Relating the Green-Schwarz and Pure Spinor Formalisms for the Superstring

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Abstract: Although it is not known how to covariantly quantize the Green-Schwarz (GS) superstring, there exists a semi-light-cone gauge choice in which the GS superstring can be quantized in a conformally invariant manner. In this paper, we prove that BRST quantization of the GS superstring in semi-light-cone gauge is equivalent to BRST quantization using the pure spinor formalism for the superstring.

Keywords: Superstrings and Heterotic Strings, BRST Symmetry.
1. Introduction

Four years ago, a new manifestly super-Poincaré covariant formalism was introduced for quantizing the superstring [1]. This formalism has been recently used for computing covariant multiloop amplitudes [2] and for quantization in an $AdS_5 \times S^5$ Ramond-Ramond background [3]. The main new ingredient in this formalism is a BRST operator constructed from fermionic Green-Schwarz (GS) constraints and a pure spinor bosonic ghost.

At the present time, a geometrical understanding of this “pure spinor” BRST operator is still lacking.\footnote{However, there are some indications that the pure spinor BRST operator can be interpreted as a twisted N=2 worldsheet supersymmetry generator using either a “twistor-superstring” formalism [4] or a WZNW model [5].} It is therefore important to try to relate the pure spinor BRST operator with BRST operators which appear in other formalisms for the superstring. In reference [6], this was done for the RNS formalism where the pure spinor BRST operator was related to the sum of the RNS BRST operator and $\eta$ ghost. It was also shown in [7] that the cohomology of the pure spinor BRST operator reproduces the light-cone GS spectrum.\footnote{There also exist “extended” versions of the pure spinor formalism in which the pure spinor constraint on the bosonic ghost is relaxed [8][9]. The BRST operator in these extended pure spinor formalisms have been related to the RNS BRST operator [10], to the light-cone GS spectrum [11], and to the original pure spinor BRST operator [12].}

Although it is not known how to covariantly quantize the GS superstring, one can choose a semi-light-cone gauge in which $\kappa$-symmetry is fixed but conformal invariance is preserved [13]. The worldsheet action is quadratic in semi-light-cone gauge, so quantization is straightforward using the BRST method. Although Lorentz invariance is not manifest in this gauge, one can construct Lorentz generators whose algebra closes up to a BRST-trivial quantity.
In this paper, the pure spinor BRST operator will be related to the semi-light-cone GS BRST operator by a similarity transformation, proving the equivalence of the cohomologies. Note that a similar result was obtained earlier for the $d = 10$ \cite{14} and $d = 11$ superparticle \cite{15} in semi-light-cone gauge. However, relating the BRST operators for the superstring is more complicated than for the superparticle because of normal-ordering subtleties. It would be interesting to try to further generalize this equivalence proof to the $d = 11$ supermembrane for which pure spinor \cite{16} and semi-light-cone gauge descriptions exist at least at the classical level.

In section 2 of this paper, we review the equivalence proof for the $d = 10$ superparticle. And in section 3, we generalize this equivalence proof to the superstring.

2. Review of Equivalence Proof for Superparticle

2.1 Brink-Schwarz superparticle in the semi-light-cone gauge

The $d = 10$ Brink-Schwarz superparticle is described by the action

$$S = \int d\tau \left( \dot{x}^m P_m - \frac{i}{2} (\dot{\theta} \gamma^m \theta) P_m + e P^m P_m \right), \quad (2.1)$$

where $P_m$ is the conjugate momentum for $x^m$, $\theta^\alpha$ is a spinor of $SO(9,1)$, $e$ is a Lagrangian multiplier which enforces the mass-shell condition, $\gamma^m_{\alpha\beta}$ are $16 \times 16$ symmetric gamma matrices which satisfy $\gamma^{(m}_{\alpha\beta} \gamma^{n)}_{\gamma\lambda} = 2 \eta^{mn} \delta^{\lambda}_{\alpha}$, and we use the metric convention $\eta^{mn} = \text{diag}(-1,1,1,\ldots)$.

The action (2.1) is spacetime supersymmetric and is also invariant under local kappa transformations, which are generated by the first-class part of the fermionic constraints

$$d_\alpha = p_\alpha + 2 P^m (\gamma^m \theta)_\alpha, \quad (2.2)$$

which satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = 4 P_m \gamma^m_{\alpha\beta}. \quad (2.3)$$

Eight of the $d_\alpha$ constraints are first-class and the other eight are second-class. There is no simple way of covariantly out the first-class constraints, preventing covariant quantization. Nevertheless, it is possible to use a non-Lorentz covariant gauge fixing to quantize this theory by imposing the condition $(\gamma^+ \theta)_\alpha = 0$ with the assumption that $P^+ \neq 0$ (where $\gamma^\pm = \gamma^0 \pm \gamma^9$ and $P^\pm = P^0 \pm P^9$). In this semi-light-cone gauge, the action takes the form

$$S = \int d\tau \left( \dot{x}^m P_m + \frac{i}{2} S_a S_a + e P^m P_m \right), \quad (2.4)$$

where $S_a = \sqrt{\frac{P^+}{2}} (\gamma^- \theta)_a$ and $a = 1$ to 8 is an $SO(8)$ chiral spinor index. Canonical quantization of (2.4) implies that $\{S_a, S_b\} = \delta_{ab}$ and therefore $\sqrt{2} S_a$ acts like a spinor version of $SO(8)$ Pauli matrices that satisfy
\[
\sigma^j_{a\dot{a}} \sigma^j_{b\dot{b}} + \sigma^j_{ba} \sigma^j_{ab} = 2 \delta_{ab} \delta_{\dot{a}\dot{b}}. 
\] 
(2.5)

Making use of the usual BRST method, the action (2.4) can be gauge-fixed to

\[
S = \int d\tau \left( \dot{x}^m P_m + \frac{i}{2} \dot{S}_a S_a - \frac{1}{2} P^m P_m + icb \right),
\]
(2.6)

where the BRST charge is

\[
Q = cP^m P_m.
\]
(2.7)

Note that \(\kappa\)-symmetry ghosts do not propagate in semi-light-cone gauge, so the BRST operator only involves the \((b, c)\) reparametrization ghosts.

Because semi-light-cone gauge is not manifestly Lorentz invariant, Lorentz transformation which change the gauge-fixing condition \((\gamma^+ \theta)_\alpha = 0\) need to include a compensating \(\kappa\)-transformation. The resulting Lorentz generators are

\[
N^{i+} = -ix^+ P^- + ix^- P^+,
\]

\[
N^{i\dot{+}} = -ix^\dot{+} P^+ + ix^\dot{+} P^-,
\]

\[
N^{i-} = -ix^i P^- + ix^- P^i - \frac{(S\sigma^i)_{\dot{a}} (S\sigma^j)_{\dot{a}} P^j}{2P^+},
\]

\[
N^{ij} = -ix^i P^j + ix^j P^i - \frac{1}{4}(S_a \sigma^i_{ab} S_b),
\]
(2.8)

where \(x^\pm, P^{\pm}\) are light-cone variables, and \(i, j = 1\) to \(8\). Using the canonical commutation relations \(\{S_a, S_b\} = \delta_{ab}\) and \([P^m, x^n] = -i\eta^{mn}\), one finds that the Lorentz generators of (2.8) satisfy the usual \(SO(9,1)\) Lorentz algebra except for \([N^{i-}, N^{j-}]\), which satisfies

\[
[N^{i-}, N^{j-}] = \left[ Q, -\frac{b(S\sigma^i)_{\dot{a}} (S\sigma^j)_{\dot{a}}}{(P^+)^2} \right].
\]
(2.9)

Since \([N^{i-}, N^{j-}]\) is BRST-trivial, the Lorentz algebra closes up to a gauge transformation when acting on BRST-invariant states. In other words, \(QV = 0\) implies that \([N^{i-}, N^{j-}]V = Q\Omega\) for some \(\Omega\).

2.2 Equivalence between Brink-Schwarz superparticle and pure spinor superparticle

In this subsection, it will be shown that the action and BRST operator for the semi-lightcone Brink-Schwarz superparticle in the previous subsection are related to the action

\[
S = \int d\tau \left( \dot{x}^m P_m - \frac{1}{2} P^m P_m + i\dot{\bar{\theta}}^\alpha p_\alpha + i\dot{\lambda}^\alpha w_\alpha \right),
\]
(2.10)

and BRST operator

\[
Q = \lambda^\alpha d_\alpha
\]
(2.11)
for the pure spinor version of the superparticle \[17\] where \((\lambda^\alpha, w^\alpha)\) are bosonic ghosts satisfying the pure spinor constraint
\[
\lambda^\alpha \gamma^m \gamma_{\alpha \beta} \lambda^\beta = 0.
\] (2.12)

To relate the actions of (2.6) and (2.10), we shall first introduce a new pair of fermionic variables \((\theta^\alpha, p^\alpha)\) which are not related with \(S_a\). And to prevent physical states from depending on these new variables, we shall also introduce a new gauge invariance which allows \(\theta^\alpha\) to be gauged to zero. This new gauge invariance will be generated by the first-class constraints
\[
\hat{d}_\alpha = d_\alpha + \frac{\gamma_m \gamma^+ S \alpha P^m}{\sqrt{P^+}},
\] (2.13)

where \(d_\alpha = p_\alpha + 2P^m(\gamma_m \theta)\alpha\). Or using \(SO(8)\) notation,
\[
\hat{d}_a = \hat{d}_\alpha + 2 S_a \sqrt{P^+},
\]
\[
\hat{d}_\dot{a} = \hat{d}_\dot{a} - 2 P^i (S \sigma^i) \dot{a} \sqrt{P^+},
\] (2.14)

where \(a, \dot{a} = 1\) to 8 are \(SO(8)\) chiral and antichiral spinor indices.

Using (2.3) and the anticommutation relation of the \(S_a\)’s, one can check that the constraints satisfy the first-class algebra
\[
\{\hat{d}_\alpha, \hat{d}_\beta\} = 2 P_m P^m \gamma_{\alpha \beta}^+ \frac{1}{P^+},
\] (2.15)

or in \(SO(8)\) notation,
\[
\{\hat{d}_a, \hat{d}_b\} = \{\hat{d}_\dot{a}, \hat{d}_\dot{b}\} = 0,
\]
\[
\{\hat{d}_a, \hat{d}_\dot{b}\} = \frac{4 P^m P_m \delta_{\dot{a} \dot{b}}}{P^+}.
\] (2.16)

So the semi-light-cone superparticle action which includes the new variables and new gauge invariances is
\[
S = \int d\tau \left( \dot{x}^m P_m + e P^m P_m + i \dot{\theta}^\alpha p_\alpha + \frac{i}{2} \dot{S} a S a + f^a \hat{d}_a \right),
\] (2.17)

where \(f^a\) are fermionic Lagrange multipliers related to the constraint (2.13). To return to the original action in the semi-light-cone gauge, \(\hat{d}_\alpha\) can be used to gauge \(\theta^\alpha = 0\), recovering (2.4). Using the usual BRST method, one can gauge fix the action above and obtain
\[
S = \int d\tau \left( \dot{x}^m P_m - \frac{1}{2} P^m P_m + i \dot{\theta}^\alpha p_\alpha + \frac{i}{2} \dot{S} a S a + i \dot{\lambda}^{\dot{a}} \tilde{w}_\dot{a} + i \dot{\bar{c}} b \right),
\] (2.18)

together with the BRST charge
\[
\hat{Q} = \hat{\lambda}_a \hat{d}_a + \hat{\lambda}_\dot{a} \hat{d}_\dot{a} + c \left( -4 P^- + \frac{4 P^i P^i}{P^+} \right) - \frac{1}{2} \hat{\lambda}_a \hat{\lambda}_a b
\] (2.19)
where $\hat{\lambda}_a$ is an unconstrained bosonic spinor ghost related to the gauge fixing $f^a = 0$, $\hat{\omega}_a$ is its conjugate momentum, and $a, \dot{a}, i = 1 \to 8$. Note that one could rescale the $(b, c)$ ghosts by factors of $P^+$ as

$$b \to \frac{b}{P^+}, \quad c \to cP^+, \quad$$

so that the $c$ ghost would multiply $P_m P^m$ in the BRST operator. But it will be more convenient for generalization to the superstring to leave the superparticle BRST operator in the form of (2.19). This means that $b$ and $c$ are not invariant under Lorentz transformations which change the value of $P^+$.

As before, Lorentz invariance is not manifest but one can define Lorentz generators which commute with the BRST operator and whose algebra closes up to a BRST-trivial quantity. The explicit expression for the Lorentz generators is

$$N^{ij} = -ix^j P^i + ix^i P^j + \frac{1}{2} (\theta \sigma^i p) \lambda + \frac{1}{2} (\lambda \sigma^i \hat{\omega} - \frac{1}{2} (\lambda \sigma^i \hat{\omega}) - \frac{1}{4} (S \sigma^i S)$$

where

$$(A \sigma^i B) = A_a \sigma^i_{ab} B_b, \quad (A \sigma^i B) = \frac{1}{2} A_a \sigma^i_{ac} \sigma^j_{bc} B_b, \quad (A \sigma^i B) = \frac{1}{2} A_a \sigma^i_{ca} \sigma^j_{cb} B_b.$$

The generators obey the usual Lorentz algebra, except for the commutator of $N^{i-}$ with $N^{j-}$ which satisfies

$$[N^{i-}, N^{j-}] = \left[ \hat{Q}, -\frac{b(S \sigma^i a)(S \sigma^j a)}{P^+} \right],$$

indicating that the algebra closes up to a gauge transformation on on-shell states.

It will now be shown that the BRST operator of (2.13) is related by a similarity transformation to the pure spinor BRST operator $Q = \lambda^a d_a$ [14]. The first step will be to show that the cohomology of the BRST operator in (2.14) is equivalent to the cohomology of $Q' = \hat{\lambda}_a \hat{d}_a + \hat{\lambda}_a \hat{d}_a$ in a Hilbert space without the $(b, c)$ ghosts and with the condition $\lambda_a \lambda_a = 0$.

Suppose that a state $V$ is in the cohomology of $Q'$, then $V$ is annihilated by the operator

$$Q'' = \hat{\lambda}_a \hat{d}_a$$

up to terms proportional to $\hat{\lambda}_a \hat{\lambda}_a$, i.e.,

$$Q'' V = \hat{\lambda}_a \hat{\lambda}_a W,$$
for some $W$. Since

$$(Q'')^2 = \frac{2\hat{\lambda}_a \hat{\lambda}_d P^m P_m}{P^+},$$

(2.24)

implies that

$$Q'' W = \frac{2P^m P_m V}{P^+}.$$  

(2.25)

which implies that the state $\hat{V} = V + 2cW$ is annihilated by $\hat{Q}$. And if $V$ is BRST-trivial up to terms proportional to $\hat{\lambda}_a \hat{\lambda}_d$, i.e.,

$$V = Q'' \Omega + \hat{\lambda}_a \hat{\lambda}_d Y$$

(2.26)

for some $\Omega$ and $Y$, then $\hat{V} = V + 2cW = \hat{Q}(\Omega - 2cY)$ is also BRST-trivial.

To prove the converse, i.e. that any state in the cohomology of $\hat{Q}$ maps to a state in the cohomology of $Q'$, suppose that the state $\hat{V}$ is in the cohomology of $\hat{Q}$. By choosing the $b$ ghost to annihilate the vacuum, one can write $\hat{V} = V + cW$ for some $V$ and $W$. Then $\hat{Q}V = \frac{1}{2} \hat{\lambda}_a \hat{\lambda}_d W$ implies $Q' V = 0$ in the reduced Hilbert space. And $\hat{V} = Q \Lambda$ where $\Lambda = \Omega + cY$ implies that $V = \hat{Q} \Omega - \frac{1}{2} \hat{\lambda}_a \hat{\lambda}_d Y$, i.e. $V = Q' \Omega$ in the reduced Hilbert space. So the cohomology of $Q'$ is equivalent to the cohomology of $\hat{Q}$.

Now, it will be shown that the cohomology of $Q' = \hat{\lambda}_a d_a + \lambda_a \hat{d}_a$ is equivalent to the cohomology of $Q = \lambda^\alpha d_\alpha$ in a Hilbert space independent of $S_a$, where $\lambda^\alpha$ is a pure spinor. To do this, it is convenient to define an antichiral spinor $r^a$ which satisfies $r^a \lambda_a = 1$ and $r_a r^a = 0$. One can then use $r^a$ to split the fields $S_a$ and $\hat{\lambda}_a$ as³

$$S_a = S^1_a + S^2_a,$$

$$\hat{\lambda}_a = \hat{\lambda}^1_a + \hat{\lambda}^2_a,$$

(2.27)

where

$$S^1_a = \frac{1}{2} (\sigma^j \lambda)_a (S \sigma^j r), \quad S^2_a = \frac{1}{2} (\sigma^j r)_a (S \sigma^j \lambda),$$

$$\hat{\lambda}^1_a = \frac{1}{2} (\sigma^j \lambda)_a (\hat{\lambda} \sigma^j r), \quad \hat{\lambda}^2_a = \frac{1}{2} (\sigma^j r)_a (\hat{\lambda} \sigma^j \lambda).$$

(2.28)

The new fields have the anticommutation relations

$$\{ S^1_a, S^2_b \} = \frac{1}{2} (\sigma^i \lambda)_a (\sigma^i r)_b,$$

$$\{ S^1_a, S^1_b \} = \{ S^2_a, S^2_b \} = 0.$$  

(2.29)

³In order that the cohomology remain non-trivial after including $r^a$, states will only be allowed to depend on $r_a$ in the combination $(\sigma^i r)_a (\sigma^i \lambda)_b$. If states could depend arbitrarily on $r_a$, the cohomology would become trivial since $Q V = 0$ implies that $Q (\theta^a r_a V) = V$. 

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And the charge $Q'$, written in terms of the new fields, reads

$$Q' = \lambda_1^a d_a + \lambda_2^a d_a + 2 \tilde{\lambda}_a^1 S_a^1 \sqrt{P^+} + 2 \tilde{\lambda}_a^2 S_a^2 \sqrt{P^+} + \lambda_\dot{a} d_{\dot{a}} - \frac{2 (S^2 \sigma^i \lambda) P^i}{\sqrt{P^+}}.$$  \hspace{1cm} (2.30)

Performing the similarity transformation

$$Q' \rightarrow e^{-\frac{d_a S_a^2}{2 \sqrt{P^+}}} Q' e^{\frac{d_a S_a^2}{2 \sqrt{P^+}}} = \lambda_1^a d_a + \lambda_2^a d_a + 2 \tilde{\lambda}_a^1 S_a^1 \sqrt{P^+},$$  \hspace{1cm} (2.31)

one obtains $Q' = \lambda^\alpha d_\alpha + 2 \tilde{\lambda}_a^1 S_a^1 \sqrt{P^+}$, where $\lambda^\alpha$ is a pure spinor defined by

$$[\lambda_{\dot{a}}, \lambda_\alpha] = [\lambda_\alpha, \tilde{\lambda}_a^1].$$  \hspace{1cm} (2.32)

Using the quartet argument, the cohomology of $Q' = Q + 2 \tilde{\lambda}_a^1 S_a^1 \sqrt{P^+}$ is equivalent to the cohomology of $Q = \lambda^\alpha d_\alpha$ in the Hilbert space independent of $\tilde{\lambda}_a^2$ and $S_a^2$, and independent of their conjugate momenta $\tilde{w}_{\dot{a}}^1$ and $S_{\dot{a}}^2$. Therefore, the action and BRST operator for the Brink-Schwarz superparticle in semi-light-cone gauge are equivalent to the action

$$S = \int d\tau \left( \dot{x}^m P_m - \frac{1}{2} P^m P_m + i \dot{\theta}^\alpha p_\alpha + i \dot{\lambda}^\alpha w_\alpha \right),$$  \hspace{1cm} (2.33)

and the BRST operator $Q = \lambda^\alpha d_\alpha$ where $(\lambda \gamma^m \lambda) = 0$.

3. Equivalence Proof for Green-Schwarz superstring

3.1 Green-Schwarz superstring in the semi-light-cone gauge

As in the covariant Brink-Schwarz superparticle action, the GS superstring action contains first-class and second-class constraints which are difficult to separate in a covariant manner. Quantization can be performed in a conformally invariant manner by gauge-fixing $\kappa$-symmetry using the condition $(\gamma + \theta)_\alpha = 0$, assuming that $\partial X^+ \neq 0$. In this semi-light-cone gauge, the GS action is written as

$$S = \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial X^m \overline{\partial} X_m + \frac{1}{2} S_a \overline{\partial} S_a + \text{anti-holomorphic terms} \right],$$  \hspace{1cm} (3.1)

where $S_a = \sqrt{\frac{1}{2} \partial X^+ (\gamma - \theta)_a}$ is a chiral $SO(8)$ spinor. The anti-holomorphic terms in (3.1) depend if one is discussing the Type II or heterotic superstring, and will be ignored in this paper.

In semi-light-cone gauge, one can construct a BRST charge in the standard manner as

$$Q = \int d\tau (c T_m + b c \partial c)$$  \hspace{1cm} (3.2)

with the action

$$S = \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial X^m \overline{\partial} X_m + \frac{1}{2} S_a \overline{\partial} S_a + b \overline{c} + \text{anti-holomorphic terms} \right],$$  \hspace{1cm} (3.3)
where
\[ T_m = -\partial X^- \partial X^+ + \partial X^i \partial X^i - \frac{1}{2} S_a \partial S_a + \frac{1}{2} \partial^2 (\log \partial X^+) \] (3.4)
is the stress tensor. The term \( \frac{1}{2} \partial^2 (\log \partial X^+) \) in the stress tensor comes from the non-covariant gauge-fixing and, as will be shown below, is necessary both for quantum conformal invariance and quantum Lorentz invariance.

Using the OPE’s
\[ X^m(y, \bar{y}) X^n(z, \bar{z}) \rightarrow \frac{1}{2} \eta^{mn} \log |y - z|^2, \]
\[ S_a(y) S_b(z) \rightarrow \delta_{ab} \frac{y - z}{|y - z|^2}, \] (3.5)
one finds that \( T_m \) has central charge \( c = 26 \), so \( Q \) is nilpotent at the quantum level.

Although Lorentz invariance is not manifest, one can construct Lorentz generators that commute with \( Q \). The holomorphic components of the currents for these generators are
\[
N^{ij} = -X^i \partial X^j + X^j \partial X^i - \frac{1}{4} (S \sigma^{ij} S), \\
N^{++} = -\frac{1}{2} X^+ \partial X^- + \frac{1}{2} X^- \partial X^+ , \\
N^{i+} = -X^i \partial X^+ + X^+ \partial X^i , \\
N^{i-} = -X^i \partial X^- + X^- \partial X^i - \frac{(S \sigma^i)_{\dot{a}} (S \sigma^j)_{\dot{a}} \partial X^j}{2 \partial X^+} .
\] (3.6)
As in the superparticle case, the algebra closes up to a BRST-trivial operator:
\[
\left[ \int dy N^{i-}(y), \int dz N^{j-}(z) \right] = \left[ Q, \int dz \left[ -\frac{b(S \sigma^i)_{\dot{a}} (S \sigma^j)_{\dot{a}} \partial X^j}{(\partial X^+)^2} \right](z) \right] . \] (3.7)

So after including the term \( \frac{1}{2} \partial^2 (\log \partial X^+) \) in \( T_m \), the Lorentz algebra closes on on-shell states up to a gauge transformation.

### 3.2 Equivalence between Green-Schwarz and pure spinor formalisms

As with the superparticle, we shall add a new pair of fermionic degrees of freedom \((\theta^\alpha, p_\alpha)\) not related to \( S_a \), and a new gauge invariance which allows \( \theta^\alpha \) to be gauged to zero. The new gauge invariance will be generated by sixteen first-class constraints \( \hat{d}_\alpha \) constructed from the \( S_a \) variables and the first and second-class GS constraints \( d_\alpha \). For the GS superstring, the holomorphic first and second-class constraints are
\[ d_\alpha = p_\alpha + (\theta \gamma^m)_{\alpha} (2 \partial X_m + (\theta \gamma_m \partial \theta)) , \] (3.8)
which satisfy the OPE’s
\[ d_\alpha(y) d_\beta(z) \rightarrow \frac{4 \gamma^m_{\alpha \beta} \Pi_m}{y - z} \] (3.9)

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4. Although there are many papers which discuss anomalies in semi-light-cone-gauge for the GS superstring, we are not aware of any discussion which uses this BRST method.
where \( \Pi^m = \partial X^m - (\theta \gamma^m \partial \theta) \).

By combining these constraints with the \( S_a \) variables satisfying the OPE's

\[
S_a(y)S_b(z) \rightarrow \frac{\delta_{ab}}{y - z} ,
\]

one can construct the sixteen first-class constraints

\[
\begin{align*}
\hat{d}_a &= d_a + 2S_a \sqrt{\Pi^+} , \\
\hat{d}_\dot{a} &= d_\dot{a} - \frac{2 \Pi^i (S\sigma^i)_{\dot{a}}}{\sqrt{\Pi^+}} + \frac{(S\sigma^j \partial \theta)(\bar{\sigma}^j \partial \theta)_{\dot{a}}}{4\Pi^+} - \frac{4 \partial^2 \theta_{\dot{a}}}{\Pi^+} + \frac{2 \partial \Pi^+ \partial \theta_{\dot{a}}}{(\Pi^+)^2} ,
\end{align*}
\]

where \( a, \dot{a} = 1 \) to 8.

The constraints of (3.11) obey the first-class algebra:

\[
\begin{align*}
\hat{d}_a(y)\hat{d}_b(z) &\rightarrow \tilde{T}(z)\delta_{ab} \frac{y}{y - z} , \\
\hat{d}_\dot{a}(y)\hat{d}_b(z) &\rightarrow \text{regular} , \\
\hat{d}_a(y)\hat{d}_\dot{b}(z) &\rightarrow \text{regular} ,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{T} &= -4 \Pi^- + \frac{8 S_a \partial \theta_a}{\sqrt{\Pi^+}} - \frac{8 \Pi^i (S\sigma^i \partial \theta)}{(\Pi^+)^{3/2}} - \frac{2 S_a \partial S_a}{\Pi^+} + \frac{4 \Pi^i \Pi^i}{\Pi^+} \\
&\quad + \frac{4 (S\sigma^j \partial \theta)(\bar{\sigma}^j \partial \theta)}{(\Pi^+)^2} - \frac{16 \partial^2 \theta_c \partial \theta_c}{(\Pi^+)^2} - \frac{2 \partial^2 (\log \Pi^+)}{\Pi^+} .
\end{align*}
\]

\( \tilde{T} \) is also a first-class quantity which satisfies the OPE's

\[
\begin{align*}
\tilde{T}(y)\tilde{T}(z) &\rightarrow \text{regular} , \\
\tilde{T}(y)\hat{d}_a(z) &\rightarrow \text{regular} , \\
\tilde{T}(y)\hat{d}_\dot{a}(z) &\rightarrow \text{regular} .
\end{align*}
\]

The following OPE's were used in these calculations:

\[
\begin{align*}
d_a(y)d_b(z) &\rightarrow -\frac{4 \delta_{ab} \Pi^+}{y - z} , \\
d_a(y)d_\dot{a}(z) &\rightarrow -\frac{4 \sigma^j_{\dot{a}} \Pi^i}{y - z} , \\
d_\dot{a}(y)d_\dot{b}(z) &\rightarrow -\frac{4 \delta_{\dot{a} \dot{b}} \Pi^-}{y - z} , \\
d_a(y)\Pi^+(z) &\rightarrow \text{regular} ,
\end{align*}
\]
\[ d_a(y)\Pi^- (z) \rightarrow \frac{4\partial \theta_a}{y-z}, \]
\[ d_a(y)\Pi^i (z) \rightarrow 2(\sigma^i \partial \theta)_a y-z, \]
\[ d_a(y)\Pi^+ (z) \rightarrow \frac{4\partial \theta_a}{y-z}, \]
\[ d_a(y)\Pi^- (z) \rightarrow \text{regular}, \]
\[ d_a(y)\Pi^i (z) \rightarrow \frac{2(\sigma^i \partial \theta)_a}{y-z}. \]  \hspace{1cm} (3.15)

Using the first-class constraints \( \widehat{d}_a \) and \( \widehat{T} \), one can construct a nilpotent BRST operator in the usual manner as
\[ \widehat{Q} = \int dz (c\widehat{T} + \widehat{\lambda}_a \widehat{d}_a + \widehat{\lambda}_a \widehat{d}_a - \frac{1}{2} \widehat{\lambda}_a \widehat{\lambda}_a b), \]  \hspace{1cm} (3.16)
with the worldsheet action
\[ S = \frac{1}{\pi} \int d^2 z \left[ \frac{1}{2} \partial X^m \overline{\partial} X_m + \frac{1}{2} S_a \overline{\partial} S_a + p_\alpha \overline{\partial} \theta^\alpha + \overline{\omega}_\alpha \overline{\partial} \lambda^\alpha + b\overline{c} \right], \]  \hspace{1cm} (3.17)
where \( (b, c) \) are the fermionic ghosts for the bosonic constraint \( \widehat{T} \), and \( (\widehat{\lambda}^\alpha, \overline{\omega}_\alpha) \) are unconstrained bosonic spinorial ghosts.

As in the superparticle, one could rescale \( b \rightarrow b \Pi^- \) and \( c \rightarrow c\Pi^+ \) so that \( c \) multiplies the standard stress tensor in the BRST operator. This can be done at the quantum level using the similarity transformation
\[ Q \rightarrow e^{-\int dz b c \log(\Pi^+) Q c + \int dz b c \log(\Pi^+) = 4 \int dz (cT_m + bc \partial c + \ldots)} \]  \hspace{1cm} (3.18)
where \( \int dz (cT_m + bc \partial c) \) is the BRST operator of (3.3) and \( \ldots \) involves the new variables \( (\theta^\alpha, p_\alpha) \) and \( (\widehat{\lambda}^\alpha, \overline{\omega}_\alpha) \). However, it will be more convenient to not rescale the \( (b, c) \) ghosts so that \( \widehat{Q} \) has the simple structure of (3.16). The usual stress tensor can be obtained from \( \widehat{Q} \) by
\[ T = \{ \widehat{Q}, 4 b \Pi^+ - \overline{\omega}_\alpha \overline{\partial} \theta^\alpha \} \]
\[ = \Pi^m \Pi_m - \frac{1}{2} S_a \overline{\partial} S_a - d_a \overline{\partial} \theta^\alpha - \overline{\omega}_\alpha \overline{\partial} \lambda^\alpha - b\partial c - \frac{1}{2} \partial^2 (\log \Pi^+) , \]  \hspace{1cm} (3.19)
so \( \frac{1}{4} b \Pi^+ - \overline{\omega}_\alpha \overline{\partial} \theta^\alpha \) plays the role of the usual \( b \) ghost. Using the OPE’s of (3.5) and (3.15), one can verify that \( T \) has no central charge. Note that the \( (b, c) \) ghosts in (3.19) have not been rescaled so they carry conformal weight \( (1, 0) \) instead of \( (2, -1) \).

As before, Lorentz invariance is not manifest but Lorentz generators can be constructed which leave \( \widehat{Q} \) invariant. The holomorphic components of the currents for these generators are
from the relation
\[ V \text{ only depends on } W \text{ for some } \hat{\lambda} \text{.} \]

It follows that
\[ \hat{\lambda} \text{ annihilated by } \hat{Q} \text{.} \]

So the Lorentz algebra closes on on-shell states up to a gauge transformation.

Once again, they obey the usual Lorentz algebra except for \( N_{i-} \) with \( N_{j-} \), which satisfies
\[ \left[ \int dy N_{i-}(y), \int dz N_{j-}(z) \right] = \left[ \hat{Q}, \int dz \left[ -\frac{b(S\sigma^i)\hat{a}(S\sigma^j)\hat{a}}{4\Pi^+} \right](z) \right] \, . \]

So the Lorentz algebra closes on on-shell states up to a gauge transformation.

The BRST operator \( \hat{Q} \) will now be related to the pure spinor BRST operator \( Q = \int dz \lambda^a d_a \). The first step is to relate the cohomology of the BRST charge (3.16) to the cohomology of a charge \( Q' = \int dz (\lambda_a \hat{d}_a + \lambda_\hat{a} \hat{d}_\hat{a}) \) where \( \lambda_{\hat{a}} \) has to satisfy \( \lambda_{\hat{a}} \lambda_{\hat{a}} = 0 \).

Suppose a state \( V \) is in the cohomology of \( Q' \), which implies that \( V \) is annihilated by
\[ Q'' = \int dz (\lambda_a \hat{d}_a + \lambda_{\hat{a}} \hat{d}_{\hat{a}}) \]

up to terms proportional to \( \lambda_{\hat{a}} \hat{\lambda}_{\hat{a}} \) or its derivatives. So
\[ Q'' V = \sum_{n=0}^{\infty} \partial^n (\lambda_a \hat{d}_a) W_{(n)} \, , \]
for some \( W_{(n)} \) for \( n = 0 \) to \( \infty \). In addition, suppose that \( V \) has no poles with \( \lambda_a \lambda_{\hat{a}} \), i.e. \( V \) only depends on \( w_{\hat{a}} \) in combinations which commute with the constraint on \( \lambda_{\hat{a}} \). Then, from the relation
\[ (Q'')^2 = \int dy \frac{(\lambda_a \hat{d}_a) \hat{d}_a}{2} \, , \]
it follows that
\[ Q'' W_{(n)} = \int dy \frac{1}{2} \hat{T}(y)V(z) \frac{(y - z)^n}{n!} \, . \]

Using the above equations, it is easy to check that the state \( \hat{V} = V + 2 \sum_n (\partial^n c) W_{(n)} \) is annihilated by \( \hat{Q} \). Also, if a state \( V \) is BRST trivial up to terms \( \partial^n (\lambda_a \hat{d}_a) \), i.e.
\[ V = Q'' \Omega + \sum_{n=0}^{\infty} \partial^n (\lambda_a \hat{d}_a) Y_{(n)} \, , \]
for some \( Y(n) \), then \( \hat{V} \) is also BRST trivial with respect to \( \hat{Q} \) since

\[
\hat{V} = V + 2 \sum_n (\partial^n c)W(n) = \hat{Q} (\Omega - 2 \sum_n (\partial^n c)Y(n)) .
\]

(3.27)

To complete the proof, one needs to show that any state in the cohomology of \( \hat{Q} \) can be mapped to a state in the cohomology of \( Q' \). To show this, first note that any ghost-number one state \( \hat{V} \) in the cohomology of \( \hat{Q} \) with non-zero \( P^+ \) momentum can be expressed as \( \hat{V} = V + cW \) for some \( V \) and \( W \) which are independent of the \((b,c)\) ghosts. This is because in light-cone gauge, the constraints \( \bar{T} \) and \( \bar{a}_a \) can be used to gauge away dependence on all variables except for \( X^i, S^a \) and the zero mode of \( X^+ \) in the integrated light-cone vertex operator \( \int d\bar{z}V_{\text{LC}}(X^i, S^a, X_0^+) \). So using the standard DDF construction, one can define a BRST-invariant vertex operator \( \int d\bar{z}V_{\text{DDF}}(X^i, S^a, X^+, \theta^a) \) such that \( V_{\text{DDF}} \) coincides with \( V_{\text{LC}} \) when \( \partial X^+ = \theta^a = 0 \). Since \( \int d\bar{z}V_{\text{DDF}} \) is BRST-invariant, \( \hat{Q}V_{\text{DDF}} = \partial \hat{V} \) for some \( \hat{V} \).

From the structure of the BRST operator \( \hat{Q} \) of (3.16), one learns that \( \hat{V} = \hat{\lambda}^a V_a + cW \) where \( V_a \) and \( W \) are the double poles of \( \bar{a}_a \) and \( \bar{T} \) with \( V_{\text{DDF}} \). Since \( \partial(QV) = \hat{Q}QV_{\text{DDF}} = 0 \) and since there are no constant worldsheet fields, \( \hat{Q}V = 0 \). Therefore, \( \hat{V} = \hat{\lambda}^a V_a + cW \) is a ghost-number one vertex operator in the BRST cohomology which represents the light-cone state \( \int d\bar{z}V_{\text{LC}} \).

Since \( \hat{Q}(V + cW) = 0 \) implies that \( Q'\hat{V} = \frac{1}{2} \hat{\lambda}^a \hat{\lambda}_a W \), \( Q'V = 0 \) in the reduced Hilbert space. And \( \hat{V} = \hat{Q} \Lambda \) where \( \Lambda = \Omega + cY \) implies that \( V = Q'\Omega - \frac{1}{2} \hat{\lambda}_a \hat{\lambda}_a Y \), i.e. \( V = Q'\Omega \) in the reduced Hilbert space. So the cohomology of \( Q' \) is equivalent to the cohomology of \( \hat{Q} \) for states with non-zero \( P^+ \) momentum.\(^5\)

To complete the equivalence proof, the relation between \( Q' \) and the pure spinor BRST operator \( Q = \int d\bar{z} \lambda^a \bar{d}_a \) with \( (\lambda^\gamma \lambda) = 0 \) has to be shown. For this purpose, it is convenient to define an antichiral spinor \( r_a \) satisfying \( r_ar_a = 0 \) and \( r_a \hat{\lambda}_a = 1 \), and to split the fields \( S_a \) and \( \hat{\lambda}_a \) into\(^6\)

\[
S_a = S_a^1 + S_a^2, \quad \hat{\lambda}_a = \hat{\lambda}_a^1 + \hat{\lambda}_a^2,
\]

(3.28)

where

\[
S_a^1 = \frac{1}{2}(\sigma^j \lambda)_a(S\sigma^j r), \quad S_a^2 = \frac{1}{2}(\sigma^j r)_a(S\sigma^j \lambda), \quad \hat{\lambda}_a^1 = \frac{1}{2}(\sigma^j \lambda)_a(\hat{\lambda}\sigma^j r), \quad \hat{\lambda}_a^2 = \frac{1}{2}(\sigma^j r)_a(\hat{\lambda}\sigma^j \lambda),
\]

(3.29)

which have the OPE’s:

---

\(^5\)The equivalence proof does not hold for states of zero momentum since such states cannot be described by light-cone vertex operators. For example, the stress tensor \( T = \{ \hat{Q}, \frac{4}{\tilde{\mu}} \partial^+ + \tilde{\omega}_a \partial^a \} \) of (3.10) is BRST trivial, but the stress tensor in the pure spinor formalism in the BRST cohomology since there are no states of negative ghost-number.

\(^6\)As in the superparticle, \( r_a \) will only be allowed to appear in the combination \( (\sigma^j r)_a(\sigma^j \lambda)_b \) so that the cohomology remains non-trivial.
\begin{align}
S^1_a(y)S^2_b(z) & \rightarrow \frac{(\sigma^i\lambda)_a(\sigma^i\rho)_b}{2(y-z)}, \\
S^1_a(y)S^1_b(z) & = S^2_a(y)S^0_b(z) \rightarrow \text{regular}.
\end{align}

(3.30)

In terms of these fields,
\begin{align}
Q' & = \int dz(\lambda_\alpha d_\alpha + \tilde{\lambda}_\alpha \tilde{d}_\alpha) \\
& = \int dz(\lambda_\alpha d_\alpha - \frac{2\Pi'(S^2\sigma^i\lambda)}{\Pi^+} + \frac{2}{\Pi^+} \frac{S^1_aS^2_b : \lambda_\alpha \partial\theta_\alpha}{\Pi^+} + \frac{(S^2\sigma^i\lambda)(S^2\sigma^i\partial\theta)}{\Pi^+} \\
& \quad - 4 \frac{\partial\theta_\alpha \lambda_\alpha}{\Pi^+} + 2 \frac{\partial\Pi^+ \partial\theta_\alpha \lambda_\alpha}{(\Pi^+)^2} + \tilde{\lambda}_1 d_\alpha + \tilde{\lambda}_2 d_\alpha \\
& \quad + 2\tilde{\lambda}_1 S^2_a \sqrt{\Pi^+} + 2\tilde{\lambda}_2 S^1_a \sqrt{\Pi^+}),
\end{align}

(3.31)

where :: denotes normal ordering, i.e. :S^1_aS^2_b(z) := \int d\gamma S^1_a(y)S^2_b(z)(y-z)^{-1}. Performing the similarity transformation Q' \rightarrow e^{-\int dzA}Q'e^{\int dzA} where A = \frac{d_\alpha S^2_a}{2\sqrt{\Pi^+}}, one obtains
\begin{align}
Q' & = \int dz(\lambda_\alpha d_\alpha - \frac{2}{\Pi^+} \frac{S^1_aS^2_b : \lambda_\alpha \partial\theta_\alpha}{\Pi^+} + \frac{4(\partial\theta_\alpha \lambda_\alpha)(\partial\lambda_\beta r_\beta)}{\Pi^+} - \frac{2\partial\Pi^+ \partial\theta_\alpha \lambda_\alpha}{(\Pi^+)^2} \\
& \quad + \tilde{\lambda}_1 d_\alpha + 2\tilde{\lambda}_2 S^1_a \sqrt{\Pi^+}).
\end{align}

(3.32)

To simplify further the BRST charge, one performs the additional similarity transformation Q' \rightarrow e^{-\int dzB}Q'e^{\int dzB} where
\begin{align}
B & = - \frac{\partial\Pi^+}{2\Pi^+} + \frac{1}{2} :S^1_aS^2_a : \log \Pi^+ + \frac{4(\partial\theta_\alpha \lambda_\alpha)(\partial\lambda_\beta r_\beta)}{\Pi^+},
\end{align}

(3.33)

to obtain
\begin{align}
Q' & = \int dz(\lambda_\alpha d_\alpha + \tilde{\lambda}_1 d_\alpha + 2\tilde{\lambda}_2 S^1_a).
\end{align}

(3.34)

So after performing these similarity transformations, Q' = \int dz(\lambda_\alpha d_\alpha + 2\tilde{\lambda}_2 S^1_a) where \lambda_\alpha is a pure spinor defined by
\begin{align}
[\lambda_\tilde{\alpha}, \lambda_\alpha] = [\lambda_\tilde{\alpha}, \tilde{\lambda}_1].
\end{align}

(3.35)

Making use of the standard quartet argument, the cohomology of Q' = Q + 2\int dz\tilde{\lambda}_\alpha S^1_a is equivalent to the cohomology of Q = \int dz\lambda_\alpha d_\alpha in a Hilbert space independent of S^1_a, \tilde{\lambda}_\alpha^2 and their conjugate momenta S^2_a and \tilde{w}_\alpha^1. So, it has been shown that the Green-Schwarz superstring action and BRST operator in semi-light-cone gauge are equivalent to the action
\begin{align}
S = \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial X^m \partial X_m + p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha + \text{anti-holomorphic terms} \right],
\end{align}

(3.36)
and BRST operator Q = \int dz\lambda_\alpha d_\alpha where \lambda_\alpha is a pure spinor.
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