Quantifying Coherence in Infinite Dimensional Systems

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We study the quantification of coherence in infinite dimensional systems, especially the infinite dimensional bosonic systems in Fock space. We show that given the energy constraints, the relative entropy of coherence serves as a well-defined quantification of coherence in infinite dimensional systems. Via using the relative entropy of coherence, we also generalize the problem to multi-mode Fock space and special examples are considered. It is shown that with a finite average particle number, increasing the number of modes of light can enhance the relative entropy of coherence. With the mean energy constraint, our results can also be extended to other infinite-dimensional systems.

Quantum coherence arising from quantum superposition principle is a fundamental aspect of quantum physics [1]. The laser [2] and superfluidity [3] are examples of quantum coherence, whose effects are evident at the macroscopic scale. However, the framework of quantification of coherence has only been methodically investigated recently. The first attempt to address the classification of quantum coherence as physical resources by T. Baumgratz et al., who have established a rigorous framework for the quantification of coherence based on distance measures in finite dimensional setting [4]. With such a foundational framework for coherence, one can find the appropriate distance measures to quantify coherence in a fixed basis by measuring the distance between the quantum state $\hat{\rho}$ and its nearest incoherent state. After the framework for coherence has been proposed, it receives increasing attentions. A. Streltsov et al. have used entanglement to quantify quantum coherence, which provides the operational quantification of coherence [8]. S. Du et al. focused on the interconversion of coherent states by means of incoherent operations using the concept of majorization relations [7]. Z. Xi et al. have given a clear quantitative analysis and operational connections between relative entropy of coherence, quantum discord and one-way quantum deficit in the bipartite quantum system [4]. T. Bromley et al. have found freezing conditions in which coherence remains unchanged during the nonunitary dynamics [5]. Up to now, all the results for quantifying the quantum coherence are assumed the finite dimensional setting, which is neither necessary nor desirable. In consideration of the relevant physical situations such as quantum optics states of light, it must require further investigations on infinite dimensional systems.

In this paper, we aim to investigate the quantification of coherence in infinite dimensional systems. Specifically, we focus on the infinite dimensional bosonic systems in Fock space [10] which are used to describe the most notable quantum optics states of light [11] and Gaussian states [12-13]. We show that when considering the energy constraints, the relative entropy of coherence serves as a well-defined quantification of coherence in infinite dimensional systems and the $l_1$ norm of coherence fails. Via using the relative entropy of coherence, we also generalize the results to multi-mode Fock space and special examples are considered. It is shown that with a finite average particle number, increasing the number of modes of light can enhance the relative entropy of coherence. Our results can also be extended to other infinite-dimensional systems with energy constraints. Our work investigates special and experimentally relevant cases and the most general and easy to use quantifiers, which is significant and essential in quantum physics as well as quantum optics.

Given the postulates presented in Ref. [4], any proper measure of the coherence $C(\hat{\rho})$ must satisfy the following conditions:

(C1) $C(\hat{\rho}) \geq 0$ for $\forall \hat{\rho} \in \mathcal{H}$ and $C(\hat{0}) = 0$ iff $\forall \hat{\delta} \in \mathcal{I}$.

(C2a) Monotonicity under all the incoherent completely positive and trace-preserving (ICPTP) maps: $C(\hat{\rho}) \geq C(\Phi_{\text{ICPTP}}(\hat{\rho}))$, where $\Phi_{\text{ICPTP}}(\hat{\rho}) = \sum_n \hat{K}_n^\dagger \hat{\rho} \hat{K}_n$ and $\{\hat{K}_n\}$ is a set of Kraus operators that satisfies $\sum_n \hat{K}_n^\dagger \hat{K}_n = I$ and $\hat{K}_n \hat{K}_n^\dagger \subset \mathcal{I}$.

(C2b) Monotonicity for average coherence under subselection based on measurement outcomes: $C(\hat{\rho}) \geq \sum_n p_n C(\hat{\rho}_n)$, where $\hat{\rho}_n = \hat{K}_n \hat{\rho} \hat{K}_n^\dagger / p_n$ and $p_n = Tr(\hat{K}_n \hat{\rho} \hat{K}_n^\dagger)$ for all $\{\hat{K}_n\}$ with $\sum_n \hat{K}_n^\dagger \hat{K}_n = I$ and $\hat{K}_n \hat{K}_n^\dagger \subset \mathcal{I}$.

(C3) Nonincreasing under the mixing of quantum states: $\sum_n p_n C(\hat{\rho}_n) \geq C(\sum_n p_n \hat{\rho}_n)$.

Two kinds of measures for coherence in finite dimensional systems [4] satisfy all the conditions mentioned above include: the relative entropy of coherence defined as

$$C_{\text{rel.ent.}}(\hat{\rho}) = S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho})$$

and the $l_1$ norm of coherence defined as

$$C_1(\hat{\rho}) = \sum_{i \neq j} |\rho_{ij}|$$

where $\hat{\rho} = \sum_{ij} \rho_{ij} |i\rangle \langle j|$ and $\hat{\rho}_{\text{diag}} = \sum_i \rho_{ii} |i\rangle \langle i|$. It has been shown that the promising fidelity of coherence does not in general satisfy $(C2b)$ under the subselection of the measurement condition [9].
Generally, the bosonic single mode Hilbert space $\mathcal{H}$ is spanned by an uncountable basis $\{|n\rangle\}_{n=0}^{\infty}$ called the Fock (number state) basis. Fock states are the eigenstates of the number operator $\hat{n} := \hat{a}^\dagger \hat{a}$ where we have $\langle n | \hat{a}^\dagger \hat{a} | n \rangle = \sqrt{n} (n - 1)$ and $\langle n | \hat{a}^\dagger | n \rangle = \sqrt{n + 1} (n + 1)$. Referring to development of entanglement theory in infinite dimensional systems, the problem of quantification of coherence can addressed by requiring energy constraints \cite{[15]}, which is experimentally relevant. Here and after, we require a new condition for this case (C4): if the first order moment, the average particle number, is finite $\langle \hat{n} \rangle < \infty$, it should fulfill $C(\hat{\rho}) < \infty$.

Given the proper definition of incoherent states, incoherent operations and maximal coherent states, the proofs of these two definitions do not require the finite dimensional setting as there are very relevant physical situations that require infinite dimensional systems for their description. The incoherent and incoherent operations defined in Ref. \cite{[4]} can be easily generalized to the case in infinite dimensional systems. In the Fock space, the set of incoherent state can be defined as $I \subset \mathcal{H}$ and all density operators $\hat{\delta} \in I$ are of the form

$$\hat{\delta} = \sum_{n=0}^{\infty} \delta_n |n\rangle \langle n|.$$  

For (C2), Kraus operators $\{\hat{K}_n\}$ satisfying $\sum \hat{K}_n^\dagger \hat{K}_n = \mathbb{1}$ and $\hat{K}_n^\dagger \hat{K}_m \in \mathcal{I}$ are $d_n \times d_m$ matrices where $d_n \rightarrow \infty$. Given these premises, our problem turns to be verifying condition (C4): whether these quantifications of coherence fulfilling (C1-3) can serve as a unit for coherence or be finite $C(\hat{\rho}) < \infty$ when the energy constraint is taken into consideration. That is, incoherent states, maximal coherent states and the maximum quantification of coherence should be well-defined.

At first, we show that relative entropy of coherence $C_{\text{rel.ent.}}$ fulfills the requirements of quantification of coherence for the states in the infinite dimensional Hilbert space. At the beginning, we show that diagonal mixed states such as thermal states have zero coherence $C_{\text{rel.ent.}} = 0$. When mean particle number is finite, we can figure out the maximal coherent state as

$$|\psi_m\rangle = \sum_{n=0}^{\infty} \frac{\tilde{\gamma}^{n/2}}{(\tilde{n} + 1)^{(n+1)/2}} e^{i \varphi_n} |n\rangle$$  

which makes (C4) saturated:

$$C_{\text{rel.ent.}}^{\text{max}} = (\tilde{n} + 1) \log (\tilde{n} + 1) - \tilde{n} \log \tilde{n} < \infty.$$  

This result can be directly obtained from the fact that the thermal state as $\hat{\rho}^{\text{th}}(\tilde{n}) = \sum_{n=0}^{\infty} (\tilde{n}/(\tilde{n} + 1)^{n+1}) |n\rangle \langle n|$ reaches the maximum von Neumann entropy given a fixed average particle number $\tilde{n}$, which is the same as the thermal state.

Given a linear phase generation $\varphi_n = n\varphi$, the state (3), a pure state with a thermal distribution (PSTD), has been shown to be the eigenstate of the SG-phase operator $\sum_{n=0}^{\infty} |n\rangle \langle n|$, which is experimentally relevant. Compared with two well-known Gaussian states, coherent state $|\alpha\rangle := D(\alpha) |0\rangle$ and squeezed vacuum state $|0, \xi\rangle := \hat{S}(\xi) |0\rangle$, the particle number distribution and coherence quantification of relative entropy are shown in Fig. $\text{(a)}$ and $\text{(b)}$, respectively. In Fig. $\text{(c)}$, the determinants of the coherence variances matrices $\gamma$ of these three states against the mean particle number are given. Since a Gaussian state is pure iff $\det \gamma = 1$ \cite{[15][12][14]}, we conclude that PSTD with form (3) is a non-Gaussian state, except for $\tilde{n} \rightarrow 0$. Therefore, PSTD can not be easily constructed by squeezing and displacement operator on vacuum state. For details, please see APPENDIX. Therefore, we conclude that relative entropy of coherence is an appropriate quantification of coherence even in infinite dimensional systems.

Next, given a fixed average particle number in Fock space, we show that no maximal coherent state can be found to maximize the $l_1$ norm of coherence. With a set of particle number distributions $\{P_n\}$ of a pure state, the identity condition: $\sum_{n=0}^{\infty} P_n = 1$ and the finite energy constraint (C4): $\sum_{n=0}^{\infty} nP_n = \tilde{n} < \infty$ should be two constraint conditions. Obviously, for any mixed state, we can find a pure state with larger $l_1$ norm of coherence. Then, $l_1$ norm of coherence of a pure state can be written as

$$C_{l_1}(\hat{\rho}) = \sum_{m,n=0}^{\infty} \sqrt{P_m P_n} - 1. \quad (5)$$

The maximum of $l_1$ norm of coherence should occur as the first variation is zero $\delta C_{l_1} = \sum_{m,n=0}^{\infty} \sqrt{P_m/P_n} \delta P_n = 0$. Via using the method of Lagrange multipliers with two Lagrange multiplier $\lambda_1$ and $\lambda_2$, a series of equations can be ob-

FIG. 1: (color online) (a) Photon number distributions of PSTD, coherent state and squeezed vacuum state against average particle number. (b) Relative entropies of coherence of PSTD, coherent state and squeezed vacuum state against average particle number. (c) Determinants of the coherence variances matrices $\gamma$ of these three states against the mean particle number.
obvious that TMSV through a 50:50 beam-splitter has a larger coherence than TMSV.

Therefore, the $l_1$ norm of coherence does not seem to be a well-defined quantification of coherence in Fock space because it does not have a well-defined maximal coherent state such that the quantification is finite. We here note that with a stronger condition (C4): the second order moment is finite $\langle \hat{n}^2 \rangle < \infty$, we can find a well-defined maximal coherent state for the $l_1$ norm of coherence and $C(\hat{\rho}) < \infty$ could be met.

Since we have shown that the relative entropy of coherence $C_{\text{rel.ent.}}$ fulfills the requirements of quantification of coherence even for the states in single mode Fock space, we then generalize this result to $d$-mode Fock space $\mathcal{H} = \bigotimes_{i=1}^d \mathcal{H}_i$. It has an uncountable basis $\{ \bigotimes_{i=1}^d |n_i\rangle \}$ and probability distributions $\{ P_n \}$ where the vector is defined as $n = (n_1, n_2, \cdots, n_d)$ and we define $|n\rangle = \sum_n P_n |n\rangle$. After simple calculations, the maximal coherent state should has a distribution as $P_n^{\text{max}} = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} C_{\text{rel.ent.}}^d$ with finite average total particle number defined as $\bar{n} = \sum_n P_n |n\rangle$. The maximum relative entropy of coherence for $d$-mode Fock space can be calculated as

$$C_{\text{rel.ent.}}^{\text{max},d} = C_{\text{rel.ent.}}^{\text{max},d=1} + S_d(\bar{n})$$

where $S_d(\bar{n}) = \sum_{n=0}^{\infty} (\bar{n}^n / (\bar{n}+1)^{n+1}) \log(C_{\text{rel.ent.}}^d)$ is a convergent series. Since $S_d(\bar{n}) > S_d(\bar{n})$ if $d > d'$, we show in Fig. 2(a) that given a fixed average total particle number $\bar{n}$, relative entropy of coherence increases as the number of modes $d$ increases. This result is significant that with a finite average particle number increasing the number of modes of light can enhance the coherence as a resource in quantum information processing. The advantages of multimode quantum optics have been recently interpreted in quantum metrology [21].

We then consider two-mode coherent state $|\alpha\rangle_1|\alpha\rangle_2$, two-mode squeezed vacuum (TMSV) state and TMSV passing a 50:50 beam-splitter as special examples. The last case has been shown to be efficient to beat the shot noise limit (SNL) in the quantum metrology [22]: TMSV can be written as $|\text{TMSV}\rangle = \sum_{n=0}^{\infty} (\bar{n}^n / (\bar{n}+1)^{n+1}) |n\rangle_1 |n\rangle_2$ and a TMSV through a 50:50 beam-splitter is written as $\{ \bigotimes_{i=1}^d |n_i\rangle \}$ where $d$ is the number of modes.

In conclusion, we investigate the quantification of coherence in infinite dimensional systems, since there are very relevant physical situations that require infinite dimensional systems for their description. A new constraint condition (C4) is suggested for this problem, with which the relative entropy of coherence is shown to be a well-defined quantification of coherence in infinite dimensional systems but the $l_1$ norm of coherence fails. We also consider quantifying coherence in the multi-mode Fock space. Given a fixed average total particle number, relative entropy of coherence increases as the

$$\hat{U}_{\text{BS}}|\text{TMSV}\rangle = \sum_{n=0}^{\infty} \frac{(\bar{n}^n)_2}{(\bar{n}+1)_2} \sum_{k=0}^{n} (-1)^k \frac{C_{\text{rel.ent.}}^2}{2n!} |2n - 2k\rangle_1 |2k\rangle_2,$$
the number of modes increases, which is significant because coherence as a resource in quantum information processing is larger when increasing the number of modes. This work investigates experimentally relevant infinite dimensional systems and the most general and easy to use quantifiers, which is important for experimental and theoretical applications in quantum physics as well as quantum optics. Moreover, our results can be easily extended to other infinite-dimensional systems.

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Appendix A: Relative entropy of coherence of coherent state and squeezed vacuum state

The well-known coherent state can be written as \(|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\) with a particle number distribution \(P_n^{\text{cs}} = e^{-\bar{n}/2} \frac{\bar{n}^n}{n!}\) and \(\bar{n} = |\alpha|^2\). The relative entropy as a quantification of coherence can be calculated as

\[
C_{\text{rel.ent.}}^{\text{cs}} = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n \log n!}{n!} - \bar{n} \log \bar{n},
\]

which is shown in Fig. 1(b). A squeezed state \(|\alpha, \xi\rangle\) may be generated by first acting with the squeeze operator \(\hat{S}(\xi)\) on the vacuum followed by the displacement operator \(\hat{D}(\alpha)\) with particle number distribution \(\langle \xi | r e^{i2\xi} | \alpha, \xi \rangle\) [18].

\[
P_n^{\text{ss}} = \exp \left[-|\alpha|^2 - \frac{1}{2} \tanh r \left(\alpha^2 e^{i\phi} + \alpha^2 e^{-i\phi}\right) \right] \tanh n r \left[H_n \left(\frac{\alpha + \alpha e^{i\phi} \tanh r}{\sqrt{2\alpha^2 e^{i\phi} \tanh r}}\right) \right]^2,
\]

where \(H_n(z)\) is the \(n\)th Hermite polynomial. For squeezed vacuum state, \(\alpha = 0\) and \(|H_n(0)| = 2^n/2(n-1)!!\) when \(n\) is even, we obtain that

\[
P_n^{\text{sv}} = \frac{\tanh^n r n!(n-1)!^2}{n! \cosh r},
\]

where \(\bar{n} = \sinh^2 r\). Then we can calculate the relative entropy using \(C_{\text{rel.ent.}}^{\text{sv}} = \sum_{n=0}^{\infty} P_n^{\text{sv}} \log P_n^{\text{sv}}\) in Fig. 1(b).

Appendix B: Covariance matrix of PSTD

Canonical variables can be written in terms of creation and annihilation operators

\[
\hat{x} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{1}{\sqrt{2i}} (\hat{a} - \hat{a}^\dagger), \quad \hbar = 1.
\]

For the one-mode state, by defining a vectorial operator \(\mathbf{R} = (\hat{x}, \hat{p})\), we can calculate the covariance matrix

\[
\gamma = 2 \begin{pmatrix} \text{Cov}_\rho(\hat{x}, \hat{x}) & \text{Cov}_\rho(\hat{x}, \hat{p}) \\ \text{Cov}_\rho(\hat{p}, \hat{x}) & \text{Cov}_\rho(\hat{p}, \hat{p}) \end{pmatrix} - i J_1
\]

\[
= 2 \begin{pmatrix} \bar{n} + \frac{1}{2} & \frac{\bar{a}}{2} \\ \frac{\bar{a}}{2} & \frac{1}{2} - \bar{a}^2 \end{pmatrix} \quad (\text{B3})
\]

where \(J_1 = \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\) and

\[
\langle \bar{a}^2 \rangle = \frac{\bar{n}}{(n+1)^2} \sum_{n=0}^{\infty} \frac{n^n}{(n+1)^n} \sqrt{n+2} \sqrt{n+1} \quad (\text{B4})
\]

\[
\langle \bar{a} \rangle = L_i \left(\frac{\bar{n}}{n+1}\right) / \sqrt{n(n+1)} \quad (\text{B5})
\]

with \(L_i(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}\) the polylogarithm function. The determinant of the covariance matrix \(\text{det}(\mathbf{R})\) is calculated numerically and is shown in Fig. 1(c) against the mean particle number.

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