Abstract

In this paper we study the nonlinear Schrödinger-Maxwell equations
\[
\begin{aligned}
-\Delta u + V(x)u + \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

If \( V \) is a positive constant, we prove the existence of a ground state solution \( (u, \phi) \) for \( 2 < p < 5 \). The non-constant potential case is treated under suitable geometrical assumptions on \( V \), for \( 3 < p < 5 \). Existence and non-existence results are proved also when the nonlinearity exhibits a critical growth.

Contents

1 Introduction 2

2 The subcritical case 6

2.1 Some preliminary results 6

2.2 The constant potential case 7

2.2.1 Proof of Theorem 1.1 9

2.3 The non-constant potential case 14

2.3.1 Proof of Theorem 1.2 16
1 Introduction

In this paper we consider the problem

\[
\begin{aligned}
-\Delta u + V(x)u + \phi u &= f'(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( V : \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( f \in C^1(\mathbb{R}^3, \mathbb{R}) \). Such a system represents the nonlinear Schrödinger-Maxwell equations in the electrostatic case. In [3], the potential \( V \) has been supposed constant, and the linear version of the problem (i.e. \( f \equiv 0 \)) has been studied as an eigenvalue problem for a bounded domain. The linear Schrödinger-Maxwell equations have been treated also in [9, 11], where the potential \( V \) has been supposed radial.

The nonlinear case has been considered in [1, 10, 13, 15, 20], where existence and multiplicity results have been stated when \( V \) is a positive constant. By means of the Pohozaev’s fibering method, a multiplicity result has been proved in [21] also in the non-homogeneous case, that is when a non-homogeneous term \( g(x) \in L^2(\mathbb{R}^3) \) is added on the right hand side of the first equation of (SM) (see also [7]). On the other hand, nonexistence results for (SM) can be found in [14, 20]. For a related problem see [18].

In this paper we will look for ground state solutions to the problem (SM), namely for couples \((u, \phi)\) which solve (SM) and minimize the action functional associated to (SM) among all possible solutions. The problem of finding such a type of solutions is a very classical problem: it has been introduced by Coleman, Glazer and Martin in [12], and reconsidered by Berestycki and Lions in [5] for a class of nonlinear equations including the Schrödinger’s one. Later on the existence and the profile of ground state solutions have been studied for a plethora of problems by many authors; of course we can not mention all these results.

In the first part of the paper, we are interested in considering pure power type nonlinearities so that the problem we will deal with becomes

\[
\begin{aligned}
-\Delta u + V(x)u + \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
where $2 < p < 5$. The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (1) are the critical points of the action functional $\mathcal{E}: H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1}.$$  

We are interested in finding a ground state solution of (1), that is a solution $(u_0, \phi_0)$ of (1) with the property of having the least action among all possible solutions of (1), namely $\mathcal{E}(u_0, \phi_0) \leq \mathcal{E}(u, \phi)$, for any solution $(u, \phi)$ of (1).

The action functional $\mathcal{E}$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [4], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

The main difficulty related with the problem of finding the critical points of the new functional, consists in the lack of compactness of the Sobolev spaces embeddings in the unbounded domain $\mathbb{R}^3$. Usually, at least when $V$ is radially symmetric, such a difficulty is overcome by restricting the functional to the natural constraint of the radial functions where compact embeddings hold. In particular, in [13] a radial solution having minimal energy among all the radial solutions has been found. However we are not able to say if that solution actually is a ground state for our equation. This is the reason why we will use an alternative method, based on a concentration-compactness argument on suitable measures, to recover compactness.

We analyze two different situations. First we assume that $V$ is a positive constant and, following an idea of Ruiz [20], we look for a minimizer of the reduced functional restricted to a suitable manifold $\mathcal{M}$. Such a manifold has two interesting features: it is a natural constraint for the reduced functional and it contains, in a sense that we will explain later (see Remark 2.2), every solution of the problem (1). The main result we get is the following

**Theorem 1.1.** If $V$ is a positive constant, then the problem (1) has a ground state solution for any $p \in [2, 5[$.

In fact, it is standard to see that such a ground state solutions does not change sign, so we can assume it positive.

Then we study (1) assuming the following hypotheses on $V$:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$;
(V2) $0 < C_1 \leq V(x) \leq C_2$, for all $x \in \mathbb{R}^3$;

(V3) $V_\infty := \liminf_{|y| \to \infty} V(y) \geq V(x)$, for all $x \in \mathbb{R}^3$, and the inequality is strict for some $x \in \mathbb{R}^3$.

These kind of hypotheses on the potential were introduced by Rabinowitz [19] to study the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f'(u) \quad \text{in} \ \mathbb{R}^3.$$  

Because of technical difficulties we are not allowed to use the same device as in the constant potential case. We study the reduced functional restricted to the Nehari manifold and we are able to prove the existence result only for $3 < p < 5$:

**Theorem 1.2.** If $V$ satisfies (V1-3) then the problem (1) has a ground state solution for any $p \in ]3, 5[$.

Theorems 1.1 and 1.2 will be proved in Section 2.

In the second part of the paper we consider the critical case, namely the case when the nonlinearity presents at infinity the same behavior of the power $t^{2^*-1}$, where $2^* = 6$ is the critical exponent for the Sobolev embeddings in dimension 3. Here a further obstacle to compactness arises, in fact, it is well known that the embedding of the space $H^1(\Omega)$ into the Lebesgue space $L^{2^*}(\Omega)$ is not compact, even if $\Omega$ is a bounded set in $\mathbb{R}^3$.

The problem becomes

$$ \begin{cases} 
-\Delta u + V(x)u + \phi u = u^5 & \text{in} \ \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in} \ \mathbb{R}^3. 
\end{cases} $$

By [14], we have the following

**Theorem 1.3** (D’Aprile & Mugnai [14]). Suppose that $V$ is a positive constant. Let $(u, \phi) \in H^1(\Omega) \times D^{1,2}(\Omega)$ be a solution of the problem (2), then $u = \phi = 0$.

We extend this nonexistence result to the case of a non-constant potential $V$. We prove the following nonexistence theorem, based on a Pohozaev-type identity.

**Theorem 1.4.** Suppose that $V$ satisfies (V2) and

(V4) $V \in C^1(\mathbb{R}^3, \mathbb{R})$;

(V5) $0 \leq (\nabla V(x) \mid x) \leq C_3$, for all $x \in \mathbb{R}^3$. 

Let \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) be a solution of the problem (2), then \(u = \phi = 0\).

Then, in the same spirit of [6] (see also [8] for the Klein-Gordon-Maxwell equation), we add a lower order perturbation to the first equation of (2), namely we look for solutions to the system

\[
\begin{cases}
-\Delta u + V(x)u + \phi u = |u|^{q-1}u + u^5 & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \(q \in [3, 5]\). The solutions \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) of (3) are the critical points of the action functional \(E^*: H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}\), defined as

\[
E^*(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 \\
- \frac{1}{q+1} \int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6} \int_{\mathbb{R}^3} u^6.
\]

The effect of the additive perturbation is to lower the energy. This causes that the ground state level of the functional falls into an interval where compactness holds. As a consequence we get the following two results, respectively for the constant and the non-constant potential case:

**Theorem 1.5.** Let \(V\) be a positive constant. Then the problem (3) has a ground state solution.

**Theorem 1.6.** Let \(V\) satisfy (V1-3). Then the problem (3) has a ground state solution.

We will prove these three last theorems in Section 3.

**NOTATION**

- For any \(1 \leq s < +\infty\), \(L^s(\mathbb{R}^3)\) is the usual Lebesgue space endowed with the norm
  \[
  \|u\|_s := \int_{\mathbb{R}^3} |u|^s;
  \]

- \(H^1(\mathbb{R}^3)\) is the usual Sobolev space endowed with the norm
  \[
  \|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2;
  \]
\( \mathcal{D}^{1,2}(\mathbb{R}^3) \) is completion of \( C^\infty_0(\mathbb{R}^3) \) with respect to the norm
\[
\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\nabla u|^2;
\]

for any \( r > 0, x \in \mathbb{R}^3 \) and \( A \subset \mathbb{R}^3 \)
\[
B_r(x) := \{ y \in \mathbb{R}^3 \mid |y - x| \leq r \},
B_r := \{ y \in \mathbb{R}^3 \mid |y| \leq r \},
A^c := \mathbb{R}^3 \setminus A;
\]

\( C, C', C_i \) are positive constants which can change from line to line;

\( o_n(1) \) is a quantity which goes to zero as \( n \to +\infty \).

2 The subcritical case

2.1 Some preliminary results

We first recall some well-known facts (see, for instance [3, 9, 10, 11, 13, 20]). For every \( u \in L^{12/5}(\mathbb{R}^3) \), there exists a unique \( \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \) solution of
\[
-\Delta \phi = u^2, \quad \text{in } \mathbb{R}^3.
\]

It can be proved that \( (u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \) is a solution of (1) if and only if \( u \in H^1(\mathbb{R}^3) \) is a critical point of the functional \( I: H^1(\mathbb{R}^3) \to \mathbb{R} \) defined as
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}, \tag{4}
\]
and \( \phi = \phi_u \).

The functions \( \phi_u \) possess the following properties (see [13] and [20])

\textbf{Lemma 2.1.} For any \( u \in H^1(\mathbb{R}^3) \), we have:

i) \( \|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leq C \|u\|^2 \), where \( C \) does not depend from \( u \). As a consequence there exists \( C' > 0 \) such that
\[
\int_{\mathbb{R}^3} \phi_u u^2 \leq C' \|u\|_{12}^\frac{4}{15};
\]

ii) \( \phi_u \geq 0 \);
iii) for any \( t > 0 \): \( \phi_t \phi_{tu} = t^2 \phi_u \);

iv) for any \( \theta > 0 \): \( \phi_u(x) = \theta^2 \phi_u(\theta x) \), where \( u_\theta(x) = \theta^2 u(\theta x) \);

v) for any \( \Omega \subset \mathbb{R}^3 \) measurable,

\[
\int_{\Omega} \phi_u u^2 = \int_{\Omega} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy.
\]

2.2 The constant potential case

In this section we will assume that \( V \) is a positive constant. Without lost of generality, we suppose \( V \equiv 1 \). It can be proved (see [14, 20]) that if \( (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) is a solution of (1), then it satisfies the following Pohozaev type identity

\[
\int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + \frac{3}{2} u^2 + \frac{5}{4} \phi u^2 - \frac{3}{p+1} |u|^{p+1} = 0.
\] (5)

As in [20], we introduce the following manifold

\[
\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid G(u) = 0 \right\},
\]

where

\[
G(u) := \int_{\mathbb{R}^3} \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3}{4} \phi u^2 - \frac{2p-1}{p+1} |u|^{p+1}.
\]

**Remark 2.2.** Observe that if \( u \in H^1(\mathbb{R}^3) \) is a nontrivial critical point of \( I \), then \( u \in \mathcal{M} \), since \( G(u) = 0 \) can be obtained by a linear combination of \( \langle I'(u), u \rangle = 0 \) and (5), with \( \phi = \phi_u \). As a consequence if \( (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) is a solution of (1), then \( u \in \mathcal{M} \).

The next lemma describes some properties of the manifold \( \mathcal{M} \):

**Lemma 2.3.** 1. For any \( u \in H^1(\mathbb{R}^3), u \neq 0 \), there exists a unique number \( \bar{\theta} > 0 \) such that \( u_{\bar{\theta}} \in \mathcal{M} \). Moreover

\[
I(u_{\bar{\theta}}) = \max_{\theta \geq 0} I(u_\theta);
\]

2. there exists a positive constant \( C \), such that for all \( u \in \mathcal{M} \), \( \|u\|_{p+1} \geq C \);

3. \( \mathcal{M} \) is a natural constraint of \( I \), namely every critical point of \( I|_{\mathcal{M}} \) is a critical point for \( I \).
Proof We refer to [20]. In particular, as regards point 3, we have to point out that Ruiz [20] has just proved that the minimum of $I|_\mathcal{M}$ is in fact a critical point of $I$: the same arguments can be adapted to prove that $\mathcal{M}$ is a natural constraint of $I$. □

By 3 of Lemma 2.3 we are allowed to look for critical points of $I$ restricted to $\mathcal{M}$.
Moreover, by 1 of Lemma 2.3, the map $\theta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+$ such that for any $u \in H^1(\mathbb{R}^3), u \neq 0$:

$$I(u_{\theta(u)}) = \max_{\theta \geq 0} I(u_{\theta})$$
is well defined.

Set

$$c_1 = \inf_{g \in \Gamma} \max_{\theta \in [0,1]} I(g(\theta));$$
$$c_2 = \inf_{u \neq 0} \max_{\theta \geq 0} I(u_{\theta});$$
$$c_3 = \inf_{u \in \mathcal{M}} I(u);$$

where

$$\Gamma = \{ g \in C([0, 1], H^1(\mathbb{R}^3)) \mid g(0) = 0, I(g(1)) \leq 0, g(1) \neq 0 \}. \quad (6)$$

Lemma 2.4. The following equalities hold

$$c := c_1 = c_2 = c_3.$$

Proof Taking into account 1 of Lemma 2.3 and the fact that for small $\|u\|$ we have (see [20, Theorem 3.2, Step1])

$$\int_{\mathbb{R}^3} \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3}{4} \phi_u u^2 > \int_{\mathbb{R}^3} \frac{2p-1}{p+1} |u|^{p+1},$$

the conclusion follows using the same arguments of [19, Proposition 3.11]. □

Remark 2.5. By point 3 of Lemma 2.3 and Remark 2.2, we argue that if $u \in \mathcal{M}$ is such that $I(u) = c$, then $(u, \phi_u)$ is a ground state solution of (1).
2.2.1 Proof of Theorem 1.1

Let \((u_n)_n \subset \mathcal{M}\) such that
\[
\lim_{n} I(u_n) = c. \tag{7}
\]

We define the functional \(J: H^1(\mathbb{R}^3) \to \mathbb{R}\) as:
\[
J(u) = \int_{\mathbb{R}^3} \frac{p-2}{2p-1} |\nabla u|^2 + \frac{p-1}{2p-1} u^2 + \frac{p-2}{2(2p-1)} \phi u^2.
\]

Observe that for any \(u \in \mathcal{M}\), by ii of Lemma 2.1 we have \(I(u) = J(u) \geq 0\). By (7), we deduce that \((u_n)_n\) is bounded in \(H^1(\mathbb{R}^3)\), so there exists \(\bar{u} \in H^1(\mathbb{R}^3)\) such that, up to a subsequence,
\[
\begin{align*}
  u_n &\rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3), \\
  u_n &\to \bar{u} \quad \text{in } L^s(B), \text{ with } B \subset \mathbb{R}^3, \text{ bounded, and } 1 \leq s < 6.
\end{align*} \tag{8}
\]

To prove Theorem 1.1, we need some compactness on the sequence \((u_n)_n\).

To this end, we use a concentration-compactness argument on the positive measures so defined: for every \(u_n \in H^1(\mathbb{R}^3)\),
\[
\nu_n(\Omega) = \int_{\Omega} \frac{p-2}{2p-1} |\nabla u_n|^2 + \frac{p-1}{2p-1} u_n^2 + \frac{p-2}{2(2p-1)} \phi u_n^2. \tag{9}
\]

By (7) we have
\[
\nu_n(\mathbb{R}^3) = J(u_n) \to c
\]
and then, by P.L. Lions [16], there are three possibilities:

\textbf{vanishing}: for all \(r > 0\)
\[
\lim_{n} \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} d\nu_n = 0;
\]

\textbf{dichotomy}: there exist a constant \(\bar{c} \in (0, c)\), two sequences \((\xi_n)_n\) and \((r_n)_n\), with \(r_n \to +\infty\) and two nonnegative measures \(\nu^1_n\) and \(\nu^2_n\) such that
\[
0 \leq \nu^1_n + \nu^2_n \leq \nu_n, \quad \nu^1_n(\mathbb{R}^3) \to \bar{c}, \quad \nu^2_n(\mathbb{R}^3) \to c - \bar{c},
\]
\[
\text{supp}(\nu^1_n) \subset B_{r_n}(\xi_n), \quad \text{supp}(\nu^2_n) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n);
\]

\textbf{compactness}: there exists a sequence \((\xi_n)_n\) in \(\mathbb{R}^3\) with the following property: for any \(\delta > 0\), there exists \(r = r(\delta) > 0\) such that
\[
\int_{B_r(\xi_n)} d\nu_n \geq c - \delta.
Arguing as in [23], we prove the following

**Lemma 2.6.** Compactness holds for the sequence of measures \((\nu_n)_n\), defined in (9).

**Proof** Vanishing does not occur

Suppose by contradiction, that for all \(r > 0\)

\[
\lim_{n} \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} d\nu_n = 0.
\]

In particular, we deduce that there exists \(\bar{r} > 0\) such that

\[
\lim_{n} \sup_{\xi \in \mathbb{R}^3} \int_{B_{\bar{r}}(\xi)} u_n^2 = 0.
\]

By [17, Lemma I.1], we have that \(u_n \to 0\) in \(L^s(\mathbb{R}^3)\), for \(2 < s < 6\). As a consequence, since \((u_n)_n \subset \mathcal{M}\) and by Lemma 2.1, we get

\[
0 \leq I(u_n) \leq \int_{\mathbb{R}^3} \frac{3}{2} |\nabla u_n|^2 + \frac{1}{2} u_n^2 + \frac{1}{4} \phi_{u_n} u_n^2 - \frac{1}{p + 1} |u_n|^{p+1} = -\frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{2p - 2}{p + 1} \int_{\mathbb{R}^3} |u_n|^{p+1} \to 0
\]

which contradicts (7).

**Dichotomy does not occur**

Suppose by contradiction that there exist a constant \(\tilde{c} \in (0, c)\), two sequences \((\xi_n)_n\) and \((r_n)_n\), with \(r_n \to +\infty\) and two nonnegative measures \(\nu_1^n\) and \(\nu_2^n\) such that

\[
0 \leq \nu_1^n + \nu_2^n \leq \nu_n, \quad \nu_1^n(\mathbb{R}^3) \to \tilde{c}, \quad \nu_2^n(\mathbb{R}^3) \to c - \tilde{c},
\]

\[
\text{supp}(\nu_1^n) \subset B_{r_n}(\xi_n), \quad \text{supp}(\nu_2^n) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n).
\]

Let \(\rho_n \in C^1(\mathbb{R}^3)\) be such that \(\rho_n \equiv 1\) in \(B_{r_n}(\xi_n)\), \(\rho_n \equiv 0\) in \(\mathbb{R}^3 \setminus B_{2r_n}(\xi_n)\), \(0 \leq \rho_n \leq 1\) and \(|\nabla \rho_n| \leq 2/r_n\).

We set

\[
v_n := \rho_n u_n, \quad w_n := (1 - \rho_n) u_n.
\]

It is easy to see that

\[
\liminf_{n} J(v_n) \geq \tilde{c},
\]

\[
\liminf_{n} J(w_n) \geq c - \tilde{c}.
\]
Moreover, denoting \( \Omega_n := B_{2r_n}(\xi_n) \ \setminus \ B_{r_n}(\xi_n) \), we have

\[ \nu_n(\Omega_n) \to 0, \quad \text{as } n \to \infty, \]

namely

\[
\int_{\Omega_n} |\nabla u_n|^2 + u_n^2 \to 0, \quad \text{as } n \to \infty, \\
\int_{\Omega_n} \phi_{u_n} u_n^2 \to 0, \quad \text{as } n \to \infty. \tag{10}
\]

By simple computations, we infer also

\[
\int_{\Omega_n} |\nabla v_n|^2 + v_n^2 \to 0, \quad \text{as } n \to \infty, \\
\int_{\Omega_n} |\nabla w_n|^2 + w_n^2 \to 0, \quad \text{as } n \to \infty.
\]

Hence, we deduce that

\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2 = \int_{\mathbb{R}^3} |\nabla v_n|^2 + v_n^2 + \int_{\mathbb{R}^3} |\nabla w_n|^2 + w_n^2 + o_n(1), \tag{11}
\]

\[
\int_{\mathbb{R}^3} |u_n|^{p+1} = \int_{\mathbb{R}^3} |v_n|^{p+1} + \int_{\mathbb{R}^3} |w_n|^{p+1} + o_n(1). \tag{12}
\]

Moreover, by point \( \nu \) of Lemma 2.1 and (10), we have

\[
\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + 2 \int_{B_{2r_n}} \int_{B_{2r_n}} \frac{u_n^2(x)u_n^2(y)}{|x-y|} \ dx \ dy + o_n(1) \\
\geq \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + o_n(1). \tag{13}
\]

Hence, by (11) and (13), we get

\[
J(u_n) \geq J(v_n) + J(w_n) + o_n(1).
\]

Then

\[
c = \lim_{n} J(u_n) \geq \liminf_{n} J(v_n) + \liminf_{n} J(w_n) \geq \bar{c} + (c - \bar{c}) = c,
\]

hence

\[
\lim_{n} J(v_n) = \bar{c}, \tag{14}
\]

\[
\lim_{n} J(w_n) = c - \bar{c}.
\]
We recall the definition of the functional $G: H^1(\mathbb{R}^3) \to \mathbb{R}$

$$G(u) = \int_{\mathbb{R}^3} \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1}$$

and that if $u \in \mathcal{M}$, then $G(u) = 0$. By (11), (12) and (13), we have

$$0 = G(u_n) \geq G(v_n) + G(w_n) + o_n(1). \quad (15)$$

By Lemma 2.3, for any $n \geq 1$, there exists $\theta_n > 0$ such that $(v_n)_\theta_n \in \mathcal{M}$, and then

$$\int_{\mathbb{R}^3} \frac{3}{2} \theta_n^2 |\nabla v_n|^2 + \frac{1}{2} \theta_n^2 v_n^2 + \frac{3}{4} \phi_{v_n} v_n^2 = \int_{\mathbb{R}^3} \frac{2p-1}{p+1} \theta_n^{2p-2} |v_n|^{p+1}. \quad (16)$$

We have to distinguish three cases.

**CASE 1:** up to a subsequence, $G(v_n) \leq 0$.

By (16) we have

$$\int_{\mathbb{R}^3} \frac{3}{2} \left( \theta_n^{2p-2} - \theta_n^2 \right) |\nabla v_n|^2 + \frac{1}{2} \left( \theta_n^{2p-2} - 1 \right) v_n^2 + \frac{3}{4} \left( \theta_n^{2p-2} - \theta_n^2 \right) \phi_{v_n} v_n^2 \leq 0,$$

which implies that $\theta_n \leq 1$. Therefore, for all $n \geq 1$

$$C \leq I((v_n)_{\theta_n}) = J((v_n)_{\theta_n}) \leq J(v_n) \to \bar{c} < C,$$

which is a contradiction.

**CASE 2:** up to a subsequence, $G(w_n) \leq 0$.

We can argue as in the previous case.

**CASE 3:** up to a subsequence, $G(v_n) > 0$ and $G(w_n) > 0$.

By (15), we infer that $G(v_n) = o_n(1)$ and $G(w_n) = o_n(1)$. If $\theta_n \leq 1 + o_n(1)$, we can repeat the arguments of Case 1. Suppose that

$$\lim_{n \to \infty} \theta_n = \theta_0 > 1.$$

We have

$$o_n(1) = G(v_n) = \int_{\mathbb{R}^3} \frac{3}{2} |\nabla v_n|^2 + \frac{1}{2} v_n^2 + \frac{3}{4} \phi_{v_n} v_n^2 - \frac{2p-1}{p+1} v_n^{p+1}$$

$$= \int_{\mathbb{R}^3} \frac{3}{2} \left( 1 - \frac{1}{\theta_n^{2p-4}} \right) |\nabla v_n|^2 + \frac{1}{2} \left( 1 - \frac{1}{\theta_n^{2p-2}} \right) v_n^2$$

$$+ \int_{\mathbb{R}^3} \frac{3}{4} \left( 1 - \frac{1}{\theta_n^{2p-4}} \right) \phi_{v_n} v_n^2.$$
and so $v_n \to 0$ in $H^1(\mathbb{R}^3)$, but we get a contradiction with (14).
Hence we conclude that dichotomy can not occur. \qed

Now we are able to yield the following

**Proof of Theorem 1.1** Let $(u_n)_n$ be a sequence in $\mathcal{M}$ such that (7) holds. We define the measures $(\nu_n)_n$ as in (9); by Lemma 2.6 there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^N$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_\delta(\xi_n)} \left( \frac{p-2}{2p-1} |\nabla u_n|^2 + \frac{p-1}{2p-1} u_n^2 + \frac{p-2}{2(2p-1)} \phi_{u_n} u_n^2 \right) < \delta. \tag{17}$$

We define the new sequence of functions $v_n := u_n(\cdot - \xi_n) \in H^1(\mathbb{R}^3)$. It is easy to see that $\phi_{v_n} = \phi_{u_n}(\cdot - \xi_n)$, and hence $v_n \in \mathcal{M}$. Moreover, by (17), we have that for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\|v_n\|_{H^1(B_\delta)} < \delta \text{ uniformly for } n \geq 1. \tag{18}$$

Since, by (8), $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, certainly there exist a subsequence (likewise labelled) and $\bar{v} \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup \bar{v} \text{ weakly in } H^1(\mathbb{R}^3), \tag{19}$$
$$v_n \to \bar{v} \text{ in } L^s(B), \text{ with } B \subset \mathbb{R}^3, \text{ bounded, and } 1 \leq s < 6. \tag{20}$$

By (18), (19) and (20), we have that, taken $s \in [2,6]$, for any $\delta > 0$ there exists $r > 0$ such that, for any $n \geq 1$ large enough

$$\|v_n - \bar{v}\|_{L^s(\mathbb{R}^3)} \leq \|v_n - \bar{v}\|_{L^s(B_r)} + \|v_n - \bar{v}\|_{L^s(B_\delta)} \leq \delta + C \left( \|v_n\|_{H^1(B_\delta)} + \|\bar{v}\|_{H^1(B_\delta)} \right) \leq (1 + 2C') \delta,$$

where $C > 0$ is the constant of the embedding $H^1(B_\delta) \hookrightarrow L^s(B_\delta)$. We deduce that

$$v_n \to \bar{v} \text{ in } L^s(\mathbb{R}^3), \text{ for any } s \in [2,6]. \tag{21}$$

Since $\phi$ is continuous from $L^{12/5}(\mathbb{R}^3)$ to $D^{1,2}(\mathbb{R}^3)$, from (21) we deduce that

$$\phi_{v_n} \to \phi_{\bar{v}} \text{ in } D^{1,2}(\mathbb{R}^3), \quad \text{as } n \to \infty,$$

$$\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2, \quad \text{as } n \to \infty. \tag{22}$$

Since $(v_n)_n$ is in $\mathcal{M}$, by 2 of Lemma 2.3 $(\|v_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (21) implies that $\bar{v} \neq 0$. Proceeding as in [20, Theorem 3.2, Step 4], by (21) and (22) we can show that $v_n \to \bar{v}$ in $H^1(\mathbb{R}^3)$ so that $\bar{v} \in \mathcal{M}$ and $I(\bar{v}) = c$. By Remark 2.5, we have that $(\bar{v}, \phi_{\bar{v}})$ is a ground state solution of (1). \qed
2.3 The non-constant potential case

In this section we suppose that the potential $V$ satisfies \((V1-3)\) and that $p \in ]3, 5].$

In order to get our result, we will use a very standard device: we will look for a minimizer of the functional \((4)\) restricted to the Nehari manifold $\mathcal{N} = \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \tilde{G}(u) = 0 \},$

where

$$\tilde{G}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 + \phi u u^2 - |u|^{p+1}.$$ 

The following lemma describes some properties of the Nehari manifold $\mathcal{N}$:

**Lemma 2.7.**

1. For any $u \neq 0$ there exists a unique number $\bar{t} > 0$ such that $\bar{t}u \in \mathcal{N}$ and $I(\bar{t}u) = \max_{t \geq 0} I(tu);$ 
2. there exists a positive constant $C$, such that for all $u \in \mathcal{N}$, $\|u\|_{p+1} \geq C;$
3. $\mathcal{N}$ is a $C^1$ manifold.

**Proof** Points 1 and 2 can be proved using standard arguments (see, for example, [19]).

3. Observe that for any $u \in H^1(\mathbb{R}^3)$ we have

$$\tilde{G}(u) = 4I(u) - \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) - \frac{p-3}{p+1} \int_{\mathbb{R}^3} |u|^{p+1},$$

and then, by point 2, for any $u \in \mathcal{N}$ we have

$$\langle \tilde{G}'(u), u \rangle = -2 \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) - (p-3) \int_{\mathbb{R}^3} |u|^{p+1} \leq -C < 0.$$ 

The Nehari manifold $\mathcal{N}$ is a natural constrained for the functional $I$, therefore we are allowed to look for critical points of $I$ restricted to $\mathcal{N}$.

In view of this, we assume the following definition

$$c_V := \inf_{u \in \mathcal{N}} I(u),$$

so that our goal is to find $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = c_V$, by which we would deduce that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of \((1)\).
First we recall some preliminary lemmas which can be obtained by using the same arguments as in [19] (see also [2]).

As a consequence of the Lemma 2.7, we are allowed to define the map
\[ t : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+ \]
such that for any \( u \in H^1(\mathbb{R}^3), u \neq 0 : \)
\[ I(t(u)u) = \max_{t \geq 0} I(tu). \]

**Lemma 2.8.** The following equalities hold
\[ c_V = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) = \inf_{u \neq 0} \max_{t \geq 0} I(tu), \]
where \( \Gamma \) is the same set defined in (6).

**Lemma 2.9.** Let \( u_n \in H^1(\mathbb{R}^3), n \geq 1, \) such that \( \|u_n\| \geq C > 0 \) and
\[ \max_{t \geq 0} I(tu_n) \leq c_V + \delta_n, \]
with \( \delta_n \to 0^+ \). Then, there exist a sequence \((y_n)_n \subset \mathbb{R}^N\) and two positive numbers \( R, \mu > 0 \) such that
\[ \lim \inf_n \int_{B_R(y_n)} |u_n|^2 \, dx > \mu. \]

**Lemma 2.10.** Let \( (u_n)_n \subset H^1(\mathbb{R}^3) \) such that \( \|u_n\| = 1 \) and
\[ I(t(u_n)u_n) = \max_{t \geq 0} I(tu_n) \to c_V, \quad \text{as } n \to \infty. \]
Then the sequence \((t(u_n))_n \subset \mathbb{R}_+\) possesses a bounded subsequence in \( \mathbb{R} \).

**Proof** We have
\[ C \geq \int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2 = t_n^2 \left( n_{p-3} \int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} \phi_{u_n}u_n^2 \right). \]
The conclusion follows from \( i \) of Lemma 2.1 and Lemma 2.9. \( \square \)

**Lemma 2.11.** Suppose that \( V, V_n \) satisfy (V1-2), for all \( n \geq 1. \) If \( V_n \to V \) in \( L^\infty(\mathbb{R}^N) \) then \( c_{V_n} \to c_V. \)

Now define
\[ I_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u}u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}, \]
\[ c_\infty := c_{V_\infty}. \]
As in [19], we have
Lemma 2.12. If \( V \) satisfies (V1-3), we get \( c_V < c_\infty \).

**Proof** By Theorem 1.1, there exists \((w, \phi_w) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) a ground state solution of the problem

\[
\begin{cases}
-\Delta u + V_\infty u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3.
\end{cases}
\]

Let \( t(w) > 0 \) be such that \( t(w)w \in \mathcal{N} \). By (V3), we have

\[
c_\infty = I_\infty(w) \geq I_\infty(t(w)w) = I(t(w)w) + \int_{\mathbb{R}^N} (V_\infty - V(x)) |t(w)w|^2 > c_V,
\]

and then we conclude. \( \square \)

### 2.3.1 Proof of Theorem 1.2

Let \((u_n)\) be a sequence in \( \mathcal{N} \) such that

\[
\lim_n I(u_n) = c_V. \tag{23}
\]

Observe that for any \( u \in \mathcal{N} \) we have

\[
I(u) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \left( \frac{1}{4} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \phi u^2,
\]

hence, by (23), we deduce that \((u_n)\) is bounded in \( H^1(\mathbb{R}^3) \). Let \( \bar{u} \in H^1(\mathbb{R}^3) \) be such that, up to a subsequence,

\[
\begin{align*}
\lim_n u_n & \rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3), \tag{24} \\
\lim_n u_n & \to \bar{u} \quad \text{in } L^s(B), \text{with } B \subset \mathbb{R}^3, \text{bounded, and } 1 \leq s < 6. \tag{25}
\end{align*}
\]

We define the measures

\[
\mu_n(\Omega) = \int_\Omega \left( \frac{1}{2} - \frac{1}{p+1} \right) |\nabla u_n|^2 + V(x)u_n^2 + \left( \frac{1}{4} - \frac{1}{p+1} \right) \phi u_n^2.
\]

Proceeding as in Lemma 2.6, we infer that compactness holds for \((\mu_n)\), namely there exists a sequence \((\xi_n)\) in \( \mathbb{R}^3 \) with the following property: for any \( \delta > 0 \), there exists \( r = r(\delta) > 0 \) such that

\[
\int_{B_{r}(\xi_n)} \left( \frac{1}{2} - \frac{1}{p+1} \right) |\nabla u_n|^2 + V(x)u_n^2 + \left( \frac{1}{4} - \frac{1}{p+1} \right) \phi u_n^2 < \delta. \tag{26}
\]
CLAIM: \((\xi_n)_n\) is bounded in \(\mathbb{R}^3\).

Suppose by contradiction that, up to a subsequence, \(|\xi_n| \to \infty\), as \(n \to \infty\).

Fix \(\hat{V} < V_\infty\) and let \(\hat{I}\) be the functional defined as \(I\) replacing \(V\) by \(\hat{V}\). For any \(n \geq 1\), let \(z_n = u_n(\cdot - \xi_n)\) and \(\hat{t}_n > 0\) such that the functions \(\hat{t}_n z_n\) are in the Nehari manifold of \(\hat{I}\).

Let \(\delta > 0\) and consider \(r > 0\) such that (26) holds. For \(n\) sufficiently large, we have

\[
V(x + \xi_n) - \hat{V} \geq 0, \quad \text{for all } x \in B_r.
\]

Hence we have

\[
c_V + o_n(1) = I(u_n) \geq \hat{I}(\hat{t}_n u_n) + \frac{\hat{t}_n^2}{2} \int_{\mathbb{R}^3} \left( V(x) - \hat{V} \right) u_n^2
\]

\[
\geq c_{\hat{V}} + \frac{\hat{t}_n^2}{2} \int_{B_r} \left( V(x + \xi_n) - \hat{V} \right) z_n^2 + \frac{\hat{t}_n^2}{2} \int_{B_{\hat{r}}} \left( V(x + \xi_n) - \hat{V} \right) z_n^2.
\]

Since by (26)

\[
\int_{B_{\hat{r}}} \left| V(x + \xi_n) - \hat{V} \right| z_n^2 \leq C\delta, \quad \text{for any } n \geq 1,
\]

and \((\hat{t}_n)_n\) is bounded (the proof is the same as in Lemma 2.10), we get that \(c_V \geq c_{\hat{V}} - C\delta\). By the arbitrariness in the choice of \(\delta > 0\), we have \(c_V \geq c_{\hat{V}}\). Using Lemma 2.11 we conclude that \(c_V \geq c_\infty\), which contradicts Lemma 2.12.

So \((\xi_n)_n\) is bounded in \(\mathbb{R}^3\) and then, by (26), for any \(\delta > 0\) there exists \(r > 0\) such that

\[
\|u_n\|_{H^1(B_r)} < \delta, \quad \text{uniformly for } n \geq 1.
\]

(27)

By (24), (25) and (27), we have that, taken \(s \in [2, 6]\), for any \(\delta > 0\) there exists \(r > 0\) such that, for any \(n \geq 1\) large enough

\[
\|u_n - \bar{u}\|_{L^s(\mathbb{R}^3)} \leq \|u_n - \bar{u}\|_{L^s(B_r)} + \|u_n - \bar{u}\|_{L^s(B_{\hat{r}})}
\]

\[
\leq \delta + C (\|u_n\|_{H^1(B_r)} + \|\bar{u}\|_{H^1(B_{\hat{r}})}) \leq (1 + 2C)\delta,
\]

where \(C > 0\) is the constant of the embedding \(H^1(B_r) \hookrightarrow L^s(B_{\hat{r}})\). We deduce that

\[
u_n \to \bar{u} \text{ in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6].
\]

(28)
Since $\phi$ is continuous from $L^{12/5}(\mathbb{R}^3)$ to $D^{1,2}(\mathbb{R}^3)$, from (28) we deduce that
\[
\phi_{u_n} \to \phi_{\bar{u}} \quad \text{in} \quad D^{1,2}(\mathbb{R}^3), \quad \text{as} \quad n \to \infty,
\]
and for any $\psi \in C_0^\infty(\mathbb{R}^3)$
\[
\int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \to \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u} \psi.
\] (30)

By (23), we can suppose (see [24]) that $(u_n)_n$ is a Palais-Smale sequence for $I|_N$ and, as a consequence, that it is easy to see that $(u_n)_n$ is a Palais-Smale sequence for $I$. By (24), (28) and (30), we conclude that $I'(\bar{u}) = 0$.

Since $(u_n)_n$ is in $N$, by 3 of Lemma 2.7 $(\|u_n\|_p)_n$ is bounded below by a positive constant. As a consequence, (28) implies that $\bar{u} \neq 0$ and so $\bar{u} \in N$. Finally, by (24), (28) and (29),
\[
c_V \leq I(\bar{u}) \leq \lim \inf I(u_n) = c_V,
\]
so we can conclude that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of (1).

## 3 The critical case

This section is devoted to the study of the critical case and in particular we will give the proofs of Theorem 1.4, Theorem 1.5 and Theorem 1.6.

### 3.1 The nonexistence result

**Proof of Theorem 1.4** Arguing as in [5, 14], we can prove that if $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of the problem (2), then $(u, \phi)$ satisfies the following Pohozaev identity:
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + 3 \int_{\mathbb{R}^3} V(x) u^2 + \int_{\mathbb{R}^3} (\nabla V(x) \cdot x) u^2 + \frac{5}{2} \int_{\mathbb{R}^3} \phi u^2 = \int_{\mathbb{R}^3} u^6. \quad (31)
\]

Multiplying the first equation of (2) by $u$ and integrating, we have
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(x) u^2 + \int_{\mathbb{R}^3} \phi u^2 = \int_{\mathbb{R}^3} u^6; \quad (32)
\]
on the other hand, multiplying the second equation of (2) by $\phi$ and integrating, we have
\[
\int_{\mathbb{R}^3} |\nabla \phi|^2 = \int_{\mathbb{R}^3} \phi u^2. \quad (33)
\]
By the combination of (31), (32) and (33), we infer that
\[
2 \int_{\mathbb{R}^3} V(x)u^2 + \int_{\mathbb{R}^3} (\nabla V(x) \cdot x)u^2 + \frac{3}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 = 0,
\]
which, together with (V2) and (V4), implies that \( u = \phi = 0 \). \( \square \)

**Remark 3.1.** In fact, the same nonexistence result would hold even if we supposed the weaker hypothesis
\[
0 < C_4 \leq 2V(x) + (\nabla V(x) \cdot x) \leq C_5,
\]
for all \( x \in \mathbb{R}^3 \), in the place of (V4).

### 3.2 The existence results

As in subsection 2.1, for every \( u \in L^{12/5}(\mathbb{R}^3) \) we denote by \( \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \) the unique solution of
\[
-\Delta \phi = u^2, \quad \text{in } \mathbb{R}^3.
\]
It can be proved that \( (u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \) is a solution of (3) if and only if \( u \in H^1(\mathbb{R}^3) \) is a critical point of the functional \( I^*: H^1(\mathbb{R}^3) \to \mathbb{R} \) defined as
\[
I^*(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{q+1} \int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6} \int_{\mathbb{R}^3} u^6,
\]
and \( \phi = \phi_u \).

The Nehari manifold of the functional \( I^* \), defined as
\[
\mathcal{N}^* := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \, \bigg| \, \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{q+1} - u^6 = 0 \right\},
\]
satisfies the equivalent of Lemma 2.7 and so it is a natural constraint for \( I^* \) and we are looking for critical points of \( I^* \) restricted to \( \mathcal{N}^* \).

Set
\[
c_1^* = \inf_{g \in \Gamma^*} \max_{t \in [0,1]} I^*(g(t));
c_2^* = \inf_{u \neq 0} \max_{t \geq 0} I^*(tu);
c_3^* = \inf_{u \in \mathcal{N}^*} I^*(u);
\]
where
\[
\Gamma^* = \left\{ g \in C([0,1], H^1(\mathbb{R}^3)) \mid g(0) = 0, I^*(g(1)) \leq 0, g(1) \neq 0 \right\}.
\]
It is standard to prove that
Lemma 3.2. The following relations hold

\[ c_1^* = c_2^* = c_3^*. \]

We denote by \( S \) the best constant for the Sobolev embedding \( D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), namely

\[ S = \inf_{u \in D^{1,2} \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}. \]

3.2.1 The constant potential case

In this section we suppose that \( V \) is a positive constant. For simplicity we assume \( V \equiv 1 \) and we denote \( c^* = c_1^*. \)

Lemma 3.3. The following inequality holds

\[ c^* < \frac{1}{3} S^\frac{3}{2}. \]

Proof Consider the one parameter Talenti’s functions \( u_\varepsilon \in D^{1,2}(\mathbb{R}^3) \) defined by

\[ u_\varepsilon := C_\varepsilon \frac{\varepsilon^{\frac{2}{3}}}{(\varepsilon + |x|^2)^{\frac{2}{3}}}, \]

where \( C_\varepsilon > 0 \) is a normalizing constant (see [22]). Let \( \varphi \) be a smooth cut off function, namely \( \varphi \in C^\infty_0(\mathbb{R}^3) \) and there exists \( R > 0 \) such that \( \varphi|_{B_R} = 1 \), \( 0 \leq \varphi \leq 1 \) and \( \text{supp} \varphi \subset B_{2R} \). Set \( w_\varepsilon := u_\varepsilon \varphi \) and \( v_\varepsilon = w_\varepsilon / \|w_\varepsilon\|_6 \). Using the estimates obtained in [6] we get

\[ \|\nabla v_\varepsilon\|_2^2 = S + O(\varepsilon^{\frac{1}{2}}), \] (34)

and, for any \( s \in [2, 6[ \),

\[ \|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{1}{s}}), & \text{if } s \in [2, 3[; \\ O(\varepsilon^{\frac{3}{s}} \log(\varepsilon))), & \text{if } s = 3; \\ O(\varepsilon^{\frac{6-s}{s}}), & \text{if } s \in ]3, 6[. \] (35)

For every \( \varepsilon > 0 \) let \( t_\varepsilon > 0 \) such that \( t_\varepsilon v_\varepsilon \in \mathcal{N}^*. \) Obviously \( (t_\varepsilon)_{\varepsilon > 0} \) is bounded below by a positive constant; otherwise there should exist a sequence \( (\varepsilon_n)_n \) such that \( \lim_n t_{\varepsilon_n} = 0 \) and then, by (34), Lemma 2.1 and (35),

\[ 0 < c^* \leq \lim_n I^*(t_{\varepsilon_n} v_{\varepsilon_n}) = 0. \]

CLAIM: For any \( \varepsilon > 0 \) small enough \( t_\varepsilon \leq \left( \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \right)^{1/4}. \)
Let $\gamma_\varepsilon(t) := \mathcal{I}^*(tv_\varepsilon)$ and set $r_\varepsilon := \left( \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \right)^{1/4}$. By (34) and (35), $(r_\varepsilon)_{\varepsilon > 0}$ is bounded below by a positive constant. Since $t_\varepsilon v_\varepsilon \in \mathcal{N}^*$, certainly $\gamma_\varepsilon'(t_\varepsilon) = 0$. On the other hand, by (i) of Lemma 2.1 and (35), for any $\varepsilon$ small enough,

$$
\gamma_\varepsilon'(t) = tr_\varepsilon^4 - t^5 + t^3 \int_{\mathbb{R}^3} \phi_\varepsilon v_\varepsilon^2 - t q \| v_\varepsilon \|_{q+1}^{q+1} \\
\leq tr_\varepsilon^4 - t^5 + C q t^3 \| v_\varepsilon \|_{q+1}^{q+1} - t q \| v_\varepsilon \|_{q+1}^{q+1} \\
= tr_\varepsilon^4 - t^5 + t^3 \left( C' O(\varepsilon) - t q - 3 O(\varepsilon^{2/3}) \right),
$$

where $O(\varepsilon)$ and $O(\varepsilon^{2/3})$ are nonnegative functions. We deduce that, for any $\varepsilon > 0$ small enough, $\gamma_\varepsilon'(t) < 0$ in $[r_\varepsilon, +\infty[$: the claim follows as a consequence.

Now, since the function

$$
t \in \mathbb{R}_+ \mapsto \frac{1}{2} t^2 r_\varepsilon^4 - \frac{1}{6} t^6
$$

is increasing in the interval $[0, r_\varepsilon]$, by (34) and (i) of Lemma 2.1 we have that

$$
\mathcal{I}^*(t_\varepsilon v_\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 + \frac{t_\varepsilon^4}{4} \int_{\mathbb{R}^3} \phi_\varepsilon v_\varepsilon^2 - \frac{t_\varepsilon^{q+1}}{q+1} \int_{\mathbb{R}^3} |v_\varepsilon|^{q+1} - \frac{t_\varepsilon^6}{6} \\
\leq \frac{1}{3} \left( \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \right)^{3/2} + C' \frac{t_\varepsilon^4}{q+1} \| v_\varepsilon \|_{q+1}^{4/3} - \frac{t_\varepsilon^{q+1}}{q+1} \| v_\varepsilon \|_{q+1}^{q+1} \\
= \frac{1}{3} \left( S + O(\varepsilon^{1/2}) + \int_{\mathbb{R}^3} v_\varepsilon^2 \right)^{3/2} + C' \frac{t_\varepsilon^4}{q+1} \| v_\varepsilon \|_{q+1}^{4/3} - \frac{t_\varepsilon^{q+1}}{q+1} \| v_\varepsilon \|_{q+1}^{q+1}.
$$

Using the inequality $(a+b)^4 \leq a^\delta + \delta(a+b)^{\delta-1}b$ which holds for any $\delta \geq 1$ and $a, b \geq 0$, by (35) and the previous chain of inequalities we get

$$
\mathcal{I}^*(t_\varepsilon v_\varepsilon) \leq \frac{1}{3} S^{3/2} + O(\varepsilon^{3/2}) + C_1(\varepsilon)O(\varepsilon) - C_2(\varepsilon)O(\varepsilon^{5/3}), \quad \text{(36)}
$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are in an interval $[\alpha, \beta]$ with $\alpha > 0$. Since $q > 3$, the conclusion follows from (36), for $\varepsilon > 0$ small enough. □

**Proof of Theorem 1.5** Let $(u_n)_n \subset \mathcal{N}^*$ such that

$$
\lim_n \mathcal{I}^*(u_n) = c^*.
$$

(37)
We easily deduce that \((u_n)_n\) is bounded in \(H^1(\mathbb{R}^3)\), so there exists \(\bar{u} \in H^1(\mathbb{R}^3)\) such that, up to a subsequence,
\[
\begin{align*}
  u_n &\rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3), \\
  u_n &\to \bar{u} \quad \text{in } L^s(B), \text{ with } B \subset \mathbb{R}^3 \text{, bounded, and } 1 \leq s < 6.
\end{align*}
\]
As in the first part of the paper, we use a concentration-compactness argument on the sequence of positive measures
\[
\mu_n^*(\Omega) = \left( \frac{1}{2} - \frac{1}{q + 1} \right) \int_\Omega |\nabla u_n|^2 + u_n^2 + \left( \frac{1}{4} - \frac{1}{q + 1} \right) \int_\Omega \phi u_n u_n^2 \\
+ \left( \frac{1}{q + 1} - \frac{1}{6} \right) \int_\Omega u_n^6.
\]
We define the functional \(J^*: H^1(\mathbb{R}^3) \to \mathbb{R}\) as:
\[
J^*(u) = \left( \frac{1}{2} - \frac{1}{q + 1} \right) \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 + \left( \frac{1}{4} - \frac{1}{q + 1} \right) \int_{\mathbb{R}^3} \phi u u^2 \\
+ \left( \frac{1}{q + 1} - \frac{1}{6} \right) \int_{\mathbb{R}^3} u^6.
\]

**Vanishing does not occur**
Suppose by contradiction, that for all \(r > 0\)
\[
\lim_{n} \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} d\mu_n^* = 0.
\]
By [17] we deduce that \(u_n \to 0\) in \(L^s(\mathbb{R}^3)\) for any \(s \in ]2, 6[\).
By \(i\) of Lemma 2.1, since \((u_n)_n \subset \mathcal{N}^*\), it follows that
\[
\lim_{n} \left[ \int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2 - \int_{\mathbb{R}^3} u_n^6 \right] = 0.
\]
By the boundedness of \((u_n)_n\) in \(H^1(\mathbb{R}^3)\), we infer that there exists \(l > 0\) such that, up to subsequence,
\[
l := \lim_{n} \int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2 = \lim_{n} \int_{\mathbb{R}^3} u_n^6.
\]
We have
\[
c^* = \lim_{n} I^*(u_n) = \frac{1}{2} l - \frac{1}{6} l = \frac{1}{3} l
\]
Ground state solutions to the nonlinear Schrödinger-Maxwell equations

and

\[ S \leq \frac{\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2}{(\int_{\mathbb{R}^3} u_n^6)^{\frac{2}{3}}} \rightarrow I^\sharp. \quad (40) \]

By (39) and (40) we get \( c^* = \frac{1}{3} l \geq \frac{1}{3} S^{\frac{3}{2}}, \) contradicting 2 of Lemma 3.3.

DICHOTOMY DOES NOT OCCUR
The proof uses similar arguments as those in the proof of Theorem 1.1.

So the measures \( \mu_n^* \) concentrate and, in particular, we have that there exists a sequence \( (\xi_n)_n \) in \( \mathbb{R}^N \) such that for any \( \delta > 0 \) there exists \( r = r(\delta) > 0 \) such that

\[ \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{B_r(\xi_n)} |\nabla u_n|^2 + u_n^2 < \delta. \quad (41) \]

From now on, we only give a sketch of the remaining part of the proof, since it is similar to that of the subcritical case. We define \( v_n := u_n(\cdot - \xi_n). \)

It is easy to see that \( (v_n)_n \subset N^*. \) From (41) we have that for any \( \delta > 0 \) there exists \( r > 0 \) such that

\[ \| v_n \|_{H^1(B_r^c)} < \delta, \quad \text{uniformly for } n \geq 1. \]

Hence we deduce

\[ v_n \to \bar{v} \text{ in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6]; \]

\[ \phi_{v_n} \to \phi_{\bar{v}} \text{ in } D^{1,2}(\mathbb{R}^3); \]

\[ \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2. \quad (43) \]

Moreover, for any \( \psi \in C_0^\infty(\mathbb{R}^3), \)

\[ \int_{\mathbb{R}^3} \phi_{v_n} v_n \psi \to \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v} \psi; \]

and, by (42),

\[ \int_{\mathbb{R}^3} v_n^5 \psi \to \int_{\mathbb{R}^3} \bar{v}^5 \psi. \]

By (37), we can suppose (see [24]) that \( (v_n)_n \) is a Palais-Smale sequence for \( I^*|_{N^*}, \) and, consequently, it is a Palais-Smale sequence for \( I^*. \) By standard arguments, we infer that \( \bar{v} \in N^*. \)
Finally, since \((v_n)_n\) and \(\bar{v}\) are in \(\mathcal{N}^*\), we have that

\[
I^*(\bar{v}) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla \bar{v}|^2 + \bar{v}^2 + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2 + \left( \frac{1}{6} - \frac{1}{q+1} \right) \int_{\mathbb{R}^3} |\bar{v}|^{q+1},
\]

\[
I^*(v_n) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla v_n|^2 + v_n^2 + \frac{1}{12} \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \left( \frac{1}{6} - \frac{1}{q+1} \right) \int_{\mathbb{R}^3} |v_n|^{q+1},
\]

so, by (37), (38), (42) and (43),

\[
c^* \leq I^*(\bar{v}) \leq \liminf I^*(v_n) = c^*.
\]

We conclude that \((\bar{v}, \phi_{\bar{v}})\) is a ground state solution of (3). □

### 3.2.2 The non-constant potential case

In this section we suppose that \(V\) satisfies hypotheses \((V1-3)\).

We define the functional \(I^*_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}\) and the Nehari manifold \(\mathcal{N}^*_\infty\) in the following way

\[
I^*_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi u u^2 - \frac{1}{q+1} \int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6} \int_{\mathbb{R}^3} u^6,
\]

\[
\mathcal{N}^*_\infty := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \phi u u^2 - |u|^{q+1} - u^6 = 0 \right\}.
\]

We set

\[
c^*_\infty = \inf_{u \in \mathcal{N}^*_\infty} I^*_\infty(u).
\]

**Lemma 3.4.** The following inequality holds

\[
c^*_V < \frac{1}{3} S^2.\]

**Proof.** By Theorem 1.5, there exists a ground state solution for (3) whenever \(V \equiv V_\infty\); so, arguing as in Lemma 2.12, we can show that \(c^*_V < c^*_\infty\). Therefore, the inequality follows by Lemma 3.3. □

Following [19], by Lemmas 3.3 and 3.4 and using a non-vanishing type argument as in the proof of Theorem 1.2, we can show that the corresponding versions of Lemmas 2.9, 2.10 and 2.11 hold for the functional \(I^*\).

**Proof of Theorem 1.6.** Let \((u_n)_n \subset \mathcal{N}^*\) such that

\[
\lim_{n} I^*(u_n) = c^*_V.
\]
We easily deduce that \((u_n)_n\) is bounded in \(H^1(\mathbb{R}^3)\), so there exists \(\bar{u} \in H^1(\mathbb{R}^3)\) such that, up to a subsequence,
\[
u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3).
\]

Let us define the positive measure
\[
\mu_n^*(\Omega) = \left(\frac{1}{2} - \frac{1}{q + 1}\right) \int_{\Omega} |\nabla u_n|^2 + V(x)u_n^2 + \left(\frac{1}{4} - \frac{1}{q + 1}\right) \int_{\Omega} \phi u_n u_n^2 + \left(\frac{1}{q + 1} - \frac{1}{6}\right) \int_{\mathbb{R}^3} u_n^6.
\]

Arguing as in the proof of Theorem 1.5, we can prove that the measures \(\mu_n^*\) concentrate and, in particular, we have that there exists a sequence \((\xi_n)_n\) in \(\mathbb{R}^N\) such that for any \(\delta > 0\) there exists \(r = r(\delta) > 0\) such that
\[
\left(\frac{1}{2} - \frac{1}{q + 1}\right) \int_{B_r(\xi_n)} |\nabla u_n|^2 + V(x)u_n^2 < \delta.
\] (44)

With the arguments similar to those of the subcritical case, we show that the sequence \((\xi_n)_n\) is bounded in \(\mathbb{R}^3\) and then by (44) we have that for any \(\delta > 0\) there exists \(r > 0\) such that
\[
\|u_n\|_{H^1(B_r)} < \delta, \quad \text{uniformly for } n \geq 1.
\]

Hence we deduce
\[
u_n \to \bar{u} \quad \text{in } L^s(\mathbb{R}^3), \quad \text{for any } s \in [2, 6].
\]

Then, arguing as in the proof of Theorem 1.5, we get the conclusion. \(\square\)

References

[1] A. Ambrosetti, D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*, preprint.

[2] A. Azzollini, A. Pomponio, *On a “zero mass” nonlinear Schrödinger equation*, preprint.

[3] V. Benci, D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal., 11 (1998), 283–293.
[4] V. Benci, D. Fortunato, A. Masiello, L. Pisani, *Solitons and the electromagnetic field*, Math. Z., **232**, (1999), 73–102.

[5] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., **82**, (1983), 313–345.

[6] H. Brezis, L. Niremberg, *Positive solutions of nonlinear elliptic problems involving critical Sobolev exponent*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[7] A.M. Candela, A. Salvatore, *Multiple solitary waves for non-homogeneous Schrödinger-Maxwell equations*, Mediterr. J. Math., **3**, (2006), 483–493.

[8] D. Cassani, *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell’s equations*, Nonlinear Anal., **58** (7-8) (2004), 733–747.

[9] G.M. Coclite, *A multiplicity result for the linear Schrödinger-Maxwell equations with negative potential*, Ann. Polon. Math., **79**, (2002), 21–30.

[10] G.M. Coclite, *A multiplicity result for the nonlinear Schrödinger-Maxwell equations*, Commun. Appl. Anal., **7**, (2003), 417–423.

[11] G.M. Coclite, V. Georgiev, *Solitary waves for Maxwell-Schrödinger equations*, Electron. J. Differential Equations 2004, No. 94, 31 pp. (electronic).

[12] S. Coleman, V. Glaser, A. Martin, *Action minima among solutions to a class of euclidean scalar field equations*, Commun. math. Phys., **58**, (1978), 211–221.

[13] T. D’Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, **134**, (2004), 893–906.

[14] T. D’Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud., **4**, (2004), 307–322.

[15] P. d’Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud., **2**, (2002), 177–192.
[16] P.L. Lions, *The concentration-compactness principle in the calculus of variation. The locally compact case. Part I*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1, (1984), 109–145.

[17] P.L. Lions, *The concentration-compactness principle in the calculus of variation. The locally compact case. Part II*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1, (1984), 223–283.

[18] L. Pisani, G. Siciliano, *Neumann condition in the Schrödinger-Maxwell system*, preprint.

[19] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., 43, (1992), 270–291.

[20] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, Journ. Func. Anal., 237, (2006), 655–674.

[21] A. Salvatore, *Multiple solitary waves for a non-homogeneous Schrödinger-Maxwell system in \( \mathbb{R}^3 \)*, Adv. Nonlinear Stud., 6, (2006), 157–169.

[22] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., 110, (1976), 353–372.

[23] X.F. Wang, B. Zeng, *On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal., 28, (1997), 633–655.

[24] M. Willem, *Minimax Theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.