MINIMAL GENERATING SETS OF NON-MODULAR INvariant RINGS OF FINITE GROUPS

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Abstract. It is a classical problem to compute a minimal set of invariant polynomials generating the invariant ring of a finite group as an algebra. We present here an algorithm for the computation of minimal generating sets in the non-modular case. Apart from very few explicit computations of Gröbner bases, the algorithm only involves very basic operations.

As a test bed for comparative benchmarks, we use transitive permutation groups on 7 and 8 variables. In most examples, our algorithm implemented in Singular works much faster than the one used in Magma, namely by factors between 50 and 1000. We also compute some further examples on more than 8 variables, including a minimal generating set for the natural action of the cyclic group of order 11 in characteristic 0 and of order up to 15 in small prime characteristic.

We also apply our algorithm to the computation of irreducible secondary invariants.

Keywords: Invariant Ring, Minimal Generating Set, irreducible Secondary Invariant, Gröbner basis.

MSC: 13A50 (primary), 13P10 (secondary)

1. Introduction

Let $G$ be a finite group linearly acting on a polynomial ring $R$ over a field, such that the characteristic of $R$ does not divide the order of $G$ ("non-modular case"). It is well known that the invariant ring $R^G = \{ r \in R : g.r = r \ \forall g \in G \}$ is a finitely generated sub-algebra of $R$. In this paper, we provide an algorithm to compute a minimal set of homogeneous invariant polynomials generating $R^G$. Such generators are also known as fundamental invariants.

In principal, this can be done as follows: First, one computes primary invariants of $R^G$ and then irreducible secondary invariants. Primary and irreducible secondary invariants together generate $R^G$ as an algebra, and (potentially after removing some primary invariants) they form an inclusion minimal generating set [5]. N. Thiéry [14] suggests another algorithm for the computation of a minimal generating set in the special case of permutation groups, i.e., of groups acting on $R$ as subgroup of the permutation group of the variables of $R$. Thiéry's algorithm is not based on the computation of primary invariants, but uses the incremental construction of so-called SAGBI-Gröbner bases. His algorithm is implemented in the library PERMuVAR of MuPAD [12]. There is extensive benchmark on MAGMA and MuPAD, using the transitive permutation groups on up to nine variables [13].

Our algorithm comes in one version for permutation groups and one version for finite matrix groups. We present comparative benchmarks based on transitive permutation groups on 7 or 8 variables. We implemented our algorithm in a library (i.e., as interpreted code) in SINGULAR [4]. In most of the examples, our algorithm
is at least 50 times, often more than 1000 times, faster than the algorithm used by MAGMA [1]. We also computed minimal generating sets for some transitive permutations groups on 9 and 10 variables. Moreover, we compute minimal generating sets for the natural action of the cyclic groups of order \( \leq 11 \) in characteristic zero and of the cyclic groups of order \( \leq 15 \) in prime characteristic (but, of course, still in the non-modular case).

We took the key ingredient for our algorithm from a previous paper [10], where we focused on the computation of secondary invariants of \( R^G \). Our algorithm does not involve solving linear algebra problems that may become rather huge, in contrast to the algorithm exposed in [2]. Instead, we use Gröbner bases. Of course, the computation of a Gröbner basis can be, in general, a very difficult business. The main feature of our algorithm is that it involves at most one computation of a Gröbner basis in each degree. It turns out that this yields a very well-performant algorithm.

Another peculiarity of our algorithm is the fact that it does not rely on \textit{a-priori} bounds for the maximal degree \( \beta(R^G) \) of elements of a minimal generating set of \( R^G \). For other algorithms, like the one presented in [14], the performance crucially depends on good estimates for \( \beta(R^G) \). Unfortunately, well known a-priori bounds like Noether’s \( \beta(R^G) \leq |G| \) are, in general, far from being optimal. In contrast, we rely on more realistic \textit{a-posteriori} bounds: While incrementally constructing the set of generators, we obtain informations allowing to estimate \( \beta(R^G) \).

We outline our algorithm. In the case of finite matrix groups, candidates for generators are found by applying the Reynolds operator to some monomials. In the case of permutation groups, candidates are found among the \textit{orbit sums}. In increasing degree \( d \), for testing whether a candidate is already contained in the algebra generated by previously found generators, one computes the normal form with respect to a homogeneous Gröbner basis up to degree \( d \) of the ideal spanned by the previously found generators. When starting in a new degree, the Gröbner basis is computed by standard procedures (e.g., Buchberger’s algorithm), and when a new generator of \( R^G \) of degree \( d \) has been found, one can directly write down a new Gröbner basis up to degree \( d \), as we showed in [10]. Eventually, the ideal spanned by the generators of \( R^G \) is 0–dimensional. Then, \( \beta(R^G) \) is bounded by the highest degree of a monomial not occurring as a leading monomial in the ideal spanned by the generators. Hence, after finishing in that degree, we can stop the quest for more generators.

A modification of our algorithm can be used to compute irreducible secondary invariants. According to our comparative benchmarks, it often performs much better than other known algorithms, including our algorithm presented in [10] and the algorithm recently implemented in MAGMA V2.13-9 that appears to be not described in a paper yet.

The rest of this paper is organized as follows. In the next section, we explain our algorithm in more detail. In Subsection 3.1 we do some benchmark tests, comparing the implementation of our algorithm in SINGULAR [4] with the function \texttt{FundamentalInvariants} of MAGMA [1]. In Subsection 3.2 we expose some additional examples that seem to be out of reach for other known algorithms. In the final section, we modify our algorithm in order to compute irreducible secondary invariants, and do some benchmarks with that algorithm.
2. The Algorithm

Let $G$ be a finite group, linearly acting on a polynomial ring $R$ with $n$ variables over some field $K$. We denote the action of $g \in G$ on $r \in R$ by $g.r \in R$.

Let $R^G = \{ r \in R : g.r = r, \forall g \in G \}$ be the invariant ring. Obviously, it is a sub-algebra of $R$, and we aim at computing a minimal set of generators for $R^G$.

We study here the non-modular case, i.e., the characteristic of $K$ does not divide the order of $G$. Note that according to [5], algorithms for the non-modular case are useful also in the modular case.

In the non-modular case, we can use the Reynolds operator $\text{Rey} : R \to R^G$, that is defined by

$$\text{Rey}(r) = \frac{1}{|G|} \sum_{g \in G} g.r$$

for $r \in R$. By construction, the restriction of the Reynolds operator to $R^G$ is the identity. The Reynolds operator does not commute with the ring multiplication. However, it does commute, if one of the factors is invariant, as in the following lemma. This is, of course, well known. We provide a proof, for completeness.

**Lemma 1.** Let $p \in R$ and $q \in R^G$. Then, $\text{Rey}(pq) = \text{Rey}(p)q$.

*Proof.* For any $g \in G$, we have $g.(pq) = (g.p)(g.q)$. But $q \in R^G$, and thus $g.(pq) = (g.p)q$. It follows

$$\text{Rey}(pq) = \frac{1}{|G|} \sum_{g \in G} g.(pq)$$

$$= \frac{1}{|G|} \sum_{g \in G} (g.p)q = \text{Rey}(p)q$$

\[\square\]

For any subset $S \subset R$, we denote by $\langle \langle S \rangle \rangle \subset R$ the sub-algebra generated by $S$, and by $\langle S \rangle \subset R$ the ideal generated by $S$. For $d > 0$, let $R^G_d$ be the set of homogeneous invariant polynomials of degree $d$. For an ideal $I \subset R$, let $\text{lm}(I)$ be the set of leading monomials occurring in $I$.

For $S \subset R$, let $\text{mon}_d(S) \subset R$ be the set of monomials of degree $d$ that are not contained in $\text{lm}(\langle S \rangle)$. This is easy to compute if $S$ is a homogeneous Gröbner basis at least up to degree $d$. Let $B_d(S) = \text{Rey}(\text{mon}_d(S))$. By Lemma 3.5.1 and Remark 3.5.3 in [2], $B_d(S)$ generates $R^G_d$ as a $K$-vector space.

So, in increasing degree $d$ starting with $d = 1$ and $S = \emptyset$, we may loop through all $b \in B_d(S)$, and add $b$ to the set $S$ of previously found generators if $b \notin \langle \langle S \rangle \rangle$. In that way, one incrementally constructs a generating set of $R^G$, consisting of homogeneous invariant polynomials. In fact, it is a minimal generating set [14].

We can test whether $b \in \langle \langle S \rangle \rangle$ according to the following lemma. The lemma is well known, but we include a proof for completeness.

**Lemma 2.** Let $S \subset R^G$ be a set of homogeneous invariant non-constant polynomials. Assume that $R^G_{d'} \subset \langle \langle S \rangle \rangle$ for all $d' < d$, and assume that we are in the non-modular case. Let $b \in R^G_d$. We have $b \in \langle \langle S \rangle \rangle$ if and only if $b \in \langle S \rangle$.

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1The notion of a Gröbner basis up to degree $d$ is well known. See, e.g., [10] for a definition.
Proof. If \( b \in \langle \langle S \rangle \rangle \) then \( b \in \langle S \rangle \). If \( b \in \langle S \rangle \) then we can write \( b \) as a finite sum,
\[
b = \sum_i p_i q_i
\]
with homogeneous polynomials \( p_i \in R \) and \( q_i \in S \). It easily follows from Lemma \[4\] that \( b = \text{Rey}(b) = \sum_i \text{Rey}(p_i) q_i \). Since the elements of \( S \) are non-constant, the \( p_i \) are of degree at most \( d - 1 \). Hence, \( \text{Rey}(p_i) \in R_{d'}^S \) for some \( d' < d \). Thus \( \text{Rey}(p_i) \in \langle \langle S \rangle \rangle \) by hypothesis. Therefore, \( b \in \langle \langle S \rangle \rangle \).
\( \square \)

As in \[10\], we test whether \( b \in \langle S \rangle \) by reduction of \( b \) with respect to a homogeneous Gröbner basis \( G \) of \( \langle S \rangle \) up to degree \( d \). After adding \( b \) to the set of generators, we easily obtain a homogeneous Gröbner basis up to degree \( d \) of \( \langle S \cup \{ b \} \rangle \), by the following result from \[10\]. Again, we provide its short proof, for completeness.

**Theorem 1.** Let \( G \subset R \) be a homogeneous Gröbner basis up to degree \( d \) of \( \langle G \rangle \). Let \( p \in R \) be a homogeneous polynomial of degree \( d \), and \( p \notin \langle G \rangle \). Then \( G \cup \{ \text{rem}(p; G) \} \) is a homogeneous Gröbner basis up to degree \( d \) of \( \langle G \cup \{ p \} \rangle \).

**Proof.** Let \( r = \text{rem}(p; G) \). Since \( p \notin \langle G \rangle \) and all polynomials are homogeneous, we have \( r \neq 0 \), \( \deg(r) = d \), and \( \langle G \cup \{ p \} \rangle = \langle G \cup \{ r \} \rangle \).

By hypothesis, the \( S \)-polynomials of pairs of elements of \( G \) are of degree \( > d \) or reduce to 0 modulo \( G \). We now consider the \( S \)-polynomials of \( r \) and elements of \( G \). Let \( g \in G \). By definition of the remainder, we have \( lm(r) \neq lm(g) \). Therefore the \( S \)-polynomial of \( r \) and \( g \) is of degree \( > d = \deg(r) \). This implies that \( G \cup \{ \text{rem}(p; G) \} \) is a homogeneous Gröbner basis up to degree \( d \).
\( \square \)

There is a problem, though. We can incrementally construct a minimal generating set of \( R^G \), in increasing degrees — but in what degree shall we stop the construction? By definition, we can stop after having found the generators in degree \( \beta(R^G) \). So, we could adopt a general estimate for \( \beta(R^G) \) like Noether’s bound \( \beta(R^G) \leq |G| \). However, such general a-priori estimates are very often far from being optimal.

Therefore, we prefer to derive an estimate for \( \beta(R^G) \) from the previously constructed generators. If \( S \) is a generating set of \( R^G \), then \( \langle S \rangle \) is zero-dimensional, as in the proof of Proposition 3.3.1 in \[2\]. Hence, there are only finitely many monomials outside \( \text{lm}(\langle S \rangle) \), of maximal degree \( d_{\text{max}} \). Since we can restrict the quest for generators of \( R^G \) of degree \( d \) to the Reynolds images of monomials of degree \( d \) outside \( \text{lm}(\langle S \rangle) \), it follows \( \beta(R^G) \leq d_{\text{max}} \).

Our strategy is to work with a homogeneous Gröbner basis of \( \langle S \rangle \) that is subject to a degree restriction, since this is easier to compute than the entire Gröbner basis. However, for testing whether \( \langle S \rangle \) is of dimension 0, one needs a Gröbner basis of \( \langle S \rangle \) without degree restriction. To avoid needless computations, we use the following trick.

By definition, in degree \( \beta(R^G) \) we will find a homogeneous generator of \( R^G \), but in degree \( \beta(R^G) + 1 \) we don’t. Hence, only if our incremental construction of \( S \) arrives at some degree \( d \), such that there is an element of \( S \) in degree \( d - 2 \) but none in degree \( d - 1 \), it makes sense to compute a Gröbner basis of \( \langle S \rangle \) without degree restriction. If \( \dim(\langle S \rangle) = 0 \), which is tested using the Gröbner basis, then we obtain an estimate for \( \beta(R^G) \) that tells us in what degree we can stop the incremental search. We thus obtain the following algorithm for the computation of a minimal generating set of \( R^G \), where \( G \) is a finite matrix group.
Algorithm Invariant Algebra

(1) Construct the Reynolds operator \( \text{Rey} : R \to R^G \).

Let \( S = G = \emptyset \). Let \( d_{\text{max}} = 0 \).

(2) For increasing degree \( d \), starting with \( d = 1 \):

(a) If \( S \) contains elements of degree \( d - 2 \) but no elements of degree \( d - 1 \):

(i) Replace \( G \) by a (complete) Gröbner basis of \( \langle S \rangle \).

(ii) If \( \dim(\langle S \rangle) = 0 \) (which is tested using \( G \)), then replace \( d_{\text{max}} \) by the maximal degree of polynomials outside \( \text{lm}(\langle S \rangle) \), and if, moreover, \( d \) exceeds the new \( d_{\text{max}} \) then break and return \( S \).

If \( S \) contains elements of degree \( d - 1 \), replace \( G \) by a homogeneous Gröbner basis \( G \) of \( \langle S \rangle \) up to degree \( d \).

(b) Compute \( B_d(S) \) using \( G \) and \( \text{Rey} \).

(c) For all \( b \in B_d(S) \):

If \( \text{rem}(b; G) \neq 0 \) then replace \( S \) by \( S \cup \{ b \} \) and \( G \) by \( G \cup \{ \text{rem}(b; G) \} \).

(d) If \( d = d_{\text{max}} \) then break and return \( S \).

By Theorem 1, in all steps \( G \) is a homogeneous Gröbner basis of \( \langle S \rangle \) at least up to degree \( d \). Of course, our algorithm has the same basic structure as many other algorithms. However, our algorithm uses much more elementary methods than the algorithm described in [2] based on linear algebra. No huge systems of linear equations occur, only few explicit Gröbner basis computations are needed (one per degree), and apart from that the most time consuming operation is the computation of normal forms. So it is not surprising that usually our implementation of Invariant Algebra in Singular [4] is much faster than the algorithm from [2] implemented in Magma [1].

In most of our examples, the computation of homogeneous Gröbner bases up to degree \( d \) is not a big deal (there are exceptions, though). However, for large group orders, the computation of the Reynolds operator exceeds the resources. So, the use of the Reynolds operator can be a problem. In the case of permutation groups, it helps to replace it by so-called orbit sums, which is also used in [14]. The orbit of a monomial \( m \in R \) is \( G.m = \{ g.m : g \in G \} \). The orbit sum of \( m \) is \( m^o = \sum_{m' \in G.m} m' \). Of course, \( m^o \in R^G \).

In contrast to the Reynolds operator, the orbit sums are defined even in the modular case, i.e., if the characteristic of \( R \) divides \( |G| \). In the non-modular case, \( m^o \) is just a scalar multiple of \( \text{Rey}(m) \). In conclusion, if \( G \) is a permutation group, we can also define \( B_d(S) \) to be the orbit sums of the monomials in \( \text{mon}_d(S) \). Note, however, that even when using orbit sums, the algorithm Invariant Algebra only works in the non-modular case, since it relies on Lemma 2.

3. Computational results

A classical test bed for the computation of minimal generating sets of invariant rings of finite groups is provided by transitive permutation groups [14], [13]. These are groups acting on a polynomial ring \( R \) over a field \( K \) by permuting variables, such that any two variables are related by the group action. The Magma function TransitiveGroups(i) provides a list of all classes of transitive permutation groups on \( i \) variables.

In our comparative benchmark in Subsection 3.1 we consider transitive permutation groups on 7 and 8 variables in characteristic 0. In Subsection 3.2 we present some more examples of transitive permutation groups, with up to 11 variables in
characteristic 0 and up to 15 variables in prime characteristic. Our benchmarks are based on a Linux x86_64 platform with two AMD Opteron 248 processors (2.2 GHz) and a memory limit of 16 Gb.

3.1. **Comparative Benchmark based on Transitive Permutation Groups.**
We study here minimal generating sets of invariant rings of transitive permutation groups on 7 and 8 variables, in characteristic 0. We compare the following algorithms.

1. Our implementation of Invariant Algebra using orbit sums. This is part of the finvar.lib library of **Singular**-3-0-3 (to be released soon) and is called invariant_algebra_perm. We test a β–version of **Singular**-3-0-3.

2. The function FundamentalInvariants of **Magma** V2.13-9 (released January 2007), which, to the best of the author’s knowledge, is either based on the algorithms described in [2] or unpublished.

Note that our implementation in **Singular** is interpreted code, without any pre-compilation. As far as known to the author, FundamentalInvariants in **Magma** is pre-compiled.

Usually (but not thoroughly) we stopped the computations of an example after two hours CPU time. Moreover, we stopped the computation by one algorithm if it took more than about 1000 times longer than by the other algorithm. The results are provided in Table 1 for the 7 transitive permutation groups on 7 variables, and in Table 2 for 45 transitive permutation groups on 8 variables. In the first column of the tables, the group is defined by its generators in disjoint cycle presentation. The rounded CPU times for **Singular** or **Magma** in seconds are provided in the next two columns. The last column of the tables indicates the number of generators of a minimal generating set of $R^G$, sorted degree-wise.

### Table 1. Transitive permutation groups on 7 variables (characteristic 0)

| Group                                      | **Singular** time [s] | **Magma** time [s] | # generators (sorted by degree) |
|--------------------------------------------|----------------------|--------------------|---------------------------------|
| (1,2,3,4,5,6,7)                            | 0.52                 | 25.3               | 1,3,8,12,12,6,6                 |
| (1,2,3,4,5,6,7),(1,6)(2,5)(3,4)              | 0.67                 | 11                 | 1,3,4,6,6,3,3                   |
| (1,2,3,4,5,6,7),(1,2,4)(3,6,5)               | 6.6                  | 239                | 1,1,4,5,8,8,6                  |
| (1,2,3,4,5,6,7),(1,2)(3,6)                  | 16.9                 | 107                | 1,1,2,2,2,2                    |
| (1,2,3,4,5,6,7),(1,3,2,6,4,5)               | 81.5                 | 600                | 1,1,2,3,4,7,5,5,1              |
| (1,2,3,4,5,6,7),(1,2,3)                    | 117                  | 474                | 1,1,1,1,1,1,1,0,0,0,0,0        |
| (1,2,3,4,5,6,7),(1,2)                      | 198                  | 0.04               | 1,1,1,1,1,1                   |
Table 2: Transitive permutation groups on 8 variables (characteristic 0)

| Group | Singular time [s] | Magma time [s] | # generators (sorted by degree) |
|-------|------------------|----------------|---------------------------------|
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7). (1,5)(2,6)(3,7)(4,8) | 0.14 | 0.07 | 1,7,7,7 |
| (1,2,3,8)(4,3,5,7,6,4) | 0.24 | 11.6 | 1,6,8,12,5 |
| (1,2,3,8)(4,5,6,7). (1,5)(2,6)(3,7)(4,8) | 0.35 | 15 | 1,5,9,16,8 |
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7). (1,5)(2,6)(3,7)(4,8). (4,5)(6,7) | 0.35 | 10.8 | 1,5,5,8,4 |
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7). (1,5)(2,6)(3,7)(4,8). (4,5)(6,7) | 0.55 | 34.6 | 1,4,4,7,3 |
| (1,2,3,8)(4,5,6,7). (1,7,3,5)(2,6,8,4) | 0.65 | 137 | 1,4,10,19,15,7 |
| (1,5)(3,7). (1,2,3,8)(4,5,6,7). (1,3,5,7)(2,4,6,8). (1,4,5,8)(2,3,6,7) | 0.77 | 73.9 | 1,4,6,11,7,2 |
| (1,5)(3,7). (1,2,3,8)(4,5,6,7). (1,3,5,7)(2,4,6,8). (1,4,5,8)(2,3,6,7) | 0.8 | 167 | 1,4,6,11,7,3 |
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7) | 1.2 | 60.3 | 1,4,4,6,4,3,2,1 |
| (4,8). (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7) | 1.4 | 7.38 | 1,4,4,6,3,1 |
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7) | 1.9 | 318 | 1,3,3,6,3,2,1 |
| (1,2,3,4,5,6,7,8). (2,6)(3,7). (1,2,3,4,5,6,7,8) | 2.2 | > 2200 | 1,4,10,18,16,8,4,4 |
| (2,6)(3,7). (1,2,3,4,5,6,7,8) | 2.3 | > 2200 | 1,3,5,8,7,7,4,4 |
| (1,2,3,8)(4,5,6,7) | 2.3 | 385 | 1,3,5,9,6,4,2,1 |
| (1,8)(2,3)(4,5)(6,7). (1,3)(2,8)(4,6)(5,7). (1,5)(2,6)(3,7)(4,8). (1,3)(4,5,6,7) | 2.4 | 649 | 1,3,3,7,6,7,5,1 |
| (1,2,3,4,5,6,7,8). (1,5)(3,7) | 2.8 | > 2800 | 1,3,7,12,13,9,4,4 |
| (1,2,3,4,5,6,7,8). (1,6)(2,5)(3,4)(7,8) | 3 | 1040 | 1,4,5,9,8,4,2,2 |
| (1,2,3,4,5,6,7,8). (1,3)(2,6)(3,7)(4,8). (1,3)(4,5,6,7) | 3.3 | > 3300 | 1,3,6,11,12,7,2,2 |
| (4,8). (1,2,3,8)(4,5,6,7) | 3.7 | 580 | 1,3,5,8,6,4,2,2 |
| (1,2,3,8)(4,5,6,7) | 3.7 | > 3600 | 1,3,5,4,4,2,2 |
| (1,2,3,8). (1,5)(2,6)(3,7)(4,8) | 4 | > 4000 | 1,3,4,7,6,4,2,2 |

Continued on the next page
| Group | SINGULAR | MAGMA | # generators (sorted by degree) |
|-------|----------|-------|--------------------------------|
| (1,2,3,5,6,7,8), (1,5)(3,7), (1,6)(2,5)(3,4)(7,8) | 4.3 | 5440 | 1,3,4,7,6,4,2,2 |
| (1,8)(2,5)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5) | 4.9 | 703 | 1,3,7,8,11,7 |
| (1,2,3,4,5,6,7,8), (1,5)(4,8), (1,7)(3,5)(4,8) | 5 | 4780,6 | 1,3,5,5,3,3,2,3,1 |
| (4,8), (1,3)(5,7), (1,2,3,8)(4,5,6,7) | 5.4 | 444 | 1,3,3,5,3,2,1,1 |
| (1,8)(2,5)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5), (2,3)(4,5) | 6.5 | 1995 | 1,3,6,4,3,1 |
| (1,3)(4,8)(5,7), (1,2,3,8)(4,5,6,7) | 7.5 | > 10800 | 1,3,3,5,3,2,3,4,3,2,1,1 |
| (1,3)(2,8)(4,6)(5,7), (1,2,3)(5,6,7), (1,4)(2,6)(3,7)(5,8) | 8.3 | 2410 | 1,3,8,7,9,6,1,1 |
| (1,8)(2,3)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5), (2,3)(4,5) | 17.5 | > 7200 | 1,2,2,5,2,5,4,3,3 |
| (1,3)(2,8), (1,2,3), (1,5)(2,6)(3,7)(4,8) | 31 | > 7200 | 1,2,2,3,2,3,2,1,1 |
| (1,8)(2,3)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5), (1,6)(2,3,5,4) | 36.5 | > 7200 | 1,2,2,4,3,5,4,2,2,1,1,1 |
| (1,8)(2,3)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5), (1,6)(2,3,5,4) | 37 | 3454 | 1,2,2,4,2,2,1 |
| (1,3,5,7)(2,4,6,8), (1,3,8)(4,5,7) | 39 | > 7200 | 1,2,4,8,11,12,7 |
| (4,8), (1,8)(4,5), (1,2,3,8)(4,5,6,7) | 39 | > 7200 | 1,2,2,3,2,2,1,1 |
| (1,8)(2,3)(4,5)(6,7), (1,3)(2,8)(4,6)(5,7), (1,5)(2,6)(3,7)(4,8), (1,2,3)(4,6,5), (4,6)(5,7) | 44 | > 7200 | 1,2,2,4,3,6,5,5,3 |
| (1,3)(2,8), (1,2,3), (1,8)(4,5), (1,5)(2,6)(3,7)(4,8) | 47 | > 7200 | 1,2,2,3,2,2,1,1,0,0,0,1 |
| (1,3)(2,8), (1,2,3), (1,8)(4,5), (1,5)(2,7,3,6)(4,8) | 50 | > 7200 | 1,2,2,3,2,2,1,1,0,0,0,0,1,1,1 |
| (4,8), (1,8)(2,3)(4,5)(6,7), (1,2,3)(5,6,7) | 51 | > 7200 | 1,2,2,3,3,5,4,3,2,1,1,1 |

Continued on the next page
In total, there are 50 classes of transitive permutation groups on 8 variables, but for five of them, neither SINGULAR nor MAGMA succeeded with the computation in the realm of our time and memory limits. Note that, according to [14], MuPAD can manage one of these five exceptions with the library PerMuVAR; with a memory limit of 500 Mb and a time limit of 2 days, it can compute 17 of the 50 examples.

In the majority of the examples, SINGULAR-3-0-3 is at least 50 times faster than MAGMA V2.13-9, in some cases even more than 1000 times faster. There appears to be only one class of exceptions: The symmetric group on \( n \) variables (the last example on Tables 1 or 2, respectively). This is a special case with a well known theoretical solution. Since MAGMA knows that TransitiveGroup(7,7) and TransitiveGroup(8,50) are symmetric groups, it seems very likely to the author that FundamentalInvariants simply returns the well known solution in this case, without computation. For our algorithm, the invariant ring of the symmetric group is particularly hard, because the a-posteriori degree bound is not very good. E.g., we find the degree bound 28 for the symmetric group on 8 variables, although a minimal generating set has maximal degree 8.

An extensive comparative benchmark of MuPAD and MAGMA on transitive permutation groups is provided by [13]. There, a different machine is used, the memory limit is more restrictive (500 Mb), and the time limit is more generous (2 days).

Note that in the case of small group orders, it sometimes turned out to be faster to use images of the reynolds operator (the function invariant_algebra_reynolds in SINGULAR-3-0-3) rather than orbit sums. However, for groups of order greater than 1000, SINGULAR is hardly able to compute the reynolds operator in reasonable time. Of course, a pre-compilation would yield a considerable speed-up of our implementation.

### 3.2. Further computational results.

In this subsection, we consider some more examples of transitive permutation groups, acting on up to 15 variables. Given the results exposed in the preceding subsection, it seems very unlikely to us that MAGMA V2.13-9 is able to compute these examples in reasonable time. Hence, we only tried with SINGULAR-3-0-3 (Beta version). Table 3 and Table 4 provide the results for some transitive permutation groups on 9 and 10 variables, in characteristic 0; here, we used orbit sums. According to [14], MuPAD can handle 5 of the transitive permutation groups on 9 variables (in total, there are 34 of them) using the library PerMuVAR, with a memory limit of 500 Mb and a time limit of 2 days.
Table 3. Some transitive permutation groups on 9 variables (characteristic 0)

| Group                                                                 | time [s] | # generators (sorted by degree) |
|------------------------------------------------------------------------|----------|----------------------------------|
| (1,2,9)(3,4,5)(6,7,8), (1,4,7)(2,5,8)(3,6,9)                         | 6.24     | 1,4,16,24,24                    |
| (1,2,3,4,5,6,7,8,9)                                                   | 38.19    | 1,4,14,26,32,18,12,6,6         |
| (1,2,9)(3,4,5)(6,7,8), (1,4,7)(2,5,8)(3,6,9), (1,2)(3,6)(4,8)(5,7) | 45.5     | 1,4,8,12,12,10                 |
| (1,2,9)(3,4,5)(6,7,8), (1,2)(4,5)(7,8), (1,4,7)(2,5,8)(3,6,9)       | 55.3     | 1,3,10,14,19,9,2               |
| (1,4,9)(3,4,5)(6,7,8), (1,4,7)(2,5,8)(3,6,9), (3,4)(6,8,7)          | 84.3     | 1,2,8,9,16,18,14,4,2           |
| (1,2,3,4,5,6,7,8,9), (1,4,9)(3,4,5)(6,7,8), (1,2)(4,5)(7,8), (1,4,7)(2,5,8)(3,6,9), (3,4)(6,8,7) | 141.6 | 1,4,7,13,16,12,6,3,3          |
| (1,2,9)(3,4,5)(6,7,8), (1,2)(4,5)(7,8), (1,4,7)(2,5,8)(3,6,9), (3,6)(4,7)(5,8) | 280.7 | 1,3,6,8,9,8,2                 |
| (1,2,3,4,5,6,7,8,9), (1,4,7)(2,5,8)(3,6,9), (3,6)(4,7)(5,8)         | 290.5    | 1,2,6,6,9,8,4                  |
| (1,2,3,4,5,6,7,8,9)                                                  | 455.1    | 1,2,6,11,20,25,26,10,8         |

Table 4. Some transitive permutation groups on 10 variables (characteristic 0)

| Group                                                                 | time [s] | # generators (sorted by degree) |
|------------------------------------------------------------------------|----------|----------------------------------|
| (1,3,5,7,9)(2,4,6,8,10), (1,4)(2,3)(5,10)(6,9)(7,8)                  | 12.3     | 1,7,14,29,28,12                 |
| (1,2,3,4,5,6,7,8,9,10)                                               | 306      | 1,5,16,36,48,32,12,8,4,4       |
| (2,7)(3,5,10)                                                        | 478      | 1,3,8,14,21,16,12,8,4,3        |
| (1,3,5,7,9)(2,4,6,8,10), (1,4,7)(2,5,8)(3,6,9), (3,6)(4,7)(5,8)     | 1294     | 1,4,9,20,31,23,8               |
| (1,2,3,4,5,6,7,8,9,10), (1,8)(2,7)(3,6)(4,5)(9,10)                  | 1425     | 1,5,8,18,24,17,6,4,2,2         |

Table 5. Natural action of $C_n$ on $n$ variables (characteristic 0)

| $n$ | time [s] | mem. [Mb] | # generators (sorted by degree) |
|-----|----------|-----------|----------------------------------|
| 6   | 0.05     | 0.746     | 1,3,6,6,2,2                      |
| 7   | 0.17     | 1.25      | 1,3,8,12,12,6,6                 |
| 8   | 1.54     | 2.25      | 1,4,10,18,16,8,4,4              |
| 9   | 35.6     | 11.92     | 1,4,14,26,32,18,12,6,6          |
| 10  | 298.3    | 54.16     | 1,5,16,36,48,32,12,8,4,4        |
| 11  | 1187     | 116       | 1,5,20,50,82,70,50,30,20,10,10  |

A rather harmlessly looking class of transitive permutation groups is the natural action of the cyclic group $C_n$ of order $n$ on $n$ variables. The maximal degree occurring in a minimal generating set is, by Noether’s bound, of course at most $|C_n| = n$, hence, quite small. However, the minimal number of generators of $R^{C_n}$ is surprisingly large. Since here the group orders are very small, we use the Reynolds operator rather than orbit sums for the generation of invariants. For $n \leq 5$ the
Table 6. Natural action of $C_n$ on $n$ variables (characteristic $p > 0$)

| $n$ | time [s] | mem. [Mb] | # generators (sorted by degree) |
|-----|----------|------------|---------------------------------|
| 6   | 0.03     | 0.746      | 1,3,6,6,2,2                     |
| 7   | 0.09     | 0.746      | 1,3,8,12,12,6,6                 |
| 8   | 0.34     | 1.25       | 1,4,10,18,16,8,4,4              |
| 9   | 1.65     | 1.86       | 1,5,16,36,48,32,12,8,4,4        |
| 10  | 12.7     | 4.48       | 1,5,20,50,82,70,50,30,20,10,10  |
| 11  | 693      | 9.33       | 1,5,20,50,82,70,50,30,20,10,10  |
| 12  | 4079     | 81.1       | 1,6,24,64,104,84,36,20,12,8,4,4 |
| 13  | 25280    | 304.3      | 1,7,32,104,216,242,162,96,42,30,18,12,6,6 |
| 14  | 99873    | 780.4      | 1,7,38,130,306,388,264,120,88,56,40,24,16,8,8 |

The computation is finished in almost no time, so we omit them in our tables. Table 5 provides the result for $n = 6, \ldots, 11$ in characteristic 0. Recall that for the timings in Tables 1–4 we used orbit sums and not the Reynolds operator — this explains the different computation times in the case of cyclic groups.

Table 6 provides the results for $n = 6, \ldots, 15$ in small prime characteristic $p > 0$, of course such that $p$ does not divide $n$ (non-modular case). Apparently this is much easier than characteristic 0. The reason is that in characteristic 0 the coefficients occurring in the Gröbner bases become very huge. By consequence, it takes too long to compute normal forms.

Note that the in all examples, the number of generators in each degree is the same in characteristic 0 and in non-modular prime characteristic. It is in fact conjectured that this is always the case [15].

To work in prime characteristic is not the only way to simplify the computations. As a last example, we study here the action of $S_5$ on pairs, which yields a 10–dimensional representation of $S_5$. One can decompose it into a 1-, a 4- and a 5–dimensional irreducible representation, and in this form, the representation is given by the matrices

$$M_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$M_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}$$

According to an advice of G. Kemper [7], we used this as an example for the computation of irreducible secondary invariants (see [10] or the benchmark in the
next section). But of course it is also a nice example for the computation of a minimal generating set.

We could describe that representation of $S_5$ by a transitive permutation group on 10 variables. However, in that formulation of the problem, our algorithm would take a very long time to find a minimal generating set. But after the decomposition, our algorithm \texttt{Invariant Algebra} executed in \texttt{Singular 3-0-2} finds a minimal generating set after 47.8 minutes using 4.4 Gb in characteristic 0 respectively after only 84.2 seconds using 81.7 Mb in characteristic 7. In both cases, there is a minimal number of 1, 2, 4, 7, 10, 13, 13, 4, 2 generators sorted by degree.

Even using the decomposition, the \texttt{Magma V2.13-9} function \texttt{FundamentalInvariants} is unable to find a minimal generating set in less than 4 hours, both in characteristic 0 and in characteristic 7.

4. Application to irreducible secondary invariants

In [10], we presented an algorithm for the computation of secondary invariants and a specialised version for the computation of irreducible secondary invariants. Shortly after the first version of [10] was posted, there was a new release of \texttt{Magma} containing a new algorithm of G. Kemper for the computation of secondary invariants. Unfortunately, to the best of the author’s knowledge, Kemper did not describe his new algorithm in a manuscript, yet. So it is not clear how it differs from the algorithm described in [5], [6] and [2] or from the algorithm described in [10].

By a slight modification, our algorithm can be used to compute irreducible secondary invariants. For this, let $P$ be a system of primary invariants. In Step (1) of algorithm \texttt{Invariant Algebra}, let $S = P$ and let $G$ be a Gröbner basis of $P$. The rest of the algorithm remains unchanged. In the end, it returns the union of $P$ with a system of irreducible secondary invariants. Note that this algorithm does not involve an application of Molien’s Theorem. So, it applies also to cases when the Molien series is difficult to compute.

In the new version of \texttt{irred_secondary_char0} in \texttt{Singular-3-0-3}, we combine both algorithms, i.e., we use the Molien series and power products as described in [10] in low degrees, and the algorithm \texttt{Invariant Algebra} in higher degrees.

For our benchmark, we use Exp. (4)–(9) from [10], and one additional example, that appeared in our study of ideal Turaev–Viro invariants (see [9] or [8] for background material). Exp. (9) is the 10–dimensional representation of $S_5$ discussed above on Page 11 primary invariants can be easily found by considering the direct summands of the decomposition separately. For the sake of brevity, we do not redefine the other examples from [10], but just provide the new example. The ring variables are called $x_1, x_2, \ldots$. Let $e_i$ be the column vector with 1 in position $i$ and 0 otherwise. In all examples of this section, we work in characteristic 0.

(10) A 20–dimensional representation of $S_3$ is given by the matrices

\[
M_1 = \begin{pmatrix}
  e_2 e_1 e_3 e_9 e_6 e_4 e_1 e_8 e_7 e_10 e_12 e_1 e_4 e_13 e_18 \\
  e_3 e_2 e_4 e_5 e_10 e_9 e_8 e_7 e_13 e_16 e_11 e_19 e_20 e_18 e_{17} e_{14} e_{15}
\end{pmatrix}
\]

\[
M_2 = \begin{pmatrix}
  e_2 e_1 e_3 e_9 e_6 e_4 e_1 e_8 e_7 e_10 e_12 e_1 e_4 e_13 e_18 \\
  e_3 e_2 e_4 e_5 e_10 e_9 e_8 e_7 e_13 e_16 e_11 e_19 e_20 e_18 e_{17} e_{14} e_{15}
\end{pmatrix}
\]

We use the following sub-optimal primary invariants:

\[
x_1 + x_2 + x_3, \ x_1 x_2 + x_3 + x_2 x_3, \ x_1 x_2 x_3, \ x_4 + x_4 + x_19, \\
x_4 x_14 + x_4 x_{19} + x_14 x_{19}, \ x_4 x_14 x_{19}, \ x_5 + x_6 + x_8 + x_9 + x_{11} + x_{13}, \\
x_8 x_9 + x_5 x_{11} + x_6 x_{13}, \ x_6 x_8 + x_5 x_9 + x_{11} x_{13},
\]
\[x_5x_8 + x_6x_9 + x_6x_{11} + x_9x_{11} + x_5x_{13} + x_8x_{13},\]
\[x_5x_6x_{11} + x_5x_8x_{11} + x_8x_9x_{11} + x_5x_6x_{13} + x_6x_9x_{13} + x_8x_9x_{13},\]
\[x^6_5 + x^6_6 + x^6_8 + x^6_9 + x^{13}_{12} + x^{16}_{12},\]
\[x^7_{7} + x^{10}_{10} + x^{15}_{15} + x^{17}_{17} + x^{18}_{18} + x^{20}_{20},\]
\[x^{10}_{10}x_{15} + x^{17}_{17}x_{18} + x^{7}_{7}x_{20},\]
\[x^7_{7}x^{10}_{10}x_{17} + x^7_{7}x^{15}_{15}x_{17} + x^7_{7}x^{10}_{10}x_{18} + x^{15}_{15}x_{17}x_{20} + x^{10}_{10}x_{18}x_{20} + x^{15}_{15}x_{18}x_{20},\]
\[x^6_{7} + x^6_{10} + x^6_{15} + x^6_{17} + x^6_{18} + x^6_{20}.\]

In this example, there are 248832 secondary invariants of maximal degree 26, among which are 283 irreducible secondary invariants of maximal degree 4. The sheer number of secondary invariants (which can be computed by Molien’s Theorem) makes the computations hardly manageable for any algorithm that is based on the generation of power products, as the one described in [5], [6] and [2], or the one described in [10]. It is in fact too much for MAGMA V2.13-9 and for SINGULAR-3-0-2. However, our new algorithm implemented in SINGULAR-3-0-3 just needs few seconds to compute all irreducible secondary invariants.

In Table 7 we compare a \(\beta\)-version of SINGULAR-3-0-3 (function \texttt{irred secondary char0}) with MAGMA V2.13-9 (function \texttt{IrreducibleSecondaryInvariants}, released in January, 2007). The only exception is Example (9), that we compute with our new algorithm, but based on SINGULAR-3-0-2. For convenience, we repeat in Table 7 the timings for SINGULAR-3-0-2 and MAGMA V2.13-8 from [10].

| Expl.       | SINGULAR 3-0-3 | MAGMA V2.13-9 | MAGMA V2.13-8 | SINGULAR 3-0-2 |
|-------------|----------------|---------------|---------------|---------------|
| (4)         | 0.07 s         | 0.09 s        | 0.48 s        | 0.32 s        |
|             | 0.91 Mb        | 7.35 Mb       | 9.09 Mb       | 2.97 Mb       |
| (5)         | 7.75 s         | 0.49 s        | 6.66 s        | 9.69 s        |
|             | 10.9 Mb        | 9.06 Mb       | 31.82 Mb      | 17.0 Mb       |
| (6)         | 1.63 s         | 2.49 s        | 118.51 s      | 16.55 s       |
|             | 6.9 Mb         | 19.8 Mb       | 54.0 Mb       | 39.0 Mb       |
| (7)         | 0.34 s         | 36.57 s       | >7 h          | 20.94 s       |
|             | 2.52 Mb        | 30.1 Mb       | >15 Gb        | 35.1 Mb       |
| (8)         | 1.05 s         | >72 min       | —             | 50.7 min      |
|             | 7.08 Mb        | >2.5 Gb       | (259.5 Gb)    | 3.36 Gb       |
| (9)*        | 17.2 min       | 29.9 min      | —             | 99.2 min      |
|             | 4.67 Gb        | 399.5 Mb      | (55.62 Gb)    | 7.35 Gb       |
| (10)        | 6.83 s         | >280 min      | —             | —             |
|             | 29.4 Mb        | >9.9 Gb       | —             | —             |

The outcome of these benchmarks is less clear than of our benchmarks on minimal generating sets. In 3 of the 7 examples, our algorithm and the one used in MAGMA V2.13-9 show more or less the same performance (by factors less than 2), in one example MAGMA is faster by a factor of about 15, whereas in 3 examples our algorithm is faster by factors between 100 and at least 4000.
Note that in Expl. (9), the critical part is the computation of a Gröbner basis of primary and irreducible secondary invariants. The rest of the computations just takes about 5 minutes. The beta version of SINGULAR-3-0-3 spends much more than 30 minutes with the computation of a Gröbner basis. Here, the old version SINGULAR-3-0-2 happens to be quicker.

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