Noise induced loss of entanglement

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The disentangling effect of repeated applications of the bit flip channel \((\mathbb{I} \otimes \sigma_x)\) on bipartite qubit systems is analyzed. It is found that the rate of loss of entanglement is not uniform over all states. The distillable entanglement of maximally entangled states decreases faster than that of less entangled states. The analysis is also generalized to noise channels of the form \(\hat{n} \cdot \hat{\sigma}\).

I. INTRODUCTION

The storage/transmission of classical data is subject to various noise processes that reduce the integrity of the data over time. One such noise process is the binary symmetric channel (Fig. 1), that flips a bit with a given probability \(1 - p\). There exist many successful strategies for dealing with this noise process [1].

![FIG. 1: The binary symmetric channel](http://kovidgoyal.cjb.net)

Entanglement is a quantum resource, essential to many applications such as teleportation, super-dense coding, etc. As such, the ability to combat noise during the storage of entanglement is essential. In this paper, we consider an instance of the binary symmetric channel, applied to bipartite qubit systems. We analyze the disentangling effect of this channel on singlet (maximally entangled) states. The choice of a qubit system is dictated by the existence of a mathematically tractable measure of entanglement for bipartite qubit systems [2].

II. THE QUANTUM BIT FLIP CHANNEL

The generalization of the symmetric bit flip channel to the case of a single qubit is straightforward. Choose the computational basis \(\{ |0\rangle, |1\rangle \}\) of the Hilbert space \(\mathcal{H}_2\). Let \(\rho\) be any density matrix acting on this space. Then the quantum bit flip channel can be defined as

\[
\rho' = p \rho + (1-p) \sigma_x \rho \sigma_x.
\]

In order to study the effect of this channel on entanglement, this definition needs to be extended for bipartite systems. We make the choice that only one of the two subsystems is affected by the noise. Then for \(\rho \in \mathcal{H}_2 \otimes \mathcal{H}_2\),

\[
\rho' = p \rho + (1-p) \mathcal{X}(\rho),
\]

\[
\mathcal{X}(\rho) := (\mathbb{I} \otimes \sigma_x) \rho (\mathbb{I} \otimes \sigma_x).
\]

Since \(\sigma_x\) is a completely positive map, \(\rho'\) is also a density matrix in \(\mathcal{H}_2 \otimes \mathcal{H}_2\). We are interested in the disentangling effect of this channel on the maximally entangled singlet state, defined as

\[
\rho_+ = \frac{1}{2} \sum_{i,j=0}^1 |i\rangle \langle j| \otimes |i\rangle \langle j|.
\]

After a single application of the channel, the resulting density matrix \(\rho_1\) has the form

\[
\rho_1 = p \rho_+ + (1-p) \mathcal{X}(\rho_+).
\]

The entanglement of this state should be a function of \(p\), which completely parameterizes the bit flip channel.

A. Entanglement of Formation

In order to calculate the entanglement of formation [2], the following definitions are required

\[
\rho = (\sigma_y \otimes \sigma_y) \rho^* \sigma_y \otimes \sigma_y,
\]

\[
\mathcal{C}(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
\]

where \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4\) are the the eigenvalues of the matrix \(\sqrt{\rho} \sqrt{\rho^*}\). Then the entanglement of formation \(E(\rho)\) is given by

\[
E(\rho) = h \left( \frac{1 + \sqrt{1 - \mathcal{C}(\rho)^2}}{2} \right),
\]

\[
h(x) = -x \log_2 x - (1-x) \log_2 (1-x).
\]

For \(\rho_1\) the concurrence is found to be

\[
\mathcal{C}(\rho_1) = 2 \left| p - \frac{1}{2} \right|.
\]

This gives an entanglement of formation

\[
E_F(\rho_1) = h \left( \frac{1}{2} + \sqrt{p(1-p)} \right).
\]

An outline of the calculations is presented in Section III. Fig. 2 shows how the entanglement varies as a function of \(p\).
from Eq. (4),

$$\rho = \rho_1$$

and

$$\sigma_\text{let state after}\text{order to answer it, we need to know the form of the sin-}$$

$$\text{plications of the channel have on the singlet state? In first few}\text{order to compare the loss of entanglement of the singlet state af-}$$

$$\text{r the fractional loss of entanglement of the singlet state af-}$$

$$\text{t least}} = 1 - h(p).$$

Evidently, $P_n$ is the sum of the even terms from the expansion of $(p + (1 - p))^n$.

$$\therefore P_n = \frac{(p + (1 - p))^n + (p - (1 - p))^n}{2}$$

$$= \frac{1}{2} + 2^{n-1} \left( \frac{p - 1}{2} \right)^n. \quad (13)$$

Now that we have obtained a general expression for $P_n$, we can calculate the entanglements as,

$$E_F(\rho_n) = h\left(\frac{1}{2} + \sqrt{P_n(1 - P_n)}\right)$$

$$E_D(\rho_n) = 1 - h(P_n). \quad (14)$$

Fig. 3 shows how the distillable entanglement decreases with $n$ for different values of $|p - \frac{1}{2}|$.

**FIG. 3: Entanglement of $\rho_{(n)}$ for $|p - \frac{1}{2}| = 0.05, 0.35, 0.42$.**

The curves have been smoothed by calculating Eq. (14) for non integral values of $n \in [0, 20]$.

**D. Combating the Disentanglement**

The form of the curves in Fig. 3 suggests that perhaps, states further along the curves lose entanglement slower than the singlet. In order to test this, first we define the fractional loss of entanglement the state $\rho_k$ after $r$ applications of the channel as

$$F(p, k, r) = -\frac{E(\rho_k) - E(\rho_{k+r})}{E(\rho_k)}; \quad (15)$$

where $E(\rho)$ is a measure of the entanglement of $\rho$. Then the fractional loss of entanglement of the singlet state after $r$ applications of the channel is given by $F(p, 0, r)$. In order to compare the loss of entanglement of the singlet state with that of $\rho_k$, define

$$R(p, k, r) = \frac{F(p, k, r)}{F(p, 0, r)}. \quad (16)$$

**FIG. 2: Entanglement of $\rho_{(1)}$ as a function of $p$**

**B. Distillable Entanglement**

While there doesn’t exist a general method for calculating the distillable entanglement of an arbitrary density matrix, we are fortunate in that $\rho_1$ can be written in the Bell diagonal form, as

$$\rho_1 = p |\Phi^+\rangle \langle \Phi^+| + (1 - p) |\Psi^+\rangle \langle \Psi^+|. \quad (10)$$

Using the one way hashing protocol [3] for distillation, it is possible to obtain a lower limit on the distillable entanglement of $1 - h(p)$. The distillable entanglement is bound above by the relative entropy of entanglement [4], which for $\rho_1$ is also [5], $1 - h(p)$. Combining the two bounds, we have

$$E_D(\rho_1) = 1 - h(p). \quad (11)$$

**C. Multiple Applications**

We now ask the question, what effect do multiple applications of the channel have on the singlet state? In order to answer it, we need to know the form of the singlet state after $n$ applications, denoted by $\rho_n$. Proceeding from Eq. (11),

$$\rho_2 = p\rho_1 + (1 - p)\mathcal{X}(\rho_1)$$

$$= (p^2 + (1 - p)^2)\rho_+ + (p(1 - p) + (1 - p)p)\mathcal{X}(\rho_+)$$

$$= P_2\rho_+ + (1 - P_2)\mathcal{X}(\rho_+);$$

$$P_2 = p^2 + (1 - p)^2.$$  

The identity $\sigma^2_\pi = I$ was used to arrive at Eq. (12). Thus $\rho_2$ has exactly the same form as $\rho_1$: repeated applications of the channel will not change this form. All that remains is to find an expression for $P_n$. By calculating $\rho_n$ for the first few $n$ explicitly, we have

$$P_0 = 1, \quad P_1 = p, \quad P_2 = p^2 + (1 - p)^2,$$

$$P_3 = p^3 + 3p(1 - p)^2.$$
Fig. 4 illustrates the behavior of $R(p, k, r)$. The most striking feature of the graphs is that the entanglement of formation and the distillable entanglement behave in a qualitatively different manner with regard to the rate of loss of entanglement of $\rho_k$. The rate of loss of entanglement of formation is higher for $\rho_k$ than for the singlet state. The reverse is true for the distillable entanglement.

It is the distillable entanglement that is of greater practical interest, and the fact that $\rho_k$ loses it slower than the singlet suggests a simple tactic to combat the disentangling action of this channel. Rather than storing entanglement as a few singlets, it should be stored as a larger number of less entangled states of the form of $\rho_k$. Since the fractional loss of entanglement for these states is less than for the singlet, there will be a smaller net loss of entanglement over time, provided that the distillable entanglement for these states is additive, that is

$$E_D(\rho_k^\otimes N) = N E_D(\rho_k). \quad (17)$$

This will ensure that the entanglement spread over $N$ copies of these states can be efficiently concentrated into singlet form again.

The second graph in Fig. 4 shows that the advantage obtained by storing the entanglement in dilute form is lost if the system is exposed to noise repeatedly. While this does impose a limit on the savings that can be made, if a sufficiently large $k$ is chosen and $r$ is bounded, there can still be significant gains.

The final graph shows, rather predictably, that the less severe the noise, the greater the gains that can be made, for a given $k$ and $r$.

### III. GENERALIZATION

Although most of the results in this paper are derived for the bit flip channel, a number of them hold for more general noise processes as well. In this section, we will analyze the general noise process

$$\rho_1 = p \rho_+ + (1-p)\mathcal{N}(\rho_+)$$

$$\mathcal{N}(\rho) : = (I \otimes \hat{n} \cdot \vec{\sigma})\rho(I \otimes \hat{n} \cdot \vec{\sigma})$$

$$= \frac{1}{2} \sum_{i,j=0}^{1} \sum_{a,b=1}^{3} n_a n_b |i \rangle \langle j | \otimes \sigma_a |i \rangle \langle j | \sigma_b. \quad (18)$$

where $\hat{n} \in \mathbb{R}^3$ is arbitrary. For $\hat{n} = (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$, this channel reduces to the bit flip, bit-phase flip and phase flip channels respectively [6].

#### A. Entanglement of Formation

Here we explicitly calculate the entanglement of formation of $\rho_1$, defined in Eq. (18). First we need to evaluate
\[ \hat{\rho}_1 = (\sigma_y \otimes \sigma_y)\rho_1(\sigma_y \otimes \sigma_y). \]  

The following identity \[7\], comes in handy

\[ (\mathbb{I} \otimes M)\rho_+(\mathbb{I} \otimes M^\dagger) = (M^T \otimes \mathbb{I})\rho_+(M^* \otimes \mathbb{I}); \]  

(19)

where \( M \) is any matrix. As a result of Eq. \[19\] we get

\[ (\sigma_y \otimes \sigma_y)\rho_+^*(\sigma_y \otimes \sigma_y) = (\sigma_y \otimes \mathbb{I})(\mathbb{I} \otimes \sigma_y)\rho_+^*(\mathbb{I} \otimes \sigma_y)(\sigma_y \otimes \mathbb{I}) = \rho_+^* = \rho_+. \]  

(20)

Define \( \hat{n}' = (n_x, -n_y, n_z) \). Then, for the second term in \( \rho_1 \)

\[ (\sigma_y \otimes \sigma_y)N(\rho_+)^*(\sigma_y \otimes \sigma_y) \]

\[ = (\sigma_y \otimes \sigma_y)(\mathbb{I} \otimes \hat{n}' \cdot \hat{\sigma})\rho_+(1 \otimes \hat{n}' \cdot \hat{\sigma})(\sigma_y \otimes \sigma_y) \]

\[ = \frac{1}{2} \sum_{a,b,i,j} \sigma_y |a\rangle \langle b| \sigma_y \otimes n'_i \sigma_y |a\rangle \langle b| n'_j \sigma_y \sigma_y \]

\[ = \frac{1}{2} \sum_{a,b,i,j} \sigma_y |a\rangle \langle b| \sigma_y \otimes (-n_i) \sigma_y |a\rangle \langle b| (-n_j) \sigma_y \sigma_j \]

\[ = (\mathbb{I} \otimes \hat{n} \cdot \hat{\sigma})(\sigma_y \otimes \sigma_y)\rho_+(\sigma_y \otimes \sigma_y)(\mathbb{I} \otimes \hat{n} \cdot \hat{\sigma}) = \mathcal{N}(\rho_+). \]  

(21)

Eq. \[20\] and Eq. \[21\] together imply that \( \hat{\rho}_1 = \rho_1 \). Thus in order to calculate the concurrence of \( \rho_1 \) we need to know only its eigenvalues. The matrix is

\[ \rho_1 = \frac{1-p}{2} \begin{bmatrix} 
 r + n_x^2 & (n_x - i n_y)n_z & (n_x + i n_y)n_z & r - n_z^2 \\
 (n_x - i n_y)n_z & n_z^2 + n_y^2 & (n_x - i n_y)^2 & -n_z^2 \\
 (n_x + i n_y)n_z & (n_x + i n_y)^2 & n_z^2 + n_y^2 & -n_z^2 \\
 r - n_z^2 & -n_z^2 & -(n_x - i n_y)n_z & r + n_z^2 
\end{bmatrix}; \]

\[ r = \frac{p}{1-p}. \]  

(22)

Amazingly enough, the eigenvalues of this matrix are \( \{p, 1-p, 0, 0\} \) giving a concurrence

\[ C = |2p-1|. \]  

(23)

This is the same result as was obtained for the bit flip channel in Eq. \[8\]. The fact that \( (\hat{n} \cdot \hat{\sigma})^2 = \mathbb{I} \) ensures that

\[ \rho_n = P_n \rho_+ + (1 - P_n) \mathcal{N}(\rho_+). \]  

(24)

It can easily be demonstrated that this \( P_n \) is the same as was obtained for the bit flip channel in Eq. \[8\]. Thus, the analysis carries over entirely for the \( \hat{n} \cdot \hat{\sigma} \) channel, in the case of entanglement of formation.

For the distillable entanglement, the situation is complicated by the absence of any method for calculating the entanglement for an arbitrary density matrix. However, for the special cases of \( \hat{n} = (1, 0, 0), (0, 1, 0) \) or \( (0, 0, 1) \) \( \rho_1 \) remains in Bell diagonal form. As a result its distillable entanglement is easily calculated to be \( 1 - h(p) \), as in Eq. \[11\].

IV. CONCLUSION

Noise reduces bipartite entanglement (of a singlet) exponentially, at a rate that depends on how non uniform the noise probability is. The greater the distance of the noise probability \( p \) from \( 1/2 \), the less severe the noise.

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