Stochastic diffusion equation with singular diffusivity and gradient-dependent noise in binary tomography

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Abstract. In this work, we use stochastic diffusion equation with a singular diffusivity and a gradient-dependent noise to improve the reconstruction of binary tomography cross-sections obtained from a small number of projections. A first reconstruction image is obtained with the Total Variation regularization method. The reconstruction is then refined with this stochastic approach. The method is applied to a noisy bone cross-section with 10 projection angles.

1. Introduction
The tomographic reconstruction problem from few projections is an ill-posed problem which has many applications in medical imaging and material science. Binary tomography methods may be proposed to set a simpler inverse problem [1]. They are associated to an under-determined linear system of equations, \( Rf = p^\delta \), with the Radon operator \( R \), the measured projection values \( p^\delta \), and the pixel values \((f_i)_{1 \leq i \leq n}\) of the image with binary constraints \( f = (f_1, ..., f_n) \in \{0, 1\}^n \). Various approaches have been studied to solve this reconstruction problem [1, 2, 3] based on convex analysis optimization [4], Markov random fields [5] or Belief Propagation [6]. Very good reconstruction results are obtained with the Total Variation regularization [7]. Some errors are still present on the boundaries of the reconstructed regions. The main contribution of this work is to use stochastic nonlinear diffusion equation with singular diffusivity and a gradient dependent noise to improve the solution obtained with the TV regularization. After the introduction, the TV regularization method for binary tomography is presented. Then, we describe the nonlinear diffusion with a new existence result. The implementation of the method and the numerical results obtained on a noisy bone CT cross-section are reported in the last section.

2. Total Variation regularization and Alternate Direction of Minimization Method
The binary tomography problem is ill-posed and must be regularized. We consider a TV regularization [8, 9] with the convex constraints \( f \in [0, 1]^n \) and the following problem:

\[
(P) \ \text{minimize} \quad \frac{\mu}{2} \|p^\delta - Rf\|^2_2 + J_{TV}(f) \quad \text{s.t.} \quad f \in [0, 1]^n
\]

Better reconstruction results were obtained with the TV regularization than with the Tikhonov regularization. The parameter \( \mu \) is the regularization parameter balancing the two terms and the noisy projections \( p^\delta \) are such that \( \|p^\delta - p\| \leq \delta \), where \( \delta \) is the noise level and \( p \)
the projections without noise. For an image \( f \in H_1(D) \), the regularization term is given by \( J_{TV}(f) = \int_D |\nabla f(r)|dr \). A fast, efficient \( TV/L_2 \) minimization algorithm based on the Alternate Direction of Minimization Method (ADMM) has been proposed \([8, 9]\). In order to include convex constraints, \( f \in C = [0, 1]^n \), the following augmented Lagrangian \( \mathcal{L}(f, (g_i), (\lambda_i), (\lambda_C)) \) is considered:

\[
\mathcal{L} = \sum_i (\|g_i\|_2 - \lambda_i^t(g_i - D_i f) + \frac{\beta}{2} \|g_i - D_i f\|_2^2) + \frac{\mu}{2} \|g - Af\|_2^2 + I_C(h) + \frac{\beta}{2} \|h - f\|_2^2 - \lambda_C^t(h - f) \tag{2}
\]

The Lagrange multipliers \((\lambda_i), \lambda_c\) are vectors in \( \mathbb{R}^{2n} \) and \( \mathbb{R}^n \) and \( \beta \) is the Lagrangian parameter. For each pixel \( i \), \( D_i f \in \mathbb{R}^2 \) represents the first-order finite difference in both horizontal and vertical directions. The sequences \((f^k, (g_i^k), h^k, (\lambda_i^k), (\lambda_C^k))_{k \geq 0} \) generated by the ADMM algorithm converges to a Kuhn-Tucker point of problem \((P)\), \((f^*, (g_i^*), h^*, (\lambda_i^*), (\lambda_C^*))_{k \geq 0} \), if \((P)\) has one. The new iterate \( f^{k+1} \) is obtained from the linear system:

\[
\sum_i D_i^tD_i + \frac{\mu}{\beta} A^tA + I) f = \sum_i D_i^t(g_i^{k+1} - \frac{1}{\beta} \lambda_i^k) + \frac{\mu}{\beta} A^t g + h^{k+1} - \frac{\lambda_C^t}{\beta} \tag{3}
\]

### 3. Singular stochastic diffusion equation with gradient dependent noise

We describe the stochastic diffusion used to improve the TV reconstruction and show a new existence result. Let \( D \) is a bounded open subset of \( \mathbb{R}^2 \), \( (V = H_0^1(D), \|\cdot\|_1) \), and a filtered probability spaces \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathcal{P})\). To refine the solution obtained with the ADMM algorithm, we consider the singular stochastic diffusion equation \((E_1)\) on \((H = L_2(D), \|\cdot\|)\) \([10]\) of the type:

\[
\begin{cases}
  dX(t) = \text{div} \left( \text{sgn}(\nabla)(X(t)) \right) dt - R^*(RX(t) - p^\delta) + \sigma(\nabla X(t)) dW(t) & \text{in } (0, \infty) \times D \\
  X(0) = x & \text{in } D \\
  X(t) = 0 & \text{in } (0, T) \times \partial D 
\end{cases} \tag{4}
\]

The noise term \( \sigma \) is a gradient dependent noise \( \sigma(\nabla X(t)) dW(t) = \frac{\partial X}{\partial x} dW_1(x, y, t) + \frac{\partial X}{\partial y} dW_2(x, y, t) \) where \((W_k(x, y, t))_{k=1,2} \) are independent Wiener random fields with continuous covariance function \((r_k(x, y))_{k=1,2} \) bounded by a constant \( r_0 \). This type of equation has been extensively studied with additive and multiplicative noise \([11]\). The new equation with a gradient-dependent noise is considered in this work because the noise is located on the boundaries of the reconstructed images. The multi-valued maximal monotone function \( u \rightarrow \text{sgn}(u) \) from \( \mathbb{R}^d \) into \( 2^\mathbb{R}^d \) is defined by:

\[
\text{sgn}(u) = \begin{cases} 
  \frac{u}{|u|} & \text{if } u \neq 0 \\
  \{v \in \mathbb{R}^d \mid |v| \leq 1\} & \text{if } u = 0
\end{cases} \tag{5}
\]

where \(|u|_d\) is the Euclidean norm. Following Barbu et al. \([11]\), to show some existence results, we consider the approximating equation \((E_2)\), \( dX(t) = A_tX(t) - R^*(RX(t) - p^\delta) + \sigma(\nabla X(t)) dW(t) \) where the operator \( A_t : H \rightarrow H \) is defined by \( A_t(u) = (1 + \epsilon A)^{-1} \text{div} \left( \beta_t(\nabla (1 + \epsilon A)^{-1}u) \right) \) with \( A = -\Delta \). The operator \( \beta_t \) is the Yosida approximation of the \( \text{sgn}(u) \) function:

\[
\beta_t(u) = \begin{cases} 
  \frac{u}{|u|} & \text{if } |u|_d \leq \epsilon \\
  \frac{u}{|u|} & \text{if } |u|_d > \epsilon
\end{cases} \tag{6}
\]

#### 3.1. Existence and unicity of a solution of equation \( E_2 \)

We first show that the approximating equation is well-defined and that it is possible to use the variational approach for monotone and coercive stochastic partial differential equations \([10]\). Let \( B = A_t + R^*R \), this operator is linear and weakly continous. We first show that:

\[
\exists \omega > 0, \omega_1 \text{ such that } \forall u \in V, \quad < Bu, u > + |\sigma(u)|^2 \leq \omega_1 |u|^2 - \omega \|u\|_1^2 \quad a.e \ (t, \omega) \tag{7}
\]
where $|\cdot|_r$ is the norm induced by the covariance operator $r$. Recalling that $\beta_\epsilon(u), u \geq |u|^2$ for $u \in V$, it is possible to choose $\epsilon$ such that

$$< A_\epsilon u, u > = -\beta_\epsilon((1 + \epsilon A)^{-1})u, (1 + \epsilon A)^{-1})u >$$

$$\leq -|\nabla(1 + \epsilon A)^{-1}|^2 \leq \frac{-|\nabla u|^2}{2}$$

We can also obtain $< R^* R u, u > \leq \|R\|^2 \|u\|^2$ and

$$|\sigma(u)|_r^2 = \sum_{j=1}^{2} \int_D r_j(x, x) \frac{\partial u}{\partial x_j} dx \leq r_0 |\nabla u|^2 \leq r_0 \|u\|^2$$

The former coercivity condition holds provided $r_0$ is small enough. We now show the monotonicity condition is satisfied:

$$\left\{ \begin{array}{ll}
\quad & \text{for some constant } \eta \ \forall u, v \in V \\
2 < B(u - v), u - v > + |\sigma(u) - \sigma(v)|_r^2 \leq \eta \|u - v\|^2 \ a.e
\end{array} \right.$$

Let $u, v \in V$, since the projection operator $R$ is injective we obtain

$$- < R^* R(u - v), u - v > = - < R(u - v), R(u - v) > \leq -\lambda_{min} \|u - v\|^2$$

where $\lambda_{min}$ is the minimal singular value of $R$. Thus, for $r_0 < 1/2$

$$2 < B(u - v), u - v > + \|\sigma(u) - \sigma(v)\|^2 \leq (r_0 - 1/2) |\nabla(u - v)|^2 - \lambda_{min} \|u - v\|^2 \leq -\lambda_{min} \|u - v\|^2 \ a.e \ (t, \omega)$$

The monotonicity condition is satisfied. By standards existence results [10], the equation (E2) has a unique solution $X_\epsilon \in C_W(0, T] ; H)$ which is path-wise $H$ continuous, where $C_W([0, T]; H)$ is the space of continuous functions from $[0, T]$ to $L^2(\Omega; H)$ which are adapted to $\{F\}_t$.

3.2. Existence of a solution of equation (E1)

In order to control the gradient noise term, we introduce a Banach space with the norm [10], $|u|_T = E\{\sup_{0 \leq t \leq T} |u|^2 + \int_0^T \|u\|^2 dt\}$. Let $X_\epsilon = (1 + \epsilon A)^{-1} X_\epsilon$, $\hat{X}_\epsilon = (1 + \lambda A)^{-1} \hat{X}_\epsilon$, by Ito's formula [11], we have:

$$\frac{1}{2} |X_\epsilon - X_\lambda|^2 + \int_0^t \int_\Omega < \beta_\epsilon(\nabla \hat{X}_\epsilon) - \beta_\lambda(\nabla \hat{X}_\lambda), \epsilon \beta_\epsilon(\nabla \hat{X}_\epsilon) - \lambda \beta_\lambda(\nabla \hat{X}_\lambda) > dxds$$

$$\leq \int_0^t (\sigma(X_\epsilon) - \sigma(X_\lambda), X_\epsilon - (X_\lambda) dW(s) + \int_0^t |\sigma(X_\epsilon) - \sigma(X_\lambda)|_r^2 ds$$

With the Davis-Burkholder-Gundy inequality, we obtain that there is a constant $c$ such that:

$$E \sup_{0 \leq t \leq T} \int_0^t (\sigma(X_\epsilon) - \sigma(X_\lambda), X_\epsilon - (X_\lambda) dW(s) \leq cE(\int_0^T (\sigma(X_\epsilon) - \sigma(X_\lambda), X_\epsilon - (X_\lambda)|_r^2 ds)^{1/2}$$

$$\leq cE \sup_{0 \leq t \leq T} |X_\epsilon - X_\lambda| \sqrt{\int_0^T |\sigma(X_\epsilon) - \sigma(X_\lambda)|_r^2 ds}$$

$$\leq \frac{1}{4} E \sup_{0 \leq t \leq T} |X_\epsilon - X_\lambda|^2 + c^2 E \int_0^T |\sigma(X_\epsilon) - \sigma(X_\lambda)|_r^2 ds$$
Since $|\beta|_\infty \leq 1$, we obtain

$$
E \sup_{0 \leq t \leq T} \frac{1}{4} |X_t - X_\lambda|^2 \leq \lambda + \epsilon + r_0(1 + c^2)E \int_0^T \|X_t - X_\lambda\|^2 ds \leq \lambda + \epsilon + r_0(1 + c^2)|X_\epsilon - X_\lambda|^2
$$

(14)

By integrating from 0 to T, there exist constants $C_1, C_2, C_3, C_4$ such that

$$
E \int_0^T \int_0^t (\sigma(X_v) - \sigma(X_\lambda)^2) ds dt \leq C_1E \int_0^T \|X_t - X_\lambda\|^2 ds dt \leq C_2E \int_0^T |X_t - X_\lambda|^2 dt
$$

(15)

$$
E \int_0^T \int_0^t (\sigma(X_v) - \sigma(X_\lambda), X_t - X_\lambda) dW(s)dv \\
\leq C_3(E \int_0^T \int_0^t (\sigma(X_v) - \sigma(X_\lambda)^2) dW(s)dt + E \int_0^T \int_0^t (X_t - X_\lambda)^2 dW(s)dt)
$$

(16)

Combining (15) and (16), we obtain: $\int_0^T |X_t - X_\lambda|^2 du \leq \lambda + \epsilon + C_2 \int_0^T |X_t - X_\lambda|^2 dt$. Combining (14) and the last inequality, with the Gronwall lemma, we deduce that $|X_t - X_\lambda|^2 \leq C_5(\epsilon + \lambda)$ $\forall \lambda, \epsilon > 0 \ a.e$ and there is a continuous $H$-valued process $X \in C_W([0, T]; H)$ such that $\lim_{t \to 0} |X_t - X|_{\infty} = 0 \ a.e.$

4. Results and discussion

4.1. Simulations details

The TV regularization and the stochastic method were tested with 10 projections angles on a 256x256 bone cross-section acquired with synchrotron micro-CT [12]. Figure 1.a displays the reconstruction obtained from 400 projections with 400 rays per projections with Filtered Back Projection. The discrete approximation of the Radon transform is the one implemented in the Matlab Toolbox. The projections were corrupted with a Gaussian noise with a PSNR of 5.7 dB. The reconstructed image $f_0$ obtained with TV after binarization and the corresponding error map are displayed in Figure 1.b and 1.c respectively. The regularization parameters were chosen with the Morozov discrepancy principle [13] applied to the grey level image. The stochastic optimization is then applied with intermittent diffusions separated by TV/ADMM minimizations. The ADMM runs are stopped when the regularization functional stagnates. At the end of each step, the image is binarized with a threshold 0.5. Numerically, equation $(E_1)$ is implemented with a stochastic noise term added at each iteration to the iterate $f_k$ given by the ADMM method (Eq.3). The discretization of the Brownian motions was performed with the classical Euler-Maruyama method[14]. For a time step $dt$, the noise term added to the iterate $f_k$ given by the ADMM method is given by $\mu Z_k \sqrt{dt}$ where the $(Z_k)_{k \geq 0}$ are i.i.d Gaussian random variables in $\mathbb{R}^n$, $Z_k \sim \mathcal{N}(0, r)$. The parameter $\mu$ controls the noise strength and $r$ is a covariance function with Fourier transform $\tilde{r}(k) = (|k|^2 + 1)^{-2}$.

4.2. Numerical results

The decrease of the data term term related to the binary image $\|R_{f_{binary}}^k - p^\delta\|$ is displayed on Fig.2, where $f_{binary}$ is the binarization of the grey-level image. In order to have more quantitative results, the evolution of the misclassification rate is displayed on Figure 3 for the iterated TV regularization or the intermittent diffusion. A decrease of the misclassification rate as a function
of the number of iterations is also obtained. The stochastic approach is more efficient that the iterated TV regularization. The image obtained with the stochastic diffusion with the lowest value of $\|R_{f_{\text{binary}}}^k - \hat{p}\|$ and the corresponding error map are shown in Fig.4a and Fig.4b. The reconstruction errors on the boundaries of the homogeneous regions are reduced.

![Figure 1](image1.png)

**Figure 1.** Reconstruction of the bone cross-section from 10 noisy projections with the TV regularization: (a) Ground truth; (b) Reconstructed image $f_0$; (c) Error map.

![Figure 2](image2.png)

**Figure 2.** Evolution of the missclassification rate (i) iterated TV (iii) iterated stochastic diffusion.

![Figure 3](image3.png)

**Figure 3.** Evolution of the missclassification rate (i) iterated TV (iii) iterated stochastic diffusion.
Figure 4. Reconstruction of the bone cross-section from 10 noisy projections with the iterated stochastic diffusion: (a) Reconstructed image; (b) Error map.

5. Conclusion
This work introduced a new stochastic method in tomographic reconstruction of binary tomography from a limited number of views. The first reconstruction image is obtained with the Total Variation regularization. The reconstruction images are then refined with a stochastic non linear diffusion equation with a singular diffusivity and a gradient dependent noise. New existence results are demonstrated. The method is applied to a bone cross-section. The stochastic algorithm leads to a decrease of the reconstruction errors localized on the boundaries. The efficiency of the method will be investigated more precisely in future works.

References
[1] G.T.Herman and A.Kuba, "Advances in discrete tomography and its applications", Birkhauser Boston (2007).
[2] K.J.Batenburg and J.Sijbers, "Generic iterative subset algorithm for discrete tomography", Discrete Applied Mathematics, vol.157, pp. 438-451, 2009.
[3] W.Cai and L.Ma, "Comparaison of approaches based on optimization and algebraic iteration for binary tomography", Computer Physics Communications, vol.181, pp. 1974-1981, 2010.
[4] T.D.Capricelli and P.L.Combettes, "A convex programing algorithm for noisy discrete tomography", Advances in discrete tomography and its applications, Boston, MA, pp. 207-226, 2007.
[5] H.Y.Liao and G.T.Herman, "Automated estimation of the parameters of the Gibbs priors to be uses in binary tomography", Discrete Applied Mathematics, pp. 249-170, 2004.
[6] E.Gouillart, F.Krzakala, M.Mezaard and L.Zdebrova, "Belief propagation reconstruction for discrete tomography", Inverse Problems, vol.29, pp. 035003, 2013.
[7] B.Sixou, L.Wang and F.Peyrin, Binary tomographic reconstruction of bone microstructure from few projections with level-set regularization, IEEE Symposium on Biomedical Imaging, San Francisco (2013).
[8] Y.Wang, J.Yang, W.Yin and Y.Zhang, "A new alternating minimization algorithm for Total Variation image reconstruction", SIAM J.Imaging Sciences, vol.1, pp. 248-272, 2008.
[9] M.Afonso, J.Bioucas-Dias and M.Figueiredo, "An Augmented Lagrangian approach to the constrained optimization formulation of imaging inverse problems", IEEE Transactions on Image Processing, vol.99, pp. 1: , 2009.
[10] C.Prevot and M.Rockner, "A concise course on stochastic partial differential equation", Lecture Notes in Mathematics 1905 , 2007.
[11] V.Barbu, G.Da Prato, M.Rekner, "Stochastic nonlinear diffusion equations with singular diffusivity", SIAM J.Math.Anal., vol.41, pp. 1106-1120, 2009.
[12] L.Apostol, V.Boudousq, O.Basset, C.Odet, S.Yot, J.Tabary, J.M.Dinten, E.Boller, P.O.Kotzki and F.Peyrin, "Relevance of 2D radiographic texture analysis for the assessment of 3D bone microarchitecture", Medical Physics, vol.33, pp. 3546-3556, 2006.
[13] H.W.Engl, M.Hanke and A.Neubauer, "Regularization of Inverse Problems", Dordrecht: Kluwer Academic, 1996.
[14] P.E.Kloeden and E.Platen, "Numerical solution of stochastic differential equations ", Applications of Mathematics., vol.23, Berlin:Springer, 1992.