Orientifolds, Mirror Symmetry and Superpotentials

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Abstract

We consider orientifolds of Calabi-Yau 3-folds in the context of Type IIA and Type IIB superstrings. We show how mirror symmetry can be used to sum up worldsheet instanton contributions to the superpotential for Type IIA superstrings. The relevant worldsheets have the topology of the disc and $\mathbb{RP}^2$. 
1. Introduction

Mirror symmetry has been proven effective in the computation of superpotentials in four dimensional $\mathcal{N} = 1$ supersymmetric theories [1,2,3,4,5,6,7]. In this paper we show how mirror symmetry can be used in the context of orientifolds of Calabi-Yau backgrounds to lead to computable superpotentials which receive corrections from worldsheet instantons. The worldsheet instantons have the topology of $\mathbb{RP}^2$ and, if there are D-branes around, the disc $\mathbb{D}^2$. In principle both contribute to the superpotential, as we will discuss. Along the way, we confirm a prediction of [8] (based on a large $N$ Chern-Simons conjecture) for the superpotential arising when we consider Type IIA superstrings propagating on an orientifold of the resolution of the conifold.

The organization of this paper is as follows: In section 2 we describe some general features of supersymmetric orientifolds of Type IIA and IIB string theories on Calabi-Yau manifolds. We describe the spacetime superpotential in the case of orientifolds of IIB. In section 3 we present examples of Type IIA orientifolds of non-compact Calabi-Yau threefolds, as well as their Type IIB mirrors. In section 4 we present further examples of non-compact CY orientifolds.

2. Superstrings and Calabi-Yau Orientifolds

Compactification of Type IIA or IIB superstring theory on a Calabi-Yau threefold $X$ gives rise to an $\mathcal{N} = 2$ supersymmetric theory in $d = 4$. We can also consider orientifolding this theory, by combining an involution symmetry $I$ on the Calabi-Yau with orientation reversal $\Omega$ on the worldsheet, i.e. we consider gauging the symmetry generated by $I\Omega$. Due to their respective orientations, this orientifold is only well defined if $I$ preserves the orientation of $X$ in the case of the IIB string and reverses it in the case of IIA. Additionally, if $I$ obeys certain further properties, the orientifold theory will be an $\mathcal{N} = 1$ supersymmetric theory in $d = 4$: For Type IIA superstrings the involution $I$ has to exchange the holomorphic 3-form $\omega$, up to a phase, with the anti-holomorphic 3-form $\overline{\omega}$. This is because the left-moving space-time supercharge corresponds to the holomorphic 3-form and the right-moving space-time supercharge corresponds to the anti-holomorphic 3-form. This means that $I$ should be an anti-holomorphic involution in the context of Type IIA superstrings. For type IIB superstrings both the left- and right-moving supercharges correspond to the holomorphic 3-form on Calabi-Yau. Thus the involution should send $\omega \to \pm \omega$. This is a holomorphic involution.
If \( I \) has fixed points \( \Sigma \) in \( X \), we find orientifold planes of topology \( \Sigma \times \mathbb{R}^4 \). For Type IIA superstrings, the fixed point of the anti-holomorphic involution will be real codimension three. In other words, it is an O6 plane. To see this, note that in local coordinates on a point on orientifold plane an anti-holomorphic involution will have three \(-1\) eigenvalues and three \(+1\) eigenvalues. For Type IIB, the orientifold plane could be O3, O5, O7 or O9, depending on the number of eigenvalues of the holomorphic involution on the tangent space of any point on the orientifold plane. Note that to have an O5 or O9 the involution takes \( \omega \to \omega \), and for O3, O7 it takes \( \omega \to -\omega \). Note that we can have one or the other of these situations if we wish to preserve supersymmetry, but not both.

In compact Calabi-Yau threefolds, if we end up with orientifold planes, we need to cancel their D-brane charge by including appropriate space-time filling, Calabi-Yau wrapped D-branes. In the language of the ‘parent’ theory before orientifolding, this corresponds to including wrapped D-brane configurations which are invariant under the involution \( I \Omega \). In the non-compact case, we do not need to cancel the D-brane charge (except for the D9 brane charge) as the flux can go off to infinity. Nevertheless we can choose to include D-branes in the background.

Note that since Type IIA and IIB superstrings on mirror Calabi-Yau pairs are equivalent, it follows that the orientifold operation of IIA is equivalent to an orientifold operation for IIB on the mirror Calabi-Yau. In this case the O6-plane gets mapped to O3- and O7-planes or O5- and O9-planes. Similarly D6-branes get mapped to D3- and D7-branes or D5- and D9-branes depending on which case we are in.

2.1. Space-time superpotential

Since we have \( \mathcal{N} = 1 \) supersymmetry in \( d = 4 \) a superpotential can in principle be generated by the orientifold operation. General arguments [9,10] which relate the tensions of domain walls to the value of the superpotential can be used to propose a formula for the superpotential itself in various classes of string compactifications. These formulae have proven successful in Calabi-Yau compactifications [9,11,12], compactifications on \( G_2 \)-manifolds, [13] and even \( Spin(7) \)-manifolds [14]. In the context of orientifolds of Type IIB string theory on Calabi-Yau threefolds, the BPS domain walls are actually D5-branes wrapped on supersymmetric 3-cycles in \( X \). The fact that such 3-cycles are in fact calibrated by \( \omega \) leads to the following formula for the superpotential,

\[
W = \int H \wedge \omega
\]  

(2.1)
where $H = dB_{RR}$ denotes the RR 3-form field strength. As a check on this formula, note that in Type I theory, $H$ also includes contributions from the Chern-Simons 3-forms constructed out of the $SO(32)$ gauge field $A$ and the spin connection $w$. Thus, $W$ contains a term which is none other that the holomorphic Chern-Simons functional,

$$
\int tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \wedge \omega
$$

(2.2)

This is the known superpotential for Type I and heterotic string theory on a Calabi-Yau threefold [15]. In fact this is simply the expected superpotential from the viewpoint of topological B-model on the 9-brane [16] [17]. Moreover, the fact that $H$ is defined with the addition of the Chern-Simons 3-form in the type I theory, reflects that fact that an instanton on the 9-brane can be viewed as a 5-brane, which is the source of $H$-flux.

Since both D5-branes and O5-planes are sources for $H$, we can think of $W$ as being “generated” by D5-brane charge. All sources for $H$ contribute. Let $D_i$ denote the locations of the D5-brane charges (which includes O5-planes). Each $D_i$ is a 2-dimensional subspace of the Calabi-Yau. This is because locally we have that

$$
dH = \sum_i \delta_i
$$

(2.3)

where the $\delta_i$ are four form currents supported at the D5 or O5 locations. These are Poincare dual to 2-cycles $D_i$.

Suppose we are considering the compact case. Then the total 5-brane charge is zero, which implies that $\sum_i [D_i] = 0$ where $[D_i]$ denotes the class in $H_2(X, \mathbb{Z})$ of $D_i$. This implies that there is a three dimensional subspace of the Calabi-Yau, $C$, such that its boundary is

$$
\partial C = \sum_i D_i
$$

Then we can view $C$ as the “flux tube” for the D5-brane charges. Of course $C$ is not unique. However the difference of any two choices gives a closed 3-cycle in the Calabi-Yau, which corresponds to turning on an integer $H$-flux in the Calabi-Yau. So for a given configuration of $H$-fluxes in the bulk Calabi-Yau, we get a class of $C$’s, whose difference is homologically trivial. The fact that $C$ is a flux tube for D5 brane charge means that

$$
\int H \wedge \alpha = \int_C \alpha
$$
for any closed 3-form $\alpha$. Applying this to (2.1) we find

$$ W = \int H \wedge \omega = \int_C \omega $$

(2.4)

where $C$ is a 3-chain with $\partial C = \sum_i D_i$. This formula was derived in the compact case, but the idea also applies to the non-compact case. In that case we do not need a net 5 brane charge being zero, as the flux can go to infinity. Put differently we can assume we have a non-compact $C$ by taking the boundary at infinity corresponding to “5-brane charge at infinity”. Thus the same formula still applies. More precisely, in that case we can view (2.4) as defining the superpotential up to an addition of a constant corresponding to how we fix the boundary condition at infinity [18,1,2].

What does this correspond to at the level of worldsheet computations? The $D_i$ correspond to charges coming from physical D5 branes as well as O5 planes carrying D5 brane charge. As far as the computations of the physical D5 branes they correspond to contributions from worldsheet geometry being a disk [17,13,20,1]. As far as the contribution to the superpotential due to the O5 planes, this should come from non-orientable worldsheets. Based on the requisite R-charge, it is possible to show that they can only come from $\mathbb{RP}^2$.

To see this, note that for worldsheet geometries being an $S^2$, in the Berkovits formalism, computation of F-type terms leads to 4 fermionic zero modes on the worldsheet which contribute to space-time action

$$ \int d^4 \theta F(t_i) $$

where $F$ is the prepotential of $N = 2$ theory in $d = 4$ and $t_i$ denote the vector multiplet. Moreover $F$ is the partition function of topological strings on Calabi-Yau where $t_i$ is identified with the moduli of Calabi-Yau. If we consider $\mathbb{RP}^2$ geometry instead, the same analysis leads to only two fermionic zero modes (two being gotten rid of by the orientifold operation). Thus we end up with an $\mathcal{N} = 1$ superpotential term [8]

$$ \int d^2 \theta W(t_i) $$

where $t_i$ denote chiral fields which survive from the corresponding vector multiplet under the orientifold operation. Here $W(t_i)$ corresponds to the partition function of topological strings on $\mathbb{RP}^2$. In this case, it corresponds to the partition function of topological B-model on $\mathbb{RP}^2$.

We can also ask how these worldsheet computations arise in the context of Type IIA orientifolds. By mirror symmetry, the worldsheets will arise in the same way: Namely
there will be contribution to the superpotential corresponding to disk amplitudes (with boundary on D6 branes) and there will also be a contribution to the superpotential from \( \mathbb{RP}^2 \). The superpotential in the IIA string theory can be computed by the topological A-model theory (The B-model computation that is relevant for the IIB superpotential ends up being equivalent to that described above \[1\]). Recall that the topological A-model, roughly speaking, counts the number of holomorphic maps from the worldsheet to the target, weighted by \( e^{-A} \) where \( A \) is the area of the image. In the context of non-orientable worldsheets geometries the same is true when we go to the covering theory. In other words we consider the worldsheets geometries with crosscaps to be the \( \mathbb{Z}_2 \) quotients of an orientable Riemann surface and we count holomorphic maps from the covering worldsheet to the target, compatible with the simultaneous \( \mathbb{Z}_2 \) actions on the worldsheet and the target (in mathematical terminology these are known as \( \mathbb{Z}_2 \) equivariant maps). In this way, once we know the mirror geometry, we can use the Type IIB result given by (2.4) to obtain the non-trivial worldsheet instanton generated superpotential in the Type IIA setup. At the level of disk amplitudes this was already done in \[1,2\]. Here we will be mostly interested in the contribution from the \( \mathbb{RP}^2 \) diagrams to the superpotential, which correspond, in the Type IIB setup, to the contribution of the O5 planes to the superpotential.

We now turn to examples.

3. Examples

In this section we present a number of examples corresponding to IIA and IIB superstrings on orientifolds of Calabi-Yau threefolds. We consider non-compact models, starting with examples of two distinct orientifolds of the resolved conifold in the IIA setup. We also study its type IIB mirror and use that to compute the contribution of worldsheet instantons of the IIA theory with the topology of \( \mathbb{RP}^2 \) to the superpotential. We then generalize the discussion to some other non-compact examples.

3.1. Anti-holomorphic orientifolds of \( O(-1) + O(-1) \to \mathbb{CP}^1 \)

We start with an example of local A-model geometry, a Calabi-Yau manifold \( X \) that is an \( O(-1) + O(-1) \) bundle over \( \mathbb{CP}^1 \).

The Calabi-Yau sigma-model can be obtained as the theory on the Higgs branch of a linear sigma model in two dimensions with \( N = (2,2) \) supersymmetry \[21\]. The linear sigma model associated to \( X \) has gauge group \( G = U(1) \) and two chiral fields \( X_{1,2} \) of
charge +1 and two fields \( X_{3,4} \) of charge \(-1\). The Higgs branch is the space of minima of D-term potential:
\[
|X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = r,
\]
modulo \( G \) action. As described in [1], \( X \) can naturally be viewed as a \( T^3 \) fibration over a toric base. The size of the \( \mathbb{CP}^1 \) is set by the Fayet-Illiopolous parameter \( r \). This is complexified by the theta angle of the gauge theory to give the complexified Kahler parameter \( t = r - i\theta \) on which the \( A \)-model amplitudes depend.

Worldsheet orientation reversal is a symmetry of topological \( A \)-model when accompanied with an anti-holomorphic involution of the target space. This is because the \( A \)-type twist correlates the chirality of the fermions and the \( U(1)_R \) charge in such a way that, e.g., \( \psi_-, \psi_+ \) have the same topological charge. Orientation reversal \( \Omega \) exchanges left and right moving fermions so in the \( A \)-model, it must be accompanied with an anti-holomorphic involution \( I \) of the target space. This is in accord with the Type IIA superstring interpretation discussed in the introduction.

As anti-holomorphic involutions we take the following possible actions on chiral superfields

\[
I^\pm : (X_1, X_2, X_3, X_4) \to (\bar{X}_2, \pm \bar{X}_1, \bar{X}_4, \pm \bar{X}_3). \quad (3.1)
\]

The involution clearly commutes with \( G \), and can be extended to a symmetry of the linear-sigma model Lagrangian for any value of the complexified Kahler parameter \( t \). If we define \( z = X_1/X_2 \), coordinatizing the \( \mathbb{CP}^1 \), then these involutions act

\[
I^\pm : z \to \pm \frac{1}{z}
\]

These are the unique, up to diffeomorphism, involutions of \( \mathbb{CP}^1 \). \( I^- \) has no fixed points and the quotient of \( \mathbb{CP}^1 \) by \( I^- \) is \( \mathbb{RP}^2 \). Therefore, \( I^- \) has no fixed points on the bundle over \( \mathbb{CP}^1 \). \( I^+ \) has a circle of fixed points in \( \mathbb{CP}^1 \) given by \( |z| = 1 \). This circle is naturally regarded as a copy of \( \mathbb{RP}^1 \subset \mathbb{CP}^1 \). Its fixed points in the \( O(-1) + O(-1) \) bundle over \( \mathbb{CP}^1 \) are easily seen to be a bundle over \( \mathbb{RP}^1 \). The fibers are all copies of \( \mathbb{R}^2 \). Thus in this case we get an O6-plane \( L \) which is this rank two real bundle over \( \mathbb{RP}^1 \). To be completely explicit, the fixed point set is given by

\[
|X_1|^2 = |X_2|^2, \quad |X_3|^2 = |X_4|^2, \quad \sum_{i=1}^{4} \theta_i = 0. \quad (3.2)
\]
and modulo $G$ action and the D-term equation. Here $\theta_i$ is the phase of $X_i$, $X_i = |X_i|e^{i\theta_i}$. By construction, the fixed point set must be a Lagrangian submanifold of the Calabi-Yau. In fact it is in the family of Lagrangians constructed in [2]. There, a 1-dimensional moduli space of Lagrangians was found. This family is $I^+$-invariant at just one point - which is exactly the 3-submanifold $L$ described here. See fig 1.

There is an alternative description of both $X$ and the orientifold action in terms of coordinates invariant under complexified gauge group $G_C = \mathbb{C}^*$. These are $x_{13}, x_{24}, x_{14}, x_{23}$, where $x_{ij} = X_i X_j$, and they satisfy one constraint

$$x_{13}x_{24} = x_{14}x_{23},$$

which is the equation of the conifold.

Introducing a coordinate $z = X_1/X_2$

$$x_{14} = zx_{24}, \quad x_{13} = zx_{23},$$

solves $x_{ij} = 0$, and defines transition functions of $O(-1) + O(-1) \to \mathbb{P}^1$, where $z$ is local coordinate on the $\mathbb{C}P^1$ and $x_{13}, x_{14}$ coordinates on fibers. The involution acts as

$$I^\pm : (x_{14}, x_{13}, x_{24}, x_{23}) \to (\pm \bar{x}_{23}, \bar{x}_{24}, \bar{x}_{13}, \pm \bar{x}_{14}), \text{ or}$$

$$I^\pm : (z, x_{13}, x_{14}) \to (\pm 1/\bar{z}, \bar{x}_{14}/\bar{z}, \pm \bar{x}_{13}/\bar{z}).$$

The involution $I^-$ has no fixed points as noted before. The fixed point set for $I^+$ is $|z| = 1$ and $\bar{z}x_{14} = \bar{x}_{13}$.

3.2. Mirror B-model geometry

Mirror symmetry exchanges the sign of the left moving $U(1)_R$ charge, so it maps the anti-holomorphic involution of the A-model into a holomorphic involution of the B-model. This is compatible with the target space interpretation where we expect for Type IIB that the orientifold operation will involve a holomorphic involution on the Calabi-Yau.

The mirror of $O(-1) + O(-1) \to \mathbb{P}^1$ [22] is a Landau-Ginzburg theory in terms of four fields $Y_i$ that are constrained by

$$Y_1 + Y_2 - Y_3 - Y_4 = t,$$

and a superpotential

$$W = e^{-Y_1} + e^{-Y_2} + e^{-Y_3} + e^{-Y_4}. \quad (3.3)$$
The fields $Y_i$ are periodic, $Y_i \sim Y_i + 2\pi i$. They are obtained by dualization of the phases of linear sigma model fields $X^i$, and there is a relation

$$Re(Y_i) = |X_i|^2. \tag{3.4}$$

Above, $t$ is the complexified Kahler parameter of the A-model $t = r - i\theta$.

At the level of the topological theory, the mirror can equivalently be thought of as a theory of variations of complex structures of a certain hypersurface $Y$. The B-model has a sigma-model description based on

$$Y : \quad xz = e^{-u} + e^{u-v-t} + e^{-v} + 1. \tag{3.5}$$

Moreover the four terms on the right-hand side above arise from the four monomials in the superpotential. More precisely, equation (3.5) arises by writing $Y_1 = u + \lambda, Y_2 = -u + v + t + \lambda, Y_3 = v + \lambda, Y_4 = \lambda$. Choosing now projective coordinates gives the right-hand side of (3.3).

The anti-holomorphic involution maps $|X_1|^2 \rightarrow |X_2|^2$ and $|X_3|^2 \rightarrow |X_4|^2$, so the mirror map (3.4) implies that the mirror holomorphic involutions $\hat{I}^\pm$ both act as

$$\hat{I}^\pm : (Y_1, Y_2, Y_3, Y_4) \rightarrow (Y_2 + i\pi, Y_1 + i\pi, Y_4 + i\pi, Y_3 + i\pi).$$

More precisely, (3.4) and holomorphy fix the action up to additions of $i\pi$. But $\hat{I}^\pm \Omega$ must be a symmetry of the theory, and for this the superpotential $W$ must be odd under $\hat{I}^\pm$. This is because $W$ enters the Lagrangian of the Landau-Ginzburg theory as $\int d\theta^+ d\theta^- W$, and since $\Omega$ acts by exchanging superspace fermionic coordinates $\theta^+$, and $\theta^-$, $\hat{I}^\pm$ must take $W \rightarrow -W$. From this we learn that the mirror involutions both act as

$$(x, z, e^{-u}, e^v) \rightarrow (-x, -z e^v, e^{u-t}, e^{-v}).$$

The action on $x, z$ follows from the projectivization of the superpotential—actually from this it follows that $xz \rightarrow xze^v$ and given the periodicity of $v$ we have to take either $x$ or $z$ to go to itself with an extra $e^v$ factor. The choice of sign on $x, z$ is not apriori determined. The choice made above turns out to be the correct one, as we shall see. The action on the other variables follows from the relation of the $u, v$ to the $Y_i$ and how the orientifold action acts on the $Y_i$. 

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The fixed point set (i.e. the IIB orientifold plane) is given by solutions to
\[ x = -x, \quad z(1 + e^v) = 0, \quad 2u - t = 0, \quad 2v = 0 \]
which has only the two components \( x = 0, v = i\pi, u = t/2, t/2 + i\pi \), generically and these are complex curves parameterized by \( z \). There is another solution when the manifold \( Y \) becomes singular, at \( t = 0 \), which is given by \( x = 0, z = 0, v = i\pi, u = 0 \). This corresponds to a new orientifold fixed plane at the conifold singularity. We will not need it for purposes of this paper.

To summarize, we have two orientifold five-planes that are two holomorphic curves parameterized by \( z \), and located at
\[ x = 0, \quad v = i\pi, \quad u = -t/2 \text{ and } -t/2 + i\pi. \]
This is consistent with the results of [1] which showed that the D6 branes wrapping \( L \) map to D5 branes wrapping holomorphic curve given by \( x = 0 \), and a choice of a point on the Riemann surface \( 0 = e^{-u} + e^{u-v-t} + e^{-v} + 1 \). This, in retrospect, justifies the choice of sign in the orientifold action on the \( x, z \) we have made above.

3.3. Proposal for Mirrors

Note that two distinct orientifold actions on the A-model geometry are mirror to a single orientifold action in the B-model. However, to fully specify the theories with orientifolds we must decide on the signs of the corresponding cross-cap states. If there is an orientifold plane, the sign determines its RR-charge. In the Type IIA side, for each of \( I^+\Omega \) and \( I^-\Omega \) there are two possibilities. For \( I^+\Omega \), in one theory \( L \) is an \( O6^-\)-plane, while it is \( O6^+ \) in the other. For \( I^-\Omega \), there is no orientifold plane but still there are two possibilities. On the other hand, in the Type IIB side there are two \( O5 \)-planes as noted above. Accordingly, there are four physically distinct possibilities corresponding to ++, −−, −+ and +− charges.

We propose that the IIB orientifold with +− and −+ charges are mirror to the two orientifolds of Type IIA by \( I^-\Omega \) and that the −− and ++ theories are respectively mirror to IIA with \( O6^- \) and \( O6^+ \). This is the only assignment of mirrors consistent with the total RR charges being the same in both theories. The total D5 charge of −− (++) is −2 (+2) which is the total D6-brane charge of \( O6^- \) (\( O6^+ \)). On the other hand, +− and −+ theories both have zero total fivebrane charge which is mirror to the statement that the orientifold of IIA by \( I^-\Omega \) has zero sixbrane charge. Note also that \( I^- \) acts on the circle \(|z| = 1\) as the shift by half period. It is known that the orientifold of \( S^1 \) by half-period shift is sent by T-duality to the orientifold of the dual \( S^1 \) with two fixed planes of the opposite signs [24]. Our proposal is consistent with this fact.
\section*{3.4. Computation of space-time superpotential}

As mentioned in section 2, the computation of the space-time superpotential for orientifolds of Calabi-Yau threefolds will involve the computation of topological string amplitudes with the topology of $\mathbb{RP}^2$. Moreover, if we add D-branes to this setup, there will be additional contributions to the superpotential which can be computed by evaluating the topological disk amplitudes as in \cite{1}. Here we will concentrate mainly on the contribution from $\mathbb{RP}^2$ to the superpotential as the disk amplitudes have been already well studied in the context of mirror symmetry \cite{1,2}.

In the context of orientifolds of the A-model \cite{8}, the relevant maps for the $\mathbb{RP}^2$ worldsheet are holomorphic maps from $\mathbb{CP}^1$ to Calabi-Yau which are $\mathbb{Z}_2$-equivariant. Let $z$ denote the coordinates of the $\mathbb{P}^1$ in the target $O(-1) + O(-1)$ geometry and $w$ denote the coordinate of the $\mathbb{P}^1$ on the worldsheet. We are thus looking for maps $z(w)$ such that

$$z(-1/w) = \pm 1/z$$

where $\pm$ depends on which orientifold $I^\pm$ we are considering. Examples of such maps include $z = w^{2n}$ for $I^+$ and $z = w^{2n+1}$ for $I^-$. It is not difficult to show that quite generally the parity of the degree of the map is correlated with $\pm$ choice in $I^\pm$ as the above representative maps suggest. Therefore for the $\mathbb{RP}^2$ amplitude one expects an infinite sum over odd or even degrees of maps depending on whether one is considering $I^-$ or $I^+$. We will now see that this follows from the mirror description we have obtained.

Mirror symmetry implies the equivalence between topological A-model amplitudes on the Calabi-Yau and topological B-model amplitudes on the mirror. In particular, as noted in section 2, the computation of the superpotential corresponds to computing the integrals of the holomorphic 3-form (2.4) on 3-chains with boundaries given by D-brane charges. In the local models under consideration it was shown in \cite{1} that this computation reduces to certain definite integrals (of the Abel-Jacobi map) on the Riemman surface

$$0 = e^{-u} + e^{-v} + e^{u-v-t} + 1.$$

The integral is of the form $\int_\gamma \lambda$ where $\lambda$ is a particular one form $vdu$ and $\partial \gamma = \text{points}$ where the D5 brane charges are localized on the Riemann surface, including the sign and multiplicity of the D5 brane charge. This charge can arise from physical D5 branes or O5 planes carrying D5 brane charge. In other words, the integral over the three chain $C$ reduces to an integral over the 1-chain $\gamma$ on the Riemann surface.
As discussed above, the mirror of $I^+$ orientifold are two $O5$ planes of the same charge, and the mirror of $I^-$ orientifold are two $O5$ planes of opposite charge. The contribution of the orientifold $++$ and $+-$ five planes to the superpotential is thus

$$W = \int_{u^*}^{u=t/2} v(u)du \pm \int_{u^*}^{u=t/2+i\pi} v(u)du.$$ 

where we have put $\pm$ on the second term because the choice of the $D5$ brane charge on the second orientifold plane depends on whether we are considering $I^\pm$. Note that each $O5$ plane carries $\pm 1$ unit of $D5$ brane charge. The above formula gives $W$ up to a choice of an arbitrary base point $u^*$ on the Riemann surface. The need to pick $u^*$ is due to non-compactness of the D-brane, and corresponds to the boundary condition at infinity \[1\]. Explicitly,

$$W = \int_{u^*}^{t/2} \log \frac{1 - e^{u-t}}{1 - e^{-u}}du \pm \int_{u^*}^{t/2+i\pi} \log \frac{1 - e^{u-t}}{1 - e^{-u}}du = -2 \sum_{n=1}^{\infty} \left\{1 \pm (-1)^n \right\} \frac{e^{-nt/2}}{n^2}. \quad (3.6)$$

The term in the sum weighted by $e^{-nt/2}$ in the language of topological A-model comes from a map that is an $n$-fold cover of $\mathbb{RP}^2$, as the action of the instanton wrapping $\mathbb{CP}^1/\mathbb{Z}_2$ once is one half the size of the $\mathbb{CP}^1$ in the covering space. Moreover we see that for $I^+$ we get contributions only from even degree whereas for $I^-$ we get contributions only from odd degrees, as was expected.

In \[8\] it is predicted that the A-model amplitudes coming from unorientable worldsheets have integrality properties analogous to A-model amplitudes on Riemann surfaces with boundaries. The two factors above then correspond to two BPS bound states of D2 branes wrapping the $\mathbb{CP}^1$ in the covering space and “ending” on the orientifold, in the sense that due to orientifolding, they propagate only in two dimensions. Note that their contributions to the partition function are exchanged, up to an overall sign, by $t \rightarrow t + 2\pi i$. This has the interpretation \[8\] that the two states differ by $1/2$ unit of $D0$ brane charge. This is so as the sum over $n$ corresponds to existence of a bound state with $n$ D0 branes for every primitive BPS state. Shifting $t$ by $2\pi$ corresponds to shifting the $B$ field through the $\mathbb{CP}^1$ by an amount corresponding to 1 unit of $D0$ brane.
Fig. 1: The mirror orientifold action has two fixed points on the Riemann surface $e^{-u} + e^{-v} + e^{-t+u-v} + 1 = 0$, at $u = t/2$ and $u = t/2 + i\pi$. This corresponds to orientifold five-planes in the full geometry. The possibilities for charge assignments – equal or opposite charges – correspond to different orientifold actions in the A-model.

We can also add D6 branes to the A-model in which case the disk amplitude will also be non-vanishing. We can compute this as in [1],

$$W_{D5 \text{ branes}} = \left(\int_{u^*}^{u} \log \frac{1-e^{u-t}}{1-e^{-u}} du + \int_{u^*}^{-u-t} \log \frac{1-e^{u-t}}{1-e^{-u}} du\right)$$

$$= -2 \sum_{n=1}^{\infty} \left\{ \frac{e^{-nu}}{n^2} + \frac{e^{-n(t-u)}}{n^2} \right\}$$

(3.7)

The two contributions correspond to a single primitive disk ending on the D6 brane and the image of this under $\Omega$ (see figure 2).
Fig. 2: The orientifold of IIA on $O(-1) + O(-1) \rightarrow \mathbb{P}^1$. The orientifold 6-plane has topology $C \times S^1$, and projects to a line in the toric base. Superpotential receives contribution from unoriented maps to the $\mathbb{CP}^1$. In presence of additional D6 brane and its image, superpotential receives contributions from the disk as well.

Note that the result for $I^{-}$ has been obtained previously by a different method [8]. Namely, this orientifold of type IIA string theory has been considered previously in the context of duality with large $N$ $SO$ and $Sp$ Chern Simons theory on $S^3$ [8], that generalized the original large $N$ conjecture of [25] for unitary groups. In this case, the free-energy of the Chern-Simons theory

$$F_{SO,Sp}(g_s, t) = \pm \frac{1}{2} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{2n \sin(n g_s/2)} e^{-nt} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-nt}}{n[2 \sin(n g_s/2)]^2}$$

provides prediction for all genus amplitudes of the $O(-1) + O(-1) \rightarrow \mathbb{P}^1$ orientifold. The first term in the above expression was interpreted in [8] as corresponding to contributions of all non-orientable Riemann surfaces with a single crosscap, as only odd powers of $g_s$ appear. Note that $\chi_{\mathbb{RP}^2} = -1$ and that three crosscaps can be traded for a handle with a single crosscap. For example, the partition function $F_{g=0, c=1}$ of the $\mathbb{RP}^2$ diagram is the coefficient of $g_s^{-1}$ and is easily seen to agree with the expression we obtained above using mirror symmetry, up to over-all normalization. The second term, on the other hand, corresponds to oriented maps, and in the superstring language corresponds to superpotential like terms generated from $\mathcal{N} = 2$ amplitudes by turning on RR flux, and is the $U(N)$ Chern-Simons amplitude, up to normalization, as discussed in [8]. It would be interesting to generalize the mirror symmetry methods to also derive these higher genus predictions of large $N$ duality for Chern-Simons theory.

4. Other Examples

It is easy to give more examples along the lines we have discussed, which involve more intricate contributions to the superpotential. Orientifold seven-planes have vanishing superpotentials, and by mirror symmetry the disk amplitude of D6 branes that are two dimensional in the toric base vanishes as well. To have a non-vanishing
superpotential, the Type IIB theory must have D5 brane charge, although this is only a necessary condition. Thus we consider here examples whose mirror on the type IIB side involve the orientifold operation $\omega \rightarrow \omega$. Moreover, analogous to what was found in [1,2], it is only the “half”-orientifolds – those whose fixed point set has topology of $R^2$ in the B-model, that give non-zero superpotential in type II B string theory.

Since the setup is very similar to what we have done above, we limit our presentation to showing on the figures in two examples how the orientifold operation acts on the toric base. In figure 3, the example depicts a case where there is no superpotential generated, and figure 4 depicts a case where there is a superpotential generated (as there is “half”-orientifold planes).

Fig.3: The orientifold of IIA on $O(K) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that acts as $I\Omega : (|z_1|,|z_2|) \rightarrow (1/|z_1|,1/|z_2|)$. The orientifold does not generate superpotential and this is related to the fact the orientifold action projected to the base, fixes a line meeting the boundary at a point in the interior.
Fig. 4: The orientifold of IIA that does generate superpotential. This has $G = U(1)^5$. In addition to the superpotential, $\mathbb{Z}_2$ action projects out 2 out of five complexified Kahler moduli.

It goes without saying that all these constructions corresponding to type IIA orientifolds can also be lifted up to M-theory involving $G_2$ holonomy metrics, possibly with singularities in geometry. It is known that the simple $O6^-$ lifts up to smooth Atiyah-Hitchin manifold \cite{26} while $O6^+$ lifts to a D4-singularity that is frozen by a discrete flux \cite{24,27}. D6-branes may add more singularity of A and D-types. In this context both the disc instanton and the $\mathbb{RP}^2$ instanton correspond to oriented Euclidean M2 brane instantons.

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