cdot \rangle_\beta \) be the thermal average defined by
\[ \langle X \rangle_\beta = \frac{\text{Tr}[X e^{-\beta H}]}{Z_\beta}, \quad Z_\beta = \text{Tr}[e^{-\beta H}]. \] \hfill (2.7)

Then we obtain
\[ \langle A \rangle_\beta \succeq 0 \] \hfill (2.8)
for all \( A \succeq 0 \text{ w.r.t. } \mathfrak{P} \).

Let \( A(s) = e^{-sH}Ae^{sH} \). If \( A_j \succeq 0 \text{ w.r.t. } \mathfrak{P} \) for all \( j = 1, \ldots, n \), we then have
\[ \langle \prod_{j=1}^{n} A_j(s_j) \rangle_\beta \succeq 0 \] \hfill (2.9)
for all \( 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \beta \), where \( \prod_{j=1}^{n} O_j = O_1O_2\cdots O_n \), the ordered product.

**Proof.** Since \( A \succeq 0 \text{ w.r.t. } \mathfrak{P} \), we have \( Ae^{-\beta H} \succeq 0 \text{ w.r.t. } \mathfrak{P} \) for all \( \beta \geq 0 \) by Remark 2.2 (iii). Thus we see that
\[ \text{Tr}[Ae^{-\beta H}] = \sum_{n=0}^{\infty} \langle x_n | Ae^{-\beta H} x_n \rangle \geq 0. \] \hfill (2.10)
This concludes (2.8).
Let $S = \left[ \prod_{j=1}^{n} A_j(s_j) \right] e^{-\beta H}$. By the assumptions, we see that

$$S = e^{-s_1 H} A_1 e^{-(s_2-s_1) H} \cdots A_n e^{-(\beta-s_n) H} \geq 0 \quad \text{w.r.t. } \mathfrak{P}. \quad (2.11)$$

Thus by (2.6), we obtain (2.9). $\blacksquare$

In the previous paper [29], we constructed several examples of the quantum Griffiths inequalities by applying Theorem 2.3. In the present paper, we discuss cases where Theorem 2.3 does not cover.

### 2.2 Reflection positivity

In this subsection, we review the reflection positivity from a viewpoint of operator inequalities.

For each $p \in \mathbb{N}$, we denote by $L^p(\mathcal{H})$ the trace ideal defined by

$$L^p(\mathcal{H}) = \{ \xi \in B(\mathcal{H}) \mid \text{Tr}[|\xi|^p] < \infty \}. \quad (2.12)$$

$L^1(\mathcal{H})$ is called the trace class, while $L^2(\mathcal{H})$ is called the Hilbert-Schmidt class. $L^2(\mathcal{H})$ becomes a Hilbert space if once we define the inner product by $\langle \eta |\xi \rangle_{L^2} = \text{Tr}[\eta^* \xi]$ for all $\eta, \xi \in L^2(\mathcal{H})$.

**Definition 2.4 (Bounded operators)** Let $A \in \mathcal{B}(\mathcal{H})$.

(i) The left multiplication operator $\mathcal{L}(A)$ is defined by $\mathcal{L}(A)\xi = A\xi$ for all $\xi \in L^2(\mathcal{H})$.

(ii) The right multiplication operator $\mathcal{R}(A)$ is defined by $\mathcal{R}(A)\xi = \xi A$ for all $\xi \in L^2(\mathcal{H})$.

Remark that $\mathcal{L}(A), \mathcal{R}(A) \in \mathcal{B}(L^2(\mathcal{H}))$, the set of all bounded operators on $L^2(\mathcal{H})$. $\diamond$

Let $\vartheta$ be an anti-linear involution on $\mathcal{H}$. Let $\Phi_\vartheta$ be an isometric isomorphism from $L^2(\mathcal{H})$ onto $\mathcal{H} \otimes \mathcal{H}$ defined by

$$\Phi_\vartheta(|x\rangle\langle y|) = x \otimes \vartheta y \quad \forall x, y \in \mathcal{H}. \quad (2.13)$$

We have the following relations:

$$\mathcal{L}(A) = \Phi_\vartheta^{-1} A \otimes \Phi_\vartheta, \quad \mathcal{R}(\vartheta A^* \vartheta) = \Phi_\vartheta^{-1} \mathbb{1} \otimes A \Phi_\vartheta \quad (2.14)$$

for each $A \in \mathcal{B}(\mathcal{H})$. We simply write these facts as

$$\mathcal{H} \otimes \mathcal{H} = L^2(\mathcal{H}), \quad A \otimes \mathbb{1} = \mathcal{L}(A), \quad \mathbb{1} \otimes A = \mathcal{R}(\vartheta A^* \vartheta), \quad (2.15)$$

if no confusion arises.

Definition 2.4 can be extended to unbounded operators by (2.14) as follows.

**Definition 2.5 (Unbounded operators)** Let $A$ be densely defined closed operator on $\mathcal{H}$.

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(i) The left multiplication operator $\mathcal{L}(A)$ is defined by $\mathcal{L}(A) = \Phi^{-1}_q A \otimes \Phi_q$.

(ii) The right multiplication operator $\mathcal{R}(A)$ is defined by $\mathcal{R}(A) = \Phi^{-1}_q \otimes \partial A^* \partial \Phi_q$.

\(\diamondsuit\)

Remark 2.6

(i) Both $\mathcal{L}(A)$ and $\mathcal{R}(A)$ are closed operators on $L^2(\mathfrak{h})$.

(ii) If $A$ is self-adjoint, then so are $\mathcal{L}(A)$ and $\mathcal{R}(A)$.

(iii) We will also use the conventional identification \(\text{(2.14)}\). \(\diamondsuit\)

Definition 2.7 A canonical cone in $L^2(\mathfrak{h})$ is defined by

$$L^2(\mathfrak{h})_+ = \{ \xi \in L^2(\mathfrak{h}) \mid \xi \geq 0 \text{ as a linear operator in } \mathfrak{h} \}. \quad (2.16)$$

$L^2(\mathfrak{h})_+$ is self-dual. \(\diamondsuit\)

In [29], the following proposition was essential to prove the quantum Griffiths inequalities for the Hubbard model at $\beta = \infty$.

Proposition 2.8 For each $A \in B(\mathfrak{h})$, we have $\mathcal{L}(A)\mathcal{R}(A^*) \succeq 0$ w.r.t. $L^2(\mathfrak{h})_+$.

Unfortunately the notion of the positivity preserving is not enough to show the quantum Griffiths inequalities at finite $\beta$. To overcome this situation, we employ a concept of the reflection positivity.

Definition 2.9 We define

$$\mathfrak{A} = \text{Coni} \left\{ \mathcal{L}(A)\mathcal{R}(A^*) \in B(L^2(\mathfrak{h})) \mid A \in B(\mathfrak{h}) \right\}^{\text{w}}. \quad (2.17)$$

where Coni$(X)$ is the conical hull of $X$ and $S^\text{w}$ means the closure of $S$ under the weak topology in $B(L^2(\mathfrak{h}))$.

If $A \in \mathfrak{A}$, then we write $A \succeq 0$ w.r.t. $L^2(\mathfrak{h})_+$. \(\diamondsuit\)

Remark 2.10

(i) $A \succeq 0 \Rightarrow A \succeq 0$.

(ii) $A \succeq 0, B \succeq 0, a, b \geq 0 \Rightarrow aA + bB \succeq 0$.

(iii) $A \succeq 0, B \succeq 0 \Rightarrow AB \succeq 0$. \(\diamondsuit\)

The following proposition is a guiding principle of the reflection positivity \[4, 8, 26\].

Proposition 2.11 (The reflection positivity) Assume that $A$ is a trace class operator on $L^2(\mathfrak{h})$, i.e., $A \in L^1(L^2(\mathfrak{h}))$. If $A \succeq 0$ w.r.t. $L^2(\mathfrak{h})_+$, then we have $\text{Tr}_{L^2}[A] \geq 0$.

Proof. It suffices to consider the case where $A = \sum_{j=1}^N \mathcal{L}(a_j)\mathcal{R}(a_j^*)$, $N \in \mathbb{N}$. In this case, we easily see that $\text{Tr}_{L^2}[A] = \sum_{j=1}^N |\text{Tr}_\mathfrak{h}[a_j]|^2 \geq 0$. \(\square\)
Theorem 2.12 Let $H$ be a self-adjoint operator on $\mathcal{L}^2(\mathcal{H})$, bounded from below. Suppose that $e^{-\beta H}$ is a trace class operator for all $\beta > 0$. Let $\langle \cdot \rangle_{\beta}$ be the thermal average defined by

$$\langle X \rangle_{\beta} = \frac{\text{Tr}[X e^{-\beta H}]}{\text{Tr}[e^{-\beta H}]} \quad \forall X \in \mathcal{B}(\mathcal{L}^2(\mathcal{H})).$$ (2.18)

Assume that $e^{-\beta H} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $\beta \geq 0$. Then we have $\langle A \rangle_{\beta} \succeq 0$ for all $A \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$.

If $A_j \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $j = 1, \ldots, n$, we then have

$$\langle \prod_{j=1}^n A_j(s_j) \rangle_{\beta} \succeq 0 \quad (2.19)$$

for all $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \beta$.

Proof. By Remark 2.10 (iii), we have $A e^{-\beta H} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $\beta \geq 0$. Thus applying Proposition 2.11, we conclude $\langle A \rangle_{\beta} \succeq 0$. Similarly since

$$\left[ \prod_{j=1}^n A_j(s_j) \right] e^{-\beta H} = e^{-s_1 H} A_1 e^{-s_2 H} A_2 \cdots e^{-s_n H} A_n \succeq 0$$ (2.20)

w.r.t. $\mathcal{L}^2(\mathcal{H})_+$, we obtain (2.19). □

To apply Theorem 2.12, it is essential to show that $e^{-\beta H} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $\beta > 0$. The following proposition is often useful to prove this condition.

Proposition 2.13 Let $H_0$ be a self-adjoint operator on $\mathcal{L}^2(\mathcal{H})$ bounded from below. Let $V \in \mathcal{B}(\mathcal{L}^2(\mathcal{H}))$ be self-adjoint. Assume the following.

(i) $e^{-\beta H_0} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $\beta \geq 0$.

(ii) $V \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$.

Let $H = H_0 - V$. We have $e^{-\beta H} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{H})_+$ for all $\beta \geq 0$.

Proof. Note that

$$e^{\beta V} = \sum_{n \geq 0} \frac{\beta^n}{n!} \sum_{s \geq 0} V^n \succeq 0 \quad \text{w.r.t.} \quad \mathcal{L}^2(\mathcal{H})_+.$$ (2.21)

Thus by the Trotter-Kato product formula, we obtain

$$e^{-\beta H} = \lim_{n \to \infty} \left( e^{-\beta H_0/n} e^{\beta V/n} \right)^n \succeq 0 \quad \text{w.r.t.} \quad \mathcal{L}^2(\mathcal{H})_+ \quad \text{for all} \quad \beta \geq 0,$$ (2.22)

where $\lim_{n \to \infty}$ means the strong limit. □
Corollary 2.14 Let $H_0 = \mathcal{L}(A) + \mathcal{R}(A)$, where $A$ is self-adjoint and bounded from below. Let
\[
V = \sum_{j=1}^{\infty} \mathcal{L}(B_j)\mathcal{R}(B_j),
\]
where $B_j \in \mathcal{B}(\mathcal{H})$ is self-adjoint and the right hand side of (2.23) is a weak convergent sum. Define $H = H_0 - V$. Then we obtain $e^{-\beta H} \succeq 0$ w.r.t. $L^2(H)$ for all $\beta \geq 0$.

Proof. Observe that $e^{-\beta H_0} = \mathcal{L}(e^{-\beta A})\mathcal{R}(e^{-\beta A}) \succeq 0$ w.r.t. $L^2(H)$ for all $\beta \geq 0$. Since $V \succeq 0$ w.r.t. $L^2(H)$, we obtain the desired assertion by Proposition 2.13. ✷

3 Quantum rotor model

3.1 Results

Let $\Lambda$ be a finite subset of $\mathbb{R}^2$. The quantum rotor model on $\Lambda$ is defined by
\[
H = \sum_{x \in \Lambda} U_x \left( -i \frac{\partial}{\partial \theta_x} \right)^2 - \sum_{x,y \in \Lambda} t_{xy} \cos(\theta_x - \theta_y).
\]

The Hilbert space is $\mathcal{H} = \otimes_{x \in \Lambda} L^2(T)$ with $T = [-\pi, \pi]$. $U_x \geq 0$ is the strength of the on-site repulsion and $t_{xy} \geq 0$ is the hopping strength. $H$ is a self-adjoint operator acting in the Hilbert space $\mathcal{H}$. For readers who want to learn the physical background, we refer to [2, 32].

Remark 3.1 In this paper, we simply write $M_f$, the multiplication operator by the function $f$, as $f(\theta)$ if no confusion occurs.

Let $T_x = e^{i\theta_x}$. For each $A = \{m_x\}_{x \in \Lambda} \in \mathbb{Z}^\Lambda$, we set
\[
T^A = \prod_{x \in \Lambda} (T_x)^{m_x}.
\]

Let
\[
\mathfrak{A} = \text{Coni}\{T^A \mid A \in \mathbb{Z}^\Lambda\}^{\text{-w}}.
\]

The thermal expectation $\langle \cdot \rangle_\beta$ is defined by
\[
\langle A \rangle_\beta = \frac{\text{Tr} \left[ A e^{-\beta H} \right]}{Z_\beta}, \quad Z_\beta = \text{Tr} \left[ e^{-\beta H} \right]
\]
for all $A \in \mathcal{B}(\mathcal{H})$.

The precise definition of $-i\frac{\partial}{\partial \theta}$ is as follows:
\[
\text{dom} \left( -i \frac{\partial}{\partial \theta} \right) = \{f \in C^1(T) \mid f(0) = f(2\pi)\},
\]
\[
-\frac{\partial}{\partial \theta} f = -if' \quad \forall f \in \text{dom} \left( -i \frac{\partial}{\partial \theta} \right).
\]

Then $-i\frac{\partial}{\partial \theta}$ is essentially self-adjoint. We still denote its closure by the same symbol.
Theorem 3.2 (First Griffiths inequality) We have $\langle A \rangle_\beta \geq 0$ for all $A \in \mathfrak{A}$.

For each $s \geq 0, m \in \mathbb{Z}$ and $x \in \Lambda$, set
$$T^m_x(s) = e^{-sH}T^mxe^{sH}. \quad (3.5)$$

We have the following variant of Theorem 3.2.

Theorem 3.3 For all $x_1, \ldots, x_n \in \Lambda, m_1, \ldots, m_n \in \mathbb{Z}$ and $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \beta$, we have
$$\langle \prod_{j=1}^n T^{m_j}_{x_j}(s_j) \rangle_\beta \geq 0. \quad (3.6)$$

To state second Griffiths inequalities, we need some preparations. We introduce an extended Hilbert space $H_{\text{ext}}$ by $H_{\text{ext}} = H \otimes H$. For each $X \in \mathscr{B}(H_{\text{ext}})$, we set
$$\langle \langle X \rangle \rangle_\beta = \frac{\text{Tr}_{H_{\text{ext}}} [X e^{-\beta H_{\text{ext}}}] / Z_\beta^2}{Z_\beta^2}, \quad (3.7)$$

$$H_{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H. \quad (3.8)$$

Let $C_x = \cos \theta_x$ and let
$$C_x(s) = e^{-sH}C_x e^{sH}. \quad (3.9)$$

Theorem 3.4 (Second Griffiths inequality) For all $x_1, \ldots, x_n \in \Lambda, 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \beta$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, we have
$$\langle \langle \prod_{j=1}^n C_{x_j}(s_j) \otimes \mathbb{1} + \varepsilon_j \mathbb{1} \otimes C_{x_j}(s_j) \rangle \rangle_\beta \geq 0. \quad (3.10)$$

From Theorem 3.4, we immediately obtain the Corollary 3.5 which has a form similar to (3.4). This is the reason why we call Theorem 3.4 as the second Griffiths inequality.

Corollary 3.5 For each $A = \{m_x\}_{x \in \Lambda} \in \mathbb{N}^\Lambda$, set
$$C^A = \prod_{x \in \Lambda} (C_x)^{m_x}. \quad (3.11)$$

For all $A, B \in \mathbb{N}^\Lambda$, we obtain
$$\langle C^A C^B \rangle_\beta \geq \langle C^A \rangle_\beta \langle C^B \rangle_\beta. \quad (3.12)$$

Let $n_x(s) = e^{-sH}n_x e^{sH}$. We have the following.

Theorem 3.6 For all $x_1, \ldots, x_n \in \Lambda, 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \beta$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, we have
$$\langle \prod_{j=1}^n \left[ n_{x_j}(s_j) \otimes \mathbb{1} + \varepsilon_j \mathbb{1} \otimes n_{x_j}(s_j) \right] \left[ n_{x_j}(s_j) \otimes \mathbb{1} + \varepsilon_j \mathbb{1} \otimes n_{x_j}(s_j) \right] \rangle_\beta \geq 0, \quad (3.13)$$
where $\varepsilon_j = -\varepsilon_j$. 

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We can construct several extensions of Theorems 3.4 and 3.6. Theorem 3.7 illustrates this fact. Let
\[\alpha^{(1)}_x(s) = C_x(s) \otimes \mathbb{1} + \mathbb{1} \otimes C_x(s)\]
\[\alpha^{(2)}_x(s) = C_x(s) \otimes \mathbb{1} - \mathbb{1} \otimes C_x(s)\]
\[\alpha^{(3)}_x(s) = \left[ n_x(s) \otimes \mathbb{1} + \mathbb{1} \otimes n_x(s) \right] \left[ n_x(s) \otimes \mathbb{1} - \mathbb{1} \otimes n_x(s) \right].\]

Theorem 3.7
For all \(x_1, \ldots, x_n \in \Lambda\), \(\mu_1, \ldots, \mu_n \in \{1, 2, 3\}\) and \(0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \beta\), we have
\[\left\langle \prod_{j=1}^{n} \alpha^{(\mu_j)}_{x_j}(s_j) \right\rangle_\beta \geq 0.\]

3.2 Proof of Theorems 3.2 and 3.3
Let \(F\) be the Fourier transformation on \(\hat{\mathcal{H}}\) and let \(\hat{H} = FHF^{-1}\). We have
\[\hat{H} = \sum_{x \in \Lambda} \frac{U_x}{2} \hat{n}_x^2 + \frac{1}{2} \sum_{x,y \in \Lambda} (-t_{xy})(\hat{T}_x \hat{T}_y^* + \hat{T}_y \hat{T}_x^*).\]

\(\hat{H}\) acts in the Hilbert space \(\hat{\mathcal{H}} = \bigotimes_{x \in \Lambda} \ell^2(\mathbb{Z})\). The definitions of \(\hat{n}_x\) and \(\hat{T}_x\) are as follows: For each \(n \in \mathbb{Z}\), set \(e_n(m) = \delta_{mn} \in \ell^2(\mathbb{Z})\). \(\{e_n\}_{n \in \mathbb{Z}}\) is a complete orthonormal system (CONS) in \(\ell^2(\mathbb{Z})\). For each \(n = \{n_x\}_{x \in \Lambda} \in \mathbb{Z}^\Lambda\), let \(e_n = \bigotimes_{x \in \Lambda} e_{n_x}\). Clearly \(\{e_n | n \in \mathbb{Z}^\Lambda\}\) is a CONS of \(\hat{\mathcal{H}}\) as well. Now we define, for each \(n = \{n_x\}_{x \in \Lambda} \in \mathbb{Z}^\Lambda\),
\[\hat{n}_x e_n = n_x e_n, \quad \hat{T}_x e_n = e_{n + \delta_x},\]
where \(\delta_x = \{\delta_{xy}\}_{y \in \Lambda} \in \mathbb{Z}^\Lambda\). In other words, \(\hat{n}_x\) is the number operator and \(\hat{T}_x\) is the creation operator at site \(x\).

Definition 3.8
Let
\[\hat{\mathcal{H}}_+ = \left\{ F = \sum_{n \in \mathbb{Z}^\Lambda} F(n) e_n \in \hat{\mathcal{H}} \mid F(n) \geq 0 \ \forall n \in \mathbb{Z}^\Lambda \right\}.\]

Remark that \(\hat{\mathcal{H}}_+\) is a self-dual cone in \(\hat{\mathcal{H}}\). ♦

Proposition 3.9
One has the following:
(i) \(\hat{T}_x \geq 0\) w.r.t. \(\hat{\mathcal{H}}_+\) for all \(x \in \Lambda\).
(ii) \(e^{-\beta H} \geq 0\) w.r.t. \(\hat{\mathcal{H}}_+\) for all \(\beta \geq 0\).

Proof. (i) Note \(\hat{\mathcal{H}}_+ = \text{Coni}\{e_n | n \in \mathbb{Z}^\Lambda\}^\circ\), where \(\text{Coni}(S)^\circ\) is the closure of \(\text{Coni}(S)\). Thus it suffices to show that \(\hat{T}_x e_n \geq 0\) w.r.t. \(\hat{\mathcal{H}}_+\) for all \(n \in \mathbb{Z}^\Lambda\). But this is trivial by (3.19).
(ii) Let
\[ -\hat{K} = \frac{1}{2} \sum_{x,y \in \Lambda} t_{xy} (\hat{T}_x \hat{T}_y + \hat{T}_y \hat{T}_x), \quad \hat{U} = \sum_{x \in \Lambda} \frac{U_x}{2} \hat{n}^2_x. \quad (3.21) \]

By (i), we see that \( -\hat{K} \geq 0 \) w.r.t. \( \hat{H} + \). Thus we have \( (-\hat{K})^n \geq 0 \) w.r.t. \( \hat{H} + \) for all \( n \in \mathbb{N} \), which implies
\[ e^{-\beta \hat{K}} = \sum_{n \geq 0} \frac{\beta^n}{n!} (-\hat{K})^n \geq 0 \text{ w.r.t. } \hat{H} + \text{ for all } \beta \geq 0. \quad (3.22) \]

On the other hand, since
\[ e^{-\beta \hat{U}} e_n = \exp \left\{ -\beta \sum_{x \in \Lambda} \frac{U_x}{2} \hat{n}^2_x \right\} e_n \text{ for all } n = \{n_x\} \in \mathbb{Z}^\Lambda, \quad (3.23) \]
we have \( e^{-\beta \hat{U}} \geq 0 \) w.r.t. \( \hat{H} + \). Thus \( (e^{-\beta \hat{K}} / \ell e^{-\beta \hat{U}} / \ell) \geq 0 \) w.r.t. \( \hat{H} + \). Taking \( \ell \to \infty \), we conclude (ii) by the Trotter-Kato formula. \( \square \)

### 3.2.1 Completion of proof of Theorems 3.2 and 3.3

By Proposition 3.9 (i), we have \( A \geq 0 \) w.r.t. \( \hat{H} + \) for all \( A \). Applying Theorem 2.3, we obtain Theorem 3.2.

Similarly since \( (\hat{T}_{xy})^{m_j} \geq 0 \) w.r.t. \( \hat{H} + \) for all \( j = 1, \ldots, n \), we can apply Theorem 2.3 again. Thus we conclude Theorem 3.3. \( \square \)

### 3.3 Proof of Theorems 3.4, 3.6, 3.7 and Corollary 3.5

Let \( \hat{H}_{\text{ext}} = \hat{H} \otimes \hat{H} \). First remark the following identification:
\[ \hat{H}_{\text{ext}} = L^2(\mathbb{T}^\Lambda \times \mathbb{T}^\Lambda, d\theta d\theta'). \quad (3.24) \]

Under the identification (3.24), we see that
\[ H_{\text{ext}} = H \otimes \mathbb{I} + \mathbb{I} \otimes H \]
\[ = \sum_{x \in \Lambda} \frac{U_x}{2} \left\{ \left( -i \frac{\partial}{\partial \theta_x} \right)^2 + \left( -i \frac{\partial}{\partial \theta'_x} \right)^2 \right\} \]
\[ - \sum_{x,y \in \Lambda} t_{xy} \left\{ \cos(\theta_x - \theta_y) + \cos(\theta'_x - \theta'_y) \right\}. \quad (3.25) \]

Next we introduce a new coordinate system \( \{\phi_x, \phi'_x\} \) by
\[ \phi_x = \frac{1}{2}(\theta_x' - \theta_x), \quad \phi'_x = \frac{1}{2}(\theta'_x + \theta_x). \quad (3.26) \]
Then we easily see that
\[ \hat{H}_{\text{ext}} = L^2(\mathbb{T}^\Lambda \times \mathbb{T}^\Lambda, d\phi d\phi'). \quad (3.27) \]
Using the following elementary fact:

\[ \cos \theta + \cos \theta' = 2 \cos \frac{\theta' + \theta}{2} \cos \frac{\theta' - \theta}{2}, \]

we obtain

\[ H_{\text{ext}} = \sum_{x \in \Lambda} \frac{U_x}{4} (\nu_x^2 + \nu_x'^2) - 2 \sum_{x,y \in \Lambda} t_{xy} \cos(\phi_x - \phi_y) \cos(\phi_x' - \phi_y'), \]  

(3.29)

where

\[ \nu_x = -i \frac{\partial}{\partial \phi_x}, \quad \nu_x' = -i \frac{\partial}{\partial \phi_x'}. \]  

(3.30)

Let \( \mathcal{X} = L^2(T^\Lambda, d\phi) \). Then by (3.27), one obtains the following identification:

\[ \mathcal{H}_{\text{ext}} = L^2(T^\Lambda, d\phi) \otimes L^2(T^\Lambda, d\phi) = \mathcal{X} \otimes \mathcal{X}. \]  

(3.31)

Moreover we obtain

**Proposition 3.10** We have \( H_{\text{ext}} = T - V \), where

\[ T = \sum_{x \in \Lambda} \frac{U_x}{4} (\nu_x^2 \otimes 1 + 1 \otimes \nu_x^2), \]  

(3.32)

\[ V = 2 \sum_{x,y \in \Lambda} t_{xy} \cos(\phi_x - \phi_y) \otimes \cos(\phi_x - \phi_y). \]  

(3.33)

Let \( \vartheta \) be the anti-linear isomorphism defined by

\[ (\vartheta f)(\phi) = \overline{f}(\phi) \, \text{ a.e., } \, f \in L^2(T^\Lambda, d\phi). \]  

(3.34)

By (2.15) and (3.31), we have the identification \( \mathcal{H}_{\text{ext}} = \mathcal{L}^2(\mathcal{X}) \) by \( \vartheta \). Moreover, by (2.15), we have the following proposition.

**Proposition 3.11** We have \( H_{\text{ext}} = T - V \), where

\[ T = \sum_{x \in \Lambda} \frac{U_x}{4} \{ \mathcal{L}(\nu_x^2) + \mathcal{R}(\nu_x'^2) \}, \]  

(3.35)

\[ V = 2 \sum_{x,y \in \Lambda} t_{xy} \mathcal{L} \left[ \cos(\phi_x - \phi_y) \right] \mathcal{R} \left[ \cos(\phi_x - \phi_y) \right]. \]  

(3.36)

By Corollary 2.14 we immediately obtain

**Corollary 3.12** We have \( \exp(-\beta H_{\text{ext}}) \succeq \sigma \) w.r.t. \( \mathcal{L}^2(\mathcal{X})_+ \) for all \( \beta \geq 0 \).
3.3.1 Proof of Theorem 3.4 and Corollary 3.5

Proposition 3.13 We have the following.

(i) \[ \cos \theta \otimes 1 + 1 \otimes \cos \theta = 2\mathcal{L}(\cos \phi)\mathcal{R}(\cos \phi) \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{X})_+. \]

(ii) \[ \cos \theta \otimes 1 - 1 \otimes \cos \theta = 2\mathcal{L}(\sin \phi)\mathcal{R}(\sin \phi) \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{X})_+. \]

Proof. (i), (ii) We apply Ginibre’s idea [11]:

\[ \cos a + \cos b = 2 \cos \frac{b + a}{2} \cos \frac{b - a}{2}, \quad (3.37) \]
\[ \cos a - \cos b = 2 \sin \frac{b + a}{2} \sin \frac{b - a}{2}. \quad (3.38) \]

Put

\[ 2V_{x}^{(v)} = C_x \otimes 1 + \varepsilon 1 \otimes C_x, \quad \varepsilon = \pm 1. \quad (3.39) \]

Then by Proposition 3.13, we have \( V_x^{(v)} \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{X})_+ \) for all \( x \in \Lambda \) and \( \varepsilon \in \{ \pm 1 \} \). Since \( \exp(-\beta H_{\text{ext}}) \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{X})_+ \) for all \( \beta \geq 0 \) by Corollary 3.12, we can apply Theorem 2.12. Thus we conclude Theorem 3.4.

For each \( A \subseteq \Lambda \), define \( [A] = \{ m_x \}_{x \in A} \in \mathbb{N}^\Lambda \) by \( m_x = 1 \) if \( x \in A \), \( m_x = 0 \) otherwise. For simplicity, we will consider the cases where \( A = [A] \) and \( B = [B] \). To prove Corollary 3.5, we note

\[ C_x \otimes 1 = V_x^{(+1)} + V_x^{(-1)}, \quad 1 \otimes C_x = V_x^{(+1)} - V_x^{(-1)}. \quad (3.40) \]

Observe that

\[ 2(C^A C^B)_\beta - 2(C^A)_\beta (C^B)_\beta \]
\[ = \left\langle \left( C^A \otimes 1 - 1 \otimes C^A \right) \left( C^B \otimes 1 - 1 \otimes C^B \right) \right\rangle_\beta \]
\[ = \sum_{A \subseteq \Lambda} \sum_{\varepsilon \in [-\mathbb{N}]} \alpha_A(1) \mathcal{L}_{\beta} \left( V_A^{(1)} V_A^{(-1)} V_B^{(1)} V_B^{(-1)} \right)_\beta \geq 0, \quad (3.41) \]

where \( V_A^{(1)} = \prod_{x \in A} V_x^{(1)} \). Hence we conclude Corollary 3.5

3.3.2 Proof of Theorem 3.6

Proposition 3.14 For all \( x \in \Lambda, \beta \geq 0 \) and \( \varepsilon \in \{ \pm 1 \} \), we have

\[ (n_x \otimes 1 + 1 \otimes n_x)(n_x \otimes 1 - 1 \otimes n_x) \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{X})_+. \quad (3.42) \]

Proof. Note that, since \( \partial \nu_x^{(v)} = -\nu_x \), we have

\[ n_x \otimes 1 + 1 \otimes n_x = 1 \otimes \nu_x = -\mathcal{R}(\nu_x), \quad (3.43) \]

while

\[ n_x \otimes 1 - 1 \otimes n_x = -\nu_x \otimes 1 = -\mathcal{L}(\nu_x). \quad (3.44) \]
Thus we have \((n_x \otimes 1 + 1 \otimes n_x)(n_x \otimes 1 - 1 \otimes n_x) = \mathcal{L}(\nu_x) R(\nu_x) \succeq 0\) w.r.t. \(\mathcal{L}^2(\mathcal{X})_+\).

By Proposition 3.14 we see that

\[
\prod_{j=1}^{n} [n_{x_j}(s_j) \otimes 1 + 1 \otimes n_{x_j}(s_j)] e^{-\beta H_{\text{ext}}}
\]

\[
\geq 0
\]

\[
\prod_{j=1}^{n} [n_{x_1} \otimes 1 + 1 \otimes n_{x_1}] e^{-(s_{2} - s_{1}) H_{\text{ext}}}
\]

\[
\geq 0
\]

\[
\prod_{j=1}^{n} [n_{x_n} \otimes 1 + 1 \otimes n_{x_n}] e^{-(\beta - s_{n}) H_{\text{ext}}}
\]

\[
\geq 0
\]

w.r.t. \(\mathcal{L}^2(\mathcal{X})_+\).

(3.45)

Therefore Theorem 3.6 follows from Proposition 2.11.

3.3.3 Proof of Theorem 3.7

By Propositions 3.13 and 3.14 we know

\[
\prod_{j=1}^{n} [n_{x_j}(s_j) \otimes 1 + 1 \otimes n_{x_j}(s_j)] e^{-\beta H_{\text{ext}}} \succeq 0 \text{ w.r.t. } \mathcal{L}^2(\mathcal{Y})_+.
\]

Thus Theorem 3.7 immediately follows from Proposition 2.11.

4 Spin-1/2 ferromagnetic Heisenberg model

4.1 Results

Let \(G = (\Lambda, E)\) be a graph with vertex set \(\Lambda\) and edge collection \(E\). An edge with end-points \(x\) and \(y\) is denoted by \(\{x, y\}\). In this paper, we assume that \(\{x, x\} \notin E\), i.e., any loops are excluded.

We consider the spin-1/2 system in \(G\) realized in the fermionic Fock space \(\mathcal{E} = \bigoplus_{n \geq 0} \Lambda^n \mathcal{Y}\) with \(\mathcal{Y} = \ell^2(\Lambda) \oplus \ell^2(\Lambda)\). Here \(\Lambda^n \mathcal{Y}\) is the \(n\)-fold anti-symmetric tensor product of \(\mathcal{Y}\). The spin operator \(S_x = \left( S_x^{(1)}, S_x^{(2)}, S_x^{(3)} \right)\) is defined by

\[
S_x^{(j)} = \frac{1}{2} \sum_{\sigma, \sigma' \in \{\uparrow, \downarrow\}} c_{x\sigma}^* s^{(j)}_{\sigma \sigma'} c_{x\sigma'}, \quad j = 1, 2, 3,
\]

(4.1)

where \(s^{(j)}\) is the Pauli matrices:

\[
s^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(4.2)

\(c_{x\sigma}\) is the fermionic annihilation operator which satisfies the standard anti-commutation relations:

\[
\{c_{x\sigma}, c_{x'\sigma'}^*\} = \delta_{xx'} \delta_{\sigma \sigma'}, \quad \{c_{x\sigma}, c_{x'\sigma'}\} = 0.
\]

(4.3)
The quantum Heisenberg model is given by
\[ H = - \sum_{\{x,y\} \in E} J_{xy} \mathbf{S}_x \cdot \mathbf{S}_y. \] (4.4)

We assume the following:

(A. 1) \(|\Lambda|\) is even.

(A. 2) \(G\) is bipartite.

(A. 3) \(J_{xy} = J_{yx} \geq 0\) for all \(\{x,y\} \in E\).

(A. 4) \(J_{xy}\) is positive semi-definite, that is,
\[ \sum_{\{x,y\} \in E} J_{xy} f_x^* f_y \geq 0 \quad \text{for all } f \in \ell^2(\Lambda). \] (4.5)

**Example 1** Here we give a typical example. Let \(\Lambda = [-L, L]^d \cap \mathbb{Z}^d\) with \(L \in \mathbb{N}\) and let \(E = \{\{x, y\} \in \Lambda^2 | |x - y| = 1\}. \) Then we obtain a graph \(G = (\Lambda, E). \) It is easy to see that \(G\) actually satisfies (A. 1) and (A. 2). Moreover if \(J_{xy} = J > 0\) for all \(\{x, y\} \in E\), then (A. 3) and (A. 4) are fulfilled. ♦

Let \(\gamma_\uparrow = (-\mathbb{1})^N\), where \(N = \sum_{x \in \Lambda} n_x\) with \(n_x = c_{x\uparrow}^* c_{x\uparrow}\). For each \(x \in \Lambda\), set
\[ b_x = c_{x\uparrow}^* \gamma_\uparrow c_{x\downarrow}. \] (4.6)

For each linear operator \(X\), we will often use the following notations:
\[ X^- = X, \quad X^+ = X^*. \] (4.7)

For each \(A \subseteq \Lambda\) and \(\#_A = \{\#_x\}_{x \in A} \in \{+, -\}^A\), we define
\[ b_{A^\#}^\# = \prod_{x \in A} b_x^{\#_x}. \] (4.8)

Let
\[ \mathfrak{A} = \text{Coni} \{ b_{A^\#}^\# \mid \#_A \in \{+, -\}^A, \ A \subseteq \Lambda \}. \] (4.9)

We use the thermal average associated with the grand canonical Gibbs state at inverse temperature \(\beta\):
\[ \langle X \rangle_\beta = \text{Tr}[X e^{-\beta H}] / \Xi_\beta, \quad \Xi_\beta = \text{Tr}[e^{-\beta H}]. \] (4.10)

For each \(\beta\), we can check that \(\langle n_x \rangle_\beta = 1\), where \(n_x = n_{x\uparrow} + n_{x\downarrow}\) with \(n_{x\sigma} = c_{x\sigma}^* c_{x\sigma}\). This means that the system at half-filling will be considered.

**Theorem 4.1 (First Griffiths inequality)** For all \(A \in \mathfrak{A}\) and \(\beta \geq 0\), we have \(\langle A \rangle_\beta \geq 0\).

\(^*\Lambda\) admits a partition into two disjoint classes such that every edge has its ends in different classes.
Theorem 4.2 For each \(x_1, \ldots, x_n \in \Lambda\), \(#_1, \ldots, #_n \in \{\pm\}\) and \(0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \beta\), we have

\[
\left\langle \prod_{j=1}^{n} b_{x_j}^{#_j}(s_j) \right\rangle_{\beta} \geq 0,
\]

(4.11)

where \( b_x^{#}(s) = e^{-sH} b_x^# e^{sH}. \)

Let

\[
S^{(+)}_x = S^{(1)}_x + iS^{(2)}_x, \quad S^{(-)}_x = S^{(1)}_x - iS^{(2)}_x.
\]

(4.12)

Since

\[
b_x^* = -S^{(-)}_x \gamma, \quad b_x = S^{(+)}_x \gamma,
\]

(4.13)

we have the following corollary.

Corollary 4.3 For each \(# \in \{+, -\}\), we set

\[
\text{sign}(#) = \begin{cases} +1 & \text{if } # = + \\ -1 & \text{if } # = - \end{cases}
\]

(4.14)

For each \(x_1, \ldots, x_n \in \Lambda\) and \(#_1, \ldots, #_n \in \{\pm\}\), we have

\[
\left\langle \prod_{j=1}^{n} \left[ \text{sign}(#_j) S^{(#_j)}_{x_j} \right] \right\rangle_{\beta} \geq 0.
\]

(4.15)

Example 2 One easily check that

\[
\left\langle S^{(+)}_{x_1} S^{(-)}_{x_2} \right\rangle_{\beta} \geq 0, \quad \left\langle S^{(+)}_{x_1} S^{(-)}_{x_2} S^{(+)}_{x_3} S^{(-)}_{x_4} \right\rangle_{\beta} \geq 0.
\]

(4.16)

To state the second quantum Griffiths inequality, we introduce the following notation:

\[
\langle Y \rangle_{\beta} = Z_{\beta}^{-2} \text{Tr}_{\epsilon \in \{ \epsilon_{\uparrow}, \epsilon_{\downarrow} \}} \left[ Y e^{-\beta H_{\text{ext}}} \right], \quad H_{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H.
\]

(4.17)

Theorem 4.4 (Second Griffiths inequality) Let \(\gamma = (-\mathbb{1})^{N_0}\) with \(N_0 = \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{x \in \Lambda} n_x \sigma.\) Choose \(x_1, \ldots, x_n \in \Lambda\) arbitrarily. For each \(0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \beta\), \(#_1, \ldots, #_n \in \{\pm\}\), \(\sigma_1, \ldots, \sigma_n \in \{\uparrow, \downarrow\}\) and \(\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}\), we have

\[
\left\langle \prod_{j=1}^{n} \left[ c_{x_j \sigma_j}(s_j) \otimes \mathbb{1} + \epsilon_j \gamma \otimes c_{x_j \sigma_j}(s_j) \right] \gamma_{\uparrow} \otimes \gamma_{\uparrow} \left[ \overline{c}_{x_j \sigma_j}(s_j) \otimes \mathbb{1} + \epsilon_j \gamma \otimes \overline{c}_{x_j \sigma_j}(s_j) \right] \right\rangle_{\beta} \geq 0,
\]

(4.18)

and

\[
\left\langle \prod_{j=1}^{n} \left[ c_{x_j \sigma_j}(s_j) \otimes \mathbb{1} + \epsilon_j \gamma \otimes c_{x_j \sigma_j}(s_j) \right] \gamma_{\uparrow} \otimes \gamma_{\uparrow} \left[ \overline{c}_{x_j \sigma_j}(s_j) \otimes \mathbb{1} - \epsilon_j \gamma \otimes \overline{c}_{x_j \sigma_j}(s_j) \right] \right\rangle_{\beta} \geq 0,
\]

(4.19)
where $c^\#_{2\sigma}(s) = e^{-sH}c^\#_{2\sigma}e^{sH}$ and

$$\sigma = \begin{cases} \uparrow & \text{if } \sigma = \downarrow, \\ \downarrow & \text{if } \sigma = \uparrow, \end{cases} \quad \# = \begin{cases} + & \text{if } # = -, \\ - & \text{if } # = +. \end{cases} \quad (4.20)$$

**Example 3** Consider the case where $n = 4$. Let

$$A_x = -S_x^{(-)}\gamma_\uparrow \otimes \gamma_\uparrow - \gamma_\uparrow \otimes S_x^{(-)}\gamma_\uparrow, \quad B_x = c_x\gamma_\uparrow \otimes \gamma_\uparrow c_x^* + \gamma_\uparrow c_x^* \otimes c_x\gamma_\uparrow, \quad C_x = -S_x^{(-)}\gamma_\uparrow \otimes \gamma_\uparrow + \gamma_\uparrow \otimes S_x^{(-)}\gamma_\uparrow, \quad D_x = -c_x\gamma_\uparrow \otimes \gamma_\uparrow c_x^* + \gamma_\uparrow c_x^* \otimes c_x\gamma_\uparrow, \quad (4.21)$$

Then one sees that

$$\begin{align*}
    &\left[ c_{x\uparrow}^* \otimes 1 + \varepsilon \gamma \otimes c_{x\uparrow}^* \right] \gamma_\uparrow \otimes \gamma_\uparrow \left[ c_{x\downarrow} \otimes 1 + \varepsilon \gamma \otimes c_{x\downarrow} \right] = A_\downarrow + \varepsilon B_\downarrow, \\
    &\left[ c_{x\uparrow} \otimes 1 + \varepsilon \gamma \otimes c_{x\uparrow} \right] \gamma_\uparrow \otimes \gamma_\uparrow \left[ c_{x\downarrow}^* \otimes 1 - \varepsilon \gamma \otimes c_{x\downarrow}^* \right] = C_\downarrow + \varepsilon D_\downarrow. \quad (4.23)
\end{align*}$$

Choose $x_1, x_2, x_3, x_4 \in \Lambda$ arbitrarily. By Theorem 4.4, we have

$$\left( A_1^* + \varepsilon_1 B_1^* \right) \left( C_2 + \varepsilon_2 D_2 \right) \left( A_3^* + \varepsilon_3 B_3^* \right) \left( C_4 + \varepsilon_4 D_4 \right) \geq 0 \quad (4.25)$$

for all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{\pm 1\}$. Here we use conventional abbreviations like $A_1 = A_{x_1}$, etc. Thus we have

$$\langle A_1^* C_2 A_3^* C_4 \rangle \geq 0. \quad (4.26)$$

By taking the facts $\langle S_0^{(+)} \rangle = 0, \langle S_0^{(+)} S_y^{(+)} \rangle = 0$, etc. into consideration, we arrive at

$$\langle S_1^{(+)} S_2^{(-)} S_3^{(+) S_4^{(-)} \rangle - \langle S_1^{(+)} S_2^{(-)} \rangle \langle S_3^{(+) S_4^{(-)} \rangle - \langle S_1^{(+)} S_4^{(-)} \rangle \langle S_2^{(-)} S_3^{(+) \rangle \geq 0. \quad \diamond \quad (4.27)$$

**4.2 Proof of Theorems 4.1 and 4.2**

For each $e = (x, y) \in E$ and $\sigma \in \{\uparrow, \downarrow\}$, set $a_{e\sigma} = c_{e\sigma}c_{y\sigma}^*$. One can express $H$ as

$$H = \sum_{e \in E} \frac{J_e}{2} (a_{e\uparrow}a_{e\downarrow}^* + a_{e\downarrow}^*a_{e\uparrow}) - \sum_{(x, y) \in E} \frac{J_{xy}}{4} (n_{x\uparrow} - n_{x\downarrow})(n_{y\uparrow} - n_{y\downarrow}), \quad (4.28)$$

where $J_e = J_{xy}$ for each $e = \{x, y\} \in E$. Since $G$ is bipartite, $\Lambda$ can be divided into two disjoint sets $\Lambda_e$ and $\Lambda_o$. Let

$$S^{(3)}_{e\text{odd}} = \sum_{x \in \Lambda_e} S^{(3)}_x. \quad (4.29)$$

Set $U = \exp \left( i\pi S^{(3)}_{e\text{odd}} \right)$. We then see that

$$UH U^{-1} = -\sum_{e \in E} \frac{J_e}{2} (a_{e\uparrow}a_{e\downarrow}^* + a_{e\downarrow}^*a_{e\uparrow}) - \sum_{(x, y) \in E} \frac{J_{xy}}{4} (n_{x\uparrow} - n_{x\downarrow})(n_{y\uparrow} - n_{y\downarrow}). \quad (4.30)$$
The hole-particle transformation is a unitary operator \( W \) such that
\[
W c_x \rightarrow W^{-1} c_x \rightarrow W^{-1}, \quad W c_x \rightarrow W^{-1} c_x, \quad W c_x \rightarrow W^{-1} c_x, \quad (4.31)
\]
where \( \mu(x) = 0 \) if \( x \in \Lambda_e \), \( \mu(x) = 1 \) if \( x \in \Lambda_o \). Note that \( W a_e \rightarrow W^{-1} a_e \) and \( W a_e \rightarrow W^{-1} a_e \). Set \( H = WUHU^{-1} + \sum_{e \in E} \frac{J_e}{2} \). We obtain
\[
\hat{H} = T - V, \quad (4.32)
\]
where
\[
T = \sum_{(x,y) \in E} \frac{J_{xy}}{2} (n_x \downarrow + n_x \uparrow) - \sum_{(x,y) \in E} \frac{J_{xy}}{4} n_x \downarrow n_y \uparrow, \quad (4.33)
\]
\[
V = \sum_{e \in \{ (x,y) \in E \}} \frac{J_e}{2} (a_e^* a_e^* + a_e^* a_e^* + n_x \downarrow n_y \uparrow), \quad (4.34)
\]
We switch to the representation \( \mathcal{E} = \mathfrak{F} \otimes \mathfrak{F} \), where \( \mathfrak{F} = \bigoplus_{n \geq 0} \wedge^n L^2(\Lambda) \). Then we have
\[
c_x \rightarrow c_x \otimes \mathbb{1}, \quad c_x \rightarrow (-\mathbb{1})^N \otimes c_x, \quad (4.35)
\]
where \( c_x \) is the annihilation operator on \( \mathfrak{F} \) and \( N = \sum_{x \in \Lambda} n_x \) with \( n_x = c_x^* c_x \). Then we see that
\[
T = K \otimes \mathbb{1} + \mathbb{1} \otimes K, \quad (4.36)
\]
\[
K = \sum_{(x,y) \in E} \frac{J_{xy}}{2} \left( n_x - \frac{1}{2} n_y \right), \quad (4.37)
\]
and
\[
V = \sum_{e \in \{ (x,y) \in E \}} \frac{J_e}{2} (a_e^* a_e^* + a_e^* a_e^* + n_x \otimes n_y), \quad (4.38)
\]
\[
a_e = c_x c_y. \quad (4.39)
\]
Let \( \vartheta \) be the anti-linear involution such that
\[
\vartheta c_x \vartheta = c_x, \quad \vartheta \Omega = \Omega. \quad (4.40)
\]
Using \( \vartheta \) and (2.15), we have the identification
\[
\mathcal{E} = L^2(\mathfrak{F}). \quad (4.41)
\]
By (2.15), we obtain the following proposition.

**Proposition 4.5** We have \( \hat{H} = T - V, \) where
\[
T = \mathcal{L}(K) + \mathcal{R}(K), \quad (4.42)
\]
\[
V = \sum_{e \in \{ (x,y) \in E \}} \frac{J_e}{2} \left\{ \mathcal{L}(a_e^*) \mathcal{R}(a_e^*) + \mathcal{L}(a_e) \mathcal{R}(a_e^*) + \mathcal{L}(n_x) \mathcal{R}(n_y) \right\}. \quad (4.43)
\]
\[\text{We here use the following identification: } \bigoplus_{n \geq 0} \wedge^n (h_1 \oplus h_2) = \left( \bigoplus_{n \geq 0} \wedge^n h_1 \right) \otimes \left( \bigoplus_{n \geq 0} \wedge^n h_2 \right).\]
Corollary 4.6 We have $\exp(-\beta \hat{H}) \succeq 0$ w.r.t. $\mathcal{L}^2(\mathfrak{F})_+$ for all $\beta \geq 0$.

Proof. We note that $V \succeq 0$ w.r.t. $\mathcal{L}^2(\mathfrak{F})_+$ by Lemma A.1. On the other hand, we have $e^{-\beta T} = \mathcal{L}(e^{-\beta K}) \mathcal{R}(e^{-\beta K}) \succeq 0$ w.r.t. $\mathcal{L}^2(\mathfrak{F})_+$. Therefore by Proposition 2.13 we conclude the result. □

The following proposition immediately follows from definitions.

Proposition 4.7 Set $S = WU$. We have the following.

(i) $S c_{x\uparrow} S^{-1} = (-1)^{n(x)} \mathcal{L}(c_x)$.

(ii) $S \gamma c_{x\uparrow} S^{-1} = i^{\mu(x)} \mathcal{R}(c_x^*)$.

Corollary 4.8 We have $S b_x S^{-1} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathfrak{F})_+$.

Proof. $S b_x S^{-1} = \mathcal{L}(c_x) \mathcal{R}(c_x^*) \succeq 0$ w.r.t. $\mathcal{L}^2(\mathfrak{F})_+$. □

4.2.1 Completion of proof of Theorems 4.1 and 4.2

By Corollary 4.8, we know that $SAS^{-1} \succeq 0$ for all $A \in \mathfrak{A}$. Thus by Proposition 2.11, we obtain Theorem 4.1.

4.3 Proof of Theorem 4.4

4.3.1 Proof of (4.18)

We introduce the extended Hilbert space $\mathfrak{E}_{\text{ext}}$ by

$$\mathfrak{E}_{\text{ext}} = \mathfrak{E} \otimes \mathfrak{E}. \quad (4.45)$$

Let

$$\phi_{x\sigma} = \frac{1}{\sqrt{2}} (c_{x\sigma} \otimes 1 + \gamma \otimes c_{x\sigma}), \quad \psi_{x\sigma} = \frac{1}{\sqrt{2}} (c_{x\sigma} \otimes 1 - \gamma \otimes c_{x\sigma}). \quad (4.46)$$

$\phi_{x\sigma}$ and $\psi_{x\sigma}$ act in $\mathfrak{E}_{\text{ext}}$. These operators satisfy the following anti-commutation relations(CARs):

$$\{\phi_{x\sigma}, \phi_{y\sigma'}^*\} = \delta_{xy} \delta_{\sigma\sigma'}, \quad \{\phi_{x\sigma}, \phi_{y\sigma'}\} = 0, \quad (4.47)$$
$$\{\psi_{x\sigma}, \psi_{y\sigma'}^*\} = \delta_{xy} \delta_{\sigma\sigma'}, \quad \{\psi_{x\sigma}, \psi_{y\sigma'}\} = 0, \quad (4.48)$$
$$\{\phi_{x\sigma}, \psi_{y\sigma'}^*\} = 0, \quad \{\phi_{x\sigma}, \psi_{y\sigma'}\} = 0. \quad (4.49)$$
Let

$$H_{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H. \quad (4.50)$$

$H_{\text{ext}}$ acts in $E_{\text{ext}}$ as well. Set $S_{\text{ext}} = WU \otimes WU$. We define a self-adjoint operator $\hat{H}_{\text{ext}}$ by

$$\hat{H}_{\text{ext}} = S_{\text{ext}} H_{\text{ext}} S_{\text{ext}}^{-1} + \sum_{e \in E} \frac{J_e}{2}. \quad (4.51)$$

It is easy to check that $\hat{H}_{\text{ext}} = \hat{H} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}$.

By using the following notations:

$$A_{e\sigma} = \phi_{x\sigma}^* \phi_{y\sigma}^* + \psi_{x\sigma}^* \psi_{y\sigma}, \quad B_{e\sigma} = \phi_{x\sigma}^* \psi_{y\sigma}^* + \psi_{x\sigma}^* \phi_{y\sigma}, \quad (4.52)$$

$$C_{e\sigma} = \phi_{x\sigma}^* \phi_{x\sigma} + \psi_{x\sigma}^* \psi_{x\sigma}, \quad D_{e\sigma} = \phi_{x\sigma}^* \psi_{x\sigma} + \psi_{x\sigma}^* \phi_{x\sigma}, \quad (4.53)$$

we can express $\hat{H}_{\text{ext}}$ as

$$\hat{H}_{\text{ext}} = T - V, \quad (4.54)$$

where

$$T = \frac{1}{2} \sum_{\{x,y\} \in E} \sum_{\sigma \in \{\uparrow, \downarrow\}} J_{xy} C_{x\sigma} - \frac{1}{8} \sum_{\{x,y\} \in E} \sum_{\sigma \in \{\uparrow, \downarrow\}} J_{xy} (C_{x\sigma} C_{y\sigma} + D_{x\sigma} D_{y\sigma}), \quad (4.55)$$

$$V = \frac{1}{2} \sum_{e \in E} J_e (A_{e\uparrow} A_{e\downarrow} + B_{e\uparrow} B_{e\downarrow}) + \frac{1}{4} \sum_{\{x,y\} \in E} J_{xy} (C_{x\uparrow} C_{y\downarrow} + D_{x\uparrow} D_{y\downarrow}). \quad (4.56)$$

Let $\{\phi_x, \psi_x \mid x \in \Lambda\}$ be new annihilation operators on $\mathfrak{X} = \mathfrak{F} \otimes \mathfrak{F}$ such that

$$\{\phi_x, \phi_y^*\} = \delta_{xy}, \quad \{\phi_x, \phi_y\} = 0, \quad (4.57)$$

$$\{\psi_x, \psi_y^*\} = \delta_{xy}, \quad \{\psi_x, \psi_y\} = 0, \quad (4.58)$$

$$\{\phi_x, \psi_y\} = 0, \quad \{\phi_x, \psi_y^*\} = 0 \quad (4.59)$$

and $\phi_x \Omega_{\mathfrak{X}} = 0 = \psi_x \Omega_{\mathfrak{X}}$, where $\Omega_{\mathfrak{X}}$ is the Fock vacuum in $\mathfrak{X}$. Under the identification $E_{\text{ext}} = \mathfrak{X} \otimes \mathfrak{X}$, we have the following relations:

$$\phi_{x\uparrow} = \phi_x \otimes \mathbb{1}, \quad \phi_{x\downarrow} = (-\mathbb{1})^N \otimes \phi_x, \quad \psi_{x\uparrow} = \psi_x \otimes \mathbb{1}, \quad \psi_{x\downarrow} = (-\mathbb{1})^N \otimes \psi_x, \quad (4.60)$$

where $N$ is the number operator defined by $N = \sum_{x \in \Lambda} (\phi_x^* \phi_x + \psi_x^* \psi_x)$.

Using (4.60), we have

$$T = \frac{1}{2} \sum_{\{x,y\} \in E} J_{xy} (C_x \otimes \mathbb{1} + \mathbb{1} \otimes C_x)$$

$$- \frac{1}{8} \sum_{\{x,y\} \in E} J_{xy} \left\{ (C_x C_y + D_x D_y) \otimes \mathbb{1} + \mathbb{1} \otimes (C_x C_y + D_x D_y) \right\}, \quad (4.61)$$

$$V = \frac{1}{2} \sum_{e \in E} J_e (A_e \otimes A_e + B_e \otimes B_e) + \frac{1}{4} \sum_{\{x,y\} \in E} J_{xy} (C_x \otimes C_y + D_x \otimes D_y), \quad (4.62)$$
where, for each \( e \in \{x, y\} \in E \),
\[
A_e = \phi_x^* \phi_y + \psi_x^* \psi_y, \quad B_e = \phi_y^* \psi_y + \psi_x^* \phi_y, \quad (4.63)
\]
\[
C_x = \phi_x^* \phi_x + \psi_x^* \psi_x, \quad D_x = \phi_y^* \psi_x + \psi_y^* \phi_x. \quad (4.64)
\]

\( A_e, B_e, C_x, D_x \) act in \( \mathfrak{X} \), while \( T \) and \( V \) act in \( \mathfrak{X} \otimes \mathfrak{X} \).

Let \( \Theta \) be the anti-linear involution on \( \mathfrak{X} \) defined by
\[
\Theta \phi_x = \phi_x, \quad \Theta \psi_x = \psi_x, \quad \Theta \Omega_X = \Omega_X. \quad (4.65)
\]

By \( \Theta \) and (2.15), we have the identification
\[
\mathfrak{X} \otimes \mathfrak{X} = L^2(\mathfrak{X}). \quad (4.66)
\]

In addition, we see that
\[
A_e \otimes 1_l = L(A_e), \quad 1_l \otimes A_e = R(A_e^*), \quad B_e \otimes 1_l = L(B_e), \quad 1_l \otimes B_e = R(B_e^*), \quad (4.67)
\]
\[
C_x \otimes 1_l = L(C_x), \quad 1_l \otimes C_x = R(C_x^*), \quad D_x \otimes 1_l = L(D_x), \quad 1_l \otimes D_x = R(D_x^*). \quad (4.68)
\]

by (2.15). Thus we arrive at the following.

**Proposition 4.9** We have
\[
\hat{H}_{\text{ext}} = T - V, \quad (4.69)
\]
where
\[
T = L(K) + R(K), \quad (4.70)
\]
\[
K = \frac{1}{2} \sum_{\{x, y\} \in E} J_{xy} C_x - \frac{1}{8} \sum_{\{x, y\} \in E} J_{xy} (C_x C_y + D_x D_y), \quad (4.71)
\]
and
\[
V = \frac{1}{2} \sum_{e = (x, y) \in E} J_e \left\{ L(A_e) R(A_e^*) + L(B_e) R(B_e^*) + \frac{1}{2} L(C_x) R(C_x^*) + \frac{1}{2} L(D_x) R(D_x^*) \right\}. \quad (4.72)
\]

**Corollary 4.10** For all \( \beta \geq 0 \), we have that \( \exp(-\beta \hat{H}_{\text{ext}}) \succeq 0 \) w.r.t. \( L^2(\mathfrak{X})_+ \).

**Proof.** We note that \( V \succeq 0 \) w.r.t. \( L^2(\mathfrak{X})_+ \) by Lemma [A.1] On the other hand, we have \( e^{-\beta T} = L(e^{-\beta K}) R(e^{-\beta K}) \succeq 0 \) w.r.t. \( L^2(\mathfrak{X})_+ \). Therefore by Proposition 2.13 we obtain the result. \( \Box \)

We easily check the following proposition. (Here we note that \( S_{\text{ext}} \gamma \otimes 1_l S_{\text{ext}}^{-1} = \gamma \otimes 1_l \) by (A. 1).)

**Proposition 4.11** One has the following.
(i) \( S_{\text{ext}} (c_x^\uparrow \otimes 1_l + \gamma \otimes c_x^\uparrow) S_{\text{ext}}^{-1} = \sqrt{2} i^{\mu(x)} L(\phi_x^*) \).
(ii) $S_{\text{ext}} (c_{x\uparrow} \otimes \mathbb{1} - \gamma \otimes c_{x\uparrow}) S_{\text{ext}}^{-1} = \sqrt{2} \mu(x) \mathcal{L}(\psi_x^\ast)$. \\
(iii) $S_{\text{ext}} \gamma \uparrow \otimes \gamma (c_{x\downarrow} \otimes \mathbb{1} + \gamma \otimes c_{x\downarrow}) S_{\text{ext}}^{-1} = \sqrt{2} \mu(x) \mathcal{R}(\phi_x^\ast)$. \\
(iv) $S_{\text{ext}} \gamma \uparrow \otimes \gamma (c_{x\downarrow} \otimes \mathbb{1} - \gamma \otimes c_{x\downarrow}) S_{\text{ext}}^{-1} = \sqrt{2} \mu(x) \mathcal{R}(\psi_x^\ast)$.

**Corollary 4.12** For all $\varepsilon \in \{\pm 1\}, \sigma \in \{\uparrow, \downarrow\}$ and $\# \in \{\pm\}$, one obtains

$$S_{\text{ext}} (c_{x\sigma}^\# \otimes \mathbb{1} + \varepsilon \gamma \otimes c_{x\sigma}^\#) \gamma \uparrow \otimes \gamma \left[ (c_{x\sigma}^\# \otimes \mathbb{1} + \varepsilon \gamma \otimes c_{x\sigma}^\#) S_{\text{ext}}^{-1} \right] \geq 0$$

w.r.t. $\mathcal{L}^2(\mathfrak{X})_+$. \\

**4.3.2 Completion of proof of (4.18)**

Let

$$Q = \prod_{j=1}^{n} \left[ c_{x_j \sigma_j}^{\#} (s_j) \otimes \mathbb{1} + \varepsilon_j \gamma \otimes c_{x_j \sigma_j}^{\#} (s_j) \right] \gamma \uparrow \otimes \gamma \left[ (c_{x_j \sigma_j}^{\#} (s_j) \otimes \mathbb{1} + \varepsilon_j \gamma \otimes c_{x_j \sigma_j}^{\#} (s_j)) S_{\text{ext}}^{-1} \right] \geq 0$$

By Corollary 4.12, we see that

$$S_{\text{ext}} Q e^{-\beta H_{\text{ext}}} S_{\text{ext}}^{-1} \geq 0$$

w.r.t. $\mathcal{L}^2(\mathfrak{X})_+$. Thus by Proposition 2.11, we obtain the desired assertion. \(\square\)

**4.3.3 Proof of (4.19)**

We provide a sketch only, since the proof of (4.19) is almost the same as that of (4.18). The point of the proof is that we choose a representation of the CARs which is different from (4.40).

We introduce a new representation of the CARs by

$$\xi_{x\uparrow} = \frac{1}{\sqrt{2}} (c_{x\uparrow} \otimes \mathbb{1} + \gamma \otimes c_{x\uparrow}), \quad \xi_{x\downarrow} = \frac{1}{\sqrt{2}} (c_{x\downarrow} \otimes \mathbb{1} - \gamma \otimes c_{x\downarrow}),$$

$$\eta_{x\uparrow} = \frac{1}{\sqrt{2}} (c_{x\uparrow} \otimes \mathbb{1} - \gamma \otimes c_{x\uparrow}), \quad \eta_{x\downarrow} = \frac{1}{\sqrt{2}} (c_{x\downarrow} \otimes \mathbb{1} + \gamma \otimes c_{x\downarrow}).$$

$\xi_{x\sigma}$ and $\eta_{x\sigma}$ satisfy the CARs in $\mathfrak{E}_{\text{ext}}$. Corresponding to (4.52) and (4.53), we set

$$A_{x\sigma} = \xi_{x\sigma}^\ast \xi_{\sigma \sigma} + \eta_{x\sigma}^\ast \eta_{\sigma \sigma}, \quad B_{x\sigma} = \xi_{x\sigma} \eta_{x\sigma} + \eta_{x\sigma}^\ast \xi_{\sigma \sigma},$$

$$C_{x\sigma} = \xi_{x\sigma}^\ast \eta_{\sigma \sigma} + \eta_{x\sigma} \xi_{\sigma \sigma}, \quad D_{x\sigma} = \xi_{x\sigma} \eta_{\sigma \sigma} + \eta_{x\sigma}^\ast \xi_{\sigma \sigma}.$$
Then $\hat{H}_\text{ext}$ can be expressed as

$$\hat{H}_\text{ext} = \mathcal{J} - \mathcal{V},$$

(4.80)

where

$$\mathcal{J} = \frac{1}{2} \sum_{\{x,y\} \in E} \sum_{\sigma \in \{\uparrow, \downarrow\}} J_{xy} \xi_x \xi_y - \frac{1}{8} \sum_{\{x,y\} \in E} \sum_{\sigma \in \{\uparrow, \downarrow\}} J_{xy} (\xi_x \xi_y + D_x D_y),$$

(4.81)

$$\mathcal{V} = \frac{1}{2} \sum_{e \in E} J_e (A^e_c A^e_c + B^e_c B^e_c) + \frac{1}{4} \sum_{\{x,y\} \in E} J_{xy} (\xi_x \xi_y + D_x D_y).$$

(4.82)

Now let $\{\xi_x, \eta_x | x \in \Lambda\}$ be new annihilation operators on $\mathcal{X} = \mathbb{F} \otimes \mathbb{F}$ such that

$$\{\xi_x, \xi^*_y\} = \delta_{xy}, \quad \{\eta_x, \eta^*_y\} = 0,$$

(4.83)

$$\{\eta_x, \eta^*_y\} = \delta_{xy}, \quad \{\xi_x, \eta_y\} = 0,$$

(4.84)

$$\{\xi_x, \eta^*_y\} = 0, \quad \{\xi_x, \eta_y\} = 0$$

(4.85)

and $\xi_x \Omega_{\mathcal{X}} = 0 = \eta_x \Omega_{\mathcal{X}}$, where $\Omega_{\mathcal{X}}$ is the Fock vacuum in $\mathcal{X}$. Under the identification $\mathcal{E}_\text{ext} = \mathcal{X} \otimes \mathcal{X}$, we have the following relations:

$$\xi_{x\uparrow} = \xi_x \otimes 1, \quad \xi_{x \downarrow} = (-\mathbb{I})^N \otimes \xi_x, \quad \eta_{x \uparrow} = \eta_x \otimes 1, \quad \eta_{x \downarrow} = (-\mathbb{I})^N \otimes \eta_x,$$

(4.86)

where $N = \sum_{x \in \Lambda} (\xi^*_x \xi_x + \eta^*_x \eta_x)$. Let $\theta$ be the anti-linear involution on $\mathcal{X}$ defined by

$$\theta \xi_x \theta = \xi_x, \quad \theta \eta_x \theta = \eta_x, \quad \theta \Omega_{\mathcal{X}} = \Omega_{\mathcal{X}},$$

(4.87)

By $\theta$ and (4.15), we have the identification $\mathcal{X} \otimes \mathcal{X} = \mathcal{L}^2(\mathcal{X})$. Moreover we obtain the following.

**Proposition 4.13** Let

$$A_e = \xi^*_x \eta_y + \eta^*_x \xi_y, \quad B_e = \xi^*_x \xi_y + \eta^*_x \eta_y,$$

$$C_x = \xi^*_x \xi_x + \eta^*_x \eta_x, \quad D_x = \xi^*_x \eta_x + \eta^*_x \xi_x.$$

(4.88)

We have $\hat{H}_\text{ext} = \mathcal{J} - \mathcal{V}$, where

$$\mathcal{J} = \mathcal{L}(\mathcal{H}) + \mathcal{R}(\mathcal{H}),$$

(4.90)

$$\mathcal{X} = \frac{1}{2} \sum_{\{x,y\} \in E} J_{xy} C_x - \frac{1}{8} \sum_{\{x,y\} \in E} J_{xy} (C_x C_y + D_x D_y)$$

(4.91)

and

$$\mathcal{V} = \frac{1}{2} \sum_{e \in \{x,y\} \in E} J_e \left\{ \mathcal{L}(A_e) \mathcal{R}(A^*_e) + \mathcal{L}(B_e) \mathcal{R}(B^*_e) + \frac{1}{2} \mathcal{L}(C_x) \mathcal{R}(C^*_x) + \frac{1}{2} \mathcal{L}(D_x) \mathcal{R}(D^*_x) \right\}.$$

(4.92)

**Corollary 4.14** For all $\beta \geq 0$, we have that $\exp(-\beta \hat{H}_\text{ext}) = \exp\{-\beta (\mathcal{J} - \mathcal{V})\} \succeq 0$ w.r.t. $\mathcal{L}^2(\mathcal{X})_+$. 

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Corresponding to Proposition 4.11, we have

**Proposition 4.15** One has the following.

(i) \( S_{\text{ext}} (c_{\uparrow} \otimes 1 + \gamma \otimes c_{\downarrow}) S_{\text{ext}}^{-1} = \sqrt{2} i \mu(x) L(\xi^* x) \).

(ii) \( S_{\text{ext}} (c_{\downarrow} \otimes 1 - \gamma \otimes c_{\uparrow}) S_{\text{ext}}^{-1} = \sqrt{2} i \mu(x) L(\eta^* x) \).

(iii) \( S_{\text{ext}} \gamma \otimes \gamma (c_{\downarrow} \otimes 1 + \gamma \otimes c_{\downarrow}) S_{\text{ext}}^{-1} = \sqrt{2} i \mu(x) R(\eta^* x) \).

(iv) \( S_{\text{ext}} \gamma \otimes \gamma (c_{\uparrow} \otimes 1 - \gamma \otimes c_{\uparrow}) S_{\text{ext}}^{-1} = \sqrt{2} i \mu(x) R(\xi^* x) \).

**Corollary 4.16** For all \( \varepsilon \in \{\pm 1\} \), \( \sigma \in \{\uparrow, \downarrow\} \) and \# \in \{\pm\}, one obtains

\[
S_{\text{ext}} \left( c_{\# x^*} \otimes 1 + \varepsilon \gamma \otimes c_{\# x} \right) \gamma \otimes \gamma \left( c_{\# x^*} \otimes 1 - \varepsilon \gamma \otimes c_{\# x} \right) S_{\text{ext}}^{-1} \succeq 0 \quad (4.93)
\]

w.r.t. \( L^2(\mathcal{X})_+ \).

In a similar way to \( \S 4.3.2 \) we obtain (4.19). \( \square \)

5 Concluding remark

Let \( \mathcal{P} \) be a self-dual cone in the Hilbert space \( \mathcal{H} \). Let \( H_0 \) and \( V \) be self-adjoint operators in \( \mathcal{H} \). \( H_0 \) is the free Hamiltonian and \( V \) is the interaction. The system’s Hamiltonian is given by \( H = H_0 - V \). Through our studies of the quantum Griffiths’ inequality, we realize that the following are model-independent properties:

(i) \( e^{-\beta H_0} \succeq 0 \) w.r.t. \( \mathcal{P} \) for all \( \beta \geq 0 \).

(ii) \( V \succeq 0 \) w.r.t. \( \mathcal{P} \).

Furthermore we can find the same structures in several areas, for example, the multi-electron systems \([22, 25, 27]\) and the open quantum systems \([23]\). More precisely, in a theory of the ferrimagnetism in the Hubbard model, the properties (i) and (ii) are fundamental. In a theory of open quantum systems, the complete positivity is closely related with (i) and (ii). These facts indicate that (i) and (ii) are universal representation of the notion of the correlations. To reinforce this hypothesis, we have to continue to study our approach of the operator inequalities.

A An auxiliary lemma

**Lemma A.1** Let \( A_j \), \( j = 1, \ldots, N \) be a bounded operator acting in \( \mathcal{H} \). Let \( M = (M_{ij}) \) be a positive semi-definite \( N \times N \) matrix. Then we have

\[
\sum_{i,j=1}^{N} M_{ij} \mathcal{L}(A_i^*) \mathcal{R}(A_j) \succeq 0 \quad \text{w.r.t. } \mathcal{L}^2(\mathcal{H})_+ . \quad (A.1)
\]

\(^6\) Even when we show the second Griffiths inequality, the properties (i) and (ii) are essential for our proof. In this case, (i) and (ii) still hold true for the extended Hamiltonian acting in the doubled Hilbert space \( \mathcal{H} \otimes \mathcal{H} \), see Sections 4 and 5.

\(^7\) (ii) is equivalent to \( -V \preceq 0 \) w.r.t. \( \mathcal{P} \). In this sense, we say that \( -V \) is attractive w.r.t. \( \mathcal{P} \).
Proof. There exists a unitary matrix $U$ such that $M = U^*DU$, where $D = \text{diag}(\lambda_j)$ is a diagonal matrix with $\lambda_j \geq 0$. Set $\tilde{A}_i = \sum_{j=1}^{N} U_{ij} A_j$. Then one sees

$$\text{L. H. S. of (A.1)} = \sum_{j=1}^{N} \lambda_j L(\tilde{A}_j^*)R(\tilde{A}_j) \geq 0 \text{ w.r.t. } L^2(\mathcal{H})^+.$$  \hspace{1cm} (A.2)

This completes the proof. \hspace{1cm} $\blacksquare$

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