The Lorentz Attractor and Other Attractors in the Economic System of a Firm

V I Shapovalov\textsuperscript{1,3} and N V Kazakov\textsuperscript{2}

\textsuperscript{1}The Volgograd Branch of Moscow Humanitarian-Economics Institute, Volgograd, Russia
\textsuperscript{2}The Volgograd State Technical University, Volgograd, Russia
\textsuperscript{3}E-mail: shavi71@rambler.ru

Abstract. A nonlinear model of the economic system of “a firm” is offered. It is shown that this model has several chaotic attractors, including the Lorentz attractor and a new attractor that, in our opinion, has not yet been described in the scientific literature. The chaotic nature of the attractors that were found was confirmed by computing the Lyapunov indicators. The functioning of our economic model is demonstrated with examples of firm behaviour that change the control parameters; these are well known in practice. In particular, it is shown that changes in the specific control parameters may change the system and avoid bankruptcy for the firm.

Keywords: chaotic attractor, synergetics, nonlinear economic model.

1. Introduction

A change of external influence corresponds to a change of control parameters in nonlinear dynamics. Let us remember that the control parameters are all constants in the evolutionary equation:

\[ \frac{dX_i}{dt} = F_i(X_1, X_2, ..., X_n) , \]

where \( X_i \) are the system variables; \( F_i \) are functions of the variables; \( t \) is time; and \( n \) is the minimum number of variables needed for a description of the system. By changing the control parameter values we can change the current steady state of the system into an unstable state, and vice versa.

In this paper we consider economic systems as self-organizing systems. We can observe the coordinated, collective movement of the elements in such systems. Therefore we understand the \( X_i \) variables in (1) to be macroscopic variables corresponding to some generalized characteristics of a collective movement of system elements. We recall that such variables are called order parameters in synergetics [1].

Creating a synergetic model usually starts with the selection of the order parameters, i.e. the macroscopic variables that quantify the main links in the system. The next step consists in drawing up the proportions to form these relationships. (The rule of drawing up proportions is described in detail in [1]. According to this rule, the increase over time in a particular value is eventually proportional to the gain in this value minus its loss). These proportions are then converted into evolutionary equations.

Below we consider a firm for which the number of employees and the working capital are average for a particular region. We also assume that, in order to develop its production base, the firm borrows money. The aim of our work is to create a synergetic model of this system. In particular, we will try to find its attractors.
2. The basic model for the firm system

Selecting the system variables for the firm is easy – they are the number of employees $X$, the working capital $Y$ and the loan amount $Z$.

We call our model the basic model if it includes the minimum number of relationships between these variables (i.e., only those relationships that are sure to arise if the company operates in the market for a long time). These relationships are the following:

1) The increase in the number of employees over time ($dX/dt$) is proportional to the number of employees joining, $J_1$, minus the number leaving, $J_2$. An increase in the number of employees at the firm depends on the firm having an attractive image. In most cases, to create an attractive image the firm must spend part of its capital $Y$ and a portion of the loan $Z$. Consequently $J_1 \sim Y, Z$. It is obvious that the number of employees who are lost due to layoffs is proportional to the number of employees. Therefore $J_2 \sim X$.

2) The capital increase over time ($dY/dt$) is proportional to the capital gains, $J_3$, minus the capital reduction, $J_4$. It is obvious that $J_3 \sim (Y+Z)$. The main losses in capital are caused by the payments of the employees’ salaries and credit service payments, so $J_4 \sim (X, Z)$.

3) The increase in the loan over time ($dZ/dt$) is proportional to the capital of the company ($\sim Y$) (we assume that the larger the firm’s capital, the greater the willingness to extend credit to the firm), minus losses that are due to the magnitude of the loans taken ($\sim Z$) (in particular, any bank is less willing to lend to a firm that has many loan obligations).

The above proportions can be represented as a system of three evolution equations [2,3]:

$$
\dot{X} = \alpha YZ - \beta X; \quad \dot{Y} = \mu(Y + Z) - \delta XZ; \quad \dot{Z} = \delta Y - \lambda Z,
$$

where $\dot{X} = dX/dt$, $\dot{Y} = dY/dt$, $\dot{Z} = dZ/dt$; and $\alpha$, $\gamma$, $\mu$, $\beta$, $\delta$, $\lambda$ are coefficients of proportionality. The values $\alpha$, $\gamma$, $\mu$, $\beta$, $\delta$, $\lambda$ are constants, and hence play a role as control parameters.

The system of equations (1) is similar in its structure to the well-known Lorenz system, which has no analytic solutions. Therefore a numerical investigation of this system is carried out for different values of the control parameters. We choose the values of the control parameters taking into account the restrictions imposed by the properties of the real economic system. For example, it is obvious that the terms of each equation in system (1) must be of similar size. Consequently, the pairs of coefficients $\alpha$ and $\gamma$, $\mu$ and $\beta$, and $\delta$ and $\lambda$ cannot be of a different order. If they differ in order (or more) then the terms with very small coefficients can be neglected.

Different types of attractors are found as a result of the calculations. To clarify the type of attractor we apply the Lyapunov indicators (recall that in the three-dimensional coordinate system, the signs of the Lyapunov indicators have the following distribution [4]: “+”, “-”, “-” for a point; “0”, “-”, “-” for a limit cycle; “0, 0, -” for the torus; and “+”, “0”, “-” for a chaotic attractor). The calculation of these parameters was performed as described in [5], and included the orthogonalization procedure on the Gramm-Schmidt method. Some typical attractors are presented in ‘Figure 1’.

The simple attractors shown in ‘Figure 1(a)’ and ‘Figure 1(b)’ correspond to the states in which the system is found as $\lambda$ increases. Recall that this control parameter reflects the factors that have an adverse effect on receiving credit. In other words, rigid actions by the bank force the company to complicate its structure. We suppose the complication of the structure to be shown in the changes to the configuration of the attractors. However, if $\lambda$ continues to increase the structure shows chaotic attractors (‘Figure 1(c)’ and ‘Figure 1(d)’).

In our opinion, the chaotic attractors in ‘Figure 1(c)’ and ‘Figure 1(d)’ correspond to the bankruptcy of the firm. However, they appear only for a certain ratio of the control parameters. Therefore, by changing the values of these parameters, we can avoid the unwanted attraction domain of the attractor. For example, we can increase $\gamma$ (i.e. fire some employees). In this case, the chaotic behaviour of the firm also changes to an ordered behaviour (‘Figure 2(e)’). Alternatively, the company can improve its management, i.e. increase $\mu$. As a consequence, it leaves the chaotic attractor system and enters the attractor “limit cycle” (‘Figure 1(f)’).

The chaotic attractor shown in ‘Figure 1(d)’ coincides in form with the famous Lorenz attractor. If
we calculate the fractal dimension $D$, using a known simplified formula \([6,7]\), we obtain the value coinciding with dimension of the Lorentz attractor: $D = 2 + \frac{L(X) + L(Y)}{L(Z)} = 2.06$, where $L(X)$, $L(Y)$ and $L(Z)$ are the Lyapunov indicators. However, the chaotic attractor shown in ‘Figure 1(c)’ is not a Lorenz attractor as it has a tape configuration. The borders of the tape remain unchanged for a long time period. During this period of time the trajectories of the system almost completely fill the space inside the tape. We believe that the attractor ‘Figure 1(c)’ is a tape modification of the Lorenz attractor. This is explained by the fact that this attractor is obtained from the system (1), which is not much different from the Lorenz system.

![Attractors](image)

**Figure 1.** Some attractors of the system (1). Changes in the parameter $\lambda$ are shown: (a) – (d) (the unchangeable parameters are: $\alpha=5$, $\gamma=1$, $\mu=2.2$, $\beta=8$, $\delta=1$). Attractors (e) and (f) follow on from the chaotic attractor represented in (d). $L(X)$, $L(Y)$ and $L(Z)$ are the Lyapunov indicators.

Attractors shown in ‘Figure 1’ are selected from a series of attractors resulting from changes of the $\lambda$ control parameter with step of 0.1. We assume that a smaller step would not be noticeable in a real economic system. However, from a mathematical point of view, such a restriction does not make sense. As an example, we explored the small $\lambda$ interval from 3.6 to 3.7 in increments of 0.01. As a result, we found that the system (1) has tape-type chaotic attractors in this range. At the same interval simple attractors appear between chaotic attractors. If we reduce the step further, we also see a
sequence of simple and chaotic attractors.

The appearance and disappearance of chaotic attractors while the control parameter changes is well known (for example, in the Lorenz system [4]). At the same time, in our opinion, a controlled change in the economic value of as little as 0.01 (and less) is difficult to implement in a real economic system. Therefore, such variations can be neglected for the average firm in practice.

3. Complex model of the firm

The basic model (1) can be complicated, depending on the economic situation. In particular, the firm can create its own attractive image using only capital equity \( Y \) or only a loan \( Z \). In order to consider these cases using the evolution equations system we have to add terms with only \( Y \) and only \( Z \) in the first equation of system (1). In addition, for some periods of time, the company may not have credit obligations (\( Z = 0 \)). Then its basic expenses will only be the cost of the employees \( X \). To account for this, a term containing only \( X \) (i.e., excluding loan servicing) should be subtracted in the second equation. As a result the evolutionary equations system for a complex model of the company will take the form:

\[
\dot{X} = \varepsilon Y + \eta Z - \gamma X; \quad \dot{Y} = \mu (Y + Z) - \beta X Z - \eta X; \quad \dot{Z} = \delta Y - \lambda Z ,
\]

where \( \varepsilon, \eta \) and \( \nu \) are coefficients of proportionality. We carried out numerical research for this system by changing the values of the control parameters \( \lambda, \gamma \) and \( \mu \). In general, the economic picture was similar to that of the base model. An increase in \( \lambda \) (i.e., reducing the possibility of receiving credit) led to a change from an ordered movement to a chaotic one (‘Figure 2’). An increase in \( \mu \) (i.e. an increase in the efficiency of the firm in the market) brings the firm out of a chaotic attractor system (‘Figure 2’). However, in contrast to the basic model, changing only \( \gamma \) (e.g., by dismissing some of the staff) is not enough to move the company away from a chaotic attractor. At the same time, this is possible by a joint change in \( \gamma \) and \( \mu \) (‘Figure 2(f)’).

4. Conclusion

1. In this paper we proposed a nonlinear mathematical model of the economic system of “a firm” (see the systems of equations (1) and (2)). We found simple attractors (limit cycles) that, in our opinion, correspond to stable states of the firm, and chaotic attractors, that correspond to the state of bankruptcy in this model. Also, we have shown that by changing the values of specific control parameters it is possible to avoid a chaotic attractor, i.e. bankruptcy. In particular, by reducing \( \lambda \) (for example, by taking a loan at a lower interest rate), we will transfer the company from a chaotic attractor to a stable state – a limit cycle (‘Figure 1’ and ‘Figure 2’). The configuration of the limit cycles presumably reflects the complexity of the firm’s economic activity that is needed to ensure sustainability with varying degrees of harshness in the credit obligations. If a company cannot change its credit obligations, it can change some other control parameters. For example, it can increase the parameter \( \gamma \) (which corresponds to dismissing some employees) and/or increase the parameter \( \mu \) (i.e. improve the management). In these cases, the chaotic behaviour firm also changes to a stable behaviour (‘Figure 1’ and ‘Figure 2’).

   We should emphasize that the mathematical models considered in this paper are fairly abstract. They describe a general methodological approach, and reveal new capabilities given by synergetic relationships for the analysis of the activity of a particular enterprise. In our opinion, these models can serve as a good methodological basis for the forecasting of bankruptcy and the search for the optimum way of managing a business.

2. We assume that we have found a new chaotic attractor. Its appearance is presented in ‘Figure 2(c)’. This attractor is seen if the values of \( \alpha=6, \, \gamma=10, \, \nu=6, \, \mu=1, \, \beta=8, \, \eta=4, \, \delta=1, \, \lambda=5 \) are substituted in the system (2). We also found chaotic attractors, which we called tape modifications of the Lorenz attractor because of their appearance (e.g., ‘Figure 1(c)’). A distinctive feature of a tape chaotic attractor is the fact that, over a long time, the trajectory of the system completely fills the space within the boundaries of the tape but does not go beyond them. In our opinion, research into the
The attractors mentioned above can be of interest to scientists dealing with problems of nonlinear dynamics.

**Figure 2.** Some attractors of system (2). Changes in the parameter \( \lambda \) are shown: (a) – (c) (the unchangeable parameters are \( \alpha=6, \gamma=1, \varepsilon=10, \nu=6, \mu=1, \beta=8, \eta=4, \delta=1 \)). Attractors (d) – (f) follow on from the chaotic attractor represented in (c).

| (a) simple attractor: \( \lambda=2 \) | (b) simple attractor: \( \lambda=3 \) | (c) chaotic attractor: \( \lambda=5 \) |
|----------------------------------------|----------------------------------------|----------------------------------------|
| \( L(X) = 0.000; L(Y) = -1.000; \) \( L(Z) = -1.000 \) | \( L(X) = 0.000; L(Y) = -0.809; \) \( L(Z) = -2.191 \) | \( L(X) = 0.961; L(Y) = 0.000; \) \( L(Z) = -5.961 \) |

| (d) simple attractor: \( \mu=3 \) | (e) simple attractor: \( \mu=4 \) | (f) simple attractor: \( \gamma=1.4, \mu=2.6 \) |
|----------------------------------------|----------------------------------------|----------------------------------------|
| \( L(X) = 0.000; L(Y) = -0.278; \) \( L(Z) = -2.722 \) | \( L(X) = 0.000; L(Y) = -1.000; \) \( L(Z) = -1.000 \) | \( L(X) = 0.000; L(Y) = -0.130; \) \( L(Z) = -3.670 \) |

References

[1] Haken H 1978 *Synergetics* (Berlin, Heidelberg, New York: Springer Verlag)
[2] Shapovalov V I, Kablov V F, Bashmakov V A and Avvakumov V E 2004 Synergetic stability model for an average firm *Synergetics and Problems in Control Theory* ed A A Kolesnikov (Moscow: FIZMATLIT Publishers) pp 454–464
[3] Shapovalov V I 2005 *Basis of Ordering and Self-Organization Theory* (Moscow: ISPO-Service)
[4] Berge P, Pomeau Y and Vidal Ch 1988 *L’ordre Dans Le Chaos* (Paris: Hermann)
[5] Naymark Yu I and Landa P S 1987 *Stochastic and chaotic oscillations* (Moscow: Nauka Publishers)
[6] Kaplan J L and Yorke J A 1979 *Lect. Notes in Math.* 730 228
[7] Ahmed I, Mu Ch and Zhang A 2014 *International Journal of Analysis and Applications* 6 27