Efficient scheme for quantum entanglement, quantum information transfer, and quantum gate with three-level SQUID qubits in cavity QED

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A novel scheme is proposed for realizing quantum entanglement, quantum information transfer and a set of universal quantum gates with superconducting-quantum-interference-device (SQUID) qubits in cavity QED. In the scheme, the two logical states of a qubit are the two lowest levels of the SQUID. An intermediate level of the SQUID is utilized to facilitate coherent control and manipulation of quantum states of the qubits. The method presented here does not create finite intermediate-level population or cavity-photon population during the operations. Thus, decoherence due to spontaneous decay from the intermediate levels is minimized and the requirement on the quality factor of the cavity is greatly loosened.

Cavity QED has been extensively studied to implement quantum information processing (QIP) with a variety of physical systems such as atoms, ions, quantum dots and Josephson junctions [1-6]. A well-known reason for this is that compared with those non-cavity proposals where significant overhead is needed for coupling distant qubits, the cavity-based schemes is preferable since the decoherence time [8-10]. In Ref. [7], the gates were performed by inducing transitions to the intermediate level (\(|a\rangle\)) [see Fig. 1(a)] via microwave pulse and cavity field. However, though the cavity mode is not populated during the operation, the population of the SQUIDs in the intermediate levels is non-zero. Thus, the operation must be done within a much shorter time than the energy relaxation time of the intermediate level to maintain coherence. Another key point is that the operation in [7] requires rapid adjustments of level spacings of SQUIDs, which might be undesirable in experiment.

In this letter, we propose a significantly improved approach to achieve entanglement, information transfer and universal gates with three-level A-type SQUID qubits in cavity QED. The new method has three major advantages: (a) during the gate operations, the intermediate level is unpopulated and thus decoherence induced by spontaneous emission from the intermediate level, is greatly suppressed; (b) no transfer of quantum information between the SQUIDs and the cavity is required, i.e., the cavity field is only virtually excited and thus the requirement on the quality factor of the cavity is relaxed; (c) there is no need to adjust the level spacings during the operation.

Let us first introduce the Hamiltonian of a SQUID qubit coupled to a single-mode cavity field and a classical microwave pulse with \(B_{\mu
u}(r, t) = B_{\mu
u}(r) \cos \omega_{\mu
u} t\). Here, \(B_{\mu
u}(r)\) is the amplitude of the magnetic component and \(\omega_{\mu
u}\) is the carrier frequency. The qubits considered in this letter are rf SQUIDs each consisting of a Josephson tunnel junction in a superconducting loop (typical size of an rf SQUID is on the order of 10 \(\mu m\) to \(100 \mu m\)). The Hamiltonian of an rf SQUID (with junction capacitance \(C\) and loop inductance \(L\)) can be written in the usual form

\[
H_s = \frac{Q^2}{2C} + \frac{(\Phi - \Phi_c)^2}{2L} - E_J \cos \left(\frac{2\pi \Phi}{\Phi_0}\right),
\]

where \(\Phi\), the magnetic flux threading the ring, and \(Q\), the total charge on the capacitor, are the conjugate variables of the system (with the commutation relation \([\Phi, Q] = i\hbar\) \(\Phi_c\) is the static (or quasistatic) external flux applied to the ring, and \(E_J \equiv I_c/\Phi_0/2\pi\) is the maximum Josephson coupling energy \(I_c\) is the critical current of the junction and \(\Phi_0 = h/2e\) is the flux quantum).

The quantized Hamiltonian of the cavity mode is given by \(H_c = \hbar \omega_c (c^+ c + 1/2)\), where \(c^+\) and \(c\) are the photon creation and annihilation operators; and \(\omega_c\) is the frequency of the cavity mode.

Consider a A-type configuration formed by the two lowest levels and an excited level of the SQUID, denoted by \(|0\rangle\), \(|1\rangle\) and \(|a\rangle\) with energy eigenvalues \(E_0\), \(E_1\), and \(E_a\), respectively [Fig. 1(a)]. For the sake of concreteness, we choose the following device and control parameters: \(C = 90 \text{ fF}\), \(L = 100 \text{ pH}\), \(I_c = 3.75 \text{ \mu A}\), \(\Phi_x = 0.4995 \Phi_0\) for the SQUID qubit in the rest of this letter. Suppose that the coupling of \(|0\rangle\), \(|1\rangle\) and \(|a\rangle\) with the other levels via the cavity mode and the microwave is negligible.
(e.g., by adjusting cavity size, microwave frequency, or level spacings of the SQUID). Under this assumption, it is easy to find that when the cavity mode is coupled to the $|0\rangle \leftrightarrow |a\rangle$ transition but far-off resonant with the $|0\rangle \leftrightarrow |1\rangle$ and $|1\rangle \leftrightarrow |a\rangle$ transitions, and when the microwave pulse is coupled to the $|1\rangle \leftrightarrow |a\rangle$ transition while far-off resonant with the $|0\rangle \leftrightarrow |1\rangle$ and $|0\rangle \leftrightarrow |a\rangle$ transitions, the Hamiltonian of the system can be written as:

$$H = E_0\sigma_{00} + E_1\sigma_{11} + E_a\sigma_{aa} + h\omega_c c^{+} c + h(ge^{+}\sigma_{0a} + h.c.) + h\left(\Omega e^{i\omega_{nu}t}\sigma_{1a} + h.c.\right), \quad (2)$$

where $g$ is the coupling constant between the cavity mode and the $|0\rangle \leftrightarrow |a\rangle$ transition; $\Omega$ is the Rabi-flopping frequency corresponding to the $|1\rangle \leftrightarrow |a\rangle$ transition; and $\sigma_{ij} = |i\rangle \langle j| \: (i, j = 0, 1, a)$. The expressions of $g$ and $\Omega$ are given by [7]

$$g = \frac{1}{2}\sqrt{\frac{\omega_c}{2\omega_0h}}\langle 0 | \Phi | a \rangle \int_S B_c(r) \cdot dS,$$

$$\Omega = \frac{1}{2Lh}\langle 1 | \Phi | a \rangle \int_S B_{\mu w}(r) \cdot dS,$$

where $S$ is any surface that is bounded by the SQUID ring, $r$ is the position vector on $S$, and $B_c(r)$ is the magnetic component of the normal mode of the cavity.

Consider a situation in which the cavity mode is largely detuned from the $|0\rangle \leftrightarrow |a\rangle$ transition, i.e., $\Delta_c = \omega_0 - \omega_c \gg g$, and the microwave pulse is largely detuned from the $|1\rangle \leftrightarrow |a\rangle$ transition, i.e., $\Delta_{\mu w} = \omega_{1a} - \omega_{\mu w} \gg \Omega$, where $\omega_{0a} = (E_a - E_0)/\hbar$ and $\omega_{1a} = (E_a - E_1)/\hbar$ [Fig. 1(a)]. Under this condition, the intermediate level $|a\rangle$ can be adiabatically eliminated [11,12]. Thus, the effective Hamiltonian in the interaction picture becomes $[11,12]$

$$H_i = h[-\frac{g^2}{\Delta_c}c^{+}\sigma_{00} - \frac{\Omega^2}{\Delta_{\mu w}}\sigma_{11} - g_{eff}e^{ist}\sigma_{01}^+ - g_{eff}e^{-ist}c^{+}\sigma_{01}], \quad (3)$$

where $\sigma_{01} = |0\rangle \langle 1|$, $\sigma_{01}^+ = |1\rangle \langle 0|$, $\delta = \Delta_c - \Delta_{\mu w}$, and $g_{eff} = \frac{\Delta_{\mu w}}{2\sqrt{\frac{1}{\Delta_c} + \frac{1}{\Delta_{\mu w}}} \cdot \frac{\Omega}{\hbar}}$. The first two terms are ac-Stark shifts of the levels $|0\rangle$ and $|1\rangle$ induced by the cavity mode and the microwave pulse, respectively. The last two terms are the familiar Jaynes-Cummings interaction, describing the Raman coupling of the two lowest levels of the SQUID.

**Effective Hamiltonian for two SQUID qubits in cavity.**

To simplify presentation, let us consider two identical SQUIDs $I$ and $II$ (the method is also applicable to non-identical SQUIDs). The two SQUIDs are coupled to the same single-mode microwave cavity and each driven by a classical microwave pulse $B_{\mu w}(r, t) = B_{\mu w}(r) \cos \omega_{\mu w} t \: (i = I, II)$ [Fig. 1(c)]. The separation of the two SQUIDs is assumed to be much larger than the linear dimension of each SQUID ring in such a way that direct interaction between the two SQUIDs is negligible. Also, suppose that the coupling of each SQUID to the cavity mode is the same (this can be readily obtained by setting the two SQUIDs on two locations $r_1$ and $r_2$ where the cavity-field magnetic components $B_c(r_1, t)$ and $B_c(r_2, t)$ are the same). In this case, it is obvious that based on Eq. (3), the Hamiltonian for the system in the interaction picture can be written as

$$H_I = \sum_{i = I, II} h[-\frac{g^2}{\Delta_c}c^{+}\sigma_{00i} - \frac{\Omega^2}{\Delta_{\mu w}}\sigma_{11i}] + h\gamma_i \left[ \sum_{i = I, II} c^{+}\sigma_{00i} + c^{+}\sigma_{11i} + \sigma_{01i}\sigma_{01i} + \sigma_{01i}\sigma_{01i}^+ \right], \quad (4)$$

Under the condition that $\delta \gg \frac{g^2}{\Delta_c}$, $\frac{\Omega^2}{\Delta_{\mu w}}$, $g_{eff}$, there is no exchange of energy between the SQUIDs and the cavity mode. The effective Hamiltonian is then given by [13-16]

$$H_{eff} = \sum_{i = I, II} h[-\frac{g^2}{\Delta_c}c^{+}\sigma_{00i} - \frac{\Omega^2}{\Delta_{\mu w}}\sigma_{11i}] + h\gamma_i \left[ \sum_{i = I, II} c^{+}\sigma_{00i} + c^{+}\sigma_{11i} + \sigma_{01i}\sigma_{01i} + \sigma_{01i}\sigma_{01i}^+ \right], \quad (5)$$

where the third and fourth terms describe the photon-number dependent Stark shifts induced by the off-resonant Raman coupling, and the last two terms describe the “dipole” coupling between the two SQUIDs mediated by the cavity mode and the classical fields. The parameter $\gamma = \frac{g_{eff}^2}{\delta}$ characterizes the strength of Stark shift and inter-qubit coupling. If the cavity is initially in the vacuum state, then the effective Hamiltonian reduces to

$$H_{eff} = -\sum_{i = I, II} h\gamma_i \left[ \sum_{i = I, II} c^{+}\sigma_{00i} + \sigma_{01i}\sigma_{01i} + \sigma_{01i}\sigma_{01i}^+ \right], \quad (6)$$

Note that the Hamiltonian (6) does not contain the operators of the cavity mode. Thus, only the state of the SQUID system undergoes an evolution under the Hamiltonian (6), i.e., no quantum information transfer occurs between the SQUIDs and the cavity mode. Therefore, the cavity mode is virtually excited.

The state $|0\rangle_I |0\rangle_{II}$ is unaffected under the Hamiltonian (6). From (6), one can easily get the following state evolution

$$|0\rangle_I |1\rangle_{II} \rightarrow e^{-i\gamma^\prime t}(\cos(\gamma t) |0\rangle_I |1\rangle_{II} - i \sin(\gamma t) |1\rangle_I |0\rangle_{II}),$$

$$|1\rangle_I |1\rangle_{II} \rightarrow e^{-i2\gamma^\prime t} |1\rangle_I |1\rangle_{II}, \quad (7)$$

where $\gamma^\prime = \gamma - \frac{\Omega^2}{\Delta_{\mu w}}$. In the following, we show that Eq. (7) can be used to create entanglement, to implement quantum information transfer, and to perform quantum gates.

2
Generation of entanglement. The two logical states of each SQUID qubit are represented by the two lowest energy states $|0\rangle$ and $|1\rangle$. From (7), one can see that if the two SQUID qubits are initially in the states $|0\rangle_I$ and $|1\rangle_{II}$, they will evolve to the following maximally entangled state after an interaction time $\pi/(4\gamma)$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(1 + i)|1\rangle_I|1\rangle_{II} - i|1\rangle_I|0\rangle_{II}, \quad (8)$$

where the common phase factor $e^{-i\pi/4} (\chi = \gamma'/\gamma)$ has been omitted.

Quantum information transfer. Suppose that the SQUID qubit $I$ is the original carrier of quantum information, which is in an arbitrary state $\alpha|0\rangle + \beta|1\rangle$. The quantum state transfer from the qubit $I$ to the qubit $II$ initially in the state $|0\rangle$ is described by

$$(\alpha|0\rangle_I + \beta|1\rangle_I)|0\rangle_{II} \rightarrow |0\rangle_I(\alpha|0\rangle_{II} + \beta|1\rangle_{II}), \quad (9)$$

which can be realized in the following two steps.

Step (i): Apply two microwave pulses to the two SQUIDs $I$ and $II$, respectively, so that the states of the two SQUIDs undergo an evolution under the Hamiltonian (6) for an interaction time $\pi/(2\gamma)$.

Step (ii): Perform a phase-shift $|0\rangle \rightarrow e^{-i(1+\chi)\pi/4}|0\rangle$ while $|1\rangle \rightarrow e^{i(1+\chi)\pi/4}|1\rangle$ on the SQUID qubit $II$ [17].

The states after each step of the above operations are listed below:

$$|0\rangle_I + \beta|1\rangle_I|0\rangle_{II} \xrightarrow{\text{Step (i)}} |0\rangle_I|0\rangle_{II} + e^{-i(1+\chi)\pi/2}\beta|1\rangle_{II},$$

$$\xrightarrow{\text{Step (ii)}} e^{-i(1+\chi)\pi/4}|0\rangle_I(\alpha|0\rangle_{II} + \beta|1\rangle_{II})$$

It is clear that the two-step operation transfers quantum information from the SQUID qubit $I$ to the SQUID qubit $II$.

Single SQUID qubit operations can be achieved via various schemes [7,17,18]. In Ref. [17], it has been shown that by applying two microwave pulses to induce two-photon Raman resonant transition between the qubit levels, any single-SQUID-qubit logic operation can be realized, without real excitation of the intermediate level. It is noted that during the present single-qubit operation inside a cavity, the cavity mode can be decoupled from the qubits without adjusting the qubits’ level spacings. The reason for this is that one can choose the frequencies of the applied microwave pulses so that two-photon Raman resonant transition between the qubit levels $|0\rangle$ and $|1\rangle$ is satisfied, while the cavity mode is highly detuned from either pulse [see Fig. 1 (b)].

Quantum logical gates. A non-trivial and universal two-qubit controlled NOT (CNOT) can be realized by combining the Hamiltonian (6) with single-qubit operations. We find that the CNOT gate $|i\rangle_I|j\rangle_{II} \rightarrow |i\rangle_I|i\oplus j\rangle_{II}$ ($i, j \in \{0,1\}$) acting on the two SQUID qubits $I$ and $II$ can be achieved through the following unitary transformations

$$U_{\text{CNOT}} = H_I^\dagger U_I U_{II} S_I S_{II} U_{II} H_I, \quad (11)$$

where $H_I$ is a two-SQUID-qubit joint unitary operation defined by $U_{II}(\gamma t) = e^{iH_{eff}t/\hbar}$ with $\gamma t = \pi/4$, $\sigma_z$ is the Pauli operator, $S$ results in a single-qubit phase-shift $|0\rangle \rightarrow e^{-i\chi\pi/8}|0\rangle$ while $|1\rangle \rightarrow e^{i\chi\pi/8}|1\rangle$, $U_I = H_I^\dagger H_{II} U_{II} H_{II}^\dagger H_I$, and $H, H^{-1}, H^{-1}$, and $H^{-1}$ are the following Hadamard transformations

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix}, \quad (12)$$

in the single-qubit Hilbert subspace formed by $|0\rangle = (0,1)^T$ and $|1\rangle = (1,0)^T$.

It is well known that at least three CNOT gates are needed [7] to construct a two-qubit SWAP gate. Note that information transfer (9) is equivalent to a transformation $|i\rangle_I|0\rangle_{II} \rightarrow |0\rangle_I|i\rangle_{II}$ ($i \in \{0,1\}$). Thus, to simplify the gate operation, a two-SQUID-qubit SWAP $|i\rangle_I|j\rangle_{II} \rightarrow |j\rangle_I|i\rangle_{II}$ ($i, j \in \{0,1\}$) can be realized through the following procedure:

$$|i\rangle_I|j\rangle_{II}|0\rangle_a \rightarrow |0\rangle_I|i\rangle_{II}|i\rangle_a \rightarrow |j\rangle_I|i\rangle_{II}|i\rangle_a \rightarrow |j\rangle_I|j\rangle_{II}|0\rangle_a.$$
is \( \omega_\text{c}/(2\pi) \approx 30 \text{ GHz} \). Hence, we choose \( \omega_\text{c}/(2\pi) = 29.7 \text{ GHz} \) as the cavity-mode frequency. For a \( 10 \times 1 \times 1 \text{ mm}^3 \) cavity and a SQUID with a \( 50 \times 50 \mu\text{m}^2 \) loop, a simple calculation shows that the coupling constant is \( g \approx 1.8 \times 10^8 \text{ s}^{-1} \), i.e., about \( 0.1\Delta_c \). By choosing the frequency and amplitude of the microwave pulse appropriately such that \( \Delta_\mu = 10\Omega \) and \( g = 1.2\Omega \) for each SQUID, we have \( \delta \approx 10\gamma_{\text{eff}} \approx 3.0 \times 10^8 \text{ s}^{-1} \). Then the typical time needed for the SQUID-cavity interaction is on the order of \( T_{\text{s-c}} = \pi\delta/(2\gamma_{\text{eff}}^2) \approx 0.5 \mu\text{s} \), which is much shorter than the level \( |a\rangle \)’s effective decay time \( T_1/P_a \geq 1.5 \times 10^3 \mu\text{s} \) for \( T_1 = 15 \mu\text{s} \), where \( P_a \leq 0.01 \) is the occupational probability of the level \( |a\rangle \) for the present case of \( \Delta_c = 10\Omega \) and \( \Delta_\mu = 10\Omega \). The photon lifetime is given by \( T_c = Q_c/\omega_c \) where \( Q_c \) is the quality factor of the cavity. In the present case, the cavity has a probability \( P_c \approx 0.01 \) of being excited during the operation. Thus, the effective decay time of the cavity is \( T_c/P_c \approx 10 \mu\text{s} \gg T_{\text{s-c}} \) for \( Q_c \approx 2 \times 10^4 \), which is realizable as demonstrated by recent experiments [20].

Note that the method described above does not require two SQUIDs with identical parameters. In the case of non-identical SQUIDs \( I \) and \( II \), one has \( \delta_I = \omega_0^I - \omega_a^I - \omega_c + \omega_{\mu I} \) and \( \delta_{II} = \omega_0^II - \omega_a^II - \omega_c + \omega_{\mu II} \), which can always be set to equal by adjusting the frequencies, \( \omega_{\mu I} \) and \( \omega_{\mu II} \), of the two microwave pulses applied to the SQUIDs.

The present scheme has the following advantages: (i) During the operation, the intermediate level is unpopulated and thus gate errors caused by energy relaxation is greatly suppressed. (ii) The cavity field is virtually excited and thus the required quality factor of the cavity is greatly loosened. (iii) No tunneling between the qubit levels \( |0\rangle \) and \( |1\rangle \) is needed and thus the rate of spontaneous decay from the level \( |1\rangle \) can be made negligibly small, by the use of higher potential barrier between the two qubit levels. (iv) No adjustment of level spacings is needed during logic operations, since the qubit-qubit interaction required for the two-qubit operations is via the cooperative actions of the cavity mode and the microwave pulses. (v) The method can be extended to perform IQP on many SQUID qubits in a cavity, because the cavity mode can mediate long-range coherent interaction between SQUID qubits. Also, the proposal can be applied to any other type of solid state qubits which have a \( \Lambda \)-type three-level configuration.

In summary, we have explicitly shown how quantum entanglement, quantum information transfer, and universal quantum gates can be realized with SQUID qubits in cavity. We stress that in our analysis, all Stark shift terms, which may significantly affect the gate fidelity, are included. In addition, we have shown that the method is feasible with experimentally demonstrated qubit and cavity parameters. Thus, it provides a realistic approach for robust quantum information processing with superconducting qubits, and we hope that this work will stimulate further theoretical and experimental activities in this emerging research field.

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**Figure Captions**

**FIG. 1.** (a) The potential and level diagram of an rf SQUID with a \( \Lambda \)-type three levels \( |0\rangle, |1\rangle \) and \( |a\rangle \). The cavity field is detuned from the classical microwave pulse by \( \delta = \Delta_c - \Delta_\mu \). (b) Illustration of single-qubit operation. The two microwave pulses with frequencies \( \omega_0 \) and \( \omega_1 \) are applied to induce two-photon Raman resonant transition between the qubit levels \( |0\rangle \) and \( |1\rangle \) with \( \omega_{01} = \omega_1 - \omega_0 \), for the purpose of single-qubit logic operation. (c) Schematic illustration of two SQUIDs (\( I, II \)) coupled to a single-mode cavity field and manipulated by microwave pulses. The two SQUIDs are placed along the cavity axis (the \( Z \) axis) and in the \( X-Z \) plane. \( B_z \), \( B_{\mu I} \) and \( B_{\mu II} \) are in \( Y \) direction.

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FIG. 1