Brake subharmonic solutions of first order Hamiltonian systems

Chong Li† Chungen Liu ‡
School of Mathematical Sciences and LPMC,
Nankai University, Tianjin 300071, P.R. China.

Abstract In this paper, we mainly use the Galerkin approximation method and the iteration inequalities of the \( L \)-Maslov type index theory in [17,19] to study the properties of brake subharmonic solutions for the first order non-autonomous Hamiltonian systems. We prove that when the positive integers \( j \) and \( k \) satisfies the certain conditions, there exists a \( jT \)-periodic nonconstant brake solution \( z_j \) such that \( z_j \) and \( z_{kj} \) are distinct.

Keywords Brake subharmonic solution; \( L \)-Maslov type index; Hamiltonian systems

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1 Introduction and the Main Results

In this paper, we consider the first order non-autonomous Hamiltonian systems

\[
\dot{z}(t) = J \nabla H(t,z(t)), \quad \forall z \in \mathbb{R}^{2n}, \quad \forall t \in \mathbb{R},
\]

where \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) is the standard symplectic matrix, \( I_n \) is the unit matrix of order \( n \), \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}) \) and \( \nabla H(t,z) \) is the gradient of \( H(t,z) \) with respect to the space variable \( z \). We denote the standard norm and inner product in \( \mathbb{R}^{2n} \) by \( | \cdot | \) and \( (\cdot,\cdot) \), respectively.

Suppose that \( H(t,z) = \frac{1}{2}(\hat{B}(t)z,z) + \hat{H}(t,z) \) and \( \hat{H} \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}) \) satisfies the following conditions:

(H1) \( \hat{H}(T + t,z) = \hat{H}(t,z) \), for all \( z \in \mathbb{R}^{2n}, t \in \mathbb{R} \),

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\[\text{†E-mail: plumechong@yahoo.com.cn} \]
\[\text{‡E-mail: liucg@nankai.edu.cn}\]
(H2) $\hat{H}(t, z) = \hat{H}(-t, Nz)$, for all $z \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$, $N = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$.

(H3) $\hat{H}''(t, z) > 0$, for all $z \in \mathbb{R}^{2n} \setminus \{0\}$, $t \in \mathbb{R}$,

(H4) $\hat{H}(t, z) \geq 0$, for all $z \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$,

(H5) $\hat{H}(t, z) = o(|z|^2)$ at $z = 0$,

(H6) There is a $\theta \in (0, 1/2)$ and $\bar{r} > 0$ such that

$$0 \leq \frac{1}{\theta} \hat{H}(t, z) \leq (z, \nabla \hat{H}(t, z)), \text{ for all } z \in \mathbb{R}^{2n}, |z| \geq \bar{r}, t \in \mathbb{R},$$

(H7) $\hat{B}(t)$ is a symmetrical continuous matrix, $|\hat{B}|_{C^0} \leq \beta_0$ for some $\beta_0 > 0$, and $\hat{B}(t)$ is a semi-positively definite for all $t \in \mathbb{R}$,

(H8) $\hat{B}(T + t) = \hat{B}(t) = \hat{B}(-t)$, $\hat{B}(t)N = N\hat{B}(t)$, for all $t \in \mathbb{R}$.

Recall that a $T$-periodic solution $(z, T)$ of $\mathbf{1}$ is called brake solution if $z(t+T) = z(t)$ and $z(t) = Nz(-t)$, the later is equivalent to $z(T/2 + t) = Nz(T/2 - t)$, in this time $T$ is called the brake period of $z$. Up to the authors’ knowledge, H. Seifert firstly studied brake orbits in second order autonomous Hamiltonian systems in [29] of 1948. Since then many studies have been carried out for brake orbits of first order and second order Hamiltonian systems. For the minimal periodic problem, multiple existence results about brake orbits for the Hamiltonian systems and more details on brake orbits one can refer the papers [1,3–6,11–13,19,23,26,31] and the references therein. S. Bolotin proved first in [5] (also see [6]) of 1978 the existence of brake orbits in general setting. K. Hayashi in [13], H. Gluck and W. Ziller in [11], and V. Benci in [3] in 1983-1984 proved the existence of brake orbits of second order Hamiltonian systems under certain conditions. In 1987, P. Rabinowitz in [26] proved the existence of brake orbits of first order Hamiltonian systems. In 1987, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [4]. In 1989, A. Szulkin in [31] proved the existence of brake orbits of first order Hamiltonian systems under the $\sqrt{2}$-pinched condition. E. van Groesen in [12] of 1985 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved the multiplicity result about brake orbits for the second order Hamiltonian systems under different pinching conditions. Without pinching conditions, in [23] (2006) Y. Long, D. Zhang and C. Zhu proved that there exist at least two geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbb{R}^n$ for $n \geq 2$. Recently, C. Liu and D. Zhang in [19] proved that there exist at least $\lceil n/2 \rceil + 1$ geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbb{R}^n$ for $n \geq 2$, and there exist at least $n$ geometrically distinct brake orbits on nondegenerate domain.
For the non-autonomous Hamiltonian systems, for periodic boundary (brake solution) problems, since the Hamiltonian function $H$ is $T$-periodic in the time variable $t$, if the system (1.1) has a $T$-periodic solution $(z_1, T)$, one hopes to find the $jT$-periodic solution $(z_j, jT)$ for integer $j \geq 1$, for example, $(z_1, jT)$ itself is $jT$-periodic solution. The subharmonic solution problem asks when the solutions $z_1$ and $z_j$ are distinct. More precisely, in the case of brake solutions, $z_1$ and $z_j$ are distinct if $kT \ast z_1(\cdot) \equiv z_1(kT + \cdot) \neq z_j(\cdot)$ for any integer $k$. In other word, $z_j(t) \neq z_1(t)$ and $z_j(t) \neq z_1(T/2 + t)$ for $t \in [0, T]$. In below we remind that the $L_0$-indices of the two solutions $z_1$ and $(kT) \ast z_1$ for any $k \in \mathbb{Z}$ in the interval $[0, T/2]$ are the same. In this paper, we first consider the brake subharmonic solution problem. We state the main results of this paper.

**Theorem 1.1.** Suppose that $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies (H1)-(H8), then for each integer $1 \leq j < 2\pi/\beta_0T$, there is a $jT$-periodic nonconstant brake solution $z_j$ of (1.1) such that $z_j$ and $z_{kj}$ are distinct for $k \geq 5$ and $kj < 2\pi/\beta_0T$. Furthermore, $\{z_{kp}|p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of (1.1) for $k \geq 5$ and $1 \leq kp < 2\pi/\beta_0T$.

Especially, if $\hat{B}(t) \equiv 0$, then $2\pi/\beta_0T = +\infty$. Therefore, one can state the following theorem.

**Theorem 1.2.** Suppose that $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ with $\hat{B}(t) \equiv 0$ satisfies (H1)-(H6), then for each integer $j \geq 1$, there is a $jT$-periodic nonconstant brake solution $z_j$ of (1.1). Furthermore, given any integers $j \geq 1$ and $k \geq 5$, $z_j$ and $z_{kj}$ are distinct brake solutions of (1.1), in particularly, $\{z_{kp}|p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of (1.1).

The first result on subharmonic periodic solutions for the Hamiltonian systems $\dot{z}(t) = J\nabla H(t, z(t))$, where $z \in \mathbb{R}^{2n}$ and $H(t, z)$ is $T$-periodic in $t$, was obtained by P. Rabinowitz in his pioneer work [27]. Since then, many new contributions have appeared. See for example [8, 9, 18, 20, 30] and the references therein. Especially, in [9], I. Ekeland and H. Hofer proved that under a strict convex condition and a superquadratic condition, the Hamiltonian system $\dot{z}(t) = J\nabla H(t, z(t))$ possesses subharmonic solution $z_k$ for each integer $k \geq 1$ and all of these solutions are pairwise geometrically distinct. In [18], the second author of this paper obtained a result of subharmonic solutions for the non-convex case by using the Maslov-type index iteration theory. We notice that in [32] T. An wants to improve the result of [18], but there is a gap in his proof when applying Theorem 2.6 there to prove his Theorem 1.3. Precisely, the formula (2.17) in [32] is not true since
the middle term $i_T(\dot{B}) + \nu_T(\dot{B}) + 1$ should be $i_{kT}(\dot{B}) + \nu_{kT}(\dot{B}) + 1$. Up to the authors’ knowledge, Theorem 1.1 and 1.2 are the first results for the brake subharmonic solution problem for the time being.

The main ingredient in proving Theorem 1.1 and 1.2 is to transform the brake solution problem into the $L_0$-boundary problem:

$$\begin{align*}
\dot{z}(t) &= J\nabla H(t, z(t)), \forall z \in \mathbb{R}^{2n}, \forall t \in [0, T/2], \\
 z(0) &\in L_0, \ z(T/2) \in L_0,
\end{align*}$$

where $L_0 = \{0\} \oplus \mathbb{R}^n \in \Lambda(n)$. $\Lambda(n)$ is the set of all linear Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega)$, here the standard symplectic form is defined by $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. A Lagrangian subspace $L$ of $\mathbb{R}^{2n}$ is an $n$ dimensional subspace satisfying $\omega_0|_L = 0$.

**Lemma 1.3.** Suppose the Hamiltonian function $H$ satisfying conditions (H1), (H2) and (H8). If $(z, T/2)$ is a solution of the problem (1.2), then $(\tilde{z}, T)$ is a $T$-periodic solution of the Hamiltonian system (1.1) satisfying the brake condition $\tilde{z}(T/2 + t) = N\tilde{z}(T/2 - t)$, where $\tilde{z}$ is defined by

$$\tilde{z}(t) = \begin{cases} z(t), & t \in [0, T/2], \\
Nz(T - t), & t \in (T/2, T]. \end{cases}$$

**Proof.** It is easy to see that $\tilde{z}(t)$ is continuous in the interval $[0, T]$. By direct computation,

$$\begin{align*}
\tilde{z}(t + T/2) &= -N\tilde{z}(T/2 - t) = JN\nabla H(T/2 - t, z(T/2 - t)) \\
&= J\nabla H(t + T/2, Nz(T/2 - t)) = J\nabla H(t + T/2, \tilde{z}(t + T/2)).
\end{align*}$$

So $(\tilde{z}, T)$ is a $T$-periodic solution of the Hamiltonian system (1.1). The brake condition is satisfied by the definition of $\tilde{z}$. The proof of Lemma 1.3 is complete. \hfill \Box

By this observation, we then use the Galerkin approximation methods to get a critical point of the action functional which is also a solution of (3.1) with a suitable $L_0$-index estimate, see Theorem 3.1 below. The $L$-Maslov type index theory for any $L \in \Lambda(n)$ was studied in [16] by the algebraic methods. In [23], Y. Long, D. Zhang and C. Zhu established two indices $\mu_1(\gamma)$ and $\mu_2(\gamma)$ for the fundamental solution $\gamma$ of a linear Hamiltonian system by the methods of functional analysis which are special cases of the $L$-Maslov type index $i_L(\gamma)$ for Lagrangian subspaces $L_0 = \{0\} \oplus \mathbb{R}^n$ and $L_1 = \mathbb{R}^n \oplus \{0\}$ up to a constant $n$. In order to prove Theorem 1.1 and 1.2 we need to consider the problem (3.1). The iteration
theory of the $L_0$-Maslov type index theory developed in [17] and [19], then help us to distinguish solutions $z_j$ from $z_{kj}$ in Theorem 1.1 and 1.2.

This paper is divided into 3 sections. In section 2, we give a brief introduction to the Maslov-type index theory for symplectic paths with Lagrangian boundary conditions and an iteration theory for the $L_0$-Maslov type index theory. In section 3, we give a proof of Theorem 1.1 and 1.2.

2 Preliminaries

In this section, we briefly recall the Maslov-type index theory for symplectic paths with Lagrangian boundary conditions and an iteration theory for the $L_0$-Maslov type index theory. All the details can be found in [15–17, 19].

We denote the $2n$-dimensional symplectic group $Sp(2n)$ by

$$Sp(2n) = \{ M \in \mathcal{L}(\mathbb{R}^{2n}) | M^T J M = J \},$$

where $\mathcal{L}(\mathbb{R}^{2n})$ is the set of all real $2n \times 2n$ matrices, $M^T$ is the transpose of matrix $M$. Denote by $\mathcal{L}_s(\mathbb{R}^{2n})$ the subset of $\mathcal{L}(\mathbb{R}^{2n})$ consisting of symmetric matrices. And denote the symplectic path space by

$$\mathcal{P}(2n) = \{ \gamma \in C([0, 1], Sp(2n)) | \gamma(0) = I_{2n} \}.$$

We write a symplectic path $\gamma \in \mathcal{P}(2n)$ in the following form

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}, \quad (2.1)$$

where $S(t), T(t), V(t), U(t)$ are $n \times n$ matrices. The $n$ vectors come from the column of the matrix $\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}$ are linear independent and they span a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$. Particularly, at $t = 0$, this Lagrangian subspace is $L_0 = \{0\} \oplus \mathbb{R}^{2n}$.

**Definition 2.1.** (see [16]) We define the $L_0$-nullity of any symplectic path $\gamma \in \mathcal{P}(2n)$ by

$$\nu_{L_0}(\gamma) \equiv \dim \ker_{L_0}(\gamma(1)) := \dim \ker V(1) = n - \text{rank } V(1)$$

with the $n \times n$ matrix function $V(t)$ defined in (2.1).

For $L_0 = \{0\} \oplus \mathbb{R}^n$, We define the following subspaces of $Sp(2n)$ by

$$Sp(2n)^*_L = \{ M \in Sp(2n) | \det V_M \neq 0 \},$$
$$Sp(2n)^0_L = \{ M \in Sp(2n) | \det V_M = 0 \},$$
$$Sp(2n)^\pm_L = \{ M \in Sp(2n) | \pm \det V_M > 0 \},$$
where $M = \begin{pmatrix} S_M & V_M \\ T_M & U_M \end{pmatrix}$ and $Sp(2n)_L^* = Sp(2n)_L^+ \cup Sp(2n)_L^-$. And denote two subsets of $P(2n)$ by

\[ P(2n)_L^* = \{ \gamma \in P(2n) | \nu_L(\gamma) = 0 \}, \]
\[ P(2n)_L^0 = \{ \gamma \in P(2n) | \nu_L(\gamma) > 0 \}. \]

We note that rank $\left( V(t) U(t) \right) = n$, so the complex matrix $U(t) \pm \sqrt{-1} V(t)$ is invertible.

We define a complex matrix function by

\[ Q(t) = (U(t) - \sqrt{-1} V(t))(U(t) + \sqrt{-1} V(t))^{-1}. \]

It is easy to see that the matrix $Q(t)$ is a unitary matrix for any $t \in [0, 1]$. We define

\[ M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \text{diag} (-1, 1, \cdots, 1). \]

For a path $\gamma \in P(2n)_L^*$, we first adjoin it with a simple symplectic path starting from $J = -M_+$, that is, we define a symplectic path by

\[ \tilde{\gamma}(t) = \begin{cases} I \cos \left( \frac{(1-2t)\pi}{2} \right) + J \sin \left( \frac{(1-2t)\pi}{2} \right), & t \in [0, 1/2], \\ \gamma(t), & t \in [1/2, 1]. \end{cases} \]

Then we choose a symplectic path $\beta(t)$ in $Sp(2n)_L^*$ starting from $\gamma(1)$ and ending at $M_+$ or $M_-$ according to $\gamma(1) \in Sp(2n)_L^+$ or $\gamma(1) \in Sp(2n)_L^-$, respectively. We now define a joint path by

\[ \bar{\gamma}(t) = \beta * \tilde{\gamma} := \begin{cases} \tilde{\gamma}(2t), & t \in [0, 1/2], \\ \beta(2t - 1), & t \in [1/2, 1]. \end{cases} \]

By the definition, we see that the symplectic path $\bar{\gamma}$ starting from $-M_+$ and ending at either $M_+$ or $M_-$. As above, we define

\[ \bar{Q}(t) = (\bar{U}(t) - \sqrt{-1} \bar{V}(t))(\bar{U}(t) + \sqrt{-1} \bar{V}(t))^{-1}, \]

for $\bar{\gamma}(t) = \begin{pmatrix} \bar{S}(t) \\ \bar{T}(t) \end{pmatrix}$. We can choose a continuous function $\bar{\Delta}(t)$ in $[0, 1]$ such that

\[ \det \bar{Q}(t) = e^{2\sqrt{-1}\bar{\Delta}(t)}. \]

By the above arguments, we see that the number $\frac{1}{\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)) \in \mathbb{Z}$ and it does not depend on the choice of the function $\bar{\Delta}(t)$. 

6
Definition 2.2. (see [16]) For a symplectic path \( \gamma \in \mathcal{P}(2n)_{L_0}^* \), we define the \( L_0 \)-index of \( \gamma \) by

\[
i_{L_0}(\gamma) = \frac{1}{\pi} (\bar{\Delta}(1) - \bar{\Delta}(0)).
\]

Definition 2.3. (see [16]) For a symplectic path \( \gamma \in \mathcal{P}(2n)_{L_0}^0 \), we define the \( L_0 \)-index of \( \gamma \) by

\[
i_{L_0}(\gamma) = \inf \{ i_{L_0}(\tilde{\gamma}) | \tilde{\gamma} \in \mathcal{P}(2n)_{L_0}^*, \text{and} \tilde{\gamma} \text{ is sufficiently close to } \gamma \}.
\]

We know that \( \Lambda(n) = U(n)/O(n) \), this means that for any linear subspace \( L \in \Lambda(n) \), there is an orthogonal symplectic matrix \( P = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \) with \( A \pm \sqrt{-1}B \in U(n) \), the unitary matrix, such that \( PL_0 = L \). \( P \) is uniquely determined by \( L \) up to an orthogonal matrix \( C \in O(n) \). It means that for any other choice \( P' \) satisfying above conditions, there exists a matrix \( C \in O(n) \) such that \( P' = P \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \) (see [24]). We define the conjugated symplectic path \( \gamma_c \in \mathcal{P}(2n) \) of \( \gamma \) by \( \gamma_c(t) = P^{-1}\gamma(t)P \).

Definition 2.4. (see [16]) We define the \( L \)-nullity of any symplectic path \( \gamma \in \mathcal{P}(2n) \) by

\[
\nu_L(\gamma) \equiv \dim \ker_L(\gamma(1)) := \dim \ker V_\gamma(1) = n - \text{rank } V_\gamma(1),
\]

where the \( n \times n \) matrix function \( V_\gamma(t) \) is defined in (2.1) with the symplectic path \( \gamma \) replaced by \( \gamma_c \), i.e., \( \gamma_c(t) = \begin{pmatrix} S_c(t) & V_\gamma(t) \\ T_c(t) & U_\gamma(t) \end{pmatrix} \).

Definition 2.5. (see [16]) For a symplectic path \( \gamma \in \mathcal{P}(2n) \), we define the \( L \)-index of \( \gamma \) by

\[
i_L(\gamma) = i_{L_0}(\gamma_c).
\]

In the case of linear Hamiltonian systems

\[
\dot{y} = JB(t)y, \quad \forall y \in \mathbb{R}^{2n},
\]

(2.2)

where \( B \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n})) \). Its fundamental solution \( \gamma = \gamma_B \) is a symplectic path starting from identity matrix \( I_{2n} \), i.e., \( \gamma = \gamma_B \in \mathcal{P}(2n) \). We denote by

\[
i_L(B) = i_L(\gamma_B), \quad \nu_L(B) = \nu_L(\gamma_B).
\]
Theorem 2.6. (see [16]) Suppose $\gamma \in \mathcal{P}(2n)$ is a fundamental solution of (2.2) with $B(t) > 0$. There holds

$$i_L(\gamma) \geq 0.$$ 

Suppose the continuous symplectic path $\gamma : [0, 2] \rightarrow Sp(2n)$ is the fundamental solution of (2.2) with $B(t)$ satisfying $B(t + 2) = B(t)$ and $B(1 + t)N = NB(1 - t)$. This implies $B(t)N = NB(-t)$. By the unique existence theorem of the differential equations, we get

$$\gamma(1 + t) = N\gamma(1-t)\gamma(1)^{-1}N\gamma(1), \quad \gamma(2 + t) = \gamma(t)\gamma(2).$$

We define the iteration path of $\gamma|_{[0,1]}$ by

$$\gamma^1(t) = \gamma(t), \quad t \in [0, 1],$$

$$\gamma^2(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \end{cases}$$

$$\gamma^3(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \gamma(t - 2)\gamma(2), & t \in [2, 3], \end{cases}$$

$$\gamma^4(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \gamma(t - 2)\gamma(2), & t \in [2, 3], \\ N\gamma(4-t)\gamma(1)^{-1}N\gamma(1)\gamma(2), & t \in [3, 4], \end{cases}$$

and in general, for $k \in \mathbb{N}$, we define

$$\gamma^{2k-1}(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \cdots \cdots \\ N\gamma(2k - 2 - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-5}, & t \in [2k - 3, 2k - 2], \\ \gamma(t - 2k + 2)\gamma(2)^{2k-4}, & t \in [2k - 2, 2k - 1], \end{cases}$$

$$\gamma^{2k}(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2], \\ \cdots \cdots \\ \gamma(t - 2k + 2)\gamma(2)^{2k-4}, & t \in [2k - 2, 2k - 1], \\ N\gamma(2k - t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-3}, & t \in [2k - 1, 2k]. \end{cases}$$

Recall that $(i_\omega(\gamma), \nu_\omega(\gamma))$ is the $\omega$-index pair of the symplectic path $\gamma$ introduced in [20], and $(i_{\omega_0}(\gamma), \nu_{\omega_0}(\gamma))$ is defined in [19].
Theorem 2.7. (see [19]) Suppose \( \omega_k = e^{\pi \sqrt{-1}/k} \). For odd \( k \) we have

\[
i_{L_0} (\gamma^k) = i_{L_0} (\gamma^1) + \sum_{i=1}^{(k-1)/2} i_{\omega_{k}^i} (\gamma^2),
\]

\[
\nu_{L_0} (\gamma^k) = \nu_{L_0} (\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu_{\omega_{k}^i} (\gamma^2),
\]

for even \( k \), we have

\[
i_{L_0} (\gamma^k) = i_{L_0} (\gamma^1) + i_{L_0} (\sqrt{-1}) \gamma^1 + \frac{k-1}{2} \sum_{i=1}^{k/2-1} i_{\omega_{k}^i} (\gamma^2),
\]

\[
\nu_{L_0} (\gamma^k) = \nu_{L_0} (\gamma^1) + \nu_{L_0} (\sqrt{-1}) \gamma^1 + \frac{k-1}{2} \sum_{i=1}^{k/2-1} \nu_{\omega_{k}^i} (\gamma^2),
\]

where \( \omega_k^{k/2} = \sqrt{-1} \).

Theorem 2.8. (see [19]) There hold

\[
i_1 (\gamma^2) = i_{L_0} (\gamma^1) + i_{L_1} (\gamma^1) + n,
\]

\[
\nu_1 (\gamma^2) = \nu_{L_0} (\gamma^1) + \nu_{L_1} (\gamma^1),
\]

where \( L_1 = \mathbb{R} \oplus \{0\} \in \Lambda(n) \).

In the following section, we need the following two iteration inequalities.

Theorem 2.9. (see [17]) For any \( \gamma \in \mathcal{P}(2n) \) and \( k \in \mathbb{N} \), there holds

\[
i_{L_0} (\gamma^1) + \frac{k - 1}{2} (i_1 (\gamma^2) + \nu_1 (\gamma^2) - n) \leq i_{L_0} (\gamma^k)
\]

\[
\leq i_{L_0} (\gamma^1) + \frac{k - 1}{2} (i_1 (\gamma^2) + n) - \frac{1}{2} \nu_1 (\gamma^{2k}) + \frac{1}{2} \nu_1 (\gamma^2), \text{ if } k \in 2\mathbb{N} - 1,
\]

\[
i_{L_0} (\gamma^1) + i_{L_0} (\sqrt{-1}) \gamma^1 + \left( \frac{k}{2} - 1 \right) (i_1 (\gamma^2) + \nu_1 (\gamma^2) - n) \leq i_{L_0} (\gamma^k) \leq i_{L_0} (\gamma^1) + i_{L_0} (\sqrt{-1}) \gamma^1
\]

\[
+ \left( \frac{k}{2} - 1 \right) (i_1 (\gamma^2) + n) - \frac{1}{2} \nu_1 (\gamma^{2k}) + \frac{1}{2} \nu_1 (\gamma^2) + \frac{1}{2} \nu_{-1} (\gamma^2), \text{ if } k \in 2\mathbb{N}.
\]

Remark 2.10. From (3.17) of [19] and Proposition B of [23], we have that

\[
i_{L_0} (B) \leq i_{L_0} (B) \leq i_{L_0} (B) + n,
\]

\[
|i_{L_0} (B) - i_{L_1} (B)| \leq n,
\]

where \( L_1 = \mathbb{R} \oplus \{0\} \in \Lambda(n) \).
3 Proof of Theorem 1.1 and 1.2

In this section, we first consider the following Hamiltonian systems

\[
\begin{aligned}
\dot{z}(t) &= J \nabla H(t, z(t)), \quad \forall z \in \mathbb{R}^{2n}, \quad \forall t \in [0, jT/2], \\
z(0) &= z_0 \\
\end{aligned}
\tag{3.1}
\]

where \(j \in \mathbb{N}\). The following result is the first part of Theorem 1.1.

**Theorem 3.1.** Suppose \(H(t, z) \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})\) satisfies (H4)-(H7), then for \(1 \leq j < 2\pi/\beta_0 T\), (3.1) possesses at least one nontrivial solution \(z_j\) whose \(L_0\)-index pair \((i_{L_0}(z_j), \nu_{L_0}(z_j))\) satisfies

\[i_{L_0}(z_j) \leq 1 \leq i_{L_0}(z_j) + \nu_{L_0}(z_j).\]

So we get a nonconstant brake solution \((\tilde{z}_j, jT)\) with brake period \(jT\) of the Hamiltonian system (1.1) by Lemma 1.3.

In order to prove Theorem 3.1 we need the following arguments. For simplicity, we suppose \(T = 2\). Let \(X := \left\{ z \in W^{1,2}([0, j], \mathbb{R}^{2n}) | z = \sum_{l \in \mathbb{Z}} e^{lt}z_l, z_l \in L_0, \|z\|_X < +\infty \right\} \) be the Hilbert space with the inner product

\[(u, v)_X = j(u_0, v_0) + j \sum_{l \in \mathbb{Z}} |l|(u_l, v_l), \quad \forall u, v \in X.\]

In the following, we use \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) to denote the inner product and norm in \(X\), respectively. It is well known that if \(r \in [1, +\infty)\) and \(z \in L_r([0, j], \mathbb{R}^{2n})\) then there exists a constant \(c_r > 0\) such that \(\|z\|_{L_r} \leq c_r\|z\|\).

We define the linear operators \(A\) and \(\hat{B}\) on \(X\) by extending the bilinear form

\[
\langle Au, v \rangle = \int_0^j (-J\dot{u}, v)dt, \quad \langle \hat{B}u, v \rangle = \int_0^j (\hat{B}(t)u, v)dt.
\]

Then \(\hat{B}\) is a compact self-adjoint operator (see [20]) and \(A\) is a self-adjoint operator, i.e.,

\[
\langle Au, v \rangle = \langle u, A^*v \rangle = \langle u, Av \rangle.
\]

Indeed, by definition

\[
\langle Au, v \rangle = \int_0^j (-J\dot{u}(t), v(t))dt = (-Ju(t), v(t))|_0^j - \int_0^j (-Ju(t), \dot{v}(t))dt
\]

\[
= (-Ju(t), v(t))|_0^j + \int_0^j (u(t), -J\dot{v}(t))dt = (-Ju(t), v(t))|_0^j + \langle u, Av \rangle.
\]
Since \((-Ju(t), v(t))\big|_0^T = \omega_0(u(T), v(T)) - \omega_0(u(0), v(0)) = 0\), so \(\langle Au, v \rangle = \langle u, Av \rangle\), i.e., \(A\) is a self-adjoint operator.

We take the spaces
\[
X_m = \left\{ z \in X | z = \sum_{l=-m}^{m} e^{\frac{\imath \pi}{J} l t} z_l, z_l \in L \right\},
\]
\[
X^+ = \left\{ z \in X | z = \sum_{l>0} e^{\frac{\imath \pi}{J} l t} z_l, z_l \in L \right\},
\]
\[
X^- = \left\{ z \in X | z = \sum_{l<0} e^{\frac{\imath \pi}{J} l t} z_l, z_l \in L \right\},
\]
\[
X^0 = L_0,
\]
and \(X_m^+ = X_m \cap X^+, X_m^- = X_m \cap X^-\). We have \(X_m = X_m^+ \oplus X^0 \oplus X_m^-\). We also know that
\[
\langle Az, z \rangle = \frac{\pi}{J} \|z\|^2, \quad \forall z \in X_m^+,
\]
\[
\langle Az, z \rangle = -\frac{\pi}{J} \|z\|^2, \quad \forall z \in X_m^-.
\]

Equalities (3.2) and (3.3) can be proved by definition and direct computation. Let \(P_m : X \to X_m\) be the corresponding orthogonal projection for \(m \in \mathbb{N}\). Then \(\Gamma = \{P_m; m \in \mathbb{N}\}\) is a Galerkin approximation scheme with respect to \(A\) (see [15]).

For any Lagrangian subspace \(L \in \Lambda(n)\), suppose \(P \in Sp(2n) \cap O(2n)\) such that \(L = PL_0\). Then we define \(X_L = PX\) and \(X_L^m = PX_m\). Let \(P^m : X_L \to X_L^m\). Then as above, \(\Gamma = \{P^m; m \in \mathbb{N}\}\) is a Galerkin approximation scheme with respect to \(A\). For \(d > 0\), we denote by \(M^*_d(Q)\), \(* = +, 0, -\), the eigenspaces corresponding to the eigenvalues \(\lambda\) of the linear operator \(Q : X_L \to X_L\) belonging to \([d, +\infty)\), \((-d, d)\) and \((-\infty, -d)\), respectively. And denote by \(M^*(Q)\), \(* = +, 0, -\), the eigenspaces corresponding to the eigenvalues \(\lambda\) of \(Q\) belonging to \((0, +\infty)\), \(\{0\}\) and \((-\infty, 0)\), respectively. For any adjoint operator \(Q\), we denote \(Q^\dagger = (Q|_{ImQ})^{-1}\), and we also denote \(P^mQ^m = (P^mQP^m)|_{X_L^m}\).

The following result is the well known Galerkin approximation formulas, it is proved in [15].

**Theorem 3.2.** For any \(B(t) \in C([0, 1], L^2_s(\mathbb{R}^{2n}))\) with its the L-index pair \((i_L(B), v_L(B))\) and any constant \(0 < d \leq \frac{1}{4} \|(A - B)\|^2\)\(^{-1}\), there exists \(m_0 > 0\) such that for \(m \geq m_0\), we
We need to truncate the function $\hat{H}$ at infinite. That is to replace $\hat{H}$ by a modified function $\hat{H}_K$ which grows at a prescribed rate near $\infty$. The truncated function was defined by P. Rabinowitz in [25]. Let $K > 0$ and select $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ for $s \leq K$, $\chi(s) = 0$ for $s \geq K + 1$, and $\chi'(s) < 0$ for $s \in (K, K + 1)$. Set

$$
\hat{H}_K(t, z) = \chi(|z|)\hat{H}(t, z) + (1 - \chi(|z|))r_K|z|^4,
$$

where $r_K = \max \left\{ \frac{\hat{H}(t, z)}{|z|^4} | K \leq |z| \leq K + 1, t \in [0, j] \right\}$. It is known that $\hat{H}_K$ still satisfies (H4)-(H6) with $\theta$ being replaced by $\hat{\theta} = \max\{\theta, 1/4\}$, and $|\nabla \hat{H}_K(t, z)| \leq (z, \nabla \hat{H}_K(t, z)) + b$, where $b > 0$ is a constant.

Define a functional $\varphi$ on $X$ by

$$
\varphi(z) = \frac{1}{2} \langle Az, z \rangle - \int_0^j \hat{H}_K(t, z(t))dt
$$

$$
= \frac{1}{2} \langle Az, z \rangle - \frac{1}{2} \langle \hat{B}z, z \rangle - \int_0^j \hat{H}_K(t, z(t))dt.
$$

Suppose $W$ is a real Banach space, $g \in C^1(W, \mathbb{R})$. $g$ is said satisfying the (PS) condition, if for any sequence $\{x_q\} \subset W$ satisfying $g(x_q)$ is bounded and $g'(x_q) \to 0$ as $q \to \infty$, there exists a convergent subsequence $\{x_{q_h}\}$ of $\{x_q\}$ (see [25]). Let $\varphi_m = \varphi|X_m$ be the restriction of $\varphi$ on $X_m$. Similar to Proposition A of [2], we have the following two lemmas.

**Lemma 3.3.** For all $m \in \mathbb{N}$, $\varphi_m$ satisfies the (PS) condition on $X_m$.

**Lemma 3.4.** $\varphi$ satisfies the $(PS)^*$ condition on $X$ with respect to $\{z_m\}$, i.e., for any sequence $\{z_m\} \subset X$ satisfying $z_m \in X_m$, $\varphi(z_m)$ is bounded and $\|\varphi'_m(z_m)\|_{(X_m)'} \to 0$ in $(X_m)'$ as $m \to +\infty$, where $(X_m)'$ is the dual space of $X_m$, there exists a convergent subsequence $\{z_{m_h}\}$ of $\{z_m\}$ in $X$.

In order to prove Theorem 3.1, we need the following definition and the saddle-point theorem.
Definition 3.5. (see [10]) Let $E$ be a $C^2$-Riemannian manifold and $D$ be a closed subset of $E$. A family $\phi(\alpha)$ of subsets of $E$ is said to be a homological family of dimensional $q$ with boundary $D$ if for some nontrivial class $\alpha \in H_q(E, D)$. The family $\phi(\alpha)$ is defined by

$$\phi(\alpha) = \{ G \subset E : \alpha \text{ is in the image of } i_* : H_q(G, D) \rightarrow H_q(E, D) \},$$

where $i_*$ is the homomorphism induced by the immersion $i : G \rightarrow E$.

Theorem 3.6. (see [10]) For above $E$, $D$ and $\alpha$, let $\phi(\alpha)$ be a homological family of dimension $q$ with boundary $D$. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies the (PS) condition. Define

$$c = \inf_{G \in \phi(\alpha)} \sup_{x \in G} f(x).$$

Suppose that $\sup_{x \in D} f(x) < c$ and $f'$ is Fredholm on

$$\mathcal{K}_c(f) \equiv \{ x \in E : f'(x) = 0, f(x) = c \}.$$

Then there exists an $x \in \mathcal{K}_c(f)$ such that the Morse index $m^-(x)$ and the nullity $m^0(x)$ of the functional $f$ at $x$ satisfy

$$m^-(x) \leq q \leq m^-(x) + m^0(x).$$

It is clear that a critical point of $\varphi$ is a solution of (3.1). For a critical point $z = z(t)$, we define the linearized systems at $z(t)$ by

$$\dot{y}(t) = JH''(t, z(t))y(t).$$

Let $B(t) = H''(t, z(t))$. Then the $L_0$-index pair of $z$ is defined by $(i_{L_0}(z), \nu_{L_0}(z)) = (i_{L_0}(B), \nu_{L_0}(B))$.

Proof of Theorem 3.1. We follow the ideas of [14] to prove Theorem 3.1. We carry out the proof in 3 steps.

Step 1 The critical points of $\varphi_m$.

Set $S_m = X^-_m \oplus X^0$. Then $\dim S_m = mn + \dim X^0 = mn + \dim \ker A = mn + n$, $\dim X^+_m = mn$.

In the following, we prove that $\varphi_m(z)$ satisfies:

(I) $\varphi_m(z) \geq \beta > 0$, $\forall z \in Y_m = X^+_m \cap \partial B_\rho(0)$,
(II) \( \varphi_m(z) \leq 0 < \beta, \forall z \in \partial Q_m, \) where \( Q_m = \{re|r \in [0, r_1]\} \oplus (B_{r_2}(0) \cap S_m) \), \( e \in X_m^+ \cap \partial B_1(0), r_1 > \rho, r_2 > 0. \)

First we prove (I). By (H5), for any \( \varepsilon > 0, \) there is a \( \delta > 0 \) such that \( \hat{H}_K(t, z) \leq \varepsilon |z|^2 \) if \( |z| \leq \delta. \) Since \( \hat{H}_K(t, z)|z|^{-4} \) is uniformly bounded as \( |z| \to +\infty, \) there is an \( M_1 = M_1(K) \) such that \( \hat{H}_K(t, z) \leq M_1|z|^4 \) for \( |z| \geq \delta. \) Hence

\[
\hat{H}_K(t, z) \leq \varepsilon |z|^2 + M_1|z|^4, \quad \forall z \in \mathbb{R}^n.
\]

For \( z \in Y_m, \) we have

\[
\int_0^j \hat{H}_K(t, z) dt \leq \varepsilon \|z\|_2^2 + M_1 \|z\|_4^4 \leq (\varepsilon c_2^2 + M_1 c_4^4 \|z\|^2) \|z\|^2. \tag{3.4}
\]

By (3.2) and (3.4)

\[
\varphi_m(z) = \frac{1}{2} \langle Az, z \rangle - \frac{1}{2} \langle \hat{B}z, z \rangle - \int_0^j \hat{H}_K(t, z(t)) dt \\
\geq \frac{\pi}{2j} \|z\|^2 - \frac{\beta_0}{2} \|z\|^2 - (\varepsilon c_2^2 + M_1 c_4^4 \|z\|^2) \|z\|^2 \\
= \frac{\pi}{2j} \rho^2 - \frac{\beta_0}{2} \rho^2 - (\varepsilon c_2^2 + M_1 c_4^4 \rho^2) \rho^2.
\]

Since \( 1 \leq j < \pi/\beta_0, \) we can choose constants \( \rho = \rho(K) > 0 \) and \( \beta = \beta(K) > 0, \) which are sufficiently small and independent of \( m, \) such that for \( z \in Y_m, \)

\[
\varphi_m(z) \geq \beta > 0.
\]

Hence (I) holds.

Next prove (II). Let \( e \in X_m^+ \cap \partial B_1 \) and \( z = z^- + z^0 \in S_m. \) By (3.2) and (3.3), there holds

\[
\varphi_m(z + re) = \frac{1}{2} \langle Az^-, z^- \rangle + \frac{1}{2} r^2 \langle Ae, e \rangle - \langle \hat{B}(z + re), z + re \rangle - \int_0^j \hat{H}_K(t, z + re) dt \\
\leq -\frac{\pi}{2j} \|z^-\|^2 + \frac{\pi}{2j} r^2 - \int_0^j \hat{H}_K(t, z + re) dt, \tag{3.5}
\]

If \( r = 0, \) by (H4), we see that

\[
\varphi_m(z + re) \leq -\frac{\pi}{2j} \|z^-\|^2 \leq 0. \tag{3.6}
\]

If \( r = r_1, \) or \( \|z\| = r_2, \) then from (H6), We have

\[
\hat{H}_K(t, z) \geq b_1|z|^\frac{j}{2} - b_2, \tag{3.7}
\]
where \( b_1 > 0, b_2 \) are two constants independent of \( K \) and \( m \). Then by \((3.7)\),

\[
\int_0^j \dot{H}_K(t, z + re) dt \geq b_1 \int_0^j |z + re|^\frac{1}{2} dt - jb_2
\]

\[
\geq b_3 \left( \int_0^j |z + re|^2 dt \right)^\frac{1}{2b} - b_4
\]

\[
\geq b_5 \left( \|z^0\|^\frac{1}{b} + r^\frac{1}{b} \right) - b_4,
\]

(3.8)

where \( b_3, b_4 \) are constants and \( b_5 > 0 \) independent of \( K \) and \( m \). Thus by \((3.8), (3.5)\) is

\[
\varphi_m(z + re) \leq -\frac{\pi}{2j} \|z\|^2 + \frac{\pi}{2j} r^2 - b_5 \left( \|z^0\|^\frac{1}{b} + r^\frac{1}{b} \right) + b_4,
\]

(3.9)

Thus we can choose large enough \( r_1 \) and \( r_2 \) independent of \( K \) and \( m \) such that

\[
\varphi_m(z + re) \leq 0, \quad \text{on } \partial Q_m.
\]

Then (II) holds.

Because \( Q_m \) is deformation retract of \( X_m \), then \( H_q(Q_m, \partial Q_m) \cong H_q(X_m, \partial Q_m) \), where \( q = \dim S_m + 1 = mn + n + 1 = \dim Q_m \), and \( \partial Q_m \) is the boundary of \( Q_m \) in \( S_m \oplus \{ \mathbb{R} \} \).

But \( H_q(Q_m, \partial Q_m) \cong H_{q-1}(S^{q-1}) \cong \mathbb{R} \). Denote by \( i : Q_m \to X_m \) the inclusion map. Let \( \alpha = [Q_m] \in H_q(Q_m, D) \) be a generator. Then \( i_* \alpha \) is nontrivial in \( H_q(X_m, \partial Q_m) \), and \( \phi(i_* \alpha) \) defined by Definition \( 3.5 \) is a homological family of dimension \( q \) with boundary \( D := \partial Q_m \) and \( Q_m \) is deformation retract of \( \partial Q_m \) of \( D \). By Lemma \( 3.3 \) \( \varphi_m \) satisfies the (PS) condition. Define \( c_m = \inf_{G \in \phi(i_* \alpha)} \sup_{z \in G} \varphi_m(z) \). We have

\[
\sup_{z \in \partial Q_m} \varphi_m(z) \leq 0 < \beta \leq c_m \leq \sup_{z \in Q_m} \varphi_m(z) \leq \frac{\pi}{2j} r_1^2.
\]

(3.10)

Since \( X_m \) is finite dimensional, \( \varphi'_m \) is Fredholm. By Theorem \( 3.6 \) \( \varphi_m \) has a critical point \( z^m_j \) with critical value \( c_m \), and the Morse index \( m^-(z^m_j) \) and nullity \( m^0(z^m_j) \) of \( z^m_j \) satisfy

\[
m^-(z^m_j) \leq mn + n + 1 \leq m^-(z^m_j) + m^0(z^m_j).
\]

(3.11)

Since \( \{c_m\} \) is bounded, passing to a subsequence, suppose \( c_m \to c \in [\beta, \frac{\pi}{2j} r_1^2] \). By the (PS)* condition of Lemma \( 3.4 \) passing to a subsequence, there exists an \( z_j \in X \) such that

\[
z^m_j \to z_j, \; \varphi(z_j) = c, \; \varphi'(z_j) = 0.
\]

Step 2 The solution of \((3.1)\).
Because the critical value \( c \) has an upper bound \( \frac{\pi^2}{2j} \) independent of \( K \), then
\[
\frac{\pi^2}{2j} \geq c = \varphi(z_j) - \frac{1}{2} \langle \varphi'(z_j), z_j \rangle 
\geq \left( \frac{1}{2} - \hat{\theta} \right) \int_0^j (z_j, \nabla \hat{H}_K(t, z_j)) dt.
\] (3.12)

Then by (3.12), \( \int_0^j (z_j, \nabla \hat{H}_K(t, z_j)) dt \) has an upper bound independent of \( K \),
\[
\int_0^j (z_j, \nabla \hat{H}_K(t, z_j)) dt \leq \hat{M}, \quad \text{for constant } \hat{M} \text{ independent of } K.
\] (3.13)

By \( \hat{H}_K(t, z_j) \leq \hat{\theta}(z_j, \nabla \hat{H}_K(t, z_j)) \), then by (3.13),
\[
\int_0^j \hat{H}_K(t, z_j) dt \leq \hat{\theta} \hat{M}.
\] (3.14)

Thus by (3.7) and (3.14),
\[
\int_0^j \left(b_1 |z_j|^{\frac{1}{2}} - b_2 \right) dt \leq \int_0^j \hat{H}_K(t, z_j) dt \leq \hat{\theta} \hat{M},
\]
i.e.,
\[
\hat{\theta} \hat{M} \geq b_1 \int_0^j |z_j|^{\frac{1}{2}} dt - b_2 j \geq b_1 \left( \int_0^j |z_j|^2 dt \right)^{\frac{1}{2}} - b_2 j.
\] (3.15)

Thus by (3.15), \( \|z_j\|_{L^2} \leq M_2 \), where \( M_2 \) is independent of \( K \), i.e.,
\[
\|z_j\|_{L^2} \leq M_3 \text{, where } M_3 \text{ is independent of } K.
\] (3.16)

Since
\[
\|z_j\|_{L^1} \leq C \|z_j\|_{L^2} \leq M_3^\prime, \quad \text{where } C > 0 \text{ is independent of } K.
\] (3.17)

Thus by (3.16) and (3.17), \( \|z_j\|_{L^1} \) has an upper bound independent of \( K \). We use Young’s inequality. For any \( w \in W^{1,2}([0, j], \mathbb{R}^{2n}) \), \( w(\tau) - w(t) = \int_t^\tau \dot{w}(s) ds \). Integrating with respect to \( t \) shows that
\[
\dot{w}(\tau) - \int_0^j w(t) dt = \int_0^j \int_t^\tau \dot{w}(s) ds dt,
\]
i.e.,
\[
|\dot{w}(\tau)| = \left| \int_0^j w(t) dt + \int_0^j \int_t^\tau \dot{w}(s) ds dt \right|
\leq \int_0^j |w(t)| dt + \int_0^j \int_t^\tau |\dot{w}(s)| ds dt
\leq \|w\|_{L^1} + j \int_0^j |\dot{w}(s)| ds
= \|w\|_{L^1} + j \|\dot{w}\|_{L^1},
\]
i.e.,

\[ |w(\tau)| \leq \frac{\|w\|_{L^1}}{J} + \|\dot{w}\|_{L^1}, \]

i.e.,

\[ \|w\|_{L^\infty} \leq \frac{\|w\|_{L^1}}{J} + \|\dot{w}\|_{L^1}. \] (3.18)

Therefore

\[
\|\dot{z}_j\|_{L^1} = \|J\dot{B}(t)z_j + J\nabla \hat{H}_K(t, z_j)\|_{L^1} \\
\leq \beta_0\|z_j\|_{L^1} + \int_0^j |\nabla \hat{H}_K(t, z_j)| dt \\
\leq \beta_0 M'_3 + \int_0^j (z_j, \nabla \hat{H}_K(t, z_j)) dt + bj \\
\leq \beta_0 M'_3 + MJ + bj \leq M_4. \tag{3.19}
\]

Thus \(\|\dot{z}_j\|_{L^1}\) has an upper bound independent of \(K\). Then from (3.17), (3.18) and (3.19), \(\|z_j\|_{L^\infty} \leq K_0\), where \(K_0\) is independent of \(K\). We choose \(K > K_0\), therefore \(\hat{H}_K(t, z_j) = \hat{H}(t, z_j)\). Consequently, \(z_j\) is a nontrivial solution of (3.1). Then by Lemma 1.3, we get a nonconstant brake solution \(\tilde{z}_j\) of the Hamiltonian system (1.1).

**Step 3** Let \(B(t) = H''_K(t, z_j(t))\), \(d = \frac{1}{4}\|A - B\|^2\|^{-1}\). Since

\[\|\varphi''(x) - (A - B)\| \to 0 \quad \text{as} \quad \|x - z_j\| \to 0,\]

there exists a \(r_3 > 0\) such that

\[\|\varphi''(x) - (A - B)\| < \frac{1}{4}d, \quad \forall x \in V_{r_3}(z_j) = \{x \in X \mid \|x - z_j\| \leq r_3\}.\]

Then for \(m\) large enough, there holds

\[\|\varphi''_m(x) - P_m(A - B)P_m\| < \frac{1}{2}d, \quad \forall x \in V_{r_3}(z_j) \cap X_m. \tag{3.20}\]

For \(x \in V_{r_3}(z_j) \cap X_m\), \(\forall u \in M_d^-(P_m(A - B)P_m) \setminus \{0\}\), from (3.20) we have

\[
\langle \varphi''_m(x)u, u \rangle \leq \langle P_m(A - B)P_mu, u \rangle + \|\varphi''_m(x) - P_m(A - B)P_m\| \cdot \|u\|^2 \\
\leq -\frac{1}{2}d\|u\|^2 < 0. \tag{3.21}\]

Thus by (3.21),

\[\dim M^-(\varphi''_m(x)) \geq \dim M_d^-(P_m(A - B)P_m), \quad \forall x \in V_{r_3}(z_j) \cap X_m. \tag{3.22}\]
Similarly, we have
\[ \dim M^+ (\varphi_m''(x)) \geq \dim M_d^+ (P_m(A - B)P_m), \quad \forall x \in V_{r_3}(z_j) \cap X_m. \]  
(3.23)

By (3.11), (3.22), (3.23) and Theorem 3.2 for large \( m \) we have
\[ mn + n + 1 \geq m^- (z_j^m) \]
\[ \geq \dim M_d^-(P_m(A - B)P_m) \]
\[ = mn + i_{L_0}(B) + n. \]  
(3.24)

We also have
\[ mn + n + 1 \leq m^- (z_j^m) + m^0 (z_j^m) \]
\[ \leq \dim M_d^- (P_m(A - B)P_m) \oplus \dim M_d^0 (P_m(A - B)P_m) \]
\[ = mn + i_{L_0}(B) + n + \nu_{L_0}(B). \]  
(3.25)

Combining (3.24) and (3.25), we have
\[ i_{L_0}(z_j) \leq 1 \leq i_{L_0}(z_j) + \nu_{L_0}(z_j). \]  
(3.26)

The proof of Theorem 3.1 is complete. \( \square \)

It is the time to give the proof of Theorem 1.1 and 1.2.

**Proof of Theorem 1.1.** For \( 1 \leq k < \pi/\beta_0 \), by Theorem 3.1, we obtain that there is a nontrivial solution \((z_k, k)\) of the Hamiltonian systems (3.1) and its \( L_0 \)-index pair satisfies
\[ i_{L_0}(z_k, k) \leq 1 \leq i_{L_0}(z_k, k) + \nu_{L_0}(z_k, k). \]  
(3.26)

Then by Lemma 1.3 \((\tilde{z}_k, 2k)\) is a nonconstant brake solution of (1.1).

For \( k \in 2\mathbb{N} - 1 \), we suppose that \((\tilde{z}_1, 2)\) and \((\tilde{z}_k, 2k)\) are not distinct. By (3.26), Theorem 2.8 and Theorem 2.9 we have
\[ 1 \geq i_{L_0}(z_k, k) \geq i_{L_0}(z_1, 1) + \frac{k - 1}{2} (i_1(\tilde{z}_1, 2) + \nu_1(\tilde{z}_1, 2) - n) \]
\[ \geq i_{L_0}(z_1, 1) + \frac{k - 1}{2} (i_{L_1}(z_1, 1) + i_{L_1}(z_1, 1) + n + \nu_{L_0}(z_1, 1) + \nu_{L_1}(z_1, 1) - n) \]
\[ = i_{L_0}(z_1, 1) + \frac{k - 1}{2} (i_{L_0}(z_1, 1) + i_{L_1}(z_1, 1) + \nu_{L_0}(z_1, 1) + \nu_{L_1}(z_1, 1)), \]  
(3.27)

where \( L_1 = \mathbb{R}^n \oplus \{0\} \in \Lambda(n) \). By (H3), (H7) and Theorem 2.6, we have \( i_{L_1}(z_1, 1) \geq 0 \). We also know that \( \nu_{L_1}(z_1, 1) \geq 0 \) and \( i_{L_0}(z_1, 1) + \nu_{L_0}(z_1, 1) \geq 1 \). Then (3.27) is
\[ 1 \geq i_{L_0}(z_1, 1) + \frac{k - 1}{2}. \]  
(3.28)
By $0 \leq i_{L_0}(z_1,1) \leq 1$, from (3.28) we have $\frac{k-1}{2} \leq 1$, i.e., $k \leq 3$. It is contradict to $k \geq 5$. Similarly, we have that for each $k \in 2\mathbb{N} - 1$, $k \geq 5$ and $kj < \frac{\pi}{\beta_0}$, $1 \leq j < \frac{\pi}{\beta_0}$, $(\tilde{z}_j, 2j)$ and $(\tilde{z}_{kj}, 2kj)$ are distinct brake solutions of (1.1). Furthermore, $(\tilde{z}_1, 2)$, $(\tilde{z}_k, 2k)$, $(\tilde{z}_{k^2}, 2k^2)$, $(\tilde{z}_{k^3}, 2k^3)$, $\cdots$, $(\tilde{z}_{kp}, 2kp)$ are pairwise distinct brake solutions of (1.1), where $k \in 2\mathbb{N} - 1$, $k \geq 5$ and $1 \leq k^p < \frac{\pi}{\beta_0}$ with $p \in \mathbb{N}$.

For $k \in 2\mathbb{N}$, as above, we suppose that $(\tilde{z}_1, 2)$ and $(\tilde{z}_k, 2k)$ are not distinct. By (3.26), Theorem 2.8 and Theorem 2.9, we have

$$1 \geq i_{L_0}(z_k, k) \geq i_{L_0}(z_1, 1) + i_{L_0}^{L_0}(z_1, 1) + \left(\frac{k}{2} - 1\right)(i_1(\tilde{z}_1, 2) + \nu_1(\tilde{z}_1, 2) - n)$$

$$\geq i_{L_0}(z_1, 1) + i_{L_0}^{L_0}(z_1, 1) + \left(\frac{k}{2} - 1\right)(i_{L_0}(z_1, 1) + i_{L_1}(z_1, 1) + n + \nu_{L_0}(z_1, 1) - n)$$

$$= i_{L_0}(z_1, 1) + i_{L_0}^{L_0}(z_1, 1) + \left(\frac{k}{2} - 1\right)(i_{L_0}(z_1, 1) + i_{L_1}(z_1, 1) + \nu_{L_0}(z_1, 1) + \nu_{L_1}(z_1, 1)).$$

Similarly, we also know that $i_{L_1}(z_1, 1) \geq 0$, $\nu_{L_1}(z_1, 1) \geq 0$, $i_{L_0}(z_1, 1) + \nu_{L_0}(z_1, 1) \geq 1$. By Remark 2.10 we have $i_{L_0}^{L_0}(z_1, 1) \geq i_{L_0}(z_1, 1) \geq 0$. Then (3.29) is

$$1 \geq i_{L_0}(z_1, 1) + \left(\frac{k}{2} - 1\right).$$

By $0 \leq i_{L_0}(z_1, 1) \leq 1$, from (3.30) we have $\frac{k}{2} - 1 \leq 1$, i.e., $k \leq 4$. It contradicts to $k \geq 5$. Similarly, we have that for each $k \in 2\mathbb{N}$, $k \geq 6$ and $kj < \frac{\pi}{\beta_0}$, $1 \leq j < \frac{\pi}{\beta_0}$, $(\tilde{z}_j, 2j)$ and $(\tilde{z}_{kj}, 2kj)$ are distinct brake solutions of (1.1). Furthermore, $(\tilde{z}_1, 2)$, $(\tilde{z}_k, 2k)$, $(\tilde{z}_{k^2}, 2k^2)$, $(\tilde{z}_{k^3}, 2k^3)$, $\cdots$, $(\tilde{z}_{kp}, 2kp)$ are pairwise distinct brake solutions of (1.1), where $k \in 2\mathbb{N}$, $k \geq 6$ and $1 \leq k^p < \frac{\pi}{\beta_0}$ with $p \in \mathbb{N}$.

In all, for any integer $1 \leq j < \frac{\pi}{\beta_0}$, $\tilde{z}_j$ and $\tilde{z}_{kj}$ are distinct brake solutions of (1.1) for $k \geq 5$ and $kj < \frac{\pi}{\beta_0}$. Furthermore, $\{\tilde{z}_{kp} | p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of (1.1) for $k \geq 5$ and $1 \leq k^p < \frac{\pi}{\beta_0}$. The proof of Theorem 1.1 is complete.

We note that Theorem 1.2 is a direct consequence of Theorem 1.1.

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