Local Stability of PD Controlled Bipedal Walking Robots

Shishir Kolathaya

Abstract

We establish stability results for PD tracking control laws in bipedal walking robots. Stability of PD control laws for continuous robotic systems is an established result, and we extend this for hybrid robotic systems, an alternating sequence of continuous and discrete events. Bipedal robots have the leg-swing as the continuous event, and the foot-strike as the discrete event. In addition, bipeds largely have underactuations due to the interactions between feet and ground. For each continuous event, we establish that the convergence rate of the tracking error can be regulated via appropriate tuning of the PD gains; and for each discrete event, we establish that this convergence rate sufficiently overcomes the nonlinear impacts by assumptions on the hybrid zero dynamics. The main contributions are 1) Extension of the stability results of PD control laws for underactuated robotic systems, and 2) Exponential ultimate boundedness of hybrid periodic orbits under the assumption of exponential stability of their projections to the hybrid zero dynamics. Towards the end, we will validate these results in a 2-link bipedal walker in simulation.

Key words: PD controllers; Walking; Robotics; Periodic motion.

1 Introduction

Despite great advances in the theory of nonlinear controls, when it comes to practical implementation, PD control laws undisputably continue to be the most popular choice. Table 1 shows a detailed account on the list of controllers used and their corresponding acceptance ratings. The popularity of this type of control laws arises from its ease of implementation and robustness due to its model independent nature. This popularity has equally pervaded robotic systems. In fact, there are formal guarantees of stability for a broad class of robots that includes manipulators [21,36]. See Table 1, which shows a list of stability results. A more detailed list of these types of control laws and the corresponding stability results are given in [5, Table 1.1].

Despite the increase in complexity of the models, PD control laws have dominated even the domain of bipedal robots. Bipedal locomotion is hybrid in nature, with alternating phases of continuous (swinging forward of the nonstance foot) and discrete events (instantaneous impact of the nonstance foot with ground). In addition, unlike the industrial robotic arms, which have a fixed base, bipedal robots are largely underactuated. Fig. 1 shows some examples of bipedal robots that used PD and PD based tracking
control laws. It is worth noting that the reference trajectories that were tracked were obtained offline. In fact, the field of locomotion has largely focused on experimental realization by dividing the problem into two parts. First, obtain reference trajectories/gaits in a simulation model by using offline optimization tools [11,31]. Second, play these trajectories in the robot by a low level tracking control law [12,32,40]. Due to uncertainties in the system, model based controllers were generally avoided, thereby giving preference to the more traditional PD based control laws. These control laws are known to be “hassle free”, since they are model-independent and easy to implement. Therefore, the main goal of this paper is to explore the stability properties of PD control laws for walking robots, which include varying levels of complexity due to the presence of impacts and underactuations.

We will be establishing stability of PD control laws by the construction of strict Lyapunov functions developed by Arimoto et al. [2], Koditschek [18], and Bayard and Wen [36] all in the same period of time 1984-1988. Local and global stability results were shown for both stationary and time varying desired configurations [15,19,38], but only for fully actuated systems. For underactuated systems, we can apply a tracking control law for the actuated states of the robot, but this does not necessarily guarantee stability, due to the coupling between the controllable and uncontrollable dynamics of the robot. In addition, the impacts due to footstrike largely have “destabilizing” effects on the tracking errors. On the other hand, if we make assumptions about the uncontrollable dynamics of the robot i.e., existence of stable periodic orbits in the hybrid zero dynamics (HZD), we can then establish local stability results. This will be the main approach of the paper.

It is important to discuss the notion of hybrid zero dynamics (HZD) in the context of bipedal walking. HZD was introduced as a feedback design method to move beyond the traditional quasi-static flat-footed walking gaits [7,37]. The goal was to realize a stable walking gait (periodic orbit) via stabilization of a subset of the states of the robot, while the remaining states of the system exhibit uncontrollable dynamics (resulting in HZD). It was shown in [37] that if there is an exponentially stable periodic orbit in the HZD, then by employing a suitable output stabilizing control law [1,28], the periodic orbit of the full hybrid dynamics can be stabilized, resulting in stable walking.

There were two main output stabilizing controllers proposed over the last ten years using this notion of HZD—feedback linearization [28] and control Lyapunov functions (CLF) [1,24]. Both of these control methodologies relied on using a user defined ε, which was, in principle, increasing the controller gain. With a sufficiently large gain, the destabilizing impact events were overcome by faster convergences of the outputs. Our focus in this paper is to realize the same behavior via PD based output stabilizing control laws, wherein the desirable convergence rates are obtained via tuning of the PD gains.

Despite their widespread use in practical robotic systems, PD control laws do not have all of the properties that are typically “taken for granted” with model based control laws. Closed loop dynamics obtained from PD control laws do not necessarily have equilibrium points. This also implies that orbits in the HZD may not necessarily be orbits in the full order dynamics. However, we can use properties of the inertia, Coriolis-centrifugal and gravity matrices, and then guarantee desirable convergence rates to an ultimate bound, i.e., exponential ultimate boundedness of the outputs. This property was utilized in [38] to establish boundedness of tracking errors for fully actuated systems.

Organization. The paper is structured as follows. Section 2 introduces the hybrid system model of locomotion and the associated control methodologies. This section also describes the concept of HZD and the associated stable periodic orbit. In Section 3, we focus on underactuation and construct the PD control law for output stabilization. We will also list a set of assumptions on the desired trajectories that will be useful to simplify the main results of the paper. Section 4 contains the main results, i.e., stability results of PD based control laws for hybrid systems with Lagrangian dynamics in a series of Lemmas and Theorems. Proofs are provided in Section 5. Finally Section 6 provides the simulation results of PD control on a 2-DOF Walker.

2 Robot walking model and control

In this section, we will discuss the hybrid model of a walking robot, and the associated notion of hybrid zero dynamics (HZD) for walking. The associated periodic orbits of the HZD will also be described.

Notation. R is the set of real numbers, R^n denotes the Euclidean space of dimension n. The open ball of radius r > 0 centered at x ∈ R^n is denoted by B_r(x). Given x ∈ R^n, |x| is the Euclidean norm of x, and given a matrix A ∈ R^{n×m}, ||A|| is the matrix norm of A. Given a set S ⊂ R^n, we denote the shortest distance between the point x ∈ R^n and the set S to be ||x||_S := inf_{y ∈ S} ||x - y||. We will sometimes denote the vector [x^T, y^T]^T ∈ R^{n+x+n_y} as the pair (x,y). Note that the Euclidean norm has the property ||(x,y)||^2 = |x|^2 + |y|^2.

2.1 Robot model

We consider an n-DOF robotic system, with the configuration manifold Q. We will specifically consider relative degree two systems for convenience. Therefore, we denote the state by x = (q,q') ∈ TQ. We will denote the torque input by u ∈ R^m, which is of dimension m. The dynamic model of walking consists of a continuous (swing) phase and a discrete (impact or foot-strike) phase. The discrete phase consists of a switch based on a guard condition (i.e., the swinging foot height h crossing zero). Each of these events will be described briefly below.
Continuous dynamics. Given the states \((q, \dot{q})\) and inputs \(u\), the Euler-Lagrangian dynamics is given by

\[
D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu,
\]

where \(D(q) \in \mathbb{R}^{n \times n}\) is the inertia matrix, \(C(q, \dot{q}) \in \mathbb{R}^{n \times n}\) is the Coriolis-centrifugal matrix, \(G(q) \in \mathbb{R}^{n}\) is the gravity vector, and \(B \in \mathbb{R}^{n \times m}\) is the mapping of the torques to the joints. Without loss of generality, we assume that the choice of \(q\) is such that the mapping of torques to actuated joints is one-to-one i.e., each column of \(B\) consists of only one element with value one and the rest are zeros. Having described (1), we have the following properties of the model [6,9,29]¹:

Property 1 \(D\) is positive definite symmetric, and \(\dot{D} - 2C\) is skew-symmetric for any \((q, \dot{q}) \in TQ\).

Property 2 There exist positive constants \(c_l, c_u > 0\) such that for any \((q, \dot{q}) \in TQ\):

- \(c_l \leq \|D(q)\| \leq c_u\)
- \(c_l \leq \|D^{-1}(q)\| \leq c_u\)
- \(\|\dot{D}(q)\| \leq c_u\)
- \(\|C(q, \dot{q})\| \leq c_u\)
- \(\|G(q)\| \leq c_u\).

Note that each of the matrices, \(D, D^{-1}, C, G\) have their own bounds. We have used the same constants for ease of notations. (1) can be represented in statespace form as

\[
\dot{x} = f(x) + g(x)u,
\]

by appropriate determination of \(f, g\) (see [23, (13)]). The continuous dynamics of the robot is defined on the set of admissible states \(\mathbb{D} \subset TQ\), defined by

\[
\mathbb{D} = \{(q, \dot{q}) \in TQ : h(q, \dot{q}) \geq 0\},
\]

where \(h : TQ \to \mathbb{R}\) is the height of the swing foot (see Fig. 2). \(\mathbb{D}\) is called the domain. Note that \(h\) is chosen such that it only depends on the configuration \(q\) i.e., \(\frac{\partial h}{\partial q} \equiv 0\)².

Discrete dynamics. Having described the continuous dynamics in statespace form (2), we can now describe the discrete dynamics of the walking robot. As observed in Fig. 2, when the height \(h\) crosses zero transversally, we have an impact. This impact is represented with the rigid contact model of [13,14]. The swinging foot is assumed to have no rebound or slip during an impact. The velocity component of the robot state experiences a jump, while the configuration component remains continuous. Since the walking is symmetric, the roles of the legs are swapped after every foot-strike. We define the guard set representing the foot-strike as

\[
\mathbb{S} = \{(q, \dot{q}) \in \mathbb{D} : h(q, \dot{q}) = 0, L_f h(q, \dot{q}) < 0\},
\]

with \(L_f h(q, \dot{q}) \equiv 0\). Here \(L_f, L_g\) are the Lie derivatives w.r.t. \(f, g\) respectively. When \(x \in \mathbb{S}\), we have a discrete event:

\[
x^+ = R(x), \quad x \in \mathbb{S},
\]

where \(x\) is the pre-impact state, \(x^+\) is the post-impact state, and \(R : \mathbb{S} \to \mathbb{D}\) is the impact map (or reset map) of the robot.

Having obtained the continuous time (2) and discrete time (5) model of walking, we obtain the hybrid control system model as

\[
\mathcal{H}^6 = \begin{cases} 
\dot{x} = f(x) + g(x)u, & x \in \mathbb{D} \setminus \mathbb{S} \\
x^+ = R(x), & x \in \mathbb{S}.
\end{cases}
\]

A pictorial representation of this hybrid control system model is given in Fig. 2. Note that based on the definition of the guard set (4), we assume that \(R(\mathbb{S}) \cap \mathbb{S} = \emptyset\) (to avoid consecutive jumps).

2.2 Hybrid zero dynamics

We will now mathematically describe the notion of HZD. Denote the relative degree two outputs \(y : Q \to \mathbb{R}^m\) of dimension \(m\) as

\[
y(q) = y^d(q) - y^d(q),
\]

where \(y^d : Q \to \mathbb{R}^m, y^d : Q \to \mathbb{R}^m\) are the actual and the desired values respectively. The desired values are chosen such that \(y(q) = 0\) for at least one \(q\), like in [37, HH4]. Similarly, denote the passive states (or the unactuated states) of dimension \(2l = 2n - 2m\) as \(z : TQ \to \mathbb{R}^{2l}\). Accordingly,
we have the output dynamics (or the transverse dynamics) and passive dynamics of the form:
\[ \ddot{y} = \dot{L}_2 y + L_q \dot{L}_f y u \]  
\[ \dot{z} = \psi(y, \dot{y}, z). \]  
(8)  
(9)

Here \( \psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^2 \) is the vector field (of \( z \)). Note that \( z \) is chosen such that \( L_q z = 0 \) and a diffeomorphism from \((q, \dot{q})\) to \((y, \dot{y}, z)\) exists (see Assumption 2 further ahead). We denote this diffeomorphism as \( \Phi : \mathbb{D} \rightarrow \mathbb{R}^{2n} \).

With a suitable output stabilizing controller, we can guarantee convergence of the outputs \((y, \dot{y})\) to zero. For example, a feedback linearizing control law of the form:
\[ u_{IO} = L_q \dot{L}_f y^{-1} (-L_2 y - 2e \dot{y} - e^2 y), \]
(10)
yields exponential convergence of the outputs to zero. If \((y, \dot{y}) = (0, 0)\), the resulting passive dynamics given by
\[ \dot{z} = \psi(0, 0, z), \]
is now called the zero dynamics of the closed loop system of (1). It is important to note that output stabilization can be realized only during the swing mode (continuous event) of the hybrid dynamics. On the other hand, if the outputs are chosen in such a way that the zero dynamics is invariant of the discrete dynamics, we have hybrid zero dynamics (HZD).

Consider the reduced dimensional surface:
\[ Z = \{(q, \dot{q}) \in \mathbb{D} : y(q) = 0, \dot{y}(q) = 0\}. \]
(12)
Also consider a post-impact map defined in terms of the transformed coordinates:
\[ \Delta(y, \dot{y}, z) := \Phi \circ R \circ \Phi^{-1}(y, \dot{y}, z), \]
(13)
which consists of two components:
\[
\begin{pmatrix}
\Delta_y(y, \dot{y}, z) \\
\Delta_z(y, \dot{y}, z)
\end{pmatrix} := \Delta(y, \dot{y}, z),
\]
(14)
corresponding to the outputs \((y, \dot{y})\) and the zero coordinates \(z\) respectively. We have HZD, if the following is satisfied:
\[ R(\mathbb{S} \cap Z) \subset Z \quad \text{or} \quad \Delta(\Phi(\mathbb{S} \cap Z)) \subset \Phi(Z). \]
(15)

With this formulation, the goal now is to obtain a periodic orbit in the HZD and, consequently, a periodic orbit in the full order hybrid dynamics.

2.3 Hybrid periodic orbits

With the initial condition \((y^*, \dot{y}^*, z^*) \in \Phi(\mathbb{S})\), and the control law (10), let \( \varphi_t(\Delta(y^*, \dot{y}^*, z^*)) \) be the resulting flow of (8), (9) represented in transformed coordinates. By assuming right continuity [35, Section II-B], this flow is, in fact, the solution for the entire hybrid system (6) that includes both the continuous and the discrete dynamics. We say that there is a periodic orbit if there exists a \( T^* > 0 \) such that
\[ \lim_{t \rightarrow T^*} \varphi_t(\Delta(y^*, \dot{y}^*, z^*)) = (y^*, \dot{y}^*, z^*), \]
(16)
and the point \((y^*, \dot{y}^*, z^*)\) is called the fixed point of the periodic orbit. We will denote this periodic orbit as
\[ \mathcal{O} := \{\varphi_t(\Delta(y^*, \dot{y}^*, z^*)) \in \Phi(\mathbb{D}) : 0 \leq t < T^*\}. \]
(17)
We will also denote the fixed point in angle-velocity coordinates as \( x^* := \Phi^{-1}(y^*, \dot{y}^*, z^*) \).

If the periodic orbit \( \mathcal{O} \) satisfies some properties (like transversality and isolated intersections with the guard \( \Phi(\mathbb{S}) \) [28, H2.4, H2.5], [35, A.6, A.7]), then we know that for an initial state \( x_0 \) in a small neighborhood of \( x^* \) i.e., \( x_0 \in \mathbb{B}_r(x^*) \cap \mathbb{S} \), the time-to-impact function is well defined with distinct lower and upper bounds, and can be obtained as
\[ T(y, \dot{y}, z) := \inf\{t > 0 : h(\Phi^{-1}(\Phi_t(\Delta(y, \dot{y}, z)))) = 0\}, \]
(18)
where \((y, \dot{y}, z) = \Phi(x_0)\). See [28,35] for more details on time-to-impact or dwell-time functions. Note that the problem formulation is constructed in such a way that undesirable behaviors like consecutive jumps and Zeno executions are avoided. Later on (Lemma 1, 2), we will show that by using sufficiently large controller gains, even PD based control laws yield well defined time-to-impact functions.

Having defined the periodic orbit and time-to-impact functions, we can define some stability properties for \( \mathcal{O} \) that will be useful throughout the paper. For the following definition, we consider the solution \( \varphi_t \) for the entire time interval \([0, \infty)\) (by assuming right continuity [35, Section II-B]).

Definition 1 \( \mathcal{O} \) is said to be locally exponentially stable (LES) if there are constants \( M, r, \lambda > 0 \) such that for all \((y, \dot{y}, z) \in \mathbb{B}_r(0, 0, z^*) \cap \Phi(\mathbb{S})\),
\[ ||\varphi_t(\Delta(y, \dot{y}, z))||_{\mathcal{O}} \leq Me^{-\lambda t} ||\Delta(y, \dot{y}, z)||_{\mathcal{O}}. \]
(19)

Similarly, \( \mathcal{O} \) is said to be locally exponentially ultimately bounded (LEUB) if there are constants \( M, r, \lambda, d > 0 \) such that for all \((y, \dot{y}, z) \in \mathbb{B}_r(0, 0, z^*) \cap \Phi(\mathbb{S})\),
\[ ||\varphi_t(\Delta(y, \dot{y}, z))||_{\mathcal{O}} \leq Me^{-\lambda t} ||\Delta(y, \dot{y}, z)||_{\mathcal{O}} + d. \]
(20)

We will be mainly establishing LEUB of \( \mathcal{O} \) via Poincaré maps (due to equivalence in stability results between periodic orbits and Poincaré maps [28,35]). Therefore, it is sufficient if we define the flow \( \varphi \) for the closed interval \([0, T]\), i.e., for only the continuous dynamics. After every impact,
the time \( t \) can be reset to zero with the resulting new initial state on the Poincaré section. More details on Poincaré maps are provided in Section 4.

**Remark 1** The notion of ultimate boundedness is valid even for systems without equilibrium points including periodic orbits [16, 4.8]. In the analysis that follows the next section, we will show that the set of points \( \mathcal{O} \) can be shown to be ultimately bounded when PD based control laws are applied.

We can have similar notions of ultimate boundedness for the output coordinates in the continuous dynamics. For convenience, we will denote the initial state as \((y_0, \dot{y}_0, z_0)\) and the resulting trajectory, post-impact, as \((y(t), \dot{y}(t), z(t))\) with \((y(0), \dot{y}(0), z(0)) = \Delta(y_0, \dot{y}_0, z_0)\). We have the following definition for boundedness of the outputs:

**Definition 2** Given \( T_\delta > 0 \), the zero values of the outputs \((y(t), \dot{y}(t))\) are said to be locally exponentially ultimately bounded (LEUB) in the interval \( t \in [0, T_\delta] \) if there are constants \( M, r, \lambda, d > 0 \) such that for all \((y_0, \dot{y}_0, z_0) \in \mathbb{B}_r(0, 0, 0) \cap \Phi(S)\),

\[
\|(y(t), \dot{y}(t))\| \leq Me^{-\lambda t}\|\Delta(0, 0, 0)\| + d.
\] (21)

The notion of ultimate boundedness (UB) and local exponential ultimate boundedness (LEUB) are applied to systems that are forward complete, i.e., \( t \in [0, \infty) \). We would still like to use this definition for shorter finite intervals, since we have a hybrid system with restrictions on dwell-time. Therefore, whenever we say that the outputs are LEUB in the interval \([0, T_\delta]\), we mean that (21) is valid for \([0, T_\delta]\).

It was shown in [1, Theorem 2] that if there is an exponentially stable periodic orbit in the HZD, then by choosing a sufficiently large enough \( \varepsilon \) in (10), we can realize an exponentially stable periodic orbit in the full order dynamics. In this paper, we will particularly focus on obtaining a similar result (boundedness) with PD control by choosing large enough gains.

3 PD tracking with underactuation

In this section, we will focus on PD based control laws for robotic systems with underactuation. Since \( m \) is the degree of actuation (DOA) and \( n \) is the degree of freedom (DOF), we have the corresponding degree of underactuation as \( l = n - m \). Accordingly, we have the following separation in the dynamics:

\[
\begin{align*}
D_{11}(q)\ddot{q} + D_{12}(q)\dot{q}^a + C_1(q, \dot{q})\dot{q} + G_1(q) &= 0 \\
D_{21}(q)\ddot{q}^a + D_{22}(q)\dot{q}^a + C_2(q, \dot{q})\dot{q} + G_2(q) &= u,
\end{align*}
\] (22)

where the terms corresponding to \( D, C, G \) are apparent from the setup. \( \dot{q}, \dot{q}^a \) are the unactuated and actuated configurations respectively. Also let \( B_c \in \mathbb{R}^{n \times l} \) be the constant matrix obtained in such a way that if the \( l \) unactuated joints had inputs, then these inputs would be mapped via this matrix. In other words, the new notations \( B_c, q^a, q^a \) are defined in such a way that

\[
q = \begin{bmatrix} B_c \quad B \end{bmatrix} \begin{bmatrix} q^a \\ q^u \end{bmatrix}.
\] (23)

Accordingly, we have that

\[
\begin{align*}
\begin{bmatrix} D_{11} & D_{12} \\
D_{21} & D_{22} \end{bmatrix} &= \begin{bmatrix} B_c^T DB_c & B_c^T DB \\
B^T DB_c & B^T DB \end{bmatrix} \\
\begin{bmatrix} C_1 \\
C_2 \end{bmatrix} &= \begin{bmatrix} B_c^T C \\
B^T C \end{bmatrix} \\
\begin{bmatrix} G_1 \\
G_2 \end{bmatrix} &= \begin{bmatrix} B_c^T G \\
B^T G \end{bmatrix}.
\end{align*}
\] (24)

The matrices \( B_c, B \) have some important properties that will be utilized in the proofs in this section. Some of them are

\[
[B_c, B] = \begin{bmatrix} B_c^T \\ B^T \end{bmatrix} = 1_{n \times n},
\]

\[
B^T B = 1_{m \times m}, \quad B_c^T B_c = 1_{l \times l}, \quad B_c^T B = 0_{n \times m}, \quad B^T B_c = 0_{m \times l},
\]

where \( 1 \) is the identity matrix, \( 0 \) is the zero matrix with appropriate dimensions. Based on these properties, \( \ddot{q}^a \) can be eliminated from (22) to obtain

\[
B^T ADB\ddot{q}^a + B^T AC\dot{q} + B^T AG = u,
\] (25)

where

\[
A := 1_{n \times n} - DB_c(B_c^T DB_c)^{-1}B_c^T,
\]

with \( 1_{n \times n} \) being the identity matrix of dimension \( n \). It can be verified that \( B^T ADB \) is nothing but the Schur complement of \( D_{11} = B_c^T DB_c \) in \( D \), and it has some important properties:

**Proposition 1** \( B^T A(q)D(q)B \) is symmetric positive definite for all \( \langle q, \dot{q} \rangle \in \mathcal{T}Q \).

**Proposition 2** There exist positive constants \( c_1, c_u \) such that for all \( \langle q, \dot{q} \rangle \in \mathcal{T}Q \),

- \( c_1 \leq \|B^T A(q)D(q)B\| \leq c_u \)
- \( c_1 \leq \|B^T A(q)D(q)B\|^{-1} \leq c_u \)
- \( \|A(q)\| \leq c_u \)
- \( \|A(q)\| \leq c_u \)
- \( \|A(q)\| \leq c_u \)
- \( \|A(q)\| \leq c_u \)
- \( \|B^T A(q)C(q, \dot{q})\| \leq c_u |\dot{q}| \)
- \( \|B^T A(q)G(q)\| \leq c_u |q| \)
- \( \|B^T A(q)G(q)\| \leq c_u |\dot{q}| \)
where the following PD control law:

\[ u = J(q) \dot{q} + K_p e - K_d \ddot{q}, \]

and \( J(q) := \frac{\partial e}{\partial q} \) is the Jacobian w.r.t. \( q \). We use the following PD control law:

\[ u_{PD}(q, \dot{q}) = -J(q)^T K_p e(q) - J(q)^T K_d \dot{e}(q, \dot{q}), \]

where \( K_p, K_d \) are the gain matrices of dimension \( m \).

**Remark 2** If the desired configuration is only a function of \( q^a \), then \( J(q) = 1 \), which reduces (27) to the familiar form of PD control law.

For simplicity, we will assume that equal gains are applied for every joint i.e., \( K_p = k_p \mathbf{1}, K_d = k_d \mathbf{1} \) for some \( k_p, k_d > 1 \) and an identity matrix \( \mathbf{1} \) of appropriate size. Having defined this PD control law (27), we have the resulting closed loop dynamics of (6) as

\[ \dot{x} = f(x) := f(x) + g(x)u_{PD}(q, \dot{q}), \]

and with the initial state \( x_0 \in S \), we have the resulting flow (integral curve) of the continuous dynamics as \( x(t) = (q(t), \dot{q}(t)) \), where \( (q(0), \dot{q}(0)) = R(x_0) \) and \( t \in [0, T] \) with \( T > 0 \) (see (18)) being the time until the next impact.

### 3.2 Hybrid zero dynamics

For underactuated robotic systems, we typically choose the zero coordinates to be the following:

\[ z(q, \dot{q}) := \begin{bmatrix} z_1(q) \\ z_2(q, \dot{q}) \end{bmatrix} := \begin{bmatrix} \dot{q}^a \\ B^T q D(q) \dot{q} \end{bmatrix}. \]

It is important to note that the coordinates \( z \) shown above are chosen based on \( e \) (see Assumption 2 ahead). Accordingly, we have the output zero coordinates as \((e, \dot{e}, z)\), and the corresponding transformation as \( \Phi(q, \dot{q}) = (e, \dot{e}, z) \). With this transformation, we have the following passive dynamics:

\[ \dot{z} = \psi(\Phi(q, \dot{q})), \quad (q, \dot{q}) \in \mathbb{D} \setminus S \]

\[ z^+ = \Delta_z(\Phi(q, \dot{q})), \quad (q, \dot{q}) \in S, \]

where \( \psi \) is given by

\[ \psi(\Phi(q, \dot{q})) := \begin{bmatrix} \dot{q}^a \\ B^T (D(q) \dot{q} - C(q, \dot{q}) \dot{q} - G(q)) \end{bmatrix}. \]

Here the input \( u \) will not appear due to underactuation. If the outputs are zero i.e., \((e, \dot{e}) = (0, 0)\), and if the hybrid invariance conditions (15) are satisfied (with the notations \( y, \dot{y} \) replaced with \( e, \dot{e} \)), we have hybrid zero dynamics (HZD). The HZD obtained lies on the reduced dimensional surface:

\[ \mathbb{Z} = \{(q, \dot{q}) \in \mathbb{D} : e(q) = 0, \dot{e}(q) = 0\}. \]

Stability of HZD has been extensively studied, where the desired values \( q^a \), and the phase \( \tau(q) \) are chosen in such a way that the HZD has an exponentially stable periodic orbit \([7,10,37]\). Therefore, given that there is a constructive way to realize an exponentially stable periodic orbit in the HZD, we make the following assumption:

**Assumption 1** The hybrid zero dynamics given by

\[ \mathcal{Z} := \begin{cases} \dot{z} = \psi(0, 0, z), & (0, 0, z) \in \Phi(\mathbb{Z} \setminus (\mathbb{Z} \cap \mathbb{S})) \\ z^+ = \Delta_z(0, 0, z), & (0, 0, z) \in \Phi(\mathbb{Z} \cap \mathbb{S}) \end{cases} \]

has an exponentially stable periodic orbit, \( \mathcal{Z} \), transverse to \( \Phi(\mathbb{Z} \cap \mathbb{S}) \).

**Remark 3** Periodic orbits on the HZD are usually obtained through an offline optimization problem \([11]\). Since the model is not accurately known, Assumption 1 may seem restrictive. Relaxation of the above assumption with uncertainties incorporated have been studied in \([22]\), where the periodic orbit was shown to be exponentially ultimately bounded. In order to keep focus on the stability analysis for PD control laws, we will continue to use Assumption 1 for the rest of the paper.
Denote the integral curve of the continuous zero dynamics as \( \Phi^t_{HZD} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). Since the HZD has a periodic orbit, we have a fixed point \( z^* \), and the associated period \( T^* \) that satisfies: \( \Phi^t_{HZD}(\Delta_e(0,0,z^*)) = z^* \). We have this periodic orbit defined as

\[
\mathcal{O}_z = \{ \Phi^t_{HZD}(\Delta_e(0,0,z^*)) \in \mathbb{R}^2 : 0 \leq t < T^* \}. \tag{34}
\]

### 3.3 Exponential stability via Poincaré maps

Exponential stability of periodic orbits is characterized by using Poincaré maps [28,35]. Therefore, we define the following Poincaré map:

\[
\rho(z_s) := \Phi^t_{HZD}(\Delta_e(0,0,z_s)). \tag{35}
\]

Here \( (0,0,z_s) \in \Phi(\mathbb{Z}) \cap \mathbb{S} \) is the initial state, and \( T_\rho \) is the reduced time-to-impact function given by

\[
T_\rho(z_s) = \inf\{ t > 0 : h(\Phi^{-1}(t(\Phi^t_{HZD}(\Delta_e(0,0,z_s)))) = 0 \}, \tag{36}
\]

obtained similar to (18). Note that we have used the subscript \( s \) in \( \dot{z} \) to distinguish it from the evolution of \( z \) of the full order dynamics. Since \( \mathcal{O}_z \) is exponentially stable, by the converse Lyapunov theorem for discrete systems, there exists an exponentially convergent Lyapunov function \( V_c \), for some \( r > 0 \), and positive constants \( c_1, c_2, c_3, c_4 \) such that for all \( z_s \in \mathbb{R}_r(z^*) \cap (\mathbb{Z} \cap \mathbb{S}) \),

\[
c_1|z_s - z^*|^2 \leq V_c(z_s - z^*) \leq c_2|z_s - z^*|^2 \tag{37}
\]

\[
|V_c(\rho(z_s) - z^*) - V_c(z_s - z^*)| = c_3|z_s - z^*|, |V_c(z_s - z^*)| \leq c_4|z_s - z^*|, |z_s - z^*| + |z' \hat{s} - z^*|.
\]

In addition, we know that there exist constants \( c_5 > 0, \gamma \in (0,1) \) such that

\[
|\rho\phi(z_s) - z^*| \leq c_5 \gamma |z_s - z^*|. \tag{38}
\]

We will be mainly using this property along with the Poincaré map of the full order hybrid system (described in the next section) to establish our main results.

### 3.4 Hybrid dynamics

We can reconstruct \( \Theta \) from \( \Theta_z \) by the canonical embedding \( t(z) := (0,0,z) \), i.e., \( \Theta = t(\Theta_z) \). Since \( \Theta_z \) is not necessarily an orbit of the full order hybrid system, we will denote the point \( (0,0,z^*) \), simply as a nominal point of \( \Theta \). Given that \( \Theta_z \) is exponentially stable, we are interested in the stability properties of \( \Theta \) when a PD control law of the form (27) is applied. Accordingly, we have the following hybrid dynamics expressed in terms of the transformed coordinates \( (e, \dot{e}, z) \):

\[
\mathcal{H} = \begin{cases} 
\ddot{e} = f_e(e, \dot{e}, z) \\
\dot{z} = \psi(e, \dot{e}, z), \quad & \text{when} \ (e, \dot{e}, z) \in \Phi(\mathbb{Z}) \cap \mathbb{S} \\
(e^+, \dot{e}^+, z^+) = \Delta(e, \dot{e}, z), \quad & \text{when} \ (e, \dot{e}, z) \in \Phi(\mathbb{S}) 
\end{cases} \tag{39}
\]

where \( f_e, g_e \) are obtained from the Lie derivatives of \( e \). For the continuous dynamics in (39), we will use \( e(t), \dot{e}(t), z(t) \) in place of the flow \( \Phi \) for convenience. The value at \( t = 0 \), i.e., \((e(0), \dot{e}(0), z(0))\), are the post-impact states of the initial state \((e_0, \dot{e}_0, z_0) := \Phi(x_0)\). We will also denote the trajectory on the orbit \( \Theta \) by \((q^*(t), \dot{q}^*(t))\), and in the transformed coordinates by \((0,0,z^*(t))\). The following additional assumptions are used to establish the main result.

### 3.5 Additional assumptions on \( \Theta_z \) and the outputs \( e \)

We make the following assumptions on \( e \). In particular, these assumptions are for a local configuration space of \( q \), i.e., we will pick a tube of radius \( r \) around the orbit \( q^*(t) \):

\[
\mathbb{N}_r = \{ q \in \mathbb{Q} : \inf_{t \in [0,r]} \| q - q^*(t) \| \leq r \}. \tag{40}
\]

We have the following first assumption on the outputs \( e \) (26):

**Assumption 2** The transformation \((q^*, e(q)) : \mathbb{Q} \rightarrow \mathbb{R}^n\) is a local diffeomorphism onto its image for every \( q \in \mathbb{N}_r \).

By Assumption 2, the Jacobian w.r.t. \( q \) is nonsingular:

\[
J_e(q) := \begin{bmatrix} 1_{t \times t} & 0_{t \times m} \end{bmatrix} \frac{\partial e(q)}{\partial q^r} f(q). \tag{41}
\]

As a result, the continuous dynamics (1) can be reformulated and represented in terms of \( q^*, e \) as

\[
D_e(q) \begin{bmatrix} \frac{\partial q^*}{\partial q} \\
\frac{\partial e}{\partial q} \end{bmatrix} + C_e(q, \dot{q}) \begin{bmatrix} \frac{\partial q^*}{\partial q} \\
\frac{\partial e}{\partial q} \end{bmatrix} + G_e(q) = J_e^{-T}(q)Bu, \tag{42}
\]

where \( D_e = J_e^{-T}DJ_e^{-1} \), \( C_e = J_e^{-T}CJ_e^{-1} + J_e^{-T}D \frac{\partial l(q)}{\partial q} \), \( G_e = J_e^{-T}G \). As a result of this change of variables, the following
functions will appear frequently throughout the paper:

\[ A_c(q) := I_{n \times n} - D_{e}(q) B_{c}(B_{e}^T D_{e}(q) B_{e})^{-1} B_{e}^T \]

\[ B_{e}(q) := I_{m \times m} + B_{e}^T D_{e}(q) B_{e}(B_{e}^T D_{e}(q) B_{e})^{-1} \frac{\partial e(q)}{\partial q} \tag{43} \]

\[ B_{e}'(q) := 1_{l \times l} - (B_{e}^T D(q) B_{e})^{-1} B_{e}^T D(q) B_{e}^{-1}(q) \frac{\partial e(q)}{\partial q}. \]

Similar to (25), we can eliminate \( \bar{q}_u \) from (42) to obtain

\[ B_{e}^T A_{e} B_{e} \bar{e} + B_{e}^T A_{e} C_{e} \begin{bmatrix} \bar{q}_u \\ \bar{e} \end{bmatrix} + B_{e}^T A_{e} C_{e} = B_{e} J^{-T} u, \tag{44} \]

where \( A_{e}, B_{e} \) are obtained from (43). Having defined \( A_{e}, B_{e} \), we now impose the following assumption:

**Assumption 3** The outputs \( e \) are chosen such that the norm \( \| \frac{\partial e(q)}{\partial q} \| \) is sufficiently small. In particular, this norm is chosen such that for all \( (q, q) \in \bigcup_{q \in \mathbb{N}} T_q Q \).

1. \( B_{e}' \) is invertible, and
2. \( A_{B_{e}} := \begin{bmatrix} B_{e} + B_{e}' & B_{e} + (1 + |e|)(B_{e}' - 1) \\ B_{e}' + (1 + |e|)(B_{e}' - 1) & (1 + |e|)(B_{e} + B_{e}') \end{bmatrix} \) is symmetric positive definite.

where \( B_{e}', B_{e} \) are dependent on \( q \) (given by (43)).

Assumption 3 only requires that \( e \) has small variations w.r.t. \( q \). This will be verified with the example biped model in Section 6. We choose \( e \) such that the entries of \( B_{e} - I_{m \times m}, B_{e}' - I_{l \times l} \) are small in magnitude. By Gershgorin’s circle theorem [4, Theorem 2.1], the diagonal entries of \( B_{e}', B_{e} \) are positive and dominate the off-diagonal entries, thereby rendering them invertible. Furthermore, since \( A_{B_{e}} \) is symmetric, the eigenvalues are real and positive, thereby ensuring positive definiteness. Note that this assumption is sufficient but not necessary, and its relaxed formulations are possible, which will be a subject of future work.

Since the norm is a continuous function, \( \| J_e(q) \| \) has nonzero upper and lower bounds on a compact set \( \mathbb{N}_r \). We can make use of this property to establish the following:

**Proposition 3** \( D_{e} \) is symmetric positive definite, and \( \tilde{D}_{e} = 2C_{e} \) is skew-symmetric. In addition, there exist \( c_l, c_u > 0 \) (possibly smaller, larger than previously determined \( c_l, c_u \)) such that for any \( (q, \tilde{q}) \in \bigcup_{q \in \mathbb{N}} T_q Q \).

1. \( c_l \leq \| D_{e}(q) \| \leq c_u \)
2. \( c_l \leq \| D_{e}^{-1}(q) \| \leq c_u \)
3. \( \| D_{e}(q) \| \leq c_u (|q^p| + |\tilde{e}|) \)
4. \( \| C_{e}(q, \tilde{q}) \| \leq c_u (|q^p| + |\tilde{e}|) \)
5. \( \| G_{e}(q) \| \leq c_u \)

Note that we have not restricted the tangent space \( T_q Q \) to a compact set. This will be useful for allowing larger velocity variations (see Remark 5 ahead). Proof of Proposition 3 is provided in Appendix B. The interested reader may also see [30, Chapter 4, Section 5.4] for more details.

Similar to Proposition 3, we have the following properties for \( B_{e} A_{e} D_{e} B_{e} \):

**Proposition 4** \( B_{e} A_{e} D_{e} B_{e} \) is symmetric positive definite. In addition, there exist positive constants \( c_l, c_u > 0 \) such that for all \( (q, q) \in \bigcup_{q \in \mathbb{N}} T_q Q \),

- \( c_l \leq \| B_{e} A_{e}(q) D_{e}(q) B_{e} \| \leq c_u \)
- \( c_l \leq \| (B_{e} A_{e}(q) D_{e}(q) B_{e})^{-1} \| \leq c_u \)
- \( \| B_{e} A_{e}(q) D_{e}(q) B_{e} + B_{e} A_{e}(q) D_{e}(q) B_{e} \| \leq c_u (|q^p| + |\tilde{e}|) \)
- \( \| B_{e} A_{e}(q) C_{e}(q) \| \leq c_u (|q^p| + |\tilde{e}|) \)
- \( \| B_{e} A_{e}(q) G_{e}(q) \| \leq c_u \)

Proof is provided in Appendix C. With these propositions, we are now ready to present the main results of the paper.

### 4 Main results

The main results will be in a series of Lemmas and Theorems. Since we are interested in a local result, we start from a small neighborhood of \( (0, 0, z^0) \) i.e., \((e_0, \dot{e}_0, z_0) \in \mathbb{B}_r(0, 0, z^0) \cap \Phi(S)\). The neighborhood radius \( r \) may, perhaps, be smaller than previously determined. Later on, \( r \) may be reduced further depending upon the gains \( k_p, k_d \). With this initial condition, we have the following time-to-impact (dwell-time) function:

\[
T(e_0, \dot{e}_0, z_0) := \begin{cases} \inf \{t > 0 : h(\Phi^{-1}(e(t), \dot{e}(t), z(t))) = 0\} & \text{if } \exists t \text{ s.t. } (e(t), \dot{e}(t), z(t)) \in \Phi(S) \\ \infty & \text{otherwise} \end{cases}
\tag{45} \]

where \((e(0), \dot{e}(0), z(0)) = \Delta(e_0, \dot{e}_0, z_0)\). Since \( R(S) \cap S = \emptyset \), we know that there is a time \( T_{\min} > 0 \) such that \( T \geq T_{\min} \) (later we will also show that upper bound for \( T \) also exists). We have the following first result. We show here that the outputs are LEUB (Definition 2).

**Lemma 1** Let the system (39) be given, and let the desired configuration \( q_0^d \) be chosen such that Assumptions 2-3 are satisfied. Then there exist sufficiently large enough gains \( k_p, k_d > 1 \), and a corresponding small enough \( r > 0 \), \( T_0 > 0 \), such that for all \((e_0, \dot{e}_0, z_0) \in \mathbb{B}_r(0, 0, z^0) \cap \Phi(S)\), the outputs \((e(t), \dot{e}(t))\) are LEUB in the interval \( t \in [0, T_0] \).

Note that [16, 3.2] establishes boundedness by Gronwall-Bellman result for any finite interval (any arbitrary \( T_0 \)) by choosing an initial point \((e_0, \dot{e}_0, z_0)\) very close to the nominal point \((0, 0, z^0)\) (i.e., by reducing \( r \)). We cannot use this...
approach due to the fact that $O$ is not necessarily an orbit for (39). On the other hand, the following Lemma will, in fact, allow us to simply pick larger $k_p,k_d$ for stretching $T_S$ to $T$, thereby establishing ultimate boundedness of the outputs for the entire step.

**Lemma 2** Let the system (39) be given, and let the desired configuration $q_d^*$ be chosen such that Assumptions 2-3 are satisfied. Given that $O$ is an orbit of the HZD, and $k_p,k_d,r,T_S > 0$ are chosen such that the outputs $(e(t),e(0))$ are LEUB in the interval $[0,T_S]$, we then have the following:

1. If $(e_0,e_0,0,0) = (0,0,z^*)$, then there exist $C_k,C_t > 0$ with $C_k$ dependent on $k_p,k_d$ such that
   \[ |z(t) - z^*(t)| \leq C_k(k_p,k_d)e^{C_tT_S}, \quad t \in [0,T_S], \]
   where $C_k$ decreases with increasing $k_p,k_d$.

2. For every $\delta > 0$, there exist $k_p,k_d > 1$, greater than previously determined, such that
   \[ |T(0,0,z^*) - T^*| < \delta. \]

3. There exist $k_p,k_d > 1$, greater than previously determined, and $r > 0$, such that for all $(e_0,e_0,0,0) \in \mathbb{R}(0,0,0)^* \cap \Phi(S)$,
   \[ T_{\min} \leq T(e_0,e_0,0,0) \leq T_{\max}, \]
   for some pre-defined constants $T_{\max} > T^* > T_{\min} > 0$.

Lemmas 1 and 2 together show that $(e(t),e(t),z(t))$ can be close to $O$ for the entire step. The ensuing lemma and the two theorems will extend this result for the entire hybrid dynamics.

For simplicity of notation, we will denote the output coordinates as $\eta := (e,e)$, and shift $z^*$ to zero. We will also drop the subscript $0$ from the initial states $\eta_0,0,0$ \footnote{Since the ensuing Lemma and Theorems are established via Poincaré maps, there is no confusion.}. Since we are analyzing periodic orbits, we will be using Poincaré maps defined as

\[ P(\eta,z) := \Phi_{T(\eta,z)}(\Delta(\eta,z)), \]  

where $T$ is the time-to-impact function (45) (which is well defined based on (48)). $P$ has two components $P_{\eta},P_z$ corresponding to the coordinates $\eta,z$ respectively. We can also establish some of the properties of impact maps. We know that $\Delta$ can be separated into two components $(\Delta_\eta,\Delta_z) = \Delta$ corresponding to $\eta$ and $z$. By local Lipschitz continuity and hybrid invariance conditions, we have the following:

\[ |z(0) - z(0)| = |\Delta_\eta(\eta,z) - \Delta_z(0,z)| \leq L_\Delta |\eta| \]
\[ |z(0) - z^*(0)| = |\Delta_\eta(\eta,z) - \Delta_z(0,0)| \leq L_\Delta |\eta| + |z|. \]

where $L_\Delta$ is the Lipschitz constant. Since $\Delta_\eta(0,z) = 0$,

\[ |(e(0),e(0))| = |\Delta_\eta(\eta,z) - \Delta_\eta(0,z)| \leq L_\Delta |\eta|. \]

There is an elegant relationship between the time-to-impact functions $T,T_p$ and the Poincaré maps $P,P_\rho$ that can be utilized to prove the main theorem. This is given in the form of the following lemma.

**Lemma 3** Let the system (39) be given, and let the desired configuration $q_d^*$ be chosen such that Assumptions 2-3 are satisfied. Given that $O$ is an LES periodic orbit of the HZD transverse to $\Phi(\mathbb{R}(0,0)^* \cap S)$, then there exist large gains $k_p,k_d$, small enough $r > 0$, and corresponding $C_T,C_p,d_T,d_p > 0$ such that for all $(\eta,z) \in \mathbb{R}(0,0)^* \cap \Phi(S)$,

\[ |P(\eta,z) - T_p(z)| \leq C_T |\eta| + dr \]
\[ |P_\rho(\eta,z) - \rho(z)| \leq C_p |\eta| + dp. \]

In addition, the gains $k_p,k_d$ can be further increased and $r$ can be further decreased in such a way that $C_T,C_p$ remain constant and $d_T,d_p$ decrease (with increasing $k_p,k_d$).

Lemmas 1-3 yield $r$ and the PD gains $k_p,k_d$, which yield the inequalities (52), (53). We will now state the following well known result for periodic orbits [35]:

**Theorem 1** Given the set of points $O$ obtained from the embedding $t$ of the periodic orbit $O$ (34) of the hybrid zero dynamics $Z$ (33), the following are equivalent:

1. $O = \tau(O)$ of the hybrid system $\mathcal{H}$ (39) is LEUB.
2. $(0,0) = \Phi(x^*)$ of the map $P$ (49) is LEUB.

Theorem 1 will be used to establish the main theorem of the paper that follows. We will not be proving this theorem, since it is directly obtained as a consequence of [35, Theorem 1] (by simply replacing the class $\mathcal{H}$ function of the disturbance with a constant $d$). The proof of [35, Theorem 1] is for a stronger property—input-to-state stability (ISS)—and similar results can be derived for boundedness. We can establish that the ultimate bound, given by $d$ in (20), can be decreased to an arbitrarily small value by choosing appropriate $k_p,k_d$. As $d \to 0$, the resulting orbit of the closed loop system coincides with $O$. Therefore, Theorem 1 will be used to establish boundedness of periodic orbits via Poincaré maps.

**Main theorem.** We will now present the main theorem.

**Theorem 2** Let the system (39) be given, and let the desired configuration $q_d^*$ be chosen such that Assumptions 2-3 are satisfied. If $O$ is an LES periodic orbit of the hybrid zero dynamics $Z$ (33) transverse to $\Phi(\mathbb{R}(0,0)^* \cap S)$, then for sufficiently large enough gains $k_p,k_d > 1$, $O$ is an LEUB periodic orbit of the full order hybrid dynamics $\mathcal{H}$ (39).

We will prove Lemmas 1-3, and also Theorem 2 in the next section.
5 Proofs of Lemmas 1-3 and Theorem 2

PROOF. [of Lemma 1] We consider the following Lyapunov candidate:

\[ V_{c}(e, \dot{e}, q) = V_{0}(e, \dot{e}, q) + V_{c}(e, \dot{e}, q) \]  \hspace{1cm} (54)

\[ V_{0}(e, \dot{e}, q) = \frac{1}{2} \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} K_{p} & 0 \\ 0 & B^{T}A_{c}(q)D_{e}(q)B \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \]  \hspace{1cm} (55)

\[ V_{c}(e, \dot{e}, q) = \alpha(e)e^{T}B^{T}A_{c}(q)D_{e}(q)B\dot{e} \]  \hspace{1cm} (56)

\[ \alpha(e) = \frac{k_{0}}{1 + |e|} = \frac{k_{0}}{1 + \sqrt{e^{T}e}}. \]  \hspace{1cm} (57)

It can be verified that \( V_{c} \) is positive definite. The addition of the cross terms \( V_{c} \) does not affect the positive definiteness as long as \( k_{0} \) is sufficiently small. For example, we can pick \( k_{0} \) that satisfies

\[ k_{0} \leq \sqrt{\frac{\|K_{p}\|\|B^{T}A_{c}D_{e}B\|}{\|B^{T}A_{c}D_{e}B\|}}. \]  \hspace{1cm} (58)

Therefore, we can choose

\[ k_{0} = \frac{\sqrt{k_{p}}}{N}, \text{ where } N > \|B^{T}A_{c}D_{e}B\|. \]  \hspace{1cm} (59)

We will be picking a larger value for \( N \) later on. Choosing this value of \( k_{0} \) also helps in separating \( k_{p} \) from the positive definite matrix in (55) to obtain the following bounds on \( V_{c} \):

\[ \lambda_{1} \left[ \begin{bmatrix} \sqrt{k_{p}}e \\ \dot{e} \end{bmatrix} \right]^{2} \leq V_{c} \leq \lambda_{0} \left[ \begin{bmatrix} \sqrt{k_{p}}e \\ \dot{e} \end{bmatrix} \right]^{2}, \]  \hspace{1cm} (60)

for some positive constants \( \lambda_{1}, \lambda_{0} \) that do not depend on \( k_{p} \). This type of inequality is useful for realizing desirable convergence rates for \( V_{c} \) (see [1, (10)-(22)] for a similar formulation).

\( V_{c} \) of the initial states. It is important to determine the sublevel sets of \( V_{c} \) that contains all the possible initial values. Since \( \mathcal{O}_{c} \) is locally stable in \( \mathcal{H}_{2} \), we can pick the same neighborhood radius \( r \) (or maybe even smaller) for the initial states \((e_{0}, \dot{e}_{0}, z_{0})\). With this initial state we know from (50) and (51) that

\[ |(e(0), \dot{e}(0))| \leq L_{\Delta}(e_{0}, \dot{e}_{0}) \leq L_{\Delta}r. \]  \hspace{1cm} (61)

Therefore, given the initial state, the maximum possible value of \( V_{c} \) is \( \lambda_{0}k_{p}L_{\Delta}r^{2} \). This will be useful for determining the gains of the controller later on. We will now solve for the dynamics of \( V_{c} \).

Derivative of \( V_{0} \). By solving for the derivative of \( V_{0} \), we get the following:

\[ V_{0} = e^{T}K_{p}\dot{e} + \frac{1}{2} \dot{e}^{T}B^{T}(\dot{A}_{c}D_{e} + A_{c}D_{e})\dot{B}\dot{e} + e^{T}B^{T}A_{c}D_{e}\dot{B}\dot{e}, \]

and after substituting (44) and (27) and using Propositions 3, 4, we have the final inequality as

\[ V_{0} = \frac{1}{2} \dot{e}^{T}B^{T}(\dot{A}_{c}D_{e} + A_{c}D_{e})\dot{B}\dot{e} - \dot{e}^{T}B^{T}A_{c}(C_{e} \begin{bmatrix} \tilde{z}_{1} \\ \dot{e} \end{bmatrix} + G_{e}) \]

\[ - k_{p}\dot{e}^{T}(B_{c} - 1)e - k_{d}\dot{e}^{T}B_{c}\dot{e}. \]  \hspace{1cm} (62)

Derivative of \( V_{c} \). By solving for the derivative of \( V_{c} \), we have

\[ \dot{V}_{c} = \alpha(e)e^{T}B^{T}A_{c}D_{e}\dot{B}\dot{e} + \alpha e^{T}B^{T}A_{c}D_{e}\dot{B}\dot{e} \]

\[ + \alpha e^{T}B^{T}(\dot{A}_{c}D_{e} + A_{c}D_{e})\dot{B}\dot{e} + \alpha \dot{e}^{T}B^{T}A_{c}D_{e}\dot{B}\dot{e} \]

\[ \preceq 2\alpha u|e|^{2} + \alpha \left( \begin{bmatrix} \tilde{z}_{1} \\ \dot{e} \end{bmatrix} |e| \right) |e| + \alpha \left( \frac{\tilde{z}_{1}}{\dot{e}} \right) |e| + \alpha u|e| \]

\[ - k_{p}\alpha \dot{e}^{T}B_{e}e - k_{d}\alpha \dot{e}^{T}B_{e}e, \]  \hspace{1cm} (63)

where we have used Propositions 3, 4, and some of the properties of \( \alpha(e) \):

\[ |\alpha(e)| \leq k_{0}, \quad |\alpha(e)| |e| \leq k_{0}, \]

\[ |\alpha(e)| |e| \leq k_{0}|e|, \quad |\alpha(e)| |e| \leq k_{0}|e|. \]

In addition, we note from (29), Property 2 and Assumptions 2, 3 that \( B_{e}^{T}\tilde{z}_{1} = D_{e}^{-1} (z_{2} - D_{12}J^{-1}\dot{e}) \), which yields

\[ \begin{bmatrix} \tilde{z}_{1} \\ \dot{e} \end{bmatrix} \]  \hspace{1cm} (64)

for some \( c_{q} > 0 \). This can be substituted for \( \tilde{z}_{1}, \dot{e} \) in the above equations. We obtain the total derivative as

\[ \dot{V}_{c} \preceq - \frac{\alpha}{2} \left[ \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^{T} \begin{bmatrix} k_{p}\dot{B}_{e} + B_{e}^{T} \\ k_{d}(B_{e} - 1) + \alpha k_{d}B_{e} \end{bmatrix} \right] \left[ \begin{bmatrix} k_{p}(B_{e} - 1) + \alpha k_{d}B_{e} \\ k_{d}(B_{e} - 1) \end{bmatrix} \right] \left[ \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \right] \]

\[ + \alpha c_{u} |z_{2}| |e| + c_{u}\left( \alpha |e| + |\dot{e}| \right) (|z_{2}|^{2} + 1) \]

\[ + 2c_{u}\left( \alpha + \alpha |e| + |\dot{e}| + |z_{2}| \right) |e|^{2}, \]  \hspace{1cm} (65)

where \( c_{q} \) is redefined to absorb \( c_{q} \). The control gains \( k_{p}, k_{d} \) must be picked in such a way that \( V_{c} \) is decreasing. For \( |z_{2}|, \)
we will pick the maximum of \( z \) in a compact neighborhood (tube) of \( \mathcal{C} \). Assume that this maximum is \( b > 0 \) (say). Similarly, the maximum possible values for \( |\dot{e}|, |e| \) in the sublevel set of \( V_e \) are given by

\[
|\dot{e}| \leq \sqrt{\frac{\lambda_0}{\lambda_4}} L_A r =: |\dot{e}|_{\text{max}}, \quad |e| \leq \sqrt{\frac{\lambda_0}{\lambda_4}} L_A r =: |e|_{\text{max}}, \quad (66)
\]

which are obtained from the maximum value of \( V_e \) for all possible initial states. Since \( |\dot{e}|_{\text{max}} \) depends on the proportional gain \( k_p \), the derivative gain \( k_d \) must be at least as high as \( \sqrt{k_p} \). Therefore, choose \( k_p = \varepsilon^2, k_d = \varepsilon k, k_0 = \varepsilon / k \) for some \( \varepsilon, k > 1 \). We require that \( \Lambda > 0 \) i.e.,

\[
\begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \Lambda \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} e e \\ \dot{e} \end{bmatrix}^T \Lambda_B \begin{bmatrix} e e \\ \dot{e} \end{bmatrix} > 0, \quad (67)
\]

for all \((q,q) \in \cup_{q \in N_t} T_q^Q\). This is true by Assumption 3. Therefore, we have the following:

\[
\dot{V}_e \leq -\frac{\alpha}{2} \lambda_{B_e} \left[ \frac{|e e|}{k e} \right]^2 + k_1 |e| |\dot{e}| + k_2 |\dot{e}|^2 \quad (68)
\]

where \( \lambda_{B_e} := \lambda_{\text{min}}(\Lambda_{B_e}) \) is the minimum eigenvalue in the compact set \( N_t \), and

\[
\begin{align*}
&k_1(|z_2|) = c_u |z_2| \\
&k_2(|e|, |\dot{e}|, |z_2|) = 2c_u \left( 1 + |e| + \frac{(|\dot{e}| + |z_2|)}{\varepsilon k^{-1}(1 + |e|)^{-1}} \right) \quad (69) \\
&k_3(|z_2|) = c_u (|z_2|^2 + 1).
\end{align*}
\]

One half of the first summmand in (68) can be used to cancel the next two summands in the following manner:

\[
-\frac{\lambda_{B_e}}{4} |(e e, k e)|^2 + k_1 |e| |\dot{e}| + k_2 |\dot{e}|^2 = \quad (70)
\]

where the summmands are collected together in the form of a matrix. We can replace \( e, \dot{e}, z_2 \) in \( k_1, k_2 \) with their maximum possible values, and by choosing a large enough \( k \), we can ensure that (70) is negative.

Note that (70) is satisfied even if \( \varepsilon \) is increased \( (|\dot{e}| \leq |\dot{e}|_{\text{max}} \) is canceled by \( \varepsilon \) in the denominator in (69)), and if the \( k \) chosen does not satisfy (58) i.e., if \( k \not\in N \), then we simply increase it further. We will be using \( \varepsilon \) as a tunable gain for obtaining desirable convergence rates. We have that

\[
\dot{V}_e \leq -\frac{\alpha}{4} \lambda_{B_e} \left[ |(e e, k e)|^2 + k_3 (|e| + |\dot{e}|) \right] \leq -\frac{\alpha}{4} \lambda_{B_e} \left[ |(e e, k e)|^2 + \frac{2 \alpha k^3}{\lambda_{B_e} k^2} + \frac{2 k^2 k^2}{\lambda_{B_e}^2} \right] \leq -2 \varepsilon \lambda_{B_e} + \frac{2 \alpha k^3}{\lambda_{B_e} k^2} + \frac{2 k^2 k^2}{\lambda_{B_e}^2} \quad (71)
\]

where \( \lambda := \lambda_{B_e}/(16(k + 1)|e|_{\text{max}} \lambda_{B_e}) \). The above inequality is satisfied as long as \( z \) remains in the tube. Let \( T_b \) be the time when \( z \) crosses this limit. Therefore, by using comparison lemma [16, Lemma 3.4] in (71), we have that

\[
V_e(t) \leq e^{-2\varepsilon t} V_e(0) + \frac{k_4}{\varepsilon^2}, \quad t \in [0, T_b], \quad (72)
\]

where \( k_4 \) is obtained by collecting all the additional terms that are independent of \( \varepsilon \). We can express the above inequality in terms of the outputs as

\[
\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} \leq e^{-2\varepsilon t} \sqrt{\frac{\lambda_{B_e}}{\lambda_{B_e}}} \begin{bmatrix} e(0) \\ \dot{e}(0) \end{bmatrix} + \frac{k_4}{\varepsilon^2} \frac{1}{\lambda_{B_e}}, \quad (73)
\]

which completes the proof.

**Remark 4** We showed that increasing \( \varepsilon \) increases the convergence rate of the outputs. It is important to note that this also results in changing shape of the Lyapunov level sets of \( V_e \). The peak velocity \( |\dot{e}|_{\text{max}} \) increases with increasing \( \varepsilon \). On the other hand, increasing \( \varepsilon \) does not increase \( |e|_{\text{max}} \). Therefore, Assumptions 2 and 3 are still valid. We will now prove Lemma 2.

**PROOF.** [of Lemma 2] Firstly, we note that by the differentiability of \( \Phi \) (diffeomorphism):

\[
|q(t) - q^*(t)| \leq L_q(|z_1(t) - z_1^*(t)| + |e(t)|), \quad (74)
\]

where \( L_q \) is the local Lipschitz constant (for the tube of radius \( r \) around the orbit). By using the definition of the zero coordinates, we have that

\[
\begin{align*}
\dot{z}_1(t) - \dot{z}_1^*(t) &= B'_e(q(t))^{-1} D_{11}(q(t))^{-1} [z_2(t) \\
&- D_{12}(q(t)) J(q(t))^{-1} \dot{e}(t)] \\
&- B'_e(q^*(t))^{-1} D_{11}(q^*(t))^{-1} \dot{z}_2^*(t).
\end{align*}
\]

Here \( B'_e \) is given by (43), and \( q^*(t), z_1^*(t) \) correspond to the trajectory on \( \mathcal{C} \). By collecting the common terms, we obtain
the following inequality:
\[ |\dot{z}_1(t) - z'_1(t)| \leq c_z(|z_2(t) - z'_2(t)| + |q(t) - q'(t)| + |\dot{e}(t)|), \]
(76)
for some \( c_z > 0 \). We can have a similar inequality for the derivative of \( z_2(t) - z'_2(t) \):
\[ |\dot{z}_2(t) - z'_2(t)| \leq c_z(|z_1(t) - z'_1(t)| + |\dot{z}_1(t) - \dot{z}'_1(t)|^2 + |\dot{e}(t)| + |\dot{e}'(t)|^2 + |q(t) - q'(t)|), \]
(77)
for, perhaps, a larger value for \( c_z > 0 \). By using (74), (76) and (77), we establish that
\[ |\dot{z}(t) - \dot{z}'(t)| \leq c_z \left( |\dot{z}(t) - \dot{z}'(t)| \right), \]
(78)
for some \( c_z > 0 \) (possibly larger than the previously determined \( c_z \)). We have used \( \eta(t) := (e(t), \dot{e}(t)) \) for convenience. We substitute (78) in the following inequality:
\[ |z(t) - z'(t)| \leq |z(0) - z'(0)| + \int_0^t |z(s) - z'(s)| ds \]
\[ \leq |z(0) - z'(0)| + c_z \int_0^t (|\eta(s)| + |\eta(s)|^2) ds \]
\[ + c_z \int_0^t (|z(s) - z'(s)| + |z(s) - z'(s)|^2) ds. \]
For a tube of radius \( r \) around \( \mathcal{O}_z \), we have that \( |z(t)|, |z'(t)| \leq b \). Therefore, by substituting \( b \) and the expression for the outputs \( \eta(t) \), we have that
\[ |z(t) - z'(t)| \leq |z(0) - z'(0)| + \frac{c_z \epsilon \lambda_u}{2 \lambda_1} |\eta(0)|^2 + \frac{c_z k_4 T_\delta}{\epsilon^2 \lambda_1} \]
\[ + \frac{c_z \lambda_1}{\epsilon^2 \lambda_1} |\eta(0)| + c_z \sqrt{ \frac{k_4}{\epsilon^2 \lambda_1} T_\delta } \]
\[ + c_z (1 + 2b) \int_0^t |z(s) - z'(s)| dt. \]
(79)
By using Gronwall-Bellman inequality [16, Lemma A.1], we obtain
\[ |z(t) - z'(t)| \leq c_z \left( |z_0| + \epsilon |\eta_0|^2 + |\eta_0| + \frac{T_\delta}{\epsilon^2} + \frac{T_\delta}{\epsilon} \right) e^{c_z (1 + 2b) t}, \]
(80)
where the pre-impact state \( (\eta_0, z_0) \) is substituted and \( c_z \) is appropriately redefined. We establish the first part of Lemma 2 by substituting \( z_0 = z'^0 = 0, \eta_0 = 0 \) and appropriately choosing \( C_k, C_i \).

To establish the second part of Lemma 2, we first note that the constant terms in (80) decrease with increasing \( \epsilon \). Therefore, \( z(t) \) gets closer to \( \mathcal{O} \) with decreasing \( r \) and increasing \( \epsilon \), implying that the states are bounded for \([0, T_\delta]\). Since \( T_\delta \) is the time when \( |z_2(t)| \) crosses its bound \( b \), i.e., leaves the compact neighborhood around \( \mathcal{O}_z \), we can use the steps followed in [16, Theorem 3.5]. \( T_\delta \) is stretched to an arbitrary constant \( T_\delta > T^* \) (say), and by choosing \( \epsilon \) (say \( \epsilon_1 \)), and \( r \) (say \( r_1 \) that is smaller than previously chosen), we ensure that Lemma 1 is valid for the entire \([0, T_{\max}]\). With this result, the next steps are similar to [35, Proof of Lemma 2], where we use implicit function theorem (in function spaces) to establish bounds for the time-to-impact function (45). We know that on the nominal orbit, we obtain the period \( T^* \) as
\[ T^* = \inf \{ t > 0 : h(\Phi^{-1}(0, 0, z^*(t))) = 0 \}, \]
(81)
which is well defined. By transversality condition (see [35, A.7]), any small perturbation of the trajectories from the nominal orbit in the guard function \( h(\Phi^{-1}(\mu_1(t), o, z^*(t) + \mu_2(t))) \) will still yield a well defined time-to-impact function, as long as the perturbations \( \mu_1(t), \mu_2(t) \) are small. We substitute (78) in the following inequality:
\[ \|z - z^*\| := \sup_{r \in [0, T_{\max}]} \|z - z^*(r)\| \]
\[ \|z - z^*\| \leq \sup_{r \in [0, T_{\max}]} \|z(t) - z^*(t)\|. \]
(82)
Both \( \|z\|_{\infty} \) and \( \|z - z^*\| \) can be evaluated from (73) and (80) respectively. Since the initial state is \((0, 0, z^*)\), we know that by increasing \( \epsilon \), \( \|z\|_{\infty} \) and \( \|z - z^*\| \) can be decreased. Given \( \delta > 0 \), if \( \delta > T_{\max} - T^* \), we will first pick a smaller \( \delta_1 > 0 \) that ensures that \( T^* + \delta_1 < T_{\max} \). Therefore, given \( \delta_1 > 0 \), we can obtain a larger \( \epsilon > \epsilon_1 \) (say \( \epsilon_2 \)) such that (47) is satisfied.

We will now establish the third part of Lemma 2. Similar to \( T_{\max} \), we can manually set some \( T_{\min} > 0 \) such that \( T_{\min} < T^* < T_{\max} \). We choose \( \delta < \min\{T_{\max} - T^*, T^* - T_{\min}\} \), and the corresponding time-to-impact satisfies \( T_{\min} < T(0, 0, z^*) < T_{\max} \), for \( \epsilon \geq \epsilon_2 \). Note that \( \epsilon \) can be increased even further to decrease the bound on \( \epsilon \). Accordingly, we can choose a small enough neighborhood (smaller than \( r_1 \)) around \((0, 0, z^*)\) in such a way that (48) is satisfied, while still satisfying the conditions of implicit function theorem and boundedness of \( z(t) \). This completes the proof.

**Remark 5** For the initial state \((0, 0, z^*)\), we have that \( \epsilon \rightarrow \infty \) if the resulting evolution of \( z(t) \) is identically equal to \( z^*(t) \). For the non-zero initial state \((e_0, \dot{e}_0, z_0)\), it can be verified that the norm \( \|z - z^*\|_{\infty} \) does not increase with increasing \( \epsilon \) if the neighborhood radius \( r \) is chosen appropriately. For example, we can choose \( r \) such that it is inversely proportional to \( \sqrt{\epsilon} \) (see (80)). In addition, the peak value of

---

4. \( b \) is not dependent on velocity \( \dot{e} \) and hence the second argument being zero still yields the time-to-impact function.

5. In [1, Lemma 1], \( T_{\min}, T_{\max} \) were chosen to be 0.9\( T^* \), 1.1\( T^* \) respectively.
We can obtain similar inequalities for the zero coordinates. We will study the progression of \( \eta(t) \) till the end of the step \( t = T \). By replacing \( T \) with its extreme values \( T_{\min}, T_{\max} \), we have the following from (73):

\[
|\eta(T)| \leq \beta(\varepsilon)|\eta| + d_\eta(\varepsilon),
\]

where the subscript 0 was omitted for convenience, and

\[
\beta(\varepsilon) = \sqrt{\frac{\lambda_0}{\lambda_1}} e^{-\varepsilon T_{\min}}, \quad d_\eta(\varepsilon) = \sqrt{\frac{k_1}{e^\varepsilon \lambda_1}}.
\]

Here the dependency of \( \beta, d_\eta \) on \( \varepsilon \) is explicitly shown. Note that \( \beta \) has an upper bound (denote it by \( \tilde{\beta} \)) that is independent of \( \varepsilon \).

We can obtain similar inequalities for the zero coordinates. Firstly, we know that \( z(t) \) is the trajectory of the zero coordinate on the full order hybrid dynamics (39) with the initial condition \( (\eta, z) \) on the guard, and \( z_s(t) \) is the trajectory on the HZD with the initial condition \( z \) on the guard. For a fixed \( \varepsilon \geq \varepsilon_2 \), we know that we can pick an \( r \) (see Remark 5) in such a way that

\[
|z(T) - z_s(T)| \leq C \varepsilon |\eta| + d_\varepsilon(\varepsilon),
\]

where \( C, d_\varepsilon \) are both obtained by collecting all the terms together. This is similar to the inequality (80). The rest of the steps are very similar to [1, Proof of Lemma 1]. We define an auxiliary time-to-impact function:

\[
T_B(\mu_\eta, \mu_z, z) = \inf_{t \geq 0} \{ t : h(\Phi^{-1}(\mu_\eta, z_s(t) + \mu_z)) = 0 \},
\]

where \( \mu_\eta \in \mathbb{R}^{2m}, \mu_z \in \mathbb{R}^{2l} \), and satisfies

\[
|T_B(\mu_\eta, \mu_z, z) - T_P(z)| \leq L_B(|\mu_\eta| + |\mu_z|),
\]

where \( L_B \) is the Lipschitz constant. In addition, \( \mu_\eta, \mu_z \) are further defined to be

\[
\mu_\eta = \eta(t)|_{t = T(\eta, z)}, \quad \mu_z = z(t) - z_s(t)|_{t = T(\eta, z)},
\]

which yields \( T_B(\mu_\eta, \mu_z, z) = T(\eta, z) \). With this result, and by substituting (83), (85), and also the bound on \( \beta(\varepsilon) \) in (87), we get the first result (52).

The second result (53) will follow exactly the steps in [1, Proof of Lemma 1] with the substitution of (52). Define

\[
C_\psi = \max_{T_{\min} \leq T \leq T_{\max}} |\psi(0, z_s(t))|.
\]

We therefore have

\[
|\mathbb{P}_z(\eta, z) - \rho(z)| \leq |z(0) - z_s(0)| + \int_0^{T(\eta, z)} |\psi(\eta(t), z(t)) - \psi(0, z_s(t))| dt + \int_0^{T(\eta, z)} |\psi(0, z_s(t))| dt.
\]

The first two terms on the RHS are substituted with (85), and the last term can be replaced with (89) and (52). This completes the proof.

We are now ready to prove the main theorem.

**Proof.** [of Theorem 2] Since we know that exponential stability of \( \dot{\sigma} \) yields a discrete time Lyapunov function \( V_c(z) \) (37), we need to construct a suitable Lyapunov function for the full order system.

**Lyapunov candidate for \( \mathbb{P} \):** Consider the following Lyapunov candidate for the Poincaré map:

\[
V(\eta, z) = \sigma |\eta|^2 + V_c(z),
\]

where \( \sigma > 0 \) is a tunable constant, and \( (\eta, z) \) is the initial state (by a slight abuse of notations).

We will analyze the two terms in RHS of (91) separately. We note in (83) that \( \beta(\varepsilon) \) can be reduced by increasing \( \varepsilon \). Hence, we can obtain \( \varepsilon > \varepsilon_2 \) such that \( \beta(\varepsilon) < 1/2 \) (say) for all \( \varepsilon \geq \tilde{\varepsilon} \). We may have to increase \( \varepsilon \) further to decrease the ultimate bound later on. We have

\[
|\mathbb{P}_\eta(\eta, z)|^2 - |\eta|^2 \leq (1 - \beta(\varepsilon)^2) |\eta|^2 + d_\eta(\varepsilon)^2.
\]

By using (37), we have the following:

\[
\mathbb{V}_c(\mathbb{P}_z(\eta, z)) - V_c(z) = V_c(\mathbb{P}_z(\eta, z)) - V_c(\rho(z)) + V_c(\rho(z)) - V_c(z) \\
\leq c_4 |\mathbb{P}_z(\eta, z) - \rho(z)| |\mathbb{P}_z(\eta, z)| + |\rho(z)| - c_3 |z|^2 \\
\leq - c_3 |z|^2 + c_4 (C \rho |\eta| + d_\rho(\varepsilon))^2 + c_4 (L_\rho + c_5 y)(C \rho |\eta| + d_\rho(\varepsilon)) |z|,
\]

where \( L_\rho \) is the Lipschitz constant of \( \rho \). The remaining steps are similar to [23, Eqns. (92)-(94)], which yields

\[
V(\mathbb{P}(\eta, z)) - V(\eta, z) \leq - \left| \frac{\eta}{|z|} \right|^T \Lambda_\psi \left[ \frac{\eta}{|z|} \right] + d(\sigma, \varepsilon),
\]

(94)
where

\[
\Lambda_{\mathcal{H}} = \left[ \sigma^{1-2\beta(\varepsilon)^2} - c_4 C_p^2 - \frac{c_4 G_{\mathcal{H}}(L_p + c_6 y)}{2} \right] (95)
\]

\[
d(\sigma, \varepsilon) = \sigma d_q(\varepsilon)^2 + \frac{2C_p^2}{\sigma} + \frac{L_p + c_4 y}{2c_3} c_4 d_p(\varepsilon)^2.
\]

By choosing a large enough \( \sigma \), it can be verified that \( \Lambda_{\mathcal{H}} \) is positive definite. We complete the proof by picking an even larger \( \varepsilon > \bar{\varepsilon} \) such that \( d(\sigma, \varepsilon) \) is sufficiently small.

### 6 Results

In this section we will discuss simulation results on a 2-link walker platform shown in Fig. 2. The biped is planar with 2 links; this model was primarily used to establish robust walking behaviors in [27]. The robot has point feet, and is thus underactuated at the ankle. Table 2 provides the list of physical parameters of the 2-link walker. More details about this model are found in [27].

#### Outputs and control. We have the configuration as \( q = (q_1, q_2) \), where \( q_1 \) is the angle between the legs i.e., angle between the stance and nonstance legs, \( q_2 \) is the stance leg angle w.r.t. the vertical axis. The output is defined as follows:

\[
e(q) = q_1 - q_0^n(\tau(q_2)),
\]

where \( q_0^n \) is a 5th order polynomial:

\[
q_0^n(\tau) = \sum_{j=0}^{5} \zeta_j \tau^{5-j}(1-\tau)^j, \quad \tau(q_2) = \frac{q_2 - q_0}{q_f - q_0},
\]

with \( \zeta_j \)'s being the parameters 0.5753, 3.1632, 0.3115, -0.0570, -1.9988, -0.5753 in increasing order, and \( q_0, q_f \) being the initial and final values of \( q_2 \) on the orbit. These parameters are obtained via an offline optimization problem [11,39,40]. The control law is given by (27). We chose the following gain values, and then studied the resulting walking:

\[
k_p = \varepsilon^2, \quad k_d = \varepsilon k, \quad \text{with} \quad \varepsilon = 5, 10, 20, \quad k = 2.
\]

\( \varepsilon = 5 \) resulted in eventual falling, while \( \varepsilon = 10, 20 \) resulted in stable walking. We verified that Assumption 3 was satisfied throughout the step, which is shown in Fig. 3. Fig. 4 shows the responses and torque profiles for these different gains. Fig. 5 shows the phase portrait of the zero coordinates for 100 steps for \( \varepsilon = 20, 40 \). It can be verified from Figs. 4 and 5 that increasing \( \varepsilon \) not only increases the convergence rate, but also reduces the ultimate bound.

### 7 Conclusions

We established that PD control laws are sufficient to realize locally stable periodic orbits in underactuated hybrid robotic systems. As an example, we have realized stable walking in a 2-link bipedal walker. The key methodology is to use derivative gains that are at least as high as the square root of the proportional gains, and then increase the proportional gains as high as possible. It is important to note that properties like existence of stable periodic orbits in HZD, and low variations of the desired trajectories w.r.t. the unactuated coordinates have been used to achieve stable walking. We are also assuming that the joint actuators have high torque limits. These are not restrictive, and can be used as guidelines for the biped design process.

### Acknowledgements

The author would like to acknowledge Sushant Veer and Wen-Loong Ma for providing useful insights and suggestions in improving the proofs in the paper.

### A Proofs of Propositions 1 and 2

**PROOF.** For convenience let \( D_s(q) := B^T A(q) D(q) B \). It is easy to see why \( D_s \) is symmetric. By [3, A.5.5], \( D_s \) is positive definite. We consider the inverse of \( D_s \), which is obtained as

\[
D_s^{-1} = \begin{bmatrix}
D_s^{-1} + D_{11}^{-1} D_{12} D_s^{-1} D_{21} D_{11}^{-1} & -D_{11}^{-1} D_{12} D_s^{-1} \\
-D_{11}^{-1} D_{21} D_s^{-1} & D_s^{-1}
\end{bmatrix}.
\]

(A.1)
Since \(\|D^{-1}\|\) has upper and lower bounds (eigenvalues of \(D^{-1}\) are the inverse eigenvalues of \(D\)), \(D\) must have upper and lower bounds. We will choose these bounds to be \(c_1, c_a\), respectively. The rest of the properties can be obtained directly from Property 2 (since \(D\) contains only sine and cosine functions of \(q\), their derivatives are bounded w.r.t. \(q\)).

\section*{B Proof of Proposition 3}

\textbf{PROOF.} Positive definiteness of \(D_q\), and skew symmetry of \(D_q - 2C_q\) are shown in [30, Lemma 4.11]. To prove the second part of the proposition, we have that

\begin{equation}
\|D_q\| \leq \|J_q^{-T}\| \|D\| \|J_q^{-1}\|, \tag{B.1}
\end{equation}

and based on boundedness and invertibility of \(J_q\), we choose a \(c_a\) (possibly larger) that bounds \(D_q\). Similarly

\begin{equation}
\|D\| \leq \|J_q^{-T}\| \|D_q\| \|J_q\|, \tag{B.2}
\end{equation}

and the lower bound can be obtained accordingly. Similar procedure follows for \(D_q^{-1}, G_q\). For \(D_e\) we have

\begin{equation}
\|D_e\| = \left\| \frac{\partial D_e}{\partial q} q^{-1} \left[ \dot{q}^u \right] \right\| \leq c_a(\|\dot{q}\| + |\dot{e}|), \tag{B.3}
\end{equation}

for some \(c_a > 0\). Similar procedure follows for \(C_e\).

\section*{C Proof of Proposition 4}

\textbf{PROOF.} Proof is very similar to Proof of Propositions 1, 2, and 3. We note that \(A_q, D_q\) are purely functions of the configuration \(q\) from a compact space \(N\). Therefore, the arguments follow directly from (A.1), (B.1), (B.2), (B.3).

\textbf{References}

[1] A. D. Ames, K. Galloway, K. Sreenath, and J. W. Grizzle. Rapidly exponentially stabilizing control Lyapunov functions and hybrid zero dynamics. \textit{IEEE Transactions on Automatic Control}, 59(4):876–891, 2014.

[2] S. Arimoto and F. Miyazaki. Asymptotic stability of feedback control laws for robot manipulator. \textit{IFAC Proceedings Volumes}, 18(16):221 – 226, 1985. 1st IFAC Symposium on Robot Control.

[3] S. Boyd and L. Vandenberghe. \textit{Convex optimization}. Cambridge university press, 2004.

[4] S. Brakken-Thal. Gershgorins theorem for estimating eigenvalues. 2007.

[5] Y. Choi and W. K. Chung. \textit{PID trajectory tracking control for mechanical systems}, volume 298. Springer, 2004.

[6] F. Ghorbel, B. Srinivasan, and M. W. Spong. On the uniform boundedness of the inertia matrix of serial robot manipulators. \textit{Journal of Robotic Systems}, 15(1):17–28, 1998.

[7] J. W. Grizzle, Christine Chevallereau, Aaron D. Ames, and Ryan W. Sinnet. 3d bipedal robotic walking: Models, feedback control, and open problems. \textit{IFAC Proceedings Volumes}, 43(14):505 – 532, 2010.

[8] J. W. Grizzle, J. Hurst, B. Morris, H. Park, and K. Sreenath. MABEL, a new robotic bipedal walker and runner. In \textit{American Control Conference}, pages 2030–2036, St. Louis, MO, USA, 2009.

[9] R. Gunawardana and F. Ghorbel. The class of robot manipulators with bounded jacobian of the gravity vector. In \textit{Proceedings of IEEE International Conference on Robotics and Automation}, volume 4, pages 3677–3682 vol.4, April 1996.

[10] K. A. Hamed, B. G. Buss, and J. W. Grizzle. Exponentially stabilizing continuous-time controllers for periodic orbits of hybrid systems: Application to bipedal locomotion with ground height variations. \textit{The International Journal of Robotics Research}, 35(8):977–999, 2016.

[11] A. Hereid, E. A. Cousineau, C. M. Hubicki, and A. D. Ames. 3d dynamic walking with underactuated humanoid robots: A direct collocation framework for optimizing hybrid zero dynamics. In 2016 \textit{IEEE International Conference on Robotics and Automation (ICRA)}, pages 1447–1454, 5 2016.
