On the classical capacity of quantum Gaussian measurement

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Abstract

In this paper we consider the classical capacity problem for Gaussian measurement channels without imposing any kind of threshold condition. We prove Gaussianity of the average state of the optimal ensemble in general and discuss the Hypothesis of Gaussian Maximizers concerning the structure of the ensemble. The proof uses an approach of Wolf, Giedke and Cirac adapted to the convex closure of the output differential entropy. Then we discuss the case of one mode in detail, including the dual problem of accessible information of a Gaussian ensemble.

In quantum communications there are several studies of the classical capacity in the transmission scheme where not only the Gaussian channel but also the receiver is fixed, and the optimization is performed over certain set of the input ensembles. These studies are practically important in view of the complexity of the optimal receiver in the Quantum Channel Coding (HSW) theorem. Our findings are relevant to such a situation where the receiver is Gaussian and concatenation of the channel and the receiver can be considered as one Gaussian measurement channel. Our efforts in this and preceding papers are then aimed at establishing full Gaussianity of the optimal ensemble (usually taken as an assumption) in such schemes.
1 Introduction

From the viewpoint of information theory measurements are hybrid communication channels that transform input quantum states into classical output data. As such, they are described by the classical information capacity which is the most fundamental quantity characterizing their ultimate information-processing performance [3], [20], [21], [7]. Channels with continuous output, such as bosonic Gaussian measurements, do not admit direct embedding into properly quantum channels and hence require separate treatment. In particular, their output entropy is the Shannon differential entropy, instead of the quantum entropy, which completely changes the pattern of the capacity formulas. The classical capacity of multimode Gaussian measurement channels was computed in [11] under so called threshold condition (which includes phase-insensitive or gauge covariant channels as a special case). The essence of this condition is that it reduces the classical capacity problem to the minimum output differential entropy problem solved in [10] (in the context of quantum Gaussian channels a similar condition was introduced and studied in [22], [8], see also references therein).

In this paper we consider the classical capacity problem for Gaussian measurement channels without imposing any kind of threshold condition. In particular, in the framework of quantum communication, this means that both (noisy) heterodyne and (noisy/noiseless) homodyne measurements [2] are treated from a common viewpoint. In this setting, we prove Gaussianity of the average state of the optimal ensemble in general and discuss the Hypothesis of Gaussian Maximizers (HGM) concerning the structure of the ensemble. The proof uses the approach of the paper of Wolf, Giedke and Cirac [17] applied to the convex closure of the output differential entropy. Then we discuss the case of one mode in detail, including the dual problem of accessible information of a Gaussian ensemble.

In quantum communications there are several studies of the classical capacity in the transmission scheme where not only the Gaussian channel but also the receiver is fixed, and the optimization is performed over certain set of the input ensembles (see [2], [5], [23], [24] and references therein). These studies are practically important in view of the enormous complexity of the optimal receiver in the Quantum Channel Coding (HSW) theorem (see e.g. [6]). Our findings are relevant to such a situation where the receiver is Gaussian and concatenation of the channel and the receiver can be considered as one Gaussian measurement channel. Our efforts in this and preceding pa-
pers are then aimed at establishing full Gaussianity of the optimal ensemble (usually taken as a key assumption) in such schemes.

2 The measurement channel and its classical capacity

An ensemble $\mathcal{E} = \{\pi(dx), \rho(x)\}$ consists of probability measure $\pi(dx)$ on a standard measurable space $\mathcal{X}$ and a measurable family of density operators (quantum states) $x \rightarrow \rho(x)$ on the Hilbert space $\mathcal{H}$ of the quantum system. The average state of the ensemble is the barycenter of this measure

$$\bar{\rho}_\mathcal{E} = \int_X \rho(x) \pi(dx),$$

the integral existing in the strong sense in the Banach space of trace-class operators on $\mathcal{H}$.

Let $M = \{M(dy)\}$ be an observable (POVM) on $\mathcal{H}$ with the outcome standard measurable space $\mathcal{Y}$. There exists a $\sigma$–finite measure $\mu(dy)$ such that for any density operator $\rho$ the probability measure $\text{Tr} \rho M(dy)$ is absolutely continuous w.r.t. $\mu(dy)$, thus having the probability density $p_\rho(y)$ (one can take $\mu(dy) = \text{Tr} \rho_0 M(dy)$ where $\rho_0$ is a nondegenerate density operator). The affine map $M : \rho \rightarrow p_\rho(\cdot)$ will be called the measurement channel.

The joint probability distribution of $x, y$ on $\mathcal{X} \times \mathcal{Y}$ is uniquely defined by the relation

$$P(A \times B) = \int_A \pi(dx) \text{Tr} \rho(x) M(B) = \text{Tr} \int_B \int_A p_{\rho(x)}(y) \pi(dx) \mu(dy),$$

where $A$ is an arbitrary Borel subset of $\mathcal{X}$ and $B$ is that of $\mathcal{Y}$. The classical Shannon information between $x, y$ is equal to

$$I(\mathcal{E}, M) = \int \int \pi(dx) \mu(dy) p_{\rho(x)}(y) \log \frac{p_{\rho(x)}(y)}{p_{\bar{\rho}_\mathcal{E}}(y)}.$$

In what follows we will consider POVMs having (uniformly) bounded operator density, $M(dy) = m(y) \mu(dy)$, with $\|m(y)\| \leq b$, so that the probability densities $p_\rho(y) = \text{Tr} \rho m(y)$ are uniformly bounded, $0 \leq p_\rho(y) \leq b$. (The
probability densities corresponding to Gaussian observables we will be dealing with possess this property). Moreover, without loss of generality \[9\] we can assume \( b = 1 \). Then the output differential entropy

\[
h_M(\rho) = - \int p_\rho(y) \log p_\rho(y) \mu(dy)
\]

is well defined with values in \([0, +\infty]\) (see \[9\] for the detail). The output differential entropy is concave lower semicontinuous (w.r.t. trace norm) functional of a density operator \( \rho \). The concavity follows from the fact that the function \( p \to -p \log p, p \in [0, 1] \) is convex. Lower semicontinuity follows by an application of the Fatou-Lebesgue lemma from the fact that this function is nonnegative, continuous and \( |p_\rho(y) - p_\sigma(y)| \leq \|\rho - \sigma\|_1 \).

Next we define the convex closure of the output differential entropy \(1\):

\[
e_M(\rho) = \inf_{\epsilon: \rho = \rho} \int h_M(\rho(x)) \pi(dx).
\]

which is the “measurement channel analog” of the convex closure of the output entropy for a quantum channel \[15\].

**Lemma 1.** The functional \( e_M(\rho) \) is convex, lower semicontinuous and strongly superadditive:

\[
e_{M_1 \otimes M_2}(\rho_{12}) \geq e_{M_1}(\rho_1) + e_{M_2}(\rho_2).
\]

As it is well known, the property \(3\) along with the definition \(2\) imply additivity: if \( \rho_{12} = \rho_1 \otimes \rho_2 \) then

\[
e_{M_1 \otimes M_2}(\rho_{12}) = e_{M_1}(\rho_1) + e_{M_2}(\rho_2).
\]

**Proof.** The lower semicontinuity follows from the similar property of the output differential entropy much in the same way as in the case of quantum channels, treated in \[15\], Proposition 4, see also \[16\], Proposition 1.

Let us prove strong superadditivity. Let

\[
\rho_{12} = \int \rho_{12}(x) \pi(dx)
\]

be a decomposition of a density operator \( \rho_{12} \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), then

\[
p_{M_1 \otimes M_2}(y_1, y_2|x)
\]

\[
= \text{Tr} \rho_{12}(x) [m_1(y_1) \otimes m_2(y_2)]
\]

\[
= \text{Tr} \rho_1(x) m_1(y_1) \text{Tr} \rho_2(y_1, x) m_2(y_2)
\]

\[
= p_{M_1}(y_1|x) p_{M_2}(y_2|y_1, x),
\]

\[
\text{Tr} \rho_{12}(x) [m_1(y_1) \otimes m_2(y_2)]
\]

\[
= \text{Tr} \rho_1(x) m_1(y_1) \text{Tr} \rho_2(y_1, x) m_2(y_2)
\]

\[
= p_{M_1}(y_1|x) p_{M_2}(y_2|y_1, x),
\]
where \( \rho_1(x) = \text{Tr}_2 \rho_{12}(x), \rho_2(y_1, x) = \frac{\text{Tr}_1 \rho_{12}(x)[m_1(y_1) \otimes I_2]}{\text{Tr}_1 \rho_{12}(x)[m_1(y_1) \otimes I_2]} \), so that

\[
\text{Tr} \rho_{12}(x)[m_1(y_1) \otimes I_2] = \text{Tr} \rho_1(x) m_1(y_1) = p_{M_1}(y_1|x),
\]

and \( \rho_2 = \int \int \rho_2(y_1, x) p_{M_1}(y_1|x) \pi(dx) \mu(dy_1) \) while \( \rho_1 = \int \rho_1(x) \pi(dx) \). It follows

\[
\begin{align*}
\text{Tr} \rho_{12}(x)[m_1(y_1) \otimes I_2] &= \text{Tr} \rho_1(x) m_1(y_1) = p_{M_1}(y_1|x) \\
&= \int \int \rho_2(y_1, x) p_{M_1}(y_1|x) \pi(dx) \mu(dy_1)
\end{align*}
\]

whence taking the infimum over decompositions \( (5) \), we obtain \( (3) \). \( \square \)

Let \( H \) be a Hamiltonian in the Hilbert space \( \mathcal{H} \) of the quantum system, \( E \) a positive number. Then the energy-constrained classical capacity of the channel \( M \) is equal to

\[
C(M, H, E) = \sup_{\mathcal{E}, \text{Tr} \bar{\rho}_E H \leq E} I(\mathcal{E}, M),
\]

where maximization is over the input ensembles of states \( \mathcal{E} \) satisfying the energy constraint \( \text{Tr} \bar{\rho}_E H \leq E \), as shown in \( [11] \), proposition 1.

If \( h_M(\bar{\rho}_E) < +\infty \), then

\[
I(\mathcal{E}, M) = h_M(\bar{\rho}_E) - \int h_M(\rho(x)) \pi(dx).
\]

Note that the measurement channel is entanglement-breaking \( [6] \) hence its classical capacity is additive and is given by the one-shot expression \( (6) \). By using \( (7), (2) \), we obtain

\[
C(M, H, E) = \sup_{\rho: \text{Tr} \rho H \leq E} [h_M(\rho) - e_M(\rho)].
\]

### 3 Gaussian maximizers for multimode bosonic Gaussian observable

Consider now multimode bosonic Gaussian system with the quadratic Hamiltonian \( H = R \xi R^\dagger \), where \( \xi > 0 \) is the energy matrix, and \( R = [q_1, p_1, \ldots, q_s, p_s] \)
is the row vector of the bosonic position-momentum observables, satisfying the canonical commutation relation

\[ [R^t, R] = i\Delta, \quad \Delta = \text{diag} \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \]

(see e.g. [4, 1]). From now on we will consider only states with finite second moments. For such states \( h_M(\rho) \leq h_M(\rho_\alpha) < +\infty \), where \( \alpha \) is the covariance matrix of \( \rho \), by the maximum entropy principle. For centered states (i.e. states with vanishing first moments) the covariance matrix and the matrix of second moments coincide and are equal to

\[ \alpha = \text{Re Tr} R^t \rho R. \]

The energy constraint reduces to \(^1\)

\[ \text{Sp} \alpha \epsilon \leq E. \quad (9) \]

We denote the set of all states \( \rho \) with the fixed covariance matrix \( \alpha \) by \( \mathcal{E}(\alpha) \) and we will study the following \( \alpha \)-constrained capacity

\[ C(M; \alpha) = \sup_{\mathcal{E}, \rho \in \mathcal{E}(\alpha)} I(\mathcal{E}, M) = \sup_{\rho \in \mathcal{E}(\alpha)} [h_M(\rho) - e_M(\rho)]. \quad (10) \]

With the Hamiltonian \( H = \text{Re} R^t \), the energy-constrained classical capacity of observable \( M \) is

\[ C(M; H, E) = \sup_{\alpha: \text{Sp} \alpha \epsilon \leq E} C(M; \alpha). \]

We will be interested in the approximate position-momentum measurement (observable, POVM)

\[ M(d^{2s}z) = D(z)\rho_\beta D(z)^\dagger \frac{d^{2s}z}{(2\pi)^s}, \quad (11) \]

where \( \rho_\beta \) is centered Gaussian density operator with the covariance matrix \( \beta \) and

\[ D(z) = \exp i \sum_{j=1}^{s} (y_j q_j - x_j p_j), \quad z = [x_1, y_1, \ldots, x_s, y_s]^t \in \mathbb{R}^{2s} \]

\(^1\)We denote Sp trace of \( s \times s \)-matrices as distinct from trace of operators on \( \mathcal{H} \).
are the unitary displacement operators. Thus \( \mu(dz) = \frac{dz}{2\pi} \) and the operator-valued density of POVM (11) is \( m(z) = D(z)\rho_\beta D(z)^* \).

In what follows we will consider \( n \) independent copies of our bosonic system on the Hilbert space \( \mathcal{H}^\otimes n \). We will supply all the quantities related to \( k \)-th copy (\( k = 1, \ldots, n \)) with upper index \( (k) \), and we will use tilde to denote quantities related to the whole collection on \( n \) copies. Thus

\[
\tilde{z} = \begin{bmatrix} z^{(1)} \\ \vdots \\ z^{(n)} \end{bmatrix}, \quad D(\tilde{z}) = D(z^{(1)}) \otimes \cdots \otimes D(z^{(n)})
\]

and

\[
M^\otimes n(d\tilde{z}) = \hat{m}(\tilde{z})\mu(d\tilde{z}) = \left[ m(z^{(1)}) \otimes \cdots \otimes m(z^{(n)}) \right] \mu(dz^{(1)}) \cdots \mu(dz^{(n)}).
\]

**Lemma 2.** Let \( O = [O_{kl}]_{k,l=1,\ldots,n} \) be a real orthogonal \( n \times n \)-matrix and \( U \) – the unitary operator on \( \mathcal{H}^\otimes n \) corresponding to the linear symplectic transformation

\[
\tilde{R} = [ R^{(1)}, \ldots, R^{(n)} ] \rightarrow \tilde{R} O,
\]

so that

\[
U^*D(\tilde{z})U = D(O \tilde{z}). \tag{12}
\]

Then for any state \( \tilde{\rho} \) on \( \mathcal{H}^\otimes n \)

\[
e_{M^\otimes n}(\tilde{\rho}) = e_{M^\otimes n}(U \tilde{\rho}U^*). \tag{13}
\]

**Proof.** The covariance matrix \( \tilde{\beta} \) of \( \rho_\beta^\otimes n \) is block-diagonal, \( \tilde{\beta} = [\delta_{kl}\beta]_{k,l=1,\ldots,n} \), hence \( O^t \tilde{\beta} O = \tilde{\beta} \). Thus we have \( U^* \rho_\beta^\otimes n U = \rho_\beta^\otimes n \), and taking into account (12),

\[
U^*\hat{m}(\tilde{z})U = D(O \tilde{z})\rho_\beta^\otimes n D(O \tilde{z})^* = \hat{m}(O \tilde{z}).
\]

Therefore for any state \( \tilde{\sigma} \) on \( \mathcal{H}^\otimes n \) the output probability density of the measurement channel \( \tilde{M} = M^\otimes n \) corresponding to the input state \( U\tilde{\sigma}U^* \) is

\[
p_{U\tilde{\sigma}U^*}(\tilde{z}) = \text{Tr} \left( U\tilde{\sigma}U^* \right) \hat{m}(\tilde{z}) = \text{Tr} \tilde{\sigma} \hat{m}(O\tilde{z}) = p_{\tilde{\sigma}}(O\tilde{z}). \tag{14}
\]

Hence, by using orthogonal invariance of the Lebesgue measure,

\[
h_{M^\otimes n}(U\tilde{\sigma}U^*) = h_{M^\otimes n}(\tilde{\sigma}).
\]
If \( \tilde{\mu} = \int_X \tilde{\mu}(x) \pi(dx) \), then \( U\tilde{\mu} U^* = \int_X (U\tilde{\mu}(x) U^*) \pi(dx) \), and taking \( \tilde{\sigma} = \tilde{\mu}(x) \) in the previous formula, we deduce
\[
\int_X h_{M \otimes n}(U\tilde{\mu}(x) U^*) \pi(dx) = \int_X h_{M \otimes n}(\tilde{\mu}(x)) \pi(dx),
\]
hence (13) follows. □

**Lemma 3.** Let \( M \) be the Gaussian measurement (11). For any state \( \rho \) with finite second moments \( e_M(\rho) \geq e_M(\rho_\alpha) \) where \( \alpha \) is the covariance matrix of \( \rho \).

**Proof.** The proof follows the pattern of Lemma 1 from the paper of Wolf, Giedke and Cirac [17]. Without loss of generality we can assume that \( \rho \) is centered. We have
\[
e_M(\rho) = \frac{1}{n} e_{M \otimes n}(\rho \otimes n) \geq \frac{1}{n} \sum_{k=1}^{n} e_M(\tilde{\rho}(k)),
\]
where \( \tilde{\rho} = U\rho \otimes n U^* \) with symplectic unitary \( U \) in \( H^{\otimes n} \), corresponding to an orthogonal matrix \( O \) as in lemma 2 and \( \tilde{\rho}(k) \) is the \( k \)-th partial state of \( \tilde{\rho} \).

Step (1) follows from the additivity (4). Step (2) follows from lemma 2, and step (3) follows from the superadditivity of \( e_M \) (lemma 1). The final step of the proof
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e_M(\tilde{\rho}(k)) \geq e_M(\rho_\alpha)
\]
uses ingeniously constructed \( U \) from [17] and lower semicontinuity of \( e_M \) (lemma 1). Namely, \( n = 2^m \), and \( U \) corresponds via (12) to the following special orthogonal matrix
\[
O = [O_{kl}]_{k,l=1,...,n} = H^{\otimes m}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]
Every row of the \( n \times n \)-matrix \( O \) except the first one which has all the elements 1, has \( n/2 = 2^{m-1} \) elements equal to 1 and \( n/2 \) elements equal to -1. Then the quantum characteristic function of the states \( \tilde{\rho}(k) \), \( k = 2, \ldots, n \) is equal to \( \phi(z/\sqrt{n})^{n/2} \phi(-z/\sqrt{n})^{n/2} \), where \( \phi(z) \) is the quantum characteristic function of the state \( \rho \). This allows to apply Quantum Central Limit Theorem [18] to show that \( \tilde{\rho}(k) \to \rho_\alpha \) as \( n \to \infty \), in a uniform way, implying (16), see [17] for detail. □

**Theorem 1.** The optimizing density operator \( \rho \) in (10) is the (centered) Gaussian density operator \( \rho_\alpha \):
\[
C(M; \alpha) = h_M(\rho_\alpha) - e_M(\rho_\alpha), \quad (17)
\]
and hence

\[ C(M, H, E) = \max_{\alpha:Sp\alpha \leq E} C(M; \alpha) = \max_{\alpha:Sp\alpha \leq E} [h_M(\rho_\alpha) - e_M(\rho_\alpha)]. \quad (18) \]

Proof. Lemma 3 implies that for any \( \rho \) with finite second moments \( e_M(\rho) \geq e_M(\rho_\alpha) \) where \( \alpha \) is the covariance matrix of \( \rho \). On the other hand, by the maximum entropy principle, \( h_M(\rho) \leq h_M(\rho_\alpha) \). Hence (17) is maximized by a Gaussian density operator. □

Remark. The proof of lemma 2 and hence of theorem 1 can be extended to a general Gaussian observable \( M \) in the sense of [6], [14], defined via operator-valued characteristic function of the form

\[ \phi_M(w) = \exp \left( i R K w - \frac{1}{2} w^t \beta w \right), \quad (19) \]

where \( K \) is a scaling matrix, \( \beta \geq \pm \frac{i}{2} K^t \Delta K \), by using this function to obtain generalization of the relation (14) for the measurement probability densities. The case (11) corresponds to the type 1 Gaussian observable with \( K = I_{2s} \).

Hypothesis of Gaussian Maximizers (HGM): Let \( M \) be an arbitrary Gaussian observable. Then the optimal ensemble for (2) and hence for (6) is Gaussian, more precisely it consists of (properly squeezed) coherent states with the displacement parameter having Gaussian probability distribution.

For Gaussian measurement channels of the type 1 (essentially of the form (11), see [14] for complete classification) and Gaussian states \( \rho_\alpha \) satisfying the “threshold condition” we have

\[ e_M(\rho_\alpha) = \min_{\rho} h_M(\rho), \quad (20) \]

with the minimum attained on a squeezed coherent state, which implies the validity of the HGM and an efficient computation of \( C(M, H, E) \), see [11]. On the other hand, the problem remains open in the case where the “threshold condition” is violated, and in particular, for all Gaussian measurement channels of the type 2, with the generic example of the energy-constrained approximate measurement of the position \( [q_1, \ldots, q_s] \) subject to Gaussian noise (see [12], where the entanglement-assisted capacity of such a measurement was computed). In the following section we discuss in some detail the HGM in this case for one mode system.
4 Gaussian measurements in one mode

Our framework in this section will be one bosonic mode described by the canonical position and momentum operators $q, p$. We recall that

$$D(x, y) = \exp i (yq - xp), \quad x, y \in \mathbb{R}$$

are the unitary displacement operators.

We will be interested in the observable

$$M(dx dy) = D(x, y)\rho_\beta D(x, y)^\dagger \frac{dx dy}{2\pi},$$

where $\rho_\beta$ is centered Gaussian density operator with the covariance matrix

$$\beta = \begin{bmatrix} \beta_q & 0 \\ 0 & \beta_p \end{bmatrix}; \quad \beta_q\beta_p \geq \frac{1}{4}. \quad (22)$$

Let $\rho_\alpha$ be a centered Gaussian density operator with the covariance matrix

$$\alpha = \begin{bmatrix} \alpha_q & 0 \\ 0 & \alpha_p \end{bmatrix}. \quad (23)$$

The problem is to compute $e_M(\rho_\alpha)$ and hence the classical capacity $C(\rho_\alpha)$ for the oscillator Hamiltonian $H = \frac{1}{2} (q^2 + p^2)$ (as shown in the Appendix of [12], we can restrict to Gaussian states $\rho_\alpha$ with the diagonal covariance matrix in this case). The energy constraint (9) takes the form

$$\alpha_q + \alpha_p \leq 2E. \quad (24)$$

The measurement channel corresponding to POVM (21) acts on the centered Gaussian state $\rho_\alpha$ by the formula

$$M \quad : \quad \rho_\alpha \rightarrow p_{p_{\rho_\alpha}}(x, y) \quad (25)$$

$$= \frac{1}{\sqrt{2\pi (\alpha_q + \beta_q)(\alpha_p + \beta_p)}} \exp \left[ -\frac{x^2}{2(\alpha_q + \beta_q)} - \frac{y^2}{2(\alpha_p + \beta_p)} \right],$$

so that

$$h_M(\rho_\alpha) = \frac{1}{2} \log (\alpha_q + \beta_q)(\alpha_p + \beta_p) + c. \quad (26)$$

In this expression $c$ is a fixed constant depending on the normalization of the underlying measure $\mu$ in (11). It does not enter the information quantities which are differences of the two differential entropies.
Assuming validity of the HGM, we will optimize over ensembles of squeezed coherent states

$$\rho_{x,y} = D(x, y) \rho_\Lambda D(x, y)^*, \quad (x, y) \in \mathbb{R}^2,$$

where $\rho_\Lambda$ is centered Gaussian state with correlation matrix $\Lambda = \begin{bmatrix} \delta & 0 \\ 0 & 1/(4\delta) \end{bmatrix}$, and the vector $(x, y)$ has centered Gaussian distribution with covariance matrix $\begin{bmatrix} \gamma_q & 0 \\ 0 & \gamma_p \end{bmatrix}$. Then the average state $\bar{\rho}_C$ of the ensemble is centered Gaussian $\rho_\alpha$ with the covariance matrix (23), where

$$\alpha_q = \gamma_q + \delta, \quad \alpha_p = \gamma_p + 1/(4\delta),$$

hence

$$\frac{1}{4\alpha_p} \leq \delta \leq \alpha_q. \quad (27)$$

For this ensemble

$$\int h_M(\rho_{x,y}) \pi(dx \, dy) = h_M(\rho_\Lambda) = \frac{1}{2} \log (\delta + \beta_q) (1/(4\delta) + \beta_p) + c.$$

Then the hypothetical value

$$e_M(\rho_\alpha) = \min_{1/(4\alpha_p) \leq \delta \leq \alpha_q} \frac{1}{2} \log (\delta + \beta_q) (1/(4\delta) + \beta_p) + c. \quad (28)$$

The derivative of the minimized expression vanishes for $\delta = \frac{1}{2} \sqrt{\frac{\beta_q}{\beta_p}}$. Thus, depending on the position of this value with respect to the interval (27), we obtain three possibilities:

| Table 1 |
|---------|
| **range** | **HGM** | **L:** $\frac{1}{2}\sqrt{\frac{\beta_q}{\beta_p}} < \frac{1}{4}\alpha_p$ | **C:** $\frac{1}{4}\alpha_p \leq \frac{1}{2}\sqrt{\frac{\beta_q}{\beta_p}} \leq \alpha_q$ | **R:** $\alpha_q < \frac{1}{2}\sqrt{\frac{\beta_q}{\beta_p}}$ |
| **$\delta_{opt}$** | open | valid | open |
| **$e_M(\rho_\alpha) - c$** | $\frac{1}{2} \log \left[ \frac{1}{4\alpha_p} + \beta_q \right]$ | $\frac{1}{2} \log \left( \sqrt{\beta_q\beta_p} + 1/2 \right)$ | $\frac{1}{2} \log \left[ \frac{1}{4\alpha_q} + \beta_q \right]$ |
| **$C(M; \alpha)$** | $\frac{1}{2} \log \frac{\alpha_q + \beta_q}{\alpha_p + \beta_q}$ | $\frac{1}{2} \log \left( \frac{\alpha_q + \beta_q}{\beta_q\beta_p + 1/2} \right)$ | $\frac{1}{2} \log \frac{\alpha_p + \beta_p}{\alpha_q + \beta_q}$ |
Here the column C corresponds to the case where the “threshold condition” holds, implying (20). Then the full validity of the HGM in much more general multimode situation was established in [11]. All the quantities in this column as well as the value of \( C(M, H, E) \) in the central column of the table 2 were obtained in that paper as an example. On the other hand, the HGM remains open in the cases of mutually symmetric columns L and R (for the derivation of the quantities in column L of tables 1, 2 see Appendix).

Maximizing \( C(M; \alpha) \) over \( \alpha_q, \alpha_p \) which satisfy the energy constraint (24) (with the equality): \( \alpha_q + \alpha_q = 2E \), we obtain \( C(M, H, E) \) depending on the signal energy \( E \) and the measurement noise variances \( \beta_q, \beta_p \):

\[
\log \left( \frac{\sqrt{1+8E\beta_q+4\beta_q^2}-1}{2\beta_q} \right)
\]

Table 2: \( C(M, H, E) \)

| L: HGM          | C: [11] | R: HGM          |
|-----------------|---------|-----------------|
| \( \beta_q \leq \beta_p; E < E(\beta_q, \beta_q) \) | \( E \geq E(\beta_q, \beta_q) \lor E(\beta_q, \beta_p) \) | \( \beta_p \leq \beta_q; E < E(\beta_q, \beta_p) \) |
| \( \log \left( \frac{\sqrt{1+8E\beta_q+4\beta_q^2}-1}{2\beta_q} \right) \) | \( \log \left( \frac{E+(\beta_q+\beta_p)/2}{\beta_q\beta_p+1/2} \right) \) | \( \log \left( \frac{\sqrt{1+8E\beta_p+4\beta_p^2}-1}{2\beta_p} \right) \) |

where we introduced the “energy threshold function”

\[
E(\beta_1, \beta_2) = \frac{1}{2} \left( \beta_1 - \beta_2 + \sqrt{\frac{\beta_1}{\beta_2}} \right).
\]

Let us stress that, opposite to column C, the values of \( C(M, H, E) \) in the L and R columns are hypothetic, conditional upon validity of the HGM. Looking into the left column, one can see that \( C(M; \alpha) \) and \( C(M, H, E) \) do not depend at all on \( \beta_p \). Thus we can let \( \beta_p \to +\infty \), and in fact set \( \beta_p = +\infty \), which corresponds to the approximate measurement of position \( q \) with Gaussian noise described by POVM

\[
M(dx) = \exp \left[ -\frac{(q-x)^2}{2\beta_q^2} \right] \frac{dx}{\sqrt{2\pi\beta_q}} = D(x, 0) e^{-q^2/2\beta_q} D(x, 0)^* \frac{dx}{\sqrt{2\pi\beta_q}}, \tag{29}
\]

which belongs to type 2 according to the classification of [14]. In other words, one makes the “classical” measurement of the observable

\[
X = q + \xi, \quad \xi \sim \mathcal{N}(0, \beta_q),
\]

with the quantum energy constraint \( \text{Tr} \rho(q^2 + p^2) \leq 2E \).
The measurement channel corresponding to POVM \( \{29\} \) acts on the centered Gaussian state \( \rho_\alpha \) by the formula

\[
M : \rho_\alpha \rightarrow p_{\rho_\alpha}(x) = \frac{1}{\sqrt{2\pi (\alpha_q + \beta_q)}} \exp \left[ -\frac{x^2}{2(\alpha_q + \beta_q)} \right].
\]

(30)

In this case we have

\[
h_M(\rho_\alpha) = \frac{1}{2} \log (\alpha_q + \beta_q) + c,
\]

(31)

\[
e_M(\rho_\alpha) = \frac{1}{2} \log (1/(4\alpha_p) + \beta_q) + c,
\]

(32)

which differ from the values in the case of finite \( \beta_p \rightarrow +\infty \) by the absence of the factor \( (\alpha_p + \beta_p) \) under the logarithms, while the difference \( C(M; \alpha) = h_M(\rho_\alpha) - e_M(\rho_\alpha) \) and the capacity \( C(M, H, E) \) have the same expressions as in that case (column L).

For \( \beta_q = 0 \) (sharp position measurement, type 3 of \[14\]) the HGM is valid with

\[
C(M, H, E) = \log 2E.
\]

This follows from the general upper bound

\[
C(M, H, E) \leq \log \left( 1 + \frac{E^{1/2}}{\beta_q + 1/2} \right) = \log \left( \frac{2(E + \beta_q)}{1 + 2\beta_q} \right)
\]

(33)

for \( \beta_q \geq 0 \) (Eq. (28) in \[4\], see also Eq. (5.39) in \[2\]).

5 The dual problem: accessible information

Let us sketch here ensemble-observable duality \[3, 20, 7\] (see \[9\] for detail of mathematically rigorous description in the infinite dimensional case).

Let \( \mathcal{E} = \{\pi(dx), \rho(x)\} \) be an ensemble, \( \mu(dy) \) a \( \sigma \)-finite measure and \( M = \{M(dy)\} \) an observable having operator density \( m(y) = M(dy)/\mu(dy) \) with values in the algebra of bounded operators in \( \mathcal{H} \). The dual pair ensemble-observable \( \{\mathcal{E'}, M'\} \) is defined by the relations

\[
\mathcal{E'} : \quad \pi'(dy) = \text{Tr} \bar{\rho}_E M(dy), \quad \rho'(y) = \frac{\bar{\rho}_E^{1/2} m(y) \bar{\rho}_E^{1/2}}{\text{Tr} \bar{\rho}_E m(y)};
\]

(34)

\[
M' : \quad M'(dx) = \bar{\rho}_E^{-1/2} \rho(x) \bar{\rho}_E^{-1/2} \pi(dx),
\]

(35)

13
Then the average states of both ensembles coincide

\[ \bar{\rho}_E = \bar{\rho}_{E'} \]  

and the joint distribution of \( x, y \) is the same for both pairs \( (E, M) \) and \( (E', M') \) so that

\[ I(E, M) = I(E', M'). \]  

Moreover,

\[ \sup_M I(E, M) = \sup_{E': \bar{\rho}_{E'} = \bar{\rho}_E} I(E', M'), \]  

where the supremum in the right-hand side is taken over all ensembles \( E' \) satisfying the condition \( \bar{\rho}_{E'} = \bar{\rho}_E \). It can be shown (\cite{9}, Proposition 4), that the supremum in the lefthand side remains the same if it is taken over all observables \( M \) (not only of the special kind with the density we started with), and then it is called the accessible information \( A(E) \) of the ensemble \( E \). Thus

\[ A(E) = \sup_{E': \bar{\rho}_{E'} = \bar{\rho}_E} I(E', M'). \]  

Since the application of the duality to the pair \( \{E', M'\} \) results in the initial pair \( \{E, M\} \), we also have

\[ A(E') = \sup_{M'} I(E', M') = \sup_{E: \bar{\rho}_E = \bar{\rho}_E} I(E, M). \]  

Coming to the case of bosonic mode, we fix the Gaussian state \( \rho_\alpha \) and restrict to ensembles \( \mathcal{E} \) with \( \bar{\rho}_\mathcal{E} = \rho_\alpha \). Let \( M \) be the measurement channel corresponding to POVM (21). Then according to formulas (34), the dual ensemble \( \mathcal{E}' = \{p'(x, y), \rho'(x, y)\} \), where \( p'(x, y) \) is the Gaussian probability density (25) and

\[ \rho'(x, y) = [p'(x, y)]^{-1} \sqrt{\rho_\alpha D(x, y) \rho_\beta D(x, y)^*} \sqrt{\rho_\alpha}. \]  

By using the formula for \( \sqrt{\rho_1 \rho_2} = \sqrt{\rho_1} \) where \( \rho_1, \rho_2 \) are Gaussian operators (see \cite{19} and also Corollary in the Appendix of \cite{13}), we obtain

\[ \rho'(x, y) = D(x', y') \rho_\alpha' D(x', y')^* = \rho_{\alpha'}(x', y'), \]  

where

\[ \alpha' = \alpha - \gamma', \quad \gamma' = \kappa (\alpha + \beta)^{-1} \kappa, \quad \left[ \begin{array}{c} x' \\ y' \end{array} \right] = \kappa (\alpha + \beta)^{-1} \left[ \begin{array}{c} x \\ y \end{array} \right]. \]  

(39)
and
\[ \kappa = \sqrt{I + (2\alpha\Delta^{-1})^{-2}} = \alpha \sqrt{I + (2\Delta^{-1})^{-2}}. \]  
(40)

Since \([x, y]^t \sim \mathcal{N}(0, \alpha + \beta)\), then from second and third equations in (39) we obtain
\[ [x', y']^t \sim \mathcal{N}(0, \kappa(\alpha + \beta)^{-1}) = \mathcal{N}(0, \gamma'). \]
By denoting \(p_{\gamma'}(x', y')\) the density of this normal distribution, we can equivalently rewrite the ensemble \(E'\) as \(E' = \{p_{\gamma'}(x', y'), \rho_{\alpha'}(x', y')\}\) with the average state \(\rho_{\alpha}\),
\[ \alpha = \alpha' + \gamma'. \]
Then HGM is equivalent to the statement
\[ A(E') = C(M; \alpha), \]
where the values of \(C(M; \alpha)\) are given in the table 1, however they should be reexpressed in terms of the ensemble parameters \(\gamma', \alpha'\). In [13] we treated the case C in multimode situation, establishing that the optimal measurement is Gaussian, and described it. Here we will discuss the case L (R is similar) and show that for large \(\beta_p\) (including \(\beta_p = +\infty\)) the HGM is equivalent to the following: the value of the accessible information
\[ A(E') = C(M; \alpha) = \frac{1}{2} \log \frac{\alpha_q + \beta_q}{4\alpha_p + \beta_q}, \]
is attained on the sharp position measurement \(M'_0(d\xi) = |\xi\rangle\langle\xi|d\xi\) (in fact this refers to the whole domain L: \(1/2\sqrt{\frac{\alpha_q}{\beta_p}} < \frac{1}{4\alpha_p}\), which however has rather cumbersome description in the new variables \(\gamma', \alpha'\), cf. [13]).

In the one mode case we are considering the matrix \(\alpha\) is given by (23), \(\beta\) by (22), and \(\Delta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\), so that \((2\Delta^{-1} \alpha)^2 = -(4\alpha_q\alpha_p) I\). Computations according to (39) and (40) give
\[ \alpha' = \begin{bmatrix} \alpha_q' & 0 \\ 0 & \alpha_p' \end{bmatrix} = \begin{bmatrix} \frac{\alpha_q(\beta_q+1/(4\alpha_p))}{\alpha_q+\beta_q} & 0 \\ 0 & \frac{\alpha_p(\beta_p+1/(4\alpha_q))}{\alpha_p+\beta_p} \end{bmatrix}. \]
(41)

But under the sharp position measurement \(M'_0(d\xi) = |\xi\rangle\langle\xi|d\xi\), one has
\[ p(\xi|x', y') = \langle \xi | \rho_{\alpha'}(x', y') | \xi \rangle = \mathcal{N}(x', \alpha_q'). \]

\[ \text{In the formulas below } p(\xi) = \mathcal{N}(m, \alpha) \text{ means that } p(\xi) \text{ is Gaussian probability density with mean } m \text{ and variance } \alpha. \]
while \( \langle \xi | \rho_\alpha | \xi \rangle = \mathcal{N}(0, \alpha_q) \) (note that \( \tilde{\rho}_\xi = \tilde{\rho}_E = \rho_\alpha \)) and

\[
I(\mathcal{E}', M'_0) = \frac{1}{2} \left[ \log (\alpha_q + \gamma_q) - \log \alpha_q \right] \nonumber \]

\[
= \frac{1}{2} \left[ \log \alpha_q - \log \frac{\alpha_q (\beta_q + 1/4\alpha_p)}{(\alpha_q + \beta_q)} \right] \nonumber \]

\[
= \frac{1}{2} \log \frac{\alpha_q + \beta_q}{(\beta_q + 1/4\alpha_p)}, \tag{42} \]

which is identical to the expression in (45).

In the case of the position measurement channel \( M \) corresponding to POVM (29) (\( \beta_p = +\infty \)) we have \( \alpha'_p = \alpha_p \), otherwise the argument is essentially the same. Thus we obtain that the HGM concerning \( e_M(\rho) \) in the case L is equivalent to the following:

The accessible information of a Gaussian ensemble \( \mathcal{E}' = \{p'(x), \rho'(x)\} \), where

\[ p'(x) = \mathcal{N}(0, \gamma_q'), \quad \rho'(x) = D(x, 0)\rho_{\alpha'}D(x, 0)^*, \]

is given by the expression (42) and attained on the sharp position measurement \( M'_0(dx) = |\xi \rangle \langle \xi | d\xi \).

6 Appendix. Case L in tables 1, 2

By taking the Gaussian ensemble parameters in (28) as

\[ \delta = 1/(4\alpha_p), \quad \gamma_p = 0, \quad \gamma_q = \alpha_q - 1/(4\alpha_p), \tag{43} \]

we get the hypothetic value

\[ e_M(\rho_\alpha) = \frac{1}{2} \log \left( \frac{1}{4\alpha_p} + \beta_q \right) (\alpha_p + \beta_p) + c, \tag{44} \]

hence taking into account (26),

\[ C(M; \alpha) = h_M(\rho_\alpha) - e_M(\rho_\alpha) = \frac{1}{2} \log \frac{\alpha_q + \beta_q}{\beta_q + 1/4\alpha_p}. \tag{45} \]

The constrained capacity is

\[
C(M, H, E) = \max_{\alpha_\alpha + \alpha_q \leq 2E} \frac{1}{2} \left[ \log (\alpha_q + \beta_q) - \log (1/(4\alpha_p) + \beta_q) \right] \tag{46} \]

\[
= \max_{\alpha_p} \frac{1}{2} \left[ \log (2E - \alpha_p + \beta_q) - \log (1/(4\alpha_p) + \beta_q) \right], \nonumber \]

16
where in the second line we took the maximal value \( \alpha_q = 2E - \alpha_p \). Differentiating, we obtain the equation for the optimal value \( \alpha_p \):

\[
4\beta_q \alpha_p^2 + 2\alpha_p - (2E + \beta_q) = 0,
\]

the positive solution of which is

\[
\alpha_p = \frac{1}{4\beta_q} \left( \sqrt{1 + 8E\beta_q + 4\beta_q^2} - 1 \right), \quad (47)
\]

whence

\[
C(M, H, E) = \log \left( \frac{\sqrt{1 + 8E\beta_q + 4\beta_q^2} - 1}{2\beta_q} \right). \quad (48)
\]

The parameters of the optimal Gaussian ensemble are obtained by substituting the value (47) into (43) with \( \alpha_q = 2E - \alpha_p \).

The above derivation concerns the measurement (21) \((\beta_p < \infty)\). The case of the measurement (29) \((\beta_p = +\infty)\) is treated similarly, with (44), (26) replaced by (32), (31). Notably, in this case the expression (48) coincides with the one obtained in [5] by optimizing the information from applying sharp position measurement to noisy optimally squeezed states.

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References

[1] Serafini A., Quantum Continuous Variables: A Primer of Theoretical Methods, CRC Press, Taylor & Francis Group, 2017.

[2] Caves C.M., Drummond P.D. Quantum limits on bosonic communication rates. Rev. Mod. Phys. 1994, vol. 68, N2, 481-537.

[3] Hall M. J. W., Quantum information and correlation bounds, Phys. Rev. A vol. 55, pp. 1050-2947, 1997.

\[\text{\textsuperscript{4}}\text{The author is indebted to M. J. W. Hall for this observation.}\]
[4] Hall M. J. W., Information exclusion principle for complementary observables, Phys. Rev. Lett. 74, 3307, 1995.

[5] Hall M. J. W., Gaussian noise and quantum optical communication, Phys. Rev. A vol. 50, pp. 3295-3303, 1994.

[6] Holevo A. S., Quantum systems, channels, information: a mathematical introduction, 2-nd ed., Berlin/Boston: De Gruyter, 2019.

[7] Holevo A. S., Information capacity of quantum observable, Problems Inform. Transmission, 48:1, 1–10 (2012). arXiv:1103.2615.

[8] Holevo A. S., On the constrained classical capacity of infinite-dimensional covariant channels J. Math. Phys. 57:1 15203 (2016).

[9] Holevo A. S., Gaussian maximizers for quantum Gaussian observables and ensembles, IEEE Trans. Inform. Theory, 2020, doi:10.1109/TIT.2020.2987789.

[10] Giovannetti V., Holevo A. S., Mari A., Majorization and additivity for multimode bosonic Gaussian channels, Theor. Math. Phys., 182:2, 284–293, (2015). arXiv:1405.4066

[11] Holevo A. S., Kuznetsova A. A., Information capacity of continuous variable measurement channel. J. Phys. A: Math. Theor. 53 (2020) 175304 (13pp.).

[12] Holevo A. S., Yashin V. I., Quantum information aspects of approximate position measurement, arXiv:2006.04383.

[13] Holevo A. S., Accessible information of a general quantum Gaussian ensemble, arXiv:2102.01981.

[14] Holevo A. S., The structure of general quantum Gaussian observable, arXiv:2007.02340.

[15] Shirokov M. E., On entropic quantities related to the classical capacity of infinite dimensional quantum channels, Theory of Probability and its Applications, Vol. 52, No. 2, (2007), 250-276. arXiv:quant-ph/0411091
[16] Shirokov M. E., On properties of the space of quantum states and their application to the construction of entanglement monotones, Izv. Math., 74:4 (2010), 849-882.

[17] Wolf M. M., Giedke G., Cirac J. I., Extremality of Gaussian quantum states, Phys. Rev. Lett. 96, 080502 (2006).

[18] Cushen C. D., Hudson R. L., A quantum mechanical central limit theorem. J. Appl. Prob. 8, (1971) 454-469.

[19] Lami L., Das S., Wilde M. M., Approximate reversal of quantum Gaussian dynamics, J. Phys. A, 51:12, 125301, 2018.

[20] Dall’Arno M., D’Ariano G. M., Sacchi M.F., Informational power of quantum measurements, Phys. Rev. A 83, 062304 (2011).

[21] Oreshkov O., Calsamiglia J., Munoz-Tapia R., Bagan E., Optimal signal states for quantum detectors, New J. Phys. 13 (2011), 073032.

[22] Schäfer J., Karpov E., García-Patrón R., Pilyavets O. V., Cerf N. J., Equivalence Relations for the Classical Capacity of Single-Mode Gaussian Quantum Channels, Phys. Rev. Lett. 111, (2013) 030503.

[23] Takeoka M., Guha S., Capacity of optical communication in loss and noise with general Gaussian receivers, Phys. Rev. A 89, 042309 (2014).

[24] Jaehak Lee, Se-Wan Ji, Jiyong Park, Hyunchul Nha, Gaussian benchmark for optical communication aiming towards ultimate capacity, Phys. Rev. A 93, 050302(R) (2016).