Periods of tropical K3 hypersurfaces

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Abstract

Let $\Delta$ be a smooth reflexive polytope in dimension 3 and $f$ be a tropical polynomial whose Newton polytope is the polar dual of $\Delta$. One can construct a 2-sphere $B$ equipped with an integral affine structure with singularities by contracting the tropical K3 hypersurface defined by $f$. We write the complement of the singularity as $\iota:\mathcal{B}_0\hookrightarrow B$, and the local system of integral tangent vectors on $\mathcal{B}_0$ as $\mathcal{T}_\mathcal{B}$. Let further $Y$ be an anti-canonical hypersurface of the toric variety associated with the normal fan of $\Delta$, and $\text{Pic}(Y)_{\text{amb}}$ be the sublattice of the Picard group of $Y$ coming from the ambient space. In this article, we give a primitive embedding $\text{Pic}(Y)_{\text{amb}}\hookrightarrow H^1(B,\iota_*\mathcal{T}_\mathcal{B})$ that preserves the pairing, and compute the radiance obstruction of $B$, which sits in the subspace generated by the image of $\text{Pic}(Y)_{\text{amb}}$.

1 Introduction

Let $M$ be a free $\mathbb{Z}$-module of rank 3 and $N:=\text{Hom}(M,\mathbb{Z})$ be the dual lattice. We set $M_\mathbb{R}:=M\otimes_\mathbb{Z}\mathbb{R}$ and $N_\mathbb{R}:=N\otimes_\mathbb{Z}\mathbb{R}=\text{Hom}(M,\mathbb{R})$. Let $\Delta \subset M_\mathbb{R}$ be a smooth reflexive polytope of dimension 3, and $\hat{\Delta}\subset N_\mathbb{R}$ be the polar polytope of $\Delta$. Let further $\Sigma$ and $\hat{\Sigma}$ be the normal fans to $\Delta$ and $\hat{\Delta}$ respectively. We choose a refinement $\hat{\Sigma}'\subset M_\mathbb{R}$ of $\hat{\Sigma}$ such that the primitive generator of any 1-dimensional cone in $\hat{\Sigma}'$ is contained in $\Delta\cap M$.

Let $A \subset N$ denote the subset consisting of all vertices of $\Delta$ and $0\in N$. We consider a tropical Laurent polynomial

$$f(x) = \max_{n \in A} \{a(n) + n_1x_1 + n_2x_2 + n_3x_3\}, \quad (1.1)$$

such that the function

$$A \rightarrow \mathbb{R}, \quad n \mapsto a(n) \quad (1.2)$$

induces a central subdivision of $\hat{\Delta}$, i.e., every maximal dimensional simplex of the subdivision has the origin $0\in N$ as its vertex. Let $V(f)$ be the tropical hypersurface defined by $f$ in the tropical toric variety associated with $\hat{\Sigma}'$. See Section 3.1 for the definition of tropical toric varieties.

We can construct a 2-sphere $B$ equipped with an integral affine structure with singularities by contracting $V(f)$. See Section 4 for details about the construction. The same construction has already been performed in Gross–Siebert program [Gro05], [GS06]. There is also another construction by Haase and Zharkov, which was discovered independently [HZ02]. It is also known that maximally degenerating families of complex K3 surfaces with Ricci-flat Kähler metrics converge to 2-spheres with integral affine structures with singularities in the Gromov–Hausdorff limit [GW00].
Let \( i : B_0 \rightarrow B \) denote the complement of singularities of \( B \). Let further \( T_\mathbb{Z} \) be the local system on \( B_0 \) of integral tangent vectors. The cohomology group \( H^1(B, i_* T_\mathbb{Z}) \) has the cup product
\[
\cup : H^1(B, i_* T_\mathbb{Z}) \otimes H^1(B, i_* T_\mathbb{Z}) \rightarrow H^2(B, i_* \wedge^2 T_\mathbb{Z}) \cong \mathbb{Z} \tag{1.3}
\]
induced by the wedge product, where the last isomorphism is determined by choosing an orientation of \( B \). Let \( Y \) be an anti-canonical hypersurface of the complex toric variety \( X_\Sigma \) associated with \( \Sigma \), and

\[
\text{Pic}(Y)_{\text{amb}} := \text{Im} (\text{Pic}(X_\Sigma) \hookrightarrow \text{Pic}(Y)) \tag{1.4}
\]

be the sublattice of \( \text{Pic}(Y) \) coming from the Picard group of the ambient space. The first main result of this paper is the following:

**Theorem 1.1.** There is a primitive embedding

\[
\psi : \text{Pic}(Y)_{\text{amb}} \hookrightarrow H^1(B, i_* T_\mathbb{Z}), \tag{1.5}
\]

that preserves the pairing.

We also compute the radiance obstruction of \( B \). Radiance obstructions are invariants of the integral affine manifolds, which were introduced in [GH84]. See Section 2 for its definition. Let \( \Sigma(1) \) be the set of 1-dimensional cones of \( \Sigma \) and \( D_\rho \) be the restriction to \( Y \) of the toric divisor on \( X_\Sigma \) corresponding to \( \rho \in \Sigma(1) \). There is a one-to-one correspondence between \( \Sigma(1) \) and the set of vertices of \( \Delta \). We write the vertex of \( \Delta \) corresponding to \( \rho \in \Sigma(1) \) as \( n_\rho \in \Delta \cap N \).

**Theorem 1.2.** The radiance obstruction \( c_B \) of \( B \) is given by

\[
c_B = \sum_{\rho \in \Sigma(1)} \{ a(n_\rho) - a(0) \} \psi(D_\rho). \tag{1.6}
\]

The period map is approximated by Schmid’s nilpotent orbit [Sch73] in the limit to the degeneration point. The leading term of the nilpotent orbit is determined by the monodromy around the degeneration point. It was also shown in [GS10] that the wedge product of the radiance obstruction corresponds to the monodromy operator around the degeneration point in the case of Calabi–Yau varieties. Hence, we can see that the radiance obstruction gives the leading term of the period map of the corresponding family of Calabi–Yau varieties.

The organization of this paper is as follows: In Section 2, we recall the definitions of integral affine manifolds and their radiance obstructions. We also recall the definition of integral affine manifolds with singularities and we define their radiance obstructions. In Section 3, we recall some notions of tropical geometry, such as tropical toric varieties and tropical modifications. In Section 4, we explain the details about how to construct integral affine spheres with
singularities from tropical K3 hypersurfaces. In Section 5, we discuss how dispersing singular points piling at one point affects the cohomology group $H^1(B, \iota_* \mathcal{T}_Z)$ and the radiance obstruction $c_B$. The results in Section 5 will be used for proofs of the main theorems. In Section 6, we give proofs of Theorem 1.1 and Theorem 1.2. In Section 7, we discuss the relation with asymptotic behaviors of period maps of complex K3 hypersurfaces.

Acknowledgment: I am most grateful to my advisor Kazushi Ueda for his encouragement and helpful advice. In particular, the problem of computing radiance obstructions of tropical K3 hypersurfaces was suggested by him. Most parts of this work was done during a visit to Yale University. I thank Sam Payne for his encouragement on this project, valuable comments on an earlier draft of this paper, and the financial support for the visit. I also stayed at University of Geneva and attended the exchange student program “Master Class in Geometry, Topology and Physics” by NCCR SwissMAP in the academic year 2016/2017. During the stay, I learned a lot about tropical geometry from Grigory Mikhalkin. I thank him for many helpful discussions and sharing his insights and intuitions. Many ideas of Section 4 of this article stem from discussions with him. I also thank NCCR SwissMAP for the financial support and the opportunity to study at University of Geneva. I am also grateful to Mark Gross for letting me know the work by Haase and Zharkov [HZ02]. This research was supported by Grant-in-Aid for JSPS Research Fellow (18J11281) and the Program for Leading Graduate Schools, MEXT, Japan.

2 Integral affine structures with singularities and radiance obstructions

Let $M$ be a free $\mathbb{Z}$-module of rank $n$ and $N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be the dual lattice of $M$. We set $M_\mathbb{R} := M \otimes \mathbb{R}$, $N_\mathbb{R} := N \otimes \mathbb{R} = \text{Hom}_\mathbb{Z}(M, \mathbb{R})$, and $\text{Aff}(M_\mathbb{R}) := M_\mathbb{R} \rtimes \text{GL}(M)$.

Definition 2.1. An integral affine manifold is a real topological manifold $B$ with an atlas of coordinate charts $\psi_i: U_i \to M_\mathbb{R}$ such that all transition functions $\psi_i \circ \psi_j^{-1}$ are contained in $\text{Aff}(M_\mathbb{R})$.

Let $B$ be an integral affine manifold. We give an affine bundle structure to the tangent bundle $TB$ of $B$ as follows: For each $U_i$ and $x \in U_i$, we set an affine isomorphism

$$\theta_{i,x}: T_x B \to M_\mathbb{R}, \quad v \mapsto \psi_i(x) + d\psi_i(x)v,$$

(2.1)

and define an affine trivializations by

$$\theta_i: TU_i \to U_i \times M_\mathbb{R}, \quad (x, v) \mapsto (x, \theta_{i,x}(v)),$$

(2.2)

where $v \in T_x B$. This gives an affine bundle structure to $TB$. We write $TB$ with this affine bundle structure as $T^{\text{aff}}B$.

Let $\mathcal{T}_Z$ be the local system on $B$ of integral tangent vectors. We set $\mathcal{T} := \mathcal{T}_Z \otimes \mathbb{Z} \mathbb{R}$.

Definition 2.2. We choose an sufficiently fine open covering $\mathcal{U} := \{U_i\}_i$ of $B$ so that there is a flat section $s_i \in \Gamma(U_i, T^{\text{aff}}B)$ for each $U_i$. When we set $c_B((U_i, U_j)) := s_j - s_i$ for each 1-simplex $(U_i, U_j)$ of $\mathcal{U}$, the element $c_B$ becomes a Čech 1-cocycle for $\mathcal{T}$. We call $c_B \in H^1(B, \mathcal{T})$ the radiance obstruction of $B$. 

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**Definition 2.3.** An integral affine manifold with singularities is a topological manifold $B$ with an integral affine structure on $B_0 := B \setminus \Gamma$, where $\Gamma \subset B$ is a locally finite union of locally closed submanifolds of codimension greater than 2. We call $\Gamma$ the singular locus of $B$.

In this article, we assume that integral affine manifolds with singularities satisfy the following condition. This was mentioned in [KS06, Section 3.1] as the fixed point property.

**Condition 2.4.** For any $x \in \Gamma$, there is a small neighborhood $U$ such that for each connected component $(U \setminus \Gamma)_i$ of $U \setminus \Gamma$, the monodromy representation $\pi_1((U \setminus \Gamma)_i) \to \text{Aff}(M,\mathbb{R})$ has a fixed vector.

Let $B$ be an integral affine manifold with singularities satisfying the above condition. We write the complement of the singular locus as $\iota: B_0 \hookrightarrow B$. Let further $T_Z$ be the local system on $B_0$ of integral tangent vectors. We set $T := T_Z \otimes \mathbb{R}$ again.

**Definition 2.5.** We choose a sufficiently fine covering $\{U_i\}_i$ of $B$ so that there is a flat section $s_i \in \Gamma(U_i \cap B_0, \text{aff} B_0)$ for each $U_i$. This is possible as long as we assume Condition 2.4. When we set $c_B((U_i, U_j)) := s_j - s_i$, the element $c_B$ becomes a Čech 1-cocycle for $\iota_* T$. We call $c_B \in H^1(B, \iota_* T)$ the radiance obstruction of $B$.

**Remark 2.6.** The inclusion $\iota: B_0 \hookrightarrow B$ induces a map $\iota^*: H^1(B, \iota_* T) \hookrightarrow H^1(B_0, T)$. Then we can see $\iota^*(c_B) = c_{B_0}$ from the definitions.

### 3 Tropical geometry

#### 3.1 Tropical toric varieties

Let $M$ be a free $\mathbb{Z}$-module of rank $n$ and $N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be the dual lattice of $M$. We set $M_\mathbb{R} := M \otimes \mathbb{R}$ and $N_\mathbb{R} := N \otimes \mathbb{R} = \text{Hom}_\mathbb{Z}(M, \mathbb{R})$. We have a canonical $\mathbb{R}$-bilinear pairing

$$\langle -, - \rangle: M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}. \quad (3.1)$$

For each cone $\sigma \in \mathcal{F}$, we set

$$\sigma^\vee := \{ m \in M_\mathbb{R} \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma \}, \quad (3.2)$$

$$\sigma^\perp := \{ m \in M_\mathbb{R} \mid \langle m, n \rangle = 0 \text{ for all } n \in \sigma \}. \quad (3.3)$$

Let $\Sigma$ be a fan in $N_\mathbb{R}$. We define the toric variety $X_\Sigma(\mathbb{T})$ associated with $\Sigma$ over $\mathbb{T}$ as follows: For each cone $\sigma \in \Sigma$, we define $X_\sigma$ as the set of monoid homomorphisms $\sigma^\vee \cap M \rightarrow (\mathbb{T}, \cdot)$

$$X_\sigma(\mathbb{T}) := \text{Hom}(\sigma^\vee \cap M, \mathbb{T}) \quad (3.4)$$

with the compact open topology. For cones $\sigma, \tau \in \Sigma$ such that $\sigma \prec \tau$, we have a natural immersion,

$$X_\sigma(\mathbb{T}) \rightarrow X_\tau(\mathbb{T}), \quad (v: \sigma^\vee \cap M \rightarrow \mathbb{T}) \mapsto (\tau^\vee \cap M \subset \sigma^\vee \cap M \xrightarrow{u} \mathbb{T}), \quad (3.5)$$

where $\sigma \prec \tau$ means that $\sigma$ is a face of $\tau$. By gluing $\{X_\sigma(\mathbb{T})\}_{\sigma \in \Sigma}$ together, we obtain the tropical toric variety $X_\Sigma(\mathbb{T})$ associated with $\Sigma$,

$$X_\Sigma(\mathbb{T}) := \left( \prod_{\sigma \in \Sigma} X_\sigma \right) / \sim. \quad (3.6)$$
We also define the torus orbit $O_\sigma$ corresponding to $\sigma$ by
\[
O_\sigma(\mathbb{T}) := \text{Hom}(\sigma^\perp \cap M, \mathbb{R}).
\] (3.7)

There is a projection map to the torus orbit
\[
p_\sigma : X_\sigma(\mathbb{T}) \to O_\sigma(\mathbb{T}), \quad (w : \sigma^\vee \cap M \to \mathbb{T}) \mapsto (\sigma^\perp \cap M \subset \sigma^\vee \cap M \xrightarrow{w} \mathbb{T}).
\] (3.8)

See [Pay09] or [Kaj08] for more details about tropical toric varieties.

### 3.2 Tropical modifications

Tropical modifications are first introduced in [Mik06]. We briefly recall the idea of it. Let $(\mathbb{T}, +, \cdot)$ be the tropical semifield, where $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and
\[
a + b := \max\{a, b\}, \quad a \cdot b := a + b,
\] (3.9) (3.10)

and $(\mathbb{T}, \oplus, \odot)$ be the tropical hyperfield, where
\[
a \oplus b := \begin{cases} 
\max\{a, b\}, & a \neq b, \\
\{t \in \mathbb{T} \mid t \leq a\}, & a = b,
\end{cases}
\] (3.11)
\[
a \odot b := a + b,
\] (3.12)

where the additions $+$ in right hand sides of (3.10) and (3.12) mean the usual addition. In this subsection, all additions $+$ and multiplications $\cdot$ mean max and $+$ respectively in the following unless otherwise mentioned.

Let $A \subset \mathbb{Z}^n$ be a finite subset. We consider a tropical polynomial
\[
f(x) = \bigoplus_{n \in A} a_n \odot x^n,
\] (3.13)

which is a multi-valued function on $\mathbb{T}^n$ defined by
\[
f(x) := \begin{cases} 
\sum_{n \in A} a_n x^n = a_{n_0} x^{n_0}, & \text{when } \exists n_0 \in A \text{ s.t. } a_{n_0} x^{n_0} > a_n x^n (\forall n \neq n_0), \\
\{t \in \mathbb{T} \mid t \leq \sum_{n \in A} a_n x^n\}, & \text{otherwise}.
\end{cases}
\] (3.14)

We consider the graph $\Gamma_f \subset \mathbb{T}^{n+1}$ of the function $f$
\[
\Gamma_f := \{(x, y) \in \mathbb{T}^{n+1} \mid y \in f(x)\}.
\] (3.15)

This coincides with the bend locus of
\[
f'(x, y) := y + \sum_{n \in A} a_n x^n
\] (3.16)
in $\mathbb{T}^{n+1}$ and has a natural balanced polyhedral structure. Let $\delta_f : \Gamma_f \to \mathbb{T}^n$ be the projection forgetting the last component.

**Definition 3.1.** We call the balanced polyhedral complex $\Gamma_f$ the **tropical modification** of $\mathbb{T}^n$ with respect to $f$. We also call the map $\delta_f : \Gamma_f \to \mathbb{T}^n$ the **contraction** with respect to $f$. 

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The graph of $\sum_{n \in A} a_n x^n$ is isomorphic to $\mathbb{T}^n$ as sets. Hence, we can think that associating $\Gamma_f$ with $\mathbb{T}^n$ corresponds to replacing $+$ and $\cdot$ of $\sum_{n \in A} a_n x^n$ with $\oplus$ and $\odot$ respectively.

We can also define tropical modifications of general tropical varieties in affine spaces. We define tropical modifications of tropical manifolds that are not necessarily embedded in ambient spaces as maps between tropical manifolds which locally coincide with a tropical modification of an affine tropical variety. For a tropical manifold $X$, we regard a tropical manifold $X'$ which relates to $X$ by tropical modifications as a tropical manifold that is equivalent to $X$. We refer the reader to [Sha15] or [Kal15] for details.

4 Contractions of tropical hypersurfaces

In this section, all additions $+$ and multiplications $\cdot$ mean $\max$ and $+$ respectively unless otherwise mentioned. We also let $M, N$ denote a free $\mathbb{Z}$-module of rank 3 and its dual lattice respectively.

4.1 A construction of focus-focus singularities

We fix basis vectors $e_1, e_2, e_3$ of $N$. Consider the cone $\sigma_k$ generated by $ke_1 + e_2, e_2 \in N\mathbb{R}$, where $k$ is some positive integer. Then the dual cone $\sigma_k^\vee$ is generated by $e_1^*, -e_1^* + ke_2^*, \pm e_3^*$ and we have

$$X_{\sigma_k}(\mathbb{T}) := \text{Hom}(\sigma_k^\vee \cap M, \mathbb{T})$$

$$= \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = z^k\}. \quad (4.1)$$

We define the space $X_{k,l}$ by

$$X_{k,l} := \{(x, y, z, w) \in X_{\sigma_k}(\mathbb{T}) \mid z = 0 + w^l\}, \quad (4.3)$$

where $l$ is also some positive integer. This space consists of two 2-dimensional faces

$$F_+ := \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = w^k, z = w^l, w > 0\}, \quad (4.4)$$

$$F_- := \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = z = 0 > w\}, \quad (4.5)$$

and a 1-dimensional face

$$L := \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = z = w = 0\}. \quad (4.6)$$

Each of these faces has an integral affine structure induced from the ambient space $\mathbb{T}^3 \times \mathbb{R}$. We extend them and construct an integral affine structure with a singular point on $X_{k,l}$ as follows:

First, we choose a point $p = (x_0, y_0, 0, 0) \in L$. We set

$$U_x := X_{k,l} \setminus \{(x, y, z, w) \in L \mid x \geq x_0\}, \quad (4.7)$$

$$U_y := X_{k,l} \setminus \{(x, y, z, w) \in L \mid x \leq x_0\}. \quad (4.8)$$

These give a covering of $X_{k,l} \setminus \{p\}$. Consider projections

$$p_x : U_x \to \mathbb{R}^2, \quad (x, y, z, w) \mapsto (x, w), \quad (4.9)$$

$$p_y : U_y \to \mathbb{R}^2, \quad (x, y, z, w) \mapsto (y, w). \quad (4.10)$$
The restrictions of $p_x$ and $p_y$ to $F_\pm$ are integral affine isomorphisms onto their images. Hence, we can extend the integral affine structures on $F_\pm$ to $U_x$ and $U_y$ so that projections $p_x$ and $p_y$ are integral affine isomorphisms onto their images. Here we have $U_x \cap U_y = F_+ \cup F_-$ and the integral affine structures on $U_x$ and $U_y$ coincide on $F_+$ and $F_-$ with each other. Hence, we can extend the integral affine structures on $U_x$ and $U_y$ to an integral affine structure on $X_{k,l} \setminus \{p\}$. We can easily calculate the monodromy of the integral affine structure around $p$.

**Lemma 4.1.** Consider a loop around the point $p$, which starts from a point in $U_x$, passes through $F_-, U_y,$ and $F_+$ in this order, and comes back to the original point. The monodromy along this loop is given by the matrix

$$
\begin{pmatrix}
1 & kl \\
0 & 1
\end{pmatrix},
$$

under the basis $e_x, e_w$ corresponding to the coordinate $(x, w)$ of $U_x$. The point $p$ is a concentration of $kl$ focus-focus singularities at one point.

**Proof.** A point $(x, w) = (x_0, w_0)$ of $U_x$ is shown as $(x, y, z, w) = (x_0, x_0^{-1}, 0, w_0)$ in $F_-$. If we see this in $U_y$, we have $(y, w) = (x_0^{-1}, w_0)$. This is shown as $(x, y, z, w) = (x_0 w^{kl}, x_0^{-1}, w_0, w_0)$ in $F_+$. If we see this in $U_x$ again, we get $(x, w) = (x_0 w^{kl}, w_0)$. Hence, the monodromy transformation is given by $e_x \mapsto e_x, e_w \mapsto (kl)e_x + e_w$. \qed

**Remark 4.2.** The monodromy invariant subspace of $p$ coincides with the tangent space of $L$.

**Remark 4.3.** In the above construction of $X_{k,l}$, there is an ambiguity in the choice of the position of $p \in L$.

**Remark 4.4.** When $k = l = 1$, the space $X_{1,1}$ is an integral affine surface with a focus-focus singularity. Here we have

$$
X_{1,1} \cong \{(x, y, w) \in \mathbb{T}^2 \times \mathbb{R} \mid xy = 0 + w\}, \quad (x, y, z, w) \mapsto (x, y, w).
$$

In [KS06 Section 8], a non-archimedean torus fibration corresponding to a surface containing a focus-focus singularity is constructed by using the algebraic surface defined by $(\alpha \beta - 1) \gamma = 1$. Here, the subtraction and multiplication mean the usual ones. When we set $\alpha = x, \beta = y, \gamma = w^{-1}$, the tropicalization of it coincides with $xy = 0 + w$, the equation defining $X_{1,1}$.

### 4.2 A tropical modification of focus-focus singularities

Consider replacing the right hand side $0 + w^l$ of the equation of (4.3) with $0 \oplus w^l$. Then the solution of the equation $z = 0 \oplus w^l$ in $X_{\sigma_k}(\mathbb{T})$ is the union of $X_{k,l}$ and the additional face

$$
F_0 := \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = z^k, z < 0, w = 0\},
$$

and this solution set coincides with the tropical hypersurface $V(f)$ defined by $f = 0 + z + w^l$ in $X_{\sigma_k}(\mathbb{T})$. The multiplicity of the facet $F_0$ is $l$ and those of others are $1$. The point $(-\infty, -\infty, -\infty, 0) \in F_0$ is a singular point of multiplicity $k$.
We can think that the surface $X_{k,l}$ is obtained by contracting the tropical hypersurfaces $V(f)$ to $x$-direction and $y$-direction at the same time. We choose a point $p = (x_0, y_0, 0, 0) \in L$ and define a contraction map $\delta_{f,p} : V(f) \to X_{k,l}$ by

$$
(x, y, z, w) \mapsto \begin{cases} 
(x, y, z, w) & (x, y, z, w) \in X_{k,l} \\
(x, x^{-1}, 0, 0) & x \geq x_0 \\
(y^{-1}, y, 0, 0) & y \geq y_0 \\
p = (x_0, y_0, 0, 0) & \text{otherwise}.
\end{cases}
$$

(4.14)

The face $F_0$ is contracted to the line $L$ by this map. The tropical hypersurface $V(f)$ and the contraction $\delta_{f,p}$ are shown in Figure 4.1.

Figure 4.1: The tropical hypersurface $V(f)$ and the contraction of the face $F_0$

Associating $V(f)$ with $X_{k,l}$ is similar to tropical modifications which we recalled in Section 3.2 in the sense that we replace operations $+$ contained in a function with hyperoperations $\oplus$. In this article, we call associating the tropical hypersurface $V(f)$ with $X_{k,l}$ a tropical modification with respect to $0 + w^l$, and the map $\delta_{f,p} : V(f) \to X_{k,l}$ the contraction with respect to $0 + w^l$.

4.3 Blowing up and focus-focus singularities

We fix basis vectors $e_1, e_2, e_3$ of $N$ again. Consider a fan $\Sigma$ consisting of all faces of a cone $\sigma_{k_1}$ generated by $k_1 e_1 + e_2, e_2 \in N$ and a cone $\sigma_{k_2}$ generated by $k_1 e_1 + e_2, (k_1 + k_2) e_2 \in N$, where $k_1$ and $k_2$ are some positive integers. This fan $\Sigma$ is a refinement of $\sigma_k$ where $k_1 + k_2 = k$. The intersection $\tau$ of $\sigma_{k_1}$ and $\sigma_{k_2}$ is the 1-dimensional cone generated by $k_1 e_1 + e_2$. The tropical affine space corresponding to the cone $\tau$ is

$$X_{\tau}(\mathbb{T}) := \text{Hom}(\tau^\vee \cap M, \mathbb{T}) \cong \mathbb{R} \times \mathbb{T} \times \mathbb{R}. \quad (4.15)$$

The tropical toric variety $X_{\Sigma}(\mathbb{T})$ associated with this fan $\Sigma$ is obtained by gluing the tropical affine spaces

$$X_{\sigma_{k_1}}(\mathbb{T}) := \text{Hom}(\sigma_{k_1}^\vee \cap M, \mathbb{T}) = \{(x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \mid xy = z^{k_1}\}, \quad (4.16)$$

$$X_{\sigma_{k_2}}(\mathbb{T}) := \text{Hom}(\sigma_{k_2}^\vee \cap M, \mathbb{T}) = \{(x', y', z', w') \in \mathbb{T}^3 \times \mathbb{R} \mid x'y' = z^{k_2}\}, \quad (4.17)$$
by the maps
\[
X_r(\mathbb{T}) \hookrightarrow X_{\sigma_1} (\mathbb{T}), \quad (y, z, w) \mapsto (y^{-1}z^{k_1}, y, z, w),
\]
\[
X_r(\mathbb{T}) \hookrightarrow X_{\sigma_2} (\mathbb{T}), \quad (y, z, w) \mapsto (yz^{k_2}, y^{-1}, z, w).
\]
(4.18)

(4.19)

The coordinate transformation between \((x, y, z, w)\) and \((x', y', z', w')\) is given by
\[
x' = x^{-1} z^{k_1 + k_2}, \quad y' = y^{-1}, \quad z' = z, \quad w' = w.
\]
(4.20)

We consider the space \(X_{k_1, k_2, l}\) defined by
\[
X_{k_1, k_2, l} := \{(x, y, z, w) \in X_\Sigma(\mathbb{T}) \mid z = 0 + w^l\},
\]
(4.21)

where \(l\) is some positive integer. This space consists of two 2-dimensional faces
\[
F_+ := \{(x, y, z, w) \in X_\Sigma(\mathbb{T}) \mid xy = w^{k_l}, z = w^l, w > 0\},
\]
(4.22)

\[
F_- := \{(x, y, z, w) \in X_\Sigma(\mathbb{T}) \mid xy = z = 0 > w\},
\]
(4.23)

and a 1-dimensional face
\[
L := \{(x, y, z, w) \in X_\Sigma(\mathbb{T}) \mid xy = z = w = 0\}.
\]
(4.24)

We construct an integral affine structure with two singular points on \(X_{k_1, k_2, l}\) as follows:

We choose two points \(p_1 = (x_1, y_1, 0, 0) \in L\) in the coordinate \((x, y, z, w)\) and \(p_2 = (x_2, y_2, 0, 0) \in L\) in the coordinate \((x', y', z', w')\) such that \(y_1 < y_2^{-1}\). We set
\[
U_x := X_{k_1, k_2, l} \setminus \{(x, y, z, w) \in L \mid y \geq y_1\},
\]
(4.25)

\[
U_y := X_{k_1, k_2, l} \setminus \{(x, y, z, w) \in L \mid y \leq y_1 \text{ or } y \geq y_2\},
\]
(4.26)

\[
U_{x'} := X_{k_1, k_2, l} \setminus \{(x, y, z, w) \in L \mid y \leq y_2\}.
\]
(4.27)

These form a covering of \(X_{k_1, k_2, l} \setminus \{p_1, p_2\}\). Consider projections
\[
p_x: U_x \to \mathbb{R}^2, \quad (x, y, z, w) \mapsto (x, w),
\]
(4.28)

\[
p_y: U_y \to \mathbb{R}^2, \quad (x, y, z, w) \mapsto (y, w),
\]
(4.29)

\[
p_{x'}: U_{x'} \to \mathbb{R}^2, \quad (x', y', z', w') \mapsto (x', w).
\]
(4.30)

As in the case of Section 4.1, we can extend the integral affine structures on \(F_+\) and \(F_-\) to an integral affine structure on \(X_{k_1, k_2, l} \setminus \{p_1, p_2\}\) by using these projections. Note that the integral affine structure on \(U_y\) induced by \(p_y\) is the same as the one induced by the projection
\[
p_{y'}: U_y \to \mathbb{R}^2, \quad (x', y', z', w') \mapsto (y', w).
\]
(4.31)

As in Lemma 4.1, we can easily see that singular points \(p_1\) and \(p_2\) are concentrations of \(k_1 l\) and \(k_2 l\) focus-focus singularities respectively, and both of the monodromy invariant subspaces of \(p_1\) and \(p_2\) are the tangent space of \(L\). Blowing up the ambient space by taking a refinement of the cone \(\sigma_k\) corresponds to dispersing the concentration of focus-focus singularities which are piled at one point to the monodromy invariant direction.

We can also think that the space \(X_{k_1, k_2, l}\) can be obtained by contracting a tropical hypersurface as in Section 4.2. Consider replacing the right hand side \(0 + w^l\) of the equation of
Example 4.5. The space $p_{p}$ with each other on the intersection. The face $x$ for $(x,y,z,w)$ in $X_{\Sigma}(T)$ is the union of $X_{k_{1},k_{2},l}$ and the additional face

$$F_{0} := \{ (x, y, z, w) \in X_{\Sigma}(T) \mid xy = z^{k_{1}}, z < 0, w = 0 \},$$

(4.32)

and this solution set coincides with the tropical hypersurface $V(f)$ defined by $f = 0 + z + w^{l}$ in $X_{\Sigma}(T)$. The multiplicity of the facet $F_{0}$ is $l$ and those of others are 1. The points $(-\infty, -\infty, -\infty, 0) \in X_{\sigma_{k_{1}}}$ and $(-\infty, -\infty, -\infty, 0) \in X_{\sigma_{k_{2}}}$ of $V(f)$ are singular points of multiplicity $k_{1}$ and $k_{2}$ respectively. We define a contraction map $\delta_{f,p_{1},p_{2}} : V(f) \rightarrow X_{k_{1},k_{2},l}$ by setting

$$(x, y, z, w) \mapsto \begin{cases} 
(x, y, z, w), & (x, y, z, w) \in X_{k_{1},k_{2},l}; \\
(x, x^{-1}, 0, 0), & (x, y, z, w) \notin X_{k_{1},k_{2},l}, \ x \geq x_{1}, \\
y^{-1}, y, 0, 0), & (x, y, z, w) \notin X_{k_{1},k_{2},l}, \ y_{1} \leq y \leq y_{2}^{-1}, \\
p_{1} = (x_{1}, y_{1}, 0, 0), & \text{otherwise}
\end{cases}$$

(4.33)

for $(x, y, z, w) \in X_{\sigma_{k_{1}}}$ such that $y \leq y_{2}^{-1}$, and

$$(x', y', z', w') \mapsto \begin{cases} 
(x', y', z', w'), & (x', y', z', w') \in X_{k_{1},k_{2},l}; \\
(x', x'^{-1}, 0, 0), & (x', y', z', w') \notin X_{k_{1},k_{2},l}, \ x' \geq x_{2}', \\
y'^{-1}, y', 0, 0), & (x', y', z', w') \notin X_{k_{1},k_{2},l}, \ y_{2}' \leq y' \leq y_{1}^{-1}, \\
p_{2} = (x_{2}', y_{2}', 0, 0), & \text{otherwise}
\end{cases}$$

(4.34)

for $(x', y', z', w') \in X_{\sigma_{k_{2}}}$ such that $y' \leq y_{1}^{-1}$. Note that contractions (4.33) and (4.34) coincide with each other on the intersection. The face $F_{0}$ is contracted to the line $L$ by this map.

Example 4.5. The space $X_{1,1,1}$ is an integral affine surface with two focus-focus singularities $p_{1}$ and $p_{2}$ whose monodromy invariant subspaces are tangent spaces of the line passing through $p_{1}$ and $p_{2}$. A contraction of the face $F_{0}$ to obtain $X_{1,1,1}$ is shown in Figure 4.2.

![Figure 4.2: A contraction to obtain $X_{1,1,1}$](image)

Remark 4.6. By taking a finer refinement of the cone $\sigma_{k}$ and the same procedure, we can construct an integral affine surface $X_{k_{1},\cdots,k_{m},l}$ with $m$ singular points $p_{1}, \cdots, p_{m}$, where $k_{1}, \cdots k_{m}$ are positive integers such that $k_{1} + \cdots + k_{m} = k$. All singular points $p_{1}, \cdots, p_{m}$ are on the same line $L$, and the $i$-th singular point $p_{i}$ is a concentration of $k_{i}l$ focus-focus singularities. The monodromy invariant subspace of any singular point is the tangent space of $L$. We can also construct a contraction map $\delta_{f,p_{1},\cdots,p_{m}} : V(f) \rightarrow X_{k_{1},\cdots,k_{m},l}$ as we did in (4.33), (4.34).
4.4 Contractions of tropical toric K3 hypersurfaces

Let $\Delta \subset M :\equiv M \otimes_{\mathbb{Z}} \mathbb{R}$ be a reflexive polytope of dimension 3, which is not necessarily smooth. We write the polar polytope of $\Delta$ as $\Delta^* \subset N :\equiv N \otimes_{\mathbb{Z}} \mathbb{R}$, and the normal fans of $\Delta$, $\Delta^*$ as $\Sigma \subset N^*$, $\Sigma^* \subset M^*$ respectively. We choose a refinement $\Sigma' \subset M_{\mathbb{R}}$ of $\Sigma$ such that the primitive generator of any 1-dimensional cone in $\Sigma'$ is contained in $\Delta \cap M$. This gives rise to a crepant resolution of the toric variety associated with $\Sigma$.

Let $A \subset N$ denote the subset consisting of all vertices of $\Delta$ and $0 \in N$. We consider a tropical Laurent polynomial

$$f(x) = \sum_{n \in A} a(n)x^n, \quad (4.35)$$

such that the function

$$A \to \mathbb{R}, \quad n \mapsto a(n) \quad (4.36)$$

induces a central subdivision of $\Delta$. Let $V(f)$ be the tropical hypersurface defined by $f$ in the tropical toric variety $X_{\Sigma'}(\mathbb{T})$ associated with $\Sigma'$.

The tropical hypersurface $V(f)$ intersects with the toric boundary as follows: Let $\rho \in \Sigma'$ be a 1-dimensional cone, and $F(\rho)$ be the face of $\Delta$ which contains the primitive generator of $\rho$ in its interior. Recall that there is a one-to-one correspondence between $k$-dimensional faces of $\Delta$ and $(2-k)$-dimensional faces of $\Delta$. Let $F(\rho)$ be the face of $\Delta$ corresponding to $F(\rho)$. On the torus orbit $O_{\rho}(\mathbb{T}) \subset X_{\Sigma'}(\mathbb{T})$, the tropical hypersurface $V(f)$ is defined by

$$\sum_{n \in A \cap F(\rho)} a(n)x^n. \quad (4.37)$$

The number of elements of $A \cap F(\rho)$ is greater than or equal to 2 if and only if $F(\rho)$ is a vertex or an edge. Hence, the tropical hypersurface $V(f)$ intersects with the torus orbit $O_{\rho}(\mathbb{T})$ if and only if $F(\rho)$ is a vertex or an edge. Let $\sigma \in \Sigma'$ be a cone of dimension greater than 1, and $\{\rho_i\}_{i=1}^m \subset \Sigma'$ be the set of 1-dimensional faces of $\sigma$. On the torus orbit $O_{\sigma}(\mathbb{T}) \subset X_{\Sigma'}(\mathbb{T})$, the tropical hypersurface $V(f)$ is defined by

$$\sum_{n \in \bigcap_{i=1}^m A \cap F(\rho_i)} a(n)x^n. \quad (4.38)$$

The number of elements of $\bigcap_{i=1}^m A \cap F(\rho_i)$ is greater than or equal to 2 if and only if the dimension of $\sigma$ is 2 and the primitive generators of $\rho_1, \rho_2$ are contained in a common edge of $\Delta$. This is when the tropical hypersurface $V(f)$ intersects with the torus orbit $O_{\sigma}(\mathbb{T})$.

We write the union of cells of $V(f)$ that do not intersect with the toric boundary as $B$. This is topologically a 2-sphere. In the following, we contract the tropical hypersurface $V(f)$ to the 2-sphere $B$, and equip $B$ with an integral affine structure with singularities.

First, we choose positions of singular points. Let $\tau$ be a 1-dimensional cell of $B$. Recall that there is a one-to-one correspondence between $k$-dimensional cells of $B$ and $k$-dimensional faces of $\Delta$. Let $\{\rho_i\}_{i=1}^{m+1}$ be the set of 1-dimensional cones in $\Sigma'$ whose primitive generators are contained in the edge of $\Delta$ corresponding to $\tau$. We write the primitive generator of $\rho_i$ as $v_i \in M$. We renumber cones $\rho_i$ ($1 \leq i \leq m+1$) so that $v_1, v_{m+1}$ are vertices of $\Delta$, and $v_i$
is nearer to \(v_1\) than \(v_{i+1}\) for any \(1 \leq i \leq l\). We choose \(m\) distinct points \(p(\tau)_i\) \((1 \leq i \leq m)\) on the interior of \(\tau\) so that the point \(p(\tau)_i\) is nearer to the vertex \(v_1\) than \(p(\tau)_{i+1}\) for any \(1 \leq i \leq m-1\). These points will be singular points of the integral affine structure of \(B\). For each 1-dimensional cell \(\tau\) of \(B\), we choose points \(p(\tau)_i\) in this way and fix them.

For each 1-dimensional cell \(\tau\) of \(B\), we take an open set \(U_\tau\) containing all of these points \(p(\tau)_i\) \((1 \leq i \leq m)\). We also take an open neighborhood \(U_v\) for each vertex \(v\) of \(B\). Here, we take these open sets so that they do not contain any other singular points or any other vertices of \(B\), and all of these open sets \(U_\tau, U_v\) and interiors of all facets of \(B\) form a covering of \(B\). We contract the tropical hypersurface \(V(f)\) to \(B\) as follows:

- **Around \(U_v\)**

  Let \(\rho \in \tilde{\Sigma}'\) be the cone whose primitive generator is the vertex of \(\Delta\) corresponding to \(v\), and \(X_\rho(\mathbb{T}) \subset X_{\tilde{\Sigma}'_p}(\mathbb{T})\) be the tropical affine toric variety corresponding to \(\rho\). Let further \(V(\tau)_v\) be the star of \(v\) in \(V(f)\). We consider the projection

  \[
  p_\rho: X_\rho(\mathbb{T}) \to O_\rho(\mathbb{T}), \quad (w: \rho^+ \cap M \to \mathbb{T}) \mapsto (\rho^+ \cap M \subset \rho^+ \cap M \xrightarrow{\rho} \mathbb{T}).
  \]

  We set \(\tilde{U}_v := p_\rho^{-1}(p_\rho(U_v)) \cap V(\tau)_v\) and defined the map \(\delta_v: \tilde{U}_v \xrightarrow{p_\rho} U_v\) as

  \[
  \delta_v: \tilde{U}_v \xrightarrow{p_\rho} p_\rho(U_v) \cong U_v,
  \]

  where \(p_\rho(U_v) \cong U_v\) is the inverse map of the bijection \(p_\rho: U_v \to p_\rho(U_v)\). We equip \(U_v\) with the integral affine structure induced by the integral affine structure of \(p_\rho(U_v) \subset O_\rho \cong \mathbb{R}^2\). The dominant part of \(f\) at \(v\) is given by

  \[
  a(0) + \sum_{n \in \mathbb{Z}^N \cap \tilde{F}(\rho)} a(n)x^n.
  \]

  By taking an appropriate coordinate, the function (4.41) can be rewritten as a function of the form \(y + f_v\), where \(f_v\) is a function on \(O_\rho\). The map \(\delta_v\) coincides with a restriction of the contraction of the hypersurface defined by \(y + f_v\) with respect to the function \(f_v\), which we considered in Section 3.2.

- **Around \(U_\tau\)**

  Let \(\{\rho_i\}_{i=1}^{m+1}\) be the set of all 1-dimensional cones in \(\tilde{\Sigma}'\) whose primitive generators are contained in the edge of \(\Delta\) corresponding to \(\tau\). We write the 2-dimensional cone whose 1-dimensional faces are \(\rho_i\) and \(\rho_{i+1}\) as \(\sigma_i \in \tilde{\Sigma}'(1 \leq i \leq m)\). Let further \(\tilde{\Sigma}'_p\) be the subfan of \(\tilde{\Sigma}\) consisting of \(\sigma_i\) \((1 \leq i \leq m)\) and all of their faces. The star \(V(\hat{\tau})\) of \(\tau\) in \(V(f)\) is contained in the subvariety \(X_{\tilde{\Sigma}'_p}(\mathbb{T}) \subset X_{\tilde{\Sigma}'_p}(\mathbb{T})\). The tropical toric variety \(X_{\tilde{\Sigma}'_p}(\mathbb{T})\) is also the ambient space that we used when we construct \(X_{k_1, \ldots, k_m, l}\) of Remark 4.6 where \(k_i\) is the integral distance between the primitive generators \(v_i, v_{i+1}\) of \(\rho_i, \rho_{i+1}\). On the other hand, the dominant part of \(f\) at \(\tau\) is given by

  \[
  a(0) + \sum_{n \in \mathbb{Z}^N \cap \tilde{F}(\rho_i) \cap \tilde{F}(\rho_{i+1})} a(n)x^n.
  \]

  The set \(A \cap \tilde{F}(\rho_i) \cap \tilde{F}(\rho_{i+1})\) consists of two elements. By taking an appropriate coordinate, the function (4.42) can be rewritten as a function \(f_\tau\) of the form \(0 + z + w^l\), where \(l\)
Remark 4.7. There is an ambiguity in the choice of the position of each singular point

However, neither the cohomology group $H^\rho$ nor the radiance obstruction $c_B$ of $B$ does not depend on this choice. We will reconsider this point in Remark 7.3.

Remark 4.8. Kontsevich and Soibelman constructed a 2-sphere with an integral affine structure with singularities by contracting a Clemens polytope of a degenerating family of K3 surfaces [KS06, Section 4.2.5]. Their contraction is quite similar to the above contraction of tropical hypersurfaces. Compare the local contraction given in [KS06, Section 4.2.5] to the contraction given in (4.3) of this article.

Example 4.9. Consider the polynomial

$$f(x, y, z) = 1 + x^3y^{-1}z^{-1} + x^{-1}y^3z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}. \quad (4.44)$$

The Newton polytope $\Delta \subset N_{\mathbb{R}}$ of $f$ is the simplex whose one side is 4. In this case, there are no further crepant refinements of $\Sigma$. We choose a point on the interior of each edge of $B$, which will be a singular point. Let $\rho_1$ and $\rho_2$ be the 1-dimensional cones in the normal fan $\Sigma$ of $\Delta$ generated by $(1, 0, 0)$ and $(0, 1, 0)$ respectively. Let further $v_1$ and $v_2$ be the vertices of $B$ that correspond to $\rho_1$ and $\rho_2$ respectively, and $\tau$ be the edge of $B$ connecting $v_1$ and $v_2$.

Around $v_1$, the tropical hypersurface $V(f)$ is locally defined by

$$1 + x^{-1}y^3z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}, \quad (4.45)$$

and the contraction $\delta_{v_1}$ coincides with a restriction of the contraction with respect to the function $f_{v_1}$ on $O_{\rho_1}(\mathbb{T})$ defined by

$$f_{v_1}(y, z) := y^3z^{-1} + y^{-1}z^3 + y^{-1}z^{-1}. \quad (4.46)$$
Around $v_2$, the tropical hypersurface $V(f)$ is locally defined by
\[
1 + x^3y^{-1}z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1},
\]
and the contraction $\delta_{v_2}$ coincides with a restriction of the contraction with respect to the function $f_{v_2}$ on $O_{\rho_2}(\mathbb{T})$ defined by
\[
f_{v_2}(x, z) := x^3z^{-1} + x^{-1}z^3 + x^{-1}z^{-1}.
\]
Around $\tau$, the tropical hypersurface $V(f)$ is locally defined by
\[
1 + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}.
\]
When we set $x' := 1x, y' := yz, z' := 1xyz, w' := z$, it is locally defined by
\[
f_{\tau}(z', w') := 0 + z' + w'^4.
\]
The contraction $\delta_\tau$ coincides with a restriction of the contraction $\delta_{f_{\tau,p}}$ of (4.14) $(k = 1, l = 4)$.

5 Dispersions of focus-focus singularities

Let $S$ be an integral affine surface with some singular points. We suppose that the monodromy around one of the singular points $p$ of $S$ is given by the matrix
\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix},
\]
(5.1)
under a local coordinate system \((x, y)\) near \(p\), where \(k\) is a non-zero integer. Here, the tangent vector \(e_x\) corresponding to the coordinate \(x\) is monodromy invariant, and the coordinate \(y\) is globally well-defined on a sufficiently small open neighborhood \(U\) of \(p\). We write the line defined by \(y = 0\) on \(U\) as \(L\). We can construct another integral affine structure with singularities on \(U\), which has just two singular points \(p_1\) and \(p_2\) on \(L\) whose monodromies are given by

\[
\begin{pmatrix}
1 & k_i \\
0 & 1
\end{pmatrix}
\]

(5.2)

under the same coordinate system \((x, y)\) respectively, where \(k_1, k_2\) are non-zero integers such that \(k_1 + k_2 = k\). By replacing the original integral affine structure with singularities on \(U\) with this new one, we can obtain another integral affine surface \(S\) with singularities, since monodromies of both integral affine structures with respect to the loop along the boundary of \(U\) are the same.

Assume that the determinant of the monodromy matrix around any singular point of \(S\) is 1. Then we have \(\iota_\ast \wedge^2 T_Z \cong \mathbb{Z}\) for both \(S\) and \(S\)', where \(\iota\) is the inclusion of the complement of singularities. The cohomology groups \(H^1(S, \iota_\ast T_Z)\) and \(H^1(S', \iota_\ast T_Z)\) have the cup product \(U \mapsto \wedge U\) induced by the wedge product. We also write the radiance obstructions of \(S\) and \(S\)' as \(c_{S}\) and \(c_{S}'\) respectively.

Let \(U' = \{ U_j \}_{j \in j'}\) be a sufficiently fine acyclic covering of \(S'\) for \(\iota_\ast T_Z\) such that each open set have one singular point at most and each singular point is contained by only one open set. Let \(U_{j_1}, U_{j_2} \in U'\) be the open sets containing \(p_1\) and \(p_2\) respectively. We set \(U_{j_1} := U_{j_1} \cup U_{j_2}, J' := J' \setminus \{ j_1, j_2 \}, \) and \(J := J' \cup \{ j_1, j_2 \}\). We replace \(U'\) if necessary so that \(U_{j_1} \cap U_{j_2}\) does not intersect with \(U_{j_1} \cap U_{j_2}\) for any \(j_1, j_2 \in J\). The set of open sets \(U := \{ U_j \}_{j \in J}\) is an acyclic covering of \(S\) for \(\iota_\ast T_Z\).

We define a map \(f : H^1(S, \iota_\ast T_Z) \to H^1(S', \iota_\ast T_Z)\) by setting

\[
f(\phi) (((U_{j_1}, U_{j_2}))) = \phi (((U'_{j_1}, U'_{j_2}))) |_{U_{j_1} \cap U_{j_2}}
\]

(5.3)

for each \(\phi \in Z^1(U, \iota_\ast T_Z)\) and \(j_1, j_2 \in J\', \)

\[
U_j' := \left\{ \begin{array}{ll}
U_j \setminus j \in J', \\
U_{j_1}, & j \in \{ j_1, j_2 \}.
\end{array} \right.
\]

(5.4)

**Lemma 5.1.** The map \(f : H^1(S, \iota_\ast T_Z) \to H^1(S', \iota_\ast T_Z)\) is well-defined.

**Proof.** Since we have

\[
\begin{align*}
\delta(f(\phi)) ( ((U_{j_1}, U_{j_2}, U_{j_1})) ) &= f(\phi) ( (U_{j_2}, U_{j_1})) - f(\phi) ( (U_{j_1}, U_{j_2})) + f(\phi) ( (U_{j_1}, U_{j_1})) \\
&= \phi ((U'_{j_2}, U'_j)) - \phi ((U'_{j_1}, U'_j)) + \phi ((U'_{j_1}, U'_j)) \\
&= (\delta(\phi)) ((U'_{j_2}, U'_j)) = 0,
\end{align*}
\]

(5.5)

(5.6)

(5.7)

\(f(\phi)\) is a cocycle. For any element \(\theta \in C^0(U, \iota_\ast T_Z)\), we take the element \(\theta' \in C^0(U', \iota_\ast T_Z)\) defined by \(\theta'(U_j) := \theta(U'_{j}) |_{U_j}\). Then we have

\[
f(\delta \theta) ((U_{j_1}, U_{j_2})) = \delta \theta ((U'_{j_1}, U'_{j_2})) |_{U_{j_1} \cap U_{j_2}} = \theta(U'_{j_2})|_{U_{j_1} \cap U_{j_2}} - \theta(U'_{j_1})|_{U_{j_1} \cap U_{j_2}},
\]

(5.8)

\[
\delta \theta' ((U_{j_1}, U_{j_2})) = \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} - \theta'(U_{j_1})|_{U_{j_1} \cap U_{j_2}} = \theta(U'_{j_2})|_{U_{j_1} \cap U_{j_2}} - \theta(U'_{j_1})|_{U_{j_1} \cap U_{j_2}}.
\]

(5.9)

Hence, we obtain \(f(\delta \theta) = \delta \theta'\).
Proposition 5.2. The map $f : H^1(S, t_s T_Z) \to H^1(S', t_s T_Z)$ is a primitive embedding that preserves the pairing.

Proof. First, we check that the map $f$ is injective. Suppose there exists $\theta \in C^0(U', t_s T_Z)$ such that $\delta(\theta') = f(\phi)$. We will construct an element $\theta \in C^0(U, t_s T_Z)$ such that $\delta(\theta) = \phi$. Here since we have

$$\theta'(U_{j_3}) - \theta'(U_{j_2}) = (\delta \theta')(U_{j_2}) = f(\phi)((U_{j_1}, U_{j_2})) = \phi((U_{j_1}, U_{j_2})) = 0,$$

there is a section $s \in \Gamma(U_{j_2}, t_s T_Z)$ such that $s|_{U_{j_2}} = \theta'(U_{j_2})$ and $s|_{U_{j_3}} = \theta'(U_{j_3})$. We define $\theta \in C^0(U, t_s T_Z)$ by setting

$$\theta((U_j)) := \begin{cases} \theta'((U_j)), & j \in J^o, \\ s_j, & j = j', \end{cases}$$

(5.11)

Then when $j_1, j_2 \in J^o$, we have

$$\delta \theta((U_{j_1}, U_{j_2})) = \theta'(U_{j_2}) - \theta'(U_{j_1}) = \delta \theta'((U_{j_1}, U_{j_2})) = f(\phi)((U_{j_1}, U_{j_2})) = \phi((U_{j_1}, U_{j_2})).$$

(5.12)

When $j_1 = j', j_2 \in J^o$, we have

$$\delta \theta((U_{j_1}, U_{j_2}))|_{U_{j_1} \cap U_{j_2}} = \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} = \theta'(U_{j_1})|_{U_{j_1} \cap U_{j_2}} = f(\phi)((U_{j_1}, U_{j_2})) = \phi((U_{j_1}, U_{j_2})).$$

(5.13)

Since we can also get $\delta \theta((U_{j_1}, U_{j_2}))|_{U_{j_1} \cap U_{j_2}} = \phi((U_{j_1}, U_{j_2}))$ in the same way, we obtain

$$\delta \theta((U_{j_1}, U_{j_2})) = \phi((U_{j_1}, U_{j_2})).$$

(5.14)

Therefore, we obtain $\delta(\theta) = \phi$.

Next, we check that the map $f$ preserves the pairing. We take a total orders of $J^o$. By adding $j_\gamma$ to $J^o$ as the minimum element, we obtain a total order of $J$. We also consider the total order of $J'$ obtained by adding $j_{\alpha}, j_\beta$ to $J^o$ as the minimum and the second minimum elements respectively. For any $\phi_1, \phi_2 \in H^1(U, t_s T_Z)$, we can calculate as follows:

$$\phi_1 \cup \phi_2 - f(\phi_1) \cup f(\phi_2) = \sum_{\substack{j_1 < j_2 \\ U_{j_1} \cap U_{j_2} \neq \emptyset}} \phi_1((U_{j_1}, U_{j_1})) \wedge \phi_2((U_{j_1}, U_{j_2})) - \sum_{\substack{j_1 < j_2 \\ U_{j_1} \cap U_{j_2} \neq \emptyset}} f(\phi_1)((U_{j_1}, U_{j_1})) \wedge f(\phi_2)((U_{j_1}, U_{j_2})) - \sum_{\substack{j_1 < j_2 \\ U_{j_1} \cap U_{j_2} \neq \emptyset}} f(\phi_1)((U_{j_1}, U_{j_1})) \wedge f(\phi_2)((U_{j_1}, U_{j_2})) - \sum_{\substack{j_1 < j_2 \\ U_{j_1} \cap U_{j_2} \neq \emptyset}} f(\phi_1)((U_{j_1}, U_{j_1})) \wedge f(\phi_2)((U_{j_1}, U_{j_2})) = 0.$$
where \( j_1, j_2 \in J^o \).

Lastly, we show that the map \( f \) is primitive. Consider the map
\[
\phi \otimes t \mapsto f(\phi) \otimes t, \quad (5.22)
\]
which we will also write as \( f \). Assume that there exists an element \( \theta' \in C^0(U', \iota_4 T) \) such that \( \delta(\theta') + f(\phi \otimes t) \in C^1(U', \iota_4 T) \). We will construct an element \( \theta \in C^0(U, \iota_4 T) \) such that \( \delta(\theta) + \phi \otimes t \in C^1(U, \iota_4 T) \).

Since the monodromy invariant directions of \( p_1 \) and \( p_2 \) are the same, we can extend sections \( \theta'(U_{j_1}), \theta'(U_{j_2}) \) to \( U_j \). Let \( s_\alpha, s_\beta \in \Gamma(U_j, \iota_4 T) \) denote the extensions of \( \theta'(U_{j_1}), \theta'(U_{j_2}) \) respectively. Since we have
\[
(\delta(\theta') + f(\phi \otimes t))((U_{j_1}, U_{j_2})) = \theta'(U_{j_1})|_{U_{j_1} \cap U_{j_2}} - \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} \in \iota_4 T(Z(U_{j_1} \cap U_{j_2})), \quad (5.23)
\]
we can see \( s_\alpha - s_\beta \in \iota_4 T(Z(U_{j_1} \cap U_{j_2})) \). Define \( \theta \in C^0(U, \iota_4 T) \) by setting
\[
\theta((U_j)) := \begin{cases} 
\theta'((U_j)), & j \in J^o, \\
s_\alpha, & j = J_2.
\end{cases} \quad (5.24)
\]

Then, in the case where \( j_1, j_2 \in J^o \), we have
\[
(\delta\theta + \phi \otimes t)((U_{j_1}, U_{j_2})) = \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} - \theta'(U_{j_1})|_{U_{j_1} \cap U_{j_2}} \in \iota_4 T(Z(U_{j_1} \cap U_{j_2})),
\]
In the case where \( j_1 = J_2, j_2 \in J^o, U_{j_1} \cap U_{j_2} \neq \emptyset \) or \( U_{j_1} \cap U_{j_2} \neq \emptyset \). When \( U_{j_1} \cap U_{j_2} \neq \emptyset \), we have
\[
(\delta\theta + \phi \otimes t)((U_{j_1}, U_{j_2})) = \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} - \theta'(U_{j_1})|_{U_{j_1} \cap U_{j_2}} + (\phi \otimes t)((U_{j_1}, U_{j_2}))|_{U_{j_1} \cap U_{j_2}}.
\]
Hence, we obtain \( (\delta\theta + \phi \otimes t)((U_{j_1}, U_{j_2})) \in \iota_4 T(Z(U_{j_1} \cap U_{j_2})) \). When \( U_{j_1} \cap U_{j_2} \neq \emptyset \), we have
\[
(\delta\theta + \phi \otimes t)((U_{j_1}, U_{j_2}))|_{U_{j_1} \cap U_{j_2}} = \theta'(U_{j_2})|_{U_{j_1} \cap U_{j_2}} - s_\alpha|_{U_{j_1} \cap U_{j_2}} + s_\beta|_{U_{j_1} \cap U_{j_2}}.
\]
Hence, we obtain \( (\delta\theta + \phi \otimes t)((U_{j_1}, U_{j_2})) \in \iota_4 T(Z(U_{j_1} \cap U_{j_2})) \). Therefore, we have \( \delta(\theta) + \phi \otimes t \in C^1(U, \iota_4 T) \).
Proposition 5.3. One has \( f(c_S) = c_{S'} \).

Proof. When we take a set of sections \( \{ s_j \in \Gamma(U_j, \iota_* T) \}_{j \in J} \), the radiance obstruction \( c_S \) of \( S \) is given by

\[
c_S((U_{j_1}, U_{j_2})) = s_{j_2}|_{U_{j_1} \cap U_{j_2}} - s_{j_1}|_{U_{j_1} \cap U_{j_2}} \in \Gamma(U_{j_1} \cap U_{j_2}, \iota_* T) \tag{5.36}
\]

for any \( j_1, j_2 \in J \). We set \( s_{j_\alpha} := s_{j_\gamma}|_{U_{j_\alpha}} \) and \( s_{j_\beta} := s_{j_\gamma}|_{U_{j_\beta}} \). We have

\[
f(c_S)((U_{j_1}, U_{j_2})) = c_S((U'_{j_1}, U'_{j_2}))|_{U_{j_1} \cap U_{j_2}} = s_{j_2}|_{U_{j_1} \cap U_{j_2}} - s_{j_1}|_{U_{j_1} \cap U_{j_2}} \in \Gamma(U_{j_1} \cap U_{j_2}, \iota_* T) \tag{5.37}
\]

for any \( j_1, j_2 \in J' \). This is just the radiance obstruction \( c_{S'} \) of \( S' \) constructed from the set of sections \( \{ s_j \in \Gamma(U_j, \iota_* T) \}_{j \in J'} \).

6 Proofs of Theorem 1.1 and Theorem 1.2

Let \( M \) be a free \( \mathbb{Z} \)-module of rank 3 and \( N := \text{Hom}(M, \mathbb{Z}) \) be the dual lattice. We set \( M_\mathbb{R} := M \otimes \mathbb{R} \) and \( N_\mathbb{R} := N \otimes \mathbb{R} = \text{Hom}(M, \mathbb{R}) \). Let \( \Delta \subset M_\mathbb{R} \) be a smooth reflexive polytope of dimension 3, and \( \hat{\Delta} \subset N_\mathbb{R} \) be the polar polytope of \( \Delta \). Let further \( \Sigma \) and \( \hat{\Sigma} \) be the normal fans to \( \Delta \) and \( \hat{\Delta} \) respectively.

Let \( A \subset N \) denote the subset consisting of all vertices of \( \hat{\Delta} \) and \( 0 \in N \). We consider a tropical Laurent polynomial

\[
f(x) = \max_{n \in A} \{ a(n) + n_1 x_1 + n_2 x_2 + n_3 x_3 \}, \tag{6.1}
\]

such that the function

\[
A \to \mathbb{R}, \quad n \mapsto a(n) \tag{6.2}
\]

induces a central subdivision of \( \hat{\Delta} \), i.e., every maximal dimensional simplex of the subdivision has the origin \( 0 \in N \) as its vertex. Let \( V(f) \) be the tropical hypersurface defined by \( f \) in the tropical toric variety \( X_\hat{\Sigma}(\mathbb{T}) \) associated with \( \hat{\Sigma} \).

Let further \( B \) be the 2-sphere with an integral affine structure with singularities obtained by contracting \( V(f) \) in the way of Section 4.4 and \( P \) be the natural polyhedral structure of it. For each 1-dimensional cell of \( B \), we choose the barycenter of it as a position of the singular point that should be on it. We write the complement of singularities of \( B \) as \( \iota: B_0 \hookrightarrow B \). Let \( T_\mathbb{Z} \) be the local system on \( B_0 \) of integral tangent vectors. We set \( T := T_\mathbb{Z} \otimes \mathbb{R} \).

In the subsequent subsections, we give proofs of Theorem 1.1 and Theorem 1.2 in this setting. Note that the statements of Theorem 1.1 and Theorem 1.2 do not depend on the choices of positions of singular points. See Remark 4.7. In Section 4.3, we also saw that blowing up the ambient toric variety corresponds to dispersing concentrations of focus-focus singularities. Proposition 5.2 and Proposition 5.3 ensure that Theorem 1.1 and Theorem 1.2 hold also when we replace \( \hat{\Sigma} \) with a refinement \( \hat{\Sigma}' \subset M_\mathbb{R} \) of \( \hat{\Sigma} \) such that the primitive generator of any 1-dimensional cone in \( \hat{\Sigma}' \) is contained in \( \Delta \cap M \), if we could prove the theorems in the above setting.
6.1 Proof of Theorem 1.1

We consider the toric variety $X_\Sigma$ associated with $\Sigma$. We write the group of toric divisors on $X_\Sigma$ as

$$\text{Div}_T(X_\Sigma) := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_{\rho},$$

where $\Sigma(1)$ is the set of 1-dimensional cones in $\Sigma$, and $D_\rho$ is the toric divisor corresponding to $\rho \in \Sigma(1)$. We take the barycentric subdivision of $\mathcal{P}$ and let $\mathcal{U} := \{U_\tau\}_{\tau \in \mathcal{P}}$ be the covering of $B$, where $U_\tau$ is the open star of the barycenter of $\tau \in \mathcal{P}$. The covering $\mathcal{U}$ is acyclic for $\iota_* T_\mathcal{Z}$ and $\iota_* T$, and

$$\check{H}^1(\mathcal{U}, \iota_* T_\mathcal{Z}) = H^1(B, \iota_* T_\mathcal{Z}), \quad \check{H}^1(\mathcal{U}, \iota_* T) = H^1(B, \iota_* T).$$

There is a one-to-one correspondence between the facets of $B$, vertices of $\check{\Delta}$, and $\Sigma(1)$. We write the facet of $B$ and the vertex of $\check{\Delta}$ that correspond to $\rho \in \Sigma(1)$ as $\sigma(\rho) \in \mathcal{P}$ and $n_\rho = (n_1(\rho), n_2(\rho), n_3(\rho)) \in \mathbb{N}$ respectively.

Let $v \in \mathcal{P}$ be a vertex, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets containing $v$. The equation defining the plane containing $\sigma(\rho_i)$ is given by

$$a(n_{\rho_i}) + n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = a(0).$$

For a divisor $D = \sum_{\rho \in \Sigma(1)} k_\rho D_\rho \in \text{Div}_T(X_\Sigma)$ and the vertex $v \in \mathcal{P}$, let $v(D)$ denote the element of $M$ defined by

$$n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k_\rho, \quad 1 \leq i \leq 3.$$

Such an element $v(D)$ always uniquely exists.

For each cell $\tau \in \mathcal{P}$, we choose an arbitrary vertex $v$ of $\tau$ and set $\tau(D) := v(D)$. Here, when $\tau$ is a vertex $v$ of $B$, we assume $\tau(D) = v(D)$. We define a map

$$\psi: \text{Div}_T(X_\Sigma) \to \check{H}^1(\mathcal{U}, \iota_* T_\mathcal{Z}), \quad D \mapsto \psi(D),$$

by setting

$$\psi(D)((U_{\tau_0}, U_{\tau_1})) := \tau_1(D) - \tau_0(D)$$

for each 1-simplex $(U_{\tau_0}, U_{\tau_1})$ of $\mathcal{U}$. We will check that this map $\psi$ gives the map of Theorem 1.1 in the following lemmas, from Lemma 6.1 to Lemma 6.5.

**Lemma 6.1.** The map $\psi$ is a well-defined group homomorphism.

**Proof.** First, we check that $\psi(D)((U_{\tau_0}, U_{\tau_1}))$ is certainly a section of $\iota_* T_\mathcal{Z}$ over $U_{\tau_0} \cap U_{\tau_1}$. Assume that we choose vertices $v_0, v_1$ for $\tau_0, \tau_1$ when we determine $\tau_0(D), \tau_1(D)$. Then we have

$$\psi(D)((U_{\tau_0}, U_{\tau_1})) = v_1(D) - v_0(D).$$

Consider the case where either of $\tau_0$ or $\tau_1$ is a facet. Assume that $\tau_1$ is a facet and the 1-dimensional cone $\rho \in \Sigma(1)$ corresponds to it. Here, we have $\tau_0 \prec \tau_1$. Since points $v_0(D)$ and $v_1(D)$ are contained in the plane defined by

$$n_1(\rho)x_1 + n_2(\rho)x_2 + n_3(\rho)x_3 = -k_\rho,$$

...
the vector $v_1(D) - v_0(D)$ is contained in the plane defined by
\[ n_1(\rho)x_1 + n_2(\rho)x_2 + n_3(\rho)x_3 = 0. \] (6.11)

On the other hand, the tangent space at $U_0 \cap U_1$ is also this subspace. Hence, we have $\psi(D) \in T_\Sigma(U_0 \cap U_1)$. If $\tau$ is an edge. Assume that $\psi$ is a vector and the point $v_0$ is its endpoint. Let $\sigma(\rho_1)$ and $\sigma(\rho_2)$ be the facets containing $\tau_1$ as their face. Since the points $v_0(D), v_1(D)$ are contained in the 1-dimensional space defined by
\[ n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k_{\rho_i}, \quad i = 1, 2, \] (6.12)
the vector $v_1(D) - v_0(D)$ is contained in the 1-dimensional subspace defined by
\[ n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = 0, \quad i = 1, 2. \] (6.13)

On the other hand, the tangent space at $U_0 \cap U_1$ contains this subspace. Hence, we have $\psi(D) \in T_\Sigma(U_0 \cap U_1)$. Next, we show that $\psi(D)$ is a cocycle. For a given 2-simplex $(U_0, U_1, U_2)$, assume that we choose vertices $v_0, v_1, v_2$ for $\tau_0, \tau_1, \tau_2$ when we determine $\tau_0(D), \tau_1(D), \tau_2(D)$. Then we have
\[ \delta(\psi(D))((U_0, U_1, U_2)) = \{v_2(D) - v_1(D)\} - \{v_2(D) - v_0(D)\} + \{v_1(D) - v_0(D)\} = 0. \] (6.14)

Lastly, we show that the map $\psi$ is a group homomorphism. We will show $\psi(D + D') = \psi(D) + \psi(D')$ for any $D = \sum_\rho k_\rho D_\rho, D' = \sum_\rho k'_\rho D_\rho \in \text{Div}_T(X_\Sigma)$. Let $v \in P$ be a vertex, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets containing $v$. Then the point $v(D + D')$ is defined by
\[ n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k_{\rho_i} - k'_{\rho_i}, \quad 1 \leq i \leq 3. \] (6.15)

The point $v(D)$ is defined by
\[ n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k_{\rho_i}, \quad 1 \leq i \leq 3, \] (6.16)
and the point $v(D')$ is defined by
\[ n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k'_{\rho_i}, \quad 1 \leq i \leq 3. \] (6.17)

Hence, we have $v(D + D') = v(D) + v(D')$. If we choose vertices $v_0, v_1$ for $\tau_0, \tau_1$ when we determine $\tau_0(D), \tau_1(D)$, we have
\[ \psi(D + D')((U_0, U_1)) = v_1(D + D') - v_0(D) \]
\[ = \{v_1(D) + v_1(D')\} - \{v_0(D) + v_0(D')\} \]
\[ = \{v_1(D) - v_0(D)\} + \{v_1(D') - v_0(D')\} \] (6.18)
\[ = \psi(D)((U_0, U_1)) + \psi(D')((U_0, U_1)) \] (6.19)
for any 1-simplex $(U_0, U_1)$ of $U$.

\[ \square \]

**Lemma 6.2.** The map $\psi$ is independent of the choice of the vertex that we choose for each $\tau \in P$ when we determine $\tau(D)$. 

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Lemma 6.3. Let $\mathcal{P} := \{\tau_i\}$ be the set of all cells of $B$. Let further $\psi$ be the map which we obtain when we choose $v_i$ for each $\tau_i$, and $\psi'$ be the map which we obtain when we choose $v'_i$ for each $\tau_i$. We show that $\psi(D) = \psi'(D)$ for any $D \in \text{Div}_T(X_\Sigma)$.

For each $D = \sum_{\rho \in \Sigma(1)} k_\rho D_\rho$ contained in the 1-dimensional space defined by (6.22), we have

$$\phi(D) = \{ \psi'(D) - v_i(D) \} = \{ \psi'(D) - v_i(D) \}$$

for each 0-simplex $(U_{\tau_i})$ of $\mathcal{U}$. We will show that the coboundary of $\phi(D)$ coincides with $\psi'(D) - \psi(D)$. First, we check that $\phi(D)$ is certainly an element of $C^0(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z})$.

When $\tau_i$ is a vertex, $v_i = v'_i = \tau$ and we have $\phi(D)(U_{\tau_i}) = 0$. Hence, $\phi(D)(U_{\tau_i}) \in \iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_i})$.

When $\tau_i$ is an edge and is contained in facets $\sigma(\rho_{i_1})$ and $\sigma(\rho_{i_2})$, vertices $v_i(D), v'_i(D)$ are contained in the 1-dimensional space defined by

$$n_1(\rho_{i_1})x_1 + n_2(\rho_{i_1})x_2 + n_3(\rho_{i_1})x_3 = -k_{\rho_{i_1}}, \quad j = 1, 2.$$  \hspace{1cm} (6.21)

Hence, the vector $v'_i(D) - v_i(D)$ is contained in the 1-dimensional subspace defined by

$$n_1(\rho_{i_1})x_1 + n_2(\rho_{i_1})x_2 + n_3(\rho_{i_1})x_3 = 0, \quad j = 1, 2.$$  \hspace{1cm} (6.22)

The section $\iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_i})$ is the lattice of integral tangent vectors that are invariant under the monodromy transformation around the singular point on $\tau_i$. That is the lattice contained in the subspace defined by (6.22). Hence, we have $\phi(D)(U_{\tau_i}) \in \iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_i})$.

When $\tau_i$ is a facet, vertices $v_i(D)$ and $v'_i(D)$ are contained in the plane defined by

$$n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -k_{\rho_i},$$  \hspace{1cm} (6.23)

where $\rho_i$ is the 1-dimensional cone corresponding to $\tau_i$. Hence, the vector $v'_i(D) - v_i(D)$ is contained in the plane defined by

$$n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = 0.$$  \hspace{1cm} (6.24)

On the other hand, the section $\iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_i})$ is the lattice of integral tangent vectors on $U_{\tau_i}$. That is the lattice contained in the subspace defined by (6.24). Hence, we have $\phi(D)(U_{\tau_i}) \in \iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_i})$. Therefore, we have $\phi(D) \in C^0(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z})$.

For any 1-simplex $(U_{\tau_1}, U_{\tau_2})$ of $\mathcal{U}$, one can get

$$\psi'(D)((U_{\tau_1}, U_{\tau_2})) - \psi(D)((U_{\tau_1}, U_{\tau_2})) = \{ v'_j(D) - v'_i(D) \} - \{ v_j(D) - v_i(D) \}$$

$$= \{ v'_j(D) - v_j(D) \} - \{ v'_i(D) - v_i(D) \}$$

$$= (\delta \phi(D))(U_{\tau_1}, U_{\tau_2}).$$

Hence, we have $\psi(D) = \psi'(D)$ in $\tilde{H}^1(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z})$. \hfill \Box

Recall that we have the exact sequence

$$M \rightarrow \text{Div}_T(X_\Sigma) \rightarrow \text{Pic}(X_\Sigma) \rightarrow 0,$$  \hspace{1cm} (6.27)

where the map $M \rightarrow \text{Div}_T(X_\Sigma)$ is given by $m \mapsto \text{div}(\chi^m)$.

Lemma 6.3. The map $\psi$ induces an injection

$$\text{Pic}(X_\Sigma) \hookrightarrow \tilde{H}^1(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z}).$$  \hspace{1cm} (6.28)
Proof. First, we check that \( \psi(\text{div}(\chi^m)) = 0 \) for any \( m = (m_1, m_2, m_3) \in M \). The facet \( \sigma(\rho) \) of \( B \) corresponding to the 1-dimensional cone \( \rho \in \Sigma \) is defined by

\[
a(n_\rho) + n_1(\rho)x_1 + n_2(\rho)x_2 + n_3(\rho)x_3 = a(0). \tag{6.29}
\]

The primitive generator of \( \rho \) is \( n_\rho = (n_1(\rho), n_2(\rho), n_3(\rho)) \in N \). Hence, we have

\[
\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, n_\rho \rangle D_\rho. \tag{6.30}
\]

Let \( v \in \mathcal{P} \) be a vertex of \( B \), and \( \{\sigma(\rho_i)\}_{i=1}^3 \) be the set of facets containing \( v \). The element \( v(\text{div}(\chi^m)) \) satisfies

\[
n_1(\rho_i)x_1 + n_2(\rho_i)x_2 + n_3(\rho_i)x_3 = -\langle m, n_{\rho_i} \rangle = -\left( \sum_{j=1}^3 m_j n_j(\rho_i) \right), \quad 1 \leq i \leq 3. \tag{6.31}
\]

Therefore, we have \( v(\text{div}(\chi^m)) = -(m_1, m_2, m_3) \in M \) for any vertex \( v \in \mathcal{P} \). From the definition of \( \psi \), we can see that \( \psi(\text{div}(\chi^m)) = 0 \).

Next, we show that the induced map \( \text{Pic}(X_\Sigma) \to \check{H}^1(\mathcal{U}, \mathcal{O}_\Sigma) \) is injective. Assume that \( \psi(D) = \delta(\phi) \) for some \( D \in \text{Div}_T(X_\Sigma) \) and \( \phi \in C^0(\mathcal{U}, \mathcal{O}_\Sigma) \). We show that \( D \) comes from \( M \).

Let \( \tau \in \mathcal{P} \) be a 1-dimensional cell, and \( v_0, v_1 \) be its endpoints. Suppose that we choose \( v_1 \) when we determine \( \tau(D) \), i.e., \( \tau(D) = v_1(D) \). Then we have

\[
\psi(D)((U_{v_0}, U_{\tau})) = v_1(D) - v_0(D) = \phi((U_{\tau})) - \phi((U_{v_0})), \tag{6.32}
\]
\[
\psi(D)((U_{v_1}, U_{\tau})) = 0 = \phi((U_{\tau})) - \phi((U_{v_1})). \tag{6.33}
\]

Here, \( \psi(D)((U_{v_0}, U_{\tau})) = v_1(D) - v_0(D) \) and \( \phi(U_{\tau}) \) are parallel to the direction which is invariant under the monodromy around the singular point on \( \tau \). Hence, from (6.32), (6.33), it turns out that \( \phi((U_{v_0})) \) and \( \phi((U_{v_1})) \) also have to be parallel to this direction.

Let \( \tau' \in \mathcal{P} \) be another 1-dimensional cell that has \( v_0 \) as its vertex. By the same argument, we can see that \( \phi((U_{v_0})) \) has to be parallel also to the direction which is invariant under the monodromy around the singular point on \( \tau' \). Since these two monodromy invariant directions are linearly independent, \( \phi((U_{v_0})) \) has to be zero. Similarly, we get \( \phi((U_{v_1})) = 0 \). Hence, by (6.32), (6.33), we obtain

\[
\phi((U_{\tau})) = 0, \tag{6.34}
\]
\[
v_1(D) = v_0(D). \tag{6.35}
\]

Since there is a sequence of edges of \( B \) connecting arbitrary two vertices of \( B \), we can conclude that the element \( v(D) \in M \) is the same for any vertex \( v \). We write this point as \( m(D) \in M \).

We set \( D = \sum_{\rho \in \Sigma(1)} k_\rho D_\rho \) and \( \text{div}(\chi^{-m(D)}) = \sum_{\rho \in \Sigma(1)} k_\rho D_\rho \). For any \( \rho \in \Sigma(1) \), we take a vertex \( v \in \mathcal{P} \) contained in the facet \( \sigma(\rho) \in \mathcal{P} \) corresponding to \( \rho \). The element \( v(D) \) satisfies

\[
n_1(\rho)x_1 + n_2(\rho)x_2 + n_3(\rho)x_3 = -k_\rho, \tag{6.36}
\]
and the element \( v(\text{div}(\chi^{-m(D)})) = m(D) \) satisfies

\[
n_1(\rho)x_1 + n_2(\rho)x_2 + n_3(\rho)x_3 = -k'_\rho. \tag{6.37}
\]

Since \( v(D) = m(D) \), we have \( k_\rho = k'_\rho \). Hence, we obtain \( D = \text{div}(\chi^{-m(D)}) \). Therefore, the induced map \( \text{Pic}(X_\Sigma) \to \check{H}^1(\mathcal{U}, \mathcal{O}_\Sigma) \) is injective. \( \square \)
Lemma 6.4. The embedding \( \psi : \text{Pic}(X_\Sigma) \to H^1(B, \iota_* \mathcal{T}_Z) \) is primitive, i.e., the image of the map
\[
\psi : \text{Pic}(X_\Sigma) \to H^1(B, \iota_* \mathcal{T}_Z) \to H^1(B, \iota_* \mathcal{T}) = H^1(B, \iota_* \mathcal{T}_Z) \otimes \mathbb{R} \quad (6.38)
\]
coincides with \( \text{Im}(\psi) \otimes \mathbb{R} \cap H^1(B, \iota_* \mathcal{T}_Z) \).

Proof. For the proof, we use another covering \( U' := \{ U'_\tau \}_{\tau \in \mathcal{P}} \) defined as follows: For each vertex \( v \) of \( B \), we choose two distinct 1-dimensional cells \( \tau_1, \tau_2 \in \mathcal{P} \) that contain \( v \) as their endpoint. We set \( U'_v := U_v \cup U_{\tau_1} \cup U_{\tau_2} \) for each vertex \( v \in \mathcal{P} \), and \( U'_\tau = U_\tau \) for each 1 or 2-dimensional cell \( \tau \in \mathcal{P} \). We can check that \( H^1(U'_v, \mathcal{T}_Z) = 0 \) for \( i \geq 0 \) and the covering \( U' \) is also acyclic for \( \iota_* \mathcal{T}_Z \) and \( \iota_* \mathcal{T} \). The covering \( U \) is a refinement of \( U' \) and we have an isomorphism \( \gamma : \hat{H}^1(U', \iota_* \mathcal{T}) \to \hat{H}^1(U, \iota_* \mathcal{T}) \). We also consider the map
\[
\psi' : \text{Div}_T(X_\Sigma) \otimes \mathbb{Z} \to \hat{H}^1(U', \iota_* \mathcal{T}), \quad (6.39)
\]
which is defined as follows: For an element \( D \otimes t \) we have an arbitrary vertex \( v \) of \( \tau \) and set \( \tau(D \otimes t) := v(D \otimes t) \). We define a map \( \psi' \) by setting
\[
\psi'(D \otimes t) \left( (U'_{\tau_{1}}, U'_{\tau_{1}}) \right) := \tau_{1}(D \otimes t) - \tau_{0}(D \otimes t) \quad (6.41)
\]
for each 1-simplex \( (U'_{\tau_{1}}, U'_{\tau_{1}}) \) of \( U' \). The arguments of Lemma 6.1 and Lemma 6.2 are applicable also to \( \psi' \) and we have \( \gamma \circ \psi' = \psi \otimes \text{id} \). This induces an injection
\[
\psi' : \text{Pic}(X_\Sigma) \otimes \mathbb{Z} \to \text{Div}_T(X_\Sigma) \otimes \mathbb{Z} / M_{\mathbb{R}} \to \hat{H}^1(U', \iota_* \mathcal{T}). \quad (6.42)
\]
We will show that there exists \( D \in \text{Div}_T(X_\Sigma) \) such that \( \psi'(D) = \lambda \) for any \( \lambda \in \text{Im}(\psi') \cap \hat{H}^1(U', \iota_* \mathcal{T}) \).

There exists \( D' \otimes t \in \text{Div}_T(X_\Sigma) \otimes \mathbb{Z} \) such that \( \psi'(D' \otimes t) = \lambda \). By adding some element \( \text{div}(\chi^m) \otimes t' \) \((t' \in \mathbb{R}, m \in M)\) to \( D' \otimes t \), we can assume that there exists a vertex \( v_0 \in \mathcal{P} \) such that \( v_0(D) \in M \), where \( D := D' \otimes t + \text{div}(\chi^m) \otimes t' \). Note that \( \psi'(D') = \psi'(D' \otimes t) = \lambda \). Since \( \lambda \) is an element of \( \hat{H}^1(U', \iota_* \mathcal{T}) \), there exists an element \( \phi \in C^0(U', \iota_* \mathcal{T}) \) such that
\[
\delta(\phi) + \psi'(D) \in C^1(U', \iota_* \mathcal{T}_Z). \quad (6.43)
\]
Let \( \tau \in \mathcal{P} \) be any 1-dimensional cell and \( v_1, v_2 \) be its endpoints. We choose \( v_2 \) when we determine \( \tau(D) \), i.e., \( \tau(D) = v_2(D) \). Since \( \iota_* \mathcal{T}(U'_{v_1}) = \iota_* \mathcal{T}(U'_{v_2}) = 0 \), we have
\[
\{ \delta(\phi) + \psi'(D) \} \left( (U'_{v_1}, U'_{v_1}) \right) = \phi(U'_{v_1}) + (v_2(D) - v_1(D)) \quad (6.44)
\]
\[
\{ \delta(\phi) + \psi'(D) \} \left( (U'_{v_2}, U'_{v_2}) \right) = \phi(U'_{v_2}). \quad (6.45)
\]
These are sections of \( \iota_* \mathcal{T}_Z \). Hence, the vector \( (v_2(D) - v_1(D)) \) is contained in \( M \). Since there is a sequence of edges of \( B \) connecting any vertex of \( B \) and \( v_0 \), we have \( v(D) \in M \) for any other vertex \( v \in \mathcal{P} \). This happens only when \( D \in \text{Div}_T(X_\Sigma) \). \( \square \)
Recall that the cohomology group $H^1(B,\iota_*\mathcal{T}_\Sigma)$ has the cup product \([33]\) induced by the wedge product. Let $Y$ be an anti-canonical hypersurface of the complex toric variety $X_\Sigma$ associated with $\Sigma$, and

$$
\text{Pic}(Y)_{\text{amb}} := \text{Im}(\text{Pic}(X_\Sigma) \hookrightarrow \text{Pic}(Y))
$$

be the sublattice of $\text{Pic}(Y)$ coming from the Picard group of the ambient space.

**Lemma 6.5.** The embedding $\psi$: $\text{Pic}(X_\Sigma) \hookrightarrow H^1(B,\iota_*\mathcal{T}_\Sigma)$ preserves the pairing.

**Proof.** We show that $D_{\rho_{i_1}} \cdot D_{\rho_{i_2}} = \psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}})$ for any $\rho_{i_1}, \rho_{i_2} \in \Sigma(1)$. First, we show this in the case where $D_{\rho_{i_1}} \neq D_{\rho_{i_2}}$. A hypersurface in $X_\Sigma$ defined by a polynomial whose Newton polytope is $\Delta$ is an anti-canonical hypersurface. $D_{\rho_{i_1}} \cdot D_{\rho_{i_2}}$ is equal to the number of intersection points $Y \cap D_{\rho_{i_1}} \cap D_{\rho_{i_2}}$ when $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$ is empty, the intersection $Y \cap D_{\rho_{i_1}} \cap D_{\rho_{i_2}}$ is also empty. When $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$ is not empty, the number of intersection points $Y \cap D_{\rho_{i_1}} \cap D_{\rho_{i_2}}$ is the integral length of the edge $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$. We write the length of $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$ as $l$.

We will show that $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}}) = l$.

When we determine $\tau(D_{\rho_{i_j}})$ for each $\tau \in \mathcal{P}$ in order to obtain $\psi(D_{\rho_{i_j}})$ where $j = 1, 2$, we choose a vertex in the following way: For $\tau(D_{\rho_{i_j}})$, if $\tau$ is not contained in $\sigma(\rho_{i_j})$ and has a vertex outside $\sigma(\rho_{i_j})$, we choose one of such vertices. If $\tau$ is contained in $\sigma(\rho_{i_j})$ and all its vertices are in $\sigma(\rho_{i_j})$, we choose one of them arbitrarily.

Here, $v(D_{\rho}) = 0$ when $v$ is not a vertex of $\sigma(\rho)$. Hence, we have $\psi(D_{\rho_{i_j}})((U_{\tau_0}, U_{\tau_1})) = 0$ if both $\tau_0$ and $\tau_1$ are not in $\sigma(\rho_{i_j})$. For each 2-simplex $(U_{\tau_0}, U_{\tau_1}, U_{\tau_2})$ of $\mathcal{U}$, one of $\tau_0, \tau_1, \tau_2$ is a vertex of $B$. Assume $\tau_i$ is that vertex. When $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}})((U_{\tau_0}, U_{\tau_1}, U_{\tau_2})) \neq 0$, either $\tau_0$ or $\tau_1$ is in $\sigma(\rho_{i_1})$, and either $\tau_1$ or $\tau_2$ has to be in $\sigma(\rho_{i_2})$. Then the vertex $\tau_i$ is contained in both $\sigma(\rho_{i_1})$ and $\sigma(\rho_{i_2})$. When $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$ is empty, this never happens. Therefore, we get $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}}) = 0$ for $\rho_{i_1}, \rho_{i_2} \in \Sigma(1)$ such that $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2}) = \emptyset$. In the following, we assume $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2}) \neq \emptyset$, and directly compute $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}})((U_{\tau_0}, U_{\tau_1}, U_{\tau_2}))$ for 2-simplexes $(U_{\tau_0}, U_{\tau_1}, U_{\tau_2})$ such that the vertex $\tau_i$ is contained in the face $\sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2})$. Those simplexes are shown in Figure 6.1 $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}})$ can be nonzero for them.

![Figure 6.1: 2-simplexes where $\psi(D_{\rho_{i_1}}) \cup \psi(D_{\rho_{i_2}})$ can be nonzero](image)

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Let \( v_0, \cdots, v_{10} \) denote the vertices of these 2-simplexes as shown in Figure 6.1. We use the ascending order of index numbers as the total order of the vertices. Let further \( \tau_j \) \( (0 \leq j \leq 10) \) be the cell of \( B \) whose barycenter is \( v_j \). For instance, \( \tau_6 = \sigma(\rho_{i_1}), \tau_4 = \sigma(\rho_{i_2}) \), and \( \tau_5 = \sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2}) \). Let \( e_j \) \((1 \leq j \leq 3)\) be the integral tangent vectors at \( v_8 \) contained in \( \mathbb{R}_{>0} \cdot \tilde{v}_8 v_9, \mathbb{R}_{>0} \cdot \tilde{v}_8 v_{10}, \mathbb{R}_{>0} \cdot \tilde{v}_8 v_7 \) respectively. Let further \( e_i \) \((i = 4, 5)\) be the integral tangent vectors at \( v_2 \) contained in \( \mathbb{R}_{>0} \cdot \tilde{v}_2 v_3, \mathbb{R}_{>0} \cdot \tilde{v}_2 v_1 \) respectively. We can assume that
\[
e_4 = e_1 + se_3, \quad e_5 = e_2 + te_3, \quad (6.47)
\]

where \( s, t \in \mathbb{Z} \).

Consider the monodromy with respect to a loop that starts at \( v_8 \), passes through \( v_4, v_2, v_6 \) in this order, and comes back to \( v_8 \). From Lemma 4.1, we can see that it is given by
\[
\begin{pmatrix}
1 & -l \\
0 & 1
\end{pmatrix},
\quad (6.48)
\]

under the basis \((e_3, e_1)\), where \( l \) is the length of \( \sigma(\rho_{i_1}) \cap \sigma(\rho_{i_2}) \). On the other hand, we can also calculate the monodromy of \( e_1 \) as follows: We have \( e_1 = -e_2 - e_3 \) in \( U_{v_8} \). Since we have \( e_2 = e_5 - te_3, e_1 \) becomes \((t - 1)e_3 - e_5\) when it arrives at \( v_2 \). We also have \( e_5 = e_3 - e_4 \) in \( U_{v_2} \), and \( e_4 = e_1 + se_3 \). Therefore, \( e_1 \) becomes \((s + t - 2)e_3 + e_1\) when it is back to \( v_2 \). Hence we obtain
\[
s + t - 2 = -l. \quad (6.49)
\]

In order to determine each \( \tau_j(D_{\rho_{i_1}}) \), we choose a vertex as follows: For \( v_0, v_1, v_4 \), we choose the endpoint of \( \tau_1 \) which is not \( v_2 \). For \( v_3, v_5, v_6 \), we choose \( v_2 \). For \( v_7, v_{10} \), we choose the endpoint of \( \tau_7 \) which is not \( v_8 \). For \( v_9 \), we choose \( v_8 \). In order to determine \( \tau_i(D_{\rho_{i_2}}) \), we choose a vertex for each \( v_i \) as follows: For \( v_0, v_3, v_6 \), we choose the endpoint of \( \tau_3 \) which is not \( v_2 \). For \( v_1, v_4, v_5 \), we choose \( v_2 \). For \( v_7, v_{10} \), we choose \( v_8 \). For \( v_9, v_{10} \), we choose the endpoint of \( \tau_9 \) which is not \( v_8 \). Then we have
\[
\begin{align*}
\psi(D_{\rho_{i_1}})((U_{j_1}, U_{j_2})) &= \begin{cases} 
-e_5 & (j_1, j_2) = (0, 2), (0, 3), (1, 2), (4, 5), \\
e_5 & (j_1, j_2) = (2, 4), \\
-e_2 & (j_1, j_2) = (4, 8), (7, 8), \\
e_5 - e_2 & (j_1, j_2) = (5, 8), (6, 8), (6, 9), \\
e_2 & (j_1, j_2) = (8, 10), (9, 10), \\
0 & \text{otherwise,}
\end{cases} \\
\psi(D_{\rho_{i_2}})((U_{j_1}, U_{j_2})) &= \begin{cases} 
-e_4 & (j_1, j_2) = (0, 1), (0, 2), \\
e_4 & (j_1, j_2) = (2, 3), (2, 6), (5, 6), \\
e_4 - e_1 & (j_1, j_2) = (4, 7), (4, 8), (5, 8), \\
é_1 & (j_1, j_2) = (6, 8), \\
e_1 & (j_1, j_2) = (7, 10), (8, 9), (8, 10), \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\quad (6.50, 6.51)
for $0 \leq j_1 < j_2 \leq 10$. We obtain

$$
\psi(D_{\rho_1}) \cup \psi(D_{\rho_2})((U_{j_1}, U_{j_2}, U_{j_3})) = \begin{cases} 
-e_5 \land e_4 & (j_1, j_2, j_3) = (0, 2, 3), \\
-e_5 \land (e_4 - e_1) & (j_1, j_2, j_3) = (4, 5, 8), \\
(e_5 - e_2) \land e_1 & (j_1, j_2, j_3) = (6, 8, 9), \\
-e_2 \land e_1 & (j_1, j_2, j_3) = (7, 8, 10), \\
0 & \text{otherwise},
\end{cases}
$$

(6.52)

$$
\sim \begin{cases} 
(2 - s - t)e_1 \land e_2 & (j_1, j_2, j_3) = (7, 8, 10), \\
0 & \text{otherwise}
\end{cases}
$$

(6.53)

for $0 \leq j_1 < j_2 < j_3 \leq 10$. We choose the orientation of $B$ so that the element in $C^2(\mathcal{U}, \mathbb{Z})$ taking $e_1 \land e_2$ for the 2-simplex $(U_7, U_8, U_{10})$ and 0 for any other 2-simplexes represents $1 \in \mathbb{Z} \cong H^2(B, \mathbb{Z})$. Then we get $\psi(D_{\rho_1}) \cup \psi(D_{\rho_2}) = 2 - s - t = l$.

Lastly, we show that $D_{\rho_0} \cdot D_{\rho_0} = \psi(D_{\rho_0}) \cup \psi(D_{\rho_0})$ for any $\rho_0 \in \Sigma$. Since there exists a primitive element $m \in M$ such that $\text{div}(\chi^m) = D_{\rho_0} + \sum_{\rho \neq \rho_0} a_\rho D_\rho$ ($a_\rho \in \mathbb{Z}$), we have

$$
D_{\rho_0} \sim - \sum_{\rho \neq \rho_0} a_\rho D_\rho.
$$

Hence, we obtain

$$
D_{\rho_0} \cdot D_{\rho_0} = D_{\rho_0} \cdot \left(- \sum_{\rho \neq \rho_0} a_\rho D_\rho \right) = - \sum_{\rho \neq \rho_0} a_\rho D_{\rho_0} \cdot D_\rho
$$

(6.55)

$$
= - \sum_{\rho \neq \rho_0} a_\rho \psi(D_{\rho_0}) \cup \psi(D_\rho) = \psi(D_{\rho_0}) \cup \psi \left(- \sum_{\rho \neq \rho_0} a_\rho D_\rho \right)
$$

(6.56)

$$
= \psi(D_{\rho_0}) \cup \psi(D_{\rho_0}).
$$

(6.57)

6.2 Proof of Theorem 1.2

For each cell $\tau \in \mathcal{P}$, we choose an arbitrary vertex $v(\tau) \in M_\mathcal{R}$ of $\tau$. Here, if $\tau$ is a vertex, then $v(\tau) = \tau$. By enlarging $U_\tau$ slightly, we assume that $U_\tau$ contains $v(\tau)$. We take each chart $\psi_\tau : U_\tau \to M_\mathcal{R}$ so that $\psi_\tau(v(\tau)) = 0 \in M_\mathcal{R}$. In order to specify the radiance obstruction $c_B$ of $B$, we choose the zero section $0 \in \Gamma(U_\tau \cap B_0, T_{\text{aff}}B_0)$ for each $U_\tau$. Then the radiance obstruction $c_B$ is represented by the element of $C^1(\mathcal{U}, \iota_* \mathcal{T})$ determined by setting

$$
c_B((U_{\tau_0}, U_{\tau_1})) := v(\tau_1) - v(\tau_0)
$$

(6.58)

for each 1-simplex $(U_{\tau_0}, U_{\tau_1})$ of $\mathcal{U}$.

Let $v \in \mathcal{P}$ be a vertex, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets of $B$ containing $v$. Then the vertex $v$ is determined by

$$
n_1(\rho_1)x_1 + n_2(\rho_1)x_2 + n_3(\rho_1)x_3 = -a(n_{\rho_1}) + a(0), \quad 1 \leq i \leq 3.
$$

(6.59)
From the definition of the map $\psi$, we can see that

$$c_B = \psi \left( \sum_{\rho \in \Sigma(1)} \{a(n_\rho) - a(0)\} D_\rho \right)$$

$$= \sum_{\rho \in \Sigma(1)} \{a(n_\rho) - a(0)\} \psi(D_\rho).$$

(6.60)

(6.61)

7 Asymptotic behaviors of period maps

Let $K := \mathbb{C}(t)$ be the convergent Puiseux series field, equipped with the standard non-archimedean valuation

$$\text{val}: K \longrightarrow \mathbb{Q} \cup \{\infty\}, \quad k = \sum_{j \in \mathbb{Z}} c_j t^j \mapsto \min \{j \in \mathbb{Q} | c_j \neq 0\}. \quad (7.1)$$

Let $M$ be a free $\mathbb{Z}$-module of rank 3 and $N := \text{Hom}(M, \mathbb{Z})$ be the dual lattice. We set $M_R := M \otimes \mathbb{Z} \mathbb{R}$ and $N_R := N \otimes \mathbb{Z} \mathbb{R} = \text{Hom}(M, \mathbb{R})$. Let $\Delta \subset M_R$ be a smooth reflexive polytope of dimension 3, and $\check{\Delta} \subset N_R$ be the polar polytope of $\Delta$. Let further $\Sigma \subset N_R, \check{\Sigma} \subset M_R$ be the normal fans of $\Delta, \check{\Delta}$ respectively. We choose a refinement $\check{\Sigma}' \subset M_R$ of $\check{\Sigma}$ which gives rise to a projective crepant resolution $X_{\check{\Sigma}'} \rightarrow X_{\check{\Sigma}}$ of toric varieties associated with $\check{\Sigma}$.

We consider a Laurent polynomial $F = \sum_{n \in A} k_n x^n \in K[x_1^\pm, x_2^\pm, x_3^\pm]$ over $K$ whose Newton polytope is $\Delta$, where $A \subset N$ denotes the subset consisting of all vertices of $\check{\Delta}$ and $0 \in N$. We assume that the function

$$A \rightarrow \mathbb{R}, \quad n \mapsto \text{val}(k_n) \quad (7.2)$$

induces a central subdivision of $\check{\Delta}$. Let $\text{trop}(F)$ be the tropicalization of $F$, which is defined as the tropical polynomial defined by

$$\text{trop}(F)(x) := \max_{n \in A} \{\text{val}(k_n) + n_1 x_1 + n_2 x_2 + n_3 x_3\}. \quad (7.3)$$

Let further $B$ be the 2-sphere with an integral affine structure with singularities obtained by contracting the tropical hypersurface defined by $\text{trop}(F)$ in the tropical toric variety $X_{\check{\Sigma}'}(\mathbb{T})$ associated with $\check{\Sigma}'$ in the way of Section 4.4. We write the radiance obstruction of $B$ as $c_B \in H^1(B, \iota_* \mathbb{T})$.

Let $D \subset \mathbb{C}$ be the open unit disk. We consider the universal covering of $D \setminus \{0\}$

$$e: \mathbb{H} \rightarrow D \setminus \{0\}, \quad z \mapsto \exp(2\pi \sqrt{-1} z), \quad (7.4)$$

where $\mathbb{H}$ is the upper half plane. We set

$$H_R := \{z \in \mathbb{H} | \text{Im} z > R\}, \quad (7.5)$$

where $R$ is a positive real number such that $e(\sqrt{1}R)$ is smaller than the radius of convergence of $k_n$ for any $n \in A$. For each element $z \in H_R$, we consider the polynomial $f_z \in \mathbb{C}[x_1^\pm, x_2^\pm, x_3^\pm]$ obtained by substituting $e(z)$ to $t$ in $F$. Let $V_z$ be the complex hypersurface defined by $f_z$ in the complex toric variety $X_{\check{\Sigma}'}$ associated with $\check{\Sigma}'$. This is a quasi-smooth K3 hypersurface.
We describe the asymptotic behavior of the period of $V_z$ in the limit $R \to \infty$ by using the radiance obstruction $c_B$. In the following, we assume $k_0 = 1$ by multiplying an element of $K$ to $F$. Some parts of the following are borrowed from [Ued14, Section 7].

For a given element $\alpha = (a_n)_{n \in \mathbb{A}_1(0)} \in (\mathbb{C}^\times)^{\mathbb{A}_1(0)}$, we associate the polynomial

$$W_\alpha(x) = 1 + \sum_{n \in \mathbb{A}_1(0)} a_n x^n. \quad (7.6)$$

We write the toric hypersurface defined by $W_\alpha$ in the complex toric variety $X_{\Sigma'}$ as $Y_\alpha$. Let $(\mathbb{C}^\times)^{\mathbb{A}_1(0)}$ be the set of $\alpha \in (\mathbb{C}^\times)^{\mathbb{A}_1(0)}$ such that $Y_\alpha$ is $\Sigma'$-regular, i.e., the intersection of $Y_\alpha$ with any torus orbit of $X_{\Sigma'}$ is a smooth subvariety of codimension one. We consider the family of $\Sigma'$-regular hypersurfaces given by the second projection

$$\varphi: \mathcal{Y} := \{(x, \alpha) \in X_{\Sigma'} \times (\mathbb{C}^\times)^{\mathbb{A}_1(0)} \mid W_\alpha(x) = 0\} \to (\mathbb{C}^\times)^{\mathbb{A}_1(0)}, \quad (7.7)$$

and the action of $M \otimes_\mathbb{Z} \mathbb{C}^\times$ to this family given by

$$t \cdot (x, \alpha) := (t^{-1} x, (t^n a_n)_{n \in \mathbb{A}_1(0)}), \quad (7.8)$$

where $t \in M \otimes_\mathbb{Z} \mathbb{C}^\times$. We write the quotient by this action as $\tilde{\varphi}: \mathcal{Y} \to \mathcal{M}_{\text{reg}}$, where $\mathcal{M}_{\text{reg}} := (\mathbb{C}^\times)^{\mathbb{A}_1(0)} / (M \otimes_\mathbb{Z} \mathbb{C}^\times)$. The space $\mathcal{M}_{\text{reg}}$ can be regarded as a parameter space of $\Sigma'$-regular hypersurfaces whose Newton polytopes are $\tilde{\Delta}$. Let $(\mathcal{H}_B, \nabla^B, H_{B\nu}, \mathcal{F}^*_\nu, Q_B)$ be the residual B-model VHS of the family $\tilde{\varphi}: \mathcal{Y} \to \mathcal{M}_{\text{reg}}$ [TRT] Definition 6.5.

When $R$ is sufficiently large, the hypersurface $V_z$ is $\Sigma'$-regular for any $z \in H_R$. We have a map $l$ given by

$$l: H_R \to \mathcal{M}_{\text{reg}}, \quad z \mapsto [(k_n(e(z)))_{n \in \mathbb{A}_1(0)}], \quad (7.9)$$

where $k_n(e(z))$ is the complex number obtained by substituting $e(z)$ to $t$ in $k_n$. We define a holomorphic form on $V_z$ by

$$\Omega_z := \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}. \quad (7.10)$$

This defines a section of $l^* \mathcal{H}_B$ over $H_R$.

Let $Y$ be an anti-canonical hypersurface of the complex toric variety $X_{\Sigma}$ associated with $\Sigma$, and $\iota: Y \hookrightarrow X_{\Sigma}$ be the inclusion. Choose an integral basis $\{p_i\}_{i=1}^r \subset \text{Pic} X_{\Sigma}$ such that each $p_i$ is nef. This determines a coordinate $q = (q_1, \cdots, q_r)$ on $\mathcal{M} := \text{Pic} X_{\Sigma} \otimes_\mathbb{Z} \mathbb{C}^\times = (\mathbb{Z}^{\mathbb{A}_1(0)}/M) \otimes_\mathbb{Z} \mathbb{C}^\times \supset \mathcal{M}_{\text{reg}}$ and a coordinate $\tau = (\tau_1, \cdots, \tau_r)$ on $H^2_{\text{amb}}(Y, \mathbb{C})$. Let $\omega_i \in H^2(X_{\Sigma}, \mathbb{Z})$ be the Poincaré dual of the toric divisor $D_{p_i}$ corresponding to the one-dimensional cone $p_i \subset \Sigma$, and $v = v_1 + \cdots + v_m$ be the anticanonical class. Givental’s $I$-function is defined as the series

$$I_{X_{\Sigma}, \nu}(q, z) = e^{p \log q/z} \sum_{d \in \text{Eff}(X_{\Sigma})} q^d \prod_{k=\infty}^{d,v} (v + k z) \prod_{j=1}^{m_k} (u_j + k z),$$

which gives a multi-valued map from an open subset of $\mathcal{M} \times \mathbb{C}^\times$ to the classical cohomology ring $H^\bullet(X_{\Sigma}, \mathbb{C}[z^{-1}])$. Here, $\text{Eff}(X_{\Sigma})$ denotes the set of effective toric divisors on $X_{\Sigma}$. We write

$$I_{X_{\Sigma}, \nu}(q, z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}). \quad (7.11)$$

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The mirror map $\varsigma: \mathcal{M} \rightarrow H_{\text{amb}}^2(Y, \mathbb{C})$ is a multi-valued map defined by

$$i^* \left( \frac{G(q)}{F(q)} \right).$$  \hfill (7.12)

The residual B-model VHS is isomorphic to the ambient A-model VHS $(\mathcal{H}_A, \nabla^A, \mathcal{F}_A^*, Q_A)$ [Iri11, Definition 6.2] via the mirror map $\varsigma$ [Iri11, Theorem 6.9]. Hence, the section of $l^* \mathcal{H}_B$ over $H_R$ defined by $\Omega_z$ can also be regarded as a section of $(\varsigma \circ l^*) \mathcal{H}_A$ via the mirror isomorphism. Here we replace the real number $R$ with a larger one if necessary.

We also choose elements $p_0 \in H_{\text{amb}}^0(Y; \mathbb{Q})$ and $p_{r+1} \in H_{\text{amb}}^4(Y; \mathbb{Q})$ so that we have $p_0 \cup p_{r+1} = 1$, and define sections

$$\tilde{p}_i := \exp(-\tau) \cup p_i, \quad 0 \leq i \leq r + 1,$$

(7.13)
of $\mathcal{H}_A$. These are flat sections with respect to the Dubrovin connection $\nabla^A$. Note that the quantum cup product coincides with the ordinary cup product, since $Y$ is a K3 surface. The sections $\{\tilde{p}_i\}_{i=0}^{r+1}$ form an integral structure of $H_{A,\mathbb{C}}$ defined in [Iri11, Definition 6.3] by a linear transformation.

We consider a map $\phi: l^* H_{A,\mathbb{C}}(H_R) \rightarrow (U \oplus \text{Pic}(Y)_{\text{amb}}) \otimes_{\mathbb{Z}} \mathbb{C}$ defined by

$$\tilde{p}_0 \mapsto 2\pi\sqrt{-1}e, \quad \tilde{p}_{r+1} \mapsto 2\pi\sqrt{-1}f, \quad \tilde{p}_i \mapsto -2\pi p_i \quad (1 \leq i \leq r),$$

(7.14)

where $U$ denotes the hyperbolic plane and $(e, f)$ is its standard basis. This is an isomorphism preserving the pairing. We obtain the period map

$$H_R \rightarrow \mathbb{P}((U \oplus \text{Pic}(Y)_{\text{amb}}) \otimes_{\mathbb{Z}} \mathbb{C})$$

(7.15)
determined by $\Omega_z$ via the mirror isomorphism and the map $\phi$. The image of this map is contained in

$$D' := \{[\sigma] \in \mathbb{P}((U \oplus \text{Pic}(Y)_{\text{amb}}) \otimes_{\mathbb{Z}} \mathbb{C}) \mid (\sigma, \sigma) = 0, (\sigma, \sigma) > 0\}.$$  \hfill (7.16)

We also set

$$D := \{\sigma \in \text{Pic}(Y)_{\text{amb}} \otimes_{\mathbb{Z}} \mathbb{C} \mid (\Re \tau, \Re \tau) > 0\}.$$  \hfill (7.17)

Here we have an isomorphism $D' \cong D$ of complex manifolds given by

$$k_1 e + k_2 f + \sigma \mapsto \frac{-\sqrt{-1}}{k_1} \sigma,$$

(7.18)

where $k_1, k_2 \in \mathbb{C}$ and $\sigma \in \text{Pic}(Y)_{\text{amb}} \otimes_{\mathbb{Z}} \mathbb{C}$. By this isomorphism, we obtain the period map $\mathcal{P}: H_R \rightarrow D$.

**Corollary 7.1.** The leading term of the period map $\mathcal{P}$ in the limit $R \rightarrow \infty$ is given by

$$-2\pi\sqrt{-1}z \cdot \psi^{-1}(c_B),$$

(7.19)

where $\psi$ denotes the map $\psi \otimes_{\mathbb{Z}} \text{id}_R: \text{Pic}(Y)_{\text{amb}} \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H^1(B, l^*\mathcal{T})$.  

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Proof. From Theorem 1.2, we can see that the radiance obstruction $c_B$ of $B$ is given by

$$c_B = \sum_{n \in A \setminus \{0\}} \text{val}(k_n)\psi(D_{\rho_n}),$$  \hspace{1cm} (7.20)

where $\rho_n \in \Sigma(1)$ is the cone corresponding to the vertex $n \in A \setminus \{0\}$ of $\Delta$.

The holomorphic form $\Omega_z$ corresponds to $F(q) \cdot 1 \in H^0_{\text{amb}}(Y, \mathbb{C})$ under the mirror isomorphism [Iri11 Theorem 6.9], where $F(q)$ is the first term of the Givental’s I-function (7.11). It turns out that the period map $\mathcal{P} : H_R \to \mathcal{D}$ is given by $z \mapsto \zeta(l(z))$. The leading term of this is given by

$$\sum_{i=1}^r p_i \log q_i(l(z)).$$  \hspace{1cm} (7.21)

Suppose $D_{\rho_n} = \sum_{i=1}^r b_{n,i}p_i$ in $\text{Pic}(X_{\Sigma})$, where $b_{n,i} \in \mathbb{Z}$. Then we have

$$q_i(l(z)) = \prod_{n \in A \setminus \{0\}} k_n(e(z))^{b_{n,i}}.$$  \hspace{1cm} (7.22)

Hence, we obtain

$$\sum_{i=1}^r p_i \log q_i(l(z)) = \sum_{i=1}^r p_i \left( \sum_{n \in A \setminus \{0\}} b_{n,i} \log (k_n(e(z))) \right)$$  \hspace{1cm} (7.23)

$$\sim - \sum_{i=1}^r p_i \left( \sum_{n \in A \setminus \{0\}} b_{n,i} \log e(z) \cdot \text{val}(k_n) \right)$$  \hspace{1cm} (7.24)

$$= - \log e(z) \sum_{n \in A \setminus \{0\}} \text{val}(k_n) \sum_{i=1}^r b_{n,i}p_i$$  \hspace{1cm} (7.25)

$$= -2\pi \sqrt{-1}z \sum_{n \in A \setminus \{0\}} \text{val}(k_n)D_{\rho_n}$$  \hspace{1cm} (7.26)

$$= -2\pi \sqrt{-1}z \cdot \psi^{-1}(c_B).$$  \hspace{1cm} (7.27)

Corollary 7.1 implies that the radiance obstruction $\psi^{-1}(c_B) \in \text{Pic}(Y)_{\text{amb}} \otimes \mathbb{Z} \otimes \mathbb{R}$ can be regarded as the period of the tropical K3 hypersurface defined by $\text{trop}(F)$. We can also obtain

$$(\psi^{-1}(c_B), \psi^{-1}(c_B)) \geq 0$$  \hspace{1cm} (7.28)

from Corollary 7.1 and the inequality $(\Re \tau, \Re \tau) > 0$ of (7.17). The following inequality (7.29) can be regarded as a tropical version of the Hodge–Riemann bilinear relation for K3 surfaces appearing in (7.16).

**Corollary 7.2.** One has

$$(\psi^{-1}(c_B), \psi^{-1}(c_B)) > 0.$$  \hspace{1cm} (7.29)
Proof. From Theorem 1.2 and the assumption that the function \( A \to \mathbb{R}, \, n \mapsto \text{val}(k_n) \) induces a central subdivision of \( \Delta \), we can see that there exists a set of negative real numbers \( \{b(n)\}_{n \in A \setminus \{0,n_0\}} \) such that
\[
\psi^{-1}(c_B) = \sum_{n \in A \setminus \{0\}} \text{val}(k_n) D_{\rho_n} \sim \sum_{n \in A \setminus \{0,n_0\}} b(n) D_{\rho_n}
\] (7.30)
for any \( n_0 \in A \setminus \{0\} \). We have
\[
(D_{\rho_{n_0}}, \psi^{-1}(c_B)) = \left(D_{\rho_{n_0}}, \sum_{n \in A \setminus \{0,n_0\}} b(n) D_{\rho_n}\right)
\] (7.31)
< 0
(7.32)
for any \( n_0 \in A \setminus \{0\} \), and hence
\[
(\psi^{-1}(c_B), \psi^{-1}(c_B)) = \left(\sum_{n \in A \setminus \{0,n_0\}} b(n) D_{\rho_n}, \psi^{-1}(c_B)\right)
\] (7.33)
\[
= \sum_{n \in A \setminus \{0,n_0\}} b(n) \left(D_{\rho_n}, \psi^{-1}(c_B)\right)
\] (7.34)
\[
> 0.
\] (7.35)

Remark 7.3. As we saw in Remark 4.3 and Remark 4.7, there are ambiguities in the choices of positions of singular points when we contract tropical toric hypersurfaces, and the radiance obstruction does not depend on these choices. This means that moving singular points to monodromy invariant directions does not change the period of the tropical K3 surface \( B \). We can infer that we should think that a tropical K3 surface which is obtained by moving singular points to monodromy invariant directions is “equivalent” to the original one.

Remark 7.4. The space
\[
\{\sigma \in \text{Pic}(Y)_{\text{amb}} \otimes_{\mathbb{Z}} \mathbb{R} \mid (\sigma, \sigma) > 0\}
\] (7.36)
is the period domain of tropical K3 hypersurfaces. This is the numerator of the moduli space of lattice polarized tropical K3 surfaces [HU18, Section 5]. In [O018, O0on], they construct Gromov–Hausdorff compactifications of polarized complex K3 surfaces by adding moduli spaces of lattice polarized tropical K3 surfaces to their boundaries.

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