ON GENERAL CONSTRUCT
OF CHAOTIC UNBOUNDED LINEAR OPERATORS
IN BANACH SPACES WITH SCHAUDEL BASES

MARAT V. MARKIN

Abstract. We utilize the idea underlying the construct of the classical weighted
backward shift Rolewicz’s operators to furnish a straightforward approach to a
general construct of chaotic unbounded linear operators in a (real or complex)
Banach space with a Schauder basis.

1. Introduction

The notions of hypercyclicity and chaoticity, traditionally considered and well stud-
ied for continuous linear operators on Fréchet spaces, in particular for bounded
linear operators on Banach spaces, and shown to be a purely infinite-dimensional
phenomena (see, e.g., [10,11,17]), in [2,3] are extended to unbounded linear
operators in Banach spaces, where also found are sufficient conditions for unbounded
hypercyclicity and certain examples of hypercyclic and chaotic unbounded linear
differential operators.

Definition 1.1 (Hypercyclicity and Chaoticity [2,3]). Let

\[ A : X \supseteq D(A) \to X \]

be a (bounded or unbounded) linear operator in a (real or complex) Banach space
\((X, \| \cdot \|)\). A nonzero vector

\[ x \in C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n) \]

\((A^0 := I, I \text{ is the identity operator on } X)\) is called hypercyclic if its orbit under \(A\)

\[ \{A^n x\}_{n \in \mathbb{Z}_+} \]

\((\mathbb{Z}_+ := \{0, 1, 2, \ldots \} \text{ is the set of nonnegative integers})\) is dense in \(X\).

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

If there exist an \(N \in \mathbb{N} \) \((\mathbb{N} := \{1,2,\ldots \} \text{ is the set of natural numbers})\) and a vector \(x \in D(A^N)\) such that \(A^N x = x\), such a vector is called a periodic point for the
operator \(A\).

Hypercyclic linear operators with a dense set of periodic points are said to be chaotic.

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Remarks 1.1.

- In the definition of hypercyclicity, the underlying space must necessarily be separable.
- For a hypercyclic linear operator $A$, the subspace $C^\infty(A)$ (see (1.1)), which contains the dense orbit of a hypercyclic vector, is also dense.
- As is easily seen, a periodic point $x$ of $A$, if any, also belongs to $C^\infty(A)$.

In [14], (bounded or unbounded) scalar type spectral operators in a complex Banach, in particular normal ones in a complex Hilbert space, (see, e.g., [4–8,16]) are proven to be non-hypercyclic.

In [17], S. Rolewicz gave the first example of a hypercyclic bounded linear operator on a Banach space (see also [10,11]), which on (real or complex) sequence space $l_p$ ($1 \leq p < \infty$) or $c_0$ (of vanishing sequences), the latter equipped with the supremum norm

$$c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\|_\infty := \sup_{k \in \mathbb{N}} |x_k|,$$

is the following weighted backward shift:

$$A(x_k)_{k \in \mathbb{N}} := \lambda(x_{k+1})_{k \in \mathbb{N}},$$

where $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is arbitrary. Rolewicz’s operators also happen to be chaotic (see, e.g., [10, Example 2.32]).

In [15, Theorem 2.1], Rolewicz’s example is modified to produce hypercyclic unbounded linear operators in $l_p$ ($1 \leq p < \infty$) or $c_0$ as follows.

**Theorem 1.1 ([15, Theorem 2.1]).** In the (real or complex) sequence space $X := l_p$ ($1 \leq p < \infty$) or $X := c_0$, the weighted backward shift

$$D(A) \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (\lambda^k x_{k+1})_{k \in \mathbb{N}},$$

where $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$, $|\lambda| > 1$ is arbitrary, with the domain

$$D(A) := \{(x_k)_{k \in \mathbb{N}} \in X \mid (\lambda^k x_{k+1})_{n \in \mathbb{N}} \in X\}$$

is a hypercyclic unbounded linear operator.

We are to generalize the above construct of unbounded weighted backward shifts to Banach spaces with Schauder bases (see, e.g., [12,13]) and show, in particular, that the operators in the prior theorem are chaotic.

### 2. Preliminaries

Here, for the reader’s convenience, we outline certain essential preliminaries concerning Banach spaces with Schauder bases.

**Definition 2.1** (Schauder Basis of a Banach Space). A Schauder basis (also a countable basis) of a (real or complex) Banach space $(X, \| \cdot \|)$ is a countably infinite set of elements $\{e_n\}_{n \in \mathbb{N}}$ in $X$ such that

$$\forall x \in X \exists! (x_k)_{k \in \mathbb{N}} \subseteq F : x = \sum_{k=1}^{\infty} x_k e_k,$$
where \( \mathbb{F} := \mathbb{R} \) or \( \mathbb{F} := \mathbb{C} \), the series called the Schauder expansion of \( x \) and the numbers \( x_k, k \in \mathbb{N} \), are called the coordinates of \( x \) relative to the Schauder basis \( \{ e_n \}_{n \in \mathbb{N}} \) (see, e.g., [12,13]).

**Examples 2.1.**

1. The set \( \{ e_n := \{ \delta_{nk} \}_{k=1}^{\infty} \}_{n \in \mathbb{N}} \), where \( \delta_{nk} \) is the Kronecker delta, is a Schauder basis for \( l_p \) \((1 \leq p < \infty)\) and \( (c_0, \| \cdot \|_{\infty}) \) \( (c_0 \) is the space of vanishing sequences\), the latter equipped with the supremum norm

\[
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \| x \|_\infty := \sup_{k \in \mathbb{N}} |x_k|,
\]

with

\[
x := (x_k)_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} x_k e_k,
\]

2. The set \( \{ e_n \}_{n \in \mathbb{Z}_+} \), where \( e_0 := (1, 1, \ldots) \) is a Schauder basis for \( (c, \| \cdot \|_{\infty}) \) \((c \) is the space of convergent sequences\) with

\[
x := (x_k)_{k \in \mathbb{N}} = \sum_{k=0}^{\infty} x_k e_k,
\]

where

\[
x_0 = \lim_{k \to \infty} x_k.
\]

3. Other examples of Banach spaces with more intricate Schauder bases are \( L_p(a, b) \) \((1 \leq p < \infty)\) and \( C[a, b] \) \((-\infty < a < b < \infty)\), the latter equipped with the maximum norm

\[
C[a, b] \ni x \mapsto \| x \|_\infty := \max_{a \leq t \leq b} |x(t)|
\]

(see, e.g., [12,18]).

**Remarks 2.1.**

- Observe that the aforementioned Schauder bases for the spaces \( l_p \) \((1 \leq p < \infty)\), \( (c_0, \| \cdot \|_{\infty}) \), and \( (c, \| \cdot \|_{\infty}) \) consist of unit vectors.

Always, without loss of generality, one can regard that, for a Schauder basis \( \{ e_n \}_{n \in \mathbb{N}} \) of a Banach space \( (X, \| \cdot \|) \),

\[
\| e_n \| = 1, \ n \in \mathbb{N},
\]

which immediately implies that, for any

\[
x = \sum_{k=1}^{\infty} x_k e_k,
\]

by the convergence of the series,

\[
|x_k| = \| x_k e_k \| \to 0, \ k \to \infty.
\]

- In particular, an orthonormal basis of a separable Hilbert space is a Schauder basis satisfying condition \( (2.2) \) (see, e.g., [1]).
• Every Banach space \((X, \| \cdot \|)\) with a Schauder basis is separable, a countable dense subset being

\[
Y := \left\{ x := \sum_{k=1}^{n} x_k e_k \mid n \in \mathbb{N}, x_1, \ldots, x_n \text{ are rational/complex rational} \right\}.
\]

The basis problem on whether every separable Banach space has a Schauder basis posed by Stefan Banach was negatively answered by Per Enflo in [9].

• The profound fact useful for our discourse is that, for each \(n \in \mathbb{N}\), the Schauder coordinate functional

\[
X \ni x = \sum_{k=1}^{\infty} x_k e_k \mapsto x_n(x) := x_n \in F
\]
is well defined, linear, and bounded (see, e.g., [12, 13]). Hence, if

\[
X \ni x^{(n)} = \sum_{k=1}^{\infty} x^{(n)}_k e_k \to x = \sum_{k=1}^{\infty} x_k e_k, \quad n \to \infty, \quad \text{in } (X, \| \cdot \|),
\]
then, for every \(k \in \mathbb{N}\),

\[
x^{(n)}_k \to x_k, \quad n \to \infty,
\]
the converse statement not being true (see, e.g., [13]).

3. Weighted Backward Shift Operators in Banach Spaces with Schauder Bases

In this section, we analyze the nature of generalized backward shift operators in Banach spaces with Schauder bases.

**Lemma 3.1.** In the (real or complex) Banach space \((X, \| \cdot \|)\) with a Schauder basis \(\{e_k\}_{k \in \mathbb{N}}\) such that

\[
\|e_k\| = 1, \quad k \in \mathbb{N},
\]
the weighted backward shift

\[
D(A) \ni x = \sum_{k=1}^{\infty} x_k e_k \mapsto Ax := \sum_{k=1}^{\infty} w_k x_{k+1} e_k,
\]
where \((w_k)_{k \in \mathbb{N}}\) is an arbitrary real- or complex-termed weight sequence such that

\[
1 \leq |w_k| \leq |w_{k+1}|, \quad k \in \mathbb{N},
\]
with the domain

\[
D(A) := \left\{ x = \sum_{k=1}^{\infty} x_k e_k \in X \mid \sum_{k=1}^{\infty} w_k x_{k+1} e_k \in X \right\}
\]
is a linear operator with a dense subspace \(C^\infty(A)\) (see (1.1)) and such that, for each \(n \in \mathbb{N}\), the operator \(A^n\) is closed and, provided

\[
\lim_{k \to \infty} |w_k| = \infty,
\]
is unbounded.
Proof. The linearity of $A$ is obvious.

For each $n \in \mathbb{N}$,

$$A^n x = \sum_{k=1}^{\infty} \left[ \prod_{j=k}^{n+k-1} w_j \right] x_{k+n} e_k,$$

$$x = \sum_{k=1}^{\infty} x_k e_k \in D(A^n) = \left\{ x = \sum_{k=1}^{\infty} x_k e_k \in X \left| \sum_{k=1}^{\infty} \left[ \prod_{j=k}^{n+k-1} w_j \right] x_{k+n} e_k \in X \right. \right\}.$$

with

$$(3.7) \quad D(A^{n+1}) \subseteq D(A^n), \quad n \in \mathbb{N},$$

the linear operator $A^n$ being densely defined since the domain $D(A^n)$ contains the countable dense in $(X, \| \cdot \|)$ subset $Y$ (see (2.3)) and hence, so does the subspace

$$C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

of all possible initial values for the orbits under $A$, which is also dense in $(X, \| \cdot \|)$.

Let

$$D(A^n) \ni x^{(m)} = \sum_{k=1}^{\infty} x_k^{(m)} e_k \rightarrow x = \sum_{k=1}^{\infty} x_k e_k, \quad m \rightarrow \infty,$$

and

$$A^n x^{(m)} = \sum_{k=1}^{\infty} \left[ \prod_{j=k}^{n+k-1} w_j \right] x_{k+n}^{(m)} e_k \rightarrow y = \sum_{k=1}^{\infty} y_k e_k, \quad m \rightarrow \infty,$$

then (see Preliminaries), for each $k \in \mathbb{N}$,

$$x_k^{(m)} \rightarrow x_k, \quad m \rightarrow \infty,$$

and

$$\left[ \prod_{j=k}^{n+k-1} w_j \right] x_{k+n}^{(m)} \rightarrow y_k, \quad m \rightarrow \infty.$$

Whence, we infer that

$$y_k = \left[ \prod_{j=k}^{n+k-1} w_j \right] x_{k+n}, \quad m \in \mathbb{N},$$

and therefore,

$$x \in D(A^n) \text{ and } y = Ax,$$

which implies that, for each $n \in \mathbb{N}$, the operator $A^n$ is closed (see, e.g., [6,13]).

In view of (3.4), with the weight sequence $(w_k)_{k \in \mathbb{N}}$ satisfying conditions (3.5) and (3.6), for every $n \in \mathbb{N}$, the operator $A^n$ is unbounded since

$$\|A^n e_{2n}\| = \left\| \left[ \prod_{j=n}^{2n-1} w_j \right] e_n \right\| = \left\| \left[ \prod_{j=n}^{2n-1} |w_j| \right] \|e_n\| \geq |w_n| \rightarrow \infty, \quad n \rightarrow \infty. \square$$
Remark 3.1. Observe that, under restriction (3.4), conditions (3.5) and (3.6) are sufficient for the unboundedness of $A^n$ ($n \in \mathbb{N}$), but, generally, may not be necessary.

4. General Construct of Chaotic Unbounded Linear Operators in Banach Spaces with Schauder Bases

Here, we furnish a straightforward approach to a general construct of chaotic unbounded linear operators in a (real or complex) Banach space with a Schauder basis utilizing the idea, which underlies the construct of the classical weighted backward shift Rolewicz's operators, i.e., the denseness in the spaces $l_p$ ($1 \leq p < \infty$) and $(c_0, \| \cdot \|_\infty)$ of the subspace $c_{00}$ of eventually zero sequences. In a Banach space $(X, \| \cdot \|)$ with a Schauder basis $\{e_n\}_{n \in \mathbb{N}}$ the natural analogue of such a subspace is that of "eventually zero" vectors relative to $\{e_n\}_{n \in \mathbb{N}}$:

$$\left\{ x := \sum_{k=1}^{\infty} x_k e_k \in X \mid \exists n \in \mathbb{N} : x_k = 0, \ k > n \right\}.$$ 

Thus, in the spaces $l_p$ ($1 \leq p < \infty$) and $(c_0, \| \cdot \|_\infty)$ relative to their standard Schauder bases described in Preliminaries, this subspace merely coincides with $c_{00}$, whereas in the space $(c, \| \cdot \|_\infty)$ relative to its standard Schauder basis described in Preliminaries, this subspace consists of all eventually constant sequences.

Theorem 4.1. In the (real or complex) Banach space $(X, \| \cdot \|)$ with a Schauder basis $\{e_k\}_{k \in \mathbb{N}}$ such that

$$\|e_k\| = 1, \ k \in \mathbb{N},$$

the weighted backward shift

$$D(A) \ni x = \sum_{k=1}^{\infty} x_k e_k \mapsto Ax := \sum_{k=1}^{\infty} w_k x_{k+1} e_k,$$

where $(w_k)_{k \in \mathbb{N}}$ is an arbitrary real- or complex-termed weight sequence satisfying (3.5) and such that

$$\sum_{k=1}^{\infty} |w_k|^{-1} < \infty,$$

with the domain

$$D(A) := \left\{ x := \sum_{k=1}^{\infty} x_k e_k \in X \mid \sum_{k=1}^{\infty} w_k x_{k+1} e_k \in X \right\}$$

is a chaotic unbounded linear operator.

Proof. Observe that, since condition (4.9), implies condition (3.6), and hence, by Lemma 3.1, $A$ is a linear operator with a dense subspace $C^\infty(A)$ (see (1.1)) and such that, for each $n \in \mathbb{N}$, the operator $A^n$ is closed and unbounded.

We proceed in a similar fashion as in the proof of [15, Theorem 2.1] (cf. [17, Theorem 1] and [10, Example 2.18]).
The right inverse of $A$

\[(4.10) \quad Bx := \sum_{k=1}^{\infty} w_k^{-1} x_k e_{k+1}, \]

is linear operator well defined on the entire space $X$ since, for any

\[x = \sum_{k=1}^{\infty} x_k e_k \in X,\]

in view of (4.8), $|x_k| = \|x_k e_k\| \to 0$, $k \to \infty$, (see Preliminaries), and hence, by the estimate

\[\|w_k^{-1} x_k e_{k+1}\| = |w_k|^{-1} |x_k e_{k+1}| \leq \left(\sup_{x \in X} |x_j|\right) |w_k|^{-1}, \quad k \in \mathbb{N},\]

and, in view of condition (4.9), the series in (4.10) converges.

Further, for any $n \in \mathbb{N}$ we have:

\[B^n x = \sum_{k=1}^{\infty} \left[ \prod_{j=k}^{n+k-1} w_j^{-1} \right] x_k e_{k+n}, \quad x = \sum_{k=1}^{\infty} x_k e_k \in X.\]

Let

\[(4.11) \quad \left\{ y^{(m)} := \sum_{k=1}^{\infty} y^{(m)}_k e_k \right\}_{m \in \mathbb{N}} \]

be an arbitrary enumeration of the countable dense subset $Y$ (see (2.3)) with

\[(4.12) \quad k_m := \max \left\{ k \in \mathbb{N} \left| y^{(m)}_k \neq 0 \right. \right\}, \quad m \in \mathbb{N}.\]

Inductively, one can choose an increasing sequence $(n_m)_{m \in \mathbb{N}}$ of natural numbers such that, for all $j, m \in \mathbb{N}$ with $m > j$,

\[(4.13) \quad n_m - n_j \geq \max(m, k_j) \quad \text{and} \quad \prod_{i=1}^{n_m} |w_i| \geq \left[ \prod_{i=n_m-n_j+1}^{n_m} |w_i| \right] |w_m| \sum_{k=1}^{k_m} |y^{(m)}_k|,\]

Let us show that the vector

\[x := \sum_{j=1}^{\infty} B^{n_j} y^{(j)}\]

is hypercyclic for $A$, the series converging in $(X, \| \cdot \|)$ by condition (4.9) since, for any $j = 2, 3, \ldots$, in view of (4.8), (3.5), and (4.13),

\[\|B^{n_j} y^{(j)}\| \leq \sum_{k=1}^{n_j} \left[ \prod_{i=k}^{n_j+k-1} w_i^{-1} \right] |y^{(j)}_k e_{k+n_j}| \leq \sum_{k=1}^{k_j} \left[ \prod_{i=k}^{n_j+k-1} |w_i|^{-1} \right] \|y^{(j)}_k\| \|e_{k+n_j}\| \leq \prod_{j=1}^{n_j} |w_j|^{-1} \sum_{k=1}^{k_j} |y^{(j)}_k| \leq |w_j|^{-1}.\]
Further, for each \( m \in \mathbb{N} \),
\[
\sum_{j=1}^{\infty} A^{n_{m}} B^{n_{j}} y(j) = \sum_{j=1}^{m-1} A^{n_{m} - n_{j}} y(j) + y(m) + \sum_{j=m+1}^{\infty} B^{n_{j} - n_{m}} y(j)
\]
since, by (4.13), for \( j = 1, \ldots, m-1 \), \( n_{m} - n_{j} \geq k_{j} \), by (4.12), \( A^{n_{m} - n_{j}} y(j) = 0 \);
\[
= y(m) + \sum_{j=m+1}^{\infty} B^{n_{j} - n_{m}} y(j).
\]

Since, for all \( m, j \in \mathbb{N} \) with \( j \geq m + 1 \), in view of (4.8), (3.5), and (4.13),
\[
\| B^{n_{j} - n_{m}} y(j) \| = \left\| \sum_{k=1}^{\infty} \left[ \prod_{i=k}^{n_{j} - n_{m} + k - 1} |w_{i}|^{-1} \right] y_{k}^{(j)} \right\| \leq \sum_{k=1}^{k_{j}} \left[ \prod_{i=k}^{n_{j} - n_{m} + k - 1} |w_{i}|^{-1} \right] \sum_{k=1}^{k_{j}} |y_{k}^{(j)}| \leq |w_{j}|^{-1}.
\]

This implies that, for any \( m \in \mathbb{N} \), the series
\[
\sum_{j=1}^{\infty} A^{n_{m}} B^{n_{j}} y(j)
\]
converges in \( (X, \| \cdot \|) \), and hence, by the closedness of \( A^{n_{m}} \) and in view of inclusion (3.7),
\[
x \in \bigcap_{m=1}^{\infty} D(A^{n_{m}}) = C^{\infty}(A)
\]
and
\[
A^{n_{m}} x = y(m) + \sum_{j=m+1}^{\infty} B^{n_{j} - n_{m}} y(j), \ m \in \mathbb{N}.
\]

Furthermore, since, by (4.14), for each \( k \in \mathbb{N} \),
\[
\left\| A^{n_{m}} x - y(m) \right\| \leq \sum_{j=m+1}^{\infty} \left\| B^{n_{j} - n_{m}} y(j) \right\| \leq \sum_{j=k+1}^{\infty} |w_{j}|^{-1} \to 0, \ k \to \infty,
\]
which, in view of the denseness of \( Y \) in \( (X, \| \cdot \|) \), shows that the orbit \( \{ A^{n} x \}_{n \in \mathbb{Z}_{+}} \) of \( x \) under \( A \) is dense in \( (X, \| \cdot \|) \) as well. This implies that the operator \( A \) is hypercyclic.

Now, let us show that the operator \( A \) is chaotic (cf. [10, Example 2.32]). Indeed, for any \( N \in \mathbb{N} \), let \( x_{1}, \ldots, x_{N} \in \mathbb{F} \) be arbitrary, then
\[
(4.15) \quad x := \sum_{m=1}^{N} x_{m} e_{m} + \sum_{k=1}^{\infty} \left[ \sum_{m=1}^{N} \left[ \prod_{i=(k-1)N+m}^{mN-1} |w_{i}|^{-1} \right] x_{m} e_{kN+m} \right],
\]
the series converging in \( (X, \| \cdot \|) \) by (4.9) since, for any \( k \in \mathbb{N} \), in view of (4.8) and (3.5),
(4.16)  
\[
\left| \sum_{m=1}^{N} \left[ kN+m-1 \prod_{i=(k-1)N+m}^{kN+m-1} w_i^{-1} \right] x_m e_{kN+m} \right| \leq \sum_{m=1}^{N} \left[ kN+m-1 \prod_{i=(k-1)N+m}^{kN+m-1} |w_i|^{-1} \right] |x_m| \left\| e_{kN+m} \right\|
\]
\[
\leq \left[ \sum_{m=1}^{N} |x_m| \right] |w_{kN}|^{-1},
\]

we have:
\[
x \in D(A^N) \text{ and } A^N x = x.
\]

For any
\[
y := \sum_{m=1}^{n} y_m e_m \in Y
\]
with some \( n \in \mathbb{N} \) (see (2.3)) and arbitrary \( N \geq n \), consider the periodic point defined by (4.15) with
\[
x_m := \begin{cases} y_m, & m = 1, \ldots, n, \\ 0, & m = n+1, \ldots, N. \end{cases}
\]

Then, in view of (4.16), by condition (4.9),
\[
\| x - y \| = \left\| \sum_{k=1}^{\infty} \sum_{m=1}^{N} \left[ kN+m-1 \prod_{i=(k-1)N+m}^{kN+m-1} w_i^{-1} \right] x_m e_{kN+m} \right\|
\]
\[
\leq \left[ \sum_{m=1}^{n} |y_m| \right] \sum_{k=1}^{\infty} |w_{kN}|^{-1} \leq \left[ \sum_{m=1}^{n} |y_m| \right] \sum_{k=N}^{\infty} |w_k|^{-1} \to 0, \ N \to \infty.
\]

Whence, in view of the denseness of \( Y \) in \((X, \| \cdot \|)\), we infer that the set of periodic points of \( A \) is dense in \((X, \| \cdot \|)\) as well, which completes the proof. \( \Box \)

The exponential weight sequence
\[
w_n := \lambda^n, \ n \in \mathbb{N},
\]
with \( \lambda \in \mathbb{R} \) or \( \lambda \in \mathbb{C}, \ |\lambda| > 1 \) satisfying conditions (3.5) and (4.9), for the sequence spaces \( l_p \ (1 \leq p < \infty) \) or \( (c_0, \| \cdot \|_{\infty}) \), we arrive at the following corollary improving the result of Theorem 1.1.

**Corollary 4.1.** In the (real or complex) sequence space \( l_p \ (1 \leq p < \infty) \) or \( (c_0, \| \cdot \|_{\infty}) \), the weighted backward shift
\[
D(A) \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (\lambda^k x_{k+1})_{k \in \mathbb{N}},
\]
where \( \lambda \in \mathbb{R} \) or \( \lambda \in \mathbb{C}, \ |\lambda| > 1 \) is arbitrary, with the domain
\[
D(A) := \{ (x_k)_{n \in \mathbb{N}} \in X \mid (\lambda^k x_{k+1})_{k \in \mathbb{N}} \in X \}
\]
is a chaotic unbounded linear operator.

**Remark 4.1.** For a detailed discourse on the weighted shifts in Fréchet sequence spaces, see [10, Chapter 4].
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Department of Mathematics
California State University, Fresno
5245 N. Backer Avenue, M/S PB 108
Fresno, CA 93740-8001

E-mail address: mmarkin@csufresno.edu