Properties of parallelotopes equivalent to Voronoi’s conjecture

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Abstract

A parallelotope is a polytope whose translation copies fill space without gaps and intersections by interior points. Voronoi conjectured that each parallelotope is an affine image of the Dirichlet domain of a lattice, which is a Voronoi polytope. We give several properties of a parallelotope and prove that each of them is equivalent to it is an affine image of a Voronoi polytope.

1 Introduction

Let us have a partition of $\mathbb{R}^n$ into equal convex polytopes (tiles) such that the partition is simultaneously a covering and a packing and the intersection of any two polytopes is empty or a common face of each. Such a partition is called tiling. Let this tiling be invariant under a group $\mathcal{T}$ of translations, and the group $\mathcal{T}$ is transitive on polytopes of the tiling. Then each tile of such a partition is called parallelotope. Here the prefix parallelo emphasizes that each tile is a parallel translation of a prototile. (Following to [MRS95] we use the word parallelotope rather than parallelohedron which was used by Voronoi in [Vo08]. Recall also that a polyhedron is a 3-dimensional polytope.)

G. Voronoi in §8 of [Vo08] defines a parallelohedron as follows. A polytope $P$ with a group of translations $\mathcal{T}$ is called a parallelohedron if the space $\mathbb{R}^n$ can be filled by non-overlapping congruent copies of $P$ using translations taken from $\mathcal{T}$.

Each parallelotope necessarily satisfies the following three conditions:

(i) a parallelotope is centrally symmetric;
(ii) each facet of a parallelotope is centrally symmetric;
(iii) for $n > 1$, the projection of a parallelotope along any $(n - 2)$-dimensional face is either a parallelogram or a centrally symmetric hexagon.

The edges of the polygon of item (iii) above are projections of four or six facets of the projected parallelotope $P$. These facets form a belt of the parallelotope $P$. Hence the property (iii) of a parallelotope $P$ has another formulation. Namely,

(iii)' for $n > 1$, each belt of a parallelotope contains four or six facets.
B. Venkov [Ve54] (and independently P. McMullen [McMu80]) proved that the above three conditions are sufficient for a polytope to be a parallelotope. A. Aleksandrov [Al54], knowing the Venkov’s result, simplified the proof of B. Venkov.

There is a special well known case of a parallelotope, namely, the Voronoi polytope related to a point of a lattice $L$. The Voronoi polytope $P_V(t_0)$ related to a point $t_0 \in L$ is the set of points of $\mathbb{R}^n$ which are at least as close to $t_0$ as to any other point of $L$.

The main conjecture of Voronoi is that any parallelotope $P$ can be mapped into a Voronoi polytope $P_V$ under an affine transformation $x \to Ax$ of the space containing $P$. Here $A$ is a non-degenerate $n \times n$ matrix, where $n$ is dimension of the space.

Call a $k$-face (= $k$-dimensional face) of a parallelotope primitive if it belongs to minimal possible number $n - k + 1$ of parallelotopes of its tiling. Obviously any facet of a parallelotope is primitive. According to Zhitomirskii [Zh29], a parallelotope is called $k$-primitive if each its $k$-face is primitive. Besides, the $k$-primitivity implies the $(k + 1)$-primitivity. A 0-primitive parallelotope is simply called primitive. Obviously, any parallelotope is $(n - 1)$-primitive.

Voronoi proved his conjecture for primitive parallelotopes. If a parallelotope is primitive, then each its belt contains 6 facets, but not conversely. On the other hand, if each belt consists of 6 facets, then the parallelotope is $(n - 2)$-primitive. This implies that each $(n - 2)$-face belongs to three parallelotopes. Zhitomirskii [Zh29] extend the result of Voronoi over $(n - 2)$-primitive parallelotopes.

P. McMullen [McMu75] proved Voronoi’s conjecture for parallelotopes which are zonotopes. Later R. Erdahl [Er99] repeated the result of McMullen. B. De launay [De29] proved Voronoi’s conjecture in complete generality for the dimensions $n \leq 4$.

We give here several conditions on a parallelotope $P$ each of which is equivalent to the Voronoi’s conjecture is true for $P$.

2 Parallelotopes

Now we consider a description of a parallelotope $P = P(0)$ with the center in origin by a system of linear inequalities. We denote by $A^T$ the transpose of a matrix $A$ and by $x^T y = y^T x$ the scalar product of two column vectors $x, y \in \mathbb{R}^n$, and set $x^T x = x^2$.

Being a convex polytope a parallelotope is described by a system of linear inequalities $\{q_i^T x \leq \alpha_i\}$. Since, by (i) of Introduction, $P$ is centrally symmetric, each facet $F_i$ of $P$ has the opposite facet $-F_i$. If $F_i$ lies in the affine hyperplane given by the equality $q_i^T x = \alpha_i$, then the opposite facet $-F_i$ lies in an affine hyperplane $q_i^T x = -\alpha_i$. The parallelotope $P$ is adjacent by the facet $F_i$ to a parallelotope $P(t_i)$ which is a parallel shift of $P$ by the translation vector $t_i \in \mathcal{T}$.

Let $\mathcal{I}_P$ be the set of indices of pairs of opposite facets of $P$. Then the set $\{t_i : i \in \mathcal{I}_P\}$ of translation vectors generates the translation group $\mathcal{T}$ and a lattice $L$. The points of $L$, i.e. the centers of parallelotopes of the tiling, are the end-points of lattice vectors. We can identify each lattice vector with an element $t \in \mathcal{T}$. By this identification origin is the zero point and simultaneously the zero lattice vector 0 of $L$.  


Obviously, the point $\frac{1}{2}t_i$ is the center of the facet $F_i$. Hence $\frac{1}{2}q_i^T t_i = \alpha_i$. We have

$$P(0) = \{ x \in \mathbb{R}^n : -\frac{1}{2}q_i^T t_i \leq q_i^T x \leq \frac{1}{2}q_i^T t_i, \; i \in \mathcal{I}_P \}. \tag{1}$$

Here the facet vectors $q_i$ are determined only up to a non-zero multiple $\beta_i$. We say that the facet vectors $q_i$ and the lattice vectors $t_i$, $i \in \mathcal{I}_P$, giving the description (1) of a parallelotope $P$, are associated.

The parallelotope $P(t)$ with the center in the point $t \in \mathcal{L}$ is a translation of $P(0)$ by the vector $t$, and therefore it is described as follows:

$$P(t) = \{ x \in \mathbb{R}^n : -\frac{1}{2}q_i^T t_i \leq q_i^T (x-t) \leq \frac{1}{2}q_i^T t_i, \; i \in \mathcal{I}_P \}. \tag{2}$$

Note that, by definition of a Voronoi polytope, a facet $F_i$ of the Voronoi polytope $P_V(0)$ is orthogonal to the lattice vector $t_i$ and bisects it. The lattice vector $t_i$ connects the centers of $P_V(0)$ and $P_V(t_i)$, where $P_V(t_i)$ is the Voronoi polytope adjacent to $P_V(0)$ by the facet $F_i$. In other words, we can set $q_i = t_i$ in the descriptions (1) and (2) of parallelotopes $P(0)$ and $P(t)$ in the case they are Voronoi polytopes.

### 3 Linear transforms of parallelotopes

Note that the usual Euclidean norm $x^2$ is used in the definition of the Voronoi polytope $P_V(0)$. But we can use an arbitrary positive quadratic form $f(x) = x^T Dx$ as a norm of $x$. Here $D$ is a symmetric positive definite $n \times n$ matrix. Then the above definition gives a parallelotope $P_f$. Call such a parallelotope the Voronoi polytope with respect to the quadratic form $f(x)$. Such a parallelotope relates to a lattice $\mathcal{L}$ (or to a translation group $T$). Consider the Voronoi polytope $P_f$ with respect to the quadratic form $x^T Dx$ in detail. By definition, we have

$$P_f(0) = \{ x \in \mathbb{R}^n : x^T Dx \leq (x-t)^T D(x-t) \text{ for all } t \in \mathcal{T} \}.\text{ (3)}$$

It is well known, that a finite set $\{ \pm t_i : i \in \mathcal{I}_f \}$ of vectors $t_i \in \mathcal{T}$ is sufficient for the description of $P_f(0)$.

Using the symmetricity of $D$ and joining the inequalities for $t_i$ and $-t_i$, we simplify this as follows

$$P_f(0) = \{ x \in \mathbb{R}^n : -\frac{1}{2}t_i^T Dt_i \leq t_i^T Dx \leq \frac{1}{2}t_i^T Dt_i, \; i \in \mathcal{I}_f \}. \tag{3}$$

For $D = I_n$, where $I_n$ is the identity matrix, we have $f(x) = x^2$ and $P_f(0) = P_V(0)$.

**Lemma 1** Let $P$ be a parallelotope given by (1). Let $A$ be an $n \times n$ non-degenerate matrix, and $D = A^T A$. The following assertions are equivalent:

(i) the affine transformation $x \rightarrow Ax$ transforms $P$ into a Voronoi polytope;

(ii) $P$ is the Voronoi polytope with respect to the quadratic form $f(x) = x^T Dx$;

(iii) the facet vectors $q_i$ satisfy the equality $q_i = Dt_i, i \in \mathcal{I}_P$.
Proof. (i)⇒(iii). Consider the affine transformation \( x \to Ax \). The new facet vector has the form \( (AT)^{-1}q \). In fact, for \( x \in F \), the point \( Ax \) belongs to a facet of \( AP \). Hence
\[
((AT)^{-1}q)^T Ax = q^T ((AT)^{-1})^T Ax = q^T A^{-1} Ax = q^T x = 0.
\]
The new center of the transformed facet is \( \frac{1}{2} At_i \). For \( AP \) to be a Voronoi polytope, we have to have
\[
(AT)^{-1}q_i = At_i, \text{ i.e. } q_i = AT At_i, \text{ i.e. } q_i = Dt_i.
\]
(ii)⇒(i) The positive definite matrix \( D \) can be represented as the product \( D = AT A \), where the matrix \( A \) is non-degenerate. Hence the form \( x^T Dx = x^T AT Ax = (Ax)^T (Ax) \) is the quadratic form \( (Ax)^2 \) in the transformed space, i.e. \( P = P_f = P^2_{(Ax)} \). Let \( y = Ax \). Then \( x = A^{-1} y \) and \( P = P_f = A^{-1} P y^2 = A^{-1} P y \). Hence \( AP = P_y \).

(iii)⇒(ii) If we set in (1) \( q_i = Dt_i \), we obtain description (3) of a parallelotope. Hence the Voronoi polytope with respect to a quadratic form is also a special case of a parallelotope, when \( q_i = Dt_i \). In other words, in this case, the parallelotope \( P(0) \) of (1) is \( P_f(0) \) for \( f(x) = x^T Dx \).

\[
\square
\]

4 A canonical representation of a parallelotope

Consider a vertex \( v \) of a facet \( F_i \). Let \( v \) be the intersection of facets \( F_j, \ j \in \mathcal{I}(v) \). Then \( i \in \mathcal{I}(v) \) and
\[
q_j^T v = \frac{1}{2} q_j^T t_j, \ j \in \mathcal{I}(v).
\]
Since each facet of \( P \) is centrally symmetric, there is a symmetric to \( v \) vertex \( v^* \). We have
\[
q_k^T v^* = \frac{1}{2} q_k^T t_k, \ k \in \mathcal{I}(v^*).
\]
Note that the point \( \frac{1}{2}(v + v^*) \) is the center \( \frac{1}{2}t_i \) of the facet \( F_i \). Hence
\[
v + v^* = t_i, \text{ i.e. } v^* = t_i - v.
\]

Recall that there are the following two types of belts in the parallelotope \( P \):
1) 3-belts containing 6 facets \( \pm F_i, \pm F_j, \pm F_k \);
2) 2-belts containing 4 facets \( \pm F_i, \pm F_j \).

We denote each belt by the set of indices of its generating facets. So, we have the following two types of belts: \( \{i,j,k\} \) and \( \{i,j\} \).

Therefore some facet vectors of the set \( \{q_j : j \in \mathcal{I}(v)\} \) and \( \{q_k : k \in \mathcal{I}(v^*)\} \) are joined into pairs of two types such that
1) either the facets \( F_i, F_j, F_k \) belong to the 3-belt \( \{i,j,k\} \);
2) or the facets \( F_j \) and \( F_k \) are opposite, i.e. \( F_k = -F_j \), and belong to the 2-belt \( \{i,j\} \).

Let \( \mathcal{B} \) be the set of all 3-belts. Consider a belt \( \{i,j,k\} \in \mathcal{B} \). The facets vectors \( q_i, q_j, q_k \) lie in a two-dimensional plane, where they are linearly dependent. Obviously, the
associated lattice vectors \(t_i, t_j, t_k\) are also linearly dependent. Moreover, this dependence has the following form
\[
t_i - \varepsilon_j t_j - \varepsilon_k t_k = 0, \quad \{i, j, k\} \in \mathcal{B},
\]  
where \(\varepsilon_j, \varepsilon_k \in \{\pm 1\}\). Since each facet vector \(q_i\) is defined up to a scalar multiple \(\beta_i\), we can choose lengths of the associated facet vectors such that the new facet vector \(\beta_i q_i\) satisfy the equality similar to (4)
\[
\beta_i q_i - \varepsilon_j \beta_j q_j - \varepsilon_k \beta_k q_k = 0, \quad \{i, j, k\} \in \mathcal{B}.
\]

Following to [Vo08] and [RR97], we say that, for the belt \(\{i, j, k\}\), the facet vectors \(q_i, q_j, q_k\) are defined canonically with respect to the 3-belt \(\{i, j, k\}\) if they satisfy the same equality as the associated lattice vectors \(t_i, t_j, t_k\).

**Definition** A parallelotope \(P\) is defined canonically by (1) if the facet vectors \(q_i, i \in I_p\), are defined canonically simultaneously with respect to all belts of \(P\).

In other words, a parallelotope \(P\) is defined canonically if the system of equations (5) determining multiples \(\beta_i, i \in I_p\), has a non-zero solution.

Voronoi proves in the first §§44 of [Vo08] that a primitive parallelotope can be defined canonically.

## 5 Relations between lattice and facet vectors

We suppose that the facet vectors \(q_i, q_j\) and \(q_k\) determine the facets \(F_i, F_j\) and \(F_k\), respectively. Hence the vector \(\varepsilon_j q_j\) defines the facet \(\varepsilon_j F_j\).

In Proposition 1 below, for simplicity sake, we suppose that \(\varepsilon_j = \varepsilon_k = 1\). For to apply the results below to the general dependences (4) and (5), it is sufficient to change \(q_j\) and \(q_k\) by \(\varepsilon_j q_j\) and \(\varepsilon_k q_k\), respectively.

So the lattice vecors, corresponding to the belt \(\{i, j, k\}\), satisfy the equality
\[
t_i = t_j + t_k.
\]  
Hence the defined canonically facet vectors satisfy the equality
\[
q_i = q_j + q_k
\]  
and the intersections \(F_i \cap F_j\) and \(F_i \cap F_k\) are non-empty and define two opposite facets of \(F_i\).

For \(i \in I_p\), let \(I_i = \{j \in I_p : F_j \cap F_i = F^{n-2}\}\), where \(F^{n-2}\) is an \((n - 2)\)-face of \(P\) (which is a facet of \(F_i\)). Let \(\mathcal{B}_i\) be the set of 3-belts containing the facet \(F_i\). Now, using the property (ii) of parallelotopes, we prove an important fact.

**Proposition 1** For all \(j \in I_i\), the vectors \(q_j\) can be defined canonically with respect to all 3-belts of \(\mathcal{B}_i\). For these canonical facet vectors we have
1) for a 3-belt \( \{i, j, k\} \in \mathcal{B}_i \) the following equalities hold:

\[
q_i^T t_j = q_j^T t_i, \quad q_i^T t_k = q_k^T t_i, \quad q_j^T t_k = q_k^T t_j;
\]

2) for a 2-belt \( \{i, j\}, j \in \mathcal{I}_i \) the following equalities hold:

\[
q_j^T t_i = q_i^T t_j = 0.
\]

**Proof.** Obviously, the vectors \( q_j, j \in \mathcal{I}_i \), can be defined canonically with respect to all 3-belts of \( \mathcal{B}_i \), since the belts of \( \mathcal{B}_i \) have only one common vector \( q_i \).

Recall that opposite vertices \( v \) and \( v^* \) of the facet \( F_i \) are determined by facet vectors some of which form belts with \( q_i \). Consider the equations

\[
q_j^T v = \frac{1}{2} q_i^T t_j, \quad q_k^T v^* = \frac{1}{2} q_k^T t_k,
\]

related either to a 3-belt \( \{i, j, k\} \) or to a 2-belt \( \{i, j\} \), and then \( q_k = -q_j \).

For the case 1), substituting the expressions \( q_k = q_i - q_j, t_k = t_i - t_j, v^* = t_i - v \), in the second equation, and using the equalities \( q_i^T v = \frac{1}{2} q_i^T t_i, q_j^T v = \frac{1}{2} q_j^T t_j \), we obtain

\[
q_i^T (t_i - v - \frac{1}{2} t_i + \frac{1}{2} t_j) = q_j^T (t_i - v - \frac{1}{2} t_i + \frac{1}{2} t_j) \Rightarrow q_i^T t_j = q_j^T t_i.
\]

Similarly, beginning with \( q_j^T v = \frac{1}{2} q_j^T t_j \) and using the equality \( v = t_i - v^* \), we obtain the equality \( q_i^T t_k = q_k^T t_i \). Now, this equality implies

\[
(q_j + q_k)^T t_k = q_j^T (t_j + t_k) \Rightarrow q_j^T t_k = q_k^T t_j.
\]

In the case 2) we have \( q_k = -q_j, t_k = -t_j \). Hence we obtain

\[
q_k^T v^* = \frac{1}{2} q_k^T t_k \Rightarrow q_j^T (v - t_i) = \frac{1}{2} q_j^T t_j \Rightarrow q_j^T (\frac{1}{2} t_j - t_i) = \frac{1}{2} q_j^T t_j \Rightarrow q_j^T t_i = 0.
\]

Using the facet \( F_j \) instead of \( F_i \), we obtain the equality \( q_j^T t_j = 0 \). Note that the equalities \( q_i^T t_i = 0 = q_i^T t_j \) do not depend on whether \( q_i \) and \( q_j \) are canonical or not. \( \square \)

Let \( |\mathcal{I}_P| = m \) and let \( Q \) and \( T \) be \( n \times m \) matrices whose columns are the vectors \( q_i \) and \( t_i \) for \( i \in \mathcal{I}_P \), respectively. Then the product \( q_i^T t_j \) is the \( (ij) \)-th element of the matrix product \( Q^T T \). If the equalities

\[
q_i^T t_j = t_i^T q_j \quad \text{hold for all pairs } i, j \in \mathcal{I}_P,
\]

then the \( m \times m \) matrix \( Q^T T \) is symmetric, i.e. \( Q^T T = (Q^T T)^T = T^T Q \).

**Lemma 2** The following assertions are equivalent:

(i) the equalities \( q_i^T t_j = t_i^T q_j \) hold for all pairs \( i, j \in \mathcal{I}_P \);

(ii) there is a unique symmetric non-degenerate \( n \times n \) matrix \( D \) such that \( q_i = Dt_i \) for all \( i \in \mathcal{I}_P \).
Recall that the centers of paralleloptopes $P(t)$ form a lattice $L$. Consider the points of $L$ (i.e. the endpoints of lattice vectors) as vertices of a graph $G_L$. Two vertices $t, t' \in L$ are adjacent in $G_L$ if and only if $t - t' \in \{\pm t_i : i \in I_P\}$. We can consider $G_L$ as an oriented graph, where the orientation of the edge $t_i$ is the orientation of the vector $t_i$. Hence edges of $G_L$ are the vectors $\pm t_i, i \in I_P$. Therefore the set of all edges of $G_L$ is partitioned into $m = |I_P|$ classes $E_i$. We suppose that all edges of $E_i$ are vectors $t_i$ (with the same sign, say $+$). In other words, all edges of $E_i$ are obtained from one by translations. For each $i \in I_P$, a vertex of $G_L$ is incident to two edges $t_i$, one of which go out and another come in the vertex.

For any collection of integers $\{z_i : i \in I_P\}$, we set $t(z) = \sum_{i \in I_P} z_i t_i$ and $q(z) = \sum_{i \in I_P} z_i q_i$. Let $I(z) = \{i \in I_P : z_i \neq 0\}$ be the support of $z = \{z_i : i \in I_P\}$.

Any two vertices $t^0$ and $t$ are connected in $G_L$ by an oriented path $P$ directed from $t^0$ to $t$. We denote such a path as a sequence $P = (t_1, t_2, \ldots, t_n)$ of vectors $t_k \in \{\pm t_i : i \in I_P\}$ corresponding to edges of the path in the natural order. Here $t_k = t_i$ or $t_k = -t_i$ according to the directions of the path and the corresponding edge coincide or not, respectively. Then, obviously, $t = t^0 + \sum_{k=1}^n t_k$. In particular, if the path is closed, i.e. it is a circuit and $i = t^0$, then $\sum_{k=1}^n t_k = 0$. We rewrite the sum $\sum_{k=1}^n t_k$ as $\sum_{i \in I_P} z_i(P) t_i = t(\{P\}) = t(P)$. So, $t = t^0 + t(P)$. We set $I(P) = I(z(P))$. Note that there are many paths with the same collection $z(P)$. All of them are obtained from $P$ by permutations the edges $t_k$.

Since the set $\{\pm t_i : i \in I_P\}$ generates the lattice $L$, any lattice vector $t$ has a (non-unique) representation $t = t(P)$, where $P$ is a path in $G_L$ connecting 0 with $t$. We associate the vector $q(P) = \sum_{t_k \in P} q_k$ to the vector $t(P)$. We call the vector $q(P)$ the vector associated to the vector $t(P)$. If the equalities $q_i t_j = q_j t_i$ hold for all $i, j \in I_P$, then using Lemma 2 we see that $q(P) = D t(P)$ does not depend on $P$. The uniqueness of $q$ in this case follows from the fact that the equality $\sum_k t_k = 0$ implies the equality $\sum_k q_k = 0$.

Consider some important subgraphs of the graph $G_L$. Let $t \in L$ and let $G(t)$ be the graph induced by all vertices $t' \in L$ adjacent to $t$. The union of $t$ and $G(t)$ is the
suspension $\nabla G(t)$. In the graph $\nabla G(t)$, the vertex $t$ is adjacent to all vertices of $G(t)$.

For $t = 0$, the vertices of $G(0)$ are endpoints of the vectors $\pm t_i$, $i \in \mathcal{I}_p$. In other words, the graph $G(0)$ is determined on two copies of the set $\mathcal{I}_p$. In fact, any edge $t_i$ of $G(0)$ belongs to a triangle $\triangle = (t_i, t_j, t_k)$ of $\nabla G(0)$ such that $t_i - \varepsilon_j t_j - \varepsilon_k t_k = 0$. This triangle corresponds to the belt $\{i, j, k\}$ of $P(0)$. Moreover, there are six such triangles in $\nabla G(0)$. These 6 triangles form a hexagon spanning a 2-dimensional plane.

Let $G(F_k)$ be the graph induced by the centers of all parallelotopes having a common $k$-face $F^k$ with $P(0)$. So, the graph $G(F_n)$ is an edge $t_i$ with end-vertices corresponding to adjacent parallelotopes $P(0)$ and $P(t_i)$. There are only two types of graphs $G(F^{n-2})$, namely, triangles and quadrangles, according to two types of belts with 6 and 4 facets, respectively.

If $P$ is $m$-primitive, then, for $m \leq k \leq n$, $G(F_k)$ is a $K_{n-k+1}$, where $K_s$ is the complete graph on $s$ vertices.

The following reformulation of canonical definitiness of a parallelotope $P$ is obvious.

**Lemma 3** For a parallelotope $P$, the following assertions are equivalent:

(i) $P$ is defined canonically;

(ii) $q(\triangle) = 0$ for all triangles $\triangle \subset G_L$ such that $\triangle = G(F^{n-2})$.

Call a 4-circuit $(t_i, t_j, -t_i, -t_j)$ by a quadrangle $Q_{ij}$. It is a parallelogram and spans a 2-dimensional plane. Among quadrangles of $G_L$ there are quadrangles $G(F^{n-2})$, where $F^{n-2}$ is an $(n-2)$-face which is common to four parallelotopes. Obviously, for each quadrangle, we have trivially $q(Q_{ij}) = q_i + q_j - q_i - q_j = 0$.

The technique used in [Ve54] can be applied for a proof of Proposition 2 below (see also Theorem 1 of [RR97]).

**Proposition 2** Any circuit of $G_L$ can be represented as a sum modulo 2 of circuits of type $G(F^{n-2})$.

Hence Lemma 3 and Proposition 2 imply the following

**Lemma 4** For a parallelotope $P$, the following assertions are equivalent:

(i) $P$ is defined canonically;

(ii) $q(C) = 0$ for all circuits $C \subset G_L$.

But we give here an explicit proof of a more weak result which we’ll use later.

**Lemma 5** Any quadrangle $Q_{ij}$ can be represented as a sum modulo 2 of triangles and quadrangles, both of type $G(F^{n-2})$.

**Proof.** We span a 2-dimensional surface $S$ on the quadrangle $Q_{ij}$ as follows. The four edges $t_i, t_j, -t_i$ and $-t_j$ form the boundary of $S$. Recall that the vertices of the quadrangle $Q_{ij}$ are centers of four parallelotopes, say $P(0)$, $P(t_i)$, $P(t_j)$ and $P(t_i + t_j)$. Hence the surface $S$ intersects a number of parallelotopes. We choose $S$ such that it intersects boundaries of these parallelotopes only by facets and $(n-2)$-faces, and these intersections
are transversal. Hence if the intersection $S \cap F$ is not empty, then it is a segment or a point depending on $F$ is a facet or an $(n - 2)$-face, respectively.

These segments and points form a planar graph $\Gamma$ drawn on $S$. This graph $\Gamma$ has 4 half-edges corresponding to the 4 facets intersected by the 4 edges of $Q_{ij}$. Vertices of $\Gamma$ have degrees 3 and 4 only. The dual of $\Gamma$ is a planar subgraph $G(\Gamma)$ of the graph $G_L$. Minimal circuits of $G(\Gamma)$ are just triangles and quadrangles of type $G(F^{n-2})$. Since $G(\Gamma)$ is planar, each edge of it (excluding the four boundary edges) belongs to two minimal circuits. So we obtain the wanted representation of the quadrangle $Q_{ij}$ as a sum of graphs $G(F^{n-2})$ modulo 2. 

\section{Pegged tilings and their duals}

McMullen in [McMu94] defines a pegged tiling as follows:

A tiling $\{Q(t) : t \in T\}$ is called pegged if with each tile $Q(t)$, $t \in T$, is associated a point $v^*(t) \in \mathbb{R}^n$, the peg of $Q(t)$, such that if the tile $Q(t')$ is adjacent to $Q(t)$, and so meets it on a facet $F$, then $v^*(t') - v^*(t)$ is an outer normal vector to $Q(t)$ at the facet $F$. The equation $x^T(v^*(t') - v^*(t)) = \alpha(t, t')$ defines the hyperplane supporting $F$.

Note that the pegs are defined up to a shift on an arbitrary vector. Hence we can suppose that $v^*(t^0) = 0$ for some $t^0 \in T$.

Recall that parallelotopes form a tiling $\{P(t) : t \in L\}$. Suppose that this tiling is pegged. Then the peg $v^*(t)$ relates to the vertex $t$ of the graph $G_L$. In particular, the pegs $v^*(0) = 0$, $v^*(\pm t_i)$ relate to the vertices $0, \pm t_i, i \in I_P$, of the graph $\nabla G(0)$. Since, by definition of pegs, $v^*(t_i) - v^*(0) = v^*(t)$ are proportional to $q_i$, and since the vectors $q_i$ are defined up to a scalar multiple, we set $q_i = v^*(t_i)$ if the tiling $\{P(t) : t \in L\}$ is pegged.

By definition of the lattice vector $t_i$, the tiles $P(t)$ and $P(t + t_i)$ are adjacent be the facet $F_i$ which is orthogonal to the facet vector $q_i$. Hence $v^*(t + t_i) - v^*(t) = \beta_i(t)q_i$ for some scalar $\beta_i(t) > 0$, where $\beta_i(0) = 1$.

In Lemma 6 below, we show that such defined facet vectors give a canonical representation of the parallelotope $P(0)$.

\textbf{Lemma 6} The following assertions are equivalent

(i) the tiling $\{P(t) : t \in L\}$ is pegged;

(ii) the parallelotope $P = P(0)$ is defined canonically.

\textbf{Proof.} (i)$\Rightarrow$(ii) Let $\{t_i, t_j, t_k\}$ be a 3-belt such that $t_i = t_j + t_k$. We show that $q_i = q_j + q_k$. Consider the hexagon of $\nabla G(0)$ corresponding to this belt. The vertices of the hexagon are $0, \pm t_i, \pm t_j$ and $\pm t_k$. The corresponding pegs are $0, v^*(t_r) = q_r$ and $v^*(-t_r)$, $r \in \{i, j, k\}$. All these pegs lie in the 2-plane spanned by the facet vectors $q_i, q_j, q_k$.

Consider the quadrangle with vertices $0 = v^*(0), v^*(t_j), v^*(t_k)$ and $v^*(t_j + t_k) = v^*(t_i)$. The pairs of opposite edges of this quadrangle are $(v^*(t_j) - v^*(0), v^*(t_i) - v^*(t_k))$ and
Lemma 7. The following assertions are equivalent

(i) a tiling $\mathcal{Q}$ is pegged;

(ii) a tiling $\mathcal{Q}$ has the dual tiling $\mathcal{Q}^\ast$.

8 Generatrissa of a tiling

For a tiling $\{P(t) : t \in L\}$, whose tiles are parallelotopes, Voronoi [Vo08] defines on the space $\mathbb{R}^n \otimes L$ a function $l(x; t)$. He calls this function generatrissa and defines it as follows:

(i) $l(x; 0) = 0$;

(ii) if $P(t')$ is adjacent to $P(t)$ by the facet $F_i$ defined by the equation $q_i^T x = \alpha_i$, then

$$l(x; t') = l(x; t) + q_i^T x - \alpha_i.$$  \hfill (9)

(Recall that, by (2), $q_i^T x \leq \alpha_i = q_i^T (t + \frac{1}{2}t_i)$ for $x \in P(t)$.) In fact, Voronoi uses vectors $-q_i, i \in \mathcal{I}_P$, and therefore defines $-l(x; t)$. Voronoi proves that for primitive canonically
defined paralleloptopes the conditions (i) and (ii) above determine uniquely \(l(x; t)\) for each \(t \in L\). The obtained generatrissa has the following property

\[
l(x; t^0) \geq l(x; t) \quad \text{for all } x \in P(t^0) \text{ and all } t^0, t \in L,
\]

with strict inequality if \(x \in \text{int}P(t^0)\).

This property implies that the function

\[
l(x) = \max_{t \in L} l(x; t)
\]

is a convex piecewise affine function on \(\mathbb{R}^n\).

Consider in \((n + 1)\)-dimensional space \(\mathbb{R}^n \oplus \mathbb{R}\) a convex surface defined as \(\{(x, z) : x \in \mathbb{R}^n, z = l(x)\}\). The main property of this surface is that its projection in the space containing the tiling \(\{P(t) : t \in L\}\) is this tiling.

But the above definition of generatrissa works for any pegged locally finite tiling (see, for example, [McMu94]).

For \(t \in \mathcal{T}\), define \(\varphi^*(t)\) as follows:

1. \(\varphi^*(t^0) = 0\);
2. \(\varphi^*(t) = \sum_{i=1}^{s} \alpha(t^i, t^{i+1})\), where \(t^s = t\), and \(P(t^0), P(t^1), \ldots, P(t^s) = P(t)\) is a chain such that \(P(t^i)\) and \(P(t^{i+1})\), \(0 \leq i \leq s - 1\), are adjacent by a facet.

Here \(P(t^i)\) lies but \(P(t^{i+1})\) does not lie in the halfspace \(\{x \in \mathbb{R}^n : x^T(v^*(t^{i+1}) - v^*(t^i)) \leq \alpha(t^i, t^{i+1})\}\).

It is proved in [McMu94] that the function \(\varphi^*(t)\) is well defined. In fact, it is sufficient to prove that if the chain \(P(t^0), P(t^1), \ldots, P(t^s) = P(t^0)\) is closed, then the sum in (2) is equal to 0. It is so if \(P(t^i)\), \(0 \leq i \leq s - 1\), have a common face \(F\), since, for \(x_0 \in F\), we have

\[
\alpha(t^i, t^{i+1}) = x_0^T(v^*(t^{i+1}) - v^*(t^i)), \quad 0 \leq i \leq s - 1.
\]

Now, any closed chain can be contracted to a point avoiding \((n - 3)\)-faces, and in contracting over an \((n - 2)\)-face we can appeal to the above reasoning.

So, the function

\[
f(x; t) = x^T v^*(t) - \varphi^*(t)
\]

is a generatrissa such that

\[
f(x; t) > f(x; t') \quad \text{for all } x \in \text{int}P(t) \text{ and all } t, t' \in \mathcal{T}.
\]

Obviously if a tiling has a generatrissa of the form (11), then this tiling is pegged. The essential part of the papers [Da59] and [Au87] is devoted to a proof of the following proposition.

**Proposition 3** The following assertions are equivalent for a tiling \(\{Q(t) : t \in \mathcal{T}\}\):

(i) the tiling is pegged with pegs \(v^*(t), t \in \mathcal{T}\);
(ii) the tiling has the generatrissa \(f(x; t) = x^T v^*(t) - \varphi^*(t)\).

The following general result is proved in [Da59] and [Au87].
Let we have an arbitrary parallelotope $P$ where additive on the variable $P$ the following generatissa $L$ related to $t$ takes the following form $v$ pegs $L(t)$.

If the case of parallelotopes $[Au87]$ constructs explicitly pegs. The proof of Voronoi: both the authors construct explicitly a generatissa. The author of Theorem 1 in $[Da59]$ is true for primitive parallelotopes. The proof of Theorem 1 in $[Vo08]$ is similar to the Theorem 1 a path connecting $t$. Then, using the recursive expression (12) and going out from $l(t)$ not supposing that $l(t)$ determines uniquely the generatissa $t$ pegs.

For parallelotopes, we know an explicit form of $\alpha_i$ in the recursion (9). In fact, since the point $t + \frac{1}{2}t_i$ belongs to the facet $F_i$, we have $\alpha_i = q_i^T(t + \frac{1}{2}t_i)$. Hence the recursion (9) takes the following form

$$l(x; t + t_i) = l(x; t) + q_i^T(x - (t + \frac{1}{2}t_i)).$$

(12)

Let we have an arbitrary parallelotope $P = P(t_0)$, and let a linear on $x$ function $l(x)$ be given. Then, using the recursive expression (12) and going out from $l(x; t) = l(x)$ by a path connecting $t^0$ and $t^1 = t^0 + t \in L$ we can, for all $t^1 \in L$, find a function $l(x; t^1)$ related to $L$.

Let $t = t(P) = \sum_{k=1}^s t_k$, where $(t_1, t_2, ..., t_s)$ is an arbitrary path $P$ in $G_L$ connecting $t^0$ with $t^1$. Let $q(P) = \sum_{k=1}^s q_k$, where each $q_k$ is associated to $t_k$. Obviously, $t(P) = t^1 - t^0$ does not depend on the path $P$.

Using (12) with $l(x; t) = l(x)$, and going along the path $P$ from $t^0$ we obtain

$$l(x; t^1) = l(x) + x^T q(P) - \phi(t^0, P),$$

where

$$\phi(t^0, P) = \sum_{k=1}^s q_k^T(t^0 + \sum_{r=1}^{k-1} t_r + \frac{1}{2} t_k),$$

(13)

and the sum $\sum_{r=1}^{k-1} t_r$ is empty for $k = 1$. (Cf., $l(x; t^1)$ with $f(x; t)$ in (11)).

Obviously, $q(P)$ is additive on $P$. It is easy to verify that the function $\phi(t^0, P)$ is additive on the variable $P$, too. In fact, let $P = (t_1, ..., t_s)$, $P' = (t_{s+1}, ..., t_w)$. We set $P + P' = (t_1, ..., t_s, t_{s+1}, ..., t_w)$. Hence the sum of two paths is its join. Let $t = t(P)$, $t' = t(P')$, $t^1 = t^0 + t = t^0 + \sum_{r=1}^{s} t_r$. Let $-P = (-t_s, -t_{s-1}, ..., -t_1)$ be the path from $t^1$ to $t^0$ opposite to $P$. Then

$$\phi(t^0, P + P') = \phi(t^0, P) + \phi(t^1, P'),$$

(14)
Lemma 8

\[ \phi(t^0, P + P') = (\sum_{k=1}^{s} + \sum_{k=s+1}^{w} )q_k^T(t^0 + \sum_{r=1}^{k-1} t_r + \frac{1}{2} t_k) = \]

\[ \phi(t^0, P) + \sum_{k=s+1}^{w} q_k^T(t^0 + \sum_{r=1}^{s} t_r + \sum_{r=s+1}^{k-1} t_r + \frac{1}{2} t_k) = \phi(t^0, P) + \phi(t^1, P'). \]

Now,

\[ \phi(t^1, -P) = \sum_{k=1}^{s} (-q_{s+1-k}^T)(t^1 - \sum_{r=1}^{k-1} t_{s+1-r} - \frac{1}{2} t_{s+1-k}). \]

Since \( t^1 = t^0 + \sum_{r=1}^{s} t_{s+1-r} \), for \( k \leq s \), we have

\[ t^1 - \sum_{r=1}^{k-1} t_{s+1-r} = t^0 + \sum_{r=1}^{s} t_{s+1-r} - \sum_{r=s+1}^{k-1} t_{s+1-r} = t^0 + \sum_{r=k}^{s} t_{s+1-r} = t^0 + \sum_{r=1}^{s+k-1} t_r. \]

If we do the substitution \( s+1-k \rightarrow k \) here and in \( \phi(t^1, -P; x) \), we obtain

\[ \phi(t^1, -P) = -\phi(t^0, P). \] (15)

Let \( C_1 = P_1 + P' \), \( C_2 = -P' + P_2 \), where \( P_1 \cap P_2 = \emptyset \). Let \( P_1 \) goes from \( t^0 \) to \( t^1 \), and the paths \( P', P_2 \) go from \( t^1 \) to \( t^0 \). Then \( C_1 \oplus C_2 = P_1 + P_2 \) is the sum modulo 2.

Lemma 8

\[ \phi(t^0, C_1 \oplus C_2) = \phi(t^0, C_1) + \phi(t^0, C_2). \]

Proof. Using (14) and (15), we have \( \phi(t^0, C_1) = \phi(t^0, P_1 + P') = \phi(t^0, P_1) + \phi(t^1, P') \) and \( \phi(t^0, C_2) = \phi(t^0, -P' + P_2) = \phi(t^0, -P') + \phi(t^1, P_2) = -\phi(t^1, P') + \phi(t^1, P_2) \). Hence

\[ \phi(t^0, C_1) + \phi(t^0, C_2) = \phi(t^0, P_1) + \phi(t^1, P_2) = \phi(t^0, P_1 + P_2) = \phi(t^0, C_1 \oplus C_2). \Box \]

Using Lemmas 8 and 5, for any quadrangle \( Q_{ij} \), we can represent \( \phi(t^0, Q_{ij}) \) as a sum of functions \( \phi(t, G(F_{n-2})). \)

Lemma 9 Let \( \phi(t^0, P) \) be given by (13). Then, for \( i, j \in I_P \) and \( P = Q_{ij} \),

\[ \phi(t^0, Q_{ij}) = q_i^T t_i - q_j^T t_j. \]

Proof. Using the equality (13) for \( P = Q_{ij} = (i, j, -i, -j) \), we obtain

\[ \phi(t^0, Q_{ij}) = q_i^T (t^0 + \frac{1}{2} t_i) + q_j^T (t^0 + t_i + \frac{1}{2} t_j) - q_i^T (t^0 + t_i + t_j - \frac{1}{2} t_i) - q_j^T (t^0 + t_i + t_j - t_i - \frac{1}{2} t_j) = q_j^T t_i - q_i^T t_j. \Box \]

Lemma 9 implies the following result.
Lemma 10 Let $i, j \in \mathcal{I}_P$, and let $\phi(t^0, \mathcal{P})$ be given by (13). The following assertions are equivalent:

(i) $\phi(t^0, Q_{ij}) = 0$;
(ii) $q^T_i t_j = q^T_j t_i$. 

Let $\mathcal{P} = (t_1, t_2, ..., t_s)$ be a path. Recall that $t(\mathcal{P}) = \sum_{k=1}^s t_k$, $q(\mathcal{P}) = \sum_{k=1}^s q_k$, and $\mathcal{I}(\mathcal{P})$ is the set of all indices $i \in \mathcal{I}_P$ of $t_i$ in the path $\mathcal{P}$.

Lemma 11 Let the equalities $q^T_i t_j = q^T_j t_i$ hold for all pairs $i, j \in \mathcal{I}(\mathcal{P})$. Then the function $\phi(t^0, \mathcal{P})$ is given by the expression

$$\phi(t^0, \mathcal{P}) = q^T(\mathcal{P})(t^0 + \frac{1}{2} t(\mathcal{P})).$$

**Proof.** Using the equalities $q^T_i t_j = t^T_i q_j$ for $i, j \in \mathcal{I}(\mathcal{P})$, we obtain the equality

$$q^T_k \sum_{r=1}^{k-1} t_r = t^T_k \sum_{r=1}^{k-1} q_r.$$

Using this equality and setting $\sum_{k=1}^s q_k = q(\mathcal{P})$, we rewrite $\phi(t^0, \mathcal{P})$ from (13) as follows

$$\phi(t^0, \mathcal{P}) = \sum_{k=1}^s q^T_k t^0 + \frac{1}{2} \sum_{k=1}^s q^T_k \sum_{r=1}^{k-1} t_r + \frac{1}{2} \sum_{k=1}^s q^T_k \left( \sum_{r=1}^{k-1} t_r + t_k \right) =$$

$$= q^T(\mathcal{P}) t^0 + \frac{1}{2} \sum_{k=1}^s t^T_k \sum_{r=1}^{k-1} q_r + \frac{1}{2} \sum_{k=1}^s q^T_k \sum_{r=1}^{k-1} t_r.$$

A permutation of the summation in the first double sum, gives

$$\frac{1}{2} \sum_{k=1}^s t^T_k \sum_{r=1}^{k-1} q_r = \frac{1}{2} \sum_{k=1}^{s-1} q^T_k \sum_{r=k+1}^s t_r.$$

Recall that $\sum_{k=1}^s t_k = t(\mathcal{P})$. Hence $\phi(t^0, \mathcal{P})$ takes the form

$$\phi(t^0, \mathcal{P}) = q^T(\mathcal{P}) t^0 + \frac{1}{2} \sum_{k=1}^s q^T_k \sum_{r=1}^s t_r = q^T(\mathcal{P})(t^0 + \frac{1}{2} t(\mathcal{P})).$$

So, we obtain the wanted expression. 

Using Lemma 11, we can prove the following important result.

Lemma 12 The following assertions are equivalent:

(i) a paralleotope $P$ is defined canonically;
(ii) $q^T_i t_j = q^T_j t_i$ for all $i, j \in \mathcal{I}_P$. 

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Proof. (i)⇒(ii). If $P$ is defined canonically, then Proposition 1 assert that $q_i^T t_j = q_j^T t_i$ for $i, j \in I(G(F^{n-2}))$. Obviously, $q(Q_{ij}) = 0$, and by Lemma 3, $q(\Delta) = 0$ if $\Delta = G(F^{n-2})$. Hence $q(G(F^{n-2})) = 0$ and Lemma 11 implies that $\phi(t^0, G(F^{n-2})) = 0$. Lemma 5 asserts that each quadrangle $Q_{ij}$ is a sum modulo 2 of circuits $G(F^{n-2})$. By Lemma 8 we obtain that $\phi(t^0, Q_{ij}) = 0$. Now, Lemma 10 gives the wanted equality for all pairs $i, j$.

(ii)⇒(i) By Lemma 2, $q_i = Dt_i$ for all $i \in I_P$. Hence any linear equality between lattice vectors implies the corresponding equality between the associated facet vectors. This implies that the facet vectors are defined canonically with respect to each belt of $P$, i.e., $P$ is defined canonically.

The obtained function $\phi(t^0, P)$ depends on the chosen path $P$. Since $\phi(t^0, P)$ satisfies the condition (15), it does not depend on $P$ if and only if $\phi(t^0, C) = 0$ for every circuit $C$. In Lemma 13 below we give conditions when this property is true.

Lemma 13 Let $C$ be a circuit and $t^0 \in C$. The following assertions are equivalent:

(i) for all $t^0$ and all circuits $C \ni t^0$, the function $\phi(t^0, C) = 0$;

(ii) the equalities $q_i^T t_j = q_j^T t_i$ hold for all pairs $i, j \in I_P$.

Proof. (i)⇒(ii) The item (i) implies that $\phi(t^0, Q_{ij}) = 0$ for any quadrangle $Q_{ij}$. Now Lemmas 10 gives wanted implication.

(ii)⇒(i) Recall that if $q_i^T t_j = q_j^T t_i$ for all $i, j \in I_P$, then $q_i = Dt_i$. Since the sum $\sum_{k=1}^n t_k = t(C) = 0$ for any circuit $C$, the associated vector $q(C) = \sum_{k=1}^n q_k = \sum_{k=1}^n Dt_k = Dt(C)$ is also equal to zero. Hence $\phi(t^0, C) = q^T(C)(t^0 + \frac{1}{2}t(C)) = 0$. □

10 Generatrissa for parallelotopes

By Lemma 13 the generatrissa $l(x; t) = q^T(t)(x - \frac{1}{2}t)$ is uniquely determined for all $x \in \mathbb{R}^n$ and $t \in L$. This is the function $q^T(P)x - \phi(0, P)$, where $q(P) = q(t)$ and $\phi(0, P) = \frac{1}{2}q^T(t)t$, both do not depend on $P$. In other words, we have the following assertion

Lemma 14 The following assertions are equivalent:

(i) the function $q^T(t)(x - \frac{1}{2}t)$, $t \in L$, is the generatrissa of the tiling obtained by translations of a parallelotope $\bar{P}$;

(ii) the equalities $q_i^T t_j = q_j^T t_i$ hold for all $i, j \in I_P$. □

Recall that the facet vector $q(t)$ is associated to the lattice vector $t$, when (8) is true. Hence, by Lemma 2, $q(t) = Dt$ and we see that $l(x; t)$ is given by

$$l(x; t) = t^T D(x - \frac{1}{2}t).$$

We see that $l(x; 0) = 0$. Using standard arguments of [Vo08] and [Zh29], one can prove that $l(x; t) \leq 0$ for all $x \in P(0)$ and all $t \in L$, with strict inequality for $x \in \text{int}P(0)$. Hence (10) (with $t^0 = 0$) is true. But Lemma 2 does not asserts that the matrix $D$ is positive definite. The next lemma proves that $D$ is positive definite if the vectors $q_i$ and $t_i$ are related to a parallelotope.

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Lemma 15 Let $P(0)$ be a parallelotope with center in origin, and let (8) is true. Let $D$ be a non-singular matrix. The following assertions are equivalent:

(i) the inequality $l(x; t) = t^T D (x - \frac{1}{2} t) < 0$ holds for all $x \in \text{int} P(0)$ and all $t \in L$, $t \neq 0$;

(ii) the matrix $D$ is positive definite.

Proof. (i)$\Rightarrow$(ii) For $x = 0$ and $t \neq 0$, we have $-l(0; t) = \frac{1}{2} t^T D t > 0$. This inequality holds for all vectors $t \in L - \{0\}$. Any rational combination of basic vectors of $L$ is equal to $\frac{1}{p} t$ for some $t \in L$ and an integer $p$. We obtain that the above inequality holds also for rational vectors. By continuity, this inequality holds for all $t \in \mathbb{R}^n$. This implies, that the matrix $D$ is positive definite.

(ii)$\Rightarrow$(i) Let $D$ be positive definite. Consider the quadratic function $x^T D x - 2 l(x; t) = (x - t)^T D (x - t) = f(x - t)$. Using the quadratic form $f(x)$ we can define the parallelotope

$$P_f(t^0) = \{ x \in \mathbb{R}^n : l(x; t^0) - l(x; t) \geq 0, \ t \in L \}.$$ 

We show that $P_f(t^0) = P(t^0)$. Then for $t^0 = 0$ we will have $l(x; t) \leq 0$, $t \in L - \{0\}$, and $l(x; t) < 0$ for interior points of $P(t^0)$. The infinite system inequalities

$$l(x; t^0) - l(x; t) \geq 0, \ t \in L,$$

contains the subsystem for $t = t^0 \pm t_i$, $i \in \mathcal{I}_P$, describing the parallelotope $P(t^0)$ of type (2). Hence $P_f(t^0) \subseteq P(t^0)$ for every $t^0 \in L$. But we have here an equality. In fact, if there is $t \in L$ such that $P_f(t) \subset P(t)$ strictly, then there is an adjacent to $P_f(t)$ parallelotope $P_f(t')$ such that $P_f(t')$ and $P(t)$ have a common interior point. Since $P(t')$ contains $P_f(t')$, the parallelotopes $P(t)$ and $P(t')$ have a common interior point. This is a contradiction.

The above proof of Lemma 15 is also a proof of the following

Lemma 16 The following assertions are equivalent

(i) the parallelotope $P$ is a Voronoi polytope with respect to the positive quadratic form $f(x) = x^T D x$;

(ii) the function $l(x; t) = t^T D (x - \frac{1}{2} t)$ is a generatrissa of the tiling $\{ P(t) : t \in L \}$.

11 Main Theorem

If we collect Lemmas 1, 2, 6, 7, Proposition 3 and Lemmas 12, 14, 16 together, we obtain

Theorem 2 Let $P$ be a parallelotope defined by facet vectors $q_i$, and defining lattice vectors $t_i$, $i \in \mathcal{I}_P$. Let $A$ be a non-degenerate $n \times n$ matrix, and $D = A^T A$. Then the following assertions are equivalent:

(i) Voronoi’s conjecture holds for $P$, i.e. the affine transformation $x \rightarrow Ax$ transforms the parallelotope $P$ into a Voronoi polytope;

(ii) the parallelotope $P$ is a Voronoi polytope with respect to the positive quadratic form $f(x) = x^T D x$;
(iii) the parallelotope $P$ is defined canonically;
(iv) the equality $q_i = Dt_i$ holds for all $i \in \mathcal{I}_P$;
(v) for all pairs $i, j \in \mathcal{I}_P$, the equalities $t_i^T q_j = q_i^T t_j$ hold;
(vi) the tiling $\{P(t) : t \in L\}$ is pegged with pegs $v^*(t) = Dt$;
(vii) the function $l(x; t) = t^T D (x - \frac{1}{2} t)$ is a generatissa of the tiling $\{P(t) : t \in L\}$;
(viii) the tiling $\{P(t) : t \in L\}$ has a dual tiling.

Theorem 2 implies a result of [MRS95] that there is a unique (up to isomorphism of $P$) map which transforms a primitive parallelorope $P$ into a Voronoi polytope. Our Theorem 2 gives an explicit matrix $A$ of the corresponding affine map. Therefore we have the following

Proposition 4 If a parallelotope $P$ is affinely equivalent to a Voronoi polytope, then this affinity is uniquely (up to the automorphism of $P$) determined by the parallelotope $P$. 

References

[Al54] A.D.Aleksandrov, *On filling of space by polytopes*, Vestnik Leningradskogo Univ., Ser. math, phys. and chem. 9 (1954) 33–43 (in Russian).

[Au87] F.Aurenhammer, *Affine equivalence of cell complexes in $\mathbb{R}^d$ and convex polyhedra in $\mathbb{R}^{d+1}$*, Discrete Comput. Geom. 2 (1987) 49–64.

[Er99] R.M.Erdahl, *Zonotopes, dicings and Voronoi’s conjecture on parallelohedra*, Europ.J.Combinatorics 20 (1999) 527–549.

[Da59] C.Davis, *The set of non-linearity of a convex piecewise-linear function*, Scripta Mathematica 24 (1959) 219–228.

[De29] B.N.Delaunay, *Sur la partition régulière de l’espace á 4 dimension*, Izvestia AN SSSR, otdel. fiz.-mat. nauk, No1 (1929) 79–110, No2 (1929) 145–164.

[MRS95] L.Michel, S.S.Ryshkov and M.Senechal, *An extension of Voronoi’s theorem on primitive parallelotopes*, Europ.J.Combinatorics 16 (1995) 59–63.

[McMu75] P.McMullen, *Space tiling zonotopes*, Mathematica 22 (1975) 202–211.

[McMu80] P.McMullen, *Convex bodies which tile space by translation*, Mathematica 27 (1980) 113–121.

[McMu94] P.McMullen, *Duality, sections and projections of certain Euclidean tilings*, Geometricae Dedicata 49 (1994) 183–202.

[RR97] S.S.Ryshkov, K.A.Rybnikov, *The theory of quality translation with applications to tilings*, Europ.J.Combinatorics 18 (1997) 431–435.

[Ve54] B.A.Venkov, *On a class of Euclidean polytopes*, Vestnik Leningradskogo Univ., Ser. math, phys. and chem. 9 (1954) 11–31 (in Russian).

[Vo08] G.F.Voronoi, *Nouvelles applications des paramètres continus à la théorie de formes quadratiques - Deuxième mémoire*, J. für die reine und angewandte Mathematik, 134 (1908) 198–287, 136 (1909) 67–178.

[Zh29] O.K.Zhitomirskii, *Verschärfung eines Satzes von Voronoi*, Zhurnal Leningradskogo Math.Obshtchestva 2 (1929) 131–151.

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