THERE ARE NO REALIZABLE 15₄- AND 16₄-CONFIGURATIONS

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ABSTRACT. There exist a finite number of natural numbers \( n \) for which we do not know whether a realizable \( n₄ \)-configuration does exist. We settle the two smallest unknown cases \( n = 15 \) and \( n = 16 \). In these cases realizable \( n₄ \)-configurations cannot exist even in the more general setting of pseudoline-arrangements. The proof in the case \( n = 15 \) can be generalized to \( n_k \)-configurations. We show that a necessary condition for the existence of a realizable \( n_k \)-configuration is that \( n > k^2 + k - 5 \) holds.

1. INTRODUCTION

Point line configurations have a long history. Levi’s book [10] about the subject starts with the remark that they can be considered a starting point for studying combinatorial geometry. It was also Levi who wrote in 1926 the first known paper [9] of pseudoline arrangements, the antecedent of the general oriented matroid concept. Within this paper we use the latter concept in the context of configurations. We assume the reader to be familiar with basic concepts from the theory of oriented matroids in the rank 3 case (see for instance [6] or [2, Chapter 6]). For an introduction to the theory of oriented matroids see also [3].

We fix our notation in Section 2, however, an intuitive impression of a realizable \( n₄ \)-configuration can be obtained from looking at the smallest known example of a realizable \( n₄ \)-configuration for \( n = 21 \) in Figure 1. A realizable \( n₄ \)-configuration consists of two \( n \) element sets, a set of \( n \) points and a set of \( n \) lines in the Euclidean plane. The defining property of an \( n₄ \)-configuration requires each element of one set to be incident with precisely 4 elements of the other.

It is known that realizable \( n₄ \)-configurations do not exist for \( n \leq 14 \). For \( 15 \leq n \leq 20 \) and for \( n = 22, 23, 26, 29, 31, 32, 34, 37, 38, 43 \), the existence of realizable \( n₄ \)-configurations is a long standing problem in this context, whereas for all other \( n \), we do have realizable configurations (see [7]). For all even values that are larger than 21, realizations with pseudolines are known. For the values \( n = 22 \) and \( n = 28 \) these can be found see [8]. For the

2000 Mathematics Subject Classification. Primary 52C30, Secondary 05B30.

Key words and phrases. configuration, oriented matroid, pseudoline arrangement.
In Section 2 we prove a general Theorem 1, which implies in particular that a realizable $15_{4}$-configuration does not exist. The proof for the case $n = 16$ in Section 3 is more involved. Our results have been achieved without using a computer in our final argument, although a foregoing result of the second author in connection with computations of Betten and Betten (see [1]) did use a computer in the case $n = 16$. In Section 4 we give an overview of this foregoing result.

2. ON GENERAL $n_k$-CONFIGURATIONS

An $n_k$-configuration (with $k \geq 3$) is a matroid $M$ of rank 3 on $n$ points such that every line of $M$ has at most $k$ points and each point is contained in exactly $k$ $k$-point lines. We say that an $n_k$-configuration is in general position, if the only number of points on a line is either $k$ or 2.

**Theorem 1.** A realizable $n_k$-configuration can only exist if $n > k^2 + k - 5$ holds. This implies that a realizable $15_{4}$-configuration does not exist.

Let $\mathcal{C}$ be a $n_k$-configuration. If an arrangement of pseudolines has exactly the incidences prescribed by $\mathcal{C}$, we say that $\mathcal{C}$ is realizable with pseudolines; or short pl-realizable. By the Folkman-Lawrence representation theorem (for an elementary proof in the rank-3-case, see [4]) this is equivalent to the fact that the matroid $M$ underlying $\mathcal{C}$ is orientable. Our definition of pl-realizability implies that every realizable configuration – that is one that can be drawn with straight lines in the projective plane – is also pl-realizable. So pl-realizability is a necessary condition for realizability in the ordinary sense.

We only have to deal with points of $\mathcal{C}$ in general position:
Remark. Let \( C \) be a pl-realizable \( n_k \)-configuration. Then there exists a pl-realizable \( n_k \)-configuration \( C' \) such that \( C' \) is in general position.

We note that we can always change to polar formulation where we switch the roles of points and lines.

\textbf{Proof of Theorem 1.} The result follows from an application of Euler’s formula. We may assume that \( C \) is in general position. We assume further that we are given a pl-realization of \( C \) on the sphere. This induces a graph embedding on the sphere.

We count the number of vertices and edges. The number of vertices is given by
\[
 f_0 = 2\left(n + \binom{n}{2} - n\binom{k}{2}\right) = n(n - k(k - 1) + 1).
\]
The number of edges is given by
\[
 f_1 = 2n(k + (n - k(k - 1) - 1)) = 2n(n - k^2 + 2k - 1).
\]
From Euler’s formula we can deduce the number of cells: \( f_2 = f_1 - f_0 + 2 \).

A pseudoline-arrangement as a pl-realization of \( C \) implies that digons are not allowed. By double-counting edge-cell incidences we get the following additional inequality: \( 3f_2 \leq 2f_1 \). Plugging in the above expressions for \( f_0 \), \( f_1 \), and \( f_2 \) our inequality becomes:
\[
 -n^2 - 5n + nk^2 + nk + 6 \leq 0.
\]
For fixed \( k \geq 3 \) and nonnegative \( n \) the expression on the left-hand-side is monotonically decreasing. For \( n = k^2 + k - 5 \) the inequality does not hold whereas for \( n = k^2 + k - 4 \) the inequality holds. \( \square \)

Remark. The proof allows us to replace \textit{realizable} with \textit{pl-realizable} in the statement of Theorem 1.

\textbf{Corollary 1.} \textit{Realizable} 154-configurations do not exist.

We are using the fact that two \( n_k \)-configurations in general position are \( \pi \)-equivalent; they have the same Poincaré polynomial (for a definition, see the book by Orlik and Terao [11]).

3. \textbf{The Case} \( n = 16 \)

The aim of this section is to prove the following theorem:

\textbf{Theorem 2.} \textit{Realizable} 164-configurations do not exist.

We start of with some convenient definitions: We call the intersection point between pseudolines \textit{crossing}. We call such a point an \textit{n-crossing} if exactly \( n \) pseudolines go through that point. As in the section above we assume that our configuration is in general position. So, we only have to deal with 2- and 4-crossings. We pick an arbitrary pseudoline as line at infinity in our arrangement and we denote it with \( \infty \).
In the case \( n = 16 \) we have to have exactly four 4-crossings and three 2-crossings on each pseudoline. This means that we have two 4-crossings that are adjacent on our pseudoline \( \infty \). Now call the elements that intersect in these two 4-crossings \( a, b, c, \) and \( d, e, f \), respectively. These six pseudolines have to have nine additional distinct crossings which we label from \( A \) to \( I \) as in Figure 2(a). We call these crossings also grid points to distinguish them from other crossings. Note that we have chosen our starting situation so that no pseudoline in our arrangement can go to \( \infty \) between the pseudolines \( c \) and \( d \) above \( A \) and no pseudoline can go to \( \infty \) between \( a \) and \( f \) below \( I \).

All the pseudolines \( a, \ldots, f \) cross \( \infty \) in a 4-crossing. This means that they contain only six further crossings in total: three 4-crossings and three 2-crossings. We further remark that at most nine and at least six of the grid points have to be 4-crossings.

However, not all nine grid points can be 4-crossings. This follows from the following Lemma.

\[\text{Figure 2.} \]

**Lemma 1.** The grid points \( A \) and \( I \) cannot be both 4-crossings.

**Proof.** Assume both \( A \) and \( I \) were 4-crossings. Then the two non-grid pseudolines leaving \( A \) have to cross the line \( b \) in two distinct crossings that both lie above \( B \). However, the two non-grid pseudolines leaving \( I \) have to cross \( b \) in two distinct crossings as well, but those crossings have to lie below \( H \). So we get four additional crossings to the crossings \( B, E, \) and \( H \) on \( b \). This is a contradiction. \( \square \)

If eight grid points are 4-crossings, we may assume that one of the points \( A \) or \( I \) is a 2-crossing. Now we can show that the case of eight 4-crossings in the grid cannot occur.

**Lemma 2.** At most seven grid points can be 4-crossings.
Proof. We cannot have nine grid points that are 4-crossings. This would contradict Lemma 1. So, assume we had eight grid points that were 4-crossings. We may then assume w.l.o.g. that \( A \) is a 4-crossing and \( I \) is not. So we are in the situation of Figure 2(a). We can then see that the new pseudolines leaving \( C \) have to cross \( b \) in at least one new crossing. The new pseudolines leaving \( G \) have to cross \( b \) in at least one new crossing as well. However, this would give us seven crossings on \( b \), which is impossible. \( \square \)

Now we deal with the case that \( A \) and \( I \) are both 2-crossings.

**Lemma 3.** The grid points \( A \) and \( I \) cannot be both 2-crossings.

**Proof.** Assume both were 2-crossings. Then at most one further grid point is a 2-crossing. So w.l.o.g. we are in the situation of Figure 2(b). No 4-crossings can lie in the bold 1-cells of \( c \), otherwise we would get too many crossings on \( f \). By symmetry the same holds for the bold 1-cells of \( g \). So we know the 1-cells in which the further 4-crossings on \( c \) resp. \( g \) lie. We count the number of lines entering the 2-cell on the right which is bordered by the dashed line. This number is ten, but only nine lines are leaving this cell to cross \( \infty \), which is a contradiction. \( \square \)

From now on we can always assume that \( A \) is a 4-crossing, \( I \) is a 2-crossing and we have at least one and at most two further grid points that are 2-crossings. First we deal with the case that precisely one further grid point is a 2-crossing.

**Lemma 4.** At most six grid points can be 4-crossings.

**Proof.** We deal with two cases seperately: First we assume \( E \) is our further 2-crossing. Then we are in the situation of Figure 3(a). The bold 1-cells

![Figure 3](image-url)
cannot contain a 4-crossing. This would lead to too many crossings on line $b$, or $e$, respectively. So one 4-crossing has to lie above the point $C$ on $e$. Now there are again ten pseudolines entering the 2-cell which is bordered by the dashed line, but again only nine pseudolines form our arrangement. This is the desired contradiction in this case.

So we may assume $E$ is a 4-crossing. So we are in the situation of Figure 3(b). By symmetry we can assume w.l.o.g. that $G$ is a 4-crossing. Now, however, $H$ has to be a 4-crossing as well, otherwise we would get a contradiction. The two lines coming from $G$ cross $b$ in two points. If one of these was not $H$, we would have eight crossings on $b$. So both $G$ and $H$ have to be 4-crossings.

Now by symmetry we may assume that $C$ is a 4-crossing. Then the two lines coming from $C$ that cross $b$ give at least one new crossing on $b$. Together with the crossing that $H$ gave, we have eight crossings in total. This is our desired contradiction. So we have shown that no seven grid points can be 4-crossings. Together with Lemma 2 we have shown the lemma. □

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure4a}
\caption{(a)}
\end{subfigure} \qquad \begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure4b}
\caption{(b)}
\end{subfigure}
\caption{Figure 4.}
\end{figure}

The following lemma reduces the number of possible positions for the 2-crossings.

**Lemma 5.** Assume $A$, $F$, and $H$ are 4-crossings and $I$ is a 2-crossing. Then $C$ is a 4-crossing.

**Proof.** This is the situation of Figure 4(a). Assume $C$ is a 2-crossing. If there was a 4-crossing on $f$ below $F$, we would get too many crossings on $b$. However, no 4-crossing can lie above $F$ on $c$ as well, we would get too many crossings on $f$. So we get one 4-crossing above $F$ on $f$ and one 4-crossing below $F$ on $c$. Now the cell bounded by the dashed line is
entered by ten pseudolines. This is a contradiction. Hence, C has to be a 4-crossing.

The next lemma reduces the possibilities further.

**Lemma 6.** The situation that A, C, F, and H are 4-crossings and G and I are 2-crossings cannot occur.

*Proof.* This is the situation of Figure 4(b). As can be seen in the figure, the lines coming from H that cross a give too many crossings on e. Hence the situation cannot occur. □

![Figure 5](image_url)

**Figure 5.**

Now only four cases remain, for the first two of them (see Figure 5) we refer only to the figure. The other two cases are considerably harder.

![Figure 6](image_url)

**Figure 6.**

**Lemma 7.** The situation, that the grid points D, E, and I are 2-crossings cannot occur.
Proof. This is the situation of Figure 6. No 4-crossings may lie on $b$ below $H$. Also no 4-crossings may lie on $d$ above $B$. Additionally, no 4-crossing lies on $a$ below $D$. So the two further 4-crossings on $a$ have to lie above $D$. This means, however, that no 4-crossing on $d$ can lie below $D$. Hence, the missing 4-crossing on $d$ has to lie between $D$ and $B$. Now we have five further lines – coming from the new 4-crossing and $G$ – that need to cross $b$ below $B$. We only have one further 4-crossing that lies on $b$, which can only lie in the segment denoted by the dotted line. This means that we only have two possible exit points for the above mentioned five lines. This, however, leads to a contradiction. The three lines coming from the new 4-crossing on $d$ need to cross $b$ in three pairwise distinct crossings. □

Now, we take on the last – and hardest – case.

![Diagram](a) ![Diagram](b)

**Figure 7.**

**Lemma 8.** The situation, that the grid points $G$, $H$, and $I$ are 2-crossings cannot occur.

*Proof.*** This is the situation of Figure 7(a). We see that no 4-crossing can lie on $a$ below $D$. No 4-crossing can lie on $b$ below $E$. By symmetry no 4-crossing can lie on $e$ below $E$, and no 4-crossing can lie on $f$ below $F$. The drawing in Figure 7(b) shows that the 4-crossings that are missing on $b$ and $e$ cannot both lie directly above $E$. We may assume that the missing 4-crossing on $e$ lies above $C$; we call it $J$. The other line coming from $A$ crosses $e$ in a 2-crossing, which means that only one further crossing lies on $e$.

Now we see that the five lines that come from $C$ and $J$ have to cross $f$ above $F$. They have the possibility to cross $f$ in the two missing 4-crossings on $f$, which have to lie above $F$; we call them $K$ and $L$. This takes care of
four of the five lines. However, one of the lines cannot go through \( K \) or \( L \), it yields a new 2-crossing on \( f \). So, all the crossings on \( f \) are now determined.

Now we take a closer look at \( K \) and \( L \). Of the two lines that come from \( A \) at least one has to cross \( f \) in \( K \) or \( L \) – the other one could go through \( J \) and leads to the above mentioned 2-crossing. Together with the four lines that come from \( C \) and \( J \) five of the six lines that go through \( K \) and \( L \) are determined. So exactly one of the lines that go through \( K \) and \( L \) does not cross \( e \) in \( C \) or \( J \); we call this line \( g \).

To determine the place where \( g \) crosses \( e \), we first look at the lines that come from \( F \). The two additional lines coming from \( F \) have to cross \( e \) below \( C \). However, we have already determined two 2-crossings on \( e \), and only one 4-crossing lies below \( C \). One of the lines leaving \( F \) has to cross \( e \) in \( E \), and the other has to cross it in a 2-crossing which either lies above \( E \) or below \( E \); we call these lines \( h \) and \( i \). If \( i \) crosses \( e \) above \( E \), we call the resulting crossing \( X \), and if \( i \) crosses \( e \) below \( E \), we call the resulting crossing \( Y \). All 2-crossings are now determined, so \( g \) has to cross \( e \) in \( E \).

Now we have two cases. If \( i \) crosses \( e \) in \( X \), \( i \) has first to cross \( g \). We call this crossing \( M \). This crossing \( M \), however, has to be a 2-crossing. We have already determined all lines that enter the 2-cell bordered by \( C \), \( E \), \( F \), and \( H \), and there are simply not enough of them to make \( M \) a 4-crossing. The 4-crossings \( E \) and \( F \) are adjacent on \( h \), no lines cross between them. And the segment between them borders two triangles which have a 2-crossing as the remaining vertex. So, if we take \( h \) to be the line at infinity, we are in the situation of Lemma 2(b) which settles this case.

If \( i \) crosses \( e \) in \( Y \), then \( i \) has to cross \( b \) in a 2-crossing between \( E \) and \( H \), we call it \( O \). Now we look at the points \( M \), \( F \), \( O \), \( E \), we are again in the situation of Lemma 2(b) using \( h \) as line at infinity.

So, with the proof of this lemma, we have settled all cases in which six grid points were 4-crossings. We already know, however that no more than six grid points can be 4-crossings (Lemma 4), and we also know that no less than six grid points can be 4-crossings. This concludes the proof of Theorem 2.

Remark. We can replace realizable with pl-realizable in Theorem 2.

4. Further Remarks

All \( n_4 \)-configurations up to \( n = 17 \) have been classified by Betten and Betten [11]. They have shown that there exist only 19 different \( 16_4 \)-configurations. This result gives us all possible matroids that can lead to (pl-)realizable \( 16_4 \)-configurations. Such a configuration can only be pl-realizable, when the matroid is orientable. Using software, written by the second author,
we can decide whether a matroid is orientable. The program tries to find a 
base orientation for the given matroid that satisfy the Grassmann-Plücker-
Relations. In all 19 cases found by Betten and Betten the matroid was not 
orientable, thus giving another proof of Theorem 2.

We are optimistic that further arguments in connection with computer 
support might lead to results in other cases as well.

After this article was written, Branko Grünbaum has sent us his pseudo-
line arrangements of Figure 8.

Figure 8. Pseudoline Realizations by Branko Grünbaum
ACKNOWLEDGEMENTS

We thank Branko Grünbaum for his permission to publish his drawings of pseudoline realizations for the cases $n = 26, 32, 34, 38$.

The second author was supported by a scholarship of the Deutsche Telekom Foundation.

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