Analysis of stress relaxation of paper tube

Ruilin Long *
School of physics, Nankai University, Tianjin, 300071, China

*Corresponding author e-mail: 1810252@mail.nankai.edu.cn.

Abstract. Roll a long paper strip into a tight tube and put it vertically on a table. Why does it often unwind in jerks? What determines the period of the jerks? In the following work, we explain the fact of why the paper tubes unwind in jerks by researching mechanical structure of paper tube theoretically and analyze the data of experiments. Finding that the basic cause of paper tube motion in jerks is stress relaxation. Successfully prove our theory by our experiments. Then we investigate the influence of several physical quantities on period of jerks, such as humidity, friction factor of paper layers. All of these factors determine the period of jerks.

1. Introduction
The title of the 2020 IPT study is described below: Roll a long paper strip into a tight tube and put it vertically on a table. Why does it often unwind in jerks? What determines the period of the jerks?

In the following work, firstly, the mechanical model of paper tube is established. Secondly, some questions about jerks were theoretically analyzed, for example, why jerks would happen, how long the period of a jerk would be, and what would determine the period of the jerk. Finally, the theory was verified through experiments, and it was discussed that what physical quantities would have influence on the period of the jerk, such as friction coefficient, humidity, etc.

2. Mechanical model of paper tube

Figure 1. A piece of straight paper slip is bent to the constraint radius $R$; after being released, it will reach its target radius $r$.

A piece of straight paper slip is bent under the action of external force, and its bending degree is described by the constraint radius $R$ (for the sake of convenience, $R$ will represent constraint radius $R$ in the following statement).

If the external force disappears, the slip will tend to get close to its original state under the action of the resilience $F$, which is produced by the slip itself. However, because paper is not perfect elastomer,
the slip can not return to its original straight state. The target radius \( r \) describes the final radius of the slip during the whole of its relaxation process. Apparently, if the slip reaches its target radius \( r \) (for the sake of convenience, \( r \) will represent target radius \( r \) in the following statement), it will not produce resilience anymore.

Each layer in the paper tube can be considered as a piece of slip. The initial roll is the unreleased roll. Obviously, not all layers return to its \( r \) after the paper tube is released; otherwise, the paper tube will not jerk.

![Figure 2. Because there are partial overlaps in the paper tube, the layers are considered as concentric ellipses.](image)

\( R_i \) and \( r_i \) is used to represent \( R \) and \( r \) of the ith layer, and \( \rho_i \) is used to represent the radius of the ith layer that changes with time when it is expanding or contracting. Actually, in a paper tube with no external force, the values of \( r_i \) and \( \rho_i \) are quite similar. The difference between them is \( \Delta \rho_i = r_i - \rho_i \), and it is assumed that \( \Delta \rho_i \ll r \).

According to elastic theory, the resilience satisfies \( F_i \propto \Delta \rho_i \), So \( \overrightarrow{R} = L \Delta \rho_i \hat{r} \), and it is assumed that the direction of \( \hat{r} \) is outwards along the radius.

As shown in the figure 2, because there are partial overlaps in the paper tube, the layers can not be seen as standard circles; therefore, they are considered as concentric ellipses.

The \( i \)th and the \((i + 1)\)th layers are taken as the object of research. Actually, as far as the \( i \)th layer is concerned, \( \rho_i \) is not always the same. But along the radius, it is easily to know that \( \rho_i < \rho_{i+1} \).

If the perimeter of the \((i + 1)\)th layer is increased by \( \Delta l \), apparently the \( i \)th layer needs to move \( \Delta l \) as well; therefore, all the inner layers will move \( \Delta l \) in tangential direction.

Based on the above analysis, it can be calculated that the angular velocity of the \((i + 1)\)th layer \( \dot{\theta}_{i+1} = \frac{\Delta l}{\rho_{i+1}} \) similarly, \( \dot{\theta}_i = \frac{\Delta l}{\rho_i} \).

![Figure 3. During a jerk, the ith layer rotates faster than the \((i + 1)\)th layer, and they will contact with each other.](image)

Because \( \rho_{i+1} > \rho_i \), it is easily to know that \( \dot{\theta}_i > \dot{\theta}_{i+1} \); namely, the \( i \)th layer rotates faster than the \((i + 1)\)th layer. As the contact points A and B shown in the figure 3, after layers’ rotating, the two end points of major axis of the \( i \)th layer will contact with non major axis parts of the \((i + 1)\)th layer. Contact points will cause pressure \( N \) between two adjacent layers. \( N \) is the reason that makes \( \rho \) of each layer get bigger with time. It is possible that \( N \) makes \( \rho \) of the outer layer exceed its \( r \) after the
paper tube is released, which leads to the production of resilience with different directions of the inner and the outer layers.

The average distance $\bar{d}_i$ between the $i$th and the $(i+1)$th layers can be determined by $\bar{\rho}_i$ and $\bar{\rho}_{i+1}$: $\bar{d}_i = \bar{\rho}_{i+1} - \bar{\rho}_i$.

It is easily to know that the smaller $\bar{d}$ is, the bigger $N$ will be.

$\bar{d}_0$ is used to describe the distance between the first and the second layers; for the first two layers don’t contact with each other very soon after the paper tube being released, it is a constant. Therefore, the distance between the $i$th and the $(i+1)$th layer can be represented as $\bar{d}_i = \bar{d}_0 - \Delta d_i$. It is assumed that $N$ has the similar form of Hooke's law

\[ \vec{N} = k\Delta \vec{d}_i \hat{r} \] (1)

The direction of $\hat{r}$ is outwards along the radius.

Therefore, the stress of $i$th layer of paper tube should include three parts: the resilience $F$ from the deformation, the push force $N_i$ from the inner layer to the outer layer, and the pressure force $N_{i+1}$ to the inner layer by the outer layer. When each layer of paper tube is in equilibrium, it can be expressed by the following equation

\[ k\Delta d_n - 0 + L\Delta \rho_n = 0 \]

\[ k\Delta d_i - k\Delta d_{i+1} + L\Delta \rho_i = 0 \]

\[ k\Delta d_3 - k\Delta d_4 + L\Delta \rho_3 = 0 \]

\[ k\Delta d_2 - k\Delta d_3 + L\Delta \rho_2 = 0 \]

\[ 0 - k\Delta d_2 + L\Delta \rho_1 = 0 \]

When the entire paper tube is considered to be in equilibrium between two jerks, add up the equations (2.2), and it can be found that

\[ \sum_{i=1}^{n} \Delta \rho_i = 0 \] (3)

Equation (2.3) means there are some layers that do not reach $r$, and other layers that is beyond $r$. If the former, layers will have a tendency to expand outward; if the latter, layers will have a tendency to contract inward. Obviously, there is a balance layer between those two parts, which is called $\beta$ layer.

It can be known that

\[ k\Delta \bar{d}_i = L \sum_{i=1}^{n} \Delta \rho_i \] (4)

Wherever $i < \beta$, it can be found that $\Delta \rho_i > 0$ ; therefore, it can be derived that

\[ \bar{d}_{min} = \bar{d}_0 - \Delta \bar{d}_{max} = \frac{L}{k} \sum_{i=1}^{\beta} \Delta \rho_i \] (5)

Equation (2.5) shows that the paper layers get denser along the radius until getting densest at the $\beta$ layer.

Because the $\beta$ layer is the most compact layer of the whole paper tube as the balance-layer, the change of the $\beta$ layer can dominate the movement of the whole paper tube. Therefore, the motion of the $\beta$ layer will be the primary discussion in the following article.

3. Theoretical analysis of jerks
In this part, the following problems will be discussed: why jerks appear? How jerks happen? How long the period of jerks? What determine the period of jerks?
3.1. Why jerk?
There exists friction between paper layers which depend on the pressure force $N$. The friction will hinder the revolving expansion of paper tube, as shown in the figure 4(a). However, due to the effect called stress relaxation, $N$ will decrease with time $t$. So the maximum static friction force will also gradually decrease, and paper tube will start to rotate and expand further, until the next time when internal and external paper layer contact and form a new balance. This repetitive process is what we called jerks. Because of stress relaxation, a paper tube will begin to expand rotationally until reaching another balance state, as shown in the figure 4(b).

![Figure 4](image)

Figure 4. (a). Friction $f$ depend on the pressure force $N$. The friction will hinder the revolving expansion of the paper tube. (b). "Jerks" is the process that paper tube motion from a stationary state to next stationary state.

3.2. How long the period of jerk is?

![Figure 5](image)

Figure 5. During a jerk, the inner layers must rotate faster than the outer layers, so they will contact.

The period of the jerks can be divided into following two parts: 1) the time of the paper tube's movement. 2) The interval time that from paper tube overcome the friction to the starting of next jerks.

The movement time of the paper tube can be seemed as the paper layer wave which are shown in the figure. Only when the phase difference between the layer $\beta$ and $\beta+1$ is an integer multiple of $\pi$, half of wavelength, the two layers can encounter again. The movement time $t_1$ can be expressed as

$$t_1 = \frac{n\lambda}{2\nu}$$

(6)
\( n \) is an integer; \( \lambda \) is the wavelength of \( \beta \) layer; \( \bar{v} \) is the average speed of the \( \beta \) layer and the \( \beta + 1 \) layer average speed difference, which we regarded as a constant in our experiment. Wavelength is depended on the radius of \( \beta \) layer, and the radius of \( \beta \) layer depended on the radius of the original first layer when we roll the paper strip to the radius of \( R_1 \).

3.3. What determine the period of jerks?
The interval time is set as \( t_2 \). It is easy to know that the interval time is related to the rate of stress relaxation.

Consider two extreme cases: (1) If the stress never decrease, the interval time will be infinity. (2) If the stress decrease rapidly, interval time will be so short that jerks will become a continuous change.

The entire period of jerks can be expressed as \( t = t_1 + t_2 \). So the three quantities that determine the jerks period are rate of stress relaxation, \( R_1 \) and the friction coefficient between layers of paper tube.

Stress relaxation is mainly related to the characteristics of the material and the outside environment, such as the temperature, humidity, inherent properties of materials, etc.

4. Experiments

4.1. Experiment method
In the experiments, a piece of paper slip was rolled around a cylindrical mold closely. The length of the slip was 15cm, and the width was 0.5cm. All the slips were from the same A4 paper (80g, 0.102mm thick).

![Figure 6](image)

**Figure 6.** The relationship between \( R \) and \( r \); the curves of different colour represent different constraint time; the purple line is the line of \( r = R \). (Temperature; humidity 91%)

The radius of cylindrical mold was changed to get the functional relationship of \( r \) and \( R \) for a certain period of constraint time, and other similar experiments with different constraint time were done as well. The actual image is shown as figure.6, from which it can be derived that each \( R \) is connected with a certain \( r \) when \( T \) is considered to be a constant.

After the paper tube was constrained for a while and being released, high speed camera was used to record the shape of the paper tube. The radius of each layer of a paper tube was measured by analyzing the photo in a software. The true distance between two points in a photo was obtained by fetching the pixels.

4.2. Mechanical model verification
Actually, the \( \beta \) layer could be spotted easily in all the experiments being done; therefore, there is a great possibility that the result of equation (2.5) is correct. As shown in figure. 7 (a) and (b), the paper layer is densest at the \( \beta \) layer.
Figure 7. (a) This is a schematic of $\beta$ layer. (b) The figure describes $\beta$ layer in real experiments (Temperature; humidity 91%).

From the red curve in Fig.8, the experimental relationship between the constraint radius $R_i$ and the radius of point $\rho_i$ on the paper tube can be known. It can be seen that the inner points of the paper tube have the relationship of $\rho_i < r_i (i < \beta)$, and they have a tendency of expanding outward; the outer points have the relationship of $\rho_i > r_i (i > \beta)$, and they have a tendency of contracting inward. As shown in Fig.8, the intersection of those two curves is both the radius of points on $\beta$ layer and the target radius $r$ of $\beta$ layer.

Figure 8. The relationship between $R$ and $r$ is shown in the black curve; the relationship between $R$ and $\rho$ is shown in the red curve. (T=1 min; Temperature 12°C; humidity 91%)

Table 1. The value of $\Delta \bar{\rho}_i$ of three paper tubes and their total (Temperature; humidity 91%)

|       | tube1 inside | tube1 outside | tube2 inside | tube2 outside | tube3 inside | tube3 outside |
|-------|--------------|---------------|--------------|---------------|--------------|---------------|
| $\Delta \bar{\rho}_i$ |              |               |              |               |              |               |
| (0.01mm) |    |         |          |          |    |          |
| 2.15  | -0.11        | 3.02          | -0.04        | 1.15          | -0.98        |
| 3.00  | -1.35        | 2.54          | -0.17        | 2.12          | -1.96        |
| 3.18  | -2.70        | 1.17          | -0.52        | 2.93          | -2.46        |
| 3.35  | -3.35        | 1.27          | -1.18        | 3.44          | -3.78        |
| 2.40  | -3.20        | 0.65          | -1.44        | 2.96          | -3.41        |
| 2.35  | -3.35        | 0.02          | -2.54        | 2.52          | -2.51        |
| 2.30  | -2.70        | -2.18         |              | 1.87          | -2.10        |
| 0.01  | -1.85        |              |              | 1.16          | -1.01        |
|       |              |               |          | 0.85          | -0.65        |
|       |              |               |          | 0.04          |              |

$\sum_{i=1} \Delta \bar{\rho}_i$  18.74  -18.61  8.69  -8.07  19.04  -18.86

Three more paper tubes were made. A jerk was observed randomly, and the value of $\bar{\rho}_i$ before and after the jerk was recorded; therefore, the value of $\Delta \bar{\rho}_i$ could be calculated. As shown in table 1, if $\Delta \bar{\rho}_i$
are summed up from $\Delta \rho_1$ to $\Delta \rho_n$, it can be derived that $\sum_{i=1}^{n} \Delta \rho_i$ is approximately equal to 0; therefore, the equation (2.3) is verified.

4.3. The movement time of the paper tube

![Figure 9](image)

**Figure 9.** Relationship between the number of jerks in quantities of experiments and per 50ms interval of time. (Temperature: humidity 91%)

| $R_1$ (mm) | $t_1$ (ms) |
|-----------|------------|
| 2.5       | 550        |
| 4.0       | 575        |
| 6.0       | 641        |
| 8.0       | 750        |
| 10.0      | —          |
| 12.0      | —          |

The relative displacement of two layers of paper may be one wavelength, two wavelength, three wavelength...(integer multiples of one wavelength). As shown in the figure 9, the time distribution of jerk was measured in a lot of experiments. The curve can be viewed as a combination of three normal distribution; therefore, equation (3.1) is verified.

The β layer of a paper tube can be abstracted as a two-phase staggered wave, and the average time required for the relative movement of one wavelength is 550ms. Wavelength depends on the radius of $\beta$ layer, and the radius of $\beta$ layer depends on $R_1$; therefore, we changed $R_1$ to measure different $t_1$. It was found that $t_1$ increased with $R_1$. If $R_1$ is more than 10mm, each point will reach its target radius and never jerk.

4.4. The interval time

There are two physical quantities that impact the interval time between jerks.

Factor 1: The friction factor.

To get a piece of paper slip with smooth surface, a pencil, which can provide graphite, was used to daub the obverse and reverse sides of the slip evenly. To get a piece of paper slip with rough surface, a blade was used to scrape the surface of the slip. Keep other physical properties of the slip unchanged, such as length, width, $R_1$, etc. The value of interval time is shown in table 3.
Table 3. With friction coefficient of the slip’s surface increasing, the interval time $t_2$ will be longer. (Temperature 12°C; humidity 91%)

|              | lubrication | normal | scrape |
|--------------|-------------|--------|--------|
| Interval time (s) | 0.16        | 1.48   | 2.93   |
|              | 0.18        | 1.58   | 3.1    |
|              | 0.24        | 1.47   | 5.81   |
|              | 0.26        | 4.11   | 24.8   |
|              | 0.51        | 7.29   |        |
|              | 0.96        |        |        |
|              | 1.24        |        |        |
|              | 1.91        |        |        |

Table 4. The drier the slip is, the longer the interval time $t_2$ will be. (Temperature 12°C; different humidity)

|              | normal | wet | dry |
|--------------|--------|-----|-----|
| Interval time (s) | 0.16 |     | 0.51 |
|              | 0.18   | 1.21|     |
|              | 0.24   | 1.49|     |
|              | 0.26   |     | 1.29 |
|              | 0.51   | Almost continuous motion | 1.76 |
|              | 0.96   |     | 1.96 |
|              | 1.24   |     |     |
|              | 1.91   |     |     |

It can be seen that with friction coefficient of the slip’s surface increasing, the interval time $t_2$ will be longer.

Factor 2: Rate of stress relaxation

The rate of stress relaxation is mainly affected by humidity. According to the mechanics of materials, the higher the humidity is, the faster the rate of stress relaxation will be. In the experiments, the spray and 95% percent alcohol was used to change the humidity of the slip. First of all, a pencil was used to lubricate both slips and ensured their friction coefficient equal. Next, one of the slips was sprayed appropriately to get wet, and 95% of the alcohol was applied to the other to make it dry. The value of the interval time is shown in table 4.

Table 5. The height of the paper tube didn’t have obvious effect on the interval time. (Temperature12°C; humidity 91%)

| Height(cm) | Interval time(s) |
|-----------|-----------------|
|           | first | second | third | fourth |
| 1         | 1.25   | 1.25   | 3.99  | —      |
| 2         | 1.48   | 1.58   | 4.11  | 7.29   |
| 3         | 1.37   | 1.30   | 3.14  | 5.90   |
| 4         | 1.87   | 1.84   | 2.09  | 6.00   |
| 5         | 1.29   | 1.78   | 1.64  | 5.01   |
| 6         | 1.62   | 1.69   | 2.48  | 3.97   |

It can be seen that the drier the slip is, the longer the interval time $t_2$ will be; namely, the faster the rate of stress relaxation is, the shorter the interval time will be.

Other physical quantities were also explored, such as the height of the paper tube and the thickness of the paper slip. Finally, it was found that they didn’t have obvious effect on the interval time. As shown in the table 5.

In conclusion, the period of the interval time is determined by the friction coefficient of the slip, the rate of stress relaxation, and the constraint radius $R_1$ of the first layer.
5. Summary
(1) In the research of Paper Tube, the physical model of the paper tube, which is the mechanical structure of "internal expansion and external pressure", was established, and a new concept, the $\beta$ layer, was defined.
(2) The main reason of paper tube’s unwinding is stress relaxation. The expansion mode of the paper tube is assumed as "rotational unrolling". The $\beta$ and $\beta'$ layers are considered as two waves. It is pointed out that jerk is phase difference jump from 0 to $n\pi$, so that the value of the movement time is integer multiples.
(3) The entire period of jerks can be expressed as $t = t_1 + t_2$. $t_1$ means the movement time of jerks, and $t_2$ means the interval time between jerks.
(4) $R_1$, the constraint radius of the first layer, determines the movement time of jerks. The rate of stress relaxation and the friction coefficient of paper determine the period of the interval time between two jerks. Other physical quantities may also have influence on different respects, for example, humidity determines the rate of stress relaxation of the paper tube.

References
[1] Lurie, A.I., Belyaev, A. (2005) Theory of Elasticity. Springer, Berlin.
[2] Linkov, A.M. (2002) Boundary Integral Equations in Elasticity Theory. Springer, Dordrecht.
[3] Slaughter, W.S. (2002) The Linearized Theory of Elasticity. Birkhäuser, Boston.
[4] Hetnarski, R.B., Ignaczak, J. (2004) Mathematical Theory of Elasticity. Taylor and Francis, Oxford.