From Twistor Actions to MHV Diagrams

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We show that MHV diagrams are the Feynman diagrams of certain twistor actions for gauge theories in an axial gauge. The gauge symmetry of the twistor action is larger than that on space-time and this allows us to fix a gauge that makes the MHV formalism manifest but which is inaccessible from space-time. The framework is extended to describe matter fields: as an illustration we explicitly construct twistor actions for an adjoint scalar with arbitrary polynomial potential and a fermion in the fundamental representation and show how this leads to additional towers of MHV vertices in the MHV diagram formalism.

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An important outcome of Witten’s twistor-string theory [1] is the “MHV formalism” [2], in which scattering amplitudes in four dimensional Yang-Mills theory are described in terms of diagrams whose vertices are the MHV vertices with two positive and arbitrarily many negative helicity gluons. Much work (see e.g. [3, 4, 5, 6]) has since gone into developing unitarity methods to compute loop amplitudes, or on extending the MHV formalism to include matter coupled to the Yang-Mills field [7, 8].

A key question here is the connection between these twistor-inspired developments and the usual, Lagrangian-based approach to gauge theory. Lagrangians have the obvious advantages of leading to a systematic perturbation theory including loops, and of making manifest the symmetry properties of a theory. Progress towards a full derivation of the MHV formalism from the standard space-time Lagrangian in lightcone gauge has been made in [9, 10, 11, 12, 13]. The key idea in these approaches is to find new field variables in which the anti-self-dual sector is linearized. These new field variables are related to the old ones by non-linear and non-local field transformations, expressed in the form of an infinite series. The possibility of such a transformation of the anti-self-dual sector into a free theory relies, in effect, on the
complete integrability of the anti-self-dual Yang-Mills equations. This leads to the triviality of the perturbative scattering theory, at least at tree level. The MHV diagram formalism then provides the perturbation theory about this linearized anti-self-dual sector.

The Ward transform [14] underlies the complete integrability of the anti-self-dual Yang Mills equations, see for example [15]. It linearizes the anti-self-dual Yang-Mills equations by reformulating them as the condition that the corresponding data on twistor space be holomorphic. Our approach [16, 17] builds on this foundation by writing an action for the complete gauge theory (not merely its anti-self-dual interactions) on twistor space rather than on space-time. It uses the Euclidean space formulation of the Ward construction [18], in which twistor space, a three dimensional complex manifold, is expressed as a 2-sphere bundle over Euclidean space. The Ward transform can then be understood in terms of the possibility of choosing different gauges for the pullback of the space-time Yang-Mills connection to twistor space. As we show explicitly in this paper, this allows us to pass between the standard and the MHV descriptions of four-dimensional Yang-Mills perturbation theory merely by making different gauge choices in the twistor action. Furthermore, we show that our actions can be extended to include fermionic or scalar matter coupled to the Yang-Mills field. The same gauge choice then leads to a MHV diagram formalism for gauge theories containing this matter in which the standard Yang-Mills MHV diagrams are supplemented by extra towers of MHV diagrams containing the matter interactions.

We start with an introduction to the twistor formulation of Yang-Mills theory, and briefly describe the gauge choice that allows us to retrieve the standard space-time formulation. We go on to discuss the gauge choice that leads to the MHV diagram formalism and show how this comes about. We provide extensions of the action to include scalar and fermionic matter fields and show how the extra terms give rise to additional towers of vertices in the MHV diagram formalism. We give a brief discussion of the connection between the standard space-time LSZ formalism and that arising from the twistor action. Questions still remain, particularly concerning the origin of the MHV loop diagram. We finish with a discussion of these and other matters.
TWISTOR YANG-MILLS

The twistor space of four-dimensional Euclidean space \( \mathbb{E} \) may be viewed as the total space of right Weyl spinors over \( \mathbb{E} \). A right Weyl spinor \( \pi_\alpha \) in four dimensions has two components, and we will be interested in the projectivized spin bundle where \( \pi_\alpha \) is defined only up to the scaling \( \pi_\alpha \sim \lambda \pi_\alpha \) for any non-zero complex \( \lambda \). Thus projective twistor space \( \mathbb{PT}' \) is a \( \mathbb{CP}^1 \) bundle over \( \mathbb{E} \), and may be described by coordinates \( (x^{\alpha\dot{\alpha}}, [\pi_\beta]) \) where \( x \in \mathbb{E} \) and \( [\pi_\beta] \) is the point on the \( \mathbb{CP}^1 \) with homogeneous coordinates \( \pi_\beta \). The indices \( \alpha, \dot{\alpha} = 0, 1 \) denote the fundamental representation of each of the two \( SL(2, \mathbb{C}) \)s in the spin group \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) of the complexified Lorentz group, and are raised and lowered using the \( SL(2, \mathbb{C}) \)-invariant tensors \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \) etc. We will often employ the notation \( \langle a b \rangle \equiv \epsilon^{\dot{\beta}\dot{\alpha}}a_\dot{\alpha}b_\beta \), \( [c d] \equiv \epsilon_{\alpha\beta}c^\alpha d^\beta \) and \( \langle a | M | c \rangle \equiv a_\alpha M^{\alpha\dot{\alpha}}c_\dot{\alpha} \) to denote these \( SL(2, \mathbb{C}) \)-invariant inner products. To preserve real Euclidean space we restrict ourselves to the subgroup \( SU(2) \times SU(2) \subset SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) which also leaves invariant the inner products \( \langle \pi \, \hat{\pi} \rangle \) and \( [\omega \, \hat{\omega}] \), where \( (\hat{\pi}_0, \hat{\pi}_1) \equiv (\pi_1, -\pi_0) \) and \( (\lambda^0, \lambda^1) \equiv (\lambda^1, -\lambda^0) \).

Twistor space \( \mathbb{PT} \) is the complex manifold \( \mathbb{P}^3 \) with holomorphic homogeneous coordinates \( (\omega^A, \pi_{A'}) = (x^{A''} \pi_{A''}, \pi_{A'}) \), \( (\omega^A, \pi_{A'}) \sim (\lambda \omega^A, \lambda \pi_{A'}) \) and \( \mathbb{PT}' \) is the subset \( \mathbb{P}^3 \setminus \mathbb{P}^1 \) on which \( \pi_{A'} \neq 0 \). We will work with coordinates \( (x, \pi_{A'}) \) on \( \mathbb{PT}' \) for the most part, and in these coordinates, the complex structure can be expressed in terms of the following basis of \( (0, 1) \)-forms and dual basis of \( (0, 1) \)-vectors adapted to the fibration over \( \mathbb{E} \)

\[
\hat{e}^0 = \frac{\hat{\pi}_1 d\hat{\pi}^\alpha}{\langle \pi \, \hat{\pi} \rangle^2} \quad \hat{e}^\alpha = \frac{\hat{\pi}_1 dx^{\alpha\dot{\alpha}}}{\langle \pi \, \hat{\pi} \rangle} \quad \hat{\partial}_0 = \langle \pi \, \hat{\pi} \rangle \pi_\alpha \frac{\partial}{\partial \hat{\pi}^\alpha} \quad \hat{\partial}_\alpha = \pi^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \tag{1}
\]

where the scale factors are chosen to ensure that the forms have only holomorphic weight \( (i.e. \) they are independent of rescalings of \( \hat{\pi} \)). The \( \hat{\partial} \) operator is defined on functions \( f \) with holomorphic weight to be \( \hat{\partial} f = \hat{e}^0 \hat{\partial}_0 f + \hat{e}^\alpha \hat{\partial}_\alpha f \), so that \( \hat{\partial} f = 0 \) implies that \( f \) is holomorphic in \( (\omega^\alpha, \pi_{\dot{\alpha}}) \).

To discuss space-time Yang-Mills theory on twistor space, we must choose a vector bundle \( E \to \mathbb{PT}' \) with vanishing first Chern class. Having \( c_1(E) = 0 \) implies that \( E \) is the pullback of some space-time bundle \( \tilde{E} \) via the projection map, \( E = \mu^* \tilde{E} \). Our action will be a functional of fields \( A \in \Omega^0_{\mathbb{PT}'}(\text{End} E) \) and \( B \in \Omega^0_{\mathbb{PT}'}(\mathcal{O}(-4) \otimes \text{End} E) \) where \( A \) is thought of as providing a \( \bar{\partial} \)-operator \( \bar{\partial}_g A = \bar{\partial} + g A \) on \( E \) and is defined up to appropriate gauge transformations \( \delta A = \bar{\partial} g A \gamma \). Here, \( g \) is the Yang-Mills coupling constant. We are not assuming that \( \bar{\partial}_g A \) be integrable, so that \( \bar{\partial}_g A = g F \) is not assumed to be zero where \( F = \bar{\partial} A + g [A, A] \).
We then consider the action \( S_{BF} + S_{B^2} \) where

\[
S_{BF} = \int_{\mathbb{P}T'} \Omega \wedge \text{tr}(B \wedge F) \quad S_{B^2} := -\frac{1}{2} \int_{E \times \mathbb{P}^1} d^4 x \, D\pi_1 \, D\pi_2 \, \langle \pi_1 \pi_2 \rangle^2 \text{tr} (B_1 K_{12} B_2 K_{21})
\]  

(2)

The first term in this action is holomorphic \( BF \) theory, with \( \Omega = \pi^\gamma d\pi_\gamma \wedge \pi^\alpha \pi^\beta dx^{\alpha\gamma} \wedge dx^{\beta\delta} \varepsilon_{\alpha\beta\gamma\delta} \), the canonical top holomorphic form of weight 4 on \( \mathbb{P}T' \). This part of the action was introduced by Witten [1] as part of a supersymmetric Chern-Simons theory. The second term is local on \( E \) but is non-local on the \( \mathbb{P}^1 \) fibres. In it, \( K_{ij} = K_{ij}(x, \pi_i, \pi_j), i, j = 1, 2 \) are Green’s functions for the restriction (\( \overline{\partial}_0 + A_0 \)) of the \( \overline{\partial}_A \) operator to these fibres described more explicitly in the next section, while \( D\pi_i = \langle \pi_i d\pi_i \rangle \) is the top holomorphic form of weight 2 on the \( i \)th fibre and \( B_i = B(x, \pi_i) \) denotes \( B \) evaluated on the \( i \)th factor. This term is the lift to twistor space of the \( B^2 \) term in the (space-time) reformulation of Yang-Mills theory by Chalmers & Siegel [16, 19].

The action is invariant under gauge transformations

\[
\delta A = \overline{\partial} \gamma + g[A, \gamma] \quad \delta B = g[B, \gamma] + \overline{\partial} \beta + g[A, \beta],
\]  

(3)

where \( \gamma \) and \( \beta \) are smooth End \( E \)-valued sections of weight 0 and -4 respectively.

In [16, 17] we obtained the classical equivalence of the twistor action with that on space-time by (partially) fixing these gauge transformations by requiring \( A_0 = 0 \) and \( B \) to be in a gauge in which it is harmonic over the \( \mathbb{P}^1 \) fibres of \( \mathbb{P}T' \rightarrow E \). We refer to this as space-time gauge. In this gauge \( A \) can be expressed directly in terms of a space-time gauge field \( A_{\alpha\dot{\alpha}} \) and \( B \) in terms of an auxiliary space-time field \( B_{\dot{\alpha}\dot{\beta}} \) and the twistor action reduces to the Chalmers and Siegel form of the usual Yang-Mills action on space-time

\[
\int_E \left( B_{\dot{\alpha}\dot{\beta}} F_{\dot{\alpha}\dot{\beta}} - \frac{1}{2} B_{\dot{\alpha}\dot{\beta}} B^{\dot{\alpha}\dot{\beta}} \right) d^4 x
\]  

(4)

with its usual space-time gauge symmetry. Here, \( F_{\dot{\alpha}\dot{\beta}} = \nabla_{(\dot{\alpha}} A_{\dot{\beta})\alpha} + A_{(\dot{\alpha}} A_{\dot{\beta})\alpha} \) is the self-dual part of the field strength of \( A_{\alpha\dot{\alpha}} \) and \( B_{\dot{\alpha}\dot{\beta}} \) is an auxiliary field that equals \( F_{\dot{\alpha}\dot{\beta}} \) on shell.

The twistor action has a natural extension to supersymmetric gauge theory up to and including \( \mathcal{N} = 4 \) and has an analogue for conformal gravity [16, 17].

**FEYNMAN RULES AND MHV DIAGRAMS**

In this section, we will impose an axial gauge - first introduced by Cachazo, Svrček & Witten [2] - to obtain the Feynman rules from (2) and we will see that these directly produce
MHV diagrams. Note that the symmetry in \([3]\), with \(\gamma\) and \(\beta\) each depending on six real variables (the real coordinates of twistor space) is larger than the gauge symmetry of the space-time Yang-Mills action; it is precisely by exploiting this larger symmetry that we are able to interpolate between the standard and MHV pictures of scattering theory.

Decomposing our fields into the basis \([1]\) as \(A = A_0 e^0 + A_\alpha e^\alpha\) etc., we seek to impose the CSW gauge condition \(\eta^\alpha A_\alpha = 0 = \eta^\alpha B_\alpha\) for \(\eta\) some arbitrary constant spinor. This is an axial gauge condition, so the corresponding ghost terms will decouple. This gauge has the benefit that the \(BAA\) vertex from the holomorphic \(BF\) theory vanishes, being the cube of three 1-forms each of which only has non-zero components in only two of the three anti-holomorphic directions. Thus, the only remaining interactions come from the non-local \(B^2\) term. This term depends on the field \(A_0\) in a non-polynomial manner because of the presence of the Green’s functions \(K = (\bar{\partial}_0 + g A_0)^{-1}\) on the \(\mathbb{P}^1\)s. To find the explicit form of the vertices, expand in powers of \(A\) using the standard formulæ

\[
\frac{\delta K_{12}}{\delta A} = \int_{\mathbb{P}^1} D\pi_3 \, K_{13} g A(x, \pi_3) K_{32} \quad \text{and} \quad K_{ij}|_{A=0} = \frac{I}{2\pi i} \frac{1}{\langle \pi_i, \pi_j \rangle}
\]

where \(I\) is the identity matrix in the adjoint representation of the gauge group. Using this repeatedly gives the expansion

\[
K_{1p} = \sum_{n=1}^{\infty} \int_{(\mathbb{P}^1)^{n-1}} \frac{1}{2\pi i \langle \pi_n, \pi_p \rangle} \prod_{r=2}^{n} \frac{g}{2\pi i} \langle \pi_{r-1}, \pi_r \rangle A_r \wedge D\pi_r
\]

This can be substituted in to give the vertices

\[
\sum_{n=2}^{\infty} \frac{g^{n-2}}{(2\pi i)^n} \int_{\mathbb{P}^1} d^4 x \left( \prod_{i=1}^{n} \frac{D\pi_i}{\langle \pi_i, \pi_{i+1} \rangle} \right) \sum_{p=2}^{n} \langle \pi_1, \pi_p \rangle^4 \text{tr} \left( B_1 A_2 \cdots A_{p-1} B_p A_{p+1} \cdots A_n \right).
\]

This expression strongly resembles a sum of MHV amplitudes, except that here we are dealing with vertices rather than amplitudes and \([7]\) is entirely off-shell.

In order to express the linear fields and propagators, it is helpful to introduce certain \((0, 1)\)-form valued weighted \(\delta\)-functions of spinor products

\[
\delta_{n-1,-n-1}(\lambda \pi) = \langle \hat{\pi} d\hat{\pi} \rangle \frac{\langle \lambda \xi \rangle^n \langle \hat{\lambda} \xi \rangle \delta^2(\langle \lambda \pi \rangle)}{\langle \pi \xi \rangle^n \langle \hat{\pi} \xi \rangle}
\]

where for a complex variable \(z = x + iy\), \(\delta^2(z) = \delta(x)\delta(y)\) and the scale factors have been chosen so that \(\delta_{n-1,-n-1}(\lambda \pi)\) has holomorphic weight only in \(\lambda\) and \(\pi\) with weights \(n - 1\), \(-n - 1\) respectively. It is independent of the constant spinor \(\xi\) as \(\lambda \propto \pi\) on the support of
the delta function; see [20] for a full discussion. With these definitions, $K_{ij}$ can be defined by

$$\left(\bar{\partial}_0 + gA_0\right)K_{ij} = \mathbb{I}_\delta_{-1,-1}\langle\pi_i, \pi_j\rangle.$$  \hfill (9)

The first term in (7) is quadratic in $B$ and involves no $A$ fields. We are always free to treat such algebraic terms either as vertices or as part of the kinetic energy of $B$. When working in this axial gauge in twistor space, it turns out to be more convenient to do the former, whereupon the only kinetic terms come from the holomorphic $BF$ theory. In this axial gauge this is

$$\int_{\mathbb{PT}'} \Omega \wedge \text{tr}(B \wedge F) = \int_{\mathbb{PT}'} \Omega \wedge \text{tr}(B \wedge \bar{\partial}A)$$  \hfill (10)

so the propagator is the inverse of the $\bar{\partial}$ operator on $\mathbb{PT}'$. Using coordinates $(x, \pi)$ on $\mathbb{PT}'$, the propagators depend on two points, $(x_1, \pi_1)$, $(x_2, \pi_2)$, but, as usual, depend only on the space-time variables through $x_1 - x_2$. We can Fourier transform the $x_1 - x_2$ to obtain the momentum space axial gauge propagator

$$\langle A(p, \pi_1)B(p, \pi_2)\rangle = \frac{\mathbb{I}}{p^2} \delta_{-2,0}[\eta|p|\pi_1] \wedge \delta_{2,-4}[\eta|p|\pi_2] + \left(\frac{\mathbb{I}}{1} \frac{\eta_\alpha e^\alpha_i}{|\eta|p|\pi_1} \delta_{2,-4}(\pi_1\pi_2) + 1 \leftrightarrow 2\right)$$  \hfill (11)

The linearized field equations obeyed by external fields are $\bar{\partial}A = 0$ and $\bar{\partial}B = 0$, while the linearized gauge transformations are $\delta A = \bar{\partial}\gamma$ and $\delta B = \bar{\partial}\beta$. Together these show that on-shell, free $A$ and $B$ fields are elements of the Dolbeault cohomology groups $H^{1,0}_{\bar{\partial}}(\mathbb{PT}', \mathcal{O})$ and $H^{1,2}_{\bar{\partial}}(\mathbb{PT}', \mathcal{O}(-4))$ representing massless particles of helicity $\mp 1$ as is well-known from the Penrose transform. Momentum eigenstates obeying the axial gauge condition are

$$A(x, \pi) = T e^{\tilde{q}_\alpha x^\alpha q_\alpha} \delta_{-2,0}(q\pi) \quad B(x, \pi) = T e^{\tilde{q}_\alpha x^\alpha q_\alpha} \delta_{2,-4}(q\pi)$$  \hfill (12)

where $T$ is some arbitrary element of the Lie algebra of the gauge group and $\tilde{q}_\alpha q_\alpha$ is the on-shell momentum. Only the components $A_0$ and $B_0$ are non-vanishing and are simple multiples of delta functions. Thus, inserting these external wavefunctions into the vertices in (7), one trivially performs the integrals over each copy of $\mathbb{P}^1$ by replacing $\pi_i^\alpha$ by $q_i^\alpha$ for each external particle. The integral over $\mathbb{E}$ then yields an overall momentum delta-function and, after colour stripping the $\text{tr}(T_1 \ldots T_n)$ factors, one obtains the standard form for an MHV amplitude, arising here from the Feynman rules of the twistor action (2).

More general Feynman diagrams arise from combining the vertices (7) with the propagators from (11) and evaluating external fields using (12). This reproduces the MHV formalism
for Yang-Mills scattering amplitudes: the delta-functions in the propagator lead to the prescrip-
tion of the insertion of $[\eta]p$ as the spinor corresponding to the off-shell momentum $p$.

We note that the vertices only couple to the components $A_0$ and $B_0$ of the fields so that the
second term in the propagator (11) does not play a role except to allow one to interchange
the role of $B_0$ and $A_\alpha$.

We note that, since $\eta$ arises here as an ingredient in the gauge condition, BRST invariance
implies that the overall amplitudes are independent of $\eta$.

**COUPLING TO MATTER**

Matter fields of helicity $n/2$ correspond to $(0, 1)$-forms $C_n$ of homogeneity $n - 2$ with
values in $\text{End} \ E$ for adjoint matter, or $E$ or $E^*$ for ‘fundamental’ matter. These are subject
to a gauge freedom $C_n \rightarrow C_n + \partial g A \kappa_n$ where $\kappa_n$ is an arbitrary smooth function of weight
$n - 2$ with values in $E$, $E^*$ or $\text{End} \ E$ as appropriate. Thus an adjoint scalar field corresponds
to a field $\phi$ on twistor space with values in $\Omega^{0,1}(\text{End} \ E \otimes \mathcal{O}(-2))$ and a fundamental massless
fermion corresponds to fields $\lambda$ in $\Omega^{0,1}(\mathcal{O}(-1)) \otimes E$ and $\tilde{\lambda}$ in $\Omega^{0,1}(\mathcal{O}(-3)) \otimes E^*$ subject to
the gauge freedom

$$(\phi, \lambda, \tilde{\lambda}) \rightarrow (\phi + \partial g A \chi, \lambda + \partial g A \xi, \tilde{\lambda} + \partial g A \tilde{\xi})$$

(13)

with $\chi, \xi, \tilde{\xi}$ of weights $-2, -1, -3$ respectively. The local part of the matter action is

$$S_{\phi, \lambda, \tilde{\lambda}, \text{Loc}} = \int_{\mathcal{P}_T} \Omega \wedge \left( \frac{1}{2} \text{tr} \phi \wedge (\partial \phi + g[A, \phi]) + \tilde{\lambda} \wedge (\partial + gA) \lambda \right)$$

(14)

It is clear that (14) is invariant under the gauge transformations (3). However, in order for
the sum $S_{BF} + S_{\phi, \lambda, \tilde{\lambda}, \text{Loc}}$ (i.e. the complete local action) to be invariant under (13), we must modify the transformation rule of $B$ to

$$B \rightarrow B + \partial \beta + g[A, \beta] + g[B, \gamma] - g[\chi, \phi] + g\lambda \tilde{\xi} - g\xi \tilde{\lambda}.$$  

(15)

This modified transformation law no longer leaves the $B^2$ term invariant, so we need to
include new terms to compensate for this additional gauge freedom. These terms may be
found either by the Noether procedure. There is some freedom in the choice of these
additional terms; we can fix this freedom by requiring that the action corresponds to matter
fields which are minimally coupled on space-time. This requirement leads to the terms

$$ S_{B\phi^2} = g \int d^4x \int_{(\mathbb{P}^1)^3} (\pi_1 \pi_2)^2 (\pi_3)^2 \text{tr} (B_1 K_{12} \phi_2 K_{23} \phi_3 K_{31}) \prod_{i=1}^3 \text{D}\pi_i $$

(16)

$$ S_{\phi^4} = g^2 \int d^4x \frac{1}{2} \int_{(\mathbb{P}^1)^4} (\pi_1 \pi_2)^2 (\pi_3 \pi_4)^2 \text{tr} \left( \prod_{i=1}^4 \phi_i K_{i,i+1} \text{D}\pi_i \right) $$

(17)

$$ S_{B\lambda\bar{\lambda}} = g \int d^4x \int_{(\mathbb{P}^1)^3} \frac{(\pi_1 \pi_2 \pi_3)^3}{(\pi_1 \pi_3)^3} \left( \tilde{\lambda}_1 K_{12} B_2 K_{23} \lambda_3 \right) \prod_{i=1}^3 \text{D}\pi_i $$

(18)

$$ S_{\lambda^2\bar{\lambda}^2} = g^2 \int d^4x \int_{(\mathbb{P}^1)^4} \frac{(\pi_2 \pi_4)^2 (\pi_1 \pi_3)^3}{(\pi_4 \pi_1)^3} \left( \tilde{\lambda}_1 K_{12} \lambda_2 \right) \left( \tilde{\lambda}_3 K_{34} \lambda_4 \right) \prod_{i=1}^4 \text{D}\pi_i $$

(19)

$$ S_{\lambda\bar{\lambda}\phi^2} = g^2 \int d^4x \int_{(\mathbb{P}^1)^4} \left[ \frac{(\pi_1 \pi_2 \pi_3)^2}{(\pi_4 \pi_1)^2} (\pi_1 \pi_3) (\pi_3 \pi_4) + 2 \leftrightarrow 3 \right] \left( \tilde{\lambda}_1 K_{12} \phi_2 K_{23} \phi_3 K_{34} \lambda_4 \right) \prod_{i=1}^4 \text{D}\pi_i $$

(20)

so that the total action is

$$ S_{\text{Full}} = S_{BF} + S_{\phi,\lambda\bar{\lambda},\text{Loc}} + S_{B^2} + S_{B\phi^2} + S_{\phi^4} + S_{B\lambda\bar{\lambda}} + S_{\lambda^2\bar{\lambda}^2} + S_{\lambda\bar{\lambda}\phi^2} $$

(21)

To see that these combinations are indeed invariant under (13) and (15), note that $S_{B^2}$ will pick up a term which is an integral over $(\mathbb{P}^1)^2$ of $g (\pi_1 \pi_2)^4 \text{tr}(B_1 K_{12}[\chi_2, \phi_2] K_{21})$ from the $\chi$ variation of $B$. On the other hand, $S_{B\phi^2}$ will pick up a term from the variation of $\phi$ that is an integral over $(\mathbb{P}^1)^3$ of $g (\pi_1 \pi_2)^2 (\pi_1 \pi_3)^2 \text{tr} \left( B_1 K_{12} \left( (\bar{\partial}_g A_2 \chi_2) K_{23} \phi_3 + \phi_2 K_{23} (\bar{\partial}_g A_3 \lambda_3) \right) K_{31} \right)$. We can integrate this last expression by parts, although care must be taken because of the singularities in the integrand, and use (9) to perform one of the $(\mathbb{P}^1)^3$ integrals to obtain $g (\pi_1 \pi_2)^4 \text{tr}(B_1 K_{12}[\chi_2, \phi_2] K_{21})$ cancelling the corresponding term in the variation of $S_{B^2}$. The $\chi$ part of the variation of $B$ in $S_{B\phi^2}$ however leads to new terms that are in turn cancelled by the corresponding variation of $S_{\phi^4}$ and so on for the $\xi$ and $\bar{\xi}$ terms.

All terms except $S_{B^2}$ vanish in space-time gauge, but are necessary for the full gauge invariance of the action. Thus the full action corresponds on space-time to Yang-Mills with a minimally coupled massless fermion $(\Lambda_\alpha, \tilde{\Lambda}_{\alpha'})$ in the (anti-)fundamental representation and a massless scalar field $\Phi$ in the adjoint representation. We note that we are free to include $\Phi^n$ interactions by including the additional gauge invariant terms

$$ S_{\Phi^n} = c_n \int d^4x \int_{(\mathbb{P}^1)^4} \text{tr} \left( \prod_{i=1}^n (\pi_i \pi_{i+1}) \phi_i K_{i,i+1} \text{D}\pi_i \right) $$

(22)
These terms are gauge invariant because the singularities in $K_{ij}$ in the integrand are cancelled by the factors of $\langle \pi_i \pi_j \rangle$. These correspond precisely to $\text{tr} \Phi^n$ terms in the space-time Yang-Mills theory by use of the standard integral formula for scalar fields $\Phi(x) = \int_{\mathbb{P}^1} H \phi H^{-1} \text{D} \pi$ where $H$ is the gauge transformation to space-time gauge.

As before, we can impose the CSW gauge condition $\eta^\alpha A_{\alpha} = 0 = \eta^\alpha B_{\alpha}$, and similarly $\eta^\alpha \phi_{\alpha} = 0 = \eta^\alpha \lambda_{\alpha} = \eta^\alpha \tilde{\lambda}_{\alpha}$. In this gauge

$$\phi = T e^{\hat{q}_a \alpha \dot{q}_a} \hat{\delta}_{0,-2} \langle q \pi \rangle \quad \tilde{\lambda} = f e^{\hat{q}_a \alpha \dot{q}_a} \hat{\delta}_{-1,-1} \langle q \pi \rangle \quad \lambda = f^* e^{\hat{q}_a \alpha \dot{q}_a} \hat{\delta}_{1,-3} \langle q \pi \rangle$$

(23)

where here $f$ is an element of the fundamental representation. The local parts of the action $S_{BF} + S_{\phi,\lambda,\tilde{\lambda}}$ become quadratic and gives rise to propagators that have the same form as (11), but with integers suitably altered to reflect the different homogeneities. Just as for $S_B^2$, each of $S_{B\phi^2}, S_{\phi^4}, S_{B\lambda\tilde{\lambda}}, S_{\lambda^2\tilde{\lambda}^2}, S_{\lambda\tilde{\lambda}^2}, S_{\phi^n}$ can be seen to be generating functions for MHV-type diagrams, now involving $\lambda, \tilde{\lambda}$ and $\phi$, by expanding out the Green’s functions $K_{ij}$ in powers of $A$ according to equation (6) and substituting into the above formulæ as we did in (7). For example, $S_{B\phi^2}$ gives rise to a sequence of MHV vertices each with one positive helicity gluon corresponding to the $B$ field, two Higgs fields $\phi$ and an arbitrary number of negative helicity gluons. The other non-local terms in (21) behave similarly. The structure of these MHV-type vertices can be seen to be the same as was analysed in (4).

**CONNECTION TO SPACE-TIME CALCULATION**

In space-time, Yang-Mills scattering amplitudes are calculated from correlation functions using the LSZ formalism. This instructs us to calculate $\langle A_{\mu_1}(p_1) \ldots A_{\mu_n}(p_n) \rangle$, isolate the residues of the single particle poles and contract with the polarization vectors. The transform of the twistor-space field $A$ to a space-time gauge field $\mathcal{A}$ is given by

$$\mathcal{A}_{\alpha\dot{\alpha}}(x) = \int k H (\bar{\partial}_\alpha + A_\alpha) H^{-1} \frac{\hat{\pi}_{\dot{\alpha}}}{\langle \pi \bar{\pi} \rangle}$$

(24)

where $H$ solves $(\partial_0 + A_0) H = 0$ and is the gauge transformation to space-time gauge, and $k$ is a volume (Kähler) form on the $\mathbb{P}^1$. This follows from solving the constraint $F_{0\alpha} = 0$ which arises by varying $B_\alpha$ in the action. Note that more than one field insertion will lead to multiparticle poles and so can be ignored at least at tree level. With this observation the
linearization of (24) can be used so that, for the application of the LSZ formalism, operators

$$A_{\alpha\dot{\alpha}} = \int k_1 \left( A_\alpha \frac{\pi_{1\dot{\alpha}}}{\langle \pi_1 \pi_1 \rangle} + \frac{\dot{\pi}_{1\dot{\alpha}} \dot{\pi}_\alpha}{\langle \pi_1 \pi_1 \rangle} \right) \int k_2 \left( \frac{\langle \pi_1 \xi \rangle}{\langle \pi_2 \xi \rangle} \frac{A_0}{\langle \pi_1 \pi_2 \rangle} \right)$$

(25)

should be inserted into the path integral. Here the second term is the first term of the expansion of $H$ in $A_0$ and $\xi$ is an arbitrary spinor reflecting the residual gauge freedom in $A_{\alpha\dot{\alpha}}$. In the axial CSW gauge, the vertices that contribute contain just fibre components of the $(0,1)$ forms. The only contractions to give $1/p^2$ poles are $\langle A_\alpha A_0 \rangle$ and $\langle A_0 B_0 \rangle$, the first of these using one power of the $B^2$ vertex. The residues of these contributions follow from (25) and are given on-shell by

$$\int k_1 \left( A_\alpha \frac{\pi_{1\dot{\alpha}}}{\langle \pi_1 \pi_1 \rangle} \right) \to \eta_\alpha [\eta \tilde{q}] q_{\dot{\alpha}} \quad \int k_1 \left( \frac{\dot{\pi}_{1\dot{\alpha}} \dot{\pi}_\alpha}{\langle \pi_1 \pi_1 \rangle} \int k_2 \left( \frac{\langle \pi_1 \xi \rangle}{\langle \pi_2 \xi \rangle} \frac{A_0}{\langle \pi_1 \pi_2 \rangle} \right) \right) \to \frac{\xi_{\dot{\alpha}} \tilde{q}_\alpha}{[\xi \tilde{q}]} [\eta \tilde{q}]^2$$

(26)

These residues must be contracted with the polarization vectors which we recall are

$$\varepsilon_{\alpha\dot{\alpha}}^{\pm} = \frac{\tilde{q}_\alpha \kappa_{\dot{\alpha}}}{\kappa \tilde{q}} \quad \text{and} \quad \varepsilon_{\alpha\dot{\alpha}}^{\mp} = \frac{\tilde{\kappa}_\alpha \tilde{q}_{\dot{\alpha}}}{\tilde{\kappa} \tilde{q}}$$

(27)

Hence it is clear that $A_\alpha$ and $A_0$ operator insertions correspond to insertions of different helicity states. In the above calculation $B_0$ could have been inserted instead of $A_\alpha$ since by the twistor transform on-shell $B_0$ is the field strength of the positive helicity gluon. Hence to insert the corresponding potential one can also use

$$\frac{p^{\alpha\dot{\alpha}} F_{\alpha\beta}}{p^2} = \frac{p^{\alpha\dot{\alpha}}}{p^2} \int k H B_0 H^{-1} \pi_{\dot{\alpha}} \pi_{\beta}$$

(28)

which can be verified to lead to the same prefactor as above. This is in effect an expression of the field equation which relates $B_0$ and $A_\alpha$. It is therefore seen that the usual calculation of Yang-Mills amplitudes is equivalent to the prescription given earlier.

**DISCUSSION**

The most important drawback of our (and indeed most other) approaches to the MHV formalism is that, while there are now a number of positive results for loop amplitudes, it is still not clear that the extension to loop level can be made systematic. This problem appears particularly acute in the non-supersymmetric case where there are one-loop diagrams that cannot be constructed from MHV vertices and propagators alone, see [13] for a full discussion.
Naïvely it would seem that such problems should not arise in our approach as the action leads to a systematic perturbation theory including loops and the ghosts decouple both in the space-time gauge and in the CSW axial gauge. Nevertheless the missing one-loop diagrams provide us with a clear problem and it is still unclear how they might arise in our derivation of the MHV formalism. One possibility is that the missing diagrams arise from regularisation difficulties: it is well-known in space-time that axial gauges require extra care for certain poles (see e.g. [21]), or the chiral nature of the action may obstruct the implementation of an efficient regularisation scheme. More likely however, in changing the path integral measure from space-time to twistor space fields one encounters a determinant in the spirit of \([12, 13]\). A potential source for this determinant is the complex nature of our gauge transformation to the CSW gauge from the space-time gauge; the path integral is only invariant under real gauge transformations and such complex ones may require an extra determinant in the path integral measure.

Nonetheless, in our opinion this paper clearly shows that the existence of the MHV diagram formalism can be understood in terms of a linear and local gauge symmetry on twistor space.

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