HILBERT-SCHMIDT AND TRACE CLASS PSEUDO-DIFFERENTIAL OPERATORS ON THE ABSTRACT HEISENBERG GROUP

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Abstract. In this paper we introduce and study pseudo-differential operators with operator valued symbols on the abstract Heisenberg group $\mathbb{H}(G) := G \times \hat{G} \times T$, where $G$ a locally compact abelian group with its dual group $\hat{G}$. We obtain a necessary and sufficient condition on symbols for which these operators are in the class of Hilbert-Schmidt operators. As a key step in proving this we derive a trace formula for the trace class $j$-Weyl transform, $j \in \mathbb{Z}^*$ with symbols in $L^2(G \times \hat{G})$. We go on to present a characterization of the trace class pseudo-differential operators on $\mathbb{H}(G)$. Finally, we also give a trace formula for these trace class operators.

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1. Introduction

The aim of this paper is to look at the pseudo-differential operators on the abstract Heisenberg group $\mathbb{H}(G)$, for locally compact abelian group $G$. In 1964, A. Weil studied certain group of unitary operators associated with a locally compact abelian group in connection with the study of the celebrated work of Seigel on quadratic form. In [16], the author introduced this group where he considered $G$ to be an adèle group or the additive group of vector space over a

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local field, which has applications in number theory. Recently Radha, et al. [1, 14] studied Weyl
multipliers and shift invariant spaces on the group $\mathbb{H}(G)$. It is a well-known fact from [17] that
pseudo-differential operators on $\mathbb{R}^n$ are based on the Plancherel formula for the Fourier trans-
form on $\mathbb{R}^n$. The Plancherel formula gives rise to the Fourier inversion formula, which says that the
identity operator for $L^2(\mathbb{R}^n)$ can be expressed in terms of the Fourier transform on $\mathbb{R}^n$. The
Fourier inversion formula, albeit useful in many situations, gives a perfect symmetry, namely,
the identity operator. The classical pseudo-differential operator $T_\sigma$ associated to a symbol $\sigma$, (a
measurable function on $\mathbb{R}^n \times \mathbb{R}^n$) is defined by

$$
(T_\sigma \phi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x, \xi) \hat{\phi}(\xi) d\xi, \ x \in \mathbb{R}^n,
$$

for all $\phi$ in the Schwartz space $S(\mathbb{R}^n)$ on $\mathbb{R}^n$, provided the integral exists. The function $\hat{\phi}$ in (1)
is the Fourier transform of $\phi$ defined by

$$
\hat{\phi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) dx, \ \xi \in \mathbb{R}^n.
$$

The formation of a pseudo-differential operator as define in (1) is mainly based on the Fourier
inversion formula given by,

$$
\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\phi}(\xi) d\xi, \ x \in \mathbb{R}^n,
$$

for all $\phi$ in $S(\mathbb{R}^n)$. By inserting a symbol, which is a suitable function on the phase space
$\mathbb{R}^n \times \mathbb{R}^n$, we break the symmetry and obtain the pseudo-differential operator. To extend pseudo-
differential operators to other settings, we first observe that $\mathbb{R}^n$ is a group and its dual is also $\mathbb{R}^n$. It is then natural to extend pseudo-differential operators to other groups which have explicit dual objects and Fourier inversion formulas. Recently, such works have been done for $\mathbb{S}^1$, $\mathbb{Z}$, finite abelian groups, locally compact abelian groups, affine groups, compact groups,
homogeneous spaces of compact groups, Heisenberg group and on general locally compact type
I groups [2, 4, 7, 12, 13, 9, 10, 15] among others.

A basic result in the theory of pseudo-differential operators on $\mathbb{R}^n$ is that if $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$,
then $T_\sigma$ can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Moreover the
resulting bounded linear operator is in Hilbert-Schmidt class as explained in [11, 18]. Recently
in [4], Dasgupta and Wong has obtained a necessary and sufficient conditions on the symbols
for which the pseudo-differential operators on Heisenberg groups are in Hilbert-Schmidt class.

Motivated by this we wish to study the boundedness property of the pseudo-differential operators
on $\mathbb{H}(G)$ and obtain conditions on the symbols for which these operators are in Hilbert-Schmidt
class. In Section 2 we recall the basics of the abstract Heisenberg group $\mathbb{H}(G)$, and define the
pseudo-differential operators on the group $\mathbb{H}(G)$, where $G$ is a locally compact abelian group. The $L^2$ boundedness property of the pseudo-differential operators on $\mathbb{H}(G)$ is given in Section 3. In Section 4 we obtained the trace formula of a trace class $j$-Weyl transform, $j \in \mathbb{Z}^*$ associated to the symbol in $L^2(G \times \hat{G})$, where $G$ is a locally compact abelian group and this result is the key step in obtaining the necessary and sufficient conditions for the pseudo-differential operators on the abstract Heisenberg group, $\mathbb{H}(G)$ to be in Hilbert-Schmidt class, which we proved in Section 5. In Section 6 trace class pseudo-differential operators on $\mathbb{H}(G)$ are given and a trace formula is obtained for them.

2. Preliminaries

Let $G$ be a locally compact abelian group such that the map $x \mapsto jx$, $j \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, is an automorphism of $G$. We denote the abstract Heisenberg group associated with $G$ by $\mathbb{H}(G) := G \times \hat{G} \times \mathbb{T}$ with the group operation given by

$$(x, \chi, \theta)(x', \chi', \theta') = (xx', \chi\chi', \theta\theta'\chi(x)).$$

The unitary dual $\hat{\mathbb{H}(G)}$ of $\mathbb{H}(G)$ can be identified with $\mathbb{Z}^*$, the set of non-zero integers, as follows: for any $j \in \mathbb{Z}^*$, the irreducible representation of $\mathbb{H}(G)$ on $L^2(G)$ is given by

$$(\rho_j(x, \chi, \theta)\varphi)(y) = \theta^j(\chi(y))^j \varphi(xy) \quad \text{for } \varphi \in L^2(G).$$

It is well known from Stone-von Neumann theorem that every infinite dimensional irreducible unitary representation of $\mathbb{H}(G)$ is unitarily equivalent to $\rho_j$, $j \in \mathbb{Z}^*$. For further details we refer Folland [5, 6].

For each $j \in \mathbb{Z}^*$, the Fourier transform, $\hat{f}$ of $f \in L^1(\mathbb{H}(G))$ at $j$ is an operator $\hat{f}(j)$ on $L^2(G)$ given by

$$\hat{f}(j)\psi := \int_{G \times \hat{G} \times \mathbb{T}} f(x, \chi, \theta) \rho_j(x, \chi, \theta) \psi \, d\mu_G \, d\mu_{\hat{G}} \, d\mu_{\mathbb{T}},$$

where $d\mu_G \, d\mu_{\hat{G}} \, d\mu_{\mathbb{T}}$ is a product of the Haar measure on $G$, $\hat{G}$ and $\mathbb{T}$. The above integral is a Bochner integral taking values in the Hilbert space $L^2(G)$. This operator $\hat{f}(j)$ is a bounded operator on $L^2(G)$ with $\|\hat{f}(j)\|_{B(L^2(G))} \leq \|f\|_{L^1(G)}$. The inverse Fourier transform, $f^j$, $j \in \mathbb{Z}^*$, of $f \in L^1(\mathbb{H}(G))$ in the $\theta$ variable is given by

$$(2) \quad f^j(x, y) = \int_{\mathbb{T}} f(x, \gamma, \theta) \theta^j \, d\mu_{\mathbb{T}}(\theta).$$

Then $f^j \in L^1(G \times \hat{G})$ and using (2), $\hat{f}$ can be written as

$$(3) \quad \hat{f}(j)\phi = \int_{G \times \hat{G}} f^j(x, \gamma) \rho_j(x, \gamma, \theta) \phi \, d\mu_G \, d\mu_{\hat{G}}, \quad \phi \in L^2(G).$$
Further the Fourier transform $\hat{f}$ of $f \in L^2(G)$ satisfies

$$||\hat{f}(j)||_{\mathcal{B}_2(L^2(G))} = (C_{j,G})^{-1}||f||^2_{L^2(G \times \hat{G})},$$

where $C_{j,G}$ is a constant depending on $j$ and $G$ such that $C_{j,G}d\mu_G(j^{-1}x) = d\mu_G(x)$, for $j \in \mathbb{Z}^*$ and $C_{0,G}$ is taken be to zero.

Since the group is $\mathbb{H}(G) = G \times \hat{G} \times \mathbb{T}$ so to obtain the Plancheral theorem we need to consider the one-dimensional representations of $\mathbb{H}(G)$ in addition to the infinite dimensional irreducible unitary representations. Then the Plancherel formula for $f \in L^2(\mathbb{H}(G))$ is given by

$$(4) \quad ||f||^2_{L^2(\mathbb{H}(G))} = ||\hat{f}||^2_{L^2(\mathbb{Z}^*,L^2_{s, \mathcal{C}};C_{j,G})} + \int_G \int_G |(\hat{f}(\gamma,y))|^2 \, d\mu_G(y) \, d\mu_{\hat{G}}(\gamma),$$

where $f_0(x,\chi) = \int_T f(x,\chi,\theta) \, d\mu_T(\theta)$. However for the case when $f - f^0$, $f \in L^2(\mathbb{H}(G))$ has mean value zero in the central variable, the Plancheral formula reduces to the following,

$$(5) \quad ||\hat{f}||^2_{L^2(\mathbb{Z} \setminus \{0\},\mathcal{B}(L^2(G);C_{j,G}))} = ||f - f^0||^2_{L^2(\mathbb{H}(G))}.$$ 

In other words, we can say there is an isometry from $L^2(\mathbb{H}(G))$ into $l^2(\mathbb{Z} \setminus \{0\},\mathcal{B}(L^2(G);C_{j,G}))$, where $l^2(\mathbb{Z} \setminus \{0\},\mathcal{B}(L^2(G);C_{j,G}))$ denotes the space of all sequences indexed in $\mathbb{Z}^*$, taking values in $\mathcal{B}(L^2(G))$ and square summable with weight $C_{j,G}$. See [11, 16] for more details.

The following Fourier Inversion formula for the Fourier transform on the abstract Heisenberg group is the starting point for the analysis of the pseudo-differential operators on the abstract Heisenberg group.

For a function with compact support, $f \in \mathcal{C}_c(\mathbb{H}(G))$, the Fourier Inversion formula is given by,

$$(6) \quad f(x,\chi,\theta) = \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x,\chi,\theta)\hat{f}(j)),$$

where $\rho_j^*(x,\chi,\theta)$ is the adjoint of $\rho_j(x,\chi,\theta)$.

Now let $\mathbb{B}(L^2(G))$ be the $C^*$-algebra of bounded linear operators on $L^2(G)$. Then we call the mapping $\sigma : \mathbb{H}(G) \times \mathbb{Z}^* \to \mathcal{B}(L^2(G))$ an operator valued symbol or simply a symbol.

We define a pseudo-differential operator $T_\sigma$ associated with the operator-valued symbol $\sigma : \mathbb{H}(G) \times \mathbb{Z}^* \to \mathcal{B}(L^2(G))$ as

$$(T_\sigma f)(x,\chi,\theta) := \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x,\chi,\theta)\sigma(x,\chi,\theta,j)\hat{f}(j)) \quad \forall f \in \mathcal{C}_c(\mathbb{H}(G)).$$
3. $L^2$-boundedness

In this section we prove $L^2$-boundedness of pseudo-differential operators on $\mathbb{H}(G)$. We also show that if two symbols with some conditions give arise to same pseudo-differential operator then symbol must be same. Before stating our results of this section we would like to recall some notation from operator theory.

Let $X$ be a complex and separable Hilbert space with the inner product denoted by $(\cdot, \cdot)$ and let $\{\psi_k, k = 1, 2, \ldots\}$ be an orthonormal basis for $X$.

An operator $T \in B(X)$ is a Hilbert-Schmidt operator if for any (hence all) orthonormal basis $\{\psi_k\}_{k=1}^{\infty}$ of $X$ we have $\sum_k \|T\psi_k\|_X < \infty$. The set of Hilbert-Schmidt operators, denoted by $S_2(X)$, is two sided ideal of $B(X)$. At times we also denote $S_2(X)$ only by $S_2$. The Hilbert-Schmidt norm on $S_2$ is given by

$$\|T\|_{HS} = \left( \sum_{k=1}^{\infty} \|T\psi_k\|_X^2 \right)^{\frac{1}{2}}.$$

Now, we are ready to state the following theorem on $L^2$-boundedness of pseudo-differential operators on $\mathbb{H}(G)$.

**Theorem 3.1.** Let $\sigma : \mathbb{H}(G) \times \mathbb{Z}^* \rightarrow S_2$ be a operator-valued symbol such that

$$\sum_{j \in \mathbb{Z}^*} \int_{G \times \widehat{G} \times T} C_{j,G}^{-1} \|\sigma(x, \chi, \theta, j)\|_{S_2}^2 \ d\mu_G \ d\mu_\theta < \infty.$$

Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{H}(G)) \rightarrow L^2(\mathbb{H}(G))$ is a bounded operator.

**Proof.** Using Minkowski integral inequality and the estimate $\|\widehat{f}\|_{\ell^2(\mathbb{Z}^*, S_2, C_{j,G})} \leq \|f\|_{L^2(\mathbb{H}(G))}$ we have, for $f \in L^2(\mathbb{H}(G))$,

$$\|T_\sigma f\|_{L^2(\mathbb{H}(G))} = \left( \int_{G \times \widehat{G} \times T} |(T_\sigma f)(x, \chi, \theta)|^2 \ d\mu_G(x) \ d\mu_\chi \ d\mu_\theta \right)^{\frac{1}{2}}$$

$$= \left( \int_{G \times \widehat{G} \times T} \left| \sum_{j \in \mathbb{Z}^*} \text{tr}(\rho_j^\theta(x, \chi, \theta)\sigma(x, \chi, \theta, j) \widehat{f}(j)) \right|^2 \ d\mu_G(x) \ d\mu_\chi \ d\mu_\theta \right)^{\frac{1}{2}}$$

$$\leq \sum_{j \in \mathbb{Z}^*} \left( \int_{G \times \widehat{G} \times T} |\text{tr}(\rho_j^\theta(x, \chi, \theta)\sigma(x, \chi, \theta, j) \widehat{f}(j))|^2 \ d\mu_G(x) \ d\mu_\chi \ d\mu_\theta \right)^{\frac{1}{2}}$$

$$\leq \sum_{j \in \mathbb{Z}^*} \left( \int_{G \times \widehat{G} \times T} \|\sigma(x, \chi, \theta, j)\|_{S_2}^2 \left\| \widehat{f}(j) \right\|_{S_2}^2 \ d\mu_G(x) \ d\mu_\chi \ d\mu_\theta \right)^{\frac{1}{2}}$$

$$= \sum_{j \in \mathbb{Z}^*} \left\| \widehat{f}(j) \right\|_{S_2} \left( \int_{G \times \widehat{G} \times T} \|\sigma(x, \chi, \theta, j)\|_{S_2}^2 \ d\mu_G(x) \ d\mu_\chi \ d\mu_\theta \right)^{\frac{1}{2}}.$$
\[ \left( \sum_{j \in \mathbb{Z}^*} C_{j,G} \| \hat{f}(j) \|_{S_2}^2 \right)^{\frac{1}{2}} \times \left( \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G} \times \mathbb{T}} C_{j,G}^{-1} \| \sigma(x, \chi, \theta, j) \|_{S_2}^2 \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_{\mathbb{T}}(\theta) \right)^{\frac{1}{2}} \]

\[ = \| \hat{f} \|_{L^2(\mathbb{H}(G))} \left( \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G} \times \mathbb{T}} C_{j,G}^{-1} \| \sigma(x, \chi, \theta, j) \|_{S_2}^2 \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_{\mathbb{T}}(\theta) \right)^{\frac{1}{2}} \]

\[ \leq \| f \|_{L^2(\mathbb{H}(G))} \left( \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G} \times \mathbb{T}} C_{j,G}^{-1} \| \sigma(x, \chi, \theta, j) \|_{S_2}^2 \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_{\mathbb{T}}(\theta) \right)^{\frac{1}{2}} < \infty. \]

Hence, \( T_{\sigma} \) is a bounded operator on \( L^2(\mathbb{H}(G)) \). \( \square \)

In the next theorem we prove that two symbols giving the same pseudo-differential operators are equal.

**Theorem 3.2.** Let \( \sigma : \mathbb{H}(G) \times \mathbb{Z}^* \to S_2 \) such that

\[ \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G} \times \mathbb{T}} C_{j,G}^{-1} \| \sigma(x, \chi, \theta, j) \|_{S_2}^2 \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_{\mathbb{T}}(\theta) < \infty. \] 

(7)

Furthermore, suppose that

\[ \sum_{j \in \mathbb{Z}^*} \| \sigma(x, \chi, \theta, j) \|_{S_2} < \infty, \quad \sup_{x,\chi,\theta,j} \| \sigma(x, \chi, \theta, j) \|_{S_2} < \infty \]

and the mapping \( (x, \chi, \theta, j) \mapsto \rho_j^* (x, \chi, \theta) \sigma(x, \chi, \theta, j) \) from \( \mathbb{H}(G) \times \mathbb{Z}^* \) into \( S_2 \) is weakly continuous. Then \( T_{\sigma} f = 0 \) \( \forall f \in L^2(\mathbb{H}(G)) \) if and only if \( \sigma(x, \chi, \theta, j) = 0 \) for almost all \( (x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^* \).

**Proof.** If \( \sigma(x, \chi, \theta, j) = 0 \) for almost all \( (x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^* \) then it is obvious that \( T_{\sigma} f = 0 \) for all \( f \in \mathbb{H}(G) \).

It remains to show that if \( T_{\sigma} f = 0 \) \( \forall f \in L^2(\mathbb{H}(G)) \) then \( \sigma(x, \chi, \theta, j) = 0 \) for almost all \( (x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^* \).

For \( (x, \chi, \theta) \in \mathbb{H}(G) \), define a function \( f_{x,\chi,\theta} \in L^2(\mathbb{H}(G)) \) as

\[ \hat{f}_{x,\chi,\theta}(j) = \sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta) \quad j \in \mathbb{Z}^*. \]
For all \((x', \chi', \theta') \in \mathbb{H}(G)\),
\[
(T_\sigma f_{x, \chi, \theta})(x', \chi', \theta') = \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x', \chi', \theta')\sigma(x', \chi', \theta', j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta)).
\]

Now, let \((x_0, \chi_0, \theta_0) \in \mathbb{H}(G)\). Then, using the weak continuity of the mapping
\[
(x, \chi, \theta, j) \mapsto \rho_j^*(x, \chi, \theta)\sigma(x, \chi, \theta, j)
\]
we get that
\[
tr(\rho_j^*(x', \chi', \theta')\sigma(x', \chi', \theta', j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta)) \to tr(\rho_j^*(x_0, \chi_0, \theta_0)\sigma(x_0, \chi_0, \theta_0, j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta))
\]
as \((x', \chi', \theta', \theta_0) \to (x_0, \chi_0, \theta_0)\) in \(\mathbb{H}(G)\). Moreover, by using \(sup_{x, \chi, \theta, j} ||\sigma(x, \chi, \theta, j)||_{S^2} < \infty\) there exists a constant \(K > 0\) such that for all \((x', \chi', \theta', j) \in \mathbb{H}(G) \times \mathbb{Z}^*\),
\[
|tr(\rho_j^*(x', \chi', \theta')\sigma(x', \chi', \theta', j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta))| \leq K ||\sigma(x, \chi, \theta, j)||_{S^2}.
\]
Now, using the assumption that \(\sum_{j \in \mathbb{Z}^*} ||\sigma(x, \chi, \theta, j)||_{S^2} < \infty\), we get, an application of Lebesgue dominated convergence theorem, that
\[
\sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x', \chi', \theta')\sigma(x', \chi', \theta', j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta)) \to \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x_0, \chi_0, \theta_0)\sigma(x_0, \chi_0, \theta_0, j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta))
\]
as \((x', \chi', \theta', \theta_0) \to (x_0, \chi_0, \theta_0)\) in \(\mathbb{H}(G)\). Therefore, \(T_\sigma f_{x, \chi, \theta}\) is continuous.

Next, by letting \((x_0, \chi_0, \theta_0) = (x, \chi, \theta)\) we get
\[
T_\sigma f_{x, \chi, \theta}(x, \chi, \theta) = \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x, \chi, \theta)\sigma(x, \chi, \theta, j)\sigma(x, \chi, \theta, j)^* \rho_j(x, \chi, \theta))
= \sum_{j \in \mathbb{Z}^*} tr(\sigma(x, \chi, \theta, j)\sigma(x, \chi, \theta, j)^*)
= \sum_{j \in \mathbb{Z}^*} ||\sigma(x, \chi, \theta, j)||_{S^2}^2 = 0.
\]
So, \(||\sigma(x, \chi, \theta, j)||_{S^2} = 0\) for almost every \(j \in \mathbb{Z}^*\) and hence \(\sigma(x, \chi, \theta, j) = 0\) for almost all \((x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^*\).
4. Trace of Weyl transform on a locally compact Abelian group

Let $X$ be a complex and separable Hilbert space in which the inner product is denoted by $(,)$ and let $A : X \to X$ be a compact operator. If we denote by $A^* : X \to X$ the adjoint of $A : X \to X$ then the linear operator $(A^*A)^{1/2} : X \to X$ is positive and compact. Let \( \{\psi_k, k = 1, 2, ..\} \) be an orthonormal basis for $X$ consisting of eigenvalues of $(A^*A)^{1/2} : X \to X$ and let $s_k(A)$ be the eigenvalue corresponding to the eigenvector $\psi_k, k = 1, 2, 3, ...$. Then $s_k(A)$ $k = 1, 2, 3, ...$, are the singular values of $A : X \to X$. If

$$
\sum_{k=1}^{\infty} s_k(A) < \infty,
$$

then the linear operator $A : X \to X$ is said to be in the trace class $S_1$. It can be shown that $S_1$ is a Banach space in which the norm $\| \cdot \|_{S_1}$ is given by

$$
\|A\|_{S_1} = \sum_{k=1}^{\infty} s_k(A), \ A \in S_1.
$$

Let $A : X \to X$ be a linear operator in $S_1$ and let \( \{\phi_k : k = 1, 2, 3, ...\} \) be any orthonormal basis for $X$. Then from [11], the series $\sum_{k=1}^{\infty} (A\phi_k, \phi_k)$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis \( \{\phi_k : k = 1, 2, 3, ...\} \). Thus the trace of any linear operator $A : X \to X$ in $S_1$ is defined by

$$
\text{tr}(A) = \sum_{k=1}^{\infty} (A\phi_k, \phi_k),
$$

where $\{\phi_k, k = 1, 2, 3...\}$ is any orthonormal basis of $X$.

The following well-known theorem describe a relation between trace class operator and Hilbert-Schmidt operators.

**Theorem 4.1.** Let $T \in B(\mathcal{H})$. Then $T$ is a trace class operator if and only if there exist two Hilbert-Schmidt operators $U$ and $V$ on $\mathcal{H}$ such that $T = UV$.

In this section we have obtained the trace formula for the trace class Weyl transform associated to a symbol in $L^2(G \times \hat{G})$, where $G$ is a locally compact abelian group.

Here first we recall that the Schrödinger representation $\rho_j, j \in \mathbb{Z}^*$ of the abstract Heisenberg group $\mathbb{H}(G)$ on $L^2(G)$ is defined by

$$
(\rho_j(x,\chi,\theta)\varphi)(y) = \theta^j \chi(y)^j \varphi(xy) \quad \varphi \in L^2(G).
$$
Using $\rho_j$, we define the $j$-Weyl transform $W^j : \sigma \mapsto W^j_\sigma$ from $C_c(G \times \widehat{G})$ into $\mathcal{B}(L^2(G))$ as

$$W^j_\sigma(\varphi)(y) = \int_{G \times \widehat{G}} \sigma(x, \chi)(\rho_j(x, \chi, 1)\varphi)(y) \, d\mu_c(\chi) \, d\mu_G(x).$$

Which is further can be written as

$$W^j_\sigma(\varphi)(y) = \int_{G \times \widehat{G}} \sigma(x, \chi)\chi(y)^j\varphi(xy) \, d\mu_G(\chi) \, d\mu_G(x) = \int_G K^j_\sigma(x, y) \, d\mu_G(x),$$

where

$$K^j_\sigma(x, y) = \int_G \sigma(xy^{-1}, \chi)\chi(y)^j \, d\mu_G(\chi).$$

Therefore, $W^j_\sigma$ is an integral operator with kernel $K^j_\sigma$. Note that for $\sigma \in L^2(G \times \widehat{G})$, the $j$-Weyl transform $W^j_\sigma$ is a Hilbert Schmidt operator. In fact,

$$\|W^j_\sigma\|_{L^2} = \|K^j_\sigma\|_{L^2} = (C_{j, G})^{-1}\|\sigma\|_{L^2}$$

and more generally, $\langle W^j_\sigma f, W^j_\sigma g \rangle = (C_{j, G})^{-1}\langle f, g \rangle_{L^2}$.

For two function $f$ and $g$ in $L^2(G \times \widehat{G})$, the $j$-twisted convolution $f \times_j g$, $j \in \mathbb{Z}^*$ of $f$ and $g$ is defined by

$$f \times_j g(x, \chi) = \int_G \int_{\mathbb{C}^G} f(x', \chi')g(xx'^{-1}, \chi\chi'^{-1})\chi(x')^j \, d\mu_G(x') \, d\mu_G(\chi').$$

Note for $j = 1$, $W^j_\sigma$ turns out to be the well known Weyl transform $W_\sigma$ and $j$-twisted convolution is nothing but twisted convolution studied, see [1][14]. In the same line of [14] it can be shown that $(L^2(G \times \widehat{G}), \times_j)$ is a Banach algebra. Further, the $j$-Weyl transform is a Banach algebra isomorphism from $L^2(G \times \widehat{G})$ onto the space of all Hilbert-Schmidt operators on $L^2(G)$ denoted by $\mathcal{S}_2(L^2(G))$ or $\mathcal{S}_2$. Therefore, for any $A \in \mathcal{S}_2(L^2(G))$, there exist a unique $\sigma \in L^2(G \times \widehat{G})$ such that $A = W^j_\sigma$. Also, $W^j_\sigma W^j_\tau = W^j_{\sigma \times_j \tau}$.

Denote the subset of all $\lambda \in L^2(G \times \widehat{G})$ such that there exist functions $\sigma, \tau \in L^2(G \times \widehat{G})$ such that $\lambda = \sigma \times_j \tau$ by $W_j$.

Now we present the following theorem on the characterization of trace class $j$-Weyl transform.

**Theorem 4.2.** Let $W^j_\sigma$ be the $j$-Weyl transform on $G$ associated with symbol $\sigma \in L^2(G \times \widehat{G})$. Then $W^j_\sigma$ is a trace class operator if and only if $\sigma \in W_j$.

**Proof.** Let $\sigma \in W_j$. Then there exist $\lambda$ and $\tau$ in $L^2(G \times \widehat{G})$ such that $\sigma = \lambda \times_j \tau$. So,

$$W^j_\sigma = W^j_{\lambda \times_j \tau} = W^j_\lambda W^j_\tau.$$

Since $\tau$ and $\lambda$ are in $L^2(G \times \widehat{G})$, it follows that $W_\lambda$ and $W_\tau$ are Hilbert Schmidt operators. Therefore, $W_\sigma$ is a trace class operator being product of two Hilbert Schmidt operators.
Conversely, suppose that $W^j_\sigma$ is a trace class operator. Then, it follows that $W_\sigma = AB$ for some Hilbert Schmidt operators $A$ and $B$ on $L^2(G)$. Since $j$-Weyl transform is an algebra isomorphism from $L^2(G \times \hat{G})$ onto $S_2$, it follows that there exist $\lambda, \tau \in L^2(G \times \hat{G})$ such that $A = W^j_\lambda$ and $B = W^j_\tau$ and therefore, $W^j_\sigma = W^j_\lambda W^j_\tau = W^j_{\lambda \tau}$. Hence, $\sigma \in W_j$. \hfill \qed

The following corollary of Theorem 4.2 is immediate once we recall that $j$-Weyl transform is a Banach algebra isomorphism from $(L^2(G \times \hat{G}), \times_j)$ onto $S_2(L^2(G))$.

**Corollary 4.3.** The $W_j$ is a subspace of $L^2(G \times \hat{G})$.

In addition to this corollary, we have the following result the space $W_j$.

**Theorem 4.4.** $W_j$ is a dense subspace of $L^2(G \times \hat{G})$.

**Proof.** In view of Corollary 4.3, we only need to prove that $W_j$ is dense in $L^2(G \times \hat{G})$. Let $F$ be the set of all functions $\sigma$ on $G \times \hat{G}$ such that $W^j_\sigma$ is a finite rank operator on $L^2(G)$. Since every element in $S_2(L^2(G))$ is the limit of a sequence of finite rank operators on $L^2(G)$ and $W^j_\sigma$ is in $S_2(L^2(G))$ if and only if $\sigma \in L^2(G \times \hat{G})$, it follows that $F$ is a dense subspace of $L^2(G \times \hat{G})$. Obviously, $F$ is a subspace of $W_j$. Therefore $W_j$ is dense in $L^2(G \times \hat{G})$. \hfill \qed

Now we calculate the trace of trace of $j$-Weyl transform $W^j_\sigma$ associated with symbol $\sigma$.

**Theorem 4.5.** Let $\sigma \in L^2(G \times \hat{G})$ such that the $j$-Weyl transform $W^j_\sigma$ is a trace class operator. Then

$$tr(W^j_\sigma) = \int_G \int_{\hat{G}} \sigma(x, \chi) d\mu_\hat{G}(\chi) d\mu_G(x).$$

**Proof.** Since $\sigma \in L^2(G \times \hat{G})$, it follows that $W^j_\sigma$ is a Hilbert Schmidt operator with kernel $K^j_\sigma(x, y)$ given by

$$K^j_\sigma(x, y) = \int_{\hat{G}} \sigma(xy^{-1}, \chi) \chi(y)^j d\mu_\hat{G}(\chi).$$

Now, as by assumption $W^j_\sigma$ is a trace class operator so $\int_G K^j_\sigma(x, x) d\mu_G(x)$ exists and

$$tr(W^j_\sigma) = \int_G K^j_\sigma(x, x) d\mu_G(x)$$

$$= \int_G \int_{\hat{G}} \sigma(e, \chi) \chi(x)^j d\mu_\hat{G}(\chi) d\mu_G(x)$$

$$= \int_G \int_{\hat{G}} \sigma(x^{-1}, \chi) \chi(e)^j d\mu_\hat{G}(\chi) d\mu_G(x)$$

$$= \int_G \int_{\hat{G}} \sigma(x, \chi) \chi(e)^j d\mu_\hat{G}(\chi) d\mu_G(x)$$

where $e$ denotes the identity element of $G$. Since $\chi(e) = 1$, we get
\[ \text{tr}(W^j_\sigma) = \int_G \int_{\hat{G}} \sigma(x, \chi) \, d\mu_G(\chi) \, d\mu_G(x). \]

We will give another formula for the trace of a trace class Weyl transform which will be useful for our study in next section.

**Theorem 4.6.** Let \( \sigma = \lambda \times \tau \) for some \( \lambda, \tau \in L^2(G \times \hat{G}) \) such that the \( j \)-Weyl transform \( W^j_\sigma \) is a trace class operator. Then

\[ \text{tr}(W^j_\sigma) = (C_{j, G})^{-1} \int_{G \times \hat{G}} \tau(x, \chi) \lambda(x^{-1}, \chi) \, d\mu_G(x) \, d\mu_G(\chi). \]  

**Proof.** First note that

\[ W^j_\sigma = W^j_\lambda W^j_\tau = W^j_{\lambda \times j \tau}. \]

Since \( W^j_\sigma \) is a trace class operator and hence a Hilbert Schmidt operator on \( L^2(G) \). Let \( \{ \varphi_k : k \in \mathbb{N} \} \) is an orthonormal basis for \( L^2(G) \). Then, by using the fact that \( (W^j_\lambda)^* = W^j_{\hat{\lambda}} \), where \( \hat{\lambda}(x, \chi) = \chi(x) \overline{\lambda(x^{-1}, \chi)} \), we have

\[ \text{tr}(W^j_\sigma) = \sum_{k \in \mathbb{N}} \langle W^j_\sigma \varphi_k, \varphi_k \rangle = \sum_{k \in \mathbb{N}} \langle W^j_\lambda W^j_\tau \varphi_k, \varphi_k \rangle = \sum_{k \in \mathbb{N}} \langle W^j_\tau \varphi_k, (W^j_\lambda)^* \varphi_k \rangle = \sum_{k \in \mathbb{N}} \langle W^j_\tau \varphi_k, W^j_{\hat{\lambda}} \varphi_k \rangle = \langle W^j_\tau, W^j_{\hat{\lambda}} \rangle_{S_2}. \]

By Using the relation \( \langle W^j_\tau, W^j_{\hat{\lambda}} \rangle_{S_2} = (C_{j, G})^{-1} \langle f, g \rangle_{L^2(G \times \hat{G})} \), we get

\[ \text{tr}(W^j_\sigma) = (C_{j, G})^{-1} \langle \tau, \hat{\lambda} \rangle_{L^2(G \times \hat{G})} \]

\[ = (C_{j, G})^{-1} \int_{G \times \hat{G}} \overline{\chi(x)} \tau(x, \chi) \lambda(x^{-1}, \chi) \, d\mu_G(x) \, d\mu_G(\chi). \]

5. **Hilbert-Schmidt pseudo-differential operators on the abstract Heisenberg groups**

In this section, we characterize the Hilbert-Schmidt pseudo-differential operators in terms of their corresponding symbols. We begin this section with following observation.

Let \( f \in L^2(\mathbb{H}(G)) \). For \( j \in \mathbb{Z}^* \), we know that \( f^j \) is defined as
Proof. We first show the sufficiency part. Let 
\[ f^j(x, \chi) = \int_{\mathbb{T}} f(x, \chi, \theta) \theta^j d\theta, \quad (x, \chi, \theta) \in G \times \hat{G} \times \mathbb{T}. \]

Note that \( f^j \) is the inverse Fourier transform of \( f \) in \( \theta \) variable or Fourier transform of \( f \) with respect to the center of \( \mathbb{H}(G) \). Therefore, it is convenient to write \( f^j \) is the following form:

\[ f^j(x, \chi) = (\mathcal{F}^{-1}_c f)(x, \chi, j) = (\mathcal{F}_c f)(x, \chi, -j), \]

where \( \mathcal{F}_c \) denote the Fourier transform with respect to center of \( \mathbb{H}(G) \).

Before stating our main theorem of this section, we would like to note \( \hat{f}(j)\varphi = W^j_{\chi}(\varphi) \) for all \( \varphi \in L^2(\mathbb{H}(G)) \). In fact,

\[
\hat{f}(j)\varphi(y) = \int_{G \times \hat{G} \times \mathbb{T}} f(x, \chi, \theta) (\rho_j(x, \chi, \theta)\varphi)(y) d\mu_G(x) d\mu_\hat{G}(\chi) d\mu_\mathbb{T}(\theta) \\
= \int_{G \times \hat{G} \times \mathbb{T}} f(x, \chi, \theta) \theta^j (\rho_j(x, \chi, 1)\varphi)(y) d\mu_G(x) d\mu_\hat{G}(\chi) d\mu_\mathbb{T}(\theta) \\
= \int_{G \times \hat{G}} f^j(x, \chi) \chi(x)^j \varphi(xy) d\mu_G(x) d\mu_\hat{G}(\chi) \\
= (W^j_{\chi}\varphi)(y).
\]

The following theorem is the main theorem of this section which characterizes Hilbert-Schmidt pseudo-differential operators on \( L^2(\mathbb{H}(G)) \).

**Theorem 5.1.** Let \( \sigma : \mathbb{H}(G) \times \mathbb{Z}^* \to S_2 \) be a symbol such that the hypothesis of Theorem 3.2 is satisfied. Then the corresponding pseudo-differential operator \( T_\sigma : L^2(\mathbb{H}(G)) \to \mathbb{H}(G) \) is Hilbert-Schmidt operator if and only if

\[ \sigma(x, \chi, \theta, j) = C_{j,G} \rho_j(x, \chi, \theta) W^j_{\alpha(x, \chi, \theta)-j} \quad (x, \chi, \theta) \in \mathbb{H}(G), \ j \in \mathbb{Z}^*, \]

where \( \alpha : \mathbb{H}(G) \to L^2(\mathbb{H}(G)) \) is a weakly continuous mapping for which

\[ \int_{G \times \hat{G} \times \mathbb{T}} \|\alpha(x, \chi, \theta)\|^2_{L^2(\mathbb{H}(G))} d\mu_G(x) d\mu_\hat{G}(\chi) d\mu_\mathbb{T}(\theta) < \infty, \]

\[ \sup_{(x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^*} \|\mathcal{F}_c \alpha(x, \chi, \theta)(\cdot, \cdot, j)\|_{L^2(\mathbb{G} \times \hat{\mathbb{G}})} < \infty \]

and

\[ \sum_{j \in \mathbb{Z}^*} \|\mathcal{F}_c \alpha(x, \chi, \theta)(\cdot, \cdot, j)\|_{L^2(\mathbb{G} \times \hat{\mathbb{G}})} < \infty. \]

**Proof.** We first show the sufficiency part. Let \( f \in C_c(\mathbb{H}(G)) \). Then for all \( (x, \chi, \theta) \in \mathbb{H}(G) \),

\[ (T_\sigma f)(x, \chi, \theta) = \sum_{j \in \mathbb{Z}^*} tr(\rho_j^*(x, \chi, \theta)\sigma(x, \chi, \theta, j)\hat{f}(j)). \]
Using the expression of $\sigma$ and the fact that $\hat{f}(j) = W^j$, we have

\[
(T_{\sigma} f)(x, \chi, \theta) = \sum_{j \in \mathbb{Z}^*} C_{j,G} \text{tr}(\rho_j^*(x, \chi, \theta) \rho_j(x, \chi, \theta) W^j_{\alpha(x, \chi, \theta) - j} W^j_{f_j})
\]

\[
= \sum_{j \in \mathbb{Z}^*} C_{j,G} \text{tr}(W^j_{\alpha(x, \chi, \theta) - j} W^j_{f_j})
\]

\[
= \sum_{j \in \mathbb{Z}^*} \text{tr}(W^j_{(\alpha(x, \chi, \theta) - j) \times j} C_{j,G} f_j).
\]

By (11), we get

\[
(T_{\sigma} f)(x, \chi, \theta) = \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G}} (\alpha(x, \chi, \theta)^{-j})(x^{-1}, \chi') f_j(x', \chi') \chi'(x^{-1}) \, d\mu_G(x') \, d\mu_{\hat{G}}(\chi')
\]

\[
= \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G}} (\alpha(x, \chi, \theta)^{-j})(x^{-1}, \chi') f_j(x', \chi') \chi'(x^{-1}) \, d\mu_G(x') \, d\mu_{\hat{G}}(\chi')
\]

\[
= \sum_{j \in \mathbb{Z}^*} \int_{G \times \hat{G}} (F_c \alpha(x, \chi, \theta))(x^{-1}, \chi') \chi'(x^{-1}) \, d\mu_G(x') \, d\mu_{\hat{G}}(\chi')
\]

\[
= \int_{G \times \hat{G} \times \mathbb{T}} (\alpha(x, \chi, \theta))(x^{-1}, \chi', \theta') f(x', \chi', \theta') \chi'(x^{-1}) \, d\mu_G(x') \, d\mu_{\hat{G}}(\chi') \, d\mu_{\mathbb{T}}(\theta').
\]

Thus, the kernel of the operator $T_{\sigma}$ is a function $k$ on $(G \times \hat{G} \times \mathbb{T}) \times (G \times \hat{G} \times \mathbb{T})$ given as

\[
k((x, \chi, \theta), (x', \chi', \theta')) = \chi'(x^{-1}) \alpha(x, \chi, \theta)(x^{-1}, \chi', \theta').
\]

Now, using Fubini theorem

\[
\int_{\mathbb{H}(G) \times \mathbb{H}(G)} |k((x, \chi, \theta), (x', \chi', \theta'))|^2 d\mu_G(x') \, d\mu_{\hat{G}}(\chi') d\mu_{\mathbb{T}}(\theta') d\mu_G(x) \, d\mu_{\hat{G}}(\chi) d\mu_{\mathbb{T}}(\theta)
\]

\[
= \int_{\mathbb{H}(G) \times \mathbb{H}(G)} |\chi'(x^{-1}) \alpha(x, \chi, \theta)(x^{-1}, \chi', \theta')|^2 d\mu_G(x') \, d\mu_{\hat{G}}(\chi') d\mu_{\mathbb{T}}(\theta') d\mu_G(x) \, d\mu_{\hat{G}}(\chi) d\mu_{\mathbb{T}}(\theta)
\]

\[
= \int_{G \times \hat{G} \times \mathbb{T}} \|\alpha(x, \chi, \theta)\|_{L^2(\mathbb{H}(G))}^2 d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_{\mathbb{T}}(\theta) < \infty.
\]

Therefore, $T_{\sigma} : L^2(\mathbb{H}(G)) \to L^2(\mathbb{H}(G))$ is a Hilbert-Schmidt operator.

Conversely, assume that $T_{\sigma} : L^2(\mathbb{H}(G)) \to L^2(\mathbb{H}(G))$ is a Hilbert-Schmidt operator. Then there exist a function $\alpha \in L^2(\mathbb{H}(G) \times \mathbb{H}(G))$ such that for $f \in L^2(\mathbb{H}(G))$

\[
(T_{\sigma} f)(x, \chi, \theta) = \int_{\mathbb{H}(G) \times \mathbb{H}(G)} \alpha((x, \chi, \theta), (x', \chi', \theta')) f(x', \chi', \theta') \, d\mu_G(x') \, d\mu_{\hat{G}}(\chi') \, d\mu_{\mathbb{T}}(\theta').
\]

Define $\alpha : \mathbb{H}(G) \to L^2(\mathbb{H}(G))$ as

\[
\alpha(x, \chi, \theta)(x', \chi', \theta') = \alpha((x, \chi, \theta), (x', \chi', \theta')), \quad (x, \chi, \theta), (x', \chi', \theta') \in \mathbb{H}(G).
\]
Then using \[10\], we see
\[
\|\sigma(x, \chi, \theta, j)\|_{S_2} = \|(F_c\alpha(x, \chi, \theta))(\cdot, j)\|_{L^2(G \times \mathbb{G})} \quad (x, \chi, \theta, j) \in \mathbb{H}(G) \times \mathbb{Z}^a.
\]

Then reversing the argument in the proof of sufficiency and using Theorem \[3.2\] the converse is proved. \hfill \Box

Now we present following corollary on a trace class pseudo-differential operator on \(\mathbb{H}(G)\) and its trace formula.

**Corollary 5.2.** Let \(\alpha \in L^2(\mathbb{H}(G) \times \mathbb{H}(G))\) be such that
\[
\int_{G \times \mathbb{G} \times \mathbb{T}} |\alpha((x, \chi, \theta), (x, \chi, \theta))| \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_T(\theta).
\]

Let \(\sigma : \mathbb{H}(G) \times \mathbb{Z}^* \rightarrow \mathcal{B}(L^2(\mathbb{H}))\) be a symbol as in the Theorem \[5.1\] Then \(T_\sigma : L^2(\mathbb{H}(G)) \rightarrow L^2(\mathbb{H}(G))\) is a trace class operator and
\[
\text{tr}(T_\sigma) = \int_{\mathbb{H}(G)} \chi(x)\alpha((x, \chi, \theta), (x^{-1}, \chi, \theta)) \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_T(\theta).
\]

**Proof.** The proof of the corollary follows immediately from the formula \[15\] on the kernel of pseudo-differential operator in Theorem \[5.1\]. \hfill \Box

6. **Trace class pseudo-differential operators on the abstract Heisenberg groups**

**Theorem 6.1.** Let \(\sigma : \mathbb{H}(G) \times \rightarrow S_2\) be a symbol satisfying the hypothesis of Theorem \[3.2\]. Then the pseudo-differential operator \(T_\sigma : L^2(\mathbb{H}(G)) \rightarrow L^2(\mathbb{H}(G))\) is a trace class operator if and only if
\[
\sigma(x, \chi, \theta, j) = C_{j, G} \rho_j(x, \chi, \theta) W_{(x, \chi, \theta) \rightarrow}^j \quad (x, \chi, \theta) \in \mathbb{H}(G), \ j \in \mathbb{Z}^*,
\]
where \(\alpha : \mathbb{H}(G) \rightarrow L^2(\mathbb{H}(G))\) is a mapping such that the conditions of Theorem \[5.1\] are satisfied and
\[
\alpha(x, \chi, \theta)(x', \chi', \theta') = \int_{G \times \mathbb{G} \times \mathbb{T}} \alpha_1(x, \chi, \theta)(x'', \chi'', \theta'') \alpha_2(x'', \chi'', \theta'')(x', \chi', \theta') \, d\mu_G(x'') \, d\mu_{\hat{G}}(\chi'') \, d\mu_T(\theta'')
\]
for all \((x, \chi, \theta), (x', \chi', \theta') \in \mathbb{H}(G)\); here \(\alpha_1 : \mathbb{H}(G) \rightarrow L^2(\mathbb{H}(G))\) and \(\alpha_2 : \mathbb{H}(G) \rightarrow L^2(\mathbb{H}(G))\) are such that
\[
\int_{G \times \mathbb{G} \times \mathbb{T}} \|\alpha_i(x, \chi, \theta)\|_{L^2(\mathbb{H}(G))} < \infty \quad i = 1, 2.
\]

Moreover, if \(T_\sigma : L^2(\mathbb{H}(G)) \rightarrow L^2(\mathbb{H}(G))\) is a trace class operator then
\[
\text{tr}(T_\sigma) = \int_{G \times \mathbb{G} \times \mathbb{T}} \alpha(x, \chi, \theta)(x, \chi, \theta) \, d\mu_G(x) \, d\mu_{\hat{G}}(\chi) \, d\mu_T(\theta)
\]
\[
\int_{H(G)} \int_{H(G)} \alpha_1(x, \chi, \theta)(x', \chi', \theta') \alpha_2(x', \chi', \theta')(x, \chi, \theta) d\mu_G(x') d\mu_G(\chi') d\mu_T(\theta') d\mu_G(x) d\mu_G(\chi) d\mu_T(\theta).
\]

**Proof.** The proof of this theorem follows from Theorem 5.1 and the fact the every trace class operator is the product of two Hilbert-Schmidt operators. \qed

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