Solitary and periodic solutions of the generalized Kuramoto - Sivashinsky equation

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Abstract

The generalized Kuramoto - Sivashinsky in the case of the power non-linearity with arbitrary degree is considered. New exact solutions of this equation are presented.

Keywords: Exact solution, nonlinear differential equation, Kuramoto - Sivashinsky equation

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1 Introduction

In this paper we present exact solutions of the generalized Kuramoto - Sivashinsky equation

\[ u_t + u^m u_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 \]  (1.1)

Nonlinear evolution equation (1.1) at \( m = 1 \) has been studied by a number of authors from various viewpoints. This equation has drawn much attention not only because it is interesting as a simple one-dimensional nonlinear evolution equation including effects of instability, dissipation and dispersion but also it is important for description in engineering and scientific problems. Equation (1.1) in work [1] was used for explanation of the origin of persistent wave propagation through medium of reaction - diffusion type. In paper [2] equation (1.1) was obtained at \( m = 1 \) for description of the nonlinear evolution of the disturbed flame front. We can meet the application of equation (1.1) for studying of motion of a viscous incompressible flowing down an inclined plane [3-5]. Mathematical model for consideration dissipative waves in plasma physics by means of equation (1.1) was presented in [6]. Elementary particles as solutions of the Kuramoto - Sivashinsky equation were studied in [7]. Equation (1.1) at \( m \neq 1 \) also can be used for the description of physical applications. For a example this equation were derived for description of nonlinear long waves in viscous - elastic tube [8].
Exact solution of equation (2.2) in the form of the solitary wave at \( m = 1 \) and at \( \beta = 0 \) were obtained in [10]. Solitary wave solutions of equation (2.2) at \( m = 1 \) in the case \( \beta \neq 0 \) were found in works [10–12] for cases:

\[
\alpha = \frac{\beta^2}{16 \gamma}, \quad \alpha = \frac{73 \beta^2}{256 \gamma}, \quad (1.2)
\]

Other forms of these solutions were presented in works [13–17]. Periodical solutions of equation (2.2) were found in [18, 19] at \( \alpha = \frac{\beta^2}{16 \gamma} \).

Exact solutions of equation (2.2) at \( m = 2 \) recently were obtained [8] for the four cases:

\[
\alpha = \frac{2 \beta^2}{25 \gamma}, \quad \alpha = \frac{71 \beta^2}{225 \gamma}, \quad \alpha = \frac{121 \beta^2}{55 \gamma}, \quad \alpha = \frac{374 \beta^2}{2025 \gamma}.
\]

(1.3)

The aim of this paper is to search for exact solutions of equation (1.1) in general case of arbitrary degree \( m \). We present exact solutions of equation (1.1) at \( m \neq -1, m \neq -3 \) and \( m \neq 0 \). Some of these solutions are new.

2 Solitary waves of equation (1.1)

Using traveling wave

\[ u(x, t) = u(z), \quad z = x - C_0 t \]

from equation (1.1) we have

\[ w_{zzz} + \sigma w_{zz} + w_z - C_0 w + \frac{1}{m+1} w^{m+1} = 0, \quad m \neq -1 \quad (2.2) \]

Assuming in (2.2)

\[ w(z) = v(z)^{\frac{1}{m+1}} \]

we have equation in the form

\[
\gamma m^3 v^2 v_{zzz} + \gamma m^2 v^2 v_{zzz} + 3 \gamma m v v_z v_{zz} + \beta m^3 v^2 v_{zz} - 2 \gamma m v z^3 -
-3 \gamma m^3 v v_z v_{zz} + \gamma v_z^3 - \gamma m^2 v_z^3 + 2 \gamma m^4 v_z^3 + \beta m v v_z^2 -
-\beta m^3 v v_z^2 + \beta m^2 v^2 v_{zz} + \alpha m^3 v^2 v_z + \alpha m^2 v^2 v_z + m^3 v^4 -
-C_0 m^4 v^3 - C_0 m^3 v^3 = 0
\]

(2.4)

Study of analytical properties of equation (2.4) allows us to determine that in the general case the meromorphic solutions of equation (2.3) can be found taking into account two cases:

\[
\alpha = \frac{\beta^2 m}{\gamma (m+3)^2}, \quad \alpha = \frac{\beta^2 (2 m^2 + 18 m + 27)}{9 \gamma (m+3)^2} \quad (2.5)
\]

2
Consider the first case. The pole order of the solution of equation (2.4) is equal to three therefore we look for exact solutions of equation (2.4) in the form
\[ v(z) = A_0 + A_1 Y(z) + A_2 Y(z)^2 + A_3 Y(z)^3, \]  
where \( Y(z) \) is solution of equation with the first order pole. Let us take simplest equation in the form \[20, 21\]
\[ Y_z = -Y^2 + b. \]  
(2.7)
Substituting (2.6) into (2.4) and taking into account equation (2.7) we have
\[ b = \frac{\beta^2 m^2}{4 \gamma^2 (m + 3)^2}. \]  
(2.8)
We have also the following values of coefficients in (2.6)
\[ A_3 = \frac{3 \gamma (2m + 3)(m + 3)(m + 1)}{m^3}, \]
\[ A_2 = -\frac{3 (2m + 3)(m + 1) \beta}{2 m^2}, \quad A_1 = -\frac{3 (2m + 3)(m + 1) \beta^2}{4 (m + 3) m \gamma}, \]  
(2.9)
\[ C_0 = \frac{2 (m + 2) \beta^3}{\gamma^2 (m + 3)}, \quad A_0 = \frac{3 (2m + 3)(m + 1) \beta^3}{8 \gamma^2 (m + 3)^2}. \]
We have solutions of equation (2.4) in the form
\[ Y(z) = \frac{\beta m}{2 \gamma (m + 3)} \tanh \left\{ \frac{\beta m (z - z_0)}{2 \gamma (m + 3)} \right\}. \]  
(2.10)
Solution of equation (2.2) takes the form
\[ w(z) = \left( \frac{3 (2m + 3)(m + 1) \beta^3}{8 \gamma (m + 3)} \right)^{\frac{1}{m}} \left( 1 - \tanh \left\{ \frac{\beta m (z - z_0)}{2 \gamma (m + 3)} \right\} \right)^{\frac{1}{m}} \]
\[ \left( 1 + \tanh \left\{ \frac{\beta m (z - z_0)}{2 \gamma (m + 3)} \right\} \right)^{\frac{1}{m}}, \quad m \neq 0, \quad m \neq -3, \quad m \neq -1. \]  
(2.11)
This is new solution of equation (1.1). Solution (2.11) at \( m = 1 \) takes the form of the solitary wave that was obtained in [10]. In the case \( m = 2 \) this solution corresponds to exact solution of equation (1.1) published in [8].
Consider the second case. Solutions of equation (2.4) we look for taking into consideration the transformation (2.6) again. Function \( Y(z) \) satisfies equation (2.8) as well. In this case we obtain the expression for the parameter \( b \) in the form
\[ b = \frac{\beta^2 m^2}{36 \gamma^2 (m + 3)^2}. \]  
(2.12)
We also obtain coefficients $A_3$, $A_2$, $A_1$, $C_0$ and $A_0$ as following

\[
A_3 = \frac{3 \gamma (2m + 3)(m + 3)(m + 1)}{m^3},
\]

\[
A_2 = -\frac{3 (2m + 3)(m + 1) \beta}{2 m^2}, \quad A_1 = \frac{(2m + 3)(m + 1) \beta^2}{4 \gamma m (m + 3)},
\] (2.13)

\[
C_0 = -\frac{(2m + 3) \beta^3}{9 (m + 3)^2 \gamma^2}, \quad A_0 = -\frac{\beta^3 (2m + 3)(m + 1)}{72 (m + 3)^2 \gamma^2}.
\]

Using these coefficients we have solution of equation (2.2). It takes the form

\[
w(z) = \left(\frac{\beta^3 (2m + 3)(m + 1)}{72 \gamma^2 (m + 3)^2}\right)^{\frac{1}{m}} \left(\tanh\left\{\frac{\beta m (z - z_0)}{6 \gamma (m + 3)}\right\} - 1\right)^{\frac{1}{m}},
\] (2.14)

\[m \neq 0, \quad m \neq -1, \quad m \neq -3.\]

Other forms of this solution were found in works [22–24]. At $m = 1$ from (2.14) we have one of the solitary solutions of work [10]. In the case $m = 2$ we get solution 8.

3 Periodical waves of equation \((1.1)\)

Let us find periodical solutions of equation (2.2) taking into consideration the formula \[18\]

\[
w(z) = A_0 + A_1 R + A_2 R_z, \quad R \equiv R(z),
\] (3.1)

taking into account the relation for the parameter $\alpha$

\[
\alpha = \frac{\beta^2 m}{\gamma (m + 3)^2}. \quad (3.2)
\]

We assume that $R(z)$ is solution of equation for the Weierstrass function

\[
R_{zz} + 6 R^2 - a R - b = 0
\] (3.3)

From equation (3.3) we have equation

\[
R_z^2 + 4 R^3 - a R^2 - 2 b R + d = 0
\] (3.4)

Substituting (3.1) into equation (2.3) and taking into account equations (3.3)
and (3.4) we have
\[ b = \frac{m^4 \beta^4 - \gamma^4 a^2 (m + 3)^2}{24 \gamma^4 (m + 3)^4}, \quad A_2 = \frac{3 \gamma (2m + 3) (m + 3) (m + 1)}{2m^3}, \]
\[ A_1 = \frac{3 \beta \gamma (2m + 3) (m + 1)}{2m^2}, \quad C_0 = \frac{2 \beta^3 (m + 2)}{\gamma^2 (m + 3)^3}, \] (3.5)
\[ A_0 = \frac{\beta (2m + 3) (m + 1) (\beta^2 m^2 - \gamma^2 (m + 3)^2 a)}{8 \gamma^2 m^2 (m + 3)^2}, \]
\[ \gamma^2 (m + 3)^2 \]
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In the case \( m = 1 \) we have \( \alpha = \frac{\beta^2}{16\gamma} \). Solution of equation (1.1) takes the form \[18\]

\[ y(z) = C_0 - \frac{5}{4} \beta a - \frac{1}{64} \frac{\beta^3}{\gamma^2} + 15 \beta R + 60 \gamma R_z \] \[(3.10)\]

where \( R(z) \) satisfies equation for the Weierstrass function

\[
R_z^2 + 4 R^3 - a R^2 - 2 \left( \frac{\beta^4}{6144 \gamma^4} - \frac{a^2}{24} \right) R + \\
+ \frac{13 \beta^6}{4423680 \gamma^6} + \frac{a \beta^4}{36864 \gamma^4} - \frac{C_0^2}{2160 \gamma^2} - \frac{a^3}{432} = 0
\] \[(3.11)\]

In the case \( m = 3 \) we have \( \alpha = \frac{\beta^2}{12\gamma} \). Periodic solution of equation (1.1) in this case can be written in the form

\[ y(z) = \left( 3C_0 - \frac{1}{2} \beta a - \frac{1}{72} \frac{\beta^3}{\gamma^2} + 6 \beta R + 12 \gamma R_z \right) \] \[(3.12)\]

where \( R(z) \) satisfies equation for the Weierstrass function in the form

\[
R_z^2 + 4 R^3 - a R^2 - 2 \left( \frac{C_0 \beta}{8 \gamma^2} - \frac{11 \beta^4}{3456 \gamma^4} - \frac{a^2}{24} \right) R - \\
- \frac{11 \beta^3 a}{20736 \gamma^4} + \frac{13 \beta^6}{373248 \gamma^6} + \frac{C_0 \beta^3}{1728 \gamma^4} + \frac{C_0 \beta a}{48 \gamma^2} - \frac{C_0^2}{16 \gamma^2} - \frac{a^3}{432} = 0
\] \[(3.13)\]

This solution of equation (1.1) is a new. Let us note that periodical solutions of equation (1.1) are appeared at \( m = 1 \) and at \( m = 3 \) when there is meromorphic solutions in the form of solitary waves of equation (1.1).

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