Regional Reconstruction of Semilinear Caputo Type Time-Fractional Systems Using the Analytical Approach

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Abstract

The aim of this paper is to investigate the concept of regional observability, precisely regional reconstruction of the initial state, for a semilinear Caputo type time-fractional diffusion system. The approaches attempted in this work are both based on fixed point techniques that leads to a successful algorithm which is tested by numerical examples.

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1. Introduction

"An apparent paradox from which one day useful consequences will be drawn", the answer that Leibniz gave when l'Hospital had asked him about the meaning of $\frac{d^n}{dx^n} f(x)$ when $n = \frac{1}{2}$, and then fractional calculus was born.

Even-though fractional calculus is approximately 300 years of age, it only caught so much attention in the last 25 years or so, the main reason being, its capability of a better describing of real world phenomena. In fact a lot of works show that non-integer order ordinary and partial differential equations present, most times, better modeling systems than integer order ones, for instance in [6] the authors presented a model of beam heating process, and with experimental setup (thermo-electrical module), theoretical results were
verified and a high degree of accuracy was obtained in the experimental ones. Also the modeling of the ultra-capacitor is given in [8], and in [7] both examples of the heat beam process and the ultra-capacitor are given. More information on fractional calculus can be found in [1, 12, 13, 14, 22, 24, 32, 35, 36, 37, 39, 40].

Control theory is an important and very active branch of mathematics which serves as a link between theoretical mathematics and its applications in the real world where most processes are modeled by nonlinear distributed parameter systems, which explains the big interest of researchers in the study (Controllability, Observability, Stability...) of nonlinear and semilinear systems. As for the study of linear systems, there exists a very wide literature for integer order system, see [9] - [34] and the references therein, whereas for fractional order systems there is a much less literature, see [17, 26, 29, 30, 38] and the references therein.

Most works in control theory precisely in observability deal with state estimation in the whole evolution domain, namely $\Omega$, of the considered system (global observability) [34], but in the 90’s the regional observability concept was born by professor El Jai et al. [31], and was after that developed by others, its purpose is to estimate the initial state of a given system only in a subregion $\omega \subset \Omega$, with positive Lebesgue measure, whereas for fractional order systems there is a much less literature, see [17, 26, 29, 30, 38] and the references therein.

We lay out, in this section, some definitions and properties, which will be used all over this manuscript. We will start with the definition of the Caputo left fractional derivative.

**Definition 2.1.** [2] The left sided fractional derivative of order $\alpha \in [0, 1]$ in Caputo’s sense of $y(x, t)$ with respect to $t$, is given by the following expression

$$\begin{align*}
\mathcal{D}_{0+}^\alpha y(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} y(x,s)ds.
\end{align*}$$

For two Banach spaces $E$ and $F$ we denote the space of all linear bounded mappings defined from $E$ to $F$ by $\mathcal{L}(E,F)$ and $\mathcal{L}(E) := \mathcal{L}(E,E)$, also we mean by $L^p(0,T;E)$ (resp. $L^p_{loc}(0,T;E)$), the space of all vector-valued functions $f$ going from the interval $[0,T]$ to $E$, which are measurable, such that $\|f(.)\|_E$ is in $L^p(0,T)$ (resp. $L^p_{loc}(0,T)$).

The two following propositions show, respectively, the existence of the convolution between an operator and a vector-valued function, and the Young inequality for this convolution [41].

**Proposition 2.2.** [41] Let $E$ and $F$ be two Banach spaces. Let’s consider $v \in L^1_{loc}(0,T;E)$ and $T : [0,T] \to \mathcal{L}(E,F)$ be strongly continuous. Then the convolution

$$\begin{align*}
(T \ast v)(t) := \int_0^t T(t-s)v(s)ds,
\end{align*}$$

exists (in the Bochner sense) and is a continuous function $(T \ast v : [0,T] \to F)$. 

2. Preliminary Notes

We lay out, in this section, some definitions and properties, which will be used all over this manuscript.

We will start with the definition of the Caputo left fractional derivative.
The previous convolution \((*)\) is called the Laplace convolution operator.

**Proposition 2.3.** [41] With the same considerations as in the previous proposition. Let’s consider \(p, q, r \geq 1\) such that \(\frac{1}{q} + \frac{1}{p} = \frac{1}{r}\).

If \(T \in L^r(0, T; \mathcal{L}(E, F))\) and \(v \in L^q(0, T; E)\), then \(T * v \in L^r(0, T; F)\) and we have
\[
\|T * v\|_{L^r(0, T; F)} \leq \|v\|_{L^q(0, T; E)} \|T\|_{L^r(0, T; \mathcal{L}(E, F))}.
\]

For more information about vector-valued analysis see [21, 41].

For a linear operator \(A\), the set \(\sigma(A) := \left\{ \lambda \in \mathbb{C} \mid (A - \lambda I)^{-1} \exists \text{ exists} \right\}\) is called the resolvent set and \(R(\lambda, A) := (A - \lambda I)^{-1}\), for all \(\lambda\) in \(\sigma(A)\), is called the resolvent of \(A\).

An important type of \(C_0\)-semigroup is the analytic one, especially in nonlinear systems. Before we give its definition, we need to define the following sector \(\Delta_{\theta}\) := \{\(z \in \mathbb{C} \mid |\arg(z)| < \theta, \ z \neq 0\}\}, where \(\theta \in ]0, \pi[\).

**Definition 2.4.** [21] Let \(\{S(t)\}_{t \geq 0}\) be a \(C_0\)-semigroup on a Banach space \(E\). We say that \(\{S(t)\}_{t \geq 0}\) is an analytic semigroup if there exists \(\theta \in ]0, \pi[\) such that \(\{S(t)\}_{t \geq 0}\) can be extended to the sector \(\Delta_{\theta}\) and it satisfies the following properties:

i- \(S(0) = \text{Id}_E\).

ii- \(S(w + z) = S(w)S(z), \forall z, w \in \Delta_{\theta} \cup \{0\}\).

iii- \(\lim_{z \to 0} S(z)x = 0, \forall x \in E\).

iv- \(z \mapsto S(z)\) is analytic from \(\Delta_{\theta}\) to \(\mathcal{L}(E)\).

Here is a useful characterization of an analytic semigroup.

**Proposition 2.5.** [28] Let \(A\) be a densely defined, linear closed operator from \(E\) to itself. If we assume that \(A\) enjoys the following conditions:

i- For some \(\theta \in ]\frac{\pi}{2}, \pi[\), we have that \(\Delta_{\theta} \subset \sigma(A)\).

ii- \(\exists M_0 > 0\) such that the resolvent \(R(\lambda, A)\) satisfies the estimate
\[
\|R(\lambda, A)\|_{\mathcal{L}(E, E)} \leq \frac{M_0}{|\lambda|}, \forall \lambda \in \Delta_{\theta}.
\]

Then, \(A\) generates an analytic semigroup on \(E\).

We now introduce the notion of fractional powers for operators.

**Definition 2.6.** [21] Let \(A : D(A) \subset E \rightarrow E\) be a linear, possibly unbounded, operator that generates a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\) on \(E\). Let’s consider \(\alpha \in ]0, 1]\), The fractional power, of order \(\alpha\), of \(A\) is defined by,
\[
(-A)^{-\alpha} : D((-A)^{-\alpha}) \subset E \rightarrow E \quad y \mapsto \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} S(t) dt,
\]
where \(D((-A)^{-\alpha}) = \left\{ x \in E \mid \int_0^{+\infty} t^{\alpha-1} S(t) dt < +\infty \right\}\) is its domain of definition. By definition \(A^0 = \text{Id}_E\).
The operator \((-A)^{-\alpha}\) is in fact injective and eventually invertibly bounded, from \(X\) into \(\text{Im}((-A)^{-\alpha})\), for all \(\alpha\) in \([0, 1]\) see [28]. Hence \((-A)^{\alpha}\) can be defined as the inverse of \((-A)^{-\alpha}\), where \(D((-A)^{\alpha}) := \text{Im}((-A)^{-\alpha})\) see [33].

Before we finish the current section, let’s introduce the following weighted Lebesgue space, for all \(q \geq 1\)
\[
L^q_{\alpha-1}[0, T] = \left\{ f : [0, T] \to \mathbb{R} \text{ measurable} \Bigg| \int_0^T |t^{\alpha-1} f(t)|^q \, dt < +\infty \right\},
\]
which is a Banach space endowed with the norm
\[
\|f\|_{L^q_{\alpha-1}[0, T]} = \left[ \int_0^T |t^{\alpha-1} f(t)|^q \, dt \right]^\frac{1}{q}.
\]

**Remark 2.7.** For all \(q \geq 1\), we have the following inclusions,
\[
L^q_{\alpha-1}[0, T] \subset L^3[0, T] \quad \text{and} \quad L^p_{\alpha-1}[0, T] \subset L^q_{\alpha-1}[0, T], \quad \forall p \geq q.
\]

### 3. Considered System

Let \(\Omega\) be an open bounded subset on \(\mathbb{R}^n\), with smooth boundary \(\Gamma = \partial \Omega\) and \([0, T]\) a time interval. Let’s denote \(Q = \Omega \times [0, T]\) and \(\Sigma = \Gamma \times [0, T]\). Consider the following fractional semi-linear evolution equation:
\[
\begin{aligned}
&\left\{ \begin{array}{l}
C D^\alpha_0 y(x, t) = A y(x, t) + N y(x, t) \quad \text{in} \ Q, \quad \alpha \in (0, 1], \\
y(\xi, t) = 0 \quad \text{in} \ \Sigma, \\
y(x, 0) = y_0(x) \in L^2(\Omega)
\end{array} \right.
\end{aligned}
\]
(2)

together with the output function:
\[
z(t) = C y(., t).
\]
(3)

Where:

* \(C D^\alpha_0\) is the left sided fractional derivative in Caputo’s sense.
* \(y_0\) is the initial state.
* \(A\) is a linear, second order, differential operator which generates a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\) on the state space \(L^2(\Omega)\).
* \(C\) is a linear operator (operator of observation) from \(L^2(\Omega)\) into \(\mathcal{O}\) (the observation space).
* \(N\) is a nonlinear operator, assumed to be defined to ensure the existence and uniqueness of a mild solution of the system (2) in \(L^q(0, T; L^2(\Omega))\) see [25, 42, 43].

Without loss of generality we denote \(y(., t) = y(t)\).

We associate with the system (2), the following linear system
\[
\begin{aligned}
&\left\{ \begin{array}{l}
C D^\alpha_0 y(t) = A y(t), \quad t \in [0, T], \\
y(\xi, t) = 0, \quad \text{in} \ \Sigma, \\
y(0) = y_0 \in L^2(\Omega).
\end{array} \right.
\end{aligned}
\]
(4)

We call a mild solution of (2) any function \(y \in C \left(0, T; L^2(\Omega)\right)\) which is written as follows [25, 42, 43]:
\[
y(t) = S_\alpha(t)y_0 + \int_0^t (t - \tau)^{\alpha-1} \mathcal{H}_\alpha(t - \tau) N y(\tau) \, d\tau, \quad \text{in} \ [0, T],
\]
(5)
Remark 3.3. If $K$ in both cases (bounded or not) the adjacent operator of linear operator from $L^2(\Omega)$ is bounded, then it is an admissible observation operator. Thus, we have the right to extend the operator $CS_a(t)$ to a bounded linear operator from $L^2(\Omega)$ to $\mathcal{O}$, see [11, 16]. In both cases (bounded or not) the adjoint operator of $K_a$, can be written

$$(K_a)^*: \mathcal{D}((K_a)^*) \subset L^2(0,T;\mathcal{O}) \rightarrow X,$$

$$z^* \mapsto \int_0^T S_a^*(s)z^*(s)ds,$$
\textbf{Definition 3.4.} \cite{19} The system (4)-(3) is said to be approximately observable in \( \omega \) (or approximately \( \omega \)-observable) if and only if
\[
\overline{\text{Im}}(\chi_\omega K_\alpha^\omega) = L^2(\omega).
\]
This definition is equivalent to \( \text{Ker}(K_\alpha^\omega) = \{0\} \).
The operator \( (K_\alpha^\omega)^\dagger := [(K_\alpha^\omega)^*(K_\alpha^\omega)]^{-1}(K_\alpha^\omega)^* \), called the pseudo inverse of \( K_\alpha^\omega \), is well defined if the system (4)-(3) is approximately \( \omega \)-observable \cite{10}.

\textbf{Definition 3.5.} We say that the system (2), augmented with the measurements (3), is initially continuously observable in \( \omega \), if it is possible to reconstruct \( y(0)|_\omega \), depending on \( z \) in a continuous way.

**Problem:** Considering any system (2), with the output (3), in \([0,T]\), can we reconstruct the initial state \( \tilde{y}_0 \) in \( \omega \)?

\section{4. Direct Approach}
For \( \omega \subset \Omega \), we assume that (4)-(3) is approximately \( \omega \)-observable. We define the following mapping
\[
\Phi : L^2(0,T;L^2(\Omega)) \rightarrow L^2(0,T;L^2(\Omega))
\]
\[
\Phi(y)(t) = S_\alpha(t)\chi^*_\omega(K_\alpha^\omega)^\dagger \left( z(.) - CS_\alpha(.)\chi^*_\Omega\omega \overline{y} - CL_\alpha(.)Ny \right) + S_\alpha(t)\chi^*_\Omega\omega \overline{y} + L_\alpha(t)Ny,
\]
for all \( t \in [0,T] \), where \( \overline{y} \) is in \( L^2(\Omega \setminus \omega) \), such that \( \chi_\omega \overline{y} = 0 \).

\textbf{Proposition 4.1.} The initial state of (2) in \( \omega \) is the restriction in \( \omega \) of a fixed point of \( \Phi \) at \( t = 0 \).

\textbf{Proof.} We have
\[
y(.) = S_\alpha(.)\chi^*_\omega \tilde{y}_0 + S_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 + L_\alpha(.)Ny.
\]
By applying the operator \( C \), we get
\[
z(.) = CS_\alpha(.)\chi^*_\omega \tilde{y}_0 + CS_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 + CL_\alpha(.)Ny.
\]
Which gives
\[
K_\alpha^\omega \tilde{y}_0 = z(.) - CS_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 - CL_\alpha(.)Ny,
\]
and, since (4)-(3) is approximately \( \omega \)-observable, by applying the pseudo inverse of \( K_\alpha^\omega \), we have
\[
\tilde{y}_0 = (K_\alpha^\omega)^\dagger \left( z(.) - CS_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 - CL_\alpha(.)Ny \right).\tag{12}
\]
Substituting (12) in (11), we get
\[
\forall t \in [0,T], \quad y(t) = S_\alpha(t)\chi^*_\omega(K_\alpha^\omega)^\dagger \left( z(.) - CS_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 - CL_\alpha(.)Ny \right) + S_\alpha(t)\chi^*_\Omega\omega \overline{y}_0 + L_\alpha(t)Ny = \Phi(y)(t).\tag{13}
\]
Thus \( y(.) \) is a fixed point of \( \Phi \), and
\[
y(0)|_\omega = (K_\alpha^\omega)^\dagger \left( z(.) - CS_\alpha(.)\chi^*_\Omega\omega \overline{y}_0 - CL_\alpha(.)Ny \right).
\]
\( \Box \)

For the next result we suppose that \( \Phi \) has a unique fixed point \( y^\ast(.) \), for example if \( \Phi \) is a strict contraction.
Proposition 4.2. If the following condition is satisfied,

\[
( z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*) \in \text{Im}(K^\omega_{\alpha}).
\] (14)

Then \( y^*(0) \in \omega \) is the estimated initial state of (2) in \( \omega \).

Proof. We have

\[
y^*(t) = \Phi(y^*) (t), \quad \forall t \in [0, T],
\]

thus

\[
Cy^* (t) = \left( K^\omega_{\alpha} (K^\omega_{\alpha})^\dagger \left( z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*) \right) (t) + CS_{\alpha}(t)\chi^*_\Omega\omega \mathcal{R} + CL_{\alpha}(t)Ny^* \right), \quad \forall t \in [0, T],
\]

by virtue of (14), we have

\[
\exists v \in L^2(\omega) \text{ such that } z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^* = K^\omega_{\alpha}v,
\]

then by applying \( (K^\omega_{\alpha})^\dagger \) we get

\[
(K^\omega_{\alpha})^\dagger \left( z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*) = (K^\omega_{\alpha})^\dagger K^\omega_{\alpha}v = v,
\]

thus

\[
K^\omega_{\alpha} (K^\omega_{\alpha})^\dagger \left( z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*) = K^\omega_{\alpha}v
\]

\[
= z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*,
\]

which gives, for all \( t \)

\[
Cy^* (t) = z(t) - CS_{\alpha}(t)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(t)Ny^* + CS_{\alpha}(t)\chi^*_\Omega\omega \mathcal{R} + CL_{\alpha}(t)Ny^* = z(t),
\]

then

\[
Cy^* (.) = z(.),
\]

and we have

\[
y^*(0) \in \omega = (K^\omega_{\alpha})^\dagger \left( z(.) - CS_{\alpha}(.)\chi^*_\Omega\omega \mathcal{R} - CL_{\alpha}(.)Ny^*) \right).
\]

In all the previous results we worked with the residual part being any function in \( L^2(\Omega \setminus \omega) \), so we can take \( \mathcal{R}_0 = 0 \) for the rest of this work.

5. Analytical Approach

In this section, we shall use another approach where we make some assumptions that will allow our dynamic, \( A \), to generate an analytic semigroup and \(-A\) to have a fractional power of order \( \alpha \in [0, 1] \). Both of these consequences play an important role in the resolution of semilinear evolution systems. In fact the benefit of working with a dynamic that generates an analytic semigroup is that it provides good information one has on the behavior of the solution at time \( t \rightarrow 0^+ \), whereas, fractional powers of \(-A\) allow us to define interpolation spaces, between \( D(A) \) and \( L^2(\Omega) \), in which might lie the solution of our system, since, for semilinear systems, the solution might not live in the evolution space, in our case \( L^2(\Omega) \).

We make the following assumptions on the operator \( A \):

1. \( \exists \theta \in (\frac{\pi}{2}, \pi], \exists b > 0 \), such that

\[
\Delta_\theta - b := \left\{ z \in \mathbb{C} \mid \arg(z + b) < \theta \right\} \subset \sigma(A).
\]
Remark 5.1. The sequence 

\[ \text{For the fractional power of } (2) \]

by \( \{L\} \) that the fractional power of the operator \(-A\), of order \( \alpha \in [0,1] \), is well defined and \( D((-A)\alpha) \) is dense in \( L^2(\Omega) \). The second one is that, using proposition (2.5), \( A \) generates an analytic semigroup, denoted again by \( \{S(t)\}_{t \geq 0} \) on the state space \( L^2(\Omega) \), in fact one can see that if (i) is satisfied then \( \Delta_\theta \subset \sigma(A) \) and if (ii) is verified, (1) is also true.

For any \( \alpha \) in \( [0,1] \) we denote, \( X_\alpha := D((-A)^\alpha) \) in which we define the following norm \( \|x\|_{X_\alpha} := \|(-A)^\alpha x\| \).

Since \((-A)^\alpha\) is bounded then \( \|\cdot\|_{X_\alpha} \) is equivalent to the graph norm. In addition to the fact that \((-A)^\alpha\) is closed, we get that \( X_\alpha \), endowed with the norm \( \|\cdot\|_{X_\alpha} \), is a Banach space.

Remark 5.2. For the fractional power of \(-A\), it is possible to choose a different value from the order of derivation of the state in (2), one can reprove the upcoming results with a slight change in the expression of some constants, we kept the same order to simplify the calculations.

For the operator \( N \) we need to make the following hypotheses:

iii- \( \exists q,p,r \geq 1 \) verifying \( \frac{1}{q} + \frac{1}{p} = 1 + \frac{1}{r} \) such that:

\[
\begin{aligned}
&\exists g_1 \in L^r_{\alpha-1}[0,T], \forall \theta \geq 0, \forall t \in [0,T], \quad \|S(t^\alpha \theta)\|_{L^r(0,T;X^\alpha)} \leq |g_1(t)|.
\end{aligned}
\]

\( \ast \) The nonlinear operator \( N : L'(0,T;X^\alpha) \rightarrow L'(0,T;L^2(\Omega)) \) is well defined and satisfies

\[
\begin{aligned}
&\bullet N(0) = 0, \\
&\bullet \|N x - N y\|_{L^p(0,T;L^2(\Omega))} \leq k(||x||,||y||)||x - y||_{L^r(0,T;X^\alpha)},
\end{aligned}
\]

\( \forall x, y \in L'(0,T;X^\alpha). \) (15)

With \( k : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( k(\theta_1,\theta_2) \frac{1}{\theta_1,\theta_2 \rightarrow 0} \rightarrow 0. \)

Even though the condition (iii)-(\( \ast \)) on the operator \( N \) might seem harsh, yet, it can actually be achieved as shown in [27], in fact it was used to obtain the exact global observability of semilinear classical systems, also in [15] we find that this condition on \( N \) is valid for an important classes of systems, such as the Burgers’ equation.

Proposition 5.3. If (iii)-(\( \ast \)) is satisfied, we have

\[ \forall t \in [0,T], \forall \theta \geq 0 \quad \|S_\alpha(t)\|_{L^r(0,T;X^\alpha)} \leq |g_1(t)| \text{ and } \|H_\alpha(t)\|_{L^r(0,T;X^\alpha)} \leq \frac{|g_1(t)|}{\Gamma(\alpha)}. \]

Proof. Let’s consider \( t \in [0,T] \) and \( \theta \geq 0 \), then for the first inequality we have

\[
\begin{aligned}
\|S_\alpha(t)\|_{L^r(0,T;X^\alpha)} &\leq \int_0^{\infty} |W_\alpha(\theta)|\|S(t^\alpha \theta)\|_{L^r(0,T;X^\alpha)} d\theta \\
&\leq |g_1(t)| \int_0^{\infty} W_\alpha(\theta)d\theta,
\end{aligned}
\]
remark (3.1) implies that \(|W_\alpha(\theta)| = W_\alpha(\theta)|\), and by using (8), we get
\[
\|S_\alpha(t)\|_{L^2(\Omega),X^\alpha} \leq |g_1(t)|.
\]
As for the second
\[
\|H_\alpha(t)\|_{L^2(\Omega),X^\alpha} \leq \alpha \int_0^\infty \theta |W_\alpha(\theta)| \|S(t^\alpha \theta)\|_{L^2(\Omega),X^\alpha} d\theta
\]
\[
\leq |g_1(t)|\alpha \int_0^\infty \theta W_\alpha(\theta) d\theta,
\]
again by (8), we deduce that \(\|H_\alpha(t)\|_{L^2(\Omega),X^\alpha} \leq |g_1(t)|/\Gamma(\alpha)\).

The goal here is to study the regional reconstruction problem for (2)-(3) in \(V = \mathcal{F}(\chi_\omega K_\alpha^\alpha)\), which is a Banach space endowed with the norm \(\|\cdot\|_V = \|K_\alpha^\alpha(\cdot)\|_{L^2(0,T,\mathcal{O})}\).

Let’s put \(f(t) = t^{\alpha-1}H_\alpha(t)\), which gives \(L_\alpha(t)x = (f * x)(t)\),
We have the following propositions.

**Proposition 5.4.** The following inequality is satisfied
\[
\|f(\cdot)\|_{L^q(0,T;L^2(\Omega),X^\alpha)} \leq \frac{|g_1(\cdot)|_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)}, \quad (16)
\]

*Proof.* We have
\[
\|f(\cdot)\|_{L^q(0,T;L^2(\Omega),X^\alpha)} = \int_0^T \|t^{\alpha-1}H_\alpha(t)\|_{L^q(X,X^\alpha)}^q dt \leq \frac{1}{\Gamma(\alpha)} \int_0^T |g_1(t)|_{L^q_{\alpha-1}[0,T]}^q dt,
\]
then
\[
\|f(\cdot)\|_{L^q(0,T;L^2(\Omega),X^\alpha)} = \left[ \int_0^T t^{q(\alpha-1)} \|H_\alpha(t)\|_{L^q(X,X^\alpha)}^q dt \right]^{\frac{1}{q}} \leq \frac{1}{\Gamma(\alpha)} \left[ \int_0^T |g_1(t)|_{L^q_{\alpha-1}[0,T]}^q dt \right]^{\frac{1}{q}},
\]
using proposition 5.3, we get
\[
\|f(\cdot)\|_{L^q(0,T;L^2(\Omega),X^\alpha)} \leq \frac{1}{\Gamma(\alpha)} |g_1(\cdot)|_{L^q_{\alpha-1}[0,T]}.
\]

**Proposition 5.5.** If the system (4) augmented by (3) is approximately observable, then the embedding \(V \hookrightarrow X(\omega) := L^2(\omega)\), is continuous.
*That is*
\[
\exists \kappa > 0, \text{ such that } \|y_0\|_{X(\omega)} \leq \kappa \|y_0\|_V, \quad \forall y_0 \in V.
\]

*Proof.* Let’s consider \(y_0 \in V\), then
\[
\|y_0\|_{X(\omega)} = \left\| \begin{pmatrix} K_\alpha^\omega \end{pmatrix} y_0 \right\|_{X(\omega)} \leq \left\| \begin{pmatrix} K_\alpha^\omega \end{pmatrix} \right\|_{L^2(0,T;\mathcal{O}),X(\omega)} \left\| K_\alpha^\omega y_0 \right\|_{L^2(0,T;\mathcal{O})},
\]
and the fact that the operator \(K_\alpha^\omega\) is bounded comes from the admissibility condition of \(C\), then
\[
\exists \kappa > 0, \forall y_0 \in V, \quad \|y_0\|_{X(\omega)} \leq \kappa \|y_0\|_V.
\]
Using (9) we set
\[ \phi_{s_0}(y(\cdot)) = S_\alpha(\cdot)\chi^{\omega}_0\tilde{y}_0 + L_\alpha(\cdot)Ny. \] (17)

We shall now, in the following proposition, show the existence of a set (ball) of admissible initial states in \( \omega \), in the sense that they give a unique solution of (2) in a ball of \( L'(0,T;X^\alpha) \).

**Theorem 5.6.** Assume that (iii)-(*) and (15) hold, the following results are satisfied.

1. There exists positive numbers \( a, m = m(a) \), such that:
   \( \forall \tilde{y}_0 \in \overline{B(0,m)} \subset V \), the function \( \phi_{s_0} \) has a unique fixed point, solution of (2), in \( \overline{B(0,a)} \subset L'(0,T;X^\alpha) \).

2. The mapping
   \[ h : \overline{B(0,m)} \rightarrow \overline{B(0,a)} \]
   (18)
   \[ \tilde{y}_0 \rightarrow y(\cdot), \]
   \( " \text{which for every initial state } \tilde{y}_0 \text{ in } \omega \text{ gives us the corresponding unique solution of (2)}" \), satisfies the Lipschitz condition.

**Remark 5.7.** The constants \( a \) and \( m(a) \) are not unique, in fact
\( \forall b \leq a, \exists m(b) \) such that the pair \( (b, m(b)) \) satisfy the last proposition.

**Proof.**
1. We have \( k(\theta_1, \theta_2) \xrightarrow{\theta_1, \theta_2 \rightarrow 0} 0 \), then \( \exists a > 0, \exists \nu > 0 \) such that
   \[ k(\theta_1, \theta_2) < \nu < \frac{\Gamma(\alpha)}{\|g_1(\cdot)\|_{L^q(0,T \to L^1)}} \forall \theta_1, \theta_2 \leq a, \]
   which gives
   \[ \sup_{\theta_1 \leq a} k(\theta_1, \theta_2) \leq \nu < \frac{\Gamma(\alpha)}{\|g_1(\cdot)\|_{L^q(0,T \to L^1)}}, \]
   then
   \[ D_1 := \frac{\|g_1(\cdot)\|_{L^q(0,T \to L^1)}}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1, \theta_2) < 1. \]
   \( \forall x, y \in \overline{B(0,a)} \), we have:
   \[ \|\phi_{s_0}(x(\cdot)) - \phi_{s_0}(y(\cdot))\|_{L'(0,T;X^\alpha)} = \|L_\alpha(\cdot)(Nx - Ny)\|_{L'(0,T;X^\alpha)} \]
   \[ = \|(f \ast (Nx - Ny))(\cdot)\|_{L'(0,T;X^\alpha)} \]
   \[ \leq \|f(\cdot)\|_{L^q(0,T;L(\ell^2(\alpha),X^\alpha))} \|Nx - Ny\|_{L^p(0,T;L^2(\Omega))}, \]
   using (16) and (15), we get
   \[ \|\phi_{s_0}(x(\cdot)) - \phi_{s_0}(y(\cdot))\|_{L'(0,T;X^\alpha)} \leq \frac{\|g_1(\cdot)\|_{L^q(0,T \to L^1)}}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1, \theta_2) \|x - y\|_{L'(0,T;X^\alpha)} \]
   \[ \leq \frac{\|g_1(\cdot)\|_{L^q(0,T \to L^1)}}{\Gamma(\alpha)} D_1 \|x - y\|_{L'(0,T;X^\alpha)}, \]
Thus $\phi_{y_0}$ is a strict contraction.

Let’s now show that $\phi_{y_0}(B(0,a)) \subset B(0,a)$.

Let $y$ be an element in $B(0,a)$, then

$$\|\phi_{y_0}(y(\cdot))\|_{L^r(0,T;X^\alpha)} \leq \|S_\alpha(\cdot)\chi_\omega^*y_0\|_{L^r(0,T;X^\alpha)} + \|L_\alpha(\cdot)Ny\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|y_0\|_{X(\omega)}^*\|g_1(\cdot)\|_{L^r[0,T]} + \|f(\cdot)\|_{L^q(\omega(0,T;L^2(\alpha),X^\alpha))}^*\|Ny\|_{L^p(0,T;L^2(\alpha))}$$

$$\leq \kappa\|y_0\|_{V^*}\|g_1(\cdot)\|_{L^r[0,T]}$$

$$+ \frac{\|g_1(\cdot)\|_{L^q_{a-1}[0,T]}}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1,0) a,$$

then if $m = \frac{a}{\|g_1(\cdot)\|_{L^r[0,T]}^\kappa} \left( 1 - \frac{\|g_1(\cdot)\|_{L^q_{a-1}[0,T]}^\kappa}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1,0) \right)$

we have

$$\tilde{y}_0 \in B(0,m) \implies \phi_{y_0}(y(\cdot)) \in B(0,a).$$

Let’s show that $m$ is positive, in fact we have

$$\frac{\|g_1(\cdot)\|_{L^q_{a-1}[0,T]}^\kappa}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1,0) \leq D_1 = \frac{\|g_1(\cdot)\|_{L^q_{a-1}[0,T]}^\kappa}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1,\theta_2) < 1,$$

which leads to

$$1 - \frac{\|g_1(\cdot)\|_{L^q_{a-1}[0,T]}^\kappa}{\Gamma(\alpha)} \sup_{\theta_1 \leq a} k(\theta_1,0) > 0,$$

hence $m > 0$.

We deduce, from the Banach fixed point theorem, that $\phi_{y_0}$ has a unique fixed point in $B(0,a)$.

2. Let $x$ and $y$ be two solutions of (2) with initial states in $\omega$, respectively, $\tilde{x}_0$ and $\tilde{y}_0$, we have:

$$h(\tilde{x}_0) - h(\tilde{y}_0) = x(\cdot) - y(\cdot) = S_\alpha(\cdot)\chi_\omega^*(\tilde{x}_0 - \tilde{y}_0) + L_\alpha(\cdot)(Nx - Ny),$$

then

$$\|h(\tilde{x}_0) - h(\tilde{y}_0)\|_{L^r(0,T;X^\alpha)} \leq \|S_\alpha(\cdot)\chi_\omega^*(\tilde{x}_0 - \tilde{y}_0)\|_{L^r(0,T;X^\alpha)}$$

$$+ \|L_\alpha(\cdot)(Nx - Ny)\|_{L^r(0,T;X^\alpha)}$$

$$\leq \|\tilde{x}_0 - \tilde{y}_0\|_{X(\omega)}^* \|g_1(\cdot)\|_{L^r[0,T]}$$

$$+ D_1\|x - y\|_{L^r[0,T;X^\alpha]}$$

$$\leq \kappa\|\tilde{x}_0 - \tilde{y}_0\|_{V^*}\|g_1(\cdot)\|_{L^r[0,T]}$$

$$+ D_1\|h(\tilde{x}_0) - h(\tilde{y}_0)\|_{L^r(0,T;X^\alpha)},$$

K. Zguaid, F.Z. El Alaoui, A. Boutoulout., Adv. Theory Nonlinear Anal. Appl. 5 (2021), 580–599. 590.
which gives
\[ ||h(\tilde{x}_0) - h(\tilde{y}_0)||_{L^r([0,\tau];X^\alpha)} \leq \frac{\kappa||g_1(.)||_{L^r([0,\tau])}}{1 - D_1}||\tilde{x}_0 - \tilde{y}_0||_V. \]
Finally, \( h \) is Lipschitz continuous.

In the next result we show that the initial state in \( \omega(\tilde{y}_0) \) is a solution of a fixed point problem, keeping in mind that the measurements are in a ball of \( L^2(0,T;\mathcal{O}) \).

The solution of (2) can be written as
\[ y(.) = S_\alpha(\cdot)\chi^*\omega \tilde{y}_0 + L_\alpha(\cdot)Ny, \]
applying the observation operator, we get
\[ z(.) = CS_\alpha(\cdot)\chi^*\omega \tilde{y}_0 + CL_\alpha(\cdot)Ny, \]
or equivalently,
\[ K_\alpha^\omega \tilde{y}_0 = z(.) - CL_\alpha(\cdot)Ny, \]
and since the system (4)-(3) is approximately \( \omega \)-observable, we obtain
\[ \tilde{y}_0 = (K_\alpha^\omega)^\dagger (z(.) - CL_\alpha(\cdot)h(\tilde{y}_0)), \]
We set
\[ \Phi_z(\tilde{y}_0) = (K_\alpha^\omega)^\dagger (z(.) - CL_\alpha(\cdot)h(\tilde{y}_0)), \quad (19) \]
them can be seen as a fixed point of \( \Phi_z(\cdot) \).

If the system (4) is approximately \( \omega \)-observable and (15) hold, we have the following result

**Theorem 5.8.** Let's assume that

**H1.** \( \forall \tilde{y}_0 \in B(0,m), \quad CL_\alpha(\cdot)(Ny) \in \text{Im}(K_\alpha^\omega), \) where \( y = h(\tilde{y}_0) \).

**H2.** \( \exists \delta > 0, \) such that
\[ ||CL_\alpha(\cdot)Ny||_{L^2(0,T;\mathcal{O})} \leq \delta ||Ny||_{L^p(0,T;L^2(\Omega))}. \]

Then we have the following assertions:

1. \( \exists l \) and \( \rho = \rho(a,l,m) > 0, \forall z \in B(0,\rho) \subset L^2(0,T;\mathcal{O}), \) \( \Phi_z(\cdot) \) has a unique fixed point in \( B(0,m) \).

2. The mapping
\[ h' : B(0,\rho) \ni z \mapsto \tilde{y}_0, \quad (20) \]
"which, for every measurement \( z \) in \( B(0,\rho) \), associates the unique fixed point of \( \Phi_z(\cdot) \)" is Lipschitzian.

**Proof.**

1. As \( k(\theta_1, \theta_2) \xrightarrow{\theta_1, \theta_2 \to 0} 0, \) \( \exists l, \exists \nu > 0, \) such that:
\[ k(\theta_1, \theta_2) < \nu < \frac{1 - D_1}{\delta \kappa||g_1(.)||_{L^r([0,\tau])}} \quad \forall \theta_1, \theta_2 \leq l, \]
which gives
\[ \sup_{0 \leq t \leq l} k(\theta_1, \theta_2) \leq \nu < \frac{1 - D_1}{\delta \kappa||g_1(.)||_{L^r([0,\tau])}}, \]
then 
\[ D_2 = \delta \text{Sup}_{\theta_1 \leq l} k(\theta_1, \theta_2) \frac{\kappa\|g_1(.)\|_{L^r([0,T])}}{1-D_1} < 1. \]

If \( a \leq l \), let \( \tilde{x}_0, \tilde{y}_0 \in B(0,m) \subset V \), then

\[ \Phi_\alpha(\tilde{y}_0) - \Phi_\alpha(\tilde{x}_0) = (K_\alpha^\omega)^\dagger [C^\alpha(.)NY -NX], \]

hence, we have

\[ \|\Phi_\alpha(\tilde{y}_0) - \Phi_\alpha(\tilde{x}_0)\|_V = \|(K_\alpha^\omega)(K_\alpha^\omega)^\dagger [C^\alpha(.)NY -NX]\|_{L^2(0,T;\mathcal{O})}, \]

using (H1), we obtain

\[ \|\Phi_\alpha(\tilde{y}_0) - \Phi_\alpha(\tilde{x}_0)\|_V = \|C^\alpha(.)NY -NX\|_{L^2(0,T;\mathcal{O})}, \]

and from (H2), we deduce that

\[ \|\Phi_\alpha(\tilde{y}_0) - \Phi_\alpha(\tilde{x}_0)\|_V \leq \delta\|NY -NX\|_{L^p(0,T;\mathcal{L}^2(\Omega))} \]
\[ \leq \delta \text{Sup}_{\theta_1 \leq l} k(\theta_1, \theta_2) \frac{\kappa\|g_1(.)\|_{L^r([0,T])}}{1-D_1} \|\tilde{y}_0 - \tilde{x}_0\|_V \]
\[ \leq D_2 \|\tilde{y}_0 - \tilde{x}_0\|_V. \]

Thus \( \Phi_\alpha(.) \) is a strict contraction on \( B(0,m) \).

We show now that \( \Phi_\alpha(B(0,m)) \subset B(0,m) \). Let \( \tilde{y}_0 \in B(0,m) \), we have

\[ \|\Phi_\alpha(\tilde{y}_0)\|_V = \|z(.) - C^\alpha(.)NY\|_{L^2(0,T;\mathcal{O})} \]
\[ \leq \|z(.)\|_{L^2(0,T;\mathcal{O})} + \delta k(\|y\|,0) \|y\|_{L^1([0,T];\mathcal{X}^\mathcal{N})}, \]

and since \( a \leq l \), we have

\[ \|\Phi_\alpha(\tilde{y}_0)\|_V \leq \|z(.)\|_{L^2(0,T;\mathcal{O})} + a.\delta \text{Sup}_{\theta_1 \leq l} k(\theta,0). \]

Let’s set \( \rho := m - a.\delta \text{Sup}_{\theta} k(\theta,0) \), which is positive. In fact

\[ \delta \text{Sup}_{\theta_1 \leq l} k(\theta,0) \frac{\kappa\|g_1(.)\|_{L^r([0,T])}}{1-D_1} \]
\[ \leq D_2 = \delta \text{Sup}_{\theta_1 \leq l} k(\theta_1, \theta_2) \frac{\kappa\|g_1(.)\|_{L^r([0,T])}}{1-D_1} < 1, \]

hence

\[ \delta \text{Sup}_{\theta_1 \leq l} k(\theta,0)\kappa\|g_1(.)\|_{L^r([0,T])} < 1 - D_1 \leq 1 - \frac{\|g_1(.)\|_{L^q_{\alpha-1}([0,T])}}{\Gamma(\alpha)} \text{Sup}_{\theta_1 \leq a} k(\theta_1,0), \]

thus

\[ \delta \text{Sup}_{\theta_1 \leq l} k(\theta,0) < \frac{1}{\kappa\|g_1(.)\|_{L^r([0,T])}} \left[ 1 - \frac{\|g_1(.)\|_{L^q_{\alpha-1}([0,T])}}{\Gamma(\alpha)} \text{Sup}_{\theta_1 \leq a} k(\theta_1,0) \right], \]

which gives

\[ \frac{1}{\kappa\|g_1(.)\|_{L^r([0,T])}} \left[ 1 - \frac{\|g_1(.)\|_{L^q_{\alpha-1}([0,T])}}{\Gamma(\alpha)} \text{Sup}_{\theta_1 \leq a} k(\theta_1,0) \right] - \delta \text{Sup}_{\theta_1 \leq l} k(\theta,0) > 0. \]
Then \( m - a \cdot \delta \sup_{\theta \leq l} k(\theta, 0) = \delta > 0. \)

If \( \|z(\cdot)\|_{L^2(0, T; \mathcal{O})} \leq \rho, \) then \( \|\Phi_z(\tilde{y}_0)\|_V \leq m. \)

Thus

\[
\Phi_z\left(B(0, m)\right) \subset B(0, m).
\]

Therefore, by the Banach’s fixed theorem, if \( z \in B(0, \rho), \) the function \( \Phi_z(\cdot) \) has a unique fixed point in \( B(0, m), \) which is the initial state in \( \omega. \)

If \( l \leq a, \) we use the remark (5.7) and reprove the proposition (5.6), for another \( a \leq l \)

2. Let’s consider \( z_1, z_2 \in B(0, \rho), \) we have

\[
h'(z_1) - h'(z_2) = \Phi_{z_1} \left( h'(z_1) \right) - \Phi_{z_2} \left( h'(z_2) \right) = \tilde{y}_01 - \tilde{y}_02.
\]

Then

\[
\|h'(z_1) - h'(z_2)\|_V = \|\Phi_{z_1} \left( h'(z_1) \right) - \Phi_{z_2} \left( h'(z_2) \right)\|_V
\]

\[
\leq \|\Phi_{z_1} \left( h'(z_1) \right) - \Phi_{z_1} \left( h'(z_2) \right)\|_V
\]

\[
+ \|\Phi_{z_1} \left( h'(z_2) \right) - \Phi_{z_2} \left( h'(z_2) \right)\|_V
\]

\[
\leq D_2\|h'(z_1) - h'(z_2)\|_V + \|z_1 - z_2\|_{L^2(0, T; \mathcal{O})}.
\]

Which gives \( \|h'(z_1) - h'(z_2)\|_V \leq \frac{1}{1 - D_2}\|z_1 - z_2\|_{L^2(0, T; \mathcal{O})}, \) thus \( h' \) is Lipschitzian.

\[ \square \]

6. Numerical Approach

For this section we adopt the same assumptions of the fourth section.

Let’s consider the following sequence,

\[
\begin{cases}
y_0^0 = 0, \\
y_0^{n+1} = \left( K_\alpha \right)^\dagger \left( z(\cdot) - CL_\alpha (\cdot) Nh(y_0^n) \right),
\end{cases}
\]

and for every \( n \in \mathbb{N}, \) we consider the following fractional system,

\[
\begin{cases}
\frac{C}{\Delta} D_t^n y^n(x, t) = Ay^n(x, t) + Ny^n(x, t) & \text{in } Q, \quad \alpha \in ]0, 1[, \\
y^n(\xi, t) = 0 & \text{in } \Sigma, \\
y^n(x, 0) = y_0^n(x) & \text{in } \Omega,
\end{cases}
\]

with the output equation,

\[
z_n(t) = Cy^n(t).
\]

**Theorem 6.1.** The sequence \( \{y_0^n\}_{n \geq 0} \) converges to the desired initial state \( y_0 \) in \( \omega. \)

**Proof.** All we need to show is that \( y_0^n \) converges to \( h'(z) \) in \( V. \) Firstly we will show that \( (y_0^n)_{n \geq 0} \) is a Cauchy sequence.

We have,

\[
\|y_0^{n+1} - y_0^n\|_V = \|\Phi_z(y_0^n) - \Phi_z(y_0^{n-1})\|_V,
\]

\[
\leq D_2\|y_0^n - y_0^{n-1}\|_V,
\]

\[
\vdots
\]

\[
\leq D_2^n\|y_0^1\|_V,
\]

...
hence, \( \forall m > n > 0 \)
\[
\|y_0^n - y_0^m\|_V \leq \|y_0^n - y_0^{m-1}\|_V + \|y_0^{m-1} - y_0^{m-2}\|_V + \cdots + \|y_0^{n+1} - y_0^n\|_V,
\]
\[
\leq D_2^{m-1}\|y_0^1\|_V + D_2^{m-2}\|y_0^1\|_V + \cdots + D_2^n\|y_0^1\|_V,
\]
\[
\leq \frac{D_2^n}{1 - D_2}\|y_0^1\|_V,
\]
which gives that \( (y_0^n)_{n \geq 0} \) is a Cauchy sequence and eventually convergent. Remark that, from the sequence definition, we have
\[
\forall n \geq 1, \quad K_\alpha^n y_0 = z - CL_\alpha(.)Nh(y_0^{n-1}).
\]
Hence,
\[
\|y_0^n - h'(z)\|_V = \|h'(z_n) - h'(z)\|_V,
\]
\[
\leq \frac{1}{1 - D_2}\|z_n - z\|_{L^2(0, T; \mathcal{O})},
\]
\[
\leq \frac{1}{1 - D_2}\|z - CL_\alpha(.)Nh(y_0^n) - K_\alpha^n y_0\|_{L^2(0, T; \mathcal{O})},
\]
\[
\leq \frac{1}{1 - D_2}\|K_\alpha^n y_0^{n+1} - K_\alpha^n y_0\|_{L^2(0, T; \mathcal{O})},
\]
\[
\leq \frac{1}{1 - D_2}\|y_0^{n+1} - y_0^n\|_V,
\]
\[
\leq \frac{D_2^n}{1 - D_2}\|y_0^1\|_V.
\]
We deduce that \( y_0^n \) converges to desired initial state in \( \omega \).

We set \( r_n(.) = z(.) - CL_\alpha(.)Nh(y_0^{n-1}) \).

we have
\[
z(.) - z_n(.) = z(.) - CL_\alpha(.)Nh(y_0^n) - K_\alpha^n y_0^n = CL_\alpha(.)Nh(y_0^{n-1}) - CL_\alpha(.)Nh(y_0^n) = r_n(.) - r_{n+1}(.),
\]
which gives
\[
r_{n+1}(.) = (z_n(.) - z(.)) + r_n(.).
\]
Thus we obtain the following algorithm,

\[
\begin{align*}
\text{Initialization of: } & \varepsilon, \alpha, \omega, \text{ Sensors} \\
\text{Repeat :} & \\
& \tilde{y}_0 = (K_\alpha^n)^\dagger r(.), \\
& \text{Solve } C D_{0+}^s y = Ay + Ny; \quad y(0) = \chi_\omega^s \tilde{y}_0. \\
& \tilde{z}(.) = Cy. \\
& r(.) = (\tilde{z}(.) - z(.)) + r(.). \\
\text{Until } & ||\tilde{z}(.) - \tilde{z}(.)||_{L^2(0,T;\mathcal{O})} \leq \varepsilon.
\end{align*}
\]

Finally \( \tilde{y}_0 \) corresponds with the initial state on \( \omega \).
Simulations

We give here numerical illustrations for the obtained algorithm. We show, with the same fractional system, both types of sensors, zonal and pointwise.

Remark 6.2. The results are related to the choice of \( \omega \), the sensor’s location also as the initial state of the system.

Let \( \Omega = [0, 1] \), \( T = 2 \) and \( \alpha \in ]0, 1[ \), we consider the following time-fractional diffusion system,

\[
\begin{cases}
  c \, D_0^\alpha y(x, t) - \frac{\partial^2}{\partial x^2} y(x, t) = \sum_{i=1}^{\infty} \langle y(\cdot, t), \varphi_i \rangle \varphi_i(x) & \text{in } \Omega \times ]0, T], \\
  y(0, t) = y(1, t) = 0 & \text{in } [0, T], \\
  y(x, 0) = y_0(x) \in \mathcal{D}(A) & \text{in } \Omega,
\end{cases}
\]

where \( \{ \varphi_i(x) = \sin(i\pi x), \ i = 1, 2, \ldots \} \) is an orthonormal basis of the state space \( L^2([0, 1]) \).

Pointwise Sensor

For this case we consider the order of derivation \( \alpha = 0.2 \) and that :

\( \checkmark \) The subregion \( \omega = [0.4, 0.75] \).

\( \checkmark \) The measurements are given by means of a pointwise sensor located in \( b = 0.8 \), which means \( z(t) = y(0.8, t) \).

\( \checkmark \) The initial state supposedly unknown in \( \omega \) is \( y_0(x) = 2x(x - 1)(2x - 1) \).

After 5 iterations of the proposed algorithm we obtain the figure (1).

The reconstruction error is : \( \|y_0 - \tilde{y}_0\|_{L^2(\omega)} = 1.21 \times 10^{-2} \).

We remark in figure (1) that the reconstructed initial state is very close to the initial one in the desired subregion \( \omega \).

The figure (2) shows the evolution of the reconstruction error in function of the sensor’s location, and it is very clear that the reconstruction error is sensitive to the position of the sensor.

Zonal Sensor

For this case we consider that :

\( \checkmark \) The order of derivation \( \alpha = 0.8 \).

\( \checkmark \) The subregion \( \omega = [0.2, 0.6] \).
The measurements are given by means of a zonal sensor, with spatial distribution equal to 1, located in $D = [0.1, 0.3]$, which means

$$z(t) = \int_{0.1}^{0.3} y(x, t) dx$$

The initial state supposedly unknown in $\omega$ is $y_0(x) = 2x(1-x)(2x-1)$.

After 7 iterations with the proposed algorithm we obtain the figure (3).

The reconstruction error is:

$$\|y_0 - \tilde{y}_0\|_{L^2(\omega)} = 9.4 \times 10^{-3}.$$  

As we can see in figure (3), the estimated initial state is quite near the actual one.

In order to show that the error changes with the choice of the geometric domain of the sensor, we give the following table:

| Geometric domain of the sensor | Error   |
|-------------------------------|---------|
| [0.3, 0.5]                    | $5.85 \times 10^{-2}$ |
| [0.5, 0.7]                    | $1.3 \times 10^{-1}$   |
| [0.7, 0.9]                    | $1.26 \times 10^{-1}$   |

Table 1: Some values of the reconstruction error for some different considerations of the geometric domain of the sensor.

Table (1) show that the reconstruction error is influenced by the placement of the geometric domain of the sensor.

7. Conclusion

In this paper we shed light on the concept of regional observability of semilinear Caputo type time-fractional diffusion systems, of order $\alpha \in [0, 1]$. The two different methods that we gave are both based on fixed point techniques, and regarding future works, we intend to investigate the same problem with the
Figure 3: Initial and estimated initial state in $\omega = [0.2 \ 0.6]$. 
Hilbert Uniqueness Method (HUM), we also plan to study the regional boundary observability for the same class of systems.

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