Cauchy problem for fractional non-autonomous evolution equations

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Abstract
This paper deals with the Cauchy problem to a class of nonlinear time fractional non-autonomous integro-differential evolution equation of mixed type via measure of noncompactness in infinite-dimensional Banach spaces. Combining the theory of fractional calculus and evolution families, the fixed point theorem with respect to convex-power condensing operator and a new estimation technique of the measure of noncompactness, we obtained the existence of mild solutions under the situation that the nonlinear function satisfy some appropriate local growth condition and a non-compactness measure condition. Our results generalize and improve some previous results on this topic, since the condition of uniformly continuity of the nonlinearity is not required, and also the strong restriction on the constants in the condition of noncompactness measure is completely deleted. As samples of applications, we consider the initial value problem to a class of time fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition at the end of this paper.

Keywords Fractional non-autonomous evolution equations · Initial value problem · Analytic semigroup · Measure of noncompactness · Mild solution

Mathematics Subject Classification 35R11 · 45K05 · 47H08

1 Introduction and main results
Let $E$ be a real Banach space with norm $\| \cdot \|$, and let $D$ be a bounded and convex set in $E$. If an operator $\mathcal{A} : D \to D$ is completely continuous, then $\mathcal{A}$ has at least fixed point

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in $D$. This is the well known Schauder’s fixed point theorem, which is a famous and important fixed point theorem and it has extremely widespread application. However, Schauder’s fixed point theorem require the operator is completely continuous, which is a very strong restriction condition. It is well known that the main difference between finite dimensional and infinite dimensional space is that the unit ball in infinite dimensional space is not compact, this fact means that Schauder’s fixed point theorem can not be applied to differential equations in infinite dimensional Banach spaces. In order to overcome this strong restriction, the condition of completely continuous operator in Schauder’s fixed point theorem has been relaxed to condensing operator based on the concept of condensing operator. In this way, the famous Sadovskii’s fixed point theorem has been obtained.

Sadovskii’s fixed point theorem is an important tool to study various differential equations and integral equations on infinite dimensional Banach spaces. In 1981, Lakshmikantham and Leela [15] studied the following initial value problem (IVP) of ordinary differential equation in Banach space $E$

$$\begin{align*}
&\begin{cases}
    u'(t) = f(t, u(t)), & t \in [0, a], \\
u(0) = u_0,
\end{cases}
\end{align*}$$

(1)

where $a > 0$ is a constant. The authors proved that, if for any constant $R > 0$, $f$ is uniformly continuous on $[0, a] \times B_R$ and satisfies the noncompactness measure condition

$$\mu(f(t, U)) \leq l\mu(U), \quad \forall \ t \in [0, a], \ U \subset B_R,$n

(2) where $B_R = \{ u \in E : \| u \| \leq R \}$, $\mu(\cdot)$ present Kuratowski measure of noncompactness, $l$ is a positive constant, then IVP (1) has a global solution provided that $l$ satisfies the condition

$$al < 1.$$ (3)

In 1989, Guo [12] studied the global solutions of the following initial value problem (IVP) for first-order nonlinear integro-differential equations of mixed type in a real Banach space $E$

$$\begin{align*}
&\begin{cases}
    u'(t) = f(t, u(t), (Tu)(t), (Su)(t)), & t \in [0, a], \\
u(0) = u_0,
\end{cases}
\end{align*}$$

(4)

where

$$(Tu)(t) = \int_0^t K(t, s)u(s)ds,$n

(5)
is a Volterra integral operator with integral kernel \( K \in C(\triangle, \mathbb{R}), \triangle = \{(t, s) \mid 0 \leq s \leq t \leq a\} \) and
\[
(Su)(t) = \int_0^a H(t, s)u(s)ds,
\] (6)
is a Fredholm integral operator with integral kernel \( H \in C(\triangle_0, \mathbb{R}), \triangle_0 = \{(t, s) \mid 0 \leq s, t \leq a\} \). Denote \( K_0 = \max_{(t, s) \in \triangle} |K(t, s)| \) and \( H_0 = \max_{(t, s) \in \triangle_0} |H(t, s)| \). Then Guo proved that IVP (4) exists at least one global solution if for any any \( R > 0 \), \( f \) is uniformly continuous on \([0, a] \times B_R \times B_R \times B_R\) and there exist positive constants \( l_i \) (\( i = 1, 2, 3 \)) such that
\[
\mu(f(t, U_1, U_2, U_3)) \leq l_1 \mu(U_1) + l_2 \mu(U_2) + l_3 \mu(U_3)
\] (7)
for \( \forall t \in [0, a] \) and bounded sets \( U_1, U_2, U_3 \subset E \), and
\[
2a(l_1 + aK_0l_2 + aH_0l_3) < 1.
\] (8)
Later, there are a large amount of authors studied ordinary differential equations in Banach spaces similar to (4) by using Sadovskii’s fixed point theorem under the assumptions analogous to (7), they also required that the constants satisfy a strong inequality similar to (8). For more details on this fact, please see Liu et al. [18,19] and the references therein.

One can easily to discover that the inequality (3) and (8) are very strong restrictive conditions, and they are difficult to be satisfied in applications. In order to remove the strong restriction on the constants in the conditions of noncompactness measure like (2) or (7), Sun and Zhang [24] generalized the definition of condensing operator to convex-power condensing operator. And based on the definition of this new kind of operator, they established a new fixed point theorem with respect to convex-power condensing operator which generalizes the famous Sadovskii’s fixed point theorem. As an application, the authors investigated the existence of global mild solutions for the initial value problem (IVP) of evolution equations in the real Banach space \( E \)
\[
\begin{aligned}
    u'(t) + Au(t) &= f(t, u(t)), \quad t \in [0, a], \\
    u(0) &= u_0,
\end{aligned}
\] (9)
the authors assume the nonlinear term \( f \) is uniformly continuous on \([0, a] \times B_R\) and satisfies a suitable noncompactness measure condition similar to (2). We should point out that the restriction condition similar to (3) has been deleted in [24]. Recently, Shi et al. [22] developed the IVP (9) to the case that the nonlinear term is \( f(t, u(t), (Tu)(t), (Su)(t)) \), and obtained the existence of global mild solutions by using the new fixed point theorem with respect to convex-power condensing operator established by Sun and Zhang [24], but they also require that the nonlinear term \( f \) is uniformly continuous on \([0, a] \times B_R \times B_R \times B_R\).

During the past 2 decades, fractional order semilinear evolution equations have been proved to be valuable tools in the investigation of many phenomena in engineering,
physics, economy, chemistry, aerodynamics and electrodynamics of complex medium. It has been found that the semilinear evolution equations involving fractional derivatives in time are more realistic to describe many phenomena in practical cases than those of integer order in time. Fractional evolution equations has attracted increasing attention in recent years and it has developed into an important branch of fractional calculus and fractional differential equations. For more details about fractional calculus and fractional evolution equations, we refer to \[1–11,16,17,20,23,27–31\] and the references therein.

However, among the previous researches, most of researchers focus on the case that the differential operators in the main parts are independent of time \(t\), which means that the problems under considerations are autonomous. On the other hand, we notice that when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time \(t\) on account of this class of operators appears frequently in the applications, for the details please see \[25\]. As a result, it is significant and interesting to investigate fractional non-autonomous evolution equations, i.e., the differential operators in the main parts of the considered problems are dependent of time \(t\). In fact, El-Borai \[8\] investigated the existence and continuous dependence of fundamental solutions for a class of linear fractional non-autonomous evolution equations in 2004. In 2010, El-Borai et al. \[9\] give some conditions to ensure the existence of resolvent operator for a class of fractional non-autonomous evolution equations with classical Cauchy initial condition.

Motivated by the above mentioned aspects, in this paper we investigate the existence of mild solutions for the following Cauchy problem to nonlinear time fractional non-autonomous integro-differential evolution equation of mixed type via measure of noncompactness in Banach space \(E\)

\[
\begin{aligned}
&C D_t^\alpha u(t) + A(t)u(t) = f(t, u(t), (Tu)(t), (Su)(t)), \quad t \in I, \\
&u(0) = A^{-1}(0)u_0,
\end{aligned}
\]  

(10)

where \(C D_t^\alpha\) is the standard Caputo’s fractional time derivative of order \(0 < \alpha \leq 1\), \(I = [0, a]\), \(a > 0\) is a constant, \(A(t)\) is a family of closed linear operators defined on a dense domain \(D(A)\) in Banach space \(E\) into \(E\) such that \(D(A)\) is independent of \(t\), \(f : I \times E \times E \times E \rightarrow E\) is a Carathéodory type function, \(u_0 \in E\), \(T\) is the Volterra integral operator defined by (5) and \(S\) is the Fredholm integral operator defined by (6).

The motives and highlights in this article are as follows:

1. We give the proper definition of mild solution for the Cauchy problem to nonlinear time fractional non-autonomous integro-differential evolution equation of mixed type (10) by introducing three operators \(\psi(t, s), \varphi(t, \eta)\) and \(U(t)\) (see definition 2.5).
2. We observed that in \[12,15,18,19,22,24\], the authors all demand that the nonlinear term \(f\) is uniformly continuous, this is a very strong assumption. As a matter of fact, if \(f(t, u)\) is Lipschitz continuous on \(I \times B_R\) with respect to \(u\), then the condition (2) is satisfied, but \(f\) may not necessarily uniformly continuous on \(I \times B_R\). Therefore, in this work we deleted the assumption that \(f\) is uniformly continuous by using
a new estimation technique of the measure of noncompactness (see Lemma 7) established by Chen and Li in 2013 [2].

3. Just as we pointed out previously, the inequality (8) is a very strong restrictive condition. How to get rid of the restriction on the constants in the conditions of noncompactness measure is a meaningful work. In this paper, we successfully used the new kind of fixed point theorem with respect to convex-power condensing operator (see Lemma 9) established in 2005 to study time fractional non-autonomous evolution equations, and completely deleted the strong restriction on the constants in the conditions of noncompactness measure.

The rest of this paper is organized as follows: in the sequel of Sect. 1, we give some general assumptions on the linear operator $-A(t)$, and also present the main results of this paper and its hypotheses. We provide in Sect. 2 some definitions, notations and necessary preliminaries on fractional derivatives, Kuratowski measure of noncompactness and fixed point theorem with respect to convex-power condensing operator. In particular, the definition of mild solution for the Cauchy problem to time fractional non-autonomous integro-differential evolution equation of mixed type (10) and the properties about the operators $\psi(t, s), \varphi(t, \eta)$ and $U(t)$ which defined in Definition 3 are also given. The proofs of the main theorems 1 and 2, are given in Sect. 3. In the final Sect. 4 we present an example of problem where the results of the previous sections apply.

Let $\mathcal{L}(E)$ be the Banach space of all linear and bounded operators in $E$ endowed with the topology defined by the operator norm. Throughout this paper, we assume that the linear operator $-A(t)$ satisfies the following conditions:

(A1) For any $\lambda$ with $\text{Re}\lambda \geq 0$, the operator $\lambda I_d + A(t)$ exists a bounded inverse operator $[\lambda I_d + A(t)]^{-1}$ in $\mathcal{L}(E)$ and

$$
\left\| [\lambda I_d + A(t)]^{-1} \right\| \leq \frac{C}{|\lambda| + 1},
$$

where $C$ is a positive constant independent of both $t$ and $\lambda$;

(A2) For any $t, \tau, s \in I$, there exists a constant $\gamma \in (0, 1]$ such that

$$
\left\| [A(t) - A(\tau)]A^{-1}(s) \right\| \leq C|t - \tau|^{\gamma},
$$

where the constants $\gamma$ and $C > 0$ are independent of both $t$, $\tau$ and $s$.

Remark 1 From Henry [14], Pazy [21] and Temam [26], we know that the assumption (A1) means that for each $s \in I$, the operator $-A(s)$ generates an analytic semigroup $e^{-tA(s)}$ ($t > 0$), and there exists a positive constant $C$ independent of both $t$ and $s$ such that

$$
\left\| A^n(s)e^{-tA(s)} \right\| \leq \frac{C}{t^n},
$$

where $n = 0, 1, t > 0, s \in I$. 
**Remark 2** In assumption (A1), if we choose $\lambda = 0$ and $t = 0$, then there exists a positive constant $C$ independent of both $t$ and $\lambda$ such that

$$\|A^{-1}(0)\| \leq C.$$ 

**Definition 3** A function $\varphi : [0, a] \times E \to E$ is said to be Carathéodory continuous if

(i) for all $u \in E$, $\varphi(\cdot, u)$ is strongly measurable,

(ii) for a.e. $t \in [0, a]$, $\varphi(t, \cdot)$ is continuous.

In order to show the existence of mild solutions to the initial value problem for nonlinear time fractional non-autonomous integro-differential evolution equation of mixed type (10), it is sufficient to impose some natural growth conditions and non-compactness measure condition on the nonlinear function $f$.

(F1) For some $r > 0$, there exist constants $0 \leq \beta < \min\{\alpha, \gamma\}$, $\rho_1 > 0$ and functions $\psi_r \in L^\frac{1}{\beta}(J, \mathbb{R}^+)$ such that for a.e. $t \in I$ and all $u \in E$ satisfying $\|u\| \leq r$,

$$\|f(t, u, Tu, Su)\| \leq \psi_r(t) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\|\psi_r\|_{L^\frac{1}{\beta}[0,a]}}{r} := \rho < +\infty.$$ 

(F2) There exist positive constants $L_1$, $L_2$ and $L_3$ such that for any bounded and countable sets $D_1$, $D_2$, $D_3 \subset E$ and a.e. $t \in I$,

$$\mu(f(t, D_1, D_2, D_3)) \leq L_1\mu(D_1) + L_2\mu(D_2) + L_3\mu(D_3).$$

For the sake of convenience, we denote by

$$\delta_1 = \left(\frac{1 - \beta}{\alpha - \beta}\right)^{1-\beta} + CB(\alpha, \gamma) a^\gamma \left(\frac{1 - \beta}{\alpha + \gamma - \beta}\right)^{1-\beta}$$

and

$$\delta_2 = 1 + Ca^\alpha \left(\frac{1}{\alpha} + a^\gamma B(\alpha, \gamma + 1)\right),$$

where

$$B(\alpha, \gamma) = \int_0^1 t^{\alpha-1}(1 - t)^{\gamma-1} dt$$

is the Beta function.

Our main results are as follows:
Theorem 1 Assume that the nonlinear function \( f : I \times E \times E \times E \rightarrow E \) is Carathéodory continuous. If the assumptions (F1) and (F2) are satisfied, then problem (10) exists at least one mild solution in \( C(I, E) \) provided that

\[
C \rho a^{\alpha - \beta} \delta_1 < 1. \tag{11}
\]

Theorem 2 Assume that the nonlinear function \( f : I \times E \times E \times E \rightarrow E \) is Carathéodory continuous. If the assumption (F2) and the following assumptions

\[(F1)^* \text{ There exist a function } \phi \in L^{\frac{1}{\beta}}(I, \mathbb{R}^+) \text{ for } 0 \leq \beta < \min\{\alpha, \gamma\} \text{ and a nondecreasing continuous function } \Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that} \]

\[
\|f(t, u, Tu, Su)\| \leq \phi(t) \Phi(\|u\|)
\]

for a.e. \( t \in I \) and all \( u \in E \),

hold, then problem (10) exists at least one mild solution in \( C(I, E) \) provided that there exists a positive constant \( R \) such that

\[
\delta_1 \Phi(R) a^{\alpha - \beta} \|\phi\| \frac{1}{L^{\frac{1}{\beta}}[0,a]} \leq \frac{R}{C} - \delta_2 \|u_0\|. \tag{12}
\]

From Theorem 2, we can obtain the following result.

Corollary 1 Assume that the nonlinear function \( f : I \times E \times E \times E \rightarrow E \) is Carathéodory continuous. If the assumptions (F1)* and (F2) are satisfied, then problem (10) exists at least one mild solution in \( C(I, E) \) provided that

\[
\liminf_{r \rightarrow +\infty} \frac{\Phi(r)}{r} < \frac{1}{C \delta_1 a^{\alpha - \beta} \|\phi\| L^{\frac{1}{\beta}}[0,a]}. \tag{13}
\]

Remark 4 In Theorems 1, 2 and Corollary 1, we did not make any restrictions on the constants \( L_1, L_2 \) and \( L_3 \) in the assumption (F2). Noticing that in [15], the constant \( l \) should satisfy the restriction condition (3) and in [12], the constants \( l_1, l_2, l_3 \) should satisfy the restriction condition (8). In this paper, we deleted the restrictions on the constants \( L_1, L_2 \) and \( L_3 \), this is a huge improvement. Furthermore, in this paper we only assume that the nonlinear function \( f \) is Carathéodory continuous, which is weaker than \( f \) is uniformly continuous required in [12,15,18,19,22,24].

2 Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts including fractional derivatives and integrals, Kuratowski measure of noncompactness, fixed point theorem with respect to convex-power condensing operator and the operators \( \psi(t, s) \), \( \varphi(t, \eta) \) and \( U(t) \), which are used throughout this paper.
Throughout this paper, we set $I = [0, a]$ denotes a compact interval in $\mathbb{R}$, where $a > 0$ is a constant. Let $E$ be a Banach space with norm $\| \cdot \|$. We denote by $C(I, E)$ the Banach space of all continuous functions from interval $I$ into $E$ equipped with the supremum norm

$$
\|u\|_C = \sup\{\|u(t)\|, \ t \in I\}, \ \forall \ u \in C(I, E),
$$

and by $\mathcal{L}(E)$ the Banach space of all linear and bounded operators in $E$ endowed with the topology defined by the operator norm. Let $L^1(I, E)$ be the Banach space of all $E$-value Bochner integrable functions defined on $I$ with the norm $\|u\|_1 = \int_0^a \|u(t)\| dt$. For any $r > 0$, denote $\Omega_r = \{u \in C(I, E) : \|u(t)\| \leq r, \ t \in I\}$, then $\Omega_r$ is a closed ball in $C(I, E)$ with center $\theta$ and radius $r$.

At first, we recall the definition of the Riemann–Liouville integral and Caputo derivative of fractional order.

**Definition 1** [6] The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f \in L^1([0, +\infty), \mathbb{R})$ is defined as

$$
I^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.
$$

Here and elsewhere

$$
\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt
$$

denotes the Gamma function.

**Definition 2** [6] The Caputo fractional derivative of order $\alpha$ with the lower limit zero for a function $f : [0, +\infty) \rightarrow \mathbb{R}$, which is at least $n$-times differentiable can be defined as

$$
C D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha}_t f^{(n)}(t),
$$

where $n - 1 < \alpha < n, n \in \mathbb{N}$.

**Lemma 1** (Bochner’s Theorem) A measurable function $f$ maps from $[0, +\infty)$ to $E$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.

**Remark 1** (i) $I^\alpha_t I^\beta_t f = I^{\alpha+\beta}_t$ for any $\alpha_1, \alpha_2 > 0$;
(ii) If $f$ is an abstract function with values in $E$, then the integrals which appear in Definitions 1 and 2 are taken in Bochner’s sense.

By the above discussion and [8, Theorem 2.6], we can get the definition of mild solutions for problem (10).
Definition 3 A function \( u \in C(I, E) \) is said to be a mild solution of the initial value problem (10) if it satisfies

\[
\begin{align*}
u(t) &= A^{-1}(0)u_0 + \int_0^t \psi(t - \eta, \eta)U(\eta)u_0d\eta \\
&+ \int_0^t \psi(t - \eta, \eta)f(\eta, u(\eta), (Tu)(\eta), (Su)(\eta))d\eta \\
&+ \int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s)f(s, u(s), (Tu)(s), (Su)(s))dsd\eta,
\end{align*}
\]

where the operators \( \psi(t, s) \), \( \varphi(t, \eta) \) and \( U(t) \) are defined by

\[
\begin{align*}
\psi(t, s) &= \alpha \int_0^\infty \theta t^{\alpha-1} \xi_\alpha(\theta)e^{-\theta x A(s)}d\theta, \\
\varphi(t, \eta) &= \sum_{k=1}^\infty \varphi_k(t, \eta)
\end{align*}
\]

and

\[
U(t) = -A(t)A^{-1}(0) - \int_0^t \varphi(t, s)A(s)A^{-1}(0)ds,
\]

\( \xi_\alpha \) is a probability density function defined on \([0, +\infty)\) such that it’s Laplace transform is given by

\[
\int_0^\infty e^{-\theta x} \xi_\alpha(\theta)d\theta = \sum_{i=0}^\infty \frac{(-x)^i}{\Gamma(1+\alpha i)}, \quad 0 < \alpha \leq 1, \ x > 0,
\]

\[
\varphi_1(t, \eta) = [A(t) - A(\eta)]\psi(t - \eta, \eta),
\]

and

\[
\varphi_{k+1}(t, \eta) = \int_\eta^t \varphi_k(t, s)\varphi_1(s, \eta)d\eta, \quad k = 1, 2, \ldots
\]

The following properties about the operators \( \psi(t, s) \), \( \varphi(t, \eta) \) and \( U(t) \) will be needed in our argument.

Lemma 2 [8] The operator-valued functions \( \psi(t - \eta, \eta) \) and \( A(t)\psi(t - \eta, \eta) \) are continuous in uniform topology about the variables \( t \) and \( \eta \), where \( t \in I, 0 \leq \eta \leq t - \epsilon \) for any \( \epsilon > 0 \), and

\[
\|\psi(t - \eta, \eta)\| \leq C(t - \eta)^{\alpha-1},
\]
where $C$ is a positive constant independent of both $t$ and $\eta$. Furthermore,

$$\|\varphi(t, \eta)\| \leq C(t - \eta)^{\gamma - 1}$$

and

$$\|U(t)\| \leq C(1 + t^\gamma).$$

By Lemma 2 and appropriate calculation, we can obtain the following result.

**Lemma 3** The integral $\int_0^t \psi(t - \eta, \eta)U(\eta)d\eta$ is uniformly continuous in the operator norm $\mathcal{L}(E)$ for any $t \in I$, and

$$\left\| \int_0^t \psi(t - \eta, \eta)U(\eta)d\eta \right\| \leq C^2 t^\alpha \left( \frac{1}{\alpha} + t^\gamma B(\alpha, \gamma + 1) \right), \quad \forall \ t \in I.$$

Using the properties of Riemann–Liouville integral of fractional order and proper integral transformation, we can obtain the following result, which will be used in the proof of our main results.

**Lemma 4** For any $t \in I$ and $g \in L^1[0, a]$, we have

$$\int_0^t \int_0^n (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1}g(s)d\eta d\eta = B(\alpha, \gamma) \int_0^t (t - \eta)^{\alpha + \gamma - 1}g(\eta)d\eta.$$

Next, we introduce the definition for Kuratowski measure of noncompactness, which will be used in the proof of our main results.

**Definition 4** [7] The Kuratowski measure of noncompactness $\mu(\cdot)$ defined on bounded set $S$ of Banach space $E$ is

$$\mu(S) := \inf \{ \delta > 0 : S = \bigcup_{k=1}^n S_k \text{ and diam}(S_k) \leq \delta \text{ for } k = 1, 2, \ldots, n \}$$

The following properties about the Kuratowski measure of noncompactness are well known.

**Lemma 5** [7] Let $E$ be a Banach space and $U, V \subset E$ be bounded. The following properties are satisfied:

1. $\mu(U) \leq \mu(V)$ if $U \subset V$;
2. $\mu(U) = \mu(\text{conv } U)$, where $\text{conv } U$ means the convex hull of $U$;
3. $\mu(U) = 0$ if and only if $\overline{U}$ is compact, where $\overline{U}$ means the closure hull of $U$;
4. $\mu(\lambda U) = |\lambda|\mu(U)$, where $\lambda \in \mathbb{R}$;
5. $\mu(U \cup V) = \max\{\mu(U), \mu(V)\}$;
6. $\mu(U + V) \leq \mu(U) + \mu(V)$, where $U + V = \{x \mid x = y + z, y \in U, z \in V\}$;
7. $\mu(U + x) = \mu(U)$, for any $x \in E$;
8. If the map $Q : D(Q) \subset E \to X$ is Lipschitz continuous with constant $k$, then $\mu(Q(S)) \leq k\mu(S)$ for any bounded subset $S \subset D(Q)$, where $X$ is another Banach space.

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In this article, we denote by $\mu(\cdot)$ and $\mu_C(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of $E$ and $C(I, E)$ respectively. For any $D \subset C(I, E)$ and $t \in I$, set $D(t) = \{ u(t) \mid u \in D \}$ then $D(t) \subset E$. If $D \subset C(I, E)$ is bounded, then $D(t)$ is bounded in $E$ and $\mu(D(t)) \leq \mu_C(D)$. For more details about the properties of the Kuratowski measure of noncompactness, we refer to the monograph [7].

The following lemmas are needed in our argument.

**Lemma 6** [7] Let $E$ be a Banach space, and let $D \subset C(I, E)$ be bounded and equicontinuous. Then $\mu(D(t))$ is continuous on $[0, a]$, and $\mu_C(D) = \max_{t \in [0, a]} \mu(D(t))$.

**Lemma 7** [2] Let $E$ be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\mu(D) \leq 2 \mu(D_0)$.

**Lemma 8** [13] Let $E$ be a Banach space. If $D = \{ u_n \}_{n=1}^\infty \subset C([0, a], E)$ is a countable set and there exists a function $m \in L^1([0, a], \mathbb{R}^+)$ such that for every $n \in \mathbb{N}$

$$\|u_n(t)\| \leq m(t), \quad a.e. \ t \in [0, a].$$

Then $\mu(D(t))$ is Lebesgue integral on $[0, a]$, and

$$\mu \left( \left\{ \int_0^a u_n(t)dt \mid n \in \mathbb{N} \right\} \right) \leq 2 \int_0^a \mu(D(t))dt.$$

The following fixed point theorem with respect to convex-power condensing operator which introduced by Sun and Zhang [24] plays a key role in the proof of our main results.

**Definition 5** Let $E$ be a real Banach space. If $\mathcal{A} : E \to E$ is a continuous and bounded operator, there exist $u_0 \in E$ and a positive integer $n_0$ such that for any bounded and nonprecompact subset $S \subset E$,

$$\mu(\mathcal{A}^{(n_0, u_0)}(S)) < \mu(S), \quad (14)$$

where

$$\mathcal{A}^{(1, u_0)}(S) \equiv \mathcal{A}(S), \quad \mathcal{A}^{(n, u_0)}(S) = \mathcal{A}(co\{ Q^{(n-1, u_0)}(S), u_0 \}), \quad n = 2, 3, \ldots$$

Then we call $\mathcal{A}$ a convex-power condensing operator about $u_0$ and $n_0$.

**Lemma 9** (Fixed point theorem with respect to convex-power condensing operator, [24]) Let $E$ be a real Banach space, and let $D \subset E$ be a bounded, closed and convex set in $E$. If there exist $u_0 \in D$ and a positive integer $n_0$ such that $\mathcal{A} : D \to D$ be a convex-power condensing operator about $u_0$ and $n_0$, then the operator $\mathcal{A}$ exists at least one fixed point in $D$.

**Remark 2** If $n_0 = 1$ in (14), then fixed point theorem with respect to convex-power condensing operator (see Lemma 9) will degrade into the famous Sadoveskii’s fixed point theorem (see [7, Chapter 2]). Noticed that Lemma 9 requires the operator $\mathcal{A}$ is
neither condensing nor completely continuous. Therefore, fixed point theorem with respect to convex-power condensing operator is the generalization of the well-known Sadoveskii’s fixed point theorem.

3 Proof of the main results

In this section, we give the proofs of Theorems 1 and 2.

Proof of Theorem 1 Define an operator $Q$ on the space of continuous functions $C(I, E)$ as follows

$$(Qu)(t) = A^{-1}(0)u_0 + \int_0^t \psi(t - \eta, \eta)U(\eta)u_0 d\eta$$

$$+ \int_0^t \psi(t - \eta, \eta) f(\eta, u(\eta), (Tu)(\eta), (Su)(\eta)) d\eta$$

$$+ \int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s) f(s, u(s), (Tu)(s), (Su)(s))dsd\eta. \quad (15)$$

By direct calculation and the properties about the operators $\psi(t, s)$, $\varphi(t, \eta)$, and $U(t)$, it is easy to know that the operator $Q$ maps $C(I, E)$ to $C(I, E)$, and it is well defined.

From Definition 3, one can easily verify that the mild solution of initial value problem (10) is equivalent to the fixed point of the operator $Q$ defined by (15). In what follows, we will prove that the operator $Q$ has at least one fixed point by applying Lemma 9.

Firstly, we prove that there exists a positive constant $R$ such that the operator $Q$ defined by (15) maps the set $\Omega_R$ to $\Omega_R$. If this is not true, then there would exist $\tau_r \in I$ and $u_r \in \Omega_r$ such that $\|Q(u_r)\| > r$ for each $r > 0$. Combining with Lemmas 2–4, the assumption (F1) and Hölder inequality, we get that

$$r < \|Q(u_r)\| \leq \|A^{-1}(0)u_0\| + \int_0^{\tau_r} \|\psi(t - \eta, \eta)U(\eta)u_0 d\eta\|$$

$$+ \|\int_0^{\tau_r} \psi(t - \eta, \eta) f(\eta, u_r(\eta), (Tu_r)(\eta), (Su_r)(\eta)) d\eta\|$$

$$+ \|\int_0^{\tau_r} \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s) f(s, u_r(s), (Tu_r)(s), (Su_r)(s))dsd\eta\|$$

$$\leq C\|u_0\| + C^2 \int_0^{\tau_r} (t_r - \eta)^{\alpha - 1}(1 + \eta^{\gamma})\|u_0\| d\eta$$

$$+ C \int_0^{\tau_r} (t_r - \eta)^{\alpha - 1}\psi_r(\eta) d\eta + C^2 \int_0^{\tau_r} \int_0^\eta (t_r - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1}\psi_r(s)dsd\eta$$

$$\leq C\|u_0\| + C^2\|u_0\|(\tau_r)^{\alpha} \left( \frac{1}{\alpha} + (\tau_r)^{\gamma} B(\alpha, \gamma + 1) \right)$$

$$+ C \int_0^{\tau_r} (t_r - \eta)^{\alpha - 1}\psi_r(\eta) d\eta + C^2 B(\alpha, \gamma) \int_0^{\tau_r} (t_r - \eta)^{\alpha + \gamma - 1}\psi_r(\eta) d\eta$$

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\[
\leq C\|u_0\| + C^2\|u_0\|\alpha a^\alpha \left(\frac{1}{\alpha} + a^\gamma B(\alpha, \gamma + 1)\right) \\
+ C \left(\int_0^{t_r} (t_r - \eta)^{\frac{\alpha-1}{1-\beta}} d\eta\right)^{1-\alpha_1} \left(\int_0^{t_r} \psi_r^\beta(\eta)d\eta\right)^\beta \\
+ C^2 B(\alpha, \gamma) \left(\int_0^{t_r} (t_r - \eta)^{\frac{\alpha+\gamma-1}{1-\beta}} d\eta\right)^{1-\beta} \left(\int_0^{t_r} \psi_r^\beta(\eta)d\eta\right)^\beta \\
\leq C\|u_0\| + C^2\|u_0\|\alpha a^\alpha \left(\frac{1}{\alpha} + a^\gamma B(\alpha, \gamma + 1)\right) \\
+ C\alpha^{-\beta} \left(\frac{1 - \beta}{\alpha - \beta}\right)^{1-\beta} \|\psi_r\|_{L^\beta[0,a]} \\\n+ C^2 B(\alpha, \gamma) a^{\alpha+\gamma-\beta} \left(\frac{1 - \beta}{\alpha + \gamma - \beta}\right)^{1-\beta} \|\psi_r\|_{L^\beta[0,a]}.
\] (16)

Dividing both side of (16) by \(r\) and taking the lower limit as \(r \to +\infty\), combined with the assumption (11) we get that

\[
1 \leq C \rho a^{\alpha - \beta} \left[\left(\frac{1 - \beta}{\alpha - \beta}\right)^{1-\beta} + C B(\alpha, \gamma) a^{\gamma} \left(\frac{1 - \beta}{\alpha + \gamma - \beta}\right)^{1-\beta}\right] = C \rho a^{\alpha - \beta} \delta_1 < 1.
\] (17)

One can easily to see that (17) is a contradiction. Therefore, we have proved that \(\mathcal{Q} : \Omega_R \to \Omega_R\).

Secondly, we prove that the operator \(\mathcal{Q} : \Omega_R \to \Omega_R\) is continuous. To this end, let \(\{u_n\}_{n=1}^\infty \subset \Omega_R\) be a sequence such that \(\lim_{n \to +\infty} u_n = u\) in \(\Omega_R\). By the Carathéodory continuity of the nonlinear function \(f\), we get that

\[
\lim_{n \to +\infty} \|f(t, u_n(t), (Tu_n)(t), (Su_n)(t)) - f(t, u(t), (Tu)(t), (Su)(t))\| = 0
\] (18)

for a.e. \(t \in I\). By (15) and Lemmas 2–4 combined with the similar calculus method with which used in (16), we have

\[
\|(\mathcal{Q}u_n)(t) - (\mathcal{Q}u)(t)\|
\leq \left\|\int_0^t \psi(t - \eta, \eta)[f(\eta, u_n(\eta), (Tu_n)(\eta), (Su_n)(\eta)) - f(t, u(\eta), (Tu)(\eta), (Su)(\eta))]d\eta\right\|
\leq \left\|\int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s)[f(s, u_n(s), (Tu_n)(s), (Su_n)(s)) - f(t, u(s), (Tu)(s), (Su)(s))]dsd\eta\right\|
\]
\[ \begin{align*}
&\leq C \int_0^t (t-\eta)^{\alpha-1} \| f(\eta, u_n(\eta), (Tu_n)(\eta), (Su_n)(\eta)) \\
&\quad - f(t, u(\eta), (Tu)(\eta), (Su)(\eta)) \| d\eta \\
&\quad + C^2 \int_0^t \int_0^\eta (t-\eta)^{\alpha-1}(\eta-s)^{\gamma-1} \| f(s, u_n(s), (Tu_n)(s), (Su_n)(s)) \\
&\quad - f(t, u(s), (Tu)(s), (Su)(s)) \| dsd\eta, \quad \forall \ t \in I.
\end{align*} \]

By the assumption (F1), we know that for every \( t \in I \),
\[ (t-\eta)^{\alpha-1} \| f(\eta, u_n(\eta), (Tu_n)(\eta), (Su_n)(\eta)) - f(t, u(\eta), (Tu)(\eta), (Su)(\eta)) \| \]
\[ \leq 2(t-\eta)^{\alpha-1} \psi_R(\eta) \] (20)
for a.e. \( \eta \in [0, t] \). By again the assumption (F1) combined with Lemma 4, we get that for every \( t \in I, 0 \leq \eta \leq t \) and a.e. \( s \in [0, \eta] \),
\[ \begin{align*}
\int_0^t \int_0^\eta (t-\eta)^{\alpha-1}(\eta-s)^{\gamma-1} \| f(s, u_n(s), (Tu_n)(s), (Su_n)(s)) \\
&\quad - f(s, u(s), (Tu)(s), (Su)(s)) \| dsd\eta \\
&\leq 2 \int_0^t \int_0^\eta (t-\eta)^{\alpha-1}(\eta-s)^{\gamma-1} \psi_R(s)dsd\eta \\
&= 2B(\alpha, \gamma) \int_0^t (t-\eta)^{\alpha+\gamma-1} \psi_R(\eta)d\eta.
\end{align*} \]
(21)

From the fact that the functions \( \eta \to 2(t-\eta)^{\alpha-1} \psi_R(\eta) \) and \( \eta \to 2B(\alpha, \gamma)(t-\eta)^{\alpha+\gamma-1} \psi_R(\eta) \) are Lebesgue integrable for a.e. \( \eta \in [0, t] \) and every \( t \in I \), combined with (18), (19), (20), (21) and the Lebesgue dominated convergence theorem, we know that
\[ \| (\mathcal{D}u_n)(t) - (\mathcal{D}u)(t) \| \to 0 \quad \text{as} \quad n \to \infty \]
for any \( t \in I \). Therefore, we get that
\[ \| \mathcal{D}u_n - \mathcal{D}u \|_C \to 0 \quad (n \to \infty), \]
which means that \( \mathcal{D} : \Omega_R \to \Omega_R \) is a continuous operator.

Now, we are in the position to demonstrate that \( \mathcal{D} : \Omega_R \to \Omega_R \) is an equicontinuous operator. For any \( u \in \Omega_R \) and \( 0 \leq t' < t'' \leq a \), by (15) and the assumption (F1), we know that
\[ \| (\mathcal{D}u)(t'') - (\mathcal{D}u)(t') \| \]
\[ \leq \left\| \int_{t'}^{t''} \psi(t'' - \eta, \eta)U(\eta)u_0d\eta \right\| \\
+ \left\| \int_0^{t'} [\psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta)]U(\eta)u_0d\eta \right\| \]

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\[ + \left\| \int_0^t \psi(t^\prime - \eta) f(\eta, u(\eta), (Tu)(\eta), (Su)(\eta)) \, d\eta \right\| \\
+ \left\| \int_0^t \left[ \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right] f(\eta, u(\eta), (Tu)(\eta), (Su)(\eta)) \, d\eta \right\| \\
+ \left\| \int_0^t \int_\eta^t \psi(t^\prime - \eta) \varphi(\eta, s) f(s, u(s), (Tu)(s), (Su)(s)) \, ds \, d\eta \right\| \\
+ \left\| \int_0^t \int_\eta^t \left[ \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right] \varphi(\eta, s) \right. \\
\cdot f(s, u(s), (Tu)(s), (Su)(s)) \, ds \, d\eta \right\| \\
\leq J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \]

where

\[ J_1 = \int_0^t \left\| \psi(t^\prime - \eta) U(\eta) \right\| \cdot \| u_0 \| \, d\eta, \]
\[ J_2 = \int_0^t \left\| \left[ \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right] U(\eta) \right\| \cdot \| u_0 \| \, d\eta, \]
\[ J_3 = \int_0^t \left\| \psi(t^\prime - \eta) \right\| \varphi_R(\eta) \, d\eta, \]
\[ J_4 = \int_0^t \left\| \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right\| \varphi_R(\eta) \, d\eta, \]
\[ J_5 = \int_0^t \int_\eta^t \left( \psi(t^\prime - \eta) \varphi(\eta, s) \right) \varphi_R(s) \, ds \, d\eta, \]
\[ J_6 = \int_0^t \int_\eta^t \left[ \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right] \varphi(\eta, s) \right. \varphi_R(s) \, ds \, d\eta. \]

Therefore, we only need to prove that \( J_k \rightarrow 0 \) independently of \( u \in \Omega_R \) as \( t^\prime - t^\prime \rightarrow 0 \) for \( k = 1, 2, 3, 4, 5, 6 \). For \( J_1 \), by Lemma 2 we know that

\[ J_1 \leq C^2 \| u_0 \| \int_0^t (t^\prime - \eta)^{\alpha - 1} (1 + \eta^\gamma) \, d\eta \rightarrow 0 \quad \text{as} \quad t^\prime - t^\prime \rightarrow 0. \]

For \( t^\prime = 0 \) and \( 0 < t^\prime \leq a \), it is easy to see that \( J_2 = 0 \). For \( t^\prime > 0 \) and \( \epsilon > 0 \) small enough, by Lemma 2 and the fact that operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for \( 0 \leq t \leq a \) and \( 0 \leq \eta \leq t - \epsilon \), we have

\[ J_2 \leq \sup_{\eta \in [0, t^\prime - \epsilon]} \left\| \psi(t^\prime - \eta) - \psi(t^\prime - \eta) \right\| \cdot C \| u_0 \| \int_0^{t^\prime - \epsilon} (1 + \eta^\gamma) \, d\eta \]
\[ + C^2 \| u_0 \| \int_0^t \left[ (t^\prime - \eta)^{\alpha - 1} + (t^\prime - \eta)^{\alpha - 1} \right] (1 + \eta^\gamma) \, d\eta \]
→ 0 as t'' - t' → 0 and ϵ → 0.

For J₃, by Lemma 2, the assumption (F1) and Hölder inequality, we get that

\[ J₃ \leq C \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1} \psi_R(\eta) d\eta \leq C \left( \int_{t'}^{t''} (t'' - \eta)^{\frac{\alpha - 1}{1 - \beta}} d\eta \right)^{1 - \beta} \left( \int_{t'}^{t''} \psi_R(\eta) d\eta \right)^{\beta} \]

\[ \leq C \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1 - \beta} \| \psi_R \|_{L^\frac{1}{\beta} [0, a]} (t'' - t')^{\alpha - \beta} \]

→ 0 as t'' - t' → 0.

For t' = 0 and 0 < t'' ≤ a, it is easy to see that J₄ = 0. For t' > 0 and ϵ > 0 small enough, by Lemma 2 and the fact that operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for 0 ≤ t ≤ a and 0 ≤ \( \eta \leq t - \epsilon \), we know that

\[ J₄ \leq (t' - \epsilon)^{1 - \beta} \| \psi_R \|_{L^\frac{1}{\beta} [0, a]} \sup_{\eta \in [0, t' - \epsilon]} \| \psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta) \| \]

\[ + C \int_{t' - \epsilon}^{t'} [(t'' - \eta)^{\alpha - 1} + (t' - \eta)^{\alpha - 1}] \psi_R(\eta) d\eta \]

→ 0 as t'' - t' → 0 and ϵ → 0.

For J₅, by Lemma 2, the assumption (F1) and the fact that the function \( \eta \to (t'' - \eta)^{\alpha - 1} I_\eta^\gamma \psi_R(\eta) \) is Lebesgue integrable, we have

\[ J₅ \leq C^2 \int_{t'}^{t''} \int_{0}^{\eta} (t'' - \eta)^{\alpha - 1} (\eta - s)^{\gamma - 1} \psi_R(s) ds d\eta \]

\[ \leq C^2 \Gamma(\gamma) \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1} I_\eta^\gamma \psi_R(\eta) d\eta \]

→ 0 as t'' - t' → 0.

For t' = 0 and 0 < t'' ≤ a, it is easy to see that J₆ = 0. For t' > 0 and ϵ > 0 small enough, by Lemma 2, the assumption (F1), the facts that the functions \( \eta \to (t'' - \eta)^{\alpha - 1} I_\eta^\gamma \psi_R(\eta) \) and \( \eta \to (t' - \eta)^{\alpha - 1} I_\eta^\gamma \psi_R(\eta) \) are Lebesgue integrable as well as the operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for 0 ≤ t ≤ a and 0 ≤ \( \eta \leq t - \epsilon \), we know that

\[ J₆ \leq \sup_{\eta \in [0, t' - \epsilon]} \| \psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta) \| C \int_{0}^{t' - \epsilon} \int_{0}^{\eta} (\eta - s)^{\gamma - 1} \psi_R(s) ds d\eta \]

\[ + C^2 \int_{t' - \epsilon}^{t'} \int_{0}^{\eta} [(t'' - \eta)^{\alpha - 1} + (t' - \eta)^{\alpha - 1}] (\eta - s)^{\gamma - 1} \psi_R(s) ds d\eta \]
\[
\begin{align*}
&\leq \left( \frac{1 - \beta}{\gamma - \beta} \right)^{1-\beta} C(t')^{\gamma} \| \psi_R \|_{L_\beta([0,a])} \sup_{\eta \in [0,t'-\epsilon]} \| \psi(t' - \eta, \eta) - \psi(t', \eta) \|
\end{align*}
\]

\[
+ C^2 \Gamma(\gamma) \int_{t'-\epsilon}^{t'} [(t' - \eta)^{\alpha-1} I^\gamma_\eta \psi_R(\eta) + (t' - \eta)^{\alpha-1} I^\gamma_\eta \psi_R(\eta)]d\eta
\]

\[
\to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \epsilon \to 0.
\]

As a result, \( \| (\mathcal{D}u)(t'') - (\mathcal{D}u)(t') \| \to 0 \) independently of \( u \in \Omega_R \) as \( t'' - t' \to 0 \), which means that the operator \( \mathcal{D} : \Omega_R \to \Omega_R \) is equicontinuous.

Next, we prove that \( \mathcal{D} : F \to F \) is a convex-power condensing operator, where \( F = \overline{\text{co}} \mathcal{Q}(\Omega_R) \) and \( \overline{\text{co}} \) means the closure of convex hull. Then one can easily to verify that the operator \( \mathcal{D} \) maps \( F \) into itself and \( F \subset C(I, E) \) is equicontinuous. Let \( u_0 \in F \). In the following, we will prove that there exists a positive integer \( n_0 \) such that for any bounded and nonprecompact subset \( D \subset F \)

\[
\mu_C \left( \mathcal{Q}^{(n_0,u_0)}(D) \right) < \mu_C(D). \quad (22)
\]

For any \( D \subset F \) and \( u_0 \in F \), by the definition of operator \( \mathcal{Q}^{(n,u_0)} \) and the equicontinuity of \( F \), we get that \( \mathcal{Q}^{(n,u_0)}(D) \subset \Omega_R \) is also equicontinuous. Therefore, we know from Lemma 6 that

\[
\mu_C \left( \mathcal{Q}^{(n,u_0)}(D) \right) = \max_{t \in I} \mu \left( \mathcal{Q}^{(n,u_0)}(D)(t) \right), \quad n = 1, 2, \ldots \quad (23)
\]

By Lemma 7, there exists a countable set \( D_1 = \{u_1^n\} \subset D \), such that

\[
\mu(\mathcal{D}(D)(t)) \leq 2\mu(\mathcal{D}(D_1)(t)). \quad (24)
\]

By the fact

\[
\int_0^a u(s)ds \in a\overline{\text{co}}\{u(s) \mid s \in I\}, \quad \forall u \in C(I, E),
\]

we know that

\[
\mu \left( \left\{ \int_0^t K(t,s)u(s)ds \mid u \in D, \ t \in I \right\} \right) \leq aK_0\mu(\{u(t) \mid u \in D, \ t \in I\}) \quad (25)
\]

and

\[
\mu \left( \left\{ \int_0^a H(t,s)u(s)ds \mid u \in D, \ t \in I \right\} \right) \leq aH_0\mu(\{u(t) \mid u \in D, \ t \in I\}). \quad (26)
\]
where \( K_0 = \max_{(t,s) \in \Delta} |K(t,s)|, H_0 = \max_{(t,s) \in \Delta_0} |H(t,s)| \). Therefore, by (15), (24), (25), (26), Lemmas 2, 4, 8 and the assumption (F2), we have

\[
\begin{align*}
\mu(\mathcal{D}(1,u_0)(D)(t)) \\
= \mu(\mathcal{D}(D)(t)) \leq 2\mu(\mathcal{D}(D_1)(t)) \\
\leq 2\mu \left( A^{-1}(0)u_0 + \int_0^t \psi(t-\eta,\eta)U(\eta)u_0 d\eta \right) \\
+ 2\mu \left( \int_0^t \psi(t-\eta,\eta)f(\eta, u_n^1(\eta), (Tu_n^1)(\eta), (Su_n^1)(\eta))d\eta \right) \\
+ 2\mu \left( \int_0^t \int_0^\eta \psi(t-\eta,\eta)\varphi(\eta, s) \\
\cdot f(s, u_n^1(s), (Tu_n^1)(s), (Su_n^1)(s))dsd\eta \right) \\
\leq 4C \int_0^t (t-\eta)^{\alpha-1} \left[ L_1 \mu(D_1(\eta)) + L_2 \mu((TD_1)(\eta)) \\
+ L_3 \mu((SD_1)(\eta)) \right] d\eta \\
+ 8C^2 \int_0^t \int_0^\eta (t-\eta)^{\alpha-1}(\eta-s)^{\gamma-1} \\
\cdot \left[ L_1 \mu(D_1(\eta)) + L_2 \mu((TD_1)(\eta)) + L_3 \mu((SD_1)(\eta)) \right] dsd\eta \\
\leq 4C \int_0^t (t-\eta)^{\alpha-1} \left[ (L_1 + aK_0L_2 + aH_0L_3) \mu(D_1(\eta)) \right] d\eta \\
+ 8C^2B(\alpha, \gamma) \int_0^t (t-\eta)^{\alpha+\gamma-1} \left[ (L_1 + aK_0L_2 + aH_0L_3) \mu(D_1(\eta)) \right] d\eta \\
\leq \frac{4CM\Gamma(\alpha)t^\alpha}{\Gamma(1+\alpha)} \mu_C(D) + \frac{8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha+\gamma}}{\Gamma(1+\gamma+\alpha)} \mu_C(D),
\end{align*}
\]  

where

\[
M := L_1 + aK_0L_2 + aH_0L_3.
\]  

Again by Lemma 7, there exists a countable set \( D_2 = \{u_n^2\} \subset \overline{\mathcal{D}(1,u_0)}(D), u_0 \}, such that

\[
\mu(\mathcal{D}(\overline{\mathcal{D}}(1,u_0)(D), u_0))(t)) \leq 2\mu(\mathcal{D}(D_2)(t)).
\]  

Therefore, by (15), (25), (26), (27), (28), (29), Lemmas 2, 4, 5 (iii), 8 and the assumption (F2), we get that

\[
\begin{align*}
\mu(\mathcal{D}(2,u_0)(D)(t)) \\
= \mu(\mathcal{D}(\overline{\mathcal{D}}(1,u_0)(D), u_0))(t)) \\
\leq 2\mu(\mathcal{D}(D_2)(t))
\end{align*}
\]
\begin{align*}
&\leq 2\mu\left(A^{-1}(0)u_0 + \int_0^t \psi(t - \eta, \eta)U(\eta)u_0 d\eta\right) \\
&+ 2\mu\left(\left\{\int_0^t \psi(t - \eta, \eta)f(\eta, u_n^2(\eta), (Tu_n^2)(\eta), (Su_n^2)(\eta))d\eta\right\}\right) \\
&+ 2\mu\left(\left\{\int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s) f(s, u_n^2(s), (Tu_n^2)(s), (Su_n^2)(s))dsd\eta\right\}\right) \\
&\leq 4CM \int_0^t (t - \eta)^{\alpha - 1} \mu(D_2(\eta))d\eta + 8C^2MB(\alpha, \gamma) \\
&\cdot \int_0^\eta (t - \eta)^{\alpha + \gamma - 1} \mu(D_2(\eta))d\eta \\
&\leq 4CM \int_0^t \mu\left(\mathcal{Q}(1, u_0)(D), u_0(\eta)\right) d\eta \\
&+ 8C^2MB(\alpha, \gamma) \int_0^\eta (t - \eta)^{\alpha + \gamma - 1} \mu\left(\mathcal{Q}(1, u_0)(D), u_0(\eta)\right) d\eta \\
&\leq 4CM \int_0^t (t - \eta)^{\alpha - 1} \left[\frac{4CM\Gamma(\alpha)t^\alpha}{\Gamma(1 + \alpha)} + \frac{8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma}}{\Gamma(1 + \gamma + \alpha)}\right] d\eta \cdot \mu_C(D) \\
&+ 8C^2MB(\alpha, \gamma) \int_0^\eta (t - \eta)^{\alpha + \gamma - 1} \left[\frac{4CM\Gamma(\alpha)t^\alpha}{\Gamma(1 + \alpha)} + \frac{8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma}}{\Gamma(1 + \gamma + \alpha)}\right] d\eta \cdot \mu_C(D) \\
&= \frac{4CM \cdot 4CM\Gamma^2(\alpha)t^{2\alpha}}{\Gamma(1 + 2\alpha)} \mu_C(D) + \frac{2 \cdot 4CM \cdot 8C^2M\Gamma^2(\alpha)\Gamma(\gamma)t^{2\alpha + \gamma}}{\Gamma(1 + \gamma + 2\alpha)} \mu_C(D) \\
&+ \frac{8C^2M \cdot 8C^2M\Gamma^2(\alpha)\Gamma^2(\gamma)t^{2\alpha + 2\gamma}}{\Gamma(1 + 2\gamma + 2\alpha)} \mu_C(D). \tag{30}
\end{align*}

If for \(\forall \ t \in I\), we assume that

\begin{align*}
\mu\left(\mathcal{Q}^{(k, u_0)}(D)(t)\right) &\leq \frac{C_k^0(4CM\Gamma(\alpha)t^\alpha)^k \cdot (8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma})^0}{\Gamma(1 + k\alpha)} \mu_C(D) \\
&+ \frac{C_k^1(4CM\Gamma(\alpha)t^\alpha)^{k-1} \cdot (8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma})^1}{\Gamma(1 + \gamma + k\alpha)} \mu_C(D) \\
&+ \frac{C_k^{k-1}(4CM\Gamma(\alpha)t^\alpha)^1 \cdot (8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma})^{k-1}}{\Gamma(1 + (k - 1)\gamma + k\alpha)} \mu_C(D) \\
&+ \frac{C_k^k(4CM\Gamma(\alpha)t^\alpha)^0 \cdot (8C^2M\Gamma(\alpha)\Gamma(\gamma)t^{\alpha + \gamma})^k}{\Gamma(1 + k\gamma + k\alpha)} \mu_C(D). \tag{31}
\end{align*}
Then by Lemma 7, there exists a countable set $D_{k+1} = \{ u_n^{k+1} \} \subset \mathcal{Q}(\mathcal{Q}([u_0^{k+1}], 2\mu(D_{k+1})((D))).

From (15), (25), (26), (28), (31), (32), Lemmas 2, 4, 5 (iii), 8, and the assumption (F2) and proper integral transformation, we get that

\[
\mu(\mathcal{Q}([u_0^{k+1}], 2\mu(D_{k+1})(D)))
\leq 4CM \int_0^t (t - \eta)^{\alpha-1} \mu(D_{k+1}(\eta)) d\eta
\]

\[
+ 8C^2 M B(\alpha, \gamma) \int_0^t (t - \eta)^{\alpha+\gamma-1} \mu(D_{k+1}(\eta)) d\eta
\]

\[
\leq 4CM \int_0^t (t - \eta)^{\alpha-1} \mu(\mathcal{Q}([u_0^{k+1}], (D))) d\eta
\]

\[
+ 8C^2 M B(\alpha, \gamma) \int_0^t (t - \eta)^{\alpha+\gamma-1} \mu(\mathcal{Q}([u_0^{k+1}], (D))) d\eta
\]

\[
\leq C_0^0(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]

\[
+ C_0^1(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]

\[
\vdots
\]

\[
+ C_0^{k}(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]

Therefore, by the method of mathematical induction, we know that for any positive integer $n$ and $t \in I$

\[
\mu(\mathcal{Q}([u_0^{n}], D(t)))
\leq C_0^0(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]

\[
+ C_0^1(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]

\[
\vdots
\]

\[
+ C_0^{n-1}(4CM \Gamma(\alpha)\Gamma(n+1) \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma)\Gamma(n+1)) \cdot \mu_C(D)
\]
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\[ + \frac{C_n^n(4CM \Gamma(\alpha) t^\alpha)^0 \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma) t^{\alpha + \gamma})^n}{\Gamma(1 + n\gamma + n\alpha)} \mu_C(D). \] (34)

Hence, by (23) and (34), we get that

\[ \mu_C \left( \mathcal{Q}^{(n,a_0)}(D) \right) = \max_{t \in I} \mu \left( \mathcal{Q}^{(n,a_0)}(D)(t) \right) \]

\[ \leq \left[ \frac{(4CM \Gamma(\alpha))^n \cdot a^{n\alpha}}{\Gamma(1 + n\alpha)} \right. \]
\[ + \frac{C_n^1(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 CM \Gamma(\alpha) \Gamma(\gamma))^1 \cdot a^{\gamma + n\alpha}}{\Gamma(1 + \gamma + n\alpha)} \]
\[ \vdots \]
\[ + \frac{C_n^{n-1}(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 CM \Gamma(\alpha) \Gamma(\gamma))^{n-1} \cdot a^{(n-1)\gamma + n\alpha}}{\Gamma(1 + (n-1)\gamma + n\alpha)} \]
\[ + \frac{(8C^2 M \Gamma(\alpha) \Gamma(\gamma))^n \cdot a^{n\gamma + n\alpha}}{\Gamma(1 + n\gamma + n\alpha)} \right] \mu_C(D). \] (35)

Thanks to the well-known Stirling’s formula

\[ \Gamma(1 + s) = \sqrt{2\pi s} \left( \frac{s}{e} \right)^s e^\theta, \quad s > 0, \quad 0 < \theta < 1, \]

we get that when \( n \to \infty \), then

\[ \frac{(4CM \Gamma(\alpha))^n \cdot a^{n\alpha}}{\Gamma(1 + n\alpha)} = \frac{(4CM \Gamma(\alpha))^n \cdot a^{n\alpha}}{\sqrt{2\pi n\alpha} (\frac{n\alpha}{e})^{n\alpha} e^{\frac{\theta}{12n\alpha}}} \to 0, \]

\[ \frac{C_n^1(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma))^1 \cdot a^{\gamma + n\alpha}}{\Gamma(1 + \gamma + n\alpha)} \]
\[ = \frac{C_n^1(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma))^1 \cdot a^{\gamma + n\alpha}}{\sqrt{2\pi (\gamma + n\alpha)} (\frac{\gamma + n\alpha}{e})^{\gamma + n\alpha} e^{\frac{\theta}{12(\gamma + n\alpha)}}} \]
\[ \to 0, \]
\[ \ldots \]
\[ \ldots \]
\[ \frac{C_n^{n-1}(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma))^{n-1} \cdot a^{(n-1)\gamma + n\alpha}}{\Gamma(1 + (n-1)\gamma + n\alpha)} \]
\[ = \frac{C_n^{n-1}(4CM \Gamma(\alpha))^{n-1} \cdot (8C^2 M \Gamma(\alpha) \Gamma(\gamma))^{n-1} \cdot a^{(n-1)\gamma + n\alpha}}{\sqrt{2\pi ((n-1)\gamma + n\alpha)} (\frac{(n-1)\gamma + n\alpha}{e})^{(n-1)\gamma + n\alpha} e^{\frac{\theta}{12((n-1)\gamma + n\alpha)}}} \]
\[ \to 0 \]

and

\[ \frac{(8C^2 M \Gamma(\alpha) \Gamma(\gamma))^n \cdot a^{n\gamma + n\alpha}}{\Gamma(1 + n\gamma + n\alpha)} = \frac{(8C^2 M \Gamma(\alpha) \Gamma(\gamma))^n \cdot a^{n\gamma + n\alpha}}{\sqrt{2\pi (n\gamma + n\alpha)} (\frac{n\gamma + n\alpha}{e})^{n\gamma + n\alpha} e^{\frac{\theta}{12(n\gamma + n\alpha)}}} \]
Therefore, there must exist a positive integer $n_0$, which is large enough, such that
\[
\frac{(4CM\Gamma(\alpha))^{n_0} \cdot a^{n_0\alpha}}{\Gamma(1 + n_0\alpha)} + \cdots + \frac{(8C^2M\Gamma(\alpha)\Gamma(\gamma))^{n_0} \cdot a^{n_0\gamma + n_0\alpha}}{\Gamma(1 + n_0\gamma + n_0\alpha)} < 1.
\] (36)

Hence, from (35) and (36) we know that (22) is satisfied, which means that $Q : F \rightarrow F$ is a convex-power condensing operator. It follows from Lemma 9 that the operator $Q$ defined by (15) has at least one fixed point $u \in F$, which is just a mild solution of initial value problem (10). This completes the proof of Theorem 1. \qed

Proof of Theorem 2 From the proof of Theorem 1, we know that the mild solution of initial value problem (10) is equivalent to the fixed point of the operator $Q$ defined by (15). In what follows, we prove that there exists a positive constant $R$ such that the operator $Q$ maps the set $\Omega_R$ to $\Omega_R$. For any $u \in \Omega_R$ and a.e. $t \in I$, by (15), (12), Lemmas 2–4, the assumption (F1)* and Hölder inequality, we know that
\[
\| (Qu)(t) \| \leq \| A^{-1}(0)u_0 \| + \left\| \int_0^t \psi(t - \eta, \eta)U(\eta)g(u)d\eta \right\|
+ \left\| \int_0^t \psi(t - \eta, \eta)f(\eta, u(\eta), (Tu)(\eta), (Su_u)(\eta))d\eta \right\|
+ \left\| \int_0^t \int_0^\eta \psi(t - \eta, \eta)\psi(\eta, s)f(s, u(s), (Tu)(s), (Su)(s))dsd\eta \right\|
\leq C\|u_0\| + C^2\int_0^t (t - \eta)^{\alpha - 1}(1 + \eta^\gamma)\|u_0\|d\eta
+ C\int_0^t (t - \eta)^{\alpha - 1}\phi(\eta)\Phi(\|u\|)d\eta
+ C^2\int_0^t \int_0^\eta (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1}\phi(s)\Phi(\|u\|)dtd\eta
\leq C\|u_0\| + C^2\|u_0\|t^{\alpha}
+ \frac{1}{\alpha} + (t)^\gamma B(\alpha, \gamma + 1)
+ C\Phi(R)\int_0^t (t - \eta)^{\alpha - 1}\phi(\eta)d\eta
+ C^2\Phi(R)B(\alpha, \gamma)\int_0^t (t - \eta)^{\alpha + \gamma - 1}\phi(\eta)d\eta
\leq C\|u_0\|\delta_2 + C\Phi(R)\left( \int_0^t (t - \eta)^{\alpha - 1}d\eta \right)^{1 - \alpha_1}\left( \int_0^t \phi^\beta(\eta)d\eta \right)^{\beta}
+ C^2\Phi(R)B(\alpha, \gamma)\left( \int_0^t (t - \eta)^{\alpha + \gamma - 1}d\eta \right)^{1 - \beta}\left( \int_0^t \phi^\beta(\eta)d\eta \right)^{\beta}
\leq C\|u_0\|\delta_2 + C\delta_1\Phi(R)a^{\alpha - \beta}\|\phi\|_{L^\beta[0,a]} \leq R.
\] (37)
Therefore, from (37) we get that the operator \( Q : \Omega \rightarrow \Omega \). By adopting a completely similar method with which used in the proof of Theorem 1, we can prove that \( Q : \Omega \rightarrow \Omega \) is continuous and equicontinuous, and also \( Q \) is a convex-power condensing operator. By Lemma 9, we know that the operator \( Q \) defined by (15) has at least one fixed point \( u \in \Omega \), which is just a mild solution of initial value problem (10). This completes the proof of Theorem 2.

**Remark 3** From (23) and (27) of Theorem 1 we know that if we assume

\[
\frac{4CM\Gamma(\alpha)a^\alpha}{\Gamma(1+\alpha)} + \frac{8C^2M\Gamma(\alpha)\Gamma(\gamma)a^{\alpha+\gamma}}{\Gamma(1+\gamma+\alpha)} < 1
\]

directly, we can apply the famous Sadoveskii’s fixed point theorem to obtain the results of Theorems 1 and 2. However, from the above arguments, one can find that we do not need this redundant condition (38) by virtue of fixed point theorem with respect to convex-power condensing operator.

### 4 Application

In this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. As an application, we consider the following initial value problem of time fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition

\[
\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \kappa(x,t)\Delta u(x,t) &= \frac{\sin(\pi t)}{1+|u(x,t)|} + e^{-t} \sin \left( \int_0^t (t-s)u(x,s)ds \right) \\
&\quad + e^{-t} \cos \left( \int_0^1 e^{-|t-s|}u(x,s)ds \right), \quad x \in \Omega, \; t \in I, \\
u(x,0) &= 0, \quad x \in \partial \Omega, \quad t \in I, \\
u(x,0) &= (\kappa(x,0))^{-1}\phi(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \frac{\partial^\alpha u}{\partial t^\alpha} \) is the Caputo fractional order partial derivative of order \( \alpha \), \( 0 < \alpha \leq 1 \), \( I = [0, 1] \), \( \Delta \) is the Laplace operator, \( \Omega \subseteq \mathbb{R}^n \) is a bounded domain with a sufficiently smooth boundary \( \partial \Omega \), the coefficient of heat conductivity \( \kappa(x,t) \) is continuous on \( \Omega \times [0, 1] \) and it is uniformly Hölder continuous in \( t \), which means that for any \( t_1, t_2 \in I \), there exist a constant \( 0 < \gamma \leq 1 \) and a positive constant \( C \) independent of \( t_1 \) and \( t_2 \), such that

\[
|\kappa(x,t_2) - \kappa(x,t_1)| \leq C|t_2 - t_1|^\gamma, \quad x \in \Omega,
\]

and \( \phi \in L^2(\Omega) \).

Let \( E = L^2(\Omega) \) be a Banach space with the \( L^2 \)-norm \( \| \cdot \|_2 \). We define an operator \( A(t) \) in Banach space \( E \) by

\[
D(A) = H^2(\Omega) \cap H^1_0(\Omega), \quad A(t)u = -\kappa(x,t)\Delta u,
\]
where $H^2(\Omega)$ is the completion of the space $C^2(\Omega)$ with respect to the norm

$$\|u\|_{H^2(\Omega)} = \left( \int_\Omega \sum_{|\theta| \leq 2} |D^\theta u(x)|^2 \, dx \right)^{1/2},$$

$C^2(\Omega)$ is the set of all continuous functions defined on $\Omega$ which have continuous partial derivatives of order less than or equal to 2, $H^1_0(\Omega)$ is the completion of $C^1(\Omega)$ with respect to the norm $\|u\|_{H^1(\Omega)}$, and $C^1_0(\Omega)$ is the set of all functions $u \in C^1(\Omega)$ with compact supports on the domain $\Omega$. Then it is well known from [10] that $-A(s)$ generates an analytic semigroup $e^{-tA(s)}$ in $E$. By (40) and (41) one can easily verify that the linear operator $-A(t)$ satisfies the assumptions (A1) and (A2).

Further, for any $t \in [0, 1]$, we define

$$u(t) = u(\cdot, t), \quad K(t, s) = t - s \quad \text{for} \quad 0 \leq s \leq t \leq 1,$$

$$H(t, s) = e^{-|t-s|} \quad \text{for} \quad 0 \leq s, t \leq 1,$$

$$(Tu)(t) = \int_0^t K(t, s)u(\cdot, s)ds, \quad (Su)(t) = \int_0^1 H(t, s)u(\cdot, s)ds,$$

$$f(t, u(t), (Tu)(t), (Su)(t)) = \frac{\sin(\pi t)}{1 + \|u(\cdot, t)\|} + e^{-t} \sin((Tu)(t)) + e^{-t} \cos((Su)(t)),$$

$$A^{-1}(0) = (\kappa(\cdot, 0))^{-1}, \quad u_0 = \varphi(\cdot).$$

Then the initial value problem of time fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition (39) can be transformed into the abstract form of initial value problem to time fractional non-autonomous integro-differential evolution equation of mixed type (10).

**Theorem 3** The initial value problem of time fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition (39) has at least one mild solution $u \in C(\Omega \times [0, 1])$.

**Proof** By the definition of nonlinear term $f$ one can easily verify that the assumption (F1) is satisfied with

$$\psi_r(t) = \sqrt{\text{mes}\Omega}(\sin(\pi t) + 2e^{-t}), \quad \text{and} \quad \beta = \rho = 0.$$

From the definition of nonlinear term $f$, we know that $f(t, u, v, w)$ is Lipschitz continuous about the variables $u, v$ and $w$ with Lipschitz constants $k_u = 1$, $k_v = 1$ and $k_w = 1$, respectively. Therefore, by Lemma 5 (8) we get that the assumption (F2) is satisfied with positive constants

$$L_1 = L_2 = L_3 = 1.$$

From the fact $\rho = 0$ one can easily verify that the condition (11) holds. Therefore, all the assumptions of Theorem 1 are satisfied. Hence, the initial value problem of time
fractional non-autonomous partial differential equation with homogeneous Dirichlet boundary condition (39) has at least one mild solution $u \in C(\Omega \times [0, 1])$ due to Theorem 1. This completes the proof of Theorem 3. □

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References

1. Bajlekova, E.G.: Fractional Evolution Equations in Banach Spaces, Ph.D. thesis, Department of Mathematics, Eindhoven University of Technology (2001)
2. Chen, P., Li, Y.: Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions. Results Math. 63, 731–744 (2013)
3. Chen, P., Zhang, X., Li, Y.: Approximation technique for fractional evolution equations with nonlocal integral conditions. Mediterr. J. Math. 14, 226 (2017)
4. Chen, P., Zhang, X., Li, Y.: Fractional non-autonomous evolution equation with nonlocal conditions. J. Pseudodiffer. Oper. Appl. (2018). https://doi.org/10.1007/s11868-018-0257-9
5. Chen, P., Zhang, X., Li, Y.: A blowup alternative result for fractional nonautonomous evolution equation of Volterra type. Commun. Pure Appl. Anal. 17, 1975–1992 (2018)
6. Corduneanu, C.: Principles of Differential and Integral Equations. Allyn and Bacon, Boston (1971)
7. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)
8. El-Borai, M.M.: The fundamental solutions for fractional evolution equations of parabolic type. J. Appl. Math. Stoch. Anal. 3, 197–211 (2004)
9. El-Borai, M.M., El-Nadi, K.E., El-Akabawy, E.G.: On some fractional evolution equations. Comput. Math. Appl. 59, 1352–1355 (2010)
10. Friedman, A.: Partial Differential Equations. Holt, Rinehart and Winston, New York (1969)
11. Gou, H., Li, B.: Local and global existence of mild solution to impulsive fractional semilinear integro-differential equation with noncompact semigroup. Commun. Nonlinear Sci. Numer. Simul. 42, 204–214 (2017)
12. Guo, D.: Solutions of nonlinear integro-differential equations of mixed type in Banach spaces. J. Appl. Math. Simul. 2, 1–11 (1989)
13. Heinz, H.P.: On the behaviour of measure of noncompactness with respect to differentiation and integration of vector-valued functions. Nonlinear Anal. 7, 1351–1371 (1983)
14. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Math, vol. 840. Springer, New York (1981)
15. Lakshmikantham, V., Leela, S.: Nonlinear Differential Equations in Abstract Spaces. Pergamon Press, New York (1981)
16. Li, M., Chen, C., Li, F.B.: On fractional powers of generators of fractional resolvent families. J. Funct. Anal. 259, 2702–2726 (2010)
17. Li, K., Peng, J., Jia, J.: Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives. J. Funct. Anal. 263, 476–510 (2012)
18. Liu, L., Wu, C., Guo, F.: Existence theorems of global solutions of initial value problems for nonlinear integro-differential equations of mixed type in Banach spaces and applications. Comput. Math. Appl. 47, 13–22 (2004)
19. Liu, L., Guo, F., Wu, C., Wu, Y.: Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. J. Math. Anal. Appl. 309, 638–649 (2005)
20. Mei, Z., Peng, J., Zhang, Y.: An operator theoretical approach to Riemann–Liouville fractional Cauchy problem. Math. Nachr. 288, 784–797 (2015)
21. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, Berlin (1983)
22. Shi, H.B., Li, W.T., Sun, H.R.: Existence of mild solutions for abstract mixed type semilinear evolution equations. Turk. J. Math. 35, 457–472 (2011)
23. Shu, X., Shi, Y.: A study on the mild solution of impulsive fractional evolution equations. Appl. Math. Comput. 273, 465–476 (2016)
24. Sun, J., Zhang, X.: The fixed point theorem of convex-power condensing operator and applications to abstract semilinear evolution equations. Acta Math. Sin. 48, 439–446 (2005). (in Chinese)
25. Tanabe, H.: Functional Analytic Methods for Partial Differential Equations. Marcel Dekker, New York (1997)
26. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edn. Springer, New York (1997)
27. Wang, J., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal. Real World Appl. 12, 262–272 (2011)
28. Wang, J., Fečkan, M., Zhou, Y.: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Part. Differ. Equ. 8, 345–361 (2011)
29. Wang, R.N., Chen, D.H., Xiao, T.J.: Abstract fractional Cauchy problems with almost sectorial operators. J. Differ. Equ. 252, 202–235 (2012)
30. Wang, J., Zhou, Y., Fečkan, M.: Abstract Cauchy problem for fractional differential equations. Nonlinear Dyn. 74, 685–700 (2013)
31. Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. 59, 1063–1077 (2010)