BANACH SPACE ACTIONS AND $L^2$-SPECTRAL GAP

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Abstract. Żuk proved that if a finitely generated group admits a Cayley graph such that the Laplacian on the links of this Cayley graph has a spectral gap $\frac{\lambda}{2}$, then the group has property (T), or equivalently, every affine isometric action of the group on a Hilbert space has a fixed point. We prove that the same holds for affine isometric actions of the group on a uniformly curved Banach space (for example an $L^p$-space with $1 < p < \infty$ or an interpolation space between a Hilbert space and an arbitrary Banach space) as soon as the Laplacian on the links has a two-sided spectral gap $1 - \epsilon$. This two-sided spectral gap condition is equivalent to the fact that the Markov operator on the links has small norm. The latter is a condition that behaves well with respect to interpolation techniques, which is a key point in our arguments.

Our criterion directly applies to random groups in the triangular model for densities $\frac{1}{3}$, partially generalizing recent results of Drutu and Mackay.

Additionally, we obtain results on the eigenvalues of $p$-Laplacians on graphs and reversible Markov chains that may be of independent interest.

1. Introduction and main results

Fixed point properties for group actions on metric spaces, e.g. Banach spaces or non-positively curved spaces, are natural rigidity properties that contribute to the understanding of both groups and the spaces on which they act. When considering actions on Banach spaces, the natural actions to consider are affine isometric actions. Given a Banach space $X$, a topological group is said to have property (F$_X$) if every continuous affine isometric action of the group has a fixed point. In this article, we deal with fixed point properties for countable discrete groups. In this setting, every affine isometric action is automatically continuous.

Property (F$_X$) was introduced by Bader, Furman, Gelander and Monod [1] as a Banach space version of Serre’s property (FH). A topological group has property (FH) if every continuous affine isometric action of the group on a Hilbert space has a fixed point. It is well known that a countable group has property (FH) if and only if it has property (T), which is a rigidity property for groups that was introduced by Kazhdan [18]. A group has property (T) if its trivial representation is isolated in the unitary dual of the group equipped with the Fell topology. Both property (T) and property (FH) have lead to striking results in several areas of mathematics, e.g. group theory, combinatorics, ergodic theory, dynamical systems, measure theory and operator algebras. We refer to [5] for a detailed account of property (T) and property (FH).

Partly because of the aforementioned connections with different areas of mathematics, recent years have shown a growing interest in Banach space versions of both fixed point properties and property (T). Alongside property (F$_X$), as recalled...
above, Bader, Furman, Gelander and Monod also defined a Banach space version of property (T), which is called property (T$_X$) and is in general weaker than property (F$_X$) (see [1, Theorem 1.3]). Another notable Banach space strengthening of property (T) is strong property (T), which Lafforgue introduced in relation to his work on the Baum-Connes Conjecture [22, 23]. He proved that if a group has strong property (T) relative to a Banach space $X \oplus \mathbb{C}$, then the group has property (F$_X$).

So far, all results that provide examples of groups with property (F$_X$) focus either on a rather specific class of groups or on a rather specific class of Banach spaces. The most straightforward non-Hilbertian Banach spaces to consider are $L^p$-spaces, with $p \neq 2$. For $1 \leq p < \infty$, a countable group is said to have property (F$_L^p$) if every affine isometric action of the group on an $L^p$-space has a fixed point. It is known that property (T) implies property (F$_L^p$) for $p \in [1, 2 + \varepsilon)$, where $\varepsilon$ may depend on the group (see [1, Theorem 1.3] (and also [11]) for the case $p \in (1, 2 + \varepsilon)$ and [2, Corollary D] for $p = 1$). In several cases, there are explicit lower bounds on $\varepsilon$ (see [7, 34, 12]). On the other hand, there are groups with property (T) that are known to fail property (F$_L^p$) for large $p$ [35, 8, 40, 10], e.g. cocompact lattices in $Sp(n,1)$. However, lattices in connected simple higher-rank Lie groups and lattices in connected simple higher-rank algebraic groups over non-Archimedean local fields have property (F$_L^p$) for all $p \in [1, \infty)$ (see [1, Theorem B] and [2, Corollary D]). Similar results have been established for universal lattices [28].

Bader, Furman, Gelander and Monod conjectured that (lattices in) connected simple higher-rank Lie groups and (lattices in) connected simple higher-rank algebraic groups over non-Archimedean local fields have property (F$_X$) for every superreflexive Banach space $X$ [1, Conjecture 1.6]. This conjecture has been proven in the non-Archimedean setting [22, 24], and in the real and complex case, partial results have been obtained [21, 20]. Other results that show fixed point properties for groups by means of establishing an appropriate strengthening of property (T) were obtained by Oppenheim [34]. His examples include certain groups acting on buildings and Kac-Moody-Steinberg groups.

Another effective way of establishing fixed point properties or property (T) for a group is by means of spectral conditions on the links of vertices of certain simplicial complexes on which the group acts. The idea of this method goes back to [15] and was further developed in [36, 41, 42, 8] in order to provide criteria to establish property (T). Nowadays, the most well-known spectral criterion for property (T) may be the one due to Żuk [42], asserting that if $\Gamma$ is a finitely generated group with finite symmetric generating set $S$ such that the smallest non-zero eigenvalue of the Laplacian of the link graph $L(S)$ associated with $S$ is strictly larger than $\frac{1}{2}$, then $\Gamma$ has property (T).

In recent years, these spectral criteria have been generalized to fixed point properties for group actions on Banach spaces, by Bourdon [7] to actions on $L^p$-spaces, and by Nowak [31] and by Oppenheim [33] to actions on reflexive spaces. Oppenheim also explains that the assumption of reflexivity is not needed in his approach, and his proof is more elementary. A key ingredient in the works of Bourdon and Nowak is a certain Poincaré inequality. In Nowak’s article, such an inequality is in fact a condition; in Bourdon’s result, the Poincaré inequality follows from estimates on the $p$-Laplacian, which is similar to Żuk’s approach.

In this article, we establish a criterion for groups that ensures that every affine isometric action of the group on a given uniformly curved Banach space has a fixed
point. Uniform curvedness is a property for Banach spaces defined by Pisier (see Section 2.4), which is stable under passing to subspaces and equivalent renormings. Examples of uniformly curved spaces are \( L^p \)-spaces and interpolation spaces between a Hilbert space and an arbitrary Banach space, i.e. strictly \( \theta \)-Hilbertian spaces. We only consider complex Banach spaces, but is is straightforward to formulate our results in the setting of real Banach spaces.

In what follows, if \( L \) is a finite graph, we denote by \( A_L \) the Markov operator of the random walk on \( L \) (see Section 3 for the definition). Our spectral criterion is as follows.

**Theorem A.** Let \( X \) be a uniformly curved Banach space. Then there exists an \( \varepsilon > 0 \) (depending on \( X \)) such that the following holds: if \( \Gamma \) is a group that admits a properly discontinuous cocompact action by simplicial automorphisms on a locally finite simplicial 2-complex \( M \) such that for all its links \( L \), we have \( \| A_L \|_{B(L^2(L,\nu))} < \varepsilon \), then \( \Gamma \) has (\( F_X \)).

If \( p \geq 2 \) and \( X \) is an \( L^p \)-space (or, more generally, a subquotient of a strictly \( \frac{p}{2} \)-Hilbertian space), then the proof of the theorem gives the value \( \varepsilon = 2p - \frac{1}{2} - \frac{1}{2} \).

If \( X \) is at Banach-Mazur distance \( C \) from such a space, then the proof gives the value \( \varepsilon = (2p - \frac{1}{2} - \frac{1}{2})/C^{\frac{1}{p} - 1} \).

Theorem A provides a widely applicable criterion for fixed point properties for finitely presented groups, since such groups naturally act on the Cayley complex associated with the presentation.

The criterion of Theorem A is a direct analogue of Żuk’s spectral criterion mentioned above, since the condition \( \| A_L \|_{B(L^2(L,\nu))} < \varepsilon \) means that the spectrum of \( A_L \), apart from a simple eigenvalue 1, is contained in \((-\varepsilon, \varepsilon)\), or equivalently, that the spectrum of the Laplacian on \( L \), apart from a simple eigenvalue 0, is contained in \((1 - \varepsilon, 1 + \varepsilon)\). This condition can be viewed as a two-sided spectral gap of the Laplacian.

Theorem A follows from a more general criterion for fixed point properties that we prove (see Theorem C), which is formulated in terms of the norm of the Markov operator acting on vector-valued \( L^p \)-spaces. In the proof of this result, working with Markov operators rather than Laplacians makes a real difference, since one can use interpolation techniques.

The use of spectral criteria is particularly beneficial when considering random groups. The framework of random groups provides ways to consider finitely presented groups in which the relators are chosen at random according to some prescribed probability measure on the set of all possible words in the generating set. It is used to study structural properties of “typical” groups. The theory of random groups goes back to [10], in which Gromov introduced what is now called the Gromov density model \( G(n, l, d) \) (see also [17]), in which the density \( d \) is a parameter that controls the number of relators. It was proven by Gromov that for \( d < \frac{1}{2} \), a random group in \( G(n, l, d) \) is infinite and hyperbolic with overwhelming probability (w.o.p.), whereas for \( d \geq \frac{1}{2} \), a group in \( G(n, l, d) \) is trivial or \( \mathbb{Z}_2 \) w.o.p. [16] (see also [32]).

The study of property (T) for random groups was initiated by Żuk [42]. By using his aforementioned criterion, he proved that for \( d > \frac{1}{2} \), a random group in the triangular model \( M(m, d) \), which is an adaptation of Gromov’s density model that is particularly suitable for the use of the spectral criterion, has property (T).
w.o.p. The fact that for \(d > \frac{1}{3}\), a group in the Gromov density model \(G(n, l, d)\) has property (T) w.o.p. was proven in detail in [19].

Our criterion (Theorem A) leads to the following result on fixed point properties for random groups.

**Theorem B.** Let \(X\) be a uniformly curved Banach space. For every density \(d > \frac{1}{3}\), a random group in the triangular model \(M(m, d)\) has property \((F_X)\) w.o.p., that is
\[
\lim_{m \to \infty} \mathbb{P}(\Gamma \in M(m, d) \text{ has } (F_X)) = 1.
\]

Theorem B partially generalizes the results on property (T) for random groups to the setting of actions of random groups on non-Hilbertian Banach spaces, but they are not the first results in this direction. As mentioned above, Bourdon [7], Nowak [31] and Oppenheim [33] already formulated spectral criteria for fixed point properties on certain non-Hilbertian Banach spaces. Nowak also applied his criterion in the setting of random groups and obtained results on fixed point properties on \(L^p\)-spaces (see [31, Section 6]). Moreover, in a very recent article, Drutu and Mackay made substantial contributions to the understanding of property \((FL^p)\) in the setting of random groups [12]. The main part of their argument consists of establishing new bounds on the first positive eigenvalue of the \(p\)-Laplacian on random graphs. By applying Bourdon’s criterion, they obtain fixed point properties of actions on \(L^p\)-spaces. For densities \(d > \frac{1}{3}\), our results generalize, with a more direct proof, the results of Drutu and Mackay in the triangular model. The significant advantage of our approach is that we can rely on well-known results on the eigenvalues of the \((2-)\)Laplacian on random graphs. For completeness, we also present an elementary proof of the fact that Poincaré inequalities give rise to fixed points (see Theorem 4.1). The line of proof is similar to Oppenheim’s proof, and we claim no originality at this point. However, we insist on the importance of the Poincaré inequality in our approach.

Theorem C. Let \(1 < p < \infty\), and let \(X\) be a superreflexive Banach space. Then there exists an \(\varepsilon' = \varepsilon'(p, X) > 0\) such that the following holds: if \(\Gamma\) is a group that admits a properly discontinuous cocompact action by simplicial automorphisms on a locally finite simplicial 2-complex \(M\) such that for all its links \(L\), we have
\[
\|A_L\|_{B(L^p(L, \nu; X))} < \varepsilon',
\]
then \(\Gamma\) has \((F_X)\).

The essential part of the proof of Theorem C is to derive a \(p\)-Poincaré inequality with small constant from the fact that the Markov operator has small norm. From that point, the result follows from the proof of the aforementioned result of Bourdon or from the result of Oppenheim. For completeness, we also present an elementary proof of the fact that Poincaré inequalities give rise to fixed points (see Theorem 4.1). The line of proof is similar to Oppenheim’s proof, and we claim no originality at this point. However, we insist on the importance of the Poincaré inequality in our approach.
The article is organized as follows. Section 2 covers some preliminaries on the geometry of Banach spaces. In Section 3, we explain how small Markov operators give rise to Poincaré inequalities. This section also includes some new results on the eigenvalues of $p$-Laplacians with applications to random graphs, and on the $p$-dependence of $p$-Poincaré inequalities. These results may be of independent interest. In Section 4, we explain how Poincaré inequalities give rise to fixed points. Theorem C and Theorem A are proven in Section 5. Fixed point properties for random groups are investigated in Section 6.

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2. Preliminaries on Banach spaces

2.1. Superreflexivity and uniform convexity. A Banach space $Y$ is said to be finitely representable in a Banach space $X$ if for every finite-dimensional subspace $U$ of $Y$ and every $\varepsilon > 0$, there exists a subspace $V$ of $X$ such that $d(U, V) < 1 + \varepsilon$, where $d$ is the Banach–Mazur distance. A Banach space $X$ is called superreflexive if every Banach space that is finitely representable in $X$ is reflexive. Equivalently, a Banach space $X$ is superreflexive if and only if all its ultrapowers are reflexive.

A Banach space $X$ is uniformly convex if $\delta_X(\varepsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} \middle| \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\} > 0$ for all $\varepsilon \in (0, 2]$. The function $\delta_X$ is called the modulus of convexity of $X$.

Every uniformly convex Banach space is superreflexive, and every superreflexive Banach space admits an equivalent uniformly convex norm [13].

Let $p \in [2, \infty)$. A Banach space $X$ is called $p$-uniformly convex if there exists a $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^p$ for all $\varepsilon \in (0, 2]$. By a famous theorem of Pisier [37], every uniformly convex Banach space has an equivalent norm with respect to which it is $p$-uniformly convex for some $p \in [2, \infty)$.

It follows from [27, Lemma 6.5] that if $X$ is a $p$-uniformly convex space, then there exists a constant $C > 0$ such that for every $X$-valued random variable $U$,

$$\|\mathbb{E}(U)\|^p + C\mathbb{E}[\|U - \mathbb{E}[U]\|^p] \leq \mathbb{E}[\|U\|^p].$$

2.2. Complex interpolation. We refer to [6] and [39] for details on complex interpolation for compatible couples of (complex) Banach spaces. We just recall that a compatible couple $(X_0, X_1)$ of Banach spaces is a pair of Banach spaces together with continuous linear embeddings from $X_0$ and $X_1$ into the same topological vector space $X$, which can always be assumed to be a Banach space. Complex interpolation is a way to assign to such a couple $(X_0, X_1)$ a family $(X_\theta)_{\theta \in [0, 1]}$ of Banach spaces (subspaces of $X$) that interpolate between $X_0$ and $X_1$. For example, if $(\Omega, \mu)$ is a measure space and $(X_0, X_1) = (L^p_0(\Omega, \mu), L^p_1(\Omega, \mu))$ (seen as
subspaces of the topological vector space of all measurable maps from $\Omega$ to $\mathbb{C}$, then $X_0$ is the space $L^p(\Omega, \mu)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. More generally, if $(X_0, X_1)$ is a compatible couple, then the complex interpolation space of parameter $\theta$ for the couple $(L^{p_0}(\Omega, \mu; X_0), L^{p_1}(\Omega, \mu; X_1))$ is $L^{p_\theta}(\Omega, \mu; X_\theta)$. A fundamental property of complex interpolation (known as Stein’s interpolation theorem) is the following: if $(X_0, X_1)$ and $(Y_0, Y_1)$ are compatible couples and if there is a holomorphic family of linear operators $T_z : X_0 + X_1 \to Y_0 + Y_1$, with $\text{Re}(z) \in [0, 1]$, such that for all $z$ with $\text{Re}(z) \in (0, 1]$, we have $\|T_z : X_{\text{Re}(z)} \to Y_{\text{Re}(z)}\| \leq 1$, then $\|T_z : X_{\text{Re}(z)} \to Y_{\text{Re}(z)}\| \leq 1$ for all $z$ with $\text{Re}(z) \in [0, 1]$.

2.3. $\theta$-Hilbertian spaces. A strictly $\theta$-Hilbertian space is a Banach space that can be written as an interpolation space $(X_0, X_1)_\theta$, where $X_1$ a Hilbert space and $\theta \in [0, 1]$ (see [39]). $L^p$-spaces are clearly strictly $\theta$-Hilbertian: if $X$ is an $L^p$-space with $p \geq 2$, then $X = (X_0, X_1)_\theta$, where $X_0 = L^\infty$, $X_1 = L^2$ and $\theta = \frac{2}{p}$.

In the setting of strictly $\theta$-Hilbertian spaces, we can derive (1) with an explicit constant $C$.

**Proposition 2.1.** If $X$ is isometric to a subquotient of a strictly $\theta$-Hilbertian space, then (1) holds with $C = 4^{1-\frac{\theta}{2}}$.

**Proof.** If $Y$ is a subquotient of $X$, then the best constant in (1) is smaller for $Y$ than for $X$. Therefore, it is sufficient to consider the case when $X$ is strictly $\theta$-Hilbertian. Consider a complex interpolation space $X = (X_0, X_1)_\theta$ between a Hilbert space $X_1$ and an arbitrary Banach space $X_0$, continuously embedded into the same Banach space $X$. Fix a probability space $(\Omega, \mu)$, and consider the holomorphic family $T_z : U \in L^1(\Omega; X) \mapsto (E(U), 2^{-1}(U - E(U))) \in X \oplus L^1(\Omega; X)$. If $\text{Re}(z) = 0$, then we have $\|T_z : L^\infty(\Omega; X_0) \to X_0 \oplus L^\infty(\Omega; X_0)\| \leq 1$, and for $\text{Re}(z) = 1$, we have $\|T_z : L^2(\Omega; X_1) \to X_1 \oplus L^2(\Omega; X_1)\| \leq 1$. By the results recalled in Section 2.2, the complex interpolation space of parameter $\theta$ between $L^\infty(\Omega; X_0)$ and $L^2(\Omega; X_1)$ (respectively $X_0 \oplus L^\infty(\Omega; X_0)$ and $X_1 \oplus L^2(\Omega; X_1)$) is $L^{p_\theta}(\Omega; X_\theta)$ (respectively $X_\theta \oplus L^{p_\theta}(\Omega; X_\theta)$, where $\frac{1}{p_\theta} = \frac{\theta}{2}$. By Stein’s interpolation theorem, we have $\|T_z : L^{p_\theta}(\Omega; X_\theta) \to X_\theta \oplus L^{p_\theta}(\Omega; X_\theta)\| \leq 1$. This is exactly (1) with constant $C = \frac{1}{\sqrt{1-\theta}} = 4^{1-\frac{\theta}{2}}$. \hfill \Box

More generally, one can consider the class of $\theta$-Hilbertian spaces, as introduced by Pisier in [39], which is a natural class of Banach spaces that includes the strictly $\theta$-Hilbertian spaces, but also certain interpolation spaces between compatible families (rather than couples) of Banach spaces. Every result that we mention for strictly $\theta$-Hilbertian spaces can be extended to the class of $\theta$-Hilbertian spaces by considering complex interpolation for families of Banach spaces.

2.4. Uniform curvedness. The notion of uniformly curved Banach space was introduced by Pisier in [39]. Let $X$ be a Banach space, and let $T : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ an operator. If $T \otimes \text{id}_X$ extends to a bounded operator $T_X$ from $L^2(\Omega_1, \mu_1; X)$ to $L^2(\Omega_2, \mu_2; X)$, then we denote by $\|T_X\|$ its norm. Otherwise, we set $\|T_X\| = \infty$. For a Banach space $X$, we set $\Delta_X(\varepsilon) = \sup \|T_X\|$, where the supremum is taken over all measure spaces $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ and operators $T : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ satisfying $\|T : L^1(\Omega_1, \mu_1) \to L^1(\Omega_2, \mu_2)\| \leq 1$, $\|T : L^\infty(\Omega_1, \mu_1) \to L^\infty(\Omega_2, \mu_2)\| \leq 1$ and $\|T : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)\| \leq \varepsilon$.

**Definition 2.2.** A Banach space $X$ is uniformly curved if $\Delta_X(\varepsilon) \to 0$ when $\varepsilon \to 0$.
Pisier proved that uniformly curved spaces are superreflexive \[39\], and hence, by the results recalled in Section 2.1, every uniformly curved space has an equivalent \(p\)-uniformly convex norm for some \(p \in [2, \infty)\). Pisier also showed that the Banach spaces \(X\) for which \(\Delta_X(\epsilon) = O(\epsilon^\alpha)\) for some \(\alpha > 0\) are exactly the spaces that are isomorphic to a subquotient of a \(\theta\)-Hilbertian space for some \(\theta > 0\).

3. Graphs, eigenvalues and Poincaré inequalities

3.1. \(p\)-Poincaré inequalities. In this article, graphs are not oriented. Unless explicitly stated otherwise, all graphs in this article are assumed to be finite, connected, and without loops or multiple edges. If \(G\) is a graph, we will write \(G = (V, E)\), where \(V\) is the vertex set of \(G\) and

\[ E = \{(s, t) \in V \times V \mid s\text{ and } t \text{ are connected by an edge}\}, \]

so \(E\) is the set of all edges of \(G\) together with orientations. The set \(E\) is a symmetric subset of \(V \times V\).

Equip \(E\) with the uniform probability measure \(P\) and \(V\) with the probability measure \(\nu(s) = \frac{\deg(s)}{\sum_{t \in V} \deg(t)}\). Note that \(\nu\) is the stationary probability measure for the random walk on \(G\).

The gradient \(\nabla f : E \to X\) of a function \(f : V \to X\) is defined by \((\nabla f)(e) = f(t) - f(s)\) if \(e = (s, t)\).

Definition 3.1. Let \(G\) be a graph, and let \(1 < p < \infty\). For a Banach space \(X\), we denote by \(\pi_{p,G}(X)\) the smallest real number \(\pi\) such that for all \(f : V \to X\), the inequality

\[ \inf_{x \in X} \|f - x\|_{L^p(V, \nu; X)} \leq \pi \|\nabla f\|_{L^p(E, P; X)} \]

holds. We call \(\pi_{p,G}(X)\) the \(X\)-valued \(p\)-Poincaré constant of \(G\).

Let \((Z_0, Z_1, \ldots)\) be the random walk on \(G\) with \(Z_0\) (and hence \(Z_n\) for all \(n \geq 0\)) distributed as \(\nu\). In this setting, the \(X\)-valued \(p\)-Poincaré constant of \(G\) is the smallest real number \(\pi\) such that the following inequality holds:

\[ \inf_{x \in X} \mathbb{E}[\|f(Z_0) - x\|^p] \leq \pi^p \mathbb{E}[\|f(Z_0) - f(Z_1)\|^p]. \]

Remark 3.2. The validity of the above inequality for every \(f\) with \(\mathbb{E}[\|f(Z_0)\|^p] < \infty\) can be used to define the \(X\)-valued \(p\)-Poincaré constant of an arbitrary irreducible Markov chain \((Z_0, Z_1, \ldots)\) on a set \(V\) with stationary \(\sigma\)-finite measure \(\nu\). In particular, we could consider infinite graphs with loops and multiple edges or weighted graphs. In the case when the measure \(\nu\) is infinite (the terminology is that the Markov chain is not positive recurrent), the definition of \(p\)-Poincaré constant becomes simpler: it is the smallest \(\pi\) such that for every \(f \in L^p(V, \nu; X)\),

\[ \mathbb{E}[\|f(Z_0)\|^p] = \int \|f\|^p_X d\nu \leq \pi^p \mathbb{E}[\|f(Z_0) - f(Z_1)\|^p]. \]

All results in this section hold in this generality, except for the interpretation in terms of the eigenvalues of \((p\)-\)Laplacians, where one needs reversibility of the Markov chain. The only adaptation in the proof of Theorem 3.4 when \(\nu\) is infinite is that in that case, \(L^p_0(V, \nu; X)\) is replaced by \(L^p(V, \nu; X)\).

Our \(p\)-Poincaré constant differs (by a factor or power) from the \(p\)-Poincaré constants in \[7\] and \[31\], neither does it exactly coincide with the conventions of \[27\].
Let \( G = (V, E) \) be a graph with \(|V| = n\). We denote by \( A_G \), or simply \( A \), the Markov operator of the random walk on \( G \), which acts on the functions on \( V \) by the formula

\[
Af(v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(w) = \mathbb{E}[f(Z_1) \mid Z_0 = v].
\]

Since the Markov chain is reversible, \( A \) is a self-adjoint operator on \( L^2(V, \nu) \), and we denote its eigenvalues by \( \mu_1(A) \geq \ldots \geq \mu_n(A) \). The largest eigenvalue is 1.

The (normalized) Laplacian on \( G \) is the operator \( \Delta_2 = \text{Id} - A \), which maps a function \( f \) to

\[
\Delta_2 f(v) = f(v) - \frac{1}{\deg(v)} \sum_{u \sim v} f(w) = \mathbb{E}[f(Z_0) - f(Z_1) \mid Z_0 = v].
\]

The following result summarizes some elementary properties of the \( p \)-Poincaré constant.

**Proposition 3.3.** For every graph \( G = (V, E) \) and \( 1 < p < \infty \),

1. \( \pi_{p,G}(X) \geq \frac{1}{p} \) for every \( X \) if \( G \) has at least two vertices,
2. \( \pi_{p,G}(L^p) = \pi_{p,G}(C) \),
3. \( \pi_{2,G}(C) \) is equal to \( \frac{1}{\sqrt{2 - 2\mu_2(A)}} \).

**Proof.** We always have \( \|\nabla f\|_{L^p(E, \nu; X)} \leq 2 \inf_{x \in X} \|f - x\|_{L^p(V, \nu; X)} \) by the triangle inequality, and therefore \( \pi_{p,G}(X) \geq \frac{1}{p} \) if \( G \) has at least two vertices. Assertion (ii) follows from Fubini’s theorem. Assertion (iii) is classical and follows from the equality \( \mathbb{E}[\|f(Z_0) - f(Z_1)\|^2] = 2((1 - A)f, f) \), which holds for all \( f \in L^2(V, \nu) \). \(\square\)

For \( 1 < p < \infty \), the constant \( \pi_{p,G}(C) \) is related to the eigenvalues of the \( p \)-Laplacian (see Section 3.3).

### 3.2. From small Markov operators to Poincaré inequalities.

The validity of a \( p \)-Poincaré inequality is a very robust property (see [29, 30]). Indeed, it is obvious that if a Banach space \( X \) is at Banach-Mazur distance \( C \) from another Banach space \( Y \), then \( \pi_{p,G}(X) \leq C \pi_{p,G}(Y) \). Also, by an argument of Matoušek [25] (see also [31, Lemma 5.5]), a \( p \)-Poincaré inequality implies the validity of a \( q \)-Poincaré inequality for all \( q < \infty \), with a multiplicative loss (\( \geq 4 \)) on the Poincaré constant. In Proposition 3.3, we establish a Banach space valued generalization of this result. We refer to [30, 29, 9] for related results.

For applications to fixed point properties, the crucial point is to prove that \( \pi_{p,G}(X) < 1 \). The aforementioned results are therefore not useful, because the property \( \pi_{p,G}(X) < 1 \) is not robust. The next result, which is one of the main points in this article, expresses that the property \( \pi_{p,G}(X) < 1 \) is a consequence of another property, which is very robust.

In what follows, \( L^p_0(V, \nu; X) = \{ f \in L^p(V, \nu; X) \mid \mathbb{E}[f(Z_0)] = 0 \} \).

**Theorem 3.4.** Let \( X \) be a \( p \)-uniformly convex Banach space. Then there exist \( \varepsilon, \delta > 0 \) (depending on \( X \)) such that for every graph \( G = (V, E) \) the following holds:

if \( \|A_G\|_{B(L^p_0(V, \nu; X))} \leq \varepsilon \), then for every \( f \in L^p_0(V, \nu; X) \), we have

\[
\|f\|_{L^p_0(V, \nu; X)} \leq (1 - \delta)\|\nabla f\|_{L^p(E, \nu; X)}.
\]

In particular, \( \|A_G\|_{B(L^p_0(V, \nu; X))} \leq \varepsilon \implies \pi_{p,G}(X) \leq 1 - \delta \).
In the random walk notation, we have to prove that
\[ \mathbb{E}[\|f(Z_0)\|^p] \leq (1 - \delta)^p \mathbb{E}[\|f(Z_0) - f(Z_1)\|^p] \]
for every \( f: V \to X \) with \( \mathbb{E}[f(Z_0)] = 0 \).

The proof is divided in several steps. The first one is standard.

**Lemma 3.5.** For every \( f \in L^p_0(V, \nu; X) \), we have \( \|f\|_p \leq (1 - \varepsilon)^{-1} \|f - Af\|_p \).

**Proof.** If \( A \) has norm \( \leq \varepsilon \), then \((1 - A)^{-1} = \sum_{n \geq 0} A^n\) has norm \( \leq (1 - \varepsilon)^{-1} \). \( \square \)

The triangle inequality implies, without any condition on \( X \), that \( \|f - Af\|_p \leq \|\nabla f\|_p \), and hence \( \|f\|_p \leq \frac{1}{1 - \varepsilon} \|\nabla f\|_p \). This is, however, not strong enough. The next lemma improves this inequality.

Recall that since \( X \) is \( p \)-uniformly convex, there exists a constant \( C \) such that \( (1) \) holds for every \( X \)-valued random variable \( U \).

**Lemma 3.6.** For every \( f \in L^p_0(V, \nu; X) \), we have
\[ \mathbb{E}[\|f - Af(Z_0)\|^p] \leq \mathbb{E}[\|f(Z_0) - f(Z_1)\|^p] - C \mathbb{E}[\|f(Z_1) - Af(Z_0)\|^p]. \]

**Proof.** Let \( U \) be the \( X \)-valued random variable \( f(Z_0) - f(Z_1) \). With this notation we have \( \mathbb{E}[U|Z_0] = f(Z_0) - Af(Z_0) \) and \( U - \mathbb{E}[U|Z_0] = Af(Z_0) - f(Z_1) \). So applying \( (1) \) conditionally to \( Z_0 \) and then averaging with respect to \( Z_0 \) proves the lemma. \( \square \)

**Proof of Theorem 3.4.** By the triangle inequality and the fact that \( Z_1 \) is distributed as \( Z_0 \),
\[ (\mathbb{E}[\|f(Z_1) - Af(Z_0)\|^p])^{\frac{1}{p}} \geq \|f\|_p - \|Af\|_p \geq (1 - \varepsilon)\|f\|_p. \]

Taking into account the two lemmas we obtain
\[ \|f\|_p \leq \frac{1}{(1 - \varepsilon)^p} (\|\nabla f\|_p^p - C(1 - \varepsilon)^p \|f\|_p^p), \]
from which we deduce
\[ \|f\|_p \leq \frac{1}{(1 + C)^{(1 - \varepsilon)}} \|\nabla f\|_p. \]

If \( \varepsilon > 0 \) is small enough, so that \((1 + C)^{(1 - \varepsilon)} > 1\), then the theorem follows for \( \delta = 1 - \frac{1}{(1 + C)^{(1 - \varepsilon)}} \).

**Remark 3.7.** If \( X \) is an \( L^p \)-space for some \( p \geq 2 \) (or more generally a subquotient of a \( \theta \)-Hilbertian space with \( \theta = \frac{2}{p} \)), then it follows from Proposition 2.1 and from the proof above that in that case, Theorem 3.4 holds when \((1 + 2^{p - 2})^{\frac{1}{p}} > 1\), for example as soon as \( \varepsilon \leq \frac{2}{2^{p - 2}} \).

### 3.3. Eigenvalues for the \( p \)-Laplacean

For \( \alpha > 0 \) and \( z \in \mathbb{C} \), we write \( \{z\}^\alpha = |z|^\alpha z \) if \( z \neq 0 \) and we extend the definition by continuity to \( \{0\}^\alpha = 0 \).

Let \( G = (V, E) \) be a connected finite graph, and let \( \nu \) be the measure on \( V \) defined at the beginning of this section. If \( 1 < p < \infty \), then the \( p \)-Laplacean on \( G \) is the non-linear map \( \Delta_p: \mathbb{R}^V \to \mathbb{R}^V \) defined by
\[ \Delta_p f(v) = \frac{1}{\deg(v)} \sum_{u \sim v} \|f(v) - f(w)\|^{p - 1}. \]
In our opinion, it would be more natural to define the $p$-Laplacian by $\Delta_p f(v) = \left( \frac{1}{d_G(v)} \sum_{w \sim v} |f(v) - f(w)|^{p-1} \right)^{\frac{1}{p-1}}$, but we will use the conventional definition here.

A scalar $\lambda \in \mathbb{R}$ is called an eigenvalue of $\Delta_p$ if there exists an $f \neq 0$ such that $\Delta_p f = \lambda |f|^{p-1}$. Bourdon proved [7, Lemme 1.3] that the eigenvalues of $\Delta_p$ coincide with the critical values of $f \mapsto \frac{\|\nabla f\|_p}{\|f\|_{L^p(V,\nu)}}$. In [7, Proposition 1.2], he proved that the smallest nonzero eigenvalue $\lambda_{1,p}(G)$ of $\Delta_p$ is related to the smallest constant $\pi$ for which the inequality

$$\inf_{c \in \mathbb{R}} \|f - c\|_{L^p(V,\nu)} \leq \pi \|\nabla f\|_p$$

holds, by the formula $\lambda_{1,p}(G) = \frac{1}{\pi^2}$. Hence, the crucial inequality $\pi < 1$ corresponds to the inequality $\lambda_{1,p}(G) > \frac{1}{\pi}$. In the case of $X = \mathbb{C}$, we can reformulate Theorem 3.8 in order to obtain spectral information on the $p$-Laplacian.

**Theorem 3.8.** Let $\mathcal{G}$ be a connected finite graph, and let $p \geq 2$. If the spectrum of $\Delta_2$ is contained in $\{0\} \cup [1 - \varepsilon, 1 + \varepsilon]$ for some $\varepsilon > 0$, then

$$\lambda_{1,p}(\mathcal{G}) \geq \left(1 - 2^{1-\frac{\varepsilon}{2}} \varepsilon \right)^p \left(\frac{1}{2} + 2^{1-p} \varepsilon \right).$$

In particular, $\lambda_{1,p}(\mathcal{G}) > \frac{1}{\pi}$ if $\varepsilon \leq 2^{-2^p \varepsilon^2} 2^{-1-2^{-1} \varepsilon}$. 

**Proof.** The assumption that the spectrum of $\Delta_2$ is contained in $\{0\} \cup [1 - \varepsilon, 1 + \varepsilon]$ means that $A_\mathcal{G}$ has norm $\leq \varepsilon$ as an operator on $L_0^2(V,\nu)$, or equivalently, that the operator $A_\mathcal{G} - P$ has norm $\leq \varepsilon$ as an operator on $L^2(V,\nu)$, where $P f = \int f d\nu$ is the projection onto the constant functions. Since $A_\mathcal{G} - P$ has norm $\leq 2$ as an operator on $L^\infty(V,\nu)$, by interpolation, this implies that $A_\mathcal{G} - P$ has norm $\leq 2^{-1-\frac{\varepsilon}{2}} \varepsilon$ as an operator on $L^p(V,\nu)$. In particular, $A_\mathcal{G}$ has norm $\leq 2^{1-\frac{\varepsilon}{2}} \varepsilon$ on $L_0^p(V,\nu)$. By Theorem 3.4 and Remark 3.7, we obtain that for every $f \in L_0^p(V,\nu)$,

$$\|f\|_p \leq (1 + 2^{2-p})^{-\frac{1}{p}} (1 - 2^{1-2^{-1} \varepsilon} \varepsilon)^{-1} \|\nabla f\|_p.$$ 

This implies that the $L^p$-Poincaré inequality holds with constant $(1 + 2^{2-p})^{-\frac{1}{p}} (1 - 2^{1-2^{-1} \varepsilon} \varepsilon)^{-1}$. The proposition follows from the relationship between the Poincaré constant and $\lambda_{1,p}(\mathcal{G})$ alluded above. 

**3.4. Application to random graphs.** We now apply Theorem 3.8 in the setting of random regular graphs. To this end, we first recall the configuration model of random graphs.

**Definition 3.9.** Let $V = \{1, \ldots, n\}$ be a set, and choose $v$ permutations $\pi_1, \ldots, \pi_v$ of $V$ independently and uniformly at random. Let $E = \{(i, \pi_k(i)) \mid i = 1, \ldots, n, k = 1, \ldots, v\}$. Then $\mathcal{G} = (V, E)$ is a graph, which we view as being undirected. (Multiple edges and loops are allowed.) Such a graph $\mathcal{G}$ is a random graph in the configuration model $\mathcal{L}(n, v)$.

A graph in the configuration model is a regular graph with degree $2v$. As above, $\mu_1(A) \geq \ldots \geq \mu_n(A)$ denote the eigenvalues of the Markov operator $A$ on a graph.

The spectral properties of graphs in the model $\mathcal{L}(n, v)$ have been studied extensively. We use the following estimate on the second up to the $n^{th}$ eigenvalue of $A$, which is a result due to Friedman [13].
Theorem 3.10. Fix $\varepsilon > 0$ and a number $v \geq 2$. For a random graph $G$ in $\mathcal{L}(n,v)$, we have that $|\mu_k(A)| \leq \frac{1}{v} \sqrt{2v - 1} + \varepsilon$ for all $k = 2,\ldots,n$ with high probability (w.h.p.), that is
\[
\lim_{n \to \infty} \mathbb{P}(G \text{ in } \mathcal{L}(n,v) \text{ satisfies } |\mu_k(A)| \leq \frac{1}{v} \sqrt{2v - 1} + \varepsilon \text{ for all } k = 2,\ldots,n) = 1.
\]
Equivalently, a random graph $G$ in $\mathcal{L}(n,v)$ satisfies w.h.p.
\[
\|A\|_{B(L^2(V,v))} \leq \frac{1}{v} \sqrt{2v - 1} + \varepsilon.
\]
A small computation shows that $\frac{1}{v} \sqrt{2v - 1} < 2p^{-\frac{q}{2}} - \frac{2}{v^2}$ if $v \geq 2p^{-1} p^p$. By Theorem 3.8 and Friedman’s theorem, we recover the following result from [12].

Corollary 3.11. Let $\varepsilon > 0$, $v \in \mathbb{N}$ and $p > 2$. If $v \geq 2p^{-1} p^p$, then a random graph $G$ in $\mathcal{L}(n,v)$ satisfies $\lambda_{1,p}(G) > \frac{1}{2}$ w.h.p.

3.5. Matoušek’s extrapolation result. In this section, we provide a Banach space valued version of Matoušek’s extrapolation result [23] (see also [4] Lemma 5.5). Our generalization shows that the validity of an $X$-valued $p$-Poincaré inequality does not really depend on $p$ in the following sense (see [30], [29], [9] for related results).

Proposition 3.12. For every $1 \leq p, q < \infty$, there is a constant $C$ such that for every Banach space $X$ and every $G$,
\[
\pi_{p,G}(X) \leq C \pi_{q,G}(X)^{\max(\frac{q}{p}, 1)}.
\]

Proof. The proof is an adaptation of the original extrapolation argument by Matoušek [25]. Suppose that a graph $G$ (or more generally a Markov chain on a finite state space) has $X$-valued $q$-Poincaré constant equal to $\pi_q$.

For $x \in X$ and $\alpha > 0$, we set $\{x\}^{\alpha} = \|x\|^{\alpha - 1} x$ if $x \neq 0$ and $\{0\}^{\alpha} = 0$. The map $M_{p,q} : L^p(\Omega;\mu;X) \to L^q(\Omega;\mu;X)$ defined by $M_{p,q}(f)(\omega) = \{f(\omega)\}^{\frac{q}{p}}$ is a version of the classical Mazur map for vector-valued $L^p$-spaces. The next lemma shows that it has the same regularity properties as in the classical case $X = \mathbb{C}$.

Lemma 3.13. For every $1 \leq p, q < \infty$, there exists a constant $C_{p,q}$, such that
\[
\|M_{p,q}(f_1) - M_{p,q}(f_2)\|_{L^q(\Omega;\mu;X)} \leq C_{p,q}\|f_1 - f_2\|_{L^p(\Omega;\mu;X)}^{\min(\frac{q}{p}, 1)}
\]
for every measure space $(\Omega, \mu)$, every Banach space $X$ and every two functions $f_1$ and $f_2$ in the unit ball of $L^p(\Omega;\mu;X)$.

Proof. For real valued functions the lemma is classical, see [29]. In particular, there exists a $C$ (depending on $p$ and $q$) such that for all $S_1, S_2 : \Omega \to \mathbb{R}^+$ with $\|S_1\|_p \leq 1$, $\|S_2\|_p \leq 1$, we have
\[
\|S_1^{\frac{q}{p}} - S_2^{\frac{q}{p}}\|_q \leq C\|S_1 - S_2\|_p^{\min(\frac{q}{p}, 1)}.
\]

Let $f_1, f_2$ be in the unit ball of $L^p(\Omega;\mu;X)$, and define $g_i$ in (the unit ball of) $L^q(\Omega;\mu;X)$ by $g_i = M_{p,q}(f_i)$. Let $\delta = \|f_1 - f_2\|_p$. We write $f_i = S_i U_i$ with $S_i(\omega) = |f_i(\omega)|$ and $U_i(\omega)$ in the unit sphere of $X$, so that $g_i = S_i^{\frac{q}{p}} U_i$. By the triangle inequality, we have $\|f_1 - f_2\|_p \geq \|S_1 - S_2\|_p$, and
\[
\|S_2(U_1 - U_2)\|_p \leq \|S_2 U_1 - S_1 U_1\|_p + \|S_1 U_1 - S_2 U_2\|_p \leq 2\|f_1 - f_2\|_p.
\]
Writing \( g_1 - g_2 = (S_1^B - S_2^B)U_1 + S_2^B (U_1 - U_2) \), we obtain
\[
\| g_1 - g_2 \|_q \leq \| S_1^B - S_2^B \|_q + \| S_2^B (U_1 - U_2) \|_q.
\]
The first term is less than \( C \delta^{\min(\frac{1}{q},1)} \) by \( 39 \). We can view the second term as the norm of \( U_1 - U_2 \) in \( L^q(\Omega, S_2^B \mu; X) \). If \( q \leq p \), then this norm is less than the norm of \( U_1 - U_2 \) in \( L^p(\Omega, S_2^B \mu; X) \), i.e. less than \( 2 \delta \). If \( q \geq p \), then by Hölder’s inequality this norm is less than the geometric mean of its norm in \( L^\infty \) and its norm is \( L^p \), i.e. less than \( 2 \delta^{\frac{1}{2}} \). The previous inequality therefore becomes
\[
\| g_1 - g_2 \|_q \leq (C + 2) \delta^{\min(\frac{1}{q},1)}.
\]
This proves the lemma, because \( \delta \) was defined as \( \| f_1 - f_2 \|_p \).

**Proof of Proposition 3.12 (continuation).** Let \( f \in L^p(V, \nu; X) \). We have to prove that
\[
(\mathbb{E} \| f(Z_0) - f(Z_1) \|_p^q)_{\frac{1}{p}} \geq \frac{1}{C_{\pi q}} \inf_{x \in X} \| f - x \|_p.
\]
By homogeneity, we may assume that \( \inf_{x \in X} \| f - x \|_p = \frac{1}{2} \), and by replacing \( f \) by \( f - x \) for a suitable \( x \), we may assume that \( \| f \|_p \leq 1 \).

Let \( g = M_{p,q}(f) \). It has norm \( \leq 1 \) in \( L^q \). By the previous lemma, we have
\[
\frac{1}{2} = \inf_{x \in X, \| x \|_1 \leq 1} \| f - x \|_p \leq C_{p,q} \inf_{x \in X, \| x \|_1 \leq 1} \| g - x \|_q^{\min(\frac{1}{q},1)}.
\]
In particular, there is a constant \( c \) (depending on \( p, q \)) such that \( \inf_{x \in X} \| g - x \|_q \geq c \).

By definition of \( \pi_q \), we have
\[
(\mathbb{E} \| g(Z_0) - g(Z_1) \|_q^q)_{\frac{1}{q}} \geq \frac{c}{\pi_q}.
\]
By the previous lemma, we obtain
\[
\frac{c}{\pi_q} \leq C_{p,q} (\mathbb{E} \| f(Z_0) - f(Z_1) \|_p^q)_{\frac{1}{q}}^{\min(\frac{1}{q},1)},
\]
or equivalently,
\[
(\mathbb{E} \| f(Z_0) - f(Z_1) \|_p^q)_{\frac{1}{q}} \geq \left( \frac{c}{C_{p,q}^{\pi_q} \min(\frac{1}{q},1)} \right)^{\max(\frac{1}{q},1)}.
\]
This concludes the proof of the result.

4. From Poincaré inequalities to fixed point properties

Recall that if \( M = (M_0, M_1, M_2) \) is a simplicial 2-complex and \( m \in M_0 \), then the link \( L(m) \) is the graph with vertex set \( V = \{ n \in M_0 \mid \{ m, n \} \in M_1 \} \) and edge set \( E = \{ (n_1, n_2) \in M_1 \mid \{ m, n_1, n_2 \} \in M_2 \} \). In the following, we give, as mentioned in the introduction, a direct proof of the fact that Poincaré inequalities give rise to fixed points. The approach is similar to the one of Oppenheim \( 33 \) and we claim no originality. We have chosen to leave out some computations.

**Theorem 4.1.** Let \( 1 < p < \infty \), let \( X \) be a Banach space, and let \( M \) be a connected and locally finite simplicial 2-complex. Suppose that \( \pi_p, L(m)(X) < 1 \) for every \( m \in M_0 \). If \( \Gamma \) is a group that admits a properly discontinuous cocompact action by simplicial automorphisms on \( M \), then \( \Gamma \) has \( (F_X) \).
Moreover, by the \( \Gamma \)-equivariance of \( \psi \) is unchanged if \( \psi \) yields by replacing \( \psi \) an element of \( \Gamma \).

We denote this quantity by \( E_{\pi_0} \) and is, in particular, nonempty.

**Lemma 4.2.** For \( \varphi, \psi \in \mathcal{E} \) and \( p \in [1, \infty) \), we have

\[
\sum_{m \in \Xi_0} a_m \| (n_1, n_2) \mapsto \varphi(n_1) - \psi(n_2) \|_{L^p(L_1(m),\mathbb{P};X)}^p = \sum_{m \in \Xi_0} a_m \| n \mapsto \varphi(n) - \psi(m) \|_{L^p(L_0(m),\mathbb{P};X)}^p.
\]

We denote this quantity by \( E(\varphi, \psi)^p \), or simply by \( E(\varphi)^p \) when \( \varphi = \psi \). Moreover, we have the inequality

\[
E \left( \frac{\varphi + \psi}{2} \right) \leq E(\varphi, \psi).
\]

**Proof of Lemma 4.2.** If \( \varphi = \psi \), then \( E(\varphi)^p \) is exactly \( E(\varphi) \). For \( \varphi \neq \psi \), the same computation proves the equality. For \( \varphi \neq \psi \), we decompose the function \( (n_1, n_2) \mapsto \varphi(n_1) - \psi(n_2) \) as \( (n_1, n_2) \mapsto \frac{\varphi(n_1) - \varphi(n_2)}{2} - \frac{\varphi(n_2) - \psi(n_2)}{2} \). By the triangle inequality, we obtain

\[
E \left( \frac{\varphi + \psi}{2} \right) \leq \frac{1}{2} \left( E(\varphi, \psi) + E(\varphi, \psi) \right).
\]

This is \( E(\varphi, \psi) \), because \( E(\varphi, \psi) = E(\psi, \varphi) \).

**Proof of Theorem 4.7.** (continuation). We now define a complete distance on \( \mathcal{E} \) by

\[
d(\varphi, \psi) = \left( \sum_{m \in \Xi_0} a_m \| \varphi(m) - \psi(m) \|_{p}^p \right)^{\frac{1}{p}}.
\]

Take \( c < 1 \) such that \( \pi_{p,L(m)}(X) < c \) for every \( m \in \Xi_0 \), and let \( \psi \in \mathcal{E} \). By definition of \( \pi_{p,L(m)}(X) \), for every \( m \in \Xi_0 \), there is a \( \psi(m) \in X \) such that

\[
\| n \mapsto \varphi(n) - \psi(m) \|_{L^p(L_0(m),\mathbb{P};X)} \leq c \| (n_1, n_2) \mapsto \varphi(n_1) - \varphi(n_2) \|_{L^p(L_1(m),\mathbb{P};X)}.
\]

Moreover, by the \( \Gamma \)-equivariance of \( \varphi \), the quantity \( \| n \mapsto \varphi(n) - \psi(m) \|_{L^p(L_0(m),\mathbb{P};X)} \) is unchanged if \( \psi(m) \) is replaced by an element of the \( \Gamma_m \)-orbit of \( \psi(m) \). Therefore, by replacing \( \psi(m) \) by the average on its \( \Gamma_m \)-orbit, we can assume that \( \psi(m) \) is \( \Gamma_m \)-invariant. This means that \( \psi \), defined so far only on \( \Xi_0 \), can be extended to an element of \( \mathcal{E} \). Summing the \( p \)-th power of the previous expression over \( m \in \Xi_0 \), yields \( E(\varphi, \psi) \leq c E(\varphi) \). By \( \Xi_0 \), this implies \( E(\varphi, \psi)^p \leq c E(\varphi)^p \).

On the other hand, using the triangle inequality, we obtain that for \( m \in \Xi_0 \),

\[
\left\| \varphi(m) - \frac{\varphi(m) + \psi(m)}{2} \right\|_{p} \leq \frac{1}{2} \| n \mapsto (\varphi(m) - \varphi(n)) + (\varphi(n) - \psi(m)) \|_{p} \leq \frac{1}{2} \| n \mapsto \varphi(m) - \varphi(n) \|_{p} + \frac{1}{2} \| n \mapsto \varphi(n) - \psi(m) \|_{p}.
\]
where \( \| \cdot \|_p \) denotes the norm on \( L^p(L_0(m), \nu; X) \). It follows that
\[
d\left( \varphi, \frac{\varphi + \psi}{2} \right) \leq \frac{1}{2} (E(\varphi) + E(\varphi, \psi)) \leq E(\varphi).
\]

The conclusion of the preceding discussion is that for every \( \varphi \in \mathcal{E} \), there is a \( \varphi' \in \mathcal{E} \) (namely \( \varphi' = \frac{\varphi + \psi}{2} \)) such that \( E(\varphi') \leq cE(\varphi) \) and \( d(\varphi', \varphi) \leq E(\varphi) \). If we start from some \( \varphi_0 \in \mathcal{E} \), by induction we obtain a sequence \( \varphi_n \) in \( \mathcal{E} \) with \( E(\varphi_n) \leq c^n E(\varphi_0) \) and \( d(\varphi_n, \varphi_{n+1}) \leq c^n E(\varphi_0) \). The sequence \( \varphi_n \) is a Cauchy sequence and therefore converges to some \( \varphi_\infty \in \mathcal{E} \) satisfying \( E(\varphi_\infty) = 0 \). For a general complex \( M \), the formula \( E(\varphi_\infty) = 0 \) means that \( \varphi_\infty \) is constant on the connected components of \( L(m) \) for every \( m \in M_0 \). Here the assumption that \( \pi_{p,L(m)}(X) < \infty \) implies that \( L(m) \) is connected, and hence the assumption that \( M \) is connected implies that \( \varphi_\infty \) is constant and is necessarily equal to a fixed point. This proves the theorem. 

\[\Box\]

5. Proofs of Theorem C and Theorem A

If \( X \) is \( p \)-uniformly convex, then Theorem C is a direct combination of Theorem 3.4 and Theorem 4.1. Moreover, if \( X \) is an \( L^p \)-space with \( p \geq 2 \) (or more generally a subquotient of a \( \theta \)-Hilbertian space with \( \theta = \frac{3}{2} \)), then we see from Remark 3.7 that Theorem C holds with \( \varepsilon = \frac{3}{2p} \).

We now prove the general case of Theorem C. The idea of the proof is to reduce to the case of \( p \)-uniformly convex Banach spaces.

**Proof of Theorem C** Let \( X \) be a superreflexive Banach space. As was recalled in Section 2 by a famous result of Pisier [37], there exists a \( q \in [2, \infty) \) and an equivalent norm \( N \) on \( X \) that is \( q \)-uniformly convex. Pisier’s proof has the feature that every isometry of \((X, \| \cdot \|)\) remains an isometry of \((X, N)\), but even if this were not the case, we could always assume this by replacing \( N \) by the equivalent norm \( N'(x) = \sum_{g \in G} N(gx) \) (see the proof of (2) \( \implies \) (3) in [1, Proposition 2.3]). Denote the Banach space \((X, N)\) by \( Y \). We now use the following interpolation result, which was already used in a similar context in [39].

**Lemma 5.1.** There exists a constant \( C \in \mathbb{R} \) and a \( \theta \in (0, 1) \) such that for every graph \( \mathcal{G} = (V, E) \), we have
\[
\| A_\mathcal{G} \|_{B(L_0^q(V, \nu; Y))} \leq C \| A_\mathcal{G} \|_{\theta}^{\theta} \| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))}^{1 - \theta}.
\]

**Proof.** Let \( r \in [1, \infty] \) and \( \theta \in (0, 1) \) such that \( \frac{1}{q} = \frac{\theta}{r} + \frac{(1 - \theta)}{r} \). The operator \( f \mapsto A(f - \int f \, d\nu) \) has norm less than \( 2\| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))} \) on \( L^r(V, \nu; X) \), since this operator is the composition of the operator \( f \mapsto f - \int f \, d\nu \in L_0^q \) with norm at most \( 2 \) with the restriction of \( A_\mathcal{G} \) to \( L_0^q \). Therefore, by interpolation we have
\[
\| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))} \leq 2\| A_\mathcal{G} \|_{\theta}^{\theta} \| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))}^{1 - \theta},
\]
which we simply bound by \( 2\| A_\mathcal{G} \|_{\theta}^{\theta} \| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))}^{1 - \theta} \). The conclusion now follows, because \( \| A_\mathcal{G} \|_{B(L_0^q(V, \nu; Y))} \) is less than the product of \( \| A_\mathcal{G} \|_{B(L_0^q(V, \nu; X))} \) and the Banach-Mazur distance between \( X \) and \( Y \). \[\Box\]

**Proof of Theorem C (continuation).** Since \( Y \) is \( q \)-uniformly convex, by the case already proved, there exists an \( \varepsilon_1 > 0 \) such that a group with a properly discontinuous cocompact action by simplicial automorphisms on a simplicial 2-complex \( M \) with all its links \( L \) satisfying \( \| A_L \|_{B(L_0^q(L, \nu; Y))} < \varepsilon_1 \) has \( (F_Y) \). Therefore, if
\( \varepsilon > 0 \) satisfies \( C \varepsilon^p \leq \varepsilon_1 \), then every such group has \((F_Y)\). In particular it has \((F_X)\) because by construction every action by affine isometries on \( X \) is an action by affine isometries on \( Y \).

Finally, we explain how Theorem \([A]\) follows from Theorem \([C]\).

**Proof of Theorem \([A]\)** Let \( X \) be a uniformly curved Banach space. As recalled in Section 2.2, the space \( X \) is superreflexive. Let \( \varepsilon' = \varepsilon'(2, X) \) be given by Theorem \([C]\) for \( p = 2 \). We claim that for a finite graph \( G \), we have the inequality

\[
\|A_G\|_{B(L^2_2(V, \nu; X))} \leq 2 \Delta_X \left( \frac{1}{2} \|A_G\|_{B(L^2_2(V, \nu))} \right).
\]

Indeed, let \( T : L^2_2(V, \nu) \to L^2_2(V, \nu) \) be the operator \( f \mapsto \frac{1}{2} (A_G f - \int f \, d\nu) \). Then \( T \) has norm \( \leq 1 \) on \( L^2_2(V, \nu) \), and \( \frac{1}{2} \|A_G\|_{B(L^2_2(V, \nu))} \) on \( L^2_2(V, \nu) \), so by definition of \( \Delta_X \), we have \( \|T_X\| \leq \Delta_X(\|T\|) \). We obtain \((6)\) by considering the restriction to \( L^2_2(V, \nu; X) \). Since \( X \) is uniformly curved, there is an \( \varepsilon > 0 \) such that \( 2 \Delta_X \left( \frac{1}{2} \right) < \varepsilon'. \) It follows from \((6)\) that Theorem \([A]\) follows for this value of \( \varepsilon \).

**Remark 5.2.** In many cases, we can give a direct proof of Theorem \([A]\) (not relying on \((57)\)) and can compute the constants explicitly.

- (i) If \( X \) is an \( L^p \)-space with \( p \geq 2 \), then Theorem \([A]\) holds with \( \varepsilon = 2 p^{-\frac{1}{2}} 2^{-\frac{p}{2}} \).
- (ii) Theorem \([A]\) also holds with \( \varepsilon = 2 p^{-\frac{1}{2}} 2^{-\frac{p}{2}} \) if \( X \) is strictly \( \frac{p}{2} \)-Hilbertian, or more generally a subquotient of a strictly \( \frac{p}{2} \)-Hilbertian space.
- (iii) If \( X \) is isomorphic to a space as in (i) or (ii), say, at Banach-Mazur distance \( d \), then Theorem \([A]\) holds with \( \varepsilon = K p^{-\frac{1}{2}} 2^{-\frac{p}{2}} d^{-\frac{p}{2p+1}} \) for a universal constant \( K \).

**Proof.** If \( p \geq 2 \) and \( X \) is an \( L^p \)-space (or more generally a strictly \( \frac{p}{2} \)-Hilbertian space), then interpolation directly gives the inequality

\[
\|f \mapsto A_G f - \int f \, d\nu\|_{B(L^p_2(V, \nu; X))} \leq 2^{1-\frac{p}{2}} \|A_G\|_{B(L^2_2(V, \nu))}^\frac{p}{2}
\]

for every graph \( G \). So, by Remark 5.2, we obtain that Theorem \([A]\) holds as soon as

\[
2^{1-\frac{p}{2}} \varepsilon^\frac{p}{2} \leq \frac{1}{p^{1-\frac{p}{2}}} \text{, i.e. as soon as } \varepsilon \leq 2^{-\frac{p}{2}} 2^{-\frac{p}{2}} \varepsilon.
\]

The same argument works if \( X \) is a subquotient of a strictly \( \frac{p}{2} \)-Hilbertian space. The additional argument is to observe that the quantity \( \|f \in L^p_2(V, \nu; X) \mapsto A_G f - \int f \, d\nu\| \) can only decrease when \( X \) is replaced by a subquotient of \( X \).

Finally, consider the case when \( X \) is at Banach-Mazur distance \( \leq d \) from a subquotient of a strictly \( \frac{p}{2} \)-Hilbertian space \( Y \). First, suppose that \((11)\) holds for \( X \) with \( C = \frac{1}{2^{1-\frac{p}{2}}} \). Second, we will reduce to this case. By the above proof for \( Y \), for every finite graph \( G \), we have \( \|A_G\|_{B(L^p_2(V, \nu; Y))} \leq 2^{1-\frac{p}{2}} \|A_G\|_{B(L^2_2(V, \nu))}^\frac{p}{2} \), and therefore

\[
\|A_G\|_{B(L^p_2(V, \nu; X))} \leq 2^{1-\frac{p}{2}} d \|A_G\|_{B(L^2_2(V, \nu))}^\frac{p}{2}.
\]

Hence, by the proof of Theorem \([54]\), we have that \( \pi_{p, G}(X, N) < 1 \) as soon as \( \|A_G\|_{B(L^p_2(V, \nu))} \leq \varepsilon \) with \( (1 + \frac{1}{p^{1-\frac{p}{2}}})^{1/2} (1 - 2^{1-\frac{p}{2}} d) > 1 \). Elementary computations show that this holds as soon as \( \varepsilon \leq K p^{-\frac{1}{2}} 2^{-\frac{p}{2}} d^{-\frac{p}{2p+1}} \) for a universal constant \( K \).

To conclude the proof, we have to explain how we reduce to the case when \((11)\) holds for \( X \) with \( C = \frac{1}{2^{1-\frac{p}{2}}} \). We identify \( X \) and \( Y \) in a way that the norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) satisfy \( \|x\|_X \leq \|x\|_Y \leq d \) for all \( x \in X \). Denote by \( O(X) \) the group
of linear isometries of \((X, \| \cdot \|_X)\) and define the norm \(N(x) = \sup_{g \in O(X)} \|gx\|_Y\), so that \(\|x\|_Y \leq N(x) \leq d\|x\|_Y\) for all \(x \in X\). By construction, every action by affine isometries on \((X, \| \cdot \|_X)\) is an action by isometries on \((X, N)\). We have to prove that (I) for \((X, N)\) holds with \(C = \frac{1}{2^{1-d/2}}\). This follows from Proposition 2.1 which asserts that (I) holds for \(\| \cdot \|_Y\) with \(C = \frac{1}{2^{1-d/2}}\). Indeed, if \(U\) is an \(X\)-valued random variable, then for every \(g \in O(X)\) we can apply (I) to the random variable \(gU\) and obtain
\[
\|gEU\|_Y^p + \frac{1}{2^{p-2}} \mathbb{E}\|g(U - EU)\|_Y^p \leq \mathbb{E}\|gU\|_Y^p.
\]
In particular, by the inequality \(N(\cdot) \leq d\| \cdot \|_Y \leq N(\cdot)\), we get
\[
\|gEU\|_Y^p + \frac{1}{2^{p-2}} \mathbb{E}N(g(U - EU))^p \leq \mathbb{E}N(U)^p.
\]
By taking the supremum over \(g \in O(X)\) we obtain
\[
N(\mathbb{E}U)^p + \frac{1}{2^{p-2}} \mathbb{E}N(g(U - EU))^p \leq \mathbb{E}N(U)^p,
\]
which finishes the proof. \(\square\)

6. Fixed point properties for random groups

As mentioned in the introduction, the theory of random groups can be used to study properties of “typical” finitely presented groups. In this section, we apply Theorem A to random groups in the triangular model.

Let \(S = \{s_1, \ldots, s_n\}\). Roughly speaking, a random group generated by \(S\) is a group given by a representation \(\langle S|R\rangle\), where \(R\) is a set of relators, i.e. words in \(S \cup S^{-1}\), that are chosen randomly with respect to some probability measure on the set of all words in \(S \cup S^{-1}\). In what follows, we only consider relators that are cylindrically reduced, i.e. if \(r = s_1\ldots s_m\) is a relator, then \(s_i \neq s_{i+1}^{-1}\) for \(i \in \{1, \ldots, m\}\), where we identify \(s_{m+1}\) and \(s_1\).

Associated with every triangular presentation \(\Gamma = \langle S|R\rangle\), i.e. a presentation in which every relator has length 3, there is a natural graph \(L(S)\). The graph \(L(S)\) has vertex set \(S \cup S^{-1}\) and edges \(\{s_x, s_y^{-1}\}\), \(\{s_y, s_x^{-1}\}\) and \(\{s_z, s_{y}^{-1}\}\) whenever \(s_x, s_y, s_z \in R\). Note that the edges come in three types, corresponding to the order in which the generators \(s_x, s_y\) and \(s_z\) occur in the relation \(s_x s_y s_z\). This order yields a decomposition of \(L(S)\) into three graphs \(L^1, L^2\) and \(L^3\), where \(L^i\) has the same vertices as \(L(S)\), but only the edges corresponding to the appropriate place in the relation.

Let us first recall the triangular model \(\mathcal{M}(m, d)\) (see [32]).

**Definition 6.1.** For a fixed density \(d \in (0, 1)\), a group in the triangular model \(\mathcal{M}(m, d)\) is a group \(\Gamma = \langle S|R\rangle\), where \(|S| = m\) and \(R\) is a set of \((2m - 1)^d\) relators that are chosen at random with uniform probability and independently from the set of relators of length 3.

Fix \(d \in (0, 1)\). A property \(P\) for groups is said to hold with overwhelming probability (w.o.p.) in the triangular model if
\[
\lim_{m \to +\infty} \mathbb{P}(\Gamma \in \mathcal{M}(m, d) \text{ has } P) = 1.
\]
It is known that if \(d < \frac{1}{2}\), then in a random group w.o.p. every relator occurs only once (see [32]).
As mentioned in the introduction, it was proven by Gromov that for \( d < \frac{1}{2} \), a random group in \( G(n, l, d) \) is infinite, hyperbolic and torsion-free w.o.p. A similar statement holds for the triangular model. Also recall that in both models, for \( d > \frac{1}{2} \), a group has property (T) w.o.p. Hence, the interesting range of densities to examine fixed point properties is the interval from \( \frac{1}{2} \) to \( \frac{3}{2} \).

We now prove Theorem B. In line with the approach of [19], we first recall the reduced permutation model for random groups, which is the most suitable model for our approach, since there is a straightforward relation between random groups in the reduced permutation model and random graphs in the configuration model.

**Definition 6.2.** Let \( S = \{s_1, \ldots, s_n\} \) be a set of generators. For a fixed number \( v \geq 1 \), choose pairs \( \{\pi_1^1, \pi_2^1\}, \ldots, \{\pi_1^v, \pi_2^v\} \) of permutations of \( S \cup S^{-1} \) uniformly at random and independently, where each \( \pi_1^i \) and \( \pi_2^i \) is chosen from the set of all permutations of \( S \cup S^{-1} \). A random group in the permutation model \( \mathcal{F}(n, v) \) is a group \( \Gamma = \langle S|R \rangle \), where \( R \) consists of the words of the form \( s_j^{\pm 1}\pi_1^i(s_j^{\pm 1})\pi_2^i(s_j^{\pm 1}) \), with \( i = 1, \ldots, v \) and \( j = 1, \ldots, n \).

Analogously, one defines the reduced permutation model \( \mathcal{F}^\text{red}(n, v) \), in which only pairs \( \{\pi_1, \pi_2\} \) of permutations with \( \pi_1(s) \neq s^{-1}, \pi_2(s) \neq \pi_1(s)^{-1} \) and \( \pi_2(s) = s^{-1} \) for all \( s \in S \cup S^{-1} \) are allowed. This condition ensures that the relations in the groups in the model \( \mathcal{F}^\text{red}(n, v) \) are cylindrically reduced.

Fix \( v \geq 1 \). A property \( P \) for groups holds w.o.p. in the permutation model if

\[
\lim_{n \to \infty} \Pr(\Gamma \text{ in } \mathcal{F}(n, v) \text{ has } P) = 1.
\]

The same probabilistic notion of overwhelming probability is used for the reduced permutation model.

We can now turn towards the proof of Theorem B, which is very similar to the proof of the fact that random groups in the triangular model have property (T), as established by Žuk.

**Proposition 6.3.** Let \( X \) be a uniformly curved Banach space. For \( v \) sufficiently large, a random group in the model \( \mathcal{F}^\text{red}(n, v) \) has property \( (F_X) \) w.o.p.

**Proof.** Let \( \Gamma = \langle S|R \rangle \) be a random group in the model \( \mathcal{F}^\text{red}(n, v) \). Consider the graph \( L(S) \) associated with this presentation and the associated subgraphs \( L^1, L^2 \) and \( L^3 \), as was explained at the beginning of this section. It follows that the norm of the Markov operator on \( L(S) \) is \( \leq \varepsilon \) if the norm of the Markov operators restricted to the graphs \( L^1, L^2 \) and \( L^3 \) is \( \leq \varepsilon \).

However, the graphs \( L^1, L^2 \) and \( L^3 \) are graphs of a “special kind” in the model \( \mathcal{L}(n, v) \). The probability that a graph in \( \mathcal{L}(n, v) \) is of this “special kind” is uniformly bounded from below by a constant independent of \( n \) (see [19 Section 3]). From (2), it follows that for sufficiently large \( v \), w.h.p. such graphs \( L \) satisfy \( \|A_L\|_{L^2(V, \nu)} \leq \varepsilon \). By Theorem A, the result follows.

**Proof of Theorem B** [2] Let \( v \) be as in Proposition 6.3. As shown in [19] Proof of Theorem A], if \( d > \frac{1}{3} \), w.o.p. we can assign to a random group \( \Gamma \) in \( \mathcal{M}(m, d) \) a random group \( \Gamma' \) in \( \mathcal{F}^\text{red}(m, v) \) such that \( \Gamma \) is a quotient of \( \Gamma' \). It now follows from Proposition 6.3 that \( \Gamma' \) has property \( (F_X) \) w.o.p., and hence \( \Gamma \) has property \( (F_X) \) w.o.p., since property \( (F_X) \) passes to quotients. This finishes the proof.
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