CLASSIFICATION OF NON-ABELIAN CHERN-SIMONS VORTICES

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Abstract

The two-dimensional self-dual Chern-Simons equations are equivalent to the conditions for static, zero-energy vortex-like solutions of the (2+1) dimensional gauged nonlinear Schrödinger equation with Chern-Simons matter-gauge coupling. The finite charge vacuum states in the Chern-Simons theory are shown to be gauge equivalent to the finite action solutions to the two-dimensional chiral model (or harmonic map) equations. The Uhlenbeck-Wood classification of such harmonic maps into the unitary groups thereby leads to a complete classification of the vacuum states of the Chern-Simons model. This construction also leads to an interesting new relationship between $SU(N)$ Toda theories and the $SU(N)$ chiral model.

The study of the nonlinear Schrödinger equation in 2+1-dimensional space-time is partly motivated by the well-known success of the 1+1-dimensional nonlinear Schrödinger equation. Here we consider a gauged nonlinear Schrödinger equation in which we have not only the nonlinear potential term for the matter fields, but also we have a coupling of the matter fields to the gauge fields. Furthermore, this matter-gauge dynamics is chosen to be of the Chern-Simons form rather than the conventional Yang-Mills form. With this choice, the
The nonlinear term in the Schrödinger equation may also be viewed as a Pauli interaction, due to the Chern-Simons relation between the magnetic field and the charge density.

The theory with an Abelian gauge field was analyzed by Jackiw and Pi [7] who found static, zero energy solutions which arise from a two-dimensional notion of self-duality. The static, self-dual matter density satisfies the Liouville equation, which is known to be integrable [10]. The gauged nonlinear Schrödinger equation with non-Abelian Chern-Simons matter-gauge dynamics has also been considered [5, 3, 4], and once again static, zero energy solutions (referred to as "self-dual Chern-Simons vortices") have been found to arise from an analogous, but much richer, two-dimensional self-duality condition. These two-dimensional self-duality equations are formally integrable and in special cases they reduce to the classical and affine Toda equations, both known integrable systems of nonlinear partial differential equations [8, 9].

Here, I classify all finite charge solutions to the self-dual Chern-Simons equations by first showing that the self-duality equations are equivalent to the classical equations of motion of the Euclidean two-dimensional chiral model (also known as the harmonic map equations), and then using a deep classification theorem due to K. Uhlenbeck [11] which classifies all \( U(N) \) and \( SU(N) \) chiral model solutions with finite chiral model action. The chiral model action is in fact proportional to the net gauge invariant charge \( Q \) in the matter-Chern-Simons system, and so the classification of all finite charge solutions is complete. I also present the explicit "uniton" decomposition of a special class of solutions to the \( SU(N) \) chiral model equations which have the remarkable property that when the matter density for these solutions is diagonalized, it satisfies the classical \( SU(N) \) Toda equations. Such a direct correspondence between the Toda equations and the chiral model equations is surprising.

The 2 + 1-dimensional nonlinear Schrödinger equation reads

\[
iD_0 \Psi = -\frac{1}{2} \tilde{D}^2 \Psi + \frac{1}{\kappa} \left[ \left[ \Psi, \Psi^\dagger \right], \Psi \right]
\]

where the covariant derivative is \( D_\mu \equiv \partial_\mu + [A_\mu, \cdot] \), and both the gauge potential \( A_\mu \) and the matter field \( \Psi \) are Lie algebra valued: \( A_\mu = A_\mu^a T^a, \Psi = \Psi^a T^a \). The main results of this paper are for the Lie algebra of \( SU(N) \), but the formulation generalizes straightforwardly to any simple Lie algebra (the noncompact case has been studied in [1]). The matter and gauge fields are coupled dynamically by the Chern-Simons equation

\[
F_{\mu\nu} = i \kappa \epsilon_{\mu\nu\rho} J^\rho
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \) is the gauge curvature, \( \kappa \) is a coupling constant and \( J^\rho \) is the covariantly conserved (\( D_\mu J^\mu = 0 \)) nonrelativistic matter current

\[
J^0 = \left[ \Psi^\dagger, \Psi \right]
\]

\(^1\)Note that there is a typographical error in this equation in [4].
\[ J^i = -\frac{i}{2} \left( [\Psi^\dagger, D_i \Psi] - [(D_i \Psi)^\dagger, \Psi] \right) \] (3)

The Schrödinger equation (1) and the Chern-Simons equation (2) are invariant under the gauge transformation

\[
\begin{align*}
\Psi &\rightarrow g^{-1}\Psi g \\
A_\mu &\rightarrow g^{-1}A_\mu g + g^{-1}\partial_\mu g
\end{align*}
\] (4)

where \( g \in SU(N) \).

In [3, 4] it has been shown that the minimum (in fact zero) energy solutions to (1,2) are given by the self-dual Ansatz

\[ D_- \Psi = 0 \] (5)

combined with the remaining Chern-Simons equation

\[ \partial_- A_+ - \partial_+ A_- + [A_-, A_+] = \frac{2}{\kappa} [\Psi^\dagger, \Psi] \] (6)

Here \( A_\pm = A_1 \pm iA_2 \), \( D_\pm = D_1 \pm iD_2 \) and with antihermitean Lie algebra generators we have \( A_\pm = -(A_\mp)^\dagger \). Equations (5,6) are collectively referred to as the self-dual Chern-Simons equations. The self-dual solutions provide static solutions to the gauged nonlinear Schrödinger equation, as can be seen from a Hamiltonian formulation [3]. Alternatively, this follows directly from the equations of motion (1,2). To see this, note that if \( D_- \Psi = 0 \), then the currents take the simple form

\[ J^+ = J^1 + iJ^2 = -\frac{i}{2} [\Psi^\dagger, D_\Psi] \] (7)

It then follows from the Chern-Simons equation (2) that \( A_0 = \frac{i}{2\kappa} [\Psi^\dagger, \Psi] \). The identity

\[
\begin{align*}
\vec{D}^2 \Psi &\equiv D_+ D_- \Psi + i[F_{12}, \Psi] \\
&= D_+ D_- \Psi - \frac{1}{\kappa} [\Psi^\dagger, \Psi], \Psi
\end{align*}
\] (8)

then implies that the Schrödinger equation reduces to

\[ i\partial_0 \Psi = -\frac{1}{2} D_+ D_- \Psi = 0 \] (9)

In fact, owing to a remarkable dynamical \( SO(2, 1) \) symmetry of the gauged nonlinear Chern-Simons-Schrödinger equations (1,2), it is possible to show that the implication holds in the reverse direction: all static solutions are self-dual [3].
Before classifying the general solutions to the self-dual Chern-Simons equations, it is instructive to consider certain special cases in which simplifying algebraic Ansätze for the fields reduce (5,6) to familiar integrable nonlinear equations. First, choose the fields to have the following Lie algebra decomposition

\[ A_i = \sum_\alpha A^\alpha_i H^\alpha, \quad \Psi = \sum_\alpha \psi^\alpha E^\alpha \]  

where the sum is over all positive, simple roots \( \alpha \) of the Lie algebra, and \( H^\alpha \) and \( E^\alpha \) are the Cartan subalgebra and step operator generators (respectively) in the Chevalley basis \([6]\). Then the self-dual Chern-Simons equations (5,6) combine to yield the classical Toda equations

\[ \nabla^2 \log \rho^\alpha = -2 \kappa K^\alpha_\beta \rho^\beta \]  

where \( \rho^\alpha \equiv |\psi^\alpha|^2 \), and \( K^\alpha_\beta \) is the classical Cartan matrix for the Lie algebra. For \( SU(2) \), (11) becomes the Liouville equation \( \nabla^2 \log \rho = -4 \kappa \rho \), which Liouville showed to be integrable and indeed ”solvable” \([10]\) - in the sense that the general real solution can be expressed in terms of a single holomorphic function \( f = f(x^-) \):

\[ \rho = \frac{\kappa}{2} \nabla^2 \log \left( 1 + f(x^-) \hat{f}(x^+) \right) \]  

Kostant \([8]\) and Leznov and Saveliev \([9]\) have shown that the classical Toda equations (11) are similarly integrable, with the general real solutions for \( \rho^\alpha \) being expressible in terms of \( r \) arbitrary holomorphic functions, where \( r \) is the rank of the algebra. For \( SU(N) \) it is possible to adapt the Kostant-Leznov-Saveliev solutions to a simpler form more reminiscent of the Liouville solution (12):

\[ \rho^\alpha = \frac{\kappa}{2} \nabla^2 \log \left( M^\dagger_\alpha(x^+) M_\alpha(x^-) \right) \]  

where \( M_\alpha \) is the \( N \times \alpha \) rectangular matrix \( M_\alpha = (u \partial_- u \partial_-^2 u \ldots \partial_-^{\alpha-1} u) \), with \( u \) being an \( N \)-component column vector

\[ u = \begin{pmatrix} 1 \\ f_1(x^-) \\ f_2(x^-) \\ \vdots \\ f_{N-1}(x^-) \end{pmatrix} \]  

For a radially symmetric \( SU(3) \) example see Figure 1. An alternative, extended, Ansatz involves the matter field choice

\[ \Psi = \sum_\alpha \psi^\alpha E^\alpha + \psi^M E^{-M} \]
Figure 1: A plot of the nonAbelian charge density $\rho_1$ for a radially symmetric SU(3) Toda-type vortex solution (13) to the self-dual Chern-Simons equations (5,6). For a radially symmetric solution, the functions $f_\alpha(x^\pm)$ appearing in (14) are chosen to be powers of $x^-$. 

where $E_{-M}$ is the step operator corresponding to minus the maximal root. With the gauge field still as in (10), the self-dual Chern-Simons equations then combine to give the affine Toda equations

$$\nabla^2 \log \rho_a = -\frac{2}{\kappa} \tilde{K}_{ab} \rho_b$$

(16)

where $\tilde{K}$ is the affine Cartan matrix. These affine Toda equations are also known to be integrable.

Having considered some special cases of solutions to the self-dual Chern-Simons equations, we now consider the general solutions by first making a gauge transformation to convert the equations (5,6) into the single equation

$$\partial_- \chi = [\chi^\dagger, \chi]$$

(17)

where $\chi$ is the gauge transformed matter field $\chi = \sqrt{\frac{2}{\kappa}} g \Psi g^{-1}$. The existence of such a gauge transformation $g^{-1}$ follows from the following zero-curvature formulation of the self-dual
Chern-Simons equations [3, 4]. Define

\[ A_+ \equiv A_+ - \sqrt{\frac{2}{\kappa}} \Psi, \quad A_- \equiv A_- + \sqrt{\frac{2}{\kappa}} \Psi^\dagger \]  

Then the self-dual Chern-Simons equations imply that the gauge curvature associated with \( A_{\pm} \) vanishes:

\[ \partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0. \]

Therefore, locally, one can write \( A_{\pm} \) as pure gauge

\[ A_{\pm} = g^{-1} \partial_\pm g \]  

for some \( g \in SU(N) \). Gauge transforming the self-dual Chern-Simons equations (5,6) with this group element \( g^{-1} \) leads to the single equation (17).

Equation (17) can be converted into the chiral model equation by defining

\[ \chi = \frac{1}{\sqrt{2}} h^{-1} \partial_+ h \]

for some \( h \in SU(N) \) (the fact that it is possible to write \( \chi \) in this manner is a consequence of (17)). The chiral model equation (13) reads:

\[ \partial_+ (h^{-1} \partial_- h) + \partial_- (h^{-1} \partial_+ h) = 0 \]

Given any solution \( h \) of the chiral model equations, or alternatively any solution \( \chi \) of (17), we automatically obtain a solution of the original self-dual Chern-Simons equations:

\[ \Psi^{(0)} = \sqrt{\frac{\kappa}{2}} \chi, \quad A^{(0)}_+ = \chi, \quad A^{(0)}_- = -\chi^\dagger. \]  

The global condition which permits the classification of solutions to the chiral model equation (20) is that the chiral model ”action functional” (also referred to in the literature as the ”energy functional”)

\[ \mathcal{E}[h] = -\frac{1}{2} \int d^2 x \text{ tr}(h^{-1} \partial_- hh^{-1} \partial_+ h) \]  

be finite. This finiteness condition is directly relevant in the related matter-Chern-Simons system because \( \mathcal{E}[h] = 2 \int d^2 x \text{ tr}(\chi \chi^\dagger) = \frac{4}{\kappa} \int d^2 x \text{ tr}(\Psi \Psi^\dagger) = \frac{4}{\kappa} Q \) where \( Q \) is the net gauge invariant matter charge. As well as being physically significant, this finiteness condition is mathematically crucial because it permits the chiral model solutions on \( \mathbb{R}^2 \) to be classified by conformal compactification to the sphere \( S^2 \) [11, 13].

**THEOREM** (K. Uhlenbeck [11]; see also J. C. Wood [14]): Every finite action solution \( h \) of the \( SU(N) \) chiral model equation (20) may be uniquely factorized as a product of ”uniton” factors

\[ h = \pm h_0 \prod_{i=1}^m (2p_i - 1) \]  

where:
a) \( h_0 \in SU(N) \) is constant;
b) each \( p_i \) is a Hermitean projector (\( p_i^\dagger = p_i \) and \( p_i^2 = p_i \));
c) defining \( h_j = h_0 \prod_{i=1}^j (2p_i - 1) \), the following linear relations must hold:

\[
(1 - p_i) \left( \partial_+ + \frac{1}{2} h_{i-1}^{-1} \partial_+ h_{i-1} \right) p_i = 0
\]

\[
(1 - p_i) h_{i-1}^{-1} \partial_- h_{i-1} p_i = 0
\]
d) \( m \leq N - 1 \).

The \( \pm \) sign in (23) has been inserted to allow for the fact that Uhlenbeck and Wood actually considered \( U(N) \) rather than \( SU(N) \).

An important implication of this theorem is that for \( SU(2) \) all finite action solutions of the chiral model have the "single uniton" form

\[
h = -h_0 (2p - 1)
\]

where \( p \) is a holomorphic projector satisfying

\[
(1 - p) \partial_+ p = 0
\]

These solutions are essentially the \( CP^1 \) model solutions of Din and Zakrzewski [4, 15].

At this point, it is not at all obvious how these types of solutions to the chiral model equations (and therefore by (21) of the self-dual Chern-Simons equations) are related to the special Toda-type solutions discussed previously. The key observation is that while the algebraic Ansätze (10,15) each lead to a non-Abelian charge density \( \rho = [\Psi^\dagger, \Psi] \) which is diagonal, the chiral model solutions (21) have charge density \( \rho^{(0)} = \frac{2}{i} [\chi^\dagger, \chi] \) which need not be diagonal. However, \( \rho \) is always hermitean, and so it can be diagonalized by a gauge transformation. It is still an algebraically nontrivial task to implement this diagonalization, but this is achieved below for the solutions of \( SU(N) \) Toda type.

It is instructive to illustrate this procedure with the \( SU(2) \) case first. Since \( p^2 = p \), the holomorphic projector condition (23) is equivalent to the condition \( \partial_+ p = 0 \). All such projectors may be written as

\[
p = M (M^\dagger M)^{-1} M^\dagger
\]

where \( M(x^-) \) is any rectangular matrix depending only on the \( x^- \) variable. For \( SU(2) \) we can only project onto a line in \( C^2 \), so we take

\[
M = \begin{pmatrix} 1 \\ f(x^-) \end{pmatrix}
\]

This then leads to

\[
p = \frac{1}{1 + \bar{f}f} \begin{pmatrix} 1 & \bar{f} \\ f & f\bar{f} \end{pmatrix}
\quad \chi = \partial_+ p = \frac{f \partial_+ \bar{f}}{(1 + f\bar{f})^2} \begin{pmatrix} -1 & 1/f \\ -f & 1 \end{pmatrix}
\]

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The corresponding matter density is
\[ [\chi^\dagger, \chi] = -\frac{\partial_+ \bar{f} \partial_- f}{(1 + f \bar{f})^3} \begin{pmatrix} 1 - f \bar{f} & 2 \bar{f} \\ 2f & -1 + f \bar{f} \end{pmatrix} \] (29)
which may be diagonalized by the unitary matrix
\[ g = \frac{1}{\sqrt{1 + f \bar{f}}} \begin{pmatrix} -\bar{f} & 1 \\ 1 & f \end{pmatrix} \]
\[ g^{-1} [\chi^\dagger, \chi] g = \partial_+ \partial_- \text{log det}(M^\dagger M) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (30)
This is precisely Liouville’s solution (12) to the classical $SU(2)$ Toda equation.

For the $SU(N)$ chiral model with $N \geq 3$ it becomes increasingly difficult to describe systematically all possible uniton factorizations consistent with the linear relations listed in Uhlenbeck’s theorem, but Wood [14] has given an explicit construction and parametrization of all $SU(N)$ solutions in terms of sequences of Grassmannian factors.

Another useful result from the chiral model literature is due to Valli:

**THEOREM** (G. Valli [12]): Let $h$ be a solution of the chiral model equation (20). Then the action $\mathcal{E}$ defined in (22) is quantized in integral multiples of $8\pi$.

As a consequence, the gauge invariant Chern-Simons charge $Q \equiv \int tr(\Psi^\dagger \Psi)$ is quantized in integral multiples of $2\pi \kappa$. A related quantization condition has been noted in [3], where the non-Abelian charges $Q_\alpha \equiv \int \rho_\alpha$ are quantized in integral multiples of $2\pi \kappa$ for the $SU(N)$ Toda-type solutions (13). (In this case, $Q = \sum_\alpha Q_\alpha$).

The relationship between the $SU(2)$ uniton solutions and the $SU(2)$ Toda solutions illustrated above (26-30) can be generalized to $SU(N)$ as follows:

**THEOREM** [4]: The following matrix
\[ h = (-1)^{\frac{1}{2} N(N+1)} \prod_{\alpha=1}^{N-1} (2p_\alpha - 1) \] (31)
where $p_\alpha$ is the hermitean holomorphic projector $p_\alpha = M_\alpha (M_\alpha^\dagger M_\alpha)^{-1} M_\alpha^\dagger$ for the matrix $M_\alpha$ in (13,14), is a solution of the $SU(N)$ chiral model equation (20). Furthermore, defining $\chi = \frac{1}{2} h^{-1} \partial_+ h$, there exists a unitary transformation $g$ which diagonalizes the charge density matrix $[\chi^\dagger, \chi]$ so that
\[ g^{-1} [\chi^\dagger, \chi] g = \sum_{\alpha=1}^{N-1} \{ \partial_+ \partial_- \text{log det}(M_\alpha^\dagger M_\alpha) \} H_\alpha \] (32)
where $H_\alpha$ are the Cartan subalgebra generators of $SU(N)$ in the Chevalley basis. This diagonal form is precisely the $SU(N)$ Toda solution (13).
This theorem is proved \[4\] by expressing the projectors \( p_\alpha \) in terms of an orthonormal basis for the space spanned by the columns of \( M_N \). The diagonalizing matrix \( g \) is also constructed from this orthonormal basis.

In conclusion, I mention some open problems suggested by these results.

1. The most important physical problem is now to make use of this complete description of the vacuum of these Chern-Simons-matter theories in order to develop a second quantized theory.
2. The fact that this quantization is possible for the \( 1+1 \)-dimensional nonlinear Schrödinger equation (NLSE) is intimately related to the integrability of the \( 1+1 \)-dimensional NLSE. Here, in \( 2+1 \)-dimensional, the situation is less clear. Is the \( 2+1 \)-dimensional gauged nonlinear Schrödinger equation (4) with Chern-Simons coupling (4) integrable?
3. Can one find time-dependent (i.e. positive energy) solutions other than those obtained via the action of the dynamical SO\((2,1)\) symmetry acting on the static solutions?
4. The work of Uhlenbeck, Wood and Ward gives a beautiful geometrical picture of the chiral model solutions for the unitary group. What is the geometrical interpretation of self-dual Chern-Simons solutions for other Lie groups? Some solutions, in the Toda form, are known, but the geometrical understanding of the corresponding chiral model solutions is not clear.

This should be particularly interesting for the self-dual Chern-Simons solutions of the affine Toda form.

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