Infinitesimal Rigidity of Strain Tensors for Shells with Mixed Type and its Applications

Liang-Biao Chen\textsuperscript{a} and Peng-Fei Yao\textsuperscript{ab}

\textsuperscript{a} Key Laboratory of Systems and Control
Institute of Systems Science, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100190, P. R. China

\textsuperscript{b} School of Mathematical Sciences
Shanxi University, Taiyuan 030006, China

e-mail: pfyao@iss.ac.cn

Abstract We derive an infinitesimal rigidity lemma for the strain tensor of surfaces with their curvatures changing sign. As an application, we obtain the optimal constant in the first Korn inequality scales like \( h^{4/3} \) for such shells of mixed type.

Keywords surface of mixed type, shell, Korn inequality, Riemannian geometry

Mathematics Subject Classifications (2010) 74K20(primary), 74B20(secondary).

1 Introduction and the Main Results

The goal of the present paper is twofold to study the rigidity of the strain tensors of surfaces and then to apply it to obtain the optimal constant in the first Korn inequality for some shells of mixed type.

The linear strain equations plays a fundamental role in the theory of thin shells, see [11, 12, 13, 14, 22]. The solvability of the strain equation is needed to prove the density of smooth infinitesimal isometries in the \( W^{2,2}(\Omega, \mathbb{R}^3) \) infinitesimal isometries and the matching property of the smooth enough infinitesimal isometries with higher order infinitesimal isometries [11, 14, 22]. This matching property is an important tool in obtaining recovery sequences (\( \Gamma \)-lim sup inequality) for dimensionally-reduced shell theories in elasticity, when the elastic energy density scales like \( h^\beta, \beta \in (2, 4) \), that is, the intermediate regime between pure bending (\( \beta = 2 \)) and the von-Karman regime (\( \beta = 4 \)). Such results have been obtained for elliptic surfaces [14], the developable surface [11], the hyperbolic surface

\hspace{1cm}

This work is supported by the National Science Foundation of China, grants no. 12071463.
[22], the degenerated surface [1], and the surface with the Gaussian curvature changing sign [2], respectively. A survey on this topic is presented in [12].

The type of strain tensors depends on the sign of the curvature of the surface: It is elliptic if the curvature is positive; it is parabolic if the curvature is zero but \( \Pi \neq 0 \); it is hyperbolic for the negative curvature. When the curvature of the surface changes its sign, the strain tensor is of mixed type.

Moreover, the lower regularity of strain tensors of surfaces represents the rigidity of geometrical shape of surfaces.

Linear strain equations have been studied in [2] for the surface with its curvature changing sign, where the density of smooth infinitesimal isometries in the \( W^{2,2} \) infinitesimal isometries is obtained and the matching property of infinitesimal isometries is proved. In particularly, [2] established the following regularity

\[
\|y\|^2_{W^{1,2}(S, \mathbb{R}^3)} \leq C \|U\|^2_{W^{2,2}(S,T_{3\text{sym}}^2)},
\]

where \( y \) is a solution to problem (1.6) later. However, the above regularity is not enough for establishing the rigidity results of the surface. Here, we derive the \( L^2 \) regularity, that is an infinitesimal rigidity lemma (Theorem 1.1) for the strain tensors. This lemma is one of the key ingredients for the optimal constant in the first Korn inequality for shells. The basic tools are some results on the generalized tensors.

Originally, Korn’s inequalities were used to prove existence, uniqueness and well-posedness of boundary value problems of linear elasticity [3, 16]. The optimal exponential of thickness in Korn’s inequalities for thin shells represents the relationship between the rigidity and the thickness of a shell when the small deformations take place since Korn’s inequalities are linearized from the geometric rigidity inequalities under the small deformations [5]. Thus it is the best Korn constant in the Korn inequality that is of central importance [4, 15, 17, 18, 19, 20]. Moreover, one has known that the best Korn constant subject to the Gaussian curvature. Under the assumption that the middle surface of the shell is given by a single principal coordinate system, the one for the parabolic shell scales like \( h^{3/2} \) [6, 7], for the hyperbolic shell, \( h^{4/3} \) [10] and for the elliptic shell, \( h \) [10]. Later the assumption of the single principal coordinate is removed in [23, 24], where the case of the closed elliptic shell is particularly included.

To the best knowledge of the authors, the present paper for the first time establishes the rigidity results for the surface with the curvature changing sign (Theorems 1.1 and 1.2) and thus yields that the optimal constant in the Korn inequality scales like \( h^{4/3} \) for some shells of mixed type (Theorem 1.3).

Let \( S \subset M \) be given by

\[
S = \left\{ \alpha(t,s) \mid (t,s) \in [0,a) \times (-b_0,b_1) \right\}, \quad a > 0, \quad b_0 > 0, \quad b > 0,
\]  
(1.1)
where \( \alpha : [0, a) \times [0, b] \to M \) is an imbedding map which is a family of regular curves with two parameters \( t, s \) such that

\[
\Pi(\alpha_t(t, s), \alpha_t(t, s)) > 0, \quad \text{for all} \quad (t, s) \in [0, a) \times [-b_0, 0],
\]

where \( \alpha(\cdot, s) \) is a closed curve with period \( a \) for each \( s \in [-b_0, b_1] \). Set

\[
S = S^+ \cup \Gamma_0 \cup S^-,
\]

where

\[
S^+ = \{ \alpha(t, s) | (t, s) \in [0, a) \times (0, b_1) \}, \quad \Gamma_0 = \{ \alpha(t, 0) | t \in [0, a) \},
\]

\[
S^- = \{ \alpha(t, s) | (t, s) \in [0, a) \times (-b_0, 0) \}.
\]

**Curvature assumptions** Let \( \kappa \) be the Gaussian curvature function on \( M \). We assume that \( S \) satisfies the following curvature conditions:

\[
\kappa(x) > 0 \quad \text{for} \quad x \in S^+ \cup \Gamma_{b_1}; \tag{1.3}
\]

\[
\kappa = 0, \quad D\kappa(x) \neq 0 \quad \text{for} \quad x \in \Gamma_0; \tag{1.4}
\]

\[
\kappa(x) < 0 \quad \text{for} \quad x \in S^- \cup \Gamma_{-b_0}, \tag{1.5}
\]

where

\[
\Gamma_{b_1} = \{ \alpha(t, b_1) | t \in [0, a) \}, \quad \Gamma_{-b_0} = \{ \alpha(t, -b_0) | t \in [0, a) \}.
\]

Let \( y \in W^{1,2}(S, \mathbb{R}^3) \) be a displacement of the middle surface \( S \). We decompose \( y \) as

\[
y = W + w\vec{n}, \quad w = \langle y, \vec{n} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the dot metric of the Euclidean space \( \mathbb{R}^3 \). The (linear) strain tensor of the middle surface (related to the displacement \( y \)) is defined by

\[
\Upsilon(y) = \text{sym}DW + w\Pi,
\]

where \( D \) is the Levi-Civita connection of the induced metric \( g \) on \( S \),

\[
\text{sym}DW = \frac{1}{2}(DW + DW^T),
\]

and

\[
\Pi(\alpha, \beta) = \langle \nabla_\alpha \vec{n}, \beta \rangle \quad \text{for} \quad \alpha, \beta \in T_xM, \quad x \in S
\]

is the second fundamental form of surface \( M \). For given \( U \in L^2(S, T^2) \), consider problem

\[
\Upsilon(y) = U \quad \text{for} \quad p \in S. \tag{1.6}
\]
Let $T^k$ denote the all $k$th-order tensor fields on $S$. Let $L^2(S,T^k)$ be the space of all $k$th-order tensor fields on $S$ with the norm

$$
(P_1, P_2)_{L^2(S,T^k)} = \int_S \langle P_1, P_2 \rangle dg,
$$

where

$$
\langle P_1, P_2 \rangle = \sum_{i_1, \cdots, i_k = 1}^2 P_1(e_{i_1}, \cdots, e_{i_k}) P_2(e_{i_1}, \cdots, e_{i_k}) \quad \text{for} \quad x \in S,
$$

and $e_1, e_2$ is an orthonormal basis of $T_x S$. Let $[W^{m,2}(S,T^k)]'$ be the dual spaces, based on the inner product $(\cdot, \cdot)_{L^2(S,T^k)}$ of $L^2(S,T^k)$.

Our main results are the following, that is, an infinitesimal rigidity lemma on the strain tensor of the middle surface with mixed type.

**Theorem 1.1.** Suppose that region $S$ of $C^5$ satisfies (1.1)-(1.5). Let $y = W + w\vec{n}$ solve problem (1.6). Then there is a constant $C > 0$, independent of $y$, such that

$$
\|W\|_{L^2(S,T)}^2 + \|w\|_{W^{1,2}(S)}^2 \leq C(\|U\|_{L^2(S,T^2)}^2 + \|W\|_{L^2(\partial S,T)}^2).
$$

(1.7)

The proof of Theorem 1.1 will be presented in the end of Section 3.

For the hyperbolic surface $S$, estimate (1.7) is given in [24]. If $S$ is elliptic, we then have

$$
\|DW\|_{L^2(S,T^2)}^2 + \|w\|_{L^2(S)}^2 \leq C(\|U\|_{L^2(S,T^2)}^2 + \|W\|_{L^2(\partial S,T)}^2),
$$

see [3, 23].

Using Theorem 1.1 and by the same argument as in [24, Proof of Theorem 1.3], we have the following.

**Theorem 1.2.** Suppose that region $S$ of $C^5$ satisfies (1.1)-(1.5). Then

$$
\|w\|_{L^2(S)}^2 \leq C(\|DW\|_{L^2(S,T)}\|\Upsilon(y)\|_{L^2(S,T^2)} + \|\Upsilon(y)\|_{L^2(S,T^2)}^2)
$$

(1.8)

for all $y = W + w\vec{n} \in W^{1,2}(S, \mathbb{R}^3)$ with $y|_{\Gamma_{-b_1}} = y|_{\Gamma_{b_1}} = 0$.

**Application to elasticity of thin shells.** Suppose that $S$ is the middle surface of the shell with thickness $h > 0$

$$
\Omega = \{ x + t\vec{n}(x) \mid x \in S, -h/2 < t < h/2 \}.
$$

By the same arguments as in [24, Proof of Theorem 1.4] we have the following results of the optimal exponential of the first Korn inequality for shells of mixed type. The details of the proofs are omitted.
Theorem 1.3. Suppose that region $S$ of $C^5$ satisfies (1.1)-(1.5). There are $C > 0$, $h_0 > 0$, independent of $h > 0$, such that
\[ \| \nabla y \|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}^2 \leq \frac{C}{h^{1/3}} \| \text{sym} \nabla y \|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}^2 \] (1.9)
for all $h \in (0, h_0)$ and $y \in W^{1,2}(S, \mathbb{R}^3)$ with $y|\Sigma = 0$, where
\[ \Sigma = \{ x + t\bar{n} \mid x \in \Gamma_{-b_0} \cup \Gamma_{b_1}, -h/2 < t < h/2 \} . \]
Moreover, the exponential of the thickness $h$ in (1.9) is optimal.

Estimates (1.8) and (1.9) have been given in [10] when the middle surface is given by one single principal coordinate, where $S$ is a hyperbolic surface. The assumption that $S$ consists of one single principal coordinate is removed in [24].

2 Generalized Tensors

Let $m \geq 0$ and $k \geq 0$ be integers. Denote by $C^m_0(S, T^k)$ all the $k$ order tensor fields with continuously $m$th order derivatives and compact supports on $\overline{S}$. Denote by $\mathcal{D}_m(S, T^k)$ the space $C^m_0(S, T^k)$ with the following convergence: a sequence $\{ \varphi_n \} \subset C^m_0(S, T^k)$ is said to converge to $\varphi_0 \in C^m_0(S, T^k)$ if

1. there exists a subset $K \subset \subset S$ such that $\text{supp} \varphi_n \subset K$ for all $n \geq 1$;
2. $\sup_{x \in K} |D_{X_1} \cdots D_{X_j}(\varphi_n - \varphi_0)| \to 0$ as $n \to \infty$ for all $X_1, \ldots, X_j \in C^m(S, T)$ where $0 \leq j \leq m$.

Let $\mathcal{D}'_m(S, T^k)$ consist of all continuously linear functionals on $\mathcal{D}_m(S, T^k)$. An element in $\mathcal{D}'_m(S, T^k)$ is said to be a generalized tensor of order $k$.

We define some generalized derivative operations as follows.

Definition 2.1. For given $w \in \mathcal{D}'_m(S)$, the generalized gradient $Dw \in \mathcal{D}'_{m+1}(S, T)$ is defined by
\[ Dw(X) = -w(\text{div } X) \quad \text{for all } X \in \mathcal{D}_{m+1}(S, T). \] (2.1)

For given $W \in \mathcal{D}'_m(S, T)$ and $y \in \mathcal{D}'_m(S, \mathbb{R}^3)$, the generalized differentials $DW \in \mathcal{D}'_{m+1}(S, T^2)$ and $\nabla y \in \mathcal{D}'_{m+1}(S, T^2)$ are defined by
\[ DW(P) = -W(\text{div } P) \quad \text{for all } P \in \mathcal{D}_{m+1}(S, T^2), \] (2.2)
and
\[ \nabla y(P) = y(-\text{div } P + \langle P, \Pi \rangle \bar{n}) \quad \text{for all } P \in \mathcal{D}_{m+1}(S, T^2), \] (2.3)
Proof. Let \( D \) be given. Suppose that \( D \in \mathcal{D}'_m(S, T) \) and \( U \in \mathcal{D}'_m(S, T^2) \), the generalized divergences \( \text{div} \, W \in \mathcal{D}'_{m+1}(S) \) and \( \text{div} \, U \in \mathcal{D}'_{m+1}(S, T) \) are defined by

\[
\text{div} \, W(z) = -W(Dz) \quad \text{for all} \quad z \in \mathcal{D}_{m+1}(S), \tag{2.4}
\]

and

\[
\text{div} \, U(Z) = -U(DZ) \quad \text{for all} \quad Z \in \mathcal{D}_{m+1}(S, T), \tag{2.5}
\]

respectively.

For given \( W \in \mathcal{D}'_m(S, T) \) and \( U \in \mathcal{D}'_m(S, T^2) \), the generalized divergences \( \text{div} \, W \in \mathcal{D}'_{m+1}(S) \) and \( \text{div} \, U \in \mathcal{D}'_{m+1}(S, T) \) are defined by

\[
\text{div} \, W(z) = -W(Dz) \quad \text{for all} \quad z \in \mathcal{D}_{m+1}(S), \tag{2.4}
\]

and

\[
\text{div} \, U(Z) = -U(DZ) \quad \text{for all} \quad Z \in \mathcal{D}_{m+1}(S, T), \tag{2.5}
\]

respectively.

For given \( y \in \mathcal{D}'_m(S, \mathbb{R}^3) \) and \( X \in \mathcal{D}_{m+1}(S, T) \), we define \( \nabla_X y \in \mathcal{D}'_{m+1}(S, T) \) by

\[
\nabla_X y(Y) = \nabla y(Y \otimes X) \quad \text{for all} \quad Y \in \mathcal{D}_{m+1}(S, T). \tag{2.6}
\]

\( \nabla_X y \in \mathcal{D}'_{m+1}(S, T) \) is said to be the generalized derivative of \( y \) along direction \( X \). Similarly, for given \( W \in \mathcal{D}'_m(S, T) \) and \( X \in \mathcal{D}_{m+1}(S, T) \), \( D_X W \in \mathcal{D}'_{m+1}(S, T) \) is defined by

\[
D_X W(Y) = DW(Y \otimes X) \quad \text{for all} \quad Y \in \mathcal{D}_{m+1}(S, T). \tag{2.7}
\]

In the above definition the right hand sides of formulas (2.1)-(2.7) are linear functionals, respectively. For instant, if \( W \in \mathcal{D}'_m(S, T) \), then we view \( W \in \mathcal{D}'_m(S, T) \) as

\[
W(Y) = \int_S (W, Y) \, dg \quad \text{for all} \quad Y \in \mathcal{D}(S, T),
\]

and so on.

**Proposition 2.1.** For given \( y \in \mathcal{D}'_m(S, \mathbb{R}^3) \) and \( X \in \mathcal{D}_{m+1}(S, T) \),

\[
\nabla_X y(Y) = -y((\text{div} X)Y + \nabla_X Y) \quad \text{for} \quad Y \in \mathcal{D}_{m+1}(S, T). \tag{2.8}
\]

If \( y \in W^{1,2}(S, \mathbb{R}^3) \), then

\[
\nabla_X y(Y) = \int_S \langle \nabla_X y, Y \rangle \, dg \quad \text{for} \quad Y \in \mathcal{D}_{m+1}(S, T). \tag{2.9}
\]

Similarly, if \( W \in W^{1,2}(S, T) \) and \( X \in \mathcal{D}_{m+1}(S, T) \), then

\[
D_X W(Y) = \int_S \langle D_X W, Y \rangle \, dg \quad \text{for all} \quad Y \in \mathcal{D}_{m+1}(S, T), \tag{2.10}
\]

where \( D_X W \) is defined by (2.7).

**Proof.** Let \( x \in S \) be given. Suppose that \( E_1, E_2 \) is a frame normal at \( x \). Then

\[
\text{div} \, (Y \otimes X) = \sum_{ij=1}^2 D(Y \otimes X)(E_i, E_j, E_j)E_i = \sum_{ij=1}^2 E_j(\langle Y, E_i \rangle \langle X, E_j \rangle)E_i
\]

\[
= \sum_{ij=1}^2 (\langle DE_j Y, E_i \rangle \langle X, E_j \rangle + \langle Y, E_i \rangle \langle DE_j X, E_j \rangle)E_i
\]

\[
= D_X Y + (\text{div} \, X)Y \quad \text{at} \quad x \quad \text{for} \quad X, Y \in \mathcal{D}_{m+1}(S, T). \tag{2.11}
\]
Thus (2.8) follows from definition (2.6).

Let \( y \in W^{1,2}(S, \mathbb{R}^3) \). Then \( y \in \mathcal{D}'_m(S, \mathbb{R}^3) \) is viewed as a linear functional on \( \mathcal{D}_m(S, T) \) by

\[
y(Y) = \int_S \langle y, Y \rangle dg \quad \text{for all} \quad Y \in \mathcal{D}_m(S, T).
\]

Noting that \( \nabla_X Y = D_X Y - \langle Y \otimes X, \Pi \rangle \vec{n} \), from (2.6), (2.3), and (2.11), we have

\[
\nabla_X y(Y) = y(-\text{div}(Y \otimes X) + \langle Y \otimes X, \Pi \rangle \vec{n}) = \int_S \langle \nabla_X y, Y \rangle dg \quad \text{for} \quad Y \in \mathcal{D}_{m+1}(S, T).
\]

A similar argument yields (2.10). \( \square \)

Let \( \mathcal{E}_m(S, T^k) \) be the space \( C^m(S, T^k) \) with the convergence: A sequence \( \{ \varphi_n \} \subset C^m(S, T^k) \) is said to converge to some \( \varphi_0 \in C^m(S, T^k) \) if for any given compact set \( K \subset S \) and any given \( X_1, \cdots, X_j \in C^m(S, T) \) where \( 0 \leq j \leq m \), there holds

\[
\lim_{n \to 0} \sup_{x \in K} |D_{X_1} \cdots D_{X_j} (\varphi_n - \varphi_0)| = 0.
\]

Denote by \( \mathcal{E}'_m(S, T^k) \) all the continuous linear functionals on \( \mathcal{E}_m(S, T^k) \).

We define some multiplier operations between generalized tensor as follows.

**Definition 2.2.** For given \( w \in \mathcal{D}'_m(S) \) and \( f \in \mathcal{E}_m(S) \), \( fw \in \mathcal{D}'_m(S) \) is defined by

\[
(fw)(z) = w(fz) \quad \text{for all} \quad z \in \mathcal{D}_m(S); \quad (2.12)
\]

for given \( W \in \mathcal{D}'_m(S, T) \) and \( f \in \mathcal{E}_m(S) \), \( fW \in \mathcal{D}'_m(S, T) \) by

\[
(fW)(Z) = W(fZ) \quad \text{for all} \quad Z \in \mathcal{D}_m(S, T); \quad (2.13)
\]

for given \( U \in \mathcal{D}'_m(S, T^2) \) and \( f \in \mathcal{E}_m(S) \), \( fU \in \mathcal{D}'_m(S, T^2) \) by

\[
(fU)(P) = U(fP) \quad \text{for all} \quad P \in \mathcal{D}_m(S, T^2); \quad (2.14)
\]

for given \( W \in \mathcal{D}'_m(S, T) \) and \( F \in \mathcal{E}_m(S, T^2) \), \( (W, F) \in \mathcal{D}'_m(S) \) by

\[
((W, F))(z) = W(zF) \quad \text{for all} \quad z \in \mathcal{D}_m(S); \quad (2.15)
\]

for given \( w \in \mathcal{D}'_m(S) \) and \( X \in \mathcal{E}_m(S, T) \), \( wX \in \mathcal{D}'_m(S, T) \) by

\[
(wX)(Z) = w((X, Z)) \quad \text{for all} \quad Z \in \mathcal{D}_m(S, T); \quad (2.16)
\]
for given \( U \in \mathcal{D}'_m(S, T^2) \) and \( R \in \mathcal{E}_m(S, T^2) \), \( \langle U, R \rangle \in \mathcal{D}'_m(S) \) by
\[
(\langle U, R \rangle)(z) = U(zR) \quad \text{for all} \quad z \in \mathcal{D}_m(S); \quad (2.17)
\]
for given \( w \in \mathcal{D}'_m(S) \) and \( R \in \mathcal{E}_m(S, T^2) \), \( wR \in \mathcal{D}'_m(S, T^2) \) by
\[
(wR)(P) = w(\langle R, P \rangle) \quad \text{for all} \quad P \in \mathcal{D}_m(S, T^2); \quad (2.18)
\]
for given \( U \in \mathcal{D}'_m(S, T^2) \), \( \text{tr} U \in \mathcal{D}'_m(S) \) by
\[
\text{tr} U(z) = U(z \text{id}) \quad \text{for all} \quad z \in \mathcal{D}_m(S); \quad (2.19)
\]
for given \( W \in \mathcal{D}'_m(S, T) \) and \( R \in \mathcal{E}_m(S, T^2) \), \( RW \in \mathcal{D}'_m(S, T) \) by
\[
(RW)(Z) = W(R^T Z) \quad \text{for all} \quad Z \in \mathcal{D}_m(S, T); \quad (2.20)
\]
for given \( U \in \mathcal{D}'_m(S, T^2) \) and \( R_1, R_2 \in \mathcal{E}_m(S, T^2) \), \( R_1 U R_2 \in \mathcal{D}'_m(S, T^2) \) by
\[
(R_1 U R_2)(P) = U(R_1^T P R_2^T) \quad \text{for all} \quad P \in \mathcal{D}_m(S, P). \quad (2.21)
\]

We need a linear operator \( Q \) ([1, 2, 22, 24]) as follows. For each point \( p \in M \), the Riesz representation theorem implies that there exists an isomorphism \( Q : T_p M \to T_p M \) such that
\[
\langle \alpha, Q \beta \rangle = \det (\alpha, \beta, \vec{n}(p)) \quad \text{for} \quad \alpha, \beta \in T_p M. \quad (2.22)
\]
Let \( e_1, e_2 \) be an orthonormal basis of \( T_p M \) with positive orientation, that is,
\[
\det (e_1, e_2, \vec{n}(p)) = 1.
\]
Then \( Q \) can be expressed explicitly by
\[
Q \alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in T_p M. \quad (2.23)
\]
Clearly, \( Q \) satisfies
\[
Q^T = -Q, \quad Q^2 = -\text{Id}.
\]
Operator \( Q \) plays an important role in our analysis.

For given \( U \in \mathcal{D}'_m(S, T) \), we define \( QU \in \mathcal{D}'_m(S, T^2) \) by
\[
(QU)(X \otimes Y) = U(X \otimes Q^T Y), \quad (UQ)(X \otimes Y) = U(QX \otimes Y) \quad \text{for} \quad X, Y \in \mathcal{D}_m(S, T).
\]
Moreover, for given \( W \in \mathcal{D}'_m(S, T) \), we define \( QW \in \mathcal{D}'_m(S, T) \) by
\[
(QW)(F) = -W(QF) \quad \text{for} \quad F \in \mathcal{D}_m(S, T). \quad (2.24)
\]
Proposition 2.2. The following formulas hold:

\[
\text{div}(wR) = R^T Dw + w\text{ div }R \quad \text{ for } \quad w \in \mathcal{D}'_m(S), \quad R \in \mathcal{E}_m(S, T^2), \quad (2.25)
\]

\[
\text{div}(RW) = \langle R, DW \rangle + \langle \text{ div }R, W \rangle \quad \text{ for } \quad W \in \mathcal{D}'_m(S, T), \quad R \in \mathcal{E}_m(S, T^2), \quad (2.26)
\]

\[
UQ + QU^T = (\text{ tr }U)Q \quad \text{ for } \quad U \in \mathcal{D}'_m(S, T^2). \quad (2.27)
\]

**Proof.** Let \( x \in S \) be given. Suppose that \( E_1, E_2 \) is a frame normal at \( x \). Then

\[
\langle R, DZ \rangle = \sum_{ij} R(E_i, E_j) \langle D_{E_j}Z, E_i \rangle
\]

\[
= \sum_{ij} \{ E_j [ R(E_i, E_j) \langle Z, E_i \rangle ] - DR(E_i, E_j) \langle Z, E_i \rangle \}
\]

\[
= \sum_{j} \{ E_j [ RZ(E_j) ] - i_z DR(E_j, E_j) \}
\]

\[
= \text{ div } (RZ) - \langle \text{ div }R, Z \rangle \quad \text{ at } \quad x.
\]

It follows from (2.5), (2.20), (2.16), and (2.18) that

\[
\text{div}(wR)(Z) = -(wR)(DZ) = -w(\langle R, DZ \rangle) = w\left( \text{ div } (RZ) \right) + w(\langle \text{ div }R, Z \rangle)
\]

\[
= Dw(RZ) + (w \text{ div }R)(Z) = (R^T Dw + w \text{ div }R)(Z)
\]

for \( Z \in \mathcal{E}_{m+1}(S, T) \).

For given \( z \in \mathcal{D}_{m+1}(S) \), we have at \( x \)

\[
R^T Dz = \sum_{ij} E_i(z) R(E_j, E_i) E_j = \sum_{ij} \{ E_i[zR(E_j, E_i)] - z DR(E_j, E_i, E_i) \} E_j
\]

\[
= \sum_{j} \{ \langle \text{ div } (zR), E_j \rangle - z \text{ tr } i_{E_j} DR \} E_j
\]

\[
= \text{ div } (zR) - z \text{ div }R. \quad (2.28)
\]

By (2.4), (2.20), (2.28), (2.2), and (2.17),

\[
\text{div}(RW)(z) = -(RW)(Dz) = -W(R^T Dz)
\]

\[
= Dw(zR) + W(z \text{ div }R)
\]

\[
= (\langle DW, R \rangle)(z) + (\langle W, \text{ div }R \rangle)(z) \quad \text{ for } \quad z \in D_{m+1}(S).
\]

Finally, we shall prove (2.27). Let \( E_1, E_2 \) be a local frame with positive orientation. Then

\[
QE_1 = -E_2, \quad QE_2 = E_1, \quad Q^T E_1 = E_2, \quad Q^T E_2 = -E_1.
\]

Then

\[
(UQ + QU^T)(E_1 \otimes E_2) = U(QE_1 \otimes E_2) + U(Q^T E_2 \otimes E_1) = (\text{ tr }U)(QE_1, E_2),
\]

\[
(UQ + QU^T)(E_2 \otimes E_1) = (\text{ tr }U)(QE_2, E_1), \quad (UQ + QU^T)(E_i \otimes E_i) = (\text{ tr }U)(QE_i, E_i)
\]

for \( 1 \leq i \leq 2 \). □
Let \( S \) be of \( C^{m+1} \). Let \( y \in D'_m(S, \mathbb{R}^3) \) be given. We define \( w \in D'_m(S), W \in D_m(S, T), v \in D_{m+1}(S), V \in D'_{m+1}(S, T), \) and \( U \in D'_{m+1}(S, T^2) \) by

\[
 w(z) = y(z\bar{n}) \quad \text{for} \quad z \in D_m(S), \quad W(Z) = y(Z) \quad \text{for} \quad Z \in D_m(S, T),
\]

\[
 v(z) = \frac{1}{2} y(QDz) \quad \text{for} \quad z \in D_{m+1}(S),
\]

\[
 V(Z) = y\left( (\text{div } QZ)\bar{n} + \nabla\bar{n}QZ \right) \quad \text{for} \quad Z \in D_{m+1}(S, T),
\]

and

\[
 U(P) = \nabla y(\text{sym } P) \quad \text{for} \quad P \in D_{m+1}(S, T^2),
\]

respectively. Then

\[
 \text{sym } DW + w\nabla\bar{n} = U \quad \text{in} \quad D'_{m+1}(S, T^2) \quad (2.33)
\]

in the sense of generalized tensors.

**Theorem 2.1.** For given \( y \in D'_m(S, \mathbb{R}^3) \), we have

\[
 v = \frac{1}{2} \text{div } QW \quad \text{for} \quad x \in S, \quad (2.34)
\]

\[
 \begin{cases} 
 Dw = \nabla\bar{n}W - QV & \text{for} \quad x \in S, \\
 \text{div } W = -(\text{tr } \Pi)w + \text{tr } U & \text{for} \quad x \in S,
\end{cases} \quad (2.35)
\]

and

\[
 \begin{cases} 
 Dv = \nabla\bar{n}V + Q \text{div } QUQ & \text{for} \quad x \in S, \\
 \text{div } V = -(\text{tr } \Pi)v - \langle Q\nabla\bar{n}, U \rangle & \text{for} \quad x \in S,
\end{cases} \quad (2.36)
\]

in the sense of generalized tensors.

**Proof.** By (2.24) and (2.4) we have

\[
 2v(z) = (W + w\bar{n})(QDz) = -(QW)(Dz) = (\text{div } QW)(z) \quad \text{for} \quad z \in D_{m+1}(S).
\]

For \( Z \in D_{m+1}(S, T) \) and by (2.1) and (2.20)

\[
 (Dw - \nabla\bar{n}W)(Z) = -w(\text{div } Z) - W(\nabla^T\bar{n}Z) = -y\left( (\text{div } Z)\bar{n} + \nabla\bar{n}Z \right)
\]

\[
 = -V(Q^TZ) = -(QV)(Z).
\]

By (2.3), (2.2), and (2.18)

\[
 \nabla y(P) = y(-\text{div } P + \langle P, \Pi \rangle \bar{n}) = -W(\text{div } P) + w(\langle P, \Pi \rangle) = (DW + w\Pi)(P)
\]

for any \( P \in D_{m+1}(S, T^2) \). Then

\[
 \nabla y = DW + w\Pi \quad \text{in} \quad D'_m(S, T^2). \quad (2.37)
\]
Since \( \text{div}(z \text{id}) = Dz, \)
from (2.18), (2.12), and (2.20), we obtain
\[
\text{div } W(z) = -W(Dz) = -W\left( \text{div}(z \text{id}) \right) = DW(z \text{id})
\]
\[
= \nabla y(z \text{id}) - (\text{tr } y)(z \text{id}) = \nabla y\left( \text{sym}(z \text{id}) \right) - w\left( (\text{tr } y)z \right)
\]
\[
= U(z \text{id}) - (\text{tr } y) w(z) = (\text{tr } U)(z) - (\text{tr } y) w(z) \quad \text{for } z \in D_{m+1}(S),
\]
that is, the second equation in (2.35).

Next, we shall prove (2.36). Let \( x \in S \) be given. Suppose that \( E_1, E_2 \) is a frame normal at \( x \) with positive orientation. Then \( E_1 = Q E_2 \). Let \( R \) be the curvature tensor. By the Ricci identity
\[
D^2 Z(X_1, X_2, X_3) = D^2 Z(X_1, X_2, X_3) + R(X_2, X_3, Z, X_1)
\]
for \( Z, X_1, X_2, X_3 \in D_{m+1}(S, T) \). Using the above formula, we have
\[
\langle \text{div } \text{sym } DZ, E_2 \rangle = E_1(\text{sym } DZ)(E_2, E_1) + E_2(\text{sym } DZ)(E_2, E_2)
\]
\[
= \frac{1}{2} E_1[DZ(E_2, E_1) + DZ(E_1, E_2)] + E_2[DZ(E_2, E_2)]
\]
\[
= \frac{1}{2}[D^2 Z(E_2, E_1, E_1) + D^2 Z(E_1, E_2, E_1)] + E_2(\text{div } Z) - D^2 Z(E_1, E_1, E_2)
\]
\[
= E_2(\text{div } Z) + \frac{1}{2}[D^2 Z(E_2, E_1, E_1) - D^2 Z(E_1, E_2, E_1)] + \kappa\langle Z, E_2 \rangle
\]
\[
= E_2(\text{div } Z) + \kappa\langle Z, E_2 \rangle + \frac{1}{2} E_1(\langle D Z, E_2 \rangle - \langle D E_2 Z, E_1 \rangle)
\]
\[
= E_2(\text{div } Z) + \kappa\langle Z, E_2 \rangle + \frac{1}{2} \langle D \text{div } QZ, E_1 \rangle
\]
\[
= \langle D \text{div } Z + \kappa Z - \frac{1}{2} QD \text{ div } QZ, E_2 \rangle \quad \text{for } Z \in D_{m+2}(S, T).
\]
A similar computation yields
\[
\langle \text{div } \text{sym } DZ, E_1 \rangle = \langle D \text{div } Z + \kappa Z - \frac{1}{2} QD \text{ div } QZ, E_1 \rangle \quad \text{at } x.
\]
Thus
\[
\text{div } \text{sym } DZ = D \text{div } Z + \kappa Z - \frac{1}{2} QD \text{ div } QZ \quad \text{for } Z \in D_{m+2}(S, T).
\]

By (2.5) and (2.38)
\[
\langle \text{div } \text{sym } DW \rangle (Z) = -DW(\text{sym } DZ) = W(\text{div } \text{sym } DZ)
\]
\[
= W(D \text{div } Z + \kappa Z - \frac{1}{2} QD \text{ div } QZ)
\]
\[
= (D \text{div } W + \kappa W)(Z) - \frac{1}{2} y(QD \text{ div } QZ)
\]
\[
= (D \text{div } W + \kappa W)(Z) - v(\text{div } QZ)
\]
\[
= (D \text{div } W + \kappa W - QDv)(Z) \quad \text{for } Z \in D_{m+2}(S, T).
\]
Using the relations

\[ \nabla \bar{n} Q \nabla \bar{n} = \kappa Q, \quad \nabla \bar{n} - (\text{tr} \Pi) \text{id} = Q \nabla \bar{n} Q, \quad \text{div} \nabla \bar{n} = D(\text{tr} \Pi), \]

we have, by (2.35), (2.25), and (2.33),

\[ \nabla \bar{n} V(Z) = V(\nabla \bar{n} Z) = QV(\nabla \bar{n} Z) = (\nabla \bar{n} W - Dw)(Q \nabla \bar{n} Z) \]
\[ = W(\nabla \bar{n} Q \nabla \bar{n} Z) - Dw \left( (\text{tr} \Pi) Q Z - \nabla \bar{n} Q \right) \]
\[ = [\kappa W + \nabla \bar{n} Dw - (\text{tr} \Pi) Dw](Q Z) \]
\[ = [\kappa W + \text{div}(w \nabla \bar{n}) - w \text{div} \nabla \bar{n} - (\text{tr} \Pi) Dw](Q Z) \]
\[ = [\text{div} U - D \text{div} W + QDv - D((\text{tr} \Pi) w)](Q Z) \]
\[ = [Dv - Q \text{div} U + QD(\text{tr} U)](Z) \]
\[ = [Dv - Q \text{div} \left( U - (\text{tr} U) \text{id} \right)](Z) \]
\[ = [Dv - Q \text{div} QUQ](Z) \quad \text{for} \quad Z \in \mathcal{D}_m(S, T), \]

where the formula \( U - (\text{tr} U) \text{id} = QUQ \) has been used, that is the first equation in (2.36). Noting that \[ \langle \Pi, Q \nabla \bar{n} \rangle = \langle Q, \Pi \rangle = 0, \quad \text{div} QDz = 0, \quad \text{div} (zQ \nabla \bar{n}) = -\nabla \bar{n} QDz, \]

\[ \text{div} \left( z(\text{tr} \Pi) Q \right) = -QD(z \text{tr} \Pi) \quad \text{for} \quad z \in \mathcal{D}_{m+1}(S), \]

we compute, by (2.37), (2.3), (2.17), and (2.26),

\[ \text{div} V(z) = -V(Dz) = -y \left( (\text{div} QDz) \bar{n} + \nabla \bar{n} QDz \right) = -W(\nabla \bar{n} QDz) \]
\[ = W \left( \text{div} (zQ \nabla \bar{n}) \right) = -DW(zQ \nabla \bar{n}) = -\nabla y(zQ \nabla \bar{n}) + w \Pi(zQ \nabla \bar{n}) \]
\[ = -\nabla y(\text{sym} Q \nabla \bar{n}) + \frac{1}{2} \nabla y(zQ \nabla \bar{n}^T - zQ \nabla \bar{n}) + w(z(\Pi, Q \nabla \bar{n})) \]
\[ = -U(zQ \nabla \bar{n}) + \frac{1}{2} \nabla y(zQ \nabla \bar{n}^T - zQ \nabla \bar{n}) \]
\[ = -U(zQ \nabla \bar{n}) - \frac{1}{2} \nabla y \left( z(\text{tr} \Pi) Q \right) \]
\[ = -(zQ \nabla \bar{n})(z) - \frac{1}{2} y \left( -\text{div} (z(\text{tr} \Pi) Q) + z(\text{tr} \Pi)(Q, \Pi) \bar{n} \right) \]
\[ = -(zQ \nabla \bar{n})(z) - v(z \text{tr} \Pi) = -(zQ \nabla \bar{n})(z) - ((\text{tr} \Pi) v)(z) \]

for \( z \in \mathcal{D}_{m+1}(Z) \), where the formula \( Q \nabla \bar{n} = (Q \nabla \bar{n})^T + (\text{tr} \Pi) Q \) has been used. Thus the second equation in (2.36) is true. \( \square \)
3 Proof of Theorem 1.1

Let $\Omega \subset M$ be a bounded Lipschitz region with boundary $\Gamma$. Consider operator $\mathcal{B} : L^2(\Omega,T) \to L^2(\Omega,T)$ given by

$$\mathcal{B} W = \text{div} DW, \quad D(\mathcal{B}) = W^{2,2}(\Omega,T) \cap W^{1,2}_0(\Omega,T).$$

Let $W \in D(\mathcal{B})$ and $x \in \Omega$ be fixed. Let $x \in \Omega$ be fixed. Let $E_1, E_2$ be a frame field normal at $x$. By [21, Theorem 1.26]

$$\langle B W, E_i \rangle = \sum_{j=1}^2 D(DW)(E_i, E_j, E_j) = \langle \sum_j D^2_{E_j} W, E_i \rangle = -\langle \Delta W, E_i \rangle + \text{Ric}(W, E_i)$$

at $x$ for $i = 1, 2$, where $\Delta$ is the Hodge-Laplacian and $\text{Ric}$ is the Ricci tensor. Thus

$$\mathcal{B} W = -\Delta W + i_W \text{Ric}.$$ 

By [?] $\mathcal{B}$ is a self-adjoint operator with the compact resolvent. We then have a direct sum

$$L^2(\Omega,T) = \mathcal{R}(\mathcal{B}) \oplus \mathcal{N}(\mathcal{B}),$$

where $\mathcal{R}(\mathcal{B})$ and $\mathcal{N}(\mathcal{B})$ are the valued field and the null space of $\mathcal{B}$, respectively. Moreover, $\dim \mathcal{N} < \infty$, and for any $Y \in \mathcal{R}(\mathcal{B})$, there exists a unique $Z \in D(\mathcal{B}) \cap \mathcal{R}(\mathcal{B})$ that solves problem

$$\text{div} DZ = Y, \quad Z|_\Gamma = 0$$

and satisfies

$$\|Z\|_{W^{2,2}(\Omega,T)} \leq C\|Y\|_{L^2(\Omega,T)}.$$

**Lemma 3.1.** There exists constant $C > 0$ such that

$$\|W\|_{L^2(\Omega,T)}^2 \leq C\left(\|DW\|_{W^{-1,2}(\Omega,T)^2}^2 + \|W\|_{L^2(\Gamma,T)}^2\right) \quad \text{for all} \quad W \in L^2(\Omega,T). \quad (3.1)$$

**Proof.** **Step 1.** First, we claim that there is a constant $C > 0$ such that

$$\|W\|_{L^2(\Omega,T)}^2 \leq C\left(\|DW\|_{W^{-1,2}(\Omega,T)^2}^2 + \|W\|_{L^2(\Gamma,T)}^2\right) \quad \text{for all} \quad W \in \mathcal{R}(\mathcal{B}). \quad (3.2)$$

Let $W \in \mathcal{R}(\mathcal{B})$ be given. For any $Y \in W^{1,2}(\Omega,T)$, consider the direct sum

$$Y = Y_0 + Y_1, \quad Y_0 \in \mathcal{N}(\mathcal{B}), \quad Y_1 \in D(\mathcal{B}).$$

Then there is $Z \in D(\mathcal{B}) \cap \mathcal{R}(\mathcal{B})$ such that $Y_1 = \text{div} DZ$. We have

$$W(Y) = W(Y_1) = \int_\Omega \langle W, \text{div} DZ \rangle dg = -\int_\Omega \langle DW, DZ \rangle dg + \int_\Gamma DZ(W, \nu)d\Gamma$$

$$\leq \|DW\|_{W^{-1,2}(\Omega,T)}\|DZ\|_{W^{1,2}(\Omega,T)} + \|DZ\|_{L^2(\Gamma,T)}\|W\|_{L^2(\Gamma,T)}$$

$$\leq C\left(\|DW\|_{W^{-1,2}(\Omega,T)}^2 + \|W\|_{L^2(\Gamma,T)}^2\right)\|Z\|_{W^{2,2}(\Omega,T)}$$

$$\leq C\left(\|DW\|_{W^{-1,2}(\Omega,T)}^2 + \|W\|_{L^2(\Gamma,T)}^2\right)\|Y_1\|_{L^2(\Omega,T)}$$

$$\leq C\left(\|DW\|_{W^{-1,2}(\Omega,T)}^2 + \|W\|_{L^2(\Gamma,T)}^2\right)\|Y\|_{L^2(\Omega,T)}$$

for all $Y \in L^2(\Omega,T)$. 

13
that is, (3.2).

**Step 2.** Let \( \Phi_1, \cdots, \Phi_k \) be an orthonormal basis of \( \mathcal{N}(\mathcal{B}) \), where \( k = \dim \mathcal{N}(\mathcal{B}) \).

Now we claim there are constants \( \sigma > 0 \) and \( C > 0 \) such that

\[
\sigma \left[ \left( \sum_{i=1}^{k} \alpha_i^2 \right)^{1/2} + \| DW \|_{W^{-1,2}(\Omega,T^2)} \right] \\
\leq \left\| \sum_{i=1}^{k} \alpha_i D\Phi_i + DW \right\|_{W^{-1,2}(\Omega,T^2)} + \| W \|_{L^2(\Gamma,T)} \\
\leq C \left[ \left( \sum_{i=1}^{k} \alpha_i^2 \right)^{1/2} + \| DW \|_{W^{-1,2}(\Omega,T^2)} + \| W \|_{L^2(\Gamma,T)} \right] \tag{3.3}
\]

for all \((\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k\) and \( W \in \mathcal{R}(\mathcal{B}) \). Clearly, the right hand side of (3.3) holds true. We prove the left hand side by contradiction. Suppose the left hand side of (3.3) is not true. Then there exist \( \{(\alpha_{j1}, \cdots, \alpha_{jk}) \} \subset \mathbb{R}^k \) and \( W_j \in \mathcal{R}(\mathcal{B}) \) such that

\[
\sigma \left[ \left( \sum_{i=1}^{k} \alpha_i^2 \right)^{1/2} + \| DW_j \|_{W^{-1,2}(\Omega,T^2)} \right] = 1,
\]

\[
\left\| \sum_{i=1}^{k} \alpha_{ji} D\Phi_i + DW_j \right\|_{W^{-1,2}(\Omega,T^2)} + \| W_j \|_{L^2(\Gamma,T)} \leq \frac{1}{j} \quad \text{for all} \quad j \geq 1.
\]

We assume that \((\alpha_{j1}, \cdots, \alpha_{jk}) \to (\alpha_1, \cdots, \alpha_k)\). Then

\[
W_j \to 0 \quad \text{in} \quad L^2(\Gamma,T), \quad DW_j \to U \quad \text{in} \quad W^{-1,2}(\Omega,T^2).
\]

By Step 1

\[
W_j \to W_0 \quad \text{in} \quad L^2(\Omega,T)
\]

with \( W_0|\Gamma = 0 \). Thus

\[
\left( \sum_{i=1}^{k} \alpha_i^2 \right)^{1/2} + \| DW_0 \|_{W^{-1,2}(\Omega,T^2)} = 1, \quad D \left( \sum_i \alpha_i \Phi_i + W_0 \right) = 0, \quad \left( \sum_i \alpha_i \Phi_i + W_0 \right)|\Gamma = 0.
\]

Then \( \sum_i \alpha_i \Phi_i + W_0 = 0 \) and, so

\[
(\alpha_1, \cdots, \alpha_k) = 0, \quad W_0 = 0.
\]

That is a contradiction.

**Step 3.** Let \( W \in L^2(\Omega,T) \) be given. Let

\[
W = \sum_i \alpha_i \Phi_i + W_0, \quad \alpha_i = (W, \Phi_i)_{L^2(\Omega,T)}, \quad W_0 \in \mathcal{R}(\mathcal{B}).
\]
For given $Y \in W^{1,2}(\Omega, T)$, we have by Steps 1 and 2
\[
W(Y) = \sum_{i} \alpha_i \beta_i + W_0(Y) \leq |\alpha| |\beta| + C(\|D W_0\|_{W^{-1,2}(\Omega,T^2)} + \|W_0\|_{L^2(\Gamma,T)}) \|Y\|_{L^2(\Omega,T)}
\]
\[
\leq C(|\alpha| + \|D W_0\|_{W^{-1,2}(\Omega,T^2)} + \|W_0\|_{L^2(\Gamma,T)}) \|Y\|_{L^2(\Omega,T)}
\]
\[
\leq C(\|D W\|_{W^{-1,2}(\Omega,T^2)} + \|W\|_{L^2(\Gamma,T)}) \|Y\|_{L^2(\Omega,T)},
\]
(3.4)
where
\[
\alpha = (\alpha_1, \cdots, \alpha_k), \quad \beta = (\beta_1, \cdots, \beta_k), \quad \beta_i = (Y, \Phi_i)_{L^2(\Omega,T)}.
\]
Finally, (3.1) follows from (3.4).

**Lemma 3.2.** The following estimate holds.
\[
\|v\|_{W^{1,2}(\Omega)'} \leq C(\|D v\|_{W^{2,2}(\Omega)'} + \|v\|_{W^{3/2,2}(\Gamma)'}) \quad \text{for} \quad v \in [W^{1,2}(\Omega)]'.
\]
(3.5)

**Proof.** For given $f \in W^{1,2}(\Omega)$, we solve problem
\[
-\Delta z = f \quad \text{for} \quad x \in \Omega, \quad z|_{\Gamma} = 0.
\]
Then
\[
\|z\|_{W^{3,2}(\Omega)} \leq C\|f\|_{W^{1,2}(\Omega)}.
\]
Thus we have
\[
(v, f)_{L^2(\Omega)} = -\int_{\Gamma} v(D z, \nu) d\Gamma + (D v, D z)_{L^2(\Omega,T)}
\]
\[
\leq \|v\|_{W^{3/2,2}(\Gamma)'} \|D z\|_{W^{3/2,2}(\Omega,T)} + \|D v\|_{W^{2,2}(\Omega)'} \|D z\|_{W^{2,2}(\Gamma,T)}
\]
\[
\leq C(\|D v\|_{[W^{2,2}(\Omega)]'} + \|v\|_{W^{3/2,2}(\Gamma)'}) \|f\|_{W^{1,2}(\Omega)}.
\]
Thus (3.5) follows.

**Lemma 3.3.** Let $\Omega \subset M$ be a region and boundary $\Gamma$ consist of finite many closed curves. Then
\[
\|\text{div div } P\|_{[W^{2,2}(\Omega)]'} \leq C\|P\|_{L^2(\Omega,T^2)} \quad \text{for} \quad P \in L^2(\Omega,T^2).
\]
(3.6)

**Proof.** Assume $P \in W^{2,2}(\Omega,T^2)$. From [1, Lemma 2.3], we have
\[
\text{div} (P Z) = \langle P, D Z \rangle + \langle \text{div } P, Z \rangle \quad \text{for} \quad x \in \Omega, \quad Z \in W^{1,2}(\Omega, T).
\]
(3.7)
For given $z \in W^{2,2}(\Omega)$, it follows from (3.7) that
\[
(\text{div div } P, z)_{L^2(\Omega)} = \int_{\Omega} [\text{div}(z \text{ div } P) - \langle \text{div } P, D z \rangle] dg
\]
\[
= \int_{\Gamma} (z \langle \text{div } P, \nu \rangle - \langle P D z, \nu \rangle) d\Gamma + \int_{\Omega} \langle P, D^2 z \rangle dg
\]
\[
\leq C(\|\text{div } P\|_{[W^{3/2,2}(\Gamma)']} \|z\|_{W^{3/2,2}(\Gamma)} + \|P\|_{[W^{1/2,2}(\Gamma,T)']} \|D z\|_{W^{1/2,2}(\Gamma,T)} + \|P\|_{L^2(\Omega,T)} \|D^2 z\|_{L^2(\Omega,T^2)}
\]
\[
\leq C\|P\|_{L^2(\Omega,T^2)} \|z\|_{W^{2,2}(\Omega)}.
\]
(3.8)
In addition, the assumption that boundary $\Gamma$ consists of finite many closed curves implies

$$[W^{\lambda,2}(\Gamma, T^k)]' = W^{-\lambda,2}(\Gamma, T^k), \quad (3.9)$$

where $\lambda > 0$ is a real number and $k \geq 0$ is an integer.

Since $W^{2,2}(\Omega, T^2)$ is dense in $[W^{2,2}(\Omega, T^2)]'$, (3.6) follows from (3.8) and (3.9).

Suppose that for given $\varepsilon > 0$ small, $\eta_0 \in C_0^\infty(-\infty, \infty)$ is given satisfying

$$0 \leq \eta_0 \leq 1; \quad \eta_0 = 0 \text{ for } s \leq \varepsilon/2; \quad \eta_0 = 1 \text{ for } s \geq \varepsilon.$$

Set

$$\eta(\alpha(t, s)) = s\eta_0(s) \text{ for } (t, s) \in (0, a) \times (-b_0, b_1).$$

For given $\varepsilon > 0$ small, let

$$\Omega_{-\varepsilon} = \{ \alpha(t, s) \mid (t, s) \in (0, a] \times (-\varepsilon, b_1) \}.$$

Suppose $X \in C^m(\Omega_{-\varepsilon}, T)$ is given in [2, Section 2, (2.34)]. We define

$$L_0V = e^{-\gamma \kappa}[(\text{div } Q\nabla \bar{n}V - \eta(V, QX))QX + (\text{div } V - \eta(V, \nabla \bar{n}X))\nabla \bar{n}X], \quad (3.10)$$

$$L_0^*V = \nabla \bar{n}QD(e^{-\gamma \kappa}(QX, V)) - D(e^{-\gamma \kappa}(QX, V)) - \eta e^{-\gamma \kappa}(\eta(V, QX)QX + (V, \nabla \bar{n}X)\nabla \bar{n}X), \quad (3.11)$$

for $V \in L^2(\Omega_{-\varepsilon}, T)$. Let $m \geq 0$ be an integer. By [2, Lemma 2.3] we have

$$(W, L_0V)_{L^2(\Omega, T)} = (V, L_0^*W)_{L^2(\Omega, T)} + \int_{\partial\Omega} (\mathcal{L}_1(V, \nu)(W, \nu) - \mathcal{L}_2(V, QX)(W, QX))d\Gamma \quad (3.12)$$

for $V, W \in T\Omega$, where

$$\mathcal{L}_1 = \frac{e^{-\gamma \kappa}}{(X, \nu)}\Pi(X, X), \quad \mathcal{L}_2 = \frac{e^{-\gamma \kappa}}{(X, \nu)}\Pi(Q\nu, Q\nu) \text{ for } x \in \partial\Omega. \quad (3.13)$$

Consider problems

$$\begin{cases} L_0V = F, \\ (V, QX)|_{\Gamma_{-\varepsilon}} = p, \quad (V, \nu)|_{\Gamma_{b_1}} = q \end{cases} \quad (3.14)$$

and

$$\begin{cases} L_0^*V = F, \\ (V, \nu)|_{\Gamma_{-\varepsilon}} = p, \quad (V, QX)|_{\Gamma_{b_1}} = q, \end{cases} \quad (3.15)$$

respectively, where $F \in W^{m,2}(\Omega_{-\varepsilon}, T)$, $p \in W^{m,2}(\Gamma_{-\varepsilon})$, and $q \in W^{m,2}(\Gamma_{b_1})$ are given.

It follows from [2, Theorems 3.1 and 3.2] that
Theorem 3.1. Let $S$ be of $C^{m+3}$. There exists a unique solution $W \in W^{m,2}(\Omega_{-\varepsilon}, T)$ to problem (3.14), or (3.15) satisfying

$$
\|V\|_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|V\|_{W^{m,2}(\partial\Omega_{-\varepsilon}, T)} \\
\leq C(\|F\|_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|q\|_{W^{m,2}(\Gamma_b)} + \|p\|_{W^{m,2}(\Gamma_{-\varepsilon})}).
$$

(3.16)

We define linear operators $L_0 : W^{m,2}(\Omega_{-\varepsilon}, T) \to W^{m,2}(\Omega_{-\varepsilon}, T)$ by

$$
L_0V = e^{-\gamma \kappa}[(\text{div } Q\nabla n)V]QX + (\text{div } V)\nabla nX - CV,
$$

(3.17)

where

$$
CV = \eta e^{-\gamma \kappa}(\langle V, QX \rangle QX + \langle V, \nabla nX \rangle \nabla nX), \quad \eta(x) = s\eta_0(s)
$$

(3.18)

for $x = \alpha(t, s) \in \Omega_{-\varepsilon}$. Moreover, set

$$
L_0^*V = \nabla nQD(e^{-\gamma \kappa}(QX, V)) - D(e^{-\gamma \kappa}(\nabla nX, V)) - CV.
$$

(3.19)

$$
D(L_0) = \{ V \in W^{m,2}(S, T) | \text{div } V \in W^{m,2}(\Omega_{-\varepsilon}), \text{div } Q\nabla nV \in W^{m,2}(\Omega_{-\varepsilon}) \}
$$

$$
\langle V, QX\rangle|_{\Gamma_{-\varepsilon}} = \langle V, \nu\rangle|_{\Gamma_{b_1}} = 0 \},
$$

By Theorem 3.1 and [2, Theorems 3.3 and 3.4] we have the following.

Theorem 3.2. (i) Operators $L_0$ and $L_0^*$ have bounded inverses on $W^{m,2}(\Omega_{-\varepsilon}, T)$ with the estimates

$$
\|L_0^{-1}V\|_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|L_0^{-1}V\|_{W^{m,2}(\partial\Omega_{-\varepsilon}, T)} \quad \text{or}
$$

$$
\|L_0^{*-1}V\|_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|L_0^{*-1}V\|_{W^{m,2}(\partial\Omega_{-\varepsilon}, T)} \leq C\|V\|_{W^{m,2}(\Omega_{-\varepsilon}, T)}
$$

(3.20)

for $V \in W^{m,2}(\Omega_{-\varepsilon}, T)$.

(ii) Operator $L_0^{-1}C$ and $L_0^{*-1}C : W^{m,2}(\Omega_{-\varepsilon}, T) \to W^{m+1,2}(\Omega_{-\varepsilon}, T)$ are compact.

Define operators $L$ and $L^* : D_m(S, T) \to D_{m+1}(S, T)$ by

$$
L^*V = L_0V + CV \quad \text{and} \quad L^*V = L_0^*V + CV,
$$

respectively. Given $Z_0 \in \mathcal{E}_m(\Omega_{-\varepsilon}, T)$, consider problem

$$
\begin{align*}
&L^*Z + \nabla nDz = Z_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
&-\Delta z = e^{-\gamma \kappa}\Pi(X, Z) \text{tr} \Pi \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
&\partial_n z|_{\partial\Omega_{-\varepsilon}} = \langle Z, \nu\rangle|_{\Gamma_{-\varepsilon}} = \langle Z, QX\rangle|_{\Gamma_{b_1}} = 0.
\end{align*}
$$

(3.21)
We define operator $A : L^2(\Omega, T) \to W^{1,2}(\Omega, T)$ by

$$AZ = \nabla uDz \text{ for } Z \in L^2(\Omega, T),$$

where $z \in W^{1,2}(\Omega, T)$ is given by

$$\begin{cases} -\Delta z = e^{-\gamma z} (Z, \nabla uX) \text{ tr } I & \text{for } x \in \Omega, \\ \partial_n z = 0 & \text{on } x \in \Gamma. \end{cases}$$

Then

$$A : L^2(\Omega, T) \to L^2(\Omega_{-\varepsilon}, T)$$

is compact. Thus by Theorem 3.2

$$\mathcal{L}_0^{-1}(C + A) : L^2(\Omega, T) \to L^2(\Omega_{-\varepsilon}, T)$$

is also compact. Set

$$\mathcal{N} = \{ W \in L^2(\Omega_{-\varepsilon}, T) | W + \mathcal{L}_0^{-1}(C + A)W = 0 \},$$

$$\mathcal{N}_* = \{ W \in L^2(\Omega_{-\varepsilon}, T) | W + \mathcal{L}_0^{-1}(C + A^*)W = 0 \}.$$

Then

$$\dim \mathcal{N} < \infty, \quad \dim \mathcal{N}_* < \infty.$$

**Proposition 3.1.** Let $S$ be of $C^{m+3}$. Problem (3.21) admits a unique solution $Z \perp \mathcal{N}$ in $L^2(\Omega_{-\varepsilon}, T)$ if and only if $Z_0 \perp \mathcal{N}_*$ in $L^2(\Omega_{-\varepsilon}, T)$. Moreover, there exists $C > 0$ such that if $Z_0 \in W^{2,2}(\Omega_{-\varepsilon}, T)$ and $(Z, z)$ solves problem (3.21) with $Z \perp \mathcal{N}$, then $(Z, z) \in W^{m,2}(\Omega_{-\varepsilon}, T) \times W^{m+2,2}(\Omega_{-\varepsilon})$ satisfying

$$\|Z\|^2_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|z\|^2_{W^{m+2,2}(\Omega_{-\varepsilon})} + \|Z\|^2_{W^{m,2}(\partial \Omega_{-\varepsilon}, T)} \leq C\|Z_0\|^2_{W^{m,2}(\Omega_{-\varepsilon}, T)}. \quad (3.22)$$

**Proof.** For given $Z_0 \in L^2(\Omega_{-\varepsilon}, T)$, it is easy to check that $Z \in L^2(\Omega_{-\varepsilon}, T)$ solves problem (3.21) if and only if $Z$ satisfies

$$Z + \mathcal{L}_0^{-1}(C + A)Z = \mathcal{L}_0^{-1}Z_0 \text{ in } L^2(\Omega_{-\varepsilon}, T).$$

By Fredholm's theorem the above problem has a solution $Z \in L^2(\Omega_{-\varepsilon}, T)$ if and only if $Z_0 \perp \mathcal{N}_*$ in $L^2(\Omega_{-\varepsilon}, T)$.

Let $Z_0 \in W^{m,2}(\Omega_{-\varepsilon}, T)$. By [2, Theorems 3.1 and 3.2] $Z \in W^{m,2}(\Omega_{-\varepsilon}, T)$. Thus by the regularity of the elliptic problem $z \in W^{m+2,2}(\Omega_{-\varepsilon})$ and the estimates

$$\|z\|^2_{W^{m+2,2}(\Omega_{-\varepsilon})} \leq C\|Z\|^2_{W^{m,2}(\Omega_{-\varepsilon}, T)}$$

follows. Moreover, using [2, Lemma 3.2 and Theorem 2.2], we have

$$\|Z\|^2_{W^{m,2}(\Omega_{-\varepsilon}, T)} + \|Z\|^2_{W^{m,2}(\partial \Omega_{-\varepsilon}, T)} \leq C\|Z_0\|^2_{W^{m,2}(\Omega_{-\varepsilon}, T)}.$$

Thus (3.22) follows. \qed
Let $\Phi_1, \cdots, \Phi_k$ be an orthonormal basis of $N$, where $k = \dim N$. Define the projection operator $\mathcal{P} : [W^{m,2}(\Omega_{-\varepsilon}, T)]' \to N$ by
$$\mathcal{P}V = \sum_{j=1}^{k} (V, \Phi_j)_{L^2(\Omega_{-\varepsilon})} \Phi_k \quad \text{for} \quad V \in [W^{m,2}(\Omega_{-\varepsilon}, T)]'.$$

Consider problem
\begin{equation}
\begin{aligned}
& Dv = \nabla \bar{n}v + F \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
& \text{div} V = - (\text{tr} \Pi)v + f \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
& \langle V, QX \rangle |_{\Gamma_{-\varepsilon}} = p, \quad \langle V, \nu \rangle |_{\Gamma_{b_1}} = q.
\end{aligned}
\tag{3.23}
\end{equation}

We have the following.

**Theorem 3.3.** Let $S$ be of $C^5$. Suppose that $(v, V)$ solves problem (3.23). Then there exists a $C > 0$ such that
\begin{equation}
\|V\|_{W^{2,2}(\Omega_{-\varepsilon}, T)}' + \|v\|_{W^{1,2}(\Omega_{-\varepsilon})}' \leq C \left( \|\mathcal{P}V\|_{W^{2,2}(\Omega_{-\varepsilon}, T)}' + I(F, f, p, q) \right),
\end{equation}

where
$$I(F, f, p, q) = \|\text{div} QF\|_{W^{2,2}(\Omega_{-\varepsilon})}' + \|F\|_{W^{2,2}(\Omega_{-\varepsilon}, T)}' + \|f\|_{W^{2,2}(\Omega_{-\varepsilon})}'.
$$

**Proof.** From (3.23) we have
\begin{equation}
\begin{aligned}
& \text{div} Q\nabla \bar{n}V = - \text{div} QF \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
& \text{div} V = - \rho v + f \quad \text{for} \quad x \in \Omega_{-\varepsilon},
\end{aligned}
\tag{3.25}
\end{equation}

where $\rho = \text{tr} \Pi$. It follows from (3.17), (3.18), and (3.25) that
\begin{equation}
\mathcal{L}_0V + CV = G,
\end{equation}

where
$$G = -e^{-\gamma \kappa} [\langle \text{div} QF\rangle QX + (\rho v - f)\nabla \bar{n}X].$$

Let
$$\Xi = \{ Z \in W^{2,2}(\Omega_{-\varepsilon}, T) \mid Z \perp N_0 \text{ in } L^2(\Omega_{-\varepsilon}, T) \}.$$

For given $Z_0 \in \Xi$, we solve problem (3.21) to have the solution $(Z, z) \in W^{2,2}(\Omega_{-\varepsilon}, T) \times W^{4,2}(\Omega_{-\varepsilon})$. Let
$$z_1 = e^{-\gamma \kappa} (Z, QX), \quad z_2 = \rho e^{-\gamma \kappa} (Z, \nabla \bar{n}X), \quad z_3 = e^{-\gamma \kappa} (Z, \nabla \bar{n}X).$$
By (3.21), (3.12), and the first equation in (3.23), we have

\[(V, Z_0)_{L^2(\Omega_{\epsilon}, T)} = (V, \mathcal{L}^*_1 Z + CZ + \nabla \tilde{n} Dz)_{L^2(\Omega_{\epsilon}, T)} = (G, Z)_{L^2(\Omega_{\epsilon}, T)}\]

\[+(Dv - F, Dz)_{L^2(\Omega_{\epsilon}, T)} + \int_{\Gamma_{\epsilon}} L_2 p(Z, QX) d\Gamma\]

\[- \int_{\Gamma_{b_1}} L_1 q(Z, \nu) d\Gamma\]

\[= -(\nabla Q F, z_1)_{L^2(\Omega_{\epsilon})} + (f, z_3)_{L^2(\Omega_{\epsilon})}\]

\[-(F, Dz)_{L^2(\Omega_{\epsilon}, T)} - (v, z_2)_{L^2(\Omega_{\epsilon})} + (Dv, Dz)_{L^2(\Omega_{\epsilon})}\]

\[+ \int_{\Gamma_{\epsilon}} L_2 p(Z, QX) d\Gamma - \int_{\Gamma_{b_1}} L_1 q(Z, \nu) d\Gamma.\]

Noting that

\[-(v, z_2)_{L^2(\Omega_{\epsilon})} + (Dv, Dz)_{L^2(\Omega_{\epsilon})} = 0,\]

by (3.22) with \(m = 2\) and Lemma 3.3 we obtain

\[
\|(V, Z_0)_{L^2(\Omega_{\epsilon}, T)}\| \leq \|\nabla Q F\|_{[W^{2,2}(\Omega_{\epsilon})]} \|z_1\|_{W^{2,2}(\Omega_{\epsilon})}
\]

\[+ \|f\|_{[W^{2,2}(\Omega_{\epsilon})]} \|z_2\|_{W^{2,2}(\Omega_{\epsilon})} + |F|_{[W^{2,2}(\Omega_{\epsilon}, T)]} \|Dz\|_{W^{2,2}(\Omega_{\epsilon}, T)}\]

\[+ C(|p|_{W^{-2,2}(\Omega_{\epsilon})} + \|q\|_{W^{-2,2}(\Omega_{\epsilon})}) \|Z\|_{W^{2,2}(\partial \Omega_{\epsilon}, T)}\]

\[\leq C I(F, f, p, q)\|Z_0\|_{W^{2,2}(\Omega_{\epsilon}, T)} \quad \text{for all} \quad Z_0 \in \Xi. \quad (3.28)\]

Since \(\Xi\) is a closed subspace of \(W^{2,2}(\Omega_{\epsilon}, T)\), (3.28) shows that there exists a function in \(\Xi\), denoted by \(B V\), such that

\[(V, Z_0)_{L^2(\Omega_{\epsilon}, T)} = (B V, Z_0)_{W^{2,2}(\Omega_{\epsilon}, T)} \quad \text{for all} \quad Z_0 \in \Xi.\]

Let \(I\) be the canonical map from \([W^{2,2}(\Omega_{\epsilon}, T)]' \to W^{2,2}(\Omega_{\epsilon}, T)\). Let \(V_0 = I^{-1} B V\).

Then \(V_0 \in [W^{2,2}(\Omega_{\epsilon}, T)]'\) satisfies

\[(V - V_0, Z_0)_{L^2(\Omega_{\epsilon}, T)} = 0 \quad \text{for all} \quad Z_0 \in \Xi.\]

Thus

\[V - V_0 \in \mathcal{N}_s. \quad (3.29)\]

Therefore,

\[V = \mathcal{P} V + V_0 \in [W^{2,2}(\Omega_{\epsilon}, T)]',\]

where \(\mathcal{P} V = V - V_0\). By (3.28) we obtain

\[
\|V\|_{[W^{2,2}(\Omega_{\epsilon}, T)]'} \leq \|\mathcal{P} V\|_{[W^{2,2}(\Omega_{\epsilon}, T)]'} + \|V_0\|_{[W^{2,2}(\Omega_{\epsilon}, T)]'}
\]

\[= \|\mathcal{P} V\|_{[W^{2,2}(\Omega_{\epsilon}, T)]'} + \sup_{Z_0 \in \Xi} \|V_0\|_{W^{2,2}(\Omega_{\epsilon}, T)} = \|V, Z_0\|_{L^2(\Omega_{\epsilon}, T)} \quad (3.30)\]

\[\leq \|\mathcal{P} V\|_{[W^{2,2}(\Omega_{\epsilon}, T)]'} + C I(F, f, p, q).\]
For given \( z_0 \in W^{1,2}(\Omega_{-\varepsilon}) \), we solve problem

\[
\Delta w = z_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \quad \langle Dw, \nu \rangle|_{\partial\Omega_{-\varepsilon}} = 0.
\]

Then

\[
\|w\|_{W^{3,2}(\Omega_{-\varepsilon})} \leq C\|z_0\|_{W^{1,2}(\Omega_{-\varepsilon})} \quad \text{for all} \quad z_0 \in W^{1,2}(\Omega_{-\varepsilon}).
\]

By the first equation in (3.23) we have

\[
\langle v, z_0 \rangle_{L^2(\Omega_{-\varepsilon})} = -(Dv, Dw)_{L^2(\Omega_{-\varepsilon},T)} = -(\nabla \overrightarrow{n} V + F, Dw)_{L^2(\Omega_{-\varepsilon},T)}
\]

\[
\leq C(\|V\|_{W^{2,2}(\Omega_{-\varepsilon},T)}' + \|F\|_{W^{2,2}(\Omega_{-\varepsilon},T)}')\|z_0\|_{W^{1,2}(\Omega_{-\varepsilon})}
\]

for all \( z_0 \in W^{1,2}(\Omega_{-\varepsilon}) \), that yields

\[
\|v\|_{W^{1,2}(\Omega_{-\varepsilon})}' \leq C(\|V\|_{W^{2,2}(\Omega_{-\varepsilon},T)}' + \|F\|_{W^{2,2}(\Omega_{-\varepsilon},T)}'). \tag{3.31}
\]

Thus (3.24) follows from (3.30) and (3.31).

**Theorem 3.4.** Let

\[
U = \text{sym} DW + w\Pi \quad \text{for} \quad y = W + w\overrightarrow{n}.
\]

Then there exists \( C > 0 \), independent of \( y = W + w\overrightarrow{n} \), such that

\[
\|W\|_{L^2(\Omega_{-\varepsilon},T)} + \|w\|_{W^{1,2}(\Omega_{-\varepsilon})}' \leq C(\|U\|_{L^2(\Omega_{-\varepsilon},T)}^2 + \|W\|_{L^2(\partial\Omega_{-\varepsilon},T)}).
\]

**Proof.** Let \( V \) be given by (2.31). By Theorem 2.1 \((W, w)\) and \((v, V)\) satisfy problems

\[
v = \frac{1}{2} \text{div} QW, \tag{3.33}
\]

\[
\begin{cases}
Dw = \nabla \overrightarrow{n} W - QV \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div} W = -\rho w + \text{tr} U \quad \text{for} \quad x \in \Omega_{-\varepsilon},
\end{cases} \tag{3.34}
\]

and

\[
\begin{cases}
Dv = \nabla \overrightarrow{n} V + Q \text{div} QUQ \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div} V = -\rho v - \langle Q \nabla \overrightarrow{n}, U \rangle \quad \text{for} \quad x \in \Omega_{-\varepsilon},
\end{cases} \tag{3.35}
\]

respectively. Then

\[
\text{div} Q\nabla \overrightarrow{n} W = -\text{div} V = \rho v - \langle Q \nabla \overrightarrow{n}, U \rangle \quad \text{for} \quad x \in \Omega_{-\varepsilon}.
\]

Applying Theorem 3.3 to problem (3.34) with

\[
F = -QV, \quad f = \text{tr} U, \quad p = \langle W, QX \rangle|_{\Gamma_{-\varepsilon}}, \quad q = \langle W, \nu \rangle|_{\Gamma_{b_1}},
\]

21
we have
\[
\|W\|_{W^{2,2}(\Omega_\epsilon,T)} + \|w\|_{W^{1,2}(\Omega_\epsilon)} \\
\leq C \left( \|PW\|_{W^{2,2}(\Omega_\epsilon,T)} + \|\text{div} V\|_{W^{2,2}(\Omega_\epsilon)} + \|V\|_{W^{2,2}(\Omega_\epsilon,T)} + \|\text{tr} U\|_{W^{2,2}(\Omega_\epsilon)} \\
+ \|\langle W, QX \rangle\|_{W^{-2,2}(\Gamma_\epsilon)} + \|\langle W, \nu \rangle\|_{W^{-2,2}(\Gamma_\epsilon)} \right). \tag{3.36}
\]

Then applying Theorem 3.3 to problem (3.35) with
\[
F = Q \text{div} QU, \quad f = -\langle Q \nabla \bar{n}, U \rangle, \quad p = \langle V, QX \rangle, \quad q = \langle V, \nu \rangle
\]
yields
\[
\|\text{div} V\|_{W^{2,2}(\Omega_\epsilon)} + \|V\|_{W^{2,2}(\Omega_\epsilon,T)} \\
\leq \|\langle Q \nabla \bar{n}, U \rangle\|_{W^{2,2}(\Omega_\epsilon)} + C \left( \|V\|_{W^{2,2}(\Omega_\epsilon,T)} + \|\nu\|_{W^{1,2}(\Omega_\epsilon)} \right) \\
\leq C \left( \|PW\|_{W^{2,2}(\Omega_\epsilon,T)} + \|U\|_{L^2(\Omega_\epsilon,T)} \\
+ \|\langle V, QX \rangle\|_{W^{-2,2}(\Gamma_\epsilon)} + \|\langle V, \nu \rangle\|_{W^{-2,2}(\Gamma_\epsilon)} \right), \tag{3.37}
\]
where Lemma 3.3 has been used. In addition, it follows from the first equation in (3.34) that
\[
\mathcal{P}V = \mathcal{P}Q(Dw - \nabla \bar{n}W). \tag{3.38}
\]

Using (3.37) and (3.46) in (3.36) and by Lemma 3.4 below, we have
\[
\|W\|_{W^{2,2}(\Omega_\epsilon,T)} + \|w\|_{W^{1,2}(\Omega_\epsilon)} \\
\leq C \left( \|PW\|_{W^{2,2}(\Omega_\epsilon,T)} \right) \\
+ \|\|U\|_{L^2(\Omega_\epsilon,T^2)} + \|W\|_{L^2(\partial\Omega_\epsilon,T)} + \|\mathcal{P}Q(Dw - \nabla \bar{n}W)\|_{W^{-2,2}(\partial\Omega_\epsilon,T)} \right). \tag{3.39}
\]

We claim that the terms \(\|PW\|_{W^{2,2}(\Omega_\epsilon,T)}\) and \(\|\mathcal{P}Q(Dw - \nabla \bar{n}W)\|_{W^{2,2}(\Omega_\epsilon,T)}\) can be removed in (3.39) to have
\[
\|W\|_{W^{2,2}(\Omega_\epsilon,T)} + \|w\|_{W^{1,2}(\Omega_\epsilon)} \\
\leq C \left( \|U\|_{L^2(\Omega_\epsilon,T^2)} + \|W\|_{L^2(\partial\Omega_\epsilon,T)} \right) \tag{3.40}
\]
by a compactness-uniqueness argument as follows.

By contradiction. Suppose that (3.40) does not hold true. Then there are \(w_k, W_k, v_k, V_k,\) and \(U_k\), which satisfy (3.33)-(3.35), respectively, such that
\[
1 = \|W_k\|_{L^2(\Omega_\epsilon,T^2)} + \|w_k\|_{W^{1,2}(\partial\Omega_\epsilon)} \\
\geq k \left( \|U_k\|_{L^2(\Omega_\epsilon,T^2)} + \|W_k\|_{L^2(\partial\Omega_\epsilon,T)} \right) \tag{3.41}
\]
for all \(k \geq 1\). Then
\[
U_k \to 0 \quad \text{in} \quad L^2(\Omega_\epsilon,T^2), \quad W_k \to 0 \quad \text{in} \quad L^2(\partial\Omega_\epsilon,T).
\]

By Lemma 3.4 below
\[
V_k \to 0 \quad \text{in} \quad W^{-2,2}(\partial\Omega_\epsilon), \quad w_k \to 0 \quad \text{in} \quad W^{-1,2}(\partial\Omega_\epsilon).
\]
Noting that \( \text{div} V_k = -\rho v_k - \langle Q \nabla \bar{n}, U_k \rangle \), by (3.37) there is a subsequence, denoted still by \((v_k, V_k)\), that converges to some \((v_0, V_0) \in [W^{2,2}(\Omega_{-\varepsilon})]' \times [W^{2,2}(\Omega_{-\varepsilon}, T)']*\) in \([W^{2,2}(\Omega_{-\varepsilon})]' \times [W^{2,2}(\Omega_{-\varepsilon}, T)']*\). It follows from (3.35) that

\[
\begin{align*}
Dv_0 &= \nabla \bar{n}V_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div} V_0 &= -\rho v_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
V_0|_{\partial \Omega_{-\varepsilon}} &= 0.
\end{align*}
\]

(3.42)

By (3.39) there exists a subsequence \((w_k, W_k)\) which satisfies

\[
(w_k, W_k) \to (w_0, W_0) \quad \text{in} \quad [W^{1,2}(\Omega_{-\varepsilon})]' \times [W^{2,2}(\Omega_{-\varepsilon}, T)']*\)

for some \((w_0, W_0) \in [W^{1,2}(\Omega_{-\varepsilon})]' \times [W^{2,2}(\Omega_{-\varepsilon}, T)']*\). It is easy to check that \((w_0, W_0)\) solves problem

\[
\begin{align*}
\|W_0\|_{L^2(\Omega_{-\varepsilon}, T)} + \|w_0\|_{[W^{1,2}(\Omega_{-\varepsilon})]'} &= 1, \\
Dw_0 &= \nabla \bar{n}W_0 - QV_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div} W_0 &= -\rho w_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
w_0|_{\partial \Omega_{-\varepsilon}} &= 0, \quad W_0|_{\partial \Gamma_{-\varepsilon}} = 0, \\
\text{sym} DW_0 + w_0 \Pi &= 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon},
\end{align*}
\]

(3.43)

and

\[
v_0 = \frac{1}{2} \text{div} QW_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon},
\]

respectively.

Consider problem (3.42). Let \( \tau = \alpha_t/|\alpha_t| \) for \( x \in \Gamma_{b_1} \). Suppose \( \nu = Q\tau \). Then \( \nu, \tau \) has positive orientation. We have

\[
2v_0 = \text{div} QW_0 = \langle D_{\nu} W_0, \tau \rangle - \langle D_{\tau} W_0, \nu \rangle = \langle D_{\nu} W_0, \tau \rangle + \langle D_{\tau} W_0, \nu \rangle - 2w_0 \Pi(\tau, \nu) = 0 \quad \text{for} \quad x \in \Gamma_{b_1}.
\]

It follows from (3.42) that

\[
\begin{align*}
\text{div} (\nabla \bar{n})^{-1} Dv_0 &= -\rho v_0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
v_0 &= Dv_0 = 0 \quad \text{for} \quad x \in \Gamma_{b_1}.
\end{align*}
\]

(3.44)

By the uniqueness of the elliptic problem

\[
v_0 = 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \quad \kappa(x) > 0.
\]

Moreover, a similar argument yields

\[
v_0 = 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}.
\]
Then we have a hyperbolic problem
\[
\begin{cases}
\text{div} \left( \nabla n \right)^{-1} Dv_0 = -\rho v_0 & \text{for } x \in \Omega_{-\varepsilon}, \quad \kappa(x) < 0, \\
v_0 = Dv_0 = 0 & \text{for } x \in \Gamma_{-\varepsilon}.
\end{cases}
\]
Since the curves
\[
\alpha(\cdot, s) \quad \text{for} \quad -\varepsilon \leq s \leq 0,
\]
are not characteristic, the uniqueness of the hyperbolic problem implies
\[
v_0 = 0 \quad \text{for } x \in \Omega_{-\varepsilon}, \quad \kappa(x) < 0.
\]
Thus we obtain
\[
v_0 = V_0 = 0 \quad \text{for } x \in \Omega_{-\varepsilon}.
\]
Moreover, by (3.43) we have
\[
\begin{cases}
\text{div} \left( \nabla n \right)^{-1} Dw_0 = -\rho w_0 & \text{for } x \in \Omega_{-\varepsilon}, \\
w_0 = Dw_0 = 0 & \text{for } x \in \partial\Omega_{-\varepsilon}.
\end{cases}
\]
By a similar argument as for \((v_0, V_0)\) we obtain
\[
w_0 = W_0 = 0 \quad \text{for } x \in \Omega_{-\varepsilon},
\]
which contradicts the first equality in (3.43).

Finally, noting that
\[
\|DW\|_{[W^{1,2}(\Omega_{-\varepsilon},T^2)]'} = \|\text{sym} DW\|_{[W^{1,2}(\Omega_{-\varepsilon},T^2)]'},
\]
by Lemma 3.1 and (3.40) we have
\[
\|W\|_{L^2(\Omega_{-\varepsilon},T)} \leq C(\|\text{sym} DW\|_{[W^{1,2}(\Omega_{-\varepsilon},T^2)]'} + \|W\|_{L^2(\partial\Omega_{-\varepsilon},T)})
\leq C(\|U\|_{[W^{1,2}(\Omega_{-\varepsilon},T^2)]'} + \|w\|_{[W^{1,2}(\Omega_{-\varepsilon})]} + \|W\|_{L^2(\partial\Omega_{-\varepsilon},T)})
\leq \left( \|U\|_{L^2(\Omega_{-\varepsilon},T^2)} + \|W\|_{L^2(\partial\Omega_{-\varepsilon},T)} \right). \tag{3.45}
\]
Thus (3.32) follows from (3.40) and (3.45).

\begin{lemma}
Suppose that \((w, W)\) and \((v, V)\) solve problem (3.34) and (3.35), respectively. Then
\[
\|V\|_{W^{-2,2}(\partial\Omega_{-\varepsilon})} + \|w\|_{W^{-1,2}(\partial\Omega_{-\varepsilon})} \leq C(\|U\|_{L^2(\Omega_{-\varepsilon},T^2)} + \|W\|_{L^2(\partial\Omega_{-\varepsilon},T)}). \tag{3.46}
\]
\end{lemma}

\textbf{Proof.} Let \(\tau = \alpha_t/|\alpha_t|\) for \(x \in \Gamma_{b_1}\). Then \(\tau, \nu\) forms an orthonormal basis along \(\Gamma_{b_1}\). We have
\[
w\Pi(\tau, \tau) = U(\tau, \tau) - DW(\tau, \tau) \quad \text{for } x \in \Gamma_{b_1}. \tag{3.47}
\]

24
Since $\Pi(\tau, \tau) \neq 0$ for all $x \in \Gamma_{B_1}$, it follows from (3.47) that

$$\|w\|_{W^{-1,2}(\partial \Omega_{-\varepsilon})} \leq C(\|U\|_{W^{-1,2}(\partial \Omega_{-\varepsilon})} + \|\tau(W, \tau)\|_{W^{-1,2}(\partial \Omega_{-\varepsilon})} + \|(W, D_\tau \tau)\|_{W^{-1,2}(\partial \Omega_{-\varepsilon})})$$

$$\leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T^2)} + \|W\|_{L^2(\partial \Omega_{-\varepsilon}, T)}). \quad (3.48)$$

Moreover, by (3.47) we obtain

$$\nu(w)\Pi(\tau, \tau) = \nu(U(\tau, \tau)) - D^2W(\tau, \tau, \nu) - (D_\tau \tau, \nu)[DW(\nu, \tau) + DW(\tau, \nu)] - w\nu(\Pi(\tau, \tau))$$

$$= \nu(U(\tau, \tau)) - D^2W(\tau, \tau, \nu) - 2(D_\tau \tau, \nu)U(\tau, \nu) + w[2\Pi(\tau, \nu) - \nu(\Pi(\tau, \tau))] \quad \text{for} \quad x \in \Gamma_{B_1}. \quad (3.49)$$

Then

$$\|\nu(w)\|_{W^{-2,2}(\Gamma_{B_1})} \leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T)} + \|w\|_{W^{-1,2}(\Gamma_{B_1})} + \|D^2W(\tau, \tau, \nu)\|_{W^{-2,2}(\Gamma_{B_1})}). \quad (3.50)$$

Next, we shall estimate $\|D^2W(\tau, \tau, \nu)\|_{W^{-2,2}(\Gamma_{B_1})}$. By Ricci’s identity, we have

$$D^2W(\tau, \tau, \nu) = D^2W(\tau, \nu, \tau) + R(\tau, \nu, W, \tau) = \tau(DW(\nu, \tau))$$

$$- (D_\tau \nu, \nu)DW(\nu, \tau) - (D_\tau \nu, \tau)DW(\tau, \nu) + R(\tau, \nu, W, \tau)$$

$$= -\tau(DW(\nu, \tau)) + 2\tau(U(\tau, \nu)) - 2\tau(w\Pi(\tau, \nu)) + R(\tau, \nu, W, \tau)$$

$$- (D_\tau \nu, \nu)U(\nu, \nu) - w\Pi(\nu, \nu) - \nu(\Pi(\tau, \tau))(U(\tau, \tau) - \nu(\Pi(\tau, \tau)))$$

which yields

$$\|D^2W(\tau, \tau, \nu)\|_{W^{-2,2}(\Gamma_{B_1})} \leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T)} + \|W\|_{L^2(\Gamma_{B_1})} + \|w\|_{W^{-1,2}(\Gamma_{B_1})}),$$

where $R(\cdot, \cdot, \cdot, \cdot)$ is the Riemannian curvature tensor. It follows from (3.50) that

$$\|\nu(w)\|_{W^{-2,2}(\Gamma_{B_1})} \leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T)} + \|W\|_{L^2(\Gamma_{B_1})} + \|w\|_{W^{-1,2}(\Gamma_{B_1})}).$$

By the first equation in (3.34) we obtain

$$\|V\|_{W^{-2,2}(\Gamma_{B_1}, T)} \leq C(\|W\|_{W^{-2,2}(\Gamma_{B_1}, T)} + \|\nu(w)\|_{W^{-2,2}(\Gamma_{B_1})} + \|\tau(w)\|_{W^{-2,2}(\Gamma_{B_1})})$$

$$\leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T)} + \|W\|_{L^2(\Gamma_{B_1})} + \|w\|_{W^{-1,2}(\Gamma_{B_1})}). \quad (3.51)$$

Combing (3.48) and (3.51), we have

$$\|V\|_{W^{-2,2}(\Gamma_{B_1}, T)} + \|w\|_{W^{-1,2}(\Gamma_{B_1})} \leq C(\|U\|_{L^2(\Omega_{-\varepsilon}, T)} + \|W\|_{L^2(\Gamma_{-\varepsilon})}).$$

A similar argument shows that the above estimates hold when $\Gamma_{B_1}$ is replaced by $\Gamma_{-\varepsilon}$. Thus (3.46) follows. □
**Proof of Theorem 1.1.** Let \( y = W + w\bar{n} \) satisfy problem (1.6) for given \( U \in L^2(S, T^2) \). Let

\[
S(-b_0, -\varepsilon) = \{ \alpha(t, s) \in S \mid (t, s) \in (0, a) \times (-b_0, -\varepsilon) \}.
\]

By (1.2) \( S(-b_0, -\varepsilon) \) is a non-characteristic region. By [24, Theorem 1.1] there is a \( C > 0 \), independent of \( y \), such that

\[
\|W\|_{L^2(S(-b_0, -\varepsilon), T)} \leq C(\|U\|_{L^2(S(-b_0, -\varepsilon), T^2)} + \|W\|_{L^2(\Gamma_{-b_0}, T)}),
\]

(3.52)

from that we obtain

\[
\|w\|_{W^{1,2}(S(-b_0, -\varepsilon))}' \leq C(\|U\|_{W^{1,2}(S(-b_0, -\varepsilon), T^2)})' + \|\text{sym}DW\|_{W^{1,2}(S(-b_0, -\varepsilon), T^2)}',
\]

\[
\leq C(\|U\|_{L^2(S(-b_0, -\varepsilon), T^2)} + \|W\|_{L^2(S(-b_0, -\varepsilon), T)})
\]

\[
\leq C(\|U\|_{L^2(S(-b_0, -\varepsilon), T^2)} + \|W\|_{L^2(\Gamma_{-b_0}, T)}),
\]

(3.53)

where the assumption

\[
\Pi(\alpha_t, \alpha_t) > 0 \quad \text{for all} \quad x \in S(-b_0, -\varepsilon)
\]

is used. Moreover, by [24, Lemma 3.6]

\[
\|W\|_{L^2(\Gamma_{-\varepsilon}, T)} \leq C(\|U\|_{L^2(S(-b_0, -\varepsilon), T^2)} + \|W\|_{L^2(\Gamma_{-b_0}, T)}).
\]

(3.54)

Thus estimate (1.7) follows from Theorem 3.4 and (3.52)-(3.54).

\[\square\]

**Ethical approval**

This article does not contain any studies with human participants or animals performed by the author.

**Declaration of competing interest**

The author declares that there is no conflict of interest.

**References**

[1] L. B. Chen and P. F. Yao, Strain tensors and matching property on degenerated hyperbolic surfaces, accepted for publication by SIAM J. Math. Anal., arXiv:2107.09269 [math.AP].

[2] —, Strain tensors and matching property on surfaces with the Gauss curvature changing sign, preprint in 2021, arXiv:2111.15189 [math.AP].

[3] P. G. Ciarlet and V. Lods, On the ellipticity of linear membrane shell equations. J. Math. Pures Appl. (9) 75 (1996), no. 2, 107-124.
[4] D. Cioranescu, O. Oleinik, and G. Tronel, On Korn’s inequalities for frame type structures and junctions. C. R. Acad. Sci. Paris Ser. I Math., 309(9):591-596, 1989.

[5] G. Friesecke, R. James, S. Muller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. Commun. Pure Appl. Math. 55, 1461-1506 (2002).

[6] Y. Grabovsky and D. Harutyunyan, Korn inequalities for shells with zero Gaussian curvature. Ann. Inst. H. Poincare Anal. Non Lineaire 35 (2018), no. 1, 267-282.

[7] —, Exact scaling exponents in Korn and Korn-type inequalities for cylindrical shells. SIAM J. Math. Anal. 46 (2014), no. 5, 3277-3295.

[8] D. Harutyunyan, On the Korn interpolation and second inequalities in thin domains, SIAM J. Math. Anal. 50 (2018), no. 5, 4964-4982.

[9] —, New asymptotically sharp Korn and Korn-like inequalities in thin domains. Journal of Elasticity, 117(1), pp. 95-109, 2014.

[10] —, Gaussian curvature as an identifier of shell rigidity. Arch. Ration. Mech. Anal. 226 (2017), no. 2, 743-766.

[11] P. Hornung, M. Lewicka, M. R. Pakzad, Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells. J. Elasticity 111 (2013), no. 1, 1-19.

[12] M. Lewicka, M. R. Pakzad, The infinite hierarchy of elastic shell models: some recent results and a conjecture. Infinite dimensional dynamical systems, 407-420, Fields Inst. Commun., 64, Springer, New York, 2013.

[13] M. Lewicka, M. G. Mora, M. R. Pakzad, Shell theories arising as low energy Γ-limit of 3d nonlinear elasticity. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), no. 2, 253-295.

[14] —, The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells. Arch. Ration. Mech. Anal. 200 (2011), no. 3, 1023-1050.

[15] M. Lewicka and S. Muller. On the optimal constants in korn’s and geometric rigidity estimates, in bounded and unbounded domains, under neumann boundary conditions. Indiana Univ. Math. J. 65 (2016), no. 2, 377-397.

[16] A. E. H. Love, A treatise on the mathematical theory of elasticity. Dover, 4th edition, 1927.

[17] S. A. Nazarov, Weighted anisotropic Korn’s inequality for a junction of a plate and a rod. Sbornik: Mathematics, 195(4):553-583, 2004.
[18] —, Korn inequalities for elastic junctions of massive bodies, thin plates, and rods. Russian Mathematical Surveys, 63(1):35, 2008.

[19] R. Paroni and G. Tomassetti, Asymptotically exact Korns constant for thin cylindrical domains. Comptes Rendus Mathematique, 350(15):749-752, 2012.

[20] —, On Kor’s constant for thin cylindrical domains. Mathematics and Mechanics of Solids, 19(3):318-333, 2014.

[21] P. F. Yao, Modeling and Control in Vibrational and Structural Dynamics. A differential geometric approach. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton, FL, 2011.

[22] —, Linear strain tensors on hyperbolic surfaces and asymptotic theories for thin shells, SIAM J. Math. Anal. 51 (2019), no. 2, 1387-1435.

[23] —, Optimal exponentials of thickness in Korn’s inequalities for parabolic and elliptic shells, Annali di Matematica Pura ed Applicata (1923 -), 2020-06-19, DOI: 10.1007/s10231-020-01000-6.

[24] —, Strain tensors on hyperbolic surfaces and their applications, J. Funct. Anal. (2021), 108986, doi: https://doi.org/10.1016/j.jfa.2021.108986.