Riemannian Geometry of Bicovariant Group Lattices

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Abstract

Group lattices (Cayley digraphs) of a discrete group are in natural correspondence with differential calculi on the group. On such a differential calculus geometric structures can be introduced following general recipes of noncommutative differential geometry. Despite of the non-commutativity between functions and (generalized) differential forms, for the subclass of “bicovariant” group lattices considered in this work it is possible to understand central geometric objects like metric, torsion and curvature as “tensors” with (left) covariance properties. This ensures that tensor components (with respect to a basis of the space of 1-forms) transform in the familiar homogeneous way under a change of basis. There is a natural compatibility condition for a metric and a linear connection. The resulting (pseudo-) Riemannian geometry is explored in this work. It is demonstrated that the components of the metric are indeed able to properly describe properties of discrete geometries like lengths and angles. A simple geometric understanding in particular of torsion and curvature is achieved. The formalism has much in common with lattice gauge theory. For example, the Riemannian curvature is determined by parallel transport of vectors around a plaquette (which corresponds to a biangle, a triangle or a quadrangle).

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1 Introduction

In a previous paper [1] we started to develop a general formalism of differential geometry of group lattices (Cayley digraphs), based on elementary notions of noncommutative geometry. The present work extends the latter to a formalism of discrete (pseudo-) Riemannian geometry of the subclass of bicovariant group lattices, as defined in Ref. 1. A group lattice, which is determined by a discrete group $G$ and a finite subset $S$ (which does not contain the unit element $e$) naturally defines a first-order differential calculus (which extends to higher orders) over the algebra $\mathcal{A}$ of functions on $G$. If $S$ generates $G$, then bicovariance of the group lattice $(G, S)$ is equivalent to bicovariance of the first-order differential calculus in the sense of Ref. 2.

“Riemannian geometry” of discrete groups in the context of noncommutative geometry has already been considered in several publications [3, 4, 5]. The present approach differs from these in particular by introducing a metric tensor as an element of a left-covariant tensor product of the space of 1-forms with itself. This tensor product is obtained from the a priori naturally given tensor product over $\mathcal{A}$ by using the special structure of group lattices and the bicovariance condition. Although this formalism has some ideas in common with the approach of Ref. 3, it crucially differs from the latter, where a left-covariant tensor product for arbitrary differential calculi on finite sets was constructed making use of a connection. The present approach is much simpler and geometrically more transparent, but restricted to bicovariant group lattices and thus a subclass of regular [7] digraphs. One should keep in mind that extensions of geometric structures from ordinary differential geometry to the framework of noncommutative geometry may be carried out in various ways and only applications can decide on their usefulness. For our choice, we will demonstrate that it leads to simple and convenient rules of discrete geometry. It is also this last aspect, namely the fact that we establish a geometric interpretation of the a priori abstract formalism, which distinguishes the present work from some previous publications on noncommutative geometry of discrete groups.

The reason why we define a metric as a left-covariant tensor is that in this case its components are “local” objects (see section 2.1 for details). More generally, the components of left-covariant tensors obey a homogeneous local transformation law under a change of basis. In this sense they are really counterparts of tensors in ordinary differential geometry. This is quite in the spirit of Wilson’s lattice gauge theory: discretization a priori moves local fields to non-local objects, but via parallel transport around a plaquette local objects are obtained. This is important in order to maintain gauge invariance, which is the main principle behind it. Similarly, we may postulate the preservation of the tensor transformation principle. This also allows to consider coordinate transformations on group lattices very much in analogy with continuum differential geometry (see section 6). The idea of constructing left- or alternatively right-covariant tensors in a noncommutative differential calculus already appeared in Ref. 3. Viewed as a map between left $\mathcal{A}$-modules, a left-covariant tensor is left $\mathcal{A}$-linear.

Discrete (pseudo-) Riemannian geometry is of relevance for numerical evaluation and also path integral quantization of classical physical models based on continuum Riemannian geometry, like mechanical and general relativistic systems (see Ref. 3 for example). The
approach based on concepts of noncommutative geometry is an alternative to Regge calculus [10]. It has the advantage, however, that its formal structure is much closer to continuum differential geometry. Similarities with previous approaches to gravity using concepts of lattice gauge theory exist [11], but there is little overlap when it comes to the details of the formalism. Of course, discrete geometry is an old subject (see Ref. 12, for example) and relations between the present work and earlier approaches can certainly be established to some extent. This will not be attempted in this work. Rather, we concentrate on what the machinery of algebraic noncommutative geometry applied in a natural way to (bicovariant) group lattices gives us and we reveal the geometric significance of discrete analogues of metric, metric-compatible linear connections, torsion and curvature.

Section 2 discusses the non-locality of the tensor product over $\mathcal{A}$ and introduces the left-covariant tensor product for bicovariant group lattices, which induces a left-covariant product in the space of forms $\Omega$. Left-covariant metrics are then introduced and a compatibility condition with a linear connection is formulated. The geometric meaning of the parallel transport determined by a metric-compatible linear connection is explored. Furthermore, we introduce the notion of a “discrete Killing vector field”.

Section 3 elaborates the torsion and the curvature of linear connections on bicovariant group lattices and also provides corresponding expressions in terms of basic vector fields (which constitute a subclass of discrete vector fields, see Ref. 1). Appendix A presents expressions of basic formulas with respect to an orthonormal coframe field.

Section 4 deals with group lattices which carry a metric and a torsion-free compatible linear connection. Several examples are treated. The conclusion is that for most group lattices a restriction to torsion-free metric-compatible linear connections too severely restricts the possible geometries.

A metric-compatible linear connection provides us with a parallel transport which maps part of the group lattice isometrically into the tangent space at some site. Torsion and curvature of the connection are, respectively, corresponding first and second order obstructions. As we demonstrate with several examples, in particular in section 5, torsion plays a much more fundamental role in this discrete framework than in ordinary continuum differential geometry. Linear connections with torsion are needed to describe even simple group lattice geometries in this framework.

In section 6 we introduce the concept of coordinates on group lattices and elaborate in particular the geometry of hypercubic lattices based on the Abelian group $\mathbb{Z}^n$. Some concluding remarks are collected in section 7.

The present work relies on the notation and results of Ref. 1. It is not self-contained. We refer to an equation in Ref. 1 in the form (I.a.b) where (a.b) is the equation number in Ref. 1. In the following we restrict our considerations to bicovariant group lattices $(G,S)$. This means that $S$ is assumed to be closed under the adjoint action of all elements of $S$ and their inverses.
2 Tensor products, metrics, and linear connections

In this section we first briefly discuss the consequences of the non-locality of the usual tensor product over \( \mathcal{A} \). Then we make use of the special structure of bicovariant group lattices in order to construct a new tensor product which is left-covariant so that the corresponding tensor components are “local” and able to carry a geometric meaning. The left-covariant tensor product of forms induces a left-covariant (generalized wedge) product in the space of forms. Left-covariant metrics are introduced and a compatibility condition with a linear connection is formulated. The latter involves an extension of the linear connection from the space of 1-forms to a left-covariant tensor product. This is a familiar procedure in the tensor calculus on manifolds, but in general not at all straightforward in noncommutative geometries (see also Ref. 3). Of particular importance for an understanding of the formalism is the observation that a metric-compatible linear connection determines an isometric map of parts of the group lattice into the tangent space at a (fixed) site. In the last subsection we define discrete Killing vector fields and invariant metrics on a (bicovariant) group lattice.

2.1 Non-locality of the tensor product over \( \mathcal{A} \)

For the differential calculus \((\Omega, d)\) determined by a group lattice \((G, S)\) there is a distinguished (left and right) \( \mathcal{A} \)-module basis \( \{ \theta^h \mid h \in S \} \) of the space of 1-forms \( \Omega^1 \) which satisfies \( \theta^h f = R^*_h f \theta^h \) for all elements \( f \) of the space of functions \( \mathcal{A} \) on \( G \), where \( R_h \) is the right action on \( G \) by an element \( h \in S \). As a consequence,

\[
(f \theta^h) \otimes_{\mathcal{A}} (f' \theta^{h'}) = f (R^*_h f') \theta^h \otimes_{\mathcal{A}} \theta^{h'}
\]

(2.1)

for all \( f, f' \in \mathcal{A} \). For each \( g \in G \) there is a function \( e^g \) such that \( e^g(g') = \delta_{g,g'} \) for all \( g' \in G \). For this function we obtain

\[
e^g \theta^h \otimes_{\mathcal{A}} \theta^{h'} = (e^g) \otimes_{\mathcal{A}} (e^{g} \theta^h \theta^{h'})
\]

(2.2)

which shows that the tensor product \( \otimes_{\mathcal{A}} \) is non-local since the two factors “sit” at different (though neighboring) points. Let us consider an object

\[
g = \sum_{h, h' \in S} \gamma_{h, h'} \theta^h \otimes_{\mathcal{A}} \theta^{h'}
\]

(2.3)

with \( \gamma_{h, h'} \in \mathcal{A} \). Under a linear change of basis \( \theta^h \mapsto \tilde{\theta}^h := \sum_{h' \in S} a^h_{h'} \theta^{h'} \) with coefficients \( a^h_{h'} \in \mathcal{A} \) we find

\[
g = \sum_{h_1, h_2, h'_1, h'_2 \in S} \gamma_{h'_1, h'_2} (a^{-1})^{h'_1 h_1} (R^*_h a^{-1})^{h'_2 h_2} \tilde{\theta}^{h_1} \otimes_{\mathcal{A}} \tilde{\theta}^{h_2} = \sum_{h_1, h_2 \in S} \tilde{\gamma}_{h_1, h_2} \tilde{\theta}^{h_1} \otimes_{\mathcal{A}} \tilde{\theta}^{h_2}
\]

(2.4)

from which we read off the coefficients with respect to the new cobasis:

\[
\tilde{\gamma}_{h_1, h_2} = \sum_{h'_1, h'_2 \in S} \gamma_{h'_1, h'_2} (a^{-1})^{h'_1 h_1} (R^*_h a^{-1})^{h'_2 h_2}.
\]

(2.5)

Here we see again the non-local character of the tensor product \( \otimes_{\mathcal{A}} \).
2.2 Left-covariant tensor product for bicovariant group lattices

By acting on each component, the maps $R_h^r$ and $R_h^{-1}$ for $h \in S$ extend to tensor products of $\Omega^1$ and to $\Omega$ as automorphisms. Then there is another tensor product with a local transformation rule. This “left-covariant” tensor product is defined via

\[
(\theta^{h_1} \otimes A \ldots \otimes A \theta^{h_r}) \otimes_L T := \theta^{h_1} \otimes A \ldots \otimes A \theta^{h_r} \otimes A R_{h_{r-1}}^* \cdots R_{h_1}^* T
\]

(2.6)

where $T$ is an arbitrary element of a tensor product of $\Omega^1$ over $A$. The inverse relation is

\[
(\theta^{h_1} \otimes A \ldots \otimes A \theta^{h_r}) \otimes A T = (\theta^{h_1} \otimes A \ldots \otimes A \theta^{h_r}) \otimes_L R_{h_1}^* \cdots R_{h_r}^* T.
\]

(2.7)

Using $R_h^* \theta^{h'} = \theta^{\mathrm{ad}(h)h'}$ we find in particular

\[
\theta^h \otimes_L \theta^{h'} = \theta^h \otimes A \theta^{\mathrm{ad}(h^{-1})h'}, \quad \theta^h \otimes A \theta^{h'} = \theta^h \otimes_L \theta^{\mathrm{ad}(h)h'}.
\]

(2.8)

Note also that

\[
(\theta^{h_1} \otimes_L \ldots \otimes_L \theta^{h_r}) \otimes_L T = (\theta^{h_1} \otimes_L \ldots \otimes_L \theta^{h_r}) \otimes_A R_{h_{r-1}}^* \cdots R_{h_1}^* T
\]

(2.9)

\[
(\theta^{h_1} \otimes_L \ldots \otimes_L \theta^{h_r}) \otimes_A T = (\theta^{h_1} \otimes_L \ldots \otimes_L \theta^{h_r}) \otimes_L R_{h_r}^* \cdots R_{h_1}^* T.
\]

(2.10)

We obtain indeed a local transformation law since the new tensor product is designed in such a way that

\[
(f_1 T_1) \otimes_L (f_2 T_2) = f_1 f_2 T_1 \otimes_L T_2
\]

(2.11)

for all $f_1, f_2 \in A$ and elements $T_1, T_2$ of tensor products of $\Omega^1$.

**Lemma 2.1** The left covariant tensor product $\otimes_L$ is associative:

\[
(T_1 \otimes_L T_2) \otimes_L T_3 = T_1 \otimes_L (T_2 \otimes_L T_3)
\]

(2.12)

for all $T_i$ in tensor products of $\Omega^1$.

**Proof:** In particular, we find

\[
(\theta^{h_1} \otimes_L \theta^{h_2}) \otimes_L T = (\theta^{h_1} \otimes_A R_{h_2}^* h_2) \otimes_L T = (\theta^{h_1} \otimes_A \theta^{\mathrm{ad}(h_2)h_2}) \otimes_L T
\]

\[
= \theta^{h_1} \otimes_A \theta^{\mathrm{ad}(h_2)h_2} \otimes_A R_{h_1}^* \cdots R_{h_1}^* T
\]

\[
= \theta^{h_1} \otimes_A \theta^{\mathrm{ad}(h_2)} \otimes_A R_{h_1}^* \cdots R_{h_1}^* T
\]

\[
= \theta^{h_1} \otimes_A R_{h_1}^* (\theta^{h_2} \otimes_A R_{h_2}^* T) = \theta^{h_1} \otimes_L (\theta^{h_2} \otimes_L T).
\]

Our more general assertion is proved in the same way. \[\square\]

**Lemma 2.2** For all $T_1, T_2$ in tensor products of $\Omega^1$,

\[
R_h^r (T_1 \otimes_L T_2) = (R_h^r T_1) \otimes_L (R_h^r T_2)
\]

(2.13)

**Proof:**

\[
R_h^r (f \theta^{h_1} \otimes_A \ldots \otimes_A \theta^{h_r}) \otimes_L T = R_h^r (f \theta^{h_1} \otimes_A \ldots \otimes_A \theta^{h_r}) R_{h_{r-1}}^* \cdots R_{h_1}^* R_{h_1}^* T
\]

\[
= R_h^r (f \theta^{h_1} \otimes_A \ldots \otimes_A \theta^{h_r}) R_{h_{r-1}}^* \cdots R_{h_1}^* \cdots R_{h_1}^* T
\]

\[
= R_h^r (f \theta^{h_1} \otimes_A \ldots \otimes_A \theta^{h_r}) \otimes_L R_h^r T
\]

for all $f \in A$ and all $T$ in a tensor product of $\Omega^1$. Now the assertion follows by linearity. \[\square\]
2.3 A left-covariant product in the space of forms

The non-locality of the tensor product $\otimes_A$ discussed above is inherited by the product in $\Omega$. For a bicovariant group lattice we can define a left-covariant product in $\Omega$ via

$$\omega_1 \cap \omega_2 = \pi(\omega_1 \otimes_L \omega_2) \quad (2.14)$$

where $\pi$ is the projection $\Omega \otimes_A \Omega \rightarrow \Omega$. The new product inherits from $\otimes_L$ left-covariance and associativity. From the definition we obtain

$$\begin{align*}
(\theta^{h_1} \cdots \theta^{h_r}) \cap \omega &= \theta^{h_1} \cdots \theta^{h_r} R^*_{h_r-1} \cdots R^*_{h_1} \omega \quad (2.15) \\
\theta^{h_1} \cdots \theta^{h_r} \omega &= (\theta^{h_1} \cdots \theta^{h_r}) \cap R^*_{h_1} \cdots R^*_{h_r} \omega \quad (2.16)
\end{align*}$$

and also

$$\begin{align*}
(\theta^{h_1} \cap \ldots \cap \theta^{h_r}) \cap \omega &= (\theta^{h_1} \cap \ldots \cap \theta^{h_r}) R^*_{h_r} \cdots R^*_{h_1} \omega \quad (2.17) \\
(\theta^{h_1} \cap \ldots \cap \theta^{h_r}) \omega &= (\theta^{h_1} \cap \ldots \cap \theta^{h_r}) \cap R^*_{h_r} \cdots R^*_{h_1} \omega \quad (2.18)
\end{align*}$$

In particular,

$$\begin{align*}
\theta^h \cap \theta^{h'} &= \theta^h \theta^{\text{ad}(h^{-1})h'} , \\
\theta^h \theta^{h'} &= \theta^h \cap \theta^{\text{ad}(h)h'} . \quad (2.19)
\end{align*}$$

The 2-form relations (see section 4 of Ref. 1) now read

$$\sum_{h,h' \in S} \delta g_{h' h} \theta^h \cap \theta^{h'} = 0 \quad \forall g \in S_{(2)} \quad (2.20)$$

and a 2-form can be decomposed using the projections

$$\begin{align*}
\pi(e)(\theta^h \cap \theta^{h'}) &= \delta_{h' h} \theta^h \cap \theta^{h'} \\
\pi(h)(\theta^{h'} \cap \theta^{h''}) &= \delta_{h'' h'} \theta^{h'} \cap \theta^{h''} \quad h \in S_{(1)} \\
\pi(g)(\theta^h \cap \theta^{h'}) &= \delta_{h' h} \theta^h \cap \theta^{h'} \quad g \in S_{(2)} \quad (2.21)
\end{align*}$$

where $S_{(1)} = S^2 \cap S$, $S_{(2)} = S^2 \setminus S_e$ and $S_e = S \cup \{ e \}$. For a cycle $h_1 h_2 = h_2 h_3 = \cdots = h_r h_1$ we obtain

$$\theta^{h_1} \theta^{h_2} + \theta^{h_2} \theta^{h_3} + \cdots + \theta^{h_r} \theta^{h_1} = \theta^{h_1} \cap \theta^{h_r} + \theta^{h_2} \cap \theta^{h_1} + \cdots + \theta^{h_r} \cap \theta^{h_{r-1}} . \quad (2.22)$$

Hence the structure of 2-form relations is preserved by the $\cap$-product.

Since $R^*_h$ commutes with $\pi$, $(2.13)$ leads to

$$R^*_h(\omega_1 \cap \omega_2) = (R^*_h \omega_1) \cap (R^*_h \omega_2) . \quad (2.23)$$

In Ref. 1 a map $\Delta : \Omega \rightarrow \Omega$ has been introduced which is a graded derivation with respect to the ordinary product in $\Omega$ and satisfies

$$\Delta(\theta^h) := \sum_{h', h'' \in S} \delta_{h' h''} \theta^{h'} \cap \theta^{h''} . \quad (2.24)$$
Lemma 2.3 \( \Delta \) is a graded derivation with respect to the \( \cap \)-product in \( \Omega \).

Proof: Using (I.4.11) and (I.4.20) we obtain

\[
\Delta(\theta^h \cap \omega) = \Delta(\theta^h R^*_h R^{-1}_h \omega) = \Delta(\theta^h R^*_h R^{-1}_h \omega) - \theta^h R^*_h R^{-1}_h \Delta(\omega)
\]

for all \( \omega \in \Omega \). This in turn implies the general derivation rule

\[
\Delta(\omega' \cap \omega) = \Delta(\omega') \cap \omega + (-1)^r \omega' \cap \Delta(\omega)
\]

where \( \omega' \) is an arbitrary \( r \)-form. \( \blacksquare \)

The map \( d \) is not a derivation with respect to the \( \cap \)-product. For an \( r \)-form \( \omega \) we obtain from (I.4.12) the formula

\[
d\omega = \sum_{h \in S} \theta^h \cap R^*_h \omega - (-1)^r \omega \cap \theta - \Delta(\omega)
\]  

(2.25)

where

\[
\theta = \sum_{h \in S} \theta^h .
\]  

(2.26)

This allows to evaluate \( d \) applied to any form in terms of expressions which only involve the \( \cap \)-product (instead of the original product in \( \Omega \)). In fact, we could have defined the left-covariant product of forms (and moreover the left-covariant tensor product) by its basic properties (without reference to the tensor product over \( A \)) and the action of \( d \) directly in terms of (2.25). Reversing some of the arguments would then demonstrate that there is a product in \( \Omega \) with respect to which \( d \) becomes a derivation.

2.4 Fixing the ambiguity of 2-form components

Given a 2-form

\[
\psi = \sum_{h, h' \in S} \psi_{h, h'} \theta^h \cap \theta^{h'}
\]  

(2.27)

the biangle and triangle coefficient functions \( \psi_{h, h'} \) are uniquely determined, but there is an ambiguity in the quadrangle coefficients as a consequence of the 2-form relations (2.20). Indeed, writing

\[
\psi(g) = p(g) \psi = \sum_{h, h' \in S} \tilde{\psi}_{(g)}(h, h') \theta^h \cap \theta^{h'}
\]  

(2.28)

for \( g \in S(2) \), there is a freedom of gauge transformations \( \tilde{\psi}_{(g)}(h, h') \mapsto \tilde{\psi}_{(g)}(h, h') + \Psi_{(g)} \delta^g_{h, h} \) with an arbitrary function \( \Psi_{(g)} \) on \( G \). [13] For any two members \( h, h' \) and \( \tilde{h}, \tilde{h}' \) of the chain \( h_1 h_1' = \ldots = h_r h_r' = g \in S(2) \), the difference

\[
\psi_{(g)\tilde{h}, \tilde{h}'; \tilde{h}, \tilde{h}'} := \tilde{\psi}_{(g)\tilde{h}, \tilde{h}'} - \tilde{\psi}_{(g)\tilde{h}, \tilde{h}'}
\]  

(2.29)
and thus also
\[
\psi(g)_{h',h} := \sum_{h,h' \in S} \psi(g)_{h',h,h} = |g| \tilde{\psi}(g)_{h',h} - \sum_{h,h' \in S} \delta^g_{h' h} \tilde{\psi}(g)_{h',h} 
\] (2.30)

is gauge invariant and hence independent of the choice of the coefficient functions \(\tilde{\psi}(g)_{h,h'}\) (from their gauge equivalence class). Here \(|g|\) denotes the length of the chain which belongs to \(g\), i.e. \(|g| = r\). Furthermore, we obtain \(\sum_{h,h'} \delta^g_{h' h} \psi(g)_{h,h'} = 0\) and
\[
\psi(g) = \frac{1}{|g|} \sum_{h,h' \in S} \delta^g_{h' h} \psi(g)_{h,h'} \theta^h \cap \theta^{h'} 
\] (2.31)
which suggests to define the functions (2.30) as the quadrangle components of the 2-form \(\psi\) (with respect to the \(\cap\)-product). The equation \(\psi(g) = 0\) (for a 2-form \(\psi\)) is equivalent to the vanishing of all the differences \(\psi(g)_{h,h',h,h'}\). Furthermore, we obtain \(\sum_{h,h'} \tilde{\psi}(g)_{h,h'} = 0\) and
\[
\psi(g) = \frac{1}{|g|} \sum_{h,h' \in S} \delta^g_{h' h} \psi(g)_{h,h'} \theta^h \cap \theta^{h'} 
\]
which suggests to define the functions (2.30) as the quadrangle components of the 2-form \(\psi\) (with respect to the \(\cap\)-product). The equation \(\psi(g) = 0\) (for a 2-form \(\psi\)) is equivalent to the vanishing of all the differences \(\psi(g)_{h,h',h,h'}\). Of course, also in the case of higher than 2-forms there is an ambiguity in the choice of coefficients and a corresponding way of fixing it.

2.5 Left-covariant metric and compatibility with a linear connection

Let us express \(g\) given in (2.3) as
\[
g = \sum_{h,h' \in S} g_{h,h'} \theta^h \otimes_L \theta^{h'} 
\] (2.32)
with \(g_{h,h'} \in A\). By comparison with (2.3), we obtain
\[
\gamma_{h,h'} = g_{h,h'} h^{-1} . \] (2.33)
We say that \(g\) is symmetric if \(g_{h,h'} = g_{h',h}\), which corresponds to \(\gamma_{h,h'} = \gamma_{h',h'}\). Furthermore, \(g\) is said to be invertible if the matrix \(g = (g_{h,h'})\) is invertible (at all sites).

An object \(g\) as considered above is a candidate for a “metric tensor”. Its components should then be expected to determine lengths of vectors and angles between vectors at a site. This interpretation clearly distinguishes the components \(g_{h,h'}\) and thus the left-covariant tensor product (see also the corresponding remarks in the introduction). Hence we define a metric tensor as an object \(g\) of the form (2.32) such that the coefficient matrix \(g\) is real, symmetric and invertible.

A metric \(g\) is called left-invariant if \(L_h^* g = g\) for all \(h \in S\), where \(L_h\) denotes the left action by \(h\) on \(G\). This is equivalent to a “constant metric”, i.e. \(g_{h,h'} \in \mathbb{R}\). A left-invariant metric is called bi-invariant if it is also right-invariant, i.e. \(R_h^* g = g\) for all \(h \in S\). This means that the metric is constant and satisfies \(g_{h_1,h_2} = g_{\text{ad}(h_1)h_1, \text{ad}(h_2)h_2}\) for all \(h_1, h_2 \in S\).

Let \(\{\ell_h \mid h \in S\}\) be the vector fields dual to \(\{\theta^h \mid h \in S\}\), so that \(\ell_h f = R_h^* f - f\) for \(f \in A\). Let \(\nabla_{\ell_h}\) be the parallel transport along the vector field \(\ell_h\) with respect to a linear connection on \(\Omega^1\) (see Ref. [1]). We write
\[
\nabla_{\ell_h} \theta^h = \sum_{h'' \in S} (R_{h' h''}^* V_{h',h''}^h) \theta^{h''} 
\] (2.34)
where \( V_h = (V^{h''}_{h,h'}) \) are matrices with entries in \( \mathcal{A} \). \( \mathcal{V} \) extends to \( \Omega^1 \otimes_L \Omega^1 \) via

\[
\mathcal{V}_{h'}(\alpha \otimes_L \beta) = \mathcal{V}_h \alpha \otimes_L \mathcal{V}_{h'} \beta.
\]

(2.35)

Then \( \mathcal{V} := \sum_{h \in S} \theta^h \otimes \mathcal{A} \mathcal{V}_h \) has the property \( \mathcal{V}(f \alpha \otimes_L \beta) = f \mathcal{V}(\alpha \otimes_L \beta) \) and thus defines a connection according to lemma 6.1 of Ref. [1].

An element \( g \in \Omega^1 \otimes_L \Omega^1 \) (e.g., a metric) is said to be compatible with the linear connection \( \mathcal{\nabla} \) if

\[
\mathcal{\nabla} g = 0 \tag{2.36}
\]

which in terms of the parallel transport operators takes the form

\[
\mathcal{V}_{h'} g = g \quad \forall h' \in S. \tag{2.37}
\]

In components, this reads

\[
\sum_{h_1,h_2 \in S} V^{h_1}_{h,h_1'} V^{h_2}_{h,h_2'} g_{h_1,h_2} = R^*_h g_{h_1',h_2'} \tag{2.38}
\]

and in matrix form

\[
R^*_h g = V^T_h g V_h. \tag{2.39}
\]

If \( g \) is a metric, this condition requires that the matrices \( V_h, h \in S \), are invertible. For a given metric, there are not always matrices \( V_h \) satisfying (2.39).

**Lemma 2.4** A linear connection compatible with a metric on a bicovariant group lattice exists if and only if the metric has the same signature at all sites.

**Proof:** This is a direct consequence of the fact that two real symmetric matrices \( A, B \) with the same rank are related by \( B = V^T A V \) with an invertible matrix \( V \) if and only if both have the same signature. ■

A bicovariant group lattice supplied with a metric of constant signature will be called a Riemannian group lattice in the following. Since we require a metric to be non-degenerate, a Riemannian group lattice \( (G, S, g) \) should be regarded as an \(|S|\)-dimensional structure.

The metric-compatibility condition determines the transport matrices, and thus the connection, only up to transformations \( V_h \mapsto J_h V_h \) with arbitrary isometries \( J_h \), which are matrices of functions on \( G \) such that

\[
J_h^T g J_h = g. \tag{2.40}
\]

### 2.6 Backward parallel transport of vector fields and geometric interpretation of metric-compatible linear connections

Vector fields are elements of the \( \mathcal{A} \)-bimodule generated by \( \{\ell_h \mid h \in S\} \). A linear connection determines a backward parallel transport of vector fields along a vector field:

\[
\tilde{\mathcal{V}}_{h'} X := \sum_{h', h'' \in S} (R^*_h X^{h''}) V^{h''}_{h,h''} \cdot \ell_{h''}, \quad \tilde{\mathcal{V}}_h X := \sum_{h \in S} X^h \tilde{\mathcal{V}}_{\ell_h} \tag{2.41}
\]
are the images in the tangent space at \( g \) of the vectors \( \ell_{h'} \) at \( gh \). If the transport is metric-compatible, the vectors \( V_{h,h'} \) at \( g \) carry the metric properties of \( \ell_{h'} \) at \( gh \), i.e.

\[
g_{h',h''}(gh) = g(\ell_{h'},\ell_{h''})(gh) = g(V_{h,h'},V_{h,h''})(g) .
\]

Of course, we can also transport tangent vectors from more remote sites to the tangent space at \( g \) by iterated application of the operators \( \hat{V}_h \):

\[
V_{h_1,...,h_{r+1}} := \hat{V}_{h_1} \cdots \hat{V}_{h_r} \ell_{h_{r+1}} .
\]

The results will, however, be path-dependent in general. But here we see very clearly the geometric significance of a metric-compatible linear connection. It maps part of the group lattice into the tangent space at a site in such a way that the metric relations are preserved, i.e. isometrically. In general, this cannot be done for the whole group lattice. Torsion and curvature are obstructions. We have already shown in Ref. [ ] that in case of vanishing torsion at least the next-neighbor part of the group lattice is mapped isometrically into the tangent space in this way, i.e. the backward parallel transport preserves the group lattice geometry to first order. Curvature is a second order obstruction. Its biangle, triangle and quadrangle parts are given, respectively, by parallel transport of a vector field around a quadrangle. There is a discrete version of a familiar formula of continuum differential geometry: the quadrangle curvature is determined by parallel transport of a vector field around a quadrangle. There are no counterparts of biangle and triangle curvature in continuum differential geometry.

Let us make more precise how an isometric tangent space picture of (part of) a group lattice is obtained if a metric-compatible linear connection is given. If \( S \) has \( n \) different elements, let \( (\ ,\ ) \) be an inner product in \( \mathbb{R}^n \) with the same signature as \( g \). At the origin in \( \mathbb{R}^n \) we choose an \( n \)-bein \( \{u_h \mid h \in S \} \) such that

\[
(u_h,u_{h'}) = g(\ell_h,\ell_{h'})(g) .
\]

Then \( \iota : \ell_h \mapsto u_h \) extends to an isomorphism of metric linear spaces. Furthermore, \( V_{h,h'} := \iota(V_{h,h'}) = \sum_{h'' \in S} V_{h'',h',h} u_{h''} \) represents the vector \( V_{h,h'} \) in \( \mathbb{R}^n \). We attach it at the tip of \( u_h \). More generally, the vector

\[
V_{h_1,...,h_{r+1}} := \iota(V_{h_1,...,h_{r+1}}) = \sum_{h \in S} u_h [V_{h_1}(g) V_{h_2}(gh_1) \cdots V_{h_r}(gh_1 \cdots h_{r-1})]^{h}_{h_{r+1}}
\]

has to be attached at the tip of \( u_{h_1} + V_{h_1,h_2} + \ldots + V_{h_1,\ldots,h_r} \).
The isometries $J_h$ act on the vectors $V_{h,h'}$ as follows,

$$J_h(V_{h,h'}) := \sum_{h_1,h_2} u_{h_1} J_{h_1,h_2}^h V_{h_2}^{h,h'}.$$ (2.48)

The isometry property of the $J_h$ then implies

$$(J_h(V_{h,h'}), J_h(V_{h,h''})) = (V_{h,h'}, V_{h,h''}).$$ (2.49)

The backward parallel transport and the isomorphism $\iota$ provide us with a convenient way to describe the action of a (metric-compatible) linear connection in $\mathbb{R}^n$ (supplied with a standard inner product). This will be used extensively in sections 4 and 5.

### 2.7 Contravariant metric tensor and compatibility with a linear connection

A left-covariant tensor product of vector fields $X, Y$ is defined as follows,

$$X \otimes_L Y := \sum_{h \in S} X^h \ell_h \otimes_{\mathcal{A}} R_{h*} Y.$$ (2.50)

Given a metric tensor in the sense of section 2.5, there is also a “contravariant” metric tensor,

$$h = \sum_{h,h' \in S} h_{h,h'} \cdot \ell_h \otimes_L \ell_{h'} = \sum_{h,h' \in S} h_{h,\text{ad}(h)h'} \cdot \ell_h \otimes_{\mathcal{A}} \ell_{h'}$$ (2.51)

where $(h(g)_{h,h'})$ is the inverse of the matrix $g$ at $g \in G$.

If the matrices $V_h$ are invertible for all $h \in S$, the corresponding linear connection on $\Omega^1$ induces a connection on the space $\mathcal{X}$ of vector fields (see Ref. 1). An element $h \in \mathcal{X} \otimes_L \mathcal{X}$ is compatible with the connection $\nabla$ if

$$\nabla h = 0$$ (2.52)

where $\nabla$ has been extended to $\mathcal{X} \otimes_L \mathcal{X}$ following the procedure in section 2.3. Using $U_h := V_h^{-1}$, this condition reads

$$R^*_{h} h_{h_1,h_2} = \sum_{h_1', h_2' \in S} (U_h)^{h_1'}_{h_1} (U_h)^{h_2}_{h_2'} h_{h_1',h_2'}$$ (2.53)

or $R^*_{h} h = U_h h U_h^T$ in matrix form.

### 2.8 Discrete Killing vector fields

Let $X = \sum_{h \in S} X^h \cdot \ell_h$ be a discrete vector field for which the map $\phi_X : G \to G$, which is determined by $\phi_{X} = I + X$ on functions, is differentiable (see Ref. 1). $X$ will be called a Killing vector field of a metric $g$ if $\mathcal{L}_X g = \phi_X^* g - g = 0$ (with the Lie derivative $\mathcal{L}$ introduced in Ref. [1]). For $X = \ell_h$ this becomes $R^*_{h} g = g$, i.e.

$$g(gh)_{h_1,h_2} = g(gh)_{\text{ad}(h)h_1,\text{ad}(h)h_2}$$ (2.54)
for all \( g \in G \). The right hand side of (2.54) can be expressed in the form \((P^T_h \, g(\cdot) \, P_h)_{h_1,h_2}\) where the matrix \( P_h \) represents a permutation.

A metric \( g \) on a bicovariant group lattice \((G,S)\) is thus right-invariant if it satisfies \( \mathcal{L}_{\ell_h} g = 0 \) for all \( h \in S \). A right-invariant metric is completely determined by its values at one site (e.g., at the unit element \( e \)).

\( V_h := P_h \) defines a linear connection which is compatible with every right-invariant metric. Each other linear connection compatible with a right-invariant metric is then obtained as \( V_h := J_h P_h \), where \( J_h \) is at each lattice site an isometry of the metric.

**Example 2.1.** Let \( G \) be a discrete group and \( S \subset G \setminus \{ e \} \) finite and Abelian. If \( \mathcal{L}_{\ell_h} g = 0 \) for some \( h \in S \), the condition (2.54) becomes \( g(gh) = g(g) \) which means that the functions \( g_{h_1,h_2}, h_1, h_2 \in S \), are constant on the orbits in \( G \) under the right action \( R_h \).

Let \( G = \mathbb{Z}_n \) or \( G = \mathbb{Z} \), and \( 1 \in S \). If \( \ell_1 \) is a Killing vector field, the metric coefficients \( g_{h_1,h_2} \) are constant on the whole group. The corresponding natural linear connection is then given by \( V_h = I \).

**Example 2.2.** Let \( G = S_3 \) and \( S = \{(12), (13), (23)\} \). If \( \ell_{(12)} \) is a Killing vector field of a metric \( g \) on this group lattice, then

\[
g((12)) = P_{(12)} \, g(g) \, P_{(12)} \quad \text{where} \quad P_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

This determines the metric at the sites \( (12), (13), (23) \) in terms of the metric at the sites \( e, (132), (123) \), respectively. If \( \ell_{(13)} \) and \( \ell_{(23)} \) are Killing vector fields of \( g \), then

\[
g((13)) = P_{(13)} \, g(g) \, P_{(13)} \quad \text{where} \quad P_{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

The right-invariant metrics on \((S_3, \{(12), (13), (23)\})\) are then given by

\[
g(e) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad g((12)) = \begin{pmatrix} a & c & b \\ c & f & e \\ b & e & d \end{pmatrix}, \quad g((13)) = \begin{pmatrix} f & e & c \\ e & d & b \\ c & b & a \end{pmatrix},
\]

\[
g((23)) = \begin{pmatrix} d & b & e \\ b & a & c \\ e & c & f \end{pmatrix}, \quad g((123)) = \begin{pmatrix} d & e & b \\ e & f & c \\ b & c & a \end{pmatrix}, \quad g((132)) = \begin{pmatrix} f & c & e \\ c & a & b \\ e & b & d \end{pmatrix}
\]

with constants \( a, b, c, d, e, f \). A linear connection compatible with this family of metrics is obtained by choosing \( V_h = P_h \). The family of right-invariant metrics includes the following bi-invariant metric:

\[
g(h) = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}
\]
with constants \(a, b\).

### 3 Torsion and curvature of linear connections as left-covariant tensors on bicovariant group lattices

The torsion 2-forms

\[
\Theta^h = d\theta^h - \pi(\nabla\theta^h) = d\theta^h - \theta \theta^h - \sum_{h', h'' \in S} V^h_{h', h''} \theta^h \theta^{h''}
\]  

(3.1)

can be rewritten in terms of the \(\cap\)-product and then decomposed into biangle, triangle and quadrangle parts as follows,

\[
\Theta^h = \sum_{h_1, h_2 \in S} Q^h_{h_1, h_2} \theta^{h_1} \cap \theta^{h_2}
\]

\[
= \sum_{h_1, h_2 \in S} \left( Q^h_{(e) h_1, h_2} + \sum_{h_0 \in S_{(1)}} Q^h_{(h_0) h_1, h_2} + \sum_{g \in S_{(2)}} \tilde{Q}^h_{(g) h_1, h_2} \right) \theta^{h_1} \cap \theta^{h_2}.
\]  

(3.2)

In this way we find the biangle components

\[
Q^h_{(e) h_1, h_2} = \delta^h_{h_1 h_2} (\delta^h_{h_1} + V^h_{h_1, h_2})
\]  

(3.3)

and the triangle components

\[
Q^h_{(h_0) h_1, h_2} = \delta^h_{h_2 h_1} (\delta^h_{h_1} - \delta^h_{h_0} + V^h_{h_1, h_1^{-1} h_2 h_1}).
\]  

(3.4)

In case of the quadrangle components, one has to take the 2-form relations (2.20) into account. As a consequence of the latter, the functions \(Q^h_{(g) h_1, h_2}\) are not uniquely determined. Following the discussion in section 2.4, it is convenient to introduce the differences

\[
Q^h_{(g) h_1, h_2; h_1, h_2} := \tilde{Q}^h_{(g) h_1, h_2} - \tilde{Q}^h_{(g) h_1, h_2}
\]

\[
= \delta^g_{h_2 h_1} (\delta^h_{h_1} - \delta^h_{h_0} + V^h_{h_1, h_1^{-1} h_2 h_1} - V^h_{h_1, h_1^{-1} h_2 h_1})
\]  

(3.5)

(cf. (2.29)) where \(\hat{h_2}, \hat{h_1}\) is any pair of elements of \(S\) which belongs to the same chain as \(h_2, h_1\) (so that \(h_2 h_1 = g = h_2 h_1\)). In particular, the vanishing of the quadrangle part of the torsion 2-form is equivalent to the vanishing of all the quantities (3.5). According to section 2.4, the quadrangle torsion components should be defined as follows,

\[
Q^h_{(g) h_1', h_1} := |g| \tilde{Q}^h_{(g) h_1', h_1} - \sum_{h_1', h_2', h_3' \in S} \delta^g_{h_1 h_1'} \tilde{Q}^h_{(g) h_1', h_1'} = \sum_{h_1', h_2', h_3' \in S} Q^h_{(g) h_1', h_1'} \quad i = 1, \ldots, |g|
\]  

(3.6)

if \(h_1 h_1' = \ldots = h_r h_1' = g\) is the corresponding chain. This does not depend on the choice of the coefficient functions \(\tilde{Q}^h_{(g) h_1', h_1'}\) which is ambiguous as a consequence of the 2-form relations.
After some manipulations like
\[
\sum_{h',h'' \in S} \theta^{h'} \theta^{h''} \otimes_L V_{\ell_{h'}} V_{\ell_{h''}} \theta^h
\]
\[
= \sum_{h_1, h'' \in S} (\theta^{h_1} \theta^{h''}) \otimes_L R^*_h R^*_h \theta^h
\]
\[
= \sum_{h_1, h'' \in S} (\theta^{h_1} \cap \theta^{h''}) \otimes_L R^*_h \theta^h
\]
\[
= \sum_{h_1, h_2 \in S} (\theta^{h_1} \cap \theta^{h_2}) \otimes_L R^*_h \left( \sum_{h', h''} (R^*_{h_1}) V^{h_1}_{h', h''} (R^*_{h_1}) V^{h_1}_{h', h''} \theta^{h''} \right)
\]
\[
= \sum_{h_1, h_2, h', h'' \in S} (\theta^{h_1} \cap \theta^{h_2}) \otimes_L V^{h_1}_{h', h''} (R^*_{h_1}) V^{h_1}_{h', h''} \theta^{h_1} \cap \theta^{h_2} \otimes_L \theta^{h'} \tag{3.7}
\]
the definition of the curvature, see (1.7.4), leads to
\[
\mathcal{R}(\theta^h) = \sum_{h', h_1, h_2} \left( \sum_{h'' \in S} V^{h_1}_{h', h''} (R^*_{h_1}) V^{h_1}_{h', h''} \theta^{h_1} \cap \theta^{h_2} \otimes_L \theta^{h'} \right)
\]
\[
= \sum_{h_1, h_2} \left( \mathcal{R}^h_{(e)} + \sum_{h_0 \in S_{(1)}} \mathcal{R}^h_{(h_0)} + \sum_{g \in S_{(2)}} \mathcal{R}^h_{(g)} \right) \theta^h \cap \theta^{h_2} \otimes_L \theta^{h'} \tag{3.8}
\]
Writing
\[
\mathcal{R}(\theta^h) = \sum_{h', h_1, h_2} \mathcal{R}^h_{(e)} \theta^h \cap \theta^{h_2} \otimes_L \theta^{h'}
\]
\[
= \sum_{h_1, h_2} \left( \mathcal{R}^h_{(e)} + \sum_{h_0 \in S_{(1)}} \mathcal{R}^h_{(h_0)} + \sum_{g \in S_{(2)}} \mathcal{R}^h_{(g)} \right) \theta^h \cap \theta^{h_2} \otimes_L \theta^{h'} \tag{3.9}
\]
we obtain the triangle components
\[
\mathcal{R}^h_{(e)} \theta^h = (V_{h_1} R^*_{h_1} V_{h_2} - I)^{h'} \tag{3.10}
\]
and the triangle components
\[
\mathcal{R}^h_{(h_0)} \theta^h = (V_{h_1} R^*_{h_1} V_{h_2} - V_{h_0})^{h'} \tag{3.11}
\]
and the differences of quadrangle components
\[
\mathcal{R}^h_{(g)} = \mathcal{R}^h_{(g)} - \mathcal{R}^h_{(g)} \theta^h \theta_{h_1} \theta_{h_2}
\]
\[
= (V_{h_1} R^*_{h_1} V_{h_2} - V_{h_1} R^*_{h_1} V_{h_2})^{h'} \tag{3.12}
\]
Again, \( \hat{h}_2, \hat{h}_1 \) is any pair with \( \hat{h}_2 \hat{h}_1 = g \in S_{(2)} \).
According to section 2.3, the quadrangle curvature components should be defined as follows,

\[ R^h_{(g)h',h'',h_i} := |g| \tilde{\mathcal{R}}^h_{(g)h',h'_i,h''} - \sum_{h''',h''''} \delta^h_{h''',h''''} \tilde{\mathcal{R}}^h_{(g)h',h''',h''''} \quad i = 1, \ldots, |g| \]  \hspace{1cm} (3.13)

if \( h_1h'_1 = \ldots = h_rh'_r = g \) is the corresponding chain. Understanding that the quadrangle part of \( \mathcal{R}^{h''}_{h',h'',h'} \) is given by the above expression, the components of a Ricci tensor can be defined without ambiguity as follows,

\[ \text{Ric}_{h,h'} := \sum_{h'' \in S} \mathcal{R}^{h''}_{h,h''} \cdot \]  \hspace{1cm} (3.14)

With the help of a metric, a curvature scalar can be built:

\[ R := \sum_{h,h' \in S} (g^{-1})^{h,h'} \text{Ric}(\ell_h, \ell_{h'}) \]  \hspace{1cm} (3.15)

There is, however, another contraction of the curvature tensor, namely

\[ \tilde{\text{Ric}}_{h,h'} := \sum_{h'' \in S} \mathcal{R}^{h''}_{h,h'',h'} \]  \hspace{1cm} (3.16)

which leads in general to a different Ricci tensor and curvature scalar. Moreover, also the contraction \( \sum_{h'' \in S} \mathcal{R}^{h''}_{h'''}h,h' \) is in general different from zero. This complicates finding a suitable analogue of the Einstein equation, for example.

### 3.1 Bianchi identities

According to Ref. [1], the first Bianchi identity can be expressed as follows,

\[ d\Theta^h + \Theta(\nabla \theta^h) = \pi \circ \mathcal{R}(\theta^h) = \sum_{h',h_1,h_2} \mathcal{R}^{h'}_{h',h_1,h_2} \theta^{h_1} \cap \theta^{h_2} \cap \theta^{h'} . \]  \hspace{1cm} (3.17)

Using \( \theta \omega = \sum_{h \in S} \theta^h \cap R^*_h \omega \) we find

\[ \Theta(\nabla \theta^h) = -\theta \Theta^h + \sum_{h',h''} V^{h'}_{h,\theta^h} \theta^{h''} \Theta^{h'} \]

\[ = -\sum_{h' \in S} \theta^{h'} \cap R^*_{h'} \Theta^h + \sum_{h',h''} V^{h'}_{h,\theta^h} \theta^{h''} \cap R^*_{h''} \Theta^{h'} \]

and thus, with the help of (2.23),

\[ d\Theta^h + \Theta(\nabla \theta^h) = -\Theta^h \cap \theta - \Delta(\Theta^h) + \sum_{h',h''} V^{h'}_{h,\theta^h} \theta^{h''} \cap (R^*_{h''} \Theta^{h'}) . \]  \hspace{1cm} (3.19)
Replacing the left hand side of (3.14) with the last expression, we obtain the first Bianchi identity in terms of the ∩-product. In case of vanishing torsion, it reduces to

$$\sum_{h',h_1,h_2 \in S} \mathcal{R}_{h',h_1,h_2}^h \theta^{h_1} \cap \theta^{h_2} \cap \theta^{h'} = 0.$$  \hspace{1cm} (3.20)

Using $V_{h'}^h := \sum_{h'' \in S} V_{h'',h'}^h \theta^{h''}$ and

$$\mathcal{R}_{h'}^h := \sum_{h_1,h_2 \in S} \mathcal{R}_{h}^{h} \text{ad}(h_2h_1)_{h'} h_1, h_2 \theta^{h_1} \cap \theta^{h_2}$$  \hspace{1cm} (3.21)

the second Bianchi identity (I.7.15) reads

$$\Delta(\mathcal{R}_{h'}^h) = \sum_{h'' \in S} (V_{h''}^h \mathcal{R}_{h'' h'} - \mathcal{R}_{h''}^h V_{h''}^h) = \sum_{h_1,h_2 \in S} V_{h_1,h_2}^h \theta^{h_1} \cap \mathcal{R}_{h_1}^e \mathcal{R}_{h_2}^h h'$$

$$- \sum_{h_1,h_2,h_3 \in S} \mathcal{R}_{\text{ad}(h_3h_2) h_1,h_2,h_3}^h \theta^{h_2} \cap \theta^{h_3} \cap \mathcal{R}_{h_3 h_2}^h V_{h_1}^h h' .$$  \hspace{1cm} (3.22)

Evaluating the left hand side with the help of lemma 2.3, this yields a three-form expression which only involves the ∩-product.

### 3.2 Integrability conditions of the metric-compatibility equation

The integrability condition for the metric-compatibility of a linear connection is $\nabla^2 g = 0$ and thus involves the curvature. After some manipulations we obtain the conditions

$$V_{h_1} R^*_h V_{h_2} = B_{h_1,h_2}$$  for a biangle $h_1 h_2 = e$  \hspace{1cm} (3.23)

$$V_{h_1} R^*_h V_{h_2} = T_{h_1,h_2} V_h$$  for a triangle $h_1 h_2 = h \in S_{(1)}$  \hspace{1cm} (3.24)

$$V_{h_1} R^*_h V_{h_2} = K_{h_1,h_2,h_1;h_2} V_{h_1}^* R^*_{h_1} V_{h_2}$$  for a quadrangle $h_1 h_2 = \hat{h}_1 \hat{h}_2 \in S_{(2)}$  \hspace{1cm} (3.25)

where for all $g \in G$ the matrices $B_{h_1,h_2}(g)$, $T_{h_1,h_2}(g)$, $K_{h_1,h_2;h_1;h_2}(g)$ are elements of the isometry group of $g(g)$. Now we obtain for biangles

$$\mathcal{R}_{h}^{h} (e) h', h_1, h_2 = \delta_{h_2 h_1}^h (B_{h_1,h_2} - I)^h h' ,$$  \hspace{1cm} (3.26)

for triangles

$$\mathcal{R}_{h_0}^{h} (h_0) h', h_1, h_2 = \delta_{h_2 h_1}^{h_0} \left( (T_{h_1,h_2} - I) V_{h_0} \right)^h h_0^{-1} h' h_0 ,$$  \hspace{1cm} (3.27)

and for quadrangles

$$\mathcal{R}_{h}^{h} (g) h', h_1, h_2; h_1, h_2 = \delta_{h_2 h_1}^h \left( (K_{h_1,h_2;h_1,h_2} - I) V_{h_1}^* R^*_{h_1} V_{h_1^{-1} h_2 h_1} \right)^h g^{-1} h' g$$  \hspace{1cm} (3.28)

where $\hat{h}_2 \hat{h}_1 = g \in S_{(2)}$. As a consequence, the essential part of the curvature tensor is given by the isometries $B, T, K$. 

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3.3 Torsion and curvature as maps on vector fields

Let \((G, S)\) be a bicovariant group lattice and \(Q^{h_{h_1,h_2}}\) the torsion tensor components introduced in (3.2) with the quadrangle part defined in (3.6). For vector fields \(X, Y\) we introduce the torsion tensor

\[
Q(X, Y) := \sum_{h \in S, h_1, h_2 \in S_e} X^{h_1} Y^{h_2} Q^{h_{h_1,h_2}} \cdot \ell_h .
\]

(3.29)

This expression obviously satisfies \(Q(f \cdot X, f' \cdot Y) = f f' Q(X, Y)\) and is therefore a (left) tensor. In the following we consider in more detail the case where \(X, Y\) are basic. The torsion tensor can then be written as

\[
Q(X, Y) = \sum_{h \in S} Q^{h_{s_X, s_Y}} \cdot \ell_h
\]

(3.30)

where the map \(s_X : G \rightarrow S\) is determined by \(X^h (g) = \delta^h_{s_X(g)}\) (see also Ref. [I]).

Below we will need the following expression for basic vector fields \(X, Y, Z\),

\[
\begin{align*}
\hat{V}_X R_{X^s} Y &= \sum_{h_1, h_2 \in S} X^{h_1} Y^{\text{ad}(h) h_2} \hat{V}_{\ell_{h_1}} \ell_{h_2} = \sum_{h_1, h_2 \in S} \delta^{h_1, s_X} \delta^{h_2, \text{ad}(s_X)^{-1} s_Y} \hat{V}_{\ell_{h_1}} \ell_{h_2} \\
&= \sum_{h \in S} V^{h}_{s_X, \text{ad}(s_X)^{-1} s_Y} \cdot \ell_h
\end{align*}
\]

(3.31)

where we used (I.5.21), (I.7.17) and (I.5.12).

If \(X, Y\) form a triangle, so that \(s_Y s_X = e\), then

\[
Q(X, Y) = \sum_{h \in S} Q^{h_{(e)s_X, s_Y}} \cdot \ell_h = \sum_{h \in S} \delta^{e}_{s_X s_Y} \left( \delta^{h}_{s_X} + V^{h}_{s_X, s_Y} \right) \cdot \ell_h = X + \sum_{h \in S} V^{h}_{s_X, s_Y} \cdot \ell_h \\
= X + \hat{V}_X R_{X^s} Y .
\]

(3.32)

If \(X, Y, Z\) form a triangle, so that \(s_Y s_X = s_Z\), we obtain

\[
Q(X, Y) = \sum_{h \in S, h_1 \in S(1)} Q^{h_{(1)s_X, s_Y}} \cdot \ell_h = \sum_{h \in S, h_1 \in S(1)} \delta^{h_1 h}_{s_X s_Y} \left( \delta^{h}_{s_X} - \delta^{h}_{s_X} + V^{h}_{s_X, \text{ad}(s_X)^{-1} s_Y} \right) \cdot \ell_h \\
= X + \hat{V}_X R_{X^s} Y - Z .
\]

(3.33)

Finally, for a quadrangle \(X, Y, \hat{X}, \hat{Y}\) (which satisfies \(s_Y s_X = s_{s_X} \notin S_e\)) we find

\[
Q(X, Y; \hat{X}, \hat{Y}) := Q(X, Y) - Q(\hat{X}, \hat{Y}) = \sum_{h \in S, g \in S(2)} Q^{h_{(g)s_X, s_Y, s_{\hat{X}}, s_{\hat{Y}}}} \cdot \ell_h \\
= \sum_{h \in S, g \in S(2)} \delta^{g}_{s_y s_x} \left( \delta^{h}_{s_X} - \delta^{h}_{s_{\hat{X}}} + V^{g}_{s_X, \text{ad}(s_X)^{-1} s_Y} - V^{h}_{s_{\hat{X}}, \text{ad}(s_{\hat{X}})^{-1} s_{\hat{Y}}} \right) \cdot \ell_h \\
= X + \hat{V}_X R_{X^s} Y - \hat{X} - \hat{V}_X R_{X^s} \hat{Y} .
\]

(3.34)

For arbitrary vector fields \(X, Y, Z\) we define the curvature tensor

\[
\mathcal{R}(X, Y)(Z) = \sum_{h \in S, h_1, h_2, h_3 \in S_e} X^{h_1} Y^{h_2} Z^{h_3} \mathcal{R}^{h_{h_1,h_2,h_3}} \cdot \ell_h
\]

(3.35)
where the ambiguity in the quadrangle components is fixed by (3.13). If \( X, Y, Z \) are basic, we obtain

\[
\mathcal{R}(X, Y)(Z) = \sum_{h \in S} \mathcal{R}^h_{sZ,sX,sY} \cdot \ell_h .
\]  

For further evaluation we need the following expressions,

\[
\tilde{V}_{R_{X,Y}Z} = \sum_{h_1} (R_{X,Y})^{h_1} \tilde{V}_{\ell_{h_1}} Z = \sum_{h_1,h_2} (R_{X,Y})^{h_1} (R_{h_1}^{*} Z^{h_2}) \tilde{V}_{\ell_{h_1}} \ell_{h_2}
\]

and

\[
\tilde{V}_X \tilde{V}_{R_{X,Y}Z} = \sum_{h_1} X^{h_1} \tilde{V}_{\ell_{h_1}} (\tilde{V}_{R_{X,Y}Z})
\]

\[
= \sum_{h_1,h_2,h_3} X^{h_1} Y^{\text{ad}(h_1)h_2} (R_{h_1}^{*} Z^{h_3}) (V_{h_1} R_{h_1}^{*} V_{h_2})^{h_3}_{h_3} \cdot \ell_h
\]

using (I.5.21) and (I.5.12). With the help of these formulas we obtain

\[
\mathcal{R}(X, Y) (Z) = \tilde{V}_X \tilde{V}_{R_{X,Y}Z} - Z \text{ for a biangle } X, Y
\]

\[
\mathcal{R}(X, Y) (Z) = (\tilde{V}_X \tilde{V}_{R_{X,Y}Z} - \tilde{V}_W) R_{W,Z} \text{ for a triangle } X, Y, W
\]

and

\[
\mathcal{R}(X, Y; \tilde{X}, \tilde{Y}) (Z) := \mathcal{R}(X, Y) (Z) - \mathcal{R}(\tilde{X}, \tilde{Y}) (Z)
\]

\[
= (\tilde{V}_X \tilde{V}_{R_{X,Y}Z} - \tilde{V}_X \tilde{V}_{R_{X,Y}Z}) R_{(R_{X,Y})*} R_{X,Z}
\]

for a quadrangle \( X, Y, \tilde{X}, \tilde{Y} \).

The Ricci tensor defined in (3.14) can also be expressed as follows,

\[
Ric(X, Y) := \sum_{h \in S} \langle \mathcal{R}(\ell_h, Y)(X), \theta^h \rangle .
\]

4 Riemannian group lattices admitting a torsion-free compatible linear connection

Let \( (G, S) \) be a bicovariant group lattice and \( (\Omega, d) \) the associated differential calculus. The formalism developed in the previous sections enables us to carry familiar constructions of continuum differential geometry over to the discrete differential geometric framework of group lattices. In particular, we may look for an analog of the Levi-Civita connection of a metric \( g \). This means we should look for torsion-free linear connections which are compatible with \( g \).
In section 2.5 a (bicovariant) group lattice supplied with a metric tensor $g$ of constant signature has been called a “Riemannian group lattice”. In this section we further demand that it admits a torsion-free metric-compatible linear connection. Unlike the continuum case, on most group lattices not every metric admits such a connection. As we shall see below, this condition indeed places severe restrictions on the components of a metric. This should not come as a big surprise. In continuum differential geometry the requirement of a smooth metric on a smooth manifold guarantees that the metric components at “neighboring” points fit together. On the other hand, given a set of points in a Euclidean space, for example, and prescribing metric components at every point, a corresponding embedded digraph does not exist, in general. This is not the whole story, however. In the case of a maximal group lattice (complete digraph), which corresponds to a maximal set $S$, vanishing torsion already determines a unique linear connection, so that no freedom is left to satisfy the metric-compatibility conditions for a “non-trivial” geometry (see subsection 4.1). Reducing $S$ to smaller sets allows for more freedom in a torsion-free connection and thus for more solutions of the metric-compatibility conditions.

If a metric-compatible linear connection is found for a given metric, it is only determined up to transformations $V_h \mapsto J_h V_h$ of the transport matrices with isometry matrices $J_h$ (see section 2.3) with coefficients $J^{h'}_{h,h''}$. Requiring vanishing torsion restricts this freedom, but in general does not fix it completely. In the following we elaborate this in more detail. More generally, we look separately at the consequences of vanishing biangle, triangle and quadrangle torsion together with the metric-compatibility condition. In the following, the matrices $J_h$ are always constrained by the isometry condition (2.40).

a) Vanishing biangle torsion. The biangle torsion vanishes for a biangle $h_1h_2 = e$ (at some lattice site) if and only if

$$V^{h}_{h_1,h_2} = -\delta^{h}_{h_1} \quad \forall h \in S$$

which is $V_{h_1,h_2} = -\ell_{h_1}$. Together with the metric-compatibility condition (2.39), this leads to

$$R^{*}_{h_1} g_{h_2,h} = -\sum_{h' \in S} g_{h_1,h'} V^{h'}_{h_1,h} \quad \forall h \in S$$

and in particular

$$R^{*}_{h_1} g_{h_2,h_1} = g_{h_1,h_1} \quad (h_1h_2 = e).$$

It is natural to assign to $g_{h_1,h_1}$ the interpretation of the square of the distance from $g$ to $gh_1$. Then the last formula tells us that this distance is equal to the reverse distance, i.e. that from $gh_1$ to $g$.

Remark. For making contact with ordinary discrete geometry, this suggests to demand vanishing biangle torsion. It should be noticed, however, that (4.3) does not necessarily require vanishing biangle torsion (see sections 5.2 and 5.3). Furthermore, in a communication network it is natural to allow the possibility of assigning different lengths (routing distances) to a direction and its inverse.
As a consequence of (4.1), only transformations of $V_{h_1}$ are allowed with an isometry matrix $J_{h_1}$ subject to

$$J^h_{h_1,h_2} = \delta^h_{h_1} \quad \forall h \in S. \quad (4.4)$$

This means $J_{h_1}\ell_{h_1} = \ell_{h_1}$, which restricts the freedom to isometries leaving the vector $V_{h_1,h_2} = -\ell_{h_1}$ invariant. These are rotations (including reflections) about $V_{h_1,h_2}$.

b) **Vanishing triangle torsion.** The vanishing of the triangle torsion for a triangle $h_1h_2 = h_0$ (at some lattice site) amounts to

$$V^h_{h_1,h_2} = \delta^h_{h_0} - \delta^h_{h_1} \quad \forall h \in S \quad (4.5)$$

which is $V_{h_1,h_2} = \ell_{h_0} - \ell_{h_1}$. Together with (2.33) this implies

$$R^h_{h_1} g_{h_2,h'} = \sum_{h' \in S} (g_{h_0,h'} - g_{h_2,h'}) V^h_{h_1,h} \quad \forall h \in S \quad (4.6)$$

and in particular

$$R^h_{h_1} g_{h_2,h_2} = g_{h_1,h_1} + g_{h_0,h_0} - 2 g_{h_0,h_1} \quad (h_1h_2 = h_0). \quad (4.7)$$

Using the standard interpretation of the metric components, this is a well-known law of Euclidean geometry, the cosine law of triangles. Hence, the requirement of a metric-compatible and triangle-torsion-free linear connection restricts the metric in such a way that triangles are always flat. If triangle torsion is admitted, however, then it is possible to curve a triangle in such a way, for example, that the parallel transport is that of a spherical triangle, see section 5.1. (4.3) restricts the freedom of isometries in the transport matrices by

$$J^h_{h_1,h_0} - J^h_{h_1,h_1} = \delta^h_{h_0} - \delta^h_{h_1} \quad \forall h \in S \quad (4.8)$$

which is $J_{h_1}(\ell_{h_0} - \ell_{h_1}) = \ell_{h_0} - \ell_{h_1}$. Hence $J_{h_1}$ corresponds to a “rotation” which leaves the vector $V_{h_1,h_2} = \ell_{h_0} - \ell_{h_1}$ fixed.

c) **Vanishing quadrangle torsion.** The vanishing of the quadrangle torsion associated with a quadrangle $h_1h_2 = \hat{h}_1\hat{h}_2 = g \not\in S$, (at some lattice site) means

$$V^h_{h_1,h_2} + \delta^h_{h_1} = V^h_{h_1,\hat{h}_2} + \delta^h_{h_1} \quad \forall h \in S \quad (4.9)$$

and thus $V_{h_1,h_2} - V_{\hat{h}_1,\hat{h}_2} = \ell_{\hat{h}_1} - \ell_{h_1}$. Together with the metric-compatibility condition this imposes restrictions on the metric components. In particular, for a positive definite metric the triangle inequalities lead to

$$\left|\|V_{h_1,h_2}\| - \|V_{h_1,\hat{h}_2}\|\right| \leq \|\ell_{h_1} - \ell_{\hat{h}_1}\| \leq \|V_{h_1,h_2}\| + \|V_{h_1,\hat{h}_2}\| \quad (4.10)$$

where $\|V_{h_1,h_2}\| = \sqrt{g(V_{h_1,h_2},V_{h_1,h_2})}$. Using (2.33), this restricts the metric components as follows,

$$\left|\sqrt{R^*_{h_1} g_{h_2,h_2}} - \sqrt{R^*_{h_1} g_{h_2,h_2}}\right| \leq \sqrt{\ell_{h_1} + \ell_{\hat{h}_1} - 2 \ell_{h_1}} \leq \sqrt{R^*_{h_1} g_{h_2,h_2}} + \sqrt{R^*_{h_1} g_{h_2,h_2}} \quad (4.11)$$
The isometries $J_h$ have to satisfy the equation
\[
\sum_{h' \in S} (J^h_{h_1,h'} - J^{h'}_{h_1,h}) V^h_{h_1,h_2} = J^h_{h_1,h_1} - \delta^h_{h_1} - J^{h'}_{h_1,h_1} + \delta^{h'}_{h_1} \quad \forall h \in S \tag{4.12}
\]
which is $J_{h_1} V_{h_1,h_2} - J_{h_1} V_{h_1,h_2} = \ell_{h_1} - \ell_{h_1}$. In particular, a rotation which leaves $V_{h_1,h_2}$ fixed, so that $J_{h_1} V_{h_1,h_2} = V_{h_1,h_2}$, together with a rotation which leaves $V_{h_1,h_2}$ fixed, so that $J_{h_1} V_{h_1,h_2} = V_{h_1,h_2}$, preserves the quadrangle and thus solves the above constraint. Another possibility is given by combined rotations $J_{h_1}$ and $J_{h_1}$ which leave the vector $V_{h_1,h_2} - V_{h_1,h_2}$ and thus $\ell_{h_1} - \ell_{h_1}$ fixed, so that $J_{h_1}(\ell_{h_1} - \ell_{h_1}) = \ell_{h_1} - \ell_{h_1}$ and $J_{h_1}(\ell_{h_1} - \ell_{h_1}) = \ell_{h_1} - \ell_{h_1}$.

The following subsections provide several examples of Riemannian group lattices which admit torsion-free linear connections. In the discussions we make use of the fact that a linear connection determines a tangent space picture of the group lattice, as described in section 2.6.

### 4.1 Maximal group lattices

A group lattice $(G, S)$ with $S = G \setminus \{e\}$ is called maximal. It is bicovariant and carries the universal differential calculus. In this case there are only biangles and triangles, but no quadrangles. The condition of vanishing torsion then determines a unique linear connection which is given by

\[
V^h_{h_1,h_2} = \begin{cases} 
-\delta^h_{h_1} & \text{if } h_1 h_2 = e \\
-\delta^h_{h_2} & \text{if } h_0 := h_1 h_2 \neq e
\end{cases} \tag{4.13}
\]

and thus constant. This implies $V_h V_{h^{-1}} = I$ and $V_{h_1} V_{h_2} = V_{h_0}$ if $h_1 h_2 = h_0$. As a consequence, the curvature of the connection vanishes.

The metric compatibility condition evaluated for this connection becomes

\[
R^*_h \mathfrak{g}_{h_1,h_2} = \begin{cases} 
\mathfrak{g}_{h,h} - \mathfrak{g}_{h,h_2} - \mathfrak{g}_{h h_1,h} + \mathfrak{g}_{h h_2,h_1} & \text{if } h h_1 \neq e, h h_2 \neq e \\
\mathfrak{g}_{h,h} - \mathfrak{g}_{h,h_2} & \text{if } h h_1 = e, h h_2 \neq e \\
\mathfrak{g}_{h,h} & \text{if } h_1 = h, h h_1 = e
\end{cases} \tag{4.14}
\]

**Example 4.1.** Let $G$ be $\mathbb{Z}_3$, the cyclic group consisting of the three elements $0, 1, 2$ with addition modulo 3 as the group composition. We choose the group lattice determined by $S = \{1, 2\}$ which is the complete digraph with three vertices. There are two biangles, $1 + 2 = 0 = 2 + 1$ (modulo 3), and two triangles, $1 + 1 = 2$ and $2 + 2 = 1$ (modulo 3). The unique torsion-free linear connection is determined by the two matrices

\[
V_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.15}
\]

A metric is given by

\[
\mathfrak{g} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{4.16}
\]
with functions $a, b, c$ and the compatibility condition with the above connection reduces to

$$R^*_1 \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a - 2b + c & a - b \\ a - b & a \end{pmatrix}.$$  \hfill (4.17)

This means that one can specify arbitrary values of the metric functions $a, b, c$ at one point. The metric at the other points is then determined by the last equation and the resulting metric on $\mathbb{Z}_3$ is compatible with the above torsion-free connection. Assigning the usual interpretation in terms of Euclidean distances and angles to the metric components, one recovers the rules of Euclidean trigonometry. In particular, in case of a constant metric, the compatibility condition restricts $g$ to

$$g = a \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$  \hfill (4.18)

This expresses metric properties of a regular Euclidean triangle. The parallel transport determined by the torsion-free connection coincides with that of the Euclidean plane. Indeed, from (2.41) we infer

$$\tilde{V}_{\ell_1} : \ell_1 \ell_2 \mapsto \ell_2 - \ell_1 - \ell_1,$$

$$\tilde{V}_{\ell_2} : \ell_1 \ell_2 \mapsto -\ell_2 - \ell_1 - \ell_2,$$

which maps the Riemannian group lattice isometrically onto a Euclidean triangle in the tangent space at a site.

**Example 4.2.** Let $G = \mathbb{Z}_4$ and $S = \{1, 2, 3\}$. The corresponding torsion-free linear connection is then given by

$$V_1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hfill (4.19)

Assuming the metric to be constant, the compatibility condition restricts it to the form

$$g = \begin{pmatrix} a & b & a - b \\ b & 2b & b \\ a - b & b & a \end{pmatrix}.$$  \hfill (4.20)

For $b = a/2$ we recover the geometry of a regular tetrahedron in Euclidean $\mathbb{R}^3$. Since we deal with a three-dimensional Riemannian group lattice, we are actually describing the tetrahedron volume. Furthermore, in the limit $b \to a$ the above geometry tends to that of a quadrate in the Euclidean plane where the vector associated with $2 \in S$ corresponds to the diagonal. Accordingly, in this limit the determinant of $g$ vanishes, so that $g$ no longer defines a metric according to our definition in section 2.3. \hfill ■
4.2 Two-dimensional Riemannian group lattices

Let $G$ be a discrete group, $S = \{a,b\}$ a subset consisting of two different elements of $G$ which generate $G$ such that $(G, S)$ is a bicovariant group lattice. Then either $aba^{-1} = a$, which contradicts $a \neq b$, or $aba^{-1} = b$ which is $ab = ba$. Hence, bicovariance requires that $G$ is Abelian. By a fundamental theorem, every finite Abelian group is isomorphic to a direct product of cyclic groups of prime power order.

The following examples in particular demonstrate that, for a given metric on a group lattice, there may not exist a metric-compatible linear connection with vanishing torsion, i.e. a Levi-Civita connection. Moreover, in contrast to ordinary continuum differential geometry, if such a connection exists, then it is not unique.

4.2.1 $\mathbb{Z}_4$ lattices

a) Let $G = \mathbb{Z}_4$ and $S = \{1,2\}$. There is one biangle, $2 + 2 = 0$ (modulo 4), one triangle, $1 + 1 = 2$, and one quadrangle, $1 + 2 = 3 = 2 + 1$, which implies the 2-form relation $\theta_1 \cap \theta_2 = -\theta_2 \cap \theta_1$. Vanishing torsion restricts the matrices $V_i$ of the linear connection to

$$V_1 = \begin{pmatrix} -1 & p \\ 1 & 1+q \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1+p & 0 \\ q & -1 \end{pmatrix} \quad (4.21)$$

with arbitrary functions $p$ and $q$. For a metric of the form (4.16), the metric-compatibility condition $R^*_1 g = V_1^T g V_1$ reads

$$R^*_1 a = a - 2b + c$$
$$R^*_1 b = p (b - a) + (1 + q)(c - b)$$
$$R^*_1 c = p^2 a + 2p (1 + q) b + (1 + q)^2 c. \quad (4.22)$$

With the help of $R^*_2 = (R^*_1)^2$, the second condition $R^*_2 g = V_2^T g V_2$ leads to

$$V_1 (R^*_1 V_1) V_2^{-1} = J \quad (4.23)$$

where $J$ is an arbitrary element of the isometry group of the metric (at each site of the group lattice). A lengthy computation, aided by computer algebra, reveals that every Levi-Civita connection on $(\mathbb{Z}_4, \{1,2\})$ is flat, i.e. its curvature vanishes. [13] The integrability condition (3.27) then enforces $J = I$ so that $V_2 = V_1 R^*_1 V_1$. As a consequence, we obtain the following representation of $\mathbb{Z}_4$:

$$R^*_1 p = -\frac{p}{1+q}, \quad R^*_1 q = -\frac{2+p+q}{1+q}. \quad (4.24)$$

This implies $R^*_2 p = -p/(1+p)$ and $R^*_2 q = q/(1+p)$ and thus $(R^*_1)^4 p = (R^*_2)^2 p = p$, $(R^*_1)^4 q = (R^*_2)^2 q = q$.

Excluding special values of $q(0)$ and $p(0)$, the geometries with a Levi-Civita connection are given by

$$a(1) = a(0) - 2b(0) + c(0)$$
$$b(1) = -p(0) a(0) + [p(0) - 1 - q(0)] b(0) + [1 + q(0)] c(0)$$
\begin{align*}
c(1) &= p(0)^2 a(0) + 2 p(0) [1 + q(0)] b(0) + [1 + q(0)]^2 c(0) \\
a(2) &= [1 + p(0)]^2 a(0) + 2 q(0) [1 + p(0)] b(0) + q(0)^2 c(0) \\
b(2) &= -[1 + p(0)] b(0) - q(0) c(0) \\
c(2) &= c(0) \\
a(3) &= [1 + p(0)]^2 a(0) + 2 [1 + p(0)] [1 + q(0)] b(0) + [1 + q(0)]^2 c(0) \\
b(3) &= p(0) [1 + p(0)] a(0) + [1 + 2 p(0)] [1 + q(0)] b(0) + [1 + q(0)]^2 c(0) \\
c(3) &= p(0)^2 a(0) + 2 p(0) [1 + q(0)] b(0) + [1 + q(0)]^2 c(0)
\end{align*}

(4.25)

and

\begin{align*}
q(1) &= -[2 + p(0) + q(0)]/[1 + p(0) + q(0)], \quad p(1) = -p(0)/[1 + p(0) + q(0)] \\
q(2) &= q(0)/[1 + p(0)], \quad p(2) = -p(0)/[1 + p(0)] \\
q(3) &= -[2 + p(0) + q(0)]/[1 + q(0)], \quad p(3) = p(0)/[1 + q(0)].
\end{align*}

(4.26)

b) Let \( G = \mathbb{Z}_4 \) again, but now we choose \( S = \{1, 3\} \). In this case, there are two biangles, \( 1 + 3 = 0 = 3 + 1 \) (modulo 4), no triangle and a quadrangle corresponding to \( 1 + 1 = 2 = 3 + 3 \) (modulo 4). The latter leads to the 2-form relation \( \theta^1 \cap \theta^1 + \theta^3 \cap \theta^3 = 0 \). The condition of vanishing torsion imposes the following restrictions on a linear connection,

\[
V_1 = \begin{pmatrix} u & -1 \\ 1 + v & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 1 + u \\ -1 & v \end{pmatrix}
\]

(4.27)

with arbitrary functions \( u \) and \( v \). For a metric of the form (4.16) the compatibility condition \( R_1^* g = V_1^T g V_1 \) reads

\[
R_1^* a = u^2 a + 2 u (1 + v) b + (1 + v)^2 c, \quad R_1^* b = -u a - (1 + v) b, \quad R_1^* c = a
\]

(4.28)

and, with the help of \( R_3^* = (R_1^*)^3 \), the second metric-compatibility condition \( R_3^* g = V_3^T g V_3 \) leads to

\[
V_1 (R_1^* V_1) [(R_1^*)^2 V_1] V_3^{-1} = J
\]

(4.29)

where \( J \) is an element of the isometry group of the metric. Further exploration with the help of computer algebra shows that every Levi-Civita connection on \( (\mathbb{Z}_4, \{1, 3\}) \) has vanishing curvature.

Since the only metric-compatible torsion-free linear connections on the above \( \mathbb{Z}_4 \) lattices have vanishing curvature, via backward parallel transport they are isometrically mapped to a closed lattice in \( \mathbb{R}^2 \) which represents the tangent space at a site. In particular, this means that we cannot model something like a tetrahedron surface in this way. To supply the \( \mathbb{Z}_4 \) group lattices with non-vanishing curvature is only possible if the condition of vanishing torsion is dropped (see sections 5.2 and 5.3).

### 4.2.2 \( \mathbb{Z}^2 \) lattices

Let us consider the group lattice \( (\mathbb{Z}^2, \{\hat{1}, \hat{2}\}) \) where \( \hat{1} := (1, 0) \) and \( \hat{2} := (0, 1) \). It has no biangles or triangles, but a quadrangle corresponding to \( \hat{1} + \hat{2} = \hat{2} + \hat{1} \). The condition of
vanishing torsion restricts the parallel transport matrices $V_i := V_i^\hat{1}$ to

$$V_1 = \begin{pmatrix} p & u \\ q & 1+v \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1+u & r \\ v & s \end{pmatrix}$$  \hspace{1cm} (4.30)

with arbitrary functions $p, q, r, s, u, v$. The following example demonstrates that there are torsion-free and metric-compatible parallel transports with non-vanishing curvature even on a two-dimensional lattice carrying the metric properties of a regular quadratic lattice in Euclidean $\mathbb{R}^2$.

**Example 4.3.** Let us choose the metric to be

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (4.31)

at all sites. The metric-compatibility condition for the above torsion-free linear connection then leads to the following two classes of solutions. The first class is given by

$$V_1 = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$  \hspace{1cm} (4.32)

with functions $\epsilon_i$ with values in $\{\pm 1\}$. The curvature only vanishes if $\epsilon_1$ and $\epsilon_2$ are constant in the $\hat{1}$ and $\hat{2}$ direction, respectively.

If the curvature vanishes, the (backward) parallel transport does not depend on the path in the lattice, see (2.43). It can thus be used to map the whole group lattice into the tangent space at one point, which is isomorphic to $\mathbb{R}^2$ in the case under consideration. Let us choose the lattice point $(0, 0)$. The tangent vectors $\ell_h$ at this site may then be identified with the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ pointing from $(0, 0)$ to $(1, 0)$ and $(0, 1)$, respectively, in $\mathbb{R}^2$. Then $\ell_1$ at the group lattice site $(1, 0)$ is mapped to the vector $V_{1,1}$ which we attach at the tip of $\mathbf{u}_1$ in $\mathbb{R}^2$ according to the prescription of section 2.6. If $\epsilon_1 = -1$ this vector points into the “wrong direction”, i.e. its tip coincides with $(0, 0)$. This means that the resulting lattice in $\mathbb{R}^2$ gets folded. Similarly, if $\epsilon_2 = -1$ the lattice gets folded in the other direction.

A particular solution is given by $V_h = I$, the unit matrix, at all sites. It corresponds to the ordinary Euclidean parallel transport. This solution certainly has a nice continuum limit. Introducing a lattice spacing parameter, we may write $V_h = I + \kappa \Gamma_h + O(\kappa^2)$. Some of the other solutions $V_h$ given above have negative determinant at some sites and cause folding in the sense described above. They are related to the above solution at those sites by an isometry $J_h$ with determinant $-1$. As a consequence, they cannot have a continuum limit. The requirement of a continuum limit may thus distinguish a certain connection and eliminate connections with folding.

The second class of solutions is given by

$$V_1 = \begin{pmatrix} 0 & -1 \\ \epsilon_1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \epsilon_2 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.33)

The curvature only vanishes if at all sites $R^*_2 \epsilon_1 = \epsilon_2$ and $R^*_3 \epsilon_2 = \epsilon_1$. An orientation-preserving connection is obtained if $\epsilon_1 = \epsilon_2 = 1$. The corresponding transports in the two directions act with rotations. 

\[\blacksquare\]
Figure 1: Levi-Civita connections on a $\mathbb{Z}^2$ group lattice exist if and only if at each lattice site the circle with radius $|V_{12}|$ around the tip of $u_1$ intersects the circle with radius $|V_{21}|$ around the tip of $u_2$.

There are metrics (with constant signature) on $(\mathbb{Z}^2, \{\hat{1}, \hat{2}\})$ which do not admit a Levi-Civita connection, although the constraints are by far not as stringent as in our previous examples. Counterexamples are easily constructed. A geometric condition for the existence of a Levi-Civita connection is given by (4.11) in the case of a positive definite metric. Let us recall its origin in the case under consideration. The tangent space at a site $a$ is isomorphic to $\mathbb{R}^2$ with the Euclidean inner product of vectors (see section 2.6). The tangent vectors $\ell_i$ are then represented by vectors $u_i \in \mathbb{R}^2$, $i = 1, 2$, such that $u_i \cdot u_j = g_{ij}(a)$ where $g_{ij} := g_{i,j}$. The parallel transport $\tilde{V}_{\ell_i}$ maps the tangent space at the site $a + \hat{i}$ into the tangent space at $a$. Metric-compatibility of the connection means

$$V_{ij} \cdot V_{ik} = g_{jk}(a + \hat{i}) \quad (4.34)$$

where $V_{ij}$ represents $\tilde{V}_{\ell_i} \ell_j$ at the site $a$. If the connection is (quadrangle) torsion free, then adjacent quadrangles are preserved by the backward parallel transport, so that $u_i + V_{ij} = u_j + V_{ji}$. The last equation has solutions if and only if $||V_{12}| - |V_{21}|| \leq |u_2 - u_1| \leq |V_{12}| + |V_{21}|$ where $|V_{12}|$ denotes the Euclidean norm of $V_{12}$ in $\mathbb{R}^2$. This is illustrated in Fig. 1. Using (4.34), this condition is expressed as

$$\left| \sqrt{g_{22}(a + \hat{1})} - \sqrt{g_{11}(a + \hat{2})} \right| \leq \sqrt{g_{11}(a) + g_{22}(a) - 2g_{12}(a)}$$

$$\leq \sqrt{g_{22}(a + \hat{1}) + g_{11}(a + \hat{2})} \quad (4.35)$$

in terms of the metric at the sites $a$, $a + \hat{1}$ and $a + \hat{2}$ (see also (4.11)). If this condition is not fulfilled, a Levi-Civita connection does not exist. If the condition is satisfied, a Levi-Civita
Figure 2: The vectors $V_{ij}$ and $V'_{ij}$ result from torsion-free metric-compatible (backward) parallel transports which differ by a reflection about some axis.

connection exists, but it is not unique. Even if equality holds in the last part of (4.35), so that the circles in Fig. 1 have exactly one point in common, we still have the freedom to choose $V_{11}$ and $V_{22}$ in two possible ways, as illustrated in Fig. 2.

The freedom in the parallel transport left by the conditions of metric-compatibility and vanishing torsion is a freedom of reflections about some axes. In Fig. 2 it shows up as reflections about the three axes $AC$, $BC$ and $AB$. In section 4.3 we show that reflections about $AB$ and $BC$ and their composition comprise the whole freedom left for $V_1$ and $V_2$ by the conditions of vanishing torsion and metric-compatibility. Such reflections lead to folding of the tangent space lattice obtained by backward parallel transport of the group lattice to the tangent space at $a$. Moreover, the orientation of some of the frames at $a$ obtained by backward parallel transport of frames of basic tangent vectors at $a + \hat{i}$, $i = 1, 2$, gets changed. This can be excluded by demanding that $\det V_1 > 0$. But we should also require that the dyad $(V_{21}, V_{12})$ has positive orientation, which is necessary in order to avoid reflections about the axis $AB$. This amounts to $V_{112} + V_{212} > 0$. In higher dimensions, the determination of the ambiguities in the Levi-Civita connections and their reduction appears to be a difficult task (see also section 4.3).

Whereas torsion is a first order quantity, curvature is of second order since it expresses features of the geometry determined by the composition of two (backward) parallel transports. In the case under consideration, the components of the curvature tensor are given in matrix form by $R_{ij} := V_i R_i^j V_j - V_j R_j^i V_i$. In Fig. 3 the vector $z$ represents $\tilde{\nabla}_i \tilde{\nabla}_j Z$ where $Z = \sum_i Z^i \cdot \ell_i$. Hence

$$z = \sum_{i,j} u_i [V_1(a)V_2(a + \hat{1})]^i_j Z^j(a + \hat{1} + \hat{2}). \quad (4.36)$$

The vector $z'$ represents $\tilde{\nabla}_j \tilde{\nabla}_i Z$, so that

$$z' = \sum_{i,j} u_i [V_2(a)V_1(a + \hat{2})]^i_j Z^j(a + \hat{2} + \hat{1}). \quad (4.37)$$

The difference gives a measure of the curvature at $a$:

$$z - z' = \sum_{i,j} u_i R_{ij}^i(a) Z^j(a + \hat{1} + \hat{2}) \quad (4.38)$$
Figure 3: The familiar effect of curvature: backward parallel transport along different paths results in different vectors in the tangent space at a point.

(see also (2.45)).

If the torsion vanishes at a site \( a \), one can draw an isometric picture of the geometry in the tangent space at \( a \) to first order. This represents the site \( a \), its first order neighbors \( a + i \), and the basic tangent vectors at these sites while preserving the metric properties at all these sites and preserving biangles, triangles and quadrangles at \( a \). If moreover the curvature vanishes at \( a \), then we can draw an isometric picture to second order.

4.3 The freedom in the choice of a Levi-Civita connection on a hypercubic \( \mathbb{Z}^n \) lattice

We already learned that, in general, there is no Levi-Civita connection for a given metric on a group lattice. If such a connection exists, it need not be unique. The corresponding freedom will be explored in this section for the case of hypercubic \( \mathbb{Z}^n \) lattices given by \( G = \mathbb{Z}^n \) and \( S = \{i|i = 1, \ldots, n\} \), where \( i := (0, \ldots, 1, \ldots, 0) \) with the 1 in the \( i \)th position. We consider only positive definite metrics and choose the standard inner product \( (u, v) = u \cdot v \) for \( u, v \in \mathbb{R}^n \) (cf. section 2.6).

In the case of a hypercubic group lattice the condition of vanishing torsion can be expressed as

\[ u_i + V_{ij} = u_j + V_{ji}. \]  

(4.39)

Together with the metric-compatibility, this determines a Levi-Civita connection up to isometries \( J_i \) which preserve the above conditions, i.e.

\[ u_i + J_i(V_{ij}) = u_j + J_j(V_{ji}) \]  

(4.40)

(see also (2.48)). Subtracting (4.39) from (4.40), we find

\[ A_{ij} = A_{ji} \quad \text{where} \quad A_{ij} := J_i(V_{ij}) - V_{ij}. \]  

(4.41)
Using the isometry condition (2.49) and the last equation, we obtain
\[ \mathbf{V}_{ij} \cdot \mathbf{V}_{ij} = J_i(\mathbf{V}_{ij}) \cdot J_i(\mathbf{V}_{ij}) = \mathbf{V}_{ij} \cdot \mathbf{V}_{ij} + \mathbf{A}_{ij} \cdot (\mathbf{V}_{ij} + \mathbf{V}_{ij}) \] (4.42)
so that
\[ \mathbf{A}_{ij} \cdot (\mathbf{A}_{ij} + 2 \mathbf{V}_{ij}) = 0 \] (4.43)
and because of (4.41) also
\[ \mathbf{A}_{ij} \cdot (\mathbf{A}_{ij} + 2 \mathbf{V}_{ji}) = 0 \] (4.44).
Subtracting the last two equations and using (4.39) leads to
\[ \mathbf{A}_{ij} \cdot (\mathbf{u}_j - \mathbf{u}_i) = 0 \] (4.45).
For \( i \neq j \) and if \( \mathbf{A}_{ij} \neq 0 \), we set \( \mathbf{A}_{ij} = \alpha_{ij} \mathbf{a}_{ij} \) with a unit vector \( \mathbf{a}_{ij} \) orthogonal to \( \mathbf{u}_j - \mathbf{u}_i \). From (4.43) we then obtain \( \alpha_{ij} = -2 \mathbf{a}_{ij} \cdot \mathbf{V}_{ij} = 0 \), so that
\[ \mathbf{A}_{ij} = -2 (\mathbf{a}_{ij} \cdot \mathbf{V}_{ij}) \mathbf{a}_{ij} \] (4.46)
and thus
\[ J_i(\mathbf{V}_{ij}) = \mathbf{V}_{ij} - 2 (\mathbf{a}_{ij} \cdot \mathbf{V}_{ij}) \mathbf{a}_{ij}, \quad J_j(\mathbf{V}_{ji}) = \mathbf{V}_{ji} - 2 (\mathbf{a}_{ij} \cdot \mathbf{V}_{ji}) \mathbf{a}_{ij}. \] (4.47)
As a consequence, the effect of \( J_i \) on \( \mathbf{V}_{ij} \) is that of a reflection with respect to the hyperplane orthogonal to \( \mathbf{a}_{ij} \) (which in turn is orthogonal to \( \mathbf{u}_j - \mathbf{u}_i \)). If \( \mathbf{A}_{ij} \neq 0 \) for all \( j \neq i \), then \( J_i \) for a fixed \( i \) reflects all the \( n - 1 \) vectors \( \mathbf{V}_{ij}, j \neq i \), with respect to the respective hyperplane (orthogonal to \( \mathbf{a}_{ij} \)). Of course, we still have to respect the remaining conditions which arise from the isometry conditions (2.49), i.e. \( J_i(\mathbf{V}_{ik}) \cdot J_i(\mathbf{V}_{il}) = \mathbf{V}_{ik} \cdot \mathbf{V}_{il} \).

Let us look at the two-dimensional case. If \( \mathbf{A}_{12} = 0 \), we have \( J_1(\mathbf{V}_{12}) = \mathbf{V}_{12} \) and \( J_2(\mathbf{V}_{21}) = \mathbf{V}_{21} \), so \( J_1 \) and \( J_2 \) are reduced to reflections about \( \mathbf{V}_{12} \) and \( \mathbf{V}_{21} \), respectively. If \( \mathbf{A}_{12} \neq 0 \), then we have \( J_1(\mathbf{V}_{12}) = \mathbf{V}_{12} - 2 (\mathbf{a}_{12} \cdot \mathbf{V}_{12}) \mathbf{a}_{12}, \ J_2(\mathbf{V}_{21}) = \mathbf{V}_{21} - 2 (\mathbf{a}_{12} \cdot \mathbf{V}_{21}) \mathbf{a}_{12} \). The effect of both is a reflection about the axis along \( \mathbf{u}_2 - \mathbf{u}_1 \). If \( H_{12} \) is such a reflection, then \( H_{12} J_1(\mathbf{V}_{12}) = \mathbf{V}_{12} \) and \( H_{12} J_2(\mathbf{V}_{21}) = \mathbf{V}_{21} \) which reduces the problem to the case \( \mathbf{A}_{12} = 0 \) for \( H_{12} J_i \).

Already in three dimensions (see Fig. 4) a classification of the various possibilities turns out to be quite involved.

## 5 Group lattice geometries with torsion

Section 4 demonstrated that Riemannian group lattices in general do not possess a Levi-Civita connection. In some cases only flat Levi-Civita connections exist so that one has to allow for non-vanishing torsion in order to get non-vanishing curvature and thus enough flexibility to assign a non-trivial geometry to the group lattice.

Relaxing the previous requirement of vanishing torsion clearly opens more possibilities for modelling discrete surfaces. In fact, the following examples demonstrate that linear connections with torsion naturally appear as properties of Riemannian group lattice geometries. The first subsection below shows how in the presence of torsion a triangle can be curved so that it fits on the surface of a sphere. The remaining subsections treat some \( \mathbb{Z}_4 \) lattice examples.
Figure 4: Tangent space picture of the nearest neighborhood for the cubic $\mathbb{Z}^3$ lattice, as determined by a Levi-Civita connection. The freedom in the choice of such a connection is due to reflections with respect to hyperplanes through $u_i - u_j$, $i \neq j$.

5.1 A $\mathbb{Z}_3$ lattice geometry with torsion

Let $G = \mathbb{Z}_3$ and $S = \{1, 2\}$. According to section 3 the components of the torsion tensor are given by

$$Q^1_{1,1} = 1 + V^1_{1,1}, \quad Q^1_{1,2} = 1 + V^1_{1,2}, \quad Q^2_{1,1} = V^1_{2,1}, \quad Q^2_{1,2} = -1 + V^1_{2,2}$$
$$Q^2_{1,1} = -1 + V^2_{1,1}, \quad Q^2_{1,2} = V^2_{1,2}, \quad Q^2_{2,1} = 1 + V^2_{2,1}, \quad Q^2_{2,2} = 1 + V^2_{2,2}. \quad (5.1)$$

If we do not require the vanishing of the whole torsion, but only of the biangle part, i.e. $Q^h_{(0)1,2} = Q^h_{(0)2,1} = 0$, then we can simulate the geometry of a spherical triangle. Setting

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5.2)$$

a particular solution of the metric-compatibility conditions is

$$V_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5.3)$$

Now (2.41) leads to

$$V_{\ell_1}^t : \ell_1 \rightarrow \ell_2, \quad \ell_2 \rightarrow -\ell_1 \quad \text{at } k + 1 \mod 3 \quad \text{at } k \mod 3$$

$$V_{\ell_2} : \ell_1 \rightarrow -\ell_2, \quad \ell_2 \rightarrow \ell_1 \quad \text{at } k + 2 \mod 3 \quad \text{at } k \mod 3$$

which matches the parallel transport along a spherical triangle.

The curvature tensor has only triangle components. Using the matrix notation $R_{h_1,h_2} = (R^h_{h_1,h_2})$, we obtain

$$R_{(2)1,1} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad R_{(1)2,2} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad (5.4)$$
Figure 5: The result of backward parallel transport of the group lattice \((\mathbb{Z}_3, \{1, 2\})\) into the tangent space at 0, using the connection given by (5.3). The points 1’ and 1’’, and also 2’ and 2’’, do not coincide because of non-vanishing torsion.

and \(R_{(0)1,2} = R_{(0)2,1} = 0\) (vanishing biangle curvature). The Ricci tensor \(Ric_{h, h'} = R^1_{h, 1, h'} + R^2_{h, 2, h'}\) is given by

\[
Ric = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\] (5.5)

in matrix notation, and the curvature scalar is \(R = -2\). The torsion 2-form is given by

\[
\Theta^1 = \theta^1 \cap \theta^1, \quad \Theta^2 = \theta^2 \cap \theta^2.
\] (5.6)

This is an example of a geometry which cannot be isometrically embedded in a Euclidean space \(\mathbb{R}^n\) for any \(n \in \mathbb{N}\), simply due to the fact that with the choice of metric (7.2) the sum of the angles of the triangle is \(3\pi/2\) and not \(\pi\) as in Euclidean geometry. This fact is taken care of by the torsion of the connection which causes the backward parallel transport of the group lattice triangle not to close to a triangle in the tangent space at a site. The resulting picture in \(\mathbb{R}^2\), which represents the tangent space at the unit element, is drawn in Fig. 5.

Here we used \(V_{11} = -V_{21} = u_2, V_{12} = -V_{22} = -u_1\) which follows from (5.3). The triangle torsion satisfies

\[
\sum_i u_i Q^1_{i,1} = u_1 + V_{11} - u_2 = u_1, \quad \sum_i u_i Q^1_{i,2} = u_2 + V_{22} - u_1 = u_2.
\] (5.7)

From (2.47) we obtain

\[
V_{111} = V_{221} = -u_1, \quad V_{112} = V_{222} = -u_2
\] (5.8)
and, using (2.45) and (3.11), the following curvature expressions:

\[
\begin{align*}
\sum_i u_i R^{(2)}_{1,1,1} &= -u_1 + u_2, \\
\sum_i u_i R^{(2)}_{2,1,1} &= -u_2 - u_1 \\
\sum_i u_i R^{(1)}_{1,2,2} &= -u_1 - u_2, \\
\sum_i u_i R^{(1)}_{2,2,2} &= -u_2 + u_1.
\end{align*}
\] (5.9)

5.2 The group lattice \((\mathbb{Z}_4, \{1, 2\})\)

Let \(G = \mathbb{Z}_4\) and \(S = \{1, 2\}\). The torsion of a linear connection has the following components:

\[
\begin{align*}
Q^{h}_{2,2} &= V^{h}_{2,2} + \delta^{h}_2 \\
Q^{h}_{1,1} &= V^{h}_{1,1} - \delta^{h}_2 + \delta^{h}_1 & \text{for the biangle } 2+2=0 \\
Q^{h}_{2,1} &= -Q^{h}_{1,2} = Q^{h}_{1,2;1} = V^{h}_{1,2} - V^{h}_{2,1} - \delta^{h}_2 + \delta^{h}_1 & \text{for the quadrangle } 1+2=2+1=3.
\end{align*}
\] (5.10)

a) If we require vanishing biangle and triangle torsion, but non-vanishing quadrangle torsion, the coefficient matrices of the parallel transport have the form

\[
V_1 = \begin{pmatrix} -1 & p \\ 1 & 1+q \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1+r & 0 \\ s & -1 \end{pmatrix}
\] (5.11)

with functions \(p, q, r, s\). As an example, choosing the constant metric

\[
g(k) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad k = 0, 1, 2, 3
\] (5.12)

(which is the metric of a regular tetrahedron surface immersed in three-dimensional Euclidean space), and assuming also constant transport matrices, the compatibility conditions with the connection given by (5.11) take the form

\[
q = p, \quad p(p+1) = 0, \quad s = -1 - r/2, \quad r(r+2) = 0
\] (5.13)

so that there are four different connections which are compatible with the metric. All solutions have vanishing biangle curvature. The solution with \(p = -1, r = 0\) has non-vanishing triangle and quadrangle curvature. The solutions with \((p, r) = (0, 0), (p, r) = (-1, -2)\) and \((p, r) = (0, -2)\) possess only non-vanishing triangle curvature. In the latter case \((p = 0, r = -2)\) we have

\[
V_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (5.14)

The only non-vanishing part of the curvature 2-form is the triangle part

\[
R^{(2)}_{1,1,1} = V_1 V_1 - V_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\] (5.15)

The development of this group lattice in the tangent space at 0 is shown in Fig. 6. Here we
Figure 6: The result of backward parallel transport of the group lattice \( (\mathbb{Z}_4, \{1, 2\}) \) into the tangent space at 0, using the connection given by (5.14).

used \( V_{11} = u_2 - u_1, \ V_{12} = u_2, \ V_{21} = -u_1 \) and \( V_{22} = -u_2 \) which follows from (5.14). The resulting surface does not exhibit folding. The quadrangle torsion is given by

\[
\sum_i u_i Q_{(3)1,2}^{i} = u_1 + V_{12} - u_2 - V_{21} = 2u_1. \tag{5.16}
\]

Using (2.47) we obtain \( V_{111} = u_1, \ V_{112} = u_2 \) and thus the following curvature expressions:

\[
\sum_i u_i R_{(2)1,1,1}^i = V_{111} - V_{21} = 2u_1, \quad \sum_i u_i R_{(2)2,1,1}^i = V_{112} - V_{22} = 2u_2. \tag{5.17}
\]

**Remark.** In general, the compatibility condition for a constant metric does not enforce a constant connection, i.e. constant transport matrices. Conversely, a constant connection may be compatible with non-constant metrics. As an example, all metrics of the form

\[
g(0) = g(2) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad g(1) = g(3) = \begin{pmatrix} a - 2b + c & c - b \\ c - b & c \end{pmatrix} \tag{5.18}
\]

are compatible with the connection (5.14).

b) If only non-vanishing biangle torsion is admitted, the coefficient matrices of the parallel transport take the form

\[
V_1 = \begin{pmatrix} -1 & p \\ 1 & 1 + q \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 + p & u \\ q & v \end{pmatrix} \tag{5.19}
\]

with functions \( p, q, u, v \). If these are taken to be constants, the compatibility conditions with the metric (5.12) reduce to

\[
p = q = 0 \\
u = 0, v = 1 \quad \text{or} \quad u = 1, v = -1
\]

or

\[
p = q = -1 \\
u = 1, v = -1 \quad \text{or} \quad u = -1, v = 0
\]
Figure 7: The group lattice \((\mathbb{Z}_4, \{1, 2\})\) mapped to the tangent space at 0 using a connection with non-vanishing biangle torsion, but vanishing triangle and quadrangle torsion.

which determines four different connections. The solution with \((p, q, u, v) = (0, 0, 0, 1)\) has the transport matrices

\[
V_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

for which the curvature 2-form vanishes. The corresponding tangent space picture obtained by backward parallel transport of the group lattice into the tangent space at 0 is drawn in Fig. 7. Indeed, from the figure we read off \(V_{22} = u_2, V_{11} = u_1 + V_{12} = u_2 + V_{21}, V_{11} + V_{12} = u_2 + V_{22} = 2u_2\). Furthermore, we have \(V_{11} = V_{21} = u_2 - u_1\) and \(V_{22} = V_{212} = u_2\).

The solution with \((p, q, u, v) = (-1, -1, 1, -1)\) has the properties \(V_1 V_1 = V_2, V_2 V_2 = V_1\) and \([V_1, V_2] = 0\), so that again the whole curvature 2-form vanishes. The remaining two solutions have vanishing biangle curvature, but non-vanishing triangle and quadrangle curvature.

c) If only triangle torsion is allowed, there is no connection compatible with the metric \((5.12)\).

5.3 The group lattice \((\mathbb{Z}_4, \{1, 3\})\)

Let \(G = \mathbb{Z}_4\) with \(S = \{1, 3\}\). The biangle components of the torsion are

\[
Q^{h,1,3} = \delta^{h,1} + V_{1,1,3}^h, \quad Q^{h,3,1} = \delta^{h,3} + V_{3,1}^h.
\]

and the quadrangle components (for \(g = 2\)) are

\[
Q^{h,3,3} = -Q^{h,1,1} = Q^{h,1,3,3} = \delta^{h,1} - \delta^{h,3} + V_{1,1,1}^h - V_{3,3}^h.
\]

a) Allowing only non-vanishing quadrangle torsion, the parallel transport matrices have the form

\[
V_1 = \begin{pmatrix} u & -1 \\ q & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & p \\ -1 & v \end{pmatrix}
\]
Figure 8: The result of backward parallel transport of the group lattice \((\mathbb{Z}_4, \{1, 3\})\) into the tangent space at 0, using the connection given by (5.26).

with functions \(p, q, u, v\). Choosing again the constant metric (5.12) and assuming constant transport matrices, the compatibility conditions for the above linear connection reduce to

\[
q = -1, \ u = 0 \quad \text{or} \quad q = 1, \ u = -1
\]

and

\[
p = -1, \ v = 0 \quad \text{or} \quad p = 1, \ v = -1
\]

which determines four different compatible connections. The two of them which satisfy \(pq = 1\) have vanishing biangle curvature (3.10). One of these, which is given by

\[
V_1 = V_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{5.26}
\]

also has vanishing quadrangle curvature (3.12), so that the whole curvature 2-form vanishes. The corresponding development in the tangent space at 0 is drawn in Fig. 8 using \(V_{11} = -u_3\), \(V_{13} = -u_1\), \(V_{31} = -u_3\) and \(V_{33} = -u_1\). There is no folding. The quadrangle torsion is given by \(\sum_i u_i Q^i_{(2)1,1} = u_1 + V_{11} - u_3 - V_{33} = 2(u_1 - u_3)\).

b) If we require vanishing quadrangle torsion, but allow for non-vanishing biangle torsion, the parallel transport matrices take the form

\[
V_1 = \begin{pmatrix} u & q \\ 1+v & p \end{pmatrix}, \quad V_3 = \begin{pmatrix} r & 1+u \\ s & v \end{pmatrix} \tag{5.27}
\]

with functions \(p, q, r, s, u, v\). If these are constants, the compatibility conditions with the constant metric (5.12) have the following solutions,

\[
\begin{align*}
u &= v = 0 \\
p &= 0, \ q = 1 \quad \text{or} \quad p = 1, \ q = -1 \\
r &= 0, \ s = 1 \quad \text{or} \quad r = 1, \ s = -1
\end{align*}
\]
and

\[
\begin{align*}
  u &= v = -1 \\
  p &= 1, q = -1 \quad \text{or} \quad p = -1, q = 0 \\
  r &= 1, s = -1 \quad \text{or} \quad r = -1, s = 0
\end{align*}
\]

which determine eight different connections. Among them there are three solutions with vanishing biangle curvature:

\[
(p, q, r, s, u, v) \in \{(0, 1, 0, 1, 0, 0), (1, -1, 1, -1, 0, 0), (1, 0, -1, 0, -1, -1)\}.
\]

For the first solution we obtain

\[
V_1 = V_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and for the third

\[
V_1 = V_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

For both also the quadrangle curvature and thus the whole curvature 2-form vanishes. The second solution has more complicated transport matrices:

\[
V_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The corresponding quadrangle curvature is

\[
R_{(2)3,3} = -R_{(2)1,1} = R_{(2)1,1;3,3} = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}.
\]

The no-folding conditions are \(\det V_1 < 0\) and \(\det V_3 < 0\). They select the transport matrices (5.28).

## 6 Group lattice geometry and coordinates

In order to explore discrete structures in close analogy with the continuum it should be of some interest to consider analogs of coordinates and coordinate transformations, as well as the associated properties of geometric objects. Moreover, if there is a continuum limit, as in the case of a hypercubic \(\mathbb{Z}^n\) lattice, one should recover the corresponding continuum structures.

Let \((G, S)\) be a group lattice with \(|S| = n\). Real functions \(x^\mu, \mu = 1, \ldots, n\), are said to be coordinates on \(G\) if \((x^\mu) : G \to \mathbb{R}^n\) is injective and the matrix \((\ell_h x^\mu)\) is invertible at all \(g \in G\). If coordinates do not exist globally, they can still be introduced on subsets of \(G\).

The first subsection below presents an example of a coordinate system on a \(\mathbb{Z}_4\) lattice. In particular, it demonstrates a relation between discrete structures and noncommutative differential calculi on the algebra of functions on \(\mathbb{R}^n\) which has not yet been sufficiently explored. The second subsection then treats in some detail Riemannian geometry of a hypercubic \(\mathbb{Z}^n\) lattice in terms of adapted coordinates.
6.1 Coordinates on \((\mathbb{Z}_4, \{1, 2\})\)

The two functions
\[
x = e^0 - e^1 + e^2 - e^3, \quad y = e^0 + e^1 - e^2 - e^3
\]
are coordinates on \(\mathbb{Z}_4\) with \(S = \{1, 2\}\). Since \((x(0), y(0)) = (1, 1), (x(1), y(1)) = (-1, 1), (x(2), y(2)) = (1, -1)\) and \((x(3), y(3)) = (-1, -1)\), the map \((x, y) : \mathbb{Z}_4 \to \mathbb{R}^2\) is obviously injective. Using
\[
R_1^* x = -x, \quad R_2^* x = x, \quad R_1^* y = xy, \quad R_2^* y = -y
\]
we obtain the Jacobian
\[
(\ell_h x^\mu) = \begin{pmatrix} -2x & 0 \\ (x - 1)y & -2y \end{pmatrix}
\]
which is indeed invertible at each lattice site. Every function on \(\mathbb{Z}_4\) can be expressed as a function of \(x\) and \(y\). They satisfy
\[
x^2 = y^2 = 1.
\]
The coordinates \(x, y\) then constitute a representation of \(\mathbb{Z}_4\). For the differentials we obtain the expressions
\[
dx = [\theta, x] = -2x \theta^1, \quad dy = [\theta, y] = (x - 1) y \theta^1 - 2y \theta^2
\]
and thus, using \(x^2 = 1\),
\[
\theta^1 = -\frac{1}{2x} dx, \quad \theta^2 = \frac{1}{4} (x - 1) dx - \frac{1}{2y} dy.
\]
Furthermore, using \(\theta^h f = R^*_h f \theta^h\) we obtain the following commutation relations between the coordinates \(x, y\) and their differentials:
\[
[dx, x] = -2x dx, \quad [dy, y] = -2y dy, \quad [dx, y] = [dy, x] = (x - 1) y dx.
\]
We have thus reached a formulation of the differential calculus on \((\mathbb{Z}_4, \{1, 2\})\) as a non-commutative differential calculus on \(\mathbb{R}^2\). Indeed, imposing the relations (6.7) on two real functions \(x, y\), the group lattice \((\mathbb{Z}_4, \{1, 2\})\) can be essentially recovered. The first two relations imply \(d(x^2) = 0 = d(y^2)\). As a consequence, \(x^2\) and \(y^2\) are “constants” for this differential calculus and commute with differentials. Using (6.7) this implies
\[
0 = [d(y^2), x] = [dx, y^2] = [dx, y] y + y [dx, y] = (x - 1) y (dx) y + (x - 1) y^2 dx
\]
\[
= (x - 1) y xy dx + (x - 1) y^2 dx = (x^2 - 1) y^2 dx
\]
and thus \(x^2 = 1\), assuming \(y^2 \neq 0\) and that \(\Omega^1\) is free with basis \(dx, dy\). The equations (6.7) are homogeneous in \(y\), so that they are not able to fix the value of \(y^2\). But the calculus is
obviously consistent with the constraint \( y^2 = 1 \). Passing over to the algebra \( \mathcal{A} \) of functions generated by the variables \( x, y \) modulo the relations (6.4) and setting
\[
e^0 = \frac{(1 + x)(1 + y)}{4}, \quad e^1 = \frac{(1 - x)(1 + y)}{4}, \quad e^2 = \frac{(1 + x)(1 - y)}{4}, \quad e^3 = \frac{(1 - x)(1 - y)}{4}
\] (6.9)
we find \( e^i e^j = \delta^{ij} e^i \) and \( \sum_i e^i = 1 \). These are the primitive idempotents of \( \mathcal{A} \).

Let us deduce some more consequences from the commutation relations (6.7). They are equivalent to
\[
d_{xx} = -x d_x, \quad dx y = xy dx, \quad dy y = -y dy, \quad dy x = x dy + (x - 1) y dx
\] (6.10)
so that
\[
d_x f(x, y) = f(-x, xy) dx, \quad dy f(x, y) = f(x, -y) dy + \frac{f(x, xy) - f(-x, xy)}{2x} (x - 1) y dx.
\] (6.11)
(6.12)
Introducing (left) partial derivatives of a function \( f \) via
\[
d f = \partial_x f \, dx + \partial_y f \, dy
\] (6.13)
we find
\[
dy f(x, y) - f(x, y) dy = [df, y] = (\partial_x f) [dx, y] + (\partial_y f) [dy, y]
\] (6.14)
which together with (6.7) and (6.12) leads to
\[
\partial_x f = \frac{1}{2x} \left( f(x, xy) - f(-x, xy) \right), \quad \partial_y f = \frac{1}{2y} \left( f(x, y) - f(x, -y) \right).
\] (6.15)
A similar calculation starting with \( dx f(x, y) - f(x, y) dx = [df, x] \) leads to an apparently different expression for \( \partial_x f \). It reduces to the above formula with the help of
\[
f(x, xy) = \frac{1}{2} \left( (x + 1) f(x, y) - (x - 1) f(x, -y) \right)
\] (6.16)
which holds as a consequence of \( x^2 = 1 \).

Of course, all geometric structures on \((\mathbb{Z}_4, \{1, 2\})\) can now be expressed in terms of the coordinates and their differentials.

### 6.2 Hypercubic group lattice geometry in coordinates

Let \( G \) be the additive group \( \mathbb{Z}^n \) and \( S = \{\hat{\mu} | \mu = 1, \ldots, n\} \) the standard basis of \( \mathbb{Z}^n \), i.e., \( \hat{\mu} = (0, \ldots, 0, 1, 0, \ldots, 0)^T \) with the 1 at the \( \mu \)th position. There are no biangles or triangles, but only quadrangles. The group lattice is the oriented hypercubic lattice and for \( a \in \mathbb{Z}^n \) the functions \( e^a \) form a basis over \( \mathbb{C} \) of \( \mathcal{A} \). Then \( (\ell_{\hat{\mu}} f)(a) = f(a + \hat{\mu}) - f(a) \) defines a basis \( \{\ell_{\hat{\mu}}\} \) of the space \( \mathcal{X} \) of vector fields. The dual basis of \( \Omega^1 \) is given by
\[
\theta^{\hat{\mu}} = \sum_{a \in \mathbb{Z}^n} e^a \, de^{a + \hat{\mu}}.
\] (6.17)
The functions \( x^\mu = \kappa \sum_{a \in \mathbb{Z}^n} a^\mu e^a \), \( \mu = 1, \ldots, n \), with a constant \( \kappa \), are coordinates on the space. Every function can be written as \( f(x) \) with \( x = (x^1, \ldots, x^n) \). Furthermore, we find
\[
\theta^\mu = \frac{1}{\kappa} dx^\mu \quad \mu = 1, \ldots, n .
\] (6.18)

Since \( \hat{\mu} + \hat{\nu} = \hat{\nu} + \hat{\mu} \), the 2-form relations \( dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \) hold for all pairs \( \mu, \nu = 1, \ldots, n \). As a consequence, every product of the form \( dx^\mu_1 \wedge \cdots \wedge dx^\mu_r \) is totally antisymmetric. Since the group is Abelian, \( dx^\mu_1 \cap \cdots \cap dx^\mu_r = dx^\mu_1 \cdots dx^\mu_r \). This implies that \( \alpha_1 \cap \cdots \cap \alpha_r \) is totally antisymmetric for arbitrary 1-forms \( \alpha_i \). It should be noticed, however, that \( \alpha_1 \cdots \alpha_r \) is not antisymmetric, in general.

Introducing (left) partial derivatives of a function \( f(x) \) via
\[
df = \sum_{\mu=1}^n (\partial_\mu f) \, dx^\mu \] (6.19)
we find
\[
\partial_\mu f = \frac{R^s_\mu f - f}{\kappa} \quad (R^s_\mu f)(x) = f(x + \kappa \hat{\mu}) .
\] (6.20)

The backward parallel transport of a linear connection with transport matrices \( V^\mu_\nu = (V^\rho_\mu_\sigma) \) acts as follows,
\[
\tilde{V} _{\partial_\mu} \partial_\nu = \frac{1}{\kappa} \sum_{\rho} V^\rho_\mu_\nu \cdot \partial_\rho .
\] (6.21)

Let us write
\[
\nabla dx^\mu = \theta \otimes A \, dx^\mu - \sum_{\nu=1}^n V^\mu_\nu \otimes A \, dx^\nu = - \sum_{\nu=1}^n \Gamma^\mu_\nu \otimes A \, dx^\nu
\] (6.22)
where
\[
V^\mu_\nu = \frac{1}{\kappa} \sum_{\rho=1}^n V^\mu_\rho_\nu \, dx^\rho , \quad \Gamma^\mu_\nu = \sum_{\rho=1}^n \Gamma^\mu_\rho_\nu \, dx^\rho .
\] (6.23)

Using \( \theta = \sum_{\mu=1}^n \theta^\mu = (1/\kappa) \sum_{\mu=1}^n \partial_\mu dx^\mu \) with \( \partial_\mu = 1 \) for \( \mu = 1, \ldots, n \), we obtain
\[
\Gamma^\mu_\rho_\nu = \frac{1}{\kappa} \left[ V^\mu_\rho_\nu - \partial_\rho \delta^\mu_\nu \right] .
\] (6.24)

For a suitable Levi-Civita connection these functions should yield the Christoffel symbols in the continuum limit.

The components of the torsion 2-form \( \Theta^\mu = (1/2) \sum_{\nu,\rho=1}^n Q^\mu_\nu_\rho \, dx^\nu \wedge dx^\rho \) are
\[
Q^\mu_\nu_\rho = \frac{1}{\kappa} (V^\mu_\nu_\rho - \partial_\nu \delta^\mu_\rho) = \Gamma^\mu_\rho_\nu
\] (6.25)
and the components of the curvature 2-form $\mathcal{R}(dx^\mu) = (1/2) \sum_{\nu,\rho,\sigma=1}^n \mathcal{R}^\mu_{\nu\rho\sigma} dx^\nu \cap dx^\sigma \otimes_L dx^\rho$ are given by

$$\mathcal{R}^\mu_{\nu\rho\sigma} = (\mathcal{R}_{\mu\sigma})_{\nu} = \frac{1}{\kappa^2} (V_\rho R^\nu_{\rho}\sigma - V_\sigma R^\nu_{\sigma}\rho)_{\mu} .$$  (6.26)

The two Bianchi identities take the form

$$\frac{1}{\kappa} \left( \sum_{\lambda=1}^n V^\nu_{[\nu|\lambda]} R^\lambda_{\rho\sigma} - Q^\nu_{[\nu\rho} R^\rho_{\nu\sigma]} \right) = \mathcal{R}^\mu_{[\nu\rho\sigma]}$$  (6.27)

and

$$V_{[\nu \rho} R^\nu_{\rho\sigma]} = \mathcal{R}_{[\nu\rho} R^\nu_{\rho\sigma]} .$$  (6.28)

The compatibility condition for the linear connection and a metric tenos or $g = \sum_{\mu,\nu=1}^n \mathfrak{g}_{\mu\nu}(x) dx^\mu \otimes_L dx^\nu$ reads

$$R^*_\rho (\mathfrak{g}_{\mu\nu}) = V^\rho_\nu (\mathfrak{g}_{\mu\nu}) V_\rho .$$  (6.29)

The integrability condition of this equation (iteration around a plaquette) implies that the matrices $K_{\mu\nu}$ which are defined by

$$V_\mu R^\nu_{\mu\nu} = K_{\mu\nu} V_\nu V^\nu_\mu$$  (6.30)

are isometries of $g$ at every point of the lattice. The curvature tensor, in matrix form, can now be written as follows,

$$\mathcal{R}_{\mu\nu} = \frac{1}{\kappa^2} (K_{\mu\nu} - I) V_\nu R^\nu_\mu .$$  (6.31)

If the torsion vanishes, the first Bianchi identity reduces to $\mathcal{R}^\mu_{[\nu\rho\sigma]} = 0$. Then there is (up to the global sign) only one definition of a Ricci tensor:

$$Ric_{\mu\nu} = \sum_{\rho=1}^n \mathcal{R}^\rho_{\mu\nu} .$$  (6.32)

The curvature scalar is given by

$$\mathcal{R} = \sum_{\mu,\nu=1}^n \mathfrak{g}^{\mu\nu} Ric_{\mu\nu}$$  (6.33)

involving the components of the inverse metric $g^{-1} = \sum_{\mu,\nu=1}^n \mathfrak{g}^{\mu\nu} \partial_{+\mu} \otimes_L \partial_{+\nu}$. 41
Let $y^\mu(x)$ be a set of $n$ real-valued functions which can be inverted to express $x^\nu$ in terms of the functions $y^\mu$ and for which the Jacobian

$$
\mathcal{J}^\mu_\nu := \partial_{+\nu} y^\mu
$$

is invertible. The functions $y^\mu$ are then new coordinates and we have

$$
dy^\mu = \sum_{\nu=1}^{n} \mathcal{J}^\mu_\nu dx^\mu, \quad dx^\mu = \sum_{\nu=1}^{n} (\mathcal{J}^{-1})^\mu_\nu dy^\nu.
$$

Note that $dy^\mu \cap dy^\nu + dy^\nu \cap dy^\mu = 0$, while $dy^\mu dy^\nu + dy^\nu dy^\mu \neq 0$, in general. Introducing (left) partial derivatives with respect to the basis $dy^\mu$ via

$$
df = \sum_{\nu=1}^{n} \partial^\nu_{+\nu} f dy^\nu
$$

we obtain

$$
\partial^\nu_{+\nu} f = \sum_{\mu=1}^{n} (\mathcal{J}^{-1})^\mu_\nu \partial_{+\mu} f
$$

and, in particular,

$$
(\mathcal{J}^{-1})^\mu_\nu = \partial^\mu_{+\nu} x^\mu.
$$

Using the coordinates $x^\mu$, the basic commutation relations of the differential calculus are

$$
[x^\mu, x^\nu] = \kappa \delta^{\mu\nu} dx^\mu.
$$

In terms of $y^\mu$ they read

$$
[dy^\mu, y^\nu] = \kappa \sum_{\rho=1}^{n} C^{\mu\nu}_\rho dy^\rho, \quad C^{\mu\nu}_\rho := \sum_{\sigma=1}^{n} \mathcal{J}^\mu_\sigma \mathcal{J}^\nu_\sigma (\mathcal{J}^{-1})^\sigma_\rho.
$$

In the limit as $\kappa \to 0$ we obtain in both coordinate systems the ordinary continuum differential calculus, as long as the coordinate transformation does not involve $\kappa$. If $f$ and $y^\mu$ are differentiable functions of $x^\mu$, then in this limit $df$ becomes $\sum_{\mu}(\partial f / \partial x^\mu) dx^\mu$ and also $\sum_{\mu}(\partial f / \partial y^\mu) dy^\mu$ with the help of the chain rule. Although the lattice differential calculus becomes particularly simple when expressed in terms of the coordinates $x^\mu$, in the continuum limit all coordinate systems are on an equal footing. The discrete calculus also allows $\kappa$-dependent coordinate transformations. But exploring the continuum limit we should require that such a transformation remains a coordinate transformation in the limit $\kappa \to 0$.

Since the metric is defined using the left-covariant tensor product, the metric components transform homogeneously with the Jacobi matrix:

$$
\mathcal{g}'_{\mu\nu}(y) = \sum_{\rho,\sigma=1}^{n} (\mathcal{J}^{-1})^{\rho}_{\mu} (\mathcal{J}^{-1})^{\sigma}_{\nu} \mathcal{g}_{\rho\sigma}(x)
$$

(6.42)
where $g = \sum_{\mu, \nu} g'_{\mu \nu}(y) dy^\mu \otimes_L dy^\nu$. This local tensor transformation property is shared by the components of the torsion and curvature, in particular. A linear connection and the associated transport matrices have a non-local character. With the help of (2.41) and (6.21) we find

$$V'_\mu(y) = \sum_{\nu=1}^{n} (J^{-1})^\nu_\mu(x) J(x) V_\nu(x) J^{-1}(x + \kappa \hat{\nu}) . \quad (6.43)$$

7 Conclusions

Starting from basic formulas of noncommutative geometry, we developed a formalism of Riemannian geometry of group lattices. More precisely, we restricted our considerations to the subclass of bicovariant group lattices. Only for this subclass there is a simple conversion between the ordinary tensor product $\otimes_A$ and the left-covariant tensor product $\otimes_L$. The latter played a crucial role in making contact with classical geometry. In particular, it allows to introduce a discrete analogue of a metric tensor with a natural geometric interpretation and, more technically, to formulate a compatibility condition with a linear connection.

In particular in the case of a $\mathbb{Z}^n$ group lattice, the discrete geometry obtained has much in common with lattice gauge theory. It yields a discretization of continuum geometry via plaquettes where the curvature results from parallel transport around a plaquette (see also the various approaches [11] to “lattice gravity” in this context). In contrast, in Regge calculus the curvature is concentrated at a hinge (which in two dimensions is a vertex).

Given a metric, the compatibility condition for a linear connection leaves us with the freedom of torsion. This is analogous to continuum differential geometry where the additional requirement of vanishing torsion uniquely determines a particular linear connection, the Levi-Civita connection (which is expressed via the Christoffel symbols in terms of the metric coefficients). The situation is much more complicated for group lattices, however.

A Levi-Civita connection need not exist for a given Riemannian group lattice. Furthermore, if such a connection exists, then it is not unique. We achieved a geometric understanding of this ambiguity through the elaboration of several examples. The deeper origin is the fact that our connections have values in a group algebra rather than a Lie algebra. The latter only feels the part of a (continuous) group which is connected with the identity. The requirement of a continuum limit in general distinguishes a certain connection.

The requirement of a Levi-Civita connection for a Riemannian group lattice strongly restricts the metric, in general. On the other hand, we learned from our examples that metric-compatible linear connections with non-vanishing torsion show up quite naturally. A convenient condition which replaces that of vanishing torsion is not available. A few general statements can nevertheless be made. Vanishing triangle torsion means assigning Euclidean properties to the respective triangle. Of course, a group lattice with Euclidean triangles, but more than three sites, may still be curved. Non-vanishing biangle torsion allows for an anisotropy of the distance relation between the respective two lattice sites, adhering to a simple interpretation of the metric coefficients. The requirement of vanishing biangle torsion would rule out this feature. But it would also eliminate geometries without such an anisotropy as we saw in sections 5.2 and 5.3.
On the other hand, a distance anisotropy may indeed appear in communication networks (with a group lattice structure), a relation which should be elaborated elsewhere. [16] The design of a communication network determines its efficiency. The broadcast time, for example, clearly depends on its geometry. [17] For such problems the geometric formalism developed in this work could be of help.

Our examples demonstrate that torsion quite naturally enters the stage. The more we depart from the continuum, the more we get away from the familiar condition of vanishing torsion of continuum (pseudo-) Riemannian geometry. Hypercubic group lattices, which only consists of quadrangles, are relatively close to the continuum in this sense. Biangles and triangles add to the rigidity of a lattice, so that torsion becomes more necessary in order to curve it. The conclusion is that, in contrast to ordinary continuum differential geometry, (non-zero) torsion is an essential ingredient of our discrete geometric formalism. Interesting field equations will have to take care of this fact and describe the dynamics of metric and torsion.

Is there a distinguished geometry associated with a (bicovariant) group lattice? Indeed, a direct consequence of the definition of a group lattice is the existence of a family of vector fields \( \ell_h, h \in S \). Requiring that these are Killing vector fields of the metric, so that their flows preserve the metric, restricts the a priori possible metrics to the class of right-invariant metrics which are completely determined by the components at one site. If \( S \) is Abelian, these are simply the constant metrics, i.e., the components are the same at all sites (which correspond to the group elements). Associated with the class of right-invariant metrics is a distinguished metric-compatible linear connection. Moreover, we have the notion of bi-invariance of a metric which determines a subclass of right-invariant metrics. Interesting relations between group structure and geometry are expected to emerge from this.

Even in the familiar hypercubic lattice case the (pseudo-) Riemannian geometry derived from the general framework of group lattice geometry appears to be new. In particular in the form presented in section 6.2, using coordinates on the lattice, the close analogy with continuum Riemannian geometry becomes transparent. This provides an alternative to the existing discretizations of gravity theories.

Representations of “intrinsic” group lattice geometries via immersions in a Euclidean space will be treated in a separate work. For two-dimensional Riemannian group lattices (where \( S \) consists of two different elements), the bicovariance condition restricts to Abelian groups, and a relatively simple formalism of immersions can be developed in analogy with that of continuum differential geometry. For immersions of higher than two-dimensional Riemannian group lattices in Euclidean \( \mathbb{R}^n \) the formalism is more complex and new features will show up.

### A Orthonormal coframe fields

Let \( g \) be a metric on a group lattice \((G, S)\) which has Euclidean (or Lorentzian) signature at each point. An orthonormal coframe field is a set of \(|S|\) linearly independent 1-forms \( E^a \)
(at each point of $G$) such that
\begin{equation}
g = \sum_{a,b=1}^{\vert S \vert} \eta_{ab} E^a \otimes_L E^b \tag{A.1}
\end{equation}

where $\eta = (\eta_{ab})$ has entries $\pm 1$ on the diagonal and zeros otherwise (according to the signature of $g$). Writing
\begin{equation}
E^a = \sum_{h \in S} E^a_h \theta^h, \quad a = 1, \ldots, \vert S \vert \tag{A.2}
\end{equation}

it follows that the matrix $(E^a_h)$ is invertible at all sites $g \in G$. Let $(\bar{E}^h_a)$ denote its inverse.

In the following, for $(\eta_{ab})$ we may take more generally an arbitrary constant symmetric matrix. Using (I.6.5) and (2.34), we find
\begin{equation}
V_{\ell_h} E^a = \sum_{b=1}^{\vert S \vert} (R^*_h \bar{L}^a_{h,b}) E^b \tag{A.3}
\end{equation}

with
\begin{equation}
L^a_{h,b} := \sum_{h',h'' \in S} E^a_{h'} V^{h',h''} R^*_h \bar{E}^{h''}_b. \tag{A.4}
\end{equation}

or $L_h = EV_h R^*_h \bar{E}$ in an obvious matrix notation. As a consequence,
\begin{equation}
\nabla E^a = \theta \otimes_A E^a - \sum_b L^a_b \otimes_A E^b \tag{A.5}
\end{equation}

with
\begin{equation}
L^a_b := \sum_{h \in S} L^a_{h,b} \theta^h. \tag{A.6}
\end{equation}

Let us introduce the dual frame field
\begin{equation}
\bar{E}_a := \sum_{h \in S} \bar{E}^h_a \cdot \ell_h \tag{A.7}
\end{equation}

which satisfies $\langle \bar{E}_a, E^b \rangle = \delta^b_a$. As a consequence of (I.7.17) and (2.41), we find
\begin{equation}
\bar{V}_{\ell_h} \bar{E}_a = \sum_b \bar{L}^b_{h,a} \cdot \bar{E}_b. \tag{A.8}
\end{equation}

The metric-compatibility condition for the connection takes the form
\begin{equation}
L^T_h \eta L_h = \eta. \tag{A.9}
\end{equation}

The matrices $L_h$ are thus isometries of $\eta$, they have values in the orthogonal group $O(\eta)$ of $\eta$. This shows that if an orthonormal coframe field is chosen, an $\eta$-compatible linear connection is equivalent to a map $G \times S \to O(\eta)$. 

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The components of the torsion 2-form with respect to the coframe \( E^a \) are
\[
\Theta^a := \Theta(E^a) = E^a \theta - \Delta(E^a) + \sum_{b=1}^{\left| S \right|} L^a_b E^b . \tag{A.10}
\]

Here we used (I.7.6), \( \Delta(f \omega) = f \Delta(\omega) \), (I.6.5) and (A.3). Writing this as
\[
\Theta^a = \sum_{h_1,h_2 \in S} \left( E^a_{h_1} - \sum_{h \in S} E^a_h \delta^h_{h_1}h_2 + \sum_{b} L^a_{h_1,b} R^*_h E^b_{h_2} \right) \theta^h_1 \theta^h_2 \tag{A.11}
\]
the condition of vanishing torsion \( \Theta^a = 0 \) yields for biangles \((h_1 h_2 = e, h_1, h_2 \in S(0))\)
\[
L_{h_1} R^*_h E_{h_2} = -E_{h_1} \tag{A.12}
\]
where, for example, \( E_{h_1} \) denotes the column with entries \( E^a_{h_1} \). For triangles \((h_1 h_2 = h \in S(1))\) it yields
\[
L_{h_1} R^*_h E_{h_2} = E_h - E_{h_1} \tag{A.13}
\]
and for quadrangles \((h_1 h_2 = \hat{h}_1 \hat{h}_2 = g \in S(2))\)
\[
L_{h_1} R^*_h E_{h_2} - L_{\hat{h}_1} R^*_\hat{h}_1 E_{h_2} = E_{\hat{h}_1} - E_{h_1} . \tag{A.14}
\]

The components of the curvature with respect to the coframe \( E^a \) are
\[
R(E^a) = \sum_{b=1}^{\left| S \right|} \mathcal{R}^a_b \otimes_A E^b \tag{A.15}
\]
where
\[
(\mathcal{R}^a_b) = L^2 - \Delta(L) - I \Delta^e, \quad L := \sum_{h \in S} L_h \theta^h . \tag{A.16}
\]
With the help of the Leibniz rule and (I.2.15), we obtain the first Bianchi identity (I.7.11) in the following form,
\[
d\Theta(E) + (L - \theta) \Theta(E) = \mathcal{R}^a_b E^b = -\Delta^e E - \Delta(L) E + L^2 E \tag{A.17}
\]
where \( E \) stands for the column with entries \( E^a \). From
\[
0 = \nabla(\mathcal{R}(E^a)) - \mathcal{R}(\nabla E^a) = \sum_b \left( -\Delta(\mathcal{R}^a_b) + \sum_c (L^a_c \mathcal{R}^c_b - \mathcal{R}^c_a L^c_b) \right) \otimes_A E^b \tag{A.18}
\]
we obtain the following version of the second Bianchi identity,
\[
\Delta(\mathcal{R}^a_b) = \sum_c \left( L^a_c \mathcal{R}^c_b - \mathcal{R}^a_c L^c_b \right) . \tag{A.19}
\]
Writing
\[ \mathcal{R}(E^a) = \sum_{b=1}^{\vert S \vert} \sum_{h_1,h_2 \in S} \mathcal{R}^a_{b,h_1,h_2} \theta^{h_1} \cap \theta^{h_2} \otimes_L E^b \] (A.20)

we find the biangle part of the curvature
\[ \mathcal{R}_{(e)} h_1,h_2 = \delta^c_{h_2 h_1} \left( L_{h_1} R^c_{h_1} L_{h_1}^{-1} - I \right), \] (A.21)

the triangle part \( (h \in S_{(1)}) \)
\[ \mathcal{R}_{(h)} h_1,h_2 = \delta^h_{h_2 h_1} \left( L_{h_1} R^h_{h_1} L_{h_1}^{-1} - L_h \right) \mathcal{E}(h) \] (A.22)

and the quadrangle curvature \( (g \in S_{(2)}) \)
\[ \mathcal{R}_{(g)} h_1,h_2; h_1,h_2 = \delta^g_{h_2 h_1} \left( L_{h_1} R^g_{h_1} L_{h_1}^{-1} - L_{h_1} R^g_{h_1} L_{h_1}^{-1} \right) \mathcal{E}(g). \] (A.23)

Here we have introduced
\[ \mathcal{E}^a_{(g)h} := \sum_{h' \in S} (R^a_{g,h'}) E^{ab'} g^{-1} b. \] (A.24)

**Example A.1.** Let \( G = \mathbb{Z}^2 \) and \( S = \{ \hat{1} = (1, 0), \hat{2} = (0, 1) \} \). We choose a metric of Euclidean signature and a corresponding orthonormal coframe \( E^a, a = 1, 2 \), so that
\[ g_{h,h'} = \sum_{a,b} \delta^{a,b} E^a_h E^b_{h'}. \] (A.25)

The metric-compatibility condition for a connection now reads \( L^T_h L_h = I \) and thus the matrices \( L_h \) have to be orthogonal \( 2 \times 2 \)-matrices. We may assume det \( L_h > 0 \) so that
\[ L_h(k, l) = \begin{pmatrix} \cos \vartheta_h(k, l) & -\sin \vartheta_h(k, l) \\ \sin \vartheta_h(k, l) & \cos \vartheta_h(k, l) \end{pmatrix} \] (A.26)

for \( (k, l) \in \mathbb{Z}^2 \), which defines a map \( \mathbb{Z}^2 \times \{ \hat{1}, \hat{2} \} \rightarrow SO(2) \). The linear connection thus associates with each arrow on the lattice a rotation angle. Since there are no biangles or triangles in the case under consideration, according to (A.23) the curvature is given by
\[ \mathcal{R}_{1,2;1} = (L_1 R^1_1 L_2 - L_2 R^2_1 L_1) \mathcal{E}_{(1,1)} \] (A.27)

where \( \mathcal{E}_{(1,1)} = (R^1_{(1,1)} E) \bar{E} \). The last factor achieves that all indices of the curvature tensor refer to the same point.

For the general metric (4.16) and an arbitrary linear connection, the curvature scalar can be expressed as follows,
\[ R = \sum_{h, h_1, h_2 \in S} (g^{-1})_{h_1, h_2} \mathcal{R}^h_{h_1, h_2} \mathcal{R}^1_{h_1, \hat{1}} \mathcal{R}^2_{h_1, \hat{2}} - (g^{-1})_{h_1, \hat{1}} R^2_{h_1, \hat{1}} \] (A.28)
where we used the antisymmetry of the curvature tensor components in the last two indices (which holds in the case under consideration), the symmetry of the metric, and anti-symmetrization brackets. With

$$R_{[1,2],i,j} = \sum_{h,h'\in S} \tilde{E}^h_{[1} \tilde{E}^{h'}_{2]} R_{h,h',i,j} = (\det \tilde{E}) R_{[1,2],i,j}$$  \hspace{1cm} (A.29)

and $(\det \tilde{E})^2 = 1/\det g$, we obtain the identity

$$R_{[1,2],i,j} = R \sqrt{\det g}$$  \hspace{1cm} (A.30)

for the Einstein-Hilbert density. ■

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[7] Regular digraphs have constant valency (degree). There are regular graphs, like the well-known Peterson graph, which are not Cayley graphs. The differential calculus associated with a Cayley digraph has the distinguishing property that there is a (left) $A$-module basis of the space of 1-forms. Moreover, the group structure organizes the arrows in a certain way which leads to the preferred basis $\{\theta^h | h \in S\}$ (see Ref. [1]).
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The ambiguity in the quadrangle coefficient functions of a 2-form is not special to the $\cap$-product, but already appears in the more general formalism using the original product in $\Omega$, see section 4.2 of Ref. 1. In the latter work, $\psi_{h,h'}$ denoted the quadrangle coefficients of a 2-form $\psi$ with respect to the original product in $\Omega$. Here it refers to the $\cap$-product. Of course, they are different in general.

Here the vanishing of the curvature is a special case of a much more general well-known result. If the torsion of a linear connection with respect to a differential calculus on some associative algebra vanishes, then $\pi \nabla = d \circ \pi$. In case of the universal differential calculus this reduces to $\nabla = d$ and the vanishing of the curvature follows from $d^2 = 0$.

A corresponding Mathematica notebook is available at the authors' homepages. Mathematica is a registered trademark of Wolfram Research. See S. Wolfram, The Mathematica Book, (Cambridge University Press, 1999).

Distances in communication networks are measured by sending a signal from one point to another and reflecting it back to the point of origin. The elapsed time read off from a clock at the point of emission, is then a measure of the distance between the two points.

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