INSEPARABLE MAPS ON $W_n$-VALUED EXT GROUPS OF NON-TAUT RATIONAL DOUBLE POINT SINGULARITIES AND THE HEIGHT OF K3 SURFACES

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Abstract. We give lower bounds of, or moreover determine, the height of K3 surfaces in characteristic $p$ admitting non-taut rational double point singularities or actions of local group schemes of order $p$ ($\mu_p$ or $\alpha_p$). The proof is based on the computation of the pullback maps by inseparable morphisms, such as Frobenius, on certain $W_n$-valued Ext groups of rational double points.

1. Introduction

The fundamental invariant of rational double point singularities (RDPs for short) is the dual graph of the exceptional divisor of the minimal resolution, which is a Dynkin diagram. In most cases the dual graph determines the isomorphism class of the singularity (in a fixed characteristic $p \geq 0$). Such RDPs are called taut. However in some special cases there are more than one isomorphism classes, in which cases the RDPs are called non-taut.

To describe them we define, for each pair of a characteristic $p \geq 0$ and a Dynkin diagram $S$ (which is $A_N$, $D_N$, or $E_N$), a non-negative integer $r_{\text{max}}(p, S) = r_{\text{max}} = r_{\text{max}}(p, S)$ as follows:

$$r_{\text{max}}(p, S) = \begin{cases} \lceil N/2 \rceil - 1 & \text{if } (p, S) = (2, D_N), \\ 1 & \text{if } (p, S) = (2, E_6), \\ 3 & \text{if } (p, S) = (2, E_7), \\ 4 & \text{if } (p, S) = (2, E_8), \\ 1 & \text{if } (p, S) = (3, E_6), (3, E_7), \\ 2 & \text{if } (p, S) = (3, E_8), \\ 1 & \text{if } (p, S) = (5, E_8), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.1 (Artin [Art77]). Let $p$ and $S$ as above. Then there exist exactly $r_{\text{max}} + 1$ isomorphism classes of RDPs in characteristic $p$ whose dual graph is a Dynkin diagram of type $S$.

When $r_{\text{max}} > 0$, the isomorphism classes are distinguished by the symbols $S^r$ ($0 \leq r \leq r_{\text{max}}$). They are ordered in the way that $r$ is lower semi-continuous in families of RDPs with the same dual graph.

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In this paper we consider $W_n$-valued Ext groups $\text{Ext}_{W_n(A)}^2(A/I, W_n(A))$ on (mainly non-taut) RDPs $A$ for some $n$ and some ideals $I \subset A$, and we compute the pullback maps by purely inseparable morphisms $\text{Spec } B \to \text{Spec } A$. The behaviors of the maps depend heavily on the isomorphism classes of the non-taut RDP.

Suppose an RDP K3 surface (a proper surface with only RDP singularities whose minimal resolution is a K3 surface in the usual sense) admits a non-taut RDP. From the local behaviors of the Frobenius maps studied above, we can deduce a relation between the isomorphism class of the non-taut RDP and the height of the K3 surface. Here the height is an invariant of K3 surfaces in positive characteristic which takes values in $\{1, 2, \ldots, 10\} \cup \{\infty\}$ and is characterized by the Frobenius actions on $W_n$-valued cohomology groups (see Theorem 5.3).

**Theorem 1.2** (see Theorem 6.4 for a detailed statement). *For each Dynkin diagram $S$ and characteristic $p$ (with $r_{\text{max}}(p, S) > 0$), we give a subsequence $(r_1, r_2, \ldots, r_l)$ of $(r_{\text{max}}(p, S), \ldots, 2, 1)$ with the following properties. Let $Y$ be an RDP K3 surface in characteristic $p$ that admits an RDP of type $S^r$.

- If $r > 0$, then $\text{ht}(Y) \leq l$ and $r = r_{\text{ht}(Y)}$.
- If $r = 0$, then $\text{ht}(Y) > l$.

In short, $\text{ht}(Y)$ determines $r$, and if $r > 0$ then $r$ determines $\text{ht}(Y)$.*

If $(p, S)$ is not $(2, D_N)$ ($N \geq 8$) nor $(2, E_8)$, then the subsequence is the whole sequence $(r_{\text{max}}(p, S), \ldots, 2, 1)$.

In Section 7, we determine which non-taut RDPs are realizable on RDP K3 surfaces (Theorem 7.1).

Now suppose $\pi: X \to Y$ is a $G$-quotient morphism between RDP K3 surfaces, where $G = \mu_p$ or $G = \alpha_p$. Then the “dual” map $\pi': Y^{(1/p)} \to X$ is also a $G'$-quotient with $G' = \mu_p$ or $G' = \alpha_p$. Again, we can use the local behavior of the pullback maps by $\pi$ and $\pi'$ to relate the singularities of $X$ and $Y$ to the height of $X$ and $Y$.

**Theorem 1.3** (see Theorem 6.6 for a detailed statement). *Let $\pi: X \to Y$ be as above. Then we have $\text{ht}(X) = \text{ht}(Y) =: h$, we determine $h$ in terms of $\text{Sing}(Y)$ and $\text{Sing}(X)$, and $h$ is always finite.*

As a consequence, we prove (Corollary 6.9) that $G$-quotient of an RDP K3 surface $X$ in characteristic $p$, with $G = \mu_p$ or $G = \alpha_p$, is an RDP K3 surface if and only if $X$ is of finite height.

This paper is organized as follows. In Section 2 we recall the definition and basic properties of the ring $W_n(A)$ of Witt vectors. In Section 3 we construct pullback morphisms on $(W_n$-valued) Ext groups with respect to inseparable morphisms of schemes, and interpret them in terms of Cech cohomology groups. In Section 4 we carry out the computations in the cases of RDPs. In Section 5 we recall the definition and basic properties of the height of K3 surfaces. The main results, connecting the height of K3 surfaces and the maps on $W_n$-valued local Ext groups, will be proved in Section 6. In Sections 7 and 8 we discuss which RDPs are realizable on RDP K3 surfaces, and give examples for all possible non-taut RDPs.
2. Rings of (truncated) Witt vectors

Let $p$ be a prime and $A$ an $\mathbb{F}_p$-algebra. The ring $W(A)$ of Witt vectors on $A$ is the set $A^\mathbb{N}$ equipped with ring structure satisfying, for each polynomial $P \in \mathbb{Z}[x,y]$,

$$P((a_0,a_1,\ldots),(b_0,b_1,\ldots)) = (P_0(a_0,b_0), P_1(a_0,b_0,a_1,b_1,\ldots),$$

where $P_i \in \mathbb{Z}[x_0,\ldots,x_{i-1},y_0,\ldots,y_{i-1}]$ is the unique collection of polynomials satisfying, for each $N \in \mathbb{N},$

$$w_N(P_0(\ldots),P_1(\ldots),\ldots,P_N(\ldots)) = P(w_N(a_0,a_1,\ldots,a_N), w_N(b_0,b_1,\ldots,b_N)),$$

where $w_N(t_0,t_1,\ldots,t_N) := \sum_{i=0}^{N} p^i t_i^{p^{N-i}}$ is the so-called $N$-th ghost component.

For example, we clearly have $(a_0) + (b_0) = (a_0 + b_0)$ and $(a_0) \cdot (b_0) = (a_0 b_0)$ on $W_1(A) \cong A$, and it follows from the equalities

$$(a_0^p + pa_1) + (b_0^p + pb_1) = (a_0 + b_0)^p + p(a_1 + b_1 - \frac{(a_0 + b_0)^p - a_0^p - b_0^p}{p})$$

$$(a_0^p + pa_1) \cdot (b_0^p + pb_1) = (a_0 b_0)^p + p(a_1 b_0^p + a_0^p b_1 + pa_1 b_1)$$

that

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - \frac{(a_0 + b_0)^p - a_0^p - b_0^p}{p})$$

$$(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_0, a_1 b_0^p + a_0^p b_1 + pa_1 b_1)$$

on $W_2(A)$.

It is known (see [Tan18, Lemma 2.4]) that, for $P(x,y) = x + y$, the $i$-th component $P_i$ of $P((a_0,a_1,\ldots),(b_0,b_1,\ldots))$ is a homogeneous polynomial of $a_0,a_1,\ldots,b_0,b_1,\ldots$ of degree $p^i$ if we declare $a_i$ and $b_i$ to be homogeneous of degree $p^i$.

$W$ is functorial: any homomorphism $f: A \to B$ of $\mathbb{F}_p$-algebras induces a morphism $f: W(A) \to W(B)$ of rings by $f(a_0,a_1,\ldots) = (f(a_0), f(a_1),\ldots)$ that is compatible with $V$ and $R$ defined below. An example is the Frobenius morphism $F: W(A) \to W(A)$ defined as $F(a_0,a_1,\ldots) = (a_0^p, a_1^p,\ldots)$.

The shift morphism, or Verschiebung, $V$ on $W(A)$ is defined as $V(a_0,a_1,\ldots) = (0,a_0,a_1,\ldots)$. The ring of Witt vectors of length $n$ is the quotient $W_n(A) = W(A)/V^n W(A)$, hence in $W_n(A)$ only the first $n$ components $(a_0,a_1,\ldots,a_{n-1})$ are considered. The Verschiebung induces $V: W_n(A) \to W_{n+1}(A)$. The restriction morphism $R: W_n(A) \to W_{n-1}(A)$ is defined as $R(a_0,\ldots,a_{n-1}) = (a_0,\ldots,a_{n-2})$ and is a ring homomorphism. We have an exact sequence

$$0 \to W_n(A) \xrightarrow{V^{n-n'}} W_n(A) \xrightarrow{R^{n'}} W_{n-n'}(A) \to 0$$

for each $0 \leq n' \leq n$.

We use the following equalities in Section 4

**Lemma 2.1.** If $x \in W_n(A)$ and $y \in W_{n+m}(A)$, then $V^m(x) \cdot y = V^m(x \cdot F^m(R^m(y)))$.

**Lemma 2.2.** Let $A$ be a $\mathbb{F}_p$-algebra.
is also Cohen–Macaulay. In particular, \( \Ext \) suffices to show that 

\[
\text{Proof.}
\]

over \( W \)

Suppose Lemma 2.4.

The latter assertion is a consequence of being Cohen–Macaulay.

\[
\text{Proof. Straightforward.}
\]

The closed immersion \( R^* : \Spec W_{n-1}(A) \to \Spec W_n(A) \) is a homeomorphism if \( n \geq 2 \). For an \( \mathbb{F}_p \)-scheme \( Z \) and \( n \geq 1 \), we define \( W_n(Z) \) to be the scheme whose underlying topological space is \( Z \) and whose structure sheaf is \( W_n(\mathcal{O}_Z) \).

**Lemma 2.3.** If \( Z \) is a scheme projective (resp. quasi-projective) over an algebraically closed field \( k \), then \( W_n(Z) \) is projective (resp. quasi-projective) over \( W_n(k) \).

\[
\text{Proof. Since } W_n(-) \text{ preserves open immersions and closed immersions, it suffices to show that } W_n(\mathbb{P}^N) \text{ is projective. We will show that } W_n(k[x_0, \ldots, x_N]) \text{ is a finitely generated } W_n(k)\text{-algebra. Indeed, it is generated by the elements } (x_i, 0, \ldots, 0) \text{ with } 0 \leq i \leq N \text{ and the elements } V^j(x_0^{i_0}, \ldots, x_N^{i_N}, 0, \ldots, 0) \text{ with } 0 < j < n \text{ and } 0 \leq i, k < p^j.
\]

\[
\text{□}
\]

**Lemma 2.4.** Suppose \( (A, m) \) is a Cohen–Macaulay local ring. Then \( W_n(A) \) is also Cohen–Macaulay. In particular, \( \Ext_{W_n(A)}^i(M', W_n(A)) = 0 \) if \( \Supp M' \subset \{m\} \) and \( i < \dim A \).

\[
\text{Proof. Cohen–Macaulayness follows from } [\text{Bor11}, \text{Proposition 16.19}]. The key point of the proof is that if } a_1, \ldots, a_N \text{ is a regular sequence in } A \text{ then the Teichmüller lifts } [a_1], \ldots, [a_N] \text{ form a regular sequence in } W_n(A). \text{ One can also show more generally that } b_1, \ldots, b_N \text{ in } W_n(A) \text{ is a regular sequence if } R^{n-1}(b_1), \ldots, R^{n-1}(b_N) \text{ is a regular sequence in } A.
\]

The latter assertion is a consequence of being Cohen–Macaulay.

\[
\text{□}
\]
3. $W_n$-valued Ext groups and morphisms

**Setting 3.1.** Throughout this section, suppose $X$ and $Y$ are Cohen–Macaulay schemes of dimension $d$ and quasi-projective over a Noetherian affine scheme, $f : X \to Y$ is a morphism that is a homeomorphism, and $\mathcal{J}$ is a sheaf of ideals of $\mathcal{O}_Y$ with $\text{Supp}(\mathcal{O}_Y / \mathcal{J})$ finite. For notational simplicity, let us write $\mathcal{O}_X$ in place of $f_*\mathcal{O}_X$.

For $? \in \{X, Y\}$, we denote by $(\text{Coh}/\mathcal{O}_?)$ the category of coherent $\mathcal{O}_?$-modules. Let $G$ be the composite functor

$$G : (\text{Coh}/\mathcal{O}_Y)^{\text{op}} \xrightarrow{G_1} (\text{Coh}/\mathcal{O}_X)^{\text{op}} \xrightarrow{G_2} (\text{Coh}/\mathcal{O}_X)$$

of two left exact functors $G_1 = - \otimes_{\mathcal{O}_Y} \mathcal{O}_X$ and $G_2 = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$. Since $X$ and $Y$ are quasi-projective, these categories have enough locally-free objects, hence $G_1$, $G_2$, and $G$ admit derived functors.

Let $\varepsilon : (\text{Coh}/\mathcal{O}_X) \to (\text{Coh}/\mathcal{O}_Y)$ be the forgetful functor. Then the morphism $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X) \to \varepsilon \circ G$ of functors (from $(\text{Coh}/\mathcal{O}_Y)^{\text{op}}$ to $(\text{Coh}/\mathcal{O}_Y)$) induces a morphism $\mathcal{E}xt_{\mathcal{O}_Y}^p(-, \mathcal{O}_Y) \to R^p(\varepsilon \circ G) = \varepsilon \circ R^pG$ of $\delta$-functors, which in turn induces (by adjoint) a collection of morphisms

$$\zeta^p : \mathcal{E}xt_{\mathcal{O}_Y}^p(-, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \to R^pG$$

of functors (from $(\text{Coh}/\mathcal{O}_Y)^{\text{op}}$ to $(\text{Coh}/\mathcal{O}_X)$) that commute with the coboundary morphisms induced from short exact sequences on $(\text{Coh}/\mathcal{O}_Y)$.

Since $G_1$ sends locally-free objects to $G_2$-acyclic objects, we have the Grothendieck spectral sequence

$$E_2^{p,q} = \mathcal{E}xt_{\mathcal{O}_X}^p (\text{Tor}_q^{\mathcal{O}_Y}(-, \mathcal{O}_X), \mathcal{O}_X) \Longrightarrow R^{p+q}G,$$

and the edge morphisms

$$\phi^p : E_2^{p,0} = \mathcal{E}xt_{\mathcal{O}_X}^p (- \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \mathcal{O}_X) \to R^pG.$$

Also we have morphisms of functors

$$\psi^p : \mathcal{E}xt_{\mathcal{O}_X}^p (- \cdot \mathcal{O}_X, \mathcal{O}_X) \to \mathcal{E}xt_{\mathcal{O}_X}^p (- \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \mathcal{O}_X)$$

on the category of sheaves of ideals of $\mathcal{O}_Y$ whose morphisms are inclusions of subsheaves of $\mathcal{O}_Y$.

**Proposition 3.2.** Let $X$, $Y$, $d$, and $\mathcal{J}$ be as in Setting 3.1 Then,

1. $\phi^p(M) : E_2^{p,0} = \mathcal{E}xt_{\mathcal{O}_X}^p (M \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \mathcal{O}_X) \to R^pG(M)$ is an isomorphism if $p \leq d$ and $M \in \{\mathcal{J}, \mathcal{O}_Y, \mathcal{O}_Y / \mathcal{J}\}$.
2. $\psi^p(\mathcal{J}) : \mathcal{E}xt_{\mathcal{O}_X}^p (\mathcal{J} \mathcal{O}_X, \mathcal{O}_X) \to \mathcal{E}xt_{\mathcal{O}_X}^p (\mathcal{J} \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \mathcal{O}_X)$ is an isomorphism if $p \leq d - 1$.
3. For $M$ as in (1) and $p \leq d$, we define $\tau'$ to be the composite

$$\mathcal{E}xt_{\mathcal{O}_Y}^p (M, \mathcal{O}_Y) \to \mathcal{E}xt_{\mathcal{O}_Y}^p (M, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{\zeta^p(M)} R^pG(M) \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_X}^p (M \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \mathcal{O}_X).$$
and let 
\[ \tau : \mathcal{E}xt^p_{O_Y}(O_Y/J, O_Y) \to \mathcal{E}xt^p_{O_X}(O_Y/J \otimes O_X, O_X) \cong \mathcal{E}xt^p_{O_X}(O_Y/(J\mathcal{O}_X), O_X), \]
\[ \tau : \mathcal{E}xt^p_{O_Y}(O_Y, O_Y) \to \mathcal{E}xt^p_{O_X}(O_X, O_X) = \mathcal{E}xt^p_{O_X}(O_X, O_X), \]
\[ \tau : \mathcal{E}xt^p_{O_Y}(J, O_Y) \to \mathcal{E}xt^p_{O_X}(J \otimes O_X, O_X) \overset{\psi^{(J)}}{\approx} \mathcal{E}xt^p_{O_X}(J\mathcal{O}_X, O_X), \]
for \( p \leq d, d, d - 1 \) respectively.

Then these morphisms commute with the long exact sequences induced by \( 0 \to J \to O_Y \to O_Y/J \to 0 \) and \( 0 \to J\mathcal{O}_X \to O_X \to O_X/(J\mathcal{O}_X) \to 0 \).

(4) Assume \( d \geq 2 \). Let \( f^\#: O_Y \to O_X \) be the natural morphism. Then the following diagram is commutative, where \( \varepsilon \) is the morphism induced by the forgetful functor.

\[
\begin{array}{ccc}
\mathcal{E}xt^1_{O_Y}(J, O_Y) & \to & \mathcal{E}xt^2_{O_Y}(O_Y/J, O_Y) \\
\downarrow \tau & & \downarrow \tau \\
\mathcal{E}xt^1_{O_X}(J\mathcal{O}_X, O_X) & \to & \mathcal{E}xt^2_{O_X}(O_X/(J\mathcal{O}_X), O_X) \\
\downarrow f^\# & & \downarrow f^\# \\
\mathcal{E}xt^1_{O_Y}(J\mathcal{O}_X, O_X) & \to & \mathcal{E}xt^2_{O_X}(O_Y/(J\mathcal{O}_X), O_X) \\
\downarrow (f^\#)^* & & \downarrow (f^\#)^* \\
\mathcal{E}xt^1_{O_Y}(J, O_X) & \to & \mathcal{E}xt^2_{O_Y}(O_Y/J, O_X) \\
\end{array}
\]

Proof. (1) It suffices to show \( E_2^{p,q} = 0 \) if \( p < d \) and \( q > 0 \). By the assumption on \( J \), \( \text{Supp}(\text{Tor}_q(M, O_X)) \) is finite if \( q > 0 \). Since \( X \) is Cohen–Macaulay, \( E_2^{p,q} = 0 \) if \( p < d \) and \( q > 0 \).

(2) The kernel \( K \) of the surjection \( J \otimes O_X \to J\mathcal{O}_X \) is equal to \( \text{Tor}_1^{O_Y}(O_Y/J, O_X) \), hence has finite support. As above, if \( p \leq d - 1 \) then \( \mathcal{E}xt^p_{O_X}(K, O_X) = \mathcal{E}xt^p_{O_X}(K, O_X) = 0 \).

(3) The commutativity of
\[
R^pG(J) \leftarrow \phi^p \mathcal{E}xt^p_{O_X}(J \otimes O_X, O_X) \leftarrow \psi^p \mathcal{E}xt^p_{O_X}(J\mathcal{O}_X, O_X) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
R^{p+1}G(O_Y/J) \leftarrow \phi^{p+1} \mathcal{E}xt^{p+1}_{O_X}((O_Y/J) \otimes O_X, O_X) \leftarrow \psi^{p+1} \mathcal{E}xt^{p+1}_{O_X}(O_X/(J\mathcal{O}_X), O_X)
\]
follows from the usual construction of the derived functors. The others follow from functoriality.

(4) Since the horizontal (coboundary) morphisms are isomorphisms and commute with the vertical morphisms, it suffices to show the commutativity \( (f^\#)^* = (f^\#)^*\circ \varepsilon \) for \( \mathcal{E}xt^1 \). This is straightforward using the interpretation of \( \text{Ext}^1 \) as the module of equivalence classes of short exact sequences: the proofs of (1), (2) shows that if a short exact sequence \( 0 \to O_Y \to N \to J \to 0 \) defined on an open subscheme of \( Y \) represent a local section of \( \mathcal{E}xt^1_{O_Y}(J, O_Y) \), then \( 0 \to O_X \to N \otimes O_X \to J \otimes O_X \to 0 \) is exact, the
inclusion $i: K \to \mathcal{J} \otimes \mathcal{O}_X$ lifts to $\tilde{i}: K \to N \otimes \mathcal{O}_X$ uniquely, and the resulting exact sequence $0 \to \mathcal{O}_X \to (N \otimes \mathcal{O}_X)/\text{Im}(\tilde{i}) \to \mathcal{J} \mathcal{O}_X \to 0$ represents the image by $\tau$ of that section. The pullback of this sequence by $f^\# : \mathcal{J} \to \mathcal{J} \mathcal{O}_X$ is isomorphic to the pushforward of the first sequence by $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$. □

**Definition 3.3.** Let $X$, $Y$, $d$, and $\mathcal{J}$ be as in Setting 3.3 and assume $d \geq 2$.

1. For an inclusion $\iota_{\mathcal{J}, \mathcal{J}'}: \mathcal{J}' \hookrightarrow \mathcal{J}$ of sheaves of ideals of $\mathcal{O}_Y$, let 

$$
\iota^*_{\mathcal{J}', \mathcal{J}'}: \text{Ext}^p_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{J}, \mathcal{O}_Y) \to \text{Ext}^p_{\mathcal{O}_Y}(\mathcal{J}, \mathcal{O}_Y),
$$

$$
\iota^*_{\mathcal{J}', \mathcal{J}'}: \text{Ext}^p_{\mathcal{O}_Y}(\mathcal{J}, \mathcal{O}_Y) \to \text{Ext}^p_{\mathcal{O}_Y}(\mathcal{J}', \mathcal{O}_Y)
$$

be the pullback by $\iota_{\mathcal{J}', \mathcal{J}}$.

2. We define morphisms 

$$
V: \text{Ext}^p_{W_n(\mathcal{O}_Y)}(\mathcal{J}, W_n(\mathcal{O}_Y)) \to \text{Ext}^p_{W_n+1(\mathcal{O}_Y)}(R^{-1}(\mathcal{J}), W_{n+1}(\mathcal{O}_Y)),
$$

$$
V: \text{Ext}^p_{W_n(\mathcal{O}_Y)}(W_n(\mathcal{O}_Y), W_{n+1}(\mathcal{O}_Y)) \to \text{Ext}^p_{W_n+1(\mathcal{O}_Y)}(W_{n+1}(\mathcal{O}_Y), W_{n+1}(\mathcal{O}_Y)),
$$

$$
V: \text{Ext}^p_{W_n(\mathcal{O}_Y)}(\mathcal{J}, W_n(\mathcal{O}_Y)) \to \text{Ext}^p_{W_n+1(\mathcal{O}_Y)}(W_{n+1}(\mathcal{O}_Y)/R^{-1}(\mathcal{J}), W_{n+1}(\mathcal{O}_Y)),
$$

to be the composite of the forgetful functor $\text{Ext}^p_{W_n(\mathcal{O}_Y)}(\mathcal{I}, \mathcal{O}_Y) \to \mathcal{O}_Y$ and the pullback by $R$, and the pushforward by $V$.

3. We denote by $f^*$ the morphisms $\tau$ given in Proposition 3.2.3.

In application, $f$ is one of the following form. Suppose $X'$ and $Y'$ are Cohen–Macaulay Noetherian schemes of dimension $d$, affine or quasi-projective over a field $k$ of characteristic $p > 0$. Let $X = W_m(X')$ and $Y = W_n(Y')$ (then $X$ and $Y$ are quasi-projective over the ring $W_n(k)$ and Cohen–Macaulay by Lemmas 2.3 and 2.4).

- $f$ is the Frobenius map (with $X' = Y'$ and $m = n$). In this case we write $F$ in place of $f^*$.
- $f = (R^*)p - m$ is the restriction (with $X' = Y'$ and $m < n$). In this case we write $R^{-m}$ in place of $f^*$.
- $f$ is induced by a morphism $f': X' \to Y'$ (with $m = n$).

If $f = F$ and $\mathcal{J} = (R^{n-1})^{-1}(\mathcal{I})$ for some ideal $\mathcal{I} \subset \mathcal{O}_Y$, then $\mathcal{J} \mathcal{O}_X = (R^{n-1})^{-1}(\mathcal{I}^{(p)})$, where $\mathcal{I}^{(p)}$ is the ideal generated by the $p$-th powers of the elements of $\mathcal{I}$. If $f = (R^*)p - m$, then $\mathcal{J} \mathcal{O}_X = R^{-m}(\mathcal{I})$.

If $Y' = \text{Spec } A$, then $\text{Ext}^p_{\mathcal{O}_Y}$ is identified with $\text{Ext}^p_{W_n(\mathcal{A})}$, and the group $\text{Ext}^p_{W_n(\mathcal{A})}(A/I, W_n(\mathcal{A}))$ and the morphisms on it admit the following description.

**Lemma 3.4.** Let $(A, \mathfrak{m}_A)$ be a normal Cohen–Macaulay 2-dimensional isolated singularity and $x_1, x_2$ a regular sequence. Let $n \geq 1$ be an integer and $I \subset A$ an $\mathfrak{m}_A$-primary ideal (i.e., $\text{Supp}(A/I) \subset \{\mathfrak{m}_A\}$). Let $J := (R^{n-1})^{-1}(1) \subset W_n(A)$, so that $W_n(A)/J \cong A/I$. Then we have an injection 

$$
h = h_I: \text{Ext}^2_{W_n(\mathcal{A})}(A/I, W_n(\mathcal{A})) \cong \text{Ext}^1_{W_n(\mathcal{A})}(J, W_n(A)) \to \text{Coker} \left( \bigoplus_{i=1,2} W_n(A[x_i^{-1}]) \to W_n(A[x_1x_2^{-1}]) \right)
$$

satisfying the following properties.

1. The image of $h_I$ is the classes annihilated by $J$. 

Conversely, suppose \( 0 \to O \to O \) is a local morphism, then \( h_I \) and \( h_{IB} : \text{Ext}^1_{\text{W}_n(B)}(B/IB, W_n(B)) \cong \text{Ext}^1_{\text{W}_n(B)}(JB, W_n(B)) \to \text{Coker} \left( \bigoplus_i W_n(B[x_i^{-1}]) \to W_n(B[(x_1 x_2)^{-1}]) \right) \) satisfies the compatibility \( h_{IB} \circ \pi^* = \pi^* \circ h_I \).

In particular, the Frobenius on \( \text{Ext}^2 \) can be computed as the \( p \)-th power.

**Proof.** \( \text{Ext}^1_{\text{W}_n(A)}(J, W_n(A)) \to \text{Ext}^2_{\text{W}_n(A)}(W_n(A)/J, W_n(A)) \) is an isomorphism, and we have \( W_n(A)/J \cong \text{A}/I \). Write \( Y_{\text{sm}} = W_n((Y')_{\text{sm}}) \). We first show that the restriction map

\[
\text{Ext}^1_{\text{W}_n(A)}(J, W_n(A)) \to \text{Ext}^1_{\text{W}_n(A)}(J|_{Y_{\text{sm}}}, W_n(A)|_{Y_{\text{sm}}}) = \text{Ext}^1_{\text{W}_n(A)}(\text{O}_Y, \text{O}_Y) \cong H^1(Y_{\text{sm}}, \text{O}_Y)
\]

is injective. Suppose the restriction to \( Y_{\text{sm}} \) of an extension \( 0 \to W_n(A) \to N \to J \to 0 \) admits a retraction \( r : N|_{Y_{\text{sm}}} \to O_{Y_{\text{sm}}} \). It suffices to show that \( r \) extends to a retraction over \( W_n(Y') \). For this it suffices to show that the restriction \( \Gamma(Y, \text{O}_Y) \to \Gamma(Y_{\text{sm}}, \text{O}_Y) \) is an isomorphism. The case \( n = 1 \) is true since \( A \) is normal, and the general case follows by induction.

\( Y_i = \text{Spec A}[x_i^{-1}], i = 1, 2 \), form an affine covering of \( (Y')_{\text{sm}} \), hence \( Y_i := W_n(Y_i) \) form an affine covering of \( Y_{\text{sm}} \), hence we have an isomorphism

\[
H^1(Y_{\text{sm}}, \text{O}_Y) \cong \text{Coker} \left( \bigoplus_i \Gamma(Y_i, \text{O}_Y) \to \Gamma(Y_1 \cap Y_2, \text{O}_Y) \right)
\]

\[
\cong \text{Coker} \left( \bigoplus_i W_n(A[x_i^{-1}]) \to W_n(A[(x_1 x_2)^{-1}]) \right).
\]

(1) The \( W_n(A) \)-module \( \text{Ext}^2_{\text{W}_n(A)}(W_n(A)/J, W_n(A)) \) is annihilated by \( J \). Conversely, suppose \( 0 \to \text{O}_{Y_{\text{sm}}} \to N' \xrightarrow{\beta} \text{O}_{Y_{\text{sm}}} \to 0 \) is an exact sequence on \( Y_{\text{sm}} \) whose class is annihilated by \( J \). Let \( J' := \text{Im}(\beta : \Gamma(Y_{\text{sm}}, N') \to \Gamma(Y_{\text{sm}}, \text{O}_{Y_{\text{sm}}}) = W_n(A) \subset W_n(A) \). Then the first sequence comes from \( \text{Ext}^1_{\text{W}_n(A)}(J', W_n(A)) \). It remains to show \( J \subset J' \). For each \( b \in J \), the pullback of the first sequence by \( \text{O}_{Y_{\text{sm}}} \xrightarrow{x_b} \text{O}_{Y_{\text{sm}}} \) is split and hence admits a section \( s : \text{O}_{Y_{\text{sm}}} \to N' \times_{\text{O}_{Y_{\text{sm}}}, b} \text{O}_{Y_{\text{sm}}} \), and then the image of \( 1 \) by \( \Gamma(Y_{\text{sm}}, \text{O}_{Y_{\text{sm}}}) \xrightarrow{s} \Gamma(Y_{\text{sm}}, N' \times_{\text{O}_{Y_{\text{sm}}}, b} \text{O}_{Y_{\text{sm}}}) \to \Gamma(Y_{\text{sm}}, N') \xrightarrow{\beta} \Gamma(Y_{\text{sm}}, \text{O}_{Y_{\text{sm}}}) \) is \( b \).

\[
\begin{array}{cccccc}
0 & \to & \text{O}_{Y_{\text{sm}}} & \to & N' & \xrightarrow{\beta} & \text{O}_{Y_{\text{sm}}} & \to & 0 \\
& & \downarrow & & \uparrow & & \downarrow & & \\
0 & \to & \text{O}_{Y_{\text{sm}}} & \to & N' \times_{\text{O}_{Y_{\text{sm}}}, b} \text{O}_{Y_{\text{sm}}} & \xrightarrow{\beta} & \text{O}_{Y_{\text{sm}}} & \to & 0
\end{array}
\]

(2) Clear.
By Proposition 3.2[3], the morphisms commute with the coboundary morphism from \( \text{Ext}^1 \) to \( \text{Ext}^2 \). The commutativity with the Cech interpretation can be checked using short exact sequences (cf. proof of Proposition 3.2[4]).

**Convention 3.5.** We will write \( \text{Ext}^2(A/I, W_n(A)) \) in place of \( \text{Ext}^2_{W_n(A)}(A/I, W_n(A)) \) and identify it with its image by \( h_I \). We often omit \( i^* \) from the notation.

**Lemma 3.6.** Let \( A \) and \( I \) be as in Lemma 3.4. Let \( n \geq 2 \) be an integer and consider a \( W_n \)-valued class of the form \( e = [(\alpha, 0, \ldots, 0)] \). Suppose \( R(e) \in \text{Ext}^2(A/I, W_{n-1}(A)) \) and \( F(R(e)) = 0 \). Then \( e \in \text{Ext}^2(A/I, W_n(A)) \).

**Proof.** Let \( I_n := (R^{n-1})^{-1}(I) = \{(a_0, a_1, \ldots, a_{n-1}) \mid a_0 \in I, a_i \in A\} \subset W_n(A) \). Then \( e \in \text{Ext}^2(A/I, W_n(A)) \) if and only if \( I_n \subset \text{Ann}(e) \). It suffices to check \( (a_0, 0, \ldots, 0) \in \text{Ann}(e) \) for \( a_0 \in I \) and \( V(b) \in \text{Ann}(e) \) for \( b \in W_{n-1}(A) \). The former assertion follows from \( R^{n-1}(e) \in \text{Ext}^2(A/I, A) \) (which follows from \( R(e) \in \text{Ext}^2(A/I, W_{n-1}(A)) \)), and the latter follows from \( F(R(e)) = 0 \) since \( V(b) \cdot e = V(b \cdot F(R(e))) \).

If \( Y \) is Cohen–Macaulay of dimension 2 and \( \mathcal{J} \) is a sheaf of ideals of \( \mathcal{O}_Y \) with \( \text{Supp}(\mathcal{O}_Y/\mathcal{J}) \) finite, then the local-to-global Ext spectral sequence implies that \( \text{Ext}^2_Y(\mathcal{O}_Y/\mathcal{J}, \ldots) \to H^0(\mathcal{O}_Y, \text{Ext}^2(\mathcal{O}_Y/\mathcal{J}, \ldots)) \) is an isomorphism. We identify these groups.

**Proposition 3.7.** Suppose \( X \) and \( Y \) are Cohen–Macaulay of dimension \( d = 2 \), \( f : X \to Y \) is a morphism that is a homeomorphism, and \( \mathcal{J} \) is a sheaf of ideals of \( \mathcal{O}_Y \) with \( \text{Supp}(\mathcal{O}_Y/\mathcal{J}) \) finite. Then the diagram

\[
\begin{array}{ccc}
\text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{J}, \mathcal{O}_Y) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \\
\downarrow f^* & & \downarrow f^* \\
\text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J} \mathcal{O}_X, \mathcal{O}_X) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \\
\end{array}
\]

is commutative. Here \( f^* \) on the right side is the one induced by \( f^* : \mathcal{O}_Y \to \mathcal{O}_X \), and \( f^* \) on the left side is the morphism induced by the one defined in Definition 3.3 using the identification described above.

If \( Y \) is \( W_n(\mathcal{O}_Y) \), then \( \alpha \circ \gamma \) commutes with \( V \) and \( i_{\mathcal{J}^*, \mathcal{J}} \) on the left side defined in Definition 3.3 and the morphisms induced by \( V \) and id on \( W_n(\mathcal{O}_Y) \) the right-hand side.

**Proof.** In the diagram

\[
\begin{array}{ccc}
\text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{J}, \mathcal{O}_Y) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \\
\downarrow f^* = \tau & & \downarrow f^* \\
\text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J} \mathcal{O}_X, \mathcal{O}_X) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \\
\downarrow e & \xrightarrow{f^*} & \downarrow e \\
\text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{J} \mathcal{O}_Y, \mathcal{O}_Y) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \\
\downarrow (f^*)^* & & \downarrow \text{id} \\
\text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{J}, \mathcal{O}_X) & \xrightarrow{\gamma} & \text{Ext}^2_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_X) \\
\end{array}
\]
Lemma 4.2. Suppose $A$ is as above, $j \geq 1$ is an integer satisfying one of the following, and $I_j \subset A$ be the ideal defined as follows.

- $j = 1$: $I_1 = m \subset A$ is the maximal ideal.
- $1 \leq j \leq [N/2] - 1$ (resp. $j = 2$) and $A$ is an RDP of type $D_N^n$ (resp. $E_8^n$) in characteristic $2$: $I_j \subset A$ consists of the elements whose vanishing order at the $2j$-th (resp. $4$-th) component of the exceptional divisor of the minimal resolution of $A$ is at least $2j$ (resp. $8$).

Fix an isomorphism $A \cong \mathbb{k}[[x, y, z]]/(f)$ with $f$ as in Table 7. Then,

1. We have $I_j = (x, y^j, z)$,
2. The class $[\alpha]$ of $\alpha := x^i y^j z$ is a generator of the $A$-module $\text{Ext}_A^2(A/I_j, A)$ with $\text{Ann}( [\alpha]) = I_j$.

Here in the case of $D_N$ or $E_8$, the components are ordered in a way that the $1$-st component is the end of the longest branch of the Dynkin diagram, and the $i$-th component ($i \leq N - 2$ or $i \leq 5$ respectively) is the unique component of distance $i - 1$ from the $1$-st component. (In the case of $D_4$...
the longest branch is not unique, but still the 2j-th component, j = 1, is well-defined.)

**Proof.** (1) Straightforward.

(2) Since A is Gorenstein, we have \( \dim_k \text{Ext}^2_A(A/I_j, A) = \dim_k A/I_j = j \).
Hence it suffices to check that, in \( \text{Coker}(A[x^{-1}] \oplus A[y^{-1}] \rightarrow A[(xy)^{-1}]) \), the classes \([x\alpha], [y\alpha], [z\alpha] \) are trivial and \([y^{j-1}\alpha] \) is nontrivial. Straightforward. \( \square \)

### 4.2. Frobenius morphisms

Let A be a non-taut RDP. In this section we compute the Frobenius maps on the groups \( \text{Ext}^2_A(A/I_j, W_n(A)) \), where \( I_j \subset A \) are the \( m \)-primary ideals introduced in Lemma 4.2.

**Proposition 4.3.** Let A be an RDP of type \( D^r_N \) in characteristic \( p = 2 \).
Let \( j \geq 1 \) and \( n \geq 1 \) be integers and assume
\[
[N/2] \geq C_1(n, j) := 2j + (2^n - 1)(2j - 1),
\]
\[
[N/2] - r \geq C_2(n, j) := j + (2^n - 1)(2j - 1).
\]
Let \( I_j \) be the ideal defined as in Lemma 4.2. Then there is an element \( e \in \text{Ext}^2_A(A/I_j, W_n(A)) \) satisfying the following two conditions:
- its restriction \( R^{n-1}(e) \) is a generator of the A-module \( \text{Ext}^2_A(A/I_j, A) \) (which is isomorphic to \( A/I_j \)), and
- its image \( F(e) \) by the Frobenius map
\[
F: \text{Ext}^2_A(A/I_j, W_n(A)) \rightarrow \text{Ext}^2_A(A/I_j^{(p)}, W_n(A))
\]
satisfies, letting \( a := (N/2) - r - C_1(n, j) = (N/2) - r - C_2(n, j) - j, \)
\[
F(e) = \begin{cases} 0 & \text{if } a \geq 0, \\
V^{n-1}(t^{*}(e')) & \text{if } a < 0 \end{cases}
\]
for some generator \( e' \in \text{Ext}^2_A(A/I_{-a}, A) \).

We will use this proposition (in the proof of Theorem 6.4) only in the following cases.
- \( [N/2] - r > 2^{n-1} \) and \( j = 1 \). In this case \( a \geq 0 \).
- \( [N/2] - r = 2^{n-1}(2j - 1) \) and \( r > 0 \). In this case \( a = -1 \).

**Proof.** Let \( \alpha \in \text{Ext}^2_A(A/I_j, A) \) be the element specified below, and consider the \( W_n \)-valued class \( e = [(\alpha, 0, \ldots, 0)] \). Then we have \( e \in \text{Ext}^2_A(A/I_j, W_n(A)) \); this is clear if \( n = 1 \), and if \( n \geq 2 \) this follows from Lemma 5.6 since we have \( F(R(e)) = 0 \) from the description of \( F(e) \) given below.

Let \( m = \lfloor N/2 \rfloor \). We may assume \( A \cong k[[x, y, z]]/(f), \)
\[
f = \begin{cases} z^2 + x^2y + xy^m + zxy^{m-r} & (D_{2m}^r), \\
z^2 + x^2y + zy^m + zxy^{m-r} & (D_{2m+1}^r), 
\end{cases}
\]
and then we have \( I_j = (x^j, y^j, z) \). Let
\[
\lambda := y^{m-r-j}, \quad \alpha := \frac{z}{xy^j}, \quad \eta := \frac{1}{y^{2j-r}}, \\
\xi := \frac{y^{m-2j}}{x}(D_{2m}^r), \quad \xi := \frac{y^{m-2j}z}{x^2}(D_{2m+1}^r),
\]

be the elements of $A[(xy)^{-1}]$. Then we have $\alpha^2 + \eta + \xi + \lambda \alpha = 0$.

As in Lemma 2.2(1), define polynomials $S_i \in k[t_1, t_2]$ by

$$(t_1 + t_2, 0, \ldots) - (t_2, 0, \ldots) = (S_0(t_1, t_2), S_1(t_1, t_2), \ldots),$$

and let $Q_i := S_i(\xi + \lambda \alpha, \eta)$ ($0 \leq i \leq n - 1$). We claim that

$$Q_i \equiv \begin{cases} 0 & (\text{if } i < n - 1), \\ \eta \cdot 2^{i-1} \cdot \lambda \alpha & (\text{if } i = n - 1) \end{cases} \pmod{A[x^{-1}]}.$$

By Lemma 2.2(1), $Q_i$ is a linear combination of monomials $\xi^{i_1}(\lambda \alpha)^{i_2}\eta^{i_3}$ with $i_1 + i_2 + i_3 = 2^{i-1}$ and $(i_1, i_2, i_3) \neq (0, 0, 2^{i-1})$. Let $c(i_1, i_2, i_3) := (m - 2j)i_1 + (m - r - 2j)i_2 + (-(2j - 1))i_3$, so that $\xi^{i_1}(\lambda \alpha)^{i_2}\eta^{i_3} \in y^{c(i_1, i_2, i_3)}A[x^{-1}]$. We shall show that $c(i_1, i_2, i_3) \geq 0$ for all such $(i_1, i_2, i_3)$ except $(0, 1, 2^{n-1} - 1)$.

- If $i_1 \geq 1$, then
  $$c(i_1, i_2, i_3) \geq c(1, 0, 2^i - 1) = m - 2j - (2^{i-1} - 1)(2j - 1) = m - C_1(i, j) \geq m - C_1(n, j),$$
  which is $\geq 0$ by assumption.

- If $i_2 \geq 1$ and $i < n - 1$, then
  $$c(i_1, i_2, i_3) \geq c(0, 1, 2^i - 1) = m - r - j - (2^i - 1)(2j - 1) - j \geq m - r - j - (2^i - 1)(2j - 1) - (2^{n-1} - 2^i)(2j - 1) = m - r - C_2(n, j),$$
  which is $\geq 0$ by assumption.

- If $i_2 \geq 2$ and $i = n - 1 \geq 1$, then
  $$c(i_1, i_2, i_3) \geq c(0, 2, 2^{n-1} - 2) = 2c(0, 1, 2^{n-2} - 1),$$
  which is $\geq 0$ by the previous case.

For the remaining term $\eta^{2^{n-1} - 1} \lambda \alpha$, which appears in $Q_{n-1}$ with coefficient 1 by Lemma 2.2(1), we have, letting $a$ be as in the statement,

$$\eta^{2^{n-1} - 1} \lambda = y^{m - r - j - (2^{n-1} - 1)(2j - 1)} = y^{m - r - C_2(n, j)} = y^a + j,$$

Hence $\eta^{2^{n-1} - 1} \lambda \alpha = x^{-1}y^a z$. Therefore we have

$$F(\alpha, 0, \ldots, 0) = (\alpha^2, 0, \ldots, 0) = (\xi + \lambda \alpha + \eta, 0, \ldots, 0) \equiv (\xi + \lambda \alpha + \eta, 0, \ldots, 0) - (\eta, 0, \ldots, 0) \pmod{W_n(A[y^{-1}])} = (Q_0, \ldots, Q_{n-1}) \equiv (0, \ldots, 0, x^{-1}y^a z) \pmod{W_n(A[x^{-1}])}.$$

If $a \geq 0$ then $x^{-1}y^a z \in A[x^{-1}]$, and if $a < 0$ then $x^{-1}y^a z$ is a generator of $\text{Ext}^2_A(A/I_{-a}, A)$. □

**Proposition 4.4.** Let $A$ be an RDP of type $E^4_2$ in characteristic $p = 2$. Then there exists an element $e \in \text{Ext}^2_A(A/I_2, A)$, where $I_2$ is defined as in Lemma 2.2, such that $F(e) = t^*(e')$ for a generator $e' \in \text{Ext}^2_A(A/m, A)$. 
Proof. We may assume $A = k[[x, y, z]]/(z^2 + x^3 + y^5 + zxy^3)$. Then $I_2 = (x, y^2, z)$. Let $e = [\alpha]$ with $\alpha := x^{-1}y^{-2}z$. Since $\alpha^2 = y^{-4}x + x^{-2}y + y\alpha \equiv y\alpha \pmod{A[x^{-1}] + A[y^{-1}]}$, we have $F(e) = e'(e')$ for the generator $e' = [y\alpha] \in \text{Ext}_A^3(A/m, A)$. 

Proposition 4.5. Suppose a prime $p$, a Dynkin diagram $S$, and a positive integer $n$ satisfy one of the following.

\begin{itemize}
  \item[(1)] $p = 2$, $S = E_7, E_8$, $n \leq 3$.
  \item[(2)] $p = 3$, $S = E_8$, $n \leq 2$.
  \item[(3)] $p = 2$, $S = E_6$, $n = 1$.
  \item[(4)] $p = 3$, $S = E_6, E_7$, $n = 1$.
  \item[(5)] $p = 5$, $S = E_8$, $n = 1$.
\end{itemize}

Let $A$ be an RDP in characteristic $p$ of type $S^r$, $0 \leq r \leq r_{\text{max}}(p, S) + 1 - n$. Then there is an element $e \in \text{Ext}_A^3(A/m, W_n(A))$ whose restriction $R^{n-1}(e)$ is a generator of $\text{Ext}_A^3(A/m, A)$ and satisfying

$$F(e) = \begin{cases} 0 & (if \ r < r_{\text{max}}(p, S) + 1 - n), \\ V^{n-1}(e') & (if \ r = r_{\text{max}}(p, S) + 1 - n) \end{cases}$$

for some generator $e' \in \text{Ext}_A^2(A/m, A)$.

Proof. As in the beginning of the proof of Proposition 4.3 it suffices to consider the Frobenius image of the $W_n$-valued class $e = [(\alpha, 0, \ldots, 0)]$, $\alpha = x^{-1}y^{-1}z$.

Case (4): $E_7^r$ (resp. $E_8^r$) in characteristic 2. We may assume $A = k[[x, y, z]]/(f)$, $f = z^2 + x^3 + xy^3 + \varepsilon$ (resp. $f = z^2 + x^3 + y^3 + \varepsilon$), with $\varepsilon$ as in Remark 4.1 (Table I). Let

$$\alpha := \frac{z}{xy}, \ \xi := \frac{y}{x} \ (\text{resp. } \xi' := \frac{y^3}{x^2}), \ \eta := \frac{x}{y^2}, \ \omega := \frac{\varepsilon}{x^2y^2},$$

so we have $\alpha^2 + \eta + \xi + \omega = 0$. We have

\begin{align*}
  \omega & = \alpha \frac{yz}{x^2}, \frac{z}{y} = 0 \quad (\text{if } A \text{ is } E_7^r), \ r = 3, 2, 1, 0 \\
  \ (\text{resp. } \omega & = \alpha \frac{yz}{x^2}, \frac{z}{x}, \frac{y^3}{x^2}, 0 \quad (\text{if } A \text{ is } E_8^r), r = 4, 3, 2, 1, 0). 
\end{align*}

Suppose $n = 1$. Then $F(\alpha) = \alpha^2 = \xi + \eta + \omega$, all monomials of which belong to $A[x^{-1}] \cup A[y^{-1}]$ except precisely for $\omega$ in the case $r = r_{\text{max}}$, which is equal to $\alpha$.

Suppose $n = 2$ and $r \leq r_{\text{max}} - 1$. We compute

\begin{align*}
  F(\alpha, 0) = (\alpha^2, 0) = (\xi + \eta + \omega, 0) & \equiv (\xi + \eta + \omega) - (\xi, 0) - (\eta, 0) \pmod{W_2(A[x^{-1}]) + W_2(A[y^{-1}])} \\
  & = (\omega, \eta + \xi \omega + \eta \omega).
\end{align*}

Then the 0-th component belongs to $A[x^{-1}]$ or $A[y^{-1}]$, and all monomials in the 1-st component belong to $A[x^{-1}] \cup A[y^{-1}]$ except precisely for $\eta \omega$ in the case $r = r_{\text{max}} - 1$, which is equal to $\alpha$. 

\[Q.E.D.\]
Suppose $n = 3$ and $r \leq r_{\text{max}} - 2$. We compute, by using Lemma 2.2(2) with $(a, b, c) = (\eta, \xi, \omega)$ (resp. $(a, b, c) = (\xi, \eta, \omega)$),

$$F((\alpha, 0, 0)) = (\alpha^2, 0, 0) = (\eta + \xi + \omega, 0, 0)$$

$$\equiv (\eta + \xi + \omega, 0, 0) - (\xi, \xi \omega, 0) \pmod{W_3(A[x^{-1}])}$$

$$= (\eta + \omega, \eta \xi, (\eta + \omega)^3 \xi + (\eta + \omega)\xi^3 + (\eta^2 + \eta \omega + \omega^2)\xi^2)$$

(resp. $F((\alpha, 0, 0)) = (\alpha^2, 0, 0) = (\xi + \eta + \omega, 0, 0)$

$$\equiv (\xi + \eta + \omega, 0, 0) - (\eta, \eta \omega, 0) \pmod{W_3(A[y^{-1}])}$$

$$= (\xi + \omega, \xi \eta, (\xi + \omega)^3 \eta + (\xi + \omega)\eta^3 + (\xi^2 + \xi \omega + \omega^2)\eta^2).$$

Then the 0-th and 1-st component belong to $A[y^{-1}]$ (resp. $A[x^{-1}]$), and all monomials in the expansion of the 2-nd component belong to $A[x^{-1}] + A[y^{-1}]$ except precisely for $\eta \omega \xi^2$ (resp. $\xi \omega \eta^2$) in the case $r = r_{\text{max}} - 2$, which is equal to $\alpha$.

Case (2): $E_8^*_{\text{L}}$ in characteristic 3. We may assume $A = k[[x, y, z]]/(f)$, $f = -z^2 + x^3 + y^5 + \lambda x^2 y^2$, where $\lambda = 1, y, 0$ for $r = 2, 1, 0$ respectively. Let $\alpha := \frac{x}{y}, \eta := \frac{y}{z}, \xi := \frac{z}{x}$, so we have $\alpha^2 = \xi + \eta + \lambda$.

Suppose $n = 1$. Then $F(\alpha) = \alpha^3 = \xi \alpha + \eta \alpha + \lambda \alpha$, all monomials of which belong to $A[x^{-1}] \cup A[y^{-1}]$ except precisely for $\lambda \alpha$ in the case $r = r_{\text{max}}$, which is equal to $\alpha$.

Suppose $n = 2$ and $r \leq r_{\text{max}} - 1$. We compute, by using Lemma 2.2(2),

$$F(\alpha, 0) = (\alpha^3, 0) = (\xi + \lambda)\alpha + \eta \alpha, 0)$$

$$\equiv (\xi + \lambda)\alpha + \eta \alpha, 0) - ((\xi + \lambda)\alpha, 0) - (\eta, 0)\pmod{W_2(A[x^{-1}]) + W_2(A[y^{-1}])}$$

$$= (0, \eta \alpha \cdot (\xi + \lambda)\alpha \cdot (\eta + \xi + \lambda)\alpha)$$

$$= (0, \alpha \eta (\xi + \lambda)(\eta + \xi + \lambda)^2).$$

Write $\lambda = by$, where $b = 1, 0$ for $r = 1, 0$. For the 1-st component, we have

$$\alpha \eta (\xi + \lambda)(\eta + \xi + \lambda)^2 = \frac{z}{x^3 y^2} \cdot (y^3 + bx^2 y) \cdot (x^3 + y^5 + bx^2 y^3)^2$$

$$= \frac{z}{x^3 y^2} (y^2 + bx^2)(2x^3 y^5 + 2bx^5 y^3 + \ldots)$$

$$= \frac{z}{x^3 y^2} (4bx^5 y^5 + \ldots)$$

$$\equiv b \alpha \pmod{A[x^{-1}] + A[y^{-1}]},$$

where $(\ldots) \in (x^6, y^6) \subset k[[x, y]]$.

The remaining cases: We may assume $A = k[[x, y, z]]/(f)$ with

$$f = \begin{cases} z^2 + x^3 + y^2 z + bxyz & (p, S) = (2, E_6), \\
-z^2 + x^3 + y^4 + bx^2 y^2 & (p, S) = (3, E_6), \\
-z^2 + x^3 + xy^3 + bx^2 y^2 & (p, S) = (3, E_7), \\
z^2 + x^3 + y^5 + (b/2) x y^4 & (p, S) = (5, E_8), \end{cases}$$

for some $b \in k$ with $b = 0$ if $r = 0$ and $b \neq 0$ if $r = 1$. Let $\alpha := x^{-1} y^{-1} z$. It suffices to show that $\alpha^6 = b \alpha$ for some $\eta \in A[y^{-1}]$ and $\xi \in A[x^{-1}]$. 


We take

\[
\eta := \begin{cases}
  y^{-2}x, & \\
y^{-3}z, & \\
y^{-3}z, & \\
y^{-5}xz, &
\end{cases} \quad \xi := \begin{cases}
x^{-2}z, & \\
x^{-3}yz, & \\
x^{-2}z, & \\
x^{-5}(y^{5} + bxy^{4} + (b^{2}/4)x^{2}y^{3} + 2x^{3})z.
\end{cases}
\]

\[\square\]

**Convention 4.6.** We abuse the notation and say that an RDP of type \( S \) is of type \( S^{0} \) if \( r_{\text{max}}(p, S) = 0 \), so that the coindex \( r \) of an RDP is always defined.

Finally we note the following relation between RDPs connected by partial resolutions (although we do not need it in this paper). If \( Z \) is an RDP surface with an RDP \( z \) of type \( S \) and \( S' \subset S \) is a subdiagram, then the minimal resolution \( \rho: \tilde{Z} \to Z \) of \( Z \) at \( z \) factors through the contraction \( \rho': \tilde{Z} \to Z_{S'} \) of \( S' \subset S = \text{Exc}(\rho) \), and \( Z_{S'} \) is an RDP surface. We say that \( Z_{S'} \to Z \) is the partial resolution corresponding to \( S' \to S \). If \( S' \) is connected and non-empty, then \( Z_{S'} \) has a single RDP above \( z \), which is of type \( S' \).

**Lemma 4.7.** Let \( S' \subset S \) be a non-empty connected subdiagram of a Dynkin diagram \( S \). Let \( Z \) be an RDP of type \( S' \) and \( Z_{S'} \to Z \) be the partial resolution corresponding to \( S' \to S \). Suppose \( z' \) is of type \( S'^{r'} \). Then we have \( r' = (r - (r_{\text{max}}(S) - r_{\text{max}}(S')))_{+} \).

Here \((q)_{+} := \max\{0, q\}\) is the positive part of \( q \in \mathbb{R} \). In other words, we have \( r_{\text{max}}(S') - r' = r_{\text{max}}(S) - r \) if this equality is achieved by a non-negative integer \( r' \), and \( r' = 0 \) otherwise.

**Proof.** We may assume that the number of components of \( S' \) is one less than that of \( S \). If \( r_{\text{max}}(S') = 0 \) then the assertion is trivial. So we may assume \( r_{\text{max}}(S') > 0 \).

If \((S, S')\) is \((E_{8}, E_{7})\) or \((E_{7}, D_{6})\), then the partial resolution is the blow-up at the closed point. In the other cases, the partial resolution is the blow-up at the ideals \((x, y^{2}, z)\), \((y, z)\), or \((x, z)\), as displayed in Table 2 with respect to the equations given in Table 1. One can check that this blow-up is dominated by the thrice blow-up \( Z_{3} \), where \( Z_{0} := Z \) and \( Z_{i+1} := \text{Bl}_{\text{Sing}(Z_{i})} Z_{i} \), hence it is indeed a partial resolution. A straightforward computation proves the assertion in each case. \(\square\)

4.3. \( \mu_{p^{n}} \) and \( \alpha_{p^{n}} \)-quotient morphisms.

**Proposition 4.8.** Suppose a prime \( p \), a group scheme \( G \), a Dynkin diagram \( S \), and a positive integer \( n \) satisfy one of the following.

1. \( p \) is arbitrary, \( G = \mu_{p} \), \( S = A_{p-1} \), \( n = 1 \).
2. \( p = 2 \), \( G = \alpha_{p} \), \( S = D_{2n} \), \( n \geq 2 \).
3. \( p = 2 \), \( G = \alpha_{p} \), \( S = E_{8} \), \( n = 4 \).
4. \( p = 3 \), \( G = \alpha_{p} \), \( S = E_{6} \), \( n = 2 \).
5. \( p = 5 \), \( G = \alpha_{p} \), \( S = E_{8} \), \( n = 2 \).

Let \( \pi: \text{Spec} B \to \text{Spec} A \) be a \( G \)-quotient map between a smooth point \( B \) and an RDP \( A \) of type \( S^{0} \) in characteristic \( p \), with \( \text{Fix}(G) = \{m_{B}\} \). Then there is an element \( e \in \text{Ext}^{2}_{A}(A/m_{A}, W_{n}(A)) \) whose restriction \( R^{n-1}(e) \) is
a generator of $\text{Ext}^2_A(A/m_A, A)$ and satisfying $\pi^*(e) = V^{n-1}(i^*(e'))$ for a generator $e' \in \text{Ext}^2_B(B/m_B, B)$.

Proof. As in the beginning of the proof of Proposition 4.3 it suffices to consider the image of $[(\alpha, 0, \ldots, 0)]$ (the assumption $F(R(e)) = 0$ follows from Propositions 4.3 and 4.5).

By applying [Mat19] Theorem 3.3(1) to $\text{Spec } A \to \text{Spec } B^{(p)}$, we may assume $A = k[[x_1, x_2, x_3]]/((f)$ and $B = k[[x_1^{1/p}, x_2^{1/p}]]$ with $f \in k[[x_1, x_2, x_3^p]]$ as given below.

Case (1): $A_{p-1}$ in characteristic $p$. We may assume $B = k[[X, Y]]$ and $A = k[[x, y, z]]/(z^p-xy)$ with $x = X^p$, $y = Y^p$, $z = XY$. Then $[x^{-1}y^{-1}z^{p-1}]$ and $\pi^*([x^{-1}y^{-1}z^{p-1}]) = [X^{-1}Y^{-1}]$ are clearly generators of $\text{Ext}^2_A(A/m_A, A)$ and $\text{Ext}^2_B(B/m_B, B)$ respectively.

Case (2): $D_2^0$ in characteristic 2. We may assume $B = k[[X, Y]]$ and $A = k[[x, y, z]]/(z^2 + x^2y + xy^2)$ with $x = X^2$, $y = Y^2$, $z = X^2Y + XY^2$. Let $\alpha = x^{-1}y^{-1}z$. We compute, by using Lemma 2.2.11 as in the proof of Proposition 4.3:

$$
\pi^*(\alpha, 0, \ldots) = \left(\frac{X^2Y + XY^2}{X^3Y^2}, 0, \ldots\right) = \left(\frac{Y^{2n-1}}{X}, 0, \ldots\right) \\
\equiv \left(\frac{Y^{2n-1}}{X}, 0, \ldots\right) - \left(\frac{1}{Y}, 0, \ldots\right) \\
= \left(\xi_0, \ldots, \xi_{n-2}, \xi_{n-1} + \frac{Y^{2n-1} - 2}{X} \left(\frac{1}{Y}\right)^{2n-1} - 1\right) \\
\equiv V^{n-1}\left(\frac{1}{XY}\right) \pmod{W_n(B[x^{-1}]) + W_n(B[y^{-1}])},
$$

for some $\xi_i \in B[x^{-1}]$. Clearly $[X^{-1}Y^{-1}]$ is a generator of $\text{Ext}^2_B(B/m_B, B)$.

Case (3): $E_0^3$ in characteristic 2. We may assume $B = k[[X, Y]]$ and $A = k[[x, y, z]]/(z^2 + x^3 + y^3)$ with $x = X^2$, $y = Y^2$, $z = X^3 + Y^3$. Let
\( \alpha = x^{-1}y^{-1}z \). We compute, by using Lemma 2.2(3),

\[
\pi^*(\alpha,0,0,0) = \left( \frac{X^3 + Y^5}{X^2Y^2}, 0, 0, 0 \right) = \left( \frac{X}{Y^2} + \frac{Y^3}{X^2}, 0, 0, 0 \right)
\]

\[
\equiv \left( \frac{X}{Y^2} + \frac{Y^3}{X^2}, 0, 0, 0 \right) - \left( \frac{Y^3}{X^2}, 0, 0, 0 \right)
\]

\[
= \left( 0, \frac{Y^7 + Y^2X^3}{X^5}, \frac{X}{Y^3}, \frac{X^{19} + Y^{14}X^3 + Y^4X^9}{X^{13}} + \frac{X^2Y^5 + X^5}{Y^{11}} \right)
\]

\[
= \left( 0, \xi_1, \frac{Y^7 + Y^2X^3}{X^5} + \frac{X}{Y^3}, \xi_3 + \eta_3 \right)
\]

\[
\equiv \left( 0, \xi_1, \frac{Y^7 + Y^2X^3}{X^5} + \frac{X}{Y^3}, \xi_3 + \eta_3 \right) - \left( 0, 0, \frac{X}{Y^3}, 0 \right)
\]

\[
= \left( 0, \xi_1, \overline{\xi}_2, \xi_3 + \eta_3 + \frac{Y^7 + Y^2X^3}{X^5} \right)
\]

\[
= V^3 \left( \frac{1}{XY} \right) \pmod{W_4(B[x^{-1}]) + W_4(B[y^{-1}])},
\]

where \( \xi_i \in B[x^{-1}] \) and \( \eta_i \in B[y^{-1}] \).

Case (1): \( E_6^0 \) in characteristic 3. We may assume \( B = k[[Y,Z]] \) and \( A = k[[x,y,z]]/(z^2 + x^3 + y^4) \) with \( x = Z^2 - Y^4, y = Y^3, z = Z^3 \). We interpret the Ext groups using the regular sequence \( x, y \). Let \( \alpha = x^{-1}y^{-1}z \).

We compute, by using Lemma 2.2(4) and the equality \( Z^2 = x + Y^4 \),

\[
\pi^*(\alpha,0) = \left( \frac{Z^3}{xY^3}, 0 \right) = \left( \frac{Z}{Y^3} + \frac{ZY}{x}, 0 \right)
\]

\[
\equiv \left( \frac{Z}{Y^3} + \frac{ZY}{x}, 0 \right) - \left( \frac{Z}{Y^3}, 0 \right) - \left( \frac{Z}{x}, 0 \right)
\]

\[
= \left( 0, \frac{Z}{Y^3} \frac{Z^3}{xY^3} \right) = \left( 0, \frac{Z(x + Y^4)^2}{x^2Y^5} \right) = \left( 0, \frac{Z \cdot (-xY^4 + \ldots)}{x^2Y^5} \right)
\]

\[
\equiv \left( 0, -\frac{Z}{xy} \right) \pmod{W_2(B[x^{-1}]) + W_2(B[y^{-1}])}
\]

\[
= V(e'),
\]

where \((\ldots) \in (x^2, Y^6) \subset k[[x,y] \] and \( e' := [\beta], \beta = -x^{-1}Y^{-1}Z \). This \( e' \) is a generator of \( \text{Ext}^2(B/m_B, B) \) since \( \text{Ann}([\beta]) = (Y,Z) = m_B \).

Case (3): \( E_6^0 \) in characteristic 5. We may assume \( B = k[[X,Z]] \) and \( A = k[[x,y,z]]/(z^2 + x^3 - y^5) \) with \( x = X^3, y = Z^2 + X^3, z = Z^3 \). We interpret the Ext groups using the regular sequence \( x, y \). Let \( \alpha = x^{-1}y^{-1}z \).
We compute, by using Lemma 2.2(5) and the equality $Z^2 = y - X^3$,

$$\pi^*(\alpha, 0) = \left( \frac{Z^5}{X^3Y}, 0 \right) = \left( \frac{Z(y - 2X^3)}{X^5} + \frac{ZX}{y}, 0 \right)$$

$$\equiv \left( \frac{Z(y - 2X^3)}{X^5} + \frac{ZX}{y}, 0 \right) - \left( \frac{Z(y - 2X^3)}{X^5}, 0 \right) - \left( \frac{ZX}{y}, 0 \right)$$

$$= \left( \frac{Z}{X^5}, \frac{Z(y - 2X^3)}{X^5} \cdot Z \cdot X^2(y^2(y - 2X^3)^2 + X^6y(y - 2X^3) + X^{12}) \right)$$

$$= \cdots = \left( 0, \frac{Z}{X^5} \cdot (-X^{18}y^3 + \ldots) \right)$$

$$\equiv \left( 0, -\frac{Z}{X^5} \right) \pmod {W_2(B[x^{-1}]) + W_2(B[y^{-1}])}$$

$$= V(e'),$$

where $(\ldots) \in (X^{21}y^2, X^{15}y^4) \subset k[[X^3, y]]$ and $e' := [\beta], \beta = -X^{-1}y^{-1}Z$. This $e'$ is a generator of $\text{Ext}^2_B(B/m_B, B)$ since $\text{Ann}([\beta]) = (X, Z) = m_B$. \hfill $\square$

5. The height of K3 surfaces

In this section we recall the definition and properties of the height of K3 surfaces.

**Theorem 5.1** (Artin–Mazur [AM77, Corollary II.4.2]). Let $Y$ be a (smooth) K3 surface. The functor $\Phi^2: \{\text{local Artinian } k\text{-algebras}\} \to \{\text{abelian groups}\}$ defined by $S \mapsto \text{Ker}(H^2_{\text{ét}}(Y \times S, \mathbb{G}_m) \to H^2_{\text{ét}}(Y, \mathbb{G}_m))$ is pro-represented by a 1-dimensional formal group.

**Definition 5.2.** This formal group is called the *formal Brauer group* of $Y$ and written $\text{Br}(Y)$. Its height is called the *Artin–Mazur height* of $Y$ and written $\text{ht}(Y)$.

Here a 1-dimensional commutative formal group is said to be of height $h \in \mathbb{Z}_{>0}$ if $[p](t) = ct^h + \ldots$ for some $c \in k^*$, and of height $\infty$ if $[p](t) = 0$, where $t$ is a uniformizer and $[p]$ is the multiplication-by-$p$ map. It follows from Proposition 5.7 that $\text{ht}(Y) \in \{1, 2, \ldots, 10\} \cup \{\infty\}$.

As before, an RDP K3 surface is a proper surface with only RDP singularities whose minimal resolution is a K3 surface in the usual sense. We define the height of an RDP K3 surface to be the height of its resolution.

To relate the height of a K3 surface with the properties of non-taut RDPs, we need the following characterization of the height.

**Theorem 5.3** (van der Geer–Katsura [vdGK00, Theorem 5.1]). Let $Y$ be an RDP K3 surface in positive characteristic and $n \geq 1$ an integer. Then $\text{ht}(Y) \leq n$ if and only if the Frobenius map on $H^2(Y, W_n(\mathcal{O}_Y))$ is nonzero.

**Proof.** In [vdGK00] this is stated for smooth K3 surfaces. The case of RDP K3 surfaces is reduced to the smooth case by using the isomorphism $H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(Y, W_n(\mathcal{O}_Y))$. \hfill $\square$

We also recall several properties that can be used to determine or bound the height of (RDP) K3 surfaces.
Proof. Let \( \omega_\mathbb{P} \) and \( \omega_X \) be the canonical sheaves. Since \( \mathcal{O}_X \otimes \mathcal{O}(\deg(f)) = \mathcal{O}_X(\sum n_i + \deg(f)) \), we have \( \deg(f) = \sum n_i \). This proves the first assertion.

The proof of the second assertion is standard and applicable to hypersurface Calabi–Yau varieties of arbitrary dimension, see for example [Hart77, Proposition IV.4.21] for the 1-dimensional case. We include the proof for the reader’s convenience. Let \( d = \deg(f) \). We have canonical isomorphisms

\[
H^2(X, \mathcal{O}_X) \xrightarrow{\sim} H^3(\mathbb{P}, f\mathcal{O}(d))
\]

where \( U_i \) (\( i \in I := \{0, 1, 2, 3\} \)) is the standard affine covering of \( \mathbb{P} \) and \( U_J := \bigcap_{i \in J} U_i \) for \( J \subset I \). The cokernel is 1-dimensional, generated by the class of \( \frac{x_0 x_1 x_2 x_3}{x_0 x_1 x_2 x_3} \). The Frobenius image of this class is the class of \( \frac{f^{p-1}}{(x_0 x_1 x_2 x_3)^p} \in \Gamma(U_1, f\mathcal{O}(d)) \), which is nontrivial if and only if the coefficient of \( (x_0 x_1 x_2 x_3)^{p-1} \) in \( f^{p-1} \) is nonzero. Apply Theorem 5.3.

Remark 5.5. In principle, it is possible to compute the Frobenius map on \( H^2(X, \mathcal{O}_X) \) in terms of \( f \).

Theorem 5.6. Let \( Y \) be a (smooth) K3 surface. Consider the crystalline cohomology group \( H^2_{\text{crys}}(Y/W(k)) \), which is an \( F \)-crystal. If \( \text{ht}(Y) < \infty \), then \( H^2_{\text{crys}}(Y/W(k)) \) has slopes \( 1 - 1/h, 1, \) and \( 1 + 1/h \), with respective multiplicity \( h, 22 - 2h, \) and \( h \). If \( \text{ht}(Y) = \infty \), then it has slope 1 with multiplicity 22.

Proof. By [AM77, Corollary II.4.3], the Dieudonné module of \( \widehat{\text{Br}}(Y) \) is isomorphic to \( H^2(Y, W\mathcal{O}_Y) \). The slope spectral sequence induces an isomorphism \( H^2(Y, W\mathcal{O}_Y) \otimes_{W(k)} K_0 = H^2_{\text{crys}}(Y/W(k)) \otimes_{W(k)} K_0 \), where \( K_0 := \text{Frac}(W(k)) \) and \( -1 \) denotes the slope < 1 part of an \( F \)-crystal. The assertion follows from this (see [Ill79, Section II.7.2]).

Proposition 5.7 ([Ill79, Proposition II.5.12]). Suppose \( Y \) is a (smooth) K3 surface of height \( h \) with Picard number \( \rho = \rho(Y) := \text{rank} \text{Pic}(Y) \). If \( h < \infty \), then \( \rho \leq 22 - 2h \), and if \( h = \infty \), then \( \rho \leq 22 \).

Suppose \( Y \) is an RDP K3 surface of height \( h \) with RDPs \( z_i \) of type \( A_{N_i} \), \( D_{N_i} \), or \( E_{N_i} \). If \( h < \infty \), then \( \sum N_i < 22 - 2h \), and if \( h = \infty \), then \( \sum N_i < 22 \).

Proof. Suppose \( Y \) is smooth. The subspace of \( H^2_{\text{crys}}(Y/W(k)) \) generated by the Picard group is of slope 1 with multiplicity \( \rho \), which should be at most \( 22 - 2h \) (resp. 22) if \( h < \infty \) (resp. \( h = \infty \)) by Theorem 5.6.

Suppose \( Y \) is an RDP K3 surface. Then the exceptional curves on the resolution \( Y \) generate a negative-definite sublattice of \( \text{Pic}(Y) \) of rank \( \sum N_i \). Since \( \text{Pic}(Y) \) is of sign \( (+1, -(\rho - 1)) \), we have \( \sum N_i < \rho \).
Proposition 5.8. Suppose $Y$ is an RDP K3 surface defined over a finite field $\mathbb{F}_q$. Define $a(m) \in \mathbb{Q}$ by $|Y(\mathbb{F}_{q^m})| = 1 + (q^m)^2 + a(m)q^m$. Let $s(j)$ be the $j$-th elementary symmetric polynomial of the indeterminates $x_1, x_2, \ldots$ satisfying $\sum x_i^m = a(m)$. Then $\text{ht}(Y) > n$ if and only if $s(1), \ldots, s(n) \in \mathbb{Z}$.

Note that $s(j)$ is expressed as a polynomial of $a(1), \ldots, a(j)$ with coefficients in $\mathbb{Q}$.

Proof. Comparing $Y(\mathbb{F}_{q^m})$ and $\tilde{Y}(\mathbb{F}_{q^m})$, the assertion is reduced to the case $Y$ is smooth.

By Theorem 5.6, it suffices to know the slopes of $H^2_{\text{crys}}(Y/W(k)) \cong H^2_{\text{crys}}(Y/W(\mathbb{F}_q)) \otimes W(\mathbb{F}_q) W(k)$. Write $q = p^b$. Then $F^b \in \text{End}(H^2_{\text{crys}}(Y/W(\mathbb{F}_q)))$ is linear (not only semilinear) and its eigenvalues coincide with the Frobenius eigenvalues on $H^2_{\text{crys}}(Y, \mathbb{Q}_l)$ (\cite{Huy16} 3.7.3), which are encoded in the sequence $a(m)$ by the Weil conjecture (more precisely, the Lefschetz trace formula). Writing down the correspondence explicitly, we obtain the assertion. \qed

Theorem 5.9 ([Ito \cite{Ito18} Theorem 1.1]). Let $X$ be a K3 surface in characteristic 0, having complex multiplication (CM) by a CM-field $E$, and defined over a number field $K$ containing $E$. Suppose $X$ has good reduction $X_v$ at a prime $v$ of $K$. Let $p$, $q$, and $p$ be respectively the primes of $E$, $F$, and $Q$ below $v$, where $F$ is the maximal totally-real subfield of $E$.

- If $q$ splits in $E$, then $X_v$ is of height $[E_p : \mathbb{Q}_p] < \infty$.
- If $q$ does not split (in other words, if it ramifies or is inert) in $E$, then $X_v$ is supersingular (i.e. of height $\infty$).

In this paper, we use this theorem only in the following situations.

- If $X$ has Picard number 20, then $X$ has CM by $E = \mathbb{Q}(\sqrt{-\text{disc}T(X)})$ (see \cite{Huy16} Remark 3.3.10), where $T(X) = \text{Pic}(X_{\overline{\mathbb{Q}}})^\perp \subset H^2(X_{\overline{\mathbb{Q}}}, \mathbb{Z})$ is the transcendental lattice. In this case $F = \mathbb{Q}$.

In this case Theorem 5.9 is proved by Shimada \cite{Shi09} Theorem 1 for all but finitely many $p$ not dividing $2\text{disc} \text{Pic}(X)$ (for each $X$). However we use Theorem 5.9 for $p = 2$.

- If $X$ admits an automorphism acting on $H^0(X, \Omega^2)$ by a primitive $m$-th root $\zeta_m$ of unity and if rank $T(X) = \phi(m)$, then $X$ has CM by the cyclotomic field $E = \mathbb{Q}(\zeta_m)$. In this case $F = \mathbb{Q}(\zeta_m + \overline{\zeta_m})$.

Jang \cite{Jan16} Corollary 4.3 proved the following related result. Suppose $Y$ is a K3 surface in characteristic $p > 2$ that admits an automorphism acting on $H^0(Y, \Omega^2)$ by a primitive $m$-th root of unity, and assume $22 - \phi(m) \leq \rho(Y)$. If $p^n \equiv -1 \pmod{m}$ for some $n$ then $Y$ is supersingular, and otherwise $\text{ht}(Y)$ is equal to the order of $p$ in $(\mathbb{Z}/m\mathbb{Z})^*$. However we use Theorem 5.9 for $p = 2$.

6. HEIGHT OF K3 SURFACES AND HEIGHT OF MORPHISMS BETWEEN RDP K3 SURFACES

6.1. Height of morphisms between RDP K3 surfaces. 

Definition 6.1. Let $\pi : X \to Y$ be a dominant morphism between RDP K3 surfaces. We define the height $\text{ht}(\pi)$ of $\pi$ to be the minimum positive integer $n$ such that the morphism $\pi^* : H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(X, W_n(\mathcal{O}_X))$ is nonzero if such $n$ exists, and to be $\infty$ if there is no such $n$. 
Clearly (Theorem 5.3), the height of the Frobenius map of an RDP K3 surface is equal to the height of the surface.

We have the following.

**Lemma 6.2.**

1. The image of \( \pi^* : H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(X, W_n(\mathcal{O}_X)) \) is equal to \( V^{ht(\pi)-1}(H^2(X, W_{n-ht(\pi)-1}(\mathcal{O}_X))) \) if \( n \geq ht(\pi) \) and to 0 if \( n < ht(\pi) \).

2. If \( \pi' : V \to X \) is another morphism with the same assumption, then \( ht(\pi) = ht(\pi') + ht(\pi') - 1 \).

3. If \( \pi \) is birational, then \( ht(\pi) = 1 \).

**Proof.** For each \( 0 \leq n' \leq n \), the exact sequence

\[
0 \to W_{n-n'}(\mathcal{O}_Y) \xrightarrow{\pi^*} W_n(\mathcal{O}_Y) \xrightarrow{R^{n-n'}} W_{n'}(\mathcal{O}_Y) \to 0
\]

induces an exact sequence

\[
0 \to H^2(W_{n-n'}(\mathcal{O}_Y)) \xrightarrow{\pi^*} H^2(W_n(\mathcal{O}_Y)) \xrightarrow{R^{n-n'}} H^2(W_{n'}(\mathcal{O}_Y)) \to 0
\]

since \( H^1(\mathcal{O}_Y) = 0 \), and this is compatible with \( \pi^* \).

\( \Box \) For \( n < ht(\pi) \), this follows from the definition of \( ht(\pi) \). Suppose \( n \geq ht(\pi) \). We have \( \text{Im}(\pi^*) \subset \text{Ker}(R^{n-h(\pi)-1}) = \text{Im}(V^{ht(\pi)-1}) \) in \( H^2(W_n(\mathcal{O}_X)) \).

Let us show \( \text{Im}(\pi^*) = \text{Im}(V^{ht(\pi)-1}) \) by induction on \( n \geq ht(\pi) \). If \( n = ht(\pi) \), then this is true since \( \text{Im}(\pi^*) \neq 0 \) and \( \text{length}(\text{Im}(V^{ht(\pi)-1})) = \text{length}(H^2(\mathcal{O}_X)) = 1 \). Suppose \( n > ht(\pi) \) and take \( x \in \text{Im}(V^{ht(\pi)-1}) \). By induction hypothesis, we have \( R(x) \in \text{Im}(V^{ht(\pi)-1}) = \text{Im}(\pi^*) = \text{Im}(\pi^* \circ R) \), hence we may assume \( R(x) = 0 \), and then \( x \in \text{Im}(\pi^*) \subset V(\text{Im}(V^{ht(\pi)-1})) = V(\text{Im}(\pi^*)) \subset \text{Im}(\pi^*) \).

\( \Box \) By (1), the image of \( \pi \circ \pi' = \pi'' \circ \pi^* : H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(V, W_n(\mathcal{O}_V)) \) is equal to 0 if \( n < (ht(\pi)-1)+(ht(\pi')-1) \) and to \( V^{ht(\pi)-1}(H^2(\mathcal{O}_X)) \) if \( n \geq (ht(\pi)-1)+(ht(\pi')-1) \).

\( \Box \) It suffices to consider the case where \( \pi \) is the minimal resolution \( \tilde{Y} \to Y \). Then since \( \pi \) has only rational singularities, we have \( H^i(\pi_* \mathcal{O}_{\tilde{Y}}) = 0 \) for \( i > 0 \), hence \( H^2(Y, \mathcal{O}_\tilde{Y}) \to H^2(Y, \mathcal{O}_Y) \) is an isomorphism. \( \square \)

In some cases we can bound \( ht(\pi) \) using local behaviors of \( \pi^* \).

Suppose \( Y \) is an RDP K3 surface in characteristic \( p \) and \( y \in Y \) is a closed point (either a smooth point or an RDP). Let \( I \subsetneq \mathcal{O}_{Y,y} \) be an \( m \)-primary ideal, let \( \mathcal{I} = \text{Ker}(\mathcal{O}_V \to \mathcal{O}_{V,y}/I) \subset \mathcal{O}_V \) (so \( \text{Supp} \mathcal{O}_V/I = \{ y \} \) and \( \mathcal{O}_V/I \cong \mathcal{O}_{Y,y}/I \) and let \( \mathcal{F} = (R^{n-1})^{-1}(I) \subset W_n(\mathcal{O}_V) \). Since \( \text{Supp} \mathcal{O}_V/I = \{ y \} \), we have \( \text{Ext}^2(\mathcal{O}_V/I, W_n(\mathcal{O}_V)) = \text{Ext}^2(\mathcal{O}/I, W_n(\mathcal{O})) \). Consider the map

\[
\gamma = \gamma_{I,n} : \text{Ext}^2(\mathcal{O}/I, W_n(\mathcal{O})) \cong \text{Ext}^2(\mathcal{O}_V/I, W_n(\mathcal{O})) \to \text{Ext}^0_{\mathcal{O}_{Y,y}/I}(\mathcal{O}/I, W_n(\mathcal{O})) = H^2(Y, W_n(\mathcal{O}))(\mathcal{O}/I, W_n(\mathcal{O})).
\]

By Proposition 5.4 this \( \gamma = \gamma_{I,n} \) commutes with the morphisms \( \iota_{I,i}^{*} \), \( V \), and \( f^* \) defined in Definition 3.3.

If \( n = 1 \) and \( I = m \) is the maximal ideal at \( y \), then \( \gamma_{m,1} \) is an isomorphism, since by Serre duality this map is the dual of \( H^0(Y, \mathcal{O}) \to H^0(Y, \mathcal{O}/m) \).

**Proposition 6.3.** Let \( \pi : X \to Y \) be as above. Let \( y \in Y \) be a point, \( \mathfrak{m}_y \subset \mathcal{O}_{Y,y} \) the maximal ideal, and \( I \subsetneq \mathcal{O}_{Y,y} \) an \( \mathfrak{m}_y \)-primary ideal. Let
In particular, in characteristic 2 (in Introduction. Define a subsequence of type $Y$).

Then we have the following. Suppose an RDP $K3$ surface and the height of the surface.

Then $\pi^*(e) = V^{n-1}(i^*(e'))$ for some generator $e' \in \text{Ext}^2_{O_{X,x}}(\mathcal{O}_{X,x}/\mathcal{O}_{X,x})$.

Then $\text{ht}(\pi) \leq n$.

$\text{Suppose } I = m_y \text{ and } R^{n-1}(e) \in \text{Ext}^2_{O_{Y,y}}(\mathcal{O}_{Y,y}/m_y, \mathcal{O}_{Y,y}) \text{ is a generator, and } \pi^*(e) = V^{n-1}(i^*(e'))$ for some generator $e' \in \text{Ext}^2_{O_{X,x}}(\mathcal{O}_{X,x}/m_x, \mathcal{O}_{X,x})$.

Then $\text{ht}(\pi) = n$.

Proof. [1] Applying $\gamma$ to $\pi^*(e) = V^{n-1}(i^*(e'))$, we obtain $\pi^*(\gamma(e)) = V^{n-1}(\gamma(e'))$.

As mentioned above, $\gamma_{m,1}$ is an isomorphism, hence $V^{n-1}(\gamma(e'))$ is nonzero. Hence $\text{ht}(\pi) \leq n$.

[2] By applying the assertion to $R(e)$ if $n > 1$, we obtain $\text{ht}(\pi) > n - 1$.

Since $\gamma_{m,1}$ is an isomorphism, $R^{n-1}(\gamma(e)) \in H^2(\mathcal{O}_Y)$ is a generator. Hence $H^2(W_n(\mathcal{O}_Y))$ is generated by $\gamma(e)$ and $V(H^2(W_n(\mathcal{O}_Y)))$, and $\pi^*$ annihilates both since $\pi^*(e) = 0$ and $\text{ht}(\pi) > n - 1$.

[3] Applying [2] to $R(e)$ if $n > 1$, we obtain $\text{ht}(\pi) > n - 1$. Applying [1] to $e$, we obtain $\text{ht}(\pi) \leq n$.

From this proposition we deduce bounds, or moreover exact values, of the height of RDP $K3$ surfaces with suitable singularities. It turns out, surprisingly, that any non-taut RDP is suitable.

6.2. Height of Frobenius maps and non-taut RDPs. Combining Proposition [23] with the computation on Frobenius maps on local RDPs given in Section [1,2], we prove the following relation between the isomorphism class of a non-taut RDP on an RDP $K3$ surface and the height of the surface. This is trivially true when $r_{\text{max}}(p, S) = 0$ (recall Convention [1,6]).

Theorem 6.4 (Precise form of Theorem [1,2]. Let $S$ be a Dynkin diagram and $p \geq 0$ be a characteristic. Let $r_{\text{max}} = r_{\text{max}}(p, S)$ be the integer defined in Introduction. Define a subsequence $(r_1, r_2, \ldots, r_l)$ of $(r_{\text{max}}(p, S), \ldots, 2, 1)$ as follows.

- $(p, S) = (2, D_N), N \geq 8 \ (r_{\text{max}} = [N/2] - 1)$:
  - If $8 \leq N \leq 9$: $(r_1, r_2) = ([N/2] - 1, [N/2] - 2)$.
  - If $10 \leq N$: $(r_1, r_2, r_3) = ([N/2] - 1, [N/2] - 2, [N/2] - 4)$.
- $(p, S) = (2, E_6) \ (r_{\text{max}} = 4): (r_1, r_2, r_3) = (4, 3, 2)$.
- all other cases: $(r_1, \ldots, r_l)$ is the whole sequence $(r_{\text{max}}(p, S), \ldots, 2, 1)$.

Then we have the following. Suppose an RDP $K3$ surface $Y$ admits an RDP of type $S^r$.

- If $r > 0$, then $\text{ht}(Y) \leq l$ and $r = r_{\text{ht}}(Y)$.
- If $r = 0$, then $\text{ht}(Y) > l$.

In particular, in characteristic 2, RDPs of type $D_N^r \ (r > 0$ and $[N/2] - r \notin \{1, 2, 4\})$ and $E_6^4$ do not occur.
Proof. Suppose $Y$ is an RDP K3 surface in characteristic $p$ having a non-taut RDP of type $S'$. If $(p, S') \neq (2, D^r_N), (2, E^r_8)$, then the assertion follows from Proposition 6.3 (3) if $r > 0$ and (2) if $r = 0$ applied to the elements $e$ given in Proposition 4.3.

Suppose $(p, S') = (2, D^r_N)$. By Proposition 6.3 (2) and (4) applied to the elements given in Propositions 1.3 and 4.4 respectively, we obtain $ht(Y) > 3$ and $ht(Y) = 1$. Contradiction.

Suppose $(p, S') = (2, D^r_N)$. We have $r_{\text{max}} + 1 = \lfloor N/2 \rfloor$. It suffices to show that

- the inequality $\lfloor N/2 \rfloor - r \leq 2^{ht(Y)-1}$ holds,
- this inequality is equality if $r > 0$, and
- $ht(Y) \leq 3$ if $r > 0$.

Let $n'$ be the (unique) non-negative integer satisfying $2^{n'-1} < \lfloor N/2 \rfloor - r \leq 2^{n'}$. By applying Proposition 6.3 (2) to the elements given in Proposition 1.3 for $(n, j) = (n', 1)$ (in which case $a \geq 0$), we obtain $ht(Y) \geq n' + 1$. Hence $\lfloor N/2 \rfloor - r \leq 2^{n'} - 2^{ht(Y)-1}$.

Suppose moreover $r > 0$. Let $(n, j)$ be the (unique) pair of positive integers with $\lfloor N/2 \rfloor - r = 2^{n-1}(2j - 1)$. By applying Proposition 6.3 (4) to the element given in Proposition 1.3 for this $(n, j)$ (in which case $a = -1$), we obtain $ht(Y) \leq n$. Hence we have $2^{ht(Y)-1} \leq 2^{n-1}(2j - 1) = \lfloor N/2 \rfloor - r$, therefore $\lfloor N/2 \rfloor - r = 2^{ht(Y)-1}$. Since $N < 22 - 2ht(Y)$ (Proposition 5.7) and $N \geq 2(\lfloor N/2 \rfloor - r) = 2^{ht(Y)}$, we have $ht(Y) \leq 3$.

6.3. Height of $\mu_p$ and $\alpha_p$-quotient morphisms. Suppose $X$ and $Y$ are RDP K3 surfaces and $\pi: X \to Y$ is a $G$-quotient morphism with $G \in \{\mu_p, \alpha_p\}$. The author proved [Mat19, Corollary 4.4] that the “dual” map $\pi' : Y^{(1/p)} \to X$ is also a $G'$-quotient morphism with $G' \in \{\mu_p, \alpha_p\}$.

Definition 6.5 ([Mat19, Definition 3.4]). We say that a $G$-quotient morphism $\pi: X \to Y$ between RDP K3 surfaces is maximal if there is no point $x \in X$ such that $x$ and $\pi(x)$ are both RDPs.

The author proved [Mat19, Corollary 3.5] that for any $G$-quotient morphism $\pi: X \to Y$ between RDP K3 surfaces there is a maximal $G$-quotient morphism $\pi_1: X_1 \to Y_1$ between RDP K3 surfaces with a birational and $G$-equivariant morphism $X_1 \to X$. Then $ht(\pi) = ht(\pi_1)$ by Lemma 6.2 [3].

Theorem 6.6 (Precise form of Theorem 1.3). Let $\pi: X \to Y$ be as above.

1. If $\pi$ is maximal (Definition 6.3), then we have

$$ht(\pi) = \begin{cases} 
1 & \text{if } G = \mu_p \text{ (in which case } p \leq 7 \text{ and } \text{Sing}(Y) = \frac{24}{p+1}A_{p-1}), \\
2 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 2D^0_9), (3, 2E^0_8), (5, 2E^0_8), \\
3 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 1D^0_9), \\
4 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 1E^0_8).
\end{cases}$$

This covers all possibilities for $G$, $p$, and $\text{Sing}(Y)$ in the maximal case ([Mat19, Theorem 4.7]).

2. We have $ht(X) = ht(Y) = ht(\pi) + ht(\pi') - 1$. In particular, $X$ and $Y$ are of finite height.
Proof. (1) Let \( y \in Y \) be a singular point. Since \( \pi \) is maximal, the inverse image \( \pi^{-1}(y) \) of \( y \) is smooth, hence \( \text{Spec} \mathcal{O}_{\pi^{-1}(y)} \to \text{Spec} \mathcal{O}_Y \) is as in Proposition 6.3. Hence we obtain \( \text{ht}(\pi) \) from Proposition 5.7.

(2) Since \( \text{Frob}_Y = \pi \circ \pi' \) and \( \text{Frob}_X = \pi' \circ \pi^{(1/p)} \), the first assertion follows from Lemma 6.2. For the second assertion, we may assume \( \pi \) is maximal, and then \( \pi' \) is also maximal, and we can apply (1) to \( \pi \) and \( \pi' \). \( \square \)

Corollary 6.7. Let \( \pi : X \rightarrow Y \) be as above.

If \( p = 5 \), then \( (G,G',\text{Sing}(X),\text{Sing}(Y)) \neq (\alpha_5,\alpha_5) \).

If \( p = 2 \) and \( \pi \) is maximal, then \( (G,G',\text{Sing}(X),\text{Sing}(Y)) \neq (\alpha_2,\alpha_2,1E_0^3,1E_0^3) \).

Proof. We may suppose \( \pi \) is maximal. (By above, this implies that if \( p = 5 \) and \( G = \alpha_5 \) then \( \text{Sing}(Y) = 2E_8^0 \).) Then the height of \( Y \) asserted in Theorem 6.6 which is 3 or 7 respectively, contradicts the latter inequality of Proposition 5.7. \( \square \)

All other \( (G,G',\text{Sing}(X),\text{Sing}(Y)) \) is realizable (see [Mat19, Examples 9.2–9.5]). Hence we have the following.

Corollary 6.8. Suppose an RDP K3 surface \( X \) in characteristic \( p \) admits an action of \( \mu_p \) or \( \alpha_p \) whose quotient is an RDP K3 surface. Then \( \text{ht}(X) \leq 6,3,2,1 \) for \( p = 2,3,5,7 \) respectively, and every such positive integer is realizable.

Furthermore we have the following criterion.

Corollary 6.9. Suppose \( X \) is an RDP K3 surface in characteristic \( p \) with a \( G \)-action, \( G \in \{ \mu_p, \alpha_p \} \). Then \( X/G \) is an RDP K3 surface if and only if \( \text{ht}(X) < \infty \), and \( X/G \) is either an RDP Enriques surface or a rational surface if and only if \( \text{ht}(X) = \infty \).

Proof. It is known ([Mat19, Proposition 4.1]) that the quotient is either an RDP K3 surface, an RDP Enriques surface, or a rational surface.

We saw in Theorem 6.6 that if \( X/G \) is an RDP K3 surface then \( X \) is of finite height.

If \( X/G \) is a rational surface or an RDP Enriques surface, then \( H^2_{\text{et}}(X/G,\mathbb{Q}_l) \) is generated by algebraic cycles, hence so is \( H^2_{\text{et}}(X,\mathbb{Q}_l) \), hence \( X \) is supersingular. \( \square \)

6.4. The case of \( \mathbb{Z}/p\mathbb{Z} \)-quotients. Suppose \( X \) is a (smooth) K3 surface and \( \pi : X \rightarrow Y \) is a \( G \)-quotient morphism with \( G = \mathbb{Z}/p\mathbb{Z} \). Suppose \( Y \) is an RDP K3 surface. The author determined all possible configurations of singularities on \( Y \) [Mat19, Theorem 7.3(1)]. In each case, the configuration contains a non-taut RDP with \( r > 0 \), hence by Theorem 6.4 we can determine the height of \( Y \), and we can show it is equal to the height of \( X \).

Theorem 6.10. Let \( \pi : X \rightarrow Y \) be as above. Then

\[
\text{ht}(X) = \text{ht}(Y) = \begin{cases} 
1 & \text{if } (p,\text{Sing}(Y)) = (2,2D_4^1), (3,2E_6^1), (5,2E_8^1), \\
2 & \text{if } (p,\text{Sing}(Y)) = (2,1D_2^5), \\
3 & \text{if } (p,\text{Sing}(Y)) = (2,1E_2^5).
\end{cases}
\]

This covers all possibilities for \( p \) and \( \text{Sing}(Y) \) ([Mat19, Theorem 7.3(1)]). In particular, \( X \) and \( Y \) are of finite height.
Proof. As explained above, it follows from Theorem 6.4 that \( \text{ht}(Y) \) is equal to the asserted value. In particular, \( \text{ht}(Y) < \infty \), hence \( H^2_{\text{crys}}(Y/W(k)) \) has slope \( 1 + \frac{1}{\text{ht}(Y)} \). The pullback \( \pi^*: H^2_{\text{crys}}(Y/W(k)) \otimes_{W(k)} K_0 \rightarrow H^2_{\text{crys}}(X/W(k)) \otimes_{W(k)} K_0 \) is a direct summand since \( \pi_* \circ \pi^* \) is a scalar. Here \( K_0 = \text{Frac}(W(k)) \). Hence \( H^2_{\text{crys}}(X/W(k)) \) has slope \( 1 + \frac{1}{\text{ht}(Y)} \). Hence \( \text{ht}(X) = \text{ht}(Y) \). \( \square \)

Remark 6.11. In the case of \( \mathbb{Z}/p\mathbb{Z} \)-quotients, we do not have an equivalence as in Corollary 6.9. There is an example of an \( \mathbb{Z}/p\mathbb{Z} \)-action on an ordinary K3 surface \( X \) with rational or Enriques quotient \( Y \), at least in characteristic 2.

7. RDPs realizable on K3 surfaces

We determine which RDPs can occur on K3 surfaces.

7.1. Non-taut RDPs. In the non-taut case, Theorem 6.4 and Proposition 5.7 give necessary conditions. We will show in Proposition 7.2 that \( D^8_{19} \) in characteristic 2 is impossible. It turns out that all remaining RDPs are realizable on RDP K3 surfaces, as we will see in Section 8.

Summarizing:

Theorem 7.1. Consider a non-taut RDP \( D^8_N \) or \( E^r_N \) in characteristic \( p \). Then it occurs on some RDP K3 surface \( Y \) in characteristic \( p \) if and only if it satisfies the following conditions.

- It does not contradict Theorem 6.4 and Proposition 7.2 (i.e. if \( p = 2 \), then it is not \( D^r_N \) with \( r > 0 \) and \( [N/2] - r \notin \{1, 2, 4\} \), nor \( D^8_{19} \), nor \( E^r_{19} \)).
- \( N < 22 - 2h \) if \( r > 0 \), where \( h \) is the height predicted in Theorem 6.4. \( N < 22 \) if \( r = 0 \).

Proposition 7.2. An RDP K3 surface in characteristic 2 cannot have an RDP of type \( D^8_{19} \).

Proof of Proposition 7.2. Suppose \( z \in Y \) is an RDP of type \( D^8_{19} \) in characteristic 2 on an RDP K3 surface \( Y \). By Theorem 6.4, \( \text{ht}(Y) = 1 \). Let \( \overset{\sim}{Y} \rightarrow Y \) be the minimal resolution. Since \( \text{ht}(\overset{\sim}{Y}) < \infty \) there exists, by [LAM18, Corollary 4.2], a K3 surface \( X \) over \( \text{Spec} W(k) \) with \( X \otimes_{W(k)} k \cong \overset{\sim}{Y} \) and \( \text{Pic}(X) \cong \text{Pic}(Y) \). Let \( X_K := X \otimes_{W(k)} K \) be the generic fiber of \( X \) over \( K := \text{Frac} W(k) \) and let \( X_{\mathbb{C}} := X_K \otimes_K \mathbb{C} \) for any embedding \( K \rightarrow \mathbb{C} \) (which we may assume to exist by replacing \( k \)). Then we have \( \text{Pic}(X_{\mathbb{C}}) \cong \text{Pic}(X) \cong \text{Pic}(\overset{\sim}{Y}) \).

Let \( L_1 \) be the sublattice of \( \text{Pic}(\overset{\sim}{Y}) \cong \text{Pic}(X_{\mathbb{C}}) \) generated by the exceptional curves above \( z \), and \( L_2 := L_1^\perp \) be its orthogonal complement. Since \( L_1 \) is negative definite, \( L_2 \) is nonzero, and since \( \rho(Y) \leq 22 - 2\text{ht}(Y) = 20 \) (Proposition 5.7), we have rank \( L_2 = 1 \). The transcendental lattice \( T = T(X_{\mathbb{C}}) = (\text{Pic}(X_{\mathbb{C}}))^\perp \) in \( H^2(X_{\mathbb{C}}, \mathbb{Z}) \) of \( X_{\mathbb{C}} \) is a rank 2 positive definite lattice, and then \( X_{\mathbb{C}} \) has complex multiplication by the imaginary quadratic field \( E := \mathbb{Q}(\sqrt{-d}) \), \( d := \text{disc} T(X_{\mathbb{C}}) \). By Theorem 5.9, the reduction \( \overset{\sim}{Y} \) of \( X_{\mathbb{C}} \) at a prime above 2 being ordinary implies that 2 is split in \( E/\mathbb{Q} \). Writing \( d = k^2 d_0 \) with \( d_0 \) square-free, this means \( d_0 \equiv -1 \pmod{8} \). By Lemma 7.3, this is impossible. \( \square \)
Lemma 7.3. Suppose $L_1$, $L_2$, and $L_3$ are lattices with

- $\text{disc}(L_1) = -4^r$,
- $L_2$ is positive definite, $\text{rank}(L_2) = 1$,
- $L_3$ is positive definite, $\text{rank}(L_3) = 2$, $\text{disc}(L_3) = k^2d_0$ with $d_0$ square-free and $d_0 \equiv -1 \pmod{8}$.

Then $L_1 \oplus L_2 \oplus L_3$ does not admit a finite index unimodular overlattice.

A non-degenerate lattice $L$ is called unimodular if the natural injection $L \hookrightarrow L^* := \text{Hom}(L, \mathbb{Z})$ is an isomorphism, equivalently if $\text{disc}(L) = \pm 1$.

Proof. Suppose $L_1 \oplus L_2 \oplus L_3$ admits a finite index overlattice $\Lambda$ with $\text{disc}(\Lambda) = \pm 1$. Take bases $e_2$ of $L_2$ and $t_1, t_2$ of $L_3$, and let $(m)$ and $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be the Gram matrices (so $m > 0$, $a > 0$, and $\text{disc}(L_3) = k^2d_0 = ac - b^2 > 0$).

Since $L_1 \oplus L_2 \oplus L_3 \subset \Lambda$ is finite index, its discriminant $\text{disc}(L_1 \oplus L_2 \oplus L_3) = \text{disc}(L_1) \cdot \text{disc}(L_2) \cdot \text{disc}(L_3) = -4^r mk^2d_0$ coincides with $\text{disc}(\Lambda) = \pm 1$ up to a square. Hence $m = n^2d_0$.

Let $g = \gcd\{a, b, c\}$. Then the discriminant group of $L_3$ is isomorphic to $\mathbb{Z}/g\mathbb{Z} \times \mathbb{Z}/gh\mathbb{Z}$, where $h = g^{-2}\text{disc}(L_3) \in \mathbb{Z}$. By lattice theory this is isomorphic to the discriminant group of the primitive closure of $L_1 \oplus L_2$ in $\Lambda$, which is a subquotient of the discriminant group of $L_1 \oplus L_2$. Hence $g$ is a power of 2.

We claim that $a$ is the norm of some ideal of $\mathcal{O}_E$, where $E = \mathbb{Q}(\sqrt{-d_0})$. It suffices to show that $\text{ord}_l(a)$ is even for any prime $l$ that is inert in $E/\mathbb{Q}$. Suppose $\text{ord}_l(a) = 2j - 1$ and $l$ is inert (then $l \neq 2$ since $-d_0 \equiv 1 \pmod{8}$).

Then, since $ac = b^2 + k^2d_0$ and since $l$ is inert, we have $\text{ord}_l(k^2d_0) \geq 2j$ and $\text{ord}_l(b^2) \geq 2j$, hence $l | c$, hence $l | g$, hence $l = 2$. Contradiction.

Since $\Lambda$ is unimodular, there is an element $v \in \Lambda$ with $e_2 \cdot v = 1$. Write $2^r v = v_1 + v_2 + v_3$ with $v_i \in L_i \otimes \mathbb{Q}$, then $v_i \in L_i^\ast$. We have $v_2 = (2^r n/m)v_2$ and hence $v_2^2 = (2^r n/m)^2m = 4^r/d_0$. We have $v_i^2 \in \mathbb{Z}$ (since $[L_i^\ast : L_i] = |\text{disc}(L_i)| = 4^r$). We have $\sum v_i^2 = (2^r n)^2v^2 \in \mathbb{Z}$. Hence we obtain $v_3^2 \equiv -4^r/d_0$ (mod $\mathbb{Z}$), hence $d_0v_3^2 \equiv -4^r$ (mod $d_0\mathbb{Z}$).

Write $v_3 = x_1t_1 + x_2t_2$ ($x_i \in \mathbb{Q}$). Then $d_0 = N_{E/\mathbb{Q}}(\sqrt{-d_0})$ and $av_3^2 = a(ax_1 + 2bx_1x_2 + cx_2^2) = N_{E/\mathbb{Q}}(ax_1 + (b + \sqrt{-k^2d_0})x_2)$ are the norms of elements of $E$. Hence $d_0v_3^2 = d_0 \cdot a^{-1} \cdot av_3^2$ is the norm of a fractional ideal of $E$. Therefore $-4^r$ and hence $-1$ are norm residues modulo $d_0$. But $-1$ cannot be a norm residue of an imaginary quadratic field. Contradiction.

7.2. Taut RDPs.

For the taut case we have the following, which is almost done by Shimada and Shimada–Zhang.

Theorem 7.4. Suppose $p \geq 0$. Suppose $S$ is a Dynkin diagram $(A_N, D_N, E_N)$ for which RDPs of type $S$ in characteristic $p$ are taut. Then such an RDP occurs on some RDP $K3$ surface $Y$ in characteristic $p$ if and only if $p$ satisfies the following respective conditions.

- If $N \leq 19$: any $p \geq 0$.
- If $S$ is $A_{20}$: $p > 0$ and $p$ is non-split in $\mathbb{Q}(\sqrt{21})$. Equivalently, either $p | 21$ or $p \equiv \pm 2, \pm 8, \pm 10 \pmod{21}$.
- If $S$ is $A_{21}$: $p = 11$.
- If $S$ is $D_{20}$ or $D_{21}$, or $N \geq 22$: no $p$. 

Proof. Suppose $N \leq 19$ and $p \neq 2$. It is known that there exists an elliptic K3 surface with a section and a singular fiber of type $I_{19}$. For $p = 0$ this is due to Shioda [Shi03, Theorem 1.1] (who used the equation given by Hall [Hal71, equation 4.29 in page 185]). It is clear from the Shioda’s equation that the same equation in characteristic $p > 3$ also gives an elliptic K3 surface with the same property. Moreover this holds for $p = 3$ using the coordinate change given by Schütt-Top [ST06, Section 2]. Then the union of a section and this singular fiber contains a configuration of type $S$.

Suppose $N \leq 20$ and $p = 2$. Then $S$ is a subset of $D_{21}$, which is realized by Theorem 7.1.

Suppose $S$ is $A_{20}$ and $p \neq 2$. If $p \nmid 2 \text{disc}(A_{20}) = 2 \cdot 3 \cdot 7$, then by [SZ15, Table 1], this is possible if and only if $\left(\frac{7}{p}\right) = -1$. If $p = 7$, then this is possible by [Shi04, Table RDP]. It remains to show that it is possible if $p = 3$. Let $L$ be the Dynkin lattice of type $A_{20}$ and $T$ be the lattice of rank 2 with basis $t_1, t_2$ and Gram matrix \(\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}\). Let $e_1, e_2, \ldots, e_{20}$ be a basis of $A_{20}$ with $e_i \cdot e_j = -2, 1, 0$ if $|i - j| = 0$, $|i - j| = 1$, $|i - j| \geq 2$ respectively.

We have $L^*/L \cong \mathbb{Z}/21\mathbb{Z}$ and $T^*/T \cong \mathbb{Z}/21\mathbb{Z}$. Let $l = \frac{1}{4}(\sum_{i=1}^{20} e_i) \in L^*$ and $t = \frac{1}{4}(t_1 + t_2) \in T^*$. They generate the prime-to-$3$ parts of $L^*/L$ and $T^*/T$ respectively. We have $l^2 \equiv -t^2 \pmod{2\mathbb{Z}}$ since $l^2 + t^2 = (\frac{1}{4})^2 \cdot (-20 \cdot (20 + 1)) + (\frac{1}{4})^2 \cdot 14 = -4 \in 2\mathbb{Z}$. We can apply Lemma 7.5 below.

Suppose $S$ is $A_{21}$. Then $Y$ is supersingular and, considering the Picard lattice, we must have $p \mid 22$. By [Shi04, Table RDP], this is possible for $p = 11$ and impossible for $p = 2$.

Suppose $S$ is $D_{20}$. Since $\text{disc}(S) = 4$ is a square, an RDP of type $S$ can be realized only in characteristic $p$ dividing $\text{disc}(S)$ by [DK09, Lemma 3.2], that is, $p = 2$. In this case $S$ is non-taut and is out of the scope of this theorem. \hfill \Box

For a finite abelian group $A$, we write its $p$-primary part (resp. prime-to-$p$ part) by $A_p$ (resp. $A_{p'}$). For a non-degenerate even lattice $L$, we define a quadratic map $q_L: L^*/L \to \mathbb{Q}/2\mathbb{Z}$ by $q_L(\bar{v}) = v^2 \mod 2\mathbb{Z}$, where the bar denotes the projection $L^* \to L^*/L$. The next lemma is a variant of [SZ15, Proposition 2.6].

Lemma 7.5. Let $p$ be an odd prime. Let $R$ be a formal finite sum of $A_N$, $D_N$, $E_N$, with $\sum N = 20$, and let $L = L(R)$ be the corresponding lattice (of rank 20). Suppose there are an even lattice $T$ of sign $(+1,-1)$ and a group isomorphism $\phi: (L^*/L)_{p'} \cong (T^*/T)_{p'}$ satisfying $\phi^*(q_T|_{T^*/T}_{p'}) = -q_L|_{(L^*/L)_{p'}}$ and $(L^*/L)_p \oplus (T^*/T)_p \cong (\mathbb{Z}/p\mathbb{Z})^2$. Then there exists an RDP K3 surface $Y$ (supersingular of Artin invariant 1) with $\text{Sing}(Y) = R$.

Proof. Let $\Lambda$ be the submodule of $L^* \oplus T^*$ consisting of the elements $(l, t)$ with $l \in (L^*/L)_{p'}$, $t \in (T^*/T)_{p'}$, and $\phi(l) = t$. Then $\Lambda$ is an even overlattice of $L \oplus T$ of sign $(+1,-21)$ with $\Lambda^*/\Lambda \cong (\mathbb{Z}/p\mathbb{Z})^2$. This means that $\Lambda$ is isomorphic to the Picard lattice of a supersingular K3 surface of Artin invariant 1. By the argument of [SZ15, Theorem 2.1, (3) $\implies$ (1)], we obtain

\footnote{The table is contained only in the preprint version available at Shimada’s website.}
Example 8.2

A supersingular RDP K3 surface $Y$ of Artin invariant 1 with $\text{Sing}(Y) = R$. □

8. Examples

Examples of maximal $G$-quotient morphisms $X \to Y$ between RDP K3 surfaces with all possible $(G, G', \text{Sing}(X), \text{Sing}(Y))$ are already given in [Mat19, Examples 9.2–9.5].

The non-taut RDPs in Examples 8.1–8.3, together with their partial resolutions, prove the existence part of Theorem 7.1.

Example 8.1 ($p = 2$).

- Schütt [Sch06, Section 6.2] gave an example of an elliptic K3 surface
  
  \[ y^2 + t^6 y = x^3 + (c^2t^4 + c^3 + \tilde{a}_6)x^2 + ct^8 x + t^{10}\tilde{a}_6, \]

  where $\tilde{a}_6 \in k[t]$ is of degree $\leq 2$, with a section and a singular fiber of type $I^*_8$. It is of height 1 since the coefficient of $t^2y$ is nonzero (Proposition 5.4 applied to $P(6,4,1,1)$). The union of a section and the singular fiber contains a configuration of type $D_{18}$ and and one of type $E_8$. The respective contractions give RDP K3 surfaces with $D_{18}$ and $E_8'$. By Theorem 6.4, we have $r = r_{\max}(2, D_{18}) = 8$ and $r' = r_{\max}(2, E_8) = 4$.

- Let $E_1$ and $E_2$ be elliptic curves, ordinary and supersingular respectively. Let $X = \text{Km}(E_1 \times E_2)$, i.e. $X$ is the minimal resolution of $(E_1 \times E_2)/\{\pm 1\}$. By [Shi74, Section 6(b)] and [Art75, Examples], $\text{Sing}(X)$ is $2D_8$. Hence by Theorem 6.4, $h^t(X) = 2$. Consider the elliptic fibrations $f_j : X = \text{Km}(E_1 \times E_2) \to (E_1 \times E_2)/\{\pm 1\} \to E_1'/\{\pm 1\} \cong P^1$. They admit sections and, by [Shi74, Section 4], the singular fibers of $f_1$ and $f_2$ are $2I^*$ and $1I^*_2$ respectively. In either case, the union of a section and the singular fiber(s) contains a configuration of type $D_{17}$ and one of type $E_8$.

- Let $X$ be the elliptic RDP K3 surface $y^2 + yxt^2 + x^3 + t^5 = 0$, $y^2 + yx + x^3 + s^2 = 0$. The singular fibers of its resolution are $I^*$ and $I_7$, hence the union of a section and the singular fibers contains a configuration of type $D_{15}$ and one of type $E_8$. We have two proofs for $h^t(X) = 3$. (1) Counting $\#(X(\mathbb{F}_p^2))$ (before taking the resolution), we obtain $\#X(F_2) = 1 + 2^2 + 2 = 5$, $\#X(F_4) = 1 + 4^2 + 4 \cdot 2$, and $\#X(\mathbb{F}_8) = 45 + 8^2 + 8 \cdot (-5/2)$, hence $h^t(X) = 3$ by Proposition 5.8. (2) Let $\tilde{X}$ be the (smooth) elliptic K3 surface in characteristic 0 defined by the same equation. Since $\tilde{X}$ admits an automorphism $((x, y, s) \mapsto (x, y, \zeta_7s))$ acting on $H^0(\tilde{X}, \Omega^2)$ by a primitive 7-th root of unity, $\tilde{X}$ has complex multiplication by $\mathbb{Q}(\zeta_7)$. Hence the mod 2 reduction $X$ of $\tilde{X}$ has $h^t(X) = 3$ by Theorem 5.9.

- The quasi-elliptic K3 surface $y^2 = x^3 + t^2x + t^{11}$ (given by Dolgachev–Kondo [DK03, Theorem 1.1]) admits a fiber of type $I^*_3$ at $t = 0$. The union of a section and the singular fibers contains a configuration of type $D_{21}$ and one of type $E_8$. Since $21 \not\in 22 - 2h$ for any $1 \leq h < \infty$, this K3 surface is supersingular.

Example 8.2 ($p = 3$).
On a stratification of the moduli of \[vdGK00\] G. van der Geer and T. Katsura, E is an RDP K3 surface with 2

Example 8.3 (p = 5). As in [Mat19, Example 9.11], \(y^2 = x^3 + at^4x + t + t^11\) is an RDP K3 surface with \(2E_8^1\) (resp. \(2E_8^0\)) if \(a \neq 0\) (resp. \(a = 0\)).

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