ON NUMBER OF PARTICLES IN
COALESCING-FRAGMENTATING WASSERSTEIN DYNAMICS

VITALII KONAROVSKYI

Abstract. Because of the sticky-reflected interaction in coalescing-fragmentating Wasserstein dynamics, the model always consists of a finite number of distinct particles for almost all times. We show that the interacting particle system must admit an infinite number of distinct particles on a dense subset of the time interval if and only if the space generated by the interaction potential is infinite-dimensional.

1. Introduction

The coalescing-fragmentating Wasserstein Dynamics is a system of interacting particles on the real line which intuitively can be described as follows. Diffusion particles start at some finite or infinite family of points move independently until their meeting. Every particle transfer a mass and its diffusion rate is inversely proportional to its mass. When particles meet they sticky-reflect from each other according to some interaction potential which is described by a non-decreasing right-continuous bounded function \( \xi : [0,1] \rightarrow \mathbb{R} \), which we call an interaction potential. The evolution of the particle system is similar to the motion of particles in the Howitt-Warren flow \([2]\). The main difference is that the motion of particles in CFWD inversely-proportionally depends on their masses. In particular, particles with “infinitesimally small” mass have “infinitely large” diffusion rate. Let \( X(u,t) \) be the position of a particle labeled by \( u \in (0,1) \) at moment \( t \geq 0 \). The evolution of such a family of particles can be defined by the following SDE

\[
dX_t = \text{pr}_{X_t} dW_t + (\xi - \text{pr}_{X_t} \xi) dt, \quad t \geq 0,
X_0 = g,
\]

in the space \( L^2_\sigma \) of all \( \sigma \)-integrable functions (classes of equivalences) \( f : (0,1) \rightarrow \mathbb{R} \) which have a non-decreasing version, where \( W_t, t \geq 0 \), is a cylindrical Wiener process in \( L_2 = L_2([0,1], du) \) and \( \text{pr}_{\sigma} \) denotes the orthogonal projection in \( L_2 \) onto its subspace \( L_2(\sigma) \) of all \( \sigma \)-measurable functions. The initial condition \( g \in L^2_\sigma \) describes the mass distribution \( \mu_0 \) of particles, that is defined as the image of the Lebesgue measure \( \text{Leb} \) on \([0,1]\) under the map \( g \).

In \([4]\), the author shows the existence of a weak solution to equation (1) for any \((2+)\)-integrable initial condition \( g \), for some \( \varepsilon > 0 \). More precisely, if \( \int_0^1 g^2 + \varepsilon(u) du < \infty \) for some \( \varepsilon > 0 \), then there exists an \( L_2 \)-valued cylindrical Wiener process \( W_t, t \geq 0 \), and a continuous \( L^2_\sigma \)-valued process \( X_t, t \geq 0 \), both defined on the same filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) such that \( \mathbb{E}\|X_t\|_2^2 < \infty, t \geq 0 \), and

\[
X_t = g + \int_0^t \text{pr}_{X_s} dW_s + \int_0^t (\xi - \text{pr}_{X_s} \xi) ds, \quad t \geq 0.
\]

Let \( D([0,1], C([0,\infty))) \) denote the Skorohod space of all càdlàg functions from \([0,1]\) to the space \( C([0,\infty]) \) of real-valued continuous functions defined on \([0,\infty)\). If the initial

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condition $g$ and the interaction potential $\xi$ are right-continuous and piecewise $(\frac{1}{2}+)$-Hölder continuous\footnote{There exist $\varepsilon > 0$ and a finite partition of the interval $[0,1]$ such that the functions are $(\frac{1}{2}+\varepsilon)$-Hölder continuous on each interval of the partition}, then equation (1) admits a weak solution which have a modification $\{X(u,t), t \geq 0, u \in [0,1]\}$ from $\mathcal{D}([0,1],C[0,\infty))$, where $X(u,\cdot), u \in [0,1]$, describes the evolution of individual particles in CFWD. Here the process $X_t := X(\cdot,t), t \geq 0$, is a weak solution to equation (1) in the space $L^+_2$ and the following properties holds:

(R1) for all $u \in [0,1], X(u,0) = g(u)$;
(R2) for each $u < v$ from $[0,1]$ and $t \geq 0, X(u,t) \leq X(u,t)$;
(R3) the process

$$M(u,t) := X(u,t) - g(u) - \int_0^t \left( \xi(u) - \frac{1}{m(u,s)} \int_{\pi(u,s)} \xi(v)dv \right) ds, \quad t \geq 0,$$

is a continuous square integrable martingale with respect to the filtration $\mathcal{F}_t = \sigma(X(v,s), u \in [0,1], s \leq t), t \geq 0$, where $\pi(u,t) = \{v: x(u,t) = X(v,t)\}$ and $m(u,t) = \text{Leb} \pi(u,t)$;
(R4) the joint quadratic variation of $M(u, \cdot)$ and $M(v, \cdot)$ equals

$$\langle M(u, \cdot), M(v, \cdot) \rangle_t = \int_0^t \frac{I\{X(u,s) = X(v,s)\}}{m(u,s)} ds, \quad t \geq 0.$$

The uniqueness of a solution to equation (1) remains an important open problem. The existence of reversible CFWD and their connection to Wasserstein diffusion\footnote{Further investigation} and the geometry of the Wasserstein space of probability measures on the real line was studied in\cite{8}.

**Remark 1.1.** Considering $f \in L^+_2$ as a function, we will always take its right continuous version on $(0,1)$ which exists and is unique according to Proposition A.1\cite{5} and Remark A.6 ibid.

We will denote by $\sharp f$ a number of distinct values of $f \in L^+_2$. By Lemma 6.1\cite{5}, the square of the Hilbert-Schmidt norm of the orthogonal projection $\text{pr}_f$ coincides with $\sharp f$, i.e.

$$\| \text{pr}_f \|_{HS}^2 := \sum_{n=1}^{\infty} \| \text{pr}_f e_n \|_{L^2}^2 = \sharp f,$$

where $\{e_n, n \geq 1\}$ is an orthonormal basis in $L^2$. Therefore, we can interpret the random variable $\| \text{pr}_{X_t} \|_{HS}^2 = \sharp X_t$ as a number of distinct particles in CFWD at time $t \geq 0$. In particular, if $X_t = X(\cdot,t), t \geq 0$, where the random element $\{X(u,t), t \geq 0, u \in [0,1]\}$ in $\mathcal{D}([0,1],C[0,\infty))$ satisfies conditions (R1)-(R4), then $\sharp X(\cdot,t)$ is exactly the number of distinct particles at time $t \geq 0$ in the CFWD. Since $X_t, t \geq 0$, is square integrable and $\xi$ is bounded, Theorem 2.4\cite{1} and equality (2) imply

$$\int_0^t \text{E}(\sharp X_s) ds < \infty$$

for all $t \geq 0$. This yields that

$$\mathbb{P} \{ \sharp X_t < \infty \text{ for a.e. } t \in [0,\infty) \} = 1,$$

i.e. the CFWD consists of a finite number of particles at almost all times. The goal of this paper is to show that with probability $1$ there exists a (random) dense subset of the time interval $[0,\infty)$, where the CFWD has infinite number of particles if and only if $\sharp \xi = \infty$. We remark that the property $\sharp \xi = \infty$ is equivalent to the fact that $L_2(\xi)$ is infinite dimensional, according to (2).

**Theorem 1.1.**

(i) If $\sharp \xi = +\infty$, then almost surely there exists a (random) dense subset $R$ of $[0,\infty)$ such that $\sharp X_t = \infty, t \in R.$
(ii) If \( \xi < \infty \), then
\[
\mathbb{P}\{\xi X_t < \infty, \ t \in [0, \infty)\} = 1.
\]

We remark that the CFWD coincides with the modified massive Arratia flow \([5, 6, 7, 9, 10]\) for \( \xi = 0 \). In this case, the claim of the theorem was proved in \([5, \text{Proposition} 6.2]\).

2. Auxiliary statements

Let \( C([a, b], L^1_2) \) denote the space of continuous functions from \([a, b]\) to \( L^1_2 \) endowed with the distance of uniform convergence. We recall that the map \( L \) the proof of Theorem
\[
\text{Lemma 2.1}.
\]

For each \( t \geq 0 \), the following lemma is needed for the measurability of events which will appear in the proof of Theorem 1.1.

**Lemma 2.1.** For each \([a, b]\), the map \( f \mapsto \sup_{t \in [a, b]} \| f \|_{HS} \) from \( C([a, b], L^1_2) \) to \( \mathbb{R} \cup \{+\infty\} \) is measurable.

**Proof.** We first note that the map \( f \mapsto \| f \|_{HS} \) from \( C([a, b], L^1_2) \) to \( \mathbb{R} \) is lower semi-continuous for each \( t \geq 0 \) as the composition of the continuous map \( C([a, b], L^1_2) \ni g \mapsto g_t \in L^1_2 \) and the lower semi-continuous map \( L^1_2 \ni h \mapsto \| h \|_{HS} \in \mathbb{R} \). This yields the claim of the lemmas due to the measurability of \( f \mapsto \| f \|_{HS} \) and the equality
\[
\{ f : \sup_{t \in [a, b]} \| f_t \|_{HS} \leq c \} = \bigcap_{t \in [a, b]} \{ f : \| f_{t} \|_{HS} \leq c \},
\]
for all \( c \geq 0 \).

The following lemma directly follows from the lower semi-continuity of the map \( t \mapsto \| f_t \|_{HS} \) for each \( f \in C([0, \infty), L^1_2) \).

**Lemma 2.2.** For every \( f \in C([0, \infty), L^1_2) \), \( c \geq 0 \) and \( 0 \leq a < b \) the set \( A^c_{f,a,b} := \{ t \in [a, b] : \| f_t \|_{HS} \leq c \} \) is closed in \([0, \infty)\).

We will also need a property of a function \( f \in C([0, \infty), L^1_2) \) if the Hilbert-Schmidt norm \( \| f_t \|_{HS} \) is a constant on an interval.

**Lemma 2.3.** Let \( f \) belong to \( C([0, \infty), L^1_2) \) and \( \| f_t \|_{HS}, t \in [a, b], \) be a constant for some \( 0 \leq a < b \).

(i) For every \( u_0 \in (0, 1) \) there exist \( u_1 < u_0 < u_2 \) and \( \alpha < \beta \) from \([a, b]\) such that \( f_t \) is a constant on \([u_1, u_0)\) and \([u_0, u_2)\) for each \( t \in [\alpha, \beta] \).

(ii) For \( u_0 = 0 \) (resp. \( u_0 = 1 \)) there exist \( u_2 > u_0 \) (resp. \( u_1 < u_0 \)) and \( \alpha < \beta \) from \([a, b]\) such that \( f_t \) is a constant on \([u_0, u_2)\) (resp. \([u_1, u_0)\)) for each \( t \in [\alpha, \beta] \).

**Proof.** Since \( \| f_t \|_{HS} \) is a constant on \([a, b]\), the function \( f_t \) takes a fixed number of distinct values, denoted by \( n \), for each \( t \in [a, b] \), by equality (2). Let
\[
f_t = \sum_{k=1}^{n} x_k(t)\|_{q_k-1(t),q_k(t)}, \quad t \in [a, b],
\]
where \( x_1(t) < \ldots < x_n(t) \) and \( 0 = q_0(t) < q_1(t) < \ldots < q_n(t) = 1 \).

We first check that the functions \( x_k \) and \( q_k \) are continuous on \([a, b]\) for each \( k \in [n-1] \). Take a sequence \( t_n \in [a, b], n \geq 1 \) converging to \( t_0 \). We can choose a subsequence \( N \subset \mathbb{N} \)
such that $x_k(t_n) \to y_k$ and $q_k(t_n) \to p_k$ along $N$ for each $k \in [n - 1]$. Moreover, if $y_k = -\infty$ ($y_k = +\infty$), then $p_k = 0$ (resp. $p_k = 1$) due to $\|f_t\|_{L^2} < \infty$, $t \in [a, b]$. We set

$$h := \sum_{k=1}^{n} y_k \|p_k\|_{L^2}.$$ 

Then, it is easy to see that $f_{t_n} \to h$ a.e. along $N$. But, by the continuity of $f$, $f_{t_n} \to f_{t_0}$ in $L^2$. Thus, $f_{t_0} = h$, that implies the equalities $y_k = x_k(t_0)$ and $p_k = q_k(t_0)$ for all $k \in [n]$. Thus, the needed continuity holds.

If there exists $l \in [n]$ such that

$$u_0 \in (q_{l-1}(t), q_l(t)) \quad \text{for some} \quad t \in (a, b),$$

then one can take $u_1 < u_0 < u_2$ and $\alpha < \beta$ from $[a, b]$ satisfying $u_1, u_2$ in $(q_{l-1}(t), q_l(t))$ for all $t \in [\alpha, \beta]$, by the continuity of $q_k, k \in [n - 1]$. This trivially implies the statement of the lemma. If $l$ satisfying (5) does not exist, then $u_0 = q_l(t)$ for some $l \in [n] \cup \{0\}$ and all $t \in [a, b]$, which also yields the statement. \hfill \Box

3. Proof of Theorem 1.1

In order to show that with probability 1 there exists a dense subset $R$ of $[0, \infty)$ such that $\sharp X_t = \infty$ for all $t \in R$, it is enough to prove that

$$\mathbb{P} \left\{ \sup_{t \in [a, b]} \sharp X_t = \infty \right\} = 1,$$

where the measurability of $\sup_{t \in [a, b]} \sharp X_t$ follows from Lemma 2.1 and equality (2). Indeed, this will imply

$$\mathbb{P} \left\{ \exists R \text{ dense in } [0, \infty) \text{ such that } \sharp X_t = \infty, \forall t \in R \right\} = \mathbb{P} \left\{ \bigcap_{a \leq t \leq b} \left\{ \sup_{t \in [a, b]} \sharp X_t = \infty \right\} \right\} = 1.$$

Let us assume the opposite, i.e.

$$\mathbb{P} \left\{ \sup_{t \in [a, b]} \sharp X_t < \infty \right\} > 0.$$

Setting $A_{n}^{a,b} := \{ t \in [a, b] : \| \pr_{X_t} \|_{HS}^2 \leq n \}$ and using equality (2), we can conclude that

$$\mathbb{P} \left\{ \bigcup_{n=1}^{\infty} A_{n}^{a,b} = [a, b] \right\} > 0.$$

By Lemma 2.2 and the Baire category theorem, we have

$$\mathbb{P} \left\{ \exists \alpha_1 < b_1 \text{ from } [a, b] \text{ and } n \in \mathbb{N} \text{ such that } \| \pr_{X_t} \|_{HS}^2 \leq n \ \forall t \in [\alpha_1, b_1] \right\} > 0.$$ 

Consequently, we can find non-random $a_1 < b_1$ from $[a, b]$ and $k_1 \in \mathbb{N}$ such that

$$\mathbb{P} \left\{ \| \pr_{X_t} \|_{HS}^2 \leq k_1 \ \forall t \in [a_1, b_1] \right\} > 0.$$

Since,

$$\mathbb{P} \left\{ \| \pr_{X_t} \|_{HS}^2 \leq k_1 \ \forall t \in [a_1, b_1] \right\} \cap \left( \bigcup_{k_1=1}^{k_1} \left\{ A_{k_1}^{a_1,b_1} \setminus A_{k_1-1}^{a_1,b_1} \neq \emptyset \right\} \right) > 0,$$

there exists $k_2 \leq k_1$ satisfying

$$\mathbb{P} \left\{ \| \pr_{X_t} \|_{HS}^2 \leq k_2 \ \forall t \in [a_1, b_1] \right\} \cap \left\{ A_{k_2}^{a_1,b_1} \setminus A_{k_2-1}^{a_1,b_1} \neq \emptyset \right\} > 0,$$
where $A_{0,1}^1 := \emptyset$. Next, since $A_{k_2}^{a_1,b_1} \setminus A_{k_2-1}^{a_1,b_1}$ is open in $A_{k_2}^{a_1,b_1}$ and non-empty with positive probability, we can find non-random $a_2 < b_2$ from $[a_1, b_1]$ satisfying

$$
P \{ \|pr_{X_t}\|^2_{H_S} = k_2 \ \forall t \in [a_2, b_2] \} > 0.
$$

Next, due to the equality $\xi_\omega = \infty$, there exists $u_0 \in [0, 1]$ such that $\xi$ takes an infinite number of distinct values in $[u_1, u_0)$ for all $u_1 < u_0$ or in $[u_0, u_2)$ for all $u_2 > u_0$. Using Lemma 2.3 and the monotonism of $X_t(\omega)$ for all $t$ and $\omega$, one can find non-random $a_3 < b_3$ from $[a_2, b_2]$ and $u < v$ such that $u = u_0$ or $v = u_0$, $\xi$ takes an infinite number of distinct values on $[u, v]$ and

$$
P \{\forall t \in [a_3, b_3] \ X_t \text{ is a constant on } [u, v] \} > 0.
$$

Let $h := I_{(u,v)/2} - I_{(u+v)/2}$. Since $X_t, t \geq 0$, solves equation (1), one has that $(X_t, h)_{L_2}, t \geq 0$, is a continuous non-negative process such that

$$
M_h(t) = (X_t, h)_{L_2} - \int_0^t (\xi - pr_{X_s}, \xi, h)_{L_2} ds, \quad t \geq 0,
$$

is a continuous square integrable $(\mathcal{F}_t)$-martingale with quadratic variation

$$
\langle M_h \rangle_t = \int_0^t \|pr_{X_s}h\|^2 ds, \quad t \geq 0.
$$

Thus, using the equalities $(X_t, h)_{L_2} = 0$, $pr_{X_s}h = 0$ and $(pr_{X_s}, \xi, h)_{L_2} = (\xi, pr_{X_s}, h)_{L_2} = 0$ for all $s \in [a_3, b_3]$ on the event $A := \{\forall t \in [a_3, b_3] \ X_t \text{ is a constant on } [u, v] \}$, we have

$$
M_h(t) = \int_0^{a_3} (\xi - pr_{X_s}, \xi, h)_{L_2} ds + \int_{a_3}^t (\xi, h)_{L_2} ds
$$

and

$$
[M_h]_t = \int_0^{a_3} \|pr_{X_s}h\|^2 ds
$$
on $A$ for all $t \in [a_3, b_3]$. The equality for the quadratic variation of $M_h$ and the representation of continuous martingales as a time shift of a Brownian motion (see [3, Theorem II.7.2]) imply that $M_h(t) = M_h(a_3), t \in [a_3, b_3]$, on $A$. But according to equality (6), $M_h(t), t \in [a_3, b_3]$, is strictly increasing on the event $A$ because $(\xi, h)_{L_2} > 0$. Since $P\{A\} > 0$, we get a contradiction. This finishes the proof of the first part of the theorem.

We next prove claim (ii). Due to $\xi_\omega < \infty$, there exists a finite partition $\pi_k, k \in [n]$, of the interval $[0, 1]$ by intervals of the form $[a, b)$ such that

$$
\xi(u) = \sum_{k=1}^n \xi_k I_{\pi_k}(u), \quad u \in [0, 1).
$$

In order to prove (ii), it is enough to show that almost surely $X_t$ takes a finite number of distinct values on every interval $[a, b)$. We fix $k \in [n]$ and consider the countable family of functions $h_{u,v} := I_{(u,v)/2} - I_{(u+v)/2}$ from $L_2, u, v \in \pi_k \cap \mathbb{Q}$, denoted by $\mathcal{R}$.

We first remark that for every $h \in \mathcal{R}$ the process $(X_t, h)_{L_2}, t \geq 0$, is a non-negative continuous supermartingale. Indeed, the non-negativity follows from the fact that $(f, h)_{L_2} \geq 0$ for every $f \in L^2_2$ and $h \in \mathcal{R}$. In order to show that $(X_t, h)_{L_2}, t \geq 0$, is a supermartingale, we use the fact that it is a weak martingale solution to equation (1). Hence for each $h \in \mathcal{R}$

$$
(X_t, h)_{L_2} = M_h(t) + \int_0^t (\xi - pr_{X_s}, \xi, h)_{L_2} ds = M_h(t) - \int_0^t (pr_{X_s}, \xi, h)_{L_2} ds, \quad t \geq 0,
$$

where $M_h$ is a martingale. According to Lemma A.2 [4], the orthogonal projection $pr_f$ maps the space $L_2^2$ into $L_2^2$ for every $f \in L^2_2$. Hence, $pr_{X_s} \xi \in L^2_2$ and, therefore, $(pr_{X_s} \xi, h) \geq 0$. This implies that $(X_t, h)_{L_2}, t \geq 0$, is a continuous supermartingale.
For every \( h \in \mathbb{R} \) we define
\[
\Omega_h = \left\{ \begin{array}{ll}
\text{for every } t \in [0, \infty) \text{ the equality } (X_t, h)_{L_2} = 0 \\
\text{implies } (X_t, h)_{L_2} = 0 \text{ for all } s \geq t
\end{array} \right\}.
\]
By Proposition II.3.4 [1], \( \mathbb{P}(\Omega) = 1 \). Thus, the event \( \Omega' := \bigcap_{h \in \mathbb{R}} \Omega_h \) has the probability 1. Take \( \omega \in \Omega' \), \( u, v \in (0, 1) \), and \( t \geq 0 \) such that \( X_t(u, \omega) = X_t(v, \omega) \). Then for every \( h \in \mathbb{R} \) one has \( (X_t(\omega), h)_{L_2} = 0 \) and, consequently, \( (X_s(\omega), h)_{L_2} = 0 \), by the choice of \( \omega \). Using the right continuity of \( X_s(\cdot, \omega) \) (see Remark 1.1), it is easily seen that \( X_s(u) = X_s(v) \).

Combining the coalescing property of \( X_t, t \geq 0 \), on every interval \( \pi_k, k \in [n] \) with equality (3), we get claim (ii) of the theorem.

References

[1] Leszek Gawarecki and Vidyadhar Mandrekar, *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR 2560625

[2] Chris Howitt and Jon Warren, *Consistent families of Brownian motions and stochastic flows of kernels*, Ann. Probab. 37 (2009), no. 4, 1237–1272. MR 2546745

[3] Nobuyuki Ikeda and Shinzo Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989. MR 1011252

[4] Vitalii Konarovskyi, *Coalescing-Fragmentating Wasserstein Dynamics: particle approach*, arXiv:1711.03011 (2017).

[5] , *On asymptotic behavior of the modified Arratia flow*, Electron. J. Probab. 22 (2017), Paper No. 19, 31. MR 3622889

[6] , *A system of coalescing heavy diffusion particles on the real line*, Ann. Probab. 45 (2017), no. 5, 3293–3335. MR 3706744

[7] Vitalii Konarovskyi and Victor Marx, *On conditioning brownian particles to coalesce*, arXiv:2008.02568 (2020).

[8] Vitalii Konarovskyi and Max von Renesse, *Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line*, arXiv:1709.02839 (2017).

[9] Vitalii Konarovskyi and Max-K. von Renesse, *Modified massive Arratia flow and Wasserstein diffusion*, Comm. Pure Appl. Math. 72 (2019), no. 4, 764–800. MR 3914882

[10] Victor Marx, *A new approach for the construction of a Wasserstein diffusion*, Electron. J. Probab. 23 (2018), Paper No. 124, 54. MR 3896861

[11] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR 1725357

[12] Max-K. von Renesse and Karl-Theodor Sturm, *Entropic measure and Wasserstein diffusion*, Ann. Probab. 37 (2009), no. 3, 1114–1191. MR 2537551

Faculty of Mathematics, Computer Science and Natural Sciences, University of Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany; Institute of Mathematics of NAS of Ukraine, Tereschenkivska st. 3, 01024 Kiev, Ukraine

Current address: Faculty of Mathematics, Computer Science and Natural Sciences, University of Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

Email address: konarovskyi@gmail.com