Finite Temperature Behavior of the ν = 1 Quantum Hall Effect in Bilayer Electron Systems

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An effective field theoretic description of ν = 1 bilayer electron systems stabilized by Coulomb repulsion in a single wide quantum well is examined using renormalization group techniques. The system is found to undergo a crossover from a low temperature strongly correlated quantum Hall state to a high temperature compressible state. This picture is used to account for the recent experimental observation of an anomalous transition in bilayer electron systems (T. S. Lay, et al. Phys. Rev. B 50, 17725 (1994)). An estimate for the crossover temperature is provided, and it is shown that its dependence on electron density is in reasonable agreement with the experiment.

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I. INTRODUCTION AND SUMMARY

The novel structure of bilayer electron systems (BLESs) has recently garnered considerable attention \cite{1,2,3}. These systems have proven to be appropriate candidates for probing quantum phases of electron systems, such as the quantum Hall effect (QHE). A BLES can be made in a double quantum well, where a high and hard-wall barrier leads to formation of two separate layers of electrons. It may also be realized in a single wide quantum well (SWQW), in which the barrier separating the electron layers originates from Coulomb repulsion in the well \cite{4}. The low energy effective theory for a BLES can be described as an XY spin model with an in-plane magnetic field, in which the spin-ordered phase corresponds to a correlated quantum Hall state \cite{5,6,7}. In this spin analogy, local spin orientations encode the coherence between the two layers, the spin stiffness is a measure of loss of exchange and correlation energy corresponding to spatial variation of the relative coherence, and the magnetic field is given by the tunneling amplitude. In the absence of the tunneling term, the system exhibits a finite temperature Kosterlitz-Thouless (KT) phase transition \cite{10}, which destroys the phase coherence and thus the QHE \cite{11,12}. For any non-zero (but small) value of the tunneling parameter, however, the transition is known to be smoothed out into a crossover, from a low temperature saturated ferromagnetic regime to a high temperature disordered regime \cite{13,14,15}.

Recently, Lay et al. have reported an experimental observation of a finite temperature quantum Hall “phase transition” in a BLES, realized in a SWQW \cite{1}. They find that collapse of the QHE gap takes place at a temperature $T^*$, whose value decreases when the electron areal density $N_e$ is increased.

Here we study the effect of thermal fluctuations on the quantum Hall behavior of a BLES in a SWQW. We sketch the derivation of the effective field theory for this system and argue that an additional Ising-like symmetry-breaking term should be added to the Hamiltonian. We provide numerical estimates for the coupling constants of the theory using Hartree-Fock (HF) variational wave-function of the many-particle system combined with the self-consistent local spin density approximation (LSDA) for single particle states. We then study finite temperature behavior of the system using renormalization group (RG) techniques. We argue that the system undergoes a crossover from a low temperature ordered regime to a high temperature disordered regime and that the experimentally observed $N_e$-dependence of $T^*$ is related to the monotonically decreasing iso-spin stiffness with increasing electron density in the SWQW.

II. EFFECTIVE HAMILTONIAN

The low energy state of a BLES at $\nu = 1$ can be well described by the following variational wave function \cite{16,17,18}

$$|\Psi\rangle = \prod_X \left( \hat{c}_{X\uparrow}^\dagger + e^{i\phi(X)} \hat{c}_{X\downarrow}^\dagger \right) |0\rangle$$

(1)

where the up or down chiral iso-spin states denote localized electrons in the left or right side of the well, respectively, and $X$ is a quantum number such as the Landau gauge orbital guiding center. The phase angle field $\phi(X)$ denotes the relative local coherence between the two electron layers, and is well known to entail lowest energy excitations of the BLES \cite{19,20}. In Eq.(1) we have neglected
the real-spin degree of freedom of the BLES. This approximation is justified by the self-consistent LSDA calculations yielding a unique, fully spin-polarized solution at $\nu = 1$ for the whole range of electron densities used in the experiment \([1]\). The calculated Zeeman splitting, $\Delta_s$, is strongly enhanced by the exchange-correlation energy and always larger than the gap between two lowest energy levels in the SWQW, as shown in Table I.

The general form of the effective Hamiltonian for a BLES in a SWQW has been obtained in Ref. [4], where it has been assumed that the BLES can be described effectively by the Hilbert space of the two lowest subbands. The effective energy functional can be written as $\mathcal{H}_{\text{eff}} = V_{\text{HF}} + T$, where $T$ is the tunneling term and the microscopic HF Coulomb energy of the BLES $V_{\text{HF}} \equiv \langle \Psi | V | \Psi \rangle$ is obtained as

$$V_{\text{HF}} = -\frac{1}{4} \sum_{X_1,X_2} V_{X_1,X_2,X_2,X_1}^\dagger (m_x(X_1)m_x(X_2))$$

+ $m_y(X_1)m_y(X_2)] + V_{X_1,X_2,X_1}^\dagger (m_x(X_1)m_x(X_2) - m_y(X_1)m_y(X_2)), \tag{2}$

with $m(X) = (\cos \phi(X), \sin \phi(X))$. The Coulomb energy has been evaluated using the Landau gauge, where

$$V_{X_1,X_2,X_2,X_1}^{\sigma_1\sigma_2\sigma_3\sigma_4} = \int \frac{d^2q}{(2\pi)^2} V_{X_1,X_2,X_2,X_1}^{\sigma_1\sigma_2\sigma_3\sigma_4} (q) e^{-q^2\ell_0^2/2} \times \delta \left( \frac{X_1 - X_2}{\ell_0} - q \right), \tag{3}$$

and

$$V_{X_1,X_2,X_2,X_1}^{\sigma_1\sigma_2\sigma_3\sigma_4} (q) = \frac{2\pi e^2}{c q} \int d\zeta \int d\bar{\zeta} \psi_{\sigma_1}^*(\zeta) \psi_{\sigma_2}^*(\bar{\zeta}) \times \psi_{\sigma_3}^*(\zeta) \psi_{\sigma_4}^*(\bar{\zeta}) e^{-q|\zeta - \bar{\zeta}|}. \tag{4}$$

are the appropriate form factors, $\epsilon$ is the dielectric constant of the host semiconductor, and $\ell_0 = \sqrt{\hbar c/eB}$ is the magnetic length.

In the continuum limit, the effective energy functional has a gradient expansion that reads

$$\mathcal{H}_{\text{eff}} = \int d^2r \left[ \frac{\rho_s}{2} (\nabla \phi)^2 - \frac{t}{2\pi \ell_0} \cos \phi - \frac{\kappa^2}{2\pi \ell_0^2} \cos 2\phi \right]. \tag{5}$$

in which we have (only) neglected higher derivative terms. The origin of the iso-spin stiffness $\rho_s$, and $\kappa^2$ that corresponds to “pair hopping”, is the HF loss of exchange energy

$$\rho_s = \frac{\ell_0^2}{32\pi^2} \int dq \ q^3 (V^{\dagger\dagger\dagger\dagger}(q) - V^{\dagger\dagger\dagger\dagger}(q)) e^{-q^2\ell_0^2/2},$$

$$\kappa^2 = \frac{1}{8\pi} \int dq q V^{\dagger\dagger\dagger\dagger}(q) e^{-q^2\ell_0^2/2}. \tag{6}$$

and $t$ is the tunneling amplitude, defined as one-half of the spacing between two lowest energy levels in the SWQW. For the BLESs with large enough overlap integral in the middle of the well $V^{\dagger\dagger\dagger\dagger}$, and thus $\kappa^2$, are significant. Values of the couplings $\rho_s$, $t$, and $\kappa^2$, of the layer separation $d$, and of the Zeeman splitting calculated for the sample used by Lay et al. [4] are presented in Table I.

The first term in Eq.\([3]\) represents the usual, rotationally invariant superfluid exchange coupling, which yields a spontaneous phase coherent state \([12,13]\). (Note that the field $\phi$ is compact.) The above Hamiltonian is thus an XY-model with two symmetry-breaking terms: the second and third terms in Eq.\([5]\) represent a uniform in-plane magnetic field and an Ising-like anisotropy, respectively. In the absence of the tunneling and the Ising-like terms, the system exhibits a finite temperature KT phase transition \([14]\), which destroys the phase coherence and thus the QHE \([15]\).

### III. RENORMALIZATION GROUP APPROACH

To study the XY model with the symmetry breaking terms, we follow closely the approach by José et al. \([16]\). Finite temperature behavior of the system is described by the partition function

$$Z = \int_{0 \leq \phi < 2\pi} D\phi \ e^{-\mathcal{H}_{\text{eff}}/k_BT}. \tag{7}$$

The essence of the symmetry-breaking terms can be captured by performing Villain expansions, which introduce discrete Coulomb-like charge species, called type-1 and type-2 vortices \([11,12]\). On the other hand, the compactness of the field $\phi$ is accounted for by introducing another type of vortex, called type-0 vortex, which describes singular behavior of an otherwise noncompact field $\phi'$; a combination that serves as a substitute for $\phi$ \([12,13]\). The latter vortex introduces a new coupling constant, namely the fugacity $y_0 = e^{-E_c/k_BT}$, which is controlled by the core energy of a type-0 vortex $E_c = 2\pi \rho_s c_{\varepsilon}$, where $c_{\varepsilon}$ is a numerical constant close to 1 \([14]\). The corresponding three-species coupled Coulomb gas can then be studied using standard renormalization group (RG) techniques.
to the leading order in the fugacities. A scaling argument shows that the coupling $y_2$ is always subleading compared to $y_1$. We thus set it to zero for the moment to simplify the RG picture of the problem, and will comment on the effects due to a nonzero $y_2$ later. The RG flow diagram for Eq. (8) is shown in Fig. 1.

The system described by Eq. (8) (for $\kappa^2 = 0$) is known to have a duality in the parameter domain where the Villain expansion is applicable. The duality maps the high temperature regions to the low temperature ones, with the roles of the fugacities being exchanged and temperature being inverted $\frac{1}{2}$. This necessitates the existence of self-dual points, i.e., those which remain invariant under the duality transformation, at some intermediate temperature. It is easy to see from Eq. (8) that $K^{-1} = 2\pi$ and $y_0 = y_1$ define the line of self-dual points, which is denoted as path 1 in Fig. 1. We note that in Fig. 1, we have only sketched the RG flow structure for $K^{-1} < 2\pi$, and that the corresponding behaviors in the $K^{-1} > 2\pi$ region can be understood using this duality.

The above RG equations show that there are three different regions in the parameter space: (I) $0 < K^{-1} < \pi/2$, in which $y_0$ is irrelevant and $y_1$ is relevant, (II) $\pi/2 < K^{-1} < 8\pi$, in which both $y_0$ and $y_1$ are relevant, and (III) $K^{-1} > 8\pi$, in which $y_0$ is relevant and $y_1$ is irrelevant. In region I, the symmetry breaking term is dominant, as illustrated by path 2 in Fig. 1 and the system is in a locked-in regime. While type-0 vortices are bound in pairs, type-1 vortices are unbound. On the other hand, region III (that is dual to region I) is a disordered regime, in which type-0 vortices are unbound and type-1 vortices are bound in pairs.

Region II consists of four different areas. Path 3 shows a typical behavior for region IIa defined by $\pi/2 < K^{-1} < 2\pi$, and $y_1 > y_0$. While $K^{-1}$ is monotonically decreasing under the RG flow, both $y_0$ and $y_1$ initially tend to increase with $y_1$ being always exponentially dominant compared to $y_0$, until the flow eventually passes onto region I. Although there are always some unbound type-0 vortices, the overwhelmingly larger number of free type-1 vortices ensures that the system is still in the locked-in regime.

Region IIb, which is defined by $\pi/2 < K^{-1} < 2\pi$, and $y_1 < y_0$, is a crossover domain. During the early stages of the RG flow, both $y_0$ and $y_1$ increase to approach the $y_0 = y_1$ surface, while $K^{-1}$ is also increasing. The paths seem to be approaching the self-dual line asymptotically as if they are being attracted to an “intermediate temperature fixed point.” However, at some point they will eventually exit the region, and find their ways to either the low temperature (such as path 4) or the high temperature (such as path 5) regions. The fact that the renormalized fugacities approach the $y_0 = y_1$ surface means that there are about as many unbound type-0 and type-1 vortices, which explains why this is a crossover region. Finally, the behavior of regions IIc ($2\pi < K^{-1} < 8\pi$, and $y_1 > y_0$) and IId ($2\pi < K^{-1} < 8\pi$, and $y_1 > y_0$), can be understood using the duality.

For an XY model in a magnetic field, a crossover from a low temperature locked-in regime to a high temperature disordered regime has indeed been suggested in the literature by many authors [14, 15]. However, it is not clear how this picture can be reconciled with the experimental observation by Lay et al., which is suggestive of a finite temperature phase transition in the same system [15]. To resolve this issue, we note that the presence of the self-dual line, and the fact that some RG flows in the crossover region are seemingly attracted to it, makes the crossover so rapid that it might “look like” a phase transition that is smeared out due to finite size effects.

To make an estimate for the temperature at which this rapid crossover takes place, it is necessary to characterize the crossover more carefully [15]. Since in region II both fugacities are relevant, we need to worry about the validity of the perturbative RG approach. Each RG equation describing exponential growth of a fugacity presumably breaks down at a length scale $\xi$ at which the fugacity is renormalized to unity. This correlation length simply characterizes the average separation between unbound vortices of the corresponding type, and is a measure of $K^{-1}$.
the number of free vortices [2]. Therefore, it is reasonable to assume that the crossover happens where the correlation length for type-0 vortices equals that of type-1 vortices.

Putting this together with the above picture for a rapid crossover, we assert that the crossover corresponds to an initial point, given by the bare parameters $K^{-1}(0)$, $y_0(0)$, and $y_1(0)$, which is going to be renormalized to $K^{-1}(l^*) \approx 2\pi$, and $y_0(l^*) \approx y_1(l^*) \approx 1$ via the RG flow, where $l^* = \ln(\xi/a)$, and $a$ is a microscopic length scale set by the core radius of a vortex. This criterion yields two equations for the two unknown parameters $\xi$ and the crossover temperature $T^*$, which could in principle be solved numerically.

Rather than elaborating on the numerical calculations that are not particularly illuminating, we attempt to make further simplifying approximations. We roughly estimate the integrals as $\int_0^{l^*} dt(1/2 - \pi K(t)) \approx c_0 l^* (1/2 - \pi K(0))$ and $\int_0^{l^*} dt(1/2 - 1/4\pi K(t)) \approx c_1 l^* (1/2 - 1/4\pi K(0))$, where the coefficients $0 < c_0 < 1$ and $0 < c_1 < 1$ describe the decay of the coupling constant $K^{-1}$. Using this approximation, we obtain an estimate for the crossover temperature as

$$T^* \approx \frac{2\pi \rho_s}{k_B} \times \frac{c_0 \ln(8\pi^2 \rho_s/t + (3 + c_1)e_c)}{(3 + c_0) \ln(8\pi^2 \rho_s/t + c_1 e_c)}.$$  

Values for $T^*$, given by Eq. (9) with $c_0 = c_1 = 0.3$ and $e_c = 1$, are shown in Fig. 2 together with experimental data from Ref. [1]. The magnitude of the crossover temperature is of the order of 1 K, and it decreases with increasing the BLES density, in reasonable agreement with the experiment.

**IV. DISCUSSION**

A nonzero value for $\kappa^2$ does not change the RG picture, because the Ising-like term is always subdominant compared to the magnetic field, except for the low temperature regime where the behavior of the system is governed by the ground state properties. While the Hamiltonian [1] has only one minimum for $t/\kappa^2 > 4$, a double-well structure with a local minimum at $\phi = \pi$, a global minimum at $\phi = 0$, and a maximum at $\phi = \cos^{-1}(-t/4\kappa^2)$ develops for $t/\kappa^2 \leq 4$. If the system is initially prepared to be at the metastable state, it will decay to the ground state with a rate $\sim \exp(-S[\phi_c]/\hbar)$ where $\phi_c$ is the classical solution, interpolating between the two minima [1].

As one may see from Table I, the condition for the existence of the metastable state ($\kappa^2 \geq t/4$) is not satisfied for a typical BLES. However, it might be possible to reduce the value of $t$ by applying a tilted magnetic field [2].

Finally, we note that a full quantitative account of the experiment can not be achieved until one correctly takes into account all the details, possibly by performing an exact diagonalization calculation. However, we believe our effective field theory approach to the problem, provides an understanding of the nature of the observed “transition” in the experiment, and is able to predict the correct order of magnitude and trend for the “transition” temperature.

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