Elliptic and hyperelliptic magnetohydrodynamic equilibria

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Abstract

The present study is a continuation of a previous one on "hyperelliptic" axisymmetric equilibria started in [Tasso and Throumoulopoulos, Phys. Plasmas 5, 2378 (1998)]. Specifically, some equilibria with incompressible flow nonaligned with the magnetic field and restricted by appropriate side conditions like "isothermal" magnetic surfaces, "isodynamicity" or $P + B^2/2$ constant on magnetic surfaces are found to be reducible to elliptic integrals. The third class recovers recent equilibria found in [Schief, Phys. Plasmas 10, 2677 (2003)]. In contrast to field aligned flows, all solutions found here have nonzero toroidal magnetic field on and elliptic surfaces near the magnetic axis.

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1 Introduction and basic equation

A generalized Grad-Shafranov equation has been derived in Ref. [1] (Eq. (22) therein) to describe axisymmetric magnetohydrodynamic equilibria with incompressible flows. This equation consisting the starting point of the present investigation is given by

\begin{equation}
(1 - M^2) \Delta^* \psi - \frac{1}{2}(M^2)' |\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + R^2 (P_S(\psi) - \frac{X F' \Phi'}{1 - M^2})' + \frac{R^4}{2} \left( \frac{\rho (\Phi')^2}{1 - M^2} \right)' = 0
\end{equation}

along with a Bernoulli relation for the pressure,

\begin{equation}
P = P_S(\psi) - \rho \left( \frac{v^2}{2} + \frac{\Phi' \Theta}{\rho} \right),
\end{equation}

where \( P_S(\psi) \) is part of the pressure which depends on \( \psi \) only, \( \psi \) being the poloidal magnetic flux function. The elliptic operator \( \Delta^* \) is defined by \( \Delta^* = R^2 \nabla \cdot (\nabla / R^2) \), \( M^2 = (F'(\psi))^2 / \rho \) where \( F(\psi) \) is the poloidal stream function and \( \rho(\psi) \) is the mass density, \( \Phi(\psi) \) is the electrostatic potential, \( \Theta / (\rho R) \) is the toroidal velocity component and \( X(\psi) = I(1 - M^2) + R^2 F' \Phi' \) where \( I / R \) is the toroidal magnetic field. \( R, \phi, z \) are the usual cylindrical coordinates with \( z \) corresponding to the axis of symmetry. As stated in Ref. [1] the surface quantities \( F(\psi), \Phi(\psi), X(\psi), \rho(\psi) \) and \( P_S(\psi) \) are free functions. For each choice of this set of five functions, Eq.(1) is fully determined and can be solved whence the boundary condition for \( \psi \) is given.

For our further investigation it is convenient to simplify Eq.(1) by introducing the following transformation

\begin{equation}
u(\psi) = \int_0^\psi (1 - M^2(g))^{1/2} dg,
\end{equation}

which reduces (1) to

\begin{equation}
(1 - M^2) \Delta^* u - \frac{1}{2}(M^2)' |\nabla u|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + R^2 (P_S(u) - \frac{X F' \Phi'}{1 - M^2})' + \frac{R^4}{2} \left( \frac{\rho (\Phi')^2}{1 - M^2} \right)' = 0,
\end{equation}

where the primes indicate now the derivatives with respect to \( u \) but \( F' = dF/d\psi \) and \( \Phi' = d\Phi/d\psi \). (See previous work e.g. Ref. [3]). Note that no quadratic term in \( |\nabla u|^2 \) appears anymore in Eq.(4).
The paper is organized as follows: section 2 addresses the question of the side conditions while in section 3 the shape of the magnetic surfaces is determined near magnetic axis. The conclusions are in section 4.

2 Side conditions

Instead of specifying the free functions mentioned above to determine Eq. (1), it may be of physical or mathematical importance to introduce side conditions on some physical quantities like the total pressure, the magnitude of the magnetic field or combinations of them. It is indeed plausible to assume isothermal magnetic surfaces in hot plasmas [1] because of the huge parallel heat conductivity or to try to eliminate neoclassical effects [2] through an isodynamic condition. Such side conditions lead, in general, to an additional relation between \((\nabla u)^2, u\) and \(R\) as already accomplished in section 4 of Ref. [1] or in Ref. [2]. It turns out that, due to the assumed incompressibility of the flow, Eq. (4) as well as the side conditions considered here have quartic \(R\) dependence on the right hand side, which together with Eq. (4) can be expressed as follows

\[
\Delta^* u = -f(u) - R^2 g(u) - R^4 h(u),
\]

\[
|\nabla u|^2 = 2[i(u) + R^2 j(u) + R^4 k(u)],
\]

where

\[
f(u) = \frac{X^2}{2(1 - M^2)},
\]

\[
g(u) = (P_S - \frac{X\Phi'F'}{1 - M^2})',
\]

\[
h(u) = (\frac{\rho\Phi'^2}{2(1 - M^2)})',
\]

and the other coefficients depend upon the specific side condition chosen. Let us consider first the case of isothermal magnetic surfaces already treated in Ref. [1] in the variable \(\psi\). Note that our present variable \(u\) defined in (3) is a special relabeling of the variable \(\psi\).
2.1 Isothermal magnetic surfaces

Setting the plasma pressure $P$ as a function of $u$ and using Ref. [1] to calculate the coefficients entering Eq.(6), one obtains

$$i(u) = -\frac{X^2}{2(1 - M^2)},$$

(10)

$$j(u) = (1 - M^2)\left[\frac{P_S - P}{M^2} - \frac{X\Phi'F'}{(1 - M^2)^2}\right],$$

(11)

$$k(u) = \frac{\rho\Phi'^2}{2(1 - M^2)}\left(\frac{1 - 2M^2}{M^2}\right).$$

(12)

To solve equations (5) and (6) simultaneously we use the method of section 4 of Ref. [1] which boils down to an ordinary differential equation on each magnetic surface

$$\frac{\partial z}{\partial x}\bigg|_u = -\frac{p}{q} = \frac{\pm \frac{1}{4}[(g + j')x + \frac{1}{2}(h + k')x^2 - d]}{\left\{2(i + jx + kx^2) - \frac{x}{4} \left[(g + j')x + \frac{1}{2}(h + k')x^2 - d\right]^2\right\}^{1/2}},$$

(13)

where $x$ stays for $R^2$ and five compatibility conditions for seven free functions including the five free functions of Eq. (1). There should be no problem, in general, to satisfy those compatibility conditions. The solutions of (13) are, in general, hyperelliptic integrals [4], which are not related to known special functions unless they can be reduced to elliptic integrals. This occurs, in particular, for field aligned flows ($\Phi' = 0$) considered in Ref. [1, 5]. The purpose of this contribution is to find other cases of elliptic reduction with nonaligned flows ($\Phi' \neq 0$). The easiest case of that kind is to annihilate the coefficient of the largest powers of $x$ in the denominator of Eq.(13)

$$h + k' = 0$$

(14)

with $h$ and $k$ both different from zero.

After introducing $J = \frac{\rho\Phi'^2}{1-M^2}$ and using (9) and (12), Eq. (14) leads to

$$J' - \frac{(M^2)'}{M^2(1 - M^2)} J = 0,$$

(15)
whose general solution is
\[ J = C \frac{M^2}{1 - M^2}, \]
(16)
with \( C \geq 0 \) and \( 0 \leq M^2 < 1 \). For \( C = 0 \), we recover the case of field aligned flows already obtained in Ref. [1].

As a byproduct of this investigation misprints have been found in the nonumbered equations for \( k(\psi) \) and \( g(\psi) \) after Eq.(35) of Ref. [1]. They should be corrected as follows:

\[ k(\psi) = \frac{1}{2} \left[ \frac{\rho (\Phi')^2}{M^2} - \frac{(F'\Phi')^2}{(1 - M^2)^2} \right] \]
(17)
and
\[ g(\psi) = \frac{M^2}{1 - M^2} \left( \frac{P_s M^2}{M^2} \right)' - \left[ \frac{X \Phi' F'}{(1 - M^2)^2} \right] + \frac{(M^2)'}{M^2(1 - M^2)} P. \]
(18)

### 2.2 Isodynamic field or \( B^2 = \text{function of } u \)

The setting of \( B^2 \) as a function of \( u \) (Ref. [1] is used for the calculations) leads to the functions \( i(u), j(u) \) and \( k(u) \) entering the side condition (6) as

\[ i(u) = -\frac{X^2}{2(1 - M^2)}, \]

(19)
\[ j(u) = (1 - M^2) \left[ \frac{B^2}{2} + \frac{X \Phi' F'}{(1 - M^2)^2} \right], \]

(20)
\[ k(u) = -\frac{\rho \Phi'^2}{2(1 - M^2)} M^2, \]

(21)
while \( f(u), g(u) \) and \( h(u) \) are still given by (7)-(9) since Eq. (5) does not change. Now the reduction equation (14) becomes

\[ J' - \frac{(M^2)'}{1 - M^2} J = 0, \]
(22)
whose solution is
\[ J = \frac{C}{1 - M^2}, \]
(23)
with \( C \geq 0 \) and \( 0 \leq M^2 < 1 \). Again we recover for \( C = 0 \) the Palumbo solution for field aligned flows [1 5 2].
2.3 \((P + B^2/2) = \text{function of } u\)

Though this side condition is not as relevant to hot plasmas as the previous cases, it is of mathematical interest since it induces a "hidden symmetry" in the equilibrium equations as discovered in Ref. [6], which leads to a rich class of solutions. After calculating \((P + B^2/2)\) and setting it a function of \(u\), we obtain the coefficients \(i(u), j(u)\) and \(k(u)\) entering condition (6) as

\[
i(u) = -\frac{X^2}{2(1 - M^2)}, \quad \text{(24)}
\]

\[
j(u) = 2[P - P_s + \frac{B^2}{2} + \frac{X\Phi F'}{(1 - M^2)}], \quad \text{(25)}
\]

\[
k(u) = -\frac{1}{2}J(u), \quad \text{(26)}
\]

while as before \(f(u), g(u)\) and \(h(u)\) are still given by (7)-(9) since Eq. (5) does not change. It turns out that, in this case, the reduction equation (14) is identically satisfied, which is reminiscent of the hidden symmetry of Ref. [6], so recovering the elliptic solutions found therein.

3 Behaviour of solutions near magnetic axis

It is instructive to analyse the properties of all possible solutions of (5) and (6) near the magnetic axis. Focussing on (5) and (6) we employ a Cartesian system \((x, y)\) centred on magnetic axis, i.e. \(R = R(0) + x\) and \(z = z(0) + y\), and expand the \(u\) surfaces in \(x\) and \(y\) around the magnetic axis up to second order:

\[
u - u(0) = ax^2 + bxy + cy^2 + \text{higher orders}. \quad \text{(27)}
\]

Also, we expand the flux functions contained in (5) and (6) up to first order in \(u - u(0)\), i.e.

\[
i(u) = i(0) + i'(0)(u - u(0)), \quad j(u) = j(0) + j'(0)(u - u(0)) \quad \text{etc}, \quad \text{(28)}
\]

and \(R^2\) and \(R^4\) up to second order in \(x\) and \(y\). On the basis of the zeroth, first and second order equations thus obtained from (5) and (6) we can determine the coefficients \(a, b\) and \(c\) of \(u - u(0)\).

It turns out that \(b = 0\) and

\[
\frac{a}{c} = \frac{1}{2} + \left[\frac{1}{4} + \frac{k(0)R^2(0)}{\gamma^2}\right]^{1/2}, \quad \text{(29)}
\]
where $\gamma = \left[ i'(0) + R(0)^2 j'(0) + R(0)^4 k'(0) \right]/(2\sqrt{2})$. For the cases of subsections 2 and 3 Eq. (29), on account of (21) and (26) implying $k(0) < 0$, means that the magnetic surfaces near the magnetic axis are elliptical with elongation being directed toward $R$. For isothermal magnetic surfaces, however, the elongation can be either parallel to $R$ if $M^2 > 1/2$ or parallel to $z$ if $M^2 < 1/2$ via (12). For $M^2 = 1/2$ the isothermal magnetic surfaces become circular. Also, in all three cases the ellipses become circles for field aligned flows, i.e. the Palumbo’s solution is recovered [5, 6, 2]. This point was overlooked in Ref. [1]. In addition, the strength of the toroidal magnetic field on the magnetic axis is proportional to $k(0)R^2(0)$, which vanishes for field aligned flows or Palumbo’s solution.

4 Conclusions

In conclusion, the reduction of the “hyperelliptic” equilibria with nonaligned flows introduced in Ref. [1] has been demonstrated for several side conditions permitting the discovery of whole classes of magnetohydrodynamic equilibria. All these equilibria have nonzero magnetic field and elliptic magnetic surfaces on magnetic axis (see Ref. [4] for a special case). The present reduction is based on the annihilation of the fourth and fifth power under the square root appearing in Eq. (13). Another possible reduction could be sought by finding the conditions under which the fifth order polynomial appearing in the denominator of Eq. (13) can be factorized with a double zero leaving a cubic polynomial under the square root. This and more details about the equilibria demonstrated here is left to future work.

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