Noncommutative X-Y model and Kosterlitz Thouless transition

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Matrix models have been shown to be equivalent to noncommutative field theories. In this work we study noncommutative X-Y model and try to understand Kosterlitz Thouless transition in it by analysing the equivalent matrix model. We consider the cases of a finite lattice and infinite lattice separately. We show that the critical value of the matrix model coupling is identical for the finite and infinite lattice cases. However, the critical values of the coupling of the continuum field theory, in the large $N$ limit, is finite in the infinite lattice case and zero in the case of finite lattice.

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1. Introduction

The BFSS [1] and IKKT [2] theories have generated recent interest in matrix models as tools for studying nonperturbative aspects of M-theory and type IIB superstring theory. In the BFSS formalism, M-theory formulated in the infinite momentum frame has been conjectured to be equivalent to the $N \to \infty$ limit of a $U(N)$ supersymmetric quantum mechanics written as a matrix model with $N \times N$ hermitian matrices. In the IKKT theory, type IIB Green Schwarz Superstring action in the Schild gauge has been related to a matrix model which has been derived as an effective theory of large $N$ reduced model of 10-dimensional Super-Yang Mills theory. In addition, there has been a recent surge of interest in noncommutative gauge theories after it has been realised that they appear as limits of D-brane open string theories in the presence of nonvanishing NSNS B fields [3], [4].

$U(N)$ gauge theories in commutative spacetime can be studied, in the ’t Hooft large $N$ limit, by equivalent $U(N)$ matrix models [3] where spacetime dependent fields are mapped to spacetime independent $U(N)$ matrices to obtain a reduced model from the gauge theory and equality is obtained at the level of correlation functions of the field theory and the matrix model. The property of factorization of correlation functions of the gauge theory in the $N \to \infty$ limit, then guarantees equivalence of the gauge theory and the matrix model, provided the matrix model is either quenched (QEK model) or twisted (TEK model) [3]. In the usual ’t Hooft limit of the TEK model, a $U(N)$ matrix model is equivalent to a $U(N)$ field theory (in commutative spacetime) in a periodic box. In some recent papers [7], however, it has been shown that there exists a limit of the twisted reduced models when they describe quantum field theory on noncommutative spacetime, in particular, a $U(mn)$ twisted reduced model is equivalent to a $U(m)_{\theta}$ gauge theory (where $\theta$ is the noncommutativity parameter) on noncommutative spacetime in the limit of $n \to \infty$. This limit is, clearly, different from the ’t Hooft large $N$ limit (in particular, nonplanar diagrams survive in this limit while they do not in the ’t Hooft limit ) and, in a sense, it converts part of the noncommutativity of the matrix model into noncommutativity of spacetime coordinates, which are generated from the spacetime independent matrices in the reduced model by expanding a general matrix around a particular classical solution which exists only in the limit $n \to \infty$. Some other aspects of the noncommutative field theory and matrix model correspondence have been considered in [8].

In this paper we use the relationship between noncommutative field theory and twisted reduced models to study Kosterlitz Thouless transition in noncommutative XY model. In
the commutative XY model, realised as a two dimensional system of interacting spins, there exist vortex configurations in which the order parameter is an integer multiple of the polar angle with the corresponding integer being the winding number of the vortex. Such vortex configurations are, however, not energetically favourable at all temperatures because of a competition between the energy required to form the vortex and the entropy of the vortex. Kosterlitz Thouless (KT) transition \[9\] refers to the transition from a phase of no vortex to a phase of vortices and the temperature at which this transition occurs is called the KT transition temperature. This transition is characteristic of two spatial dimensions, since, in this case, both the vortex energy and the entropy of a single vortex scale as the logarithm of the system size. In noncommutative space one can, formally, construct the action for an XY model that reduces, in the appropriate limit, to the action for commutative XY model. Going to the reduced model one can, then, show that there exist vortex configurations for which the energy and entropy in two spatial dimensions scale as the logarithm of the system size. One can, by a calculation of the free energy similar to the commutative case, find an expression for the KT transition temperature, which turns out to be independent of the noncommutativity parameter \( \theta \). A brief comment on the nature of divergence of the noncommutative vortex action is, probably, in order here. The action scales as the sum of two terms one linear and the other a logarithm of the system size. By a suitable regularization, one can extract the logarithm piece for the action which gives the desired behaviour for the vortex energy.

We organise the paper as follows. In section 2 we develop our notation and review the relation between noncommutative gauge theories and twisted reduced models. In section 3 we discuss the noncommutative XY model and map it to the equivalent matrix model. In section 4 we find the vortex solution to the XY model and discuss Kosterlitz Thouless transition in the continuum limit of the model with infinite lattice size. In section 5 we study the same for the case of a finite lattice. Section 6 ends with some concluding remarks.

2. Description of noncommutative field theory in terms of reduced model

Field theories in noncommutative spaces can be studied by twisted reduced models obtained by a mapping of the fields on the noncommutative space to matrix valued operators. The usual matrix multiplication involving these operators then defines the way in which fields are multiplied in the corresponding noncommutative space. Integration of
fields over noncommutative coordinates is then equivalent to tracing over the corresponding matrix operators. On a noncommutative space of dimension $D$, the local coordinates $x_\mu$ are replaced by hermitian operators $X_\mu$ obeying the commutation relations

$$[X_\mu, X_\nu] = i \theta_{\mu\nu}$$

(2.1)

where $\theta_{\mu\nu}$ are dimensionful real valued c-numbers. Fields are defined as functions on this coordinate space $x_\mu$. The reduced model prescription is equivalent to finding matrix representation of the coordinate space, that respects the basic commutation relation (2.1), and of fields on this space. However, it turns out, that (2.1) does not have finite dimensional representations. For the purpose of doing explicit calculations one puts the field theory of interest on a lattice with some lattice spacing. Noncommutativity then implies that the lattice is periodic [10], [11], [12].

As an example, let us consider a scalar field defined on the noncommuting space $X_\mu$ with lattice spacing $a$ and linear size $L$. We will specialize to two dimensions and hence $\mu = 1, 2$. We impose periodic boundary condition on the lattice and construct, formally, the matrices $\Gamma_1 \equiv e^{iaB X^2}, \Gamma_2 \equiv e^{iaB X^1}$ (where $B = -\frac{1}{\theta_{12}}$), which generate translations by one lattice spacing in the directions $X^1$ and $X^2$ respectively. In a periodic lattice it is possible to obtain finite dimensional representation of the $\Gamma_\mu$ s even though the $X_\mu$ s themselves are infinite dimensional. We consider an $N$ dimensional representation of the $\Gamma_\mu$ s which satisfies the following relation,

$$\Gamma_1 \Gamma_2 = e^{i\alpha} \Gamma_2 \Gamma_1$$

(2.2)

$$\Gamma_\mu^N = 1, \quad (\mu = 1, 2)$$

(2.3)

where $\alpha = \frac{2 \pi}{N}$. For an explicit construction of these matrices we refer to [13]. All operators in this space can be written as linear combinations of integral powers of these matrices (since the set of all translations generate the full quantum space)

$$\bar{\phi} = \sum_{n_x, n_y = 0}^{N-1} \phi_{n_x n_y} \Gamma_1^{n_x} \Gamma_2^{n_y} e^{-\frac{1}{2} n_x n_y \alpha} = \sum_{k_x k_y} \bar{\phi}(k_x, k_y) e^{-i k_\mu X^\mu}$$

(2.4)

where $\phi_{n_x n_y} = \bar{\phi}(k_x, k_y)$ and the $k$ s belong to the discrete (because the lattice size is finite) dual lattice. The corresponding field in noncommutative space is then defined as a fourier tranform of the scalar valued coefficient $\bar{\phi}(k)$

$$\phi(x, y) = \sum_{k_x k_y} \bar{\phi}(k_x, k_y) e^{-i k_\mu x^\mu}$$

(2.5)
This gives a new coordinate space $x^{\mu}$ which one interprets as the semiclassical limit of $X^{\mu}$.

We label the lattice, with $N_0$ ($L = N_0 a$) lattice sites along each dimension, by the set of dimensionless numbers $(\tilde{x}, \tilde{y})$. The dimensional position coordinates on the lattice are then given by $x = \tilde{x} a$, $y = \tilde{y} a$. The finite lattice size implies that the dimensional momenta $k_x$ and $k_y$ are quantized in inverse units of the lattice size,

$$k_x = \frac{2\pi n_x}{N_0 a}, \quad k_y = \frac{2\pi n_y}{N_0 a}$$

where $n_x$ and $n_y$ are integers. We choose $N = N_0$ and, hence, fourier expand the field $\phi$ in terms of dimensionless coordinates $(\tilde{x}, \tilde{y})$ and dimensional coordinates $(x, y)$ on the lattice as follows,

$$\phi(\tilde{x}, \tilde{y}) = \sum_{n_x, n_y} \phi_{n_x, n_y} e^{-i\alpha(n_x \tilde{x} + n_y \tilde{y})} = \phi(x, y) = \sum_{k_x, k_y} \tilde{\phi}(k_x, k_y) e^{-i(k_x x + k_y y)}$$

where $\phi_{n_x, n_y} = \tilde{\phi}(k_x, k_y)$.

To go to the continuum, we take the limit of vanishing lattice spacing while keeping the dimensional noncommutativity parameter $\theta$ fixed. One can determine how $\theta$ scales with the lattice spacing $a$ and the lattice size $L = N_0 a = Na$ as follows. We consider the product of two matrix model operators $\tilde{\phi}_1$ and $\tilde{\phi}_2$ given by

$$\tilde{\phi}_1 \tilde{\phi}_2 = \sum_{n_x, n_y=0}^{N-1} \phi_{n_1, n_1} \phi_{n_2, n_2} \Gamma_{1}^{n_1 y + n_2 y} \Gamma_{2}^{n_1 x + n_2 x} e^{-\frac{i}{2}(n_1 x + n_2 x)(n_1 y + n_2 y)} \alpha e^{-i\alpha(n_1 x n_2 y - n_2 x n_1 y)}$$

Recalling the map from the reduced model to the noncommutative field theory we obtain the following equality

$$\theta^{12} (k_1 x k_2 y - k_1 y k_2 x) = \alpha(n_1 x n_2 y - n_1 y n_2 x)$$

from which we obtain the following scaling equation for $\theta$

$$\theta^{12} \equiv \theta = \frac{La}{2\pi} = \frac{Na^2}{2\pi}$$

where we have used the relation $L = Na$. (2.10) is also verified from (2.1) and (2.2).

The continuum limit is taken such that $a \to 0$. Then, in order to keep $\theta$ fixed, one has to take $N \to \infty$ with $La = Na^2$ fixed. Thus, in the present case (in section 5 we will take a different continuum limit such that the dimensional lattice size is fixed), the continuum
limit forces one to go to the limit of infinite lattice size. In the continuum limit, the map
between the noncommutative field φ and the matrix model operator ̃φ remains the same
as in (2.4) and (2.5) with the sum over the discrete set k being changed to integration
over continuous values of k. The derivative of the field φ is then mapped to the following
commutator in the reduced model

\[ i \partial_\mu \phi \leftrightarrow [P_\mu, \phi] \]  (2.11)

where \( P_\mu = (\theta^{-1})_{\mu \nu} X^\nu \equiv B_{\mu \nu} X^\nu \) and vector indices are raised or lowered with the metric
\( g_{\mu \nu} = \delta_{\mu \nu} \).

Finally, we mention that the following relation holds between the integrals of fields on
the noncommutative space and traces of operators in the reduced model,

\[ Tr \bar{\phi} = N \phi_{00} = \frac{1}{N} \sum_{\bar{x} \bar{y}} \phi(\bar{x}, \bar{y}) = \frac{1}{2\pi \theta} \int dxdy\phi(x, y) \]  (2.12)

3. Non commutative XY model

In this section we will consider the noncommutative XY model and try to understand
the nature of Kosterlitz - Thouless transiton in it. The continuum action for the model is
given by

\[ S = \frac{1}{g_{NC}^2} \int d^2x (\partial_\mu U^+ \partial^\mu U) \]  (3.1)

where

\[ U^*(x) * U(x) = 1 \]  (3.2)

Here \( U \) ( \( U^* \) is the complex conjugate of \( U \) ) is a scalar field on the two dimensional
noncommutative space \( (x^1, x^2) \) and product of fields implies the star product. To see that
this model is indeed the noncommutative analogue of the XY model one first uses (3.2) to
prove that \( U(x) \) can be written as \( U(x) = (e^{i\theta(x)})^* \), where \( \theta(x) \) is a new field. One can
easily check that the action in (3.1) then takes the form

\[ S = \frac{1}{g_{NC}^2} \int d^2x (\partial_\mu \theta * \partial^\mu \theta + \frac{i^2}{2!} (\theta * \partial_\mu \theta * \partial^\mu \theta - \partial_\mu \theta * \partial^\mu \theta * \theta) + O(\theta^4) + \cdots) \]  (3.3)

In the commutative limit, when star products are replaced by ordinary multiplication of
functions, the action takes the form \( S = \frac{1}{g_{NC}^2} \int d^2x (\partial_\mu \theta)^2 \) which, one easily recognizes,
is the form for the dimensionless energy in the corresponding statistical mechanical XY model with $\theta(x)$ denoting the order parameter.

The corresponding reduced model is obtained by replacing derivatives by commutators, integrals by traces and star products by ordinary matrix products according to the prescription given above and is, therefore, given by the following action

$$S = -\frac{1}{g^2} Tr[P_\mu, U^+][P^\mu, U]$$

(3.4)

where

$$U^+ U = U U^+ = 1 \quad \text{and} \quad g^2 N a^2 = g_{NC}^2$$

(3.5)

We define creation and annihilation operators (we will see later that $B$ is negative in our convention) as follows

$$a = \frac{1}{\sqrt{-2B}} (P_1 + i P_2) \quad \text{and} \quad a^+ = \frac{1}{\sqrt{-2B}} (P_1 - i P_2)$$

(3.6)

where

$$[P_1, P_2] = -i B_{12} \equiv -i B$$

(3.7)

so that $a$ and $a^+$ satisfy the usual commutation relation of the annihilation-creation operators of a one dimensional harmonic oscillator

$$[a, a^+] = 1$$

(3.8)

In terms of these variables the action takes the following form

$$S = \frac{2B}{g^2} Tr([a^+, U^+][a, U] + [a, U^+][a^+, U])$$

(3.9)

and the equation of motion for the field $U$ is given by

$$[a^+, U^+ [a, U]] + [a, U^+ [a^+, U]] = 0$$

(3.10)

4. Vortex Solution in noncommutative XY model

There exists a solution in this model which behaves as a vortex solution in the commutative limit. The solution is given by

$$U_0 = a^n \frac{1}{\sqrt{a^+ n a^n}}$$

(4.1)
This is the vortex solution for the higgs field in the abelian higgs model with an appropriate
gauge field [14] but also happens to be the vortex solution in our model in the absence of
gauge field. To see that this is a vortex solution in the commutative limit we recall the
definition of the creation and annihilation operators $a$ and $a^+$,
\[
a = \sqrt{-\frac{B}{2}}(X^2 - iX^1), \quad a^+ = \sqrt{-\frac{B}{2}}(X^2 + iX^1)
\] (4.2)

Now $X^\mu$ maps to a new coordinate space $x^\mu$ interpreted as the semiclassical limit of $X^\mu$.
In this new coordinate space the solution $U_0$ looks like
\[
U_0(z, \bar{z}) = \frac{z^n}{(z\bar{z})^{\frac{n}{2}}}
\] (4.3)

where $z = (x^2 - ix^1)$ and $\bar{z}$ is the complex conjugate of $z$. Writing $z$ in polar coordinates
as $z = r\exp(i\phi)$, we regain the following behaviour of the above solution, namely,
\[
U_0(r, \phi) = e^{in\phi}
\] (4.4)
which is a vortex in two dimensions with winding number $n$. In the commutative XY model
in two dimensions it can be easily shown, by a straightforward calculation, that the action
for the vortex behaves as $lnL$ where $L$ is the linear dimension of the two dimensional space .
We evaluate the noncommutative action in the following way. We first consider a complete
set of states of the number operator $a^+a$ and denote the individual states by $|m\rangle$, with
$m$ running from 0 to $\infty$. We then evaluate the action as a trace over this complete set of
states [15]. However, it turns out that a complete analytic behavior of this action is not
possible to obtain because of the infinite sum over the oscillator states. We regularize by
introducing an upper cutoff $R$ on the oscillator number $m$ and evaluate the action as a
sum only over states ranging from $|0\rangle$ to $|R\rangle$. This prescription for regulating the action
can be interpreted as an infrared regulation in the noncommutative theory because of the
following reason. The spread ( root mean square deviation ), in position space ($x^1, x^2$),
of the wavefunction corresponding to the state $|m\rangle$ can be calculated to be $\sqrt{m\theta}$. The
spread must be bounded by the system size $L = \sqrt{2\pi N\theta}$. Equating the maximum spread
with the system size we obtain $R = 2\pi N$. Since the spread depends on the oscillator
number, imposing an infrared cutoff ( or fixing a particular system size ) implies a system
size dependent cutoff on the harmonic oscillator level number $m$ and is expressed by the
relation $L = \sqrt{R\theta}$. Without the cutoff, we calculate the action for this vortex solution as follows,

$$S = \frac{2B}{g^2} \text{Tr}[a^\dagger U_0^\dagger aU_0 + a^\dagger U_0 aU_0^\dagger - (aa^\dagger + a^\dagger a)]$$

$$= \frac{2B}{g^2} \left[ \sum_{m=n}^\infty \sqrt{m(m-n)} + \sum_{m=0}^\infty \sqrt{m(m+n)} - \sum_{m=0}^\infty (2m+1) \right]$$

$$= \frac{2B}{g^2} \left[ 2 \sum_{m=0}^\infty \sqrt{m(m+n)} - \sum_{m=0}^\infty (2m+1) \right] \quad (4.5)$$

Here the limits on the first sum in the second line comes because

$$\sum_{m=0}^\infty < m|a^\dagger U_0^\dagger aU_0|m > = \sum_{m=n}^\infty \sqrt{m(m-n)} \quad (4.6)$$

and we have shifted the summed over integer to go from the second to the third line and used the following relations,

$$a|m >= \sqrt{m}|m-1 > , \quad a^+|m >= \sqrt{m+1}|m+1 > \quad (4.7)$$

$$U_0|m >= |m-n > , \quad U_0^+|m >= |m+n > \quad (4.8)$$

Since we are interested in the way the action scales with the system size or, in other words its large distance behaviour, we shall try to investigate the behaviour, for large oscillator numbers, of the sum over oscillator levels in the action.

Expanding the first term in the action as

$$\sqrt{m(m+n)} = m(1 + \frac{1}{2} \frac{n}{m} - \frac{1}{8} \frac{n^2}{m^2} + \cdots) \quad (4.9)$$

we get

$$S = \frac{2B}{g^2} \sum_{m=1}^\infty [(n-1) - \frac{1}{4} \frac{n^2}{m} + \cdots] \quad (4.10)$$

The first term is linearly divergent and the second term is logarithmically divergent.

One way to make sense of these sums is to use zeta function regularization, where

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \quad (4.11)$$

While this is divergent for $s = 0$ the analytic continuation of $\zeta(z)$ for arbitrary complex $z$ is finite and equal to $-\frac{1}{2}$. Therefore, it is possible to assign a finite value to the first
term. The second term is genuinely divergent and is in fact log divergent. Introducing the
cutoff \( R \) on oscillator level \( m \) at this stage shows that the action diverges as \( \ln R = \ln \frac{L^2}{\theta} \).
Thus we recover the usual behaviour that the action (or the dimensionless energy in the
statistical mechanical viewpoint) for a vortex scales as the logarithm of the system size.

One can follow a more rigorous method for finding the asymptotic expansion in \( R = 2\pi N \) of the action by using the Euler - MacLaurin formula,

\[
\sum_{m=0}^{R} f(m) = \frac{1}{2} f(R) + \int_{0}^{R} dt f'(t) + C + \left[ \frac{B_2}{2!} f^{(1)}(R) - \frac{B_3}{3!} f^{(2)}(R) + \frac{B_4}{4!} f^{(3)}(R) - \cdots \right]
\] (4.12)

where \( C \) is a constant and \( B_n \) are Bernoulli numbers. Using \( f(m) = \sqrt{m(m+n)} \) in the
above formula, we get for our action,

\[
S = \frac{2B}{g^2} [R(n - 1) - \frac{n^2}{4} \ln R + O(1/R) + \cdots]
\] (4.13)

This is the same expression as obtained from (4.10).

A better regularization [16] is obtained by using the q analogue of the harmonic
oscillator operators \( a \) and \( a^+ \) denoted by \( A \) and \( A^+ \) respectively,

\[
A = a \sqrt{[M]_q}, \quad A^+ = a^+ \sqrt{[M]_q}
\] (4.14)

where

\[
[M]_q = \frac{q^M - q^{-M}}{q - q^{-1}}
\] (4.15)

where \( q \) is \( 2R \)-th root of unity and the usual oscillator limit is the one for \( q \to 1 \) or \( R \to \infty \).
Replacing \( a, a^+ \) by \( A, A^+ \) respectively, and evaluating the sum over oscillators we get the
following expression for the action,

\[
S = \frac{2B}{g^2} \sum_{m=0}^{\infty} (2\sqrt{[m][m+n]} - [m + 1] - [m])
\] (4.16)

The infinite sum is automatically cutoff because \( [R]_q = 0 \) for \( q^{2R} = 1 \) and, hence, one
does not have to put in a cutoff by hand. The action can, in principle, be evaluated, as a
function of the cutoff \( R \), by numerical methods.

A small note on the Kosterlitz - Thouless (KT) transition for this model is, probably,
in order here. As noted earlier the field theory action of the vortex varies with \( R \) as

\[
S = -\frac{Bn^2}{2g^2} \ln R = \frac{\pi n^2}{g_{NC}} \ln R = \frac{\pi n^2}{g_{NC}^2} \ln \frac{L^2}{\theta}
\] (4.17)
The apparent negative sign in the first expression for the action may be surprising. However, we note that with our choice of conventions

\[
\theta^{\mu\nu} = \frac{N a^2}{2\pi} \epsilon^{\mu\nu}, \quad B^{\mu\nu} \equiv (\theta^{-1})^{\mu\nu} = -\frac{2\pi}{N a^2} \epsilon^{\mu\nu}
\]  

(4.18)

from where it easily follows that \( B \) is a negative number

\[
B \equiv B^{12} = -\frac{1}{\theta^{12}} = \frac{1}{\theta}
\]

(4.19)

One can, of course, obtain a positive \( B \) by a change of convention which involves interchanging the roles of \( X^1, X^2 \) but since the action (3.1) is symmetric in \( x^1 \) and \( x^2 \) it still remains positive. The entropy \( \bar{S} \) of a vortex depends on the number of available locations for the vortex. Because of the noncommutativity of the coordinates, a vortex cannot be localized in any region with area smaller than \( \theta \), since, the product of uncertainties of the two noncommuting coordinates is of order \( \theta \). This is true because from the relation

\[
[X^1, X^2] = i\theta
\]

(4.20)

it follows that

\[
\Delta X^1 \Delta X^2 \geq \frac{\theta}{2}
\]

(4.21)

where \( \Delta X^i \) denote the root mean square deviations of the coordinates \( X^i \) taken over any state of the system. Therefore, for an area \( A = X^1 X^2 \) of order \( \theta \), the minimum uncertainty, \( \Delta A \), is of order \( \theta \) which implies that the centre of a vortex cannot be localized to a region of area less than order \( \theta \). Hence, the number of positions available to a vortex in a region of size \( L^2 \) is of order \( \frac{L^2}{\theta} \), which implies that the entropy should scale as \( \ln \frac{L^2}{\theta} \). Therefore, the dimensionless Helmholtz free energy is given, in terms of the couplings \( g \) and \( g_{NC} \) by

\[
F \equiv S - \bar{S} = -\frac{B n^2}{2g^2} \ln \frac{L^2}{\theta} - \ln \frac{L^2}{\theta} = \frac{\pi n^2}{g_{NC}^2} \ln \frac{L^2}{\theta} - \ln \frac{L^2}{\theta}
\]

(4.22)

Hence, the couplings \( g_c \) and \((g_{NC})_c\) at which it is first favourable to produce vortices is given by

\[
g_c^2 = -\frac{B n^2}{2}, \quad (g_{NC})_c^2 = \pi n^2
\]

(4.23)

The critical temperature is obtained by using the following relationship between the field theory coupling \( g_{NC}^2 \) and the the statistical mechanical coupling \( J \), namely,

\[
\frac{J}{T} = \frac{1}{g_{NC}^2}
\]

(4.24)
where $J$ is the strength of the spin-spin interaction that occurs in the expression for the spin-spin interaction energy $E$ as follows

$$E = J \sum_{ij} \cos(\theta_i - \theta_j) \quad (4.25)$$

where $\theta_i$ refers to the order parameter at position $i$ in a two dimensional lattice and the pair $ij$ refers to nearest neighbours. With the correspondence (4.24) the critical temperature is obtained as

$$T_c = J\pi n^2 \quad (4.26)$$

5. Effect of finite lattice size on KT transition

In this section we will modify our previous calculations and take the continuum limit while keeping the lattice size fixed. This limit has been considered in [11], [17]. We shall use the same $N$ dimensional representation of $\Gamma_\mu$ as in section 2. We expand the matrix model field $\bar{\phi}$ as follows,

$$\bar{\phi} = \frac{(N-1)/\beta}{\sum_{n_x,n_y=0} \phi_{n_x n_y} \Gamma_1^{\beta n_y} \Gamma_2^{\beta n_x} e^{-\frac{i}{2} n_x n_y \alpha \beta^2}} \equiv \sum_{k_x k_y} \tilde{\phi}(k_x, k_y) e^{-ik\mu X^\mu} \quad (5.1)$$

where $\phi_{n_x n_y} = \tilde{\phi}(k_x, k_y)$ and $\beta \equiv N N_0$ is an integer and $N_0$ is the number of lattice sites along one dimension in the lattice. The upper limit on the sum above follows from the fact that $\Gamma_\mu^N = 1$. For a lattice of spacing $a$ and size $L = N_0 a$, the dimensional momenta $k_x$ and $k_y$ are again given as in (2.6). The fourier expansion of the field $\phi$ in dimensionless and dimensional coordinates is then as follows

$$\phi(\tilde{x}, \tilde{y}) = \sum_{n_x, n_y} \phi_{n_x n_y} e^{-i\alpha \beta (n_x \tilde{x} + n_y \tilde{y})} = \phi(x, y) = \sum_{k_x, k_y} \tilde{\phi}(k_x k_y) e^{-i(k_x x + k_y y)} \quad (5.2)$$

The product of two matrix model operators $\bar{\phi}_1$ and $\bar{\phi}_2$ is given by

$$\bar{\phi}_1 \bar{\phi}_2 = \frac{(N-1)/\beta}{\sum_{n_1, n_2, n_y=0} \phi_{n_1 x n_1 y} \phi_{n_2 x n_2 y} \Gamma_1^{\beta N_x} \Gamma_2^{\beta N_y} e^{-\frac{i\alpha^2}{2} N_x N_y e^{-\frac{i\beta^2}{2} (n_1 x n_2 y - n_2 x n_1 y)}} \quad (5.3)$$

where

$$N_x = (n_1 x + n_2 x), \quad N_y = (n_1 y + n_2 y) \quad (5.4)$$
Recalling the map from the reduced model to the noncommutative field theory we obtain the following equation for $\theta$

$$\theta = \frac{Na^2}{2\pi}$$

(5.5)

The continuum limit ($a \to 0$) is taken such that the system size $L = N_0a$ and the dimensional noncommutativity parameter $\theta$ are kept constant. This implies the following scaling relations

$$a = \sqrt{\frac{2\pi\theta}{N}}, \quad \beta = \frac{1}{L}\sqrt{2\pi\theta N}$$

(5.6)

In this case, the following relation holds between the integrals of fields and traces of corresponding operators,

$$\text{Tr} \bar{\phi} = N\phi_{00} = \frac{N}{N_0^2} \sum_{\tilde{x}, \tilde{y}} \phi(\tilde{x}, \tilde{y}) = \frac{N}{L^2} \int dxdy\phi(x, y)$$

(5.7)

We now consider the noncommutative XY model action given by (3.1). As usual, the corresponding reduced model is obtained by replacing derivatives by commutators, star products by matrix products and integrals by traces. Noticing that the map between derivatives and commutators remains the same as in (2.11), we get the following form for the reduced action

$$S = -\frac{1}{g^2} \text{Tr}[P_\mu, U^+]\text{[}P^\mu, U\text{]}$$

(5.8)

where $g^2 = \frac{N}{L^2}g_{NC}^2$. An evaluation of the action along lines analogous to sections (3) and (4) then gives

$$S = \frac{2B}{g^2} \left[2 \sum_{m=0}^{L^2/\theta} \sqrt{m(m+n)} - \sum_{m=0}^{L^2/\theta} (2m+1)\right]$$

(5.9)

The upper limit on the sum follows from the fact that the spreads of the harmonic oscillator levels $|m >$ (which scale as $\sqrt{m\theta}$) should be bounded by the size $L$ of the system. Evaluating the sum in (5.9) we obtain (after subtracting the spurious linear dependence on $L^2/\theta$), for large $L^2/\theta$, the following expression for the action,

$$S = -\frac{Bn^2}{2g^2} \ln \frac{L^2}{\theta}$$

(5.10)

The entropy is given by the same expression as in the infinite lattice case. The dimensionless Helmholtz free energy is, therefore, given by

$$F = -\frac{Bn^2}{2g^2} \ln \frac{L^2}{\theta} - \ln \frac{L^2}{\theta} = -\frac{Bn^2L^2}{2Ng_{NC}} \ln \frac{L^2}{\theta} - \ln \frac{L^2}{\theta}$$

(5.11)
from which it follows that the critical values $g_c$ and $(g_{NC})_c$ of the couplings above which it is favourable to produce vortices is given by

$$
g_c^2 = -\frac{Bn^2}{2}, \quad (g_{NC})_c^2 = -\frac{Bn^2L^2}{2N} \quad (5.12)
$$

From (4.23) and (5.12) it is clear that the critical coupling $g_c$ is the same in both the finite and infinite lattice cases. However, the critical coupling $g_{NC}$, in the large $N$ limit, is finite in the infinite lattice case and zero in the case of the finite lattice. It, therefore, follows that the transition to the vortex phase occurs at finite temperature for infinite lattice and at zero temperature for the finite lattice.

6. Conclusion

The noncommutative $X - Y$ model we have considered is a generalization of the commutative $X - Y$ model for nonzero noncommutativity parameter $\theta$. By mapping the field theory to its equivalent matrix model we have been able to construct vortex solutions in it. Kosterlitz Thouless transition from a phase of no vortex to a phase of vortices occurs when the field theory coupling or, equivalently, the matrix model coupling, reaches a critical value. We have shown that the critical matrix model coupling is the same when the system is put in either a finite or an infinite lattice. The critical values of the coupling in the continuum field theory is finite in the case of an infinite lattice and zero in the case of a finite lattice.

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