Integral Representations of Holomorphic Functions on Coverings of Pseudoconvex Domains in Stein Manifolds

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Abstract

The classical integral representation formulas for holomorphic functions defined on pseudoconvex domains in Stein manifolds play an important role in the constructive theory of functions of several complex variables. In this paper we construct similar formulas for certain classes of holomorphic functions defined on coverings of such domains.

1. Introduction.

1.1. The method of integral representations for holomorphic functions works successfully in various problems of the theory of functions of several complex variables, e.g., in problems of uniform estimates for solutions of the Cauchy-Riemann equations, uniform estimates for extensions of holomorphic functions from submanifolds, uniform approximation of holomorphic functions that are continuous on the boundary, etc. (We refer to the book of Henkin and Leiterer [HL] devoted to this subject.) In the present paper we use this method to study holomorphic functions of slow growth defined on unbranched coverings of pseudoconvex domains in Stein manifolds. In particular, we will show that many known results for holomorphic functions on such domains can be extended to similar results for holomorphic functions of slow growth defined on their coverings. Our approach is based on new integral representation formulas for holomorphic functions of slow growth on coverings of pseudoconvex domains. These formulas generalize the classical Leray integral formula and certain of its developments. (Note that the classical integral formulas are usually applied to bounded domains with rectifiable boundaries, while infinite...
coverings of such domains may have not these properties.) The application of our integral formulas allows to reduce some problems for holomorphic functions on the covering of a domain to analogous problems for Banach-valued holomorphic functions on the domain itself. In our proofs we exploit some ideas previously used in [Br1], [Br2] in the area of the Corona problem and based on infinite-dimensional versions of Cartan’s A and B theorems originally proved by Bungart [B].

In [Br3] we apply our technique to extend and strengthen certain results of Gromov, Henkin and Shubin [GHS] on holomorphic $L^2$-functions on coverings of pseudoconvex manifolds in the case of coverings of Stein manifolds.

To formulate our main results we first introduce basic notation and definitions. Throughout this paper we consider complex manifolds $M$ satisfying the condition

$$M \subset \subset \tilde{M} \subset N \quad \text{and} \quad \pi_1(M) = \pi_1(N) \quad (1.1)$$

where $M$ and $\tilde{M}$ are open connected subsets of a complex manifold $N$, and $\tilde{M}$ is Stein. (Here $\pi_1(X)$ denotes the fundamental group of $X$.)

For instance, this condition is valid for $M$ a strictly pseudoconvex domain or an analytic polyhedra in a Stein manifold.

Let $r : N' \to N$ be an unbranched covering of $N$. By $M' = r^{-1}(M)$ we denote the corresponding unbranched covering of $M$. Condition (1.1) implies that $M'$ is an open connected subset of $N'$ and $\pi_1(M') = \pi_1(N')$. Let $\psi : N' \to \mathbb{R}_+$ be such that $\log \psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$. We introduce the Banach space $\mathcal{H}_{p,\psi}(M')$, $1 \leq p \leq \infty$, of functions $f$ holomorphic on $M'$ with norm

$$|f|_{p,\psi} := \sup_{x \in M} \left( \sum_{y \in r^{-1}(x)} |f(y)|^p \psi(y) \right)^{1/p} \quad (1.2)$$

(Here $\mathcal{H}_{\infty,\psi}(M')$ is the Banach space of bounded holomorphic functions on $M'$.)

**Example 1.1** Let $d$ be the path metric on $N'$ obtained by the pullback of a Riemannian metric defined on $N$. Fix a point $o \in M'$ and set

$$d_o(x) := d(o,x) \quad , \quad x \in N' .$$

It is easy to show by means of the triangle inequality that as the function $\psi$ one can take, e.g., $(1 + d_o)^\alpha$ or $e^{\alpha d_o}$ with $\alpha \in \mathbb{R}$. (For instance, if $N'$ is a strip \{ $z = x + iy$ : $|y| < L$ \}$ \subset \mathbb{C}$ with the action of group $\mathbb{Z}$ given by translations along $\mathbb{R}$, i.e., $N'$ is a regular covering of an annulus, one can take as $\psi$ either the functions $1 + |x|^\alpha$ or $e^{\alpha |x|}$, $\alpha \in \mathbb{R}$.)

**Remark 1.2** Let $dV_{M'}$ be the Riemannian volume form on the covering $M'$ obtained by a Riemannian metric pulled back from $N$. Note that every $f \in \mathcal{H}_{p,\psi}(M')$ also belongs to the Banach space $\mathcal{H}^p_{\psi}(M')$ of holomorphic functions $g$ on $M'$ with norm

$$\left( \int_{z \in M'} |g(z)|^p \psi(z) dV_{M'}(z) \right)^{1/p} .$$

Moreover, one has a continuous embedding $\mathcal{H}_{p,\psi}(M') \hookrightarrow \mathcal{H}^p_{\psi}(M')$. 


Let \( x \in M \) and \( r : M' \to M \) be an unbranched covering of \( M \). We introduce the Banach space \( l_{p,\psi,x}(M') \), \( 1 \leq p \leq \infty \), of functions \( g \) on \( r^{-1}(x) \) with norm

\[
|g|_{p,\psi,x} := \left( \sum_{y \in r^{-1}(x)} |g(y)|^p \psi(y) \right)^{1/p}.
\]  

Also, for Banach spaces \( E \) and \( F \), by \( B(E, F) \) we denote the space of all linear bounded operators \( E \to F \) with norm \( || \cdot || \).

**Theorem 1.3** Suppose that \( M \) satisfies condition (1.1). Then for any \( p \in [1, \infty] \) there exists a family \( \{L_z \in B(l_{p,\psi,z}(M'), H_{p,\psi}(M'))\}_{z \in M} \) holomorphic in \( z \) such that

\[
(L_z h)(x) = h(x) \quad \text{for any} \quad h \in l_{p,\psi,z}(M') \quad \text{and} \quad x \in r^{-1}(z).
\]

Moreover,

\[
\sup_{z \in M} ||L_z|| < \infty.
\]

1.2. Theorem 1.3 allows to obtain integral representation formulas for holomorphic functions from \( H_{p,\psi}(M') \) by means of known integral formulas for holomorphic functions on \( M \). As an example we will show how to get such formulas from the classical Leray integral formula, the basis of many other integral formulas. We first recall this formula itself.

For vectors \( \xi, \eta \in \mathbb{C}^n \) we set

\[
<\eta, \xi> = \sum_{j=1}^{n} \eta_j \cdot \xi_j.
\]

Also, we set

\[
\omega(\xi) = d\xi_1 \wedge \cdots \wedge d\xi_n \quad \text{and} \quad \omega'(\eta) = \sum_{k=1}^{n} (-1)^{k-1} \eta_k d\eta_1 \wedge \cdots \wedge d\eta_{k-1} \wedge d\eta_{k+1} \wedge \cdots \wedge d\eta_n.
\]

Let \( M \subset \mathbb{C}^n \) be a domain and \( z \in M \) be a fixed point. Consider in the domain \( Q = \mathbb{C}^n \times M \) with coordinates \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n \) and \( \xi = (\xi_1, \ldots, \xi_n) \in M \) the hypersurface of the form

\[
P_z = \{(\eta, \xi) \in Q : <\eta, \xi - z> = 0\}.
\]

Let \( h_z \) be a \((2n - 1)\)-dimensional cycle in the domain \( Q \setminus P_z \) such that its projection onto \( M \setminus \{z\} \) is homologous to \( \partial M \). Then for any holomorphic function \( f \) defined on \( M \) we have (see [L])

\[
f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{h_z} f(\xi) \frac{\omega'(\eta) \wedge \omega(\xi)}{<\eta, \xi - z>}.
\]  

(1.4)

Note that since the integral kernel in this formula is bounded and continuous on \( h_z \), similar formulas are valid for Banach-valued holomorphic functions defined on \( M \).

Now from Theorem 1.3 we obtain
Corollary 1.4 Suppose that $M \subset \mathbb{C}^n$ satisfies condition (1.1). Then under the assumptions of the Leray integral formula for any holomorphic function $f \in \mathcal{H}_{p,\psi}(M')$ we have

$$f(x) = \frac{(n-1)!}{(2\pi i)^n} \int_{(x)} L_\xi(f|_{r^{-1}(\eta)})(x) \frac{\omega'(\eta) \wedge \omega(\xi)}{\eta, \xi - x} \quad \text{for any} \quad x \in r^{-1}(z) . \quad (1.5)$$

Similarly to (1.5) we obtain extensions of other known integral formulas. To get such an extension replace only the integrand function $f(\xi)$ in an integral formula by $L_\xi(f|_{r^{-1}(\xi)})(x)$. In the same way one obtains multi-dimensional Cauchy-Green and Koppelman-Leray type formulas for some classes of differential forms defined on coverings of $M$ satisfying (1.1). Several such formulas are presented in the Appendix (see also [HL] for an exposition of known integral formulas).

1.3. Let us formulate some other applications of Theorem 1.3.

Let $r : M' \to M$ be a covering. Suppose that $\{x_n\}_{n \geq 1} \subset M$ converges to $x \in M$. Then for sufficiently big $n$ we can arrange $r^{-1}(x_n)$ and $r^{-1}(x)$ in sequences $\{y_{in}\}_{i \geq 1}$ and $\{y_i\}_{i \geq 1}$ such that every $\{y_{in}\}$ converges to $y_i$ as $n \to \infty$. For such $n$ we define maps $\tau_n(x) : r^{-1}(x) \to r^{-1}(x_n)$ so that $\tau_n(y_i) = y_{in}$, $i \in \mathbb{N}$. Below, $\tau_n$ denotes the transpose map generated by $\tau_n$ on functions defined on $r^{-1}(x_n)$ and $r^{-1}(x)$.

Definition 1.5 Let $X \subset M$ be a subset. We say that a function $f$ on $r^{-1}(X)$ belongs to the class $C_{p,\psi}(r^{-1}(X))$ if

1. $f|_{r^{-1}(x)} \in l_{p,\psi,x}(M')$ for any $x \in X$ and

2. for any $x \in X$ and any sequence $\{x_n\} \subset X$ converging to $x$ the sequence of functions $\{\tau_n^*(f|_{r^{-1}(x_n)})\}$ converges to $f|_{r^{-1}(x)}$ in the norm of $l_{p,\psi,x}(M')$.

Suppose that $X$ belongs to a complex manifold $X_a \subset M$ with $\dim_{\mathbb{C}} X_a \geq 1$ and any other complex manifold of dimension $d$, $1 \leq d < \dim_{\mathbb{C}} X_a$, does not contain $X$. In what follows the interior of $r^{-1}(X)$ is defined with respect to $r^{-1}(X_a)$.

Definition 1.6 By $\mathcal{H}_{p,\psi}(r^{-1}(X))$ we denote the Banach space of functions $f \in C_{p,\psi}(r^{-1}(X))$ holomorphic in interior points of $r^{-1}(X)$ with norm

$$|f|_{p,\psi}^X := \sup_{x \in X} |f|_{r^{-1}(x)}|_{p,\psi,x} . \quad (1.6)$$

Remark 1.7 (a) Arguments used in the proof of Proposition 2.4 stated below show that if $X \subset X_a$ is open then any function $f$ holomorphic on $r^{-1}(X)$ satisfying part (1) of Definition 1.5 and such that $|f|_{p,\psi}^X < \infty$ belongs to $\mathcal{H}_{p,\psi}(r^{-1}(X))$.

(b) Suppose that $M$ satisfies condition (1.1) and that the function $\psi$ in the definition of $\mathcal{H}_{p,\psi}(M')$ is such that $1 \in \mathcal{H}_{p,\psi}(M')$. Then it is easy to see that any function on $r^{-1}(X)$ uniformly continuous with respect to the path metric pulled back from $N$ satisfies part (2) of Definition 1.5. In particular, this is always true for $p = \infty$.

Suppose that a domain $M \subset \mathbb{C}^N$ satisfies (1.1). Let $D \subset M$ be a strictly pseudoconvex open set (not necessarily $D \neq \emptyset$), and $\rho$ be a strictly plurisubharmonic $C^2$-function in a neighbourhood $O$ of $\partial D$ such that

$$D \cap O = \{z \in O : \rho(z) < 0\} \quad \text{and} \quad N(\rho) := \{z \in O : \rho(z) = 0\} \subset \subset O . \quad (1.7)$$

We set $C := D \cup N(\rho)$. 

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Theorem 1.8 There is a neighbourhood $\Omega$ of $C$ such that every $f \in \mathcal{H}_{p,\psi}(r^{-1}(C))$ can be uniformly approximated in the norm of $\mathcal{H}_{p,\psi}(r^{-1}(C))$ by functions from $\mathcal{H}_{p,\psi}(r^{-1}(\Omega))$.

Example 1.9 Let $C$ be a (connected) compact real-analytic manifold. By the Grauert theorem [Gr] we can assume that $C$ is an analytic submanifold of some $\mathbb{R}^n$. It is easy to show that $C$ is the zero set of a non-negative strictly plurisubharmonic $C^2$-function defined in a neighbourhood of $C$. Let $C_{\epsilon}$ be an $\epsilon$-neighbourhood of $C$ in $\mathbb{C}^n$ where $\epsilon$ is sufficiently small. Then $C_{\epsilon}$ satisfies condition (1.1). Let $r : C' \to C$ be a covering of $C$, and $r : C'_{\epsilon} \to C_{\epsilon}$ be the covering of $C_{\epsilon}$ such that $C' \subset C'_{\epsilon}$ (i.e., for this covering $\pi_1(C') = \pi_1(C'_{\epsilon})$). Now, Theorem 1.8 implies that for a sufficiently small $\epsilon$ any $C_{p,\psi}$-function on $C'$ can be uniformly approximated in the norm of $\mathcal{H}_{p,\psi}(C')$ by functions from $\mathcal{H}_{p,\psi}(C'_{\epsilon})$.

Finally, let us formulate a result on bounded extension of holomorphic functions from complex submanifolds.

Suppose that $M$ is a strictly pseudoconvex open set (with not necessarily smooth boundary) in a Stein manifold satisfying condition (1.1). Let $Y$ be a closed complex submanifold of some neighbourhood of $\overline{M}$. Consider a covering $r : M' \to M$. Since $M$ satisfies condition (1.1), there exists a covering $r : N' \to N$ such that $M'$ is an open subset of $N'$. By $\overline{M'}$ we denote the closure of $M'$ in $N'$. Also, we set $X := Y \cap M$.

Theorem 1.10 (1) For every function $f \in \mathcal{H}_{p,\psi}(r^{-1}(X))$, there exists a function $F \in \mathcal{H}_{p,\psi}(M')$ such that $F = f$ on $r^{-1}(X)$.

(2) For every function $f \in \mathcal{H}_{p,\psi}(r^{-1}(\overline{X}))$, there exists a function $F \in \mathcal{H}_{p,\psi}(\overline{M'})$ such that $F = f$ on $r^{-1}(\overline{X})$.

(Recall that $\mathcal{H}_{p,\psi}(r^{-1}(\overline{X}))$ is the space of $C_{p,\psi}$-functions on $r^{-1}(\overline{X})$ holomorphic in $r^{-1}(X)$ with the norm defined by (1.6).)

Analogous results hold for a connected component $Z$ of $r^{-1}(X)$. In this case we define $\mathcal{H}_{p,\psi}(Z)$ as the space of functions on $Z$ whose extensions to $r^{-1}(X)$ by 0 belong to $\mathcal{H}_{p,\psi}(r^{-1}(X))$. One defines $\mathcal{H}_{p,\psi}(\overline{Z})$ similarly.

2. Auxiliary Results.

For the standard facts about bundles see, e.g., Hirzebruch’s book [Hi]. In what follows, all topological spaces are assumed to be finite or infinite dimensional.

2.1. Let $X$ be a complex analytic space and $S$ be a complex analytic Lie group with unit $e \in S$. Consider an effective holomorphic action of $S$ on a complex analytic space $F$. Here holomorphic action means a holomorphic map $S \times F \to F$ sending $s \times f \in S \times F$ to $sf \in F$ such that $s_1(s_2f) = (s_1s_2)f$ and $ef = f$ for any $f \in F$. Efficiency means that the condition $sf = f$ for some $s$ and any $f$ implies that $s = e$. 
**Definition 2.1** A complex analytic space $W$ together with a holomorphic map (projection) $\pi : W \to X$ is called a holomorphic bundle on $X$ with structure group $S$ and fibre $F$, if there exists a system of coordinate transformations, i.e., if

1. there is an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ and a family of biholomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times F$, that map “fibres” $\pi^{-1}(u)$ onto $u \times F$;
2. for any $i, j \in I$ there are elements $s_{ij} \in \mathcal{O}(U_i \cap U_j, S)$ such that
   $$(h_i h_j^{-1})(u \times f) = u \times s_{ij}(u)f \quad \text{for any} \quad u \in U_i \cap U_j, \ f \in F.$$ 

In particular, a holomorphic bundle $\pi : W \to X$ whose fibre is a Banach space $F$ and the structure group is $\text{GL}(F)$ (the group of linear invertible transformations of $F$) is called a holomorphic Banach vector bundle.

A holomorphic section of a holomorphic bundle $\pi : W \to X$ is a holomorphic map $s : X \to W$ satisfying $\pi \circ s = \text{id}$.

Let $\pi_i : W_i \to X$, $i = 1, 2$, be holomorphic Banach vector bundles. A holomorphic map $f : W_1 \to W_2$ satisfying

(a) $f(\pi_1^{-1}(x)) \subset \pi_2^{-1}(x)$ for any $x \in X$;

(b) $f|_{\pi_1^{-1}(x)}$ is a linear continuous map of the corresponding Banach spaces,

is called a homomorphism. If, in addition, $f$ is a homeomorphism, then $f$ is called an isomorphism.

We will use the following construction of holomorphic bundles (see, e.g. [Hi, Ch.1]):

Let $S$ be a complex analytic Lie group and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $X$. By $Z^1_{\mathcal{O}}(\mathcal{U}, S)$ we denote the set of holomorphic $S$-valued $\mathcal{U}$-cocycles. By definition, $s = \{s_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, S)$, where $s_{ij} \in \mathcal{O}(U_i \cap U_j, S)$ and $s_{ij} s_{jk} = s_{ik}$ on $U_i \cap U_j \cap U_k$. Consider the disjoint union $\sqcup_{i \in I} U_i \times F$ and for any $u \in U_i \cap U_j$ identify the point $u \times f \in U_i \times F$ with $u \times s_{ij}(u)f \in U_j \times F$. We obtain a holomorphic bundle $W_s$ on $X$ whose projection is induced by the projection $U_i \times F \to U_i$. Moreover, any holomorphic bundle on $X$ with structure group $S$ and fibre $F$ is isomorphic (in the category of holomorphic bundles) to a bundle $W_s$.

**Example 2.2 (a)** Let $M$ be a complex manifold. For any subgroup $H \subset \pi_1(M)$ consider the unbranched covering $r : M(H) \to M$ corresponding to $H$. We will describe $M(H)$ as a holomorphic bundle on $M$.

First, assume that $H \subset \pi_1(M)$ is a normal subgroup. Then $M(H)$ is a regular covering of $M$ and the quotient group $G := \pi_1(M)/H$ acts holomorphically on $M(H)$ by deck transformations. It is well known that $M(H)$ in this case can be thought of as a principle fibre bundle on $M$ with fibre $G$ (here $G$ is equipped with the discrete topology). Namely, let us consider the map $R_G(g) : G \to G$, $g \in G$, defined by the formula

$$R_G(g)(q) = q \cdot g^{-1}, \quad q \in G.$$ 

Then there is an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $M$ by sets biholomorphic to open Euclidean balls in some $\mathbb{C}^n$ and a locally constant cocycle $c = \{c_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, G)$
such that $M(H)$ is biholomorphic to the quotient space of the disjoint union $V = \sqcup_{i \in I} U_i \times G$ by the equivalence relation: $U_i \times G \ni x \times R_G(c_{ij})(q) \sim x \times q \in U_j \times G$. The identification space is a holomorphic bundle with projection $r : M(H) \to M$ induced by the projections $U_i \times G \to U_i$. In particular, when $H = e$ we obtain the definition of the universal covering $M_u$ of $M$.

Assume now that $H \subset \pi_1(M)$ is not necessarily normal. Let $X_H = \pi_1(M)/H$ be the set of cosets with respect to the (left) action of $H$ on $\pi_1(M)$ defined by left multiplications. By $[Hq] \in X_H$ we denote the coset containing $q \in \pi_1(M)$. Let $A(X_H)$ be the group of all homeomorphisms of $X_H$ (equipped with the discrete topology). We define the homomorphism $\tau : \pi_1(M) \to A(X_H)$ by the formula:

$$\tau(g)([Hq]) := [Hqg^{-1}], \quad q \in \pi_1(M).$$

Set $Q(H) := \pi_1(M)/\ker(\tau)$ and let $\overline{g}$ be the image of $g \in \pi_1(M)$ in $Q(H)$. By $\tau_{Q(H)} : Q(H) \to A(X_H)$ we denote the unique homomorphism whose pullback to $\pi_1(M)$ coincides with $\tau$. Consider the action of $H$ on $V = \sqcup_{i \in I} U_i \times \pi_1(M)$ induced by the left action of $H$ on $\pi_1(M)$ and let $V_H = \sqcup_{i \in I} U_i \times X_H$ be the corresponding quotient set. Define the equivalence relation $U_i \times X_H \ni x \times \tau_{Q(H)}(\overline{c}_{ij})(h) \sim x \times h \in U_j \times X_H$ with the same $\{c_{ij}\}$ as in the definition of $M(e)$. The corresponding quotient space is a holomorphic bundle with fibre $X_H$ biholomorphic to $M(H)$.

**2.2** We retain the notation of example (a). Let $B$ be a complex Banach space and $GL(B)$ be the group of invertible bounded linear operators $B \to B$. Consider a homomorphism $\rho : G \to GL(B)$. Without loss of generality we assume that $\ker(\rho) = e$, for otherwise we can pass to the corresponding quotient group. The holomorphic Banach vector bundle $E_\rho \to M$ associated with $\rho$ is defined as the quotient of $\sqcup_{i \in I} U_i \times B$ by the equivalence relation $U_i \times B \ni x \times \rho(c_{ij})(w) \sim x \times w \in U_j \times B$ for any $x \in U_i \cap U_j$. Let us illustrate this construction by an example.

Let $X_H = \pi_1(M)/H$ be the fibre of the covering $r : M(H) \to M$. Consider a function $\phi : X_H \to \mathbb{R}_+$ satisfying

$$\phi(\tau_{Q(H)}(h)(x)) \leq c_h \phi(x), \quad h \in Q(H), \quad x \in X_H,$$

where $c_h$ is a constant depending on $h$. By $l_{p,\phi}(X_H)$, $1 \leq p \leq \infty$, we denote the Banach space of complex functions $f$ on $X_H$ with norm

$$\|f\|_{p,\phi} := \left( \sum_{x \in X_H} |f(x)|^p \phi(x) \right)^{1/p}. \tag{2.2}$$

Then according to (2.1) the map $\rho$ defined by the formula

$$[\rho(h)(f)](x) := f(\tau_{Q(H)}(h)(x)), \quad h \in Q(H), \quad x \in X_H,$$

is a homomorphism of $Q(H)$ into $GL(l_{p,\phi}(X_H))$. By $E_{p,\phi}(X_H)$ we denote the holomorphic Banach vector bundle associated with this $\rho$. 

**2.2** Let $r : M' \to M$ be a covering where $M' = M(H)$ (i.e., $\pi_1(M') = H$). In this part we establish a connection between Banach spaces $H_{p,\psi}(M')$ defined in section 1.1 and certain spaces of holomorphic sections of bundles $E_{p,\phi}(X_H)$.  


We retain the notation of Example 2.2. Assume that $M$ satisfies condition (1.1), i.e., $M \subset N$ and $\pi_1(M) = \pi_1(N)$. Then $M(H)$ is embedded into $N(H)$. (Without loss of generality we consider $M(H)$ as an open subset of $N(H)$.) Let $\{V_i\}_{i \in I}$ be a finite acyclic open cover of $\overline{M}$ by relatively compact sets. We set $U_i := V_i \cap M$ and consider the open cover $U = \{U_i\}_{i \in I}$ of $M$. Then as in Example 2.2 (a) we can define $M(H)$ by a cocycle $\tilde{c} = \{\tilde{c}_{ij}\} \in Z^1_0(U, Q(H))$ where $Q(H) = \pi_1(M)/Ker \tau$.

Further, let $\psi : N(H) \to \mathbb{R}_+$ be a function such that $\log \psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$. Fix a point $z_0 \in M$. Identifying $r^{-1}(z_0)$ with $X_H$ define the function $\phi : X_H \to \mathbb{R}_+$ by the formula

$$
\phi(x) := \psi(x) , \quad x \in X_H \quad (= r^{-1}(z_0)) .
$$

Lemma 2.3 The function $\phi$ satisfies inequality (2.1).

Proof. We recall some facts from the theory of covering spaces (see, e.g., [Hu, Chapter III]. Let $h \in Q(H)$. Then there exists a closed path $\gamma \subset M$ passing through the point $z_0$ such that its lifting $\gamma_x \subset M(H)$ with the initial point $x \in r^{-1}(z_0)$ has the endpoint $\tau_{Q(H)}(h)(x)$. By the definition of the metric on $M(H)$, the length of every such $\gamma_x$ is the same. This and uniform continuity of $\log \psi$ with respect to the path metric defined by a Riemannian metric pulled back from $N$ imply that every $\gamma_x$ can be covered by $k$ metric balls $\{N_j(x)\}_{j=1}^k$ such that

$$
\frac{1}{2} \psi(y) \leq \psi(z) \leq 2 \psi(y) \quad \text{for any} \quad y, z \in N_j(x), \quad 1 \leq j \leq k .
$$

In particular, we have

$$
\frac{\psi(\tau_{Q(H)}(h)(x))}{\psi(x)} \leq 2^k \quad \text{for every} \quad x \in r^{-1}(z_0) . \quad \Box
$$

According to this lemma, the bundle $E_{p,\phi}(X_H)$ is well defined. By definition, any holomorphic section of this bundle can be determined by the family $\{f_i(z, x)\}_{i \in I}$ of holomorphic functions on $U_i$ with values in $l_{p,\phi}(X_H)$ satisfying

$$
f_i(z, \tau_{Q(H)}(\tilde{c}_{ij})(x)) = f_j(z, x) \quad \text{for any} \quad z \in U_i \cap U_j .
$$

We introduce the Banach space $B_{p,\phi}(X_H)$ of bounded holomorphic sections $f = \{f_i\}_{i \in I}$ of $E_{p,\phi}(X_H)$ with norm

$$
|f|_{p,\phi} := \sup_{i \in I, z \in U_i} ||f_i(z, \cdot)||_{p,\phi} . \quad (2.3)
$$

(Here $|| \cdot ||_{p,\phi}$ is the norm on $l_{p,\phi}(X_H)$, see (2.2).)

Further, let $f \in H_{p,\psi}(M(H))$ (see section 1.1 for the definition). We identify $M(H)$ with the quotient set of $V_H = \sqcup_{i \in I} U_i \times X_H$ as in Example 2.2 (a). This gives local coordinates on $M(H)$. Using these coordinates we define the family $\{f_i\}_{i \in I}$ of functions on $U_i$ with values in the space of functions on $X_H$ by the formula

$$
f_i(z, x) := f(z, x) , \quad z \in U_i , \quad i \in I , \quad x \in X_H . \quad (2.4)
$$

Then the following result holds.
**Proposition 2.4** The correspondence \( f \mapsto \{ f_i \}_{i \in I} \) determines an isomorphism of Banach spaces \( D : \mathcal{H}_{p,\psi}(M(H)) \to B_{p,\psi}(X_H) \).

**Proof.** First we will prove that every \( f_i(z, x) = f(z, x) \) determines a holomorphic map of \( U_i \) into \( l_{p,\phi}(X_H) \). Since by definition \( f_i(z, \tau_Q(H)(\tilde{c}_{ij}(x))) = f_j(z, x) \) for any \( z \in U_i \cap U_j \), this will show that \( \{ f_i \} \) is a holomorphic section of \( E_{p,\phi}(X_H) \).

Check that \( f_i(z, \cdot) \in l_{p,\phi}(X_H) \) for any \( z \in U_i \). We will assume that \( z_0 \) in the definition of \( \phi \) from Lemma 2.3 belongs to some \( U_k \) and so \( r^{-1}(z_0) = z_0 \times X_H \) in the above coordinates on \( r^{-1}(U_k) \subset M(H) \). Let us fix some points \( z_i \in U_i \). Since \( M \) is compactly embedded into \( N \) and the path metric on \( N(H) \) is defined by a Riemannian metric pulled back from \( N \), every \( (z,x) \in r^{-1}(U_i) \), \( i \in I \), there exists a path \( \gamma_i(x) \) joining \( (z_i, x) \) with \( (z, x) \) such that the lengths of all these paths are bounded from above by a constant \( L \). Further, by the covering homotopy theorem for every \( (z_0, x) \in r^{-1}(z_0) \) there exist a point \( (z_i, s_i(x)) \in r^{-1}(U_i) \) and a path \( \tilde{\gamma}_i(x) \) joining these two points in \( M(H) \) such that \( s_i(x) = \tau_Q(H)(h_i(x)) \) for some \( h_i \in Q(H) \) (independent of \( x \)) and the lengths of all \( \tilde{\gamma}_i(x) \) (for all \( i \in I \) and \( x \in X_H \)) are bounded from above by \( L_1 \). Since \( \log \psi \) is uniformly continuous with respect to the path metric obtained by a Riemannian metric pulled back from \( N \), from here arguing as in the proof of Lemma 2.3 and using the boundedness of lengths of all the paths \( \tilde{\gamma}_i(x) \) and \( \gamma_i(x) \) we obtain that there is a constant \( C \) depending on \( M \) and \( z_0 \) such that

\[
\frac{1}{C} \psi(z, x) \leq \phi(x) \leq C \psi(z, x) \quad \text{for any} \quad z \in U_i, \ i \in I, \ x \in X_H. \tag{2.5}
\]

Now by the definition of the norm on \( \mathcal{H}_{p,\psi}(M(H)) \) we have

\[
\sup_{i \in I, x \in U_i} \left( \sum_{x \in X_H} |f(z, x)|^p \psi(z, x) \right)^{1/p} = |f|_{p,\psi}.
\]

Combining this with (2.5) we get

\[
(1/C)^{1/p} \cdot |f|_{p,\psi} \leq \sup_{i \in I, x \in U_i} ||f(z, \cdot)||_{p,\phi} \leq C^{1/p} \cdot |f|_{p,\psi}. \tag{2.6}
\]

This shows that every \( f_i(z, \cdot) \in l_{p,\phi}(X_H) \).

Let us prove now that \( f_i(\cdot, \cdot) : U_i \to l_{p,\phi}(X_H) \) is holomorphic. Without loss of generality we identify \( U_i \) with an open subset of a certain \( \mathbb{C}^n \). Let \( P \subset U_i \), \( P = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - x_j| \leq r, \ 1 \leq j \leq n\} \), be a polydisk with the center at a point \( x = (x_1, \ldots, x_n) \in U_i \) and of radius \( r \). By \( P_t \) we denote its boundary torus (i.e., \( P_t = \{z \in P : |z_j - x_j| = r, \ 1 \leq j \leq n\} \)). Let us introduce \( c_{\alpha_1,\ldots,\alpha_n}(y) \) by the formula

\[
c_{\alpha_1,\ldots,\alpha_n}(y) = \left( \frac{1}{2\pi i} \right)^n \int_{P_t} \frac{f_i(w, y)}{w_1^{\alpha_1+1} \cdots w_n^{\alpha_n+1}} dw_1 \cdots dw_n.
\]

We will show that

\[
c_{\alpha_1,\ldots,\alpha_n} \in l_{p,\phi}(X_H) \quad \text{and} \quad ||c_{\alpha_1,\ldots,\alpha_n}||_{p,\phi} \leq C_{\alpha_1,\ldots,\alpha_n}^{1/p} ||f||_{p,\psi}. \tag{2.7}
\]
In what follows $\mathbb{T}^n$ denotes the standard torus in $\mathbb{R}^n$ with coordinates $t = (t_1, \ldots, t_n)$. Now, using the definition of $c_{\alpha_1, \ldots, \alpha_n}(y)$, the triangle inequality for the norm $\| \cdot \|_{p, \phi}$, inequality (2.6) and the Lebesgue monotone convergence theorem we have

$$
\| c_{\alpha_1, \ldots, \alpha_n} \|_{p, \phi} := \left( \sum_{y \in X_H} \left| c_{\alpha_1, \ldots, \alpha_n}(y) \right|^p \phi(y) \right)^{1/p} 
$$

$$
\leq \left( \sum_{y \in X_H} \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{T}^n} \left| f_i(x + rt, y) \right|^p \frac{dt_1 \cdots dt_n}{r^{\alpha_1 + \ldots + \alpha_n}} \right)^{1/p} \cdot \phi(y) 
$$

$$
\leq \frac{1}{r^{\alpha_1 + \ldots + \alpha_n}} \cdot \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{T}^n} \left( \sum_{y \in X_H} |f_i(x + rt, y)|^p \phi(y) \right)^{1/p} dt_1 \cdots dt_n \leq C^{1/p} \| f \|_{p, \psi}.
$$

Inequality (2.7) implies, in particular, that

$$
\sum_{\alpha_1 + \ldots + \alpha_n = 0}^{\infty} c_{\alpha_1, \ldots, \alpha_n} (z_1 - x_1)^{\alpha_1} \cdots (z_n - x_n)^{\alpha_n}
$$

converges uniformly in any polydisk $P'$ centered at $x$ of radius $< r$ to a holomorphic $l_{p, \phi}(X_H)$-valued function $F$. By definition, evaluation of this function at every point $y \in X_H$ coincides with $f_i(\cdot, y)$ on any such $P'$. Thus $F(z) = f_i(z, \cdot)$ for $z \in P'$. Since $x$ is an arbitrary point of $U_i$, this shows that $f_i(\cdot, \cdot) : U_i \to l_{p, \phi}(X_H)$ is holomorphic.

Therefore by (2.6) we obtain that the map $D : \mathcal{H}_{p, \psi}(M(H)) \to B_{p, \phi}(X_H)$, $f \mapsto \{f_i\}_{i \in I}$, is well defined and continuous.

Conversely, let $\{f_i(z, x)\}_{i \in I}$ be a family of holomorphic $l_{p, \phi}(X_H)$-valued functions on the cover $\{U_i\}_{i \in I}$ determining an element from $B_{p, \phi}(X_H)$. We set

$$
f(z, x) := f_i(z, x), \quad z \in U_i, \quad x \in X_H.
$$

Then by the definition $f$ is a holomorphic function on $M(H)$. Now inequality (2.6) easily implies that $|f|_{p, \psi} \leq C^{1/p} \cdot |f|_{p, \phi}$. 

The map $D$ from Proposition 2.4 will be called the direct image map.

**Remark 2.5** One can easily see that the direct image map $D$ is an isometry in the case $\psi \equiv 1$.

**3. Proof of Theorem 1.3.**

**3.1.** Assume that $M$ satisfies condition (1.1). Let $r : M' \to M$ be an unbranched covering. By $R_x$ we denote the restriction map of functions from $\mathcal{H}_{p, \psi}(M')$ to the fibre $r^{-1}(x)$, $x \in M$. The definitions of the norms on $\mathcal{H}_{p, \psi}(M')$ and $l_{p, \psi, x}(M')$ (see section 1.1) imply that $R_x : \mathcal{H}_{p, \psi}(M') \to l_{p, \psi, x}(M')$ is a linear continuous operator.
Proposition 3.1 For every \( x \in M \) there exists a linear continuous operator \( C_x : l_{p,\psi}(M') \rightarrow \mathcal{H}_{p,\psi}(M') \) such that
\[
R_x \circ C_x = id.
\]

Proof. As before we think of \( M' \) as an open subset of \( N' \) where \( r : N' \rightarrow N \) is a covering such that \( \pi_1(N') = \pi_1(M') \). We also set \( M' := r^{-1}(\bar{M}) \). (Here by our assumption \( M \subset \bar{M} \subset N \) and \( \bar{M} \) is Steinitz.) Recall that the fibre of the covering \( N' \) is \( X_H := \pi_1(N)/H \) where \( H := \pi_1(N') \) and the quotient is taken with respect to the left action of \( H \) on \( \pi_1(N) \). In what follows, according to Proposition 2.4, we identify \( \mathcal{H}_{p,\psi}(M') \) with the space \( B_{p,\phi}(X_H) \) of bounded holomorphic sections of the Banach vector bundle \( E_{p,\phi}(X_H) \rightarrow M \) where \( \phi = \psi|_{r^{-1}(z_0)} \) for some \( z_0 \in M \) (see Example 2.2).

Let us introduce the Banach space \( B \) of complex functions \( F \) defined on \( X_H \times X_H \) with norm
\[
|F(h, g)|_B := \max \left\{ \sup_{h \in X_H} \left( \sum_{g \in X_H} |F(h, g)| \cdot \frac{\phi(g)}{\phi(h)} \right) , \sup_{g \in X_H} \left( \sum_{h \in X_H} |F(h, g)| \right) \right\}.
\]

It is easy to see by means of (2.1) that the formula
\[
[q(q, s)(F)](h, g) := F(\tau_{Q(H)}(q)(h), \tau_{Q(H)}(s)(g)), \quad h, g \in X_H, \quad q, s \in Q(H),
\]
determines a homomorphism of the group \( Q(H) \times Q(H) \) into \( GL(B) \) (see Example 2.2 (a) for the definitions of \( \tau_{Q(H)} \) and \( Q(H) \)). Note that \( Q(H) \times Q(H) \) is the quotient of \( \pi_1(N \times N) = \pi_1(N) \times \pi_1(N) \). Thus the associated with \( \rho \) holomorphic Banach vector bundle \( t : E_{\rho} \rightarrow N \times N \) on \( N \times N \) is defined. We identify the fibre \( t^{-1}(x \times x) \) with \( B \). Further, let \( \delta_h \) be a function on \( X_H \) such that \( \delta_h(g) = 1 \) if \( h = g \) and \( \delta_h(g) = 0 \) if \( h \neq g \). We define a function \( \Delta \) on \( X_H \times X_H \) by the formula
\[
\Delta(h, g) := \delta_h(g).
\]

Check that \( \Delta \in B (= t^{-1}(x \times x)) \). Indeed,
\[
|\Delta|_B := \max \left\{ \sup_{h \in X_H} \left( \sum_{g \in X_H} |\delta_h(g)| \cdot \frac{\phi(g)}{\phi(h)} \right) , \sup_{g \in X_H} \left( \sum_{h \in X_H} |\delta_h(g)| \right) \right\} = 1.
\]

Now, according to Bungart [B, Lemma 3.3]) there is a holomorphic section \( S \) of \( E_{\rho} \) such that \( S(x \times x) = \Delta \). Since \( \overline{M} \subset N \) is compact, \( S|_{M \times M} \) is a bounded holomorphic section of \( E_{\rho}|_{M \times M} \), cf. (2.3).

Let \( \{V_i\}_{i \in I} \) be a finite acyclic open cover of \( \overline{M} \). We set \( U_i = V_i \cap M \) and consider the open cover \( \mathcal{U} = \{U_i \times U_j\}_{i,j \in I} \) of \( M \times M \). According to Example 2.2 (a) \( M' \) is defined on the cover \( \{U_i\}_{i \in I} \) of \( M \) by a cocycle \( \bar{c} = \{\bar{c}_{ij}\} \in Z^1_{\partial}((U_i, Q(H)) \). Thus \( M' \times M' \) is defined on \( \mathcal{U} \) by the cocycle \( \bar{c} \times \bar{c} = \{\bar{c}_{ij} \times \bar{c}_{ij}\} \in Z^1_{\partial}(\mathcal{U}, Q(H) \times Q(H)) \).

Using this construction one represents the restriction of the section \( S \) to \( \mathcal{U} \) by a family \( \{S_{ij}(z, h, w, g) : h, g \in X_H, z \times w \in U_i \times U_j\}_{i,j \in I} \) of holomorphic functions on \( U_i \times U_j \) with values in \( B \) satisfying for \( z \times w \in (U_i \times U_j) \cap (U_l \times U_m) \)
\[
S_{ij}(z, \tau_{Q(H)}(\bar{c}_{il})(h), w, \tau_{Q(H)}(\bar{c}_{mj})(g)) = S_{lm}(z, h, w, g).
\]

(3.1)
Since \( S \) is bounded on \( M \times M \),
\[
\sup_{i,j \in I, x \times w \in U_i \times U_j} |S_{ij}(z, \cdot, w, \cdot)|_B = C < \infty .
\] (3.2)

Also, by the definition we have for \( x \times x \in M \times M \) (say, e.g., \( x \in U_{i_0}, i_0 \in I \))
\[
S_{i_0i_0}(x, h, x, g) = \delta_h(g) .
\] (3.3)

Next, consider the restriction of \( S \) to \( x \times M \) and the open cover \( \{U_{i_0} \times U_i\}_{i \in I} \) of
\( x \times M \). We define a family \( \{T_i\}_{i \in I} \) of linear operators on the set of complex functions \( v \) on \( X_H \) by the formula
\[
T_i(v)(z, g) := \sum_{h \in X_H} v(h)S_{i_0i}(x, h, z, g) , \quad z \in U_i, g \in X_H .
\] (3.4)

Lemma 3.2 (a) \( T_i \) is a bounded operator from \( l_{p,\phi}(X_H) \) into the Banach space
\( H_{p,\phi}(U_i) \) of bounded \( l_{p,\phi}(X_H) \)-valued holomorphic functions \( f \) on \( U_i \) with norm
\[
||f||_{p,\phi} := \sup_{z \in U_i} ||f(z, \cdot)||_{p,\phi}
\]
(here \( || \cdot ||_{p,\phi} \) is the norm on \( l_{p,\phi}(X_H) \), see (2.2)).

(b) For every \( z \in U_i \cap U_j \)
\[
T_i(v)(z, \tau Q_i)(g)) = T_j(v)(z, g) , \quad v \in l_{p,\phi}(X_H) .
\]

Proof. (a) Let us first check the above statement for \( p = 1 \) and \( p = \infty \). In fact, for
\( p = 1 \) from (3.4) and (3.2) we have
\[
||T_i(v)(z, \cdot)||_{1,\phi} := \sum_{g \in X_H} \left| \sum_{h \in X_H} v(h)S_{i_0i}(x, h, z, g) \frac{\phi(g)}{\phi(h)} \right| \leq
\]
\[
\sum_{g \in X_H} \left| \sum_{h \in X_H} v(h)\phi(h) \right| \cdot ||S_{i_0i}(x, h, z, g)||_{1,\phi} \frac{\phi(g)}{\phi(h)} \leq
\]
\[
\left( \sum_{h \in X_H} |v(h)| \frac{\phi(h)}{\phi(h)} \right) \cdot \sup_{h \in X_H} \left( \sum_{g \in X_H} ||S_{i_0i}(x, h, z, g)||_{1,\phi} \frac{\phi(g)}{\phi(h)} \right) \leq
\]
\[
||v||_{1,\phi} \cdot ||S_{i_0i}(x, \cdot, \cdot)||_B \leq C \cdot ||v||_{1,\phi} .
\]

From here we have (see (2.3))
\[
|T_i(v)|_{U_i}^{U_i} \leq C \cdot ||v||_{1,\phi} .
\]
Thus \( T_i(v) \in H_{1,\phi}(U_i) \), and \( T_i : l_{1,\phi}(X_H) \to H_{1,\phi}(U_i) \) is a linear bounded operator.

Similarly, for \( p = \infty \) we have
\[
|T_i(v)|_{\infty,\phi} := \sup_{z \in U_i} \left( \sup_{g \in X_H} \left| \sum_{h \in X_H} v(h)S_{i_0i}(x, h, z, g) \right| \right) \leq
\]
\[
||v||_{\infty,\phi} \sup_{z \in U_i} \left( \sup_{g \in X_H} \left( \sum_{h \in X_H} |S_{i_0i}(x, h, z, g)| \right) \right) \leq C \cdot ||v||_{\infty,\phi} .
\]
So, \( T_i : l_{\infty,\phi}(X_H) \to H_{\infty,\phi}(U_i) \) is well defined and continuous.

Let us prove the similar statement for \( 1 < p < \infty \). Consider the evaluation \( T_{z,i}(v) \) of \( T_i(v) \) at \( z \in U_i \):

\[
[T_{z,i}(v)](g) := T_i(v)(z,g) = \sum_{h \in X_H} v(h) S_{i_0i}(x,h,z,g).
\]

From the above arguments it follows that \( T_{z,i} \) is a linear continuous map of \( l_{1,\phi}(X_H) \) to \( l_{1,\phi}(X_H) \) and of \( l_{\infty,\phi}(X_H) \) to \( l_{\infty,\phi}(X_H) \), and in both these cases its norm is bounded by \( C \). Now, by the M. Riesz interpolation theorem (see, e.g., [R]), \( T_{z,i} \) maps also each \( l_{p,\phi}(X_H) \) to \( l_{p,\phi}(X_H) \) and its norm there is bounded by \( C \), as well. Taken the supremum of norms of \( T_{z,i} \) over \( z \in U_i \) we obtain that \( T_i \) is a linear continuous operator from \( l_{p,\phi}(X_H) \) into \( H_{p,\phi}(U_i) \) for any \( p \).

(b) Using (3.1) with \( i = i_0, j = i \) and \( l = i_0, m = j \) we get

\[
S_{i_0i}(x,h,z,\tau_Q(H)(\tilde{c}_{ij})(g)) = S_{i_0j}(x,h,z,g).
\]

This and (3.4) produce the result.  \( \square \)

From Lemma 3.2 we obtain that for every \( v \in l_{p,\phi}(X_H) \) the family \( \{T_i(v)\}_{i \in I} \) represents a section \( T(v) \) from \( B_{p,\phi}(X_H) \), see (2.3). Moreover, the correspondence \( v \mapsto T(v) \) determines a linear bounded operator \( T : l_{p,\phi}(X_H) \to B_{p,\phi}(X_H) \). Note also that (3.3) implies that \( T_{i_0}(v)(x,\cdot) = v \). From this identifying \( l_{p,\phi}(X_H) \) with the fibre of \( E_{p,\phi}(X_H) \) over \( x \) (by means of the coordinates on \( U_{i_0} \)) we obtain that \( T \) maps the sections of \( E_{p,\phi}(X_H)|_{\{x\}} \) to \( B_{p,\phi}(X_H) \) and \( T(v)(x) = v \). To complete the proof of the proposition it remains to identify \( B_{p,\phi}(X_H) \) with \( H_{p,\phi}(M') \) and the space of sections of \( E_{p,\phi}(X_H)|_{\{x\}} \) with \( l_{p,\psi,x}(M') \).  \( \square \)

3.2. We proceed with the proof of the theorem. According to condition (1.1) there exists a connected Stein neighbourhood \( \tilde{M} \) of \( \overline{M} \) such that \( \tilde{M} \subset \subset N \). Let \( \tilde{M}' = r^{-1}(\tilde{M}) \) be the corresponding covering of \( \tilde{M} \). Let us consider the space \( H_{p,\psi}(\tilde{M}') \). By \( E_0(\tilde{M}) := \tilde{M} \times H_{p,\psi}(\tilde{M}') \) we denote the trivial holomorphic Banach vector bundle on \( \tilde{M} \) with fibre \( H_{p,\psi}(\tilde{M}') \). Also, by \( E_{p,\phi}(\tilde{M}) \) we denote the bundle \( E_{p,\phi}(X_H) \) (see Example 2.2 (b)). (Here \( \phi \) is defined as in Proposition 3.1.)

For \( z \in \tilde{M} \) let \( R_z \) be the restriction map of functions from \( H_{p,\psi}(\tilde{M}') \) to the fibre \( r^{-1}(z) \). If we identify \( H_{p,\psi}(\tilde{M}') \) with the Banach space \( B_{p,\phi}(\tilde{M}) \) of bounded holomorphic sections of the bundle \( E_{p,\phi}(\tilde{M}) \) by the direct image map (cf. Proposition 2.4), then \( R_z \) will be the evaluation map of sections from \( B_{p,\phi}(\tilde{M}) \) at \( z \in \tilde{M} \). In particular, one can define a homomorphism of bundles \( R : E_0(\tilde{M}) \to E_{p,\phi}(\tilde{M}) \) which maps \( z \times v \in E_0(\tilde{M}) \) to the vector \( R_z(v) \) in the fibre over \( z \) of the bundle \( E_{p,\phi}(\tilde{M}) \) (see Definition 2.1). Now, by Proposition 3.1 with \( M \) replaced by \( \tilde{M} \), every \( R_z \) is surjective and, moreover, there exists a linear continuous map \( C_z \) of the fibre \( E_{p,\phi,z}(\tilde{M}) \) of \( E_{p,\phi}(\tilde{M}) \) over \( z \) to the fibre \( E_{0,z}(\tilde{M}) \) of \( E_0(\tilde{M}) \) over \( z \) such that \( R_z \circ C_z = \text{id} \). Finally, by \( \text{Ker} R := \bigcup_{z \in \tilde{M}} \text{Ker} R_z \subset E_0(\tilde{M}) \) we denote the kernel of \( R \).

Let \( U \subset \tilde{M} \) be an open set biholomorphic to a Euclidean ball. Using some holomorphic trivializations of \( E_0(\tilde{M})|_U \) and \( E_{p,\phi}(\tilde{M})|_U \) we may assume that \( E_0(\tilde{M})|_U = U \times B_{p,\phi}(\tilde{M}) \) and \( E_{p,\phi}(\tilde{M})|_U = U \times l \), where \( l = l_{p,\phi}(X_H) \) is the fibre of \( E_{p,\phi}(\tilde{M}) \).
In these trivializations we have \( R(z \times v) := z \times R_z(v) \), \( z \times v \in U \times B_{p,\varphi}(\tilde{M}) \), where \( R_z(v) := v(z) \), and \( C_z(z \times f) := z \times C_z(f) \), \( z \times f \in U \times l \), where \( C_z : l \to B_{p,\varphi}(\tilde{M}) \) is a linear continuous map.

Next, take \( x \in U \). Then in the above trivializations \( R_z \circ C_z : l \to l \), \( z \in U \), is a family of linear continuous operators, holomorphic in \( z \), such that \( R_x \circ C_x = id \). Thus by the inverse function theorem (which in this case follows easily from the Taylor expansion of \( R_z \circ C_x \) at \( x \)) we obtain that there exists a neighbourhood \( U_x \) of \( x \) such that \( R_z \circ C_x \) is invertible for every \( z \in U_x \).

We set \( P_z := (R_z \circ C_x)^{-1} \), and \( \tilde{C}_z := C_x \circ P_z : l \to B_{p,\varphi}(\tilde{M}) \), \( z \in U_x \). Then \( P_z \) and \( \tilde{C}_z \) are holomorphic in \( z \in U_x \), and by definition

\[
R_z \circ \tilde{C}_z = id.
\]

It is easy to see (by the inverse function theorem) that this identity implies that there is a neighbourhood \( V_x \subset U_x \) of \( x \) such that \( Ker R|_{V_x} \) is biholomorphic to \( V_x \times Ker R_x \) and this biholomorphism is linear on every \( Ker R_z \) and maps this space onto \( z \times Ker R_z \), \( z \in V_x \).

Taking different \( U \) and \( x \) we then obtain from here that \( Ker R \) is a holomorphic Banach vector subbundle of \( E_0(\tilde{M}) \) and locally, for every \( V_x \) as above, we have an isomorphism of bundles \( E_0(\tilde{M})|_{V_x} \cong E_{p,\varphi}(\tilde{M})|_{V_x} \oplus (Ker R)|_{V_x} \) given in the above trivializations by the formula

\[
z \times v \mapsto (z \times R_z(v), \ z \times (id - \tilde{C}_z \circ R_z)(v)).
\]

Let \( U = \{U_i\}_{i \in I} \) be an open cover of \( \tilde{M} \) by Stein sets such that every quotient bundle \( E_{p,\varphi}(\tilde{M})|_{U_i} \) is complemented in \( E_0(\tilde{M})|_{U_i} \). Let \( C_i : E_{p,\varphi}(\tilde{M})|_{U_i} \to E_0(\tilde{M})|_{U_i} \) be the complement homomorphism (i.e., \( R|_{U_i} \circ C_i \) is the identity homomorphism of \( E_{p,\varphi}(\tilde{M})|_{U_i} \)). We set

\[
C_{ij}(z) = C_i(z) - C_j(z), \quad z \in U_i \cap U_j.
\]

Then \( \{C_{ij}\} \) is a holomorphic 1-cocycle with values in \( Ker R \). Since \( U \) is acyclic and \( \tilde{M} \) is Stein, by the Bungart theorem [B] and the Leray lemma one can find a family \( \{H_i\}_{i \in I} \) of holomorphic sections of \( Ker R \) over \( U_i \) such that

\[
H_i(z) - H_j(z) = C_{ij}(z), \quad z \in U_i \cap U_j.
\]

In particular, setting \( F|_{U_i} := C_i - H_i \) we obtain that \( F : E_{p,\varphi}(\tilde{M}) \to E_0(\tilde{M}) \) is a (holomorphic) homomorphism of bundles such that \( R \circ F = id \). Since by the definition \( \overline{M} \subset \tilde{M} \) is compact, \( F|_{\overline{M}} \) is bounded. Next, by \( R_{\overline{M}} \) we denote the restriction homomorphism of trivial bundles \( E_0(\tilde{M})|_{\overline{M}} \to E_0(\tilde{M}) \) defined by

\[
z \times v \mapsto z \times v|_{\overline{M}}, \quad z \in \overline{M}, v \in \mathcal{H}_{p,\varphi}(\tilde{M}').
\]

Finally, we set

\[
L_z := R_{\overline{M}} \circ F(z), \quad z \in \overline{M}.
\]

Then by definition, every \( L_z \) is a linear continuous map of \( l_{p,\varphi,z}(M') \) into \( \mathcal{H}_{p,\varphi}(M') \) (see section 1.1), the family \( \{L_z\} \) is holomorphic in \( z \in M \), \( R_z \circ L_z = id \), and \( \sup_{z \in M} ||L_z|| < \infty \).

This completes the proof of the theorem. □
Remark 3.3 Using some modification of the arguments of the above proof one can show that for a fixed \( \psi \) there exists a family \( \{L_z\} \) satisfying the assumptions of Theorem 1.3 such that \( \sup_{z \in M} ||L_z|| \leq C < \infty \) where \( C \) does not depend on the class \( H_{p,\psi}(M') \) (cf. the construction in the proof of Theorem 1.4. of \( \text{Br2} \)).

4. Proofs.

Proof of Corollary 1.4. The result follows straightforwardly from formula (1.4) where \( f \) is a holomorphic function with values in \( H_{p,\psi}(M') \) and from the properties of the operators \( L_z \). \( \Box \)

Proof of Theorem 1.8. Let \( f \in H(r^{-1}(C)) \) be a function satisfying the assumptions of the theorem. Consider the function

\[
    h(z) := L_z(f|_{r^{-1}(z)}), \quad z \in C .
\]

By Definition 1.5, Proposition 2.4 and by the properties of \( \{L_z\} \) we have that \( h \) is a \( H_{p,\psi}(M') \)-valued continuous function on \( C \) holomorphic in \( D \). (It can be written as the scalar function of the variables \( (z,w) \in C \times M' \).) Therefore it suffices to prove an approximation theorem for such Banach-valued functions. Namely, it suffices to show that every such \( h \) can be uniformly approximated on \( C \) by \( H_{p,\psi}(M') \)-valued holomorphic functions defined in a neighbourhood \( \Omega \) of \( C \).

Further, if \( \{h_i(z,w) : z \in \Omega, w \in M'\}_{i \geq 1} \) is such an approximation sequence for \( h \), then \( \{f_i(y) : y \in r^{-1}(\Omega)\}_{i \geq 1} \) with \( f_i(y) := h_i(r(y),y) \) is the approximation sequence for \( f \) satisfying the required statement of the theorem.

The proof of the above approximation theorem for functions \( h \) repeats word-for-word the proof of Theorem 3.5.1 in \( \text{HL} \) where in all integral formulas we replace the scalar functions by Banach-valued ones. We leave the details to the readers. \( \Box \)

Proof of Theorem 1.10. Let us consider a function \( f \) satisfying either (1) or (2). Define the function

\[
    h(z) := L_z(f|_{r^{-1}(z)}), \quad z \in X \text{ (or } z \in X) .
\]

Then according to Definition 1.5, \( h \) is a holomorphic function on the submanifold \( X \subset M \) with values in \( H_{p,\psi}(M') \) (and in case (2) it is also continuous on \( X \)). Thus it suffices to prove the extension theorem for Banach-valued holomorphic functions on \( X \) (extending them to \( M \)). Evaluating the extended Banach-valued functions at the points \( (r(y),y), y \in M' \) (cf. the proof of Theorem 1.8 above), we get the required result.

The scalar case of the required extension theorem is proved in Theorem 4.11.1 of \( \text{HL} \). The Banach-valued case repeats literally the arguments of the proof of Theorem 4.11.1 where in all integral formulas we replace scalar functions by Banach-valued ones. Also, instead of the classical Cartan B theorem for Stein manifolds, we use in the proof its Banach-valued generalization due to Bungart. \( \Box \)
Theorem 1.3 (with the same \( p \))

Below we use the notation of the Leray integral formula (1.4) (see section 1.2).

We present a multi-dimensional analog of Cauchy-Green formulas on coverings of similar ones on the domains \( M \). We denote the bundle \( \mathcal{E} \) of \( \mathbb{R}^n \)-valued \( 0 \)-forms on \( M \) such that \( \psi \in \partial M \). By \( \mathcal{H}_{p,\psi}(r^{-1}(X)) \) we denote the family of operators given by \( \mathcal{H}_{p,\psi}(r^{-1}(X)) \equiv \mathcal{H}_{p,\psi}(M) \) such that \( \psi \in \partial M \) and \( \mathcal{H}_{p,\psi}(r^{-1}(X)) \equiv \mathcal{H}_{p,\psi}(M) \).

5. Appendix.

We present a multi-dimensional analog of Cauchy-Green formulas on coverings of domains \( M \subset \mathbb{C}^n \) satisfying condition (1.1). These formulas are obtained from similar ones on the domains \( M \) (see, e.g., [H]) by the application of Theorem 1.3. Below we use the notation of the Leray integral formula (1.4) (see section 1.2).

Let \( M \subset \mathbb{C}^n \) be a domain with a rectifiable boundary. Assume that \( M \) satisfies condition (1.1), that is, \( M \subset \subset N \subset N \subset \mathbb{C}^n \), \( \pi_1(N) = \pi_1(M) \) and \( M \) is Stein. Let \( \eta = \xi(z) = (\eta_1(z), \ldots, \eta_n(z)) \) be a smooth \( \mathbb{C}^n \)-valued function of the variable \( \xi \in \overline{M} \) such that \( < \eta(\xi,z), \xi-z > = 1 \) for \( \xi \in \partial M \). Consider a covering \( M' \) of \( M \). Then \( M' \) is an open subset of the covering \( r : N' \to N \) such that \( \pi_1(N') = \pi_1(M') \).

Let \( \psi : N' \to \mathbb{R}_+ \) such that \( \log \psi \) is uniformly continuous with respect to the path metric obtained by a Riemannian metric pulled back from \( N \). By \( E_{p,\psi}(\hat{M}) \) we denote the bundle \( E_{p,\psi}(X_H) \) on \( \hat{M} \) (see Example 2.2 (b)). Here \( \phi \) is defined as in Proposition 3.1. Also, by \( L = \{L_z\} \) we denote the family of operators given by Theorem 1.3 (with the same \( p \) and \( \psi \)) where \( z \) varies in a small neighbourhood \( \Omega \) of \( \overline{M} \) (in this case \( L_z \) maps \( \mathcal{H}_{p,\psi}(r^{-1}(\Omega)) \) into \( \mathcal{H}_{p,\psi}(r^{-1}(\Omega)) \)).

Suppose that a function \( f \) defined on \( \overline{M} \) is such that its direct image \( r_*(f) \) with respect to \( r \) is a continuous section of \( E_{p,\psi}(\hat{M}) \text{ on } \overline{M} \) and \( \overline{r}_*(f) \) is a continuous section of \( E_{p,\psi}(\hat{M}) \text{ on } \overline{M} \).

We set

\[
(Lf)(z) := L_z(f|_{r^{-1}(z)}) , \quad z \in \overline{M} .
\]

Then from the construction of the family \( L \) and from the definition of \( f \) it follows that \( Lf \) is a continuous \( \mathcal{H}_{p,\psi}(r^{-1}(\Omega)) \)-valued function on \( \overline{M} \) such that \( \overline{r}Lf \) is a continuous \( \mathcal{H}_{p,\psi}(r^{-1}(\Omega)) \)-valued \( (0,1) \)-form on \( \overline{M} \).

Now, for the function \( f \) we have the following integral representations:

\[
f(z) = \tilde{K}^s f(z) + \tilde{H}^s(\overline{r}f)(z) , \quad z \in M' ,
\]

where, for \( y := r(z) \),

\[
\tilde{K}^s f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\xi \in \partial M} L_\xi(f|_{r^{-1}(\xi)})(z) \omega'(\eta(\xi,y)) \wedge \omega(\xi) ,
\]

Remark 4.1 One can show (cf. the remark after the proof of Theorem 4.1.1 of [HL]) that under the hypothesis (1) of Theorem 1.10 there exists a linear continuous operator \( E : \mathcal{H}_{p,\psi}(r^{-1}(X)) \to \mathcal{H}_{p,\psi}(M') \) such that \( Ef = f \) on \( r^{-1}(X) \) for all \( f \in \mathcal{H}_{p,\psi}(r^{-1}(X)) \). Moreover, the norm \( ||E|| \) of \( E \) is bounded by a constant depending only on \( \psi : M' \to \mathbb{R}_+ \) and \( X \). In particular, if \( \psi \equiv 1 \), then \( ||E|| \) depends only on \( X \) (and not on the covering \( M' \)). However, it is not clear whether the extension in Theorem 1.10 (2) can be made by a linear continuous operator, as well.
\[
\tilde{H}^0(\partial f)(z) = 
\frac{(n-1)!}{2\pi i^n} \left( \int_{(\xi,\lambda_0) \in \partial M \times [0,1]} (\overline{\partial L}_{\xi}(f|_{r^{-1}(\xi)}))(z) \wedge \omega' \left( (1 - \lambda_0) \frac{\overline{\xi} - \overline{y}}{|\xi - y|^2} + \lambda_0 \eta(\xi, y) \right) \wedge \omega(\xi) \right) 
\]

\[
+ \int_{\xi \in M} (\overline{\partial L}_{\xi}(f|_{r^{-1}(\xi)}))(z) \wedge \omega' \left( \frac{\overline{\xi} - \overline{y}}{|\xi - y|^2} \right) \wedge \omega(\xi) \right).
\]

And for \( s > 0 \),
\[
\tilde{K}^s f(z) = 
\frac{(n+s-1)!}{(s-1)!(2\pi i)^n} \int_{\xi \in M} L_{\xi} (f|_{r^{-1}(\xi)})(z) (1 - <\eta(\xi, y), \xi - y>)^{s-1} \cdot \omega(\eta(\xi, y)) \wedge \omega(\xi),
\]

\[
\tilde{H}^s (\partial f)(z) = 
\frac{(n+s-1)!}{(s-1)!(2\pi i)^n} \int_{(\xi,\lambda_0) \in \partial M \times [0,1]} (\overline{\partial L}_{\xi}(f|_{r^{-1}(\xi)}))(z) \times (\lambda_0 (1 - <\eta(\xi, y), \xi - y>)^{s-1} \times 
\]

\[
\omega' \left( (1 - \lambda_0) \frac{\overline{\xi} - \overline{y}}{|\xi - y|^2} + \lambda_0 \eta(\xi, y) \right) \wedge \omega(\xi) \right).
\]

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