Abrupt transitions of zonal jets in two-dimensional turbulent shear flow

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We study the complex transitions of zonal jets in the turbulent regime of a forced shear flow using direct numerical simulations of the Navier-Stokes equations. The behaviour of the zonal mean flow involves bifurcations of the probability density function (PDF) of the largest scale mode which exhibits two types of transitions over the turbulent regime. One transition is from Gaussian to bimodal PDF which signifies the emergence of reversals of the zonal mean flow where the flow fluctuates between two turbulent states. The other transition is from bimodal to unimodal PDF which designates the disappearance of the reversals of the largest scale mode. We attribute the later bifurcation to the long-time memory that the large scale flow exhibits related to low frequency $1/f$ type of noise. We also demonstrate the a minimal model with 15 modes, that was devised using the truncated Euler equations, is able to capture the large scale dynamics of the Navier-Stokes equations.

Keywords: bifurcations, bistability, truncated Euler equations, $1/f$ noise

I. INTRODUCTION

Low-frequency variability in climate exhibits recurrent patterns that are directly linked to dynamical processes of the governing dissipative system [1, 2]. The study of the persistence of these large scale patterns and the transitions between them plays a crucial role in understanding climate change [3]. Climate variability is often associated with dynamic transitions between different regimes, each represented by local attractors. Examples of climate phenomena where such transitions have been investigated experimentally as well as numerically are the transitions between different mean flow patterns of the Kuroshio Current in the North Pacific [4, 5], the transitions between blocked and zonal flows in the midlatitude atmosphere [6], and the transition to oscillatory behaviour in models of the El Niño/Southern Oscillation phenomenon in the equatorial Pacific [7].

While much of the progress in fluid dynamics has been in hydrodynamic instabilities, only a few studies exist on instabilities that occur when a control parameter is varied within the turbulent regime, which is the focus of this paper. These types of transition resemble more closely phase transitions in statistical mechanics where an order parameter [8] deviates continuously from zero. These transitions describe closely the extreme weather transitions described above between two turbulent states.

In laboratory experiments bifurcations over a turbulent background has been observed in a wide range of experiments, such as the Rayleigh-Bénard convection [9], flow past a cylinder [10], the von Karman flow [11], reversals in a dynamo experiment [12] and experiments of two-dimensional turbulence [13]. The two-dimensional turbulence experiment showed the formation of a large scale condensate which displays a series of bifurcations. These were also seen in numerical simulations that mimic the experimental set-up and were captured by the truncated Euler equations [14].

Large scale zonal flows are found in the atmospheres of planets and in oceans. Their existence motivates the study of jets formation in a turbulent flow driven at smaller scales. Anisotropy helps in the formation of these jet structures, they are usually introduced with a β-plane which models the variation of the rotation with altitude or a non-unity aspect ratio of the domain [15] or due to confinement and boundary conditions [16]. These zonal jets are known to be quite stable, evolving over longer time scales than the underlying turbulence. Such seemingly stable states do undergo bifurcations and transitions at times [17], and modelling such phenomena remains an open question.

When studying naturally occurring phenomena like those we already mentioned one cannot control experimental conditions and study their effect on the behaviour of the system. On the other hand, numerical and experimental studies of such type of transitions, in which the parameters can be systematically controlled, can provide important insights for our understanding of the behaviour observed in the natural counterparts. In this paper we study the
abrupt transitions of zonal jets in a two-dimensional flow confined in the latitudinal direction. Using direct numerical simulations of the Navier-Stokes equations we quantify the behaviour of the large scale mode and the bifurcations that occur in the turbulent regime. Finally, we devise a minimal model by using the truncated Euler equations to capture the behaviour of the Navier-Stokes equations.

II. PROBLEM SET-UP

We consider the two-dimensional Navier-Stokes equations for an incompressible velocity field \( \mathbf{u} = \nabla \times \psi \hat{z} \) forced by a Kolmogorov type forcing in a anisotropic domain of dimensions \((0, 2\pi L_x) \times (0, \pi L_y)\) as illustrated in Fig. 1. The governing equation written in terms of the streamfunction \( \psi(x, y, t) \) is given by,

\[
\partial_t \psi + \nabla^{-2} \{ \nabla^2 \psi, \psi \} = \nu \nabla^2 \psi - \mu \psi + f_0 \sin(k_f y)
\]  

where \( \{f, g\} = f_x g_y - g_x f_y \) is the standard Poisson bracket (subscripts here denote differentiation), \( \nu \) is the kinematic viscosity, \( \mu \) is the friction coefficient, \( f_0 \) is the amplitude of the Kolmogorov forcing and \( k_f \) is the forcing wavenumber. Our choice of the external driving force can be physically justified. In an atmospheric model the Kolmogorov forcing \( f_0 \sin(k_f y) \) may represent the transfer of angular momentum into midlatitudes due to the tropical Hadley cell [18]. In the case of the ocean, the forcing term is the curl of the wind stress \( \nabla \times \mathbf{\tau} = f_0 \sin(k_f y) \). A wind stress of the form \( \mathbf{\tau} = f_0 / k_f (\cos(k_f y), 0) \) mimics the annually averaged zonal wind distribution over the North Atlantic and North Pacific with westward winds over the mid-latitudes and eastward winds in the tropics and polar latitudes [19].

The boundary conditions are taken to be periodic in the \( x \) (or zonal) direction and free-slip in the \( y \) (or latitudinal) direction, i.e. \( \psi_{yy} = \psi_x = 0 \) at \( y = 0, \pi L_y \). We are interested on the effect of the friction coefficient \( \mu \) on the dynamics of large scale jets. Thus, we vary \( \mu \) by keeping fixed the ratio \( \nu / (f_0^{1/2} L_x) = 10^{-3} \), the forcing wavenumber with respect to the height \( k_f L_y = 4 \) and the aspect ratio of the domain \( 2\pi L_x / (\pi L_y) = 2 \). We choose to non-dimensionalise all the quantities with the rms velocity \( u_{\text{rms}} = \langle |\mathbf{u}|^2 \rangle^{1/2} \), the length scale \( L_x \) and the time scale \( L_x / u_{\text{rms}} \). Here, the angle brackets \( \langle \cdot \rangle \) indicate integration over the domain and time. Then, the Reynolds number can be defined as \( Re = u_{\text{rms}} L_x / \nu \) and the friction Reynolds number as

\[
Rh = u_{\text{rms}} / (\mu L_x),
\]

which is the ratio of the inertial term to the friction term in Eq. 1.

We perform direct numerical simulations (DNS) by integrating Eq. 1 using the pseudospectral method. We can decompose the streamfunction into Fourier modes in the \( x \) direction and sine modes in the \( y \) direction,

\[
\psi(x, y, t) = \sum_{m=-N_x}^{N_x-1} \sum_{n=1}^{N_y} \hat{\psi}_{m,n}(t) e^{im\pi x} \sin(ny),
\]

with \( \hat{\psi}_{m,n} \) being the amplitude of the mode \((m, n)\) and \( (N_x, N_y) \) denote the number of grid points in the \( x, y \) coordinates respectively. A third-order Runge-Kutta scheme is used for time advancement and the aliasing errors are removed with the 2/3 dealiasing rule which implies that the minimum and maximum wavenumbers are \( k_x^{\text{min}} = k_y^{\text{min}} = 1, \)
$k_x^{\text{max}} = N_x/3$ and $k_y^{\text{max}} = 2N_y/3$, respectively. The resolution was fixed to $(N_x,N_y) = (512,128)$ for all the simulations done in this study. Note that our resolution is limited in this study by the very long time integrations that are required to accumulate reliable statistics for the zonal mean flow transitions, which become rare as $Rh$ increases (see below).

III. ZONAL MEAN FLOW TRANSITIONS

We are interested in quantifying the transitions of the zonal mean flow as a function of the control parameter $Rh$, where the zonal mean flow can be defined as the energy in the largest mode,

$$\tilde{\psi}_{0,1}(t) = \frac{1}{2\pi^2} \int_0^{2\pi L_x} \int_0^{\pi L_y} \psi(x,y,t) \sin(y) \, dy \, dx.$$  \hspace{1cm} (4)

Due to the symmetry of the forcing and the domain with respect to the line $y = \pi L_y/2$, a non-zero value of $\tilde{\psi}_{0,1}(t)$ implies a departure from symmetry. For $Rh \ll 1$, the forcing drives a laminar flow (see the red profile shown in Fig. 1) which essentially consists of two westward and two eastward zonal jets, with a zero projection onto the largest mode in the system ($\tilde{\psi}_{0,1} = 0$). This laminar flow becomes unstable \cite{20} above a critical value of $Rh = Rh_{c}^{(1)} \approx 1.686$ leading to a codimension-two bifurcation which we report in detail elsewhere \cite{21}. This saturated state above $Rh_c$ still has zero projection onto the largest mode. As $Rh$ increases even further the system undergoes another Hopf bifurcation at $Rh_{c}^{(2)}$ with the largest mode directly forced due to the nonlinearity. Then, for $Rh > Rh_{c}^{(2)}$ the system undergoes a sequence of bifurcations until it reaches a 2D turbulent state like in \cite{22}. In this study we focus on the turbulent regime where we show that there are bifurcations in the behaviour of the zonal mean flow on top of a turbulent background.

In the turbulent regime $Rh$ still plays the role of the bifurcation parameter but in this case the relevant quantities are statistical, such as the moments or the probability density function (PDF) of $\tilde{\psi}_{0,1}$. In Fig. 2a we show the time series of $\tilde{\psi}_{0,1}$ and in Fig. 2b their corresponding PDFs for different values of $Rh$. At $Rh = 3.13$, the time series is turbulent with the amplitude of $\tilde{\psi}_{0,1}$ fluctuating randomly around the zero mean implying that planar symmetry is restored for the mode in a statistical sense. The PDF of this time series is close to Gaussian. For $Rh = 42.95$ we start to see longer duration of time where $\tilde{\psi}_{0,1}$ has a non-zero amplitude, the corresponding PDF is bimodal with two distinct peaks. This represents the first bifurcation where the system transitions from a Gaussian distribution to a bimodal distribution with two symmetric maxima. This behaviour is related to the emergence of two symmetric states where the time series is characterised by random reversals between these two states. One state is when $\tilde{\psi}_{0,1} > 0$ and the other symmetric state is when $\tilde{\psi}_{0,1} < 0$ for long duration. For $Rh = 59.45$ we get random reversals of the zonal

![Figure 2](image-url)
mean flow with a PDF which has more pronounced bimodality. As we keep increasing $Rh$ the reversals become rarer until we get to a state where there are no more zonal mean flow reversals indicating another bifurcation. This is seen in the case of $Rh = 89.36$ where the system was never observed to reverse, the corresponding PDF is unimodal with a non-zero mean. A similar sequence of bifurcations has been observed in laboratory experiments of quasi-2D flow \[13\] and also in numerical simulations of 2D turbulence \[13\]. In their studies the boundary conditions were no-slip or free-slip in both $x, y$ boundaries which led to a formation of a large scale circulation which then displayed a series of bifurcations similar to what is reported here.

IV. MINIMAL MODEL: TRUNCATED EULER EQUATIONS

Now we seek for a minimal model to capture the dynamics of the abrupt and random transitions of the zonal mean flow. To do this we consider the incompressible Euler equations truncated at a maximum wavenumber $k_{\text{max}}$ using a Galerkin truncation, which in Fourier space take the form

$$
\partial_t \hat{\psi}_k = \sum_{p,q} A_{k,p,q} \hat{\psi}_p \hat{\psi}_q
$$

with the interaction coefficients to be given by $A_{k,p,q} = (p \times q)(q^2 - p^2)k^{-2}\delta_{k,p+q}$, where $p \times q = p_x q_y - p_y q_x$ and $\delta_{i,j}$ stands for the Kronecker delta. This truncated system of ODEs conserves exactly the two quadratic invariants of the Euler Equations (i.e. Eq. \[1\] with the RHS equal to zero), the kinetic energy $E$ and the enstrophy $\Omega$. Using this formalism, Kraichnan \[24, 25\] derived the absolute equilibria of the energy spectrum $\hat{E}(k)$ and the enstrophy spectrum $\hat{\Omega}(k)$, which are

$$
\hat{E}(k) = \frac{k}{\alpha + \beta k^2} \quad \text{and} \quad \hat{\Omega}(k) = \frac{k^3}{\alpha + \beta k^2}.
$$

These spectra exhibit three regimes, distinguished by the signs of $\alpha$ and $\beta$ and by the value of the wavenumbers $k_{\text{min}}$, $k_{\text{max}}$ and $k_c = (\Omega/E)^{1/2}$. According to Kraichnan \[24, 25\] these regimes are

I) $k_{\text{min}}^2 < k_c^2 < k_{\alpha}^2$, $-\beta k_{\text{min}}^2 < \alpha < 0$, $\beta > 0$ \hspace{1cm} (8)

II) $k_c^2 < k_{\alpha}^2 < k_{\beta}^2$, $\alpha > 0$, $\beta > 0$ \hspace{1cm} (9)

III) $k_{\beta}^2 < k_c^2 < k_{\text{max}}^2$, $\alpha > 0$, $-\alpha < \beta k_{\text{max}}^2 < 0$ \hspace{1cm} (10)

where $k_{\alpha}^2 = \frac{1}{2}(k_{\text{max}}^2 - k_{\text{min}}^2)/\ln(k_{\text{max}}/k_{\text{min}})$ if $\alpha = 0$ and $k_{\beta}^2 = \frac{1}{2}(k_{\text{max}}^2 + k_{\text{min}}^2)$ if $\beta = 0$. These three regimes are illustrated in Fig. \[3\].

Regime I is the energy dominated regime, where $\hat{E}(k)$ peaks at the low end of the spectrum. This is in other words the well known regime of condensed states due to the inverse energy cascade \[24, 27\]. Regime III is the enstrophy dominated regime, where $\hat{E}(k)$ peaks at the high end of the spectrum denoting a direct energy cascade due to the enstrophy conservation. Finally, regime II is the intermediate regime, where $E(k)$ peaks at an intermediate wavenumber. This regime is bounded by the energy equipartition state $\beta = 0$ and the enstrophy equipartition state $\alpha = 0$.

The energy spectra from our DNS are peaked at the intermediate wavenumber $k_f$ when the system displays bimodal behaviour. Thus, this behaviour falls in the domain of regime II of the truncated system. This is in contrast to the study \[14\] where they focused on regime I because their DNS spectra peaked at the lowest wavenumber $k_{\text{min}} = 1$ in the bimodal regime. For the truncated Euler equations (TEE) the control parameter is the ratio of initial enstrophy to initial energy defined as the wavenumber

$$
k_c = (\Omega/E)^{1/2}.
$$

[13]

[14]
So far, we do not have a theoretical explanation for these scalings. The condition which reversals are not possible in our truncated system is \( k < k_{\text{crit}} = \sqrt{2} \). This can be obtained by

\[ |\alpha| = \beta k_{\min}^2 \]

\[ \beta = \frac{k_{\max}^2}{k_{\min}^2} \]

\[ \alpha = |\beta| k_{\max}^2 \]

FIG. 3: (Color online) Regimes for the initial conditions of the truncated Euler equations

In the full Navier-Stokes equations the equivalent wavenumber is controlled by the friction coefficient (i.e. the value of \( Rh \)) when the viscosity is fixed. In Fig. 4, we show the dependence of \( k_c \) on \( Rh \) for the Navier-Stokes equations and we see that as \( Rh \) increases the wavenumber \( k_c \) decreases like \( k_c \propto Rh^{-1/10} \). This is because we observe the energy to scale like \( E \propto Rh^{4/5} \) and the enstrophy to scale like \( \Omega \propto Rh^{3/5} \) for the values of \( Rh \) in the bimodal regime.

The initial conditions for the TEE are then chosen such that \( k_c \) is as close as possible to the DNS results and within the range \( k^2_\alpha < k^2_c < k^2_\beta \) to make sure we fall in regime II. We then integrate Eq. (5) using a numerical scheme similar to the one used for the Navier-Stokes equations (see section I). The Fourier amplitudes for the TEE satisfy \( \hat{\psi}_k = 0 \) for \( |k_x| > k^\max_x \) and \( |k_y| > k^\max_y \) and \( k^\max_x = k^\min_y = 1 \). Note that \( k^\max_x \) and \( k^\max_y \) are our free parameters for the truncation of the Euler equations. The starting point to obtain a minimal model from the TEE was to set \( k^\max_x = k^\max_y = 4 \), which corresponds to the value of the forcing wavenumber \( k_f \) we chose for the Navier-Stokes equations.

The basic principle behind this choice is that the dynamics of the large scales, i.e. \( k < k_f \), are not affected by viscosity and hence they are governed by the dynamics of the Euler equations as it has been demonstrated in previous studies [14, 28, 29]. After further truncation of the Euler equations we determined that the minimal model which captures qualitatively the bifurcations between the different turbulent regimes observed by our DNS consists of 15 ODEs for the complex large scale modes, while the minimal model that can produce zonal mean flow reversals but not the bifurcations consists of 11 ODEs.

In Fig. 4, we show the time series of the amplitude of the large scale mode \( \hat{\psi}_{0,1} \) for different values of \( k_c \) for the 15 mode minimal model and the inset of Fig. 4 shows the corresponding PDF distributions of the time series. For \( k_c = 3.5 \) the TEE system exhibits a turbulent time series with a Gaussian PDF. At \( k_c = 2.375 \) the system shows abrupt and random reversals of the zonal mean flow with a bimodal PDF. Thus we observe a first bifurcation between these two values of \( k_c \) when the system transitions from a Gaussian to a bimodal distribution. As \( k_c \) decreases the bimodality in the distribution becomes more pronounced and the reversals become less frequent. Finally, for \( k_c = 1.58 \) there is no longer any reversal for the duration of the simulation that lasted tens of thousands of turnover times. Here, the flow bifurcates to a one-sided unimodal PDF. Clearly this minimal set of ODEs captures the bifurcations between the different turbulent regimes observed in the full Navier-Stokes equations. It is worth mentioning that we even have an approximate quantitative agreement for the values of \( k_c \) between the TEE and the DNS. Note that canonical distributions with quadratic invariants are Gaussian [26]. In order to have a statistical theory of the transitions with non-Gaussian PDFs one has to use the microcanonical formalism [26]. In fact, it has been shown that the microcanonical ensemble predicts accurately the PDFs of the large scale mode and its transitions [13].

In the simulations we observe two different scenarios of emergence and disappearance of zonal mean flow reversals. At high values of \( k_c \) (or low \( Rh \) values for the DNS), zonal mean flow reversals are absent because the time series are very fluctuating and one cannot identify the two states anymore. At low \( k_c \) (or high \( Rh \) for DNS), zonal mean flow reversals become less and less probable and eventually are no longer observed in the duration of the simulation. In other words, the zero probability measure for reversals to occur becomes very large as \( k_c \) decreases (or \( Rh \) increases). The condition which reversals are not possible in our truncated system is \( k_c < k^\text{crit}_c = \sqrt{2} \). This can be obtained by
expanding the energy and enstrophy as sums of the Fourier modes, viz.

\[ E = \langle |\nabla \psi|^2 \rangle = \sum_{m=-N_y/2}^{N_y/2} \sum_{n=1}^{N_y} (m^2 + n^2)|\hat{\psi}_{m,n}|^2 \]

\[ \Omega = \langle |\nabla^2 \psi|^2 \rangle = \sum_{m=-N_y/2}^{N_y/2} \sum_{n=1}^{N_y} (m^2 + n^2)^2|\hat{\psi}_{m,n}|^2 \]

which in short can be written as \( E = (|\hat{\psi}_{0,1}|^2 + 2|\hat{\psi}_{1,1}|^2 + \ldots) \) and \( \Omega = (|\hat{\psi}_{0,1}|^2 + 4|\hat{\psi}_{1,1}|^2 + \ldots) \). Then, notice that the condition \( k_\tau^2 = \Omega/E < 2 \) is satisfied if and only if \( \hat{\psi}_{0,1} \neq 0 \) so that both energy and enstrophy are conserved. The disappearance of reversals can be attributed to broken ergodicity which may be viewed as “delayed ergodicity” in fluids \([30, 31]\) where very long-time correlations (see also section V below) delay ergodically to cover the whole phase space and therefore the zero probability measure becomes very large. In this case, the system remains attracted in one of the two possible states depending on the initial conditions, keeping a given sign of the zonal mean flow. These two scenarios of emergence and disappearance of reversals have also been observed in other contexts \([14, 22, 32, 33]\) and hence we believe that they are generic.

A quantity of interest is the mean waiting time \( \tau \) between successive sign changes of \( \hat{\psi}_{0,1} \) for both the Navier-Stokes equations and the minimal model of the TEE in the bimodal regime. Figure 5 shows \( \tau \) as a function of \( Rh \) for the Navier-Stokes equations and in the inset it shows how \( 1/\tau \) varies as a function of \( k_c \) for the TEE. Figure 5 shows an exponential divergence, i.e. \( \tau \propto e^{Rh} \) for the Navier-Stokes equations, while \( \tau \propto (k_c - k_{c}^{crit})^{-1} \) with a critical \( k_{c}^{crit} \approx 1.6 \) for our minimal model (see inset of Fig. 5). The existence of a critical \( k_{c}^{crit} \) in order for reversals to occur comes from the conservation of energy and enstrophy of the Euler equation. As we already discussed for \( k_c < \sqrt{2} \) zonal mean flow reversals are not possible in the TEE minimal model. On the other hand, the exponential divergence cannot be explained from the relation \( k_c \propto Rh^{-1/10} \) (see Fig. 4b). In contrast, the Navier-Stokes equations do not involve conserved quantities that prevent reversals, even for \( Rh \gg 1 \). Moreover, all the wavenumbers \( k > k_f \) that are suppressed in the truncated Euler model can act as an additional source of noise for the Navier-Stokes equations and trigger reversals. This is why we observe a different behaviour between the Navier-Stokes equations and the TEE.

V. \( 1/f^{\alpha} \) NOISE AND LONG-TIME MEMORY

In this section, we study the spectral properties of the large scale mode \( \hat{\psi}_{0,1} \) comparing the results of the Navier-Stokes equation with the TEE minimal model of 15 modes. Figure 6a shows the power spectrum \( S(f) \) of the large...
scale mode $\hat{\psi}_{0,1}$ for the Navier-Stokes equations with $Rh = 48.51$, $k_c = 3.13$ and Fig. 6b the power spectrum for the TEE system with $k_c = 1.8$. Both spectra are compared with a power-law $S(f) \propto f^{-1.5}$ which fits well the data.

The $1/f^{1.5}$ noise is between the well understood white noise with $f^0$ power spectrum and the random walk (Brownian motion) noise with $1/f^2$ power spectrum. However, direct differentiation or integration of such convenient signals does not give the required $1/f^{1.5}$ spectrum. Understanding of such low frequency noise is of great importance because power spectra of the type

$$S(f) \propto 1/f^\alpha$$

with $0 < \alpha < 2$ have been observed in many systems ranging from voltage and current fluctuations in vacuum tubes [34], atmosphere and oceans [35, 36], astrophysical magnetic fields [37] and turbulent flows [31, 33, 38]. These systems often display an intermittent regime with bursts occurring after random waiting times $\tau$. For this kind of processes, it has been shown that the $1/f^\alpha$ spectrum is related to a power-law distribution of the mean waiting time $\tau$ of the
form

\[ P(\tau) \propto 1/\tau^\beta \]  

(15)

with the exponents \( \alpha \) and \( \beta \) satisfying the relation \( \alpha + \beta = 3 \) \[32, 39\]. This relation is also valid for signals fluctuating between two symmetric states with a random waiting time \( \tau \) in each state when \( 1 < \beta < 3 \) \[32\].

The corresponding PDF of the waiting times \( P(\tau) \) are shown as insets in Fig. [3a] for the Navier-Stokes equations and in Fig. [6b] for the TEE model. Both PDFs show a power-law distribution with \( P(\tau) \propto \tau^{-1.5} \) fitting well with the data. These waiting times, distributed as a power law, reflect the scale invariant nature of the statistics, and they are associated with durations spent by the system in two different states. The scaling of the spectra and of the distribution of the mean waiting times are in agreement with the relation \( \alpha + \beta = 3 \) both for the Navier-Stokes equations and the minimal TEE model.

Note that the process for generating \( 1/f \) noise involves a large number of degrees of freedom with a large network of triads interacting nonlinearly and hence it differs strongly from low-dimensional dissipative dynamical systems. Our results indicate that the coherent structures of our turbulent flows are responsible for the long-term dynamical memory of the zonal mean flow direction and hence the emergence of the \( 1/f^\alpha \) noise.

VI. CONCLUSION

We have studied in detail the bifurcations of the zonal mean flow for a turbulent shear flow driven by a Kolmogorov forcing. The domain studied here is anisotropic with an aspect ratio of 2, and so are the boundary conditions that were taken to be free-slip in the latitudinal direction, and periodic in the zonal direction. The geometry and the forcing lead to the formation of zonal jets, with the largest mode in the system \( \psi_{0,1} \) corresponding to two counter propagating zonal jets. The mode \( \psi_{0,1} \) is not directly excited by the forcing but the energy injected into the forcing scale is transferred to the largest mode \( \psi_{0,1} \) by an inverse cascade. A non-zero value of \( \psi_{0,1} \) implies symmetry breaking about the mid-plane \( y = \pi L_y/2 \).

With the friction coefficient as a control parameter in the system, we see that the mode \( \psi_{0,1} \) first displays a Gaussian behaviour in the turbulent regime. Then, as we increase the friction Reynolds number \( Rh \), the large scale mode transitions to a bimodal behaviour denoting the first bifurcation. For larger \( Rh \), we get to another bifurcation with a unimodal distribution where the \( \psi_{0,1} \) mode no longer reverses for the whole duration of the simulation. Using the truncated Euler equations we are able to construct a minimal model, which consists of 15 modes, that effectively captures these bifurcations which occur on top of a turbulent background flow. The control parameter in the TEE model is the ratio of the enstrophy and energy at time \( t \) captures these bifurcations which occur on top of a turbulent background flow. The control parameter in the TEE model is the ratio of the enstrophy and energy at time \( t \) is now replaced by \( k \) for the Navier-Stokes equations.

We also compared the power spectra from the time series of \( \psi_{0,1} \) and the PDF of the mean waiting times \( \tau \) between the two symmetric turbulent states for both the Navier-Stokes and the TEE systems. We found the power spectra to scale as \( S(f) \propto 1/f^{1.5} \) and the PDFs of the mean waiting times to obey a power law \( P(\tau) \propto \tau^{-1.5} \) in the bimodal regime. The TEE model demonstrates that a small number of modes is enough to reproduce the scaling laws for the power spectrum and PDF of the mean waiting time. The origin of such a spectrum still remains to be fully understood.

In a broader context of understanding the behaviour of zonal jet like structures in atmospheric and oceanic flows, the minimal model from the TEE seems to have all the necessary ingredients to capture the bifurcations of the large scale mode. It would be of interest therefore to test whether the TEE can capture the bifurcations observed in the turbulent regime of flows from more realistic models, such as more advanced quasi-geostrophic models and/or shallow water models for the dynamics of the atmosphere and ocean.

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