Evolution of cooperation is a robust outcome in the prisoner’s dilemma on dynamic networks

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Abstract

Dynamics of evolutionary games strongly depend on underlying networks. We study the coevolutionary prisoner’s dilemma in which players change their local networks as well as strategies (i.e., cooperate or defect). This topic has been increasingly explored by many researchers. On the basis of active linking dynamics [J. M. Pacheco et al., J. Theor. Biol. 243, 437 (2006), J. M. Pacheco et al., Phys. Rev. Lett. 97, 258103 (2006)], we show that cooperation is enhanced fairly robustly. In particular, cooperation evolves when the payoff of the player is normalized by the number of neighbors; this is not the case in the evolutionary prisoner’s dilemma on static networks.

PACS numbers:

I. INTRODUCTION

Existing studies have mainly focused on the mechanisms underlying other-regarding behavior in social dilemma situations. A prototypical model for studying this subject is the prisoner’s dilemma game, where each player either cooperates or defects. From an egoistic perspective, defection is more lucrative and is the unique Nash equilibrium. However, mutual cooperation is more profitable for a population. Maintenance and emergence of cooperation in the prisoner’s dilemma are often explained in terms of evolutionary dynamics, where individuals imitate successful others in the social dynamics nomenclature \cite{1,2}.

Upshots of evolutionary game dynamics generally depend on the network structure underlying interaction between players. In particular, it was recently discovered that heterogeneous networks enable cooperation in the evolutionary prisoner’s dilemma \cite{3,4}. In this scheme, a hub (i.e., a player directly connected to many others) tends to earn higher payoffs than a player with a small degree (i.e., the number of neighbors for a player). Cooperation on hubs can be stabilized, whereas defection on hubs cannot. Cooperation then propagates from hubs to the periphery players. A less-noticed fact underlying these results is that the payoff for each player is defined as the sum of the payoffs earned by playing against all the neighbors. We consider two types of collective payoff. The first is the summed payoff scheme, in which well-connected players tend to obtain a large payoff. In contrast, the present study examines the coevolutionary prisoner’s dilemma in a more adverse condition for cooperators: the average payoff scheme. We show that even under the average payoff scheme, cooperation emerges through coevolutionary dynamics, with the exception of a specific update rule that is known to disfavor cooperation in evolutionary games on static networks.

II. MODEL

A. Prisoner’s dilemma dynamics

We consider the prisoner’s dilemma game in which each player at a node is either a cooperator (C) or a defector (D) in each round. We use \( s_i \in \{C, D\} \) to denote the strategy selected by the \( i \)th player.

We set the payoff matrix to

\[
\begin{pmatrix}
C & D \\
C & 0.5 & -0.5 \\
D & 1 & 0
\end{pmatrix}
\] (1)

The values represent the payoff that the row player obtains. We assume the symmetric prisoner’s dilemma game; the payoff of the column player is determined analogously. For example, if player \( i \) cooperates and player \( j \) defects, \( i \) and \( j \) gain \(-0.5\) and \(1\), respectively.

In one round, each player plays the prisoner’s dilemma against all the neighbors. We consider two types of collective payoff. The first is the summed payoff \( P_i \) defined for the \( i \)th player as the summation of the payoff gained over all the neighbors. The second is the average payoff \( P_i / (\langle k_i \rangle / \langle k \rangle) \), where \( k_i \) is the degree of node \( i \) and
dependent Poisson processes. Because we focus on the strategy update and the AL dynamics are defined as independent Poisson processes. We assume the players alter the strategy according to the so-called Fermi update rule \(10, 22, 23\), unless otherwise stated. According to Fermi rule, each player is selected at the rate \(1/T_a\). The selected player, denoted as \(i\), chooses one of his/her neighbors, denoted as \(j\). The strategy of the \(j\)th player replaces that of the \(i\)th player with probability \(1/(1 + \exp[\beta(P_i - P_j)])\). Otherwise, the strategy of the \(i\)th player replaces that of the \(j\)th player. We set \(\beta = 1\). In the summed payoff scheme, the fraction of cooperators is large for a small \(\beta\) \(17\). The main focus of the following numerical simulations is the average payoff scheme.

### B. Dynamics of networks

The network is assumed to evolve in accordance with the active linking (AL) model proposed by Pacheco and colleagues \(14, 15\). We implement the AL model as follows:

1. Each pair of players \(i\) and \(j\) is selected at the rate \(1/T_a\).
2. If players \(i\) and \(j\) are not adjacent, we connect them with probability \(\alpha_s, \alpha_s\). If nodes \(i\) and \(j\) are adjacent, we disconnect them with probability \(\gamma_{s,s_j}\). If \(k_i = 1\) or \(k_j = 1\), we do not disconnect them to keep the network connected.

We set \(\alpha_C = \alpha_D = 0.15, \gamma_{CC} = 0.1, \gamma_{CD} = \gamma_{DC} = 0.8\), and \(\gamma_{DD} = 0.32\), unless otherwise stated. The values of \(\alpha_C\) and \(\alpha_D\) are smaller than those used in \(14, 15\). As a result, relatively sparse networks emerge. This is aimed at emphasizing heterogeneity in the degree and examining its differential effects in the summed and average payoff schemes.

At the meanfield level, the AL dynamics are described by

\[
\dot{X}_{s_1,s_2} = \frac{1}{T_a}\{\alpha_s, \alpha_s (X_{s_1,s_2}^{\text{max}} - X_{s_1,s_2}) - \gamma_{s_1,s_2}X_{s_1,s_2}\}, \tag{2}
\]

where \((s_1, s_2) = (C, C), (C, D), \) or \((D, D)\), \(X_{s_1,s_2}\) is the number of links that have \(s_1\) and \(s_2\) at the two ends, and \(X_{s_1,s_2}^{\text{max}} = N(N - 1)/2\).

### C. Setup for numerical simulations

We assume \(N = 100\) players. The players initially form the complete graph, unless otherwise stated. The strategy update and the AL dynamics are defined as independent Poisson processes. Because we focus on the stationary state of the coevolutionary dynamics, we set \(T_a = 1\) without loss of generality \(15\). We stop each realization at \(T^\infty = 200 \times \max\{T_a, T_b\}\) and regard the final state as an approximate stationary state. The quantities shown in the following sections are the averages over \(10\) realizations.

### III. RESULTS

#### A. Evolution of cooperation for different payoff schemes

The final fraction of cooperators is shown in Fig. \(1\) for the summed and average payoff schemes, respectively. The results for the summed payoff scheme (Fig. \(1a\)) are consistent with those in \(14, 15\): cooperators flourish if the AL is fast enough relative to the strategy update and many of them exist initially. We find that cooperation also survives in the average payoff scheme (Fig. \(1b\)), although it is slightly more difficult to maintain than in the case of the summed payoff scheme.

Because the two payoff schemes coincide when the network is regular (i.e., homogeneous in the degree), we examine the dispersion in the degree. The mean degree of the nodes in the final networks is shown in Fig. \(1\) for the summed (Fig. \(1c\)) and average (Fig. \(1d\)) payoff schemes, respectively. The coefficient of variance \((CV)\) of the degree, defined by \(\sum_{i=1}^N (k_i - \langle k \rangle)^2 / \langle k \rangle N\), in the final networks is shown for the summed and average payoff schemes in Figs. \(1e\) and \(1f\), respectively. We obtain \(CV \approx 1\) for both payoff schemes, suggesting that the dispersion of the degree is comparable to that implied by the Poisson distribution.

Our main finding in this section is that cooperation is maintained regardless of the payoff scheme if coevolutionary dynamics are considered. We examine the robustness of this finding in the following.

#### B. Heterogeneous networks

Networks with heterogeneous degree distributions do not enhance cooperation if they are static and the average payoff scheme is used \(14, 15\). Although coevolutionary dynamics with the average payoff scheme enhance cooperation, the Poisson degree distribution revealed in Fig. \(1\) may not be heterogeneous enough to sufficiently distinguish between the consequence of the summed payoff scheme and that of the average payoff scheme. Therefore, we perform additional numerical simulations with a modified AL model that yields more heterogeneous networks.

We use a variation of the static network model proposed by Goh and colleagues \(24\). In the original network model, nodes \(i\) and \(j\) \((1 \leq i, j \leq N)\) are connected with probability proportional to \(w_i w_j\), where \(w_i = i^{-\alpha}\). The obtained network has degree distribution \(p(k) \propto k^{-\gamma}\),
where $\gamma = 1 + 1/\alpha$. On the basis of this model, we modify the AL dynamics by replacing the rate at which a link is created between players $i$ and $j$, i.e. $\alpha_s \alpha_j$, by $w_i w_j \alpha_s \alpha_j / N$. We set the normalization constant $N = 0.0015$ so that the average degree in the final network is comparable to that for the original AL dynamics. The rule for removing links remains unchanged.

The numerical results for this variant of the AL model are shown in Fig. 2. The fraction of cooperation is approximately the same as that of the original AL model (Fig. 1(a), (b)); cooperation emerges in both the summed (Fig. 2(a)) and average (Fig. 2(b)) payoff schemes when AL is fast and sufficient cooperators exist initially. As planned, the average degree in the final networks (Fig. 1(c), (d)) is comparable to that obtained from the original AL model. As shown in Fig. 2(e, f), the degree distribution is considerably more heterogeneous than in the case of the original AL model (Fig. 1(e, f)). Coevolutionary dynamics in the average payoff scheme enhance cooperation for both homogeneous and heterogeneous networks.

### C. Different update rules

The difference in update rules can drastically affect evolutionary dynamics on static networks. In this section, we examine the effects of different update rules on coevolutionary dynamics.

We implement birth-death (BD) update rule, death-birth (DB) update rule, and a variant of the DB rule proposed by Nowak. In BD rule, a player is selected for reproduction with probability proportional to the payoff, and his/her strategy is transmitted to a randomly chosen neighbor. In DB rule, a player, selected with equal probability $1/N$, dies, and the neighbors compete for reproduction on this node, such that the reproduction probability is proportional to the payoff. In Nowak’s rule, the reproduction probability of the neighbors in the DB rule is assumed to be proportional to $P_i$ and $(P_i / \langle k_i \rangle)^\delta$ for the summed and average payoff schemes, respectively. The case $\delta = 1$ is equivalent to DB rule. We set $\delta = 20$.

The final fractions of cooperators for the BD rule are shown in Fig. 3(a) and 3(b) for the summed and average payoff schemes, respectively. In this case, it is difficult to achieve cooperation. This is consistent with the fact that cooperation is generally not likely for BD rule in static networks, as compared to other update rules. In contrast, in DB rule (Fig. 3(c, d)) and Nowak’s rule (Fig. 3(e, f)), cooperation emerges when AL is fast and sufficient cooperators exist initially.

### D. Fragile C-C links

We have set $\gamma_{CC} < \gamma_{DD}$ in the previous numerical simulations on the basis of the intuition that neighboring Cs
FIG. 2: (Color online) Results for the modified AL dynamics. See the caption of Fig. 1 for the legends.

may be more willing to remain connected than neighboring Ds. In this case, Cs tend to have larger degrees than Ds. To show that this assumption is not needed for enhancing cooperation, we perform numerical simulations by swapping the values of $\gamma_{CC}$ and $\gamma_{DD}$. We use Fermi update rule and the complete graph as the initial network. The results for the two payoff schemes are shown in Fig. 3. Although the parameter region for enhanced cooperation is relatively small, cooperation is viable when AL dynamics are fast enough and sufficient Cs exist initially.

FIG. 3: (Color online) Final fraction of cooperators for various update rules. We use (a, c, e) summed and (b, d, f) average payoff schemes. We use BD rule in (a) and (b), DB rule in (c) and (d), and Nowak's rule with $\delta = 20$ in (e) and (f).
E. Initially sparse networks

In the previous sections, we performed simulations with the all-to-all connection initially. In this section, we examine the case in which the network is sparse in the beginning. For Fermi update rule, the final fraction of cooperators when the initial network is a random graph with mean degree 8 is shown in Fig. 5. The results are almost the same as those shown in Fig. 1(a, b).

IV. CONCLUSIONS

We have demonstrated that the coevolutionary prisoner’s dilemma game promotes cooperation in the average payoff scheme by extending the results for the summed payoff scheme with AL dynamics [14, 15]. The results are robust against the introduction of heterogeneous connectivity inherent in the players, changes in the update rule, an increase in the rate of removing C-C links, and the density of links in the initial network.

We remark that Fu and colleagues obtained the results similar to ours. Using different link dynamics, they showed that cooperation is enhanced under both summed and average payoff schemes if the link dynamics are fast [34]. In comparison, we have shown that the comparable results also hold true for AL dynamics and that the results are robust with respect to the heterogeneity in the degree, the update rule, and an increased probability of pruning C-C links.

Acknowledgements

We thank Takehisa Hasegawa for fruitful discussions. N.M. acknowledges the support through Grants-in-Aid for Scientific Research (Nos. 20760258, 20540382, and 20115009) from MEXT, Japan.

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