ABSTRACT: Quantum amplitudes for Euclidean gravity constructed by sums over compact manifold histories are a natural arena for the study of topological effects. Such Euclidean functional integrals in four dimensions include histories for all boundary topologies. However, a semiclassical evaluation of the integral will yield a semiclassical amplitude for only a small set of these boundaries. Moreover, there are sequences of manifold histories in the space of histories that approach a stationary point of the Einstein action but do not yield a semiclassical amplitude; this occurs because the stationary point is not a compact Einstein manifold. Thus the restriction to manifold histories in the Euclidean functional integral eliminates semiclassical amplitudes for certain boundaries even though there is a stationary point for that boundary. In order to incorporate the contributions from such semiclassical histories, this paper proposes to generalize the histories included in Euclidean functional integrals to a more general set of compact topological spaces. This new set of spaces, called conifolds, includes the nonmanifold stationary points; additionally, it can be proven that sequences of approximately Einstein manifolds and sequences of approximately Einstein conifolds both converge to Einstein conifolds. Consequently, generalized Euclidean functional integrals based on these conifold histories yield semiclassical amplitudes for sequences of both manifold and conifold histories that approach a stationary point of the Einstein action. Therefore sums over conifold histories provide a useful and self-consistent starting point for further study of topological effects in quantum gravity.
1. INTRODUCTION

An interesting property of our observed universe is, that on scales ranging from fermis to parsecs, its spatial topology is Euclidean. This fact is not explained by the dynamics of classical relativity; classically all three manifolds admit initial data satisfying the constraints and this initial data has a finite evolution that produces a classical spacetime.\(^1\) Moreover, the spatial topology of the initial 3-manifold cannot change under evolution.\(^2\) Though classically not allowed, topology change may occur when the quantum mechanics of gravity is considered. At distances near the planck scale, one expects that metric fluctuations will become important and potentially lead to degeneracies in the metric; such degeneracies can be argued to signal topology change. This leads naturally to the question, what does quantum mechanics of gravity predict for the topology of our universe?

A formulation of quantum mechanics especially suited to addressing this question is that given by Feynman’s sum over histories. In particular, Euclidean sums over histories weighted by the Einstein action provide a versatile method of constructing amplitudes and states that very naturally incorporates histories corresponding to different topologies.\(^3\) A quantum amplitude is constructed by summing over all physically distinct histories that satisfy the appropriate boundary conditions weighted by the Euclidean action. In Euclidean gravity, such a history consists of both a manifold \(M^n\) and a Euclidean metric \(g\) on the manifold. In terms of such histories, an amplitude for topology change between a set of boundary manifolds \(\Sigma^{n-1}\) is heuristically

\[
G[\Sigma^{n-1}, h] = \sum_{(M^n, g)} \exp(-I[g]) \quad (1.1)
\]

where the sum is over all physically distinct histories \((M^n, g)\) that satisfy the appropriate conditions and that have the correct induced metric \(h\) on each boundary. Such a sum over histories will produce an amplitude for topology change between boundary manifolds in the
same cobordism class. Additionally, it naturally incorporates contributions from histories of different topologies. Thus, Euclidean sums over histories provide a method of incorporating topology into the quantum mechanics of gravity. Indeed, they have frequently been used as a starting point for studying the qualitative effects of quantum gravity and topology change. In fact many interesting investigations of the quantum mechanics of gravity have been carried out in certain specialized forms; for example by evaluating expressions such as (1.1) in semiclassical approximation in terms of certain known Euclidean instantons\textsuperscript{3,4} or in minisuperspace models in which the histories summed over are restricted to a limited set of metrics on a fixed manifold.\textsuperscript{5,6} However, in order to address the full consequences of topology on the quantum mechanics of gravity, the sum over histories with all topologies, not those in a restricted set really must be considered. It turns out that two important issues arise when carefully considering the precise meaning of the heuristic expression (1.1); what are the allowed topologies of the histories that should be included in a more precise sum over histories and how does one formulate in implementable terms a sum over histories that have different topologies.

The first issue may seem to be a moot point given that the formal description of a history to be summed over in (1.1) is one with the topology of a smooth manifold. However, this is not the case; it is well known that formal descriptions of histories are based on the classical configurations of the theory and they do not necessarily correspond to the precise mathematical definition of the space of histories needed in order to make Euclidean sums over histories both well defined and yield the correct quantum mechanics. For example, the formal description of a history for a single particle in Euclidean mechanics is a smooth path. However, it is well known that the paths summed over in a Euclidean functional integral to produce an amplitude include nondifferentiable paths. The contribution of these paths is important; indeed it is well known that the differentiable ones form a set of measure zero. Similarly, analogous nondifferentiable field configurations occur in functional integrals for
field theories. Thus one expects that histories with some sort of suitably nondifferentiable metrics and matter fields occur in gravity. However, there is an important difference between histories in field theories and those in Einstein gravity; a history in Einstein gravity is specified by both a manifold and a metric and the manifold nature of the histories is not changed by including nondifferentiable or distributional metrics in the space of histories. In light of this, it is natural to ask whether or not more general topological spaces should be included in the sum over histories as the topological analog of including nondifferentiable paths and if so, what sort of topological spaces besides manifolds should occur.

The second issue arises even for sets of histories restricted to be manifolds. In order to proceed from the heuristic expression (1.1), it is necessary to provide an implementable description of how to take the sum over histories. Naively, one imagines implementing a path integral for gravity by tabulating all distinct manifolds with the given boundary \( \Sigma^{n-1} \), calculating the sum over some appropriately defined space of all physically distinct metrics on these manifolds and finally summing over the contributions for each distinct manifold in the tabulation weighted perhaps by some phase factor. However, no matter how reasonable it sounds, such a scheme is not generally possible because of the first step. There is no way to tabulate all physically distinct manifolds in dimension \( n \geq 4 \). Even worse, it is an open problem to find a method to determine if a space is a manifold in four dimensions and it is proven that there there is no way to do so for \( n \geq 5 \). Moreover, it is completely independent of such other issues in the concrete implementation of Euclidean integrals for gravity such as conformal rotation and the perturbative nonrenormalizability of the theory.

Finally, the first issue of what topological spaces should be included as histories is strongly coupled to this problem of finding an implementable method of summing over distinct spaces; changing the set of spaces summed over will change the properties of the sum. Thus, as first suggested by Hartle, the inclusion of more general topological spaces
in the sum over histories may provide an avenue towards finding an expression of the form of (1.1) that is better defined. Thus the twin issues of what set of topological spaces should be included as histories and of defining a sum over topologies for Euclidean gravitational integrals is as much at hand as the calculation of possible effects from topology and topology change. These issues will be examined in a two part paper; Part I will concentrate on the question of what set of topological spaces should be included as histories. Part II (Ref. [9]) will discuss the issues involved in defining a sum over topologies for both manifolds and the set of more topological spaces, conifolds, proposed in Part I.

The starting point toward finding a candidate for a more general set of topological spaces than manifolds is to examine the properties of the formal expression (1.1) in terms of semiclassical approximation. Even though (1.1) is not fully defined, semiclassical approximations to it are computable as they are closely related to the classical solution space of the theory. The semiclassical evaluation of a Euclidean sum over histories such as (1.1) is constructed from the solution of the Einstein equations \( g \) on a manifold \( M^n \) that has boundary \( \Sigma^{n-1} \) with the appropriate induced boundary data \( h \). However, although the expression (1.1) includes histories for any boundary \( \Sigma^{n-1} \) cobordant to a \((n-1)\)-sphere, there may not be a stationary history for that boundary. One can demonstrate that the requirement that the Euclidean Einstein equations be satisfied on a compact manifold eliminates a large set of possible boundary manifolds. Therefore in the semiclassical limit, transition amplitudes constructed from Euclidean sums over histories make strong predictions about the allowed spatial topology of the universe.

Finding that most boundary manifolds are suppressed in the semiclassical limit might be considered a positive result. However, as discussed in detail in this paper, further investigation shows that for certain \((\Sigma^{n-1}, h)\) that do not have a semiclassical amplitude, there is a set of smooth histories consisting of compact manifolds with metrics that approach a stationary point of the Einstein equations. Nonetheless, there is no limiting Einstein man-
ifold; the topology of the compact limit space exhibits nonmanifold points. Therefore a semiclassical evaluation of (1.1) does not yield a semiclassical amplitude, precisely because of the restriction to manifolds. On the other hand, it is reasonable to expect that a semiclassical evaluation of a Euclidean integral that contains a set of histories that approach a stationary point yields a semiclassical amplitude. Thus this feature of the semiclassical approximation to Euclidean sums over histories for gravity indicates that their formulation should be generalized: the sum over histories should include histories corresponding to more general topological spaces than manifolds.

Given these results, it is natural to investigate the properties and consequences of extending the sum over manifolds in Euclidean sums over histories for gravity to a sum over a more general set of topological spaces. This paper will concentrate on the properties of a particular set of such topological spaces, conifolds, whose definition is motivated by the study of the semiclassical approximation. Section 2 will discuss topological aspects of the manifold histories used in the formulation of Euclidean sums over histories for Einstein gravity in terms of the explicit example of the Hartle-Hawking functional integral for the initial state of the universe. It will be manifestly apparent that the general properties of this particular Euclidean functional integral are common to all such amplitudes. Section 3 will discuss in detail why the Euclidean functional integrals do not provide semiclassical wavefunctions for all boundary topologies. As an illustrative example, the case of the Hartle-Hawking wavefunction for \( \mathbb{RP}^3 \) with round metric will be studied. Section 4 will be devoted to a discussion of the properties desired in a more general set of topological spaces in order to define and implement a generalized sum over histories, namely that the sum over histories can actually be formulated for the set. Section 5 will propose a new set of topological spaces that satisfy the criteria of section 4; these spaces will be called conifolds. Both their topology and geometry will be defined, examples of conifolds will be presented and various useful results for both understanding conifolds and using them will
be derived. Section 6 will outline the proof that sequences of approximately stationary histories converge to a Einstein conifold. It will discuss the implications this set has in the semiclassical evaluation of Euclidean functional integrals and propose this set as suitable generalized histories.

The issue of defining the sum over these topological spaces will be discussed in Part II, reference [9]. Indeed, it will be seen that conifolds can be described by a simple algorithm in four or fewer dimensions. Thus certain problems with the sum over histories present in the case of manifolds will be avoided. Additionally it will be seen that the simplicial version of conifolds provide a useful method of formulating discrete forms of (1.1) using Regge calculus suitable for the computation of topological effects.

2. THE SPACE OF HISTORIES

It is useful to begin by discussing the elements entering into a heuristic expression such as (1.1) in terms of an illustrative example; the Hartle-Hawking wavefunction for the initial state of the universe.\textsuperscript{5,10} The configuration space for the Hartle-Hawking wavefunction is the space of all closed smooth \((n-1)\)-manifolds \(\Sigma^{n-1}\) with three metrics \(h\). These closed \((n-1)\)-manifolds may consist of more than one disconnected component. Then the Euclidean sum over histories defining the Hartle-Hawking state for the case of positive cosmological constant \(\Lambda\) is given by

\[
\Psi[\Sigma^{n-1}, h] = \sum_{M^n} \int Dg \exp\left(-I[g]\right)
\]

\[
I[g] = -\frac{1}{16\pi G} \int_{M^n} (R - 2\Lambda)d\mu(g) - \frac{1}{8\pi G} \int_{\Sigma^{n-1}} Kd\mu(h)
\]

(2.1)

where \(d\mu\) is the covariant volume element corresponding to the indicated metrics. The boundary conditions that specifically determine this state are given in terms of the histories
included in the sum; formally, these histories consist of physically distinct metrics $g$ on compact manifolds $M^n$ that have the correct induced metric $h$ on the boundary $\Sigma^{n-1}$.

Note that, as typically done in the formulation of Euclidean sums over histories for gravity, the fact that the set of all compact smooth $n$-manifolds is countable has been utilized to write the sum over histories in (2.1) in terms of a sum over distinct manifolds $M^n$ and a functional integration over physically distinct metrics on each distinct manifold $M^n$.

This formal description of the histories is the starting point for a concrete definition of the space of histories and the measure of the path integral. A complete and detailed specification of this space and measure is unknown as it is equivalent to demonstrating the existence of the quantum theory. Indeed a properly defined quantum theory of Einstein gravity may not exist given its well known problems such as perturbative nonrenormalizability. However, it is possible to discuss the topological aspects of the space and measure as this information is encoded into the classical histories. These aspects are important as they enter into both the semiclassical evaluation of the theory and discrete approximations of the Hartle-Hawking integral. Thus these topological aspects are directly relevant to the qualitative study of Einstein gravity. Moreover, as the topology is not coupled to the metric, one anticipates that the topological properties of expressions such as (2.1) are directly relevant to other theories that include a sum over topologies such as theories that include gravity or topological field theories.

It is useful to begin by giving the mathematical definitions corresponding to the formal description of the histories; such histories will be called Riemannian histories to clearly indicate their correspondence with classical Riemannian geometry.

2.1 Riemannian Histories

A Riemannian history consists of two quantities; a metrizable space corresponding to
a smooth manifold and a Riemannian metric on that manifold. A metrizable topological space is one for which the open sets of the space can be defined in terms of a distance function.\textsuperscript{11} A distance function is a real valued symmetric function for which 1) given any two points \(x, y\) in the space, \(d(x, y) = d(y, x) \geq 0\) and \(d(x, y) = 0\) if and only if \(x = y\), 2) the triangle inequality holds, \(d(x, y) + d(y, z) \geq d(x, z)\). Then

**Definition (2.1).** A metrizable space \(M^n\) is a smooth manifold if it satisfies the following conditions:

1) Every point has a neighborhood \(U_\alpha\) which is homeomorphic to an open subset of \(\mathbb{R}^n\) via a mapping \(\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\).

2) Given any two neighborhoods with nonempty intersection, then the mapping

\[
\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)
\]

is a smooth mapping between subsets of \(\mathbb{R}^n\).

The set \(\{(U_\alpha, \phi_\alpha)\}\) is called an atlas of the manifold; the set \(\{U_\alpha\}\) is the cover. A manifold for which every cover can be reduced to a finite subcover is called a *compact manifold*. A manifold that satisfies the first condition but not the second is called a *topological manifold*. The second condition requires the existence of a smooth structure on the manifold. The smooth structure of a manifold is given in terms of its atlas. The equivalence of two smooth structures is thus a question of whether or not two manifolds that are homeomorphic are actually diffeomorphic; that is given that there is a homeomorphism \(h : M^n \rightarrow M'^n\), is the homeomorphism a differentiable, invertible map whose inverse is also differentiable. This question is not an issue in two and three dimensions; it can be proven that all smooth structures on a given closed connected manifold are equivalent in two and three dimensions. Indeed, all topological closed manifolds have a unique smooth structure in two and three dimensions. In four or more dimensions, there are topological manifolds that do
not admit any smooth structure;\textsuperscript{12} the example of $\|E8\|$ illustrates the deduction of this result. Additionally, it can be demonstrated that there are compact 4-manifolds which have more than one smooth structure; in fact there exist compact manifolds for which there are a countably infinite number of smooth structures. For example, the 4-manifold $CP^2 \# 9(-CP^2)$, a connected sum of complex projective space with nine copies of itself with the opposite orientation, has a countably infinite number.\textsuperscript{12} Even more interesting is the result that the number of different smooth structures is uncountable for certain open manifolds. In particular, $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ both have an uncountable number of distinct smooth structures.\textsuperscript{12} In five or more dimensions, the number of inequivalent smooth structures on compact manifolds is finite.\textsuperscript{13} Even open manifolds have a finite number of smooth structures in this case provided that their homology groups are finitely generated.

However, even though topological manifolds occur in 4 or more dimensions, it is not necessary to consider them. A smooth structure is necessary for physical reasons; derivatives of fields can only be defined on manifolds with smooth structures and such quantities are fundamental in the discussion of physical theories. Therefore topological manifolds that are not smooth are not physically interesting. Finally, one can prove that any $C^1$ atlas is $C^1$ diffeomorphic to a smooth atlas.\textsuperscript{14} Therefore, without loss of generality, it is sufficient to assume that the atlas is smooth in the definition of a Riemannian history as done above.

The definition of a smooth manifold with boundary differs from that of a smooth manifold by replacing condition 1) by the requirement that every point has a neighborhood $U_\alpha$ which is homeomorphic to an open subset of the upper half space, $\mathbb{R}^n_+$. The case of smooth manifolds without boundary is contained in this definition as open subsets of $\mathbb{R}^n$ are open subsets of the upper half space. The boundary of the manifold is given by the set of points that are mapped to the boundary of the upper half space. From this definition it follows that the boundary of a smooth $n$-manifold is a $(n-1)$-manifold without boundary. It
is important to note that the boundary of a smooth n-manifold is a topological invariant. Finally, a compact manifold without boundary is called a \textit{closed manifold}.

The geometry of a Riemannian history is carried by a metric $g$. Smooth metrics can be found on all smooth manifolds $M^n$ with a smooth atlas. However, it is clear that the set of smooth metrics is too restrictive as many physically interesting spaces have $C^k$ metrics. Thus the geometry should include $C^k$ metrics with some appropriate choice of $k$. Given that the action should be defined for a Riemannian history, the metric $g$ should be at least $C^2$ so that the scalar curvature of the manifold will be a well defined function. In the case of $M^n$ with boundary, the metric is restricted by the requirement that it induce the correct specified metric $h$ on the compact boundary $\Sigma^{n-1}$. Of course, the degree of differentiability of a given $h$ will constrain the differentiability of $g$.

Thus the previous paragraphs provide the mathematical foundation for the formal description of the histories included in the Hartle-Hawking integral: A \textit{Riemannian history} is a pair $(M^n, g)$ where $M^n$ is a smooth compact manifold and $g$ is at least a $C^2$ metric with the specified induced metric $h$ on the boundary $\Sigma^{n-1}$. It is important to stress that the definition of a smooth manifold does not in any way require the presence of a smooth metric even though a smooth metric on a manifold can be used to construct the neighborhoods and maps in Def.(2.1). This point is sometimes overlooked because of this strong connection between geometry and topology for Riemannian manifolds. However, it is especially important to keep it in mind when working with Euclidean functional integrals for gravity as one should anticipate that all histories will not be classical Riemannian manifolds. Furthermore, care in separating the issues of topology and metric can be invaluable in resolving certain confusions that may arise regarding what constitutes a Riemannian history for the Hartle-Hawking wavefunction. For example, an $n$-ball is an allowed history with boundary $S^{n-1}$. However, an $n$-ball minus a small $q$-ball around one of its interior points where $q < n - 1$ is not; this manifold is not compact. Even so, its
boundary is still $S^{n-1}$; no points in the neighborhood of the excised q-ball are mapped to the boundary of $\mathbb{R}^n_+$. Note that this example holds even for the case of a point. This example is obvious when phrased in terms of the topology; however, the issue can become less clear if one does not isolate the topology from the metric. It is apparent that a careful recall of these mathematical definitions is useful and necessary for deciding if a given history is to be included in the Hartle-Hawking integral.

Although the histories in the Euclidean integrals are formally Riemannian histories, it is important to remember that the correct space of histories for Einstein gravity very likely includes not only Riemannian histories but more general histories. For example, in Euclidean functional integrals in quantum mechanics, recall that a classical history for a one dimensional free particle is a differentiable path $x(t)$. However, it is well known that the space of histories for the path integral for a transition amplitude $G(a, t; b, t')$ is the space of all continuous paths between the endpoints $a, b$. This space is much larger than the set of all differentiable paths between the endpoints. One finds that it is necessary to use this larger space space in order for there to be a well defined measure, the conditional Wiener measure. This measure replaces both the formal sum over paths and the weighting by the classical Euclidean action in the path integral; thus it is not necessary that the classical action by itself be well defined on all paths in the space. The conditional Wiener measure has the appropriate properties such that integration over the space of continuous paths yields the correct quantum mechanical amplitudes for this system. Similarly, the appropriate space of histories for the functional integral for a free scalar field includes distributional fields $\phi(x)$ such that $\int_{\mathbb{R}^n} f(x)\phi(x)dx$ is finite where $f$ is a smooth test function with compact support. Of course, the correct space of histories and measure for a general interacting field theories are unknown; such a formulation is equivalent to a solution of the full quantum field theory and is therefore a highly nontrivial matter. However it is expected that the space of histories for any general field theory is larger than
the set of classical histories of that theory. Therefore, one anticipates that the space of histories in the Hartle-Hawking wavefunction will include not only all $C^2$ metrics $g$ but also less regular, distributional metrics defined in a manner similar to that for distributional fields. Again, having a well defined action $I(g)$ for a distributional history is not directly relevant for the issue of defining the space of histories. As for quantum mechanics, one anticipates that the measure on this space replaces both the integration over physically distinct metrics and the weighting factor of the action in (2.1).

Given that such distributional metrics are likely to be included in the space of histories, it is important to stress that a distributional metric does not imply that the underlying manifold is somehow singular. The smooth structure on the manifold is used in the definition of the space of distributional fields or metrics through the integration against a smooth test field. What is true is that the classical correspondence of a smooth metric with the topology of the manifold no longer holds for distributional fields. This is illustrated by the familiar example of the one parameter minisuperspace model in which the radius of the (n-1)-sphere is the only degree of freedom.\(^5\) The histories for the Hartle-Hawking integral for this model consist of the set of spherically symmetric metrics on the n-ball. As there is one degree of freedom, the integral is of the same form as one for a quantum system with one degree of freedom. Thus, one anticipates that the space of histories will include continuous paths. Continuous spherically symmetric histories can written in gauge fixed form as $ds^2 = d\tau^2 + a^2(\tau)d\Omega^2$ where $a$ is a continuous function on the interval $[0, 1]$ that vanishes at 0 and $d\Omega^2$ is the round (n-1)-sphere metric. Note that even though the metrics are continuous not differentiable, the topology of the underlying manifold is still that of a smooth n-ball. Also, even though the metric may not be differentiable at the point $\tau = 0$, this point is not a boundary of the n-ball by definition of boundary. Therefore a minisuperspace path integral based on this set of metrics does not change the topology of the n-ball or have any boundaries other than those supplied as boundary conditions,
even though the connection between the distance function, topology and smooth structure
does not hold for some of the metrics in the set. Similarly, the inclusion of distributional
metrics in the space of histories does not change the topological aspects of this space. It
again emphasizes the point that the topology of a manifold is specified independently of
its metric. Additionally, it implies that the topological aspects of defining the space of
physically distinct histories can be discussed in terms of the Riemannian histories alone.

Given that the topology of a Riemannian history is any smooth compact manifold $M^n$
which has as its boundary the closed manifold $\Sigma^{n-1}$, an immediate question is whether or
not the set of allowed $M^n$ is empty for a given closed $\Sigma^{n-1}$. The answer to this question
depends on the dimension and topology of the boundary manifold. In two dimensions,
the set of $M^2$ is not empty for any closed boundary manifold. The closed boundary $\Sigma^1$
simply consists of the disjoint union of circles. It is easy to see that the desired $M^2$ can
be constructed from any closed 2-manifold; excise the interiors of the required number of
disjoint discs from the manifold. In higher dimensions, the answer to this question can be
determined from the cobordism class of the boundary manifold; two closed (n-1)-manifolds
in the same cobordism class are the boundary of some compact n-manifold. Consequently,
any closed (n-1)-manifold cobordant to an (n-1)-sphere is itself the boundary of a compact
n-manifold: A (n-1)-sphere is the boundary of an n-ball and thus the cobordism between
these two boundary manifolds can be capped off at the (n-1)-sphere to form a compact
manifold with the desired boundary. The set of closed 2-manifolds has two cobordism
classes; closed 2-manifolds with even Euler characteristic and closed 2-manifolds with odd
Euler characteristic. As the Euler characteristic of a 2-sphere is even, all closed 2-manifolds
with even Euler characteristic are the boundaries of some compact 3-manifold. Therefore
the set of allowed histories $M^3$ is not empty for boundary $\Sigma^2$ with even Euler characteristic.
However, closed 2-manifolds with odd Euler characteristic are not cobordant to the 2-
sphere; therefore, there are no allowed histories in the Hartle-Hawking wavefunction for
these boundaries. All closed 3-manifolds are in the same cobordism class; therefore, the set of allowed histories $M^4$ for any given $\Sigma^3$ is not empty. A similar analysis can be done in 5 or more dimensions to determine whether or not the set of n-manifolds with a specified boundary (n-1)-manifold is empty.

Indeed, the cobordism properties of manifolds imply that it is inconsistent to restrict the topology of the n-manifolds included in the set of allowed histories without strong restrictions on the allowed boundary (n-1)-manifolds and conversely. For example, consider restricting the allowed boundary manifolds to be 3-spheres. As any arbitrary 3-manifold is cobordant to $S^3$, an arbitrary history in Hartle-Hawking wavefunction for two $S^3$ boundaries will include an intermediate hypersurface with arbitrary topology $\Sigma^3$. Therefore, the set of histories with $S^3$ boundary can be used to generate a set of histories with arbitrary boundary $\Sigma^3$. Thus if the set of all $M^4$ cobordant to $S^3$ are allowed histories, then the configuration space of the wavefunction must include all $\Sigma^3$; conversely if arbitrary boundary 3-manifolds are allowed, the allowed histories must include all cobordisms. This argument can easily be extended to any dimension by discussing everything in terms of cobordism classes.

If the set of manifolds $M^n$ cobordant to a boundary manifold $\Sigma^{n-1}$ is not empty, then the next step is to discuss how to select physically distinct histories for inclusion in the sum in (2.1).

2.2 Physically Distinct Histories and the Problem of Decidability

There are two parts to finding the space of physically distinct histories for integrals such as (2.1); finding the space of physically distinct metrics on each manifold $M^n$ and finding the set of physically distinct manifolds $M^n$. The issues involved in finding the space of physically distinct metrics are familiar from the studies of gauge theories; it is well known
that the metric $g$ does not uniquely determine the geometry of the Riemannian history. As Einstein gravity is a diffeomorphism invariant theory, metrics that are equivalent under a diffeomorphism of the manifold correspond to the same physical space: given two histories $(M^n, g)$ and $(M^n, g')$ if $g = f^* g'$ under a diffeomorphism $f : M^n \rightarrow M^n$ of the manifold, then the two histories are equivalent. Note that the diffeomorphism $f$ is not required to be smooth; thus a smooth metric $g$ may be equivalent to a $C^k$ one. In order to find a set of physically distinct histories for a given manifold $M^n$, it is necessary to restrict to physically distinct metrics. This can be done formally on each manifold by taking the space of all suitably defined metrics that are inequivalent under diffeomorphisms. Of course an implementation of this space will encounter many of the same difficulties as found in Yang Mills theories. For example, additional complications are introduced by the appearance of $\theta$ sectors for the space of metrics that must be properly handled. Nonetheless, the issues encountered are isolated from those relating to the sum over topologies in (2.1) and thus can be put aside when addressing topological issues.

The problem of finding the set of physically distinct manifolds $M^n$ is not one encountered in the study of gauge theories as the background manifold that the gauge fields are defined on is typically fixed. Thus the question arises as to whether or not two smooth manifolds are physically equivalent, that is whether or not they have the same topology and smooth structure. This question can be separated into three distinct issues. The first is whether or not it is possible to show that there is a finite method of determining that a given topological space satisfies the definition of a smooth manifold. The second is whether or not it is possible to show that the two manifolds are homeomorphic to each other. The third is whether or not two manifolds that are homeomorphic have equivalent smooth structures. The second issue turns out to be related to the first.

A full and detailed discussion of these three issues will be given in Part II of this paper as it is necessary to introduce additional mathematical tools in order to adequately address
them. Essentially, a finite representation of a smooth manifold is needed for a discussion of finite methods for determining properties of topological spaces. However, it is useful to have a brief introduction to the issues and the results for the purposes of this paper.\textsuperscript{16}

The first issue is called the algorithmic decidability of n-manifolds: whether or not there is an algorithmic description of the set of all n-manifolds. The algorithmic description of n-manifolds follows the same logic as the algorithmic description of other things such as flora and fauna. For example, given the set of all fauna on earth, one can look for an algorithmic description of a bird that will select out the subset of all birds on earth. This algorithmic description is a set of rules that can be implemented to determine whether or not a given animal in the set of all fauna is in fact a bird. Such a set of rules can be a series of questions such as does the animal have feathers, does it have a beak, does it have wings, and so on. By applying these rules, the subset of all birds can be selected out from the set of all fauna. Such an algorithm must be known to take a finite number of steps in order to be useful. In the case of birds, it is clear that there exists some finite set of rules by which this can be done simply because there are a finite number of fauna on Earth and ipso facto, a set of rules can be divided to divide this finite set into two, birds and not birds. Therefore the set of all birds is algorithmically decidable.

Clearly, any finite set is algorithmically decidable. Equally clearly, any uncountable set is not algorithmically decidable. Therefore the question of algorithmic decidability is nontrivial only in the case of countably infinite sets and there are examples of sets with and of sets without algorithmic descriptions. As the set of all compact n-manifolds is countably infinite, their algorithmic decidability is nontrivial and it turns out to depend on the dimension n. As discussed in Part II, in one, two and three dimensions, the set of all compact manifolds is algorithmically decidable. In four dimensions, whether or not there is an algorithmic description of all compact 4-manifolds is an open problem; the existence of such a description relies on whether or not there is an algorithm for recognizing a 4-
ball. The existence of such an algorithm relies on solutions to the Poincare conjecture and to the word problem for the fundamental groups of 3-manifolds, both open problems in topology. In five or more dimensions, n-manifolds are not algorithmically decidable; it has been proven that there is no algorithm for recognizing a n-ball for \( n \geq 5 \). Thus in five or more dimensions, there is no algorithmic method of constructing the space of histories formally summed over in the Hartle-Hawking integral.

The second issue is called the classifiability of n-manifolds; does there exist a method of determining whether or not a given n-manifold is homeomorphic to another n-manifold. Again, the classification of n-manifolds follows the same logic used in classification of other things. In order to classify birds, for example, one needs a set of rules by which it can be determined whether or not a given bird is in fact a member of a previously identified type or whether it is a distinct, new bird to be added to the list. This set of rules is based on a finite algorithm for determining whether or not the given bird is the same as another specific bird, say a robin. Such an algorithm can be a series of questions such as is the bird the same length as a robin, is its breast the same color as a robin’s, is its beak the same shape as a robin’s, and so on. If it can be established in a finite number of steps that a bird is not a robin, then one can go on to the next bird, say a crow, and use a similar procedure and continue until the bird is classified as either being a previously identified type or a distinct new bird.

Again, it is clear that the issue of classifiability is only nontrivial for countably infinite sets such as n-manifolds. It turns out that closed connected 1-manifolds are classifiable; the only closed connected 1-manifold is a circle. Whether or not two closed connected 2-manifolds are in fact the same can be determined by comparing their orientations and Euler characteristics. Thus closed connected 2-manifolds are classifiable. The classifiability of closed connected 3-manifolds is an open problem; the existence of an algorithm again depends on several open conjectures in the topology of 3-manifolds. Finally, it can be
proven that closed connected n-manifolds are not classifiable in four or more dimensions using the unsolvability of the word problem for finitely presented groups. It is important to note, as discussed in more detail in Part II, that this is a problem in practice, not just in principle; one can find explicit finite representations of manifolds with explicit undecidable groups as their fundamental groups. Thus in four dimensions, there is no algorithm for providing a list of physically distinct 4-manifolds necessary for constructing the space of physically distinct histories.

The third issue is the equivalence of smooth structures on a manifold $M^n$. In four or more dimensions, a given closed connected manifold can have a countable number of smooth structures. These structures are physically distinct and thus also must be included separately in the sum over histories. Again, in order to concretely implement such a sum one needs a method of deciding whether or not two manifolds $M^n$ and $M'^n$ that are homeomorphic are also diffeomorphic. However, by the results of the second issue, one immediately runs into trouble; as there is no method of classifying manifolds in four or more dimensions up to homeomorphism, it is clear that there is no finite algorithm for finding representatives of distinct smooth structures.

These three issues have several important implications for the Hartle-Hawking functional integral. First, they imply that it is not possible to define the space of physically distinct histories in four or more dimensions; there is no way to enumerate all physically distinct compact $M^n$ with boundary $\Sigma^{n-1}$. Second, there is no way to algorithmically construct a Riemannian history in five or more dimensions and a problem with doing so in four dimensions. Therefore, there is no way to construct the space of histories in these dimensions let alone the space of physically distinguishable ones. Third, the formal split between manifolds and metrics in (2.1) ignores the point that different smooth structures on a given manifold $M^n$ of four or more dimensions correspond to physically distinct histories and should be included independently in the sum. As the number is countable, it
can be formally included by replacing the sum over $M^n$ (2.1) with a sum over $(M^n, s)$ where $s$ is an integer labeling the different smooth structures on $M^n$. However, though this formal addition to the measure for the Hartle-Hawking wavefunction is important, it is also undecidable. Moreover, it in no way affects the problems of classifiability and algorithmic decidability.

Thus, the sum over physically distinct manifolds in (2.1) is formally well defined only in two dimensions; it is clearly not well defined in four or more dimensions and may or may not be so in three. These problems for a concrete formulation of the space of physically distinct histories and the measure for Hartle-Hawking integral persist even in terms of a finite approximation as discussed in Part II. Thus the problems with the topological aspects of the Hartle-Hawking wavefunction cannot be avoided by simply going to discrete models. Therefore, if an expression of the form (2.1) is to be replaced by a well defined meaningful sum over topologies, the fundamental question of what a physically distinct history is must be addressed.

As first observed by Hartle, one way to make such a sum over topologies well defined is by finding an appropriate set of more general topological spaces which are algorithmically decidable. Such generalized spaces can be thought of as the topological analog of nondifferentiable paths. Thus it is reasonable to explore the option of summing over more general topological spaces in order to make expressions such as (2.1) well defined. However, this abstract criterion is not enough; there are many algorithmically decidable spaces. Therefore it is useful to have a more physical motivation for what more general topological spaces should be used. Such motivation can be provided by a study of the semiclassical approximation. In semiclassical approximation, only histories which correspond to the stationary points of the action contribute to the evaluation of a functional integral; thus semiclassical approximations do not require a precise definition of the space of physically distinct Riemannian histories in order to be carried out. However, as seen in
the next section, such stationary points need not be manifolds. Thus they provide models for more general topological spaces for inclusion in the sum over histories.

3. EUCLIDEAN FUNCTIONAL INTEGERS
IN SEMICLASSICAL APPROXIMATION

A semiclassical evaluation of the Euclidean functional integral for a wavefunction of the form (2.1) involves finding an appropriately differentiable metric corresponding to an extremum of the action on some compact manifold $M^n$. Such a semiclassical history typically consists of a metric that is Euclidean at small geometries and is Lorentzian at large geometries. If there is more than one extremum of the action, the semiclassical approximation will consist of a superposition of the extrema although in practice often one keeps only the dominant contribution. As a consequence of the continuity of the extremizing metric, the resulting transition amplitude will be continuous with one continuous functional derivative on the space of three geometries.\(^{17}\)

A now familiar illustration of the semiclassical approximation is provided by the case of the Hartle-Hawking wavefunction for a boundary 3-sphere with round metric of radius $a_0$.\(^{5,10}\) The unit 3-sphere metric can be written

$$d\Omega^2 = \left( d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right) \quad (3.1)$$

$$0 \leq \chi \leq \pi \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi. \quad (3.2)$$

For $Ha_0 < 1$, a particularly simple extremum of the gravitational action (2.1) is the Euclidean de Sitter metric

$$ds^2 = d\tau^2 + a^2(\tau) d\Omega^2$$

$$a(\tau) = \frac{1}{H} \sin(H\tau) \quad (3.3)$$
where $3H^2 = \Lambda$. The topology of this solution is $S^4$; indeed it is metrically a round 4-sphere. The scale factor $a(\tau)$ explicitly satisfies the Einstein equations, which reduce to

$$\frac{\partial^2 a}{a} - \left(\frac{\partial a}{a}\right)^2 + \frac{H^2}{a^2} = 0$$

(3.4)

for the metric (3.3). There are two possible positions for the 3-sphere boundary in the Euclidean solution (3.3) that yield the correct induced metric on the 3-sphere; they correspond to filling either less than or more than half the 4-sphere. According to Hartle and Hawking, the Euclidean extremum that dominates in the steepest descents evaluation is that with least action corresponding to filling less than half the 4-sphere. The wavefunction is thus

$$\Psi_E(S^3, a_0) \sim \exp -I^{-}(a_0)$$

$$I^{-}(a_0) = -\frac{1}{3H^2 k^2}[(1 - H^2 a_0^2)^{\frac{3}{2}} - 1].$$

(3.5)

For $Ha_0 > 1$, there are no real Euclidean extrema. Instead, there are two complex extrema corresponding to the Lorentzian de Sitter solution

$$ds^2 = -dt^2 + a^2(t)d\Omega^2$$

$$a(t) = \frac{1}{H} \cosh(Ht).$$

(3.6)

The topology of the Lorentzian de Sitter solution is $\mathbb{R} \times S^3$. The extrema contribute equally to the stationary phase approximation. An analysis of the contour of steepest descents leads to the phase for the semiclassical wavefunction:

$$\Psi_L(S^3, a_0) \sim \cos(S(a_0) - \frac{\pi}{4})$$

$$S(a_0) = \frac{1}{3H^2 k^2}(H^2 a_0^2 - 1)^{\frac{3}{2}}.$$

(3.7)

This simple example concretely illustrates three generic features about the semiclassical approximation to the Hartle-Hawking wavefunction and more generally, semiclassical approximations of gravitational Euclidean functional integrals:
1) There may exist regions of the configuration space of all \((n-1)\)-metrics on \((n-1)\)-manifolds 
\((\Sigma^{n-1}, h)\) for which the extremum of the action is a Einstein manifold. These regions 
correspond to boundaries with diameter smaller than that given by Myers’ Theorem.\(^{11}\) 
The diameter of a closed \(n\)-manifold is defined as the least upper bound of the distance 
between any two points; equivalently it is the length of the longest globally minimizing 
geodesic between any two points. Then

**Theorem (Myers) (3.1).** Let \(M^n\) be a complete Riemannian manifold with metric \(g\) 
which satisfies \(R_{ab} \geq \kappa^2 g_{ab}\) and \(\kappa\) is a nonzero constant. Then \(M^n\) is compact with 
diameter \(d \leq \frac{\pi}{\kappa}\).

This is a general theorem, but can of course be applied to Riemannian manifolds that 
are solutions of the Euclidean Einstein equations provided that their Riemannian curvature 
is manifestly bounded in the required fashion. For positive cosmological constant, the 
Euclidean Einstein equations reduce to precisely the form required in Thm.(3.1) with \(\kappa^2 = \frac{2}{n-2} \Lambda\) for \(n > 2\). Thus if the diameter of \((\Sigma^{n-1}, h)\) is less than \(\pi/\kappa\), there may be a Einstein 
manifold with that boundary. However, there is no existence theorem guaranteeing that 
such an extremum exists for all boundary data and boundary manifolds.

2) There is a region of configuration space \((\Sigma^{n-1}, h)\) for which there is no extremum of 
(2.1) corresponding to a real Euclidean solution. Given any Einstein manifold with positive 
curvature, Myers’ theorem determines its maximum diameter in terms of the curvature. 
Consequently, if the diameter of \(\Sigma^{n-1}\) is larger than this maximum, there is no classical 
Euclidean solution for this boundary data. However, although there are no stationary 
Euclidean paths for such large 3-geometries, there are complex stationary paths. Examples of such paths are Lorentzian solutions of the Lorentzian Einstein equations. Initial 
data which can be evolved to a Lorentzian solution to the Einstein equations exist on 
all 3-manifolds.\(^1\) For example, suppose \(\Sigma^3\) is a hyperbolic 3-manifold, that is one admit-
ting metrics with negative curvature. Pick an initial $h$ such that $3R = -6H^2$ and let $K^L_{ij} = K_{ij} = \sqrt{2}H h_{ij}$. This initial data set satisfies the Hamiltonian constraints for the Lorentzian Einstein equations with positive cosmological constant,

$$K^{ij}K_{ij} - K^2 - 3R + 2\Lambda = 0$$

$$\nabla_i(K^{ij} - Kh^{ij}) = 0.$$  

(3.8)

Therefore, by standard existence theorems for solutions to the Einstein equations, it can be evolved for a finite distance. The same argument can be used to show the existence of initial data sets for 3-manifolds admitting positive or zero curvature by appropriate changes in sign and constants. The action in (2.1) when evaluated for Lorentzian solutions of the Einstein equations is purely imaginary; therefore the wavefunctional is oscillatory in the corresponding regions of $(\Sigma^{n-1}, h)$.

3) The wavefunction and its functional derivative are continuous everywhere in the configuration space. In the $S^3$ example, it can be easily checked that this property holds even at the point $Ha_0 = 1$ between the Lorentzian and Euclidean regions. This continuity was enforced by the contour of steepest descents but it also follows directly from basic properties of the Wheeler de Witt equation. In lowest order semiclassical approximation, it is equivalent to matching conditions for the wavefunctionals at certain points in the space of all three geometries, i.e. at boundary three geometries between Euclidean and Lorentzian wavefunctionals. Thus on each component of the boundary $\Sigma^{n-1}_b$ with metric $h^b$ between the Euclidean and Lorentzian wavefunctionals

$$\delta \frac{\delta}{\delta h_{ij}} \log \Psi_E[\Sigma^{n-1}_b, h^b] = \delta \frac{\delta}{\delta h_{ij}} \log \Psi_L[\Sigma^{n-1}_b, h^b].$$  

(3.9)

For generic semiclassical wavefunctionals, these boundary conditions are satisfied for $\pi^L_{ij} = \pi_E^{ij} = 0$ where $\pi^L_{ij} = \frac{\delta S}{\delta h_{ij}}$ and $\pi_E^{ij} = \frac{\delta I}{\delta h_{ij}}$ are the classical Lorentzian and Euclidean momenta respectively. By the standard relation between the extrinsic curvature and momenta

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in gravity, this boundary condition implies that the extrinsic curvature of the boundary vanishes,

$$K^{L}_{ij} = 0 = K^{E}_{ij}. \quad (3.10)$$

For special cases of the steepest descents evaluation, the Lorentzian wavefunction or the Euclidean wavefunction may have values allowing (3.9) to be satisfied for boundaries with nonvanishing extrinsic curvature. However, one is lead to believe that these special cases are the exception, and that the generic situation requires (3.10) at the boundary hypersurface.

The semiclassical approximation does not explicitly restrict the allowed boundary topologies in the Hartle-Hawking wavefunction. However, its implementation does indeed lead to such restrictions. This result follows from the three general properties of the semiclassical approximation to the functional integral discussed above. Most obviously, the vanishing of the extrinsic curvature requires, from (3.8), that the scalar curvature of the boundary manifold is positive definite, $3R = 2\Lambda$. Thus the boundary manifold must be one that admits positive curvature. Such manifolds are $S^3$, $S^2 \times S^1$, $S^3/\Gamma$ where $\Gamma$ is a finite group and connected sums of these manifolds. This is a small subset of the countably infinite number of three manifolds as can be demonstrated by a simple argument using results due to Thurston. Similar results hold for any matter source with positive stress energy. Thus the boundary conditions (3.10) imply that the kinds of topology change allowed in semiclassical approximation are limited. However, semiclassical topology changing amplitudes may not be allowed even for boundary manifolds in this limited set due to the first and second properties discussed above. For example, the Hartle-Hawking proposal does not yield a semiclassical amplitude for $RP^3$ with round 3-sphere metric.

The manifold $RP^3$ with its round metric can be constructed from $S^3$ with (3.1) by identifying antipodal points;

$$\chi = \pi - \chi; \quad \theta = \pi - \theta; \quad \phi = \phi + \pi \quad (3.11)$$

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Thus the unit metric and range of coordinates on $RP^3$ is

$$d\Omega^2 = \left( d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2 \right)$$  \hspace{1cm} (3.12)

$$0 \leq \chi \leq \pi \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq \pi$$  \hspace{1cm} (3.13)

differing from the ranges (3.2) for the $S^3$ round metric by a factor of 1/2 for the $\phi$ component. Locally, the manifolds $S^3$ and $RP^3$ have the same metrics; they are only distinguished by their global properties. Thus, it follows that the equations of motion are the same for the $RP^3$ case as for the $S^3$ case because they are local. Consequently a Lorentzian solution of the Einstein equations corresponding to $RP^3$ with round metric is identified de Sitter,

$$ds^2 = -dt^2 + a^2(t)d\Omega^2$$

$$a(t) = \frac{1}{H} \cosh(Ht)$$  \hspace{1cm} (3.14)

with topology $\mathbb{R} \times RP^3$. It is a nonsingular Lorentzian spacetime; it has the same local metric as the $S^3$ de Sitter spacetime (3.6), although the global properties of the spatial hypersurfaces of the two spacetimes are different. The Lorentzian extrema of (2.1) for $RP^3$ de Sitter correspond to complex stationary points of the Euclidean action. Thus they would be expected to contribute to the stationary phase approximation of the Hawking Hartle wavefunction for $Ha_0 > 1$ with action

$$\Psi_L(RP^3, a_0) \sim \cos(\bar{S}(a_0) + \alpha)$$

$$\bar{S}(a_0) = \frac{1}{6H^2\ell^2}(H^2a_0^2 - 1)^{\frac{3}{2}}$$  \hspace{1cm} (3.15)

with the phase $\alpha$ to be determined either by steepest descents contour or by matching conditions (3.9) at $Ha_0 = 1$. The action (3.15) differs by a factor of 1/2 from that of the $S^3$ case (3.7) due to the difference in volume of the boundary metrics.

The difficulty with constructing a semiclassical wavefunction occurs when looking for compact manifolds with real Euclidean metrics that satisfy the Einstein equations and
match the boundary data on $RP^3$. Locally, the equations of motion for explicitly spherically symmetric solutions are the same as those for the $S^4$ Euclidean de Sitter solution (3.4); thus one locally unique solution in a neighborhood of the initial $RP^3$ with round metric is

$$ds^2 = d\tau^2 + a^2(\tau)d\Omega^2$$

$$a(\tau) = \frac{1}{H} \sin(H\tau)$$

(3.16)

This metric is well defined between two $RP^3$ hypersurfaces with $a(\tau_0) = a_0$ and $a(\tau_1) = a_1$; the corresponding range of $\tau$ is $0 < \tau < \frac{2\pi}{H}$. The manifold of the solution has product topology $R \times RP^3$. The Euclidean action between the two $RP^3$ hypersurfaces is well defined

$$\bar{I}(a_0, a_1) = -\frac{1}{6H^2\ell^2}[(1 - H^2a_0^2)^{\frac{3}{2}} - (1 - H^2a_1^2)^{\frac{3}{2}}].$$

(3.17)

However, there is no compact Riemannian manifold with this solution as its metric. If the range of $\tau$ is extended to $\tau = 0$ in (3.16) then indeed $a = 0$ at this point. The corresponding topological space is compact. However, the global structure of this compact topological space is not that of a manifold by Def.(2.1); rather it is a cone over $RP^3$. In order to discuss this issue, it is necessary to introduce the definitions of a join, a cone and a suspension.\textsuperscript{19}

**Definition (3.2).** Let $U$ and $V$ be topological spaces. Their join, $U \ast V$ is the space formed by the cartesian product of $U$, $V$ and the unit interval $I$ modulo an equivalence relation, $U \ast V = (U \times V \times I)/\sim$ where

$$(u, v, t) \sim (u', v', t') \begin{cases} t = t' = 0 \text{ and } u = u' \text{ or } \\
 t = t' = 1 \text{ and } v = v'.
\end{cases}$$

**Definition (3.3).** The cone $C(V)$ is the join of the topological space $V$ with a point $\{p\}$, $C(V) = V \ast p$. 27
**Definition (3.4).** The suspension $S(V)$ is the join of the topological space $V$ with the zero dimensional circle, $S(V) = V \ast S^0$.

Figure 1 provides an example of a join and Figure 2 that of a suspension. Note that the suspension of a topological space is equivalent to gluing two cones of the space together at their boundary. For example, the cone of $S^n$ is the $(n+1)$-ball $B^{n+1}$, the suspension of $S^n$ is $S^{n+1}$ and it is equivalent to gluing two $B^{n+1}$ together along their $S^n$ boundary. In general, cones and suspensions of arbitrary topological spaces $V$ will not produce manifolds.

The compact topological space formed by the Euclidean solution (3.16) for $0 \leq \tau < \frac{2\pi}{\mathcal{H}}$ is $C(RP^3)$. This is obvious from Defs.(3.2-4); the metric (3.16) is homeomorphic to the cartesian space $(RP^3, \{p\}, \tau)$ with all points at $\tau = 0$ identified to a single point. The compact space formed by the Euclidean solution (3.16) for $0 \leq \tau \leq \frac{2\pi}{\mathcal{H}}$ is $S(RP^3)$; a representation of this suspension is given in Figure 3. The space $C(RP^3)$ is compact by definition. However, it is not a compact manifold. In order to prove this, note that given any $n$-manifold $M^n$ with $n \geq 3$, one can readily demonstrate that $\pi_1(M^n - \{p\}) = \pi_1(M^n)$ for any point $p \in M^n$. This is due to the fact that in three or more dimensions, curves can always be moved around an isolated point without ever crossing the point itself. Now assume that $C(RP^3)$ is a manifold and take the point $p$ to be the apex of the cone, $\tau = 0$. Note that by construction $C(RP^3) - \{p\} = I \times RP^3$. Hence $\pi_1(C(RP^3) - \{p\}) = \pi_1(I \times RP^3) = \mathbb{Z}_2$. However, note that $C(RP^3)$ is contractible which implies that $\pi_1(C(RP^3)) = 1$. Therefore, by contradiction, it follows that $C(RP^3)$ is not a manifold. Consequently, there is no classical Euclidean solution for $RP^3$ corresponding to the Euclidean de Sitter solution for $S^3$. By construction, it follows that there is no semiclassical Euclidean wavefunction for $RP^3$ with round metric corresponding to that given by the contribution of the Euclidean de Sitter for $S^3$. Therefore there is no semiclassical approximation to the Hartle-Hawking wavefunction for a boundary $RP^3$ with round metric even though there is a semiclassical Lorentzian wavefunction for this boundary geometry.
One objection to this conclusion is that this model is too restrictive; it implicitly assumes that the Euclidean solution is of the form (3.16). However, it can be argued that there is no Einstein manifold with $RP^3$ boundary with round metric that allows the continuity conditions (3.9) to be satisfied at $H a_0 = 1$. Note that $H a_0 = 1$ is a maximal slice in the Lorentzian identified de Sitter metric (3.14). Consequently, $K^L_{ij} = 0$. Therefore, given that the Lorentzian wavefunction is of the form (3.15), the matching conditions (3.9) imply that $K^E_{ij}$ must also vanish identically. Thus the compact Euclidean Einstein manifold with $RP^3$ boundary sought has vanishing extrinsic curvature and intrinsic metric (3.12). This manifold with boundary can be doubled over at the $K^E_{ij} = 0$, that is at the maximal slice, to form a closed Einstein manifold. Next, the locally unique evolution of this maximal slice is the Euclidean solution (3.16). Therefore the Weyl tensor vanishes, $C_{abcd} = 0$ by explicit calculation in a neighborhood of the maximal slice. Therefore, the scalar function given by the square of the Weyl tensor vanishes as well, $C^{abcd}C_{abcd} = 0$, in this neighborhood. Now, Einstein manifolds are analytic. Therefore if a function vanishes on an open set of the manifold, it must vanish everywhere. As $C^{abcd}C_{abcd}$ is positive definite, it follows that it must vanish everywhere. This implies that $C_{abcd} = 0$ everywhere. Einstein manifolds with vanishing Weyl tensor are space-forms and there are two space-forms with constant positive curvature; $S^4$ and $RP^4$. But $RP^3$ does not divide either of these manifolds into two manifolds with boundary $RP^3$. A proof of this assertion can be derived using the Mayer-Vietoris homology sequence for the decomposition of a topological space; the full argument is given in appendix A. Therefore, there are no Einstein manifolds with $RP^3$ boundary with round metric suitable for constructing the semiclassical wavefunction in the classically forbidden region. Thus the Hartle-Hawking initial condition does not produce a semiclassical wavefunction corresponding to Lorentzian $RP^3$ de Sitter.

What is the physical significance of this result? First, it indicates that in semiclas-
sical approximation to the Hartle-Hawking wavefunction, the geometry of the universe is limited to a very special class of 3-manifolds that admit positive scalar curvature: ones that divide Einstein manifolds. What is disturbing is that, as explicitly seen for $RP^3$, the obstruction to having a classical Euclidean path with $RP^3$ boundary occurs at one point. This suggests that there are manifolds that are approximately Einstein occurring in the Euclidean integral; that is paths with metric of form (3.16) up to a $RP^3$ boundary of radius $a_0$ at $\tau_0$ that then become nonextremal paths on some manifold $G$ that smoothly caps off this boundary. Note that the diameter of this boundary $RP^3$ can be made arbitrarily small by taking $a_0 \to 0$. Also, the volume of $G$ can be made arbitrarily small by a suitable conformal transformation. Thus if the curvature of the metric on $G$ can be controlled such that its integral over $G$ can be made small then the action would approach an extremum.

Therefore, it appears that there are sequences of manifolds histories that approach a limiting extremum; however, this limit point is not contained in the space of histories itself. Thus from a physical point of view, the contribution from the stationary point that is supposed to approximate the contribution from these paths is being suppressed because of a mathematical technicality; the limit point is not in the space of histories. Therefore this semiclassical result suggests that the properties of semiclassical histories considered in the Euclidean functional integral should be generalized. Furthermore, it immediately follows that the generalization should be carried over to the space of histories for the Euclidean functional integral itself.

From the topological nature of the example, it is logical to consider generalizing the topology of the histories to be included in the Euclidean integral. However, it is useful to first discuss the most immediate generalization; to allow some set of complex metrics, that is metrics which are neither Lorentzian or Euclidean as paths in the Euclidean functional integral.\textsuperscript{21} This proposal is appealing as it is intimately connected to unresolved issues such as conformal rotation\textsuperscript{22} needed to concretely implement the Euclidean functional
integral beyond the semiclassical approximation. Although the study of this approach is certainly worthwhile, it must be defined and implemented. A complex metric can be considered to be a complex symmetric tensor defined on a smooth manifold. A general complex symmetric tensor is not invertible and will not have well defined curvature, so one immediately expects that the generalization must be restricted to some smaller set of complex metrics for which the action and curvature can be made well defined. How to do so is not straightforward; the Einstein equations for a general complex metric are neither elliptic nor hyperbolic and thus standard existence theorems for solutions to initial data do not apply. Such an existence theorem cannot be constructed for general complex metrics and thus a specification of the allowed complex metrics must be made and an existence theorem proven for this set of complex metrics. Such a specification can be made well defined in simple models, however, it remains to be seen how to do so for some general family of complex metrics.

The immediate effect of such a generalization is that the constraint on the scalar curvature of the boundary 3-manifold is slightly relaxed as the continuity condition on the wavefunction (3.9) no longer involves purely real and imaginary actions. Thus, one might hope to produce semiclassical amplitudes for a broader class of 3-manifolds. However it is important to note that though it may provide more general semiclassical wavefunctions, it does not provide a method of including stationary limit points for all sequences of histories. In particular it does not provide such a stationary limit point for the $RP^3$ boundary with round metric. For example, allowing for complex solutions of the form (3.16) does not produce a manifold in the $RP^3$ case. Take the metric to be a one parameter complex metric by allowing the variable $\tau$ in (3.16) to be complex. A direct computation of the curvature leads to the same equations of motion (3.4) as for the real $\tau$ case. The solution to the equations of motion is analytic; consequently, the topological space is always pinched off at a non-manifold point. Again, as in the real case, more general arguments can be
made to show that the non-manifold point persists for complex metrics. Therefore complex metrics do not provide a generic solution to the problem; indeed this is to be expected as the topology and metric of a history are specified independently. Thus one is brought back to the idea that the topology of the histories should be generalized.

A natural starting point is to generalize the topology in the minimal way needed to produce an amplitude for $R\mathbb{P}^3$. For example, semiclassical calculations of the Hartle-Hawking wavefunction for a given boundary 3-manifold could be performed on the covering space of the boundary manifold, with the result being pulled back to the original space. This proposal abandons the manifold restriction on the classical Euclidean solution, replacing it with a restrictive generalization. It immediately yields semiclassical amplitudes for round $R\mathbb{P}^3$; the covering space of $R\mathbb{P}^3$ is simply $S^3$ which does have a semiclassical amplitude. However, it is not true in general that the covering space of an arbitrary compact 3-manifold is also a compact 3-manifold. For example, let $\Sigma^3$ be a $K(\pi,1)$ manifold, that is a manifold whose only nonvanishing homotopy group is the fundamental group. The covering space of this manifold is an open contractible 3-manifold which is not compact; therefore its covering space is not the boundary of a compact 4-manifold. Therefore, the Hartle Hawking wavefunction is not defined for the covering space and thus the calculation cannot be performed. Of course one might argue that this case is not interesting because in general $\Sigma^3$ will have negative $^3R$ and thus cannot be the boundary of an Einstein manifold with $K_{ij} = 0$. However, it is unsatisfactory to have a prescription that must be applied on an ad hoc basis. Therefore, an appropriate generalization of the topology of the history should be formulated in more generic terms. Certain results on the moduli space of Einstein metrics on manifolds support such a viewpoint. Under certain conditions on the volume and curvature of the manifold, it can be proven that nonmanifold Einstein spaces occur as boundaries of the moduli space of Einstein metrics on manifolds. Such spaces have points locally related to covering spaces; however, globally their structure is
different. Therefore, heuristically, one expects that such spaces must be included in the semiclassical approximation to the sum over histories. In order to begin the definition of an appropriate set of topological spaces to allow as histories, it is useful to summarize the properties needed for use in Euclidean functional integrals for gravity.

4. REQUIREMENTS ON THE CLASS OF GENERALIZED HISTORIES

Given the motivation for including histories formed of more general topological spaces in Euclidean functional integrals, the next step is to address the issue of what set of topological spaces should be included. In order to do so, the purpose must be kept in mind, namely formulating Euclidean functional integrals for Einstein gravity. Recall that Riemannian histories played a critical role in the formulation of the Euclidean functional integral (2.1); indeed these classical histories carry the topological properties of the space of histories. It is clear that a useful starting point for finding a more general space of histories is to find a description of the generalized histories that are the analogs of the Riemannian histories; that is to find well behaved classical histories formulated on more general topological spaces. It is obvious that these more generalized histories must contain all manifolds. The requirement that the generalized topological spaces have a notion of classical geometry places certain restrictions on the kinds of topological spaces allowed. Thus this section will discuss the geometrical properties that are required of candidates for generalized histories; histories and their weighting must be implementable.

A generalized history in the Euclidean gravity consists of a topological space $X$. This history is weighted by an action, which in the case $X$ is a smooth manifold, is simply the Euclidean action $I(g)$. In order to concretely define the corresponding action for the generalized topological space $X$, distance, volume, and scalar curvature must be defined.
This restriction limits the set of generalized topological spaces.

In order to define distance, the topological spaces \( X \) must be metrizable.\(^{19,24} \) This alone is not enough of a restriction because metrizable topological spaces can have regions of different dimension. For example, a sphere with a flagpole attached at the north pole is a rather nice metrizable space. However, how to weight the contributions of such spaces in a sum over histories is not clear as the form and properties of the action depend on the dimension of the space. Therefore, it is reasonable to require that the dimension of the metrizable topological space be well defined and more specifically, that it have uniform dimension. When the dimension \( n \) is well defined, the space will be denoted by \( X^n \). Again, one can find examples which satisfy this new condition but which are unsatisfactory for other reasons. So before proceeding with the selection of the generalized space it will be useful to review some facts about manifolds. Furthermore, by stating some of these properties in a more abstract language, one is able to decide how to define things such as distance, geodesics, integration, and curvature on the generalized spaces. Once armed with these concepts the choice of suitable topological spaces is simplified.

The length of a smooth curve on a Riemannian manifold is easily defined using the metric and tangent vectors to the curve. Since the metric is positive definite, the length can be used to define a distance function on the manifold in the following way:\(^{11} \)

**Definition (4.1).** Given a connected Riemannian manifold \( M^n \), a distance \( d(x, y) \) for \( x, y \in M^n \) can be defined by

\[
d(x, y) = \inf_{c \in \Omega[x,y]} \ell(c) \tag{4.1}
\]

where \( \Omega[x,y] \) is the set of all \( C^1 \) curves between the points \( x \) and \( y \),

\[
\Omega[x,y] = \{ c : [0, 1] \to M^n | c \text{ is } C^1 \text{ with } c(0) = x \text{ and } c(1) = y \}
\]

and \( \ell(c) \) is the length of the curve,

\[
\ell(c) = \int_0^1 \left( g_{ab} \dot{c}^a \dot{c}^b \right)^{1/2} ds. \tag{4.2}
\]
The distance function $d$ defined above is compatible with the topology of the manifold. This means that open sets and all other topological properties are determined entirely by $d$. Of course, the choice of different Riemannian metrics yields different distance functions; however all the topological properties of the manifold will remain equivalent for all choices of metric. Furthermore, the critical points of $\ell$ are geodesics. This suggests a relationship between geodesics and topology which is explicitly given in the following well known theorem.

**Theorem (Hopf-Rinow) (4.2).** Given a connected Riemannian manifold $M^n$, the following are equivalent:

i) The distance function $d(x,y)$ is Cauchy complete.

ii) $M$ is geodesically complete.

iii) Any two points can be joined by a minimizing geodesic.

One can see that this theorem is useful for deciding when a Riemannian manifold is geodesically complete. For example, given a closed Riemannian manifold one can easily show it is complete as a metric space using a compactness argument. Then direct application of the above theorem proves it is geodesically complete.

One might wonder how much of the geodesic properties are described by $d$. In fact, geodesics can be characterized entirely in terms of $d$. To see this, one can prove that the length of a geodesic is the distance between its endpoints. Furthermore, given any point of the geodesic between the endpoints, the length of the geodesic is the sum of the distances from each endpoint to the third point. This property of the addition of distances is captured in the more concise definition:

**Definition (4.3).** Given a topological metric space $X$ with distance function $d$, a segment
is a continuous map $c : [a, b] \rightarrow X$ such that

$$d(c(t_1), c(t_2)) + d(c(t_2), c(t_3)) = d(c(t_1), c(t_3))$$

whenever $a \leq t_1 \leq t_2 \leq t_3 \leq b$.

Although not relevant to this discussion, note that Def.(4.3) applies to metric spaces of nonuniform dimension.

A geodesic is therefore a segment. The following theorem proves the converse; it is proven using the fact that every point has neighborhood such that any two points in the neighborhood are connected via a geodesic. The interesting feature of the following theorem is that a segment is only assumed to be continuous by Def.(4.3).

**Theorem (4.4).** Let $M^n$ be a Riemannian manifold with Riemannian metric $g$ and induced topological metric $d$, the segments of $d$ are geodesics (up to parameterization) of $g$.

The notion of equivalence of Riemannian manifolds is useful both mathematically and physically. Recall, two Riemannian manifolds are equivalent if there is a diffeomorphism between them such that the metrics are pullbacks of one another. In order to address this issue, one can see what happens to $d$ via the pullback. Locally, this is just a change of coordinates and the two distances will be related via an isometry as defined below.

**Definition (4.5).** An isometry between metric spaces $X$ and $X'$ is a map $f : X \rightarrow X'$ such that $d(x, y) = d'(f(x), f(y))$.

Again, note that Def.(4.5) holds for metric spaces of nonuniform dimension. Conversely, given any isometry of the two distance functions as defined in (4.1), the following is true:
**Theorem (4.6).** Given a map $f$ of a Riemannian manifold $M^n$ onto a Riemannian manifold $M'^n$, such that $d(x, y) = d'(f(x), f(y))$, then $f$ is a diffeomorphism and $g = f^*g'$.

Furthermore, one does not need to assume continuity of $f$.

The above definitions and theorems all reflect properties of Riemannian manifolds that are utilized in constructing the geometry of histories for Euclidean functional integrals. Any candidate for a set of generalized histories will also need to satisfy these properties in order for the geometry of a history to be well behaved. Thus at this point the set of topological spaces can be greatly restricted by imposing the requirement that the above theorems and definitions or a direct generalization of them hold. Such a set of topological spaces are polyhedra or spaces homeomorphic to a polyhedra.\(^{24}\)

In order to define these spaces, a little background is needed. The notion of the join of topological spaces was given in Def.(3.2); however, this definition is only continuous and does not impose any further conditions. It useful to define a more restricted version of joins in order to obtain a nicer set of spaces. This definition is the *piecewise linear* or *PL* version. Let two subspaces $X$ and $Y$ of $\mathbb{R}^n$ be positioned so that for any distinct points $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ the line segments connecting $x_1$ to $y_1$ and $x_2$ to $y_2$ do not intersect. For spaces $X$ and $Y$ positioned in such a way, the *PL join* is the union of all line segments joining points of $X$ to points of $Y$. The join will be denoted $XY$. One can show that as topological spaces, the PL join when it exists is homeomorphic to the topological join $X * Y$. This means that as topological spaces, there is no difference between the PL join and the topological join. The difference is that the PL join has more structure and can only be defined when the two space can be positioned in this nice way. Using the PL join, a PL cone is defined to be PL join of space $X$ with a point $a$. It will be denoted by $aX$. Again a PL cone $aX$ is homeomorphic to a topological cone $C(X)$.

**Definition (4.7).** A polyhedron $P$ is a subset of $\mathbb{R}^n$ such that each point $p$ has a cone

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neighborhood $N = aL \subseteq P$ where $L$ is compact topological space. $L$ is called the link of the neighborhood $N$.

It is important to note that the dimension of $P$ is not necessarily that of the space $\mathbb{R}^n$. Like a smooth manifold, the space $P$ can be triangulated. This is connected to the fact that the spaces $L$ are not pathological and that the PL join is linear. The spaces $P$ given by Def.(4.7) may seem limited at first; however, any topological space which has a triangulation is always homeomorphic to a polyhedron. Therefore the spaces $P$ are quite general. Since this paper is concerned with topological properties of spaces alone, it is useful at the present time to introduce the convention that the term polyhedron refers to any space homeomorphic to a polyhedron.

There are many examples of polyhedra: Smooth manifolds are a subset of polyhedra; in this case each point has a cone neighborhood where $L \approx S^{n-1}$. An example of a non-manifold polyhedron is the surface of a $n$-dimensional hypercube with a line segment attached at one of its vertices; in this case the vertex where the line segment is attached has a cone neighborhood where $L$ is the disjoint union of $S^{n-1}$ and a point, all other points in the hypercube have cone neighborhoods with $L \approx S^{n-1}$ and all other points in the line segment have cone neighborhoods with $X \approx S^0$. It is clear that all $L$ are compact topological spaces even though they are not all connected manifolds.

It is obvious from these examples that polyhedra include metrizable spaces of different dimension; however it is easy to restrict to the subset of pure polyhedra: A pure polyhedron is one for which every point has a cone neighborhood $aL$ with the same dimension $n$. Other well defined subsets of the set of all polyhedra can be formed as well by applying the usual definitions of closed and connected to these topological spaces.\textsuperscript{14}

Next the requirement that the action be defined is now more appropriately addressed as it can be restricted to the case of pure polyhedra. As pure polyhedra are subsets
of \( \mathbb{R}^n \), Lebesgue integration of integrands with the appropriate behavior is well defined, namely the Lebesgue integral is well defined if the integrand is defined except on sets of measure zero. Secondly, if the notion of a stationary point of the action is to carry over to generalized histories, it must be possible to associate a well defined scalar curvature with each point of the polyhedra. Thus the task is to find a subset of pure polyhedra for which the scalar curvature has the appropriate behavior.

Recall that the Riemann curvature of a Riemannian manifold is defined in terms of parallel transport around infinitesimal closed curves on the manifold. Such curves lie in a neighborhood diffeomorphic to \( \mathbb{R}^n \). Given the Riemann curvature, the scalar curvature is then computed by taking the appropriate contraction of the indices with the metric. Thus, turning to the case of pure polyhedra, it is clear that scalar curvature can be defined exactly as for manifolds for all points in the polyhedra that have neighborhoods diffeomorphic to \( \mathbb{R}^n \). Thus the task is to extend this definition of curvature to points of the polyhedra which do not have such neighborhoods. Note that one only needs to extend the scalar curvature to all points in the pure polyhedra in order that the action of the history be well defined. As the scalar curvature must be well defined at all points in the pure polyhedra, the set of measure zero must not have Hausdorff dimension greater than zero; i.e. it must not be a line segment, triangle, or other \((n-1)\)-dimensional set. The reason is that for such sets of measure zero, the value obtained from taking the limit of the scalar curvature onto the set will be direction dependent for arbitrary metrics. For example, consider a polyhedra consisting of two 2-spheres with a line segment in each identified as in Figure 4. The space consisting of the polyhedra minus the line segment is two disjoint 2-spheres, each minus a line segment. The scalar curvature on each two sphere is well defined, even though the metric on the space is incomplete. However, the scalar curvature does not approach a well defined value as points on the missing line segment are approached for a general metric on the space. The value of the scalar curvature will depend both on the direction of approach
and on the parameterization of the removed line segment. Thus a limiting procedure will not yield a well defined scalar curvature on each point of the line segment. Therefore the subset of pure polyhedra must be restricted to a subset for which there is a set of isolated points whose cone neighborhoods \( aL \) are not homeomorphic to \( \mathbb{R}^n \); additionally, the closed spaces \( L \) must be manifolds. In this case, the limit of the scalar curvature onto the singular points is controlled by the relation of the metric on the space to the distance function; away from the singular point \( a \), the curvature is continuous as the cone neighborhood \( aL \) minus the point \( a \) is a manifold and as the point \( a \) is approached, the curvature must approach a fixed value from all directions.

Even though this subset of pure polyhedra forms a relatively nice set of topological spaces, the scalar curvature still cannot necessarily be defined at all points for all polyhedra in this subset. For example, consider the compact polyhedron formed by taking two \( C(S^3) \) and identifying the two vertices as illustrated in Figure 4. Let the metric on the first cone be \( dt^2 + \alpha^2 t^2 d\Omega^2 \) and that on the second be \( d\rho^2 + \beta^2 \rho^2 d\Omega^2 \) with constants \( \alpha \neq \beta \) and the vertex at \( t = \rho = 0 \). The scalar curvature can be computed on each cone away from the vertex; it is \( 6/\alpha^2 \) on the first and \( 6/\beta^2 \) on the second. However, as the scalar curvature is not the same constant on both cones, it cannot be defined at the vertex by an extension of the function on both cones to this point. Therefore, the scalar curvature is not necessarily well defined at the vertex. This example indicates that the subset of pure polyhedra should be further restricted to be those for which the isolated set of points have cone neighborhoods \( aL \) where \( L \) is a closed connected manifold.

Thus, geometrical properties greatly restrict the set of topological spaces that can be considered as candidates for generalized histories. It is now possible to propose and discuss a new set of generalized histories that is a subset of these polyhedra, contains all manifolds and contains the suspension of \( RP^3 \) that appeared in the study of the semiclassical approximation. The geometry of the above set of polyhedra will be done by using the properties
which only depend on the distance function. In particular, given a Riemannian metric at
the manifold points of the polyhedra one will choose a distance function on the polyhedra
which is the Cauchy completion of the distance induced by the given Riemannian metric.
The allows natural extensions of geometry to the new set of spaces. For example, geodesics
on these spaces will be defined as segments. By the above theorems, this will agree with
the usual definition of geodesics at the manifold points. Similarly, these polyhedra will be
considered to be geometrically equivalent when they are isometric via an isometry of the
distance function. Again, at manifold points this is equivalent to diffeomorphism equiv-
almence of Riemannian metrics as mentioned above. Integration of functions on this set
of spaces will be defined via the measure constructed from the distance function. If one
deletes all of the nonmanifold points, this is equivalent to the integration of functions us-
ing the volume element of the given Riemannian metric. Hence, one can define square
integrable functions and other useful objects from analysis on this set of polyhedra. Thus
the study of the geometry of spaces in this new set of spaces is akin to that of manifolds.

5. GENERALIZED TOPOLOGICAL
SPACES FOR QUANTUM GRAVITY

This section will present a new set of topological spaces for use as generalized histories.
This set has not been previously defined or studied in the literature. These spaces will be
called conifolds. Smooth conifolds are a subset of the special set of pure polyhedra for which
the neighborhood of every point is a homeomorphic to a PL cone over a closed connected
manifold. By definition this subset clearly includes all manifolds. As for manifolds, it is
the subset of topological conifolds that admit a smooth structure, smooth conifolds, that
are appropriate for physics.

At this point it is useful to define a slightly more general set of spaces than the above
polyhedra by weakening the PL cone neighborhoods to be only the topological cones $C(L)$. This generalization allows the following definitions to parallel the analogous definitions for manifolds. The name of this more general set, conifolds, is due to the observation that the neighborhood of every point is a cone.

**Definition (5.1).** A $n$-dimensional conifold $X^n$, $n \geq 2$, is a metrizable space such that given any $x_0 \in X^n$ there is an open neighborhood $N_{x_0}$ and some closed connected $(n-1)$-manifold $\Sigma_{x_0}^{n-1}$ such that $N_{x_0}$ is homeomorphic to the interior of $C(\Sigma_{x_0}^{n-1})$ with $x_0$ mapped to the apex of the cone. Any neighborhood homeomorphic to such cones will be referred to as conical neighborhoods.

A zero dimensional conifold will defined as a collection of discrete points. This is the same definition as that of a zero dimensional manifold. The definition of a one dimensional conifold is almost the same as that of the above $n$-dimensional conifold except that the links are assumed to have two disconnected components. The reason for this follows from the case of 1-manifolds: Every point of a 1-manifold has a neighborhood that is a cone over the zero sphere $S^0$. Since $S^0$ consists of two points it has two disconnected components; consequently the neighborhood of any point of a 1-manifold is a cone over two disconnected points. Thus it is natural to define a 1-conifold by requiring that the links consist of two disconnected components. Using this definition, 1-conifolds are just 1-manifolds. Therefore, zero and one dimensional conifolds are manifolds.

In two dimensions, the general $n$-dimensional definition applies, however all 2-conifolds are manifolds because the only closed connected 1-manifold is the circle $S^1$. Since there are a countably infinite number of closed connected 2-manifolds, 3-conifolds are not just 3-manifolds but more general metrizable spaces. The same will be true in all dimensions larger than three. The set of all $n$-conifolds includes all topological $n$-manifolds as the neighborhood of every point in a manifold is homeomorphic to a cone over $S^{n-1}$. Secondly
it follows from Def.(3.4) that the set of n-conifolds includes the suspensions of all closed connected (n-1)-manifolds. For example, the suspension of $RP^3$ described in section 3 and illustrated in Figure 3 is also included in the set of 4-conifolds; the neighborhoods of the singular points at the north and south poles are cones over $RP^3$ by construction and all other points have neighborhoods homeomorphic to cones over $S^3$. This space is an example of a conifold with a finite number of nonmanifold points. Closed n-conifolds which are neither manifolds nor suspensions of manifolds can also be readily constructed. Two examples are given below and Figure 5 provides a visualization of another.

The first example is $K^n = T^n / \mathbb{Z}_2$ for $n \geq 3$ where the $\mathbb{Z}_2$ action is defined as follows: The n-torus can be thought of as $T^n = \{(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \mid -\pi \leq \theta_n < \pi\}$. Define a self-homeomorphism $c : T^n \rightarrow T^n$ by $c(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) = (e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n})$. Clearly, $c$ is a self-homeomorphism and $c^2$ is the identity map. Hence $c$ gives an action of the cyclic group $\mathbb{Z}_2$ on $T^n$. Finally $K^n$ is defined as the quotient space $T^n / \mathbb{Z}_2$ where $\mathbb{Z}_2$ is represented by $c$.

$K^n$ as just defined is a closed conifold with $2^n$ singular points for $n \geq 3$ and each singular point has a conical neighborhood homeomorphic to a cone over $RP^{n-1}$. In order to prove this, the observation that the identifying map $c$ has fixed points is utilized. In general, such fixed points indicate that $K^n$ will not be a manifold. Let $F = \{x \in T^n \mid c(x) = x\}$, i.e. the set of fixed points of the $\mathbb{Z}_2$ action. Next pick some $x_0 \in F$ and a small closed neighborhood $B^n_{x_0}$ of $x_0$ in $T^n$ such that $B^n_{x_0} \cap F = \{x_0\}$ and $B^n_{x_0}$ is homeomorphic to a n-ball with a (n-1)-sphere as boundary. Next, define a new neighborhood $N^n_{x_0}$ of $x_0$ by $B^n_{x_0} \cap c(B^n_{x_0})$ where $c(B^n_{x_0})$ is the image of $B^n_{x_0}$ with respect to $c$. By construction, $c(N^n_{x_0}) = N^n_{x_0}$ and $N^n_{x_0}$ is homeomorphic to a n-ball. Now let $Y^n = T^n - \bigcup_{x \in F} \text{int}(N^n_x)$. The $\mathbb{Z}_2$ group action acts freely on $Y^n$ because all fixed points are removed. Hence $Y^n / \mathbb{Z}_2$ is a compact manifold with boundary. Since the boundary of $Y_n$ is a disjoint union of (n-1)-spheres and $\mathbb{Z}_2$ acts freely on each, it follows that $Y^n / \mathbb{Z}_2$ is a compact manifold.
with boundary equal to the disjoint union of $2^n$ copies of $\mathbb{R}P^{n-1} = S^{n-1}/\mathbb{Z}_2$. Finally, $K^n = Y^n/\mathbb{Z}_2 \cup N(S)$ where $N(S) = \bigcup_{x \in F} N^n_x/\mathbb{Z}_2$. Each $N^n_x/\mathbb{Z}_2$ is homeomorphic to $\mathbb{Z}_2$ acting on a n-ball with one fixed point and it follows that $N^n_x/\mathbb{Z}_2$ is a cone over $\mathbb{R}P^{n-1}$. Consequently $K^n$ is a conifold for $n > 2$; $K^2$ is a manifold because $\mathbb{R}P^1 = S^1$.

The second example is $L^{2n} = \mathbb{C}P^n/\mathbb{Z}_3$ for $n \geq 2$ where $\mathbb{C}P^n$ is complex projective space and the $\mathbb{Z}_3$ action will be defined below. $L^{2n}$ is a closed conifold with one singular point. $\mathbb{C}P^n$ is the set of complex lines in $\mathbb{C}^{n+1}$; however, a more useful description of it for present purposes is the following: Let the $(2n+1)$-sphere $S^{2n+1}$ be defined by $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) | z_k \in \mathbb{C}, \sum_{k=0}^n \bar{z}_k z_k = 1\}$. Now define the equivalence relation $(z_0, z_1, \ldots, z_n) \sim (w_0, w_1, \ldots, w_n)$ if and only if $(z_0, z_1, \ldots, z_n) = (e^{i\theta}w_0, e^{i\theta}w_1, \ldots, e^{i\theta}w_n)$ for some real $\theta$. Then $\mathbb{C}P^n = S^{2n+1}/\sim$.

Next let $a : \mathbb{C}P^n \to \mathbb{C}P^n$ be defined by $a([z_0, z_1, \ldots, z_n]) = [az_0, z_1, \ldots, z_n]$ where $a = e^{2\pi i}$ and $[z_0, z_1, \ldots, z_n]$ is the equivalence class in $S^{2n+1}/\sim$. Clearly, $a$ is a self-homeomorphism and $a^3$ is the identity map. Using the above definitions combined with some basic algebra, it is possible to show that the only fixed point is the point $[1, 0, \ldots, 0]$ in $\mathbb{C}P^n$. Repeating the same type of analysis as done in the case of $K^n$, one can show that $L^{2n}$ is a conifold with one singular point with neighborhood $S^{2n-1}/\mathbb{Z}_3$.

Examples of n-conifolds with an infinite number of nonmanifold points can also be constructed: Pick a countably infinite number of disjoint n-balls in $\mathbb{R}^n$ with $n \geq 3$, and continuously embed a (n-1)-torus in each ball. Now, remove the interior region bound by each torus. The resulting space is a n-manifold with a countable number of (n-1)-tori as boundary. Finally, cap off the boundaries on this manifold with the cone over each torus. The resulting space is a n-conifold.

In the above examples, the conifolds are manifolds except at a discrete set of points. In general, this will be true for all conifolds. In order to see this, let $S$ be the set of points in a n-conifold $X^n$ which do not have neighborhoods homeomorphic to the interior of a
cone over $S^{n-1}$. The set $S$ is called the singular set of $X^n$. It is the set of points at which the conifold is not a manifold. In order to prove that $S$ only consists of a discrete set of points, one needs to show that $S$ has no limit points, i.e. that one can find a collection of disjoint neighborhoods around all of the points in $S$ simultaneously. Each $x_0 \in S$ has a conical neighborhood $N_{x_0}$ homeomorphic to $C(\Sigma^{n-1}_{x_0})$. Since the only nonmanifold point of the cone is the apex, it follows that the only nonmanifold point of $N_{x_0}$ is $x_0$. Hence, as each $x_0$ in $S$ has a neighborhood that contains no other point of $S$, no point of $S$ is a limit point of the set $S$. Now, one must show that the neighborhoods can be chosen to be simultaneously disjoint; this is equivalent to showing that there are no limit points of $S$ in $X^n$. Let $x_1 \in X^n$ be some point not in $S$. This means $x_1$ has a conical neighborhood homeomorphic to a cone of a $(n-1)$-sphere. Hence all points in this neighborhood are manifold points. Hence, as $x_1$ has a neighborhood that contains no point in $S$, $x_1$ is not a limit point of $S$. Therefore, $S$ has no limit points. Immediately it follows that $S$ consists of a discrete set of points.

The set $S$ of singular points is always countable for each connected component of a $n$-conifold. In order to see this, let $X^n$ be a connected conifold. Let $N(S)$ be the disjoint union of connected conical neighborhoods about each of the singular points. From the above discussion, $X^n - N(S)$ is a topological connected n-manifold with boundary and therefore can be continuously embedded in $\mathbb{R}^{2n+1}$. If one continuously pinches off each of the boundaries of $X^n - N(S)$, one obtains a space homeomorphic to $X^n$. Hence, $n$-conifolds can be continuously embedded in $\mathbb{R}^{2n+1}$. Now, the topology of $\mathbb{R}^{2n+1}$ has a basis which consists of a countable number of open sets with compact closure. It follows that $X^n$ has the same property because it is a subspace. Now suppose that $S$ consisted of an uncountable number of points; as the basis of $X^n$ is countable it follows that there would have to be an infinite number of points of $S$ in one of the these sets with compact closure. This would mean that $S$ has limit points; however by the proof of the previous
paragraph, $S$ can have no limit points. Therefore, the set $S$ can have only a countable number of points.

Def.(5.1) defines topological n-conifolds without boundary; it is also useful to define n-conifolds with boundary. The boundary of an n-conifold will be required to be a (n-1)-manifold without boundary. This restriction is consistent with the definition of n-manifolds with boundary; additionally it makes the mathematics of these spaces less pathological. It is also motivated by the requirements of a physical application of conifolds to Euclidean functional integrals as the argument of the amplitudes typically consists of a metric on a closed (n-1)-manifold. Thus, specifically, a \textit{n-conifold with boundary} is a metrizable space $X^n$ such every point has a conical neighborhood as in Def. (5.1) or an open neighborhood homeomorphic to open subset of the half-space $\mathbb{R}^n_+$. The \textit{boundary} of $X^n$, $\partial X^n$, is defined as the set of points which are mapped to boundary points of $\mathbb{R}^n_+$. Again, the singular set $S$ is defined as the set of points in $X^n$ whose conical neighborhoods are not homeomorphic to a n-ball. The definition of a n-conifold with boundary implies that all the singular points are on the interior of $X^n$. Again, one can show that on each connected component of $X^n$, $S$ will be a countable set. Similarly, $X^n - S$ will be a manifold with boundary and $\partial X^n = \partial(X^n - S)$. This property implies that the boundary of a conifold is topologically well defined; it is preserved under homeomorphisms of the space. Finally, in analogy with the manifold case, a \textit{closed conifold} is defined as a compact conifold without boundary.

Given that three dimensions is the lowest dimension allowing nontrivial examples of conifolds, it is interesting to see how different the set of 3-conifolds is from that of 3-manifolds. Also, it would be useful to have a simple test that describes the subset of 3-conifolds which are 3-manifolds. Recall that the Euler characteristic of a topological space is just the alternating sum of its Betti numbers.$^{25}$ Thus calculation of the Euler characteristic is reduced to simple arithmetic for such spaces. The observation that the Euler characteristic is zero for closed 3-manifolds suggests that it provides a simple test
for determining whether or not a 3-conifold is a 3-manifold. Indeed, the following theorem will be of great value in Part II of this paper to discuss the algorithmic decidability of conifolds.

**Theorem (5.2).** Let $X^3$ be a closed 3-conifold. Then $X^3$ is a 3-manifold iff $\chi(X^3) = 0$.

First $X^3$ is a 3-manifold except at a finite set of singular points $S$. At each singular point in $S$, choose a small connected neighborhood such that it is homeomorphic to a cone over some closed 2-manifold. Let the collection of all these neighborhoods be $N(S)$. Then $M_0 \equiv X^3 - N(S)$ is a compact 3-manifold with boundary. Using the Mayer-Vietoris sequence for homology one can show that

$$\chi(X^3) - \chi(M_0) + \chi(\partial M_0) - b_0(S) = 0 \quad (5.1)$$

and also as demonstrated in Appendix A that

$$2\chi(M_0) = \chi(\partial M_0). \quad (5.2)$$

Combining these two equations yields $\chi(X^3) + \frac{1}{2}\chi(\partial M_0) = b_0(S)$. Now, assume that $\chi(X^3) = 0$; then

$$\chi(\partial M_0) = 2b_0(S). \quad (5.3)$$

Next $b_0(\partial M_0) = b_0(S)$ as $b_0(\partial M_0)$ is the number of connected components of $\partial M_0$. Furthermore, $b_2(\partial M_0) \leq b_0(S)$ because on each connected component of $\partial M_0$, $b_2$ is either one or zero. Hence, using these two results and $\chi(\partial M_0) = b_0(\partial M_0) - b_1(\partial M_0) + b_2(\partial M_0)$ in (5.3),

$$-b_1(\partial M_0) = b_0(S) - b_2(\partial M_0) \geq 0.$$

The only solution is if $b_1(\partial M_0) = 0$. This implies that $b_2(\partial M_0) = b_0(S)$. Finally, it follows that the Euler characteristic of each connected component of $\partial M_0$ is equal to 2. The only closed connected 2-manifold with Euler characteristic equal to 2 is a 2-sphere. Hence $N(S)$ is the disjoint union of 3-balls. Therefore, $X^3$ is a 3-manifold.
Conversely, it is easily proven that the Euler characteristic of any odd-dimensional manifold is zero. Q.E.D.

The proof of this theorem can be applied directly to compute the Euler characteristic for explicit constructions of 3-conifolds; for example one can show from the construction of $K^3 = T^3/\mathbb{Z}_2$ that it is a 3-conifold. Recall that $K^3 = Y^3/\mathbb{Z}_2 \cup N(S)$ where $S$ is the set of 8 singular points and $Y^3/\mathbb{Z}_2$ is a compact manifold with boundary consisting of the disjoint union of eight $RP^2$ manifolds. Eqn.(5.2) can be applied to find that $2\chi(Y^3/\mathbb{Z}_2) = 8\chi(RP^2)$. Then, using $\chi(RP^2) = 1$, (5.1) immediately yields $\chi(K^3) = -4\chi(RP^2) + b_0(S) = 4$. Thus the proof of Thm.(5.2) provides a very useful method of computing the Euler characteristic for certain 3-conifolds as well as a characterization of the subset of 3-manifolds.

In four dimensions, conifolds are more complicated than in three dimensions. For example, there is no simple test to determine which 4-conifolds are 4-manifolds as in Thm.(5.2) as there is no known algorithmic description of 4-manifolds. Another problem in four or more dimensions is that there exist conifolds which are not homeomorphic to polyhedra. This may seem strange until one recalls that there are closed topological 4-manifolds which are not homeomorphic to polyhedra. Such examples will be discussed more fully in Part II of this paper after the appropriate machinery is introduced; however, the point is that examples of poorly behaved conifolds exist in four or more dimensions and this means that one needs to place extra conditions on topological conifolds in order to use them in physics. This is not a conceptual problem as such extra conditions are indeed imposed in the case of manifolds already as discussed in section 2; some sort of smoothness or regularity is required if geodesics, triangulations, and other geometric objects are to be definable. Thus such extra conditions are natural. Therefore given the topological definition of conifolds, one would like to discuss the smoothness and geometry of conifolds. Again as in the case of manifolds, conifolds with boundary and conifolds without boundary
can be discussed at the same time.

One can define an atlas on a conifold in exactly the same manner as one defines an atlas on a manifold:

**Definition (5.3).** An atlas on a n-conifold $X^n$ is a collection \{$(U_\alpha, \varphi_\alpha)$\}_\alpha_\Lambda of open sets and homeomorphisms indexed by a set $\Lambda$ satisfying the following:

1. The sets $U_\alpha$ cover $X^n$.
2. $X^n - S = \bigcup_{\alpha \in \Lambda_0} U_\alpha$ for some subset $\Lambda_0 \subset \Lambda$.
3. For $\alpha \in \Lambda_0$, $\varphi_\alpha$ is a homeomorphism of $U_\alpha$ to an open set in $\mathbb{R}_+^n$.
4. For each $\alpha \in \Lambda - \Lambda_0$, $U_\alpha$ is a conical neighborhood of a singular point and $\varphi_\alpha$ is homeomorphism onto the interior of a cone.

Using the above notion of atlas, one can define a smooth conifold by analogy with the definition of a smooth manifold Def.(2.1). The main difference lies in defining the smoothness near the singular points. This is done by requiring that there is a neighborhood of each singular point such that removing it yields a smooth manifold with boundary. More precisely,

**Definition (5.4).** A n-conifold is smooth $(C^k)$ if and only if there is an atlas \{$(U_\alpha, \varphi_\alpha)$\}_\alpha_\Lambda such that

\[
\varphi_\beta \varphi^{-1}_\alpha : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)
\]

is a smooth $(C^k)$ map on $\mathbb{R}_+^n$ for $\alpha \in \Lambda_0$ and the remaining sets $U_\alpha$ are connected conical neighborhoods of the singular points such that $X^n - \bigcup_{\alpha \in \Lambda - \Lambda_0} U_\alpha$ is a smooth $(C^k)$ submanifold of $X^n - S$ with respect to the differential structure given by \{$(U_\alpha, \varphi_\alpha)$\}_\alpha_\Lambda.

It is important to note that an atlas defining a smooth conifold has a subset of neighborhoods that are open sets in $\mathbb{R}_+^n$ as noted in its definition. The reason for this condition is
that smoothness for manifolds is defined in terms of the images of the overlap of the open sets on $\mathbb{R}^n$. Thus Def.(5.4) is a close parallel of the definition of smoothness for manifolds.

One might try to define an atlas as a cover of a conifold by conical neighborhoods, however such an approach encounters technical problems avoided by Def.(5.4). For example, consider the closed conifold of section 3 consisting of the suspension of $RP^3$. This conifold can be covered with two open sets, each homeomorphic to a cone over $RP^3$. This is a cover of the space; however, the intersection of the two open sets is homeomorphic to a product of an open interval with $RP^3$ instead of to an open neighborhood in $\mathbb{R}^4$. Thus the smooth structure cannot be defined in the usual way using this cover. In order to define a smooth structure using these sets, one would have provide a new procedure to do so. For example one could pick a smooth structure on the product manifold given by the intersection of the sets and then require the composition of maps on this overlap be smooth. However, there is no natural way to do this in general as it relies on the choice of a smooth structure on the intersection and there is no unique or well established way of making this choice on an arbitrary product manifold. One difficulty is that there may be more than one smooth structure on an arbitrary product manifold; for example, it has been shown that the product of an open interval with a 3-sphere has an uncountable number of nondiffeomorphic smooth structures. Another difficulty is that even if one selects a particular smooth structure to use, one must show that the use of this choice leads to all inequivalent smooth structures on the conifold. Therefore, such an alternate procedure involves unnecessary complications that can be avoided by choosing an appropriate atlas on the conifold. Thus it is safe to say that Def.(5.4) is a reasonable and pragmatic definition of a smooth conifold.

For two or fewer dimensions, all conifolds are manifolds. Hence, all n-conifolds with $n \leq 2$ admit smooth structures. Similarly every 3-conifold has a smooth structure. In order to see this, remove a conical neighborhood around each singular point of the 3-conifold,
so that the resulting space is a 3-manifold with boundary. Now, all 3-manifolds admit a smooth structure so pick one such smooth structure on the manifold. Next smoothly pinch off the boundary. One produces a smooth 3-conifold this way which is homeomorphic to the original space. Thus this generates a smooth structure on the 3-conifold. In higher dimensions, one can construct examples of conifolds which do not admit smooth structures. Recalling the results for manifolds from section 2, this should be no surprise; as there are $n$-manifolds for $n \geq 4$ which admit no smooth structures, it follows immediately that there are $n$-conifolds for $n \geq 4$ which admit no smooth structures. Again smooth structures will be discussed in more detail in Part II of this paper after the appropriate machinery is introduced.

Having defined the notion of a smooth $n$-conifold, its differential geometry is done by studying the differential geometry of the manifold $X^n - S$ and then requiring that the objects in question extend to $X^n$ in a continuous way. For example, the notion of metric on a manifold can be extended to a metric on a conifold as below;

**Definition (5.5).** A Riemannian metric on a conifold $X^n$ is a Riemannian metric $g$ on $X^n - S$ and a Cauchy complete distance function $d$ on $X^n$ such that the distance function associated with $g$ is equal to $d$ when restricted to $X^n - S$.

In other words, $d$ in the above definition is just the Cauchy completion of the distance associated with $g$. Given the link between geodesic completeness and Cauchy completeness of the distance on Riemannian manifolds, it is reasonable to define the Riemannian metric on the conifold by using a similar connection. For notational convenience, when referring to Riemannian metrics on conifolds only the metric on the manifold will be mentioned. It will be understood that the distance at singular points is given by the Cauchy completion.

Having defined a Riemannian metric, defining geodesics is next.
**Definition (5.6).** A geodesic on a conifold is a continuous curve which is a segment as defined in Def.(4.3).

Recall that segments are equivalent to geodesics in Riemannian manifolds; thus this definition is a natural generalization of the definition of geodesics to conifolds. As is often the case, it is very useful to have other equivalent characterizations of geodesics in conifolds. Observe that if the segment never passes through a singular point of the conifold $X^n$, then it must be a geodesic in the usual sense for the Riemannian manifold $X^n - S$. Thus, the issue of interest is to have an equivalent characterization of geodesics in conifolds which pass through singular points. In order to do this, one can define a class of curves on conifolds for which the usual length as used for curves on manifolds makes sense. Since the usual definition of length (4.2) involves a derivative of the curve, it is useful to use an equivalent characterization of length in order to allow for the fact that derivatives will not exist in a classical sense as the singular point is approached. Before proceeding, a trick using embeddings of conifolds in Euclidean space will be introduced. This embedding will be used to produce an easily visualized equivalent characterization of length and distance.

It has already been shown that all connected n-conifolds embed continuously in $\mathbb{R}^{2n+1}$. One can similarly show that all smooth n-conifolds embed smoothly in $\mathbb{R}^{2n+1}$. This is done just as in the continuous case by removing neighborhoods of the singular points and then applying the embedding theorem for smooth manifolds, namely that every n-manifold smoothly embeds in $\mathbb{R}^{2n+1}$. Finally, the boundaries can be smoothly pinched off so that the resulting space is equivalent to the original conifold. These embeddings are not necessarily isometric, that is an arbitrary metric on the conifold is not in general induced by the standard metric on $\mathbb{R}^{2n+1}$ by the embedding. However, a very impressive theorem for Riemannian manifolds is the Nash embedding theorem:26 Given any connected Riemannian manifold $M^n$ there is some $m$ such that $M^n$ isometrically embeds in $\mathbb{R}^m$. Using this theorem, one can show that every connected n-conifold isometrically embeds in
some $\mathbb{R}^m$; that is the conifold can be considered as a closed subset of $\mathbb{R}^m$ such that the Riemannian metric on the conifold is induced by the usual Riemannian metric on $\mathbb{R}^m$. To see this, remove all singular points $S$ in the conifold and apply the Nash embedding theorem to the resulting manifold $X^n - S$. Now, $X^n - S$ is isometrically embedded. However, note that the distance function as defined by (4.1) is no longer Cauchy complete because the singular points have been removed. Hence, by taking the completion of the distance function, the singular points $S$ are added. Once the singular points are included, it follows that this embedding is an isometric embedding of the conifold $X^n$.

Since every connected Riemannian conifold can be isometrically embedded in $\mathbb{R}^m$, one can assume a connected Riemannian n-conifold $X^n$ is a closed subset of $\mathbb{R}^m$. Now the set of rectifiable curves, that is all curves in $\mathbb{R}^m$ for which the integral defining their length converges, can be used to define the length of curves in $X^n$ in the following way: Given any such rectifiable curve which is contained entirely in $X^n$, then its length is equal to its length in $\mathbb{R}^m$. This will agree with the usual length (4.2) for all curves that do not pass through singular points but more importantly it is well defined even for those that do. Furthermore, this is an intrinsic property of the geometry of conifolds; it does not depend on the embedding, even though the embedding trick is a convenient way of describing these rectifiable curves.

Moreover, as a conifold can be considered to be a closed subset of Euclidean space and the length functional of any rectifiable curve is bounded below, there will be a minimizing curve between any two points which can be joined by a rectifiable curve. Thus, if one can show that any two points in a connected Riemannian conifold can be joined by at least one rectifiable curve, then it will immediately follow that any two points can be joined by a minimizing curve.

Now the first step toward this result will be derived: Given any two points in a connected Riemannian conifold $X^n$ with distance function $d$, there is a rectifiable curve con-
necting them. For convenience, assume that there is one singular point \( p \); the generalization to a countable set of singular points \( S \) follows immediately. First observe that there is always a smooth curve connecting any two nonsingular points because \( X^n - \{ p \} \) is a smooth manifold. Next assume that there is no rectifiable curve passing through \( p \). Then any two points in any neighborhood of \( p \) can be connected by a geodesic in the manifold \( X^n - \{ p \} \); otherwise there would be a family of smooth curves of decreasing length which limit onto a rectifiable curve passing through \( p \). This implies that \( X^n - \{ p \} \) is geodesically complete; thus by the correspondence between geodesic completeness and Cauchy completeness for smooth manifolds, \( d \) must be complete on \( X^n - \{ p \} \). It follows that the singular point \( p \) is not the completion of the distance function on \( X^n - S \), a contradiction to the definition of the conifold metric (5.5). Hence, the singular point \( p \) must have at least one rectifiable curve passing through it. Finally, it follows that any two points can be connected by a rectifiable curve. Thus, from the previous discussion, any two points on a conifold \( X^n \) can be joined by a minimizing curve.

Thus it follows from the above paragraph that

**Theorem (5.7).** *The distance between two points in a Riemannian conifold \( X^n \) is the length of a rectifiable curve joining them which has minimal length.*

Furthermore, using this result, the next theorem follows by the same argument as in the case of smooth manifolds and gives an equivalent characterization of geodesics in conifolds.

First note that a curve is said to be a *locally minimizing* curve if the curve is minimizing for any two points on the curve that are sufficiently close to each other. Then

**Theorem (5.8).** *A segment is a locally minimizing curve.*

Let \( x(t), \ t_0 \leq t \leq t_1 \) be a segment. Note that \( x(t) \) is also a segment on \( t_\alpha \leq t \leq t_\beta \) for any \( t_\alpha \) and \( t_\beta \) between the endpoints \( t_0 \) and \( t_1 \). Now take \( t_\alpha \) and \( t_\beta \) such that \( t_\beta - t_\alpha \) is
a sufficiently small interval. By choosing this interval to be small enough, the endpoints \( x(t_\alpha) \) and \( x(t_\beta) \) can be connected with a unique minimizing curve by the previous result. Suppose for some \( t' \) between the endpoints, \( t_\alpha \leq t' \leq t_\beta \), \( x(t') \) is not on the minimizing curve. Then the distance between the two endpoints must be strictly less than the sum of the distances from each endpoint to \( x(t') \);

\[
d(x(t_\alpha), x(t_\beta)) < d(x(t_\alpha), x(t')) + d(x(t'), x(t_\beta)).
\]

However, this contradicts the assumption that \( x(t) \) is a segment. Therefore \( x(t) \) must be a minimizing curve between \( x(t_\alpha) \) and \( x(t_\beta) \). Finally note that this argument applies to all sufficiently close points \( x(t_\alpha) \) and \( x(t_\beta) \) on the segment. Thus \( x(t) \) is a locally minimizing curve. Q.E.D.

Conversely, a locally minimizing curve is a segment. Therefore, geodesics in conifolds are locally minimizing curves. Also, the above implies the following theorem:

**Theorem (5.9).** Any two points of a conifold \( X^n \) with a Riemannian metric can be joined not only by a geodesic but by a minimizing geodesic.

Observe that this theorem is consistent with the condition that the distance function associated with the Riemannian metric on a conifold was taken to be Cauchy complete. Finally, it is important to stress that although Thms.(5.7-9) were proven using an embedding trick, they actually reflect intrinsic properties of the conifold. With a little more work, these theorems can all be proven intrinsically using the definition of the generalized derivative on the conifold. However, the embedding trick allows for a more intuitive understanding of these results.

It is worth mentioning another useful observation related to the embedding technique; namely, one can define the tangent space at any point of a conifold using the definition of a tangent cone for any subset \( E \) of Euclidean space:
**Definition (5.10).** Given any subset $E \subseteq \mathbb{R}^m$, the tangent cone at a point $x_0$ is

$$\text{Tan}(x_0, E) = \{x \in \mathbb{R}^m | x = cr \text{ where } c \in \mathbb{R} \text{ with } c \geq 0 \text{ and } r \in \overline{T}\}$$

where $\overline{T}$ is the closure of $T$ given by

$$T = \cap_{\epsilon > 0} \{ \frac{x - x_0}{|x - x_0|} | x, x_0 \in E \text{ and } 0 < |x - x_0| < \epsilon \}.$$ 

At manifold points this definition yields the usual tangent space, however, at singular points it yields a cone. Also, observe that the tangent cone is not a vector space at singular points. One can show that the tangent cones of conifolds have many properties in common with the tangent spaces of manifolds; for example, even though the tangent cone is not a vector space at the singular points, it always has the same dimension as the conifold. Also, given nice mappings between two conifolds, the respective tangent cones are mapped to one another via the generalized derivative of the map.

Finally note that spin structures and thus spinors can be defined on a conifold $X^n$ by defining a spin structure on $X^n - S$ where $S$ is the singular set and requiring that each $\Sigma_x^{n-1}$ for $x \in S$ have an induced spin structure consistent with $X^n - S$. The actual bundle of spinors on $\Sigma_x^{n-1}$ form a subspace of the bundle of spinors on the conical neighborhood $N_x = C(\Sigma_x^{n-1})$. Just as in the case of manifolds, not all conifolds admit a spin structure. Finally, for $X^n$ that do admit spin structures, once a spin structure is chosen on the conifold, the Dirac operator can be defined by defining it in the usual way on $X^n - S$. These properties of conifolds and their uses in the study of the geometry of conifolds are not used in the present paper so details will be presented elsewhere.

6. **EINSTEIN CONIFOLDS**

Given the definition of the topology and geometry of conifolds it is now possible to discuss Einstein conifolds and their relevance to semiclassical approximations of sums over
histories for Euclidean gravity. Recall from Def.(5.5) that the geometry of a conifold is determined from the metric $g$ on $X^n - S$ by completion. Similarly, other quantities such as scalar fields and scalar curvature can be defined on conifolds in a similar fashion: The quantities are defined as in the case of manifolds on $X^n - S$ and their values are then extended to the singular points $S$. A definition of tensor quantities on conifolds can be provided in terms of the tangent cones of Def.(5.10). However, an explicit definition is not necessary for the purposes of this paper as the discussion of Einstein conifolds can be done entirely using the techniques of completion and embedding as utilized in the previous section.

Integration on conifolds is defined using the measure induced by the volume element associated with the Riemannian metric $g$ at manifold points and extending it to singular points using the distance function. Equivalently, this Lebesgue integral could be defined in terms of an isometric embedding in Euclidean space. The action on a conifold is defined in terms of Lebesgue integration of the scalar curvature. Again, the scalar curvature is defined by extending the scalar curvature at manifold points to the singular points. For simplicity, the following definition assumes no boundary and compactness:

**Definition (6.1).** Let $X^n$ be a closed conifold with smooth metric $g$. The Einstein action is

$$I[g] = -\frac{1}{16\pi G} \int_{X^n} (R - 2\Lambda) d\mu(g)$$

where $R$ is the scalar curvature on $X^n - S$.

In the case of n-conifolds with boundary, the appropriate boundary term is needed; as the boundary is a (n-1)-manifold, it follows that the required boundary term is exactly the same as that in the corresponding n-manifold case (2.1).

An *Einstein conifold* is a closed conifold for which the metric $g$ restricted to $X^n - S$ is Einstein. The Euclidean Einstein equations are elliptic in an appropriate gauge; using
regularity of elliptic partial differential equations this means the metric is analytic at manifold points. Hence, the metric of an Einstein conifold is particularly nice at non-singular points. The suspension of $RP^3$ with its round metric (3.16) is an example of an Einstein conifold; this example provides an excellent illustration of the analytic properties of an Einstein conifold as discussed in section 3. Furthermore, it will be shown that there is no loss of generality by considering only closed conifolds when discussing Einstein conifolds by Thm.(6.3) and Thm.(6.5). Finally, one can show using variations with compact support on $X^n - S$ that the following is true:

**Theorem (6.2).** The extremum of the action $I(g)$ on $X^n$ is an Einstein metric.

In other words, away from the singular points of the conifold, the variation of the action yields the Einstein equations. A detailed proof that shows that $I[g]$ is a differentiable functional involves picking the right space of metrics and will be given elsewhere. As conifolds become manifolds when a countable number of points are removed, it would not be surprising if generalizations of various useful theorems in Riemannian geometry carry over. As mentioned in section 3, an important result for manifolds with positive Ricci curvature, in particular Einstein manifolds, is Myers’ theorem, Thm.(3.1). Recall that it puts an upper bound on the diameter of the manifold, which for a complete manifold is the length of a longest minimizing geodesic. Since conifolds have a well defined notion of geodesics, the diameter can again be defined in terms of longest minimizing geodesics. This can be used to prove a theorem for conifolds similar to Myers’ theorem:

**Theorem (6.3).** Let $X^n$ be a conifold with metric $g$ such that the restriction of $g$ to $X^n - S$ is a metric with strictly positive Ricci curvature, i.e. $R_{ab} \geq (n - 1)k^2 g_{ab}$ where $k$ is a nonzero constant. Then the diameter of $X^n$ obeys the relation $d(X^n) \leq \frac{\pi}{k}$.

A sketch of the proof follows: First, one shows that there is no minimizing geodesic of length greater than $\frac{\pi}{k}$ in $X^n - S$. To do this, assume that there is a minimizing geodesic $\gamma$
of length \( L > \frac{\pi}{k} \). Next, parallel transport an orthonormal frame denoted by \( \{ e_i \}_{i=1}^n \) along the curve \( \gamma \) and let \( e_1 \) denote the unit tangent vector of \( \gamma \). Define a new set of \( n \) vectors by multiplying the original ones by \( \sin(\kappa t) \), \( w_i = (\sin(\kappa t)) e_i \) where \( \kappa = \frac{\pi}{L} \) and \( 0 \leq t \leq L \) is the parameterization of \( \gamma \). Since \( \gamma \) is minimizing, the first variation of the length functional of \( \gamma \) must vanish, and the second variation must be nonnegative. Explicitly calculating the second variation of the length in terms of the vectors \( w_i \) and then summing the result from \( i = 2 \) to \( n \) implies

\[
\sum_{i=2}^{n} D^2 \ell(\gamma)(w_i, w_i) = \int_{0}^{L} (\sin(\kappa t))^2 [(n - 1)\kappa^2 - R_{ab} e_1^a e_1^b] dt.
\]

Using \( R_{ab} \geq (n - 1)k^2 g_{ab} \) and \( \kappa^2 < k^2 \), it follows that

\[
\sum_{i=2}^{n} D^2 \ell(\gamma)(w_i, w_i) < 0.
\]

This implies that the second variation for some \( w_i \) must be negative. However, this is a contradiction because it must be nonnegative for minimizing curves. Therefore, minimizing curves in \( X^n - S \) must have length less than or equal to \( \frac{\pi}{k} \).

The second part is to assume that there is a minimizing curve in \( X^n \) with length greater than \( \frac{\pi}{k} \). If it contains no singular points, then the argument presented above implies it cannot be a minimizing curve. Therefore assume it contains singular points. Since the curve is finite, it can contain at most a finite number of singular points. One can show that by continuously perturbing the geodesic a small amount, a new geodesic containing no singular points of length greater than \( \frac{\pi}{k} \) can be constructed. However, no smooth minimizing curves of length greater than \( \frac{\pi}{k} \) exist. Hence, all minimizing curves have length less than or equal to \( \frac{\pi}{k} \). Therefore, \( d(X^n) \leq \frac{\pi}{k} \). Q.E.D.

In particular, this theorem applies to Einstein conifolds. The bound on the diameter is the same bound as for Einstein manifolds (see Thm.(3.1)). Thm.(6.3) can also be combined with several simple observations in order prove that the topology of Einstein conifolds is restricted. Before doing so, it is necessary to define the universal cover of a conifold.
Suppose $X$ is a path connected metric space such that every point has a simply connected neighborhood. Then, the universal covering space of $X$ is defined to be any space $\tilde{X}$ which is a simply connected covering space of $X$. The universal covering space is unique. Under the above conditions, the universal covering space can be constructed using all continuous paths starting at some fixed point $x_0$ in the following way: Given the set of paths $P = \{ c : [0,1] \to X | c(0) = x_0 \}$, a continuous projection map $p : P \to X$ is defined by $p(c(t)) = c(1)$. Let $\tilde{X}$ be $P$ modulo the equivalence relation that $c_1 \sim c_2$ if and only if $c_1(1) = c_2(1)$ and one path can be deformed continuously into the other while holding the endpoints fixed. The projection map $p$ is also well defined as a map on $\tilde{X}$, $p : \tilde{X} \to X$. Also, the fundamental group $\pi_1(X, x_0)$ acts naturally on $\tilde{X}$ by composition of paths. One can use this group action to prove that $X$ can be constructed from $\tilde{X}$ via identifying points under the action of $\pi_1$. Since for any path between $x_0$ and $x \in X$, one can add a loop in $\pi_1(X, x_0)$ to obtain a new path between $x_0$ and $x$, it follows that $p^{-1}(x)$ has the same cardinality as $\pi_1(X, x_0)$. Clearly, if $X$ is simply connected, then there is a one to one correspondence between equivalence classes of paths and points in $X$. Hence, $\tilde{X} = X$ if and only if $X$ is simply connected. Even if $X$ is not simply connected, by assumption each point of $X$ has a simply connected neighborhood $U$. Using these neighborhoods, one can show that the projection is locally a homeomorphism, namely, each point in $X$ has a simply connected neighborhood $U$ such that $p$ is a homeomorphism from each path connected component of $p^{-1}(U)$ and $U$. Hence, locally $\tilde{X}$ looks like $X$. Finally, one can show $\tilde{X}$ as constructed is always simply connected by verifying that $\tilde{\tilde{X}} = \tilde{X}$. Therefore, $\tilde{X}$ is the universal covering space.

Examples of universal covering spaces are $\mathbb{R}^n$, which is the universal covering space of $n$-torus and the group $SU(2)$, which is the universal covering space of $SO(3)$. Since the universal covering space $\tilde{X}$ of a space $X$ is locally homeomorphic, the two spaces will share many of the same local properties. In particular, if $X$ is a manifold, its universal cover is
also a manifold. Similarly, as Def.(5.1) is a local definition,

**Lemma (6.4).** The universal covering space of a conifold is also a conifold.

Furthermore, if the above construction is performed for a smooth manifold and smooth curves are used, the universal covering space will be a smooth manifold. Likewise, the same is true for smooth conifolds. Given any local geometric structure, it will be carried by the universal covering space because the two spaces are locally the same. The lemma (6.4) and this observation are crucial parts of the next result.

**Theorem (6.5).** Let $X^n$ be a conifold which admits a Riemannian metric with strictly positive Ricci curvature. Then $X^n$ is compact. Furthermore, $\pi_1 X^n$ is a finite group.

Since the Ricci curvature is strictly positive, it follows from the previous theorem that diameter of $X^n$ is bounded by a finite constant. This means that no minimizing curve can be longer than this constant. Recall from an earlier argument that $X^n$ can be considered as a closed subset of Euclidean space $\mathbb{R}^m$ (with its geometry induced by its embedding in Euclidean space). Hence, the diameter of $X^n$ as a subset of Euclidean space is bounded by a constant. In other words, $X^n$ is a closed bounded subset of Euclidean space. However, all closed bounded subsets of $\mathbb{R}^m$ are compact. Therefore, $X^n$ is compact.

Let $\tilde{X}^n$ be the universal covering space of $X^n$. Since the two spaces are locally equivalent, $\tilde{X}^n$ must admit a metric with strictly positive Ricci curvature. Hence, $\tilde{X}^n$ is also compact by the above argument. Let $x \in \tilde{X}^n$. If $\pi_1 X^n$ is infinite, then there is an infinite sequence of distinct points in $\tilde{X}^n$ generated by acting on $x$ with elements of $\pi_1 X^n$. Since $\tilde{X}^n$ is compact, the above sequence must have a convergent subsequence $x_k$ with limit $l$. All of the points $x_k$ in the convergent subsequence are equivalent to the same point $x$ in the space $X^n$ because they are all constructed from each other by acting on that point by elements of $\pi_1 X^n$, thus by definition $p(x) = p(x_k)$. Using the continuity of the projection
map \( p \), it follows that \( \lim_{k \to \infty} p(x_k) = p(l) \). These two observations immediately imply that \( p(x) = p(l) \). Hence, the limit \( l \) is equivalent to \( x \) and all other \( x_k \) via the group action of \( \pi_1 X^n \). This means that there is an open neighborhood \( U \) of \( p(l) \) in \( X^n \) such that \( p^{-1}(U) \) is a collection of disjoint sets, one for each member \( x_k \) of the sequence. Since \( l \) is the limit point of the sequence, this is a contradiction as there can be no disjoint open neighborhood around \( l \). Therefore, \( \pi_1 X^n \) must have been finite. Q.E.D.

This theorem verifies the earlier assertion that there was no loss of generality by assuming that Einstein conifolds are closed. It also obviously puts restrictions on the topology of Einstein conifolds. Since the first homology group of a space is the abelization of its fundamental group, theorem (6.2) implies that the first homology is a finite group. This means there can be no free part to the group which implies the first Betti number is zero. Thus immediately

**Corollary (6.6).** Let \( X^n \) be a conifold that admits a Riemannian metric with strictly positive Ricci curvature. Then \( b_1(X^n) = 0 \).

This corollary is a type of Bochner vanishing theorem for conifolds. It yields an easy necessary condition to apply to conifolds in order to decide if a particular conifold admits an Einstein metric.

Having developed the topology and geometry of conifolds, it is now possible to discuss the inclusion of conifolds in the sum over histories formulation of Euclidean gravity. From the earlier discussion of section 3, the relevant question is whether or not Einstein conifolds arise in some natural way in semiclassical quantum amplitudes. Recall that the motivation for the inclusion of generalized histories lay in the intuitive picture that there were sequences of Riemannian manifolds with metrics that were almost Einstein that approached an Einstein conifold; that is the Einstein conifold arises as the limit point of a sequence of almost stationary paths in the Euclidean gravitational integral. Indeed, given
the mathematical formulation of conifolds, this can now be proven to be the case. The last part of this section will present the definitions used in the proof of the main theorem and an outline of the proof. The detailed proof of the theorem will be presented elsewhere it is rather technically involved.

It is again useful to use the example of the suspension of $RP^3$ to illustrate how the main result is proven. One can remove the singular points of the $RP^3$ conifold by removing conical neighborhoods of each of the two singular points. The resulting space is a manifold $I \times RP^3$ with two $RP^3$ boundaries. As all 3-manifolds are cobordant, the boundaries can be capped off by adding two 4-manifolds $G$ with $RP^3$ boundaries to obtain a closed 4-manifold $M^4$. The Einstein metric (3.16) on the $I \times RP^3$ can be smoothly matched to a smooth metric on each $G$ to produce a smooth metric $g$ on the manifold. Furthermore, by removing smaller and smaller conical neighborhoods around the singular points of the Einstein conifold and repeating the capping off procedure, one can construct a sequence of Riemannian manifolds $M^4_k$ each diffeomorphic to $M^4$ such the each of the caps $G_k$ is becoming smaller with respect to the metric $g_k$. If one chooses the sequence so that the diameter of the caps is going to zero, then the sequence is approximating an Einstein metric on $M^4$. Intuitively, the caps are pinching off as one takes the limit and the diameter condition restricts the caps from pinching off other than at a point. The set of points on which the metrics $g_k$ on the sequence of manifolds $M^4_k$ are non-Einstein shrinks in this limit of zero diameter because these points are all contained within the caps. The reason one expects to obtain the $RP^3$ Einstein conifold as the limit of this sequence is that the caps $G_k$ are not balls for the $RP^3$ case but rather some complicated 4-manifold. From this description of the convergence of the sequence of manifolds to the $RP^3$ Einstein conifold, it is clear that the steps involved in a general proof are to first define the sequence of manifolds that have the needed properties and then to define what is meant by the convergence of this sequence and finally to prove that an Einstein conifold results from a convergent sequence
of this form. Thus

**Definition (6.7).** A sequence of Riemannian manifolds \((M^n_k, g_k)\) is approximately Einstein if and only if there is a sequence of open sets \(G_k\) such that

1) \(\bar{G}_k\) is compact.

2) \(M^n_k = M^n\) and \(G_{k+1} \subseteq G_k\).

3) \(d_{k+1}(G_{k+1}) < d_k(G_k)\).

4) \(d_k(G_k) \to 0\) as \(k \to \infty\).

5) \(g_k\) is Einstein on \(M^n - G_k\).

The following definition gives part of the needed definition of convergence for the theorem.

**Definition (6.8).** A sequence of Riemannian manifolds \(\{(M^n_k, g_k)\}\) converges uniformly on compact sets to the Riemannian manifold \((M^n_\infty, g_\infty)\) if and only if for any compact domain \(D \subseteq M^n_\infty\) and sufficiently large \(k\) there are compact domains \(D_k \subseteq M^n_\infty\) and diffeomorphisms \(F_k : D \to D_k\) such that the pullbacks \(F^*_k g_k\) converge to the metric \(g_\infty\) on \(D \subseteq M^n_\infty\).

This definition can applied to the sequence of approximately Einstein manifolds of Def.(6.7) to produce the limiting manifold with its Einstein metric. This limiting manifold is not geometrically complete; however the completion of this space is the desired Einstein conifold. It is this more general definition of convergence that is needed in order to prove the theorem. Such a definition for any compact metric space indeed can be provided, but it involves more technical detail. It can be understood if one recalls that convergent sequences define a topology on the appropriate underlying space; it is this fact that is used to construct the definition of general convergence for compact metric spaces.

Given the above definitions (with the appropriate generalization) the following theorem can be proven as described:\textsuperscript{27}
Theorem (6.9). If \( \{(M^n_k, g_k)\} \) is approximately Einstein, then \( (X^n, g_\infty) \) is a conifold. Furthermore, \( g_k \) converges uniformly on compact sets to \( g_\infty \) on \( X^n - S \) where \( S \) is the singular set of the conifold.

Also note that the definition of an approximately Einstein sequence of Riemannian manifolds and its generalization can be directly extended to yield the definition of an approximately Einstein sequence of conifolds. Consequently,

Theorem (6.10). If a sequence of conifolds with their metrics, \( \{(X^n_k, g_k)\} \), is approximately Einstein, then \( (X^n_\infty, g_\infty) \) is a conifold. Furthermore, \( g_k \) converges uniformly on compact sets to \( g_\infty \).

This result follows as a trivial extension of Thm.(6.9). The full proof of these theorems will be given in a further paper that provides the necessary results on the convergence of the sequence.

The connection between the requirements on the sequence given in Def.(6.7) and the result that the topology of the limit of a sequence of manifolds is that of a conifold is made clear by observing the relationship of Thm.(6.9) to the result of Gao on the moduli space of Einstein metrics. Gao considers sequences of Einstein metrics on a given fixed manifold subject to certain conditions analogous to those of Def.(6.7) and an additional requirement on injective radius. Roughly speaking, the injective radius is the size for which a normal coordinate neighborhood is defined at each point of the manifold. The requirement is that the injective radius of every manifold in the sequence is bounded below by the same constant. With such a sequence he proves that the moduli space is compact and the topology of the manifold does not change; the bound on the injective radius acts to keep the topology of the manifold from pinching off in the sequence. Thm.(6.9) has no such restriction on the injective radius and thus the size of normal coordinate neighborhoods can collapse and the topology of the sequence of manifolds can be pinched off. Thus
the topology of the limiting space can differ from that of the manifolds in the sequence. However, the Ricci curvature and metric are still finite on the resulting conifold. Therefore the Einstein conifold is a well behaved topological space that arises naturally as the limit of this sequence.

Further qualitative understanding of these results can be gained by comparing them to the more familiar case of Yang-Mills theory. The curvature tensor is the analog of the field strength of Yang-Mills and the moduli space of Einstein metrics like that of self dual Yang-Mills theory is finite dimensional; thus the study of the moduli space of Einstein metrics is analogous to the study of the space of self dual Yang-Mills fields. An important result in the study of Yang-Mills theories is Uhlenbeck’s theorem that singularities of self dual Yang-Mills fields can be removed. These singularities are purely gauge and are not coupled to the topology of the underlying manifold. Gao’s result can be considered the analog of Uhlenbeck’s theorem for Einstein gravity for a special set of manifolds with injective radius bounded below. This special set of manifolds is needed in order to remove topological singularities; in gravity the singularities of the metric fields in general involve not only the connection but the topology of the space because of the strong coupling of topology and Riemannian geometry. Therefore a further restriction is needed in order to make the singularities appear only in the connection. Thm.(6.9) removes Gao’s restriction; the consequence of its removal is that topological singularities occur, in particular, conifolds occur as boundary points of the moduli space of Einstein manifolds. However, these boundary points still exhibit a regular geometry. Thus Thm.(6.9) can be viewed as a generalization of Uhlenbeck’s theorem to Einstein gravity.

7. CONIFOLDS AS HISTORIES IN
Given that Einstein conifolds occur as limit points of sequences of approximately Einstein metrics on manifolds, one is compelled to include their contribution to the semiclassical approximation of Euclidean functional integrals for gravity. One expects the classical space of histories used as a starting point for formulating Euclidean functional integrals should be reasonably well behaved. A space of classical histories that does not include its limit points is not; such spaces are not complete and generally do not exhibit the mathematical properties expected of spaces for quantum amplitudes. Given that the limit points of these sequences are well behaved spaces both geometrically and topologically, it is natural to complete the space of histories by including these points. Thus Thm.(6.9) provides a precise mathematical motivation for including Einstein conifolds in the semiclassical approximation.

An immediate consequence to allowing Einstein conifolds as classical extrema of the Einstein action is that there will be semiclassical amplitudes for a more general set of boundary topologies. An example follows from the discussion of section 3; the Hartle-Hawking wavefunction for $RP^3$ boundary with round metric yields a semiclassical wavefunction. For $Ha_0 < 1$, there is an Einstein conifold, the suspension of $RP^3$ with metric (3.16) as defined in Def.(5.5). The action of this extremum can be computed by Def.(6.1); the curvature of (3.16) is well defined and constant everywhere on the suspension of $RP^3$ minus the two singular points and thus the Lebesgue integral can be performed. There are two possible positions for the $RP^3$ boundary with radius $a_0$ in this solution corresponding to filling either less than or more than half of the conifold. Again using the prescription of Hartle and Hawking, the Euclidean conifold extremum that dominates in the steepest descents evaluation is that with least action corresponding to filling less than half of the
suspension of \( \mathbb{RP}^3 \). The wavefunction in the Euclidean region is thus

\[
\Psi_E(\mathbb{RP}^3, a_0) \sim \exp -\tilde{I}^-(a_0)
\]

\[
\tilde{I}^-(a_0) = -\frac{1}{6H^2\ell^2} [(1 - H^2a_0^2)^{\frac{3}{2}} - 1].
\]

(7.1)

where the action is simply (3.17) with \( a_1 = 0 \). It is immediately apparent from the discussion of section 3 that this wavefunction matches continuously with continuous derivative onto \( \Psi_L(\mathbb{RP}^3, a_0) \) (3.15) at the point \( Ha_0 = 1 \) for the phase \( \alpha = -\pi/4 \). Therefore this wavefunction is a semiclassical solution of the Hartle-Hawking boundary condition when the Einstein conifolds are allowed as Euclidean extrema.

It is clear that a similar set of semiclassical Hartle-Hawking solutions can be constructed for any boundary of the form \( S^3/\Gamma \) with round metric where \( \Gamma \) is a finite subgroup of the rotation group. However, one should note that just as not all Einstein 4-conifolds are topologically suspensions of 3-manifolds, they are also not geometrically such suspensions. Secondly, although the inclusion of Einstein conifolds into semiclassical approximations allows semiclassical amplitudes for a larger set of boundary topologies, this set is still restricted. The requirements (3.10) on the intrinsic curvature of the boundary manifold still limit its topology to \( S^3, S^2 \times S^1, S^3/\Gamma \) and connected sums of these manifolds as discussed in section 3. Furthermore, the geometry of Einstein conifolds restricts the topology as well; Thm.(6.5) and its Cor.(6.6) restrict the topology of Einstein conifolds in a manner completely analogous to the restriction of the topology of Einstein manifolds by Bochner’s theorem. Thus allowing Einstein conifolds as extrema in semiclassical approximations to Euclidean functional integrals does not radically change their properties but rather enlarges the number of semiclassical amplitudes in a rational and arguably desirable way.

Note that allowing Einstein conifolds as classical extrema is self-consistent. Indeed, Thm.(6.10) proves that if one begins with a sequence of approximately Einstein conifolds, one does not end up with a more general topological space, but rather another Einstein
conifold. This self-consistency obviously does not occur for sequences of approximately Einstein manifolds by Thm.(6.9). Therefore the inclusion of Einstein conifolds leads to a complete moduli space and is thus a reasonable extension to the moduli space of Einstein manifolds.

Finally, given the inclusion of Einstein conifolds in the space of classical Euclidean solutions, it is eminently reasonable to propose the set of compact conifolds as generalized histories for Euclidean functional integrals for gravity. Clearly, topological spaces that occur as extrema of the Euclidean action are suitable as spaces for more general smooth histories as well. In addition, the appropriate set of such topological spaces is the set of all conifolds rather than just those that admit Einstein metrics; the arguments given in section 2 for the need to include all manifolds as histories can be extended to show that a similar set of conifolds must be used as well. For example, the generalized Hartle-Hawking wavefunction (2.1) is expressed as

$$\Psi[\Sigma^{n-1}, h] = \sum_{X^n} \int Dg \exp \left( -I[g] \right)$$

$$I[g] = -\frac{1}{16\pi G} \int_{X^n} (R - 2\Lambda) d\mu(g) - \frac{1}{8\pi G} \int_{\Sigma^{n-1}} K d\mu(h) \quad (7.2)$$

where the mathematical description of the histories is: A *generalized history* is a pair $(X^n, g)$ where $X^n$ is a smooth compact conifold and $g$ is at least a $C^2$ metric on $X^n - S$ with the specified induced metric $h$ on the boundary $\Sigma^{n-1}$. This definition includes all Riemannian histories. Like Riemannian histories, these generalized histories are classical histories of the theory and consequently, one expects that they provide the underlying topology for an appropriate set of distributional histories for the Hartle-Hawking wavefunctional.

One consequence of allowing conifold histories in the Euclidean functional integral is that the principle of equivalence, that is that spacetime is locally $\mathbb{R}^n$, is no longer automatically enforced by the set of histories in the sum. Therefore, as emphasized by
Hartle, it is now possible to consider the issue of whether or not the principle of equivalence holds for a given quantum amplitude calculated from the generalized sum over histories. As the principle of equivalence is a property of classical Lorentzian spacetime, it is reasonable to expect it to appear only in quantum amplitudes corresponding to classical spacetimes. In order to fully address this issue, one must have a method of determining when a given quantum amplitude corresponds to a classical spacetime. Therefore this issue is closely tied to that of interpreting quantum amplitudes for the universe. This question of the interpretation of the quantum mechanics of gravity is complex and unresolved and this paper will not touch on it. However, given the close connection of classical solutions and semiclassical wavefunctions, it is useful to provide a discussion of principle of equivalence in the context of semiclassical Hartle-Hawking wavefunctions.

There are two simplistic methods of associating a classical spacetime with a given semiclassical amplitude. The first is to associate the stationary path used to construct the semiclassical amplitude with a classical spacetime. The second is to use the the semiclassical wavefunction itself to provide initial data for the classical spacetime. In both approaches, classical spacetime is only associated with a complex extremum of the Euclidean action; such complex extrema are typically Lorentzian solutions to the Einstein equations. Therefore questions about the principle of equivalence can only be asked in regions of configuration space \((\Sigma^{n-1}, h)\) where the semiclassical wavefunction is formed from such extrema.

In the first approach, the question about the principle of equivalence can be rephrased into two questions, are there Lorentzian spacetimes which exhibit non-manifold singularities and do these spacetimes occur in the semiclassical approximation to the Euclidean functional integral for a given boundary geometry. The answer to the first question is well known to be yes; it follows immediately from the singularity theorems. In fact, explicit examples of singular Lorentzian spacetimes with isolated singular points can be constructed.
The answer to the second question is maybe; in order to answer this question, one has to decide whether or not a Lorentzian solution to the Einstein equations with such a nonmanifold singularity is an complex extrema to the Euclidean functional integral (7.2). To do so, one must provide a set of regularity conditions on the curvature and metric suitable for application in Lorentzian conifolds similar to those given for Euclidean conifolds. Given such a set of regularity conditions, one could then see if there was or was not a singular Lorentzian solution for the specified boundary geometry. In any case, the important point to stress is that if such a Lorentzian spacetime exists, it also exists classically. Therefore, the properties of the spacetime derived from these semiclassical amplitudes are completely determined by the classical theory. Therefore, the question about whether or not the principle of equivalence holds is really a question about a particular classical Lorentzian spacetime.

In the second approach, regions of classical spacetime correspond to regions of configuration space for which the quantum amplitude becomes oscillatory. Classical spacetime is retrieved from the semiclassical limit of the wavefunction $\Psi[\Sigma^{n-1}, h] \sim \cos(S[h] + \alpha)$ as the evolution along the normals to the surfaces of constant phase $\pi_{ij} = \frac{\delta S}{\delta h_{ij}}$. That is the metric and its conjugate momenta $(\pi, h)$ derived from a given wavefunction provide the initial data for a family of Lorentzian spacetimes with topology $\Sigma^{n-1} \times R$. Given sufficiently regular initial data, it is well known that it can be evolved for a finite distance. In the semiclassical approximation, $\pi$ is a continuous differentiable tensor field for actions evaluated on continuous differentiable solutions of the Einstein equations. Thus the principle of equivalence holds for semiclassical wavefunctions for which the complex stationary path is sufficiently regular. Note that this means that the principle of equivalence holds automatically for any semiclassical wavefunction where the initial data is appropriate initial data for a Lorentzian spacetime, even if there are nonmanifold points to the past of the boundary $(\Sigma^{n-1}, h)$ either in the Lorentzian or Euclidean regions. Of course, the evolution
of this initial data may result in a Lorentzian solution containing nonmanifold points to the future. Whether or not it does depends on the explicit form of the initial data. Therefore, whether or not the principle of equivalence holds in a classical spacetime associated with a given semiclassical wavefunction is again equivalent to the same question for Lorentzian solutions of the Einstein equations.

Thus, the issue of whether or not the principle of equivalence holds in the classical limit for a set of generalized histories is completely determined by the properties of the Lorentzian solutions themselves in these simplistic interpretations. Note that both of these simplistic approaches completely ignore the issue of how probable a given Lorentzian spacetime is; in fact it follows from the above discussion that it is this issue that is really the one of interest. It is clear that both a more detailed computation of the quantum amplitude and a more sophisticated interpretation of the amplitude is needed to really address this issue. In any case, Euclidean quantum amplitudes using generalized histories provide a viable starting point for such a study because in the most simplistic interpretation, their semiclassical limit corresponds to Lorentzian spacetimes.

8. CONCLUSION

This paper proposes a new set of topological spaces called conifolds and argues that the set of smooth compact conifolds form a suitable and necessary set of topological spaces for generalized histories for Euclidean functional integrals for gravity. It can be proven that Einstein conifolds arise as the limit of a sequence of approximately Einstein manifolds. Thus Einstein conifolds correspond to the boundaries of the moduli space of Einstein manifolds. Additionally, sequences of approximately Einstein conifolds also converge to Einstein conifolds. Therefore it is natural to include such spaces as histories for Euclidean functional integrals for gravity.
The immediate benefit of using conifold histories in Euclidean functional integrals for gravity such as (7.2) is that semiclassical amplitudes corresponding to Einstein conifolds follow immediately. However there are additional benefits as discussed in Part II of this paper. As the set of conifolds is larger than the set of manifolds, Euclidean sums over histories for gravity formulated using conifolds have the additional advantage of being algorithmically decidable in four or fewer dimensions. Thus, unlike for the case of sums over manifold histories, these sums can be implemented in a systematic way. Additionally, the sums can be explicitly carried out in finite approximations to expressions such as (7.2) formulated in terms of Regge calculus in four dimensions. Obviously many difficulties with the formulation of Euclidean integrals for Einstein gravity will remain for any generalization of the histories to any set of more general topological spaces that includes all manifolds. However, these problems are not any more severe for the set of conifolds than for manifolds. Moreover, the topological issues addressed are relevant to many theories involving sums over topological spaces. Therefore the above proposal provides a starting point for addressing further issues regarding the formulation of Euclidean integrals for gravity.

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Given a topological space built from the union of other topological spaces, a useful result for calculating the relation of the homology of the pieces to that of the whole space is the Mayer-Vietoris theorem.

**Theorem (A.1).** Given a topological space $Y$ and two open subsets $U$ and $V$ such that $Y = U \cup V$ then the following sequence of homology groups is exact

\[
\ldots \xrightarrow{\partial_*} H_k(U \cap V) \xrightarrow{\alpha_*} H_k(U) \oplus H_k(V) \xrightarrow{\beta_*} H_k(Y) \xrightarrow{\partial_*} H_{k-1}(U \cap V) \xrightarrow{\alpha_*} \ldots
\]

\[
\xrightarrow{\partial_*} H_1(U \cap V) \xrightarrow{\alpha_*} H_1(U) \oplus H_1(V) \xrightarrow{\beta_*} H_1(Y) \xrightarrow{\partial_*} H_0(U \cap V) \xrightarrow{\alpha_*} H_0(U) \oplus H_0(V) \xrightarrow{\beta_*} H_0(Y)
\]

where the homology groups are taken to have coefficients in any abelian group.

Recall that exact means that the composite of any two homomorphisms in the above sequence is zero, e.g. $\alpha_* \beta_* = 0$, and the kernel each homomorphism is equal to the image of the previous one, e.g. $\ker(\beta_*) = \text{im}(\alpha_*)$. If real coefficients are used, the homology groups are vector spaces over the real numbers and the above exact sequence is a sequence of vector spaces. If the integer coefficients are used, the homology groups are direct products of vector spaces over the integers and cyclic groups.

The Mayer-Vietoris sequence can be applied to show that $RP^3$ cannot be embedded in $S^4$ or $RP^4$ so that it divides them in half. The homology groups of all of these manifolds can be computed (for example via the application of the Mayer-Vietoris sequence to a decomposition of the manifold); thus two of the three homology groups in the sequence are
known. This information is sufficient to determine the homology groups of the remaining space. Namely,

**Theorem (A.2).** There is no compact topological 4-manifold $M$ with boundary $\mathbb{RP}^3$ so that $M \cup M$ is $S^4$ or $\mathbb{RP}^4$.

First, consider the case of $\mathbb{RP}^4$. For this case it is sufficient to use real homology. Assume a manifold $M$ as above exists. Writing the Mayer-Vietoris sequence starting at $H_4(\mathbb{RP}^4)$ gives

$$H_4(\mathbb{RP}^4) \longrightarrow H_3(\mathbb{RP}^3) \xrightarrow{\alpha_*} H_3(M) \oplus H_3(M) \xrightarrow{\beta_*} H_3(\mathbb{RP}^4).$$

Since $\mathbb{RP}^4$ is nonorientable, $H_4(\mathbb{RP}^4) = 0$ and by an explicit calculation, $H_3(\mathbb{RP}^4) = 0$ with real coefficients. Also, $H_3(\mathbb{RP}^3) = \mathbb{R}$ because $\mathbb{RP}^3$ is orientable. Hence, the exact sequence is

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha_*} H_3(M) \oplus H_3(M) \longrightarrow 0.$$

This is a contradiction because the vector space $\mathbb{R}$ is one dimensional and it cannot be decomposed into the sum of two identical vector spaces. Therefore, $\mathbb{RP}^3$ can not divide $\mathbb{RP}^4$.

Second, consider the case of $S^4$. A similar contradiction can be obtained by using the lower part of the Mayer-Vietoris sequence with integer coefficients

$$H_2(S^4) \xrightarrow{\partial_*} H_1(\mathbb{RP}^3) \xrightarrow{\alpha_*} H_1(M) \oplus H_1(M) \xrightarrow{\beta_*} H_1(S^4).$$

Now $H_1$ of any space is the abelization of its fundamental group. Since $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$ and $S^4$ is simply connected, $H_1(\mathbb{RP}^3) = \mathbb{Z}^2$ and $H_1(S^4) = 0$. Additionally, by explicit calculation, $H_2(S^4) = 0$. Hence,

$$0 \xrightarrow{\partial_*} \mathbb{Z}_2 \xrightarrow{\alpha_*} H_1(M) \oplus H_1(M) \xrightarrow{\beta_*} 0.$$
This is a contradiction because a cyclic group of order two can not be the direct sum of two groups. Therefore, $RP^3$ does not divide $S^4$. Q.E.D.

The Mayer-Vietoris sequence can be used to prove a useful relationship between the Euler characteristic of a manifold and that of its boundary in odd dimensions.

**Theorem (A.3).** Let $M$ be a compact manifold of odd dimension $n$ and with boundary $\partial M$. Then

$$2\chi(M) = \chi(\partial M).$$

First, form the manifold $N = M \cup M$ by doubling over $M$ on its boundary $\partial M$. By construction, $N$ is a closed connected odd dimensional manifold. Applying Thm.(A.1) to this decomposition of $N$ using real coefficients yields

$$0 \xrightarrow{\partial} H_n(\partial M) \xrightarrow{\alpha^*} 2H_n(M) \xrightarrow{\beta^*} H_n(N) \rightarrow H_{n-1}(\partial M) \xrightarrow{\alpha^*} 2H_{n-1}(M) \xrightarrow{\beta^*} H_{n-1}(N) \rightarrow \ldots$$

$$\xrightarrow{\partial^*} H_1(\partial M) \xrightarrow{\alpha^*} 2H_1(M) \xrightarrow{\beta^*} H_1(N) \rightarrow H_0(\partial M) \xrightarrow{\alpha^*} 2H_0(M) \xrightarrow{\beta^*} H_0(N)$$

where the factors of 2 in the above denote the sum of the two identical vector spaces, $2H_q(M) = H_q(M) \oplus H_q(M)$. It can be proven that the alternating sum of the dimensions of the vector spaces $H_k$ in the Mayer-Vietoris sequence sum to zero; consequently

$$\sum_{k=0}^{n} (-1)^k [\dim(H_k(\partial M)) - 2\dim(H_k(M)) + \dim(H_k(N))] = 0. \quad (7.3)$$

Now $H_k(\partial M) = 0$ for $k > n - 1$ as $n - 1$ is the dimension of the boundary $\partial M$. Also the betti numbers $b_k = \dim H_k$ and the alternating sum of the betti numbers is the Euler characteristic. Thus (7.3) is equal to $\chi(\partial M) - 2\chi(M) + \chi(N) = 0$. Finally, note that the Euler characteristic of closed odd dimensional manifolds is zero. Thus $\chi(\partial M) = 2\chi(M)$. Q.E.D.
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17. Continuity of the wavefunction and its derivative also follows directly from the presumed equivalence of the sum over histories formulation and the Wheeler de Witt equation. The Wheeler de Witt equation $H\Psi(h) = 0$ is a functional equation where $H = \Box + ^3R(h) - 2\Lambda$ is the operator corresponding to the Hamiltonian constraint of the theory, (3.8). The kinetic term $\Box$ is formally the wave operator with respect to the metric on superspace; thus the Wheeler de Witt equation is formally a hyperbolic second order differential equation. In minisuperspace models, this equation is finite dimensional. Given a finite dimensional second order hyperbolic differential equation of the form of the Wheeler de Witt equation and differentiable initial data, one can easily show that a solution exists in a neighborhood of the initial hypersurface and is also differentiable. In particular, semiclassical solutions corresponding to Euclidean instantons provide such differentiable initial data by the analyticity properties of Einstein manifolds. Therefore one can extend the semiclassical wavefunction from areas
of configuration space where there is a Euclidean stationary point to areas where there is not. Thus it follows that the wavefunction and its derivative must be continuous for minisuperspace models, at least in a neighborhood of the Euclidean region. It is also reasonable to expect the same in general, at least for semiclassical solutions; a proof for the functional equation of course requires a definition of the functional derivative, a concrete definition of the Wheeler de Witt operator, and an appropriate choice of function space, all of which are problematic.

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FIGURE CAPTIONS

**Figure 1:** The product space $X \times Y \times I$ and the join of $X$ and $Y$ obtained by identifying all points in $Y$ at $X \times Y \times \{0\}$ and all points in $X$ at $X \times Y \times \{1\}$.

**Figure 2:** The suspension of $\mathbb{R}P^2$ represented by a solid 3-ball with identifications. Each cross section of the ball is a 2-ball with the indicated identifications on its $S^1$ boundary.

**Figure 3:** A sequence of slicings of the suspension of $\mathbb{R}P^3$. The $\mathbb{R}P^3$ manifold is represented by a solid 3-ball subject to the condition that antipodal points on its 2-sphere boundary are identified. Note that no identification occurs on interior points of the 3-ball.

**Figure 4:** Two illustrations of polyhedra with badly behaved curvatures: two 2-spheres connected along a line segment and two cones with their apexes identified.

**Figure 5:** A 4-conifold produced by removing three disks from a 2-manifold of genus 4, taking the product of the result with a 2-sphere and then coning off the boundaries. Two dimensions have been suppressed; each nonsingular point is a 2-sphere but the singular points $a$, $b$, and $c$ are just points.