A Gauge-invariant Analysis of Magnetic Fields in General Relativistic Cosmology

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Abstract

We provide a fully general-relativistic treatment of cosmological perturbations in a universe permeated by a large-scale primordial magnetic field using the Ellis-Bruni gauge-invariant formalism. The exact non-linear equations for general relativistic magnetohydrodynamic evolution are derived. A number of applications are made: the behaviour of small perturbations to Friedmann universes are studied; a comparison is made with earlier Newtonian treatments of cosmological perturbations and some effects of inflationary expansion are examined.

1 Introduction

The study of cosmological magnetic fields has many motivations. Any primordial magnetic field could provide a seed field for dynamo amplification in disc galaxies, and fields of order $10^{-10}$ gauss would create significant magnetic fields in galaxy clusters through adiabatic compression alone, [1]-[8]. It would also introduce new ingredients into the standard, but necessarily uncertain, picture of the very early universe. Large-scale magnetic fields introduce anisotropies into the expansion dynamics. They would not survive an epoch of inflation although it is conceivable that large-scale fields and magnetic inhomogeneities with a constant curvature spectrum could be generated at the end of inflation. A number of proposals of this sort have been made, [9]-[13]. They involve speculative changes to the nature of the electromagnetic interactions in the universe at the time of inflation which, as yet, have no other motivation other than to allow magnetic field generation to take place. While such changes are not altogether unreasonable, and may reveal important aspects of fundamental physics, it is hard to test them independently. Ways of generating magnetic fields at cosmological phase transitions have also been explored by a number of authors, [14]-[16].

Any primordial magnetic field must be consistent with a number of astrophysical constraints upon its strength in the early universe. Since it provides an additional form of relativistic energy density at the epoch of cosmological nucleosynthesis, it increases the expansion rate of the universe; hence the neutron-proton freeze-out of weak interactions occurs at higher temperature, with a higher

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value, leading to an increase in the synthesised abundance of helium-4, [17]-[20]. Therefore helium-4 observations (extrapolated to zero metallicity) lead to an upper limit on the energy density of any cosmological magnetic field at the epoch of nucleosynthesis. If the field is spatially homogeneous over large scales then the COBE microwave background data provide the strongest limits on the magnetic field strength at last scattering of the microwave background photons, [21], [22]. The strength of this limit arises from the subtle evolution of cosmological magnetic fields in expanding universes during the radiation era. There is a non-linear coupling between the magnetic density and the anisotropic distortion of the expansion needed to support it. The anisotropic pressure created by the magnetic field dominates the evolution of the shear anisotropy and it decays far more slowly than if the pressure is isotropic, [23], [22]. This shear distortion determines the microwave background anisotropy directly. If the field is inhomogeneous, then the limits weaken, and on small scales the fluctuations can be dissipated. A number of discussions of the damping processes have recently appeared, [24], [26], and there is a possibility that a field of sufficient strength might have observable effects on the pattern of Doppler peaks expected in the microwave background on small scales, [25], but the field strengths necessary may not be compatible with the nucleosynthesis and COBE limits [21], [22].

In this paper we shall focus upon the problem of establishing a formalism for studying the general-relativistic evolution of magnetic inhomogeneities in an expanding universe. We do this by extending the gauge-invariant treatment of cosmological perturbations, introduced by Ellis and Bruni for perfect fluid cosmologies, to the case of electric and magnetic fields in a universe which also contains a perfect fluid. This formalism identifies a combination of gauge-invariant variables which specify the evolution of the perturbations in an invariant manner. It offers several advantages over the gauge-invariant approach of Bardeen, most notably by virtue of its transparent physical interpretation.

The study of small perturbations to an homogeneous and isotropic Friedmann-Robertson-Walker (FRW) universe is beset by subtleties of interpretation because of gauge dependencies. The problems arise because two different spacetimes are employed: a ‘smooth’ spacetime, \( W \), corresponding to the unperturbed background universe, and a ‘lumpy’ spacetime, \( W' \), that is ‘close’ to \( W \) both dynamically and kinematically, and represents the ‘perturbed’ universe we live in. These two spacetimes must be related in a way that permits the unique recovery of \( W \) from \( W' \) and vice-versa. This is impossible when the only information provided is that the ‘distance’ between them is small in some suitable sense [27]. Instead, one must establish a one-to-one map, \( \Phi : W \rightarrow W' \), so that every point, \( P \), in \( W' \) has a unique image, \( P' \), in \( W \) and vice-versa. Establishing such a correspondence, that is, a ‘gauge’ (or a diffeomorphism) between the two manifolds \( W \) and \( W' \), is equivalent to introducing a fictitious smooth background model into the universe. The perturbation in the density \( \mu \) at some point \( P \) of \( W \) will be \( \delta \mu = \mu - \overline{\mu} \), with \( \overline{\mu} \) the density at \( \overline{P} = \Phi^{-1}(P) \) in \( W' \). Clearly, there exists a relation between the gauge choice and the value of the perturbation itself. Any variation in the correspondence between \( W \) and \( W' \), which keeps the background model fixed, is called a gaug transformation, and it is to be distinguished from a coordinate transformation which merely relabels events in the two spacetimes. A gauge transformation not only induces a coordinate transformation, but also changes the point in \( W' \) that corresponds to a given point in \( W \). As a result, the perturbation value will depend on the gauge choice if the perturbed quantity is non-zero and its spacetime coordinates will depend on the background universe [28]. Even scalars which depend on spacetime coordinates, such as the density itself (i.e. \( \mu = \mu(t) \) in \( W' \)), are affected.
by changes in the gauge. This creates a problem, since by varying $\Phi$ one could give the perturbation any desired value at any particular point of $W$ \([27]\).

There exist natural choices of gauge (e.g. fundamental fluid flow lines in both $\overline{W}$ and $W$, hypersurfaces of simultaneity and homogeneity in $\overline{W}$). Unfortunately, some residual freedom remains, which is enough to create the aforementioned arbitrariness in the perturbation value. We could deal with this gauge freedom simply by keeping track of its consequences, but this is rather inefficient. Alternatively, we could specify the gauge completely at the beginning and perform all calculations in the chosen gauge. In practice however, difficulties emerge because different gauges are best suited for different applications. A third approach, which side-steps the gauge problem altogether, is to tackle cosmological perturbations by means of quantities that are entirely independent of the way the two spacetimes $\overline{W}$ and $W$ are associated with each other.

A covariant gauge-invariant analysis of density fluctuations was introduced by Ellis and Bruni, and initially applied to a dust-dominated almost-Friedmann universe \([27]\). Their geometrical approach is simple and physically transparent. In contrast to the standard approach, which compares the evolution of the perturbed quantities along the worldlines of two observers, one in $\overline{W}$ and the other in $W$, brought together via some specific gauge, the covariant gauge-invariant technique compares the evolutions of neighbouring observers in the same universe, $W$. It provides exact, fully non-linear, propagation equations for all variables, which can be linearized about a “variety” of background models. This formalism has already been extended to the study of density irregularities within almost-FRW spacetimes with pressure \([29]\), \([30]\), or with a multi-component fluid \([31]\), as well as to an almost-Bianchi type I universe \([32]\). Here, we shall develop the covariant gauge-invariant technique for a universe that contains a perfect fluid and large-scale magnetic and electric fields. This provides, for the first time, a fully general relativistic treatment of electromagnetic perturbations for use in cosmological investigations.

The outline of our analysis is as follows: in sections 2 and 3 the covariant formalism is introduced. In section 4, the key gauge-invariant variables are defined. Three of them, representing spatial changes in the energy density, the pressure, and the expansion within an almost FRW universe, were first introduced in \([27]\). The fourth gauge-invariant variable is new: it determines variations in the magnetic field between neighbouring fundamental observers in almost Friedman-nian space-times. The medium, a single perfect fluid of infinite conductivity, is specified in section 5, together with the relevant equations that characterize its behaviour. In section 6, Einstein’s field equations are employed to derive the fully non-linear exact propagation formulae for the basic kinematic quantities and for all the gauge-invariant variables. Some aspects of the spatial geometry are briefly discussed in section 7, and in section 8 we linearize the general equations about a FRW universe containing a homogeneous magnetic field. Such an assumption is a valid approximation provided that the field is “weak”, with an energy density much smaller than that of the isotropic perfect fluid, which does not destroy the model’s isotropy. Our results are compared with earlier non-relativistic treatments by Ruzmaikina and Ruzmaikin \([33]\) and by Wasserman \([34]\). When the pressure terms are removed from our equations, the standard results of the Newtonian approach appear naturally. We consider several particular applications of the formalism: the case of a dust universe; the effect of a magnetic field on a period of de Sitter inflation; and the growth of isocurvature perturbations.
2 Spacetime Splitting

Assume that there exists a well defined average velocity vector in the universe. As a result, spacetime is supplied with a preferred vector field \( u_i \) corresponding to a congruence of worldlines, known as the fundamental fluid-flow lines, that carry a family of special observers, namely the fundamental observers.\(^1\) As usual, the velocity 4-vector is normalized so that \( u_i u^i = -c^2 \).

The existence of the timelike vector field \( u_i \), defined by the motion of the matter in the universe, generates a unique splitting of spacetime. To every observer there corresponds, at any instant, a tangent spacelike three-dimensional hypersurface \( \Sigma_\perp \), orthogonal to \( u_i \). Such 3-surfaces are the observers’ instantaneous rest-spaces and they are generally different from each other. All these 3-spaces mesh together to create a single hyperplane, the common rest-space for all the fundamental observers, only in a non-rotating universe.

The metric of \( \Sigma_\perp \) is provided by a second-order symmetric spacelike tensor, namely the observer’s “projection tensor”, defined by

\[
    h_{ij} = g_{ij} + \frac{1}{c^2} u_i u_j, \quad (1)
\]

and satisfying

\[
    h^j_i = h^k_i h^j_k, \quad (2)
\]

\[
    h_{i}^i = 3. \quad (3)
\]

The projection tensor, together with the velocity 4-vector, decompose tensor fields and tensor equations into their spatial and temporal parts. They split the covariant derivative of \( u_i \) into irreducible basic kinematic quantities, \( \sigma_{ij}, \omega_{ij}, \Theta \),

\[
    u_{i;j} = \sigma_{ij} + \omega_{ij} + \frac{\Theta}{3} h_{ij} - \frac{1}{c^2} \dot{u}_i u_j, \quad (4)
\]

where \( \sigma_{ij} = h^{(i} k h^{j)} q u_{k;q} - \Theta h_{ij}/3 \), \( \omega_{ij} = h^{[i} k h^{j]} q u_{k;q} \), \( \Theta = u_{i ;i} \) and \( \dot{u}_i = u_{i;j} u^j \) are respectively the shear tensor, the vorticity tensor, the expansion scalar and the 4-acceleration. The first three terms in (4) comprise the second order tensor \( v_{ij} = h^{k} i h^{j} q u_{k;q} \), which in turn determines the relative spatial velocity between neighbouring worldlines. The expansion scalar is used to introduce a representative length scale \( (S) \) along the observer’s worldline. In particular, we define

\[
    \frac{\dot{S}}{S} = \frac{\Theta}{3}, \quad (5)
\]

for the volume expansion.

\(^1\)Hereafter, an observer will always be a fundamental one unless stated otherwise.
The Weyl tensor, $C_{ijkl}$, is decomposed into a pair of symmetric traceless and completely spacelike second-order tensors, $E_{ij}$ and $H_{ij}$, respectively known as the “electric” and the “magnetic” parts of $C_{ijkl}$ [37, 38].

$$E_{ij} = \frac{1}{c^2} C_{ikjq} u^k u^q,$$  \hspace{1cm} (6)

$$H_{ij} = \frac{1}{2c^2} \eta_{kp} C_{kqjs} u^p u^s.$$

(7)

where, $\eta_{ijkl}$ is the covariant permutation tensor of the spacetime.

3 The Electromagnetic Field

3.1 The Electromagnetic Field Tensor

The electromagnetic field is represented by the second-order antisymmetric Maxwell tensor $F_{ij}$, satisfying

$$F_{[ij;k]} = F_{ij;k} + F_{jk;i} + F_{ki;j} = 0.$$  \hspace{1cm} (8)

The components of $F_{ij}$ are derived from a potential, $V_i$, according to

$$F_{ij} = V_{ji} - V_{ij} = \frac{\partial V_j}{\partial x^i} - \frac{\partial V_i}{\partial x^j},$$

(9)

where $V_i = (V_\mu, -\Psi/c)$ are the covariant components of the 4-potential relative to a freely-falling frame. The quantities $V_\mu$ and $\Psi$ are the vector and the scalar potentials respectively.

3.2 The Electromagnetic Field Components

As seen by an observer moving with 4-velocity $u_i$, the electromagnetic field tensor decomposes into an electric ($E_i$) and a magnetic ($H_i$) part, defined by [36],

$$E_i = F_{ij} u^j = -F_{ij} u^j,$$

(10)

and

$$H_i = \frac{1}{2c} \eta_{ijkl} u^l F^{kj},$$

(11)

respectively. From definitions (10), (11) and the total skewness of $F_{ij}$ and $\eta_{ijkl}$, we obtain
\[ E_i u^i = 0, \]  
\[ H_i u^i = 0, \]

to ensure that both \( E_i \) and \( H_i \) are 3-vectors in the observer's rest space.

The projection of \( F_{ij} \) onto the 3-surface \( \Sigma_\perp \), the observer’s instantaneous rest space, provides the decomposition

\[ F_{ij} = \frac{1}{c^2} (u_i E_j - E_i u_j) - \frac{1}{c} \eta_{ijkq} u^k H^q, \]

and shows how \( E_i \) and \( H_i \) completely determine \( F_{ij} \).

### 3.3 Maxwell’s Equations

The first of Maxwell's equations has the form

\[ F_{ij}{}^{;j} = \frac{1}{c} J^i, \]

where \( J_i \) is the 4-current which generates the electromagnetic field and obeys the conservation law \( J_i{}^{;i} = 0 \). The second equation,

\[ F_{[ij; k]} = 0, \]

is a direct consequence of the existence of the 4-potential.

As measured by a fundamental observer equipped with the projection tensor \( h_{ij} \), equations (13) and (16) decompose into temporal and spatial parts as

\[ E_i{}^{;i} + 2 \omega^i H_i - \frac{1}{c^2} E^i \dot{u}_i = \epsilon c, \]

\[ \left( \sigma^i_j + \omega^j - \frac{2\Theta}{3} h^i_j \right) E^j + \frac{1}{c} \eta^{ijkq} u^k H_q - c \eta^{ijkq} u_j H_{k; q} - c \chi^j J^i = h^i_j \dot{E}^j, \]

\[ H_i{}^{;i} - \frac{2}{c^2} \omega^i E_i - \frac{1}{c^2} H^i \dot{u}_i = 0, \]

\[ \left( \sigma^i_j + \omega^j - \frac{2\Theta}{3} h^i_j \right) H^j - \frac{1}{c^2} \eta^{ijkq} u^k u_j E_q + \frac{1}{c^2} \eta^{ijkq} u_j E_{k; q} = h^i_j \dot{H}^j, \]

where the vorticity vector \( (\omega_i) \), and the charge density \( (\epsilon = -J^i u_i/c^2) \) are new variables. Note that \( \omega_{ij} \) and \( \omega_i \) are related by the formula
\[ \omega_{ij} = \frac{1}{c} \eta_{ijkq} \omega^k u^q, \quad (21) \]

which ensures that \( \omega_i \omega^{ij} = 0. \)

### 3.4 The Energy-Momentum Tensor of the Electromagnetic Field

The electromagnetic energy-momentum tensor \( T_{em}^{ij} \) is a symmetric and trace-free second-order tensor of the following general form

\[ T_{em}^{ij} = F^{ki} F_{kj} - \frac{1}{4} g^{ij} F_{kq} F^{kq}. \quad (22) \]

Relative to a fundamental observers, \( T_{em}^{ij} \) splits as [36],

\[ T_{em}^{ij} = \frac{1}{2c^2} \left( \frac{E^2}{c^2} + H^2 \right) u^i u^j + \frac{1}{6} \left( \frac{E^2}{c^2} + H^2 \right) h^{ij} + \frac{2}{c^3} u^{(i} \eta_{j)kq} u_k E_q H_s + M^{ij}, \quad (23) \]

where \( E^2 = E_i E^i \) and \( H^2 = H_i H^i \) are the magnitudes of the field’s electric and the magnetic components respectively; \( M_{ij} \) is a traceless and completely spacelike symmetric tensor given by

\[ M^{ij} = \frac{1}{3} \left( \frac{E^2}{c^2} + H^2 \right) h^{ij} - \frac{1}{c^2} E^i E^j - H^i H^j. \quad (24) \]

The expression (23) allows a direct comparison to be made between \( T_{em}^{ij} \) and the energy-momentum tensor, \( *T_{ij} \), of a general imperfect fluid possessing viscous and heat conduction contributions:

\[ *T_{ij} = *\mu u^i u^j + *p h^{ij} + \frac{2}{c^2} *q^{(i} u^j) + *\pi^{ij}. \quad (25) \]

The resulting correspondence, which provides a “fluid description” for the electromagnetic field, is

\[ \mu_{em} = \frac{1}{2c^2} \left( \frac{E^2}{c^2} + H^2 \right), \quad (26) \]
\[ p_{em} = \frac{1}{6} \left( \frac{E^2}{c^2} + H^2 \right), \quad (27) \]
\[ q_{em}^i = \frac{1}{c} \eta^{ijkq} u_j E_k H_q, \quad (28) \]
\[ \pi_{em}^{ij} = M^{ij}, \quad (29) \]

and leads to the familiar equation of state of radiation \( p_\gamma = \mu_\gamma c^2/3 \). The last expression suggests
that $M_{ij}$ contributes an anisotropic electromagnetic pressure.

4 The Key Gauge-Invariant Variables

4.1 The $D_i$, $Y_i$ and $Z_i$ Spatial Gradients

In their covariant and gauge-invariant study of cosmological perturbations Ellis and Bruni \[27\] describe spatial variations in the energy density ($\mu c^2$), the pressure ($p$) and the expansion ($\Theta$) by projecting their gradients onto the instantaneous rest space of an observer comoving with the expanding fluid. Assuming that the unperturbed background universe is represented by a FRW spacetime, they consider the following basic variables

$$X_i = \kappa h^j_{\mu} \mu_{j\mu} c^2 = \kappa (3) \nabla_i \mu c^2,$$

$$Y_i = \kappa h^j_{\mu} p_{j\mu} = \kappa (3) \nabla_i p,$$

$$Z_i = h^j_{\mu} \Theta_{ij} = (3) \nabla_i \Theta,$$

where $\kappa = 8\pi G/c^4$ is the Einstein gravitational constant. All three vectors $X_i$, $Y_i$ and $Z_i$ vanish in a FRW model, thus ensuring their gauge-invariance.\[2\] Ellis and Bruni then define the following “comoving” variables

$$D_i = \frac{SX_i}{\kappa \mu c^2},$$

namely the comoving fractional orthogonal spatial gradient of the energy density, and

$$Z_i = SZ_i,$$

which is the comoving orthogonal spatial gradient of the expansion. Besides being invariant under gauge transformations, $D_i$ and $Z_i$ also describe the spatial variations of $\mu c^2$ and $\Theta$ respectively, within a perturbed FRW universe \[27\].

4.2 A Gauge-Invariant Variable for the Magnetic Field

The assumption of a large-scale primordial magnetic field requires that the fictitious background universe must include some degree of anisotropy and a spatially homogeneous non-rotating Bianchi-I spacetime provides the simplest example of such a universe. However, later we shall primarily

\[2\]The simplest gauge-invariant quantities are scalars that are constant in the background, or tensors which are zero there. The only other possibility is a tensor written as a linear combination of products of Kronecker deltas with constant coefficients (Stewart and Walker lemma \[38\]).
consider the case of an unperturbed FRW spacetime. This is an acceptable simplification provided that the field is too weak to affect the model’s isotropy and allows us compare our approach with the existing (Newtonian) results directly.

Although Ellis and Bruni restricted their study to almost-FRW geometries, the gauge-invariance of their key variables also holds in any spatially homogeneous, non-rotating background spacetime, such as a Bianchi-I universe. Furthermore, the same variables can describe spatial gradients in μc², p and Θ within an almost Bianchi-I model, as it has been argued by Dunsby [32].

Let us consider a spatially homogeneous universe permeated by a large-scale magnetic field (i.e. \( H_i = H_i(t) \)) with no accompanying electric field (i.e. \( E_i \equiv 0 \)). Following Ellis and Bruni [27], we define the second-order tensor

\[
\mathcal{H}_{ij} = \kappa h^k h^q H_{kq} = \kappa (3) \nabla_j H_i, \tag{35}
\]

namely the orthogonal spatial gradient of the magnetic field. When the vorticity is zero, \( \mathcal{H}_{ij} \) vanishes (see Appendix A.1), and so provides a new gauge-invariant variable. Moreover, the quantity

\[
\mathcal{M}_{ij} = S \mathcal{H}_{ij}, \tag{36}
\]

namely the comoving orthogonal spatial gradient of the magnetic field, describes the spatial variations of \( H_i \) between neighbouring world lines in a perturbed FRW universe (see Appendix A.2). Following Dunsby [32], we can extend this result to a nearly Bianchi-I spacetime if required.

The quantity \( \mathcal{H}_{ij} \) is a completely spacelike and traceless tensor

\[
u^i \mathcal{H}_{ij} = \mathcal{H}_{ij} u^j = 0, \tag{37}
\]

\[
\mathcal{H}_i^i = 0, \tag{38}
\]

where the last result is derived from (19) under the assumption of a zero electric field.

5 Specifying the Medium

5.1 The Case of Infinite Conductivity

In order to specify the material content of the universe, we consider the covariant form of Ohm’s law [39],

\[
J_i + \frac{1}{c^2} J^j u_j u_i = \frac{1}{c} \sigma E_i, \tag{39}
\]

where \( u_i \) is the fluid velocity and \( \sigma \) represents the conductivity of the medium. Projecting onto the instantaneous rest space of a fundamental observer, we obtain
Thus, non-zero spatial current densities (i.e. \( h^i_j J_j \neq 0 \)), are compatible with a vanishing electric field as long as the conductivity of the medium is infinite (i.e. \( \sigma \rightarrow \infty \)). Under this assumption, formulae (17)-(20) become

\[
2 \omega^i H_i = \epsilon c, \quad (41)
\]

\[
\eta^{ijkl} u_j \left( \dot{u}_k H_q - c^2 H_{kq} \right) = c^2 h^i_j J^j, \quad (42)
\]

\[
H^i_{;i} - \frac{1}{c^2} H^i \dot{u}_i = 0, \quad (43)
\]

\[
\left( \sigma^i_j + \omega^i_j - \frac{2\Theta}{3} h^i_j \right) H^j = h^i_j \dot{H}^j. \quad (44)
\]

Equation (43) can be rearranged as

\[
h^i_j H^i_{;j} = (3) \nabla_i H^i = 0, \quad (45)
\]

to provide the familiar 'vanishing 3-divergence' of the magnetic field. Some useful expressions can be obtained from (44); in particular,

\[
h^i_j \left( S^2 H^j \right) = \left( \sigma^i_j + \omega^i_j \right) S^2 H^j, \quad (46)
\]

is the covariant analogue of the 'induction' equation and verifies that within an exactly FRW universe the magnetic field decays adiabatically as the inverse square of the scale factor. An additional formula, which will be used later to simplify the energy-density conservation law, is obtained after contracting (44) with \( H_i \)

\[
\sigma_{ij} H^i H^j = \frac{2\Theta H^2}{3} + \frac{(H^2)^2}{2}. \quad (47)
\]

The introduction of a perfectly conducting medium, as opposed to the assumption of a pure magnetic field with no electric field and zero spatial currents, is important in this approach. As we shall see, it allows perturbations in the energy density of the fluid to be coupled with magnetic irregularities in a straightforward and natural way (see further discussion in Appendix B).

### 5.2 Magnetized Perfect Fluid with Infinite Conductivity

In a medium with infinite conductivity (i.e. \( E_i = 0 \)), the electromagnetic energy-momentum tensor (see [23]) takes the form

\[
h^i_j J_j = \frac{1}{c} \sigma E_i. \quad (40)
\]
\[ T^{ij}_{em} = \frac{H^2}{2c^2} u^i u^j + \frac{H^2}{6} h^{ij} + M^{ij}, \]  

(48)

where

\[ M^{ij} = \frac{H^2}{3} h^{ij} - H^i H^j. \]  

(49)

The energy-momentum tensor of a single perfect fluid (i.e. zero energy-flux and no anisotropic stresses) is

\[ T^{ij}_m = \mu u^i u^j + ph^{ij}. \]  

(50)

Thus, the energy-momentum tensor that describes a single magnetized perfect fluid of infinite conductivity has the form

\[ T^{ij} = \left( \mu + \frac{H^2}{2c^2} \right) u^i u^j + \left( p + \frac{H^2}{6} \right) h^{ij} + M^{ij}, \]  

(51)

with trace \( T = 3p - \mu c^2 \).

A comparison between (51) and (25), the stress tensor of an imperfect fluid, shows that a single magnetized perfect fluid of infinite conductivity can be represented as an imperfect fluid with the following properties

\[ \* \mu = \mu + \frac{H^2}{2c^2}, \]  

(52)

\[ \* p = p + \frac{H^2}{6}, \]  

(53)

\[ \* q = 0, \]  

(54)

\[ \* \pi^{ij} = M^{ij} = \frac{H^2}{3} h^{ij} - H^i H^j. \]  

(55)

We can now derive the conservation laws that characterize the medium. We start by setting the divergence of (51) equal to zero. The resulting formula is then split into a temporal and a spatial part, respectively expressing the conservation of the energy and of the momentum densities. More precisely, contracting with the observer’s 4-velocity and using (47) we find

\[ \dot{\mu} c^2 + \left( \nu c^2 + p \right) \Theta = 0, \]  

(56)

which coincides with the energy-density conservation law for a single non-magnetized perfect fluid. The conservation of the momentum-density is obtained after projecting (51) onto the observer’s instantaneous rest space,
\( (\mu + \frac{p}{c^2} + \frac{H^2}{c^2}) \dot{u}^i + h^{ij} \left( p + \frac{H^2}{2} \right)_j - h^{ij}h^j_kH^k - H^iH^j = 0, \) 

(57)

and can be rewritten, in order to involve the spatial gradients \( Y_i \) and \( h_{ij} \), as

\[
\kappa \left( \mu + \frac{p}{c^2} + \frac{2H^2}{3c^2} \right) \dot{u}_i + Y_i - 2h_{ij}H^j + \frac{\kappa}{c^2} \ddot{M}_{ji} = 0.
\]

(58)

Equation (57) (or (58)) naturally connects spatial fluctuations in the magnetic field, with the acceleration and gradients in the pressure, and subsequently with spatial variations in the energy density (see also Appendix B).

### 6 The General Propagation Equations

#### 6.1 The Field Equations

The general form of Einstein’s field equations is

\[
R^{ij} = \kappa T^{ij} - \frac{\kappa T}{2} g^{ij} + \Lambda g^{ij},
\]

(59)

where \( R^{ij} = R^{ij}_{jk} \) is the Ricci tensor, \( g^{ij} \) is the spacetime metric and \( \Lambda \) is the cosmological constant. The Ricci scalar \( (R = R^{ii}) \) and the trace \( (T) \) of the energy-momentum tensor, that describes the material content, are related by

\[
R = 4\Lambda - \kappa T.
\]

(60)

Using the observer’s projection tensor, the Ricci tensor splits as

\[
R^{ij} = h^{i}_k h^{j}_q R^{kq} - \frac{1}{c^2} \left( h^{i}_q R^{kq} u_k u^j + u^i u_k R^{kq} h^j_q \right) + \frac{1}{c^4} R^{kq} u^i u^j u_k u^j,
\]

(61)

which shows that the decomposition is entirely determined by the sums \( h^{i}_k h^{j}_q R^{kq} \), \( h^{i}_j R^{jk} u_k \) and \( R^{ij} u_i u_j \).

When dealing with a single perfect fluid of infinite conductivity, permeated by a large-scale magnetic field, \( T_{ij} \) is given by (51) and consequently we find

\[
h^{i}_k h^{j}_q R^{kq} = \left( \frac{\kappa}{2} \left( \mu c^2 - p + \frac{H^2}{3} \right) + \Lambda \right) h^{ij} + \kappa M^{ij},
\]

(62)

\[
h^{i}_j R^{jk} u_k = 0,
\]

(63)

\[
R^{ij} u_i u_j = \frac{\kappa c^2}{2} \left( \mu c^2 + 3p + H^2 \right) - \Lambda c^2.
\]

(64)
Thus, the Ricci tensor associated with our cosmological model, decomposes into purely spatial and temporal parts only,
\[ R^{ij} = \left( \frac{\kappa}{2} \left( \mu c^2 - p + \frac{H^2}{3} \right) + \Lambda \right) h^{ij} + \kappa M^{ij} + \left( \frac{\kappa}{2} \left( \mu + \frac{3p}{c^2} + \frac{H^2}{c^2} \right) - \frac{\Lambda}{c^2} \right) u^i u^j, \]  
(65)
while the Ricci scalar becomes
\[ R = \kappa \left( \mu c^2 - 3p \right) + 4\Lambda. \]  
(66)

These equations allow us to derive the formulae that determine the time-evolution of the expansion, the shear and the vorticity.

6.2 Propagation of the Kinematic Quantities

The evolution of basic kinematic variables such as the expansion scalar \( (\Theta) \), the shear tensor \( (\sigma_{ij}) \) and the vorticity vector \( (\omega_i) \), along the world line of a fundamental observer, is governed by their respective propagation equations. These are derived from the following formula, describing the evolution of the relative velocity between neighbouring fundamental fluid-flow lines
\[ h^{ik} h^{jq} \left( \dot{u}^k + \frac{1}{c^2} \dot{u}^k \dot{u}_k - (3) \nabla_j \dot{u}_i + v_{ik} v_j^k + R_{ikjq} u^k u^q \right) = 0. \]  
(67)
Contracting (67), recalling that \( v_{ij} = \sigma_{ij} + \omega_{ij} + \Theta h_{ij}/3 \), and using (64), we find that under the influence of a cosmological magnetic field Raychaudhuri’s relation becomes
\[ \dot{\Theta} - A + 2 \left( \sigma^2 + \omega^2 \right) + \frac{\Theta^2}{3} + \frac{\kappa c^2}{2} \left( \mu c^2 + 3p + H^2 \right) - \Lambda c^2 = 0, \]  
(68)
where \( A = \dot{u}^i \dot{u}_i \), \( \sigma^2 = \sigma_{ij} \sigma_{ij}/2 \) and \( \omega^2 = \omega_{ij} \omega_{ij}/2 \) by definition.

The shear propagation formula is obtained by taking the symmetric, trace-free, part of (67). In order to derive the final expression, we employ (68), the definition of the Weyl tensor along with equations (62), (64) and (66). The outcome is
\[ \dot{\sigma}_{ij} - A - 2 \left( \sigma^2 - \omega^2 \right) + \frac{\Theta^2}{3} + \frac{\kappa c^2}{2} \left( \mu c^2 + 3p + H^2 \right) - \Lambda c^2 = 0, \]  
(69)
where \( E_{ij} \) is the electric part of the Weyl tensor and \( M_{ij} \) is given by (49). In deriving (69) we have used the result \( \omega_{ij} \omega^k_j = \omega_{ij} \omega^j_k - \omega^2 h_{ij} \), which relates \( \omega^2 \) to the vorticity tensor and vector. Comparing (69) with the corresponding formula in the case of a non-magnetized imperfect fluid (see [36]), we notice again that the distortion generated by the field is analogous to that induced by anisotropic stresses. The proportional to the projection tensor term, simply subtracts off the trace.
Finally, the skew part of (67), provides the equation governing the rate of change of the vorticity vector

\[ h^i_j \left( S^2 \omega^j \right) = \frac{S^2}{2c} \eta^{ijkq} u_j \dot{u}_{kq} + S^2 \omega^j \sigma^j, \]  

(70)

where, \( \omega^i = \eta^{ijkq} u_j \omega_{kq}/2c \). Notice that the magnetic effects are felt indirectly through the shear and the acceleration.

### 6.3 Propagation Equations for the Key Variables

From definitions (30)-(34) we can derive the propagation formulae for the various spatial gradients. The most important equation determines the evolution of the comoving fractional orthogonal spatial energy-density gradient (\( D_i \))

\[ h^i_j \dot{D}_j = -\frac{p \Theta}{\mu_c^2} D_i - D_j \left( \sigma^j_i + \omega^j_i \right) - \left( 1 + \frac{p}{\mu_c^2} \right) Z_i - \frac{2 \Theta}{\kappa \mu_c^2} M_{[ij]} H^j + \]

\[ \frac{2 S \Theta H^2}{3 \mu_c^4} \dot{u}_i + \frac{S \Theta}{\mu_c^4} \dot{u}^j M_{ji}. \]  

(71)

It is obtained by taking the time derivative of (33) and then using the conservation laws (56) and (58). Notice that the field gradients are already connected with those in the energy density, unlike the Newtonian approach, where the coupling occurs at the next level. This coupling arises from the presence of the quantity \( \dot{u}_i \), which will then bring \( D_i \) and \( M_{ij} \) together via the momentum-density conservation law.

The vector \( Z_i \), the comoving orthogonal spatial gradient of the expansion, has a rate of change determined by the following propagation formula,

\[ h^i_j \dot{Z}_j = -\frac{2 \Theta}{3} Z_i - Z_j \left( \sigma^j_i + \omega^j_i \right) - \kappa \mu_c^4 \left( \frac{1}{2} D_i + \frac{1}{\kappa \mu_c^2} M_{ji} H^j \right) - 3 c^2 M_{[ij]} H^j + \]

\[ S R \dot{u}_i + \frac{3 \kappa S}{2} \dot{u}^j M_{ji} + S A_i - 2 S (3) \nabla_i \left( \sigma^2 - \omega^2 \right), \]  

(72)

where \( A_i = h^j_i A_{ij} = (3) \nabla_i A \), and

\[ R = \left( \kappa \left( \mu_c^2 + \frac{H^2}{2} \right) - \frac{\Theta}{3 c^2} + \frac{\sigma^2}{c^2} - \frac{\omega^2}{c^2} + A \right) + \frac{A}{c^2} - \frac{3}{c^2} \left( \sigma^2 - \omega^2 \right), \]  

(73)

with the quantity in brackets representing, as we shall see in section 7, the 3-Ricci scalar of the observer’s instantaneous rest space. To derive (72) we start with the time derivative of (34), and then use (58), together with (58).
The third key variable is the comoving orthogonal spatial gradient of the magnetic field \( \mathcal{M}_{ij} \). In order to derive the propagation formula for \( \mathcal{M}_{ij} \), we first split the covariant derivative of the field with respect to irreducible kinematic quantities, employing the observer’s projection tensor and equation (44) to obtain,

\[
H_{ij} = \frac{1}{\kappa} H_{ij} - \frac{2}{c^2} H_k \sigma_{[i}^k u_{j]} + \frac{2}{c^2} H_k \omega_{(i}^k u_{j)} + \frac{\Theta}{3c^2} \left( 2H_i u_j + u_i H_j \right) - \frac{1}{c^4} \dot{u}^k H_k u_i u_j. \tag{74}
\]

The time evolution of the spatial gradient of the magnetic field can now be obtained by applying the Ricci identity to the field vector, in connection with (74). The outcome,

\[
S^{-2} h_i^k h_j^q \left( S^2 \mathcal{M}_{kq} \right) = -\mathcal{M}_{ik} \left( \sigma_{kj}^i + \omega_{kj}^i \right) + \left( \sigma_{ij}^i + \omega_{ij}^i \right) \mathcal{M}_{kj} - \frac{2\kappa}{3} H_i Z_j - \frac{2\kappa S}{c^2} H_k \omega_{(i}^k \dot{u}_{j)} + \frac{2\kappa S}{c^2} H_k \sigma_{[i}^k \dot{u}_{j]} + \frac{\kappa S H^k(3)}{c^2} \nabla_j \left( \sigma_{ik} + \omega_{ik} \right) - \frac{\kappa \Theta S}{3c^2} \left( 2H_i \dot{u}_j + \dot{u}_i H_j \right) + \kappa S h_i^k R_{kqjs} H^q u^s, \tag{75}
\]

shows how the Riemann curvature tensor \( R_{ijkq} \) acts as an additional source for magnetic inhomogeneities. Notice that so far there has been no formula determining the rate of change of spatial gradients in the pressure. When an equation of state for the non-magnetic fluid is introduced, the propagation of \( Y_i \), along the observer’s worldline, can be obtained from (71).

Dunsby [31] applied the covariant and gauge-invariant technique to the study of cosmological perturbations in an almost FRW universe filled with an imperfect non-magnetized fluid. As mentioned earlier, our stress can be written as corresponding to an imperfect fluid with the special properties given by (52)-(55). When Dunsby’s fully non-linear propagation formulae for \( D_i \) and \( Z_i \) are applied to this particular imperfect fluid, we recover equations (71) and (72). Of course, there is no expression equivalent to (75) in Dunsby’s analysis.

### 7 Aspects of the Spatial Geometry

The three-dimensional Riemann tensor \( (3) R_{ijkq} \), that determines the curvature of a fundamental observer’s instantaneous rest space, \( \Sigma_{\perp} \), is defined via the commutator for the 3-D gradients of any spacelike vector. In particular, provided that \( v_i u^i = 0 \), we have [30],

\[
(3) \nabla_i (3) \nabla_j v_k \left( - (3) \nabla_i (3) \nabla_j v_k = - \frac{2}{c^2} \omega_{ij} h_q^q v_i + (3) R_{qkji} v^q, \right. \tag{76}
\]

where, by definition

\[
(3) R_{ijkq} = h_i^p h_j^s h_k^t \frac{1}{R_{psrt}} - \frac{1}{c^2} v_i k v_{jq} + \frac{1}{c^2} v_i q v_{jk}. \tag{77}
\]
with \( v_{ij} = (3) \nabla_j u_i = \sigma_{ij} + \omega_{ij} + \Theta h_{ij}/3 \). The successive contractions of (77) provide the 3-Ricci tensor \((3) R_{ij}\) and the 3-Ricci scalar \((3) R\) of \(\Sigma_\perp\), by

\[
(3) R_{ij} = h_i^k h_j^q R_{kq} + \frac{1}{c^2} R_{ikjq} u^k u^q + \frac{1}{c^2} v_{ik} v_{lj} - \frac{\Theta}{c^2} v_{ij},
\]

and

\[
(3) R = R + \frac{2}{c^2} R_{ij} u^i u^j - \frac{2\Theta^2}{3c^2} + \frac{2\sigma^2}{c^2} - \frac{2\omega^2}{c^2}.
\]

Notice the vorticity term that appears in equation (76). In a rotating spacetime, this term prevents the commutation between the 3-gradients of scalars. In particular, the formula

\[
(3) \nabla_i (3) \nabla_j f = (3) \nabla_j (3) \nabla_i f = -\frac{2}{c^2} \omega_{ij} \dot{f},
\]

holds for any scalar quantity \( f \). This non-commutativity, reflects the fact that the fluid velocity does not consist a hypersurface orthogonal vector field if \( \omega_{ij} \neq 0 \) (see Appendix in [30] for more details).

Equations (78) and (79) characterize the curvature of the observer’s rest space within a general spacetime. We shall rewrite (78) and (79) for a magnetic universe containing a single perfect fluid of infinite conductivity. Using (62), (67) and (68), we obtain

\[
(3) R_{ij} = \frac{2}{3} \left( \kappa \left( \mu c^2 + \frac{H^2}{2} \right) - \frac{\Theta^2}{3c^2} + \frac{\sigma^2}{c^2} - \frac{\omega^2}{c^2} + \Lambda \right) h_{ij} - \frac{A}{3c^2} h_{ij} + \kappa M_{ij} + \frac{1}{c^4} \dot{u}_i \dot{u}_j - \frac{1}{c^2} h_i^k h_j^q \left( S^{-3} \left( S^3 \sigma_{kq} \right) \dot{\cdot} \dot{u}_{(k|q)} \right) - \frac{1}{c^2} h_i^k h_j^q \left( S^{-3} \left( S^3 \omega_{kq} \right) \dot{\cdot} \dot{u}_{[k|q]} \right).
\]

Moreover, from (64), (68) and (77), we find the following 3-curvature scalar

\[
K \equiv (3) R = 2 \left( \kappa \left( \mu c^2 + \frac{H^2}{2} \right) - \frac{\Theta^2}{3c^2} + \frac{\sigma^2}{c^2} - \frac{\omega^2}{c^2} + \Lambda \right).
\]

Combining (81) with (82), we can now decompose the 3-Ricci tensor into its trace and its trace-free parts

\[
(3) R_{ij} = \frac{K}{3} h_{ij} - \frac{A}{3c^2} h_{ij} + \kappa M_{ij} + \frac{1}{c^4} \dot{u}_i \dot{u}_j - \frac{1}{c^2} h_i^k h_j^q \left( S^{-3} \left( S^3 \sigma_{kq} \right) \dot{\cdot} \dot{u}_{(k|q)} \right) - \frac{1}{c^2} h_i^k h_j^q \left( S^{-3} \left( S^3 \omega_{kq} \right) \dot{\cdot} \dot{u}_{[k|q]} \right).
\]

The evolution of \( K \), along the observer’s worldline, is governed by the following propagation equation, derived from (79), (44), (56) and (68)
\[
\left( K - \frac{2}{c^2} (\sigma^2 - \omega^2) \right). = -\frac{2\Theta}{3} \left( K + \frac{2}{c^2} A \right) + \frac{4\Theta}{3} (\sigma^2 - \omega^2) - 2\kappa\sigma_{ij} M^{ij}.
\]

(84)

The 3-curvature scalar qualifies as an additional gauge invariant variable, provided that it is constant in the unperturbed cosmological model. In a spatially homogeneous irrotational background, the 3-gradient \( K_i \) is always independent of the gauge choice, as is implied by the relation

\[
K_i \equiv (3) \nabla_i K = \frac{2\kappa\mu c^2}{S} \left( D_i + \frac{1}{\kappa\mu c^2} M_{ji} H^j \right) - \frac{4\Theta}{3c^2} Z_i + \frac{2}{c^2} (3) \nabla_i (\sigma^2 - \omega^2),
\]

(85)

which will be useful in discussing the case of isocurvature perturbations.

It should be emphasized that all the relations given above refer to the instantaneous rest space of an individual fundamental observer. For non-rotating spacetimes, the same formulae (without the vorticity terms) determine the geometry of the observers’ common rest space.

8 Approximations

8.1 Linearization About a FRW universe

Let us assume that the background universe is described by a FRW spacetime containing a perfect fluid and a weak large-scale magnetic field (i.e. \( H^2 \ll \mu c^2 \)). The gauge-invariant variables are the shear \( (\sigma_{ij}) \), the vorticity \( (\omega_{ij}) \), the acceleration \( (\dot{u}_i) \), the divergence of the acceleration \( (A) \), its spatial gradient \( (A_i) \), the electric and magnetic parts of the Weyl tensor \( (E_{ij}, H_{ij}) \), the spatial gradients of the energy density \( (X_i, D_i) \), the spatial gradient of the pressure \( (Y_i) \), the spatial gradient of the expansion \( (Z_i, Z^i_j) \) and of the magnetic field \( (\mathcal{H}_{ij}, \mathcal{M}_{ij}) \), together with the spatial gradient \( (K_i) \) of the 3-curvature scalar. All these quantities vanish in a Friedmann universe. According to (84), the 3-curvature scalar is independent of the gauge choice, provided that it vanishes in the background spacetime.

The formulae above are the exact, fully non-linear, propagation equations. We shall linearize them about the FRW model. In the process we shall retain the mass density \( (\mu) \), the pressure \( (p) \), the expansion \( (\Theta) \) and the magnetic field \( (H_i) \), which are all spatially independent in the background, as zero-order quantities. Variables that vanish in the FRW background, together with their derivatives, will be treated as first order. The anisotropic pressure \( (M_{ij}) \) generated by the primordial field will be also considered as of order one, since it must disappear within a FRW spacetime. Finally, terms of order higher than the first will be disregarded. Under these conditions, the conservation law (56) for the energy-density remains unaffected, whereas law (58), for the momentum-density, changes. In particular, we have

\[
\dot{\mu} c^2 + (\mu c^2 + p) \Theta = 0,
\]

(86)

and

\[
\kappa \left( \mu + \frac{p}{c^2} \right) \dot{u}_i + Y_i - 2\mathcal{H}_{[ij]} H^j = 0.
\]

(87)

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Raychaudhuri’s equation, (68), becomes
\[ \dot{\Theta} - A + \frac{\Theta^2}{3} + \frac{\kappa c^2}{2} \left( \mu c^2 + 3p \right) - \Lambda c^2 = 0, \] (88)
and the time-evolution formula for the vorticity vector, (70), reduces to
\[ \dot{\omega}^i + \frac{2\Theta}{3} \omega^i = \frac{1}{2c} \eta^{ij}k_q u_j \dot{u}_k, \] (89)

Linearization permits the omission of the projection tensor from the left-hand side of the remaining propagation equations and the weakness of the field implies that \( D_i/2 + M_{ij}H^j/\kappa \mu c^2 \simeq D_i/2 \). Hence, equations (71), (72) and (75), which describe the evolution of \( D_i \), \( Z_i \) and \( M_{ij} \) respectively, become
\[ \dot{D}_i = \frac{\mu \Theta}{\mu c^2} D_i - \left( 1 + \frac{p}{\mu c^2} \right) Z_i - \frac{2\Theta}{\kappa \mu c^2} M_{[ij]} H^j + \frac{2S \Theta H^2}{3\mu c^2} \dot{u}_i, \] (90)
\[ \dot{Z}_i = -\frac{2\Theta}{3} Z_i - \frac{\kappa \mu c^4}{2} D_i - 3c^2 M_{[ij]} H^j + S R \dot{u}_i + S A_i, \] (91)

where now \( R = \mathcal{K}/2 \), with \( \mathcal{K} = 2(\kappa \mu c^2 - \Theta^2/3c^2 + \Lambda) \), and
\[ \dot{M}_{ij} = -\frac{2\Theta}{3} M_{ij} - \frac{2\kappa}{3} H_i Z_j + \kappa S H^k(3) \nabla_j (\sigma_{ik} + \omega_{ik}) - \frac{\kappa \Theta S}{3c^2} (2H_i \dot{u}_j + \dot{u}_i H_j) + \frac{\kappa \Theta S}{3c^2} \dot{h}_k h_{ij} - \kappa S h_i k R_{kqjs} H_q u_s. \] (92)

Linearization leaves Maxwell’s equations unaffected: hence,
\[ H^i_{\, \, \, ;i} - \frac{1}{c^2} H^i u_i = 0, \] (93)
and
\[ \left( \sigma^i_j + \omega^j_i - \frac{2\Theta}{3} h^i_j \right) H^i = h_j \ddot{H}^j. \] (94)

The key role played by the acceleration (\( \dot{u}_i \)), by its divergence (\( A = \dot{u}^i_i \)) and by the latter’s spatial gradient (\( A_i = (3) \nabla_i A \)) will become clear below. Therefore, it is useful to derive expressions for these three variables. The momentum-density conservation law (87), immediately provides a useful formula for \( \dot{u}_i \), that leads to the following expression for \( A \)
\[ A = -\frac{1}{\kappa (\mu + p/c^2)} \left( (3) \nabla_i Y^i - \frac{\kappa H^2 \mathcal{K}}{3} + \kappa (3) \nabla^2 H^2 \right). \] (95)
This is derived by applying the commutator (74) to the spacelike vector $H_i$, and substituting the zero-order part of the Ricci tensor associated with $\Sigma_\perp$ from (83). Furthermore, the spatial derivative of (95), combined with (44), (56), (76) and (80) provides an expression for $A_i$ within a perturbed FRW universe:

$$A_i = \frac{1}{\kappa (\mu + p/c^2)} \left( \frac{\kappa}{3} (3) \nabla^2 \right) Y_i + \frac{1}{2 (\mu + p/c^2)} (3) \nabla^2 B_i + \frac{H^2}{3 (\mu + p/c^2)} K_i - 2 \Theta \left( \frac{c_s^2}{c^2} + \frac{2H^2}{3 (\mu c^2 + p)} \right) (3) \nabla j \omega^j_i, \quad (96)$$

where $c_s^2 \equiv \dot{p}/\dot{\mu}$ is the sound speed and $B_i = (3) \nabla_i H^2$ is an additional first-order variable.

We make one further comment about the 3-curvature scalar $K$. Its linearized propagation formula,

$$\dot{K} = -\frac{2\Theta}{3} \left( K + \frac{2}{c^2} A \right), \quad (97)$$

contains a first-order term generated by the divergence of the acceleration. Thus, to zero-order, we find

$$K = \frac{6k}{S^2}, \quad (98)$$

where $k$ is the 3-curvature constant (i.e. $\dot{k} = 0$) of the background FRW spacetime [27]. One can replace $K$ by (98) whenever the 3-Ricci scalar is coupled to a quantity of first or higher order. Such a substitution further simplifies equations (91) and (96). We shall adopt this approach in what follows. Finally, equation (83) reduces to

$$K_i = \frac{2\kappa \mu c^2}{S} D_i - \frac{4\Theta}{3Sc^2} Z_i, \quad (99)$$

which can be used to modify (96), if required.

### 8.2 Pressureless Fluid

Let us assume that the perturbed FRW universe, described in the last section, is dominated by a single non-relativistic (i.e. $p = 0$) perfect fluid. Under this restriction the equation of continuity, (86), becomes

$$\dot{\mu} + \mu \Theta = 0, \quad (100)$$

and leads to the familiar evolution formula for the mass density of a dust-dominated cosmological model.
\[ \mu = \frac{M}{S^3}, \]  
(101)

with \( \dot{M} = 0 \).

An important result emerges from the conservation of the momentum-density. In particular, the contraction of the pressure-free form of (87) with the magnetic-field vector provides the relation

\[ \dot{u}_i H^i = 0. \]  
(102)

Hence, within a magnetized dust-dominated nearly FRW universe, the acceleration must always be normal to the magnetic field, a result consistent with the Lorentz force law of special relativity. Furthermore, equation (102) may be rearranged to give

\[ \dot{H}_i u^i = 0, \]  
(103)

and ensures that the field’s derivative actually lies in the observer’s rest space. Also, \( p = 0 \) in (87) leads to the following expression for the acceleration

\[ \dot{u}_i = \frac{2}{\kappa \mu} \mathcal{H}_{[ij]} H^j. \]  
(104)

This demonstrates how the geodesic motion of the matter is disturbed by the spatial variations of the magnetic field. We can express (104) in a more familiar form: \( \mathcal{H}_{[ij]} \) is an antisymmetric 3-tensor, the generalized curl of \( H_i \) projected onto the observer’s rest space. Such a tensor has only three independent components and corresponds to a 3-vector, namely the covariant spatial curl of \( H_i \). In particular (see Appendix C.1), we find

\[ \mathcal{H}_{[ij]} = -\frac{\kappa}{2} \epsilon_{ijk} \text{curl} H^k, \]  
(105)

where \( \epsilon_{ijk} = \eta_{ijkq} u^q / c \) is the covariant spatial permutation tensor. When substituted into (104), (103) provides the following expression for the acceleration, familiar from classical magnetohydrodynamics

\[ \dot{u}_i = \frac{1}{\mu} \epsilon_{ijk}((\text{curl} H^j) H^k). \]  
(106)

Notice that results (102), (103) and (104), or equivalently (106), also hold within a pressure-free almost Bianchi-I universe.

An additional familiar result is obtained when we adapt the vorticity propagation formula, (70), to our cosmological model. From \( \omega^i = -\text{curl} u^i / 2 \) and equation (106), we find that

\[ (\text{curl} u^i) + \frac{2\Theta}{3} \text{curl} u^i = \frac{1}{\mu} \epsilon_{ijk(3)} \nabla_j (\epsilon_{ksq} (\text{curl} H^q) H^s), \]  
(107)
in agreement with Wasserman’s analysis [34].

The adaptation of Maxwell’s equations to this cosmological model gives

\[ H^i_{;i} = 0, \]  

and

\[ \left( \sigma^i_j + \omega^i_j - \frac{2\Theta}{3} h^i_j \right) H^j = \dot{H}^i, \]  

with the latter providing the standard expression \( H^2 \propto S^{-4} \) for the evolution of the energy density of the magnetic field with the expansion scale factor \( S(t) \).

We shall now proceed with some additional simplifications. Suppose that spacetime is flat (i.e. \( R_{ijkl} = 0 \)) and that there is no cosmological constant (i.e. \( \Lambda = 0 \)), then Raychaudhuri’s equation \((88)\) reduces to

\[ \dot{\Theta} = A + \frac{Kc^2}{2} - \frac{3\kappa\mu c^4}{2}, \]  

since \( K = 2(\kappa\mu c^2 - \Theta^2/3c^2) \). Using \((102)\) and \((104)\), the propagation equations for the spatial gradients \((90), (91)\) and \((92)\) become

\[ \dot{D}_i = -Z_i - \frac{2\Theta}{\kappa\mu c^2} M_{[ij]} H^j, \]  

\[ \dot{Z}_i = -\frac{2\Theta}{3} Z_i - \frac{\kappa\mu c^4}{2} D_i - 3c^2 M_{[ij]} H^j + SA_i + \frac{6k}{S^2\kappa\mu} M_{[ij]} H^i, \]  

\[ \dot{M}_{ij} = -\frac{2\Theta}{3} M_{ij} - \frac{2\kappa}{3} H_i Z_j + \kappa S H^k(\sigma^j_{ik} + \omega^j_{ik}) + \frac{2\Theta H^2}{9\mu c^2} M_{[ij]}, \]  

respectively, where the pressure-free expression for the quantity \( A_i \), that appears in \((112)\), is

\[ A_i = \frac{1}{2\mu} \left( \frac{6k}{S^2} - (3)\nabla^2 \right) B_i + \frac{H^2}{3\mu} K_i - \frac{4\Theta H^2}{3\mu c^2} (3)\nabla_j \omega^j_i, \]  

\( B_i = (3)\nabla_i H^2 \), and \( K_i \) is given by \((13)\).

The antisymmetry of \( M_{[ij]} \) means that the indices \( i \) and \( j \) in \((111)\) must take different values \((i \neq j)\). Thus, irregularities in the energy density are not influenced by the component of the field that acts parallel to them, a result which is also apparent in the analysis given by Ruzmaikina and Ruzmaikin [33]. Combining equations \((111)-(114)\) and ignoring the non-linear terms, we obtain the following second-order differential equation for the time-evolution of the comoving fractional orthogonal spatial gradient of the energy density,
\[ \ddot{D}_i = -\frac{2\Theta}{3} \dot{D}_i + \frac{\kappa \mu c^4}{2} D_i - \frac{S}{2\mu} \left( \frac{6k}{S^2} - (3) \nabla^2 \right) B_i + \frac{6k}{S^2 \kappa \mu} \mathcal{M}_{ij} H^j + \]
\[ \frac{4\Theta S H^2}{3\mu c^2} (3) \nabla_j \omega_i^j - \frac{2\Theta S}{\mu c^2} (3) \nabla_i \dot{H}_j H^j. \] 

(115)

In a perturbed Einstein de Sitter universe (i.e. \( k = 0 \) and \( \Lambda = 0 \)) this reduces to
\[ \ddot{D}_i = -\frac{2\Theta}{3} \dot{D}_i + \frac{\kappa \mu c^4}{2} D_i + \frac{S}{2\mu} (3) \nabla^2 B_i + \frac{4\Theta S H^2}{3\mu c^2} (3) \nabla_j \omega_i^j - \frac{2\Theta S}{\mu c^2} (3) \nabla_i \dot{H}_j H^j \] 

(116)

The first three terms on the right hand side of (116) also appear in the Newtonian treatments presented by Ruzmaikina and Ruzmaikin [33], and by Wasserman [34]; the first two are immediately identifiable, while the third is obtained after a transformation (see Appendix C.2). The latter becomes obvious, when the gauge-invariant approach is applied within the non-relativistic context, as it has been shown by Ellis in [40]. The resulting second-order differential equation contains only the three aforementioned standard terms. Consequently, our relativistic approach provides the classical equation plus two extra relativistic terms. The first term links the vorticity to the time evolution of linear density gradients through the 3-divergence of the vorticity tensor. It appears because in a general spacetime 3-D surfaces need not be orthogonal to the fluid-flow. However, the term’s coefficient suggests a weak effect since \( H^2 \ll \mu c^2 \), or no effect at all if the universe is static. Such a ‘correction term’ does not appear in the Newtonian treatment, because there the hyperplanes orthogonal to the fluid flow are always tangent to the 3-surfaces of absolute time [30].

Also, using (21) and the covariant definition of a vector’s spatial curl we find
\[ \frac{4\Theta S H^2}{3\mu c^2} (3) \nabla_j \omega_i^j = \frac{4\Theta S H^2}{3\mu c^2} \text{curl}\omega_i, \] 

(117)

where \( \omega^i = -\text{curl} u^i/2 \).

Let us focus on the last term in (116). It emerges from the coupling between \( D_i \) and \( \mathcal{M}_{ij} \) in (71). We see that its presence also depends on the universal expansion, but it also creates only a weak effect, since it contains the ratio \( H^2/\mu c^2 \ll 1 \). Similarly to the acceleration case above, the quantity \( (3) \nabla_j \dot{H}_i \) represents the generalized curl of the field’s derivative projected onto the instantaneous rest space of a fundamental observer. Thus, it can be mapped onto the covariant spatial curl of \( \dot{H}_i \). As a result, the second relativistic ‘correction term’ can be reshaped into the following easily identifiable form
\[ \frac{2\Theta S}{\mu c^2} (3) \nabla_j \dot{H}_i^j H^k = \frac{\Theta S}{\mu c^2} \epsilon_{ijk} (\text{curl}\dot{H}^j) H^k. \] 

(118)

Thus, according to (117) and (118), equation (116) becomes
\[ \ddot{D}_i = -\frac{2\Theta}{3} \dot{D}_i + \frac{\kappa \mu c^2}{2} D_i + \frac{S}{2\mu} (3) \nabla^2 B_i + \frac{4\Theta S H^2}{3\mu c^2} \text{curl}\omega_i - \frac{\Theta S}{\mu c^2} \epsilon_{ijk} (\text{curl}\dot{H}^j) H^k. \] 

(119)
8.3 Long-Wavelength Solutions

The gradient $D_i$ describes spatial variations of the energy density orthogonal to the fluid flow. One can extract the information contained in $D_i$ by considering the following local decomposition [30],

$$ S^{(3)} \nabla_i D_j \equiv \Delta_{ij} = \Sigma_{ij} + W_{ij} + \frac{1}{3} \Delta h_{ij}, \quad (120) $$

where $\Sigma_{ij} \equiv \Delta^{(ij)} - \Delta^{h_{ij}}/3$, $W_{ij} \equiv \Delta^{[ij]}$ and $\Delta \equiv S^{(3)} \nabla^i D_i$ are all gauge-invariant quantities. The symmetric, traceless tensor, $\Sigma_{ij}$, describes the evolution of anisotropic structures (e.g. pancake or cigar-like structures). On the other hand, the skew tensor $W_{ij}$ characterizes the rotational behaviour of $D_i$. Hence, the scalar $\Delta$ contains the information regarding the spatial aggregation of matter. When dealing with the problem of structure formation, the latter quantity is the crucial one [11], [12]. The propagation formula for $\Delta$ is found by taking the 3-divergence of (115) and using the properties of the 3-D gradients as well as the fact that, in the linear approximation, the total divergence of the vorticity tensor is zero (see section 7 and also [30]). We obtain

$$ \ddot{\Delta} + \frac{2}{3} \dot{\Delta} - \frac{\kappa \mu c^4}{2} \Delta = -\frac{S^2}{2\mu} \left( \frac{4k}{S^2} - (3) \nabla^2 \right) B + \frac{S^2}{2\mu} \left( \frac{3k}{S^2} + \frac{2\Theta^2}{3c^2} \right) \left( \frac{4k}{S^2} - (3) \nabla^2 \right) H^2, \quad (121) $$

where $B = (3) \nabla^i B_i = (3) \nabla^2 H^2$. The final form of (121) is obtained only after treating the field’s energy density ($H^2$) as a perturbation itself.

Assuming that spatial and temporal dependences are separable, we express every gauge-invariant first-order quantity in (121) as a sum of time-independent scalar harmonics $Q^{(n)}$, defined as eigenfunctions of the Laplace-Beltrami operator, [37],

$$ (3) \nabla^2 Q^{(n)} = -\frac{n^2}{S^2} Q^{(n)}, \quad (122) $$

where the eigenvalue $n$ directly corresponds to physical wavelengths only if $k = 0$ [13]. Thus, in an almost Einstein de Sitter universe, and for low frequency (i.e. $n \to 0$) fluctuations, we may ignore all the terms in the right-hand side of (121). Consequently, the large-scale evolution of matter perturbations is determined by

$$ \ddot{\Delta}^{(n)} + \frac{2}{3} \dot{\Delta}^{(n)} - \frac{\kappa \mu c^4}{2} \Delta^{(n)} = 0, \quad (123) $$

where $\Delta^{(n)}$ is the nth harmonic component of $\Delta$, with $(3) \nabla_i \Delta^{(n)} \approx 0$. Its solution,\footnote{Although it has not been explicitly used yet, the characterization of $H^2$ as a gauge-invariant perturbative term is in line with our treatment of $M_{ij}$. Nevertheless, up to this point, we have only used the restriction that the ratio $H^2/\mu c^2$ is very small.}
\[ \Delta^{(n)} = \Delta_{+}^{(n)} \tau^{2/3} + \Delta_{-}^{(n)} \tau^{-1}, \]  

(124)

contains a growing and decaying modes, exactly as in the case of a dust-dominated non-magnetized universe and in agreement with [33]. The variable \( \tau \) measures the observer’s proper time, while the quantities \( \Delta_{+}^{(n)} \) and \( \Delta_{-}^{(n)} \) remain constant along its worldline.

9 Two Special Cases

9.1 The False Vacuum Assumption

The introduction of a medium that obeys the “inflationary” false vacuum condition (i.e. \( p + \mu c^2 = 0 \)), implies that the spatial gradients \( X_i \) and \( Y_i \), defined by (30)-(31), are related by the simple formula

\[ Y_i = -X_i. \]

(125)

The exact propagation equation for the orthogonal spatial gradient of the energy density, \( X_i \), is obtained by taking the time derivative of (30). From the conservation laws (56) and (58), we have

\[ S^{-4} h^j_i \left( S^4 X_j \right) = -X^j \left( \sigma_i^j + \omega_i^j \right) - \kappa \left( \mu c^2 + p \right) Z_i - 2\Theta \mathcal{H}_{ij} H^j + \]

\[ \frac{2\kappa \Theta \dot{H}^2}{3c^2} \dot{u}_i + \frac{\kappa \Theta}{c^2} \dot{u}^j M_{ji}, \]

(126)

where \( Z_i \) has been defined in (32). In an almost-FRW magnetized universe, the false vacuum assumption, together with the momentum-density conservation law, reduces (126) to the following evolution formula

\[ X_i = \frac{\dot{X}}{S}, \]

(127)

with \( \dot{X} = 0 \). Hence, spatial gradients in the energy density die away as \( S^{-1} \), independent of their wavelength and in line with the cosmic “no-hair” theorems for expanding inflationary universes.

According to (30), however, the energy density of the material component remains constant along the observers worldline. Consequently, definition (33), along with (127), suggests that the fractional density gradients do not change on comoving scales

\[ D_i = \text{const.} \]

(128)

In other words, fundamental observers experience no variation in \( D_i \) within their event horizons: density contrasts remain “frozen-in” as long as the equation of state \( p = -\mu c^2 \) holds.
9.2 Isocurvature Perturbations

We define isocurvature perturbations as fluctuations evolving in a non-rotating (i.e. $\omega_{ij} = 0$) spacetime of constant spatial curvature (i.e. $K_i = 0$). Within an almost-FRW universe, equation (99) provides a condition for the occurrence of isocurvature inhomogeneities:

$$X_i = \frac{2\Theta}{3c^2} Z_i.$$  \hspace{1cm} (129)

The consistency condition for this kind of disturbance is obtained by linearizing the time derivative of (129) about the background FRW spacetime, giving

$$\frac{6k}{S^2} \left( Z_i + \frac{\Theta}{c^2} \dot{u}_i \right) + \frac{2\Theta}{c^2} A_i = 0,$$  \hspace{1cm} (130)

where $A_i$ is given by (96).

Provided that the smooth model has $k = 0$, we can ignore the first term in (130), which means that isocurvature disturbances cannot survive in a non-static universe so long as $A_i \neq 0$. On the other hand, in the absence of pressure, formula (96) implies that the remaining term is unimportant on large scales. Consequently, the matter era seems capable of preserving long-wavelength isocurvature fluctuations.

10 Discussion

We have pursued the study of cosmological density perturbations in a universe that contains a large-scale primordial magnetic field by means of the Ellis-Bruni covariant and gauge-invariant method. We have defined a new variable that is independent of the gauge choice and describes the spatial variations of the magnetic field. Assuming a universe that contains a single perfect fluid of very high conductivity, we have derived the exact, and fully non-linear, equations that determine the model’s time evolution. When the linearized equations are applied to the simple case of pressure-free matter, familiar results are recovered. The energy density of the matter is found to evolve as the inverse cube of the scale factor and the energy density of the magnetic field as $S^{-4}$. The acceleration is provided by the well known expression of classical magnetohydrodynamics, and is always normal to the magnetic field in accord with the Lorentz force law.

We also identify the general-relativistic corrections to the Newtonian treatment. These are introduced by the two extra terms (117) and (118) on the right-hand side of the propagation equation, (116), of the density gradient. The first term is due to universal rotation and does not affect the gravitational clumping of matter. A similar vorticity term also appears in the treatment of a perfect fluid with non-vanishing pressure, (31). Here the fluid pressure is zero, but there exists a residual isotropic pressure, induced into the model by the magnetic field (see (53)). Technically, the second relativistic correction, which modifies the evolution of both $D_i$ and $\Delta$, appears because everything is projected onto the observer’s instantaneous rest space (where all measurements take place). Physically, its source is the anisotropic pressure, also introduced by the field (see (53)). In Dunsby’s imperfect fluid analysis, (31), such a term is incorporated into the time derivative $(\nabla_j \pi_i^j)_{ik} u_k$. 

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By focussing upon the spatial aggregation of matter, we have obtained long-wavelength solutions for (115), which reveal the relative unimportance of the magnetic field on the growth of large-scale density fluctuations. We also provide solutions during a period of de Sitter inflation. The results show that density gradients decay, but slower than in the non-magnetized case investigated by Ellis and Bruni, [27]. Furthermore, within the immediate neighbourhood of a comoving observer perturbations freeze out. Our analysis permits a transparent approach to isocurvature inhomogeneities. We have obtained a criterion for the occurrence of this kind of disturbances as well as a consistency condition. The latter shows that a dust-dominated almost-FRW universe, can sustain long-wavelength isocurvature perturbations.

A major advantage of the Ellis and Bruni technique is that the non-linear propagation equations can be often linearized about a variety of background models. For example, we can linearize our exact equations about a smooth Bianchi-I universe. This cosmological model is anisotropic. The shear no longer behaves as a first-order variable and the model’s evolution becomes more complicated. [22]. The formalism we have established here, will enable a full gauge-invariant analysis to be carried out to determine the effects of cosmological magnetic and electric fields on the early universe, and on the temperature anisotropy of the microwave background.

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APPENDICES

A The Spatial Gradient of the Magnetic Field

A.1 Proof of the Gauge-Invariance for \( M_{ij} \)

The metric associated with an exact FRW or with an exact Bianchi-I spacetime can always be written in the diagonal form \( g_{ij} = \text{diag}(g_{00}, g_{11}, g_{22}, g_{33}) \) with the quantities \( g_{ii} \) (no sum) functions of time only (i.e. \( g_{ij,\mu} = 0 \)). As a result the associated spatial Christoffel symbols vanish

\[
\Gamma^\mu_{\nu\kappa} = 0. \tag{131}
\]

Following definition (35), we split the spatial gradient of the magnetic field as

\[
\mathcal{H}_{ij} = \kappa h^\mu_i h^\nu_j \left( H_{\mu,\nu} - \Gamma^\kappa_{\mu\nu} H_\kappa \right) - \kappa h^\mu_i h^\nu_j \Gamma^0_{\mu\nu} H_0 + \kappa h^\mu_i h^0_j H_{\mu,0} + \kappa h^0_i h^q_j H_{0,q}. \tag{132}
\]

\[^4\text{Latin indices run from 0 to 3, whereas Greek ones from 1 to 3.}\]
In a spatially homogeneous universe $H_{i,\mu} = 0$. Also, in a comoving frame, equation (13) means that $H_0 = 0$. Also, when there is no rotation, $u_i = -c\delta_i^0$, so $h_i^0 = 0$. Thus, via (131), relative to a comoving frame, equation (132) gives

$$H_{ij} = 0.$$  \hfill (133)

Therefore, definition (33) provides a spatial gradient for the magnetic field that satisfies the requirements for gauge invariance, when the background universe is a FRW or a Bianchi-I spacetime.

### A.2 Description of Magnetic Spatial Variations by $\mathcal{M}_{ij}$

Let us consider two neighbouring fundamental observers $P$ and $P'$; \{\textit{x}\} is a coordinate system comoving with the expanding fluid. Relative to this frame, the timelike worldlines associated with the two observers, which are assumed to be close enough to measure the same proper time ($\tau$), are labelled by $(x^\mu = \text{const}, c\tau)$ and by $(x^i + \delta x^i, c\tau)$ respectively. With respect to a general frame $y^i = y^i(x^j)$, the vector $(\delta x^\mu, 0)$, which connects the two worldlines at all times, becomes

$$\delta y^i = \frac{\partial y^i}{\partial x^\nu} \delta x^\nu.$$  \hfill (134)

As far as the observer $P$ is concerned, the relative position vector between the two events is the projection of $\delta y^i$ onto its own 3-D instantaneous rest space $\Sigma_{\perp}$

$$\delta_{\perp} y^i = h^i_j \delta y^j.$$  \hfill (135)

In order to examine the field’s spatial variation, the observer at $P$ must parallel transport its own vector $(H_i)_P$ to $P'$ and compare it to the one defined there by the field itself. The resulting difference should then be projected onto $\Sigma_{\perp}$. The field vector at $P'$ as seen from $P$ is

$$(H_i)_{P'} = H_i (y^j + \delta_{\perp} y^j) = (H_i)_P + (H_{i,j})_P \delta_{\perp} y^j.$$  \hfill (136)

The parallel transport of $(H_i)_P$ to $P'$, along the infinitesimal displacement $\delta_{\perp} y^i$, generates the vector

$$(H_i')_{P'} = (H_i)_P + \left(\Gamma^k_{ij}\right)_P (H_k)_P \delta_{\perp} y^j.$$  \hfill (137)

The spacelike part of the difference between (137) and (136) gives the variation of the magnetic field at $P'$ as measured by the observer at $P$. Since $P$ is arbitrary, we have
\[ \delta \perp H_i = h^j_i \left( H_{j,k} - \Gamma^q_{jk} H_q \right) \delta \perp y^k \]
\[ = \frac{1}{\kappa} H_{ij} \delta \perp y^j. \] (138)

Within an almost FRW spacetime, the relative position vector evolves as \( \delta \perp y^i = S (\delta \perp y^i)_0 \), where \( (\delta \perp y^i)_0 = \text{const} \), [27]. Hence, we arrive at the formula
\[ \delta \perp H_i = \frac{1}{\kappa} S H_{ij} \left( \delta \perp y^j \right)_0, \] (139)
and conclude that the second-order tensor \( \mathcal{M}_{ij} = S H_{ij} \) determines the spatial variations of the magnetic field between two neighbouring fundamental observers.

B  The Case of a Pure Magnetic Field

The adoption of an infinitely conducting medium has led to the omission of the electric field from Maxwell’s equations, while preserving the spatial currents. An alternative approach is to assume a pure magnetic field with no electric field and zero currents. This assumption reduces (18) to
\[ \eta^{ij} \eta_{jk} u_j \left( \dot{u}_k H_q - c^2 H_{k;i} \right) = 0, \] (140)
and subsequently to
\[ \dot{u}_i H_j = \frac{c^2}{\kappa} H_{[ij]}. \] (141)

Superficially, (141) seems to provide a relation between the acceleration and the field gradients. Yet, when inserted into (58), it causes the magnetic terms to cancel out, thus effectively disconnecting inhomogeneities in the field from those in the energy density of the medium. Some coupling between \( D_i \) and \( \mathcal{M}_{ij} \) can still be achieved via the third term in the right-hand side of (72). However, this coupling is only significant when \( H^2 \sim \mu c^2 \), but is negligible in the case of a weak magnetic field discussed here.

C  Aspects of the Pressureless Case

C.1  An Expression for \( \mathcal{H}_{[ij]} \)

From the definition of the covariant spatial curl of a 3-vector (see [43]) and the total antisymmetry of \( \epsilon_{ijk} \) we have

\[ \mathcal{H}_{[ij]} = \frac{1}{\kappa} \mathcal{H}_{ij} \]
\[ \text{curl}H^i = \epsilon^{ijk} \nabla_j H_k \]
\[ = -\frac{1}{\kappa} \epsilon^{ijk} \mathcal{H}_{[jk]} . \]  

Thus, contracting with the \( \epsilon \)-tensor, and using the identity \( \epsilon^{ijk} \epsilon_{psk} = 2 h^{[i}_p h^{j]}_s \), we obtain
\[ \mathcal{H}_{[ij]} = -\frac{\kappa}{2} \epsilon^{ijk} \text{curl}H^k. \]  

C.2 Comparison with the Newtonian Approach

In Newtonian theory, and in comoving coordinates, the evolution of the density contrast \( \delta = \delta \mu/\mu \) is determined by,
\[ \frac{\partial^2 \delta}{\partial t^2} = -\frac{2\Theta}{3} \frac{\partial \delta}{\partial t} + 4\pi G \mu \delta + \frac{H^2}{\mu S^2} \nabla^2 \delta H. \]  

By contrast, the covariant gauge-invariant approach provides the following propagation formula for the comoving fractional orthogonal spatial energy-density gradient,
\[ \ddot{D}_i = -\frac{2\Theta}{3} \dot{D}_i + \frac{\kappa \mu c^2}{2} D_i + \frac{S}{2\mu} (\nabla H^2 B_i + \frac{4\Theta S^2 H^2}{3\mu c^2} (3) \nabla_j \omega^j_i - \frac{2\Theta S}{\mu c^2} (3) \nabla_j [H^j_i H^j]. \]  

Relative to a comoving frame \( \dot{D} = \partial D/\partial t \), \[ \text{References} \]
\[ \begin{align*}
[1] & \quad \text{Pudritz R. and Silk J. (1989), Ap. J., 342, 650.} \\
[2] & \quad \text{Kim K.T., Tribble P.C. and Kronberg P.P. (1991), Ap. J., 379, 80.} 
\end{align*} \]

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