AF EMBEDDINGS AND THE NUMERICAL COMPUTATION OF SPECTRA IN IRRATIONAL ROTATION ALGEBRAS

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Abstract. The spectral analysis of discretized one-dimensional Schrödinger operators is a very difficult problem which has been studied by numerous mathematicians. A natural problem at the interface of numerical analysis and operator theory is that of finding finite dimensional matrices whose eigenvalues approximate the spectrum of an infinite dimensional operator. In this note we observe that the seminal work of Pimsner-Voiculescu on AF embeddings of irrational rotation algebras provides a nice answer to the finite dimensional spectral approximation problem for a broad class of operators including the quasiperiodic case of the Schrödinger operators mentioned above. Indeed, the theory of continued fractions not only provides good matrix models for spectral computations (i.e. the Pimsner-Voiculescu construction) but also yields sharp rates of convergence for spectral approximations of operators in irrational rotation algebras.

1. Introduction

In this paper we address the problem of finding numerical approximations to the spectrum of a bounded linear operator on a complex, separable Hilbert space. This is a very difficult problem in general and there are various ways to attack it as well as various quantities that one may wish to use to approximate the spectrum (e.g. pseudospectra or numerical ranges). A very natural approach is to start with a given operator $T$ acting on a Hilbert space $H$ and “compress” it to a finite dimensional subspace of $H$. This compression is a finite dimensional matrix whose eigenvalues can hopefully be numerically computed and one further hopes that these eigenvalues will somehow approximate the spectrum of the infinite dimensional operator $T$. This approach has been studied by numerous authors (see, for example, [6], [2] and the references therein) and a number of interesting results have been obtained for large classes of operators.

Though the present work grew naturally out of the author’s own study of the method described above (cf. [4, Section 6]) it turns out that an absurdly abstract approach to numerical approximations of spectra actually proves useful in at least one important case. The main result of this paper simply observes how some very specialized and technical work in C*-algebra theory can be used to get excellent approximations to spectra of some important operators. In particular, this applies to discretized one-dimensional Schrödinger operators with quasiperiodic potential. See [1] for a nice treatment of the spectral theory of Almost Mathieu operators (up to 2001) and [3] (and its references) for a recent contribution to the more general case of quasiperiodic Schrödinger operators.

When presented abstractly the basic idea of this note becomes quite simple, but will require a bit of C*-algebra theory to explain. The first basic fact we need is that all finite dimensional C*-algebras have a special form – they are just finite direct sums of finite dimensional matrix...
algebras. We will also need the definition of an AF (“approximately finite dimensional”) algebra; A C*-algebra C is AF if there exist finite dimensional C*-subalgebras C_1 \subset C_2 \subset \cdots \subset C whose union is dense in C. The observation below is the trivial, but key, idea of this paper.

**Observation:** If T ∈ B(H) is an element in an AF C*-subalgebra of B(H) then there exist finite dimensional matrices whose spectral quantities (e.g. pseudospectra or numerical ranges) approximate those of T.

Indeed, if T ∈ C \subset B(H) with C an AF algebra then we can find operators T_n such that \|T - T_n\| \to 0 (hence spectral quantities of T_n are close to those of T) and each T_n is contained in a finite dimensional C*-algebra (hence can be identified with a finite dimensional matrix). There is one subtlety that must be mentioned. Namely, the actual spectrum of T need not be close to the actual spectrum of T (unless T happens to be normal) but singular values, numerical ranges and pseudospectra will be close. Moreover, the actual spectrum of T is equal to the intersection of all its pseudospectra and hence we still get reasonable approximations to the spectrum of T by looking at pseudospectra of the T_n’s. In the case both T and T_n are normal (e.g. self-adjoint Schrödinger operators) then the actual spectra are close in a very strong sense; the Hausdorff distance between their spectra is bounded by \|T - T_n\|.

AF-embeddability (i.e. studying which operators belong to AF algebras) has been considered by numerous authors but most of this work is too general to be of much help in finding explicit matrices which approximate given operators. For example, it is an easy consequence of the spectral theorem that every normal operator is contained in an AF-subalgebra of B(H) but this gives no clue how to actually find the right matrix approximations. However, returning to the seminal paper on AF-embeddability (cf. [8]) one finds that not only are explicit matrix models provided (for operators in irrational rotation algebras) but computable and sharp rates of convergence can be proved.

In section 2 we will review the necessary aspects of the theory of continued fractions. We also state a few simple lemmas that will be needed later.

In section 3 we review the construction of Pimsner-Voiculescu which provides explicit matrix models for AF-embeddings of the irrational rotation algebras. We will not reproduce the proofs of any estimates as these can be found either in the original paper [8] or in [5, Chapter VI].

In section 4 we state and prove the main result for spectral approximations of operators in irrational rotation algebras. The proof really amounts to working out a few estimates as the hard part was accomplished more than 20 years ago in [8].

Finally, in section 5 we show how results of Haagerup-Rørdam can be used to give general “one-sided” rates of convergence. Though the results of this section are not as good as previous sections they have the advantage of always being numerically implementable.

## 2. Preliminaries

In this section we will review some basic facts about continued fractions and point out a few technical (but very elementary) inequalities that we will need later. Though there are a number of excellent books on continued fractions, a classic reference is [7]. The facts we will need, however, can be found in any book on the subject. We also state the definition of pseudospectrum at the very end of this section.
Given an irrational number \( \theta \in (0, 1) \) there exist unique positive integers \( a_1, a_2, \ldots \) such that
\[
\theta = \lim_{n \to \infty} [a_1, \ldots, a_n] = \lim_{n \to \infty} \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}.
\]

Note that if \( \theta \) is given then one can compute the integers \( a_n \) via the following recursive procedure: Define \( \theta_1 = \frac{1}{\theta}, a_1 = \lfloor \theta_1 \rfloor, \theta_{n+1} = \frac{1}{\theta_n - a_n}, a_{n+1} = \lfloor \theta_{n+1} \rfloor \), where \( \lfloor x \rfloor \) is the integer part of a real number \( x \). As is customary, we use \( p_n, q_n \) to denote the numerator and denominator, respectively, of the \( n^{th} \) convergent:
\[
[a_1, \ldots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = \frac{p_n}{q_n}.
\]

The \( p_n \)'s and \( q_n \)'s satisfy the recursive relations
\[
p_n = a_np_{n-1} + p_{n-2}, \quad q_n = a_nq_{n-1} + q_{n-2},
\]
where \( p_0 = 0, p_1 = 1, q_0 = 1 \) and \( q_1 = a_1 \). The following remarkable fact can be found in any book on continued fractions.

**Theorem 2.1.** For every \( n \) one has \( |\theta - \frac{p_n}{q_n}| < \frac{1}{q_nq_{n+1}} < \left( \frac{1}{q_n} \right)^2 \).

We now record a few trivial lemmas that will be used in later estimates.

**Lemma 2.2.** If \( F(k) \) is the \( k^{th} \) Fibonacci number (where \( F(0) = 1 = F(1), F(2) = 2, \ldots \) then for each pair of positive integers \( n \) and \( k \) we have \( q_{n+k} \geq F(k)q_n \).

**Proof.** Use induction and the recursion formula. \( \Box \)

**Lemma 2.3.** \( \sum_{k=0}^{\infty} \frac{1}{F(k)} \leq \frac{2\sqrt{5}}{\sqrt{5}-1} \)

**Proof.** Induction shows that \( F(k) \geq \left( \frac{1+\sqrt{5}}{2} \right)^{k-1} \) and hence the result follows by taking reciprocals and applying the formula for a geometric series. \( \Box \)

**Lemma 2.4.** \( \sum_{k=0}^{\infty} \frac{1}{q_{n+k}} \leq \frac{1}{q_n} \frac{2\sqrt{5}}{\sqrt{5}-1} \)

**Proof.** Apply the two previous lemmas. \( \Box \)

We close this section with the definition of pseudospectra.

**Definition 2.5.** For \( \varepsilon > 0 \), the \( \varepsilon \)-pseudospectrum of \( T \in B(H) \) is
\[
\sigma^{(\varepsilon)}(T) = \{ \lambda \in \mathbb{C} : \| (\lambda - T)^{-1} \| \geq \frac{1}{\varepsilon} \},
\]
where \( \| X^{-1} \| = \infty \) if \( X \in B(H) \) is a non-invertible operator. Note that the usual spectrum \( \sigma(T) \) is contained in \( \sigma^{(\varepsilon)}(T) \) for every \( \varepsilon > 0 \) and, moreover,
\[
\sigma(T) = \bigcap_{\varepsilon > 0} \sigma^{(\varepsilon)}(T).
\]

See [http://web.comlab.ox.ac.uk/projects/pseudospectra](http://web.comlab.ox.ac.uk/projects/pseudospectra) for more, including software for computing pseudospectra.
3. The Pimsner-Voiculescu Construction

In this section we state the technical aspects of the Pimsner-Voiculescu construction that we will need. We see no reason to reproduce proofs as all the details can be found in [5, Sections VI.4 and VI.5].

Throughout this note, $M_k(\mathbb{C})$ will denote the $k \times k$ complex matrices. Let $0 < p < q$ be integers, $\omega = e^{2\pi i \frac{p}{q}}$ and define matrices $u_{\frac{p}{q}}, v_{\frac{p}{q}} \in M_q(\mathbb{C})$ as follows.

$$u_{\frac{p}{q}} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

$$v_{\frac{p}{q}} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \omega & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \omega^2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \omega^3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \omega^{q-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \omega^{q-1}
\end{pmatrix}$$

If we are given an irrational number $\theta \in (0, 1)$ and we let $\frac{p_n}{q_n}$ denote the $n^{th}$ convergent (as in the previous section) then to ease notation we will let $\theta_n = \frac{p_n}{q_n}$, $u_{\theta_n} = u_{\frac{p_n}{q_n}}$ and $v_{\theta_n} = v_{\frac{p_n}{q_n}}$.

**Theorem 3.1.** (Pimsner-Voiculescu) For each natural number $n$ there is a $\ast$-monomorphism $\varphi_n : M_{q_{n-1}}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C}) \hookrightarrow M_{q_n}(\mathbb{C}) \oplus M_{q_{n+1}}(\mathbb{C})$ such that

$$\|\varphi_n(u_{\theta_{n-1}} \oplus u_{\theta_n}) - u_{\theta_n} \oplus u_{\theta_{n+1}}\| < \frac{2\pi}{q_{n-1}}$$

and

$$\|\varphi_n(v_{\theta_{n-1}} \oplus v_{\theta_n}) - v_{\theta_n} \oplus v_{\theta_{n+1}}\| < \frac{\pi}{q_{n-1}} + \frac{4\pi}{q_n}.$$

4. Computing Spectra in Irrational Rotation Algebras

For an arbitrary number $\theta \in (0, 1)$ there is a universal $C^*$-algebra, denoted by $A_\theta$, which is generated by two unitaries $U_\theta, V_\theta$ subject to the commutation relation

$$U_\theta V_\theta = e^{2\pi i \theta} V_\theta U_\theta.$$ 

Universality means that if $\tilde{U}_\theta, \tilde{V}_\theta$ are any other unitary operators satisfying the same relation then there exists a surjective $\ast$-homomorphism $A_\theta \to C^*(\tilde{U}_\theta, \tilde{V}_\theta)$ such that $U_\theta \mapsto \tilde{U}_\theta$ and $V_\theta \mapsto \tilde{V}_\theta$. One crucial fact which we will need is that if $\theta$ is irrational then $A_\theta$ is a simple $C^*$-algebra (i.e. has no non-trivial, closed, two-sided ideals) and hence any $\ast$-homomorphism will be injective in this case (cf. [1, Theorem 1.10]).
In the theorem below \( d_H(\cdot, \cdot) \) denotes the Hausdorff distance between two compact subsets of the complex plane:

\[
d_H(\Sigma, \Lambda) = \max\{\sup_{\sigma \in \Sigma} d(\sigma, \Lambda), \sup_{\lambda \in \Lambda} d(\lambda, \Sigma)\},
\]

where \( d(\sigma, \Lambda) = \inf_{\lambda \in \Lambda} |\sigma - \lambda| \).

**Theorem 4.1.** Let \( \theta \in (0, 1) \) be irrational, \( \theta_n = \frac{\nu_n}{q_n} \), \( \alpha_{\pm 1}, \beta_{\pm 1} \in \mathbb{C} \) be four complex numbers and \( M = \max\{|\alpha_{\pm 1}|, |\beta_{\pm 1}|\} \). If we let \( H_\theta = \alpha_1 U_\theta + \alpha_{-1} U_\theta^* + \beta_1 V_\theta + \beta_{-1} V_\theta^* \) and \( h_\theta_n = \alpha_1 u_\theta_n + \alpha_{-1} u_\theta_n^* + \beta_1 v_\theta_n + \beta_{-1} v_\theta_n^* \) then:

1. (cf. Definition 2.5) For each \( \varepsilon > 0 \) we have

\[
\left( \sigma^{(c)}(h_{\theta_{n-1}}) \cup \sigma^{(c)}(h_\theta_n) \right) \subset \sigma^{(\varepsilon + \varepsilon_n)}(H_\theta) \subset \left( \sigma^{(\varepsilon + 2\varepsilon_n)}(h_{\theta_{n-1}}) \cup \sigma^{(\varepsilon + 2\varepsilon_n)}(h_\theta_n) \right),
\]

where \( \varepsilon_n = 204M\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right) \).

2. (Normal Case) If \( h_{\theta_{n-1}}, h_\theta_n \) and \( H_\theta \) all happen to be normal operators (e.g. self-adjoints) then

\[
d_H(\sigma(h_{\theta_{n-1}}) \cup \sigma(h_\theta_n), \sigma(H_\theta)) \leq 204M\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right).
\]

**Proof.** It follows from the Pimsner-Voiculescu construction that there is an AF algebra \( \mathfrak{A}_\theta \) and a sequence of \(*\)-monomorphisms \( \psi_n : M_{q_n-1}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C}) \rightarrow \mathfrak{A}_\theta \) with the property that

\[
\|\psi_n(u_{\theta_{n-1}} \oplus u_{\theta_n}) - \psi_{n+1}(u_{\theta_n} \oplus u_{\theta_{n+1}})\| < \frac{2\pi}{q_{n-1}}
\]

and

\[
\|\psi_n(v_{\theta_{n-1}} \oplus v_{\theta_n}) - \psi_{n+1}(v_{\theta_n} \oplus v_{\theta_{n+1}})\| < \frac{\pi}{q_{n-1}} + \frac{4\pi}{q_n}.
\]

Due to the exponential growth of the \( q_n \)'s we have that both \( \{\psi_n(u_{\theta_{n-1}} \oplus u_{\theta_n})\} \) and \( \{\psi_n(v_{\theta_{n-1}} \oplus v_{\theta_n})\} \) are Cauchy sequences of unitaries and hence converge to some unitaries in \( \mathfrak{A}_\theta \) which satisfy the same commutation relation as \( U_\theta \) and \( V_\theta \). By universality there exists a (necessarily faithful – by simplicity) \(*\)-homomorphism \( \Psi : \mathfrak{A}_\theta \rightarrow \mathfrak{A}_\theta \) with the property that \( \psi_n(u_{\theta_{n-1}} \oplus u_{\theta_n}) \rightarrow \Psi(U_\theta) \) and \( \psi_n(v_{\theta_{n-1}} \oplus v_{\theta_n}) \rightarrow \Psi(V_\theta) \). More importantly, we have the following estimates:

\[
\|\Psi(U_\theta) - \psi_n(u_{\theta_{n-1}} \oplus u_{\theta_n})\| \leq \sum_{k=0}^{\infty} \|\psi_{n+k}(u_{\theta_{n+k-1}} \oplus u_{\theta_{n+k}}) - \psi_{n+k+1}(u_{\theta_{n+k}} \oplus u_{\theta_{n+k+1}})\|
\]

\[
\leq \sum_{k=0}^{\infty} \frac{2\pi}{q_{n+k-1}}
\]

\[
= 2\pi\left(\frac{1}{q_{n-1}} + \frac{1}{q_n} + \sum_{k=0}^{\infty} \frac{1}{q_{n+k+1}}\right)
\]

\[
\leq 2\pi\left(\frac{1}{q_{n-1}} + \frac{1}{q_n} + \frac{4\pi\sqrt{5}}{\sqrt{5} - 1} \frac{1}{q_{n+1}}\right).
\]

Note that we applied Lemma 2.4 to get the last inequality. A similar argument shows that

\[
\|\Psi(V_\theta) - \psi_n(v_{\theta_{n-1}} \oplus v_{\theta_n})\| \leq \frac{\pi}{q_{n-1}} + \frac{5\pi}{q_n} + \frac{10\pi\sqrt{5}}{\sqrt{5} - 1} \frac{1}{q_{n+1}}.
\]
With these estimates in hand it is immediate that
\[
\|\Psi(H_\theta) - \psi_n(h_{\theta_{n-1}} \oplus h_{\theta_n})\| \leq (|\alpha_1| + |\alpha_2|)(2\pi(\frac{1}{q_{n-1}} + \frac{1}{q_n}) + \frac{4\pi\sqrt{5}}{\sqrt{5} - 1} \frac{1}{q_{n+1}})
+ (|\beta_1| + |\beta_2|)(\frac{\pi}{q_{n-1}} + \frac{5\pi}{q_n} + \frac{10\pi\sqrt{5}}{\sqrt{5} - 1} \frac{1}{q_{n+1}}).
\]

The estimate above, though a bit ugly, is the one which should be used in practice as the constants are smaller than those claimed in the theorem. To get the cleaner statement of the theorem we first note that the recursion formula \(q_{n+1} = a_{n+1}q_n + q_{n-1}\) implies
\[
\frac{1}{q_{n+1}} \leq \frac{1}{q_{n-1}} + \frac{1}{q_n},
\]
and thus
\[
\|\Psi(H_\theta) - \psi_n(h_{\theta_{n-1}} \oplus h_{\theta_n})\| \leq 2M\left(2\pi + \frac{4\pi\sqrt{5}}{\sqrt{5} - 1}\right)\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)
+ 2M\left(5\pi + \frac{10\pi\sqrt{5}}{\sqrt{5} - 1}\right)\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)
= 2M\left(7\pi + \frac{14\pi\sqrt{5}}{\sqrt{5} - 1}\right)\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)
= 14\pi M\left(\frac{3\sqrt{5} - 1}{\sqrt{5} - 1}\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)\right)
\leq 204M\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right).
\]

Now we quote some general results concerning approximation of spectral quantities. Given two operators \(S, T \in B(H)\) one has:

1. (cf. [6, Theorem 3.27]) For every \(\varepsilon > 0\), \(\sigma^{(\varepsilon)}(S) \subset \sigma^{(\varepsilon + \|S - T\|)}(T) \subset \sigma^{(\varepsilon + 2\|S - T\|)}(S)\).
2. If both \(S\) and \(T\) happen to be normal then \(d_H(\sigma(S), \sigma(T)) \leq \|S - T\|\).

With these general facts and the norm estimates above one easily completes the proof by observing that the spectral quantities of \(S \oplus T\) (i.e. pseudo or actual spectrum) are just the union of the spectral quantities for \(S\) and \(T\). \(\square\)

**Remark 4.2.** *Almost tri-diagonality.* Note that \(h_{\theta_n}\) is “almost” a tri-diagonal matrix. It is non-zero in the upper-right and lower-left entries but otherwise is tri-diagonal. In particular, these matrix models are sparse.

**Remark 4.3.** *Discretized Schrödinger operators with polynomial potential.*

The theorem above is stated for a class of operators that includes all of the examples discussed in [1] (i.e. Almost Mathieu operators and a few non-self-adjoint examples). However, similar results hold for a broader class of examples that includes all the one-dimensional discretized Schrödinger operators with polynomial potential. In fact, if \(R(X, Y)\) is any polynomial in non-commuting variables \(X\) and \(Y\) then
\[
\|\Psi(R(U_\theta, V_\theta)) - \psi_n(R(u_{\theta_{n-1}}, v_{\theta_{n-1}}) \oplus R(u_{\theta_n}, v_{\theta_n}))\| = O\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right).
\]
Unfortunately, the constants involved get quite complicated quite quickly but they can be explicitly worked out for a given polynomial should one so desire. Having this norm estimate
one gets similar convergence results for spectral quantities in this setting. Finally, we should point out that the spectral quantities of any operator in an irrational rotation algebra (e.g. Schrödinger operators with arbitrary quasiperiodic potential) can, in principle, be approximated with this method – first approximate by a polynomial and then proceed – however, it seems impossible to control rates of convergence in this generality.

Remark 4.4. These rates of convergence are sharp.

To see this we consider the case that $H_\theta = U_\theta$ and $\theta$ is an irrational number for which the $a_n$'s are uniformly bounded. Then $d_H(\sigma(H_\theta), \sigma(h_{\theta_{n-1}}) \cup \sigma(h_{\theta_n})) \leq 292(\frac{1}{q_{n-1}} + \frac{1}{q_n})$. It is easy to check that $\frac{1}{q_{n-1}} + \frac{1}{q_n} \leq \frac{4 + q_n}{q_{n-1} + q_n}$ and hence if one could prove that $d_H(\sigma(H_\theta), \sigma(h_{\theta_{n-1}}) \cup \sigma(h_{\theta_n})) = o\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)$ then it would follow that $d_H(\sigma(H_\theta), \sigma(h_{\theta_{n-1}}) \cup \sigma(h_{\theta_n})) = o\left(\frac{1}{q_{n-1} + q_n}\right)$. However this is impossible as it would imply that the Lebesgue measure of $\sigma(H_\theta)$ is zero (which it isn’t: $\sigma(U_\theta) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ since there are at most $q_{n-1} + q_n$ points in $\sigma(H_{\theta_{n-1}}) \cup \sigma(H_{\theta_n})$.

Remark 4.5. Implementability. It is, of course, true that for some irrational numbers the strategy proposed in this paper for approximating spectra can’t reasonably be implemented. For example, it can happen that $q_1$ is very small while $q_2$ is far larger than any computer can handle (just take $a_2$ to be huge). On the other hand, the set of irrationals for which the associated sequences $\{a_n\}$ stay bounded is a dense set in $(0,1)$. For example, all irrationals which are the roots of quadratic equations have this property; hence $\sqrt{r}$ for any rational number and these are already dense in $(0,1)$. In other words, for a dense set of irrationals this program is reasonable and hence the question becomes how spectra behave as the parameter $\theta$ changes. But it is well known that spectra of operators in irrational rotation algebras vary continuously in $\theta$ (see the next section) and hence the program suggested in this paper is about as practical as could be hoped for.

Remark 4.6. Uniqueness of convergents. Another basic, but remarkable, property of continued fractions is that if $\theta$ is irrational and $|\theta - \frac{p}{q}| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is necessarily one of the convergents in the continued fraction decomposition of $\theta$ (i.e. if $\theta = [a_1, a_2, \ldots]$ then there is an $n$ such that $\frac{p}{q} = [a_1, \ldots, a_n] = \frac{p_n}{q_n}$ – see [7, Theorem 184]). The point is that if one starts with a rational $\frac{p}{q}$ and only looks at the irrationals $\theta$ which are a distance at most $\frac{1}{2q^2}$ away from $\frac{p}{q}$ then the spectral quantities of $h_{\frac{p}{q}}$ will provide good approximations to those of $H_\theta$ (since $\frac{p}{q}$ necessarily arises in the continued fraction decomposition of all such $\theta$ and hence the theorem above applies). Of course, rates of convergence become trickier in this setting but our only point is that one can simultaneously approximate a small interval of operators with a single matrix model.

5. GENERAL ONE-SIDED RATES OF CONVERGENCE

In our final section we will record a consequence of the following theorem of Haagerup and Rørdam (cf. [1, Corollary 3.5]). Among other things this shows that if one fixes a polynomial in the canonical generators then the spectral quantities vary continuously in $\theta$.

Theorem 5.1. For every $\theta, \theta' \in [0,1)$ there exists a Hilbert space $H$ and injective $\ast$-homomorphisms $\pi_\theta : A_\theta \to B(H)$, $\pi_{\theta'} : A_{\theta'} \to B(H)$ such that

$$\|\pi_{\theta'}(U_{\theta'}) - \pi_\theta(U_\theta)\|, \|\pi_{\theta'}(V_{\theta'}) - \pi_\theta(V_\theta)\| \leq 9\sqrt{6}\pi|\theta - \theta'|.$$
As in the last section we will restrict attention to operators of the form \( H_\theta = \alpha_1 U_\theta + \alpha_1 U_\theta^* + \beta_1 V_\theta + \beta_1 V_\theta^* \) as the estimates are cleaner in this setting. However, the same techniques handle more general polynomials as well. In the theorem below we only get “one-sided” rates of convergence (as opposed to estimates on the Hausdorff distance). The modest advantage, however, is that this strategy is implementable for any irrational number.

We find the following notation convenient: For compacts \( \Lambda, \Sigma \subset \mathbb{C} \) and \( \delta > 0 \) we write \( \Lambda \subset^\delta \Sigma \) if for each \( \lambda \in \Lambda \) there exists \( \sigma \in \Sigma \) such that \( |\lambda - \sigma| < \delta \). Note that \( d_H(\Lambda, \Sigma) < \delta \) if and only if \( \Lambda \subset^\delta \Sigma \) and \( \Sigma \subset^\delta \Lambda \).

**Theorem 5.2.** Let \( \theta \in (0, 1) \) be irrational and for each \( n \in \mathbb{N} \) choose an integer \( p_n \in \{0, 1, \ldots, n - 1\} \) such that \( |\theta - \frac{p_n}{n}| \leq \frac{1}{2n} \). If \( H_\theta \) is an operator polynomial as above, \( h_n = \alpha_1 u_{\frac{p_n}{n}} + \alpha_1 u_{\frac{p_n}{n}}^* + \beta_1 v_{\frac{p_n}{n}} + \beta_1 v_{\frac{p_n}{n}}^* \) is the associated matrix model and \( M = \max\{|\alpha_{\pm 1}|, |\beta_{\pm 1}|\} \) then the following statements hold:
1. For every \( \varepsilon > 0 \), \( \sigma^{(e)}(h_n) \subset \sigma^{(e + \frac{C_1}{\sqrt{n}})}(H_\theta) \), where \( C_1 \leq 36M\sqrt{3\pi} \).
2. If \( H_\theta \) happens to be normal then \( \sigma(h_n) \subset C_1 \sigma(H_\theta) \), where \( C_1 \leq 36M\sqrt{3\pi} \).

**Proof.** To prove this result we first observe that all of the spectral quantities associated to \( h_n \) are contained in the corresponding quantities for \( H_{\frac{p_n}{n}} = \alpha_1 U_{\frac{p_n}{n}} + \alpha_1 U_{\frac{p_n}{n}}^* + \beta_1 V_{\frac{p_n}{n}} + \beta_1 V_{\frac{p_n}{n}}^* \) (since universality gives us a \( * \)-homomorphism taking \( H_{\frac{p_n}{n}} \mapsto h_n \)). The \( \text{Lip}^{1/2} \)-continuity theorem of Haagerup-Rørdam then gives upper bounds on the distance between the spectral quantities of \( H_\theta \) and \( H_{\frac{p_n}{n}} \) which completes the proof. \( \Box \)

**Remark 5.3.** Hausdorff Convergence. It can be shown that \( d_H(\sigma^{(e)}(h_n), \sigma^{(e)}(H_\theta)) \to 0 \) (for every \( \varepsilon > 0 \)). The proof is similar to the proof of [4, Theorem 3.5] and hence will be omitted. However, it does not seem possible to control the rate of convergence in the Hausdorff metric. For example, it is certainly not the case that rates of convergence are controlled by \( |\theta - \frac{p_n}{n}|^\sigma \) for any \( \sigma > 0 \) because many irrational numbers (e.g. transcendental numbers) have extremely good rational approximations. To be more precise, it can happen that a particular \( \theta \) has the property that there are infinitely many co-prime solutions to the inequality \( |\theta - \frac{p}{q}| < \frac{1}{q^k} \) where \( k \) is some fixed natural number. If the Hausdorff distance between spectra could be bounded above by \( |\theta - \frac{p}{q}|^\sigma \) then we would also get an upper bound of \( (\frac{1}{q})^{k\sigma} \). However, when \( k\sigma > 1 \) it would then follow that the spectrum of \( H_\theta \) has Lebesgue measure zero since there are at most \( q \) eigenvalues for the matrix \( h_{\frac{p}{q}} \). Since these operators need not have spectra of measure zero it follows that an upper bound of the form \( |\theta - \frac{p_n}{n}|^\sigma \) is impossible.

**References**
1. F.P. Boca, *Rotation C*-algebras and almost Mathieu operators*, Theta Series in Advanced Mathematics, 1. The Theta Foundation, Bucharest, 2001.
2. A. Böttcher, *C*-algebras in numerical analysis, Irish Math. Soc. Bull. No. 45 (2000), 57–133.
3. J. Bourgain and S. Jitomirskaya, *Absolutely continuous spectrum for 1D quasiperiodic operators*, Invent. Math. 148 (2002), 453–463.
4. N.P. Brown, *Quasidiagonality and the finite section method*, preprint.
5. K.R. Davidson, $C^*$-algebras by example, Fields Institute Monographs 6, American Mathematical Society, Providence, RI, 1996.
6. R. Hagen, S. Roch and B. Silbermann, $C^*$-algebras and numerical analysis. Monographs and Textbooks in Pure and Applied Mathematics, 236. Marcel Dekker, Inc., New York, 2001.
7. G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Fifth edition, The Clarendon Press, Oxford University Press, New York, 1979.
8. M. Pimsner and D. Voiculescu, Imbedding the irrational rotation $C^*$-algebra into an AF-algebra, J. Operator Theory 4 (1980), 201–210.

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