ABSTRACT. We use the integral by parts to get a Liouville type theorem for a class quasilinear $p$-Laplace type equation on the sphere, this $p$-Laplace type equation arises from the study of asymptotic behaviour near the origin for the semilinear $p$-Laplace equation on the puncture ball $B_1(o) \subset \mathbb{R}^n$. This give a positive answer to L. Véron’s question in a paper [15] and his book [16] at page 440.

1. INTRODUCTION

In Gidas-Spruck[9], they studies the Liouville type theorem for the nonnegative solution on the following semilinear elliptic equation

$$\Delta u + u^q = 0 \quad \text{in} \quad \mathbb{R}^{n+1}$$

(1.1)

in the range of $1 < q < 2^* - 1$ where $2^* = \frac{2(n+1)}{n-1}$, and they obtained the unique solution is trivial solution. Gidas-Spruck[9] proved their results via the method of vector fields and integral by parts motivated by Obata identity [11].

In order to study the asymptotic behaviour near the origin for the above equation (1.1) on the puncture ball $B_1(o) \setminus \{o\} \subset \mathbb{R}^{n+1}$, their studied the following equation on sphere $S^n$.

$$\Delta u + u^q - \lambda u = 0 \quad \text{in} \quad S^n,$$

(1.2)

and their also got a Liouville type theorem under certain condition on $q, \lambda$. In a more genaral setting, Veron-Veron [14] got the following theorem.

**Theorem 1.1.** (Véron-Véron [14]) Assume $(M, g)$ is a compact Riemannian manifold without boundary of dimension $n \geq 2$, $\Delta$ is the Laplace-Beltrami operator on $M$, $q > 1, \lambda > 0$ and $u$ is a positive solution of

$$\Delta u + u^q - \lambda u = 0 \quad \text{on} \quad M^n.$$ 

(1.3)

Assume also that the spectrum $\sigma(R(x))$ of the Ricci tensor $R$ of the metric $g$ satisfies

$$\inf_{x \in M} \min \sigma(R(x)) \geq \frac{n-1}{n}(q-1)\lambda, \quad q \leq \frac{n+2}{n-2}. \quad (1.4)$$

Moreover, assume that one of the two inequalities is strict if $(M, g)$ is conformally diffeomorphic to $(S^n, g)$. Then $u$ is constant with the value $\lambda^\frac{1}{q-1}$. 

For the following semilinear $p$-Laplace equation

$$\Delta_p u + u^q = 0 \quad \text{in} \quad \mathbb{R}^{n+1}. \quad (1.5)$$
In Serrin-Zou[13], they got a Liouville type theorem for the nonnegative solution of equation (1.5), in the range of $1 < p < n + 1$ and $p - 1 < q < p^* - 1$ where $p^* = \frac{(n+1)p}{n+1-p}$, they got the unique solution is trivial solution.

In order to study the asymptotic behaviour near the origin for the above equation (1.5) on the puncture ball $B_r(0) \setminus \{0\} \subset \mathbb{R}^{n+1}$. As in Gidas-Spruck[9], Véron [16] made the following observation. With the spherical coordinate $(r, \sigma)$, separable solutions of (1.5) under the form $u(x) = u(r, \sigma) = r^{-\alpha} \omega(\sigma)$ exist, then $\omega$ satisfies

$$
\text{div}((\lambda_{p,q}^2 \omega^2 + |\nabla \omega|^2)^{\frac{p-2}{2}} \nabla \omega) + |\omega|^{q-1} \omega - \lambda_{p,q}(\alpha_{p,q}^2 \omega^2 + |\nabla \omega|^2)^{\frac{p-2}{2}} \omega = 0 \quad \text{on} \quad S^n, \quad (1.6)
$$

where

$$
\lambda_{p,q} = \alpha_{p,q}(n + 1 - \alpha_{p,q}), \quad \alpha = \alpha_{p,q} = \frac{p}{q+1-p},
$$

div and $\nabla$ are operators under the canonical metric on $S^n$.

If $\lambda_{p,q} < 0$, i.e. $p - 1 < q \leq \frac{(n+1)(p-1)}{n+1-p}$, integrating equations (1.6) shows that there exists no nontrivial solution to (1.6).

For $q = \frac{np-n-2p}{n-p}$, which is the Sobolev critical exponent, it is well known that (1.5) admits nonconstant solutions.

For $\lambda_{p,q} > 0$, $\frac{(n+1)(p-1)}{n+1-p} < q < \frac{np-n+2p}{n-p}$, in a paper [15] and his book[16] at page 440, L. Véron asked if all positive solutions of (1.5) are constant. In this paper we confirm it:

**Theorem 1.2.** For $1 < p < n$ and $\frac{(n+1)(p-1)}{n+1-p} < q < \frac{np-n+2p}{n-p}$ with $\alpha_{p,q} = \frac{p}{q+1-p}$ and $\lambda_{p,q} = \alpha_{p,q}(n + 1 - \alpha_{p,q}q)$, any positive solution to (1.6) is constant.

**Remark 1.3.** Using the similar computation, our proof also work on the closed Riemannian manifold $(M, g)$ with $\text{Ric}_{ij} \geq (n-1)g_{ij}$.

We give some reviews on the this related subject. Based on the technique developed in Dolbeault-Esteban-Loss[7], Dolbeault-Esteban-Loss[8] finally solve the famous problem of the characterization of the optimal symmetry breaking region in Caffarelli-Kohn-Nirenberg inequalities[3] completely. As in Gidas-Spruck [9], Ma-Ou [10] get similar Liouville results on Heisenberg group $\mathbb{H}^n$. Inspired by Serrin-Zou[13], Ciraolo-Figalli-Roncoroni[5] classify the positive energy finite solutions to (1.5) when $q = \frac{(n+1)p}{n+1-p} - 1$ in convex cones with the help of some prior estimates. For the $p$ version Caffarelli-Kohn-Nirenberg inequalities, there are some partial results in Ciraolo-Corso[4]. See the recent result for the critical $p$-Laplace equation in $\mathbb{R}^n$ by Q.Ou [12]. In Ciraolo-Figalli-Roncoroni[5], an important Lemma by [1] and [6] for the research of $p$-Laplacian equations has been used.

In the above $p$-Laplace equation papers, the authors always introduce only one parameter to get their result. In the proof of our theorem 1.2, we introduce three parameters and use the Lemma from [1] or [6] to complete our proof.

The paper is organized as follows. In section 2, we introduce some notations and prove an integral equality. Then we use the integral equality through choosing these parameters to prove the Theorem 1.2 in section 3.

2. AN INTEGRAL EQUALITY

In this section, we drive a useful equality.
Let \( \omega = v^{-\beta}, \beta \neq 0 \). We denote \( k = (\beta + 1)(p - 1) - \beta q, Q = (\alpha^2 v^2 + \beta^2 |\nabla v|^2)^{\frac{1}{2}}, \)
\( X^i = Q^{p-2}v_i, X^i_j = (Q^{p-2}v_i)_j, E^i_j = X^i_j - \frac{div_a(X^i)}{n}g_{ij}, L^i_j = Q^{p-2}(\frac{v_iv_j}{v} - \frac{|\nabla v|^2}{nv}g_{ij}). \)

Then our equation (1.2) becomes
\[
E^i_j X^i_j = X^i_j X^j_i - \frac{X^i_i X^j_j}{n}. \tag{2.1}
\]
and
\[
|L^i_j|^2 = \frac{n - 1}{n}Q^{2p-4}v^{-2}|\nabla v|^4. \tag{2.2}
\]

We modify \( E^i_j \) to deal with the subcritical case. We set \( F^i_j = E^i_j + \varepsilon div_a(X^i)g_{ij}, \) for some \( \varepsilon \neq 0, \) which is \( F^i_j = X^i_j + (\varepsilon - \frac{1}{n})div_a(X^i)g_{ij}. \)

Using the fact that \( L^i_j \) is trace free, we have
\[
F^i_j F^j_i = X^i_j X^j_i + (n\varepsilon^2 - \frac{1}{n})(X^i_i)^2, \tag{2.3}
\]
\[
F^i_j L^j_i = E^i_j L^j_i = Q^{p-2} \left( (Q^{p-2}v_i)_j \frac{v_iv_j}{v} - (Q^{p-2}v_i)_i \frac{|\nabla v|^2}{nv} \right). \tag{2.4}
\]

Then our equation (1.2) becomes
\[
X^i_j - (\beta + 1)(p - 1)v^{-1}|\nabla v|^2Q^{p-2} - \beta^{-1}v^k + \beta^{-1}\lambda Q^{p-2}v = 0. \tag{2.5}
\]

Multiplying (2.5) with \( v^a X^j_j \) and integrating on \( S^n, a \) shall be determined in later, we get
\[
\int v^a X^i_j X^j_i - (\beta + 1)(p - 1) \int v^{a-1}|\nabla v|^2Q^{p-2}X^j_j - \beta^{-1} \int v^{k+a}X^j_j + \beta^{-1}\lambda \int v^{a+1}Q^{p-2}X^j_j = 0. \tag{2.6}
\]

As by integral by parts, for the third term in (2.6) we have
\[
-\beta^{-1} \int v^{k+a}X^j_j = \beta^{-1}(k+a) \int v^{k+a-1}|\nabla v|^2Q^{p-2}. \tag{2.7}
\]

Note that
\[
(Q^{p-2})_j = \left[ (\alpha^2 v^2 + \beta^2 |\nabla v|^2)^{\frac{p-2}{2}} \right]_j = (p-2)Q^{p-4}(\alpha^2 v v_j + \beta^2 v_l v_{lj}). \tag{2.8}
\]

Now we set \( f = vv_j v_l v_{lj} - |\nabla v|^4, \) then the last term in (2.6) becomes
\[
\beta^{-1}\lambda \int v^{a+1}Q^{p-2}Q^{p-2}v_j_j \]
\[
= -\beta^{-1}\lambda(a+1) \int v^a|\nabla v|^2Q^{2p-4} - \beta^{-1}\lambda \int v^{a+1}(Q^{p-2})_j Q^{p-2}v_j \]
\[
= -\beta^{-1}\lambda(a+p-1) \int v^a|\nabla v|^2Q^{2p-4} - (p-2)\beta\lambda \int v^aQ^{2p-6}f. \tag{2.9}
\]
As for the first term in (2.6), we observe that \((Q^{p-2}v_j)_i = (Q^{p-2}v_j)_{ij} - R_{ij}v_j Q^{p-2}\) where \(R_{ij}\) is the Ricci curvature. So we have

\[
\int v^a (Q^{p-2}v_i)_j X_j^i = -a \int v^{a-1} Q^{p-2} |\nabla v|^2 X_j^i - \int v^a Q^{p-2} v_i X_j^i \\
= -a \int v^{a-1} Q^{p-2} |\nabla v|^2 X_j^i - \int v^a (Q^{p-2}v_j)_i j Q^{p-2} v_i + \int v^a Q^{2p-4} R_{ij} v_j v_i \\
= -a \int v^{a-1} Q^{p-2} |\nabla v|^2 X_j^i + \int v^a (Q^{p-2}v_j)_i (Q^{p-2}v_j)_i \\
+ a \int v^{a-1}(Q^{p-2}v_j)_i Q^{p-2} v_i v_j + \int v^a Q^{2p-4} R_{ij} v_j v_i.
\]

Invoking (2.3), it follows that the first term in (2.6) becomes

\[
\int v^a X_i^i X_j^j = -\frac{na}{n-1 + n^2 \varepsilon^2} \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i + \frac{n}{n-1 + n^2 \varepsilon^2} \int v^a R_{ij} v_j v_i \\
+ \frac{na}{n-1 + n^2 \varepsilon^2} \int v^{a-1}(Q^{p-2}v_j)_i Q^{p-2} v_i v_j + \frac{n}{n-1 + n^2 \varepsilon^2} \int v^a F_j^i F_i^j.
\]  

(2.10)

To deal with the term in (2.7), we times the equation (2.5) with \(|\nabla v|^2 v^{a-1} Q^{p-2}\). Then for the third term in (2.6), we get

\[
-\beta^{-1} \int v^{k+a} X_j^j = \beta^{-1}(k+a) \int v^{a+k-1} |\nabla v|^2 Q^{p-2} \\
=(k+a) \int v^{a-1} |\nabla v|^2 Q^{p-2} X_i^i - (k+a)(\beta+1)(p-1) \int v^{a-2} |\nabla v|^4 Q^{2p-4} \\
+ (k+a)\beta^{-1} \lambda \int v^a |\nabla v|^2 Q^{2p-4}.
\]  

(2.11)

Recalling that \(k = (\beta+1)(p-1) - \beta q\) and \(R_{ij} = (n-1) g_{ij}\), combining (2.9), (2.10) and (2.11) with (2.6), we arrive at the following integral identity.

**Proposition 2.1.** If \(v\) is a positive solution for the equation (2.5), then we have

\[
\left( -\beta q + \frac{-a + a n^2 \varepsilon^2}{n-1 + n^2 \varepsilon^2} \right) \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i + \frac{na}{n-1 + n^2 \varepsilon^2} \int v^{a-1} Q^{p-2} v_i v_j (Q^{p-2} v_j)_i \\
+ \left[ \frac{n(n-1)}{n-1 + n^2 \varepsilon^2} + \lambda (p-1-q) \right] \int v^a |\nabla v|^2 Q^{2p-4} + \frac{n}{n-1 + n^2 \varepsilon^2} \int v^a F_j^i F_i^j \\
- (k+a)(\beta+1)(p-1) \int v^{a-2} |\nabla v|^4 Q^{2p-4} - \beta \lambda (p-2) \int v^a Q^{2p-6} f = 0.
\]

(2.12)

To address the last term for \(p \neq 2\), we need the following lemma.
Lemma 2.2. We have
\[ \int v^{a-1}Q^{p-2}|\nabla v|^2(Q^{p-2}v_i)_i = -(a - 1) \int v^{a-2}|\nabla v|^4Q^{2p-4} - \frac{p}{p-1} \int v^{a-1}(Q^{p-2}v_j)_iQ^{p-2}v_i v_j - \frac{p-2}{p-1} \alpha^2 \int v^aQ^{2p-6}f. \] (2.13)

Proof. Combining
\[ L.H.S = \int v^{a-1}Q^{p-2}v_j v_j(Q^{p-2}v_i)_i, \]
\[ = -(a - 1) \int Q^{2p-4} v^{a-2} |\nabla v|^4 - \int v^{a-1}(Q^{p-2}v_j)_iQ^{p-2}v_i v_j - \int v^{a-1}Q^{2p-4}v_i v_j v_{ij} \]
and
\[ (Q^{p-2}v_j)_iQ^{p-2}v_i v_j = Q^{2p-4}v_{ij} v_{ij} + (p - 2)Q^{2p-6}(\alpha^2v|\nabla v|^4 + \beta^2|\nabla v|^2v_i v_k v_k) \]
\[ = Q^{2p-4}v_{ij} v_{ij} - (p - 2)\alpha^2vQ^{2p-6}f + (p - 2)Q^{2p-6}v_i v_k v_k (\alpha^2v^2 + \beta^2|\nabla v|^2) \]
we get (2.13). \[\square\]

Therefore the last term in (2.12) becomes
\[ -\beta \lambda(p-2) \int v^aQ^{2p-6}f = \frac{\beta \lambda(p-1)}{\alpha^2} \int v^{a-1}Q^{p-2} |\nabla v|^2 X_i^i \]
\[ + \frac{\beta \lambda(p-1)(a-1)}{\alpha^2} \int v^{a-2} |\nabla v|^4 Q^{2p-4} \]
\[ + \frac{\beta \lambda p}{\alpha^2} \int v^{a-1}(Q^{p-2}v_j)_iQ^{p-2}v_i v_j. \] (2.14)

It follows that we get the following important integral identity.

Proposition 2.3. If \( v \) is a positive solution for the equation (2.5), then for any constants \( \varepsilon, \beta, \alpha \) we have
\[ 0 = \left[ -\beta q + \frac{-a + an^2 \varepsilon^2}{n - 1 + \varepsilon^2 n^2} + \frac{\beta \lambda(p-1)}{\alpha^2} \right] \int v^{a-1}Q^{p-2} |\nabla v|^2 X_i^i \]
\[ + \left[ \frac{n(n-1)}{n - 1 + n^2 \varepsilon^2} + \lambda(p - 1 - q) \right] \int v^a |\nabla v|^2 Q^{2p-4} \]
\[ + \left( \frac{\beta \lambda p}{\alpha^2} + \frac{na}{n - 1 + n^2 \varepsilon^2} \right) \int v^{a-1}Q^{p-2}v_i v_j (Q^{p-2}v_i)_j \]
\[ + \left[ \frac{\beta \lambda(p-1)(a-1)}{\alpha^2} - (k + a)(\beta + 1)(p - 1) \right] \int v^{a-2} |\nabla v|^4 Q^{2p-4} \]
\[ + \frac{n}{n - 1 + n^2 \varepsilon^2} \int v^a F_i^j F_j^i. \] (2.15)
3. PROOF OF THE THEOREM 1.2

In this section, through the choice of the constants $\varepsilon, \beta, a$, we analyze the coefficients in (2.15), and we prove $|\nabla v| = 0$, then $|\nabla \omega| = 0$ so we complete the proof of our Theorem 1.2. Using (2.4) in the third term in (2.15), we can rewrite (2.15)

$$0 = \left[ -\beta q + \frac{-a + an^2 \varepsilon^2}{n - 1 + \varepsilon^2 n^2} + \frac{\beta \lambda (p - 1)}{\alpha^2} \right] \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i$$

$$+ \left[ \frac{n(n - 1)}{n - 1 + n^2 \varepsilon^2} + \lambda (p - 1 - q) \right] \int v^a |\nabla v|^2 Q^{2p-4}$$

$$+ \left( \frac{\beta \lambda p}{\alpha^2} + \frac{na}{n - 1 + n^2 \varepsilon^2} \right) \int v^a F_j^i F_i^j$$

$$+ \left( \frac{\beta \lambda p}{na^2} + \frac{a}{n - 1 + n^2 \varepsilon^2} \right) \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i$$

$$+ \left[ \frac{\beta \lambda (p - 1)(a - 1)}{\alpha^2} - (k + a)(\beta + 1)(p - 1) \right] \int v^{a-2} |\nabla v|^4 Q^{2p-4}$$

$$+ \frac{n}{n - 1 + n^2 \varepsilon^2} \int v^a F_j^i F_i^j.$$

To be convenient, we let

$$M = \left( \frac{\beta \lambda p}{\alpha^2} + \frac{na}{n - 1 + n^2 \varepsilon^2} \right).$$

By (2.2), we get the following crucial integral identity.

$$0 = \left[ -\beta q + \frac{-a + an^2 \varepsilon^2}{n - 1 + \varepsilon^2 n^2} + \frac{\beta \lambda (p - 1)}{\alpha^2} \right] \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i$$

$$+ \left[ \frac{n(n - 1)}{n - 1 + n^2 \varepsilon^2} + \lambda (p - 1 - q) \right] \int v^a |\nabla v|^2 Q^{2p-4}$$

$$+ \frac{n}{n - 1 + n^2 \varepsilon^2} \int v^a (F_j^i + ML^i_j)(F_i^j + ML_i^j)$$

$$+ \left[ -\frac{1}{4} \left( \frac{\beta \lambda p}{\alpha^2} + \frac{na}{n - 1 + n^2 \varepsilon^2} \right)^2 \frac{(n - 1 + n^2 \varepsilon^2)(n - 1)}{n} + \frac{\beta \lambda (p - 1)(a - 1)}{\alpha^2} \right.$$

$$- (k + a)(\beta + 1)(p - 1) \right] \int v^{a-2} |\nabla v|^4 Q^{2p-4}.$$ (3.2)

Here we have only four terms and but three parameters $\beta, a, \varepsilon$, we shall choose them properly to cancel three terms.

First, we choose $\varepsilon$ to make

$$\frac{n(n - 1)}{n - 1 + n^2 \varepsilon^2} + \lambda (p - 1 - q) = 0,$$ (3.3)

from (3.3) we know the coefficient of the second term in identity (3.2) is zero.

To see this is possible, we show that
Lemma 3.1. If the constant $p, q, \alpha$ and $\lambda$ satisfy the condition in the Theorem 1.2, then we have $n + \lambda(p - 1 - q) > 0$.

Proof. Recall that $\lambda = \alpha(n + 1 - \alpha q)$ and $\alpha = \frac{p}{q + 1 - p}$, then we need to show

$$n - p \left( n + 1 - \frac{pq}{q + 1 - p} \right) > 0,$$

which holds iff

$$\frac{pq}{q + 1 - p} > \frac{np + p - n}{p}.$$

The above inequality is reduced to

$$(1 - p)(n - p)q > (1 - p)(np + p - n),$$

which is from the subcritical exponent of $q$,

$$q < \frac{(n + 1)(p - 1) + 1}{n - p}.$$

□

Now we take $\varepsilon = \left[ \frac{n - \lambda(q + 1 - p)}{\lambda(q + 1 - p)} \right]^\frac{1}{2} (n - 1)^\frac{1}{2} n^{-1}$ then we get

$$n^2 \varepsilon^2 = \frac{\left[ n - \lambda(q + 1 - p) \right]}{\lambda(q + 1 - p)} (n - 1),$$

and

$$\frac{n}{n - 1 + n^2 \varepsilon^2} = \frac{\lambda(q + 1 - p)}{n - 1}.$$ (3.5)

Second, we let $\alpha = t\beta$, and take $t$ to make

$$\frac{\lambda p}{n \alpha^2} - q + \frac{tn^2 \varepsilon^2}{n - 1 + n^2 \varepsilon^2} + \frac{\lambda(p - 1)}{\alpha^2} = 0.$$ (3.6)

From (3.6) we know the coefficient of the first term in the identity (3.2) is zero.

By substituting $\varepsilon$, we take

$$t = \left( q - \frac{\lambda(p - 1)}{\alpha^2} - \frac{\lambda p}{n \alpha^2} \right) \frac{n}{n - \lambda(q + 1 - p)}.$$ (3.7)

Now we simplify it.

Lemma 3.2. In fact $t = \frac{n + 1}{\alpha}.$

Proof. First we have

$$n - \lambda(q + 1 - p) = \frac{1}{q + 1 - p}[(q + 1 - p)n - p(q + 1 - p)(n + 1) + p^2 q]$$

$$= \frac{1}{q + 1 - p}[qn + (1 - p)n - (p - 1)(q + 1 - p)(n + 1) - (q + 1 - p)(n + 1) + q + (p^2 - 1)q]$$

$$= \frac{p - 1}{q + 1 - p}[1 - (q + 1 - p)(n + 1) + (p + 1)q].$$
And we can get

\[
n \left( q - \frac{\lambda(p - 1)}{\alpha^2} - \frac{\lambda p}{n \alpha^2} \right)
\]

\[= nq - \frac{n(n + 1 - \alpha q)(p - 1)}{\alpha} - \frac{(n + 1 - \alpha q)p}{\alpha}
\]

\[= nq - \frac{n + 1 - \alpha q}{\alpha} - \frac{n(n + 1 - \alpha q)(p - 1)}{\alpha} - \frac{(n + 1 - \alpha q)(p - 1)}{\alpha}
\]

\[= nq + \frac{n + 1 - \alpha q}{\alpha} - \frac{(n + 1 - \alpha q)(p - 1)}{\alpha}
\]

\[= \frac{(q \alpha - 1)(n + 1)}{\alpha} - \frac{(n + 1 - \alpha q)(p - 1)(n + 1)}{\alpha}
\]

\[= \frac{n + 1}{\alpha} [q \alpha - 1 - (n + 1 - \alpha q)(p - 1)]
\]

\[= \frac{n + 1}{\alpha} \left( \frac{(q + 1)(p - 1)}{q + 1 - p} - (n + 1 - \alpha q)(p - 1) \right)
\]

\[= \frac{n + 1}{\alpha} \frac{(p - 1)}{(q + 1 - p)} [q + 1 - (n + 1)(q + 1 - p) + pq].
\]

\[\square\]

Now we substitute \( \varepsilon \) and \( a = \frac{n + 1}{\alpha} \beta \) into the coefficient of \( \int v^{\alpha - 2} |\nabla v|^4 Q^{2p - 4} \), and we shall find \( \beta \) such that the coefficient of the last term in the identity (3.2) is zero.

First we get the coefficient of \( \int v^{\alpha - 2} |\nabla v|^4 Q^{2p - 4} \) in (3.2) is \( g(\beta) \), where

\[g(\beta) = - \frac{1}{4} \left( \frac{\beta \lambda p}{\alpha^2} + \frac{n a}{n - 1 + n^2 \varepsilon^2} \right)^2 \frac{(n - 1 + n^2 \varepsilon^2)(n - 1)}{n} \frac{\lambda(p - 1)(a - 1)}{\alpha^2}
\]

\[- (k + a)(\beta + 1)(p - 1) - \frac{1}{4n} \frac{\lambda p}{\alpha^2} + \frac{n + 1}{\alpha} \frac{\lambda(q + 1 - p)}{n - 1} \frac{(n - 1)^2}{\lambda(q + 1 - p)} \beta^2 + \frac{\lambda(p - 1)}{\alpha^2} \frac{(n + 1)}{\alpha} \beta^2
\]

\[- (p - 1)^2 \beta^2 + q(p - 1) \beta^2 - \frac{(n + 1)(p - 1)}{\alpha} \beta^2
\]

\[+ q(p - 1) \beta - \frac{(n + 1)(p - 1)}{\alpha} \beta - \frac{\lambda(p - 1)}{\alpha^2} \beta - 2(p - 1)^2 \beta
\]

\[- (p - 1)^2.
\]

We have the following lemma.

**Lemma 3.3.** \( \exists ! \beta_0 \), such that \( g(\beta_0) = 0. \)
Proof. Since $g(\beta)$ is a quadratic function, we show that its determinant identically vanishes, which is

\[
\begin{align*}
&\left[-\frac{\lambda}{\alpha^2} - 2(p-1) + q - \frac{n+1}{\alpha}\right]^2(p-1)^2 \\
&+ 4(p-1)^2\left[-\frac{1}{4n}\left(\frac{\lambda p}{\alpha^2} + \frac{n+1}{\alpha} \frac{\lambda(q+1-p)}{n-1}\right)^2 \frac{(n-1)^2}{\lambda(q+1-p)} + \frac{\lambda(p-1)}{\alpha^2} + \frac{n+1}{\alpha}\right] = 0.
\end{align*}
\]

(3.8)

We simplify it term by term, first we have

\[
-\frac{\lambda}{\alpha^2} - 2(p-1) + q - \frac{n+1}{\alpha} = -(n+1)(q+1-p) + \frac{pq}{p} - 2(p-1) + q - \frac{(n+1)(q+1-p)}{p} \\
= -\frac{2(n+1)(q+1-p)}{p} + 2(q+1-p) \\
= \frac{2(p-n-1)(q+1)}{p}.
\]

And we also have

\[
\frac{\lambda p}{\alpha^2} + \frac{(n+1)}{\alpha} \frac{\lambda(q+1-p)}{n-1} = (n+1 - \alpha q)(q+1-p)[1 + \frac{n+1}{n-1}] \\
= (n+1 - \alpha q)(q+1-p)\frac{2n}{n-1}.
\]

We simply the following term

\[
-\frac{1}{4n}\left(\frac{\lambda p}{\alpha^2} + \frac{n+1}{\alpha} \frac{\lambda(q+1-p)}{n-1}\right)^2 \frac{(n-1)^2}{\lambda(q+1-p)} \\
= -\frac{1}{4n}(n+1 - \alpha q)^2(q+1-p)^2 \frac{4n^2}{(n-1)^2} \frac{(n-1)^2}{\lambda(q+1-p)} \\
= -\frac{n(n+1 - \alpha q)(q+1-p)^2}{p}.
\]

Then

\[
\frac{\lambda(p-1)(n+1)}{\alpha^3} = \frac{(n+1 - \alpha q)(p-1)(n+1)(q+1-p)^2}{p^2},
\]

and

\[
-\frac{(n+1)(p-1)}{\alpha} = -\frac{(n+1)(p-1)(q+1-p)}{p}.
\]
It follows that its determinant (3.8) is equivalent to
\[
\frac{(p - n - 1)^2(q + 1 - p)}{p^2} - \frac{n(n + 1 - \alpha q)(q + 1 - p)^2}{p} + \frac{(n + 1 - \alpha q)(p - 1)(n + 1)(q + 1 - p)^2}{p^2} - \frac{(n + 1)(p - 1)(q + 1 - p)}{p} + (q + 1 - p)(p - 1) = 0.
\]

Multiplying \(\frac{p^2}{q+1-p}\) and using \((n + 1 - \alpha q)(q + 1 - p) = (q + 1 - p)(n + 1) - pq\), it is equivalent for us to show
\[
(p - n - 1)^2(q + 1 - p) - pn(q + 1 - p)(n + 1) + p^2 n(q + 1 - p) + p^2 n(p - 1) + (n + 1)(p - 1) + (n + 1)(p - 1)p(1 - p) - p(p - 1)(n + 1) + p^2 (p - 1) = 0,
\]
iff
\[
(q + 1 - p)[(p - 1 - n)^2 -pn(n + 1) + p^2 n + (n + 1)^2(p - 1) + (n + 1)(p - 1)p] = 0,
\]
which is correct by direct computation. \(\square\)

From the expression \(g(\beta)\), we can take \(\beta_0\) such that \(g(\beta_0) = 0\). It follows that the first term, the second term and the last term in (3.2) is zero. At last we get the following result.

**Proposition 3.4.** If \(v\) is a positive solution for the equation (2.5), then for the above determined constants \(\varepsilon, \beta\), a we have
\[
0 = \frac{n}{n - 1 + n^2 \varepsilon^2} \int v^a (F^i_j + ML^i_j)(F^j_i + ML^j_i).
\]

(3.9)

To deduce the desired results, we cite a key Lemma from [1] or [6].

**Lemma 3.5.** Let the matrix \(A\) be symmetric with positive eigenvalues and let \(\lambda_{\text{min}}\) and \(\lambda_{\text{max}}\) be its smallest and largest eigenvalue, respectively; let \(B\) be a symmetric matrix, then
\[
\text{trace}(AB(AB)^T) \leq n \left(\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}\right)^2 \text{trace}((AB)^2).
\]

Now we show

**Lemma 3.6.** \(F^i_j + ML^i_j = (AB)_{ij}\) where \(A, B\) satisfy the conditions of the above Lemma.

**Proof.** From the definition of \(F^i_j, L^i_j\) in the beginning of section 2, we have
\[
F^i_j + ML^i_j = (Q^{p-2}v_i) + \left(\varepsilon - \frac{1}{n}\right) X^i_l g_{lj} + M \frac{v_i v_j}{v} Q^{p-2} - \frac{M |\nabla v|^2}{nv} Q^{p-2} g_{ij}
\]
\[
= Q^{p-4}[(p - 2)\beta^2 v_i v_j v_{ij} + (\alpha^2 v^2 + \beta^2 |\nabla v|^2)v_{ij}]
\]
\[
+ (p - 2)Q^{p-4} \alpha^2 v_i v_j + \left(\varepsilon - \frac{1}{n}\right) X^i_l g_{lj} + M \frac{v_i v_j}{v} Q^{p-2} - \frac{M |\nabla v|^2}{nv} Q^{p-2} g_{ij}
\]
\[
= (N_1 + N_2)_{ij},
\]
where \((N_1)_{ij} = Q^{p-1}[(p-2)\beta^2 v_i v_j + (\alpha^2 v^2 + \beta^2 |\nabla v|^2) v_{ij}]\).

We rewrite

\[ N_1 = N_3 N_4, \]

where \((N_4)_{ij} = Q^{p-2} v_{ij}, (N_3)_{ij} = (p-2)\frac{\beta^2 |\nabla v|^2}{\alpha^2 v^2 + \beta^2 |\nabla v|^2} v_{ij} + \delta_{ij}, N_3 \) is positive define with eigenvalues 1 and \(1 + (p-2)\frac{\beta^2 |\nabla v|^2}{\alpha^2 v^2 + \beta^2 |\nabla v|^2}\). From basic linear algebra we have

\[ (N_3^{-1})_{ij} = \delta_{ij} - (p-2)\frac{\beta^2 |\nabla v|^2}{\alpha^2 v^2 + (p-1)\beta^2 |\nabla v|^2} v_{ij}. \]

Then

\[ N_1 + N_2 = N_3(N_4 + N_3^{-1} N_2). \]

By direct calculations, \(N_3^{-1} N_2\) is also a symmetric matrix.

Setting \(A = N_3, B = N_4 + N_3^{-1} N_2\) and we have done.

Now we prove the following last lemma.

**Lemma 3.7.** \(|\nabla v| = 0\)

**Proof.** By Lemma 3.5, Lemma 3.6, Proposition 3.4, we have

\[ F_j^i + M L_j^i = 0, \]

which is

\[ E_j^i + \varepsilon X_l^i g_{ij} + M L_j^i = 0. \]

By taking trace we have

\[ X_l^l = 0. \]

Then

\[ 0 = \int X_i^i X_j^j = \frac{n}{n-1} \int E_j^i E_i^j + \frac{n}{n-1} \int R_{ij} v_i v_j Q^{2p-4}. \]

Following the method of Lemma 3.6, one can show that \(\int E_j^i E_i^j \geq 0\), it forces that \(|\nabla v| = 0\). Then we get \(v\) is constant and complete the proof of Theorem 1.2.

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