THE ABSOLUTE ORDER OF A PERMUTATION
REPRESENTATION OF A COXETER GROUP

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Abstract. A permutation representation of a Coxeter group $W$ naturally
defines an absolute order. This family of partial orders (which includes the
absolute order on $W$) is introduced and studied in this paper. Conditions
under which the associated rank generating polynomial divides the rank
generating polynomial of the absolute order on $W$ are investigated when $W$
is finite. Several examples, including a symmetric group action on perfect
matchings, are discussed. As an application, a well-behaved absolute order
on the alternating subgroup of $W$ is defined.

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1. Introduction

The Bruhat order on a Coxeter group $W$ is a key ingredient in understanding
the structure of $W$. This order involves both the set of simple reflections $S$ and
the set of all reflections $T$ of $W$: it may be defined by the condition that $u \in W$
is covered by $v \in W$ if there exists $t \in T$ such that $v = tu$ and $\ell_S(v) = \ell_S(u) + 1$,
where $\ell_S : W \to \mathbb{N}$ is the length function with respect to the generating set
$S$. There are two “more coherent” closely related concepts. Replacing the role
of $T$ by $S$ determines an order which was extensively studied in the past three
decades, namely the weak order on $W$. Replacing the role of $S$ by $T$ determines

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the absolute order. The study of maximal chains in the absolute order on the symmetric group is traced at least back to Hurwitz [15]; see also [11, 28]. However, the growing interest in the absolute order is relatively recent and followed the discovery [6, 9] that distinguished intervals in the absolute order, known as the noncrossing partition lattices, are objects of importance in the theory of finite-type Artin groups. For further information on the absolute order, the reader is referred to [1, Section 2.4] [2, 16].

Consider a transitive action of \( W \) on a set \( X \). Motivated by recent work of Rains and Vazirani [20], which introduces and studies the Bruhat order on \( X \), a naturally defined absolute order on \( X \) is introduced in this paper. Our goal is to find conditions under which important enumerative and structural properties of the absolute order on the acting group \( W \) carry over to the absolute order on \( X \); in particular, conditions under which the associated rank generating polynomial divides the rank generating polynomial of the absolute order on \( W \). Several examples, including the symmetric group action on ordered tuples and its conjugation action on fixed point free involutions, are discussed. As an application, a well-behaved absolute order on the alternating subgroup of \( W \) is defined and studied.

2. Basic concepts

Let \( W \) be a Coxeter group with set of reflections \( T \) (for background on Coxeter groups the reader is referred to [7, 8, 14]). The minimum length of a \( T \)-word for an element \( w \in W \) is denoted by \( \ell_T(w) \) and called the absolute length of \( w \). The absolute order on \( W \), denoted by \( \text{Abs}(W) \), is the partial order \( (W, \leq_T) \) defined by letting \( u \leq_T v \) if \( \ell_T(vu^{-1}) = \ell_T(v) - \ell_T(u) \), for \( u, v \in W \). Equivalently, \( \leq_T \) is the reflexive and transitive closure of the relation on \( W \) consisting of the pairs \((u, v)\) of elements of \( W \) for which \( \ell_T(u) < \ell_T(v) \) and \( v = tu \) for some \( t \in T \). For basic properties of \( \text{Abs}(W) \), see [1, Section 2.4].

We will be concerned with the following generalization of the absolute order on \( W \). Consider a transitive action \( \rho \) of \( W \) on a set \( X \). We will write \( wx \) for the result \( \rho(w)(x) \) of the action of \( w \in W \) on an element \( x \in X \).

**Definition 2.0.1.** Fix an arbitrary element \( x_0 \in X \).

(a) The absolute length of \( x \in X \) is defined as \( \ell_T(x) := \min \{ \ell_T(w) : x = wx_0 \} \).

(b) The absolute order on \( X \), denoted \( \text{Abs}(X) \), associated to \( \rho \) is the partial order \( (X, \leq_T) \) defined by letting \( x \leq_T y \) if there exists \( w \in W \) such that \( y = wx \) and \( \ell_T(w) = \ell_T(y) - \ell_T(x) \), for \( x, y \in X \). Equivalently, \( \leq_T \) is the reflexive and transitive closure of the relation on \( X \) consisting of the pairs \((x, y)\) of elements of \( X \) for which \( \ell_T(x) < \ell_T(y) \) and \( y = tx \) for some \( t \in T \).

The present section discusses elementary properties and examples of \( \text{Abs}(X) \). We begin with some comments on Definition 2.0.1.
Remark 2.0.2. (a) A different way to describe the relation \( \leq_T \) on \( X \) is the following. Let \( x_0 \in X \) be fixed, as before, and consider the simple graph \( \Gamma = \Gamma(W, \rho) \) on the vertex set \( X \) whose (undirected) edges are the sets of the form \( \{x, tx\} \) for \( t \in T \) and \( x \in X \). Then for every \( x \in X \), the absolute length \( \ell_T(x) \) is equal to the distance between \( x_0 \) and \( x \) in the graph \( \Gamma \) and for \( x, y \in X \), we have \( x \leq_T y \) if and only if \( x \) lies in a geodesic path in \( \Gamma \) with endpoints \( x_0 \) and \( y \). This description implies that \( \leq_T \) is indeed a partial order on \( X \) and that it coincides with the reflexive and transitive closure of the relation on \( X \) described in Definition 2.0.1 (b).

(b) The isomorphism type of \( \text{Abs}(X) \) is independent of the choice of \( x_0 \in X \). Indeed, consider another base point \( y_0 \in X \) and let \( \text{Abs}(X, x_0) \) and \( \text{Abs}(X, y_0) \) denote the absolute orders on \( X \) with respect to \( x_0 \) and \( y_0 \), respectively. Choose \( w_0 \in W \) so that \( y_0 = w_0 x_0 \) and define a map \( f : X \rightarrow X \) by letting \( f(x) = w_0 x \) for \( x \in X \). Clearly, \( f \) is a bijection and satisfies \( f(x_0) = y_0 \). Moreover, since \( T \) is closed under conjugation, the map \( f \) is an automorphism of the graph \( \Gamma \) considered in part (a). These properties imply that \( f : \text{Abs}(X, x_0) \rightarrow \text{Abs}(X, y_0) \) is an isomorphism of partially ordered sets.

(c) The order \( \text{Abs}(X) \) has minimum element \( x_0 \).

(d) As an easy consequence of the definition of absolute length, we have \( \ell_T(wx) \leq \ell_T(w) + \ell_T(x) \) for all \( w \in W \) and \( x \in X \). \( \square \)

Since the action \( \rho \) is transitive, the set \( X \) may be identified with the set of left cosets of the stabilizer of \( x_0 \in X \) in \( W \). This identification leads to the following reformulation of Definition 2.0.1 which we will often find convenient.

Definition 2.0.3. Let \( H \) be a subgroup of \( W \) and let \( X = W/H \) be the set of left cosets of \( H \) in \( W \).

(a) The absolute length of \( x \in X \) is defined as \( \ell_T(x) := \min \{ \ell_T(w) : w \in x \} \).

(b) The absolute order on \( X \), denoted \( \text{Abs}(X) \), is the partial order \( (X, \leq_T) \) defined by letting \( x \leq_T y \) if there exists \( w \in W \) such that \( y = wx \) and \( \ell_T(w) = \ell_T(y) - \ell_T(x) \), for \( x, y \in X \).

We recall that a partially ordered set (poset) \( P \) with a minimum element \( \hat{0} \) is said to be locally graded with rank function \( \text{rk} : P \rightarrow \mathbb{N} \) if for each \( x \in P \), every maximal chain in the closed interval \([\hat{0}, x]\) of \( P \) has exactly \( \text{rk}(x) + 1 \) elements (for background and terminology on posets we refer to [26, Chapter 3]). We note the following elementary property of \( \text{Abs}(X) \).

Proposition 2.0.4. The absolute order \( \text{Abs}(X) \) is locally graded, with minimum element \( \hat{0} = x_0 \) and rank function given by the absolute length.

Proof. We have already noted that \( x_0 \) is the minimum element of \( \text{Abs}(X) \). Thus, it suffices to show that \( \ell_T(y) = \ell_T(x) + 1 \) whenever \( y \) covers \( x \) in \( \text{Abs}(X) \). This is an easy consequence of Definition 2.0.1 \( \square \)
We recall that when $W$ is finite, the rank (or length) generating polynomial of $\text{Abs}(W)$ satisfies

$$W_T(q) := \sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{d} (1 + e_i q),$$

where $d$ is the Coxeter rank of $W$ and $e_1, e_2, \ldots, e_d$ are its exponents. The rank generating polynomial

$$X_T(q) := \sum_{x \in X} q^{\ell_T(x)}$$

of $\text{Abs}(X)$ is well-defined when $X$ is finite. The following question provided much of the motivation for this paper.

**Question 2.0.5.** For which $W$-actions $\rho$ does $X_T(q)$ divide $W_T(q)$?

We now list examples, some of which will be studied in detail in later sections.

**Example 2.0.6.** (a) The order $\text{Abs}(W)$ occurs by letting $\rho$ be the left multiplication action of $W$ on itself and choosing $x_0$ as the identity element $e \in W$ in Definition 2.0.1, or by choosing $H$ as the trivial subgroup $\{e\}$ of $W$ in Definition 2.0.3.

(b) Let $H$ be the subgroup of $W$ generated by a given reflection $t_0 \in T$. The set $X = W/H$ of left cosets of $H$ in $W$ is in bijection with the alternating subgroup $W^+$ of $W$ and hence $\text{Abs}(X)$ gives rise to an absolute order on $W^+$. This order will be studied in Section 5.

(c) Let $\lambda$ be an integer partition of $m$ and let $X$ consist of the set partitions of $\{1, 2, \ldots, m\}$ whose block sizes are the parts of $\lambda$. The symmetric group $S_m$ acts transitively on $X$ and thus defines an absolute order. This order will be studied in Section 4.3 in the motivating special case in which $m = 2n$ is even and all parts of $\lambda$ are equal to 2. The resulting absolute order is a partial order on the set of perfect matchings of $\{1, 2, \ldots, 2n\}$. The stabilizer of this action is the natural embedding of the hyperoctahedral group $B_n$ in $S_{2n}$. 
(d) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition of $n$ and let $X$ consist of the ordered set partitions (meaning, set partitions in which the order of the blocks matters) of $\{1, 2, \ldots, n\}$ whose block sizes are $\lambda_1, \lambda_2, \ldots, \lambda_r$, in this order. The symmetric group $S_n$ acts transitively on $X$ and the stabilizer is a Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ of $S_n$. The resulting absolute order will be discussed in Section 6 in the special case in which $\lambda = (n - k, 1, \ldots, 1)$, where $k \in \{1, 2, \ldots, n - 1\}$.

Then $X$ can be identified with the set of $k$-tuples of pairwise distinct elements of $\{1, 2, \ldots, n\}$.

(e) Consider the special case $n = 4$, $\lambda = (2, 2)$ and $x_0 = (\{1, 2\}, \{3, 4\})$ of the example of part (d). Equivalently, let $W$ be the symmetric group $S_4$ and let $H$ be the four element subgroup generated by the commuting reflections $(1 2)$ and $(3 4)$. Then $X = W/H$ has six elements. The Hasse diagram of $\text{Abs}(X)$ is shown on Figure 2.1.

Remark 2.0.7. It is possible that not all edges of the graph $\Gamma = \Gamma(W, \rho)$, defined in Remark 2.0.2 (a), are edges of the Hasse diagram of $\text{Abs}(X)$. For instance, consider the action of $S_4$ on the set $X$ of perfect matchings of $\{1, 2, 3, 4\}$, discussed in Example 2.0.6 (c). Then $X$ has three elements and $\Gamma$ is the complete graph on these three vertices. On the other hand, $\text{Abs}(X)$ has a minimum element $x_0$ which is covered by the other two elements of $X$. Thus exactly one of the edges of $\Gamma$ is not an edge of the Hasse diagram of $\text{Abs}(X)$.

3. Modular subgroups

This section investigates a natural condition on a subgroup of a Coxeter group, called modularity, and shows that under this condition, the corresponding absolute order is well-behaved in several ways. Enumerative (Proposition 3.0.11) and order-theoretic (Theorem 3.0.16) characterizations, as well as examples, of modularity are given. Throughout this section, $W$ is a Coxeter group with identity element $e$, $T$ is the set of reflections, $H$ is a subgroup of $W$ and $X = W/H$ is the set of left cosets of $H$ in $W$. The Coxeter rank of $W$ will be denoted by $\text{rank}(W)$.

The following elementary properties of absolute length (proofs are left to the reader) will be frequently used throughout this paper.

Fact 3.0.8. For $u, v, w \in W$ we have:

(a) $\ell_T(w) = 0 \iff w = e$,
(b) $\ell_T(w) = 1 \iff w \in T$,
(c) $\ell_T(w^{-1}) = \ell_T(w)$,
(d) $\ell_T(uv) \leq \ell_T(u) + \ell_T(v)$,
(e) $\ell_T(uvw^{-1}) = \ell_T(u)$.

The main definition of this section is as follows.

Definition 3.0.9. We say that $H$ is a modular subgroup of $W$ if every left coset of $H$ in $W$ has a minimum in $\text{Abs}(W)$. 

We note that for \( x \in X \) and \( w_0 \in x \), the element \( w_0 \) is the minimum of \( x \) in \( \text{Abs}(W) \) if and only if we have \( \ell_T(w_0 h) = \ell_T(w_0) + \ell_T(h) \) for every \( h \in H \).

**Example 3.0.10.** (a) Let \( H \) be a subgroup of \( W \) generated by a single reflection \( t \in T \). Then every left coset \( x \in X \) consists of two elements \( w \) and \( wt \), which are comparable in \( \text{Abs}(W) \). This implies that \( H \) is a modular subgroup of \( W \).

(b) Let \( H \) be the symmetric group \( S_{n-1} \), naturally embedded in \( S_n \). It will be shown in Example 3.0.26 (and can be verified directly) that \( H \) is a modular subgroup of \( S_n \). The corresponding absolute order consists of the minimum element \( H \) and the left cosets \((i\, n)\) \( H \) for \( i \in \{1, 2, \ldots, n-1\} \), each of which covers \( H \).

(c) The subgroup \( H \) of \( S_4 \) in part (e) of Example 2.0.6 is not modular. Indeed, there is a single left coset \( wH \in X \), that with \( w = (1 \, 3)(2 \, 4) \), which does not have a minimum in \( \text{Abs}(S_4) \). As an induced subposet of \( \text{Abs}(W) \), this coset has \( w \) and \((1 \, 4)(2 \, 3) \) as minimal elements, \((1 \, 4 \, 2 \, 3) \) and \((1 \, 3 \, 2 \, 4) \) as maximal elements and all four possible Hasse edges among these elements.

(d) It is possible for a subgroup \( H \) of a finite Coxeter group \( W \) to have a left coset which has a unique element of minimum absolute length but no minimum in \( \text{Abs}(W) \) (clearly, such a subgroup \( H \) cannot be modular). Consider, for instance, the hyperoctahedral group \( B_n \) for some \( n \geq 4 \) and write \( \langle a \, b \rangle \) for the reflection in \( W \) with cycle form \( (a \, b)(-a \, -b) \). Let \( H \) be the subgroup of order 16 generated by the pairwise commuting reflections \( t_1 = \langle 1 \, 2 \rangle \), \( t_2 = \langle 1 \, -2 \rangle \), \( t_3 = \langle 3 \, 4 \rangle \) and \( t_4 = \langle 3 \, -4 \rangle \) and let \( t = \langle 1 \, 3 \rangle \) and \( h = t_1 t_2 t_3 t_4 \in H \). Then \( tH \) contains a unique reflection, namely \( t \), but has no minimum element in \( \text{Abs}(W) \), since \( t \) is not comparable to \( th \).

The following proposition explains the significance of modularity with respect to Question 2.0.5.

**Proposition 3.0.11.** Assume that \( W \) is finite. Then the subgroup \( H \) is modular if and only if \( W_T(q) = H_T(q) \cdot X_T(q) \).

**Proof.** Let \( w_x \in x \) be an element of minimum absolute length in \( x \in X \). Thus, we have \( \ell_T(w_x) = \ell_T(x) \) for every \( x \in X \) and hence \( \ell_T(w_x h) \leq \ell_T(w_x) + \ell_T(h) = \ell_T(x) + \ell_T(h) \) for all \( x \in X \) and \( h \in H \). As a result, we find that

\[
W_T(q) = \sum_{w \in W} q^{\ell_T(w)} = \sum_{x \in X} \sum_{h \in H} q^{\ell_T(w_x h)} \leq \sum_{x \in X} \sum_{h \in H} q^{\ell_T(x) + \ell_T(h)} = X_T(q) \cdot H_T(q),
\]

where \( \preceq \) stands for the reverse lexicographic order on the set of polynomials with nonnegative integer coefficients, i.e., for \( f(q), g(q) \in \mathbb{N}[q] \) we write \( f(q) < g(q) \) if the highest term of \( g(q) - f(q) \) has positive coefficient. Equality holds if and only if \( \ell_T(w_x h) = \ell_T(w_x) + \ell_T(h) \), that is \( w_x \preceq_T w_x h \), for all \( x \in X \) and \( h \in H \).
The latter holds if and only if \( w_x \) is the minimum element of \( x \) in \( \text{Abs}(W) \) for every coset \( x \in X \) and the proof follows. \( \square \)

A subgroup of \( W \) generated by reflections is called a reflection subgroup. The absolute length function on such a subgroup \( K \) is defined with respect to the set of reflections \( \mathcal{T} \cap K \). When \( W \) is finite, this function coincides with the restriction of \( \ell_{\mathcal{T}} : W \to \mathbb{N} \) on \( K \). As a result, the corresponding absolute order on \( K \) coincides with the induced order from \( \text{Abs}(W) \) on \( K \).

**Proposition 3.0.12.** Assume that \( W \) is finite. If \( K \) is a modular reflection subgroup of \( W \) and \( H \) is a modular subgroup of \( K \), then \( H \) is a modular subgroup of \( W \).

**Proof.** Let \( x \) be any left coset of \( H \) in \( W \). Clearly, \( x \) is contained in a left coset \( y \) of \( K \) in \( W \). Since \( K \) is modular in \( W \), the coset \( y \) has a minimum element \( w_o \) in \( \text{Abs}(W) \). We leave it to the reader to check that the map \( f : K \to \mathbb{N} \) defined by \( f(w) = w_o w \) for \( w \in K \), is a poset isomorphism, where \( K \) and \( y \) are considered as induced subposets of \( \text{Abs}(W) \). Thus \( x \) is isomorphic to its preimage \( f^{-1}(x) \) in \( K \) under \( f \), which is a left coset of \( H \) in \( K \). Since \( H \) is modular in \( K \), this preimage has a minimum element in \( \text{Abs}(K) \), therefore in \( \text{Abs}(W) \), and hence so does \( x \). It follows that \( H \) is modular in \( W \). \( \square \)

**Remark 3.0.13.** The absolute length function on \( K \) with respect to \( \mathcal{T} \cap K \) coincides with the restriction of \( \ell_{\mathcal{T}} : W \to \mathbb{N} \) on \( K \) even if \( W \) is infinite, provided \( K \) is a parabolic reflection subgroup of \( W \) (meaning that \( K \) is conjugate to a subgroup generated by simple reflections) \cite[Corollary 1.4]{12}. Thus, the transitivity property of modularity in Proposition \ref{thm:3.0.12} holds in this situation as well.

**Proposition 3.0.14.** Assume \( H \) is modular in \( W \) and let \( \sigma(x) \) be the minimum element of \( x \in X \) in \( \text{Abs}(W) \). Then the map \( \sigma : X \to W \) induces a poset isomorphism from \( \text{Abs}(X) \) onto an order ideal of \( \text{Abs}(W) \).

**Proof.** We need to show that (i) \( x \leq_{T} y \Leftrightarrow \sigma(x) \leq_{T} \sigma(y) \) for all \( x, y \in X \) and that (ii) \( \sigma(X) \) is an order ideal of \( \text{Abs}(W) \). For \( x, y \in X \) we have

\[
\begin{align*}
\text{(i)} & \quad x \leq_{T} y \quad \iff \quad y = wx \quad \text{for some} \quad w \in W \quad \text{with} \quad \ell_{T}(w) = \ell_{T}(y) - \ell_{T}(x) \\
& \quad \iff \quad w\sigma(x) \in \sigma(y)H \quad \text{for some} \quad w \in W \quad \text{with} \quad \ell_{T}(w) = \ell_{T}(\sigma(y)) - \ell_{T}(\sigma(x)) \\
& \quad \iff \quad w\sigma(x) = \sigma(y) \quad \text{for some} \quad w \in W \quad \text{with} \quad \ell_{T}(w) = \ell_{T}(\sigma(y)) - \ell_{T}(\sigma(x)) \\
& \quad \iff \quad \sigma(x) \leq_{T} \sigma(y),
\end{align*}
\]

where the third equivalence is because \( \sigma(y) \) is the unique element of minimum absolute length in its coset and \( \ell_{T}(w\sigma(x)) \leq \ell_{T}(w) + \ell_{T}(\sigma(x)) = \ell_{T}(\sigma(y)) \). This proves (i).

For (ii), given elements \( u, w \in W \) with \( u \leq_{T} w \) and \( w \in \sigma(X) \), we need to show that \( u \in \sigma(X) \). We set \( v = u^{-1}w \), so that \( vw = w \) and \( \ell_{T}(w) = \ell_{T}(u) + \ell_{T}(v) \). Since \( w \) is the minimum element of \( wH \) in \( \text{Abs}(W) \), we have \( \ell_{T}(wh) = \ell_{T}(w) + \ell_{T}(h) \).
for every \( h \in H \). Thus, for \( h \in H \) we have

\[
\ell_T(wh) = \ell_T(w) = \ell_T(w) + \ell_T(h) = \ell_T(u) + \ell_T(v) + \ell_T(h)
\]

\[
\ell_T(wh) = \ell_T(uh \cdot h^{-1}vh) \leq \ell_T(uh) + \ell_T(h^{-1}vh) = \ell_T(uh) + \ell_T(v).
\]

We conclude that \( \ell_T(uh) \geq \ell_T(u) + \ell_T(h) \), hence that \( \ell_T(uh) = \ell_T(u) + \ell_T(h) \), for every \( h \in H \). This means that \( u \) is the minimum element of \( uH \) in \( \text{Abs}(W) \), so that \( u \in \sigma(X) \), and the proof follows. \( \square \)

**Remark 3.0.15.** Part (i) of the proof of Proposition 3.0.14 shows that \( \text{Abs}(X) \) is isomorphic to an induced subposet of \( \text{Abs}(W) \) (moreover, covering relations are preserved). For that we only needed that each left coset of \( H \) in \( W \) has a unique element of minimum absolute length. \( \square \)

Next we give a characterization of modularity (which explains our choice of terminology) for the class of parabolic reflection subgroups of \( W \).

First we need to recall some background and notation on finite Coxeter groups. Such a group \( W \) acts faithfully on a finite-dimensional Euclidean space \( V \) by its standard geometric representation \([3] \, \S V.4 \) \([14] \, \S V.3 \). This representation realizes \( W \) as a group of orthogonal transformations on \( V \) generated by reflections.

Let \( \Phi \) be a corresponding root system. For \( \alpha \in \Phi \), we denote by \( H_\alpha \) the linear hyperplane in \( V \) which is orthogonal to \( \alpha \) and by \( t_\alpha \) the orthogonal reflection in \( H_\alpha \), so that \( T = \{ t_\alpha : \alpha \in \Phi \} \). We denote by \( \mathcal{L}_A \) the intersection lattice \([19] \, \S 2.1 \) \([27] \, \S 1.2 \) of the Coxeter arrangement \( A = \{ H_\alpha : \alpha \in \Phi \} \) and by \( \mathcal{L}_W \) the geometric lattice of all linear subspaces of \( V \) (flats) spanned by subsets of \( \Phi \), partially ordered by inclusion. Thus \( \mathcal{L}_A \) and \( \mathcal{L}_W \) are isomorphic as posets and the map which sends an element of \( \mathcal{L}_A \) to its orthogonal complement in \( V \) is a poset isomorphism from \( \mathcal{L}_A \) onto \( \mathcal{L}_W \).

Given a reflection subgroup \( H \) of \( W \), we will denote by \( V_H \) the linear span of all roots \( \alpha \in \Phi \) for which \( t_\alpha \in H \), so that \( V_H \in \mathcal{L}_W \). Then \( H \) is parabolic if and only if \( t_\alpha \in H \) for every \( \alpha \in \Phi \cap V_H \) (see, for instance, \([4] \) ). Finally, we recall that an element \( Z \) of a geometric lattice \( \mathcal{L} \) is called modular \([19] \, \text{Definition 2.25} \) \([25] \) \([27] \, \text{Definition 4.12} \) if we have

\[
\text{rk}(Y) + \text{rk}(Z) = \text{rk}(Y \wedge Z) + \text{rk}(Y \vee Z)
\]

for every \( Y \in \mathcal{L} \), where \( \text{rk} : \mathcal{L} \to \mathbb{N} \) denotes the rank function of \( \mathcal{L} \) and \( Y \wedge Z \) (respectively, \( Y \vee Z \)) stands for the greatest lower bound (respectively, least upper bound) of \( Y \) and \( Z \) in \( \mathcal{L} \).

**Theorem 3.0.16.** Assume that \( W \) is finite and that \( H \) is a parabolic reflection subgroup of \( W \). Then \( H \) is a modular subgroup of \( W \) if and only if \( V_H \) is a modular element of the geometric lattice \( \mathcal{L}_W \).

We will give two proofs of Theorem 3.0.16. We first need to establish two crucial lemmas. We recall \([1] \, \S 2.4 \) that to any \( w \in W \) are associated the spaces \( \text{Fix}(w) \in \mathcal{L}_A \) and \( \text{Mov}(w) \in \mathcal{L}_W \), where \( \text{Fix}(w) \) is the set of points in \( V \) which are fixed by the action of \( w \) and \( \text{Mov}(w) \) is the orthogonal complement of \( \text{Fix}(w) \).
in $V$. For instance, for every $\alpha \in \Phi$ the space $\text{Mov}(t_\alpha)$ is the one-dimensional subspace of $V$ spanned by $\alpha$. The maps $\text{Fix} : W \mapsto \mathcal{L}_A$ and $\text{Mov} : W \mapsto \mathcal{L}_W$ are surjective and we have $\dim \text{Mov}(w) = \ell_T(w)$ for every $w \in W$. Moreover (see the proof of [1, Theorem 2.4.7]), if $w = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k}$ is a reduced T-word for $w$, then \{$\alpha_1, \alpha_2, \ldots, \alpha_k$\} is an $\mathbb{R}$-basis of $\text{Mov}(w)$. In particular, $u \leq_T v \Rightarrow \text{Mov}(u) \subseteq \text{Mov}(v)$ for $u, v \in W$.

**Lemma 3.0.17.** Assume that $W$ is finite and that $H$ is a reflection subgroup of $W$ and let $w_0 \in W$. Then $w_0$ is the minimum of $w_0 H$ in $\text{Abs}(W)$ if and only if $\text{Mov}(w_0) \cap V_H = \{0\}$.

**Proof.** Let $w_0 = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k}$ be a reduced T-word for $w_0$. Thus $\ell_T(w_0) = k$ and \{$\alpha_1, \alpha_2, \ldots, \alpha_k$\} is an $\mathbb{R}$-basis of $\text{Mov}(w_0)$.

Suppose first that $\text{Mov}(w_0) \cap V_H = \{0\}$. We need to show that $w_0 \leq_T w_0 h$ for every $h \in H$. Let $h = t_{\beta_1} t_{\beta_2} \cdots t_{\beta_\ell}$ be a reduced T-word for $h \in H$. Then $\ell_T(h) = \ell$ and \{$\beta_1, \beta_2, \ldots, \beta_\ell$\} is a linearly independent subset of $V_H$. Our hypothesis implies that \{\$\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$\} is a linearly independent subset of $V$. We may infer from Carter’s Lemma [1, Lemma 2.4.5] that $t_{\alpha_1} \cdots t_{\alpha_k} t_{\beta_1} \cdots t_{\beta_\ell}$ is a reduced T-word for $w_0 h$. Therefore $\ell_T(w_0 h) = \ell_T(w_0) + \ell_T(h)$, which means that $w_0 \leq_T w_0 h$.

Conversely, suppose that $w_0$ is the minimum of $w_0 H$ in $\text{Abs}(W)$. We choose an $\mathbb{R}$-basis \{\$\beta_1, \beta_2, \ldots, \beta_\ell$\} of $V_H$ consisting of roots $\beta_i$ with $t_{\beta_i} \in H$ and set $h = t_{\beta_1} t_{\beta_2} \cdots t_{\beta_\ell} \in H$. By assumption, we have $\ell_T(w_0 h) = \ell_T(w_0) + \ell_T(h)$. This equation and Carter’s Lemma imply that \{\$\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$\} is linearly independent or, equivalently, that $\text{Mov}(w_0) \cap V_H = \{0\}$. \hfill $\square$

**Lemma 3.0.18.** Assume that $W$ is finite and that $H$ is a parabolic reflection subgroup of $W$ and let $w \in W$. Then $w$ is a minimal element of $w H$ in $\text{Abs}(W)$ if and only if $\text{Mov}(w) \cap V_H = \{0\}$ holds in $\mathcal{L}_W$.

**Proof.** We recall that every element of $\mathcal{L}_W$ is of the form $\text{Mov}(u)$ for some $u \in W$ and that $\text{Mov}(u)$ is nonzero if and only if it contains $\text{Mov}(t)$ for some $t \in T$. Moreover, we have $\text{Mov}(t) \subseteq \text{Mov}(u) \iff t \leq_T u$ [1, Theorem 2.4.7] for $t \in T$ and since $H$ is parabolic, we have $t \in H$ for every reflection $t \in T$ for which $\text{Mov}(t) \subseteq V_H$. From these facts we conclude that $\text{Mov}(w) \cap V_H \neq \{0\}$ holds in $\mathcal{L}_W$ if and only if there exists $t \in H \cap T$ such that $t \leq_T w$. The latter holds if and only if $w t <_T w$ for some $t \in H \cap T$ or, equivalently, if and only if $w$ is not a minimal element of $w H$ in $\text{Abs}(W)$. \hfill $\square$

**First proof of Theorem 3.0.16.** We will use the following characterization of modularity in $\mathcal{L}_W$: An element $Z \in \mathcal{L}_W$ is modular if and only if $Y \cap Z \in \mathcal{L}_W$ for every $Y \in \mathcal{L}_W$. This statement follows directly from [19, Lemma 2.24], which implies that an element $Z \in \mathcal{L}_A$ is modular if and only if $Y + Z \in \mathcal{L}_A$ for every $Y \in \mathcal{L}_A$.

We first assume that $H$ is modular in $W$ and consider any element $Y \in \mathcal{L}_W$. We need to show that $Y \cap V_H \in \mathcal{L}_W$. Since $Y \in \mathcal{L}_W$, we have $Y = \text{Mov}(w)$ for
some \( w \in W \). By our assumption, the coset \( wH \) has a minimum element, say \( w_0 \), in \( \text{Abs}(W) \). We claim that \( Y \cap V_H = \text{Mov}(w_0^{-1}w) \). Since \( \text{Mov}(w_0^{-1}w) \in \mathcal{L}_W \), it suffices to prove the claim. Indeed, since \( w_0 \leq T w \), we also have \( w_0^{-1}w \leq T w \) and hence \( \text{Mov}(w_0^{-1}w) \subseteq \text{Mov}(w) = Y \). Similarly, since \( w \in w_0H \), we have \( w_0^{-1}w \in H \) and hence \( \text{Mov}(w_0^{-1}w) \subseteq V_H \), so we may conclude that \( \text{Mov}(w_0^{-1}w) \subseteq Y \cap V_H \).

For the reverse inclusion, we recall [11, p. 25] that

\[
Y = \text{Mov}(w) = \text{Mov}(w_0) \oplus \text{Mov}(w_0^{-1}w).
\]

By our choice of \( w_0 \) and Lemma 3.0.17, we have \( \text{Mov}(w_0) \cap V_H = \{0\} \). As we already know that \( Y \cap V_H \supseteq \text{Mov}(w_0^{-1}w) \), it follows that \( Y \cap V_H = \text{Mov}(w_0^{-1}w) \).

Suppose now that \( V_H \) is a modular element of \( \mathcal{L}_W \) and consider any left coset \( x \) of \( H \) in \( W \). We need to show that \( x \) has a minimum in \( \text{Abs}(W) \). Let \( w_0 \) be any minimal element of \( x \) in \( \text{Abs}(W) \). Since \( \text{Mov}(w_0) \cap V_H \subseteq \mathcal{L}_W \), by modularity of \( V_H \), the greatest lower bound \( \text{Mov}(w_0) \cap V_H \) of \( \text{Mov}(w_0) \) and \( V_H \) in \( \mathcal{L}_W \) must be equal to \( \text{Mov}(w_0) \cap V_H \). This statement and Lemmas 3.0.17 and 3.0.18 imply that \( w_0 \) is the minimum element of \( x \) in \( \text{Abs}(W) \) and the proof follows.

**Remark 3.0.19.** The assumption in Theorem 3.0.16 that the reflection subgroup \( H \) is parabolic was not used in the proof of the only if direction of the theorem. However, it is essential for the other direction. Indeed, let \( W \) be the dihedral group of symmetries of a square \( Q \) and let \( H \) be the subgroup of order 4 generated by the reflections on the lines through the center of \( Q \) which are parallel to the sides. The unique left coset of \( H \) in \( W \), other than \( H \), has no minimum element in \( \text{Abs}(W) \) and hence \( H \) is not modular in \( W \). On the other hand, \( V_H = V \) is trivially a modular element of the lattice \( \mathcal{L}_W \).

For the second proof of Theorem 3.0.16 we recall the following definition. Let \( \mathcal{L} \) be a geometric lattice of rank \( d \), with rank function \( \text{rk} : \mathcal{L} \to \mathbb{N} \). The characteristic polynomial of \( \mathcal{L} \) is defined by the formula

\[
\chi_{\mathcal{L}}(q) := \sum_{Y \in \mathcal{L}} \mu_{\mathcal{L}}(\hat{0}, Y) q^{\text{rk}(Y)},
\]

where \( \mu_{\mathcal{L}} \) stands for the Möbius function [26 §3.7] of \( \mathcal{L} \) and \( \hat{0} \) is the minimum element of \( \mathcal{L} \). We now let \( \mathcal{L} = \mathcal{L}_W \) and recall that \( \text{rk}(Y) = \dim(Y) \) and (see, for instance, [18 Lemma 4.7])

\[
(-1)^{\text{rk}(Y)} \mu_{\mathcal{L}}(\hat{0}, Y) = \#\{ w \in W : \text{Mov}(w) = Y \}
\]

for \( Y \in \mathcal{L} \) and that \( \dim \text{Mov}(w) = \ell_T(w) \) for \( w \in W \). As a result, the characteristic polynomial of \( \mathcal{L}_W \) is related to the rank generating polynomial of \( \text{Abs}(W) \) by the well known equality

\[
W_T(q) = (-q)^d \chi_{\mathcal{L}}(-1/q).
\]

**Second proof of Theorem 3.0.16** Let us write \( \mathcal{L} = \mathcal{L}_W \), as before, and set \( Z = V_H \in \mathcal{L} \). By the Modular Factorization Theorem for geometric lattices [25]
[27] Theorem 4.13] and its converse (see [17] Section 8) we have that \( Z \) is a modular element of \( L \) if and only if

\[
\chi_L(q) = \chi_{[\hat{0}, Z]}(q) \sum_{Y \in L, Y \wedge Z = \hat{0}} \mu_L(\hat{0}, Y) q^{d - \text{rk}(Y) - \text{rk}(Z)},
\]

where \([\hat{0}, Z]\) denotes a closed interval in \( L \) and \( \hat{0} = \{0\} \) is the minimum element of \( L \). Replacing \( q \) by \( -1/q \) and taking (4) and (5) into account, we see that (6) can be rewritten as

\[
W_T(q) = H_T(q) \sum_{\text{Mov}(w) \wedge Z = \hat{0}} q^{\ell(w)}.
\]

We recall that every finite partially ordered set has at least one minimal element. Assume first that \( Z \) is a modular element of \( L \). Setting \( q = 1 \) in (7) and using Lemma 3.0.18 we conclude that every left coset of \( H \) in \( W \) has exactly one minimal (and hence a minimum) element in \( \text{Abs}(W) \). By definition, this means that \( H \) is a modular subgroup of \( W \). Conversely, suppose that \( H \) is a modular subgroup of \( W \). Then, by Lemma 3.0.18, the sum in the right-hand side of (7) is equal to \( X_T(q) \) and hence (7) holds by Proposition 3.0.11. Thus \( Z \) is a modular element of \( L \) and the proof follows.

**Proposition 3.0.20.** Assume that \( W \) is finite. Then every modular reflection subgroup of \( W \) is a parabolic reflection subgroup.

**Proof.** Let \( H \) be a modular reflection subgroup of \( W \) and let \( K \) be the unique parabolic reflection subgroup of \( W \) with \( V_K = V_H \). Thus \( K \) is generated by all reflections \( t \in T \) with \( \text{Mov}(t) \subseteq V_H \) and contains \( H \) as a reflection subgroup. We need to show that \( H = K \). Since \( H \) is modular in \( W \), it is also modular in \( K \). Thus, without loss of generality we may assume that \( K = W \), so that \( \text{rank}(H) = \text{rank}(W) \). We note that \( H_T(q) \) and \( W_T(q) \) are both polynomials of degree \( \text{rank}(W) \). Therefore, Proposition 3.0.11 implies that \( X_T(q) \) is a constant. Since this can only happen if \( X \) is a singleton, we conclude that \( H = W \) and the proof follows.

**Question 3.0.21.** Does there exist a modular subgroup of a Coxeter group which is not a reflection subgroup?

We recall that a poset \( P \) is said to be graded of rank \( d \) if every maximal chain in \( P \) has exactly \( d + 1 \) elements. The following proposition generalizes the fact that \( \text{Abs}(W) \) is graded with rank equal to \( \text{rank}(W) \).

**Proposition 3.0.22.** The order \( \text{Abs}(X) \) is graded of rank \( \text{rank}(W) - \text{rank}(H) \) for every finite Coxeter group \( W \) and every modular reflection subgroup \( H \) of \( W \).

**Proof.** Since \( \text{Abs}(X) \) has a minimum element and is locally graded with rank function given by absolute length (Proposition 3.0.14), it suffices to show that
for every element \( x \in X \) there exists \( y \in X \) of absolute length \( \operatorname{rank}(W) - \operatorname{rank}(H) \) such that \( x \leq_T y \).

Consider any \( x \in X \) and let \( u_o \) be the minimum element of \( x \) in \( \text{Abs}(W) \). Thus we have \( \text{Mov}(u_o) \cap V_H = \{0\} \) by Lemma 3.0.17 and \( \ell_T(x) = \ell_T(u_o) = \dim \text{Mov}(u_o) \). Let \( u_o = t_{\alpha_k} \cdots t_{\alpha_2} t_{\alpha_1} \) be a reduced \( T \)-word for \( u_o \), so that \( \ell_T(x) = k \). We extend \( \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) to a maximal linearly independent set of roots \( \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) whose linear span intersects \( V_H \) trivially and set \( w_o = t_{\alpha_r} \cdots t_{\alpha_2} t_{\alpha_1} \) and \( y = w_o H \in X \). Clearly, we have \( r = \dim(V) - \dim(V_H) = \operatorname{rank}(W) - \operatorname{rank}(H) \). Since \( \text{Mov}(w_o) \) is the linear span of \( \alpha_1, \alpha_2, \ldots, \alpha_r \), we have \( \text{Mov}(w_o) \cap V_H = \{0\} \) by construction. Lemma 3.0.17 implies that \( w_o \) is the minimum element of \( y \) in \( \text{Abs}(W) \) and hence that \( \ell_T(y) = \ell_T(w_o) = r \). Finally, setting \( v = w_o u_o^{-1} t_{\alpha_r} \cdots t_{\alpha_{k+1}} \) we have \( w_o = vu_o \) and hence \( y = vx \). By Carter’s Lemma [1] Lemma 2.4.5] we also have \( \ell_T(v) = r - k = \ell_T(y) - \ell_T(x) \). Definition 2.0.7 implies that \( x \leq_T y \) and the proof follows. \( \Box \)

Question 3.0.23. Does there exist a subgroup \( H \) of a Coxeter group \( W \) for which \( \text{Abs}(X) \) is not graded?

A reflection subgroup \( H \) of \( W \) is said to be of almost maximal rank if \( \operatorname{rank}(H) = \operatorname{rank}(W) - 1 \). Modular parabolic reflection subgroups of this kind can be characterized as follows.

Proposition 3.0.24. Assume that \( W \) is finite and that \( H \) is a parabolic reflection subgroup of \( W \), other than \( W \). The following are equivalent:

(i) \( H \) is a modular subgroup of \( W \) of almost maximal rank.

(ii) Every left coset of \( H \), other than \( H \), contains a reflection.

(iii) Every left coset of \( H \), other than \( H \), contains a unique reflection.

Proof. Suppose that (i) holds. We then have \( W_T(q) = H_T(q) \cdot X_T(q) \) by Proposition 3.0.11. Since the degrees of \( W_T(q) \) and \( H_T(q) \) are equal to the Coxeter ranks of \( W \) and \( H \), respectively, it follows that the degree of \( X_T(q) \) is equal to one. This means that every left coset \( x \in X \) of \( H \), other than \( H \), contains an element of absolute length one, so that (ii) is satisfied. We have shown that (i) implies (ii).

Suppose that (ii) holds and let \( x \in X \) be a left coset of \( H \) in \( W \), other than \( H \). Choose a reflection \( t \in x \). Since \( H \) is parabolic and does not contain \( t \), we have \( \text{Mov}(t) \cap V_H = \{0\} \). Lemma 3.0.17 implies that \( t \) is the minimum element of \( x \) in \( \text{Abs}(W) \). In particular, \( x \) contains a unique reflection. We conclude that (ii) implies both (i) and (iii). The implication (iii) \( \Rightarrow \) (ii) is trivial. \( \Box \)

Question 3.0.25. Does there exist a non-parabolic (necessarily non-modular) reflection subgroup \( H \) of a finite Coxeter group every left coset, other than \( H \), of which contains a unique reflection?

Example 3.0.26. For \( k \leq n \) and under the natural embedding, the symmetric and hyperoctahedral groups \( S_k \) and \( B_k \) are modular subgroups of \( S_n \) and \( B_n \), respectively. This follows from Theorem 3.0.16 and known facts on the modular elements of the geometric lattice \( \mathcal{L}_W \) in these cases; see, for instance. [3] Theorem
2.2]. Alternatively, one can check directly that for $1 \leq i \leq n - 1$, the transpositions $(i \; n)$ are representatives of the left cosets of $S_{n-1}$ in $S_n$, other than $S_{n-1}$. Proposition 3.0.24 implies that $S_{n-1}$ is modular in $S_n$. The transitivity property of Proposition 3.0.12 implies that $S_k$ is modular in $S_n$ for each $k \leq n$. A similar argument works for the hyperoctahedral groups. □

We end this section with two more open questions.

**Question 3.0.27.** Do infinite modular subgroups exist?

**Question 3.0.28.** For which subgroups $H$ of $W$ does $\text{Abs}(X)$ have a maximum element?

4. **Quasi-modular subgroups**

This section introduces a condition on a subgroup of a Coxeter group, termed quasi-modularity, which is broader than modularity and guarantees an affirmative answer to Question 2.0.5. Examples of quasi-modular subgroups which are not modular are discussed. Throughout this section, the set of reflections of a Coxeter group $H$ will be denoted by $T(H)$.

**4.1. Quasi-modularity.** The main definition of this section is as follows.

**Definition 4.1.1.** A subgroup $H$ of a finite Coxeter group $W$ is **quasi-modular** if $H$ is isomorphic to a Coxeter group and

$$W_T(q) = H_{T(H)}(q) \cdot X_T(q),$$

where $T = T(W)$.

Proposition 3.0.11 implies that for reflection subgroups of $W$, quasi-modularity is equivalent to modularity. However, this is not the case for general subgroups as $T(H)$ may not be equal to $H \cap T(W)$.

**Example 4.1.2.** We list two families of examples of quasi-modular subgroups which are not modular.

(a) Let $W$ be the Weyl group of type $D_n$, considered as a group of signed permutations of $\{1, 2, \ldots, n\}$ with an even number of sign changes. Let $H$ be the subgroup consisting of all $w \in W$ satisfying $w(n) \in \{n, -n\}$. Then $H$ is isomorphic to the hyperoctahedral group $B_{n-1}$ and the identity element $e \in W$ together with the reflections $(i \; n)(-i \; -n)$ for $1 \leq i \leq n - 1$ form a complete list of coset representatives of $H$ in $W$. As a result, we have $X_T(q) = 1 + (n - 1)q$, where $X = W/H$ and $T = T(W)$. Using this fact and (1), it can be easily verified that (5) holds in this situation and hence that $H$ is a quasi-modular subgroup of $W$. On the other hand, it is also easy to verify that $H_T(q)$ has degree $n$, as does $W_T(q)$. Thus $H$ is not a modular subgroup of $W$ by Proposition 3.0.11.

(b) Consider the symmetric group $S_{2n}$ as the group of permutations of the set $\{1, -1, 2, -2, \ldots, n, -n\}$ and the natural embedding of the hyperoctahedral group $B_n$ in $S_{2n}$, mapping the Coxeter generators of $B_n$ to the transposition
(n − n) and the products (i i + 1)(−i − i − 1) for 1 ≤ i ≤ n − 1. Several combinatorial interpretations to the poset \text{Abs}(S_{2n}/B_n) will be given in Section 4.3, where the following statement will also be proved.

**Theorem 4.1.3.** The group \( B_n \) is a non-modular, quasi-modular subgroup of \( S_{2n} \) for every \( n ≥ 2 \).

### 4.2. Balanced complex reflections

Before proving Theorem 4.1.3 we introduce an absolute order on balanced complex reflections. Recall that the wreath product of the cyclic group \( \mathbb{Z} \) by the symmetric group \( S_n \) is defined as

\[
G(r, n) = \mathbb{Z}_r \wr S_n := \{((c_1, \ldots, c_n); \pi) : c_i \in \mathbb{Z}_r, \pi \in S_n\}
\]

with group operation

\[
((c_1, \ldots, c_n); \pi) \cdot ((c'_1, \ldots, c'_n); \pi') := ((c_1 + c'_{\pi(1)} - 1, \ldots, c_n + c'_{\pi(n)} - 1); \pi\pi').
\]

We think of the elements of \( \mathbb{Z}_r \) as colors and denote by \( \psi : G(r, n) \to S_n \) the canonical map, defined by \( \psi((c; \pi)) := \pi \). Via this map, the elements of \( G(r, n) \) inherit a cycle structure from those of \( S_n \).

**Definition 4.2.1.** A cycle of an element of \( G(r, n) \) is balanced if the sum of the colors of its elements is zero modulo \( r \). An element \( w \in G(r, n) \) is balanced if all cycles of \( w \) are balanced. We denote by \( C(r, n) \) the set of balanced elements of \( G(r, n) \).

For example, there are three balanced elements in \( G(2, 2) = B_2 \), namely the identity and the reflections \(((0, 0); (1 2)) = (1 2)(-1 - 2) \) and \(((1, 1); (1 2)) = (1 - 2)(-1 2) \).

**Remark 4.2.2.** Balanced cycles generalize the notion of paired cycles, introduced by Brady and Watt \cite{9} in the study of the absolute order of types \( B \) and \( D \) and further studied in \cite{10}.

The wreath product \( G(r, n) \) acts naturally on the vector space \( V = \mathbb{R}^n \). The set of pseudoreflections \( T(r, n) \subseteq G(r, n) \) consists of all elements fixing a hyperplane. The absolute length function \( \ell_{T(r, n)} : G(r, n) \to \mathbb{N} \) is defined with respect to the generating set \( T(r, n) \).

**Definition 4.2.3.** The absolute order on \( C(r, n) \), denoted \( \text{Abs}(C(r, n)) \), is the reflexive and transitive closure of the relation consisting of the pairs \((u, v)\) of elements of \( C(r, n) \) for which \( v = \tau u \) for some \( \tau \in T(r, n) \cap C(r, n) \) and \( \ell_{T(r, n)}(u) < \ell_{T(r, n)}(v) \).

The order \( \text{Abs}(C(r, n)) \) is the subposet induced on \( C(r, n) \) from Shi’s absolute order on \( G(r, n) \); see \cite{22, 23}.

**Proposition 4.2.4.** (a) The canonical map \( \psi : G(r, n) \to S_n \) induces a rank preserving poset epimorphism from the order \( \text{Abs}(C(r, n)) \) onto \( \text{Abs}(S_n) \).

(b) Every maximal interval in \( \text{Abs}(C(r, n)) \) is mapped isomorphically by \( \psi \) onto a maximal interval in \( \text{Abs}(S_n) \).
Proof. (a) By definition, \( \psi \) is a group epimorphism and \( \psi(T(r,n) \cap C(r,n)) = T(S_n) \). Hence we have \( \ell_t(r,n)(w) = \ell_T(\psi(w)) \) for every \( w \in C(r,n) \). Furthermore, for \( u,v \in C(r,n) \) we have \( v = \tau u \) for some \( \tau \in T(r,n) \cap C(r,n) \) and \( \ell_{T(r,n)}(u) < \ell_{T(r,n)}(v) \) if and only if \( \psi(v) = t\psi(u) \) for some \( t \in T \) and \( \ell_T(\psi(u)) < \ell_T(\psi(v)) \). In other words, \( u \) is covered by \( v \) in \( \text{Abs}(C(r,n)) \) if and only if \( \psi(u) \) is covered by \( \psi(v) \) in \( \text{Abs}(S_n) \).

(b) We first check that \( \psi \) maps maximal elements of \( \text{Abs}(C(r,n)) \) to maximal elements of \( \text{Abs}(S_n) \). Indeed, since \( \psi \) is rank preserving, the rank of an element \( w \) in \( \text{Abs}(C(r,n)) \) is equal to \( n-k \), where \( k \) is the number of cycles. Thus, if \( \psi(w) \) is not maximal in \( \text{Abs}(S_n) \), then \( \psi(w) \) has at least two cycles and one can check that there exists \( \tau \in T(r,n) \cap C(r,n) \) such that \( \tau w \) has fewer cycles than \( w \), so \( w \) is not maximal either. We next observe that \( \psi \) has the following property: given \( w \in C(r,n) \) and \( t \in T(S_n) \) such that \( tw \) is covered by \( \psi(w) \) in \( \text{Abs}(S_n) \), there is a unique complex reflection \( \tau \in \psi^{-1}(t) \) such that \( \tau w \) is balanced. As a result, the map \( \psi \) induces a bijection between elements covered by \( w \) in \( \text{Abs}(C(r,n)) \) and those covered by \( \psi(w) \) in \( \text{Abs}(S_n) \). By induction on the rank of the top element, it follows that intervals in \( \text{Abs}(C(r,n)) \) are mapped isomorphically by \( \psi \) to intervals in \( \text{Abs}(S_n) \). In particular, every maximal interval in \( \text{Abs}(C(r,n)) \) is mapped isomorphically by \( \psi \) onto a maximal interval in \( \text{Abs}(S_n) \).

\[ \sum_{w \in C(r,n)} q^{\ell_{T(r,n)}(w)} = \prod_{i=1}^{n-1} (1 + riq). \]

Proof. By Proposition 4.2.4, for every \( w \in C(r,n) \) we have \( \ell_t(r,n)(w) = \ell_T(\pi) = n-k \), where \( k \) is the number of cycles of \( \pi := \psi(w) \). Since all elements in the preimage \( \psi^{-1}(\pi) \) are balanced, we have

\[ |\psi^{-1}(\pi)| = r^{n-k} = r^{\ell_T(\pi)} \]

and thus

\[ \sum_{w \in C(r,n)} q^{\ell_{T(r,n)}(w)} = \sum_{\pi \in S_n} |\psi^{-1}(\pi)| q^{\ell_T(\pi)} = \sum_{\pi \in S_n} r^{\ell_T(\pi)} q^{\ell_T(\pi)} = \prod_{i=1}^{n-1} (1 + riq). \]

\[ \square \]

4.3. Perfect matchings. A partition of set \( \Omega \) into two-element subsets is called a perfect matching. Throughout this section we will denote by \( \mathcal{M}_n \) the set of perfect matchings of \( \Omega_n := \{1, -1, 2, -2, \ldots, n, -n\} \). Consider the simple graph \( \Delta_n \), introduced in \cite{13}, on the set of nodes \( \mathcal{M}_n \) in which two perfect matchings are adjacent if their symmetric difference is a cycle of length 4. The diameter
and the enumeration of geodesics of this graph were studied in [5]; the induced subgraph on non-crossing perfect matchings was studied earlier in [13].

**Definition 4.3.1.** Fix an arbitrary element \( x_0 \in M_n \). The absolute order on \( M_n \), denoted \( \text{Abs}(M_n) \), is the poset \( (M_n, \preceq) \) defined by letting \( x \preceq y \) if \( x \) lies in a geodesic path in \( \Delta_n \) with endpoints \( x_0 \) and \( y \), for \( x, y \in M_n \).

The symmetric group \( S_{2n} \) of permutations of \( \Omega_n \) acts naturally on \( M_n \) (this action may be identified with the conjugation action of \( S_{2n} \) on the set of fixed point free involutions on a \( 2n \)-element set). The stabilizer is the natural embedding of the hyperoctahedral group \( B_n \) in \( S_{2n} \) and hence we get the following statement.

**Observation 4.3.2.** The poset \( \text{Abs}(M_n) \) is isomorphic to \( \text{Abs}(S_{2n}/B_n) \).

In particular, the isomorphism type of \( \text{Abs}(M_n) \) is independent of the choice of \( x_0 \). Without loss of generality, for the remainder of this section we will assume that \( x_0 \) consists of the sets (arcs) \( \{-i, i\} \) for \( 1 \leq i \leq n \).

**Proposition 4.3.3.** The poset \( \text{Abs}(M_n) \) is isomorphic to \( \text{Abs}(C(2, n)) \).

**Proof.** The proof generalizes a construction from [13].

Given a perfect matching \( x \in M_n \), consider the union \( x \cup x_0 \), consisting of the arcs of \( x \) and \( \{-i, i\} \) for \( 1 \leq i \leq n \). This is a disjoint union of nontrivial cycles and isolated arcs. We orient the nontrivial cycles in the following way: Given any such cycle \( C \), we let \( k \) be the minimum positive integer such that \( \{-k, k\} \) is an arc of \( C \) and choose the cyclic orientation of \( C \) in which this edge is directed from \(-k\) to \( k\). We associate to \( x \) a signed permutation \( f(x) = \pi \in B_n \) as follows. For \( i \in \Omega_n \), we set \( \pi(i) = i \) if \( \{-i, i\} \in x \). Otherwise we set \( \pi(i) = -j \) if either \( (i, j) \) or \( (-i, -j) \) is a directed edge in the above orientation, and \( \pi(i) = j \) if either \( (-i, j) \) or \( (i, -j) \) is a directed edge in the orientation. We will show that \( f : M_n \to C(2, n) \) is a well-defined map which is an isomorphism of the corresponding absolute orders.

We first observe that the map \( f : M_n \to B_n \) is well-defined. Indeed, this holds since \( \{-i, i\} \in x \cup x_0 \) for \( 1 \leq i \leq n \) and hence at most one of \( i \) and \( -i \) can be the initial vertex of a directed arc in the above orientation. Moreover, since the number of arcs of any nontrivial cycle of \( x \cup x_0 \) is even, the number of arcs with vertices of same sign in such a cycle must also be even. This implies that every nontrivial cycle of the signed permutation \( f(x) \) is balanced and hence we have a well-defined map \( f : M_n \to C(2, n) \).

To show that \( f : M_n \to C(2, n) \) is a bijection, it suffices to describe the inverse map. Given a balanced signed permutation \( \pi \in C(2, n) \), we construct \( g(x) \in M_n \) as follows. First, we include in \( g(x) \) the arc \( \{-i, i\} \) for each \( i \in \Omega_n \) with \( \pi(i) = i \). Second, let \( (a_1, a_2, \ldots, a_k) \) be any nontrivial cycle of \( \pi \) and assume that \( a_1 \) is the minimum positive element of this cycle and its negative. We then include in \( g(x) \) the arcs \( \{a_1, -a_2\}, \{a_2, -a_3\}, \ldots, \{a_k, -a_1\} \). We leave it to the reader to verify that \( g \) is the inverse map of \( f \).
Finally we prove that $f : \mathcal{M}_n \to C(2, n)$ induces an isomorphism of absolute orders. We consider the simple graph $\Gamma_n$ on the node set $C(2, n)$ in which two permutations $\pi, \sigma \in C(2, n)$ are adjacent if $\pi^{-1}\sigma \in T(2, n)$. Since $f$ maps $x_0$ to the identity element of $C(2, n)$, it suffices to show that $f$ induces a graph isomorphism from $\Delta_n$ to $\Gamma_n$. Indeed, two matchings $x_1, x_2 \in \mathcal{M}_n$ are adjacent in $\Delta_n$ if and only if there exist four distinct elements $i, j, k, l \in \Omega_n$ such that $x_1 \setminus \{(i, j),\{k, l\}\} = x_2 \setminus \{(i, k),\{j, l\}\}$. Without loss of generality, we may assume that $(i, j)$ and $(k, l)$ are directed edges in the orientation of $x_1 \cup x_0$. By considering the eight cases determined by the signs of $i, j, k, l$, one can verify that this happens if and only if there exists a reflection $\tau \in \{(j, k), (-j, -k)\} \subseteq T(2, n)$ such that $f(x_2) = \tau f(x_1)$ and the proof follows.

\textbf{Corollary 4.3.4.} There is a poset epimorphism from $\text{Abs}(\mathcal{M}_n)$ to $\text{Abs}(S_n)$ which maps every maximal interval in $\text{Abs}(\mathcal{M}_n)$ isomorphically onto a noncrossing partition lattice of type $A_{n-1}$.

\textit{Proof.} This follows from Propositions 4.3.3 and 4.2.4 and the fact that every maximal interval in $\text{Abs}(S_n)$ is isomorphic to the lattice of noncrossing partitions of the set $\{1, 2, \ldots, n\}$.

The previous corollary implies [13, Corollaries 1.6 and 2.2] and [5, Theorem 3.20].

\textbf{Corollary 4.3.5.} For every $n \geq 1$ we have

$$\mathcal{M}_n T(q) = \prod_{i=0}^{n-1}(1 + 2iq).$$

\textit{Proof.} This follows from Proposition 4.3.3 and Corollary 4.2.5.

\textbf{Proof of Theorem 4.1.3.} That $B_n$ is a quasi-modular subgroup of $S_{2n}$ follows from Observation 4.3.2, Corollary 4.3.5 and the known formulas for the rank generating functions of $\text{Abs}(S_{2n})$ and $\text{Abs}(B_n)$.

Suppose that $B_n$ were a modular subgroup of $S_{2n}$ for some $n \geq 2$. Then, according to Proposition 4.3.3 and Observation 4.3.2, we should have $S_{2n} T(q) = B_n T(q) \cdot \mathcal{M}_n T(q)$, where $T := T(S_{2n})$, and hence $B_n T(q)$ should have degree $n$. This is not correct, since there exist elements of the natural embedding of $B_n$ in $S_{2n}$ which are cycles in $S_{2n}$ of absolute length $2n - 1$.

\textbf{Remark 4.3.6.} The $B_{2n}$-conjugation action on involutions of cycle type $2^n$ (as well as the $B_n$-action on cosets of $S_n$) and the $S_n$-conjugation action on involutions with fixed points have nicely factorized rank generating functions which do not divide the rank generating function of the acting Coxeter group.

\textbf{Question 4.3.7.} For $r \geq 3$, is there a Coxeter group action whose associated absolute order is isomorphic to $\text{Abs}(C(r, n))$?
The cardinality of \( C(r, n) \) is equal to the product \( \prod_{i=1}^{n-1} (1 + ri) \) (by Corollary 4.2.5) and hence to the number of \((r + 1)\)-ary increasing trees of order \( n \); see, for instance, [24].

**Question 4.3.8.** For \( r \geq 1 \), is there a (natural) Coxeter group action on these trees whose associated absolute order is isomorphic to \( \text{Abs}(C(r, n)) \)?

5. **An application to alternating subgroups**

Throughout this section \((W, S)\) will be a Coxeter system with set of reflections \( T = \{ ws w^{-1} : w \in W, s \in S \} \). The **alternating subgroup** \( W^+ \) is defined as the kernel of the sign character on \( W \), which maps every element of \( S \) to \(-1\). We will show that a natural absolute order on \( W^+ \) can be defined in a way which is compatible with the general construction of Section 2.

Choose any element \( s_0 \in S \). Then \( S_0 := \{ s_0 s : s \in S \} \) is a generating set for \( W^+ \) which carries a simple presentation [8, §IV.1, Ex. 9] and a Coxeter-like structure [10]. Let us write

\[
T_0 := \{ s_0 t : t \in T \}.
\]

Given a pair \((G, A)\) of a group \( G \) and generating set \( A \), we say that an element \( g \in G \) is an **odd palindrome** if there is an \((A \cup A^{-1})^\ell\)-word \( (a_1, \ldots, a_\ell) \) for \( g \) such that \( \ell \) is odd and \( a_i = a_{\ell-i+1} \) for every index \( i \). For example, the set of odd palindromes for \((W, S)\) is equal to \( T \).

**Claim 5.0.9.** The set of odd palindromes for \((W^+, S_0)\) is equal to \( T_0 \).

**Proof.** Let \( w \) be an odd palindrome in \((W^+, S_0)\). Then \( s_0 w \) is an odd palindrome in \((W, S)\) and hence a reflection in \( T \). Conversely, since \( s_0 \) is an involution, for every reflection \( t = s_{i_1} s_{i_2} s_{i_3} s_{i_4} \cdots s_{i_k} \in T_0 \),

\[
s_0 t = (s_0 s_{i_1}) (s_{i_2} s_0) (s_0 s_{i_3}) (s_{i_4} s_0) \cdots (s_{i_k} s_0) (s_0 s_{i_k}) (s_0 s_0) (s_0 s_{i_1})
\]

is an odd palindrome in \((W^+, S_0)\). \( \square \)

Odd palindromes in alternating subgroups play a role which is analogous to that played by reflections in Coxeter groups [10] §2.5, §3.5]. This leads to the following definition of absolute order to alternating subgroups.

**Definition 5.0.10.** Given a simple reflection \( s_0 \in S \), the **(left) absolute order** \( \leq_{T_0} \) on the alternating subgroup \( W^+ \) of \( W \) is defined as the reflexive and transitive closure of the relation consisting of the pairs \((u, v)\) of elements of \( W^+ \) for which \( \ell_{T_0}(u) < \ell_{T_0}(v) \) and \( v = \tau u \) for some \( \tau \in T_0 \).

The absolute order on \( W^+ \), which we will denote by \( \text{Abs}_0(W^+) \), depends on the choice of \( s_0 \): non-conjugate simple reflections determine non-isomorphic absolute orders on \( W^+ \). For example, the absolute order on \( H_3^+ \) which is determined by the choice of the adjacent transposition \( s_0 = (1 2)(-1 - 2) \) is not isomorphic to the one determined by the choice \( s_0 = (1 - 1) \). However, the rank generating
function is independent of the choice of $s_0$. This will be proved by considering the action of $W$ on cosets of $\langle s_0 \rangle$, the subgroup generated by $s_0$.

Here are some basic lemmas on the absolute lengths $\ell_T$ and $\ell_{T_0}$ which will be used in the proof.

**Lemma 5.0.11.** For every $w \in W^+$ we have

$$\ell_{T_0}(w) = \begin{cases} \ell_T(w), & \text{if $\ell_{T_0}(w)$ is even;} \\ \ell_T(w) - 1, & \text{if $\ell_{T_0}(w)$ is odd.} \end{cases}$$

**Proof.** Let $w = t_1 \cdots t_\ell$ be a $T$-word for $w$ of length $\ell := \ell_T(w)$. Since $w \in W^+$, the number $\ell$ is even and we may write

$$w = t_1 s_0 t_2 s_0 t_3 \cdots t_\ell - 1 s_0 t_\ell,$$

where $t^0 := s_0 t s_0 \in T$ for $t \in T$. This proves that

$$\ell_{T_0}(w) \leq \ell = \ell_T(w).$$

Suppose that $\ell_{T_0}(w) = 2m$ is even. Then we may write

$$w = s_0 t_1 \cdots s_0 t_{2m} = \prod_{i=1}^{m} t^0_{2i-1} \cdot t_{2i},$$

with $t_i \in T$ for each index $i$. Thus $\ell_T(w) \leq 2m = \ell_{T_0}(w)$ and the proof follows in this case. Finally, if $\ell_{T_0}(w) = 2m + 1$ is odd, then we may write

$$w = s_0 t_1 \cdots s_0 t_{2m+1} = s_0 t_1 \prod_{i=1}^{m} t^0_{2i} \cdot t_{2i+1},$$

with $t_i \in T$ for each $i$. This shows that

$$\ell_T(w) \leq 2m + 2 = \ell_{T_0}(w) + 1.$$ Combining (9) with (10) yields $\ell_{T_0}(w) \leq \ell_T(w) \leq \ell_{T_0}(w) + 1$. Since $\ell_T(w)$ and $\ell_{T_0}(w)$ have distinct parities, we conclude that $\ell_T(w) = \ell_{T_0}(w) + 1$ and the proof follows in this case too. \hfill $\square$

**Lemma 5.0.12.** For every $w \in W^+$, the following conditions are equivalent:

(i) $\ell_{T_0}(w)$ is even.

(ii) $\ell_T(w) < \ell_T(s_0 w)$.

(iii) $\ell_T(w) < \ell_T(w s_0)$.

**Proof.** Since the absolute length is invariant under conjugation (Fact 3.0.3(e)), we have $\ell_T(s_0 w) = \ell_T(w s_0)$ and hence it suffices to prove that (i) $\iff$ (ii).

Suppose first that $\ell_T(w) > \ell_T(s_0 w)$. We note that $\ell_T(s_0 w)$ is an odd number, since $w \in W^+$, say $\ell_T(s_0 w) = 2m + 1$, and let $t_1 \cdots t_{2m+1}$ be a reduced $T$-word for $s_0 w$. Then $s_0 t_1 \cdots t_{2m+1}$ is a reduced $T$-word for $w$ and $\ell_T(w) = 2m + 1$. Since

$$w = s_0 t_1 \prod_{i=1}^{m} (s_0 t^0_{2i}) (s_0 t_{2i+1}),$$


we have $\ell_{T_0}(w) \leq 2m+1$. On the other hand, we have $\ell_{T_0}(w) \geq \ell_T(w) - 1 = 2m+1$ by Lemma 5.0.11. Thus $\ell_{T_0}(w) = 2m+1$ and, in particular, $\ell_{T_0}(w)$ is odd. This proves the implication (i) $\Rightarrow$ (ii).

Conversely, suppose that $\ell_{T_0}(w)$ is odd. Then the proof of Lemma 5.0.11 shows that there is a reduced $T$-word for $w$ which starts with $s_0$. This implies that $\ell_T(w) > \ell_T(s_0 w)$ and hence (ii) $\Rightarrow$ (i). □

Let us denote by $\langle s_0 \rangle$ the two-element subgroup of $W$ generated by $s_0$. We recall that the absolute length function on $W/\langle s_0 \rangle$ is determined by Definition 2.0.3.

**Corollary 5.0.13.** We have $\ell_T(w\langle s_0 \rangle) = \ell_{T_0}(w)$ for every $w \in W^+$.

**Proof.** By definition of $\ell_T(w\langle s_0 \rangle)$ and Lemma 5.0.12 we have

$$\ell_T(w\langle s_0 \rangle) = \min \{ \ell_T(w), \ell_T(ws_0) \} = \begin{cases} 
\ell_T(w), & \text{if } \ell_{T_0}(w) \text{ is even;} \\
\ell_T(w) - 1, & \text{if } \ell_{T_0}(w) \text{ is odd}
\end{cases}$$

and the result follows from Lemma 5.0.11. □

**Proposition 5.0.14.** The orders $\text{Abs}_0(W^+)$ and $\text{Abs}(W/\langle s_0 \rangle)$ are isomorphic.

**Proof.** We set $w^{s_0} := s_0 ws_0$ for $w \in W$ and consider the map $\varphi : W^+ \to W/\langle s_0 \rangle$ defined by

$$\varphi(w) := \begin{cases} 
w\langle s_0 \rangle, & \text{if } \ell_{T_0}(w) \text{ is even;} \\
w^{s_0}\langle s_0 \rangle, & \text{if } \ell_{T_0}(w) \text{ is odd}
\end{cases}$$

for $w \in W^+$. We will show that $\varphi$ is the required isomorphism of absolute orders.

We first note that conjugation by $s_0$ is an automorphism on both $W$ and $W^+$ which preserves the lengths $\ell_T$ and $\ell_{T_0}$, respectively. Corollary 5.0.13 then implies that

$$\ell_T(\varphi(w)) = \ell_{T_0}(w)$$

for every $w \in W^+$. Since the map $\pi : W^+ \to W/\langle s_0 \rangle$ defined by $\pi(w) = w\langle s_0 \rangle$ is a bijection, we may conclude that $\varphi$ is a bijection as well. Thus, it remains to show that the following conditions are equivalent for $u, v \in W^+$:

(a) $u$ is covered by $v$ in $\text{Abs}_0(W^+)$,

(b) $\varphi(u)$ is covered by $\varphi(v)$ in $\text{Abs}(W/\langle s_0 \rangle)$.

Using the definitions of the relevant absolute orders, we find that

(a) $\Leftrightarrow v = \tau u$ for some $\tau \in T_0$ and $\ell_{T_0}(u) < \ell_{T_0}(v)$

(a) $\Leftrightarrow v = s_0 tu$ for some $t \in T$ and $\ell_{T_0}(u) < \ell_{T_0}(v)$

(a) $\Leftrightarrow v^{s_0}\langle s_0 \rangle = tu\langle s_0 \rangle$ for some $t \in T$ and $\ell_{T_0}(u) < \ell_{T_0}(v)$

(a) $\Leftrightarrow v\langle s_0 \rangle = t^{s_0}u^{s_0}\langle s_0 \rangle$ for some $t \in T$ and $\ell_{T_0}(u) < \ell_{T_0}(v)$

and

(b) $\Leftrightarrow \varphi(v) = t\varphi(u)$ for some $t \in T$ and $\ell_T(\varphi(u)) < \ell_T(\varphi(v))$.

The claim that (a) $\Leftrightarrow$ (b) follows from the previous equivalences, (11) and the definition of the map $\varphi$. □
The following statement extends [21, Theorem 7.2] from the case of symmetric groups to that of all finite Coxeter groups.

**Corollary 5.0.15.** For every finite Coxeter group $W$ we have

$$
\sum_{w \in W^+} q^{\ell_0(w)} = \frac{W_T(q)}{1+q} = \prod_{i=2}^{d} (1 + e_i q),
$$

where $d$ is the Coxeter rank and $1 = e_1, e_2, \ldots, e_d$ are the exponents of $W$.

**Proof.** Proposition 5.0.14 implies that

$$
\sum_{w \in W^+} q^{\ell_0(w)} = (W/\langle s_0 \rangle)_T(q).
$$

Since $\langle s_0 \rangle$ is a modular subgroup of $W$ (see Example 3.0.10 (a)), we have

$$
(W/\langle s_0 \rangle)_T(q) = \frac{W_T(q)}{\langle s_0 \rangle_T(q)} = \frac{W_T(q)}{1+q}
$$

by Proposition 3.0.11 and the first equality in (12) follows. The second equality is a restatement of (1).

Another description of $\text{Abs}_0(W^+)$ can be given as follows. Let us write $R_0 := \{ w \in W : \ell_T(ws_0) > \ell_T(w) \}$. The proof of Proposition 3.0.14 shows that $R_0$ is an order ideal of $\text{Abs}(W)$. 

**Corollary 5.0.16.** The absolute order $\text{Abs}_0(W^+)$ is isomorphic to $(R_0, \leq_T)$.

**Proof.** The proof of Proposition 3.0.14 shows that $\text{Abs}(W/\langle s_0 \rangle)$ is isomorphic to $(R_0, \leq_T)$. The result follows from this statement and Proposition 5.0.14.

### 6. Remarks on ordered tuples

This section briefly discusses the action of the symmetric group $S_n$ on the set $X_{n,k}$ of ordered $k$-tuples of pairwise distinct elements of $\{1, 2, \ldots, n\}$, as well as a generalization. The stabilizer $S_{n-k}$ of this action is a modular reflection subgroup of $S_n$ (see Example 3.0.26). Therefore, by Proposition 3.0.11 we have

$$
X_{n,k_T}(q) = \frac{S_{nT}(q)}{S_{n-k_T}(q)} = \prod_{i=n-k}^{n-1} (1 + iq).
$$

By a classical result of Hurwitz [15] (see also [11, 28]), there is a one-to-one correspondence between the maximal chains of any maximal interval of $\text{Abs}(S_n)$ and labeled trees of order $n$. The following generalization of this statement on the enumeration of maximal chains of $X_{n,k}$ is possible. We will denote by $d_{\Gamma}(v)$ the valency (i.e., number of neighbors) of a node $v$ of a labeled tree $\Gamma$ of order $n$. 

Proposition 6.0.17. For all integers \(1 \leq k < n\), the number of maximal chains of \(\text{Abs}(X_{n,k})\) is equal to

\[
  k! \sum_\Gamma (n - k)^{d_\Gamma(v_0)},
\]

where the sum runs over all trees \(\Gamma\) on the node set \(\{v_0, v_1, \ldots, v_k\}\).

The proof of this statement will be given elsewhere. The special case \(k = n - 1\) is equivalent to Hurwitz’s theorem.

Combining Propositions 5.0.14 and 6.0.17, we get the following statement.

Corollary 6.0.18. The number of maximal chains of the absolute order on the alternating group of \(S_n\) is equal to

\[
  (n - 2)! \sum_\Gamma 2^{d_\Gamma(v_0)},
\]

where the sum runs over all trees \(\Gamma\) on the node set \(\{v_0, v_1, \ldots, v_{n-2}\}\).

The previous setting has a natural extension to wreath product actions on ordered colored tuples. Recall from [22, 23] the absolute order on the complex reflection group \(G(r, n) = \mathbb{Z}_r \wr S_n\); absolute length and order are naturally defined with respect to the set \(T\), consisting of all elements (pseudoreflections) of finite order fixing a hyperplane. Let \(X_{r,n,k} := \{(a_1, \ldots, a_k) : \forall i a_i \in \mathbb{Z}_r \times \mathbb{Z}_n\}\) be the set of ordered \(k\)-tuples of letters in an alphabet of size \(n\) which are \(r\)-colored. Then \(G(r, n)\) acts naturally on \(X_{r,n,k}\), with stabilizer \(G(r, n - k)\). By extending Propositions 3.0.12 and 3.0.24, one can prove that the subgroup \(G(r, n - k)\) is a modular subgroup of \(G(r, n)\) for \(1 \leq k \leq n\). Hence, by (the extension of) Proposition 3.0.11 we have

\[
  X_{r,n,k,T}(q) = \frac{G(r, n)_T(q)}{G(r, n - k)_T(q)} = \prod_{i=n-k}^{n-1} (1 + r iq).
\]

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