Zero Modes and Conformal Anomaly in Liouville Vortices

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The partition function of a two dimensional Abelian gauge model reproducing magnetic vortices is discussed in the harmonic approximation. Classical solutions exhibit conformal invariance, that is broken by statistical fluctuations, apart from an exceptional case. The corresponding “anomaly” has been evaluated. Zero modes of the thermal fluctuation operator have been carefully discussed.

I. INTRODUCTION

In the recent years, a great interest has been devoted to the study of magnetic vortices, which are intimately connected to the study of superconductivity. A magnetic vortex can be represented as an infinitely long magnetic flux tube in three spatial dimensions or, equivalently, as a two dimensional localised magnetic field source. The first vortex solutions were discussed in 1957 by Abrikosov in the framework of the Ginzburg-Landau model, whereas Nielsen and Olesen in 1973 were the first to recognize that the same type of vortices are also present in a classical relativistic model, i.e. the Abelian Higgs model. Several years later Hong, Kim and Pac and, independently, Jackiw and Weinberg, discussed vortex solutions in the same (planar) model, but with dynamics for the gauge fields governed by a Chern–Simons (CS) term. In a CS system, the magnetic field is proportional to the charge density, so that any excitation carrying magnetic flux is necessarily charged, contrary to the vortices in the Abelian Higgs model, that are electrically neutral.

Unfortunately, in both cases the classical solutions are known either asymptotically or by series with complicated recursion formulas. Consequently, quantization or thermal fluctuations of the system around the classical background are very difficult to study.

More recently, a different type of magnetic vortex was proposed as a solution of 2 dimensional euclidean scalar electrodynamics with topological coupling. In such (Liouville) vortices, the magnetic field satisfies the Liouville equation, all of whose regular solutions can be easily expressed in terms of an arbitrary analytic function. In spite of the seeming resemblance of this model with the Abelian Higgs system, Liouville vortices are very different from the ones in the Ginzburg Landau theory. In the Ginzburg Landau case, the potential of the Higgs field leads to a spontaneously symmetry breaking. Due to this fact the classical vortex solutions, besides the expected long range tail, have an exponentially decreasing component, and the coefficient of the quadratic term in the Higgs potential is related to the characteristic length of the exponential behavior. On the contrary, in Liouville vortices the potential of the scalar field is that of a pure $|\phi|^4$ theory, without quadratic-mass term. In addition, there is also a non-minimal...
(topological) interaction that couples the matter density directly to the magnetic field. As a consequence of these two facts, Liouville vortices exhibit conformal invariance, and their asymptotic behavior is always inverse power-like.

The model leading to Liouville vortices shares important properties with two other models: the Jackiw and Pi model \(8\) and the non linear sigma model (NLSM) with \(O(3)\) local symmetry \(9\). Concerning the first, the profile of the magnetic field of these Liouville vortices is identical to that discussed by Jackiw and Pi as the static soliton solution of the gauged non linear Schrödinger equation on the plane, in strong analogy with the Abelian Higgs model, that is a classical field theory whose equations of motion coincides with the non linear Schrödinger equation governing the Ginzburg-Landau theory for type II superconductors. Concerning the NLSM, in ref. \(9\) it was shown that all the solutions of the above mentioned 2 dimensional euclidean scalar electrodynamics can be obtained from the solutions of NLSM with local symmetry. In addition, there are other important features shared with the euclidean NLSM in 2 dimensions \(9\) that will be discussed below.

Statistical mechanics is a natural framework for topologically non trivial euclidean theories and, on the other hand, many important physical features that the model in ref. \(9\) should hopefully exhibit, are strictly related to statistical mechanics. As an example, if these Liouville vortices are really related to superconductivity, a phase transition of the system should occur at some critical temperature.

In euclidean models, with positive defined actions \(S\) and admitting topologically non trivial solutions, the partition function can be defined as the path integral over the configuration space of the Boltzmann factor \(e^{-S}\). In this context, the action plays the role of potential energy and the free energy is usually interpreted as interaction energy due to “thermal” fluctuations. However, in these cases, the definition of temperature in not always straightforward.

For instance, in the NLSM it is common to consider the classical action already implemented by an overall factor \(1/e^2\), i.e.

\[
S = \frac{1}{2e^2} \int d^2x \partial_\mu N^a \partial_\mu N^a, \quad N^a N^a = 1,
\]

and such a factor is eventually interpreted as “thermal bath” \(\beta = 1/e^2\).

As we shall see, in our model it will be possible to rescale the fields in such a way that the action depends on the \(U(1)\) coupling \(e\) only through an overall factor \(1/e^2\), just like in the NLSM above, allowing a temperature-like interpretation of the coupling constant.

The great advantage of such Liouville vortices is that the classical solutions are given by relatively simple expressions, so that the problem of the thermal fluctuations of the classical solutions can be faced. To provide a first insight in this direction is the aim of the present paper.

In Sect. II we shall briefly review the model. The solutions of this model are conformally invariant and can be classified according to their vorticity, which is proportional to the topological invariant of the model and is nothing but the degree of the arbitrary analytic function \(\omega(z)\) upon which the solutions depend.

In Sect. III we begin the study of the thermal fluctuations of the model. After noticing some remarkable similarities with the euclidean NLSM in 2D, we consider the partition function in the so called harmonic approximation: fields are parametrized as classical solutions plus small thermal fluctuations (small coupling regime), and then the action in the Boltzmann factor is replaced by its expansion around the classical solutions up to the second order in the thermal fluctuations. All the zero modes of the corresponding fluctuation operator can be explicitly evaluated, for any choice of the arbitrary analytic function \(\omega(z)\) characterising the background.

If the vorticity \(N\) of the background is different from the minimal one (i.e. \(N \neq 1\)), the conformal symmetry of the classical solutions is broken by thermal fluctuations, and in Sect. IV the corresponding “conformal anomaly” is explicitly calculated.

In Section V we discuss the special case \(N = 1\) and we comment on future developments.

II. THE CLASSICAL MODEL

Let us consider the 2-dimensional euclidean action of a \(|\phi|^4\) complex scalar field minimally coupled to a Maxwell gauge potential with an additional topological coupling

\[
S = \int d^2x \mathcal{L} = \int d^2x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D_\mu \phi + e \phi^* \phi \epsilon_{\mu\nu} F_{\mu\nu} + \frac{e^2}{2} (\phi^* \phi)^2 \right]
\]

where \(D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi\) is the \(U(1)\) covariant derivative and \(e\) is the Abelian coupling constant that in 2D has dimensions 1 in mass units. The topological coupling \(e \phi^* \phi \epsilon_{\mu\nu} F_{\mu\nu}\) is consistent with both \(U(1)\) gauge invariance and \(ISO(2)\) invariance of the action \(9\). Actually, any general coupling of the type \(e f(\phi^* \phi) \epsilon_{ij} F_{ij}\) and any general scalar...
potential \( V(\phi^* \phi) \), \( f \) and \( V \) being arbitrary functionals of the scalar density \( \phi^* \phi \), would be also allowed, due to the fact that in 2D the scalar field \( \phi \) is dimensionless. Such a general case has been considered in ref. \[7\] and, with a suitable choice of the two functionals, it is possible to obtain classical solutions of the generalised version of the action \[2\] satisfying a wide variety of 2 dimensional conformal equations. Here we shall restrict to the most interesting case given by \[2\].

Up to a total derivative, \( S \) can be rewritten as

\[
S = \int d^2x \mathcal{L} = \int d^2x \left[ \left| (D_x \pm i \, D_y) \phi \right|^2 + \frac{1}{2} (B \mp e \phi^* \phi)^2 \right] \geq 0 ,
\]

\( B = F_{12} \) being the magnetic field. As a consequence, \[3\] is extremized by field configurations satisfying the following self-duality (anti self-duality) conditions

\[
(D_\mu \pm i \, \varepsilon_{\mu \nu} D_\nu) \, \phi = 0 ,
B = \pm e \, \phi^* \phi .
\]

It is easy to verify that any configuration satisfying \[4\] also solves the classical Euler-Lagrange equations. Combining together eqs. \[4\], the scalar density \( \phi^* \phi \) satisfies the Liouville equation

\[
\triangle \ln (\phi^* \phi) = -2 \, e^2 (\phi^* \phi) ,
\]

all of whose regular, positive definite solutions are given by

\[
\phi^* \phi = \frac{4}{e^2} \frac{\left| \omega' \right|^2}{\left| 1 + |\omega(z)|^2 \right|^2} .
\]

In eq. \[6\], \( \omega(z) \) is an arbitrary meromorphic function and \( \omega'(z) = d\omega/dz \). Explicit solutions of the self-duality conditions \[4\] in the covariant gauge \( \partial_\mu A_\mu = 0 \) are \[10\]

\[
e \, A_\mu = -\varepsilon_{\mu \nu} \partial_\nu \ln \left[ 1 + |\omega(z)|^2 \right] ,
e \, \phi = \frac{2 \omega'(z)}{1 + |\omega(z)|^2} .
\]

The above field configuration describes a magnetic vortex. As a matter of fact, for any choice of the arbitrary meromorphic function \( \omega(z) \) the magnetic field is always localised. The magnetic flux \( \Phi(B) \) is quantized, as the integral of the magnetic field over the whole space is proportional to the degree of the analytic function \( \omega \), i.e. the number of solutions \( z_i = z_i(\omega) \) of the equation \( \omega = \omega(z) \), each multiplied by the appropriate multiplicity \( b_i \), namely

\[
\Phi(B) = \frac{4\pi \mathcal{N}}{e} , \quad \mathcal{N} = \frac{1}{\pi} \int d^2x \frac{|\omega'(z)|^2}{(1 + |\omega(z)|^2)^2} = \sum_i b_i .
\]

The integer \( \mathcal{N} \) is usually denoted as vorticity, and is the topological invariant associated to the classical configuration. For the self-dual configurations we are considering, \( \mathcal{N} > 0 \). A parity transformation on the solutions maps self-dual into anti self-dual configurations and the vorticity changes sign.

In 2 dimensions the natural flux units are \( \Phi_0 = 2\pi/Q \), \( Q \) being the electric charge. Consequently, if we identify \( e \) with the electric charge \( Q \), the flux is an even multiple of \( \Phi_0 \). Alternatively, eq. \[8\] gives a magnetic flux which is an arbitrary integer \( \mathcal{N} \) in terms of the natural flux units \( \Phi_0 \), provided the Abelian coupling \( e \) is twice the electric charge \( Q \), suggesting the idea that the matter field we are considering should be somehow related to an electron-pair condensate (Cooper’s pair).

In the remaining part of this Section, we shall discuss the classical symmetries of the solutions \[7\], as they will play a crucial role in the thermal fluctuations the model \[7\]. Besides the obvious gauge and Poincaré ISO(2) symmetries, solutions \[7\] possess also conformal invariance. A conformal transformation

\[
z = x + iy \rightarrow \rho(z) = \tilde{x}(x, y) + i \tilde{y}(x, y) ,
\]

where \( \rho \) is an arbitrary analytic function, connects different solutions of the action \[2\]. Under the action of the conformal redefinition \[8\], gauge and matter fields transform as \[9\].
\[ \phi(z, \bar{z}) \rightarrow \tilde{\phi}(z, \bar{z}) = \frac{dp}{dz} \phi(\rho, \bar{\rho}) \]
\[ A_\mu(r) \rightarrow \tilde{A}_\mu(r) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} A_\nu(\tilde{r}) \]

so that the matter density \(\phi^* \phi\), the magnetic field and the self-dual derivatives transform as densities with weight \(J = \det(\partial \tilde{x}^\mu / \partial x^\nu) = |\rho'(z)|^2\), namely

\[
\phi^* \phi \rightarrow \tilde{\phi}^* \phi(\tilde{z}, \bar{\tilde{z}}) = J \phi^* \phi \big|_{(\rho, \bar{\rho})},
\]

\[
[(D_\mu + i\varepsilon_{\mu\nu} D_\nu) \phi](r) \rightarrow \left[ (\tilde{D}_\mu + i\varepsilon_{\mu\nu} \tilde{D}_\nu) \tilde{\phi} \right] (r) = J \left[ (D_\mu + i\varepsilon_{\mu\nu} D_\nu) \phi \right] (\tilde{r}),
\]

\[
B(z, \bar{z}) \rightarrow \tilde{B}(z, \bar{z}) = JB \big|_{(\rho, \bar{\rho})}.
\]

(11)

\[ T_{\mu\nu} = \frac{1}{2} (B + e \phi^* \phi) (B - e \phi^* \phi), \quad T^\pm = 2 (D_1 \phi^* \pm i D_2 \phi^*) (D_1 \phi \pm i D_2 \phi), \]

(12)

and we see that the self-duality conditions (3), that solve the equations of motion, also make vanishing the whole energy momentum tensor, and in particular its trace.

Nonetheless, the action \(S\) is not conformally invariant. At first sight, conformal symmetry of the solutions could seem at odd with the fact that the action explicitly depends on a dimensional parameter (the Abelian coupling \(e\)), and with the fact that the action is not conformally invariant. However, to preserve the stationary points of the action, the Lagrangian density transforms covariantly under conformal transformations,

\[ S = \int d^2x \mathcal{L}(x) \rightarrow \int d^2x J \mathcal{L}(x). \]

(13)

The reason of such a nice transformation property relies in the fact that it is always possible to rescale the fields in such a way that the action \(\Phi\) does not contain any mass scale, apart from an overall multiplying factor, just like in the NLSM \(\Phi\). As a matter of fact, if we rescale the fields, collectively denoted by \(\Phi\), as \(\tilde{\Phi} = e \Phi\), the classical action \(\Phi\) written in terms of \(\tilde{\Phi}\) becomes

\[ S[\Phi; e] = \frac{1}{e^2} S[\tilde{\Phi}; 1], \]

(14)

and, classically, all the dependence on dimensional parameters in the action \(\Phi\) can be ruled out.

### III. Statistical Fluctuations

The partition function of an euclidean system admitting topologically non trivial solutions is defined as the path integral over the configuration space of the Boltzmann factor \(e^{-S[\Phi]}\).

There are some important analogies between the model we have described in the previous Section and the NLSM in eq. \(\Phi\), that are definitely worth mentioning, also in view of the fact that for the NLSM the partition function can be estimated in the small coupling regime \(\Phi\).

In both the models, all the classical solutions are expressed in terms of an arbitrary analytic function \(\omega(z)\), and the topological invariant expressed in terms of \(\omega(z)\) is identical for the two models. In addition, due to eq. \(\Phi\), also in our model all the dependence on the coupling constant can be factorized in an overall factor \(1/e^2\), allowing a temperature-like interpretation of the coupling constant in the evaluation of the partition function, just like in the NLSM.

Finally, the most remarkable analogy: following \(\Phi\), in order to exploit the renormalization group of the system \(\Phi\), we decompose the \(N^a\) variables of the NLSM according to

\[ N^a(x) = (1 - |\phi|^2)^{1/2} N_0^a(x) + \phi^i e_i^a, \]

(15)
where $a = 1, 2, 3$ and $i = 1, 2$. $N^a_0$ is the so-called “slowly varying” vector and $e^a_i$ are orthogonal to it, whereas $\phi^i$ represents the “fast fluctuations”. As a consequence of $N^0_0 e_i^0 = 0$ and $e^a_i e^a_j = \delta_{ij}$, we have

$$
\partial_\mu N^a_0 = B^a_\mu e^a_i,
\partial_\mu e^a_i = -\epsilon_{ij} A^a_\mu + B^a_\mu N^a_0. \quad (16)
$$

The variables $A_\mu$ and $B^a_\mu$ have to be considered as auxiliary variables characterizing the slowly varying fields. Substituting eqs. $(13, 16)$ in eq. $(1)$ and selecting terms up to the second order in $\phi^i$ one gets

$$
S^{(II)} = \frac{1}{\epsilon^2} \int d^2 x \left[ (D_\mu \phi)^* D_\mu \phi - \frac{1}{2} \phi^a \phi^a F_{\mu\nu} \right], \quad (17)
$$

where $\phi = (\phi^1 + i \phi^2)/\sqrt{2}$ and $F_{\mu\nu}$ is the “field strength” of the auxiliary variable $A_\mu$. Notice that eq. $(17)$ is exactly of the same type of the quadratic term in $\phi$ of the action $(3)$; in particular, the same topological coupling is reproduced. However, its coefficient is one half of the one in eq. $(3)$, and therefore eq. $(17)$ is nothing but the first term in the r.h.s. of eq. $(3)$, up to an inessential total derivative. Consequently, extremizing the action $S^{(II)}$ is equivalent to impose the first set of self-duality conditions $(4)$. The remaining term of the action in $(3)$ (and the second set of self-duality conditions $(3)$) provides the “dynamics” for the field $A_\mu$ that, instead, in eq. $(14)$ has to be considered as a background field.

Thus, a first very rough estimate of the partition function of the model $(3)$ could be obtained in the following way. In the path integral, one could integrate only over matter fluctuations $\varphi = \phi - \phi^c$ induced by the first term in eq. $(3)$, keeping the magnetic field fixed and equal to its classical value $B = \phi^c \phi^c$. Then, one immediately gets Polyakov’s results and the partition function turns out to be that of a Coulomb gas in its plasma phase, with Debye screening, and therefore a mass gap.

On the one hand, the fact that the original system is equivalent to a set of massive fermions is definitely a positive result, that confirms previous conjectures $(3)$. On the other hand, a physical interpretation of this picture in terms of vortices is still obscure and, more important, it is not clear to what extent the approximation of considering only matter fluctuations is reliable.

To proceed, one should consider fluctuations of the whole set of self-interacting fields (matter and gauge fields). Clearly, such a problem is much more complicated and an explicit evaluation of the partition function becomes a formidable problem. However, even without an explicit knowledge of the partition function, something can be done and interesting results can be obtained.

### A. Harmonic approximation

In the evaluation of the partition function of our model, the most appropriate approximation is the so called harmonic approximation, where the action in the Boltzmann factor is expanded up to the second order around the classical solutions. This approximation is frequently used also in quantum mechanics (stationary phase approximation) where it leads to the one loop effective potential.

Clearly, in the harmonic approximation, it is assumed that trajectories deviating significantly from the classical solutions have a negligible weight in the path integral. In our case, due to eq. $(14)$, this assumption is certainly true in the small coupling regime. Thus, we shall consider the case $\beta = 1/\epsilon^2 \gg 1$.

Let us decompose the fields $\hat{\Phi} = \{ A_1, A_2, \phi_1, \phi_2 \}$, with $\phi = (1/\sqrt{2}) (\phi_1 + i \phi_2)$, as the sum of the classical solutions $\Phi^c = \{ A^c_1, A^c_2, \phi^c_1, \phi^c_2 \}$ plus “thermal” fluctuations $\eta = \{ a_1, a_2, \varphi_1, \varphi_2 \}$ and let us consider the formal expansion of the action $(3)$ around the classical solutions up to the second order in the field fluctuations. The first two terms of this expansion vanish, due to the Euler-Lagrange equations and to the fact that $(3)$ vanishes on the classical solutions. We have therefore

$$
S \left[ \hat{\Phi} \right] \simeq \frac{1}{2} \int d^2 \mathbf{x} d^2 \mathbf{y} \, \eta_i(\mathbf{x}) \, M_{ij}(\mathbf{x}, \mathbf{y}) \, \eta_j(\mathbf{y}) \quad ,
$$

$$
M_{ij}(\mathbf{x}, \mathbf{y}) = \frac{\delta^2 S}{\delta \Phi_i(\mathbf{x}) \delta \Phi_j(\mathbf{y})} \bigg|_{cl} \quad (18)
$$

Fluctuations $\eta$ are required to be sufficiently regular and normalizable,

$$
\| \eta \|^2 = \int d^2 \mathbf{x} \left[ a_i a_i + \varphi_i \varphi_i \right] < \infty. \quad (19)
$$
Applying eq. (18) to our model, after straightforward calculations, we get

\[
S \left[ \Phi \right] \simeq \frac{1}{2} \int d^2 x \, d^2 y \, \eta_i (x) \, \delta^2 (x - y) \, M_{ij} (y) \, \eta_j (y) \\
= \frac{1}{2} \int d^2 y \, \eta_i (y) \, M_{ij} (y) \, \eta_j (y) \\
= \frac{1}{2} \int d^2 y \, \{ \, a_\mu \, [ \, ( - \Delta + e^2 (\phi_i^2 + \phi_j^2) \, ) \, \delta_{\mu \nu} + \partial_\mu \partial_\nu \, ] \, a_\nu + \\
+ \phi_1 \, \frac{1}{2} \, \left[ \, 2 \, \phi_1 \, \partial_\mu \partial_\nu \phi_1 - \phi_2 \, \partial_\mu \partial_\nu \phi_2 + 2 \, e^2 A_\mu \phi_1 - 2 \, e^2 A_\nu \phi_2 \, \right] \, \phi_1 + \\
+ \phi_2 \, \frac{1}{2} \, \left[ \, 2 \, \phi_2 \, \partial_\mu \partial_\nu \phi_2 - \phi_1 \, \partial_\mu \partial_\nu \phi_1 + 2 \, e^2 A_\mu \phi_2 - 2 \, e^2 A_\nu \phi_1 \, \right] \, \phi_2 + \\
\left[ \, - \Delta + \frac{1}{2} \, e^2 (\phi_1^2 - \phi_2^2) + e^2 A^2 \, \right] \, \phi_1 + \phi_2 \left[ \, - \Delta - \frac{1}{2} \, e^2 (\phi_1^2 - \phi_2^2) + e^2 A^2 \, \right] \, \phi_2 \} ;
\]

where \( A^2 = A_\mu A_\mu \) and we omitted, for brevity, the superscript \( \text{cl} \) on the classical fields in the square brackets. In order to factorize the \( \delta^2 (x - y) \) term in the first equality of (20), some integrations by parts have been performed and, in so doing, the fluctuation operator \( M_{ij} \) does not look manifestly self-adjoint. This is not a problem, as the normalizability condition (19) always permits to write \( M_{ij} \) as in (20).

Evidently, in the harmonic approximation the evaluation of the partition function is equivalent to the calculation of the determinant of \( M_{ij} \) and, in turn, the solutions of the eigenvalue problem

\[
M_{ij} (y) \, \eta_{ij}^{(n)} (y) = \lambda_n \, \eta_{ij}^{(n)} (y)
\]

would completely solve the problem.

Clearly, this approach is practically impossible, due to the complicated form of the operator \( M_{ij} \). Nonetheless, even without the explicit knowledge of the determinant, quite often it is possible to calculate expectation values. This is the case of the conformal anomaly, that we shall evaluate in Sect. V with the help of the zeta-function regularization technique.

B. Zero modes of the thermal fluctuation operator

From eq. (33), \( S \) is manifestly positive definite and, on the classical solutions, (33) achieves its vanishing minima. Consequently, \( M_{ij} \) has no negative eigenvalues, and the lowest eigenvalues are the zero modes. In our formulation, there are many operators \( M_{ij} \), depending on the choice of the classical background that, in turn, is completely specified by the choice of the arbitrary meromorphic function \( \omega (z) \). For a given choice of \( \omega \) one can associate a topological invariant \( \mathcal{N} \) (vorticity) to the classical solutions. The topological invariant classifies classical vortex solutions into distinct inequivalent classes. Obviously, in each class there exist infinite analytic functions leading to the same vorticity.

Zero modes will strongly depend on the specific choice of the arbitrary analytic function \( \omega \) characterizing the classical background. Nevertheless, the number of normalizable zero modes of the operator \( M_{ij} \) will depend only on the homotopy class to which the function \( \omega \) belongs.

Zero modes satisfy a first order equation which is simpler than eq. (21) with \( \lambda_i = 0 \), and therefore is worth mentioning. Such an equation is a consequence of the peculiar form (33) of the action. Due to the self-duality conditions (34), the action vanishes when evaluated on the classical background (33). On the other hand, from eq. (34) the action is written as the sum of squares, so that the requirement that the action vanishes up to the second order in the “thermal” expansion \( \Phi = \Phi^\text{cl} + \eta \) (zero-modes), is equivalent to the requirement that zero modes \( \eta \) solve the self-duality conditions (34) expanded up to the first order, i.e.

\[
(D^\mu_{\mu} + i \, e_{\mu \nu} D^\nu_{\nu} ) \, \phi = ie \, (a_\mu + i \, e_{\mu \nu} a_\nu ) \, \phi^\text{cl} \\
b = e \, (\phi^* \phi^\text{cl} + \phi^\text{cl} \phi^*)
\]

(22)
where $D^\mu_{cl} = \partial_\mu - ieA^\mu_{cl}$, $b = \varepsilon_{\mu\nu}\partial_\mu a_\nu$. It can be verified by direct inspection that eqs. (22) are indeed equivalent to eq. (21) with $\lambda_i = 0$. Moreover, eqs. (22) can be decoupled and rewritten as a Schrödinger type problem. Taking eqs. (4) and (5) into account, it is not difficult to check that the magnetic field $b$ of the zero modes has to satisfy the equation

$$\Delta \left( \frac{b}{B^{cl}} \right) = -2eb$$

that, written in terms of $b/B^{cl} = \psi$, becomes the zero-energy Schrödinger equation for a unit mass wavefunction $\psi$ moving in the classical potential $V = -eB^{cl}$, i.e.

$$-\frac{1}{2}\Delta \psi + V\psi = 0, \quad V = -eB^{cl}.$$  

Consequently, by finding the zero modes of our system one gets, as a by-product, the zero energy solutions of the above Schrödinger equation.

Zero modes of the fluctuation operator $M_{ij}$ are associated to the continuous symmetries of the classical solutions. Such symmetries are: translations, rotations, gauge transformations, conformal transformations and variations of the arbitrary parameters upon which the classical solutions may depend. Practically, once a gauge choice has been picked, all such symmetries are included in conformal transformations.

It can be shown that, starting from a classical solution $\Phi^{cl}$, the zero mode of the operator $M_{ij}$ associated to the continuous symmetry $\Sigma$ with infinitesimal parameter $\sigma$ is given by the variation of $\Phi^{cl}$ under the action of the continuous symmetry $\Sigma$, i.e. if $\Phi^{cl} \to \Phi^{cl} + \delta_\Sigma\Phi^{cl}$, then $\eta^\Sigma = \delta_\Sigma\Phi^{cl}/\delta\sigma$ is the zero mode of $M_{ij}$ associated to the symmetry $\Sigma$. The proof can be easily obtained by performing a transformation $\Sigma$ on the Euler-Lagrange equations.

The same procedure can be also generalised to local symmetries, like the gauge and the conformal ones. In this case, zero modes are obtained by performing an infinitesimal transformation on the classical fields: the obtained result is a zero mode also when the function specifying the continuous transformation is no longer infinitesimal. Several examples will be provided below.

C. Explicit evaluation of zero modes

We begin by evaluating the zero mode associated to the gauge symmetry. By performing an (infinitesimal) gauge transformation on the classical solutions we get

$$\eta^G = \begin{pmatrix} \frac{1}{e} \partial_\mu \alpha; & -\alpha \phi_2^{cl}; & \alpha \phi_1^{cl} \end{pmatrix}$$

(25)

It can be verified that (25) is a solution of the coupled equations (22), and therefore a zero mode of $M_{ij}$ for any arbitrary function $\alpha(x)$, not necessarily infinitesimal. However, the simultaneous requirement of normalizability (19) and the gauge condition $\partial_\mu A_\mu = 0$ forces $\alpha$ to be a constant. Introducing for later convenience complex notation $a = a_1 + i a_2$ for the fluctuations of the gauge potential, the only normalizable zero mode associated to residual gauge symmetry is then

$$\eta^G = (\phi, a) \equiv (i\alpha\phi^{cl}, 0).$$

(26)

We now consider the remaining symmetry transformations of the classical solutions. Such symmetries are all included in the conformal one, since translations, rotations and variations of the parameters upon which the classical solutions may depend can be always seen as particular cases of conformal transformations. Actually, even the zero mode (26) arises from a particular conformal transformation.

Since the classical solutions are completely determined in terms of the arbitrary analytic function $\omega(z)$ (see eq. (7)), we can write a general expression for the zero mode associated to a given symmetry $\Sigma$ in terms of $\Delta\omega$ and $\omega$, $\Delta\omega$ being the variation of $\omega$ under the action of $\Sigma$. By direct calculation we obtain the following general form of the zero modes $\eta = (\phi, a)$:

$$e \phi = \frac{2(\Delta\omega)'}{1 + |\omega|^2} - \frac{2\omega'(\omega\Delta\bar{\omega} + \bar{\omega}\Delta\omega)}{(1 + |\omega|^2)^2},$$

$$e \ a = 2i\bar{\partial} \left( \frac{\omega\Delta\bar{\omega} + \bar{\omega}\Delta\omega}{1 + |\omega|^2} \right).$$

(27)
Now it is immediate to check that the zero mode (26) is a particular case of (27), with $\Delta \omega = i \omega$. In turn, such a $\Delta \omega$ can be always obtained through a conformal transformation, so that hereafter the zero mode (27) will be classified among the conformal ones.

The modes (27) are already in the Lorentz gauge ($\partial a + \bar{\partial} a = 0$ in complex notation). Among them, we have to select only the normalizable ones by imposing eq. (19). In turn, normalizability condition (19) can also be written in terms of $\omega$ and $\Delta \omega$. After straightforward calculations, one can see that such a requirement is equivalent to the convergence condition of the two following integrals

$$I_1 = \int d^2x \left| \frac{(\Delta \omega)^{\prime}}{1 + |\omega|^2} \right|^2 < \infty$$

$$I_2 = \int d^2x \left| \frac{\omega(\Delta \omega)^{\prime} - \omega^{\prime} \Delta \omega}{1 + |\omega|^2} \right|^2 < \infty$$

Equations (27) and (28) define the zero modes of the fluctuation operator associated to the continuous symmetry $\Sigma$. It should be noticed that (27) are indeed eigenvectors of the operator $M_{ij}$ with vanishing eigenvalue even without specifying neither the form of $\omega$, nor its variation $\Delta \omega$. On the contrary, the normalizability criterion (28) will depend on the particular choice of arbitrary function $\omega$ as well as on the variation $\Delta \omega$ associated to the zero mode. As a consequence, to continue, we have to provide some specific examples.

Let us fix the topological number of the classical solution to be $N$. Clearly, there are infinite functions $\omega$ with such a degree. Here, we shall consider two limiting examples, in such a way that the evaluation of zero modes in all the other possible choices of $\omega$ will be easily understood as intermediate between these two cases.

The first example is the totally degenerate case, where all the vorticity $N$ is carried by a single vortex that, for convenience, will be located at the origin. Then,

$$\omega(z) = \left( \frac{z}{z_0} \right)^N$$

where $z_0$ is a scale introduced to render $\omega$ dimensionless, as required. The choice (29) corresponds to the radially symmetric classical solutions

$$e A^{cl} = 2iN \left[ 1 + \left( \frac{r_0}{r} \right)^{2N} \right]^{-1} z^{-1}$$

$$e \phi^{cl} = \frac{2N}{r_0^N} \left[ 1 + \left( \frac{r}{r_0} \right)^{2N} \right]^{-1} z^{N-1}$$

$$e B^{cl} = \frac{4N^2}{r^2} \left[ \left( \frac{r}{r_0} \right)^N + \left( \frac{r_0}{r} \right)^{-N} \right]^{2}$$

where $r = |z|$ and $r_0 = |z_0|$. In this totally degenerate case, all the zero modes are easily obtained by performing a conformal transformations of the type $z \rightarrow z + \chi(z)$ on the classical solutions. The corresponding variation on $\omega$ is thus

$$\Delta \omega = \omega^{\prime}(z) \chi(z) = \frac{N}{z_0} \left( \frac{z}{z_0} \right)^{N-1} \chi(z).$$

We have now to investigate on the form of the function $\chi(z)$. Since the classical solutions (30) vanish only at the origin and at infinity, one can easily realize that the only possible singularities of the function $\chi(z)$ can be at the origin and at infinity: a singularity of $\chi$ in any other point $\zeta \neq 0$ would necessarily render the corresponding zero mode singular and not normalizable in $z = \zeta$. On the contrary, singularities of $\chi$ at the origin are admissible, provided that when inserted in (27) and (28) they cancel against the corresponding zeros of the classical solutions and of the integration measure. Thus, in this totally degenerate case, it is not restrictive to consider functions $\chi$ as pure powers of the type $\chi_n = g_n (z/r_0)^n$, with $g_n$ complex coefficients, and count the number of independent functions $\chi_n$ that render the zero modes normalizable. For a given $\chi_n(z)$ and with $\omega(z)$ of the form (29), the zero modes (27) take the form

$$e a = \frac{2iN}{r_0^2} \frac{(r/r_0)^{2N}}{1 + (r/r_0)^{2N}} \left( \frac{\bar{z}}{r_0} \right)^{-1} \chi(z).$$

8
Considering the asymptotic behaviour of the zero modes (32) at the origin and at infinity, one can easily see that eq. (32) define normalizable zero modes as long as 1 − \( \mathcal{N} \leq n \leq 1 \). Thus, there are \( \mathcal{N} + 1 \) admissible values of \( n \). Since the coefficients \( g_n \) are complex, two linearly independent zero modes (32) correspond to each value of \( n \), and eq. (32) with 1 − \( \mathcal{N} \leq n \leq 1 \) defines the 2\( \mathcal{N} \) + 2 zero modes of the totally degenerate case.

Notice that zero modes associated to ISO(2) symmetry and to a variation of the parameter \( z_0 \) in (28) are included in eq. (32) with \( n = 0 \) (translations) and \( n = 1 \) (rotations if \( g_1 \) is purely imaginary, and variation of the parameter \( z_0 \) if \( g_1 \) is purely real). In addition, from eq. (31), the gauge zero mode (26) is also a particular case of eq. (32) with \( n = 1 \).

As a second limiting example we shall consider the totally non degenerate case, i.e. the choice of \( \omega \) that, for a given value \( \mathcal{N} \) of vorticity, depends on as many free parameters as possible. Such an \( \omega(z) \) is (33), for instance,

\[
\omega(z) = \omega_0 \prod_{i=1}^{\mathcal{N}} \frac{(z - a_i)}{(z - b_i)},
\]

with \( a_i \neq a_j, \ b_i \neq b_j \) if \( i \neq j \) and \( a_i \neq b_j \) for any \( i, j \). It depends on \( 4\mathcal{N} + 2 \) real parameters (the constants \( \omega_0, \ {a_i} \) and \( \{b_i\} \) are complex) and it corresponds to a classical \( \mathcal{N} \)-vortex configuration, where each vortex carries the minimum vorticity. The explicit location of the vortices is irrelevant to our purposes, but in general it will be a function of all the parameters \( \{a_i, b_i\} \).

Let us define

\[
\mathcal{A} = \prod_{i=1}^{\mathcal{N}} (z - a_i), \quad \mathcal{B} = \prod_{i=1}^{\mathcal{N}} (z - b_i),
\]

in such a way that \( \omega(z) \) and its variation under a conformal transformation \( z \rightarrow z + \chi(z) \) can be rewritten as

\[
\omega(z) = \omega_0 \frac{\mathcal{A}}{\mathcal{B}}, \quad \Delta \omega = \omega'(z) \chi(z) = \omega_0 \frac{\mathcal{A}}{\mathcal{B}} \sum_{i=1}^{\mathcal{N}} \left( \frac{1}{z - a_i} - \frac{1}{z - b_i} \right) \chi(z).
\]

We have to investigate on the possible form of the arbitrary function \( \chi(z) \) associated to the conformal variation \( \Delta \omega \) that makes the integrals (29) convergent. We first notice that, if \( \chi(z) \) is regular in \( z = a_i \), then also \( (\Delta \omega)' \) is regular in \( a_i \). Thus, integrability of \( I_1 \) and \( I_2 \) in \( z = a_i \) only requires \( \chi(z) \) to be regular at \( z = a_i \). In \( z \sim b_i \), \( (\Delta \omega)' \) behaves like \( \chi(b_i)/(z - b_i)^3 \). On the other hand, the quantity \( (1 + |\omega|^2)^{-2} \) goes to zero for \( z \sim b_i \) like \( |z - b_i|^4 \). Thus, in order to render converge \( I_1 \), \( \chi(z) \) must have at least simple zeros in \( z = b_i \), and therefore it has to be of the type \( \chi(z) = B f(z) \), with \( f \) regular in \( b_i \) and \( a_i \). Finally, integrability of \( I_1 \) at infinity requires \( f(z) \sim 1/z^q \) with \( q \geq \mathcal{N} - 1 \). There is only one possibility to have such a behaviour without introducing extra singularities in \( \Delta \omega \), that is when the poles of \( f(z) \) exactly cancel with the zeros of \( \omega'(z) \). Consequently, \( f(z) \) and \( \chi(z) \) have to be of the form

\[
f_\mathcal{N}(z) = \frac{P_\mathcal{N}(z)}{AB \sum_{i=1}^{\mathcal{N}} \left( \frac{1}{z - a_i} - \frac{1}{z - b_i} \right)}, \quad \chi_\mathcal{N}(z) = B f_\mathcal{N}(z),
\]

where \( P_\mathcal{N}(z) \) is an arbitrary polynomial of degree \( \mathcal{N} \). It is not difficult to verify that for such a choice of \( \chi(z) \) also \( I_2 \) is convergent. Thus, \( \Delta \mathcal{N} \omega = \omega'(z) \chi_\mathcal{N}(z) \), with \( \chi_\mathcal{N}(z) \) defined as in (36), is the most general form of conformal variations defining normalizable zero modes. Since an arbitrary polynomial of degree \( \mathcal{N} \) depends on \( \mathcal{N} + 1 \) arbitrary complex parameters, the number of normalizable zero modes associated to conformal symmetry in the totally non degenerate case is \( 2\mathcal{N} + 2 \), just like in the totally degenerate case.
We conclude this section by observing that all the zero modes in the totally non degenerate case could have also been obtained through infinitesimal variations of the parameters defining $\omega(z)$. Under a variation of the parameters $b_i$, the corresponding variation of $\omega$ is $\Delta_{\omega/b_i} \equiv \delta\omega/\delta b_i = \omega(z)/(z-b_i)$. However, such a variation renders divergent $I_1$ due to a non integrable singularity in $z = b_i$, and infinitesimal variations of $b_i$ do not define normalizable zero modes. Under a variation of the remaining parameters $\omega_0$ and $a_i$ we have

$$
\Delta_{\omega_0} \omega = \delta\omega(z)/\delta\omega_0 = \omega(z)/\omega_0,
\Delta_{a_i} \omega = \delta\omega(z)/\delta a_i = -\omega(z)/(z-a_i).
$$

(37)

Substituting (37) in (38), we have that both $I_1$ and $I_2$ are convergent, and eqs. (37) define $2N+2$ normalizable zero modes. Obviously, these modes are not independent as they are precisely of the form (35), (36). Actually, any linear combination of the zero modes associated to (37) can be used as a basis for the arbitrary functions $\chi_{\mathcal{N}}(z)$ in (36).

Finally, also in this case one can easily check that the zero modes associated to $ISO(2)$ symmetry as well as the gauge zero mode (20) are contained in eqs. (37), (38) (or, equivalently, in eqs. (37)).

Starting from these two limiting examples, it is easy to extract zero modes from any other intermediate choice of the arbitrary function $\omega(z)$. In all the cases, the number of the normalizable zero modes is $2N+2$ and it is thus only a function of the topological sector associated to the classical background.

The thermal correction to the magnetic field due to the zero modes is given by

$$
b = \varepsilon_{\mu\nu} \partial_\mu a_\nu = \frac{4}{e} \partial_\mu \left( \frac{\omega \Delta\omega + \bar{\omega} \Delta\bar{\omega}}{1 + |\omega|^2} \right).
$$

(38)

For instance, in the radially symmetric case, eq. (38) reads

$$
b = \frac{8N^2}{erv_0^{n-2}} \left[ \left( \frac{z}{r_0} \right)^N + \left( \frac{r}{r_0} \right)^N \right] \Re \left[ g_n z^{n-1} \left( n - N - 1 + \frac{2N}{1 + \left( \frac{r}{r_0} \right)^N} \right) \right],
$$

(39)

with $1-N \leq n \leq 1$. It can be shown that the integral over the whole plane of eq. (39) vanishes for any $1-N \leq n \leq 1$, so that zero modes corrections to the magnetic field do not modify the vorticity of the classical solutions, as expected on general ground.

Having the zero modes corrections to the magnetic field, eq. (38) or (39), the solutions of the zero energy sector of the Schroedinger equation (24) straightforwardly follow.

IV. CONFORMAL ANOMALY

Although the overall factor $1/e^2$ in eq. (14) is clearly irrelevant at the classical level, one expects that statistically such a term is no longer inessential, as it introduces an explicit scale in the path integral. In this case the coupling constant will acquire a non trivial dependence on the arbitrary mass scale $\mu$ that any regularization procedure entails [36], and an “anomaly” is expected: the expectation value of the trace of the energy momentum is likely to be non-zero. Here we shall evaluate this expectation value by using the $\zeta$-function regularization technique.

The determinant of the thermal fluctuation operator $M$, defined as the product of its non vanishing eigenvalues $|14\rangle$

\begin{equation}
\text{Det} M = \prod_n \lambda_n[M],
\end{equation}

is divergent due to the unbounded nature of its eigenvalues $\lambda_n$. A popular way to circumvent this problem, is to define the regularised determinant through the $\zeta$ function. The $\zeta$ function appropriate to our problem is defined by

$$
\zeta(s|M) \equiv \sum_n \lambda_n^{-s} |M| = \text{Tr} \left( M^{-s} \right).
$$

The $\zeta$ function is regular for $\text{Re} \ s > d/m$, where $m$ is the order of the differential operator $M$ and $d$ the dimension of the manifold. However, $\zeta$ can be analytically continued to the point $s = 0$ and the regularised determinant is defined as

$$
\left[ \ln \text{Det} M \right]_\zeta \equiv \lim_{s \to 0} \left[ -\frac{d}{ds} \zeta(s|M) \right] = -\zeta'(0|M).
$$

(40)

Let us introduce the kernel
\[ \zeta (t; x, y | M) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \left[ h(t; x, y | M) - P(0)(x, y) \right], \]  
(41)

where \( P(0) \) is the projector into the zero mode space and \( h(t; x, y | M) \) the heat kernel satisfying the differential equation \((\partial/\partial t + M_p)h(t; x, y | M) = 0\) with boundary condition \( h(0; x, y | M) = I_\delta(x-y) \). The heat kernel at \( x = y \) can be expanded in powers of \( t \) through the so-called Seeley - de Witt \([17]\) coefficients

\[ h(t; x, x | M) = \frac{1}{(4\pi t)^{d/m}} \left[ a_0(x | M) + a_1(x | M) t + \ldots \right]. \]  
(42)

Finally, standard manipulations \([18]\) permits to express the value of \( \zeta(s = 0 | M) \) in terms of the Seeley - de Witt coefficient \( a_{d/m} \) and the number \( N \) of zero modes of the operator \( M \) as

\[ \zeta(s = 0 | M) = \frac{1}{(4\pi)^{d/m}} \text{Tr} \int d^d x a_{d/m}(x) - N. \]  
(43)

In our case, \( d/m = 1 \) and the total number of zero modes is \( N = 2N + 2 \), as seen in the previous section.

It can be shown \([19]\) that the integral of the expectation value of the trace of the energy momentum tensor is given by

\[ \int d^2 x \langle T_{\rho \rho} \rangle = -\frac{\delta W}{\delta \log \mu} \]  
(44)

\[ W = -\frac{1}{2} \lim_{x \to 0} \text{Tr} \left[ \frac{d}{ds} \left( \frac{M}{\mu^2} \right)^s \right] = \frac{1}{2} [\zeta(0) + \ln \mu^2 \zeta(0)] \]

where \( \mu \) is the usual mass parameter introduced to render the operator \( (M/\mu^2) \) dimensionless, as required. Consequently, the integrated trace anomaly is just given by \( \zeta(0) \), up to a sign,

\[ \int d^2 z \langle T_{\rho \rho} \rangle = -\zeta(0). \]  
(45)

In turn, from eq. \([18]\), \( \zeta(0) \) can be expressed in terms of the total number of zero modes of the operator \( M \) and in term of the integrated \( a_1 \) Seeley - de Witt coefficient. Thus, we only need to evaluate such a coefficient.

If a (matrix valued) differential operator \( M \) is written in the form

\[ M = D_\mu^+ D_\mu + X^+ , \quad X = X^+ \]  
(46)

with \( D_\mu = 1 \partial_\mu - iC_\mu \), \( D_\mu^+ = -1 \partial_\mu + iC_\mu^+ \), \( C_\mu = C_\mu^+ \), \( C_\mu \) and \( X \) being matrices whose entries are solely classical fields (not operators), then the coefficient \( a_1 \) is just \(-X\). In our case the operator \( M \) was already introduced in eq. \([20]\), and after algebraic manipulations and taking the gauge \( \partial_\mu A_\mu = 0 \) and the self-duality conditions into account, it follows that \( M \) can be indeed written in the form \([20]\), with

\[ C_1 = i e \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & -\phi_1 & -\phi_2 \\
 0 & \phi_1 & 0 & A_1 \\
 0 & \phi_2 & -A_1 & 0
\end{pmatrix}, \]  
(47)

\[ C_2 = i e \begin{pmatrix}
 0 & 0 & \phi_1 & \phi_2 \\
 0 & 0 & 0 & 0 \\
 -\phi_1 & 0 & 0 & A_2 \\
 -\phi_2 & 0 & -A_2 & 0
\end{pmatrix}, \]  
(48)

\[ X = \begin{pmatrix}
 0 & 0 & L_{11} & L_{12} \\
 0 & 0 & L_{21} & L_{22} \\
 L_{11} & L_{21} - \frac{3}{2} e^2 \phi_1^2 & L_{12} & L_{22} \\
 L_{12} & L_{22} & -e^2 \phi_1 \phi_2 & -\frac{i}{2} e^2 \phi_1^2 - \frac{3}{2} e^2 \phi_2^2
\end{pmatrix}, \]  
(49)

where
$$L_{\mu a} = -e\varepsilon_{ab}D_\mu \phi_b,$$

$$D_\mu \phi_a = \partial_\mu \phi_a + eA_\mu \varepsilon_{ab} \phi_b$$ being the $U(1)$ covariant derivative of the matter fields $\phi_a$. Consequently, the trace of $a_1$ is

$$\text{tr} \ a_1(z) = -\text{tr} \ X = 2 e^2 \left( \phi_1^2 + \phi_2^2 \right) = 4 e B(z)$$

so that its integral is a topological invariant. From eqs. (45), (43) it follows that

$$\int d^2 z \langle T_\mu \phi(z) \rangle = 2(1 - \mathcal{N})$$

manifesting a trace anomaly for any $\mathcal{N} \neq 1$.

**V. DISCUSSION**

Among the local gauge theories admitting classical vortex solutions, the one we have presented is particularly interesting. First of all its classical solutions are very simple. All the solutions can be expressed in terms of an arbitrary analytic function, which is the only degree of freedom of the analytic function. The topological invariant is the same as that characterizing the solutions of the 2D euclidean Liouville theory model, all of whose solutions can be also expressed in terms of an arbitrary analytic function.

The trace of the energy momentum tensor vanishes on the classical solutions and, consequently, conformal transformations are a symmetry of the classical solutions: a conformal reparametrization relates two different solutions of the model. Related to this fact there is also the factorization property (14) of the coupling constant $\beta$, that is the only dimensional parameter of the model. Such a factorization is important also because it permits, for topological models, a temperature-like interpretation of the coupling constant ($\beta = 1/e^2$). In this context, the euclidean action should be interpreted as the potential energy of the system whereas the free energy should represent the energy fluctuations due to the interaction with the thermal bath $\beta$. In the limit of small coupling (large $\beta$), the partition function of the system is just the determinant of operator obtained by expanding the classical action around the classical solutions up to the second order in the thermal fluctuations. The evaluation of the determinant requires a regularization procedure and, therefore, an arbitrary scale. Consequently, one expects that thermal fluctuations destroys scale (and conformal) symmetry, and in fact there is a conformal “anomaly”: the expectation value of the trace of the energy momentum tensor does not vanish, except for the special value $\mathcal{N} = 1$ (see eq. (51)). When the classical background carries the minimum vorticity ($\mathcal{N} = 1$), conformal symmetry seems to survive, at least in the harmonic approximation.

This is a very surprising property that certainly deserves a deeper analysis. At present, we do not know the exact reason of such a phenomenon. Nonetheless, there are indeed some properties that make the case $\mathcal{N} = 1$ different from the others: for example, the trace of the $a_1$ Seeley -- De Witt coefficient or, alternatively, the shape of the potential $V = -eB^{cd}$ felt by the Schroedinger particle, dramatically changes in the cases $\mathcal{N} = 1$ and $\mathcal{N} \neq 1$. Let us consider for convenience the radially symmetric case. Then, if $\mathcal{N} = 1$, the potential $V = -eB^{cd} = -e^2 \phi^{cd} \phi^{cd}$ is a monotonic radial function, and therefore it has a single, non degenerate minimum at $r = 0$. On the contrary, if $\mathcal{N} \neq 1$, the potential $V$ has a maximum point at $r = 0$, whereas its minima are degenerate and located around the circle $r = R_0[(\mathcal{N} - 1)/(\mathcal{N} + 1)]^{1/2\mathcal{N}}$. Thus, there is the intriguing possibility that the occurrence of conformal anomaly could be related to the vacuum degeneracy of the free energy.

Another feature that makes the case $\mathcal{N} = 1$ different from all the others is the following: the most general solution carrying vorticity $1$ is obtained by choosing the arbitrary analytic function $\omega(z) = \omega_0(z - a)/(z - b)$. It is easily recognized that such a function is in a one to one correspondence with an arbitrary transformation of the group $SL(2, \mathbb{C})$, $z \to \zeta = (az + b)/(cz + d)$, $ad - bc = 1$. This is a very special subgroup of the conformal group: it defines the projective transformations, that are the only conformal transformations providing invertible mappings of the whole complex plane onto itself. As a consequence of this fact, $SL(2, \mathbb{C})$ is a kind of “residual” symmetry one has, once the the number of vortices and the total vorticity has been fixed: let the pair $(Q, \mathcal{N})$ denote an arbitrary background field configuration of $Q$ vortices with total vorticity $\mathcal{N}$; then, applying an arbitrary $SL(2, \mathbb{C})$ transformation, the background $(Q, \mathcal{N})$ is mapped into another background but with the same pair $(Q, \mathcal{N})$. By contrast, if a conformal transformation $z \to \zeta(z)$ not belonging to $SL(2, \mathbb{C})$ is applied to a field configuration $(Q, \mathcal{N})$, it necessarily increases vorticity, i.e. $(Q, \mathcal{N}) \to (Q', Q, \mathcal{N'} > \mathcal{N})$. Clearly, if $\mathcal{N} = 1$ (minimum vorticity) then necessarily $Q = 1$, and only in this case it happens that the most general element $\omega(z)$ characterizing the background $(Q = 1, \mathcal{N} = 1)$ belongs to the same group of transformations leaving the background $(Q = 1, \mathcal{N} = 1)$ unchanged. This observation is at the root to understand why the $\mathcal{N} = 1$ case is not anomalous.
Besides the conformal anomaly, we have also evaluated the zero modes of the thermal fluctuation operator. For a given fixed vorticity $N$, we have derived the zero modes in two limiting cases of the classical background: the totally degenerate and the totally non degenerate cases, corresponding to a single vortex located at the origin and carrying vorticity $N$, and to $N$ distinct vortices carrying each the minimum vorticity, respectively. All the other possible cases can be easily derived as intermediate between these two. Clearly, the explicit form of the zero modes dependson the classical background. However, the total number of normalizable zero modes is only a function of the vorticity of the classical solutions. In addition, the contribution to the thermal fluctuations given by the zero modes does not change the vorticity of the classical solution.

There are several aspects related to this model that deserve consideration for future investigations. An important problem is certainly a deeper understanding of the persistence of conformal symmetry in the $N = 1$ case, and its possible relation with the non-degenerateness of the free energy vacuum.

Concerning the eigenvalue problem (24) or, equivalently, the determinant of the operator $M$, an exact evaluation seems very difficult. However, some approximate method should be available to investigate the system beyond the zero-mode sector, perhaps reducing the number of degrees of freedom by introducing some collective coordinates. Alternatively, one could try to investigate some particular limits that simplify the form of the matrix $M$. To this purpose, two particular cases should be mentioned: if $r_0 \to 0$, Liouville vortices becomes Aharonov-Bohm vortices, i.e. $A^{cl} \sim \nabla \theta$ and $B^{cl} \sim \delta(r)$. This limit greatly simplify the form of the fluctuation operator, although in this case the classical solutions become singular. Another interesting limit is the large $N$ limit; in this case the magnetic field has a significant non vanishing contribution only in a neighbourhood of $r = r_0$, and it could be easier to evaluate the determinant.

Another interesting issue is the possibility that the model (2) could be an ‘effective action’ of another, more elementary, model. This hypothesis could be supported by the fact that Liouville vortices always have an even vorticity in terms of the elementary flux quanta $\Phi_0 = 2\pi/e$ when the Abelian coupling $e$ is interpreted as electric constant. Consequently, it could be that the scalar field $\phi$ is related to an electron pair condensate. A further point towards this direction is the one discussed at beginning of Section 3: in the very drastic approximation considered there, the partition function is the one of a Coulomb gas in its Debye phase that, in turn, can be equally described by a system of massive fermions.

Finally, at the purely classical level, it could be interesting to investigate on the possibility of constructing non linear superpositions of Liouville vortices: just like the ‘t Hooft Polyakov monopole can be seen as a non linear superposition of infinite instantons equally separated in time (20), a non linear superposition of Liouville vortices could provide new soliton solutions of some lower dimensional gauge theory.

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[10] From now on, without loosing generality, we shall restrict ourselves to self-dual configurations, i.e. upper sign in eqs. (4).

This is clearly not restrictive as anti-self-dual solutions can be always obtained through a parity transformation on self-dual solutions.

[11] To understand the transformation properties of the fields under (4), one can consider the solutions (4) with $\omega \equiv z$, and then interpret the general form (2) as the conformally transformed fields under the redefinition $z \to \omega(z)$. Then, eqs. (10) follow.

[12] In spite of the fact that zero modes are defined up to an overall constant, $g_n$ and $i g_n$, define two distinct zero modes due to the fact that in eq. (27) zero modes depend linearly on $\Delta \omega$ and its complex conjugate.

[13] The choice of $\omega$ is not unique. For instance, an alternative choice is a finite Mittag-Leffler sum of the type $\omega(z) = \omega_0 + \sum_{n=1}^{N} a_i/(z-b_i)$, depending on $2N+1$ complex parameters. We choose the finite product expansion because in this
case is much easier to check the normalizability conditions of the zero modes. It is important to remark that, whatever is
the choice of $\omega$ when the vorticity is fixed, the maximum number of real parameters upon which $\omega$ depends is fixed and
equal to $4N+2$.

[14] The position of the vortexes can be easily found by studying the extrema of the magnetic field $eB = 4|\omega'(z)|^2/(1+|\omega(z)|^2)^2$;
for example, in the $N = 1$ case the position of the single vortex is at $z_1 = (a_1|\omega_0|^2 + b_1)/(1 + |\omega_0|^2)$.

[15] In this case, a regularization in the evaluation of the partition function is required due to the infinite number of degrees of
freedom of the system.

[16] Following the standard notation, the prime index means that the vanishing eigenvalues have to be omitted from sums
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